ON SPINOR REPRESENTATIONS IN THE WEYL GAUGE THEORY

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Abstract

A spinor current-source is found in the Weyl non-Abelian gauge theory which does not contain the abstract gauge space. It is shown that the searched spinor representation can be constructed in the space of external differential forms and it is a 16-component quantity for which a gauge-invariant Lagrangian is determined. The connexion between the Weyl non-Abelian gauge potential and the Cartan torsion field and the problem of a possible manifestation of the considered interactions are considered.

1 Introduction

In Ref. 1 it has been shown that the congruent transference introduced by Weyl \(^2\) in 1921 defines a non-Abelian gauge field. The Weyl gauge theory is a realization of abstract theory of gauge fields in the framework of classical differential geometry which does not assume separation between space-time and a gauge space. At the same time, contemporary gauge models assume an exact local separation between space-time and a gauge field. It is just this point at that the Weyl theory opens a new possibility.

It is shown that the space of all covariant antisymmetric tensor fields is a spinor representation of the Weyl gauge group and allows the construction of a spinor current-source in a gauge theory of that type. Status of the Cartan torsion field within the Weyl gauge theory is considered from different points of view.
2 Gauge potential

The Weyl connexion which defines the congruent transference of a vector is of the form

\[ \Gamma^i_{jk} = \{i_{jk}\} + F^i_{jk}, \]

(1)

where \( \{i_{jk}\} \) are components of the Levi-Civita connexion of the metric \( g_{ij} \) usually called the Christoffel symbols:

\[ \{i_{jk}\} = \frac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}), \]

(2)

and \( F^i_{jk} = F_{jkl}g^{il} \) are components of the Weyl non-Abelian gauge potential that is a covariant third-rank tensor, skew-symmetric in the last two indices

\[ F_{jkl} + F_{jlk} = 0. \]

(3)

According to (1), a vector \( v^i \) under the congruent transference changes by the law

\[ dv^i = -\{i_{jk}\} dx^j v^k - F^i_{jk} dx^j v^k, \]

(4)

which includes the displacement belonging to the Riemann geometry and the rotation determined by the metric \( g_{ij} \) and the bivector \( F_{jkl} dx^j \). Denote by \( \nabla_i \) the covariant derivative with respect to the Weyl connexion (1). Then with allowance for (3) we obtain

\[ \nabla_i g_{jk} = 0. \]

(5)

Thus, the Weyl connexion is metric and this is in agreement with results obtained by Hayashi \(^3\), who has found that in macrophysical and microphysical systems the affine connexion cannot be nonmetric but is very likely to be metric.

The Weyl geometric construction presented above has a simple group-theoretical meaning. Let \( S^i_j \) be components of a tensor field \( S \) of type (1,1) obeying the condition \( det(S^i_j) \neq 0 \). In this case there exists a tensor field \( S^{-1} \) with components \( P^i_j \) such that \( S^i_k P^k_j = \delta^i_j \). It is obvious that the tensor field \( S \) can be regarded as a linear transformation

\[ \bar{v}^i = S^i_j v^j \]

(6)

in the space of vector fields; \( S^{-1} \) is the inverse transformation. Since under congruent transference the length of a vector remains constant,
among the transformations (6) we distinguish those that do not change the length of a vector; they are given by the equations

\[ g_{ik} S^k_j = g_{jk} P^k_i. \]  

(7)

Transformations of the form (6) and (7) form a group that is a gauge group, as will be shown below; we denote it by \( G_W \). The gauge group establishes an equivalence relation in the spaces of different fields. It can be shown that if a vector \( v^i \) in an equivalence class undergoes congruent transference, then any vector \( \bar{v}^i = S^i_j v^j \) equivalent to it in the sense of the group \( G_W \), also undergoes congruent transference. In other words, if for a \( v^i \) we have (4), then for \( \bar{v}^i = S^i_j v^j \) the formula

\[ d\bar{v}^i = -\{ i_{jk} \} dx^j \bar{v}^k - \bar{F}^i_{jk} dx^j \bar{v}^k, \]

takes place, where

\[ \bar{F}_{ikm} = F_{ij} P^i_k P^j_m + g_{ij} P^i_k \nabla_i P^j_m. \]  

(8)

In (8) and what follows \( \nabla_i \) is the covariant derivative with respect to the Levi-Civita connexion (2). From (7) it follows that the tensor \( \bar{F}_{ikm} \) obeys equation (3), and hence, the Weyl connexion

\[ \Gamma^i_{jk} = \{ i_{jk} \} + F^i_{jk} \]

is also metric. Thus, with a given metric connexion we have an entire class of equivalent metric connexions.

Consider an infinitesimal gauge transformation \( S^i_j = \delta^i_j + u^i_j \),  \( P^i_j = \delta^i_j - u^i_j \), which upon substitution into (7) gives \( g_{ik} u^k_j + g_{jk} u^k_i = 0 \). Hence it follows that any antisymmetric covariant tensor field of second rank (2-form) \( u_{ij} = -u_{ji} \) determines an infinitesimal gauge transformation since \( u^i_j = u_{jk} g^{ik} \). From (8) we obtain that an infinitesimal gauge transformations of the potential have the form

\[ \delta F_{ijk} = \nabla_i u_{jk} + F_{ijl} u^l_k - F_{ikl} u^l_j. \]  

(9)

Let us now construct the strength tensor of the gauge field

\[ B_{ijkl} = \nabla_i F_{jkl} - \nabla_j F_{ikl} + F_{ikm} F^m_{jl} - F_{jkm} F^m_{il} + R_{ijkl}, \]

(10)

where \( R_{ijkl} \) is the Riemann curvature tensor of the metric \( g_{ij} \). From (9) it follows that the strength tensor is gauge-transformed by the law

\[ \delta B_{ijkl} = B_{ikmn} u^m_l - B_{ijln} u^m_k. \]
Let us interpret the Riemann curvature tensor in expression (10) from a group-theoretical and geometric point of view. We set $B_{ijkl} = H_{ijkl} + R_{ijkl}$ and from (9) and (10) we get

$$\delta H_{ijkl} = H_{ijkm} u^m_l - H_{ijlm} u^m_k + (\nabla_i \nabla_j - \nabla_j \nabla_i) u_{kl}.$$ 

According to the Ricci identities, we find

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) u_{kl} = R_{ijkm} u^m_l - R_{ijlm} u^m_k,$$

which clearly shows the role of the Riemann curvature tensor under gauge transformations. The tensor (10) has a simple geometric meaning. It can be shown that the curvature tensor of the Weyl connexion (1) coincides with the strength tensor (10) of the Weyl non-Abelian gauge field whereas the gauge potential is considered as a deformation tensor of the Levi-Civita connexion (2).

Thus the tensor field $F_{ijk}$ entering into the Weyl connexion is a gauge field, and the tensor $B_{ijkl}$ is the strength tensor of that field. We stress that the gauge group in the case under consideration is defined by the metric, while the gauge field has a direct geometrical meaning (congruent transference) and no extra internal or gauge space is to be introduced. Here gauge symmetry reflects the fact that there does not exist any objective property that could distinguish the geometry defined by the connexion $\Gamma^i_{jk} = \{i_{jk}\} + F^i_{jk}$ from the one defined by the connexion $\bar{\Gamma}^i_{jk} = \{i_{jk}\} + \bar{F}^i_{jk}$.

### 3 Gauge-field equations

We write the gauge-invariant Lagrangian in the form

$$L = -\frac{1}{16} B_{ijkl} B^{ijkl} + \frac{1}{4} F_{ijk} S^{ijk},$$

where $S^{ijk}$ is an unknown current-source of the gauge field that should be a quadratic function of components of the quantity defining a spinor representation of the gauge group $G_W$. Variational procedure results in the following equations of the gauge field:

$$\nabla_i B^{ijkl} + F^k_{im} B^{jiml} - F^l_{im} B^{ijmk} + S^{jkl} = 0.$$ 

From these equations we derive the equations for the gauge-field current-source

$$\nabla_i S^{ijkl} + F^k_{im} S^{jiml} - F^l_{im} S^{ijmk} = 0.$$ 

4
Next, consider the current vector

$$Q^i = \frac{1}{2} v_{kl} (F^{k}_{im} B^{jml} - F^{l}_{im} B^{ijmk} + S^{jkl}).$$

From the field equations it follows that the current is conserved if the bivector $v_{ij}$ obeys the equation $\nabla_{i} v_{jk} = 0$. However, the corresponding conserved quantity is not gauge-invariant. The same holds true also in the abstract theory of gauge fields. In all the previous formulas it was assumed that the gauge potential is of dimension of the inverse of the inverse length. To introduce the constant of interaction with the gauge field, we should make the substitution $F_{ikl} \Rightarrow (\varepsilon/\bar{h}c) F_{ikl}$. In the limit $\varepsilon \Rightarrow 0$ the Lagrangian (11) transforms into the pure gravitational one

$$L = -\frac{1}{16} R_{ijkl} R^{ijkl},$$

which is known to be renormalizable.

Let us now compare the Weyl non-Abelian gauge theory with the abstract theory of gauge fields. The latter is based on an arbitrary semisimple Lie group with structure constants $f^{a}_{bc}$ and a set of vector fields. Space-time indices are raised and lowered with the metric tensor $g_{ij}$, whereas parametric indices, with the group metric $g_{ab} = f^{a}_{mn} f^{m}_{bn}$. In the Weyl non-Abelian gauge theory, the metric tensor is also a group tensor and structure constants are absent. The reason is that for some Lie groups, and for the group in question as well, the coordinates on a group can be regarded as tensor fields in space-time, which just leads to the situation when space-time and gauge space are not separated like in the abstract theory.

## 4 Spinor representation

Let us consider the field that is a source of the Weyl non-Abelian gauge field and defines a spinor representation of the group $G_W$. Consider a 16-component object which can be defined as space of all covariant anti-symmetric tensor fields $f_{i_{1}...i_{p}} (p = 0, 1, 2, 3, 4)$ on a space-time manifold with the metric $g_{ij}$. Mathematically, a shorted notation ‘differential form’ is adopted. So, the form is the following quantity

$$F = (f, f_{i}, f_{ij}, f_{ijk}, f_{ijkl}).$$
Objects of that sort were first considered in Ref.7 (see also Refs.8 - 11).

The spinor representation of the Weyl gauge group is the field of type (14). To prove this statement, we determine the natural Lagrangian for the field (14) and show that it is invariant under gauge transformations which define the symmetry aspect of the Weyl non-Abelian gauge field. We define the scalar bracket of two fields of the type (14) as follows

$$(F, H) = \bar{f}h + \bar{f}_i h^i + \frac{1}{2!} \bar{f}_{ij} h^{ij} + \frac{1}{3!} \bar{f}_{ijk} h^{ijk} + \frac{1}{4!} \bar{f}_{ijkl} h^{ijkl},$$

where the bar means complex conjugation. If $F$ is a form, the generalized curl operator $d$ is given as follows

$$dF = (0, \partial_i f, 2\partial_j f_{ij}, 3\partial_k f_{ijk}, 4\partial_l f_{ijkl}). \quad (15)$$

Here square brackets denote alternation; $\partial_i = \partial/\partial x^i$. The simplest Lagrangian for the field $F$ that can be constructed in terms of the operator $d$ is of the form

$$L_d(F) = (F, dF) + (dF, F) + m(F, F), \quad (16)$$

where $m$ is the mass of a particle ($c = \hbar = 1$). Note that the operator of external differentiation (15) is the only linear operator of first order that commutes with transformations of the group of diffeomorphisms, the group of symmetry of gravitational interactions. Therefore, the Lagrangian (16) is defined uniquely. If $\nabla_i$ is a covariant derivative with respect to the Levi-Civita connexion of the metric $g_{ij}$, defined by relations (2), then partial derivatives in (15) can be replaced by covariant derivatives. The Lagrangian (16) is not suitable for the investigation since the operator $d$ is not self-conjugate with respect to the scalar product

$$< F|H > = \int (F, H) \sqrt{-g} d^4 x.$$

Using the identity

$$\sum_{p=0}^{4} \frac{1}{p!} f_{k_{i_1} \ldots i_p} h^{k_{i_1} \ldots i_p} = \sum_{p=0}^{4} \frac{1}{p!} (p f_{i_1 \ldots i_p}) h^{i_1 \ldots i_p}$$

we can easily verify that the operator $\nabla = \delta + d$, possesses the required property, where $\delta$ is the operator of generalized divergence

$$\delta F = ( -\nabla^m f_m, -\nabla^m f_{mi}, -\nabla^m f_{mij}, -\nabla^m f_{mijk}, 0).$$
The Lagrangian (16) in terms of the operator $\nabla = \delta + d$ reads

$$L_d(F) = \frac{1}{2}(F, \nabla F) + \frac{1}{2} (\nabla F, F) + m(F, F) + \nabla_i T^i,$$

where

$$T^k = \sum_{p=0}^{4} \frac{1}{p!} (\bar{f}_{i_1 \cdots i_p} f^{k i_1 \cdots i_p} + \bar{f}^{k i_1 \cdots i_p} f_{i_1 \cdots i_p}).$$

So, the Lagrangian (16) is equivalent to the Lagrangian

$$L(F) = \frac{1}{2}(F, \nabla F) + \frac{1}{2} (\nabla F, F) + m(F, F), \quad (17)$$

which will be now analyzed. We define a numerical operator $\Lambda$, by setting

$$\Lambda F = (f, -f_i, f_{ij}, -f_{ijk}, f_{ijkl}).$$

It is not difficult to verify the validity of the following relations

$$\Lambda^2 = 1, \quad \Lambda d + d\Lambda = 0, \quad \nabla\Lambda + \Lambda \nabla = 0. \quad (18)$$

Since $\nabla d + d\nabla = \nabla^2$, then we have

$$\nabla\left(\frac{1}{2}\nabla - d\right) + \left(\frac{1}{2}\nabla - d\right)\nabla = 0. \quad (19)$$

From (18) and (19) it follows that the operator

$$\nabla^* = (\nabla - 2d)\Lambda \quad (20)$$

commutes with the operator $\nabla$, whereas their squares are equal

$$\nabla \nabla^* = \nabla^* \nabla, \quad \nabla^2 = \nabla^* \nabla^*.$$

We will call the operator $\nabla^*$ dual to the operator $\nabla$. This duality property of the field (14) allows us to introduce the important operators in the following way. In accordance with the principle of 'minimal electromagnetic interaction', we make the substitution $\nabla_i \Rightarrow \nabla_i - \frac{ie}{\hbar c} A_i$, in the operators $\nabla$ and $\nabla^*$, denote the new operators by $D$ and $D^*$, respectively, and determine their squares. We have

$$D^* \nabla^2 = \nabla^2 - \frac{ie}{\hbar c} Q(F_{ij}) + \frac{2ie}{\hbar c} A' \nabla_i + \frac{e^2}{\hbar^2 c^2} A_i A' + \frac{ie}{\hbar c} \nabla_i A' \nabla \nabla^2.$$
where $F_{ij}$, is a bivector of the electromagnetic field. A similar formula follows for the dual operator $D$ with the change of the operator $Q(F_{ij})$ by the dual operator $\ast Q(F_{ij})$. The operators $Q(F_{ij})$ and $\ast Q(F_{ij})$ are defined by antisymmetric tensor fields of second rank (2-forms). Let us write the operators $Q(u_{ij}), \ast Q(u_{ij})$ in an explicit form

\[
Q(u_{ij}) F = \left( \frac{1}{2} u^{mn} f_{mn}, \frac{1}{2} u^{mn} f_{mni} + u_{mi} f^m, \right.
\left. \frac{1}{2} u^{mn} f_{mnij} + 2 u_{m[i} f^m_{j]} - u_{ij} f, \right.
\left. 3 u_{m[i} f^m_{jk]} - 3 u_{[ij} f_{k]}, -6 u_{[ij} f_{k]}, \right)
\]

\[
\ast Q(u_{ij}) F = \left( -\frac{1}{2} u^{mn} f_{mn}, -\frac{1}{2} u^{mn} f_{mni} + u_{mi} f^m, \right.
\left. -\frac{1}{2} u^{mn} f_{mnij} + 2 u_{m[i} f^m_{j]} + u_{ij} f, \right.
\left. 3 u_{m[i} f^m_{jk]} + 3 u_{[ij} f_{k]}, 6 u_{[ij} f_{k]}, \right).
\]

It can be shown that the operators $Q(u_{ij})$ and $\ast Q(u_{ij})$ commute and this is another manifestation of the duality. Algebra of the operators $J(u_{ij}) = \frac{1}{2} Q(u_{ij})$ is closed with respect to the Lie bracket operation, i.e.

\[
[J(u_{ij}), J(v_{ij})] = J(w_{ij}),
\]

where

\[
w_{ij} = u_{im} v^m_j - u_{jm} v^m_i.
\]

From (23) it follows that the operators $J(u_{ij})$ define a representation of the considered Weyl group $G_W$ in the space of the fields (14). Since

\[
(F, J(u_{ij}) H) = -(J(u_{ij}) F, H),
\]

then the Lagrangian (17) will be invariant under the gauge transformations

\[
F \Rightarrow \tilde{F} = exp(J(u_{ij})) F,
\]

provided that

\[
[J(u_{ij}), \nabla] = 0.
\]

The relation (26) holds valid if the bivector $u_{ij}$ satisfies the equations

\[
\nabla_i u_{jk} = 0.
\]
The conditions of integrability of equations (27) follow from the Ricci identities and are of the form $R_{ijkm} u_{ml} + R_{ijkl} u_{km} = 0$. When $R_{ijkl} = K (g_{il} \delta^m_j - g_{jl} \delta^m_i)$, equations (27) will not have solutions at all. Thus, the Lagrangian (17) in the space of constant curvature will be invariant under the transformations (25) only upon introducing a gauge field of a definite type. The latter can be determined as follows. Consider variations of the type $\delta F = J(u_{ij}) F$. This class of variations, up to the Lagrange derivative, yields for the Lagrangian (17)

$$\delta L(F) = \frac{1}{4} \nabla_i u_{jk} S^{ijk},$$

where $S^{ijk}$ is a tensor field of third rank antisymmetric in the last two indices

$$S^{jkl} = \sum_{p=0}^{4} \frac{1}{p!} (\bar{f}^j_{i_1 \cdots i_p} f^{kli_1 \cdots i_p} + 2 g^{[k} f_{i_1 \cdots i_p} f^{i_1 \cdots i_p]} - 2 \bar{f}^{i_1 \cdots i_p} [k f_{i_1 \cdots i_p} f^{jkl}] + c.c.) + \frac{c.c.}{24}.$$ (28)

So, the Lagrangian (17) is to be supplemented with a term of the form

$$L_I = \frac{1}{4} F_{jkl} S^{jkl}$$

to ensure gauge invariance. We added the same term to the Lagrangian (11) of the gauge field. Thus, the explicit form of the current source of the gauge field is determined uniquely. From (28) it follows that under transformations $\delta F = J(u_{ij}) F$, the tensor $S^{jkl}$ is transformed by the law

$$\delta S^{jkl} = u^k_m S^{ijm} - u^i_m S^{jm}.$$  

Hence we obtain that the gauge field $F_{ijk}$ is transformed as follows:

$$\delta F_{ijk} = \nabla_i u_{jk} + F_{ijm} u^m_k - F_{ikm} u^m_j.$$  

According to (9), the field $F_{ijk}$ is the Weyl non-Abelian gauge field, whereas the field $F$ is shown to be its spinor source. That the transformations (25) define the spinor representation of the group $G_W$ can easily be verified by comparing them with the transformations (6). Let $a_i$ and $b_i$ be the unit orthogonal covectors, $(a, a) = (b, b) = 1$, $(a, b) = 0$, where $(a, b) = a_i b_j g^{ij}$, and $u_{ij} = \alpha (a_i b_j - a_j b_i)$, $(u = \alpha a \wedge b)$. If $Rv^i = (a_j b_i - a_i b_j) v^j$, then after some calculations we have

$$\left\{ \exp(\alpha R) \right\} v^i = v^i - a^i (a, v) - b^i (b, v) +$$
$$+\{(\cos \alpha)a^i + (\sin \alpha)b^i\}(a, v) + \{(\cos \alpha)b^i - (\sin \alpha)a^i\}(b, v).$$

For the operator $J(u_{ij})$ we have $J^2 = -(1/4)\alpha^2$ and hence

$$\exp(\alpha J(a \wedge b)) = \cos \frac{\alpha}{2} + (\sin \frac{\alpha}{2})J(a \wedge b).$$

By setting $\alpha = 0, 2\pi$, it is not difficult to verify that the vector fields are a tensor representation of the group $G_W$ whereas the space of fields (14) is the carrier space of a spinor representation of the gauge group in question.

Theory of the field $F_{ijk}$ has been already formulated above, and in the next section we dwell upon the relation between the Weyl gauge potential and the Cartan torsion. That this relation does exist follows from both the fields being tensor fields of the same type.

## 5 Torsion and gauge symmetry

At present, the torsion discovered by Cartan is the subject of numerous studies aimed at establishing its physical meaning and the connexion of general relativity with the physics of microworld. We will consider this question in the framework of the Weyl non-Abelian gauge theory. Let an affine connexion be given, and $\Gamma^i_{jk}$ be its components. Then, as it was first shown by Cartan $^{12}$, the affine connexion uniquely defines a tensor field $T^i_{jk} = (\Gamma^i_{jk} - \Gamma^i_{kj})/2$, that is called the torsion tensor. The Riemann-Cartan space-time $U_4$ is a paracompact, Hausdorff, connected $C^\infty$ 4-dimentional manifold endowed with a locally Lorentzian metric $g_{ij}$ and an affine connexion $\Gamma^i_{jk}$ which is metric

$$\partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} = 0.$$  

The Riemann-Cartan space-time has both the curvature tensor and the torsion tensor. In terms of the torsion tensor the solution of the last equation may be represented in the form

$$\Gamma^i_{jk} = \{^i_{jk}\} + T^i_{jk} + g^{il}T^m_{lj} g_{mk} + g^{il}T^m_{lk} g_{mj}. \quad (29)$$

Thus, the connexion of the Riemann-Cartan space-time is defined unambiguously.

We will show that the Cartan torsion cannot be introduced in the framework of Weyl non-Abelian gauge theory in a gauge-covariant manner. As it follows from (1), the torsion tensor of the Weyl connexion is
equal to
\[ T_{jk}^i = (F_{jk}^i - F_{kj}^i)/2 \quad (30) \]
or
\[ T_{ijk} = (F_{ijk} - F_{jik})/2, \quad (31) \]
where \( T_{ijk} = T_{ij}^l g_{lk} \). Similarly the \((31)\), for the Weyl connexion \( \Gamma_{jk}^i = \{^i_{jk}\} + \bar{F}_{jk}^i \), equivalent \((1)\), we have
\[ \bar{T}_{ijk} = (\bar{F}_{ijk} - \bar{F}_{jik})/2. \quad (32) \]

In view of \((8)\) it is entirely natural to pose the question on the relationship between the tensors \(\bar{T}_{ijk}\) and \(T_{ijk}\). However, such a relationship that contains only the tensors \(\bar{T}_{ijk}\), \(T_{ijk}\) and the elements of the gauge group does not exist. It is easy to see from the formulas
\[ \bar{F}_{ijk} = F_{ilm} P_{l}^i P_{m}^j + g_{lm} P_{l}^i \nabla_i P_{m}^j, \]
\[ \bar{F}_{jik} = F_{jlm} P_{l}^i P_{m}^k + g_{lm} P_{l}^i \nabla_j P_{m}^k. \]

Indeed, since the tensor \(F_{ijk}\) is a skew-symmetric with respect to the second and third indices, whereas the torsion tensor is skew-symmetric with respect to the first two indices, in the relation \((31)\) the index that participates in the gauge transformation \((8)\) and an index that is not affected by it are confused. Now it is clear why the torsion determines the metric connexion uniquely.

The conclusion is that the torsion tensor is not a geometrical quantity from the point of view of gauge symmetry. The tensor \(T_{jk}^i\) does not define a representation of the gauge group. It may be said that the concept of the torsion tensor is not gauge-covariant. Specifying the torsion tensor, we definitely fix the gauge. Thus, from the point of view of symmetry, the fundamental geometrical object is the tensor \(F_{ijk}\) that determines the congruent transport. It is for this tensor that the gauge-invariant equations \((12)\) are written down, which are in fact determined uniquely by the gauge symmetry. It is now easy to understand why for the torsion tensor all possible Lagrangians are encountered and investigated in literature with equal success. If one does pose the question of equations for the torsion, then it is most natural to this end to fix the gauge in accordance with what was said earlier.

We note an interesting connexion between gauge transformations and Riemannian geometry. The second term on the right-hand side of relation \((8)\) vanishes if \(\nabla_i P_{m}^i = 0\). In the standard theory of gauge fields, this
corresponds to transition from local to global transformations. In the considered case, the equations \( \nabla_i P^i_m = 0 \) may not have any nontrivial solutions at all, for example, in the case when \( g_{ij} \) is the metric of a space of constant curvature. Thus, a Riemannian geometry in general requires a local (gauge) symmetry. We note also that geometrical relationships, like physical laws, depend neither on the choice of the coordinate system nor on the choice of the basis in the studied vector spaces, so that all the relations that have been established above can be expressed in any coordinate system and in any basis, including an orthogonal one.

6 Field equations and Riemann-Cartan geometry

In this section we give an explicit example of the relation of the Weyl non-Abelian gauge field to the torsion tensor in definite gauge that is defined as follows. We take the gauge-invariant Lagrangian

\[
L = L(F) + \frac{1}{4} F_{ijk}^4
\]

and determine its variation with respect to \( F \). As a result, we have

\[
\delta L = \delta L(F) + \frac{1}{2} \sum_{p=0}^{4} \frac{1}{p!} \delta \bar{f}^{i_1 \ldots i_p} (-F^m_{m_{i_1 \ldots i_p}} + pF_{[i_1 i_2 \ldots i_p]} + \frac{p(p-1)}{2} D_{[i_1 i_2 m]} f_{m_{i_3 \ldots i_p}} - \frac{1}{3!} C_{i_1 i_2 i_3} f_{i_4 \ldots i_p} - \frac{1}{3!} p(p-1)(p-2) C_{[i_1 i_2 i_3 f_{i_4 \ldots i_p}]} + \text{c.c.})
\]

where \( F_m = g^{jk} F_{jkm} \),

\[
C_{ijk} = 3 F_{[ijk]}, \quad D_{ijk} = -C_{ijk} + 2 F_{ijk}.
\]

Indices sandwiched between vertical lines are not subject to the operation of alternation.

Next we replace the covariant derivative \( \nabla_i \) with respect to the Levi-Civita connexion in the Lagrangian \( L(F) \) by the covariant derivative \( \hat{\nabla}_i \) with respect to the connexion (29) of the Riemann-Cartan space-time. This peculiar substitution introduces the torsion field into the
Lagrangian (17). A new Lagrangian will be denoted by $L_K$. According to (29), this Lagrangian for the field $F$ in the Riemann-Cartan space can be represented as a sum of the Lagrangian (17) and an extra term to be denoted as $L_A(F)$, $L_T = L(F) + L_A(F)$. With this notes we vary the Lagrangian $L_T$ with respect to $F$ and get

$$\delta L_T(F) = \delta L(F) + \frac{1}{p!} \sum_{p=0}^{4} \delta f^{i_1 \cdots i_p} (T^m f_{m i_1 \cdots i_p}) - p T^{[i_1 f_{i_2 \cdots i_p]} - p T^{[i_1 m] f_{m i_2 \cdots i_p]} - p(p-1) T^{[i_1 i_2 f_{i_3 \cdots i_p]} + c.c.,} \quad (35)$$

where $T_i = T^m_i$ is the covector of torsion. When varying the Lagrangian $L_T$, we should take into account that

$$\nabla_i A^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} A^i) + 2 T_i A^i,$$

where $g$ is the determinant of the metric tensor. Now we raise the question about the connexion of the Euler-Lagrange equations for the field $F$ that follows from the lagrangians $L = L(F) + \frac{1}{4} F_{ijk} S^{ijk}$ and $L_T = L(F) + L_A(F)$. From comparison of (34) and (35) it can be seen that these expressions will coincide if the gauge condition $C_{ijk} = 0$ is imposed on the field $F_{ijk}$, i.e. if we set

$$F_{ijk} + F_{jki} + F_{kij} = 0$$

and then set that $F_{ijk} = -2 T_{jki}$.

Thus, we have shown that the field-$F$ equations derived by varying the Lagrangian (33) can, in a certain gauge, be represented as equations in the Riemann-Cartan space with the following constraint on the torsion tensor

$$T_{ijk} + T_{jki} + T_{kij} = 0.$$

We write the equations for the field $F$ in the Riemann-Cartan space-time

$$-\hat{\nabla}^i f_{i_1 \cdots i_p} + p \hat{\nabla}^{[i_1 f_{i_2 \cdots i_p]} + m f_{i_1 \cdots i_p} = 0,$$

where $\hat{\nabla}_i = \nabla_i - T_i$, and $p = 0, 1, 2, 3, 4$. These equations coincide with the gauge invariant equations for the field $F$ under the conditions described above.
7 Conclusions

We summarize the obtained results and present some problems. The interpretation of congruent transport given here makes it possible to establish a deep connexion between classical differential geometry and the theory of gauge fields. It is important to emphasize once more the fundamental significance of this relationship, which is that in the considered case it is not necessary to introduce an abstract gauge space. The equations for interacting fields can in fact be uniquely derived. The relations established for the Weyl non-Abelian gauge field and the Cartan torsion make it possible to consider, from a new point of view, the problem of physical interpretation of the torsion in the framework of the gauge principle.

The existence of the spinor source of the Weyl gauge field is an interesting feature of this field that dictates the question about possible physical manifestations of this kind of interactions. In the Minkowski space-time equations (27) are quite integrable. Thus, the gauge symmetry can be considered in this case as a global one. With respect to this global symmetry a space of forms (14) is reducible. Associated reduction of the space of forms (14) gives the Dirac theory in which we find only well known interactions. In contrast with this case, there is a more interesting possibility, when equations (27) have no solutions at all. As it was mentioned above, this situation occurs in the space of constant curvature, where the appearance of the Weyl non-Abelian gauge field in a definite sense becomes necessary because of the full absence of global internal symmetry. A very interesting space-time of this kind is the de Sitter one which is usually considered as a cosmological model. So, the Weyl non-Abelian forces could be manifested on the cosmological scale. Of course, this do not close the realm of microphysics. The general remark is that all questions and problems discussed in literature in relation to the physical interpretation of torsion can be investigated in a more suitable framework of the Weyl non-Abelian gauge theory.

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