Dynamically encircling an exceptional curve by crossing diabolic points: A programmable multimode switch

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We propose a programmable multimode switch based on dynamically encircling an exceptional curve (EC) while crossing diabolic points in non-Hermitian systems. ECs are the one-dimensional extension of exceptional points (EPs). Recent studies for two-mode systems have shown that, by dynamically winding around EPs, mode switching can be realized by simply changing the encircling direction. However, for multimode systems with higher-order EPs, the situation can be more involved. That is, due to the breakdown of adiabadicity, the ability to directly swap between two desired modes on demand can be impeded. Here we demonstrate that this difficulty can be overcome by winding around ECs, whose trajectories can additionally cross diabolic points, allowing, thus, to implement a programmable multimode switch. We propose a four-mode $\mathcal{PT}$-symmetric bosonic system as a platform for experimental realization of this multimode switch. Our work opens new perspectives for light manipulations in non-Hermitian photonic setups.

I. INTRODUCTION

Physical systems that are described by non-Hermitian Hamiltonians (NHHs), have attracted much research interest during the last two decades thanks to their peculiar spectral properties. Namely, such systems can possess exotic spectral singularities referred to as exceptional points (EPs). While in classical and semiclassical systems EPs are associated with the coalesce of both the eigenvalues and the corresponding eigenmodes of an NHH (thus, referred to as Hamiltonian EPs) [1, 2], in quantum systems they are associated with eigenvalue degeneracies and the coalescence of the corresponding eigenmatrices of a Liouvillian superoperator (hence, Liouvillian EPs) [3]. The latter takes into account the effects of decoherence, quantum jumps, and associated quantum noise.

In addition to EPs, physical systems can also exhibit diabatic point (DP) spectral degeneracies where eigenvalues coalesce but the corresponding eigenstates remain orthogonal. Although they are often referred to as Hermitian spectral degeneracies and studied in Hermitian systems, it is well-known that DPs can emerge in non-Hermitian systems, too.

The term DP was coined in Ref. [4] referring to the degeneracies of energy levels of two-parameter real Hamiltonians. Graphically, such a typical DP corresponds to a double-cone connection between energy-level surfaces resembling a diabolo toy, which justifies the DP notion.

Analogously to EPs, this original definition of DPs was later generalized to the eigenvalue degeneracies of non-Hermitian Hamiltonians (see, e.g., [5]) in case of DPs of classical or semiclassical systems, and of Liouvillians [3] in case of quantum systems. Note that quantum jumps are responsible for a fundamental difference between semiclassical and quantum EPs/DPs, and the effect of quantum jumps can be experimentally controlled by postselection [6].

EPs have been predicted and observed in different experimental platforms [1, 6–24]. It seems that DPs in non-Hermitian systems have been attracting relatively less interest than EPs in recent years (see, e.g., [1, 20, 25, 26]). The reported demonstrations of a Berry phase (with a controlled phase shift), acquired by encircling a DP [27–29], can lead to applications in topological photonics [30], quantum metrology [31], and geometric quantum computation in the spirit of Refs. [32–35]. Note that the Berry curvature (i.e., the “curvature” of a certain subspace) can be nonzero for non-Hermitian systems and, thus, can be used for simulating effects of general relativity [36–38].

The emergence of geometric Berry phases is quite common in non-Hermitian systems, but the acquired phases can be largely enhanced by encircling DPs or EPs [39–41]. Moreover, DPs and EPs are useful in testing and classifying phases and phase transitions [42, 43]. For example, a Liouvillian spectral collapse in the standard Scully-Lamb...
Recent studies on EPs have also shown that by exploiting a nontrivial topology in the vicinity of EPs in the energy spectrum can lead to a swap-state effect, where the initial state does not come back to itself after a round trip around an EP. Such phenomenon has been predicted theoretically \cite{46, 47} and observed experimentally in \cite{23, 39, 48–50}, while performing ‘static’, i.e., independent, measurements at various locations in the system parameter space. However, when encircling an EP dynamically, another intriguing effect can be invoked; namely, a chiral mode behavior, such that a starting state, after a full winding period, can eventually return to itself \cite{51–54}. The latter effect stems from the breakdown of the adiabatic theorem in non-Hermitian systems \cite{51, 55}. This asymmetric mode switching phenomenon has also been experimentally confirmed in various experimental platforms \cite{40, 56–60}. A number of studies have demonstrated the practical feasibility to observe the chiral light behavior on a pure quantum level \cite{61} and even in a so-called hybrid mode \cite{62}, where by exploiting various measurement protocols, one can switch between the system dynamics described by a quantum Liouvillian and the corresponding classical-like effective NHH.

Other works, both theoretical \cite{53} and experimental \cite{63}, have pointed that crucial ingredient in detecting a dynamical flip-state asymmetry is the very curved topology near EPs. In other words, it is not necessary to wind around EPs in order to observe such phenomena. More recently, much effort is put on studying the behavior of modes while encircling high-order or multiple EPs in a parameter space. Indeed, the presence of high-order or multiple low-order EPs in a system spectrum, along with the non-Hermitian breakdown of adiabaticity, can impose a substantial difficulty to manipulate the mode-switching behavior on demand, i.e., to implement a programmable mode switch \cite{54, 64}.

In this work we demonstrate that winding around exceptional curves (ECs), whose trajectories can additionally cross diabolic curves (DCs), provides a feasible and easy to implement route to realize a programmable multimode switch. We use a four-mode parity-time ($PT$)-symmetric bosonic system, which is governed by an effective NHH, as an example platform to demonstrate this programmable switch. At the crossing of EC and DC a new type of a spectral singularity is formed, referred to as diabolically degenerate exceptional points (DDEPs) \cite{65}. Remarkably, by exploiting the presence of DDEPs in dynamical loops of the system parameter space, one can restore the swap-state symmetry, which breaks down along the breakdown of adiabaticity in two-mode non-Hermitian systems. This implies that the initial state can eventually return to itself after a state flip in a double cycle. In other words, one can control the chiral behavior of modes in such systems by crossing or avoiding a DDEP.

II. THEORY

We start from the construction of a four-mode NHH, possessing both exceptional and diabolic degeneracies. For this, we follow the procedure described in \cite{65}, where one can construct a matrix, whose spectrum is a combination of the spectra of two other matrices by exploiting Kronecker sum properties. Namely, by taking two $PT$-symmetric matrices

\[
M_1 = \begin{pmatrix}
    i\Delta & k \\
    k & -i\Delta
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
    0 & g \\
    g & 0
\end{pmatrix}, \quad (1)
\]

one can form a $PT$-symmetric $4 \times 4$ non-Hermitian matrix $H$, produced by the Kronecker sum of two matrices $M_{1,2}$, namely,

\[
H = M_1 \otimes I_2 + I_2 \otimes M_2, \quad (2)
\]

where $I_2$ is the $2 \times 2$ identity matrix. Explicitly, the matrix $H$ reads

\[
H = \begin{pmatrix}
    i\Delta & g & k & 0 \\
    g & i\Delta & 0 & k \\
    k & 0 & -i\Delta & g \\
    0 & k & g & -i\Delta
\end{pmatrix}. \quad (3)
\]

The $PT$-symmetry operator is expressed via the parity operator $P = \text{antidiag}[1, 1, 1, 1]$, and $T$ as the complex conjugate operation, corresponding to the time-reversal action, thus implying $PTHPT^{-1} = H$. The matrix $H$ can be related to a linear four-mode NHH operator $\hat{H}$, written in the mode representation, i.e.,

\[
\hat{H} = \sum \hat{a}_i^\dagger H \hat{a}_k,
\]

where $\hat{a}_i$ ($\hat{a}_i^\dagger$) are the annihilation (creation) operators of bosonic modes $i = 1, \ldots, 4$. Such an NHH can be associated, e.g., with a system of four coupled cavities or waveguides [see Fig. 1(a)]. A somewhat similar scheme, based on two lossy and two amplified subsystems, has been proposed in Ref. \cite{66} to generate high-order EPs. However that scheme has different coupling configuration and spectrum which does not possess DPs.

The peculiarity of such a non-Hermitian Hamiltonian $H$ is that its eigenvalues are just sums of the eigenvalues of $M_1 (\pm \sqrt{k^2 - \Delta^2})$ and $M_2 (\pm g)$ \cite{65, 67}. Namely,

\[
E_{1,2,3,4} = \mp \sqrt{k^2 - \Delta^2} \mp g. \quad (4)
\]

In what follows, we always list eigenvalues in ascending order, i.e.,

\[
\text{Re}(E_1) \leq \text{Re}(E_2) \leq \text{Re}(E_3) \leq \text{Re}(E_4).
\]

The corresponding eigenvectors of $H$ are simply formed by the tensor products of eigenvectors of $\psi_{j}^{M_1}$ and $\psi_{k}^{M_1}$ ($j, k = 1, 2$) of the two matrices $M_1$ and $M_2$, respectively,

\[
\psi_{1,2}^{M_1} = \left( \pm \exp(\pm i\phi) \begin{array}{c} 1 \\ 1 \end{array} \right), \quad \psi_{1,2}^{M_2} = \left( \begin{array}{c} \pm 1 \\ 1 \end{array} \right). \quad (5)
\]
FIG. 1. (a) Schematic representation of a four-mode \( \mathcal{PT} \)-symmetric non-Hermitian Hamiltonian \( \hat{H} \), given in Eq. (3). The red (blue) balls represent cavities with gain (loss) rate \( i\Delta \) (\(-i\Delta \)). Various mode couplings are depicted by double arrows. (b) The encircling trajectory is described by a loop in the 3D parameter space defined by the dissipation strength \( \Delta \), perturbation \( \delta \), and coupling \( g \). The counterclockwise (clockwise) direction is determined by \(+\omega\) (\(-\omega\)). The encircling starts at \( t_0 \) at a point in the exact \( \mathcal{PT} \)-phase (the orange ball). The loop winds around an exceptional curve, EC (red vertical line), determined by the condition \( \Delta = 1 \) and \( \delta = 0 \).

The trajectory may cross a diabolic curve, DC (green horizontal line), at some point when \( g = 0 \), i.e., a diabolic point, DP. Moreover, at \( g = 0 \), a diabolically degenerate exceptional point, DDEP, is formed, at the intersection of EC and DC. Note that in this 3D parameter space, the DC is presented as a line.

Namely, the eigenvector \( \psi_{jk}^H = \psi_{j1}^{M_1} \otimes \psi_{k2}^{M_2} \) corresponds to the eigenvalue \( E_{jk}^H = E_{j1}^{M_1} + E_{k2}^{M_2} \) of the matrix \( \hat{H} \).

The spectrum of this \( \mathcal{PT} \)-symmetric \( \hat{H} \) has two types of degeneracies:

- a pair of second-order ECs at \( k = \Delta \), determined by \( \pm g \) (\( g \neq 0 \)),
- and a pair of DCs at \( g = 0 \), defined by \( \pm \sqrt{k^2 - \Delta^2} \).

III. ENCIRCLING AN EXCEPTIONAL CURVE BY CROSSING DIABOLIC POINTS IN A FOUR-MODE \( \mathcal{PT} \)-SYMMETRIC BOSONIC SYSTEM

In order to implement the dynamical winding around ECs, one may apply a perturbation \( \delta(t) \) to the NHH \( \hat{H} \) in the following form:

\[
\hat{H}(\delta) = \begin{pmatrix}
(i\Delta(t) + \delta(t)) & g(t) & 1 & 0 \\
g(t) & i\Delta(t) & 0 & 1 \\
1 & 0 & -i\Delta(t) & g(t) \\
0 & 1 & g(t) & -i\Delta(t) - \delta(t)
\end{pmatrix},
\]

where we set \( k = 1 \), i.e., the coupling \( k \) determines a unit of the system energy. The time-dependent parameters are:

\[
\Delta(t) = 1 + \cos(\omega t + \phi_0), \\
g(t) = g_0 \sin^2(\omega t/2 + \phi_0/2), \\
\delta(t) = \sin(\omega t + \phi_0).
\]

The angular (winding) frequency is \( \omega = 2\pi/T \), with period \( T \), and an initial phase \( \phi_0 \). The perturbation \( \delta \) can play the role of the frequency detuning in the first and fourth cavities. Note that other choices of perturbation are also allowed, although they can lead to a different energy distribution in the perturbed parameter space.

The energy spectrum of \( H(\delta) \) consists of two pairs of Riemann sheets. For real-valued energies, these pairs may or may not intersect, depending on the system parameters, as shown in Figs. 2(a) and 3. Whereas for imaginary-valued \( E \), these pairs always coincide, as follows from Eq. (4) [see also Fig. 2(b)].

FIG. 2. Real (a) and imaginary (b) parts of the spectrum of the non-Hermitian Hamiltonian, NHH, \( H(\delta) \) in Eq. (6). For real-valued energies, the spectrum of the NHH is formed by two pairs of Riemann surfaces. Whereas for the imaginary-valued spectrum, those two pairs coincide. Each pair of Riemann sheets, for a given value of \( g \), has a branch cut at an EP determined by the conditions \( \Delta = 1 \) and \( \delta = 0 \). The system parameters are: \( k = 1 \) and \( g = 2 \).
In order to determine the time evolution of a wave function $\psi$, during a dynamical cycle, we solve the time-dependent Schrödinger equation

$$i\frac{\partial \psi(t)}{\partial t} = \hat{H}(t)\psi(t). \quad (8)$$

Here, we focus solely on the mode switching behavior in the stable exact $\mathcal{PT}$-phase, where the eigenvalues $E_k$ are real-valued, thus, representing propagating fields without losses. That is, the dynamical encircling starts in the exact $\mathcal{PT}$-phase (i.e., $\Delta < 1$). We note here that our results can also be extended to purely lossy modes. This can be achieved by mapping the NHH in Eq. (6) to its anti-$\mathcal{PT}$-symmetric counterpart, i.e., $H' = -iH$. In this case, the role of freely propagating fields in $\mathcal{PT}$-symmetric systems is played by purely decoherent fields in the corresponding anti-$\mathcal{PT}$-symmetric systems [59].

The encircling loop moves in a 3D-parameter space, spanned by the dissipation rate $\Delta$, the detuning perturbation $\delta$, and the coupling $g$ [see Fig. 1(b)]. Encircling one of the ECs (e.g., $+g$) automatically ensures that EC ($-g$) is also encircled due to the system symmetry. For a given fixed value $g$, there is a distance [2$g$] between two EPs, belonging to the two ECs [see, e.g., Fig. 3(b)].

The initial point ($t_0 = 0$), from which the encircling trajectory starts, is located in the exact $\mathcal{PT}$-phase, i.e., $\phi_0 = \pi$ [see Figs. 1(b) and 3(a)]. The winding process can be performed in the counterclockwise ($+\omega$) or clockwise ($-\omega$) direction. By appropriately modulating $g(t)$, one can make the encircling trajectory to pass through the DC at some point, (i.e., a DP), when $g = 0$ in the broken $\mathcal{PT}$-phase ($\Delta > 1$) [as shown in Fig. 1(b)]. Note that the two DCs coincide in the broken $\mathcal{PT}$-phase for real-valued energies [see Fig. 3(c)]. A single dynamical loop, thus, corresponds to the splitting-crossing-splitting behaviour for the two pairs of the Riemann energy sheets [as shown in panels (b)-(d) in Fig. 3]. The intersection of the sheets occurs at the DC.

Figure 3(b) depicts the initial state at $t = 0$, when $g \neq 0$, and the spectrum of the NHH $\hat{H}$ consists of two disconnected pairs of Riemann sheets (for real-valued $E$), where each pair is formed around a second-order EP (with characteristic branch cuts) [see Fig. 3(b)]. As mentioned above, depending on the coupling $g \neq 0$, these Riemann pair sheets can cross [as shown in Fig. 3(b)]. The state can be initialized in one of the four different eigenmodes in the exact $\mathcal{PT}$-phase.

If $g \neq 0$ is either modulated such that the two separated pairs of the real-valued Riemann sheets do not cross at the DC [as presented in Fig. 3(b)] or it is kept fixed, then the dynamical loop is similar to the case of two independent two-mode systems, for which the dynamical winding around an EP results in the well-known two-mode asymmetric switching [56]. This means that, in this specific case, the eigenmodes $\psi_1 \leftrightarrow \psi_3$ and $\psi_2 \leftrightarrow \psi_4$, which belong to the separated pairs of Riemann sheets [61], are swapped.

Interestingly, in order to realize any desired mode
FIG. 4. Programmable four-mode switch by dynamically encircling an exceptional curve, EC, while crossing a diabolic curve, DC, as indicated by the fidelity of the initial, \( \psi_k \), and time-evolving, \( \psi(t) \), states during a double period \( 2T \). The initial eigenmodes \( \psi_k, k = 1, \ldots, 4 \), are located in the exact \( PT \)-phase [see also Fig. 3(b)]. Counterclockwise [panels (a)-(d)] and clockwise [panels (e)-(h)] encircling directions. Depending on the winding direction and the number of times the loop encircles the EC with the crossed DC, one can realize various mode-switching combinations. These illustrated various mode-switching combinations are summarized in Table I. The system parameters are: \( \phi_0 = \pi \), \( \omega t = \pi t/40 \), and \( g_0 = 0.5 \).

TABLE I. Programmable four-mode switch, a summary of Fig. 4. By initializing a state in one of the system eigenmodes \( \psi_k \) (first column), one can switch to any final eigenstate \( \psi_j \) (first row) of the system by appropriately choosing an encircling trajectory, which winds around the EC and traverse the DC. The order and meaning of the values and symbols in each cell is the following: the values \( 1, 2 \) denote the number of times one winds around the EC and the symbols \( \Leftarrow \) and \( \Rightarrow \) denote counterclockwise and clockwise encircling directions, respectively (see Fig. 4).

| Initial | Final  |
|---------|--------|
| \( \psi_1 \) | \( \psi_1 \) | \( \psi_2 \) | \( \psi_3 \) | \( \psi_4 \) |
| \( \psi_2 \) | \( \Leftarrow \) | \( \Rightarrow \) | \( \Leftarrow \) | \( \Rightarrow \) |
| \( \psi_3 \) | \( \Leftarrow \) | \( \Rightarrow \) | \( \Leftarrow \) | \( \Rightarrow \) |
| \( \psi_4 \) | \( \Leftarrow \) | \( \Rightarrow \) | \( \Leftarrow \) | \( \Rightarrow \) |

Therefore, by choosing an appropriate winding loop, one can switch between various desired eigenmodes on demand. The latter fact is in striking contrast with high-order EPs, where arbitrary mode switching is hard to realize either due to the breakdown of adiabaticity [58], or because the mode-shift can be performed only in the ordered and/or stroboscopic (static) way [54, 64]. The above conclusions are confirmed numerically in Fig. 4, where the values of the fidelity \( F = |\langle \psi_k(0) | \psi(t) \rangle|^2 \) of the initial \( |\psi_k(0)\rangle \) and the time-evolving \( |\psi(t)\rangle \) states are shown. We also summarize our results in Table I, demonstrating the realization of programmable mode switching in the \( PT \)-symmetric four-mode bosonic system.

Note that mode switching between \( \psi_1 \leftrightarrow \psi_3 \) and \( \psi_2 \leftrightarrow \psi_4 \) [as depicted in Fig. 3(b)] can also be realized without traversing the DC. In that case the situation is similar to the ‘independent’ pairs of two-mode \( PT \)-symmetric dimers [61]. However, and most remarkably, by exploiting the presence of a diabolic degeneracy, that particular mode switching can become symmetric, in striking contrast to the flip-state asymmetry in a \( PT \)-symmetric dimer. That is, by crossing the DC one can restore the swap-state symmetry in the otherwise asymmetric two-mode setups.

IV. DISCUSSION AND CONCLUSIONS

The observed programmable mode switching mechanism is enabled by the interplay between the presence...
shown that, by exploiting the nontrivial interplay between exceptional and diabolic point degeneracies and non-adiabaticity in non-Hermitian systems, it is possible to swap between any desired pairs of the system eigenvalues. Our findings are not limited to free propagating fields in $\mathcal{PT}$-symmetric systems, but are also applicable to purely lossy fields in anti-$\mathcal{PT}$-symmetric setups. Our work, thus, opens new perspectives for light manipulations in various photonic setups.

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