A remark on the simple cuspidal representations of $GL(n, F)$

Peng Xu

Abstract

Let $F$ be a non-archimedean local field of residue characteristic $p$, and let $G$ be the group $GL(n, F)$. In this note, under the assumption $(n, p) = 1$, we show that a simple cuspidal representation $\pi$ (that is with normalized level $\frac{1}{n}$) of $G$ is determined uniquely up to isomorphism by the local constants of $\chi \circ \det \otimes \pi$, for all characters $\chi$ of $F^\times$.

Keywords. simple cuspidal representations; local constants

1 Introduction

Let $F$ be a non-archimedean local field with integer ring $\mathfrak{o}_F$ and maximal ideal $\mathfrak{p}_F$ and assume its residue field $k_F = \mathfrak{o}_F / \mathfrak{p}_F$ is of order $q$ and of characteristic $p$. Fix a prime element $\varpi$ and a root of unity $\eta$ of order $q - 1$ in $F$. Let $G$ be the general linear group $GL(n, F)$. In this short note, we investigate some aspect of simple cuspidal representations of $G$, especially the behaviour of their local constants under twists by characters of $F^\times$.

The main result is the following Theorem 1.1, which in particular verifies a very special case of Jacquet’s conjecture on the local converse theorem of $G$ ([CPS94]).

We fix a level one additive character $\psi$ (i.e., $\psi$ is trivial on $\mathfrak{p}_F$ but non-trivial on $\mathfrak{o}_F$) of $F$.

In this note, for a cuspidal representation $\pi$ of $G$ to be simple, we mean it has normalized level $l(\pi) = \frac{1}{n}$. Denote $\chi \circ \det \otimes \pi$ by $\chi\pi$ as usual.

**Theorem 1.1.** Assume $(n, p) = 1$. Let $\pi_1$ and $\pi_2$ be two cuspidal representations of $G$, such that

$$\varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi),$$

for all characters $\chi$ of $F^\times$ and $s \in \mathbb{C}$, which forces $\pi_1$ and $\pi_2$ to have the same normalized level $l$. If further $l = \frac{1}{n}$, then
Remark 1.2. The tameness condition \((n, p) = 1\) is crucially used in the argument, but it is reasonable to believe the result should hold without it.

Remark 1.3. In a recent preprint [AL], Moshe Adrian and Baiying Liu have also obtained the same result as Theorem 1.1, via a different method.

2 Preliminary facts

In this section, we recall some well-known facts, for which we also include a sketched proof and detailed references.

Proposition 2.1. Let \(\pi_1\) and \(\pi_2\) be two cuspidal representations of \(G\), such that
\[
\varepsilon(\chi \pi_1, s, \psi) = \varepsilon(\chi \pi_2, s, \psi),
\]
for all characters \(\chi\) of \(F^\times\) and \(s \in \mathbb{C}\). Then

(i) The identity (2) holds when \(\psi\) is replaced by any additive character of \(F\).

(ii) \(\pi_1\) and \(\pi_2\) have the same central characters, i.e., \(\omega_{\pi_1} = \omega_{\pi_2}\).

(iii) \(\pi_1\) and \(\pi_2\) have the same normalized level, i.e., \(l(\pi_1) = l(\pi_2)\).

Proof. (i) is direct from the definition, combined with (2). For an irreducible cuspidal representation of \(G\), the identity
\[
\varepsilon(\pi, s, \psi) = q^{n(l(\pi))(\frac{1}{2} - s)} \varepsilon(\pi, \frac{1}{2}, \psi)
\]
holds (see 6.1.2 in [BHK98] and note that \(\psi\) is chosen to be level one), from which (iii) follows.

(ii) follows from the following Lemma, as in 27.4 of [BH06].

Lemma 2.2. Let \(\pi\) be a cuspidal representation of \(G\) and let \(\chi\) be a character of \(F^\times\), such that,
\[
m = l(\chi) > 2l(\pi),
\]
where \(l(\chi)\) is the level of \(\chi\). Let \(c\) be an element in \(F^\times\) such that \(\chi(1 + x) = \psi(c \cdot x)\) for \(x \in p^{\lfloor \frac{l(\pi)}{2} \rfloor + 1}\), then
\[
\varepsilon(\chi \pi, s, \psi) = \omega_{\pi}(c)^{-1} \varepsilon(\chi \circ \det, s, \psi).
\]

Proof. This is a minor refinement of a Lemma of Jacquet-Shalika [JS85]. We include a detailed proof in Appendix 4 for the reader’s convenience, following [BH06].
This completes the proof of Proposition 2.1.

Let \( P_n(F) \) be the set of isomorphism classes of admissible pairs of degree \( n \), and \( A_n^\text{ct}(F) \) be the set of isomorphism classes of essentially tame cuspidal representations of \( G \). For the exact definitions of admissible pairs and essentially tame cuspidal representations, see [BH05].

**Theorem 2.3.** There is a natural bijection between \( P_n(F) \) and \( A_n^\text{ct}(F) \).

**Proof.** Theorem 2.3, [BH05].

Denote by \( \pi_{E,\vartheta} \) the cuspidal representation which arises from an admissible pair \( (E/F, \vartheta) \), via the above theorem. The content we need from Theorem 2.3 of [BH05] is summarized in the following proposition:

**Proposition 2.4.** (i) \( l(\pi_{E,\vartheta}) = \frac{l(\vartheta)}{e(E/F)} \), where \( e(E/F) \) is the ramification index of \( E/F \).
   (ii) \( \omega_{\pi_{E,\vartheta}} = \vartheta|_{E^\times} \).
   (iii) \( \chi \cdot \pi_{E,\vartheta} \cong \pi_{E,\chi E \cdot \vartheta} \), where \( \chi \) is a character of \( E \times \) and of level \( 2k+1 \) for some \( k \geq 0 \).

**Proof.** (ii) and (iii) are the contents of Proposition 2.4, [BH05]. (i) can be easily concluded from the constructions in 2.3, [BH05].

For the argument in Section 3, we need to recall the local constant of a simple cuspidal representation \( \pi \) arising from an admissible pair \( (E/F, \theta) \).

We assume \((n,p) = 1\) in the following Proposition.

**Proposition 2.5.** Let \( \pi_{E,\theta} \) be a cuspidal representation arising from an admissible pair \( (E/F, \theta) \), where \( E/F \) is a totally ramified extension of degree \( n \) (hence \((n,p) = 1\)) and \( \theta \) is a character of \( E^\times \) and of level \( 2k+1 \) for some \( k \geq 0 \). Choose \( \alpha \in F_{E}^{(2k+1)/k} \), such that \( \theta(1+x) = \psi_{E/F}(\alpha x) \) for \( x \in F_{E}^{k+1} \), where \( \psi_{E/F} \) is \( \psi \circ \text{tr}_{E/F} \). Then,

\[
\varepsilon(\pi_{E,\theta}, 1, 1, \psi) = \vartheta(\alpha)^{-1} \psi_{E/F}(\alpha).
\]

**Proof.** This follows directly from Proposition 1 of 6.3 in [BH99] and the construction of \( \pi_{E,\theta} \) from an admissible pair \( (E/F, \theta) \) (2.3, [BH05]).

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1To be consistent, we use the notations of 2.3 of [BH05]. In 2.3 of [BH05], the cuspidal representation \( \varphi \pi_{\xi} \) constructed from an admissible pair \( (E/F, \xi) \) contains some simple character arising from a simple stratum \([\mathfrak{A}, l, 0, \beta]\), where \( l = l(\phi) \), and \( \phi \) is a character of \( E^\times \) chosen to satisfy \( \xi \mid U_{E}^{\phi} = \phi \circ N_{E'/E} \). As \( E'/E \) is unramified, one has \( l(\phi) = l(\xi)(\geq 1) \). Recall the definition of normalized level, the construction in 2.3 of [BH05] implies \( l(\varphi \pi_{\xi}) = \frac{l(\phi)}{e(E/F)} \).
3 Proof of Theorem 1.1

The strategy of using admissible pairs was suggested by Professor Guy Henniart.

3.1 Reduction to the same field extension

From now on, we assume \((n,p) = 1\).

Let \(\pi_1\) and \(\pi_2\) be two cuspidal representations of \(G\) of level \(\frac{2k+1}{n}\), where \(k \geq 0\) and \((n,\frac{2k+1}{n}) = 1\), which satisfy \(\omega_{\pi_1} = \omega_{\pi_2}\). For \(i = 1, 2\), assume \((E_i/F, \theta_i)\) is an admissible pair associated to \(\pi_i\) via Theorem 2.3. Then from Proposition 2.4, \(E_i/F\) is a totally ramified extension of degree \(n\) and \(\theta_i\) is a character of \(E_i^\times\) of level \(2k+1\). Also, from Proposition 2.4, the restrictions of \(\theta_1\) and \(\theta_2\) to \(F^\times\) coincide.

Proposition 3.1. Assume \(\pi_1\) and \(\pi_2\) satisfy the condition (1) in Theorem 1.1. Then \(E_1\) is isomorphic to \(E_2\) over \(F\).

Proof. From Chapter 16 of [Has80], there are in all \(e = (n,q-1)\) different totally tamely ramified extension of degree \(n\) over \(F\), which can be described as:

\[F(\sqrt[n]{\wp^r}), \quad 0 \leq r < e.\]

Hence we assume \(e > 1\). Assume \(E_1\) and \(E_2\) are different over \(F\). Without loss of generality, we may then assume further that

\[E_1 = F(\sqrt[n]{\wp}), \quad E_2 = F(\sqrt[n]{\wp^a\eta^b}),\]

where \(0 < a < e\).

We note \(\wp_1 = \sqrt[n]{\wp}\) and \(\wp_2 = \sqrt[n]{\wp^a\eta^b}\) are respectively prime elements in \(E_1\) and \(E_2\).

Write \(2k+1 = a'n + b\), for \(a' \geq 0, \quad 0 < b \leq n - 1\). Note that \(b\) is coprime to \(n\). The additive character \(\psi_{E_i} = \psi \circ \text{tr}_{E_i/F}\) of \(E_i\) is also of level one, as \(E_i/F\) is a tame extension and \(\psi\) is of level one.

As \(\theta_i\) is of level \(2k+1\), there is a unique \(\alpha_i + p_{E_i}^{-k} \in p_{E_i}^{-(2k+1)} / p_{E_i}^{-k}\) such that

\[\theta_i(1 + x) = \psi_{E_i/F}(\alpha_i x), \quad \text{for} \quad x \in p_{E_i}^{k+1}.\]

Write \(\alpha_i\) as \(\omega^{-a'}\omega_i^{-b}\eta^a\beta_i\), for some \(0 \leq a_i < q\) and some \(\beta_i \in U_{E_i}^1 = 1 + p_{E_i}\).

Then, using the assumption on the local constants of \(\pi_i\) for twists by level zero characters \(\chi\) of \(F^\times\), we are given a family of identities from Proposition 2.5

\[\theta_1 \cdot \chi_{E_i}(\alpha_1)^{-1}\psi_{E_1/F}(\alpha_1) = \theta_2 \cdot \chi_{E_2}(\alpha_2)^{-1}\psi_{E_2/F}(\alpha_2).\] (4)
We emphasize that $\chi$ is chosen to be of level zero, which is the reason that one can still use $\alpha_i$ in both sides of (4).

Then from (4), we get

$$
\chi(\eta^{n(a_1-a_2)-ab}) = \theta_1(\alpha_1)^{-1}\psi_{E_1/F}(\alpha_1)\psi_{E_2/F}(-\alpha_2)\theta_2(\alpha_2). \tag{5}
$$

The left hand side of (5) cannot be constant when $\chi$ goes through all the level zero characters of $F^\times$, as under our assumption $q-1$ does not divide $n(a_1-a_2) - ab$. We get a contradiction. \hfill \Box

3.2 The case of simple cuspidal representations ($k = 0$)

With the same assumptions as in the last subsection, we carry on to prove the admissible pairs of $\pi_1$ and $\pi_2$ are isomorphic over $F$ when $k = 0$. Hence we prove in this case that $\pi_1 \sim \pi_2$ by Theorem 2.3.

From Proposition 3.1, one can take $E_1 = E_2 = E = F(\sqrt[ n]{\eta})$. Denote $\sqrt[ n]{\eta}$ by $\eta_E$.

Now we repeat a bit more from the last section. Choose $\alpha_i + \sigma_E \in p_E^{-1}/\sigma_E$ such that

$$
\theta_i(1 + x) = \psi_{E/F}(\alpha_i x), \text{ for } x \in p_E.
$$

We note the choice of $\alpha_i$ is up to multiplication by $U_E^1$. In writing $\alpha_i$ as $\bar{\eta}_E^{a_i}\beta_i$, for some $0 \leq a_i < q$ and some $\beta_i \in U_E^1 = 1 + p_E$, we may assume $\beta_i = 1$. Also we know $\text{tr}_{E/F}(\bar{\eta}_E^c) = 0$ when $n \nmid c$. Hence,

$$
\psi \circ \text{tr}_{E/F}(\alpha_i) = 1.
$$

In all, we get a simplified version of (4)

$$
\chi(\eta)^{n(a_1-a_2)} = \theta_1(\bar{\eta})^{-1}\theta_2(\bar{\eta})\xi^{a_1-a_2},
$$

where $\xi_\eta = \theta_1(\eta) = \theta_2(\eta)$. The left hand side of the above equation is constant for all $\chi$ of level zero, only if $q-1$ divides $n(a_2-a_1)$; as a result $\eta^{a_1-a_2}$ is an $n$-th root of unity in $F$.

Denote by $\sigma$ the automorphism of $E$ over $F$, determined by sending $\bar{\eta}_E$ to $\bar{\eta}_E \cdot \eta^{a_1-a_2}$ (which is a conjugate of $\bar{\eta}_E$). Then one can easily check

$$
\theta_1 = \theta_2 \circ \sigma.
$$

This complete the proof of Theorem 1.1.

Remark 3.2. As we have seen, under the assumption $k = 0$ the situation is essentially simplified, which makes the final argument completely elementary. However, once $k$ becomes larger than zero, it is not clear (to the author) what one should expect for the relations between $\theta_1$ and $\theta_2$, even involving $\chi$ of levels bigger than zero.
4 Appendix A: proof of Lemma 2.2

In this appendix we carry out the proof of Lemma 2.2, following the process in 25.7 of [BH06]. The only difference here is that we include some details on the local constant $\varepsilon(\chi \circ \det, s, \psi)$ of the one-dimensional character $\chi \circ \det$ of $G = GL(n, F)$, for a character $\chi$ of $F^\times$ of level $l \geq 1$. When $n = 2$, it is indeed an exercise in the excellent book [BH06]. On the one hand, we have the following first:

**Lemma 4.1.** $\varepsilon(\chi \circ \det, s, \psi) = \varepsilon(\chi, s, \psi)^n$.

**Proof.** By writing $\chi \circ \det$ as $Q(\chi \cdot | \cdot |_F^{-1}, \ldots, \chi \cdot | \cdot |_F^{-n})$ in the Langlands classification, the Lemma is a special case of 3.1.4 in [Kud94].

Choose any principal hereditary order $A$ in $A = M_n(F)$ of ramification index $e_A$, with Jacobson radical $\mathfrak{p}$. Then the restriction of $\chi \circ \det$ to $K_A$ is of level $e_A l$, where $K_A$ is the normalizer of $A$ in $G$. Choose $c \in \mathfrak{p}^{-l/2} + 1$, such that $\chi(1 + x) = \psi(cx)$ for $x \in \mathfrak{p}^{-l/2} + 1$. Then one may check directly that $\chi \circ \det | U_A^{[e_A l/2] + 1} = \psi_c$, where $\psi_c$ is the additive character on $U_A^{[e_A l/2] + 1}$:

$$\psi_c(1 + y) = \psi_A(cy),$$

where $\psi_A$ goes through $U_A^{[e_A l/2] + 1}$.

**Lemma 4.2.** One has

$$\varepsilon(\chi \circ \det, s, \psi) = q^{n(l+1)/2} \tau_A(\chi, \psi),$$

where $\tau_A(\chi, \psi)$ is the Gauss sum defined as follows,

$$\tau_A(\chi, \psi) = \sum_{y \in U_A^{[e_A l/2] + 1}} \chi^{-1}(\det(cy)) \psi_A(cy),$$

which simplifies to

$$\tau_A(\chi, \psi) = (A : \mathfrak{p})^{[e_A l/2] + 1} \sum_{y \in U_A^{[e_A l/2] + 1}} \chi^{-1}(\det(cy)) \psi_A(cy),$$

where $y$ goes through $U_A^{[e_A l/2] + 1} / U_A^{[e_A l/2] + 1}$.

**Proof.** By the remarks proceeding the Lemma, it is purely formal (and standard) to arrive at (8) from (7).

We first simplify the RHS of (6).

Write $e_A$ as $e$ for short. In fact, the following identity is well-known, although its proof is scattered in the literature:

$$(A : \mathfrak{p})^{[e l/2] + 1} \tau_A(\chi, \psi) = q^{-n(l+1)/2} \tau(\chi, \psi)^n$$

(9)
where \( \tau(\chi, \psi) \) is the classical Gauss sum in Tate’s thesis (23.6.4 of \([BH06]\)).

From 1.8 in \([Bus87]\), the index of \( \mathfrak{P} \) in \( \mathfrak{A} \) is \( q^{n^2/e} \). Hence, it suffices to simplify the sum appearing in \( \tau_\mathfrak{A}(\chi, \psi) \). Denote respectively by \( c', c'' \) the integers \([\lceil e\ell + 1 \rceil / 2] \) and \([\ell/2] + 1 \). When \( e\ell + 1 \) is even (hence \( l \) is odd), the sums in both sides of (9) become one term, and one can check the equation holds immediately, by taking \( y = \text{Id} \). We assume \( e\ell + 1 \) is odd, i.e., \( 2 \mid e\ell \).

We check the case \( l = 2m \) in detail; where the situation when \( e \) is even and \( l \) is odd follows in the same manner. Clearly, one has \( c' = em \), \( c'' = em + 1 \). For \( y \in U_{\mathfrak{A}} / U_{\mathfrak{A}}^c \), we write \( y = 1 + \varpi^m a \), for some \( a = (a_{ij})_{1 \leq i, j \leq n} \in \mathfrak{A} \). From the description of \( \mathfrak{A} \) in (2.5) of \([BK93]\) as an \( e \times e \) block matrix, we see

\[
\det(y) = \prod_{1 \leq i \leq n} (1 + \varpi^m a_{ii}) + \sum_{s=1}^{e} \sum_{1 < i < j} u_{ij}^s a_{ij} a_{ji} \varpi^{2m} + \sum \text{(remaining terms)},
\]

where the second inner sum runs through all the integer pairs \((i, j)\) in \(((s - 1)n/e, sn/e)\], and \( u_{ij}^s \) is some unit in \( \mathfrak{o}_E^\times \). We note that there are in all \( e \cdot \frac{n^2 \cdot ne}{2e} / e - 1) / 2 = \frac{n^2 - ne}{e} \) terms in the second sum of (10). Note also that the terms in the third sum of (10) will be killed by \( \chi \), as \( \chi \) is of level \( l = 2m \).

We are now able to verify (9) easily:

\[
\sum_y \chi^{-1}(\det y) \psi_A(cy) = t \prod_{1 \leq i \leq n} \sum_{a_{ii} \in \mathfrak{o}/p} \chi^{-1}(1 + \varpi^m a_{ii})) \psi(c(1 + \varpi^m a_{ii})),
\]

where \( t \) is the following quantity:

\[
t = \prod_{1 \leq s \leq e, (s - 1)n/e < i < j \leq sn/e} \sum_{a_{ij}, a_{ji} \in \mathfrak{o}/p} u_{aij} a_{ji}.
\]

The following easy identity shows that the value of \( t \) is \( q^{\frac{n^2 - ne}{e}} \), which completes the proof of (9) in the case that \( l \) is even: for any unit \( u \in \mathfrak{o}_E^\times \), one has

\[
\sum_{a, b \in \mathfrak{o}/p} \psi(uab) = q,
\]

where \( \psi \) is a Gauss sum. We have indeed verified that the RHS of (6) does not depend on the choice of \( \mathfrak{A} \). (6) is reduced to the following:

\[
\varepsilon(\chi \circ \det, s, \psi) = q^{n(l^2 - s)} q^{-n(l+1)/2} \tau(\chi, \psi)^n
\]

(13)

Note the RHS of (13) is \( (q^{n(l^2 - s)} q^{-n(l+1)/2} \tau(\chi, \psi))^n \), which is just \( \varepsilon(\chi, s, \psi)^n \) by 23.6.2 of \([BH06]\). We are done, by Lemma 4.1. \( \square \)

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We now complete the proof of Lemma 2.2; actually based on Lemma 4.2 this is just to repeat the argument in 25.7 of [BH06]. We will use the language of [BH99] freely.

Let $\Lambda$ be a central type contained in $\pi$, say $\Lambda \in \mathcal{CC}(\mathfrak{A}, \beta)$, for some principal hereditary order $\mathfrak{A}$ and some element $\beta \in \mathfrak{A}$. Then the level $l(\Lambda)$ of $\Lambda$ is $e_\mathfrak{A} \cdot l(\pi)$, where $e = e_\mathfrak{A}$ is the ramification index of $\mathfrak{A}$. Then, one has

$$\varepsilon(\pi, \frac{1}{2}, \psi) = (\mathfrak{A} : \mathfrak{P}^{1+ l(\Lambda)})^{-\frac{1}{2}} \tau(\Lambda, \psi),$$

(14)

where $\tau(\Lambda, \psi)$ is the Gauss sum defined in [BH99], and can be simplified as:

$$\tau(\Lambda, \psi) = c_1 \sum_{y \in U^{(l(\Lambda)+1)/2}/U^{l(\Lambda)/2}+1} \text{tr} \Lambda^\vee(\beta y) \psi_A(\beta y),$$

(15)

in which $c_1 = \frac{(U^{l(\Lambda)/2}+1)\dim \Lambda}{\dim \mathfrak{A}}$.

Now for a character $\chi$ of level $m > 2l(\pi)$, the cuspidal representation $\chi \pi$ contains the central type $\chi \Lambda$, which is of level $em$. More precisely, $\chi \circ \det \otimes \Lambda \in \mathcal{CC}(\mathfrak{A}, c + \beta)^2$. As $m > 2l(\pi)$, $\Lambda$ is trivial on $U^{(em+1)/2}$, and hence $\chi \circ \det \otimes \Lambda \mid U^{(em+1)/2} = \chi \circ \det$. The identity in Lemma 2.2 follows by using (3), (14), (15), combing Lemma 4.2 (note that $1+c^{-1} \beta \in \mathfrak{P}^{em/2+1}$ under the assumption). We are done.

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\footnote{One indeed needs to check that the group $J_{\beta+c}$ arising from the simple stratum $(\mathfrak{A}, em, 0, \beta + c)$ coincides with the $J_{\beta}$ arising from $(\mathfrak{A}, l(\Lambda), 0, \beta)$, but this is directly from [BK93].}
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SCHOOL OF MATHEMATICS, UNIVERSITY OF EAST ANGLIA, NORWICH, NR4 7TJ, UK
E-mail address: xupeng2012@gmail.com