GENERATING GEOMETRY AXIOMS FROM POSET AXIOMS

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Abstract. Two axioms of order geometry are the poset axioms of transitivity and antisymmetry of the relation 'is in front of' when looking from a point. From these axioms, by looking from an interval instead of a point, further well-known axioms of order geometry are generated in the following sense: Transitivity when looking from an interval is equivalent to [4, §10, Assioma XIII]. Assuming this axiom, antisymmetry when looking from an interval is equivalent to [3, §1, VIII. Grundsatz]. Further equivalences, with some of the implications well-known, are proved along the way.

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1. INTRODUCTION

Let $X$ be a vector space over a totally ordered field $K$, for example $K = \mathbb{R}$ and $X = \mathbb{R}^n$ for an $n \in \mathbb{Z}_{\geq 1}$. The vector interval relation on $X$ is the ternary relation $\langle \cdot, \cdot, \cdot \rangle$ defined by

$$\langle x, y, z \rangle : \iff \text{There is a } \lambda \in K \text{ such that } 0 \leq \lambda \leq 1 \text{ and } y = x + \lambda (z - x),$$

$X$ together with this relation satisfies the following conditions:

- For $a \in X$, the binary relations $\langle \cdot, \cdot, a \rangle$ and $\langle a, \cdot, \cdot \rangle$ are reflexive on $X$.
- For $a \in X$, the binary relation $\langle \cdot, a, \cdot \rangle$ is symmetric.
- For $x, y \in X$, $\langle x, y, x \rangle$ implies $y = x$.

An interval space is a pair consisting of a set $X$ and a ternary relation $\langle \cdot, \cdot, \cdot \rangle$ on $X$ such that these conditions are satisfied. Thus, a vector space $X$ over a totally ordered field $K$ together with its vector interval relation is an interval space. The concept of an interval space has been taken from [9] chapter I, 3.1].

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An interval space \((X, \langle \cdot, \cdot, \cdot \rangle)\) is also simply denoted by \(X\) when it is clear from the context whether the interval space or only the set is meant.

An interval space \(X\) is called point-transitive iff for each \(a \in X\), the binary relation \(\langle a, \cdot, \cdot \rangle\) is transitive, i.e. for all \(x, y, z \in X\), \((\langle a, x, y \rangle \text{ and } \langle a, y, z \rangle) \implies \langle a, x, z \rangle\). A vector space with its vector interval relation is point-transitive and satisfies the equivalent conditions of the following theorem. Condition (1) is obtained from the point-transitivity condition that \(\langle a, \cdot, \cdot \rangle\) is transitive by replacing the point \(a\) by an interval. Condition (2) is the interval relation version of the strict interval relation condition [4, §10, Assioma XIII].

The definitions of the interval space concepts and notations follow immediately afterwards. The proof is given below. For counter-examples and more examples and history of the concepts, see [6, sections 1.4, 1.5] and the references given there. For a set \(X\), \(P(X)\) denotes the power set of \(X\), i.e. the set of all subsets of \(X\).

Theorem. 2.3 (interval-transitivity criterion) Let \(X\) be an interval space. Then the following conditions are equivalent:

1. \(X\) is interval-transitive.
2. For all \(a, b, c \in X\), \([\{a\}, [b, c]] \subseteq [[a, b], \{c\}]\).
3. For all \(a, b, c \in X\), \([\{a\}, [b, c]] = [[a, b], \{c\}]\).
4. \(P(X)\) with the binary operation \([\cdot, \cdot]\) is a semigroup.
5. \(P(X)\) with the binary operation \([\cdot, \cdot]\) is a commutative semigroup.
6. \(X\) is interval-convex, and for each convex set \(A\), the binary relation \(\langle A, \cdot, \cdot \rangle\) is transitive.
7. For all convex sets \(A, B\), \([A, B]\) is convex.
8. For all \(a, b, c \in X\), \([[a, b], \{c\}]\) is convex.
9. For all \(a, b, c \in X\), \(\text{co} (\{a, b, c\}) = [[a, b], \{c\}]\).

Let \(X\) be an interval space.

For \(A \subseteq X\) and \(b, c \in X\),

\[\langle A, b, c \rangle : \iff \text{There is an } a \in A \text{ such that } \langle a, b, c \rangle.\]

For \(A, C \subseteq X\) and \(b \in X\),

\[\langle A, b, C \rangle : \iff \text{There are } a \in A, c \in C \text{ such that } \langle a, b, c \rangle.\]

For \(a, c \in X\), the interval between \(a\) and \(c\) is the set

\[[a, c] := \langle a, \cdot, c \rangle\]

\[= \{ x \in X \mid \langle a, x, c \rangle \}.\]

For \(A, C \subseteq X\), the interval between \(A\) and \(C\) is the set
\[ [A, C] := \langle A, \cdot, C \rangle = \{ x \in X \mid \langle A, x, C \rangle \} . \]

A subset \( C \) of \( X \) is called \textit{convex} iff \([C, C] \subseteq C\), i.e. for all \( x, y, z \in X \), if \( \langle x, y, z \rangle \) and \( x, z \in C \), then \( y \in C \).

For \( A \subseteq X \), the \textit{convex closure} or \textit{convex hull} of \( A \) in \( X \) is the set
\[ \co (A) := \bigcap \{ B \subseteq X \mid B \supseteq A \text{ and } B \text{ is convex.} \} . \]

It is the smallest convex set in \( X \) containing \( A \).

\( X \) is called \textit{interval-transitive} iff for all \( a, b \in X \), the binary relation \( \langle [a, b], \cdot, \cdot \rangle \) is transitive. Each interval-transitive interval space is point-transitive.

\( X \) is called \textit{interval-convex} iff for all \( a, b \in X \), \([a, b] \) is convex. The concept of an interval-convex interval space generalizes the concept of an interval monotone graph in [2, 1.1.6]. The term ‘interval-convex’ has been introduced in [6, section 1.5].

An interval space \( X \) is called \textit{point-antisymmetric} iff for each \( a \in X \), the binary relation \( \langle a, \cdot, \cdot \rangle \) is antisymmetric, i.e. for all \( x, y \in X \setminus [a, b] \), \( \langle a, x, y \rangle \) and \( \langle a, y, x \rangle \) imply \( x = y \). A vector space with its vector interval relation is point-antisymmetric. It is also interval-transitive and satisfies the equivalent conditions of the following theorem. Condition (1) is obtained from the point-antisymmetry condition that \( \langle a, \cdot, \cdot \rangle \) is point-antisymmetric by replacing the point \( a \) by an interval. Condition (2) is the interval relation version of the strict interval relation condition [3 §1, VIII. Grundsatz]. The definitions of the interval space and closure space concepts follow immediately afterwards. The proof is given below. For counter-examples and more examples and history of the interval space concepts, see [6 sections 1.4, 1.5] and the references given there.

**Theorem.** 3.5 (interval-antisymmetry criterion) Let \( X \) be an interval-transitive interval space. Then the following conditions are equivalent:

1. \( X \) is interval-antisymmetric.
2. \( X \) is stiff.
3. For each convex set \( A \), the binary relation \( \langle A, \cdot, \cdot \rangle \) is antisymmetric on \( X \setminus A \).
4. The pair consisting of \( X \) and the set of convex sets is an antiexchange space.
5. The pair consisting of \( X \) and the set of convex sets is an antimatroid.

Let \( X \) be an interval space.

\( X \) is called \textit{interval-antisymmetric} iff for all \( a, b \in X \) the binary relation \( \langle [a, b], \cdot, \cdot \rangle \) is antisymmetric on \( X \setminus [a, b] \). In general, there will be no chance that this relation is antisymmetric on the whole of \( X \).

\( X \) is called \textit{stiff} iff for all \( a, b, c, d \in X \), \( \langle a, b, c \rangle \) and \( b \neq c \) and \( \langle b, c, d \rangle \) imply \( \langle a, b, d \rangle \).

As noted above, this condition is the interval relation version of the strict interval relation condition [3 §1, VIII. Grundsatz]. In [6 section 3.2] a stiff interval space has been called \textit{one-way}.

\[ \begin{array}{cccc}
  a & b & c & d \\
\end{array} \]
Let $X$ be a set. A closure system or Moore family on $X$ is a set $C$ of subsets of $X$ such that $X \in C$ and for each non-empty $D \subseteq C$, $\bigcap D \in C$.

A closure space is a pair consisting of a set $X$ and a closure system $C$ on $X$. A set $A \subseteq X$ is called closed iff $A \in C$. When $(X, O)$ is a topological space, then the pair consisting of $X$ and the set of closed sets in $(X, O)$ is a closure space. When $(X, \langle \cdot, \cdot, \cdot \rangle)$ is an interval space, then the pair consisting of $X$ and the set of convex sets is a closure space. The concept of a closure space as defined here is slightly more general than in [8, chapter I, 1.2], where it is required that $\emptyset \in C$ and a closure system is called a protopology.

A closure space $(X, C)$ is also simply denoted by $X$ when it is clear from the context whether the closure space or only the set is meant.

Let $(X, C)$ be a closure space.

For $A \subseteq X$, the closure of $A$ is the set

$$\text{cl} (A) := \bigcap \{B \subseteq X | B \supseteq A \text{ and } B \in C\}$$

It is the smallest closed superset of $A$. When $X$ is an interval space and $C$ is the system of convex sets in $X$, then for $A \subseteq X$, the closure of $A$ is the convex closure of $A$.

For $A \subseteq X$, the entailment relation of $C$ relative to $A$ or $A$-entailment relation is the binary relation $\vdash_A$ on $X$ defined by

$$x \vdash_A y :\iff y \in \text{cl} (A \cup \{x\}) .$$

$(X, C)$ is called an antiexchange space iff for each closed $A \subseteq X$, one and therefore all of the following conditions hold, which are equivalent by [6, Proposition 3.1.1]:

- The relation $\vdash_A$ is antisymmetric on $X \setminus A$.
- The restriction $\vdash_A \mid (X \setminus A)$ is a partial order on $X \setminus A$.

$(X, C)$ is called algebraic or combinatorial iff for each chain $D \subseteq C$, $\bigcup D \in C$. [8, chapter I, 1.3] states the equivalence of this definition with other well-known definitions. A combinatorial (i.e. algebraic) closure space is also just called a combinatorial space. When $X$ is an interval space and $C$ is the system of convex sets in $X$, then $(X, C)$ is a combinatorial closure space.

An antimatroid (anti-matroid) or Dilworth space is a combinatorial exchange space with $\emptyset$ closed. This concept has been taken from [8, chapter I, 2.24].

2. Interval-Transitivity Criterion

Part (1) of the following proposition is cited from [5, Theorem 2.3]. Parts (2) and (3) are cited from [5, Theorem 2.1].

**Proposition 2.1.** (set interval operator) Let $X$ be an interval space. The binary operation $\lbrack, \rbrack$ on $P(X)$ has the following properties:

1. $\lbrack, \rbrack$ is commutative, i.e. for $A, B \subseteq X$, $[A, B] = [B, A]$.
2. For $C \subseteq X$, the unary operation $\lbrack, C \rbrack$ is increasing, i.e. for $A, B \subseteq X$, $A \subseteq B \implies [A, C] \subseteq [B, C]$.
3. For $C \subseteq X$, the unary operation $[C, \cdot]$ is increasing, i.e. for $A, B \subseteq X$, $A \subseteq B \implies [C, A] \subseteq [C, B]$.
(4) The binary operation $\cdot$ is increasing, i.e. for $A, B, C, D \subseteq X$, $A \subseteq B$ and $C \subseteq D \Rightarrow [A, C] \subseteq [B, D]$.

**Proof.**

(1) [5, Theorem 2.3]
(2) [5, Theorem 2.1]
(3) [5, Theorem 2.1]
(4) follows from (3) and (4)

Let $X$ be an interval space and $a, b \in X$. If the binary relation $\langle [a, b], \cdot, \cdot \rangle$ is transitive, then for $x, y, c \in X$,

$$\langle [a, b], x, y \rangle \text{ and } \langle [a, b], y, c \rangle \Rightarrow \langle [a, b], x, c \rangle.$$

Substituting $a \in [a, b]$ and $b \in [a, b]$,

$$\langle a, x, y \rangle \text{ and } \langle b, y, c \rangle \Rightarrow \langle [a, b], x, c \rangle,$$

i.e.

$$\langle a, x, y \rangle \text{ and } y \in [b, c] \Rightarrow \langle [a, b], x, c \rangle.$$

Consequently, for $x, c \in X$,

$$\langle a, x, [b, c] \rangle \Rightarrow \langle [a, b], x, c \rangle,$$

i.e. for $c \in X$,

$$[[a], [b, c]] \subseteq [[a, b], \{c\}].$$

The last condition says: For $x \in X$, if $x$ is between $a$ and $[b, c]$, then $x$ is also between $[a, b]$ and $c$:

![Diagram](image)

Summarizing,

**Proposition 2.2.** (interval spaces transitive from a base-interval) Let $X$ be an interval space and $a, b \in X$. If the binary relation $\langle [a, b], \cdot, \cdot \rangle$ is transitive, then for $c \in X$, $[a, [b, c]] \subseteq [[a, b], c]$.

Condition (2) in the following theorem is the interval relation version of the strict interval relation condition [4, §10, Assioma XIII]. In [8, chapter I, 4.9] it has been called the *Peano Property*. In [6, section 1.5], its equivalence with conditions (8) and (9) has been stated, and accordingly, $X$ has been called *triangle-convex* iff one and therefore each of these three equivalent conditions is satisfied. The implication (3) \(\Rightarrow\) (4) has been proved in [5, Theorem 2.4]. In [8, chapter I, 4.10
it has been shown that (2) implies interval-convexity in (6). In [5, Theorem 4.45 (a)] it has been demonstrated that (3) implies the strict interval relation version of transitivity in (6). The implication (2) \(\Rightarrow\) (7) has been proved in [5, Theorem 2.12].

**Theorem 2.3.** *(interval-transitivity criterion)* Let \(X\) be an interval space. Then the following conditions are equivalent:

1. \(X\) is interval-transitive.
2. For all \(a, b, c \in X\), \([\{a\}, [b, c]] \subseteq [[a, b], \{c\}]\).
3. For all \(a, b, c \in X\), \([[a, b], \{c\}] = [[a, b], \{c\}]\).
4. \(P(X)\) with the binary operation \([\cdot, \cdot]\) is a semigroup.
5. \(P(X)\) with the binary operation \([\cdot, \cdot]\) is a commutative semigroup.
6. \(X\) is interval-convex, and for each convex set \(A\), the binary relation \(\langle A, \cdot, \cdot \rangle\) is transitive.
7. For all convex sets \(A, B\), \([A, B]\) is convex.
8. For all \(a, b, c \in X\), \([[a, b], \{c\}]\) is convex.
9. For all \(a, b, c \in X\), \(\text{co}(\{a, b, c\}) = [[a, b], \{c\}]\).

**Proof.**

Step 1. \((1) \Rightarrow (2)\) follows by 2.2 (interval spaces transitive from a base-interval).

Step 2. \((2) \Rightarrow (3)\). For \(a, b, c \in X\) it remains to be proved that \(=[[a, b], c] \subseteq [a, b, c]\). The assumption (2) implies by 2.1(1) (set interval operator):

\[
[[a, b], \{c\}] = \{c\} \cdot [b, a] \\
\subseteq [[c, b], \{a\}] \\
= \{a\} \cdot [b, c].
\]

Step 3. \((3) \Rightarrow (4)\). [5, Theorem 2.4]

Step 4. \((4) \Rightarrow (5)\) follows by 2.1(1) (set interval operator).

Step 5. \((5) \Rightarrow (6)\).

Step 5.1. Proof that \(X\) is interval-convex, i.e. for \(a, b \in X\), \([[a, b], [a, b]] \subseteq [a, b]\). The assumption (5) implies by generalized associativity and commutativity:

\[
[[a, b], [a, b]] \\
= [[\{a\}, \{b\}], \{\{a\}, \{b\}\}] \\
= [[\{a\}, \{a\}], \{\{b\}, \{b\}\}] \\
= \{a\} \cdot \{b\} \\
= [a, b].
\]

Step 5.2. Proof that for each convex set \(A\), the binary relation \(\langle A, \cdot, \cdot \rangle\) is transitive, i.e. for \(x, y, z \in X\), \(\langle A, x, y \rangle\) and \(\langle A, y, z \rangle\) implies \(\langle A, x, z \rangle\), i.e. for \(y, z \in X\), \(y \in [A, \{z\}]\) implies \([A, \{y\}] \subseteq [A, \{z\}]\). The assumption that \(A\) is convex says:

\[
[A, A] \subseteq A.
\] (2.1)
From the assumptions $y \in [A, \{z\}]$, i.e. $\{y\} \subseteq [A, \{z\}]$, and (5) and (2.1) it follows by (2.1)(2) and (3) (set interval operator):

$$
[A, \{y\}] \\
\subseteq [A, [A, \{z\}]] \\
= [[A, A], \{z\}] \\
\subseteq [A, \{z\}].
$$

Step 6. (6) $\Rightarrow$ (11). The assumption that $X$ is interval-convex entails:

$$
[a, b] \text{ is convex.} \tag{2.2}
$$

From (2.2) and the assumption that for each convex set $A$ the binary relation $\langle A, \cdot, \cdot \rangle$ is transitive it follows that $\langle [a, b], \cdot, \cdot \rangle$ is transitive.

Step 7. (5) $\Rightarrow$ (7). The assumption that $A, B$ are convex says:

$$
[A, A] \subseteq A, \tag{2.3} \\
[B, B] \subseteq B. \tag{2.4}
$$

The assumption (5), (2.3) and (2.4) imply by by generalized associativity and commutativity: and (2.1.4) (set interval operator):

$$
[[A, B], [A, B]] \\
= [[A, A], [B, B]] \\
\subseteq [A, B].
$$

Step 8. (7) $\Rightarrow$ (8).

$$
\{a\}, \{b\}, \{c\} \text{ are convex.} \tag{2.5}
$$

From (2.5) and the assumption (7) it follows that $\{\{a\}, \{b\}\}$ is convex, i.e.

$$
[a, b] \text{ is convex.} \tag{2.6}
$$

From (2.6) and the assumption (7) imply that $[\{a\}, c]$ is convex.

Step 9. (8) $\Rightarrow$ (9). The assumption (8) implies that it suffices to prove that for $C$ a convex set,

$$
C \supseteq \{a, b, c\} \text{ iff } C \supseteq [[a, b], c].
$$

From the assumption that $C$ is convex it follows:

$$
C \supseteq \{a, b, c\} \\
\iff (C \supseteq \{a\} \text{ and } C \supseteq \{c\}) \\
\iff (C \supseteq [a, b] \text{ and } C \supseteq \{c\}) \\
\iff C \supseteq [[a, b], c]
$$
Step 10. (2) ⇒ (3). The assumption (2) implies by (3.1) (set interval operator):

\[
[a], [b, c] \\
= [b, c], \{a\} \\
= \text{co}\{b, c, a\} \\
= \text{co}\{a, b, c\} \\
= ([a, b], \{c\}).
\]

□

3. INTERVAL-ANTISYMMETRY CRITERION

Let \(X\) be point-transitive interval space and \(a, d \in X\). If the binary relation \(\langle [a, d], \cdot, \cdot \rangle\) is antisymmetric on \(X \setminus [a, d]\), then for \(b, c \in X\),

\[
(c, b \notin [a, d] \text{ and } \langle [a, d], b, c \rangle \text{ and } \langle [a, d], c, b \rangle) \implies b = c.
\]

Substituting \(a \in [a, d]\) in the second and \(d \in [a, d]\) in the third condition,

\[
(c, b \notin [a, d] \text{ and } \langle a, b, c \rangle \text{ and } \langle d, c, b \rangle) \implies b = c.
\]

Equivalently,

\[
(\langle a, b, c \rangle \text{ and } \langle d, c, b \rangle \text{ and } b \neq c) \implies (c \in [a, d] \text{ or } b \in [a, d]).
\]

Rewriting,

\[
(\langle a, b, c \rangle \text{ and } b \neq c \text{ and } \langle b, c, d \rangle) \implies (\langle a, c, d \rangle \text{ or } \langle a, b, d \rangle).
\]

The condition \(\langle a, b, c \rangle\) on the left, the first possibility \(\langle a, c, d \rangle\) on the right and the assumption that \(X\) is point-transitive imply the second possibility \(\langle b, c, d \rangle\) on the right. Consequently,

\[
(\langle a, b, c \rangle \text{ and } b \neq c \text{ and } \langle b, c, d \rangle) \implies \langle a, b, d \rangle.
\]

Summarizing,

**Proposition 3.1.** (interval spaces antisymmetric from a base-interval) Let \(X\) be a point-transitive interval space and \(a, d \in X\). If the binary relation \(\langle [a, d], \cdot, \cdot \rangle\) is antisymmetric on \(X \setminus [a, d]\), then for \(b, c \in X\), \((\langle a, b, c \rangle \text{ and } b \neq c \text{ and } \langle b, c, d \rangle) \implies \langle a, b, d \rangle\).

The following proposition is similar to [1] chapter II, proposition 10.

**Proposition 3.2.** (stiff interval spaces) Let \(X\) be a stiff interval space. Then for each convex set \(A\), the binary relation \(\langle A, \cdot, \cdot \rangle\) is antisymmetric on \(X \setminus A\).

**Proof.** It is to be proved that \(b, c \in X \setminus A, \langle A, b, c \rangle \text{ and } \langle A, c, b \rangle\) implies \(b = c\). The assumptions \(\langle A, b, c \rangle\) and \(\langle A, c, b \rangle\) say that there are

\[
a, d \in A
\]

such that

\[
\langle a, b, c \rangle
\]

(3.1)

(3.2)
and \( \langle d, c, b \rangle \), i.e.

\[ \langle b, c, d \rangle . \]  \hspace{1cm} (3.3)

Seeks a contradiction, suppose

\[ b \neq c . \]  \hspace{1cm} (3.4)

(3.2), (3.4), (3.3) and the assumption (2) imply:

\[ \langle a, b, d \rangle . \]  \hspace{1cm} (3.5)

From (3.1), (3.5) and the assumption that \( A \) is convex it follows that \( b \in A \), contradicting the assumption that \( b \in X \setminus A \).

\[ \square \]

**Proposition 3.3.** (interval-transitve interval spaces) Let \( X \) be an interval-transitive interval space and \( A \) a convex set. Then the relative entailment relation \( \vdash_A \) is the reverse relation of the binary relation \( \langle A, \cdot, \cdot \rangle \).

**Proof.** It is to be proved that for \( b, c \in X \), \( c \vdash_A b \) iff \( \langle A, b, c \rangle \).

\[ \{c\} \text{ is convex.} \]  \hspace{1cm} (3.6)

From (3.6) and the assumptions that \( A \) is convex and \( X \) is interval-transitive it follows by 2.3 (interval-transitivity criterion) that \( [A, \{c\}] \) is convex. Therefore, \( \text{co} (A \cup \{c\}) = [A, \{c\}] \). Consequently, the following equivalences hold:

\[ c \vdash_A b \]
\[ \iff b \in \text{co} (A \cup \{c\}) \]
\[ \iff b \in [A, \{c\}] \]
\[ \iff \langle A, b, c \rangle . \]

\[ \square \]
The following proposition is a particular case of a more general principle for relational structures.

**Proposition 3.4.** *(interval spaces are combinatorial spaces)* Let $X$ be an interval space. Then the closure space consisting of $X$ and the set of convex sets is combinatorial.

**Proof.** For a chain $D$ of convex sets is to be proved that $\bigcup D$ is convex, i.e. for $a, b, c \in X$, if $a, c \in \bigcup D$ and $\langle a, b, c \rangle$, then $b \in \bigcup D$. The assumption that $a, c \in \bigcup D$ says that there are $A, C \in D$ such that

\begin{align*}
a &\in A, \quad (3.7) \\
c &\in C. \quad (3.8)
\end{align*}

From the assumptions that $D$ is a chain and $A, C \in D$ it follows that $A \subseteq C$ or $C \subseteq A$. Suppose without loss of generality that

\begin{equation}
A \subseteq C. \quad (3.9)
\end{equation}

(3.7) and (3.9) imply

\begin{equation}
a \in C. \quad (3.10)
\end{equation}

From (3.10), (3.8) and the assumptions that $\langle a, b, c \rangle$ and $C$ is convex it follows:

\begin{equation}
b \in C. \quad (3.11)
\end{equation}

(3.11) and the assumption that $C \in D$ imply that $b \in \bigcup D$. \hfill $\Box$

**Theorem 3.5.** *(interval-antisymmetry criterion)* Let $X$ be an interval-transitive interval space. Then the following conditions are equivalent:

1. $X$ is interval-antisymmetric.
2. $X$ is stiff.
3. For each convex set $A$, the binary relation $\langle A, \cdot, \cdot \rangle$ is antisymmetric on $X \setminus A$.
4. The pair consisting of $X$ and the set of convex sets is an antiexchange space.
5. The pair consisting of $X$ and the set of convex sets is an antimatroid.

**Proof.** Step 1. (1) $\Rightarrow$ (2). From the assumption that $X$ is interval-transitive it follows:

\begin{equation}
X \text{ is point-transitive.} \quad (3.12)
\end{equation}

From (3.12) and the assumption (1) it follows (3.1)(interval spaces antisymmetric from a base-interval) that $X$ is stiff.

Step 2. (2) $\Rightarrow$ (3) follows from (3.2)(stiff interval spaces).

Step 3. (3) $\Rightarrow$ (1). For $a, d \in X$ it is to be proved that the binary relation $\langle [a, d], \cdot, \cdot \rangle$ is antisymmetric on $X \setminus [a, d]$. From the assumption that $X$ is interval-transitive it follows by (2.3) (interval-transitivity criterion) that $X$ is interval-convex. In particular,

\begin{equation}
[a, d] \text{ is convex.} \quad (3.13)
\end{equation}

(3.13) and the assumption (3) imply that the binary relation $\langle [a, d], \cdot, \cdot \rangle$ is antisymmetric on $X \setminus [a, d]$.

Step 4. (3) $\Leftrightarrow$ (4). It is to be proved iff that for each convex set $A$, the binary relation $\langle A, \cdot, \cdot \rangle$ is antisymmetric on $X \setminus A$. for each convex set $A$, the relative entailment relation $\models_A$ is
antisymmetric on $X \setminus A$. Antisymmetry being preserved under passing to the reverse relation, it suffices to prove that for each convex set $A$, the relation $\vdash_A$ is the reverse relation of the relation $\langle A, \cdot, \cdot \rangle$. This claim follows by \textbf{3.3} (interval-transitive interval spaces) from the assumption that $X$ is interval-transitive.

Step 5. \textbf{(4)} $\iff$ \textbf{(5)} follows by \textbf{3.4} (interval spaces are combinatorial spaces) because $\emptyset$ is convex.

4. Conclusion

Two well-known geometry axioms have been generated from the poset axioms of transitivity and antisymmetry, passing from a base-point to a base-interval in the following sense: By \textbf{2.3} (interval-transitivity criterion), interval-transitivity is equivalent to the axiom \textbf{[4] \S10, Assioma XIII}. By \textbf{3.5} (interval-antisymmetry criterion), assuming interval-transitivity, interval-antisymmetry is equivalent to the axiom \textbf{[3] \S1, VIII. Grundsatz}. Beyond the conditions defining an interval space, these are the first two axioms in the incremental buildup of order geometry as developed in \textbf{[1]}. Sticking to a base-point, but passing from the interval relation to the incidence relation, which is the symmetrized interval relation, and passing from poset axioms to the axioms for an equivalence relation, this theme of generating geometry axioms is continued in \textbf{[6, Theorem 3.3.2]}: Assuming the two axioms above and one further axiom, incidence-transitivity is equivalent to the conjunction of the next two axioms in the incremental buildup in \textbf{[1, 7 (5.2)]} and \textbf{[3] \S1, VII. Grundsatz}. These results together reinforce the choice of geometric axioms as well as the order of the above-mentioned incremental buildup. Research is under way on the question how many of the other axioms of order geometry fit into this scheme.

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