An explanation of the quantum speed up

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Abstract
In former work, we showed that a quantum algorithm requires the number of operations (oracle’s queries) of a classical algorithm that knows in advance 50% of the information that specifies the solution of the problem. We gave a preliminary theoretical justification of this "50% rule" and checked that the rule holds for a variety of quantum algorithms. Now, we make explicit the information about the solution available to the algorithm throughout the computation. The final projection on the solution becomes acquisition of the knowledge of the solution on the part of the algorithm. Backdating to before running the algorithm a time-symmetric part of this projection, feeds back to the input of the computation 50% of the information acquired by reading the solution.

1 Introduction
We provide an explanation of the quantum speed up – the fact that quantum problem solving requires fewer operations than its classical counterpart.

Let us consider, for example, the quantum data base search problem. We resort to a visualization. With data base size \( N = 4 \), we have a chest of 4 drawers, numbered 00, 01, 10, and 11, a ball, and two players. The first player, the oracle, hides the ball in drawer number \( k \equiv k_0, k_1 \) and gives to the second player, Alice – the algorithm that solves the problem – the chest of drawers. This is represented by a black box that, given in input a drawer number \( x \equiv x_0, x_1 \), computes the Kronecker function \( f_k(x) = \delta(k, x) \) (which is 1 if \( k = x \), 0 otherwise). Alice should find the number of the drawer with the ball, and this is done by computing \( \delta(k, x) \) for different values of \( x \) – by opening different drawers (we also say: by querying the oracle with different values of \( x \)). A classical algorithm requires 2.25 computations of \( \delta(k, x) \) on average, 3 computations if one wants to be a priori certain of finding the solution. The quantum algorithm yields the solution with certainty with just one computation.

The reason for the quantum speed up is not well understood. For example, recently Gross et al. [1] asserted that the exact reason for it has never been pinpointed.
Let I be the information acquired by reading the solution produced by the quantum algorithm. The explanation we gave in Ref. [2] and [3] is that a quantum algorithm requires fewer oracle’s queries because it knows in advance 50% of I. The computation performed by the quantum algorithm is a superposition of all the computation histories of a classical algorithm that knows in advance 50% of I. Each history corresponds to a possible way of taking 50% of I and a possible result of computing the missing information. We called this the "50% rule".

A key step was extending the representation of the quantum algorithm to the random generation of a value of k. Consequently, the end of the unitary part of the algorithm is a state of maximal entanglement between oracle’s choice k and solution of the problem. Measuring the content of the computer registers produces both a value of k at random and the corresponding solution – namely the value of x such that \( \delta(k, x) = 1 \). The 50% rule is derived from the assumption that causality between these two correlated measurement outcomes is “mutual and symmetrical” – see also Ref. [4].

In this paper, we represent the mutual determination between oracle’s choice and solution in a different way. The quantum algorithm is seen under two time-symmetric perspectives. In *Alice’s perspective*, given the value of k, the algorithm finds the value of x such that \( \delta(k, x) = 1 \). In the time-symmetric *oracle’s perspective*, the input and the output of the computation are exchanged: given the value of x, the algorithm finds the value of k such that \( \delta(k, x) = 1 \).

Furthermore, these two perspectives are represented in a relational way. In Alice’s perspective the observer is Alice, in the oracle’s perspective the observer is the oracle. In the relational representation in Alice’s (the oracle’s) perspective, we make explicit the information about the solution available to Alice (the oracle) throughout the computation. The final projection on the solution becomes acquisition of the knowledge of the solution on the part of Alice (the oracle).

In either perspective, backdating to before running the algorithm a percentage of the final projection on the solution, makes available at the input of the computation the same percentage of I. Since the two perspectives must yield the same speed up, the projection on the solution must share out evenly between the two perspectives. This means that, in either perspective, the quantum algorithm knows in advance 50% of I.

The potential applications of the 50% rule look interesting. According to the rule, the quantum speed up in terms of number of oracle’s queries comes from comparing two classical algorithms, with and without advanced information. Therefore the rule could be used for a characterization of the problems solvable with a quantum speed up in an entirely computer science framework, with no physics involved.

Section 2 provides the new derivation of the 50% rule in the case of Grover’s algorithm. In section 3, we check that the rule holds for a class of quantum algorithms that yield an exponential speed up. In section 4, by way of exemplification, we develop a new quantum algorithm out of the 50% rule. In section 5 we draw the conclusions.
2 Explaining the quantum speed up

We explain the speed up on Grover’s [5] quantum data base search algorithm, first for data base size $N = 4$, then for $N > 4$.

2.1 Reviewing the extended representation of Grover’s algorithm

In Ref. [2], we extended the representation of Grover’s algorithm to the random generation of a value of $k$. We review this representation, which is the kernel of the relational representations of the quantum algorithm developed in the next section.

We have three computer registers: (i) a two-qubit register $X$ contains the argument $x$ to query the oracle with (namely the input of the computation of $\delta (k,x)$), (ii) a one-qubit register $V$ is meant to contain the result of the computation, modulo 2 added to its initial content for logical reversibility, and (iii) a two-qubit register $K$ (just a conceptual reference) contains the oracle’s choice of a value of $k$.

The initial state of the three registers is:

$$\frac{1}{4\sqrt{2}} (|00\rangle_K + |01\rangle_K + |10\rangle_K + |11\rangle_K) (|00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X)$$

(1)

$$|0\rangle_V - |1\rangle_V ,$$

(2)

The computation of $\delta (k,x)$ is performed in quantum parallelism on each term of the superposition. For example, the input term $- |01\rangle_K |01\rangle_X |1\rangle_V$ means that the input of the black box is $k = 01$, $x = 01$ and that the initial content of register $V$ is 1. The computation yields $\delta (01,01) = 1$, which modulo 2 added to the initial content of $V$ yields the output term $- |01\rangle_K |01\rangle_X |0\rangle_V$ ($K$ and $X$ keep the memory of the input for logical reversibility). In the overall, the computation of $\delta (k,x)$ sends state (1) into:

$$\frac{1}{4\sqrt{2}} \left[ \begin{array}{c} |00\rangle_K \langle 00| - |01\rangle_K \langle 01| + |10\rangle_K \langle 10| + |11\rangle_K \langle 11| \\ |00\rangle_K \langle 01| - |01\rangle_K \langle 00| + |10\rangle_K \langle 11| + |11\rangle_K \langle 10| \end{array} \right] (|0\rangle_V - |1\rangle_V) ,$$

(2)

a maximally entangled state where four orthogonal states of $K$, each corresponding to a single value of $k$, are correlated with four orthogonal states of $X$. This means that the information about the value of $k$ has propagated to $X$.

A suitable rotation of the measurement basis of $X$ (or $K$ – see further below), transforms entanglement between $K$ and $X$ into correlation between the outcomes of measuring their contents, sending state (2) into:
\[
\frac{1}{\sqrt{2}} (\langle 00 \rangle_K |00\rangle_X + \langle 01 \rangle_K |01\rangle_X + \langle 10 \rangle_K |10\rangle_X + \langle 11 \rangle_K |11\rangle_X)
\]
\[
(\langle 0 \rangle_V - |1\rangle_V).
\]

Let us call \([X]\) the content of register \(X\) and \([K]\) the content of register \(K\). Measuring \([X]\) and/or \([K]\) in the output state (3) yields the solution \(x = k\), also generating the oracle’s choice at random. In fact, since the algorithm is the identity in the Hilbert space of register \(K\) (having rotated the basis of register \(X\), nothing changes if we measure \([K]\) in the initial state (1) – which generates the oracle’s choice at random – and \([X]\) in the output state (3).

Until now we have seen Grover’s algorithm in ”Alice’s perspective”: the oracle chooses a value of \(k\) at random and Alice finds it, first by computing \(\delta(k,x)\) for all the values of \(x\) in quantum superposition, then by rotating the basis of register \(X\), and finally by measuring \([X]\). In the time-symmetric oracle’s perspective, Alice chooses a value of \(x\) at random and the oracle finds it, first by computing \(\delta(k,x)\) for all the values of \(k\) in quantum superposition, then by rotating the basis of register \(K\), and finally by measuring \([K]\). We note that states (1) through (3) are common to the two perspectives.

### 2.2 Relational representation of the quantum algorithm

We develop two relational [6] representations of the quantum algorithm. In one the observer is Alice, who is in control of register \(X\) (from now on, by Alice’s perspective we mean the relational representation of the quantum algorithm with respect to Alice). In the other the observer is the oracle, who is in control of register \(K\) (this is the oracle’s perspective).

By definition, initially Alice does not know the oracle’s choice – to fix ideas, say that this choice (which does not need to be random) is \(k = 00\). This does not affect the initial state of the algorithm in Alice’s perspective, which is anyhow:

\[
\frac{1}{4\sqrt{2}} (\langle 00 \rangle_K + e^{i\varphi_{01}} \langle 01 \rangle_K + e^{i\varphi_{10}} \langle 10 \rangle_K + e^{i\varphi_{11}} \langle 11 \rangle_K)
\]
\[
(\langle 0 \rangle_V + \langle 01 \rangle_V + \langle 10 \rangle_V + \langle 11 \rangle_V) (\langle 0 \rangle_V - |1\rangle_V),
\]

where the \(\varphi_{ij}\) are random phases with uniform distribution in \([0,2\pi]\). We use the random phase representation of density operators to keep the state vector representation of the quantum algorithm (the density operator is the average over \(\varphi\) of the product of the ket by the bra). Register \(K\) is in a maximally mixed state. The two bits von Neumann entropy of the state of register \(K\) represents Alice’s initial ignorance of the oracle’s choice.

In Alice’s perspective, the algorithm is the identity in the Hilbert space of register \(K\), thus state (4) develops like state (1). Computing \(\delta\) and rotating the basis of register \(X\) sends (4) into the output state:
\[ \frac{1}{2\sqrt{2}} (|00\rangle_K |00\rangle_X + e^{i\varphi_{01}} |01\rangle_K |01\rangle_X + e^{i\varphi_{10}} |10\rangle_K |10\rangle_X + e^{i\varphi_{11}} |11\rangle_K |11\rangle_X ) \] (5)

\[ (|0\rangle_V - |1\rangle_V ) . \]

The measurement of \([X]\) (or, indifferently, \([K]\)) projects the output state on

\[ \frac{1}{\sqrt{2}} |00\rangle_K |00\rangle_X (|0\rangle_V - |1\rangle_V ) \] (6)

and yields the solution \(x = k = 00\). This projection is random to Alice but, seen from outside, it occurs on the eigenstate of the eigenvalue of \(k\) chosen by the oracle (\(k = 00\)). It is reduction of ignorance of the solution on the part of Alice – her acquisition of the knowledge of the oracle’s choice. Correspondingly, the entropy decreases from the two bits of states (4) and (5) to the value zero of state (6).

In the oracle’s perspective, the initial state of the algorithm is:

\[ \frac{1}{4\sqrt{2}} (|00\rangle_K + |01\rangle_K + |10\rangle_K + |11\rangle_K ) \]

\[ (|00\rangle_X + e^{i\varphi_{01}} |01\rangle_X + e^{i\varphi_{10}} |10\rangle_X + e^{i\varphi_{11}} |11\rangle_X ) (|0\rangle_V - |1\rangle_V ) . \] (7)

Note that the two perspectives transforms into one another by swapping the labels \(K\) and \(X\). Computing \(\delta\) and rotating the basis of register \(K\) sends state (7) into the output state (5), invariant under the swapping of \(K\) and \(X\) and common to both perspectives.

### 2.3 Taking advantage of backdated projection on the solution

Let us break down \([X]\), the content of register \(X\), into content of first qubit \([X_0]\) and content of second qubit \([X_1]\) (other ways of halving \([X]\) will be considered in the next section). Let \(x_0\) (\(x_1\)) be the eigenvalue obtained by measuring \([X_0]\) (\([X_1]\)) in the output state (5). We define in a similar way \([K_0]\), \([K_1]\), \(k_0\), and \(k_1\). In the assumption that \(x = k = 00\), measuring, say, \([X_0]\) in (5) yields \(x_0 = k_0 = 0\), measuring \([K_1]\) yields \(x_1 = k_1 = 0\). Together, the two measurements project the output state (5) on the eigenstate corresponding to the solution.

The measurement of \([X_0]\) (or, indifferently, \([K_0]\)) projects the output state on:

\[ \frac{1}{2} (|00\rangle_K |00\rangle_X + e^{i\varphi_{01}} |01\rangle_K |01\rangle_X ) (|0\rangle_V - |1\rangle_V ) . \] (8)
We backdate the related projection along Alice’s perspective. This projects the initial state of Alice’s perspective, (4), on:

$$\frac{1}{4} \left( |00\rangle_K + e^{i\phi_{01}} |01\rangle_K \right) \left( |00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X \right) \left( |0\rangle_V - |1\rangle_V \right). \quad (9)$$

We have applied to state (8) the inverse of the time forward unitary transformation (the result is evident keeping in mind that, in Alice’s perspective, the algorithm is the identity in the Hilbert space of register $K$). State (9) says that, "after" the backdated projection, Alice knows before running the algorithm that the oracle’s choice is either $k = 00$ or $k = 01$, namely that $k_0 = 0$. Correspondingly, the entropy representing Alice’s ignorance of the solution has decreased from two to one bit.

The measurement of $[K_1]$ (or, indifferently, $[X_1]$) projects the output state on:

$$\frac{1}{2} \left( |00\rangle_K |00\rangle_X + e^{i\phi_{10}} |10\rangle_K |10\rangle_X \right) \left( |0\rangle_V - |1\rangle_V \right). \quad (10)$$

We backdate this projection along the oracle’s perspective. This projects the initial state of the oracle’s perspective, (7), on:

$$\frac{1}{4} \left( |00\rangle_K + |01\rangle_K + |10\rangle_K + |11\rangle_K \right) \left( |00\rangle_X + e^{i\phi_{10}} |10\rangle_X \right) \left( |0\rangle_V - |1\rangle_V \right) \quad (11)$$

(now the algorithm is the identity in the Hilbert space of $X$). This means that the oracle knows in advance that $x_1 = k_1 = 0$, in fact the other bit of information about the solution.

This is what is needed, because the amount of advanced knowledge of the solution should be the same in either perspective (the two perspectives must yield the same speed up).

We discuss the other ways of backdating the projection on the solution. Backdating along Alice’s perspective the projection due to measuring $[X_1]$ and along the oracle’s perspective the projection due to measuring $[K_0]$, just exchanges the two bits known in advance by respectively Alice and the oracle. Backdating along Alice’s (the oracle’s) perspective the entire projection on the solution, would leave nothing to backdate along the oracle’s (Alice’s) perspective, in conflict with the requirement that the projection on the solution shares out evenly between the two perspectives.

The above shows that: (i) the advanced knowledge of 50% of $I$, highlighted in Ref. [2], is backdated projection on the solution and (ii) the algorithm cannot exploit in advance more than 50% of the projection on the solution.

In view of what will follow, we highlight another way of seeing Alice’s advanced knowledge. In Alice’s perspective, the reduced density operator of register $K$ is $\frac{1}{2} \left( |00\rangle_K + e^{i\phi_{01}} |01\rangle_K + e^{i\phi_{10}} |10\rangle_K + e^{i\phi_{11}} |11\rangle_K \right)$ throughout the unitary part of the quantum algorithm (which is the identity on this operator). The measurement of $[X_0]$ (or $[K_0]$) in the output state (5) projects it on $\frac{1}{\sqrt{2}} \left( |00\rangle_K + e^{i\phi_{01}} |01\rangle_K \right)$. This projection goes back unaltered to before running the algorithm, yielding Alice’s advanced knowledge.
2.4 Superposition of the histories

We show how the quantum algorithm exploits the advanced knowledge of the solution. We put ourselves in Alice’s perspective. Until now we have considered two ways of halving the projection on the solution: measuring the binary observable \([X_0]\), which tells whether the oracle’s choice \(k\) belongs to \(\{00, 01\}\) or \(\{10, 11\}\), or measuring \([X_1]\), which tells whether \(k\) belongs to \(\{00, 10\}\) or \(\{01, 11\}\). There is a third binary observable, say \([X_+]\), whose measurement tells whether \(k\) belongs to \(\{00, 11\}\) or \(\{01, 10\}\). Measuring any pair of these observables projects the output state (5) on the solution. We note that the results of the former section remain unaltered if we replace either \([X_0]\) or \([X_1]\) by \([X_+]\) and, correspondingly, either \([K_0]\) or \([K_1]\) by \([K_+]\).

In the overall, there are 6 halved projections, or ways of knowing in advance one bit of \(I\): knowing that \(k\) belongs to \(\{00, 01\}\), or \(\{10, 11\}\), ..., or \(\{01, 10\}\). Each halved projection originates 8 classical computation histories, as follows.

We assume that Alice knows in advance that \(k\) belongs to \(\{00, 01\}\). To compute the missing bit, she should query the oracle with either \(x = 00\) or \(x = 01\). We assume that she queries with \(x = 00\). If the outcome of the computation is \(\delta = 1\), this means that \(k = 00\). This originates two histories, depending on the initial state of register \(V\). History # 1: initial state \(|00\rangle_K |00\rangle_X |0\rangle_V\), state after the computation \(|00\rangle_K |00\rangle_X |1\rangle_V\). History # 2: initial state \(|00\rangle_K |00\rangle_X |1\rangle_V\), state after the computation \(|00\rangle_K |00\rangle_X |0\rangle_V\). If the outcome of the computation is \(\delta = 0\), this means that \(k = 01\). This originates two histories, depending on the initial state of register \(V\). History # 3: initial state \(|01\rangle_K |00\rangle_X |0\rangle_V\), state after the computation \(|01\rangle_K |00\rangle_X |0\rangle_V\). History # 4: initial state \(|01\rangle_K |00\rangle_X |1\rangle_V\), state after the computation \(|01\rangle_K |00\rangle_X |1\rangle_V\). If she queries the oracle with \(x = 01\) instead, this originates other 4 histories. Etc.

If we sum together all the different histories (some histories are originated more than once), each with a suitable phase, and normalize, we obtain the transformation of state (1) into (2), namely the oracle’s query stage of Grover’s algorithm. For simplicity, we consider the kernel of the quantum algorithm.

Rotating the basis of register \(X\) transforms state (2) into state (3). Correspondingly, each classical history in quantum notation branches into four histories, the branches of different histories interfere with one another to give state (3).

The 50% rule only establishes that the quantum algorithm can be broken down into a superposition of such histories, the history phases and the final rotation of the basis of register \(X\) are what is needed for the breaking down.

2.5 Synthesizing the quantum algorithm out of the advanced information classical algorithm

As from Ref. [2], the history phases that reconstruct the quantum algorithm also maximize the entanglement between \(K\) and \(X\) after the computation of \(\delta\) – see state (2). Then the rotation of the basis of \(X\) transforms this entanglement into correlation between the outcomes of measuring \([K]\) and \([X]\) – see state (3).
This, in principle, allows to synthesize the quantum algorithm out of the advanced information classical algorithm. We should choose history phases and rotation of the basis of $X$ in such a way that they maximize: (i) correlation between the outcomes of measuring $[K]$ and $[X]$, or (ii) interference between histories, or (iii) the information about the solution readable in $X$ at the end of the algorithm.

2.6 Generalizing to $N > 4$

We check that the explanation of the quantum speed up holds also for $N = 2^n > 4$. Registers $K$ and $X$ are $n$-qubit each. Register $V$ is one-qubit. Given the advanced knowledge of $n/2$ bits, in order to compute the missing $n/2$ bits we should compute $\delta(k, x)$ for all the values of $x$ in quantum superposition and rotate the basis of $X$ (in Alice’s perspective) an $O(2^{n/2})$ times. The output state (3) becomes:

$$\frac{1}{2^{(n+1)/2}} \left( \sum_{k=0}^{2^n-1} |k\rangle_K |k\rangle_X \right) (|0\rangle_V - |1\rangle_V),$$

(12)

we have considered for simplicity only the kernel of the quantum algorithm. Measuring $[X]$ (or $[K]$) projects $|X\rangle$ on the solution. According to the rationale of section 2.3, we should halve the final projection on the solution in all possible ways; for example, by measuring $[X_0]$, ..., $[X_{2^n-1}]$ (or, indifferently, $[K_0]$, ..., $[K_{2^n-1}]$). Evidently, the considerations of section 2.3 apply also here: backdating a half projection in one perspective, makes available at the input of the algorithm the corresponding 50% of $I$. The other half projection (in the example, that due to measuring $[X_0]$, ..., $[X_{n-1}]$) should be backdated in the other perspective. Instead, backdating more than half projection in one perspective, would leave less to backdate in the other, violating the symmetry between the two perspectives.

The quantum algorithm can still be seen as a superposition of all the possible ways of taking 50% of $I$ and all the possible results of computing the missing information. It suffices to track each individual term of the superposition throughout the computation. However, after each computation of $\delta(k, x)$, each history branches into $2^n$ histories because of the rotation of the basis of register $X$. The history superposition picture still explains how the quantum algorithm exploits the advanced information, however it becomes very complex.

3 Checking the 50% rule on other quantum algorithms

We check the 50% rule on Deutsch&Jozsa’s, Simon’s, and the hidden subgroup algorithms, see Ref. [2]. Problem solving is still seen as a game between two players. Given a set of functions $f_k : \{0, 1\}^n \rightarrow \{0, 1\}^m$ known to both players, the oracle chooses a function $f_k(x)$ and gives to Alice a black box that, given in
input a value of $x$, computes $f_k(x)$. Alice should find a property of the function by computing $f_k(x)$ for various values of $x$.

### 3.1 Deutsch&Jozsa’s algorithm

In Deutsch&Jozsa’s algorithm, the set of functions is all the constant and ”balanced” functions (with an even number of zeroes and ones) $f_k: \{0,1\}^n \rightarrow \{0,1\}$. Table 13 gives this set of functions for $n = 2$. The string $k \equiv k_0, k_1, ..., k_{2^n-1}$ is both the suffix and the table of the function – the sequence of function values for increasing values of the argument.

| $x$  | $f_{0000}(x)$ | $f_{1111}(x)$ | $f_{0011}(x)$ | $f_{1100}(x)$ | $f_{0101}(x)$ | $f_{1010}(x)$ | $f_{0110}(x)$ | $f_{1001}(x)$ |
|------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 00   | 0            | 1            | 0            | 1            | 0            | 1            | 0            | 1            |
| 01   | 0            | 1            | 0            | 1            | 1            | 0            | 1            | 0            |
| 10   | 0            | 1            | 1            | 0            | 0            | 1            | 1            | 0            |
| 11   | 0            | 1            | 1            | 0            | 1            | 0            | 0            | 1            |

One should find whether the function chosen by the oracle is balanced or constant, by computing $f_k(x) = f(k,x)$. In the classical case this requires, in the worst case, a number of computations of $f_k(x)$ exponential in $n$; in the quantum case one computation – see Ref. [7].

For simplicity, we develop only the kernel of the quantum algorithm in Alice’s perspective. The initial state is:

$$\frac{1}{8} (|0000\rangle_K + |1111\rangle_K + |0011\rangle_K + |1100\rangle_K + ...)$$

$$\left(|00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X\right) (|0\rangle_V - |1\rangle_V). \quad (14)$$

Computing $f(k,x)$ and modulo 2 adding the result to the former content of $V$, yields the state of maximal entanglement:

$$\frac{1}{8} \left[ (|0000\rangle_K - |1111\rangle_K)(|00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X) + 
(\langle 0011\rangle_K - \langle 1100\rangle_K)(\langle 00\rangle_X + \langle 01\rangle_X - \langle 10\rangle_X - \langle 11\rangle_X) + ... \right] (|0\rangle_V - |1\rangle_V). \quad (15)$$

Performing Hadamard on register $X$ yields:

$$\frac{1}{4} \left[ (|0000\rangle_K - |1111\rangle_K)|00\rangle_X + (\langle 0011\rangle_K - \langle 1100\rangle_K)|10\rangle_X + ... \right] (|0\rangle_V - |1\rangle_V). \quad (16)$$

Measuring $[K]$ and $[X]$ in the output state (16) yields the oracle’s choice and the solution found by Alice: all zeroes if the function is constant, not so if it is balanced.

We check that the quantum algorithm requires the number of oracle’s queries of a classical algorithm that knows in advance 50% of $I$. Since the solution is
a function of $k$, we can define the advanced information as any 50% of the information about the solution contained in $k$, namely in the table of $f_k(x)$. If $f_k(x)$ is constant, for reasons of symmetry, the advanced information is any 50% of the table of the function – see table (13). If the function is balanced, still for reasons of symmetry, it is any 50% of the table that does not contain different values of the function – for each balanced function there are two such half tables. In fact, the half tables that contain different values of the function already tell that the function is balanced and thus contain 100% of $I$. For the good half tables, that do not contain different values of the function, the solution (whether the function is constant or balanced) is always identified by computing $f_k(x)$ for any value of $x$ outside the half table. Thus, both the quantum algorithm and the advanced information classical algorithm require just one oracle’s query.

We show that the advanced information, as defined above, is backdated projection. With respect to Grover’s algorithm, we have used a different way of representing both $k$ and the advanced information. It is instructive to retrofit Grover’s algorithm in this way.

The equivalent of table (13) for Grover’s algorithm is:

| $x$ | $\delta(00,x) = f_{1000}(x)$ | $\delta(01,x) = f_{0100}(x)$ | $\delta(10,x) = f_{0010}(x)$ | $\delta(11,x) = f_{0001}(x)$ |
|-----|----------------------------|----------------------------|----------------------------|----------------------------|
| 00  | 1                          | 0                          | 0                          | 0                          |
| 01  | 0                          | 1                          | 0                          | 0                          |
| 10  | 0                          | 0                          | 1                          | 0                          |
| 11  | 0                          | 0                          | 0                          | 1                          |

(17)

The advanced information is any half table of $f_k(x)$ that does not contain the value 1 of the function. The reduced density operator of register $K$, throughout the unitary part of the algorithm in Alice’s perspective, is:

$$\frac{1}{2} \left( |1000\rangle_K + e^{i\varphi_{01}} |0100\rangle_K + e^{i\varphi_{10}} |0010\rangle_K + e^{i\varphi_{11}} |0001\rangle_K \right).$$  (18)

Let us assume that the advanced information (a good half table) is $f_k(10) = 0$ and $f_k(11) = 0$, which means that the function chosen by the oracle is either $f_{1000}(x)$ or $f_{0100}(x)$. This corresponds to projecting the reduced density operator (18) on $\frac{1}{\sqrt{2}} \left( |1000\rangle_K + e^{i\varphi_{01}} |0100\rangle_K \right)$ (by the way, this projection corresponds, in the representation of section 2.3, to measuring $|K_0\rangle$ in the output state (15) and finding $k_0 = 0$). The outcome of the projection goes back unaltered to before running the algorithm, where it becomes Alice’s advanced knowledge of the solution.

Furthermore, we can see that this advanced knowledge cannot exceed 50% of $I$. In fact, increasing any good half table by one row, projects the output state on the solution, leaving to the oracle nothing to backdate.

We show that the oracle’s query stage of Deutsch&Jozsa’s algorithm is a superposition of the histories of the advanced information classical algorithm. Let us assume that the advanced information is $f(k,00) = 0$ and $f(k,01) = 0$, namely the first two rows of either $f_{0000}(x)$ or $f_{0011}(x)$ – see
The quantum algorithm is iterated until finding \( n \) qubit, given that \( k \) part of this problem, namely finding a string \( s \) \( \mathcal{O} \left( n \right) \). In present knowledge, a classical algorithm requires a number of computations of \( f \) exponential in \( n \). The quantum algorithm solves the hard part of this problem, namely finding a string \( s \) orthogonal to \( h \), with one Facebook. Running the quantum algorithm yields one of these strings at random (see further below). The quantum algorithm is iterated until finding \( n - 1 \) different strings. This allows to find \( h \) by solving a system of modulo 2 linear equations. The black box, given \( k \) and \( x \), computes \( f_k(x) = f(k, x) \). Register \( K \) is now \( 2^n \) \((n - 1)\)-qubit, given that \( K \) is the sequence of \( 2^n \) fields each on \( n - 1 \) bits.

\[ 1 \] The modulo 2 addition of the bits of the bitwise product of the two strings should be zero.

| x   | \( f_{001}(x) \) | \( f_{100}(x) \) | \( f_{010}(x) \) | \( f_{010}(x) \) | \( f_{010}(x) \) | \( f_{010}(x) \) | \( f_{100}(x) \) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 00  | 0               | 0               | 1               | 0               | 1               | 0               | 1               |
| 01  | 0               | 1               | 1               | 0               | 1               | 0               | 0               |
| 10  | 1               | 0               | 0               | 1               | 1               | 0               | 0               |
| 11  | 1               | 0               | 1               | 0               | 0               | 0               | 1               |
We develop the kernel of the quantum algorithm in Alice’s perspective. The initial state, with register $V$ prepared in the all zeroes string (just one zero for $n = 2$), is:

$$\frac{1}{2\sqrt{6}} (|0011\rangle_K + |1100\rangle_K + |0101\rangle_K + |1010\rangle_K + ...)$$

(20)

$$|00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X |0\rangle_V .$$

Computing $f(k, x)$ changes the content of $V$ from zero to the outcome of the computation, yielding the entangled state:

$$\frac{1}{2\sqrt{6}} \left[ (|0011\rangle_K + |1100\rangle_K) [|00\rangle_X + |01\rangle_X |0\rangle_V + (|10\rangle_X + |11\rangle_X |1\rangle_V] + 
\quad (|0101\rangle_K + |1010\rangle_K) [(|00\rangle_X + |10\rangle_X |0\rangle_V + (|01\rangle_X + |11\rangle_X |1\rangle_V] + ... \right].$$

(21)

Performing Hadamard on $X$ yields:

$$\frac{1}{2\sqrt{6}} \left[ (|0011\rangle_K + |1100\rangle_K) [(|00\rangle_X + |10\rangle_X |0\rangle_V + (|00\rangle_X - |10\rangle_X |1\rangle_V] + 
\quad (|0101\rangle_K + |1010\rangle_K) [(|00\rangle_X + |01\rangle_X |0\rangle_V + (|00\rangle_X - |01\rangle_X |1\rangle_V] + ... \right].$$

(22)

where, for each value of $k$, register $X$ (no matter the content of $V$) hosts even weighted superpositions of the $2^{n-1}$ strings $s_j^{(k)}$ orthogonal to $h^{(k)}$. By measuring $[K]$ and $[X]$ in state (22), we obtain at random the oracle’s choice $k$ and one of the $s_j^{(k)}$.

We leave $K$ in its after-measurement state, thus fixing $k$, and iterate the "right part" of the algorithm (preparation of registers $X$ and $V$, computation of $f(k, x)$, and measurement of $[X]$) until obtaining $n − 1$ different $s_j^{(k)}$.

We check that the quantum algorithm requires the number of oracle’s queries of a classical algorithm that knows in advance 50% of $I$. Any $s_j^{(k)}$ is a solution of the problem addressed by the quantum part of Simon’s algorithm. The advanced information is any 50% of the information about the solution contained in $k$. For reasons of symmetry, this is any 50% of the table of the function that does not contain the same value of the function twice. In fact, the half tables that contain a same value twice already specify the value of $h^{(k)}$ and thus the value of any $s_j^{(k)}$. For the half tables that do not contain the same value of the function twice, the solution is always identified by computing $f(k, x)$ for any value of $x$ outside the half table. The new value of the function is necessarily a value already present in the half table, which identifies $h^{(k)}$ and all the $s_j^{(k)}$. Thus, both the quantum algorithm and the advanced information classical algorithm require just one oracle’s query.

As in section 3.1, the above defined advanced information is backdated projection on the solution.

We show that the oracle’s query stage of the quantum algorithm is a superposition of the histories of the advanced information classical algorithm. For
example, let us assume that the advanced information is \( f(k,00) = 0 \) and \( f(k,11) = 1 \), namely the first and last row of either \( f_{0011}(x) \) or \( f_{0101}(x) \) – see table (19). To find the value of \( k \), Alice should query the oracle with either \( x = 01 \) or \( x = 10 \). We assume that she queries with \( x = 01 \). If the result of the computation is 0, this means that \( k = 0011 \). This originates two histories. History # 1: initial state \( |0011\rangle_K |01\rangle_X |0\rangle_V \), state after the computation \( |0011\rangle_K |01\rangle_X |0\rangle_V \). History # 2: initial state \( |0011\rangle_K |01\rangle_X |1\rangle_V \), state after the computation \( |0011\rangle_K |01\rangle_X |1\rangle_V \). If the result of the computation is 1, this means that \( k = 0101 \). This originates two histories. History # 3: initial state \( |0101\rangle_K |01\rangle_X |0\rangle_V \), state after the computation \( |0101\rangle_K |01\rangle_X |0\rangle_V \). History # 3: initial state \( |0101\rangle_K |01\rangle_X |1\rangle_V \), state after the computation \( |0101\rangle_K |01\rangle_X |1\rangle_V \). If she queries the oracle with \( x = 10 \) instead, this originates other 4 histories, etc.

To synthesize the quantum algorithm out of the advanced information classical algorithm, we should choose history phases and rotation of the basis of register \( X \) in such a way that the information about the solution readable in that register at the end of the algorithm is maximized.

The 50% rule also applies to the generalized Simon’s problem and to the hidden subgroup problem. In fact the corresponding algorithms are essentially the same as the algorithm that solves Simon’s problem. In the hidden subgroup problem, the set of functions \( f_k : G \to W \) map a group \( G \) to some finite set \( W \) with the property that there exists some subgroup \( S \leq G \) such that for any \( x, y \in G \), \( f_k(x) = f_k(y) \) if and only if \( x + S = y + S \). The problem is to find the hidden subgroup \( S \) by computing \( f_k(x) \) for various values of \( x \). Now, a large variety of quantum problems can be re-formulated in terms of the hidden subgroup problem [9]. Among these we find: Deutsch’s problem, Bernstein&Vazirani problem, finding orders, finding the period of a function (thus the problem solved by the quantum part of Shor’s factorization algorithm), discrete logarithms in any group, hidden linear functions, self shift equivalent polynomials, Abelian stabilizer problem, graph automorphism problem.

4 Applying the 50% rule to the search of quantum speed ups

In hindsight, the quantum algorithms examined are skillfully designed around the 50% rule. In unstructured data base search, the advanced knowledge of 50% of \( I \) yields a quadratic speed up, given that the number of oracle’s queries goes from \( O(2^n) \) to \( O(2^n/2) \). Thus, the possibility of a quadratic speed up is established by the 50% rule, one does not need to know Grover’s algorithm. Similarly, in the structured algorithms that yield an exponential speed up, the problem is chosen in such a way that, if one knows in advance 50% of \( I \), computing \( f_k(x) \) for a single value of \( x \) outside the advanced information yields the solution. Thus, the possibility of an exponential speed up is established by the 50% rule before knowing the quantum algorithm.
One way of searching for new quantum speed ups is thus looking for problems solvable with a single computation of $f_k(x)$ once that 50% of $I$ is known. We provide an example – see also Ref. [2]. The set of functions is the 4! functions $f_k : \{0,1\}^2 \rightarrow \{0,1\}^2$ such that the sequence of function values is a permutation of the values of the argument – see table (23).

We have chosen this set because, if we know 50% of the rows of one table, we can identify the corresponding $k$ with a single computation of $f_k(x)$, for any value of $x$ outside the advanced information. Without advanced information, three computations of $f_k(x)$ are required. Thus there is room for a speed up in terms of number of oracle’s queries. We build a quantum algorithm over this possibility. Register $K$ is 8 qubits, registers $X$ is 2 qubits, and register $V$ is 2 qubits, denoted $V_0$ and $V_1$. The first (second) bit of the result of the computation of $f_k(x) = f(k,x)$ is modulo 2 added to the former content of $V_0$ ($V_1$). The initial state is

$$\frac{1}{8\sqrt{6}} (|00011110\rangle_K + |00110110\rangle_K + |00011011\rangle_K ... )$$

$$= (|00\rangle_X + |01\rangle_X + |10\rangle_X + |11\rangle_X ) (|0\rangle_{V_0} - |1\rangle_{V_0} ) (|0\rangle_{V_1} - |1\rangle_{V_1} ) .$$

Performing one computation of $f(k,x)$ then Hadamard on $X$, yields

$$\frac{1}{4\sqrt{6}} [ (|00011110\rangle_K + ... |01\rangle_X + (|00110110\rangle_K + ... ) |10\rangle_X + (|00011011\rangle_K + ... ) |11\rangle_X ]$$

$$= (|0\rangle_{V_0} - |1\rangle_{V_0} ) (|0\rangle_{V_1} - |1\rangle_{V_1} ) ,$$

an entangled state where three orthogonal states of $K$ (each a superposition of 8 values of $k$, corresponding to a partition of the set of 24 functions) are correlated with, respectively, $|01\rangle_X$ , $|10\rangle_X$ , and $|11\rangle_X$ . Measuring $X$ in the above state tells which of the three partitions the function belongs to. In the case of a classical algorithm, identifying the partition requires three computations of $f(k,x)$, as readily checked. There is thus a quantum speed up.

With the 50% rule, one can figure out any number of these speed ups in terms of number of oracle’s queries. Thus, this rule provides a playground for studying the engineering of quantum algorithms.

5 Conclusions

Let $I$ be the information acquired by reading the solution of the problem. The 50% rule establishes that a quantum algorithm requires the number of oracle’s
queries of a classical algorithm that knows in advance 50% of \( I \). The advanced knowledge of the solution is due to backdating, to before running the algorithm, a time symmetric part of the final projection on the solution. The computation performed by the quantum algorithm is a superposition of classical computations that exploit the advanced knowledge of the solution to reach the solution with fewer oracle’s queries. We have checked that the rule holds for a variety of quantum algorithms yielding both quadratic and exponential speed up.

This article should be considered work in progress. For example, it would be desirable to check the 50% rule on other known quantum algorithms and to demonstrate that the rule holds in a more general way, for example for the generic quantum computation network.

This rule would have an important practical consequence: the speed up in terms of number of oracle’s queries comes from comparing two classical algorithms, with and without advanced information. This allows to characterize the problems solvable with a quantum speed up in an entirely computer science framework, with no physics involved. By way of exemplification, we have produced a new quantum speed up on the basis of the 50% rule.

The possibility that quantum algorithms use backdated information about the solution they will find in the future to reduce the number of operations required to find the solution, involves a causality loop that could be interesting also outside quantum computation.

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