Improved transport equations including correlations for electron-phonon systems. Comparison with exact solutions in one dimension

J. Fricke, V. Meden, C. Wöhler, and K. Schönhammer
Institut für Theoretische Physik der Universität Göttingen,
Bunsenstraße 9, 37073 Göttingen, Germany
(July 4, 1996)

Abstract

We study transport equations for quantum many-particle systems in terms of correlations by applying the general formalism developed in an earlier paper to exactly soluble electron-phonon models. The one-dimensional models considered are the polaron model with a linear energy dispersion for the electrons and a finite number of electrons and the same model including a Fermi sea (Tomonaga-Luttinger model). The inclusion of two-particle correlations shows a significant and systematic improvement in comparison with the usual non-Markovian equations in Born approximation. For example, the improved equations take into account the renormalization of the propagation by the self-energies to second order in the coupling.

I. INTRODUCTION

In a recent work [1] one of the authors proposed a new method for deriving transport equations in terms of correlations. In this paper we test these transport equations and gain more insight into their abilities to describe physical effects for some exactly soluble electron-phonon models.

In their study of the Jaynes-Cummings model [2] Zimmermann and Wauer used the same decoupling procedure as ours one order higher than the Born approximation in order to reduce unphysical results such as fermionic occupation numbers outside the range [0, 1]. We extended this procedure systematically to all orders in the correlations, thus obtaining an exact, infinite set of equations of motion in terms of correlations. For this simple model it is easy to solve these equations numerically to very high order in order to test our method [3].

Here, we focus on one-dimensional versions of more realistic models, i.e. the polaron model and the same model with a Fermi sea of electrons (Tomonaga-Luttinger model). For the polaron model with a linearized electronic energy dispersion exact results for non-equilibrium expectation values were presented in Ref. [4], as well as for the Tomonaga-Luttinger model with coupling to phonons in Ref. [5]. Similar models have recently been discussed to study the relaxation of highly excited semiconductors and insulators [6, 7].
For these models non-Markovian quantum-kinetic equations in Born approximation were numerically solved assuming isotropic distributions in $k$-space. It became clear that damping factors for the electron propagation had to be introduced in order to reduce the occurrence of unphysical results, e.g. negative occupation numbers. Renormalized free propagators including the damping were obtained by the generalized Kadanoff-Baym ansatz \cite{10,11,12}, which in turn was justified by Schoeller \cite{12} as a partial resummation in a diagrammatic perturbation theory. For the Tomonaga-Luttinger model including phonons it turned out that the renormalization of the phonon frequencies plays a major role for the relaxation process \cite{5}. In this work we study in which way these important renormalizations of the free propagators appear in our method, which contains only single-time quantities without using advanced and retarded Green’s functions. This is quite interesting as the Hamiltonian dynamics is local in time and memory effects only appear by truncating the infinite hierarchy of quantities.

II. THE MODELS

As in the first paper, we consider systems described by a general statistical operator $\rho_0$ at the initial time $t_0$. The dynamics is determined by a Hamiltonian $H$, and the expectation value of an operator $A$ at time $t$ is given by $\langle A \rangle_t = \text{Tr}[\rho_0 A(t)]$, where $A(t)$ is the operator in the Heisenberg representation with $A(t_0) = A$.

The one-dimensional polaron model is described by the following Hamiltonian

\begin{equation}
H = \sum_k \epsilon_k \psi_k^\dagger \psi_k + \sum_q \omega_q B_q^\dagger B_q + \sum_{q,k} g_q \left( \psi_k^\dagger \psi_{k+q} B_q + B_q^\dagger \psi_{k+q}^\dagger \psi_k \right) .
\end{equation}

The operator $\psi_k$ ($B_q$) annihilates an electron (phonon) in the state with momentum $k$ ($q$) and energy $\epsilon_k$ ($\omega_q$). The electron-phonon coupling strength is denoted by $g_q$. In the case of a linear energy dispersion $\epsilon_k = v_F k$ for the electrons this model is exactly soluble for the initial one-electron state $\rho_0 = |k_0\rangle \langle k_0|$ \cite{4} or more generally for $\rho_0 = |k_1, \ldots, k_N\rangle \langle k_1, \ldots, k_N|$ where $|k_1, \ldots, k_N\rangle$ is an anti-symmetrized $N$-electron state \cite{13}. The time evolution of the electron distribution function $n_k(t) = \langle \psi_k^\dagger \psi_k \rangle_t$ for the initial state $\rho_0 = |k_0\rangle \langle k_0|$ can, e.g., be obtained by numerically solving the exact differential equation derived in Ref. \cite{4}:

\begin{equation}
\frac{d}{dt} n_k(t) = 2 \sum_{q>0} g_q \frac{\sin[(\omega_q - v_F q)(t-t_0)]}{\omega_q - v_F q} \left[ n_{k+q}(t) - n_k(t) \right] .
\end{equation}

As in Ref. \cite{4} we assume that only phonons with $q > 0$ exist in accordance with the fact that each electron is moving “right” with the velocity $v_F$ (chiral model). Even for the initial state $\rho_0 = |k_0; FS\rangle \langle k_0; FS|$ ($k_0 > k_F$) with a Fermi sea $|FS\rangle = \sum_{k\leq k_F} \psi_k^\dagger |\text{Vac}\rangle$ of electrons it turns out that the exact time evolution of the electron and phonon distribution functions can be numerically determined \cite{4}. Here we use the exact solutions for a comparison with the results of our approximate transport equations.

The exact solutions are obtained for the case of an infinite range of momenta $k = 2\pi n/L$ with $n = \ldots, -1, 0, 1, \ldots$ whereas the approximate numerical calculations are for a system with a lower and upper momentum cut-off ($k_\leq \leq k \leq k_\geq$). These differences are of little
importance if the cut-offs are chosen to be far away from the Fermi momentum resp. the initial state at \( k_0 \). In all numerical calculations we use a constant electron-phonon coupling with a momentum cut-off \( q_c \), i.e. \( g_q = g \Theta(q) \Theta(q_c - q) \), and a constant phonon frequency \( \omega_0 \equiv \omega_q \) as a model for longitudinal optical phonons. We define the “resonant” phonon momentum \( q_B \) by \( \omega_0 = v_F q_B \) and choose \( q_c \gg q_B \). In the plots of the numerical results all momenta \( k \) will be given dimensionless as \( k/k_B \) and all times \( t \) as \( \omega_0 t \). The system is described by the following dimensionless parameters: the ratio of the one-electron level spacing and the phonon energy \( \nu = (v_F 2\pi/L)/\omega_0 = (2\pi/L)/q_B \), the coupling constant \( \Gamma = g^2/(\omega_0 \cdot v_F 2\pi/L) \), and \( q_c/q_B \).

### III. QUANTUM KINETIC EQUATIONS

We now present the equations of motion (EOM) for the correlations derived using the general formalism described in Ref. \[1\]. First, the transport equations one order beyond the Born approximation are studied. The form of the equations is independent of \( \rho_0 \), which only determines the initial values. The relevant correlations are \((q,q' > 0)\):

\[
n_k(t) = \langle \psi_k^{\dagger} \psi_k \rangle_t \, , \tag{3}
\]

\[
N_q(t) = \langle B_q^\dagger B_q \rangle_t \, , \tag{4}
\]

\[
h_{k,q}(t) = i \left( \langle B_q^\dagger \psi_{k-q}^{\dagger} \psi_k \rangle_t - \langle B_q \psi_{k-q}^{\dagger} \psi_k \rangle_t \right) \, , \tag{5}
\]

\[
H_{k,q}(t) = \langle B_q^\dagger \psi_{k-q}^{\dagger} \psi_k \rangle_t + \langle B_q \psi_{k-q}^{\dagger} \psi_k \rangle_t \, , \tag{6}
\]

\[
n_{k,q}(t) = 2 \langle \psi_{k-q}^{\dagger} \psi_{k-q} \rangle_t \, , \tag{7}
\]

\[
n_{k,q,q'}(t) = \langle \psi_{k-q-q'}^{\dagger} \psi_{k-q} \psi_{k-q'} \rangle_t + \langle \psi_{k-q}^{\dagger} \psi_{k-q-q'} \psi_{k-q'} \rangle_t \, , \tag{8}
\]

\[
N_{k,q,q'}(t) = i \left( \langle \psi_{k-q-q'}^{\dagger} \psi_{k-q} \psi_{k-q'} \rangle_t - \langle \psi_{k-q}^{\dagger} \psi_{k-q-q'} \psi_{k-q'} \rangle_t \right) \, , \tag{9}
\]

\[
k_{k,q,q'}(t) = \langle B_q^\dagger B_q \psi_{k-q}^{\dagger} \psi_{k-q} \rangle_t + \langle B_q^\dagger B_q \psi_{k-q}^{\dagger} \psi_{k-q} \rangle_t \, , \tag{10}
\]

\[
K_{k,q,q'}(t) = i \left( \langle B_q^\dagger B_q \psi_{k-q}^{\dagger} \psi_{k-q} \rangle_t - \langle B_q^\dagger B_q \psi_{k-q}^{\dagger} \psi_{k-q} \rangle_t \right) \, , \tag{11}
\]

\[
l_{k,q,q'}(t) = \langle B_q^\dagger B_q \psi_{k-q-q'}^{\dagger} \psi_{k-q} \rangle_t + \langle B_q^\dagger B_q \psi_{k-q-q'}^{\dagger} \psi_{k-q} \rangle_t \, , \tag{12}
\]

\[
L_{k,q,q'}(t) = i \left( \langle B_q^\dagger B_q \psi_{k-q-q'}^{\dagger} \psi_{k-q} \rangle_t - \langle B_q^\dagger B_q \psi_{k-q-q'}^{\dagger} \psi_{k-q} \rangle_t \right) \, . \tag{13}
\]

The indices are chosen such that the correlations with three indices are either symmetric or anti-symmetric in \( q \) and \( q' \). The transport equations read

\[
\frac{d}{dt} n_k(t) = \sum_{q > 0} g_q \left( h_{k,q}(t) - h_{k+q,q}(t) \right) \, , \tag{14}
\]
\[
\frac{d}{dt} N_q(t) = -g_q \sum_k h_{k,q}(t) ,
\]

(15)

\[
\frac{d}{dt} h_{k,q}(t) = \left( \epsilon_k - \epsilon_{k-q} - \omega_q \right) H_{k,q}(t) + 2g_q \left[ N_q(t) (n_{k-q}(t) - n_k(t)) - n_k(t) (1 - n_{k-q}(t)) \right] - g_q [n_{k,q}(t) + \sum_{q' > 0} (n_{k,q,q'}(t) + n_{k+q',q,q'}(t))] + \sum_{q' > 0} g_q' \left( k_{k,q,q'}(t) - k_{k+q',q,q'}(t) - l_{k,q,q'}(t) + l_{k+q',q,q'}(t) \right) ,
\]

(16)

\[
\frac{d}{dt} H_{k,q}(t) = -\left( \epsilon_k - \epsilon_{k-q} - \omega_q \right) h_{k,q}(t) - g_q \sum_{q' > 0} (N_{k,q,q'}(t) - N_{k+q',q,q'}(t)) - \sum_{q' > 0} g_q' \left( K_{k,q,q'}(t) - K_{k+q',q,q'}(t) - L_{k,q,q'}(t) + L_{k+q',q,q'}(t) \right) ,
\]

(17)

\[
\frac{d}{dt} n_{k,q}(t) = 2g_q \left( n_{k-q}(t) - n_k(t) \right) h_{k,q}(t) ,
\]

(18)

\[
\frac{d}{dt} n_{k,q,q'}(t) = \left( \epsilon_{k-q-q'} + \epsilon_k - \epsilon_{k-q} - \epsilon_{k-q'} \right) N_{k,q,q'}(t) + \{ g_q \left[ (n_{k-q-q'}(t) - n_{k-q}(t)) h_{k,q}(t) + (n_{k-q}(t) - n_k(t)) h_{k-q,q'}(t) \right] - (q \leftrightarrow q') \} ,
\]

(19)

\[
\frac{d}{dt} N_{k,q,q'}(t) = -\left( \epsilon_{k-q-q'} + \epsilon_k - \epsilon_{k-q} - \epsilon_{k-q'} \right) n_{k,q,q'}(t) + \{ g_q \left[ (n_{k-q-q'}(t) - n_{k-q}(t)) H_{k,q}(t) - (n_{k-q}(t) - n_k(t)) H_{k-q,q'}(t) \right] - (q \leftrightarrow q') \} ,
\]

(20)

\[
\frac{d}{dt} K_{k,q,q'}(t) = \left( \epsilon_{k-q} + \omega_q - \epsilon_{k-q} - \omega_{q'} \right) K_{k,q,q'}(t) + \{ g_q \left[ (N_q(t) + n_{k-q}(t)) h_{k-q,q'}(t) - (1 + N_q(t) - n_{k-q}(t)) h_{k,q'}(t) \right] + (q \leftrightarrow q') \} ,
\]

(21)

\[
\frac{d}{dt} K_{k,q,q'}(t) = -\left( \epsilon_{k-q} + \omega_q - \epsilon_{k-q} - \omega_{q'} \right) K_{k,q,q'}(t) + \{ g_q \left[ (N_q(t) + n_{k-q}(t)) H_{k-q,q'}(t) - (1 + N_q(t) - n_{k-q}(t)) H_{k,q'}(t) \right] - (q \leftrightarrow q') \} ,
\]

(22)
\[
\frac{d}{dt} l_{k,q,q'} (t) = (\omega_q + \omega_{q'} + \epsilon_{k-q-q'} - \epsilon_k) L_{k,q,q'} (t)
\]
\[
\quad + \{ g_q \left[ (N_q(t)+n_k(t)) h_{k-q,q'} (t) + (N_q(t)-n_{k-q-q'}(t)) h_{k,q,q'} (t) \right] \\
\quad + (q \leftrightarrow q') \} ,
\]
\[
\frac{d}{dt} L_{k,q,q'} (t) = - (\omega_q + \omega_{q'} + \epsilon_{k-q-q'} - \epsilon_k) L_{k,q,q'} (t)
\]
\[
\quad + \{ g_q \left[ (N_q(t)+n_k(t)) H_{k-q,q'} (t) - (N_q(t)-n_{k-q-q'}(t)) H_{k,q,q'} (t) \right] \\
\quad + (q \leftrightarrow q') \} .
\]

In the graphical representation of the EOM we often use fermionic (solid) and bosonic (curly) lines without arrows in order to reduce the number of diagrams. Arrows and momentum labels are supposed to be added according to the conservation of the number of electrons and of the momentum. The interaction vertices are given in Fig. 1. The graphs for the distribution functions \(n_k\) and \(N_q\) are those shown in Fig. 2, the diagrams for the “phonon assisted polarizations” \(h_{k,q}\) and \(H_{k,q}\) are shown in Fig. 3, the graph for \(k_{k,q,q'}\), \(K_{k,q,q'}\), \(l_{k,q,q'}\) and \(L_{k,q,q'}\) is given in Fig. 4 and the graph for \(n_{k,q}, n_{k,q,q'}\) and \(N_{k,q,q'}\) is the one shown in Fig. 5.

By restriction to the correlations \(n_k, N_q, h_{k,q}\) and \(H_{k,q}\) the transport equations in Born approximation are obtained. For the case of vanishing initial correlations they read after integrating the differential equations for \(h_{k,q}\) and \(H_{k,q}\):

\[
\frac{d}{dt} n_k (t) = 2 \sum_{q>0} g_q^2 \int_{t_0}^t dt' \left\{ \cos \left( (\epsilon_k - \epsilon_{k-q} - \omega_q) (t - t') \right) \right. \\
\left. \times \left[ N_q (n_{k-q} - n_k) - n_k (1 - n_{k-q}) \right]_{t'} \right. \\
\left. - \cos \left( (\epsilon_{k+q} - \epsilon_k - \omega_q) (t - t') \right) \left[ N_q (n_{k+q} - n_k) - n_{k+q} (1 - n_k) \right]_{t'} \right\} ,
\]
\[
\frac{d}{dt} N_q (t) = - 2 g_q^2 \sum_k \int_{t_0}^t dt' \cos \left( (\epsilon_k - \epsilon_{k-q} - \omega_q) (t - t') \right) \\
\times \left[ N_q (n_{k-q} - n_k) - n_k (1 - n_{k-q}) \right]_{t'} .
\]

### A. The polaron model

We first discuss the transport equations for the one-electron polaron model with the initial state \(\rho_0 = |k_0\rangle \langle k_0|\). A renormalization of the free propagators, i.e. the cosine factors in Eqs. (23) and (26), is of importance for the Born approximation to yield acceptable results for the first phonon replica at larger times. In the one-electron case there is no significant renormalization of the phonon propagators and the most important renormalization is the inclusion of a damping factor \(\exp[-2\gamma (t - t')]\) for the electronic propagations. For the determination of \(\gamma\) from the second order electron self-energy in equilibrium we refer to Ref. [3].
We solved numerically the equations including the two-particle correlations as described in Eqs. (14)-(24). The initial states discussed so far do not have initial correlations. Fig. 7 shows the time evolution of \( n_k \) for the first phonon replica at \( k = k_0 - q_B \) with \( q_B = \omega_0/v_F \). The importance of the damping of the electronic propagation in the Born approximation can be seen clearly. The approximation including the two-particle correlations seems to include the damping in the dynamics as it follows the exact evolution for a long time.

We now explain how the dynamical equations incorporate the damping of the electron propagation. In the Keldysh formalism [14] the renormalization of the free propagators is usually justified by the Kadanoff-Baym ansatz [10–12], which substitutes the free propagators in Eqs. (24) and (26) by the exact retarded/advanced Green’s functions. As these Green’s functions are not known they are, in a first approximation, taken in lowest order perturbation theory for the equilibrium. In our case, the electron propagators are renormalized by the self-energy in second order Born approximation shown in Fig. 6. Diagrammatically these self-energy insertions arise from the third diagram in Fig. 3. Therefore, in order to explain the renormalization we can neglect for a moment the two-electron correlations \( n_{k,q} \), \( n_{k,q,q'} \) and \( N_{k,q,q'} \). Then the differential equations for the phonon assisted density matrices read

\[
\left( \frac{d}{dt} + i(\epsilon_k - \epsilon_{k-q} - \omega_q) \right) (h_{k,q} + iH_{k,q})(t) = 2g_q \left[ N_q(t) (n_{k,q}(t) - n_k(t)) - n_k(t) (1 - n_{k-q}(t)) \right] + \sum_{q' > 0} g_{q'} \left[ (k_{k,q,q'} - iK_{k,q,q'})(t) - (k_{k+q',q,q'} - iK_{k+q',q,q'})(t) \right] - \left( l_{k,q,q'} - iL_{k,q,q'} \right)(t) + \left( l_{k+q',q,q'} - iL_{k+q',q,q'} \right)(t) .
\]

The terms in the time derivative of \( (k_- - iK_-) \) and \( (l_- - iL_-) \) that yield the renormalization of the electron propagation for \( (h_{k,q} + iH_{k,q}) \) are the “coherent terms”, i.e. those proportional to \( h_{k,q} \) and \( H_{k,q} \):

\[
\left( \frac{d}{dt} - i(\epsilon_{k-q} + \omega_q - \epsilon_{k-q'} - \omega_{q'}) \right) (k_{k,q,q'} - iK_{k,q,q'})(t) = -g_{q'} \left( 1 + N_q(t) (n_{k,q}(t) - n_k(t)) \right) (h_{k,q} + iH_{k,q})(t) + \ldots
\]

\[
\left( \frac{d}{dt} - i(\epsilon_{k+q'-q} + \omega_q - \epsilon_k - \omega_{q'}) \right) (k_{k+q',q,q'} - iK_{k+q',q,q'})(t) = g_q \left( N_q(t) + n_{k+q'-q}(t) \right) (h_{k,q} + iH_{k,q})(t) + \ldots
\]

and the analogous expressions for \( (l_- - iL_-) \). If there are no initial correlations we obtain the following integro-differential equation for \( (h_{k,q} + iH_{k,q}) \) neglecting the “incoherent terms”:

\[
\left( \frac{d}{dt} + i(\epsilon_k - \epsilon_{k-q} - \omega_q) \right) (h_{k,q} + iH_{k,q})(t) = 2g_q \left[ N_q(t) (n_{k,q}(t) - n_k(t)) - n_k(t) (1 - n_{k-q}(t)) \right] - \int_{t_0}^t dt' \Pi_{k,q}(t, t')(h_{k,q} + iH_{k,q})(t') ,
\]

\( 6 \)
where
\[
\Pi_{k,q}(t, t') = \Theta(t - t') \sum_{q' > 0} g_q^2 \left[ (1 + N_{q'} - n_{k-q'}) (t') e^{i(\epsilon_{k-q} + \omega_q - \epsilon_{k-q'} - \omega_{q'})(t-t')} ight. \\
+ (N_{q'} + n_{k+q'}) (t') e^{i(\omega_q + \omega_{q'} + \epsilon_{k-q} - \epsilon_{k+q'})(t-t')} \\
+ (1 + N_{q'} - n_{k-q-q'}) (t') e^{i(\omega_q + \omega_{q'} + \epsilon_{k-q} - \epsilon_{k-q-q})(t-t')} \\
+ (N_{q'} + n_{k+q-q}) (t') e^{i(\epsilon_{k+q-q} + \omega_q - \epsilon_{k-q})(t-t')}.\]  
(31)

In this context, the time-dependence of the distribution functions \(n_k\) and \(N_q\) is of little importance, as can be checked numerically. If we therefore replace them by their initial values \(n_k(t_0)\) and \(N_q(t_0)\), the “self-energy” \(\Pi_{k,q}(t, t')\) depends only on the relative time \(t - t'\). Then the differential equation (30) is easily solved:
\[
(h_{k,q} + iH_{k,q})(t) = 2g_q \int_{t_0}^{t} dt' D_{k,q}(t - t') \left[ N_q(t') (n_{k-q}(t') - n_k(t')) - n_k(t') (1 - n_{k-q}(t')) \right]
\]  
(32)

where \(D_{k,q}\) is the retarded solution to the Dyson equation
\[
D_{k,q}(t - t') = D_{k,q}^{(0)}(t - t') - \int_{t'}^{t} dt_1 \int_{t_1}^{t_2} dt_2 D_{k,q}^{(0)}(t - t_1) \Pi_{k,q}(t_1 - t_2) D_{k,q}(t_2 - t')
\]
(33)

with the “free propagation” \(D_{k,q}^{(0)}(t - t') = \Theta(t - t') e^{-i(\epsilon_k - \epsilon_{k-q} - \omega_q)(t-t')}\). Except for the fact that \(D_{k,q}\) is the “propagator” for an electron-hole pair plus a phonon, this is an ordinary Dyson equation, which can be easily solved by Fourier transformation
\[
- i\tilde{D}_{k,q}(\omega) = \left( \omega - (\epsilon_k - \epsilon_{k-q} - \omega_q) + i\tilde{\Pi}_{k,q}(\omega) + i0 \right)^{-1}
\]
(34)

and leads to a damping for both electronic propagations.

As can be seen from Fig. 8, the time evolution of the occupation at the second phonon replica at \(k = k_0 - 2q_B\) is much better described by the approximate equations including correlations (long dashed line). This is easily explained by the fact that contrary to the Born approximation these equations contain coherent two-phonon processes by the inclusion of expectation values \(\langle B^{(i)}(\hat{B}^{(i)}(\psi^\dagger(\psi)\rangle\rangle\). By an analogous argument to the one used to explain a damping factor in the Born approximation it can be justified to damp the time evolution of \(n_{k,q}, n_{k,q,q'}, N_{k,q,q'}, k_{k,q,q'}, K_{k,q,q'}, l_{k,q,q'},\) and \(L_{k,q,q'},\) i.e. replacing \(\frac{d}{dt}\) by \(\left( \frac{d}{dt} + 2\gamma \right)\) in the corresponding differential equations. Why the correlations with four electron operators \(\langle \psi^\dagger \psi^\dagger \psi^\dagger \psi \rangle\) are damped with \(2\gamma\) instead of \(4\gamma\) as one might expect will be discussed later when we extend the equations by “half an order”. The inclusion of this damping for the two-particle correlations yields an improvement of the solution for longer times in the same way as the damping of the Born approximation does one order higher (dot-dashed line in Fig. 8).

Figs. 9 and 10 show the electron distribution function at two different times. In the range of the first phonon replica, the distribution function is well described by the Born approximation with a damping factor for the free electron propagation (dashed lines) and by the kinetic equations including two-particle correlations (long dashed lines), whereas the Born approximation without damping (dotted lines) yields unphysical results like negative occupation numbers. For the second phonon replica only the kinetic equations including two-particle correlations are in excellent agreement with the exact solution (solid lines).
B. The Tomonaga-Luttinger model

We now turn to the model with a Fermi sea (Tomonaga-Luttinger model). The initial state $\rho_0 = |k_0; \text{FS}\rangle \langle k_0; \text{FS}|$ again contains no correlations. The most important effect of the Fermi sea (apart from the Pauli blocking) is the renormalization of the phonon propagator. Due to the linear energy dispersion of the electrons, which also enables the exact solution by bosonization [5], the phonon mode splits into two modes in the equilibrium case and the second order self-energy (electron-hole bubble) already gives the exact result for the equilibrium phonon Green’s function. In Ref. [5] the importance of the phonon renormalization for the kinetic equations in Born approximation was shown. In our dynamical equations this renormalization is incorporated by the second diagram in Fig. 3. By extending the discussion following Eq. 27 to the case of the renormalization of both the electron and phonon propagation, i.e. by inclusion of the electron correlations $n_{k,q}$, $n_{k,q,q'}$, and $N_{k,q,q'}$, we obtain the “self-energy”

$$-i\tilde{\Pi}_{k,q}(\omega) = \frac{g_q^2 \sum_{k'}(n_{k'-q} - n_{k'})(t_0)}{\omega + i0} - i2\gamma$$  \hspace{1cm} (35)$$

for the “propagator” $\tilde{D}_{k,q}(\omega)$ in Eq. (34), where we approximated the electron self-energies by a constant damping factor. But this termination of the “continued fraction expansion” for the propagator $\tilde{D}_{k,q}(\omega)$ is not the best one. Better results are obtained if the damping of the propagation of the scattered electron in $n_{k,q}$, $n_{k,q,q'}$, and $N_{k,q,q'}$ is included:

$$-i\tilde{\Pi}_{k,q}(\omega) = \frac{g_q^2 \sum_{k'}(n_{k'-q} - n_{k'})(t_0)}{\omega + i2\gamma} - i2\gamma.$$  \hspace{1cm} (36)$$

Now $\sum_{k'}(n_{k'-q} - n_{k'})(t_0) = \frac{L^2\pi}{2\gamma}q$ for $0 < q \leq q_c$, so that the propagator $D_{k,q}$ is given as described in Ref. [3] by

$$D_{k,q}(t-t') = \Theta(t-t') \left\{ \frac{\lambda_+}{\lambda_+ - \lambda_-} e^{-(i\lambda_+ - 2\gamma)(t-t')} - \frac{\lambda_-}{\lambda_+ - \lambda_-} e^{-(i\lambda_- - 2\gamma)(t-t')} \right\}$$  \hspace{1cm} (37)$$

where

$$\lambda_{\pm} = \frac{1}{2} \left\{ (v_F q - \omega_q) \pm \sqrt{(v_F q - \omega_q)^2 + 4g_q^2 \frac{L}{2\gamma}q} \right\}.$$  \hspace{1cm} (38)$$

The splitting of the phonon frequency into two modes as well as the above mentioned cancellation of certain corrections for the electron-hole bubble are special effects of the linear electronic energy dispersion. The terms proportional to $k_{k,q,q'}$, $K_{k,q,q'}$, $l_{k,q,q'}$, and $L_{k,q,q'}$ in the time evolution of the phonon assisted density matrices $h_{k,q}$ and $H_{k,q}$ (Eqs. (16)-(17)), which led to a damping for $h_{k,q}$ and $H_{k,q}$, do not contribute to the time evolution of $N_q$ (Eq. (15)) due to the summation of $h_{k,q}$ over $k$. (There are contributions from electron states near the momentum cut-offs which do not cancel exactly. But in the spirit discussed above they may be neglected.) This cancellation occurs only due to the linear energy dispersion which makes the non-interacting energies $\epsilon_k - \epsilon_{k-q} - \omega_q = v_F q - \omega_q$ of $h_{k,q}$ and $H_{k,q}$ independent
of \( k \). Therefore, the damping of \( h_{k,q} \) and \( H_{k,q} \) is only “existent” for their appearance in the differential equation (14) for \( n_k \), but not in that for \( N_q \). Although for the parameters chosen this remark is of little importance, it should have been taken into account for the equations in Born approximation with damping.

The same kind of exact cancellation is responsible for the damping of only “half the electrons” in \( n_{k,q} \), \( n_{k,q,q'} \), and \( N_{k,q,q'} \). These correlations enter the differential equations for correlations of lower order only in the form \( \tilde{n}_{k,q}(t) := n_{k,q}(t) + \sum_{q'>0}(n_{k,q,q'}(t) + n_{k+q',q''}(t)) = \sum_{k'} \left\{ \langle \psi_{k'}^{\dagger} \psi_{k-q}^{\dagger} \psi_{k-q'} \psi_{k-q''} \rangle_t + \langle \psi_{k'}^{\dagger} \psi_{k} \psi_{k-k-q} \psi_{k-k-q'} \rangle_t \right\} \) and \( \tilde{N}_{k,q}(t) := - \sum_{q'>0}(N_{k,q,q'}(t) - N_{k+q',q''}(t)) = i \sum_{k'} \left\{ \langle \psi_{k'}^{\dagger} \psi_{k-q}^{\dagger} \psi_{k-q'} \psi_{k-q''} \rangle_t - \langle \psi_{k'}^{\dagger} \psi_{k} \psi_{k-k-q} \psi_{k-k-q'} \rangle_t \right\} \). As the non-interacting energy is the same for all these correlations, i.e. vanishes exactly due to the linear energy dispersion where momentum and energy conservation are identical for the electrons, only the quantities \( \tilde{n}_{k,q} \) and \( \tilde{N}_{k,q} \) must be considered. This fact was also used in the numerical calculations since it considerably reduces the number of differential equations to be solved. It also allows the extension of the transport equations by “half an order”, i.e. in the next order we take into account only certain sums of the correlations over momenta such that the number of indices does not increase. To be specific, we consider the correlations which arise from \( K_{k,q,q'} \), \( k_{k,q,q'} \), \( L_{k,q,q'} \), and \( l_{k,q,q'} \) by the substitution of one or both phonon operators \( B_q \) by the density operator \( \rho_q = \sum_{k'} \psi_{k'}^{\dagger} \psi_{k'} \). These correlations as well as the extended transport equations are given in the Appendix.

Figs. 11-14 show the numerical results for the Tomonaga-Luttinger model. As discussed in Ref. 3 the renormalization of the phonon propagation (Eq. (37)) is important for the Born approximation to describe the relaxation process adequately. The time evolution of the electron occupation number at the first phonon replica shown in Fig. 11 demonstrates that the number of indices does not increase. To be specific, we consider the correlations which arise from \( K_{k,q,q'} \), \( k_{k,q,q'} \), \( L_{k,q,q'} \), and \( l_{k,q,q'} \) by the substitution of one or both phonon operators \( B_q \) by the density operator \( \rho_q = \sum_{k'} \psi_{k'}^{\dagger} \psi_{k'} \). These correlations as well as the extended transport equations are given in the Appendix.

Figs. 11-14 show the numerical results for the Tomonaga-Luttinger model. As discussed in Ref. 3 the renormalization of the phonon propagation (Eq. (37)) is important for the Born approximation to describe the relaxation process adequately. The time evolution of the electron occupation number at the first phonon replica shown in Fig. 11 demonstrates that the number of indices does not increase. To be specific, we consider the correlations which arise from \( K_{k,q,q'} \), \( k_{k,q,q'} \), \( L_{k,q,q'} \), and \( l_{k,q,q'} \) by the substitution of one or both phonon operators \( B_q \) by the density operator \( \rho_q = \sum_{k'} \psi_{k'}^{\dagger} \psi_{k'} \). These correlations as well as the extended transport equations are given in the Appendix.

C. Initial correlations

Finally, we discuss the role of initial correlations for the dynamics. The initial correlations enter as initial values of the kinetic equations. In order to study the capability of the kinetic equations (13)-(24) to describe the effect of initial correlations we consider different initial states \( \rho_0 \) inducing identical initial distribution functions \( n_k(t_0) \) and \( N_q(t_0) \). Since the
dynamics is exactly soluble for $\rho_0 = |k_1, \ldots, k_N\rangle\langle k_1, \ldots, k_N|$ the initial states were taken as convex combinations of states of this form. The exact solution is described elsewhere \[13\]. In the example shown in Fig. [13] the two initial states considered are

\begin{align*}
\text{(a)} \quad & \rho_0 = \frac{1}{2} \left\{ |k_0, k_0 - q_B, k_0 - 2q_B, k_0 - 3q_B\rangle\langle \ldots | \\
& + |k_0 - \frac{1}{2} q_B, k_0 - \frac{3}{2} q_B, k_0 - \frac{5}{2} q_B, k_0 - \frac{7}{2} q_B\rangle\langle \ldots | \right\}
\end{align*}

\begin{align*}
\text{(b)} \quad & \rho_0 = \frac{1}{2} \left\{ |k_0, k_0 - \frac{1}{2} q_B, k_0 - q_B, k_0 - \frac{3}{2} q_B\rangle\langle \ldots | \\
& + |k_0 - 2q_B, k_0 - \frac{5}{2} q_B, k_0 - 3q_B, k_0 - \frac{7}{2} q_B\rangle\langle \ldots | \right\}
\end{align*}

with a very small “resonant” phonon momentum $q_B = 2 \cdot \frac{2\pi}{L}$ in order to make the effect more pronounced. As the electron and phonon distribution functions $n_k(t_0)$ and $N_q(t_0)$ as well as the phonon assisted density matrices $h_{k,q}(t_0)$ and $H_{k,q}(t_0)$ are the same for both cases the Born approximation yields identical results whereas the kinetic equations including two-particle correlations provide an excellent approximation for the exact time evolution (Fig. [13]).

### IV. SUMMARY

We studied kinetic equations for electron-phonon systems beyond the Born approximation by the inclusion of many-particle correlations. The dynamics is described by a system of differential equations for the distribution functions and single-time correlations. The results from the approximate kinetic equations were compared with the exact solutions for one-dimensional electron-phonon models. In contrast to real-time Green’s function techniques, which have definite advantages in the context of quantum field theories, equations for retarded/advanced Green’s functions are not needed.

For systems with intrinsic damping, like the one-dimensional electron-phonon models studied, already the inclusion of two-particle correlations leads to a significant improvement in comparison with the Born approximation. The renormalization of the electron and phonon propagation, which is important to reduce the occurrence of unphysical results in the Born approximation, is automatically incorporated in the dynamics. There is no need for the Kadanoff-Baym ansatz. In addition to the renormalization of the propagators, the improved kinetic equations also correctly describe coherent two-phonon processes. We studied in detail the one-electron polaron model and the Tomonaga-Luttinger model.

Finally, the ability of the improved kinetic equations to describe the effect of initial correlations was shown by comparison with exact solutions for the many-electron case.

### ACKNOWLEDGMENTS

This work was financially supported by the Deutsche Forschungsgemeinschaft (SFB 345 “Festkörper weit weg vom Gleichgewicht”). Parts of the numerical calculations were done on machines of the RRZN (Regionales Rechenzentrum für Niedersachsen).

10
APPENDIX:

In this Appendix we present the extension of the kinetic equations mentioned in Sec. III B. In addition to the two-particle correlations, we consider the following correlations

\[ p_{k,q,q'}(t) = i \sum_{k'} \left( \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c - \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c \right) \]  

(A1)

\[ P_{k,q,q'}(t) = \sum_{k'} \left( \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c + \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c \right) \]  

(A2)

\[ r_{k,q,q'}(t) = i \sum_{k'} \left( \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c - \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c \right) \]  

(A3)

\[ R_{k,q,q'}(t) = \sum_{k'} \left( \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c + \langle B_{q'}^\dagger \psi_{k'-q'}^\dagger \psi_{k'-q}^\dagger \psi_{k-q} \rangle_t^c \right) . \]  

(A4)

which arise from \( K_{k,q,q'}, k_{k,q,q'}, L_{k,q,q'}, \) and \( l_{k,q,q'} \) by the substitution of one phonon operator \( B_q \) by the density operator \( \rho = \sum_{k'} \psi_{k'-q}^\dagger \psi_{k'} \), and

\[ s_{k,q,q'}(t) = \sum_{k',k''} \left( \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c + \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c \right) \]  

(A5)

\[ S_{k,q,q'}(t) = i \sum_{k',k''} \left( \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c - \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c \right) \]  

(A6)

\[ t_{k,q,q'}(t) = \sum_{k',k''} \left( \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c + \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c \right) \]  

(A7)

\[ T_{k,q,q'}(t) = i \sum_{k',k''} \left( \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c - \langle \psi_{k'-q}^\dagger \psi_{k'-q}^\dagger \psi_{k''-q}^\dagger \psi_{k''-q} \psi_{k'-q} \psi_{k''-q} \psi_{k'-q} \rangle_t^c \right) , \]  

(A8)

in which the second phonon operator is also substituted by the corresponding density operator.

The extension of our transport equations by these correlations reads

\[ \frac{d}{dt} \tilde{n}_{k,q} = \ldots + \sum_{q' > 0} g_{q'} \left\{ p_{k,q,q'} - p_{k+q',q,q'} + r_{k,q,q'} - r_{k+q',q,q'} \right\} \]  

(A9)

\[ \frac{d}{dt} \tilde{N}_{k,q} = \ldots + \sum_{q' > 0} g_{q'} \left\{ P_{k,q,q'} - P_{k+q',q,q'} - R_{k,q,q'} + R_{k+q',q,q'} \right\} \]  

(A10)

\[ \left( \frac{d}{dt} + 2\gamma \right) k_{k,q,q'} = \ldots - g_{q} P_{k,q',q} - g_{q} P_{k,q,q'} \]  

(A11)

\[ \left( \frac{d}{dt} + 2\gamma \right) K_{k,q,q'} = \ldots + g_{q} P_{k,q,q'} - g_{q} P_{k,q,q'} \]  

(A12)

\[ \left( \frac{d}{dt} + 2\gamma \right) l_{k,q,q'} = \ldots + g_{q} r_{k,q',q} + g_{q} r_{k,q,q'} \]  

(A13)

\[ \left( \frac{d}{dt} + 2\gamma \right) L_{k,q,q'} = \ldots - g_{q} R_{k,q,q'} - g_{q} R_{k,q,q'} \]  

(A14)

where the dots stand for the terms in these equations written down so far.
\[
\left( \frac{d}{dt} + 2\gamma \right) P_{k,q,q'} = -(\omega_q' - v_F q') P_{k,q,q'} \\
+ g_q \left\{ -(1 + N_{q'} - n_{k-q'}) \bar{n}_{k,q} + (N_q' + n_{k-q}) \bar{n}_{k-q,q'} \right\} \\
+ g_q \frac{1}{2} \left\{ (h_{k-q,q'} - h_{k,q'}) \sum_{k'} h_{k',q'} - (H_{k-q,q'} - H_{k,q'}) \sum_{k'} H_{k',q'} \right\} \\
+ g_q \sum_{k'} (n_{k'-q} - n_{k'}) k_{k,q,q'} - g_q s_{k,q,q'} \\
\tag{A15}
\]

\[
\left( \frac{d}{dt} + 2\gamma \right) P_{k,q,q'} = (\omega_q' - v_F q') P_{k,q,q'} \\
+ g_q \left\{ -(1 + N_{q'} - n_{k-q'}) \bar{N}_{k,q} + (N_q' + n_k) \bar{n}_{k-q,q'} \right\} \\
+ g_q \frac{1}{2} \left\{ (h_{k-q,q'} - h_{k,q'}) \sum_{k'} H_{k',q'} + (H_{k-q,q'} - H_{k,q'}) \sum_{k'} h_{k',q'} \right\} \\
+ g_q \sum_{k'} (n_{k'-q} - n_{k'}) K_{k,q,q'} + g_q' S_{k,q,q'} \\
\tag{A16}
\]

\[
\left( \frac{d}{dt} + 2\gamma \right) r_{k,q,q'} = -(\omega_q' - v_F q') R_{k,q,q'} \\
+ g_q \left\{ -(1 + N_{q'} - n_{k-q'}) \bar{N}_{k,q} + (N_q' + n_k) \bar{n}_{k-q,q'} \right\} \\
+ g_q \frac{1}{2} \left\{ (h_{k-q,q'} - h_{k,q'}) \sum_{k'} h_{k',q'} - (H_{k-q,q'} - H_{k,q'}) \sum_{k'} H_{k',q'} \right\} \\
- g_q \sum_{k'} (n_{k'-q} - n_{k'}) l_{k,q,q'} - g_q t_{k,q,q'} \\
\tag{A17}
\]

\[
\left( \frac{d}{dt} + 2\gamma \right) R_{k,q,q'} = (\omega_q' - v_F q') r_{k,q,q'} \\
+ g_q \left\{ (1 + N_{q'} - n_{k-q'}) \bar{N}_{k,q} - (N_q' + n_k) \bar{n}_{k-q,q'} \right\} \\
+ g_q \frac{1}{2} \left\{ -(h_{k-q,q'} - h_{k,q'}) \sum_{k'} H_{k',q'} + (H_{k-q,q'} - H_{k,q'}) \sum_{k'} h_{k',q'} \right\} \\
+ g_q \sum_{k'} (n_{k'-q} - n_{k'}) L_{k,q,q'} + g_q' T_{k,q,q'} \\
\tag{A18}
\]

and last but not least

\[
\left( \frac{d}{dt} + 2\gamma \right) s_{k,q,q'} = g_q \left\{ \sum_{k'} (n_{k'-q} - n_{k'}) p_{k,q',q} \right\} \\
+ \frac{1}{2} \left\{ (\bar{N}_{k-q,q'} - \bar{N}_{k,q'}) \sum_{k'} H_{k',q'} + (\bar{n}_{k-q,q'} - \bar{n}_{k,q'}) \sum_{k'} h_{k',q'} \right\} + (q \leftrightarrow q') \\
\tag{A20}
\]

\[
\left( \frac{d}{dt} + 2\gamma \right) S_{k,q,q'} = g_q \left\{ \sum_{k'} (n_{k'-q} - n_{k'}) P_{k,q',q} \right\} \\
+ \frac{1}{2} \left\{ (\bar{N}_{k-q,q'} - \bar{N}_{k,q'}) \sum_{k'} h_{k',q'} - (\bar{n}_{k-q,q'} - \bar{n}_{k,q'}) \sum_{k'} H_{k',q'} \right\} - (q \leftrightarrow q') \\
\tag{A21}
\]
\[
\left( \frac{d}{dt} + 2\gamma \right) t_{k,q,q'} = g_q \left\{ \sum_{k'} (n_{k'-q} - n_{k'}) r_{k,q',q} \right\} + \frac{1}{2} \left[ -(\tilde{N}_{k-q,q'} - \tilde{N}_{k,q'}) \sum_{k'} H_{k',q'} + (\tilde{n}_{k-q,q'} - \tilde{n}_{k,q'}) \sum_{k'} h_{k',q'} \right] \right\} + (q \leftrightarrow q')
\]

\[
\left( \frac{d}{dt} + 2\gamma \right) T_{k,q,q'} = g_q \left\{ -\sum_{k'} (n_{k'-q} - n_{k'}) R_{k,q',q} \right\} + \frac{1}{2} \left[ (\tilde{N}_{k-q,q'} - \tilde{N}_{k,q'}) \sum_{k'} h_{k',q'} + (\tilde{n}_{k-q,q'} - \tilde{n}_{k,q'}) \sum_{k'} H_{k',q'} \right] \right\} + (q \leftrightarrow q').
\]

The same discussion as for the damping of the phonon assisted density matrices (Eqs. (27)-(34)) shows that the terms \(p_{...}\), \(P_{...}\), \(r_{...}\), and \(R_{...}\) in the time evolution of \(\tilde{n}_{k,q}\) and \(\tilde{N}_{k,q}\) lead to a damping of \(\tilde{n}_{k,q}\) and \(\tilde{N}_{k,q}\) by the same factor \(2\gamma\), as mentioned in Sec. III A. In general, in all correlations containing a density operator \(\rho_q\), i.e. \(\sum_{k'} \langle \psi_{k'-q}^\dagger \psi_{k'} \rangle\), these electron propagations are not damped.
REFERENCES

[1] J. Fricke, to be published.
[2] R. Zimmermann and J. Wauer, J. Lumin. 58 (1994), 271.
[3] J. Fricke, doctor thesis, University of Göttingen, 1996.
[4] V. Meden, J. Fricke, C. Wöhler, K. Schönhammer, Z. Phys. B 99 (1996), 357.
[5] V. Meden, C. Wöhler, J. Fricke, and K. Schönhammer, Phys. Rev. B 52 (1995), 5624.
[6] R. Zimmermann, Phys. Stat. Sol. 159 (1990), 317.
[7] H. Haug, Phys. Stat. Sol. 173 (1992), 139; L. Bányai et al., Phys. Stat. Sol. 173 (1992), 149.
[8] J. Schilp et al., Phys. Rev. B 50 (1994), 5435.
[9] Ph. Daguzan et al., Phys. Rev. B 52 (1995), 17099.
[10] P. Lipavský et al., Phys. Rev. B 34 (1986), 6933.
[11] L. P. Kadanoff and G. Baym, “Quantum Statistical Mechanics,” W. A. Benjamin, New York, 1962.
[12] H. Schoeller, Ann. Phys. (N.Y.) 229 (1994), 273.
[13] K. Schönhammer and C. Wöhler, to be published.
[14] J. Rammer and H. Smith, Rev. Mod. Phys. 58 (1986), 323.
FIGURES

FIG. 1. The interaction vertices for the polaron model.

FIG. 2. The diagrams for the electron and the phonon distribution function.

FIG. 3. The graphs for the time evolution of $h_{k,q}$ and $H_{k,q}$.

FIG. 4. The graph for the correlations $K_{k,q,q'}$, $k_{k,q,q'}$, $L_{k,q,q'}$, and $l_{k,q,q'}$. 
FIG. 5. The graph for the correlations \( n_{k,q} \), \( n_{k,q,q'} \), and \( N_{k,q,q'} \).

FIG. 6. Second order self-energy for the electron propagator.

The occupation at the first phonon replica (\( k = k_0 - q_B \))

FIG. 7. Time evolution of the electron occupation at the first phonon replica (\( k = k_0 - q_B \)) for the one-electron polaron system with \( \nu = 1/16 \), \( \Gamma = 0.005 \), and \( q_c/q_B = 5 \). The exact solution is compared with the results from the kinetic equations in Born approximation with and without damping of the electron propagation and the improved equations including two-particle correlations.
The occupation at the second phonon replica ($k = k_0 - 2q_B$)

FIG. 8. Time evolution of the electron occupation at the second phonon replica ($k = k_0 - 2q_B$) for the system of Fig. 7. In addition to Fig. 7 the improved equations with damping are considered.

The electron distribution function at $\omega_0 t = 20$.

FIG. 9. The electron distribution function at $\omega_0 t = 20$ for the system of Fig. 7. Shown are the first and second phonon replica. The electron was initially at $k_0 = 7.5q_B$. 
The electron distribution function at $\omega_0 t = 40$.

![Graph showing the electron distribution function at $\omega_0 t = 40$.]

FIG. 10. The same as in Fig. but for the time $\omega_0 t = 40$.

The occupation at the first phonon replica ($k = k_0 - q_B$)

![Graph showing the occupation at the first phonon replica.]

FIG. 11. Time evolution of the electron occupation at the first phonon replica ($k = k_0 - q_B$) for the Tomonaga-Luttinger model with $\nu = 1/20$, $\Gamma = 0.003$, and $q_c/q_B = 3$. The excited electron was initially at $k_0 = 6q_B$ ($k_F = 0$). The exact solution is compared with the results from the kinetic equations in Born approximation with and without renormalization of the phonon propagation and the improved equations including two-particle correlations.
FIG. 12. Time evolution of the electron occupation at the second phonon replica \((k = k_0 - 2q_B)\) for the system of Fig. [1]. In addition to Fig. [1] the kinetic equations are extended by “half an order”.

FIG. 13. The electron distribution function at \(\omega_0 t = 35\) for the system of Fig. [1]. Shown are the first and second phonon replica. The electron was initially at \(k_0 = 6q_B\).
The electron distribution function at $\omega_0 t = 100$. 

FIG. 14. The same as in Fig. 13, but for the time $\omega_0 t = 100$. The Born approximation without renormalization of the phonon propagation is omitted as it does not yield acceptable results at larger times (see Fig. 11).

The electron distribution function at $\omega_0 t = 5$. 

FIG. 15. The electron distribution function at $\omega_0 t = 5$ for a four-electron system with $\nu = 1/2$, $\Gamma = 0.01$, and $q_e/q_B = 10$. The system was in two different initial states (a) and (b) described in the text with the same initial distribution functions. At time $t = 0$ the electron distribution function was 0.5 at $k = 33-36.5q_B$ and vanished elsewhere. The exact solution is compared with the results from the kinetic equations in Born approximation and from the improved equations including two-particle correlations.