On unconventional integrations and cross ratio on supermanifolds

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The conventional integration theory on supermanifolds had been constructed so as to possess (an analog of) Stokes’ formula. In it, the exterior differential \( d \) is vital and the integrand is a section of a fiber bundle of finite rank. Other, not so popular, but, nevertheless, known integrations are analogs of Berezin integral associated with infinite dimensional fibers. Here I offer other unconventional integrations that appear thanks to existence of several versions of traces and determinants and do not allow Stokes formula. Such unconventional integrations have no counterpart on manifolds except in characteristic \( p \).

Another type of invariants considered are analogs of the cross ratio for “classical superspaces”.

As a digression, homological fields corresponding to simple Lie algebras and superalgebras are described.

For the basics on Linear Algebra in Superspaces and Supermanifold theory see [1]; for notations and useful facts see [2], [3]. At the talk I also considered related issues partly collected in [4]. As compared with the talk, §§2, 3 are new; they are a part of the talk given 10 years earlier [5] but yet unpublished. Encouraged by Manin’s selected examples [6] and recent results in classification of simple Lie superalgebras [7], I decided to draw attention to these issues.

1. INTEGRATION

1.1. Integration with Stokes’ formula

In mid 1970’s J. Bernstein and I discussed how to construct an analog of integration theory on supermanifolds. We had at our disposal (1) the differential forms, i.e., functions polynomial in differentials of the coordinates, the coefficients of these polynomials being usual functions and (2) volume forms, the latter constituted a rank one module \( \text{Vol} \) over the algebra of functions and under the change of coordinates the generator \( \text{vol}(x(y)) \) of \( \text{Vol} \) accrued the Berezinian (superdeterminant) of the Jacobi matrix as the factor. Each of the above notions had to be carefully reconsidered in super setting because even the most innocent-looking notions and theorems (e.g., the Foubini theorem) displayed, in super-setting, funny signs at unexpected places, see [8], v. 31.

On manifolds, one can integrate differential forms; on supermanifolds, one can not: their transformation rule yields no analog of determinant, except in the absence of odd parameters. On the other hand, one, clearly, can integrate elements of \( \text{Vol} \), provided they are with compact support, of course. But we wanted to have some analog of Stokes’ formula, and, therefore, needed (1) elements of “degrees” lesser than that of volume forms, and (2) the notion of the supermanifold with boundary to overcome the puzzle demonstrated by “Rudakov’s example”; for solution see [9].

To have integration theory, one needs not only what to integrate (the integrand), but over what (cycle), and orientation. The latter two notions turned out to be more involved than we originally thought; Shander clarified this in his development of integration theory, see [10] and the details in [8]. Actually, what we had had was sufficient to construct the integration theory desired: by setting \( \Sigma_{-i} = \text{Hom}_F(\Omega^i, \text{Vol}) \), where \( F \) is the superspace of functions, we obtain a complex dual to
the de Rham one with \( \Sigma_0 = \text{Vol} \) as forms of the highest degree. We called the elements of \( \Sigma \) *integrable* forms (the ones one can integrate) and described how to integrate such forms in [11].

### 1.2. Veblen’s problem and Rudakov

We wondered for a while if there is another integration theory with Stokes’ formula, and to investigate the options, considered the following problem: *describe all differential operators acting in the spaces of tensor fields and invariant with respect to any changes of variables*. Indeed, the exterior differential (instrumental in Stokes’ formula) is, evidently, an invariant and, as is proven in [11], this is the only invariant unary differential (perhaps, [15] leads to it). Such a theory does exist and in [16] the calculus of variations is defined.

This problem (to list all invariant differential operators) goes back to O. Veblen (see [4] for a review). For unary operators on manifolds it was solved by Rudakov [11] as a part of another problem (description of irreducible vacuum vector modules over simple Lie algebras of formal or polynomial vector fields).

### 1.3. Unconventional integrations

A. Shwarts and his students [12] attempted to integrate densities and objects depending on higher jets of the diffeomorphism but all their examples boil down to either pseudodifferential ([14]) or integral forms. Having obtained an analog of Rudakov’s result for the general vectorial differential forms ([13]), we can be sure that there is only one integration theory on supermanifolds provided the integration involves tensors

with irreducible finite dimensional fibers. (*)

The result of [13] do not preclude, however, unconventional integrations. For tensors other than (*) constructions à la Shwarts may lead to an integration theory (perhaps, [15] leads to it). Such a theory does exist and in [13] the calculations from [13] are used to consider infinite dimensional fibers snubbed at in [13] for no reason except tradition. It turns out that in the spaces of such tensors there act invariant operators similar to Berezin integral. Next, observe that having stated that it is impossible to integrate differential forms on supermanifolds, we almost immediately published a paper [14] showing, nevertheless, how to do it if one is very eager to. More exactly, one has to consider *pseudodifferential forms*, i.e., functions nonpolynomial in differentials. Of course, there are no such functions on manifolds. Certain types of pseudoforms lead to new invariants — semi-infinite cohomology of supermanifolds; quite criminally, no examples are calculated yet.

Here I consider still another type of “integrations”.

### 1.4. Supertraces and superdeterminants

From the very beginning I wondered what if we stop insisting on having an analog of the Stokes’ formula? What remains of the integration then? Only the Jacobian, one can say. Since it is easier to deal with Lie algebras than with groups, let me list analogs of trace for Lie superalgebras. Then, if the Lie superalgebra \( \mathfrak{g} \) can be exponentiated to a Lie supergroup, we can consider the analog of the determinate defined via the formula

\[
\det \exp X = e^{\text{tr}(X)} \quad \text{for any } X \in \mathfrak{g}. \tag{1}
\]

In other words, I mean:

1) Let us consider the Lie superalgebra \( \mathfrak{g} \) with a trace also denoted by \( \text{tr} \) (i.e., \( \text{tr}([x, y]) = 0 \) for any \( x, y \in \mathfrak{g} \); then for the role of \( \text{Vol} \) we can take tensor fields of type \( \text{tr} \), its infinitesimal transformations being the Cartan prolongation (see [2]) of the pair \( \text{id}, \mathfrak{g} \), where \( \text{id} \) is the “standard” or “identity” representation of \( \mathfrak{g} \).

The prime example is provided by the Poisson Lie superalgebra \( \mathfrak{g} = \mathfrak{po}(0|2n) \). Indeed, there is a parametric family (quantization) of Lie superalgebras \( \mathfrak{g}_t \) which at \( t = 0 \) coincides with \( \mathfrak{po}(0|2n) \) and \( \mathfrak{g}_t \simeq \mathfrak{gl}((2^{n-1}|2^{n-1}) \) for \( t \neq 0 \), see [17].

2) From various points of view it is clear that \( \mathfrak{gl}(n) \) has at least two superanalog: the “simple-minded” one, \( \mathfrak{gl}(n|m) \), and the “queer” one, \( \mathfrak{q}(n) \). On \( \mathfrak{q}(n) \), the supertrace vanishes identically but there are specially designed for it its particular, queer, trace and determinant. Regrettably, the queer trace is odd and, therefore, to describe the
corresponding representation, we need odd parameters, which causes extra difficulties.

So, still another versions of integration theory, if exist, are related with the queertrace and its "quasiclassical limit" as $t \to 0$: the restriction of the above quantization is a parametric family $\mathfrak{g}_t$ which at $t = 0$ coincides with $\mathfrak{po}(0|2n - 1)$ and $\mathfrak{g}_t \simeq \mathfrak{q}(2^{n-1})$ for $t \neq 0$. In 1) and 2) the identity representation of $\mathfrak{po}(0|m)$ is the adjoint one and the Berezin integral serves as $\text{tr}$.

3) The analog of trace on the general vectorial algebra $\text{vect}(m|n)$ is the divergence. I do not know how to generalize formula (1) with divergence serves as $\text{tr}$, so let me mention two other, more obvious, analogs of $\text{tr}$: (a) in characteristic $p$ such analog exists, e.g., for Lie algebras of contact vector fields (but not only), see [13]; another one is provided by (b) "superconformal" algebras of divergence-free series $\text{vect}(1|N)$ and the exceptions related with $N = 4$ and $N = 5$ extended Neveu-Schwarz algebras, see [14]; these traces are of the same parity as $N$.

2. Jordan superalgebras

I consider here certain algebraic structures associated with certain selected "classical superdomains". So far, nobody knows yet (as far as I know) even a "right" definition of this basic notion, to start with: for finite dimensional manifolds all is clear, for supermanifolds we just consider the most easy to handle simple Lie supergroups and their Lie superalgebras whereas the elusive "right" definition requires, perhaps, semisimple or almost simple Lie superalgebras. So we take the road of least resistance:

Unless otherwise mentioned the ground field is $\mathbb{C}$, the classical superdomains are considered as quotients of simple or close to them "classical" Lie supergroups modulo certain maximal parabolic subsupergroups; for the list see [24].

This paper is an attempt to tackle the following questions: What are the criteria for selecting the above-mentioned subgroups among other maximal ones? Why cosets modulo other parabolic subsupergroups are seldom considered in Differential Geometry whereas only these "other" cosets are the main topic of study in analytical mechanics of nonholonomic dynamical systems and in supergravity ([21], [23])?

In these questions "super" is beside the point, so we can very well begin with manifolds. The classical domains are distinguished among symmetric spaces by the fact that the Lie algebra of the symmetry group of any classical domain (Hermitian symmetric space) $M$ is a simple complex Lie algebra of the form

$$\mathfrak{g} = \bigoplus_{|i| \leq 1} \mathfrak{g}_i; \quad (d = 1)$$

the tangent space to $M$ at a fixed point can be identified with the $\mathfrak{g}_{-1}$ and, on it, one can always define a Jordan algebra structure, by fixing any element $p \in \mathfrak{g}_1$ and setting

$$x \circ y = [[p, x], y] \quad \text{for any } x, y \in \mathfrak{g}_{-1}. \quad (2)$$

Recall that a Jordan algebra is a commutative algebra $J$ with product $\circ$ satisfying, instead of associativity, the identity

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x). \quad (JI)$$

In a very inspiring paper [23] McCrimmon gave an account of some applications of Jordan algebras from antiquity to nowadays, see also refs. in [24]. The paper and books strengthen my prejudice that general Jordan algebras are, bluntly speaking, useless. Contrariwise, simple Jordan algebras give rise to several notions important in various problems. For simple Jordan algebras the so-called general norm [26] should be nondegenerate which imposes additional constraints on the parabolic subalgebra. This answers the above questions but tempts one to make use of the other coset spaces as well.

I wish to make similar use of simple Jordan superalgebras, especially infinite dimensional ones, associated with infinite dimensional classical superdomains listed in [3]; for convenience I reproduce these tables.

In the '60s a remarkable correspondence between Jordan algebras and certain $\mathbb{Z}$-graded Lie algebras became explicit, cf. [27], [28] and [29]. Kantor used this correspondence to list simple Jordan algebras (over $\mathbb{C}$ and $\mathbb{R}$) by the, so far, simplest known method. He clarified the mysterious relation of Jordan algebras with classical...
domains and actively studied certain generalizations of Jordan algebras associated with \( \mathbb{Z} \)-graded simple Lie algebras of finite depth \( d \) (since all of them are of the form \( \mathfrak{g} = \bigoplus_{i \leq d} \mathfrak{g}_i \), their length is equal to \( d \)). Supersymmetry, supertwistors etc. are related with gradings \( d > 1 \) almost without exceptions; so it is interesting to find generalizations of Jordan algebras (or, rather, useful related structures).

Following Freudental and Springer, Kantor generalized products (2) to several arguments which is natural for \( d > 1 \). I suggest, contrariwise, to stick to formula (2), even for \( d > 1 \), with the minimal modification: fix \( p \in \mathfrak{g}_1 \) and for \( \mathfrak{g}_- = \bigoplus_{i \leq 0} \mathfrak{g}_i \) set

\[
x \circ y = [[p, x], y] \quad \text{for any } x, y \in \mathfrak{g}_-. \tag{3}
\]

In this way we obtain noncommutative generalizations of Jordan algebras (with unknown relations instead of (JI)) and it is interesting to investigate what type of integrable systems are associated with them under the Sokolov-Svinolupov's approach, cf. \([32], [31] \).

Kac \([32] \) has already applied this correspondence to list simple \emph{finite} dimensional Jordan superalgebras.

Tables (borrowed from \([3]) provide with a list of simple Jordan superalgebras associated with the known in 1991 simple \( \mathbb{Z} \)-graded Lie superalgebras of polynomial growth (SZGLSAPGs for short), cf. \([22], [17], [33] \), namely, with \( \mathbb{Z} \)-gradings of depth 1 of SZGLSAPGs \emph{including} finite dimensional ones. (This is the place where \([22] \) contains an omission — cf. Kac’ exceptional Jordan superalgebra \( K \) with our series \( \mathfrak{h} \) discovered by Serganova in 1983, see \([3] \), and later rediscovered several times.)

\subsection{2.0.1. Tits–Kantor–Köcher’s functor } \( \text{tan} \)

Let \( J \) be a Jordan superalgebra and \( p \) the tensor that determines the product in \( J \), i.e., \( p(x, y) = x \circ y \). To \( J \), Kantor assigns (see \([22] \)) a \( \mathbb{Z} \)-graded Lie superalgebra \( \text{tan}(J) = \bigoplus_{i \leq 1} \text{tan}(J)_i \), a \( \mathbb{Z} \)-graded Lie subalgebra in \( \text{vect}(J) \) such that (here \( L_a(x) = a \circ x \))

\[
\text{tan}(J)_- = \text{vect}(J)_- - \text{vect}(J)_-
\]

Conversely, for any \( \mathbb{Z} \)-graded Lie superalgebra of the form \( \mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}_i \) we define a Jordan superalgebra structure on \( \mathfrak{g}_- \) if \( (\mathfrak{g}_1)_0 \neq 0 \) in the following way. Take \( p \in (\mathfrak{g}_1)_0 \) and for \( x, y \in \mathfrak{g}_- \) set

\[
x \circ y = [[p, x], y]. \tag{J}
\]

\subsection{2.0.2. Digression: on homological fields}

I do not know what structure is related with an arbitrary odd \( p \) but if \( p \) is \emph{homologic}, i.e., \( [p, p] = 0 \), then the formula

\[
[x, y]' = [[p, x], y] \tag{L}
\]

determines a Lie superalgebra structure on \( \Pi(\mathfrak{g}_-) \). This structure had been first noticed, perhaps, by M. Gerstenhaber in ‘60s and rediscovered many times since then. It seemed interesting to describe in intrinsic terms the \( p \)'s which determine \emph{simple} Lie (super)algebras. Homological vector fields were first introduced, in connection with the problem of integration of differential equations on supermanifolds, by V. Shander \([34] \) who gave a normal form for the nonsingular fields. However, Shander did not consider singularities of the fields in that work; this was recently done by Vaintrob in a series of articles (e.g., \([33] \) in which he showed that the study of singularities of homological fields, and their classification, turns out to be rather similar to the case of singularities of smooth functions. Regrettably, the answer for \( p \) corresponding to the simple algebras is more trivial than expected, as we have recently established with Grozman.

Let, first, \( \mathfrak{g} = \mathfrak{g}(n) \); denote the matrix units by \( \partial^i_j \), let \( (\partial^i_j)^* = x^i_j \) be the dual basis. Then \( p = \sum \varepsilon_i^j \delta^i_k \delta^i_1 \), where \( \varepsilon_i^j \) is the odd copy of \( x^i_j \) and \( \delta^i_k = \partial^i_j \). Similarly, if the \( X_i \) form a basis of \( \mathfrak{g} \) and \( [X_i, X_j] = \sum c^k_{ij} X_k \), then the operator \( p \in \text{vect}(0) (\dim \mathfrak{g}) \) is of the form \( \frac{1}{2} \sum c^k_{ij} X^*_i X^*_j X_k \), where \( X^*_i \) is the dual of \( X_i \).
Having observed that every simple finite dimensional Lie algebra (over $\mathbb{C}$) possesses a non-degenerate symmetric bilinear form, we see that for such algebras $p$ is a Hamiltonian vector field; to find the corresponding generating function is easy: it is the sum of all elements of degree 1 and weight 0 with respect to the Cartan subalgebra of $\mathfrak{po}(0\dim\mathfrak{g})$.

Generalization to Lie superalgebras is straightforward. Still, observe that some simple Lie superalgebras have no form at all, some (e.g., $\mathfrak{f}_4$) possess an odd nondegenerate symmetric bilinear form in which case $p$ belongs to the antibracket algebra, not to the Poisson one.

2.1. SZGLSAPGs of depth 1 and length 1

All possible $\mathbb{Z}$-gradings of SZGLSAPGs $\mathfrak{g}$ are listed in [22] for $\dim\mathfrak{g} < \infty$ and in [8] for most of the other cases. Our job is to pick those of them which are of depth 1, in particular, of the form $\bigoplus_{|i|\leq 1} \mathfrak{g}_i$, see Tables.

Albert’s notation for Jordan algebras were given in accordance with Cartan’s notations for the corresponding Lie algebras. As follows from Serganova’s classification of systems of simple roots of simple Lie superalgebras [36], Cartan’s notations are highly inappropriate for Lie superalgebras.

2.1.1. Matrix Jordan superalgebras

Let $B_{m,2n} = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}$, where $J_{2n} = \begin{pmatrix} 1_n \\ -1_n \\ 0 \end{pmatrix}$. Set

$$\mathbf{Mat}(m|n) = \{X \in \mathbf{Mat}(m|n)\},$$

$$\mathbf{Q}(n|n) = \{X \mid [X, J_{2n}] = 0\},$$

$$\mathbf{OSp}(m|2n) = \{X \mid [X^{st}B_{m,2n} = B_{m,2n}X\},$$

$$\mathbf{P}(n|n) = \{X \mid X^{st}J_{2n} = (-1)^{p(X)}J_{2n}X\}.$$

In the first two of these spaces the Jordan product is given by the formula

$$X \circ Y = XY + (-1)^{p(X)p(Y)}YX.$$

I leave it as an excersise to figure out the formula in the other two cases; for the answer see [22].

2.1.2. Jordan algebras from bilinear forms

Set $Q_{m,2n} = C^{m|2n}$ with a nondegenerate even symmetric bilinear form $(\cdot, \cdot)$ and the product

$$x \circ y = (e, x)y + x(e, y) - (x, y)e$$

$(Q)$

where $e \in (Q_{m,2n}) \circ \circ$ satisfies $(e, e) = 1$.

Set $\mathbf{H}Q_{m|2n} = \Pi(\mathbb{C}[p, q, \Theta])$, where $m > 0$, $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, $\Theta = (\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_r)$ for $m = 2r$, of $\Theta = (\xi, \eta, \Theta)$ for $m = 2r + 1$ with the Jordan product defined with the help of the symplectic form $\omega$ on the supermanifold with coordinates $p, q, \Theta$.

$$x \circ y = \omega(e, x)y + x\omega(e, y) - \omega(x, y)e$$

$(H')$

where $e \in (\mathbf{H}Q_{m|2n}) \circ \circ$ satisfies $\omega(e, e) = 1$.

To explicitly give the product, consider the space $\mathbb{C}[p, q, \Theta, \alpha, \beta]$ with two extra odd indeterminates and the Poisson bracket such that $p$ and $q$, $\xi$ and $\eta$, and $\alpha, \beta$ are dual. Setting $\deg \alpha = -\deg \beta = -1$ the degrees of the other indeterminates being 0, we obtain the $\mathbb{Z}$-grading of the Poisson algebra, and its quotient modulo center, of the form $(d = 1)$. On $\mathfrak{g}_- = \mathbb{C}[p, q, \Theta]$, define the product

$$H_{f_\alpha} \circ H_{g_\beta} = \{(H_{f_\beta}H_{f_\alpha}), H_{g_\beta} = (-1)^{p(f)}H_{(f, g)_\alpha}\}.$$

In other words, on the superspace of functions with shifted parity, we set

$$f \circ g = (-1)^{p(f)+1}\{f, g\}.$$  

$(H, K)$

2.1.3. Exceptional Jordan superalgebras

There are two of them associated with the gradings of $\mathfrak{osp}(4|2; \alpha)$ and $\mathfrak{ab}_3$ from Table 1 and the corresponding loops.

2.1.4. Stringy Jordan superalgebras

These are obtained from $\mathfrak{t}^{(1|n)}$ for $n > 2$ and $\mathfrak{t}^{(M|1|n)}$ (see [8]) for $n > 3$ by formula $(J)$ with the grading from Table 1. They will be denoted, respectively, by

$$K^l\mathfrak{q}_{1|n} \cong \Pi(\mathbb{C}[\tau^{-1}, \tau, \theta_1, \ldots, \theta_n]),$$

$$K^n\mathfrak{q}_{1|n} \cong \Pi(\mathbb{C}[\tau^{-1}, \tau, \theta_1, \ldots, \sqrt{\theta}_n]).$$

The product is given by formula $(H, K)$.
2.1.5. Loop Jordan superalgebras

For a finite-dimensional Jordan superalgebra $J$ denote by $J^{(1)} = J \otimes \mathbb{C}[t^{-1}, t]$ the loops with values in $J$ and point-wise product.

2.1.6. Twisted loop Jordan superalgebras

These are associated by formula $(J)$ with the Lie superalgebras from Table 2.

3. Cross ratios

In [27] Kantor generalized the cross ratio of four points on $\mathbb{P}^1$ to most of the quotients $G/P$, where $G$ is a simple Lie group and $P$ is its parabolic subgroup. The Lie algebra $\mathfrak{g} = \text{Lie}(G)$ in these cases is of the form $\mathfrak{g} = \bigoplus_{i \leq d} \mathfrak{g}_i$. I do not know any paper referring to [27], so Kantor’s studies drew no attention at all. His constructions, however, naturally appear in supersetting [6]; this prompts me to try to decipher a part of [27]. I will consider here the simplest case, when $d = 1$ and the corresponding Jordan algebra is simple. In this case one can generalize the cross ratio from $\mathbb{P}^1 = \text{Gr}_1^2$ to a collection of $\mathfrak{gl}(2m|2n)$-invariants of four points on $\mathfrak{g}^*_{m|n}$. First, consider

$$(A, B, C, D) = (A - B, A - B)(C - D, C - D)(A - D, A - D).$$

(CR)

Let $mn = 0$. Now, replace the rhs of (CR) — call it $X$ — with $\det(X - \lambda E)$. The collection of all coefficients of the powers of $\lambda$ is the analog of the cross ratio.

By dimension considerations these are all the invariants of four points for the general, orthogonal and Lagrangian grassmannians.

For their super counterparts we take the Berezinian (superdeterminant) and the amount of polynomially independent invariants is infinite, cf. [8]. If, however, we consider rational dependence, which is natural in super setting, the coefficients of the first $n + m$ powers of $\lambda$ generate the algebra of invariants and is a natural candidate for the cross ratio.

On $\mathfrak{q}(n|n)$, we should take the queerdeterminant, yet, instead of $\det$; the collection obtained is finite.

For loop Jordan superalgebras we consider matrix-valued functions and $\det(X - \lambda E)$ returns a collection of functions, rather than numbers.

I do not know the complete cross ratio for Jordan superalgebras related to quadrics and do not know at all what they are for twisted loops and in stringy cases, most interesting to me. One invariant is obvious (but there should be several if $\text{dim} \mathcal{Q} > 1$): given the form $(\cdot, \cdot)$, or the symplectic form $\omega$ in the curved case, set (for the $\Lambda$-points (see [1]) of the Jordan algebra $J$, i.e., for $A, B, C, D \in (J \otimes \Lambda)_{\bar{0}}$

$$(A, B, C, D) = (A - B, A - B)(C - D, C - D)(A - D, A - D).$$

(CRQ)

For curved quadrics, take $\omega(H_{A-B}, H_{A-B})$ instead of $(A - B, A - B)$, etc.

Perhaps, other invariants (in non-super case) can be dug out from Reichstein’s results.

I almost forgot to add refs. [28] that studies four-point functions in $N = 2$ superconformal field theories and [4], where matrix cross ratio is applied to Riccati equation; together with [1] they provide a wide setting for applications of our cross ratios.

4. Tables

Everywhere we assume the notational conventions of [3] and definitions adopted there.

In Table 1 we say that the homogeneous superspace $G/P$, where $G$ is a simple Lie supergroup, $P$ its parabolic subgroup corresponding to several omitted generators of a Borel subalgebra (description of these generators can be found in [28]), of depth $d$ and length $l$ if such are the depth and length of $\mathfrak{g} = \text{Lie}(G)$ in the $\mathbb{Z}$-grading compatible with that of $\text{Lie}(P)$. Note that all superspaces of Table 1 possess an hermitian structure (hence are of depth 1) except $\text{PeGr}$ (no hermitian structure), $\text{PrQ}$ (no hermitian structure, length 2), $\text{CGr}^{0,n}_{0,k}$ and $\text{SCGr}^{0,n}_{0,k}$ (no hermitian structure, lengths $n - k$ and, resp., $n - k - 1$).

Let $\mathfrak{sl}(\mathfrak{g})$ be the traceless part of $\mathfrak{g}$ and $\mathfrak{p}(\mathfrak{g}) = \mathfrak{g}/\text{center}$; let $\mathfrak{g}_{\varphi}^{(m)}$ be the stationary subalgebra of the loop algebra with values in $\mathfrak{g}$ singled out by the degree $m$ automorphism $\varphi$ of $\mathfrak{g}$; for $G = \mathfrak{g}_{\varphi}^{(m)}$ with the $\mathbb{Z}$-grading of type $(d = 1)$ the last column of Table 1 contains $G_0$: the map $-\text{st}$, “minus supertransposition”, sends $X$ to $-X^{*t}$, the map $\Pi$ sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, and $\delta_\varphi$ sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} a & xb \\ xc & d \end{pmatrix}$; the automorphism $A$ of
is defined on monomials \( f(\theta) \) as id if \( \frac{\partial f}{\partial \theta_1} = 0 \) and otherwise; \( \text{irr}(\ldots) \) is any of the two irreducible components; \( LGr \) and \( OGr \) stand for the Lagrangian and orthogonal Grassmannian, respectively; the dual domain is endowed with an asterisk as a left superscript.

Table 2: for the lack of space I give an interpretation of the supergrassmannians here, linewise:

| Dimension | Description |
|-----------|-------------|
| \( p \mid q \)-dimensional subspaces in \( \mathbb{C}^m \mid n \) and same for \( n = m, p = q \); superquadric of 1|0-dimensional isotropic (wrt a non-degenerate even form) lines in \( \mathbb{C}^m \mid n \); ortolagrangian supergrassmannian; queergrassmannian; “odd” superquadric (wrt a non-degenerate even form) of 1|0-lines in \( \mathbb{C}^n \mid n \); odd-lagrangian supergrassmannian; curved supergrassmannian of 0|1-dimensional subsupermanifolds in \( \mathbb{C}^0 \mid n \); curved superquadric; two exceptions. |

REFERENCES

1. Deligne P. et al (eds.) Quantum fields and strings: a course for mathematicians. Vol. 1, 2. AMS, Providence, RI, 1999
2. I. Shchepochkina, Represent. Theory (electronic journal of AMS), v. 3, 1999, 3 (1999) 373
3. A. Sergeev, Michigan Math. J. 49 (2001) no. 1, 113 [math.RT/9810113]
4. P. Grozman, D. Leites and I. Shchepochkina, In: M. Olshanetsky and A. Vainshtein (eds.) Multiple facets of quantization and supersymmetry Michael Marinov Memorial Volume, World Sci., to appear [ESI preprint 1111 (2001)]
5. D. Leites, V. Serganova and G. Vinel. In: C. Bartocci, U. Bruzzo and R. Cianci (eds.) Differential Geometric Methods in Theoretical Physics Proc. DGM-XIX, 1990, Springer, LN Phys. 375 (1991) 286
6. Yu. Manin, Topics in noncommutative geometry, Princeton Univ. Press, Princeton, NJ, 1991
7. D. Leites and I. Shchepochkina Classification of simple Lie superalgebras of vector fields, to appear
8. D. Leites (ed.), Seminar on supermanifolds, vV. 1–34, 1986–90, Reports of Dept. of Math. University of Stockholm, 2100 pp.
9. J. Bernstein and D. Leites, Functional Anal. Appl. 11 (1977) no. 1, 45
10. V. Shander, Funct. Anal. Appl. 22 (1988) no. 1, 80
11. A. Rudakov, Math. USSR Izvestiya, v. 38 (1974) n. 4, 835
12. A. Shvarts, Nuclear Phys. B 171 (1980), no. 1-2, 154; A. Gaiduk, V. Romanov and A. Shvarts, Comm. Math. Phys. 79 (1981), no. 4, 507; A. Gaiduk, O. Khudaverdyan and A. Shvarts, Theoret. and Math. Phys. 52 (1982), no. 3, 862
13. J. Bernstein and D. Leites, Selecta Math. Soviet. 1 (1981) no. 2, 143
14. J. Bernstein and D. Leites, Functional Analysis and Its Applications 11 (1977) 219
15. O. Khudaverdian, Comm. Math. Phys. 198 (1998), no. 3, 591
16. D. Leites, Yu. Kochetkov and A. Vaintrob, In: S. Andima et. al (eds.) General topology and its applications, LN in pure and applied math, v. 134, Marcel Decker, NY (1991) 217
17. D. Leites and I. Shchepochkina, Theor. and Math. Physics, v. 126 (2000) no. 3, 339
18. H. Strade and R. Farnsteiner, Modular Lie algebras and their representations. Marcel Dekker, NY, 1988
19. P. Grozman, D. Leites and I. Shchepochkina, [hep-th 9702120]. Acta Mathematica Vietnamica, v. 26 (2001) no. 1, 27
20. V. Serganova, Functional Anal. Appl. 17 (1983), no. 3, 200
21. Yu. Manin, Gauge field theory and complex geometry. Second edition. Springer-Verlag, Berlin, 1997
22. P. Grozman and D. Leites In: J. Wess, E. Ivanov (eds.), Supersymmetries and quantum symmetries, (SQS’97, 22–26 July, 1997), Lecture Notes in Phys., 524, 1999, 58
23. K. McCrimmon, Bull. Amer. Math. Soc., v.84 (1978) no. 4, 612
24. H. Upmeier, Jordan algebras in analysis, operator theory, and quantum mechanics. CBMS Regional Conference Series in Mathematics, 67. AMS, Providence, RI, 1987.
25. M. Koecher (edited by A. Krieg and
Table 1
Gradings of twisted loop (super)algebras corresponding to hermitian superdomains.

| \(\mathfrak{sl}_m\) | \(\mathfrak{psl}_m\) | grading elements from \(\mathfrak{h}\) | \(\mathfrak{psl}_m\) |
|----------------|----------------|-----------------|----------------|
| \(\mathfrak{sl}(2m)\) | \(\mathfrak{psl}(2m)\) | \(\mathfrak{sl}(2m)\) | \(\mathfrak{psl}(2m)\) |
| \(\mathfrak{sl}(2m+1)\) | \(\mathfrak{psl}(2m+1)\) | \(\mathfrak{sl}(2m+1)\) | \(\mathfrak{psl}(2m+1)\) |

Table 2
Classical superspaces of depth 1.

| \(\theta_0\) | \(\theta_1\) | Underlying domain | Name of the superdomain |
|-------------|-------------|-------------------|-------------------------|
| \(\mathfrak{sl}(m)\) | \(\mathfrak{psl}(m)\) | \(\mathfrak{sl}(m)\) | \(\mathfrak{psl}(m)\) |
| \(\mathfrak{psl}(2m)\) | \(\mathfrak{psl}(2m)\) | \(\mathfrak{psl}(2m)\) | \(\mathfrak{psl}(2m)\) |
| \(\mathfrak{sp}(2m)\) | \(\mathfrak{sp}(2m)\) | \(\mathfrak{sp}(2m)\) | \(\mathfrak{sp}(2m)\) |
| \(\mathfrak{so}(2m)\) | \(\mathfrak{so}(2m)\) | \(\mathfrak{so}(2m)\) | \(\mathfrak{so}(2m)\) |

The curved superscript has infinite dimensional ‘stringy’ counterpart with \(\mathfrak{h}(0|m)\) replaced with the centerless N-extended Neveu-Schwarz algebra \(\mathfrak{t}^M(1|N)\) or Ramond algebra \(\mathfrak{t}^M(1|N)\).

S. Walcher) The Minnesota Notes on Jordan Algebras and Their Applications, Lect. Notes Math., 1710, Springer, 1999
26. I. Kantor, Proc. of P. K. Rashevsky’s seminar on vector and tensor analysis. v. 14 (1968), 114 (in Russian)
27. J. Tits, Indag. Math. v. 24 (1962) 530
28. I. Kantor, Soviet. Math. Doklady v. 5 (1964) 1404
29. I. Kantor, Proc. of P. K. Rashevsky’s seminar on vector and tensor analysis. v. 13 (1966), 310 (in Russian)
30. S. Svinolupov and V. Sokolov, Theoret. and Math. Phys. 108 (1996) no. 3, 1160; id., Acta Appl. Math. 41 (1995) no. 1–3, 323; id., Math. Notes 53 (1993) no. 1–2, 201
31. I. Habibullin, V. Sokolov and R. Yamilov. In: Nonlinear physics: theory and experiment (Lecce, 1995) World Sci., River Edge, NJ, 1996, 139
32. V. Kac, Commun. Alg. 5 (13) (1977) 1375; L. Hogben and V. Kac, Erratum Comm. Algebra 11 (1983) no. 10, 1155
33. V. Kac, C. Martinez and E. Zelmanov, Graded simple Jordan superalgebras of growth one. Mem. Amer. Math. Soc. 150 (2001) no. 711
34. V. Shander, Functional Anal. Appl. 14 (1980), no. 2, 160
35. A. Vaintrob, J. Math. Sci. 82 (1996), no. 6, 3865
36. V. Serganova, Comm. Algebra 24 (1996) no.
13. 4281
37. I. Kantor, Proc. of P. K. Rashevsky’s seminar on vector and tensor analysis. v. 17 (1974) 250; v. 18 (1975) 234 (in Russian)
38. P. Grozman, D. Leites and E. Poletaeva, In: E. Ivanov et. al. (eds.) *Supersymmetries and Quantum Symmetries* (SQS’99, 27–31 July, 1999), Dubna, JINR, 2000, 387
39. B. Eden, P. Howe, A. Pickering, E. Sokatchev, P. West, Nuclear Phys. B 581 (2000), no. 1-2, 523–558
40. M. Zelikin, *Control theory and optimization. I. Homogeneous spaces and the Riccati equation in the calculus of variations*. Encyclopaedia of Mathematical Sciences, 86. Springer-Verlag, Berlin, 2000. xii+284 pp.