BRST approach to Lagrangian construction for fermionic massless higher spin fields

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Abstract

We develop the BRST approach to Lagrangian formulation for all massless half-integer higher spin fields on an arbitrary dimensional flat space. General procedure of Lagrangian construction describing the dynamics of fermionic field with any spin is given. It is shown that in fermionic case the higher spin field model is a reducible gauge theory and the order of reducibility grows with the value of spin. No off-shell constraints on the fields and the gauge parameters are used. We prove that in four dimensions after partial gauge fixing the Lagrangian obtained can be transformed to Fang-Fronsdal form however, in general case, it includes the auxiliary fields and possesses the more gauge symmetries in compare with Fang-Fronsdal Lagrangian. As an example of general procedure, we derive the new Lagrangian for spin 5/2 field containing all set of auxiliary fields and gauge symmetries of free fermionic higher spin field theory.

1 Introduction

Construction of the self-consistent Lagrangian theory of interacting higher spin fields is one of the longstanding problems of the theoretical physics. First success in the theory of massless higher spin fields was the formulation of Lagrangians for free bosonic \cite{1} and fermionic \cite{2} fields in four dimensions. Since then the various approaches to higher spin fields problem were
developed (see e.g. [3] for reviews and [4, 5] for recent developments) however the general problem is still open. We would like to point out two modern approaches.

An approach, called the unfolded formalism, was developed by Vasiliev et al. (see e.g. [6] and references therein). It allows to construct both the theory of free higher spin fields and the theory of higher spin fields coupled to AdS_D background (see [7] for the bosonic case and [8] for the fermionic case and references therein). Also this formalism turned out to be fruitful for constructing the consistent equations of motion for interacting higher spin fields.

Another approach to higher spin field problem, called BRST\textsuperscript{1} approach [10], was initiated by development of string field theory where the interacting model of open strings was constructed (see e.g. [11]) on the base of BRST techniques\textsuperscript{2}. Higher spin BRST approach is analogous to string field theory however it contains two essential differences related with structure of constraints, which are used in construction of BRST charge, and with presence only massless fields in the spectrum of higher spin field model. If ones try to consider the tensionless limit of the string field theory (see e.g. for free string theory [13, 14]) we expect to get the theory of interacting massless fields. Since the string field theory contains lesser number of constraints on the fields than we need to construct an irreducible representation of Poincare group, fields in string spectrum do not belong to irreducible representations with fixed spin and their equations of motion describe rather propagation of Regge trajectories, instead of one spin mode. For the equations of motion to describe propagation of one spin mode the additional, in compare with string, off-shell constraints on the fields must be imposed. In order to get Lagrangian which contains the additional constraints as equations of motion we have to include these additional constraints into the set of constraints which is used in constructing the BRST charge and then try to get the Lagrangian of the higher spin field theory. Using this approach one can hope to construct the theory of interacting higher spin fields analogously to the string field theory. (An attempt to do that was undertaken in [15].)

The first natural step of constructing the massless higher spin interacting model in the BRST approach is a formulation of corresponding free model. This problem was studied in [10] and finally solved for the bosonic massless higher spin fields both on the flat [16, 17] and the AdS [18, 19] backgrounds. However, the BRST approach to fermionic fields has not been developed at all so far.

The present paper is devoted to formulation of BRST approach to derivation the Lagrangian for free fermionic massless higher spin fields on the flat Minkowski space of arbitrary dimension. The method which we use here slightly differs from the one in the bosonic case, but application of our method in the bosonic case leads to the same final result for the Lagrangian describing propagation of all spin fields. The difference of the methods is rather technical and consists in that we do not use the similarity transformation like in [17] and therefore one can construct Lagrangian for the field of one fixed value of spin while the approach in [17] demands to use fields of all spins together. As a future purpose we hope, using our method, to construct the free theory of massive higher fermionic fields (see e.g. [20] and references therein), to get the Lagrangian describing propagation of fermionic higher spin fields through AdS background, and to consider an application of BRST approach to supersymmetric higher superspin models (for recent development in these directions see e.g. [21] for massive higher spin field in AdS background and [22] for supersymmetric higher spin field models).

The paper is organized as follows. In Section 2 we investigate the superalgebra generated

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\textsuperscript{1}BRST construction was discovered at first in context of Yang-Mills theories [9].

\textsuperscript{2}Also we point out the approach [12] to finding the gauge invariant actions for arbitrary representations of the Poincare group.
by the constraints which are necessary to define an irreducible half-integer spin representation of Poincare group. It is shown that an naive use of the BRST charge, constructed on the base of these constraints, leads us to the equations of motion only for spin-1/2 fields. We argue, to overcome this difficulty we should reformulate the constraint algebra and find a new representation for the constraints.

Section 3 is devoted to actual formulation of new representation for constraints.

In Section 4 we construct Lagrangian describing a propagation of field with any fixed half-integer spin. We find that in the case of arbitrary spin fermionic field, the theory has reducible gauge symmetry with the finite order of reducibility which increases with the spin value. Next, in Section 5 we construct Lagrangian describing propagation of all half-integer spin fields simultaneously.

Then in Section 6 we show that in four dimensions found Lagrangian may be transformed, after partial gauge fixing, to the Fang & Fronsdal Lagrangian [2].

In Section 7 we illustrate the general procedure of Lagrangian construction by finding the Lagrangian and gauge transformations for the spin 5/2 field model without gauge fixing, keeping all auxiliary fields and higher spin gauge symmetries.

This paper is devoted to the memory of our friend and collaborator, a remarkable human being and scientist A.I. Pashnev who tragically passed away on March 30, 2004.

2 Algebra of the constraints

It is well known that the totally symmetrical tensor-spinor field $\Psi_{\mu_1\ldots\mu_n}$ (the Dirac index is suppressed), describing the irreducible spin $s = n + 1/2$ representation must satisfy the following constraints (see e.g. [23])

$$\gamma^\nu \partial_\nu \Phi_{\mu_1\ldots\mu_n} = 0, \tag{1}$$
$$\gamma^\mu \Phi_{\mu\mu_2\ldots\mu_n} = 0. \tag{2}$$

Here $\gamma^\mu$ are the Dirac matrices $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = (+, -, \ldots, -)$.

In order to describe all higher tensor-spinor fields together it is convenient to introduce Fock space generated by creation and annihilation operators $a^+_{\mu}, a_{\mu}$ with vector Lorentz index $\mu = 0, 1, 2, \ldots, D - 1$ satisfying the commutation relations

$$[a_{\mu}, a^+_{\nu}] = -\eta_{\mu\nu}. \tag{3}$$

These operators act on states in the Fock space

$$|\Phi\rangle = \sum_{n=0}^{\infty} \Phi_{\mu_1\ldots\mu_n}(x)a^{+\mu_1}\ldots a^{+\mu_n}|0\rangle \tag{4}$$

which describe all half-integer spins simultaneously if the following constraints are taken into account

$$T_0|\Phi\rangle = 0, \quad T_1|\Phi\rangle = 0, \tag{5}$$

where $T_0 = \gamma^\mu p_\mu$, $T_1 = \gamma^\mu a_\mu$. If constraints (5) are fulfilled for the general state (4) then constraints (1), (2) are fulfilled for each component $\Phi_{\mu_1\ldots\mu_n}(x)$ in (4) and hence the relations (5) describe all free higher spin fermionic fields together. In order to construct hermitian BRST charge we have to take into account the constraints which are hermitian conjugate
Let us introduce the BRST charge for the enlarged system of constraints

\[ Q' = q_0 T_0 + q_1 T_1 + q_1 T_1^+ + \eta_0 L_0 + \eta_1 L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta G_0 + i(\eta_1^+ q_1 - \eta_1 q_1^+)p_0 - i(\eta_G q_1 + \eta_2 q_1^+)p_1^+ + i(\eta_G q_1^+ + \eta_2^+ q_1)p_1 + (q_0^2 - \eta_2^+ \eta_2 - \eta_1^+ \eta_1)P_0 + (2q_1 q_1^+ - \eta_2^+ \eta_2)P_G + (\eta_G \eta_1^+ + \eta_2^+ \eta_1 - 2q_0 q_1^+)P_1 + (\eta_1^+ \eta_G + \eta_1^+ \eta_2 - 2q_0 q_1)P_1^+ + 2(q_1 q_1^+ - q_1^2)P_2 + 2(\eta_2 \eta_G - q_1^2)P_2^+. \]

(6)

to \( T_0 \) and \( T_1 \). Since \( T_0^+ = T_0 \) we have to add only one constraint \( T_1^+ = \gamma^\mu a_\mu^+ \) to the initial constraints \( T_0 \) and \( T_1 \).

Algebra of the constraints \( T_0, T_1, T_1^+ \) is not closed and in order to construct the BRST charge we must include in the algebra of constraints all the constraints generated by \( T_0, T_1, T_1^+ \). The resulting constraints and their algebra are written in Table 1. The constraints \( T_0, T_1, T_1^+ \) are fermionic and the constraints \( L_0, L_1, L_1^+, L_2, L_2^+, G_0 \) are bosonic. All the commutators are graded, i.e. graded commutators between the fermionic constraints are anticommutators and graded commutators which include any bosonic constraint are commutators. In Table 1 the first arguments of the graded commutators and explicit expressions for all the constraints are listed in the left column and the second argument of graded commutators are listed in the upper row. It is worth pointing out that this algebra involves all the bosonic constraints \( L_0, L_1, L_1^+, L_2, L_2^+, G_0 \) which were used to describe the irreducible bosonic representation in [16] as a subalgebra.

|     | \( T_0 \) | \( T_1 \) | \( T_1^+ \) | \( L_0 \) | \( L_1 \) | \( L_1^+ \) | \( L_2 \) | \( L_2^+ \) | \( G_0 \) |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \( T_0 = \gamma^\mu p_\mu \) | \(-2L_0\) | \(2L_1\) | \(2L_1^+\) | \(0\)     | \(0\)     | \(0\)     | \(0\)     | \(0\)     | \(0\)     |
| \( T_1 = \gamma^\mu a_\mu \) | \(2L_1\) | \(4L_2\) | \(-2G_0\) | \(0\)     | \(0\)     | \(-T_0\)  | \(0\)     | \(-T_1^+\) | \(T_1\)   |
| \( T_1^+ = \gamma^\mu a_\mu^+ \) | \(2L_1^+\) | \(-2G_0\) | \(4L_2^+\) | \(0\)     | \(T_0\)   | \(0\)     | \(T_1\)   | \(0\)     | \(-T_1^+\) |
| \( L_0 = -p^2 \) | \(0\)    | \(0\)    | \(0\)    | \(0\)    | \(0\)    | \(0\)    | \(0\)    | \(0\)    | \(0\)    |
| \( L_1 = p^\mu a_\mu \) | \(0\)    | \(0\)    | \(-T_0\) | \(0\)    | \(0\)    | \(L_0\)  | \(0\)    | \(-L_1^+\) | \(L_1\)   |
| \( L_1^+ = p^\mu a_\mu^+ \) | \(0\)    | \(T_0\)  | \(0\)    | \(0\)    | \(-L_0\) | \(0\)    | \(L_1\)  | \(0\)    | \(-L_1^+\) |
| \( L_2 = \frac{1}{2} a_\mu a^{\mu \mu} \) | \(0\)    | \(0\)    | \(-T_1\) | \(0\)    | \(0\)    | \(-L_1\) | \(0\)    | \(G_0\)  | \(2L_2\)  |
| \( L_2^+ = \frac{1}{2} a_\mu^+ a^{\mu \mu^+} \) | \(0\)    | \(T_1^+\) | \(0\)    | \(0\)    | \(L_1^+\) | \(0\)    | \(-G_0\) | \(0\)    | \(-2L_2^+\) |
| \( G_0 = -a_\mu^+ a^{\mu \mu} + \frac{D^2}{2} \) | \(-T_1\) | \(T_1^+\) | \(-L_1\) | \(L_1^+\) | \(-2L_2\) | \(2L_2^+\) | \(0\)    |         |          |

Table 1: Algebra of the constraints
Here \( q_0, q_1^+, q_1 \) are the bosonic ghosts corresponding to the fermionic constraints \( T_0, T_1, T_1^+ \) respectively and \( \eta_0, \eta_1^+, \eta_1, \eta_2^+, \eta_2, \eta_G \) are fermionic ghosts corresponding to the bosonic constraints. The momenta for these ghosts are \( p_0, p_1, p_1^+ \) for bosonic and \( P_0, P_1, P_1^+, P_2, P_2^+, P_G \) for fermionic ones. They satisfy the usual commutation relations

\[
\{\eta_0, P_0\} = \{\eta_G, P_G\} = \{\eta_1, P_1^+\} = \{\eta_1^+, P_1\} = \{\eta_2, P_2^+\} = \{\eta_2^+, P_2\} = 1,
\]

\[
[q_0, p_0] = [q_1, p_1^+] = [q_1^+, p_1] = i
\]

and act on the vacuum state as follows

\[
p_0|0\rangle = q_1|0\rangle = p_1|0\rangle = P_0|0\rangle = P_G|0\rangle = \eta_1|0\rangle = P_1|0\rangle = \eta_2|0\rangle = P_2|0\rangle = 0.
\]

The BRST charge \( Q \) acts in enlarged space of state vectors depending both on \( a^{+\mu} \) and on the ghost operators \( q_0, q_1^+, p_1^+, \eta_0, \eta_G, \eta_1^+, \eta_1, \eta_2^+, \eta_2, P_1^+, P_2^+ \) and having the structure

\[
|\Psi\rangle = \sum_{k_i} (q_0)^{k_1} (q_1^+)^{k_2} (p_1^+)^{k_3} (\eta_0)^{k_4} (\eta_G)^{k_5} (\eta_1^+)^{k_6} (\eta_1)^{k_7} (\eta_2^+)^{k_8} (\eta_2)^{k_9} \times
\]

\[
\times a^{+\mu_1} \cdots a^{+\mu_{k_0}} \Psi^{k_1 \cdots k_9} (x)|0\rangle
\]

The corresponding ghost number is 0. The sum in \( (10) \) is assumed over \( k_0, k_1, k_2, k_3 \) running from 0 to infinity and over \( k_4, k_5, k_6, k_7, k_8, k_9 \) running from 0 to 1. It is evident that the state vectors \( (1) \) are the partial cases of the above vectors.

The physical states are defined in the BRST approach by the equation

\[
Q'|\Psi\rangle = 0
\]

which is treated as an equation of motion. Besides, if \( |\Psi\rangle \) is a physical state, then due to nilpotency of the BRST operator, the state \( |\Psi\rangle + Q'|\Lambda\rangle \) will also be physical for any \( |\Lambda\rangle \). It means we have the gauge transformations

\[
\delta|\Psi\rangle = Q'|\Lambda\rangle.
\]

Let us decompose the BRST charge \( Q' \), the state vector \( |\Psi\rangle \) and the parameter of the gauge transformations \( |\Lambda\rangle \) as follows

\[
Q' = Q_0 + \eta_G \tilde{G}_0 + (2q_1^+ q_1 - \eta_2^+ \eta_2) P_G, \quad \tilde{G}_0, Q_0 = 0,
\]

\[
\tilde{G} = G_0 - i q_1 p_1^+ + i q_1^+ p_1 + i \eta_1^+ P_1 - \eta_1 P_1^+ + 2 \eta_2^+ P_2 - 2 \eta_2 P_2^+, \quad (14)
\]

\[
|\Psi\rangle = |\Psi_0\rangle + \eta_G |\Psi_G\rangle,
\]

\[
|\Lambda\rangle = |\Lambda_0\rangle + \eta_G |\Lambda_G\rangle.
\]

Here the state vectors \( |\Psi_0\rangle, |\Psi_G\rangle \) and the gauge parameters \( |\Lambda_0\rangle, |\Lambda_G\rangle \) are independent of \( \eta_G \). Then the equations of motion and the gauge transformations take the form

\[
Q_0|\Psi_0\rangle + (2q_1^+ q_1 - \eta_2^+ \eta_2)|\Psi_G\rangle = 0,
\]

\[
\tilde{G}_0|\Psi_0\rangle - Q_0|\Psi_G\rangle = 0,
\]

\[
\delta|\Psi_0\rangle = Q_0|\Lambda_0\rangle + (q_1^+ q_1 - \eta_2^+ \eta_2)|\Lambda_G\rangle,
\]

\[
\delta|\Psi_G\rangle = \tilde{G}_0|\Lambda_0\rangle - Q_0|\Lambda_G\rangle.
\]

Now ones try to simplify these equations. First, we decompose the state vector \( |\Psi\rangle \) and the gauge parameter \( |\Lambda\rangle \) in the eigenvectors of operator \( \tilde{G}_0 \): \( |\Psi\rangle = \sum |\Psi_n\rangle, |\Lambda\rangle = \sum |\Lambda_n\rangle \),

\[
\sum (q_1^+ q_1 - \eta_2^+ \eta_2)|\Lambda_n\rangle = 0.
\]
with \( \tilde{G}_0 |\Psi_n\rangle = (n + \frac{D-4}{2}) |\Psi_n\rangle \), \( \tilde{G}_0 |\Lambda_n\rangle = (n + \frac{D-4}{2}) |\Lambda_n\rangle \), \( n = 0, 1, 2, \ldots \). Then using the gauge transformation we can make all \( |\Psi_{Gn}\rangle = 0 \) choosing \( |\Lambda_n\rangle = \frac{-1}{n+(D-4)/2} |\Psi_G\rangle \) except the case \( n + \frac{D-4}{2} = 0 \). When \( D = 4 \) we have \( n = 0 \) and the field \( |\Psi_G\rangle \) after this gauge transformation is reduced to

\[
|\Psi_G\rangle \rightarrow |\Psi_{G0}\rangle = \psi(x) |0\rangle,
\]

i.e. it contains only spin-1/2 field. Substituting (21) in the equations of motion (17), (18) ones get

\[
\sum_{n=0}^{\infty} Q_0 |\Psi_{0n}\rangle = 0,
\]

\[
\sum_{n=0}^{\infty} \left( n + \frac{D-4}{2} \right) |\Psi_{0n}\rangle = Q_0 |\Psi_{G0}\rangle.
\]

Acting by the operator \( \tilde{G}_0 \) on both sides of equation (23) ones obtain

\[
\sum_{n=0}^{\infty} \left( n + \frac{D-4}{2} \right)^2 |\Psi_{0n}\rangle = 0.
\]

Since all the states vectors \( |\Psi_{0n}\rangle \) are linear independent, equation (24) means that all \( |\Psi_{0n}\rangle = 0 \), except \( n = -\frac{D-4}{2} = 0 \). Thus analogously to (21) one can write

\[
|\Psi_0\rangle \rightarrow |\Psi_{00}\rangle = \psi(x) |0\rangle.
\]

Ultimately we have two independent equations of motion

\[
T_0 |\Psi_{00}\rangle = 0, \quad T_0 |\Psi_{G0}\rangle = 0
\]

both for \( |\Psi_{00}\rangle \) and \( |\Psi_{G0}\rangle \). These equations in component form read

\[
\gamma^\mu p_\mu \psi(x) = 0, \quad \gamma^\mu p_\mu \psi_G(x) = 0.
\]

So, we see that the above construction leads to double number of equations and only for spin-1/2 fields. Hence, such a procedure is unsatisfactory.

To clarify a situation we pay attention to two points.

First, if we suppose that the state vectors and the gauge parameters do not depend on the ghost field \( \eta_G \) then we have only one Dirac equation and avoid the doubling the physical component states.

Second, the above construction has led us to the equations only for spin-1/2 fields. This happens because of \( \tilde{G}_0 \) has the structure

\[
\tilde{G}_0 = \tilde{N}_0 + \frac{D-4}{2},
\]

where \( \tilde{N}_0 \) is proportional to the ”particle” number operators associated with the operators \( a^+ \mu, q_1^+, p_1^+, \eta_1^+, \mathcal{P}_1^+, \eta_2^+, \mathcal{P}_2^+ \) and therefore if we want \( G_0 \) to be considered as a constraint, we get that there are no ”particles” (in \( D = 4 \) case) in the physical states and hence only equations of motion for the field with spin 1/2 arise.

We note that if we had instead of constraint \( \tilde{G} \) another constraint \( \tilde{G}_0 + h \) with \( h \) being an arbitrary constant, we could get equations of motion for fields of any spin by choosing the
arbitrary parameter \( h \) in the proper way for each spin. As a result, we could put \( n \) to any integer number since instead of condition \( n + \frac{D-4}{2} = 0 \) we had condition \( n + \frac{D-4}{2} + h = 0 \). However, if we simply change the constraint \( G_0 \rightarrow G_0 + h \) we break the algebra of the constraints which is given in Table 1. Thus, introducing of this arbitrary parameter \( h \) must be carry out in such a way that the algebra of the constraints will not be broken. This discussion shows that the representation for the constraints we used is too naive and improper and we have to find another representation.

Such a new representation may be realized as follows. We enlarge the number of creation and annihilation operators and extend the expressions for the constraints using the prescription: the new expressions for the constraints should have the general structure

\[
\text{New constraint} = \text{Old constraint} + \text{Additional part},
\]

with some additional parts which will be found in Section 3 in explicit form. This new representation must be constructed in such a way that the arbitrary parameter \( h \) appears in constraint \( G_{0\text{new}} \) as follows

\[
G_{0\text{new}} = G_0 + N_{\text{add}} + C + h,
\]

where \( N_{\text{add}} \) is proportional to the number operators of additional ”particles”, associated with extra annihilation and creation operators, \( C \) is some fixed constant which may arise.

It is evident that we must construct these additional parts first of all for those constraints whose commutators give \( G_0 \). The corresponding operators are \( T_1, T_1^+, L_2, L_2^+ \). Then we must go on and construct additional parts for those constraints whose commutators give \( T_1, T_1^+, L_2, L_2^+ \). Fortunately, they are \( G_0, T_1, T_1^+, L_2, L_2^+ \) and we may construct no additional parts for the other operators. Note that these operators form a subalgebra. Thus we have to construct a representation for the subalgebra of the constraints \( G_0, T_1, T_1^+, L_2, L_2^+ \) only. In the next Section we describe construction of a representation for such an subalgebra.

Certainly, we can construct additional parts for all the operators of the algebra given in Table 1, but in this case we must use some massive parameter which is absent in the true massless theory. Of course, one can try to introduce such a massive parameter to constraints by hand. However in this case we expect to get a massive higher spin field theory [24].

3 New representation of the constraints

In this Section we construct a new representation for the algebra of the constraints so that the new expressions for the constraints have the structure (29) and the parameter \( h \) appears in the constraint \( G_0 \) in the proper way (30). Algebra of the new constraints still has the form given by Table 1. As was explained at the end of the previous Section, for this purpose it is enough to construct the additional parts only for \( G_0, T_1, T_1^+, L_2, L_2^+ \) and the new expressions for the constraints should be

\[
\begin{align*}
L_{2\text{new}} &= L_2^+ + L_{2\text{add}}, & L_{2\text{new}} &= L_2 + L_{2\text{add}}, \\
T_{1\text{new}}^+ &= T_1^+ + T_{1\text{add}}, & T_{1\text{new}} &= T_1 + T_{1\text{add}}, \\
G_{0\text{new}} &= G_0 + G_{0\text{add}}
\end{align*}
\]

(all the other constraints do not change). Since the constraints \( G_{0\text{new}}, T_{1\text{new}}, T_{1\text{new}}^+, L_{2\text{new}}, L_{2\text{new}}^+ \) form a subalgebra and since the old and additional expressions of the constraints commute, the \( G_{0\text{add}}, T_{1\text{add}}, T_{1\text{add}}^+, L_{2\text{add}}, L_{2\text{add}}^+ \) form a subalgebra too with the same commutation
relation among them as for the old expressions for the constraints \( G_0, T_1, T_1^+, L_2, L_2^+ \). Thus it is enough to find a representation of the subalgebra in terms of new creation and annihilation operators which will be introduced later.

Let us turn to the construction of the subalgebra representation.

Note that the commutation relations between \( G_0 \) and the other operators of the subalgebra resemble the commutation relations between a number operator and creation and annihilation operators. Therefore let us consider the representation of the subalgebra of the constraints with the state vector \(|0\>_V\) annihilated by the operators \( T_1 \) and \( L_2 \)

\[
T_1|0\>_V = L_2|0\>_V = 0
\]  

(34)

and being the eigenvector of the operator \( G_0 \)

\[
G_0|0\>_V = h|0\>_V,
\]

(35)

where \( h \) is an arbitrary constant.\(^3\) It is the relation (35) where the vacuum state \(|0\>_V\) is an eigenvector of the number operator \( G_0 \) with eigenvalue \( h \) gives us the desired structure of the operator \( G_{\text{new}} \). Since \((T_1^+)^2 = 2L_2^+\) we can choose the basis vectors in this representation as follows

\[
|0, m\>_V = (L_2^+)^m|0\>_V, \quad |1, m\>_V = T_1^+(L_2^+)^m|0\>_V.
\]

(36)

The next step is to find the action of the operators \( T_1, T_1^+, L_2, L_2^+, G_0 \) on the basis vectors (36). The result is

\[
\begin{align*}
L_2^+|0, n\>_V &= |0, n + 1\>_V, & L_2^+|1, n\>_V &= |1, n + 1\>_V, \\
L_2|0, n\>_V &= (n^2 - n + nh)|0, n - 1\>_V, & L_2|1, n\>_V &= (n^2 + nh)|1, n - 1\>_V, \\
T_1^+|0, n\>_V &= |1, n\>_V, & T_1^+|1, n\>_V &= 2|0, n + 1\>_V, \\
T_1|0, n\>_V &= -|1, n - 1\>_V, & T_1|1, n\>_V &= -2(n + h)|0, n\>_V, \\
G_0|0, n\>_V &= (2n + h)|0, n\>_V, & G_0|1, n\>_V &= (2n + 1 + h)|1, n\>_V.
\end{align*}
\]

(37) \(\cdots\) (41)

Now, in order to construct the new representation for the subalgebra ones introduce the additional creation and annihilation operators. The number of pairs of these operators is equal to the number of the mutually conjugate pairs of the constraints. So we introduce one pair of fermionic \( d^+ \), \( d \) (corresponding to the constraints \( T_1^+ \), \( T_1 \)) and one pair of bosonic \( b^+ \), \( b \) (corresponding to the constraints \( L_2^+ \), \( L_2 \)) creation and annihilation operators with the standard commutation relations

\[
\{ d, d^+ \} = 1, \quad [b, b^+] = 1.
\]

(42)

Making use of the map of the basis vectors (36) and the basis vectors of the Fock space of the operators \( d^+, b^+ \)

\[
|0, n\>_V \leftrightarrow (b^+)^n|0\> \equiv |n\>, \quad |1, n\>_V \leftrightarrow d^+|n\>
\]

(43)

we can construct a representation of the subalgebra. From (37)–(41) and (43) ones find

\[
\begin{align*}
L_{2\text{add}} &= b^+, & L_{2\text{add}} &= (b^+b + d^+d + h)b, \\
T_{1\text{add}} &= 2b^+d + d^+, & T_{1\text{add}} &= -2(b^+b + h)d - d^+b, \\
G_{0\text{add}} &= 2b^+b + d^+d + h.
\end{align*}
\]

(44) \(\cdots\) (46)

\(^3\)The representation which is given by (44) and (45) is called in the mathematical literature the Verma module. It explains the subscript \( V \) at the state vectors.
It is easy to see, the operators (44), (45) are not hermitian conjugate to each other

\[(T_{1\text{add}})^+ \neq T_{1\text{add}}^+, \quad (L_{2\text{add}})^+ \neq L_{2\text{add}}^+\]  

(47)

if we use the usual rules for hermitian conjugation of the additional creation and annihilation operators

\[(d)^+ = d^+, \quad (b)^+ = b^+. \]  

(48)

The reason is that the map (43) does not preserve the scalar product. If we have two vectors |Φ₁⟩ and |Φ₂⟩ and corresponding them two vectors in the Fock space |Φ₁⟩ ↔ |Φ₁⟩, |Φ₂⟩ ↔ |Φ₂⟩ then in general

\[V⟨Φ₁|Φ₂⟩ ≠ ⟨Φ₁|Φ₂⟩, \]  

(49)

where we assumed that \(V⟨0|0⟩ = 1\). In order to improve the situation we change the scalar product in the Fock space so that

\[V⟨Φ₁|Φ₂⟩ = ⟨Φ₁|Φ₂⟩_{\text{new}} = ⟨Φ₁|K|Φ₂⟩, \]  

(50)

with some operator \(K\).

This operator may be found as follows. If we have a map between two bases

\[|n⟩_V \leftrightarrow |n⟩, \]  

(51)

then

\[|Φ⟩_V = \sum c_n |n⟩_V \leftrightarrow \sum c_n |n⟩ = |Φ⟩. \]  

(52)

Therefore if we preserve the scalar product for the basis vectors \(V⟨m|n⟩ = ⟨m|K|n⟩\) then it will be preserved for all vectors.

In the case of orthogonal basis in the Fock space, \(⟨m|n⟩ = C_n δ_{mn}\) with \(C_n\) being some constants (as we have in our case) it may be proved by direct substitution to (50) that the operator \(K\) is

\[K = \sum |m⟩ V⟨m|n⟩_V C_m C_n ⟨n|. \]  

(53)

Hence, in the case under consideration we get

\[K = \sum_{n=0}^{∞} \frac{1}{n!} \left( |n⟩⟨n| C(n, h) - 2d^+ |n⟩⟨n| d C(n + 1, h) \right), \]  

(54)

\[C(n, h) = \prod_{k=0}^{n-1} (k + h), \quad C(0, h) = 1. \]  

(55)

It is a simple exercise to check that operators (44) and (45) are now mutually conjugate in the following sense

\[KT_{1\text{add}} = (T_{1\text{add}}^+) K, \quad K T_{1\text{add}}^+ = (T_{1\text{add}})^+ K, \]  

(56)

\[KL_{2\text{add}} = (L_{2\text{add}}^+) K, \quad K L_{2\text{add}}^+ = (L_{2\text{add}})^+ K. \]  

(57)
Thus we have found the new representation of the algebra of constraints which is given by (31)–(33) with (44)–(46). (Remind that all the other constraints of the algebra do not change.) Since the algebra of the constraints have not been changed, a new BRST charge is constructed substituting in (6) the new constraints instead of old. As a result ones get

\[
\hat{Q} = q_0T_0 + q_1^+(T_1 - 2(b^+b + h)d - bd^+) + q_1(T_1^+ + 2b^+d + d^+)
\]

\[
+ \eta_0L_0 + \eta_1^L_1 + \eta_1^L_2(L_2 + (b^+b + h + d^+)d)b + \eta_2(L_2^+ + b^+)
\]

\[
+ \eta_G(G_0 + 2b^+b + d^+d + h)
\]

\[
+ i(\eta_1^+ q_1 - \eta_1 q_1^+)p_0 - i(\eta_2 q_1 + \eta_2 q_1^+)p_1^+ + i(\eta_G q_1^+ + \eta_G q_1)p_1
\]

\[
+ (q_0^2 - \eta_1^+ \eta_1)p_0 + (2q_1 q_1^+ - \eta_G \eta_G)P_G
\]

\[
+ (\eta_G \eta_1^+ - \eta_1^+ \eta_1 - 2q_0 q_1^+)P_1 + (\eta_G \eta_2 + \eta_1^+ \eta_2 - 2q_0 q_1)P_1^+
\]

\[
+ 2(\eta_G \eta_2^+ - q_1^2)P_2 + 2(\eta_2 \eta_G - q_1^2)P_2^+.
\]

(58)

Let us notice that the new BRST charge \((58)\) is selfconjugate in the following sense

\[
\hat{Q}^+ K = K \hat{Q},
\]

(59)

with operator \(K \) \((54)\). Now we turn to the construction of the Lagrangians for free fermionic higher spin fields.

### 4 Lagrangians for the free fermionic fields of single spin

In this Section we construct the Lagrangians for free higher spin fermionic fields using the BRST charge \((58)\). Unlike the bosonic case \([17, 18]\) we use here slightly another procedure.

First, let us extract the dependence of the new BRST charge \((58)\) on the ghosts \(\eta_G, P_G\)

\[
\hat{Q} = Q + \eta_G(\pi + h) + (2q_1 q_1 - \eta_2 \eta_2)P_G,
\]

(60)

\[
Q^2 = (\eta_2^+ \eta_2 - 2q_1^+ q_1)(\pi + h), \quad [Q, \pi] = 0
\]

(61)

with

\[
\pi = G_0 + 2b^+b + d^+d - iq_1 p_1^+ + iq_1^+ p_1 + \eta_1^+ P_1 - \eta_1 P_1^+ + 2\eta_2^+ P_2 - 2\eta_2 P_2^+,
\]

(62)

\[
Q = q_0(T_0 - 2q_1^+ P_1 - 2q_1 P_1^+) + i(\eta_1^+ q_1 - \eta_1 q_1^+)p_0 + \eta_0 L_0 + (q_0^2 - \eta_1^+ \eta_1)P_0
\]

\[
+ q_1^+(T_1 - 2(b^+b + h)d - bd^+) + q_1(T_1^+ + 2b^+d + d^+)
\]

\[
+ \eta_1^L_1 + \eta_1^L_2(L_2 + (b^+b + h + d^+)d)b + \eta_2(L_2^+ + b^+)
\]

\[
- i\eta_2 q_1^+ p_1^+ + i\eta_2^+ q_1 p_1 + \eta_2^+ \eta_1 P_1 + \eta_2 \eta_2 P_1^+ - 2q_1^+ q_1 P_2 - 2q_1 q_2^+ P_2 - 2q_1^2 P_2^+.
\]

(63)

Second, in order to avoid the doubling the physical component states as it was in Section 2 eq. \((26)\) we suppose that the state vectors are independent of \(\eta_G\), i.e. \(P_G|\chi\rangle = 0\). Now the general structure of the states looks like

\[
|\chi\rangle = \sum k_i (q_0)^{k_1} (q_1^+)^{k_2} (p_1)^{k_3} (\eta_0)^{k_4} (\eta_1^+)^{k_5} (d^+)^{k_6} (P_1)^{k_7} (\eta_2^+)^{k_8} (P_2)^{k_9} (b^+)^{k_{10}} \times
\]

\[
\times a^{+\mu_1} ... a^{+\mu_{k_0}} \chi^{k_1 ... k_{10}} \mu_1 ... \mu_{k_0} (x)|0\rangle.
\]

(64)

The corresponding ghost number is 0, as usual. The sum in \((64)\) is assumed over \(k_0, k_1, k_2, k_3, k_{10}\) running from 0 to infinity and over \(k_4, k_5, k_6, k_7, k_8, k_9\) running from 0 to 1.
After this assumption the equation on the physical states in the BRST approach \( \tilde{Q}|\chi\rangle = 0 \) yields two equations

\[
Q|\chi\rangle = 0, \quad (\pi + h)|\chi\rangle = 0.
\] (65)  (66)

Equation (66) is the eigenvalue equation for the operator \( \pi \) (62) with the corresponding eigenvalues \(-h\)

\[-h = n + \frac{D - 4}{2}, \quad n = 0, 1, 2, \ldots .\] (67)

The numbers \( n \) are related with the spin \( s \) of the corresponding eigenvectors as \( s = n + 1/2 \). Let us denote the eigenvectors of the operator \( \pi \) corresponding to the eigenvalues \( n + \frac{D - 4}{2} \) as \( |\chi\rangle_n \)

\[
\pi|\chi\rangle_n = \left( n + \frac{D - 4}{2} \right) |\chi\rangle_n.
\] (68)

Then solutions to the system of equations (65), (66) are enumerated by \( n = 0, 1, 2, \ldots \) and satisfy the equations

\[
Q_n|\chi\rangle_n = 0,
\] (69)

where in the BRST charge (63) we substitute \( n + \frac{D - 4}{2} \) instead of \(-h\) for each given equation on spin \( s = n + 1/2 \). Thus we get that the BRST charge depends on \( n \)

\[
Q_n = q_0(T_0 - 2q_1^+P_1 - 2q_1P_1^+) + i(\eta_1^+q_1 - \eta_1q_1^+)p_0 + \eta_0L_0 + (q_0^2 - \eta_1^+\eta_1)P_0 + q_1^+(T_1 - 2b^+bd - bd^+) + q_1(T_1^+ + 2b^+d + d^+)
+ \eta_1^+L_1 + \eta_1L_1^+ + \eta_2^+(L_2 + (b^+b + d^+d)b) + \eta_2(L_2^+ + b^+)
- i\eta_2Q_1^+p_1^+ + i\eta_2q_1p_1 + \eta_2^+\eta_1p_1 + \eta_1^+\eta_2P_1 + 2q_1^+2P_2 - 2q_1P_2^+
+ (2q_1^+d - \eta_2^+b)(n + \frac{D - 4}{2}).
\] (70)

Let us rewrite the operators \( Q_n \) (70) in the form independent of \( n \). This may be done by replacing \( n + \frac{D - 4}{2} \) in (70) by the operator \( \pi \) (62). Then we obtain

\[
Q_\pi = q_0(T_0 - 2q_1^+P_1 - 2q_1P_1^+) + i(\eta_1^+q_1 - \eta_1q_1^+)p_0 + \eta_0L_0 + (q_0^2 - \eta_1^+\eta_1)P_0 + q_1^+(T_1 - 2b^+bd - bd^+) + q_1(T_1^+ + 2b^+d + d^+)
+ \eta_1^+L_1 + \eta_1L_1^+ + \eta_2^+(L_2 + (b^+b + d^+d)b) + \eta_2(L_2^+ + b^+)
- i\eta_2Q_1^+p_1^+ + i\eta_2q_1p_1 + \eta_2^+\eta_1p_1 + \eta_1^+\eta_2P_1 + 2q_1^+2P_2 - 2q_1P_2^+
+ (2q_1^+d - \eta_2^+b)\pi,
\] (71)

where \( Q_\pi = Q_{n|n + \frac{D - 4}{2}} \). Operator (71) analogous to the BRST operator which obtained in the bosonic case (17), (18) after the dependence on the ghost fields \( \eta_G \), \( P_G \) was removed. Let us note that operator \( Q_\pi \) is nilpotent identically.

Now we can rewrite the set of equations (62) in the equivalent form as one equation for all half-integer spins. Since the operators \( Q_\pi \) and \( \pi \) commute then vectors \( Q_\pi|\chi\rangle_n \) belong to different eigenvalues of the operator \( \pi \) and consequently are linear independent. Therefore we may write the set of equations (62) as one equation summing them

\[
\sum_{n=0}^{\infty} Q_\pi|\chi\rangle_n = Q_\pi \sum_{n=0}^{\infty} |\chi\rangle_n = Q_\pi|\chi\rangle = 0,
\] (72)
where we denote

$$|\chi\rangle = \sum_{n=0}^{\infty} |\chi\rangle_n.$$  \hspace{1cm} (73)

Thus we obtain that the equation

$$Q\pi |\chi\rangle = 0$$  \hspace{1cm} (74)

with $|\chi\rangle$ defined by (73) describes propagation of all half-integer spin fields simultaneously.

Let us turn to the gauge transformations. Analogously we suppose that the parameters of the gauge transformations are also independent of $\eta_G$. Due to eq. (61) we have the following tower of the gauge transformations and the corresponding eigenvalue equations for the gauge parameters

$$\delta |\chi\rangle = Q |\Lambda\rangle,$$

$$\delta |\Lambda\rangle = Q |\Lambda^{(1)}\rangle,$$

$$\delta |\Lambda^{(i)}\rangle = Q |\Lambda^{(i+1)}\rangle,$$

where $h$ has already been determined for each spin. Since the ghost number of the gauge parameters is reduced with the stage of reducibility $gh(|\Lambda^{(i)}\rangle) = -(i + 1)$ we get that for each $n$ (and for the spin $s = n + 1/2$ respectively) the tower of the gauge transformations must be finite. Thus in case of fermionic higher spin fields we have gauge symmetry with reducible generators.

Doing the same procedure as above for the equations of motion we may write the gauge transformations (75)–(77) for each given spin

$$\delta |\chi\rangle_n = Q_n |\Lambda\rangle_n,$$

$$\delta |\Lambda\rangle_n = Q_n |\Lambda^{(1)}\rangle_n,$$

$$\delta |\Lambda^{(i)}\rangle_n = Q_n |\Lambda^{(i+1)}\rangle_n,$$

and for all half-integer spins simultaneously

$$\delta |\chi\rangle = Q\pi |\Lambda\rangle,$$

$$\delta |\Lambda\rangle = Q\pi |\Lambda^{(1)}\rangle,$$

$$\delta |\Lambda^{(i)}\rangle = Q\pi |\Lambda^{(i+1)}\rangle,$$

Next step is to extract the zero ghost mode from the operator $Q\pi$ (74). This operator has the structure

$$Q\pi = \eta_0 L_0 + (q_0^2 - \eta_1^+ \eta_1) P_0 + q_0(T_0 - 2q_1^+ P_1 - 2q_1 P_1^+) + i(\eta_1^+ q_1 - \eta_1 q_1^+) p_0 + \Delta Q\pi,$$

where $\Delta Q\pi$ is independent of $\eta_0$, $P_0$, $q_0$, $p_0$. Also we may decompose the state vector and the gauge parameters as

$$|\chi\rangle = \sum_{k=0}^{\infty} q_0^k (|\chi^k_0\rangle + \eta_0 |\chi^k_1\rangle),$$

$$|\Lambda^{(i)}\rangle = \sum_{k=0}^{\infty} q_0^k (|\Lambda^{(i)k}_0\rangle + \eta_0 |\Lambda^{(i)k}_1\rangle),$$

$$gh(|\chi^k_m\rangle) = -(m + k),$$

$$gh(|\Lambda^{(i)k}_m\rangle) = -(i + k + m + 1).$$
Following the procedure described in [14] we get rid of all the fields except two $|\chi_0^0\rangle$, $|\chi_0^1\rangle$ and equation (74) is reduced to

\[ \Delta Q_\pi |\chi_0^0\rangle + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\chi_0^1\rangle = 0, \tag{87} \]

\[ \tilde{T}_0 |\chi_0^0\rangle + \Delta Q_\pi |\chi_0^1\rangle = 0, \tag{88} \]

where $\tilde{T}_0 = T_0 - 2q^+_1P_1 - 2q_1^+P_1^+$, and $\{A,B\} = AB + BA$.

To be complete we show how equations (87) and (88) can be derived from the (74). First we extract the zero ghost modes from the BRST charge $Q_\pi$ (84), the state vector $|\chi\rangle$ (85) and the gauge parameter $|\Lambda\rangle$ (86). After this the gauge transformation for the fields $|\chi_0^k\rangle$, $k \geq 2$ are

\[ \delta |\chi_0^k\rangle = \Delta Q_\pi |\Lambda_0^k\rangle + \eta_1^+ \eta_1 |\Lambda_0^k\rangle + (k + 1)(\eta_1^+ q_1 - \eta_1 q_1^+) |\Lambda_0^{k+1}\rangle + \tilde{T}_0 |\Lambda_0^{k-1}\rangle + |\Lambda_1^{k-2}\rangle. \tag{89} \]

We see that we can make all fields $|\chi_0^k\rangle$, $k \geq 2$ to be zero using the gauge parameters $|\Lambda_0^k\rangle$.

Second step is to consider the equations of motion at coefficients $(q_0)^k$, $k \geq 3$. Taking into account that all fields $|\chi_0^k\rangle = 0$, $k \geq 2$, these equations are reduced to

\[ |\chi_1^{k-2}\rangle = \eta_1^+ \eta_1 |\chi_1^k\rangle, \quad k \geq 3 \tag{90} \]

and we find that all $|\chi_1^k\rangle = 0$, $k \geq 1$.

Finally we consider the equation at coefficient $(q_0)^2$ and express $|\chi_0^0\rangle$ field from $|\chi_0^1\rangle$

\[ |\chi_0^0\rangle = -\tilde{T}_0 |\chi_0^1\rangle. \tag{91} \]

So it remained only two fields $|\chi_0^0\rangle$ and $|\chi_0^1\rangle$ and the independent equations of motion for them are (87) and (88).

Since the operators $Q_\pi$, $\tilde{T}_0$, $\{ \tilde{T}_0, \eta_1^+ \eta_1 \}$ commute with the operator $\pi$, then from equations (87), (88) we may get equations of motion for fixed spin fields

\[ \Delta Q_\pi |\chi_0^0\rangle_n + \frac{1}{2} \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\chi_0^1\rangle_n = 0, \tag{92} \]

\[ \tilde{T}_0 |\chi_0^0\rangle_n + \Delta Q_\pi |\chi_0^1\rangle_n = 0, \tag{93} \]

These field equations can be deduced from the following Lagrangian

\[ \mathcal{L}_n = n |\chi_0^0\rangle_n |K_n \tilde{T}_0 |\chi_0^0\rangle_n + \frac{1}{2} n |\chi_0^1\rangle_n |K_n \{ \tilde{T}_0, \eta_1^+ \eta_1 \} |\chi_0^1\rangle_n + n |\chi_0^0\rangle_n |K_n \Delta Q_\pi |\chi_0^1\rangle_n + n |\chi_0^1\rangle_n |K_n \Delta Q_\pi |\chi_0^0\rangle_n, \tag{94} \]

where the standard scalar product for the creation and annihilation operators is assumed.

This Lagrangian is now written for fields with given spin which are defined by $n$ chosen according to (65), (67)

\[ \pi |\chi_0^0\rangle_n = (n + (D - 4)/2) |\chi_0^0\rangle_n, \quad \pi |\chi_0^1\rangle_n = (n + (D - 4)/2) |\chi_0^1\rangle_n \tag{95} \]

and the operator $K_n$ is the operator $K$ (54) where the following substitution is assumed be done $h \rightarrow -(n + (D - 4)/2)$

\[ K_n = \sum_{k=0}^{\infty} \frac{1}{k!} \left( |k\rangle \langle k| \ C(k, -n - \frac{D - 4}{2}) - 2d^+ |k\rangle \langle d \ C(k + 1, -n - \frac{D - 4}{2}) \right). \tag{96} \]

\[ 4 \text{The Lagrangian is defined up as usual to an overall factor} \]
Thus the operators $K_n$ depend on the spin of the fields. Note also that we can write $\Delta Q_n$ instead of $\Delta Q_\pi$ in the equations of motion (92), (93), in the Lagrangian (94) and in the gauge transformations (97)–(100) for fixed spin fields.

The equations of motion (92), (93) and the Lagrangian (94) are invariant under the gauge transformations

$$\delta |\chi^0_0\rangle_n = \Delta Q_\pi |\Lambda^0_0\rangle_n + \frac{1}{2}\{\bar{T}_0, \eta^\dagger_1 \eta_1\} |\Lambda^1_0\rangle_n, \quad (97)$$
$$\delta |\chi^0_0\rangle_n = \bar{T}_0 |\Lambda^0_0\rangle_n + \Delta Q_\pi |\Lambda^0_0\rangle_n, \quad (98)$$

which are reducible

$$\delta |\Lambda^{(i)0}_0\rangle_n = \Delta Q_\pi |\Lambda^{(i+1)0}_0\rangle_n + \frac{1}{2}\{\bar{T}_0, \eta^\dagger_1 \eta_1\} |\Lambda^{(i+1)1}_0\rangle_n, \quad |\Lambda^{(0)0}_0\rangle_n = |\Lambda^0_0\rangle_n, \quad (99)$$
$$\delta |\Lambda^{(i)1}_0\rangle_n = \bar{T}_0 |\Lambda^{(i+1)0}_0\rangle_n + \Delta Q_\pi |\Lambda^{(i+1)1}_0\rangle_n, \quad |\Lambda^{(0)1}_0\rangle_n = |\Lambda^1_0\rangle_n, \quad (100)$$

with finite number of reducibility stages $i_{\text{max}} = n - 1$ for spin $s = n + 1/2$. In Section 6 we show that the Lagrangian (94) is transformed to the Fang & Fronsdal Lagrangian [2] in four dimensions after eliminating the auxiliary fields. So, we construct the Lagrangian for arbitrary fixed spin fermionic fields using the BRST approach. Now we turn to construction of Lagrangian describing propagation of all half-integer spin fields simultaneously.

## 5 Lagrangian for all half-integer spin fields

In this section we construct the Lagrangian which describes all half-integer spin fields simultaneously, i.e. we construct the Lagrangian in terms of the fields containing all half-integer spins

$$|\chi^i_0\rangle = \sum_{n=0}^{\infty} |\chi^i_n\rangle_n, \quad i = 0, 1. \quad (101)$$

As we mentioned above the operator $\pi$ commutes with the operators $Q_\pi$, $\bar{T}_0$, $\{\bar{T}_0, \eta^\dagger \eta_1\}$ and moreover it commutes with each term of these operators. Therefore we can write all the operators $K_n$ in the Lagrangians (94) in the same form for any spin. This is done analogously to that when we transformed $Q_n$ into $Q_\pi$. Namely, we stand all $h$ to the right (or to the left position) in the expression for $K$ (93) and substitute $\pi$ instead of $-h$. As a result we have

$$K_\pi = \sum_{n=0}^{\infty} \frac{1}{n!} \left( |n\rangle \langle n| \ C(n, -\pi) - 2d^+ |n\rangle \langle n| \ d \ C(n + 1, -\pi) \right). \quad (102)$$

Thus we can substitute $K_\pi$ (102) instead of $K_n$ (93) in the expression for the Lagrangian corresponding to one fixed spin field (94).

Evidently that Lagrangian describing all half-integer spin fields simultaneously should be a sum of all the Lagrangians for each spin (94)

$$\mathcal{L} = \sum_{n=0}^{\infty} \mathcal{L}_n = \sum_{n=0}^{\infty} \left[ n \langle \chi^0_0| K_\pi \bar{T}_0 |\chi^0_0\rangle_n + \frac{1}{2} n \langle \chi^1_0| K_\pi \{\bar{T}_0, \eta^\dagger_1 \eta_1\} |\chi^1_0\rangle_n \right. \right.$$
$$\left. \left. + n \langle \chi^0_0| K_\pi \Delta Q_\pi |\chi^1_0\rangle_n + n \langle \chi^1_0| K_\pi \Delta Q_\pi |\chi^0_0\rangle_n \right]. \quad (103)$$
Now we rewrite this Lagrangian in terms of two concise fields \( \chi^0_0 \) and \( \chi^1_0 \) containing all half-integer spin fields. Using that \( n\langle \chi^i_0|\chi^{i'}_0 \rangle_n' \sim \delta^{i'i} \delta_{nn'} \), we transform each term in Lagrangian (103) as

\[
\sum_{n=0}^{\infty} n\langle \chi^0_0|K_0^i\tilde{T}_0^i|\chi^0_0 \rangle_n = \left( \sum_{n=0}^{\infty} n\langle \chi^0_0 \rangle \right) K_0^i\tilde{T}_0^i \left( \sum_{n=0}^{\infty} |\chi^0_0 \rangle_n \right) = \langle \chi^0_0|K_0^i\tilde{T}_0^i|\chi^0_0 \rangle
\]

and find

\[
\mathcal{L} = \langle \chi^0_0|K_0^i\tilde{T}_0^i|\chi^0_0 \rangle + \frac{1}{2}\langle \chi^1_0|K_0^i\left\{\tilde{T}_0^i, \eta_1^i \right\}|\chi^1_0 \rangle \\
+ \langle \chi^0_0|K_0^i\Delta Q^i_0|\chi^1_0 \rangle + \langle \chi^1_0|K_0^i\Delta Q^i_0|\chi^0_0 \rangle
\]

where we have used (101). Thus we have constructed the Lagrangian describing propagation of all half-integer spin fields simultaneously (105), the equations of motion which are derived from it are (87), (88).

Let us turn to the gauge transformations for the fields (101). Summing up the gauge transformation for the fields of fixed spins (97), (98) over all \( n \) we get the gauge transformations for the fields containing all half-integer spins (101)

\[
\delta|\chi^0_0 \rangle = \Delta Q^i_0|\Lambda^0_0 \rangle + \frac{1}{2}\left\{\tilde{T}_0^i, \eta_1^i \right\}|\Lambda^1_0 \rangle, \\
\delta|\chi^1_0 \rangle = \tilde{T}_0^i|\Lambda^0_0 \rangle + \Delta Q^i_0|\Lambda^1_0 \rangle
\]

which are also reducible

\[
\delta|\Lambda^{(i)}_0 \rangle = \Delta Q^i_0|\Lambda^{(i+1)}_0 \rangle + \frac{1}{2}\left\{\tilde{T}_0^i, \eta_1^i \right\}|\Lambda^{(i+1)}_0 \rangle, \\
|\Lambda^{(0)}_0 \rangle = |\Lambda^0_0 \rangle, \\
|\Lambda^{(1)}_0 \rangle = |\Lambda^1_0 \rangle
\]

where we introduced

\[
|\Lambda^{(j)}_0 \rangle = \sum_{n=i+j+1}^{\infty} |\Lambda^{(j)}_0 \rangle_n 
\]

Since the fields (101) contain infinite number of spins and since the order of reducibility grows with the spin value, then the order of reducibility of the gauge symmetry for fields (101) will be infinite.

It should be noted that the procedure developed here for constructing the Lagrangians both for fixed spin fields (94) and for all half-integer spin fields (105) may be used for constructing Lagrangians for bosonic fields as well. Also this procedure may be generalized for constructing Lagrangians with mixed symmetry tensor-spinor fields as it was done in the bosonic case [17].

Now we turn to the reduction of (94) to the Fang & Fronsdal Lagrangian [2].

6 Reduction to Fang & Fronsdal Lagrangian

In this Section we show that Lagrangian (94) is transformed to the Fang & Fronsdal Lagrangian [2] in four dimensions after elimination of the auxiliary fields.
Let us consider Lagrangian (94) with some fixed $n$. In this case the gauge symmetry (99), (100) is reducible with $i_{max} = n - 1$. We can write down the dependence of the fields and the gauge parameters on the ghost fields explicitly. For the lowest gauge parameters we have

$$|\Lambda^{(n-1)0}\rangle_0/n = (p^+_1)^{n-1} \left\{ \mathcal{P}^+_1 |\omega\rangle_0 + p^+_1 |\omega_1\rangle_0 \right\},$$  (111)

$$|\Lambda^{(n-1)1}\rangle_0/n = 0.$$  (112)

Here we have taken into account that the gauge parameters are the eigenvectors of the operator $\pi$ with the eigenvalue $-\left( n + \frac{d-4}{2} \right)$ and that they have the ghost numbers $-n$ and $-(n+1)$ respectively. Let us recall that the subscripts at the state vectors are associated with the eigenvalues of the corresponding state vectors (68).

With the help of these parameters we can get rid of the dependence on the ghost $\mathcal{P}^+_2$ in the parameters $|\Lambda^{(n-2)0}\rangle_0/n$. (The parameter $|\Lambda^{(n-2)1}\rangle_0/n$ has no dependence on the ghost $\mathcal{P}^+_2$.) We may go on and get rid of any dependence on the ghost $\mathcal{P}^+_2$ in all the fields and the parameters. The restriction which appears on the fields and the parameters is that they can depend on the ghost $p^+_1$ maximum in the first power. (They must be annihilated by the operator $q^2_1$.) Due to this restriction the remain gauge symmetry is reducible with the first stage reducibility.

Then we can get rid of the dependence on the ghost $\eta^+_2$ in all the remain fields and the gauge parameters. The restriction appeared is that the fields and the gauge parameters must be annihilated by the operator $L^2 \equiv L_2 + (b^+b + h)b + d^+ db + \eta_1 \mathcal{P}_1 + i q_1 p_1$.

After this we get rid of the gauge transformation parameter $|\Lambda^0_0\rangle_0/n$ with the help of the parameter $|\Lambda^0_1\rangle_0/n$. Now we write down the remain fields and the gauge parameter explicitly

$$|\chi^0_0\rangle_0/n = |\Psi\rangle_0/n + \eta^+_1 \mathcal{P}^+_1 |\Psi_1\rangle_{n-2} + q^+_1 p^+_1 |\Psi_2\rangle_{n-2} + p^+_1 \eta^+_1 |\Psi_3\rangle_{n-2} + q^+_1 \mathcal{P}^+_1 |\Psi_4\rangle_{n-2} + q^+_1 \mathcal{P}^+_1 |\Psi_5\rangle_{n-4} + q^+_1 \mathcal{P}^+_1 |\Psi_6\rangle_{n-4},$$  (113)

$$|\chi^1_0\rangle_0/n = \mathcal{P}^+_1 |\chi\rangle_{n-1} + p^+_1 |\chi_1\rangle_{n-1} + p^+_1 \eta^+_1 \mathcal{P}^+_1 |\chi_2\rangle_{n-3} + q^+_1 p^+_1 \mathcal{P}^+_1 |\chi_3\rangle_{n-3},$$  (114)

$$|\Lambda^0_0\rangle_0/n = \mathcal{P}^+_1 |\xi\rangle_{n-1} + p^+_1 |\xi_1\rangle_{n-1} + q^+_1 \mathcal{P}^+_1 |\xi_2\rangle_{n-3} + p^+_1 \eta^+_1 \mathcal{P}^+_1 |\xi_3\rangle_{n-3} + 2q^+_1 p^+_1 |\xi_4\rangle_{n-5} + q^+_1 \mathcal{P}^+_1 \langle\xi_5|_{n-4} + q^+_1 \mathcal{P}^+_1 \langle\xi_6|_{n-4},$$  (115)

with the restriction $L_2^2 |\chi^0_0\rangle_0/n = L_2^2 |\chi^1_0\rangle_0/n = L_2^2 |\Lambda^0_0\rangle_0/n = 0$. Here $|\Psi\rangle_0$, $|\chi\rangle_k$, $|\xi\rangle_k$ do not depend on the ghost fields. Using the gauge transformations we first get rid of the fields $|\Psi_2\rangle_{n-2}$, $|\Psi_4\rangle_{n-2}$, $|\Psi_5\rangle_{n-4}$, $|\Psi_6\rangle_{n-4}$ after which the gauge parameters are restricted by $T_1^0 |\xi\rangle_{n-1} = T_1^0 |\xi_1\rangle_{n-1} = T_1^0 |\xi_2\rangle_{n-3} = 0$, with $T_1^0 \equiv T_1 - 2(b^+b + h)d - d^+b$. Now we can see that $|\chi_3\rangle_{n-3} = 0$ and then $|\Psi_5\rangle_{n-4} = 0$ as the equation of motion. Then we eliminate one after another the fields $|\chi_2\rangle_{n-3}$, $|\Psi_3\rangle_{n-2}$, $|\Psi_1\rangle_{n-2}$ and $|\chi_1\rangle_{n-1}$. The new restrictions on the gauge parameters are $T_0 |\xi_3\rangle_{n-3} = L_1 |\xi_1\rangle_{n-1} = T_0 |\xi_1\rangle_{n-1} = 0$. The remain gauge freedom is enough to get rid of the dependence on $b^+$ and $d^+$ in $|\Psi\rangle_{n}$, $|\Psi_1\rangle_{n-2}$ and $|\chi_1\rangle_{n-1}$. After this the remain equations of motion and the gauge transformation are

$$T_0 |\Psi_0\rangle_{n} + L^+_1 |\chi_0\rangle_{n-1} = 0,$$  (116)

$$T_0 |\chi_0\rangle_{n-1} - L_1 |\Psi_0\rangle_{n} + L^+_1 |\Psi_10\rangle_{n-2} = 0,$$  (117)

$$T_0 |\Psi_10\rangle_{n-2} + L_1 |\chi_0\rangle_{n-1} = 0,$$  (118)

$$\delta |\Psi_0\rangle_{n} = L^+_1 |\xi_0\rangle_{n-1},$$  (119)

$$\delta |\chi_0\rangle_{n-1} = -T_0 |\xi_0\rangle_{n-1},$$  (120)

Here subscript 0 means that the corresponding fields and the gauge parameter do not depend on $b^+$ and $d^+$. Besides, the gauge parameter is restricted as in the Fang and Fronsdal theory.
First we find that
\[ \gamma_l \] for arbitrary dimensional spacetime

Using the restrictions on the fields which stand in the right column of (116), (117) we can express the fields \(|\psi_0|_{n-2}\) and \(|\chi_0|_{n-1}\) through \(|\psi_0|_n\) and substitute them in the Lagrangian (121). As a result ones get the Lagrangian which is generalization of the Fang and Fronsdal Lagrangian [2] for arbitrary dimensional spacetime

with the vanishing triple \(\gamma\)-trace \((T_1)^3|\psi_0|_n = 0\) and gauge transformation (119) with the constrained gauge parameter \(T_1|\xi|_{n-1} = 0\).

To see that the Lagrangian (122) indeed coincides with the Lagrangian of Fang and Fronsdal [2] we calculate it (122) explicitly for an arbitrary spin field \(s = n + 1/2\)

First we find that

\[ n(\psi_0|T_0|\psi_0)_n = (-1)^n \hbar \cdot \hbar, \]

\[ n(\psi_0|T_1+T_0T_1|\psi_0)_n = n(-1)^n n \hbar \cdot \hbar', \]

\[ n(\psi_0|L_2^+T_0L_2|\psi_0)_n = (-1)^n \frac{n(n-1)}{4} \hbar'' \cdot \hbar''', \]

\[ n(\psi_0|L_1^+T_1|\psi_0)_n = -(-1)^n n \hbar \cdot p \cdot h, \]

\[ n(\psi_0|L_2^+L_1^+L_2|\psi_0)_n = (-1)^n \frac{n(n-1)}{2} \hbar'' \cdot p \cdot h', \]

where we have used the notation of [2]. Substituting the found relations in (122) ones get

\[ \mathcal{L} = (-1)^n \left( \hbar \cdot \hbar + n \hbar \cdot \hbar' - \frac{1}{4} n(n-1) \hbar'' \cdot \hbar''' \right) - n(\hbar \cdot p \cdot h + \text{H.c.}) + \frac{1}{2} n(n-1) (\hbar'' \cdot p \cdot h' + \text{H.c.}) \].

The Lagrangian (122) coincides in \(D = 4\) with the Lagrangian of Fang and Fronsdal up to overall factor \((-1)^n\). It can be treated as Fang-Fronsdal Lagrangian for arbitrary \(D\)-dimensional space.

Thus we showed that Lagrangian (124) is reduced to the Fang and Fronsdal Lagrangian (122) with all necessary conditions on the field and the gauge parameter.

We point out that from the Lagrangian (122) we may get one Lagrangian describing propagation of all half-integer spin fields simultaneously. Summing up the Lagrangians (122) over all half-integer spins and noticing that \(n(\psi_0|\psi_0)_n \sim \delta_{nn'}\) we get this Lagrangian in the form

\[ \mathcal{L} = \langle \psi_0| \{ T_0 - T_1^+T_0T_1 - L_2^+T_0L_2 + T_1^+L_1^+L_1T_1 + L_2^+L_1^+L_2 \} |\psi_0 \rangle, \]
where we used the notation

$$|\Psi_0\rangle = \sum_{n=0}^{\infty} |\Psi_0\rangle_n.$$  \hfill (130)

In the next Section, we apply our procedure for derivation a new Lagrangian for spin-5/2 field model, where, unlike Fang-Fronsdal Lagrangian, all auxiliary fields stipulated by general Lagrangian construction, are taken into account.

## 7 Construction of the Lagrangian for field with spin 5/2

In this Section we show how the generic Lagrangian construction \[\text{[69]}\], given in terms of abstract state vectors, is transformed to standard space-time Lagrangian form. We explicitly derive a Lagrangian for the field with spin 5/2 which contains the auxiliary fields and more gauge symmetries in compare with Fang-Fronsdal Lagrangian. Of course, it can be reduced to Fang-Fronsdal Lagrangian after partial gauge-fixing and putting \(D = 4\). However, this new Lagrangian possesses the interesting properties, in particular it has a reducible gauge symmetry.

Let us start. Since \(s = n + 1/2 = 5/2\) we have \(n = 2\) and \(h = -D/18.\) Then we first extract the ghost fields dependence of the fields and the gauge parameters

\[
\begin{align*}
|\chi_0\rangle_2 &= |\Psi_2 + \eta_1^+ \mathcal{P}_1^+ |\Psi_1\rangle_0 + q_1^+ p_1^+ |\Psi_2\rangle_0 + p_1^+ \eta_2^+ |\Psi_3\rangle_0 + q_1^+ \mathcal{P}_2^+ |\Psi_4\rangle_0, \\
|\chi_0\rangle_2 &= \mathcal{P}_1^+ |\chi_1 + p_1^+ |\chi_1\rangle_1 + \mathcal{P}_2^+ |\chi_4\rangle_0, \\
|\Lambda_0\rangle_2 &= \mathcal{P}_1^+ |\xi_1 + p_1^+ |\xi_1\rangle_1 + \mathcal{P}_2^+ |\xi_4\rangle_0, \\
|\Lambda_0\rangle_2 &= p_1^+ \mathcal{P}_1^+ |\lambda_0\rangle_0 + (p_1^+)^2 |\lambda_1\rangle_0, \\
|\Lambda_0\rangle_2 &= p_1^+ \mathcal{P}_1^+ |\omega_0\rangle_0 + (p_1^+)^2 |\omega_1\rangle_0.
\end{align*}
\]

Here the ghost numbers of the fields and the gauge parameters are also taken into account. In the following we omit the subscripts at the state vectors associated with the eigenvalues of the operator \(\pi\) \[68\].

Substituting the fields in this concise form in the Lagrangian \[\text{[94]}\] one finds

\[
\mathcal{L}_2 = \langle \Psi | K_2 \{ T_0 | \Psi \} + L_1^+ | \chi \rangle + i T_1^+ | \chi 1 \rangle + L_2^+ | \chi 4 \rangle \\
+ \langle \Psi_1 | K_2 \{ -T_0 | \Psi_1 \} - 2 i | \Psi_3 \rangle - L_1 | \chi \rangle + | \chi 4 \rangle \\
+ \langle \Psi_2 | K_2 \{ T_0 | \Psi_2 \} - 2 | \Psi_3 \rangle + T_1^\prime | \chi 1 \rangle - i | \chi 4 \rangle \\
+ \langle \Psi_3 | K_2 \{ 2 i | \Psi_1 \} - 2 | \Psi_2 \rangle + i T_0 | \Psi_4 \rangle + i T_1^\prime | \chi \rangle \\
+ \langle \Psi_4 | K_2 \{ -i T_0 | \Psi_3 \} + i L_1 | \chi_1 \rangle \\
+ \langle \chi_1 | K_2 \{ -T_0 | \chi \} - i | \chi 1 \rangle + L_1 | \Psi \rangle - L_1^+ | \Psi_1 \rangle - i T_1^+ | \Psi_3 \rangle \\
+ \langle \chi_1 | K_2 \{ i | \chi \} - i T_1^\prime | \Psi \rangle + T_1^+ | \Psi_3 \rangle - i L_1^+ | \Psi_4 \rangle \\
+ \langle \chi_4 | K_2 \{ L_2^\prime | \Psi \rangle + | \Psi_1 \rangle + i | \Psi_2 \rangle, \tag{136}\end{align*}
\]

where we have used that the ghost fields commute with the operator \(K_n\) \[96\]. Next we find the gauge transformations \[\text{[77]}\], \[\text{[85]}\]

\[
\begin{align*}
\delta |\Psi\rangle &= L_1^+ |\xi\rangle + i T_1^+ |\xi_1\rangle + L_2^+ |\xi_4\rangle, \quad \delta |\Psi_1\rangle = L_1 |\xi\rangle - |\xi_4\rangle + i |\lambda\rangle, \tag{137} \\
\delta |\Psi_2\rangle &= T_1^\prime |\xi_1\rangle - i |\xi_4\rangle - |\lambda\rangle, \quad \delta |\Psi_3\rangle = L_1 |\xi\rangle - T_0 |\lambda\rangle - 2 i |\lambda_1\rangle, \tag{138} \\
\delta |\Psi_4\rangle &= -T_1^\prime |\xi\rangle, \quad \delta |\lambda\rangle = -T_0 |\xi\rangle - 2 i |\xi_1\rangle - i T_1^+ |\lambda\rangle, \tag{139} \\
\delta |\chi_1\rangle &= T_0 |\xi_1\rangle + L_1^+ |\lambda\rangle + 2 i T_1^+ |\lambda_1\rangle, \quad \delta |\chi_4\rangle = -T_0 |\xi_4\rangle + 4 |\lambda_1\rangle. \tag{140}
\end{align*}
\]
and the gauge for gauge transformations (99), (100)

\[
\delta|\xi\rangle = -iT_1^+|\omega\rangle, \quad \delta|\xi_1\rangle = L_1^+|\omega\rangle + 2iT_1^+|\omega_1\rangle, \quad \delta|\xi_4\rangle = 4|\omega_1\rangle, \quad (141)
\]
\[
\delta|\lambda\rangle = -T_0|\omega\rangle - 4i|\omega_1\rangle, \quad \delta|\lambda_1\rangle = T_0|\omega_1\rangle. \quad (142)
\]

in the concise form.

Now in order to derive Lagrangian (94) (or (136)) in component form we write the fields \(|\Psi_i\rangle\) and \(|\chi_i\rangle\) entering into (133) and (134) explicitly (taking into account the field’s eigenvalues associated with the operator \(\pi\))

\[
|\Psi\rangle = \left\{ \frac{1}{2} \gamma^\mu a^\nu \psi_{\mu\nu}(x) + d^+ a^\mu \psi_{\mu}(x) + b^+ \psi(x) \right\}|0\rangle, \quad (143)
\]
\[
|\Psi_1\rangle = \psi_1(x)|0\rangle, \quad |\Psi_2\rangle = \psi_2(x)|0\rangle, \quad (144)
\]
\[
|\Psi_3\rangle = \psi_3(x)|0\rangle, \quad |\Psi_4\rangle = \psi_4(x)|0\rangle, \quad (145)
\]
\[
|\chi\rangle = \left\{ a^+ \chi_\mu(x) + d^+ \chi(x) \right\}|0\rangle, \quad (146)
\]
\[
|\chi_1\rangle = \left\{ a^+ \chi_\mu(x) + d^+ \chi_1(x) \right\}|0\rangle, \quad |\chi_4\rangle = \chi_4(x)|0\rangle. \quad (147)
\]

and substitute them into (136). As a result we get the Lagrangian (94) for the field with spin 5/2 in the explicit form

\[
\mathcal{L} = -i\bar{\psi}_{\mu\nu}\left\{ \frac{1}{2} \gamma^\sigma \partial_\sigma \psi_{\mu\nu} + \partial_\mu \chi_\nu - \gamma_\mu \chi_1 \nu + \frac{i}{2} \eta_{\mu\nu} \chi_4 \right\} - iD\bar{\psi}_{\mu}\left\{ \gamma^\nu \partial_\nu \psi_{\mu} - \partial_\mu \chi + \chi_1 - \gamma_{\mu} \chi_4 \right\} + \frac{i}{2} D\bar{\psi}\left\{ \gamma_\mu \partial_\mu \psi - 2\chi_1 + i\chi_4 \right\} \]
\[
+ i\bar{\psi}_1\left\{ \gamma_\mu \partial_\mu \psi_1 - 2\psi_3 - \partial_\mu \chi_1 - i\chi_4 \right\} \]
\[
- i\bar{\psi}_2\left\{ i\gamma_\mu \partial_\mu \psi_2 + 2\psi_3 + \gamma_\mu \chi_4 - D\chi_1 + i\chi_4 \right\} \]
\[
+ \bar{\psi}_3\left\{ 2i\psi_3 + 2\gamma_\nu \partial_\nu \psi_4 - i\gamma_\mu \chi_4 + iD\chi_4 \right\} - \bar{\psi}_3\left\{ \gamma_\mu \partial_\mu \psi_3 + \partial_\mu \chi_1 \right\} \]
\[
- i\bar{\chi}_4\left\{ \gamma_\sigma \partial_\sigma \chi_\mu - \chi_1 \mu + \partial_\mu \chi_4 + \partial_\mu \psi_1 - \gamma_\mu \psi_3 \right\} - D\bar{\chi}_4\left\{ \gamma_\mu \partial_\mu \chi + \chi_1 - \partial_\mu \psi_3 + \psi_3 \right\} \]
\[
+ i\bar{\chi}_1\left\{ \chi_\mu + \chi_\nu \partial_\nu \psi_4 - D\psi_4 - i\gamma_\mu \psi_3 + 2i\partial_\mu \psi_1 \right\} + iD\bar{\chi}_3\left\{ \chi - \gamma_\mu \psi_4 + \psi - i\psi_2 \right\} \]
\[
+ i\bar{\chi}_4\left\{ \frac{1}{2} \psi_{\mu\nu} - D_2 \psi + \psi + i\psi_2 \right\}. \quad (148)
\]

Here \(D\) is dimension of the space-time, \(\psi_{\mu\nu}\) is basic spin 5/2 field and all other fields are auxiliary. In order to write the gauge transformations in the explicit form we write the gauge fields \(|\xi_i\rangle\) and \(|\lambda_i\rangle\) entering into (133) and (134) as follows (also taking into account the gauge parameters’ eigenvalues associated with the operator \(\pi\))

\[
|\xi\rangle = \left\{ a^+ \xi_\mu(x) + d^+ \xi(x) \right\}|0\rangle, \quad |\xi_1\rangle = \left\{ a^+ \xi_1 \mu(x) + d^+ \xi_1(x) \right\}|0\rangle, \quad (149)
\]
\[
|\xi_4\rangle = \xi_4(x)|0\rangle, \quad (150)
\]
\[
|\lambda\rangle = \lambda(x)|0\rangle, \quad |\lambda_1\rangle = \lambda_1(x)|0\rangle. \quad (151)
\]

and substitute them into (137)–(140). As a result we get the gauge transformations for the
field associated with spin 5/2 in the explicit form
\[
\begin{align*}
\delta \psi_{\mu\nu} &= -i(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) + i(\gamma_\mu \xi_{1\nu} + \gamma_\nu \xi_{1\mu}) + \eta_{\mu\nu} \xi_4, \\
\delta \psi_\mu &= -i \partial_\mu \xi + i \xi_{1\mu} - i \gamma_\mu \xi_1, \\
\delta \chi &= i \partial^\mu \xi_\mu - \xi_4 + i \lambda, \\
\delta \psi_2 &= -\gamma^\nu \xi_{1\nu} + D \xi_1 - i \xi_4 - \lambda, \\
\delta \psi_3 &= i \partial^\mu \xi_\mu + \gamma^\mu \partial_\mu \lambda - 2i \lambda_1, \\
\delta \chi &= -i \gamma^\mu \partial_\mu \xi - 2i \xi_1 - i \lambda, \\
\delta \chi_1 &= -i \gamma^\sigma \partial_\sigma \xi_{1\mu} - i \partial_\mu \lambda + 2i \gamma_\mu \lambda_1, \\
\delta \lambda &= i \gamma^\mu \partial_\mu \xi + 4 \lambda_1.
\end{align*}
\]

Finally we get in the explicit form the gauge for gauge transformations (141), (142).

Writing the gauge for gauge parameters as
\[
|\omega\rangle = \omega(x)|0\rangle, \quad |\omega_1\rangle = \omega_1(x)|0\rangle.
\]

ones find (141), (142) in the explicit form
\[
\begin{align*}
\delta \xi_\mu &= -i \gamma_\mu \omega, \\
\delta \xi_{1\mu} &= -i \partial_\mu \omega + 2i \gamma_\mu \omega_1, \\
\delta \lambda &= i \gamma^\mu \partial_\mu \omega - 4i \omega_1.
\end{align*}
\]

Thus, following the general procedure described in Section 4 we have constructed the Lagrangian (91), the gauge transformations (97), (98) and the gauge for gauge transformations (99), (100) for the field model of spin 5/2 in the explicit form (148), (152)–(158), (160)–(162) respectively. Unlike Fang-Fronsdal construction, we obtained the Lagrangian containing all proper set of auxiliary fields.

8 Summary

We have developed the new BRST approach to derivation of Lagrangians for fermionic mass-
less higher spin models in arbitrary dimensional Minkowski space. We investigated the superalgebra generated by the constraints which are necessary to define an irreducible massless half-integer spin representation of Poincare group and constructed the corresponding BRST charge. We found that the model is reducible gauge theory and the order of reducibility linearly grows with the value of spin. It is shown that this BRST charge generates the correct Lagrangian dynamics for fermionic fields of any value of spin. We construct Lagrangians in the concise form for the fields of any fixed spin in arbitrary space-time dimension and show that our Lagrangians are reduced to the Fang-Fronsdall Lagrangians after partial gauge-fixing. As an example of general scheme we obtained the Lagrangian and the gauge transformations for the field of spin 5/2 in the explicit form without any gauge fixing.

The main results of the paper are given by the relations (94), where Lagrangian for the field with arbitrary half-integer spin is constructed, and (97)–(100) where the gauge transformations for the fields and the gauge parameters are written down. In the case when ones consider all half-integer spin fields together, the analogous relations are (105) for the Lagrangian and (106)–(109) for the gauge transformations. Our formulation does not impose any off-shell constraints on the fields and the gauge parameters\(^5\) (see the discussion of this point in [14]).

\(^5\)The possibility to formulate a higher spin field theory without restrictions on traces of the fields and the gauge parameters was considered in [4].
The procedure for Lagrangian construction developed here for higher spin massless fermionic field can be also applied to bosonic higher spin massless theories and leads to the same results as in [16]. There are several possibilities for extending our approach. This approach may be applied to Lagrangian construction for mixed symmetry tensor-spinor fields (see [17] for corresponding bosonic case), for Lagrangian construction for fermionic fields in AdS background, for massive higher spin fields using the dimensional reduction and for supersymmetric higher spin models.

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