Abstract

Studies related to fairness in machine learning have recently gained traction due to its ever-expanding role in high-stakes decision making. For example, it may be desirable to ensure that all clusters discovered by an algorithm have high gender diversity. Previously, these problems have been studied under a setting where sensitive attributes, with respect to which fairness conditions impose diversity across clusters, are assumed to be observable; hence, protected groups are readily available. Most often, this may not be true, and diversity or individual interests can manifest as an intrinsic or latent feature of a social network. For example, depending on latent sensitive attributes, individuals interact with each other and represent each other’s interests, resulting in a network, which we refer to as a representation graph. Motivated by this, we propose an individual fairness criterion for clustering a graph $G$ that requires each cluster to contain an adequate number of members connected to the individual under a representation graph $R$. We devise a spectral clustering algorithm to find fair clusters under a given representation graph. We further propose a variant of the stochastic block model and establish our algorithm’s weak consistency under this model. Finally, we present experimental results to corroborate our theoretical findings.

1 Introduction

Machine learning has penetrated several application domains and now exhibits a significant influence on society. Notable examples include content recommendations, targeted advertising, ranking search results, and assessment of credit risks, to name a few. As stakes rise, it is essential to ensure that machine learning algorithms are fair, reliable, and transparent. For example, a service clustering news articles by topics must provide a diversity of opinions within each cluster to counter polarization. Towards this end, several recent approaches have presented fair variants of commonly used supervised (Dwork et al., 2012; Hardt et al., 2016; Caton & Haas, 2020) and unsupervised machine learning algorithms (Chierichetti et al., 2017; Celis et al., 2018a,b; Samadi et al., 2018).

Of particular interest in the context of this paper are the fairness notions for clustering (Chierichetti et al., 2017; Bercea et al., 2019; Bera et al., 2019; Kleindessner et al., 2019a; Jung et al., 2020; Mahabadi & Vakilian, 2020). Intuitively, the objective of clustering seems to be at odds with fairness (Chierichetti et al., 2017). If the data has an inherent bias, a clustering algorithm will benefit from exploiting it. However, the solution favored by the algorithm may not be ethical, as pointed in the example above. Fortunately, several recent studies have shown that it is often possible to find fair clusters while only paying a relatively small price for the clustering algorithm’s increased objective function value (Kleindessner et al., 2019b; Bercea et al., 2019).

Several fairness notions have been recently proposed to quantify the fairness of clusters. (Chierichetti et al., 2017) use the disparate impact doctrine to suggest that all protected groups (such as gender or race) must have approximately equal representation in all clusters. Others have followed up with the idea of proportional representation in various forms (Ahmadian et al., 2019; Backurs et al., 2019; Bera et al., 2019; Kleindessner et al., 2019b; Ahmadian et al., 2020). While these approaches assume that sensitive attributes under consideration are binary or categorical, others have worked with real-valued
attributes (Abraham et al., 2020).

All approaches listed above assume that the algorithm can directly observe the sensitive attributes that provide a notion of diversity, hence assuming that protected groups are readily available. On the other hand, in the real world, diversity can be an intrinsic or latent feature of individuals connected by a social network. Further, in most cases, fairness is dealt with at the level of population. However, it is also important to consider a fairness notion at the individual level. Motivated by this, we address the following questions. How can we define fairness when the sensitive attributes are not observable, but protected groups underlie in a representation graph? Can the spectral graph methods solve this problem, especially with individual fairness constraints? If so, is the resultant spectral algorithm consistent?

Central to our fairness definition is the notion of a representation graph. This graph contains a node for each individual in the dataset, and two nodes are connected if they are similar with respect to sensitive attributes or represent each others’ interests. We define our fairness notion from each individual’s viewpoint and require the individual’s neighbors in the representation graph to be approximately proportionally represented in all clusters. In Section 3, we show that this notion generalizes the case of fully observed sensitive attributes and is compatible with both discrete and real-valued attributes.

Most common fairness notions enforce statistical fairness that applies at the population level but may not consider individuals’ interests. For example, the proportional representation of protected groups (Chierichetti et al., 2017) implicitly assumes that every pair of members from a protected group can represent each others’ interests. On the other hand, the representation graph allows us to define fairness from the individuals’ perspective while also being equivalent to statistical fairness in a specific setting, as shown in Section 3. While approaches that consider individual fairness for clustering do exist, they do not use sensitive attributes but instead, rely on examples being close to cluster centroids (Mahabadi & Vakilian, 2020; Jung et al., 2020; Chen et al., 2019).

We propose a spectral clustering algorithm to partition or cluster a graph $\mathcal{G}$, where clusters are fair for each individual under a representation graph $\mathcal{R}$. Following Kleindessner et al. (2019b), we add the fairness objective as a constraint to the spectral clustering optimization problem. However, as noted before, our fairness notion is more expressive than the one used by Kleindessner et al. (2019b). We also propose a variant of the well-known stochastic block model (Holland et al., 1983) with respect to the representation graph $\mathcal{R}$ and establish that our algorithm recovers fair clusters with high probability under this model (weak consistency). Note that spectral clustering solves the relaxation of an NP-hard optimization problem (von Luxburg, 2007). Thus, as in the case of Kleindessner et al. (2019b), we cannot guarantee in general that the clusters returned by the algorithm will be fair. However, our experiments show that incorporating the fairness constraint does yield relatively more fair clusters from an individual perspective in practice.

Our main contributions are, (i) the notion of individual fairness with respect to a representation graph that does not require observable sensitive attributes and generalizes statistical fairness; (ii) a spectral clustering algorithm that tries to find clusters that are fair from individuals’ perspective; (iii) Theoretical guarantees on the performance of our algorithm under the proposed variant of SBM; and (iv) Empirical validation of our algorithm.

2 Notation and Preliminaries

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ be a set of nodes (e.g., set of individuals) connected by a graph (or network) $\mathcal{G}$. Our aim is to cluster or partition $\mathcal{G}$ while satisfying the fairness criterion. Let $\mathcal{R}$ be a graph defined over $\mathcal{V}$ that represents how similar the individuals are with respect to some sensitive attributes. Another interpretation of $\mathcal{R}$ can be that nodes are connected if they represent each others’ interests. We refer to $\mathcal{R}$ as the Sensitive Attribute Representation graph of $\mathcal{V}$ or simply representation graph. We use $A$ and $R \in \{0, 1\}^{N \times N}$ to denote the adjacency matrices of graphs $\mathcal{G}$ and $\mathcal{R}$, respectively. We assume that $\mathcal{G}$ and $\mathcal{R}$ are undirected, and $\mathcal{G}$ additionally has no self-loops. Thus, $A$ and $R$ are symmetric and $A_{ii} = 0$ for all $i \in [N]$, where $[n] := \{1, 2, \ldots, n\}$ for any integer $n$. Such graphs are often naturally available in many practical scenarios or can be constructed from observed features via kernels. For example, $\mathcal{G}$ can be a graph constructed based on non-sensitive attributes, and $\mathcal{R}$ can be a graph constructed based on sensitive attributes. We often use the terms nodes, vertices, and individuals interchangeably.

The aim is to find $K \geq 2$ clusters $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_K \subseteq \mathcal{V}$. Following Kleindessner et al. (2019b), we add the fairness objective as a constraint to the spectral clustering optimization problem. However, as noted before, our fairness notion is more expressive than the one used by Kleindessner et al. (2019b). We also propose a variant of the well-known stochastic block model (Holland et al., 1983) with respect to the representation graph $\mathcal{R}$ and establish that our algorithm recovers fair clusters with high probability under this model (weak consistency). Note that spectral clustering solves the relaxation of an NP-hard optimization problem (von Luxburg, 2007). Thus, as in the case of Kleindessner et al. (2019b), we cannot guarantee in general that the clusters returned by the algorithm will be fair. However, our experiments show that incorporating the fairness constraint does yield relatively more fair clusters from an individual perspective in practice.
Towards this end, we modify the spectral clustering algorithm in Section 4 to include the fairness criterion as a constraint. In this section, we recall the optimization problem solved by spectral clustering.

The ratio-cut objective given by

\[ \text{RatioCut}(C_1, \ldots, C_K) = \sum_{i=1}^{K} \frac{\text{Cut}(C_i, \mathcal{V} \setminus C_i)}{|C_i|}, \]

provides a measure of the goodness of clusters \( C_1, \ldots, C_K \) (von Luxburg, 2007). Here, \( \text{Cut}(A, B) = \frac{1}{2} \sum_{u \in A, v \in B} A_{uv} \) counts the edges between subsets \( A, B \subseteq \mathcal{V} \). A good clustering has a low ratio-cut value.

Define \( H \in \mathbb{R}^{N \times K} \) as

\[ H_{ij} = \begin{cases} \frac{1}{\sqrt{|C_j|}} & \text{if } v_i \in C_j \\ 0 & \text{otherwise.} \end{cases} \]

Further, let \( L = D - A \) denote the Laplacian of the similarity graph \( G \), where \( D \in \mathbb{R}^{N \times N} \) is a diagonal matrix and \( D_{ii} = \sum_{j=1}^{N} A_{ij} \) for all \( i \in [N] \). It can be easily verified that \( \text{RatioCut}(C_1, \ldots, C_K) = \text{trace}\{H^\top LH\} \) (von Luxburg, 2007). Thus, to find good clusters, one can minimize \( \text{trace}\{H^\top LH\} \) where \( H \) is of the form (1).

However, this is computationally hard due to the combinatorial nature of the constraint requiring \( H \) to be of the form (1). Spectral clustering instead solves the following relaxed optimization problem:

\[ \min_{H \in \mathbb{R}^{N \times K}} \text{trace}\{H^\top LH\} \quad \text{s.t.} \quad H^\top H = I. \]

Note that if \( H \) is of the form (1), then indeed \( H^\top H = I \).

### 3 Fairness Criterion

Consider the topic of a “good” economy, where individual opinions lie on a broad spectrum. Suppose the goal is to partition the population into clusters that can frame their separate economic policies. However, as members can trade across clusters, one cluster’s policies will likely also affect others’ prosperity. Thus, every individual wishes to have other like-minded people representing their viewpoint in other clusters.

But, being like-minded is often a complex function of several factors. For example, individuals having the same perspective on the economy may not always trust each other due to, for instance, personality differences. A representation graph specifies pairs of individuals who believe that they can represent each other’s viewpoint in different clusters.

The proposed fairness criterion is defined as follows.

**Definition 3.1** (Fairness Criterion). Given a representation graph \( \mathcal{R} \), a vertex \( v_i \) finds clusters \( C_1, \ldots, C_K \) of \( \mathcal{G} \) fair, if all its neighbors in \( \mathcal{R} \) are proportionately represented in each cluster. That is

\[ \frac{|\{ j : R_{ij} = 1 \} \cap \mathcal{C}_k |}{| \mathcal{C}_k |} = \frac{| \{ j : R_{ij} = 1 \} |}{N}, \quad \forall k \in [K]. \]

Clusters \( C_1, \ldots, C_K \) are said to be fair with respect to a representation graph \( \mathcal{R} \), if all vertices in \( \mathcal{V} \) find the clusters fair.

The next lemma specifies a condition on the matrix \( H \) (defined in (1)), which ensures that the corresponding clusters satisfy the fairness criterion given above.

**Lemma 3.2.** Let \( H \in \mathbb{R}^{N \times K} \) have the form specified in (1). The condition

\[ R(I - \frac{1}{N} \mathbf{1}^\top)H = 0_{N \times K} \]

implies that the corresponding clusters \( C_1, \ldots, C_K \) are fair under the representation graph \( \mathcal{R} \). Here, \( I \) is the \( N \times N \) identity matrix and \( \mathbf{1} \) is a \( N \) dimensional all-ones vector.

The next few remarks highlight the important properties of our fairness notion.

### Statistical fairness as a special case:

Several approaches assign each vertex to one of the \( P \) protected groups (Chierichetti et al., 2017, Kleindessner et al., 2019b) and require these protected groups to have proportional representation in all clusters, i.e., \( \frac{|C_k \cap P_i|}{|P_i|} = \frac{|C_k|}{N}, \quad \forall i \in [P], \: j \in [K], \quad \text{where} \quad P_i \subseteq \mathcal{V} \) is the \( i^{th} \) protected group. This is an example of statistical fairness. A representation graph, where \( R_{ij} = 1 \) if and only if \( v_i \) and \( v_j \) belong to the same protected group reduces the fairness criterion in Definition 3.1 to the statistical fairness criterion given...
above. Thus, statistical fairness is a special case of our fairness notion. In particular, we strictly generalize the approach presented in Kleindessner et al. (2019b). Also noteworthy is the assumption made by statistical fairness, namely that every pair of vertices in a protected group can represent each others’ interests, or they are very similar with respect to some sensitive attributes (notice the choice of \( R \) above). This assumption becomes unreasonable as protected groups grow in size.

Sensitive attributes and Protected groups:
The proposed fairness criterion only requires a representation graph \( R \). It has two advantages over the existing fairness criteria: (i) it does not require observable sensitive attributes, (ii) even if sensitive attributes are provided, one need not specify the number of protected groups or explicitly compute them. For example, individuals in \( R \) may be connected if their age difference is less than five years and if they went to the same school. Crucially, the sensitive attributes used to construct \( R \) may be numerical, binary, categorical, etc.

Realizability: Clearly, not all representation graphs admit a fair solution. In particular, because the columns of \( H \in \mathbb{R}^{N \times K} \) in (1) are orthogonal, (1) cannot be satisfied if the dimension of \( \text{null}(R(I - \mathbf{1} \mathbf{1}^\top/N)) \) is less than \( K \). Intuitively, as \( \text{rank}(R(I - \mathbf{1} \mathbf{1}^\top/N)) \) increases, it becomes harder to satisfy (1). In particular, any clustering is fair when \( R = \mathbf{1} \mathbf{1}^\top \) (i.e., every individual can represent every other individual) and hence \( \text{rank}(R(I - \mathbf{1} \mathbf{1}^\top/N)) = 0 \). We impose additional constraints on \( R \) in our consistency analysis in Section 5 to ensure that a fair clustering always exists. While those constraints are sufficient, they are not necessary, and hence our fairness notion is applicable more generally.

In the next section, we modify the spectral clustering algorithm to discover clusters that approximately satisfy our fairness criterion.

4 Algorithm

Recall from Section 2 that spectral clustering approximately minimizes the ratio-cut objective by relaxing an NP-hard optimization problem to the one given in (2). Lemma 3.2 provides a sufficient condition that a matrix \( H \) of the form (1) must satisfy to ensure that the corresponding clusters are fair. Ideally, we would like to solve the following optimization problem to get fair clusters,

\[
\min_H \quad \text{trace}\{H^\top LH\}
\]

\[
\text{s.t.} \quad H \text{ is of the form (1): } R(I - \frac{1}{N} \mathbf{1} \mathbf{1}^\top)H = 0_{N \times K}.
\]

However, as noted before, the constraint on \( H \) makes this problem NP-hard. Following Kleindessner et al. (2019b), we instead solve the following relaxed problem,

\[
\min_H \quad \text{trace}\{H^\top LH\}
\]

\[
\text{s.t.} \quad H^\top H = I; \quad R(I - \frac{1}{N} \mathbf{1} \mathbf{1}^\top)H = 0_{N \times K}. \tag{5}
\]

Clearly, the columns of \( H \) must belong to the null space of \( R(I - \mathbf{1} \mathbf{1}^\top/N) \). Thus, any feasible \( H \) can be expressed as \( H = YZ \) for some matrix \( Z \in \mathbb{R}^{N-r \times K} \), where \( Y \in \mathbb{R}^{N \times N-r} \) is an orthonormal matrix containing the basis vectors of the null space of \( R(I - \mathbf{1} \mathbf{1}^\top/N) \) as its columns. Here, \( r \) is the rank of \( R(I - \mathbf{1} \mathbf{1}^\top/N) \). Because \( Y^\top Y = I \), \( H^\top H = Z^\top Y^\top Y Z = Z^\top Z \). Thus, \( H^\top H = I \iff Z^\top Z = I \). The following optimization problem is equivalent to (5) by setting \( H = YZ \).

\[
\min_Z \quad \text{trace}\{Z^\top Y^\top LY Z\}
\]

\[
\text{s.t.} \quad Z^\top Z = I. \tag{6}
\]

As in standard spectral clustering (von Luxburg 2007), the solution to (6) is given by the \( K \) leading eigenvectors of \( Y^\top LY \). Of course, \( N-r \) must be at least \( K \) for \( K \) eigenvectors to exist as \( Y^\top LY \) has dimensions \( N-N \times N-r \). The clusters can then be recovered by using k-means clustering to cluster the rows of \( H = YZ \). Algorithm 1 summarizes this procedure, which we refer to as GEN-FairSC.

Relation to Kleindessner et al. (2019b):
Kleindessner et al. (2019b) also follow the same strategy and introduce their fairness criterion as an equality constraint in the spectral clustering’s optimization problem. As noted in Section 3, a particular configuration of \( R \) reduces our fairness criterion to

\[\text{Standard spectral clustering also follows a similar procedure. It first finds a matrix } U \in \mathbb{R}^{N \times K} \text{ whose columns are the first } K \text{ eigenvectors of laplacian } L. \text{ Then, it uses k-means on the rows of } U \text{ to obtain the clusters.}\]
Algorithm 1 Gen-FairSC

1: **Input:** Adjacency matrix $A$, number of clusters $K$, representation matrix $R$
2: **Output:** Clusters $C_1, C_2, \ldots, C_K$
3: Compute $Y$ containing orthonormal basis vectors of the null space of $R(I - \frac{1}{N}11^T)$
4: Compute Laplacian $L = D - A$
5: Compute leading $K$ eigenvectors of $Y^TLY$. Let $Z$ contain these vectors as its columns.
6: Apply $k$-means clustering to rows of $H = YZ$

Thus, their algorithm is a special case of Gen-FairSC, and hence our algorithm’s name is justified as it is also compatible with more general representation graphs for which $\text{rank}(R(I - \frac{1}{N}11^T/N)) \leq N - K$.

**Fairness of the clusters:** Note that the constraint $R(I - \frac{1}{N}11^T/N)H = 0$ implies the fairness of clusters only when $H$ has the form given in (1). Thus, a feasible solution to the relaxed optimization problem in (5) may not necessarily result in fair clusters. In fact, as noted in (Kleindessner et al., 2019b), there are no general guarantees that bound the difference between the optimal solution of (2) and the optimal solution of the original NP-hard ratio-cut problem. Hence, we cannot guarantee the fairness of clusters discovered by solving (6) instead of solving (5) in the general case. Nonetheless, we show in Section 5 that the discovered clusters are indeed fair under certain assumptions.

**Computational complexity:** Algorithm 1 has a time complexity of $O(N^3)$ and space complexity of $O(N^2)$. Finding the null space in Line 3 and computing the eigenvectors in Line 5 are the computationally dominant steps. This matches the worst-case complexity of both the standard spectral clustering algorithm and the algorithm given in Kleindessner et al. (2019b). For small $K$, several approximations can reduce complexity, but most such techniques work only when $K = 2$ (Yu & Shi, 2004; Xu et al., 2009).

# 5 Theoretical Guarantees

In this section, we show that Algorithm 1 recovers the ground truth clusters with a high probability under mild assumptions on the representation graph for networks sampled from a modified variant of the Stochastic Block Model (SBM) (Holland et al., 1983). The ground truth clusters are fair by construction as described in Section 5.2.

## 5.1 $\mathcal{R}$-SBM

The well known Stochastic Block Model (SBM) (Holland et al., 1983) takes, as input, a function $\pi : [N] \rightarrow [K]$ that assigns each vertex $v_i \in V$ to one of the $K$ clusters. Then, independently, for all node pairs $(v_i, v_j)$ such that $i > j$,

$$P(A_{ij} = 1) = B_{\pi(i)\pi(j)},$$

where $B \in [0, 1]^{K \times K}$ is a symmetric matrix. The entry $B_{kk}$ specifies the probability of a connection between two nodes that belong to clusters $C_k$ and $C_l$ respectively. A commonly used variant of SBM assumes $B_{kk} = \alpha$ and $B_{k\ell} = \beta$ for all $k, \ell \in [K]$ such that $k \neq \ell$. We define a SBM with respect to a representation graph $\mathcal{R}$ as follows.

**Definition 5.1 ($\mathcal{R}$-SBM).** A $\mathcal{R}$-SBM is defined by the tuple $(\pi, \mathcal{R}, p, q, r, s)$, where $\pi : [N] \rightarrow [K]$ maps vertices in $V$ to clusters, $\mathcal{R}$ is a representation graph, and $1 \geq p \geq q \geq r \geq s \geq 0$ are probabilities used for sampling edges. Under this model,

$$P(A_{ij} = 1) = \begin{cases} p & \text{if } \pi(i) = \pi(j) \text{ and } R_{ij} = 1, \\ q & \text{if } \pi(i) \neq \pi(j) \text{ and } R_{ij} = 1, \\ r & \text{if } \pi(i) = \pi(j) \text{ and } R_{ij} = 0, \\ s & \text{if } \pi(i) \neq \pi(j) \text{ and } R_{ij} = 0. \end{cases}$$

(7)

When vertices $v_i$ and $v_j$ belong to the same cluster, they have a higher probability of connection between them for a fixed value of $R_{ij}$ ($p > q$ and $r > s$). However, the vertices also have a higher tendency to connect if they are connected in $\mathcal{R}$, even if they do not belong to the same cluster ($q > r$). When $\mathcal{R}$ itself has a community structure, there are two natural ways to cluster the vertices: (i) based on the ground-truth clusters $C_1, C_2, \ldots, C_K$ specified by $\pi$; and (ii) based on the clusters in $\mathcal{R}$.

The clusters based on communities in $\mathcal{R}$ are likely to not satisfy the fairness criterion in Definition 3.1. To see this, note that tightly connected vertices in $\mathcal{R}$ will be assigned to the same cluster in this case rather than being spread evenly across clusters as demanded by the fairness criterion. On the other
hand, the ground-truth clusters can be constructed to be fair, as we demonstrate in Section 5.2 after enforcing more constraints on \( R \). Assuming that the ground-truth clusters are fair, the goal is to show that Algorithm 1 recovers the ground-truth clusters with high probability rather than returning any other natural but unfair clusters.

5.2 Consistency Result

As noted in Section 3, some representation graphs lead to fairness constraints that cannot be satisfied. For example, consider an \( R \), where a vertex \( v_i \) has only two neighbors. It is not possible for this vertex to have a representative in all clusters if \( K > 2 \). Therefore, some additional assumptions are required on \( R \) to ensure that a fair clustering is indeed possible.

**Assumption 1.** \( \mathrm{rank}(R) \leq N - K \).

Note that \( I - 11^T/N \) is a projection matrix and \( 1 \) is its eigenvector with eigenvalue 0. Any vector orthogonal to \( 1 \) is also an eigenvector with eigenvalue 1. Thus, \( \mathrm{rank}(I - 11^T/N) = N - 1 \). Because \( \mathrm{rank}(R(I - 11^T/N)) \leq \min(\mathrm{rank}(R), \mathrm{rank}(I - 11^T/N)) \), requiring \( \mathrm{rank}(R) \leq N - K \) ensures that \( \mathrm{rank}(R(I - 11^T/N)) \leq N - K \) which is necessary for \( \Theta \) to have a solution.

**Assumption 2.** All ground truth clusters have the same number of vertices. Thus, \( |C_i| = \frac{K}{d} \) for all \( i \in [K] \). This implicitly requires \( N \) to be a multiple of \( K \).

This assumption, together with the next one, allows us to compute the smallest \( K \) eigenvalues of the laplacian matrix in the expected case. This is a crucial step in proving Theorem 5.3.

**Assumption 3.** \( R \) is a \( d \)-regular graph for some \( K \leq d \leq N \). Moreover, \( R_{ii} = 1 \) for all \( i \in [N] \), and each vertex in \( R \) is connected to \( d/K \) vertices from cluster \( C_i \) for all \( i \in [K] \) (including the self-loop). As in Assumption 2, this requires \( d \) to be a multiple of \( K \).

Recall that the function \( \pi : [N] \to [K] \) assigns each node to a cluster in \( R \)-SBM. Assumption 3 ensures the existence of a \( \pi \) for which the corresponding ground-truth clusters are fair under \( R \). Namely, let \( \pi(i) = k \), if \( k\frac{N}{K} \leq i \leq (k + 1)\frac{N}{K} \) for all \( i \in [N] \), and \( k \in [K] \). It can be easily verified that the resulting clusters \( C_k = \{ v_i : \pi(i) = k \}, k \in [K] \) are fair under \( R \).

**Remark 5.2.** It is important to note that in practice, Algorithm 2 only requires Assumption 1 to ensure that \( Y^\top L Y \) has at least \( K \) orthonormal eigenvectors. Assumptions 2 and 3 are only needed for our theoretical analysis.

The next theorem states our main result. It places a high-probability upper bound on the fraction of misclustered vertices for similarity graphs \( G \) drawn from a \( R \)-SBM, where the representation graph \( R \) satisfies Assumptions 1-3. Let \( \Theta \in \{0, 1\}^{N \times K} \) indicate the ground-truth cluster memberships, i.e., \( \Theta_{ij} = 1 \iff v_i \in C_j \). Similarly, \( \hat{\Theta} \in \{0, 1\}^{N \times K} \) indicates the clusters of \( G \) predicted by Algorithm 1, i.e., \( \hat{\Theta}_{ij} = 1 \) if and only if Algorithm 1 assigns \( v_i \) to the \( j \)-th cluster. Further, let \( J \) be the set of all \( K \times K \) permutation matrices. The fraction of misclustered nodes is given by \( M(\Theta, \hat{\Theta}) = \min_{J \in J} \frac{1}{N} \| \Theta - \hat{\Theta} J \|_F \).

Recall that the ground truth clusters \( C_1, \ldots, C_K \) are fair by construction. Thus, a low \( M(\Theta, \hat{\Theta}) \) indicates that the clusters returned by the algorithm are also fair. Theorem 5.3 uses the eigenvalues of the Laplacian matrix in the expected case, which is defined as \( L = D - A \), where \( A = E[A] \) is the expected adjacency matrix for a graph sampled from \( R \)-SBM and \( D \in \mathbb{R}^{N \times N} \) is a diagonal matrix such that \( D_{ii} = \sum_{j=1}^{N} A_{ij} \) for all \( i \in [N] \).

**Theorem 5.3.** Let \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r} \) denote the eigenvalues of \( Y^\top L Y \). Define \( \gamma = \mu_{K+1} - \mu_K \). Under assumptions 1-3, there exists a universal constant \( \text{const}(C, \alpha) \), such that if \( \gamma \) satisfies

\[
\gamma^2 \geq \text{const}(C, \alpha)(2 + \epsilon)pNK \log N,
\]

and \( p \geq C \log N/N \) for some \( C > 0 \), then,

\[
M(\Theta, \hat{\Theta}) \leq \text{const}(C, \alpha)\frac{(2 + \epsilon)}{\gamma^2}pN \log N,
\]

for every \( \epsilon > 0 \) with probability at least \( 1 - 2N^{-\alpha} \) when a \((1 + \epsilon)\)-approximate algorithm for \( k \)-means clustering is used in Step 6 in Algorithm 2.

Theorem 5.3 implies that \( M(\Theta, \hat{\Theta}) = o(1) \) with probability \( 1 - o(1) \) if \( \gamma = \omega(\sqrt{pNK \log N}) \). Thus, \( \text{A note on the asymptotic notations: If } f(n) = o(g(n)) \text{ then } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0. \) Similarly, if \( f(n) = \omega(g(n)) \), then \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty. \) Clearly, if \( f(n) = \omega(g(n)) \) then \( g(n) = o(f(n)). \)
in this case Algorithm [1] is weakly consistent (Abbe et al., 2018). This condition on $\gamma$ is often satisfied in many interesting cases. For example, when there are $P$ protected groups as in Kleindessner et al. (2019b), the equivalent representation graph has $P$ cliques that are not connected to each other (see the relation with statistical fairness in Section 3). One can show that $\gamma = \theta(N/K)$ in this case (Kleindessner et al., 2019b) which satisfies the criterion given above if $K$ is not too large.

Theorem 5.3 requires a $(1+\epsilon)$-approximate solution to the $k$-means clustering problem. Several efficient algorithms have been proposed in the literature for this task (Kumar et al., 2004; Arthur & Vassilvitskii, 2007; Ahmadian et al., 2017). Such algorithms are also available in commonly used software packages like MATLAB and scikit-learn. The assumption $p \geq C \log N/N$ controls the sparsity of the graph and is required to prove the consistency of normal spectral clustering as well (Lei & Rinaldo, 2015).

The proof of Theorem 5.3 is given in Appendix 3. It follows the commonly used template for such results (Rohe et al., 2011; Lei & Rinaldo, 2015). We first compute the expected Laplacian matrix $\mathcal{L}$ under $\mathcal{R}$-SBM and show that its top $K$ eigenvectors can be used to recover the ground-truth clusters. We then show that these top $K$ eigenvectors lie in the null space of $R(I - 11^\top/N)$, and hence are also the top $K$ eigenvectors of $Y^\top L Y$. This implies that Algorithm 1 returns the ground truth clusters in the expected case. We then use matrix perturbation arguments as in Lei & Rinaldo (2015) to establish the high probability consistency result in the general case when the graph is sampled from $\mathcal{R}$-SBM.

Our analysis recovers the consistency result in Kleindessner et al. (2019b) where $\gamma = \theta(N/K)$, as mentioned earlier. However, as opposed to them, our arguments apply more generally to $d$-regular representation graphs instead of being limited to the case where $\mathcal{R}$ is a block-diagonal matrix.

6 Related Work

Our work is closely related to constrained clustering (Basu et al., 2008) as fairness requirements restrict the set of acceptable clusters. Most methods for constrained clustering apply to variations of pairwise must-link or cannot-link constraints, which specify which pairs of nodes must or must not belong to the same cluster (Yu & Shi, 2001; 2004; Xu et al., 2009; Wang & Davidson, 2010; Rangapuram & Hein, 2012; Cucuringu et al., 2016). In contrast, fairness notions for clustering are usually concerned with the distribution of examples within clusters, for example, to enforce diversity (Chierichetti et al., 2017).

As noted before, several fairness notions for clustering have been proposed. While most study statistical fairness (Chierichetti et al., 2017; Rösner & Schmidt, 2018; Bercea et al., 2019; Backurs et al., 2019; Bera et al., 2019; Kleindessner et al., 2019b), a few study individual fairness as well (Chen et al., 2019; Mahabadi & Vakilian, 2020; Jung et al., 2020; Kleindessner et al., 2020; Anderson et al., 2020). Chierichetti et al. (2017) assumes that examples belong to one of the two protected groups and require each protected group to have a proportional representation in all clusters. Rösner & Schmidt (2018) extend this idea to multiple protected groups. Bercea et al. (2019) relax the requirement of exact proportions by allowing them to lie in a range. Bera et al. (2019) allow the protected groups to overlap with each other. Efficient algorithms for finding fair clusters under these criteria have also been proposed (Schmidt et al., 2018; Ahmadian et al., 2019; Kleindessner et al., 2019b).

Individual fairness criteria require each point to be sufficiently close to a cluster centroid (Mahabadi & Vakilian, 2020). For example, Chen et al. (2019) allow a group of points to form their own cluster if there is another center that is closer to all the points. Taking a different approach, Anderson et al. (2020) adapt the individual fairness criterion from supervised learning (Dwork et al., 2012) to the clustering setting. Our fairness notion is conditioned on the representation graph and can enforce both statistical and individual fairness for different configurations of this graph.

We develop a spectral clustering algorithm to find approximately fair clusters under the proposed fairness criterion. Spectral clustering a popular choice among clustering algorithms (von Luxburg, 2007). It not only works well in practice but is also backed by strong theoretical guarantees on its performance (von Luxburg, 2007; Rohe et al., 2011; Lei & Rinaldo, 2015). It is especially suited for the practical case when one observes the pairwise similarity between examples rather than directly observing the features. Most statistical guarantees for spectral clustering use the Stochastic Block Model (Holland et al., 1983).

Kleindessner et al. (2019b) proposed fair spectral clustering (FAIRSC) for the proportionality fairness
criterion. (Chierichetti et al., 2017). We follow a similar approach but our algorithm considers the more general fairness criterion proposed in Section 3. Notably, we show that the algorithm in Kleindessner et al. (2019b) and their consistency analysis can be recovered as a special case of our setting. Moreover, our experiments demonstrate a few cases where our algorithm outperforms their algorithm.

7 Experiments

We perform three types of experiments. In the first two cases, we use synthetic data to validate our theoretical results using d-regular representation graphs and show that Algorithm 1 works well in practice even when the representation graph is not d-regular. Finally, we demonstrate the effectiveness of the proposed algorithm on a real-world dataset described in Section 7.2.

FairSC (Kleindessner et al., 2019b) assumes that each vertex belongs to one of the P protected groups \( P_1, \ldots, P_P \subseteq V \). Our algorithm reduces to FairSC by using a representation graph where nodes are connected if and only if they belong to the same protected group. Thus, we only experiment with representation graphs \( R \) that are not of the above-mentioned form. Naturally, FairSC is not directly applicable in such settings. Nonetheless, to compare with FairSC, we cluster \( R \) using standard spectral clustering and use these clusters as protected groups for FairSC.

Figure 1: Comparing GEN-FairSC with FAIRSC (Kleindessner et al., 2019b) and standard spectral clustering (SC) on d-regular representation graphs. \( d = N/10, K = 5 \), and FAIRSC uses \( P = 5 \) groups as explained before Section 7.1.

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Figure 2: Comparison between GEN-FairSC and FAIRSC (Kleindessner et al., 2019b) on representation graphs sampled from an SBM. Varying \( P \) in FAIRSC does not significantly improve the fairness or quality of clusters. However, the rank \( r \) can be tuned in GEN-FAIRSC to a trade-off between cluster quality and fairness.

7.1 Experiments using Synthetic Data

Figure 1 compares the performance of GEN-FAIRSC with FAIRSC and standard spectral clustering. We sample d-regular representation graphs under Assumptions 2 and 3 for various values of \( N \) using \( K = 5 \) and \( d = N/10 \). We then sample adjacency matrices corresponding to the representation graphs using \( R \)-SBM with parameters: \( p = 0.4, q = 0.3, r = 0.2 \), and \( s = 0.1 \). Figure 1 shows the average accuracy of each algorithm across 10 independent runs for different values of \( N \). GEN-FAIRSC is able to recover fair clusters (ground-truth clusters are fair), whereas both spectral clustering and FAIRSC have inferior performance. This confirms our theoretical analysis from Theorem 5.3.

In the second case, the nodes are divided into \( P = 5 \) protected groups as in Kleindessner et al. (2019b), and \( R \) is sampled from an SBM. Nodes in the same (different) protected group(s) are connected with probability \( p_{in} = 0.8 \) (\( p_{out} = 0.2 \)). Conditioned on this \( R \), we sample an adjacency matrix from \( R \)-SBM as before. Note that such an \( R \) is not d-regular. Moreover, a graph \( R \) generated this way may violate Assumption 4, which is crucial to ensure that (6) is feasible. Thus, instead of using \( R \) directly, we use the best rank \( r \) approximation of its adjacency matrix \( \hat{R} \) in (6). This approximation need not have binary elements, but it works well in practice.

Let \( N_i = \{ j : R_{ij} = 1 \} \) be the set of neighbors of node \( v_i \) in \( R \). We compute \( \rho_i = \min_{k, \ell \in [K]} \frac{\sum_{j \in N_i} R_{ij}}{\sum_{j \in N_i} R_{ij}} \) for
each node $i \in [N]$ to measure the fairness of proposed clusters $C_1, \ldots, C_K \subseteq [N]$. While $\rho_i$ looks similar to the notion of balance defined from the perspective of protected groups in [Chierichetti et al. (2017)], in this case, it is associated with an individual $v_i$. Hence, we refer to it as individual balance for node $i$. Clearly, $0 \leq \rho_i \leq 1$, and higher values are desirable. We use average balance $\bar{\rho} = \frac{1}{N} \sum_{i=1}^{N} \rho_i$ to measure the fairness of clusters.

Figure 2 uses $N = 500$, $K = 5$, $P = 5$, and the same values of $p$, $q$, $r$, and $s$ as before. The $x$-axis corresponds to the number of protected groups (for FAIRSC$^3$) and the rank $r$ used in the approximation above (for GEN-FairSC). The $y$-axis represents average balance $\bar{\rho}$ and modularity score for the clusters. Figure 2 shows the trade-off between clustering accuracy and fairness. One can choose an appropriate value of $r$ (for example, $r = 300$ in Figure 2), and use GEN-FairSC to get good quality clusters with a high balance. Note that both modularity score and balance are low for FAIRSC.

### 7.2 Experiments using Real Data

We use the FAO trade network ([De Domenico et al., 2013](#)), a multiplex network based on data from the Food and Agriculture Organization (FAO) of the United Nations. It has 214 nodes representing countries and 364 layers corresponding to commodities like coffee, banana, barley, etc. An edge between two countries in a layer indicates the volume of the corresponding commodity traded between them.

We convert the weighted graph in each layer to an unweighted graph by connecting every node with its five nearest neighbors and make all the edges undirected. We use the first 182 layers to construct the representation graph $\mathcal{R}$. Nodes in $\mathcal{R}$ are connected if they are linked in either of these layers. Similarly, the next 182 layers are used to construct the graph $\mathcal{G}$. Note that $\mathcal{R}$ constructed this way is not $d$-regular. The goal is to find clusters in $\mathcal{G}$ that are fair with respect to $\mathcal{R}$.

Clusters based on $\mathcal{G}$ consider the trade of commodities 183–364 only. However, countries also have trade relations in $\mathcal{R}$, leading to shared economic interests. Assume that the members of each cluster

![Figure 3: Balance and modularity scores for different values of $r$ in GEN-FairSC and $P$ in FAIRSC for the FAO trade network. Our algorithm finds better quality clusters with higher balance.](#)

jointly formulate the economic policies for that cluster. However, these policies affect everyone as they all share a global market, incentivizing the countries to influence the economic policies of all clusters. Being fair with respect to $\mathcal{R}$ entails that each country has members in other clusters with shared interests. This enables a country to indirectly shape the policies of other clusters.

As before, we use the low-rank approximation for the representation graph in GEN-FairSC. Figure 3 compares various algorithms and has the same semantics as Figure 2. Not only is GEN-FairSC able to achieve a higher balance, it does so with a better modularity score as compared to FAIR-SC. In practice, a user would assess the relative importance of modularity and balance to select $r$. Some additional experiments are provided in Appendix D.

### 8 Conclusion

In this paper, we proposed an individual fairness criterion with respect to a representation graph $\mathcal{R}$. The term “protected groups” has been overloaded here. One set of protected groups are used to sample $\mathcal{R}$ from an SBM, as explained above. Then, we cluster $\mathcal{R}$ as explained before Section 7.1 to get the second set of protected groups which are used by FAIRSC.
We showed that our criterion could enforce both statistical and individual fairness under different configurations. We also developed a companion spectral clustering algorithm to find approximately fair clusters, proposed a variant of SBM called $R$-SBM, and established the weak consistency of this algorithm under $R$-SBM. Our experiments validate our theoretical claims. The theoretical guarantee in Section 5 applies only to $d$-regular representation graphs. Showing guarantees for more general graphs is a promising direction for future work. It would also be interesting to explore other algorithms for finding fair clusters under the criterion proposed in Section 3.

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The notation used in this paper is summarized below.

| Symbol | Description |
|--------|-------------|
| $\mathcal{V}$ | Set of nodes |
| $v_i$ | An element of set $\mathcal{V}$ |
| $\mathcal{G}$ | Given graph or network in which we need to find clusters |
| $A$ | $N \times N$ adjacency matrix of $\mathcal{G}$ |
| $\mathcal{R}$ | Representation graph |
| $R$ | $N \times N$ adjacency matrix of $\mathcal{R}$ |
| $N$ | Number of nodes in the graph |
| $K$ | Number of ground-truth clusters in $\mathcal{G}$ |
| $d$ | Degree of each node in the $d$-regular representation graph $\mathcal{R}$ |
| $C_k$ | Set of nodes in the $k$th cluster |
| $p, q, r, s$ | Parameters used by $\mathcal{R}$-SBM |
| $\mathcal{A}$ | Expected adjacency matrix of a graph sampled from $\mathcal{R}$-SBM |
| $\mathcal{A} + pI$ | Degree matrix corresponding to $\mathcal{A}$ and $\mathcal{D}$: $\mathcal{L} = \mathcal{D} - \mathcal{A}$ |
| $G_k$ | $N \times N$ diagonal matrix where $(G_k)_{ii} = 1$ if $v_i \in C_k$ and 0 otherwise |
| $||X||$ | $\sqrt{\lambda_{\text{max}}(X^TX)}$: Square-root of the maximum eigenvalue of $X^TX$ |
| $Y$ | $N \times N - r$ matrix containing orthonormal basis for null($R(I - 11^T/N)$) |
| $y_i$ | $i$th column of $Y$ |
| $Z$ | $Z \in \mathbb{R}^{N-r \times K}$; solution to (6) in the observed case |
| $\mu_i$ | $i$th-smallest eigenvalue of $Y^T\mathcal{L}Y$ |
| $\gamma$ | $\mu_{K+1} - \mu_K$; eigen-gap |

A Proof of Lemma 3.2

Fix an arbitrary node $v_i \in \mathcal{V}$ and $k \in [K]$. Recall that $H$ is such that

$$H_{ik} = \begin{cases} \frac{1}{\sqrt{|C_k|}} & \text{if } v_i \in C_k \\ 0 & \text{otherwise.} \end{cases}$$

Because $R(I - 11^T/N)H = 0$,

$$\sum_{j=1}^{N} R_{ij}H_{jk} = \frac{1}{N} \left( \sum_{j=1}^{N} R_{ij} \right) \left( \sum_{j=1}^{N} H_{jk} \right) \Rightarrow \frac{1}{\sqrt{|C_k|}} \{|v_j \in \mathcal{V}: R_{ij} = 1 \land v_j \in C_k\} = \frac{1}{N} \{|v_j \in \mathcal{V}: R_{ij} = 1\} \frac{|C_k|}{\sqrt{|C_k|}}$$

$$\Rightarrow \frac{|\{v_j \in \mathcal{V}: R_{ij} = 1 \land v_j \in C_k\}|}{|C_k|} = \frac{|\{v_j \in \mathcal{V}: R_{ij} = 1\}|}{N}.$$
B Proof of Theorem 5.3

We prove Theorem 5.3 via a series of technical lemmas. See Appendix C for the proofs of these lemmas. We begin by showing that Algorithm 1 recovers the ground-truth clusters when the expected adjacency matrix of graph $G$ is specified as input. For the remainder of this proof, we will assume that Assumptions 1-3 are satisfied.

The first lemma shows that certain vectors that can be used to recover the ground-truth clusters indeed satisfy the fairness criteria in (4).

Lemma B.1. The $N$-dimensional vector of all ones, denoted by $1$, is an eigenvector of $R$ with eigenvalue $d$. Define $u_k \in \mathbb{R}^N$ for $k \in [K-1]$ as,

$$(u_k)_i = \begin{cases} 1 & \text{if } v_i \in \mathcal{C}_k \\ -\frac{1}{K-1} & \text{otherwise}, \end{cases}$$

where $(u_k)_i$ is the $i$th element of $u_k$. Then, $1, u_1, \ldots, u_{K-1} \in \text{null}(R(I - \frac{1}{N}11^T))$. Moreover, $1, u_1, \ldots, u_{K-1}$ are linearly independent.

We use $A$ to denote the expected adjacency matrix of graph $G$. Clearly, $A = \tilde{A} - pI$, where $\tilde{A}$ is such that $\tilde{A}_{ij} = P(A_{ij} = 1)$ for $i \neq j$ (see (7)) and $\tilde{A}_{ii} = p$ otherwise. Note that

$$\tilde{A}x = \lambda x \iff Ax = (\lambda - p)x.$$  (8)

Simple algebra shows that $\tilde{A}$ can be written as

$$\tilde{A} = qR + s(11^T - R) + (p - q) \sum_{k=1}^{K} G_k R G_k + (r - s) \sum_{k=1}^{K} G_k (11^T - R) G_k,$$  (9)

where for all $k \in [K]$, $G_k \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that $(G_k)_{ii} = 1$ if $v_i \in \mathcal{C}_k$ and 0 otherwise. The next lemma shows that $1, u_1, \ldots, u_{K-1}$ defined in Lemma B.1 are eigenvectors of $\tilde{A}$.

Lemma B.2. Let $1, u_1, \ldots, u_{K-1}$ be as defined in Lemma B.1. Then,

$$\tilde{A}1 = \lambda_1 1 \text{ where } \lambda_1 = qd + s(N-d) + \frac{(p-q) d}{K} + \frac{(r-s) N - d}{K}, \text{ and}$$

$$\tilde{A}u_k = \lambda_{1+k} u_k \text{ where } \lambda_{1+k} = \frac{(p-q) d}{K} + \frac{(r-s) N - d}{K}.$$  (10)

Let $L = D - A$ be the expected Laplacian matrix, where $D$ is a diagonal matrix with $D_{ii} = \sum_{j=1}^{N} A_{ij}$ for all $i \in [N]$. It is easy to see that $D_{ii} = \lambda_1 - p$ for all $i \in [N]$ as $A1 = (\lambda_1 - p)1$ by (8) and Lemma B.2. Thus, $D = (\lambda_1 - p)I$ and hence any eigenvector of $\tilde{A}$ with eigenvalue $\lambda$ is also an eigenvector of $L$ with eigenvalue $\lambda_1 - \lambda$. That is, if $\tilde{A}x = \lambda x$,

$$Lx = (D - A)x = ((\lambda_1 - p)I - (\tilde{A} - pI))x = (\lambda_1 - \lambda)x.$$  (10)

Hence, the eigenvectors of $L$ corresponding to the $K$ smallest eigenvalues are the same as the eigenvectors of $\tilde{A}$ corresponding to the $K$ largest eigenvalues.

Recall that the columns of $Y$ in (6) contain the orthonormal basis for the null space of $R(I - 11^T/N)$. In the absence of a fairness constraint (or equivalently if $Y = I_{N \times N}$), the optimal solution to (6) would have been the first $K$ eigenvectors of $L$. However, with the fairness constraint, we only need to optimize over vectors that belong to the null space of $R(I - 11^T/N)$. By Lemma B.1, $1, u_1, \ldots, u_{K-1} \in \text{null}(R(I - 11^T/N))$ and are linearly independent. Let $y_1 = 1/\sqrt{N}$ and $y_2, \ldots, y_K$ be orthonormal vectors that span the same space as $u_1, \ldots, u_{K-1}$. The next lemma computes such $y_2, \ldots, y_K$. 

Lemma B.3. Define \( y_{1+k} \in \mathbb{R}^N \) for \( k \in [K-1] \) as

\[
(y_{1+k})_i = \begin{cases}
0 & \text{if } v_i \in C_{k'} \text{ s.t. } k' < k \\
\frac{-K-k}{\sqrt{\frac{1}{N} (K-k)(K-k+1)}} & \text{if } v_i \in C_k \\
\frac{1}{\sqrt{\frac{1}{N} (K-k)(K-k+1)}} & \text{otherwise}.
\end{cases}
\]

Then, for \( k \in [K-1] \), \( y_{1+k} \) are orthonormal vectors that span the same space as \( u_1, u_2, \ldots, u_{K-1} \) and \( y_1^T y_{1+k} = 0 \).

Let \( T \in \mathbb{R}^{N \times K} \) be such that it has \( y_1, \ldots, y_K \) as its columns. If two nodes belong to the same cluster, the rows corresponding to these nodes in \( TU \) will be the same for any \( U \in \mathbb{R}^{K \times K} \) such that \( U^T U = I \). Thus, any \( K \) orthonormal vectors belonging to the span of \( y_1, \ldots, y_K \) can be used to recover the ground truth clusters.

Let \( Z \in \mathbb{R}^{N-r \times K} \) be a solution to the optimization problem \( (6) \) in the expected case with \( A \) as input. The next lemma shows that columns of \( YZ \) indeed lie in the span of \( y_1, \ldots, y_K \). Thus, the k-means clustering step in Algorithm I will return the correct ground truth clusters when \( A \) is passed as input.

Lemma B.4. Let \( y_1 = 1/\sqrt{N} \) and \( y_{1+k} \) be as defined in Lemma B.3 for \( k \in [K-1] \). Further, let \( Z \) be the optimal solution to the optimization problem in \( (6) \) with \( L \) set to \( L \). Then, the columns of \( YZ \) lie in the span of \( y_1, y_2, \ldots, y_K \).

Now that we have established that Algorithm I returns the correct ground truth clusters in the expected case, we will use arguments from matrix perturbation theory to show a high-probability bound on the number of mistakes made by the algorithm. In particular, we need an upper bound on \( ||Y^T LY - Y^T LY|| \) where \( L \) is the Laplacian matrix for a graph randomly sampled from \( \mathcal{R} \)-SBM and \( ||X|| = \sqrt{\lambda_{\max}(X^T X)} \) for any matrix \( X \).

Note that \( ||Y|| = ||Y^T|| = 1 \) as \( Y^T Y = I \). Thus,

\[
||Y^T LY - Y^T LY|| \leq ||Y^T|| ||L - L|| ||Y|| = ||L - L||. \tag{11}
\]

Moreover,

\[
||L - L|| = ||D - A - (D - A)|| \leq ||D - D|| + ||A - A||.
\]

The next two lemmas bound the two terms on the right hand side of the inequality above, thus providing an upper bound on \( ||L - L|| \), and hence on \( ||Y^T LY - Y^T LY|| \) by \( (11) \).

Lemma B.5. Assume that \( p \geq C \frac{\log N}{N} \) for some constant \( C > 0 \). Then, for every \( \alpha > 0 \), there exists a constant \( \text{const}_1(C, \alpha) \) that only depends on \( C \) and \( \alpha \) such that

\[
||D - D|| \leq \text{const}_1(C, \alpha) \sqrt{pN \log N}
\]

with probability at-least \( 1 - N^\alpha \).

Lemma B.6. Assume that \( p \geq C \frac{\log N}{N} \) for some constant \( C > 0 \). Then, for every \( \alpha > 0 \), there exists a constant \( \text{const}_2(C, \alpha) \) that only depends on \( C \) and \( \alpha \) such that

\[
||A - A|| \leq \text{const}_2(C, \alpha) \sqrt{pN}
\]

with probability at-least \( 1 - N^\alpha \).

From Lemma B.5 and B.6, we conclude that there always exists a constant, denoted by \( \text{const}_3(C, \alpha) = \max\{\text{const}_1(C, \alpha), \text{const}_2(C, \alpha)\} \), such that for any \( \alpha > 0 \), with probability at least \( 1 - 2N^{-\alpha} \),

\[
||Y^T LY - Y^T LY|| \leq ||L - L|| \leq \text{const}_3(C, \alpha) \sqrt{pN \log N}. \tag{12}
\]

Let \( Z \) and \( Z \) denote the optimal solution of \( (6) \) in the expected \((L \) replaced with \( L)\) and observed case respectively. We will use \( (12) \) to show a bound on \( ||YZ - YZ||_F \) in Lemma B.7. This will be used to argue that Algorithm I makes a small number of mistakes when the graph is sampled from \( \mathcal{R} \)-SBM.
Lemma B.7. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r}$ be eigenvalues of $Y^\top LY$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-r}$ be eigenvalues of $Y^\top LY$. Further, let the columns of $Z \in \mathbb{R}^{N-r \times K}$ and $Z \in \mathbb{R}^{N-r \times K}$ correspond to the leading $K$ eigenvalues of $Y^\top LY$ and $Y^\top LY$ respectively. Define $\gamma = \mu_{K+1} - \mu_K$. Then, with probability at least $1 - 2N^{-\alpha}$,

$$
\inf_{U \in \mathbb{R}^{K \times K} : UU^\top = U^\top U = I} \|YZ - YZU\|_F \leq \text{const}(C, \alpha) \frac{4\sqrt{2K}}{\gamma} \sqrt{pN \log N},
$$

where $\text{const}(C, \alpha)$ is from (12).

Let $T \in \mathbb{R}^{N \times K}$ be a matrix containing the first $K$ columns of $Y$, and let $T_i$ denote the $i^{th}$ row of $T$. Simple calculation using Lemma B.3 shows that,

$$
||T_i - T_j||_2 = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ belong to the same community} \\ \sqrt{\frac{2K}{N}} & \text{otherwise.} \end{cases}
$$

By Lemma B.4, $Z$ can be chosen such that $YZ = T$. Let $U$ be the matrix that solves $\inf_{U \in \mathbb{R}^{K \times K} : UU^\top = U^\top U = I} \|YZ - YZU\|_F$. As $U$ is orthogonal, $||T_iU - T_jU||_2 = ||T_i - T_j||_2$. The following lemma is a direct consequence of Lemma 5.3 in Lei & Rinaldo (2015).

Lemma B.8. Let $T$ and $U$ be as defined above. For any $\epsilon > 0$, let $\hat{\Theta} \in \mathbb{R}^{N \times K}$ be the assignment matrix returned by a $(1 + \epsilon)$-approximate solution to the $k$-means clustering problem when rows of $YZ$ are provided as input features, and let $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_K \in \mathbb{R}^K$ be the estimated cluster centroids. Define $T = \hat{\Theta}\hat{\mu}$ where $\hat{\mu} \in \mathbb{R}^{K \times K}$ contains $\hat{\mu}_1, \ldots, \hat{\mu}_K$ as its rows. Further, define $\delta = \sqrt{\frac{2K}{N}}$, and $S_k = \{v_i \in C_k : ||\hat{T}_i - T_i|| \geq \delta/2\}$. Then,

$$
\delta^2 \sum_{k=1}^{K} |S_k| \leq 8(2 + \epsilon)||TU^\top - YZ||^2_F. \tag{13}
$$

Moreover, if $\gamma$ from Lemma B.7 satisfies $\gamma^2 > \text{const}(C, \alpha)(2 + \epsilon)pNK \log N$ for a universal constant $\text{const}(C, \alpha)$, there exists a permutation matrix $J \in \mathbb{R}^{K \times K}$ such that

$$(\hat{\Theta}J)_i = \Theta_i, \quad \forall i \in [N] \setminus (\cup_{k=1}^{K} S_k). \tag{14}$$

Here, $(\hat{\Theta}J)_i$ and $\Theta_i$ represent the $i^{th}$ row of matrix $\hat{\Theta}J$ and $\Theta$ respectively.

Recall that $M(\Theta, \hat{\Theta}) = \min_{J \in \mathcal{J}} \frac{1}{N}||\Theta - \hat{\Theta}J||_0$, where $\mathcal{J}$ is the set of all $K \times K$ permutation matrices. In particular, for the matrix $J$ used in Lemma B.8, $M(\Theta, \hat{\Theta}) \leq \frac{1}{N}||\Theta - \hat{\Theta}J||_0$. But, according to Lemma B.8

$$
||\Theta - \hat{\Theta}J||_0 \leq 2 \sum_{k=1}^{K} |S_k|.
$$

Using Lemma B.7 and B.8 we get:

$$
M(\Theta, \hat{\Theta}) & \leq \frac{1}{N}||\Theta - \hat{\Theta}J||_0 \\
& \leq \frac{2}{N} \sum_{k=1}^{K} |S_k| \\
& \leq \frac{16(2 + \epsilon)}{N\delta^2} ||TU^\top - YZ||^2_F \\
& \leq \text{const}(C, \alpha)^2 \frac{512(2 + \epsilon)}{N\delta^2\gamma^2} pNK \log N.
$$

Noting that $\delta = \sqrt{\frac{2K}{N}}$ and setting $\text{const}(C, \alpha) = 256 \times \text{const}(C, \alpha)^2$ finishes the proof.

\[^4\hat{\Theta}_{ik} = 1\text{ if the algorithm assigns vertex } v_i \text{ to the } k^{th} \text{ cluster and } \hat{\Theta}_{ik} = 0 \text{ otherwise.}\]
C Proof of Technical Lemmas from Appendix B

\section{Proof of Lemma B.1}

Because \( \mathcal{R} \) is a \( d \)-regular graph, it is easy to see that \( R1 = d1 \). Recall from Section 5.2 that \( I - \frac{1}{N}11^T \) is a projection matrix that removes the component of any vector \( x \in \mathbb{R}^N \) along the all ones vector \( 1 \). Thus, \( (I - \frac{1}{N}11^T)1 = 0 \) and hence \( 1 \in \text{null}(RI - \frac{1}{N}11^T)) \). Moreover, using Assumption 2.

\[
1^T u_k = \sum_{i=1}^N (u_k)_i = \sum_{i:v_i \in C_k} (u_k)_i + \sum_{i:v_i \not\in C_k} (u_k)_i = \frac{N}{K} - \frac{1}{K-1} \left( \frac{N}{K} - \frac{N}{K} \right) = 0.
\]

Thus, \( R(I - \frac{1}{N}11^T)u_k = Ru_k \). Let us compute the \( i \)th element of the vector \( Ru_k \) for an arbitrary \( i \in [N] \).

\[
(Ru_k)_i = \sum_{j=1}^N R_{ij}(u_k)_j = \sum_{j: R_{ij} = 1, & v_j \in C_k} 1 - \sum_{j: R_{ij} = 1, & v_j \not\in C_k} \frac{1}{K-1} = \frac{d}{K} - \frac{1}{K-1} \left( d - \frac{d}{K} \right) = 0.
\]

Here, the second last equality follows from Assumptions 2 and 3. Thus, \( Ru_k = 0 \) and hence \( u_k \in \text{null}(RI - \frac{1}{N}11^T) \).

Because \( 1^T u_k = 0 \) for all \( k \in [K-1] \), to show that \( 1, u_1, \ldots, u_{K-1} \) are linearly independent, it is enough to show that \( u_1, \ldots, u_{K-1} \) are linearly independent. Consider the \( i \)th component of \( \sum_{k=1}^{K-1} \alpha_k u_k \) for arbitrary \( \alpha_1, \ldots, \alpha_{K-1} \in \mathbb{R} \) and \( i \in [N] \). If \( v_i \in C_K \), then

\[
\left( \sum_{k=1}^{K-1} \alpha_k u_k \right)_i = -\frac{1}{K-1} \sum_{k=1}^{K-1} \alpha_k.
\]

Similarly, when \( v_i \in C_{k'} \) for some \( k' \in [K-1] \), we have,

\[
\left( \sum_{k=1}^{K-1} \alpha_k u_k \right)_i = \alpha_{k'} - \frac{1}{K-1} \sum_{k=1, k \not= k'}^{K-1} \alpha_k.
\]

Thus, \( \sum_{k=1}^{K-1} \alpha_k u_k = 0 \) implies that \( -\frac{1}{K-1} \sum_{k=1}^{K-1} \alpha_k = 0 \) and \( \alpha_{k'} - \frac{1}{K-1} \sum_{k=1, k \not= k'}^{K-1} \alpha_k = 0 \) for all \( k' \in [K-1] \). Subtracting the first equation from the second gives \( \alpha_{k'} + \frac{1}{K-1} \alpha_k = 0 \) for all \( k' \in [K-1] \). Thus, \( 1, u_1, \ldots, u_{K-1} \) are linearly independent.

\section{Proof of Lemma B.2}

Using the representation of \( \tilde{A} \) from [9], Lemma B.1 and Assumption 2, we get,

\[
\tilde{A}1 = qR1 + s(11^T - R)1 + (p - q) \sum_{k=1}^K G_k RG_k 1 + (r - s) \sum_{k=1}^K G_k (11^T - R)G_k 1 \]

\[
= qd1 + sN1 - sd1 + (r - s) \sum_{k=1}^K G_k 11^T G_k 1 + [(p - q) - (r - s)] \frac{d}{K} \sum_{k=1}^K G_k RG_k 1 \]

\[
= qd1 + s(N - d)1 + (r - s) \frac{N}{K} 1 + [(p - q) - (r - s)] \frac{d}{K} \frac{N}{K} 1 \]

\[
= \left[ qd + s(N - d) + (p - q) \frac{d}{K} + (r - s) \frac{N}{K} \right] 1.
\]
Similarly, for any $k \in [K]$,
\[
\tilde{A}u_k = qR u_k + s(\mathbf{1}\mathbf{1}^T - R)u_k + (p-q) \sum_{k=1}^{K} G_k R G_k u_k + (r-s) \sum_{k=1}^{K} G_k (\mathbf{1}\mathbf{1}^T - R) G_k u_k \\
= 0 + 0 + (r-s) \sum_{k=1}^{K} G_k \mathbf{1}\mathbf{1}^T G_k u_k + [(p-q) - (r-s)] \sum_{k=1}^{K} G_k R G_k u_k \\
= \left[ (r-s) \frac{N}{K} + [(p-q) - (r-s)] \frac{d}{K} \right] u_k \\
= \left[ (p-q) \frac{d}{K} + (r-s) \frac{N-d}{K} \right] u_k.
\]

### C.3 Proof of Lemma B.3

Recall that $y_1 = 1/\sqrt{N}$. We start by showing that $y_1^T y_{1+k} = 0$.

\[
y_1^T y_{1+k} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (y_k)_i \]
\[
= \frac{1}{\sqrt{N}} \left[ \sum_{i:v_i \in C_k} (K-k)q_k - \sum_{i:v_i \in C_{k'}, k'>k} q_{k'} \right] \\
= \frac{1}{\sqrt{N}} \left[ \frac{N}{K} (K-k)q_k - \left( N - k \frac{N}{K} \right) q_k \right] = 0.
\]

Here, $q_k = \frac{1}{\sqrt{\frac{N}{K} (K-k)(K-k+1)}}$. Now consider $y_1^T y_{1+k}$ for $k_1, k_2 \in [K-1]$ such that $k_1 \neq k_2$. Assume without loss of generality that $k_1 < k_2$.

\[
y_1^T y_{1+k_1} y_{1+k_2} = \sum_{i:v_i \in C_{k_2}} (-q_{k_1})(K-k_2)q_{k_2} + \sum_{i:v_i \in C_{k_1}, k_1, k_2} (-q_{k_1})(-q_{k_2}) \\
= -q_{k_1}q_{k_2}(K-k_2) \frac{N}{K} + q_{k_1}q_{k_2}(N-k_2 \frac{N}{K}) = 0.
\]

Finally, for any $k \in [K-1]$,
\[
y_1^T y_{1+k} = \sum_{i:v_i \in C_k} (K-k)^2 q_{k_1}^2 + \sum_{i:v_i \in C_{k'}, k'>k} q_{k'}^2 = q_{k_1}^2 \left[ \frac{N}{K} (K-k)^2 + N - k \frac{N}{K} \right] = 1,
\]

where the last equality follows from the definition of $q_k$.

### C.4 Proof of Lemma B.4

Note that columns of $Z$ are the eigenvectors of $Y^T \tilde{A} Y$ corresponding to its $K$ largest eigenvalues as $Z$ is the solution to $[k]$ with $L$ set to $L$. The calculations below show that for all $k \in [K]$, $e_k \in \mathbb{R}^{N-r}$, the $k^{th}$ standard basis vector, is an eigenvector of $Y^T \tilde{A} Y$ with eigenvalue $\lambda_k$ where $\lambda_1, \ldots, \lambda_K$ were defined in Lemma B.3.

\[
Y^T \tilde{A} Y e_k = Y^T \tilde{A} y_k = \lambda_k Y^T y_k = \lambda_k e_k.
\]

The third equality follows from Lemma B.3 because $y_2, \ldots, y_K \in \text{span}(u_1, \ldots, u_{K-1})$, and $u_1, \ldots, u_{K-1}$ are all eigenvectors of $\tilde{A}$ with the same eigenvalue. To show that columns of $Y Z$ lie in the span of $y_1, \ldots, y_K$, it is enough to show that $e_1, \ldots, e_K$ are the top $K$ eigenvectors of $Y^T \tilde{A} Y$.

Let $\alpha \in \mathbb{R}^{N-r}$ be an eigenvector of $Y^T \tilde{A} Y$ such that $\alpha \notin \text{span}(e_1, \ldots, e_K)$ and $||\alpha||_2^2 = 1$. Then, because $Y^T \tilde{A} Y$ is symmetric, $\alpha^T y_1 = 0$, i.e. $\alpha_1 = 0$, where $\alpha_i$ denotes the $i^{th}$ element of $\alpha$. The eigenvalue corresponding to $\alpha$ is given by
\[
\lambda_\alpha = \alpha^T Y^T \tilde{A} Y \alpha.
\]
Let \( x = Y\alpha = \sum_{i=1}^{N-r} \alpha_i y_i \), then \( \lambda_\alpha = x^T \hat{A} x \). Using the definition of \( \hat{A} \) from (9), we get,

\[
x^T \hat{A} x = (q-s)x^T R x + s x^T 11^T x + [(p-q) - (r-s)] \sum_{k=1}^{K} x^T G_k R G_k x + (r-s) \sum_{k=1}^{K} x^T G_k 11^T G_k x.
\]

We will consider each term in (15) separately. Before that, note that adding the four terms we get the following bound. For eigenvector \( \alpha \),

\[
\text{Thus, the columns of } \lambda_\alpha = x^T \hat{A} x \text{.}
\]

Thus, \( \lambda_\alpha = x^T \hat{A} x \). Now consider the first term in (15).

\[
x^T Rx = \sum_{i=1}^{N-r} \sum_{j=1}^{N-r} \alpha_i \alpha_j y_i^T R y_j = \alpha_1^2 y_1^T R y_1 = 0.
\]

Here, the second equality follows from (16), and the third equality follows as \( \alpha_1 = 0 \). Similarly, for the second term in (15),

\[
x^T 11^T x = N x^T y_1^T y_1 = N \sum_{i=1}^{N-r} \sum_{j=1}^{N-r} \alpha_i \alpha_j y_i y_j = N \alpha_1^2 (y_1^T y_1)^2 = 0.
\]

Note that \( G_k = G_k G_k \) because it is a diagonal matrix with either 0 or 1 on its diagonal. For the third term in (15),

\[
x^T G_k R G_k x = x^T G_k G_k R G_k G_k x = x_{[k]}^T R_{[k]} x_{[k]} \leq \frac{d}{K} ||x_{[k]}||^2_2,
\]

where \( x_{[k]} \in \mathbb{R}^{N/K} \) contains elements of \( x \) corresponding to vertices in \( C_k \). Similarly, \( R_{[k]} \in \mathbb{R}^{N/K \times N/K} \) contains the submatrix of \( R \) restricted to rows and columns corresponding to vertices in \( C_k \). The last inequality holds because \( R_{[k]} \) is a \( d/K \)-regular graph by Assumption 3, hence its maximum eigenvalue is \( d/K \). Thus,

\[
\sum_{k=1}^{K} x^T G_k F G_k x \leq \frac{d}{K} \sum_{k=1}^{K} ||x_{[k]}||^2_2 = \frac{d}{K} ||x||^2_2 = \frac{d}{K}.
\]

Similarly, for the fourth term in (15),

\[
x^T G_k 11^T G_k x = x^T G_k G_k 11^T G_k G_k x = x_{[k]}^T 1_{N/K} 1_{N/K} x_{[k]} \leq \frac{N}{K} ||x_{[k]}||^2_2.
\]

Here, \( 1_{N/K} \in \mathbb{R}^{N/K} \) is an all-ones vector and the last inequality holds because \( 1_{N/K} 1_{N/K} \) is a \( N/K \)-regular graph. Because \( x_{[k]} \notin \text{span}(y_1, \ldots, y_K) \), there is at least one \( k \in [K] \) for which \( x_{[k]} \) is not a constant vector (if this was not true, \( x_{[k]} \) will belong to span of \( y_1, \ldots, y_K \)). Thus, at least for one \( k \in [K] \),

\[
x^T G_k 11^T G_k x < \frac{N}{K} ||x_{[k]}||^2_2.
\]

Summing over \( k \in [K] \), we get,

\[
\sum_{k=1}^{K} x^T G_k 11^T G_k x < N \sum_{k=1}^{K} ||x_{[k]}||^2_2 = \frac{N}{K} ||x||^2_2 = \frac{N}{K}.
\]

Adding the four terms we get the following bound. For eigenvector \( \alpha \) of \( Y^T \hat{A} Y \) such that \( \alpha \notin \text{span}(e_1, \ldots, e_K) \) and \( ||\alpha||^2_2 = 1 \),

\[
\lambda_\alpha = x^T \hat{A} x < [(p-q) - (r-s)] \frac{d}{K} + (r-s) \frac{N}{K} = \lambda_K.
\]

Thus, \( \lambda_1, \ldots, \lambda_K \) are the highest \( K \) eigenvalues of \( Y^T \hat{A} Y \) and hence \( e_1, \ldots, e_K \) are the top \( K \) eigenvectors. Thus, the columns of \( YZ \) lie in the span of \( y_1, \ldots, y_K \).
C.5 Proof of Lemma B.5

As $D$ and $\mathcal{D}$ are diagonal matrices, $||D - \mathcal{D}|| = \max_{i \in [N]} |D_{ii} - \mathcal{D}_{ii}|$. Applying union bound, we get,

$$P(\max_{i \in [N]} |D_{ii} - \mathcal{D}_{ii}| \geq \epsilon) \leq \sum_{i=1}^{N} P(|D_{ii} - \mathcal{D}_{ii}| \geq \epsilon).$$

We consider an arbitrary term in this summation. For any $i \in [N]$, note that $D_{ii} = \sum_{j \neq i} A_{ij}$ is a sum of independent Bernoulli random variables such that $E[D_{ii}] = \mathcal{D}_{ii}$. We consider two cases depending on the value of $p$.

Case 1: $p > \frac{1}{2}$: By Hoeffding’s inequality,

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \epsilon) \leq 2 \exp \left( - \frac{2\epsilon^2}{N} \right).$$

Setting $\epsilon = \sqrt{2(\alpha + 1)\sqrt{pN} \log N}$, we get for any $\alpha > 0$,

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \sqrt{2(\alpha + 1)\sqrt{pN} \log N}) \leq 2 \exp \left( - \frac{4p(\alpha + 1)N \log N}{N} \right) \leq N^{-(\alpha + 1)}.$$

Case 2: $p \leq \frac{1}{2}$: By Bernstein’s inequality, as $|A_{ij} - A_{ij}| \leq 1$ for all $i, j \in [N],$

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \epsilon) \leq 2 \exp \left( - \frac{\epsilon^2/2}{\sum_{i=1}^{N} E[(A_{ij} - A_{ij})^2] + \epsilon/3} \right).$$

Also note that,

$$E[(A_{ij} - A_{ij})^2] \leq A_{ij}(1 - A_{ij})^2 + (1 - A_{ij})(-A_{ij})^2 = A_{ij}(1 - A_{ij}) \leq A_{ij} \leq p.$$ 

Thus,

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \epsilon) \leq 2 \exp \left( - \frac{\epsilon^2/2}{Np + \epsilon/3} \right).$$

Let $\epsilon = c\sqrt{pN \log N}$ for some constant $c > 0$ and assume that $p \geq C\frac{\log N}{N}$ for some $C > 0$. We get,

$$2 \exp \left( - \frac{\epsilon^2/2}{Np + \epsilon/3} \right) = 2 \exp \left( - \frac{c^2pN \log N}{2(Np + c\sqrt{pN \log N}/3)} \right) = 2 \exp \left( - \frac{c^2 \log N}{2(1 + c/3\sqrt{C})} \right) \leq 2 \exp \left( - \frac{c^2 \log N}{2(1 + c/3\sqrt{C})} \right).$$

Let $c$ be such that $\frac{c^2}{2(1+c/3\sqrt{C})} \geq 2(\alpha + 1)$ then,

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \epsilon) \leq N^{-(\alpha + 1)}.$$ 

Thus, there always exists a constant $\text{const}_1(C, \alpha)$ that depends only on $C$ and $\alpha$ such that for all $\alpha > 0$ and for all values of $p \geq C \log N/N$,

$$P(|D_{ii} - \mathcal{D}_{ii}| \geq \text{const}_1(C, \alpha) \sqrt{pN \log N}) \leq N^{-(\alpha + 1)}.$$ 

Applying the union bound over all $i \in [N]$ yields the desired result. 

\footnote{Note that such a $c$ can always be chosen as $\lim_{c \to \infty} \frac{c^2}{2(1+c/3\sqrt{C})} = \infty$.}
C.6 Proof of Lemma \[B.6\]

Note that \(\max_{i,j \in [N]} A_{ij} = p\). Define \(g = pN = N \max_{i,j \in [N]} A_{ij}\). Note that, \(g \geq C \log N\) as \(p \geq C\frac{\log N}{N}\). By Theorem 5.2 from [Lei & Rinaldo (2015)] for any \(\alpha > 0\), there exists a constant \(\text{const}_2(\alpha, C)\) such that,

\[\|A - A\| \leq \text{const}_2(\alpha, C) \sqrt{g} = \text{const}_2(\alpha, C) \sqrt{pN}\]

with probability at least \(1 - N^{-\alpha}\).

C.7 Proof of Lemma \[B.7\]

Because \(Y^T Y = I\), for any orthonormal matrix \(U \in \mathbb{R}^{K \times K}\) such that \(U U^T = U^T U = I\),

\[\|Y Z - Y Z U\|_F^2 = \|Y(Z - ZU)\|_F^2 = \text{trace}((Z - ZU)^T Y Y (Z - ZU)) = \|Z - ZU\|_F^2.\]

Thus, it is enough to show an upper bound on \(\|Z - ZU\|_F\), where recall that columns of \(Z \in \mathbb{R}^{N-r \times K}\) and \(Z \in \mathbb{R}^{N-r \times K}\) contain the \(K\) leading eigenvectors of \(Y^T Y\) and \(Y^T L Y\) respectively. Thus,

\[\inf_{U \in \mathbb{R}^{K \times K}, U^T U = U U^T = I} \|Y Z - Y Z U\|_F = \inf_{U \in \mathbb{R}^{K \times K}, U^T U = U U^T = I} \|Z - ZU\|_F.\]

By equation (2.6) and Proposition 2.2 in [Vu & Lei (2013)],

\[\inf_{U \in \mathbb{R}^{K \times K}, U^T U = U U^T = I} \|Z - ZU\|_F \leq \sqrt{2}\|Z Z^T (I - ZZ^T)\|_F.\]

Moreover, \(\|Z Z^T (I - ZZ^T)\|_F \leq \sqrt{K}\|Z Z^T (I - ZZ^T)\|\) as \(\text{rank}\{ZZ^T (I - ZZ^T)\} \leq K\). Thus, we get,

\[\inf_{U \in \mathbb{R}^{K \times K}, U^T U = U U^T = I} \|Z - ZU\|_F \leq \sqrt{2K}\|ZZ^T (I - ZZ^T)\|.\] \tag{19}

Let \(\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r}\) be eigenvalues of \(Y^T L Y\) and \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-r}\) be eigenvalues of \(Y^T L Y\). By Weyl’s perturbation theorem,

\[|\mu_i - \alpha_i| \leq \|Y^T L Y - Y^T L Y\|, \quad \forall i \in [N-r].\]

Define \(\gamma = \mu_{K+1} - \mu_K\) as the eigen-gap between the \(K\)th and \((K+1)\)th eigenvalues of \(Y^T L Y\).

**Case 1:** \(\|Y^T L Y - Y^T L Y\| \leq \frac{\gamma}{2}\) if \(\|Y^T L Y - Y^T L Y\| \leq \frac{\gamma}{2}\), then \(|\mu_i - \alpha_i| \leq \frac{\gamma}{2}\) for all \(i \in [N-r]\) by the inequality given above. Thus, \(\alpha_1, \alpha_2, \ldots, \alpha_K \in [0, \mu_K + \frac{\gamma}{4}]\) and \(\alpha_{K+1}, \alpha_{K+2}, \ldots, \alpha_{N-r} \in [\mu_{K+1} - \frac{\gamma}{4}, \infty)\). Let \(S = [0, \mu_K + \frac{\gamma}{4}]\), then \(\mu_1, \ldots, \mu_K \in S\) and \(\alpha_{K+1}, \ldots, \alpha_{N-r} \notin S\). Define \(\delta\) as,

\[\delta = \min\{|\alpha_i - s|, \alpha_i \notin S, s \in S\}.\]

Then, \(\delta \geq [\mu_{K+1} - \gamma/4] - [\mu_K + \gamma/4] = \gamma/2\). By Davis-Kahan sin \(\Theta\) theorem,

\[\|ZZ^T (I - ZZ^T)\| \leq \frac{1}{\delta}\|Y^T L Y - Y^T L Y\| = \frac{\gamma}{2}\|Y^T L Y - Y^T L Y\|.\]

**Case 2:** \(\|Y^T L Y - Y^T L Y\| > \frac{\gamma}{2}\): Note that \(\|ZZ^T (I - ZZ^T)\| \leq 1\) as,

\[\|ZZ^T (I - ZZ^T)\| \leq \|ZZ^T\| \|I - ZZ^T\| = 1.1 = 1.\]

Thus, if \(\|Y^T L Y - Y^T L Y\| > \frac{\gamma}{4}\), then,

\[\|ZZ^T (I - ZZ^T)\| \leq \frac{4}{\gamma}\|Y^T L Y - Y^T L Y\|.\]

In both cases, \(\|ZZ^T (I - ZZ^T)\| \leq \frac{4}{\gamma}\|Y^T L Y - Y^T L Y\|\). Using (11), (12) and (19), we get with probability at least \(1 - 2N^{-\alpha}\),

\[\inf_{U \in \mathbb{R}^{K \times K}, U^T U = U U^T = I} \|Z - ZU\|_F \leq \text{const}_3(C, \alpha) \frac{4\sqrt{2K}}{\gamma} \sqrt{pN \log N}.\]
C.8 Proof of Lemma B.8

(13) directly follows from Lemma 5.3 in Lei & Rinaldo (2015). We only need to show that
\[
\frac{8(2 + \epsilon)}{\delta^2} \|TU - YZ\|^2_F < \frac{N}{K}.
\]

(14) then follows from Lemma 5.3 in Lei & Rinaldo (2015). Recall that \(\delta = \sqrt{\frac{2K}{N}}\). Using Lemma B.7 we get
\[
\frac{8(2 + \epsilon)}{\delta^2} \|TU - YZ\|^2_F = \text{const}_3(C, \alpha) \cdot 128(2 + \epsilon) \cdot pN^2 \log N < \frac{N}{K},
\]
Here, the last inequality follows from the assumption that \(\gamma^2 > \text{const}_3(C, \alpha)^2 \cdot 128(2 + \epsilon)pNK \log N\).

D Additional Experiments

In this section, we study the variation in the performance of our algorithm as a function of quantities like number of nodes, number of clusters, network sparsity, etc. Similar to Section 7, this section has been divided into two parts: synthetic and real-world networks.

D.1 Synthetic Networks

For the first set of experiments, we sample \(d\)-regular representation graphs that satisfy Assumptions 2 and 3. Figure 1 in the paper compares the performance of GEN-FAIRSC with FAIRSC (Kleindessner et al., 2019b) and spectral clustering without fairness (SC) for various values of \(N\) by fixing \(K = 5\) and \(d = \frac{N}{10}\). Recall that \(N\) is the number of nodes, \(K\) is the number of clusters, and \(d\) is the common degree of the nodes in the sampled \(d\)-regular representation graphs.

Figure 4a also shows accuracy as a function of \(N\). However, as opposed to the previous case, here we set \(d = 40\) instead of using \(d = 0.1N\). Thus, as \(N\) increases, the representation graph becomes increasingly sparse. As before, \(K = 5\). We run FAIRSC with \(P = 0.1N\) protected groups that are discovered using the observed adjacency matrix as explained in Section 7. We also run the approximate variant of GEN-FAIRSC that was used for real-data experiments in Section 7. To do so, we find the best rank \(r\) approximation of the representation graph and use it instead of the original \(R\). Figure 1a uses \(r = 0.1N\). It reports the mean and standard deviation in accuracy across 10 independent runs in each case. Recall that the ground-truth clusters are fair by construction. Hence, high accuracy implies high fairness. GEN-FAIRSC outperforms all other algorithms and GEN-FAIRSC-(APPROX.) performs better than FAIRSC and SC.
Figure 5: Comparison between Gen-FairSC-(approx.), FairSC, and spectral clustering without fairness (SC) using representation graphs generated from a SBM. In all cases, rank $r$ can be chosen for Gen-FairSC-(approx.) to get clusters with high balance and low ratio-cut. However, FairSC yields poor performance across various values of $P$. The difference in performance is especially significant when $K$ is large. These plots use $p_{in} = 0.8$, $p_{out} = 0.2$, and five protected groups in the SBM used for sampling the representation graphs (see Section 7.1).

Figure 4b plots accuracy against the number of ground-truth clusters $K$. It uses $N = 2000$, $d = 40$, $P = 0.1N$ (for FairSC), and $r = 0.1N$ (for Gen-FairSC (approx.)). Crucially, the performance of Gen-FairSC does not change as $K$ increases, whereas the performance of all other algorithm degrades substantially. It appears in Figure 4a that even normal spectral clustering will return fair clusters for a large enough graph. However, Figure 4b shows that this is not true if the number of clusters also increases, which is more common in practice.

Figure 4c shows the variation in performance with the degree $d$ of the representation graph. We used $N = 2000$, $K = 5$, $P = 0.1N$ (for FairSC), and $r = 0.1N$ (for Gen-FairSC (approx.)) to generate this figure. Once again, both Gen-FairSC and its approximate variant outperform other algorithms and are also robust to changes in degree of the representation graph. While the performance of SC also appears to be robust to changes in $d$, FairSC has poor performance for higher $d$.

It may be tempting to think that FairSC may perform well with a more carefully chosen value of $P$, the number of protected groups. Figure 6 shows that this is not the case.
It plots the performance of FairSC as a function of the number of protected groups \( P \). Also shown is the performance of the approximate variant of GEN-FairSC for various values of rank \( r \) used in approximating the representation graph. As expected, the accuracy increases with \( r \) as GEN-FairSC-(APPROX.) gets to use a better approximation for \( R \).

The next set of experiments sample representation graphs from an SBM as described in Section 7.1. As noted before, representation graphs sampled this way are not \( d \)-regular. We always use the approximate variant of GEN-FairSC in these experiments as a sampled \( R \) may violate Assumption 1 which is necessary to ensure that eq. (7) has a solution. Figure 2 in the paper shows the variation in the performance of GEN-FairSC-(APPROX.) and FairSC with \( P \) and \( r \) for a fixed value of \( N \) and \( K \) (\( P \) and \( r \) have the same meaning as above. See Section 7 for more details about the notation and the sampling process). Figure 5 compares GEN-FairSC-(APPROX.) and FairSC for more combinations of \( N \) and \( K \). Every sub-plot shows the ratio of average balance \( \bar{\rho} \) to the ratio-cut objective defined in Section 2 on \( y \)-axis. We want to find clusters with high balance and low ratio-cut. Thus, higher values on \( y \)-axis are more desirable.

D.2 Real-World Network

We use the FAO trade network for the next set of experiments. See Section 7.2 for details about the construction of this network. Figure 3 fixes \( K = 2 \) and compares GEN-FairSC-(APPROX.) with FairSC for different values of \( r \) and \( P \) used by GEN-FairSC-(APPROX.) and FairSC respectively. Figure 7 makes the same comparison for other values of \( K \). On the \( y \)-axis is the ratio of average balance \( \bar{\rho} \) to the value of ratio-cut. The \( x \)-axis corresponds to different values of \( P \) and \( r \). As before, higher values on \( y \)-axis are desirable.

E Individual vs Statistical Fairness: A Toy Example

In this section, we demonstrate a toy example to distinguish between statistical and individual fairness. Recall that a commonly used notion of statistical fairness, called proportional representation (Chierichetti et al., 2017; Kleindessner et al., 2019b), assumes that the nodes belong to protected groups \( P_1, P_2, \ldots, P_P \) and requires,

\[
\frac{|C_k \cap P_p|}{|C_k|} = \frac{|P_p|}{N}, \quad \text{for all } p \in [P], k \in [K].
\]

The algorithm FairSC proposed in Kleindessner et al. (2019b) tries to achieve statistical fairness. In contrast, our algorithm GEN-FairSC supports the fairness notion specified in Definition 3.1 which includes statistical fairness as a special case (see Section 3), but can more generally enforce individual fairness.

Consider the representation graph specified in Figure 8. All nodes have a self-loop associated with them, which has not been shown for clarity. In this example, \( N = 24 \), \( K = 2 \), and every node is connected to \( d = 6 \) nodes (including the self-loop). The color of nodes indicates their ground-truth cluster membership. It is easy to verify that the ground-truth clusters are fair with respect to this representation graph as per
Figure 7: Comparing the performance of GEN-FAIRSC-(APPROX.) and FAIRSC on the FAO trade network for different values of $K$. GEN-FAIRSC-(APPROX.) outperforms FAIRSC.

Definition 3.1 as each node has three neighbors in both red and blue clusters. Our consistency result shows that GEN-FAIRSC recovers the fair ground-truth clusters for such $d$-regular graphs with high probability.

On the other hand, to apply FAIRSC to this problem, we first need to cluster nodes in the representation graph to find the protected groups $P_1, P_2, \ldots, P_P$. A natural choice is to have two protected groups $P_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and $P_2 = \{13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\}$. However, clustering nodes based on these protected groups can produce clusters where, for instance, all nodes on the yellow background are in $C_1$, and all nodes on the green background are in $C_2$. It is easy to verify that this clustering satisfies the statistical fairness criterion given above. However, it is very unfair from the perspective of
Figure 8: A small 24-node representation graph. Circles represent nodes and lines represent edges. The color of circles indicate the ground-truth cluster membership. The rectangles enclose nodes that can be possibly clustered together by FAIRSC. See Appendix E for a description of why the clusters recovered by FAIRSC can be statistically fair but highly unfair from each individual’s perspective.

each individual. For example, only one of the six neighbors of node 1 is in $C_2$ despite the equal size of both the clusters. Thus, node 1 does not have enough representation in $C_2$. A similar argument can be made for every node in this graph.

This example illustrates how a clustering can be highly unfair from each individual’s perspective despite being statistically fair. GEN-FAIRSC can recover the individually fair clusters but FAIRSC cannot.