Charges in gauge theories

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Abstract. In this article we investigate charged particles in gauge theories. After reviewing the physical and theoretical problems, a method to construct charged particles is presented. Explicit solutions are found in the abelian theory and a physical interpretation is given. These solutions and our interpretation of these variables as the true degrees of freedom for charged particles, are then tested in the perturbative domain and are demonstrated to yield infra-red finite, on-shell Green's functions at all orders of perturbation theory. The extension to collinear divergences is studied and it is shown that this method applies to the case of massless charged particles. The application of these constructions to the charged sectors of the standard model is reviewed and we conclude with a discussion of the successes achieved so far in this programme and a list of open questions.

Keywords. Gauge theories; infra-red; charged particles; confinement.

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1. Introduction

One of the most significant advances in particle physics was the definition of what a particle should be. Wigner [1] identified a particle with an irreducible representation of the Poincaré group or its covering. Such representations have a well defined mass and spin. Building upon free creation and annihilation operators, and invoking [2] the cluster decomposition theorem, one can construct the paradigm quantum field theoretic description of particles scattering into other particles.

Physical particles generally carry additional quantum numbers such as isospin, electric and colour charge. These particles are described by the gauge theories of the standard model. Due to the masslessness of the gauge bosons, these field theories are plagued by infra-red divergences which dramatically alter the singularities of the Green's functions of the matter fields. The states no longer form irreducible representations of the Poincaré group [3] and it is therefore widely believed [4] that there is no particle description of charges such as the electron. This review will describe the physics underlying this problem, the manifold consequences of this breakdown and how a particle description of charged fields may be recovered.

The standard picture of a particle's journey to a detector is as follows. When it is shot out of a scattering event it is, after enough time, a long way away from any other particle
and the interaction between it and the other particles may be neglected. The free Hamiltonian then describes its dynamics and a particle description holds. This, the cornerstone of the interaction picture, breaks down for charged particles interacting via the gauge theories of the standard model.

The importance of the large distance interactions, which mean that the residual interactions between particles at large separations cannot be neglected, is most obvious in quantum chromodynamics (QCD). Experimentalists observe hadrons and not quarks which are professed to be eternally confined. Although hadrons are colourless (chargeless as far as the strong nuclear force is concerned) their masses are rather well described in terms of building blocks which are the original Gell–Mann quarks. However, the current quarks of the QCD Lagrangian are not the constituent quarks of hadronic spectroscopy. This dichotomy is an unsolved puzzle and shows itself in many ways. For the light quark flavours the masses of the current and constituent quarks are very different (roughly two orders of magnitude for the u and d flavours). The division of the spin of the proton amongst its constituents led to the ‘proton spin crisis’ a name which shows the depth of our difficulties in understanding the experimental data. Finally, since constituent quarks are presumably constructs made from surrounding a Lagrangian matter field with a cloud of coloured glue, it is initially at least highly unclear how the constituents obtain a well-defined colour.

A major aim of the programme of research [5] described here is to understand how constituent quarks arise in QCD, their mutual interactions and finally how it is that these effective degrees of freedom are confined.

Electrons are not confined and so it may seem initially less obvious that the interaction picture paradigm breaks down here too. Yet the S-matrix elements of quantum electrodynamics (QED) are afflicted by infra-red divergences, as are the on-shell Green’s functions of the theory with external legs corresponding to charged particles (which we will sometimes generically refer to as electrons). The infra-red problems change the form of the Green’s functions such that we cannot associate a pole to the external legs. There are two different sorts of divergence here: soft divergences, which show up in Green’s functions and S-matrix elements, and phase divergences which occur in the phase. These latter divergences are often ignored in QED, but are of importance in QCD (see § 3.4 of ref. [6] and [7]).

The masslessness of the photon is the underlying cause of the infra-red problem [7a]. This vanishing of the mass means that photons can travel over a large distance and indeed that an infinite number of soft photons can be created for any finite amount of energy. Recognition of this led to the Bloch–Nordsieck answer to the infra-red problem in QED: since any finite experimental resolution does not restrict the number of photons which may accompany any charged particle, an experimental cross-section must come from summing over all these possibilities. In this sense QED is taken to be a theory defined only at the level of (measurable) cross-sections and not in terms of (unobservable) S-matrix elements. Although this is in no way incompatible with the experiment, it is a radical conclusion.

The survival of large distance interactions is responsible for the claim [4] that charged particles cannot be incorporated into relativistic quantum field theory. This conclusion followed from noting that as the usual free Hamiltonian does not determine the asymptotic dynamics, it must be modified. This leads to a description in terms of coherent states and there is no pole structure. Is particle physics then a misnomer?
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In fact we will see below that these problems can be resolved by realising that, as the asymptotic interaction Hamiltonian is not zero, electric charges are surrounded by an electromagnetic cloud, just as quarks are by glue. It will then be seen that such systems of dressed charges must be gauge invariant. We will demonstrate that the Green’s functions of such dressed charges are infra-red finite and have a good pole structure.

1.1 General properties of charges

The gauge dependence of the Lagrangian degrees of freedom means that it is hard to associate any physical meaning to them. This hinders any attempt to understand how phenomenological models and concepts can arise from the underlying theory. As an example of this we recall that the highly non-trivial gauge dependence of the Lagrangian vector potential in non-abelian gauge theories has put difficulties in the way of extracting such phenomenologically useful ideas as effective gluon masses from lattice calculations [8]. Constrained dynamics is the mathematical tool appropriate to finding gauge invariant degrees of freedom. The true degrees of freedom then correspond to locally gauge invariant constructs (which obey Gauss’ law). There have been many attempts to obtain such variables in gauge theories (see e.g., [9] and references therein). We now want to sketch some general properties that any description of charges must fulfill.

Local gauge transformations in QED have the form

\[ A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \theta(x), \quad \text{and} \quad \psi(x) \rightarrow e^{ie\theta(x)} \psi(x), \]

so if the coupling \( e \) could be switched off, then the Lagrangian fermion would be locally gauge invariant. Similarly Gauss’ law

\[ \partial F_{\mu 0} = -e J_0, \]

where \( J_0 \) is the charge density, would, in the \( e = 0 \) limit, imply that only the transverse components of the field strength were physical. However, as we have noted the coupling does not vanish and so the matter fields cannot be identified with physical particles. Similarly Gauss’ law shows that there is an intimate link between the matter fields and the electromagnetic cloud which surrounds them. An immediate consequence of this is that any description of a charged particle cannot be local since the total charge can be written as a surface integral at infinity.

We also see that objects which are invariant under global gauge transformations are chargeless, as the charge density is the generator of such global transformations. The Gauss law constraint generates local gauge transformations, so we demand invariance under local but not global gauge transformations of any description of a charged particle.

The form of the cloud around a charge determines the electric and magnetic fields surrounding the charge. This implies a fundamental non-covariance: the velocity of any charged particle will determine the nature of the cloud.

The implications of these inevitable properties of non-locality [10] and non-covariance [1] for any description of charged particles for the general properties of gauge theories have been investigated both for scattering theory [12, 13] and in axiomatic approaches [14, 15].
1.2 Dirac’s dressings

To the best of our knowledge the interplay between a charged matter field and the electromagnetic cloud which inevitably surrounds it was first used by Dirac [16] to construct a specific description of the electron. He suggested that one should use

\[ \psi_D(x) \equiv \exp\left( -ie \frac{\partial A_i}{\nabla^2} \right) \psi(x). \]  

(3)

This he motivated in the following way: it is locally gauge invariant [15a]. Using the fundamental equal-time commutator, \[ [E_i(x), A_j(y)] = i\delta_{ij}\delta(x - y), \] and the representation

\[ \partial_i A_i(x) = -\frac{1}{4\pi} \int d^3y \frac{\partial_i A_i(x_0, y)}{|x - y|} \]  

(4)

then the electric field of the state \[ \psi_D(x)|0\rangle \] is found to be

\[ E^i(x_0, y)\psi_D(x)|0\rangle = -\frac{e}{4\pi} \frac{x_i - y_i}{|x - y|^3} \psi_D(x)|0\rangle, \]  

(5)

which is what one would expect for a static charge. He further pointed out that this is actually a member of an entire class of composite fields

\[ \psi_f(x) \equiv \exp\left( -ie \int d^4z f^\mu(x - z)A_\mu(z) \right) \psi(x), \]  

(6)

which, he argued are gauge invariant for all \( f^\mu \) if it is demanded that \[ \partial_\mu f^\mu(w) = \delta^{(4)}(w) \] holds. These constructs are all evidently non-local. We refer to the cloud around the matter field as a dressing. It is clear that the dressing suggested by Dirac (3) is also non-covariant. This review will be concerned with generalising and refining such descriptions of charges.

Such pictures have been rediscovered by various authors since Dirac and there have been many attempts to use certain examples of this wide class of dressings over the years [17–30].

A natural extension of this is to generalize eq. 6 to systems involving more than one matter field. Two opposite charges at different points can be made gauge invariant by including a dressing which keeps the entire system gauge invariant. Such a description could correspond to a positronium state, a hydrogen atom or a meson. If the cloud factorises so that each of the matter fields together with its part of the dressing is gauge invariant, then we can clearly speak of constituents. A dressing which does not factorise at all would mean that we had an effective ‘meson’ field but could not really speak of constituent particles. The success of the constituent quark picture of hadrons, taken together with the rough equality of the constituent masses in mesons and baryons indicates that there must be at least a rough factorisation of the dressings in some dynamical domains of QCD.

Once one has an ansatz for the dressing, it may be used to study the interaction between charges. The easiest manner to produce a gauge invariant description is to link the two matter fields by a path ordered exponential along some line. The potential is then obtained [30a] by taking the commutator of the Hamiltonian with this description: that part of the energy which depends on the separation is the potential. The string ansatz
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leads already in QED to a linear, confining potential. Furthermore the overall coefficient is, as a result of the infinite thinness of the string, divergent. This string description has no physical relevance, even in QCD the potential between two heavy quarks should have a Coulombic form at short distances [31] and the finite string tension implies a cigar shaped dressing.

Since the string model corresponds to an (infinitely) excited state, it is unstable [21]. If we consider two extremely heavy, fixed charges and neglect pair creation, we may use the free Hamiltonian and thus solve the time development of the system starting from the string ansatz initial state. The electric and magnetic fields immediately broaden out and, of course, lead to the usual Coulombic field. An animation and detailed discussion of this can be found at the web site: http://www.ifae.es/~roy/qed.html

After this fly-by tour of the subject, it is time to get to grips with the physics of charges. The structure of this article is as follows. In § 2 the form of the interaction Hamiltonian at large times is investigated. It is shown not to vanish for the matter fields but we demonstrate that, for suitably dressed fields, the asymptotic interaction does indeed vanish. This result is then used to construct explicit charged fields. In § 3 an alternative account of the charged fields in the heavy scalar theory is presented. These results and our interpretation of them are then tested in perturbation theory in § 4. It is shown that the soft and phase divergences cancel and a pole structure is obtained. The massless electron limit is considered in § 5 as a testing ground for the study of collinear divergences. The form of the asymptotic interaction Hamiltonian is shown to be such that the solutions of the dressing equation will remove these mass singularities as well. Subtleties of the dressing equation in this limit are investigated in this section. Finally § 6 reviews what we have learned and presents a list of outstanding problems.

2. Charged particles

As we have seen, the masslessness of the photon implies that the interactions between charges cannot be 'switched off' in the remote past or future. It is this long range nature of electromagnetic interactions that lies at the heart of the infra-red problem in QED. In this section we will review how the asymptotic dynamics found in such a gauge theory deviates from that of a free theory. As a consequence of this we will see that the correct identification of asymptotic, charged particle states can be made through a process of dressing the matter of the theory.

2.1 Asymptotic dynamics

The S-matrix codifies the intuitive picture of a scattering experiment whereby in-coming particles get sufficiently close to interact, resulting (after the dust settles) in some set of out-going particles. The in-coming and out-going particle regimes are identified with elements of the Fock space constructed out of the creation and annihilation operators found in the free theory. As discussed in the introduction, the reason for this is that it is these states that can be identified with (tensor products) of irreducible representations of the Poincaré group. As such, they are particles!

The key assumption, then, in the S-matrix description of particle scattering is that in-and out-regimes where the dynamics is that of the free system exist, i.e., the interacting
Hamiltonian must tend to zero in the remote past and future. It is this assumption that fails in a gauge theory \[32, 4\].

To see how this arises in QED, we start with the gauge fixed Lagrangian density:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu - ieA_\mu) \psi - m \bar{\psi} \psi + \frac{1}{2} \xi B^2 + \partial_\mu A^\mu B.$$  (7)

In this we are working in the Lorentz class of gauges (Feynman gauge corresponding to the choice \(\xi = 1\)). The gauge invariance of the physical states is encoded in the condition that \(B^{+}\text{phys}) = 0\). For the present, we do not allow the mass of the electron to be zero \((m \neq 0)\).

The interaction Hamiltonian is given by

$$\mathcal{H}_{\text{int}}(t) = -e \int d^3x A_\mu(t, x) J^\mu(t, x),$$  (8)

where the conserved matter current is \(J^\mu(t, x) = \bar{\psi}(t, x) \gamma^\mu \psi(t, x)\). In order to construct the \(S\)-matrix, the fields that enter this part of the Hamiltonian are taken to be in the interaction picture. We recall that this means that the time evolution of the states is described by (8) while the fields evolve under the free Hamiltonian. Thus in (8) we should insert the free field expansions for the matter and gauge fields, this will then allow us to study in detail the large \(t\) behaviour of the interaction.

We take as our free field expansions

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ b(p, s) u^s(p) e^{-ip\cdot x} + d^s(p, s) v^s(p) e^{ip\cdot x} \right\},$$  (9)

and

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ a_\mu(k) e^{-ik\cdot x} + a_\mu^+(k) e^{ik\cdot x} \right\},$$  (10)

where \(E_p = \sqrt{p^2 + m^2}\) and \(\omega_k = |k|\). Inserting these into (8) results in eight terms which we group according to the positive and negative frequency components of the fields. Each of these pieces will have a time dependence of the form \(e^{i\alpha t}\) where \(\alpha\) involves sums and differences of energy terms. As \(t\) tends to plus or minus infinity, only terms with \(\alpha\) tending to zero can survive and thus contribute to the asymptotic Hamiltonian. After performing the spatial integration, and using the resulting momentum delta function, only terms of the form \(e^{\pm iE(t+k)\pm \pm \omega_k}\) have a large \(t\)-limit; there are four of them. The requirement that \(E_{p+k} = E_p ± \pm \omega_k \approx 0\) can only be met in QED because the photon is massless, in which case it implies that \(\omega_k \approx 0\), i.e., only the infra-red regime contributes to the asymptotic dynamics. From this observation it is straightforward to see that the full interacting Hamiltonian (8) has the same asymptotic limit \([32a]\) as

$$\mathcal{H}_{\text{int}}^\text{as}(t) = -e \int d^3x A_\mu(t, x) J^\mu_{\text{as}}(t, x)$$  (11)

with

$$J^\mu_{\text{as}}(t, x) = \frac{1}{(2\pi)^3} \frac{p^\mu}{E_p} \rho(p) \delta^3 \left( x - \frac{p}{E_p}.t \right).$$  (12)
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The operator content of this current is only contained in the charge density

\[ \rho(p) = \sum_s (b^\dagger(p,s)b(p,s) - d^\dagger(p,s)d(p,s)) \]  

(13)

which implied that the asymptotic current satisfies the trivial space-time commutator relation

\[ [J^\mu_{as}(x), J^\nu_{as}(y)] = 0. \]  

(14)

As such, this current can be interpreted as effectively as the integral over all momenta of the current associated with a charged particle moving with velocity \( p^\mu / E_p \). Such a current does not vanish as \( t \to \infty \). We thus see that the asymptotic dynamics of QED is not that of a free theory.

The non-triviality of the asymptotic dynamics dramatically alters the form of the in- and out-matter states. In particular, their propagator no longer has a pole-like structure, but instead behaves near its mass-shell like \( (p^2 - m^2)^{\beta-1} \) where the exponent \( \beta \) is gauge dependent; in the Lorentz class of gauges it is given by \[ \beta = \frac{e^2}{8\pi^2}(\xi - 3). \]  

(15)

It is this observation that lies at the heart of the statement [4] that there is no relativistic concept of a charged particle.

2.2. Charges as dressed matter

The persistence of the asymptotic dynamics in QED means that we cannot set the electromagnetic coupling to zero for the in-coming and out-going particle. As an immediate consequence of this we see that the matter field, \( \psi(x) \), cannot be viewed as the field which creates or annihilates charges since it is not gauge invariant in the remote past or future. An equivalent statement of this fact is that the matter field, \( \psi(x) \), does not satisfy Gauss’ law at any time.

To construct a charged field we need to be able to find a functional of the fields, \( h^{-1}(x) \), such that, under a gauge transformation described by the group element \( U(x) = e^{i\phi(x)} \), we have

\[ h^{-1}(x) \to h^{-1}(x)U(x). \]  

(16)

Then, from (1) the charged field will be given by the gauge invariant product

\[ \Psi(x) = h^{-1}(x)\psi(x). \]  

(17)

It is this product which makes precise how we associate charges with dressed matter: a (chargeless) functional of the fields which transforms as in (16) is what we mean by a dressing; the product (17) we then identify as charged matter. We continue to refer to the matter terms that enter directly into the Lagrangian simply as matter.

There are many distinct ways to construct dressings, reflecting the fact that our identification (17) of charged matter should be viewed as a minimal requirement. In specific applications the form of the dressing must be tailored to the physics at hand: charges that enter into bound states will have a very different structure to those that describe a particle. In our programme to construct charges explicitly we have so far dealt
exclusively with the construction of charged particles [33a]: it is an open and immensely interesting problem to extend our approach to bound states.

In order for the charged matter to describe a particle, we demand that the dressing is such that the charged field creates a state which is an eigenstate of the energy-momentum tensor, i.e., that it has a sharp momentum. For the matter field, this was not possible due to the infra-red problem [3]. For our charged field, though, we shall see that the form of the dressing can be chosen so that there are no infra-red divergences.

The interacting Hamiltonian (8) is derived from the coupling term in the matter part of the Lagrangian density for QED:

\[ i \bar{\psi}(x) \gamma^\mu (\partial_\mu - i e A_\mu(x)) \psi(x). \] (18)

We can rewrite this in terms of the physical charged fields \( \Psi(x) \) as

\[ i \bar{\Psi}(x) \gamma^\mu (\partial_\mu - i e A^h_\mu(x)) \Psi(x). \] (19)

where

\[ A^h_\mu = h^{-1} A_\mu h + \frac{1}{ie} \partial_\mu (h^{-1}) h, \] (20)

which we recognise as a (field dependent) gauge transformation [35a] of the vector potential. Thus, written in terms of the charged fields, the interacting Hamiltonian is

\[ H_{\text{int}}(t) = -e \int d^3x A^h_\mu(t,x) J^\mu(t,x), \] (21)

which asymptotically becomes

\[ H_{\text{int}}(t) \rightarrow -e \int d^3x A^h_\mu(t,x) J^\mu_{\text{as}}(t,x) \] (22)

\[ = -e \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{A^h_\mu(t,x)p^\mu}{E_p} \rho(p) \delta^3 \left( x - \frac{p}{E_p} t \right). \] (23)

This would vanish if we could construct the dressing such that \( A^h_\mu(t,x)p^\mu = 0 \).

To investigate the extent to which this asymptotic interaction can vanish, we note that the momentum, \( p^\mu \), is an arbitrary on-shell, four vector. Now through a gauge transformation, such an algebraic condition can be imposed on the vector potential at one point in the mass shell, but not for the whole mass shell. So we cannot expect to be able to construct a dressing such that the asymptotic interaction Hamiltonian (22) vanishes for the whole mass-shell. However, we can insist that our charged field creates an in-coming or out-going particle with a definite momentum.

From this discussion we see that if we want the charged field (17) to asymptotically create or annihilate a charged particle moving with four-velocity \( u^\mu \), then the dressing must satisfy the additional kinematical condition that

\[ u^\mu A^h_\mu(t,x) = 0. \] (24)

This will then ensure that, at the point in the mass-shell where \( p^\mu = mu^\mu \), the asymptotic interaction Hamiltonian (22) vanishes and thus the state created by the field will have the appropriate sharp momentum.
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What this means in practical terms is that each charged particle must be constructed out of the matter fields with a different dressing—reflecting the velocity of the particle concerned. At first sight this looks very peculiar: we are used to particles being put on-shell as the mass of a particle is a well defined quantum number, but a particle’s velocity is not usually thought of as a quantum number of the system. For charged particles, though, velocity is a well defined quantum number. In order not to interrupt our account of how to construct charges we shall postpone a discussion of this interesting point until §3 where the connection with the heavy charge sector will also be discussed.

2.3 Construction of charges

In order to construct an asymptotic charged particle moving with four-velocity $u^\mu$, we need to be able to find a dressing field $h^{-1}$ such that (16) and (24) hold. The dressing equation (24) is, for this purpose, best written as

$$ (\eta + v)^\mu \partial_\mu (h^{-1}) = -ieh^{-1}(\eta + v)^\mu A_\mu, \quad (25) $$

where we have written the four-velocity $u^\mu$ as $\gamma(\eta + v)^\mu$ where $\eta^\mu$ is the time-like vector $(1, 0)$, $v^\mu$ is the space-like vector $(0, v)$ with $v$ the three-velocity of the charged particle we wish to construct and $\gamma$ is just the standard relativistic factor $(1 - v^2)^{-1/2}$.

To see how to construct the dressing, we first restrict ourselves to lowest order in the coupling. That is, we take $h^{-1}(x) = 1 - ieR(x)$ and, to this order, the dressing equation is

$$ (\eta + v)^\mu \partial_\mu R(x) = (\eta + v)^\mu A_\mu(x). \quad (26) $$

This equation can now be easily solved in terms of an integral along the world line of a particle moving with the given velocity:

$$ R(x) = \int_a^x (\eta + v)^\mu A_\mu(x_t)dt + \chi(x_a). \quad (27) $$

In this expression we have, for convenience, parameterized the world line not by the proper time but by $x_t^\mu = x^\mu + (t - x^0)(\eta + v)^\mu$ and we have introduced an arbitrary reference time $a$. The term $\chi(x_a)$ is, for the moment, an unspecified field configuration in the kernel of the differential operator $(\eta + v)^\mu \partial_\mu$. It is useful here to group the $a$-dependent terms together under the integral, and write $R(x)$ as

$$ R(x) = \int_a^x \left[(\eta + v)^\mu A_\mu(x_t) - \frac{d\chi}{dt}(x_t)\right]dt + \chi(x) = \int_a^x (\eta + v)^\mu \left(A_\mu(x_t) - \frac{\partial\chi}{\partial x_t^0}(x_t)\right)dt + \chi(x). \quad (28) $$

In addition to the dressing equation, we also need to ensure the correct gauge transformation properties for the dressing. To this order in the coupling, this means that under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \theta$, we must have

$$ R(x) \rightarrow R(x) + \theta(x). \quad (29) $$
This, in turn, implies that
\[ \chi(x) \rightarrow \chi(x) + \theta(x). \]  

(30)

This is precisely the type of transformation that Dirac investigated in (6) and we would be tempted to write
\[ \chi(x) = \int d^4z f^\mu(x - z)A_\mu(z), \]

(31)

where \( f^\mu(x - z) \) satisfies \( \partial_\mu f^\mu(x - z) = \delta^\mu(x - z) \). But this only implies (30) if no surface terms arise when we integrate by parts. The restriction on the local gauge transformations to those that vanish at spatial infinity is quite natural as finite energy restrictions impose a \( 1/r \) fall-off on the potential. However, no such restriction exists on the fields at temporal infinity. Thus the form of \( f^\mu(x - z) \) must be such that no such surface terms arise.

As it stands, we can only infer from this that \( f^0(x - z) \) should be zero outside of some bounded region in the \( z^0 \)-direction. To get more from this, we make the ansatz that in fact
\[ \chi(x) = \int d^4z G(x - z)G^\mu A_\mu(z), \]

(32)

where \( G^\mu \) is a first order differential operator and \( G \cdot \partial G(x - z) = \delta(x - z) \). In order to avoid the surface terms that would obstruct the gauge transformation properties of the dressing, we must have that the operator \( G \cdot \partial \) cannot involve any time derivatives. Given this restriction, we see that \( G(x - z) = \delta(x^0 - z^0)G(x - z) \). We shall, for convenience, write \( \chi \) as
\[ \chi(x) = \frac{G \cdot A}{G \cdot \partial}(x). \]

(33)

The temporal parameter \( \alpha \) which enters into the form of the dressing (27) has been introduced by hand and thus should not directly affect any physical results. Taking the derivative of (28) with respect to \( \alpha \) yields the gauge invariant term
\[ -(\eta + v)^\mu (A_\mu(x_\alpha) - \partial_\alpha \chi(x_\alpha)) = -\frac{G^\mu A_\mu}{G \cdot \partial}(x_\alpha). \]

(34)

This we can write in a form where the gauge invariance is manifest as
\[ -(\eta + v)^\mu \left( \frac{G^\nu \partial_\nu A_\mu}{G \cdot \partial}(x_\alpha) - \frac{G^\nu \partial_\nu A_\mu}{G \cdot \partial}(x_\alpha) \right) = -(\eta + v)^\mu \frac{G^\mu F_{\nu\mu}}{G \cdot \partial}(x_\alpha). \]

(35)

The only common condition on physical states is that they are annihilated by the \( B \)-field, thus any physical observable must commute with \( B(x) \). The relevant equation of motion that follows from (7) is, to this order, \( \partial^\nu F_{\nu\mu} = \partial_\mu B \). This tells us that, in order for (35) to act trivially on physical states, it must be equal to
\[ (\eta + v)^\mu \frac{\partial^\nu F_{\nu\mu}}{G \cdot \partial}(x_\alpha). \]

(36)

This now allows us to find the form for \( G^\mu \) and hence the dressing to this order.

The first order operator \( G^\mu \) is constructed out of the vectors \( \partial^\mu \), \( \eta^\mu \) and \( v^\mu \) that characterise the theory. The anti-symmetry of the field strength \( F_{\nu\mu} \) in (35) implies that (36) will
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follow if
\[
G' = -\partial^\nu + (\eta + v)^\nu (\alpha (n \cdot \partial) + \beta (v \cdot \partial)).
\]  
(37)

In which case the second order operator \( G \cdot \partial \) is
\[
G \cdot \partial = \nabla^2 + \beta (v \cdot \partial)^2 + (\alpha - 1) (\eta \cdot \partial)^2 + (\alpha + \beta) (v \cdot \partial) (\eta \cdot \partial),
\]
(38)
where \( \nabla = \partial_i \partial_i \). We have seen that it is essential for the gauge invariance of the charges that this operator has no time derivatives in it. This follows only if \( \alpha = 1 \) and \( \beta = 1 \). Hence we see that
\[
G' = (\eta + v)^\mu (n - v) \cdot \partial - \partial^\mu.
\]
(39)
Thus, to the lowest order in the coupling, we have seen that \( h^{-1}(x) = 1 - ieR(x) \) where \( R(x) \) is the sum of two terms:
\[
R(x) = - \int_a^x ds (\eta + v) \frac{\partial^\beta F_{\mu
u}(s, x)}{G \cdot \partial}(x) + G' A_\mu A_\mu(x).
\]
(40)

As will become apparent when we perform perturbative test of our construction, this decomposition of the dressing into two terms reflects the two manifestations of the infrared in QED that were discussed in the introduction: the soft divergence and the phase divergence. Indeed, the first (gauge invariant) part of \( R(x) \) will be shown to be responsible for controlling the phase structure of charged matter; while the second term will be responsible for the soft dynamics of the charged particle.

It is, perhaps, helpful to specialise to the static situation in order to get a better feel for the form of the dressing we have been constructing. In that case \( v = 0 \) and we get
\[
R(x) = - \int_a^x ds \frac{\partial F_{\mu\nu}(s, x)}{\nabla^2} + \frac{\partial A_\mu}{\nabla^2}(x),
\]
(41)
where \( 1/\nabla^2 \) has been defined in (4). The expected spatial non-locality of the charge is now manifest. More generally, the inverse to \( G \cdot \partial \) is given by
\[
\frac{1}{G \cdot \partial} f(t, x) = - \frac{1}{4\pi} \int d^3y \frac{f(t, y)}{||x - y||_v},
\]
(42)
where
\[
\frac{1}{||z||_v} := \frac{1}{2\pi^2} \int d^3k \frac{\delta^{kz}}{V^v \cdot k},
\]
(43)
and \( V^\nu_\mu = (\eta + v)_\mu (\eta - v) \cdot k - k_\mu \).

In order to go beyond this first order result we must take into account the fact that the fields which enter into the dressing are operators, and thus we need to know their commutation properties. Equal time commutators between the fields can be directly read off from the Lagrangian (7). However, the non-locality displayed in (40) means that we need general space-time commutators between the fields. For the fully interacting theory there is no general approach to finding these commutators without first solving the whole theory. Charges as particles, though, only make physical sense in the asymptotic regime governed by the interacting Hamiltonian (11). In this regime, the relevant dynamics of the...
The gauge field is described by the simpler set of equations:

\[ \partial^{\nu} F_{\nu\mu} = \partial_{\mu} B - e J_{\mu}^{as}, \tag{44} \]
\[ \partial^{\nu} A_{\mu} = -\xi B, \tag{45} \]

where the matter coupling is now through the asymptotic current (12). For this theory the space-time commutators can be explicitly constructed, as we will now show.

In Feynman gauge \((\xi = 1)\) the potential satisfies

\[ \Box A_{\mu} = -e J_{\mu}^{as}. \tag{46} \]

The general solution to this is

\[ A_{\mu}(x) = A_{\mu}^{\text{free}}(x) - e \int d^4y D_R(x - y) J_{\mu}^{as}(y), \tag{47} \]

where \(D_R(x - y)\) is the retarded Green's function for the \(\Box\) operator and \(A_{\mu}^{\text{free}}\) is a solution of the homogeneous equation \(\Box A_{\mu}^{\text{free}} = 0\). From the form of this solution for the potential, and given the triviality of the asymptotic current commutations (14), the space-time commutator for the potential in this interacting theory are seen to be the same as that for the free theory.

The free field equation \(\Box A_{\mu} = 0\) implies that

\[ A_{\mu}(x) = \int d^3z (\partial_{0}^2 D(x - z) A_{\mu}(z) - D(x - z) \partial_{0} A_{\mu}(z)) \tag{48} \]

where \(D(x - z)\) is the commutator function for free fields:

\[ D(x - y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} e^{ik(x - y)} \sin(\omega_k(x^0 - y^0)). \tag{49} \]

The identification in (48) is made by first observing that the right hand side is independent of \(z^0\): setting \(z^0 = x^0\) then implies the result. Exploiting this \(z^0\)-independence, the commutator \([A_{\mu}(x), A_{\nu}(y)]\) is simply calculated by using (48) with \(z^0 = y^0\). Then the equal time commutation relations \([A_{\nu}(y), A_{\nu}(z)]_{et} = -i\eta_{\mu\nu}\delta(y - z)\) can be used.

This results in the space-time commutators in Feynman gauge being:

\[ [A_{\mu}(x), A_{\nu}(y)] = -ig_{\mu\nu} D(x - y). \tag{50} \]

Using this, the first order result (40) can be exponentiated (while still preserving the \(a\)-independence of the construction on physical states) to give

\[ h^{-1}(x) = e^{-iaK(x)} e^{-iex(x)}, \tag{51} \]

where

\[ x(x) = \frac{G_{\mu} A_{\mu}}{G \cdot \partial}(x), \tag{52} \]

and

\[ K(x) = -\int_{-\infty}^{x^0} ds(\eta + v)^{\mu} \frac{\partial^{\nu} F_{\nu\mu}}{G \cdot \partial}(x_s) + e \int d\eta \frac{J_{\mu}^{as}}{G \cdot \partial}(x_s) - \frac{1}{2} e\gamma^{-1} u \cdot x \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - (k \cdot v)^2}. \tag{53} \]

\[ \frac{1}{2} e\gamma^{-1} u \cdot x \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 - (k \cdot v)^2}. \tag{54} \]
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These results are, by construction, independent of $a$. Setting $a = -\infty$ gives (modulo the field independent, tadpole term)

$$K(x) = -\int_{-\infty}^{\eta} d\eta (\eta + v)^{\mu} \nabla F_{\mu \nu} (x),$$

(55)

which is a form adapted to performing perturbative calculations. The full derivation of these results will be presented elsewhere.

In summary, we have seen that charged fields corresponding to charged particles moving with momentum, $p^\mu = mv^\mu$ may be described by

$$\Psi_p(x) = e^{-ieK_p(x)} e^{-ieX_p(x)} \psi(x),$$

(56)

where the dressing satisfies both the gauge transformation property (16) and the dressing equation (25) with the four velocity appropriate for the particular momentum $p$. Our construction has shown that the on-shell Green's functions of these dressed fields, taken at the correct points on their mass shells, should not suffer from any infra-red divergences. Such formal arguments must be tested: in §4 we will show that this prediction [5] is true to all orders in perturbation theory [36].

We now conclude this construction of charges in QED with an observation and a comment. Note that in the static limit the charge field is given by

$$\psi_D(x) = e^{ie\int_{-\infty}^{x} ds \frac{\partial F_{\mu \nu}}{\nabla^2} (s, x)} \psi(x),$$

(57)

where $\psi_D$ was Dirac’s proposal (3) for a static charged field based on the form of the electric field produced by such a charge. The additional term in (57), that follows from the more fundamental dressing equation approach presented here, does not alter the form of the electromagnetic field and was thus missed by Dirac. Its role in the cancellation of the phase divergence will be presented in §4.

The construction of the charge above has been done in Feynman gauge. It is essential to verify that the results hold for the full Lorentz class of gauges. While it is true that the very construction of the charges (17) ensures gauge invariance, it is not immediately clear that this is equivalent to the independence of the construction from the gauge fixing parameter $\xi$. In particular, changing $\xi$ from one will modify the form of the space-time commutators, (50), and thus could have a potentially non-trivial impact on the construction of the dressing. A full proof of the gauge invariance of these dressed fields will be presented elsewhere. However, in §4, we shall demonstrate perturbatively that the $n$-point Green’s functions constructed out of the charged fields are manifestly $\xi$-independent. Before making these tests, we will now present an alternative approach to the physical restrictions on dressings.

3. Heavy charges

We have seen in the last section that the construction of the dressing for a charged particle depends on the velocity of the charge. In this section we wish to discuss, in more general
terms, the role the velocity of a charged particle plays in the structure of the states of QED. We will then look at heavy charges, where velocity is also singled out, and see how the asymptotic regime, governed by the asymptotic Hamiltonian (11), can be characterised in terms of the mass of the charge. In addition, we will also see a further derivation for the dressing equation (25).

3.1 Moving charges

It is not possible [37] to talk about the space of states of QED without first identifying the observables of the theory and their algebra. The dual requirements of gauge invariance and locality on any physical observable have immediate and quite striking implications for the structure of QED.

A familiar example of this is the fact that charge is superselected: the state space is a direct sum of different charge sectors and the action of any physical observable cannot change the charge. In order to derive this physically important result we recall that Gauss' law (2) implies that on physical (gauge invariant) states there is a precise relation between the electric field and the charge density. In particular, the total charge $Q$ is given on such states by the spatial integral

$$Q = \int d^3 x \, \partial_t E_i(t, x). \quad (58)$$

Using Gauss' theorem we can relate this expression for the charge to a surface integral and, in particular, to one at spatial infinity. That is, if we define the electric flux in the direction $\hat{x}$ by

$$E_i(\hat{x}) := \lim_{R \to \infty} R^2 E_i(x + R \hat{x}), \quad (59)$$

then the total charge is given by the 'flux at infinity' through

$$Q = \int_{S^2_\infty} \mathbf{s} \cdot \mathbf{E}. \quad (60)$$

The superselection properties of the charge then follow immediately from the obvious fact that any local observable will always commute with the flux at spatial infinity.

This simple argument can be extended to show why velocity plays such a central role in the identification of charged particles. Recall that the electric field of a charge whose present position is $y$ and moving with velocity $v$ is

$$E_i(x) = -\frac{e \gamma}{4\pi \left(\gamma^2 (v \cdot (x - y))^2 + |x - y|^2\right)^{3/2}}. \quad (61)$$

The electric flux at spatial infinity is then

$$E_i(\hat{x}) = -\frac{e \gamma}{4\pi \left(\gamma^2 (v \cdot \hat{x})^2 + 1\right)^{3/2}}. \quad (62)$$

Just as above for the total charge, this flux will commute with all local observables and thus can be used to characterise distinct sectors of the theory. As this flux only depends
on the velocity of the charge we see that, in the asymptotic regime where charged particles exist, their velocity is a well defined quantum number.

Within the class of local observables, this means that velocity is superselected. However, the very non-locality of the construction of the charges implies that we have to extend the algebra of observables to include non-local constructions such as dressings. As we will see in the next section, the perturbative construction of Green's functions for charged matter will involve non-local interactions between different charges and thus allow charges with different velocities to interact. This does not mean, though, that the total charge $Q$ is no-longer superselected: the dressings are all chargeless and hence the interactions they induce commute with the charge.

3.2 Heavy matter

In our discussion of the asymptotic interactions found in QED, we have simply taken the large $t$-limit of the interaction Hamiltonian in the interaction picture. A disadvantage of studying this limit, though, is that it does not supply a description of the scale for the onset of the asymptotic dynamics and thus of the domain within which velocity is a valid quantum number. More properly, then, we should investigate the limit as some dimensionless parameter gets large. In QED with massive matter, the natural parameter is the product of $t$ and the mass scale of the theory set by $m$ the lightest mass of the system. In this sense, the dynamics at large time is equivalent to that of a theory with heavy charges. Given the central role of the asymptotic dynamics in our programme, it is important to understand precisely how the asymptotic interactions emerge in the heavy sector, and to understand the significance of the dressing equation for heavy matter.

It is well known [2] that the infra-red structures found in QED are independent of the spin of the charged particles, thus it is instructive to see how the interaction (11) also emerges from the heavy sector of scalar QED. The matter part of the QED Lagrangian is now

$$ (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi. \quad (63) $$

The heavy limit [38] can only be taken at specific points on the mass-shell of the particle. To this end, one introduces the rescaled fields

$$ \tilde{\phi}(x) := \sqrt{2m}e^{imux} \phi(x), \quad (64) $$

where we have chosen a four-velocity $u^\mu (u \cdot u = 1)$ that will ultimately describe the velocity of the heavy charge. In terms of these new fields, the matter part of the Lagrangian becomes

$$ i\tilde{\phi}^\dagger u^\mu D_\mu \tilde{\phi} + \frac{1}{2m} (D_\mu \tilde{\phi})^\dagger (D^\mu \tilde{\phi}). \quad (65) $$

In the large $m$-limit only the first term survives and the equations of motion for the heavy matter become

$$ u^\mu D_\mu \tilde{\phi} = 0. \quad (66) $$
The interaction Hamiltonian is easily identified in this limit and is constructed out of the current

\[ J_\text{heavy}^\mu(x) = u^\mu \phi^\dagger(x) \bar{\phi}(x), \]

which has, as expected, precisely the same form as the asymptotic current (12) for the specific point on the mass-shell described by \( u^\mu \).

Heavy charged matter also needs to be constructed out of the heavy matter through the process of dressing. Thus the heavy scalar charge is given by the gauge invariant field

\[ \tilde{\phi}(x) = h^{-1}(x) \bar{\phi}(x), \]

where the dressing is that appropriate to a charge moving with the given four velocity \( u^\mu \):

\[ u^\mu \partial_\mu(h^{-1}) = -ieh^{-1}u^\mu A_\mu. \]

This dressing equation, in conjunction with the equation of motion for the heavy matter (66) implies that

\[ u^\mu \partial_\mu \tilde{\phi} = 0. \]

That is, the state created by the field \( \varphi = e^{-imu \cdot x} \tilde{\phi} \) is an eigenstate of the momentum operator:

\[ P_\mu \varphi(0) = mu_\mu \varphi(0), \]

i.e., as claimed, it is a particle with momentum \( mu^\mu \).

Having seen that the heavy sector characterises the regime described by the asymptotic Hamiltonian (11), we can now identify the onset of the asymptotic dynamics, and thus the emergence of a charged particle interpretation, with the kinematical regime where the lightest mass \( m \) of the matter dominates the momentum flow in any process. This is precisely the domain of the eikonal approximation in soft photon amplitudes [39] whereby the photon has a spin-independent coupling to the matter, which acts as an external point-like source. It is also the domain where scattering theory applies and we will now test our construction.

### 4. Perturbation theory

Our aim now is to show how we may apply the physical, dressed fields we derived in the last section to remove infra-red divergences already at the level of Green's functions. We will start with a one-loop example, where we will classify the various sorts of mass singularities present in an unbroken abelian gauge theory. Then we will demonstrate [36] their explicit cancellation when physical, dressed fields are used. Finally we will show that this cancellation is in fact general and occurs at all orders. In this review we will solely concern ourselves with the IR-structure and not with the renormalisation of UV-singularities – full calculations of the renormalisation constants associated with the physical propagators of spinor and scalar QED were, however, given in references [40] and [41] respectively. References [2, 42, 39] offer introductions to infra-red divergences in perturbation theory.
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\[ \frac{i}{p^2 - m^2}, \quad \frac{-ig_{\mu\nu}}{k^2}, \quad \frac{q}{p = ie(p + q)^\mu} \]

Figure 1. The Feynman rules.

4.1 One loop calculations

The full spectrum of infra-red divergences that characterise gauge theories where the charged particles are massive all occur in pair creation. We thus start by briefly reviewing this process for undressed matter fields. We will use the theory of scalar QED since these divergences are insensitive to the spin of the particles and we may so avoid the technical complications of Diracology. The Feynman rules are shown in figure 1.

We have chosen to work in Feynman gauge, but the physical results will be shown to be gauge invariant below. The seagull (\(\phi^*\phi AA\)) vertex does not alter the spin independent IR-behaviour and is therefore irrelevant to our purpose.

4.1.1 One-loop pair creation: Consider now pair creation from a classical source [42a] as shown in figure 2.

The vertex function corresponding to figure 2 is given in Feynman gauge by

\[
\Gamma = \frac{-ie^2}{[p^2 - m^2][q^2 - m^2]} \int \frac{d^4k}{(2\pi)^4} \frac{(2q + k)^\mu(2p - k)^\nu}{[(q + k)^2 - m^2 + ie][(p - k)^2 - m^2 + ie]k^2 + ie} \cdot g_{\mu\nu}.
\]

Naive power counting tells us that this does not have an infra-red divergence when the outgoing particles are off-shell. However, if we extract a simple pole for each of our outgoing particles, then the on-shell residue may be seen to be IR-divergent

\[
\Gamma_{IR} = \frac{ie^2}{[p^2 - m^2][q^2 - m^2]} \int \frac{d^4k}{(2\pi)^4} \frac{q^\mu p^\nu}{(q \cdot k + ie)(p \cdot k - ie)k^2 + ie} \cdot g_{\mu\nu}.
\]

(72)

where we have dropped higher powers of \(k\) as they do not lead to IR-divergences (this means that we are henceforth restricting ourselves to loop momenta smaller than some cutoff). The easiest way to calculate these divergences is to perform the \(k_0\) integral using

Figure 2. Covariant pair production diagram.
Cauchy’s theorem. The four poles are:

\[ k_0 = \pm |k| \mp i\epsilon, \quad k_0 = q \cdot k/q_0 - i\epsilon, \quad k_0 = p \cdot k/p_0 + i\epsilon. \tag{74} \]

The results of the two different sets of poles have different physical interpretations. To see this distinction, let us take them one at a time.

The contribution from the \( k^2 \) poles to (73) is easily found to be

\[ \frac{e^2}{(p^2 - m^2)(q^2 - m^2)} \cdot \frac{1}{8\pi^2 |v_{rel}|} \int \frac{d|k|}{|k|} \log \left( \frac{1 + |v_{rel}|}{1 - |v_{rel}|} \right) \int |u| \cdot d|k|. \tag{75} \]

where \( v_{rel} \) is the relative velocity between the two charged particles [2]. In (75) there is a logarithmic divergence in the small \(|k|\) limit, i.e., there is a divergent contribution from soft virtual photons. This is called a soft divergence.

The other poles are often neglected as they correspond to divergences in the (unobservable) phase of the Green’s function, which cancel in cross-sections. These structures yield in (73)

\[ \frac{-e^2}{(p^2 - m^2)(q^2 - m^2)} \cdot \frac{1}{4\pi} \int_{-1}^{1} \frac{du}{|v_{rel}|u - i\epsilon} \int \frac{d|k|}{|k|}. \tag{76} \]

An IR-divergence is again visible. To perform the \( u \)-integral we can now employ the relation

\[ \frac{1}{u - i\epsilon} = \text{PV} \frac{1}{u} + i\pi\delta(u), \tag{77} \]

where \( \text{PV} \) denotes the principle value. Only the last term in (77) contributes as the other leads to an odd integral in \( u \). We so obtain

\[ \frac{-ie^2}{(p^2 - m^2)(q^2 - m^2)} \cdot \frac{1}{4\pi |v_{rel}|} \int \frac{d|k|}{|k|}. \tag{78} \]

The extra factor of \( i \) here, compared to (75), betrays the fact that this divergence occurs in the (unobservable) phase. Such singularities are called phase divergences. (Note that this is why we chose a pair creation process: if we had taken a scattering process then both of the poles in \( p_i \cdot k \) would be in the same half-plane and could be neglected.) Now we want to demonstrate that the use of the correct, gauge invariant fields removes both of these divergences.

4.1.2 Perturbation theory with dressed fields: Using dressed fields means calculating Green’s functions of the fields given in (56). Since the dressings explicitly depend on the coupling, \( e \), we must also expand the dressings as well as including the usual interaction vertices. Thus we introduce new vertices and hence new diagrams. The diagrammatic rules for dressed Green’s functions are then just the standard ones augmented by the extra vertices which come from expanding the dressings in powers of the coupling. The two factors in the dressing each yield a different vertex structure. The Feynmann rules corresponding to the dressings are given in figure 3.

The dressings are, of course, dependent upon the momentum of the particle being dressed and so here we have defined

\[ V^\mu_p := (\eta + v)^\mu(\eta - v) \cdot k - k^\mu, \quad W^\mu_p := \frac{(\eta + v) \cdot kk^\mu - (\eta + v)^\mu k^2}{k \cdot (\eta + v)} \tag{79} \]

where \( v = (0, v) \) is the velocity of a particle with momentum \( p = mv\gamma(1, v) \).
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\[ p - k = \frac{eV_p^\mu}{V_p \cdot k}, \quad p - k = \frac{eW_p^\mu}{V_p \cdot k}. \]

**Figure 3.** The Feynman rules from expanding the dressings. The first vertex comes from the soft (\(\chi\)) part of the dressing, and the latter corresponds to the phase (\(K\)) term.

**Figure 4.** All one-loop Feynman diagrams in the pair creation process which contain IR-divergences. Diagrams (a)–(c) are covariant; (d)–(f) involve the perturbative expansion of the \(\chi\) term in dressing; (g)–(i) comes from expanding the \(K\) term; finally the diagrams (j) and (k) are cross terms from expanding both dressing structures.
Figure 5. Classes of diagrams which do not contribute IR-divergences to the residue. The square vertex here and below signifies that the generic contributions of both parts of the dressing are meant.

Since the dressed fields are, by construction, gauge invariant, so are their (connected) Green's functions. In the connected vertex function, there are two further diagrams with extra interaction vertices, figures 4(b–c). The remaining diagrams of figure 4 come from expanding both parts of the dressing.

Our procedure is to extract the IR-divergences in the on-shell residue after we have taken out a pole for each external leg. For simplicity we work in Feynman gauge, however, the gauge invariance of our dressed Green's functions will be fully apparent in our final results. We will employ dimensional regularisation since it preserves gauge invariance. We may neglect the sorts of diagram shown in figure 5: those of type (a) are massless tadpoles (and so vanish in dimensional regularisation), type (b) do not yield IR-divergences when we extract the poles, which reflects the spin independence of the infrared structure, and diagrams of type (c) are also found not to yield IR-divergences in the residue.

Let us now consider the contribution from the χ part of the dressing. Since the other term in the dressing is itself gauge invariant, this part, taken together with the covariant diagrams, must be gauge invariant. The diagrams 4(e–f) have two poles already, but figure 4(d) appears not to have any poles; it is, however, already IR-divergent even off-shell. This has nothing to do with the usual IR-divergences which, we recall, arise when we go on-shell. We can, however, extract poles from such diagrams as we now describe.

4.1.3 Factorisation: This sort of off-shell IR-divergence appears when one or more photons are exchanged between two dressings. We will call such diagrams rainbow diagrams. When we extract poles associated with the external legs, we find the residue has an on-shell IR-divergence. Together with the other diagrams of figure 4, the divergences from the rainbow diagrams will then (if our predictions of the IR-finiteness of the physical Green’s functions are correct) cancel the soft and phase divergences of the covariant (dressing independent) diagrams. This procedure makes much use of the following algebraic identity:

\[
\frac{1}{(p - k)^2 - m^2} = \frac{1}{p^2 - m^2} \left[ 1 + \frac{2p \cdot k - k^2}{(p - k)^2 - m^2} \right].
\]
The integrand from the Feynman rules for figure 4d (may be thus rewritten as we do not write superfluous factors)

\[ \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2} \frac{1}{[(p + k)^2 - m^2 + ie][(q - k)^2 - m^2 + ie]} \]

\[ = \int \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2} \frac{1}{(p + k)^2 - m^2 + ie} \frac{1}{(q - k)^2 - m^2 + ie} \left[ -\frac{2p \cdot k + k^2}{(p + k)^2 - m^2 + ie} \right]. \]  

(81)

The first term in the square bracket here is the only relevant one. Before we consider it, let us show that the second term does not contribute to the residue: extracting the pole in \(1/(q^2 - m^2)\) we obtain from this term

\[ \frac{1}{[p^2 - m^2][q^2 - m^2]} \int \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2} \frac{k^2 - 2p \cdot k}{(p + k)^2 - m^2 + ie} \]

\[ \times \left[ 1 + \frac{2q \cdot k - k^2}{(q - k)^2 - m^2 + ie} \right]. \]  

(82)

We have extracted the two poles and may now go on-shell: but the square bracket here, and hence the contribution of this second term, vanishes on-shell. Returning now to (81) the contribution of the first term is easily found to be

\[ \frac{1}{[p^2 - m^2][q^2 - m^2]} \int \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2} \left[ 1 + \frac{2q \cdot k - k^2}{(q - k)^2 - m^2 + ie} \right]. \]  

(83)

However, the one in the square bracket here is just a massless tadpole and thus vanishes. On-shell the second term becomes \(-1\) and we obtain the final result that, as far as IR-divergent terms in the residue are concerned, we have

\[ \frac{1}{[p^2 - m^2][q^2 - m^2]} \times (-1) \int \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2}, \]  

(84)

i.e., the rainbow line has been stripped off the original integral to yield a factor of

\[ C_{pq} = -\int \frac{d^4k}{(2\pi)^4} \frac{V_p \cdot V_q}{V_p \cdot k V_q \cdot k k^2}, \]  

(85)

times the Feynman rules for the diagram without this line. It may be demonstrated that this property of the factorisation of rainbow lines is completely general. For a diagram with \(n\) rainbow lines, as far as the soft divergences in the on-shell residue of the poles are concerned, the rainbow lines may be stripped off and replaced by \((C_{pq})^n\) A diagrammatic proof of the factorisation property will be published elsewhere. We will make extensive use of this property in the all orders proof below.
4.1.4 Tadpoles: With the factorisation property we are able to calculate all of the one-loop diagrams. However, it seems worthwhile to first remark on the identification of IR- and UV-singularities since, as is well known, in dimensional regularisation we can exchange IR- and UV-divergences because massless tadpoles vanish in this scheme. We have used this cancellation above in (83). Using this identity the badly defined, soft, off-shell divergences were removed and replaced by terms which yield IR divergences only when we went on-shell. These on-shell singularities, as we will soon show, help to cancel the soft on-shell divergences of the other diagrams. This replacement of ill-defined, off-shell singularities in rainbow diagrams is the only place in our calculations where we drop tadpoles. At no stage do we alter structures which first develop soft or phase divergences only on-shell.

However, there are other massless tadpole diagrams associated with dressed Green’s functions, as well as the tadpoles which we dropped above in eq. (83). Let us now consider their interplay—firstly for the dressed propagator. Figure 6 shows the generic diagrams which enter here:

Here there are massless tadpole diagrams, figure 6(d), and rainbow diagrams, figure 6(c). The propagator has, of course, no phase divergence, so let us solely consider the contribution of the soft part of the dressing here [42b]. Using (80) to strip off the rainbow line we obtain from this diagram (for the soft part of the dressing) the massless tadpole integral

\[ \frac{ie^2}{p^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{V_p^2}{(V_p \cdot k)^2 - k^2}, \]

which we can drop in dimensional regularisation. However, the contribution of the two massless tadpole diagrams, of the type shown in figure 6(d), is rapidly found from the Feynman rules to be equal to (86) but with the opposite sign and so cancels it exactly [42c]. The remaining integrals conspire to yield an infra-red finite propagator as has been shown elsewhere in detail [40,41].

This exact cancellation does not carry through to higher Green’s functions. In general a gauge invariant subset of tadpoles survives. This appears to be a subtlety concerned with taking products of distributions. It has been suggested that such problems may perhaps be removed by using the Hertz formulation of QED [17]. The removal of these tadpoles in a general fashion is a worthy object for further study, here we content ourselves with noting that they vanish in dimensional regularisation and that we do not drop tadpoles connected with on-shell infra-red divergences.

4.1.5 Cancellation of the soft and phase divergences: Putting together the one-loop diagrams relevant to the soft structure, i.e., figures 4 (a–f), extracting a pole for each of
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the external legs and going on-shell, we obtain the following terms in the residue which are by power counting IR-divergent:

\[ I_{\text{IR}} = \int \frac{d^4k}{(2\pi)^4} \left\{ \left( \frac{p^\mu}{p \cdot k} - \frac{V_q^\mu}{V_q \cdot k} \right) \frac{g_{\mu\nu}}{k^2} \left( \frac{V_p^\nu}{V_p \cdot k} - \frac{q^\nu}{q \cdot k} \right) 
- \left( \frac{p^\mu}{p \cdot k} - \frac{q^\mu}{q \cdot k} \right) \frac{g_{\mu\nu}}{k^2} \left( \frac{p^\nu}{p \cdot k} - \frac{q^\nu}{q \cdot k} \right) \right\}, \]

where we have removed a factor of \( ie^2/[(p^2 - m^2)(q^2 - m^2)] \). This is a gauge invariant result: replacing the Feynman gauge propagator by any more general form necessarily brings in a factor of \( k^\mu \) or \( k^\nu \) which vanishes on multiplication into the brackets in (87). Similar gauge invariant structures are obtained in the study of the dressed propagator and in other physical vertex functions. We now have to show that the soft divergences seemingly apparent in (87) in fact cancel.

The cancellation may be demonstrated by performing the integrals explicitly, using the methods outlined above. However, it is simpler to realise that

\[ \frac{V_p^\mu}{V_p \cdot k} = \frac{(\eta + v)^\mu(\eta - v) \cdot k - k^\mu}{k^2 - (k \cdot v)^2}, \]

which, at the soft pole, \( k^2 = 0 \), we may write as

\[ \frac{V_p^\mu}{V_p \cdot k} = \frac{(\eta + v)^\mu(\eta - v) \cdot k - k^\mu}{(k \cdot \eta)^2 - (k \cdot v)^2}. \]

We may drop the \( k^\mu \) term in the numerator here (cf., the argument for the gauge invariance of eq. 87). So effectively we have

\[ \frac{V_p^\mu}{V_p \cdot k} = \frac{(\eta + v)^\mu(\eta - v) \cdot k}{(k \cdot \eta)^2 - (k \cdot v)^2} = \frac{(\eta + v)^\mu}{(\eta + v) \cdot k} = \frac{p^\mu}{p \cdot k}, \]

where it is important to note that we have taken \( p \) to be at the correct point on the mass shell, \( p = m\gamma(\eta + v) \), which defines the appropriate dressing. (If we were to go on-shell at a different point, this equality would not hold and the soft divergences do not cancel.)

In all cases of gauge invariant Green's functions, we are thus able to replace

\[ \frac{V_p^\mu}{V_p \cdot k} = \frac{p^\mu}{p \cdot k}, \quad \text{and} \quad \frac{V_p^\mu}{V_p \cdot k} = \frac{q^\mu}{p \cdot k}, \]

as far as the soft divergences are concerned. This equality makes the cancellation of the soft divergences in (87) and other dressed Green's functions immediately apparent.

Up till now we have not included the other gauge invariant part of the dressing. It is straightforward to check that the contribution of this structure to the pair creation process is gauge invariant. It may also be demonstrated that it does not spoil the above cancellation of soft divergences. (Some of the individual diagrams which involve this part of the dressing, figures 4(g–k), contain soft divergences, but overall they cancel.) The role of this structure is, we expect, to cancel the phase divergence of figure 4(a). The only phase divergent contribution from the entire dressing comes from a single diagram
involving only the $K$-structure: figure 4(g). From the Feynman rules we obtain from this figure after factorisation

$$\int \frac{d^4k}{(2\pi)^4} W^\mu_p W^\nu_q g_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{(\eta + v) \cdot (\eta + v') k^2}{V_p \cdot k V_q \cdot (\eta + v) \cdot V_k \cdot V_k} \cdot \frac{1}{(2\pi)^4} \frac{1}{V_p \cdot k V_q \cdot k}$$

(92)

where we have not written the two poles $-ie^2/([p^2 - m^2][q^2 - m^2])$ and have thrown away terms which do not yield phase divergences. This result may be readily seen to cancel exactly with (78) when we go on-shell at the right point.

To summarise the one-loop results: the dressed Green’s functions are, by construction, gauge invariant. If we go on-shell at the points on the mass shells such that the momenta of the external lines and the velocities that define the various dressings agree, then the soft and phase divergences cancel. We now want to show that this holds at all orders.

4.2 All orders

The physical dressed propagator was shown elsewhere [40, 41] to be IR-finite at one loop. However, we can show that this holds at all orders through a slight extension of the work of Jackiw and Soloviev [43].

These authors used low energy theorems and spectral representations to study the behaviour of the propagator around the pole, $p^2 = m^2$, in various gauges. In covariant gauges they regained the usual result (15), while in non-covariant gauges there is a pole if the following integral vanishes (this is their eq. 3.41)

$$F_R = \frac{e^2}{(2\pi)^3} \int k e^{ik'k} \theta(k_0) \delta(k^2) \frac{r^\mu r^\nu}{(r \cdot k)^2} \Pi_R^\mu \nu$$

(93)

here $r$ corresponds to the momentum of the matter field and $\Pi$ is the photon propagator up to a factor of $i/k^2$. Now our dressed propagator, for a charge with momentum $r = m\gamma(\eta, v)$ corresponds to the usual matter propagator in the gauge where

$$\Pi_R^\mu \nu = -g^\mu \nu + \frac{k^\mu (\eta + v)^\nu + (\eta + v)^\mu k^\nu}{k \cdot (\eta + v)} - \frac{k^\mu k^\nu}{(k \cdot (\eta + v))^2} [V_r^2 + 2 V_r \cdot k].$$

(94)

taking the $\delta(k^2)$ factor of the integrand into account, it is easy to see that at the right point on the mass shell (93) vanishes and the propagator has a pole. This was pointed out explicitly for the static case (Coulomb gauge) in ref. [43].

4.2.1 The dressed propagator: Since we now know that the dressed propagator has a pole, we see that the wave function renormalisation constants are IR-finite. We can therefore multiply the dressed Green’s functions by them without introducing new IR-divergences. We will use this freedom to show the cancellation of soft and phase singularities in dressed Green’s functions at all orders. Our approach is diagrammatic. First we shall consider the types of diagrams which can introduce IR-divergences in the propagator, in this way we can describe the structure of the wave function renormalisation constants.
The non-dressed propagator, i.e., just the covariant diagrams, can be represented at all orders in the following way:

\[
\frac{i}{p^2 - m^2 - \sum}.
\]

When we take dressings into account, these chains must be supplemented by various possible structures which we now enumerate. The first is

\[
\frac{2i\sum}{p^2 - m^2 - \sum}.
\]

where the 2 accounts for the possibility that dressing corrections, \(\sum\), can be indistinguishably attached at either end in the scalar theory.

There may of course also be such dressing corrections at both ends. This is diagrammatically

\[
\frac{i\sum^2}{p^2 - m^2 - \sum}.
\]

Taking all of these possibilities together we see that these contributions to the propagator have the form

\[
\frac{i(1 + \sum)^2}{p^2 - m^2 - \sum}.
\]  

(95)

Of course we can also exchange one or more photons from one dressing to the next. This can be done for each and every one of the above sets of diagrams. We so obtain the following rainbow diagrams

\[
\frac{i(1 + \sum)^2}{p^2 - m^2 - \sum} \exp(-C_{pq}),
\]

where \(-C_{pq}\) was defined in eq. 85.

Of course there can be other diagrams such as
where the dressing corrections overlap in a non-rainbow form, i.e., diagrams which are not one-particle irreducible even after rainbow lines are stripped off. Such diagrams do not yield poles and so we can drop them here. We neglect loops of matter fields since these also remove the soft divergences. Eq. 96 is therefore our final result for the infra-red divergences in and the pole structure of the physical propagator.

We see that the dressings factorise into a covariant and a non-covariant part. For the full non-covariant wave function renormalisation constants, we thus have the attractive result that

$$\mathcal{Z}_2^p = \mathcal{Z}_2^{\text{cov}}(1 + \tilde{\Sigma})^2 \exp(-C_{pp}).$$

(97)

Although only a partial exponentiation of the dressing contribution is apparent, we know that it is in fact IR-finite at all orders. However, (97) will suffice to show that the pair creation vertex is finite to all orders.

4.2.2 *All orders pair creation:* The general class of diagrams with double poles and possible IR-divergences in the pair creation process is

where we have chosen to multiply the diagrams by an (IR-finite) factor: the inverse square root of the appropriate wave function renormalisation constant (as given by eq. 97) for each of the legs. This is useful as it makes the exponentiation of the infra-red divergences apparent.

The diagrams we have retained are those which can yield IR-divergences, i.e., where there are possible covariant vertex corrections from one leg to another, covariant corrections on the external legs, dressing corrections at the end of the external legs and possible rainbow corrections from one dressing to the other. The use of a black blob, $\bullet$, here denotes that there may or may not be a dressing correction ($\tilde{\Sigma}$) at the ends of one or both of the lines.

Other diagrams (e.g., where a line from a dressing connects to a covariant interaction vertex on the other line) will not give two poles or will be infra-red finite.

We now factorise the rainbow dressings, this means (summing over all possible rainbow lines and including the $1/n!$ symmetry factor for a diagram with $n$ such lines) that we may write

$$\times \exp(-C_{pq}) \frac{1}{\sqrt{\mathcal{Z}_2^p \mathcal{Z}_2^q}}.$$
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Since there may or may not be dressing correction at the end of these lines, we may write these end factors as \((1 + \hat{\Sigma})\). They clearly then cancel the \(\hat{\Sigma}\) dependence from each of the external leg wave function renormalisation factors as given by (97). Diagrammatically we thus have

\[
\times \exp \left(-C_{pq} + \frac{1}{2} C_{pp} + \frac{1}{2} C_{qq}\right) \frac{1}{Z_{\text{cov}}}
\]

and we see that the dressing effects have exponentiated. Since the covariant diagrams which are left over are well known to exponentiate, we see that all the soft and phase effects exponentiate in the residue of the double poles. We have seen that they cancel at one loop, this now trivially implies that the dressed process is IR-finite at all orders at the level of Green's functions. We stress that this holds for both soft and phase divergences. The extension of this approach to scattering and to higher vertices is straightforward.

We have thus seen that the dressings we have introduced in §2 remove soft and phase divergences in massive electrodynamics already at the level of Green's functions and permit a particle interpretation. The immediate question now is whether these perturbative successes can be extended to QCD. In particular, we need to clarify the generalisation to a different class of mass singularities, collinear divergences, which arise in QCD. Since collinear divergences also characterise QED with massless charged particles, the next section is dedicated to a study of that theory.

5. Massless charges

Massless charges are an essential ingredient of the standard model. In QCD the colour charge of the massless gluon results in its self-interaction which in turn leads to asymptotic freedom. In addition, gluonic bremsstrahlung plays a dominant role in jet formation [44]. However, such massless charges are poorly understood, even at the level of cross-sections, since the Block–Nordsieck procedure of summing over degenerate final states fails to yield finite results [45, 46]. Massless charges can be modelled in QED by taking the limit of vanishing electron mass. As one would expect, QED then has many new and unusual features. In particular, massless electron pairs can now be created by expending arbitrarily small amounts of energy. This freedom leads to a new type of singularity, the collinear divergences. In perturbation theory these express themselves through the divergence of on-shell massless propagators. The inverse propagator, \((p - k)^2 - m^2\), vanishes in the \(m \to 0\) limit if \(p\) is on-shell and \(k\) is parallel to \(p\). (Note that \(k\) is not necessarily small). For further details see [2]. The matter pairs, which can now be easily created, screen the initial charge and the effective coupling vanishes [47]. Here we will see that the argument of §2 for the vanishing of the asymptotic interaction Hamiltonian, if correctly dressed matter is used, also applies in the massless limit. Finally we discuss some aspects of the construction of dressings in this limit.
5.1 Collinear asymptotic dynamics

The asymptotic behaviour of the interaction Hamiltonian for massive QED was discussed in §2. In the massless case we will show that the interaction Hamiltonian in the distant past or future has a far richer structure than in the massive case.

The interaction Hamiltonian, $\mathcal{H}_{\text{int}}(t)$, is given by (8), where the conserved matter current and the free field expansions for matter and gauge fields are still those of (9) and (10). The all-important difference is that, since $m = 0$, the energy in the massless case is given by $E_p = |p|$.

When the expansions (9) and (10) are substituted into (8), then, of the eight possible terms, the two involving $a^\mu \bar{u} \gamma^\mu u$ and $a^\dagger_\mu \bar{u} \gamma^\mu v$ will have a time dependence of the form $e^{i\alpha t}$ with $\alpha$ being either positive or negative. These cannot survive for large values of $t$ and so do not appear in the asymptotic limit. This leaves six terms which are potential survivors, of which two vanish in the massive case. A closer look at some of these structures will show how to evaluate all the terms.

The first term to be considered in detail is one of the 'off-diagonal' (in the spinors) terms:

$$-e \int \frac{d^3 x d^3 k d^3 p d^3 q}{(2\pi)^6} \frac{a_\mu(k)}{2\omega_k \sqrt{4E_p E_q}} b^\dagger(q, s) d^\dagger(p, r) \bar{u}^\dagger(q) \gamma^\mu \nu^\dagger(p) e^{-i\omega x} e^{iq x} e^{ip x}.$$  

(98)

Integrating out the $x$-integral will give a term involving $\delta(q + p - k)$. Following this by integrating out the $p$ integral, the expression (98) becomes

$$-e \int \frac{d^3 k d^3 q}{(2\pi)^6} \frac{a_\mu(k)}{2\omega_k \sqrt{4E_p E_{k-q}}} b^\dagger(q, s) d^\dagger(k-q, r) \bar{u}^\dagger(q) \gamma^\mu \nu^\dagger(k-q) e^{i(E_q + E_{k-q} - \omega_k)}.$$  

(99)

If this is to survive as $t \to \infty$ then the coefficient in the exponent must vanish. This is equivalent to demanding that $\omega_k = E_q + E_{k-q}$, or $|k| = |q| + |k - q|$. Since these three vectors represent the three sides of a triangle, plainly a solution to this equation will exist if and only if the vectors $k$ and $q$ are parallel and $|k| \geq |q|$. In this region, where the photon is collinear with the matter field, this contribution to the asymptotic interaction Hamiltonian does not vanish. The momentum, $k$, is no longer restricted to the soft region, $k : 0$, but must be larger than the matter momenta, since this term corresponds to pair creation.

The second term to be examined is one of the 'diagonal' terms and has the form

$$-e \int \frac{d^3 x d^3 k d^3 p d^3 q}{(2\pi)^9} \frac{a_\mu(k)}{2\omega_k \sqrt{4E_p E_q}} d(q, s) d^\dagger(p, r) \bar{u}^\dagger(q) \gamma^\mu \nu^\dagger(p) e^{-i\omega x} e^{-iq x} e^{ip x}.$$  

(100)

Integrating out the $x$-integral again gives a delta function, this time $\delta(p - k - q)$. Integrating out the $q$ integral now yields

$$-e \int \frac{d^3 k d^3 p}{(2\pi)^6} \frac{a_\mu(k)}{2\omega_k \sqrt{4E_p E_{p-k}}} d(p-k, s) d^\dagger(p, r) \bar{u}^\dagger(p-k) \gamma^\mu \nu^\dagger(p) e^{i(E_p - E_{p-k} - \omega_k)}.$$  

(101)
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The exponent in (101) must vanish for large $t$ and this implies that $E_p - E_{p-k} - \omega_k$ must vanish. This is similar to the previous case except that now we must have $|p| = |k| + |p - k|$. Reasoning as before, we find that $p$ must be parallel to $k$ but now $|p| \geq |k|$, i.e., $E_p \geq \omega_k$. This clearly corresponds to photon production.

If the six terms are evaluated using (128) then the final form of the asymptotic interaction Hamiltonian in the massless case is [48]

$$\mathcal{H}_{\text{int}}^{\text{as}}(t) = -e \int \frac{d^3k}{2\omega_k} \left[ a_{\mu}(k) J_{\text{as}}^{\mu}(k,t) + \text{h.c.} \right],$$

(102)

where 'h.c.' denotes the hermitian conjugate of the first term in brackets, and

$$J_{\text{as}}^{\mu}(k,t) = \int \frac{d^3p}{(2\pi)^3 E_p} \frac{p^\mu}{E_p} \left[ \rho_{\text{scatt}}(p,k) + \rho_{\text{prod}}(p,k) \right].$$

(103)

The two structures in the asymptotic current are respectively

$$\rho_{\text{scatt}}(p,k) = \sum_s \left[ b^s(p,s)b(p-k,s) - d^s(p,s)d(p-k,s) \right] e^{it(E_p - E_{p-k} - \omega_k)}$$

(104)

and

$$\rho_{\text{prod}}(p,k) = \sum_s b^s(p,s)d^s(k-p,r)\xi_s^{\dagger}\xi^r e^{it(E_p + E_{p+k} - \omega_k)}$$

(105)

in the region $\omega_k \leq E_p$, and

in the region $\omega_k \geq E_p$. The two terms in (102) have different physical interpretations: the $\rho_{\text{scatt}}$ part corresponds to photon radiation and includes the soft region which was responsible for the non-vanishing of the usual soft asymptotic interaction Hamiltonian, (11). However, the photon momenta is only required to be collinear and not extremely soft. The other term, $\rho_{\text{prod}}$, is completely new and corresponds to the production of massless matter pairs with momenta less than that of the initial photon. Even though photons cannot radiate other photons, these two structures in this model theory already show the basic processes underlying the collinear production of gluons and quark-antiquark pairs in jet creation.

We now note that form the form of (102), and in particular the explicit $p^\mu$ factor in the asymptotic current, the solution to the dressing equation, (24) or (25), will, also here in the massless theory, correspond to a particle since the asymptotic interaction Hamiltonian vanishes at the correct point on the mass shell. We thus predict that the Green's functions of the dressed fields will also be free of collinear divergences.

In the context of this discussion of the asymptotic interaction Hamiltonian, it should be noted that, for massive electrons, if we allow the photon a small mass there is no momentum configuration such that the exponential survives at asymptotic times. In perturbation theory it may be easily seen that a small photon mass regulates the infrared singularities. However, if the mass of the photon becomes large enough to open decay channels into matter pairs, then once again the interaction picture breaks down and this results in apparently gauge dependent S-matrix elements (see, e.g., [49]).

We now turn to the construction of the solutions of the dressing equation in the massless limit.
5.2 Collinear dressings

In the massless theory, the dressing must still fulfill the requirements of gauge invariance (16) and the dressing equation (25), where now \((\eta + v)^2 = 0\). Following the procedure outlined in §2 for solving these equations, we will require the commutators of the potential in the theory described by the asymptotic current, (103). As we saw for the massive case, the potential can now be written as in (47). However, in this massless theory, the asymptotic currents no longer commute and the commutators of the theory are no longer those of the free theory. This makes the construction of the dressing of massless charges somewhat more difficult. Rather than discuss the details of how to extend the analysis of §2, we will now examine the \(v \to 1\) limit of the massive case and then study, in this limit, the route taken by Dirac to the dressings.

The naive massless limit of a dressed charge (56) moving in the \(x^1\)-direction results in integrals of the form

\[
\frac{1}{4\pi} \int d^3z \frac{\partial_2 A_2 + \partial_3 A_3 - E_1(x^0, z)}{|x^1 - z^1|}.
\]

Clearly this is ill-defined, and although \(1/|x^1 - z^1|\) can be defined in terms of generalised functions [50], the serious problems with this naive limit cannot be circumvented in this fashion. Further evidence that a naive approach to the massless theory will not suffice, comes from the detailed perturbative calculation of the one-loop propagator [41, 40]. The wave function renormalisation constants found in these papers diverge as \(v \to 1\).

To obtain a deeper insight into the form of the dressings, we can follow Dirac's lead and construct a dressing that reproduces the electric and magnetic fields for a massless charge. The electric field of an ultra-relativistic charge contracts in the direction of motion. For a massless charge this contraction leads to a singular field configuration. It has been argued that, for motion in the \(x^1\)-direction, the electromagnetic fields are [51, 52]

\[
E_1 = 0, \quad E_i = \frac{-\epsilon \delta(x^0 - x^1)}{2\pi r^2},
\]

\[
B_1 = 0, \quad B_i = \frac{\epsilon \delta(x^0 - x^1)}{2\pi r^2}.
\]

where \(i, j = 2, 3\) and \(r^2 = (x^2)^2 + (x^3)^2 = |x_\perp|^2\). This result should follow from the soft dressing term.

We argue that this part of the dressing should have the form

\[
\chi_1(x^0, x) = \frac{1}{2\pi} \int d^2z^\perp (\partial_2 A_2 + \partial_3 A_3 - E_1)(x^0, x^1, z^\perp) \log |x^\perp - z^\perp| + g(x)
\]

where \(g(x)\) is a function which is harmonic in \(x^\perp\), i.e., \(g(x)\) is in the kernel of the two dimensional Laplacian. Although this is not the naive limit (106), it can be understood as arising from that part of the dressing equation which describes the soft dynamics, \(G \cdot \partial(\chi) = G \cdot A\). For this configuration, the equation becomes

\[
((1 - v^2)\partial_1^2 + \partial_2^2 + \partial_3^2)\chi_1(x^0, x) = ((1 - v^2)\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - vE_1)(x^0, x).
\]
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Now in the $v \to 1$ limit, one sees that the limiting value of $\chi_v$, which we write $\chi_1$, must satisfy

$$(\partial_2^2 + \partial_3^2)\chi_1(x^0, \mathbf{x}) = (\partial_2 A_2 + \partial_3 A_3 - E_1)(x^0, \mathbf{x}),$$

with solution (108). It may be checked that the dressing constructed out of this soft term, with $g(x)$ set to zero, indeed yields the electric and magnetic fields, (107). This result strengthens our claim that a careful analysis of the dressing equation will allow us to construct massless charges.

6. Discussion

In this review we have demonstrated that it is indeed possible to construct relativistic charged particles. Let us now recall what we have seen.

Our starting point was the non-vanishing of the asymptotic interaction Hamiltonian which characterises gauge theories. The misidentification of the asymptotic interactions of the free Hamiltonian with the true asymptotic dynamics of gauge theories causes the infra-red problem. Since the interaction Hamiltonian does not vanish at large times, Gauss' law tells us that charged particles are not just the matter fields of the Lagrangian: physical particles like the electron are always accompanied by an electromagnetic dressing. In practical calculations this expresses itself in the lack of a pole structure in the on-shell Green's functions of the Lagrangian matter fields and in divergences in $S$-matrix elements.

Any physical degree of freedom must be gauge invariant. For charged particles this means that the dressing together with the matter field at its core must be locally gauge invariant—this translates into an equation for the dressing, (16). This requirement, though, does not suffice to construct charged particles: any gauge invariant solution is in principle a physical degree of freedom, but it is not necessarily one that physics chooses to use. To restrict ourselves to the solutions which are physically relevant, we require a second equation, (25). This latter relation was deduced from demanding that the velocity of an incoming or outgoing charged particle is well defined and the dressing must take this into account. The form of the asymptotic interaction Hamiltonian is such that it vanishes in the propagator of a correctly dressed particle, i.e., one which satisfies (16) and (25). It follows that dressed charges are then described by the free Hamiltonian and a relativistic particle description is indeed possible. In §2 dressings which solve these equations were explicitly constructed. These are physical degrees of freedom with a specific physical interpretation. The dressings factorise into two parts, each of which was interpreted as playing a different role in the infra-red physics of QED. Section 3 then illuminated the role of velocity in any description of charged particles and showed how the form of the dressing could be derived from the theory of heavy charges.

These arguments have been tested by explicit perturbative calculations described in §4. The interpretations of the two different terms in the dressing were such that they were expected to each cancel a different type of infra-red divergence. The constructions passed these one-loop tests with flying colours and a gauge invariant pole structure was obtained. This was then generalised to an all orders proof of the cancellation of
infra-red divergences in the on-shell Green's functions of QED with dressed fields. These results give us great confidence in the requirements (16) and (25), in the solutions (56) and also in the interpretation we associate with these physical degrees of freedom.

Finally in §5 collinear divergences were studied in the framework of QED with massless fermions. Here the structure of the interaction Hamiltonian which survives at large times is much richer than in QED with massive matter. However, it still vanishes if correctly dressed matter, satisfying (25), is used. This implies that the on-shell Green's functions of the solutions to (16) and (25) in the massless theory will also be free of all infra-red singularities, including collinear divergences. The construction of these dressings was considered.

6.1 Charges in the standard model

Most of this paper has been given over to QED. The other interactions in the standard model of particle physics are also described by gauge theories, however, the charged particles in these theories are very different to those of the unbroken abelian theory. In this subsection we will sketch some of the most important differences which characterise the charged particles of the weak and strong nuclear forces.

The asymptotic Hamiltonian in QCD also does not reduce to the free one. However, there is a new problem with defining charges in non-abelian gauge theories such as QCD. The colour charge

\[ Q^a = \int d^3x (J_0^a(x) - f_{bc}^{\alpha} E_b^\alpha(x) A_c^\alpha(x)), \]  

(111)
in sharp contrast to that of QED, is not invariant under gauge transformations. It is therefore natural to wonder how we may speak of colour charged particles. But on physical states, where Gauss' law holds, the charge may be written as

\[ Q^a = \frac{1}{g} \int d^3x \partial_i E_i^a(x). \]  

(112)

Using Gauss' theorem it follows that the colour charge expressed in this way is in fact invariant under gauge transformations which reduce, in a directionally independent manner, to elements of the centre of the group at spatial infinity. Thus the concept of colour charge is only meaningful for locally gauge invariant fields and then only if this restriction on the allowed gauge transformations is imposed. Since the Lagrangian matter fields are not gauge invariant, coloured quarks in QCD are necessarily dressed by glue [3, 4, 5].

The dressing equation (25) of QED may be directly generalised [36] to QCD. The non-abelian solutions will be asymptotically described by a free Hamiltonian provided that the field transformations required are, in fact, admissible. To the extent that they may be constructed, it follows that the various mass singularities would cancel at the correct points in the mass shell of the dressed quark fields.

However, there is a fundamental obstruction to the construction of dressed quarks (and gluons) which follows from the nature of non-abelian gauge transformations. Any gauge
invariant description of a quark could be used to construct a gauge fixing. However, the boundary conditions which must be imposed on the allowed gauge transformations in order for colour to be associated with the physical degrees of freedom, are such that the Gribov ambiguity [53,54] holds. There is then a fundamental, non-perturbative limit on the construction of gauge-invariant, coloured charges. Thus the true degrees of freedom in QCD outside of the perturbative domain do not include quarks and gluons. Colourless gauge-invariant fields can of course be constructed, but quarks and gluons are confined.

The picture of confinement which emerges from studying the construction of coloured charges is as follows. When a $Q\bar{Q}$-system is separating, but the matter fields are still at a short distance from each other, the interaction potential is essentially Coulombic and the dynamics is described by perturbative dynamics. The short-distance, Coulombic inter-quark potential may be described using low-order perturbative solutions to the dressing equations for each of the individual quarks. Deviations from this such that a confining potential arises in a $Q\bar{Q}$-system are expected to come from a 'mesonic dressing' which does not factorise into two parts.

Finally, it might be objected that the non-abelian theories which underly the weak interaction do not lead to confinement of weakly charged particles such as the $W$ or indeed the electron. This is easily understood: in spontaneously broken gauge theories we are entitled to use the Higgs sector to dress charged particles [55]. (The ability to choose the unitary gauge circumvents the gauge fixing ambiguity.) In this way gauge invariant solutions corresponding to weakly charged particles may be constructed.

6.2 Open questions

This review has sketched out a systematic approach to the construction of charged particles. The qualitative results to date of the program are fully in accord with phenomenology and the calculational tests to which the charged particles have been subjected have all supported the methods. But there are still a large number of unanswered questions ranging from calculational procedures to the extension of the applicability of this programme to physics at finite temperature and density. We will now conclude by listing some of the most pressing tasks.

(a) The above proof of the infra-red finiteness of the dressed on-shell Green's functions needs to be extended to full calculations of these Green's functions, in particular the UV renormalisation of the $n$-point functions of these composite operators must be performed. The dressed propagators of both fermionic and scalar QED have been carried out at one-loop. This now needs to be extended to higher loops and to vertex functions.

(b) The perturbative tests need to be extended to collinear divergences. Massless QED is the natural testing ground here. First the dressing equation needs to be solved within the framework of the remarks of § 5. Then the solutions of the equivalent dressing equation for QCD should be constructed and tested.

(c) Although a brute force, direct perturbative solution of the dressing equations for the non-abelian theory is feasible at low orders, a systematic and practical approach to the construction of a gauge invariant dressed quark fulfilling the dressing equation of QCD for an arbitrary velocity is needed. In the appendix to ref. [5] an algorithm was
presented with which the calculation of a particular gauge invariant dressed quark solution to any order in perturbation theory became rather simple and some of the first terms in this perturbative construction were given. In this context we also refer to reference [56] where all orders expression apparently corresponding to the gauge invariant extension of this term were presented. This corresponded to the gauge invariant extension of the soft term in the QED dressing of a static charge. Such work needs to be generalised to both terms and indeed to arbitrary velocities. Furthermore the success of phenomenological constituent quark models in describing hadronic structure strongly indicates that, within the overall constraint of confinement, some non-perturbative in-put into the construction of dressed quarks should be possible. Here we are thinking especially of the role of instantons and condensates in chiral symmetry breaking. We stress that it will not be possible to incorporate all the non-perturbative, topological aspects of QCD into dressings of individual quarks or gluons. This, we have argued above, is how confinement makes itself manifest.

(d) The importance of the perturbative chromo-electric and chromo-magnetic dressings is that they determine the short distance interactions between quarks and have implications for jet physics. We would urge, e.g., a comparative study of the distribution of glue in the dressings around quarks and gluons which could shed light on the different development of such jets. The fruitful concept of parton-hadron duality [57] could we suggest be potentially replaced by a refined version of a duality between (perturbatively) gauge invariant, dressed colour charges and the resulting physical hadrons.

(e) The qualitative proof of quark confinement which was described above needs to be quantified. How can the hadronic scale be found in this way? That quarks are not part of the physical degrees of freedom of QCD will only become apparent at larger distances where that part of the non-perturbative dynamics which is sensitive to the Gribov ambiguity becomes first significant and finally dominant. This is, of course, a tough non-perturbative calculation. One natural technique is the lattice, where there has been some work on studying the gauge fixing problem. It will be important here to distinguish between lattice artifacts and the true, physical limitation on gauge fixing [58, 59]. We also suggest here that the construction of dressed charges in monopole backgrounds be studied.

(f) In view of the difficulties inherent in non-perturbative calculations, phenomenological modelling of dressings is desirable. Thus we feel that the construction of dressed quarks with phenomenologically desirable properties, e.g., running masses should be investigated. A corollary of this last point is the construction of mesonic dressings which lead to phenomenological inter-quark potentials. The stability of such model dressings should then be tested using variational methods. We recall from the introduction the instability of the (confined) $e^+e^-$-system where the two matter fields are linked by a string [21, 30].

(g) One of the most important questions in particle physics is how mass is generated. The construction of charges in theories with spontaneous symmetry breaking deserves further, quantitative investigation. Both perturbative and non-perturbative effects should be studied. A major question closely related to this is how one should describe unstable charged particles, such as those occurring in the weak interaction. It is well
known that there are problems with obtaining gauge invariant results in the presence of such fields [49, 60, 61]. We have seen, at the end of § 5.1, the non-vanishing of the asymptotic interaction Hamiltonian in such theories. We urge the development of, in some sense dressed, admixtures of fields which would fulfill the following properties: (i) if the coupling is set by hand to zero it should reduce to the Lagrangian field, (ii) they must be gauge invariant and (iii) the asymptotic interaction Hamiltonian should vanish for the propagator of these constructs.

To what extent can we talk about charged particles beyond the standard model? The construction of descriptions of charged particles in theories such as technicolour and with (broken) supersymmetry could cast light on their experimental signatures. Here it is important to see if the gauge invariant, dressed charged particles are still related by supersymmetries, or if this just holds for the Lagrangian fields. Furthermore we strongly suggest a study of the construction and dynamics of charged particles in unbroken supersymmetric models so as to clarify the role of (supersymmetrically) charged particles in the Seiberg–Witten description of confinement in such theories [62].

Constrained dynamics is the mathematical framework for extracting physical degrees of freedom in theories such as QED and QCD. In modern formulations BRST symmetry is used to single out gauge invariant, local fields. However, as will have become apparent above, gauge invariance is not enough: one must still make a clear identification between the true degrees of freedom and the observed particles. Quark confinement in QCD is merely the most obvious example; even in quantum electrodynamics the infra-red problem, and the associated superselection structures labelling charges with different velocities, show that this identification is not direct. The interplay between the asymptotic interaction Hamiltonian and the physical observables needs further study. It should be noted that the BRST method is a local construction and, as we have seen, charged fields are necessarily non-local. In this context we point out a further symmetry of QED which was noted in [63] and which singled out a subset of gauge invariant fields. We believe that this is only one representative of a class of symmetries with whose aid the true degrees of freedom may be interpreted. Finally, the incorporation of non-perturbative effects is the outstanding question in mathematical physics today. In constrained dynamics this entails extending the usual reduction procedure to theories with superselection sectors characterised by the existence of non-trivial surface terms and other global structures. A proper understanding of these effects is essential if constrained dynamics is to be able to discuss non-perturbative physics whether it be the quark structure of low energy QCD or string phenomenology.

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Appendix A

To obtain the asymptotic Hamiltonians of § 2 and § 5 we must evaluate the terms $\bar{u}'(p)\gamma^\mu u'(q)$, $\bar{v}'(p)\gamma^\mu v'(q)$, $\bar{v}'(p)\gamma^\mu u'(q)$ and $\bar{u}'(p)\gamma^\mu v'(q)$. The results we will obtain, and indeed some generalisations, are quoted in Appendix J of ref. [64].

Given an on-shell 4-vector, $p$, so that $p^2 = m^2$, then $E_p$ is given by $E_p = \sqrt{|p|^2 + m^2}$. We introduce the notation

$$N_p = \frac{1}{E_p + m} \quad \text{and} \quad \hat{p} = N_p p. \quad (113)$$

Let $\xi^1 = (1, 0)$ and $\xi^2 = (0, 1)$. The Dirac spinors for $r = 1, 2$ are taken as

$$u'(p) = \frac{1}{\sqrt{N_p}} \left( \begin{array}{c} \xi^r \\ \sigma \cdot \hat{p} \xi^r \end{array} \right), \quad v'(p) = \frac{1}{\sqrt{N_p}} \left( \begin{array}{c} \xi^r \\ \xi^r \end{array} \right). \quad (114)$$

The $u'(p)$ and $v'(p)$ are, respectively, positive and negative energy solutions to the Dirac equation. Let us define

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (115)$$

Then we have

$$u'(p) = \Lambda v'(p), \quad (116)$$

and

$$\bar{u}'(p) = \bar{v}'(p)\gamma^0 \Lambda \gamma^0. \quad (117)$$

Our gamma matrix convention is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (118)$$

where $\sigma^i$ are the Pauli matrices. It is now straightforward to show that

$$\bar{u}'(p)\gamma^\mu u'(q) = \bar{v}'(p)\gamma^\mu v'(q), \quad (119)$$

$$\bar{u}'(p)\gamma^\mu v'(q) = \bar{v}'(p)\gamma^\mu u'(q). \quad (120)$$

This observation is based upon the easily verifiable identity

$$\gamma^0 \Lambda \gamma^0 \gamma^\mu \Lambda = \gamma^\mu. \quad (121)$$

Two other useful observations for the evaluation of the above expressions are the following. The first is

$$\bar{u}'(p)\gamma^\mu u'(q) = \frac{1}{2m} \bar{u}'(p)(\hat{p} \gamma^\mu + \gamma^\mu \hat{q})u'(q) \quad (122)$$

which uses the fact that $u'(p)$ is a positive frequency solution to the Dirac equation; a similar identity can be obtained for $\bar{v}'(p)\gamma^\mu u'(q)$. The second is the observation that

$$\hat{p} \gamma^\mu + \gamma^\mu \hat{q} = (p + q)^\mu + i(p - q)_\nu \sigma^{\mu\nu}, \quad (123)$$
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where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. Routine calculations now lead to the following results.

$$\bar{v}'(p)\gamma^0 u'(q) = \frac{1}{\sqrt{N_p N_q}} \xi^r \sigma \cdot (\hat{p} + \hat{q}) \xi^s,$$

$$\bar{u}'(p)\gamma^1 u'(q) = \frac{1}{\sqrt{N_p N_q}} \xi^r \left[ ((\sigma \cdot \hat{p}) \hat{q} + (\sigma \cdot \hat{q}) \hat{p} - ip \times q) \xi^s + (1 - \hat{p} \cdot \hat{q}) \sigma^f \right],$$

$$\bar{u}'(p)\gamma^2 u'(q) = \frac{1}{\sqrt{N_p N_q}} \xi^r \left[ (1 + \hat{p} \cdot \hat{q} + i\sigma \cdot \hat{p} \times \hat{q}) \xi^s, \right]$$

(124)

$$\bar{u}'(p)\gamma^3 u'(q) = \frac{1}{\sqrt{N_p N_q}} \xi^r \left( p + q + i\sigma \times (\hat{p} - \hat{q}) \right) \xi^s.$$  

The latter pair of results is sufficient to calculate the asymptotic interaction Hamiltonian (11) of the massive theory.

In the $m = 0$ limit, these results simplify in an attractive way. Recall from the discussion of §5 that the asymptotic interaction Hamiltonian only receives a contribution from the region where the momenta are collinear. As an illustration, we shall examine the first of the terms in (124), in the collinear region where we have $|k| = |q| + |k - q|$ (c.f. the paragraph after eq. 99). In the massless case we have $N_p = 1/E_p$, so that $\hat{p} = p/E_p$. It is also clear that $p^2 = 1$ and $\hat{p} \cdot p = E_p$. We thus obtain for massless charges for this momentum configuration

$$\bar{v}'(q)\gamma^0 u'(k - q) = -\frac{1}{\sqrt{N_{k-q} N_q}} \xi^r \sigma \cdot (\hat{q} + (k - q)) \xi^s,$$

$$= \sqrt{E_q E_{k-q}} \xi^r \sigma \cdot \left( \frac{q}{E_q} + \frac{k-q}{E_{k-q}} \right) \xi^s. \quad (125)$$

Now in the asymptotic interaction Hamiltonian this spinor combination only occurs if the momenta are such that $k$ and $q$ are parallel light-like vectors, and using this in (125), we see that

$$\bar{v}'(q)\gamma^0 u'(k - q) = \sqrt{E_q E_{k-q}} \xi^r \sigma \cdot \left( \frac{q}{E_q} + \frac{1}{E_k - q} \right) \xi^s,$$

$$= 2\sqrt{E_q E_{k-q}} \xi^r \hat{q} \cdot \sigma \xi^s. \quad (126)$$

The other terms in (124) can be found in a similar manner. We confine ourselves to listing them. They yield the simple results:

$$\bar{v}'(q)\gamma^1 u'(k - q) = 2\hat{q} \sqrt{E_q E_{k-q}} \xi^r \hat{q} \cdot \sigma \xi^s$$

$$\bar{u}'(q)\gamma^2 u'(k - q) = 2\hat{p} \sqrt{E_q E_{k-q}} \xi^s. \quad (128)$$

These results may be used to extract the asymptotic interaction Hamiltonian for massless QED.

References

[1] E P Wigner, Ann. Math. 40, 149 (1939)
[2] S Weinberg, The quantum theory of fields (Cambridge University Press, Cambridge, 1995)
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Charges in gauge theories

[41] E Bagan, B Fiol, M Lavelle, and D McMullan, *Mod. Phys. Lett.* A12, 1815 (1997), hep-ph/9706515, Erratum-ibid, A12 (1997) 2317

[42] T Muta, *Foundations of quantum chromodynamics* (World Scientific, Singapore, 1987)

[42a] The Feynman rule for the vertex with the source is here taken to be unity for simplicity, it could also be given the renormalisation group invariant form, $m^2 \phi^* \phi$

[42b] It is easy to check that the phase part of the dressing does not bring in any soft divergences

[42c] Note that the massless tadpole diagrams of figure 5(b) do not cancel, but they, of course, do not alter the on-shell dependence of the IR-structure

[43] R Jackiw and L Soloviev, *Phys. Rev.* 27, 1485 (1968)

[44] Y L Dokshitser, V A Khoze, A H Mueller, and S I Troian, *Basics of perturbative QCD* (Ed. Frontieres, Gif-sur-Yvette, France, 1991)

[45] R Doria, J Frenkel, and J C Taylor, *Nucl. Phys.* B168, 93 (1980)

[46] D Espriu and R Tarrach, *Phys. Lett.* B383, 482 (1996), hep-ph/9604431

[47] F Havemann, *Collinear divergences and asymptotic states*, Zeuthen Report No. PHE-85-14, 1985 (unpublished), scanned at KEK

[48] M Nowakowski and A Pilaftsis, *Z. Phys.* C60, 121 (1993), hep-ph/9305321

[49] G S Jones, *Generalised functions*, First ed. (McGraw-Hill, Berkshire, 1966)

[50] R Jackiw, D Kabat and M Ortiz, *Phys. Lett.* B277, 148 (1992)

[51] I Robinson and D Rozga, *J. Math. Phys.* 25, 499 (1984)

[52] C Parrinello, S Petrarca, and A Vladikas, *Phys. Lett.* B268, 236 (1991)

[53] P de Forcrand and J E Hetrick, *Nucl. Phys. Proc. Suppl.* 42, 861 (1995), hep-lat/9412044.

[54] R G Stuart, *Unstable particles*, (1995), hep-ph/9504308

[55] J Papavassiliou and A Pilaftsis, *Phys. Rev. Lett.* 75, 3060 (1995), hep-ph/9506417

[56] N Seiberg and E Witten, *Nucl. Phys.* B426, 19 (1994), hep-th/9407087

[57] M Lavelle and D McMullan, *Phys. Rev. Lett.* 71, 3758 (1993), hep-th/9306132

[58] I Bialynicki-Birula and Z Bialynicka-Birula, *Quantum electrodynamics* (Pergamon Press, Oxford, 1975)