Stability and related zero viscosity limit of steady plane Poiseuille-Couette flows with no-slip boundary condition

Song Jiang† Chunhui Zhou‡*

† Department of Mathematics, Southeast University, Nanjing, 210096, China. Email: zhouchunhui@seu.edu.cn
‡ Institute of Applied Physics and Computational Mathematics, Beijing, China. Email: jiang@iapcm.ac.cn

Abstract

We prove the existence and stability of smooth solutions to the steady Navier-Stokes equations near plane Poiseuille-Couette flow. Consequently, we also provide the zero viscosity limit of the 2D steady Navier-Stokes equations to the steady Euler equations. First, in the absence of any external force, we prove that there exist smooth solutions to the steady Navier-Stokes equations which are stable under infinitesimal perturbations of plane Poiseuille-Couette flow. In particular, if the basic flow is the Couette flow, then we can prove that the flow is stable for any finite perturbation $o(1)$. Moreover, we also show that for any smooth shear flow satisfying (1.27), if we put a proper external force to control the flow, then we can also obtain a smooth solution of the steady Navier-Stokes equations which is stable for infinitesimal perturbation of the external force. Finally, based on the same linear estimates, we establish the zero viscosity limit of all the solutions obtained above to the solutions of the Euler equations.

Keywords: Steady Navier-Stokes equations, stability of shear flows, zero viscous limit, weak boundary layers, strong solutions.

1 Introduction

Hydrodynamic stability has been recognized as one of the central problems in fluid mechanics. There have been plenty of works including theoretical, experimental and numerical results on the stability characteristics of different classes of basic flows.

*Corresponding author
These include flows in channels, boundary layers, jets and shear layers. Concerning the stability of viscous unsteady shear flows, the authors in [2, 3, 4, 5, 27, 31, 32] show that the 2D or 3D Couette flow and pipe Poiseuille flow are linear stable for infinitesimal perturbations at high Reynolds number. In [15], the authors prove that generic plane shear profiles other than the plane Couette flow are linearly unstable for sufficiently large Reynolds number. More recently, the authors in [28] use a new energy method to prove the instability of plane Couette flow for some perturbation of size $\varepsilon^{\frac{1}{2}}-\delta$ with any small $\delta > 0$.

In this paper, we shall study the stability of shear flows in the steady setting. First we prove that, in contrast with the unsteady case, a steady plane Poiseuille flow is stable under infinitesimal perturbations in the absence of external forces. In fact, we can also prove that the basic flows of plane Poiseuille-Couette family defined in (1.1) are stable under infinitesimal perturbations. In particular, if the basic flow is Couette flow, then through a formal asymptotic expansion including the Euler correctors and weak boundary layer correctors, we can show that the flow is stable for any finite perturbation $o(1)$. Moreover, we also show that any shear flow $u_0 = (\mu(y), 0)$ satisfying (1.27), which is a solution to the steady Navier-Stokes equations with an external force $f_0 = (-\mu''(y), 0)$, is stable under an infinitesimal perturbation of the external force. Finally, based on the same linear estimates, we prove the zero viscosity limit of all the obtained solutions to the solutions of the Euler equations.

Let $U_0 = (U(y), 0)$ be the basic flow of Poiseuille-Couette family with

$$U(y) = \alpha_1 y + \alpha_2 y(2 - y), \quad 0 \leq y \leq 2,$$

where $\alpha_1, \alpha_2 \geq 0$ are given constants and assumed always to satisfy $\alpha_1 + \alpha_2 > 0$. If $\alpha_1 \equiv 0$, then we get the plane Poiseuille flow, while $\alpha_2 \equiv 0$, the flow is called the plane Couette flow.

For the characteristic quantities of the flow: the density $\rho^*$, the viscosity $\mu$, the characteristic velocity such as the free stream velocity $V$, and the characteristic length $d$, we consider the dimensionless spatial coordinates $x = x^* / d$, $y = y^* / d$, $(x, y) \in \Omega = (0, L) \times (0, 2)$, the dimensionless velocity $u^\varepsilon = u^*/V$ and the dimensionless normal stress $P^\varepsilon = P^*/\rho^*V^2$. The steady incompressible Navier-Stokes equations can be written into the following dimensionless form:

$$\begin{cases}
    u^\varepsilon \cdot \nabla u^\varepsilon + \nabla P^\varepsilon - \varepsilon \Delta u^\varepsilon = f^\varepsilon, \\
    \nabla \cdot u^\varepsilon = 0,
\end{cases}$$

where $u^\varepsilon = (u^\varepsilon, v^\varepsilon)$, $\varepsilon = 1/Re$, $Re = \frac{V^* x^*}{\mu}$ is the Reynolds number, $P^\varepsilon$ is the pressure, and $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon)$ is the external force.

Besides, we consider an Euler flow:

$$u^0 = (\mu(y), 0),$$

(1.3)
which, for any sufficiently smooth function $\mu$, satisfies the stationary Euler equations:

$$\begin{align*}
&\mathbf{u}^0 \cdot \nabla \mathbf{u}^0 + \nabla P^0 = 0 \\
&\nabla \cdot \mathbf{u}^0 = 0 \\
&\mathbf{u}^0 \cdot \mathbf{n}|_{y=0,y=2} = 0.
\end{align*}$$

(1.4)

Here the pressure $P^0$ is a constant, $\mathbf{n}$ denotes the unit normal vector on $\{y = 0\}$ and $\{y = 2\}$, and the boundary condition $\mathbf{u}^0 \cdot \mathbf{n} = 0$ is known as the no-penetration condition.

We are interested in the asymptotic behavior of $\mathbf{u}^\varepsilon$ as $\varepsilon \to 0$ with no-slip boundary condition on $y = 0$ and $y = 2$. In the presence of boundaries, if there is a mismatch between the no-slip boundary condition for the Navier-Stokes equations and the no-penetration boundary condition satisfied by the Euler equations, then the limit as $\varepsilon \to 0$ would be rectified by the presence of the Prandtl boundary layer. There have been many works on the well-posedness of the Prandtl boundary layer equations. Concerning the instability of the Prandtl boundary layer in the non-steady setting, we can refer, for instance, to [1, 9, 10, 12, 14, 16, 26, 29] and the references therein. Considering the well-posedness of steady Prandtl boundary layers, we refer to [11, 18, 19, 17, 21, 22, 23] and the references therein.

In this article, we shall study the situation when the Euler flow $\mathbf{u}^0 = (\mu(y), 0)$ itself is a shear flow satisfying the no-slip boundary condition:

$$\mu(0) = 0, \quad \mu(2) = 2\alpha_1,$$

(1.5)

and is a small perturbation of the basic flow $\mathbf{U}_0$ (cf. (1.1)).

In the 2D steady setting, under the assumption that $\mu''/\mu$ vanishes at high order at $y = 0, 2$, and in the absence of external forces, the authors in [24] investigate the inviscid limit of weak solutions to the steady Navier-Stokes equations in a channel around the shear flows $(\mu(y), 0)$ with homogeneous no-slip boundary condition on the rigid wall and the stress-free condition on the outflow part of the boundary. The weak solutions in [24] can not be strong ones since one has no adequate regularity near the corners of the domain due to the stress free boundary condition. In [11], under the proper control of the external forces, the authors establish the $H^1$ stability of shear flows of Prandtl type with periodic boundary condition in $x$-direction.

The aim of this paper is to study the existence and stability of smooth solutions to the steady Navier-Stokes equations around the plane Poiseuille-Couette family under the perturbations of both viscous stresses and external forces. Except for the no-slip boundary condition on the rigid walls $y = 0$ and $y = 2$, we assume further that the curl of the flow is well controlled on the inflow and outflow parts of the boundary.

Before stating the main results, we introduce the notation used throughout this paper.

**NOTATION:** Let $G$ be an open domain in $\mathbb{R}^N$. We denote by $L^p(G)$ ($p \geq 1$) the
where simplicity, in the following we shall assume a number, then $M$ terms in the a priori estimates. In this paper we shall take

$$\| \cdot \|_{k,p}$$

where $u$ here.

We use $| \cdot |$ for the standard norm in $W^{k,p}(G)$ and $\| \cdot \|$ for the norm in $L^2(G)$.

We also use $| \cdot |_{L^\infty}$ to denote $| \cdot |_{L^\infty(\Omega)} = \text{ess sup}_\Omega | \cdot |$. The symbol $\lesssim$ means that the left side is less than the right side multiplied by some constant. Let $a \in \mathbb{R}$ be a real number, then $a^+$ means any real number a little bigger than $a$.

We also define a smooth cut-off function $\chi(t) \in C^\infty([0, \infty))$ satisfying $|\chi| \leq 1$, $|\chi|_{C^4} \leq C$ and

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq 1/2, \\ 0, & t \geq 1, \end{cases}$$

(1.6)

here $C > 0$ is a finite constant.

Next, we give the boundary condition considered in this paper. Precisely, we impose the following boundary condition:

$$v_x^\varepsilon|_{x=0} = a_1; \; u_x^\varepsilon|_{x=0} = \mu(y) + a_2; \; v_x^\varepsilon|_{x=L} = a_3; \; \partial_x \text{curl} u_x^\varepsilon|_{x=L} = a_4;$$

$$u_x^\varepsilon|_{y=0} = v_x^\varepsilon|_{y=0} = v_y^\varepsilon|_{y=0} = v_x^\varepsilon|_{y=2} = v_y^\varepsilon|_{y=2} = 0, \; u_x^\varepsilon|_{y=2} = 2a_1, \quad (1.7)$$

where $a_i \ (i = 1, \ldots, 4)$ are pre-scribed functions which are sufficiently small. For simplicity, in the following we shall assume $a_i = 0, \ i = 1, \ldots, 4$.

To study the asymptotics as $\varepsilon \to 0$, we shall use the following ansatz of asymptotic expansion:

$$\begin{cases} u_x^\varepsilon = u_s + \varepsilon M_0 u \\ v_x^\varepsilon = v_s + \varepsilon M_0 v \\ P_x^\varepsilon = P_s + \varepsilon M_0 P, \end{cases} \quad (1.8)$$

where $u_s, P_s$ are given function to be determined later, $u_s = (u_s, v_s)$ satisfying $\text{div} u_s = 0, \ M_0$ is a constant sufficiently large, such that we can control the nonlinear terms in the a priori estimates. In this paper we shall take $M_0 = \frac{11}{8} +$. Putting (1.8) into (1.2), we obtain that the remainder solutions $(u, v, P)$ satisfy the following system:

$$\text{div} u = 0, \quad (1.9)$$

$$u_s u_x + u_s v_y + u_s v_x + v_s u_y - \varepsilon \Delta u + \partial_x P = N_1(u, v) + F_1 \quad (1.10)$$

$$\begin{align*}
    u_s v_x + v_s v_y + u_s v_x + v v_s - \varepsilon \Delta v + \partial_y P = N_2(u, v) + F_2,
\end{align*} \quad (1.11)$$

where $u = (u, v), \ N_1 = -\varepsilon M_0(v u_y + u u_x), \ N_2 = -\varepsilon M_0(v v_y + u v_x), \ F = (F_u, F_v), \ F = (F_1, F_2) = F + \varepsilon^{-M_0} F^\varepsilon$ and

$$\begin{align*}
    F_u &= -\varepsilon^{-M_0}[u_s \partial_x u + v_s \partial_y u + \partial_x P_s - \varepsilon \Delta u_s] \quad (1.12) \\
    F_v &= -\varepsilon^{-M_0}[u_s \partial_x v + v_s \partial_y v + \partial_y P_s - \varepsilon \Delta v_s]. \quad (1.13)
\end{align*}$$

4
Finally, let us define the space $\mathcal{X}$:

$$
\mathcal{X} = \{ u = (u, v) \in H^3(\Omega) \times H^3(\Omega) \mid \text{div} u = 0, \| u \|_X < \infty, u|_{y=0} = v|_{y=0} = 0, u|_{y=2} = v|_{y=2} = u|_{x=0} = v|_{x=0} = 0|_{x=L} = v|_{x=L} = 0 \},
$$

(1.14)

where the norm $\| \cdot \|_X$ is defined by

$$
\| u \|_X := \| \sqrt{u_s} \nabla v \| + \varepsilon^{\frac{1}{2}} \| \sqrt{u_s} \nabla^2 q \| + \varepsilon^{\frac{3}{2}} \| \nabla^3 u \| + |u_s q_y(0, \cdot)|,
$$

with $q = v/u_s$.

### 1.1 Main theorems

In this section we state the main results. First, we shall show that in the absence of any external force, if $u_0 = (\mu(y), 0)$ is a small perturbation of the basic flow $U_0$ within order $O(\varepsilon^{\frac{2}{8}+})$, and

$$
\mu(y) > 0 \text{ for } 0 < y < 2, \mu'(0) > 0,
$$

(1.15)

then there exists a unique solution to the steady Navier-Stokes equations around $u_0$ which will converge to $U_0$ as $\varepsilon \to 0$. More precisely, if we denote

$$
u^\varepsilon = u_0 + \varepsilon^{1+\frac{2}{8}+}\gamma u,$$

(1.16)

where $\gamma > 0$ is a sufficiently small constant, then our first main result reads as follows.

**Theorem 1.1.** Assume that $f^\varepsilon \equiv 0$ in (1.2), $0 < L \ll 1$ is a given constant, $\mu \in C^4([0,2])$ satisfies (1.5), (1.15), then there is a constant $\alpha_0 > 0$, such that if

$$
|\mu(y) - U(y)|_{C^4([0,2])} \leq \alpha_0 \varepsilon^{\frac{2}{8}+\gamma},
$$

(1.17)

there exists a unique solution $(u^\varepsilon, P^\varepsilon) \in H^4(\Omega) \times H^3(\Omega)$ to the system (1.2) with the remainder solution $u = (u, v)$ defined in (1.16) satisfying

$$
\| v \|_{L^2} + \varepsilon^{\frac{1}{2}} \| \nabla^2 v \|_{L^2} + \varepsilon \| u_{yy} \|_{L^2} + \varepsilon^{\frac{3}{2}} \| \nabla^3 u \|_{L^2} + \varepsilon^{\frac{5}{2}} \| \nabla^4 u \|_{L^2} \leq C\alpha_0,
$$

(1.18)

and consequently

$$
\varepsilon^{-\frac{1}{2}} |u^\varepsilon - \mu|_{L^\infty(\Omega)} + \varepsilon^{-\gamma} \| u^\varepsilon - \mu \|_{H^2(\Omega)} \leq C \varepsilon^{\frac{3}{8}},
$$

(1.19)

$$
\varepsilon^{-\frac{1}{4}} |u^\varepsilon|_{L^\infty(\Omega)} + \varepsilon^{-\gamma} \| v^\varepsilon \|_{H^2} \leq C \varepsilon^{\frac{7}{8}},
$$

(1.20)

where the constant $C$ does not depend on $\varepsilon$. 

5
Next, in addition to (1.15), if we assume that $\mu''$ degenerates rapidly, when $y \to 0$, i.e., we have to take $\alpha_2 \equiv 0$ and thus the basic flow is Couette flow, then the flow is stable within any finite disturbance $o(1)$ of the basic flow.

We denote

$$u^\varepsilon = u_s + \varepsilon^{1+\frac{3}{2}+\gamma} u,$$

(1.21)

where $u_s$ is a known function defined in (2.2). Our second result reads as

**Theorem 1.2.** In addition to the assumptions in Theorem 1.1, we also assume that $\alpha_2 = 0$ in (1.1), $k > 0$ is a suitably large integer, and

$$|\frac{\mu''}{\mu}|_{L^\infty} \text{ is sufficiently small},$$

(1.22)

then there is a constant $\alpha_0 > 0$, such that if

$$|\frac{\mu''}{\mu}|_{C^k([0,2])} \leq \alpha_0,$$

(1.23)

there exists a unique solution $(u^\varepsilon, P^\varepsilon) \in H^4(\Omega) \times H^3(\Omega)$ to the system (1.2) satisfying

$$\|u^\varepsilon\|_{H^2} + \|P^\varepsilon\|_{H^1} \leq C \alpha_0,$$

(1.24)

and

$$\|v\|_{L^2} + \varepsilon^{\frac{3}{2}} \|
abla^2 v\|_{L^2} + \varepsilon \|u_{yy}\|_{L^2} + \varepsilon^{\frac{3}{2}} \|
abla^3 u\|_{L^2} + \varepsilon^{\frac{3}{2}} \|
abla^4 u\|_{L^2} \leq C \alpha_0,$$

(1.25)

where $u = (u, v)$ is the remainder solution defined in (1.21). Moreover, it holds that

$$|u^\varepsilon - \mu|_{L^\infty} + |v^\varepsilon|_{L^\infty} \leq C \varepsilon,$$

(1.26)

where $C$ is a constant independent of $\varepsilon$.

Finally, if we put a suitable external force $f^\varepsilon$ to “rewind” the flow, then we can obtain a unique smooth solution to the system (1.2) around any basic shear flow $u^0(y) = (\mu(y), 0)$, satisfying

$$\mu'(0) > 0, \mu(y) > 0, \text{ for } 0 < y < 2, \ |\mu|_{C^4} \leq K_0 \text{ in } \Omega,$$

(1.27)

where $K_0 > 0$ is a fixed constant. More precisely, setting

$$g^\varepsilon = f^\varepsilon + \varepsilon (u^0)'', \ u^\varepsilon = u^0 + \varepsilon^{1+\frac{3}{2}+\gamma} u,$$

where $u = (u, v)$, $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon)$, $g^\varepsilon = (g_1^\varepsilon, g_2^\varepsilon)$. Our last theorem reads as

**Theorem 1.3.** Let $0 < L \ll 1$ be a given constant, $\mu(y)$ is a smooth function satisfying (1.27). Then, there exists a constant $\alpha_0 > 0$, such that if

$$\|g^\varepsilon\|_{H^2} \leq \alpha_0 \varepsilon^{\frac{11}{8}+\gamma},$$

(1.28)

there exists a unique solution $(u^\varepsilon, P^\varepsilon) \in H^4(\Omega) \times H^3(\Omega)$ to the system (1.2), satisfying (1.18)-(1.20).
Remark 1.4. The zero viscosity limit of the solutions in Theorems 1–3 to the solutions of the steady Euler equations follow immediately from (1.19), (1.20) and (1.26). In particular, we have
\[ u^\varepsilon \to u_0 \quad \text{in} \quad H^2(\Omega) \]
in Theorems 1 and 3, while in Theorem 2, we only have
\[ u^\varepsilon \to u_0 \quad \text{in} \quad H^s(\Omega), \quad 0 < s < 2. \]  
(1.29)
However, if we further assume
\[ \left| \frac{\mu'''}{\mu} \right|_{C^k([0,2])} \leq \alpha_0 \varepsilon^0, \]
then we can take \( s = 2 \) in (1.29).

Remark 1.5. In Theorem 1.2, \( u_0 = (\mu, 0) \) could be any shear flow with \( \mu \) be a smooth function satisfying (1.15), (1.22) and (1.23). In particular, if \( \mu(0) = \mu(2) = 0 \), we need one more condition that \( \mu'' \) degenerates rapidly near \( y = 2 \). In this case, we can use the same scale \( Y = \varepsilon^{-\frac{1}{2}} \tilde{y} \) (\( \tilde{y} = y \) near \( y = 0 \), \( \tilde{y} = 2 - y \) near \( y = 2 \)) in the construction of the weak boundary layer correctors in Section 2.

Remark 1.6. In Theorem 1.2, although the remainders \((F_u, F_v)\) defined in (1.12)–(1.13) can be arbitrary small, we can only obtain the uniform-in-\( \varepsilon \) boundedness of \( \|u^\varepsilon\|_{H^2} \) as a result of the weak boundary layer correctors in the approximate solution \( u_s \).

Let us give a few comments on our results. Considering solutions to the steady Navier-Stokes equations in the absence of external forces, the class of strictly parallel flows \( U_0 = (\mu(y), 0) \) is limited, since \( \mu(y) \) has to satisfy the equation of motion:
\[ \varepsilon \frac{d^2\mu}{dy^2} = \frac{dP}{dx}. \]
This includes two important special cases:

**Plane Couette flow:** \( \mu(y) = y, \) \( P = \text{constant}, \ 0 < y < 2, \)

and

**Plane Poiseuille flow:** \( \mu(y) = y(2 - y), \) \( \frac{dP}{dx} = \text{constant}, \ 0 < y < 2. \)

We have therefore obtained a family of strictly parallel flows, say the plane Poiseuille-Couette flow \( U_0 = (U(y), 0) \) with \( U(y) \) defined in (1.1).

Compared with [11, 18], where the authors studied the stability of the Prandtl expansions for the steady Navier-Stokes equations, the Euler flow \( u^0 = (\mu(y), 0) \) in
this paper satisfies the no-slip boundary condition: \( \mu(0) = 0, \mu(2) = 2\alpha_1 \). Thus, there would be no strong boundary layers around the rigid walls \( y = 0 \) and \( y = 2 \). In the absence of any external force, we could construct an Euler corrector to balance the perturbation of the internal viscous stress of the fluid. Due to the no-slip boundary condition \( \mu(0) = 0 \), the equation for the Euler corrector would have a degenerate coefficient. It seems that for Poiseuille-Couette flows with \( \alpha_2 > 0 \), we can not continue the expansion to obtain arbitrary small remainders due to the eigenvalue problem of the Laplace operator. However, if \( \alpha_2 = 0 \), and \( \mu'' \) degenerates rapidly near \( y = 0 \), then we can continue the expansion to obtain arbitrary small remainders by using weak boundary layer correctors to rectify the mismatch between the no-slip boundary condition for the Navier-Stokes equations and the no-penetration boundary condition for Euler correctors. So, we can prove that the Couette flow is stable under finite small perturbation.

Our proof is inspired by the method in [18]. Compared with the boundary layer profiles in [18], we would have one more boundary \( y = 2 \) with no-slip boundary condition, and as mentioned above, the equations for the Euler correctors would have degenerate coefficients near the rigid walls. If the basic flow is a Couette flow, considering the weak boundary layer correctors in the expansion, the dominating equation is a linear parabolic equation with a degenerate coefficient \( \mu(y) \sim y \) near the boundary \( y = 0 \) and non-degenerate coefficients near \( y = 2 \). So in the process of multi-scale analysis, we have to use different scales to balance the weak boundary layer terms, i.e., we utilize the scale \( Y_- := y/\epsilon^{\frac{3}{2}} \) near \( y = 0 \) and \( Y_+ := (2-y)/\epsilon^{\frac{2}{3}} \) near \( y = 2 \). Consequently, the boundary conditions for the corresponding Euler correctors \( u_{i,\pm} \) are also different near the rigid walls. Moreover, the linear estimates in [18] failed in obtaining the third order derivative \( v_{yy} \) as we do not have a good sign for \( u_{sy} \) on the upper bound \( y = 2 \). Instead, we shall use the theory of the Dirichlet boundary value problem for biharmonic equations as well as the Sobolev imbedding theory to close the estimates. Finally, let us describe the main steps in the proof of the theorems.

In Section 2, we give the construction of the asymptotic solution \( u_s \) when \( \alpha_2 = 0 \) in the absence of any external force. First, we use an Euler corrector \( u_1^e \) to offset the perturbation of the viscous stress of the flow. Then we use weak boundary layer correctors to rectify the mismatch between the no-slip condition for Navier-Stokes equations and the no-penetration condition for Euler correctors. Thanks to the condition that \( \mu'' \) degenerates rapidly when \( y \to 0 \), we can continue to make as many expansions as we want and the remainders \( (F_u, F_v) \) can be arbitrary small. For the weak boundary layer correctors in the expansion, as mentioned above, we use the scale \( Y_- := y/\epsilon^{\frac{1}{2}} \) near \( y = 0 \) and the scale \( Y_+ := (2-y)/\epsilon^{\frac{2}{3}} \) near \( y = 2 \). Consequently, we have to construct the weak boundary layer correctors \( u_{i,\pm} \) separately.

In Section 3, we study the linearized system. First by introducing the stream function, we prove the existence of solutions to the linearized system. Then, we derive the uniform-in-\( \epsilon \) estimates of solutions to the linearized system. We first use
v as a multiplier to act on the curl equation to obtain the $L^2$ norm of $v$ bounded in order $O(1)$. Then, we use the multiplier $q_x = (v/u_s)_x$ to act on the curl equation to get the estimates for second order derivatives. Broadly speaking, the multiplier $q_x$ works well in controlling the convection terms, but also brings complicated calculations in controlling the viscous terms. Next, as mentioned above, we can not use the method in \[18\] to obtain the estimates for the third order derivatives of the solutions as we do not have good signs for $u_{sy}$ on the upper bound $y = 2$. Instead, we shall employ the multiplier $v_{xxx}$ to act on the curl equation to obtain a bound for $\|\nabla^2 v_x\|$ which implies the boundedness of the convection terms in the biharmonic equation for $v$. Finally, by the compatibility condition from the curl equation, we can obtain the estimates of fourth order derivatives of $v$ by employing the theory of the Dirichlet boundary value problem for biharmonic equations. In this way, we can close the estimates by employing a careful bootstrap argument and the Sobolev imbedding theory.

Finally in Section 4, based on the linear estimates in Section 3, we prove the main theorems by the contract mapping principle. We point out here that the only difference between the proofs of Theorems 1-3 lies in the remainders $\mathcal{F} = (\mathcal{F}_u, \mathcal{F}_v)$ in the linearized system. Besides, the restriction on the exponent $M_0$ in the expansion \[18\] mainly comes from controlling the nonlinear terms in Subsection 3.2.

2 Formal asymptotic expansion around the Couette flow

The goal of this section is to construct approximate solutions $(u_s, P_s)$ in \[18\]. In the case when $\alpha_2 > 0$ in the absence of any external force, or there is a suitable external force $f^\epsilon$ to control the flow, we will directly take $u_s = u^0$. So in this section we will only study the special case when $\alpha_2 = 0$ in the absence of any external force, i.e., the basic flow is Couette flow. Without loss of generality, we will assume $\alpha_1 = 1$. Due to the fact that the Euler profile $u^0$ satisfies the no-slip boundary condition, there would be no strong boundary layers near the boundaries $y = 0$ and $y = 2$.

First, we shall use an Euler corrector $u_1^e$ to offset the perturbation of the viscous stress of the fluid. Then there would be, similar to the case of Prandtl boundary layers, a mismatch between the no-slip condition for Navier-Stokes equations and the no-penetration condition for the Euler corrector. This mismatch can be rectified by the presence of the weak boundary layer corrector. As the dominating equation for the weak boundary layer corrector is a linear parabolic equation including a degenerate term near $y = 0$, in the process of multi-scale analysis near the boundaries, we will broke the boundary layer profile into two components, one supported near $y = 0$ with scale $Y_- := y/\varepsilon^3$, $u_p^{1-,0}$, and one supported near $y = 2$ with scale $Y_+ := (2 - y)/\varepsilon^3$, $u_p^{1+,0}$. Thanks to the degeneration of $\mu''$ near $y = 0$, we can continue the construction of $u_p^{1,\pm}$ and $u_p^{2,\pm}$ till the remainders defined in \[1.12\]-\[1.13\] are small enough.

In what follows, the Eulerian profiles are functions of $(x, y)$, whereas the bound-
ary layer profiles are functions of \((x,Y)\), where

\[
Y = \begin{cases} 
  \frac{2 - y}{\varepsilon^2} & \text{if } 1 \leq y \leq 2, \\
  \frac{y}{\varepsilon^3} & \text{if } 0 \leq y \leq 1.
\end{cases}
\]  

As we have used different scales \(Y_\pm\) in the construction of the weak boundary layer profiles, we will have to use different Euler correctors \(u^{i,\pm}_c\) in the following expansion. The expansion will be continued till the remainders are small enough. More precisely, we expand the solution in \(\varepsilon\) as:

\[
\begin{align*}
  u^\varepsilon &= \mu + \varepsilon u^1_c + \varepsilon u^{1,+}_p + \varepsilon u^{1,-}_p + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} (u^i_c + u^i_p) \\
  &\quad + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} (u^i_c - u^i_p) + \varepsilon^M u \\
  &\triangleq u^E_c + \varepsilon^M u, \\
  v^\varepsilon &= \varepsilon v^1_c + \varepsilon^2 v^1_p + \varepsilon^4 v^1_p - \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} (v^i_c + \varepsilon^2 v^i_p) \\
  &\quad + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} (v^i_c - \varepsilon^2 v^i_p) + \varepsilon^M v \\
  &\triangleq v^E_c + \varepsilon^M v, \\
  P^\varepsilon &= C + \varepsilon P^1_c + \sum_{i=1}^{5} \varepsilon^{1+\frac{i-1}{2}} P^i_{p,a,-} + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} P^i_{p,a,+} \\
  &\quad + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} P^i_{e} + \varepsilon^M P \\
  &\triangleq P^E_c + \varepsilon^M P,
\end{align*}
\]  

where \(C\) is any constant, \(M\) is an integer large enough, for example, \(M = 10\) would be enough.

We shall also introduce the notation

\[
\begin{align*}
  u^E_s &= \mu + \varepsilon u^1_s + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} u^i_e + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} u^i_p, \\
  v^E_s &= \varepsilon v^1_s + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} v^i_e + \sum_{i=2}^{M} \varepsilon^{1+\frac{i-1}{2}} v^i_p.
\end{align*}
\]  

The main result of this section reads as:

**Theorem 2.1.** Assume that \(\mu\) is a smooth function satisfying conditions in Theorem \(2.2\) then we can construct a triple \((u^E_s, v^E_s, P^E_s)\) with formula defined in \((2.2)\) satisfying:

\[
\text{div} u_s = 0 \text{ in } \Omega, \quad u_s = v_s = 0, \text{ on } y = 0; \quad u_s = 2\alpha_1, \quad v_s = 0, \text{ on } y = 2,
\]  

and

\[
\begin{align*}
  |\partial_x u_s| + |\partial_y v_s| + |u_s - \mu| &\lesssim \min\{O(\varepsilon^{\frac{1}{2}})(2-y), O(\varepsilon)\}, \\
  \varepsilon^{\frac{1}{2}} |\partial_y u_s - \mu| + |u_{syg}| + |u_{sygg}\lambda(y)| + \varepsilon^{\frac{1}{2}} |u_{sygg}(2-y)| &\lesssim \alpha_0, \\
  |\partial_x^l v_s| &\lesssim \min\{\varepsilon y, \varepsilon(2-y)\} \text{ for } l \geq 0,
\end{align*}
\]
where $\chi(y)$ is the cut-off function defined in (1.6). Moreover, the remainders defined in (1.12) and (1.13) satisfying

$$
\varepsilon^M_0 \mathcal{F}_u = \mathcal{T}_{u,\varepsilon^2} + \mathcal{F}_{u,\varepsilon^3},
$$

(2.8)

$$
\varepsilon^M_0 \mathcal{F}_v = \mathcal{T}_{v,\varepsilon^2} + \mathcal{F}_{v,\varepsilon^3},
$$

(2.9)

with estimate:

$$
\|\mathcal{T}_{u,\varepsilon^2}\|_{H^2} + \|\mathcal{T}_{v,\varepsilon^2}\|_{H^2} + \|\mathcal{F}_{u,\varepsilon^3}\|_{H^2} + \|\mathcal{F}_{v,\varepsilon^3}\|_{H^2} \lesssim \varepsilon^2 \alpha_0.
$$

(2.10)

**Remark 2.2.** In the case when $\alpha_2 > 0$ in the absence of any external force, as the Euler profile $u^0$ satisfies the no-slip boundary condition, we have $v = 0$ on $y = 0$ and $y = 2$. For this case we can still construct the first Euler corrector $u^1_e$ and the corresponding boundary layer correctors similar to the case $\alpha_2 = 0$. However, as the equations for the Euler corrector have degenerate coefficients near $y = 0$, we cannot continue the expansion due to the eigenvalue problem of the Laplace operator. In fact, if $v_e \neq 0$ on $y = 0$, since $\mu'' \sim -2\alpha_2$, we shall have $\mu''/\mu \in (-\infty, 0)$ in $\Omega$. Thus, the equation (2.13) is not well-posed. So in this case, the first weak boundary layer corrector is also the finally one, which do not degenerate when $Y$ tends to infinity. As a result, we can not obtain smaller remainders. So, we have to take $u_s = u^0$ directly when $\alpha_2 > 0$.

### 2.1 Euler correctors

In this subsection we will study the construction of Euler correctors. The equations satisfied by the first Euler corrector are obtained by collecting the $O(\varepsilon)$ order Euler terms from (1.12) - (1.13), and is now shown:

$$
\begin{align*}
\mu \partial_x u^1_e + \mu' v^1_e + \partial_x P^1_e &= \mu''(y) \\
\mu \partial_x v^1_e + \partial_y P^1_e &= 0, \\
\partial_x u^1_e + \partial_y v^1_e &= 0, \\
\partial_y v^1_e|_{x=0} &= 0, v^1_e|_{x=L} = 0, \\
v^1_e|_{y=0} &= v^1_e|_{y=2} = 0.
\end{align*}
$$

(2.11)

By going to the vorticity formulation, we arrive at the following problem:

$$
-\mu \Delta v^1_e + \mu'' v^1_e = \mu'''(y), \quad u^1_e := -\int_0^y v^1_{ey}(s, y) ds.
$$

(2.12)

Then we divide (2.12) by $\mu$ to obtain:

$$
-\Delta v^1_e + \frac{\mu''}{\mu} v^1_e = \frac{\mu'''}{\mu}.
$$

(2.13)

Thanks to the degeneration of $\mu''$ near $y = 0$, If we take $|\mu'''|_{L^\infty}$ small enough, equation (2.13) will be well-posed. Next we turn to the construction of the rest
Lemma 2.3. Assume that \( \mu(y) \) satisfies the conditions in Theorem 2.1, then there exists a unique triple \((u_e^1, v_e^1, P_e^1)\) satisfying system (2.14), and unique triples \((u_e^{i, \pm}, v_e^{i, \pm}, P_e^{i, \pm})\), \(i = 2, ..., M\), satisfying system (2.15) and (2.16). Besides, we have

\[
\begin{align*}
  u_e^{i,+} &= v_e^{i,+} = 0 \text{ on } y = 0; \quad u_e^{i,-} = v_e^{i,-} = 0 \text{ on } y = 2, \quad i = 2, ..., M, \quad (2.19)
\end{align*}
\]

Euler correctors. Due to the different scales of the weak boundary layer correctors near \( y = 0 \) and \( y = 2 \), we will have to use different Euler correctors to rectify the velocity \( v_p^{i, \pm} \) from weak boundary layers correctors. The system satisfied by the rest Euler correctors are shown as \((i = 2, ...M)\):

\[
\begin{align*}
  \mu \partial_x u_e^{i,+} + \mu' v_e^{i,+} + \partial_x P_e^{i,+} &= 0, \\
  \mu \partial_x v_e^{i,+} + \partial_y P_e^{i,+} &= 0, \\
  \partial_x u_e^{i,+} + \partial_y v_e^{i,+} &= 0, \\
  \partial_x v_e^{i,+}|_{y=0} = 0, \quad v_e^{i,+}|_{y=0} = 0, \quad \partial_y v_e^{i,+}|_{y=2} = 0.
\end{align*}
\]

and

\[
\begin{align*}
  \mu \partial_x u_e^{i,-} + \mu' v_e^{i,-} + \partial_x P_e^{i,-} &= 0, \\
  \mu \partial_x v_e^{i,-} + \partial_y P_e^{i,-} &= 0, \\
  \partial_x u_e^{i,-} + \partial_y v_e^{i,-} &= 0, \\
  \partial_x v_e^{i,-}|_{y=0} = 0, \quad v_e^{i,-}|_{y=0} = 0, \quad \partial_y v_e^{i,-}|_{y=2} = 0.
\end{align*}
\]

Going to vorticity produces the homogeneous system:

\[
-\Delta v_e^{i, \pm} + \frac{\mu''}{\mu} v_e^{i, \pm} = 0, \quad u_e^{i, \pm} := - \int_0^x v_e^{i, \pm}(s, y) ds. \quad (2.16)
\]

This procedure contributes the error terms to the remainder

\[
\begin{align*}
  C_{\text{Euler},u} := (u_e^E - \mu) \partial_x (u_s^E - \mu) + v_s^E \partial_y (u_s^E - \mu) - \varepsilon \Delta (u_s^E - \mu), \quad (2.17)
  C_{\text{Euler},v} := (u_e^E - \mu) \partial_x v_s^E + v_s^E \partial_y v_s^E - \varepsilon \Delta v_s^E, \quad (2.18)
\end{align*}
\]

where \( u_s^E, v_s^E \) are defined in (2.3). We can observe that \( C_{\text{Euler},u}, C_{\text{Euler},v} \) are in order \( O(\varepsilon^2) \).

We point out that as we put the boundary conditions \( \partial_y v_e^{i,-} = 0 \) on \( y = 2 \), and \( \partial_y v_e^{i,+} = 0 \) on \( y = 0 \) for \( i = 2, 3, ..., M \), by the divergence free condition and the maximum principle of elliptic equations we can prove that \( u_e^{i,+} = v_e^{i,+} = 0 \) on \( y = 0 \), i.e. the Euler correctors \( u_e^{i,-} \) will not cause the mismatch of the velocity on \( y = 0 \). On the other hand, we have \( u_e^{i,-} = v_e^{i,-} = 0 \) on \( y = 2 \), i.e. the Euler correctors \( u_e^{i,-} \) will not cause the mismatch of the velocity on \( y = 2 \).

Precisely, we have the following Lemma for Euler correctors:

**Lemma 2.3.** Assume that \( \mu(y) \) satisfies the conditions in Theorem 2.1 then there exists a unique triple \((u_e^1, v_e^1, P_e^1)\) satisfying system (2.14), and unique triples \((u_e^{i, \pm}, v_e^{i, \pm}, P_e^{i, \pm})\), \(i = 2, ..., M\), satisfying system (2.15) and (2.16). Besides, we have

\[
\begin{align*}
  u_e^{i,+} &= v_e^{i,+} = 0 \text{ on } y = 0; \quad u_e^{i,-} = v_e^{i,-} = 0 \text{ on } y = 2, \quad i = 2, ..., M, \quad (2.19)
\end{align*}
\]
and
\[
\|\partial_x^l \partial_y^m \{u_e^1, v_e^1, P_e^1\}\| + \|\partial_x^l \partial_y^m \{u_e^{i,\pm}, v_e^{i,\pm}, P_e^{i,\pm}\}\| \leq C(m,l)\frac{\|\mu\|}{\mu} |C_k|, \tag{2.20}
\]
where \(m, l \geq 0\) are integers satisfying \(m + l \leq k + 2\).

**Proof.** First of all, under the assumption that \(|\mu|_{L^\infty}\) is small enough, the existence of \((u_e^1, v_e^1, P_e^1), (u_e^{i,\pm}, v_e^{i,\pm}, P_e^{i,\pm})\) and estimates (2.20) can be easily obtained by theory of elliptic equations.

Then we need to prove \(u_e^{i,\pm} = v_e^{i,\pm} = 0\) on \(y = 0\) and \(u_e^{i,-} = v_e^{i,-} = 0\) on \(y = 2\).

We will first prove \(u_e^{i,\pm} = v_e^{i,\pm} = 0\) on \(y = 0\) by weak maximum principle of elliptic equations.

First, by (2.16), we easily have
\[
u_e^{i,\pm}(x, 0) = - \int_0^x v_e y dx = 0.
\]
Next, for fixed \(\sigma > 0\) small enough, since we have already known by (2.20) that \(v_e^{i,\pm}\) is bounded, we can define a barrier function
\[\phi^\sigma = K(y + \sigma)^\alpha, \quad 0 < \alpha < 1,
\]
where \(K\) is a constant satisfying
\[K \geq C(\|\mu\|_{L^\infty} + \alpha_0)
\]
with \(C\) being a sufficiently large constant. Then we have
\[\Delta \phi^\sigma = \alpha(\alpha - 1)K(y + \sigma)^{\alpha - 2} < 0. \tag{2.21}
\]
Now we cut \(v_e^{i,\pm}\) by the cut-off function \(\chi(y)\) defined in (1.6). Thanks to the boundary conditions \(\partial_x v_e^{i,\pm}|_{x=0} = 0\), we can make an even extension of \(\chi(y)v_e^{i,\pm}\) with respect to \(x = 0\) and the new function, still denoted by \(\chi(y)v_e^{i,\pm}\), belong to \(C^2(\Omega^*) \cap C^0(\bar{\Omega}^*)\), here \(\Omega^* = (-L, L) \times (0, 2)\). Recalling that \(K\) is large enough, we have by (2.16) and (2.21)
\[\Delta (\chi v_e^{i,\pm} - \phi^\sigma) \geq 0 \text{ in } \Omega^*.
\]
Besides, direct computation shows that
\[\partial_y (\chi v_e^{i,\pm} - \phi^\sigma)|_{y=0} < 0.
\]
So we can not obtain the maximum value of \(\chi v_e^{i,\pm} - \phi^\sigma\) on \(y = 0\). More over, we also have
\[(\chi v_e^{i,\pm} - \phi^\sigma)|_{y=2} \cup (x=L) \cup (x=-L) < 0.
\]
The weak maximum principle for elliptic equations implies that
\[ \chi v^{i,+}_e - \phi^\sigma \leq 0 \quad \text{in} \quad \Omega^*. \]
If we replace \( \chi v^{i,+}_e \) by \(-\phi^\sigma\), and \( \phi^\sigma \) by \(-\chi v^{i,+}_e \) above, we can obtain
\[ -\phi^\sigma \leq -\chi v^{i,+}_e \quad \text{in} \quad \Omega^*. \]
Combing above, we have proved that for any \( 0 < \alpha < 1 \)
\[ |v^{i,+}_e| \leq K(y + \sigma)^\alpha, \quad \text{when} \quad 0 \leq y \leq \frac{1}{2}, \]
for any \( \sigma > 0 \) small enough, which implies immediately \( v^{i,+}_e = 0 \) on \( y = 0 \). Similarly we can also prove \( v^{i,-}_e = v^{i,-}_e = 0 \) on \( y = 2 \) and we complete the proof of the Lemma.

### 2.2 Weak boundary layers correctors

In this subsection, we will only give the construction for \( u^{i,+}_p \) near \( y = 2 \). The construction of \( u^{i,-}_p \) near the boundary \( y = 0 \) is exactly the same as the process in [25]. For the completeness of the article, we would give the sketch of the construction of \( u^{i,-}_p \) in Appendix.

#### 2.2.1 Construction of \( u^{i,+}_p \)

As there would be no degeneration in the coefficients of the linear parabolic equations for the weak boundary layer correctors near \( y = 2 \), we would use the scale \( Y_+ = \frac{2-y}{\epsilon^2} \) here. Using (2.11), the leading \( O(\epsilon) \) order boundary layer terms from (1.12) are:
\[ \mu \partial_x u^{1,+}_p + \partial_x P^{1,+}_p - \partial_{Y_+} Y_+ u^{1,+}_p = 0. \]
The leading \( O(\epsilon^{\frac{1}{2}}) \) order boundary layer terms from (1.13) is:
\[ \partial_{Y_+} P^{1,+}_p = 0. \]

It is easy to obtain that \( P^{1,+}_p \equiv 0 \), and we consider the following system (as \( \mu(2) = 2 \))
\[
\begin{align*}
2 \partial_x u^{1,0,+}_p - \partial_{Y_+} Y_+ u^{1,0,+}_p &= 0, \\
|u^{1,0,+}_p|_{x=0} &= 0, \\
|u^{1,0,+}_p|_{Y_+=0} &= -u^{1,0,+}_p|_{y=2}, \\
\partial_{Y_+} u^{1,0,+}_p|_{Y_+ \to \infty} &= 0.
\end{align*}
(2.22)
\]

We now cut-off \( u^{1,0,+}_p = (u^{1,0,+}_p, v^{1,0,+}_p) \) to obtain the first boundary layer corrector \( u^{1,+}_p = (u^{1,+}_p, v^{1,+}_p) \) near \( y = 2 \):
\[
u^{1,+}_p = \chi(\sqrt{\epsilon} Y_+^{1/2}) u^{1,0,+}_p - \frac{\sqrt{\epsilon}}{a_0} \chi' \left( \frac{\sqrt{\epsilon} Y_+^{1/2}}{a_0} \right) \int_0^y v^{1,0,+}_p = \chi\left( \frac{\sqrt{\epsilon} Y_+^{1/2}}{a_0} \right) v^{1,0,+}_p, \quad v^{1,+}_p := \chi\left( \frac{\sqrt{\epsilon} Y_+^{1/2}}{a_0} \right) v^{1,0,+}_p, \quad (2.23)
\]
where $\chi$ is the cut-off function defined in (1.6), $a_0 > 0$ is a constant small enough. After cutting off $f^{2,3}_p$, we have the contribution with $O(\epsilon^{\frac{7}{2}})$ order to the next layer

$$
C_{cut}^{1,+} = \frac{\mu}{a_0} \varepsilon^2 \chi' v_p^{1,0,+} + 3 \frac{1}{a_0} \varepsilon^2 \chi' \partial Y_+ u_p^{1,0,+} + 3 \frac{1}{a_0} \varepsilon^2 \chi'' u_p^{1,0,+} - \frac{1}{a_0} \varepsilon^2 \chi'' \int_0^x v_p^{1,0,+}(s, Y_+) ds. \tag{2.24}
$$

We also have higher order terms that contribute to the error:

$$
C_{quad}^{1,+} := \varepsilon^2 (u_e^{1} + u_p^{1,+}) \partial_x u_p^{1,+} + \varepsilon^2 v_e^{1,+} \partial_x u_e^{1,+} + \varepsilon^2 v_p^{1,0,+} + \varepsilon^2 v_e^{1,0,+} + \varepsilon^2 v_p^{1,+} + \varepsilon^2 v_e^{1,+} - \varepsilon^2 u_p^{1,+}. \tag{2.25}
$$

Besides, due to the approximation of $\mu(y)$ by $\mu(2) = 2$ in the support of the cut-off function $\chi'\frac{2}{a_0}$, we have another contribution with $O(\epsilon^{\frac{1}{2}})$ order to the error defined by

$$
C_{approx}^{1,+} := (2 - \mu(y)) \left[ \chi (\frac{2 - y}{a_0}) \partial_x u_p^{1,0,+} + \frac{1}{a_0} \varepsilon^2 \chi' v_p^{1,0,+} \right]. \tag{2.26}
$$

For the higher order terms in the second momentum equation, we will use our auxiliary pressure to move it to the top equation. This is achieved by defining the first auxiliary pressure, $P_p^{1,a,+}$ to zero out the terms contributed from

$$
\varepsilon^2 (\mu + \varepsilon u_e^{1}) v_p^{1,+} + \varepsilon u_p^{1,+} (\varepsilon v_e^{1} + \varepsilon^2 v_p^{1,+}) + \varepsilon^2 v_e^{1,0,+} + \varepsilon^2 v_p^{1,0,+} + \varepsilon^2 v_e^{1,+} v_p^{1,0,+} - \varepsilon^2 v_p^{1,0,+} - \varepsilon^2 v_p^{1,+} Y_+, Y_e + \varepsilon^2 P_{pY_+}^{1,a,+} = 0,
$$

which therefore motivates our definition of

$$
-\varepsilon^2 P_p^{1,a,+} : = \int_{Y_+}^{\infty} \left( \varepsilon^2 (\mu + \varepsilon u_e^{1}) v_p^{1,+} + \varepsilon u_p^{1,+} (\varepsilon v_e^{1} + \varepsilon^2 v_p^{1,+}) + \varepsilon^2 v_e^{1,0,+} + \varepsilon^2 v_p^{1,0,+} + \varepsilon^2 v_e^{1,+} v_p^{1,0,+} - \varepsilon^2 v_p^{1,0,+} - \varepsilon^2 v_p^{1,+} Y_+, Y_e - \varepsilon^2 v_p^{1,+} Y_+ - \varepsilon^2 v_p^{1,+} Y_e \right) dY'. \tag{2.27}
$$

As a result, we can define the forcing for the next order weak boundary layer via

$$
F^{2,+} := -\varepsilon^2 \left( -\varepsilon^2 C_{cut}^{1,+} + \varepsilon^2 C_{approx}^{1,+} + \varepsilon^2 C_{quad}^{1,+} + \varepsilon^2 \partial_x P_p^{1,a,+} \right). \tag{2.28}
$$

Computing $F^{i,+}$ when $i = 3, \ldots, M$ are almost in the same manner. Since we will put the terms with $H^2$ norm less than $\varepsilon^{\frac{7}{2} + \gamma}$ into the remainders ($F_u, F_v$), we only need the auxiliary pressure $P_p^{i,a,+}$ for $i = 1, 2, 3$. When $i > 3$, we will take $P_p^{i,a,+} = 0$. Please also notify that the interaction terms containing $u_e^{i} - u_p^{i,+}$ in $F^{i,+}$ with different scale would not cause any trouble here. In fact, we only have $\varepsilon^2 u_e^{i} - u_p^{i,+}$ in $C_{quad}^{i,+}$ and $\varepsilon^2 v_e^{i} - v_p^{i,+}$ in $P_p^{i,a,+}$. Both of the two terms containing a
boundary layer corrector near \( y = 2 \), which would decay rapidly when \( Y_+ \to \infty \).

Other interaction terms with form \( u_i^{i-1} u_p^{i+} \) will be put into the remainders \((F_u, F_v)\)

We thus derive the general equation that we study for the boundary layer correctors when \( 1 \leq i \leq M - 1 \) (that is, all but the last layer):

\[
2 \partial_x u_p^{i,0,+} - \partial_{Y_+} Y_+ u_p^{i,0,+} = F^{i,+},
\]

\[
u_p^{i,0,+} \big|_{x=0} = 0, \quad u_p^{i,0,+} \big|_{Y_+=0} = -u_e^{i,+} \big|_{y=2}, \quad u_p^{i,0,+} \big|_{Y_+ \to \infty} = 0
\]

Let us therefore consider the abstract problem (dropping indices):

\[
2 \partial_x u - \partial_{Y}^2 u = F, \quad (x, Y) \in (0, L) \times (0, \infty) \tag{2.29a}
\]

\[
v := \int_{Y}^{\infty} \partial_x u \, dY', \tag{2.29b}
\]

\[
u \big|_{x=0} = 0, \quad u \big|_{Y=0} = g(x), \quad u \big|_{Y \to \infty} = 0. \tag{2.29c}
\]

For this abstract problem, we have the following Lemma:

**Lemma 2.4.** Assume that \( F(x, Y) \) decays rapidly at infinity, i.e., for any \( m \geq 0 \), there is a constant \( M > 0 \), such that

\[
\|(1 + Y)^m \partial_x^m \partial_Y F\| \leq M \text{ for } 0 \leq 2n + l \leq K,
\]

where \( K > 0 \) is a constant sufficiently large. Then there exists a unique solution \((u, v)\) to \((2.29a) - (2.29c)\) that satisfies the following estimate:

\[
\|(1 + Y)^m \partial_x^m \partial_Y \{u, v\}\| \leq C(m, n, l)(M + \|g\|_{H^{2K+1}}) \text{ for any } 2n + l \leq K + 2, \tag{2.30}
\]

where the constant \( C \) does not depend on \( Y \).

**Proof.** First, a standard homogenization enables us to consider the Dirichlet problem, \( g = 0 \), up to modifying the forcing \( F \). Indeed, fixing a cut-off function \( \chi(Y) \) so that \( \chi(0) = 1, \chi(Y) = 0 \), when \( Y \geq 2 \), and \( \int_0^\infty \chi(Y') \, dY' = 0 \), we can consider the unknowns

\[\tilde{u} := u - \chi(Y)g(x),\]

which will satisfy the system

\[
2 \partial_x \tilde{u} - \partial_Y^2 \tilde{u} = \tilde{F}, \quad (x, Y) \in (0, L) \times (0, \infty) \tag{2.31}
\]

\[
\tilde{u} \big|_{x=0} = 0, \quad \tilde{u} \big|_{Y=0} = 0, \quad \tilde{u} \big|_{Y \to \infty} = 0. \tag{2.32}
\]

Above, the modified forcing

\[
\tilde{F} := F - 2\chi(Y)g'(x) + g(x)\chi''(Y). \tag{2.33}
\]
We now drop the $\tilde{u}, \tilde{v}$ notation, and simply consider the homogenized problem above.

First we multiply (2.31) by $u_x$ to obtain
\[
\frac{\partial_x}{2} \int_0^\infty u_Y^2 \, dY + \int_0^\infty |\partial_x u|^2 \, dY = \int_0^\infty F u_x \, dY. \tag{2.34}
\]

Consequently, we have
\[
\int_\Omega u_x^2 \, dY \, dx + \sup_{x \in [0, L]} \int_0^\infty u_Y^2 \, dY \leq C \|F\|_{L^2(\Omega)}^2. \tag{2.35}
\]

From equation (2.31), we have $u_x|_{x=0} = \frac{1}{2} F(0, Y)$. Similarly, we can obtain from equation (2.31)
\[
|\partial^m_x u(0, \cdot)|_{L^2} \leq C(n) \|F\|_{H^{n+1}(\Omega)},
\]

for any $n \geq 0$.

Now we take the derivative of equation (2.31) with respect to $x$ for $n$ times. Repeating the above process and using the resulting equation, we have
\[
\int_\Omega (\partial_x^m u_x)^2 \, dY \, dx + \int_\Omega (\partial_x^m u_Y Y)^2 \, dY \, dx + \sup_{x \in [0, L]} \int_0^\infty (\partial_x^m u_Y)^2 \, dY \leq C \|F\|_{H^{n+1}}^2.
\]

To get weighted estimates, we multiply equation (2.31) with $u_x (1 + Y)^{2m}$ to have
\[
2 \int_0^\infty (1 + Y)^{2m} u_x^2 \, dy + \frac{d}{dx} \int_0^\infty (1 + Y)^{2m} (u_Y)^2 \, dy = \int_0^\infty (1 + Y)^{2m} u_x F \, dy - 2m \int_0^\infty (1 + Y)^{2m-1} u_x u_Y \, dy \leq C \|(1 + Y)^m u_x(x, \cdot)\|_{L^2_Y}\{\|(1 + Y)^m F(x, \cdot)\|_{L^2_Y} + \|(1 + Y)^m u_Y(x, \cdot)\|_{L^2_Y}\}.
\]

Consequently
\[
\|(1 + Y)^m u_x\|_{L^2(\Omega)} + \sup_{x \in [0, L]} \|(1 + Y)^m u_Y(x, \cdot)\|_{L^2_Y} \leq C(m) \|(1 + Y)^m F\|_{L^2(\Omega)}. \tag{2.36}
\]

Similarly, we can obtain
\[
\|(1 + Y)^m \partial_x^n u_x\|_{L^2(\Omega)} + \|(1 + Y)^m \partial_x^n u_Y Y\|_{L^2(\Omega)} + \sup_{x \in [0, L]} \|(1 + Y)^m \partial_x^n u_Y(x, \cdot)\|_{L^2_Y} \leq C(m, n) (M + \|g\|_{H^{K+1}}). \tag{2.37}
\]

Finally, for any $m, n, l \geq 0$, we use the differential operator $\partial_x^n \partial_Y^l$ to act on equation (2.31), and then multiply the resulting equation with $(1 + Y)^{2m} \partial_x^n \partial_Y^l u_x$, to get
\[
\|(1 + Y)^m \partial_x^n \partial_Y^l u_x\|_{L^2(\Omega)} + \|(1 + Y)^m \partial_x^n \partial_Y^{l+2} u\|_{L^2(\Omega)} + \sup_{x \in [0, L]} \|(1 + Y)^m \partial_x^n \partial_Y^{l+1} u(x, \cdot)\|_{L^2_Y} \leq C(m, n, l) (M + \|g\|_{H^{K+1}}). \tag{2.38}
\]

Then (2.30) follows immediately.
For $i = M$, we consider now
\begin{align*}
2\partial_x u - \partial^2_Y u &= F, \quad (x, Y) \in (0, L) \times (0, \infty) \quad (2.39a) \\
v &= -\int_0^Y \partial_x u \, dY', \\
u_{|x=0} &= 0, \quad u_{|Y=0} = g(x), \quad \partial_Y u_{|Y \to \infty} = 0. \quad (2.39c)
\end{align*}
Compared to (2.29a) – (2.29c), the main difference here is $v(x, 0) = 0$. We would have the following lemma

**Lemma 2.5.** Assume that $F(x, Y)$ decays rapidly at infinity, i.e., for any $m \geq 0$, there is a constant $M > 0$, such that
\[\|(1 + Y)^m \partial^n_x \partial^l_Y F\| \leq M \quad \text{for} \quad 0 \leq 2n + l \leq K.\]
Then, there exists a unique solution $(u, v)$ of (2.39a) – (2.39c), satisfying the following estimates:
\begin{align*}
\|(1 + Y)^m \partial^n_x \partial^l_Y u\| &\leq C(m, n, l) (M + \|g\|_{H^{K+1}}) \quad \text{for any} \quad 0 \leq 2n + l \leq K + 2, \\
\|(1 + Y)^m \partial^n_x \partial^l_Y v\| &\leq C(m, n, l) (M + \|g\|_{H^{K+1}}) \quad \text{for any} \quad 0 \leq 2n + l \leq K + 1, \\
\|\partial^n_x \{\frac{v}{Y}\}\| &\leq C(m, n, l) (M + \|g\|_{H^{K+1}}) \quad \text{for any} \quad 0 \leq 2n \leq K + 2.
\end{align*}

The proof of Lemma 2.5 would be almost the same as Lemma 2.4, and we omit the details here.

### 2.3 Proof of Theorem 2.1

Combining Lemma 2.3-Lemma 2.5, Lemma A.2, Lemma A.3, we have the following Lemma:

**Lemma 2.6.** Let $u_p^{i, \pm}, v_p^{i, \pm}$ be solutions obtained in the above process, then we have for any $m \geq 0$,
\begin{align*}
\|(1 + Y)^m \partial^n_x \partial^l_Y \{u_p^{i, \pm}, v_p^{i, \pm}\}\| &\leq C_{m, n, l, \alpha_0} \quad \text{for} \quad 1 \leq i \leq M - 1, \\
\|(1 + Y)^m \partial^n_x \partial^l_Y u_p^{M, \pm}\| &\leq C_{m, n, l, \alpha_0} \quad \text{for} \quad l \geq 1, \\
\|\partial^n_x \{u_p^{M, \pm}, v_p^{M, \pm}\}/Y\| &\leq C_{m, n, l, \alpha_0},
\end{align*}
where $n, l$ are integers satisfying $2n + l \leq k + 2$, $(u_p^{M, \pm}, v_p^{M, \pm})$ are the last boundary layer correctors.

**Proof of Theorem 2.1**
First of all, the divergence free condition of \( u_s \) and the boundary conditions in (2.4) follow directly from the process of construction of \( u_p^{i,\pm} \).

Then recalling the definition of \( Y_\pm \) and the expression of \( u_s \) in (2.2), estimates (2.5)-(2.7) follows immediately from Lemma 2.6.

Next, using definition (1.12)-(1.13), \( F_u, F_v \) can be divided into two parts, i.e.

\[
\varepsilon M_0 F_u = C_{Euler,u} + C_{prandtl,u},
\]
\[
\varepsilon M_0 F_v = C_{Euler,v} + C_{prandtl,v},
\]

where \( C_{Euler,u}, C_{Euler,v} \) are defined in (2.17)-(2.18). By Lemma 2.3, the Euler contributions to the error in (2.17) and (2.18) can be written as

\[
\begin{cases}
C_{Euler,u} = \varepsilon^2 (u_e^1 \partial_x u_e + v_e^1 \partial_y v_y - \Delta u_e^1) + O(\varepsilon^3), \\
C_{Euler,v} = \varepsilon^2 (u_e^1 \partial_x v_e + v_e^1 \partial_y u_e - \Delta v_e^1) + O(\varepsilon^3),
\end{cases}
\]

(2.40)

while the weak boundary layer contributions to the error can be arbitrarily small if we expanding enough sub-layers. Here \( M = 10 \) would be sufficient. If we define

\[
T_{u,\varepsilon^2} = C_{Euler,u}, \quad T_{v,\varepsilon^2} = C_{Euler,v}, \quad F_{u,\varepsilon^3} = C_{prandtl,u}, \quad F_{v,\varepsilon^3} = C_{prandtl,v},
\]

then (2.10) follows immediately by Lemma 2.3 and Lemma 2.6 and we finish the proof of Theorem 2.1.

3 Existence and uniform-in-\( \varepsilon \) estimates of solutions to the linearized system

In this section, we will study the existence as well as the uniform-in-\( \varepsilon \) estimates of the solutions to the linearized system of (1.2).

For given \((\bar{u}, \bar{v})\) and \( F = (F_1, F_2) \), we consider the linear system

\[
\text{div} \mathbf{u} = 0, \quad (3.1)
\]
\[
u_s u_x + u_s y v + u_s x u + v_s y u - \varepsilon \Delta u + \partial_x P = N_1(\bar{u}, \bar{v}) + F_1 \quad (3.2)
\]
\[
u_s v_x + v_s y v + v_s x u + v v_y - \varepsilon \Delta v + \partial_y P = N_2(\bar{u}, \bar{v}) + F_2, \quad (3.3)
\]

with boundary condition:

\[
v_x|_{x=0} = 0; \quad u|_{x=0} = 0; \quad v|_{x=0} = 0; \quad v_{xx}|_{x=0} = 0; \quad v|_{x=L} = 0; \quad v_{xx}|_{x=L} = 0;
\]
\[
u|_{y=0} = v|_{y=0} = v|_{y=0} = u|_{y=2} = v|_{y=2} = v|_{y=2} = 0, \quad (3.4)
\]

where \( u_s, v_s \) are known functions defined in (1.8).

First we differentiate (3.2) with \( y \) and (3.3) with \( x \), using the divergence free condition to obtain the following equation for \( \text{curl} \mathbf{u} = u_y - v_x \):

\[
- u_s v_{yy} + u_{syy} v - u_s v_{xx} + u_{sxx} v + S(u, v) - \varepsilon \Delta (u_{yy} - v_x) = \text{curl} \mathbf{N} + \text{curl} \mathbf{F}, \quad (3.5)
\]
where \( S(u, v) = \partial_y(u_{sx}u + v_{sy}) - \partial_x(v_{sy} + v_{sx}). \)

Recalling that \( q = \frac{v}{u_x}, \) equation (3.5) can be written as

\[-\partial_y(u^2_qy) - \partial_x(u^2_qx) + S(u, v) - \varepsilon \Delta (u_y - v_x) = \text{curl } N + \text{curl } F. \tag{3.6}\]

3.1 Existence of solutions to the linearized system

In this subsection, we will prove the existence of solutions to the linear system (3.7)-(3.9) with boundary condition (3.4). Due to the boundary condition \( v_{xx} = 0 \) on \( x = L \), we cannot obtain the existence of the solutions directly from the second order equations (3.7)-(3.9). Instead, by introducing the stream function, we will first prove the existence of solutions to a fourth order equation for the stream function. Then by an elliptic equation of second order for the pressure, we can prove the existence of solutions to the second order linear system (3.7)-(3.9).

For given \( f = (f_1, f_2) \), we will first consider the following linear system:

\[
\text{div } u = 0, \tag{3.7}
\]
\[
u_{sx}u_{x} + v_{sy}v_{x} - \varepsilon \Delta u + \partial_x P = f_1, \tag{3.8}
\]
\[
u_{sx}v_{y} + v_{sy}v_{y} + \varepsilon \Delta v + \partial_y P = f_2, \tag{3.9}
\]

with boundary condition (3.4). The main theorem for this subsection reads as

**Theorem 3.1.** For any given \( f = (f_1, f_2) \in H^1(\Omega) \times H^1(\Omega), \) there exists a unique solution \((u, v, \nabla P)\) to the system (3.7)-(3.9) satisfying the boundary condition (3.4) and the following estimate:

\[
\|u\|_{H^3} + \|v\|_{H^3} + \|\nabla P\|_{H^1} \leq C(\varepsilon)\|f\|_{H^1}. \tag{3.10}
\]

To prove Theorem 3.1, we first consider the following equation for \( \text{curl } u \):

\[
- u_{sx}v_{yy} + u_{sy}v_{x} - \varepsilon \Delta u_{x} - \partial_x P = \text{curl } f, \tag{3.11}
\]

Then we introduce the stream-function \( \psi \) of \((u, v)\):

\[
\psi_x = -v, \quad \psi_y = u. \tag{3.12}
\]

If we assume that \( \psi(0, 0) = 0 \), then \( \psi \) satisfies the following system

\[
u_{sx}\psi_{yy} + u_{sy}\psi_{xx} - \Delta u_{sx}\psi_{x} + S(\psi_y, -\psi_x) - \varepsilon \Delta^2 \psi = \text{curl } f \quad \text{in } \Omega; \tag{3.13}
\]

\[
\psi_{xx}|_{x=0} = \psi_{y}|_{x=0} = 0; \quad \psi_{x}|_{x=L} = 0; \quad \psi_{xxx}|_{x=L} = 0; \tag{3.14}
\]

\[
\psi_{y}|_{y=0} = \psi_{y}|_{y=2} = \psi_{y}|_{y=2} = 0. \tag{3.15}
\]

To prove the existence of solutions to system (3.13)-(3.15), we first consider the following biharmonic equation

\[
\Delta^2 \psi = f, \quad \text{in } \Omega, \tag{3.16}
\]
with boundary condition \((3.14)-(3.15)\).

We define the function space via:

\[
H^m_0 = \{ v \in H^m(\Omega) \mid (3.14) - (3.15) \text{ are satisfied} \}.
\]

For any \(\psi, \phi \in H^2_0(\Omega)\), we define the bilinear form

\[
B[\psi, \phi] = \int_{\Omega} (\psi_{xx} \phi_{xx} + 2\psi_{xy} \phi_{xy} + \psi_{yy} \phi_{yy}) \, dx \, dy.
\]

**Definition 3.2.** We say that \(\psi \in H^2_0\) is a weak solution to \((3.16)\) with boundary conditions \((3.14)-(3.15)\) if

\[
B[\psi, \phi] = (f, \phi) \quad \text{for all } \phi \in H^2_0. \tag{3.17}
\]

**Lemma 3.3.** For any \(f \in L^2(\Omega)\), there exists a unique solution \(\psi \in H^4_0(\Omega)\) to equation \((3.16)\) satisfying boundary condition \((3.14)-(3.15)\), and the following estimate:

\[
\|\psi\|_{H^4} \leq C \|f\|_{L^2}, \tag{3.18}
\]

where \(C\) is a constant depending on \(\Omega\).

**Proof.** First, it is easy to prove the existence of weak solution by Lax-Milgram Theorem. Obviously, there exist constants \(C_1, C_2 > 0\) such that for any \(\psi, \phi \in H^2_0(\Omega)\),

\[
B[\psi, \phi] \leq C_1 \|\psi\|_{H^2} \|\phi\|_{H^2},
\]

and

\[
B[\psi, \psi] \geq C_2 \|\psi\|_{H^2}^2.
\]

So \(B\) is a coercive bilinear operator. The Lax-Milgram Theorem implies that there exists a unique weak solution \(\psi \in H^2_0(\Omega)\) to system \((3.14)-(3.16)\) satisfying \((3.17)\) and

\[
\|\psi\|_{H^2} \leq C \|f\|_{L^2}, \tag{3.19}
\]

Moreover, since \(f \in L^2(\Omega)\), we have \((3.16)\) holds a.e. in \(\Omega\).

Next, to obtain the global \(H^4\) regularity of \(\psi\) in \(\Omega\), we multiply \((3.16)\) with \(\psi_{xx}\). Using the boundary condition \((3.14)-(3.15)\) and estimate \((3.19)\), we have

\[
\int_{\Omega} [\psi_{xxx}^2 + 2\psi_{xxy}^2 + \psi_{xyy}^2] \, dx \, dy = \int_{\Omega} f \psi_{xx} \, dx \, dy \leq C \|f\|^2. \tag{3.20}
\]

Then we make the odd extension of \(\psi\) and \(f\) with respect to \(x = 0\), and denote by

\[
\psi_{\text{odd}}(x, y) = \begin{cases} 
\psi(x, y) & \text{if } x > 0, \\
-\psi(-x, y) & \text{if } x < 0,
\end{cases} \quad
f_{\text{odd}}(x, y) = \begin{cases} 
f(x, y) & \text{if } x > 0, \\
-f(-x, y) & \text{if } x < 0.
\end{cases}
\]
By \( (3.20) \) and the boundary condition on \( x = 0 \), we have \( \partial_x \psi_{\text{odd}} \in H^2(\Omega) \) and

\[
\int_{\Omega} \left[ (\partial_{xxx} \psi_{\text{odd}})^2 + 2(\partial_{xxy} \psi_{\text{odd}})^2 + (\partial_{xyy} \psi_{\text{odd}})^2 \right] dx dy \leq C \| f \|_{L^2(\Omega)}^2, \tag{3.21}
\]

here \( \tilde{\Omega} = [-L, L] \times [0, 2] \).

Next we define a smooth cutoff function \( 0 \leq \eta(x) \leq 1 \) satisfying

\[
\eta(x) = \begin{cases} 
1 & \text{if } -\frac{3}{4} L \leq x \leq \frac{3}{4} L, \\
0 & \text{if } -L \leq x \leq -\frac{4}{5} L, \text{ or } \frac{4}{5} L \leq x \leq L.
\end{cases}
\]

If we denote \( \tilde{\psi} = \eta(x) \psi_{\text{odd}} \) and \( \tilde{\Omega} \subset \tilde{\Omega} \) is a smooth domain containing \( [-\frac{3}{4} L, \frac{4}{5} L] \times [0, 2] \), then it is easy to check that \( \tilde{\psi} \) is the weak solution to the following bihamornic system with Dirichlet boundary condition in \( \tilde{\Omega} \):

\[
\begin{aligned}
\Delta^2 \tilde{\psi} &= \tilde{f}, & \text{in } \tilde{\Omega} \\
\tilde{\psi} &= \partial_n \tilde{\psi} = 0, & \text{on } \partial \tilde{\Omega},
\end{aligned}
\]

where \( \tilde{f} = \eta' \partial_{xxx} \psi_{\text{odd}} + \eta'' \partial_{xx} \psi_{\text{odd}} + \eta''' \partial_x \psi_{\text{odd}} + \eta'''' \partial_{yy} \psi_{\text{odd}} + \eta' \partial_{xyy} \psi_{\text{odd}} + \eta'' \partial_{yy} \psi_{\text{odd}} \).

Using \( (3.19) \) and \( (3.21) \), we have

\[
\| \tilde{f} \|_{L^2} \leq C \| f \|_{L^2}.
\]

Then by Theorem 2.20 in [8], we have \( \tilde{\psi} \in H^4(\tilde{\Omega}) \) satisfying

\[
\| \tilde{\psi} \|_{H^4(\tilde{\Omega})} \leq C \| f \|_{L^2(\tilde{\Omega})}.
\]

Similarly we can make the even extension of \( \psi \) and \( f \) with respect to \( x = L \) and repeat the process above to obtain

\[
\| \psi \|_{H^4(\Omega)} \leq C \| f \|_{L^2(\Omega)}.
\]

Thus, we finish the proof of the Lemma. \( \square \)

Next we turn to the following system

\[
\begin{aligned}
u_{xx} \psi_{xyy} + u_{x} \psi_{xxx} - \Delta u_{xx} \psi_x + S(\psi_y, -\psi_x) - \varepsilon \Delta^2 \psi &= f & \text{in } \Omega; \\
\psi_{xx}|_{x=0} = \psi|_{x=0} &= 0; & \psi_x|_{x=L} = 0; & \psi_{xxx}|_{x=L} = 0; \\
\psi|_{y=0} = \psi|_{y=0} &= \psi|_{y=0} = \psi|_{y=2} = \psi|_{y=2} = 0.
\end{aligned}
\tag{3.22-3.24}
\]

By Lemma 3.3 and Leray-Schauder fixed point theory, which can be found in [13, Chapter 11], we have the following lemma:

**Lemma 3.4.** For any \( f \in L^2(\Omega) \), there exists a unique solution \( \psi \in H^4(\Omega) \) of the boundary value problem \( (3.22)-(3.24) \) satisfying

\[
\| \psi \|_{H^4} \leq C(\varepsilon) \| f \|_{L^2}.
\]
Then by Lemma 3.3, $T\in B\times [0,1] \rightarrow B$ by the solution of the following system:

\[ -\varepsilon \Delta^2 \psi = t \{ f - [u_s \phi_{xyy} - \Delta u_s \phi_x + u_s \phi_{xxx} + S(\phi_y, -\phi_x)] \}, \quad (3.26) \]
\[ \psi_{xx}|_{x=0} = \psi|_{x=0} = 0; \quad \psi_x|_{x=L} = 0; \quad \psi_{xxx}|_{x=L} = 0; \quad \psi|_{y=2} = \psi|_{y=2} = 0. \quad (3.27) \]
\[ \psi|_{y=0} = \psi|_{y=0} = 0. \quad (3.28) \]

Then by Lemma 3.3 $T$ is a compact operator and $T(\phi, 0) = 0$ for all $\phi \in B$.

Next, remember that $\psi_x = -v, \quad \psi_y = u$, $q = \frac{1}{u_s}$, if we replace $\phi$ by $\psi$, then (3.26) can be written as

\[ -t[\partial_y(u_s^2 q_y) - \partial_x(u_s^2 q_x) + S(\psi_y, -\psi_x)] - \varepsilon \Delta(u_y - v_x) = tf. \quad (3.29) \]

Multiplying (3.29) with $tv$ to obtain

\[ -\int_\Omega t^2 v[\partial_y(u_s^2 q_y) - \partial_x(u_s^2 q_x) + S(\psi_y, -\psi_x)]dxdy - \varepsilon \int_\Omega \Delta(u_y - v_x)tvdxdy = \int_\Omega tvfdxdy. \quad (3.30) \]

By similar process in Lemma 3.9, we have

\[ t^2 \int_\Omega u_s(v_y^2 + v_x^2)dxdy + t^2 \int_\Omega (\frac{1}{2}u_{syy}v^2 + u_{sxx}v^2 + u_{sx}v v_x)dxdy \geq t^2 \||u_s v_y||^2 + t^2 \||u_s v_x||^2, \quad (3.31) \]

and

\[ t^2 \int_\Omega S(u, v)dxdy \geq t^2 \int_\Omega [-\partial_y(u_s x u + v_s u_y) + \partial_x(v_s v_y + v_s v_x)]v dxdy \]
\[ = t^2 \int_\Omega (u_{sx} u + v_s u_y)vdxdy + \int_\Omega (v_s v_{xy} + v_{sxx} u)v dxdy \]
\[ \leq t^2 \{ L^2 \||u_s v_y||^2 + L^2 \||u_s v_y|| \|v_{yy}|| + ||v||_L^2 (\|v_{xy}|| + L^2 ||u_s v_y||) \}
\[ \leq t^2 L^\frac{1}{2} (\||u_s v_y||^2 + \|u_{sxx} v_x||^2) + \varepsilon^2 ||u_s \nabla^2 q||^2, \quad (3.32) \]

while

\[ -\varepsilon \int_\Omega \Delta(u_y - v_x)tvdxdy \]
\[ = t^2 \varepsilon \int_0^2 u_y^2 |x=L + t\varepsilon \int_0^2 v_y^2 |x=0 - t\varepsilon \int_0^2 v_{xx} v |x=0 - \frac{1}{2} t\varepsilon \int_0^2 v_x |x=L \]
\[ \geq t\varepsilon |u_y(L, \cdot)|^2 + t\varepsilon |v_y(0, \cdot)|^2 - t\varepsilon u_s^\frac{1}{2} v_{xx}(0, \cdot)|u_s^\frac{1}{2} v(0, \cdot)| - \varepsilon ||u_s v_x|| \sqrt{u_s \nabla^2 q}, \]

By Lemma 3.3, $T$ is a compact operator and $T(\phi, 0) = 0$ for all $\phi \in B$.
where
\[\int t^2 \varepsilon^2 |u_x^2 v(0, \cdot)|^2 |u_x^{-1} v_{xx}(0, \cdot)|^2 \]
\[= t^2 \varepsilon^2 \left\{ \int_{\Omega} (2u_x^2 v_{ux} + \frac{1}{2} u_x^{-1} u_{xx} v^2) dx \right\} \int_{\Omega} \left[ u_x^{-1} v_{xx} v_{xxx} - \frac{1}{2} u_x^{-1} u_{xx} v_{xx}^2 \right] dx dy \]
\[\leq t^2 \varepsilon^2 \left( \| \sqrt{u_x v_x} \|_\varepsilon + \varepsilon^\frac{1}{2} \| v \|_\varepsilon \right)^2 \left( \| \sqrt{u_x q_{xx}} \| + \| q_x \| \right) \| v_{xxx} \| + \varepsilon^\frac{1}{2} \| \sqrt{u_x q_{xx}} \| + \| q_x \|_\varepsilon \right)^2 \]
\[\lesssim L^\frac{1}{2} t^2 \| \sqrt{u_x v_x} \|^2 (\varepsilon \| \sqrt{u_x v_x} \| + \varepsilon^3 \| v_{xxx} \|). \quad (3.33)\]

Combining \((3.30)-(3.33)\) and \((3.49)\), we have
\[t^2 \| \sqrt{u_x v_y} \|^2 + t \| \sqrt{u_x v_x} \|^2 \leq C[L^{\frac{1}{2}} (\varepsilon \| \sqrt{u_x v_x} \| + \varepsilon^3 \| v_{xxx} \|) + \| f \|^2], \quad (3.34)\]

where the constant \(C\) does not depend on \(t, \varepsilon\).

Then, we multiply \((3.29)\) with \(q_x\) to get
\[\int_{\Omega} \left[ -t \partial_y (u_x^2 \partial_y q) - t \partial_x (u_x^2 q_x) + t S(u, v) - \varepsilon \Delta (u_y - v_x) \right] q_x dx dy = \int_{\Omega} f q_x dx dy. \quad (3.35)\]

Similar to the process in Lemma 3.10, we obtain
\[
\int_{\Omega} t q_x \left[ - \partial_y (u_x^2 \partial_y q) - \partial_x (u_x^2 q_x) \right] dx dy
\]
\[= -\frac{1}{2} t \int_{\Omega} u_x^2 q_y^2 |x=0| dy - \frac{1}{2} t \int_{\Omega} u_x^2 q_x^2 |x=L| dy + \frac{1}{2} t \int_{\Omega} u_{xx}^2 q_x^2 |x=0| dy
\]
\[= -t \int_{\Omega} u_x u_{xx} (q_x^2 + q_y^2) dx dy
\]
\[\lesssim -t (|u_x q_y(0, \cdot)|^2 + |u_x q_x(L, \cdot)|^2) + L t \varepsilon \int_{\Omega} (u_x^2 q_{xx} + u_{xx}^2 q_{xy}) dx dy, \quad (3.36)\]

and
\[t \int_{\Omega} S(u, v) q_x dx dy|\]
\[= t \int_{\Omega} \left[ - \partial_y (u_x s u + v v_y + v s u) + \partial_x (v s v_y + v s u) \right] q_x dx dy|\]
\[= t \int_{\Omega} \left[ (u s u_x + v v_y) q_{xy} + (v s v_y + v s u) q_{xx} \right] dx dy|\]
\[\leq L^{\frac{3}{2}} \| \sqrt{u_x v_{xy}} \| (\varepsilon^{\frac{1}{2}} \| \sqrt{u_x v_{xy}} \| + \varepsilon^{\frac{1}{2}+3} \| v_{xxx} \|), \quad (3.37)\]

while by virtue of \((3.68)\),
\[-\varepsilon \int_{\Omega} \Delta (u_y - v_x) q_x dx dy \lesssim -\varepsilon \| \sqrt{u_x v_x} \|_\varepsilon + \varepsilon^{\frac{1}{2}+3} \| v_{xxx} \|. \quad (3.38)\]
Putting (3.35)-(3.38) together, we conclude 

$$\varepsilon \| \sqrt{u_s} \nabla^2 q \|^2 \leq C(\varepsilon)\| f \|^2 + C_0(L_4^2 t^2 \| \sqrt{u_s} \nabla v \|^2 + \varepsilon^{1/3} \| v_{xxx} \|^2),$$

(3.39)

where the constant $C(\varepsilon)$ does not depend on $t$, while $C_0$ does not depend on $t, \varepsilon$.

Next, we multiply (3.29) with $\varepsilon^2 v_{xxx}$ to find that

$$-\varepsilon^3 \int_\Omega v_{xxx} \Delta (u - v) \, dx \, dy = \varepsilon^3 \int_\Omega (v_{xxx}^2 + v_{xyy}^2 + v_{xxy}^2) \, dx \, dy$$

$$\leq \varepsilon^2 \int_\Omega t|f - u s v_{yy} + u s v_{yy} v - u s v_{xxx} + v (u, v)| v_{xxx} \, dx \, dy$$

$$\leq \varepsilon^2 (\| v_{xxx} \| + \| v_{xyy} \|)(\| f \| + \| \sqrt{u_s} \nabla^2 q \|),$$

which implies

$$\varepsilon^3 \| \nabla^2 v_x \|^2 \leq C(\| f \|^2 + \varepsilon \| \sqrt{u_s} \nabla^2 q \|^2),$$

(3.40)

where the constant $C$ does not depend on $\varepsilon, t$.

Combing (3.34) and (3.39) with (3.40), we obtain

$$\varepsilon \| \sqrt{u_s} \nabla^2 q \|^2 + \varepsilon^3 \| \nabla^2 v_x \|^2 \leq C(\varepsilon)\| f \|^2 \quad \text{for any } 0 \leq t \leq 1,$$

where the constant $C(\varepsilon)$ does not depend on $t$. Recalling that $u = \psi_y, v = -\psi_x$, and the divergence free condition of $u$, we have

$$\| t \{ f - [u s \psi_{yy} - u s v_{yy} \psi_x + u s \psi_{xxx} + S(\psi_y, -\psi_x)] \} \| \leq C(\varepsilon)\| f \|_{L^2} + \| v_s u_{yy} \|.$$

Using Lemma 3.3 again and recalling that $|v_s|_{L^\infty} \leq \varepsilon\alpha_0$, we have

$$\varepsilon \| \psi \|_{H^4} \lesssim C(\varepsilon)\| f \|_{L^2} + \| v_s u_{yy} \| \lesssim C(\varepsilon)\| f \|_{L^2} + \varepsilon\alpha_0 \| \psi_{yyy} \|.$$

Taking $\alpha_0$ small enough, then

$$\| \psi \|_{H^4} \lesssim C(\varepsilon)\| f \|_{L^2},$$

(3.41)

where the constant $C(\varepsilon)$ does not depend on $t$.

The Leray-Schauder fixed point theorem implies that there exists a unique $\psi \in B$ with estimate (3.41), such that $\psi = T(\psi, 1)$. Then the proof is completed. □

**Proof of Theorem 3.1:**

To prove Theorem 3.1 we also have to show the existence of the pressure $P$. We first differentiate (3.8) with respect to $x$ and differentiate (3.9) with respect to $y$ to have

$$\Delta P = \partial_x f_1 + \partial_y f_2 - (2u_{sy} v_x + 4v_{sy} v_y + 2v_{sx} v_y).$$

(3.42)
Moreover, we assume the following boundary condition for \( P \) from equations (3.8) and (3.9):
\[
P_x = f_1 - [u_s u_x + u_s v_x + u_{sx} + v_x u_y - \varepsilon \Delta u] \quad \text{on} \quad \{x = 0\} \cup \{x = L\}, \quad (3.43)
\]
\[
P_y = f_2 - [u_s v_x + v_s v_y + v_{sx} + u v_{sy} - \varepsilon \Delta v] \quad \text{on} \quad \{y = 0\} \cup \{y = 2\}. \quad (3.44)
\]

Then, by the theory for elliptic equations, there exists a unique solution \( P \) up to a constant to the system (3.42)-(3.44) satisfying the estimate
\[
\|\nabla P\|_{H^1} \leq C(\varepsilon)(\|f\|_{H^1} + \|u\|_{H^3} + \|v\|_{H^3}) \leq C(\varepsilon)\|f\|_{H^1}. \quad (3.45)
\]

Therefore, we have obtained a triple \((u, v, \nabla P)\) satisfying (3.10). Besides, the equations (3.8)-(3.9) follow immediately from (3.11), (3.42) and the boundary conditions (3.43)-(3.44).

### 3.2 Uniform-in-\( \varepsilon \) estimates

In this subsection, we will work on the uniform-in-\( \varepsilon \) estimates of the linear system (3.1)-(3.3). The main result of this subsection reads as follows.

**Theorem 3.5.** For given \( \bar{u} = (\bar{u}, \bar{v}) \in X \), \((F_1, F_2) \in H^1(\Omega) \times H^1(\Omega)\), if \( u = (u, v) \) is the solution to system (3.1)-(3.4), then we have the following estimate:
\[
\|u\|_X^2 \leq C(\varepsilon^\gamma)\|\bar{u}\|_X^4 + \|F\|_{H^1}^2 + \varepsilon^3\|F\|_{H^2}^2 + (\text{curl} F, q_x), \quad (3.46)
\]
where the constant \( C \) does not depend on \( \varepsilon, L \).

#### 3.2.1 Preparation

Before proving Theorem 3.5, we first introduce the following notations:
\[
A_1^1 = \|\sqrt{u_s} v_y\|^2 + \|\sqrt{u_s} v_x\|^2, \quad (3.47)
\]
\[
A_2^2 = \varepsilon(\|\sqrt{u_s} q_{xx}\|^2 + \|\sqrt{u_s} q_{xy}\|^2 + \|\sqrt{u_s} q_{yy}\|^2) + |u_s q_y(0, \cdot)|^2 + |u_s q_x(L, \cdot)|^2,
\]
\[
A_3^3 = \varepsilon^3(\|v_{xx}\|^2 + \|v_{xy}\|^2 + \|v_{yy}\|^2). \quad (3.48)
\]

**Lemma 3.6.** Let \( u = (u, v) \) be the solution to system (3.7)-(3.8), then we have the following estimates:
\[
\|v\|_{L^2} \lesssim L^{\frac{1}{2}}(\|\sqrt{u_s} v_y\| + \|\sqrt{u_s} v_x\|), \quad (3.49)
\]
\[
\varepsilon^{\frac{1}{5}}\|q_x\| \lesssim A_1 + A_2, \quad (3.50)
\]
\[
\varepsilon^{\frac{1}{4}}\|v\|_{H^1} + \varepsilon\|u_{yy}\|_{L^2} + \varepsilon^{\frac{1}{4}+\tau}\|u\|_{L^\infty} + \varepsilon^{\frac{1}{4}+\tau}\|v\|_{L^\infty} + \varepsilon^{\frac{1}{4}+\tau}\|v_y\|_{L^\infty} \lesssim \|u\|_X, \quad (3.51)
\]

where \( \tau \) is any constant satisfying \( 0 < \tau \ll \gamma \).
Remark 3.7. In fact, \((3.49)\) holds for any \(v \in H^1(\Omega)\) satisfying \(v|_{x=0} = 0\), or \(v|_{x=L} = 0\). For example, we also have

\[
\|\nabla q\| \lesssim L^\frac{1}{4}\|\sqrt{u_s \nabla^2 q}\|.
\]

Proof. First of all, if \(\alpha_1 > 0\), then \(u_s > 0\) on \(y = 2\). Let \(\chi\) be the cutoff function defined in \((1.6)\), \(\delta > 0\) is a small constant to be determined, direct computation shows that

\[
\int_{\Omega} v^2 \, dxdy = \int_{\Omega} \chi^2 \left(\frac{y}{\delta}\right) v^2 \, dxdy + \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\delta}\right)\right] v^2 \, dxdy
= \int_{\Omega} \partial_y(y) \chi^2 \left(\frac{y}{\delta}\right) v^2 \, dxdy + \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\delta}\right)\right] (\int_{L}^x v_s ds)^2 \, dxdy
= -\int y \left[2 \chi^2 v v_y + 2 \delta \chi v' \right] \, dxdy + \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\delta}\right)\right] (\int_{L}^x v_s ds)^2 \, dxdy
\leq 8\delta \|\frac{1}{\sqrt{\varepsilon}} \chi v_y\|^2 + \frac{1}{2} \|\chi v\|^2 + \frac{L}{\delta} \|\sqrt{u_s v_x}\|^2.
\]

(3.52)

If we take \(\delta = L^\frac{1}{4}\) in \((3.52)\) we obtain

\[
\|v\| \lesssim L^\frac{1}{4}(\|\sqrt{u_s v_y}\| + \|\sqrt{u_s v_x}\|).
\]

Secondly

\[
\begin{align*}
\varepsilon^\frac{1}{4}\|q_x\|^2 &= \varepsilon^\frac{1}{4} \int_{\Omega} q_x^2 \, dxdy = \varepsilon^\frac{1}{4} \int_{\Omega} \chi^2 \left(\frac{y}{\varepsilon^\frac{1}{4}}\right) q_x^2 \, dxdy + \varepsilon^\frac{1}{4} \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\varepsilon^\frac{1}{4}}\right)\right] q_x^2 \, dxdy \\
&= \varepsilon^\frac{3}{4} \int_{\Omega} \partial_y(y) \chi^2 \left(\frac{y}{\varepsilon^\frac{1}{4}}\right) q_x^2 \, dxdy + \varepsilon^\frac{1}{4} \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\varepsilon^\frac{1}{4}}\right)\right] (\frac{v_x}{u_s} - \frac{v u_{s x}}{u_s^2})^2 \, dxdy \\
&= -\varepsilon^\frac{3}{4} \int_{\Omega} \left[2 y \chi^2 q_x q_{xy} + 2 y \varepsilon^{-\frac{1}{4}} \chi q_x^2 \right] + \varepsilon^\frac{1}{4} \int_{\Omega} \left[1 - \chi^2 \left(\frac{y}{\varepsilon^\frac{1}{4}}\right)\right] (\frac{v_x}{u_s} - \frac{v u_{s x}}{u_s^2})^2 \\
&\lesssim \varepsilon^\frac{1}{4} \|\sqrt{u_s q_{xy}}\|^2 \|\chi q_x\| + \|\sqrt{u_s v_x}\|^2
\end{align*}
\]

(3.53)

which implies that

\[
\varepsilon^\frac{3}{4}\|q_x\| \lesssim A_1 + A_2.
\]

Next, if \(\alpha_1 = 0\), then we have \(u_s = 0\) on \(y = 2\). The estimate of \(v\) and \(q_x\) around \(y = 2\) can be obtained by arguments similar to the case near \(y = 0\). For \(u_s = \mu\) satisfying \((1.27)\), we can obtain the estimate in a similar manner. Thus, we obtain \((3.49)\) and \((3.50)\).

Finally, recalling the definition of \(X'\), the estimate \((3.51)\) follows immediately from the divergence free condition of \(u\) and the Sobolev imbedding theorem.
Remark 3.8. Let’s remind here that in the following subsection, we often use the fact that if \( u_s = \mu(y), \ v_s = 0 \), then
\[
|u_s|_{C^k} \leq C;
\]
while if \((u_s, v_s)\) is constructed in section 2, we have
\[
|u_s|_{L^\infty} + |u_{syy}|_{L^\infty} \lesssim \alpha_0,
\]
and \(u_s, v_s, v_{sy}\) degenerate near \(y = 0\) and \(y = 2\) as shown in (2.5).

3.2.2 Proof of Theorem 3.5

Let \( A_i, i = 1, 2, 3 \), be defined in (3.47)-(3.48), the proof of Theorem 3.5 will be broken up into several lemmas as well as the method of bootstrap. First we multiply (3.5) with \( v \) to have the following weighted estimates for the first order derivatives:

Lemma 3.9. Under the assumption of Theorem 3.5, the solution \((u, v)\) to system (3.1)-(3.4) satisfying:
\[
A_i^2 \leq C(\varepsilon^{\frac{3}{2}} + 2\|u\|_X\|\bar{u}\|_X^2 + L^\frac{1}{2}(A_2^2 + A_3^2) + \|F\|_{H^1}^2),
\]
where \( C \) is a constant independent of \( \varepsilon \).

Proof. We multiply (3.5) with \( v \) to see that
\[
\int_{\Omega} [-u_s v_{yy} + u_{syy} v - u_s v_{xx} + u_{sxx} v + S(u, v) - \varepsilon \Delta (u_y - v_x)] v dx dy
= \int_{\Omega} (\text{curl} N + \text{curl} F) v dx dy,
\]
whence by virtue of (3.49) and the boundary condition (3.4),
\[
\int_{\Omega} [-u_s v_{yy} + u_{syy} v - u_s v_{xx} + u_{sxx} v] v dx dy
= \int_{\Omega} u_s (v_y^2 + v_x^2) dx dy + \int_{\Omega} u_{syy} v_y v dx dy + \int_{\Omega} (u_{sxx} + u_{syy}) v^2 dx dy + \int_{\Omega} u_{sxx} v_x v dx dy
\geq \|\sqrt{u_s} v_y\|^2 + \|\sqrt{u_s} v_x\|^2.
\]
Using the divergence free condition, one gets

\[-\varepsilon \int_\Omega \Delta(u_y - v_x) \, v \, dx \, dy\]

\[= \varepsilon \int_\Omega u_{yy} v_y \, dx \, dy - \varepsilon \int_0^2 v_{xx} v|_{x=0} \, dy - \varepsilon \int_\Omega v_{xy} \, v \, dx \, dy - 2\varepsilon \int_\Omega v_y \, v \, dx \, dy\]

\[= \varepsilon \int_\Omega u_y u_{xy} \, dx \, dy - \varepsilon \int_0^2 v_{xx} v|_{x=0} - \frac{1}{2} \varepsilon \int_0^2 v_{x}^2 \, L \, dy + \varepsilon \int_\Omega v_y \, v|_{x=0} - \frac{1}{2} \varepsilon \int_0^2 v_{x}^2 \, L \, dy\]

\[\geq \varepsilon |u_y(L, \cdot)|^2 + \varepsilon |v_y(0, \cdot)|^2 - \varepsilon |u_{-\frac{1}{2}} \, v_{xx}(0, \cdot)|^2 - \varepsilon |u_{\frac{1}{2}} \, v(0, \cdot)| - \varepsilon |u_{s} q_{x}(L, \cdot)|^2,\]

where

\[|u_{-\frac{1}{2}} \, v(0, \cdot)|^2 = \int_0^2 u_{-\frac{1}{2}} \, v^2(0, y) \, dy \]

\[= \int_{\Omega} (2u_{\frac{1}{2}} \, v v_x + \frac{1}{2} u_{-\frac{1}{2}} \, u_{s} \, v^2) \, dx \, dy\]

\[\leq \| \sqrt{A} v_x \| \| v \| + \varepsilon \| v \|^2 \leq L^2 A^2_1,\]

and

\[\varepsilon^2 |u_{-\frac{1}{2}} \, v_{xx}(0, \cdot)|^2 = \varepsilon^2 \int_0^2 u_{-\frac{1}{2}} \, v_{xx}^2 (0, y) \, dy \]

\[= \varepsilon^2 \int_{\Omega} \left[ 2u_{\frac{1}{2}} \, u_{s} \, v_{xx} v_{xx} - \frac{1}{2} u_{-\frac{1}{2}} \, u_{s} \, v_{xx}^2 \right] \, dx \, dy\]

\[\leq \varepsilon^2 \left[ (\| \sqrt{A} q_{xx} \| + \| q_{x} \|) \| v_{xx} \| + \varepsilon \frac{1}{2} (\| \sqrt{A} q_{xx} \| + \| q_{x} \|)^2 \right]\]

\[\leq \varepsilon \| \sqrt{A} v_{xx}^2 q \|^2 + \varepsilon^3 \| v_{xx} \|^2.\] (3.56)

Consequently, we conclude

\[-\varepsilon \int_\Omega \Delta(u_y - v_x) \, v \, dx \, dy \geq \varepsilon |u_y(L, \cdot)|^2 + \varepsilon |v_y(0, \cdot)|^2 - L^2 (A^2_1 + A^2_2 + A^2_3).\] (3.57)

On the other hand, it is easy to see that

\[| \int_\Omega S(u, v) \, v \, dx \, dy | = | \int_\Omega [-\partial_y (u_{s} x u + v_s u_y) + \partial_x (v_s v_y + v_{s} x u)] \, v \, dx \, dy |\]

\[= | \int_\Omega (u_{s} x u + v_s u_y) \, v_y \, dx \, dy + \int_\Omega (v_s v_{xy} + v_{s} x u) \, v \, dx \, dy |\]

\[\leq L \varepsilon^\frac{1}{2} \| \sqrt{A} v_y \|^2 + L \varepsilon \| \sqrt{A} v_y \| \| v_{xy} \| + \| v \| \| v \|_L^2 (\varepsilon \| v_{xy} \| + L \varepsilon \| \sqrt{A} v_y \|)\]

\[\leq L^\frac{1}{2} (\| \sqrt{A} v_y \|^2 + \| \sqrt{A} v_x \|^2) + \varepsilon^2 \| \sqrt{A} v_{xx}^2 q \|^2.\] (3.58)
Finally, by (3.51) we arrive at
\[
| \int_\Omega (\text{curl} N + \text{curl} F) v dxdy |
= | - \varepsilon^{\frac{11}{8} + \gamma} \int_\Omega [(\bar{v} \bar{u}_y + \bar{u} \bar{v}_x)_y] - (\bar{v} \bar{v}_y + \bar{u} \bar{v}_x)_x v dxdy + \int_\Omega \text{curl} F v dxyz |
= \varepsilon^{\frac{11}{8} + \gamma} \int_\Omega (\bar{v} \bar{u}_y - \bar{u} \bar{v}_x)_y v dxdy + \int_\Omega \text{curl} F v dxyz |
\leq \varepsilon^{\frac{11}{8} + \gamma} \| \bar{v} \|_{L^\infty} \| \bar{u}_y \| + \| v \| \| \bar{v}_x \| \| \bar{v}_y \| + \| \bar{u} \| \| \bar{v}_x \| \| \bar{v}_y \| + \| v \| \| \nabla \bar{v} \| \| \nabla \bar{v} \| ^2 \\
+ L^\frac{1}{2} \| \text{curl} F \| A_1 \\
\leq \varepsilon^{\frac{3}{8} + \frac{2}{3}} \| u \| \| \bar{u} \| \| \bar{v} \| \| \bar{v}_x \| ^2 + \| F \| _{H^1} ^2 + L^\frac{1}{2} A_1 ^2 .
\] (3.59)
Putting (3.51)-(3.59) together, we conclude
\[
A_1 ^2 \lesssim \varepsilon^{\frac{3}{8} + \frac{2}{3}} \| u \| \| \bar{u} \| \| \bar{v} \| \| \bar{v}_x \| ^2 + \| F \| _{H^1} ^2 + L^\frac{1}{2} A_1 ^2 ,
\] (3.60)
which completes the proof of Lemma 3.9

Next, to well control the convection terms, we shall use \( q_x \) as a multiplier to obtain the second-order derivative estimates. More precisely, we have the following lemma.

**Lemma 3.10.** Under the assumptions of Theorem 3.7, the solution \((u, v)\) of the system (3.1)-(3.3) satisfies
\[
A_2 ^2 \lesssim L A_1 ^2 + \varepsilon^{\frac{1}{8}} A_3 ^2 + \varepsilon^{\frac{7}{8}} \| u \| \| \bar{u} \| \| \bar{v} \| \| \bar{v}_x \| ^2 + (\text{curl} F, q_x)
\] (3.61)

**Proof.** We multiply (3.6) with \( q_x \) to have
\[
\int_\Omega [ - \partial_y (u_x ^2 \partial_y q) - \partial_x (u_y ^2 q_x) + S(u, v) - \varepsilon \Delta (u_y - v_x) ] q_x dxdy \\
= \int_\Omega (\text{curl} N + \text{curl} F) q_x dxdy.
\]
First, the convection terms can be bounded as follows.
\[
\int_\Omega q_x [ - \partial_y (u_x ^2 \partial_y q) - \partial_x (u_y ^2 q_x) ] dxdy \\
= \int_\Omega u_x ^2 q_y yq dx dy - \int_\Omega u_x ^2 q_x q_x dx dy - \int_\Omega 2 u_x u_x q_x ^2 dx dy \\
= - \frac{1}{2} \int_\Omega u_x ^2 q_x ^2 | x = 0 | dx dy - \frac{1}{2} \int_\Omega u_y ^2 q_x ^2 | x = L | dx dy + \frac{1}{2} \int_\Omega u_x q_x ^2 | x = 0 | dx dy \\
- \int_\Omega u_x u_x q_x ^2 + q_y ^2 dx dy \\
\lesssim - \frac{1}{2} \int_\Omega u_x ^2 q_x ^2 | x = 0 | dx dy - \frac{1}{2} \int_\Omega u_y ^2 q_x ^2 | x = L | dx dy + L^\frac{1}{2} A_2 ^2 .
\] (3.62)
Now, we turn to controlling the viscous terms:

\[-\varepsilon \int_\Omega q_s \Delta (u_y - v_x) dxdy\]

\[= -\varepsilon \int_\Omega u_{yyy} q_s dxdy + \varepsilon \int_\Omega (2v_{xyy} + v_{xxx}) q_s dxdy\]

\[= \varepsilon \int_\Omega u_{yy} q_s dxdy - 2\varepsilon \int_\Omega v_{xyy} q_s dxdy - \varepsilon \int_\Omega v_{xxy} q_s dxdy - \varepsilon \int_\Omega^2 v_{xx} q_s |_{x=0} ddy\]

\[= -\varepsilon \int_\Omega u_{yy} q_y dxdy - 2\varepsilon \int_\Omega (u_s q_{xy} + u_{sy} q_y + 2 u_{sx} q_{xx}) q_y dxdy\]

\[= \varepsilon \int_\Omega v_{yyy} q_y dxdy - \varepsilon \int_\Omega (2 u_s q_{xy}^2 + u_s q_{xx}^2) q_y dxdy + \varepsilon \int_\Omega u_{yy} q_y^2 dxdy\]

\[= -\varepsilon \int_\Omega (2 u_s q_{xy} q_x + u_{sx} q_{xx} + 2 u_{sy} q_y q_{xy}) dxdy - \varepsilon \int_\Omega^2 u_s q_{xx} q_y |_{x=0} ddy, \quad (3.63)\]

where from Lemma 3.6 and (3.56) one gets

\[|\varepsilon \int_\Omega [u_{yy} q_y^2 - (2 u_s q_{xy} q_x + u_{sx} q_{xx} + 2 u_{sy} q_y q_{xy})] dxdy - \varepsilon \int_\Omega^2 u_s q_{xx} q_y |_{x=0} ddy|\]

\[\lesssim \varepsilon \left\|q_x\right\|^2 + \varepsilon \frac{3}{2} \left\|\sqrt{u_{xx} q_x}\right\|^2 + \varepsilon \frac{3}{2} \left\|\sqrt{u_{xyy} q_y}\right\|^2 + \varepsilon \frac{\alpha}{2} \left\|u_{xyy} q_{xy}\right\|_{L^\infty(0, t)} \left\|u_{x}^{\frac{1}{2}} q_{xx}(0, \cdot)\right\| q_x\]

\[\lesssim \varepsilon L^\frac{1}{2} \left\|\sqrt{u_{xx} \nabla^2 q}\right\|^2 + \varepsilon \frac{3}{2} \left\|\sqrt{u_{xyy} \nabla^2 q}\right\|^2 + \varepsilon \frac{3}{2} \left\|v_{xxx}\right\| \left\|\sqrt{u_{xx} \nabla^2 q}\right\|

\[\lesssim L^\frac{1}{2} \left\|\sqrt{u_{xx} \nabla^2 q}\right\|^2 + \varepsilon \frac{3}{2} \left\|v_{xxx}\right\|^2, \quad (3.64)\]

while

\[\varepsilon \int_\Omega v_{yyy} q_y dxdy\]

\[= -\varepsilon \int_\Omega^L (u_s q_{yy} + 2 u_{sy} q_y) q_y |_{y=0} dx + \varepsilon \int_\Omega^L (u_s q_{yy} + 2 u_{sy} q_y) q_y |_{y=2} dx\]

\[= -\varepsilon \int_\Omega^L (u_s q_{yy} + 2 u_{sy} q_y + u_{sy} q_y) q_{yy} dxdy\]

\[= -\varepsilon \int_\Omega^L (u_s q_{yy} + u_{sy} q_y) q_y |_{y=0} dx + \varepsilon \int_\Omega^L (u_s q_{yy} + u_{sy} q_y) q_y |_{y=2} dx\]

\[= -\varepsilon \int_\Omega u_s q_{yy}^2 + \varepsilon \int_\Omega (u_{sy} q_{yy}^2 - u_{sy} q_{yy}) q_{yy} dxdy. \quad (3.65)\]

If \(u_s > 0\) on \(y = 2\), then we have \(q_y = 0\), while when \(u_s = 0\) on \(y = 2\), we have \(u_{sy} \leq 0\) on \(y = 2\). So, we always have

\[\varepsilon \int_\Omega^L (u_s q_{yy} + u_{sy} q_y) q_y |_{y=2} dx \leq 0, \quad (3.66)\]
Finally, from Lemma 3.6 one gets

\[ \varepsilon \int_{\Omega} v_{yy} q_y dxdy \]

\[ \leq -\varepsilon \int_{0}^{L} u_{sy} q^2_y |y=0| dx - \varepsilon \int_{\Omega} u_{s} q^2_{yy} + \varepsilon \int_{\Omega} (u_{sys} q_y - u_{sy} q_{yy}) dxdy \]

\[ \leq -\varepsilon \int_{0}^{L} u_{sy} q^2_y |y=0| dx - \varepsilon \int_{\Omega} u_{s} q^2_{yy} + \varepsilon \|q_y\|^2 + \varepsilon \|u_{s} q_{yy}\|_{L^2} \|u_s^{-\frac{1}{2}} q\| \]

\[ \leq -\varepsilon \int_{0}^{L} u_{sy} q^2_y |y=0| dx - \varepsilon \int_{\Omega} u_{s} q^2_{yy} + \varepsilon L^{-\frac{1}{2}} \|u_s q_{yy}\|. \quad (3.67) \]

Putting (3.63)-(3.67) together, we have

\[ -\varepsilon \int_{\Omega} q_x \Delta (u_y - v_x) dxdy \lesssim -\varepsilon \|u_s q_{yy}^2\|^2 - \varepsilon \|q_y(\cdot,0)_{L^2}^2 + \varepsilon^{\frac{4}{3}+3} \|v_{xxx}\|^2. \quad (3.68) \]

Furthermore, by virtue of (2.5) and Lemma 3.6

\[ |\int_{\Omega} S(u, v) q_x dxdy| \]

\[ = |\int_{\Omega} [-\partial_y (u_{sx} u + v_{sx} u) + \partial_x (v_{s} v_{y} + v_{sx} u)] q_x dxdy| \]

\[ = |\int_{\Omega} [(u_{sx} u + v_{sx} u) q_{xy} + v_{sx} u q_x - v_{sy} v_{x} q_x - v_{s} v_{x} q_{xy}] dxdy| \]

\[ \lesssim \|u_s q_{xy}\| (\varepsilon^{\frac{2}{3}} \|u\| + \varepsilon \|u_y\| + \varepsilon \|v_x\|) + \varepsilon \|q_x\|(\|u\| + \|v_x\|) \]

\[ \lesssim L(A_2 + A_1^2). \quad (3.69) \]

Finally, from Lemma 3.6 one gets

\[ |\int_{\Omega} \text{curl} N q_x dxdy| = \varepsilon^{\frac{4}{3}+\gamma}|\int_{\Omega} q_{xy} [\partial_y (\bar{v} u_y - \bar{u} \bar{v}_y) - \partial_x (\bar{v} \bar{v}_y + \bar{u} v_x)] dxdy| \]

\[ = \varepsilon^{\frac{4}{3}+\gamma} - \int_{\Omega} q_{xy} \bar{u} \bar{v}_y dxdy - \int_{\Omega} q_x (\bar{u} \bar{v}_y + \bar{u} \bar{v}_y + \bar{v} \bar{v}_x) dxdy | \]

\[ \lesssim \varepsilon^{\frac{4}{3}+\gamma} \left\{ \|\sqrt{u_s q_{xy}}\|_{L^2} \|u_y\|_{L^2} \|\bar{v}_y\|_{L^2} \|\bar{v}\|_{L^2} \|\bar{u}\|_{L^2} \|\bar{v}\|_{L^2} \right\} \]

\[ \lesssim \varepsilon^{\frac{2}{3}} \|u\| X \|\bar{u}\|^{\frac{2}{3}}. \quad (3.70) \]

Putting (3.62) - (3.70) together, we obtain the estimate (3.61). \hfill \Box

To obtain the estimates for the third order derivatives, we shall first multiply (3.5) with \( \varepsilon^2 v_{xxx} \) to obtain the estimate for \( \|\nabla^2 v_x\| \). This will be done in the following lemma.
Lemma 3.11. Under the assumptions of Theorem 3.5, the solution \((u, v)\) to the system (3.1)-(3.4) satisfies

\[
A_3^2 \leq C \{ A_2^2 + \varepsilon^{2\gamma} \| u \|^2_{X} + \varepsilon \| u \|^2_{X} + \varepsilon \| F \|^2_{H^1} \},
\]

(3.71)

where the constant \(C\) does not depend on \(\varepsilon\).

Proof. We multiply (3.5) with \(\varepsilon^2 v_{xxx}\) to have

\[
\varepsilon^2 \int_{\Omega} v_{xxx}(-u_s v_{yy} + u_{sy} v - u_s v_{xx} + u_{sx} v + S(u, v)) dxdy
- \varepsilon^3 \int_{\Omega} \Delta(u_y - v_x) v_{xxx} dxdy = \varepsilon^2 \int_{\Omega} (\text{curl} N + \text{curl} F) v_{xxx} dxdy.
\]

First of all,

\[
\varepsilon^2 \left| \int_{\Omega} (-u_s v_{yy} + u_{sy} v - u_s v_{xx} + u_{sx} v) v_{xxx} dxdy \right|
\leq C \varepsilon^2 \| \nabla^2 v \| \| v_{xxx} \| \leq \delta A_3^2 + C(\delta) A_2^2,
\]

where \(\delta > 0\) is a sufficiently small constant. By Lemma 3.6 we see that

\[
\varepsilon^2 \left| \int_{\Omega} S(u, v) v_{xxx} dxdy \right|
= \varepsilon^2 \left| \int_{\Omega} \left[ -\partial_y (u_s v_{xy}) + \partial_x (v_s v_y + v_{xy} v) \right] v_{xxx} dxdy \right|
= \varepsilon^2 \left| \int_{\Omega} \left[ v_{sy} u - v_s u_{yy} + v_{xy} v + v_{xx} u \right] v_{xxx} dxdy \right|
\leq \varepsilon^2 \left\{ \varepsilon^2 \| u \| + \varepsilon \| u_y \| + \varepsilon \| u_{xy} \| + \varepsilon \| v_{xy} \| \right\} (\| v_{xxx} \| + \| v_{xy} \|)
\leq \delta A_3^2 + \varepsilon^{\frac{3}{2}} A_2^2,
\]

where \(\delta > 0\) is a constant small enough.

Next, we turn to estimating the viscous terms. In view of the boundary condition (3.4), we see that

\[
\varepsilon^3 \int_{\Omega} (-u_{yy} v_{xx} + 2 v_{xyy}) v_{xxx} dxdy
= -\varepsilon^3 \int_{\Omega} v_{yy} v_{xx} dxdy + \varepsilon^3 \int_{\Omega} v_{xx}^2 dxdy + 2 \varepsilon^3 \int_{\Omega} v_{xy}^2 dxdy
= \varepsilon^3 \left( v_{xxx}^2 + 2 v_{xyy}^2 + v_{xxy}^2 \right) dxdy.
\]
For the nonlinear terms, we have

\[ \varepsilon^2 |(\text{curl } N)v_{xxx}| = \varepsilon^2 \frac{11}{8} + \frac{11}{8} \gamma |(\text{curl } N)v_{xxx}| = \varepsilon^2 \frac{11}{8} + \gamma \left| \int \text{curl } N \right| v_{xxx} \left| \left| \begin{array}{c} \bar{v}u_{yy} - \bar{v}u_{yy} - \bar{v}v_{xy} - \bar{v}v_{xx} \end{array} \right| \right| \leq \varepsilon^2 \frac{11}{8} + \gamma \left\| \int \text{curl } N \right\| v_{xxx} \left\| \left\| \begin{array}{c} \bar{v} \bar{u}_{yy} - \bar{v}u_{yy} - \bar{v}v_{yy} - \bar{v}v_{xx} \end{array} \right\| \right\| \leq \varepsilon^2 \frac{11}{8} + \gamma \left\| u \right\| X \left\| \bar{u} \right\| X. \]

Finally,

\[ \varepsilon^2 |(\text{curl } F, v_{xxx})| \leq \varepsilon^2 \left\| F \right\|_{H^1} \left\| v_{xxx} \right\| \leq \delta A_3^2 + \varepsilon \left\| F \right\|_{H^1}^2. \]

Thus, (3.65) follows from combing the above estimates together. \( \square \)

**Remark 3.12.** In the case \( \alpha_1 = 0 \), we can control \( \left\| \sqrt{u_s q_{yyy}} \right\| \), and consequently control \( \left\| v_{yyy} \right\| \) by the method used in [18]. For the general case, however, we cannot control \( \left\| \sqrt{u_s q_{yyy}} \right\| \) as we do not always have good sign for \( u_{sy} \) on the upper bound \( y = 2 \).

Next, we use the forth-order estimates of \( v \) as well as Sobolev imbedding to get the bound of \( \left\| v_{yyy} \right\| \). To this end, differentiating (3.5) with respect to \( x \), we can obtain the following biharmonic equation for \( v \)

\[ \varepsilon \Delta^2 v = G, \quad (3.72) \]

where

\[ G = \partial_x \text{curl } N + \partial_x F - \partial_x [-u_s \Delta v + \Delta u_s v + S(u, v)]. \]

Recalling that we already have the following boundary condition for \( v \):

\[ v_{x|x=0} = v_{x|x=L} = v_{xx|x=L} = 0, \quad v = v_y = 0 \quad \text{on} \quad \{ y = 0 \} \cup \{ y = 2 \}, \]

we only need one more condition for \( v \) on \( x = 0 \). Using equation (3.5) and boundary condition \( u|_{x=0} = v|_{x=0} = 0 \), we can obtain:

\[ \varepsilon v_{xxx}(0, y) = [\text{curl } N + \text{curl } F + u_s \Delta v - \Delta u_s v - S(u, v)]|_{x=0}. \]

Besides, it is easy to find that \( v_{xxx}(0, 0) = v_{xxx}(0, 2) = 0 \).

In the following, we shall first homogenize the boundary condition for \( v_{xxx} \) on \( x = 0 \). Then, making use of the even extension of \( v \) with respect to \( x = 0 \), and the odd extension of \( v \) with respect to \( x = L \), we obtain a Dirichlet problem for the biharmonic equation (3.72) in a new domain \( \Omega^* \).

To homogenize the boundary condition of \( v_{xxx} \) on \( x = 0 \), we construct a function \( v_0 \), such that

\[ \partial_x v_0(0, y) = 0, \quad \partial_{xxx} v_0(0, y) = v_{xxx}(0, y), \quad \partial_y v_0 = 0, \quad \text{on} \quad \{ y = 0 \} \cup \{ y = 2 \}. \]
In fact, we can first define a function $W$ satisfying
\[
\begin{cases}
\Delta W = 0, & \text{in } \Omega, \\
W = 0, & \text{on } \{x = L\} \cup \{y = 0\} \cup \{y = 2\}, \\
\partial_x W = v_{xxx}(0,y), & \text{on } \{x = 0\}.
\end{cases}
\] (3.73)

Thanks to the compatibility condition $v_{xxx}(0,0) = v_{xxx}(0,2) = 0$, we have $W \in H^2(\Omega)$ satisfying
\[\|W\|_{H^2} \leq C|v_{xxx}(0,\cdot)|_{H^1([0,2])} \] (3.74)

Next, we define $v_0$ by solving the following system:
\[
\begin{cases}
\Delta v_0 = W, & \text{in } \Omega, \\
\partial_y v_0 = 0, & \text{on } \{y = 0\} \cup \{y = 2\}, \\
\partial_x v_0 = 0, & \text{on } \{x = 0\}, \\
v_0 = 0, & \text{on } \{x = L\}.
\end{cases}
\] (3.74)

Then, the system (3.74) has a unique solution $v_0 \in H^4(\Omega)$ with the estimate
\[\|v_0\|_{H^4} \leq C\|W\|_{H^2} \leq C|v_{xxx}(0,\cdot)|_{H^1([0,2])} \]

Combining the system (3.73) with (3.74), we have
\[\partial_{xxx} v_0(0,y) = \partial_x \Delta v_0(0,y) - \partial_{yyy} v_0(0,y) = v_{xxx}(0,y), \\
v_x(0,y) = \partial_{xxx} v_0(0,0) = \partial_{xxx} v_0(0,2) = 0.\]

**Remark 3.13.** Here we do not need $v_0 = 0$ on $\{y = 0\} \cup \{y = 2\}$.

Let $\chi(x)$ be the cutoff function defined in (1.6), we denote by $\tilde{v} = v - v_0 \chi(\frac{x}{L/2})$. Then $\tilde{v}$ satisfies the following system:
\[
\begin{cases}
\Delta^2 \tilde{v} = F, & \text{in } \Omega, \\
\tilde{v}_x = \tilde{v}_{xxx} = 0, & \text{on } x = 0, \\
\tilde{v} = \tilde{v}_{xx} = 0, & \text{on } x = L, \\
\tilde{v} = -v_0 \chi(\frac{x}{L/2}), & \text{on } \{y = 0\} \cup \{y = 2\}, \\
\tilde{v}_y = 0, & \text{on } \{y = 0\} \cup \{y = 2\},
\end{cases}
\] (3.75)

where $F = \varepsilon^{-1}G - \Delta^2(v_0 \chi(\frac{x}{L/2}))$.

Now, if we take the even extension of $\tilde{v}$ with respect to $x = 0$ and denote the new function by $v_{even}$, then thanks to the boundary condition $\partial_y v_0 = 0$ on $\{y = 0\} \cup \{y = 2\}$, as well as the compatibility condition $\partial_{xxx} v_0(0,0) = \partial_{xxx} v_0(0,2) = \partial_x v_0(0,0) = \partial_x v_0(0,2) = 0$, we have
\[\partial_y v_{even}(\cdot,0) = \partial_y v_{even}(\cdot,2) \equiv 0,\]
and
\[|v_{\text{even}}(\cdot, 0)|_{H^{3+\frac{1}{2}}(-L,L)} + |v_{\text{even}}(\cdot, 2)|_{H^{3+\frac{1}{2}}(-L,L)} \leq \|v_0\|_{H^4(\Omega)} \]

Next, following a process similar to that in the proof of Lemma 3.3, we get
\[\|\bar{v}\|_{H^4} \leq \|v_0\|_{H^4} + \|F\|_{L^2} \leq |v_{xxx}(0, \cdot)|_{H^{\frac{5}{2}}((0,2))} + \varepsilon^{-1}\|G\|_{L^2},\]
and consequently
\[\varepsilon^\frac{5}{2}\|v\|_{H^4} \lesssim \varepsilon^\frac{5}{2}\|\bar{v}\|_{H^4} + \varepsilon^\frac{5}{2}\|v_0\|_{H^4} \lesssim \varepsilon^\frac{5}{2}|v_{xxx}(0, \cdot)|_{H^{\frac{5}{2}}((0,2))} + \varepsilon^\frac{5}{2}\|G\|_{L^2}. \quad (3.76)\]

Recalling the expression of \(v_{xxx}(0, y)\) and \(G\), we have
\[\varepsilon^\frac{5}{2}\|v\|_{H^4} \lesssim \varepsilon^\frac{5}{2}\|\text{curl} N + \text{curl} F + u_s \Delta v + \Delta u_s v - S(u, v)\|_{H^1} \lesssim \varepsilon^\frac{3}{2}\|\text{curl}(N + F)\|_{H^1} + \varepsilon^\frac{3}{2}\|v\|_{H^3} + \varepsilon\|v\|_{L^2} + \varepsilon^\frac{5}{2}\|u_{yy}\|_{H^1}. \quad (3.77)\]

The Sobolev imbedding theory implies
\[\varepsilon^\frac{5}{2}\|v\|_{H^3} \leq \varepsilon^\frac{3}{2}(\delta \varepsilon\|v\|_{H^4} + C(\delta)\varepsilon^{-1}\|v\|_{H^2}) = \delta \varepsilon^\frac{3}{2}\|v\|_{H^4} + C(\delta)\varepsilon^\frac{5}{2}\|v\|_{H^2} \quad (3.78)\]
here \(\delta > 0\) is a constant small enough. Besides,
\[\varepsilon^\frac{5}{2}\|\text{curl} N\|_{H^1} = \varepsilon^{\frac{13}{8} + \gamma^2 + \frac{7}{8}}\|(\bar{v} u_y + \bar{\bar{v}} u_x)_{y} - (\bar{v} u_y + \bar{\bar{v}} u_x)_{x}\|_{H^1} \leq \varepsilon^{\frac{13}{8} + \gamma^2 + \frac{7}{8}}\|\bar{u} u_{yy} + \bar{\bar{u}} u_{yy} - \bar{v} u_{xy} + \bar{\bar{v}} u_{xx}\|_{H^1} \lesssim \varepsilon^{\frac{5}{8} + \frac{1}{2}}\|\bar{u}\|_{H^2}. \quad (3.79)\]

Putting (3.78) and (3.79) into (3.77), we have proved the following Lemma:

**Lemma 3.14.** *Under the assumptions of Theorem 3.5, the solution \((u, v)\) to the system (3.1)-(3.4) satisfies*
\[\varepsilon^\frac{5}{2}\|v\|_{H^3} \leq C(A_2 + \varepsilon\|u\|_{H^1} + \varepsilon^{\frac{5}{8} + \frac{7}{8}}\|\bar{u}\|_{H^2} + \varepsilon^\frac{3}{2}\|F\|_{H^2}), \quad (3.80)\]
*where the constant \(C\) does not depend on \(\varepsilon\).*

**Proof of Theorem 3.5.** First by Lemmas 3.9, 3.10, and the fact that \(L\) is sufficiently small, we find
\[A_1^2 + A_2^2 \lesssim L^\frac{5}{2} A_3^2 + \varepsilon^\frac{7}{2}\|\bar{u}\|_{H^1} + (\text{curl} F, q_x) + \|F\|_{H^1}^2, \quad (3.81)\]
which, together with Lemma 3.11 implies
\[A_1^2 + A_2^2 \lesssim \varepsilon^\frac{7}{2}\|\bar{u}\|_{H^1} + (\text{curl} F, q_x) + \|F\|_{H^1}^2, \quad (3.82)\]
Next, combining (3.78) with (3.81), we see that
\[\varepsilon^\frac{5}{2}\|v\|_{H^3} \lesssim A_2 + \varepsilon\|u\|_{H^1} + \varepsilon^{\frac{5}{8} + \frac{7}{8}}\|\bar{u}\|_{H^2} + \varepsilon^\frac{3}{2}\|F\|_{H^2}. \quad (3.83)\]

36
Finally, using equation (3.5), we find
\[ \varepsilon^2 \| u_{yy} \| \lesssim \varepsilon \| v \|_{L^2} + \varepsilon^2 \| \nabla^2 v_x \|_{L^2} + \varepsilon^2 \| \text{curl} N \|_{L^2} + \varepsilon \| F \|_{H^1}, \] (3.84)
which together with (3.83) gives
\[ \varepsilon^2 \| u \|_{H^3} \leq A_2 + \varepsilon \| u \|_{X} + \varepsilon^2 \| \bar{u} \|_{X} + \varepsilon \| F \|_{H^1} + \varepsilon^2 \| F \|_{H^2}. \] (3.85)

Now, if we put (3.82) and (3.85) together, we conclude
\[ \| u \|_{X} \lesssim \varepsilon^7 \| \bar{u} \|_{X}^4 + \| \text{curl} F, q_x \|_{L^2} + \| F \|_{H^1}^2 + \varepsilon^2 \| F \|_{H^2}^2. \]

This completes the proof of Theorem 3.5.

4 Proof of the main theorems

Based on the uniform-in-\( \varepsilon \) estimates for the linearized system as well as the contract mapping principle, we are now ready to prove our main theorems. Obviously, the main difference between the proofs of the main theorems lies in the term (\( \text{curl} F, q_x \)).

**Proof of the main theorems:**

Step I: Estimates of the term (\( \text{curl} F, q_x \))

(i) In the Poiseuille-Couette flow case with \( \alpha_2 > 0 \) in the absence of external forces, we take
\[ u_s = u^0, \quad P_s = \varepsilon U''(y)x = -2\varepsilon \alpha_2 x, \]
where \( U \) is defined in (1.11). Then, by (1.12) and (1.13),
\[ F_u = -\varepsilon^{-\frac{1}{2}}\gamma (\varepsilon \mu'' + \partial_x P_s) = \varepsilon^{-\frac{1}{2}}\gamma \varepsilon (\mu'' - U''), \]
\[ F_v = 0. \]

Thus, from the condition (1.17) in Theorem 1.1 we get
\[ |(\text{curl} F, q_x)| = |(\text{curl} F, q_x)| \]
\[ = |\varepsilon^{-\frac{1}{2}}\gamma \int_0^2 \mu''(y)q_x dxdy| = |\varepsilon^{-\frac{1}{2}}\gamma \int_0^2 \mu''q(0, y)dy| \]
\[ \leq C \varepsilon^{-\frac{1}{2}}\gamma \| \mu'' \|_{L^\infty}(\| u_s q_y(0, \cdot) \|_{L^2} + \| u_s v_x \|_{L^2}), \]
\[ \leq C\alpha_0 \| u \|_{X}, \] (4.1)

where the constant \( C \) does not depend on \( \varepsilon \).
(ii) In the case when $\alpha_2 = 0$ in the absence of external forces, i.e., the flow is of Couette type, $(u_s, P_s)$ is defined in (2.2) and $(F_u, F_v)$ can be written as (2.8) and (2.9). Therefore, by virtue of (2.10) and (3.50), one has

$$\|(\text{curl } F, q_x)\| = \|(\text{curl } F, q_x)\| \leq \|\text{curl } F\||q_x| \leq C\varepsilon^{-\frac{1}{8}}\varepsilon^{-\frac{3}{8}}\alpha_0\|u\|_X = C\varepsilon^{-\frac{1}{8}}\varepsilon^{-\frac{3}{8}}\alpha_0\|u\|_X, \quad (4.2)$$

with the constant $C$ being independent of $\varepsilon$.

(iii) In the case when there is a proper external force $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon)$ to control the flow, we take $u_s = u_0^\varepsilon, P_s = C$, where $C$ is any constant. By (1.12) and (1.13) we have

$$F_1 = \varepsilon^{-M_0}f_1^\varepsilon + F_u = \varepsilon^{-M_0}(\varepsilon\mu'' - f_1^\varepsilon) = \varepsilon^{-M_0}g_1^\varepsilon;$$

$$F_2 = \varepsilon^{-M_0}f_2^\varepsilon + F_v = \varepsilon^{-M_0}f_2^\varepsilon = \varepsilon^{-M_0}g_2^\varepsilon.$$

Then, from (1.28) in Theorem 1.3 we get

$$\|(\text{curl } F, q_x)\| = \varepsilon^{-\frac{1}{8}}\gamma\|(\text{curl } g^\varepsilon, q_x)\| = \varepsilon^{-\frac{1}{8}}\gamma\left\|\int_0^2 \text{curl } g^\varepsilon q|_{x=0} dy\right\| \leq \varepsilon^{-\frac{1}{8}}\gamma\|g^\varepsilon\|_{H^2}(|u_s q_y(0, \cdot)|_{L^2} + \|\sqrt{u_s} v_x\|_{L^2}) \leq C\alpha_0\|u\|_X, \quad (4.3)$$

where the constant $C$ does not depend on $\varepsilon$.

Step II: Solutions to the nonlinear systems.

First of all, let $(u_s, P_s)$ be one of the cases in Step I, then we see that

$$\|F\|_{H^1} + \varepsilon^3\|F\|_{H^2} + |(\text{curl } F, q_x)| \leq C(\alpha_0^2 + \alpha_0\|u\|_X) \quad (4.4)$$

with the constant $C$ being independent of $\varepsilon$. So we have from Theorem 3.5

$$\|u\|_X^2 \leq C(\varepsilon^\gamma\|u\|_X^\frac{1}{4} + \alpha_0^\frac{3}{2}). \quad (4.5)$$

In view of Theorems 3.1 and (4.5), we see that for any $\bar{u} = (\bar{u}, \bar{v}) \in X$, there is a unique solution $(u, P)$ satisfying (3.1)-(3.4) with the following estimate:

$$\|u\|_X \leq C(\varepsilon^\frac{a}{2}\|\bar{u}\|_X^\frac{1}{4} + \alpha_0). \quad (4.6)$$

Now, for $\alpha_0$ small enough, we define

$$V = \{u \in X|\|u\|_X < 2C\alpha_0\}.$$

Then we can define a mapping $T: V \rightarrow V$ by $T(\bar{u}) = u$. 

38
Now, for any \( \tilde{u}_1, \tilde{u}_2 \in V \), we denote \( u_1 = (u_1, v_1) = T(\tilde{u}_1) \) and \( u_2 = (u_2, v_2) = T(\tilde{u}_2) \). Then, \( u_1 - u_2 \) satisfies the following system:

\[
-u_x \Delta (v_1 - v_2) + \Delta u_s (v_1 - v_2) + S(u_1 - u_2, v_1 - v_2) - \varepsilon \Delta \text{curl}(u_1 - u_2) = 0
\]

\[
\begin{aligned}
&= \text{curl} N(\tilde{u}_1, \tilde{v}_1) - \text{curl} N(\tilde{u}_2, \tilde{v}_2) \\
&= \partial_y [(\tilde{v}_1 - \tilde{v}_2) u_{1y} + (\tilde{u}_1 - \tilde{u}_2) u_{1x}] - \partial_x [(\tilde{v}_1 - \tilde{v}_2) v_{1y} + (\tilde{u}_1 - \tilde{u}_2) v_{1x}] \\
&+ \partial_y [\tilde{v}_2 (\tilde{u}_1 - \tilde{u}_2)_y + \tilde{u}_2 (\tilde{u}_1 - \tilde{u}_2)_x] - \partial_x [\tilde{v}_2 (\tilde{v}_1 - \tilde{v}_2)_y + \tilde{u}_2 (\tilde{v}_1 - \tilde{v}_2)_x]. \tag{4.7}
\end{aligned}
\]

If we further denote \( \tilde{q} = \frac{u_1 - u_2}{\varepsilon} \), and

\[
\begin{aligned}
B_1^2 &= \|\sqrt{u_s}(v_1 - v_2)_y\|^2 + \|\sqrt{u_s}(v_1 - v_2)_x\|^2, \\
B_2^2 &= \varepsilon \|\sqrt{u_s q_{xx}}\|^2 + \varepsilon \|\sqrt{u_s q_{xy}}\|^2 + \varepsilon \|\sqrt{u_s q_{yy}}\|^2 + |u_s q_y(0, \cdot)|^2,
\end{aligned}
\]

then we have in a manner similar to (3.82) that

\[
B_1^2 + B_2^2 \leq \frac{\varepsilon}{2} \|u_1 - u_2\|_X \|\tilde{u}_1 - \tilde{u}_2\|_X (\|\tilde{u}_1\|_X + \|\tilde{u}_2\|_X). \tag{4.8}
\]

Similarly to the proof of Lemma 3.14, we obtain

\[
\varepsilon \frac{\rho}{2} \|v_1 - v_2\|_{H^4} \lesssim B_2 + \varepsilon \|u_1 - u_2\|_X + \varepsilon \frac{\rho}{2} \|\tilde{u}_1 - \tilde{u}_2\|_X (\|\tilde{u}_1\|_X + \|\tilde{u}_2\|_X). \tag{4.9}
\]

Combing (4.8) with (4.9), we argue similarly to the proof of Theorem 3.5 to deduce

\[
\|u_1 - u_2\|_X \lesssim \varepsilon \frac{\rho}{2} \|\tilde{u}_1 - \tilde{u}_2\|_X (\|\tilde{u}_1\|_X + \|\tilde{u}_2\|_X).
\]

Thus, \( T \) is a compact mapping if \( \alpha_0 \) is taken to be small enough. We finish the proof of the main theorems.

### A Sketch of the construction of \( u_p^{i,-} \)

In this section, we sketch the construction of \( u_p^{i,-} \), and refer to [25] for the details.

As the equations for the weak boundary layer correctors are linear parabolic equations including degenerate terms \( \mu(y) \partial_x u_p^{i,-} \) near \( y = 0 \), we would have to use the scale \( Y_- = \varepsilon^{-\frac{4}{3}} \) near \( y = 0 \). Recalling that \( \mu(y) \sim y \) as \( y \to 0 \), and that the boundary layer correctors would degenerate rapidly when \( Y_- > 1 \), we first collect from (1.12) the leading \( O(\varepsilon^{\frac{4}{3}}) \) terms for the boundary layer profiles near \( y = 0 \):

\[
R_{u_i}^{1,-} = \varepsilon^{-\frac{4}{3}} \mu \partial_x u_p^{1,-} + v_p^{1,-} \mu' + \varepsilon^{-\frac{4}{3}} \partial_x P_p^{1,-} - \partial_{Y_-} u_p^{1,-}, \tag{A.1}
\]

and the leading \( O(\varepsilon^{\frac{4}{3}}) \) terms for (1.13) is

\[
\partial_{Y_-} P_p^{1,-} = 0. \tag{A.2}
\]
Then to construct \( u_{p_1}^{1,-} \), we first consider the following system (using the fact that \( \mu(y) \sim y \) when \( y \to 0 \)):

\[
\begin{aligned}
Y_-' \partial_x u_{p_1}^{1,0,-} + v_{p_1}^{1,0,-} + \varepsilon^4 \partial_x P_{p_1}^{1,0,-} - \partial Y_- Y'_- u_{p_1}^{1,0,-} = 0, & \quad \partial Y_- P_{p_1}^{1,0,-} = 0, \\
u_{p_1}^{1,0,-} |_{x=0} = 0, & \quad u_{p_1}^{1,0,-} |_{Y_-=0} = -u_e^1(x, 0), & \quad u_{p_1}^{1,0,-} |_{Y_- \to \infty} = 0,
\end{aligned}
\]

(A.3)

It is easy to obtain \( P_{p_1}^{1,-} \equiv 0 \), and thus,

\[
\begin{aligned}
Y_- \partial_x u_{p_1}^{1,0,-} + v_{p_1}^{1,0,-} - \partial Y_- Y'_- u_{p_1}^{1,0,-} = 0, & \quad u_{p_1}^{1,0,-} |_{x=0} = 0, & \quad u_{p_1}^{1,0,-} |_{Y_-=0} = -u_e^1(x, 0), & \quad u_{p_1}^{1,0,-} |_{Y_- \to \infty} = 0,
\end{aligned}
\]

(A.4)

Note that we construct \( u_{p_1}^{1,0,-}, v_{p_1}^{1,0,-} \) on \((0, L) \times (0, \infty)\). We now cut-off these layers and make an \( O(\varepsilon^{4/3}) \)-order error:

\[
u_{p_1}^{1,-} = \chi_1 Y_- \partial_x u_{p_1}^{1,0,-} + 3 \varepsilon_0 \chi' Y_- \partial_x u_{p_1}^{1,0,-} + \varepsilon^2 \mu \chi'' u_{p_1}^{1,0,-} + \varepsilon^3 \chi'' u_{p_1}^{1,0,-}, \quad v_{p_1}^{1,-} := \chi_1 Y_- u_{p_1}^{1,0,-}
\]

(A.5)

where \( a_0 > 0 \) is a fixed constant small enough, and \( \chi \) is defined in (1.6).

When \( i = 2 \), let \( u_{p_1}^{1,0,-}, v_{p_1}^{1,0,-} \) be solutions to the system (A.4), then after cutting off (A.3), the contribution to the next layer is

\[
C_{cut}^{1,-} = \frac{1}{a_0} \varepsilon^4 \chi Y_- v_{p_1}^{1,0,-} + 3 \varepsilon_0 \chi' Y_- v_{p_1}^{1,0,-} + 3 \varepsilon^2 \varepsilon_0 \chi'' v_{p_1}^{1,0,-} + \frac{1}{a_0} \varepsilon^3 \chi'' v_{p_1}^{1,0,-}.
\]

(A.6)

We also obtain another error, due to approximating \( \varepsilon^{4/3} \mu \) by \( Y_- \) in the support of the cut-off function \( \chi_1 \) and by approximating \( \mu' \) by 1. This error is given by

\[
C_{approx}^{1,-} := (\varepsilon^{4/3} \mu(y) - Y_-) (\chi Y_-) (\partial_x u_{p_1}^{1,0,-} + \frac{1}{a_0} \varepsilon^4 \chi' u_{p_1}^{1,0,-}) + (\mu' - 1) (\chi Y_-) v_{p_1}^{1,0,-}.
\]

(A.7)

Finally, we have higher order terms that contribute to the error:

\[
C_{quad}^{1,-} := \varepsilon^2 (u_e^1 + u_{p_1}) \partial_x u_{p_1}^{1,-} + \varepsilon^2 u_{p_1}^{1,-} \partial_x u_e^1 + \varepsilon^{4/3} v_{p_1}^{1,-} u_{p_1}^{1,-} + \varepsilon^{4/3} v_{p_1}^{1,-} v_{p_1}^{1,-} + \varepsilon^{4/3} v_{p_1}^{1,-} v_{p_1}^{1,-}
\]

(A.8)

For the higher order terms in the second equation, we shall use our auxiliary pressure to move it to the top equation. This is achieved by defining the first auxiliary pressure \( P_{p_1}^{1,0,-} \) to zero out the terms contributed from

\[
\varepsilon^4 \mu + \varepsilon u_e^1 v_{p_1}^{1,-} + \varepsilon u_{p_1}^{1,-} (\varepsilon v_{p_1}^{1,-} + \varepsilon^4 v_{p_1}^{1,-}) + \varepsilon^2 v_{p_1}^{1,-} u_{p_1}^{1,-} + \varepsilon^2 v_{p_1}^{1,-} v_{p_1}^{1,-} + \varepsilon^2 v_{p_1}^{1,-} v_{p_1}^{1,-}
\]

\[
- \varepsilon^2 \chi Y_- v_{p_1}^{1,-} - \varepsilon^2 v_{p_1}^{1,-} + \varepsilon^2 P_{p_1}^{1,0,-} = 0,
\]

(A.9)
which therefore motivates our following definition:

\[
-\varepsilon \frac{4}{3} P_{p}^{1,a,-} := \int_{Y} \left( \varepsilon \left( V_{1}^{1} + \varepsilon V_{1}^{1} \right) + \varepsilon V_{1}^{1} \right) dY.
\]  

(A.10)

As a result, we can define the force for the next order weak boundary layer via

\[
F_{2}^{1,a,-} := \varepsilon \frac{4}{3} \left( C_{cut}^{1} + C_{approx}^{1} - \varepsilon \frac{5}{3} \partial_{x} P_{p}^{1,a,-} \right).
\]  

(A.11)

Remark A.1. For \( i = 3, \ldots, M \), \( F_{i}^{1,a,-} \) can be constructed similarly. Since we shall put the terms with \( H^{2} \)-norm smaller than \( \varepsilon \frac{7}{4} + \gamma \) into the remainders \( (F_{u}, F_{v}) \), we only need the auxiliary pressure \( P_{i,a,-} \) for \( i = 1, \ldots, 5 \). When \( i > 5 \), we shall take \( P_{i,a,+} = 0 \). We remark here that all the interaction terms with form \( u_{i}^{+} + u_{i}^{-} \) will be put into the remainders \( (F_{u}, F_{v}) \).

Let us therefore consider the abstract problem (dropping indices):

\[
\begin{align*}
Y \partial_{x} u + v - \partial_{y}^{2} u &= F, \quad (x, Y) \in (0, L) \times (0, \infty) \quad (A.12) \\
v &= \int_{Y} \partial_{x} u dY', \quad (A.13) \\
u|_{x=0} &= 0, \quad u|_{Y=0} = g(x), \quad u|_{Y=\infty} = 0. \quad (A.14)
\end{align*}
\]

For this abstract problem, we have the following estimates:

Lemma A.2. Assume that \( F(x, Y) \) decays rapidly at infinity, i.e., for any \( m \geq 0 \), there is a constant \( M > 0 \), such that

\[
\|(1 + Y)^{m} \partial_{x}^{n} \partial_{Y}^{l} F\| \leq M \quad \text{for } 0 \leq 2n + l \leq K,
\]

where \( K > 0 \) is a sufficiently large constant. Then, there exists a unique solution \((u, v)\) to (A.12)–(A.14) satisfying

\[
\|(1 + Y)^{m} \partial_{x}^{n} \partial_{Y}^{l} \{u, v\}\| \leq C(m, n, l)(M + \|g\|_{H^{K+1}} \sum_{n+l} \) for any \( 2n + l \leq K + 2, \)

where the constant \( C \) does not depend on \( Y \).

For \( i = M \), we need to slightly modify the abstract problem. Consider

\[
\begin{align*}
Y \partial_{x} u + v - \partial_{y}^{2} u &= F, \quad (x, Y) \in (0, L) \times (0, \infty) \\
v &= -\int_{0}^{Y} \partial_{x} u dY', \quad (A.15) \\
u|_{x=0} &= 0, \quad u|_{Y=0} = g(x), \quad \partial_{Y} u|_{Y=\infty} = 0,
\end{align*}
\]

and we have
Lemma A.3. Assume that $F(x,Y) \text{ decays rapidly at infinity, i.e., for any } m \geq 0,$ there is a constant $M > 0,$ such that

\begin{equation}
\|(1 + Y)^m \partial_x^n \partial_Y^l F\| \leq M \text{ for } 0 \leq 2n + l \leq K, \tag{A.16}
\end{equation}

then there exists a unique solutions $(u,v)$ of (A.15), satisfying

\begin{align*}
\|(1 + Y)^m \partial_x^n \partial_Y^l u\| &\leq C(m,n,l)(M + \|g\|_{H^{K+1}}) \text{ for any } l \geq 1, 1 \leq 2n + l \leq K + 2, \\
\|\partial_x^n \{u, \frac{v}{Y}\}\| &\leq C(m,n,l)(M + \|g\|_{H^{K+1}}) \text{ for any } 0 \leq 2n \leq K + 2.
\end{align*}

Remark A.4. The main difference between the proofs of Lemma A.2 and Lemma A.3 lies in that one does not have the decay of $v$ as $Y \to \infty$ in Lemma A.3.

Acknowledgements. Chunhui Zhou would like to thank Prof. Yan Guo for many fruitful discussions. The research of Song Jiang is supported by National Key R&D Program (2020YFA0712200), National Key Project (GJXM92579), and the Sino-German Science Center (Grant No. GZ 1465) and the ISF-CNSFC joint research program (Grant No. 11761141008). Chunhui Zhou is supported by the NSFC under the contract 11871147 and the Zhishan scholarship of Southeast University.

References

[1] Alexandre, R., Wang, Y.-G., Xu, C.-J., Yang, T.: Well-posedness of the Prandtl equation in Sobolev spaces. J. Am. Math. Soc. 28(3), 745-784, 2015

[2] Bedrossian, Jacob; Germain, Pierre; Masmoudi, Nader Stability of the Couette flow at high Reynolds numbers in two dimensions and three dimensions. Bull. Amer. Math. Soc. (N.S.) 56 (2019), no. 3, 373-414.

[3] Bedrossian, Jacob; Germain, Pierre; Masmoudi, Nader, On the stability threshold for the 3D Couette flow in Sobolev regularity. Ann. of Math. (2) 185, no. 2, 541-608 (2017).

[4] Chen, Qi; Li, Te; Wei, Dongyi; Zhang, Zhifei Transition threshold for the 2-D Couette flow in a finite channel. Arch. Ration. Mech. Anal. 238 (2020), no. 1, 125-183.

[5] Chen, Qi, Wei Dongyi, Zhang Zhifei, Linear Stability of Pipe Poiseuille Flow at High Reynolds Number Regime. Commun. Pure Appl. Math. 2022(online)

[6] A. Cherhabili and U. Ehrenstein, Finite-amplitude equilibrium states in plane Couette flow, J. Fluid Mech. 342 (1997), 159-177.

[7] Drazin, P., Reid, W.: *Hydrodynamic Stability*. Cambridge Monographs Mech. Appl. Math. Cambridge University Press, New York 1981
[8] Gazzola, Filippo; Grunau, Hans-Christoph; Sweers, Guido. Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains. Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.

[9] Gerard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. J. Am. Math. Soc. 23(2), 591-609, 2010

[10] Gerard-Varet, D., Maekawa, Y., Masmoudi, N.: Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. Duke Math. J. 167(13), 2531-1631, 2018

[11] Gerard-Varet, David; Maekawa, Yasunori: Sobolev stability of Prandtl expansions for the steady Navier-Stokes equations. Arch. Ration. Mech. Anal. 233 (2019), no. 3, 1319-1382.

[12] Gerard-Varet, D., Masmoudi, N.: Well-posedness for the Prandtl system without analyticity or monotonicity. Ann. Scient. Ec. Norm. Sup., 48(4), 1273-1325, 2015

[13] D. Gilbarg, N.S. Trudinger: Elliptic Partial Differential Equations of Second Order, 2nd Edition, Springer-Verlag, 1983.

[14] Grenier, E.: On the nonlinear instability of Euler and Prandtl equations. Commun. Pure Appl. Math. 53(9), 1067-1091, 2000

[15] Grenier, Emmanuel; Guo, Yan; Nguyen, Toan T. Spectral instability of general symmetric shear flows in a two-dimensional channel. Adv. Math. 292 (2016), 52-110.

[16] Grenier, E., Guo, Y., Nguyen, T.: Spectral instability of characteristic boundary layer flows. Duke Math. J. 165, 3085-3146, 2016

[17] Y. Guo, and T. Nguyen. Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. Ann. PDE. 3 no. 1, Art. 10, (2017) 58pp.

[18] Guo, Yan; Iyer, Sameer. Regularity and expansion for steady Prandtl equations. Comm. Math. Phys. 382 (2021), no. 3, 1403-1447. 35Q30 (76D10)

[19] Yan Guo, Sameer Iyer. Validity of Steady Prandtl Layer Expansions. arXiv:1805.05891

[20] Hains, F. D.. Stability of plane Couette-Poiseuille flow. Phys. Fluids 10, 2079-80. [p.223.] (1967)

[21] S. Iyer. Steady Prandtl Boundary Layer Expansion of Navier-Stokes Flows over a Rotating Disk. Arch. Ration. Mech. Anal., 224 (2017), no. 2, 421-469.
[22] Sameer Iyer, Nader Masmoudi. Boundary Layer Expansions for the Stationary Navier-Stokes Equations. arXiv:2103.09170

[23] Sameer Iyer, Nader Masmoudi. Global-in-x Stability of Steady Prandtl Expansions for 2D Navier-Stokes Flows. arXiv:2008.12347

[24] Iyer, Sameer; Zhou, Chunhui Stationary inviscid limit to shear flows. J. Differential Equations 267 (2019), no. 12, 7135-7153.

[25] Iyer, Sameer; Zhou, Chunhui Corrigendum to ”Stationary inviscid limit to shear flows” [J. Differ. Equ. 267 (12) (2019) 7135–7153]. J. Differential Equations 289 (2021), 279-288.

[26] Masmoudi, N., Wong, T.K.: Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. Commun. Pure Appl. Math. 68(10), 1683-1741, 2015.

[27] Masmoudi, Nader; Zhao, Weiren, Stability threshold of two-dimensional Couette flow in Sobolev spaces. Ann. Inst. H. Poincare Anal. Non Lineaire 39, no. 2, 245-325 (2022).

[28] H Li, N Masmoudi, W Zhao, New energy method in the study of the instability near Couette flow. DOI:10.48550/arXiv.2203.10894 (2022)

[29] Li, W.-X., Yang., T.: Well-posedness in Gevrey space for the Prandtl equations with non-degenerate critical points. To appear in J. Eur. Math. Soc.

[30] Oleinik, O.A. and Samokhin, V.N, Mathematical models in boundary layer theory. Applied Mathematics and Mathematical Computation, 15. Champan and Hall/ CRC, Boca Raton, FL, 1999.

[31] Romanov, V. A. Stability of plane-parallel Couette flow. Funkcional Anal, i Prolozen. 1, no. 2, 62-73. Translated in Functional Anal. Its Applies 7, 137-46(1973). [p. 213.](1973).

[32] Wei, Dongyi, Zhang, Zhifei, Transition Threshold for the 3D Couette Flow in Sobolev Space. Commun. Pure Appl. Math 74(11), 2398-2479,(2021)