Scenario Aggregation using Binary Decision Diagrams for Stochastic Programs with Endogenous Uncertainty

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Abstract Modeling decision-dependent scenario probabilities in stochastic programs is difficult and typically leads to large and highly non-linear MINLPs that are very difficult to solve. In this paper, we develop a new approach to obtain a compact representation of the recourse function using a set of binary decision diagrams (BDDs) that encode a nested cover of the scenario set. The resulting BDDs can then be used to efficiently characterize the decision-dependent scenario probabilities by a set of linear inequalities, which essentially factorizes the probability distribution and thus allows to reformulate the entire problem as a small mixed-integer linear program. The approach is applicable to a large class of stochastic programs with multivariate binary scenario sets, such as stochastic network design, network reliability, or stochastic network interdiction problems. Computational results show that the BDD-based scenario representation reduces the problem size, and hence the computation time, significant compared to previous approaches.

Keywords multistage stochastic optimization, exact reformulation, scenario aggregation, reliability optimization

1 Introduction

When modeling a stochastic optimization problem as a stochastic program, one usually assumes that the scenario probabilities are fixed and given input the problem. In this setup, decisions can influence the outcome in each scenario but not their probability of realizing. This standard way of reflecting the effect of uncertainty in the model has been referred to as exogenous uncertainty, while the term endogenous uncertainty has been introduced to describe situations where decisions can actually influence the stochastic process itself and not only its outcomes [28].

While endogenous uncertainty is straightforward to express in the framework of Markov decision processes, its use in stochastic programming has remained very rare, because the resulting models appear very cumbersome to formulate...
and notoriously hard to solve. [19,20] distinguish two types of endogenous uncertainty, according to whether the decisions can influence the temporal unfolding of events or the probability distribution. Models of the first type are usually formulated by enumerating the various scenario trees that correspond to the different possible sequencing of events and couple them via disjunctive formulations of non-anticipativity constraints (see e.g. [12]. Solution approaches include implicit enumeration schemes [28], branch-and-bound coupled with Lagrange duality [20], branch-and-cut [11], Lagrange relaxation of the non-anticipativity constraints [22], decomposition techniques [21] or heuristics [26].

In this paper we focus on problems of the second type of endogenous uncertainty, where decisions can influence the probability distribution of the scenarios. The straightforward way to model this involves products or other non-linear expressions of decision variables thus leads to highly non-linear models that are very hard to solve [37,29]. [1] as well as [17] have used convexification techniques to deal with these polynomials, while [37] have relied on linear approximations. [42] solve the non-linear stochastic program directly with a new constraint programming approach using new type of extreme resource constraint in combination with an efficient propagation algorithm.

In order to deal with the usually exponentially large scenario sets, most approaches had to resort to scenario sampling and work with a representative subset of scenarios. Recently [38] and [39] proposed an exact scenario bundling approach, where scenarios that are equivalent with respect to the recourse function are merged. In [42] scenarios are grouped according to the shortest surviving paths in the underlying network.

An important class of stochastic optimization problems that typically include endogenous uncertainty comes from the related areas of stochastic network design, network reliability, and network interdiction. Common to these problems are an underlying graph whose edges or nodes are subject to random failures. The decisions to be made are in order to change their failure (or survival) probabilities of edges or nodes such that the resulting failure process has some favorable properties, such as connectivity, shortest path lengths or other performance indicators related to the function or process which the graph expresses. Such problems can naturally be formulated as two-stage stochastic programs, where those tactical or “design” decisions constitute the first stage, the failure process is represented by a (usually exponentially) large set of scenarios, and the resulting performance of the network in each failure scenario is captured by a recourse function (which might involve auxiliary second-stage decisions, e.g., for determining flows or shortest paths).

We consider a general setting where scenarios can be formally described as random binary vectors. This holds in particular for stochastic network design and network reliability problems described above. In many such cases, the recourse function is monotonous with respect to the scenario. Take for example the shortest path length between a given pair of nodes in a graph: if scenarios indicate a set of surviving edges after some disaster, then the shortest path length can never increase when fewer edges survive. Our main idea is exploit this structure for scenario aggregation, and our main contribution is a new method for scenario aggregation by means of binary decision diagrams. The approach not only enables a substantial reduction of the (initially exponentially large) scenario set without any accuracy loss, but also to efficiently characterize the decision-dependent scenario probabilities by a set of linear inequalities. This essentially factorizes the
probability distribution and thus allows to reformulate the entire problem as a small mixed-integer linear program.

Our motivation and running example will be the shortest path problem hinted at above:

**Example 1.** Let $G = (V, E)$ be a graph whose edges have some (independent) failure probabilities $p_e \in [0, 1]$ ($e \in E$). After a disastrous event some edges will have failed, and each failure scenario is described by a set of surviving edges $\xi \subseteq E$. We are interested answering the following two questions:

1. What is the expected value of the shortest path lengths between a set of distinguished nodes?
2. If we can take some actions to modify the failure probabilities $p_e$, how should this be done optimally in order to to minimize the expected shortest path length?

If the graph is a road network, and edges fail due to an earthquake, this is the setting of [37]. The set of all failure scenarios can be totally ordered by comparing the lengths of the shortest path between two (fixed) nodes: If in failure scenario there exists an $st$-path of length at most $\alpha$, then in all scenarios where only a subset of these edges fails, this path still exists. Scenarios with the same shortest path length will be considered equal in the total order. Of course, depending on the application, many other scenario orderings are conceivable, e.g., according to longest $st$-path, number of disjoint $st$-paths, or size of the largest connected component of the graph.

The successive shortest path problem is closely related to the classic $st$-reliability problem introduced by [33,34] and [46] and has been studied by many authors in numerous variants since then. Here the network is a system of circuits with edges corresponding to components which have some probability of failing. The system as a whole operates if there exists some $st$-path, and the probability of having such a path is called $st$-reliability. This fundamental problem and its various extensions have a vast number of applications in reliability engineering [6].

The question also naturally arises in the setting of interdiction problems, where typically one considers decisions to be actions of an attacker to weaken the network structure (to increase failure probability of edges, reduce capacity, etc.) in order to hamper the network’s operational capability. For a survey on typical network interdiction problems and corresponding modeling and solution approaches see [13].

The approach we develop in this paper is not limited to combinatorial stochastic optimization problems. It can also be used in computing the expected objective function value for linear programs with varying right-hand-side coefficients, where each coefficient independently attains one of two values with a given probability. Consider a maximization problem with less than or equal constraints, and assume that: (i) in the nominal problem all right-hand-side coefficients are at the larger of the two values; (ii) the scenarios are given by sets of rows where the coefficient takes the smaller value. Then the objective function is monotonously decreasing when taking supersets among the scenarios. In a similar way, irreducible inconsistent linear systems and maximal consistent subsets of LPs [36] can be studied.
2 Problem Setting

We consider a two-stage stochastic optimization problem

\[
\begin{align*}
\min & \mathbb{E}_{\xi|x}[f(\xi)] \\
C x & \leq d \\
x e & \in \{0, 1\} \quad e \in E \\
\xi &= (\xi e)_{e \in E} \in \{0, 1\}^{|E|}
\end{align*}
\]

(1)

where we minimize the expected value (over all realized failure scenarios) of the recourse function \(f(\xi) : 2^E \to \mathbb{R}\) conditioned on first-level decisions \(x e\), where \(e \in E\) has an associated elementary failure event \(\xi e\) whose probability may be influenced by decision \(x e\). The probability distribution of the scenario distribution thus changes depending on the first-level decisions. The first-level decisions are to be taken subject to some set of linear constraints, e.g., a budget restriction \(\sum_{e \in E} x e \leq \beta\). Typically the evaluation of the recourse function \(f(\xi)\) will itself amount to solving an optimization problem. We will interpret scenarios as sets of ‘positive events’, e.g., the sets of surviving edges in a network after some disaster.

In order to make our assumption on the problem structure more precise, we define the following properties and objects related to the recourse function.

**Definition 1 (aggregable recourse function)** The recourse function \(f\) of a 2-stage stochastic optimization problem is called *aggregable* if

1. \(f\) does not depend on the first-stage decision \(x\),
2. \(f\) is counter-monotone with taking subsets of scenarios,

\[
\xi_1 \subseteq \xi_2 \Rightarrow f(\xi_1) \geq f(\xi_2)
\]

(2)

for all \(\xi_1, \xi_2 \in 2^E\),
3. the probabilities of events \(e \in E\) are independent.

We denote the *critical values* of \(f\), i.e. the different values \(f\) takes, by \(\mathcal{C}(f) = \{\alpha : \alpha = f(\xi), \xi \in 2^E\}\). Moreover we denote by \(\mathcal{T}\mathcal{M}_\alpha(f) = \{\xi \in 2^E : f(\xi) > \alpha, \xi = 1^E - \xi\}\) the sets of edge failures that forces \(f\) to take values strictly greater than \(\alpha\). Finally we define the set of minimal survivable scenarios for each critical value by \(\mathcal{M}_\alpha(f) = \{\xi \in 2^E : f(\xi) = \alpha, \forall \xi' \subset \xi : f(\xi') > f(\xi)\}\), with \(\alpha \in \mathcal{C}(f)\).

The elements of \(\mathcal{M}_\alpha(f)\) are the transversal of the clutter given by the minimal members of \(\mathcal{T}\mathcal{M}_\alpha(f)\).

**Example 1 (continued)** In our running example, computation of \(f\) amounts to computing a shortest path length \(f_{SP}\) in a graph whose edge set depends on the scenario: the edges that failed are missing in the scenario compared to the initial graph \(G\). Clearly, shortest path length is counter-monotone with removing supersets of edges, i.e., considering subsets of the surviving edges of some other scenario. Another way of looking at this is to say that the sublevel sets \(L_\alpha^{-1}(f) = \{\xi : f(\xi) \leq \alpha\}\) of \(f\) are co-monotone (wrt. inclusion) with the natural ordering of real numbers \(\alpha_i < \alpha_j\). Each set \(\mathcal{M}_\alpha(f)\) consists of the simple paths of length \(\alpha\) in the graph.
The set \( C(f) \) allows us to rewrite the objective function as follows:

\[
E_{\xi \mid x}[f(\xi)] = \sum_{\alpha \in C(f)} \alpha \text{Prob}[f(\xi) = \alpha \mid x].
\]

We are thus interested in computing the probabilities \( \text{Prob}[f(\xi) = \alpha] \), first without taking into account possible decision variables \( x_e \), and then integrating them.

3 Aggregation of Nested Scenario Covers

In this section, we develop our approach to provide an effective scenario aggregation by using nested scenario covers. We first describe how scenario sets can be encoded as BDDs in such a way that these sets represent the sublevel sets of the recourse function. We then proceed to show how scenario probabilities can be represented as a flow of probability on the DAGs represented by the BDDs. Finally, we describe how those probability flows can be shaped using the binary first-stage decision variables via linear inequalities and a derive the resulting MIP formulation.

3.1 Scenario Aggregation using BDDs

We will from now on consider the set \( C(f) \) as an ordered set \((\alpha_0, \ldots, \alpha_N)\) with the natural ordering \(<\) of the real numbers. This induces an ordering of the set \( M(f) \) as \((M_{\alpha_0}(f), \ldots, M_{\alpha_N}(f))\).

Each \( M_{\alpha}(f) \) induces a monotone Boolean function \( \Phi_{\leq \alpha}^\leq \) on the scenarios whose minimal true points are the members of \( M_{\alpha}(f) \) by “\( \Phi_{\leq \alpha}^\leq(\xi) = 1 \) if and only if \( f(\xi) \leq \alpha \).” It is well-defined because of (2). Note that \( \Phi_{\leq \alpha}^\leq(\xi) \) describes the support of the cumulative distribution function \( \text{Prob}[f(\xi) \leq \alpha] \). We refer to [14] for more details on Boolean functions. In the reliability analysis setting the function \( \Phi_{\leq \alpha}^\leq \) is called system state function, and we can assume that the system is a coherent binary system. Then \( M_{\alpha}(f) \) are the path sets of the system and \( \text{Prob}[\Phi_{\leq \alpha}^\leq(\xi) = 1] \) is the reliability, see [4].

We propose to encode each function \( \Phi_{\leq \alpha}^\leq \) for \( \alpha \in C(f) \) in the form of a (reduced, ordered) binary decision diagram (BDD), see [30][8]. Intuitively, a BDD can be understood as a directed acyclic graph with a single source and a single sink, in which each source-sink path encodes one or more feasible points of a Boolean function. The graph is layered, with the layers indexed by the variables of the function. The key property is that every node has at most 2 children, and for each sub-dag that includes a sink there are no isomorphic copies in the BDD. This can make them much more compact than conventional decision trees encoding the same feasible points. Although BDDs can be of exponential size compared to the function they encode, for various classes of functions one can find compact BDD encodings. Note that finding a minimal size BDD is \( \mathcal{NP} \)-hard, even for monotone Boolean functions [45], and not every BDD construction algorithm will be output-polynomial. Despite all of these caveats, BDDs have been highly successful in many applications, see [92] for an overview. Furthermore, good software for BDD
For each node \( w \) and all arcs \( \delta \) and \( \Phi \) dual. If

**Remark 1** Let \( \phi(x) \) be a monotone Boolean function and \( \phi^D(x) = \neg\phi(\neg x) \) its dual. If \( B = (E, V, A, \epsilon, l) \) is a BDD representing \( \phi(x) \), then \( B' = (E, V, A, \epsilon, l') \) with \( l'(u,v) = 1 - l((u,v)) \) represents \( \neg\phi^D(x) \), i.e. except for the labeling, \( \phi(x) \) and \( \phi^D \) have the same BDD.

**Definition 2 (BDD)** Let \( E = (e_1, \ldots, e_{|E|}) \) be a finite linearly ordered set. A binary decision diagram (BDD) \( B = (E, V, A \subseteq V \times V, \epsilon : V \rightarrow \{1, \ldots, |E|\}, l((u,v)) : A \rightarrow \{0,1\}) \) is either degenerate, \( B = (E, \{\bot\}, \emptyset, \epsilon, l) \) or \( B = (E, \{\top\}, \emptyset, \epsilon, l) \), or consists of a directed acyclic graph on node set \( V \) with exactly one root \( u^* \), two leaves \( \top, \bot \), arc set \( A \), arc label function \( l : A \rightarrow \{0,1\} \) and a layering function \( \epsilon : V \rightarrow \{0,1\} \). Unless \( B \) is degenerate, it must satisfy the following conditions:

- \( B \) is a layered digraph, i.e. its node set partitions into layer sets \( L_i = \{u \in V : \epsilon(u) = i\}, i = 1, \ldots, |E| \) and \( L_{|E|+1} = \{\top\} \), such that \( |L_1| = 1 \) (the root layer) and \( L_{|E|+1} \) is the terminal layer.
- Arcs only extend to layers with higher index, i.e. \( \forall (u,v) \in A, \epsilon(u) < \epsilon(v) \).
- Every node \( u \in V \setminus \{\bot, \top\} \) is the tail of exactly two differently labeled arcs, i.e. \( \forall u \in V \setminus \{\bot, \top\} \exists \nu_0, \nu_1 \in V : (u, \nu_0) \in A, (u, \nu_1) \in A, l((u, \nu_0)) \neq l((u, \nu_1)) \).
- For any two distinct nodes \( u, v \) of \( V \), the sub-BDDs rooted at \( u \) and \( v \) are not isomorphic, i.e. \( B_u \not\cong B_v \).

Here \( B_u \) denotes the sub-BDD of \( B \) defined by the sub-dag of \( (V, A) \) rooted at \( u \) with node and arc label functions obtained by restriction of \( \epsilon \) and \( l \) to \( V(B_u) \) and \( A(B_u) \), respectively. BDD-isomorphism \( \cong \) is defined as isomorphism of directed graphs with identical variable set and identically evaluating functions \( \epsilon \) and \( l \).

For each node \( u \in V \), the outgoing arc labeled by \( 1 \) is called the True-arc, and the outgoing arc labeled by \( 0 \) is called the False-arc.

Note that BDDs according to this definition are called reduced, ordered BDD in the classical literature [30,32]. (In drawing BDDs one often suppresses the \( \bot \)-node and all arcs \( \delta^0(\bot) \), since these can be easily reconstructed.)

For \( i = 1, \ldots, |E| \), the layer width \( w_i \) of layer \( L_i \) is defined as \( w_i = |L_i| \). The BDD-width is then \( w = \max_{i=1,\ldots,|E|} w_i \). The total size of a BDD is \( |V| + 1 = \sum_{i=1}^{|E|+1} w_i \).

Each BDD \( B \) represents a Boolean function:

\[
\Phi_B(x) = \bigvee_{u^*: \neg \top \text{ path}} \left( \bigwedge_{(u,v) \text{ edge of } \delta^u(\bot)} x_{\epsilon(u)} \wedge \bigwedge_{(u,v) \text{ edge of } \delta^v(\bot)} \neg x_{\epsilon(u)} \right).
\]  

(3)

Furthermore, every Boolean function has a BDD representation (which is essentially unique given the ordering of \( E \)).
This is a direct consequence of the fact that the BDD not only encodes all feasible points of the monotone Boolean function but also all infeasible points (as paths from \( u^* \) to \( \perp \)) and formula (3).

The isomorphism between BDDs for a monotone Boolean function and its dual function is useful if the set \( M_\alpha \) are given as a list of minimal true elements: this list provides a covering-type formulation for the dual function and the output-polynomial time top-down BDD construction rule using matrix minors of \([25]\).

### 3.2 Integrating Scenario Probabilities

**Lemma 1** Given a BDD \( B \) encoding a Boolean function \( \Phi : 2^E \rightarrow \{0,1\} \) and a scenario distribution such that the events \( e \in E \) are independently distributed random variables, the value \( \text{Prob}[B] := \text{Prob}[\Phi(\xi) = 1] \) can be computed in linear time.

Note that linear time here refers to linear in the size of the BDD \( B \) (and \( |E| \)); in the proof we only have to show that we can obtain a system of equations computing the probability that is shaped like the BDD and does not have excessively large subexpressions.

**Proof** The construction is similar to the classical computation of probabilities in a scenario tree, with the difference being that recurring computations in subtrees are avoided because the BDD structure (‘no isomorphic sub-BDDs’) merges these cases.

Let \( p_1 \in [0,1] \) be the probability of event \( e_1 \in E \). We proceed by induction: If \( B \) is the BDD consisting of only the \( \top \)-terminal, it encodes the entire cube \( \{0,1\}^{|E|} \) and \( \text{Prob}[\Phi(\xi) = 1] = 1 \). Similarly, if it consists only of the \( \perp \) terminal, in encodes the empty set, so \( \text{Prob}[\Phi(\xi) = 1] = 0 \).

Assume now that the statement has been proven for BDDs with \( k-1 \geq 0 \) layers. Let \( B \) be a BDD with \( k \) layers. The unique root node \( u^* \) has exactly 2 children that we’ll call \( \text{True}-\)child and \( \text{False}-\)child; let the event for the root layer be \( \epsilon(u^*) = e_1 \).

Initially assume that the sub-BDDs have their roots in the layer \( L_2 \) directly following \( e_1 \). Then, denoting the probabilities computed for the sub-BDDs by \( p_{\text{True-child}} \) and \( p_{\text{False-child}} \),

\[
\text{Prob}[\Phi(\xi) = 1] = p_1 p_{\text{True-child}} + (1 - p_1) p_{\text{False-child}}, \tag{4}
\]

since scenarios contain/don’t contain event \( e_1 \) with these probabilities and the probabilities of the completions have been computed inductively for the sub-BDDs of size at most \( k-1 \).

If a sub-BDD, say the one for \( \xi_1 = 1 \), has its root in a layer \( l > 2 \), then this sub-BDD encodes the set of solutions \( \{1, \star, \ldots, \star, \xi'\} \) where \( \star \) denotes that 0 or 1 can be chosen arbitrarily for every \( (\xi') \in \{0,1\}^{l-1} \). For simplicity assume that only the \( \text{True}-\)child of \( u^* \) starts at layer \( l \). Then

\[
\text{Prob}[\Phi(\xi) = 1] = p_1 \left( \prod_{i=2}^{l-1} (p_i + (1 - p_i)) \right) p_{\text{True-child}} + (1 - p_1) p_{\text{False-child}} \tag{5}
\]

\[
= p_1 p_{\text{True-child}} + (1 - p_1) p_{\text{False-child}},
\]
since the scenarios do not depend on events on layer 2, . . . , l − 1 for the choice of $\xi_1 = 1$.

### 3.3 Shaping the Scenario Distribution

Equations (4) and (5) give a recursive set of linear equations (starting with $p_T = 1, p_\perp = 0$ for the leaves) that can be used to directly turn a BDD encoding $M_\alpha(f)$ into a set of linear constraints to compute $\text{Prob}[\Phi_\alpha^T(\xi) = 1]$ using as many variables and equations as the BDD has nodes (each node has exactly one defining equation). We now show how decisions influencing the probabilities $p_i$ can be incorporated into these formulas as well.

**Lemma 2**

Given a BDD $B$ encoding a Boolean function $\Phi : 2^E \rightarrow \{0, 1\}$ and a scenario distribution such that the events $e_i \in E$ are independently distributed random variables depending on decisions $(x_i)_{i=1,...,|E|}$, the value $\text{Prob}[\xi|x] [\Phi(\xi) = 1]$ can be computed in linear time.

**Proof**

This follows immediately from (5) because the events are assumed to be independent: Let $p_i(x)$ denote the conditioned probabilities, and $p_{\text{TRUE-sub}}(x)$, $p_{\text{FALSE-sub}}(x)$ the conditioned sub-BDD probabilities. Then

$$\text{Prob}[\Phi(\xi) = 1] = p_1(x)p_{\text{TRUE-sub}}(x) + (1 - p_1(x))p_{\text{FALSE-sub}}(x),$$

just as in (4).

Again, equation (6) gives a recursive set of equations (starting with $p_T = 1, p_\perp = 0$ for the leaves), but depending on how $p_i(x)$ is defined they may not be linear. In the next section we will show that if the decisions are binary (or can take only a fixed number of values), the problem can be formulated as a compact mixed-integer programming problem.

### 3.4 MIP Model for Aggregated Scenario Probabilities

Let us now consider for ease of presentation the case where $|E|$ binary decision variables $x_i \in \{0, 1\}$ influence the nominal event probability $p_e$ such that

$$p_i(x) = \begin{cases} p_i & \text{if } x_i = 0 \\ p_i + \Delta_i & \text{if } x_i = 1 \end{cases}$$

where $\Delta_i \in [-p_i, 1 - p_i]$ is the boost or decrease of the probability of event $e_i \in E$ when taking decision $x_i = 1$.

With this definition, and in light of equation (6), we define for each arc $(u, v)$ in the BDD $B_0^\alpha = (E, A, \epsilon, l)$

$$p_{(u,v)}(x) = \begin{cases} p_i(x) & \text{if } (u, v) \in A, \epsilon(u) = i, l((u, v)) = 1 \\ (1 - p_i(x)) & \text{if } (u, v) \in A, \epsilon(u) = i, l((u, v)) = 0, \end{cases}$$

that is, depending on the decision $x_i$ all arcs in the BDD that start at node $u$ associated with event $e_i$ are assigned the appropriate probability depending on $x_i$. 
and whether they correspond to the event being in the scenario \((l((u, v)) = 1)\) or not \((l((u, v)) = 0)\). Introducing intermediate variables for each BDD node we can thus formulate the following MIP constraints to compute the probability of all scenarios of \(B^<_A\) subject to decision variables \(x = (x_i)_{i=1,\ldots,|E|}\):

\[
P_{\alpha} = \begin{cases} 
    p^<_A(u) = 1 & u = \top \in V(B^<_A) \\
    p^<_A(u) = 0 & u = \bot \in V(B^<_A) \\
    p^<_A(u) = p(u, v)(x)p^<_A(v) + p(u, w)(x)p^<_A(w) & (u, v), (u, w) \in A(B^<_A), v \neq w \\
    p^<_A \in [0, 1]^{V(B^<_A)} & x \in \{0, 1\}^E
\end{cases}
\]

(7)

It is easy to see that the constraints of \(P_{\alpha}\) can be written as linear inequalities, which we avoided above to unclutter the formulation. For example, in the case \((u, v), (u, w) \in A(B^<_A), v \neq w, l((u, v)) = 1, l((u, w)) = 0\) where \(\epsilon(u) = i:\)

\[
p^<_A(u) = p(u, v)(x)p^<_A(v) + p(u, w)(x)p^<_A(w)
\]

can be written using inequalities

\[
\begin{align*}
    p^<_A(u) &\leq (p_i + \Delta_i)p^<_A(v) + (1 - p_i - \Delta_i)p^<_A(w) + (1 - x_i) \\
    p^<_A(u) &\leq p_i p^<_A(v) + (1 - p_i)p^<_A(w) + x_i \\
    p^<_A(u) &\geq (p_i + \Delta_i)p^<_A(v) + (1 - p_i - \Delta_i)p^<_A(w) - (1 - x_i) \\
    p^<_A(u) &\geq p_i p^<_A(v) + (1 - p_i)p^<_A(w) - x_i.
\end{align*}
\]

Then the formulation of \(P_{\alpha}\) contains \(|V(B^<_A)| + |E|\) variables and \(4|V(B^<_A)| + 1\) constraints (plus box constraints for all variables).

Clearly, similar models can be built if \(x_i\) is allowed to take a finite number of discrete values, using 2 constraints per possible value of \(x_i\), or disjunctive, or constraint programming techniques.

To obtain a MIP model for aggregable problems of the form (1) we can simply combine blocks of the form (7) using inclusion-exclusion on the scenario sets, exploiting:

\[
\begin{align*}
    \min_{\alpha \in C(f)} & \quad \alpha p^w_{\alpha} \\
    p^w_{\alpha_0} = & \quad p^<_A(u^*) & u^* \text{ root node of } B^<_A \\
    p^w_{\alpha_i} = & \quad p^<_A(u^*) - p^w_{\alpha_{i-1}} & u^* \text{ root node of } B^<_A, \alpha_i \in C(f) \setminus \{\alpha_0\} \\
    Cx &\leq d \\
    (p^<_A, x) &\in P_{\alpha_i} & \alpha_i \in C(f)
\end{align*}
\]

(8)

Note that the \(x\) variables will be shared between \(P_{\alpha_0}\) and \(P_{\alpha_i}\) for \(\alpha_i, \alpha_j \in C(f)\).

When can we expect a polynomial-size MIP formulation using aggregated scenarios? This depends on two prerequisites: We need to be sure that the set \(C(f)\) and the BDDs \(B^<_A\) are small. The first condition may be satisfied automatically for some problem classes, or may be a consequence of the way that the instance is
encoded, e.g., giving an explicit list of relevant objective values. The second condition is hard to control in general, but often one can identify parametric problem subclasses such that the width of $B_0^x$ is appropriately bounded for each fixed value of the parameter: Bounds on the BDD width have been studied widely. For our purposes we will use the results of [25] and [35].

To actually construct the formulation we also need to ensure that the construction of each $B_0^x$ can be performed in (output-)polynomial time. The easiest way to ensure this is to specify a top-down-compile rule, such that each BDD node is touched only once in the construction, and decisions about whether to merge the child subtrees are made immediately in the node in polynomial time. For monotone Boolean functions, which encode the members of an independence system, this can be achieved if equivalence of minors of the associated circuit system can be checked efficiently, see [25]. A prominent example is that of the graphic matroid [43], i.e. when scenarios are forests or simple cycles of a graph. By Remark [1] this is also always the case if the sets $M_0^x$ (or the dual sets) are explicitly given. Thus if $M_0^x$ is explicitly given or can be enumerated in polynomial time, efficient BDD construction reduces to the question whether the BDD width is polynomially bounded.

In light of this reformulation technique we make the following definition.

**Definition 3 (polynomially aggregable 2-stage stochastic optimization problem)** An aggregable 2-stage stochastic optimization problem is called **polynomially aggregable** if

- the number of critical values $C(f)$ of $f$ is polynomial, and
- all BDDs $B_0^x(f)$ can be constructed in polynomial time.

It follows from [25 Thm. 6] that whenever the matrix containing the incidence vectors of the elements of $\mathcal{T}M_0(f)$ has bandwidth $k$, then $B_0^x(f)$ has size at most $n2^{2k-1}$, which is polynomial if $k \in O(\log n)$). Hence 2-stage stochastic optimization problems with bandwidth-limited sets $\mathcal{T}M_0(f)$ and a polynomial number of critical values $C(f)$ are polynomially aggregable:

**Lemma 3** 2-stage stochastic optimization problems with input size $\sigma$, a polynomial number of critical values $C(f)$, and an explicitly given sets $\mathcal{T}M_0(f)$ whose incidence matrix has bandwidth $k \in \log(\sigma)$ are polynomially aggregable.

Note that our definition of (8) contains as a special case the union of products (UPP) problem (this is the case where the decisions $x_e$ have no influence on the probabilities). [4] show that UPP is $\mathcal{NP}$-hard, even when the sets $\mathcal{M}(f)$ are explicitly given – thus unless $\mathcal{P} = \mathcal{NP}$ the BDD construction will be exponential in these cases.

4 Applications

4.1 Pre-disaster Planning Problem

The pre-disaster planning problem we consider first is an instance of the running example [1] we used throughout. As mentioned before, the algorithmic challenges are
| O-D-pair | Dist.-Limit | #bundles | MIP size | # BDDs | MIP size |
|----------|-------------|----------|----------|--------|----------|
|          | (in Prestwich ‘13) | (using BDD-bundles) |          |        |          |
| 14–20    | 31          | 39       | 4        | 237 × 89 |
| 14–7     | 31          | 29       | 6        | 333 × 113 |
| 12–18    | 28          | 56       | 4        | 237 × 89 |
| 9–7      | 19          | 26       | 4        | 164 × 71 |
| 4–8      | 35          | 73       | 6        | 421 × 135 |
|          | ∑           | 223      | 14174 × 6221 | 24     | 1466 × 454 |

MIP solution 1s

(a) With path length cutoffs, penalty 120 for excessively long paths/unconnectedness.

| O-D-pair | #bundles | MIP size | # BDDs | MIP size |
|----------|----------|----------|--------|----------|
|          | (in Prestwich ‘13) | (using BDD-bundles) |          |        |          |
| 14–20    | 378      | 14       | 2609 × 682 |
| 14–7     | 712      | 30       | 13097 × 3304 |
| 12–18    | 233      | 8        | 997 × 1026 |
| 9–7      | 266      | 8        | 1137 × 314 |
| 4–8      | 305      | 12       | 2301 × 605 |
|          | ∑        | 1894     | 123682 × 56851 | 72     | 20137 × 5064 |

MIP solution 36s

(b) Without path length cutoffs.

Table 1: Scenario partition bundles vs. BDD bundles in the Istanbul road network problem instance of [87]. MIP size is number of constraints/number of rows, excluding [0,1]-box constraints before preprocessing of the solver. All MIPs have 30 binary decision variables and one budget constraint. Total path enumeration and BDD construction time is < 1s. CPLEX 12.5 in single-thread mode on a dedicated 24-core, 4-CPU Intel X5650/2.67 GHz system with 96Gb RAM running Linux 3.14.17.

easy to answer: The set of all shortest paths \( M(f_{SP}) \) between a pair of nodes can be enumerated in time \( O((|V| + |E|)|M(f_{SP})) \) using the classic restricted backtracking technique of [41] or the recent output-linear method of [7]. One could alternatively enumerate the elements of \( TM(f_{SP}) \) using the method of [40], which may be preferable if one is interested in displaying the minimal sets of simultaneously failing edges characterizing a bundle of scenarios to a user. Actually, since \( TM(\alpha)(f_{SP}) \) may be exponential in the size of \( M_\alpha(f_{SP}) \) (a graph consisting of \( k \) edge-disjoint \( st \)-paths with at most \( \alpha \) edges each has \( \alpha^k \) minimal sets of edge failures defining \( M_\alpha(f_{SP}) \)), it may be preferable to start computing both sets concurrently, and stop when one is complete. From both sets one can construct identically-sized BDD (which differ only by the arc labels), since the sets define a pair of dual monotone Boolean functions.

We can apply Lemma 3 to this problem as follows: For fixed \( s \) and \( t \) and each \( \alpha \) the set \( TM_\alpha^\alpha \) contains the minimal sets of edges that need to be removed to destroy all shortest \( st \)-paths of length at most \( \alpha \) (sometimes called \( \alpha \)-length-bounded cuts [87]). Clearly, each such set is a subset of some minimal \( st \)-cut. Hence, if the incidence matrix of all minimal \( st \)-cuts has bandwidth at most \( k \), so do all incidence matrices for the clutters \( TM_\alpha^\alpha \). Note that for the incidence matrix of all
| n arcs | arcs | # O-D- pairs | size | min | median | 25% | 75% | 99% | max |
|--------|------|--------------|------|-----|--------|-----|-----|-----|-----|
| 1      | 5    | 32           | 64   | 2   | 3      | 3   | 5   | 5   | 5   |
| 2      | 11   | 2048         | 384  | 2   | 3      | 4   | 5   | 14  | 26  |
| 3      | 19   | 524288       | 1280 | 2   | 3      | 5   | 6   | 33  | 126 |
| 4      | 30   | 1.1 × 10^9   | 3200 | 2   | 4      | 6   | 9   | 79  | 623 |
| 5      | 43   | 8.8 × 10^12  | 6720 | 2   | 5      | 7   | 17  | 125 | 921 |
| 6      | 59   | 5.8 × 10^17  | 12544| 2   | 6      | 9   | 26  | 258 | 2778|
| 7      | 77   | 1.5 × 10^23  | 21504| 2   | 6      | 11  | 35  | 518 | 22967|

- **Table 2**: BDD bundles for random road networks on grid of \((n + 1)^2\) nodes with density approximately 1.2, see [31]. Experiments repeated for 32 different networks and for all origin-destination pairs in each network, subject to length cutoff \(\alpha \cdot d\), where \(d\) is the shortest path distance for the origin-destination pair under consideration. Number and size of BDDs is reported for minimum, maximum, 25/50/75/99% quantiles. Average CPU time per experiment across all 411264 experiments is 2.5s. Compare to [39, Table 7].

(a) Length cutoff \(\alpha = 1.1\). (b) Length cutoff \(\alpha = 1.5\). (c) Length cutoff \(\alpha = \infty\).

**Fig. 1**: BDD size comparison across instances of Table 2.
minimal st-cuts to have bandwidth at most \( k \), in particular the cardinality of each cut must be \( \leq k \). Lemma 4 shows a class of such graphs.

**Lemma 4** Let \( G_1, \ldots, G_l \) be graphs for which the incidence matrices of minimal \( s_i - t_i \)-cuts (\( s_i, t_i \in N(G_i) \)) all have bandwidth at most \( k \). Let \( v \) be a new node, i.e. \( v \notin \bigcup_i N(G_i) \). Then all graphs \( G = (N, E) \) of the form \( N = \{v\} \cup \bigcup_i N(G_i), E = E_v \cup \bigcup_i E(G_i) \) with \( E_v = \bigcup_i E'_v \) such that \( E'_v \subset \{ (x, v) : x \in N(G_i) \} \) and \( |E'_v| \leq 1 \) have incidence matrices of minimal st-cuts (for all \( s, t \in N \)) with bandwidth at most \( k \).

**Proof** If \( s \) and \( t \) are in an original graph \( G_i \) then the statement is true by assumption. If \( s \in G_i \) and \( t = v \) then all minimal cuts are either the edge \( (v, x) \in E'_v \), or \( s - x \)-cuts in \( G_i \), so have bandwidth at most \( k \). Otherwise there is a path from \( s \) to \( t \) in which \( v \) is a separating node, so cuts either separate \( s \) and \( v \) or \( v \) and \( t \) and the previous case applies.

This recursive construction yields tree-like graphs composed of smaller bandwidth-bounded subgraphs and connecting nodes that are separators.

We conjecture that the pre-disaster planning problem has polynomial-size exact MIP reformulation if the line graph of \( G = (V, E) \) has logarithmically bounded pathwidth \( k \in O(\log(|V| + |E|)) \), by exploiting few vertex cuts in the line graph and translating them to few edge cuts in the original graph, but refer this to further work.

From a practical perspective, we obtained very small BDDs for the relatively sparse graphs arising in road networks using the following heuristic. For each \( \alpha \in \mathcal{C}(f_{SP}) \) use the following edge ordering: Count the number of occurrences of edge \( e \) in the sets of \( M_\alpha(f_{SP}) \) and sort them by increasing value. The intuition is that edges which are in no element of \( M_\alpha(f_{SP}) \) are 'don’t care' edges for every scenario, and will thus not require introduction of any node in the BDD, while putting central edges (participating in many minimal scenarios) at the end will not lead to many branching decisions in the BDD until the very last layers. Furthermore, these edges will be included to complete many scenarios, so when merging isomorphic sub-dags in the BDD construction it is good to find them in the bottom layers. The ordering obtained will be similar, at least for graphs with few central nodes, to a heuristic low-bandwidth ordering obtained by the Cuthill-McKee algorithm ([15]). The circuit system minor equivalence check amounts to a direct matrix minor comparison, since we assume that the circuit system is explicitly given by the list of minimal scenarios.

This allows us to give an exact formulation of the pre-disaster planning problem of [37] as a MIP of very small size, an order of magnitude smaller than in the recent work of [38,39]. Table 1 shows the MIP sizes for the Istanbul instance, and Table 2 shows number of BDDs and sizes for randomly constructed networks of similar density, as done in [39] (see also [31] for the specifics on sampling random connected graphs). Again, the BDD sizes directly correspond to MIP sizes.

We remark that the partitioning scheme and 'molded distribution' concept of [38,39] can be seen as a special case of our construction where all BDDs are simply paths, not arbitrary DAGs.
4.2 Independence Systems with failing elements

Let $I = (E, S)$ with $S \subseteq 2^E$ be an independence system, i.e. $\forall S_1 \subseteq S : S_2 \subseteq S_1 \Rightarrow S_2 \in S$. Let $w : S \rightarrow \mathbb{R}$ with $w(S_i) = \sum_{e \in S_i} w_e$ with $w_e \geq 0$ be a nonnegative linear weight function. Consider a setting where elements $e \in E$ can fail independently at random. Let $x_e \in \{0, 1\}$ be decision variables such that the element failure probabilities are $p_e(x_e) = \begin{cases} p_e^1 & x_e = 1 \\ p_e^0 & x_e = 0. \end{cases}$

Consider the 2-stage stochastic programming problem of the form (1) where for a scenario of failing edges $\xi \subseteq E$ the inner optimization problem is given by $f(\xi) = \max_{S_i \cap \xi \in S} w(S_i)$, i.e. by optimizing $w$ over the restriction of $I$ to $\xi$. Then this setting covers the following problems:

- Expected maximum weight forests in a graph $G = (V, E)$
  The independence system considered is the graphic matroid with ground set elements $E$, non-negative edge weights $w_e$. Edges can fail (independently) with some failure probability that can be influenced by decisions $x_e$. The goal is to minimize or maximize the expected weight of the maximal forest under edge decisions.
- Expected stability number in a graph $G = (V, E)$
  The independence system considered is the set of stable sets in $G$.
- Matroid Steiner Problems [10]

As mentioned before, BDD construction is output-polynomial in the graphic matroid. We point out that in all of the cases we believe that relevant graph classes with bounded BDD width can be characterized, but each may be a separate and intricate topic of investigation.

4.3 Stochastic Flow Network Interdiction

The flow value function of a capacitated network flow problems is monotone wrt. the network capacities. Consider a setting where arc capacity failures (or reductions) occur randomly. Decisions allow us to increase or decrease reliability of the links. If we assume complete failure of arcs, the fundamental theorem of network flow theory about path decomposition of flows lets us link this problem to a classic maximum flow problem. It is more challenging to consider the case of (discrete) arc capacity changes; see [16] and papers citing that for structural statements on the lattice of flow distributions under capacity changes.

In particular, the network interdiction problem SNIP(IB) introduced in [13] fits this framework:

**Definition 4 (stochastic network interdiction problem SNIP(IB))** Let $G = (V, A)$ be a (directed) graph, $s, t \in V$ a source and a sink node, and arc capacities $\{c_e\}_{e \in A}$ be given. For each arc $e \in A$ let a probability $p_e$ of an interdiction attempt being successful be given, where success means that the arc capacity becomes 0. The problem SNIP(IB) for $(G, \{p_e\}_{e \in A}, s, t)$ is then the problem of selecting a set of arcs to interdict, such that the expected maximum flow in the remaining network is minimized.
(a) SNIP(IB)-4x9: 38 nodes, 67 arcs (24 targetable), interdiction success rate 75%. Network has 10 different flow values; ∑ BDD sizes: 207 nodes, construction time 3.3s (3969 max flow oracle calls), MIP size 1771 × 434

(b) SNIP(IB)-7x5: 37 nodes, 72 arcs (22 targetable), interdiction success rate 75%. Network has 19 different flow values; ∑ BDD sizes: 2302 nodes, construction time 5.9s (61988 max flow oracle calls), MIP size 22867 × 5133

(c) SNIP(IB)-10x10: 10 × 10 grid, 35% interdictable arcs, vertical arc orientation unif. random up/down 50 : 50, capacities 10..100 uniformly random multiples of 10, 75% interdiction success rates.

Fig. 2: SNIP(IB) Instances of [13]. Computations performed using CPLEX on a single node 16-core Intel(R) Xeon(R) CPU E5-2698 v3 @ 2.30GHz.
To solve an SNIP(IB) problem using our framework we need to compute the set of all $st$-flows arising under edge removal from the graph $G$. By the max flow min cut theorem the number of different flow values appearing depends on the number of possible cut values, so in particular a rough estimate shows that if the number of different arc capacities is $k$ there are at most $\binom{|E|}{k}$ possible cut values. Having a limited number of arc capacities is very natural in many applications where the capacity is determined by a technological parameter of the arc type, e.g., link speed or pipe diameter.

For the purpose of showing practicality of our approach we resort to computing the sets of all minimal edge sets whose removal reduces the maximal $st$-flow in the residual graph to less than $\alpha$ using a technique successfully applied to determining minimal cut sets in metabolic flux networks [24,5]. In particular, we solve the max-flow problem using the lemon-1.3.1 library [9] in the membership oracle of the joint generation procedure [18,23]. In Figure 2 we use the benchmark SNIP(IB)-instances of [13]. Due to the fact that the MIP only depends on the network structure and the set of interdictable arcs we can seamlessly change the budget value or interdiction success probabilities without re-generating the MIP. This separation of structure and data is a major advantage to other methods that take these parameters into account throughout the solution procedure like [27], and makes it possible to do a parameter study accounting for all conceivable budget values.

In summary, we find that we obtain MIPs that allow us to solve the exact reformulation, rather than resorting to solving approximations, and for all conceivable budgets, but that current MIP solvers – we’ve also done limited trials with FICO Xpress, Gurobi and SCIP – struggle with the particular problem structure more than we expected. The formulation with few, binary variables coupling many continuous variables in equations seems like a natural fit for decomposition approaches, which we are planning to investigate in future research.

Acknowledgements This work was partially supported by EU-FP7-ITN 289581 ‘NPLast’ (while the second author was at ETH Zürich) and EU-FP7-ITN 316647 ‘MINO’ (while the first author was at ETH Zürich).

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