Abstract

We suggest a new asymptotic representation for the solutions to the 2-D wave equation with variable velocity with localized initial data. This representation is a generalization of the Maslov canonical operator and gives the formulas for the relationship between initial localized perturbations and wave profiles near the wave fronts including the neighborhood of backtracking (focal or turning) and selfintersection points. We apply these formulas to the problem of a propagation of tsunami waves in the frame of so-called piston model. Finally we suggest the fast asymptotically-numerical algorithm for simulation of tsunami wave over nonuniform bottom. In this first part we present the final formulas and some geometrical construction. The proofs concerning analytical calculations will be done in the second part.

1 Main equations and a simple example: the wave field in the case of constant bottom

1.1 Some notation

Let us introduce the notations used in this paper. A two dimensional vector can be written with capital or small letters $X = (X_1, X_2)$ or $x = (x_1, x_2)$. The vector can be written also as a column vector $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Two dimensional vectors $X$ and $Y$ can form a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ with four rows. The real scalar product between two vectors $X$ and $Y$, with real components, is indicated by $<X, Y>$, the complex scalar product among bi-dimensional vectors $Z, W$, with complex components, is written as $<Z, W>_{c}$, the two by two matrix generated by two bi-dimensional vectors $X, Y$ is written as $(X,Y)$ where in the first column there are the components of the vector $X$ and in the second column those of the vector $Y$; the transposed matrix of $C$ is denoted by $^tC$. 

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1.2 Main equations

Let us remind the statements of problems used in tsunami wave problems ([1]-[10]) as well as in general linear water wave theory ([11]-[17].)

Let us assume that the bottom of the basin is moving \( H = H_0(x) + H_1(x, t). \) We assume also that the perturbation \( H_1(x, t) \) is small with respect to \( H_0 \)

\[ |H_1| \ll H_0(x), \]

and that \( H_1 \) is localized in a neighborhood of some given point \( x_0. \) If \( L \) is the dimension of the region where the wave phenomena is studied, and \( l \) is the dimension of the perturbed region, then our hypothesis implies that \( l \ll L. \) Another assumption is that the bottom "changes slowly", i.e. that \( \nabla H_0 \sim \mu, \) where \( \mu \) is some small ("adiabatic parameter"). We discuss below its meaning. Introducing the scaled variables \( x' = \frac{x}{L}, \) then \( H = H_0(x') + \chi H_1(\frac{x'}{\mu}, t), \) where \( \mu = \frac{l}{L} \ll 1. \)

The equation for the velocity potential \( \Phi \) in the water \(-H \leq z \leq \eta, \) where \( \eta(x, t) \) is the sea elevation in the linear approximation, has the form, in dimensional variables:

\[
\Delta \Phi = 0, \tag{1.1}
\]

\[
\eta_t - \frac{\partial \Phi}{\partial z} \big|_{z=0} = 0, \tag{1.2}
\]

\[
\Phi_t + g\eta \big|_{z=0} = 0, \tag{1.3}
\]

\[
\frac{\partial \Phi}{\partial n} \equiv \frac{\partial \Phi}{\partial z} + \langle \nabla H, \nabla \Phi \rangle = v(x, t) \big|_{z=-H}. \tag{1.4}
\]

where \( v(x, t) \) is the normal component of the velocity of the motion of the bottom in the point \( x. \) The velocity \( v \) can be expressed by means of the derivative \( \frac{\partial H_1}{\partial t} \) by:

\[
\frac{\partial H_1}{\partial t} / \sqrt{(\nabla H)^2 + 1}, \]

since \( V \) is the projection of the velocity on the vector \( \frac{1}{\sqrt{(\nabla H)^2 + 1}} (\nabla H, 1) \) normal to the surface \( z = -H. \) If we consider \( \nabla H_0 \) to be small (because of the slow variation of the bottom relief), and that also \( \nabla H_1 \) is small (because of the small amplitude \( H_1 \)), then we have

\[
v = \frac{\partial H_1}{\partial t}. \]

1.3 The solution in the form of the Fourier transform

Let us begin considering the system \([14]-[17]\) in the case of constant bottom. In this case the velocity potential and its derivatives are zero for \( t = 0. \) We make the Fourier transform of the system \([1.1]-[1.4]\) with respect to the variables \( x_1, x_2. \) The dual variables will be denoted with \( p_1, p_2 \) and the Fourier transform of the corresponding function will be considered as a "wave". Then \([1.1]-[1.4]\) gets the form

\[
\Phi_{zz} - p^2 \Phi = 0, \tag{1.5}
\]

\[
\tilde{\eta}_t - \frac{\partial \Phi}{\partial z} \big|_{z=0} = 0, \tag{1.6}
\]

\[
(\tilde{\Phi}_t + g\tilde{\eta}) \big|_{z=0} = 0, \tag{1.7}
\]

\[
\frac{\partial \Phi}{\partial z} \big|_{z=-H} = \tilde{v} \equiv \frac{\partial H_1}{\partial t}. \tag{1.8}
\]
Solving (2.7)-(1.8), we find

\[ \Phi = \cosh((z + H)p) \varphi + \frac{\sinh(z|p|)}{|p| \cosh(H|p|)} \frac{\partial \tilde{H}_1}{\partial t} \]  

(1.9)

and

\[ \Phi |_{z=0} = |p| \tanh(H|p|) \varphi + \frac{1}{\cosh H|p|} \frac{\partial \tilde{H}_1}{\partial t} \]  

(1.10)

Thus the equations (2.8)-(3.2) take the form

\[ \frac{\partial \tilde{\eta}}{\partial t} - |p| \tanh(H|p|) \varphi - \frac{1}{\cosh(H|p|)} \frac{\partial \tilde{H}_1}{\partial t} = 0 \]

\[ \frac{\partial \tilde{\varphi}}{\partial t} + g\tilde{\eta} = 0 \]  

(1.11)

Where \( \tilde{\varphi} = \Phi |_{z=0} \), and we have the initial conditions \( t = 0 \)

\[ \tilde{\varphi} |_{t=0} = 0, \quad \tilde{\varphi} |_{t=0} = 0 \iff \tilde{\eta} |_{t=0} = 0. \]  

(1.12)

These conditions define the so called Cauchy-Poisson problem for the system (1.11). They are compatible with the perturbation of the bottom only if we suppose that the earthquake starts at a time different from zero. So we assume that the bottom has an "instantaneous" movement at a small time \( t = \varepsilon \):

\[ H_1(x, t) = \theta(t - \varepsilon)V(x), \]  

(1.13)

then we send \( \varepsilon \) to zero at the end of the calculation; the smooth function \( V(x) \) decays rapidly at infinity.

Differentiating the first equation in (1.11) with respect to \( t \) and substituting \( \frac{\partial \tilde{\varphi}}{\partial t} \) with \(-g\tilde{\eta}\) we get the equation for \( \tilde{\eta} \):

\[ \tilde{\eta}_{tt} + \mathcal{L} \tilde{\eta} - \frac{1}{\cosh(H|p|)} \frac{\partial^2 \tilde{H}_1}{\partial t^2} = 0, \quad \mathcal{L} = g|p| \tanh(H|p|). \]  

(1.14)

Differentiating the second equation of the system (1.11) with respect to \( t \) and substituting the derivative \( \eta_t \) with the expression of the first equation and considering the condition that the source is active at the moment \( t = \varepsilon > 0 \), we get \( \varphi_t |_{t=0} = -g|p| \tanh(H|p|) \tilde{\varphi} |_{t=0} = 0 \) and the initial condition for (1.14)

\[ \eta_t |_{t=0} = 0 \quad \eta_t |_{t=0} = 0. \]  

(1.15)

It is easy to find the solution \( \tilde{G} \) of the homogeneous equation associated with (1.14):

\[ \tilde{G}_{tt} + \mathcal{L}(p, H)\tilde{G} = 0, \quad \tilde{G} |_{t=\tau} = 0, \quad \tilde{G} |_{t=\tau} = 1, \]

\[ \tilde{G}(t, \tau, p) = \frac{e^{i\sqrt{\mathcal{L}}(t-\tau)} - e^{-i\sqrt{\mathcal{L}}(t-\tau)}}{2i\sqrt{\mathcal{L}}} = \frac{\sin \sqrt{\mathcal{L}}(t - \tau)}{\sqrt{\mathcal{L}}}. \]

In this way the solution of the non homogeneous equation (1.14) is

\[ \tilde{\eta} = \int_0^t \tilde{G}(t, \tau, p) \frac{1}{\cosh(H|p|)} \frac{\partial^2 \tilde{H}_1(\tau, p)}{\partial t^2} d\tau. \]  

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The inverse Fourier transform of the function \( \tilde{\eta} \) gives the elevation of the free surface. Under our assumption of instantaneous motion at time \( \varepsilon \) we have \( \frac{\partial^2 H_t(x,p)}{\partial \varepsilon^2} = \delta'(t-\varepsilon)\tilde{V} \) and so:

\[
\tilde{\eta} = \int_0^t \tilde{G}(t,\tau,p) \frac{1}{\cosh(H|p|)} \frac{\partial^2 \tilde{H}_1(\tau,p)}{\partial \tau^2} d\tau = \frac{\tilde{V}}{\cosh H|p|} \int_0^t \sin \sqrt{\mathcal{L}}(t-\tau) \delta'(t-\varepsilon)d\tau = -\frac{\tilde{V}}{\cosh H|p|} \frac{\partial}{\partial \tau} \left( \frac{\sin \sqrt{\mathcal{L}}(t-\tau)}{\sqrt{\mathcal{L}}} \right) \bigg|_{\tau=\varepsilon} = \frac{\tilde{V}}{\cosh H|p|} \cos \sqrt{\mathcal{L}}(t-\varepsilon).
\]

We send now \( \varepsilon \) to zero so we get the function \( \tilde{\eta} = \frac{\tilde{V}}{\cosh(H|p|)} \cos \sqrt{\mathcal{L}}(t) \). It is evident that \( \tilde{\eta} \) is the solution of the equation (1.14) with the following Cauchy conditions

\[
\tilde{\eta}|_{t=0} = \eta_0(p), \quad \tilde{\eta}'|_{t=0} = 0. \tag{1.16}
\]

We shall discuss the meaning of such initial conditions for the function \( \eta \) later.

### 1.4 Solution of the Cauchy problem for constant bottom and instantaneous source

Let us study the solution \( \eta \) corresponding to (1.16). It is not restrictive to assume that the center of the source is located in the origin of the coordinates \( x_0 = 0 \) and that the perturbation decays rapidly with the distance from the origin and that it has a maximum in a small neighborhood of the origin. We use also dimensionless variables:

\[
\tilde{V} = V\left(\frac{p}{l}\right),
\]

where \( l \) is the size of the shifted region and

\[
\tilde{V} = \frac{1}{2\pi} \int V(\xi)e^{-ip\xi}d\xi = \frac{1}{2\pi} \int V(y)e^{-ip\sqrt{y^2-y_l^2}}dy = l\tilde{V}(pl), \quad \tilde{\eta}_0(p) = \frac{l}{\cosh(|p|H)}\tilde{V}(pl),
\]

where we made the substitution \( y = \frac{\xi}{l}, \quad \xi = yl \) and \( \tilde{V}(p) \) is the usual Fourier transform of the function \( V(y) \). We assume that \( V(y) \) is smooth function rapidly decaying as \( |y| \to \infty \).

Then we can make the inverse Fourier transform:

\[
\eta = \frac{1}{4\pi} \sum_{\pm} \int e^{\pm it\sqrt{\mathcal{L}(p,H)+ip\sqrt{|p|^2-\eta_0(p)^2}}} \frac{1}{\cosh(|p|H)}\tilde{V}(pl)dp.
\]

Changing the variables \( pl = p', p = p'/l \), we get

\[
\eta = \frac{1}{4\pi} \sum_{\pm} \int e^{\pm i\sqrt{\frac{2\mathcal{L}(p,H)}{1+\cosh(|p|^2/H^2)}}+ip\sqrt{|p|^2-\eta_0(p)^2}}} \frac{1}{\cosh(|p|^2/H^2)}\tilde{V}(p)dp.
\]

In this way the problem is reduced to the computation of the asymptotic behavior of the integral.

We will study the asymptotic values for \( |x| >> l \). We change to polar coordinates \((\rho, \varphi)\) in the integral, where \( \varphi \) is defined as the angle among \( p \) and \( x - x_0 \). Thus \( p = \rho \Theta(\varphi) \frac{x}{|x|} \), where \( \Theta(\varphi) \) is the two dimensional matrix defining the rotation of an angle \( \varphi \).

\[
\Theta(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}
\]

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Then the last integral has the form

$$\eta = \frac{1}{4\pi} \sum_\pm \int_0^\infty \rho d\rho \int_0^{2\pi} d\varphi \exp \left( \pm it \sqrt{\frac{g\rho}{l}} \tanh(\rho \frac{H}{l}) \right) \exp \left( i \rho \frac{|x|}{l} \cos \varphi \right) \frac{1}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} \hat{V}(\rho \frac{x}{|x|}).$$

The internal integral can be computed using the method of stationary phase. The phase has the form: \( \Phi = \frac{d\varphi}{d\rho} \cos \varphi \), the equation \( \frac{d\Phi}{d\rho} = 0 \) gives \( \varphi = 0, \varphi = \pi \); furthermore it is not possible to apply the method of the stationary phase in the point \( \rho = 0 \). Nevertheless one can take a sufficiently small neighborhood of the saddle points of the variable \( \varphi \) and show that, \[18, 19, 20, 24\], it is smaller than the contribution of the terms that we neglect. The result is:

$$\rho \int_0^{2\pi} d\varphi e^{i\rho |x| \cos \varphi} \frac{1}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} \hat{V}(\rho \frac{x}{|x|}) \approx \frac{\sqrt{2\pi}}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} e^{-i\pi/4} \sqrt{\frac{l}{|x|}} e^{i\rho |x|} \hat{V}(\rho \frac{x}{|x|}) + e^{i\pi/4} \sqrt{\frac{l}{|x|}} e^{-i\rho |x|} \hat{V}(-\rho \frac{x}{|x|})$$

and

$$\eta \approx \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{l}{|x|}} \sum_\pm \int_0^\infty d\rho \frac{\sqrt{g}}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} \exp \left( \pm it \sqrt{\frac{g\rho}{l}} \tanh(\rho \frac{H}{l}) \right) \left( e^{i\pi/4} e^{i\rho |x|} \hat{V}(\rho \frac{x}{|x|}) + e^{-i\pi/4} e^{-i\rho |x|} \hat{V}(-\rho \frac{x}{|x|}) \right).$$

Let us consider the last integral. Its complete phases are:

$$\Phi_{\pm, l} = \pm (t \sqrt{gl \rho \tanh(\rho \frac{H}{l})} \pm \rho |x|) / l.$$

For \( t > 0, \rho > 0 \) the derivative \( \frac{\partial \Phi_{\pm, l}}{\partial \rho} \) is strictly positive, this implies the absence of critical points for the functions \( \Phi_{\pm, l} \). It follows that these terms give, for \( t > 0 \), a contribution to the wave field which is asymptotically small with respect to the other contributions and so it can be dropped. Furthermore since \( V \) is a real function then \( \hat{V}(\rho \frac{x}{|x|}) \) and \( \hat{V}(-\rho \frac{x}{|x|}) \) are complex conjugates so the last integral may be written in the form

$$\eta \approx \frac{1}{\sqrt{2\pi}} \sqrt{\frac{l}{|x|}} x \text{Re} \int_0^\infty d\rho \frac{\sqrt{g}}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} \hat{V}(\rho \frac{x}{|x|}) e^{-i\pi/4} \exp \left( \frac{i}{1} (g |x| - t \rho \sqrt{gH} \sqrt{\frac{1}{\rho H} \tanh(\rho \frac{H}{l})}) \right).$$

Since the ratio \( \frac{H}{l} \) is rather small, the source is localized, and the function \( \hat{V}(\rho \frac{x}{|x|}) \) decays rapidly as a function of \( \rho \), then the main contribution to the last integral is coming from the small values of \( \rho \). Then we get that the functions \( \frac{1}{\cosh(\rho \frac{x}{|x|} \frac{H}{l})} \) and \( t \rho \sqrt{gH} \sqrt{\frac{1}{\rho H} \tanh(\rho \frac{H}{l})} \) can be expanded in Taylor series. If we substitute the first function with \( 1 \) we neglect a term of the order of \( O(\frac{H}{l})^2 \). The second function can be approximated by the first two non zero terms of its expansion \( t \rho \sqrt{gH} \left( \frac{1}{l} - \frac{1}{3} (\frac{\rho H}{l})^2 \right) \)
making an error of the order of \( t \sqrt{gH} \left( \frac{H}{l} \right)^4 \). It is clear from the previous estimates that these terms are small and so we obtain

\[
\eta \approx \frac{l}{\sqrt{2\pi} |x|} \mathrm{Re} \int_0^\infty d\rho \sqrt{g} \tilde{V}(\rho |x|) e^{-i\pi/4} \exp \left( \frac{i}{1} (\rho |x| - t \rho \sqrt{gH}(1 - \frac{\rho^2}{6} \left( \frac{H}{l} \right)^2)) \right).
\]

It will be explained below that the integral gets its larger values in the neighborhood of the front, i.e. near the curve (circle) \( |x| = \sqrt{gH}t \). In this way the dispersion effects can have an influence on the asymptotic values in the far wave field under the condition that the coefficient of \( \rho^3 \) in the exponent is larger or equal to one. Thus we obtain different behaviors, putting \( \sqrt{gH}t \) equal to \( |x| \) in this coefficient, according to different relations among \( |x|, H, l \) (compare [1]-[10], [24]):

a) For \( |x| >> \frac{l^3}{H^2} \) the dispersion has an important influence in the neighborhood of the front, and the asymptotic can be expressed by means of a function similar to the Airy function. In this case the behavior of the function \( V \) is not important for the definition of the profile of the front.

b) For \( |x| \sim \frac{l^3}{H^2} \) the weak dispersion and the function \( \tilde{V} \) have equal influence on the formation of the wave profile;

c) For \( |x| << \frac{l^3}{H^2} \) the dispersion is not important. If the term with \( \rho^3 \), is dropped from the phase of the integral an error of the order of \( |x|H^2/l^3 \) is done.

Let us consider the example where \( H = 4 \text{ km} \), \( l = 40 \text{ km} \), thus \( l^3/H^2 = 4000 \text{ km} \). Thus a (weak) effect of the dispersion starts at 4000 km. If the size of the source increases twice this distance increases 8 times and becomes 32000 km, a distance larger than any ocean. Thus we will start analyzing the point c (it possible to neglect the effect of the dispersion).

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Thus, assuming that the inequality \( |x| << \frac{l^3}{H^2} \) is satisfied, we have

\[
\eta \approx \frac{l}{\sqrt{2\pi} |x|} \mathrm{Re} \int_0^\infty d\rho \sqrt{g} \tilde{V}(\rho |x|) e^{-i\pi/4} \exp \left( \frac{i}{1} (\rho |x| - t \rho \sqrt{gH}(1 - \frac{\rho^2}{6} \left( \frac{H}{l} \right)^2)) \right) =
\]

\[
\frac{l^{1/2}}{\sqrt{|x|}} \mathrm{Re}(e^{-i\pi/4} F(\frac{\Phi(x, t)}{l}, \frac{x}{|x|})) \quad \Phi(x, t) = |x| - t \sqrt{gH},
\]

where

\[
F(z, n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iz\rho} \sqrt{\rho} \tilde{V}(\rho n) d\rho \quad n = \frac{x}{|x|}.
\]

(1.17)

Here \( n \) is a unit vector. It is natural to introduce its angle \( \psi \) in such a way that \( \psi = 0 \) corresponds to the axis \( x_1 \). Then

\[
n = n(\psi) = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix}.
\]

(1.18)

Hence the function \( \tilde{V}(\rho, n(\psi)) \) depends on \( (\rho, \psi) \) and the function \( F(z, n(\psi)) \) depends on \( (z, \psi) \). Not to complicate notation we use the same symbols \( \tilde{V} \) and \( F \) for them and sometimes write \( \tilde{V}(\rho, \psi) \) and \( F(z, \psi) \) instead \( \tilde{V}(\rho, n(\psi)) \) and \( F(z, n(\psi)) \) respectively.
We note, that the function \( F(z, n) \) decreases for \( |z| \to \infty \) as an inverse power. Indeed, let us change variable in the last integral \( \rho = \frac{y^2}{2} \); then

\[
F(z, n) = \frac{1}{\sqrt{2\pi}} \text{Re} \int_0^\infty y^2 e^{i\pi(x - \frac{y^2}{2})} \tilde{V}(\frac{y^2}{2}) \text{d}y.
\]

Let us use the method of the stationary phase, we get, because of the presence of the factor \( y^2 \) under the integral, \( F(z, \omega) \sim \frac{1}{|z|^2} \), if \( \tilde{V}(0) \neq 0 \). Thus for \( |x - \sqrt{g\Theta t}| \gg l \) and \( |x| \gg l \), we have that \( \eta \sim \frac{1}{|x|^2} \tilde{V}(0) \).

**Example 1.** Let us give some example of the function \( F(z, \omega) \). We choose for the function \( V \), defining the source, the function

\[
V(y) = \tilde{V} \cos(a_1Y_1 + a_2Y_2 + \chi)e^{-b_1Y_1^2 - b_2Y_2^2}, \quad Y = \Theta(\theta)y,
\]

where \( \tilde{V}, a_1, a_2, b_1, b_2 > 0, \theta, \chi \) are parameters. In this case the function \( F(z, \psi) \) can be expressed in terms of parabolic cylinder functions \( D_{-\frac{3}{2}} \) or confluent hypergeometric functions \( \Gamma \) (see [21], 3.462, page. 351)

\[
\tilde{V}(\rho, \psi) = \frac{V \sqrt{b_1b_2}}{2\sqrt{2\pi}} e^{-\alpha - \beta \rho^2} \cosh(i\delta + \gamma \rho),
\]

\[
F(z, \psi) = \tilde{V} \sqrt{\frac{b_1b_2}{2\sqrt{2\pi}}} \text{Re}(e^{-\frac{i\pi}{4}} \int_0^\infty \sqrt{\rho(e^{-\frac{\rho^2}{2} + \gamma \rho \cos \theta} + e^{-\frac{\rho^2}{2} - \gamma \rho \cos \theta})} \text{d}\rho) \equiv \tilde{V} \sqrt{\frac{b_1b_2}{2\sqrt{2\pi}}} \text{Re}([Q_+ + Q_-]),
\]

\[
Q_{\pm}(z, \psi) = \frac{e^{-i\frac{\pi}{4}} - \alpha e^{i\delta}}{(5 \pm 3)\beta^{3/4}} \left( \mp \sqrt{\beta} \Gamma \left( \frac{3}{4}, \frac{1}{2}, \frac{w_{\pm}^2}{4\beta} \right) + w_{\pm} \Gamma \left( \frac{1}{4}, \frac{3}{4}, \frac{w_{\pm}^2}{4\beta} \right) \right),
\]

where \( \Gamma \) is a gamma function, and

\[
w_{\pm} = \gamma + i\beta, \quad \alpha = (b_1a_2^2 + b_2a_1^2)/(4b_1b_2),
\]

\[
\beta = (b_1\sin^2(\psi - \theta) + b_2\cos^2(\psi - \theta))/(4b_1b_2),
\]

\[
\gamma = (b_1a_2 \sin(\psi - \theta) + b_2a_1 \cos(\psi - \theta))/(2b_1b_2),
\]

The figures Fig. 1, Fig. 2 present some types of profiles.

**Main conclusion:** the phase in the neighborhood of the front defines completely a one parameter family of trajectories which generate the front. Further we remark that, since the function \( F \) decreases in a neighborhood of the front, we can expand in the formula \( \text(L.17) \) \( |x| \) in a neighborhood of the front, keeping in the expansion only the zero order term, and that we can substitute the factor \( \frac{1}{\sqrt{|x|}} \) (the amplitude of the wave) with the term \( \frac{1}{\sqrt{g\Theta t}} \). We want to find analogous formulae for the wave field in the case of negligible small dispersion and for variable bottom.
### 2 Asymptotic behavior of the wave field over nonuniform bottom for very small dispersion

#### 2.1 The wave equation, rays and wave fronts

In this section we start the analysis of the behavior of the amplitude of the wave when the bottom is not constant. We use here well known objects and their characteristics which one can find in books connected with the semiclassical asymptotics and ray method, geometrical optics and wave fronts, Hamiltonian mechanics, catastrophe theory etc. We try to collect here all necessary objects and give their description in elementary form. More complete form and details one can find in [25]-[51]. It is clear that in practice we have studied the solution of the wave equation in the previous section. In order to be accurate we introduce in the calculations the characteristic depth \( H_0 \) of the basin, the small parameter

\[
\mu = \frac{l}{L},
\]

expressing the relationship among the characteristic size of the source and the characteristic size of the basin. We begin introducing non dimensional variables in the equations. Then after a suitable change of variables

\[
x' = x/L, \quad t' = t\sqrt{gH_0/L}, \quad H = H_0H'(x'),
\]

our equations and initial data will take the form:

\[
\begin{align*}
\frac{\partial^2 \eta}{\partial t^2} &= g <\nabla, H'(x')\nabla> \eta, \\
\eta|_{t=0} &= V(x/\mu), \quad \eta_t|_{t=0} = 0.
\end{align*}
\]

Our asymptotic expansions will be done in term of this parameter under assumption \( \mu \ll 1 \) and that the domain in \( \mathbb{R}^2 \) and time interval \([0,t] \in \mathbb{R}_t \) while asymptotic expansion are working independent of \( \mu \). To come back to original variables it is enough to change in final asymptotic formulas \( \mu \) by \( l \).

We assume that the source of the perturbation is localized in \( x = 0 \). It is easy to see that finding the field far from the source, \(|x| \gg l \), is similar to find the asymptotic values for \( \mu \rightarrow 0 \) in the problem (2.2). The problem now is to study the wave equation with variable coefficient. The asymptotic values of the wave amplitude \( \eta \) can be expressed by means of the wave front formed by rays. It is a known fact that instead of the straight rays one has to introduce curved rays and characteristics which are the one dimensional family of trajectories \( P(\psi, t), X(\psi, t) \) of an appropriate Hamiltonian system. The ends of the rays again form the wavefront, but now it can be different from the circle, a more complicated closed curve probably with cusps and self intersection points. In the considered situation these rays and characteristics are determined in the following way.

We introduce the function \( C(x) = \sqrt{gH(x)} \) and, as before, let \( n \) be the unit vector (1.18) directed as the external normal to the unit circle. Then the Hamilton system is:

\[
\begin{align*}
\dot{x} &= p |p| C(x), \quad \dot{p} = -|p| \frac{\partial C}{\partial x}, \quad x|_{t=0} = 0, \quad p|_{t=0} = n(\psi),
\end{align*}
\]

i.e. the family of trajectories \( P(t, \psi), X(t, \psi) \) going out from the point \( x = 0 \) with unit impulse \( p = n(\psi) \). Let us indicate \( C(0) = C_0 \). The Hamiltonian corresponding to (2.4) is \( \mathcal{H} = C(X)|p| \). From the conservation of the Hamiltonian on the trajectories we have the important equation

\[
|P|C(X) = C_0.
\]
The projections \( x = X(t, \psi) \) of trajectories on the plane \( \mathbb{R}^2_x \) are called the rays. Recall that the *front* in the plane \( \mathbb{R}^2_x \) at the time \( t > 0 \) is the curve \( \gamma_t = \{ x \in \mathbb{R}^2 | x = X(\psi, t) \} \). The points on this curve are parameterized by the angle \( \psi \in (0, 2\pi] \). If in each point \( x \) of the front \( \gamma_t \), \( \frac{\partial X}{\partial \psi} \neq 0 \), then the front is a smooth curve. The points where \( \frac{\partial X}{\partial \psi} = 0 \) are named *focal* (backtracking or turning points), in these points the front looses its smoothness. In the situation in which the focal points appear, (they are very interesting from the point of view of tsunami), it is reasonable to introduce the concept of the front in the phase space \( \mathbb{R}^4_{p,x} \) at the moment \( t > 0 \), i.e. the curve \( \Gamma_t = \{ p = P(\psi, t), x = X(\psi, t), \psi \in [0, 2\pi] \} \). We note that at least one of the component of the vector \( P_{\psi}, X_{\psi} \) is different from zero, see Lemma 3; and also the rays \( x = X(t, \psi) \) are orthogonal to the front \( \gamma_t \): \( \langle X, X_{\psi} \rangle = 0 \) see Lemma 3.

### 2.2 The wave field before critical times.

It is not difficult to check that a (possibly sufficiently small) \( t_1 \) exists such that, for any \( t, t_1 \geq t > \delta > 0 \), there are no focal points in \( \gamma_t \). The first instant of time \( t_{cr} \), in which focal points are formed is called *critical*. Let us first write the solution before critical times, larger than \( \delta \), when the front is already defined. In this case the asymptotic solution is defined in the following way. We define a neighborhood of the front for sufficiently small (but independent of \( h \)) coordinates \( \psi, y \), where \( |y| \) is the distance among the point \( x \) belonging to a neighborhood of the front and the front. For this aim we will take \( y \geq 0 \) for the external subset of the front and \( y \leq 0 \) and for the internal subset of the front. Then a point \( x \) of the neighborhood of the front is characterized by two coordinates: \( \psi(t, x) \) and \( y(t, x) \), where \( \psi(t, x) \) is defined by the condition that the vector \( y = x - X(\psi, t) \) is orthogonal to the vector tangent to the front in the point \( X(\psi, t) \). Thus we have the condition \( \langle y, X_{\psi}(\psi, t) \rangle = 0 \). Let us find the phase

\[
S(t, x) = \langle P(\psi(t, x), t), x - X(\psi(t, x), t) \rangle = \frac{C(0)}{C(X(\psi(t, x), t))} y = \sqrt{\frac{H(0)}{H(X(\psi(t, x), t))}} y
\]

The second equality is a consequence of the equation (2.5).

Now we state the first important theorem of this paper connecting the wave amplitude with the initial perturbation \( V(x) \) and the profile of the bottom and the integration over the characteristics.

**Theorem 1.** For \( t_{cr} > t > \delta > 0 \) in some neighborhood of the front \( \gamma_t \), not depending on \( \mu, \eta \), the asymptotic elevation of the free surface, has the form:

\[
\eta = \sqrt{\frac{\mu}{|X_{\psi}(\psi, t)|}} \sqrt{\frac{H(0)}{H(X(\psi, t))}} \text{Re} \left[ e^{-i\pi/4} F\left(\frac{S(t, x)}{\mu}, n(\psi)\right)\right]_{\psi=\psi(t, x)} + O(\mu^{3/2}). \tag{2.6}
\]

Outside this region \( \eta = O(\mu^{3/2}) \). The function \( F(z, n) \) is defined in (1.17).

In this way till the critical time the asymptotic elevation of the free surface is completely defined by means of the trajectory, which forms the front of the wave, and of the function \( V \), corresponding to the source of the perturbation. Despite of the simple and natural form of the asymptotic of \( \eta \), the proof of the formula (2.6) is not trivial at all; the main step is the computation of the function \( V \), more exactly the proof of the fact that the formula is the same as in the case of constant bottom, if the right choice of the rays is made. We will give below a constructive approach of the proof of this formula, in the meantime we now show some elementary consequence of
the equation (2.6). Since the phase $S(x,t)$ is equal to zero on the front and $S(x,t)/\mu$ gets large going out from the front, then $\eta$, as one could expect, decreases enough quickly and the maximum of $|\eta|$ is attained in a neighborhood of the front. As a consequence, $\eta$ can have some oscillations depending on the form of the source. The second factor in (2.6) is the two dimensional analogue of the Green law, well known in the theory of water waves in the channels: the amplitude $\eta$ increases when the depth decreases as the inverse of the fourth root of the depth $1/\sqrt{C(x)} = 1/\sqrt{H(x)}$; the factor $1/\sqrt{|X_\psi|}$ is connected to the divergence of the rays, in other words if a smaller number of rays goes through a neighborhood of the point $X(\psi,t)$, the smaller will be the amplitude of the wave field. The factor $\frac{\partial}{\partial X(\psi,t)}$ appearing in the formula of the phase expresses the phenomena, also well known, of the “contraction” of the wave profile as the depth decreases and the increase of its amplitude. In fact the amplitude increases because of the factor in front of the function $V$ but also the phase $S(x,t)$ increases and this makes the wave profile narrower. This result explains the well know fact that the wave length of the tsunami decreases when the wave approaches the coast and that its amplitude increases. The same profile (i.e. a section of $\eta(x,t)$ for fixed $t$ and $\psi$) can depend on the way the trajectory (ray) intersects the initial perturbation of the bottom at $t = 0$. It is just this fact to give the dependence of the diagram of the directions on two factors: the shape of the source and the angle of its intersection with the ray passing through a given point of the front. For this reason, depending on the form of the bottom, two rays going out with two very different angles, can arrive near a point of the front and contribute to the profile with very different amplitudes. These effects can be well seen in the figure Fig.3

Remark 1. The main argument usually used for deriving the the analytical and and asymptotic formula of the solution of (2.2) is that, after the front is formed, the problem is essentially one dimensional in space till the appearance of focal points and its dynamic is described by the one dimensional wave equation with non uniform velocity. But it is possible to construct different particular wave solution, localized in different neighborhoods of the same front but having completely different profiles. The question is the right choice of the function describing this profile, i.e. a function connected already with the construction of the solution of bi-dimensional problems, containing mathematical difficulties such as the presence in the solutions of the wave equation ( with variable coefficients) of effects of intersection of the characteristics (which in quantum mechanics are also called terms). This happens for very small wave vectors, i.e. only for very long wave lengths. A general approach allowing to treat difficulties of this type were developed in [27] and, in particular for the Cauchy problem with localized initial conditions for the wave equation and hyperbolic systems, in the papers [22, 23]. These approach is basically founded on the following argument. The solution has two contributions, the first, corresponding to “very long” waves, can be found only directly by numerical methods. It has a very small amplitude and, in the case of the problem of the tsunami does not have a very great importance. The second has a wave length long compared with the depth of the basin of the wave but sufficiently small compared with oceanic scale, in such a way that it is possible to apply effective asymptotic methods as the expansion in rays. These arguments for the given problem (2.2) were given accurately in the papers [22, 23], but the final formulas, based on the asymptotic for the system with constant coefficients [47], are not very efficient from the point of view of practical applications. The basic of the derivation of the formula (2.6) of [47] consists of two steps: 1) the construction of the asymptotic expansion in the smoothness of the fundamental solutions of the problem (parametriz) and 2) the evaluation of the asymptotic of the convolution of this solution with the initial function $\eta_0$. Basically
our main observation (missing in the papers [47]) is that this last asymptotic may be represented in a simple and useful way

\[ (4.1) \], from which the representation \( (2.6) \), as well as representations \( (2.9), (2.16) \) working after appearance of critical points follow immediately.

2.3 The structure and metamorphosis of wave profiles.

2.3.1 The Maslov index and metamorphosis of the wave profile.

For \( t > t_{cr} \) when the focal points appear, as it is well known in the wave theory, the front can have the “angles” and sometimes the front lines can have self intersection points. The ends of the arcs corresponding to these angles are the focal points (or backtracking or turning points). For \( t > t_{cr} \) the front divides in some arcs \( \gamma^t_j \), indexed by the number \( j \), separated by focal points. The internal points of these arcs are the ends of the trajectories \( P(\psi, t), X(\psi, t) \) with the same topological structure. Namely these equivalent trajectories cross the same numbers of focal points at times \( t^F \) before \( t \), \( t^F < t \). They are characterized, from the topological point of view, by the Maslov index, an integer number \( m(\psi, t) \) depending on \( \psi, t \). The Maslov index \( m \) can be defined on the regular points of the front in different ways, we give below a more practical definition of this important concept by a simple definition of its increments in the problem under examination. The index \( m \) is related to the sign of the Jacobian \( J = \frac{\partial X}{\partial (t, \psi)} \equiv (X_X, X_\psi) \). The function \( J \) is equal to zero in the focal points and only in these points. Thus moving along the front \( \gamma_t \) or along the trajectory \( (P, X) \) after crossing the focal point, the Jacobian can change its sign. Actually the Maslov index prescribes a receipt for assigning the correct sign to the square root of \( J \) and it can be defined in a way independent from the trajectories. But if we move along a trajectory there is, in this problem, the nice and useful fact that the Maslov index coincides with the simpler Morse index. So, considering the trajectories arriving to \( \gamma^t_1 \), we have that the Morse index \( m(\psi, t) \) of the point \( x = X(\psi, t) \in \mathbb{R}^2 \) is equal to the number of focal points on the trajectory \( p = P(\psi, \tau), X(\psi, \tau), \tau \in (0, t) \) arriving to \( x = X(\psi, t) \). Note also that, as the time \( t \) changes, the ends of the arcs \( \gamma^t_j \) produce the entire set of focal points. It is also a well known fact that these sets constitute the (space-time) caustics which are the singularities of the projections of some Lagrangian manifold (we denote it \( M^2 \)) from the phase space \( \mathbb{R}^4_{p,x} \) to the plane (configuration space) \( \mathbb{R}^2_x \).

Example 2.

Let us illustrate the concepts explained above by the example [] about the waves on an axially symmetrical bank described by the depth function

\[ H = H(\rho), \rho = \sqrt{x_1^2 + x_2^2}. \] (2.7)

In this case there exists an additional integral

\[ p_\varphi = x_1 p_2 - x_2 p_1 \] (2.8)

and the Hamiltonian system \( (2.7) \) is completely integrable.

We assume that the source is located in a neighborhood of the point \( x_1 = 0, x_2 = -\rho_0 \). After the appearance of the focal points one has the following picture (see fig.4) For each fixed time \( t \) the front \( \gamma_t \) is separated into two arcs: the first, a long one, is \( \gamma_t^1 \) with self-intersection, and the second, a short one, is \( \gamma_t^2 \), located between the angles on the fronts. The union of the ends of the arc \( \gamma_t^2 \) for different times \( t \) gives a caustic. The arc \( \gamma_t^1 \) consists of the ends of trajectories (rays) without focal points on them (except
Thus the Jacobian $J(\psi, t) = \det(\dot{X}, X_\psi)(\psi, \tau) > 0$ for fixed $\psi$ and for each $\tau \in (0, t]$; hence the Morse index $m(x \in \gamma_1^i) = 0$. On the contrary the arc $\gamma_2^i$ consists of the final points of the trajectories (rays) which cross one focal point at some time $t = t_F(\psi), 0 < t_F(\psi) < t$ when they touch the caustic. In this case before $t_F(\psi)$ $J > 0$, $J(\psi, t_F(\psi)) = 0$, and $J < 0$ for $t > t_F(\psi)$. Hence $m(x \in \gamma_2^i) = 1$.

Now let us fix the time $t$ and move along the front $\gamma_t$. Then after the passage through the focal points the phase $-\pi/4$ in formula (2.6) increases by a quantity $-\pi/4 + \pi/2$, where $\pm 1$ is the jump of the Maslov index. Finally after passing through several focal points instead of the factor $e^{-i\pi/4}$ one has the factor $e^{-i\pi m(\psi, t)/2}$. The number $m$ is defined mod 4. The appearance of this new factor produces crucial changes of the form of the wave profile in the formula (2.9) i.e. in the function $\text{Re}(e^{-i\pi m(\psi, t)/2} F)$. This fact is analogous to the well known metamorphosis of the discontinuity in the theory of hyperbolic systems, and the formula (2.9) describes explicitly the appearance of the same fact in the case of localized initial perturbations.

Example 3. Let us give the examples of the transformation (metamorphosis) of the wave profile depending on the index $m$ and on the source of gaussian type. We have the following pictures for $m = 0, 1, 2, 3, \mod 4$ (see fig ??). Thus in the cases $m = 2, 3$ we have the “overturned” profiles with respect to the cases $m = 0, 1$. In the considered example of formula (2.9) one gets the profile (Fig.) for the long arc $\gamma_1^i$ and the profile (Fig.2) for the short arc $\gamma_2^i$.

Let us present the formula for the wave amplitude in a neighborhood of the front but outside of some neighborhood of the focal points. As we have just seen in the previous example, points of self-intersection can appear for $t > t_{cr}$. The amplitude of the wave in a point $x$ belonging to a neighborhood of these points now is the sum of the contributions coming from different $\psi_j(x, t), y_j(x, t)$, and $S_j(x, t)$ with index $j$, and with the Maslov index $m(\psi_j(x, t), t)$.

Theorem 2. In a neighborhood of the front but outside of some neighborhood of the focal points the wave field is the sum of the fields

$$\eta = \sum_j \left\{ \sqrt{\frac{\mu}{|X_\psi(x, t)|}} \sqrt{\frac{H(0)}{H(X_\psi(x, t))}} \text{Re} \left[ e^{-i\pi m(\psi, t)/2} F(\frac{S_j(x, t)}{\mu}, n(\psi)) \right] \right\}_{|\psi_j(x, t)} + O(\mu^{3/2}).$$

(2.9)

Outside this neighborhood of the front $\gamma_t$, $\eta(x, t) = O(\mu^{3/2})$. Again the function $F(z, n(\psi))$ is determined in (1.17).

Let us emphasize that the number $m$ has a pure topological and geometrical character and can be calculated without any relation with the asymptotic formulas for the wave field. From the theorem 2 it follows that, in order to construct the wave field at some time $t$ and in a point $x$, one has to know only the initial values $\eta|_{t=0}$ and $\eta_t|_{t=0}$ and has not to know the wave field $\eta$ for all previous time between 0 and $t$. The trajectories and the Maslov (Morse) index take into account all metamorphosis of the wave field during the evolution from zero time until time $t$.

Remark 2. Finally let us note that it is possible to define the Maslov index in the singular (focal) points also (see [?]), actually this definition is associated with a chain of covering maps of the front and gives the correct wave field asymptotic in a neighborhood of the focal point. It is useful to distinguish these two types of Maslov indices; we also meet in our calculations the second one, denoted by $m$, but after the discussion of the index $m$ (see subsection (3.3)).
2.4 Wave field asymptotic in a neighborhood of focal point

2.4.1 Completely nongenerated focal points and coordinate system

Now we consider the situation when for some \( t \) the point \((P^F, X^F) = (P(\psi^F(t), t), X(\psi^F(t), t))\) corresponding to the angle \( \psi^F(t) \) is a focal one. In this point \( X_\psi = 0 \) and one has to use another asymptotic representation for the solution. Roughly speaking the neighborhood of the point \( X(\psi^F(t), t) \) on the plane \( \mathbb{R}^2 \) can include several arcs of \( \gamma_t \) with the angles \( \psi \) far from \( \psi^F(t) \). This means that one has to take into account contribution of all of these arks in the final formulas for \( \eta \) in the neighborhood of the point \( x = X(\psi^F(t), t) \). The influence of nonsingular points are defined by formula (2.9) and the influence of the points from the neighborhood of the focal points are described by formulas (2.10) given below. Thus it is necessary to innumerate the focal points with the closed projection and write \( P(\psi^F_j(t), t), X(\psi^F_j(t), t) \). These points have the closed position \( X^F = X(\psi^F_j(t), t) \), but different momentum \( P^F = P(\psi^F_j(t), t) \). To simplify the notation we discuss here the influence into \( \eta \) only one focal point omitting subindex \( j \) but keeping \( P^F \).

We present the corresponding formula under the assumption that some derivative.

\[
X^{(n)}_{\psi} = \frac{\partial^n X}{\partial \psi^n}(\psi^F(t), t) = \frac{\partial^n X}{\partial \psi^n}(\psi^F(t), t) \neq 0, \tag{2.10}
\]

and the derivatives \( X^{(k)}_{\psi} = 0 \) for \( 1 \leq k < n \). It means that this focal point is not completely degenerate. For future it is convenient to introduce the "mixed" Jacobian

\[
\tilde{J} = \det(\tilde{X}, P_{\psi})(\psi, t) = \frac{C^2(X) \det(P, P_{\psi})}{C_0}(\psi, t) \tag{2.11}
\]

and some characteristics of the focal point \((P^F, X^F)\):

\[
C_F = C(X^F), \quad \tilde{X}^F = \tilde{X}(\psi^F(t), t) = \frac{P^F C^2_F}{C_0}, \quad P^F = P_{\psi}(\psi^F(t), t),
\]

\[
\tilde{J}_F = \det(\tilde{X}^F, P^F_{\psi}) = \frac{C^2_F \det(P, P_{\psi})}{C_0}, \quad \tilde{J}^{(n)}_{\psi} = \det(\tilde{X}^F, X^{(n)}_{\psi}). \tag{2.12}
\]

Again the topological characteristic appears, i.e. the Maslov index of this focal point or its neighborhood (it is the same), but now it depends on the choice of the coordinates in the neighborhood of \((P^F, X^F)\). It is natural to choose the new coordinates \( (x_1', x_2') \) associated with the nonzero vector \( \tilde{X}^F = \tilde{X}(\psi^F(t), t) \); namely we assume that the direction of the new vertical axis \( x_2' \) coincides with the vector \( \tilde{X}^F \). We put \( k_2 = t(k_{21}, k_{22}) = \tilde{X}^F / |\tilde{X}^F| = \tilde{X}^F / C_F = P^F C_F / C_0, k_1 = t(k_{11}, k_{12}) = (k_{22}, -k_{21}) \) and introduce the new coordinates \( p_1', x_1' \) in the neighborhood of \((P^F, X^F)\) in the phase space \( \mathbb{R}^4_{p,x} \) by formulas:

\[
x_1' = \langle k_1, x - X^F \rangle = -\frac{\det(\tilde{X}^F, x - X^F)}{C_F} = -\frac{C_F}{C_0} \det(P^F, x - X^F),
\]

\[
x_2' = \langle k_2, x - X^F \rangle = \frac{\langle \tilde{X}^F, x - X^F \rangle}{C_F} = \frac{C_F}{C_0} \langle P^F, x - X^F \rangle,
\]

\[
p_1' = \langle k_1, p \rangle, \quad p_2' = \langle k_2, p \rangle. \tag{2.13}
\]

Easy to see that

\[
\det \left( \begin{array}{cc}
P^F_{\psi} & P^F_{\psi} \\
\tilde{X}_2' & X_2'
\end{array} \right) = \tilde{J}_F. \tag{2.14}
\]
2.4.2 Maslov index of the focal point.

As the determinant $\tilde{J} \neq 0$ in the focal point $(P^F, X^F)$, hence the same inequality takes place in some its neighborhood, thus $\tilde{J}$ has a constant sign. On the contrary the Jacobian $J$ changes sign in this neighborhood. We define the Maslov index $m(P^F, X^F)$ of the non (completely) degenerate focal point $(P^F, X^F) = (P, X)(\psi(t), t)$ as the index $m(\tilde{P}, \tilde{X})(\psi, t)$ of a regular point $(\tilde{P}, \tilde{X}) = (P, X)(\tilde{\psi}, \tilde{t})$ in the neighborhood of $(P^F, X^F)$ such that the signs of the determinants $J(\tilde{\psi}, \tilde{t})$ and $\tilde{J}(\tilde{\psi}, \tilde{t})$ coincide. For instance one can choose $\tilde{\psi} = \psi^F(t), \tilde{t} = t \pm \delta$, where delta is small enough. This means that we compare the sign of $J$ with the sign of $\tilde{J}$ on the trajectory $(P, X)$ crossing the curve $\Gamma_t$ in the focal point $(P^F, X^F)$ before and after this crossing.

**Example 4.** Let us illustrate this definition for the focal points of the example with the axial symmetric bank (2.7) us find the sign of $\tilde{J}$ in the focal point. (2.8) Theorem 3. We put $\psi, t \in \int_0^\infty \sqrt{\rho d\rho \tilde{f}(\rho)}(\rho n)^{\xi n+1} \exp\{i\rho(z_2 - \xi z_1 - \frac{\xi n+1}{(n+1)!})\}$.

$$\langle P^F, P^F \rangle = 0, \quad X^F_2 P^F_1 - X^F_1 P^F_2 = \rho_0 \sin \psi^F(t) \quad \text{for} \quad \psi = \psi^F(t).$$

The solution of this equation is

$$\left(\frac{P^F_1}{P^F_2}\right) = \frac{\rho_0 \sin \psi^F(t)}{\langle P^F, X^F \rangle}(\psi^F(t), t) \left(\frac{P^F_2}{P^F_1}\right).$$

and $\langle P^F, P^F \rangle = \det(P^F, P^F) = -\frac{\rho_0 P^2}{\langle P^F, X^F \rangle} \sin \psi^F(t)$. Obviously the angle $\psi^F(t)$ belongs to the interval $(0, \pi)$ and $\langle P^F, X^F \rangle > 0$, hence $\tilde{J} < 0$ and $m(P^F, X^F) = 1$.

2.4.3 The model functions and the wave profile in neighborhood of the focal point.

Now we present the formulas for the wave field in the neighborhood of a focal point $x = X^F$. Let us put $\sigma = \text{sign}(\tilde{J} F J_F^{(n)})$ and introduce the function (or more precisely the linear operator acting to the source function $V(y_1, y_2)$)

$$g_\sigma(z_1, z_2, \psi) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \rho d\rho \tilde{f}(\rho n)(\psi) \exp\{i\rho(z_2 - \xi z_1 - \frac{\xi n+1}{(n+1)!})\} =$$

$$\int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} \sqrt{\rho d\rho \tilde{f}(\rho n)(\psi)} \exp\{i\rho(z_2 - \xi z_1 - \frac{\xi n+1}{(n+1)!})\}. \quad (2.15)$$

We put

$$z^F_1 = \frac{x_1}{\mu^{n+1}} \tilde{J} F J_F^{(n)} \mu^{n+1} \mu^{n+1} \frac{J_F^{(n)}}{C_F} \equiv \frac{\det(P^F, x - X^F)}{C_0 C_F^{n+1} C_F^{n+1}} \frac{J_F^{(n)}}{C_F}, \quad z^F_2 = \frac{x_2}{\mu} C_0 \mu C_F \equiv \frac{\langle P^F, x - X^F \rangle}{\mu}.$$ 

**Theorem 3.** Each focal point $(P^F, X^F)$ on the front $\gamma_t$ gives in its neighborhood the following contribution

$$\eta^F = \mu^{n+1} \{\sqrt{C_0 |J_F^{(n)}}^{n+1} \text{Re} e^{-i\frac{\sigma}{2} m(P^F, X^F)} g_\sigma(z^F_1, z^F_2, \psi^F) + O(\mu)\} \quad (2.16)$$

into the asymptotic of solution $\eta$. If the several arcs of $\gamma_t$ belong to the neighborhood of the point $x$, then one need to summarize all corresponding functions 2.16 and 2.9.
3 The geometric base of asymptotics: Lagrangian manifolds, the Maslov and Morse indices.

The aim of the next section is to prove Theorems 1-3. But let us first recall the geometrical objects and the important properties of the Hamiltonian system (2.4), giving an uniform asymptotic solution to problem (2.4) including the behavior in a neighborhood of focal points, initial moment of time, calculation of the Maslov and Morse indices etc. The majority of these constructions and properties are well known, we present them in the most simple form and collect them in our paper for giving a self-contained treatment. An exhaustive description of the wave fronts and the focal points, their connection with the ray method and the semiclassical asymptotic, can be found for instance in mentioned above monographs and papers. Maslov, Fed, Berry. There exist different equivalent definitions of the Maslov index; one of the aims of the next subsection is to recall the definition from [?, 37] which, in our opinion, is more suitable for concrete calculations.

3.1 Lagrangian manifolds (“bands”) and their properties.

As we have already said, taking into account the fact that after the appearance of the focal points the front line can intersect itself, it is convenient to add to the point $x = X(\psi, t)$ the corresponding momentum component $p = P(\psi, t)$, and consider the point $r = r(\psi, t) = (P(\psi, t), X(\psi, t))$ in the 4 dimensional phase space $\mathbb{R}^4_{p,x}$. Each point $r(\psi, t)$ is completely defined by its coordinates, which are the angle $\psi$ (defined mod $2\pi$) and the “proper time” $t$.

Fixing the angle $\psi$ we obtain the trajectories (bi-characteristics) of Hamiltonian system (2.4) in the phase space $\mathbb{R}^4_{p,x}$, and, fixing the time $t$, we obtain the front $\Gamma_t$ in the in the phase space $\mathbb{R}^4_{p,x}$. The projections of the trajectories from $\mathbb{R}^4_{p,x}$ to the configuration space (plane) $\mathbb{R}^2_2$ are the rays. The projection of the curve $\Gamma_t$ from $\mathbb{R}^4_{p,x}$ to the configuration space (plane) $\mathbb{R}^2_2$ are the fronts $\gamma_t$. Different points $r(\psi_j, t)$ on $\Gamma_t$ can have the same projection $x = X(\psi_j, t)$ on $\gamma_t$, but now we distinguish them by different angles $\psi_j$.

Let us fix some small but independent of $\mu$ number $\delta$. According to [?, ?] changing both the angle $\psi$ and the time $\tau \in (t - \delta, t + \delta)$ on the cylinder $\mathbb{S} \times (t - \delta, t + \delta)$ we obtain the 2-D Lagrangian manifold (with the boundary) $M^2_\delta = \{p = P(\psi, \tau), x = X(\psi, \tau) | \psi \in \mathbb{S}, \tau \in (t - \delta, t + \delta)\}$: the angle $\psi$ from the unit circle $\mathbb{S}$ and the time $t$ from $(t - \delta, t + \delta) \in \mathbb{R}$ are the coordinates on $M^2_\delta$, sometimes we shall use the notation $\alpha = \tau - t$ instead of the time $t$. Actually the manifold $M^2_\delta$ has a structure of a cylindrical “band” (or closed strip) with the width $2\delta$, thus we call it Lagrangian band; of course it depends on $\delta$, we omit this dependence to simplify the notation. The family of Lagrangian bands $M^2_\delta$ is invariant with respect to the phase flow $g^t_{\mathbb{H}}$ generated by the system (2.4). This means that the point $r(\psi_j, \tau)$ from $M^2_{\delta_0}$ shifted by the action of the flow $g^t_{\mathbb{H}}$ gives again the point $r(\psi_j, \tau + t)$ on $M^2_{\delta_0 + t}$ but with the shifted time $\tau + t$. Due to definition the coordinate $\alpha$ does not change. That is why the coordinate $\tau$ (corresponding to $t$) is called the proper time. Sometimes it is possible to choose $\delta$ arbitrary large, even infinity (e.g. in the case $C = \text{const}$). But in many situation the set $\{p = P(\psi, \tau), x = X(\psi, \tau) | \psi \in \mathbb{S}, \tau \in -\infty\}$ has the intersection points (e.g. if the trajectories $P(\psi, \tau), x = X(\psi, \tau)$ belong to the Liouville tori), and this set is not even the manifold. But for our purpose it is enough to work with the “Lagrangian band”
$M^2_t$ only. Along with the general properties of Lagrangian manifolds, the band $M^2_t$ has very useful additional ones. Let us present them all for the completeness.

Let us introduce the matrices

$$B = \frac{\partial P}{\partial (t, \psi)} \equiv (\dot{P}, P_\psi), \quad C = \frac{\partial X}{\partial (t, \psi)} \equiv (\dot{X}, X_\psi).$$

It is easy to see also that each column-vector $(\dot{P}, \dot{X}, P_\psi, X_\psi)$, and $(P, 0)$ satisfies the variational system

$$\delta x = H_{pp} \delta p + H_{px} \delta x, \quad \delta p = -(H_{xp} \delta p + H_{xx} \delta x) \quad (3.1)$$

Easy to check that these vectors are linearly independent and obviously two first vectors are tangent to $M^2_t$.

**Lemma 1.** (see e.g.?[?], ?) The following properties are true:

1) the rank of the matrix $(B, C)$ is equal to 2 which actually means that dimension of $M^2_t$ is 2.
2) $BC = CB$ which means that $M^2$ is Lagrangian,
3) for any positive $\varepsilon$ the matrix $C - i\varepsilon B$ is not degenerate.

**Proof.** The first two propositions follow from the properties of the variational system. It is easy to check them for $t = 0$ because $B = (-\nabla C(0), n_\perp), C = (C(0)n, 0)$ where $n_\perp = (-\sin \psi, \cos \psi)$. In this argument we use the definition of the trajectories $(P, X)$, namely the property $P|_{t=0} = n(\psi), X|_{t=0} = 0$, $n = (\cos \psi, \sin \psi)$. Thus according to the variational system $(3.1)$ the vector columns $(\dot{P}, \dot{X})$ and $(P_\psi, X_\psi)$ are linearly independent for each $t$ which gives 1). Also a simple calculation based on the variational system $(3.1)$ gives that $\frac{d}{dt}(BC - CB) = 0$, which gives 2). To prove 3) assume that $C - i\varepsilon B$ is degenerate, then there exists a 2-D vector $\xi \neq 0$ such that $C\xi = i\varepsilon B\xi$. Consider the (complex) scalar product $0 =<\xi, (BC - CB)\xi>$.

From this equation it follows that both $B\xi = 0, C\xi = 0$ which contradicts 1). □

The same consideration allows one to obtain the following closed result.

**Lemma 2.** The propositions of the previous Lemma concerning the matrices $B, C$ is true if one changes the matrix $B$ by the matrix

$$\tilde{B} = (\dot{P} - \lambda P, P_\psi),$$

where $\lambda = \langle C_x(0), n(\psi) \rangle$.

Let us recall that the points $x = X(\psi^F, t) = X^F$ on $M^2_t$ where the Jacobian

$$J \equiv \det C \equiv \det(\dot{X}, X_\psi) = 0$$

are the focal points$^1$. Since the manifold $M^2_t$ is generated by the curves $\Gamma_t$ as well as by the trajectories $(P, X)$ each focal point of one of theses objects simultaneously is

$^1$Note that using the Hamiltonian system we can change $\dot{X}$ by $P$ in last formula as well as in many formulas containing $\dot{X}$.  

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a focal point for the others. Little later we shall show that this definition of the focal points coincides with the definition, based on equality $X_\psi =$, used in the previous sections.

Let us fix some time $t$ and consider the smooth curve $\Gamma_t = \{ p = P(\psi, t), x = X(\psi, t) \}$ on $M_t^2 \in \mathbb{R}^{p,x}$ (the “time cut” of $M^2$). Then obviously the front $\gamma_t = \{ x = X(\psi, t) \}$ is nothing but the projection of $\Gamma_t$ to $\mathbb{R}^2$. Hence the focal points on the front are also the focal points of the manifold $M_t^2$, and from this point of view the caustics of $M_t^2$ are called space-time ones.

**Lemma 3.** The vector-functions $\dot{X}$ and $X_\psi$ as well as vector-functions $P$ and $X_\psi$ are orthogonal: $\langle \dot{X}, X_\psi \rangle = \langle P, X_\psi \rangle = 0$.

**Proof.** According to system (2.4) the vectors $P$ and $\dot{X}$ are parallel and it is enough to prove the second equality. Let us differentiate $\langle P, X_\psi \rangle$ along the trajectories of the system (2.4). We have

$$\frac{d}{dt} \langle P, X_\psi \rangle = \langle \dot{P}, X_\psi \rangle + \langle P, \dot{X} \psi \rangle = \text{using 2.5} = -|P| \langle C_x, X_\psi \rangle + \frac{C^2}{C_0} \langle P, P \psi \rangle + \frac{\partial C^2}{\partial \psi} \frac{1}{C_0} \langle P, P \rangle = -|P| \frac{\partial C}{\partial \psi} + \frac{1}{2C_0} \partial |P| \frac{\partial C}{\partial \psi} = -|P| \frac{\partial C}{\partial \psi} + \frac{1}{2C_0} \partial |P| \frac{\partial C}{\partial \psi} = 0.$$

But $X|_{t=0} = 0$, thus $\langle P, X_\psi \rangle|_{t=0} = 0$ and Lemma is proved. $\square$

**Corollary.** 1) The following equality is true $J = \det(\dot{X}, X_\psi) = \pm |\dot{X}| |X_\psi|$. 2) The point $x = X(\psi, t)$ on the front $\gamma_t$, or the point $r = (p = P(\psi, t), x = X(\psi, t)$ on $\Gamma_t$ is a focal point if and only if $J = \det(\dot{X}, X_\psi) = 0$.

According to the equality $|\dot{X}| = C(X)$ $J = \det(\dot{X}, X_\psi)$ as well as the Jacobian in some neighborhood of $\gamma_t$ can be equal to zero if and only if $X_\psi = 0$. Thus the last equation really determines the focal points from the point of view of the Lagrangian manifold also.

**Lemma 4.** In the focal point $x = x^F = X(\psi^F, t)$ 1) $\langle P^F, P_\psi^F \rangle = 0$, but 2) $P_\psi^F \neq 0$, 3) $\frac{dJ}{dt} = \frac{C^2}{C_0} \det(\dot{X}^F, P_\psi^F)$, where as it was before $C_0 = C(0)$ and $C_F = C(X^F)$.

**Proof.** According to the conservation law 2.5 $\langle P, P_\psi \rangle(\psi, t) = \langle \nabla(\frac{C^2}{C(\psi)}), X_\psi \rangle(\psi, t) = 0$. To prove the second inequality one can mention that the vector-function $(P_\psi, X_\psi)^T$ satisfies the linear (variational) system with non-zero initial condition. Thus both components of the solution cannot be equal to zero. To prove 3) we write $\frac{dJ}{dt}|_{\psi = \psi^F} = [\det(\dot{X}, X_\psi) + \det(\dot{X}, X_\psi)]|_{\psi = \psi^F} = \det(\dot{X}, P < \nabla(\frac{C^2(X)}{C_0}), X_\psi >)|_{\psi = \psi^F} + \det(\dot{X}, (\frac{P \cdot C^2(X)}{C_0}))|_{\psi = \psi^F} = \frac{C^2(X)}{C_0} \det(\dot{X}, P_\psi)|_{\psi = \psi^F}$. $\square$

**Corollary.** In the focal point 1) $\frac{dJ}{dt} = \frac{C^2}{C_0} \det(\dot{X}, P_\psi) = \pm |\frac{C^2(X^F)}{C_0} \dot{X}||P_\psi|(\psi^F, t) \neq 0$; 2) during the passage through the focal point the Jacobian $J$ changes its sign from $+$ to $-$ if $\det(\dot{X}, P_\psi)|_{\psi = \psi^F} > 0$ and from $+$ to $-$ if $\det(\dot{X}, P_\psi)|_{\psi = \psi^F} < 0$; 3) There exists $t_{cr}$ such that $J(\psi, t) > 0$ for $0 < t < t_{cr}$.

**Proof.** To prove 3) it is enough to note that $\det(\dot{X}, P_\psi)|_{t=0} = C(0) \det(\mathbf{n}(\psi), \mathbf{n}(\psi)_J) = C(0)$. $\square$
3.2 The Maslov and Morse indices.

As we said before the front $\gamma_t$ as well as the curve $\Gamma_t$ is partitioned into arcs with the focal points at their ends and it is formed by the ends of trajectories having the same topological structure. This means that they have similar crossing (on $M^2_t$) with the focal points and the same topological characteristic, i.e. the Maslov index. But it coincides with the the Morse index for the considered situation (see subsection [2.3.1]). Let us prove this proposition. Let us remind some necessary definitions and constructions. It is needless to say that there exist different definitions of the Maslov index. The original definition [?] is based on calculation of indices of inertia of matrices $\frac{\partial(x_1,x_2)}{\partial(x_1,x_2)}|_{M^2_t}$, $\frac{\partial(x_1,x_2)}{\partial(x_1,x_2)}|_{M^2_t}$, $\frac{\partial(p_1,p_2)}{\partial(x_1,x_2)}|_{M^2_t}$ etc. It is not very convenient in practical calculation. Thus we want to present below one [?, ?, ?] which, from our point of view, is more pragmatic for computer calculations. We already pointed out that the Maslov index of the points on $x \in \gamma_t$ is the index of the nonsingular point $r(\psi,t) = (P(\psi,t),X(\psi,t))$ on the Lagrangian band $M^2_t$: According to the procedure from [?, ?, ?, ?] one needs to fix the index $m^0$ in some marked nonsingular point $p = P(\psi,\zeta), x = X(\psi,\zeta)$ on $M^2_0$ and then to find the change of the argument of the determinant of the $2 \times 2$ matrix $C_\varepsilon^{(1,2)} = (C - i\varepsilon B) \equiv (\dot{X} - i\varepsilon \dot{P}, X_\psi - i\varepsilon P_\psi)$ along one of the paths described below joining the marked point $p = P(\psi,\zeta), x = X(\psi,\zeta)$ with the given nonsingular point $p = P(\psi,t), x = X(\psi,t)$, more precisely

$$m(\psi,t) = m(\psi,0) + \Delta m, \quad \Delta m = \frac{1}{\pi} \lim_{\varepsilon \to 0} \arg \det(\dot{X} - i\varepsilon \dot{P}, X_\psi - i\varepsilon P_\psi)(\psi,t)|_{\psi,0}^{\psi,t}.$$  \hspace*{1cm} (3.2)

From definition [3.2] it follows the fact that we used before: the index can change (jump) only crossing a focal point. In fact, if the point $(\psi,t)$ is a regular point then $\det(\dot{X}, X_\psi)$ is different from zero so the increment of the argument of the determinant goes to zero when $\varepsilon$ goes to zero, otherwise, if the determinant of $(\dot{X}, X_\psi) = 0$, as it happens in a focal point, then the increment of the determinant in [3.2] is different from zero when $\varepsilon$ goes to zero. We know (see Corollary from Lemma 4) that the Jacobian $J = \det C(\psi,\zeta) > 0$ for small enough positive $\zeta$. Thus all the points on the front $\gamma_t$ are nonsingular. So we choose one of the point $(P(\psi,\zeta), X(\psi,\zeta))$ and put Maslov index $m(\psi,0) = 0$. It is possible (and natural) to use one of the of the following paths. 1) To move first along the trajectory $P(\psi,\tau), X(\psi,\tau)$ starting from $\tau = \zeta$ until $\tau = t$, then to move from the point $P(\psi,t), X(\psi,t)$ along the curve $\Gamma_t$ to the point $P(\psi,t), X(\psi,t)$ changing the angle $\psi$. As we will see a little later this choice is not very convenient from the point of view of computer realization. The choice 2) is to move first from the point with the angle $\psi_0$ to the point with the angle $\psi$ along the curve $\Gamma_{\zeta}$, and then to move along the trajectory with the angle $\psi$, changing time from $\zeta$ to $t$. It could happen that during the motion along some closed path on $M^2_{\zeta}$ one can get nontrivial increment of argument of the determinant of the matrix $C_\varepsilon^{(1,2)} = (C - i\varepsilon B) \equiv (\dot{X} - i\varepsilon \dot{P}, X_\psi - i\varepsilon P_\psi)$ and nontrivial Maslov index of this closed path. The following lemma shows that it is not so.

**Lemma 5.** The increment of argument of the determinant of the matrix $C_\varepsilon^{(1,2)} = (C - i\varepsilon B) \equiv (\dot{X} - i\varepsilon \dot{P}, X_\psi - i\varepsilon P_\psi)$ along any closed path on $M^2_t$ is equal to zero. Thus the Maslov index of any closed path on $M^2_{\zeta}$ is also equal to zero.

**Proof** First let us show that Lemma is true for $t = 0$. Obviously all closed paths on $M^2_0$ are homotopic one to another. Let us choose as a path the curve $\Gamma_{\zeta}, \delta > \zeta > 0$. According to Corollary from Lemma 4 the Jacobian $J = \det C(\psi,\zeta) > 0$ for $\zeta$ small
enough. Thus the increment of argument of determinant of matrix $C - i\varepsilon C$ over $\Gamma_\zeta$ is zero and Maslov index of this path is also zero. Hence according to the property of Maslov index it is equal to zero for any other closed path on $M_0^2$. Any path on $M_0^2$ could be obtained from some path from $M_0^2$ by means of the canonical transform (the flow) $g^i_{\zeta t}$. But this transform preserves the Maslov index of closed path. □

**Remark 3.** Thanks to this statement the Bohr-Sommerfeld quantization rule does not appear in asymptotic solutions, although the paths $\Gamma_t$ on $M^2_t$ are not contractible. This fact is also natural for Lagrangian manifolds associated with the asymptotics of Green functions for evolution equations in Euclidean space (like van Vleck formula for the nonstationary Schrödinger equation).

Taking into account this Lemma, choosing the the second way for the calculation of the Maslov index and putting $m(\psi_0, \zeta) = 0$ for some small positive $\zeta < \delta$ and some fixed fixed $\psi_0$, (and hence for any $\psi$) we can write for the index $m(\psi, t)$:

$$m(\psi, t) = \Delta m, \quad \Delta m = \frac{1}{\pi} \lim_{\varepsilon \to +0} \text{Arg det}(\hat{\chi} - i\varepsilon \hat{P}, X_\psi - i\varepsilon P_\psi)(\psi, t)_{\psi, \zeta}. \quad (3.3)$$

The Corollary of Lemma 3 allows us (in the considered problem) to simplify the definition (3.2). Let us analyze the determinant

$$\text{det}(\hat{\chi} - i\varepsilon \hat{P}, X_\psi - i\varepsilon P_\psi)(t, \psi) = \text{det}(\hat{\chi}, X_\psi) - i\varepsilon \text{det}(\hat{\chi}, P_\psi) - i\varepsilon \text{det}(\hat{P}, X_\psi) - \varepsilon^2 \text{det}(\hat{P}, P_\psi). \quad (3.4)$$

Since $\text{det}(\hat{\chi}, P_\psi)$ is not equal to zero and the term $\text{det}(\hat{P}, X_\psi) = 0$ in the focal point, then the third term of equation (3.4) can be omitted. Similarly taking into account the fact that the jump of the index is an integer number it easy to show that the term $\varepsilon^2 \text{det}(\hat{P}, P_\psi)$ does not play any role in the calculation of $\Delta m$ and also can be omitted. Thus instead of the determinant $\text{det}(\hat{\chi} - i\varepsilon \hat{P}, X_\psi - i\varepsilon P_\psi)(t, \psi)$ we can use the determinant $\text{det}(\hat{\chi}, X_\psi - i\varepsilon P_\psi)(\psi, t)$.

Thus one needs to find the jumps of $m$ during the crossing of the focal points and find

$$\Delta m = \frac{1}{\pi} \lim_{\varepsilon \to +0} \text{Arg det}(\hat{\chi}, X_\psi - i\varepsilon P_\psi)(\psi, \tau)_{\tau=t_\psi+\Delta t}^{\tau=t_\psi-\Delta t}, \quad (3.5)$$

where $\Delta t > 0$ is small enough, and $t_\psi$ is the time at which the trajectory crosses the focal point (with coordinates $\psi, t_\psi$) with the angle $\psi$. There may exist several such $t_\psi$, but all of them, according to the point 1 of the Corollary of Lemma 3, are isolated with respect to $t$. In fact the derivative of the Jacobian is different from 0 in a focal point so the Jacobian has an isolated zero and so those zeros cannot accumulate in one point. Now we use again this Corollary. Obviously we can change $\tau$ with $t^F$ in the right hand side of (3.5). But the term $\text{det}(\hat{\chi}, P_\psi)(\psi^F, t^F)$ characterizes the increasing or decreasing of the first term. Hence if $\text{det}(\hat{\chi}, P_\psi)(\psi^F, t^F) < 0$, then the argument of the complex vector in (3.5) changes on the upper half plane from $O(\varepsilon)$ to $\pi + O(\varepsilon)$ and $\Delta m = 1$. If $\text{det}(\hat{\chi}, P_\psi)(\psi^F, t^F) > 0$ then the argument of the vector in (3.5) changes from $\pi + O(\varepsilon)$ to $2\pi + O(\varepsilon)$ and again $\Delta m = 1$. Thus we obtain the following important result.

**Lemma 6.** The Maslov index $m(\psi, t)$ of any nonsingular point $(p = P(\psi, t), x = X(\psi, t)) \in \Gamma_t$ with the projection $x = X(\psi, t)$ on the front is equal to number of focal points on the trajectory $P(\psi, \zeta), X(\psi, \zeta), \zeta \in (+0, t)$, i.e. it coincides with the Morse index of this trajectory.
The behavior of the front near the focal points.

For the future developments it is useful to have the description of the wave front \( \gamma_t \) in a neighborhood of the focal points. The focal points which are ends of arcs \( \gamma^j_t \) of the wavefront belong to the caustics, a well known concept in geometrical optics and in the space-time wave theory. It is an important fact that in our problem the caustics do not depend on the time because the family of manifolds (bands) \( M^2_t \) is invariant with respect to the phase shift \( y^j_H \). One may distinguish two types of sets organized by the focal points with stable and unstable structures with respect to small changes of the Lagrangian manifold \( M^2_T \) (or of functions \( X(\psi,t), P(\psi,t) \)). The first type is under the so-called “general position”, there exist only a finite numbers of them and in the considered 2-D situation namely there are only two types: the so-called fold and cusp[?]. These curves are presented in the Fig.1. Sometimes there exist also different focal sets with unstable structure. For instance the circle \( \Gamma = p = \mathbf{n}(\psi), x = 0 \) on \( M^2 \), has the point \( x = 0 \) as the projection \( \Gamma_0 \) from \( M^2_T \) (or \( \mathbb{R}^4_{p,x} \)) to \( \mathbb{R}^2_x \). Rotating a little the coordinate system in \( \mathbb{R}^4_{p,x} \) one can obtain a small ellipse on \( \mathbb{R}^2_x \) instead of the point \( x = 0 \). But we show below that namely this “unstable singular” circle determine localized functions in the asymptotic constructions.

Taking into account the smoothness of the vector-functions \( X(\psi,t) \) one can easily describe the behavior of the wavefront near the focal points. Namely let us fix the time \( t \) and let \( \psi^F = \psi^F(t) \) define the angle (coordinate on \( \Gamma_t \)) of the focal point \( X^F = X(\psi^F,t) \). We put \( y = \Delta \psi = \psi - \psi^F \). Let \( n \geq 2 \) be the minimum degree of the Taylor expansion of the function \( X \) around \( \psi \) with increment \( y \). We say that the focal point \( X^F \) is not completely degenerate if \( n \neq \infty \). From the point of view of this definition the point \( p = P_0 = \mathbf{n}(\psi), x = X_0 = 0 \) is a complete degenerate one.

Shifting the origin into the focal point and rotating the coordinates we can “kill” the second component of the \( n \)-th derivative vector \( X^{(n)}(\psi^F(t),t) \) and write \( x'_1 = ay^n + O(y^{n+1}), x'_2 = by^k + O(y^{k+1}) \). Here \( a \neq 0, b \neq 0 \) are Taylor coefficients, the integer \( k > n \) and the prime indicates the new coordinates; actually \( a \) and \( b \) depend on the time \( t \), but now for us it is not important. It is clear that the previous Lemmas are true also in the new coordinates.

**Lemma 7.** In the non degenerate case only one opportunity is possible: \( k = n + 1 \).

**Proof.** In new coordinates

\[
X' = \left( ay^n + O(y^{n+1}) \right), \quad X'_\psi = \left( any^{n-1} + O(y^n) \right),
\]

Since the vector \( P' \) is orthogonal to \( X'_\psi \) everywhere, and, according to the conservation of the Hamiltonian \( \mathcal{H} \), \( P' \neq 0 \), then we can write

\[
P' = q \left( -bk y^{k-n} + O(y^{k-n+1}) \right) \quad \text{and} \quad P'_\psi = q \left( -bk(k-n) y^{k-n-1} + O(y^{k-n}) \right),
\]

where the factor \( q(t) \neq 0, \) \( q(t) \) is proportional to the Taylor coefficient just after \( qa(t) \). Taking into account point 1) of Lemma 4 we immediately find \( \tilde{a}(t) = 0 \). But from this it follows that if \( n,k \neq \infty \) and \( k > n + 1 \) then in the focal point \( P'_\psi = 0 \) which contradicts point 2) of Lemma 4. \( \square \).

**Corollary.** Let \( n \neq \infty \) then in the neighborhood of the focal point \( x = X^F \) in new
coordinates
\[ X' = \left( ay^n + O(y^{n+1}) \right), \quad X'_\psi = \left( an^{n-1} + O(y^n) \right), \]
\[ P' = q \left( -b(n+1)y + O(y^2) \right), \quad P'_\psi = q \left( -b(n+1) + O(y) \right), \]
and
\[ \det(\dot{X}, X) = -\frac{qan}{|q|} C(X^F) y^{n-1} + O(y^n), \]
\[ \det(\dot{X}, P) = |q|b(n+1)C_F + O(y), \]
where it was before \( C_F = C(X^F) \). The last two equalities do not depend on the choice of the coordinates.

Thus in agreement with the Lemma 3 the determinant \( \det(\dot{X}, P) \) does not change its sign in a neighborhood of a non degenerate focal point.

Finally in the non completely degenerate case, in the neighborhood of the focal point, we have \( X'_1 = ay^n + O(y^{n+1}), \) \( X'_2 = by^{n+1} + O(y^{n+2}), \) \( P'_1 = -qb(n+1)y + O(y^2), \) \( P'_2 = qan + O(y^2). \)

Omitting the higher corrections we find the equation for the part of the front \( x^F: x'_1 = ay^n, \) \( x'_2 = by^{n+1}. \) The sign of \( ab \) for odd \( n \) and the sign of \( a \) for even \( n \) defines the direction of the passage from higher to lower leaves for odd \( n \) and from left to right leaves for even \( n \), see Fig.5. Let us note also that in the general case \( n \) is equal to 2 or 3 only.

It is convenient to express the coefficients \( a, b, q \) via \( P, X \) and their derivatives in the focal point \( \psi^F(t) \). Putting in formulas (3.11) \( y = 0 \) we find
\[ P'_1 = 0, \quad P'_2 = qan, \quad P'_{1\psi} = -bq(n+1), \text{ for } \psi = \psi^F(t), \]
and
\[ a = \frac{X'_1\langle n \rangle}{n!} \equiv \frac{1}{n!} \frac{\partial^n X'_1}{\partial \psi^n}, \quad b = -\frac{P'_{1\psi}X'_1\langle n \rangle}{(n-1)!(n+1)P'_2}, \quad q = \frac{P'_2(1-1)!}{X'_1\langle n \rangle} \text{ for } \psi = \psi^F(t). \]

The directions of the vectors \( P \) and \( \dot{X} \) coincide in the focal points. Thus we see that the coordinates with index “prime” could be chosen the coordinates introduced in (2.13). This gives: \( P'_2 = |P^F| \equiv \frac{C_F}{C_F}, \quad P'_1 = \langle k_1, P^F \rangle \equiv -\det(\dot{X}^F, P^F)/C_F \equiv -\tilde{J}_F/C_F, \)
\[ X'_1\langle n \rangle = \langle k_1, X^{(n)F} \rangle \equiv -\det(\dot{X}^F, X^{(n)F})/C_F \equiv -\tilde{J}_F^{(n)}/C_F \quad \text{and} \]
\[ a = \frac{\tilde{J}_F^{(n)}}{n!C_F}, \quad b = -\frac{n\tilde{J}_F^{(n)}}{(n+1)!C_FC_0}, \quad q = -\frac{C_0(n-1)!}{\tilde{J}_F^{(n)}}. \]

3.4 The jumps of the Maslov index along the front.

Let us find the jumps \( \Delta m \) of the Maslov index during the passage through the focal points along the front. We fix the time \( t > 0 \) and consider the path to cross non degenerate focal points (studied above) starting from the angle \( \psi_0 - \delta \) and ending at the angle \( \psi_0 + \delta \).

Lemma 8. The following equalities are true (see Fig. 3): for odd \( n \) \( \Delta m = 0 \) for even \( n \) and \( \text{sign}(\tilde{J}_F^{(n)}) = \pm 1 \) \( \Delta m = \pm 1. \)
Proof. Similarly to the proof of Lemma 4, taking into account the inequality det(\(\tilde{X}, P_{\psi}\)) \(\neq 0\) instead of \((3.2)\) we can write:

\[
\Delta m = \frac{1}{\pi} \lim_{\varepsilon \to +0} \text{Arg det}(\tilde{X}, X_{\psi} - i\varepsilon P_{\psi})(\psi, t)_{|\psi_0 + \delta, t} = \frac{1}{\pi} \lim_{\varepsilon \to +0} \text{Arg} \left[ -qaC(X^F)y^{n-1} + O(y^n) - i\varepsilon(q^2C(X^F)b(n+1) + O(y)) \right]_{y = -\delta} = (3.10)
\]

\[
\Delta m = \frac{1}{\pi} \lim_{\varepsilon \to +0} \text{Arg} \left[ -y^{n-1} - i\varepsilon\left(\frac{q b(n+1)}{a}\right) \right]_{y = -\delta}. \quad (3.11)
\]

We see now that the complex vector-function \(-y^{n-1} - i\varepsilon\left(\frac{q b(n+1)}{a}\right)\) lies in one half plane for even values of \(n\) for each \(y\) and in one quadrant for odd values of \(n\). So for \(n\) even one has \(\Delta m = -1\) if \(q a b > 0\) and \(\Delta m = 1\) if \(q a b < 0\) while for \(n\) odd \(\Delta m = 0\). To finish the proof it is enough to take into account formulas \((3.8)\).

Coming back to the original variables we can make the following conclusion.

**Lemma 9.** During the motion along the front \(\gamma_t\)

1) the Maslov index does not change if the path does not cross the focal points or if the Jacobian \(J = \text{det} (\tilde{X}, X_{\psi})\) does not change the sign after the passage through the focal point;

2) let the Jacobian \(J = \text{det} (\tilde{X}, X_{\psi})\) change sign after the passage through the focal point then \(\Delta m = 1\) if the signs of \(J = \text{det} (\tilde{X}, X_{\psi})\) and \(\tilde{J} = \text{det} (\tilde{X}, P_{\psi})\) coincide in the end of the path and \(\Delta m = -1\) if the signs of \(J = \text{det} (\tilde{X}, X_{\psi})\) and \(\tilde{J} = \text{det} (\tilde{X}, P_{\psi})\) are different.

**Example 5.**

Let us check the correspondence of this conclusion with the index \(m(x \in \gamma^2_t)\) for the example with the axial symmetric bank. Obviously \(J > 0\) on the arc \(\gamma^1_t\). Since the trajectories coming to arc \(\gamma^2_t\) meet the focal point only one time and \(\tilde{J} \neq 0\) in this point, then \(J < 0\) on the arc \(\gamma^2_t\). We have shown in the Example \((4)\) that \(\tilde{J} < 0\) in the focal point.

Thus \(\Delta m = 1\) if we move from the arc \(\gamma^1_t\) to the arc \(\gamma^2_t\) and \(\Delta m = -1\) if we move in the opposite direction. Finally we find \(m(x \in \gamma^2_t) = 1\), which agrees with our previous conclusion. In the critical moment of time when the first focal point just appears the Jacobian \(J(\psi, t_{cr})\) does not change the sign after crossing the focal point. Thus the index \(m\) also does not change and equals to 0 in all regular points of the front \(\gamma^{t_{cr}}\). So we see the difference in the jump of the Maslov index depends on the direction of crossing the focal points on the manifold \(M^2_t\).

### 3.5 Canonical planes in the phase space, nonsingular and singular maps.

To construct the asymptotic solution of problem \((2.2), (2.3)\) in the neighborhood of the focal points we need additional construction, related with the fronts, maps covering Lagrangian bands \(M^2_t\); indices of these maps etc. Let us describe them also briefly, using notation introduced in \((11)\).

The 2-D planes with the focal coordinates \(x^{(1,2)} = (x_1, x_2)\), \(x^{(1,0)} = (x_1, p_2)\), \(x^{(0,2)} = (p_1, x_2)\), \(x^{(0,0)} = (p_1, p_2)\) in the phase space \(\mathbb{R}^4_{p,x}\) are called symplectic canonical planes.

It is convenient to introduce the multi-indices \(I = (1, 2)\) corresponds to the canonical plane \((x_1, x_2)\), \(I = (1, 0)\) to \((x_1, p_2)\), \(I = (0, 2)\) to \((p_1, x_2)\), \(I = (0, 0)\) to \((p_1, p_2)\).

These multi-indices indicate the replacement of the coordinate \(x_j\) corresponding to the entry zero of the pair \((a_1, a_2)\) by the momentum \(p_j\) \((\text{with the same number } j)\).
denote also \( p^{(1,2)} = (p_1, p_2), p^{(1,0)} = (p_1, -p_2), p^{(0,2)} = (-x_1, p_2), p^{(0,0)} = (-x_1, -p_2) \). We call \( I \) the index of singularity. It is convenient to mark the canonical plane by the corresponding index \( I \) and write \( \mathcal{R}_I^2 \).

According to general property of Lagrangian manifold one can cover \( M_t^2 \) by the maps \( \Omega^I_j \) with the numbers \( j \) such that there exist one-to-one map from \( \Omega^I_j \) to its projection to the canonical plane \( \mathbb{R}_I^2 \). This means the following. Along with the matrices \( \mathcal{B}^{(1,2)} = \mathcal{B}, \mathcal{C}^{(1,2)} = \mathcal{C} \) it is convenient to introduce the matrices

\[
\begin{align*}
\mathcal{B}^{(0,2)}(\psi, \tau) &= \begin{pmatrix}
-\frac{\partial X_1}{\partial t} & -\frac{\partial X_2}{\partial \psi} \\
-\frac{\partial X_2}{\partial t} & -\frac{\partial X_1}{\partial \psi}
\end{pmatrix}, \\
\mathcal{B}^{(1,0)}(\psi, \tau) &= \begin{pmatrix}
\frac{\partial X_1}{\partial t} & \frac{\partial X_2}{\partial \psi} \\
\frac{\partial X_2}{\partial t} & \frac{\partial X_1}{\partial \psi}
\end{pmatrix}, \\
\mathcal{C}^{(0,2)}(\psi, \tau) &= \begin{pmatrix}
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi} \\
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi}
\end{pmatrix}, \\
\mathcal{C}^{(1,0)}(\psi, \tau) &= \begin{pmatrix}
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi} \\
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi}
\end{pmatrix}.
\end{align*}
\] (3.12)

\[
\mathcal{B}^{(0,0)}(\psi, \tau) = -\mathcal{C} = \begin{pmatrix}
\frac{\partial X_1}{\partial t} & -\frac{\partial X_2}{\partial \psi} \\
-\frac{\partial X_2}{\partial t} & \frac{\partial X_1}{\partial \psi}
\end{pmatrix}, \\
\mathcal{C}^{(0,0)}(\psi, \tau) = \mathcal{B} = \begin{pmatrix}
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi} \\
\frac{\partial P_1}{\partial \psi} & \frac{\partial P_2}{\partial \psi}
\end{pmatrix}.
\] (3.13)

The matrices \( \mathcal{C} \) give the Jacobians \( J^I(\psi, \tau) = \det \mathcal{C} \). Then in each map \( \Omega^I_j(\tau) \) \( J^I \neq 0 \). The maps with the indices \( I_j = (1,2) \) are nonsingular ones, all others are singular ones with the focal coordinates \( x^I \). Note that for practical application sometimes it useful to choose some rotated coordinates \( (x'_1, x'_2) \) and \( (p'_1, p'_2) \) in some maps \( \Omega^I_j \). It is important to remember, that the Jacobians \( J = J^{(1,2)} \) and \( J^{(0,0)} \) are invariant with respect to the rotation, but the Jacobians \( J^{(0,2)}, J^{(1,0)} \) are not. Actually in the considered problem the maps with index \( I = (0,0) \) are not needed.

**Lemma 10.** For any time \( t \) there exists a finite covering of the neighborhood of \( \Gamma_t \) from Lagrangian manifold \( M_t^2 \) by the maps \( \Omega^I_j \) with the indices \( I_j = (1,2), I_j = (1,0), I_j = (0,2) \).

**Proof.** It is enough to prove that at least one of the Jacobians \( J^{(0,2)} \) or \( J^{(1,0)} \) is not equal to zero in each focal point. Assume that both \( J^{(0,2)} = 0 \) and \( J^{(1,0)} = 0 \) in the point \( \psi^F, t^F \). Since \( X_\psi = 0 \) in the focal point, this means that \( x_1 P_{\psi} = x_2 P_{1,\psi} = 0 \) and \( \det(\dot{X}, P_\psi) = 0 \). But according to Lemmas 2,3 the vector \( P_\psi \neq 0 \) in the focal point and \( P_\psi \) is orthogonal to \( \dot{X} \), which is nonzero everywhere. This contradiction proves this lemma. \( \Box \)

**Remark 4.** 1) It is possible to prove a similar proposition about the whole manifold \( M_t^2 \), but, since \( M_t^2 \) is noncompact, the number of maps is infinite, and we do not need this fact. 2) For practical applications sometimes it useful to choose rotated coordinates \( (x'_1, x'_2) \) and \( (p'_1, p'_2) \) in some maps \( \Omega^I_j \). For instance in the neighborhood of non degenerate focal points \( X(\psi^F(t), t) \) in the situation considered in subsection it is convenient to choose the axis \( x_2' \) coinciding with the vector \( \dot{X}(\psi^F(t), t) \) (see subsection focal).

**Example 6.**

1) Construct first the covering of the neighborhood of the “initial” curve \( \Gamma_0 = \{ p = n(\psi), x = 0 \} \) on the Lagrangian manifold \( M_t^2 \). The projection of \( \Gamma_0 \) from the phase space \( \mathbb{R}^4_{ps} \) on the space \( \mathbb{R}^2_\psi \) is shrunk to the point \( x = 0 \), in this point \( \dot{X}_\psi = 0 \) for each \( \psi \). Thus according to our classification the point \( x = 0 \) is a completely degenerate point of
the “front” \( \gamma_0 \). Using the definition of \( P(\tau, \psi), X(\tau, \psi) \) as solution of the system (2.4) we find for \( \tau = t = 0 \):

\[
\dot{P} = -C'_0 \equiv -\nabla C(x)|_{x=0}, \quad \dot{X} = n(\psi)C(x), \quad P_\psi = n_\perp(\psi), \quad X_\psi = 0,
\]

where \( C'_0 \equiv -\nabla C(x)|_{x=0} \) is a vector with components \( C_{10} = \frac{\partial C}{\partial x_1}(0), \quad C_{20} = \frac{\partial C}{\partial x_2}(0) \). Hence \( J^{(0,0)}|_{\Gamma_0} = J|_{\tau=0} = \det(\dot{X}, X_\psi)|_{\tau=0} = 0 \). But

\[
J^{(1,0)}|_{\Gamma_0} = \det \left( \begin{array}{cc} C_0 \cos \psi & 0 \\ -C_{02} \cos \psi & \end{array} \right) = C_0 \cos^2 \psi, \quad J^{(2,0)}|_{\Gamma_0} = \det \left( \begin{array}{cc} -C_{10} & -\sin \psi \\ C_0 \sin \psi & 0 \end{array} \right) = C_0 \sin^2 \psi.
\]

(3.14)

Thus the neighborhood of the curve \( \Gamma_0 \) can be covered by four singular maps (see Fig. . . .): the map \( \Omega^{(1,0)}_1 \), covering the arc with the angle \(-\pi - \zeta < \psi < \pi + \zeta\), the map \( \Omega^{(0,2)}_2 \), covering the arc with the angle \( -\zeta < \psi < \frac{\pi}{2} - \zeta \), the map \( \Omega^{(1,0)}_3 \), covering the arc with the angle \( -\frac{\pi}{4} - \zeta < \psi < \frac{5\pi}{4} + \zeta \), and the map \( \Omega^{(1,2)}_4 \), covering the arc with the angle \( \frac{\pi}{4} - \zeta < \psi < \frac{7\pi}{4} + \zeta \). Here and furthermore in these examples \( \zeta \) is a small enough positive number. The choice of these maps is not unique but the final results does not depend on the particular choice it is necessary that they are 4.

2) For \( t \) less than \( t_{cr} \) one can cover the neighborhood of \( \Gamma_t \) by two nonsingular maps (see Fig. . . .) e.g. \( \Omega^{(1,2)}_1 \) covering the arc with the angle \(-\zeta < \psi < \pi + \zeta\), and \( \Omega^{(1,2)}_2 \) covering the arc with the angle \( \pi - \zeta < \psi < 2\pi + \zeta \). Actually these maps disappear in the final formula for wave field because both of them are nonsingular; they are used like temporary objects. From the other side changing these maps by their union gives the tubular neighborhood of \( \Gamma_t \) which is not a map.

3) Let us construct the covering of the front \( \Gamma_t \) corresponding to the example of the axial symmetric case and for time \( t = t_{cr} \), when the first focal point has just appeared. Then we have only one focal point on \( \Gamma_{t_{cr}} \), the one with the angle \( \psi^F(t_{cr}) = \pi/2 \) (which follows from the symmetry of the problem). Obviously we can choose covering consisting of two maps: nonsingular \( \Omega^{(1,2)}_1 \) covering the arc with the angle \( \pi/2 + 2\zeta < \psi < 2\pi + \pi/2 - 2\zeta \) and singular \( \Omega^{(0,2)}_2 \) covering the arc with the angle \( \pi/2 - \zeta < \psi < \pi/2 + \zeta \).

4) Now we describe the covering of the front \( \Gamma_t \) but for \( t > t_{cr} \). Denote by \( \psi^F(t) \) the angles of the left and right focal points. We can choose the maps in the following way: the nonsingular map \( \Omega^{(1,2)}_1 \), covering the arc with the angle \( -\frac{\pi}{4} - \zeta < \psi < \psi^F(t) - \zeta \), the singular map \( \Omega^{(0,2)}_2 \), covering the arc with the angle \( \psi^F(t) - 2\zeta < \psi < \psi_+^F(t) + 2\zeta \), the nonsingular map \( \Omega^{(1,2)}_3 \), covering the arc with the angle \( \psi^F(t) + \zeta < \psi < \psi^F_-(t) - \zeta \), the singular map \( \Omega^{(0,2)}_4 \), covering the arc with the angle \( \psi_+^F(t) - 2\zeta < \psi < \psi^F_-(t) + 2\zeta \), and the nonsingular map \( \Omega^{(1,2)}_5 \), covering the arc with the angle \( \psi_+^F(t) + \zeta < \psi < \frac{3\pi}{2} + \zeta \).

**Remark 5.** 1) We mentioned that in the singular maps covering the isolated focal point with the angle \( \psi^F(t) \) it is convenient to rotate the coordinates in such a way that the new axis \( x'_2 \) has the same direction as vector \( \hat{X}(\psi^F(t)) \). In the new coordinate system the singular map always has a singular index \( I = (0, 2) \) and focal (rotated) coordinates \( (p^F_1, x'^F_2) \). 2) One needs to use the covering from item 3) of the previous example not only for the critical time \( t_{cr} \) but also times for near times \( t \).
3.6 The Maslov index of a singular map.

The last object we need is the Maslov index of chains of maps \( \{ \Omega^{I_j}_j(t) \} \). To find it one has to fix some nonsingular point \( r(\tilde{\psi}, \tilde{\tau}) \) in the corresponding map \( \Omega^{I_j}_j \) and construct their one of the following matrices \( C^{(1,0)}_\epsilon, C^{(0,2)}_\epsilon \) with the elements of matrices \( B, C \) defined in (3.11): \[
C^{(1,2)}_\epsilon = C - i\epsilon B = \begin{pmatrix}
C_{11} - i\epsilon B_{11} & C_{12} - i\epsilon B_{12} \\
C_{21} - i\epsilon B_{21} & C_{22} - i\epsilon B_{22}
\end{pmatrix}
\]

\[
C^{(1,0)}_\epsilon = \begin{pmatrix}
C_{11} - i\epsilon B_{11} \\
(C_{21} - i\epsilon B_{21}) \cos \eta + (B_{21} - i\epsilon C_{21}) \sin \eta
\end{pmatrix}
\]

\[
C^{(0,2)}_\epsilon = \begin{pmatrix}
C_{12} - i\epsilon B_{12} \\
(C_{22} - i\epsilon B_{22}) \cos \eta + (B_{22} + i\epsilon C_{22}) \sin \eta
\end{pmatrix}
\]

\[
C^{(0,0)}_\epsilon = \begin{pmatrix}
(C_{11} - i\epsilon B_{11}) \cos \eta + (B_{11} + i\epsilon C_{11}) \sin \eta \\
(C_{21} - i\epsilon B_{21}) \cos \eta + (B_{21} + i\epsilon C_{21}) \sin \eta
\end{pmatrix}
\]

\[
C^{(0,0)}_\epsilon = \begin{pmatrix}
(C_{12} - i\epsilon B_{12}) \cos \eta + (B_{12} + i\epsilon C_{12}) \sin \eta \\
(C_{22} - i\epsilon B_{22}) \cos \eta + (B_{22} + i\epsilon C_{22}) \sin \eta
\end{pmatrix}
\]

These matrices are not generated for any \( \eta \in [0, \pi/2] \) and any positive \( \epsilon \) in the maps with index \((1,0), (0,2)\) and \((0,0)\) respectively \(^3\). Obviously \( C^{(1,0)}_{\epsilon|\eta=0} = C^{(1,2)}_\epsilon \equiv C - i\epsilon B \) and \( C^{(0,0)}_{\epsilon|\eta=0} = C^I - i\epsilon B^I \) and these matrices determine a continuous non degenerate transition from the matrix \( C^{(1,2)}_\epsilon \) to the matrix \( C^I - i\epsilon B^{(1,2)} \). The corresponding determinant \( J^{(1,0)}_\epsilon = \det C^{(1,0)}_\epsilon \) or \( J^{(0,1)}_\epsilon = \det C^{(0,2)}_\epsilon \) is not equal to zero. Let \( m(\tilde{\psi}, \tilde{\tau}) \) be the Maslov index of the point \( r(\tilde{\psi}, \tilde{\tau}) \). Then the index \( m(\Omega^{I_j}_j) \) of the map \( \Omega^{I_j}_j \) is

\[
m(\Omega^{I_j}_j) = m(\tilde{\psi}, \tilde{\tau}) + \frac{1}{\pi} \lim_{\epsilon \to +0} \text{Arg} J^{(1,0)}_\epsilon|_{\eta=\pi/2}.
\] (3.15)

This definition does not depend \( \mod 4 \) on the choice of the point \((\tilde{\psi}, \tilde{\tau})\) in a given map. The calculation of the index \( m(\Omega^{I_j}_j) \) could be technically complicated even in quite simple situations. But taking into account the fact that we can restrict ourselves to maps with \( I = (1,0) \) and \( I = (0,2) \) it is possible to simplify the application of this formula.

**Lemma 11.** One can always find a nonsingular point \( r(\tilde{\psi}, \tilde{\tau}) \) in the map \( m(\Omega^{I_j}_j) \) with \( I_j = (1,0) \) or \((0,2)\) such that the sign Jacobian \( J(\tilde{\psi}, \tilde{\tau}) \) coincides with the sign of the Jacobian \( J^{I_j}(\psi, \tau) \) in this map. Then the second term in (3.15) is equal to zero and

\[
m(\Omega^{I_j}_j) = m(\tilde{\psi}, \tilde{\tau}).
\]

As the sign of the Jacobian \( J^{I_j} \) does not depend on a point in the \( \Omega^{I_j}_j \), thus one can find it in any point, for instance in a focal one.

**Proof.** It is obvious that the second term in (3.15) can be equal to 0, 1 or \(-1\) only. Consider for \( \epsilon \) the Jacobian \( J^I_0 \). A simple calculation gives \( J^I_0 = J \cos \eta + J^I \sin \eta \). In

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\(^3\)As we mentioned above the objects with singular index \((0,0)\) are not needed in the considered problem we present \( C^{(0,0)}_\epsilon \) for completeness.
the interval \([0, \pi/2]\) this function has no zero if \(J\) and \(J^I\) have the same signs and one zero in the opposite situation. Hence including the parameter \(\varepsilon\) gives only the rule of bypassing of the zero point on the complex plane, and it is not necessary to use this rule in the case when \(JJ^I > 0\) Thus one obtains \(\lim_{\varepsilon \to +0} \text{Arg}_{\psi|\varepsilon|_{I=0}}^{I|\varepsilon|/\pi/2} = 0\) if a point \(r(\tilde{\psi}, \tau)\) is chosen in the way prescribed in the Lemma. From the other side, according to Lemma 3 and its Corollary, the existence of the focal point in any focal map means the existence of nonsingular points with positive and negative signs of the Jacobian \(J\).

\[\square\]

**Corollary.** The index \(m(\Omega^I_J(t))\) of the singular map \(\Omega^I_J\) coincides with the index of any nonsingular point \(m(\psi, t)\) on the front \(\Gamma_t\) where the Jacobians \(J\) and \(J^I\) have the same sign.

Let us illustrate this lemma by the calculations of indices of maps from the examples considered above.

**Example 7.** 1) Consider the maps from the example 6. Let us take the points on the maps \(\Omega^I_{j, j}, j = 1, 2, 3, 4\) with coordinates \(\psi, \tau, \tau > 0\). We put \(m(\psi, \tau) = 0\). According to Corollary from Lemma 4 \(J(\psi, \tau) > 0\). From this and (3.14) we find \(m(\Omega^I_{j, j}) = 0\) for each \(j = 1, 2, 3, 4\).

**Example 8.** 2) Consider the maps from 3), 4) of example 6. Before let us come back to formulas (3.11). As we mentioned in remark 8.5 it is convenient to rotate the coordinates. Then in the new coordinates in the focal point \(X^F\) with coordinates \((\psi^F(t), t)\)

\[
\dot{X'} = P' C^2(X^F)/C_0 \left(\begin{array}{c} 0 \\ q_{an} \end{array}\right), \quad \text{and} \quad P'_\psi = \begin{cases} -qb(n+1) & \text{if a point} \\
0 & \text{else} \end{cases}
\]

Hence \(J^{(0,2)} = abC^2(X^F)q^2\psi n(n+1)/C_0\) and the sign of \(J^{(0,2)}\) coincides with the sign of the product \(ab\). For both examples 3), 4) the sign is negative (see Fig. 7) thus the index of the map \(\Omega^{(0,2)}_{1, 2}\) in 3 of example 6, and indices of the maps \(\Omega^{(0,2)}_{2, 2}, \Omega^{(0,2)}_{4, 2}\) in 3 of example 6 are \(-1\).

### 3.7 Germs of Lagrangian manifolds and their properties.

The geometrical objects described above one can meet in many asymptotical problems having fast oscillation solution. We see that the solution of considered problem decays quit rapidly outside of some neighborhood of the front \(\gamma_t\). This gives the opportunity to use the ideas of boundary layer expansions [52]-[54] and the “complex germ theory” [29]. Their geometrical realization contains in a change Lagrangian band \(M_t^2\) by its linearization (germ) near the curve \(\Gamma_t\).

**Definition 1.** For each fix \(t\) we call the linear germ corresponding to the manifold \(M_t^2\) a vector fiber bundle in the phase space with base coinciding with the front \(\Gamma_t = (x = X(\psi, t), p = P(\psi, t))\) and fibers generated by the vectors \(\dot{X}, \dot{P}\).

Denote \(\alpha \in \mathbb{R}\) the coordinate on the bundle (which is a linear analog to a proper time \(\tau\)), then we can define a family of manifolds \(A_t^2\) as a strip in a neighborhood of the front \(\Gamma_t\) in the phase space \(\mathbb{R}^{4,p,x}\)

\[
A_t^2 = \{p = P(\psi, t, \alpha) \equiv P(\psi, t) + \dot{P}(\psi, t)\alpha, \quad x = X(\psi, t, \alpha) \equiv X(\psi, t) + \dot{X}(\psi, t)\alpha\}, \quad (3.16)
\]
where $\psi \in S^1 = [0, 2\pi]$, $|\alpha| < \alpha_0$ are the coordinates in $\Lambda^2_1$. It is easy to see that $M^2_t$ can be approximated by $\Lambda^2_t$, and that the parameter $\alpha$ is used for linearizing the functions $X(\psi, t + \alpha), P(\psi, t + \alpha)$ and that it defines a shift of time near the front $\Gamma_t$. Taking into account this fact it is easy to prove the following proposition.

**Lemma 12.** 1) With an error of the order $O(\alpha^2)$ the manifold $\Lambda^2_1$ is obtained from $\Lambda^2_0$ by means of a shift of time $t$ along the trajectories of the phase flow with the Hamiltonian $\mathcal{H} = H[p]C(x)$. 2) For the matrices $B(\psi, t, \alpha) = \frac{\partial p}{\partial (\alpha, \psi)}$, $C(\psi, t, \alpha) = \frac{\partial x}{\partial (\alpha, \psi)}$ the following equalities are true $B = B + O(\alpha), C = C + O(\alpha); \quad CB = BC + O(\alpha)$, where as before $B = \frac{\partial p}{\partial (\psi, \alpha)}$, $\mathcal{C} = \frac{\partial x}{\partial (\psi, \alpha)}$. The last equality means that the manifold (band) $\Lambda^2_1$ is (almost) Lagrangian mod $O(\alpha)$, the statement 1) means that it is (almost) invariant mod $O(\alpha)$.

**Proof.** Consider the Hamilton equations $\dot{x} = \mathcal{H}_p, \dot{p} = -\mathcal{H}_x$. We expand the derivatives of the Hamiltonian $\mathcal{H}$ around the point $X(t, \psi), P(t, \psi)$

$$
\mathcal{H}_p = \mathcal{H}_p(X(\psi, t), P(\psi, t)) + \mathcal{H}_{pp} \dot{\alpha} \dot{P} + \mathcal{H}_{px} \dot{\alpha} \dot{X} + O(\alpha^2)
$$

and substitute in the Hamilton equations and we get the result. The second proposition is obvious and use the variational system for the evolution of $(x, p)$:

$$
\dot{X} + \alpha \ddot{X} - (\mathcal{H}_p(X(\psi, t), P(\psi, t)) + \mathcal{H}_{pp} \dot{\alpha} \dot{P} + \mathcal{H}_{px} \dot{\alpha} \dot{X}) = O(\alpha^2)
$$

and we get the result. $\square$

As the germ $\Lambda^2_1$ is some approximation of $M^2_t$ almost all previous proposition, geometrical definition and construction (like Maslov and Morse index) related to the band $M^2_t$ is true for the band (germ) $\Lambda^2_1$. From the other side obviously one does not need any additional objects besides the family of curves (fronts in the phase space) $\Gamma_t$ to construct both the Lagrangian bands $M^2_t$ and their germs $\Lambda^2_t$. It also follows from formulas (2.6) (2.9) (2.16) that leading term of the solution $\eta$ based also only on these objects. Nevertheless the proof of (2.6) (2.9) (2.16) needs something more, and it seems that sometimes technically instead of the germ $\Lambda^2_1$ it is convenient to consider also the other germ (fiberbundle) namely

$$
\tilde{\Lambda}^2_1 = \{p = \tilde{P}(\psi, t, \alpha) \equiv P(\psi, t) + (\tilde{\dot{P}}(\psi, t) - \lambda(\psi)P)\alpha, \quad x = X(\psi, t, \alpha) \equiv X(\psi, t) + \tilde{\dot{X}}(\psi, t)\alpha\}
$$

where (see Lemma 2) $\lambda = \left(\frac{\partial C}{\partial x}(0), n(\psi)\right)$. This germ also implies the matrices $\tilde{B}(\psi, t, \alpha) = \frac{\partial p}{\partial (\alpha, \psi)}, \quad \tilde{C}(\psi, t, \alpha) = C(\psi, t, \alpha) = \frac{\partial x}{\partial (\alpha, \psi)}$. But now $\tilde{B} = \tilde{B} + O(\alpha)$, where $\mathcal{B} = (\tilde{\dot{P}} - \lambda P, P)$. After analysis of our previous consideration it is possible to prove the following statement.

**Lemma 13.** All previous proposition concerning the germ $\Lambda^2_1$ and matrices $\mathcal{B}, \mathcal{C}$ are true for the germ $\tilde{\Lambda}^2_1$ and matrices $\tilde{\mathcal{B}}, \mathcal{C}$.

### 4 Geometrical asymptotic solution and the Maslov canonical operator

The central mathematical result of our paper is the observation that the asymptotic solution of the problem (2.22) can be represented as an integral over $dp$ of the canonical
Maslov operator with “semiclassical” parameter \( h = l/\rho \), defined on the appropriate family of Lagrangian manifolds \( \Lambda_\mu^2 \), and acting on the function \( V \) \( (1.19) \) defining the initial localized perturbation. In some neighborhood of the front line \( \gamma_t \) the final formula has the form:

\[
\eta = \text{Re} \left( \sqrt{\frac{\mu C_0}{2\pi i}} \int_0^\infty K^{\mu/\rho}_\Lambda t \left( \sqrt{\rho V} \right) d\rho + o(\mu), \right.
\]

and \( \eta = o(\mu) \) outside of this neighborhood. The initial data \( (1.19) \), the representation of the asymptotic solution \( (2.6), (2.9) \) out of the neighborhood of the focal points as well as the future representation of the solution in the neighborhood of the focal points is only a realization of \( (4.1) \) in the corresponding domain of \( \mathbb{R}^2 \). As we said before from the geometrical point of view it means that we can use germs \( \Lambda_\mu^2 \) instead of the Lagrangian bands \( M_t^2 \). In next subsections we shall describe the functions and other objects determining the operator \( K^{\mu/\rho}_\Lambda \).

4.1 The functions on Lagrangian bands \( M_t^2 \).

\( \text{a. The action-function.} \) The Lagrangian property allows one to define on the family \( M_t^2 \), function \( s(\psi, t, \alpha) \) satisfying the equation \( ds = \langle P, dX \rangle \big|_{M_t^2} \).

**Lemma 14.** The phases \( (\psi, \tau) \) on \( M_t^2 \) is equal to \( C_0 \alpha \).

**Proof.** First let us find \( s \) on \( M_0^2 \). Then the coordinate \( \alpha \) is the proper time: \( \tau = \alpha \). But in this case we have

\[
\int_{(0,0)}^{(\psi, \tau)} \langle P, dX \rangle = \int_{(0,0)}^{(0, \tau)} \langle P, \dot{X} \rangle dt + \int_{(0, \tau)}^{(\psi, \tau)} \langle P, X_\psi \rangle d\psi.
\]

The second term in the last expression is equal to zero according to Lemma 2. Changing \( \dot{X} \) by the right hand side from system \( (2.3) \) and using the integral of motion \( (2.5) \), we find (using the proper time \( \tau \)) \( s(\psi, \tau) = C_0 \tau = C_0 \alpha \). According to \( 25, 28, 41 \) to construct the action on the band \( M_t^2 \) one has to add \( \int_0^t \mathcal{L} dt \) to \( C_0 \alpha \), where the Lagrangian \( \mathcal{L} = \langle p, \mathcal{H} \rangle - \mathcal{H} \). But for the wave equation \( \mathcal{L}' = 0 \), which gives the first proposition of Lemma.  □

The phase \( s \) implies the phases associated with projection to singular maps \( \Omega_j^2 \):

\[
s^{(1,0)}(\psi, \tau) = C_0 \tau - P_2(\psi, \tau) X_2(\psi, \tau), s^{(0,2)}(\psi, \tau) = C_0 \tau - P_1(\psi, \tau) X_1(\psi, \tau).
\]

The initial source function \( V(y) \) implies on \( M_t^2 \) the function (more precisely the family of function depending on parameter \( \rho \in (0, \infty) \))

\[
f(\rho, \psi) = \sqrt{\rho V} \langle \rho \eta(\psi) \rangle.
\]

We need also the smooth cut off function \( e_0(\alpha) \), \( e_0(\alpha) = 1 \) for \( |\alpha| < \alpha_0 \), and \( e_0(\alpha) = 0 \) for \( |\alpha| > 2\alpha_0 \) where \( \alpha_0 \) is some small enough positive number. The product \( f(\rho, \psi) e_0(\alpha) \) gives the finite function on the band \( M_0^2 \). We continue it on all family \( M_t^2 \) assuming that it does not depend on time \( t \).
b. Functions in the maps $\Omega^I_j$. In each map $\Omega^I_j$ the Jacobians $J^I$ are not equal to zero. This means that one can construct the smooth solutions $(\psi^I_j(x^I), \tau^I_j(x^I)) \in \Omega^I_j$ of the system of equations

$$X^I(\psi, \tau) = x^I.$$  

(4.3)

Let us emphasize that it could exists several angles $\psi^I_j$, corresponding to one vector $x^I$. Although the manifold $M^2_I$ is invariant with respect to the shift $g^2_H$ it does not mean that there is no dynamics on $M^2_I$. It only means that turning on the time dependence we transform the objects related to $M^2_I$ in a special way. Namely let us use from the beginning the coordinate $\alpha$ instead of the proper time $\tau$. We have already mentioned that the points $r(\psi, \alpha) \in M^2_I$ after the action of the transform $g^2_H$ passes to the points $r(\psi, \alpha) \in M^2_I$ with shifted coordinate $\tau = \alpha + t$, but with the same angle $\psi$. Thus the equations (4.3) are changed by the equations

$$X^I(\psi, \alpha + t) = x^I;$$  

(4.4)

The following trivial proposition is very useful.

Lemma 15. Let $\psi_j(x^I), \tau_j(x^I)$ be the solution of the equations (4.3) in the map $\Omega^I_j(t)$ and let the point $r(\psi, \alpha + t) \in M^2_I$ with coordinates $\psi, \alpha + t$ belong to the same map $\Omega^I_j$. Then the angle component $\psi_j$ of the solution of the equation (4.4) does not depend on $t$: $\psi_j = \psi_j(x^I)$ and $\alpha_j = \alpha_j(x^I_j, t)$ is

$$\psi_j(x^I_j, t) = \psi_j(x^I), \quad \alpha_j(x^I_j, t) = \tau_j(x^I_j) - t,$$  

(4.5)

Proof is obvious $\Box$.

Using (4.4) and (4.5) we can rewrite action-functions and the Jacobians in the coordinates $x^I$. The behavior of the functions on $M^2_I$ is different with respect to the shift $g^2_H$. Namely the function $s = \alpha$, and the functions $f, e$ are constant, this means that for each $t$ in the coordinates $\psi, \alpha$ one has the same form. On the contrary the functions $s^I$ and all the Jacobians $^I J$ take the forms:

$$s^{(1,0)}(\psi, \alpha, t) = C_0 \alpha - P_2(\psi, \alpha + t)X_2(\psi, \alpha + t),$$  

$$s^{(0,2)}(\psi, \alpha + t) = C_0 \alpha - P_1(\psi, \alpha + t)X_1(\psi, \alpha + t),$$  

(4.6)

$$J^I = J^I(\psi, \alpha + t).$$  

(4.7)

Now in the maps $\{\Omega^I_j(t)\}$ we want to pass from coordinates $\psi, \alpha$ to coordinates $x^I_j$.

This gives us the actions, Jacobians etc. in the coordinates $x^I_j$:

$$S^{(1,2)}_j(x_1, x_2) = \tau_j(x_1, x_2) - C_0 t,$$

$$S^{(1,0)}_j(x_1, p_2) = \tau_j(x_1, p_2) - p_2 X_2(\psi_j(x_1, p_2), \tau_j(x_1, p_2)) - C_0 t,$$

$$S^{(0,2)}_j(p_1, x_2) = \tau_j(p_1, x_2) - p_1 X_1(\psi_j(p_1, x_2), \tau_j(p_1, x_2)) - C_0 t,$$

$$J^{I_j}_j(x^I_j) = J^I_j(\psi_j(x^I_j), \tau_j(x^I_j)),$$

$$e^t = e(\tau_j(x^I_j) - t).$$  

(4.8)

Let us emphasize that the complicated notations only reflect the situation: each map has it own number $j$ and index of singularity $I_j$. (See examples below).

Finally we need to introduce the partition of unity with the maps $\Omega^I_j(t)$ covering $\Gamma_t$: the set of smooth functions $e_j(\psi, t)$ associated with the covering $\{\Omega^I_j(t)\}$: $\text{supp} e_j(\psi) \in \Omega^I_j(t), \sum_j e_j(\psi) = 1$.

\[\text{To simplify notation we do not introduce a new symbol for time-shifted Jacobian.}\]
4.2 The time-dependent canonical Maslov operator on the invariant manifold \( M^2_t \).

Now everything is ready to determine the canonical Maslov operator \( K^h_{M^2_t} \), acting on the function \( f(\rho, \psi)e(\alpha) \), which is constant one on the trajectories of system \( (2.2) \) and depending on the parameter \( h > 0 \). It means that this function is the same in all points \( P(\psi, t + \alpha), X(\psi, t + \alpha) \). Let \( \{\Omega^J_j\} \) be a covering of the curve \( \Gamma_t \). Let us divide the set of indices \( \{j\} \) into three parts \( \{j(1,2), j(1,0), j(0,2)\} \) corresponding to the maps with indices of singularity \( (1,2) \),\( (1,0) \) and \( (0,2) \) respectively. We put

\[
\Psi(\rho, x_1, x_2, t) = K^h_{M^2_t}(f e) \equiv
\]

\[
\sum_{j \in \{j(1,2)\}} e^{-\frac{i\pi}{2}m(\psi_j(x_1, x_2))} \exp\frac{iS^{(1,2)}_j(x_1, x_2, t)}{\sqrt{|S^{(1,2)}_j(x_1, x_2)|}} f(\rho, \psi)e_{j}(\psi)|_{\psi=\psi_j(x_1, x_2)}e(\tau_j(x_1, x_2, t) - t) +
\]

\[
\sum_{j \in \{j(1,0)\}} e^{-\frac{i\pi}{2}m(\tau^{(1,0)}_j)} \frac{i}{\sqrt{2\pi h}} \int_{-\infty}^{+\infty} \exp\frac{i(S^{(1,0)}_j(x_1, p_2, t) + x_2 p_2)}{h} f(\rho, \psi)e_{j}(\psi)|_{\psi=\psi_j(x_1, p_2)}e(\tau_j(x_1, p_2, t) - t)dp_2 +
\]

\[
\sum_{j \in \{j(0,2)\}} e^{-\frac{i\pi}{2}m(\psi^{(0,2)}_j)} \frac{i}{\sqrt{2\pi h}} \int_{-\infty}^{+\infty} \exp\frac{i(S^{(0,2)}_j(p_1, x_2, t) + x_1 p_1)}{h} f(\rho, \psi)e_{j}(\psi)|_{\psi=\psi_j(p_1, x_2)}e(\tau_j(p_1, x_2, t) - t)dp_1
\]

(4.9)

where \( S^j \) and \( J^j \) are defined in (4.8). Now we can construct the asymptotic solution \( \eta \) to the problem \( (2.2) \). We put in the last formula \( h = \rho/l \).

**Theorem 4.** 1) For any \( T \) independent of \( \mu = l/L \), the solution \( \eta \) to the problem \( (2.2) \) in the interval \( t \in [0, T] \) has the form:

\[
\eta = \eta_{as} + o(\mu), \quad \eta_{as} = \sqrt{\frac{\mu C_0}{2\pi}} \text{Re}(e^{-\frac{i\pi}{4}} \int_{0}^{\infty} \Psi(\rho, x_1, x_2) d\rho).
\]

(4.10)

This asymptotic, apart from terms of the order \( O(\mu) \), does not depend on the choice of the covering \( \{\Omega^J_j\} \), and functions \( e_j, e \).

2) For each time \( t \in [0, T] \) the function \( \eta \) is localized in a neighborhood of the front: the function \( \eta \) is equal \( O(\mu) \) outside some neighborhood of the front \( \gamma_t \).

**Sketch of Half of Proof.** Using the results \( [25, 28, 27, 11, 22] \) one can show that the function \( (2.2) \) is a leading term of some asymptotic solution \( \Psi^k \mod O(h^k) \) to original equation \( (2.2) \), where \( k \) is an arbitrary big by fixed integer number. We introduce the smooth cut off function \( g(y) \): \( g(y) = 0 \) for \( y \leq 1/2 \) and \( g(y) = 1 \) for \( y \geq 1 \). Multiplying \( \Psi^k \) by \( g(\rho/\mu^{1/2}) \) and integrating the product by \( d\rho \) we obtain that the result is asymptotic solution \( (2.2) \mod O(\mu^2) \). Then as in \( [22, 23] \) we show that the influence of the term \( \int_{0}^{1}(1 - g(\rho/\mu^{1/2}))|\Psi^k|t=0 d\rho \) into the solution \( (2.2) \) is \( o(\mu) \), and hence the function \( \eta_{as} \) from is a leading term of some asymptotic solution of \( (2.2) \). Now we need to check the conditions \( (2.2) \). But it is better to do after simplification of function and we shall do it in the next subsection.

4.3 The germ \( A^2_t \) of the manifold \( M^2_t \) and the simplification of the asymptotic.
Since the function \( (4.2) \) decays quite rapidly when the point \( x \) goes away from the front, it is possible to change the functions \( S_{ij}^I, |J_{ij}^I| \) in a neighborhood of the front by their Taylor expansions. The nice fact is that one does not change the accuracy \( O(\mu) \) in formula \( (4.2) \) using only the zero, first terms and sometimes second terms of the Taylor expansions of the phases, and zero terms in the other functions. All these expansions are expressed via the vector functions \((P_1(\psi,t),X(\psi,t))\) and matrices \(B(\psi,t),C(\psi,t)\).

We need the Taylor expansion in the following form. Let the equations \((y_1,y_2) = (Y_1(\psi),Y_1(\psi))\) determine a smooth curve \(Y\) in some domain in \(R^2\) and \(\Phi(y_1,y_2)\) be a smooth function in some neighborhood \(D\) of \(Y\). Let \(q(\psi)\) be the smooth family of nonzero vectors with components \((q_1,q_2)\) transversal to the curve \(Y\). This means that the vectors \(q(\psi)\) and \(Y\psi(\psi)\) are not parallel. The parameter \(\psi\) on \(Y\) and the family of vectors \(q(\psi)\) define the curvilinear system in some neighborhood of \(Y\): each point \(y\) in this neighborhood can be characterized by two values: \(\psi(y)\) and the length (with the sign) \(z = \langle y - Y(\psi(y)), q(\psi(y)) \rangle / (q(\psi(y)))^2\) of the vector \(y - Y(\psi(y))\). To find the value of \(\psi(y)\) one has to solve the equation

\[
\langle y - Y(\psi), q_\perp(\psi) \rangle \equiv (y_1 - Y_1(\psi))q_2(\psi) - (y_2 - Y_2(\psi))q_1(\psi) = 0. \tag{4.11}
\]

**Lemma 16.** The following expansion is valid:

\[
\Phi(y) = \Phi(Y) + \left( \frac{\partial \Phi}{\partial y}(Y), (y - Y) \right) + \frac{1}{2} \left( (y - Y), \frac{\partial^2 \Phi}{\partial y^2}(Y)(y - Y) \right)_{Y=Y(\psi(y))} + O((y - Y(\psi(y)))^3)
\]

\[
\tag{4.12}
\]

**Proof** follows from the 1-D Taylor expansion of the function \(\Phi(Y + qz)\) with respect to variable \(z\) \(\Box\).

Now we want to apply this lemma to the phases \(S_{ij}^I\) and Jacobians \(J_{ij}^I\) in \((4.8)\) and \((4.12)\). The variable \(y\) are \(x^I\), the curve \(Y = \{ x^I = X^I(\psi,t) \}\), thus the solution \(\psi\) will depend also on time \(t\). We need the first and second derivatives of \(S^I\) in the points \(x^I = X^I(\psi,t)\). From the general theory of Hamilton-Jacobi equation \([?, ?, ?]\) it follows

\[
\frac{\partial S^I}{\partial x^I} = P^I(\psi,t), \quad \frac{\partial^2 S^I}{\partial (x^I)^2} = \frac{\partial P^I}{\partial x^I}(\psi,t) = B^I(\psi,t)(C^I(\psi,t))^{-1},
\]

where the matrices \(B^I,C^I\) are defined in \((4.13)\). Now let us choose the vector \(q\) as following. In the case \(I = (1,2)\) \(q = (X(\psi,t) = (X(\psi))_\perp\), then equation \((4.11)\) is equation \((4.14)\). In the case \(I = (0,2)\) \(q = t(0,1)\), then equation \((4.11)\) is

\[
P_1(\psi,t) = p_1. \tag{4.13}
\]

In the case \(I = (1,0)\) \(q = t(1,0)\), then Eq.\((4.11)\) is

\[
P_2(\psi,t) = p_2. \tag{4.14}
\]

We denote \(\psi_j^I(x^I,t)\) the solution of these equations in the map with the number \(j\). Then after some algebra we obtain the following formulas in the maps with numbers \(j\)

\[
S_{ij}^I(x^I,t) = \{ S_{ij}^I(\psi,t) + O(x^I - X^I(\psi,t))^3 \}_{\psi = \psi_j^I(x^I,t)}, \tag{4.15}
\]

\[
J_{ij}^I(x^I,t) = \{ J_{ij}^I(\psi,t) + O(x^I - X^I(\psi,t))^3 \}_{\psi = \psi_j^I(x^I,t)}.
\]

These solutions are different from the solutions \(\psi_j(\psi^I)\) of equation \((3.5)\), we use almost the same symbol to simplify the notation.
here

\[ S_j^{(1,2)}(\psi, t) = (P(\psi, t), (x - X(\psi, t)) - \frac{1}{2}(P(\psi, t), C_x(X(\psi, t)))(x - X(\psi, t))^2, \]

\[ J_j^{(1,2)} = \det(\dot{X}, X_\psi)(\psi, t) \] (4.16)

\[ S_j^{(0,2)}(\psi, t) = -P_1(\psi, t)X_1(\psi, t) + P_2(\psi, t)(x_2 - X_2(\psi, t)) + \]

\[ \frac{1}{2}(x_2 - X_2(\psi, t))^2 \frac{\dot{P}_1P_2 - \dot{P}_2P_1}{\dot{P}_1X_2 - \dot{X}_2P_1} \]

\[ J_j^{(0,2)}(\psi, t) = \det C^{0,2}(\psi, t) = (\dot{P}_1X_2 - \dot{X}_2P_1)(\psi, t) \] (4.17)

\[ S_j^{(1,0)}(\psi, t) = -P_2(\psi, t)X_2(\psi, t) + P_1(\psi, t)(x_1 - X_1(\psi, t)) + \]

\[ \frac{1}{2}(x_1 - X_1(\psi, t))^2 \frac{\dot{P}_1P_2 - \dot{P}_2P_1}{X_1P_2 - \dot{P}_2X_1} \]

\[ J_j^{(1,0)}(\psi, t) = \det C^{1,0}(\psi, t) = (\dot{X}_1P_2 - \dot{P}_2X_1\psi)(\psi, t, t) \] (4.18)

**Remark 6.** It is important that the last formulas do not depend on the choice of the vector \( q \) with the same accuracy they are valid.

**Theorem 5.** The proposition of Theorem 1 is valid if one changes in the formulas \( S_j^{I_j}, J_j^{I_j}, \) by \( S_j, J_j \), \( \psi_j(x^{I_j}), \psi_j(x^{I_j}, t), e(\gamma_j(x^{I_j}, t) - t) \) by \( e(|x^{I_j} - x^{I_j}(\psi_j(x^{I_j}, t)|. \) In the singular maps in formulas \( 1.19 \) one can change the integration over \( p_j \in (-\infty, \infty) \) by the integration over the angle \( \psi \in \Omega_j^{I_j} \), putting \( p_j = P_j(\psi, t) \) and \( dp_j = \frac{\partial P_j}{\partial \psi}(\psi, t) d\psi \) adjusting the limits in the integral with these change.

**The idea of Proof.** The proof in regular maps is no more but the Taylor expansion of regular components in \( (1.9), (1.2) \) with respect to distance from \( \gamma_t \). The proof in the focal maps based on Taylor expansions but also on estimates of some rapidly oscillating integrals.

### 4.4 Derivation of formulas from Theorems 1-3

Now let us apply this Theorem in different cases.

**a. A verification of initial data** We shall start from \( t = 0 \). It was shown in Example \( 3 \) that one can cover the neighborhood of the curve \( \Gamma_0 \) by maps \( \Omega_j^{1,0} \) and \( \Omega_j^{0,2} \). Taking into account the definition of the functions \( e_j \) the equalities \( P(\psi, 0) = n(\psi), X(\psi, 0) = 0, \dot{P}(\psi, 0) = -C_x(0) = -C_0, \dot{X}(\psi, 0) = C_0n(\psi) \) and \( m(\Omega_j^{1,0}) = \)

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The following asymptotic equality is true:

\[
\text{Sketch of Proof.}
\]

Now let us put \( h = \rho/\mu \) and integrate this expression over \( \rho \) from 0 to \( \infty \).

**Lemma 17.** The following asymptotic equality is true:

\[
\sqrt{\frac{\mu C_0}{2\pi}} \text{Re}(e^{-i\pi/4} \int_0^\infty d\rho \sqrt{\rho} K_{\mu C_0}^{1/\rho} \tilde{V}(\rho n(\psi)e)|_{t=0} = V(x/\mu) + O(\mu).}
\]

**Sketch of Proof.** Let us first consider the last integrals without the factors \( \exp\left(-i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \sin^2 \psi}\right)e(x_1) \) and \( \exp\left(i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \cos^2 \psi}\right)e(x_2) \). Then taking into account the equality \( \sum_j e_j = 1 \) we obtain the integral

\[
\frac{i}{\sqrt{2\pi \mu C_0}} \int_0^\infty \int_0^{2\pi} \frac{e^{ip(x_1 \cos \psi + x_2 \sin \psi)}}{n} (\tilde{V}(\rho n(\psi)) \rho d\psi dp,
\]

which without the factor \( \frac{i}{\sqrt{\mu C_0}} \) is no more but the inverse Fourier transform of the function \( \tilde{V}(p) \). Thus this integral is \( V(x/\mu) \). Now we need to proof that the factors \( \exp\left(-i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \sin^2 \psi}\right)e(x_1) \) and \( \exp\left(i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \cos^2 \psi}\right)e(x_2) \) change the function \( \tilde{V}(p) \) only by \( O(\sqrt{\mu}) \). Consider for instance the expression corresponding to the first integral in (4.19)

\[
\int_0^\infty \int_{-\pi/4-\varepsilon}^{\pi/4+\varepsilon} e^{i(x_1 \cos \psi + x_2 \sin \psi)} \tilde{V}(\rho n(\psi))(1 - \exp\left(-i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \sin^2 \psi}\right))e_j(\psi) \rho d\psi dp
\]

\[m(\Omega_{j}^{0,2}) = 0\] we easily find in these maps

\[
K_{\mu C_0}^{1/\rho} \left| \begin{array}{c}
\int_{\pi/4-\varepsilon}^{\pi/4+\varepsilon} e^{i(x_1 \cos \psi + x_2 \sin \psi)} f(\rho, \psi) \exp\left(-i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \sin^2 \psi}\right)e_j(\psi) \rho d\psi dp
\end{array} \right|_{t=0} = \frac{i}{\sqrt{2\pi hC_0}} \times
\]

\[
\int_{-\pi/4-\varepsilon}^{5\pi/4+\varepsilon} e^{i(x_1 \cos \psi + x_2 \sin \psi)} f(\rho, \psi) \exp\left(-i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \sin^2 \psi}\right)e_j(\psi) \rho d\psi dp
\]

\[
\int_{3\pi/4-\varepsilon}^{\pi/4+\varepsilon} e^{i(x_1 \cos \psi + x_2 \sin \psi)} f(\rho, \psi) \exp\left(i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \cos^2 \psi}\right)e_j(\psi) \rho d\psi dp
\]

\[
\int_{\pi/4-\varepsilon}^{7\pi/4+\varepsilon} e^{i(x_1 \cos \psi + x_2 \sin \psi)} f(\rho, \psi) \exp\left(i \frac{x_2^2(C(0), n(\psi))}{2hC_0 \cos^2 \psi}\right)e_j(\psi) \rho d\psi dp
\].
We divide this integral into two parts with the help of the cut off functions $e_p(\rho/\sqrt{h})$ and $1 - e_p(\rho/\sqrt{h})$. Here $e_p(z) = 1$ for $z < 1/2$ and $e_p(z) = 0$ for $z > 1$. It is simple to estimate the first part, which gives $O(h)$. The second part we can integrate by part which gives the required estimate. The estimates of other integrals are similar.

The verification of the second condition in (4.3) established by differentiation and similar calculations.

**Remark 7.** The constructed solution could be decomposed into two parts. The first part is based on the used construction but with the different function $f$. The second part is similar, but based on characteristics and family of invariant Lagrangian bands associated with the Hamiltonian $-|p|C(x)$. But all calculations related with this part is equivalent to complex conjugation of the second one, which allows one not go beyond to this negative Hamiltonian.

**b. Asymptotics corresponding to regular points.** Now consider the neighborhood of the regular point. Then each component from the first sum in (4.4) gives the following component in the solution (4.2)

$$
\frac{\sqrt{2\pi IC_0}}{\sqrt{|J_j^{(1,2)}(\psi, t)|}} \text{Re}\{e^{-i\pi/4 - i\frac{\pi}{2}m(\Omega_j^{(1,2)})} \int_0^\infty \sqrt{\rho} \tilde{V}(\rho n(\psi)) \exp \frac{ipS_j^{(1,2)}(x_1, x_2, t)}{\mu} d\rho \times e_j(\psi)e^\ell(x - X(\psi, t))|_{\psi = \psi^j(x_1, x_2, t)}.
$$

The Jacobian $|J_j^{(1,2)}(\psi, t)|$ is $|\dot{X}|X_\psi = C(X)|X_\psi|$, index $m(\Omega_j^{(1,2)}) = m(\psi^j(x, t), t)$ and the integral can be presented as a function $F(S_j^{(1,2)}(x_1, x_2, t), \psi^j(x, t)))$. The function $F(z, \psi)$ decays at least as $|z|^{-3}$ as $|z| \to \infty$. Thus the solution is localized in the neighborhood of the front $\gamma$ and with the accuracy $O(\mu^{3/2})$ one can omit the cut off functions in this neighborhood. Also without disturbing of this accuracy one can change $S_j^{(1,2)}(x_1, x_2, t)$ by its linear part $(P(\psi, t), (x - X(\psi, t)))|_{\psi = \psi^j(x, t)}$; the quadratic correction to the phase changes the solution by $O(\mu^{3/2})$ only. This gives the proof of Theorems 1 and 2.

**c. Asymptotics corresponding to singular maps.** Finally let us consider briefly and without rigorous estimates the behavior of the solution in the neighborhood of a nondegenerated focal point. To this end we fix the time $t$ and the angle $\psi^F(t)$ determining the focal point $P^F = P(\psi^F(t), t), X^F = X(\psi^F(t), t)$ and rotated coordinates choosing the direction of the new axis $x_2'$ coinciding with the vector $X(\psi^F, t^F)$. It gives us opportunity to make all consideration in the map with $I = (0, 2)$ and use formulas (4.8). Thus we need to investigate and to simplify the integral

$$
e^{-\frac{i\pi}{4}m(\Omega_j^{(0,2)})} \frac{i}{\sqrt{2\pi h}} \int_{-\infty}^{+\infty} \exp\left(\frac{i(S_j^{(0,2)}(p_1', x_2', t) + x_2'p_1')}{h}\right) f(\rho, \psi)e_j(\psi)|_{\psi = \psi_j(p_1')}e^\ell(x_2 - X_2') dp_1.
$$

Like in the case for $t = 0$ we change the integration over $dp_1'$ by the integration over $d\psi = dy$. Taking into account formulas (2.13) we find:

$$S_j^{(0,2)}(p_1', x_2', t) + x_2'p_1'|_{p_1' = P_1'} =
$$

$$P_1'(x_1' - X_1') + P_2'(x_2' - X_2') + \frac{1}{2}(x_2' - X_2'(\psi, t))^2 \frac{P_1'P_2' - P_1P_2'}{P_1'X_2' - X_2'P_2'} =
$$

$$-q(n + 1)y x_1' + qan x_2' + qaby^{n+1} + O(y^{n+2}) + O(y^2|x'|) + O(x_2' - by^{n+1})^2
$$
According to (2.14) the Jacobian $J_j^{(0,2)}(p'_1, x'_2) = \tilde{J}_F + O(y)$. We put $dP'_1 = -qb(n + 1)dy$, which gives in the integral the factor $|qb(n + 1)|$ and the same limits $-\infty, \infty$. It is possible to prove that in the neighborhood of the focal point (depended on $\mu$) after integration over $\rho$ the corrections denoted as $O(\cdot)$ gives the small correction to the leading term of asymptotic of $\eta$. Taking into account this fact and also the fact of fast decaying of $\eta$ outside of the neighborhood of the front $\gamma_t$ we can omit the cut-off functions $e_j$, $e^t$. Finally we come to the following formula in the neighborhood of the focal point $X^F$

$$\eta_j^F = \frac{|qb(n + 1)|\sqrt{\mu C_0}}{\sqrt{2\pi |\tilde{J}_F|}} \times \Re\{e^{-i\pi/4-i\pi/8}m(0,2)} \int_0^\infty d\rho \int_{-\infty}^{\infty} dy \tilde{V}(\rho n(\psi^F)) \times$$

$$\exp(i\rho(-qb(n + 1)y x'_1 + qan x'_2 + gaby{n+1}) \mu(4.21)$$

Now we express the coefficients $a, b, q$ via $J^{(n)}, \tilde{J}_F, C_0, C_F$ using formulas (3.8) and instead $y$ introduce the variable $\xi = \frac{|\tilde{J}_F J^{(n)}|}{\mu C_F} y$. After some algebra we obtain formula (2.15).

**Remark 8.** About the quadratic terms (corrections) in the phases and regularization of asymptotic formulas. One can see that actually the quadratic terms disappear in the final formulas for the leading term of asymptotics. Nevertheless they play important role in our construction. First, without this term it is impossible to show that the asymptotic solution satisfy the original equation with necessary accuracy. Second the passage from representation acting in the regular maps to the representation acting in the singular maps is based on the partial Fourier transform and stationary phase method (see [25, 28, 41, 19]). It is well known that the application of the second one based under assumption of nondegeneracy of second derivatives. Thus it is necessary to preserve the quadratic terms in proofs, and it is possible to omit the only in final formulas. It is also possible to proof that instead of the germ $\Lambda^2_t$ one can choose the germ $\tilde{\Lambda}^2_t$. This gives another asymptotic solution, but with the same leading term. More over this choice can be simplify the verification of the initial data because the exponential function containing $x'_1$ and $x'_2$ disappear (there are no quadratic with respect to $x_1$ and $x_2$ corrections in the phases). Thus this choice can be viewed as the other regularization, but all of them are natural in such a sense that they based on the more precise asymptotic construction.

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