On the stability of quantum holonomic gates

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Abstract. We provide a unified geometrical description for analyzing the stability of holonomic quantum gates in the presence of imprecise driving controls (parametric noise). We consider the situation in which these fluctuations do not affect the adiabatic evolution but can reduce the logical gate performance. Using the intrinsic geometric properties of the holonomic gates, we show under which conditions of the noise’s correlation time and strength the fluctuations in the driving field cancel out. In this way, we provide theoretical support for previous numerical simulations. We also briefly comment on the error due to the mismatch between the real and the nominal time of the period of the driving fields and show that it can be reduced by suitably increasing the adiabatic time.

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1. Introduction

In recent years, there has been renewed interest in geometric phases and their application to quantum information [1–7], including several solid state experiments [8–11]. These new results could open the way to the realization of ‘holonomic quantum computation’ in which the quantum information is only manipulated by means of geometric operators. This interesting line of research was opened by the seminal paper of Wilczek and Zee [12] and later on by Zanardi and Rasetti [13, 14].

Holonomic computation is based on a Hamiltonian \( H \) depending adiabatically and periodically on time via a set of time-dependent parameters \( x(t) \equiv (x_1(t), \ldots, x_n(t)) \). Interesting implementation proposals were initially based on atoms and ions [15, 16] and later based on many other quantum systems [17–22]. All these models have a Hamiltonian of the form

\[
H(x) = f(r) \hat{H}(\hat{x}),
\]

where \( f \) is a real-valued function of \( r \), the norm of \( x \in \mathbb{R}^n \) and \( \hat{x} \) is the unit vector \( x/r \). This structure in atomic physics is usually referred to as a tripod Hamiltonian.

The logical operation associated with (1) depends only on the shadow that the curve \( x = x(t) \) projects onto the surface of the unit sphere \( S^{n-1} \) (more specifically on the solid angle spanned by it); moreover, the dependence of the Hamiltonian on \( f(r) \) can be transformed away by a suitable (time-dependent) projective transformation in the Hilbert space of the system. Thus the norm \( r = r(t) \) does not need to be periodic—only periodicity of \( \hat{x} = \hat{x}(t) \) is required—and this, in turn, ensures that any mismatch between \( r(0) \) and \( r(T) \) does not affect the performance of the gate.

For the sake of concreteness, one may think of the specific implementation proposal for a quantum dot driven by ultrafast lasers [20], modulated in amplitude and phase with the quantum information stored in the excitonic degree of freedom. However, the same formalism and results hold for all other proposals [15–22].

For this purpose, the relevant Hamiltonian is

\[
H(t) = x(t) \cdot b,
\]

with \( x(t) \in \mathbb{R}^3 \) and a suitable matrix-valued vector \( b \). More precisely, we have the following structure.

1. The quantum dot has a level structure of three excited degenerate states \( |i\rangle \) \((i = 1, 2, 3)\) at energy \( \epsilon \), and a ground state \( |0\rangle \), set for convenience at energy 0.
2. The system is driven by time-dependent laser fields, with frequency in resonance with $\epsilon$, inducing transitions between ground and excited states. In the interaction representation, the Hamiltonian governing the dynamics of the system is (2), where $b$ is the matrix-valued vector with components

$$b_i = |0\rangle\langle i| + |i\rangle\langle 0|.$$  

(3)

3. The components of the vector $x = x(t)$ represent the amplitudes of the three laser driving fields which are slowly varying functions of time. Moreover, the motion of the unit vector $\hat{x}(t)$ is periodic of period $T$, i.e.

$$\hat{x}(0) = \hat{x}(T) \equiv \hat{x}_0.$$  

(4)

Note that we are not assuming periodicity of $r(t)$.

We refer to the final geometric transformation as the gate operator $G$, which is defined as the adiabatic limit of the evolution generated by (2), for a suitable logical space $L \subset \mathbb{C}^4$ (we recall that ‘logical space’ stands here for ‘qubit’ or, equivalently, for ‘two-dimensional complex space $\mathbb{C}^2$’):

$$G = \text{ad-lim} \ U(T)|_L,$$  

(5)

where $U(t)$ is the evolution operator in $\mathbb{C}^4$ generated by (2), i.e. the solution of ($\hbar = 1$ from now on)

$$i \frac{dU}{dt} = H(t) \ U, \quad U(0) = I,$$  

(6)

where $I$ is the identity operator.

The main advantage in using the gate operator $G$ to manipulate quantum information is that it has an intrinsic robustness under different kinds of error. In fact, its performance can be optimized in the presence of environmental effects and decoherence [23–31]. Similar properties and geometric interpretations have been discussed for the Abelian geometric phases for which a full analytical treatment exists [32]. These include environmentally induced geometric [33] and non-adiabatic non-Markovian contributions [34]. Moreover, numerical investigations show that geometric operators are robust against fluctuations of the driving fields $x(t)$ [35–37], the so-called parametric noise. However, no analytic treatment has been provided so far.

Here we provide a first-principles explanation of such a robustness. We analyze the stability of the gate operator $G$ under different sources of error which can decrease the logical gate performance. We identify two kinds of error induced by the fluctuations of the driving fields: the error in switching off the driving fields and that accumulated during the evolution. These errors are supposed to be weak enough to avoid any loss of adiabaticity, i.e. they do not produce any transition between non-degenerate states, but they can decrease the logical gate performance. Using the intrinsic geometric properties of the holonomic gates, we show that both errors can be reduced in the adiabatic limit, thus recovering the robustness of the holonomic transformation.

The paper is organized as follows. In section 2, we set the stage for the study of the stability of $G$. In particular, we study the adiabatic regime using the language of scaling limits and introduce the dimensionless adiabatic scaling parameter $\epsilon$. In section 3, we highlight the geometrical aspects of the problem; in particular, we stress the utility of adopting a geometrical extrinsic point of view in describing the geometrical features of $G$. In section 4, we discuss the robustness of the holonomic gate operator in the presence of parametric noise. In section 5, we present the conclusions.
2. Adiabatic regime

In this section, we solve (6) in three steps.

First, we observe that \( H(t) \) in (2) is a unitary transformation of \( H(0) \) multiplied by a scale factor, i.e.

\[
H(t) = \alpha(t) R(t)^{-1} H(0) R(t). \tag{7}
\]

To see how this comes about, consider the time dependence of the unit vector \( \hat{x} = \hat{x}(t) \) expressed in terms of spherical coordinates \( \theta = \theta(t) \) and \( \phi = \phi(t) \) with respect to an orthonormal basis \( i \equiv (i_1, i_2, i_3) \) in the space \( \mathbb{R}^3 \) of the driving parameters, i.e.

\[
\hat{x}(t) = \sin \theta \cos \phi i_1 + \sin \theta \sin \phi i_2 + \cos \theta i_3, \tag{8}
\]

and regard \( \mathbb{R}^3 \) as embedded in \( \mathbb{C}^4 \) according to the identifications \( i_1 = |1\rangle, i_2 = |2\rangle, i_3 = |3\rangle \), to which one may add, for uniformity of notations, the stipulation \( i_0 = |0\rangle \).

Let

\[
D(t) = \begin{bmatrix}
\cos \theta(t) \cos \phi(t) & \cos \theta(t) \sin \phi(t) & -\sin \theta(t) \\
-\sin \phi(t) & \cos \phi(t) & 0 \\
\sin \theta(t) \cos \phi(t) & \sin \theta(t) \sin \phi(t) & \cos \theta(t)
\end{bmatrix}, \tag{9}
\]

in the \((i_1, i_2, i_3)\) basis, and

\[
D(t) = \begin{bmatrix}
1 & 0 \\
0 & \mathcal{D}(t)
\end{bmatrix} \tag{10}
\]

in the \((i_0, i_1, i_2, i_3)\) basis. Then one may easily check that (7) is satisfied for

\[
R(t) \equiv D(0)^{-1} D(t) \tag{11}
\]

and scale factor

\[
\alpha(t) = r(t)/r(0). \tag{12}
\]

The second step consists in writing the equation of motion in the moving frame associated with \( R(t) \). This is realized by the change of variables

\[
V(t) = R(t) U(t), \tag{13}
\]

whence by (6) and (7),

\[
\frac{dV}{dt} = A(t)V + B(t)V, \tag{14}
\]

with

\[
A(t) \equiv i \frac{dR(t)}{dt} R(t)^{-1} \quad \text{and} \quad B(t) \equiv \alpha(t) H(0). \tag{15}
\]

The third step exploits the physical assumption that the external fields are slowly varying functions of time. Since the energy scale of (2) is determined by \( r \) to characterize the evolution, we introduce the adiabatic parameter

\[
\varepsilon = \frac{1}{\bar{r} T}, \tag{16}
\]

where \( \bar{r} \) is an estimate of the size of \( r(t) \) (say, the minimum value of \( r(t) \) during the drive). The parameter \( \varepsilon \) must, of course, be small in order to avoid transitions between non-degenerate states.
For such an adiabatic regime the time dependence of the curve in $\mathbb{R}^3$ should be better regarded as given by $\mathbf{x} = \mathbf{x}(\varepsilon t)$, with the function $\mathbf{x}(t)$ having a scale of variation of the order of one. The time for the application of the geometric transformation is of the order of $1/\varepsilon$. Note that the derivative of $R(t)$ leads to the rescaling $A(t) \rightarrow \varepsilon A(\varepsilon t)$, so that the dynamical problem in the adiabatic regime becomes that of solving the rescaled equations of motions for $V(t)$

$$i \frac{dV(t)}{dt} = \varepsilon A(\varepsilon t)V(t) + B(\varepsilon t)V(t)$$

(17)

for $\varepsilon \ll 1$.

At any given time, the Hamiltonian (2) can be diagonalized (the explicit time dependence plays no role in diagonalization and we shall omit it in the notations). Consider a vector $u$ in $\mathbb{C}^4$ written as $u = u_0 \mathbf{i}_0 + u$, where $u = u_1 \mathbf{i}_1 + u_2 \mathbf{i}_2 + u_3 \mathbf{i}_3$ and $\mathbf{x} = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + x_3 \mathbf{i}_3$. From the definitions of $\mathbf{i}_j$ and $b_j$ in (3), we obtain $b_j \mathbf{i}_k = \delta_{jk} \mathbf{i}_0 + \delta_{0k} \mathbf{i}_j$. Immediately from (2), the eigenvalues equation reads

$$H[u_0 \mathbf{i}_0 + u] = x \cdot u \mathbf{i}_0 + u_0 x$$

(18)

from which eigenvalues and the eigenvectors of $H$ follow: taking $u = \hat{x}$, then from (18)

$$H[u_0 \mathbf{i}_0 + \hat{x}] = r[\mathbf{i}_0 + u_0 \hat{x}]$$

Thus, $u_0 = 1$ gives the eigenvalue $\lambda_+ = r$, and $u_0 = -1$ gives the eigenvalue $\lambda_- = -r$. The corresponding normalized eigenvectors are, respectively,

$$e_\pm = (1/\sqrt{2})(\hat{x} \pm \mathbf{i}_0).$$

(19)

Finally, for $u_0 = 0$ and $u$ orthogonal to $\hat{x}$ (as vectors in $\mathbb{R}^3$), we find from (18) that $\lambda_0 = 0$ is a doubly degenerate eigenvalue. Thus, we have recovered the well-known properties of the tripod Hamiltonian (2): there are two degenerate states, at zero energy, called ‘dark states’ and two others, called ‘bright states’, with one excited state ($e_+$ in (19)) and one ground state ($e_-$ in (19)) with energy $r$ and $-r$, respectively. In the following, we shall denote by $P_\beta$, $\beta = +1, -1, 0$, the spectral projectors of $H(0)$ corresponding respectively to the eigenvalues $\lambda_+ = r(0)$, $\lambda_- = -r(0)$ and $\lambda_0 = 0$. $P_0$ projects onto the plane $L$ orthogonal to $\hat{x}(0)$ which will soon be identified with the logical space in (5).

Consider now the interaction representation of the time evolution operator $V(t)$, whose dynamics is governed by (17), with respect to the free dynamics generated by $B$

$$V_I(t) = W(t)V(t),$$

(20)

with

$$W(t) = e^{\int_0^t B(\varepsilon \tau') d\tau'} = e^{\frac{\varepsilon}{2}\int_0^t H(0)} = \sum_\beta e^{\frac{\varepsilon}{2}H(0)\lambda_\beta} P_\beta,$$

(21)

where (see (12))

$$h(t) = \int_0^t \alpha(\tau) d\tau.$$

(22)

Then from (17) it follows that $V_I(t)$ satisfies

$$i \frac{dV_I(t)}{dt} = \varepsilon W(t)A(\varepsilon t)W(t)^{-1}V_I(t),$$

(23)
or, equivalently,

\[ V_I(t) = I + \int_0^{\varepsilon t} W(s/\varepsilon)A(s)W(s/\varepsilon)^{-1}V_I(s/\varepsilon) \, ds. \tag{24} \]

Equation (24) shows that the effect of \( A \) on the evolution manifests itself only on the adiabatic time scale \( t_\varepsilon = t/\varepsilon \). Thus, setting \( V_I^\varepsilon(t) = V_I(t/\varepsilon) \) one obtains

\[ V_I^\varepsilon(t) = I + \sum_{\beta, \beta'} \int_0^{\varepsilon t} e^{i\frac{\hbar}{\varepsilon}(\lambda_\beta - \lambda_{\beta'}) P_\beta A(s) P_{\beta'} V_I^\varepsilon(s) \, ds. \tag{25} \]

Since \( h(t) \) is positive and never equal to zero, standard stationary phase approximation gives \( \beta' = \beta \) and

\[ V_I^\varepsilon(t) = I + \sum_{\beta} \int_0^{\varepsilon t} P_\beta A(s) P_\beta V_I^\varepsilon(s) \, ds + O(\varepsilon). \tag{26} \]

By multiplying both sides of the above equation by \( P_\alpha \) we see that the evolution separates into autonomous spectral components. In particular, since \( P_0 \) projects onto the plane \( L \), (26) defines a non-trivial dynamics on it. In other words, the operator

\[ G(t) \equiv P_0 V_I(t) P_0 \tag{27} \]

evolves autonomously in the adiabatic limit (modulo corrections of the order of \( \varepsilon \)) according to the equation

\[ i \frac{dG}{dt} = A(t) G, \tag{28} \]

where, recalling (15),

\[ A(t) = i P_0 \frac{dR(t)}{dt} R(t)^{-1} P_0. \tag{29} \]

3. Geometry

Equation (29) can be solved analytically using a geometric approach. By defining the operator-valued vectors \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{a}^\ast = (a_1^\ast, a_2^\ast, a_3^\ast) \), having components \( a_i = \langle 0 | i \rangle \) and \( a_i^\ast = \langle i | 0 \rangle \), \( i = 1, 2, 3 \), respectively. The Hamiltonian in (2) can be written as \( H(t) = \mathbf{a} \cdot \mathbf{x}(t) + \mathbf{a}^\ast \cdot \mathbf{x}(t) \). Another operator-valued vector that it is useful to introduce is \( \mathbf{J} = \mathbf{a}^\ast \times \mathbf{a} \), whose components

\[ J_1 = a_2^\ast a_3 - a_3^\ast a_2, \]
\[ J_2 = a_3^\ast a_1 - a_1^\ast a_3, \]
\[ J_3 = a_1^\ast a_2 - a_2^\ast a_1 \tag{30} \]

are indeed the generators of an \( SO(3) \) algebra with commutation relations \([J_i, J_j] = -\varepsilon_{ijk} J_k\) and in terms of which \( D(t) \) in (10) can be expressed as

\[ D(t) = e^{-\theta(t)J_2} e^{-\phi(t)J_3}. \tag{31} \]
The frame \( e(t) = (e_0(t), e_\phi(t), e_r(t)) \), where

\[
e_0 = i_0, \\
e_r(t) = \hat{x}(t), \\
e_\phi(t) = \cos \theta(t) \cos \phi(t) i_1 + \cos \theta(t) \sin \phi(t) i_2 - \sin \theta(t) i_3, \\
e_\phi(t) = -\sin \phi(t) i_1 + \cos \phi(t) i_2
\]

is indeed a moving frame adapted to the surface of the unit sphere \( S^2 \) on which \( \hat{x}(t) \) moves in the course of time, i.e. \( e_\phi(t) \) and \( e_\phi(t) \) are tangent to \( S^2 \), and \( e_r(t) \) is perpendicular to it. Then (31) is the operator transforming the frame \( i = (i_0, i_1, i_2, i_3) \) into the frame \( e(t) \),

\[
D(t)^{-1} i_1 = e_\phi(t), \quad D(t)^{-1} i_2 = e_\phi(t), \quad D(t)^{-1} i_3 = e_r(t).
\]

Note that \( D(0)^{-1} \) is not the identity and, in particular, that \( D(0)^{-1} i_3 = \hat{x}(0) \), whence from (11)

\[
\hat{x}(t) = e_r(t) = D(t)^{-1} D(0) \hat{x}(0) = R(t)^{-1} \hat{x}(0).
\]

Then (7) is nothing but an expression of the usual duality between the action of the operators on vectors and on operators, i.e.

\[
\hat{x}(t) \cdot b = [R(t)^{-1} \hat{x}(0)] \cdot b = \hat{x}(0) \cdot [R(t) b R(t)^{-1}],
\]

and \( \hat{x}(0) \cdot b = D(0)^{-1} b_3 D(0) \).

The unit vectors \( e_\phi(t) \) and \( e_\phi(t) \) (the ‘dark states’) are a natural basis in the moving degenerate space—the plane orthogonal to \( e_r(t) = \hat{x}(t) \), which is, according to section 2, the degenerate eigenspace of the eigenvalue 0 of \( H(t) \). Let \( P_0(t) \) denote the projector onto such a plane; then \( P_0(0) \) is the projector \( P_0 \) onto the \( \mathbb{L} \)-plane. Note that \( P_0 \) is just a \( D(0) \)-rotation of the projector \( P(i_1 i_2) = a_1^* a_1 + a_2^* a_2 \) onto the \( i_1 - i_2 \) plane; thus, \( P_0 = D(0)^{-1} P(i_1 i_2) D(0) \).

Putting all the pieces together, from (11), (28) and (29) we obtain

\[
A(t) = i D(0)^{-1} P(i_1 i_2) \frac{dD(t)}{dt} D(t)^{-1} P(i_1 i_2) D(0),
\]

with (see (31))

\[
\frac{dD(t)}{dt} D(t)^{-1} = -\dot{\theta} J_2 - \dot{\phi} (\sin \theta J_1 + \cos \theta J_3).
\]

Using (31) and the above definition of \( P(i_1 i_2) \), we can calculate the projection \( P(i_1 i_2) J_k P(i_1 i_2) = \delta_{k3} J_k \) and we are left with

\[
A(t) = i (\dot{\phi} \cos \theta) D(0)^{-1} J_3 D(0).
\]

Observing (31) at time \( t = 0 \) and using the rotational properties of \( J_k \), we have

\[
D(0)^{-1} J_3 D(0) = J \cdot \hat{x}(0)
\]

and, finally, we arrive at a geometric expression

\[
A(t) = i (\dot{\phi} \cos \theta) J \cdot \hat{x}(0).
\]

We have at our disposal all the ingredients to determine the geometric operator that is used to manipulate the quantum state. Now, since \( A(t) \) at different times commutes, equation (28) can be solved by direct exponentiation

\[
G(t) = \exp \left( i \int_0^t \cos \theta(t') \dot{\phi}(t') \, dt' \, J \cdot \hat{x}(0) \right).
\]
One may recognize that $G(t)$ is a rotation in the $\mathbb{L}$-plane of an angle given by the integral multiplying $J \cdot \hat{x}(0)$ in (36). At this point, starting from (27) and going back from (36) to the moving frame by means of equation (20) is immediate: $W(t)$ is just the identity on $\mathbb{L}$ since the corresponding eigenvalue $\lambda_0$ is zero. To go back to the laboratory frame, we follow (13) and obtain

$$\text{ad-lim} \ U(t)|_{\mathbb{L}} = R(t)^{-1}G(t).$$  

(37)

From the assumption (4) of periodicity of $\hat{x}(t)$, it follows that $R(T) = I$. Therefore

$$G = \text{ad-lim} \ U(T)|_{\mathbb{L}} = G(T).$$  

(38)

If we choose a logical basis directly in $\mathbb{L}$ we can write $G(t)$ in (36) as a $2 \times 2$ matrix. Thus (modulo a change of basis in $\mathbb{L}$),

$$G = G(T) = \begin{bmatrix} \cos \Omega & \sin \Omega \\ -\sin \Omega & \cos \Omega \end{bmatrix},$$

(39)

where

$$\Omega = \int_0^T \cos \theta(t) \phi(t) \, dt.$$  

(40)

This is the expected result: the $G(T)$ operator depends only on the solid angle spanned by $x(t)$ on the Hamiltonian parameter space. Note that this result does not rely on the assumption of full periodicity ($x(0) = x(T)$) made by Wilczek–Zee and Zanardi–Rasetti, although it holds only for a restricted class of Hamiltonians.

4. Stability

We shall now consider the effects of external perturbations on the system, in order to understand the robustness of quantum evolution in the presence of noise. There are several sources of undesired energy exchange due to coupling with impurities, and/or external environments, or due to imprecise control of the system parameters during the evolution. In the following, we focus on this parametric error. First of all, we observe that the good performance of the gate $G$ relies on the validity of the adiabatic limit, and since the dimensionless constant $\varepsilon$ is finite, this fact alone introduces an error of the order of $\varepsilon$, i.e. $O(\varepsilon)$, which is the order of magnitude of the off-diagonal terms ($\beta \neq \beta'$) in (25) which are neglected in the limit $\varepsilon \to 0$. Then we shall say that the gate $G$ is stable if the perturbations produce corrections of higher order in $\varepsilon$.

Setting aside corrections not following a power law, we can say that stability is ensured if the corrections on $G$ due to perturbations are $O(\varepsilon^r)$, with $r > 1$. To simplify the analysis, we will work with dimensionless quantities: energy scale in units of $\hbar$, so that (16) becomes $\varepsilon = 1/T$. We consider the error induced by the inaccuracy of the control field. We start with the case in which the actual curve in the parameter space is not $x(t)$ but instead $x'(t) = x(t) + \delta x(t)$, still fulfilling the periodicity requirement (4) on the unit vector $\hat{x}'(0) = \hat{x}'(T)$. The error $\delta x(t)$ is assumed small with respect to $x(t)$ in the sense of some suitable functional norm. It should be regarded as a rapidly fluctuating random process whose scale of variation is very small on the adiabatic scale.

From the above periodicity it follows that the holonomic operator $U'(T)$, related to $x'(t)$, is still of the form

$$\text{ad-lim} \ U'(T)|_{\mathbb{L}} = G'(T) = G[x'(t)].$$

(41)
The geometric operator $G[x'(t)]$ is now a functional of $x'(t)$ and then the error is accumulated during the whole evolution.

In the following we evaluate at lowest order in $\delta x(t)$ the error on the holonomic operator $U'(T) = G[x(t) + \delta x(t)]$ induced by the fluctuations of the driving fields along the path. We have

$$U'(T) = G[x(t) + \delta x(t)] = G[x(t)] + \delta G + o(\sigma),$$

(42)

where $\sigma$ is a measure of the size of the (mean) variation of $\delta x(t)$. To do that, let us start rewriting the solid angle in (40) for the path $x'(t)$ as

$$\Omega[x'(t)] = \oint_{\hat{y}} \cos \theta \, d\phi = \int_{\Sigma} \sin \theta \, d\theta \, d\phi,$$

(43)

where $\hat{y}$ is the shadow on the unit sphere $S^2$ of the curve $y$ in $\mathbb{R}^3$ given by the parametric equations $x' = x'(t)$ and satisfying conditions (4) of partial periodicity, and $\Sigma$ is the surface on $S^2$ bounded by $\hat{y}$, i.e. $\hat{y} = \partial \Sigma$.

Thus, $\Omega$ is the area of $\Sigma$, that is, the solid angle spanned by curve $y$ (i.e. the solid angle from which $y$ is seen from the origin in $\mathbb{R}^3$). Accordingly, different unitary geometric transformations and then quantum logical gates can be constructed traversing different loops in the parameters space.

Since the north pole $\theta = \phi = 0$ is a singularity of spherical coordinates, one is naturally lead to consider the one-form on $S^2$, locally defined by $\omega = \cos \theta \, d\phi$, and extend it to all $S^2$ in a coordinate-independent way. Accordingly, (43) should be replaced by the Stokes theorem on $S^2$ expressed in an ‘intrinsic’ geometrical way (i.e. coordinate independent),

$$\Omega = \int_{\partial \Sigma} \omega = \int_{\Sigma} d\omega,$$

(44)

This is the approach usually adopted in holonomic computation [13, 14]. To take care of the singularity problem, one can rewrite (43) in terms of the auxiliary vector field $A = e_{\phi}(1 - \cos \theta)/(r \sin \theta)$

$$\Omega[x'(t)] = \oint_{\hat{y}} A \cdot dr = \iint_{\Sigma} B \cdot dS,$$

(45)

where

$$B = \nabla \times A = \frac{1}{r^2} e_r.$$

(46)

We can now rewrite (45) as an integral over time

$$\Omega[x'(t)] = \int_{0}^{T} A(\hat{x}'(t)) \cdot \hat{x}'(t) \, dt.$$

(47)

Noting that the right-hand side is analogous to the Lagrangian of a particle in a magnetic field given by (46), we have (see (36) and (39))

$$\delta G = i[GJ \cdot \hat{x}(0)] \delta \Omega,$$

(48)

with

$$\delta \Omega = \int_{0}^{T} B(\hat{x}(t)) \times \hat{x}(t) \cdot \delta x(t) \, dt.$$

(49)
Note, as expected, that radial fluctuations give no contribution to the variation since \( B \times \hat{x} \) is tangent to the sphere.

We now define the statistical properties of \( \delta x(t) \), which describes the parametric noise perturbing the external field. As already discussed, it should have a scale of variation very small on the adiabatic scale; however, if we wish to ensure the validity of the adiabatic approximation, we should demand, at the same time, that its scale of variation be sufficiently long on the microscopic scale. The simplest possibility to ensure this is to regard the components of \( \delta x(t) \) as independent mean-zero stationary Gaussian processes with a ‘white-noise’ correlation function, i.e.

\[
(\delta x_i(t) \delta x_j(t')) = \delta_{ij} \tau_i \sigma_i^2 \delta(t - t'),
\]

with \( i, j = 1, 2, 3 \) and where \( \sigma_i^2 \) is a measure of the strength of the noise components (time independent, as the processes are stationary) and \( \tau_i \)s are the correlation times of the noise components. Since the Gaussian processes are stationary, we can take \( \sigma_i^2 \equiv \langle \delta x_i(0)^2 \rangle \).

Again, it is convenient to consider \( \tau_i \) and \( \sigma_i \) as functions of \( \varepsilon \), e.g.

\[
\tau_i = O(\varepsilon^p), \quad \sigma_i = O(\varepsilon^q),
\]

with suitable exponents \( p > 0 \) and \( q > 0 \). Recalling that the intrinsic error of the adiabatic approximation is \( O(\varepsilon) \), we should then inquire whether there is a range of \( p, q \)-values for which the mean error

\[
\Delta = \sqrt{\langle \delta \Omega^2 \rangle - \langle \delta \Omega \rangle^2}
\]

is below the \( O(\varepsilon) \) upper bound, i.e. \( O(\varepsilon^r) \) with \( r > 1 \). Of course, \( \Delta \) would then provide an estimate of the (mean) first-order correction \( \delta \mathcal{G} \) in (42).

Since the fluctuations \( \delta x_i \) have zero mean, \( \langle \delta \Omega \rangle = 0 \). Moreover, from (49) and (50) it follows

\[
\langle \delta \Omega^2 \rangle = \sum_{i=1}^{3} \tau_i \sigma_i^2 \int_0^T \langle B \times \hat{x} \rangle_i^2 \, dt.
\]

Recalling (46), we find that

\[
\Delta^2 = \sum_{i=1}^{3} \tau_i \sigma_i^2 \int_0^T \dot{x}_i^2(t) \, dt.
\]

Since the velocity \( \dot{x}_i \) scales as \( 1/T \), the integral in the right-hand side of (54) is \( O(T \times 1/T^2) = O(1/T) \), i.e. \( O(\varepsilon) \), and the mean error is then

\[
\Delta = O(\varepsilon^r), \quad r = p/2 + q + 1/2.
\]

From (51) it follows that with this constraint there are many solutions \( p/2 + q + 1/2 > 1 \), in the desired range of \( p, q \)-values, which ensure stability of the gate. Moreover, from (55), we also read the answer to the questions about the smallness of \( \tau_i \) and the largeness of \( \sigma_i \), namely that the fluctuations can indeed be quite large with respect to \( \varepsilon \), provided that the \( \tau_i \)s are small on the macroscopic scale (but large on the microscopic scale). This is the cancelation effect already discussed in [35, 36]. For example, let \( \varepsilon = 10^{-4} \). Then a noise with correlation time of order, say, \( \tau = 10^{-2} \) \((p = 1/2)\) can have fluctuations of order, say, \( \sigma = 10^{-2} \) \((q = 1/2)\) while producing, at the same time, an error on the gate of the order of \( 10^{-5} \) \((p/2 + q + 1/2 = 5/4)\), well within the bound \( \varepsilon = 10^{-4} \) of the adiabatic regime.
We conclude this section with two observations. The first concerns the geometric interpretation of solid angle perturbation, with a simple geometrical formula for the right-hand side of (53). We focus on the case of noise components $\delta x_i(t)$ with the same statistical properties, i.e. $\sigma_i = \sigma$ and $\tau_i = \tau$ (a condition which is indeed quite reasonable from a physical point of view). Since $\Omega$ is invariant under re-parameterization of time, it is convenient to use as an invariant parameter the arc length $s$ with the origin in $\hat{x}_0$ and rewrite (47) as

$$\Omega[x'(s)] = \int_0^L A(\hat{x}'(s)) \cdot \hat{x}'(s) \, ds,$$

where $L$ is the length of the curve and $\dot{x} = dx/ds$. Then (53) becomes

$$\langle \delta \Omega^2 \rangle = \ell \sigma^2 \sum_{i=1}^3 \int_0^L [\mathbf{B} \times \dot{\hat{x}}]^2_i \, ds,$$

where $\ell$ is the correlation length of the noise. (Note that the integral is now $O(1)$, since $L$ is the length of the curve on $S^2$ and therefore $\ell = O(\varepsilon^{p+1})$). But, from (46), $\mathbf{B} = \mathbf{n}$, the normal to the curve lying on $S^2$, i.e.

$$\langle \delta \Omega^2 \rangle = \ell \sigma^2 \int_0^L |\mathbf{n} \times \dot{x}| \, ds,$$

and one recognizes

$$\delta A \equiv \sigma \int_0^L |\mathbf{n} \times \dot{x}| \, ds$$

as the area of a thick boundary along the curve of width $\sigma$. Thus, $\Delta^2 = \ell \sigma \delta A$.

Let us now turn to the second observation. It concerns how to deal with the difference between the nominal and the real values of the period of the laser. We recall that in the tripod model two control fields are turned off at the initial and final configurations $[15–22]$. However, in a practical implementation there is uncertainty in the real time $T$ in which the fields are turned off since, in general, this does not correspond to the nominal time $T_0$, with $T = T_0 + \Delta T$, in which $\Delta T$ is a statistical fluctuation time. This can be due, for example, to latent times or delays in the control fields of the experimental setup. In the parameter space this error corresponds to an evolution that, if referred to the nominal time, has $R^{-1}(T_0) \neq I$, since the periodicity requirement is fulfilled at time $T$, e.g. $R^{-1}(T) = I$. This means a path non-periodic at the nominal time. Thus, from (37)

$$U(T_0) = R^{-1}(T_0)G(T_0),$$

and therefore at the lowest order in $\Delta T$

$$R^{-1}(T_0) = R^{-1}(T - \Delta T) \approx I - \frac{\partial R^{-1}}{\partial T} \bigg|_T \Delta T. \quad (57)$$

For the partial derivative in the right-hand side of the previous equation, one has

$$\frac{\partial R^{-1}}{\partial T} \bigg|_T = \frac{\partial R^{-1}}{\partial \hat{x}} \bigg|_T \cdot \dot{x}(T). \quad (58)$$

Note that the possibility of having fluctuations at the final time corresponds to an open path in the parameter space. The more general definition of non-Abelian holonomy for open paths and its possible use in quantum computation have been addressed in [38].

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The $\partial R^{-1}/\partial \hat{x}$ depends only on geometrical properties of the path and thus is of the order of one, while the velocity is of the order of $1/T$ (since we are analyzing the system in the adiabatic regime). Thus the error due to the mismatch between the real and the nominal time of the period of the laser is of the order of $1/T \approx 1/T_0$ and can therefore be reduced by suitably increasing the nominal time $T_0$.

5. Conclusions

We have provided a unified geometrical description for analyzing the stability of holonomic quantum gates in the presence of parametric noise affecting their time evolution. We have identified two main critical parameters: the correlation time of the noise and its strength with respect to the driving field. In this way, we have recovered what was already obtained numerically for Abelian [35] and non-Abelian gates [36], namely that even strong fluctuations in the driving field can lead to accurate logical gates if the correlation time is short enough to let the fluctuations cancel out. This is an effect of the geometric dependence of the holonomic operator. In addition, we have shown that the error due to the mismatch between the real and the nominal time of the period of the laser can be reduced by suitably increasing the adiabatic time.

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