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Hawking radiation from a spherical loop quantum gravity black hole

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We introduce quantum field theory on quantum space-times techniques to characterize the quantum vacua as a first step towards studying black hole evaporation in spherical symmetry in loop quantum gravity and compute the Hawking radiation. We use as quantum space time the recently introduced exact solution of the quantum Einstein equations in vacuum with spherical symmetry and consider a spherically symmetric test scalar field propagating on it. The use of loop quantum gravity techniques in the background space-time naturally regularizes the matter content, solving one of the main obstacles to back reaction calculations in more traditional treatments. The discreteness of area leads to modifications of the quantum vacua, eliminating the trans-Planckian modes close to the horizon, which in turn eliminates all singularities from physical quantities, like the expectation value of the stress energy tensor. Apart from this, the Boulware, Hartle–Hawking and Unruh vacua differ little from the treatment on a classical space-time. The asymptotic modes near scri are reproduced very well. We show that the Hawking radiation can be computed, leading to an expression similar to the conventional one but with a high frequency cutoff. Since many of the conclusions concern asymptotic behavior, where the spherical mode of the field behaves in a similar way as higher multipole modes do, the results can be readily generalized to non spherically symmetric fields.

I. INTRODUCTION

The evaporation of a black hole is one of the most fascinating problems in fundamental theoretical physics today, as can be attested by the surge of activity related to “firewalls” in the last few months. A complete treatment of the evaporation requires a theory of quantum gravity. Loop quantum gravity is a contender for such a theory, but a complete treatment of the evaporation has proved elusive. Here we take an incremental step in its study by considering a quantum field theory on quantum space-time approach, studying a spherically symmetric scalar field propagating on the recently introduced exact solution for the quantum space-time of a vacuum spherically symmetric black hole. We will treat the matter field as a test field, as a first step towards a perturbative treatment of the evaporation via back-reaction. We consider the quantum states to be a direct product of states of gravity and states of matter. For the states of gravity we take the physical states constructed in that are annihilated by the Hamiltonian and diffeomorphism constraints of vacuum spherically symmetric gravity. We take the expectation value of the matter part of the Hamiltonian constraint on the exact quantum states of the gravitational field, and write it in terms of parameterized Dirac observables of the gravitational field. The resulting operator acts on the states of matter as a true Hamiltonian yielding quantum gravity corrected equations for the propagation of matter. The main quantum gravity correction consists in the discretization of the equations of motion as a consequence of the discrete structure of space in loop quantum gravity. We study the impact of the resulting changes on the various usual vacua for quantum fields in a Schwarzschild space-time. All of them suffer small modifications due to the discreteness. Also, all issues involving singularities of physical quantities at horizons are resolved by the discreteness. We study the Hawking radiation in terms of two point functions taking into account the discreteness induced by the quantization and show that the black body radiation is a robust property.

The organization of this paper is as follows. In section II we review the recent developments concerning the solution of vacuum spherically symmetric loop quantum gravity and its corresponding space of physical states. In section III we review the use of parameterized Dirac observables to represent variables in totally constrained systems and how they are applied in this case. In section IV discuss the Hamiltonian for the matter fields and how to realize it as a quantum operator on the space of physical states discussed in section II. Section V discusses the resulting equations of motion corrected due to the quantum background space-time. Section VI proceeds to discuss the various quantum vacua that are usually considered in the context of black holes in light of the corrected evolution equations. Section VII discusses the Hawking radiation. We end in section VIII with conclusions.

II. SPHERICALLY SYMMETRIC VACUUM GRAVITY

We briefly review spherically symmetric vacuum gravity, referring the reader to our previous work for further references. The Ashtekar-like variables adapted to spherical symmetry yield two pairs of canonical variables $E^r$, $K^r$ and $E^\phi$, $K^\phi$, that are related to the traditional canonical variables in spherical symmetry $ds^2 = \Lambda^2 dx^2 + R^2 d\Omega^2$.
by $\Lambda = E^x / \sqrt{|E^x|}$, $P_\Lambda = -\sqrt{|E^x|} K_x$, $R = \sqrt{|E^x|}$ and $P_R = -2\sqrt{|E^x|} K_x - E^x K_\varphi / \sqrt{|E^x|}$ where $P_\Lambda, P_R$ are the momenta canonically conjugate to $\Lambda$ and $R$ respectively, $x$ is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. We will take the Immirzi parameter equal to one.

As discussed in [3], rescaling and combining the Hamiltonian and diffeomorphism constraints leads to the following total Hamiltonian,

$$H_T = \int dx \left[ -N' \left( -\sqrt{E^x} (1 + K_\varphi^2) + \frac{(E^x')^2 \sqrt{E^x}}{4 (E^x)^2} + 2GM \right) \right.$$

$$\left. + N \left( \frac{1}{8} \frac{(E^x')^2 P_\phi^2}{\sqrt{E^x}} \right) + \frac{(E^x') (E^x)^{3/2} (\phi')^2}{2 (E^x)^2} - \frac{K_\varphi \sqrt{E^x} P_\phi \phi'}{E^x} \right) + N_r \left[ -(E^x') K_x + E^x K_\varphi' + P_\phi \phi' \right] \right]$$

(1)

where $N$ and $N_r$ are combinations of the original lapse and shift, i.e.,

$$N^{\text{orig}} = \frac{N (E^x')}{E^x},$$

$$N_r^{\text{orig}} = N_r - \frac{2K_\varphi \sqrt{E^x}}{(E^x)^2} N^{\text{orig}},$$

(2)

and we have added a scalar field characterized by the canonical pair of variables $\phi, P_\phi$.

The constraints introduced above close a Lie algebra, and this allows us to complete the Dirac quantization in closed form. For the quantum treatment of the vacuum theory we refer the reader to [4]. We would like to consider a state that approximates well a classical metric with a given value of the mass $M_0$, it will be given by a superposition

$$|\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = \int |\tilde{g}, \tilde{k}, M\rangle \psi(M) dM$$

(4)

with $\psi(M) = c \exp \left( \frac{-(M-M_0)^2}{2\sigma} \right)$ with $\sigma$ small compared to $M_0^2$ and $c$ a normalization constant. As a first step, we will assume that the spread of the mass is negligible and all expressions will be analyzed only at leading order in it, that is, we neglect fluctuations in the mass. The state $|\tilde{g}, \tilde{k}, M\rangle$ is an exact solution of the constraints as constructed in [3]. On such states $M$, a Dirac observable, acts as an operator multiplicatively. We recall that $\tilde{g}$ is a family of graphs related by diffeomorphisms and $\tilde{k}$ are the valences of the links in the spin nets. The gravitational part of the Hamiltonian vanishes exactly on such a state. We will discuss the choice of $\tilde{k}$ later on which achieves in the simplest terms a semiclassical behavior in the exterior and near the horizon. We are choosing a state with well defined values of $\tilde{k}$, one could have considered superpositions. Either considering superpositions of the mass or the $\tilde{k}$’s does not change the results discussed here provided one is superposing states that approximate a classical geometry well. It should be noted that obtaining these physical states does not involve any gauge fixing (apart from using coordinates adapted to spherical symmetry).

Our general philosophy is to treat the scalar field as a test field in a background defined by a semiclassical state that approximates the geometry of a Schwarzschild black hole in vacuum. To this aim, we will assume that the quantum states are a tensor product of states of vacuum gravity and states of matter. We will evaluate the expectation value of the matter part of the Hamiltonian on a state of vacuum gravity. The resulting quantity is an operator acting on the matter variables and we will interpret it as the quantum Hamiltonian of matter living on a quantum space-time.

### III. PARAMETERIZED DIRAC OBSERVABLES

We need to write the matter part of the Hamiltonian as a Dirac observable to compute its expectation value in the physical space of states corresponding to a black hole in vacuum. That would make it a well defined quantity to promote to an operator on the space of physical states of the vacuum gravity theory and it would make it commute with the pure gravity part of the constraint. To this aim, we will use the technique that Rovelli [5] calls “evolving constants of motion”, which can perhaps be better characterized in this context as “parameterized Dirac observables” [6]. It was originally developed to address the time evolution of constrained mechanical systems but can be extended to any constrained field theory. In that case it can be used to discuss local properties of dynamical variables that are gauge dependent. When one is dealing with constrained field theories, the dynamical variables of the theory are generically not defined as operators acting on the physical space of states. However, they can be written as functions of the Dirac observables and some (functional) parameters and then can be viewed as acting on the physical space of...
states. This in particular applies in the gravitational case. Although there is a tendency to believe that the only well
defined quantum operators in the gravitational case, due to diffeomorphism invariance, will be global quantities, it is
possible to describe local properties of the space-time as a function of the (global) observables and parameters.

Let us recall the definition of parameterized Dirac observables. They are functions of the canonical variables and
parameters that have vanishing Poisson brackets with all the constraints of the system. We illustrate the technique
with a simple mechanical example: the parameterized free particle. One has canonical variables \( x_0, x_1, p_0, p_1 \) and a
constraint \( C = p_0 + p_1^2/(2m) = 0 \). A pair of independent Dirac observables are \( X = x_1 - x_0 p_1/m \) and \( p_1 \) (or \( p_0 \)).
The physical states, annihilated by the constraints are given by \( \psi(p_0, p_1) = f(p_1) \delta(p_0 + p_1^2/(2m)) \) in the momentum
representation, with \( f(p_1) \) an arbitrary function. The inner product on the phase of physical states obtained via
refined algebraic quantization is given by

\[
\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_1 \delta(p_0 + p_1^2/(2m)) f_1^*(p_1) f_2(p_1) = \int_{-\infty}^{\infty} dp_1 f_1^*(p_1) f_2(p_1),
\]

yielding the ordinary inner product of the non-relativistic particle in quantum mechanics. A dynamical variable like
\( x_1 \) cannot be directly represented on the space of physical states. However, one can construct a parameterized Dirac
observable \( X_1(\lambda) = X + (p_1/m)\lambda \) with \( \lambda \) a parameter. This has the property that \( X_1(\lambda = x_0) = x_1 \), and in that sense
we consider \( X_1 \) to be a representation of \( x_1 \). Notice that the quantization obtained could have been derived using a
gauge fixing \( x_0 = \tau \), which imposed as a constraint leads to the fixing of the Lagrange multiplier and one would obtain
in the Heisenberg representation for \( x_1 \) the same quantum representation as the one obtained here, with a particular
value of the parameter \( \lambda = \tau \). It is important to emphasize that the parameterized Dirac observables must be self
adjoint operators. For instance, if one had tried to build a parameterized Dirac observable \( X_0(\lambda) \) one would find it
is not self-adjoint. The condition of self adjointness in this case leads to the identification of a good time variable for
the system. Further details can be seen in reference [3].

At a classical level, in vacuum spherically symmetric gravity there are two constraints and two (functional) parameters,
with \( M \) the Dirac observable. We can define a parameterized Dirac observable associated with \( E^x \) guided by the expression of the Hamiltonian constraint,

\[
E^\varphi(K_\varphi, E^x) = \frac{(E^x)'}{2\sqrt{1 + K_\varphi^2}} - \frac{2GM}{\sqrt{E^x}}
\]

and taking \( K_\varphi \) and \( E^x \) as parameters (we write them in calligraphic to emphasize that point, they are not canonical
variables anymore). One has, using the expression of the mass function in terms of the canonical variables given by
the Hamiltonian constraint, that \( E^\varphi(K_\varphi) = K_\varphi, E^z = E^\varphi(x,t) = E^\varphi(x) \).

Similarly, guided by the expression of the diffeomorphism constraint, one can write a parameterized Dirac observable
associated with \( K_x \),

\[
K_x(K_\varphi, E^x) = \frac{K_\varphi'}{2\sqrt{1 + K_\varphi^2} - \frac{2GM}{\sqrt{E^x}}}
\]

with \( K_x(K_\varphi, E^x) = E^x(x,t) = K_x(x,t) \) using the explicit expression of the observable \( M \) in terms of the dynamical
variables.

As we discussed in the example of the free particle, if one wishes to obtain the Lagrange multipliers, one can impose as constraints \( K_\varphi = K_\varphi(x,t) \) and \( E^x = E^x(x,t) \) with \( K_\varphi \) and \( E^x \) given functions. The consistency in time of
these constraints will determine the Lagrange multipliers. In particular if one chooses the given functions to be time
independent one gets that the shift vanishes and the Lapse is a constant. It should be emphasized that there are no
further consistency conditions that need to be checked, given the fact that the parameterized Dirac observables are
written in terms of Dirac observables.

The quantization is a bit more delicate in this case than in the simple example we considered before, since Dirac
observables arise at the quantum level that do not have a classical counterpart. This in particular will imply that the
parameterized Dirac observables are not functions of \( K_\varphi \) and \( E^x \) but of slightly different variables. Let us recall
that on the space of states annihilated by the Hamiltonian constraint of vacuum gravity (but not necessarily by the
diffeomorphism constraint) \( \hat{E}^x \) is a well defined operator,

\[
\hat{E}^x(x)|g, k, M\rangle = \ell_{\text{Planck}}^2 k(x)|g, k, M\rangle,
\]

where \( k(x) \) is the value of the valence of the spin network between the two consecutive vertices of the spin network
that contain within them the point \( x \). To solve the diffeomorphism constraint, one group averages. Although on
the space of solutions of all the constraints $k(x)$ is not well defined, the succession of values $\vec{k}$ is. This allows us to introduce the observable $\hat{O}(z|\vec{k},g)\text{phys} = \ell_P^2\text{phys}k_{\text{Int}(Vz)}|\vec{k},g\rangle\text{phys}$ with $z \in [0,1]$ and where $V$ is the number of vertices in the spin network. $\text{Int}(Vz)$ is the integer part of the number of vertices times $z$. This observable allows us to encode the information in $E^x$ in terms of it and a functional parameter $z(x)$ (a function from the real line into the interval $[0,1]$) as $\hat{E}^x(x) = \hat{O}(z(x))$. With this definition, $E^x$ becomes a Dirac observable. For a choice of $z(x)$, the action of $E^x$ on a state of the physical space $|g,\vec{k},M\rangle$ is the same as the action of $E^x$ on the states that only solve the Hamiltonian constraint $|g,\vec{k},M\rangle$. To put it differently, the freedom present in $z(x)$ corresponds to the freedom of spatial diffeomorphisms and to choose a $z(x)$ corresponds to a choice of coordinates.

So the operators associated with the parameterized Dirac observables, acting on the space of physical states are,

$$\hat{E}^x(K_{\varphi},z(x)) = \frac{\hat{O}(z(x))'}{2 \sqrt{1 + K_{\varphi}^2 - \frac{2GM}{\sqrt{\hat{O}(z(x))}}}}, \tag{9}$$

$$\hat{K}_x(K_{\varphi},z(x)) = \frac{K_{\varphi}'}{2 \sqrt{1 + K_{\varphi}^2 - \frac{2GM}{\sqrt{\hat{O}(z(x))}}}}. \tag{10}$$

With this, we can rewrite the coefficients of the matter Hamiltonian dependent on the gravitational variables as parameterized Dirac observables acting on the space of physical states as functions of the functional parameters $K_{\varphi},z(x)$ and the observables $M$ and $O(z)$. We have therefore succeeded in expressing the matter Hamiltonian entirely in terms of Dirac observables of the vacuum gravitational theory and (functional) parameters. The functional freedoms represented by $z(x)$ and $K_{\varphi}$ correspond to the freedoms associated with spatial diffeomorphisms and the foliation choice. Choosing these functions is equivalent to fixing a gauge and no further conditions are needed since everything is already expressed in terms of Dirac observables. This is a main advantage of working with parameterized Dirac observables, one works directly on the space of physical states and gauges can be fixed easily, without additional consistency conditions.

From now on we will assume (except for a brief discussion in section V.E) that $K_{\varphi}$ is time independent, further partially fixing the gauge, again in order to ultimately deal with coordinates in which the background is manifestly static.

IV. THE HAMILTONIAN FOR THE SCALAR FIELD

A. A choice of the background quantum state

We need to make a choice of the background quantum space-time. We wish to have a quantum state that approximates a semiclassical geometry well. We also make some additional assumptions to simplify calculations. For instance, one could consider states that involve superpositions of $\vec{k}$’s but we will choose not to do so. Since all the parameterized Dirac observables only depend on $O(z(x))$ and not its canonically conjugate variable, states with well defined $\vec{k}$’s are eigenstates of the observables.

We also need to make a choice of the labels of the spin network $\vec{k}$. Let us recall that the kinematically, $k_i$’s are the eigenvalues of the $E^x$ operator and that classically $E^x = r^2 / \sqrt{g}$, that is, it is proportional to the area of the spheres of symmetry. Therefore if one is to approximate a semiclassical space-time one should choose $k_i$’s that monotonically grow and such that the “step” between successive $k_i$’s is small compared to their values. Obviously, there are many possible choices of $\vec{k}$ that accomplish this, and they will correspond to semiclassical states with relatively similar properties, at least when probed at large scales compared to the spacing of the $k_i$’s.

Whatever the choice of $\vec{k}$’s, they are constrained by the fact that in loop quantum gravity areas are quantized. If the radial coordinate is $r$, the difference in values for it in two successive points of the lattice is bounded below by $\ell_P^2 / (2r)$. So in the exterior of the black hole we can choose, for instance, a uniform spacing with separation $\Delta > \ell_P^2 / (4GM)$ in the $r$ coordinate, with $r$ the usual Schwarzschild radial coordinate. Notice that for macroscopic black holes this is much smaller than Planck’s length. This immediately will limit the existence of some trans-Planckian modes when computing the vacua. This is in line with what has been observed by many authors in models, for instance in analogue black holes [2]. In order to have a semiclassical spin network one could also choose to have a distribution for the values of $\vec{k}$ such that the separation in the $r$ variable between successive vertices will be given by a fixed value $\Delta$. 
We will work with a spin network with a finite number of vertices $V$. For simplicity, we will choose the function $z(x)$ such that the coordinate $x$ is identified with the coordinate $r$. The spin network vertex $i$ will therefore be located at

$$x_i = (i + i_H)\Delta$$

with $i_H = \text{Int}(2GM/\Delta)$, so the point $i = 0$ is the closest vertex in the spin network to the horizon. We also choose $\Delta$ in such a way that it is a multiple of $\ell^2_{\text{Planck}}$. With such choices, the eigenvalues of $E^x_i$ with the choice of $z(x)$ we made are $k_i = x_i^2/\ell^2_{\text{Planck}}$, that is $E^x_i |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}} = \ell^2_{\text{Planck}} k_i |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}} = x_i^2 |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}}$. These choices for the gauge imply $z = x/(V\Delta)$, so $z_i \equiv z(x_i) = (i + i_H)/V$.

One can penetrate into the black hole by considering negative values of $i$ and also allowing a non-vanishing extrinsic curvature $K^x$ from the horizon inwards, since that corresponds to a coordinate choice that makes the horizon nonsingular. The analysis will be simpler if $2GM = i_H\Delta$, that is, we are putting a vertex of the spin network at the horizon. That is not a generic situation, it just simplifies calculations. Alternatively, one can consider that the point is not on the horizon but the separation is negligible compared to the separation to the next vertex on the spin net and then the results will very approximately hold.

We will see that the discreteness has implications for the types of vacua one gets. This is a priori surprising since Hawking radiation is a phenomenon usually associated entirely with the exterior of the black hole, where for non Planck-sized black holes, one expects quantum gravity effects to be negligible.

It should be emphasized that the specific spin network chosen is mostly in order to simplify the calculations. Many other spin networks, more refined in the separation of the vertices, could be chosen approximating even better the classical background. However, as we discussed, certain degree of discreteness will always be present due to the quantization of $\tilde{k}$ and this will always have implications near the horizon and in particular, for the behavior of the vacua at the horizons.

### B. The matter Hamiltonian as a parameterized Dirac observable

On the quantum states considered $|\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}}$, the operator associated with $E^x_i$ is,

$$\hat{E}^x_i |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}} = \ell^2_{\text{Planck}} k_i |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}},$$

(recall that we made a specific choice for the function $z(x)$).

To realize the Hamiltonian as a quantum operator we need to realize the operator $(\hat{E}^x)'$ and its inverse. The realization of spatial derivatives of operators in loop quantum gravity has been considered only a few times before, and never for a momentum operator like $\hat{E}^x$. When we analyzed the vacuum case, only $(E^x)'$ appeared and we represented it by a finite difference. Because the Hamiltonian is a scalar given by an integral of a density, the final expression is independent of the spacing used in the finite difference. In this case we have a further challenge, since we will need to differentiate $\phi$, which only exists in the vertices of the spin net. This leads us to propose that we realize all derivatives as a finite difference between nearest vertices of the spin network. Again, because the Hamiltonian is the integral of a density, the actual spacing drops off from the calculation at the end of the day. It should be noted that this is at variance, for instance, with the spirit of the realization of the Hamiltonian constraint proposed for the three dimensional case, where the resulting operator does not connect neighboring vertices [3], but adds extraordinary vertices adjacent to the vertex it is acting on. This is more in the spirit of “Algebraic Quantum Gravity” [9]. We have conducted extensive tests to see that our proposal leads to excellent agreement with the classical theory in the semiclassical regime for suitable spin networks. We define $(E^x)' \equiv (E^x)'/\Delta x$, with

$$(E^x)'_i |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}} = \ell^2_{\text{Planck}} [k_{i+1} - k_i] |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}}.$$ (13)

And for $1/\sqrt{(E^x)'} = \sqrt{\Delta x}/\sqrt{(E^x)_i}$, we have, following similar steps to those used in loop quantum cosmology [10], one rewrites the term on the left as a Poisson bracket classically and then promotes the Poisson bracket to a quantum commutator to yield,

$$\frac{\text{sgn}((E^x)')}{\sqrt{(E^x)'_i}} |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}} = \frac{1}{\ell^2_{\text{Planck}}} \left[ \sqrt{|\Delta k_i + 1|} - \sqrt{|\Delta k_i - 1|} \right] |\psi, \tilde{\gamma}, \tilde{k}\rangle_{\text{grav}}.$$ (14)
where $\Delta k_i \equiv k_{i+1} - k_i$. Notice that this particular representation of the operator will avoid introducing singularities in the matter part of the Hamiltonian in the interior of the black hole. In this paper we are concentrating on the exterior and there one can simply talk of $1/(E^x)$ as an operator directly, both definitions yield extremely close results in that region. With this, the Hamiltonian is a sum of contributions at the vertices $H = \sum_i H_i$, with,

$$
\hat{H}_i = \hat{A}_i P^2_{\phi,i} + \hat{B}_i (\phi_{i+1} - \phi_i)^2 + \hat{C}_i P_{\phi,i} (\phi_{i+1} - \phi_i)
$$

(15)

where $P_{\phi,i}$ is the value of the momentum of the scalar field at the vertex $i$ and similarly for $\phi_i$, and these two quantities are operators acting on the quantum states of matter. The coefficients are

$$
\hat{A}_i |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = \frac{1}{2} \frac{1}{(E^x)_i} \frac{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_i}}}{\sqrt{\ell^2_{\text{Planck}} k_i}} |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}},
$$

(16)

$$
\hat{B}_i |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = 2 \frac{1}{(E^x)_i} \frac{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_i}}}{\sqrt{\ell^2_{\text{Planck}} k_i}} k_i^{3/2} \ell^3_{\text{Planck}} |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}},
$$

(17)

$$
\hat{C}_i |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = 2 \frac{1}{(E^x)_i} \frac{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_i}}}{\sqrt{\ell^2_{\text{Planck}} k_i}} K_{\phi,i} \sqrt{\ell^2_{\text{Planck}} k_i} |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}}.
$$

(18)

Notice that all dependence on $\Delta x$ has canceled out. Strictly speaking, these expressions are valid for the action of the operators on states of the form $|\tilde{g}, \tilde{k}, M\rangle$. Acting on states $|\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}}$ these expressions are only true at leading order in the dispersion of the mass $\sigma$. From now on all expressions should be interpreted as leading order in the dispersion of the mass. We also have dropped the subindex 0 from $M_0$ for simplicity.

We will work in the exterior region where we can choose $K_{\phi} = 0$ without incurring coordinate singularities, so the above expressions simplify a bit, in particular $\hat{C}_i = 0$ and,

$$
\hat{A}_i |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = \frac{1}{2} \frac{1}{(E^x)_i} \frac{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_i}}}{\sqrt{\ell^2_{\text{Planck}} k_i}} |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}},
$$

(19)

$$
\hat{B}_i |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}} = 2 \frac{1}{(E^x)_i} \frac{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_i}}}{\sqrt{\ell^2_{\text{Planck}} k_i}} k_i^{3/2} \ell^3_{\text{Planck}} |\psi, \tilde{g}, \tilde{k}\rangle_{\text{grav}},
$$

(20)

where,

$$
\frac{1}{(E^x)_i} \equiv \frac{1}{\ell^2_{\text{Planck}}} \left[ \sqrt{|\Delta k_i + 1|} - \sqrt{|\Delta k_i - 1|} \right]^2.
$$

(21)

Taking the expectation value of the Hamiltonian in the normalized state considered leads to an expression (recall that in the exterior $K_{\phi} = 0$) for the quantum Hamiltonian of matter given by,

$$
H = \sum_j A_j P^2_{\phi,j} + B_j (\phi_{j+1} - \phi_j)^2,
$$

(22)

with $A_j$ and $B_j$ the eigenvalues obtained above.

V. THE EQUATIONS OF MOTION

We will now consider the equations of motion for the scalar field stemming from the classical version of the Hamiltonian derived in the previous section. The associated classical equation of motion for the field is,

$$
\dot{\phi}_i = \{\phi_i, H\} = 2A_i P_{\phi,i},
$$

(23)

with $H$ given by (22) with $P_{\phi,j}$ and $\phi_j$ classical fields. One can introduce the associated Lagrangian,

$$
L = \sum_j P_{\phi,j} \dot{\phi}_j - H = \sum_j \frac{\dot{\phi}_j^2}{2A_j} - B_j (\phi_{j+1} - \phi_j)^2,
$$

(24)
which can be recognized as a discrete version of,

\[ L = \int \sqrt{-g} \left( g^{00} (\partial_0 \phi)^2 + g^{xx} (\partial_x \phi)^2 \right) dx. \tag{25} \]

To see this, let us consider its discretization,

\[ L = \sum_j \sqrt{-g(x_j)} \left( g^{00}(x_j) (\partial_0 \phi(x_j))^2 + g^{xx}(x_j) \frac{(\phi(x_{j+1}) - \phi(x_j))^2}{\Delta^2} \right) \Delta. \tag{26} \]

And from it we can read off the components of the discrete metric,

\[ \sqrt{-g(x_j)} g^{00}(x_j) = \frac{1}{2A_j \Delta}, \tag{27} \]

\[ \sqrt{-g(x_j)} g^{xx}(x_j) = B_j \Delta. \tag{28} \]

We should recall that these expressions are valid in the exterior only \((j > 0)\). The resulting expressions for the covariant form of the metric are,

\[ g^{00}(x_j) = -1 + \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_j}}, \tag{29} \]

\[ g^{xx}(x_j) = \frac{1}{1 - \frac{2GM}{\sqrt{\ell^2_{\text{Planck}} k_j}}}, \tag{30} \]

which can be seen to coincide with the expectation value of the metric, viewed as a parameterized Dirac observable, on the gravitational states considered using the results of section II, recalling that \(g^{xx} = (E^x)^2/E^x\) and \(g^{00} = -(N^{\text{orig}})^2\).

We recognize the usual form of the Schwarzschild metric. To obtain the above expressions we used that \((E^x)^i = 2x^i \Delta\) in the spin network chosen.

The equation of motion for the field becomes a spatially discretized version of the Klein–Gordon equation in a curved space-time,

\[ (\sqrt{-gg^{ab}} \phi_{,a})_{,b} = 0. \tag{31} \]

For the kind of quantum states here considered for the gravitational field one recovers what would be a lattice version of the equations of a quantum field in a background space time. If one considered states that involve superpositions the situation is more involved.

\section{The Quantum Vacua on a Quantum Background}

\subsection{Introduction}

The construction of quantum vacua on curved space-times is carried out by considering modes that solve the wave equation on the curved space-time and in terms of them constructing the creation and annihilation operators \cite{11}. These constructions are slicing dependent. Accordingly, different vacua have been defined in the literature through the choice of different slicings. When one computes physical quantities, like the expectation value of the stress energy tensor in the vacuum, one may encounter singularities at various points of space-time, particularly at the horizons. The Boulware vacuum is associated with the tortoise radial coordinate and the time of an observer at infinity. It leads to singularities of the physical quantities at the past and future horizons. The Hartle–Hawking vacuum is based on the Kruskal coordinates and does not lead to singularities in the physical quantities anywhere. The Unruh vacuum is associated to a foliation in which the Cauchy surfaces approach the past horizon and the past null infinity asymptotically and leads to singularities in the physical quantities at the past horizon. The Hawking radiation can be computed by comparing the modes of the Unruh and Boulware vacua and computing the expectation value of the number operator associated with one vacuum on the other. The Hartle–Hawking vacuum has ingoing and outgoing modes and is therefore associated with a black hole with incoming radiation in addition to the Hawking radiation.

We will start with a discussion of the coordinate systems used in the definition of the vacua.
B. Coordinate systems

The first coordinate change we want to consider is to pass to a tortoise radial coordinate. In the discrete case this corresponds to,

\[
x_j^* = x_j + 2GM \ln \left( \frac{x_j}{2GM} - 1 \right).
\]

(32)

The inverse transformation can be cast in terms of the Lambert function \( W \),

\[
x_j = 2GM \left( W \left( \exp \left( \frac{x_j^* - 2GM}{2GM} \right) \right) + 1 \right).
\]

(33)

Choosing a function \( z(x(x^*)) \) with \( x(x^*) \) given by (33), reworking expressions (27,28) leads to a metric,

\[
g_{00}(x_j) = -1 + \frac{2GM}{\sqrt{\ell_{\text{Planck}}^2 k_j}},
\]

(34)

\[
g_{xx}(x_j) = 1 - \frac{2GM}{\sqrt{\ell_{\text{Planck}}^2 k_j}}.
\]

(35)

We will also need null coordinates,

\[
u_i = t - x_i^*, \quad x_i^* = \frac{v_i - u_i}{2},
\]

(36)

\[
v_i = t + x_i^*, \quad t = \frac{u_i + v_i}{2},
\]

(37)

in terms of which the metric has the non-vanishing component,

\[
g_{uv}(u_j, v_j) = -\frac{1}{2} \left( 1 - \frac{2GM}{\sqrt{\ell_{\text{Planck}}^2 k_j}} \right),
\]

(38)

where \( \ell_{\text{Planck}}^2 k_j^2 = x_j^2 \) with the right hand side given as function of \( u_j, v_j \) via the Lambert function.

We will also need the Kruskal coordinates,

\[
U_i = -\exp \left( \frac{u_i}{4GM} \right),
\]

(39)

\[
V_i = \exp \left( \frac{v_i}{4GM} \right),
\]

(40)

in terms of which the metric has a non-vanishing component,

\[
g_{UV}(U_j, V_j) = -4 \left( \frac{2GM}{\sqrt{\ell_{\text{Planck}}^2 k_j}} \right)^3 \exp \left( \frac{\sqrt{\ell_{\text{Planck}}^2 k_j}}{2GM} \right),
\]

(41)

and we also have that

\[
U_j V_j = -\exp \left( -\frac{x_j}{2GM} \right) \left( \frac{x_j}{2GM} - 1 \right).
\]

(42)

These coordinate changes can all be done at the level of (27,28) defining appropriate functions \( z(x) \) (in general \( z(x,t) \)). We just proceeded directly since the changes are simpler to do that way, remembering that the \((t,r)\) portion of the metric is explicitly conformally flat, and the changes preserve that nature.

A final comment about coordinates is that we will be studying behaviors at scri. Strictly speaking, one cannot study null infinity with our canonical framework based on a spatial spin network with a finite number of points, so we will be really making statements about behaviors at null surfaces far away from the black hole and into the past or future, rather than scri itself.

Notice that these coordinate manipulations just involve a relabeling of the spin network vertices, not a change in their physical distance. When the relabellings involve time, one is considering the fields at different Schwarzschild times.
C. Boulware vacuum

The Boulware vacuum is constructed on a foliation determined by coordinates $t, x^s$. In such a foliation, the Lagrangian we identified for the discrete theory becomes,

$$L = -\sum_j \left( (\partial_0 \phi(x_j^s, t))^2 - \frac{(\phi(x_{j+1}^s, t) - \phi(x_j^s, t))^2}{\Delta_j^2} \right) \Delta_j$$  \hspace{1cm} (43)

where $\Delta_j$ is the separation between vertices in the tortoise coordinate, which is non-uniform. Here we have made the simplifying assumption of going to the $1+1$ dimensional case, ignoring the centrifugal and gravitational potential that appear in $3+1$ dimensions with spherical symmetry on a curved background. For the calculation of the Hawking radiation one needs the modes close to the horizon and infinity only, and there the potential vanishes and the discussion of the modes in both cases is equivalent. The equation of motion that follows from this Lagrangian is, after some rescalings,

$$\partial_0^2 \phi(x_j^s, t) - \left[ \frac{\phi(x_{j+1}^s, t) - \phi(x_j^s, t)}{\Delta_j} - \frac{\phi(x_j^s, t) - \phi(x_{j-1}^s, t)}{\Delta_j \Delta_{j-1}} \right] = 0.$$  \hspace{1cm} (44)

The separation in the tortoise coordinate is given by,

$$\Delta_j = \frac{dx^s}{dx} \bigg|_{x_j} = \frac{x_j \Delta}{x_j - 2GM} = \Delta + \frac{2GM}{j},$$  \hspace{1cm} (45)

where we took into account that $j \Delta = x_j - 2GM$. Asymptotically far from the black hole $\Delta_j \to \Delta$ and one therefore recovers a very refined equally spaced lattice. That leads to an excellent lattice approximation to the traditional B-frequencies, where the dispersion relation is approximately linear.

Let us consider the case of waves with the shortest wavelength permitted by the lattice and the case of long wavelengths. The maximum frequency allowed by the lattice is $\omega = \frac{2}{j} \Delta$. In that case the first term on the right hand side dominates. Setting a boundary condition $B(x_1^s) = B_0 = \text{constant}$, the equation becomes, for the two next points,

$$B(x_{j+1}^s) = -\Delta_j^2 \omega^2 B(x_j^s) + B(x_j^s) \left( \frac{2j - 1}{j} \right) - B(x_{j-1}^s) \left( \frac{j - 1}{j} \right).$$  \hspace{1cm} (51)

Let us consider the case of waves with the shortest wavelength permitted by the lattice and the case of long wavelengths. The maximum frequency allowed by the lattice is $\omega = \frac{2}{j} \Delta$. In that case the first term on the right hand side dominates. Setting a boundary condition $B(x_1^s) = B_0 = \text{constant}$, the equation becomes, for the two next points,
so we see that the values for successive lattice points alternate in sign and increase rapidly as one moves away from the horizon. That means that ingoing, these modes are heavily suppressed and one faces no trans-Planckian problem in the vicinity of the horizon. For sub-Planckian frequencies, the modes behave as in the continuum starting at scri and then get modified close to the horizon. The modification starts closer to the horizon for smaller $\Delta$'s and/or smaller frequencies. There is a maximum frequency that corresponds to $\omega = 2/\Delta$. On the other hand, for large wavelengths $\omega = 2\pi/\lambda$ with $\lambda \gg 2GM$, one can see that the solution agrees very well with the continuum solution with small deviations, as one expects from a lattice approximation with small lattice spacing compared to the wavelength.

The fact that the high frequency modes suffer significant changes in the region close to the horizon(s) has consequences for physical quantities. In particular, physical quantities computed with the quantum states stop being singular there. In fact they are non-singular everywhere. The singularities in the usual quantum field theory in curved space time framework manifest themselves in the Feynman propagators and the expectation values of the energy momentum tensor. The propagators have singular derivatives on the horizon(s) and as most physical properties of the matter field —like the energy— involve derivatives, their physics becomes singular there. This is due to the fact that in the continuum the radiation never enters the horizon in these coordinates and there are trans-Planckian modes due to the infinite blueshift of the radiation at the horizon. This is the origin of the singularities associated with the Boulware vacuum at the horizon in the traditional treatment. In our treatment there are no trans-Planckian modes, the dispersion relation is modified in a sub-luminal way (it does not affect the horizon structure) and there are no singularities. Also the radiation reaches the horizon in a finite time, since the last point before the horizon is reached in a finite time and from the horizon to the interior one can choose a non-vanishing $K_{\phi}$ and transition seamlessly without coordinate singularities. The appearance of a fundamental discrete lattice, in addition to eliminating the singularities in the physical quantities, also eliminates the ambiguities present in the continuum in the regularization of quantities, like for instance in the energy-momentum tensor. This has been a major obstacle to performing back reaction calculations, which would not be present in the current approach.

D. The Hartle–Hawking vacuum

As in the usual treatment, the modes are perfectly well behaved near the horizon and they are not infinitely blue shifted as there is no coordinate singularity at the horizon. In this case the uniform spacing in the radial coordinate translates itself into a spacing that grows when one approaches scri. The Hartle–Hawking modes are formulated in terms of a chart $T, R$ given in terms of the Kruskal coordinates by $R = V - U$ and $T = V + U$. In a surface $T = T_0 = \text{constant}$, we have that

$$R^2 - T_0^2 = \exp\left(\frac{r}{2GM}\right)\left(\frac{r}{2GM} - 1\right), \quad (53)$$

which implies

$$R = \sqrt{4 \exp\left(\frac{r}{2GM}\right)\left(\frac{r}{2GM} - 1\right) + T_0^2}, \quad (54)$$

and therefore,

$$\Delta R = \frac{\partial R}{\partial r} \Delta = \frac{\sqrt{2r} \exp\left(\frac{r}{2GM}\right)}{(2GM)^{3/2}} \Delta, \quad (55)$$

which grows exponentially with $r$, no matter how small one takes $\Delta > \ell_{\text{Planck}}/(2r)$. As a consequence, some modes of the continuum cannot be reproduced. However, such modes are not physically relevant, since they would imply trans-Planckian frequencies at scri and there is no good motivation to consider such frequencies there.

E. The Unruh vacuum

Finally, for the Unruh vacuum one chooses a foliation with coordinates $\tilde{x}^U = \ln(V/2) - U/2$, $\tilde{t}^U = \ln(V/2) + U/2$. Here we will make a slightly different choice of congruence in order to enhance the behavior at the past horizon,

$$\tilde{x}^U = \frac{V + 1}{V} \ln(V/2) - U/2, \quad (56)$$

$$\tilde{t}^U = \frac{V + 1}{V} \ln(V/2) + U/2. \quad (57)$$
The additional factor in front of the logarithm has the role of keeping the \( \tilde{r}^U = \) constant surfaces closer to the past horizon when one discretizes. Otherwise one would, in the first discrete vertex away from the horizon allowed by the quantization of area condition, be already very far away from it, and the region close to the horizon would contain very few vertices, spoiling the continuum approximation. The extra factor does not alter the global behavior of the slicing, in particular near the future horizon one has incoming modes that come from scri- with positive frequency in terms of the standard Schwarzschild time \( t \).

The metric takes the following form,

\[
\begin{align*}
  g^U_{00} &= -\frac{32(GM)^3}{r} \frac{V^2}{V + 1 - \ln \frac{r}{2GM}} \exp \left( -\frac{r}{2GM} \right), \\
  g^U_{xx} &= -g^U_{00},
\end{align*}
\]

where \( r = r(U, V) \) given by inverting (42). The extra factor involving \( V \) tends exponentially to infinity at scri+ and compensates the exponentially decreasing term in \( r \), the metric does not go to zero at scri+, as was the case in the slicing of the Hartle–Hawking vacuum. This will lead to reproducing the continuum modes at scri+. The separation between vertices tends to \( \Delta \) at scri+, \( \delta x^U \to 2\Delta/(2GM) \) when \( r \to \infty \). Since the metric is conformal in these coordinates, the modes coincide with those of flat space-time in the \( \tilde{r}^U, \tilde{x}^U \) coordinates. When written in terms of the Kruskal coordinates \( U, V \), the modes are \( \exp(i\omega U)/\sqrt{2\pi\omega} \) and \( \exp(2\ln(V/2))/\sqrt{2\pi\omega} \).

Since we have no problems at infinity, we will study the behavior at the past horizon. For this, we consider a Cauchy surface for the outgoing modes \( \tilde{x}^U = t_0 \ll 0 \). For \( V \sim 1, \tilde{r}^U \) starts to quickly depart from the past horizon. Since for large \( V \) we have that \( V = 2\exp(i(\tilde{x}^U + \tilde{r}^U)/2) \) and \( U = \tilde{r}^U - \tilde{x}^U \), we have that \( V \sim 1 \) occurs for \( \tilde{x}^U \sim -t_0 \). The variable \( \tilde{x}^U \) ranges from \( \tilde{x}^U = t_0 \) in which case \( U = 0 \) and one is at the future horizon, all the way to \( \tilde{x}^U = -t_0 \) where one is at the past horizon. To understand the behavior at the past horizon we have to look at very negative values of \( t_0 \). An important property of the Unruh coordinates is that the potential term in the equation for the modes at \( \tilde{r}^U = t_0 \) is only significantly different from zero at \( \tilde{x}^U = -\tilde{r}^U \), where we have that \( V = 1 \) and the initial surface quickly departs from the past horizon. In the limit \( \tilde{r}^U \to -\infty \) the complete horizon is covered by the surface and the potential tends to zero. However, that limit corresponds to \( V \to 0 \) and one immediately sees the metric is singular there. This is the origin of the singularities in physical quantities of the Unruh vacuum at the past horizon. In our case, the spin network never reaches the horizon so this problem is avoided.

To study the influence of the discrete structure in the behavior of the solutions at \( \tilde{r}^U = t_0 \) we note that the vertices of the spin network lie on curves \( U_j V_j = \) constant. We therefore have

\[
U_j V_j = -\exp \left( \frac{x_j}{2GM} \right) \left( \frac{x_j}{2GM} - 1 \right),
\]

so at \( \tilde{r}^U = t_0 \) one has that,

\[
\frac{V_j(t) + 1}{V_j(t)} \ln (V_j(t)) - \frac{1}{V_j(t)} \exp \left( \frac{x_j}{2GM} \right) \left( \frac{x_j}{2GM} - 1 \right) = t_0.
\]

From here one can read off the distribution of \( V_j(t) \) and from \( U_j V_j = \) constant one can see that the points \( U_j(t) \) are approximately uniformly distributed between \( U = 0 \) and \( U = 2t_0 \), which is the region in which \( \tilde{r}^U = t_0 \) approximates the past horizon. As a consequence, the spacing between vertices of the spin network in terms of \( U \) is given by \( \delta U = 2t_0 \Delta/(2GM) \). The spacing therefore is a function of time. To study the propagation of modes, we consider a lattice with spacing that is time dependent but spatially uniform.

So we start from equation (44) considering a time dependent spacing (notice that such equation would not follow from the action (43), a complete treatment would have required re-deriving (43) with \( K_{\phi} \) and \( z \) functions of \( x \) and \( t \). This in particular would have led to a complicated dependence of the lattice spacing with space and time, and we could not find a way to treat the situation analytically, so we are using here an approximation in which the lattice spacing only depends on time),

\[
\partial_t^2 \phi(\tilde{x}^U_j, \tilde{r}^U) - \left[ \frac{\phi(\tilde{x}^U_{j+1}, \tilde{r}^U) - 2\phi(\tilde{x}^U_j, \tilde{r}^U) + \phi(\tilde{x}^U_{j-1}, \tilde{r}^U)}{a(\tilde{r}^U)^2} \right] = 0,
\]

where in our case \( a(\tilde{r}^U) = 2\tilde{r}^U \Delta/2GM) \). It is worthwhile reminding that we are working on a lattice with a finite number of points \( N \) in the region close to the horizon and up to the maximum of the centrifugal potential in order to characterize these modes. So the total size of the lattice will change as the spacing changes with time. We note that the above equation can be solved by the ansatz,

\[
\phi(\tilde{x}^U_j, \tilde{r}^U) = f(\tilde{r}^U) \exp \left( i\tilde{x}^U_j \frac{2\pi n}{Na(t)} \right) = f(\tilde{r}^U) \exp \left( \frac{2\pi nj}{N} \right),
\]

(63)
with \( f(\tilde{t}^U) \) satisfying,

\[
a (\tilde{t}^U)^2 \ddot{f} = 2 \cos \left( \frac{2\pi n}{N} \right) - 2 = -K_n^2,
\]

which with the definition of \( a(\tilde{t}^U) \) yields,

\[
\ddot{f} + \frac{L_n^2}{\tilde{t}^U} f = 0
\]

with \( L_n^2 = (GMK_n)^2/\Delta^2 \) (and for a stellar sized black hole \( L_n \sim 10^{67} \)). Solving the equation we get,

\[
f(\tilde{t}^U) = A \exp \left( \frac{1}{2} \pm \frac{i}{2} \sqrt{4L_n^2 - 1} \right) \ln (-\tilde{t}^U),
\]

To understand this solution we set ourselves in an asymptotic past time \( \tilde{t}_1^U \) so it takes a large negative value and we study how the solution evolves for a period of time \( \tau \) much smaller than \( \tilde{t}_1^U \),

\[
f(\tilde{t}_1^U + \tau) = A(\tilde{t}_1^U) \exp \left( \pm i \frac{L_n}{\tilde{t}_1^U} \tau + \frac{\tau}{2\tilde{t}_1^U} \right),
\]

with \( A(\tilde{t}_1) \) a constant. So apart from a term that changes the amplitude, it has an oscillation with a frequency that varies with time. For \( \tilde{t}_1^U \to -\infty \) the frequency is zero, as the metric vanishes on the past horizon. Computing the physical separation between points in the lattice as one goes near the horizon, it tends to the Planck length (choosing the most refined spin network subject to the condition of quantization of area). There exists a cutoff frequency that is coordinate independent and trans-Planckian modes are suppressed. The presence of the discrete structure modifies the dispersion relations. One can work them out explicitly expanding for small momenta equation (64) the quantity \( 2\pi n/N \) plays the role of momentum in the lattice, whereas \( 2\pi n/(Na(\tilde{t}^U)) \) is the physical momentum.

**VII. HAWKING RADIATION**

In order to compute the Hawking radiation we will use the technique of computing the two point functions and from them compute the expectation value of the number operator \([12]\). We will not strictly carry out the discrete calculations associated with the spin network background. We will sketch the usual calculations on a classical space time and point out at key instances what differs in the discrete case.

The main ingredient is the calculation of the Bogoliubov coefficients, which connect the modes of the “in” and “out” states, corresponding respectively to the state of the field at the past horizon and at future scri, and are defined as,

\[
\beta_{i_1,k} = (u_{i_1}^{\text{out}}, u_k^{\text{in}}^*),
\]

and we introduce the Klein–Gordon inner products among modes \( u_k \) defined by

\[
\begin{align*}
(u_{i_1}^{\text{out}}, u_k^{\text{in}}) &= \int d\Sigma^\mu u_{i_1}^{\text{out}} (x) \nabla^\mu u_k^{\text{in}} (x), \\
(u_{i_2}^{\text{out}}, u_k^{\text{in}}) &= \int d\Sigma^\mu u_{i_2}^{\text{out}} (x) \nabla^\mu u_k^{\text{in}} (x),
\end{align*}
\]

where the integrals are calculated at scri+ \( \mathcal{I} \). Since we are in the spherical case we will consider the modes,

\[
\begin{align*}
u_{\omega,\ell,m}^{\text{in}} &= \frac{1}{\sqrt{4\pi \omega}} \exp \left( -i\omega U \right) \frac{Y_{\ell}^m (\theta, \varphi)}{r}, \\
u_{\omega,\ell,m}^{\text{out}} &= \frac{1}{\sqrt{4\pi \omega}} \exp \left( -i\omega \tilde{U} \right) \frac{Y_{\ell}^m (\theta, \varphi)}{r}
\end{align*}
\]
where \( \omega, \ell, m \) play the role of \( k^3 \). Given the relation between the creation and annihilation operators out and in given by the Bogoliubov coefficients,

\[
a^\text{out}_{11} = \sum_k \left( \beta_{11,k} a^\text{in}_k + \beta^*_{11,k} a^\text{in*}_k \right),
\]

we can compute the expectation value of the number operator associated with the out modes in the in states, which is non-vanishing,

\[
\langle \text{in}|N^\text{out}_{11,22}|\text{in}\rangle = \sum_k \beta_{11,k} \beta^*_{11,k} = - \sum_k \left( a^\text{out}_{11}, a^\text{in*}_k \right) \left( a^\text{out*}_{12}, a^\text{in}_k \right),
\]

We note that the sum in \( k \) that appears is the two point function,

\[
\langle \text{in}|\phi(x_1) \phi(x_2)|\text{in}\rangle = \sum_k u^\text{in}_k(x_1) u^\text{in*}_k(x_2)
= \int_0^\infty d\omega \sum_{\ell,m} \exp(-i\omega U_1) Y^m_\ell(\theta_1, \varphi_1) \exp(-i\omega U_2) Y^{m*}_\ell(\theta_2, \varphi_2).
\]

So we can rewrite the expectation value of the number operator as,

\[
\langle \text{in}|N^\text{out}_{11,22}|\text{in}\rangle = \int_0^\infty dU_1 d\Omega_1 r_1^2 dU_2 d\Omega_2 \times \exp(-i\omega u(U_1)) Y^m_\ell(\theta_1, \varphi_1) \exp(-i\omega u(U_2)) Y^{m*}_\ell(\theta_2, \varphi_2) \partial_{U_1} \partial_{U_2} \langle \text{in}|\phi(x_1) \phi(x_2)|\text{in}\rangle \quad \text{at} \quad \omega \to \infty,
\]

where we have integrated by parts in \( U_1, U_2 \). The factors \( t_\ell(\omega) \) are the “transmission coefficients” given by the fact that one has a potential in the wave equation and therefore not the entirety of the modes get transmitted to infinity. The angular integrals can be computed to give \( 4 \delta_{\ell_1, \ell_2} \delta_{m_1, m_2} \). In the case of quantum field theory on a classical space-time the calculation continues as follow,

\[
\partial_{U_1} \partial_{U_2} \langle \text{in}|\phi(x_1) \phi(x_2)|\text{in}\rangle \quad \text{at} \quad \omega \to \infty,
\]

Although the integral is immediate, its expression has a problem at \( \omega \to \infty \), where it oscillates. To compute it in the continuum, one introduces a small purely imaginary term \( i\epsilon \), that ensures an exponential falloff for \( \omega \to \infty \) and can be removed by taking the limit \( \epsilon \to 0 \),

\[
\lim_{\epsilon \to 0} i\epsilon \partial_{U_1} \int_0^\infty d\omega \frac{\exp(-i\omega (U_1 - U_2))}{4\pi} = \int_0^\infty d\omega \frac{\omega}{4\pi} \exp(-i\omega (U_1 - U_2)).
\]

In our quantum treatment, due to the discreteness introduced by the spin network one gets similar expressions but there exists a maximum \( \omega = 4\pi GM/\ell^2_\text{Planck} \). One cannot take the limit \( \epsilon \to 0 \) and the parameter must take a small finite value. We need \( \exp(-i\omega \epsilon) \) to be small, so that leads to a value for \( \epsilon \sim \ell_\text{Planck} \) to avoid getting frequencies where the dispersion relation differs from the continuum. This may sound ad-hoc presented this way, but there is a good motivation for it. We are considering only the leading contributions in the dispersion of the mass. In reality one will have states that are superpositions of states with different values of the mass. The natural value for the mass uncertainty is given by the Planck scale. This introduces a fuzziness in the lattice, that leads to the elimination of frequencies higher than the Planck one where the dispersion relation differs from the continuum one.

This leads to an expression for the expectation value of the number operator is,

\[
\langle \text{in}|N^\text{out}_{11,22}|\text{in}\rangle = A \int_{I^+} dU_1 dU_2 \frac{\exp(-i\omega_1 u_1(U_1) + i\omega_2 u_2(U_2))}{(U_1 - U_2 - i\epsilon)^2},
\]

\footnote{To simplify expressions, in this section we switched conventions and are using a variable \( U \) that has dimensions of length, as does \( u \). So this \( U \) is \( 4GM \) times the previously defined one.}
with
\[ A = \frac{t_{\ell_1}(\omega_1) t_{\ell_2}(\omega_2)}{4\pi^2 \sqrt{\omega_1 \omega_2}} \delta_{\ell_1, \ell_2} \delta_{m_1, m_2}. \] (81)

We now perform a change of variables,
\[ U_1 = -4GM \exp \left( \frac{u_M + z}{4GM} \right), \] (82)
\[ U_2 = -4GM \exp \left( \frac{u_M + z}{4GM} \right), \] (83)
where \( u_M = u_1 + u_2 \) and \( z = u_2 - u_1 \), and the calculation reduced to an integral in \( z \) that can be computed,
\[ \langle \text{in} | N^\text{out}_{\ell_1, \ell_2} | \text{in} \rangle = \frac{|t_{\ell}(\omega)|^2}{\exp(2\pi \omega/k) - 1}. \] (84)

where \( \kappa = 1/(4GM) \). Performing the integral in \( z \) yields the Hawking formula,
\[ \langle \text{in} | N^\text{out}_{\ell_1, \ell_2} | \text{in} \rangle = \frac{|t_{\ell}(\omega)|^2}{\exp(2\pi \omega/k) - 1}. \] (85)

In the quantum background case, the spin network introduces a cutoff in length \( \ell^2_{\text{Planck}}/(4GM) \), but as we mentioned, we are considering wavelengths much larger than such cutoff to avoid running into the modified dispersion relation, this implies a cutoff of the order \( \ell_{\text{Planck}} \) in the \( u \) variable. One therefore has
\[ (u_1 - u_2)^2 \geq \ell^2_{\text{Planck}} \] (86)
which is the same as that of [12]. Notice that the cutoff emerging in the \( u \) variable is important since otherwise Lorentz invariance would be violated. This leads to the same formula found in [12],
\[ \langle \text{in} | N^\text{out}_{\ell_1, \ell_2} | \text{in} \rangle = \frac{|t_{\ell}(\omega)|^2}{\exp(2\pi \omega/k) - 1} - \frac{\kappa^2 \ell^2_{\text{Planck}}}{96\pi^3 \omega} \] (87)
with the second term much smaller than the first one at least for black holes with Schwarzschild radius bigger than the Planck length and typical frequencies. A more complete study considering non-vanishing dispersion of the mass of the background state, which will lead to a formula valid for all frequencies, with significant modifications at high frequencies, will be presented in a forthcoming publication.

Let us recall that in the canonical framework we are using, spin networks live on Cauchy surfaces, so strictly speaking the treatment we have done in this section takes place on a null surface far into the future and far from the black hole, but not technically on scri, which cannot be reached by the spin networks we consider.

VIII. CONCLUSIONS

We have studied the quantization of a scalar field on a quantum space time that approximates well the geometry of a Schwarzschild black hole. The treatment reproduces the results of quantum field theory on a classical space-time well, with some interesting differences. The presence of a discrete structure for the space-time eliminates the divergences associated with the Boulware and Unruh vacua arising from the trans-Planckian modes and only slight modifications for the Hartle–Hawking vacuum. All the different vacua’s modes change considerably on the horizons where all the singularities present in the usual analysis disappear.

We have carried out the analysis for a given spin network, but it is valid and can be extended without significant changes (except the one we will mention next) for generic refinements of the given spin network that include more vertices such that \( k_n \) grows monotonically with \( n \). The main difference with the behavior of the vacua in the continuum is the loss of local Lorentz invariance, that in the state we considered, would lead to corrections in the propagator in the asymptotic region of scale \( \Delta x^2 p^4 \). It is worthwhile pointing out that there exist states in the physical space of states where the corrections are much smaller than the Planck scale and therefore it is possible that the argument of Collins et al. [13] that implies that interactions in the perturbative treatment will lead to unacceptably large corrections to observable quantities may not be applicable. As it was emphasized in [14, 15], the Collins et al. argument requires
that the corrections be large at the Planck scale. To see that they are not large we note that \( r^2 = \ell_{\text{Planck}}^2 k \) with \( k \) an integer. One can have discretizations where \( k \) differ at most in one unit. In that case the radial distance goes from \( r = \ell_{\text{Planck}} \sqrt{k} \) to \( \ell_{\text{Planck}} \sqrt{k+1} \). Since \( \sqrt{k} = r/\ell_{\text{Planck}} \) we have that \( \Delta r = \ell_{\text{Planck}}^2/r < \ell_{\text{Planck}}^2/(2GM) \). These are corrections of order \( \ell_{\text{Planck}}^2 \) and therefore are small in the sense of the Collins et al. argument. It is premature to try to draw conclusions about Lorentz invariance at this stage, a proper study will require introducing interactions in the quantum fields and study the radiative corrections.

The cutoff the type of discreteness here considered introduces is similar in nature to the one considered by [12] and leads to a similar calculation of the Hawking radiation, which does not suffer significant modifications with respect to the continuum, at least for large black holes and typical frequencies.

Summarizing, we have shown that the midisuperspace formulation of loop quantum gravity with spherical symmetry is able to reproduce many features of standard analysis of quantum vacua in black hole space-times in the limit in which one considers a quantum test field living on a quantum space time. The discreteness of the quantum space-time has implications for some of the vacua even in regions of low curvature, in particular eliminating singularities. An interesting task ahead is to study in detail the type of regularization that the background introduces on the two point functions and to understand the origin of the Hadamard conditions. This opens the possibility of contemplating enhancing these computations with back reaction. We have only taken the first steps towards computing Hawking radiation in loop quantum gravity. A more complete treatment, including superpositions of the quantum spin network states and a more complete discussion of the properties of the Green’s functions of the theory, in particular their Lorentz invariance will be pursued in further publications.

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