Criticality in self-dual sine-Gordon models

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We discuss the nature of criticality in the $\beta^2 = 2\pi N$ self-dual extension of the sine-Gordon model. This field theory is related to the two-dimensional classical XY model with a N-fold degenerate symmetry-breaking field. We briefly overview the already studied cases $N = 2, 4$ and analyze in detail the case $N = 3$ where a single phase transition in the three-state Potts universality class is expected to occur. The $Z_3$ infrared critical properties of the $\beta^2 = 6\pi$ self-dual sine-Gordon model are derived using two non-perturbative approaches. On one hand, we map the model onto an integrable deformation of the $Z_4$ parafermion theory. The latter is known to flow to a massless $Z_3$ infrared fixed point. Another route is based on the connection with a chirally asymmetric, $su(2)_4 \otimes su(2)_1$ Wess-Zumino-Novikov-Witten model with anistropic current-current interaction, where we explore the existence of a decoupling (Toulouse) point.

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I. INTRODUCTION

The emergence of a non-trivial criticality in a conformal field theory (CFT) perturbed by several competing relevant operators has attracted much interest in recent years in the context of two-dimensional statistical mechanics or one-dimensional quantum systems 1, 2, 3, 4. When acting

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separately, each perturbation yields a massive field theory, but the interplay between them may
give rise to a second-order phase transition at intermediate coupling. The lack of integrability
in such models and the inapplicability of perturbation theory in this situation makes it difficult
to analyse the vicinity of the intermediate fixed point that separates physically different, strong-
coupling massive phases. However, a good understanding of such criticality can still be reached
in some special cases. A concrete example is the double-frequency sine-Gordon model (DSG)[1],
which is the Gaussian model of a bosonic field $\Phi$ perturbed by two scalar vertex operators with
the ratio of their scaling dimensions equal to 4. An Ising critical point occurs when the two
perturbations cannot be minimized simultaneously, i.e. when the two massive phases generated by
each perturbation cannot be connected by a continuous path in the parameter space of the model.
The nature of the resulting phase transition has been clarified by general arguments concerning the
excitation spectrum of the DSG model [1]. A non-perturbative treatment of the Ising critical point
was proposed in Ref. [2]. This approach is based on a quantum lattice version of the model that
enables one to clearly identify the fast and slow degrees of freedom of the problem. The existence of
the Ising criticality has also been confirmed using the truncated conformal space approach [3]. The
DSG model has interesting applications in one-dimensional quantum magnetism (a spontaneously
dimerized spin-1/2 Heisenberg chain in a staggered magnetic field [2], a two-leg spin-1 ladder [6])
and one-dimensional models of interacting electrons (the half-filled Hubbard model with alternating
chemical potential [2, 7], the quarter-filled electron system with dimerization [8]).

In this paper, we investigate the critical properties of another deformation of the sine-Gordon
model obtained by adding a second relevant vertex operator which depends on the field $\Theta$ dual to
the field $\Phi$. The action of this model is given by

$$S = \frac{1}{2} \int d^2 r \left( \partial_{\mu} \Phi \right)^2 + g \int d^2 r \cos (\beta \Phi) + \tilde{g} \int d^2 r \cos (\tilde{\beta} \Theta).$$

(1)

Notice that the two perturbations in Eq. (1), with scaling dimensions $\Delta_g = \beta^2/4\pi$ and $\Delta_{\tilde{g}} = \tilde{\beta}^2/4\pi$,
are mutually nonlocal and, therefore, cannot be minimized simultaneously. When $\Delta_g \neq \Delta_{\tilde{g}} < 2$,
the low-energy physics is governed by the most relevant operator, and the problem effectively
reduces to the standard sine-Gordon model, either for the field $\Phi$ ($\Delta_g < \Delta_{\tilde{g}}$) or for the dual field
$\Theta$ ($\Delta_{\tilde{g}} < \Delta_g$). In this case the resulting infrared (IR) theory is fully massive. However, the most
interesting situation arises when the two perturbations are both relevant and have the same scaling
dimension: $\beta = \tilde{\beta}$, $\Delta_g < 2$. Then the competition between these two antagonistic terms can lead
to a non-perturbative critical point (or line) at a finite coupling. Note that at $\beta = \tilde{\beta}$ and $g = \tilde{g}$ the
action (1) becomes invariant under the duality transformation $\Phi \leftrightarrow \Theta$. In what follows, such model
will be referred to as the self-dual sine-Gordon (SDSG) model. Its self-duality opens a possibility for the existence of a critical point. From the renormalization group (RG) point of view, the SDSG model will then be characterized by a massless flow from the ultraviolet (UV) Gaussian fixed point with central charge $c_{UV} = 1$ to a conformally invariant IR fixed point with a smaller central charge $c_{IR} < 1$ according to the Zamolodchikov’s $c$-theorem [9]. The underlying CFT will necessarily be a member of the minimal model series, since these are the only unitary CFTs with central charge $c < 1$ [11].

Much insight on the critical properties of the SDSG model can be gained by considering the related two-dimensional classical XY model with a N-fold symmetry-breaking field. The lattice Hamiltonian of this system reads:

$$\mathcal{H}_N = -J \sum_{<r,r'>} \cos (\theta_r - \theta_{r'}) + h \sum_r \cos (N \theta_r),$$

where $\theta_r$ is the angle of the unit-length rotor at site $r$ of a square lattice, and the symbol $<r,r'>$ indicates summation over the nearest-neighbor sites. The last term in Eq. (2) breaks the continuous $O(2)$ symmetry of the XY model down to a discrete one, $Z_N$ ($N = 2, 3, ...$).

The model (2) has a long history and has been extensively studied. At $N \geq 5$ there are two phase transitions as a function of the temperature [11]. For $T < T_{c1}$ the symmetry breaking field is dominant, and the system occurs in the broken symmetry phase in which one of the global directions $\theta = 2\pi n/N, n = 0, 1, ..., N - 1$ is preferred. At $T > T_{c2}$ the system is in a paramagnetic phase with unbroken $Z_N$ symmetry and short-range order. In the temperature window $T_{c1} < T < T_{c2}$ the system is in a Gaussian, XY-like phase with power-law correlations. However, for $N \leq 4$ such intermediate massless phase is absent, and a direct transition from the “ferromagnetic” phase to the paramagnetic one takes place. The type of the emerging criticality is determined by the symmetry of the model. Thus the criticality should belong to the Ising and three-state Potts universality classes at $N = 2$ and $N = 3$, respectively. A perturbative RG analysis predicts that the $N = 4$ criticality is characterized by nonuniversal, continuously varying exponents. Finally, in the $N = 1$ case, the $O(2)$ symmetry is broken at all temperatures. Similar conclusions have been reached for the $Z_N$ clock model [12] which is equivalent to the Hamiltonian (2) in the large-$h$ limit. The critical properties of the clock models have been further investigated by means of series expansions [12, 13] and exact diagonalization calculations on finite samples [14, 15].

Close to the transition point, all universal (long-distance) properties of the classical lattice model (2), or the related $Z_N$ clock model with $2 \leq N \leq 4$, are adequately described within a
In continuum description based on the effective action [16]:

\[ S = \frac{K}{2} \int d^2 r \left( \partial_\mu \Phi \right)^2 + g \int d^2 r \cos(2\pi \Theta) + h \int d^2 r \cos(N\Phi), \]  

(3)

where \( K \) is the stiffness of the Bose field \( \Phi \). The first two terms in Eq. (3) constitute the effective action of the O(2)-symmetric XY model, with the cosine of the dual field accounting for topological vortices (for a review see e.g. Refs. [17, 18, 19]), whereas the last term represents the symmetry breaking perturbation. By a simple rescaling of the Bose field, actions (3) and (1) become identical, with \( \beta = N/\sqrt{K} \) and \( \tilde{\beta} = 2\pi\sqrt{K} \), and the self-duality condition becomes \( \beta = \tilde{\beta} = \sqrt{2\pi N} \).

In this paper, we discuss the critical properties of the \( \beta^2 = 2\pi N \) SDSG model

\[ S_{SDSG} = \frac{1}{2} \int d^2 r \left( \partial_\mu \Phi \right)^2 + g \int d^2 r \left( \cos\left(\sqrt{2\pi N} \Phi\right) + \cos\left(\sqrt{2\pi N} \Theta\right)\right), \]  

(4)

at \( 2 \leq N \leq 4 \). The fact that at \( N = 2 \) and \( 4 \) the criticality belongs to the \( Z_N \) universality class is already well known and can be reproduced using the standard bosonization/refermionization techniques. For completeness, we review these cases in Section II. To the best of our knowledge, the case \( N = 3 \) is still lacking a consistent non-perturbative analytical description. The main difficulty stems from the fact that the underlying field theory, Eq. (4), does not admit a simple free field representation and, as opposed to the DSG model [2], it does not suggest any clear decomposition between the fast and slow degrees of freedom. The resulting IR fixed point is strongly non-perturbative in this respect. In what follows, the three-state Potts universality class of the criticality in the \( \beta^2 = 6\pi \) SDSG model will be derived using two different routes. First, we shall map the \( N=3 \) action (4) onto an integrable deformation of the \( Z_4 \) parafermion theory [20] and exploit the existence of a massless flow from this model to the three-state Potts criticality. On the other hand, we shall also relate the \( \beta^2 = 6\pi \) SDSG model to a chirally asymmetric version of a Wess-Zumino-Novikov-Witten (WZNW) model with \( su(2)_1 - su(2)_1 \) current-current interaction. That theory is integrable and, in the IR limit, displays the properties of the so-called chirally stabilized liquids [21]. From the symmetry of this IR fixed point, the three-state Potts criticality of the \( \beta^2 = 6\pi \) SDSG model will be deduced by considering a special anisotropic version of the above WZNW model. Finally, this approach will enable us to address the UV-IR transmutation of some fields of the \( \beta^2 = 6\pi \) SDSG model.

The rest of the paper is organized as follows. In section II we review the properties of the \( \beta^2 = 2\pi N \) SDSG model [3] in the simplest cases \( N = 1, 2, 4 \). The mapping of the \( \beta^2 = 6\pi \) SDSG model onto an integrable deformation of the \( Z_4 \) parafermion theory is presented in Section III. In
Section IV we study an anisotropic version of the WZNW model with $\text{su}(2)_4$ and $\text{su}(2)_1$ current-current interaction and address its relationship to the $\beta^2 = 6\pi$ SDSG model. Our concluding remarks are summarized in Section V. The paper is supplied with an Appendix which provides some details on the bosonization approach to the $\mathbb{Z}_4$ parafermion theory.

II. REVIEW OF SOME SIMPLE CASES

In this section, we overview the IR properties of the $\beta^2 = 2\pi N$ SDSG model (4) in the simple cases $N = 1, 2, 4$. The $\beta^2 = 2\pi$ SDSG model describes a massive field theory since the related lattice model (2) for $N = 1$ has no magnetic phase transition. In contrast, the $\beta^2 = 4\pi$ SDSG model displays critical properties in the Ising universality class, whereas the $\beta^2 = 8\pi$ case corresponds to a Gaussian CFT with central charge $c = 1$.

A. The $\beta^2 = 2\pi$ SDSG model

Let us first consider the case $N = 1$. The lattice Hamiltonian (2) describes the two-dimensional classical XY model in a magnetic field along the x-axis. The effective field theory associated with this system is given by the $\beta^2 = 2\pi$ SDSG model whose quantum Hamiltonian reads:

$$H_{\beta^2=2\pi} = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + g \left[ \cos \left( \sqrt{2\pi} \Phi \right) + \cos \left( \sqrt{2\pi} \Theta \right) \right].$$

The fact that no magnetic phase transition takes place in the $N = 1$ lattice model (2) can be inferred from the relationship between the field theory (3) and an explicitly dimerized spin-1/2 antiferromagnetic Heisenberg chain in a staggered magnetic field along the x-direction. The Hamiltonian of this quantum spin chain is

$$H = J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \Delta \sum_i (-1)^i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + h \sum_i (-1)^i S^x_i,$$

where $\mathbf{S}_i$ is the spin-1/2 operator at the lattice site $i$. In the low-energy limit, the standard spin-1/2 Heisenberg chain, given by the first term in Eq. (3), is described by the critical $\text{su}(2)_1$ WZNW model (22, 23) (with a marginally irrelevant current-current perturbation). This model, in turn, can be bosonized and recast as a simple Gaussian model with compactification radius $R = 1/\sqrt{2\pi}$ of the bosonic field $\Phi$:

$$H_0 = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right]$$

(7)
(for simplicity, we have set the velocity \( v = 1 \)). In the continuum description, the local spin density \( S(x) \) separates into the smooth and staggered parts:

\[
S(x) = J(x) + (-1)^{x/a} n(x),
\]

where \( a \) is the lattice spacing. The bosonized expression for the staggered magnetization is \[24\]

\[
n = \frac{\lambda}{\pi a} \left[ \cos \left( \sqrt{2\pi} \Theta \right), \sin \left( \sqrt{2\pi} \Theta \right), -\sin \left( \sqrt{2\pi} \Phi \right) \right],
\]

\[\text{(9)}\]

\( \lambda \) being a nonuniversal constant. Similarly, the dimerization operator transforms to

\[
(-1)^i S_i \cdot S_{i+1} \rightarrow \epsilon(x) = \frac{\lambda}{\pi a} \cos \left( \sqrt{2\pi} \Phi \right).
\]

\[\text{(10)}\]

Using the expressions \[\text{(8)}\] and \[\text{(10)}\] and properly fine tuning the coupling constants, one establishes the correspondence between the Hamiltonian \[\text{(6)}\] and the \( \beta^2 = 2\pi \) SDSG model, Eq. \[\text{(5)}\].

The next step is to perform a spin rotation along the \( y \)-axis in the Hamiltonian \[\text{(6)}\], so that the staggered magnetic field becomes applied in the \( z \)-direction. This transformation does not affect the dimerization term (which is SU(2) invariant) but otherwise makes the new continuum model dependent only on a Bose field \( \Phi' \) (no dual field in the interaction):

\[
H'_{\beta^2=2\pi} = \frac{1}{2} \left[ (\partial_x \Phi')^2 + (\partial_x \Theta')^2 \right] + g \left[ \cos \left( \sqrt{2\pi} \Phi' \right) - \sin \left( \sqrt{2\pi} \Phi' \right) \right].
\]

\[\text{(11)}\]

This Hamiltonian reduces to the standard \( \beta^2 = 2\pi \) sine-Gordon model by a shift \( \Phi' \rightarrow \Phi' - \sqrt{\pi}/4\sqrt{2} \). The latter is a fully massive integrable field theory. Thus, there is no quantum critical point in the phase diagram of the \( \beta^2 = 2\pi \) SDSG model.

**B. The \( \beta^2 = 4\pi \) SDSG model**

We now turn to the analysis of the \( N = 2 \) case when the Hamiltonian takes the form

\[
H_{\beta^2=4\pi} = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + g \left[ \cos \left( \sqrt{4\pi} \Phi \right) + \cos \left( \sqrt{4\pi} \Theta \right) \right].
\]

\[\text{(12)}\]

This model is well known (see e.g. Ref. \[23\]) and has a number of applications; for instance, it appears in the context of weakly coupled Heisenberg spin chains \[21, 27\]. It can be exactly diagonalized even in a more general case when the two cosine terms in Eq. \[\text{(12)}\] have independent amplitudes, \( g \) and \( \tilde{g} \). Since each of the two perturbation has scaling dimension 1 and, given the fact that the Hamiltonian does not possess any continuous symmetry, the model can be refermionized.
by introducing two Majorana fields, $\xi^{1,2}$. This procedure is nothing but the standard bosonization of two Ising models \[28, 29, 30\]. The bosonization rules are given by

$$
\xi_R^1 + i \xi_R^2 = \frac{1}{\sqrt{\pi}} : \exp\left(i\sqrt{4\pi} \Phi_R\right) :,
$$

$$
\xi_L^1 + i \xi_L^2 = \frac{1}{\sqrt{\pi}} : \exp\left(-i\sqrt{4\pi} \Phi_L\right) :,
$$

(13)

where $\Phi_{R,L}$ are the chiral components of the Bose field: $\Phi_L = (\Phi + \Theta)/2$ and $\Phi_R = (\Phi - \Theta)/2$.

These fields are normalized according to

$$
\langle \Phi_L(z) \Phi_L(w) \rangle = -\ln(z - w)/4\pi \quad \text{and} \quad \langle \Phi_R(\bar{z}) \Phi_R(\bar{w}) \rangle = -\ln(\bar{z} - \bar{w})/4\pi \quad \text{with} \quad z = \tau + ix \quad \text{and} \quad \bar{z} = \tau - ix \quad (\tau \text{ being the Euclidean time}).
$$

In the present work, all chiral Bose fields will be normalized in this way. One can easily check that the bosonic representation (13) is consistent with the standard operator product expansion (OPE) for the Majorana fields:

$$
\xi^a_L(z) \xi^b_L(w) \sim \frac{\delta^{ab}}{2\pi(z - w)},
$$

$$
\xi^a_R(\bar{z}) \xi^b_R(\bar{w}) \sim \frac{\delta^{ab}}{2\pi(\bar{z} - \bar{w})}.
$$

(14)

Strictly speaking, the doublet of identical critical Ising copies is not equivalent to the CFT of a free massless Dirac fermion since the two Majorana fermions in Eq. (13) are not independent but constrained to share the same type of boundary conditions. The Dirac CFT can be bosonized using a free massless scalar field compactified on a circle with radius $R = 1/\sqrt{4\pi}$ in our notation. In contrast, two decoupled Ising models are described by a Bose field living on the orbifold line with the same radius (for a review on this identification, see for instance Refs. \[31, 32\]). Nonetheless, as far as the bulk properties of the $\beta^2 = 4\pi$ SDSG model (12) are concerned, this subtlety does not manifest itself and the correspondence (13) can be safely applied. The self-dual Hamiltonian (12) can then be expressed in terms of these Majorana fermions:

$$
\mathcal{H}_{\beta^2 = 4\pi} = -\frac{i}{2} \sum_{a=1}^2 (\xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L) + im\xi^2_R\xi^2_L,
$$

(15)

with $m = 2\pi g$. One thus observes that the Hamiltonian of the $\beta^2 = 4\pi$ SDSG model separates into two commuting pieces. One of the decoupled degrees of freedom corresponds to an effective off-critical Ising model described by the massive Majorana fermion $\xi^2_{R,L}$, whereas the second Majorana field $\xi^1_{R,L}$ remains massless. (In the two-parameter model with $g \neq \tilde{g}$ both sectors are massive: $m_1 = \pi(g - \tilde{g})$ and $m_2 = \pi(g + \tilde{g})$.) The existence of a massless Majorana mode signals the $\mathbb{Z}_2$ (Ising) criticality of the $\beta^2 = 4\pi$ SDSG model (12).
C. The $\beta^2 = 8\pi$ SDSG model

Here we briefly review the remaining simple case, $\beta^2 = 8\pi$, when the Hamiltonian has the form:

$$\mathcal{H}_{\beta^2=8\pi} = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + g \left[ \cos \left( \sqrt{8\pi} \, \Phi \right) + \cos \left( \sqrt{8\pi} \, \Theta \right) \right]. \quad (16)$$

The self-dual interaction is now marginal, so it is natural to expect a critical behavior. The perturbative RG approach is applicable to this case and indicates the existence of a line of $c=1$ fixed points \cite{11, 16, 18, 33, 34}. The $\beta^2 = 8\pi$ SDSG model emerges in the problem of the one-dimensional Fermi gas with backscattering and spin-nonconserving processes \cite{35}, and it also describes critical properties of weakly coupled Luttinger chains \cite{36}. The model (16) is also related to the quantum Ashkin-Teller model (two identical quantum Ising chains coupled by a self-dual interchain coupling, see e.g. Ref. \cite{2}).

A simple way to clarify the nature of the criticality of the $\beta^2 = 8\pi$ SDSGM is to map the Hamiltonian (16) onto an anisotropic version of the su(2)$_1$ WZNW model with a current-current interaction. This can be done by exploiting the fact that the chiral currents of the su(2)$_1$ Kac-Moody (KM) algebra have a free-field representation in terms of a massless Bose field $\Phi$ (see Ref. \cite{37}):

$$J^a_L = \frac{i}{\sqrt{2\pi}} \partial \Phi_L,$$

$$J^a_L = \frac{1}{2\pi} \exp \left( i\sqrt{8\pi} \Phi_L \right),$$

$$J^a_R = \frac{-i}{\sqrt{2\pi}} \partial \Phi_R,$$

$$J^a_R = \frac{1}{2\pi} \exp \left( -i\sqrt{8\pi} \Phi_R \right), \quad (17)$$

with $J^a_{L,R} = J^a_{L,R} \pm i J^a_{L,R}$. Representation (17) correctly reproduces the su(2)$_1$ KM algebra ($a = x, y, z$):

$$J^a_L (z) J^b_L (w) \sim \frac{\delta^{ab}}{8\pi^2 (z - w)^2} + \frac{ie^{abc} J^c_L (w)}{2\pi (z - w)},$$

$$J^a_R (\bar{z}) J^b_R (\bar{w}) \sim \frac{\delta^{ab}}{8\pi^2 (\bar{z} - \bar{w})^2} + \frac{ie^{abc} J^c_R (\bar{w})}{2\pi (\bar{z} - \bar{w})}, \quad (18)$$

where $\epsilon^{abc}$ is the totally antisymmetric tensor. The Hamiltonian (16) can then be entirely expressed in terms of the vector currents:

$$\mathcal{H}_{\beta^2=8\pi} = \frac{2\pi}{3} (J_L \cdot J_L + J_R \cdot J_R) - 8\pi^2 g \, J^x_L J^x_R. \quad (19)$$
Due to the SU(2) symmetry of the unperturbed Hamiltonian, the interaction can alternatively be expressed in terms of the z-component of the currents. The resulting Hamiltonian can be further bosonized using the correspondence (17):

$$\mathcal{H}_{\beta^2=8\pi} = \frac{v}{2} \left[ (\partial_x \Phi)^2 + \frac{1}{K} (\partial_x \Phi)^2 \right],$$

where the role of interaction $g$ in Eq. (19) is exhausted by renormalization of the velocity and compactification radius: $v^2 = 1 - 4\pi^2 g^2$ and $K^2 = (1 + 2\pi g)/(1 - 2\pi g)$. We thus deduce that the $\beta^2 = 8\pi$ SDSG model displays Gaussian critical properties parameterized by the exponent $K$. Correlation functions in this model are characterized by continuously varying ($K$ dependent) critical exponents, which is a distinctive feature of the Luttinger-liquid universality class [38].

### III. MAPPING OF THE $\beta^2 = 6\pi$ SDSG MODEL ONTO AN INTEGRABLE DEFORMATION OF THE $Z_N$ PARAFERMION THEORY

Now we turn to our main problem and discuss the $\beta^2 = 6\pi$ SDSG model. To determine the nature of the IR fixed point, in this Section we exploit a mapping onto an integrable deformation of the $Z_N$ parafermion CFT, discovered some time ago by Fateev and Zamolodchikov [20]. The $Z_N$ parafermion CFT [39] has central charge $c_N = 2(N - 1)/(N + 2)$ and, in addition to the chiral components of the stress-energy tensor, is characterized by a chiral algebra containing $N - 1$ left (respectively, right) parafermionic currents $\Psi_{kL}, k = 1, \ldots, N - 1$ (respectively, $\Psi_{kR}$). At $N > 2$ these currents generalize the Majorana fermion of the Ising ($Z_2$) model and are primary fields of the Virasoro algebra with fractional spin $h_k = k(N - k)/N$. Two integrable deformations of the $Z_N$ CFT are known [20]:

$$\mathcal{H}_1 = \mathcal{H}^0 (Z_N) + \lambda \left( \Psi_{1L} \Psi_{1R} + \Psi^\dagger_{1R} \Psi^\dagger_{1L} \right),$$

$$\mathcal{H}_2 = \mathcal{H}^0 (Z_N) + \lambda \left( e^{i\pi/N} \Psi_{1L} \Psi_{1R} + e^{-i\pi/N} \Psi^\dagger_{1R} \Psi^\dagger_{1L} \right),$$  \hspace{1cm} (21)

where $\Psi^\dagger_{(1L,R)} = \Psi_{N-1(L,R)}$ and $\mathcal{H}^0 (Z_N)$ stands for the Hamiltonian density of the $Z_N$ CFT which has no explicit form in the general $N$ case. Both quantum field theories have been studied in details and several exact results have been derived [10, 41]. In what follows, we shall adopt the prescription that the left and right parafermionic fields commute between themselves.

It has been shown in Ref. [42] that the first model in Eq. (21) with $N = 4$ is equivalent to the $\beta^2 = 6\pi$ sine-Gordon model, which is a massive field theory. The identification follows from an
observation that the two theories share the same S-matrix. On the other hand, the second model in Eq. (21) describes an integrable deformation of the $Z_N$ CFT characterized by a massless flow from the UV $Z_N$ fixed point to the IR one corresponding to the minimal model series $\mathcal{M}_{N+1}$ with central charge $1 - 6/(N + 1)(N + 2)$. A massless thermodynamic Bethe ansatz (TBA) system associated with this flow has been conjectured in Ref. [20], which in the large-$N$ limit reduces to that of the O(3) non-linear sigma model with a topological term $\theta = \pi$. For $N = 4$, the massless TBA equations correspond to the $D_4$ Dynkin graph with driving terms at two legs of the graph connected via the respective incidence matrix. Using the appropriate dilogarithm identities, we have verified that the above TBA system interpolates between $c_{UV} = 1$ and $c_{IR} = 4/5$, as stated in Ref. [20]. Thus, the Hamiltonian $\mathcal{H}_2$ with $N = 4$ displays the IR critical properties falling into the three-state Potts universality class ($\mathcal{M}_5$). It is then tempting to conjecture that the Hamiltonian $\mathcal{H}_2$ with $N = 4$ should be related to that of the $\beta^2 = 6\pi$ SDSG model. In the rest of this Section we show that they are indeed equivalent.

It proves instructive to bosonize the two models in Eq. (21). For arbitrary $N$, the parafermionic currents can be expressed in terms of a suitable set of Bose fields using the Feigin-Fuchs or coset constructions [43, 44, 45]. However, the case $N = 4$ is special since the $Z_4$ theory is characterized by central charge $c = 1$ and so it should be possible to realized it by a single free Bose field. In fact, the identification on the level of the partition function has been done by Yang [46], and the $Z_4$ parafermion CFT with a diagonal modular invariant partition function turns out to be equivalent to a Bose field living on the orbifold line at radius $R = \sqrt{3/2\pi}$ in our normalization. However, it is still possible to bosonize the $Z_4$ parafermionic currents $\Psi_{1,2L,R}$ with a simple (periodic) Bose field defined on the circle with radius $R = \sqrt{3/2\pi}$, as argued in the Appendix. In particular, a bosonic representation for the first left $Z_4$-parafermionic current reads

$$\Psi_{1L} = \frac{1}{\sqrt{2}} \left[ : \exp \left( i\sqrt{6\pi} \Phi_L \right) : + e^{i\sqrt{3\pi/2} p_L} : \exp \left( -i\sqrt{6\pi} \Phi_L \right) : \right], \quad (22)$$

where the zero mode momentum operator $p_L$ has the following discrete spectrum: $p_L = n\sqrt{2\pi/3} + m\sqrt{6\pi}$, $n$ and $m$ being integers as a consequence of the compactification of the Bose field at radius $R = \sqrt{3/2\pi}$. A similar construction can be done in the right sector by introducing a right Bose field $\Phi_R$. So $\mathcal{H}_1$ and $\mathcal{H}_2$ in Eq. (21) can, in turn, be expressed in terms of the total Bose field, $\Phi = \Phi_L + \Phi_R$, and its dual, $\Theta = \Phi_L - \Phi_R$. For the first model we find

$$\mathcal{H}_1 = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + 2\lambda : \cos \left( \sqrt{6\pi} \Phi \right) :, \quad (23)$$

which is nothing but the $\beta^2 = 6\pi$ sine-Gordon model, in full agreement with the findings of Ref.
In contrast, we obtain a generalized $\beta^2 = 6\pi$ sine-Gordon model for $H_2$:

$$H_2 = \frac{1}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + \lambda \sqrt{2} \left[ \cos \left( \sqrt{6\pi} \Phi \right) + i e^{i\sqrt{3\pi/2}p_L} \cos \left( \sqrt{6\pi} \Theta \right) \right]. \quad (24)$$

The cocycle operator that enters this equation takes the values $e^{i\sqrt{3\pi/2}p_L} = \pm 1$; this follows from the discrete spectrum of the zero-mode momentum operator. Moreover, one should note that this cocycle operator anticommutes with $\cos(\sqrt{6\pi}\Theta)$, as it can be easily seen from the mode decomposition; see Eq. (A10) of the Appendix. This ensures hermiticity of the Hamiltonian (24). The cocycle operator in Eq. (24) can be absorbed in a redefinition of the dual Bose field. Therefore, as far as the bulk properties of the model are concerned, the Hamiltonian (24) is equivalent to that of the $\beta^2 = 6\pi$ SDSG model (4). From this correspondence and from the existence of the massless flow of the integrable deformation of the $Z_4$ CFT ($H_2$ in Eq. (21)), we finally conclude that the $\beta^2 = 6\pi$ SDSG model displays the three-state Potts criticality.

IV. THE $\beta^2 = 6\pi$ SDSG MODEL AND CHIRALLY STABILIZED LIQUIDS

In this Section, the nature of the critical point of the $\beta^2 = 6\pi$ SDSG model will be studied using a different approach. Namely, we will make contact with a weakly perturbed, chirally asymmetric WZNW model. Field theories of this kind have recently attracted much interest in connection with the universality class of chirally stabilized liquids, introduced by Andrei, Douglas, and Jerez [21].

A. Chirally stabilized liquids

Consider an $[\text{su}(2)_N]_R \otimes [\text{su}(2)_k]_L$ invariant WZNW model perturbed by a marginal current-current interaction. The condition $N > k$ makes the model chirally asymmetric. The Hamiltonian density reads:

$$H = \frac{2\pi v}{N+2} J_R \cdot J_R + \frac{2\pi v}{k+2} J_L \cdot J_L + g J_R \cdot J_L, \quad (25)$$

where $J_R$ and $J_L$ are the right $\text{su}(2)_N$ and left $\text{su}(2)_k$ chiral vector currents, $v$ being the velocity. The OPE for the $\text{su}(2)_N$ current in the right sector is conventionally defined as

$$J_R^a(\bar{z}) J_R^b(\bar{w}) \sim \frac{N \delta^{ab}}{8\pi^2 (\bar{z} - \bar{w})^2} + \frac{i \epsilon^{abc}}{2\pi (\bar{z} - \bar{w})} J_R^c(\bar{w}), \quad (26)$$

with an analogous expression for the holomorphic (left) current ($N$ to be replaced by $k$). A simple one-loop RG analysis shows that the interaction in Eq. (25) is marginally relevant for a positive
coupling constant $g$, so that the UV fixed point is unstable. When the chiral symmetry is restored ($N = k$), the Hamiltonian $H$ is nothing but the standard $\text{su}(2)_N$ WZNW model with a marginally relevant current-current interaction. In that case, it is well known that this model has a spectral gap generated by the interaction. The low energy excitations consist of massive kinks and antikinks. Since the one-loop beta function does not depend on the levels of the $\text{su}(2)$ KM algebras in the right and left sectors, one might conclude that at $N > k$ the chiral asymmetry does not play any role, and in the IR limit the system will enter a massive phase. However, this naive picture is not correct. Instead there is a massless flow towards an conformally invariant fixed point. The existence of this non-trivial criticality has been discovered by Polyakov and Wiegmann who studied a related fermionic model in a special limit where $N$ and $k$ are sent to infinity at a fixed difference $N - k > 0$. In particular, they argued that the criticality results from the chiral excess of particles in the problem and belongs to the $\text{su}(2)_{N-k}$ WZNW universality class. Physically, this means that a finite excess of the right movers over the left movers makes it impossible to bind all chiral particles into a gapped state, so that, in the IR limit, some degrees of freedom ought to remain massless.

For finite values of $N$ and $k$, the symmetry of the IR criticality in the model $H$ was shown to be

$$[\text{su}(2)_{N-k}]_R \otimes \left[\frac{\text{su}(2)_k \times \text{su}(2)_{N-k}}{\text{su}(2)_N}\right]_L.$$  

(27)

This result is consistent with several facts. First of all, the global SU(2) symmetry should remain intact at the IR fixed point. The difference between the right and left central charges, as well as the difference between the levels of the chiral KM algebras, should be preserved under the RG flow. The model is integrable and the IR central charge can be extracted by means of the TBA approach. This has been done in Ref. and it was found that $c_{IR}$ coincides with that of the CFT given by Eq. (27). The fixed point with the symmetry (27) has been checked for the two channel case ($N = 2, k = 1$) by utilizing the existence of a decoupling (Toulouse) point where the mapping onto Majorana fermions solves the problem. Finally, Leclair has recently obtained the all-orders beta-function for the model confirming the existence of the stable IR fixed point at $N \neq k$.

It is worth mentioning here that the fixed point with the symmetry (27) provides an example of a non-Fermi-liquid which is characterized by universal critical exponents and in this respect differs from the standard Luttinger liquid. This state has been dubbed the chirally stabilized liquid.
since the very existence of the critical fixed point follows from the chiral asymmetry of the model. Several strongly correlated systems have been found displaying similar properties in the low-energy limit. Chiral asymmetry can, of course, result from broken time reversal invariance. An example is edge states in a paired sample of integer quantum Hall systems with different filling factors [21]. On the other hand, it is also possible that the field-theoretical Hamiltonian, describing universal properties of a lattice model invariant under time reversal, decomposes into two commuting and chirally asymmetric parts, $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not separately time-reversal invariant but transform to each other under $t \rightarrow -t$. This scenario is realized in certain one-dimensional quantum systems, such as the three-leg spin ladder with crossings [50] and the Kondo-Heisenberg chain [50, 51, 53, 54, 55].

Relevant to our purposes is the four-channel case ($N = 4$, $k = 1$) for which the symmetry of the IR fixed point is

$$[su(2)_3]_R \otimes \left[ \frac{su(2)_1 \times su(2)_3}{su(2)_4} \right]_L = [su(2)_3]_R \otimes [M_5]_L,$$

(28)

where we have used, in the left sector, the coset construction [50] of the minimal model $M_5$ with central charge $c = 4/5$. Expression (28) can be further simplified using the coset $[39] su(2)_3/u(1)_3 \sim Z_3$, where $u(1)_3$ is a rational $c = 1$ CFT [57] realized by a Bose field with compactification radius $R = \sqrt{3/2\pi}$. Finally, it is known that the $Z_3$ CFT describes the three-state Potts model (see for instance Ref. [32]), so that the symmetry of the IR fixed point in the four-channel case is given by

$$[u(1)_3]_R \otimes Z_3,$$

(29)

where $Z_3$ stands for the full, chirally invariant, three-state Potts model. In the approach suggested in this Section, we first need to find a procedure to extract those bosonic degrees of freedom from the original Hamiltonian (25) which account for the “redundant” $u(1)$ criticality in the right sector of Eq. (29). The remaining degrees of freedom will then describe the three-state Potts universality class. The last step of the procedure will be to relate these degrees of freedom to the $\beta^2 = 6\pi$ SDSG model.

**B. Decoupling bosonic degrees of freedom**

Let us consider the original model (25) at $N = 4$ and $k = 1$ with anisotropic current-current interaction:

$$\mathcal{H} = \frac{\pi v}{3} \mathbf{J}_R \mathbf{J}_R + \frac{2\pi v}{3} \mathbf{J}_L \mathbf{J}_L + g_\parallel J^z_R J^z_L + \frac{g_\perp}{2} \left( J^+_R J^-_L + H.c. \right).$$

(30)
We will assume that $g_\parallel > 0$. This condition ensures that the model flows towards strong coupling in the IR limit. The sign of the transverse coupling constant, $g_\perp$, is arbitrary ($g_\perp \to -g_\perp$ under the transformations $J_\pm^R \to -J_\pm^R, J_0^R \to J_0^R$). The important feature of the anisotropic model (30) is that it displays the same universal behavior (29) as in the fully SU(2)-symmetric case. Restoration of the SU(2) symmetry at IR fixed point can be inferred from the recently computed all-orders beta function of the model (30) and, since the anisotropic version is still integrable, by means of the Bethe-ansatz approach. The advantage of the anisotropic model is that it becomes particularly simple in the so-called Toulouse limit, similar to the decoupling (or Luther-Emery point) in the sine-Gordon model. There exists a special line in the parameter space of the coupling constants along which certain bosonic degrees of freedom decouple from the rest of the spectrum and remain massless in the IR limit. Toulouse type solutions proved extremely fruitful in recent years, especially in quantum impurity problems and, most notably, in the two-channel Kondo problem.

The starting point of the Toulouse solution is the introduction of a bosonized description for the su(2)$_4$ KM current. Such identification can be derived from the coset $su(2)$_4$/u(1)$_4 \sim Z_4$ which relates the su(2)$_4$ KM algebra to the $Z_4$ parafermion CFT through the u(1)$_4$ rational CFT, corresponding to a Bose field living on the circle at the radius $R = \sqrt{2/\pi}$. The right su(2)$_4$ current ($J_R^R$) can then be expressed in terms of a chiral Bose field $\Phi_{sR}$ and the first $Z_4$ parafermion current $\psi_1^R$ via

$$

case{a}{J^+_R} = \frac{1}{\pi} \psi_{1R} : \exp \left( -i\sqrt{2\pi} \Phi_{sR} \right) :,

case{b}{J^-_R} = \frac{1}{\pi} : \exp \left( i\sqrt{2\pi} \Phi_{sR} \right) \psi_{1R}^+ :,

case{c}{J^z_R} = \frac{1}{\sqrt{2\pi}} \partial_x \Phi_{sR} = \frac{1}{\sqrt{2\pi}} \partial_x \Phi_{sR},
$$

with the prescription that the parafermionic fields commute with the bosonic ones. Then, using the parafermionic algebra, it is not difficult to show that the correspondence (31) reproduces OPE (26) with $N = 4$. In the left sector, we introduce a Bose field at the self-dual radius $R_0 = 1/\sqrt{2\pi}$ to obtain a bosonized description of the su(2)$_1$ KM current as in Eq. (17)

$$

case{a}{J^+_L} = \frac{1}{2\pi} : \exp \left( i\sqrt{8\pi} \Phi_{0L} \right) :,

case{b}{J^-_L} = \frac{i}{\sqrt{2\pi}} \partial \Phi_{0L} = \frac{1}{\sqrt{2\pi}} \partial_x \Phi_{0L},
$$

(we are working here with left and right moving Bose fields that commute with themselves).
The Hamiltonian (30) can be expressed in terms of the bosonic and parafermionic fields as follows:

\[
H = v \left[ (\partial_x \Phi_0 L)^2 + (\partial_x \Phi_{sR})^2 \right] + \frac{g_\parallel}{\pi} \partial_x \Phi_0 L \partial_x \Phi_{sR} + H^0_R (Z_4) + \frac{g_\perp}{4\pi^2} \left[ \psi_{1R} : \exp \left( -i\sqrt{2\pi} \Phi_{sR} - i\sqrt{8\pi} \Phi_0 L \right) : + H.c. \right],
\]

where \( H^0_R (Z_4) \) is the right-moving piece of the Hamiltonian density associated to the \( Z_4 \) CFT. The cross derivative terms in Eq. (33) can be eliminated by performing a canonical transformation of the Bose fields (this is the standard procedure that solves the Luttinger model; see e.g. Ref. [38]):

\[
\begin{pmatrix} \Phi_0 L \\ \Phi_{sR} \end{pmatrix} = \begin{pmatrix} \text{ch} \alpha & \text{sh} \alpha \\ \text{sh} \alpha & \text{ch} \alpha \end{pmatrix} \begin{pmatrix} \Phi_2 L \\ \Phi_1 R \end{pmatrix},
\]

with

\[
\text{th} 2\alpha = - \frac{g_\parallel}{2\pi v}.
\]

Under this transformation, the argument of the vertex operator in Eq. (33) becomes

\[
\sqrt{2\pi} \Phi_{sR} + \sqrt{8\pi} \Phi_0 L \rightarrow \sqrt{2\pi} \left[ (2 \text{ch} \alpha + \text{sh} \alpha) \Phi_2 L + (\text{ch} \alpha + 2 \text{sh} \alpha) \Phi_1 R \right].
\]

We then observe that for a special value of \( \alpha \) determined by the condition

\[
\text{th} \alpha = - \frac{1}{2},
\]

(the corresponding (nonuniversal) value of the coupling \( g_\parallel \) is \( g_\parallel^* = 8\pi v / 5 \)) the Bose field \( \Phi_1 R \) decouples from the rest of the Hamiltonian, so that the degrees of freedom described by this field will remain critical. This is the main feature of the Toulouse point. In the new basis, the Hamiltonian (38) transforms to

\[
H = u \left[ (\partial_x \Phi_{1R})^2 + (\partial_x \Phi_{2L})^2 \right] + H^0_R (Z_4) + \frac{g_\perp}{4\pi^2} \left[ \psi_{1R} : \exp \left( -i\sqrt{6\pi} \Phi_{2L} \right) : + H.c. \right],
\]

with the renormalized velocity \( u = 3v / 5 \). At the Toulouse point, the transformation (34) of the chiral Bose fields is simplified:

\[
\begin{align*}
\Phi_0 L &= \frac{1}{\sqrt{3}} (2\Phi_{2L} - \Phi_{1R}) \\
\Phi_{sR} &= \frac{1}{\sqrt{3}} (2\Phi_{1R} - \Phi_{2L}),
\end{align*}
\]

the inverse transformation being

\[
\begin{align*}
\Phi_{1R} &= \frac{1}{\sqrt{3}} (2\Phi_{sR} + \Phi_0 L) \\
\Phi_{2L} &= \frac{1}{\sqrt{3}} (2\Phi_0 L + \Phi_{sR}).
\end{align*}
\]
C. Three-state Potts criticality

At this point, let us pause to discuss the situation at hand. Starting from the anisotropic model (30), the decoupling of the right-moving bosonic degree of freedom described by the field $\Phi_{1R}$ has been achieved at the Toulouse point. Moreover, this Bose field is compactified with radius $R = \sqrt{\frac{3}{2\pi}}$, as can be seen from Eq. (40) and the values of the radii for $\Phi_0$ and $\Phi_s$. We thus deduce that the Bose field $\Phi_{1R}$ describes a chiral $[u(1)_3]_R$ rational CFT. The question is how to interpret the remaining degrees of freedom described by the Bose field $\Phi_{2L}$ and the $Z_4$ parafermionic field $\psi_{1R}$, which, according to Eq. (38), are nontrivially coupled. As already stressed in the preceding subsection, in the anisotropic model (30) the SU(2) symmetry is asymptotically restored in the IR limit and the symmetry of the IR fixed point is thus still given by Eq. (29). This observation leads us to conclude that the Hamiltonian

$$H_{\text{eff}} = u \left( \partial_x \Phi_{2L} \right)^2 + H_0^R (Z_4) + \frac{g_{\perp}}{4\pi^2} \left[ \psi_{1R} : \exp \left( -i\sqrt{6\pi}\Phi_{2L} \right) : + \text{H.c.} \right],$$

(41)

describes critical properties of the three-state Potts model.

Thus, we are left to relate the Hamiltonian (41) to the $\beta^2 = 6\pi$ SDSG model. To this end, we introduce a right-moving Bose field $\varphi_R$ to bosonize the $Z_4$ parafermion fields which enter Eq. (41). Using representation (A12) of the Appendix in the right sector, we transform the Hamiltonian (41) to

$$H_{\text{eff}} = u \left[ (\partial_x \Phi_{2L})^2 + (\partial_x \varphi_R)^2 \right] + \frac{g_{\perp} \sqrt{2}}{4\pi^2} \left[ \cos \left( \sqrt{6\pi} (\varphi_R - \Phi_{2L}) \right) : \right.

$$

$$- i e^{-i\sqrt{3\pi/2}p_R} : \sin \left( \sqrt{6\pi} (\varphi_R + \Phi_{2L}) \right) : \right],$$

(42)

where we have neglected the velocity anisotropy between the two Bose fields. The zero mode momentum $p_R$ associated with the chiral Bose field $\varphi_R$ has a discrete spectrum: $p_R = n \sqrt{2\pi/3} - m \sqrt{6\pi}$, so that the cocycle operator in Eq. (42) takes the values: $e^{i\sqrt{3\pi/2}p_R} = \pm 1$. The two chiral Bose fields in Eq. (42) can be combined into a total Bose field $\Phi = \varphi_R + \Phi_{2L}$ and its dual field $\Theta = \Phi_{2L} - \varphi_R$. The above Hamiltonian then simplifies as

$$H_{\text{eff}} = \frac{u}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + \frac{g_{\perp} \sqrt{2}}{4\pi^2} \left[ : \cos \left( \sqrt{6\pi} \Theta \right) : - i e^{-i\sqrt{3\pi/2}p_R} : \sin \left( \sqrt{6\pi} \Phi \right) : \right].$$

(43)

By absorbing the cocycle operator into the Bose field, the resulting Hamiltonian shares the same bulk properties as

$$H_{\text{eff}} = \frac{u}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + \frac{g_{\perp} \sqrt{2}}{4\pi^2} \left[ : \cos \left( \sqrt{6\pi} \Theta \right) : + : \cos \left( \sqrt{6\pi} \Phi \right) : \right].$$

(44)
which is nothing but the Hamiltonian of the $\beta^2 = 6\pi$ SDSG model. Thus we arrive at the same conclusion as in the end of section III i.e. the $\beta^2 = 6\pi$ SDSG model displays critical properties in the three-state Potts universality class.

D. The UV-IR transmutation of the fields

The next important point is the determination of the UV-IR transmutation associated with the massless flow of the $\beta^2 = 6\pi$ SDSG model. To this end, one has to find out how the operators, originally defined in the vicinity of the UV fixed point, transmute into the three-state Potts fields when going from the UV limit to the IR limit. Unfortunately, this is not an easy task since the $\beta^2 = 6\pi$ SDSG model is a non-trivial field theory. In particular, it does not admit any simple decomposition between the massive and critical degrees of freedom which was crucial a step to perform the UV-IR transmutation of the DSG model [2]. Furthermore, the UV-IR correspondence of the integrable deformation of the $\mathbb{Z}_4$ parafermion theory [21], equivalent to the $\beta^2 = 6\pi$ SDSG model, is unknown to the best of our knowledge. However, we shall here present some conjectures on the UV-IR transmutation of some fields using the massless flow of the chirally asymmetric $\text{su}(2)_4 \otimes \text{su}(2)_1$ WZNW model [25] and the Toulouse limit solution.

Let us first discuss more precisely the UV limit of the $\beta^2 = 6\pi$ SDSG model (44). It corresponds to a $c = 1$ CFT described by a bosonic field living on the circle at radius $R = \sqrt{3/2\pi}$. At this special radius, the resulting $\text{u}(1)_3$ CFT exhibits an extended symmetry algebra $\mathcal{A}_3$ [57] which is generated in the left sector by the standard left $\text{u}(1)$ current $J_L = i\partial \Phi_L$ together with extra left currents with spin 3: $\Gamma^\pm_L =: \exp\left(\pm i\sqrt{21\pi/3}\Phi_L\right) :$. Under this extended algebra, the $\text{u}(1)_3$ CFT has a finite number of primary fields and is thus an example of a rational CFT. The partition function of this CFT on the torus is given by Eq. (A8) of the Appendix and the six primary fields are vertex operators $V_\lambda$ mutually local with the currents of the extended symmetry:

$$V_\lambda =: \exp\left(i\lambda\sqrt{2\pi/3}\Phi\right) :, \quad (45)$$

with $\lambda = 0, \pm 1, \pm 2, 3$ and conformal weights ($\lambda^2/12, \lambda^2/12$). Interestingly, one can associate a $\mathbb{Z}_3$ charge $q$ to the vertex operators $V_\lambda$ [15] through: $q \equiv \lambda \text{ (mod 3)}$, $\lambda$ being integer. This charge is additive under fusion of these vertex operators. This $\mathbb{Z}_3$ symmetry is simply generated by a shift on the Bose field: $\Phi \rightarrow \Phi + \sqrt{2\pi/3}$. We note that the perturbing field of the $\beta^2 = 6\pi$ SDSG model is neutral with respect to this $\mathbb{Z}_3$ symmetry. In fact, the explicit description of the $\mathbb{Z}_3$ symmetry of the $\beta^2 = 6\pi$ SDSG model can also be derived by considering its lattice version i.e. the two-dimensional
classical XY model with a three-fold symmetry breaking field (see Eq. (2) with $N = 3$). The $Z_3$ symmetry of this model is described by the following transformation on the lattice variable $\theta_r$: $\theta_r \to \theta_r + 2\pi/3$ so that in the continuum limit it will correspond to $\Phi \to \Phi + \sqrt{2\pi/3}$ as can be easily seen.

We now turn to the analysis of the UV-IR correspondence of the $\beta^2 = 6\pi$ SDSG model by considering the chirally asymmetric $su(2)_4 \otimes su(2)_1$ WZNW model (25). As already discussed, this latter model is characterized by the following massless flow:

$$[su(2)_4]_L \otimes [su(2)_1]_R \to [su(2)_3]_R \otimes \frac{su(2)_1 \times su(2)_3}{su(2)_4},$$

The resulting UV-IR transmutation can be analysed from the conservation of the $su(2)$ spin and one has the following correspondence [21, 54]:

$$\Phi^{(l_1)}_1 \Phi^{(l_2)}_4 \sim \sum_{|l_1-l_2| \leq l_3 \leq l_1+l_2} \Phi^{(l_3)}_3 \varphi^{l_1l_3}_{l_2L},$$

where the $su(2)_k$ primaries are denoted by $\Phi^{(l)}_k$ with $l = 0, 1, \ldots, k$ and carry spin $l/2$. In Eq. (47), $\varphi^{l_1l_3}_{l_2L}$ ($0 \leq l_1 \leq 1, 0 \leq l_2 \leq 4, 0 \leq l_3 \leq 3$) are the left fields of the coset in Eq. (46) with conformal weight $l_1(l_1+2)/12 + l_3(l_3+2)/20 - l_2(l_2+2)/24$. Branching selection rules restrict $l_1 + l_2 + l_3$ to be even and one has also the field identification: $\varphi^{l_1l_3}_{l_2L} \sim \varphi^{1-l_13-l_2}_{4-l_2L}$.

The strategy to determine the UV-IR correspondence of the $\beta^2 = 6\pi$ SDSG model is to choose special values of $l_1$ and $l_2$ so that the primary fields in the lhs of Eq. (47) have a simple free-field representation. The next step is to make use of the Toulouse basis (39) to express these fields in terms of the $\beta^2 = 6\pi$ SDSG UV operators and vertex operators built from the $\Phi_{1R}$ bosonic field. By extracting the IR $u(1)$ criticality in Eq. (46) associated with the field $\Phi_{1R}$, one can expect to obtain a representation of some IR operators of the three-state Potts model in terms of the original UV fields of the $\beta^2 = 6\pi$ SDSG model. For instance, the current-current perturbation of the $su(2)_4 \otimes su(2)_1$ WZNW model (25) is known [33, 54] to have conformal weights $(7/5, 7/5)$ in the IR limit and thus identifies to the neutral X field of the three-state Potts model. As described in the Toulouse limit approach of this model, the UV perturbation reduces to the self-dual contribution of Eq. (44) so that one expects the following UV-IR correspondence:

$$: \cos \left( \sqrt{6\pi} \Phi \right) : + : \cos \left( \sqrt{6\pi} \Theta \right) : \sim X.$$

One should note that this result is consistent with the fact that these two operators are neutral with respect to the $Z_3$ symmetry and the charge conjugation. In fact, the transmutation (48) can also be justified by an additional argument. Indeed, it is known [20] that the perturbation of $\mathcal{H}_2$
in Eq. (24) degenerates, in the IR limit, into the irrelevant operator \( \Phi_{31} \) of the \( \mathcal{M}_{N+1} \) minimal model with conformal weights \( (N+3)/N+1, (N+3)/N+1) \). In the special \( N = 4 \) case, the perturbation of \( \mathcal{H}_2 \) is nothing but the SDSG perturbing field as seen in Section III so that one recovers the correspondence (48).

Further progress can be made by considering Eq. (47) with \( l_1 = 0 \) and \( l_2 = l_3 = 2 \). In that case, the su(2)\(_4\) primary field \( \Phi_{4}^{(2)} \), transforming in the spin 1 representation, has a free-field description in terms of two Bose fields using the conformal embedding su(2)\(_4\) \( \in \) su(3)\(_1\). On the other hand, the su(2)\(_3\) primary field \( \Phi_{3}^{(2)} \) can be expressed in terms of the Z\(_3\) spin field and the chiral Bose field \( \Phi_{1R} \) using Eq. (A1). With help of the Toulouse basis, we have obtained that the leading behavior in the IR limit of the vertex operators \( V_{\pm 1} \) (45), which carry a \( q = \pm 1 \) Z\(_3\) charge as discussed above, identifies to the two Z\(_3\) spin fields \( \sigma \) and \( \sigma^\dagger \) with conformal weights \((1/15, 1/15)\) and \( q = \pm 1 \) Z\(_3\) charge:

\[
: \exp \left( i \sqrt{2\pi/3} \Phi \right) : \sim \sigma
\]

\[
: \exp \left( -i \sqrt{2\pi/3} \Phi \right) : \sim \sigma^\dagger.
\]

(49)

A similar approach can be applied for \( l_1 = 1 \) and \( l_2 = 2 \) where now two terms in the rhs of Eq. (47) contribute with \( l_3 = 1 \) and \( l_3 = 3 \). As for the left su(2)\(_1\) current (32), the left su(2)\(_1\) primary field \( \Phi_{1L} \) has a simple free-field representation in terms of the Bose field \( \Phi_{0L} \). In this case, we have obtained the following UV-IR correspondence:

\[
: \exp \left( -i 2 \sqrt{2\pi/3} \Phi \right) : \sim \sigma + \psi_{1R}\psi_{1L}
\]

\[
: \exp \left( i 2 \sqrt{2\pi/3} \Phi \right) : \sim \sigma^\dagger + \psi_{1R}^\dagger\psi_{1L}^\dagger.
\]

(50)

where \( \psi_{1R} \) (respectively \( \psi_{1L} \)) is the right (respectively left) Z\(_3\) parafermionic current with conformal weights \((0, 2/3)\) (respectively \((2/3, 0)\)). The primary field \( \psi_{1R}\psi_{1L} \) with conformal weights \((2/3, 2/3)\), also denoted by \( Z_1 \) in the book, has a \( q = 1 \) Z\(_3\) charge. We observe that the result (50) is consistent with the Z\(_3\) charge of the vertex operators (47) through the transformation \( \Phi \rightarrow \Phi + \sqrt{2\pi/3} \).

V. CONCLUDING REMARKS

In this paper, we have discussed the critical properties of the \( \beta^2 = 2\pi N \) SDSG model which provides a continuum description of the two-dimensional classical XY model with an N-fold symmetry breaking field. This system has a single phase transition for \( N = 2 \) and \( N = 3 \) which
falls into the Ising and three-state Potts universality class, respectively. The $N = 4$ case exhibits continuously varying critical exponents typical for the Luttinger-liquid behavior. The $N = 2$ and $N = 4$ criticalities can be clearly understood and described starting from the corresponding SDSG model and treating it by standard methods, like bosonization or perturbative RG approaches. The case $N = 3$ is exceptional for being strongly non-perturbative and resistant to any simple-minded free-field treatment.

The three-state Potts universality class of the IR fixed point of the $\beta^2 = 6\pi$ SDSG model has been determined in this paper by two independent approaches. We have first mapped this model onto an integrable deformation of the $Z_4$ parafermion CFT [20], which has a massless flow to a three-state Potts IR fixed point. The second approach was based on establishing a relationship between the $\beta^2 = 6\pi$ SDSG model and an anisotropic, chirally asymmetric version of the WZNW model with $\text{su}(2)_4$ and $\text{su}(2)_1$ current-current interaction. This model exhibits critical properties of the chirally stabilized liquid universality class [21]. From the nature of the IR fixed point of the latter model we have deduced the same $Z_3$ IR properties of the $\beta^2 = 6\pi$ SDSG model.

Regarding perspectives, the $\beta^2 = 6\pi$ SDSG model may be analyzed using the form factor perturbation theory [61] with help of the form factors of topologically charged operators in the sine-Gordon model [62]. It will be very interesting to determine the complete UV-IR transmutation of the fields of the $\beta^2 = 6\pi$ SDSG model and in particular to check the conjectures presented in this paper. Finally, there are specific physical realizations of the $\beta^2 = 6\pi$ SDSG model in one-dimensional quantum spin/electron systems and in two-dimensional statistical mechanics. In this respect, Delfino [63] has recently proposed that the $\beta^2 = 6\pi$ SDSG model describes the field theory corresponding to the crossover from antiferromagnetic to ferromagnetic three-state Potts behavior. The connection between the $\beta^2 = 6\pi$ SDSG model and the chirally stabilized liquids, discussed in this paper, leads us to a conclusion that this model also accounts for the $Z_3$ critical properties of the four-channel underscreened Kondo-Heisenberg chain with incommensurate fillings. We hope that other applications of the $\beta^2 = 6\pi$ SDSG model will be found in the future.

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APPENDIX A: BOSONIZATION OF THE Z₄ PARAFERMION THEORY

In this Appendix, a bosonization approach to the Z₄ parafermion CFT is presented. This theory has central charge $c = 1$, which suggests that it can be brought to correspondence with a suitably defined free Bose field. The precise identification requires full knowledge of the operator content of the Z₄ parafermion CFT.

1. Identification of the Bose field

The spectrum of the Z₄ parafermion theory can be obtained from the $su(2)_N/u(1)_N$ coset model. In the holomorphic sector, the $su(2)_N$ primaries $(\Phi^l_m)$ are related to the Z₄ parafermionic ones $(f^l_m)$ by

$$\Phi^l_m = f^l_m : \exp \left( im \sqrt{\frac{2\pi}{N}} \Phi_L \right) :,$$

where $l = 0, \ldots, N$ and $-N+1 \leq m \leq N$ with the constraint: $l \equiv m \mod 2$. The operator content of the Z₄ CFT is obtained by constructing different modular invariants of this series which can be determined with help of the coset $su(2)_N/u(1)_N$:

$$Z(Z_N) = \left| \frac{\eta}{\eta} \right|^2 \sum_{L,l=0}^{N} \sum_{M,m=-N+1}^{N} L_{l,l} M_{m,m} c^l_m c^l_{m} \left( 1 + (-1)^{l-m} \right) \left( 1 + (-1)^{l-m} \right) \frac{1}{4},$$

where $\eta$ is the Dedekind function: $\eta(q) = q^{1/24} \prod_{n=1}^{+\infty} (1 - q^n)$. The coefficients $c^l_m$ are the so-called level-N string functions of the current algebra (see for instance Ref. [32]) and verify the following properties: $c^l_m = c^l_{-m}$, $c^l_m = c^{N-l}_{N-m}$. In Eq. (A2), $L_{l,l}$ and $M_{m,m}$ are two positive integers which define different modular invariants of the su(2)$_N$ and u(1)$_N$ theories, respectively. The simplest modular invariant of the Z₄ CFT is the diagonal one with $L_{l,l} = \delta_{l,l}$ and $M_{m,m} = \delta_{m,m}$ so that the resulting partition function reads:

$$Z_{\text{diag}}(Z_4) = \left| \frac{\eta}{\eta} \right|^2 \left( |c^0_0|^2 + 2|c^0_1|^2 + |c^0_2|^2 + 2|c^1_1|^2 + 2|c^1_2|^2 + |c^2_0|^2 + |c^2_2|^2 \right).$$

At the next step we use the following identities for the string functions first obtained by Yang [46]:

$$\eta c^0_0 + \eta c^0_4 = K^{(6)}_0$$
$$\eta c^0_0 - \eta c^0_4 = \frac{1}{\eta} \sum_{n=-\infty}^{+\infty} (-1)^n q^n$$
$$\eta c^0_2 = K^{(6)}_1$$
$$\eta c^2_0 = K^{(6)}_2$$
Here
\[ K^{(N)}_{\lambda} = \frac{1}{\eta} \sum_{n=-\infty}^{+\infty} q^{N/2(n+\lambda/N)^2}, \quad (A5) \]
represent the generalized characters of a Bose field living on a circle at a rational radius \[ \frac{3}{2}, \frac{5}{2} \]. The partition function \((A3)\) takes then the form
\[
\mathcal{Z}_{\text{diag}}(Z_4) = 
\sum_{\lambda=1}^{2} |K^{(6)}_{\lambda}|^2 + 2|K^{(6)}_{\lambda}|^2 + 2 \sum_{\lambda=1,3} |K^{(8)}_{\lambda}|^2 
+ \left( \frac{1}{2\eta} \sum_{n=-\infty}^{+\infty} q^{3n^2} \right) + \left( \frac{1}{2\eta} \sum_{n=-\infty}^{+\infty} q^{3n^2} \right) (-1)^n q^{n^2} \right] 
+ \left( \frac{1}{2\eta} \sum_{n=-\infty}^{+\infty} q^{2n^2} \right) - \left( \frac{1}{2\eta} \sum_{n=-\infty}^{+\infty} q^{2n^2} \right) (-1)^n q^{n^2}. \quad \text{(A6)}
\]
This expression is nothing but the partition function of a Bose field on the orbifold line at the special radius \( R = \sqrt{3/2\pi} \) (see for instance Ref. \[ 32 \]). The twisted sector of the orbifold model corresponds to the states with conformal weights \((h, \bar{h}) = (1/16+n, 1/16+n)\) or \((9/16+n, 9/16+n)\) and thus identifies with the third term in Eq. \((A6)\). From the identification \((A4)\), one then deduces that the \(Z_4\) parafermion fields characterized by an odd-integer \(l\) belong to the twisted sector of the orbifold theory. In contrast, the fields with even \(l\) have representations in the untwisted sector and thus can be described by a free boson living on a circle of radius \( R = \sqrt{3/2\pi} \).

In fact, this result can be better seen by considering another modular invariant of the \(Z_4\) CFT. Indeed, one can replace in the general partition function \((A2)\) the diagonal \(\text{su}(2)_4\) modular invariant by the non-diagonal one which is diagonal under a larger algebra \((\text{su}(3)_{1})\). In that case, the new partition function reads
\[
\mathcal{Z}'(Z_4) = |\eta|^2 \left( |c_0^0 + c_0^1|^2 + 4|c_2^0|^2 + 2|c_0^2|^2 + 2|c_2^2|^2 \right). \quad \text{(A7)}
\]
We thus observe that only the parafermionic fields with even \(l\) appear in this modular invariant and all the states with odd \(l\) have been projected out. Using identities \((A4)\), the partition function \((A7)\) can then be expressed in terms of the characters of the bosonic theory:
\[
\mathcal{Z}'(Z_4) = |K^{(6)}_0|^2 + 2|K^{(6)}_1|^2 + 2|K^{(6)}_2|^2 + |K^{(6)}_3|^2 
= \sum_{\lambda=0}^{5} |K^{(6)}_{\lambda}|^2. \quad \text{(A8)}
\]
which is precisely the partition function of a Bose field living on the circle at radius \( R = \sqrt{3/2\pi} \).

In this paper, we are only concerned with the bosonization of some \( Z_4 \) parafermionic fields with even \( l \) so that one can safely work with a free boson on the circle with radius \( R = \sqrt{3/2\pi} \).

2. Bosonization of the \( Z_4 \) parafermionic currents

We give now a bosonized description of the \( Z_4 \) parafermionic currents \( \psi_{1L}, \psi_{2L} = \psi_{2L}^\dagger, \) and \( \psi_{1L}^\dagger = \psi_{3L}, \) which act on the left sector and have dimensions \( 3/4, 1, 3/4 \), respectively. These fields appear in the parafermionic modules characterized by \( l = 0 \) or \( l = 4 \), so that it is possible to find their bosonic representation in terms of a compactified boson at radius \( R = \sqrt{3/2\pi} \), as discussed above. An explicit realization of the algebra of the \( Z_4 \) parafermionic currents [39] in terms of a Bose field and some appropriate cocycles can be derived. The latter degrees of freedom are of utmost importance to achieve a faithful representation of the algebra. One way to formally implement the cocycles is to introduce extra degrees of freedom like, for instance, the Pauli matrices \( \sigma_{x,y,z} \).

In this respect, we have checked that a faithful representation of the \( Z_4 \) parafermionic currents in terms of a chiral bosonic field \( \varphi_L \) is given by

\[
\psi_{1L} = e^{i\pi/4} \sqrt{2} \left( \sigma_y : \exp \left( i\sqrt{6\pi} \varphi_L \right) : + \sigma_x : \exp \left( -i\sqrt{6\pi} \varphi_L \right) : \right)
\]

\[
\psi_{1L}^\dagger = e^{-i\pi/4} \sqrt{2} \left( \sigma_y : \exp \left( -i\sqrt{6\pi} \varphi_L \right) : + \sigma_x : \exp \left( i\sqrt{6\pi} \varphi_L \right) : \right)
\]

\[
\psi_{2L} = \sigma_z i\sqrt{4\pi} \partial \varphi_L.
\]

(A9)

On the other hand, the cocycles can be directly expressed in terms of the zero mode of the compactified Bose field \( \varphi \). To this end, we turn to the mode expansion of its chiral components \( \varphi_{L,R} \):

\[
\varphi_L(z) = q_L - \frac{i p_L}{4\pi} \ln z + i \sum_{k \neq 0} \frac{\alpha_{Lk}}{k \sqrt{4\pi}} z^{-k}
\]

\[
\varphi_R(\bar{z}) = q_R - \frac{i p_R}{4\pi} \ln \bar{z} + i \sum_{k \neq 0} \frac{\alpha_{Rk}}{k \sqrt{4\pi}} \bar{z}^{-k},
\]

(A10)

where \( \alpha_{L(R)k} \) are the oscillator operators in the quantization of the free boson and \( q_{L,R} \) and \( p_{L,R} \) are canonically conjugate operators: \( [q_L, p_L] = [q_R, p_R] = i \). Since the Bose field \( \varphi \) is compactified with radius \( R = \sqrt{3/2\pi} \), the zero mode momentum \( p_{L,R} \) has the following discrete spectrum [32]

\[
p_L = \sqrt{\frac{2\pi}{3}} n + \sqrt{6\pi} m
\]

\[
p_R = \sqrt{\frac{2\pi}{3}} n - \sqrt{6\pi} m,
\]

(A11)
$n$ and $m$ being integers which correspond, respectively, to the electric and magnetic charges associated with the Bose field. The cocycles can then be expressed in terms of the zero mode $p_{L,R}$, and the $Z_4$ parafermionic algebra is realized by the following representation:

$$\psi_{1L} = \frac{1}{\sqrt{2}} \left( \exp \left( i \sqrt{6} \pi \varphi_L \right) : + e^{i \sqrt{3} \pi/2} : \right)$$
$$\psi_{1L}^\dagger = \frac{1}{\sqrt{2}} \left( \exp \left( -i \sqrt{6} \pi \varphi_L \right) : + : \exp \left( i \sqrt{6} \pi \varphi_L \right) : e^{-i \sqrt{3} \pi/2} p_L \right)$$
$$\psi_{2L} = -e^{i \sqrt{3} \pi/2} p_L \sqrt{4 \pi} i \partial \varphi_L. \tag{A12}$$

From the spectrum of the left zero-mode momentum \([A11]\), one observes that the cocycle term appearing in the above identification takes two values: $e^{i \sqrt{3} \pi/2} p_L = \pm 1$. It is worth noting that the parafermionic representation of the su(2)$_4$ currents \([31]\), together with the formula \([A12]\), provide a faithful explicit representation of the su(2)$_4$ current operators in terms of two free Bose fields. The fact that such construction should, in principle, be possible was already anticipated in Ref. \([37]\). An explicit representation was worked out in Ref. \([65]\). The latter construction, however, suffers from the neglect of the cocycles, thus resulting in an incorrect OPE for $J^\pm_L(z)J^\pm_L(w)$ (this circumstance, however, did not affect final conclusions for the impurity problem studied in Ref. \([65]\) for which only the sub-set $J^\pm_L(z)J^\mp_L(w)$ and $J^Z_L(z)J^Z_L(w)$ of the current OPE was actually needed). Thus Eqs. \([31, A12]\) complement the representation found in Ref. \([65]\).

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