General conversion method for constrained systems

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Abstract

We reformulate in a systematic way the convolutional approach in its most general and compact form. We present a new definition of generalized Dirac bracket directly in terms of the super-observables commuting with the basic BFV-BRST charge.

Keywords: Constrained systems, conversion method, Dirac bracket, BFV-BRST charge

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1 Introduction

In the Dirac theory of Hamiltonian constraint dynamics, all constraints are split naturally into the two classes \([1, 2]\). First-class constraints are in Poisson-bracket involution among themselves. So, they do serve naturally as a gauge symmetry generators. Second-class constraints have their Poisson-bracket matrix invertible. So, they do reduce effectively an original phase space to the second-class constraint hypersurface. Locally, in their Abelian form, first-class constraints do commute among themselves, so that they are similar, say, to a set of momenta. Second-class constraints, in their local Abelian form, have their Poisson-bracket matrix invertible and constant. So, they are similar to a set of canonical pairs of co-ordinates and conjugate momenta.

The famous Dirac bracket concept provides for a natural projection to the Poisson bracket to a tangential subspace with respect to the second-class constraint hypersurface. However, it appears a rather difficult problem as to how to reformulate the Dirac bracket concept within a consistent quantum theory. In a series of papers \([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\) the so-called conversional approach to the quantization of dynamical systems with second-class constraints has been developed. This approach is based on the idea of converting the second-class-constraints into effective first-class ones by introducing extra degrees of freedom. Initial first-class constraints (if they are present in the system) and the initial Hamiltonian should also be converted into the corresponding modified objects, depending on the extra variables, so that we have, as a result, a new Hamiltonian in involution with new constraints of the first class only.

Thus, within the framework of the conversional approach, the problem of quantizing the system with general constraints is in fact reduced to the case of first-class constraints only, for which the scheme of generalized canonical quantization, which is operating well, does exist \([4, 5, 15, 16, 17]\). Thereby, the unification proposed does resolve the operator quantization problem, whereas one has to make use of the canonical commutation relations, only.

In the present article, we reformulate systematically the conversional approach in its most general and compact form. We present a new definition for the Dirac bracket directly in terms of the super-observables commuting with the basic BFV-BRST charge.

NOTATIONS: \(\{A, B\}\) and \([A, B]\) denotes the Poisson (super)bracket and the (super)commutator, respectively. \(\varepsilon(A)\) and \(gh(A)\) denotes the Grassmann parity and the ghost number, respectively. Other notation is clear from the context.

2 Conversion of constraints in its most general form

Let

\[
Z =: (p, q); \quad \varepsilon(p) = \varepsilon(q), \quad gh(p) = -gh(q) =: 0
\]  

(2.1)
be a set of initial canonical variables, and let
\[ \phi^\alpha; \quad \varepsilon(\phi^\alpha) =: \varepsilon_\alpha, \quad \text{gh}(\phi^\alpha) = 0, \quad (2.2) \]
be the conversion variables commuting as
\[ \{\phi^\alpha, \phi^\beta\} =: \omega^{\alpha\beta} = \text{const}, \quad (2.3) \]
with even invertible metric \( \omega^{\alpha\beta} \),
\[ \varepsilon(\omega^{\alpha\beta}) = \varepsilon_\alpha + \varepsilon_\beta. \quad (2.4) \]
In turn, let
\[ C^A, P_B; \quad \varepsilon(C^A) = \varepsilon(P_A) =: \varepsilon_A + 1, \quad \text{gh}(C^A) = -\text{gh}(P_A) =: 1, \quad (2.5) \]
be the ghost canonical variables
\[ \{C^A, P_B\} =: \delta^A_B, \quad (2.6) \]
Define the "BFV - BRST" charge,
\[ Q =: Q(Z, \phi, C, P), \quad \varepsilon(Q) =: 1, \quad \text{gh}(Q) =: 1, \quad (2.7) \]
to satisfy the master equation,
\[ \{Q, Q\} = 0, \quad (2.8) \]
and the boundary condition
\[ Q = C^A T_A(Z, \phi) + ..., \quad (2.9) \]
where ellipses mean higher powers in ghosts \( (2.5) \).
If one expands the \( Q \) to the first order in ghost momenta \( P \),
\[ Q = C^A T_A(Z, \phi) + \frac{1}{2}(-1)^{\varepsilon_B} C^B C^A U_{AB}^C(Z, \phi) P_C (-1)^{\varepsilon_C} + ..., \quad (2.10) \]
then the involution relations follow from the master equation \( (2.8) \),
\[ \{T_A, T_B\} = U_{AB}^C T_C. \quad (2.11) \]
These relations show us that the coefficients \( T_A(Z, \phi) \) are effective (converted) first-class constraints in the "extended original phase space" spanned with the phase variables \( (Z, \phi) \). These effective first-class constraints can be split as
\[ T_A = (T_a(Z, \phi); \Theta_a(Z, \phi)), \quad (2.12) \]
where $\varepsilon(\Theta_\alpha) := \varepsilon_\alpha$, the second in (2.2), with

$$T_a(Z, 0) := t_a(Z) \text{ and } \Theta_\alpha(Z, 0) := \theta_\alpha(Z)$$

(2.13)

being original first-class and second-class constraints, respectively.

Define an observable $A(Z, \phi, C, \mathcal{P})$ as to satisfy the standard equation,

$$\{Q, A\} = 0.$$  (2.14)

For two observables, $A$ and $B$, the generalized Dirac bracket, $\{\ , \}_D$, is then defined as

$$\{A_0, B_0\}_D =: \{A, B\}_0,$$  (2.15)

where it is denoted for an arbitrary $X$,

$$X_0 =: X|_{\phi=0}.$$  (2.16)

By expanding the $A, B$ and $Q$ in power series in $\phi$,

$$A = A_0 + \phi^\alpha A_\alpha + \ldots, \quad B = B_0 + \phi^\alpha B_\alpha + \ldots,$$  (2.17)

$$Q = Q_0 + \phi^\alpha Q_\alpha + \ldots,$$  (2.18)

we rewrite (2.15) as

$$\{A_0, B_0\}_D = \{A_0, B_0\} + A_\alpha \omega^{\alpha\beta} B_\beta (-1)^{(\varepsilon(A)+1)\varepsilon_\alpha},$$  (2.19)

where $A_\alpha, B_\alpha$ and $Q_\alpha$ should satisfy the equations

$$Q_\alpha \omega^{\alpha\beta} A_\beta = -\{Q_0, A_0\},$$  (2.20)

$$Q_\alpha \omega^{\alpha\beta} B_\beta = -\{Q_0, B_0\},$$  (2.21)

$$\{Q_0, Q_0\} + Q_\alpha \omega^{\alpha\beta} Q_\beta = 0.$$  (2.22)

These equations rewrite themselves in a natural way in terms of the definition (2.19)

$$\{Q_0, A_0\}_D = 0, \quad \{Q_0, B_0\}_D = 0, \quad \{Q_0, Q_0\}_D = 0.$$  (2.23)

Due to the ghost number conservation, these equations are uniquely resolvable, certainly. Indeed, in the Abelian second-class constraint basis, we have

$$A_\alpha = -\{\Upsilon_\alpha(Z), A_0(Z)\}(-1)^\varepsilon_\alpha,$$  (2.24)

$$B_\alpha = -\{\Upsilon_\alpha(Z), B_0(Z)\}(-1)^\varepsilon_\alpha,$$  (2.25)

$$\{\Upsilon_\alpha, \Upsilon_\beta\} = \omega_{\alpha\beta}(-1)^\varepsilon_\alpha,$$  (2.26)
and
\[ Q_0 = C^\alpha Y_\alpha(Z) + \text{terms independent of } C^\alpha, \quad (2.27) \]
\[ Q_\alpha = -\omega_{\alpha\beta} C^\beta + \text{terms independent of } C^\alpha. \quad (2.28) \]

In order to cover the case of the general basis of second-class constraints \( \theta_\alpha(Z) \), we define an even matrix
\[ V^\alpha_\beta(Z), \quad \varepsilon(V^\alpha_\beta) = \varepsilon_\alpha + \varepsilon_\beta, \quad (2.29) \]
so as to satisfy the equation
\[ \{ \theta_\alpha, \theta_\beta \} = V^\gamma_\alpha(-1)^{\varepsilon_\gamma} \omega_{\gamma\delta} V^\delta_\beta(-1)^{\varepsilon_\beta+1}\varepsilon_\delta. \quad (2.30) \]
In terms of the latter matrix \((2.29)\), the equations \((2.24)-(2.28)\) modify as
\[ \{ \theta_\alpha, A_0 \} = -V^\alpha_\beta(-1)^{\varepsilon_\beta} A_\beta, \quad (2.31) \]
\[ \{ \theta_\alpha, B_0 \} = -V^\alpha_\beta(-1)^{\varepsilon_\beta} B_\beta, \quad (2.32) \]
\[ \{ \theta_\alpha, \theta_\beta \} D^{\beta\gamma} = \delta_\gamma, \quad (2.33) \]
\[ Q_0 = C^\alpha \theta_\alpha(Z) + \ldots, \quad (2.34) \]
\[ Q_\gamma = C^\alpha V^\alpha_\beta(-1)^{\varepsilon_\beta} \omega_{\beta\gamma} + \ldots. \quad (2.35) \]

It follows then the standard formula for the Dirac bracket \((2.19)\),
\[ \{ A_0, B_0 \}_D = \{ A_0, B_0 \} - \{ A_0, \theta_\alpha \} D^{\alpha\beta}\{ \theta_\beta, B_0 \}. \quad (2.36) \]
Here in \((2.31), (2.32), (2.36)\), we have assigned zero values as for all ghost variables, which means the lowest order in ghosts. Also, here we do assume, for the sake of simplicity, that the lowest structure coefficients \( \mathcal{U}^\alpha_{\alpha\beta} \) and \( \mathcal{U}^c_{\alpha\beta} \) in \((2.10)\) are zero at \( \phi^\alpha = 0 \) (Abelian conversion).

It follows directly from \((2.30)-(2.33)\) that
\[ -\{ A_0, \theta_\alpha \} D^{\alpha\beta}\{ \theta_\beta, B_0 \} = A_\delta \tilde{V}^\delta_\alpha(-1)^{\varepsilon_A+1}\varepsilon_\delta (\tilde{V}^{-1})^\alpha_\mu \omega^{\mu\nu}(-1)^{\varepsilon_\nu}(V^{-1})^\beta_\nu \omega^\gamma(-1)^{\varepsilon_\gamma} B_\gamma, \quad (2.37) \]
where
\[ \tilde{V}^\alpha_\mu =: V^\alpha_\mu(-1)^{\varepsilon_\mu+1}, \quad (2.38) \]
is a super-transposed to \( V \). Now, the \( V \) drops out completely from \((2.37)\), and we arrive at \((2.36)\).

If the coefficients \( \mathcal{U}^\gamma_{\alpha\beta} \) and/or \( \mathcal{U}^c_{\alpha\beta} \) are non-zero at \( \phi^\alpha = 0 \), then one should shift in \((2.30), (2.33)\):
\[ \{ \theta_\alpha, \theta_\beta \} \to \{ \theta_\alpha, \theta_\beta \} - \mathcal{U}^\gamma_{\alpha\beta} \theta_\gamma - \mathcal{U}^c_{\alpha\beta} t_c, \quad (2.39) \]
which means a symptom of a non-Abelian conversion.

The standard conversion procedure has been considered perturbatively via \( \phi \)-power series expansion in Refs. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], as applied to the simplest particular cases of linear and Abelian conversions, and then to the general case of non-Abelian conversion. The latter allows one to deal with non-scalar constraints, as well.

### 3 Operator formulation

In the previous Section 2, we did consider constraint dynamics at the classical level, in terms of the canonical Poisson brackets. Now, we are in a position as to consider in short how to apply the Dirac formal quantization rule. First, we change all classical quantities for respective operators. Then, we change all Poisson brackets for respective (super) commutators,

\[
\{ , \} \rightarrow (i\hbar)^{-1}[ , ], \quad [A, B] =: AB - BA(-1)^{\varepsilon(A)\varepsilon(B)}. \quad (3.1)
\]

In this way, we reformulate our basic master equation (2.8) as

\[
[Q, Q] = 0, \quad [C^A, P_B] = i\hbar \delta^A_B 1, \quad (3.2)
\]

Further, we consider the (2.10) as a \( \mathcal{CP} \) normal ordered power series expansion for the operator \( Q \). Coefficients in (2.10) are operator valued functions of the operators (2.1), (2.2), now commuting as

\[
[q_j^i, p_k] =: i\hbar \delta^i_j 1, \quad (3.3)
\]

\[
[\phi^\alpha, \phi^\beta] =: i\hbar \omega^{\alpha\beta} 1. \quad (3.4)
\]

Here, we are not interested, so far, as to which type of normal ordering is chosen for those operators (2.1), (2.2). To the zeroth order in ghost momenta, it follows from (3.2)

\[
[\mathcal{T}_A, \mathcal{T}_B] = i\hbar \mathcal{U}_{ABC} \mathcal{T}_C, \quad (3.5)
\]

which looks quite similar to the classical involution (2.11). The latter similarity holds because the \( \mathcal{CP} \) normal ordering chosen does respect the ghost numbers of \( C \) and \( \mathcal{P} \). Consider, however, the Jacobi relations that follow from (3.2) to the first order in ghost momenta \( \mathcal{P} \),

\[
((i\hbar)^{-1}\mathcal{U}_{AB}^E \mathcal{T}_C)(-1)^{\varepsilon_E \varepsilon_E} + \mathcal{U}_{AB}^D \mathcal{U}_{DC}^E (-1)^{\varepsilon_{AE}} + \]

\[
+ \text{cyclic permutations (}A, B, C\text{)} + \frac{1}{2} \mathcal{U}_{ABC} \Pi_{DF}^E = 0. \quad (3.6)
\]

Here in (3.6), the operator \( \mathcal{U}_{ABC}^{FD} \) enters the \( \mathcal{CCC\mathcal{P}\mathcal{P}} \) order in (2.10),

\[
\frac{1}{12} C^C C^B C^A (-1)^{\varepsilon_{AEC} + \varepsilon_B} \mathcal{U}_{ABC}^{FD} \mathcal{P}_D \mathcal{P}_F (-1)^{\varepsilon_D}, \quad (3.7)
\]
while the operator
\[
\Pi_{DF}^E =: \mathcal{T}_D \delta_F^E - (D \leftrightarrow F)(-1)^{\varepsilon_D F} - i\hbar \mathcal{U}_{DF}^E,
\]
annihilates the constraint operators,
\[
\Pi_{DF}^E \mathcal{T}_E = 0,
\]
due to the (3.5). Thereby, we have confirmed the compatibility of the operator valued involution relations (3.5). All higher compatibility relations can be confirmed subsequently by making use of the generating Jacobi identity,
\[
\mathcal{Q} \mathcal{Q} \mathcal{Q} = 0.
\]
(3.10)
Now, we can see from (3.8) that, in contrast to the involution (3.5), the first Jacobi relation (3.6) has acquired an actual quantum correction (the third term in (3.8)), as compared to the classical counterpart to the (3.6). Also, it seems worthy to mention again that, in general, actual quantum corrections could appear already in the involution of constrains when using another normal ordering for ghosts, such as the Weyl or the Wick ordering.

If one defines the $Q$-invariant converted constraints (they are similar to the BRST-invariant constraints [18] in relativistic field theory),
\[
T_A =: (i\hbar)^{-1}[\mathcal{P}_A, Q](-1)^{\varepsilon_A}, \quad [T_A, Q] = 0,
\]
then their gauge algebra is generated by the relations via the procedure of [19]
\[
(i\hbar)^{-1}[T_A, T_B] = (i\hbar)^{-3}[(\mathcal{P}_A(-1)^{\varepsilon_A}, \mathcal{P}_B(-1)^{\varepsilon_B})_Q, Q],
\]
(3.12)
where the general quantum antibracket, $(A, B)_Q$, is defined by
\[
(A, B)_Q =: \frac{1}{2}([A, [Q, B]] - (A \leftrightarrow B)(-1)^{(\varepsilon_A + 1)(\varepsilon_B + 1)}),
\]
(3.13)
as for any two operators $A$ and $B$. It follows from (3.13)
\[
[Q, (A, B)_Q] = [[Q, A], [Q, B]].
\]
(3.14)
By choosing in (3.14) $A = \mathcal{P}_A$, $B = \mathcal{P}_B$, one arrives at (3.12).

4 Intrinsic Weyl symbols as for conversion variable operators

Let us proceed from the master equation (2.8), to consider its Weyl symbol representation with respect to the conversion variable operators $\phi^a$ commuting as in (3.5). We do proceed from the Weyl representation as for any operator $X(Z, \phi, C, \mathcal{P})$,
\[
X \leftrightarrow \bar{X}, \quad XY \leftrightarrow \bar{X} \star \bar{Y},
\]
(4.1)
with $\tilde{X}$ being a Weyl symbol as for an operator $X$,

$$X =: \exp \left\{ \phi^\alpha \frac{\partial}{\partial \phi^\alpha} \right\} \tilde{X}(Z, \tilde{\phi})|_{\tilde{\phi}=0},$$  \hspace{1cm} (4.2)

where in (4.2), $\tilde{\phi}^\alpha$ means ordinary classical variables. It follows then from (2.8)

$$\tilde{Q} \star \tilde{Q} = 0.$$ \hspace{1cm} (4.3)

In particular, as for the quantum involution (3.5), it follows

$$\tilde{T}_A \star \tilde{T}_B - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B} = i\hbar \tilde{U}_{AB} \star \tilde{T}_C.$$ \hspace{1cm} (4.4)

Here, in the second in (4.1), (4.3), (4.4), the $\star$ means the Weyl symbol multiplication,

$$\star =: \exp \left\{ i \frac{\hbar}{2} \frac{\partial}{\partial \tilde{\phi}^\alpha} \omega^{\alpha\beta} \frac{\partial}{\partial \tilde{\phi}^\beta} \right\},$$ \hspace{1cm} (4.5)

Similarly to (4.4), the symbol representation can easily be derived as for the first Jacobi relation (3.6), as well as for all higher Jacobi relations. By using the symbol representations, one can also expand easily the respective relations in power series in the classical variables $\tilde{\phi}$, as to derive the relations required for their tensor valued coefficient operators.

In terms of a symbol super-commutator,

$$[\tilde{A}, \tilde{B}]_\star := \tilde{A} \star \tilde{B} - \tilde{B} \star \tilde{A} (-1)^{\varepsilon(\tilde{A})\varepsilon(\tilde{B})},$$ \hspace{1cm} (4.6)

one can consider the equations for symbols of physical observables, $\tilde{A}, \tilde{B},$

$$[\tilde{Q}, \tilde{A}]_\star = 0, \quad [\tilde{Q}, \tilde{B}]_\star = 0,$$ \hspace{1cm} (4.7)

so as to define the symbol Dirac’s bracket,

$$[\tilde{A}_0, \tilde{B}_0]_D := ([\tilde{A}, \tilde{B}]_\star)_0,$$ \hspace{1cm} (4.8)

where we have denoted,

$$X_0 =: X|_{\tilde{\phi}=0}, \quad \text{for any } X.$$ \hspace{1cm} (4.9)

5 Discussion

It is an important aspect of the conversion method, what is the relativistic status of the conversion variables. So far, the latter question remains open in its general meaning. In principal, if one proceeds from relativistic covariant Lagrangian theory, it seems natural to expect the relativistic covariance group to be represented in the Hamiltonian formalism, in the form of the respective algebra in terms of Poisson brackets. However, when converting
second-class constraints, one introduces extra conversion variables, quite new with respect to
the original theory. So, their relativistic status expected is also unclear originally. Moreover,
the relativistic status expected is also unclear originally. Moreover, it remains unclear originally,
which type and form of the effective (converted) gauge algebra we could expect to be compatible
with required relativistic covariance. Another open question concerns the boundary condi-
tion for converted constraints. Usually, we do assume the natural boundary conditions
requiring the converted constraints to coincide with the original second-class constraints
at zero value of the conversion variables. However, it is unknown if such boundary
covariance. Besides, it is worthy to mention that taking the zero value of the conversion
variables is by itself a particular case of second-class constraints, although very simple.
To avoid that point, when expanding the converted constraints in power series in the
conversion variables, we identify directly the zeroth order term with the original second-class
constraints. Of course, we find ourselves rather far from being able to provide for
general answers to even some of the questions mentioned. Our main conjecture is the follow-
ing. Being the conversion variables introduced in an appropriate way, so that they have
their relativistic status well-defined, one can expect the relativistic covariance
transformations to be realized in the form of canonical transformations, typical for all other
symmetry transformations. In particular, we do not insist on being the natural boundary
conditions the only possibility. It seems natural to expect that one should apply some
canonical transformation to the naturally converted constraints, as to make them well-defined
in their relativistic status. Now, we are in a position to try to demonstrate what we mean
by considering a simple example.

Consider first the second-class constraints in the Proca model \([24]\),

\[ \Theta_P =: (\pi^0, \pi^i + m^2 A_0), \quad (5.1) \]

with \( \pi^0 \) and \( \pi^i \) being canonical momenta conjugate to \( A_0 \) and \( A_i \), respectively. The first-class
constraints converted from (5.1) under natural boundary conditions are

\[ T_P =: \Theta_P + m(\phi; p), \quad (5.2) \]

with \( \phi \) and \( p \) being the conversion field and its conjugate momentum, respectively. On the
other hand, consider the original first-class constraints in the Stueckelberg model \([25]\),

\[ T_S =: (\pi^0; \pi^i + mp). \quad (5.3) \]

Here in (5.3), we have identified the Stueckelberg scalar field and the Proca conversion field in
(5.1). Regrettably, the (5.2) does not coincide with the (5.3). However, it follows immediately
that

\[ U^{-1} T_S U = T_P, \quad (5.4) \]

with \( U \) being a unitary transformation of the form

\[ U =: \exp \left\{ \frac{i}{\hbar} mA_0 \phi \right\}. \quad (5.5) \]
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