ON GRIFFITHS CONJECTURE

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Abstract. By using techniques of holomorphic jets and Jacobian fields, we devise a non-equidistribution theory of holomorphic curves into complex projective varieties intersecting normal crossing divisors. Based on this theory established, we prove the Griffiths conjecture and the Green-Griffiths conjecture in Nevanlinna theory and algebraic geometry.

1. Introduction

1.1. Motivations.

In 1972, Carlson and Griffiths [2] made a significant progress in the study of Nevanlinna theory, who devised an equidistribution theory of holomorphic mappings into complex projective varieties. The theory was generalized soon by Griffiths and King [11] to the case where dimension of domains is greater than that of targets, but little is known about the non-equidimensional value distribution theory. What is a Second Main Theorem of holomorphic curves into complex projective varieties intersecting normal crossing divisors? Griffiths for that posed a conjecture in his paper [10], which is called the Griffiths conjecture. To state this conjecture, we need to introduce some notations in Nevanlinna theory (see [15, 21, 22, 24, 26]). Let $f : \mathbb{C} \to X$ be a holomorphic curve into a smooth complex projective variety $X$ and $L$ be a positive line bundle over $X$. Equip $L$ with a Hermitian metric $h$ such that the first Chern form $c_1(L, h) := (i\partial \bar{\partial} / 2\pi) \log h > 0$. The characteristic function $T_f(r, L)$ of $f$ with respect to $L$ is defined by

$$T_f(r, L) = \int_1^r \frac{dt}{t} \int_{|z| < t} f^* c_1(L, h)$$

up to a bounded term. Let $D \in |L|$, where $|L|$ is the complete linear system of all effective divisors $(s)$ for $s \in H^0(X, L)$ (see [2]). Write $f^* D = \sum \lambda a_\lambda$. 

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The counting function \( N_f(r, D) \) of \( f \) with respect to \( D \) is defined by
\[
N_f(r, D) = \int_1^r \frac{n(t, f^*D)}{t} dt,
\]
where
\[
n(t, f^*D) = \sum_{f^{-1}(D) \cap \{|z|<t\}} \mu_\lambda.
\]
More general, the \( k \)-truncated counting function \( N_f^{[k]}(r, D) \) of \( f \) with respect to \( D \) is defined by
\[
N_f^{[k]}(r, D) = \int_1^r \frac{n^{[k]}(t, f^*D)}{t} dt,
\]
where
\[
n^{[k]}(t, f^*D) = \sum_{f^{-1}(D) \cap \{|z|<t\}} \min \{k, \mu_\lambda\}.
\]
Clearly, \( N_f(r, D) = N_f^{[\infty]}(r, D) \). We define the proximity function \( m_f(r, D) \) of \( f \) with respect to \( D \) by
\[
m_f(r, D) = -\frac{1}{2\pi} \int_{|z|=r} \log \|s_D \circ f(re^{i\theta})\| d\theta,
\]
where \( s_D \) is the canonical section of \( L \) over \( X \). Assume that \( f(\mathbb{C}) \not\subseteq D \), using Green-Jensen formula and Poincaré-Lelong formula (see [15, 22, 24]), we get the First Main Theorem as follows
\[
T_f(r, L) + O(1) = m_f(r, D) + N_f(r, D).
\]

In 1972, Griffiths [10] conjectured the following Second Main Theorem

**Conjecture 1.1** (Griffiths). Let \( X \) be a smooth complex projective variety. Let \( D \in |L| \) be a simple normal crossing divisor on \( X \), where \( L \) is a positive line bundle over \( X \). Let \( f : \mathbb{C} \to X \) be an algebraically non-degenerate holomorphic curve. Then for any \( \delta > 0 \),
\[
T_f(r, L) + T_f(r, K_X) \leq N_f(r, D) + O\left( \log^+ T_f(r, L) + \delta \log r \right)
\]
holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

If \( X = \mathbb{P}^n(\mathbb{C}) \), then Conjecture [10] turns out to be the following form

**Conjecture 1.2** (Griffiths). Let \( H \) be the hyperplane line bundle over \( \mathbb{P}^n(\mathbb{C}) \). Let \( D_1, \ldots, D_q \in |H^d| \) (\( d \geq 1 \)) such that \( D_1 + \cdots + D_q \) has simple normal crossings. Let \( f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \) be an algebraically non-degenerate holomorphic curve. Then for any \( \delta > 0 \),
\[
(qd - n - 1)T_f(r, H) \leq \sum_{j=1}^q N_f(r, D_j) + O\left( \log^+ T_f(r, H) + \delta \log r \right)
\]
holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

We consider a defect relation. The defect \( \delta_f(D) \) of \( f \) with respect to \( D \) is defined by

\[
\delta_f(D) = 1 - \limsup_{r \to \infty} \frac{N_f(r, D)}{T_f(r, L)},
\]

and the \( k \)-truncated defect \( \delta^k_f(D) \) of \( f \) with respect to \( D \) is defined by

\[
\delta^k_f(D) = 1 - \limsup_{r \to \infty} \frac{N^k_f(r, D)}{T_f(r, L)}.
\]

For two holomorphic line bundles \( L_1, L_2 \) over \( X \), set

\[
\left[ \frac{c_1(L_2)}{c_1(L_1)} \right] = \inf \left\{ t \in \mathbb{R} : t \eta_2 < t \eta_1, \exists \eta_1 \in c_1(L_1), \exists \eta_2 \in c_1(L_2) \right\}.
\]

Conjecture 1.2 yields that Conjecture 1.3 (Defect relation). Assume the same conditions as in Conjecture 1.1. Then

\[
\delta_f(D) \leq \left[ \frac{c_1(K^*_X)}{c_1(L)} \right] .
\]

In particular, for \( X = \mathbb{P}^n(\mathbb{C}) \)

\[
\sum_{j=1}^{q} \delta_f(r, D_j) \leq \frac{n + 1}{d} .
\]

Conjecture 1.3 has attracted attention of many authors. In 1987, Siu [27] confirmed the conjecture in the case \( X = \mathbb{P}^2(\mathbb{C}) \); In 1996, Siu-Yeung [33, 34] confirmed the conjecture in the case that \( X \) is an Abelian variety; In 2009, Ru [23] obtained a defect relation provided that \( X = \mathbb{P}^n(\mathbb{C}) \) and \( D \) is a sum of hypersurfaces in general position. This conjecture includes a conjecture, i.e., the Green-Griffiths conjecture raised by Green-Griffiths [9].

We say that \( X \) is of general type if \( K_X \) is pseudo ample (see [9]), i.e.,

\[
\limsup_{m \to +\infty} \frac{\log h^0(X, K_X^m)}{\log m} = \dim_{\mathbb{C}} X .
\]

A well-known fact is that \( X \) is of general type if \( K_X \) is ample. Green-Griffiths conjecture [9] states that

Conjecture 1.4 (Green-Griffiths). Let \( X \) be a smooth complex projective variety of general type. Then there are no algebraically non-degenerate holomorphic curves in \( X \).
In 1987, Lang [18] put forward Conjecture 1.4 again, as a stronger version, and which was later called the Green-Griffiths-Lang conjecture asserting that there exists a proper algebraic subvariety \( Z \subseteq X \) such that the image of each non-constant holomorphic curve in \( X \) is contained in \( Z \). This conjecture was announced by Siu [30] at the 2002 ICM. Many efforts were made to promote the progress of the study of Conjecture 1.4, see, e.g., Demailly [6, 7], Green [8], Green-Griffiths [9], Lang [17, 18], Lu [19], Nadel [20], Shiffman [25], Siu [27, 28, 29, 31] and Siu-Yeung [33, 34], etc.

Conjecture 1.4 includes a conjecture of Lang [17] as follows

**Conjecture 1.5 (Lang).** Let \( X \) be a smooth complex projective variety. If \( X \) and all its subvarieties are of general type, then \( X \) is Brody hyperbolic.

### 1.2. Techniques.

In this paper, motivated by those conjectures above, we are committed to the study of non-equidistribution theory of holomorphic curves into complex projective varieties intersecting a normal crossing divisor. The central goal of the present paper is confirming the Griffiths conjecture. To do so, our main ideas are to design the Second Main Theorem of holomorphic curves, and one then employs it to prove the Griffiths conjecture and other conjectures. The major approach is the use of holomorphic full jets introduced by Wong-Stoll [35] and Jacobian sections introduced by Stoll [26]. Stoll’s basic thoughts of Jacobian sections can help establish the Second Main Theorem with an error term, but this error term is very hard to estimate since it relies heavily on the length of a so-called holomorphic field. As a result, Stoll finally gave up such attempt. Along the line of Stoll, the second named author had made efforts (see, e.g., [13, 14, 15]) in developing techniques of Stoll, who contributed an important theorem (see Lemma 3.1) which plays a necessary role in showing our main theorem in the paper. However, the estimate problem of the error term has not yet been settled at all. To receive an expected estimate of this error term, we come up with an original technique so that one can construct a holomorphic \( \mathbb{C}^* \)-bundle mapping from the restricted \( k \)-jet bundles into the full \( k \)-jet bundles over a complex projective variety. By means of this bundle mapping, one can define an effective holomorphic field \( \varphi \) over \( \mathbb{C} \). Then, with the aid of the techniques of Chandler-Wong-Stoll (see [4, 35]) in dealing with \( k \)-jet bundles, the expected estimate of the error term is finally obtained.

We give a short introduction to our main technique. To do so, we review a theorem of Hu-Yang [14]. Let \( J^k X \) denote the \( k \)-jet bundle and \( TX \) denote the holomorphic tangent bundle over an \( n \)-dimensional complex projective variety \( X \). Let \( D \) be a simple normal crossing divisor on \( X \), over which there exists a positive line bundle \( L \). Let \( f : \mathbb{C} \to X \) be a non-constant holomorphic curve with \( f(\mathbb{C}) \not\subset \text{supp}(D) \). Suppose that there exist holomorphic bundle
mappings

\[ t_k : J^k X \rightarrow TX, \quad 2 \leq k \leq n \]
such that

\[ f' \wedge t_2(j_2f) \wedge \cdots \wedge t_n(j_nf) \not\equiv 0 \]

and

\[ \| t_k(j_kf) \| \leq c_k(\rho_k(j_kf))^\alpha_k, \quad 2 \leq k \leq n \]

for some positive constants \( c_2, \ldots, c_n \) and \( \alpha_1, \ldots, \alpha_n \), where \( \rho_2, \ldots, \rho_n \) are non-negative functions defined by (20). With these assumptions, Hu-Yang ([14], Theorem 4.2) showed that for any \( \delta > 0 \),

\[ T_f(r, L) + T_f(r, K_X) \leq N_f(r, D) - N_{\text{Ram}}(r, f) + O\left( \log^+ T_f(r, L) + \delta \log r \right) \]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

Hu-Yang didn’t give arguments about the existence of holomorphic bundle mappings \( t_k \), they left a question as follows

**Question.** *Do such holomorphic bundle mappings* \( t_k \) *exist?*

It seems very difficult to provide a positive answer to the above question. In this paper, we construct the following holomorphic \( \mathbb{C}^* \)-bundle mappings

\[ t_k : J^k X \rightarrow T^k X, \quad k \geq 1, \]

where \( T^k X \) is the Wong-Stoll jet bundle (the full \( k \)-jet bundle). Through this construction, one defines a key quantity, i.e., the so-called "holomorphic field" \( \varphi \) over \( \mathbb{C} \), which plays an important role in establishing the Second Main Theorem of holomorphic curves into complex projective varieties intersecting normal crossing divisors.

1.3. Main results.

In what follows, we introduce the main results of the paper.

**Theorem 1.6.** *Let* \( X \) *be a smooth complex projective variety. Let* \( D \in |L| \) *be a simple normal crossing divisor on* \( X \), *where* \( L \) *is a positive line bundle over* \( X \). *Let* \( f : \mathbb{C} \rightarrow X \) *be an algebraically non-degenerate holomorphic curve. Then for any* \( \delta > 0 \),

\[ T_f(r, L) + T_f(r, K_X) \leq N_f(r, D) - N(r, (F_\varphi)) + O\left( \log^+ T_f(r, L) + \delta \log r \right) \]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure, where \( F_\varphi \) is the Jacobian section (see (5)) of \( f \) defined by the holomorphic field

\[ \varphi := (t_2(j_2f)) f \wedge \cdots \wedge (t_n(j_nf)) f, \]

where \( t_k \) is given by Definition 2.3.
We may see that the Griffiths conjecture is a consequence of Theorem 1.6. Note its two canonical applications, the Green-Griffiths conjecture and Lang conjecture shall also be confirmed (see Section 3.2.3 and Section 3.2.4). Not only that, we obtain a result stronger than Theorem 1.6 (see Theorem 3.4). Moreover, Theorem 1.6 derives a stronger form of Griffiths conjecture (see Corollary 3.5).

The consequence below is also called the "small Griffiths conjecture" (see, e.g., \([10, 15, 25]\)).

**Corollary 1.7.** Let \(H\) be the hyperplane line bundle over \(\mathbb{P}^n(\mathbb{C})\). Let \(D_1, \ldots, D_q \in |H^d|\) \((d \geq 1)\) such that \(D_1 + \cdots + D_q\) has simple normal crossings. Let \(f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})\) be an algebraically non-degenerate holomorphic curve. Then for any \(\delta > 0\),

\[(qd-n-1)T_f(r, H) \leq \sum_{j=1}^{q} N_f(r, D_j) - N(r, (W_f)) + O\left(\log^+ T_f(r, H) + \delta \log r\right)\]

holds outside a set \(E_\delta \subset (0, \infty)\) of finite Lebesgue measure, where \(W_f\) is the Wronskian determinant of \(f\).

When \(d = 1\), Corollary 1.7 (see Remark 3.3 and Theorem 3.4) agrees with the Second Main Theorem of Cartan (see \([3]\)). The ramification divisor \((F_\varphi)\) gives an upper bound of the truncated multiplicity of divisor \(f^*D\) (see \([17]\)). By combining this with Theorem 1.6, we obtain a defect relation

**Corollary 1.8** (Defect relation). Assume the same conditions as in Theorem 1.6. Then

\[\delta_f^{[\varpi]}(D) \leq \left\lfloor \frac{c_1(K_X^*)}{c_1(L)} \right\rfloor\]

for some integer \(\varpi\) with an upper bound \(n(n+1)/2\). In particular, for \(X = \mathbb{P}^n(\mathbb{C})\)

\[\sum_{j=1}^{q} \delta_f^{[\varpi]}(r, D_j) \leq \frac{n+1}{d}.\]

It is clear that Conjecture 1.3 is a consequence of Corollary 1.8. Finally, we still obtain the following consequences

**Corollary 1.9.** Let \(X\) be a smooth complex projective variety. Let \(K\) be the canonical divisor, and \(D\) be a normal crossing divisor on \(X\). If \(K + D\) is pseudo ample, then there are no algebraically non-degenerate holomorphic curves in \(X \setminus D\).

**Corollary 1.10** (Picard-type theorem). There are no linearly non-degenerate holomorphic curves in \(\mathbb{P}^n(\mathbb{C}) \setminus D\) for a normal crossing hypersurface \(D\) of degree at least \(n + 2\).
Corollary 1.11 (Green conjecture). Let $V \subseteq \mathbb{P}^n(\mathbb{C})$ be smooth hypersurface of degree at least $n + 2$. Then there are no algebraically non-degenerate holomorphic curves in $V$.

Corollary 1.9 was stated as a conjecture (see [13], Conjecture 2.100), and both Corollary 1.10 and Corollary 1.11 were also put forward as conjectures by Green [8] in 1975.

2. Jacobian sections and holomorphic fields

2.1. Jacobian sections.

The notion of Jacobian sections was introduced by Stoll [26]. Let $M, X$ be complex manifolds of complex dimensions $m, n$, respectively. Let $f : M \to X$ be a holomorphic mapping and $\pi : E \to X$ be a holomorphic vector bundle. The pull-back bundle $\tilde{\pi} : f^*E \to M$ is defined naturally up to an isomorphism $\tilde{f} : f^*E_x \to E_{f(x)}$ such that $\tilde{f} \circ \tilde{\pi} = \pi \circ f$ for each $x \in M$. It is equivalent to be formulated as follows

$$f^*E := \{(x, \xi) \in M \times E : f(x) = \pi(\xi)\}$$

satisfying that $\tilde{\pi}(x, \xi) = x$ and $\tilde{f}(x, \xi) = \xi$. Let $H^0(X, E)$ denote the space of all holomorphic sections of $E$ over $X$. Given a section $s \in H^0(X, E)$, the lifted section $s_f \in H^0(M, f^*E)$ of $s$ via $f$ is defined by

$$s_f(x) = \tilde{f}^{-1}(s \circ f(x)), \quad x \in M.\tag{1}$$

If $s \neq 0$ and $f(M) \not\subseteq s^{-1}(0)$, then $s_f \neq 0$, and the pull-back divisor $f^*(s)$ of the zero divisor $(s) := (s = 0)$ exists. The multiplicity functions for divisors $(s_f)$ and $f^*(s)$ satisfy $\mu(s_f) \geq \mu_f(s)$.

The Jacobian bundle (see Stoll [26]) of $f$ is defined by

$$K(f) := K_M \otimes f^*K_X^*,$$

where $K_X^*$ is the dual of canonical line bundle $K_X$. A holomorphic section $F$ of $K(f)$ over $M$ is called a Jacobian section of $f$, and which is said to be effective if $F^{-1}(0)$ is thin. If $F$ is effective, we call the zero divisor $(F)$ the ramification divisor of $f$ with respect to $F$.

Let $F$ be a Jacobian section of $f$. The natural interior product (see [15], Page 37) on $K_X^* \otimes K_X$ pulls back to an interior product on $f^*K_X^* \otimes f^*K_X$. Further, it induces a $K_M$-valued interior product

$$\angle : K(f) \oplus f^*K_X \to K_M.$$

For an open subset $V \subseteq X$ with $\tilde{V} = f^{-1}(V) \neq \emptyset$, $F$ defines a linear mapping

$$\mathcal{F} : H^0(V, K_X) \to H^0(\tilde{V}, K_M)\tag{2}$$
by $\mathcal{F}[\Psi] = F \angle \Psi_f$ for all $\Psi \in H^0(V, K_X)$. Take any open subset $W \subseteq X$ such that $\tilde{W} = f^{-1}(W) \neq \emptyset$. Let $(U; z_1, \ldots, z_m)$, $(V; w_1, \ldots, w_n)$ be holomorphic coordinate charts of $M, X$ respectively such that $V \subseteq W$ and $f(U) \subseteq V$. For $\Psi \in H^0(W, K_X)$, we write $\Psi|_V = \tilde{\Psi} dw$ and $\tilde{F}|_U = \tilde{F} dz \otimes d^*w_f$, where $\tilde{\Psi}, \tilde{F}$ are holomorphic functions on $V, U$ respectively and

$$dz = dz_1 \wedge \cdots \wedge dz_m, \quad d^*w = \frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_n}.$$ 

According to the definition of $\mathcal{F}$,

$$(\mathcal{F}[\Psi]) \cap U = (F) \cap U$$

for every $\Psi \in H^0(W, K_X)$ such that $\Psi$ vanishes nowhere on $W$. Let $A^{2n}(W)$ denote the space of smooth $(n, n)$-forms on $W$. We can extend $\mathcal{F}$ linearly to $\mathcal{F}: A^{2n}(W) \rightarrow A^{2m}(\tilde{W})$ in a natural way: if $\Omega \in A^{2n}(W)$ is locally expressed as $\Omega|_V = i_{n_1} dw \wedge d\overline{w}$, where

$$i_p = \left(\frac{-1}{2\pi}\right)^p (-1)^{\frac{p(p-1)}{2}} p!, \quad p \geq 1,$$

then

$$\mathcal{F}[\Omega]|_U = i_{m_1} \tilde{\rho} \circ f \cdot \mathcal{F}[dw] \wedge \overline{\mathcal{F}[dw]} = i_{m_1} \tilde{\rho} \circ f \cdot |\tilde{F}|^2 dz \wedge d\overline{z}.$$ 

It is trivial to check that

$$\mathcal{F}[g_1 \Omega_1 + g_2 \Omega_2] = g_1 \circ f \cdot \mathcal{F}[\Omega_1] + g_2 \circ f \cdot \mathcal{F}[\Omega_2]$$

for $g_1, g_2 \in A^0(W)$ and $\Omega_1, \Omega_2 \in A^{2n}(W)$. Endowing $K(f)$ with a Hermitian metric $\kappa$ and setting $\tilde{\kappa}|_U = ||dz \otimes d^*w_f||^2_\kappa$, then we have $||F||^2_\kappa = \tilde{\kappa}|\tilde{F}|^2$. Take $\Theta \in A^{2m}(\tilde{W})$ such that

$$\Theta|_U = i_{m_1} \tilde{\rho} \circ f \cdot \tilde{\kappa}^{-1} dz \wedge d\overline{z},$$

one obtains $\mathcal{F}[\Omega] = ||F||^2_\kappa \Theta$. It is clear that $\Theta > 0$ if and only if $\Omega > 0$.

### 2.2. Holomorphic fields.
2.2.1. Holomorphic jet bundles.

We introduce two classes of important holomorphic jet bundles.

a) Green-Griffiths jet bundles

The Green-Griffiths jet bundles (or restricted jet bundles) were introduced by Green-Griffiths [9]. It is defined as follows. Take $x \in X$, denote by $H_x$ the sheaf of germs of holomorphic curves $f : \Delta_r \to X$ with $f(0) = x$, where $\Delta_r$ is the disc centered at 0 with radius $r$ in $\mathbb{C}$. Take $k \geq 1$, one defines an equivalence relation $\sim_k$ by designating two germs $f, g \in H_x$ as $k$-equivalent, denoted by $f \sim_k g$, if $f_j^{(i)} = g_j^{(i)}$ for $1 \leq i \leq k, 1 \leq j \leq n$, where $f_j = w_j \circ f$ and $w_1, \ldots, w_n$ are local holomorphic coordinates near $x$. A Green-Griffiths $k$-jet bundle $J^k X$ over $X$ is the sheaf of parametrized $k$-jets defined by

$$J^k X := \bigsqcup_{x \in X} H_x / \sim_k.$$ 

According to the definition of $k$-jets, for an element $\xi \in J^k X$, there exists a holomorphic curve $f : \Delta_r \to X$ such that $\xi = j_k f(0)$, i.e., $\xi$ is expressed as $\xi = (f(0), f'(0), \ldots, f^{(k)}(0))$.

It is clear that $J^1 X = TX$, here $TX$ is the holomorphic tangent bundle over $X$. In general, the nonlinearity of the change of coordinates shows that $J^k X$ is not locally free for $k \geq 2$, and thus not a vector bundle for $k \geq 2$.

Let $f : \Delta_r \to X$ be a holomorphic curve, the lifted $k$-jet $j_k f : \Delta_{r/2} \to J^k X$ is defined by

$$j_k f(z) = j_k g(0), \quad z \in \Delta_{r/2},$$

where $g(t) = f(t+z)$ is holomorphic for $t \in \Delta_{r/2}$. That is to say, if $f : \mathbb{C} \to X$ is a holomorphic curve, then $f$ defines a $k$-jet $j_k f(z)$ for every $z \in \mathbb{C}$.

A $\mathbb{C}^*$-action on $J^k X$ is defined via parametrization as follows. For $\lambda \in \mathbb{C}^*$ and $f \in H_x$, a mapping $f_\lambda \in H_x$ is defined by $f_\lambda(t) = f(\lambda t)$, i.e.,

$$j_k f_\lambda(0) = (f(0), f'(0), \ldots, f^{(k)}(0)).$$

In other words, a $\mathbb{C}^*$-action of $k$-jets is defined by

$$(4) \quad \lambda j_k f(0) = (f(0), \lambda f'(0), \ldots, \lambda^k f^{(k)}(0)).$$

Let $(V; w_1, \ldots, w_n)$ be a holomorphic coordinate chart of $X$. The dual of the restricted $k$-jet bundle $J^k X$ (i.e., the sheaf associated to the presheaf consisting of holomorphic mappings $\omega : J^k X|_V \to \mathbb{C}$ with $\omega(\lambda j_k f) = \lambda^m \omega(j_k f)$ for $\lambda \in \mathbb{C}^*$ and an integer $m > 0$) shall be referred to as the sheaf of germs
of k-jet differentials of weight m, we denote it by $\mathcal{J}^m_k X$. A k-jet differential $\omega$ of weight $m$ is of the form

$$\omega = \sum_{|I_1| + \cdots + |I_k| = m} a_{I_1 \cdots I_k} dw^{i_1} \cdots d^k w^{i_k},$$

where the coefficients $a_{I_1 \cdots I_k}$ are holomorphic functions which are symmetric in indices $I_1, \cdots, I_k$, here $I_j = (i_{1j}, \cdots, i_{nj})$, $|I_j| = i_{1j} + \cdots + i_{nj}$ and

$$dw^{i_j} = dw^{i_{1j}} \cdots dw^{i_{nj}}.$$

b) Wong-Stoll jet bundles

We introduce the concept of Wong-Stoll jet bundles (or full jet bundles) introduced by Wong-Stoll [35] (or see Chandler-Wong [4]).

**Definition 2.1** ([4, 35]). Let $X$ be a complex manifold of complex dimension $n$. The sheaf of germs of holomorphic k-jets (differential operators of order $k$), denoted by $T^k X$, is the subsheaf of the sheaf of germs of holomorphisms $\text{Hom}_C(\mathcal{O}_X, \mathcal{O}_X)$ consisting of elements (differential operators) of the form

$$\sum_{j=1}^k \sum_{i_1, \cdots, i_j \in \mathbb{N}} D_{i_1} \circ \cdots \circ D_{i_j},$$

where $D_{i_j} \in TX$. In terms of local holomorphic coordinates $w_1, \cdots, w_n$ of $X$, an element of $T^k X$ is expressed as

$$\sum_{j=1}^k \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} a_{i_1 \cdots i_j} \frac{\partial^j}{\partial w^{i_1} \cdots \partial w^{i_j}},$$

where the coefficients $a_{i_1 \cdots i_j}$ are holomorphic functions.

By the above definition, it is clear that $TX \subseteq T^2 X \subseteq T^3 X \subseteq \cdots$

The dual of the full k-jet bundle $T^k X$ is referred to as the sheaf of germs of holomorphic k-jet forms and denoted by $\mathcal{J}^*_k X$. Let $(V, w_1, \cdots, w_n)$ be a local holomorphic coordinate chart of $X$. Then

$$\left(\frac{\partial}{\partial w^{i_1}}\right)_{1 \leq i_1 \leq n}, \cdots, \left(\frac{\partial^k}{\partial w^{i_1} \cdots \partial w^{i_k}}\right)_{1 \leq i_1 \leq \cdots \leq i_k \leq n}$$

is a basis of $T^k X|_V$. The dual basis shall be denoted, formally, by

$$\left(dw^{i_1}\right)_{1 \leq i_1 \leq n}, \cdots, \left(dw^{i_1} \cdots dw^{i_k}\right)_{1 \leq i_1 \leq \cdots \leq i_k \leq n}.$$
An element of $T_k^* X$ is then of the form

$$\omega = \sum_{j=1}^{k} \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n} b_{i_1 \cdots i_j} dw_{i_1} \cdots dw_{i_j},$$

where the coefficients $b_{i_1 \cdots i_j}$ are holomorphic functions.

**Theorem 2.2 ([4, 35]).** Let $X$ be a complex manifold of complex dimension $n$. Then $T^k X$ is a holomorphic vector bundle over $X$ of rank $\sum_{j=1}^{k} \binom{n+j-1}{j}$, whose transition function is of the form

$$
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
* & A_2 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& \vdots & \ddots & \ddots & \ddots \\
* & \cdots & * & A_{k-1} & 0 \\
* & \cdots & * & * & A_k
\end{pmatrix},
$$

where the $((n+j-1), (n+j-1))$-matrix $A_j$ is the transition function of the bundle $\otimes^j T X$, i.e., the $j$-fold symmetric product of $T X$.

The linear (and invertible) nature of the transition function implies that $T^k X$ is locally free, which can also be seen by observing that $T^{k-1} X$ injects into $T^k X$ and there is an exact sequence of sheaves that

$$0 \longrightarrow T^{k-1} X \longrightarrow T^k X \longrightarrow T^k X/T^{k-1} X \longrightarrow 0.$$  

### 2.2.2. Holomorphic $\mathbb{C}^*$-bundle mappings.

Let $f : \Delta_r \to X$ be a holomorphic curve into a complex manifold $X$. Set

$$\frac{d}{dz} \left( \frac{\partial^p}{\partial w_{i_1} \cdots \partial w_{i_p}} \right) = \sum_{j=1}^{n} (w_j \circ f)' \frac{\partial^{p+1}}{\partial w_j \partial w_{i_1} \cdots \partial w_{i_p}},$$

$$= \sum_{j=1}^{n} f_j' \frac{\partial^{p+1}}{\partial w_j \partial w_{i_1} \cdots \partial w_{i_p}},$$

where $w_1, \cdots , w_n$ are local holomorphic coordinates of $X$ and $z$ is holomorphic coordinate of $\Delta_r$. For any $z \in \Delta_r/2$, by a direct computation, we have
the following canonical correspondences:

\[
\begin{align*}
\frac{d}{dz} & \mapsto \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} \\
\frac{d^2}{dz^2} & \mapsto \frac{d}{dz} \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} = \sum_{j=1}^{n} f_j'' \frac{\partial}{\partial w_j} + \sum_{i,j=1}^{n} f_i' f_j' \frac{\partial^2}{\partial w_i \partial w_j}, \\
\frac{d^3}{dz^3} & \mapsto \frac{d^2}{dz^2} \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} = \sum_{j=1}^{n} f_j^{(3)} \frac{\partial}{\partial w_j} + 3 \sum_{i,j=1}^{n} f_i' f_j'' \frac{\partial^2}{\partial w_i \partial w_j} + \sum_{i,j,k=1}^{n} f_i' f_j' f_k' \frac{\partial^3}{\partial w_i \partial w_j \partial w_k}, \\
\frac{d^4}{dz^4} & \mapsto \frac{d^3}{dz^3} \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} = \sum_{j=1}^{n} f_j^{(4)} \frac{\partial}{\partial w_j} + \sum_{1 \leq i,j \leq n} \left( 4 f_i' f_j^{(3)} + 3 f_i'' f_j'' \right) \frac{\partial^2}{\partial w_i \partial w_j} + 6 \sum_{i,j,k=1}^{n} f_i' f_j' f_k' \frac{\partial^3}{\partial w_i \partial w_j \partial w_k} + \sum_{i,j,k,l=1}^{n} f_i' f_j' f_k' f_l' \frac{\partial^4}{\partial w_i \partial w_j \partial w_k \partial w_l}, \\
\cdots \\
\frac{d^k}{dz^k} & \mapsto \frac{d^{k-1}}{dz^{k-1}} \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} = \sum_{j=1}^{n} f_j^{(k)} \frac{\partial}{\partial w_j} + \cdots + \sum_{i_1, \cdots, i_k=1}^{n} f_{i_1}' \cdots f_{i_k}' \frac{\partial^k}{\partial w_{i_1} \cdots \partial w_{i_k}}, \\
\cdots
\end{align*}
\]

Since \( \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} \) is independent of the choices of local holomorphic coordinates, i.e., for another set of local holomorphic coordinates \( \zeta_1, \cdots, \zeta_n \)

\[
\sum_{j=1}^{n} (w_j \circ f)' \frac{\partial}{\partial w_j} = \sum_{j=1}^{n} (\zeta_j \circ f)' \frac{\partial}{\partial \zeta_j},
\]

then we see that \( (d^k/dz^k) \sum_{j=1}^{n} f_j' \frac{\partial}{\partial w_j} \) is holomorphic and it does not depend on the choices of local holomorphic coordinates.
Definition 2.3. Let \( f : \Delta_r \to X \) be a holomorphic curve into a complex manifold \( X \). The holomorphic bundle mapping
\[
t_k : J^kX \to T^kX, \quad k \geq 1
\]
is defined by
\[
t_k(j_kf) = t_k((f,f',\ldots,f^{(k)})) = \frac{d^{k-1}}{dz^{k-1}} \sum_{j=1}^n f'_j \frac{\partial}{\partial w_j},
\]
where
\[
d \frac{\partial^p}{\partial w_{i_1} \cdots \partial w_{i_p}} \bigg|_{w_j \circ f} = \sum_{j=1}^n (w_j \circ f)' \frac{\partial^{p+1}}{\partial w_j \partial w_{i_1} \cdots \partial w_{i_p}}
\]
in local holomorphic coordinates \( w_1, \ldots, w_n \).

According to the definition of \( \mathbb{C}^* \)-actions, it yields immediately that
\[
t_k(\lambda.j_kf) = \lambda^k t_k(j_kf), \quad \lambda \in \mathbb{C}^*.
\]
Therefore, \( t_k \) is a holomorphic \( \mathbb{C}^* \)-bundle mapping.

Remark 2.4. Differ from Definition 2.3, Wong-Stoll (see [35], (11)) defined a holomorphic \( \mathbb{C}^* \)-bundle mapping as follows
\[
p_k : J^kX \to T^kX
\]
by
\[
p_k(j^k f) = \sum_{j=1}^k \sum_{i=1}^n \left( f^{(j)}_{i_1,\ldots,i_{k-j} \neq i} \prod_{i_{k-j} \neq i} f'_i \right) \frac{\partial^{k-j+1}}{\partial w_{i_1} \partial w_{i_1} \cdots \partial w_{i_{k-j}}}
\]
in local holomorphic coordinates \( w_1, \ldots, w_n \).

2.2.3. Holomorphic fields.

We introduce the notion of holomorphic fields in order to construct Jacobian sections. Let \( M, X \) be complex manifolds of complex dimensions \( m, n \), respectively, with \( m < n \). Let \( f : M \to X \) be a holomorphic mapping, then there exists a unique homomorphism
\[
\hat{f} : f^* \bigwedge^m T^*X \to \bigwedge^m T^*M
\]
such that \( \hat{f}(\xi) = f^* \xi \) for \( \xi \in H^0(V, \wedge^m T^*X) \), where \( V \) is an open set in \( X \) such that \( \tilde{V} = f^{-1}(V) \neq \emptyset \). Note that
\[
d w_j \left( \frac{\partial^k}{\partial w_{i_1} \cdots \partial w_{i_k}} \right) = \frac{\partial^k w_j}{\partial w_{i_1} \cdots \partial w_{i_k}} = 0, \quad k \geq 2.
\]
Then there is a natural interior product (see [15], Page 37)
\[
\angle : K_X \oplus \bigwedge^{n-m} T^*X \to \bigwedge^m T^*X
\]
which pulls back to an interior product
\[
\angle : f^* K_X \oplus f^* \bigwedge^{n-m} T^n X \to f^* \bigwedge^m T^* X.
\]

A section \( \varphi \in H^0(M, f^* \bigwedge^{n-m} T^n X) \) is called a \textit{holomorphic field of } \( f \) \textit{over } \( M \) \textit{of degree } \( n-m \). Remark: this notion of holomorphic fields extends the one of holomorphic fields defined by Stoll (see [26]). A holomorphic field \( \varphi \) defines a Jacobian section \( F_\varphi \) of \( f \) by
\[
F_\varphi|_{\tilde{V}} = \hat{f}(\Psi \angle \varphi) \otimes \Psi_f,
\]
where \( \Psi \in H^0(V, K_X) \) vanishes nowhere on \( V \), and \( \Psi^* \) is the dual of \( \Psi \). Or equivalently, \( F_\varphi \) can be described as follows (see (2))
\[
F_\varphi[\Psi] = \hat{f}(\Psi \angle \varphi),
\]
where \( \Psi \in H^0(V, K_X) \) with \( \tilde{V} \neq \emptyset \) for \( V \) open in \( X \). We say that \( \varphi \) is effective if \( F_\varphi \) is effective. By (3), it yields \( (F_\varphi[\Psi]) \cap U = (F_\varphi) \cap U \) for \( \Psi \in H^0(U, K_X) \) satisfying that \( \Psi \) vanishes nowhere on \( U \subseteq X \). Similarly as before, \( F_\varphi \) can be linearly extended to \( \mathcal{F}_\varphi : A^{2n}(V) \to A^{2m}(\tilde{V}) \) with \( \tilde{V} = f^{-1}(V) \).

Take an integer \( k \) satisfying that \( 1 \leq k \leq n \). Denote by \( J_{1,k}^n \) the set of all increasing injective mappings
\[
\lambda : \mathbb{Z}[1,k] \to \mathbb{Z}[1,n],
\]
where \( \mathbb{Z}[r,s] \) \( (r \leq s) \) denotes the set of integers \( p \) with \( r \leq p \leq s \). If \( \lambda \in J_{1,k}^n \), then \( \lambda^\perp \) is uniquely defined such that \( (\lambda, \lambda^\perp) \) is a permutation of \( \{1, \ldots, n\} \). Clearly, \( \perp : J_{1,k}^n \to J_{1,n-k}^n \) is a bijective mapping.

**Lemma 2.5** ([12], Corollary 5.6.3). Let \( E \) be a holomorphic vector bundle over a Stein manifold \( M \). Then for every \( x_0 \in M \) and every \( s_0 \in E_{x_0} \), there exists a holomorphic section \( s \) of \( E \) over \( M \) such that \( s(x_0) = s_0 \).

**Theorem 2.6.** Let \( M, X \) be complex manifolds of complex dimensions \( m, n \), respectively, with \( m < n \). Let \( f : M \to X \) be a holomorphic mapping. Assume that \( M \) is Stein, then there exists a holomorphic field \( \varphi \) of \( f \) over \( M \) of degree \( n-m \) such that \( \varphi \) is effective if and only if \( f \) has rank \( m \).

**Proof.** Let \( S \) denote the set of all \( x \in M \) such that the rank of the Jacobian of \( f \) at the point \( x \) is smaller than \( m \). Take an arbitrary point \( x_0 \in M \), there are local holomorphic coordinate charts \( (U; z_1, \ldots, z_m) \) of \( x_0 \) and \( (V; w_1, \ldots, w_n) \) of \( f(x_0) \) such that \( f(U) \subseteq V \). Set
\[
d^* w_j = \frac{\partial}{\partial w_j}, \quad 1 \leq j \leq n.
\]
Again, set $q = n - m$ and
\[
\begin{align*}
&dz = dz_1 \wedge \cdots \wedge dz_m, \\
&dw = dw_1 \wedge \cdots \wedge dw_n, \\
&dw_\nu = dw_{\nu(1)} \wedge \cdots \wedge dw_{\nu(m)}, \\
&d^\ast w_\lambda = d^\ast w_{\lambda(1)} \wedge \cdots \wedge d^\ast w_{\lambda(q)},
\end{align*}
\]
where $\nu \in J_{k,m}^n, \lambda \in J_{l,q}^n$ are defined by (6). Write
\[
(7) \quad f^\ast dw_\nu = A_\nu dz, \quad \nu \in J_{k,m}^n.
\]
It is clear that
\[
S \cap U = \bigcap_{\nu \in J_{k,m}^n} A_\nu^{-1}(0).
\]
a) Necessity

Assume that $\varphi$ is an effective field of $f$ over $M$ of degree $n - m$, we prove that $f$ has rank $m$. By (5), $\varphi$ defines an effective Jocabian section $F_\varphi$. Write $\varphi$ as two parts
\[
(8) \quad \varphi = \varphi' + \varphi'',
\]
where
\[
(9) \quad \varphi' = \sum_{\lambda \in J_{k,q}^n} \varphi_\lambda' d^\ast w_\lambda f
\]
denotes the first order derivative part of $\varphi$, and $\varphi''$ denotes the higher order derivative part of $\varphi$. Since
\[
dw_j \left( \frac{\partial^k}{\partial w_{i_1} \cdots \partial w_{i_k}} \right) = \frac{\partial^k w_j}{\partial w_{i_1} \cdots \partial w_{i_k}} = 0, \quad k \geq 2,
\]
then
\[
dw_f \angle \varphi = dw_f \angle \varphi' = \sum_{\lambda \in J_{k,q}^n} \text{sign}(\lambda^\perp, \lambda) \varphi_\lambda' dw_{\lambda^\perp} f,
\]
where $(\lambda, \lambda^\perp)$ is a permutation of $\{1, \cdots, n\}$. This implies that
\[
F_\varphi[dw] = f(dw_f \angle \varphi) = \sum_{\lambda \in J_{k,q}^n} \text{sign}(\lambda^\perp, \lambda) \varphi_\lambda' A_{\lambda^\perp} dz,
\]
where $F_\varphi$ is defined by $F_\varphi$ (see (2)). Thus, it follows from (3) that
\[
(10) \quad (F_\varphi) \cap U = \left( \sum_{\lambda \in J_{k,q}^n} \text{sign}(\lambda^\perp, \lambda) \varphi_\lambda' A_{\lambda^\perp} \right) \cap U \supseteq S \cap U.
\]
Thus, $S$ is thin for that the support of zero divisor $(F_\varphi)$ is thin. This implies that $f$ has rank $m$.

b) Sufficiency
Assume that \( f \) has rank \( m \), we prove that there exists a holomorphic field \( \varphi \) of \( f \) over \( M \) of degree \( n - m \). Since \( f \) is of rank \( m \), then \( S \) is thin. Taking \( x_0 \in M \setminus S \), then there is \( \iota \in J_{1,m}^n \) such that \( A_i(x_0) \neq 0 \), where \( A_i \) is defined by (7). Since \( M \) is Stein, then by Lemma 2.5 there are holomorphic sections \( s_j \in H^0(M, f^*TX) \subseteq H^0(M, f^*T^nX) \) such that

\[
s_j(x_0) = d^*w_{jf}(x_0), \quad 1 \leq j \leq n.
\]

Set \( \varphi := s_{\iota^\perp} \), where \( s_{\iota^\perp} = s_{\iota(1)} \land \cdots \land s_{\iota(m)} \). Then

\[
\varphi = \sum_{\lambda \in J_{1,q}^n} \varphi_{\lambda} d^*w_{\lambda f} \in H^0\left(M, \bigwedge^q f^*TX \right) \subseteq H^0\left(M, \bigwedge^q f^*T^nX \right),
\]

where

\[
\varphi_{\lambda}(x_0) = \begin{cases} 1, & \text{if } \lambda = \iota^\perp; \\ 0, & \text{if } \lambda \neq \iota^\perp. \end{cases}
\]

Next, we show that \( \varphi \) is effective. Note that

\[
F_{\varphi}[dw] = \sum_{\lambda \in J_{1,q}^n} \text{sign}(\lambda^\perp, \lambda) \varphi_{\lambda} A_{\lambda^\perp} dz
\]

and

\[
F_{\varphi}[dw](x_0) = \text{sign}(\iota, \iota^\perp) A_\iota(x_0) dz \neq 0.
\]

Again, by (4)

\[
(F_{\varphi}[dw]) \cap U = (F_{\varphi}) \cap U,
\]

then \( x_0 \) is not a zero of \( F_{\varphi} \). This implies that \( \text{supp}((F_{\varphi})) \) is thin. Thus, \( \varphi \) is effective. \( \square \)

3. Proofs of the theorems, corollaries and conjectures

3.1. Lemmas.

Let \( X \) be a smooth complex projective variety and let \( D \in |L| \) be a simple normal crossing divisor on \( X \), where \( L \) is a positive line bundle over \( X \). Write \( D = D_1 + \cdots + D_q \), where \( D_j \) (\( 1 \leq j \leq q \)) are irreducible components of \( D \). Taking the canonical section \( s_j \) of holomorphic line bundle \( \mathcal{O}_X(D_j) \) (defined by \( D_j \)) over \( X \), such that we have \( D_j = (s_j) \) for \( 1 \leq j \leq q \). Equip Hermitian metrics \( h_j \) on \( \mathcal{O}_X(D_j) \) (\( 1 \leq j \leq q \)) such that

\[
0 \leq ||s_j|| < 1, \quad 1 \leq j \leq q.
\]

One gets the induced Hermitian metric \( h = h_1 \otimes \cdots \otimes h_q \) on \( L \). Since \( L > 0 \), then one may assume that \( c_1(L, h) > 0 \). Let \( \Omega \) be a smooth volume form on \( X \) such that \( [\text{Ric}(\Omega)] = c_1(K_X) \), where the Ricci form \( \text{Ric}(\Omega) \) is defined in the sense of Griffiths, i.e., if \( \Omega = adw_1 \land \cdots \land dw_n \) is written in local holomorphic
coordinates \( w_1, \cdots, w_n \), in which, \( a \) is a local smooth positive function, then \( \text{Ric}(\Omega) = dd^c \log a \). Define the Carlson-Griffiths form as follows

\[
\Psi = \prod_{j=1}^q \| s_j \|^2 (\log \| s_j \|^2)^2.
\]

If \( c_1(L, h) + \text{Ric}(\Omega) > 0 \), then there is a volume form \( \Omega \) being satisfied with

\[
\text{Ric}(\Psi) > 0, \quad \text{Ric}(\Psi)^n \geq \Psi, \quad \int_{X \setminus D} \text{Ric}(\Psi)^n < \infty
\]
on \( X \setminus D \) (see Carlson-Griffiths [2], Proposition 2.1). Set

\[
\chi := c_1(L, h) + \text{Ric}(\Omega).
\]

A direct computation gives that (see Carlson-Griffiths [2], (2.7))

\[
\text{Ric}(\Psi) = \chi - 2 \sum_{j=1}^q \frac{dd^c \log \| s_j \|^2}{\log \| s_j \|^2} + \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^q \frac{\partial \log \| s_j \|^2 \wedge \bar{\partial} \log \| s_j \|^2}{(\log \| s_j \|^2)^2},
\]

where

\[
d = \partial + \bar{\partial}, \quad dd^c = \sqrt{-1}(\bar{\partial} - \partial)/4\pi.
\]

**Lemma 3.1** ([13, 14, 15]). Let \( X \) be a smooth complex projective variety. Let \( D = D_1 + \cdots + D_q \in |L| \) be a simple normal crossing divisor, where \( L \) is a positive line bundle over \( X \). Let \( f : \mathbb{C} \to X \) be a holomorphic curve such that \( f(\mathbb{C}) \not\subseteq \text{supp}(D) \). Assume that there is an effective Jacobian section \( F \) of \( f \). Let \( g \) be a non-negative function defined by

\[
F[\text{Ric}(\Psi)^n] = g^{2} f^{*} \text{Ric}(\Psi),
\]

where \( F \) is defined by \( F \) and \( \Psi \) is defined by (12). Then for any \( \delta > 0 \),

\[
T_f(r, L) + T_{f, K_X}(r) \leq N_f(r, D) - N(r, (F)) + \frac{1}{2\pi} \int_0^{2\pi} \log g(re^{i\theta}) d\theta + O\left( \log^{+} T_f(r, L) + \delta \log r \right)
\]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

**Lemma 3.2** ([15], Lemma 2.80). Let \( X \) be an \( n \)-dimensional smooth complex projective variety. Let \( A \) be a very ample line bundle, \( L \) be a pseudo ample line bundle over \( X \). Then

\[\limsup_{j \to +\infty} \frac{1}{j^n} \dim H^0(X, L^j \otimes A^*) > 0.\]

3.2. Proofs of the theorems, corollaries and conjectures.
3.2.1. *Proof of Theorem 1.*

For \( n = 1 \), the theorem was confirmed by Chern \[1\] and Carlson-Griffiths-King \[2, 11\]. In the following, one only needs to prove the theorem for \( n \geq 2 \). According to Lemma 3.1, it suffices to prove the following estimate

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log^+ g(re^{i\theta}) d\theta \leq O\left( \log^+ T_f(r, L) \right)
\]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure, where \( g \) is defined by

\[
F_\varphi[Ric(\Psi)^n] = g^2 f^* Ric(\Psi)
\]

with \( \Psi \) being defined by (12).

First of all, we construct an effective holomorphic field \( \varphi \) over \( \mathbb{C} \) so that one can define an effective Jacobian section \( F_\varphi \) which fits to the assumptions of Lemma 3.1.

Consider the following holomorphic \( \mathbb{C}^* \)-bundle mappings

\[
t_k : j^k X \longrightarrow T^k X, \quad k \geq 1
\]

given by Definition 2.3. Since \( f \) is algebraically non-degenerate, then it has to be

\[
f' \wedge t_2(j_2 f) \wedge \cdots \wedge t_n(j_n f) \neq 0.
\]

To see this fact, write

\[
t_k(j_k f) = t'_k(j_k f) + t''_k(j_k f),
\]

where, in local holomorphic coordinates \( w_1, \cdots, w_n \)

\[
t'_k(j_k f) = \sum_{j=1}^{n} f^j_k \frac{\partial}{\partial w_j},
\]

which stands for the first order derivative part of \( t_k(j_k f) \), and \( t''_k(j_k f) \) stands for the higher order derivative part of \( t_k(j_k f) \). Locally, we have

\[
f' \wedge t_2(j_2 f) \wedge \cdots \wedge t_n(j_n f)
\]
\[
= f' \wedge t'_2(j_2 f) \wedge \cdots \wedge t'_n(j_n f) + \text{higher order derivative terms}
\]
\[
= W_f \frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_n} + \text{higher order derivative terms},
\]

where

\[
W_f = \begin{vmatrix}
  f'_1 & f'_2 & \cdots & f'_n \\
  f''_1 & f''_2 & \cdots & f''_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(n)}_1 & f^{(n)}_2 & \cdots & f^{(n)}_n
\end{vmatrix}
\]

with \( f_j = w_j \circ f \) for \( 1 \leq j \leq n \). To prove (15), it suffices to show \( W_f \neq 0 \).
If $W_f \equiv 0$, then a property of the determinants points out that there exist constants $a_1, \cdots, a_n$ that are not all zeros, such that $f$ satisfies the following system of differential equations

$$a_1 f_1^{(k)} + \cdots + a_n f_n^{(k)} = 0, \quad 1 \leq k \leq n.$$ 

Integrating the above differential equation of order $k = 1$, we get

$$a_1 f_1 + \cdots + a_n f_n = a_0$$

for some constant $a_0$. This means that the curve $f$ is algebraically degenerate in $X$, but it is a contradiction with the assumption for $f$. Thus, (15) holds.

Theorem 2.6 gives that there exists an effective holomorphic field $\varphi$ of $f$ over $\mathbb{C}$ of degree $n-1$. In fact, we can choose the following holomorphic field

(18) \[ \varphi := (t_2(j_2 f)) \cdots (t_n(j_n f)) f. \]

By (15), $\varphi$ is effective. Consequently, $\varphi$ defines an effective Jacobian section $F_\varphi$ of $f$ (see (5)). Now write $\varphi$ as two parts

$$\varphi = \varphi' + \varphi'',$

where $\varphi'$ stands for the first order derivative part of $\varphi$ and $\varphi''$ stands for the higher order derivative part of $\varphi$. Then we have

(19) \[ \varphi' = t'_2(j_2 f)f \wedge \cdots \wedge t'_n(j_n f)f = \sum_{j=1}^n f_j''(\frac{\partial}{\partial w_j})f \wedge \cdots \wedge \sum_{j=1}^n f_j^{(n)}(\frac{\partial}{\partial w_j})f, \]

where $t'_k(j_k f)f$ is the lift of $t''_k(j_k f)$ via $f$ (refer to (1)).

By the arguments of Chandler-Wong-Ru (see [4], proof of Theorem 6.1; or [24], Page 231-233), there is a finite number of rational functions $R_1, \cdots, R_p$ on $X$ such that the logarithmic jet differentials

$$\left( \frac{d^j R_i}{R_i} \right)^{m/j}, \quad 1 \leq i \leq p, \quad 1 \leq j \leq k$$

span the fibers of sheaf $\mathcal{J}_k^m X(\log D)$ of germs of $k$-jet differentials of weight $m$ over every point of $X$. Define a non-negative function (see [4], (6.1))

$$\rho_k : J^k X(-\log D) \to [0, +\infty]$$

by

(20) \[ \rho_k(\xi) = \sum_{i=1}^p \sum_{j=1}^k \left| \frac{d^j R_i}{R_i}(\xi) \right|^2, \quad \xi \in J^k X(-\log D). \]

It is clear that

(21) \[ \rho_k(\lambda \xi) = \lambda^{2m} \rho_k(\xi), \quad \lambda \in \mathbb{C}^*. \]
Claim: there exists a positive constant $c$ such that
\begin{equation}
\tag{22}
g \leq c \left( \rho_2(j_2 f) \right)^{2/2m} \cdots \left( \rho_n(j_n f) \right)^{n/2m}.
\end{equation}

First, we show that
\begin{equation}
\tag{23}
g \leq \| \varphi' \|,
\end{equation}
where the norm $\| \cdot \|$ is induced from the singular metric $\text{Ric}(\Psi)$ defined by (13). We first consider those points $z_0 \in \mathbb{C}$ such that the metric $\text{Ric}(\Psi)$ has no singularity at $f(z_0)$, Take a local holomorphic frame $(\psi, \ldots, \psi_n)$ of $T^*X$ near $f(z_0)$ such that
\begin{equation}
\text{Ric}(\Psi) = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^n \psi_j \wedge \overline{\psi}_j.
\end{equation}

In what follows, we show $g(z_0) \leq \| \varphi'(z_0) \|$ under the frame $(\psi, \ldots, \psi_n)$. Let $\psi_1^*, \ldots, \psi_n^*$ be the dual frame of $\psi_1, \ldots, \psi_n$. Set $q = n - 1$ and
\begin{equation}
\psi^*_{\lambda} = \psi^*_{\lambda(1)} \wedge \psi^*_{\lambda(2)} \wedge \cdots \wedge \psi^*_{\lambda(q)},
\end{equation}
where $\lambda \in J^n_{1,q}$ is defined by (6). Define a non-negative function $u$ by
\begin{equation}
f^*\text{Ric}(\Psi) = u \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z},
\end{equation}
where $z$ is the holomorphic coordinate of $\mathbb{C}$. Set
\begin{equation}
f^*\psi_j = A_j dz, \quad 1 \leq j \leq n.
\end{equation}

Then
\begin{equation}
u(z_0) = \sum_{j=1}^n |A_j(z_0)|^2 \neq 0.
\end{equation}

Again, set
\begin{equation}
\Psi = \psi_1 \wedge \cdots \wedge \psi_n.
\end{equation}

According to (9), it yields that
\begin{equation}
\Psi_f \varphi = \sum_{\lambda \in J^n_{1,q}} \text{sign}(\lambda^\perp, \lambda) \varphi_{\lambda}^f \psi_{\lambda^\perp}.
\end{equation}

Thus, we get
\begin{equation}
\mathcal{F}_\varphi[\Psi] = \hat{f}(\Psi_f \varphi) = \sum_{\lambda \in J^n_{1,q}} \text{sign}(\lambda^\perp, \lambda) \varphi_{\lambda}^f A_{\lambda^\perp} dz.
\end{equation}

Combining the above, one obtains
\begin{equation}
g^2(x_0) u(z_0) = \left| \sum_{\lambda \in J^n_{1,q}} \text{sign}(\lambda^\perp, \lambda) \varphi_{\lambda}^f(x_0) A_{\lambda^\perp}(z_0) \right|^2.
\end{equation}
By Schwarz’s inequality, it is trivial to verify that
\[
\left| \sum_{\lambda \in \mathcal{J}_n^k} \text{sign}(\lambda^\perp, \lambda) \varphi'_\lambda(z_0) A_{\lambda^\perp}(z_0) \right|^2 \leq u(z_0) \|\varphi'(z_0)\|^2.
\]
Hence, we obtain \( g(z_0) \leq \|\varphi'(z_0)\| \). When \( z_0 \) is a point such that the metric \( \text{Ric}(\Psi) \) has singularity at \( f(z_0) \), it is clear that \( g(z_0) \leq \|\varphi'(z_0)\| \) since \( z_0 \) is a singularity of \( \|\varphi'\| \). By the arbitrariness of \( z_0 \), we show that (24) holds.

Then, we show the following inequality
\[
\|\varphi'\| \leq c \left( \rho_2(j_2f) \right)^{2/2m} \cdots \left( \rho_n(j_n f) \right)^{n/2m}.
\]
By (19), there exists a positive constant \( c_1 \) such that (see [15], Lemma 1.55)
\[
\|\varphi'\| \leq c_1 \left\| t'_2(j_2 f) \right\| \cdots \left\| t'_n(j_n f) \right\| = c_1 \left\| t'_2(j_2 f) \right\| \cdots \left\| t'_n(j_n f) \right\|.
\]
To prove (24), it suffices to show that there are positive constants \( c_k = c_{k,m} \) such that
\[
\|t'_k(j_k f)\| \leq c_k (\rho_k(j_k f))^{k/2m}, \quad 2 \leq k \leq n.
\]
The \( \mathbb{C}^* \)-action of \( k \)-jets says that
\[
t'_k(\lambda j_k f) = \lambda^k t'_k(j_k f), \quad \lambda \in \mathbb{C}^*.
\]
Set
\[
\mathcal{H}_k(\xi) = \frac{\|t'_k(\xi)\|}{(\rho_k(\xi))^{k/2m}}, \quad \xi \in J^k X(-\log D).
\]
Clearly, \( \mathcal{H}_k(\lambda \xi) = \mathcal{H}_k(\xi) \) for \( \lambda \neq 0 \) due to (21) and (26). By observing (13) and (20), we note that the singularities of \( \|t'_k\| \) are not worse than those of \( \rho_k^{k/2m} \), since that the singularity of \( \|t'_k\| \) occurs only along \( D \) and the order of singularities of \( \|t'_k\| \) is not greater than that of singularities of \( \rho_k^{k/2m} \). Thus, \( \mathcal{H}_k : \mathbb{P}(J^k X(-\log D)) \to [0, \infty) \) is a continuous function, where \( \mathbb{P}(J^k X(-\log D)) = (J^k X(-\log D) \setminus \{0\}) / \mathbb{C}^* \). By the compactness of \( \mathbb{P}(J^k X(-\log D)) \), \( \mathcal{H}_k \) is bounded from above by some positive constant \( c_k \). Namely, the claim (25) is showed. By the arbitrariness of \( k \), we confirm (24). Combining (23) with (24), we prove the claim (22).

Since \( R_1, \cdots, R_p \) are rational functions on \( X \), then
\[
\log T_{R_i \circ f}(r) \leq O(\log T_f(r, L)), \quad 1 \leq i \leq p
\]
hold outside a set of finite Lebesgue measure in \( \mathbb{R}_+ \), and
\[
\left( \frac{d^j R_i}{R_i} \right)^{m/j}(j_k f) = \left( \frac{R_i \circ f}{R_i \circ f} \right)^{m/j}(j_k f), \quad 1 \leq i \leq p, \quad 1 \leq j \leq k
\]
are meromorphic functions on \( \mathbb{C} \). Applying the classical lemma of logarithmic derivative (see [15], Lemma 2.50), we conclude that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{R_i \circ f}{R_i \circ f} \right) (re^{i\theta}) d\theta \leq O\left( \log^+ T_f(r, L) \right)
\]

holds outside a set of finite Lebesgue measure in \( \mathbb{R}_+ \). Therefore, the estimate (14) holds. By Lemma 3.1, we show that for any \( \delta > 0 \),

\[
T_f(r, L) + T_f(r, K_X) \leq N_f(r, D) - N(r, (F_\varphi)) + O\left( \log^+ T_f(r, L) + \delta \log r \right)
\]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure. Q.E.D.

**Remark 3.3.** By observing the proof of Theorem 1.6, we shall notice that the condition of the algebraical non-degeneracy of \( f \) is not necessary. In fact, this condition can be replaced by (15), which is weaker than the assumption of the algebraical non-degeneracy of \( f \).

Therefore, we obtain

**Theorem 3.4.** Let \( X \) be a smooth complex projective variety. Let \( D \in |L| \) be a simple normal crossing divisor on \( X \), where \( L \) is a positive line bundle over \( X \). Let \( f : \mathbb{C} \to X \) be a holomorphic curve such that \( f(\mathbb{C}) \not\subseteq \text{supp}(D) \) and

\[
f' \wedge t_2(j_2 f) \wedge \cdots \wedge t_n(j_n f) \neq 0,
\]

where \( t_k \) is given by Definition 2.3. Then for any \( \delta > 0 \),

\[
T_f(r, L) + T_f(r, K_X) \leq N_f(r, D) - N(r, (F_\varphi)) + O\left( \log^+ T_f(r, L) + \delta \log r \right)
\]

holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure, where \( F_\varphi \) is the Jacobian section of \( f \) defined by the holomorphic field

\[
\varphi := (t_2(j_2 f))_f \wedge \cdots \wedge (t_n(j_n f))_f.
\]

Note that Conjecture 1.1 is an immediate consequence of Theorem 1.6.

From (16), we see that

\[
t'_k(j_k f) = \sum_{j=1}^{n} f_j^{(k)} \frac{\partial}{\partial w_j}, \quad 1 \leq j \leq n.
\]

Set

\[
\mathcal{P} = f' \wedge t_2'(j_2 f) \wedge \cdots \wedge t_n'(j_n f).
\]

Then

\[
\mathcal{P}_f = \mathcal{W}_f \left( \frac{\partial}{\partial w_1} \right)_f \wedge \cdots \wedge \left( \frac{\partial}{\partial w_n} \right)_f f',
\]

where \( \mathcal{W}_f \) is defined by (17). Since \( f^*dw_j = f'_j dz \), where \( z \) is the holomorphic coordinate of \( \mathbb{C} \), then we combine (7) and (9) with (10) to get

\[
(F_\varphi) = (\mathcal{P}_f) = (\mathcal{W}_f).
\]
Observing the expression of \( W_f \), it is easy to see that there exists an integer \( \omega \) satisfying \( \omega \leq n(n+1)/2 \), such that
\[
N_f(r, D) - N(r, (F_\varphi)) \leq N[\omega](r, D).
\]

Therefore, it deduces that

**Corollary 3.5 (Strong Griffiths conjecture).** Let \( X \) be a smooth complex projective variety. Let \( D \in |L| \) be a simple normal crossing divisor on \( X \), where \( L \) is a positive line bundle over \( X \). Let \( f : \mathbb{C} \to X \) be an algebraically non-degenerate holomorphic curve. Then for any \( \delta > 0 \), there is an integer \( \omega \) with an upper bound \( n(n+1)/2 \), such that
\[
T_f(r, L) + T_f(r, K_X) \leq N_f^{[\omega]}(r, D) + O(\log^+ T_f(r, L) + \delta \log r)
\]
holds outside a set \( E_\delta \subseteq (0, \infty) \) of finite Lebesgue measure.

### 3.2.2. Proof of Corollary 3.7

Set \( f_j = \zeta_j \circ f \) \((0 \leq j \leq n)\), where \( \zeta_0, \ldots, \zeta_n \) are homogeneous coordinates of \( \mathbb{P}_n(\mathbb{C}) \). Assume without loss of generality that \( f_0 \neq 0 \). Set
\[
\tilde{f}_j = \frac{f_j}{f_0}, \quad 1 \leq j \leq n.
\]

By (17), we get
\[
W_f = \begin{vmatrix}
\tilde{f}_1' & \tilde{f}_2' & \cdots & \tilde{f}_n' \\
\tilde{f}_1'' & \tilde{f}_2'' & \cdots & \tilde{f}_n'' \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{f}_1^{(n)} & \tilde{f}_2^{(n)} & \cdots & \tilde{f}_n^{(n)}
\end{vmatrix}
\]
in local holomorphic coordinates \( \zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0 \). It is not difficult to certify that \( (W_f) \cap \{ f_0 \neq 0 \} = (W_f) \cap \{ f_0 \neq 0 \} \) due to
\[
W_f = \begin{vmatrix}
f_0 & f_1 & \cdots & f_n \\
f_0' & f_1' & \cdots & f_n' \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{f}_n^{(n)} & \tilde{f}_1^{(n)} & \cdots & \tilde{f}_n^{(n)}
\end{vmatrix} = f_0^{n+1} \begin{vmatrix}
1 & \tilde{f}_1 & \cdots & \tilde{f}_n \\
0 & \tilde{f}_1' & \cdots & \tilde{f}_n' \\
\vdots & \vdots & \ddots & \vdots \\
0 & \tilde{f}_n' & \cdots & \tilde{f}_n' 
\end{vmatrix} = f_0^{n+1} W_f
\]
(see [22], Page 124). This implies that \( (F_\varphi) = (W_f) \). By \( K_{\mathbb{P}_n(\mathbb{C})} = -(n+1)H \) and Theorem 1.6, then we prove the corollary. Q.E.D.

**Corollary 1.8** is a simple consequence of Corollary 3.5 and Corollary 3.7

**Corollary 3.6 (Weak Green-Griffiths conjecture).** Let \( X \) be a smooth complex projective variety with ample \( K_X \). Then there are no algebraically non-degenerate holomorphic curves in \( X \).
3.2.3. Proof of Conjecture 1.4

Since $X$ is projective algebraic, then there exists a very ample line bundle $L$ over $X$. Note from Lemma 3.2 that there is also a non-trivial holomorphic section $s \in H^0(X, L^* \otimes K_X^d)$ for an integer $d$ big enough. Equip a Hermitian metric on $L^* \otimes K_X^d$ such that $\|s\| < 1$. Suppose the holomorphic curve $f$ in $X$ is algebraically non-degenerate, then the First Main Theorem shows that

$$N_f(r, (s)) \leq T_f(r, L^* \otimes K_X^d) + O(1) = dT_f(r, K_X) - T_f(r, L) + O(1),$$

which yields that

$$T_f(r, L) \leq dT_f(r, K_X) + O(1).$$

By Corollary 1.8 we conclude that

$$\delta_f(D) \leq \left[ \frac{c_1(K_X^d)}{c_1(L)} \right] \leq -\frac{1}{d} < 0.$$

But, it is a contradiction with $\delta_f(D) \geq 0$. Thus, the curve $f$ is algebraically degenerate in $X$. Q.E.D.

Corollary 1.9 can be showed similarly by Lemma 3.2 and Corollary 1.8

3.2.4. Proof of Conjecture 1.5

Suppose $f(\mathbb{C})$ is not a point for a holomorphic curve $f : \mathbb{C} \to X$, then $f(\mathbb{C})$ is a subvariety of general type in $X$. By Conjecture 1.4 $f(\mathbb{C})$ is algebraically degenerate in itself, but it is a contradiction. Hence, $X$ is Brody hyperbolic. Q.E.D.

3.2.5. Proof of Corollary 1.10

If $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}) \setminus D$ is a linearly non-degenerate holomorphic curve, then (15) holds. Using Theorem 3.4 (or Corollary 1.7 and Remark 3.3), it follows that (if necessary, a proper modification of $X$ will be used)

$$(d - n - 1)T_f(r, H) \leq O\left( \log^+ T_f(r, H) + \delta \log r \right),$$

which contradicts with $d \geq n + 2$. This proves the corollary. Q.E.D.

3.2.6. Proof of Corollary 1.11

A theorem of Lang (see [17], Page 196) asserts that a smooth hypersurface $V \subseteq \mathbb{P}^n(\mathbb{C})$ of degree at least $n + 2$ has ample canonical class. By Conjecture 1.4 proved, the corollary follows immediately. Q.E.D.
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