Strict decomposition of diffusions associated to degenerate (sub)-elliptic forms

Jiyong Shin

Abstract. For given strongly local Dirichlet forms with possibly degenerate symmetric (sub)-elliptic matrix, we show the existence of weak solutions to the stochastic differential equations (associated with the Dirichlet forms) starting from all points in \( \mathbb{R}^d \). More precisely, using heat kernel estimates, stochastic calculus, and Dirichlet form theory, we obtain the pointwise existence of weak solutions to the stochastic differential equations which have possibly unbounded and discontinuous drift. We also present some conditions that the weak solutions become pathwise unique strong solutions and provide a new non-explosion criterion.

2010 Mathematics Subject Classification. Primary 31C25, 35J70, 47D07; Secondary 31C15, 60J35, 60J60.

Key words: Subelliptic operators, intrinsic metric, strong existence, Fukushima decomposition, degenerate forms.

1 Introduction

In this paper, we are concerned with a symmetric Dirichlet form (given as the closure of)

\[
\mathcal{E}^D(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} \langle D \nabla f, \nabla g \rangle \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^d)
\]

on \( L^2(\mathbb{R}^d, dx) \) and the corresponding stochastic differential equation (hereafter SDE)

\[
X_t = x + \int_0^t \sigma(X_s) \, dW_s + \int_0^t b(X_s) \, ds, \quad x \in \mathbb{R}^d,
\]

where the conditions on the (possibly) degenerate diffusion matrix \( D = (d_{ij})_{1 \leq i, j \leq d} \) are formulated in (A1) and (A2) in Section 2 and in (A3), (A4), and (A5)' in Section 3 (for \( \sigma, b \) see Theorem 2.11 and Theorem 3.12).

Given the bilinear form (1), it is well known from Dirichlet form theory (Fukushima decomposition) and localization method that one may derive a weak solution to the SDE (2) for any starting point \( x \in \mathbb{R}^d \setminus N \), where \( N \) is some capacity zero set w.r.t. \( \mathcal{E}^D \) (see [5]). For the Dirichlet form \( \mathcal{E}^D \) with uniformly elliptic matrix \( D \), it is shown in [4, Example] that the weak solution to the corresponding SDE (2) exists for all starting points in \( \mathbb{R}^d \). However for the Dirichlet form \( \mathcal{E}^D \) with the possibly degenerate matrix \( D \) there is in general no characterization of \( D \) which allows to give rise to a weak solution to the corresponding SDE (2) for explicitly specified starting points in \( \mathbb{R}^d \).

In this point of view, the main aim of this article is to construct a Hunt process associated to \( \mathcal{E}^D \) (degenerate (sub)-elliptic form) which satisfies the Fukushima’s absolute continuity condition (cf. [5] (4.2.9) and Theorem 5.5.5) and in the sequel to identify it to the solution of the associated SDE (2) for any starting point \( x \in \mathbb{R}^d \). The identification of the process to the SDE (2) and the explicit specification of the capacity zero set are of central interest in Dirichlet form...
theory. Following the tools and techniques developed in [13] and [14], we construct a Hunt process satisfying the absolute continuity condition and identify it to the solution of the SDE in pointwise under some additional assumptions, namely (A1), (A2) in Section 2 and (A3), (A4), and (A5) in Section 3. In [13] and [14], the (strong) equivalence between the intrinsic metric (derived from the Dirichlet form $E$ there) and the Euclidean metric plays a crucial role throughout the articles. In this paper we show that the local equivalence between the intrinsic metric (derived from $E'_{\mu}$) and the Euclidean metric is enough to obtain similar results (see (4), (5) and Lemma 3.2). Therefore this paper is basically a continuation of [14]. To our knowledge, however, it is first time to consider the essential degenerate matrix $D$ in bilinear form (1) and identify the SDE for any starting points in $\mathbb{R}^d$ (see Section 4).

The contents of this paper are organized as follows. In Section 2, we consider a symmetric diffusion matrix $A = (a_{ij})_{i,j \in \mathbb{N}_0}$ satisfying the subelliptic estimate. We first present analytic background based on the results from [1, 3, 6, 7, 12, 15, 16]. Then using local equivalence between the intrinsic metric and the Euclidean metric, we show that the Dirichlet form $(E^A, D(E^A))$ is conservative and even recurrent in the case of $d = 2$ (see Theorem 2.4). In order to construct the Hunt process satisfying the absolute continuity condition we apply the Dirichlet form method developed in [13] and finally identify it to the solution of the SDE (2). In Section 3, we consider a different degenerate (locally uniformly) elliptic matrix $B$ and do the same as in Section 2. In this case, however, unlike Section 2, we can show that the associated semigroup is Feller in classical sense. Section 4 is devoted to pathwise uniqueness and strong solution. We also provide a new non-explosion criterion.

Notations:
For an open set $E \subset \mathbb{R}^d$, $d \geq 2$ with Borel $\sigma$-algebra $\mathcal{B}(E)$ we denote the set of all $\mathcal{B}(E)$-measurable $f : E \to \mathbb{R}$ which are bounded by $\mathcal{B}_b(E)$. The usual $L^q$-spaces $L^q(E, \mu), q \in [1, \infty]$ are equipped with $L^q$-norm $\| \cdot \|_{L^q(E, \mu)}$ with respect to the measure $\mu$ on $E$, $\mathcal{A}_b := \mathcal{A} \cap \mathcal{B}_b(E)$ for $\mathcal{A} \subset L^1(E, \mu)$, and $L^q_{loc}(E, \mu) := \{ f \mid f \cdot 1_U \in L^q(E, \mu), \forall U \subset E, U \text{ relatively compact open} \}$, where $1_{A}$ denotes the indicator function of a set $A$. If $\mathcal{A}$ is a set of functions $f : E \to \mathbb{R}$, we define $\mathcal{A}_b := \{ f \in \mathcal{A} \mid \text{supp}(f) : = \text{supp}(|f|\mu) \text{ is compact in } E \}$. Let $\nabla f := (\partial_1 f, \ldots, \partial_d f)$ where $\partial_j f$ is the $j$-th weak partial derivative of $f$ and $\partial_{ij} f := \partial_i (\partial_j f), i, j = 1, \ldots, d$. We denote the set of continuous functions on $E$, the set of continuous bounded functions on $E$, the set of compactly supported continuous functions in $E$ by $C(E), C_0(E), C_0^d(E)$, respectively. The space of continuous functions on $E$ which vanish at infinity is denoted by $C_0(E)$. The set of all infinitely differentiable functions on $E$, the set of all infinitely differentiable functions with compact support in $E$, and the set of all infinitely differentiable bounded functions on $E$ are denoted by $C^{\infty}(E), C_0^{\infty}(E)$, and $C_0^d(E)$, respectively. As usual we denote the Lebesgue measure on $\mathbb{R}^d$ by $dx$ and equip $\mathbb{R}^d$ with the Euclidean norm $\| \cdot \|$ and the corresponding inner product $(\cdot, \cdot)$. We write $B_r(x) := \{ y \in \mathbb{R}^d \mid \| x-y \| < r \}, x \in \mathbb{R}^d, r > 0$. For $A \subset \mathbb{R}^d$, the closure of $A$ in $\mathbb{R}^d$ is denoted by $\overline{A}$. 


2 Preliminaries and degenerate subelliptic forms with Lebesgue measure

Throughout this paper, we consider a symmetric matrix $D = (d_{ij})_{1 \leq i, j \leq d}$, $d_{ij} \in L^1_\text{loc}(\mathbb{R}^d, dx)$ and a symmetric bilinear form

$$E^D(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} \langle D \nabla f, \nabla g \rangle \, dx, \quad f, g \in C^0_\text{loc}(\mathbb{R}^d).$$

For the time being, suppose to define some notations that the symmetric bilinear form $(E^D, C^0_\text{loc}(\mathbb{R}^d))$ is closable in $L^2(\mathbb{R}^d, dx)$ and its closure $(E^D, D(E^D))$ is a strongly local, regular, symmetric Dirichlet form (cf. [5]). The Dirichlet form $(E^D, D(E^D))$ can be written as

$$E^D(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} d\Pi^D(f, g), \quad f, g \in D(E^D),$$

where $\Gamma^D$ is a symmetric bilinear form on $D(E^D) \times D(E^D)$ with values in the signed Radon measures on $\mathbb{R}^d$ (called energy measures). The nonnegative definite measure $\Gamma^D(f, f)$ can be defined by the formula

$$\int_{\mathbb{R}^d} \phi \, d\Pi^D(f, f) = 2E^D(f, \phi f) - E^D(f^2, \phi),$$

for every $f \in D(E^D) \cap L^\infty(\mathbb{R}^d, dx)$ and every $\phi \in D(E^D) \cap C_0(\mathbb{R}^d)$. Let $D(E^D)_{\text{loc}}$ be the set of all measurable functions $f$ on $\mathbb{R}^d$ for which on every compact set $K \subset \mathbb{R}^d$ there exists a function $g \in D(E^D)$ with $f = g \, dx$-a.e on $K$. By an approximation argument we can extend the quadratic form $f \mapsto \Gamma^D(f, f)$ to $D(E^D)_{\text{loc}} = \{ f \in L^2_{\text{loc}}(\mathbb{R}^d, dx) \mid \Gamma^D(f, f) \text{ is a Radon measure} \}$. By polarization we then obtain for $f, g \in D(E^D)_{\text{loc}}$ a signed Radon measure

$$\Gamma^D(f, g) = \frac{1}{4} \{ \Gamma^D(f + g, f + g) - \Gamma^D(f - g, f - g) \}.$$

For these properties of energy measures we refer to [5], [9 Proposition 1.4.1], and [11] (cf. [15], Appendix). The energy measure $\Gamma^D$ defines in an intrinsic way a pseudo metric $d$ on $\mathbb{R}^d$ by

$$d(x, y) = \sup \{ f(x) - f(y) \mid f \in D(E^D)_{\text{loc}} \cap C(\mathbb{R}^d), \, \Gamma^D(f, f) \leq d\zeta \text{ on } \mathbb{R}^d \},$$

where $\Gamma^D(f, f) \leq d\zeta$ means that the energy measure $\Gamma^D(f, f)$ is absolutely continuous w.r.t. the reference measure $d\zeta$ with Radon-Nikodym derivative $d\frac{\Gamma^D(f, f)}{d\zeta} \leq 1$ (cf. [11]). We define the balls w.r.t. the intrinsic metric by

$$\tilde{B}_r(x) = \{ y \in \mathbb{R}^d \mid d(x, y) < r \}, \quad x \in \mathbb{R}^d, \quad r > 0.$$

**Definition 2.1.**

(i) We say the completeness property holds, if for all balls $\tilde{B}_r(x)$, $x \in \mathbb{R}^d$, $r > 0$, the closed balls $\overline{\tilde{B}}_r(x)$ are complete (or equivalently, compact).

(ii) We say the doubling property holds for a given measure $\mu$, if there exists a constant $N = N(d)$ such that for all balls $\tilde{B}_r(x) \subset \mathbb{R}^d$

$$\mu(\tilde{B}_r(x)) \leq 2^N \mu(\tilde{B}_{2r}(x)).$$
(iii) We say the (scaled) weak Poincaré inequality holds, if there exists a constant \( c = c(d) \) such that for all balls \( B_2(x) \subset \mathbb{R}^d \)

\[
\int_{\mathbb{R}^d} |f - \tilde{f}_x|^2 \, dy \leq c \int \mathcal{R}(f, f), \quad f \in D(\mathcal{R}),
\]

where \( \tilde{f}_x = \frac{1}{|B_2(x)|} \int_{B_2(x)} f \, dy. \)

(iv) A strongly local, symmetric Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) is called strongly regular if it is regular and if \( d(\cdot, \cdot) \) (defined by \( 1 \)) is a metric on \( \mathbb{R}^d \) whose topology coincides with the original one.

A positive Radon measure \( \mu \) on \( \mathbb{R}^d \) is said to be of finite energy integral if

\[
\int_{\mathbb{R}^d} |f(x)| \mu(dx) \leq c \sqrt{\mathcal{E}^\mu(f, f)} = c \int \mathcal{R}(f, f), \quad f \in D(\mathcal{R}) \cap C_0(\mathbb{R}^d),
\]

where \( c \) is some constant independent of \( f \) and \( \mathcal{E}^\mu(f, f) := \mathcal{E}(f, f) + \int_{\mathbb{R}^d} |f|^2 \, dx. \) A positive Radon measure \( \mu \) on \( \mathbb{R}^d \) is of finite energy integral, if and only if there exists a unique function \( U_1 \mu \in D(\mathcal{E}) \) such that

\[
\mathcal{E}(U_1 \mu, f) = \int_{\mathbb{R}^d} f(x) \mu(dx),
\]

for all \( f \in D(\mathcal{E}) \cap C_0(\mathbb{R}^d) \). \( U_1 \mu \) is called the 1-potential of \( \mu \). The measures of finite energy integral are denoted by \( S_0 \). We further define \( S_{\infty} := \{ \mu \in S_0 \mid \mu(\mathbb{R}^d) < \infty, \| U_1 \mu \|_{L^\infty(\mathbb{R}^d)} < \infty \}. \)

In this section, we consider the following assumption:

(A1) \( A = (a_{ij})_{1 \leq i, j \leq d} \) is a symmetric matrix such that

\[
a_{ij} \in C^0_b(\mathbb{R}^d), \quad i, j = 1, \ldots, d,
\]

and \( A \) satisfies the degenerate elliptic condition (positive semidefinite), i.e. for \( dx \)-a.e. \( x \in \mathbb{R}^d \)

\[
0 \leq \langle A(x) \xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^d.
\]

From now on we fix a symmetric matrix \( A = (a_{ij})_{1 \leq i, j \leq d} \) satisfying (A1) and consider the symmetric bilinear form

\[
\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} \langle A \nabla f, \nabla g \rangle \, dx, \quad f, g \in C^0_b(\mathbb{R}^d).
\]

Furthermore we assume:

(A2) The symmetric matrix \( A \) satisfies the following subelliptic estimate, i.e. there exist constants \( \varepsilon > 0 \) and \( \delta > 0 \) such that

\[
\delta \| u \|^2_{H^\varepsilon} \leq \mathcal{E}(u, u) + \| u \|^2_{L^2(\mathbb{R}^d \setminus 0)}, \quad \forall u \in C^0_b(\mathbb{R}^d).
\]

Here \( \| u \|^2_{H^\varepsilon} := \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \cdot (1 + \| \xi \|^2)^\varepsilon \, d\xi \) for any \( \varepsilon > 0 \) and \( \hat{u} \) is the Fourier transform of \( u \) and \( H^\varepsilon(\mathbb{R}^d) := \{ u \in L^2(\mathbb{R}^d, dx) \mid \| u \|_{H^\varepsilon} < \infty \} \) is the fractional Sobolev space of order \( \varepsilon > 0 \).

Remark 2.2. We refer to [7] for some operators satisfying the subelliptic estimate (A2) (more precisely, see [7] (1.3), Theorem 2.1, 2.2)).
By \[5\] Section 3.1 (1')) \((\mathcal{E}^A, C^0_0(\mathbb{R}^d))\) is then closable in \(L^2(\mathbb{R}^d, dx)\) and its closure \((\mathcal{E}^A, D(\mathcal{E}^A))\) is a strongly local, regular, symmetric Dirichlet form. Furthermore, it is known that the intrinsic metric \(d(\cdot, \cdot)\) derived from the Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) satisfies (see [16] Theorem 4.2)

\[
d(x, y) \geq c_0^{-1}||x - y||, \quad \forall x, y \in \mathbb{R}^d,
\]

where \(c_0 \geq 1\) is some constant and there exist \(r_0 > 0, C_0 > 0\) such that

\[
d(x, y) \leq C_0||x - y||, \quad \forall x, y \in \mathbb{R}^d \text{ with } ||x - y|| < r_0,
\]

where \(c \in (0, 1)\) is the constant as in (A2) (see [1, Section 1. (b)] and [6]). Hence the Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) is strongly regular. In particular \(\widehat{B}_s(x) \in \mathcal{B}(\mathbb{R}^d)\) for all \(x \in \mathbb{R}^d, r > 0\).

**Remark 2.3.** The topology induced by the intrinsic metric coincides with the Euclidean topology (see [11] Section 1. (b) and [12]). Hence the Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) is strongly regular. In particular \(\widehat{B}_s(x) \in \mathcal{B}(\mathbb{R}^d)\) for all \(x \in \mathbb{R}^d, r > 0\).

**Theorem 2.4.**

(i) Let \(d = 2\). Then the Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) is strongly regular.

(ii) The Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) is conservative.

**Proof.**

(i) Let \(d = 2\). Then by [3]

\[
\int_1^\infty \frac{r}{dx(\widehat{B}(0))} dr \geq \int_1^\infty \frac{r}{dx(\widehat{B}_{r_0}(0))} dr = \infty,
\]

where \(c_0\) is the constant as in [3]. Therefore by [16] Theorem 3.4, \((\mathcal{E}^A, D(\mathcal{E}^A))\) is recurrent.

(ii) Similarly, using [16] one can show that for any \(d \geq 2\)

\[
\int_1^\infty \frac{r}{\log (dx(\widehat{B}(0)))]} dr \geq \int_1^\infty \frac{r}{\log (dx(\widehat{B}_{r_0}(0))]} dr = \infty.
\]

Hence by [16] Theorem 3.6, \((\mathcal{E}^A, D(\mathcal{E}^A))\) is conservative. \(\square\)

By [4], the completeness property holds and the doubling property holds since the reference measure is the Lebesgue measure. The weak Poincaré inequalities on intrinsic balls is also satisfied (see [1, Section 1. (b)], [6], and [7]). Hence the properties (Ia)-(Ic) of [15] are satisfied. Therefore by [15] p. 286 A) there exists a jointly continuous transition kernel density \(p_t(x, y)\) such that

\[
P_t f(x) := \int_{\mathbb{R}^d} p_t(x, y) f(y) dy, \quad t > 0, x, y \in \mathbb{R}^d, f \in \mathcal{B}(\mathbb{R}^d)
\]

is a \(dx\)-version of \(T_t f\) if \(f \in L^2(\mathbb{R}^d, dx)_0\). Throughout this paper we set \(P_0 := id\). Taking the Laplace transform of \(p_t(x, y)\), we obtain a \(\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)\) measurable non-negative resolvent kernel density \(r_\alpha(x, y)\) such that

\[
R_\alpha f(x) := \int_{\mathbb{R}^d} r_\alpha(x, y) f(y) dy, \quad \alpha > 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d),
\]

is a \(dx\)-version of \(G_\alpha f\) if \(f \in L^2(\mathbb{R}^d, dx)_0\). For a signed Radon measure \(\mu\) on \(\mathbb{R}^d\), let us define

\[
R_\alpha \mu(x) = \int_{\mathbb{R}^d} r_\alpha(x, y) \mu(dy), \quad \alpha > 0, x \in \mathbb{R}^d.
\]
whenever this makes sense. In particular, \( R_t \mu \) is a version of \( U_t \mu \) (see e.g. Example 4.2.2]).

It follows from [15 Corollary 4.2 and Remarks (ii) in p.286] that for \( x, y \in \mathbb{R}^d, \ t > 0, \ any \ \delta > 0 \)

\[
p_t(x,y) \leq c \ dz(\tilde{B}_\sigma(x))^{-1/2} dz(\tilde{B}_\sigma(y))^{-1/2} \exp \left( -\frac{d(x,y)^2}{4(\delta + \mu)} \right), \tag{6}
\]

where \( c \) is some constant.

**Lemma 2.5.** Let \( r > 0 \) and \( t > 0 \). Then

\[
\sup_{x \in B_r(0)} p_t(x, \cdot) \in L^1(\mathbb{R}^d, dz).
\]

**Proof.** Let \( x, y \in \mathbb{R}^d \) and \( t, r > 0 \). Note that by [5, \( \inf_{x \in B_r(0)} dz(\tilde{B}_\sigma(x)) =: M_{x,r} > 0 \). Putting [4] into [5] we obtain for \( x, y \in \mathbb{R}^d, \ t > 0, \ any \ \delta > 0 \)

\[
p_t(x,y) \leq c \ dz(\tilde{B}_\sigma(x))^{-1/2} dz(\tilde{B}_\sigma(y))^{-1/2} \exp \left( -\frac{\|x-y\|^2}{c_0(\delta + \mu)} \right), \tag{7}
\]

Using the doubling property, [7] can be rewritten as

\[
p_t(x,y) \leq c_1 \frac{1}{dz(\tilde{B}_\sigma(x))} \exp \left( -\frac{\|x-y\|^2}{c_0(\delta + \mu)} \right),
\]

where \( c_1 \) is some constant (cf. [13 proof of Lemma 3.2] and [15 p. 287]). Therefore

\[
\sup_{x \in B_r(0)} p_t(x, \cdot) \in L^1(\mathbb{R}^d, dz).
\]

\( \Box \)

Using Lemma 2.5 we show that \((P_t)_{t \geq 0}\) is strong Feller:

**Proposition 2.6.** \((P_t)_{t \geq 0}\) (resp. \((R_\alpha)_{\alpha > 0}\)) is strong Feller, i.e. for \( t > 0, \ P_t(B_\delta(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d) \) (resp. for \( \alpha > 0, \ R_\alpha(B_\delta(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)\))

**Proof.** Let \( x_n \to x \) in \( \mathbb{R}^d \) as \( n \to \infty \). For \( f \in B_\delta(\mathbb{R}^d) \) and \( t > 0 \)

\[
|P_t f(x_n) - P_t f(x)| \leq \int_{\mathbb{R}^d} |p_t(x_n, y) - p_t(x, y)| |f(y)| \ dy
\]

which converges to 0 by Lebesgue and Lemma 2.5 and the continuity of \( p_t(\cdot, y) \). Note that clearly for \( f \in B_\delta(\mathbb{R}^d), \ t > 0, \ P_t f \) is bounded. Therefore \((P_t)_{t \geq 0}\) is strong Feller. Since \( R_\alpha f(x) = \int_0^\infty e^{-t} P_t f(x) \ dt \) and \( \|P_t f\|_{L^\infty(\mathbb{R}^d; dx)} \leq \|f\|_{L^\infty(\mathbb{R}^d; dx)} \) for any \( f \in \mathbb{B}_\delta(\mathbb{R}^d) \), \((R_\alpha)_{\alpha > 0}\) is clearly also strong Feller by Lebesgue.

\( \Box \)

**Remark 2.7.** We do not know whether the transition function \((P_t)_{t \geq 0}\) is a Feller semigroup or not.
Now we adopt the construction method of a Hunt process associated with a given Dirichlet form as introduced in [13] Section 2]. In [13], Section 2] we considered a symmetric, strongly local, regular Dirichlet form \((\mathcal{E}, D(\mathcal{E}))\) on \(L^2(E, \mu)\) with generator \((L, D(L))\) admitting carré du champ, where \(E\) is a locally compact separable metric space and \(\mu\) is a positive Radon measure on \((E, \mathcal{B}(E))\) with full support on \(E\).

There, with the corresponding semigroup \((T_t)_{t\geq 0}\), the transition function \((P_t)_{t\geq 0}\), the resolvent kernel \(R_t\) w.r.t. \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\), and so on, we assumed:

\begin{itemize}
  \item [(H1)] There exists a \(\mathcal{B}(E) \times \mathcal{B}(E)\) measurable non-negative map \(p_t(x, y)\) such that
  \[
  p_t(x, y) = \int_E p_t(x, y) f(y) \mu(dy), \quad t > 0, \quad x \in E, \quad f \in \mathcal{B}_b(E),
  \]
  is a (temporally homogeneous) sub-Markovian transition function (see [13, Lemma 2.9]) and an \(\mu\)-version of \(T_t f\) if \(f \in L^2(E, \mu)_b\).
  \item [(H2')] We can find \(u_n \mid n \geq 1\) \(\in \mathcal{D}(L) \cap C_0(E)\) satisfying:
    \begin{enumerate}
      \item [(i)] For all \(x \in E \cap (0, 1)\) and \(y \in D,\) where \(D\) is any given countable dense set in \(E,\)
          there exists \(n \in \mathbb{N}\) such that \(u_n(z) \geq 1\), for all \(z \in \overline{B}_E(y)\) and \(u_n \equiv 0\) on \(E \setminus B_{\frac{1}{2}}(y)\).
      \item [(ii)] \(R_1 \left(\{1 - L u_n\}^+\right), R_1 \left(\{1 - L u_n\}^-\right), R_1 \left(\{1 - L_1 u_{n_1}^0\}^+\right), R_1 \left(\{1 - L_1 u_{n_1}^0\}^-\right)\) are continuous on \(E\) for all \(n \geq 1\) where \(L_1\) denotes the \(L^1(E, \mu)-\)generator of \((\mathcal{E}, \mathcal{D}(\mathcal{E})).\)
      \item [(iii)] \(R_1 C_0(E) \subset C(E)\).
      \item [(iv)] For any \(f \in C_0(E)\) and \(x \in E,\)
          the map \(t \mapsto P_t f(x)\) is right-continuous on \((0, \infty)\).
    \end{enumerate}
\end{itemize}

Under \((H1)\) and \((H2)\)' we showed that there exists a Hunt process with \((P_t)_{t\geq 0}\) as transition function (see [13, Lemma 2.9]).

We intend to do the same here in our concrete situation, i.e. we first show that \((\mathcal{E}^A, D(\mathcal{E}^A))\)

satisfies \((H1)\) and \((H2)'\) and so finally can construct a Hunt process

\[
\mathcal{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \xi, (X_t)_{t \geq 0}, (\mathbb{P}_t)_{t \in \mathbb{R}_+})
\]

satisfying the absolute continuity condition (as stated in [5, p. 165]) with the transition function \((P_t)_{t\geq 0}\).

Here \(\Delta\) is the cemetery point, \(\mathbb{R}^d_\Delta := \mathbb{R}^d \cup \{\Delta\}\)

and the lifetime \(\xi := \inf\{t \geq 0 \mid X_t \in \{\Delta\}\}\)

and \(P_t(x, B) := P_t 1_B(x) = \mathbb{P}_t(X_t \in B)\) for any \(x \in \mathbb{R}^d, B \in \mathcal{B} (\mathbb{R}^d), t \geq 0\).

Let \((L^A, D(L^A))\) be the generator of \((\mathcal{E}^A, D(\mathcal{E}^A)).\)

Since \(a_{ij}, \partial_i a_{ij} \in \mathcal{L}_{\infty}^\mu (\mathbb{R}^d, dx)\), we have for \(f \in C^0_b (\mathbb{R}^d)\)

\[
f \in D(L^A) \quad \text{and} \quad L^A f = \frac{1}{2} \sum_{i,j=1}^d \left( a_{ij} \partial_i f + \partial_a a_{ij} \partial_i f \right) \in L^\infty (\mathbb{R}^d, dx)_b.
\]

**Theorem 2.8.** There exists a Hunt process \(\mathcal{M}\) satisfying the absolute continuity condition with transition function \((P_t)_{t\geq 0}\).

**Proof.** Using the transition density estimate [7], we can see as in [13, Proposition 3.3 (ii)] that \((H1)\) and \((H2)'\) (iii), (iv) hold. Clearly we can find \((u_n)_{n \geq 1} \subset C^0_b (\mathbb{R}^d) \subset D(L^A)\) such that \((H2)'\) (i) is satisfied. Furthermore \((H2)'\) (ii) for \((u_n)_{n \geq 1}\) satisfying \((H2)'\) (i) follows from [8] and Proposition [2.6].
Remark 2.9. By Theorem 2.4 and Proposition 2.6 \( \mathbb{P}_x(\zeta = \infty) = 1 \) for all \( x \in \mathbb{R}^d \).

Lemma 2.10. Assume (A1) and (A2) hold. For any relatively compact open set \( G \subset \mathbb{R}^d \),

\[
1_G \cdot a_i \, dx \in S_{00}, \quad 1_G \cdot |\partial_j a_{ij}| \, dx \in S_{00}.
\]

Proof. For any relatively compact open set \( G \subset \mathbb{R}^d \) \( 1_G \cdot a_i \, dx \) and \( 1_G \cdot |\partial_j a_{ij}| \, dx \) are positive finite measures on \( \mathbb{R}^d \). Furthermore by (A1) and Proposition 2.6, \( R_1(1_G \cdot a_i \, dx) \in C_b(\mathbb{R}^d) \) and \( R_1(1_G \cdot |\partial_j a_{ij}| \, dx) \in C_b(\mathbb{R}^d) \). Consequently \( 1_G \cdot a_i \, dx \in S_{00}, \ 1_G \cdot |\partial_j a_{ij}| \, dx \in S_{00} \) (see [14, Proposition 2.12]).

\[ \square \]

We will refer to [5] till the end, hence some of its standard notations may be adopted below without definition. Let \( f^i(x) := x_i, \ i = 1, \ldots, d, \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), be the coordinate functions. Then \( f^i \in D(E^d)_{b,loc} \) and for any \( g \in C^\infty(\mathbb{R}^d) \) the following integration by parts formula holds:

\[
- \mathcal{E}^A(f^i, g) = \frac{1}{2} \int_{\mathbb{R}^d} (\sum_{j=1}^d \partial_j a_{ij}) g \, dx, \quad 1 \leq i \leq d.
\]

The proof of next theorem is basically similar to [14, Theorem 3.9]. But we add the proof for the convenience of readers.

Theorem 2.11. Assume (A1)-(A2) hold. Then it holds \( \mathbb{P}_x \)-a.s. for any \( x \in \mathbb{R}^d, \ i = 1, \ldots, d \)

\[
X_i^t = x_i + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_s) \, dW^j_s + \frac{1}{2} \sum_{j=1}^d \int_0^t \partial_j a_{ij}(X_s) \, ds, \quad t \geq 0,
\]

where \( \sqrt{A} = (\sigma_{ij})_{1 \leq i,j \leq d} \) is the square root of the matrix \( A \), \( W = (W^1, \ldots, W^d) \) is a standard \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \).

Proof. By Lemma 2.10 (2), and [5, Theorem 5.5.5], the strict continuous additive functional, locally of zero energy and corresponding to the coordinate function \( f^i \in D(E^d)_{b,loc} \), is given by

\[
N^t_i(f^i) = \frac{1}{2} \int_0^t \left( \sum_{j=1}^d \partial_j a_{ij} \right)(X_s) \, ds, \quad t \geq 0, \quad 1 \leq i \leq d.
\]

The energy measure of \( f^i \) denoted by \( \mu_i(f^i) \) satisfies \( \mu_i(f^i) = a_i \, dx \). By Lemma 2.10 for any relatively compact open set \( G \subset \mathbb{R}^d \), \( 1_G \cdot \mu_i(f^i) \) is \( S_{00} \) and so the positive continuous additive functional in the strict sense corresponding to the Reuvz measure \( \mu_i(f^i) \) is given by

\[
\langle M^i(f^i) \rangle_t = \int_0^t a_i(X_s) \, ds,
\]

where \( M^i(f^i) \) is the continuous local martingale additive functional in the strict sense corresponding to \( f^i \). Furthermore since the covariation is

\[
\langle M^i(f^i), M^j(f^j) \rangle_t = \int_0^t a_{ij}(X_s) \, ds,
\]
we can construct a \(d\)-dimensional Brownian motion \(W\) (on a possibly enlarged probability space \((\Omega, \mathcal{F}, \mathbb{P})\)), see \([8\) Chapter 3, Theorem 4.2]), that we call again w.l.o.g. \((\Omega, \mathcal{F}, \mathbb{P})\)) such that

\[
M_t^{(\mathcal{F})} = \sum_{j=1}^d \int_0^t \sigma_j(X_s) \, dW_s^j,
\]

where \((\sigma_j)_{1 \leq j \leq d} = \sqrt{A}\) is the square root of the matrix \(A\). Note that the equation \((10)\) holds for all \(t \geq 0\) because \((\mathcal{E}^\lambda, D(\mathcal{E}^\lambda))\) is conservative (see Remark 2.9). \(\Box\)

3 Degenerate elliptic forms with Lebesgue measure

In this section we consider the following assumption:

(A3) Let \(B := (b_{ij})_{1 \leq i,j \leq d}\) be an elliptic symmetric matrix on \(\mathbb{R}^d\), i.e. there exist \(\lambda_1, \lambda_2 \in C(\mathbb{R}^d)\) with \(0 < \lambda_1 \leq \lambda_2\) such that for \(dx\)-a.e. \(x \in \mathbb{R}^d\)

\[
\lambda_1(x) |\xi|^2 \leq (B(x)\xi, \xi) \leq \lambda_2(x) |\xi|^2, \quad \forall \xi \in \mathbb{R}^d.
\]

(11)

We say this matrix \(B\) is degenerate (or locally uniformly elliptic) since it can not be uniformly bounded away from zero in \((11)\). Now we fix a matrix \(B = (b_{ij})_{1 \leq i,j \leq d}\) satisfying (A3) and consider the symmetric bilinear form

\[
\mathcal{E}^B(f, g) := \frac{1}{2} \int_{\mathbb{R}^d} (B \nabla f, \nabla g) \, dx, \quad f, g \in C_0^\infty(\mathbb{R}^d).
\]

By \([10\) Chapter II. 2. b)] \((\mathcal{E}^B, C_0^\infty(\mathbb{R}^d))\) is closable in \(L^2(\mathbb{R}^d, dx)\) and its closure \((\mathcal{E}^B, D(\mathcal{E}^B))\) is a strongly local, regular, symmetric Dirichlet form. As before in Section 2 we denote the \(L^2(\mathbb{R}^d, dx)\)-semigroup and resolvent associated to \((\mathcal{E}^B, D(\mathcal{E}^B))\) by \((T_t)_{t \geq 0}\) and \((G_a)_{a \geq 0}\). Correspondingly, we can define the intrinsic metric \(d(\cdot, \cdot)\) and the intrinsic balls \(\tilde{B}_r(x), x \in \mathbb{R}^d, r > 0\) relevant to \((\mathcal{E}^B, D(\mathcal{E}^B))\) as introduced in Section 2. Furthermore, we assume the (scaled) weak Poincaré inequality:

(A4) There exists a constant \(c > 0\) such that

\[
\int_{\tilde{B}_r(x)} |u - \bar{u}_r|^2 \, dy \leq cr^2 \int_{\tilde{B}_2(x)} (B \nabla u, \nabla u) \, dy, \quad \forall u \in C_0^\infty(\mathbb{R}^d), x \in \mathbb{R}^d, r > 0,
\]

where \(\bar{u}_r = \frac{1}{\mu(\tilde{B}_r(x))} \int_{\tilde{B}_r(x)} u \, dy\).

Remark 3.1. Suppose that the symmetric matrix \(B = (b_{ij})_{1 \leq i,j \leq d}\) satisfies (A1) and (A2). Then this \(B\) satisfies (A4) (see Section 2).

Lemma 3.2. For any \(x, y \in \mathbb{R}^d\)

\[
d(x, y) \geq \frac{1}{\sqrt{c_2}} \|x - y\|, \quad c_2 := \sup_{x \in \mathbb{R}^d} \lambda_2(x), \quad (12)
\]

and for any bounded set \(D \in \mathcal{B}(\mathbb{R}^d)\)

\[
d(x, y) \leq \frac{1}{\sqrt{c_D}} \|x - y\|, \quad x, y \in D,
\]

(13)

where \(c_D := \inf_{x \in D} \lambda_1(x)\).
Proof. We basically follow the ideas in the proof of [16, Theorem 4.1]. For any $z \in \mathbb{R}^d$ the map $u : x \mapsto \langle x, z \rangle$ lies in $D(E^1) \cap C(\mathbb{R}^d)$. For fixed $y, y' \in \mathbb{R}^d, y \neq y'$, choose $z = \frac{(y - y')}{\sqrt{c_2 \|y - y\|}} \in \mathbb{R}^d, \ c_2 := \sup_{x \in \mathbb{R}^d} \lambda_2(x)$. Then by (11) $\int_A dt^\theta(u, u) = \int_A \langle B\nabla u, \nabla u \rangle dx \leq c_2 \int_A \|\nabla u\|^2 dx = \int_A dx, \ \forall A \in \mathcal{B}(\mathbb{R}^d)$. Hence $\Gamma^\theta(u, u) \leq dx$. Furthermore $u(y) - u(y') = \frac{1}{\sqrt{c_2}} \|y - y\|$. Therefore for any $x, y \in \mathbb{R}^d$ $d(x, y) \geq \frac{1}{\sqrt{c_2}} \|x - y\|$. Conversely, let $u \in D(E^1) \cap C(\mathbb{R}^d)$ with $\Gamma^\theta(u, u) \leq dx$. Let $(u_n)_{n \geq 1} \subset C_0^\infty(\mathbb{R}^d)$ be a sequence that converges to $u$ locally in $\mathcal{E}_0^1$-norm and locally uniformly. Then by (11), $(\partial_t u_n)_{n \geq 1}$ is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^d, dx)$. Therefore there exists $v \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ as $n \to \infty$. Let $D \in \mathcal{B}(\mathbb{R}^d)$ be a bounded set. Then by (11) $\lim_{n \to \infty} \int_D \langle B\nabla u_n, \nabla u_n \rangle dx = \int_D \langle Bv, v \rangle dx$. Then $\int_D dt^\theta(u, u) = \lim_{n \to \infty} \int_D dt^\theta(u_n, u_n) = \lim_{n \to \infty} \int_D \langle B\nabla u_n, \nabla u_n \rangle dx = \int_D \langle Bv, v \rangle dx,$ where $v = (v_1, \ldots, v_d)$. Together with (14) this implies that $\int_D \langle Bv, v \rangle dx \leq \int_D 1 \ dx.$ In particular, by (11) $c_D \|v\|^2 \leq 1$ $dx$-a.e. on $D$ where $c_D := \inf_{x \in D} A_1(x)$. Now following the proof of [16, Theorem 4.1, p.264] one can show that $|u(x) - u(y)| \leq \frac{1}{\sqrt{c_D}} \|x - y\|, \ \forall x, y \in D.$ \hfill $\square$
Remark 3.3. By Lemma 3.2 the topology induced by the intrinsic metric coincides with the Euclidean topology. Hence the Dirichlet form \((\mathcal{E}^\mu, D(\mathcal{E}^\mu))\) is strongly regular.

By (12), the completeness property holds. The doubling property holds since the reference measure is the Lebesgue measure. By the assumption (A4) the weak Poincaré inequality on intrinsic balls is also satisfied. Hence the properties (Ia)-(Ic) of [15] are satisfied. Therefore likewise Section 2 by [15, p. 286 A]) there exists a jointly continuous transition kernel density \(p_t(x, y)\) such that

\[
P_t f(x) := \int_{\mathbb{R}^d} p_t(x, y) f(y) \, dy, \quad t > 0, \quad x, y \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d)
\]

is a \(dx\)-version of \(T_t f\) if \(f \in L^2(\mathbb{R}^d, dx)\). Furthermore, it follows from [15] Corollary 4.2 and Remarks (ii) in p.286 that for \(x, y \in \mathbb{R}^d, t > 0, \) any \(\delta > 0\)

\[
p_t(x, y) \leq c \, dx(B_t(x))^{-1/2} dx(B_t(y))^{-1/2} \exp \left(-\frac{d(x,y)^2}{4 + \delta^2/4}\right).
\]

where \(c\) is some constant. Similarly, \(R_{\alpha} f\) and \(R_{\alpha} \mu\) can be defined as in Section 2.

Theorem 3.4. (i) Let \(d = 2\). Then the Dirichlet form \((\mathcal{E}^\mu, D(\mathcal{E}^\mu))\) is recurrent.

(ii) The Dirichlet form \((\mathcal{E}^\mu, D(\mathcal{E}^\mu))\) is conservative.

Proof. Using (12) the proof is similar to Theorem 2.4. \(\square\)

Lemma 3.5. Let \(t, r > 0\). Then

\[
\sup_{x \in B_r(0)} p_t(x, \cdot) \in L^1(\mathbb{R}^d, dz).
\]

Proof. Using (12), (15), and the doubling property, the proof is similar to Lemma 2.5. \(\square\)

Proposition 3.6. \((P_t)_{t \geq 0}\) and \((R_{\alpha})_{\alpha > 0}\) are strong Feller (cf. Proposition 2.6).

Proof. Using Lemma 3.5 the proof is similar to Proposition 2.6. So we omit it. \(\square\)

In contrast to the case of the subelliptic Dirichlet form \((\mathcal{E}^A, D(\mathcal{E}^A))\) considered in Section 2 we obtain:

Theorem 3.7. The transition function \((P_t)_{t \geq 0}\) satisfies:

(i) \(\lim_{t \to 0} P_t f(x) = f(x)\) for each \(x \in \mathbb{R}^d\) and \(f \in C_0(\mathbb{R}^d)\).

(ii) \(P_tC_0(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)\) for each \(t > 0\).

In particular, \((P_t)_{t \geq 0}\) is a Feller semigroup.
Proof. By (13) there exists a constant $c_x > 0$ (depending on $t$ and $x$) such that
\[ d(B_{t,x}(y)) = d(\mathcal{B}_x(y)) \]
and a constant $c_y > 0$ (depending on $t$ and $y$) such that
\[ d(\mathcal{B}_{t,x}(y)) < d(\mathcal{B}_x(y)). \]
Therefore together with the doubling property (15) can be rewritten as
\[ p_t(x,y) \leq c_1 \frac{1}{d(B_{t,x}(y))} \exp\left(-\frac{\|x-y\|^2}{c_1(4+\delta)t}\right), \tag{16} \]
and using symmetry of $p_t(\cdot,\cdot)$
\[ p_t(x,y) \leq c_1 \frac{1}{d(B_{t,y}(x))} \exp\left(-\frac{\|x-y\|^2}{c_1(4+\delta)t}\right). \]
where $c_1$ is some constant and $c_2$ is the constant as in (12). Note that since $(\mathcal{E}_0, D(\mathcal{E}_0))$ is conservative and $(P_t)_{t \geq 0}$ is strong Feller, we have $P_t(1) = 1$ for all $x \in \mathbb{R}^d$, $t > 0$. Then for $f \in C_0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t > 0$
\[ |P_t f(x) - f(x)| = \int_{\mathbb{R}^d} p_t(x,y)(f(y) - f(x)) \ dy \leq c_1 \int_{\mathbb{R}^d} \frac{1}{d(B_{t,x}(y))} \exp\left(-\frac{\|x-y\|^2}{c_1(4+\delta)t}\right) |f(y) - f(x)| \ dy, \]
which converges to zero as $t$ tends to zero. Furthermore for $f \in C_0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t > 0$ by Proposition 3.6
\[ P_t f \in C(\mathbb{R}^d) \]
and
\[ P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y) \ dy \leq c_1 \int_{\mathbb{R}^d} \frac{1}{d(B_{t,y}(x))} \exp\left(-\frac{\|x-y\|^2}{c_1(4+\delta)t}\right) f(y) \ dy, \]
which converges to zero as $\|x\|$ goes to infinity. In particular, by (13) Lemma 2.3] $(P_t)_{t \geq 0}$ is a Feller semigroup.

Remark 3.8. Under the assumptions of (A1) and (A2) in Section 2, we do not know whether the transition function $(P_t)_{t \geq 0}$ associated with $(\mathcal{E}_0, D(\mathcal{E}_0))$ in Section 2 is a Feller semigroup or not (cf. Remark 2.7). However Theorem 3.7 says that if we add the assumption (A3), the transition function $(P_t)_{t \geq 0}$ is a Feller semigroup (cf. Remark 3.7).

According to Theorem 3.7 and the classical Feller theory, there exists a Hunt process
\[ M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \xi, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}) \]
satisfying the absolute continuity condition with the transition function $(P_t)_{t \geq 0}$.

Remark 3.9. By Theorem 3.4 and Proposition 3.6 $\mathbb{P}_x(\xi = \infty) = 1$ for all $x \in \mathbb{R}^d$.

For the time being, we consider a rather strong assumption:
For each \( i, j = 1, \ldots, d \), \( \partial_j b_{ij} \in L^{\infty}_{\text{loc}}(\mathbb{R}^d, dx) \).

**Lemma 3.10.** Assume (A3)-(A5). Then for any relatively compact open set \( G \subset \mathbb{R}^d \),
\[
1_G \cdot b_{ii} \, dx \in S_{\infty}, \quad 1_G \cdot |\partial_j b_{ij}| \, dx \in S_{\infty}.
\]

**Proof.** For any relatively compact open set \( G \subset \mathbb{R}^d \), \( 1_G \cdot b_{ii} \, dx \) and \( 1_G \cdot |\partial_j b_{ij}| \, dx \) are positive finite measures on \( \mathbb{R}^d \). Furthermore by (A3), (A5), and Proposition 3.6, \( R(1_G \cdot b_{ii} \, dx) \in C^b(\mathbb{R}^d) \) and \( R(1_G \cdot |\partial_j b_{ij}| \, dx) \in C^b(\mathbb{R}^d) \). Therefore \( 1_G \cdot b_{ii} \, dx \in S_{\infty} \) and \( 1_G \cdot |\partial_j b_{ij}| \, dx \in S_{\infty} \) (see [14, Proposition 2.12]).

**Remark 3.11.** It is not possible to obtain the resolvent density estimate by taking Laplace transform in (16) w.r.t. \( t \) because the constant \( c x \) in (16) depends on \( t \). This is the reason why we assume local boundedness of \( \partial_j b_{ij} \) as in (A5). However, the assumption (A5) can be relaxed in the next subsection.

Finally, we obtain:

**Theorem 3.12.** Assume (A3)-(A5). It holds \( \mathbb{P}_x \)-a.s. for any \( x \in \mathbb{R}^d \), \( i = 1, \ldots, d \)
\[
X^i_t = x_i + \frac{1}{2} \sum_{j=1}^d \int_0^t \rho_{ij}(X_s) \, dW_j^s + \frac{1}{2} \sum_{j=1}^d \int_0^t \partial_j b_{ij}(X_s) \, ds, \quad t \geq 0,
\]
where \( \sqrt{B} = (\rho_{ij})_{1 \leq i,j \leq d} \) is the positive square root of the matrix \( B \), \( W = (W^1, \ldots, W^d) \) is a standard \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \).

**Proof.** The proof is similar to Theorem 2.11. So we omit it.

### 3.1 Nash-type inequality and part processes

This subsection is devoted to relaxing the strong assumption (A5) to a rather weak assumption (A5)’ below. We mainly use the Nash-type inequality and part Dirichlet forms of \( (\mathcal{E}^B, D(\mathcal{E}^B)) \). For the notations (especially concerning part forms and part processes) which appear in this subsection we refer to [13, Section 2] (cf. [5]). Let
\[
B_k := \{ x \in \mathbb{R}^d \mid \|x\| < k \}, \quad k \geq 1,
\]
and for any \( G \subset \mathbb{R}^d \)
\[
C^\infty(G) := \{ f : G \rightarrow \mathbb{R} \mid \exists g \in C^0_0(\mathbb{R}^d), g|_G = f \}.
\]

According to (11) the closure of
\[
\mathcal{E}^B_{\frac{1}{2}}(f,g) := \frac{1}{2} \int_{B_k} \langle B \nabla f, \nabla g \rangle \, dx, \quad f,g \in C^\infty(\overline{B_k}),
\]
in \( L^2(B_k, dx) \equiv L^2(\overline{B_k}, dx), k \geq 1 \), denoted by \( (\mathcal{E}^B_{\frac{1}{2}}, D(\mathcal{E}^B_{\frac{1}{2}})) \), is a regular Dirichlet form on \( \overline{B_k} \).

**Lemma 3.13.** (i) The following Nash-type inequality holds:
(a) if \( d \geq 3 \), then for \( f \in \mathcal{D}(\mathcal{E}^{b,\overline{B}_k}) \)
\[
\|f\|_{L^2,\underline{L}^2}^{2+\frac{2}{d}} \leq c_k \left[ \mathcal{E}^{b,\overline{B}_k}(f, f) + \|f\|_{L^2,\underline{L}^2}^2 \right]^{\frac{1}{2+\frac{2}{d}}},
\]
(b) if \( d = 2 \), then for \( f \in \mathcal{D}(\mathcal{E}^{b,\overline{B}_k}) \) and any \( \delta > 0 \)
\[
\|f\|_{L^2,\underline{L}^2}^{2+\frac{2}{d}} \leq c_k \left[ \mathcal{E}^{b,\overline{B}_k}(f, f) + \|f\|_{L^2,\underline{L}^2}^2 \right]^{\frac{1}{2+\frac{2}{d}}}.\]

Here \( c_k > 0 \) is a constant which goes to infinity as \( k \to \infty \).

(ii) We obtain for m.a.e. \( x, y \in B_k \)
(a) if \( d \geq 3 \), then
\[
r_1^{B_k}(x, y) \leq c_1 \frac{1}{\|x - y\|^{d-2}},
\]
(b) if \( d = 2 \), then for any \( \delta > 0 \)
\[
r_1^{B_k}(x, y) \leq c_2 \frac{1}{\|x - y\|^{d-2}}.
\]

Here \( c_1, c_2 > 0 \) are some constants.

Proof. We can apply Sobolev’s inequality on each \( B_k \). Using (11) we derive the Nash type inequalities in (i) (see [13, Lemma 5.4]). Following the proof of [13, Proposition 5.5, Corollary 5.6] the assertion (ii) follows. \( \square \)

Now we replace (A5) by
\((A5)'\) \( \partial_j b_{ij} \in L^{4+\varepsilon}_{\text{loc}}(\mathbb{R}^d, dx) \) for some \( \varepsilon > 0 \) and each \( i, j = 1, \ldots, d \).

Lemma 3.14. Assume (A3), (A4), and (A5)'. Let \( f \in L^{4+\varepsilon}(B_k, dx) \) for some \( \varepsilon > 0 \). Then
\[
1_{B_k} \cdot |f|dx \in S_{1,0}^{\overline{B}_k}.
\]
In particular
\[
1_{B_k} \cdot b_{ij}dx \in S_{1,0}^{\overline{B}_k}, \quad 1_{B_k} \cdot |\partial_j b_{ij}|dx \in S_{1,0}^{\overline{B}_k}.
\]

Proof. Using the estimate of resolvent density as in Lemma 3.13 (ii) and (A5)', the proof is similar to the proof of [13, Lemma 5.8]). So we omit it. \( \square \)

The following integration by parts formula holds for the coordinate functions \( f^i \in \mathcal{D}(\mathcal{E}^{b,\overline{B}_k})_{b,\text{loc}}, \ i = 1, \ldots, d \) and \( g \in C_0^\infty(B_k) \):
\[
-\mathcal{E}^{b,\overline{B}_k}(f^i, g) = \frac{1}{2} \int_{B_k} \left( \sum_{j=1}^d \partial_j b_{ij} \right) g \, dx. \tag{18}
\]

Let \( D_{\overline{B}_k} := \inf \{ t \geq 0 | X_t \in B_k \} \).

Proposition 3.15. Assume (A3), (A4), and (A5)'. Then the process \( \overline{B}_k \) satisfies
\[
X_t^i = x_i + \sum_{j=1}^d \int_0^t \rho_j(X_s) \, dW_s^j + \frac{1}{2} \sum_{j=1}^d \int_0^t \partial_j b_{ij}(X_s) \, ds, \quad t < D_{\overline{B}_k}, \tag{19}
\]
\( \mathbb{P}_x \)-a.s. for any \( x \in B_k, \ i = 1, \ldots, d \) where \( W \) is a standard \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \).
Proof. Applying [5, Theorem 5.5.5] to \((E^R, D(E^R))\), the assertion then follows from Lemma 3.14 and (18) (see Theorem 2.11 for details). □

Lemma 3.16. For all \(x \in \mathbb{R}^d\)
\[
P_x\left(\lim_{k \to \infty} D_{B^k} = \infty\right) = 1.
\]

Proof. The proof is similar to [13, Lemma 5.10]. So we omit it. □

Theorem 3.17. Assume (A3), (A4), and (A5)'. Then the process \(M\) satisfies (17) for all \(x \in \mathbb{R}^d\).

Proof. Let \(k \to \infty\) in (19). Then by Lemma 3.16 the result follows. □

Remark 3.18. The strict decomposition associated to the Dirichlet form with the uniformly elliptic matrix is presented in [4, Example]. Note that the uniformly elliptic matrix clearly satisfies (A3) and (A4). Furthermore the assumptions (A5)' is weaker than the assumption that \(\partial_j b_{ij}\) is locally bounded as in [4, Example]. Therefore the Dirichlet form \((E^B, D(E^B))\) includes the case in [4, Example].

4 Pathwise unique and strong solutions

In this section we present the conditions with which the weak solutions appearing in Section 2, 3 can be pathwise unique and strong solutions. We additionally assume in the case of \((E^A, D(E^A))\)

(A6) for each \(1 \leq i, j \leq d\),

(i) There exists a constant \(c_A > 0\) such that \(c_A \|\xi\|^2 \leq \langle A(x) \xi, \xi \rangle\) for all \(x, \xi \in \mathbb{R}^d\),

(ii) \(\|\nabla \sigma_{ij}\| \in L_{loc}^{2(d+1)}(\mathbb{R}^d, dx)\),

and in the case of \((E^B, D(E^B))\) (in Section 3)

(A6)' for each \(1 \leq i, j \leq d\),

(i) \(\rho_{ij}\) is continuous on \(\mathbb{R}^d\),

(ii) There exists a constant \(c_B > 0\) such that \(c_B \|\xi\|^2 \leq \langle B(x) \xi, \xi \rangle\) for all \(x, \xi \in \mathbb{R}^d\),

(iii) \(\|\nabla \rho_{ij}\| \in L_{loc}^{2(d+1)}(\mathbb{R}^d, dx)\),

(iv) \(\partial_j b_{ij} \in L_{loc}^{2(d+1)}(\mathbb{R}^d, dx)\).

Remark 4.1. (i) The assumption (A6) implies (A2) with \(\varepsilon = 1\).

(ii) The assumption (A6)' implies (A3), the the weak Poincaré inequality (A4), and (A5)'.

Theorem 4.2. Assume that (A1) and (A6) (resp. (A6)') hold. Then the (weak) solution in Theorem 2.11 (resp. Theorem 3.12) is strong and pathwise unique. In particular, it is adapted to the filtration \((\mathcal{F}_t^W)_{t \geq 0}\) generated by the Brownian motion \((W_t)_{t \geq 0}\) as in (10) (resp. (17)) and its lifetime is infinite.
Proof. Assume that (A1) and (A6) (resp. (A6)′) hold. Then it follows from [17, Theorem 1.1] that for given Brownian motion \((W_t)_{t \geq 0}, x \in \mathbb{R}^d\) as in (10) (resp (17)) there exists a pathwise unique strong solution to (10) (resp (17)) up to its explosion time. Therefore the (weak) solution in Theorem 2.11 (resp. Theorem 3.12) is strong, pathwise unique and its lifetime is infinite by Remark 2.9 (resp. Remark 3.9).

□

Remark 4.3. For unique strong solutions to the SDE (2) up to lifetime, [17, Theorem 1.1] presents two non-explosion conditions. The two non-explosion conditions are not satisfied by the assumptions (A1) and (A6). On the other hand, with the assumptions (A1) and (A6), by Theorem 4.2 and its proof, we know that the solution to (10) up to its lifetime fits into the frame of [17, Theorem 1.1] and so is unique strong. Furthermore its lifetime is infinite. Therefore, the redundant condition to show strong unique solution of (10), i.e.

\[ a_{ij} \in C^\infty_b(\mathbb{R}^d) \text{ in (A1)} \]

provides another non-explosion criterion in [17, Theorem 1.1].

References

[1] M. Biroli, U. Mosco, A Saint-Venant type principle for Dirichlet forms on discontinuous media, Ann. Mat. Pura Appl. (4) 169 125-181 (1995).

[2] K. L. Chung, J. B. Walsh, Markov processes, Brownian motion, and time symmetry, Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 249. Springer, New York, 2005.

[3] C. Fefferman, D. H. Phong, Subelliptic eigenvalue problems, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 590-606, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.

[4] M. Fukushima, On a decomposition of additive functionals in the strict sense for a symmetric Markov process. Dirichlet forms and stochastic processes (Beijing, 1993), 155-169, de Gruyter, Berlin, (1995).

[5] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet forms and symmetric Markov processes, Second revised and extended edition. de Gruyter Studies in Mathematics, 19. Walter de Gruyter Co., Berlin (2011).

[6] D. Jerison, The Poincaré inequality for vector fields satisfying an Hörmander’s condition, Duke Math. J., 53, 2 (1986), 503-523.

[7] D. Jerison, A. Sanchez-Calle, Subelliptic second order differential operators, Lecture Notes in Math., 1277, Springer-Verlag, Berlin-Heidelberg-Newyork (1987), 46-77.

[8] I. Karatzas, S. Shreve, Brownian Motion and Stochastic Calculus (second edition), Springer-Verlag (1991).

[9] Y. Lejan, Mesures associées une forme de Dirichlet. Applications, Bull. Soc. Math. France 106 (1978), no. 1, 61-112.
[10] Z. M. Ma, M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms*, Berlin, Springer (1992).

[11] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. 123 (1994), no. 2, 368-421.

[12] A. Nagel, E. M. Stein, S. Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. 155 (1985), no. 1-2, 103-147.

[13] J. Shin, G. Trutnau, *On the stochastic regularity of distorted Brownian motions*, arXiv:1405.7585 to appear in Trans. Amer. Math. Soc. doi:10.1090/tran/6887.

[14] J. Shin, G. Trutnau, *Pointwise weak existence for diffusions associated to degenerate elliptic forms with 2-admissible weights*, arXiv:1508.02278, to appear in J. Evol. Equ, doi:10.1007/s00028-016-03453.

[15] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) 75 (1996), no. 3, 273-297.

[16] K. T. Sturm, *The geometric aspect of Dirichlet forms. New directions in Dirichlet forms*, 233-277, AMS/IP Stud. Adv. Math., 8, Amer. Math. Soc., Providence, RI, 1998.

[17] X. Zhang, *Strong solutions of SDES with singular drift and Sobolev diffusion coefficients*, Stochastic Process. Appl. 115 (2005), no. 11, 1805-1818.

Jiyong Shin
School of Mathematics
Korea Institute for Advanced Study
85 Hoegiro Dongdaemun-gu,
Seoul 02445, South Korea,
E-mail: yonshin2@kias.re.kr