Opinion formation in a locally interacting community with recommender

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Abstract

We present a user of model interaction based on the physics of kinetic exchange, and extend it to individuals placed in a grid with local interaction. We show with numerical analysis and partial analytical results that the critical symmetry breaking transitions and percolation effects typical of the full interaction model do not take place if the range of interaction is limited, allowing for the co-existence of majority and minority opinions in the same community.

We then introduce a peer recommender system in the model, showing that, even with very local iteration and a small probability of appeal to the recommender, its presence is sufficient to make both symmetry breaking and percolation reappear. This seems to indicate that one effect of a recommendation system is to uniform the opinions of a community, reducing minority opinions or making them disappear. Although the recommender system does uniform the community opinion, it doesn’t constrain it, in the sense that all opinions have the same probability of becoming the dominating one. We do a partial study, however, that suggests that a “mischievous” recommender might be able to bias a community so that one opinion will emerge over the opposite with overwhelming probability.

1. Introduction

Standard models of opinion formation have generally been from economic models based on game theory [1]: free agents interact with each other exchanging opinions in lieu of wealth. The main result of this work is that opinion exchange systems present a Nash equilibrium [2] in which each agent holds the “best” opinion possible (that is, the one that leads to more fruitful engagement) given the opinions held by the rest of the group.

More recently, an alternative model has emerged in the form of kinetic exchange models [3,4] in which the agents are considered as free wandering particles (viz. one assumes full interaction) that, upon meeting, influence each other, exchanging, in part, their opinions. The advantage of this model is that it is formally very similar to gas dynamic models [5] to obtain analytical results on the behavior of the model.

These models assume no external influence on the opinion of the individuals: the opinion of an individual changes only through a process in which individuals reach a fair compromise after exchanging opinions [3,4]. Their most interesting characteristic is a symmetry breaking transition for a specific value of a conviction parameter, at which point the generally neutral opinion is transformed into a strongly polarized unanimous opinion [4]. For low values of the conviction parameters, several clusters of different opinions can coexist in the community, but around the critical value the main cluster percolates into the whole community, unifying the opinion.

In this paper, we extend this model in two ways. First, we place the individuals in a rectangular grid, and allow only interactions at a limited range. We show that with this change, the percolation of the main cluster disappear if the range is less than about half the linear size of the community. The symmetry breaking transition doesn’t disappear, but it becomes less and less polarized, allowing the coexistence of dissenting opinions even though the average opinion is not neutral. That is, limited interactions allow the coexistence of a majority opinion and a minority one. (Such a phenomenon has been observed in other types of models, such as [5] and [6].) We then show that the presence of the recommender system is sufficient to restore uniformity of opinion even with very limited interaction range. The symmetry breaking transition and the percolation of the main cluster take place as in the fully connected system even for very limited interaction ranges and for very limited interactions with the recommender.

Finally, we study the consequence of the presence of a mischievous recommender system, one that tries to steer opinion towards one of the extremes.

2. The basic model

Assume a set of $N$ individuals, each one having, at time $t$, an opinion $O_i(t) \in [-1,1]$ ($i = 1, \ldots, N$). When individuals $i$ and $j$ interact, the opinion of each one changes as a consequence of a negotiation process that tends to make them more similar to one another. If $\lambda_i \in [0,1]$ is the conviction of individual $i$, that is, the strength with which an individual holds her opinions and if we define the boxing operator

$$[x] = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases} \quad (1)$$
then we can model the interchange between \(i\) and \(j\) as:

\[
O_i(t+1) = \left[ \lambda_i O_i + \lambda_e \epsilon_i O_j(t) \right] \\
O_j(t+1) = \left[ \lambda_j O_j + \lambda_e \epsilon'_j O_i(t) \right]
\]

(2)

where \(\epsilon\) and \(\epsilon'\) are annealed (time-varying) variables: uncorrelated stochastic processes with uniform distribution in \([0, 1]\).

The basic model is assumed to be fully connected, that is, any agent can interact with any other agent. This system is described by an order parameter, which is simply the average conviction of an individual is equal to her proselytizing power, viz. to her capacity to convince others. There is a more complex model in which different parameters are introduced for conviction (\(\lambda_e\) and proselytism (\(\mu_i\)) but whose behavior is similar to the simpler one. If we assume uniformity of conviction (viz. a community without a strong leader) then the one-side interaction can be written as:

\[
O_i(t+1) = \left[ \lambda_i O_i + \epsilon_i O_j(t) \right]
\]

(3)

The basic model is assumed to be fully connected, that is, any agent can interact with any other agent. This system is described by an order parameter, which is simply the average opinion among all individuals:

\[
O(t) = \frac{1}{N} \sum_{i=1}^{N} O_i(t)
\]

(4)

The analytical study of the model [3] is in general impossible, although mean-field solutions can be found in special cases [9].

In the hypothesis of full interaction, the time evolution of the order parameter is described adequately by the equation

\[
O(t+1) = \left[ \lambda (1 + \epsilon O(t)) \right]
\]

(5)

We can study the stochastic map [5] in terms of random walks. Defining \(S(t) = \log |O(t)|\), eq. [5] can be written as

\[
S(t+1) = S(t) + \nu
\]

(6)

where \(\nu = \log \lambda (1 + \epsilon)\). The presence of the boxing function entails that we are actually describing a random walk with a reflecting boundary at \(S = 0\) (when \(S = 0\) the box function will make it bounce back into negative values). Depending on the value \(\lambda\) the random walk can be biased towards positive or towards negative values, and there is a critical value \(\lambda_c\) at which it is unbiased. Averaging independently over the two terms of the sum, one can estimate the critical point [3][10]. The walk is unbiased if \(\lambda = 0\), viz.

\[
\int_0^1 \log \lambda (1 + \epsilon) d\epsilon = 0
\]

(7)

giving \(\lambda_c = \epsilon/4\). We can estimate the dependence of \(O_o\) on \(\lambda\) (at steady state, as an ensemble average over all trajectories) by first estimating the average "return time" \(T\), that is, the time between two consecutive bounces at the reflecting boundary. Since \(\epsilon\) is uniformly distributed, after a bounce the walker will go on a verage to a position \((\lambda + 1)/2\). The average contribution of each step of the walk is given by 

\[
\int_0^1 \log[\lambda(1 + \epsilon)] d\epsilon = \log(\lambda/\lambda_c)
\]

This is a measure of the bias of the walk which, for

Figure 1: \(|O_o|\) plotted against \(\lambda\). The data points are obtained from a numerical simulation, the solid line is given by eq. (12).

\[
\lambda \to \lambda_c \text{ varies as } (\lambda - \lambda_c). \ T \text{ of these steps will take us back (on average) to the boundary, so multiplying this value by itself } T \text{ time we should go back from the bounce position to 1, that is:}
\]

\[
\frac{\lambda + 1}{2} \left( \frac{\lambda}{\lambda_c} \right)^T = 1
\]

(8)

which yields

\[
T = -\frac{\log \lambda}{\log \lambda - \log \lambda_c} \approx -\frac{\log \lambda}{\lambda - \lambda_c}
\]

(9)

where the approximation is valid for \(\lambda \to \lambda_c\). The steady state average of \(S\) is expected to be

\[
S_a \sim \sqrt{T} \log \lambda
\]

(10)

that is

\[
S_a = k \sqrt{T} \log \lambda
\]

(11)

where \(k\) is a constant to the determined by fitting the data. This gives an approximation

\[
|O_o| = \exp(-k |\log \lambda|^2 (\lambda - \lambda_c)^2)
\]

(12)

Figure 1 shows the results of a simulation calculating \(O\) as a function of \(\lambda\) at steady state, and the prediction of eq. (12), which is in excellent agreement with the data for \(k = 0.7\). Up until \(\lambda_c\), the system is in a symmetric (disordered) state \((O_o = 0)\) in which the opinions average out. At \(\lambda_c\) the system undergoes a critical symmetry breaking, and becomes quickly completely polarized, either on a positive opinion \((O_o \approx 1)\) or a negative one \((O_o \approx -1)\), the two occur with equal probability.
3. Lattice model with local interaction

The model presented in the previous section assumed that each individual could interact indifferentily with any other individual. In this case, spontaneous symmetry breaking occurs for \( \lambda_c = \varepsilon/4 \). A richer model can be obtained by considering placed individuals, which interact with their neighbors. We ask whether in this case a similar symmetry breaking transition occurs and what are its characteristics. We consider a grid of \( N \times N \) individuals where, as in the previous model, the individual \((i,j)\) holds an opinion \( O_j(t) \) at time \( t \). The interaction of individual \((i,j)\) with individual \((h,k)\) is given, as before, by

\[
O_{i,j}(t+1) = \left[ \lambda O_{i,j} + \varepsilon_i O_{h,k}(t) \right]
\]  
(13)

(Note that we use the asymmetric version of the evolution equation: in each interaction, only one individual change opinion; this choice doesn’t change the steady state of the interaction, although it multiply by two the transition time.) However, unlike the previous model, the individuals \((i,j)\) and \((h,k)\) can interact only if \(|i-h| \leq r\) and \(|j-k| \leq r\), where \( r \) is the range of the interaction. Note that if \( r \geq N \) the system is again a fully interconnected one. The parameter that characterizes the system is the localization parameter \( \rho = r/N \), which determine how far, relative to the size of the lattice, can an individual interact.

The presence of a finite interaction range changes the characteristics of the system, as shown in figure 2 in which \(|O(t)|\) is plotted against \( \lambda \) for various values of the interaction parameter. For \( \rho > \rho_c \approx 1/2 \), the behavior of the local interaction system is practically equivalent to that of the fully connected, with a critical symmetry breaking transition at \( \lambda_c \).

For \( \rho < \rho_c \), the symmetry still break for \( \lambda = \lambda_c \), but the order parameter doesn’t reach the value 1, stabilizing upon a value that depends on \( \rho \). For \( \rho < \rho_c \), this value decreases quite rapidly, as shown in figure 3 which shows the value of \(|O(t)|\) for various \( \lambda > \lambda_c \) as a function of \( \rho \). These lower values of the order parameter correspond to the coexistence of various groups of individuals of different opinion. This is shown in figure 4, which shows the results of a percolation study (see [11] for a similar study on the full interaction model). Given an edge value \( \Omega \), we consider a cluster as a group of adjacent \((4\text{-neighborhood})\) individuals that have opinion \( O_j > \Omega \) or \( O_j < \Omega \). The measure is significant mainly for extreme opinions, that is, for \( \Omega \approx 1 \).

Below the critical conviction \( \lambda_c \), the system is in disorder, and there is practically no formation of clusters. As \( \lambda \) increases beyond \( \lambda_c \), we see the formation of groups with mutually reinforcing opinions, but the locality of the interaction permits the creation of stable solutions with groups holding different opinions. The same conclusion can be reached by looking at the formation of clusters as a function of the localization parameter (figure 5). From these results we observe what appears to be a more gradual transition. For \( \lambda \approx 1 \) we assist to the sudden break of the symmetry around a critical point \( \rho_c \approx 0.55 \). For \( \lambda \approx \lambda_c \) we are in the pre-percolation area (cf. figure 4), and the clusters of opinion are consistently small. In an intermediate area (\( \lambda \approx 0.75 \)) there appears to be a phase of instability, in which the formation of clusters varies widely (it must be noted that this is the area in which the variance of the cluster size is higher).

We can have a little insight into the behavior of the system by considering an extremely simplified case, that of a one-dimensional continuous system in which the opinion of the individual \( x \) at time \( t \) is a continuous function of \( x \) \( O(x,t) \), \( x \in [-1,1] \) (we still assume that time is discrete, so no continuity in time can not be imposed). Consider a stable configuration, \( O(x) \), for \( t \to \infty \), and assume that the individual at the origin has opinion \( O(0) = 1 \). We are asking whether the configuration with a single cluster, that is, the configuration \( O(x) = 1 \)
is a stable one. If \( r \) is the interaction radius (note that the interaction span is equal to \( 2r \) and the individual space has size two, so that \( \rho = r \)), we can write the equilibrium as

\[
O(0) = 1 \\
O(x) - \left[ \lambda O(x) + \lambda(1 - \int_{x-r}^{x+r} O(u) du) \right] = 0
\]

the solution \( O(x) = 1 \) is stable if and only if

\[
\lambda + \lambda(1 - \int_{x-r}^{x+r} du) = \lambda(1 + r) > 1
\]

that is, if \( r > 1/(1 + \lambda) \). This would give us, for \( \lambda = 1 \) point, that the formation of complete clusters is possible only for \( \rho > 0.5 \), viz. \( \rho_\ast = 0.5 \), a bit less than the value that we observe in the numerical simulations. We argue that this discrepancy might be due to corner effects in the two-dimensional grid: the two-dimensional equivalent of our one-dimensional analysis would be a layout with radial symmetry, while the numerical analysis was carried out on a square grid.

### 4. Interaction with recommendation

So far we have dealt with the free exchange of opinion among individual, without the introduction of a recommender. Here we shall consider a very simple form of recommendation, but which should be enough to capture the essential characteristics of this kind of systems. Given an individual \( O_i \) that is about to interact, we model the recommendation by looking for the individual \( O_{r(i)} \) closer to \( O_i \), that is

\[
 r(i) = \arg \min_j |O(j) - O(i)|
\]

The value \( O_{r(i)} \) is the recommendation made to the individual \( i \), which interacts with it using the standard interaction model of eq. (3). When individual \( i \) is about to interact, it will choose to interact with the recommender system with probability \( p \) and with an individual in its neighborhood with probability \( 1 - p \). We have found that the results are quite stable even for rather low values of \( p \), so we have set \( p = 0.05 \), meaning that 5% of the interactions will be with the recommender system.

Figure 5 shows the value \( |O_{r(i)}| \) plotted against \( \lambda \) for several values of the localization parameter \( \rho \). Note that the dependence on \( \rho \) has basically disappeared, and the system has reverted to its full connection behavior. This is confirmed in figure 7 where \( |O_{r(i)}| \) is plotted as a function of \( \rho \). The variations of \( |O_{r(i)}| \) with \( \rho \) are extremely limited (the y axis of the figure covers only the range \([0.8, 1]\)), which makes the irregularities seem stronger then they actually are. Even at this low level of interaction with the recommender (5%), and even though the recommender allows interaction only of individuals with similar opinions, it appears that the long range interactions are sufficient to impede formation of “pockets” of minority opinion.

This is confirmed if we analyze the size of the largest cluster in the system (figures 8-9). Figure 8 shows a percolation effect for \( \lambda \approx 0.7 \) analogous to that of figure 4 and virtually independent of the localization parameter \( \rho \). The effects of the recommendation system are very evident if we compare figure 9 (maximum cluster size as a function of the localization parameter) with figure 5. In the absence of recommender, there is, for \( \lambda > \lambda_c \), a sharp transition in the cluster size corresponding to \( \rho \approx 0.5 \). With the recommender system, no such transition is present. There is, as in the case of absence of recommender, an instability zone corresponding to \( \lambda \approx 0.75 \) (right in the middle of the symmetry breaking transition of the conviction parameter), but the overall behavior is quite independent of \( \rho \).
The results of the model indicate that the presence of a recommender system, even one used in only 5% of the interaction is capable of inducing a global consensus in a community with only local interactions, leading to a symmetry breaking transition of the order parameter and to a percolation effect of the opinion clusters similar to those characteristics of a fully connected system.

We must remark that a fair recommender system like the one used so far induces a global opinion but it doesn’t constrain the nature of this consensus. In the community with recommender, just like in the ones without it, after the symmetry break ($\lambda = 1$) the values $O_u = 1$ and $O_u = -1$ occur with probability $1/2$.

4.1. Mischievous Recommenders

In order to determine in what condition the breaking of symmetry may be accompanied by a polarization of opinions, we create a system with a mischievous recommender system, that is, a system that tries to polarize the community towards the opinion $O_\gamma = 1$. In order to do this, given an individual $i$ that is interacting with the recommender, the recommender determines the value $r(i)$ as in (16). If $O_{\nu 0}(t) > O_i(t)$, then the recommender uses $O_{\nu 0}$ as in the fair recommended case. If $O_{\nu 0}(t) < O_i(t)$, then the recommender “flips” $O_{\nu 0}$ around $O_i$, replacing it with $2O_i - O_{\nu 0}$. This means that individual $i$ will always receive at time $t$ a recommendation with a value greater than $O(t)$.

The results are synthetized in figure 10 Whenever $\lambda < \lambda_c$, except for very low values of $\rho$, with probability 1, the outcome of the symmetry break is $O_u = 1$, that is, the mischievous recommender always succeeds in influencing the dominant opinion of the community. As the opinion parameter moves beyond the critical value $\lambda_c$, the probability that the recommender may influence the community becomes smaller, until the system reverts to the completely random behavior.

From these results it appears therefore that the presence of a recommender system has always the effect of inducing a common consensus, but that its capacity of directing this consensus is limited to the disorganized area in which, in the absence of the recommender, no opinion would prevail.

5. Conclusion

In this paper we have studied a model of opinion formation based on kinetic exchange. We have first shown that the symmetry breaking and cluster percolation transitions, characteristics of the fully connected system are lost if the interactions between individuals are restricted to a neighborhood with a locality parameter $\rho < 0.5$. However, the presence of a recommender system is capable of creating a system with the same characteristics of the fully connected even with a probability of use of recommendation as little as 0.01 and with a locality as little as 0.1.

Although the presence of the recommender does restore the symmetry breaking transition and therefore creates a unique opinion in the whole community, it does not, by its mere presence, determine the nature of this opinion. A mischievous system, one that would try to direct the opinion to a particular state would only succeed in pre-critical communities, that is, in communities in which the conviction parameter $\lambda$ is less than the critical value necessary for the symmetry breaking transition.

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Maximum cluster vs. $\lambda$ ($\Omega = 0.99$), recommender, $p = 0.05$

Figure 8: Maximum cluster size as a function of the conviction parameter for various values of the localization parameter and $\Omega = 0.99$ in a system with recommender.

Figure 9: Maximum cluster size as a function of the localization parameter $\rho$ for various values of the conviction parameter $\lambda$ and $\Omega = 0.99$ in a system with recommender. The value of $\lambda$ for the five curves, from top to bottom, are $0.99, 0.9, 0.8, 0.75, 0.7$.

Figure 10: Probability of the symmetry breaking transition resulting in $O_a = 1$ with a “mischievous” recommender as a function of the conviction parameter $\lambda$ for various values of the locality parameters $\rho$.

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