Symmetries of asymptotically flat electrovacuum spacetimes and radiation

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Symmetries compatible with asymptotic flatness and admitting gravitational and electromagnetic radiation are studied by using the Bondi-Sachs-van der Burg formalism. It is shown that in axially symmetric electrovacuum spacetimes in which at least locally a smooth null infinity in the sense of Penrose exists, the only second allowable symmetry is either the translational symmetry or the boost symmetry. Translationally invariant spacetimes with in general a straight "cosmic string" along the axis of symmetry are non-radiative although they can have a non-vanishing news function. The boost-rotation symmetric spacetimes are radiative. They describe "uniformly accelerated charged particles" or black holes which in general may also be rotating - the axial and an additional Killing vector are not assumed to be hypersurface orthogonal. The general functional forms of both gravitational and electromagnetic news functions, and of the mass aspect and total mass of asymptotically flat boost-rotation symmetric spacetimes at null infinity are obtained. The expressions for the mass are new even in the case of vacuum boost-rotation symmetric spacetimes with hypersurface orthogonal Killing vectors. In Appendices some errors appearing in previous works are corrected.

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I. INTRODUCTION AND SUMMARY

There is only one class of explicitly known radiative solutions which are asymptotically flat in the sense that null infinity is global, i.e., admits spherical sections, though its generators are not complete: boost-rotation symmetric spacetimes describing "uniformly accelerated particles" of various kinds [1]. A unique role of these solutions is exhibited by a theorem which roughly states that in axially symmetric, asymptotically flat spacetimes (in the sense that at least a local null infinity exists) the only additional symmetry that does not exclude radiation is the boost symmetry [2].

The boost-rotation symmetric spacetimes have been used in various contexts. From a mathematical point of view, these solutions contain the only known spacetimes in which arbitrarily strong initial data with the given symmetry can be chosen on a hyperboloidal hypersurface which lead to the complete, smooth null infinity and regular timelike infinity in future. From a more physical point of view these are - and most probably will long remain - the only exact solutions of Einstein's equations for which one can find such quantities as angular distribution of gravitational radiation emitted by particles (represented by singularities) or by uniformly accelerated black holes [3]. In numerical calculations these spacetimes have been employed as important test beds in the null cone version of numerical relativity [4,5], and, most recently, also in the standard approach based on a spacelike initial hypersurface [6]. In the context of quantum gravity the boost-rotation symmetric spacetimes (as "generalized C-metric") have been used to describe production of black-hole pairs in strong background fields (see e.g. [7]). We refer to the recent review [8] of exact radiative spacetimes and to the comprehensive analysis of the general structure of the boost-rotation symmetric spacetimes [1] for more details and references on both the history and recent developments in these issues.

Until now all work on the general properties of the boost-rotation symmetric spacetimes has concentrated on the vacuum case with two non-null, hypersurface orthogonal Killing vectors - the rotational (axial) Killing vector and the boost Killing vector. The metric of such spacetimes, in suitable coordinates, contains just two functions. It can be constructed starting from the appropriately behaved boost-rotation symmetric solution of the flat-space wave equation with sources which is satisfied by one of the functions. The other metric function can then be determined by

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a quadrature (see [1]). In the present paper we start the systematic study of more general boost-rotation symmetric spacetimes. We assume that an electromagnetic field coupled to gravity may also be present; and we consider Killing vectors which need not be hypersurface orthogonal. Even if only one of these extensions is taken into account, the field equations become fully nonlinear - there is no flat-space wave equation available. The inclusion of electromagnetic field is also of interest because there exist the analogous features of gravitational fields due to uniformly accelerated masses in general relativity and of electromagnetic fields due to uniformly accelerated charges in special relativity. The study of boost-rotation symmetric fields within the Einstein-Maxwell theory treats both gravitational and electromagnetic fields from a unified point of view.

It is known that the example of a boost-rotation symmetric electrovacuum solution with Killing vectors which are not hypersurface orthogonal exists - this is the charged "rotating" C-metric [2]. Here we consider general solutions. Our main result is the theorem which roughly states that even under the presence of electromagnetic field and "rotating sources", the boost symmetry is the only one which can be combined with axial symmetry and radiation exists. However we shall also obtain other results (e.g. for the news function and total mass) some of which are new even in the vacuum case with hypersurface orthogonal Killing vectors.

In the following Section II we start out from the general form of axially symmetric metric and electromagnetic field in Bondi-Sachs coordinates \( \{u, r, \theta, \phi\} \), where \( u = \text{const} \) labels null hypersurfaces (in flat space \( u = t - r \) is the usual retarded time), \( r \) is a luminosity distance along null rays \( u = \text{const}, \theta = \text{const}, \phi = \text{const} \), and \( \theta, \phi \) are standard spherical angles. We consider their asymptotic expansions at \( r \to \infty \) as they follow from the Einstein-Maxwell equations under the assumption of asymptotic flatness. Since we wish to study spacetimes in which the axial Killing vector is in general not hypersurface orthogonal we cannot employ the original Bondi’s et al work [3] but have to start from its generalization by Sachs [4] or, rather, from van der Burg’s [5] explicit treatment of the asymptotic behaviour of the coupled Einstein-Maxwell fields in the Bondi-Sachs coordinates. Now van der Burg’s work involves complicated equations in which many errors and misprints appear. These do not change basic conclusions of his paper but we need correct forms. Therefore, in Appendix A we first summarize all Einstein-Maxwell’s equations in the Bondi-Sachs coordinates in general spacetime and then give asymptotic forms of their solutions under the assumption of axial symmetry. (All equations were checked by using MAPLE V.) The structure of the field equations (the splitting into twelve main equations and five supplementary conditions), the total quantities (the mass and the charge) and their fluxes given in terms of two gravitational and two electromagnetic news functions are briefly reviewed also in Appendix A. The resulting formulas are used extensively in Section II to prove the theorem which states the following:

Suppose that an axially symmetric electrovacuum spacetime admits a "piece" of null infinity. If this spacetime admits an additional Killing vector forming with the axial Killing vector a 2-dimensional Lie algebra, then the additional Killing vector either generates a supertranslation or it is the boost Killing vector.

The theorem is proven by first decomposing the additional Killing vector field \( \eta^a \) in the null (Sachs) tetrad and then solving the Killing equations asymptotically in the leading terms. In order to illustrate the form of the Killing equations, the lengthiest among them, \( \mathcal{L}_\eta g_{00} = 0 \), is written down in Appendix B.

In Section III the case of the supertranslational Killing field is considered further. By solving Killing equations in the higher orders in \( r^{-k} \) and considering also asymptotic solutions of equations \( \mathcal{L}_\eta F_{\mu\nu} = 0 \) (assuming thus that the electromagnetic field shares the same symmetry), we show that the supertranslational Killing field has in fact to be the generator either of translations along the axis of axial symmetry (\( z \)-axis) or time translations. Resulting spacetimes are non-radiative. Somewhat surprisingly perhaps, the news functions of the system need not necessarily be vanishing. This is not because of cylindrical waves which of course are symmetric under translations along \( z \)-axis. Cylindrical waves are excluded from our analysis if we assume that the local null infinity exists at \( \theta = \pi/2 \); there are no cylindrical spacetimes which admit a regular cross-section of null infinity. (We refer to the papers [6,7,8] for the study of null infinity of cylindrically symmetric spacetimes and to our forthcoming work in which the Killing equations will be considered in a greater detail in further orders in \( r^{-k} \) for translational Killing vectors.) Here we consider spacetimes with a straight cosmic string along \( z \)-axis which, as shown by Bičák and Schmidt [9], have a non-vanishing news function independent of time. If we assume that electromagnetic field is regular at the axis of symmetry, only the gravitational news function is non-vanishing. We give the explicit form of the translational Killing vector in this case. This translational Killing vector in the case of the string has not been given in Ref. [2] because of a sign error. The error and misprints appearing in [2] are corrected in Appendix C. In Appendix D the translations in Bondi’s coordinates in a spacetime with a straight, non-rotating cosmic string are analyzed in detail. At the end of Section III we give explicitly the asymptotic form of both gravitational and electromagnetic fields in the Bondi-Sachs coordinates for stationary spacetimes without a string.

Section IV is devoted to the case when the additional Killing vector is the boost Killing vector. By analyzing the Killing equations in further orders in \( r^{-k} \) and considering also equations \( \mathcal{L}_\eta F_{\mu\nu} = 0 \), we discover that both gravitational and electromagnetic news functions must have the same functional form - they depend on \( u \) and \( \theta \) so that the news function \( f(\sin\theta/u)u^{-2} \), where \( f \) is an arbitrary function of its argument. We also derive the general form of the mass aspect and total (Bondi) mass for boost-rotation symmetric spacetimes. The mass aspect is given
by

\[ M(u, \theta) = \frac{1}{2 \sin \theta} \left( w^2 K_w \right)_w + \frac{\mathcal{L}}{u^3}, \]

where \( w = \sin \theta/u, K(w) \) is an arbitrary function, and \( \mathcal{L}(w) = \lambda(w)/w^3 \), where \( \lambda(w) \) satisfies the simple equation \( \Box \) in the main text. \( \lambda(w) \) can be determined if the news functions are known. The total Bondi mass is then given by

\[ m(u) = \frac{1}{4} \int_0^\pi (w^2 K_w)_w \, d\theta + \frac{1}{2} \int_0^\pi \frac{\mathcal{L}_w}{u^2} \, d\theta. \]

The formulas for the mass aspect and total mass are new even in the case of hypersurface orthogonal Killing vectors studied in \( \Box \). Also, the boost Killing vector is here expanded to further orders than in Ref. \( \Box \) (see Eq. (122)).

Consider an axially symmetric electrovacuum spacetime with circular group orbits; denote the corresponding Killing vector field by \( \partial/\partial \phi \). Assume that at least the "piece of \( I^+ \) exists in the sense of \( \Box \). Then one can introduce the Bondi-Sachs coordinate system \( \{ u, r, \theta, \phi \} \equiv \{ x^0, x^1, x^2, x^3 \} \) in which the metric satisfying the Einstein-Maxwell equations has the form \( \Box \)

\[ ds^2 = \left( \frac{V}{r} e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 U W \sinh 2\delta \right) du^2 \]
\[ + 2e^{2\beta} du dr + 2r^2 (e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta) du \sin \theta d\theta d\phi + 2r^2 (e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta) \sin \theta d\theta d\phi \]
\[ - r^2 \left[ \cosh 2\delta (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta \, d\phi^2) + 2 \sinh 2\delta \sin \theta \, d\theta d\phi \right], \]

where six metric functions \( U, V, W, \beta, \gamma, \delta \) and the Maxwell field \( F_{\mu\nu} \) do not depend on \( \phi \) because of axial symmetry.

The complete set of the Einstein-Maxwell equations and the asymptotic expansions of the metric functions and electromagnetic field components at large \( r \) are given in Appendix A. We shall need these expansions, in which also the field equations are used, up to the following orders (all "coefficients" \( c, d, M, e, f, \ldots \) below being in general functions of \( u \) and \( \theta \))

\[ \gamma = \frac{c}{r} + O(r^{-3}) , \]
\[ \delta = \frac{d}{r} + O(r^{-3}) , \]
\[ \beta = -\frac{1}{4} (c^2 + d^2) \frac{1}{r^2} + O(r^{-4}) , \]
\[ U = -(c_{,\theta} + 2c \cot \theta) \frac{1}{r^2} + O(r^{-3}) , \]
\[ W = -(d_{,\theta} + 2d \cot \theta) \frac{1}{r^2} + O(r^{-3}) , \]
\[ V = r - 2M + O(r^{-1}) . \]

The leading terms in the electromagnetic field tensor are given by

\[ F_{12} = \frac{e}{r^2} + (2E + ec + fd) \frac{1}{r^3} + O(r^{-4}) , \]
\[ F_{13} = \left( \frac{f}{r^2} + (2F + ed - fc) \frac{1}{r^3} + O(r^{-4}) \right) \sin \theta , \]
\[ F_{01} = -\frac{e}{r^2} + (e_{,\theta} + e \cot \theta) \frac{1}{r^3} + O(r^{-4}) , \]
$$F_{23} = \left(-\mu - (f, \theta + f \cot \theta) \frac{1}{r} + O(r^{-2})\right) \sin \theta ,$$
$$F_{02} = X + (\epsilon, \theta - \epsilon, u) \frac{1}{r} + O(r^{-2}) ,$$
$$F_{03} = \left(Y - \frac{f, u}{r} + O(r^{-2})\right) \sin \theta .$$

The mass aspect $M(u, \theta)$ is connected with the two gravitational news functions $e, f$, $\epsilon, \mu$ and $X, Y$ by the relation

$$M_{su} = -(c_{su}^2 + d_{su}^2) - (X^2 + Y^2) + \frac{1}{2}(c_{s, \theta \theta} + 3c_{s, \theta} \cot \theta - 2c_{su}) .$$

As a consequence of the Einstein-Maxwell equations the electromagnetic functions $e, f, \epsilon, \mu$ and $X, Y$ are connected by the relations

$$e_{su} = \frac{1}{2} \epsilon_{s, \theta} - (cX + dY) ,$$
$$f_{su} = -\frac{1}{2} \mu_{s, \theta} - (-cY + dX) ,$$
$$\epsilon_{su} = -X_{s, \theta} - X \cot \theta ,$$
$$\mu_{su} = -Y_{s, \theta} - Y \cot \theta .$$

The gravitational news functions enter the evolution equations for $e$ and $f$. The evolution equations for functions $E, F$ are given in Appendix A. There the evolution of the whole Einstein-Maxwell system is summarized and the interpretation of the mass and the charges of the system is recalled. Here let us only notice that both the gravitational news functions $e, f$ and the electromagnetic news functions $X, Y$ are freely specifiable functions.

Now we wish to emphasize that we assume that Eqs. (3) - (4) are valid for all $\theta \in [0, 2\pi]$, however, they need not to be true for all $\theta \in [0, \pi]$. Since we want to admit spacetimes with only "local" $\mathcal{I}^+$, we assume Eqs. (3) - (4) to be satisfied in some open interval of $\theta$, i.e., not necessarily on the whole sphere. In particular, the "axis of symmetry" ($\theta = 0, \pi$) may contain some nodal singularities and need not thus be regular. The regularity conditions on the axis (which can be obtained by transforming (3) into local Minkowskian coordinates) on functions $\gamma, \nu, \beta$ ... need not be satisfied for any $u$. If the axis is singular then at least two generators of $\mathcal{I}^+$ would be missing so that $\mathcal{I}^+$ would not be topologically $S^2 \times \mathbb{R}$ (this is exactly the case which can occur in the boost-rotation symmetric spacetimes discussed in Sec. IV).

Let us now assume that another Killing vector field $\eta$ exists which forms together with $\xi = \partial/\partial \phi$ a two-parameter group. In Ref. [1] it is proved (see Lemma in Sec. 2) that in the case of $\xi$ with circles as integral curves, $\xi$ and $\eta$ determine an abelian Lie algebra so that we can assume $[\eta, \xi] = 0$. Hence, the components of $\eta^\alpha$ are independent of $\phi$.

Introduce the standard null tetrad field $\{k^\alpha, m^\alpha, t^\alpha, \bar{t}^\alpha\}$, with bar denoting the complex conjugate, where $k^\alpha = \partial_u u$, $m^\alpha m_\alpha = 1$, and the complex vector $t^\alpha = t^\alpha_R + it^\alpha_I$ (subscripts $R$ and $I$ denoting the real and imaginary parts) obeys $t^\alpha t_\alpha = -1$, $t^\alpha t_\alpha = t^\alpha k_\alpha = t^\alpha m_\alpha = 0$. A convenient choice of the tetrad reads

$$k^\alpha = \left[1, 0, 0, 0\right] , \quad m^\alpha = \left[\frac{1}{2} V r^{-1} e^{2\beta}, e^{2\beta}, 0, 0\right] ,$$
$$t_\alpha = \frac{1}{2} r^2 (e^{2\delta})^{-1/2} \left[(1 + \sinh 2\delta) e^{\gamma} U + \cosh 2\delta e^{-\gamma} W + i[(1 - \sinh 2\delta)e^{\gamma}U - \cosh 2\delta e^{-\gamma}W] , \right.$$
$$0 , - (1 + \sinh 2\delta + i(1 - \sinh 2\delta))e^{\gamma} , -(1 - i) \cosh 2\delta \sin \theta e^{-\gamma} \right] .$$

This choice indeed implies $g_{\mu\nu} = 2k_{(\mu}m_{\nu)} - 2\bar{t}_{(\mu}t_{\nu)}$, with $g_{\mu\nu}$ given by (3).

We now decompose the Killing vector field $\eta^\alpha$ in the null tetrad,

$$\eta^\alpha = Ak^\alpha + Bm^\alpha + Ct^\alpha + \bar{C}\bar{t}^\alpha ,$$

or

$$\eta^\alpha = Ak^\alpha + Bm^\alpha + \tilde{f}(t^\alpha_R + t^\alpha_I) + \tilde{g}(t^\alpha_R - t^\alpha_I) ,$$

where $A, B, C = C_R + iC_I$, $f = C_R - C_I$, $g = C_R + C_I$ are general functions of $u, r, \theta$. The Killing equations
\[ \mathcal{L}_\eta g_{\alpha\beta} = 0 \]  

can then be rewritten in the form

\[ 0 = \mathcal{L}_\eta g_{\alpha\beta} = A\mathcal{L}_k g_{\alpha\beta} + B\mathcal{L}_m g_{\alpha\beta} + 2A_{(\alpha} k_{\beta)} + 2B_{(\alpha} m_{\beta)} \]
\[ + \tilde{f}[(\mathcal{L}_t g_{\alpha\beta})_R + (\mathcal{L}_t g_{\alpha\beta})_I] + 2\tilde{f}_{(\alpha} [t_{R\beta)} + t_{I\beta}] \]
\[ + \tilde{g}[(\mathcal{L}_t g_{\alpha\beta})_R - (\mathcal{L}_t g_{\alpha\beta})_I] + 2\tilde{g}_{(\alpha} [t_{R\beta)} - t_{I\beta}] . \]

The easiest is the equation

\[ \mathcal{L}_\eta g_{11} = 2e^{2\beta} B, = 0 , \]

which implies

\[ B = B(u, \theta) . \]

In the hypersurface orthogonal case, analyzed in Ref. [2], also the equations \( \mathcal{L}_\eta g_{03} = \mathcal{L}_\eta g_{13} = \mathcal{L}_\eta g_{23} = 0 \) can easily be solved. This is not so if \( \partial / \partial \phi \) is not hypersurface orthogonal. All Killing equations other than \( \mathcal{L}_\eta g_{11} = 0 \) become now complicated. For illustration, equation \( \mathcal{L}_\eta g_{00} = 0 \) is written down in Appendix B; the other equations can be found in [13].

We assume that the coefficients \( A, \tilde{f}, \tilde{g} \) can be expanded in powers of \( r^{-k} \) and solve the Killing equations asymptotically. Denoting by \( A^{(k)}, f^{(k)}, g^{(k)} \) the coefficients of \( r^{-k} \) in the expansions, we first notice that \( \mathcal{L}_\eta g_{22} = 0, \mathcal{L}_\eta g_{12} = 0, \mathcal{L}_\eta g_{13} = 0 \) imply

\[ A = A^{(-1)} r + A^{(0)} + \frac{A^{(1)}}{r} + O(r^{-2}) , \]
\[ \tilde{f} = f^{(-1)} r + f^{(0)} + \frac{f^{(1)}}{r} + O(r^{-2}) , \]
\[ \tilde{g} = g^{(-1)} r + g^{(0)} + \frac{g^{(1)}}{r} + O(r^{-2}) , \]

where \( A^{(k)}, f^{(k)}, g^{(k)} \) are functions of \( u \) and \( \theta \). Remaining nine Killing equations imply the conditions on functions \( B, A^{(k)}, f^{(k)}, g^{(k)} \). The procedure of their solutions is similar to that in Ref. [2] (cf. Eqs. (20)-(24) therein). However, the equations are now more complicated and there are three additional equations to be satisfied.

In the leading orders in \( r^{-k} \) the Killing equations can be written down easily (we omit equations \( \mathcal{L}_\eta g_{12} = 0, \mathcal{L}_\eta g_{13} = 0 \) since they do not restrict the leading terms in \( r^0 \)):

\[ \mathcal{L}_\eta g_{00} = 0 \quad (r^1) : \quad A^{(-1)}\, u = 0 , \]
\[ \mathcal{L}_\eta g_{01} = 0 \quad (r^0) : \quad B, = A^{(-1)} = 0 , \]
\[ \mathcal{L}_\eta g_{02} = 0 \quad (r^2) : \quad f^{(-1)}\, u = 0 , \]
\[ \mathcal{L}_\eta g_{03} = 0 \quad (r^2) : \quad g^{(-1)}\, u = 0 , \]
\[ \mathcal{L}_\eta g_{22} = 0 \quad (r^2) : \quad f^{(-1)}\, , A^{(-1)} = 0 , \]
\[ \mathcal{L}_\eta g_{23} = 0 \quad (r^2) : \quad -g^{(-1)}\, , g^{(-1)}\, \cot \theta = 0 , \]
\[ \mathcal{L}_\eta g_{33} = 0 \quad (r^2) : \quad f^{(-1)}\, \cot \theta + A^{(-1)} = 0 . \]

The system of equations (19) - (25) is, at this order, identical to Eqs. (20) - (24) in Ref. [2]. The solutions are thus identical, reading

\[ A^{(-1)} = k \cos \theta , \]
\[ f^{(-1)} = -k \sin \theta , \]
\[ B = -ku \cos \theta + a(\theta) , \]

where \( k = \text{const} \) and \( a \) is an arbitrary function of \( \theta \). Eqs. (22) and (24) imply

\[ g^{(-1)} = h \sin \theta , \]
where \( h = \text{const.} \).

Regarding Eqs. (11), (13) and (18) we find \( \eta^\phi = h + O(r^{-1}) \). Since the contribution of \( h \) to the vector field \( \eta^\alpha \) is just \( \eta^\phi = \text{const.} \), which is a constant multiple of the axial Killing vector \( \partial/\partial \phi \), we may, without loss of generality, put \( h = 0 \). Therefore, in the lowest order in \( r^{-1} \) the general asymptotic form of the Killing vector \( \eta \) turns out to be

\[
\eta^\alpha = [-ku \cos \theta + \alpha(\theta) , \; kr \cos \theta + O(r^0) , \; -k \sin \theta + O(r^{-1}) , \; O(r^{-1})] ,
\]

where \( k \) is a constant, \( \alpha \) - an arbitrary function of \( \theta \). Thus, in the leading order of the asymptotic expansion the presence of electromagnetic field satisfying the boundary conditions (5) and the fact that the axial Killing vector need not be hypersurface orthogonal do not change the conclusion obtained in Ref. [2] in the vacuum case with no singularity arises for \( \hat{A} \) functions and taking into account the metric expansions (4), we find the following restrictions on the expansion coefficients of \( \hat{A} \).

The asymptotic form of the Killing vector field \( \eta \) is

\[
\eta^\alpha = [-u \cos \theta , \; r \cos \theta + O(r^0) , \; -\sin \theta + O(r^{-1}) , \; O(r^{-1})] ,
\]

which is the boost Killing vector. It generates the Lorentz transformations along the axis of axial symmetry.

We have thus proven the following

**Theorem:** Suppose that an axially symmetric electrovacuum spacetime admits a "piece" of \( I^+ \) in the sense that the Bondi-Sachs coordinates can be introduced in which the metric takes the form (3) - (4) and the asymptotic form of the electromagnetic field is given by (5). If this spacetime admits an additional Killing vector forming with the axial Killing vector a 2-dimensional Lie algebra, then the additional Killing vector has asymptotically the form (28). For \( k = 0 \) it generates a supertranslation; for \( k \neq 0 \) it is the boost Killing field.

On \( I^+ \) in the coordinates \( u, l = r^{-1}, \theta, \) and \( \phi \) the boost Killing vector reads (cf. 2)

\[
\eta^\alpha_{I^+} = [-u \cos \theta , \; 0 , \; -\sin \theta , \; 0] .
\]

### III. THE SUPERTRANSLATIONAL KILLING FIELDS

In this section we shall show that if the Killing field (28) for \( k = 0 \) is the supertranslational Killing field, it has, in fact, to be the generator of translations and the resulting spacetime is thus non-radiative.

Assuming \( k = 0 \), and considering the Killing equations (14) in the higher orders in \( r^{-k} \) than in Eqs. (19) - (22), and taking into account the metric expansions (4), we find the following restrictions on the expansion coefficients of functions \( A, \tilde{f} \) and \( \tilde{g} \):

\[
\mathcal{L}_\eta g_{00} = 0 \quad (r^0) : A^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{01} = 0 \quad (r^1) : A^{(1)}_{uu} - f^{(0)}_{uu} (c_{,\theta} + 2c \cot \theta) - g^{(0)}_{uu} (d_{,\theta} + 2d \cot \theta) = 0 ,
\]

\[
\mathcal{L}_\eta g_{02} = 0 \quad (r^2) : -B (c_{uu} + dd_{,u}) - BM - A^{(1)} + f^{(0)}_{uu} (c_{,\theta} + 2c \cot \theta) + g^{(0)}_{uu} (d_{,\theta} + 2d \cot \theta) = 0 ,
\]

\[
\mathcal{L}_\eta g_{03} = 0 \quad (r^3) : f^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{12} = 0 \quad (r^1) : f^{(0)} + B_{,\theta} = 0 ,
\]

\[
\mathcal{L}_\eta g_{13} = 0 \quad (r^0) : g^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{14} = 0 \quad (r^{-1}) : f^{(0)} - B (c_{,\theta} + 2c \cot \theta) + g^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{15} = 0 \quad (r^{-1}) : g^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{16} = 0 \quad (r^{-1}) : g^{(0)}_{uu} = 0 ,
\]

\[
\mathcal{L}_\eta g_{17} = 0 \quad (r^{-1}) : g^{(0)}_{uu} = 0 .
\]
\[ \mathcal{L}_{\eta} g_{22} = 0 \quad (r \, 1): \quad f^{(0)}_{\theta \phi} + B c_{u} + A^{(0)} - \frac{1}{2} B = 0 , \]
\[ \mathcal{L}_{\eta} g_{23} = 0 \quad (r \, 0): \quad f^{(1)}_{\theta \phi} + B (c_{u} + 2 d_{u}) - B (c_{\phi} + 2 c \cot \theta)_{\phi} + A^{(1)} + B M + 2 d (g^{(0)}_{\phi \phi} - g^{(0)} \cot \theta) = 0 , \]
\[ \mathcal{L}_{\eta} g_{33} = 0 \quad (r \, 1): \quad f^{(0)} \cot \theta - B c_{u} + A^{(0)} - \frac{1}{2} B = 0 , \]
\[ \mathcal{L}_{\eta} g_{33} = 0 \quad (r \, 0): \quad \cot \theta \left[ -B (c_{\phi} + 2 c \cot \theta) + f^{(1)} - f^{(0)} c \right] - c_{\phi} f^{(0)} + c (A^{(0)} - \frac{1}{2} B) + A^{(1)} + 2 B d d_{u} + B M = 0 . \]

First notice that Eqs. (31), (32), (37), (33), (41) and (45) imply
\[ A^{(0)} = A^{(0)}(\theta) , \quad f^{(0)} = f^{(0)}(\theta) = -B_{\phi} \cot \theta , \quad g^{(0)} = 0 , \quad d = d(\theta) . \]

Using these results in the remaining equations we find
\[ A^{(0)} = \frac{1}{2} (B_{\phi \phi} + B_{\phi} \cot \theta + B) , \]
\[ c_{u} = \frac{1}{2 B} (B_{\phi \phi} - B_{\phi} \cot \theta) , \]
\[ c = \frac{u}{2 B} (B_{\phi \phi} - B_{\phi} \cot \theta) + \omega(\theta) , \]
\[ A^{(1)} = A^{(1)}(\theta) , \]
\[ f^{(1)} = B (c_{\phi} + 2 c \cot \theta) , \]
\[ g^{(1)} = B (d_{\phi} + 2 d \cot \theta) - B_{\phi} d , \]

where \( \omega(\theta) \) is an arbitrary function of \( \theta \); \( \omega \) can be transformed away by a supertranslation \( u = \tilde{u} + \tilde{\alpha}(\theta) \) with \( \tilde{\alpha} \) satisfying \( \omega + (-\tilde{\alpha}_{\phi} + \tilde{\alpha}_{\phi} \cot \theta) / 2 = 0 \). Equations (36) and (44) are now satisfied identically. Equations (34), (44) and (48) lead to only one further independent condition:
\[ M = -cc_{u} - B^{-1} [A^{(1)} + B_{\phi} (c_{\phi} + 2 c \cot \theta)] . \]

Important results follow from Eqs. (19) and (51). Eq. (19) shows that the news function \( d_u = 0 \). Since \( B = B(\theta) \), Eq. (51) implies that the time-derivative of the news function \( c_u \) must vanish. Therefore, with the supertranslational Killing field, we arrive at the Weyl tensor (see e.g. (13))
\[ C_{\alpha \beta \gamma \delta} m^{\alpha} t^{\beta} m^{\gamma} t^{\delta} = [(c + i d)_{\alpha \beta}, \frac{1}{r} + O(r^{-2}) , \]
which is non-radiative since the first term proportional to \( r^{-1} \) vanishes.

Substituting the expansion of the metric functions (44) into the null tetrad (11) and coefficients \( A, B, \tilde{f} \) and \( \tilde{g} \) (given by Eqs. (19) and (20)) into Eq. (13), we find the expansion of the supertranslational Killing vector to be
\[ \eta^{\mu} = \left[ B(\theta), \frac{1}{2} (B_{\phi \phi} + B_{\phi} \cot \theta) + (-B_{\phi \phi} - 2 B_{\phi} B_{\phi \phi} + 2 B_{\phi}^{2} B_{\phi \phi} B^{-1} - 2 B_{\phi}^{3} \cot \theta B^{-1}
+ B_{\phi}^{2} (3 \cot \theta^{2} - \frac{2}{\sin^{2} \theta}) B^{-1} \frac{u}{4 r} + O(r^{-2}), \right.
\left. -B_{\phi} \frac{1}{r} + B_{\phi} \frac{c}{r^{2}} + O(r^{-3}), B_{\phi} d \frac{1}{r^{2}} \sin \theta + O(r^{-3}) \right] . \]

Let us now turn to the asymptotic properties of electromagnetic field. We assume \( \mathcal{L}_{\eta} F_{\alpha \beta} = 0 \) where \( \eta \) is now the supertranslational Killing field (8). From the general expression of the Lie derivative of \( F_{\mu \nu} \) with respect to general \( \eta^{\mu} \) decomposed into the null tetrad according to Eq. (13),
\[ \mathcal{L}_{\eta} F_{\alpha \beta} = A \mathcal{L}_{\eta} F_{\alpha \beta} + A_{\alpha} k^{\gamma} F_{\beta \gamma} + A_{\beta} k^{\gamma} F_{\alpha \gamma} + B \mathcal{L}_{\eta} m F_{\alpha \gamma} + B_{\alpha} m^{\gamma} F_{\beta \gamma} + B_{\beta} m^{\gamma} F_{\alpha \gamma} + \frac{\tilde{f}}{\mathcal{L}_{\eta} F_{\alpha \beta}} (R_{\alpha \beta} + (\mathcal{L}_{\eta} F_{\alpha \beta}) I) + \frac{\tilde{g}}{\mathcal{L}_{\eta} F_{\alpha \beta}} (R_{\alpha \beta} - \mathcal{L}_{\eta} F_{\alpha \beta}) I + \tilde{f}_{\alpha} (t_{\gamma} + t_{\gamma}^{\phi}) F_{\gamma \beta} + \tilde{g}_{\beta} (t_{\gamma} + t_{\gamma}^{\phi}) F_{\alpha \gamma} + \tilde{g}_{\alpha} (t_{\gamma} - t_{\gamma}^{\phi}) F_{\gamma \beta} + \tilde{g}_{\beta} (t_{\gamma} - t_{\gamma}^{\phi}) F_{\alpha \gamma} . \]
and after substituting for \( A, B, \tilde{f}, \) and \( \tilde{g} \) in accordance with Eq. (58), we find that in the first orders in \( r^{-k} \) the Lie equations imply

\[
\mathcal{L}_\eta F_{01} = 0 \quad (r^{-2}) : -\epsilon_{\alpha u} B + X B_{,\theta} = 0 ,
\]

\[
\mathcal{L}_\eta F_{02} = 0 \quad (r^0) : X_{,\alpha} B = 0 ,
\]

\[
\mathcal{L}_\eta F_{03} = 0 \quad (r^0) : (Y \sin \theta)_{,u} B = 0 ,
\]

\[
\mathcal{L}_\eta F_{12} = 0 \quad (r^{-2}) : e_{,u} B + \epsilon B_{,\theta} = 0 ,
\]

\[
\mathcal{L}_\eta F_{13} = 0 \quad (r^{-2}) : (f \sin \theta)_{,u} B - \mu \sin \theta B_{,\theta} = 0 ,
\]

\[
\mathcal{L}_\eta F_{23} = 0 \quad (r^0) : - (\mu \sin \theta)_{,u} B + Y \sin \theta B_{,\theta} = 0 .
\]

From Eq. (61) we get \( X = X(\theta) \). Combining Eq. (60) and Maxwell equation (9) we get equation for \( X, (X B \sin \theta)_{,\theta} = 0 \), the solution being

\[
X = \frac{x_0}{B \sin \theta} , \quad x_0 = \text{const} .
\]

Regarding the last result and integrating Eq. (60), we obtain

\[
\epsilon = \frac{x_0 B_{,\theta}}{B^2 \sin \theta} u + \epsilon_1(\theta) ,
\]

where \( \epsilon_1(\theta) \) is an arbitrary integration function. Eq. (63) then implies

\[
\epsilon = -\frac{x_0 B_{,\theta}^2}{2 B^3 \sin \theta} u^2 - \frac{\epsilon_1 B_{,\theta}}{B} u + \epsilon_1(\theta) ,
\]

where \( \epsilon_1(\theta) \) is again an integration function.

Analogously, using Eqs. (62), (64), (65) and Maxwell equation (10) we find

\[
Y = \frac{y_0}{B \sin \theta} , \quad y_0 = \text{const} ,
\]

\[
\mu = \frac{y_0 B_{,\theta}}{B^2 \sin \theta} u + \mu_1(\theta) ,
\]

\[
f = \frac{y_0 B_{,\theta}^2}{2 B^3 \sin \theta} u^2 + \frac{\mu_1 B_{,\theta}}{B} u + f_1(\theta) ,
\]

where \( \mu_1, f_1 \) are arbitrary functions of \( \theta \).

Comparing Eqs. (68) and (71) with Maxwell equations (7) and (8) we obtain restrictions on \( \epsilon_1 \) and \( \mu_1 \):

\[
\epsilon_{1,\theta} + \frac{2 \epsilon_1 B_{,\theta}}{B} - \frac{2 y_0 d}{B \sin \theta} = 0 ,
\]

\[
\mu_{1,\theta} + \frac{2 \mu_1 B_{,\theta}}{B} + \frac{2 x_0 d}{B \sin \theta} = 0 .
\]

Now we assume that the Killing field \( \eta \) is bounded at the axis, i.e. \( B(\theta) \) is bounded for \( \theta = 0, \pi \) and the electromagnetic field (5), i.e. \( X \) and \( Y \) have to be bounded there, too. Therefore constants entering \( X \) and \( Y \) must vanish:

\[
x_0 = y_0 = 0 .
\]

Taking the time-derivative of Eq. (56), regarding Eqs. (11), (13), (15), and comparing with Eq. (3), we arrive at the following equation for function \( B \):

\[
\left[ \frac{\sin^3 \theta}{2B} \left( \frac{B_{,\theta}}{\sin \theta} \right)_{,\theta} \right]_{,\theta} = 0 .
\]

The general solution can be seen to be

\[
B = a \sin \theta \left( \frac{\sin \theta}{\cos \theta + 1} \right)^C + b \sin \theta \left( \frac{\sin \theta}{\cos \theta + 1} \right)^{-C} ,
\]
where \( a, b, C > 0 \) are constants. If we assume \( C \in [0, 1] \) then \( B \) is bounded at the axis.

The news function is obtained from (51): \( c_{ua} = (C^2 - 1)/(2\sin^2 \theta) \). As described in Ref. [14] and in Appendix D it corresponds to the news function of a string; we exclude \( C = 0 \). Function \( B \) given in Eq. (76) corresponds to translations along \( z \)-axis and \( t \)-axis in the spacetime of an infinite thin cosmic string described by the deficit angle \( 2\pi(1 - C) \), \( C \in (0, 1] \); in the weak-field limit, \( C = 1 - 4\mu \), where \( \mu \) is the mass per unit length of the string. (Notice that \( \mu \geq 0 \) for \( C \in (0, 1] \).) See Appendix D for details.

In Appendix E, part 1, the explicit form of electromagnetic field for general \( B \), Eq. (76), representing translations in asymptotically flat spacetimes with a straight string is given.

If \( C = 1 \), there is no string extending to infinity. Eq. (76) gives

\[
B = (b - a) \cos \theta + (b + a) .
\]

Then \( c_{ua} = 0 \) and from Eqs. (11) and (14) we get

\[
f^{(0)} = (b - a) \sin \theta , \quad A^{(0)} = \frac{1}{2} \left[ -(b - a) \cos \theta + b + a \right] .
\]

Consequently, regarding Eqs. (11), (14) and (18), we find the Killing vector field \( \eta^a \) to be asymptotically of the form

\[
\eta^a = [(b - a) \cos \theta + b + a , -(b - a) \cos \theta + O(r^{-2}) , (b - a) \sin \theta \frac{1}{r} + O(r^{-3}) , O(r^{-3})] . \tag{79}
\]

We thus see that the Killing vector field generates translations: with \( b - a = 0 \) this is the time translation, with \( a + b = 0 \) the translation along the \( z \)-axis.

Since there is no string extending to infinity, both functions \( c = c(\theta) \) and \( d = d(\theta) \) are independent of time and both news functions thus vanish; there is no radiation. By employing two transformations from the Bondi-Metzner-Sachs group, functions \( c \) and \( d \) can be transformed away.

For an illustration let us write down the asymptotic form of both gravitational and electromagnetic fields in the case of the Killing vector representing timelike translations in asymptotically flat spacetime without a string; we thus assume \( B = \text{const} \) and \( c = d = 0 \). The resulting axially symmetric stationary metric and electromagnetic field then have the asymptotic form

\[
ds^2 = \left( 1 - \frac{2M(\theta)}{r} + O(r^{-2}) \right) du^2 + 2(1 + O(r^{-4}))dudr + 2O(r^{-1})dud\theta + 2O(r^{-1}) \sin \theta dud\phi
- r^2 \left[ (1 + O(r^{-3}))d\theta^2 + (1 + O(r^{-3})) \sin^2 \theta d\phi^2 + 2O(r^{-3}) \sin \theta d\theta d\phi \right] ,
\]

\[
F_{01} = -\frac{e_1}{r^2} + (e_{1,\theta} + e_1 \cot \theta) \frac{1}{r^3} + O(r^{-4}) ,
F_{02} = O(r^{-2}) ,
F_{03} = O(r^{-2}) ,
F_{12} = \frac{e_1}{r^2} + O(r^{-3}) ,
F_{13} = f_1 \sin \theta \frac{1}{r^2} + O(r^{-3}) ,
F_{23} = -\mu_1 \sin \theta - (f_1 \sin \theta)_\theta \frac{1}{r} + O(r^{-2}) .
\]

Using the consequence of Einstein’s equations (A40) and assuming \( N_{ua} = 0 \) (which, as will be proven in our forthcoming publication, follows from further terms in \( r^{-k} \) in the Killing equations), we see that \( M_{,\theta} = 0 \), i.e., \( A^{(1)}(\theta) = \text{const} \). Constants \( M, e_1 \) and \( \mu_1 \) represent the total mass, electric charge and magnetic charge, respectively.

**IV. THE BOOST KILLING VECTOR**

In this section we shall find the form of the gravitational news functions \( c_{ua} \) and \( d_{ua} \) for the case of the boost Killing vector by expanding the Killing equations in further orders in \( r^{-1} \). We shall also obtain the form of the electromagnetic news functions \( X \) and \( Y \).

We thus assume the asymptotic form of the Killing vector to be given by Eq. (23). Expanding now the Killing equations (14) in higher orders of \( r^{-1} \) we obtain
This equation can be solved by introducing a new variable $\rho$.

Using now Eqs. (90)-(92) in Eq. (89) we find that additive function $f$.

Comparing Eqs. (83) and (87) we obtain an equation for $L$.

From Eqs. (81) and (85) we immediately get

$$A^{(0)} = \frac{u}{2} \cos \theta + \rho(\theta),$$

where $\rho(\theta)$ is an arbitrary function of $\theta$. Eq. (83) is just the $u$-derivative of Eq. (85). Eq. (87) implies

$$c_u = -u - \frac{f^{(0)}}{\sin \theta},$$

Using now Eqs. (90)-(92) in Eq. (89) we find that additive function $\rho(\theta)$ in Eq. (90) must vanish, i.e.

$$A^{(0)} = \frac{u}{2} \cos \theta.$$

Comparing Eqs. (83) and (87) we obtain an equation for $f^{(0)}$,

$$uf^{(0)^2} + \tan \theta f^{(0)^2} + 2u \sin \theta = 0.$$

This equation can be solved by introducing a new variable

$$w = \frac{\sin \theta}{u}.$$

Eq. (94) then becomes

$$\partial_u f^{(0)}(u, w) + 2uw = 0,$$

so that the general solution is

$$f^{(0)} = -u^2w + K(w) = -u \sin \theta + K(\sin \theta/u),$$

where $K(w)$ is an arbitrary function of $w$. Consequently, Eq. (91) leads to

$$c(u, \theta) = -\frac{K(w)}{uw},$$

where $w$ is given by Eq. (95). The news function thus reads

$$c_{u}(u, \theta) = \frac{K(w)}{u^2w}.$$
Before writing down the result for the second news function, let us compare the expression (99) with the news function given for the case of the vacuum boost-rotation symmetric spacetimes with the hypersurface orthogonal Killing vectors in Ref. [3]. There the news function is obtained in the form \( c_{u} = F(U/\sin \theta) / \sin^{2} \theta \), where \( F \) is an arbitrary function of \( U/\sin \theta \) and the flat-space retarded time \( \bar{U} \) satisfies the equation \( \bar{U}_{,u} \cot \theta = \bar{U} \cot \theta - \bar{U}_{,\theta} \).

This equation can be rewritten as the equation \( u \bar{U}_{,u} + \tan \theta \bar{U}_{,\theta} - \bar{U} = 0 \), the solution of which is \( \bar{U} = A(u)u, A \) being function of \( w = \sin \theta / u \). We can write \( c_{u} = F(A(w)/w) / \sin \theta = F(A(w)/w) / w^{2}u^{2} = K_{,w} / u^{2} \), where \( K_{,w} = F(A(w)/w) / w^{2} \). Therefore, our result (99) for the general form of the news function is in agreement with Eq. (59) given in Ref. [2].

The second news function, \( d_{u} \), can be found analogously. Eq. (84) is the \( u \)-derivative of Eq. (80) which gives

\[
d = -\frac{g^{(0)}}{2 \sin \theta}.
\]

Substituting this result and \( d_{u} \) from Eq. (84) into Eq. (83), we obtain the equation for \( g^{(0)} \),

\[
u g^{(0)} + \tan \theta g^{(0)} = 0 ,
\]

which in terms of variables \( u \) and \( w \) simply yields

\[
g^{(0)} = g^{(0)}w.
\]

The second news function is thus given by

\[
d_{u} (u, \theta) = \frac{g^{(0)}w}{2u^{2}} ,
\]

where \( g^{(0)} \) is an arbitrary function of \( w = \sin \theta / u \).

In order to obtain the mass aspect and the total mass at null infinity we have to expand the Killing equations in higher orders in \( r^{-1} \). Straightforward though rather lengthy calculations lead to the following system of equations (in which \( A^{(-1)}, A^{(0)}, B, f^{(-1)}, f^{(0)}, g^{(0)}, c \) and \( d \) are already known but are left unspecified for the sake of compactness):

\[
L_{\eta}g_{00} = 0 \quad (r^{-1}) : \quad A^{(1)}_{,u} - B_{,u} M + A^{(-1)}(c_{,u} + dd_{,u} + M) - f^{(-1)}M_{,\theta} - (d_{,\theta} + 2d \cot \theta)(g^{(0)}_{,u} - 2f^{(-1)}d_{,u}) - (c_{,\theta} + 2c \cot \theta)(f^{(0)}_{,u} - f^{(-1)}c_{,u}) = 0 ,
\]

\[
L_{\eta}g_{01} = 0 \quad (r^{-2}) : \quad (c_{,\theta} + 2c \cot \theta)(f^{(0)} - f^{(-1)}c) - f^{(-1)}(dd_{,\theta} + cc_{,\theta}) - \frac{1}{2} B_{,u} (c^{2} + d^{2}) - A^{(1)} - BM - B(dd_{,u} + cc_{,u}) = 0 ,
\]

\[
L_{\eta}g_{02} = 0 \quad (r^{-2}) : \quad f^{(-1)}(cc_{,u} + 2dd_{,u}) - f^{(-1)}(c_{,\theta} + 2c \cot \theta)_{,\theta} - f^{(-1)}(c_{,\theta} + 2c \cot \theta)
\]

\[
+ A^{(0)}_{,\theta} + \frac{1}{2} B_{,\theta} - f^{(-1)}c_{,\theta} - f^{(-1)}c_{,\theta} - 2dg^{(0)} = 0 ,
\]

\[
L_{\eta}g_{03} = 0 \quad (r^{-2}) : \quad -g^{(0)}c_{,u} + g^{(0)}_{,u} c - g^{(-1)}_{,u} + 2f^{(0)}d_{,u} + 2f^{(-1)}(2dc_{,u} - cd_{,u}) - f^{(-1)}(d_{,\theta} + 2d \cot \theta)_{,\theta} - f^{(-1)}(d_{,\theta} + 2d \cot \theta) \cot \theta = 0 ,
\]

\[
L_{\eta}g_{12} = 0 \quad (r^{-1}) : \quad 2f^{(1)} - -2B(c_{,\theta} + 2c \cot \theta) - f^{(-1)}(c^{2} + d^{2}) + 2dg^{(0)} = 0 ,
\]

\[
L_{\eta}g_{13} = 0 \quad (r^{-1}) : \quad g^{(1)} - B(d_{,\theta} + 2d \cot \theta) - d(f^{(0)} + f^{(-1)}c) = 0 ,
\]

\[
L_{\eta}g_{22} = 0 \quad (r^{-1}) : \quad 2d(g^{(0)}_{,\theta} - g^{(-1)}c \cot \theta) + A^{(-1)} \frac{3d^{2}}{2} + 4f^{(-1)}d^{2} \cot \theta + A^{(1)} + BM + f^{(1)}_{,\theta}
\]

\[
\quad + B(2dd_{,u} + cc_{,u}) - B(c_{,\theta} + 2c \cot \theta) = 0 ,
\]

\[
L_{\eta}g_{23} = 0 \quad (r^{-1}) : \quad -2A^{(0)}d + Bd + 4f^{(-1)}d(c_{,\theta} + c \cot \theta) - 4f^{(0)}d \cot \theta
\]

\[
\quad + g^{(0)}_{,\theta} c - g^{(-1)}(c_{,\theta} + c \cot \theta) - g^{(1)}_{,\theta} + g^{(1)}c \cot \theta
\]

\[
\quad + B(d_{,\theta} + 2d \cot \theta)_{,\theta} - B(d_{,\theta} + 2d \cot \theta) \cot \theta = 0 ,
\]

\[
L_{\eta}g_{33} = 0 \quad (r^{-1}) : \quad A^{(1)} - A^{(-1)}c + \frac{1}{2} Bc + \frac{1}{2} A^{(-1)}(c^{2} + d^{2}) + BM - f^{(-1)}d^{2} \cot \theta
\]

\[
\quad + 2f^{(-1)}d(d_{,\theta} + 2d \cot \theta) - B \cot \theta(c_{,\theta} + 2c \cot \theta) + 2B(c_{,\theta} + dd_{,u})
\]

\[
\quad + f^{(1)} \cot \theta + (c_{,\theta} + 2c \cot \theta)(-f^{(-1)} + 3f^{(-1)}c) - c \cot \theta(f^{(0)} + \frac{3}{2} f^{(-1)}c) = 0 .
\]
Substituting for $c$, $d$ and $g^{(0)}$ from Eqs. (98), (100) and (102) into Eqs. (108) and (109) we obtain
\[
 f^{(1)}(u) = \frac{1}{4\sin \theta} (-2K^2 + g^{(0)})^2 + u \cot^2 \theta (K_{,w} w + K),
\]
and
\[
 g^{(1)}(u) = \frac{u}{2} g^{(0)} - \frac{g^{(0)}K}{\sin \theta} + \frac{u}{2} \cot^2 \theta (g^{(0)}_{,w} w + g^{(0)}).
\]
Using the results for $f^{(-1)}$, $f^{(0)}$, $B$, ..., already obtained, we find that Eqs. (106), (107) and (111) are just identities; Eqs. (110), (112) become also identities if Eq. (105) is used.

Comparing the remaining Eq. (104) with the $u$-derivative of Eq. (105), we finally arrive at the equation for the mass aspect $M$:
\[
u M_{,u} + \tan \theta M_{,\theta} + 3M - u \tan \theta (c_{,u} + 2c \cot \theta)_{,u} = 0,
\]
where $c_{,u}$ is given by Eq. (103). This equation can be solved by the substitution (99) to yield
\[
 M(u, \theta) = \frac{1}{2u}(K_{,w} w + 2K_{,w}) + \frac{\mathcal{L}(w)}{u^3},
\]
where $\mathcal{L}(w)$ is an arbitrary function of $w$. Substituting $M$ into Eq. (105) we obtain function $A^{(1)}$ in the form
\[
 A^{(1)} = \cos \theta \left(-\frac{1}{2} K_{,w} w + 2K_{,w} + \frac{K}{w}\right) - \frac{\cos \theta}{8 \sin^2 \theta} \left(4K^2 + g^{(0)}\right)^2 + \frac{\cos \theta}{u^2} \mathcal{L}(w).
\]
The last expression for $M$ can be written as
\[
 M(u, \theta) = \frac{1}{2\sin \theta}(w^3 K_{,w})_{,w} + \frac{\mathcal{L}(w)}{u^3}.
\]
Comparing this form of $M(u, \theta)$ with the consequence of Einstein’s equations (4), we find
\[
 \mathcal{L}(w) = \frac{\lambda(w)}{w^3},
\]
where $\lambda$ has to satisfy the equation
\[
 \lambda(w)_{,w} = w^2(K_{,w}^2 + \frac{1}{4} g^{(0)}_{,w} + \mathcal{E}^2 + \mathcal{B}^2) - \frac{1}{2w}(w^3 K_{,w})_{,w}.
\]
Here, $K$ and $g^{(0)}$ determine the gravitational news functions, $c_{,u}$ and $d_{,u}$, by relations (99) and (103), $\mathcal{E}$ and $\mathcal{B}$ determine the electromagnetic news functions, $X$ and $Y$, given below by relations (129) and (130). Hence, solving the last equation for $\lambda$ for given $K$, $g^{(0)}$, $\mathcal{E}$ and $\mathcal{B}$, we find $\mathcal{L}(w)$ and thus the mass aspect $M(u, \theta)$ in the form of Eq. (118). The total mass at $\tilde{T}^+$ is then given by integrating Eq. (118) over the sphere:
\[
 m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta d\theta = \frac{1}{4} \int_0^\pi (w^2 K_{,w})_{,w} d\theta + \frac{1}{2} \int_0^\pi \frac{w \mathcal{L}}{u^3} d\theta.
\]
Substituting the expansions of the metric functions, Eq. (4), into the null tetrad, Eq. (11), and coefficients $A$, $B$, $\tilde{f}$ and $g$, Eqs. (53), (77), (98), (100), (102), (113), (114), (116), (117), into Eq. (13), we find the expansion of the boost Killing vector to be
\[
 \eta^\mu = \left[-u \cos \theta, \ r \cos \theta + u \cos \theta + \cos \theta \left(K_{,w} + \frac{K}{w}\right) \frac{1}{r} + O(r^{-2}),
\right.
\[
\left. - \sin \theta - u \sin \theta \frac{1}{r} + uc \sin \theta \frac{1}{r^2} + O(r^{-3}), \ \frac{u \eta^\mu}{r^2} + O(r^{-3})\right]
\]
It can easily be seen that the boost Killing vector generating Lorentz transformations along the $z$-axis in Minkowski space, $\eta^\mu_M = [z, \ 0, \ t, \ 0]$, in $\{u, r, \theta, \phi\}$ coordinates reads $\eta^\mu_M = [-u \cos \theta, \ r(1 + u/r) \cos \theta, \ -\sin \theta(1 + u/r), \ 0]$ so that both vectors differ only in terms proportional to $K$, $c$ and $d$. 

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We thank Vojtěch Pravda for discussions and help with calculations. We are grateful to Piotr Chruściel for reading
Using again variable $w$ given by Eq. (65), Eqs. (124) and (125) can easily be solved to yield
$$\begin{align*}
X(u, \theta) &= \frac{\mathcal{E}(w)}{u^2}, \\
Y(u, \theta) &= \frac{\mathcal{B}(w)}{u \sin \theta} = \frac{\mathcal{B}(w)}{wu^2} = \frac{\tilde{\mathcal{B}}(w)}{u^2},
\end{align*}$$
where functions $\mathcal{E}(w)$ and $\mathcal{B}(w)$ are arbitrary integration functions and $\tilde{\mathcal{B}}(w) = \mathcal{B}(w)/w$. Substituting these results into Eqs. (123) and (128) we find
$$\begin{align*}
\epsilon &= \frac{\mathcal{E}(w)}{u} \cot \theta + \frac{\mathcal{F}(w)}{u^2}, \\
\mu &= \frac{\mathcal{B}(w)}{u} \cot \theta + \frac{\mathcal{C}(w)}{u^2} = \frac{\tilde{\mathcal{B}}(w) - \tilde{\mathcal{C}}(w)}{u^2},
\end{align*}$$
where functions $\mathcal{F}(w)$ and $\mathcal{C}(w)$ are arbitrary integration functions and $\tilde{\mathcal{C}}(w) = \mathcal{C}(w)/w$. Then the solution of Eqs. (126) and (127) reads
$$\begin{align*}
e &= \frac{\mathcal{E}(w)}{2} - \frac{\mathcal{F}(w)}{u} \cot \theta + \frac{\mathcal{G}(w)}{u^2}, \\
f &= -\frac{\mathcal{B}(w)u}{2 \sin \theta} + \frac{\mathcal{C}(w)}{\sin \theta \cot \theta} + \frac{\mathcal{D}(w)}{u \sin \theta} = -\frac{\tilde{\mathcal{B}}(w)}{2} + \frac{\tilde{\mathcal{C}}(w) - \tilde{\mathcal{D}}(w)}{u^2},
\end{align*}$$
where $\mathcal{G}(w)$ and $\mathcal{D}(w)$ are again integration functions and $\tilde{\mathcal{D}}(w) = \mathcal{D}(w)/w$. Comparing Eqs. (123) and (128) with Maxwell equations (1) and (2), and using previous results, we get equations for $\mathcal{F}(w)$ and $\mathcal{C}(w)$: $\mathcal{F} + 2\mathcal{F} = 0$ and $\mathcal{C} - \mathcal{w} + \mathcal{C} = 0$. Their solutions are
$$\begin{align*}
\mathcal{F} &= \frac{l}{w^2}, & l &= \text{const}, \\
\mathcal{C} &= \frac{p}{w}, & p &= \text{const}.
\end{align*}$$
Since $w = \sin \theta/u$, the regularity of the electromagnetic field implies $l = p = 0$.

Finally, comparing Maxwell equations (3) and (4) with Eqs. (126) and (127), we get restrictions on functions $\mathcal{D}(w)$ and $\mathcal{G}(w)$ in the form
$$\begin{align*}
\mathcal{E}_{\mathcal{w}} w - \mathcal{E} + 2w^2(\mathcal{G}_{\mathcal{w}} w + 2\mathcal{G}) + 2w\mathcal{K} + \mathcal{B} g^{(0)} = 0, \\
-(\mathcal{B}_{\mathcal{w}} w - 2\mathcal{B}) + 2w^2(\mathcal{D}_{\mathcal{w}} w + \mathcal{D}) - 2w\mathcal{B} + w^2 g^{(0)} \mathcal{E} = 0.
\end{align*}$$

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APPENDIX A: THE EINSTEIN-MAXWELL EQUATIONS
IN THE BONDI-SACHS COORDINATES

In our convention (see the end of Introduction) the Einstein-Maxwell equations read

\[ K_{\mu\nu} \equiv R_{\mu\nu} + 8\pi T_{\mu\nu} = 0 , \] (A1)

where the electromagnetic stress tensor is given by

\[ T_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\sigma} F_{\nu}^\sigma - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho}) , \] (A2)

and \( T^{\mu}_{\mu} = R = 0 \); the Maxwell equations are

\[ G_{\mu\nu\lambda} \equiv F[\mu,\lambda]_{\text{cykl.}} = 0 , \] \( \lambda \)

and

\[ J^\mu \equiv F^{\mu \nu} ;_{\nu} = 0 . \] (A4)

In paper [11] the tensor \( E_{\mu\nu} = 4\pi T_{\mu\nu} \) is introduced; however, the field equations (A1) are erroneously written without the factor 2 at \( E_{\mu\nu} \). This can be "cured" by considering \( F_{\mu\nu} = \sqrt{2} F_{\mu\nu\text{(real)}} \). Following [11], eighteen equations (A1), (A3) and (A4) can be divided into twelve main equations, one trivial equation, and five supplementary conditions. The main equations are

\[ K_{11} = K_{12} = K_{13} = K_{22} = K_{23} = K_{33} = 0 , \] (A5)

\[ G_{123} = G_{012} = G_{013} = 0 , \]

\[ J^0 = J^2 = J^3 = 0 . \] (A6)

If these are satisfied, then also

\[ K_{01} = 0 . \] (A6)

The only further equations to be satisfied are the supplementary conditions

\[ (r^2 K_{00})_{,r} = (r^2 K_{02})_{,r} = (r^2 K_{03})_{,r} = 0 , \] (A7)

\[ G_{023} = 0 , \]

\[ J^{1}_{,r} = 0 , \]

which imply that \( r^2 (K_{00}, K_{02}, K_{03}) \), \( G_{023} \), and \( J^1 \) are functions of \( u, \theta, \phi \) only.

Since in [11] all Einstein equations contain errors due to the factor 2 mentioned above, and equations (5), (6), (7), (14), (15), (17), (18), (19) in [11] contain additional errors, we write all field equations explicitly here (we checked them by using MAPLE V). Starting from the metric of the form (3) we find that the main field equations are (denoting \( ch = \cosh 2\delta, sh = \sinh 2\delta \)

\[ \frac{r}{4} K_{11} = 0 : \] (A8)

\[ \beta_{,r} = \frac{r}{2} \left( \gamma_r^2 ch^2 + \delta_r^2 \right) + \frac{1}{2r} \left( e^{-2\gamma} ch F_{12}^2 + e^{2\gamma} ch F_{13}^2 \csc^2 \theta - 2 sh F_{12} F_{13} \csc \theta \right) , \]

\[ 2r^2 K_{12} = 0 : \] (A9)

\[ \left\{ r^4 e^{-2\beta} (e^{2\gamma} U_r ch + W_r sh) \right\}_{,r} = \]

\[ = 2r^2 \left\{ \beta_{,r\theta} - \frac{2}{r} \beta_{,r} + 2\delta_{,r} \delta_{,\theta} - 4\gamma_{,r} \delta_{,\theta} \sinh ch - (\gamma_{,r}\theta + 2\gamma_{,r} \cot \theta - 2\gamma_{,r} \gamma_{,\theta}) ch^2 \right\} \]

\[ + 2r^2 e^{2\gamma} \csc \theta \left\{ - \delta_{,r\phi} - 2\delta_{,r} \delta_{,\phi} + 2\gamma_{,r} \delta_{,\phi} (1 + 2 sh^2) + (\gamma_{,r\phi} + 2\gamma_{,r} \gamma_{,\phi}) shc h \right\} \]

\[ - 4r^2 e^{-2\beta} (F_{01} - UF_{12} - W F_{13} \csc \theta) F_{12} - 4 (sh F_{12} - e^{2\gamma} ch F_{13} \csc \theta) F_{23} \csc \theta , \]

\[ \frac{2r^2}{\sin \theta} K_{13} = 0 : \] (A10)

\[ \left\{ r^4 e^{-2\beta} (e^{-2\gamma} W_r ch + U_r sh) \right\}_{,r} = 2r^2 e^{-2\gamma} \left\{ - \delta_{,r\theta} + 2\delta_{,r} \gamma_{,\theta} - 2\delta_{,r} \cot \theta \right. \]

\[ - (\gamma_{,r\theta} + 2\gamma_{,r} \cot \theta - 2\gamma_{,r} \gamma_{,\theta}) shc h - 2\gamma_{,r} \delta_{,\theta} (1 + 2 sh^2) \right\} \]
\[
+ 2r^2 \csc \theta \left\{ \beta_{,\phi} - \frac{2}{r} \beta_{,\phi} + 2\delta_{,\phi} \delta_{,\phi} + 4\gamma_{,r} \delta_{,\phi} \sinh \theta + (\gamma_{,r,\phi} + 2\gamma_{,r} \gamma_{,\phi}) \cosh \theta \right\} \\
- 4r^2 e^{-2\beta} (F_{01} - U F_{12} - W F_{13} \csc \theta) F_{13} \csc \theta - 4(e^{-2\gamma} \cosh F_{12} - \sinh F_{13} \csc \theta) F_{23} \csc \theta ,
\]
\[
\frac{1}{2} e^{2\beta} \left\{ (e^{-2\gamma} K_{22} + e^{2\gamma} \csc^2 \theta K_{33}) \csc \theta - 2 \csc \theta K_{23} \csc \theta \right\} = 0 \quad \text{(A11)}
\]
\[
V_{,r} = 2 e^{2\beta} \csc \theta \left\{ (\beta_{,\phi} + \beta_{,\phi} + 2\delta_{,\phi} \delta_{,\phi}) \sinh \theta + (\delta_{,\phi} + \delta_{,\phi} \cot \theta + \delta_{,\phi} \gamma_{,\phi} - \gamma_{,\phi} \delta_{,\phi} + \delta_{,\phi} \beta_{,\phi} + \beta_{,\phi} \delta_{,\phi}) \csc \theta \right\} \\
- e^{2(\beta - \gamma)} \left\{ \beta_{,\phi} + \beta_{,\phi} + 2\gamma_{,\phi} + 2\delta_{,\phi} \gamma_{,\phi} + 2\beta_{,\phi} \gamma_{,\phi} \csc \gamma + (\delta_{,\phi} \delta_{,\phi} + 2\delta_{,\phi} \delta_{,\phi} + 4\gamma_{,\phi} \delta_{,\phi}) \csc \theta \right\} \csc^2 \theta \\
- \frac{r^3}{4} e^{-2\beta} \left\{ (e^{2\gamma} U_{,r}^2 + e^{-2\gamma} W_{,r}^2) \csc \theta + 2U_{,r} \csc \theta \right\} \\
+ \frac{r}{2} (r U_{,r} + r U_{,\phi} \cot \theta + 4U_{,\phi} + 4U \cot \theta) + \frac{r}{2} \csc \theta (r W_{,r} + 4W_{,\phi}) \\
- \frac{1}{4} e^{2\beta} \left\{ (r^2 e^{-2\beta} (F_{01} - U F_{12} - W F_{13} \csc \theta))^2 + (F_{23} \csc \theta)^2 \right\} ,
\]
\[
\frac{1}{4r} e^{2\beta} \left\{ (e^{-2\gamma} K_{22} + e^{2\gamma} \csc^2 \theta) \csc \theta \right\} = 0 \quad \text{(A12)}
\]
\[
(r \gamma)_{,rr} = 2r (\gamma_{,rr} \delta_{,r} + \gamma_{,r} \gamma_{,r} \csc \theta) \csc \theta = \frac{1}{2} (\gamma_{,rr} \csc \theta + \gamma_{,r} V_{,r} + \gamma_{,r} V + \frac{1}{r} \gamma_{,r} V) \csc \theta + 2\gamma_{,r} \delta_{,r} V \csc \theta \\
+ \frac{r^3}{8} e^{-2\beta} (e^{2\gamma} U_{,r}^2 - e^{-2\gamma} W_{,r}^2) + \frac{1}{2r} e^{2(\beta - \gamma)} (\beta_{,\phi} + \beta_{,\phi} - \beta_{,\phi} \cot \theta) \\
- \frac{1}{2r} e^{2(\beta + \gamma)} (\beta_{,\phi} + \beta_{,\phi} \csc^2 \theta + \frac{1}{r} e^{2\beta} (\beta_{,\phi} \delta_{,\phi} - \beta_{,\phi} \delta_{,\phi}) \csc \theta \\
+ \frac{r}{4} e^{2\gamma} \csc \theta \left\{ (U_{,r} + \frac{2}{r} U_{,\phi}) \csc \theta + 4\delta_{,r} U_{,\phi} \csc \theta \right\} \\
- \frac{r}{4} e^{-2\gamma} \left\{ (W_{,r} - W_{,\phi} \csc \theta + \frac{2}{r} W_{,\phi} - \frac{2}{r} W \csc \theta) \csc \theta + 4\delta_{,r} (W_{,\phi} \csc \theta) \csc \theta \right\} \\
- \frac{1}{4} (r U_{,\phi} + 2U_{,\phi} - r U_{,\phi} \csc \theta - 2U \csc \theta + 4\gamma_{,\phi} U \\
+ 4r \gamma_{,\phi} U_{,r} + 2r \gamma_{,\phi} U_{,\phi} + 2r \gamma_{,\phi} U \csc \theta) \csc \theta \\
+ 2r \delta_{,r} U_{,\phi} + 2r \delta_{,r} U_{,\phi} - 2r \gamma_{,\phi} U_{,\phi} \csc \theta \\
+ 2r \gamma_{,\phi} U_{,\phi} \csc \theta + 2r \gamma_{,\phi} W_{,\phi} \csc \theta \\
+ r \csc \theta (\delta_{,r} W_{,\phi} - 2\delta_{,r} \gamma_{,\phi} W - 2r \gamma_{,\phi} W_{,\phi}) \csc \theta \\
- \frac{1}{2r} \left\{ 2(e^{-2\gamma} F_{02} F_{12} - e^{2\gamma} F_{03} F_{13} \csc^2 \theta) - \frac{1}{r} V (e^{-2\gamma} F_{12}^2 - e^{2\gamma} F_{13}^2 \csc^2 \theta) \\
- 2(e^{-2\gamma} W F_{12} + e^{2\gamma} U F_{13} \csc \theta) F_{23} \csc \theta \right\} ,
\]
\[
\frac{1}{4r} e^{2\beta} \left\{ (e^{-2\gamma} K_{22} + e^{2\gamma} \csc \theta K_{33}) \csc \theta - 2 \csc \theta K_{23} \csc \theta \right\} = 0 \quad \text{(A13)}
\]
\[
(r \delta)_{,rr} = 2r \gamma_{,r} \gamma_{,r} \csc \theta \csc \theta = \frac{1}{2} (\delta_{,r} V_{,r} + \delta_{,r} V + \frac{1}{r} \delta_{,r} V - 2\gamma_{,r}^2 V \csc \theta) \\
+ \frac{r^3}{8} e^{-2\beta} \left\{ (e^{2\gamma} U_{,r}^2 + e^{-2\gamma} W_{,r}^2) \csc \theta + 2U_{,r} \csc \theta \right\} \\
- \frac{1}{2r} e^{2(\beta - \gamma)} (\beta_{,\phi} + \beta_{,\phi} \csc \gamma + \frac{1}{r} e^{2(\beta + \gamma)} (\beta_{,\phi} + \beta_{,\phi} \csc \gamma) \csc^2 \theta \\
- \frac{1}{r} e^{2\beta} (-\beta_{,\phi} \beta_{,\phi} + \beta_{,\phi} \beta_{,\phi} \csc \csc \theta + \beta_{,\phi} \gamma_{,\phi} - \beta_{,\phi} \gamma_{,\phi}) \csc \theta \\
- \frac{r}{2} \left\{ 2\delta_{,\phi} U + 2 \frac{\delta_{,\phi} U + \delta_{,\phi} U_{,r} + \delta_{,\phi} U_{,r} + \delta_{,\phi} U \cot \theta - 2\gamma_{,r} (U_{,\phi} - U \csc \theta + 2\gamma_{,\phi} U) \csc \theta \right\} 
\]
the absence of logarithmic terms. (We thus do not consider polyhomogeneous null infinity of Ref. [19].) Then twelve

where

Since we assume the axial symmetry, hereafter we put all derivatives \( \partial / \partial \phi \) equal to zero.

Now following [11], we assume functions \( \gamma, \delta, F_{12} \) and \( F_{13} \) to have the expansions at large \( r \) on a hypersurface \( u = u_0 \) of the form

\[
\gamma = \frac{c}{r} + \left( C - \frac{1}{6} c^3 - \frac{3}{2} a^2 \right) \frac{1}{r^3} + \frac{D}{r^4} + O(r^{-5}) ,
\]

\[
\delta = \frac{d}{r} + \left( H - \frac{1}{6} a^3 + \frac{1}{2} c^2 d \right) \frac{1}{r^3} + \frac{K}{r^4} + O(r^{-5}) ,
\]

\[
F_{12} = \frac{e}{r^2} + \left( 2E + ec + f d \right) \frac{1}{r^3} + O(r^{-4}) ,
\]

\[
F_{13} = \left( \frac{f}{r^2} + \left( 2F + ed - fc \right) \frac{1}{r^3} + O(r^{-4}) \right) \sin \theta ,
\]

where \( c, C, d, \ldots \) are prescribed functions of \( \theta \) on \( u = u_0 \). This corresponds to the outgoing radiation condition and the absence of logarithmic terms. (We thus do not consider polyhomogeneous null infinity of Ref. [14].) Then twelve
main equations determine the other eight functions $\beta, U, V, W, F_{01}, F_{02}, F_{03}$ and $u$-derivatives of the four prescribed functions $\gamma, \delta, F_{12}$ and $F_{13}$ on the hypersurface $u = u_0$. Denoting by $N, \ P, M, \epsilon, \mu, \ X$ and $Y$ arbitrary functions of $\theta$ on $u = u_0$, we find

$$\beta = -\frac{1}{4}(c^2 + d^2)\frac{1}{r^2} + O(r^{-4}),$$  \hspace{1cm} (A24)$$

$$U = -(c_{,\theta} + 2c \cot \theta)\frac{1}{r^2} + \left(2N + 3(c_{,\theta} + dd_{,\theta}) + 4(c^2 + d^2) \cot \theta \right)\frac{1}{r^3},$$

$$+ \frac{1}{2} \left(3(C_{,\theta} + 2C \cot \theta) - 6(cN + dP) - 4(2c^2c_{,\theta} + cdd_{,\theta} + c\theta d^2) \right)\frac{1}{r^4},$$

$$+ \frac{1}{2} \left(-8c(c^2 + d^2) \cot \theta + 2(ee - f\mu) \right)\frac{1}{r^4} + O(r^{-5}),$$  \hspace{1cm} (A25)$$

$$W = -(d_{,\theta} + 2d \cot \theta)\frac{1}{r^2} + \left(2P + 2(c_{,\theta} d - cd_{,\theta}) \right)\frac{1}{r^3},$$

$$+ \frac{1}{2} \left(3(H_{,\theta} + 2H \cot \theta) + (cP - dN) - 4(2d^2d_{,\theta} + cdc_{,\theta} + c^2d_{,\theta}) \right)\frac{1}{r^4},$$

$$+ \frac{1}{2} \left(-8d(c^2 + d^2) \cot \theta + 2(\mu e + e\theta) \right)\frac{1}{r^4} + O(r^{-5}),$$  \hspace{1cm} (A26)$$

$$V = r - 2M - \left(N_{,\theta} + N \cot \theta - \frac{1}{2}(c^2 + d^2) \right)\frac{1}{r} - \left(-(c_{,\theta} + 2c \cot \theta)^2 - (d_{,\theta} + 2d \cot \theta)^2 - (e^2 + \mu^2) \right)\frac{1}{r},$$

$$- \frac{1}{2} \left(C_{,\theta\theta} + 3C_{,\theta} \cot \theta - 2C + 6N(c_{,\theta} + 2c \cot \theta) + 6P(d_{,\theta} + 2d \cot \theta) \right)\frac{1}{r^2},$$

$$- \frac{1}{2} \left(4c_{,\theta}c_{,\theta} - 3c_{,\theta} + ddd_{,\theta} - cd_{,\theta} \right) + 8(2c_{,\theta} d^2 + 3c^2c_{,\theta} + cd_{,\theta} \cot \theta) \frac{1}{r^2},$$

$$- \frac{1}{2} \left(16c(c^2 + d^2) \cot \theta + 2c(ee + e \cot \theta) - 2\mu(f_{,\theta} + f \cot \theta) \right)\frac{1}{r^2} + O(r^{-3}),$$  \hspace{1cm} (A27)$$

$$F_{01} = \frac{e}{r^2} + \left(e_{,\theta} + e \cot \theta \right)\frac{1}{r^3} + O(r^{-4}),$$  \hspace{1cm} (A28)$$

$$F_{23} = \left(-\mu - (f_{,\theta} + f \cot \theta) \right)\frac{1}{r} + O(r^{-2}) \sin \theta,$$  \hspace{1cm} (A29)$$

$$F_{02} = X + \left(e_{,\theta} - e_{,u} \right)\frac{1}{r} - \left(E + \frac{1}{2}(ec + fd)_{,u} + \frac{1}{2}(e_{,\theta} + e \cot \theta)_{,u} \right)\frac{1}{r^2} + O(r^{-3}),$$  \hspace{1cm} (A30)$$

$$F_{03} = \left(Y - \frac{f_{,u}}{r} - \left(\left[F + \frac{1}{2}(ed - fc)_{,u} \right)\right)\frac{1}{r^2} + O(r^{-3}) \right) \sin \theta,$$  \hspace{1cm} (A31)$$

and

$$2e_{,u} = e_{,\theta} - 2(cX + dY),$$  \hspace{1cm} (A32)$$

$$2f_{,u} = -\mu_{,\theta} - 2(-cY + dX),$$  \hspace{1cm} (A33)$$

$$4E_{,u} = - \left(\frac{\partial}{\partial \theta} + 2 \cot \theta \right) \left(\frac{\partial}{\partial \theta} \cot \theta \right) e + 2(ce + d\mu) \right),$$  \hspace{1cm} (A34)$$

$$4F_{,u} = - \left(\frac{\partial}{\partial \theta} + 2 \cot \theta \right) \left(\frac{\partial}{\partial \theta} \cot \theta \right) f + 2(ce - c\mu) \right),$$  \hspace{1cm} (A35)$$

$$4C_{,u} = 2(c^2 - d^2)c_{,u} + 4dcd_{,u} + 2cM + d \left(\frac{\partial}{\partial \theta} + 2 \cot \theta \right) (d_{,\theta} + 2d \cot \theta) \right) \left(\frac{\partial}{\partial \theta} \right) N + 2(eX - fY) \right),$$  \hspace{1cm} (A36)$$

$$4H_{,u} = -2c(c^2 - d^2)d_{,u} + 4dcd_{,u} + 2dM - c \left(\frac{\partial}{\partial \theta} + 2 \cot \theta \right) (d_{,\theta} + 2d \cot \theta) \right) \left(\frac{\partial}{\partial \theta} \right) P + 2(eY + fX) \right),$$  \hspace{1cm} (A37)$$

$$4D_{,u} = \left(\frac{\partial}{\partial \theta} + 3 \cot \theta \right) \left[ \left(\frac{\partial}{\partial \theta} - 2 \cot \theta \right) C + 2(cN - dP) \right] \]
In order to simplify the last equations one can, following \[11\], introduce ten new quantities in terms of which equations (A32)-(A39) become

\[
\begin{align*}
\frac{2}{3} \left( \frac{\partial}{\partial \theta} - \cot \theta \right) e + \frac{2}{3} \mu \left( \frac{\partial}{\partial \theta} - \cot \theta \right) f \\
= \frac{4}{3} \epsilon (\epsilon^2 + \mu^2) + \frac{8}{3} (EX - YF) ,
\end{align*}
\]

\[
4K_{\mu u} = \left( \frac{\partial}{\partial \theta} + 3 \cot \theta \right) \left[ - \left( \frac{\partial}{\partial \theta} - 2 \cot \theta \right) H + 2(cP + dN) \right] \]

\[
= \frac{2}{3} \left( \frac{\partial}{\partial \theta} - \cot \theta \right) f - \frac{2}{3} \mu \left( \frac{\partial}{\partial \theta} - \cot \theta \right) e \\
= \frac{4}{3} \epsilon (\epsilon^2 + \mu^2) + \frac{8}{3} (FX + YE) .
\]

In order to simplify the last equations one can, following \[11\], introduce ten new quantities

\[
\begin{align*}
c^* &= c + id , \\
X^* &= X + iY , \\
C^* &= C + iH , \\
e^* &= e + if , \\
D^* &= D + iK , \\
E^* &= E + iF , \\
N^* &= N + iP , \\
M^* &= M + i \left( \frac{\partial}{\partial \theta} + \cot \theta \right) \left( d,\theta + 2d \cot \theta \right) , \\
\mathcal{L}_p &= - \left( \frac{\partial}{\partial \theta} - p \cot \theta \right) , \\
p &= -3, -2, 2 ,
\end{align*}
\]

in terms of which equations (A32)-(A39) become

\[
\begin{align*}
2e_{\mu u}^* &= - \mathcal{L}_0 e^* - 2c^* X^* , \\
4E_{\mu u}^* &= - \mathcal{L}_{-2}(L_1 e^* - 2c^* e^*) , \\
4C_{\mu u}^* &= 2c^* e^* + 2c^* M^* + L_1 N^* + 2e^* X^* , \\
4D_{\mu u}^* &= - \mathcal{L}_{-3}(L_2 C^* + 2c^* N^*) + \frac{2}{3} \epsilon \mathcal{L}_1 e^* - \frac{4}{3} c^* e^* + \frac{8}{3} E^* X^* .
\end{align*}
\]

The supplementary conditions (A7) have the form

\[
\begin{align*}
M_{\mu u} &= - (c_{\mu u}^2 + d_{\mu u}^2) - (X^2 + Y^2) + \frac{1}{2} (c_{,\theta \theta} + 3 c_{,\theta} \cot \theta - 2c)_{\mu u} , \\
3N_{\mu u} &= - M_{,\theta} - 2c \left( \frac{\partial}{\partial \theta} + 2 \cot \theta \right) c_{\mu u} - 2d \left( \frac{\partial}{\partial \theta} + 2 \cot \theta \right) d_{\mu u} \\
&\ - (c_{\mu u, \theta})_\theta - (d_{\mu u, \theta})_\theta - 2(\epsilon X + \mu Y) , \\
3P_{\mu u} &= \frac{1}{2} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} + \cot \theta \right) (d,\theta + 2d \cot \theta) + 2c \left( \frac{\partial}{\partial \theta} + 2 \cot \theta \right) d_{\mu u} - 2d \left( \frac{\partial}{\partial \theta} + 2 \cot \theta \right) c_{\mu u} \\
&\ + (c_{\mu u, \theta})_\theta - (d_{\mu u, \theta})_\theta - 2(\epsilon Y - \mu X) , \\
\epsilon_{\mu u} &= - X_{,\theta} - X \cot \theta , \\
\mu_{\mu u} &= - Y_{,\theta} - Y \cot \theta ,
\end{align*}
\]

which in terms of quantities (A40) simplify to

\[
\begin{align*}
M^*_{\mu u} &= - c_{\mu u} c^* - X^* X^* + \frac{1}{2} \mathcal{L}_{-1} \mathcal{L}_{-2} c_{\mu u} , \\
3N^*_{\mu u} &= \mathcal{L}_0 M^* + 2c^* \mathcal{L}_{-2} c_{\mu u} + \mathcal{L}_0 (c^* e^*)_{\mu u} - 2e^* X^* , \\
e^*_{\mu u} &= \mathcal{L}_{-1} X^* .
\end{align*}
\]

The structure of the field equations is thus following. Nine functions \( \gamma, \delta, F_{12}, F_{13}, M, N, P, \epsilon \) and \( \mu \) are prescribed on an "initial" hypersurface \( u = u_0 \), and four functions \( c_{\mu u}, d_{\mu u}, X \) and \( Y \) have to be prescribed for all \( u \). Then the time evolution of gravitational and electromagnetic fields is fully determined. Functions \( c_{\mu u} \) and \( d_{\mu u} \) are the well-known gravitational news functions, functions \( X \) and \( Y \) – the electromagnetic news functions. Non-vanishing news functions \( d_{\mu u} \) and \( Y \) correspond to a "rotation" of a radiating source. In vacuum spacetimes with hypersurface orthogonal Killing vector \( \partial / \partial \phi \) we find \( d = 0 \).

The total mass of the system at a given retarded time \( u \) is defined by
\[ m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta d\theta . \]  

(A53)

(Notice that the definition of the total mass given in \[ \text{[1]} \] is different; for example, in the Schwarzschild case it gives \( 4\pi m_{\text{Schw.}} \)). The time derivative of the total mass is equal to

\[ m_{,u} = -\frac{1}{2} \int_0^\pi (e^{,u} c^{,u} + X^* X^*) \sin \theta d\theta \]

(A54)

\[ = -\frac{1}{2} \int_0^\pi (e^{,u} + d_a^2 + X^2 + Y^2) \sin \theta d\theta \leq 0 . \]

Hence, if any of the news functions is non-vanishing, the waves are radiated out and the mass of the system necessarily decreases.

There exist two quantities which are always conserved:

\[ \frac{\partial}{\partial u} \int_0^\pi \frac{1}{2} e^* \sin \theta d\theta = 0 . \]  

(A55)

These are "the electric" and "magnetic" charges of the source.

The rate of loss of the electromagnetic energy radiated out from the system is given by

\[ \frac{1}{2} \frac{\partial}{\partial u} \int_0^\pi X^* X^* \sin \theta d\theta , \]

(A56)

which also implies the loss of mass as seen from Eq. \([A54]\).

**APPENDIX B: THE KILLING EQUATION**

For an illustration we write down the expression for the Lie derivative component \( \mathcal{L}_\eta g_{00} = 0 \), where the metric has the form \([3]\) and the vector \( \eta \) is given by Eqs. \([11]\), \([13]\):

\[
\mathcal{L}_\eta g_{00} = -\frac{1}{2r^3(2ch^2 - 1)^2} e^{-3\gamma - 2\beta}
\]

\[
\left\{ 2r^2(2ch^2 - 1)e^{2(\beta + \gamma)} \left[ -\sqrt{2ch^2 - 1} e^\gamma (2A_{,u} r + B_{,u} V e^{2\beta}) 
- 2e^{2\gamma} (2\tilde{g}_{,u} shch + \tilde{f}_{,u}) U r^2 - (2ch^2 - 1) 2r^2 \tilde{g}_{,u} W \right]
+ \tilde{g} r^4 e^{2(\beta + \gamma)} \left[ W (2ch^2 - 1) + 2U shch e^{2\gamma} \right] [2\delta_{,u} shch - (2ch^2 - 1) \gamma_{,u}]
+ \tilde{f} 2r^2 e^{2\beta} \left[ 2r^3 W (2ch^2 - 1)^2 \left( 2W \delta_{,u} shch + (2ch^2 - 1) (W_{,u} + \gamma_{,u}) W \right) - e^{2(\beta + \gamma)} (2ch^2 - 1)^2 (V_{,u} + 2\beta_{,u} V) 
+ 4r^3 e^{2\gamma} (2ch^2 - 1) \left( (2ch^2 - 1)^2 UW \delta_{,u} + (2ch^2 - 1) shch (U_{,u} W + UW_{,u} + 2W_{,u}) + W\delta_{,u} \right)
+ 2U r^3 e^{4\gamma} \left( (2ch^2 - 1)^3 (U_{,u} + U\gamma_{,u}) + (2ch^2 - 1) (2ch^2 - 1)^2 - 2\gamma_{,u} \right)
+ 2U \delta_{,u} shch (2ch^2 - 1)^2 + 2\delta_{,u} shch \right]
\]

+ 2Ac^2 e^{2(\gamma - 2 - 1)^2} \left( 2r^2 W \left( (2ch^2 - 1)(-rW_{,u} + W + r W_{,u}) + 2rW \delta_{,u} shch \right)
+ 2r^2 U e^{4\gamma} \left( (2ch^2 - 1) (rU_{,u} + U + rU_{,u}) + 2rU \delta_{,u} shch \right) + e^{2(\beta + \gamma)} (-V_{,u} + \frac{v}{r} - 2V\beta_{,u} + 4r\gamma_{,u})
+ 4r^2 c_{2\beta} \left( (2ch^2 - 1) rUW \delta_{,u} + shch (rU_{,u} W + 2UW + rUW_{,u}) \right)\right]
+ 2Be^{2\beta + \gamma}(2ch^2 - 1)^2 \left[ e^{2(\beta + \gamma)} \left( V (V_{,u} - \frac{v}{r} + 2V\beta_{,u}) - 4V r^2 \beta_{,u} - 2Ur^2 (2\beta_{,u} V + V_{,u}) \right)
+ 2r^4 e^{4\gamma} \left( (2ch^2 - 1) (2U^2 (U_{,u} + U\gamma_{,u} + \gamma_{,u}) - VU (\frac{1}{r} U + U_{,u} + U\gamma_{,u} )) + 2U^2 r shch (2U \delta_{,u} - \frac{v}{r} \delta_{,u} + 2\delta_{,u} \right) \right] \right\}
\]
The Killing equation \( L_{\eta}g_{00} \) is the lengthiest among all Killing equations \( L_{\eta}g_{\mu\nu} = 0 \).

**APPENDIX C: CORRECTIONS OF KILLING EQUATIONS GIVEN IN REF. [2]**

In paper [2] a number of misprints appear, and there is also a sign error which does not change the conclusions of the paper but makes wrong Eqs. (53)-(57), including the pathological solution given by Eq. (57) for \( \eta^u \). Below we give the correct forms of all equations from \([2]\) in which a misprint appears. First we give the correct forms of translations. The second possibility leads to

\[
\text{In the main text in }[2] \text{ the following equations contain misprints and are here corrected:}
\]

- \( B_{,u} + 2A_{(0),u} = 0 \),
- \( f^{(1),\theta} + B_{,\theta} c + A^{(1)} - c(A^{(0)} - \frac{1}{2}B) + BM - B(c_{,\theta} + 2c \cot \theta),_\theta = 0 \),
- \( f^{(1)} \cot \theta + B_{,\theta} (c_{,\theta} + c \cot \theta) + c(A^{(0)} - \frac{1}{2}B) + A^{(1)} + BM - B c \cot \theta(c_{,\theta} + 2c \cot \theta) \),
- \( [B_{,\theta} \cot \theta + Bc_{,u} - (A^{(0)} - \frac{1}{2}B)c] = 0 \).

As with Eq. (48) in \([2]\) (which contains a sign error), the correct equation \([28]\) implies that either \( c = 0 \) or the expression in the square brackets has to vanish. The first possibility gives vanishing news function and leads to translations. The second possibility leads to

\[
A^{(0)} = \frac{1}{2}(B_{,\theta} + B_{,\theta} \cot \theta + B) ,
\]

i.e. to Eq. (50) of the present paper. As it is shown below equation (50) in the main text, this case corresponds to the translational Killing vector under the presence of a straight cosmic string along \( z \)-axis.

**APPENDIX D: TRANSLATIONS IN SPACETIMES WITH A STRAIGHT COSMIC STRING**

In this Appendix we show that function \( B \) in Eq. (76) corresponds to translations along \( z \)-axis and \( t \)-axis in the spacetime of an infinite thin cosmic string. Let us first recall some results from Ref. [14]. The metric outside a straight non-rotating cosmic string along \( z \)-axis can be written in cylindrical coordinates as

\[
ds^2 = dt^2 - d\rho^2 - dz^2 - C^2 \rho^2 d\phi^2 ,
\]
where \( \phi \in [0, 2\pi) \) and \( C \in (0, 1] \) is a constant. (Eq. [D1] represents also the asymptotic form of the metric corresponding to a spatially bounded system and the cosmic string along \( z \)-axis.) Introducing spherical flat-space coordinates \( \{ R, \vartheta, \phi \} \) by \( \rho = R \sin \vartheta, z = R \cos \vartheta, \phi = \phi \), and flat-space retarded time \( U = t - R \), we get

\[
ds^2 = dU^2 + 2dUdR - R^2 (d\vartheta^2 + C^2 \sin^2 \vartheta d\phi^2) .
\] (D2)

We can now go over to Bondi’s coordinates \( \{ u, r, \theta, \phi \} \) – in which the metric has asymptotically Bondi’s form – by assuming expansions

\[
U = \varpi (u, \theta) + O(r^{-1}) ,
\]
\[
R = q(u, \theta) r + O(r^0) ,
\]
\[
\vartheta = \varphi (u, \theta) + O(r^{-1}) .
\] (D3)

Comparing then the resulting metric with the general form of Bondi’s metric one obtains the expressions for functions \( \varpi, q, \varphi, \ldots \) (see [14] for details) and the expression for the news function \( c_{\varpi u} \). Here we need only the following results:

\[
q(\theta) = \frac{\sin \theta}{C \sin \theta} , \quad \varphi_{\vartheta \vartheta} = \pm \frac{1}{q} ,
\] (D4)

and the news function is

\[
c_{\varpi u} = \frac{C^2 - 1}{2 \sin^2 \theta} .
\] (D5)

Taking the \( \theta \)-derivative of \( q \), using \( \varphi_{\vartheta \vartheta} = \pm 1/q \), and excluding the original flat-space \( \vartheta \), we obtain equation

\[
q_{\vartheta \vartheta} - 2qq_{\vartheta} \cot \theta + q^2 \left( \cot^2 \theta - \frac{C^2}{\sin^2 \theta} \right) + 1 = 0 ,
\] (D6)

which implies

\[
q_{\vartheta \vartheta} = q \cot \theta \pm \sqrt{\frac{q^2 C^2}{\sin^2 \theta} - 1} .
\] (D7)

The solution is

\[
q = \frac{\sin \theta}{2C} \left[ \left( \chi \frac{\sin \theta}{\cos \theta + 1} \right)^C + \left( \chi \frac{\sin \theta}{\cos \theta + 1} \right)^{-C} \right] , \quad \chi = \text{const} .
\] (D8)

Consider now translations along \( z \)-axis and \( t \)-axis. In coordinates \( \{ t, \rho, z, \phi \} \) they have the form \( c^\mu_{(z)} = [0, 0, a_0, 0] \) and \( \zeta^\mu_{(t)} = [b_0, 0, 0, 0] \), with \( a_0 \) and \( b_0 \) constant, so that their general combination is just \( \zeta^\mu_{(z)} + \zeta^\mu_{(t)} \).

The asymptotic form of this linear combination in Bondi’s coordinates reads

\[
\zeta^\mu_{(z)} + \zeta^\mu_{(t)} = b_0 q + \frac{a_0}{C} \sin^2 \theta \left( \frac{q}{\sin \theta} \right)_{\vartheta} + O(r^{-1}) ,
\]
\[
\zeta^\mu_{(z)} + \zeta^\mu_{(t)} = b_0 \frac{2q}{2C} \left[ q_{\vartheta \vartheta} - q^2 + 1 \right] + \frac{a_0}{2Cq^2} \left[ (q_{\vartheta \vartheta} + 2q_{\vartheta} + 1)(q_{\vartheta \vartheta} - q_{\vartheta} \sin \theta) + 2q_{\vartheta} \sin \theta \right] + O(r^{-1}) ,
\]
\[
\zeta^\vartheta_{(z)} + \zeta^\vartheta_{(t)} = -\frac{b_0 q_{\vartheta \vartheta}}{r} + \frac{a_0}{Cq^2} \left[ q_{\vartheta} (q_{\vartheta \vartheta} - q_{\vartheta} \sin \theta) - \sin \theta \right] + O(r^{-2}) ,
\]
\[
\zeta^\varphi_{(z)} + \zeta^\varphi_{(t)} = 0 .
\] (D9)

Substituting for \( q \) from Eq. (D8) we obtain

\[
\zeta^\mu_{(z)} + \zeta^\mu_{(t)} = \sin \theta \left( \frac{\sin \theta}{\cos \theta + 1} \right)^C \left( b_0 + a_0 \right) + \frac{\sin \theta}{2C} \left( \frac{\sin \theta}{\cos \theta + 1} \right)^{-C} \left( b_0 - a_0 \right)
\]
\[
= \sin \theta \left( \frac{\sin \theta}{\cos \theta + 1} \right)^C \left( b_0 + a_0 \right) \chi^C + \sin \theta \left( \frac{\sin \theta}{\cos \theta + 1} \right)^{-C} \left( b_0 - a_0 \right) \chi^{-C} ,
\] (D10)
and similar expressions for $\zeta^t_{(z)} + \zeta^i_{(t)}$ and $\zeta^0_{(z)} + \zeta^i_{(t)}$; these are somewhat lengthy and are thus not written here explicitly. Comparing now the last results (including those for $\zeta^r_{(z)} + \zeta^i_{(t)}$ and $\zeta^0_{(z)} + \zeta^i_{(t)}$) with a general asymptotic form of the Killing vector representing a supertranslation, i.e. with Eq. (58) with $B$ given by Eq. (76), we find that this Killing vector is just equal to $\zeta^\mu_{(t)}$ if the constant parameters are related by

$$a = \frac{(b_0 + a_0)\chi C}{2C},$$  \hspace{1cm} (D11)

$$b = \frac{(b_0 - a_0)\chi C}{2C}. \hspace{1cm} (D12)$$

Therefore the supertranslational Killing vector is in fact the translational Killing vector in the asymptotically flat spacetime with an infinite cosmic string along $z$-axis; $a + b = 0$, $b_0 = 0$ corresponds to a translation along $z$-axis, and $a - b = 0$, $a_0 = 0$ - along $t$-axis.

For the special case $C = 1$ (no string) the spacetime is asymptotically flat, and

$$q = \cos\theta \left(\frac{1}{2\chi} - \frac{\chi}{2}\right) + \frac{1}{2\chi} + \frac{\chi}{2} \hspace{1cm} (D13)$$

corresponds to a standard translation where $B$ is given by Eq. (77).

**APPENDIX E: ELECTROMAGNETIC FIELD**

1. Translational case

The electromagnetic tensor reads:

$$F_{01} = -\epsilon_1(\theta) \frac{1}{r^2} \left[u B^{-2}(-\epsilon_1 BB,_{\theta \theta} + \epsilon_1 B,_{\theta} B - \epsilon_1 BB,_{\theta} \cot \theta - \epsilon_1,_{\theta} BB,_{\theta}) + \epsilon_1,_{\theta} + \epsilon_1 \cot \theta\right] \frac{1}{r^3} + O(r^{-4}) \hspace{1cm} (E1)$$

$$F_{02} = \left[\epsilon_1,_{\theta} + \epsilon_1 B,_{\theta} B^{-1}\right] \frac{1}{r^2} + O(r^{-2}) \hspace{1cm} (E2)$$

$$F_{03} = -\mu_1 \sin \theta B,_{\theta} B^{-1} \frac{1}{r} + O(r^{-2}) \hspace{1cm} (E3)$$

$$F_{12} = -\epsilon_1 B,_{\theta} B^{-1} u + \epsilon_1 \frac{1}{r^2} + O(r^{-3}) \hspace{1cm} (E4)$$

$$F_{13} = \left[\mu_1 \sin \theta B,_{\theta} B^{-1} u + f_1 \sin \theta\right] \frac{1}{r^2} + O(r^{-3}) \hspace{1cm} (E5)$$

$$F_{23} = -\mu_1 \sin \theta - \left\{B^{-2} \left[\left(\mu_1 \sin \theta,_{\theta} BB,_{\theta} + \mu_1 \sin \theta BB,_{\theta \theta} - \mu_1 \sin \theta B,_{\theta}\right) B + \left(f_1 \sin \theta,_{\theta}\right) B\right] \frac{1}{r} + O(r^{-2})\right\} \hspace{1cm} (E6)$$

2. Boost case

Assuming the boost-rotation symmetry, the electromagnetic tensor has the form:

$$F_{01} = -\frac{\mathcal{E}}{u} \cot \theta \frac{1}{r^2} + \left[\frac{\cot \theta}{2} (\mathcal{E},_{\theta} w + \mathcal{E}) + \frac{\cot \theta}{u^2} (\mathcal{G},_{\theta} w + \mathcal{G})\right] \frac{1}{r^3} + O(r^{-4}) \hspace{1cm} (E7)$$

$$F_{02} = \frac{\mathcal{E}}{u^2} + \left[\frac{1}{u \sin \theta} (\mathcal{E},_{\theta} w - \mathcal{E}) - \frac{\mathcal{E},_{\theta} \sin \theta}{u^2} + \frac{1}{u^3} (\mathcal{G},_{\theta} w + \mathcal{G})\right] \frac{1}{r} + O(r^{-2}) \hspace{1cm} (E8)$$

$$F_{03} = \frac{\mathcal{B}}{u} + \left[-\frac{1}{2} (B,_{\theta} w - \mathcal{B}) + \frac{1}{u^2} (\mathcal{D},_{\theta} w + \mathcal{D})\right] \frac{1}{r} + O(r^{-2}) \hspace{1cm} (E9)$$

$$F_{12} = \left[\frac{\mathcal{E}}{2} + \frac{\mathcal{G}}{u^2}\right] \frac{1}{r^2} + O(r^{-3}) \hspace{1cm} (E10)$$

$$F_{13} = \left[-\frac{1}{2} B,_{\theta} + \frac{\mathcal{D}}{u}\right] \frac{1}{r^2} + O(r^{-3}) \hspace{1cm} (E11)$$

$$F_{23} = -B \cot \theta + \cos \theta \left[\frac{1}{2} B,_{\theta} - \frac{\mathcal{D},_{\theta}}{u^2}\right] \frac{1}{r} + O(r^{-2}) \hspace{1cm} (E12)$$

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Let us yet illustrate the general asymptotic forms (E8), (E9) for boost-rotation symmetric electromagnetic fields by specific examples. As follows from Eqs. (E8), (E9), the first non-vanishing terms are given by

\[ F_{02} = E(w)/u^2, \]

\[ F_{03} = B(w)/u, \]

with \( w = \sin \theta/u \). Denote by \( e, p, m \) electric charge, electric dipole and magnetic dipole moments respectively, and by \( \alpha^{-1} \) the magnitude of acceleration. For a uniformly accelerated electric monopole (producing Born’s solution), we find

\[ E(w) = e\alpha^2 w/(1 + \alpha^2 w^2)^{3/2}, \]

\[ B(w) = 0, \]

whereas for a magnetic dipole (see [21] for the complete field), one finds

\[ E(w) = 0, \]

\[ B(w) = -\alpha mw^2(2 - \alpha^2 w^2)/\left(1 + \alpha^2 w^2\right)^{5/2}. \]

We can easily check that \( \int E \cos \theta/u d\theta = 0 \) in the case of electric dipole and \( \int B \cot \theta d\theta = 0 \) in the magnetic case - total charges indeed vanish.

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