Construction of a Novel Superstring in Four Dimensions

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A string in four dimensions is constructed by supplementing it with forty four Majorana fermions. The central charge is 26. The fermions are grouped in such a way that the resulting action is supersymmetric. The super-Virasoro algebra is constructed and closed by the use of Jacobi identity. The tachyonic ground state decouples from the physical states. GSO projections are necessary for proving modular invariance and space-time supersymmetry is shown to exist for modes of zero mass. The symmetry group of the model descends to the low energy group $SU(3) \times SU(2) \times U(1) \times U(1)$.

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I. INTRODUCTION

String theory was invented [1] as a sequel to dual resonance models [2] to explain the properties of strongly interacting particles in four dimensions. Assuming the string to live in a background gravitational field and demanding Weyl invariance, the Einstein equations of general relativity could be deduced. It was believed that about these classical solutions one can expand and find the quantum corrections. But difficulties arose at the quantum level. Even though the string interaction amplitude obeyed crossing, it was no longer unitary. There were anomalies and ghosts. Due to these compelling reasons it was necessary for the open string to live in 26 dimensions [3,16]. At present the most successful theory is the ten dimensional superstring on a Calabi-Yau manifold or an orbifold. However, in order to realise the programme of string unification of all the four particle interaction, one must eventually arrive at a theory in four flat space-time dimensions, with N=1 supersymmetry and chiral matter fields. This paper is an attempt in that direction.

A lot of research has been done in the construction of four dimensional strings [4] specially in the latter half of the eighties. Antoniadis et al [5] have considered a four dimensional superstring supplemented by eighteen real fermions in trilinear coupling. The central charge of the construction is 15. Chang and Kumar [6] have considered Thirring fermions, but again with the central charge at 15. Kawai et al [7] have considered four dimensional model in a different context than the model proposed here. None of these makes contact with a standard like model.

In section II, we give the details of the supersymmetric model. Section III gives the usual quantization and super-Virasoro algebra is deduced in the section IV. Bosonic states are constructed in Section V. Fadeev- Popov ghosts are introduced and the BRST charge is explicitly constructed in section VI. Ramond states are constructed in section VII. In section VIII the mass spectrum of the model and the necessary GSO projections to eliminate the half integral spin states are introduced. In section IX, we show that these projections are necessary to prove the modular invariance of the model. Space-time supersymmetry is shown to exist for the zero mass modes in section X. In section XI we show how the chain $SO(44) \rightarrow SO(11) \rightarrow SO(6) \times SO(5) \rightarrow SU(3) \times SU(2) \times U(1) \times U(1)$ is possible in this model.

The literature on string theory is very vast and exist in most text books on the subject. The references serve only as a guide to elucidate the model.

II. THE MODEL

The model essentially consists of 26 vector bosons of an open (closed) string in which there are the four bosonic coordinates of four dimensions and there are forty four Majorana fermions representing the remaining 22 bosonic coordinates [8]. We divide them into four groups. They are labelled by $\mu = 0, 1, 2, 3$ and each group contains 11 fermions. These 11 fermions are again divided into two groups, one containing six and the other five. For convenience, in one group we have $j = 1, 2, 3, 4, 5, 6$, and in the other, $k = 1, 2, 3, 4, 5$.

The string action is

$$S = -\frac{1}{2\pi} \int d^2 \sigma \left[ \partial_\alpha X^\mu \partial^\alpha X_\mu - i \bar{\psi}_{\mu,j} \gamma^\alpha \partial_\alpha \psi_{\mu,j} - i \phi_{\mu,k} \gamma^\alpha \partial_\alpha \phi_{\mu,k} \right],$$

(1)

$\gamma^\alpha$ are the two dimensional Dirac matrices.
\( \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) \( \) (2) 

and obey

\[ \{ \rho^\alpha, \rho^\beta \} = -2\eta_{\alpha\beta} . \] (3)

String co-ordinates \( X^\mu \) are scalars in \((\sigma, \tau)\) space and vectors in target space. Similarly \( \psi^{\mu, j} \) are spinors in \((\sigma, \tau)\) space and vectors in target space.

In general we follow the notations and conventions of reference \[9\] whenever omitted by us. \( X^\mu(\sigma, \tau) \) are the string coordinates. The \( \psi \)'s are the odd indexed and \( \phi \)'s the even indexed Majorana fermions decomposed in the basis

\[ \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} . \] (4)

The nonvanishing commutation and anticommutations are

\[ [\dot{X}^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i \delta(\sigma - \sigma')\eta^{\mu\nu} \] (5)

\[ \{ \psi^\mu_A(\sigma, \tau), \psi^\nu_B(\sigma', \tau) \} = \pi \eta^{\mu\nu} \delta_{AB} \delta(\sigma - \sigma') \] (6)

\[ \{ \phi^\mu_A(\sigma, \tau), \phi^\nu_B(\sigma', \tau) \} = \pi \eta^{\mu\nu} \delta_{AB} \delta(\sigma - \sigma') \] (7)

The action is invariant under infinitesimal transformations

\[ \delta X^\mu = \bar{\epsilon} \left( \sum_j \psi^{\mu, j} + i \sum_k \phi^{\mu, k} \right) \] (8)

\[ \delta \psi^{\mu, j} = -i \rho^\alpha \partial_\alpha X^\mu \epsilon \] (9)

\[ \delta \phi^{\mu, k} = +\rho^\alpha \partial_\alpha X^\mu \epsilon \] (10)

where \( \epsilon \) is an infinitesimally constant anticommuting Majorana spinor. The commutator of the two supersymmetry transformations gives a spatial translation, namely

\[ [\delta_1, \delta_2] X^\mu = a^\alpha \partial_\alpha X^\mu \] (11)

and

\[ [\delta_1, \delta_2] \Psi^\mu = a^\alpha \partial_\alpha \Psi^\mu \] (12)

where

\[ a^\alpha = 2i \bar{\epsilon}_1 \rho^\alpha \epsilon_2 \] (13)

and

\[ \Psi^\mu = \sum_j \psi^{\mu, j} + i \sum_k \phi^{\mu, k} \] (14)

In deriving this, the Dirac equation for the spinors have been used. The Noether super-current is

\[ J_\alpha = \frac{1}{2} \rho^\beta \rho_\alpha \Psi^\mu \partial_\beta X_\mu \] (15)

We now follow the standard procedure. The light cone components of the current and energy momentum tensors are
\[ J_+ = \partial_+ X_\mu \Psi_+^\mu \]  
(16)

\[ J_- = \partial_- X_\mu \Psi_-^\mu \]  
(17)

\[ T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2} \psi^\mu_+ j \partial_+ \psi_{+\mu, j} + \frac{i}{2} \phi^\mu_+ k \partial_+ \phi_{+\mu, k} \]  
(18)

\[ T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2} \psi^\mu_- j \partial_- \psi_{-\mu, j} + \frac{i}{2} \phi^\mu_- k \partial_- \phi_{-\mu, k} \]  
(19)

where \( \partial_\pm = \frac{1}{2} (\partial_\tau \pm \partial_\sigma) \).

To proceed further we note that in equation (8) and (14) we could have taken \(-i\) instead of \(+i\). We now introduce a phase factor \( \eta_\phi \) to replace \('i'\) in these equation. \( \eta_\phi \) depends on the number \( n_\phi \) of \( \phi \)'s (or its quantar), in a given individual term. Explicitly \( \eta_\phi = (-1)^{1/4 n_\phi (n_\phi - 1)} \). \( \eta_\phi = i \) if \( n_\phi = 1 \) reproducing \('i'\) in the above equations. But \( \eta_\phi^2 = -1 \) if \( n_\phi = 0 \) where two \( \phi \)'s have been contracted away and \( \eta_\phi^2 = 1 \) if \( n_\phi = 2 \).

One now readily calculates the algebra

\[ \{ J_+ (\sigma), J_+ (\sigma') \} = \pi \delta (\sigma - \sigma') T_{++} (\sigma) \]  

\[ \{ J_-(\sigma), J_- (\sigma') \} = \pi \delta (\sigma - \sigma') T_{--} (\sigma) \]  

\[ \{ J_+ (\sigma), J_- (\sigma') \} = 0 \]  
(20)

The time like components of \( X^\mu \) are eliminated by the use of Virasoro constraints \( T_{++} = T_{--} = 0 \). In view of equation (20), we postulate that

\[ 0 = J_+ = J_- = T_{++} = T_{--} \]  
(21)

\( J_+ \) is a sum of a real and imaginary term, The real term, is further, a sum of six mutually independent \( \psi_{+\mu,j} \)'s and the imaginary term, the five mutually independent \( \phi_{+\mu,k} \)'s. It will be shown in Section V, that \( J_+ = 0 \) constraint excludes all the eleven time like components of \( \psi \)'s and \( \phi \)'s from the physical space.

### III. QUANTIZATION

As usual the theory is quantized \( (\alpha^\mu = p^\mu) \), with

\[ X^\mu = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \exp^{-in\tau} \cos(n\sigma), \]  
(22)

or

\[ \partial_\pm X^\mu = \frac{1}{2} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau \pm \sigma)} \]  
(23)

\[ [\alpha_m^\mu, \alpha_n^\nu] = m \delta_{m+n} \eta^{\mu\nu} \]  
(24)

While discussing the mass spectrum, it will be more illuminating to consider the closed string. The related additional quantas here and in wherever occurs will be denoted by attaching a tilde. For instance

\[ \partial_- X^\mu_R = \sum_{-\infty}^{+\infty} \tilde{\alpha}_n^\mu e^{-2in(\sigma - \tau)} \]  
(25)

\[ \partial_+ X^\mu_L = \sum_{-\infty}^{+\infty} \tilde{\alpha}_n^\mu e^{-2in(\sigma + \tau)} \]  
(26)
The transition formulas for closed strings can be easily effected. We consider the open string. We first choose the Neveu-Schwarz (NS) [10] boundary condition. Then the mode expansions of the fermions are

$$\psi_{\pm}^{\mu,j}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{\mu,j}^{r} e^{-ir(\tau \pm \sigma)}$$

(27)

$$\phi_{\pm}^{\mu,k}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_{\nu,j}^{r} e^{-ir(\tau \pm \sigma)}$$

(28)

$$\Psi_{\pm}^{\mu,j}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} B_{r} e^{-ir(\tau \pm \sigma)}$$

(29)

The sum is over all the half-integer modes.

$$\{b_{\mu,j}^{r}, b_{\nu,j}^{s}\} = \eta^{\mu\nu} \delta_{j,j'} \delta_{r+s}$$

(30)

$$\{b_{\mu,k}^{r}, b_{\nu,k}^{s}\} = \eta^{\mu\nu} \delta_{k,k'} \delta_{r+s}$$

(31)

$$\{B_{r}^{\mu}, B_{s}^{\nu}\} = \eta^{\mu\nu} \delta_{r+s}$$

(32)

**IV. VIRASORO ALGEBRA**

Virasoro generators [11] are given by the modes of the energy momentum tensor $T_{++}$ and Noether current $J_{+}$,

$$L_{m}^{M} = \frac{1}{\pi} \int_{-\pi}^{+\pi} d\sigma e^{im\sigma} T_{++}$$

(33)

$$G_{r}^{M} = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{+\pi} d\sigma e^{ir\sigma} J_{+}$$

(34)

‘$M$’ stands for matter. In terms of creation and annihilation operators

$$L_{m}^{M} = L_{m}^{(\alpha)} + L_{m}^{(b)} + L_{m}^{(b')}$$

(35)

where

$$L_{m}^{(\alpha)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\alpha_{-n} \cdot \alpha_{m+n} :$$

(36)

$$L_{m}^{(b)} = \frac{1}{2} \sum_{r=-\infty}^{\infty} (r + \frac{1}{2} m) :b_{-r} \cdot b_{m+r} :$$

(37)

$$L_{m}^{(b')} = \frac{1}{2} \sum_{r=-\infty}^{\infty} (r + \frac{1}{2} m) :b'_{-r} \cdot b'_{m+r} :$$

(38)

In each case normal ordering is required. The single dot implies the sum over all qualifying indices. The current generator is
\[
G_r^M = \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot (b_{r+n} + \eta_0 b'_{r+n}) = \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot (b_{r+n} + i b'_{r+n}) = \sum_{n=\infty}^{\infty} \alpha_n \cdot B_{r+n}
\]  

(39)

Following from eqn. (33) the Virasoro algebra is

\[
[L^M_m, L^M_n] = (m-n)L^M_{m+n} + A(m) \delta_{m+n}
\]  

(40)

Using the relations

\[
[L^M_m, \alpha^\mu_n] = -n\alpha^\mu_{n+m}
\]  

(41)

\[
[L^M_m, B^\mu_n] = -(n + \frac{m}{2})B^\mu_{n+m}
\]  

(42)

we get, also

\[
[L^M_m, G^M_r] = \left(\frac{1}{2}m - r \right) G^M_{m+r}
\]  

(43)

The anticommutator \( \{G^M_r, G^M_s\} \) is obtained directly or by the use of the Jacobi identity

\[
\{[G^M_r, G^M_s], L^M_m\} + \{[L^M_m, G^M_r], G^M_s\} + \{[L^M_m, G^M_s], G^M_r\} = 0
\]  

(44)

which implies, consistent with equations (34) and (35),

\[
\{G^M_r, G^M_s\} = 2L^M_{r+s} + B(r)\delta_{r+s}
\]  

(45)

A\( (m) \) and B\( (r) \) are normal ordering anomalies. Taking the vacuum expectation value in the Fock ground state |0, 0\rangle with four momentum \( p^\mu = 0 \) of the commutator \([L_1, L_{-1}]\) and \([L_2, L_{-2}]\), it is easily found that

\[
A(m) = \frac{26}{12} (m^3 - m) = \frac{C}{12} (m^3 - m)
\]  

(46)

and using the Jacobi identity

\[
B(r) = \frac{A(2r)}{2r}
\]  

(47a)

\[
B(r) = \frac{26}{3} \left( r^2 - \frac{1}{4} \right) = \frac{C}{3} \left( r^2 - \frac{1}{4} \right)
\]  

(47b)

The central charge \( C = 26 \). This is what is expected. Each bosonic coordinate contribute 1 and each fermionic ones contribute \( 1/2 \), so that the total central charge is +26.

For closed strings there will be another set of tilded generators satisfying the same algebra.

V. BOSONIC STATES

A physical bosonic state \( \Phi \) which should have \( SO(6) \times SO(5) \) internal symmetry satisfies

\[
L^M_m \mid \Phi \rangle = 0 \quad m > 0
\]  

(48)

\[
G^M_r \mid \Phi \rangle = 0 \quad r > 0
\]  

(49)

The real and imaginary parts of \( G^M_r \mid \Phi \rangle \) separately vanish for \( r > 0 \). Specialising to a rest frame we write the conditions as

\[
\frac{1}{2} p^\mu \alpha^\mu_m \mid \Phi \rangle + \text{(terms quadratic in oscillators)} \mid \Phi \rangle = 0
\]  

(50)
\[ p^0(b_{r}^{0.1} + \cdots + b_{r}^{0.6}) | \Phi \rangle + (\text{terms containing both } \alpha \text{ and } b \text{ oscillators}) | \Phi \rangle = 0 \] (51)

\[ p^0(b_{r}^{0.1} + \cdots + b_{r}^{0.5}) | \Phi \rangle + (\text{terms containing both } \alpha \text{ and } b' \text{ oscillators}) | \Phi \rangle = 0 \] (52)

The resulting states in the second terms of equations (50), (51), (52) are different from the first terms. So we concentrate our attention for the vanishing of the first term only. To satisfy equation (50), as usual the time like component \( \alpha^0_m \) is excluded from \( \Phi \). As a result the second terms of equation (51) and (52) contain only transverse oscillators. \( b_{r}^{0.1} \) to \( b_{r}^{0.6} \) or \( b_{r}^{0.1} \) to \( b_{r}^{0.5} \) are all independent anihilation operators for \( r > 0 \) and there is no relation between them. Therefore \( \Phi \) should not contain any of the eleven time like components \( b_{r}^{c} \)'s or \( b_{r}^{c} \)'s, for, otherwise the equality to zero in equations (51) and (52) cannot be achieved. Thus the vanishing of the energy-momentum tensor and the current excludes all the time like components from the physical space. No negative norm state will show up in the physical spectrum and at the same time preserve \( SO(6) \times SO(5) \) internal symmetry.

Let us make a more detailed investigation to ensure that there are no negative norm physical states. We shall do this by constructing the zero norm states or the ‘null’ physical states. Due to GSO condition, which we shall study later, the physical states will be obtained by operation of the product of even number of \( G \)'s. So the lowest state above the tachyonic state is

\[ | \Psi \rangle = L_{-1} | \chi_1 \rangle + G_{-1/2}G_{-1/2} | \chi_2 \rangle \]

But \( G_{-1/2}G_{-1/2} = \frac{1}{2}\{G_{-1/2}, G_{-1/2}\} = L_{-1} \). Without loss of generality, the state is

\[ | \Psi \rangle = L_{-1} | \chi \rangle \] (53)

This state to be physical, it must satisfy \( (L_0 - 1) | \Psi \rangle = 0 \) which is true if \( L_0 | \chi \rangle = 0 \). The norm \( \langle \Psi | \Psi \rangle = (\chi | L_1L_{-1} | \chi) = 2(\chi | L_0 | \chi) = 0 \). Let us consider the next higher mass state

\[ | \Psi \rangle = L_{-2} | \chi_1 \rangle + L_{-1}^2 | \chi_2 \rangle + (G_{-3/2}G_{-1/2} + \lambda G_{-1/2}G_{-3/2}) | \chi_3 \rangle + G_{-1/2}G_{-1/2}G_{-1/2} | \chi_4 \rangle + \cdots \]

It can be shown that \( G_{-3/2}G_{-1/2} | \chi \rangle = (\beta_1 L_{-1}^2 + \beta_2 L_{-2}) | \chi \rangle \). The coefficients \( \beta_1 \) and \( \beta_2 \) can be calculated by evaluating \( [L_1, G_{-3/2}G_{-1/2}] | \chi \rangle \) and \( [L_2, G_{-3/2}G_{-1/2}] | \chi \rangle \). \( G_{-1/2} \) is proportional to \( L_{-1}^2 \). So, in essence, we have the next excited state as

\[ | \Psi \rangle = (L_{-2} + \gamma L_{-1}^2) | \chi \rangle \] (54)

The condition \( (L_0 - 1) | \Psi \rangle = 0 \) is satisfied if \( (L_0 + 1) | \chi \rangle = 0 \). Further the physical state condition \( L_1 | \Psi \rangle = 0 \) gives the value of \( \gamma = 3/2 \). The norm is easily obtained as

\[ \langle \Psi | \Psi \rangle = \frac{1}{2}(C - 26) \] (55)

This is negative for \( C < 26 \) and vanishes for \( C = 26 \). So the critical central charge is 26. It is easily checked that \( L_2 | \Psi \rangle \) also vanishes for \( C = 26 \).

To find the role of \( b \) and \( b' \) modes, let us calculate the norm of the following state with \( p^2 = 2 \)

\[ (L_{-2} + 3/2 L_{-1}^2) | 0, p \rangle = \left( L_{-2}^{(b)} + \frac{3}{2} L_{-1}^{(b)} \right) | 0, p \rangle + \left( L_{-2}^{(b')} + \frac{3}{2} L_{-1}^{(b')} \right) | 0, p \rangle \] (56)

The norm of the first term equal \( -11 \) as calculated in reference 3.

Noting that \( L_{-2}^{(b')} | 0, p \rangle = L_{-2}^{(b')} | 0, p \rangle = 0; L_{-2}^{(b)} = \frac{1}{2} b_{-3/2} \cdot b_{-1/2} \) and \( L_{-2}^{(b')} = \frac{1}{2} b'_{-3/2} \cdot b'_{-1/2} \) the norms of the second and third terms are \( \frac{1}{4} (\delta_{\mu \nu} \delta_{i j}) = 6 \) and \( \frac{1}{4} (\delta_{\mu \nu} \delta_{kk}) = 5 \) respectively. The norm of the state given in equation (56) is \( -11 + 6 + 5 = 0 \)

Since \( L_1 = G_{1/2}^{1/2}, L_1 | \Psi \rangle = 0 \) implies \( G_{1/2} | \Psi \rangle = 0 \). \( G_{3/2} \) can be expressed as a commutator of \( L_1 \) and \( G_{1/2} \), so that \( G_{3/2} | \Psi \rangle = 0 \). Further \( L_2 | \Psi \rangle = \frac{1}{2} \{G_{3/2}, G_{1/2}\} | \Psi \rangle = 0 \) and so on satisfying all the physical state conditions.
VI. GHOSTS

For obtaining a zero central charge so that the anomalies cancel out and natural ghosts are isolated, Faddeev-Popov (FP) ghosts \[12\] are introduced. The FP ghost action is

\[
S_{FP} = \frac{1}{\pi} \int (c^+ \partial_- b_{++} + c^- \partial_+ b_{--}) d^2 \sigma \tag{57}
\]

where the ghost fields \(b\) and \(c\) satisfy the anticommutator relations

\[
\{b_{++}(\sigma, \tau), c^+ (\sigma', \tau)\} = 2\pi \delta(\sigma - \sigma') \tag{58}
\]

\[
\{b_{--}(\sigma, \tau), c^- (\sigma', \tau)\} = 2\pi \delta(\sigma - \sigma') \tag{59}
\]

and are quantized with the mode expansions

\[
c^\pm = \sum_{-\infty}^{\infty} c_n e^{-in(\tau \pm \sigma)} \tag{60}
\]

\[
b_{\pm\pm} = \sum_{-\infty}^{\infty} b_n e^{-in(\tau \pm \sigma)} \tag{61}
\]

The canonical anticommutator relations for \(c_n\’s\) and \(b_n\’s\) are

\[
\{c_m, b_n\} = \delta_{m+n} \tag{62}
\]

\[
\{c_m, c_n\} = \{b_m, b_n\} = 0 \tag{63}
\]

Deriving the energy momentum tensor from the action and making the mode expansion, the Virasoro generators for the ghosts (G) are

\[
L^G_m = \sum_{n=-\infty}^{\infty} (m-n) b_{m+n} c_{-n} - a \delta_m \tag{64}
\]

where \(a\) is the normal ordering constant. These generators satisfy the algebra

\[
[L^G_m, L^G_n] = (m-n) L^G_{m+n} + A^G(m) \delta_{m+n} \tag{65}
\]

The anomaly term is deduced as before and give

\[
A^G(m) = \frac{1}{6} (m - 13m^3) + 2a m \tag{66}
\]

With \(a = 1\), this anomaly term becomes

\[
A^G(m) = -\frac{26}{12} (m^3 - m) \tag{67a}
\]

\[
B^G(r) = -\frac{26}{3} \left( r^2 - \frac{1}{4} \right) \tag{67b}
\]

The central charge is \(-26\) and cancels the normal order \(A(m)\) and \(B(r)\) of the \(L\) and \(G\) generators. Noting that

\[
[L^G_m, c_n] = -(2m+n)c_{n+m} \tag{68}
\]

it is possible to construct an equation for the generator for the current of the ghost sector,
\[ G_{r}^{gh} = \sum_{p} \left( \frac{p}{2} - r \right) c_{-p} G_{p+r}^{gh} \] (69)

such that
\[ [L_{m}^{G}, G_{r}^{gh}] = (m/2 - r) G_{m+r}^{gh} \] (70)

Let us examine the possibility of an expression for the current as
\[ G_{r}^{gh} = \sum_{n} b_{n+r} c_{-n} \] (71)

\( n \) takes only integral values whereas \( r \) is half integral. \( b_{n+r} \) is therefore outside the ghost sector. It has a conformal dimension of \( 5/2 \) so that \( G_{r}^{gh} \) has the correct conformal dimension \( 3/2 \). The commutator
\[ [L_{m}^{g}, b_{n+r}] = \left( \frac{3}{2} m - n - r \right) b_{n+r+m} \] (72)

is easily evaluated demanding that equation (70) be satisfied. From Jacobi identity (65)
\[ \{ G_{r}^{gh}, G_{s}^{gh} \} = 2 L_{r+s}^{G} + \delta_{r+s} B^{G}(r) \] (73)

The total current generator is
\[ G^{r} = G^{M}_{r} + G^{gh}_{r} \] (74)

thus we have the anomaly free Super Virasoro algebra,
\[ [L_{m}, L_{n}] = (m - n) L_{m+n} \] (75)
\[ [L_{m}, G_{r}] = (m/2 - r) G_{m+r} \] (76)
\[ [G_{r}, G_{s}] = 2 L_{r+s} \] (77)

Thus from the usual conformal field theory we have obtained the algebra of a superconformal field theory. This is the novelty of the present formulation. The BRST charge operator is
\[ Q_{BRST} = \sum_{-\infty}^{\infty} L_{-m}^{M} c_{m} - \frac{1}{2} \sum_{-\infty}^{\infty} (m - n) : c_{-m} c_{-n} b_{m+n} : -a c_{0} \] (78)

and is nilpotent for \( a = 1 \). The physical states are such that \( Q_{BRST} |phys \rangle = 0 \).

**VII. FERMIONIC STATES**

The above deductions can be repeated for Ramond sector \([14]\). We write the main equations. The mode expansion for the fermions are
\[ \psi_{\mu,j}^{\pm}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} d_{m}^{\mu,j} e^{-im(\tau \pm \sigma)} \] (79)

\[ \phi_{\mu,j}^{\pm}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{-\infty}^{\infty} d_{m}^{\mu,j} e^{-im(\tau \pm \sigma)} \] (80)

The generators of the Virasoro operators are
\[ L_{m}^{M} = L_{m}^{(\alpha)} + L_{m}^{(d)} + L_{m}^{(d')} \] (81)
\[ L_m^{(d)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (n + \frac{1}{2} m) : d_{-n} \cdot d_{m+n} : \]  

(82) 

\[ L_m^{(d')} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (n + \frac{1}{2} m) : d'_{-n} \cdot d'_{m+n} : \]  

(83) 

and the fermionic current generator is

\[ F^M_m = \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot (d_{n+m} + id'_{n+m}) = \sum_{n=-\infty}^{\infty} \alpha_{-n} \cdot D_{n+m} \]  

(84) 

The Ramond sector Virasoro algebra is the same as the NS-sector with the replacement of G’s by F’s. It is necessary to define \( L_0 \) suitably to keep the anomaly equations the same [9].

In this Ramond sector, a physical state \( | \Phi \rangle \) should satisfy

\[ F_n | \Phi \rangle = L_n | \Phi \rangle = 0 \quad \text{for} \quad n > 0 \]  

(85) 

The normal order anomaly constant in the anticommutables of the Ramond current generators has to be evaluated with care, because the definition of \( F_0 \) does not have a normal ordering ambiguity. So \( F^2_0 = L_0 \). Using commutation relation (43) with \( G \) replaced by \( F \) and the Jacobi Identity we get

\[ \{ F_r, F_{-r} \} = 2 r \{ [L_r, F_0], F_{-r} \} = 2 L_0 + 4 \frac{r}{r} A(r) \] 

So

\[ B(r) = \frac{4}{r} A(r) \]  

(86) 

\[ B(r) = \frac{C}{3} (r^2 - 1), \quad r \neq 0 \]  

(87) 

A physical state in the fermionic sector satisfies

\[ (L_0 - 1) | \Psi \rangle = 0 \]  

(88) 

It follows that

\[ (F^2_0 - 1) | \Psi \rangle = (F_0 - 1)(F_0 + 1) | \Psi \rangle = 0 \]  

The construction of ‘null’ physical states becomes much simpler because all \( F_{-m} \) terms can be assigned to \( L_{-m} \) terms by the commutation ration \( F_{-m} = 2[F_0, L_{-m}]/m \) and \( F_0 \) has eigen values which are roots of eigen values of \( L_0 \) acting on the generic states or states constructed out of the quadratic states. Thus the zero mass null physical state with \( L_0 | \tilde{\chi} \rangle = F^2_0 | \tilde{\chi} \rangle = 0 \) is simply

\[ | \Psi \rangle = L_{-1} | \tilde{\chi} \rangle \]  

(89) 

with \( L_1 | \Psi \rangle = F_1 | \Psi \rangle = 0 \). The next excited state with \( (L_0 + 1 \tilde{\chi}) \rangle \) becomes the same as in the bosonic sector. Obtained from the condition \( L_1 | \Psi \rangle = 0 \),

\[ | \Psi \rangle = (L_{-2} + \frac{3}{2} L_{-1}) | \tilde{\chi} \rangle \] 

The norm \( \langle \Psi | \Psi \rangle = (C - 26)/2 \) and vanishes for \( C = 26 \). It is easy to check that all physical state conditions are satisfied. \( F_1 | \Psi \rangle = 2 [L_1, F_0] | \Psi \rangle = 0 \) since \( L_1 | \Psi \rangle = 0 \) and \( F_0 | \Psi \rangle = | \Psi \rangle \) \( L_2 | \Psi \rangle = F_1 F_1 | \Psi \rangle = 0 \) and \( F_2 | \Psi \rangle = [L_2, F_0] | \Psi \rangle = 0 \). For \( C = 26 \), there are no negative norm states in the Ramond sector as well.

The ghost current in the Ramond sector satisfies the equation
\[ F^g_m = \sum_p (\frac{p}{2} - m) c_{-p} F^g_{m+p} \]  

so that

\[ [L^G_m, F^g_n] = \left( \frac{m}{2} - n \right) F^g_{m+n} \]  

we can construct \( F^g_0 \) with the help of an anti commuting object \( \Gamma_n \) which satisfy

\[ \{ \Gamma_n, \Gamma_m \} = 2 \delta_{m,n} \]  

It is important to write \( L^G_0 \) in terms of positive integrals as

\[ L^G_0 = \sum_{n=1}^{\infty} n(b_{-n}c_n + c_{-n}b_n) \]  

It is found that

\[ F^g_0 = \sum_{n=1}^{\infty} \sqrt{n} \Gamma_n (b_{-n}c_n + c_{-n}b_n) \]  

All other \( F \)'s can be constructed by the use of the equations of super Virasoro algebra.

From equation (67) and (86), the ghost current anomaly constant is \( B^G(r) = -\frac{26}{3}(r^2 - 1) \) and cancels out the \( B(r) \) of equation(87). The total current anomalies in both the sectors vanish.

\section*{VIII. THE MASS SPECTRUM}

The ghosts are not coupled to the physical states. Therefore the latter must be of the form (up to null state) \[ |\{n\} p\rangle_M \otimes c_1 |0\rangle_G \]  

\[ |\{n\} p\rangle_M \] denotes the occupation numbers and momentum of the physical matter states. The operator \( c_1 \) lowers the energy of the state by one unit and is necessary for BRST invariance. The ghost excitation is responsible for lowering the ground state energy which produces the tachyon.

\[ (L^M_0 - 1) |\text{phys}\rangle = 0 \]  

Therefore, the mass shell condition is

\[ \alpha' M^2 = N^B + N^F - 1 \]  

where

\[ N^B = \sum_{m=1}^{\infty} \alpha_{-m} \alpha_m \]  

or

\[ N^F = \sum_{r=1/2}^{\infty} r (b_{-r} \cdot b_r + b'_{-r} \cdot b'_r) \ (NS). \]  

Due to the presence of Ramond and Neveu-Schwartz sectors with periodic and anti-periodic boundary conditions, we can effect a GSO projection \[ \text{GSO} \] on the mass spectrum on the NSR model \[ \text{NSR} \]. Here the projection should refer to the unprimed and primed quanta separately. The desired projection is

\[ G = \frac{1}{4} (1 + (-1)^F)(1 + (-1)^{F'}) \]
where \( F = \sum b_r \cdot b_r \); \( F' = \sum b'_r \cdot b'_r \). This eliminates the half integral values from the mass spectrum by choosing \( G=1 \).

For closed strings we have a similar separation as in Eq. (95), namely the left-handed states will be in the form
\[
|\{\tilde{\eta}\} p\rangle_M \otimes \tilde{c}_1|0\rangle_G
\]
The mass spectrum can be written as
\[
\frac{1}{2} a'M^2 = N + \tilde{N} - 1 - \tilde{1}
\] (102)

**IX. MODULAR INVARIANCE**

The GSO projection is necessary for the modular invariance of the theory. We follow the notation of Seiberg and Witten [18]. Following Kaku [17], the spin structure \( \chi(-,\tau) \) for a single fermion is given by
\[
\chi(-,\tau) = q^{-1/24} Tr q^{n} \psi_{-n} \psi_{n} = q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \sqrt{\frac{\Theta_3(\tau)}{\eta(\tau)}},
\] (103)
where \( \Theta \)'s will be the Jacobi Theta functions \( \Theta(\theta,\tau) \) \([18]\), \( q = e^{i\pi\tau} \) and \( \eta(\tau)(2\pi) = \Theta^{1/3}(\tau) \). The path integral functions of Seiberg and Witten for the twenty four unprimed oscillators are
\[
A((-\cdot),\tau) = (\Theta_3(\tau)/\eta(\tau))^{12},
\] (104)
This is normalised to one.
\[
A((+\cdot),\tau) = A((-\cdot),\tau) = -(\Theta_2(\tau)/\eta(\tau))^{12},
\] (105)
\[
A((+\cdot),\tau) = A(+\cdot,\tau) = -(\Theta_4(\tau)/\eta(\tau))^{12},
\] (106)
\[
A((+\cdot),\tau) = 0.
\] (107)
It is easily checked that the sum
\[
A(\tau) = (\Theta_3(\tau)/\eta(\tau))^{12} - (\Theta_2(\tau)/\eta(\tau))^{12} - (\Theta_4(\tau)/\eta(\tau))^{12}
\] (108)
is modular invariant, using the properties of the theta functions given in [18].

For the twenty primed oscillators it is not so straightforward because of the ambiguity of fractional powers of unity. If we prescribe a normalization \( 1 = 1^{1/2} = \sqrt{e^{2i\pi}} \), then
\[
A'((-\cdot),\tau) = (\Theta(\tau)/\eta(\tau))^{12},
\] (109)
\[
A'((+\cdot),\tau) = \sqrt{e^{i\pi}} (\Theta_2(\tau)/\eta(\tau))^{12},
\] (110)
\[
A'((+\cdot),\tau) = \sqrt{e^{i\pi}} (\Theta_4(\tau)/\eta(\tau))^{12},
\] (111)
\[
A'((+\cdot),\tau) = 0.
\] (112)
The sum \( (\Theta_3^{12}(\tau) + \sqrt{e^{i\pi}} \Theta_2^{12}(\tau) + \sqrt{e^{i\pi}} \Theta_4^{12}(\tau))/\eta^{10}(\tau) \) is also modular invariant up to the factor of cube root and fractional roots of unity. The sum of the modulii is, of course, modular invariant. It is easy to construct the modular invariant partition function for the two physical bosons, namely
\[
\mathcal{P}_B(\tau) = \frac{(Im \tau)^{-2} \Delta^{-2}(\tau)}{\Delta^{-2}(\tau)},
\] (113)
in four dimensions \([20]\). For normal ordering constant is \( -1 \) in \( q^{2(L_0-1)} \), \(-2/12 \) comes from the bosons and \(-44/24 \) comes from the fermions adding to \(-1 \) for both NS and R sectors.
X. SPACE-TIME SUPERSYMMETRY

Let us construct the vector state of zero mass as
\[ \alpha_1^\mu(0)|e_\mu(p)\rangle. \] (114)
The condition \( p^\mu p_\mu = 0 \) and \( L_1|\phi\rangle = 0 \) gives the equation of motion and Lorentz condition,
\[ p^2 e_\mu = 0 \quad \text{and} \quad p^\mu e_\mu = 0 \] (115)
The Ramond fermion give a state
\[ D^\mu_{-1}(0)\gamma^\mu \psi(p). \] (116)
\( \psi(p) \) is a dirac spinor of zero mass, \( \gamma^\mu \) are the usual Dirac Gamma matrices. The condition \( F_1|\phi\rangle = 0 \) gives the Dirac equation
\[ \gamma \cdot p \psi(p) = 0. \] (117)
The bosonic degrees of freedom match the fermionic degrees of freedom.

For closed strings a similar analysis can be made with \( \alpha_1^\mu \tilde{\alpha}_{-1}^\nu|0\rangle F^{\mu\nu} \) and \( \alpha_1^\mu \tilde{D}_{-1}^\nu|0\rangle \gamma^\nu \psi_\mu \) following the details given by G.S.O., we get a massless antisymmetric tensor, massless scalar and a massless symmetric tensor (spin 2) adding upto 4 degrees of freedom. The fermionic sector contains a massless spin 3/2 and a massless 1/2 adding upto a total of four degrees of freedom as has been shown by G.S.O. for the four dimensional case.

XI. APPROACH TO STANDARD MODEL

The main motivation of constructing this theory is to see if the internal symmetry group makes contact with the Standard Model which explains all experimental data upto date with a high degree of accuracy. The internal symmetry of the group was \( SO(44) \). We have divided them into four \( SO(11) \)'s and further subdivided to \( SO(6) \) and \( SO(5) \). According to Slansky \[2\] we can have the breaking
\[ SO(11) \supset SO(6) \times SO(5) \supset G^{st} \] (118)
where
\[ G^{st} = SU_C(3) \times SU_L(2) \times U_Y(1) \times U'(1). \] (119)
This has the content of the Standard Model with an additional gauge boson. The 15 gauge fields of \( SO(6) \) and 10 gauge fields of \( SO(5) \) are contained in the states \( H_{jj'}^{\mu\nu}(p) \) and \( H_{kk'}^{\mu\nu}(p) \) which are massless \( (p^2 = 0) \) with the structure
\[ b_{-1/2}^{\mu,j} b_{-1/2}^{\nu,j'}|0\rangle H_{jj'}^{\mu\nu}(p) \] (120)
and
\[ b_{-1/2}^{\mu,k} b_{-1/2}^{\nu,k'}|0\rangle H_{kk'}^{\mu\nu}(p) \] (121)
From the condition \( F_1|\phi\rangle = 0 \) we have \( p^\mu H^{\mu\nu} = p^\nu H^{\mu\nu} = p^\nu H^{\nu\mu} = p^\nu H^{\mu\nu} = 0 \). There are many vectors and scalars. The vectors are like \( p^\mu A_{jj'}^{\mu} + p^\nu A_{jj'}^{\nu} \) and \( p^\mu A_{kk'}^{\mu} + p^\nu A_{kk'}^{\nu} \). The potentials satifies the equation \( \square A_{jj'}^{\mu} = 0 \), \( p^\nu A_{jj'}^{\nu} = 0 \) and \( \square A_{kk'}^{\nu} = 0 \), \( p^\nu A_{kk'}^{\nu} = 0 \). The \((i,j)\) and \((k,k')\) antisymmetric parts are the 15 and 10 of \( SO(6) \) and \( SO(5) \) respectively.

There are also the massless Ramond spinors \( d_{-1/2}^{\mu,j} |0\rangle \gamma^\nu \psi_j^\nu \) six in number, and \( d_{-1/2}^{\mu,k} |0\rangle \gamma^\nu \psi_k^\nu \), five in number. So in the product space there are 30 chiral spinors which exactly correspond to the 15 chiral fermions and 15 chiral antifermions in one generation of the Standard Model. The particles, \( e_R, u_R, d_R, (\nu, e)_L \) and \( (u, d)_L \) and their antiparticles can be assigned the usual quantum numbers under \( G^{st} \). The \( G^{st} \) with fifteen fermions is exactly space-time super-symmetric has been proven in reference \[2\] recently.

The fifteen Ramond fermions ca be placed in five groups each containing three particles
with the help of the diagonal $SU(3)$ Gallmann matrices $\lambda^B$ and $\lambda^Y$ and Isospin $I_3$, one can construct a matrix operator $Q_c = I_3 + \frac{1}{4}(\sqrt{3}\lambda^B - \lambda^Y)$ which has three eigen values for the first four sets and zero eigen values for the colourless leptons.

It is of interest to note that Higgs boson is classically tachyonic. We cannot show that there are three generations of fermions. Perhaps it is related to the number of ways the 30 chiral fermions can be placed under $G^{st}$. There are many aspects which are not clear to us, but we have made out a case to study this model further to relate Gravity and Standard Model through string theory with a unique vacuum.

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