Combinatorial Approaches in Quantum Information Theory

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CERTIFICATE

This is to certify that the dissertation entitled “Combinatorial Approaches in Quantum Information theory”, submitted by Mr. Sudhir Kumar Singh to the Department of Mathematics, Indian Institute of Technology, Kharagpur, in partial fulfillment of the requirements of the degree of Master of Science in Mathematics and Computing, is an authentic record of the work carried out by him under our supervision and guidance.

In my opinion, this work fulfills the requirements for which it has been submitted and has not been submitted to any other Institution for any degree.

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Abstract

Quantum entanglement is one of the most remarkable aspects of quantum physics. If two particles are in an entangled state, then, even if the particles are physically separated by a great distance, they behave in some respects as a single entity. Entanglement is a key resource for quantum information processing and spatially separated entangled pairs of particles have been used for numerous purposes such as teleportation, superdense coding and cryptography based on Bell’s theorem.

Just as two distant particles could be entangled, it is also possible to entangle three or more separated particles. A well-known application of multipartite entanglement is in testing nonlocality from different directions. Recently, it has also been used for many multi-party computation and communication tasks and multi-party cryptography. One of the major issues in dealing with multi-partite entangled states is of purification. Distilling pure maximally entangled state in this case may not be as simple as that of bipartite case. But if it is possible to create multi-partite entangled states from the bipartite ones then we can first distill pure maximal bipartite states and can then prepare the multi-partite ones.

To this end, we consider the problem of creating maximally entangled multi-partite states out of Bell pairs distributed in a communication network from a physical as well as from a combinatorial perspective. We investigate the minimal combinatorics of Bell pairs distribution required for this purpose and discuss how this combinatorics gives rise to resource minimization for practical implementations. We present two protocols for this purpose. The first protocol enables to prepare a GHZ state using two Bell pairs shared amongst the three users with help of two cbits of communication and local operations. The protocol involves all the three users dynamically and thus can find applications in cryptographic tasks. Second protocol entails the use of $O(n)$ cbits of communication and local operations to prepare an $n$ partite maximally entangled state in a distributed network of bell pairs along a spanning tree of EPR graph of the $n$ users. We show that this spanning tree structure is the minimal combinatorial requirement. We also characterize the minimal combinatorics of agents in the creation of pure maximal
multi-partite entanglement amongst the set $N$ of $n$ agents in a network using apriori multi-partite entanglement states amongst subsets of $N$.

Another major and interesting issue is of quantifying multi-partite entangled states. Multi-partite entangled states, unlike the bipartite ones, lack convenient mathematical properties like Schmidt decomposition and therefore it becomes difficult to characterize them. Some approaches, essentially using the generalization of Schmidt decomposition, have been taken in this direction; however a general formulation in this case is still an outstanding unresolved problem. State transformations under local operations and classical communication (LOCC) are very important while quantifying entanglement because LOCC can at the best increase classical correlations and therefore a good measure of entanglement is not supposed to increase under LOCC. All the current approaches to study the state transformation under LOCC are based on entropic criterion. We present an entirely different approach based on nice combinatorial properties of graphs and set systems. We introduce a technique called *bicolored merging* and obtain several results about such transformations. We demonstrate a partial ordering of multi-partite states and various classes of incomparable multi-partite states. We utilize these results to establish the impossibility of doing *selective teleportation* in a case where the apriori entanglement is in the form of a GHZ state. We also discuss the minimum number of copies of a state required to prepare another state by LOCC and present bounds on this number in terms of *quantum distance* between the two states. The ideas developed in this work continues the combinatorial setting mentioned above and can been extended to incorporate other new kinds of multi-partite states. Moreover, the idea of *bicolored merging* may also be appropriate to some other areas of information sciences.

Key distribution is a fundamental problem in secure communication and quantum key distribution (QKD) protocols for key distribution between two parties on the account of quantum uncertainty and no-cloning principles was realized two decades ago, however the more rigorous and comprehensive proofs of this task, taking into consideration source, device and channel noise as well as an arbitrarily powerful eavesdropper, have been only recently studied by various authors. We consider QKD between two parties extended to that between $n$ trustful parties, that is, how the $n$ parties may share an identical secret key among themselves. We propose a protocol for this purpose and prove its unconditional security. The protocol is simple in the
sense that the proof of its security is established on the basis of the already proven security of the bipartite case. Our protocol works in two broad steps. In the first step, the $n$-partite problem is reduced to a two-party problem. In the second step, the Lo-Chau protocol or Modified Lo-Chau protocol is invoked to prove the unconditional security of sharing nearly perfect EPR pairs between two parties. The first step essentially utilizes the spanning tree combinatorics mentioned above.

An other interesting aspect of multi-party cryptography is to split information through secret sharing and splitting the quantum information has also been studied recently. In conventional quantum secret sharing (QSS) schemes, it is often implicitly assumed that all share-holders carry quantum information. This requirement can be relaxed, whereby some share holders may carry only classical information and no quantum information. Such a hybrid (classical-quantum) QSS that combines classical and quantum secret sharing brings a significant improvement to the implementation of QSS, in as much as quantum information is much more fragile than classical information. The practical implementation of a quantum secret sharing scheme is facilitated if it can be compressed to an equivalent scheme with fewer quantum information carrying players, the reduction compensated by players who carry only classical information. Conversely, a given quantum secret sharing scheme may be inflated by adding only classical information carrying players. To this end, we explore some generalizations of such quantum secret sharing (QSS). We study an extended QSS scheme, wherein some shares may be retained by the share dealer, that enables the construction of access structures with disjoint authorized sets, similar to classical secret sharing. We also propose a hybrid (classical-quantum) generalization of the threshold scheme. Our schemes are based on two interesting ideas. First one is the idea of qubit encryption and encryption key as the relevant classical information. However, in principle any classical data whose suppression leads to maximal ignorance of the secret is also good. The second, while more restricted, is interesting because it is not directly based on quantum erasure correction, but on information dilution via homogenization, in contrast to current proposals of QSS.

QKD involves sharing a random key amongst trustworthy parties where as QSS splits quantum information amongst untrusted parties. We discuss situations where some kind of mutual trust may be present between sets of
parties while parties being individually mistrustful. This way we take a step towards combining the essential features of QKD and QSS. We discuss two problems where this idea is applicable. The first problem is of secure key distribution between two trustful groups where the individual group members may be mistrustful. The two groups retrieve the secure key string, only if all members should cooperate with one another in each group. In the second case, we consider several such groups. Members of the same group trust each other whereas members from different groups do not and the problem is to establish a common shared random key amongst the $n$ untrustful parties. We present protocols for these cases and discuss the proof of their unconditional security. Finally, we conclude briefly with some open research directions based on our research.
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Chapter 1  
Introduction  

1.1 Stepping in to the Quantum World

_The most incomprehensible thing about the world is that it is comprehensible!_  
— Albert Einstein

_I am not happy with all the analyses that go with just classical theory, because Nature isn’t classical, dammit, and if you want to make a simulation of Nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem!_  
— Richard Feynman

This wonderful observation of Feynman in the early 1980’s [Fey82, Fey96] that no classical computer could simulate quantum mechanical systems without incurring exponential slowdown but it might be possible provided the simulator was itself quantum mechanical, stemmed the widespread interest in the field of quantum computation and quantum information. There are three very important features of quantum mechanics which provides us the way to exploit such powerful processors:

1. A quantum particle can exist simultaneously in many incompatible states.
2. We can operate on a quantum particle while it is in a superimposed state and affect all the states at once.
3. One quantum system can influence another far away quantum system instantaneously.
This is the kind of parallelism inherent in quantum world and combined with the present day information processing gives birth to exciting developments: Quantum computation, quantum error correction, quantum entanglement and teleportation and quantum cryptography. Quantum computers are able to solve some problems intractable to conventional computation (problems like prime factorization and discrete logarithm). Quantum error correcting techniques enable us to do quantum computation and communication in present of noise. Quantum cryptosystems provide guaranteed secure communication using no-cloning theorem and uncertainty principle and quantum teleportation provides the way to do quantum communication in absence of a quantum channel using prior quantum entanglement and classical communication.

1.2 Quantum Mechanical Model of Computation

In any model of information processing there are at least three requirements:

1. The representation of the information
2. The operations to be applied on the information
3. The way for extracting result after the operation

In the quantum mechanical model of information processing the information is mathematically represented by a ray in a Hilbert Space, operations are the unitary operators in the space and result is the measurement of an observable described by a hermitian operator in the space.

1.2.1 Representing Quantum Information: Qubits and Quantum Registers

The first postulate of quantum mechanics sets up the arena in which quantum mechanics takes place. The arena is our familiar friend from linear algebra, Hilbert space.

*Postulate 1:* Associated to any isolated physical system is a Hilbert space known as the *state space* of the system. The system is completely described
by its state vector, which is a unit (ray) in the system’s state space.

The simplest non-trivial Hilbert space is of dimension two and a state vector in the state space of dimension two is called a qubit (stands for a quantum bit). Suppose $|0\rangle$ and $|1\rangle$ form an orthonormal basis for that state space. Then an arbitrary state vector in the state space can be written

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$

where $a$ and $b$ are complex numbers. The condition that $|\psi\rangle$ be a unit (ray), $<\psi|\psi> = 1$, is therefore equivalent to $|a|^2 + |b|^2 = 1$.

Nature is not so simple and a qubit is not sufficient to deal with its complexity. We must be interested in composite system made of two (or more) distinct physical systems and we must also have a mathematical way of playing around with them. In analogy to the classical terminology, a composite quantum system i.e. a set of qubits is called a quantum register. The following postulate describes how the state space of a composite system (quantum register) is built up from the state spaces of the component systems (the qubits).

**Postulate 2:** The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through $n$, and a number $i$ is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$, where $\otimes$ denotes tensor product.

### 1.2.2 Unitary Evolution: Quantum Gates

Time evolution of a quantum state is unitary; it is generated by a self-adjoint (Hermitian) operator, called the Hamiltonian of the system. In the Schrödinger picture of dynamics, the vector describing the system moves in time as governed by the Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle$$

where $H$ is the Hamiltonian. We may express this equation, to first order in the infinitesimal quantity $dt$, as

$$|\psi(t + dt)\rangle = (1 - iHdt)|\psi(t)\rangle.$$ 

Clearly, the operator $U(dt) \equiv 1 - iHdt$ is unitary. Thus the time evolution over a finite interval is unitary given by
\[ |\psi(t)\rangle = U(t)|\psi(0)\rangle. \]

**Postulate 3:** The evolution of a closed quantum system is described by a unitary transformation. That is, the state \( |\psi\rangle \) of the system at time \( t_1 \) is related to the state \( |\psi'\rangle \) of the system at time \( t_2 \) by a unitary operator \( U \) which depends only on the times \( t_1 \) and \( t_2 \),

\[ |\psi'\rangle = U|\psi\rangle. \]

Now that a quantum system evolves according to a unitary operator which is always invertible, quantum gates must be reversible. In fact, quantum gates are nothing but these unitary operations. Following are some commonly used one qubit and two qubit gates in terms of their unitary operations represented by matrices in the computational basis.

**Pauli’s Gates (Operators):**

**X**

\[
\begin{pmatrix}
|0\rangle & |1\rangle \\
|0\rangle & 0 & 1 \\
|1\rangle & 1 & 0 \\
\end{pmatrix}
\]

**Y**

\[
\begin{pmatrix}
|0\rangle & |1\rangle \\
|0\rangle & 0 & -i \\
|1\rangle & i & 0 \\
\end{pmatrix}
\]

**Z**

\[
\begin{pmatrix}
|0\rangle & |1\rangle \\
|0\rangle & 1 & 0 \\
|1\rangle & 0 & -1 \\
\end{pmatrix}
\]

**Hadamard Gate:**

\[
\begin{pmatrix}
|0\rangle & |1\rangle \\
|0\rangle & 1/\sqrt{2} & 1/\sqrt{2} \\
|1\rangle & 1/\sqrt{2} & -1/\sqrt{2} \\
\end{pmatrix}
\]

**Controlled- NOT (CNOT) Gate:** A two qubit gate

\[
\begin{pmatrix}
|00\rangle & |01\rangle & |10\rangle & |11\rangle \\
|00\rangle & 1 & 0 & 0 & 0 \\
|01\rangle & 0 & 1 & 0 & 0 \\
|10\rangle & 0 & 0 & 0 & 1 \\
|11\rangle & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
The Postulate 3 requires that the system being described be closed. That is, it is not interactive in any way with other systems. In reality, of course, all systems (except the Universe as a whole) interact at least somewhat with the other systems. Nevertheless, there are interesting systems which can be described by unitary evolution to some good approximation. Furthermore, at least in principle every open system can be described as part of a larger closed system (the Universe) which is undergoing unitary evolution.

1.2.3 Quantum Measurement: Observables

An observable is a property of a physical system that in principle can be measured. In quantum mechanics, an observable is a hermitian operator. We also know that a hermitian operator in a Hilbert space $H$ has a spectral decomposition—its eigenstates form a complete orthonormal basis in $H$. We can express a hermitian operator $A$ as

$$ A = \sum_n a_n P_n. $$

Here each $a_n$ is an eigen value of $A$, and $P_n$ is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalues $a_n$. (If $a_n$ is non-degenerate, then $P_n = |n\rangle\langle n|$; it is the projection onto the corresponding eigenvector.) The $P_n$ satisfy

$$ P_n P_m = \delta_{n,m} P_n, $$

$$ P_n^\dagger = P_n. $$

**Postulate 4:** In quantum mechanics, the numerical outcome of a measurement of the observable $A$ is an eigenvalue of $A$; right after the measurement, the quantum state is an eigenstate of $A$ with the measured eigenvalue. If the quantum state just prior to the measurement is $|\psi\rangle$, then the outcome $a_n$ is obtained with the probability

$$ \text{Prob}(a_n) = \| P_n|\psi\rangle \|^2 = \langle \psi|P_n|\psi\rangle; $$

If the outcome attained is $a_n$, then the (normalized) quantum state becomes

$$ \frac{P_n|\psi\rangle}{\sqrt{\langle \psi|P_n|\psi\rangle}}. $$

(Note that if the measurement is immediately repeated, then according to this rule the same outcome is attained again, with probability one.)
1.3 Quantum Entanglement

Quantum mechanics builds systems out of subsystems in a remarkable, holistic way. The states of the subsystems do not determine the state of the system. Schrödinger, commenting on the EPR paper [EPR35] in 1935, the year it appeared, coined the term *entanglement* for this aspect of quantum mechanics.

Consider a system consisting of two subsystems. Quantum mechanics associates to each subsystem a Hilbert space. Let $H_A$ and $H_B$ denote these two Hilbert spaces; let $|i_A⟩$ (where $i=1,2,...$) represent a complete orthonormal basis for $H_A$, and $|i_B⟩$ (where $i=1,2,...$) represent a complete orthonormal basis for $H_B$. Quantum mechanics associates to the system—the two subsystem taken together—the Hilbert space $H_A \otimes H_B$, namely the Hilbert space spanned by the states $|i_A⟩ \otimes |i_B⟩$. In the following we will drop the tensor product symbol $\otimes$ and write $|i_A⟩ \otimes |i_B⟩$ as $|i_A⟩|i_B⟩$.

Any linear combinations of the basis states $|i_A⟩|i_B⟩$ is a state of the system, any state $|ψ⟩_{AB}$ of the system can be written as

$$|ψ⟩_{AB} = \sum_{i,j} c_{i,j} |i_A⟩|i_B⟩,$$

where the $c_{i,j}$ are complex coefficients, we take $|ψ⟩_{AB}$ to be normalized, hence $\sum_{i,j} |c_{i,j}|^2 = 1$.

A special case of the above state is a direct product state in which $|ψ⟩_{AB}$ factors into (a tensor product of) a normalized state $|ψ^{(A)}⟩_A = \sum_i c_i^{(A)} |i⟩_A$ in $H_A$ and a normalized state $|ψ^{(B)}⟩_B = \sum_j c_j^{(B)} |j⟩_B$ in $H_B$.

$$|ψ⟩_{AB} = |ψ^{(A)}⟩_A |ψ^{(B)}⟩_B = (\sum_i c_i^{(A)} |i⟩_A)(\sum_j c_j^{(B)} |j⟩_B).$$

Not every state in $H_A \otimes H_B$ is a product state. Take, for example, the state $(|1⟩_A|1⟩_B + |2⟩_A|2⟩_B)/\sqrt{2}$; if we try to write it as a direct product of states of $H_A$ and $H_B$, we will find that we can not.

If $|ψ⟩_{AB}$ is not a product state, we say that it is entangled.

Thus when two quantum subsystems are entangled, we may have a complete knowledge of the composite system as a whole but not of the individual subsystems. Technically speaking, the system as a whole may be in a pure state while individual subsystems still being in mixed states.

Entanglement is a key resource for quantum information processing and
spatially separated entangled pairs of particles have been used for numerous purposes like teleportation [BBJPW93], superdense coding [BW92] and cryptography based on Bell’s Theorem [Ekert91], to name a few. We shall see some novel and interesting characterization and applications of entanglement in this thesis.
Chapter 2

Characterizing the Combinatorics of Distributed EPR Pairs for Multi-partite Entanglement

2.1 Introduction

Quantum entanglement is one of the most remarkable aspects of quantum physics. Two particles in an entangled state behave in some respects as a single entity even if the particles are physically separated by a great distance. The entangled particles exhibit what physicists call non-local effects. Such non-local effects were alluded to in the famous 1935 paper by Einstein, Podolsky, and Rosen [EPR35] and were later referred to as spooky actions at a distance by Einstein. In 1964, Bell [Bell64] formalized the notion of two-particle non-locality in terms of correlations amongst probabilities in a scenario where measurements are performed on each particle. He showed that the results of the measurements that occur quantum physically can be correlated in a way that cannot occur classically unless the type of measurement selected to be performed on one particle affects the result of the measurement performed on the other particle. The two particles thus correlated maximally are called EPR pairs or Bell pairs. Non-local effects, however, without being supplemented by additional quantum or classical communication, do not convey any signal and therefore the question of faster than light communi-
Entanglement is a key resource for quantum information processing and spatially separated entangled pairs of particles have been used for numerous purposes like teleportation [BBCJPW93], superdense coding [BW92] and cryptography based on Bell’s Theorem [Ekert91], to name a few. An EPR channel (a bipartite maximally entangled distributed pair of entangled particles) can be used in conjunction with a classical communication channel to send an unknown quantum state of a particle to a distant particle. The original unknown quantum state is destroyed in the process and reproduced at the other end. The process does not copy the original state; it transports a state and thus does not violate the quantum no-cloning theorem [WZ82]. This process is called teleportation and was proposed by Bennett et al. in their seminal work [BBCJPW93]. In teleportation, a quantum communication channel is simulated using a classical channel and an EPR channel. A classical channel can also be simulated using a quantum channel and an EPR channel using superdense coding as proposed by Bennett and Wiesner [BW92]. Two qubits (classical bits) are compressed into one and sent through an EPR channel to the distant party, who then recovers the qubits by local operations.

The use of EPR pairs for cryptography was first proposed by Ekert in 1991 [Ekert91]. He proposed a protocol based on generalized Bell’s theorem for quantum key distribution between two parties. The two parties share an EPR pair in advance. They do a computational basis measurement on their respective qubits and the measurement result is then used as the one bit shared key. While the measurement result is maximally uncertain, the correlation between their results is deterministic. Based on similar principles, a multiparty quantum key distribution protocol using EPR pairs in a distributed network and its proof of unconditional security has been proposed by Singh and Srikanth [SS03A] (Chapter 4). Apart from these applications, entanglement has been used in several other applications such as cheating bit commitment [LC97], broadcasting of entanglement [BVPKH97] and testing Bell’s inequalities [Bell64, CHSH69, GP92].

Just as two distant particles could be entangled forming an EPR pair, it is also possible to entangle three or more separated particles. One example (called GHZ state) is due to Greenberger, Horne and Zeilinger [GHZ89];
three particles are entangled. A well-known manifestation of multipartite entanglement is in testing nonlocality from different directions [GHZ89, Mer90, HM95, DP97]. Recently, it has also been used for many multi-party computation and communication tasks [BCD01, BDHT99, BBT03, Grov97, PSK03] and multi-party cryptography [HBB99, SG01, BVK98]. Buhrman, Cleve and van Dam [BCD01], make use of three-party entanglement and demonstrate the existence of a function whose computation requires strictly lesser classical communication complexity compared to the scenario where no entanglement is used. Brassard et al. in [BBT03], show that prior multipartite entanglement can be used by n agents to solve a multi-party distributed problem, whereas no classical deterministic protocol succeeds in solving the same problem with probability away from half by a fraction that is larger than an inverse exponential in the number of agents. Buhrman et al. [BDHT99] solves an n-party problem which shows a separation of n versus Θ(n log n) bits between quantum and classical communication complexity. For one round, three-party problem this article also proved a difference of (n + 1) versus ((3/2)n + 1) bits between communication with and without initial entanglement. In [PSK03], Pal et al. present a fair and unbiased leader election protocol using maximal multi-partite entanglement.

Quantum teleportation strikingly underlines the peculiar features of the quantum world. All the properties of the quantum state being transferred are retained during this process. So it is natural to ask whether one qubit of an entangled state can be teleported while retaining its entanglement with the other qubit. The answer is, not surprisingly, in the affirmative. This is called entanglement swapping. Yurke and Stoler [YS92] and Zukowski et al [ZZHE93] have shown that through entanglement swapping one can entangle particles that do not even share any common past. This idea has been generalized to the tripartite case by Zukowski et al. in [ZHWZ95] and later to the multipartite case by Zeilinger et al. [ZHWS97] and Bose et al. [BVK98]. Zeilinger et al. presented a general scheme and realizable procedures for generating GHZ states out of two pairs of entangled particles from independent emissions. They also proposed a scheme for observing four-particle GHZ state and their scheme can directly be generalized to more particles. To create a maximally entangled state of (n + m − 1) particles from two groups, one of n maximally entangled particles and the other of m maximally entangled particles, it is enough to perform a controlled operation between a particle from the first group and a particle from the second group and then a mea-
measurement of the target particle. We observe that if particles are distributed in a network then a single cbit of communication is required for broadcasting the measurement result to construct the desired \((n + m - 1)\) maximally entangled state using local operations. In \cite{BVK98}, Bose et al. have generalized the entangled swapping scheme of Zukowski et al. in a different way. In their scheme, the basic ingredient is a projection onto a maximally entangled state of \(N\) particles. Each of the \(N\) users needs to share a Bell pair with a central exchange. The central exchange then projects the \(N\) qubits with it on to an \(N\) particle maximally entangled basis, thus leaving the \(N\) users in an \(N\) partite maximally entangled state. However, we note that in order to get a desired state, the measurement result obtained by the central exchange must be broadcast so that the end users can appropriately apply requisite local operations. This involves \(N\) cbits of communication.

In this chapter, we consider the problem of creating pure maximally entangled multi-partite states out of Bell pairs distributed in a communication network from a physical as well as a combinatorial perspective. We investigate and characterize the minimal combinatorics of the distribution of Bell pairs and show how this combinatorics gives rise to resource minimization in long-distance quantum communication. We present protocols for creating maximal multi-partite entanglement. The first protocol (see Theorem 1) enables us to prepare a GHZ state using two Bell pairs shared amongst the three agents with the help of two cbits of communication and local operations with the additional feature that this protocol involves all the three agents dynamically. Such a protocol with local dynamic involvement in creating entanglement may find applications in cryptographic tasks. The second protocol (see Theorem 4) entails the use of \(O(n)\) cbits of communication and local operations to prepare a pure \(n\)-partite maximally entangled state in a distributed network of Bell pairs; the requirement here is that the pairs of nodes sharing EPR pairs must form a connected graph. We show that a spanning tree structure (see Theorem 3) is the minimal combinatorial requirement for creating multi-partite entanglement. We also characterize the minimal combinatorics of agents in the creation of pure maximal multi-partite entanglement amongst the set \(N\) of \(n\) agents in a network using apriori multi-partite entanglement states amongst subsets of \(N\). This is done by generalizing the EPR graph representation to an entangled hypergraph and the requirement here is that the entangled hypergraph representing the entanglement structure must be connected.
Figure 2.2.1: A shares an EPR pair with each of B and C.

The chapter is organized as follows. In Section 2.2 we present our protocol I to prepare a GHZ state from two Bell pairs involving all the three agents dynamically and compare our protocol in the light of existing schemes. Section 2.3 is devoted to characterizing the spanning tree combinatorics of Bell pairs for preparing a pure $n$-partite maximally entangled state. We develop our protocol II for this purpose. In Section 2.4 we generalize the results of Section 2.3 to the setting where subsets of agents in the network share apriori pure multi-partite maximally entangled states. Finally in Section 2.5 we compare our scheme of Section 2.3 with the multipartite entanglement swapping scheme of Bose et al. [BVK98], observe the similarity between Helly-type theorems and the combinatorics developed in Sections 2.3 and 2.4 and conclude with a few remarks on open research directions.

2.2 Preparing a GHZ State from Two EPR Pairs Shared amongst Three Agents

In this section we consider the preparation of a GHZ state from two EPR pairs shared amongst three agents in a distributed network. We establish the following theorem.
Figure 2.2.2: Entangling qubits $a3$ with the EPR pair between $A$ and $B$.

**Theorem 1** If any two pairs of the three agents $A$ (Alice), $B$ (Bob) and $C$ (Charlie) share EPR pairs (say the state $(|00⟩ + |11⟩)/\sqrt{2}$) then we can prepare a GHZ state $(|000⟩ + |111⟩)/\sqrt{2}$ amongst them with two bits of classical communication, while involving all the three agents dynamically.

**Proof:** The proof follows from the Protocol I.

*The Protocol I:* Without loss of generality let us assume that the sharing arrangement is as in Figure 2.2.1. $A$ shares an EPR pair with $B$ and another EPR pair with $C$ but $B$ and $C$ do not share an EPR pair. This means that we have the states $(|0_a10_b⟩ + |1_a11_b⟩)/\sqrt{2}$ and $(|0_a20_c⟩ + |1_a21_c⟩)/\sqrt{2}$ where subscripts $a1$ and $a2$ denote the first and second qubits with $A$ and subscripts $b$ and $c$ denote qubits with $B$ and $C$, respectively.

Our aim is to prepare $(|0_a10_b0_c⟩ + |1_a21_b1_c⟩)/\sqrt{2}$ or $(|0_a20_b0_c⟩ + |1_a21_b1_c⟩)/\sqrt{2}$. We need three steps to do so.

**Step 1:** $A$ prepares a third qubit in the state $|0⟩$. We denote this state as $|0_{a3}⟩$ where the subscript $a3$ indicates that this is the third qubit of $A$.

**Step 2:** $A$ prepares the state $(|0_a10_b0_c⟩ + |1_a21_b1_c⟩)/\sqrt{2}$ using the circuit in Figure 2.2.2.

**Step 3:** $A$ sends her third qubit to $C$ with the help of the EPR channel $(|0_a20_c⟩ + |1_a21_c⟩)/\sqrt{2}$. A straightforward way to do this is through teleportation. This method however, does not involve both of $B$ and $C$ dynamically. By a party being dynamic we mean that the party is involved in applying
the local operations for the completion of the transfer of the state of third qubit to create the desired GHZ state.

We use our new and novel teleportation circuit as shown in Figure 2.2.3 where both $B$ and $C$ are dynamic. The circuit works as follows. $A$ has all her three qubits with her and can do any operation she wants to be performed on them. Initially the five qubits are jointly in the state $|\phi_1\rangle$. $A$ first applies a controlled NOT gate on her second qubit controlling it from her third qubit changing $|\phi_1\rangle$ to $|\phi_2\rangle$. Then she measures her second qubit yielding measurement result $M_2$ and bringing the joint state to $|\phi_3\rangle$. She then applies a Hadamard gate on her third qubit and the joint state becomes $|\phi_4\rangle$. A measurement on the third qubit is then done by her yielding the result $M_1$ and bringing the joint state to $|\phi_5\rangle$. She then applies a NOT (Pauli’s $X$ operator) on her first qubit, if $M_2$ is 1. Now she sends the measurement results $M_2$ to $B$ and $M_1$ to $C$. $B$ applies an $X$ gate on his qubit if he gets 1 and $C$ applies a $Z$ gate (Pauli’s $Z$ operator) if he gets 1. The order in which $B$ and $C$ apply their operations does not matter. The final state is $|\phi_7\rangle$. The circuit indeed produces the GHZ state between $A$, $B$ and $C$ as can be seen from the detailed mathematical explanation of the circuit given below. It can be noted that this protocol requires two cbits of communication.

The above circuit can be explained as follows:

$$|\phi_1\rangle = (|0_a1_0_b0_a3\rangle + |1_a1_1_b1_a3\rangle)(|0_a20_c\rangle + |1_a21_c\rangle)/2,$$

$$|\phi_2\rangle = [|0_a1_0_b0_a3\rangle(|0_a20_c\rangle + |1_a21_c\rangle) + |1_a1_1_b1_a3\rangle(|1_a20_c\rangle + |0_a21_c\rangle)]/2.$$  

**Case 1: $M_2 = 0$**

$$|\phi_3\rangle = (|0_a1_0_b0_a30_a20_c\rangle + |1_a1_1_b1_a30_a21_c\rangle)/\sqrt{2}$$

$$= (|0_a1_0_b0_a30_c\rangle + |1_a1_1_b1_a31_c\rangle)|0_a2\rangle/\sqrt{2},$$

$$|\phi_4\rangle = (|0_a1_0_b0_a30_c\rangle + |0_a1_0_b1_a30_c\rangle + |1_a1_1_b0_a31_c\rangle - |1_a1_1_b1_a31_c\rangle)|0_a2\rangle/2$$

$$= [(|0_a1_0_b0_c\rangle + |1_a1_1_b1_c\rangle)|0_a3\rangle + (|0_a1_0_b0_c\rangle - |1_a1_1_b1_c\rangle)|1_a3\rangle]|0_a2\rangle/2.$$  

When $M_1 = 0$,

$$|\phi_5\rangle = (|0_a1_0_b0_c\rangle + |1_a1_1_b1_c\rangle)|0_a2\rangle/\sqrt{2},$$

$$16.$$
Figure 2.2.3: Circuit for creating a GHZ state from two EPR pairs with dynamic involvement of both $B$ and $C$. 
\[ |\phi_0\rangle = (|0_a10_b0_c\rangle + |1_a11_b1_c\rangle)|0_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_7\rangle = (|0_a10_b0_c\rangle + |1_a11_b1_c\rangle)|0_{a2}\rangle/\sqrt{2}. \]

When \( M_1 = 1, \)
\[ |\phi_5\rangle = (|0_a10_b0_c\rangle - |1_a11_b1_c\rangle)|0_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_6\rangle = (|0_a10_b0_c\rangle - |1_a11_b1_c\rangle)|1_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_7\rangle = (|0_a10_b0_c\rangle + |1_a11_b1_c\rangle)|1_{a2}\rangle/\sqrt{2}. \]

Case 2: \( M_2 = 1 \)
\[ |\phi_3\rangle = (|0_a10_b0_a31_c\rangle + |1_a11_b1_a31_c\rangle)/\sqrt{2} = (|0_a10_b0_a31_c\rangle + |1_a11_b1_a31_c\rangle)|0_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_4\rangle = (|0_a10_b0_a31_c\rangle + |0_a10_b1_a31_c\rangle + |1_a11_b0_a30_c\rangle - |1_a11_b1_a30_c\rangle)|0_{a2}\rangle/2 \]
\[ = [(|0_a10_b1_c\rangle + |1_a11_b0_c\rangle)|0_{a3}\rangle + (|0_a10_b1_c\rangle - |1_a11_b0_c\rangle)|1_{a3}\rangle]|0_{a2}\rangle/2. \]

When \( M_1 = 0, \)
\[ |\phi_5\rangle = (|0_a10_b1_c\rangle + |1_a11_b0_c\rangle)|0_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_6\rangle = (|1_a11_b1_c\rangle + |0_a10_b0_c\rangle)|0_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_7\rangle = (|1_a11_b1_c\rangle + |0_a10_b0_c\rangle)|0_{a2}\rangle/\sqrt{2}. \]

When \( M_1 = 1, \)
\[ |\phi_5\rangle = (|0_a10_b1_c\rangle - |1_a11_b0_c\rangle)|1_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_6\rangle = (|1_a11_b1_c\rangle - |0_a10_b0_c\rangle)|1_{a2}\rangle/\sqrt{2}, \]
\[ |\phi_7\rangle = (-|1_a11_b1_c\rangle - |0_a10_b0_c\rangle)|1_{a2}\rangle/\sqrt{2} \]
\[ = (|1_a11_b1_c\rangle + |0_a10_b0_c\rangle)|1_{a3}\rangle|1_{a2}\rangle/\sqrt{2}. \]
The roles of $B$ and $C$ are symmetrical. Nevertheless, there is a condition on what operations they should perform when they get a single cbit from $A$. $B$ performs an $X$ and $C$ performs a $Z$ operation, as required. We set a cyclic ordering $A \rightarrow B \rightarrow C \rightarrow A$. Let $A$ be the one sharing EPR pairs with the other two; $A$ is the first one in the ordering. The second one is $B$, and he must perform an $X$ operation when he gets a single cbit from $A$. The third one is $C$, and he must perform a $Z$ operation on his qubit when he gets a single cbit from $A$. If $B$ is the one sharing EPR pairs with the other two then $C$ applies an $X$ on his qubit after getting a cbit from $B$ and, $A$ applies $Z$ on her qubit after getting a cbit from $B$ and so on.

As we mentioned in the introduction, methods for creating a GHZ state from Bell pairs have also been discussed by Zukowski et al. [ZZW95] and Zeilinger et al. [ZHWZ97]. First one uses three Bell pairs for this purpose, therefore our protocol seems better than theirs in the sense that it uses only two Bell pairs. The later, however, uses only two Bell pairs and only one cbit of communication and seems to be better than our method at first sight. However, the most interesting fact and the motivation for developing our protocol is the dynamic involvement of both $B$ and $C$ which was lacking in the above methods. It might be highly desired in many multi-party interactive quantum protocols and multi-party cryptography (viz. secret sharing) that both $B$ and $C$ take part actively, say for fairness. By fairness we mean that every party has an equal chance for participating and effecting the protocol in probabilistic sense. It should be interesting to implement this in practical situation.

2.3 Preparing a Pure $n$-partite Maximally Entangled State from EPR Pairs Shared amongst $n$ Agents

**Definition 1** EPR graph: Suppose there are $n$ agents. We denote them as $A_1, A_2, ..., A_n$. Construct an undirected graph $G = (V, E)$ as follows:

$$V = \{A_i : i = 1, 2, 3, ..., n\},$$
$$E = \{\{A_i, A_j\} : A_i \text{ and } A_j \text{ share an EPR pair}, 1 \leq i, j \leq n; i \neq j\}.$$ 

We call the graph $G = (V, E)$, thus formed, the **EPR graph** of the $n$ agents.
Our definition should not be confused with the *entangled graph* proposed by Plesch and Buzek [PB03A, PB03B]. In entangled graph edges represent any kind of entanglement and not necessarily maximal entanglement and therefore there is no one to one correspondence between graphs and states. EPR graph is unique up to different EPR pairs. Moreover, we are not concerned with classical correlations which are also represented by different kind of edges in entangled graphs. In an EPR graph, two vertices are connected by an edge if and only if they share an EPR pair.

**Definition 2** Spanning EPR Tree: We call an *EPR Graph* $G = (V, E)$ as *spanning EPR tree* when the undirected graph $G = (V, E)$ is a spanning tree [CLR90].

We are now ready to develop protocol-II to create the $n$-partite maximally entangled state $(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$ along a spanning EPR tree. The protocol uses only $O(n)$ cbits of communication and local operations. We do not use any qubit communication after the distribution of EPR pairs to form a spanning EPR tree.

**Protocol II:** Let $G = (V, E)$ be the spanning EPR tree. Since $G$ is a spanning EPR tree, it must have a vertex say $T = A_i$ which has degree one (the number of edges incident on a vertex is called its degree). Note that vertices of $G$ are
denoted $A_i$, where $1 \leq i \leq n$. Let $S = A_s$ be the unique vertex connected to $T$ by an edge in $G$ and $L$ be the set of all vertices of $G$ having degree one. The vertex $T$ and its only neighbor $S$ are vertices we start with. We eventually prepare the $n$-partite maximally entangled state using the three steps summarized below. In the first step, one cbit (say 1) is broadcasted by $S$ to signal other $(n-1)$ parties that the protocol for the preparation of the $n$-partite entangled state is about to commence. The second step creates the GHZ state between $S = A_s$, $T = A_t$ and another neighbour $R = A_r$ of $S$. The third step is the main inductive step where multi-partite entanglement states are created in a systematic manner over the spanning EPR tree. At the end of step 3, when the $n$-partite entangled state is ready, one cbit of broadcasting from all the $(k-1)$ elements of $L \setminus \{T\}$ (terminal or degree one vertices of $G$) is expected. The $(k-1)th$ such cbit indicates that the protocol is over and that maximally entangled state is ready. The details are stated below.

**Step 1:** $S$ broadcast one classical bit to signal the other $(n-1)$ agents that the preparation of an $n$-partite entangled state is going to be started and they must not use their EPR pairs for a qubit teleportation amongst themselves. In other words, they must save their EPR pairs in order to use them for the preparation of the $n$-partite entangled state.

**Step 2:** Clearly $S$ must be connected to a vertex $R$ (say $A_r$) other than $T$ by an edge in $G$, otherwise $G$ will not be a spanning EPR tree. A GHZ state among $S$, $T$ and $A_r$ is created. The GHZ state can be prepared either by using usual teleportation circuit, the symmetric circuit of protocol I or by the Zeilinger et al. scheme. Thus using the EPR pairs $((|0_s,0_t,0_r\rangle + |1_s,1_t,0_r\rangle)/\sqrt{2}$ and $(|0_s,0_r,0_t\rangle + |1_s,0_r,1_t\rangle)/\sqrt{2}$, we prepare the GHZ state $(|0_s,0_t,0_r\rangle + |1_s,1_t,1_r\rangle)/\sqrt{2}$. Here the double subscript $i, j$ denotes that in preparing the given state, EPR pairs among the agents $A_i$ and $A_j$ have been used. Here, $A_s = S, A_r = R$ and $A_t = T$.

**Step 3:** Suppose we are currently at vertex $A_i$ and we have already prepared the $m$-partite maximally entangled state, say

$$((|0_{i_1,j_1},0_{i_2,j_2},0_{i_3,j_3},...,0_{i_m,j_m}\rangle + |1_{i_1,j_1},1_{i_2,j_2},1_{i_3,j_3},...,1_{i_m,j_m}\rangle)/\sqrt{2},$$

where $i_1 = s, j_1 = t, i_2 = t, j_2 = s, i_3 = r, j_3 = s$ and $i = ir$ for some $1 \leq r \leq m$.

The vertex $A_i$ starts as follows. As soon as he gets two cbits from one of his neighbors, he completes the operations required for the success of
teleportation and starts processing as follows. If \( A_i \in L \) then \( A_i \) broadcast a single cbit. Otherwise, (when \( A_i \notin L \)) let \( A_{k1}, A_{k2}, ..., A_{kp} \) be the vertices connected to \( A_i \) by an edge in \( G \) such that \( k1, k2, ..., kp \) are not in the already entangled set with vertex indices \( \{i1, i2, i3, ..., im\} \). \( A_i \) takes an extra qubit and prepares this qubit in the state \(|0\rangle\) denoted by \(|0_i\rangle\). He then prepares the state
\[
(|0_{i1,j1}0_{i2,j2}0_{i3,j3}...0_{im,jm}0_i\rangle + |1_{i1,j1}1_{i2,j2}1_{i3,j3}...1_{im,jm}1_i\rangle)/\sqrt{2}
\]
using the circuit in Figure 2.3.4. Finally, he teleports his extra qubit to \( A_{k1} \) using the EPR pair \((|0_{i,k1}0_{k1,i}\rangle + |1_{i,k1}1_{k1,i}\rangle)/\sqrt{2}\), thus enabling the preparation of the \((m + 1)\)-partite maximally entangled state:
\[
(|0_{i1,j1}0_{i2,j2}0_{i3,j3}...0_{im,jm}0_{k1,i}\rangle + |1_{i1,j1}1_{i2,j2}1_{i3,j3}...1_{im,jm}1_{k1,i}\rangle)/\sqrt{2}.
\]

\( A_i \) repeats this until no other vertex, which is connected to it by an edge in \( G \), is left.

Step 3 is repeated until one cbit each from the elements of \( L \) (except for \( T \)) is broadcasted, indicating that all vertices in \( L \) as well as in \( V \setminus L \) have got entangled.

Note that more than one vertex might be processing Step 3 at the same time. This however does not matter since local operations do not change the reduced density matrix of other qubits. Moreover, while processing the Step 3 together, such vertices will no longer be directly connected by an edge in \( G \).

Now we determine the communication complexity of protocol II, the number of cbits used in creating the \( n \)-partite maximally entangled state. Step 1 involves one cbit broadcast by \( S \) to signal the initiation of the protocol. To create the GHZ state in Step 2, atmost 2 cbits is required. In Step 3, teleportation is used to create an \((m + 1)\)-partite maximally entangled state from that of \( m \)-partite. Such \((n - 3)\) teleportation steps are used in this Step entailing \(2(n - 3)\) cbits of communication. Finally, \((k - 1)\) cbits are broadcast by terminal vertices (except \( T \)). Thus the total cbits used in protocol II is \(1 + 2 + 2(n - 3) + k - 1 = 2n + k - 4 \leq 2n + n - 1 - 4 = 3n - 5 = O(n)\).

Protocol-II leads to the following interesting theorem.

**Theorem 2** If the combinatorial arrangement of distributed EPR pairs amongst \( n \) agents forms a spanning EPR tree, then the \( n \)-partite maximally entangled
state $(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$ can be prepared amongst them with $O(n)$ bits of classical communication.

Theorem 2 thus gives a sufficient condition for preparing a maximally entangled $n$-partite state in a distributed network of EPR pairs. In order to prove this sufficiency, we have also developed two more protocols which require $O(n)$ cbits of communication. The first two steps of these protocols are essentially the same as that of Protocol II. The first protocol involves all the agents already entangled in each iteration in Step 3, where, a circuit very similar to the symmetric teleportation circuit (Figure 2.2.3) of Protocol I is used. The classical communication cost is $(2n - 4)$ bits. The second protocol uses a generalization of the method of Zeilinger et al. in each iteration of Step 3 and requires $(2n - 3)$ cbits of communication. In this paper we have presented only Protocol II instead of these two protocols because of simplicity and the direct use of teleportation.

The question of interest now is that of determining the minimal structure or combinatorics of the distribution of EPR pairs necessary for creating the $n$-partite maximally entangled state. In other words, we wish to characterize necessary properties to be satisfied by the EPR graph for this purpose. We argue below that the EPR graph, indeed, must contain a spanning EPR tree, and must therefore be connected. We assume for the sake of contradiction that the EPR graph $G$ is not connected. Then, it must have at least two components, say $C_1$ and $C_2$. No member of $C_1$ is connected to any member of $C_2$ by an edge in $G$. This means that no member of $C_1$ is sharing an EPR pair with any member of $C_2$. Suppose a protocol $P$ can create a pure $n$-partite maximally entangled state starting from the disconnected EPR graph $G$ of $n$ agents. If we are able to create an $n$-partite maximally entangled state using protocol $P$ with this structure using only classical communication and local operations, it is easy to see that we will also be able to create an EPR pair between two parties that were not earlier sharing any EPR pair, using just local operations and classical communication. This can be done as follows. Let $A$ be the first party that possesses all the qubits of his group (say $C_1$) and $B$ be the second party that possesses all the qubits of his group (say $C_2$). Now the protocol $P$ is run on this structure to create the $n$-partite maximally entangled state. Then, $A$ ($B$) disentangles all of his qubits except one by reversing the circuit in Figure 2.3.4; this leaves $A$ and $B$ sharing an EPR pair. This means that two parties which were never sharing an EPR
pair are able to share it just by local operations and classical communication (LOCC). This is forbidden by fundamental laws in quantum information theory (LOCC cannot increase the expected entanglement \cite{Ve02}), hence $G$ must be connected. Note that no qubit communication is permitted after the formation of EPR graph $G$. We present this necessary condition in the following theorem.

**Theorem 3** A necessary condition that the $n$-partite maximally entangled state $(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$ be prepared in a distributed network permitting only EPR pairs for pairwise entanglement between agents is that the EPR graph of the $n$ agents must be connected.

It can be noted that after the preparation of the state $(|000\ldots0\rangle + |111\ldots1\rangle)/\sqrt{2}$, any other pure $n$-partite maximally entangled state can also be prepared by just using local operations. We also know that any connected undirected graph contains a spanning tree \cite{CLR90}. Thus a connected EPR graph will contain a spanning EPR tree. With this observations, we combine the above two theorems in the following theorem.

**Theorem 4** Amongst $n$ agents in a communication network permitting only pairwise entanglement in the form of EPR pairs, a pure $n$-partite maximally entangled state can be prepared if and only if the EPR graph of the $n$ agents is connected.

### 2.4 Entangling a Set of Agents from Entangled States of Subsets: Combinatorics of General Entanglement Structure

In the previous section we have presented the necessary and sufficient condition for preparing a pure multi-partite maximally entangled state in a distributed network of EPR pairs (see Theorem 3). However, agents may not be connected by EPR pairs in a general network. We assume that subsets of agents may be sharing pure maximally entangled states. So, some triples of agents may be GHZ entangled, some pairs of agents may share EPR pairs and some subsets of agents may share even higher dimensional entangled states.
Now we develop the combinatorics of multi-partite entanglement within subsets of agents required to prepare multi-partite entanglement between all the agents. When we were dealing only with EPR pairs in the case of EPR graphs or spanning EPR trees, we used the simple graph representation. Now subsets of the set of all agents may be in multi-partite entangled states and therefore we use a natural representation for such entanglement structures with hypergraphs as follows.

Let $S$ be the set of $n$ agents in a communication network. Let $E \subseteq S$, $|E| = k$. Suppose $E$ is such that the $k$ agents in $E$ are in a $k$-partite pure maximally entangled state. Let $E_1, E_2, ..., E_m$ be such subsets of $S$, each having a pure maximally entangled shared state amongst its agents. Note that the sizes of these subsets may be different. Consider the hypergraph $H = (S, F)$ such that $F = \{E_1, E_2, ..., E_m\}$. We call such a hypergraph $H$, an entangled hypergraph of the $n$ agents. In standard hypergraph notation the elements of $F$ are called hyperedges of $H$. Now we present the necessary and sufficient condition for preparing a $n$-partite pure maximally entangled state in such networks, given entanglements as per the entangled hypergraph. We need the definition of a hyperpath in a hypergraph: a sequence of $j$ hyperedges $E_1, E_2, ..., E_j$ in a hypergraph is called a hyperpath from a vertex $a$ to a vertex $b$ if (i) $E_i$ and $E_{i+1}$ have a common vertex (agent) for all $1 \leq i \leq j - 1$ (ii) $a$ and $b$ are agents in $S$ (iii) $a \in E_1$ and (iv) $b \in E_j$. If there is a hyperpath between every pair of vertices of $S$ in a hypergraph $H$ then we say that $H$ is connected.

**Theorem 5** Given $n$ agents in a communication network and an entangled hypergraph, a pure $n$-partite maximally entangled state can be prepared amongst the $n$ agents if and only if the entangled hypergraph is connected.

The proof of this theorem is based on the following Protocol-III.

The Protocol III: We assume without loss of generality that $n > |E_1| \geq |E_2| \geq \ldots \geq |E_m|$. We maintain the set $F$ and $R = S \setminus F$ where $F$ contains the agents already entangled in the $|F|$-partite pure maximally entangled state. Initially, $F = E_1$ and $R = \{E_2, E_3, ..., E_m\}$. We repeat the following steps until $F = S$. Choose $E_i \in R$ with minimum $i$ such that $F$ and $E_i$ have at least one common agent and $E_i$ is not in $F$; let the smallest index common element between $E_i$ and $F$ be the agent $A_j$. (Since the entangled hypergraph is connected, there is always such a hyperedge $E_i$.) We can
now use the method of Zeilinger et al. to create an \((N + M - 1)\)-partite maximum entangled state from two groups, one containing \(N = |F|\) agents and the other containing \(M = |E_i|\) agents. The measurement is processed by \(A_j\). So, an \((|F| + |E_i| - 1)\)-partite entanglement state is prepared from amongst the members of \(F\) and \(E_i\). If \(F\) and \(E_i\) share only one common agent then we are done. Otherwise, each member common to \(F\) and \(E_i\) other than \(A_j\) will have two qubits each from the \(|F| + |E_i| - 1\) entangled qubits. These qubits must be disentangled using a circuit same as the reverse of circuit in Figure 2.3.4. Now the members of \(F\) and \(E_i\) remain entangled in \((|F| + |E_i| - |F \cap E_i|)\)-partite state, each holding exactly one qubit. Finally, we set \(F = F \cup E_i\) and \(R = R \setminus E_i\).

The proof of necessity is similar to that of the proof of necessity in Theorem 3. For the sake of contradiction assume that the entangled hypergraph \(H\) is not connected. Then there is no hyperpath between two agents (say \(a\) and \(b\)), implying the existence of at least two components \(C_1\) and \(C_2\) in \(H\), with no member of \(C_1\) sharing a hyperedge of entanglement with any member of \(C_2\). Suppose a protocol \(P\) can create a pure \(n\)-partite maximally entangled state starting from the disconnected entangled hypergraph \(H\) of \(n\) agents. If we are able to create an \(n\)-partite maximally entangled state using protocol \(P\) with this structure using only classical communication and local operations, it is easy to see that we will also be able to create an EPR pair between two parties that were not earlier sharing any EPR pair, using just local operations and classical communication. This is forbidden by fundamental laws in quantum information theory (LOCC cannot increase the expected entanglement \(\text{Ved02, Hor01}\)). Hence \(H\) must be connected. This completes the proof of Theorem 5.

### 2.5 Concluding Remarks

We compare our method (Protocol-II) of generating multipartite maximally entangled states with that of Bose et al. \(\text{BVK98}\). The scheme of Bose et al. works as follows. Each agent needs to share a Bell pair with a central exchange in the communication network of \(n\) agents. The central exchange then projects the \(n\)-qubits with him, on to the \(n\)-partite maximally entangled basis. This leaves the \(n\) agents in a \(n\)-partite maximally entangled state. Thus, the two basic requirements of their scheme are a central exchange and a projective measurement on a multi-partite maximally entangled basis. The
central exchange essentially represents a star topology in a communication network and allows certain degree of freedom to entangle particles belonging to any set of users only if the necessity arises. However, a real time communication network may not always be a star network, in which case, we may need to have several such central exchanges. Of course, one will also be interested in setting up such a network with minimum resources, especially in the case of a long distance communication network. Issues involved in the design of such central exchanges such as minimizing required resources, are of vital interest while dealing with real communication networks. Such networks may be called Quantum Local Area Network (Q-LAN or Non-LAN, a Non-Local LAN) or Quantum Wide Area Network (Q-WAN or Non-WAN). Our scheme presented in Section 2.3 addresses these issues. We have shown in Theorem 4 that the spanning EPR tree is the minimal combinatorial requirement for this purpose. The star topology is a special case of the spanning EPR tree where the central exchange is one of the agents. It is therefore clear that the star network requirements of the scheme of Bose et al. provides a sufficient condition where as the requirement in our spanning EPR tree scheme is the most general and minimal possible structure.

Our scheme also helps in minimizing resources. Our spanning tree topology has been used by Singh and Srikanth [SS03A] (Chapter 4) for this purpose. They assign weights to the edges of the EPR graph based on the resources (such as quantum repeaters, etc.) needed to build that particular edge. Then, a minimum spanning EPR tree represents the optimized requirement. They also use this topology for multi-party quantum cryptography to minimize the size of the sector that can be potentially controlled by an eavesdropper. Thus our topology seems to be a potential candidate for building a long distance quantum communication network (such as in a Non-LAN or Non-WAN).

The second basic ingredient of the scheme of Bose et al. is the projection on a multipartite maximally entangled basis. As they point out, the circuit for such a measurement is an inverse of the circuit that generates a maximally entangled state from a disentangled input in the computational basis. In a communication network involving a large number of agents, this entails a lot of work to be done on part of the central exchange while the agents are idle. In our scheme, work is distributed amongst the agents. Moreover, the \( n \)-qubit joint measurement on the entangled basis in the scheme of Bose
et al. seems to be well high impossible from a practical standpoint given the current technology, whereas all the practical requirements of our scheme (Protocol II) can be met using current technology (using telecom cables to distribute entanglement etc).

The projection used by the central exchange in the scheme of Bose et al. may lead to any of the $2^n$ possible $n$-partite maximally entangled states. For practical purposes, one might be more interested in a particular state. To get the desired state, the measurement result must be broadcast by the central exchange. The $2^n$ possible states can be represented by a $n$ bit number and thus the communication complexity involved in their scheme is $n$ cbits, essentially the same as that of ours asymptotically. Therefore, our scheme is comparable to their scheme also in terms of communication complexity. It can also be noted at this point that, in our topology, even the method of Zeilinger et al. for creating $(m+1)$-partite maximally entangled state from a $m$-partite maximally entangled state becomes applicable. The use even reduces the communication complexity by some cbits but still requires $2n - 3$ cbits which is $O(n)$. As it can be observed, all these schemes require $O(n)$ cbits of communication. Whether there is an $\Omega(n)$ lower bound on the cbit communication complexity for preparing $n$-partite a pure maximally entangled state given a spanning EPR tree remains open for further research.

The results in Theorem 4 and Theorem 5 are similar to the classical theorem by Helly [Val64] in convex geometry. Helly’s theorem states that a collection of closed convex sets in the plane must have a non-empty intersection if each triplet of the convex sets from the collection has a non-empty intersection. In one dimension, Helly’s theorem ensures a non-empty intersection of a collection of intervals if each pair of intervals has a non-empty intersection. In our case (Theorems 4 and 5), there is similar combinatorial nature; if $n$ agents are such that each pair has a shared EPR pair, then (with linear classical communication cost) a pure $n$-partite state with maximum entanglement can be created entangling all the $n$ agents. As stated in Theorem 2, the case is stronger because just $(n - 1)$ EPR pairs suffice. Due to this similarity in combinatorial nature, we call our results in Theorem 4 and Theorem 5 quantum Helly-type theorems.
Chapter 3

A Combinatorial Approach to Study the LOCC Transformations of Multipartite States

3.1 Introduction

Given the extensive use of quantum entanglement as a resource for quantum information processing [BEZ00 NC00], its quantification has become one of the central topics of quantum information theory and, of late, a lot of research has been going on in this direction. However, apart from simple cases (for example, low-dimensions, few particles, pure states etc.) the mathematical structure of entanglement is not yet fully understood. In particular, the entanglement properties of bipartite states have been widely explored (see [Brus02 Hor01] for a comprehensive review). Fortunately, bipartite states possess a nice mathematical property in the form of the Schmidt decomposition [NC00] which encompasses their all non-local peroperties. However, the entangled states involving more than two parties lack such convenient form and so it is difficult to characterize them. Some approaches, essentially using the generalization of Schmidt decomposition, have been taken in this direction [BPRST00 Kempe99 Par04]; however a general formulation in this case is still an outstanding unresolved problem.
State transformations under local operations and classical communication (LOCC) are very important while quantifying entanglement because LOCC can at the best increase classical correlations and therefore a good measure of entanglement is not supposed to increase under LOCC. A necessary and sufficient condition for such transformation to be possible with certainty in the case of bipartite states was given by Nielsen [Nielsen99] and an immediate consequence of his result was the existence of incomparable states (the states which can not be obtained by LOCC from one another). Bennett et al. [Bennett00] formalized the notions of reducibility, equivalence and incomparability to multi-partite states and gave a sufficient condition for incomparability based on partial entropic criteria.

All the current approaches to study the state transformation under LOCC are based on entropic criterion. In this work, we present a entirely different approach based on nice combinatorial properties of graphs and set systems. We introduce a technique called bicolored merging and obtain several results about such transformations. We demonstrate a partial ordering of multi-partite states and various classes of incomparable multi-partite states. We utilize these results to establish the impossibility of doing selective teleportation in a case where the apriori entanglement is in the form of a GHZ state. We also discuss the minimum number of copies of a state required to prepare another state by LOCC and present bounds on this number in terms of quantum distance between the two states. The ideas developed in this work continues the combinatorial setting developed in Chapter 2 and can been extended to incorporate other new kinds of multi-partite states. Moreover, the idea of bicolored merging may also be appropriate to some other areas of information sciences.

3.2 The Combinatorial Framework

In this section we first revise the combinatorics developed in Chapter 2 and introduce the framework for deriving our results.

**Definition 3** EPR Graph: For $n$ agents $A_1, A_2, \ldots, A_n$ an undirected graph $G = (V, E)$ is constructed as follows:
\[ V = \{ A_i : i = 1, 2, \cdots, n \} , \ E = \{ \{ A_i, A_j \} : A_i \text{ and } A_j \text{ share an EPR pair, } 1 \leq i, j \leq n; i \neq j \}. \]

The graph \( G = (V, E) \) thus formed is called the EPR graph of the \( n \) agents.

**Definition 4** Spanning EPR Tree: An EPR graph \( G = (V, E) \) is called a spanning EPR tree if the undirected graph \( G = (V, E) \) is a spanning tree.

**Definition 5** Entangled Hypergraph: Let \( S \) be the set of \( n \) agents and \( F = \{ E_1, E_2, \cdots, E_m \} \), where \( E_i \subset S ; i = 1, 2, \cdots, m \) and \( E_i \) is such that its elements (agents) are in \(|E_i|\)-partite pure maximally entangled state. The hypergraph (set system) \( H = (S, F) \) is called an entangled hypergraph of the \( n \) agents.

**Definition 6** Connected Entangled Hypergraph: A sequence of \( j \) hyper-edges \( E_1, E_2, \ldots, E_j \) in a hypergraph \( H = (S, F) \) is called a hyperpath (path) from a vertex \( a \) to a vertex \( b \) if

1. \( E_i \) and \( E_{i+1} \) have a common vertex for all \( 1 \leq i \leq j - 1 \),
2. \( a \) and \( b \) are agents in \( S \),
3. \( a \in E_1 \), and
4. \( b \in E_j \).

If there is a hyperpath between every pair of vertices of \( S \) in the hypergraph \( H \), we say that \( H \) is connected.

**Definition 7** Entangled Hypertree: A connected entangled hypergraph \( H = (S, F) \) is called an entangled hypertree if it contains no cycles, that is, there do not exist any pair of vertices from \( S \) such that there are two paths between them.

**Definition 8** \( r \)-Uniform Entangled Hypertree: An entangled hypertree is called a \( r \)-uniform entangled hypertree if all of its hyperedges are of size \( r \).

**Theorem 6** If any two pairs of the three agents \( A, B \) and \( C \) share EPR pairs (say the state \((|00\rangle + |11\rangle)/\sqrt{2})\) then we can prepare a GHZ state \((|000\rangle + |111\rangle)/\sqrt{2})\) amongst them with two bits of classical communication, while involving all the three agents dynamically.

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We would henceforth refer the protocol developed in [SKP03] (Chapter 2) in order to establish Theorem 6 as SKP-1.

**Theorem 7** Amongst \( n \) agents in a communication network permitting only pairwise entanglement in the form of EPR pairs, a pure \( n \)-partite maximally entangled state can be prepared if and only if the EPR graph of the \( n \) agents is connected.

We shall use the name SKP-2 to refer to the protocol suggested in [SKP03] (Chapter 2) to prove the sufficiency of Theorem 7.

**Theorem 8** Given \( n \) agents in a communication network and an entangled hypergraph, a pure \( n \)-partite entangled state can be prepared amongst the \( n \) agents if and only if the entangled hypergraph is connected.

We shall use the name SKP-3 to refer to the protocol suggested in [SKP03] (Chapter 2) to prove the sufficiency of the above theorem.

### 3.3 Bicolored Merging

Monotonicity is easily the most natural characteristic that ought to be satisfied by all entanglement measures [Hor01]. This means that any appropriate measure of entanglement must not change by local unitary operations and more generally the expected entanglement must not increase under LOCC. We should note here that in LOCC, LO involves unitary transformations, additions of ancillas (that is, enlarging the Hilbert Space), measurements, and throwing away parts of the system, each of these actions performed by one party on his or her subsystem. CC between the parties allows local actions by one party to be conditioned on the outcomes of the earlier measurements performed by the other parties.

Apart from monotonicity, there are certain other characteristics required to be satisfied by the entanglement measures. It is interesting to note (as we show in this work) that monotonicity itself restricts a large number of state transformations and gives rise to several classes of incomparable (multipartite) states. Thus, to study the possible state transformations of (multipartite) states under LOCC, it would be interesting to look at the kind of state transforms under LOCC which monotonicity does not allow. We can
observe that monotonicity does not allow the preparation of \( n + 1 \) or more EPR pairs between two parties starting from only \( n \) EPR pairs between them. In particular, it is not possible to prepare two or more EPR pairs between two parties starting only with a single EPR pair and only LOCC. This is an example of impossible state transformation in bipartite case as dictated by the monotonicity postulate. Thus, we might anticipate that a large class of multi-partite states could also be shown to be incomparable just by using impossibility results in bipartite case through a suitable reduction. For example, consider transforming (under LOCC) the state represented by a spanning EPR tree, say \( T_1 \), to that of the state represented by the spanning EPR tree, say \( T_2 \). (See the Figure 3.3.1). This transformation can be shown to be impossible by reducing to the bipartite case as follows: Let us assume that there exists a protocol \( P \) which can perform the required transformation. It is easy to see that the protocol \( P \) is also applicable in the case when a party \( A \) possesses all the qubits of parties 4, 5, 6, and 7 and all the qubits of the parties 1, 2, and 3 are possessed by another party \( B \). This means that party \( A \) is playing the role of all the parties 4, 5, 6, and 7 and \( B \) is playing the role of all the parties 1, 2, and 3. Therefore for the protocol \( P \), these two parties represent the complete EPR spanning tree \( T_1 \). It is indeed reasonable as any LOCC actions done amongst \{1, 2, 3\} (\{4, 5, 6, 7\}) is reduced to just LO done by \( B \) (\( A \)) and any CC done between one party from \{1, 2, 3\} and the other from \{4, 5, 6, 7\} is managed by CC between \( B \) and \( A \). Therefore, starting only with one edge \( (e_3) \) they eventually construct \( T_1 \) just by LO (by local creation of EPR pairs representing the edges \( e_1, e_2, e_4, e_5, \) and \( e_6; \{e_1, e_2\} \) by \( B \) and \( \{e_4, e_5, e_6\} \) by \( A \)). They then apply protocol \( P \) to obtain \( T_2 \) with the edges \( f_1, f_2, f_3, f_4, f_5 \) and \( f_6 \). (Refer to the Figure 3.3.2). All edges except \( f_2 \) and \( f_3 \) are local EPR pairs (that is, both qubits are with the same party). Now the parties \( A \) and \( B \) share two EPR pairs in the form of the edges \( f_2 \) and \( f_3 \), though they started with sharing only one EPR pair. This is actually an impossible state transformation under LOCC in the bipartite case. Hence, we can conclude that such a protocol \( P \) can not exist! The complete reduction process is shown in Figure 3.3.2 below.

In general, suppose we want to show that the multi-partite state \( |\psi\rangle \) can not be converted to the multi-partite state \( |\phi\rangle \) by LOCC. This can be done by showing an assignment of the qubits (of all parties) only to two parties such that \( |\psi\rangle \) can be obtained from \( n \) \( (n = 0, 1, 2, \cdots) \) EPR pairs between the two parties by LOCC while \( |\phi\rangle \) can be converted to more than \( n \) EPR
Figure 3.3.1: The spanning EPR trees $T_1$ and $T_2$.
Local creation of \( \{e_1, e_2\} \) by B and \( \{e_4, e_5, e_6\} \) by A

\[ \text{One EPR pair between A and B} \]

\[ \text{Two EPR pairs between A and B} \]

\[ \text{An impossible state transformation under LOCC} \]

Figure 3.3.2: Converting \( T_1 \) to \( T_2 \) under LOCC through \( P \)
pairs between the two parties by LOCC. This is equivalent to saying that each party is given either of two colors (say A or B). Finally all qubits with parties colored with color A are assigned to the first party (say A) and that with parties colored with second color to the second party (say B). This coloring is done in such a way that the state $|\psi\rangle$ can be obtained by LOCC from less number of EPR pairs between A and B than that can be obtained from $|\phi\rangle$ by LOCC. Local preparation (or throwing away) of EPR pairs is what we call merging in combinatorial sense. Keeping this idea in mind, we now formally introduce the idea of bicolored merging for such reductions in the case of the multi-partite states represented by EPR graphs and entangled hypergraphs.

Suppose that there are two EPR graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set $V$ (meaning that the two multi-partite states are shared amongst the same set of parties) and we want to show the impossibility of transforming $G_1$ to $G_2$ under LOCC, then this is reduced to a bipartite LOCC transformation which violates monotonicity, as follows:

1. Bicoloring: Assign either of the two colors A or B to every vertex, that is, each element of $V$.

2. Merging: For each element $\{v_i, v_j\}$ of $E_1$, merge the two vertices $v_i$ and $v_j$ if and only if they have been assigned the same color during the bicoloring stage and assign the same color to the merged vertex. Call this graph obtained from $G_1$ as BCM (Bicolored-Merged) EPR graph of $G_1$ and denote it by $G_{1 bcm}$. Similarly, obtain the BCM EPR graph $G_{2 bcm}$ of $G_2$.

3. The bicoloring and merging is done in such a way that the graph $G_{2 bcm}$ has more number of edges than that of $G_{1 bcm}$.

4. Give all the qubits possessed by the vertices with color A to the first party (say, party A) and all the qubits possessed by the vertices with color B to the second party (say, party B). Combining this with the previous steps, it is ensured that in the bipartite reduction of the multi-partite state represented by $G_2$, the two parties A and B share more number of EPR pairs (say, state $|\psi_2\rangle$) than that for $G_1$ (say, state $|\psi_1\rangle$).

Now if there exits a protocol $P$ which can transform $G_1$ to $G_2$ by LOCC, then $P$ can also transform $|\psi_1\rangle$ to $|\psi_2\rangle$ just by LOCC as follows: A (B) will
play the role of all vertices in $V$ which were colored as $A$ ($B$). The edges which were removed due to merging can easily be created by local operations (local preparation of EPR pairs) by the party $A$ ($B$) if the color of the merged end-vertices of the edge was assigned color $A$ ($B$). This means that starting from $|\psi_1\rangle$ and only LO, $G_1$ can be created. This graph is virtually amongst $|V|$ parties even though there are only two parties. The protocol $P$ then, can be applied to $G_1$ to obtain $G_2$ by LOCC. Subsequently $|\psi_2\rangle$ can be obtained by the necessary merging of vertices by LO, that is by throwing away the local EPR pair represented by the edges between the vertices being merged. Since the preparation of $|\psi_2\rangle$ from $|\psi_1\rangle$ by LOCC violates monotonocity postulate, such a protocol $P$ can not exist! An example of bicolored merging for EPR graphs has been illustrated in Figure 3.3.3.

The bicolored merging in the case of entangled hypergraphs is essentially the same as that for EPR graphs. For the sake of completeness, we present it here. Suppose there are two entangled hypergraphs $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ on the same vertex set $S$ (that is, the two multi-partite states are shared amongst the same set of parties) and we want to show the impossibility of transforming $H_1$ to $H_2$ under LOCC. Transformation of $H_1$ to $H_2$ can be reduced to a bipartite LOCC transformation which violates monotonocity thus proving the impossibility. The reduction is done as follows:

1. Bicoloring: Assign either of the two colors $A$ or $B$ to every vertex, that is, each element of $S$.

2. Merging: For each element $E = \{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$ of $F_1$, merge all vertices with color $A$ to one vertex and those with color $B$ to another vertex and give them colors $A$ and $B$ respectively. This merging collapses each hyperedge to either a simple edge or a vertex and thus the hypergraph reduces to a simple graph with vertices assigned with either of the two colors $A$ or $B$. Call this graph obtained from $H_1$ as BCM EPR graph of $H_1$ and denote it by $H_1^{bcm}$. Similarly obtain the BCM EPR graph $H_2^{bcm}$ of $H_2$.

3. The bicoloring and merging is done in such a way that the graph $H_2^{bcm}$ has more number of edges than that of $H_1^{bcm}$.

4. Give all the qubits possessed by the vertices with color $A$ to the party one (say party $A$) and all the qubits possessed by the vertices with color $B$ to the second party (say party $B$).
Figure 3.3.3: Bicolored Merging of EPR Graphs

EPR Graphs

G1

1
2 6
3
4 5
7

Bicoloring

1(B)

2(B) 6(B)
3(B)
4(A) 5(B)
7(A)

Merging

A{4,7} —— B{1,2,3,5,6}
only one edge

G2

1
2 6
3 5
7

B{6}

1(B)

2(B) 5(B)
3(B)
4(A) 6(B)
7(A)

A{4,7} —— B{1,2,3}
three edges
Rest of the discussion goes exactly as in the EPR graph case. In the Figure 3.3.4 below, we demonstrate the bicolored merging of entangled hypergraphs.

It is interesting to note at this point that the LOCC incomparability shown by using the method of bicolored merging is in fact strong incomparability \cite{BRS02}. We would also like to stress that any kind of reduction (in particular, various possible extensions of bicolored merging) which leads to the violation of any of the properties of a potential entanglement measure, is pertinent to show the impossibility of many multi-partite state transformations under LOCC. Since the bipartite case has been extensively studied, such reductions can potentially provide many ideas about multi-partite case by just exploiting the results from bipartite case. In particular, the definitions of EPR graphs and entangled hypergraphs could also be suitably extended to capture more types of multi-partite pure states and even mixed states and a generalization of the idea of bicolored merging as a suitable reduction for this case could also be worked out. It would be interesting to investigate such issues.

### 3.4 Irreversibility of SKP-1 and Selective Teleportation

We know that a GHZ state amongst three agents $A$, $B$ and $C$ can be prepared from EPR pairs shared between any two pairs of the three agents using only LOCC \cite{SKP03}. We consider the problem of reversing this operation, that is, whether it is possible to construct two EPR pairs between any two pairs of the three agents from a GHZ state amongst the three agents, using only LOCC. By using the method of bicolored merging, we show below that this is not possible.

Suppose there exists a protocol $P$ for reversing a GHZ state into two EPR pairs using only LOCC. More precisely, protocol $P$ starts with a GHZ state amongst the agents $A$, $B$ and $C$, and prepares EPR pairs between any two pairs of $A$, $B$ and $C$ (say, $A$ and $C$, and $B$ and $C$). Since we can prepare the GHZ state from EPR pairs between any two pairs of the three agents, we can prepare the GHZ state starting from EPR pairs between $A$ and $B$,
Entangled Hypergraphs

BICOLORING

MERGING

B\{1,2,3,4,5,6,7\} → A\{8,9,10\}

only one edge

Figure 3.3.4: Bicolored Merging of Entangled Hypergraphs
and $A$ and $C$. Once the GHZ state is prepared, we can apply protocol $P$ to construct EPR pairs between $A$ and $C$ and between $B$ and $C$ using only LOCC. So, we can use only LOCC to convert a configuration where EPR pairs exist between $A$ and $C$ and between $A$ and $B$, to a configuration where EPR pairs are shared between $A$ and $C$ and between $B$ and $C$. It can be noted that the configuration where the two EPR pairs are shared between $A$, $B$ and $A$, $C$ (say, EPR graph $G_1$) is symmetrical with respect to the GHZ state amongst $A$, $B$ and $C$ to the configuration where the two EPR pairs are shared between $A$, $C$ and $B$, $C$ (say, the EPR graph $G_2$). Apply the bicolored merging by giving the color $A$ to parties $A$ and $B$ and the color $B$ to the party $C$. We can observe that $A$ and $C$ start with a single EPR pair between themselves and (by only LOCC) end up sharing two EPR pairs between themselves (Figure 3.4.3). The same result could also be achieved by similar bicolored merging directly applied on the GHZ state and any of $G_1$ or $G_2$ but we prefer the above proof for stressing the argument on the symmetry of $G_1$ and $G_2$ with respect to the GHZ. Moreover, this proof gives an intuition about possibility of incomparability amongst spanning EPR trees as $G_1$ and $G_2$ are two distinct spanning EPR trees on three vertices. We prove this general result in the Theorem 17.

By repeatedly applying the protocol $P$ (if possible), we can indeed prepare as many EPR pairs between $A$ and $C$ (using only LOCC) as we wish, starting from a single shared EPR pair. This is impossible and so our assertion is proved. We summarize this result in the following theorem.

**Theorem 9** Starting from a GHZ state shared amongst three parties in a communication network, two EPR pairs can not be created between any two sets of two parties using only LOCC.

The above theorem motivates us to think of some kind of comparison between a GHZ state and two pairs of EPR pairs in terms of the non-local correlations they possess. In this sense, therefore, a GHZ state is strictly less than two EPR pairs. This is also easy to see that an EPR pair between any two parties can be obtained starting only from a GHZ state shared amongst the three parties and LOCC. The third party will just do a measurement and send the result to other two. By applying the corresponding suitable operations they get the required EPR pair. Using the necessary conditions
Figure 3.4.5: Irreversibility of SKP-1
presented in [SKP03] (Chapter 2) to prepare a multi-partite entangled state starting from only bi-partite entanglement in a distributed network, we observe that an EPR pair between any two of the three parties is not sufficient for preparing a GHZ state amongst the three parties using only LOCC. These arguments can be summarized in the following theorem.

**Theorem 10** 1-EPR pair $<_{\text{LOCC}}$ a GHZ state $<_{\text{LOCC}}$ 2-EPR pairs

An interesting problem in quantum information theory is that of selective teleportation [Samal]. Given three agents $A$, $B$ and $C$, and two qubits of unknown quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ with $A$, the problem is to send $|\psi_1\rangle$ to $B$ and $|\psi_2\rangle$ to $C$ selectively, using only LOCC and apriori entanglement between the three agents. A simple solution to this problem is by applying standard teleportation [BBCJPW93], in the case where $A$ shares EPR pairs with both $B$ and $C$. An interesting question is whether any other form of apriori entanglement can help achieving selective teleportation. In particular, is it possible to perform selective teleportation where the apriori entanglement is in the form of a GHZ state amongst the three agents. Following theorem answers this question using the result of the Theorem 9.

**Theorem 11** With a prior entanglement given in the form of a GHZ state shared amongst three agents, two qubits can not be selectively teleported by ane of the three parties to the other two parties.

Proof: Suppose there exists a protocol $P$ which can enable one of the three parties (say $A$) to teleport two qubits $|\psi_1\rangle$ and $|\psi_2\rangle$ selectively to the other two parties (say $B$ and $C$). Now $A$ takes four qubits; she prepares two EPR pairs one from the first and second qubits and the other from the third and fourth qubits. He then teleports the first and third qubits selectively to $B$ and $C$ using $P$ (consider first qubit as $|\psi_1\rangle$ and the third qubie as $|\psi_2\rangle$). We can note here that in this way $A$ is able to share one EPR pair each with $B$ and $C$. But this is impossible for it enables $A$ to prepare two EPR pairs starting from a GHZ state and only using LOCC which already we have proved (Theorem 9) not to be possible. Hence follows the result. \hfill $\Box$
3.5 A Partial Ordering of Entangled Hypergraphs

In this section, we investigate whether some kind of comparison and ordering can be made between various multi-partite entangled states based on the non-local content contained in them and establish the following theorems.

Theorem 12 None of the inductive steps of SKP-2 can be reversed. Hence the non-local content contained in the multi-partite state represented by the structure in a step is strictly less than that of the multi-partite states represented by the structure in the subsequent steps.

Proof: The proof goes much similar as the proof of irreversibility of SKP-1. Suppose in an inductive step the \( m \) parties \( \{i_1, i_2, \ldots, i_m\} \) already entangled in a \( m \)-partite maximally entangled state, with help of EPR pair shared between \( i_m \) and \( i_{m+1} \) (call this configuration \( C_1 \)) prepare an \((m+1)\)-partite maximally entangled state amongst \( \{i_1, i_2, \ldots, i_m, i_{m+1}\} \) (call this configuration \( C \)). We are interested in proving the impossibility of reversing this step. For the sake of contradiction assume that such a protocol \( P \) for this reversing exists. Then it is also easy to see that starting with the \((m+1)\)-partite maximally entangled state amongst \( \{i_1, i_2, \ldots, i_m, i_{m+1}\} \) and using LOCC one can get a configuration (call \( C_2 \)) where the parties \( \{i_1, i_2, \ldots, i_{m+1}\} \) are in \( m \)-partite maximally entangled state and parties \( i_m \) and \( i_{m+1} \) are sharing an EPR pair (use the symmetry argument as in the proof of irreversibility of SKP-1). Thus it is possible to convert state represented by \( C_1 \) to that by \( C_2 \) just using LOCC by first converting \( C_1 \) to \( C \) by the inductive step of SKP-2 and subsequently applying \( P \) to \( C \). This state transformation under LOCC from \( C_1 \) to \( C_2 \) is shown impossible by bicolored merging where the color \( A \) is assigned to the parties \( \{i_1, i_2, \ldots, i_m\} \) and the color \( B \) to the party \( i_{m+1} \). The detail is suggested in the Figure 3.5.6.

Theorem 13 None of the inductive steps of SKP-3 can be reversed. Hence the non-local content contained in the multi-partite state represented by the entangled hypergraph in a step is strictly less than that of the multi-partite state represented by the entangled hypergraphs in the subsequent steps.
Figure 3.5.6: Inductive steps of SKP-2 are irreversible
Proof: The proof is exactly same as the proof of the last theorem using bicolored merging and we just depict it by Figure 3.5.7.

It is worth noting at this point that the above theorems gives partial orderings of multi-partite states. This order thus relatively quantifies the entanglement for these states.

3.6 Classifying Multi-partite Entanglement.

An immediate result comparing an n-CAT state with EPR pairs follows from Theorem 10 and Theorem 12.

Theorem 14
\[1 - EPR <_{LOCC} n - CAT <_{LOCC} (n - 1) - EPR.\]

On similar lines we can argue that an n-CAT state amongst \(n\)-parties can not, by just using LOCC, be converted to any form of entanglement structure which possess EPR pairs between any two or more different sets of two parties. Let this be possible then the two edges could be in either of the two forms:

1. \(\{i_1, i_2\}\) and \(\{j_1, j_2\}\)
2. \(\{i_1, i_2\}\) and \(\{i_2, j_2\}\)

where \(i_1, i_2, j_1, j_2\) are all distinct.

In bicolor-merging assign the colors as follows:

In case (1), give color \(A\) to \(i_2\) and \(j_2\) and give the color \(B\) to the rest of the vertices.

In case (2), give color \(A\) to \(i_2\) and color \(B\) to the rest of the vertices.

Thus our above assertion follows. Moreover, from the necessity condition in [SKP03] (Chapter 2) for preparing an \(n\)-CAT state, no disconnected EPR graph would be able to yield \(n\)-CAT just by LOCC. These two observations combined together leads to the following theorem which signifies the fact that these two multi-partite states can not be compared.

Theorem 15 A CAT state amongst \(n\) agents in a communication network is LOCC incomparable to any disconnected EPR graph associated with the \(n\) agents having more than one edge.
Figure 3.5.7: Inductive steps of SKP-3 are irreversible
The above result indicates that there are many possible forms of entanglement structures (multi-partite states) which cannot be compared at all in terms of non-local contents they deserve to have and this simple result was just an implication of the necessity combinatorics required for the preparation of the CAT states. One more interesting question while still in the domain of that combinatorics is to compare an spanning EPR tree and a CAT state. An spanning EPR tree is a sufficient combinatorics to prepare the CAT state and thus seems to entail more non-local content than in a CAT state but whether in a strict sense is still required to be investigated. It is easy to see that an EPR pair between any two parties can be obtained starting from a CAT state shared amongst the n agents just by LOCC (Theorem 14). Therefore, given \( n - 1 \) copies of the CAT state we can build all the \( n - 1 \) edges of any spanning EPR tree just by LOCC. But whether this is the lower bound on the number of copies of \( n \)-CAT required to obtain an spanning EPR tree is even more interesting. The following theorem shows that this indeed is the lower bound.

**Theorem 16** Starting with only \( n - 2 \) copies of \( n \)-CAT state shared amongst its \( n \) agents, any spanning EPR tree of the \( n \) agents can not be obtained just by LOCC.

**Proof:** Suppose it is possible to create a spanning EPR tree \( T \) from \( (n - 2) \) copies of \( n \)-CAT states. As we know, an \( n \)-CAT state can be prepared from any spanning EPR tree by LOCC (use SKP-2). Thus if \( (n - 2) \) copies of \( n \)-CAT can be converted to \( T \) then \( (n - 2) \) copies of any spanning EPR tree can be converted to \( T \) just by LOCC. In particular, \( (n - 2) \) copies of a chain EPR graph (which is clearly a spanning EPR tree) can be converted to \( T \) just by LOCC. Now, we know that any tree is a bipartite connected graph with \( n - 1 \) edges across the two parts. Let \( i_1, i_2, \ldots, i_m \) be the members of first group and the rest are in the other group. Construct a chain EPR graph where the first \( m \) vertices are \( i_1, i_2, \ldots, i_m \) in a sequence and the rest of vertices are from the other group in the sequence (Figure 3.6.8). As in our usual proofs, we give the color \( A \) to the parties \( \{i_1, i_2, \ldots, i_m\} \) and the rest of the parties are given the color \( B \). This way we are able to create \( (n - 1) \) EPR pairs (Note that there are \( n - 1 \) edges in \( T \) across the two groups) between \( A \) and \( B \) starting only from \( (n - 2) \) EPR pairs. Therefore, we conclude that \( (n - 2) \) copies of \( n \)-CAT can not be converted to any spanning EPR tree just by LOCC. See Figure 3.6.8 for illustration of required bicolored merging. The proof could also be achieved by similar kind of bicolored merging.
directly applied on \(n\)-CAT and \(T\).

In the preceding results we tried to compare an spanning EPR trees with CAT states. What about two different spanning EPR trees? Are they comparable? The following theorem targets to answer these questions.

**Theorem 17** Any two distinct spanning EPR trees are LOCC-incomparable.

**Proof:** Let \(T_1\) and \(T_2\) be the two respective spanning EPR trees. By the way of number of edges \((n - 1)\) in the spanning tree, there exists two vertices (say \(i\) and \(j\)) which are connected by an edge in \(T_2\) but not in \(T_1\). Also by virtue of connectedness of spanning trees, there will be a path between \(i\) and \(j\) in \(T_1\). Let this path be \(i k_1 k_2 \cdots k_m j\) with \(m > 0\) (See figure 3.6.9). Since \(m > 0\), \(k_1\) must exist.

Let \(T_{1i}^1\) = subtree in \(T_1\) rooted at \(i\) except for the branch which contains the edge \(\{i, k_1\}\).

\(T_{1j}^1\) = subtree in \(T_1\) rooted at \(j\) except for the branch which contains the edge \(\{j, k_m\}\).

\(T_{kr} =\) subtree in \(T_1\) rooted at \(k_r\) except for the branches which contain either of the edges \(\{K_{r-1}, k_r\}\) and \(\{K_r, k_{r+1}\}\) \((k_0 = i, k_{m+1} = j)\).

Let \(T_{2i}^2\) = subtree in \(T_2\) rooted at \(i\) except for the branch which contains the edge \(\{i, j\}\).

\(T_{2j}^2\) = subtree in \(T_2\) rooted at \(j\) except for the branch which contains the edge \(\{i, j\}\).

It is easy to see that the set \(T_{1i}^1 \cup T_{1j}^1\) is not empty for \(T_1\) and \(T_2\) being distinct must contain more than two vertices. Also \(T_{i}^2\) and \(T_{j}^2\) are disjoint otherwise there will be a path between \(i\) and \(j\) in \(T_2\) which does not contain the edge \(\{i, j\}\), and there will be thus two paths between \(i\) and \(j\) contradicting the fact that \(T_2\) is a spanning EPR tree (Figure 3.6.9). With these two characteristics of \(T_{i}^2\) and \(T_{j}^2\), it is clear that \(k_1\) will lie either in \(T_{i}^2\) or in \(T_{j}^2\). Without loss of generality let us assume that \(k_1 \in T_{i}^2\). Now we do bicolored merging where the color \(A\) is assigned to \(i\) and all vertices in \(T_{i}^1\) and rest of the vertices are assigned the color \(B\). Refer to Figure 3.6.9 for illustration.
Figure 3.6.8: \(n-2\) copies of \(n\)-CAT are not sufficient to prepare an \(A\{i_1, i_2, ..., i_m\} \rightarrow B\{i_{m+1}, i_{m+2}, ..., i_n\}\) EPR tree.

- **BICOLORING**:\(\text{(n-2) copies}\)
  - \((A) i_1, (A) i_3, ..., (A) i_m\)
  - \((B) i_{m+1}, (B) i_{m+2}, ..., (B) i_n\)

- **MERGING**:\(\text{(n-2) copies}\)
  - \(A\{i_1, i_2, ..., i_m\} \rightarrow B\{i_{m+1}, i_{m+2}, ..., i_n\}\)

In total \((n-2)\) edges one from each copy

- **n-CAT**:\(\text{(n-2) copies}\)
  - \((A) i_1, (A) i_2, ..., (A) i_m\)
  - \((B) i_{m+1}, (B) i_{m+2}, ..., (B) i_n\)

- **LOCC** (if possible)
  - \(i_1 \rightarrow i_{m+1}\)
  - \(i_2 \rightarrow i_{m+2}\)
  - \(i_{m-1} \rightarrow i_h\)

- **same as above**
  - In total \((n-1)\) edges, all from single copy of \(T\)

The set of \(n-1\) edges across the two parts
Figure 3.6.9: Spanning EPR trees are LOCC incomparable.
Since $T_1$ and $T_2$ were chosen arbitrarily, the same arguments also imply that there can not exist a protocol which can convert $T_2$ to $T_1$. Hence we lead to conclusion that any two distinct spanning EPR trees are LOCC incomparable.

**Corollary 1** There are exponentially many LOCC-incomparable pure multipartite entangled states.

*Proof:* We know from results in graph theory [Deo74] that on a labelled graph on $n$ vertices, there are $n^{n-2}$ different spanning trees possible. Hence there are $n^{n-2}$ different spanning EPR trees in a network of $n$ agents. From the theorem 17 all these spanning EPR trees are LOCC incomparable. Hence the result.

Since entangled hypergraphs represent more general entanglement structures than that represented by the EPR graphs (in particular spanning EPR trees are nothing but 2-uniform entangled hypertrees), it is likely that there will be even more classes of incomparable multi-partite states and this motivates us for generalizing the theorem 17 for entangled hypertrees, however remarkably this intuition does not work directly and there are entangled hypertrees which are not incomparable. An immediate contradiction comes from the partial ordering of entangled hypergraphs dictated by the theorem 13. However, there are still a large number of entangled hypertrees which do not fall under any such partial ordering and thus remains incomparable. We investigate such states here below. We need this important definition.

**Definition 9** Pendant Vertex: A vertex of a hypergraph $H = (S,F)$ such that it belongs to only one hyperedge of $F$ is called a pendant vertex in $H$. Vertices which belong to more than one hyperedge of $H$ are called non-pendant.

We are now ready to present our first incomparability result on entangled hypergraphs.

**Theorem 18** Let $H_1 = (S,F_1)$ and $H_2 = (S,F_2)$ be two entangled hypertrees. Let $P_1$ and $P_2$ be the set of pendant vertices of $H_1$ and $H_2$ respectively. If the sets $P_1 \setminus P_2$ and $P_2 \setminus P_1$ are both non-empty then the multi-partite states represented by $H_1$ and $H_2$ are necessarily LOCC-incomparable.
Proof: First we show by using bicolored merging that \( H_1 \) can not be converted to \( H_2 \) under LOCC. Impossibility of the reverse conversion will also be immediate. From the hypothesis, \( P_1 \setminus P_2 \) is non-empty, therefore there exists \( u \in S \) such that \( u \in P_1 \setminus P_2 \). This means to say that \( u \) is pendant in \( H_1 \) but non-pendant in \( H_2 \).

The bicolored merging is then done where the color \( A \) is assigned to the vertex \( u \) and all other vertices are assigned the color \( B \). This way \( H_1 \) reduces to a single EPR pair shared between the two parties \( A \) and \( B \). The complete becolored merging is shown in figure 3.6.10.

We can note that this proof does not utilize the fact that \( H_1 \) and \( H_2 \) are entangled hypertrees, and thus the theorem is indeed true even for entangled hypergraphs satifying the conditions specified on the set of pendant vertices.

The conditions specified on the set of pendant vertices in the theorem 18 cover a very small fraction of the entangled hypergraphs. However this condition is not necessary and thus there may be other classes and characterization which can decipher such incomparable classes of entangled hypergraphs. In passing we first give some examples, for whenever the above conditions are not satisfied \( H_1 \) and \( H_2 \) may or may not be incomparable.

**Example-1:** (Figures 3.6.11 and 3.6.12) \( P_1 \neq P_2 \) but either \( P_1 \subset P_2 \) or \( P_2 \subset P_1 \).

**Example-2:** (Figure 3.6.13) \( P_1 = P_2 \)

Theorem 17 shows that two EPR spanning trees are LOCC incomparable and the spanning EPR trees are nothing but 2-uniform entangled hypertrees. Therefore, a natural generalization of this theorem would be to \( r \)-uniform entangled hypertrees for any \( r \geq 3 \). As we show below the generalization indeed holds. It should be noted that the theorem 18 does not necessarily capture such entanglement structures (multi-partite states) (Figure 3.6.14).

However, while proving the fact that two different \( r \)-uniform entangled hypertrees are LOCC incomparable we shall need an important result about \( r \)-uniform hypertrees. We state this result in the following theorem for sake of continuity and completeness, however we defer its proof to the appendix.
Figure 3.6.10: Entangled hypergraphs with $P_1 \setminus P_2$ non-empty
Figure 3.6.11: Comparable with $P_1 \neq P_2$ and $P_1 \subset P_2$
Figure 3.6.12: InComparable with $P_1 \neq P_2$ and $P_1 \subset P_2$
Not possible due to irreversibility of SKP−1

Incomparable entangled hypergraphs with same set of pendant vertices

$H_1$ and $H_2$ being distinct spanning EPR trees are LOCC incomparable

Figure 3.6.13: $P_1 = P_2$
Figure 3.6.14: $r$-uniform entangled hypertrees not captured in theorem
THEOREM 19 Given two different $r$-uniform hypertrees $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ with $r \geq 3$, there exists vertices $u, v \in S$ such that $u$ and $v$ belong to same hyperedge in $H_2$ but necessarily in different hyperedges in $H_1$.

Now we prove our one of the main result on LOCC incomparability of multi-partite entangled states and establish the following theorem.

THEOREM 20 Any two distinct $r$-uniform entangled hypertrees are LOCC-incomparable.

Proof: Let $H_1 = (S, F_1)$ and $H_2 = (S, F_2)$ be the two $r$-uniform entangled hypertrees. If $r = 2$ then $H_1$ and $H_2$ happens to be two different spanning EPR trees and the proof follows from the theorem 17. Therefore, let $r \geq 3$.

Now from theorem 19 there exits $u, v \in S$ such that $u$ and $v$ belong to same hyperedge in $H_2$ but necessarily in different hyperedges in $H_1$. Let the same hyperedge in $H_2$ be $E \in F_2$. Also, since $H_1$ being hypertree is connected, there exists a path between $u$ and $v$ in $H_1$. Let this path be $uE_1E_2 \cdots E_{k+1}v$. Clearly $k > 0$ because $u$ and $v$ necessarily do not belong to the same hyperedge in $H_1$.

We keep the following notations (figure 3.6.15).

$T^1_u$: sub-hypertree rooted at $u$ in $H_1$ except that branch which contains $E_1$.

$T^1_v$: sub-hypertree rooted at $v$ in $H_1$ except that branch which contains $E_{k+1}$.

$T_{w_i}$: sub-hypertree rooted at $w_i$ in $H_1$ except that branches which contain $E_i$ and $E_{i+1}$.

$T_{E_i}$: Collection of all sub-hypertrees in $H_1$ rooted at some vertices in $E_i$ other than $w_{i-1}$ and $w_i$ (where $w_0 = u$ and $w_{k+1} = v$) except for the branches
The path between $u$ and $v$ in $H_1$

$H_2$

the vertices $u$ and $v$ in the same hyperedge $E$

Figure 3.6.15: Two distinct $r$-uniform entangled hypertrees
which contain \( E_i \).

\[
T = ((E_1 \cup E_2 \cup \cdots \cup E_{k+1}) \cup (T_{E_1} \cup T_{E_2} \cup \cdots \cup T_{E_{k+1}}) \cup (T_{w_1} \cup T_{w_2} \cup \cdots \cup T_{w_k})) \setminus \{u, v\}
\]

= set of all vertices from \( S \setminus \{u, v\} \) which are not contained in \( T_u \cup T_v \).

\( T^2_u \): sub-hypertree rooted at \( u \) in \( H_2 \) except that branch which contains \( E \).

\( T^2_v \): sub-hypertree rooted at \( v \) in \( H_2 \) except that branch which contains \( E \).

\( T_E \): Collection of all sub-hypertrees in \( H_2 \) rooted at some vertices in \( E \setminus \{u, v\} \) except for the branches which contain \( E \).

We break the proof in to the various cases:

**CASE \( S_1 \):** \( \exists w \in T \) such that \( w \in (T^2_u \cup T^2_v) \)

Without loss of generality let us take \( w \in T^2_u \). Now since \( w \in T \), \( w \) is exactly one of \( E_i \), \( T_{w_1} \), or \( T_{E_i} \) for some \( i \). Accordingly there will be three subcases.

**Case \( S_{11} \):** \( w \in E_i \) for some \( i \).

Do bicolored merging where the vertex \( u \) along with all the vertices in

\( T_u, E_1, E_2, \cdots, E_{i-1}, T_{w_1}, T_{w_2}, \cdots, T_{w_{i-1}}, T_{E_1}, T_{E_2}, \cdots, T_{E_{i-1}} \)

are given the color \( A \) and the rest of the vertices are given the color \( B \).

**Case \( S_{12} \):** \( w \in T_{w_i} \) for some \( i \).

Do the bicolored merging while assigning the colors as in the above case.

**Case \( S_{13} \):** \( w \in T_{E_i} \) for some \( i \).

Bicolored merging in this case is also same as in Case \( S_{11} \).

**CASE \( S_2 \):** There does not exist any \( w \in T \) such that \( w \in T^2_u \cup T^2_v \).

Clearly, \( T^2_u \cup T^2_v \subset T^1_u \cup T^1_v \) and \( T \subset T_E \cup (E \setminus \{u, v\}) \). Note that whenever, we are talking of set relations like union, containments etc., we are considering the trees, edges etc. as sets of appropriate vertices from \( S \) which
make them. First we establish the following claim.

**Claim**: \( \exists t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \) such that \( t \in T_E \).

We have \( k > 0 \) therefore, both \( E_1 \) and \( E_2 \) exist and since \( H_1 \) is \( r \)-uniform \( |E_1| = |E_2| = r \). Also \( (E_1 \setminus \{u, w_1\}) \cap (E_2 \setminus \{w_1, w_2\}) \) is empty otherwise there will be a cycle in \( H_1 \) which is not possible as \( H_1 \) is a hypertree [GGL95, Berge89]. Therefore,

\[
|(E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})| = |(E_1 \setminus \{u, w_1\})| + |E_2 \setminus \{w_1, w_2\}| = (r - 2) + (r - 2) = 2r - 4.
\]

Also \( |E| = r \) implying that \( |E \setminus \{u, v\}| = (r - 2) \).

It is clear that \( u, v \notin (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \).

\[
|(E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})| - |E \setminus \{u, v\}| = (2r - 4) - (r - 2) = r - 2 \geq 1
\]

since \( r \geq 3 \).

Also \( (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \subset T \subset T_E \cup (E \setminus \{u, v\}) \),

therefore, by Pigeonhole principle [LW92],

\[
\exists t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\})
\]

and

\[
t \in T_E(\notin (E \setminus \{u, v\})).
\]

Hence our claim is true.

Now we have \( t \in (E_1 \setminus \{u, w_1\}) \cup (E_2 \setminus \{w_1, w_2\}) \) such that \( t \in T_E \). Since \( t \in T_E \), by the definition of \( T_E \) it is clear that there must exist \( w \in E \setminus \{u, v\} \) such that \( t \in T_w \), the sub-hypertree in \( H_2 \) rooted at \( w \) except for the branch containing \( E \). Depending on whether \( t \in E_1 \setminus \{u, w_1\} \) or \( t \in E_2 \setminus \{w_1, w_2\} \), we break this case into several subcases and further in sub-subclasses depending on the part in \( H_1 \) where \( w \) lie.

**CASE S_{21}**: \( t \in E_1 \setminus \{u, w_1\} \) (Figure 9.6.16).

**CASE S_{211}**: \( w \in T_u \).
The entangled hypertree $H_1$

\[ T \]

The entangled hypertree $H_2$

Figure 3.6.16: CASE $S_2$: CASE 1
Do the bicolored merging where \( u \) and the vertices in \( T_u \) are assigned the color \( A \) and the rest of the vertices from \( S \) are given the color \( B \).

**CASE** \( S_{212} \): \( w \in T_v \).
Bicolored merging is done where \( v \) as well as all the vertices in \( T_v \) are assigned the color \( B \) and rest of the vertices from \( S \) are given the color \( A \).

**CASE** \( S_{213} \): \( w \in T \).
Here in this case, depending on whether \( w \) is in \( T_t \) or not there can be two cases.

- **case** \( S_{2131} \): \( w \in T_t \).

  Bicolored merging is done where all the vertices in \( T_t \) are given the color \( A \) and rest of the vertices are assigned the color \( B \).

- **case** \( S_{2132} \): \( w \notin T_t \).

  \( w \notin T_t \) implies that either \( w \in E_i \) for some \( i \) or \( w \in T_q \) where \( q \in E_i \) for some \( i \) and \( q \neq t \). In any of these possibilities the bicolored merging is same and is done as follows.

  Assign the color \( A \) to \( u \) as well as all vertices in 
  \( T_u \cup E_1 \cup T_{E_1} \cup T_{w_1} \cup \cdots \cup E_{i-1} \cup T_{E_{i-1}} \cup T_{w_i-1} \cup (E_i \setminus \{q, w, w_i\}) \cup (T_{E_i} \setminus T_q) \)
  and rest of the vertices are assigned the color \( B \).  

**CASE** \( S_{22} \): \( i \in E_2 \setminus \{w_1, w_2\} \) (Figure 3.6.17).

**CASE** \( S_{221} \): \( w \in T_u \cup E_1 \cup T_{E_1} \cup T_{w_1} \).
Do the bicolored merging where all the vertices in \( T_u \cup E_1 \cup T_{E_1} \cup T_{w_1} \) including \( u \) are given the color \( A \) and rest of the vertices are assigned the color \( B \).

**CASE** \( S_{222} \): \( w \in T_v \cup T_{E_{k+1}} \cup E_{k+1} \cup T_{w_k} \cup \cdots \cup E_{3} \cup E_{3} \cup T_{w_2} \).
In bicolored merging give the color \( B \) to all the vertices (including \( v \)) in 
\( T_v \cup T_{E_{k+1}} \cup E_{k+1} \cup T_{w_k} \cup \cdots \cup T_{E_3} \cup E_{3} \cup T_{w_2} \)
and color \( A \) to the rest of the vertices.

**CASE** \( S_{223} \): \( w \in E_2 \cup T_{E_2} \).
In this case depending on whether \( w \in T_t \) or \( w \notin T_t \) the bicolored merging will be different.
The entangled hypertree $H_1$

The entangled hypertree $H_2$

Figure 3.6.17: CASE $S_2$: CASE 2
case $S_{2_{23}}^1$: $w \in T_i$.
Bicolored merging is done where all the vertices in $T_i$ are given the color $A$ and rest of the vertices are assigned the color $B$.

case $S_{2_{23}}^2$: $w \notin T_i$.
$w \notin T_i$ implies that either $w \in E_2$ or $w \in T_q$ for some $q(\neq) \in E_2$. In any case do the bicolored merging where the color $A$ is assigned to all the vertices in
\[ T_u \cup E_1 \cup T_{E_1} \cup T_{w_1} \cup T_{E_1} \cup (E_2 \setminus \{w, q, w_2\}) \cup (T_{E_2} \setminus T_q) \]
and rest of the vertices are assigned the color $B$.

Now that we have exhausted all the cases and shown by clever method of bicolored merging that the $r$-uniform entangled hypertree $H_1$ can not be LOCC converted to the $r$-uniform entangled hypertree $H_2$, the same arguments also work for showing that $H_2$ can not be LOCC converted to $H_1$ by interchanging the roles of $H_1$ and $H_2$. Hence the theorem follows.

Before ending our section on LOCC incomparability of multi-partite states represented by EPR graphs and entangled hypergraphs we note that Bennett et. al partial entropic criteria [BPRST00] which gives a sufficient condition for LOCC incomparability of multi-partite states do not capture the LOCC-incomparability of spanning EPR tree or spanning entangled hypertrees in general. Consider two spanning EPR trees $T_1$ and $T_2$ on three vertices (say $1, 2, 3$). $T_1$ is such that the vertex pairs $1, 2$ and $1, 3$ are forming the two edges where as in $T_2$ the vertex pairs $1, 3$ and $2, 3$ are forming the two edges. It is easy to see that $T_1$ and $T_2$ are not marginally isentropic.

### 3.7 Quantum Distance between Multi-partite Entangled States

In the proof of theorem 17 we have utilized the fact that there exists at least two vertices which are connected by an edge in $T_2$ but not in $T_1$ which follows because $T_1$ and $T_2$ are different as well as they have equal number of edges (namely $n - 1$ if there are $n$ vertices). In fact, in general there may exist several such pair of vertices depending on the structures of $T_1$ and $T_2$. Fortunately there is some nicety in the number of such pair of vertices and it gives
rise to a metric on the set of spanning (EPR) trees with fixed vertex set and hence a concept of distance [Deo74]. The distance between any two spanning (EPR) trees $T_1$ and $T_2$ denoted by $QD_{T_1,T_2}$ on the same vertex set is defined as the number of edges in $T_1$ which are not in $T_2$. Let us call this distance to be the quantum distance between $T_1$ and $T_2$. Now that we have proved in theorem 17 obtaining $T_2$ from $T_1$ is not possible just through LOCC, we need to do quantum communication. The minimum number of qubit communication required for this purpose should be an interesting parameter related to state transformations amongst multi-partite states represented by spanning EPR trees; let us denote this number by $q_{T_1,T_2}$. We can note that $q_{T_1,T_2} \leq QD_{T_1,T_2}$. This is because each edge not present in $T_2$ can be created by only one qubit communication. The exact value of $q_{T_1,T_2}$ will depend on the structures of $T_1$ and $T_2$ and, as we can note, on the number of edge disjoint paths in $T_1$ between the vertex pairs which form an edge in $T_2$ but not $T_1$.

But this is not all about the quantum distance. Let us recall the theorem 16 where we prove that $n-1$ is a lower bound on the number of copies of $n-CAT$ to prepare a spanning EPR tree by LOCC. Can we obtain some lower bound like this in the case of two spanning EPR trees and relate it to the quantum distance? Answer is indeed yes. Let $C_{T_1,T_2}$ denote the minimum number of copies of the spanning EPR tree $T_1$ to obtain $T_2$ just by LOCC. We claim that $2 \leq C_{T_1,T_2}, C_{T_2,T_1} \leq QD_{T_1,T_2} + 1$. The lower bound follows from theorem 17. The upper bound is also true because of the following reason. $QD_{T_1,T_2}$ is the number of (EPR pairs) edges present in $T_2$ but not in $T_1$. For each such edge in $T_2$ if $u,v$ are the vertices forming the edge, while converting many copies of $T_1$ to $T_2$ by LOCC, an edge between $u$ and $v$ must be created. Since $T_1$ is a spanning tree and therefore connected, there must be a path between $u$ and $v$ in $T_1$ and this path can be well converted (using entanglement swapping) to an edge between them (i.e. EPR pair between them) only using LOCC. Hence one copy each will suffice to create each such edges in $T_2$. Thus $QD_{T_1,T_2}$ copies of $T_1$ will be sufficient to create all such $QD_{T_1,T_2}$ edges in $T_2$. One more copy will supply all the edges common in $T_1$ and $T_2$. Even more interesting point is that both these bounds are saturated. This means to say that there do exist spanning EPR trees satifying these bounds (Figure 3.7.18).
Figure 3.7.18:
3.8 Appendix

Proof of Theorem 19. We first establish the following claim.

claim: \( \exists E_1 \in F_1, E_2 \in F_2 \) such that \( E_1 \neq E_2, E_1 \cap E_2 \neq \phi \) and \( E_2 \notin F_1 \cap F_2 \).

Proof of the claim: We first show that on same vertex set, the number of hyperedges in any \( r \)-uniform hypertree is always same. Let \( n \) and \( m \) be the number of vertices and hyperedges in a \( r \)-uniform hypertree then we show by induction on \( m \) that \( n = m \cdot (r - 1) + 1 \).

For \( m = 1 \), \( n = 1 \cdot (r - 1) + 1 = r \) which is true because all possible vertices (since no one can be isolated) fall in the single edge and it has exactly \( r \) vertices.

Let us assume that this relation between \( n \) and \( m \) for a fixed \( r \) holds for all values of the induction variable up to \( m - 1 \) then we are to show that it holds good for \( m \).

Now take a \( r \)-uniform hypertree with \( m \) hyperedges. Remove any of the hyperedges to get another hypergraph (which may not be connected) which has only \( m - 1 \) edges. This removal may introduce \( k \) connected components (sub-hypertrees) where \( 1 \leq k \leq r \). Let these components has respectively \( m_1, m_2, \ldots, m_k \) number of hyperedges. Therefore, \( \sum_{i=1}^{k} m_i = m - 1 \). Total number of vertices in the new hypergraph (with the \( k \) sub-hypertrees as components), \( n_1 = \sum n_i \) where \( n_i \) is the number of vertices in the component \( i \).

Therefore, \( n_1 = \sum n_i = \sum_{i=1}^{k} \{m_i(r - 1) + 1\} = (m - 1)(r - 1) + k \).

Now the number of vertices in the original hypertree, \( n = n_1 + (r - k) \) because \( k \) vertices were already covered, one each in the \( k \) components. Therefore, \( n = (m - 1)(r - 1) + k + (r - k) = (m - 1)(r - 1) + r = (m - 1)(r - 1) + (r - 1) + 1 = m(r - 1) + 1 \). The result is thus true for \( m \) and hence for any number of hyperedges by induction. This result implies that any \( r \)-uniform hypertree on the same vertex set will always have the same number of hyperedges.

Let \( F = F_1 \cap F_2 \) and \( m = |F_1| = |F_2| \). Obviously \( m > |F| \) otherwise \( H_1 = H_2 \) implying that \( \exists E \in F_2 \) such that \( E \notin F \).

Let \( U = \bigcap_{A \in F} A \) is the vertex set on which the common hyperedges relies. Depending on the characteristics of \( U \) we break the proof in two cases.
CASE 1: ∃w ∈ E such that w /∈ U.
Since w /∈ U we get E /∈ F. Now since H₁ is a hypertree w can not be an
isolated vertex and therefore there exists an edge say E¹ ∈ F₁ but E¹ /∈ F
(otherwise w ∈ U). Take E₂ = E and E₁ = E¹.
CASE 2: A ⊂ U ∩ A ∈ F₂
We have |E| = r and E ⊂ U, say E = \{e₁, e₂, \ldots, e_r\}.
No pair of (eᵢ, eⱼ) can be connected by the hyperedges only in F (i.e.
through common hyperedges) otherwise eᵢ (path in F) eⱼEeᵢ will be a cycle
in H₂ (mean to say that there will be two paths between eᵢ and eⱼ in H₂
one in F and another one eᵢEeⱼ both being in same hyperedge E) which is
absurd given that H₂ is a hypertree.
Now H₁ is a hypertree, therefore must be connected and there must be
a path between eᵢ and eⱼ in H₁. Say this path be eᵢG₁G₂\cdots Gₗeⱼ where
Gₖ ∈ F₁ for 1 ≤ k ≤ l. Take E₁ = G₁ and E₂ = E
Thus in all cases, we have proved that ∃E₁ ∈ F₁ and E₂ ∈ F₂ such that
E₁ ≠ E₂, E₁ ∩ E₂ ≠ φ and E₂ /∈ F₁ ∩ F₂ and hence follows our claim.

Now switch over to prove the theorem. Choose E₁ and E₂ so as to
satisfy the above claim. Let E₁ = \{u₁, u₂, \ldots, uₗ, wₗ₊₁, wₗ₊₂, \ldots, w_r\} and
E₂ = \{u₁, u₂, \ldots, uₗ, vₗ₊₁, vₗ₊₂, \ldots, v_r\}. Since E₁ ∩ E₂ ≠ φ , l ≥ 1 and
E₁ ≠ E₂ implies that l ≤ r − 1. Hence 1 ≤ l ≤ r. Now based on the value of
l we consider different cases below.

CASE 1 : l > 1

case 1₁: ∃vᵢ such that u₁ and vᵢ are not in same hyperedge in H₁.
Take u = u₁ and v = vᵢ in the statement of the theorem.
case 1₂: All vᵢ are respectively in some hyperedges in H₁ in which u₁ also
lies.
None of these vᵢ can belong to the same hyperedge as of u₂ in H₁. This
is because if say vⱼ happen to be in same hyperedge as of u₂ in H₁ then
u₁u₂vⱼu₁ will be a cycle in H₁ which is absurd as H₁ is a hypertree. Note
that at least one such vᵢ must exist as l < r.
Take u = u₂ and v = any vᵢ.

CASE 2: l = 1
case 2₁: ∃vᵢ such that u₁ and vᵢ are not in same hyperedge in H₁.
Take u = u₁ and v = vᵢ.

case 2₂: All vᵢ are respectively in some hyperedges in H₁ in which u₁ also lies.

Since there are r − 1 vᵢ in number and E₂ ∉ F₁ ∩ F₂, these vᵢ will be distributed in at least two different hyperedges in H₁ in which u₁ also lies. Therefore, ∃vᵢ, vⱼ such that they are in same hyperedge in H₂ (namely in E₂) but in necessarily different edges in H₁ otherwise (i.e. if they lie in the same hyperedge in H₁) u₁vᵢvⱼu₁ will be a cycle in H₁ which is absurd as H₁ is a hypertree. Also note that both vᵢ and vⱼ will exist as r ≥ 3.

Take u = vᵢ and v = vⱼ.

Thus we have proved the above theorem in all possible cases. □

We would like to point out that this very result might be some direct implication of standard results in combinatorics however for sake of completeness we have proved it in our own way, moreover keeping in appendix section!
Chapter 4

Unconditionally Secure Multipartite Quantum Key Distribution

4.1 Introduction

With the growing use of the internet and other forms of electronic communication, the question of secure communication becomes one of considerable importance. Modern cryptographic techniques, based on the availability of ever increasing computational power, and the invention of public key cryptography, provide practical solutions for information security in various situations. But invariably these techniques are only computationally– and not unconditionally– secure, that is, they depend on the (unproven) hardness of certain mathematical problems. As a result, it cannot be guaranteed that future advances in computational power will not nullify their cryptographic protection. Nevertheless, there does exist a form of encryption with unconditional security: the use of one-time key pads. These are strings of random numbers added by the information sender to encode the message, to be subtracted by the receiver to decode. Provided that the key material is truly random and used only once, this system is unbreakable in the sense described by Shannon in the 1940s [Shannon45]. It is critically important that the pad is only used once, i.e., an encryption key can never be used twice. This restriction translates into the practical one of key distribution (KD). This need to securely distribute the key between the users makes it impractical in many
applications. Where it is used in real life (eg., in confidential communications between governments), the one-time key pads are actually delivered in person by some trusted third party, an arrangement prohibitively expensive for common usage and moreover not truly secure. Fortunately, recent advances in quantum information theory have shown that unconditionally secure key distribution is possible in principle.

That quantum information can, on account of quantum uncertainty and the no-cloning principle [WZ82], be used to distribute cryptographic keys was realized two decades ago [BB84, Ekert91]. More rigorous and comprehensive proofs of this task, generally called quantum key distribution (QKD), taking into consideration source, device and channel noise as well as an arbitrarily powerful eavesdropper, have been studied by various authors [May01, BBBMR00, LC99, IRV01, SP00]. Recently, the issues of efficiency [HWMKL03], security in the presence of an uncharacterized source [KP03] and high bit-error rate tolerance [Wang04] of QKD have been considered. In particular, Lo and Chau [LC99] showed that, given fault-tolerant quantum computers, quantum key distribution over an arbitrarily long distance of a realistically noisy channel can be made unconditionally secure. This is a heartening development, since QKD is, among quantum information applications, relatively easy to implement, and some large scale implementations have already been achieved [GR1Z02, PGUWZ03]. The above mentioned works consider QKD between two parties (ie., 2-QKD). It is of interest to consider its extension to more than two parties (ie., n-QKD).

The problem of n-QKD is to determine how n parties, who are able to communicate quantally, may share an identical and unconditionally secure, secret key among themselves in the presence of eavesdroppers. (A different generalization of 2-QKD gives multipartite quantum secret-sharing [SG01], which we do not consider here). In this work, we propose a protocol for this purpose and prove its unconditional security. We note that a simple extrapolation of 2-QKD to n-QKD would suggest that the agents should begin by sharing an n-partite entangled state. However, this proposition suffers from two drawbacks: from a practical viewpoint, preparing n-partite entanglement is no easy task; from a theoretical viewpoint, proving the security of secure extraction of n separated copies of a bit-string may not be simple, even given the existing proof of security of the bipartite case. Our main result is that it is sufficient if some pairs of agents share bipartite entanglement along any spanning tree connecting the n agents, who are taken to be vertices on a graph. In this way, n-QKD is reduced to a 2-QKD problem. Existing 2-QKD proto-
can be invoked to prove the unconditional security of sharing nearly perfect Einstein-Podolsky-Rosen (EPR) pairs between two parties. They prescribe procedures for reliably sharing EPR pairs, and thence sharing randomness, by virtue of fault-tolerant quantum computers, quantum error correction and suitable random sampling. In the interests of brevity, we will not elaborate these protocols here, and only point them out as subroutines for the general n-QKD task. Our protocol is fairly simple in the sense that the proof of its security is built on top of the already proven security of the bipartite case. However, it is important to know the necessary and sufficient conditions on the network topology for our proposed protocol to work.

4.2 Classical Reduction of n-KD to 2-KD

As in 2-KD, the goal of n-KD is to show that n trustful parties can securely share random, secret classical bits, even in the presence of noise and eavesdropping. It is assumed that the n agents can share authenticated classical communication. It is convenient to treat the problem graph theoretically as in Chapter 2. The n agents $A_i$ ($1 \leq i \leq n$) are considered as the vertices (or nodes) of an undirected graph. An instance of a secure bi-partite channel being shared between two parties is considered as an undirected edge between the two corresponding vertices. A graph so formed is called a security graph. It is obvious that if the security graph has a star topology (a hub vertex with all edges radiating from it to the other vertices), a simple n-KD protocol can be established. The agent at the hub vertex (say, called, Lucy) generates a random bit string and transmits it to every other agent along the edges to each of them. This will create a secure, identical random bit string with each agent.

In real life situations, because of practical and geographical constraints, the n agents may not form a security graph with star topology. We describe a simple protocol that allows for more general secure bi-partite connectivity between the agents. In particular, from among the secure bipartite channels suppose a spanning tree (a graph connecting all vertices without forming a loop) can be constructed. This construction can be formalized in order to determine an optimal spanning tree. Some useful definitions are given below.

**Definition 10** Weighted security graph: Given n parties treated as nodes on a graph, we extend the definition of a security graph to the *weighted*
security graph. A weight is associated with every edge and is defined to be some suitable measure of the cost of communicating by means of the channel corresponding to the edge.

**Definition 11** Minimum spanning security tree: Consider the weighted security graph $G = (V, E)$. A spanning tree selected from $G$, given by $G_1 = (V, E_1), E_1 \subseteq E$ is called the minimum spanning security tree if it minimizes the total weight of the graph.

Minimum spanning security tree need not be unique and can be obtained using Kruskal’s or Prim’s algorithm [CLR90]. The minimum spanning security tree minimizes the resources needed in the protocol as well as the size of the sector eavesdroppers can potentially control.

**Definition 12** Terminal agent: An agent that corresponds to a vertex of degree one (ie., with exactly one edge linked to it). On the other hand, an agent that corresponds to a vertex of degree greater than one is called a non-terminal agent.

We now present a classical subroutine that allows $n - 1$ pair-wise shared random bits to be turned into a single random bit shared between the $n$ parties.

1. 2-KD: Along the $n - 1$ edges of a minimum spanning security tree, $n - 1$ random bits are securely shared by means of some secure 2-KD protocol.

2. Each non-terminal agent $A_i$ announces his uniformly randomized record: this is the list of edges emanating from the vertex along with the corresponding random bit values, to all of which a fixed random bit $x(i)$ is added.

3. This information is sufficient to allow every player, in conjunction with her/his own random bit record, to reconstruct the random bits of all parties. The protocol leader (say, Lucy) decides randomly on the terminal agent whose random bit will serve as the secret bit shared among the $n$ agents.
This subroutine consumes \( n - 1 \) pair-wise shared random numbers to give a one-bit secret key shared amongst the \( n \)-parties. To generate an \( m \)-bit string shared among the \( n \) agents, the subroutine is repeated \( m \) times.

Given that the initial bipartite sharing of randomness is secure, we will show that the above protocol subroutine allows some randomness to be shared between the \( n \) agents without revealing anything to an eavesdropper. It involves each non-terminal agent announcing his uniformly randomized record. Suppose one such, \( A_i \), has the random record 0,1,1 on the three edges linked to his vertex. He may announce 0,1,1 (for \( x(i) = 0 \)) or 1,0,0 (for \( x(i) = 1 \)). All the three agents linked to him can determine which the correct string is by referring to their shared secret bit. It is a straightforward exercise to see that each of other agents linked to these three can determine the right bit string. Therefore, each agent can determine the random bits of all others. Eavesdroppers, on the other hand, lacking knowledge of any of the \( n - 1 \) shared random bits, can only work out the relative outcomes of all parties. The result is exactly two possible configurations for each secret bit, which are complements of each other. The eavesdropper “Eve” is thus maximally uncertain about which the correct configuration is. Hence, Lucy’s choice of a party to fix the secret bit reveals little to Eve. Insofar as the \( n \)-parties are able to communicate authenticated classical messages, the subroutine protocol is as secure as the underlying procedure for 2-KD.

It is obvious that the above protocol works for any spanning security tree. Clearly, a sufficient condition for turning shared bipartite randomness into randomness shared between \( n \) parties is that the weighted security graph should contain at least one spanning tree. On the other hand, if the security graph is disconnected, one easily checks that it is impossible to arrive at a definite random bit securely shared between both the disconnected pieces. Therefore, the existence of at least one spanning tree in the weighted security graph is both a necessary and sufficient condition for the required task.

The amount of securely shared randomness may be quantified by the length of shared random bit string multiplied by the number of sharing agents. In the above protocol, the \( n - 1 \) instances of pair-wise shared randomness is consumed to produce exactly one instance of a random bit shared between the \( n \) parties. We can then define the ‘random efficiency’ of the above protocol by \( \eta = (n \times 1)/((n - 1) \times 2) \), which tends to \((1/2)\) as \( n \to \infty \). Unconditional security of the above subroutine can in principle only be guaranteed in a protocol which includes in step 1 a quantum sub-routine that implements 2-QKD. In the following Section, we will present one such, based
on the Shor-Preskill protocols [SP00], as an example.

### 4.3 Quantum Protocol

As in 2-QKD, the goal of the proposed $n$-QKD protocol is to show that $n$ trustful parties can securely distil random, shared, secret classical bits, whose security is to be proven inspite of source, device and channel noise and of Eve, an eavesdropper assumed to be as powerful as possible, and in particular, having control over all communication channels. From the result of the preceding Section, it follows that a quantum protocol is needed only in step 1 above. It will involve establishing 2-QKD along a minimum spanning tree in order to securely share pair-wise randomness along spanning tree’s edges and thence proceed to $n$-QKD. We assume as given the security of establishing pair-wise randomness along a spanning tree by means of a quantum communication network, based on a secure 2-QKD protocol [May01, BBBMR00, LC99, SP00, IRV01, HWMKL03, KP03, Wang04]. In principle, these protocols guarantee security under various circumstances.

In an $n$-QKD scheme, the insecurity of even one of the players can undermine all. Hence additional classical processing like key reconciliation and privacy amplification of the final key may be needed at the $n$-partite level. In the full $n$-QKD protocol that we present below, following Ref. [SP00] we exploit the connection of error correction codes [MS77] with key reconciliation and privacy amplification. These procedures have been extensively studied by classical cryptographers [GRTZ02], and other possibilities exist.

In particular, we adopt a quantum protocol wherein pair-wise randomness is created by means of sharing EPR pairs (this follows the pattern set by the Ekert [Ekert91], Lo-Chau [LC99] and Modified Lo-Chau [SP00] protocols, but entanglement is not necessary, as seen in the original BB84 protocol). The basic graph theoretic definitions introduced above apply also for the quantum case, except that now the security channels correspond to shared EPR pairs. In place of a secure bipartite channel, an instance of EPR pair shared between two parties is considered as an undirected edge between the two corresponding vertices. A graph so formed is called an EPR graph (Chapter 2). The analog of the weighted security graph is the *weighted EPR graph*, and that of the minimum spanning security tree is the *minimum spanning EPR tree*. Let us enumerate the $n$ parties as $A_1, A_2, \cdots, A_n$. Suppose that only $A_{i_1}, A_{i_2}, \cdots, A_{i_s}$ ($i_1, i_2, \cdots, i_s \in \{1, 2, \cdots, n\}$) are capable of
producing EPR pairs and \( S \equiv \{A_{i_1}, \cdots , A_{i_s}\} \) is the set of all such vertices, with \( S \neq \emptyset \). We construct a weighted undirected graph \( G = (V, E) \) as one whose every edge must contain a vertex drawn from the set \( S \), as follows: \( V \equiv \{A_i; \ i = 1, 2, \cdots , n\} \) and \( E \equiv \{(A_i, A_j) \ \forall \ A_i \in S \ \text{and} \ \forall \ A_j \in V; \ i \neq j\} \). And the weight of edge \( (A_i, A_j) \) is defined to be \( w_{i,j} \propto \) number of quantum repeaters \[ \text{BDCZ98} \] (more generally: entanglement distilling resources \[ \text{DLCZ01} \]) required to be put between \( A_i \) and \( A_j \). Usually, the larger the distance between two agents, the larger is the weight. The minimum spanning EPR tree minimizes the number of quantum repeaters needed, and, in general, the resources needed in the protocol (EPR pairs, etc.) subject to the constraint of available EPR sources. Apart from improving efficiency in terms of costs incurred, this optimization is also important from the security perspective in that it minimizes the size of the sector that Eve can potentially control.

Let \( C \) be a classical \( t \)-error correcting \( [m, k] \)-code \[ \text{MS77} \]. We now present a protocol that consumes \( n - 1 \) pair-wise securely shared sets of EPR pairs to create random bits shared between the \( n \) parties with asymptotic efficiency \( \eta = (1/2)k/m \), where \( k/m \) is the rate of the code. The classical subroutine described in the previous Section is adapted to include key reconciliation and privacy amplification at the \( n \)-partite level, that uses the group theoretic properties of \( C \).

1. **EPR protocol:** Along the \( n - 1 \) edges of the minimum spanning EPR tree, EPR pairs are shared (using eg., the Lo-Chau \[ \text{LC99} \] or Modified Lo-Chau protocols \[ \text{SP00} \]). Let the final, minimum number of EPR pairs distilled along any edge of the minimum spanning EPR tree be \( 2m \). A projective measurement in the computational basis is performed by all the parties on their respective qubits to obtain secure pair-wise shared randomness along the tree edges (making due adjustments according to whether the entangled spins are correlated or anti-correlated).

2. **Classical subroutine of Section 4.2:** All non-terminal vertices announce their uniformly randomized outcome record. This information in principle allows every party, in conjunction with her/his outcome, to reconstruct the outcomes of all other parties, save for some errors of mismatch.

3. For each set of \( n - 1 \) shared EPR pairs, protocol leader Lucy decides
randomly on the terminal party whose outcome will serve as the secret bit.

4. Lucy decides randomly a set of $m$ bits to be used as check bits, and announces their positions.

5. All parties announce the value of their check bits. If too few of these values agree, they abort the protocol.

6. Lucy broadcasts $c_i \oplus v$, where $v$ is the string consisting of the remaining code (non-check) bits, and $c_i$ is a random codeword in $C$.

7. Each member $j$ from amongst the remaining $n - 1$ parties subtracts $c_i \oplus v$ from his respective code bits, $v \oplus \epsilon_j$, and corrects the result, $c_i \oplus \epsilon_j$, to a codeword in $C$. Here $\epsilon_j$ is a possibly non-vanishing error-vector.

8. The parties use $i$ as the key.

A rigorous proof of the security of the $n$-QKD scheme requires: (a) the explicit construction of a procedure such that whenever Eve’s strategy has a non-negligible probability of passing the verification test by the $n$ parties, her information on the final key will be exponentially small. (b) the shared, secret randomness is robust against source, device and channel noise. By construction, our scheme combines a 2-QKD scheme to generate pair-wise shared randomness and a classical scheme to turn this into multipartite-shared randomness. The security of the latter (in its essential form) was proven in Section 4.2. Therefore the security of the protocol with respect to (a) and (b) reduces to that of the 2-QKD in step 1. For various situations, 2-QKD can be secured, as proven in Refs [May01, BBBMR00, LC99, IRV01, SP00]. For example, Lo and Chau [LC99] and Shor and Preskill [SP00] have proved that EPR pairs can be prepared to be nearly perfect, even in the presence of Eve and channel noise. Their proofs essentially relies on the idea that sampling the coherence of the qubits allows one to place an upper bound on the effects due to noise and information leakage to Eve. Yet, subject to the availability of high quality quantum repeaters and fault-tolerant quantum computation, in principle 2-QKD can be made unconditionally secure [LC99].

In regard to the key reconciliation part: in step (3), each non-terminal vertex party announces his uniformly randomized outcome record. Here this consumes $m$ instances of $n - 1$ pair-wise shared random bits into $k$ random
bits shared between the $n$ agents while revealing little to Eve. The random efficiency is given by $\eta = (k \times n \times 1)/(m \times (n - 1) \times 2)$, which tends to $(1/2)k/m$ as $n \to \infty$. The check bits, whose positions and values are announced in steps (4) and (5), are eventually discarded. Steps (7) and (8) involve purely local, classical operations. If security of step (1) against Eve is guaranteed, the string $v$, and thereby the string $c_i \oplus v$ announced by Lucy in step (6), are completely random, as far as Eve can say. So, she (Eve) gains nothing therefrom. Hence her mutual information with any of the $n - 1$ (sets of) random bits does not increase beyond what she has at the end of the EPR protocol.

Finally, step (5) permits with high probability to determine whether the key can be reconciled amongst the $n$ players. The check bits that the parties measure behave like a classical random sample of bits \cite{SP00}. We can then use the measured error rates in a classical probability estimate. For any two parties, the probability of obtaining more than $(\delta + \epsilon)n$ errors on the code bits and fewer than $\delta n$ errors on the check bits is asymptotically less than $\exp[-0.25\epsilon^2n/(\delta - \delta^2)]$. Noting that the errors on the $n$ check vectors are independent, it follows that probability that the check vectors are all scattered within a ball of radius $\delta n$ but one or more code vectors fall outside a scatter ball of radius $(\delta + \epsilon)n$ is exponentially small, and can be made arbitrarily small by choosing sufficiently small $\delta$. The decision criterion adopted in step (5) is calculated so that the Hamming weight of the error vectors $e_j$ estimated in the above fashion will be less than $t$ with high probability. Hence all parties correct their results to the same codeword $c_i$ in step (8) with high probability. This completes the proof of unconditionally security of $n$-QKD.
5.1 Introduction

Suppose the president of a bank, Alice, wants to give access to a vault to two vice-presidents, Bob and Charlie, whom she does not entirely trust. Instead of giving the combination to any one of them, she may desire to distribute the information in such a way that no vice-president alone has any knowledge of the combination, but both of them can jointly determine the combination. Cryptography provides the answer to this question in the form of secret sharing \cite{Sch96,Grus97}. In this scheme, some sensitive data is distributed among a number of parties such that certain authorized sets of parties can access the data, but no other combination of players. A particularly symmetric variety of secret splitting (sharing) is called a threshold scheme: in a \((k,n)\) classical threshold scheme (CTS), the secret is split up into \(n\) pieces (shares), of which any \(k\) shares form a set authorized to reconstruct the secret, while any set of \(k-1\) or fewer shares has no information about the secret. Blakely \cite{Blakeley79} and Shamir \cite{Shamir79} showed that CTS’s exist for all values of \(k\) and \(n\) with \(n \geq k\). By concatenating threshold schemes, one can construct arbitrary access structures, subject only to the condition of monotony (ie., sets containing authorized sets should also be authorized) \cite{BL90}. Hillery \textit{et al.} \cite{HBB99} and Karlsson \textit{et al.} \cite{KK199} proposed methods for implementing CTSs that use quantum information to transmit shares securely in the presence of eavesdroppers.
Subsequently, extending the above idea to the quantum case, Cleve, Gottesman and Lo \cite{CGL99} proposed a \((k,n)\) quantum threshold scheme (QTS) as a method to split up an unknown secret quantum state \(|S\rangle\) into \(n\) pieces (shares) with the restriction that \(k > n/2\) (for if this inequality were violated, two disjoint sets of players can reconstruct the secret, in violation of the quantum no-cloning theorem \cite{WZ82}). The notion of QTS is based on quantum erasure correction \cite{CS96, GBP97}. QSS has been extended beyond QTS to general access structures \cite{Got99, Smi00}, but here the no-cloning theorem implies that none of the authorized sets shall be mutually disjoint. Potential applications of QSS include creating joint checking accounts containing quantum money \cite{Wis83}, or share hard-to-create ancilla states \cite{Got99}, or perform secure distributed quantum computation \cite{CGS02}.

In conventional QSS schemes, it is often implicitly assumed that all share-holders carry quantum information. Sometimes it is possible to construct an equivalent scheme in which some share holders carry only classical information and no quantum information \cite{NMI01}. Such a hybrid (classical-quantum) QSS that combines classical and quantum secret sharing brings a significant improvement to the implementation of QSS, inasmuch as quantum information is much more fragile than classical information. Moreover, hybrid QSS can potentially avail of features available to classical secret sharing such as share renewal \cite{HJKY96}, secret sharing with prevention \cite{Bet90} and disenrolment \cite{Mar93}. The essential method to hybridize QSS is to somehow incorporate classical information that is needed to decrypt or prepare the quantum secret as classical shares. A simple instance of such classical information is the ordering information of the shares. In QTS, it is implicitly assumed that the share-holders know the the coordinates of the shares in the secret, i.e., they know who is holding the first qubit, who the second and so on. This ordering information is necessary to reconstruct the secret, without which successful reconstruction of the secret is not guaranteed. If we wish to make use of this ordering information in the above sense, then only quantum error correction based secret sharing where lack of ordering information leads to maximal ignorance can be used. In particular, the scheme should be sensitive to the interchange of two or more qubits. For example, let us consider a \((2,3)\)-QTS. The secret here is an arbitrary qutrit and the encoding maps the secret qutrit to three qutrits as:

\[
\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \longrightarrow \alpha(|000\rangle + |111\rangle + |222\rangle) + \beta(|012\rangle + |120\rangle + |201\rangle) + \gamma(|021\rangle + |210\rangle + |102\rangle),
\]

(5.1)
and each qutrit is taken as a share. While from a single share no information can be obtained, two shares, with ordering information, suffice to reconstruct the encoded state. However, the lack of ordering information does not always lead to maximal ignorance about the secret. Note that the structure of the above code is such that any interchange of two qubits leaves an encoded $|0\rangle$ intact but interchanges $|1\rangle$ and $|2\rangle$. Thus, a secret like $|0\rangle$ or $(1/\sqrt{2})(|1\rangle + |2\rangle)$ can be entirely reconstructed without the ordering information. Therefore, only the subset of quantum error correction codes admissible in QSS that do not possess such symmetry properties can be used if the scheme is to be sensitive to ordering information.

Theoretically simpler but practically somewhat more difficult is an interesting idea proposed by Nascimento et al. \cite{NMI01}, based on qubit encryption \cite{MTW00}. In Sections 5.2 and 5.3 we adopt this method to generate the relevant encrypting classical information. However, in principle any classical data whose suppression leads to maximal ignorance of the secret is also good. Elsewhere, in Section 5.5 we consider another way. Quantum encryption works as follows: suppose we have a $n$-qubit quantum state $|\psi\rangle$ and random sequence $K$ of $2n$ classical bits. Each sequential pair of classical bit is associated with a qubit and determines which transformation $\hat{\sigma} \in \{\hat{I}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ is applied to the respective qubit. If the pair is 00, $\hat{I}$ is applied, if it is 01, $\hat{\sigma}_x$ is applied, and so on. The resulting $|\tilde{\psi}\rangle$ is a complete mixture and no information can be extracted out of it because the encryption leaves any pure state in a maximally mixed state, that is: 

\[
(1/4)(|I\rangle\langle S| + |\tilde{\sigma}_x\rangle\langle S| + |\tilde{\sigma}_y\rangle\langle S| + |\tilde{\sigma}_z\rangle\langle S|)|\tilde{\sigma}_2\rangle = (1/2)\hat{I}.
\]

However, with knowledge of $K$ the sequence of operations can be reversed and $|\psi\rangle$ recovered. Therefore, classical data can be used to encrypt quantum data.

In analogy with classical secret sharing, it is customary to consider that all quantum shares must be distributed among the players. A simple generalization (which we call ‘assisted QSS’ schemes), that is helpful from both theoretical and practical considerations, is to allow some shares to remain with the share dealer. This simple device will, rather surprisingly, enable us to implement QSS schemes in which authorized sets may be mutually disjoint. In Sections 5.3 and 5.4 we study such assisted schemes.
5.2 Inflating Quantum Secret Sharing Schemes

In hybrid QSS, the quantum secret is split up into quantum and classical shares of information. We call the former q-shares, and the latter c-shares. A player holding only c-shares is called a c-player or c-member. Otherwise, she or he is a q-player or q-member.

**Definition 13** A QSS scheme realizing an access structure \( \Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) among a set of players \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \) is said to be compressible if fewer than \( n \) q-shares are sufficient to implement it.

Here the \( \alpha_i \)'s are the minimal authorized sets of players. Knowledge of compressibility helps us decide how to minimize quantum resources needed for implementing a given QSS scheme. As an example of compression by means of hybrid QSS, suppose we want to split a quantum secret \( |S\rangle \) among a set of players \( \mathcal{P} = \{A, B, C, D, E, F\} \) realizing the access structure \( \Gamma = \{ABC, AD, AEF\} \). That is, the only sets authorized to reconstruct the secret are \( \{A, B, C\} \), \( \{A, D\} \) and \( \{A, E, F\} \) and sets containing them, whilst any other set is unauthorized to do so. For distributing the secret, we encrypt \( |S\rangle \) using the quantum encryption method (described above) with classical key \( K \) into a new state \( |\tilde{S}\rangle \) and give \( |\tilde{S}\rangle \) to A. We then split up \( K \) using a CSS scheme that realizes \( \Gamma \). Player A cannot recover \( |S\rangle \) from \( |\tilde{S}\rangle \) because he cannot unscramble it without \( K \). Only the \( \alpha_j \)'s, and sets containing them, can recover the classical key \( K \), and thence decrypted secret state. In this way, by means of a hybrid (classical-quantum) secret-sharing scheme, we can compress the original QSS scheme into an equivalent one in which fewer players need to handle quantum information.

The question, how to augment or “inflate” a given QSS scheme keeping the quantum component fixed, is considered herebelow. This is of practical relevance if we wish to expand a given QSS scheme by including new players who do not have (reliable) quantum information processing capacity. To this end, we now define an inflatable QSS.

**Definition 14** A QSS(\( \Gamma \)) scheme realizing an access structure \( \Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) among a set of players \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \) using a total of \( m \) q-shares is inflatable if \( n \) can be increased for fixed \( m \) to form a new QSS(\( \Gamma' \)) such that \( \Gamma'|_\mathcal{P} = \Gamma \), where \( \Gamma'|_\mathcal{P} \) denotes the restriction of \( \Gamma' \) to \( \mathcal{P} \).

Clearly, inflation involves the addition of classical information carrying c-players. The additional shares required for them will be c-shares, so that
q-shares may remain fixed at \( m \). The following theorem answers the question when a QSS scheme can be inflated.

**Theorem 21** A QSS scheme realizing an access structure \( \Gamma = \{ \alpha_1, \alpha_2, \cdots, \alpha_r \} \) among a set of players \( P = \{ P_1, P_2, \cdots, P_n \} \) using a total of \( m \) q-shares can always be inflated.

**Proof.** Consider the addition of a single player, \( P_{n+1} \). The new set of players are \( P' \equiv \{ P_1, P_2, \cdots, P_{n+1} \} \). A new access structure \( \Gamma' \) can be obtained by arbitrarily adding \( P_{n+1} \) to any of the \( \alpha_i \)'s. Clearly, \( \Gamma' \) will not violate the no-cloning theorem [WZ82], since \( \Gamma \) does not. As a result, the augmented scheme can be realized as a conventional QSS scheme using (say) \( m' > m \) q-shares. Therefore, by construction, there is one \( P_i \) (namely, that for \( i = n+1 \) in the above case) such that \( \Gamma'|_{P' - P_i} \) does not violate the no-cloning theorem, where \( \Gamma'|_{P' - P_i} \) denotes the restriction of \( \Gamma' \) to \( P' - P_i \). Therefore, according to Theorem 1 of Ref. [NMI01], the new QSS scheme obtained by adding \( P_{n+1} \) is compressible, meaning that \( P_{n+1} \) can be a c-player. Therefore, the new scheme \( \text{QSS}(\Gamma') \) is an inflation of the given scheme \( \text{QSS}(\Gamma) \). It is clear that the process of addition of new c-players can be continued indefinitely without restriction. \( \square \)

The above theorem only says that that any QSS scheme can be inflated in some way. A specific problem is whether a given \((k, n)\)-QTS can be inflated. This is considered in the following two theorems.

**Theorem 22** A \((k, n)\)-QTS cannot be inflated at constant threshold.

**Proof.** Suppose \((k, n)\)-QTS can be inflated at constant threshold. Then there exists a \((k, n')\)-QTS, consistent with the no-cloning theorem and with \( n' > n \), whose restriction leads to \((k, n)\)-QTS. Let \( n' - n \equiv \gamma \). The restriction of \((k, n')\)-QTS by \( \gamma \) players will lead to a \((k - \gamma, n)\)-QTS [NMI01], whose access structure is different from \((k, n)\)-QTS. This contradicts our original assumption. \( \square \)

**Theorem 23** A \((k, n)\)-QTS can be inflated conformally, i.e., to newer threshold schemes having the form \((k + \gamma, n + \gamma)\), for any positive integer \( \gamma \).
Proof. If the given \((k, n)\)-QTS satisfies the no-cloning theorem, then clearly so will the \((k + \gamma, n + \gamma)\)-QTS. Further, according to Lemma 1 of Ref. [NMI01], a restriction of the \((k + \gamma, n + \gamma)\)-QTS by \(\gamma\) players yields a \((k, n)\)-QTS. Therefore, an expansion of a \((k, n)\)-QTS to a \((k + \gamma, n + \gamma)\)-QTS is possible adding only \((\gamma)\) c-players.

5.3 Assisted Quantum Secret Sharing

Because of the no-cloning theorem, secret sharing requires \(q\)-shares being converged to some site in order to reconstruct the secret, which could be the secret dealer or some designated reconstructor for final processing. For example, in the first example, access is allowed by the vault (which can be thought of as the dealer) if the secret reconstructed from the vice-presidents’ shares is the required password. In this case, by definition, the secret dealer is a trusted party in the secret sharing. In such cases, there appears to be little loss of generality in leaving some shares obtained from splitting the secret with the dealer, the other shares being shared among the players, each of whom receives at least one share. Where the dealer may be distinct from the reconstructor, as in multiparty secure computation, the dealer transmits his shares to the latter, once the latter is identified. When shares of an authorized set converge at the reconstructor, the latter simply adds his own shares before processing the verification. In practice, the dealer can simply be a computer program that stores passwords of bank accounts or a central quantum computer in a multi-party secure computation procedure. We refer to this generalization of quantum secret sharing as ‘assisted quantum secret sharing’ (AQSS).

In classical secret sharing, such “share assistance” (from the dealer) does not appear to offer any new advantage. However, the situation is quite different in QSS. First, as we point out below, the only restriction on the access structure \(\Gamma\) in AQSS is monotony. The members of \(\Gamma\) are not required to have mutual overlaps, in order to satisfy the no-cloning theorem. Further, as we show later, it can considerably reduce the amount of quantum communication and the number of quantum information carrying players required in a QSS, in ways clarified below. This is quite important from the viewpoint of implementation, considering that quantum information processing is extremely difficult. Shares which are dealt out to players are called ‘player
shares'; that/those which remain(s) back with the dealer (to be transmitted to the reconstructor directly, if necessary) is/are called ‘resident share(s)’. A conventional QSS scheme is a special cases of the assisted scheme, in which the set of resident shares is empty. We note that share assistance is needed only when the members of an access structure do not have pairwise overlap.

**Theorem 24** Given an access structure \( \Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \) among a set of players \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \), an assisted quantum secret sharing scheme exists iff \( \Gamma \) is monotone.

**Proof:** It is known that if the members of \( \Gamma \) all overlap, then there exists a conventional QSS to realize it [Got99]. Suppose the members of \( \Gamma \) do not overlap. (It is instructive to look at the classical situation. Suppose \( \Gamma = \{ABC, DE\} \), which can be written in the normal form \( \{(A \text{ AND } B \text{ AND } C) \text{ OR } (D \text{ AND } E)\} \). The AND gate corresponds to a \( (|\alpha_j|, |\alpha_j|) \) threshold scheme, while OR to a \( (1,2) \) threshold scheme. By concatenating these two layers, we get a construction for \( \Gamma \).) In the conventional QSS, the above fails for two reasons, because by the no-cloning theorem: the members of \( \Gamma \) should not be disjoint and further there is no \( ((1, 2)) \) scheme (here, following Ref. [Got99], double (single) brackets denote the quantum (classical) scheme.) However, we replace \( ((1, 2)) \) by a \( ((2, 3)) \) scheme, which corresponds to a majority function of OR. In general, we replace a \( ((1, r)) \) scheme by a \( ((r, 2r - 1)) \) scheme. \( r \) of the shares correspond to individual authorized sets in \( \Gamma \), and the other \( r - 1 \) shares will remain as resident shares with the dealer. Any authorized set, by combining its second layer (AND) shares can reconstruct its first layer (OR) share, which, combined with the resident share, can reconstruct the secret. Since the necessity of the resident share by itself fulfils the no-cloning theorem, authorized sets are not required to be mutually overlapping. \( \square \)

Another way to view this is that given any arbitrary \( \Gamma \), including disjoint members, we systematically add the same player to all authorized sets to obtain a new \( \Gamma' \) which is compatible with the no-cloning theorem in the usual sense. Thus, we can turn \( \Gamma = \{ABC, DE\} \) into \( \Gamma' = \{ABCX, DE\} \), by adding member \( X \), whose share is the resident share deposited with the dealer. Thereby, the structure \( \Gamma = \Gamma'|_{\mathcal{P}} \), which denotes a restriction of \( \Gamma' \) to members other than \( X \), is effectively realized among the players (excluding the dealer).

For example, for the access structure \( \Gamma = \{ABC, DE\} \), in the first layer, a \( ((2, 3)) \) scheme is employed to split \( |S\rangle \) into three shares, with one share
designated to $ABC$ and the other to $DE$. The last remains with the dealer (cf. Eq. (5.2)). In the second layer, the first two block rows are $((|\alpha_j|,|\alpha_j|))$ schemes.

\[
((2,3)) \begin{cases}
((3,3)) : & A, B, C \\
((2,2)) : & D, E \\
((1,1)) : & \text{dealer}
\end{cases}
\]  

(5.2)

In contrast, without share assistance, the share corresponding to the last block would, recursively, be split according to a maximal scheme containing $\Gamma$ (which should of course not contain any disjoint members), for which a pure state scheme exists \cite{Got99}. Note that either $ABC$ or $DE$ can reconstruct only one of the $((2,3))$ shares. Thus the disjointness of these sets does not violate the no-cloning theorem. When the share from an individual authorized set is submitted, the secret can be reconstructed at the reconstructor station, when combined with the resident share.

**Theorem 25** For a $((k,n))$ scheme, with $1 \leq k \leq n$, there exists an equivalent assisted quantum threshold scheme. If $k \leq n/2$, the equivalent assisted scheme is given by $((k+\gamma,n+\gamma))$, where $\gamma = n - 2k + 1$.

**Proof:** If $k > n/2$, then the theorem stands proved by the known result \cite{CGL99} that a quantum erasure code exists equivalent to a $((k,n))$ scheme. Suppose that $k \leq n/2$. Consider a $((k+\gamma,n+\gamma))$ scheme, where $\gamma$ is increased until the no-cloning condition $2(k+\gamma) > n+\gamma$ is met, which is at $\gamma = n - 2k + 1$. The dealer employs this scheme, retains $\gamma$ shares as resident shares, giving one of the remaining $n$ shares to each player. Any $k$ player shares, combined with the resident share, suffice to reconstruct the secret. Hence this method effectively realizes the required $((k,n))$ scheme over the players. \hfill $\square$

Once again we note that there is no contradiction with the non-cloning theorem because the procedure basically realizes a $((k+\gamma,n+\gamma))$ scheme in which the $\gamma$ shares with the dealer forms a common element in all authorized sets. As an example, suppose a $((2,10))$ assisted scheme is required. From Theorem 25 we find $\gamma = 7$, so that the required assisted version is a $((9,17))$ scheme, which can be realized as a quantum erasure code. Of these, $\gamma = 7$ shares are resident shares, the remaining ten each being given to a player. Any two players can reconstruct the secret by submitting their two shares jointly to the reconstructor, which adds its seven shares to reconstruct the
Consider an access structure \( \Gamma = \{ABC, AD, EFG\} \). The associated assisted structure is \( \Gamma' = \{ABCX, ADX, EFGX\} \), where share \( X \) is designated to be a resident share. A q-share \( q \) is ‘important’ if there is an unauthorized set \( T \) such that \( T \cup \{q\} \) is authorized. The size of an important share of a conventional QSS that realizes \( \Gamma' \) cannot be smaller than the dimension of the secret \( \text{Got99, IMNTW03} \). Generalizing this argument gives the following easy theorem.

**Theorem 26** The dimension of each important share of an assisted quantum secret sharing scheme must be at least as large as the dimension of the secret.

### 5.4 Compressing Assisted Quantum Secret Sharing Schemes

In a conventional \( (k,n) \) scheme, a compression of shares is possible only if \( 2k > n + 1 \) \( \text{[NM01]} \), in which case, the scheme can be compressed into a \( (k-\gamma, n-\gamma) \) scheme combined with a \( (k, n) \) scheme. A general access structure \( \Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_r\} \) can be realized by a first layer of \( (1, r) \)-threshold scheme. In the quantum case, since this violates the no-cloning theorem, it is replaced by the majority function \( (r, 2r - 1) \)-QTS \( \text{[Got99]} \). This, again, is incompressible. However, in the second layer of the construction, the \( (|\alpha_i|, |\alpha_i|) \) schemes can be replaced with a \( ((1, 1)) \) schemes combined with \( (|\alpha_i|, |\alpha_i|) \) schemes. As seen from the results below, further saving on q-players and q-shares becomes possible under share assistance. Here compression refers to the player shares and not the resident shares. Note that there is a trivial compression for a share assisted scheme realizing an access structure \( \Gamma \), in which the entire secret simply remains with the dealer, and only the encryption information is split-shared according to a classical scheme realizing \( \Gamma \). This is ruled out by requiring that every \( \alpha_i \) must have at least one important q-share allocated to it.

**Theorem 27** A conventional QSS scheme realizing an access structure \( \Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_r\} \) among a set of players \( \mathcal{P} = \{P_1, P_2, \cdots, P_n\} \) can always be compressed to an assisted QSS scheme requiring no more than \( M \equiv |\mathcal{M}(\Gamma)| \) quantum players, where \( \mathcal{M}(\Gamma) \) is the smallest hitting set for the collection \( \Gamma \). Further compression is impossible.
Proof. A ‘hitting set’ \( S(\Gamma) \) for the collection of sets \( \Gamma \) is a set of players such that \( S \cap \alpha_i \neq \emptyset \ \forall \ i \ (1 \leq i \leq r) \). Let \( M(\Gamma) \) be the smallest hitting set for \( \Gamma \). \( M(\Gamma) \) may or may not be unique. Having found \( M(\Gamma) \), designate its members to be q-players and all others as c-players. This guarantees that there is at least one q-player in each \( \alpha_j \) (which is necessary in a quantum scheme). In the first layer of QSS, a \(((M, 2M - 1))\) majority function scheme is used to divide \(|S|\) into \(2M - 1\) shares. \( M\) of these shares are encrypted using keys \( K_i \ (1 \leq i \leq M) \) and given one per q-member. The respective keys are shared classically using a \(|\alpha_j|, |\alpha_j|\) scheme in each authorized set. The remaining \( M - 1 \) are retained by the dealer as resident shares. Together the players of any \( \alpha_j \) can reconstruct \( K_j \). This is used to decrypt the q-share(s) of q-player(s) in \( \alpha_j \). The decrypted share, in conjunction with the resident shares, suffices to reconstruct the secret. This proves that \( M \) members, provided they belong to \( M(\Gamma) \), are sufficient to effectively enact the QSS. However, no further compression is possible. For, if we have fewer than \( M \) q-players, then there is at least one \( \alpha_j \) with no q-player in it, which would render reconstruction of the quantum secret impossible for that member of \( \Gamma \).

The significance of the theorem lies in saving quantum communication and minimizing q-players, which are required to be only \( M \leq n \) in number. Let us consider Theorem 27 applied to the access structure \( \Gamma = \{ABC, DE\} \), the example considered in Eq. (5.2). The two authorized sets are disjoint, so \( M = 2 \). We choose \( M = \{A, D\} \), who are the two required q-players (instead of five q-players, required in the uncompressed version). The first layer will employ a \((2, 3)\)-QTS to split \(|S|\). One share is encrypted using \( K_1 \) and given to \( A \), another using \( K_2 \) and given to \( D \). \( K_1 \) is shared using a classical \((3, 3)\) scheme among \( ABC \), and \( K_2 \) using a \((2, 2)\) scheme among \( DE \). The remaining share remains resident at the dealer. This is depicted in Eq. (5.3). The authorized set on any row suffices to unscramble and reconstruct one first layer share, which in conjunction with the resident share permits reconstruction of the whole secret.

\[
\begin{align*}
((2,3)) & \quad A \quad \rightarrow (3, 3) : \quad A, B, C \\
D & \quad \rightarrow (2, 2) : \quad D, E
\end{align*}
\]

If some \( \alpha_j \)'s have a common element, as for example in \( \Gamma = \{ABC, DE, AFG\} \), then some q-players are chosen to belong to more than one authorized set;
in this case, e.g., \( \mathcal{M} = \{A, D\} \) or \( \mathcal{M} = \{A, E\} \). If all members of \( \Gamma \) happen to have one or more common players, then only one q-share is needed, given to one of the common players, and no share assistance is required, as in the case of \( \Gamma = \{ABC, AD, AEF\} \) that we encountered earlier. For a general access structure involving a large number of players, computing \( \mathcal{M} \) is a provably hard problem (Infact its decision version is shown to be NP-Complete [GJ79]). Nevertheless, a particularly simple case is the symmetric one of a threshold scheme.

**Theorem 28** An assisted \((k, n)\) scheme, with \(1 \leq k \leq n\), can be maximally compressed to one requiring no more than \(n - k + 1\) quantum players.

**Proof:** If \(k > n/2\), we retain the scheme as such, but if \(k \leq n/2\), we replace the \((k, n)\) scheme by \((k + \gamma, n + \gamma)\) scheme, where by Theorem 25, \(\gamma = n - 2k + 1\) and \(\gamma\) shares are resident at the dealer. In either case, the requirement that any authorized set should have a q-share implies that any set of \(k\) players must have at least one q-player. This is possible only if \(M \geq n - k + 1\). Minimally, \(\mathcal{M}\) can be chosen to be any \(n - k + 1\) players, who are designated as q-players. Thus, a further \(n - (n - k + 1) = k - 1\) shares are designated to remain with the dealer, bringing a total of \(\gamma + k - 1\) resident shares. The remaining \(n - k + 1\) shares are encrypted and given one each to \(n - k + 1\) q-players. The encryption key is shared among the players using a \((k, n)\) scheme. Any \(k\) players will have at least one q-share among them, which they can decrypt and transmit to the reconstructor. This will suffice to reconstruct the secret using the \((k + \gamma, n + \gamma)\) scheme. If a set of \(k\) players has \(y > 1\) q-shares among them, then reconstruction can proceed in any of \(2^y\) ways, by transmitting any subset of the \(y\) shares to the reconstructor. \(\Box\)

For example, we return to the earlier example to realize a \((2, 10)\) assisted scheme via a \((9, 17)\) scheme. The uncompressed version requires 7 resident shares and 10 q-players. In the compressed scheme, only \(10 - 2 + 1 = 9\) q-players are required, with \(17 - 9 = 8\) resident shares. The q-shares with the players are encrypted using a classical \((2, 10)\) scheme. Now, a \((k, 2k - 1)\) is incompressible without assistance [NM01]. Theorem 28 implies that with assistance it can be further compressed to one involving only \(k\) q-shares (rather than \(2k - 1\) q-shares) among the players. For example suppose a \((2, 3)\) scheme is to be realized among players \(A, B, C\). The authorized sets are \(\{AB, BC, AC\}\). Going by Theorem 28 we require only 2 q-players, i.e., any 2 players fully construct \(\mathcal{M}\). Let them be \(A, B\). \(C\) remains a c-player. The dealer \(D\) encrypts the secret \(|S\rangle\) using data \(K\) and then splits
the encrypted secret $|S'\rangle$ according to $((2, 3))$. He gives one of the resulting three shares to $A$, another to $B$, retaining the third himself. Then he splits-shares $K$ according to a $(2, 3)$ scheme. Any two members can reconstruct the secret by submitting the reconstructed $K$ and taking share assistance of one q-share from the dealer, if necessary. No fewer than two players can reconstruct the secret. As members of $\mathcal{M}$, $A$ and $B$ do not require share assistance, but the other two combinations do.

### 5.5 Twin-threshold Quantum Secret Sharing Schemes

In a conventional or compressed $(k, n)$-QTS, the threshold $k$ applies to all members taken together. Now suppose that we have separate thresholds for $c$-members and $q$-members, namely $k_c$ and $k_q$, with $k = k_c + k_q$. We now extend the definition of a conventional QTS to a $(k_c, k_q, n)$ quantum twin-threshold scheme $(Q2TS)$ and a $(k_c, k_q, n, C)$ quantum twin-threshold scheme with common set $(Q2TS+C)$, where a quantum secret $|S\rangle$ is split into $n$ pieces (shares) according to some pre-agreed procedure and distributed among $n$ players. These $n$ share-holders consist of members of set $Q$ of $q$-players and set $\bar{Q}$ of $c$-players. We denote $q \equiv |Q|$, so that $|\bar{Q}| = n - q$. Obviously, in a quantum scheme, $Q \neq \emptyset$.

**Definition 15** A QSS scheme is a $(k_c, k_q, n)$ quantum twin-threshold scheme $(Q2TS)$ among $n$ players, of which $q$ are $q$-players and the remaining are $c$-players, if at least $k_c$ $c$-players and at least $k_q$ $q$-players are necessary to reconstruct the secret.

**Definition 16** A QSS scheme is a $(k_c, k_q, n, C)$ quantum twin-threshold scheme with common set $(Q2TS+C)$ among $n$ players, of which $q$ are $q$-players and the remaining are $c$-players, if: (a) at least $k_c$ $c$-players and at least $k_q$ $q$-players are necessary to reconstruct the secret; (b) All members of the set $C$ are necessary to reconstruct the secret.

The idea behind distinguishing between the classical threshold $k_c$ and the quantum threshold $k_q$ is to obtain a simple generalization that combines the properties of the CTS and QTS. Practically speaking, it is best to minimize $k_q$, at fixed $k$. However, one can in principle consider situations of potential
use for a twin-threshold scheme, when a sufficiently large number of members are able to process quantum information safely. Further, some of the shareholders, while not entirely trust-worthy, may yet be more trust-worthy than others. The share-dealer (say Alice) may prefer to include all such shareholders during any reconstruction of the secret. This is the requirement that motivates the introduction of set $C$. In general, $C$ can contain members drawn from $Q$ and/or $\overline{Q}$ or may be a null set. By definition, $Q2TS+C$ with $C = \emptyset$ is $Q2TS$.

In the following sections we present two methods to realize in varying degrees the generalized quantum secret splitting scheme. The first of these is the general version of $Q2TS+C$. The second, while more restricted, is interesting because it is not directly based on quantum erasure correction, but on information dilution via homogenization, in contrast to current proposals of QSS.

### 5.5.1 Quantum Error Correction and Quantum Encryption

We now give protocols that realizes the twin-threshold scheme based on quantum encryption.

**Scheme 1.** Protocol to realize $(k_c, k_q, n)$-$Q2TS$.

*Distribution phase.* (1) Choose a random classical encryption $K$. Encrypt the quantum secret $|S\rangle$ using the encryption algorithm described in Section 5.1. The encrypted state is denoted $|\tilde{S}\rangle$; (2) Using a conventional $(k_q, q)$-QTS, split-share $|\tilde{S}\rangle$ among the members of $Q$; to not violate no-cloning, $q$ should satisfy $k_q > (q/2)$; (3) Using a $(k_c, n-q)$-CTS, split-share $K$ among the members of $\overline{Q}$.

*Reconstruction phase.* (1) Collect any $k_q q$-shares from members of $Q$ and reconstruct $|\tilde{S}\rangle$; (2) Collect any $k_c$ shares from members of $\overline{Q}$ and reconstruct $K$; (3) Reconstruct $|S\rangle$ using $|\tilde{S}\rangle$ and $K$.

Now consider the case $C \neq \emptyset$ and the $Q2TS$ scheme becomes the more general $Q2TS+C$ scheme. We now give a protocol that realizes this more general twin-threshold scheme. We denote $\lambda_q \equiv |Q \cap C|$ and $\lambda_c \equiv |\overline{Q} \cap C|$. Clearly, $\lambda_c + \lambda_q = |C|$. If there are no q-players in $C$, set $\lambda_q = 0$, and if there are no c-players in $C$, set $\lambda_c = 0$. Note that by definition, q-players may also carry classical information, but c-players don’t carry quantum information.

**Scheme 2.** Protocol to realize $(k_c, k_q, n, C)$-$Q2TS+C$. 

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Distribution phase. (1) Choose a random classical encryption $K$. Encrypt the quantum secret $|S\rangle$ using the encryption algorithm described in Section 5.1. The encrypted state is denoted $|\tilde{S}\rangle$; (2) Using a $(2,2)$-QTS, divide $|\tilde{S}\rangle$ into two pieces, say $|\tilde{S}_1\rangle$ and $|\tilde{S}_2\rangle$; (3) Using a $(\lambda_q, \lambda_q)$-QTS, split $|\tilde{S}_1\rangle$ among the q-members in $\mathcal{C}$; (4) Using a conventional $(k_q - \lambda_q, q - \lambda_q)$-QTS, split $|\tilde{S}_2\rangle$ among the q-members not in $\mathcal{C}$; to not violate no-cloning, $q$ should satisfy $(k_q - \lambda_q) > (q - \lambda_q)/2$; (5) Using a $(2,2)$-CTS, divide $K$ into two shares, say $K_1$ and $K_2$; (6) Part $K_1$ is split among the members of $\mathcal{C}$ using a $(|\mathcal{C}|, |\mathcal{C}|)$-CTS. Alternatively, it can be split using a $(\lambda_c, \lambda_c)$-CTS among the c-players in $\mathcal{C}$; (7) Using a $(k_c - \lambda_c, n - q - \lambda_c)$-CTS, split $K_2$ among the members of $\bar{\mathcal{Q}} - \mathcal{C}$.

Reconstruction phase. (1) Collect all $\lambda_q$ shares from all members of $\mathcal{Q} \cap \mathcal{C}$ and reconstruct $|\tilde{S}_1\rangle$; (2) Collect any $k_q - \lambda_q$ q-shares from $\mathcal{Q} - \mathcal{C}$ to reconstruct $|\tilde{S}_2\rangle$; (3) Combining $|\tilde{S}_1\rangle$ and $|\tilde{S}_2\rangle$, reconstruct $|\tilde{S}\rangle$; (4) Collect all $|\mathcal{C}|$ c-shares from members of $\mathcal{C}$ and reconstruct $K_1$. Alternatively, collect all $\lambda_c$ c-shares from members of $\bar{\mathcal{Q}} \cap \mathcal{C}$ and reconstruct $K_1$; (5) Collect any $k_c - \lambda_c$ shares from $\bar{\mathcal{Q}} - \mathcal{C}$ and reconstruct $K_2$; (6) Combining $K_1$ and $K_2$, reconstruct $K$. (7) Reconstruct $|S\rangle$ using $|\tilde{S}\rangle$ and $K$.

5.5.2 Quantum Twin-threshold Scheme Based on Information Dilution via Homogenization

The second, more restrictive scheme, is based on the procedure for information dilution in a system-reservoir interaction, proposed by Ziman et al.\cite{ZSBHSG02}. The novelty of the scheme lies in the fact that it is not directly based on a quantum error-correction code. However, it is applicable only to QSS with $\mathcal{C} \neq \emptyset$. Ref. \cite{ZSBHSG02} present a universal quantum homogenizer, a machine that takes as input a system qubit initially in the state $\rho$ and a set of $N$ reservoir qubits initially prepared in the identical state $\xi$. In the homogenizer the system qubit sequentially interacts with the reservoir qubits via the partial swap operation. The homogenizer realizes, in the limit sense, the transformation such that at the output each qubit is in an arbitrarily small neighborhood of the state $\xi$ irrespective of the initial states of the system and the reservoir qubits. Thus the information contained in the unknown system state is distributed in the correlations amongst the system and the reservoir qubits. As the authors point out, this process can be used as a quantum safe with a classical combination. Now we show how
this particular feature can be turned into a special case of the \((k_c, k_q, n, C)\) threshold scheme, subject to the restriction that \(Q \subseteq C\), so that \(k_q = q\), i.e. all q-players must be present to reconstruct the secret.

The homogenization is reversible and the original state of the system and the reservoir qubits can be unwound. Perfect unwinding can be performed only when the system particle is correctly identified from among the \(N + 1\) output qubits, and it and the reservoir qubits interact via the inverse of the original partial swap operation. Therefore, in order to unwind the homogenized system, the classical information (denoted \(K\)) about the sequence of the qubit interactions is essential. Now, of the \((N + 1)!\) possible orderings, only one will reverse the original process. The probability to choose the system qubit correctly is \(1/(N + 1)\). Even when the particle is chosen successfully, there are still \(N!\) different possibilities in choosing the sequence of interaction with the reservoir qubits. Thus, without the knowledge of the correct ordering, the probability of successfully unwinding the homogenization transformation is \(1/((N + 1)!)\), which is exponentially small in \(N\). So for sufficiently large value of \(N\), hardly any information about the system qubit can be deduced without this classical information.

If \(K\) is split up among the \(q\) members holding the system and reservoir qubits according to a \((q, q)\)-CTS, it is easy to observe that this realizes a \((q, q)\)-QTS not based directly on a quantum error-correction code. In terms of the generalized definition, this corresponds to a \((k_c, k_q, n, C)\)-scheme in which \(k_c = 0\), \(Q = C\) and \(n = k_q = q\). The classical layer of information sharing is necessary in order to strictly enforce the threshold: if prior ordering information were openly available, then for example the last \(q - 1\) participants could collude to obtain a state close to \(\rho\). We now present the most general twin-threshold scheme possible based on homogenization. It will still be more restricted than that obtained via quantum encryption, requiring that \(Q \subseteq C\), so that \(k_q = q\). If \(n\) is not too large, it is preferable for prevention of partial information leakage to choose the number \(N\) of reservoir qubits such that \(N \gg n\). The general protocol is executed recursively as follows. Alice takes \(N\) \((\gg 1)\) reservoir qubits, where \(N + 1 = \sum_i m_i\) and integers \(m_i \geq 1\) \((\forall i : 1 \leq i \leq n)\), and performs the process of homogenization to obtain states \(\xi_0, \xi_1, \ldots, \xi_N\) on the system qubit and the \(N\) reservoir qubits.

**Scheme 3:** Protocol to realize a a restricted \((k_c, k_q, n, C)\)-Q2TS+C, with \(k_q = q \leq n\).

**Distribution phase:** (1) Any \(m_i\) qubits from \(N + 1\) qubits are given to the \(i\)th member of \(Q\); (2) \(K\) is divided into two parts, \(K_1\) and \(K_2\), according
to a $(2,2)$-CTS; (3) Let $\lambda_c \equiv |\bar{Q} \cap C| \geq 0$. $K_1$ is further split among the members of $Q$ and $\bar{Q} \cap C$ using a $(q + \lambda_c, q + \lambda_c)$-CTS; (4) $K_2$ is split among the members of $\bar{Q} - C$ using a $(k_c - \lambda_c, n - q - \lambda_c)$-CTS.

Reconstruction phase: (1) Collect all $q$-shares from members of $\bar{Q}$; (2) Collect all $|C|$ $c$-shares from members of $C$ and reconstruct $K_1$; (3) Collect any $k_c - \lambda_c$ shares from members of $\bar{Q} - C$ to reconstruct $K_2$; (4) Using $K_1$ and $K_2$, reconstruct $K$; (5) Using the $q$-shares and $K$, unwind the system state to restore the secret $|S\rangle$. 

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Chapter 6

A Step Towards Combining the Features of QKD and QSS

6.1 Introduction

Regarding the cryptographic use of multipartite entanglement, three broad issues may be discerned. In the first case, the parties may choose to simply obtain random bits shared between pairs of the $n$ parties. In the second case, the entanglement may be used to generate random shared bits between the $n$ trustful parties. Finally, the $n$ parites may not be entirely trusting and we may wish to use entanglement to obtain a secret sharing protocol. First case is essentially QKD between two trustful parties and the later two cases are respectively $n$-QKD and QSS on which we have discussed a great deal in the last two chapters. $n$-QKD involves sharing a random key amongst $n$ trustworthy parties where as QSS splits quantum information amongst untrusted parties. It would be an interesting extension to consider situations where some kind of mutual trust may be present between sets of parties while parties being individually mistrustful. This way it could be possible to combine the essential features of QKD and QSS. In this chapter, we discuss two such extensions.

Firstly, we consider the problem of secure key distribution between two trustful groups where the invidual group members may be mistrustful. The two groups retrieve the secure key string, only if all members should cooperate with one another in each group. That is, how the two groups one of
size $k$ and the other of size $n - k$ may share an identical secret key among themselves while an evesdropper may co-operate with several (of course not all) dishonest members from any of the groups. This task is trivially a classical secret sharing scheme if we involve a trusted third party, say, Lucy. Lucy will simply generate a random classical bit string. Since it is just a classical information she makes two copies of it and split one each amongst the two groups. Principles of quantum physics allows us, as in the case of 2-QKD, to do away with the third party. Adopting an idea similar to that in Chapter 4, we present a quantum key distribution protocol for this purpose based on entanglement purification, which can be proven secure by reducing the problem to the bipartite case using combinatorics developed in Chapter 2.

We can observe that the above problem essentially seems to be a combination of

1. 2-QKD between the two groups, each group being considered as a single party and

2. Secret sharing in each group amongst their parties.

In the second possible extension, we consider several such groups. Members of the same group trust each other whereas members from different groups do not and the problem is to establish a common shared random key amongst the $n$ untrustful parties. This problem, as above, could also be tackled by similar reduction to bipartite case. We discuss a necessary condition for such schemes to exist and also present one such scheme. Another step in combining QKD and QSS could be the generalization of the first case with a setting just opposite to that in the second case. Members of the same group do not trust each other whereas members from different groups combined together do trust the members of other group combined together. The problem is to establish a common shared random key amongst the different groups.

6.2 Quantum Key Distribution between Two Groups

In this section we develop a simple protocol for QKD between two groups. Our protocol works in two broad steps. In the first step, the $n$-partite problem
is reduced to a two-party problem by means of SKP-2 (Chapter 2, Protocol-II). This creates a pure $n$-partite maximally entangled state among $n$-parties, starting from $n-1$ EPR pairs shared along a spanning EPR tree using only $\mathcal{O}(n)$ bits of classical communication. The SKP-2 exploits the combinatorial arrangement of EPR pairs to simplify the task of distributing multipartite entanglement. In the second step, as in case of $n$-QKD, the Lo-Chau protocol [LC99] or Modified Lo-Chau protocol [SP00] is invoked to prove the unconditional security of sharing nearly perfect EPR pairs between two parties.

To this end, we will be using a state of the form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle),$$

(6.1)
a maximally entangled $n$-partite state, represented in the computational basis.

Our protocol is motivated by a simple mathematical property possessed by multi-partite states, unlike EPR pairs, which forces them to behave differently when measured in computational or diagonal basis. Here below we develop this mathematics.

Let $H$, $\oplus$ and $\otimes$ denote the Hadamard gate, the XOR operation and the tensor product respectively then (assume the presence of a proper normalizing factor in each expression),

$$H\otimes^n |1\rangle\otimes^n = \sum_{x_1,x_2,\cdots,x_n} (-1)^{x_1 \oplus x_2 \oplus \cdots \oplus x_n} |x_1 x_2 \cdots x_n\rangle$$

and

$$H\otimes^n |0\rangle\otimes^n = \sum_{x_1,x_2,\cdots,x_n} |x_1 x_2 \cdots x_n\rangle$$

therefore,

$$H\otimes^n(|1\rangle\otimes^n + |0\rangle\otimes^n) = \sum_{x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0} |x_1 x_2 \cdots x_n\rangle$$

$$= \left( \sum_{x_1 \oplus x_2 \cdots \oplus x_s = 0} |x_1 x_2 \cdots x_s\rangle \right) \left( \sum_{x_{s+1} \oplus x_{s+2} \cdots \oplus x_n = 0} |x_{s+1} x_{s+2} \cdots x_n\rangle \right)$$

$$+ \left( \sum_{x_1 \oplus x_2 \cdots \oplus x_s = 1} |x_1 x_2 \cdots x_s\rangle \right) \left( \sum_{x_{s+1} \oplus x_{s+2} \cdots \oplus x_n = 1} |x_{s+1} x_{s+2} \cdots x_n\rangle \right).$$
We can observe by symmetry that the above factoring can be in fact done for any two groups of sizes \( s \) and \( n - s \) respectively.

We are now ready to develop our protocol which involves the following steps:

1. **EPR protocol:** Along the \( n - 1 \) edges of the minimum spanning EPR tree, EPR pairs are created using Lo-Chau \([\text{LC99}]\) or Modified Lo-Chau protocol \([\text{SP00}]\) (by leaving out the final measurement step). This involves pairwise quantum and classical communication between any two parties connected by an edge. Successful completion ensures that each of the two parties across a given edge share a nearly perfect singlet state \( \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \). At the end of the run, let the minimum number of EPR pairs distilled along any edge of the minimum spanning EPR tree be \( 2^m \).

2. The \( 2^m \) instances of the singlet state are then converted to the triplet state \( \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \), by the Pauli operator \( XZ \) being applied by the second party (called \( \mathcal{Y} \)) on his qubit.

3. For each edge, the party \( \mathcal{Y} \) intimated the protocol leader (say “Lucy”) of the completion of step (2). Lucy is the one who starts and directs the SKP protocol (Chapter 2, Protocol-II) used below. Note that Lucy can be from any of the two groups.

4. **SKP protocol:** Using purely local operations and classical communication (LOCC), the \( n \) parties execute the SKP protocol, which consumes the \( n - 1 \) EPR pairs to produce one instance of the state (6.1) shared amongst them.

5. A projective measurement in the diagonal basis is performed by all the parties on their respective qubits.

6. Lucy decides randomly a set of \( m \) bits to be used as check bits, and announces their positions.

7. All parties from a group assist (cooperate) to get one cbit corresponding to each check bit position by XORing their corresponding check bits. This gives an effective check bit corresponding to each check bit position. The two group then announce the value of their effective check bits. If
too few of these values agree, they abort the protocol. We can note from
the mathematics developed above that the effective check bits should
agree after the diagonal basis measurement. Effective non-check bits
are also calculated as above by XORing the non-check bits of the group
members.

(8) Error correction is done as in for quantum key distribution between two
trustful parties.

Proof of unconditional security: The proof of unconditional security of
the above protocol is almost the same as for n-QKD (Chapter 4). However,
we would like to stress the role of fault tolerant quantum computation and
of quantum error correcting codes during the execution of SKP protocol for
the following reason: Suppose the probability of error on a bit is \( p \). Then
the probability of an error on the bit obtained by XORing all the \( s \) group
members’ bits may be larger, given by:

\[
P = \sum_{r=1,3,5...} C(s, r) p^r (1-p)^{s-r},
\]

where \( C(s, r) \) is the number of all possible way selecting \( r \) elements from a set of \( s \) distinct elements.

If \( P \) is too close to 0.5, then the effective channel capacity \( Ch \) for the
protocol (given by \( Ch = 1 - H(P) \), where \( H(.) \) is Shannon entropy) will
almost vanish. Therefore, the quantum part of the protocol implementation
should be very good to ensure that \( P \) is not too close to 0.5.

Of the XORed \( 2m \) raw bits, \( m \) bits are first used for getting an estimate of
\( P \), by obtaining the Hamming distance \( \delta \) between each group’s \( m \)-bit check
string. If they are mutually too distant, the protocol run is aborted. If \( \delta \nocr
6.3 $n$-QKD amongst Untrustful Parties

We now consider the second case where there are trustful groups but different set of these groups may not trust each other. For example, suppose there are 10 parties \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \{1, 2, 3\} trust each other but any of \{1, 2\} do not trust any of the parties \{4, 5, 6, 7, 8, 9, 10\}. \{3, 4, 6, 7\} trust each other but 6 does not trust any other party. \{4, 5, 7, 8, 9, 10\} trust each other but any of \{5, 8, 9, 10\} do not trust any of \{1, 2, 3, 6\}. We mean to say that there can be several trustful groups and they may have some common parties. Within a group every one trust each other but a party does not trust another party who is not in any group he belongs. In the above example the trustful groups are \{1, 2, 3\}, \{3, 4, 6, 7\} and \{4, 5, 7, 8, 9, 10\}. Our aim in this case will be to obtain an unconditionally secure QKD scheme amongst the $n$-parties.

Using our $n$-QKD scheme for trustful parties presented in Chapter 4 and hypergraph combinatorics developed in Chapter 2 the above problem, along with necessary and sufficient conditions, can be easily tackled as follows:

Let $S$ denotes the set of parties. Then the trustful groups can be represented as subsets of $S$. Now let these groups are $E_1, E_2, \cdots$, and $E_m$ and $F$ is collection of these subsets. This structure can be represented by a hypergraph $H = (S, F)$ and we call it a security hypergraph of the $n$ parites. This representation is inline with that of entangled hypergraph in Chapter 2. The necessary and sufficient condition is same as dictated by Theorem 5 that is, the QKD between the $n$ parties can be successfully done if and only if the security hypergraph is connected.

The protocol is to first reduce the security hypergraph to a simple security graph (Chapter 4) and then to apply a slight modification of the scheme for trustful $n$-QKD. In the reduced simple security graph only those vertices will survive which belongs to at least two hyperedges (that is, two trustful groups) and edges will be between those vertices that trust each other.
Chapter 7
Open Research Directions

We conclude briefly with some open questions based on our research.

1. *Lower bound on classical communication complexity for preparing multipartite entanglement from bi-partite entanglement under LOCC.*

   In Chapter 2, we observed that, all the schemes for creating a pure $n$-partite maximally entangled state from the distributed network of EPR pairs amongst $n$ agents, require $O(n)$ cbits of communication. An obvious open problem is to determine whether there is an $\Omega(n)$ lower bound on the cbit communication complexity for preparing a pure $n$-partite maximally entangled state given a spanning EPR tree of $n$ agents. We hope that criteria using quantum information theory may help settle this issue.

2. *Partial secret sharing*

   Our Protocol-I in Chapter 2 was motivated by the dynamic and symmetric involvement of all the players in the preparation of GHZ state from two EPR pairs. We pointed out there that it should be interesting to investigate the ramifications of the protocol. One possible approach in this direction could be a secret sharing scheme with relaxed conditions as discussed below.
In usual secret sharing schemes we have mainly two constraints as specified in their definition:

1. Any authorized set will have the complete information about the secret.

2. Any unauthorized set will have no information about the secret, that is, any unauthorized set is maximally uncertain about the secret.

Now let us relax the second constraint. Let the unauthorized sets have partial information about the secret but not the complete information, that is, they may not be maximally uncertain about the secret though they may have partial knowledge of the secret. How much information an unauthorized set should have could be specified as a part of the specification of the scheme and could be better given in terms of Shannon’s or Von-Neumann entropy respectively for classical and quantum case.

For example, one way to relax the threshold scheme \((k, n)\) may be as follows: The ignorance about the secret which any \(m\) members will have is given by Shannon entropy \(H(m) = 0\) for \(m \geq k\) and \((k - m)/k\) for \(m < k\). We could call such schemes to be *partial* secret sharing schemes (PSS). There may be some practical applications of such schemes especially in the quantum case for say multi-party secure computation. For example, let there are two players \(B\) and \(C\). The dealer \(A\) wants that \(B\) and \(C\) should be able to do full proof computation only when both of \(B\) and \(C\) co-operate. She also wants to allow a computation by \(B\) up to a phase factor and that by \(C\) up to a bit-flip factor. Such a scheme may be obtained by relaxing a \(((2, 2))\) QTS. This is where we might use the above mentioned symmetric teleportation circuit of Chapter 2 (Protocol-I). Such schemes may also be important in the situation wherein some members die (in computer network language these are called faulty nodes), we have partial information and can do reliable (probabilistic) computation. It should also be interesting to investigate the compression and inflation of partial secret sharing schemes and to investigate how the relaxation constraint relates with such compression and inflation.

Quantitatively, for PSS, one would have to show that the weakening does not compromise the SS itself. This means that we have to obtain a lower bound on the number of dishonest colluders (who can sabotage
the protocol for the remaining players) as a function of the weakening (quantified using some parameter). One could then say that the protocol is $Y$-level secure at $X$-degree weakness, or suchlike.

Another approach to implement PSS could be like our QSS scheme (Chapter 5) based on method of information dilution via homogenization, where we had to scramble the ordering information in order to prevent partial information gain. Not scrambling this information would appear to allow partial information reconstruction in a way that seems compatible with PSS. Therefore, perhaps this method affords a possible approach to PSS.

3. Distributed quantum error correction and multi-party problems based on secure execution of SKP protocol

In Chapter 6, the protocol for 2-group QKD utilized SKP protocol as a subroutine. If we have fault tolerant quantum computers, then from the security proof of the protocol (similar as for $n$-QKD in Chapter 4), it can be noted that a pure $n$-partite maximally entangled state could be securely prepared. Given the slow pace in practical realization of quantum computers than that of quantum communication systems, it would be very pragmatic to do away with the requirement of the fault tolerant quantum computers in the secure execution of SKP protocol. An interesting way of doing this may be to do quantum error correction in a distributed manner while determining syndromes using local measurements. In the bipartite case, this has been achieved by Modified Low-Chou protocol. In the case of multi-partite states, however it seems to be a subtle task to achieve and remains an unresolved problem for future research.

Nevertheless, we could also get a simpler protocol for $n$-QKD which did not use SKP protocol at all, though it enjoyed the spanning tree combinatorics. Therefore, it should also be interesting to investigate the class of secure multi-party computational and cryptographic problems which can be or can not be executed without secure execution of SKP.
4. **Combining the features of QKD and QSS**

Chapter 6 takes the first step towards exploiting the essential features of QKD and QSS together, however, we have only dealt with two cases, namely, 2-group QKD and QKD amongst untrustful parties. It remains to obtain more interesting situations, such as a simple generalization of 2-group QKD to \( n \)-group QKD, which could essentially combine the features of QKD and QSS.

5. **Generalizing the concept of bicolored merging**

In Chapter 3, we have developed the idea of bicolored merging and utilized it to show various results on the possible or impossible state transformations of the multi-partite states represented by EPR graphs and entangled hypergraphs by utilizing only the monotonicity postulate for appropriate entanglement measures. However, we would also like to stress that any kind of reduction, which leads to the violation of any of the properties of a potential entanglement measure, is pertinent to show the impossibility of many multi-partite state transformations under LOCC. Since the bipartite case has been extensively studied, such reductions can potentially provide many ideas about multi-partite case by just exploiting the results from bipartite case. In particular, the definitions of EPR graphs and entangled hypergraphs could also be suitably extended to capture more types of multi-partite pure states and even mixed states and a generalization of the idea of bicolored merging as a suitable reduction for this case could also be worked out. It would be interesting to investigate such issues. For example, one possible variation of bicolored merging could be to relax the third constraint, that is the constraint on the number of EPR pairs in the BCM EPR graphs. One could relax this condition such that the BCM EPR graphs need only to be distinct and use entropic criterion (restrictions directly put only by monotonicity postulate may not help here) to show possibility or impossibility of various state transformations.
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