LAPLACIAN COMPARISON
FOR ALEXANDROV SPACES

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Dedicated to Professor Karsten Grove on the occasion of his sixtieth birthday.

Abstract. We consider an infinitesimal version of the Bishop-Gromov relative volume comparison condition as a generalized notion of Ricci curvature bounded below for Alexandrov spaces. We prove a Laplacian comparison theorem for Alexandrov spaces under the condition. As an application we prove a topological splitting theorem.

1. Introduction

In this paper, we study singular spaces of Ricci curvature bounded below. For Riemannian manifolds, having a lower bound of Ricci curvature is equivalent to an infinitesimal version of the Bishop-Gromov volume comparison condition. Since it is impossible to define the Ricci curvature tensor on Alexandrov spaces, we consider such the volume comparison condition as a candidate of the conditions of the Ricci curvature bounded below.

In Riemannian geometry, the Laplacian comparison theorem is one of the most important tools to study the structure of spaces with a lower bound of Ricci curvature. A main purpose of this paper is to prove a Laplacian comparison theorem for Alexandrov spaces under the volume comparison condition. As an application, we prove a topological splitting theorem of Cheeger-Gromoll type.

Let us present the volume comparison condition. For $\kappa \in \mathbb{R}$, we set

$$s_\kappa(r) := \begin{cases} 
\frac{\sin(\sqrt{|\kappa|}r)}{\sqrt{|\kappa|}} & \text{if } \kappa > 0, \\
\frac{\sinh(\sqrt{|\kappa|}r)}{\sqrt{|\kappa|}} & \text{if } \kappa < 0.
\end{cases}$$

The function $s_\kappa$ is the solution of the Jacobi equation $s_\kappa''(r) + \kappa s_\kappa(r) = 0$ with initial condition $s_\kappa(0) = 0$, $s_\kappa'(0) = 1$.

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Let $M$ be an Alexandrov space of dimension $n \geq 2$. For $p \in M$ and $0 < t \leq 1$, we define a subset $W_{p,t} \subset M$ and a map $\Phi_{p,t} : W_{p,t} \to M$ as follows. $x \in W_{p,t}$ if and only if there exists $y \in M$ such that $x \in py$ and $d(p,x) : d(p,y) = t : 1$, where $py$ is a minimal geodesic from $p$ to $y$ and $d$ the distance function. For a given point $x \in W_{p,t}$ such a point $y$ is unique and we set $\Phi_{p,t}(x) := y$. The Alexandrov convexity (cf. §2.2) implies the Lipschitz continuity of the map $\Phi_{p,t}$. Let us consider the following.

**Condition BG($\kappa$) at a point $p \in M$:** We have
\[
\text{d}(\Phi_{p,t}^* \mathcal{H}^n)(x) \geq \frac{t s_\kappa(t d(p,x))^{n-1}}{s_\kappa(d(p,x))^{n-1}} \text{d} \mathcal{H}^n(x)
\]
for any $x \in M$ and $t \in (0,1]$ such that $d(p,x) < \pi/\sqrt{\kappa}$ if $\kappa > 0$, where $\Phi_{p,t}^* \mathcal{H}^n$ means the push-forward by $\Phi_{p,t}$ of the $n$-dimensional Hausdorff measure $\mathcal{H}^n$ on $M$.

If $M$ satisfies BG($\kappa$) at any point $p \in M$, we simply say that $M$ satisfies BG($\kappa$).

The condition, BG($\kappa$), is an infinitesimal version of the Bishop-Gromov inequality. For an $n$-dimensional complete Riemannian manifold, BG($\kappa$) holds if and only if the Ricci curvature satisfies $\text{Ric} \geq (n-1)\kappa$ (see Theorem 3.2 of [20] for the ‘only if’ part). We see some studies on similar (or same) conditions to BG($\kappa$) in [7, 33, 15, 16, 29, 20, 38] etc. BG($\kappa$) is sometimes called the Measure Contraction Property and is weaker than the curvature-dimension condition introduced by Sturm [34, 35] and Lott-Villani [17]. Any Alexandrov space of curvature $\geq \kappa$ satisfies BG($\kappa$). However we do not necessarily assume $M$ to be of curvature $\geq \kappa$. For example, a Gromov-Hausdorff limit of closed $n$-manifolds of $\text{Ric} \geq (n-1)\kappa$, sectional curvature $\geq \kappa_0$, diameter $\leq D$, and volume $\geq v > 0$ is an Alexandrov space with BG($\kappa$) and of curvature $\geq \kappa_0$.

To state the Laplacian comparison theorem, we need some notations and definitions. If $M$ has no boundary, we define $M^*$ as the set of non-$\delta$-singular points of $M$ for a number $\delta$ with $0 < \delta \ll 1/n$. If $M$ has nonempty boundary, we refer to Fact 2.6 below for $M^*$. All the topological singularities of $M$ are entirely contained in $M \setminus M^*$ and $M^*$ has a natural structure of $C^\infty$ differentiable manifold. We have a canonical Riemannian metric $g$ on $M^*$ which is a.e. continuous and of locally bounded variation (locally BV for short). See §2.2 for more details. We set $\cot_\kappa(r) := s'_\kappa(r)/s_\kappa(r)$ and $r_p(x) := d(p,x)$ for $p, x \in M$. The **distributional Laplacian $\bar{\Delta} r_p$ of $r_p$ on $M^*$** is defined by the usual formula:
\[
\bar{\Delta} r_p := -D_i(\sqrt{\text{g}^{ij}} \partial_j r_p),
\]
on a local chart of $M^*$, where $D_i$ is the distributional derivative with respect to the $i$th coordinate. Then, $\bar{\Delta} r_p$ becomes a signed Radon
measure on $M^*$ (see §4). An main theorem of this paper is stated as follows.

**Theorem 1.1** (Laplacian Comparison Theorem). Let $M$ be an Alexandrov space of dimension $n \geq 2$. If $M$ satisfies $\text{BG}(\kappa)$ at a point $p \in M$, then we have

$$ d\Delta r_p \geq -(n-1) \cot \kappa, \quad \text{on } M^* \setminus \{p\}. $$

**Corollary 1.2.** If $M$ is an Alexandrov space of dimension $n \geq 2$ and $\text{curvature} \geq \kappa$, then (1.1) holds for any $p \in M$.

Even if $M$ is a Riemannian manifold, $\Delta r_p$ is not absolutely continuous with respect to $\mathcal{H}^n$ on the cut-locus of $p$ (see Remark 5.10). Different from Riemannian, the cut-locus of an Alexandrov space is not necessarily a closed subset. In fact, we have an example of an Alexandrov space for which the singular set and the cut-locus are both dense in the space (cf. Example (2) in §0 of [22]). The Riemannian metric $g$ on $M^*$ is not continuous on any singular point and has at most the regularity of locally BV. Therefore, the Laplacian of a $C^\infty$ function does not become a function, only does a Radon measure in general. In particular, considering a Laplacian comparison in the barrier sense is meaningless. In this reason, for Theorem 1.1 a standard proof for Riemannian does not work and we need a more delicate discussion using BV theory.

In [26], Petrunin claims that the Laplacian of any $\lambda$-semiconvex function is $\geq -n\lambda$ from the study of gradient curves. This implies Corollary 1.2. However we do not know the details. After Petrunin, Renesse [36] proved Corollary 1.2 in a different way under some additional condition. Our proof is based on a different idea from them.

We do not know if the converse to Theorem 1.1 is true or not, i.e., if (1.1) implies $\text{BG}(\kappa)$ at $p$. For $C^\infty$ Riemannian manifolds, this is easy to prove.

As an application to Theorem 1.1, we have

**Theorem 1.3** (Topological Splitting Theorem). If an Alexandrov space $M$ satisfies $\text{BG}(0)$ and contains a straight line, then $M$ is homeomorphic to $N \times \mathbb{R}$ for some topological space $N$.

We do not know if the isometric splitting in the theorem is true, i.e., if $M$ is isometric to $N \times \mathbb{R}$ for some Alexandrov space $N$. If we replace ‘$\text{BG}(0)$’ with ‘$\text{curvature} \geq 0$’, then the isometric splitting is well-known ([19]) as a generalization of the Toponogov splitting theorem. For Riemannian manifolds, $\text{BG}(0)$ is equivalent to $\text{Ric} \geq 0$ and the isometric splitting was proved by Cheeger-Gromoll [8]. In our case, we do not have the Weitzenböck formula, so that we cannot obtain the isometric splitting at present.

If the metric of $M$ has enough $C^\infty$ part, we prove the isometric splitting.
Corollary 1.4. Let $M$ be an Alexandrov space. Assume that the singular set of $M$ is closed and the non-singular set is an (incomplete) $C^\infty$ Riemannian manifold of $\text{Ric} \geq 0$. If $M$ contains a straight line, then $M$ is isometric to $N \times \mathbb{R}$ for some Alexandrov space $N$.

For Riemannian orbifolds, Borzellino-Zhu\cite{Borzellino-Zhu} proved an isometric splitting theorem. Corollary 1.4 is more general than their result.

In our previous paper\cite{previous-paper}, we proved for an Alexandrov space $M$ the existence of the heat kernel of $M$ and the discreteness of the spectrum of the generator (Laplacian) of the Dirichlet energy form on a relatively compact domain in $M$. As another application to Theorem 1.1, we have the following heat kernel and first eigenvalue comparison results, which generalize the results of Cheeger-Yau\cite{Cheeger-Yau} and Cheng\cite{Cheng}.

$B(p, r)$ denotes the metric ball centered at $p$ and of radius $r$ and $M^n(\kappa)$ an $n$-dimensional complete simply connected space form of curvature $\kappa$.

Corollary 1.5. Let $M$ be an $n$-dimensional Alexandrov space which satisfies $BG(\kappa)$ at a point $p \in M$, and $\Omega \subset M$ an open subset containing $B(p, r)$ for a number $r > 0$. Denote by $h_t : \Omega \times \Omega \to \mathbb{R}$, $t > 0$, the heat kernel on $\Omega$ with Dirichlet boundary condition, and by $\bar{h}_t : B(\bar{p}, r) \times B(\bar{p}, r) \to \mathbb{R}$ that on $B(\bar{p}, r)$ for a point $\bar{p} \in M^n(\kappa)$. Then, for any $t > 0$ and $q \in B(p, r)$ we have

$$h_t(p, q) \geq \bar{h}_t(\bar{p}, \bar{q}),$$

where $\bar{q} \in M^n(\kappa)$ is a point such that $d(\bar{p}, \bar{q}) = d(p, q)$.

Corollary 1.6. Let $M$ be an $n$-dimensional Alexandrov space which satisfies $BG(\kappa)$ at a point $p \in M$, and $r > 0$ a number. Denote by $\lambda_1(B(p, r))$ the first eigenvalue of the generator (Laplacian) of the Dirichlet energy form on $B(p, r)$ with Dirichlet boundary condition, and by $\bar{\lambda}_1(B(\bar{p}, r))$ that on $B(\bar{p}, r)$ for a point $\bar{p} \in M^n(\kappa)$. Then we have

$$\lambda_1(B(p, r)) \leq \bar{\lambda}_1(B(\bar{p}, r)).$$

Once we have the Laplacian Comparison Theorem (see Corollary 5.11), the proofs of Corollaries 1.5 and 1.6 are the same as of Theorem II and Corollary 1 of Renesse’s paper\cite{Renesse}. We can carefully verify that the local $(L^1, 1)$-volume regularity is not needed in the proof of Theorem II of\cite{Renesse}.

We also obtain a Brownian motion comparison theorem in the same way as in\cite{Renesse}. The detail is omitted here.

Remark 1.7. All the results above are true even in the case where $M$ has non-empty boundary. In Corollaries 1.5 and 1.6, we implicitly assume the Neumann boundary condition on the boundary of $M$ for the heat kernel and the first eigenvalue. In particular, the results hold for any convex subset of an Alexandrov space.
Let us briefly mention the idea of the proof of Theorem 1.1. One of the important steps is to prove the Green formula on a region $E \subset M^*$ with piecewise smooth boundary:

$$\bar{\Delta} r_p(E) = \int_{\partial E} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1},$$

where $\nu_E$ is the inward normal vector field along $\partial E$ of $E$ (Theorem 4.1). For the proof of the Green formula, it is essential to prove that $\text{div}_{g(h)} Y \to \text{div}_g Y$ weakly $*$ as $h \to 0$ (Lemma 4.9), where $Y$ is any $C^\infty$ vector field on $M^*$, $g(h)$ the $C^\infty$ mollifier of the Riemannian metric $g$ on $M^*$, and $\text{div}_g$ (resp. $\text{div}_{g(h)}$) the distributional divergence with respect to $g$ (resp. $g(h)$). Remark that to obtain this, we need some geometric property of singularities of $M$ (see the proofs of Lemmas 4.8 and 4.9) besides the BV property of $g$.

Using the Green formula, we prove the Laplacian Comparison Theorem, 1.1. Our idea is to approximate any region $E$ with piecewise smooth boundary by the union of finitely many regions $A_k$, where each $A_k$ forms the intersection of some concentric annulus centered at $p$ of radii $r_k^- < r_k^+$ and a union of minimal geodesics emanating from $p$. See Figure 1.

![Figure 1](approximate_E.png)

**Figure 1.** Approximate $E$ by $\bigcup_k A_k$.

Set $B_k^- := \partial B(p, r_k^-) \cap \partial A_k$ and $B_k^+ := \partial B(p, r_k^+) \cap \partial A_k$. We assume that each $A_k$ is very thin, i.e., the diameters of $B_k^\pm$ are very small. We note that $\bigcup_k (B_k^- \cup B_k^+)$ approximates $\partial E$ and $\partial E$ has a division corresponding to $\{A_k\}$. $B_k^\pm$ are all perpendicular to $\nabla r_p$ and the area of $B_k^\pm$ is close to that of the corresponding part of $\partial E$ multiplied by $\langle \nu_E, \nabla r_p \rangle$. Since the cut-locus of $p$ could be very complex (e.g. could be a dense subset), we need a delicate discussion. Using BG($\kappa$), we estimate the difference between the areas of $B_k^-$ and $B_k^+$ by the volume of $A_k$. Summing up this for all $k$, we have an estimate of the right-hand
side of the Green formula for $E$ by the volume of $E$, that is,
\[ \Delta r_p(E) \geq -(n - 1) \sup_{x \in E} \cot_\kappa(r_p(x)) \mathcal{H}^n(E). \]
This implies the Laplacian Comparison Theorem.

The organization of this paper is as follows. In §2 we prepare Alexandrov spaces and BV functions. In §3 we prove some basic properties for Condition BG($\kappa$). In §4 we perform some serious BV calculus on Alexandrov spaces and prove the Green formula. In §5 we give a proof of the Laplacian Comparison Theorem. In the final section, §6, we prove Theorem 1.3 following the method of Cheeger-Gromoll [8].

2. Preliminaries

2.1. Notation. Let $\theta(x)$ be some function of variable $x \in \mathbb{R}$ such that $\theta(x) \to 0$ as $x \to 0$, and $\theta(x|y_1, y_2, \ldots)$ some function of variable $x \in \mathbb{R}$ depending on $y_1, y_2, \ldots$ such that $\theta(x|y_1, y_2, \ldots) \to 0$ as $x \to 0$. We use them like Landau’s symbols.

2.2. Alexandrov spaces and their structure. In this section, we present basics for Alexandrov spaces. Refer [3, 6, 22, 24] for the details.

Let $M$ be a geodesic space, i.e., any two points $p, q \in M$ can be joined by a length-minimizing curve, called a minimal geodesic $pq$. Note that for given $p, q \in M$ a minimal geodesic $pq$ is not unique in general. A triangle $\triangle pqr$ in $M$ means a set of three points $p, q, r \in M$ (vertices), and of three geodesics $pq, qr, rp$ (edges). For a number $\kappa \in \mathbb{R}$, a $\kappa$-comparison triangle of a triangle $\triangle pqr$ in $M$ is defined to be a triangle $\triangle \bar{p}\bar{q}\bar{r}$ in a complete simply connected space form of curvature $\kappa$ with the property that $d(p, q) = d(\bar{p}, \bar{q}), d(q, r) = d(\bar{q}, \bar{r}), d(r, p) = d(\bar{r}, \bar{p})$. We denote by $\angle pqr$ the angle $\angle \bar{p}\bar{q}\bar{r}$ between $\bar{q}\bar{r}$ and $\bar{r}\bar{p}$ at $\bar{q}$ of $\triangle \bar{p}\bar{q}\bar{r}$. $\angle pqr$ is determined only by $d(p, q), d(q, r), d(r, p)$, and $\kappa$.

**Definition 2.1** (Alexandrov Convexity). A subset $\Omega \subset M$ is said to satisfy the ($\kappa$-)Alexandrov convexity if for any triangle $\triangle pqr \subset \Omega$, there exists a $\kappa$-comparison triangle $\triangle \bar{p}\bar{q}\bar{r}$ such that for any $x \in pq, y \in pr, \bar{x} \in \bar{p}\bar{q}, \bar{y} \in \bar{p}\bar{r}$ with $d(p, x) = d(\bar{p}, \bar{x}), d(p, y) = d(\bar{p}, \bar{y})$ we have
\[ d(x, y) \geq d(\bar{x}, \bar{y}). \]

**Definition 2.2** (Lower Bound of Curvature, $\kappa$). For a subset $\Omega \subset M$, we denote by $\kappa(\Omega)$ the supremum of $\kappa \in \mathbb{R}$ for which $\Omega$ satisfies the $\kappa$-Alexandrov convexity. $\kappa(\Omega)$ may be $+\infty$ or $-\infty$. For a point $x \in M$ we set $\kappa(x) := \sup_U \kappa(U)$, where $U$ runs over all neighborhoods of $x$.

The function $\kappa : M \to [-\infty, +\infty]$ is lower semi-continuous.

**Definition 2.3** (Alexandrov Space). We say that $M$ is an Alexandrov space if

1. $M$ is a complete geodesic space,
2. $\kappa(x) > -\infty$ for any $x \in M$,.
(3) the Hausdorff dimension of $M$ is finite.

An Alexandrov space $M$ is said to be of curvature $\geq \kappa$ if $\kappa(M) \geq \kappa$.

We usually assume the connectedness for Alexandrov spaces. However, we agree that a two-point space $M = \{p, q\}$ is an Alexandrov space of curvature $\geq \pi^2/d(p, q)^2$.

Let $M$ be an Alexandrov space. Then, $M$ is proper, i.e., any bounded subset is relatively compact. If $M$ is of curvature $\geq \kappa > 0$, then $\text{diam } M \leq \pi/\sqrt{\kappa}$ and $M$ is compact. By the globalization theorem, for any bounded subset $\Omega \subset M$, there exists $R > 0$ such that $\kappa(\Omega) \geq \inf_{x \in B(\Omega, R)} \kappa(x) > -\infty$,

where $B(\Omega, R)$ is the $R$-neighborhood of $\Omega$. In particular we have $\kappa(M) = \inf_{x \in M} \kappa(x) (\geq -\infty)$.

The Hausdorff dimension of (any open subset of) $M$ is a non-negative integer and coincides with the covering dimension. A zero-dimensional Alexandrov space is a one-point or two-point space. A one-dimensional Alexandrov space is a one-dimensional complete Riemannian manifold possibly with boundary.

Let $n$ be the dimension of $M$ and assume $n \geq 1$. We take any point $p \in M$ and fix it. Denote by $\Sigma_p M$ the space of directions at $p$, and by $K_p M$ the tangent cone at $p$ (see [6]). $\Sigma_p M$ is an $(n-1)$-dimensional compact Alexandrov space of curvature $\geq 1$ and $K_p M$ an $n$-dimensional Alexandrov space of curvature $\geq 0$. If $M$ is a Riemannian manifold, $\Sigma_p M$ and $K_p M$ are identified respectively with the unit tangent sphere and the tangent space.

**Definition 2.4** (Singular Point, $\delta$-Singular Point). A point $p \in M$ is called a singular point of $M$ if $\Sigma_p M$ is not isometric to the unit sphere $S^{n-1}$. Let $\delta > 0$. We say that a point $p \in M$ is $\delta$-singular if $\mathcal{H}^{n-1}(\Sigma_p M) \leq \text{vol}(S^{n-1}) - \delta$. Let us denote the set of singular points of $M$ by $S_M$ and the set of $\delta$-singular points of $M$ by $S_\delta$.

We have $S_M = \bigcup_{\delta > 0} S_\delta$. Since the map $M \ni p \mapsto \mathcal{H}^n(\Sigma_p M)$ is lower semi-continuous, the $\delta$-singular set $S_\delta$ is a closed set and so the singular set $S_M$ is a Borel set. For a sufficiently small $\delta > 0$, any point in $M \setminus S_\delta$ has some Euclidean neighborhood. For any geodesic segment $pq$ and any $x, y \in pq \setminus \{p, q\}$, $\Sigma_x M$ and $\Sigma_y M$ are isometric to each other ([27]). Therefore, a geodesic joining two points in $M \setminus S_M$ is entirely contained in $M \setminus S_M$.

**Definition 2.5** (Boundary). The boundary of an Alexandrov space $M$ is defined inductively. If $M$ is one-dimensional, then $M$ is a complete Riemannian manifold and the boundary of $M$ is defined as usual. Assume that $M$ has dimension $\geq 2$. A point $p \in M$ is a boundary point of $M$ if $\Sigma_p M$ has non-empty boundary.
Any boundary point of $M$ is a singular point. More strongly, the boundary of $M$ is contained in $S_\delta$ for a sufficiently small $\delta > 0$, which follows from the Morse theory in [23, 25].

The doubling theorem (§5 of [23]; 13.2 of [6]) states that if $M$ has non-empty boundary, then the double of $M$ (i.e., the gluing of two copies of $M$ along their boundaries) is an Alexandrov space without boundary and each copy of $M$ is convex in the double.

Denote by $\hat{S}_M$ (resp. $\hat{S}_M$) the set of singular (resp. $\delta$-singular) points of $\text{dbl}(M)$ contained in $M$, where we consider $M$ as a copy in $\text{dbl}(M)$. We agree that $\hat{S}_M = S_M$ and $\hat{S}_M = S_\delta$ provided $M$ has no boundary.

**Fact 2.6.** For an Alexandrov space $M$ of dimension $n \geq 2$, we have the following (1)–(5).

1. There exists a number $\delta_n > 0$ depending only on $n$ such that $M^* := M \setminus \hat{S}_M$ is a manifold (with boundary) ([6, 23, 25]) and have a natural $C^\infty$ differentiable structure (even on the boundary) ([14]).
2. The Hausdorff dimension of $S_M$ is $\leq n - 1$ ([6, 22]), and that of $\hat{S}_M$ is $\leq n - 2$ ([6]).
3. We have a unique Riemannian metric $g$ on $M^* \setminus \hat{S}_M$ such that the distance function induced from $g$ coincides with the original one of $M$ ([22]).
4. For any $\delta$ with $0 < \delta \leq \delta_n$, there exists a $C^\infty$ Riemannian metric $\hat{g}_\delta$ on $M \setminus \hat{S}_M$ such that
   \[ |g - \hat{g}_\delta| < \theta(\delta|n) \] on $M^* \setminus \hat{S}_M$
   ([14]), where $\theta(\delta|n)$ is defined in §2.1.
5. A $C^\infty$ differentiable structure on $M^*$ satisfying (4) is unique ([14]). In this meaning, the $C^\infty$ structure is canonical.

**Remark 2.7.** In [14] we construct a $C^\infty$ structure only on $M \setminus B(S_{\delta_n}, \epsilon)$. However this is independent of $\epsilon$ and extends to $M^*$. The $C^\infty$ structure is a refinement of the structures of [22, 21, 24]. In particular, it is compatible with the DC structure of [24].

Note that the metric $g$ is defined only on $M^* \setminus \hat{S}_M$ and does not continuously extend to any other point of $M$. In general the non-singular set $M^* \setminus \hat{S}_M$ is not a manifold because $\hat{S}_M$ may be dense in $M$.

**Fact 2.8.** $g$ is of locally bounded variation ([24]; see §2.4 below for functions of bounded variation). The tangent spaces at points in $M \setminus S_M$ is isometrically identified with the tangent cones ([22]). The volume measure on $M^*$ induced from $g$

\[ d\text{vol} = d\text{vol}_g := \sqrt{|g|} \, dx \]
coincides with the n-dimensional Hausdorff measure $\mathcal{H}^n$ ([22]), where $dx := dx^1 \cdots dx^n$ is the Lebesgue measure on a chart. $g$ is uniformly elliptic ([22]), i.e., there exists a chart around each point in $M^*$ on which

$(\text{UE})$ the eigenvalues of $(g_{ij})$ are bounded away from zero and bounded from above.

We assume that all charts of $M^*$ satisfy (UE).

**Definition 2.9** (Cut-locus). Let $p \in M$ be a point. We say that a point $x \in M$ is a cut point of $p$ if no minimal geodesic $py$ from $p$ contains $x$ as an interior point. The set of cut points of $p$ is called the cut-locus of $p$ and denoted by $\text{Cut}_p$.

For the $W_{p,t}$ defined in §1 we have $\bigcup_{0 < t < 1} W_{p,t} = X \setminus \text{Cut}_p$. Since $W_{p,t}$ is a closed set, the cut-locus $\text{Cut}_p$ is a Borel set. We have $\mathcal{H}^n(\text{Cut}_p) = 0$ (Proposition 3.1 of [22]).

By Lemma 4.1 of [22], $r_p := d(p, \cdot)$ is differentiable on $M \setminus (S_M \cup \text{Cut}_p \cup \{p\})$. At any $x \in M \setminus (S_M \cup \text{Cut}_p \cup \{p\})$ the gradient vector $\nabla r_p(x)$ coincides with the tangent vector to the minimal geodesic from $p$ passing through $x$. The gradient vector field $\nabla r_p$ is continuous at all differentiable points.

### 2.3. Analysis on Alexandrov spaces.

Let $M$ be $n$-dimensional Alexandrov space and $L^2(M)$ the Hilbert space consisting of all real valued $L^2$ functions on $M$ with inner product

$$(u,v)_{L^2} := \int_M uv \, d\mathcal{H}^n, \quad u,v \in L^2(M).$$

We indicate the locally $L^2$ by $L^2_{\text{loc}}$. For a (uniformly elliptic) chart $(U; x^1, \ldots, x^n)$ on $M^*$, we denote by $D_i$ the distributional partial derivative with respect to the coordinate $x^i$. If $D_i u$ is a function for a function $u$, we write it by $\partial_i u$. Define $W^{1,2}(M)$ to be the set of all $u \in L^2(M)$ such that on each chart $(U; x^1, \ldots, x^n)$ of $M^*$, all $D_i u$, $i = 1, \ldots, n$, are locally $L^2$ functions, $\partial_i u$, and $\langle du, du \rangle := g^{ij} \partial_i u \partial_j u$ belongs to $L^1(M^*)$, where we follow Einstein’s convention and $\langle du, du \rangle$ is determined independent of the chart $U$. $W^{1,2}_{\text{loc}}(M)$ denotes the set of $u \in L^2_{\text{loc}}(M)$ such that all $D_i u$, $i = 1, \ldots, n$, are locally $L^2$ functions. We define the symmetric bilinear form $\mathcal{E}$ on $W^{1,2}(M)$ by

$$\mathcal{E}(u,v) := \int_{M^*} \langle du, dv \rangle \, d\mathcal{H}^n, \quad u,v \in W^{1,2}(M).$$

We call $\mathcal{E}$ the Dirichlet energy form of $M$. $W^{1,2}(M)$ is a Hilbert space with inner product $(u,v)_{L^2} + \mathcal{E}(u,v)$. The pair $(\mathcal{E}, W^{1,2}(M))$ becomes a strongly local regular Dirichlet form in the sense of [12] (Theorem 4.2 and Proposition 7.2 of [14]). $\mathcal{E}(u,v)$ is defined for $u,v \in W^{1,2}_{\text{loc}}(M)$ such that $\text{supp} \, v$ is compact in the same manner.
Definition 2.10 (Sub-(super-)harmonicity). A function $u \in W^{1,2}_{\text{loc}}(M)$ is said to be $\mathcal{E}$-subharmonic (resp. $\mathcal{E}$-superharmonic) if for any $v \in C_0^\infty(M^*)$ with $v \geq 0$ we have $\mathcal{E}(u,v) \leq 0$ (resp. $\geq 0$).

Remark 2.11. Theorem 3.1 of [14] implies that $M \setminus M^*$ is an almost polar set in $M$. Therefore, the $\mathcal{E}$-sub(super)harmonicity defined here is compatible with the terminology in [13].

By Theorem 1.3 of [13] and Theorem 3.1 of [14], we have

Lemma 2.12 (Maximum Principle; [13]). Let $u \in W^{1,2}_{\text{loc}}(M)$ be continuous and $\mathcal{E}$-subharmonic. If $u$ attains its maximum in $M$, then $u$ is constant on $M$.

2.4. BV functions. We mention basics for BV functions needed in this paper. For more details we refer to [1]. Let $U \subset \mathbb{R}^n$ be an open subset.

Definition 2.13 (Approximate Limit). We say that locally $L^1$ function $u : U \to \mathbb{R}$ has approximate limit at $x \in U$ if there exists $z \in \mathbb{R}$ such that

$$
\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - z| \, dy = 0,
$$

where $B(x,r)$ is the Euclidean ball centered at $x$ of radius $r$ and $|B(x,r)|$ its Lebesgue measure. Denote by $S_u$ the set of $x \in U$ where $u$ does not have approximate limit. For $x \in U \setminus S_u$, the above $z$ is unique and set $\tilde{u}(x) := z$. The function $\tilde{u} : U \setminus S_u \to \mathbb{R}$ is called the approximate limit of $u$.

$S_u$ is a Borel set and satisfies $\mathcal{H}^n(S_u) = 0$. $\tilde{u}$ is a Borel function.

Lemma 2.14 (cf. Proposition 3.64 of [1]). (1) For any bounded locally $L^1$ functions $u_1, u_2 : U \to \mathbb{R}$ we have

$$
S_{u_1+u_2} \subset S_{u_1} \cup S_{u_2}, \quad S_{u_1 u_2} \subset S_{u_1} \cup S_{u_2},
$$

and $\tilde{u_1 + u_2} = \tilde{u_1} + \tilde{u_2}$, $\tilde{u_1 u_2} = \tilde{u_1}\tilde{u_2}$ on $U \setminus (S_{u_1} \cup S_{u_2})$.

(2) For any Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and locally $L^1$ function $u : U \to \mathbb{R}$ we have

$$
S_{f \circ u} \subset S_u, \quad \tilde{f \circ u} = f \circ \tilde{u} \quad \text{on } U \setminus S_u.
$$

Definition 2.15 (Approximate Jump Point). For a locally $L^1$ function $u : U \to \mathbb{R}$, a point $x \in U$ is called an approximate jump point of $u$ if there exist $a, b \in \mathbb{R}$ and $\nu \in S^{n-1}$ such that $a \neq b$ and

$$
\lim_{\rho \to 0} \frac{1}{|B^+(\nu,x,\rho)|} \int_{B^+(\nu,x,\rho)} |u(y) - a| \, dy = 0,
$$

and

$$
\lim_{\rho \to 0} \frac{1}{|B^-(\nu,x,\rho)|} \int_{B^-(\nu,x,\rho)} |u(y) - b| \, dy = 0,
$$
where \( B^+_v(x, \rho) := \{ y \in B(x, \rho) \mid \langle y - x, \nu \rangle > 0 \} \) and \( B^-_v(x, \rho) := \{ y \in B(x, \rho) \mid \langle y - x, \nu \rangle < 0 \} \). Denote by \( J_u \) the set of approximate jump point of \( u \).

\( J_u \) is a Borel set and satisfies \( J_u \subset S_u \) (cf. Proposition 3.69 of [1]).

**Definition 2.16 (BV Function).** An \( L^1 \) function \( u : U \rightarrow \mathbb{R} \) is of BV (bounded variation) if the distributional derivatives \( D_i u, i = 1, \ldots, n \), are all finite Radon measures. \( |D_i u| \) denotes the total variation measure of \( D_i u \).

**Lemma 2.17** (cf. Lemma 3.76 of [1]). Let \( u : U \rightarrow \mathbb{R} \) be a BV function and \( B \subset U \) a Borel set.

1. If \( \mathcal{H}^{n-1}(B) = 0 \), then \( |D_i u|(B) = 0 \).
2. If \( \mathcal{H}^{n-1}(B) < +\infty \) and \( B \cap S_u = \emptyset \), then \( |D_i u|(B) = 0 \).

**Lemma 2.18** (Federer-Vol’pert; cf. Theorem 3.78 of [1]). For any BV function \( u : U \rightarrow \mathbb{R} \) we have \( \mathcal{H}^{n-1}(S_u \setminus J_u) = 0 \).

**Lemma 2.19** (cf. Theorem 3.96 of [1]). (1) For any BV functions \( u_1, u_2 : U \rightarrow \mathbb{R} \) and \( c_1, c_2 \in \mathbb{R} \), the linear combination \( c_1 u_1 + c_2 u_2 \) is also of BV and satisfies

\[
D_i(c_1 u_1 + c_2 u_2) = c_1 D_i u_1 + c_2 D_i u_2.
\]

(2) Assume \( |U| < \infty \). If \( u : U \rightarrow \mathbb{R} \) is a BV function with \( \inf u > 0 \), \( \sup u < +\infty \), and \( \mathcal{H}^{n-1}(J_u) = 0 \), and if \( f : (0, +\infty) \rightarrow \mathbb{R} \) is a \( C^1 \) function, then \( f \circ u \) is of BV and

\[
D_i(f \circ u) = (f' \circ u) D_i u.
\]

**Lemma 2.20** (Leibniz Rule; cf. Example 3.97 in §3.10 of [1]). For any bounded BV functions \( u_1, u_2 : U \rightarrow \mathbb{R} \) we have the following (1) and (2).

1. \( u_1 u_2 \) is of BV.
2. If \( \mathcal{H}^{n-1}(J_{u_1} \cap J_{u_2}) = 0 \), then

\[
D_i(u_1 u_2) = \tilde{u}_1 D_i u_2 + \tilde{u}_2 D_i u_1
\]

and \( J_{u_1 u_2} \) is contained in \( (J_{u_1} \setminus S_{u_2}) \cup (J_{u_2} \setminus S_{u_1}) \) up to an \( \mathcal{H}^{n-1} \)-negligible set.

2.5. **DC functions.** Let \( \Omega \) be an open subset of an Alexandrov space. (\( \Omega \) is allowed to be an open subset of \( \mathbb{R}^n \).) A function \( u : \Omega \rightarrow \mathbb{R} \) is said to be convex if \( u \circ \gamma \) is a convex function for any geodesic \( \gamma \) in \( \Omega \).

**Definition 2.21** (DC Function). A locally Lipschitz function \( u : \Omega \rightarrow \mathbb{R} \) is of DC if it is locally represented as the difference of two convex functions, i.e., for any \( p \in \Omega \) there exists two convex functions \( v \) and \( w \) on some neighborhood \( U \) of \( p \) in \( \Omega \) such that \( u|_U = v - w \) on \( U \).
Lemma 2.22 (cf. §6.3 of [11]). For any DC function $u$ on $\Omega \subset \mathbb{R}^n$, the partial derivatives $\partial_i u$, $i = 1, 2, \ldots, n$, are all locally bounded and locally BV functions.

Lemma 2.23 (Perelman; §3 of [24]). Let $M$ be an $n$-dimensional Alexandrov space. For any chart $(U, \varphi)$ in $M^*$ and any DC function $u$ on $U$, $u \circ \varphi^{-1}$ is of DC on $\varphi(U) \subset \mathbb{R}^n$. In particular, $\partial_i u := \partial_i (u \circ \varphi^{-1})$, $i = 1, 2, \ldots, n$, are all locally bounded and locally BV functions.

Lemma 2.24 (cf. [24]). For any point $p$ in an Alexandrov space $M$, $r_p(x) := d(p, x)$ is a DC function on $M \setminus \{p\}$.

3. Condition $\text{BG}(\kappa)$

In this section, we mention some elementary and obvious properties of Condition $\text{BG}(\kappa)$. Let $M$ be an $n$-dimensional Alexandrov space and $p \in M$ a point. For a subset $C \subset M$, let $A_p(C) \subset M$ be the union of images of minimal geodesics from $p$ intersecting $C$. For $r > 0$ and $0 \leq r_1 \leq r_2$, we set

\[
a_C(r) := \mathcal{H}^{n-1}(A_p(C) \cap \partial B(p, r)),
A_{r_1, r_2}(C) := \{x \in A_p(C) \mid r_1 \leq r_p(x) \leq r_2\}.
\]

Lemma 3.1. Let $0 < r_1 \leq r_2 \leq R$ and $t := r_1/r_2$. Then we have

\[
a_C(r_1) \geq (1 - \theta(t - 1|R, \kappa_0)) a_C(r_2),
\]

where $\kappa_0$ is a lower bound of curvature on $B(p, 2R)$, i.e., $\kappa_0 := \kappa(B(p, 2R))$, and $\theta(\cdots)$ is defined in [2.7]. In particular, if $a_C(r_0) = 0$ for a number $r_0 > 0$, then $a_C(r) = 0$ for any $r \geq r_0$. Moreover, we have $a_C(r^+) \leq a_C(r) \leq a_C(r^-)$ for any $r > 0$ and $a_C$ has at most countably many discontinuity points.

Proof. To prove the first assertion, we assume that $A_p(C) \cap \partial B(p, r_2)$ is nonempty. The map $\Phi_{p,t} : A_p(C) \cap \partial B(p, r_1) \cap W_{p,t} \rightarrow A_p(C) \cap \partial B(p, r_2)$ defined in [11] is surjective and Lipschitz continuous by the Alexandrov convexity. If $t$ is close to 1, then so is the Lipschitz constant of $\Phi_{p,t}$. Therefore we have the first assertion of the lemma, which proves the rest. \(\square\)

Lemma 3.1 implies the integrability of $a_C(r)$. The same proof as for a Riemannian manifold leads to

\[
(3.1) \quad \mathcal{H}^n(A_{r_1, r_2}(C)) = \int_{r_1}^{r_2} a_C(r) \, dr
\]

for any $0 \leq r_1 \leq r_2$.

For a function $f : (\alpha, \beta) \rightarrow \mathbb{R}$, we set

\[
f^t(x) := \limsup_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \in \mathbb{R} \cup \{-\infty, +\infty\}, \quad \alpha < x < \beta.
\]

Lemma 3.2. The following (1)–(3) are equivalent to each other.
(1) BG(κ) at p.
(2) For any subset $C \subset M$ and $0 < r_1 \leq r_2$ (with $r_2 < \pi/\sqrt{\kappa}$ if $\kappa > 0$), we have

$$a_C(r_1) \geq \frac{s_\kappa(r_1)^{n-1}}{s_\kappa(r_2)^{n-1}} a_C(r_2).$$

(3) For any $C \subset M$ and $r > 0$ with $a_C(r) > 0$ (and with $r < \pi/\sqrt{\kappa}$ if $\kappa > 0$), we have

$$\tilde{a}_C^*(r) \leq (n-1) \cot_\kappa(r) a_C(r).$$

Proof. (1) $\implies$ (2): We fix $0 < r_1 \leq r_2$. Assume that $r_2 < \pi/\sqrt{\kappa}$ if $\kappa > 0$. For a sufficiently small $\delta > 0$, we set $t_\delta := r_1/(r_2 - \delta)$. Since $\Phi_{p,t\delta}^{-1}(A_{r_2-\delta,r_2}(C)) \subset A_{r_1,t\delta r_2}(C)$, we have by BG(κ) at p,

$$\mathcal{H}^n(A_{r_1,t\delta r_2}(C)) \geq \min_{r_2-\delta \leq r \leq r_2} \frac{t_\delta s_\kappa(t_\delta r)^{n-1}}{s_\kappa(r)^{n-1}} \mathcal{H}^n(A_{r_2-\delta,r_2}(C)).$$

We multiply the both sides of this formula by $1/(t_\delta \delta)$ and take the limit as $\delta \to 0$. Then, remarking Lemma 3.1 we obtain (2).

(2) $\implies$ (1): Let $E \subset M$ be a compact subset and set $r_E^- := \inf_{x \in E} r_p(x)$, $r_E^+ := \sup_{x \in E} r_p(x)$. We assume $r_E^+ < \pi/\sqrt{\kappa}$ if $\kappa > 0$. For $r \in [r_E^-, r_E^+]$, we set $C := \partial B(p,r) \cap E$, $r_2 := r$, and $r_1 := tr$. (2) implies

$$\mathcal{H}^n(\Phi_{p,t\delta}^{-1}(\partial B(p,r) \cap E)) \geq \frac{s_\kappa(tr)^{n-1}}{s_\kappa(r)^{n-1}} \mathcal{H}^n(\partial B(p,r) \cap E).$$

Integrating this with respect to $r$ over $[r_E^-, r_E^+]$ yields

$$\Phi_{p,t\delta} \mathcal{H}^n(E) = \mathcal{H}^n(\Phi_{p,t\delta}^{-1}(E)) \geq \min_{r_E^- \leq r \leq r_E^+} \frac{t s_\kappa(tr)^{n-1}}{s_\kappa(r)^{n-1}} \mathcal{H}^n(E),$$

which implies BG(κ) at p.

(2) $\iff$ (3): We set $a(r) := a_C(r)$ for simplicity. Let $0 < r_1 \leq r_2$ be any numbers such that $r_2 < \pi/\sqrt{\kappa}$ if $\kappa > 0$. In (2) we may assume that $r_1 < r_2$ and $a(r_1), a(r_2) > 0$, so that (2) is equivalent to

$$\frac{\log a(r_2) - \log a(r_1)}{r_2 - r_1} \leq (n-1) \frac{\log s_\kappa(r_2) - \log s_\kappa(r_1)}{r_2 - r_1}.$$

This is also equivalent to that for any $r > 0$ with $a(r) > 0$ (and with $r < \pi/\sqrt{\kappa}$ if $\kappa > 0$),

$$\frac{\log a(r)}{r} \leq (n-1) \cot_\kappa(r).$$

The left-hand side of this is equal to $\tilde{a}'(r)/a(r)$. \hfill \square

**Corollary 3.3.** If $\kappa > 0$ and if $M$ satisfies BG(κ) at $p \in M$, then

$$r_p \leq \frac{\pi}{\sqrt{\kappa}}.$$
Proof. By Lemma 3.2(2), we have \( a_M(r) \to 0 \) as \( r \to \pi/\sqrt{\kappa} \). By Lemma 3.1 \( a_M(r) = 0 \) for any \( r \geq \pi/\sqrt{\kappa} \). Using (3.1) yields that \( H^n(M \setminus B(p, \pi/\sqrt{\kappa})) = 0 \). This completes the proof. \( \square \)

Lemma 3.2(2) leads to the following.

**Corollary 3.4 (Bishop-Gromov Inequality).** If \( M \) satisfies BG(\( \kappa \)) at a point \( p \in M \), then

\[
\frac{\mathcal{H}^n(B(p, r_1))}{\mathcal{H}^n(B(p, r_2))} \geq \frac{v_\kappa(r_1)}{v_\kappa(r_2)} \quad \text{for } 0 < r_1 \leq r_2,
\]

where \( v_\kappa(r) \) is the volume of an \( r \)-ball in an \( n \)-dimensional complete simply connected space form of curvature \( \kappa \).

**Proposition 3.5 (Stability for BG(\( \kappa \))).** Let \( M_i, i = 1, 2, \ldots \), and \( M \) be \( n \)-dimensional compact Alexandrov spaces of curvature \( \geq \kappa_0 \) for a constant \( \kappa_0 \in \mathbb{R} \). If all \( M_i \) satisfy BG(\( \kappa \)) and if \( M_i \) Gromov-Hausdorff converges to \( M \), then \( M \) satisfies BG(\( \kappa \)).

Proof. By §3 of [32] or 10.8 of [6], \( (M_i, \mathcal{H}^n) \) measured Gromov-Hausdorff converges to \( (M, \mathcal{H}^n) \). The rest of the proof is omitted (cf. [7]). \( \square \)

The following proposition and corollary are proved by standard discussions (cf. Theorem 3.5 in Chapter IV of [30]).

**Proposition 3.6.** Let \( M \) be an Alexandrov space with BG(\( \kappa \)), \( \kappa > 0 \). Then we have \( \text{diam} \, M \leq \pi/\sqrt{\kappa} \). If \( \text{diam} \, M = \pi/\sqrt{\kappa} \) then \( M \) is homeomorphic to the suspension over some topological space.

**Corollary 3.7.** Let \( M \) be an Alexandrov space such that the singular set \( S_M \) is closed. If \( M \setminus S_M \) is an (incomplete) \( C^\infty \) Riemannian manifold of Ric \( \geq \kappa > 0 \) and if \( \text{diam} \, M = \pi/\sqrt{\kappa} \), then \( M \) is isometric to the spherical suspension over some compact Alexandrov space.

The corollary is a generalization of Cheng’s maximal diameter theorem [10]. Compare also [3].

### 4. Green Formula

Throughout this section, let \( M \) be an \( n \)-dimensional Alexandrov space. The purpose of this section is to prove the following Green formula, which is needed for the proof of the Laplacian Comparison Theorem, 1.1.

**Theorem 4.1 (Green Formula).** Let \( p \in M \) be a point and \( E \subset M^* \setminus \{p\} \) a region satisfying Assumption 4.2 below and \( \mathcal{H}^{n-1}(\text{Cut}_p \cap \partial E) = 0 \). Then, for any \( C^\infty \) function \( f : M^* \to \mathbb{R} \) we have

\[
\int_E f \, d \Delta_p = \int_E \langle \nabla f, \nabla r_p \rangle \, d\mathcal{H}^n + \int_{\partial E} f \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1},
\]
where \( \nu_E \) denotes the inward normal vector field along \( \partial E \) of \( E \). In particular,

\[
\bar{\Delta} r_p(E) = \int_{\partial E} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1}.
\]

**Assumption 4.2.** \( E \) is a compact region in \( M^* \) with piecewise \( C^\infty \) boundary such that \(|D_k g_{ij}|(\partial E \cap U) = 0\) for any \( i, j, k = 1, 2, \ldots, n \) and for any chart \( U \) of \( M^* \).

Here, the **piecewise \( C^\infty \) boundary** means that the boundary \( \partial E \) is divided into two disjoint subsets \( \tilde{\partial E} \) and \( \hat{\partial E} \) such that \( \tilde{\partial E} \) is an \((n-1)\)-dimensional \( C^\infty \) submanifold of \( M^* \) and that \( \hat{\partial E} \) is a closed set with \( \mathcal{H}^{n-1}(\hat{\partial E}) = 0 \).

To define the distributional Laplacian \( \bar{\Delta} \), let us consider the distributional divergence of locally BV vector fields. Let \( \Omega \) be an open subset of \( M^* \). A **locally BV vector field** \( X \) on \( \Omega \) is defined as a linear combination \( X = X^i \partial_i \) of locally BV functions \( X^1, \ldots, X^n \) on each chart in \( \Omega \) with the compatibility condition under chart transformations. For a locally \( L^1 \) function \( u \) on \( \Omega \), the approximate jump set \( J_u \) on a chart is defined. It is easy to prove that \( J_u \) is independent of the chart, so that \( J_u \) is defined as a subset of \( \Omega \). For a locally \( L^1 \) tensor \( T \) on \( M^* \), the approximate jump set \( J_T \subset \Omega \) is defined to be the union of the approximate jump sets of all coefficients of \( T \).

**Definition 4.3 (Distributional Divergence).** For a locally bounded and locally BV vector field \( X \) on \( \Omega \), the **distributional divergence of** \( X \) is defined by

\[
\bar{\text{div}} X = \bar{\text{div}}_g X := D_i(\sqrt{|g|} X^i),
\]

where \( X = X^i \partial_i \) on a chart. By Fact \([2,3]\), Lemmas \([2,19]\) and \([2,20]\), \( \sqrt{|g|} \) is a locally bounded and locally BV function. Since \( \mathcal{H}^{n-1}(J_{\sqrt{|g|}}) \leq \mathcal{H}^{n-1}(S_M \cap M^*) = 0 \) and by the Leibniz rule (Lemma \([2,20]\) ), \( \text{div} X \) is determined independent of the local chart. \( \bar{\text{div}} X \) is a Radon measure on \( \Omega \).

**Remark 4.4.** \( \bar{\text{div}} X \) is a generalization of \( \text{div} X \, d\text{vol} \) on a \( C^\infty \) Riemannian manifold, where \( \text{div} X \) is the usual divergence of a \( C^\infty \) vector field \( X \). \( \bar{\text{div}} X/\sqrt{|g|} \) is corresponding to \( \text{div} X \, dx \) and is not an invariant under chart transformations, where \( dx \) is the Lebesgue measure on the chart.

**Definition 4.5 (Distributional Laplacian).** For a DC function \( u \) on \( \Omega \), the partial derivatives \( \partial_i u, i = 1, 2, \ldots, n \), are locally bounded and locally BV functions (see Lemmas \([2,22]\) and \([2,23]\) ) and so the gradient vector field \( \nabla u := g^{ij} \partial_j u \partial_i \) is a locally bounded and locally BV vector field on \( \Omega \). \( \nabla u \) is independent of the local chart. The **distributional Laplacian of** \( u \)

\[
\Delta u := -\bar{\text{div}} \nabla u = -D_i(\sqrt{|g|} g^{ij} \partial_j u)
\]
is defined as a Radon measure on $\Omega$.

$\bar{\Delta}u$ is corresponding to $\Delta u d\text{vol}$, where $\Delta$ is the usual Laplacian. By Lemma 2.24, $\bar{\Delta}p$ is defined on $M^* \setminus \{p\}$.

**Lemma 4.6.** For any bounded BV vector field $X$ on $M^*$ with compact support in $M^*$, we have

$$\int_{M^*} d\text{div} X = 0.$$

**Proof.** There is a finite covering $\{U_k\}$ of $\text{supp} X$ consisting of charts of $M^*$ with compact closure $\bar{U}_k$. We take a $C^\infty$ partition of unity $\{\rho_k : M^* \to [0,1]\}_k$ associated with the covering, i.e., $\text{supp} \rho_k \subset U_k$ and $\sum_k \rho_k = 1$ on $\text{supp} X$. Since $X = \sum_k \rho_k X$ we have

$$\int_{M^*} d\text{div} X = \sum_k \int_{U_k} D_i(\sqrt{|g|}\rho_k X^i) = 0.$$

□

For a locally $L^1$ vector field $X = X^i \partial_i$ on $\Omega$, we define

$$e(\nabla u, X) := \sqrt{|g|} \tilde{X}^i D_i u,$$

where $\tilde{X}^i$ is the approximate limit of $X^i$ (see Definition 2.13).

**Lemma 4.7.** Let $f : \Omega \to \mathbb{R}$ be a $C^\infty$ function, $u$ a bounded BV function with compact support in $\Omega$, and $v$ a DC function on $\Omega$. If $H^{n-1}(J_u \cap J_dv) = 0$, then

1. $\text{div}(fu \nabla v) = f \text{e}(\nabla u, \nabla v) + \tilde{u} \text{div}(f \nabla v)$,
2. $\int_{\Omega} f \text{e}(\nabla u, \nabla v) = -\int_{\Omega} \tilde{u} d\text{div}(f \nabla v)$.

**Proof.** (2) is obtained by integrating (1) on $\Omega$ and using Lemma 4.6.

We prove (1). We fix a relatively compact chart $(U; x^1, \ldots, x^n)$ with $\bar{U} \subset \Omega$. The uniform ellipticity of $(U; x^1, \ldots, x^n)$ implies that $|U| < +\infty$. Since $g_{ij}$ are continuous on $M \setminus S_M$, we have $S_{g_{ij}} \subset S_M$ and $g_{ij} = g_{ij}$ on $M \setminus S_M$. By Lemma 2.14 the same is true for $g^{ij}$ and $\sqrt{|g|}$. Since the Hausdorff dimension of $S_M \cap M^*$ is $\leq n - 2$ and since $g_{ij}$, $g^{ij}$, and $\partial_j v$ are all BV functions on $U$, Lemmas 2.14 and 2.20 show

$$\text{div}(fu \nabla v) = D_i(\sqrt{|g|} u f g^{ij} \partial_j v)$$

$$= \sqrt{|g|} f g^{ij} \partial_j v D_i u + \tilde{u} D_i(\sqrt{|g|} f g^{ij} \partial_j v)$$

$$= f \text{e}(\nabla u, \nabla v) + \tilde{u} \text{div}(f \nabla v).$$

□

Let us study the $C^\infty$ mollifier $g^{(h)}$ of the Riemannian metric $g$ on $M^*$. Let $\{U_{\lambda}\}$ be a locally finite covering of $M^*$ consisting of relatively compact charts with $\bar{U}_{\lambda} \subset M^*$, and $\{\rho_{\lambda} : M^* \to [0,1]\}$ an associated
partition of unity, i.e., each supp \( \rho_\lambda \) is a compact subset of \( U_\lambda \) and \( \sum_\lambda \rho_\lambda = 1 \). Let \( \eta \in C^0_0(\mathbb{R}^n) \) be such that \( \eta \geq 0 \), \( \eta(-x) = \eta(x) \), supp \( \eta \subset B(o, 1) \), and \( \int_{\mathbb{R}^n} \eta dx = 1 \). We set \( \eta_\epsilon := \epsilon^{-n} \eta(x/\epsilon) \), \( \epsilon > 0 \).

Denote by \( g_{\lambda;ij} \) the coefficients of \( g \) with respect to the coordinate of \( U_\lambda \). For each \( \lambda \), there exists \( \epsilon_\lambda > 0 \) such that for any \( \epsilon \) with \( 0 < \epsilon \leq \epsilon_\lambda \),

\[
g_{\lambda;ij} * \eta_\epsilon(x) := \int_{U_\lambda} \eta_\epsilon(x-y)g_{\lambda;ij}(y) \, dy
\]

is a \( C^\infty \) Riemannian metric on some neighborhood of supp \( \rho_\lambda \). For \( 0 < h \leq 1 \), \( g_h^{(\lambda)} \) denotes the metric tensor defined by \( g_{\lambda;ij} * \eta_{\lambda,h}(x) \).

Define the \( C^\infty \) Riemannian metric \( g^{(h)} \) on \( M^* \) by

\[
g^{(h)} := \sum_\lambda \rho_\lambda g^{(h)}_\lambda.
\]

On each relatively compact chart \( U \), \( g^{(h)}_{ij} \) is uniformly bounded. As \( h \to 0 \), \( g^{(h)}_{ij} \to g_{ij} \) pointwise on \( U \setminus S_M \) and \( D_k g^{(h)}_{ij} = \partial_k g^{(h)}_{ij} \, dx \to D_k g_{ij} \) weakly * (cf. Proposition 3.2 of [1]).

**Lemma 4.8** (Compare Fact 2.6(4) (or Theorem 6.1 of [14])). For any \( \epsilon, \delta > 0 \) with \( \delta \leq \delta_n \), we have

\[
\limsup_{h \to 0} \left( \sup_{U \setminus (S_M \cup B(S_\delta, \epsilon))} |g^{(h)}_{ij} - g_{ij}| < \theta(\delta|U),
\right.
\]

where \( \delta_n \) is that in Fact 2.6 and \( \theta(\delta|U) \) depends also on the coordinates of \( U \).

**Proof.** The same proof as of Lemma 3.2(1) of [22] yields that for any \( p, q \in M \) and \( z \in U \setminus B(S_\delta, \epsilon) \),

\[
\sup_{x \in B(z, t)} |\angle pxq - \angle pizq| < \theta(\delta) + \theta(t|p, q, z).
\]

Hence, looking at the definition of \( g \) in [22], we have for any \( z \in U \setminus B(S_\delta, \epsilon) \),

\[
\sup_{x,y \in U \cap B(z,t) \setminus S_M} |g_{ij}(x) - g_{ij}(y)| < \theta(\delta|U) + \theta(t|z, U).
\]

This and the relative compactness of \( U \setminus B(S_\delta, \epsilon) \) imply the lemma. \( \square \)

**Lemma 4.9.** For any \( C^\infty \) vector field \( Y \) on \( M^* \) we have

\[
\overline{\text{div}}_{g^{(h)}} Y \to \overline{\text{div}}_{\hat{g}} Y \quad \text{weakly *},
\]

**Proof.** We take any relatively compact chart \( (U; x^1, \ldots, x^n) \) with \( \hat{U} \subset M^* \) and fix it. Let \( Y = Y^i \hat{\partial}_i \). For \( \hat{g} = g, g^{(h)} \) we have on \( U \),

\[
\overline{\text{div}}_{\hat{g}} Y = \frac{Y^i}{2\sqrt{|\hat{g}|}} \left( \sum_{k=1}^n \left| (\hat{g}_{j1}, \ldots, \hat{g}_{jk,k-1}, D_i \hat{g}_{jk}, \hat{g}_{j,k+1}, \ldots, \hat{g}_{jn})_{j=1,\ldots,n} \right| \right.
\]

\[
+ \partial_i Y^i \sqrt{|\hat{g}|} \, dx,
\]

which forms

\[
F^{ijk}(\hat{g}) D_i \hat{g}_{jk} + \partial_i Y^i \sqrt{|\hat{g}|} \, dx,
\]
where \( F^{ijk}(g^{(h)}) \) is a \( C^\infty \) function and \( F^{ijk}(g) \) is a BV function which is continuous on \( U \setminus S_M \).

We fix \( i, j, k \) and set \( f_h := F^{ijk}(g^{(h)}) \), \( f := F^{ijk}(g) \), \( \mu_h := D_ijg^{(h)} \) \( \partial_k g^{(h)} \, dx \), and \( \mu := D_ijg_{jk} \). It suffices to prove that \( f_h\mu_h \to f\mu \) weakly *.

Denote the positive part of \( \mu_h \) by \( \mu^+_h \) and the negative part by \( \mu^-_h \). There are a sequence \( h \to 0 \) and non-negative Radon measures \( \nu^+ \) and \( \nu^- \) on \( U \) such that \( \mu^+_h \to \nu^+ \) weakly *. It holds that \( \mu = \nu^+ - \nu^- \). We do not know the positive (resp. negative) part of \( \mu \) coincides with \( \nu^+ \) (resp. \( \nu^- \)). For simplicity we write \( h \) by \( h \). Since \( f_h \mu^+_h - f\nu^+ = (f_h - f)\mu^+_h + \mu^-_h - f\nu^+ \), we have for any \( \varphi \in C_0(U) \),

\[
\left| \int_U \varphi f_h \, d\mu^+_h - \int_U \varphi f \, d\nu^+ \right| \leq \int_U |\varphi| |f_h - f| \, d\mu^+_h + \int_U \varphi f \, d\mu^-_h - \int_U \varphi f \, d\nu^-.
\]

(4.1)

Take any \( \epsilon, \delta > 0 \) with \( \delta \leq \delta_h \). If \( h \ll \delta, \epsilon \), then \( |f_h - f| < \theta(\delta) \) on \( U \setminus (S_M \cup B(S_\delta, \epsilon)) \) by Lemma 3.8. By the uniform boundedness of \( g^{(h)}_{ij} \) and \( g_{ij} \) on \( U \setminus S_M \), we have \( |f_h|, |f| \leq c \) on \( U \setminus S_M \), where \( c \) is some constant independent of \( h \). The limit-sup as \( h \to 0 \) of the first term of the right-hand side of (4.1) is

\[
\limsup_h \int_U |\varphi| |f_h - f| \, d\mu^+_h
\]

\[
\leq \limsup_h \int_{U \cap B(S_\delta, \epsilon)} 2c|\varphi| \, d\mu^+_h + \limsup_h \int_{U \setminus B(S_\delta, \epsilon)} \theta(\delta)|\varphi| \, d\mu^+_h
\]

\[
\leq \int_{U \cap B(S_\delta, \epsilon)} 2c|\varphi| \, d\nu^+ + \theta(\delta) \int_U |\varphi| \, d\nu^-.
\]

To estimate the first term of the right-hand side, we prove:

**Sublemma 4.10.** We have \( |\nu|(U \cap S_\delta) = 0 \) for any \( \delta > 0 \), where \( |\nu| := \nu^+ + \nu^- \).

**Proof.** By remarking the uniform ellipticity of the charts, a direct calculation shows that

\[
d\mu_h \leq c\,dx + c' \sum_{\lambda,l,m,a} \rho_\lambda \, d|D_{\lambda;a}g_{\lambda;lm} * \eta_{\lambda,h}|,
\]

where \( c' \) is some positive constant, \( l, m, a \) run over all \( 1, 2, \ldots, n \), and \( D_{\lambda;a} \) means \( D_a \) for the coordinate of \( U_\lambda \). According to Proposition 3.7 of [1] we have, as \( h \to 0 \), \( \rho_\lambda \, d|D_{\lambda;a}g_{\lambda;lm} * \eta_{\lambda,h}| \to \rho_\lambda \, d|D_{\lambda;a}g_{\lambda;lm}| \) weakly * on \( U_\lambda \), and hence

\[
d|\nu| \leq c\,dx + c' \sum_{\lambda,l,m,a} \rho_\lambda \, d|D_{\lambda;a}g_{\lambda;lm}|.
\]

Since the Hausdorff dimension of \( S_\delta \cap M^* \) is \( \leq n - 2 \), and by Lemma 2.17(1), this proves the sublemma. \( \square \)
By the sublemma, taking $\delta \to 0$ after $\epsilon \to 0$ in (4.2), we have
\[
\lim_h \int_{\mathcal{U}} |\varphi||f_h - f| \, d\mu_h^\pm = 0.
\]

We are going to estimate the other term of (4.1). There is a continuous function $\psi_{\delta,\epsilon} : \mathcal{U} \to [0, 1]$ such that $\psi_{\delta,\epsilon} = 1$ on $\mathcal{U} \cap B(S_\delta, \epsilon)$, $\psi_{\delta,\epsilon} = 0$ on $\mathcal{U} \setminus B(S_\delta, 2\epsilon)$. Set $\bar{\psi}_{\delta,\epsilon} := 1 - \psi_{\delta,\epsilon}$ and take a number $h_0$ with $0 < h_0 \ll \delta$. Since $\bar{\psi}_{\delta,\epsilon} \varphi f_{h_0}$ is continuous, we have $\lim_{h \to 0} \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f_{h_0} \, d\mu_h^\pm = \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f_{h_0} \, d\nu^\pm$. Moreover, by Lemma 4.8, $|f_{h_0} - f| < \theta(\delta)$ on $\mathcal{U} \setminus (S_M \cup B(S_\delta, \epsilon))$ and therefore
\[
\limsup_h \left| \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\mu_h^\pm - \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\nu^\pm \right| 
\leq \limsup_h \left| \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\mu_h^\pm - \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f_{h_0} \, d\mu_h^\pm \right| 
+ \left| \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f_{h_0} \, d\nu^\pm - \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\nu^\pm \right| 
\leq \theta(\delta) \limsup \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} |\varphi| \, d\mu_h^\pm + \theta(\delta) \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} |\varphi| \, d\nu^\pm \leq \theta(\delta).
\]

We also have
\[
\limsup_h \left| \int_{\mathcal{U}} \psi_{\delta,\epsilon} \varphi f \, d\mu_h^\pm \right| \leq \nu^\pm(B(S_\delta, 3\epsilon)) \sup_{\mathcal{U}} |\varphi f| \leq \theta(\epsilon|\delta),
\]
(4.4)
\[
\limsup_h \left| \int_{\mathcal{U}} \psi_{\delta,\epsilon} \varphi f \, d\nu^\pm \right| \leq \theta(\epsilon|\delta).
\]
(4.5)

Combining (4.3), (4.4), and (4.5) yields
\[
\limsup_h \left| \int_{\mathcal{U}} \varphi f \, d\mu_h^\pm - \int_{\mathcal{U}} \varphi f \, d\nu^\pm \right| 
\leq \limsup_h \left| \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\mu_h^\pm - \int_{\mathcal{U}} \bar{\psi}_{\delta,\epsilon} \varphi f \, d\nu^\pm \right| 
+ \limsup_h \left| \int_{\mathcal{U}} \psi_{\delta,\epsilon} \varphi f \, d\mu_h^\pm \right| \leq \theta(\delta) + \theta(\epsilon|\delta).
\]

Thus we obtain $f_h \mu_h^\pm \to f \nu^\pm$ and so $f_h \mu_h = f_h \mu_h^+ - f_h \mu_h^- \to f \nu^+ - f \nu^- = f \mu$. This completes the proof. \hfill \Box

We need Lemma 4.9 to prove:

**Lemma 4.11.** Let $E \subset M$ be a region satisfying Assumption 4.2. Define $I_E(x) := 1$ for $x \in E$, $I_E(x) := 0$ for $x \in M \setminus E$. Then we have
\begin{align*}
(1) & \quad D_i I_E = |g|^{-1/2} g_{ij} \nu^j_E \mathcal{H}^{n-1}|_{\partial E}, \\
(2) & \quad e(\nabla I_E, X) = \langle \nu_E, \check{X} \rangle \mathcal{H}^{n-1}|_{\partial E}
\end{align*}
for any bounded measurable vector field $X$ on $M^*$, where $\nu_E = \nu_E^j \partial_j$ is the inward normal vector field along $\partial E$ of $E$ and $\mid$ indicates the restriction of a measure.

**Proof.** (1): We take any $C^\infty$ vector field $Y$ on $M^*$. On the $C^\infty$ Riemannian manifold $(M^*, g^{(h)})$, the divergence formula implies

$$\int_E \text{div}_{g^{(h)}} Y = - \int_{\partial E} \langle \nu^{(h)}_E, Y \rangle_{g^{(h)}} \text{dvol}_{g^{(h)}}$$

where $\nu^{(h)}_E$ is the inward normal vector field along $\partial E$ with respect to the metric $g^{(h)}$. It follows from Assumption 4.2 that $|\text{div}_{g} Y|_{\partial E} = 0$. Lemma 4.9 shows that the left-hand side of the above converges to $\int_E \text{div}_g Y$. Since $g^{(h)} \to g$ on $M^* \setminus S_M$, the right-hand side converges to $- \int_{\partial E} \langle \nu_E, Y \rangle_g \text{dH}^{n-1}_{\partial E}$. Therefore we have

$$\int_E \text{div}_{g} Y = - \int_{\partial E} \langle \nu_E, Y \rangle_g \text{dH}^{n-1}_{\partial E},$$

which implies (1).

(2) follows from (1) by a direct calculation. \hfill $\square$

With the help of Lemma 4.11, we finally prove the Green Formula.

**Proof of Theorem 4.1.** By Lemma 4.7(1), we have

$$\int_E \text{div}(f \nabla r_p) = \langle \nabla f, \nabla r_p \rangle \text{dH}^n - f \Delta r_p,$$

which implies

$$\int_E \text{ddiv}(f \nabla r_p) = \langle \nabla f, \nabla r_p \rangle \text{dH}^n - \int_E f \Delta r_p.$$

By $J_{dr_p} \subset \text{Cut}_p$, by the assumption for $E$, and by applying Lemmas 4.7(2), 4.11(2), the left-hand side of the above is equal to

$$\int_{M^*} I_E \text{ddiv}(f \nabla r_p) = - \int_{E \setminus \{p\}} f \text{d}(\nabla I_E, \nabla r_p)$$

$$= - \int_{\partial E} f \langle \nu_E, \nabla r_p \rangle \text{dH}^{n-1}_{\partial E}.$$ 

Since $\text{H}^{n-1}(\text{Cut}_p \cap \partial E) = 0$, we have $\nabla r_p = \nabla r_p \text{H}^{n-1}$-a.e. on $\partial E$. This completes the proof of the theorem. \hfill $\square$

5. LAPLACIAN COMPARISON

We prove Theorem 4.1 by using the Green Formula (Theorem 4.1). Let $a_C(r)$ and $A_{r_1, r_2}(C)$ be as defined in 33. Lemma 3.2(3) implies the following.

**Lemma 5.1.** If $M$ satisfies $BG(\kappa)$ at a point $p \in M$, then for any $C \subset M$ and $0 < r_1 \leq r_2$,

$$a_C(r_2) - a_C(r_1) \leq (n - 1) \cot \kappa(r_1) \text{H}^n(A_{r_1, r_2}(C)).$$
Denote by $M^3(\kappa_0)$ the three-dimensional complete simply connected space form of curvature $\kappa_0$.

**Fact 5.2** (Wald Convexity; [37,2]). Let $p_1, p_2, q_1, q_2 \in M$ be four points. Take a sufficiently large domain $\Omega$ containing $p_1, p_2, q_1, q_2$ and set $\kappa_0 := \min\{\kappa(\Omega), 0\}$. Then there exist four points $\tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2 \in M^3(\kappa_0)$ and $i_0, j_0 = 1, 2$ with $(i_0, j_0) \neq (1, 2)$ such that

$$d(p_1, p_2) = d(\tilde{p}_1, \tilde{p}_2), \quad d(q_1, q_2) \geq d(\tilde{q}_1, \tilde{q}_2),$$

$$d(p_1, q_j) = d(\tilde{p}_i, \tilde{q}_j) \quad \text{for } (i, j) \neq (i_0, j_0),$$

$$d(p_{i_0}, q_{j_0}) \geq d(\tilde{p}_{i_0}, \tilde{q}_{j_0}).$$

Moreover, for any $x \in p_i q_i, i = 1, 2$, if we take $\tilde{x}_i \in \tilde{p}_i \tilde{q}_i$ such that $d(p_i, x_i) = d(\tilde{p}_i, \tilde{x}_i) = d(\tilde{p}_1, \tilde{q}_i)$, then we have

$$d(x_1, x_2) \geq d(\tilde{x}_1, \tilde{x}_2).$$

For $a, b \in \mathbb{R}$ (depending on a number $\delta > 0$), we define $a \doteq b$ as $|a - b| < \theta(\delta)$.

**Fact 5.3** (5.6 of [6]). Take four points $p_1, p_2, q_1, q_2 \in M$ and set $\kappa_0 := \min\{\kappa(\Omega), 0\}$ for a sufficiently large domain $\Omega$ containing $p_1, p_2, q_1, q_2$. If

$$d(q_1, q_2) < \delta \min\{d(p_1, q_1), d(p_2, q_1)\} \quad \text{and} \quad \angle p_1 q_1 p_2 > \pi - \delta,$$

then we have

$$\angle p_1 q_1 q_2 + \angle p_2 q_1 q_2 \doteq \pi, \quad \angle p_1 q_1 q_2 \doteq \angle p_1 q_1 q_2, \quad \angle p_2 q_1 q_2 \doteq \angle p_2 q_1 q_2,$$

where $\angle$ indicates the angle of a $\kappa_0$-comparison triangle.

**Corollary 5.4.** Under the same assumption as in Fact 5.3, if we take a point $x \in p_1 q_1$ such that $d(q_1, x) < \delta \min\{d(p_1, q_1), d(p_2, q_1)\}$, then

1. $\angle x q_1 q_2 \doteq \angle x q_1 q_2$,
2. $\angle q_1 x q_2 \doteq \angle q_1 x q_2$.

**Proof.** (1): The Alexandrov convexity implies that

$$\angle x q_1 q_2 \geq \angle x q_1 q_2 \geq \angle p_1 q_1 q_2 \doteq \angle p_1 q_1 q_2 = \angle x q_1 q_2.$$

(2): The points $p_1, p_2, x, q_2$ satisfy the assumption of Fact 5.3. Take a point $q'_2 \in p_2 x$ with $d(x, q'_2) = d(x, q_2)$ and use (1). Then,

$$\angle p_2 x q_2 \doteq \angle q'_2 x q_2.$$

Since $\angle p_1 x p_2 \geq \angle p_1 x p_2 \doteq \pi$, we have $\angle q_1 x q'_2 \leq \angle q_1 x p_2 = \pi - \angle p_1 x p_2 \doteq 0$. Therefore $\angle p_2 x q_2 \doteq \angle q_1 x q_2$ and $\angle q'_2 x q_2 \doteq \angle q_1 x q_2$, which imply (2).

Let $E$ be a region satisfying the following.
Assumption 5.5. \( E \) is a region in \( M^* \setminus \{ p \} \) satisfying Assumption 4.2 and \( H^{n-1}(\text{Cut}_p \cap \partial E) = 0 \). The smooth part of \( \partial E \) is transversal to \( \nabla r_p \).

Recall that \( \partial E \) is divided into the smooth part \( \tilde{\partial} E \) and the non-smooth part \( \hat{\partial} E \). By \( H^n(\text{Cut}_p) = 0 \), we have a lot of \( E \)'s satisfying Assumption 5.5.

For \( \rho > 0 \) we set
\[
D := \tilde{\partial} E \setminus (S_M \cup \text{Cut}_p \cup A_p(\hat{\partial} E \setminus \text{Cut}_p)),
\]
\[
D_{\rho} := \{ x \in D \mid \text{there is } y \in M \text{ such that } x \in p y \text{ and } d(x,y) \geq \rho \}.
\]
Namely, \( x \in D \) if and only if the following (1) and (2) hold.

1. \( x \in \tilde{\partial} E \setminus (S_M \cup \text{Cut}_p) \).
2. If we extends \( p x \) to a minimal geodesic from \( p \) hitting \( \hat{\partial} E \), then it cannot be extended any more.

It is obvious that \( \bigcup_{\rho > 0} D_{\rho} = D \).

Lemma 5.6.

1. \( D \) and \( D_{\rho} \) are Borel subsets.
2. We have \( H^{n-1}(\partial E \setminus D) = 0 \).

Proof. (1): For \( \rho > 0 \), we set
\[
W_{\rho} := \{ x \in M \mid \text{there is } y \in M \text{ such that } x \in p y \text{ and } d(x,y) \geq \rho \}.
\]
Since \( D_{\rho} = D \cap W_{\rho} \) and \( W_{\rho} \) is closed, it suffices to prove that \( D \) is a Borel set. In fact, \( A_p(\hat{\partial} E \cap W_{\rho}) \) is closed, monotone non-increasing in \( \rho \), and satisfies
\[
(5.1) \quad \bigcup_{\rho > 0} A_p(\hat{\partial} E \cap W_{\rho}) = A_p(\hat{\partial} E \setminus \text{Cut}_p),
\]
which is a Borel set. Since \( \tilde{\partial} E \), \( S_M \), \( \text{Cut}_p \) are all Borel, so is \( D \).

(2): We take any points \( p_1, p_2 \in \partial E \cap A_p(\hat{\partial} E \cap W_{\rho}) \). For each \( i = 1, 2 \), we extend \( p_i \) to a minimal geodesic from \( p \) hitting \( \hat{\partial} E \cap W_{\rho} \) and denote the hitting point by \( x_i \). We further extends the geodesic beyond \( x_i \) to the point, say \( q_i \), such that \( d(x_i,q_i) = \rho \). Such the points \( x_i \) and \( q_i \) necessarily exist because of \( p_1, p_2 \in A_p(\hat{\partial} E \cap W_{\rho}) \). For the points \( p_i, q_i, x_i, i = 1, 2 \), we apply the Wald convexity (Fact 5.2) and have \( d(p_1, p_2) \leq c d(x_1, x_2) \), where \( c \) is a constant independent of \( p_1, p_2 \). Therefore, \( H^{n-1}(\partial E \cap A_p(\hat{\partial} E \cap W_{\rho})) \leq c^{n-1} H^{n-1}(\partial E \cap W_{\rho}) = 0 \), which together with (5.1) implies \( H^{n-1}(\partial E \cap A_p(\hat{\partial} E \setminus \text{Cut}_p)) = 0 \). Combining this, \( H^{n-1}(S_M) = 0 \), and \( H^{n-1}(\text{Cut}_p \cap \partial E) = 0 \), we obtain (2). \( \square \)

For two points \( x, y \in D_{\rho} \), we define the point \( \pi_x(y) \) to be the intersection point of a minimal geodesic from \( p \) passing through \( y \) and \( \partial B(p, r_p(x)) \) (if any). Since \( \nabla r_p \) and \( \hat{\partial} E \) are transversal to each other, if \( d(x,y) \) is small enough compared with a given \( x \in \tilde{\partial} E \), such the intersection point \( \pi_x(y) \) exists.
Lemma 5.7. For any subset $A \subset B(x, \delta) \cap D_\rho$ with $\mathcal{H}^{n-1}(A) > 0$, we have
\[ \left| \frac{\langle \nu_E(x), \nabla r_p(x) \rangle \mathcal{H}^{n-1}(A)}{\mathcal{H}^{n-1}(\pi_x(A))} - 1 \right| < \theta(\delta|x, \rho). \]

Proof. We fix $x$ and $\rho$, then we write $\theta(\delta) = \theta(\delta|x, \rho)$. Assume $\delta \ll \rho$. For $a, b \in \mathbb{R}$, we define $a \simeq b$ as $|a - b| \leq \theta(\delta)|a|.$

Let $y, z \in B(x, \delta) \cap D_\rho$ be two different points. We take a minimal geodesic, say $\sigma$ (resp. $\tau$), from $p$ containing $py$ (resp. $pz$) which has maximal length. It follows that $L(\sigma), L(\tau) \geq r - \delta + \rho \geq r + \rho/2,$ where we set $r := r_p(x)$.

Sublemma 5.8. We have $d(\sigma(t_1), \tau(t_1)) \simeq d(\sigma(t_2), \tau(t_2))$ for any $t_1$ and $t_2$ with $\sigma(t_1), \tau(t_1) \in B(x, \delta)$.

Proof. The Alexandrov convexity implies
\[ d(\sigma(t_1), \tau(t_1)) \geq (1 - \theta(\delta))d(\sigma(t_2), \tau(t_2)). \]

An inverse estimate follows from applying the Wald convexity (Fact 5.2) to $p_1 := \sigma(t_1), p_2 := \tau(t_1), q_1 := \sigma(r + \rho/2), q_2 := \tau(r + \rho/2), x_1 := \sigma(t_2), x_2 := \tau(t_2).$ \hfill $\square$

Setting $y' := \tau(r_p(y))$ and $z' := \sigma(r_p(z))$ we have, by Sublemma 5.8,
\[ d(\pi_x(y), \pi_x(z)) \simeq d(y, y') \simeq d(z, z'). \]
Let $\alpha := \angle zy\!\!z'$. By Corollary 5.4 $\alpha \div \angle zy\!\!z'$ and hence
\[ d(y, z) \sin \alpha \simeq d(\pi_x(y), \pi_x(z)). \]
We also have
\[ |r_p(y) - r_p(z)| = d(y, z') \simeq d(y, z) \cos \alpha. \]
We assume that $\delta$ is small enough compared with $x$. Then, there is a chart $(U, \varphi)$ of $M^s$ containing $B(x, \delta)$ such that $\varphi(\partial E)$ is a hyper-plane in $\varphi(U) \subset \mathbb{R}^n$ and $g_{ij}(x) = \delta_{ij}.$ Let $c$ be a curve from $y$ to $z$ such that $\varphi \circ c$ is a Euclidean line segment in $\varphi(U)$. Since $\langle \dot{c}(s), \nabla r_p(c(s)) \rangle \div \langle \dot{c}(0), \nabla r_p(y) \rangle$ and $L(c) \simeq d(y, z)$, the first variation formula leads to
\[ r_p(y) - r_p(z) \simeq d(y, z)\langle \dot{c}(0), \nabla r_p(y) \rangle. \]
and so $\cos \alpha \div |\langle \dot{c}(0), \nabla r_p(y) \rangle|$. Therefore,
\[ d(y, z)\sqrt{1 - \langle \dot{c}(0), \nabla r_p(y) \rangle^2} \simeq d(\pi_x(y), \pi_x(z)). \]
Take a hyper-plane $H \subset \mathbb{R}^n$ containing $\varphi(x)$ and perpendicular to $\nabla r_p(x)$. Denoting the orthogonal projection by $P : \varphi(\partial E) \to H$, we see
\[ d_{\mathbb{R}^n}(P(\varphi(y)), P(\varphi(z))) = d(y, z)\sqrt{1 - \langle \dot{c}(0), \nabla r_p(y) \rangle^2} \simeq d(\pi_x(y), \pi_x(z)), \]
23
which implies that \( \mathcal{H}^{n-1}(\pi_x(A)) \simeq \mathcal{H}^n(P(\varphi(A))) \). Since \( g_{ij} \equiv \delta_{ij} \) on \( U \), we have \( d(y, z) \simeq d_{\mathbb{R}^n}(\varphi(y), \varphi(z)) \) and \( \mathcal{H}^{n-1}(A) \simeq \mathcal{H}^{n-1}(\varphi(A)) \). This completes the proof.

\[ \Box \]

**Proof of Theorem 1.1.** By the Green Formula (Theorem 4.1), it suffices to prove the theorem that

\[
\int_{\partial E} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1} \geq -(n - 1) \sup_{x \in E} \cot_n(r_p(x)) \mathcal{H}^n(E)
\]

for any region \( E \) satisfying Assumption 5.5.

We define

\[
D^- := \{ x \in D \mid \langle \nu_E(x), \nabla r_p(x) \rangle < 0 \},
\]

\[
D^+ := \{ x \in D \mid \langle \nu_E(x), \nabla r_p(x) \rangle > 0 \},
\]

\[
D^\pm := D_\rho \cap D^\pm.
\]

They are all \( \mathcal{H}^{n-1} \)-measurable sets. Take any \( \epsilon > 0 \) and fix it for a moment. Let \( \delta_{x, \rho} > 0 \) be a number small enough compared with \( x, \rho \), and \( \epsilon \). For the \( \theta(\delta|x, \rho) \) of Lemma 5.7, we assume \( \theta(\delta_{x, \rho}|x, \rho) \leq \epsilon \). For a point \( x \in \partial D^- \), let \( \pi(x) \) be the intersection point of \( px \) and \( \partial E \). From the definition of \( D \) we have \( \pi(D^-) \subset D^+ \). Take a countable dense subset \( \{ x_k^- \}_k \subset D_\rho^- \) and set \( x_k^+ := \pi(x_k^-) \). It holds that \( x_k^+ \in D^+ \). We find a number \( \delta_k \) in such a way that \( 0 < \delta_k < \delta_{x_k^+, \rho} \) and \( \pi(B(x_k^+, \delta_k) \cap D^-) \subset B(x_k^+, \delta_{x_k^+, \rho}) \). It follows from \( D_\rho^+ \subset \bigcup_k B(x_k^-, \delta_k) \) that there are disjoint \( \mathcal{H}^{n-1} \)-measurable subsets \( B_k^- \subset B(x_k^-, \delta_k) \) with \( D_\rho^- = \bigcup_k B_k^- \). Setting \( B_k^+ := \pi(B_k^-) \) we have \( B_k^+ \subset D_\rho^+ \). The definition of \( \delta_k \) and Lemma 5.7 lead to

\[
\left| \frac{|\langle \nu_E(x_k^\pm), \nabla r_p(x_k^\pm) \rangle| \mathcal{H}^{n-1}(B_k^\pm)}{\mathcal{H}^{n-1}(\pi_x^\pm(B_k^\pm))} - 1 \right| < \epsilon.
\]

Taking \( \delta_k \) small enough, we assume that

\[
|\langle \nu_E(x_k^\pm), \nabla r_p(x_k^\pm) \rangle - \langle \nu_E, \nabla r_p \rangle| < \epsilon \quad \text{on} \quad B_k^\pm.
\]

Thus we have

\[
\int_{D_\rho^+ \cup \pi(D_\rho^-)} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1}
\]

\[
= \sum_k \left\{ \int_{B_k^-} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1} + \int_{B_k^+} \langle \nu_E, \nabla r_p \rangle \, d\mathcal{H}^{n-1} \right\}
\]

\[
\geq \sum_k \left\{ |\langle \nu_E(x_k^-), \nabla r_p(x_k^-) \rangle| \mathcal{H}^{n-1}(B_k^-) + |\langle \nu_E(x_k^+), \nabla r_p(x_k^+) \rangle| \mathcal{H}^{n-1}(B_k^+) \right\}
\]

\[
- 2\epsilon \mathcal{H}^{n-1}(\partial E)
\]

\[
\geq \sum_k \left\{ \mathcal{H}^{n-1}(\pi_x^+(B_k^+)) - \mathcal{H}^{n-1}(\pi_x^-(B_k^-)) \right\} - 4\epsilon \mathcal{H}^{n-1}(\partial E).
\]
By Lemma 5.1, this is
\[ \geq -(n-1) \left( \sup_{E} \cot \kappa \circ r_{p} \right) \sum_{k} \mathcal{H}^{n}(A_{r_{k}^{+},r_{k}^{-}}(\pi_{x_{k}}(B_{k}^{-}))) - 4\epsilon \mathcal{H}^{n-1}(\partial E). \]
where we set \( r_{k}^{\pm} := r_{p}(x_{k}^{\pm}) \). Let \( A(D_{\rho}^{-}) \) be the region in \( A_{p}(D_{\rho}^{-}) \) between \( D_{\rho}^{-} \) and \( \pi(D_{\rho}^{-}) \). We assume the division \( \{B_{k}^{-}\}_{k} \) of \( D_{\rho}^{-} \) to be so fine that
\[ \left| \sum_{k} \mathcal{H}^{n}(A_{r_{k}^{+},r_{k}^{-}}(\pi_{x_{k}}(B_{k}^{-}))) - \mathcal{H}^{n}(A(D_{\rho}^{-})) \right| < \epsilon. \]
Therefore,
\[ \int_{D_{\rho}^{-} \cup \pi(D_{\rho}^{-})} \langle \nu_{E}, \nabla r_{p} \rangle \, d\mathcal{H}^{n-1} \]
\[ \geq -(n-1) \left( \sup_{E} \cot \kappa \circ r_{p} \right) (\mathcal{H}^{n}(A(D_{\rho}^{-})) + \bar{\epsilon}) - 4\epsilon \mathcal{H}^{n-1}(\partial E), \]
where \( \bar{\epsilon} \) is either \( \epsilon \) or \( -\epsilon \). We define \( A(D^{-}) \) as in the same manner as \( A(D_{\rho}^{-}) \). After \( \epsilon \to 0 \) we take \( \rho \to 0 \) and then have
(5.3)
\[ \int_{D^{-} \cup \pi(D^{-})} \langle \nu_{E}, \nabla r_{p} \rangle \, d\mathcal{H}^{n-1} \geq -(n-1) \left( \sup_{E} \cot \kappa \circ r_{p} \right) \mathcal{H}^{n}(A(D^{-})). \]
Set \( D' := D^{+} \setminus \pi(D^{-}) \). The set of \( x \in \partial E \) such that \( px \) passes through \( D' \) is of \( \mathcal{H}^{n-1} \)-measure zero. Therefore, the same discussion as above leads to
(5.4)
\[ \int_{D'} \langle \nu_{E}, \nabla r_{p} \rangle \, d\mathcal{H}^{n-1} \geq -(n-1) \left( \sup_{E} \cot \kappa \circ r_{p} \right) \mathcal{H}^{n}(A(D')). \]
where \( A(D') \) is the intersection of \( E \) and the union of images of minimal geodesics from \( p \) intersecting \( D' \). By (5.3) and (5.4) we obtain (5.2). This completes the proof. \( \square \)

By using Theorem 1.1, a direct calculation implies

**Corollary 5.9.** Under the same assumption as in Theorem 1.1, for any \( C^{2} \) function \( f : \mathbb{R} \to \mathbb{R} \) with \( f' \geq 0 \), we have
\[ \Delta f \circ r_{p} \geq \frac{(s_{\kappa}^{n-1} f')'}{s_{\kappa}^{n-1}} \circ r_{p} \, d\mathcal{H}^{n} \quad \text{on } M^{*} \setminus \{p\}. \]

**Remark 5.10.** \( \Delta r_{p} \) is not absolutely continuous with respect to \( \mathcal{H}^{n} \) on the cut-locus of \( p \). In fact, let \( M \) be an \( n \)-dimensional complete Riemannian manifold without boundary and \( N \subseteq M \) a \( k \)-dimensional submanifold without boundary which is contained in \( \text{Cut}_{p} \) for a point \( p \in M \). (We do assume the completeness of \( N \).) Denote by \( \nu(N) \) the normal bundle over \( N \) and by \( \nu_{\kappa}(N) \) the set of vectors in \( \nu(N) \) with length \( \leq \kappa \). We assume that there exists a number \( \epsilon_{0} > 0 \) such that \( \exp(\nu_{\epsilon_{0}}(N)) \cap \text{Cut}_{p} = N \). For \( x \in N \), let \( V_{x} \) be the set of unit vectors
at $x$ tangent to minimal geodesics from $x$ to $p$. $V_x$ is isometric to a $(n - k - 1)$-sphere of radius $\in (0, 1]$. The angle between $u$ and $V_x$, $\alpha(x) := \inf_{v \in V_x} \angle(u, v)$, is constant for all $u \in \nu_1(N) \cap T_xM$. Applying Theorem 4.1 to $\nu_1(N')$, $N' \subset N$, $0 < \epsilon \leq \epsilon_0$, we have

$$d \tilde{\Delta} r_p|_N(x) = \omega_{n-k-1} \cos \alpha(x) \, d\mathcal{H}^k|_N(x),$$

where $\omega_{n-k-1}$ is the volume of a unit $(n - k - 1)$-sphere. In particular, $\Delta r_p$ is not absolutely continuous with respect to $\mathcal{H}^n$ on $N$.

The following is needed in the proofs of Theorem 1.3, Corollaries 1.5 and 1.6

**Corollary 5.11.** For any $f \in C_0^\infty(M^* \setminus \{p\})$ with $f \geq 0$, we have

$$\int_{M^*} \langle \nabla r_p, \nabla f \rangle \, d\mathcal{H}^n \geq \int_{M^*} f \, d \tilde{\Delta} r_p \geq -(n - 1) \int_{M^*} f \cot \omega r_p \, d\mathcal{H}^n.$$

**Proof.** Theorem 1.1 says the second inequality of the corollary. In the case where $M^*$ has no boundary, the Green Formula (Theorem 4.1) tells us that the first term is equal to the second. We prove the first inequality in the case where $M^*$ has non-empty boundary. Assume that $M^*$ has non-empty boundary. Since $M^*$ is a $C^\infty$ manifold with $C^\infty$ boundary $\partial M^*$, we can approximate $\partial M^*$ by a $C^\infty$ hypersurface $N \subset M^*$ with respect to the $C^1$ topology such that $\mathcal{H}^{n-1}(\text{Cut}_p \cap N) = 0$ and $|D_k g_{ij}|(N \cap U) = 0$ for any $i, j, k$ and for any chart $U$ of $M^*$. Let $V_N$ be the closed region in $M^*$ bounded by $N$ and not containing the boundary of $M^*$. We find a compact region $E \subset M^* \setminus \{p\}$ satisfying the assumption of the Green Formula (Theorem 4.1) such that $V_N \cap \text{supp} f \subset E \subset V_N$. The Green Formula implies

$$\int_{V_N} \langle \nabla r_p, \nabla f \rangle \, d\mathcal{H}^n = \int_{V_N} f \, d \tilde{\Delta} r_p - \int_{N \cap \text{supp} f} f \, \langle \nu_N, \nabla r_p \rangle \, d\mathcal{H}^{n-1},$$

where $\nu_N$ is the inward unit normal vector fields on $N$ with respect to $V_N$. Since $M$ is convex in the double of $M$, as $N$ converges to the boundary of $M^*$ in the $C^1$ topology, any limit of $\langle \nu_N, \nabla r_p \rangle$ is non-positive, which together with Fatou’s lemma shows

$$\limsup_{N \rightarrow \partial M^*} \int_{N \cap \text{supp} f} f \, \langle \nu_N, \nabla r_p \rangle \, d\mathcal{H}^{n-1} \leq 0.$$

This completes the proof. \qed

6. **Splitting Theorem**

We prove the Topological Splitting Theorem, following the idea of Cheeger-Gromoll [8].

Let $M$ be a non-compact Alexandrov space and $\gamma$ a ray in $M$, i.e., a geodesic defined on $[0, +\infty)$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for any $s, t \geq 0$. 


Definition 6.1 (Busemann Function). The Busemann function $b_\gamma : M \to \mathbb{R}$ for $\gamma$ is defined by

$$b_\gamma(x) := \lim_{t \to +\infty} \{t - d(x, \gamma(t))\}, \quad x \in M.$$ 

It follows from the triangle inequality that $t - d(x, \gamma(t))$ is monotone non-decreasing in $t$, so that the limit above exists. $b_\gamma$ is a 1-Lipschitz function.

Definition 6.2. We say that a ray $\sigma$ in $M$ is asymptotic to $\gamma$ if there exist a sequence $t_i \to +\infty$, $i = 1, 2, \ldots$, and minimal geodesics $\sigma_i : [0, l_i] \to M$ with $\sigma_i(l_i) = \gamma(t_i)$ such that $\sigma_i$ converges to $\sigma$ as $i \to \infty$, (i.e., $\sigma_i(t) \to \sigma(t)$ for each $t$).

For any point in $M$, there is a ray asymptotic to $\gamma$ from the point. Any subray of a ray asymptotic to $\gamma$ is asymptotic to $\gamma$. By the same proof as for Riemannian manifolds (cf. Theorem 3.8.2(3) of [31]), for any ray $\sigma$ asymptotic to $\gamma$ we have

$$b_\gamma \circ \sigma(s) = s + b_\gamma \circ \sigma(0) \quad \text{for any } s \geq 0. \quad (6.1)$$

For a complete Riemannian manifold, $b_\gamma$ is differentiable at $\sigma(s)$ for any $s > 0$, which seems to be true also for Alexandrov spaces, but we do not need it for the proof of Theorem 1.3.

Lemma 6.3. Let $f : M \to \mathbb{R}$ be a 1-Lipschitz function and $u, v \in \Sigma_p M$ two directions at a point $p \in M$. If the directional derivative of $f$ to $u$ is equal to 1 and that to $v$ equal to $-1$, then the angle between $u$ and $v$ is equal to $\pi$.

Proof. There are points $x_t, y_t \in M$, $t > 0$, such that $d(p, x_t) = d(p, y_t) = t$ for all $t > 0$ and that the direction at $p$ of $px_t$ (resp. $py_t$) converges to $u$ (resp. $v$) as $t \to 0$. The assumption for $f$ tells us that

$$\lim_{t \to 0} \frac{f(x_t) - f(p)}{t} = 1 \quad \text{and} \quad \lim_{t \to 0} \frac{f(y_t) - f(p)}{t} = -1,$$

which imply

$$\lim_{t \to 0} \frac{d(x_t, y_t)}{t} \geq \lim_{t \to 0} \frac{f(x_t) - f(y_t)}{t} = 2.$$

This completes the proof. \qed

Lemma 6.4. Assume that a ray $\sigma : [0, +\infty) \to M$ is asymptotic to a ray $\gamma : [0, +\infty) \to M$, and let $s$ be a given positive number. Then, among all rays emanating from $\sigma(s)$, only the subray $\sigma|_{[s, +\infty)}$ of $\sigma$ is asymptotic to $\gamma$.

Proof. Look at (6.1) and use Lemma 6.3 for $f := b_\gamma$. \qed

Lemma 6.5. Let $\gamma$ be a straight line in $M$. Denote by $b_+$ the Busemann function for $\gamma_+ := \gamma_{|[0, +\infty)}$ and by $b_-$ that for $\gamma_- := \gamma_{|(-\infty, 0]}$. If
$b_+ + b_- \equiv 0$ holds, then $M$ is covered by disjoint straight lines bi-asymptotic to $\gamma$. In particular, $b_+^{-1}(t)$ for all $t \in \mathbb{R}$ are homeomorphic to each other and $M$ is homeomorphic to $b_+^{-1}(t) \times \mathbb{R}$.

**Proof.** Take any point $p \in M$ and a ray $\sigma : [0, +\infty) \to M$ from $p$ asymptotic to $\gamma_+$. For any $s > 0$, the directional derivatives of $b_+$ to the two opposite directions at $\sigma(s)$ tangent to $\sigma$ are $-1$ and $1$ respectively. Since $b_- = -b_+$ and by Lemma 6.3, a ray from $\sigma(s)$ asymptotic to $\gamma_-$ is unique and contains $\sigma([0, s])$. By the arbitrariness of $s > 0$, $\sigma$ extends to a straight line bi-asymptotic to $\gamma$. Namely, for a given point $p \in M$, we have a straight line $\sigma_p$ passing through $p$ and bi-asymptotic to $\gamma$. By Lemma 6.4, any ray from a point in $\sigma_p$ asymptotic to $\gamma_\pm$ is a subray of $\sigma_p$. In particular, $\sigma_p$ is unique (upto parameters) for a given $p$. $M$ is covered by $\{\sigma_p\}_{p \in M}$ and this completes the proof. $\square$

**Lemma 6.6.** Assume that $M$ satisfies $\text{BG}(0)$ at any point on a ray $\gamma$ in $M$. Then, the Busemann function $b_\gamma$ is $\mathcal{E}$-subharmonic.

See Definition 2.10 for the definition of $\mathcal{E}$-subharmonicity.

**Proof.** We take a sequence $t_i \to +\infty$, $i = 1, 2, \ldots$. Since $r_{\gamma(t_i)}$, $b_\gamma$ are $1$-Lipschitz, they are $\mathcal{H}^n$-a.e. differentiable. Let $x \in M^*$ be any point where $r_{\gamma(t_i)}$ and $b_\gamma$ are all differentiable. We have a unique minimal geodesic $\sigma_{x,i}$ from $x$ to $\gamma(t_i)$ and $\nabla r_{\gamma(t_i)}(x)$ is tangent to it. A ray $\sigma_x$ from $x$ asymptotic to $\gamma$ is unique and $-\nabla b_\gamma(x)$ is tangent to it. Since $\sigma_{x,i} \to \sigma_x$ as $i \to \infty$, we have $\nabla r_{\gamma(t_i)}(x) \to -\nabla b_\gamma(x)$. Therefore, the dominated convergence theorem and Corollary 5.11 show that for any $u \in C^0_0(M^*)$ with $u \geq 0$,

$$- \int_{M^*} \langle \nabla b_\gamma, \nabla u \rangle \, d\mathcal{H}^n = \lim_i \int_{M^*} \langle \nabla r_{\gamma(t_i)}, \nabla u \rangle \, d\mathcal{H}^n \geq -(n-1) \lim_i \int_{M^*} \frac{u}{r_{\gamma(t_i)}} \, d\mathcal{H}^n = 0.$$

This completes the proof. $\square$

**Remark 6.7.** In general, $b_\gamma$ is not of DC and $\tilde{\Delta} b_\gamma$ does not exist as a Radon measure.

**Proof of Theorem 1.3.** By Lemma 6.6, $b := b_+ + b_-$ is $\mathcal{E}$-subharmonic. It follows from the triangle inequality that $b \leq 0$. We have $b \circ \gamma \equiv 0$ by the definition of $b$. The maximum principle (Lemma 2.12) proves $b \equiv 0$. Lemma 6.5 implies the theorem. $\square$

**Proof of Corollary 1.4.** We denote by $\Delta$ the usual Laplacian induced from the $C^\infty$ Riemannian metric on $M \setminus S_M$. It follows from $b_+ + b_- = 0$ that $b_\pm$ is $\mathcal{E}$-subharmonic and $\mathcal{E}$-superharmonic, so that $b_\pm$ is a weak solution of $\Delta u = 0$ on $M \setminus S_M$. By the regularity theorem of elliptic differential equation, $b_+$ is $C^\infty$ on $M \setminus S_M$ and satisfies $\Delta b_+ = 0$ pointwise on $M \setminus S_M$. By using Weitzenböck formula and by $\text{Ric}(\nabla b_+, \nabla b_+) \geq 0$, 28
the Hessian of $b_+$ vanishes on $M \setminus S_M$, namely $b_+$ is a linear function along any geodesic in $M \setminus S_M$. Since any geodesic joining two points in $M \setminus S_M$ is contained in $M \setminus S_M$, the set of geodesic segments in $M \setminus S_M$ is dense in the set of all geodesic segments. Therefore, $b_+$ is linear along any geodesic in $M$. Since $M$ is covered by straight lines bi-asymptotic to $\gamma$, $b_+$ is averaged $D^2$ in the sense of [18]. The corollary follows from Theorem A of [18].

\[ \square \]

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