Representability of $\text{Aut}_F$ and $\text{End}_F$

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May 30, 2018

Abstract

Recently N. Nitsure showed that for a coherent sheaf $F$ on a noetherian scheme the automorphism functor $\text{Aut}_F$ is representable if and only if $F$ is locally free. Here we remove the noetherian hypothesis and show that the same result holds for the endomorphism functor $\text{End}_F$ even if one asks for representability by an algebraic space.

MSC2000: 14A25

1 Statement of results

1.1

Let $X$ be a scheme and $F$ a quasi-coherent $O_X$-module of finite presentation. We are interested in the representability of the following two functors on the category of $X$-schemes:

$$\text{Aut}_F(X') := \text{Aut}_{O_{X'}}(f^*F)$$

$$\text{End}_F(X') := \text{End}_{O_{X'}}(f^*F)$$

for $f : X' \to X$ an $X$-scheme.

The result is as follows:
Theorem 1 Let $X$ be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then the following are equivalent:

1) $\mathcal{F}$ is locally free.
2) $\text{Aut}_\mathcal{F}$ is representable by a scheme.
2') $\text{End}_\mathcal{F}$ is representable by a scheme.

If $X$ is locally noetherian, these conditions are also equivalent to the following:

3) $\text{Aut}_\mathcal{F}$ is representable by an algebraic space.
3') $\text{End}_\mathcal{F}$ is representable by an algebraic space.

1.2

The equivalence of 1) and 2) in theorem in case $X$ is noetherian is the main result of [N]. Our proof follows the ideas of loc.cit. closely. The main steps are contained in the following two lemmas:

Lemma 1 Let $A$ be a local ring and $M$ a finitely presented $A$-module which is not free. Then there is a local homomorphism $A \to B$ such that

$$M \otimes_A B \cong B^n \oplus (B/b)^m,$$

for some $0 \neq b \in B, b^2 = 0$ and $m \geq 1, n \geq 0$.

If $A$ is noetherian, $B$ can be chosen to be artin.

We observe that in the last statement of the lemma the noetherian hypothesis is dispensable: let $(B, \mathfrak{m})$ be a local ring such that there is $0 \neq b \in \bigcap_{n \geq 1} \mathfrak{m}^n$. Clearly $(b^2) \subsetneq (b)$, so after dividing out $(b^2)$ one gets a ring $B$ as in the lemma but for any local homomorphism $f : B \to C$ with $C$ noetherian one clearly has $f(b) = 0$.

Lemma 2 Let $S$ be a scheme and $S_0 \subseteq S$ a closed subscheme defined by a nilpotent ideal sheaf. Assume $X$ is a flat $S$-scheme and $f : X \to Y$ is an $S$-morphism such that $f \times \text{id}_{S_0}$ is an isomorphism. Then $f$ is an isomorphism.
1.3

In order to treat the representability of \( \text{End}_F \) we will use the following observation:

**Lemma 3** Under the assumptions of 1.1 the obvious natural transformation of (set-valued) functors \( \text{Aut}_F \to \text{End}_F \) is relatively representable by an open immersion.

For completeness we also include a proof of the next lemma which is essentially lemma 5 of [N] and shows the relative representability of a “parabolic” sub-group functor:

Let \( X \) be a scheme and

\[
0 \to F' \to F \to F'' \to 0
\]

a short exact sequence of quasi-coherent \( \mathcal{O}_X \)-modules with \( F' \) finitely presented and \( F'' \) locally free. For any morphism \( f : Y \to X \), the sequence \( f^*(\mathbb{1}) \) is exact because \( F'' \) is in particular \( \mathcal{O}_X \)-flat and it makes sense to consider

\[
P(Y) := \{ \alpha \in \text{Aut}_{\mathcal{O}_Y}(f^*F) \mid \alpha(f^*F') \subseteq f^*F' \} \subseteq \text{Aut}_F(Y).
\]

**Lemma 4** In the above situation, the natural transformation \( P \hookrightarrow \text{Aut}_F \) is relatively representable by a closed immersion.

For basic facts about (relative) representability we refer to [BLR], 7.6.

2 Proofs

2.1

In this subsection we dispense with the easy implications of theorem \([\mathbb{1}]\) the assumptions and notations of which we now assume:
As $\text{Aut}_F$ and $\text{End}_F$ are clearly Zariski sheaves the problem of representing them is Zariski local on $X$, i.e. we can assume that $X$ is affine and $F$ corresponds to a free module of finite rank. In this case, representability of both $\text{Aut}_F$ and $\text{End}_F$ is obvious; we have proved the implications $1) \Rightarrow 2)$ and $1) \Rightarrow 2')$. Finally, the implications $2) \Rightarrow 3)$ and $2') \Rightarrow 3')$ are trivial.

2.2

Proof of lemma Let $(A, \mathfrak{m})$ be a local ring and $M$ a finitely presented $A$-module which is not free. We will find the required local homomorphism $A \to B$ as a suitable quotient of $A$:

Let

$$A^m \xrightarrow{\alpha} A^n \xrightarrow{\beta} M \to 0$$

be a minimal presentation of $M$, i.e. $n = \dim_k(M/\mathfrak{m}M)$ where $k := A/\mathfrak{m}$ is the residue field of $A$. Then $M$ is free if and only if $\alpha = 0$: clearly $\alpha = 0$ is sufficient for freeness of $M$ and conversely, if $M$ is free, it is necessarily so of rank $n$, hence $\beta$ is a surjective endomorphism of $A^n$ which must be an isomorphism by a standard application of Nakayama’s lemma, c.f. [M], thm. 2.4., hence $\alpha = 0$.

For any $J \subseteq \mathfrak{m}$, $(2) \otimes_A A/J$ is a minimal presentation of the $A/J$-module $M/\mathfrak{m}M$. If we denote by $I \subseteq A$ the ideal generated by the coefficients of any matrix representation of $\alpha$ and note that the minimality of $(2)$ implies $I \subseteq \mathfrak{m}$ we find that $M/\mathfrak{m}M$ is $A/J$-free if and only if $\alpha \otimes \text{id}_{A/J} = 0$ if and only if $I \subseteq J$. As $M$ is not $A$-free we have $I \neq 0$ and as $I$ is finitely generated we get $\mathfrak{m}I \subseteq I$, again by Nakayama’s lemma. By Zorn’s lemma, using again that $I$ is finitely generated, there is an ideal $J$ with $\mathfrak{m}I \subseteq J \subseteq I$ and which is maximal subject to these conditions (indeed, any ascending chain of such ideals admits its union as an upper bound because $I$ is finitely generated).

We claim that $B := A/J$ is as required:

By the maximality of $J$ the ideal $J := I/J$ is non-zero principal: $J = (b), 0 \neq b \in B$ and we necessarily have $b^2 = 0$: if not, we would have $b \in (b^2)$, i.e. $b = xb^2$ or $b(1 - xb) = 0$ for some $x \in B$. As $b \in \mathfrak{m} := \mathfrak{m}/J$, the maximal ideal of $B$, $1 - xb$ was a unit of $B$, so we would have $b = 0$.

We now show that $M \otimes_A B$ has the desired structure: any coefficient $\alpha_{ij}$ of
a matrix representation of \( \alpha \otimes \text{id}_B \) is of the form \( \alpha_{ij} = bu_{ij}, u_{ij} \in B \). As by construction \( \bar{m}b = 0 \) we see that if \( \alpha_{ij} \neq 0 \), then \( u_{ij} \in B^* \). We get a matrix equation \( (\alpha_{ij}) = b(u_{ij}) \) and \( (u_{ij}) \) can be chosen with \( u_{ij} = 0 \) or \( u_{ij} \in B^* \), all \( i,j \). Then the usual Gauß-algorithm can be applied to \( (u_{ij}) \), showing that indeed \( M \otimes_A B \cong B^n \oplus (B/b)^m \) for some \( m,n \geq 0 \). As, by construction, \( M \otimes_A B \) is not \( B \)-free, we finally see that \( m \geq 1 \).

If \( A \) is noetherian we can start the construction of \( B \) by first dividing out a suitable high power of \( m \): Indeed, if \( M/m^nM \) was free for all \( n \geq 1 \) we would have \( I \subseteq \bigcap_{n \geq 1} m^n = (0) \). Then the ring \( B \) we obtain in the above construction is noetherian local with \( \bar{m} \) nilpotent, hence zero-dimensional, i.e. \( B \) is artin local.

**Proof of lemma 2**  We can assume that the ideal sheaf \( I \) of \( S_0 \subseteq S \) satisfies \( I^2 = 0 \). Our assertion is local on \( S,X \) and \( Y \) and thus reduces to the following:

Given a ring \( k \) and an ideal \( I \subseteq k \) of square zero, if \( f : A \rightarrow B \) is a morphism of \( k \)-algebras with \( B \) \( k \)-flat and such that \( f \otimes_k \text{id}_{k/I} \) is an isomorphism, then \( f \) is an isomorphism:

1) \( f \) is surjective: any \( b \in B \) can be written

\[
b = f(a) + \sum_j \alpha_j b'_j , \text{some } \alpha_j \in I, b'_j \in B, a \in A.
\]

Applying this to the \( b'_j \) we get (for some \( \alpha_{ij} \in I, b''_{ij} \in B, a_j \in A \)):

\[
b = f(a) + \sum_j \alpha_j(f(a_j) + \sum_{ij} \alpha_{ij} b''_{ij}) = f(a + \sum_j \alpha_j a_j).
\]

2) \( f \) is injective: For \( K := \ker(f) \) the \( k \)-flatness of \( B \) implies \( K/IK = 0 \) and the same argument as in 1) shows that \( K = 0 \).

**Proof of 2) \( \Rightarrow 1) \) in theorem 1**  Under the notations of 1.4 we assume that \( \text{Aut}_F \) is representable by a scheme and, by contradiction, that \( F \) is not locally free. Note that the assumption on representability is stable under base-change \( Y \rightarrow X \). So, base-changing to a suitable local ring of \( X \), we can assume \( X = \text{Spec}(A) \) with \( A \) local and \( F \) corresponding to a finitely presented \( A \)-module \( M \) which is not free. According to lemma 1 we can assume \( M \cong A^n \oplus (A/a)^m \) for some \( 0 \neq a \in A \) with \( a^2 = 0 \) and \( m \geq 1 \). Let \( G \rightarrow S := \text{Spec}(A) \) be the group-scheme representing \( \text{Aut}_M \) and put
$S_0 := \text{Spec} (A/a)$. The sub-functor $G' \hookrightarrow G$ of automorphisms preserving (base-changes of) the direct summand $(A/a)^m$ is represented by a closed sub-group scheme (still to be denoted $G'$) according to lemma 4.

Let $P \subseteq \text{Gl}_{n+m,S}$ denote the standard parabolic sub-group of automorphisms preserving the rank $m$ direct summand. $P$ is flat over $S$, as can be seen over $\text{Spec} (\mathbb{Z})$. There is a morphism of $S$-groups $f : P \to G'$ which on points is given by sending $\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$ to $\begin{pmatrix} \alpha & 0 \\ \pi \beta & \gamma \end{pmatrix}$, where $\alpha \in \text{Aut}_A(A^n), \gamma \in \text{Aut}_A(A^m), \beta : A^n \to A^m$ and $\overline{\gamma} \in \text{Aut}_{A/a}((A/a)^m)$ denotes the reduction of $\gamma$ and $\pi : A^m \to (A/a)^m$ is the natural map. This “point-wise” description of $f$ is immediately checked to be functorial and a homomorphism and hence does indeed define a morphism of $S$-groups. Obviously, $f \times \text{id}_{S_0}$ is an isomorphism, hence so is $f$ by lemma 2. This is however a contradiction, because $f(S)(\text{id}_{A^n} \oplus (1-a)\text{id}_{A^m}) = 1$ and $a \neq 0, m \geq 1$.

2.3

**Proof of lemma 3**: Given a scheme $X$, a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation and some $\varphi \in \text{End}_{\mathcal{O}_X}(\mathcal{F})$ we have to show that there is an open sub-scheme $X_0 \subseteq X$ such for all $f : Y \to X$, $f^*(\varphi) \in \text{End}_{\mathcal{O}_Y}(f^*\mathcal{F}) \subseteq \text{End}_{\mathcal{O}_Y}(f^*\mathcal{F})$ if and only if $f$ factors through $X_0$. Consider $\mathcal{G} := \text{coker}(\varphi)$ and the exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{F} \to \mathcal{G} \to 0.$$  

We claim that $f^*(\varphi)$ is an automorphism if and only if $f^*(\mathcal{G}) = 0$: as $f^*(\mathcal{G})$ is exact, necessity is obvious. If, conversely, $f^*(\mathcal{G}) = 0$ then for any $y \in Y$ $f^*(\varphi)_y$ is a surjective endomorphism of the finitely generated $\mathcal{O}_{Y,y}$-module $\mathcal{F}_y$, hence is an isomorphism, hence so is $f^*(\varphi)$.

So the sought for $X_0 \subseteq X$ is the complement of the support of $\mathcal{G}$ which is open, because $\mathcal{G}$ is finitely presented.

**Proof of lemma 4**: Given a scheme $X$ and a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{F}'$ finitely presented and $\mathcal{F}''$ locally free and some $\alpha \in \text{Aut}_{\mathcal{O}_X}(\mathcal{F})$, we have to show the representability by a closed sub-scheme of $X$ of the following functor on $X$-schemes:

$$F(Y \xrightarrow{f} X) := \begin{cases} * & : f^*(\alpha)(f^*\mathcal{F}') \subseteq f^*\mathcal{F}' \\ \emptyset & , \text{ otherwise} \end{cases}$$
Clearly, $F$ is a Zariski sheaf, so the problem is local on $X$, i.e. we can assume that $X = \text{Spec}(A)$ is affine, $\mathcal{F}''$ corresponds to some $A^n$, $\mathcal{F}'$ corresponds to some $A$-module $M$ for which there is a presentation $A^a \to A^b \to M \to 0$ and $\mathcal{F}$ corresponds to some $A$-module $N$. The exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ then becomes an exact sequence $0 \to M \xrightarrow{\iota} N \xrightarrow{\pi} A^n \to 0$ of $A$-modules and we are given some $\alpha \in \text{Aut}_A(N)$. Consider $\nu := \pi \alpha \iota$. As all the above sequences are exact after any base-change, we have $F(Y \xrightarrow{f} X) \neq \emptyset \iff f^*(\nu) = 0$.

We have a diagram (defining $\psi$):

$\begin{array}{ccc}
A^a & \xrightarrow{\psi} & A^n \\
\downarrow \psi & & \downarrow \nu \\
M & \rightarrow & 0
\end{array}$

which is exact after any base-change, hence $f^*(\nu) = 0 \iff f^*(\psi) = 0$, for any $f : Y \to X$. So the closed sub-scheme of $X$ we are looking for is the one defined by the ideal of $A$ generated by the coefficients of any matrix representation of $\psi$.

**Proof of 2') $\Rightarrow$ 1) in theorem** 11 Under the notations of 1.1 we assume that $\text{End}_\mathcal{F}$ is representable by a scheme. Then so is $\text{Aut}_\mathcal{F}$ by lemma 3, hence $\mathcal{F}$ is locally free by what has been shown in 2.2.

2.4

**Proof of 3) $\Rightarrow$ 1) and 3') $\Rightarrow$ 1) in theorem** 11 Under the notations of 1.1 we assume that $X$ is locally noetherian and that either 3) or 3') holds as well as, by contradiction, that $\mathcal{F}$ is not locally free. By lemma 3, we know in either case that $\text{Aut}_\mathcal{F}$ is representable by an algebraic space. Using the last assertion of lemma 1, we can assume $X = \text{Spec}(A)$ with $A$ artin local. Then $\text{Aut}_\mathcal{F}$ is representable by a scheme according to [K], p. 25, 7) contradicting what we proved in 2.2.

**Acknowledgements.** I would like to thank H. Frommer and M. Volkov for interesting discussions and G. Weckermann for excellent type-setting.
References

[BLR] S. Bosch, W. Lütkebohmert, M. Raynauld, Néron Models, Ergebnisse der Mathematik, 3. Folge, Band 21, Springer, Heidelberg 1990.

[K] D. Knutson, Algebraic Spaces, Springer LNM 203.

[M] H. Matsumura, Commutative ring theory, Cambridge studies in advanced mathematics 8, 1997.

[N] N. Nitsure, Representability of $\text{Gl}_E$, arXiv:math.AG/0204047

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