Turbulent viscosity variability in self-propelled body wake model.

Katya Dubrovin, Michael Gedalin, Ephim Golbraikh and Alex Soloviev

1Physics Department, Ben-Gurion University of the Negev, Beer-Sheva, Israel
2Oceanographic Center, NOVA SouthEastern University, USA

Abstract

We study the influence of turbulent viscosity variability on the properties of self-propelled body wake model. In addition to the already known integrals of motion obtained with constant turbulent viscosity, we obtain new ones. The presence of new integrals of motion leads, in particular, to changes in the behavior of the width and profile of the wake leading to its conservation.

Study the behavior of long-lived turbulent wakes behind moving bodies is not only important from a scientific but also practical point of view. On the one hand the evolution of such wakes can affect the operation of airports, and on the other — mixing processes in the surface layer of the sea or ocean. The turbulent wakes behind ships have attracted special interest, largely due to the need to interpret the radar observations of ship wakes.

One of the most important models of the turbulent wakes behavior, from a practical point of view, is the model of self-propelled body wake, and the main characteristic of this wake is turbulent viscosity, $\nu_t$. Analytical approaches so far assumed a constant viscosity $\nu_t = \text{const}$, which leads to the wake width at the distance $x$ equal to $W(x) \sim x^\alpha$ (where $\alpha = 1/5$ for an axisymmetric wake and for $\alpha = 1/4$ a plane one). However, in real wakes, turbulent viscosity may be a function of the coordinates. In this paper we study the consequence of this dependence.

Consider the axisymmetric wake in which the mean flow in planes passing through the axis of symmetry is identical and weakly depends on the distance $x$. Simultaneously, we examine flows whose principal mean velocity component is directed along the x-axis. This means that radial component of the mean velocity $|U_r| < |U_x|$. If $L$ is a typical spatial scale of inhomogeneity in the direction of the wake (x-direction) while $l$ is a typical spatial scale of inhomogeneity in the transverse direction (r), $l << L$ (in this paper we use the notations in conformity with [13]). Then we denote by the mean flow velocity in x direction, and by $U_s$ the maximum value of $|U_0 - U_x|$ ($U_s << U_0$ far from the body.) of the cross-wake variation of the mean velocity component in direction. Following [13], an equation for the momentum transfer in x direction can be written as:

$$U_0 \frac{\partial \tilde{U}}{\partial x} + \frac{1}{r} \frac{\partial rq_{xr}}{\partial r} = 0$$  \hspace{1cm} (1)

where $\tilde{U} = U_x - U_0$, $q_{xr} = <u_xu_r>$, $u_i$ is a turbulent component of the velocity field, and $<...>$ denotes averaging over an ensemble. Taking into account self-preservation hypothesis, the velocity defect and the Reynolds stress become
invariant with respect to x, and they are expressed in terms of the local length and velocity scales \(l(x)\) and \(U_s(x)\). That is, \(\frac{\tilde{U}}{U_s} = f(r/l)\) and \(\frac{r q_{xx}}{U_s} = -g(r/l)\).

We substitute these expressions into the equation of motion (1):

\[
- \frac{U_0 l}{U_s^2} dU_s dx f + \frac{U_0}{U_s} \frac{dl}{dx} \xi f' = g' + \frac{g}{\xi}
\]

where \(\xi = r/l(x)\). To satisfy this equation for all \(x\) we obtain \(l = Ax^\alpha, U_s = Bx^{\alpha-1}\), where \(A\) and \(B\) are constants. Substitution into (2) gives

\[
s(\alpha - 1)f - s\alpha \xi f' = \frac{1}{\xi} \frac{d}{d\xi} (\xi g)
\]

where \(s = (U_0/B)\). The functions \(f(\xi)\) and \(g(\xi)\) are still to be found.

Since the turbulent viscosity is defined by the relation

\[
- q_{xx} = \nu_t \frac{\partial \tilde{U}}{\partial r},
\]

the equation of motion acquires the following form:

\[
U_0 \frac{\partial \tilde{U}}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \nu_t \frac{\partial \tilde{U}}{\partial r} \right)
\]

where \(\nu_t = Bx^{2\alpha-1}(g/f')\), and prime denotes \(\xi\) derivative.

The integral \(I_0 = \int_0^\infty \tilde{U} r dr\) is proportional to the total momentum across the wake and is conserved. Indeed, \(\frac{d}{dx} \int_0^\infty U_0 \tilde{U} r dr = - \int_0^\infty \frac{\partial}{\partial r} (r q_{xx}) dr = 0\). The conservation of momentum, together with the assumption that \(\tilde{U} \propto x f(\xi)\), immediately gives \(I_0 \propto x^{3\alpha-1} \int_0^\infty f(\xi) \xi d\xi\), and therefore we obtain a well-known value \(\alpha = 1/3\).

However, it fails for self-propelled body wake, where the total momentum vanishes \((I_0 = 0)\). In this case, multiplying by \(r^{m+1}\) and assuming the convergence, one has \(\frac{d}{dx} I_m = - \int_0^\infty r^m \frac{\partial}{\partial r} (r q_{xx}) dr\), where \(I_m = \int_0^\infty U_0 \tilde{U} r^{m+1} dr\). Using turbulent viscosity, we obtain:

\[
\frac{d}{dx} I_m = \int_0^\infty r^m \frac{\partial}{\partial r} \left( r \nu_t \frac{\partial \tilde{U}}{\partial r} \right) dr
\]

Integrating in parts twice and taking into account that \(\tilde{U} \to 0\) and \(\frac{\partial \tilde{U}}{\partial r} \to 0\) as \(r \to \infty\):

\[
\frac{d}{dx} I_m = m \int_0^\infty \tilde{U} \frac{\partial}{\partial r} (r^m \nu_t) dr
\]

If \(\frac{\partial}{\partial r} (r^m \nu_t) \propto r\) then \(\int_0^\infty \tilde{U} \frac{\partial}{\partial r} (r^m \nu_t) dr \propto I_0 = 0\) and \(I_m\) is conserved.
Figure 1: Wake profiles for certain parameters $m$. $m = 2$ corresponds to $\nu_t = \text{const.}$

In this case, $\nu_t = K(x)\xi^{2-m}$, where $K(x)$ is some function on $x$. Conservation of $I_m$, together with self-preservation hypothesis, gives

$$\alpha = \frac{1}{m+3}$$

It should be noted that similar results can be obtained for the plane flow. In this case we obtain $\alpha = \frac{1}{m+2}$.

Thus, with turbulent viscosity definition and conserved $I_m$, we obtain $K(x)\xi^{2-m} = Bx^{2\alpha-1}g/f'$, and therefore $K(x) = Bx^{2\alpha-1}$ and $g = \xi^{2-m}f'$.

The latter relation can be now substituted into (3). Eventually, we arrive at the following equation:

$$f'' + \left[\frac{3 - m}{\xi} + \frac{s}{m + 3} \xi^{m - 1}\right] f' + \frac{s(m + 2)}{m + 3} \xi^{m - 2} f = 0$$

The solution of this equation for certain values of $m$ (and $s = 1$) is shown in Fig. 1.

It is worth mentioning that although the turbulence viscosity coefficient, formally defined by (4), diverges as $\xi \to 0$, physical quantities remain well-defined. Really, $\bar{U}(0)$ is constant and nonzero, near $\xi = 0: f' \propto \xi^{m-1} \exp(-a\xi^m)$ (where $a$ is a constant), and $q_{xx} \propto g \propto \xi^{2-m} f' \propto \xi$.

As we can see in Fig. 1, the growth of $m$ (more rapid decrease of the turbulent viscosity with $\xi$) leads to an effective reduction of the wake width. At the same time the zone characterized by $\bar{U}/U_s = \text{const}$ increases.

Thus, in the present work we have suggested to get rid of the assumption of the uniform turbulent viscosity for a self-propelled body wake. Instead, we
consider a power-law dependence \( \nu_T \propto r^{2-m} \), which allows us to construct a new integral of motion, \( I_m = \int_0^\infty r^m \hat{U}dr \) instead of the vanishing \( I_0 = \int_0^\infty \hat{U}dr \).

Each of these integrals leads to the formation of a corresponding flow profile. Simultaneously, the increase in the parameter \( m \) leads to an increase in the flow "core" and slower growth of its width.

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