Rotation Number of Interval Contracted Rotations

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Abstract. Let $0 < \lambda < 1$. We consider the one-parameter family of circle $\lambda$-affine contractions $f_\delta : x \in [0,1) \mapsto \lambda x + \delta \mod 1$, where $0 \leq \delta < 1$. Let $\rho$ be the rotation number of the map $f_\delta$. We will give some numerical relations between the values of $\lambda, \delta$ and $\rho$, essentially using Hecke-Mahler series and a tree structure. When both parameters $\lambda$ and $\delta$ are algebraic numbers, we show that $\rho$ is a rational number. Moreover, in the case $\lambda$ and $\delta$ are rational, we give an explicit upper bound for the height of $\rho$ under some assumptions on $\lambda$.

1. Introduction

Let $I = [0,1)$ be the unit interval.

Definition 1. Let $0 < \lambda < 1$ and $\delta \in I$. We call the map defined by

$$f = f_{\lambda,\delta} : x \in I \mapsto \{\lambda x + \delta\},$$

where the symbol $\{\cdot\}$ stands for the fractional part, a contracted rotation of $I$.

In particular, if $\lambda + \delta > 1$, $f$ is a 2-interval piecewise contraction on the interval $I$ (see Figure 1).

![Figure 1](image-url)

Figure 1. A plot of $f_{\lambda,\delta} : I \to I$, where $\lambda + \delta > 1$

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Many authors have studied the dynamics of contracted rotations, as a dynamical system or in applications, amongst others [4, 5, 6, 7, 8, 9, 11, 14]. It is known that every contracted rotation map $f$ has a rotation number $\rho = \rho_{\lambda, \delta}$, satisfying $0 \leq \rho < 1$. The classical definition of $\rho$ will be recalled in Section 5. If $\rho$ takes an irrational value, then the closure $C$ of the limit set $C := \cap_{k\geq1}f^k(I)$ of $f$ is a Cantor set and $f$ is topologically conjugated to the rotation map $x \in I \mapsto x + \rho \mod 1$ on $C$. When the rotation number is rational, the map $f$ has at most one periodic orbit (see [10] or Section 5.2 for a simple proof) and the limit set $C$ equals the periodic orbit if it does exist.

The goal of this article is to study the value of the rotation number $\rho_{\lambda, \delta}$ according to the diophantine nature of the parameters $\lambda$ and $\delta$. Applying a classical transcendence result, which is stated as Theorem 5 below, we obtain the

**Theorem 1.** Let $0 < \lambda, \delta < 1$ be algebraic real numbers. Then, the rotation number $\rho_{\lambda, \delta}$ is a rational number.

In view of Theorem 1, a natural problem that arises is to estimate the height of the rational rotation number $\rho_{\lambda, \delta}$ in terms of the algebraic numbers $\lambda$ and $\delta$. We provide a partial solution for this issue when $\lambda$ and $\delta$ are rational. Note that Theorem 2 includes the case where $\lambda$ is the reciprocal of an integer.

**Theorem 2.** Let $\lambda = a/b$ and $\delta = r/s$ be rational numbers with $0 < \lambda, \delta < 1$. Assume that $b > a^\gamma$, where $\gamma = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio. Then, the rotation number $\rho_{\lambda, \delta}$ is a rational number $p/q$ where

$$0 \leq p < q \leq \gamma^{\frac{\log(\gamma b)}{\log b - \gamma \log a}}.$$

It should be interesting to extend the validity of Theorem 2 to a larger class of ratios $a/b$. The exponent $\gamma = \frac{1+\sqrt{5}}{2}$ is best possible with regard to the tools employed in its proof, as explained in Section 3.4 below. However, Theorem 2 should presumably be improved. We propose the following

**Conjecture.** Let $\gamma$ be any real number greater than 1. Theorem 2 holds true assuming that $b > a^\gamma$, with a possible larger upper bound for $q$ depending only on $\gamma, a, b$ and $s$.

Our proofs of Theorems 1 and 2 are based on an arithmetical analysis of formulae giving the rotation number $\rho_{\lambda, \delta}$ in terms of the parameters $\lambda$ and $\delta$. As far as we are aware, it is in the works of E. J. Ding and P. C. Hemmer [8] and Y. Bugeaud [5] that appears the first complete description of the relations between the parameters $\lambda, \delta$ and the rotation number $\rho_{\lambda, \delta}$. For $0 < \lambda < 1$ fixed, these papers deal with the variation of the rotation number in the one-dimensional family of contracted rotations $f_{\lambda, \delta}$ as $\delta$ runs through the interval $(0, 1)$. We summarize the results that we need in the following

**Theorem 3.** Let $0 < \lambda < 1$ be given. Then the application $\delta \mapsto \rho_{\lambda, \delta}$ is a continuous non-decreasing function sending $I$ onto $I$ and satisfying the following properties:

(i) The rotation number $\rho_{\lambda, \delta}$ vanishes exactly when $0 \leq \delta \leq 1 - \lambda$.

(ii) Let $\frac{p}{q}$ be a positive rational number, where $0 < p < q$ are relatively prime integers. Then $\rho_{\lambda, \delta}$ takes the value $\frac{p}{q}$ if and only if $\delta$ is located in the interval

$$\frac{1 - \lambda}{1 - \lambda q} c\left(\lambda, \frac{p}{q}\right) \leq \delta \leq \frac{1 - \lambda}{1 - \lambda q} \left(c\left(\lambda, \frac{p}{q}\right) + \lambda q - 1 - \lambda^q\right),$$
where
\[ c(\lambda, \frac{p}{q}) = 1 + \sum_{k=1}^{q-2} \left( \left[ \frac{(k+1)p}{q} \right] - \left[ \frac{kp}{q} \right] \right) \lambda^k \]
and the above sum equals 0 when \( q = 2 \).

(iii) For every irrational number \( \rho \) with \( 0 < \rho < 1 \), there exists one and only one real number \( \delta \) such that \( 0 < \delta < 1 \) and \( \rho_{\lambda,\delta} = \rho \) which is given by the formula
\[
\delta = \delta(\lambda, \rho) = (1 - \lambda) \left( 1 + \sum_{k=1}^{+\infty} \left( \left[ (k+1)\rho \right] - \left[ k\rho \right] \right) \lambda^k \right).
\] (1)

The proof of Theorem 2 deeply relies on a tree structure, introduced by Y. Bugeaud and J.-P. Conze in [6], which is parallel to the classical Stern-Brocot tree of rational numbers. It enables us to handle more easily the complicated intervals occuring in Theorem 3 (ii). Finally, using the same tools, we give a simple proof that

**Theorem 4.** Let \( 0 < \lambda < 1 \) be given. Then
\[
\{ \delta \in I : \rho_{\lambda,\delta} \text{ is an irrational number} \}
\]
has zero Hausdorff dimension.

The paper is organized as follows. In Section 2, we prove Theorem 1 and give another claim (Corollary 7) about the diophantine nature of the number \( \delta(\lambda, \rho) \). To that purpose, we use some results coming from Transcendental Number Theory which are briefly recalled. The proof of Theorem 2 is achieved in Section 3. It combines the tree structure already mentioned with an application of Liouville’s inequality in our context (Lemma 8). In Section 4 we provide a proof of Theorem 4 based again on an analysis of the same tree structure. For completeness, we present in Section 5 various known results in connexion with Theorem 3. We show how the formula (1) is obtained and describe the dynamics of \( f_{\lambda,\delta} \) in the case of an irrational rotation number. This section also includes a proof that \( f_{\lambda,\delta} \) has at most one periodic orbit, as a consequence of a general statement of unicity. Finally, we give some useful properties of Liouville numbers in an Appendix.

## 2. Transcendental numbers

Let us begin with the following result on transcendence due to Loxton and Van der Poorten [12, Theorem 7].

**Theorem 5.** Let \( p(x) \) be a non-constant polynomial with algebraic coefficients and \( \rho \) an irrational real number, then the power series
\[
\sum_{k=1}^{\infty} p([k\rho]) \lambda^k
\]
takes a transcendental value for any algebraic number \( \lambda \) with \( 0 < |\lambda| < 1 \).

We only need Theorem 5 for a polynomial \( p(x) \) of degree one, in which case the statement is equivalent to the transcendency of the so-called Hecke-Mahler series
\[
S_\rho(\lambda) := \sum_{k \geq 1} [k\rho] \lambda^k.
\]
A proof can also be found in Sections 2.9 and 2.10 of the monograph [15]. See also the survey article [13].
2.1. **Proof of Theorem 1.** We use the fact that the number $\delta(\lambda, \rho)$ in (1) may be expressed in terms of the Hecke-Mahler series, thanks to the useful identity

$$\sum_{k \geq 0} \left( [(k+1)\rho] - [k\rho] \right) \lambda^k = \left( \frac{1}{\lambda} - 1 \right) S_\rho(\lambda), \quad (2)$$

relating the Hecke-Mahler series with the power series associated to Sturmian sequences. Using (2), we rewrite formula (1) in the form

$$\delta(\lambda, \rho) = 1 - \lambda + \frac{(1-\lambda)^2}{\lambda} \sum_{k=1}^{\infty} [k\rho] \lambda^k = \frac{(1-\lambda)^2}{\lambda} \sum_{k=1}^{\infty} ([k\rho] + 1) \lambda^k. \quad (3)$$

Applying Theorem 5 to the polynomial

$$p(z) = \frac{(1-\lambda)^2}{\lambda} z + \frac{(1-\lambda)^2}{\lambda}$$

with algebraic non-zero coefficients, we obtain that $\delta(\lambda, \rho)$ is a transcendental number. As a consequence, if $\delta$ is an algebraic number, the rotation number $\rho_{\lambda,\delta}$ cannot be an irrational number. It is therefore a rational number. □

2.2. **Liouville numbers.** We are concerned here with the diophantine nature of the transcendental number $\delta(\lambda, \rho)$ when $\lambda$ is the reciprocal of an integer. Let us start with the following

**Theorem 6.** Let $b \geq 2$ be an integer and let $0 < \rho < 1$ be an irrational number. Then, the number

$$S_\rho \left( \frac{1}{b} \right) = \sum_{k=1}^{\infty} \frac{[k\rho]}{b^k}$$

is a Liouville number if, and only if, the partial quotients of the continued fraction expansion of the irrational number $\rho$ are unbounded.

**Proof.** By formula (2) and Lemma 11 below, we have to prove equivalently that the Sturmian number in base $b$

$$\sum_{k \geq 1} \frac{([(k+1)\rho] - [k\rho])}{b^k}$$

is a Liouville number exactly when $\rho$ has unbounded partial quotients. P.E. Böhmer has proved in [3] that this number is a Liouville number when the partial quotients of $\rho$ are unbounded. For the converse, notice that the sequence $([(k+1)\rho] - [k\rho])_{k \geq 1}$ is a Sturmian sequence with values in $\{0,1\}$ and slope $\rho$. Now, it follows from [1, Proposition 11.1] that the associated real number

$$\sum_{k=1}^{\infty} \frac{([(k+1)\rho] - [k\rho])}{b^k}$$

in base $b$ is transcendental and it has a finite irrationality exponent exactly when the slope $\rho$ has bounded partial quotients. □

**Corollary 7.** Let $\lambda = \frac{1}{b}$, where $b \geq 2$ is an integer, and let $\rho$ be an irrational number with $0 < \rho < 1$. Then the transcendental number $\delta \left( \frac{1}{b}, \rho \right)$ is a Liouville number if and only if the partial quotients of the continued fraction expansion of $\rho$ are unbounded.
Proof. Using formula (3), we have
\[ \delta \left( \frac{1}{b}, \rho \right) = \frac{b-1}{b} + \frac{(b-1)^2}{b} \sum_{k=1}^{\infty} \frac{[k\rho]}{b^k}. \]

It follows from Lemma 11 that \( \delta \left( \frac{1}{b}, \rho \right) \) is a Liouville number if and only if \( S_\rho \left( \frac{1}{b} \right) \) is a Liouville number. According to Theorem 6, this happens exactly when \( \rho \) has unbounded partial quotients.

\[ \square \]

3. Rational parameters

We prove Theorem 2 in this Section. To that purpose, we make use of results due to Y. Bugeaud and J.-P. Conze [6], who have shown that the numbers \( \frac{1 - \lambda}{1 - \lambda q} c \left( \lambda, \frac{p}{q} \right) \) occurring in Theorem 3 (ii) share a combinatorial structure parallel to the classical Stern-Brocot tree of rational numbers (called the Farey tree in [6]).

3.1. On the Stern-Brocot tree. The Stern-Brocot tree, restricted to the interval \([0, 1]\), may be described as an infinite sequence of finite rows, indexed by \( k \geq 1 \), consisting of ordered rational numbers \( \frac{p}{q} \in [0, 1] \) and starting with
\[
\begin{align*}
  k = 1 : & \quad \frac{0}{1}, \frac{1}{1} \\
  k = 2 : & \quad \frac{0}{1}, \frac{1}{1 + \lambda}, \frac{1}{1 + \lambda} \\
  k = 3 : & \quad \frac{0}{1}, \frac{1}{1 + \lambda}, \frac{1}{1 + \lambda + \lambda^2}, \frac{1}{1 + \lambda + \lambda^2}, \frac{1}{1 + \lambda + \lambda^2 + \lambda^3}, \frac{1}{1 + \lambda + \lambda^2 + \lambda^3}, \ldots
\end{align*}
\]

The \((k+1)\)-th row is constructed from the \( k \)-th row by inserting between any pair of consecutive elements \( \frac{p}{q} < \frac{p'}{q'} \) on the \( k \)-th row, their mediant \( \frac{p + p'}{q + q'} \). Moreover, every rational number \( 0 \leq \frac{p}{q} \leq 1 \) appears in (4).

Let \( \lambda \) be a real number with \( 0 < \lambda \leq 1 \). We consider the analogous tree
\[
\begin{align*}
  k = 1 : & \quad \frac{0}{1}, \frac{1}{1} \\
  k = 2 : & \quad \frac{0}{1}, \frac{1}{1 + \lambda}, \frac{1}{1 + \lambda} \\
  k = 3 : & \quad \frac{0}{1}, \frac{1}{1 + \lambda}, \frac{1}{1 + \lambda + \lambda^2}, \frac{1}{1 + \lambda + \lambda^2}, \frac{1}{1 + \lambda + \lambda^2 + \lambda^3}, \frac{1}{1 + \lambda + \lambda^2 + \lambda^3}, \ldots
\end{align*}
\]

Any element \( \frac{P}{Q} \) of the tree is a rational function in \( \lambda \), where the numerator \( P = P(\lambda) \) and the denominator \( Q = Q(\lambda) \) are polynomials in \( \lambda \). The tree is constructed by a similar process with the new mediant rule
\[
\frac{P' + \lambda q' P}{Q' + \lambda q' Q}
\]
between consecutive elements \( \frac{P}{Q} < \frac{P'}{Q'} \) where \( q' = Q'(1) \). In the case \( \lambda = 1 \), we recover the Stern-Brocot tree (4). Notice that for every \( \frac{P}{Q} \) appearing in (5), the rational number \( \frac{p}{q} = \frac{P(1)}{Q(1)} \) appears at the same place in (4). It is proved in [6] that the rational fractions \( \frac{P}{Q} \) appearing in (5) are precisely those which are involved in Theorem 3 (ii); namely that we have the explicit formula:

\[
\frac{P}{Q} = \frac{1 - \lambda}{1 - \lambda^q} c\left(\lambda, \frac{p}{q}\right) = \frac{c\left(\lambda, \frac{p}{q}\right)}{1 + \lambda + \cdots + \lambda^{q-1}},
\]

where \( \frac{p}{q} = \frac{P(1)}{Q(1)} \) and \( c\left(\lambda, \frac{p}{q}\right) \) is defined in Theorem 3 (ii). We further define \( c\left(\lambda, \frac{0}{1}\right) = 0 \) and \( c\left(\lambda, \frac{1}{1}\right) = 1 \) in order to extend the validity of the formula to the endpoints 0 and 1. Moreover, if \( \frac{P}{Q} < \frac{P'}{Q'} \) are adjacent elements on some row of (5), then

\[
\frac{P'}{Q'} - \frac{P}{Q} = \frac{\lambda^{q-1} - \lambda^q}{Q'Q} > \frac{\lambda^{q-1}(1 - \lambda)}{Q},
\]

where \( q = Q(1) \). It follows from (6) that

\[
\frac{P}{Q} \leq \frac{P + \lambda^{q-1} - \lambda^q}{Q} < \frac{P'}{Q'}.
\]

The assertions (i) and (ii) of Theorem 3 mean that the rotation number \( \rho_{\lambda, \delta} \) is equal to \( \frac{p}{q} \) exactly when \( \delta \) is located in the interval

\[
\left[\frac{P}{Q}, \frac{P + \lambda^{q-1} - \lambda^q}{Q}\right] \subset \left[\frac{P}{Q}, \frac{P'}{Q'}\right].
\]

3.2. A lemma. From now on, put \( \lambda = a/b \) and \( \delta = r/s \) as in Theorem 2. The following result, based on Liouville’s inequality, will be our main tool for proving Theorem 2.

Lemma 8. Let \( 0 \leq \frac{P}{q} < \frac{P'}{q'} \leq 1 \) be consecutive rational numbers on some row of the Stern-Brocot tree (4). Put

\[
P = c\left(\lambda, \frac{p}{q}\right), \quad Q = 1 + \cdots + \lambda^{q-1}, \quad P' = c\left(\lambda, \frac{p'}{q'}\right) \quad \text{and} \quad Q' = 1 + \cdots + \lambda^{q'-1}.
\]

If we assume that

\[
\frac{P + \lambda^{q-1} - \lambda^q}{Q} < \delta < \frac{P'}{Q'},
\]

then we have the inequality

\[
\delta^{\max(q, q')} \leq sb\theta^{q+q'}.
\]
Proof. It follows from (6) and (7) that
\[
\max \left( \frac{P' - \delta}{Q'} - \frac{P + \lambda^{q-1} - \lambda^q}{Q} \right) < \frac{P' - \delta}{Q'} - \frac{P + \lambda^{q-1} - \lambda^q}{Q} = \frac{\lambda^{q-1} - \lambda^q}{Q} \tag{8}
\]
since \( Q' = 1 + \cdots + \lambda^{q-1} = 1 - \lambda^q \). On the other hand, we have the lower bound
\[
\frac{P' - \delta}{Q'} = \frac{sP' - rQ'}{sQ'} = \frac{b^{q'-1}(sP' - rQ')}{b^{q'-1}sQ'} \geq \frac{1}{b^{q'-1}sQ'}, \tag{9}
\]
observing that the numerator
\[
b^{q'-1}(sP' - rQ') = s \left( b^{q'-1} + \sum_{j=1}^{q'-2} \left( ([j+1]p'/q') - [jp'/q'] \right) a^j b^{q'-1-j} \right) - r \sum_{j=0}^{q'-1} a^j b^{q'-1-j}
\]
is an integer, which is necessarily \( \geq 1 \) since it is the numerator of a positive rational number. In a similar way, we find that
\[
\delta - \frac{P + \lambda^{q-1} - \lambda^q}{Q} \geq \frac{1}{b^q sQ}. \tag{10}
\]
Combining the upper bound (8) with the lower bounds (9) and (10), we obtain the estimates
\[
\frac{b^q \leq \frac{s}{Q} \leq s^{q+q'-1}}{Q} \text{ and } \frac{b^{q'-1} \leq \frac{s}{Q}}{Q} \leq s^{q+q'-1},
\]
which yield the slightly weaker inequality \( b_{\max(q,q')} \leq s b^{q+q'}, \) since \( b > a \geq 1 \). \( \square \)

3.3. Proof of Theorem 2. Let \( \gamma \) be the golden ratio and set
\[
k = \left\lceil \frac{\gamma \log(sb)}{\log b - \gamma \log a} \right\rceil \geq 1. \tag{11}
\]
We claim that there exists a rational \( 0 \leq \frac{p}{q} < 1 \) appearing on the \( (k+1) \)-st row of (4) such that
\[
c \left( \frac{\lambda, p}{q} \right) \leq \delta \leq c \left( \frac{\lambda, p}{q} \right) + \lambda^{q-1} - \lambda^q \tag{12}
\]
Then, the assertions (i) and (ii) of Theorem 3 yield that \( \rho_{\lambda, \delta} = p/q \).

In order to prove the claim, we first focus to the \( k \)-th row of (5). There exist adjacent elements \( \frac{P}{Q} < \frac{P'}{Q'} \) on the \( k \)-th row of (5) such that \( \delta \) is located in the interval
\[
\frac{P}{Q} \leq \delta < \frac{P'}{Q'}.
\]
Put \( \frac{p}{q} = \frac{P(1)}{Q(1)} \) and \( \frac{p'}{q'} = \frac{P'(1)}{Q'(1)} \). If \( \delta \) falls in the subinterval
\[
\frac{P}{Q} \leq \delta \leq \frac{P + \lambda^{q-1} - \lambda^q}{Q},
\]
then (12) is established. We may therefore assume that
\[
\frac{P + \lambda^{q-1} - \lambda^q}{Q} < \delta < \frac{P'}{Q'}.
\]
Then, Lemma 8 gives the estimate
\[
b^{\max(q,q')} \leq sba^{q+q'}.
\]
We now consider the mediant \(\frac{P''}{Q''} = \frac{P' + \lambda^q P}{Q' + \lambda^q Q} \) of \(\frac{P}{Q}\) and \(\frac{P'}{Q'}\) which is an element of the \((k + 1)\)-th row of (5). Put \(p'' = P''(1) = p + p'\) and \(q'' = Q''(1) = q + q'\). Since \(\frac{p}{q} < \frac{p''}{q''}\) and \(\frac{p''}{q''} < \frac{p'}{q'}\) are two pairs of consecutive rational numbers on the \((k + 1)\)-th row of the Stern-Brocot tree (4), we have the inequalities
\[
\frac{P}{Q} < \frac{P + \lambda^{q-1} - \lambda^q}{Q} < \frac{P''}{Q''} < \frac{P'' + \lambda^{q''-1} - \lambda^{q''}}{Q''} < \frac{P'}{Q'}.
\]
If \(\delta\) belongs to the subinterval
\[
\frac{P''}{Q''} \leq \delta \leq \frac{P'' + \lambda^{q''-1} - \lambda^{q''}}{Q''},
\]
then (12) holds true with \(p/q\) replaced by \(p''/q''\). It thus remains to deal with the intervals
\[
\frac{P + \lambda^{q-1} - \lambda^q}{Q} < \delta < \frac{P''}{Q''} \quad \text{and} \quad \frac{P'' + \lambda^{q''-1} - \lambda^{q''}}{Q''} < \delta < \frac{P'}{Q'}.
\]
In both cases, we shall reach a contradiction, due to the fact that the level \(k\) has been selected large enough in (11).

Assume first that
\[
\frac{P + \lambda^{q-1} - \lambda^q}{Q} < \delta < \frac{P''}{Q''}.
\]
Then, Lemma 8 applied to the pair of adjacent rationals \(\frac{p}{q} < \frac{p''}{q''}\), gives now the estimate
\[
b^{q+q'} = b^{q''} = b^{\max(q,q'')} \leq sba^{q+q'} = sba^{2q+q'}.
\]
Raising the inequalities (13) and (14) respectively to the powers \(1/(q + q')\) and \(1/(2q + q')\), we immediately obtain the upper bound
\[
b^{\max\left(\frac{q}{q+q'}, \frac{q}{q+q'}; \frac{q}{q+q}; \frac{q}{q+q} + \frac{q'}{2q+q'}\right)} \leq (sb)^{\frac{1}{(q+q')}} a.
\]
Putting \(\xi = \frac{q}{q'}\), we bound from below the exponent
\[
\max\left(\frac{q}{q+q'}, \frac{q'}{q+q'}; \frac{q}{2q+q'}\right) \geq \max\left(\frac{q}{q+q'}, \frac{q}{2q+q'}\right) = \max\left(\frac{1}{1 + \xi}; \frac{1 + \xi}{1 + 2\xi}\right) \geq \frac{1}{\gamma}'.
\]
For the last inequality, it suffices to note that
\[
\frac{\xi}{1 + \xi} \geq \frac{1}{\gamma} \iff \xi \geq \frac{1}{\gamma - 1} \quad \text{and} \quad \frac{1 + \xi}{1 + 2\xi} \geq \frac{1}{\gamma} \iff \frac{\gamma - 1}{2 - \gamma} = \frac{1}{\gamma - 1} \geq \xi.
\]
Observe that for each pair of adjacent numbers $\frac{p}{q} < \frac{p'}{q'}$ on the $k$-th row of (4), we have the lower bound $q + q' \geq k + 1$. We then deduce from (15) that
\[
\frac{1}{b} \leq \left( \frac{sb}{a} \right)^{\frac{1}{k+1}},
\]
or equivalently that $k + 1 \leq \frac{\gamma \log(sb)}{\log b - \gamma \log a}$, in contradiction with our choice (11) for $k$.

The second interval
\[
\frac{P'' + \lambda q'' - 1 - \lambda q''}{Q''} < \delta < \frac{P'}{Q'}
\]
is treated in a similar way. In this case, the analogue of (14) writes now
\[
\frac{b^{q+q'} = b''}{q''} = b^{\max(q', q'')} \leq sba^{q+q''} = sba^{q+2q'}.
\]

Permuting $q$ and $q'$, we are led to the same computations. The claim is proved.

It is well-known that each rational number on the $(k + 1)$-th row of the Stern-Brocot tree (4) has a denominator $q$ bounded by the Fibonacci number
\[
F_{k+2} = \frac{1}{\sqrt{5}} \left( \gamma^{k+2} - (-1/\gamma)^{k+2} \right) \leq \frac{1}{\sqrt{5}} \gamma^{2+ \frac{\gamma \log(sb)}{\log b - \gamma \log a}} + \frac{1}{\sqrt{5}} \leq \gamma^{2+ \frac{\gamma \log(sb)}{\log b - \gamma \log a}},
\]
since $k \leq \frac{\gamma \log(sb)}{\log b - \gamma \log a}$ by (11). Theorem 2 is proved.

\[\square\]

3.4. A remark of optimality. We have considered in the preceding argumentation two consecutive rows of index $k$ and $k + 1$ in the tree (5). Using the subsequent rows of indices $k + 2, \ldots$ does not necessarily produce a substantially better result, as the following construction shows.

Let
\[
\frac{\sqrt{5} - 1}{2} = \frac{1}{\gamma} = [0; 1, 1, 1, \ldots]
\]
be the continued fraction expansion of the irrational number $\frac{\sqrt{5} - 1}{2}$ whose sequence of convergents is $0 = \frac{F_0}{F_1}, 1 = \frac{F_1}{F_2}, \frac{F_2}{F_3}, \ldots$, where we recall that $(F_{l})_{l \geq 0}$ denotes the Fibonacci sequence. For every $l \geq 0$, set
\[
P_l = c \left( \lambda \frac{F_l}{F_{l+1}} \right) \quad \text{and} \quad Q_l = 1 + \cdots + \lambda^{F_{l+1}-1},
\]
so that $\frac{P_l}{Q_l}$ is the element in the tree (5) associated to the convergent $\frac{F_l}{F_{l+1}}$. Now,
\[
\frac{F_{l+2}}{F_{l+3}} = \frac{F_{l+1} + F_l}{F_{l+2} + F_{l+1}}
\]
is the mediant of $\frac{F_{l+1}}{F_{l+2}}$ and $\frac{F_l}{F_{l+1}}$ in the Stern-Brocot tree (4). Accordingly, $\frac{P_{l+2}}{Q_{l+2}}$ is the mediant of $\frac{P_{l+1}}{Q_{l+1}}$ and $\frac{P_l}{Q_l}$ in the tree (5). It follows that $\frac{P_l}{Q_l} < \frac{P_{l+1}}{Q_{l+1}}$ (resp. $\frac{P_{l+1}}{Q_{l+1}} < \frac{P_l}{Q_l}$) are adjacent elements on the $(l + 1)$-th row of the tree (5) when $l$ is even (resp. odd). Therefore, the open intervals
\[
I_l := \begin{cases} 
\left( \frac{P_l \lambda^{F_{l+1} - 1} - \lambda^{F_{l+1}}}{Q_l}, \frac{P_{l+1}}{Q_{l+1}} \right) & \text{if } l \text{ is even} \\
\left( \frac{P_{l+1} \lambda^{F_{l+2} - 1} - \lambda^{F_{l+2}}}{Q_{l+1}}, \frac{P_l}{Q_l} \right) & \text{if } l \text{ is odd}
\end{cases}, \quad l \geq 0,
\]

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form a decreasing sequence of nested intervals whose intersection reduces to the transcendental number $\delta(\lambda, 1/\gamma)$. Now, assuming that the rational $\delta = \frac{r}{s}$ belongs to the interval $I_l$, Lemma 8 gives us the inequality

$$b \leq (sb)^{\frac{F_{l+1} + F_{l+2}}{F_{l+2}}}.$$

Observe that the exponent $\frac{F_{l+1} + F_{l+2}}{F_{l+2}}$ converges to $\gamma$ as $l$ tends to infinity. Thus, the constraint $b > a^\gamma$ assumed in Theorem 2 cannot be essentially relaxed by using Lemma 8. However, we have no reason to believe that Lemma 8 itself is optimal.

4. Proof of Theorem 4

Let $0 < \lambda < 1$. We prove that the set

$$\mathcal{E} = \{\delta \in I : \rho_{\lambda, \delta} \text{ is irrational}\}.$$

has null Hausdorff dimension, using properties of the tree (5) displayed in Section 3. It turns out that $\mathcal{E}$ has a natural structure of Cantor set induced by the tree (5).

For any $k \geq 1$, both trees (4) and (5) have $2^{k-1} + 1$ elements on their $k$-th row. Let us number

$$0 = p_{0,k} < q_{0,k} < \cdots < p_{2^k-1,k} = 1,$$

their respective elements by increasing order, and put

$$I_{j,k} = \left(\frac{p_{j,k} + \lambda q_{j,k} - \lambda q_{j,k}}{q_{j,k}}, \frac{p_{j+1,k}}{q_{j+1,k}}\right) \subset I, \quad 0 \leq j \leq 2^k - 1, k \geq 1,$$

$$\mathcal{E}_k = \bigcup_{j=0}^{2^{k-1}-1} I_{j,k} \subset I, \quad k \geq 1.$$

Each interval $I_{j,k}$ in $\mathcal{E}_k$ contains two intervals from $\mathcal{E}_{k+1}$:

$$I_{j,k} \cap \mathcal{E}_{k+1} = I_{j',k+1} \cup I_{j'+1,k+1},$$

where $j'$ and $j' + 1$ are the respective indices of $\frac{p_{j,k}}{q_{j,k}}$ and of the mediant $\frac{p_{j,k} + p_{j+1,k}}{q_{j,k} + q_{j+1,k}}$ on the $(k+1)$-th row of the Stern-Brocot tree (4). Hence, for every $k \geq 1$, we have the inclusion

$$\mathcal{E}_{k+1} \subset \mathcal{E}_k.$$

It follows from Theorem 3 that

$$\mathcal{E} = \bigcap_{k \geq 1} \mathcal{E}_k = \bigcap_{k \geq 1} \bigcup_{j=0}^{2^{k-1}-1} I_{j,k}.$$

Lemma 9. The upper bounds

$$\max_{0 \leq j \leq 2^{k-1}-1} |I_{j,k}| \leq \lambda^k \quad \text{and} \quad \sum_{j=0}^{2^{k-1}-1} |I_{j,k}|^\sigma \leq \sum_{n \geq k} n \lambda^{\sigma n},$$

hold for any integer $k \geq 1$ and any real number $\sigma > 0$, where the bars $| \cdot |$ indicate here the length of an interval.
Proof. It follows from (6) and (8) that

\[ |I_{j,k}| = \frac{\lambda q_j + q_{j+1,k} - 1}{Q_j,k Q_{j+1,k}} \leq \frac{\lambda q_j + q_{j+1,k} - 1}{Q_j,k Q_{j+1,k}}, \quad k \geq 1, 0 \leq j \leq 2^{k-1} - 1. \]

As \( \frac{p_j}{q_j} < \frac{p_{j+1,k}}{q_{j+1,k}} \) are consecutive elements on the \( k \)-th row of the Stern-Brocot tree, the lower bound \( q_j + q_{j+1,k} \geq k + 1 \) holds for \( 0 \leq j \leq 2^{k-1} - 1 \). The first upper bound of Lemma 9 immediately follows. For the second one, observe that

\[ \sum_{j=0}^{2^{k-1}-1} |I_{j,k}|^\sigma \leq \sum_{n \geq k+1} \text{Card} \{ j : q_j + q_{j+1,k} = n \} \lambda^{\sigma(n-1)} \leq \sum_{n \geq k+1} (n-1) \lambda^{\sigma(n-1)}, \]

the last inequality coming from the fact that \( q_j + q_{j+1,k} \) is the denominator of the mediant \( \frac{p_{j+1,k} + p_{j+1,k}}{q_{j+1,k} + q_{j+1,k}} \) located on the \((k+1)\)-th row and that for any \( n \geq 2 \) a given row of the Stern-Brocot tree contains at most \( n - 1 \) elements having denominator \( n \).

Observe that the series \( \sum_{n \geq 1} n \lambda^{\sigma n} \) converges for any \( \sigma > 0 \). Then, Theorem 4 follows from Lemma 9 by using standard arguments on the \( \sigma \)-dimensional Hausdorff measure for \( 0 < \sigma \leq 1 \). See for instance Corollary 2 in [2].

5. On the Rotation Number \( \rho_{\lambda,\delta} \)

We have collected in this section further information concerning the basic Theorem 3 and the dynamics of the transformation \( f_{\lambda,\delta} \).

Let us begin with the definition of the rotation number. In order to compute the rotation number of \( f \), it is convenient to introduce a function \( F : \mathbb{R} \rightarrow \mathbb{R} \) called suspension or lift of \( f \) on \( \mathbb{R} \). Our lift \( F \) will be defined by

\[ F = F_{\lambda,\delta} : x \in \mathbb{R} \mapsto \lambda \{x\} + \delta + [x], \]

where \([x]\) is the integer part of the real number \( x \). Its name comes from the fact that \( F \) satisfies the following properties:

(i) Let \( \pi : x \in \mathbb{R} \mapsto \{x\} \) be the canonical projection of \( \mathbb{R} \) on \( I \), then, for every \( x \in \mathbb{R} \),

\[ \pi(F(x)) = f(\pi(x)). \]

(ii) \( F(x+1) = F(x) + 1 \), for every \( x \in \mathbb{R} \).

(iii) \( F \) is an increasing function on \( \mathbb{R} \) which is continuous on each interval of \( \mathbb{R} \setminus \mathbb{Z} \) and right continuous on \( \mathbb{Z} \).

Let \( x_0 \in \mathbb{R} \). It is known (see [17]) that the limit

\[ \rho_{\lambda,\delta} = \lim_{k \to \infty} \frac{F_{\lambda,\delta}^k(x_0)}{k} \]

exists and does not depend on the initial point \( x_0 \). The number \( \rho_{\lambda,\delta} \) is called rotation number of the map \( f = f_{\lambda,\delta} \) and \( 0 \leq \rho_{\lambda,\delta} < 1 \).

5.1. The Irrational Rotation Number Case. Let \( 0 < \lambda < 1 \) be fixed. As indicated in Theorem 3, it is known that the graph of the application \( \delta \mapsto \rho_{\lambda,\delta} \) is a Devil’s staircase, meaning that this map is a continuous non-decreasing function sending [0,1) onto [0,1) which takes the rational value \( p/q \) on the whole interval

\[ \frac{1 - \lambda}{1 - \lambda^q} \left( \frac{p}{q} \right) \leq \delta \leq \frac{1 - \lambda}{1 - \lambda^q} \left( c \left( \frac{p}{q} \right) + \lambda^{q-1} - \lambda^q \right). \]
Figure 2. Plot of $F_{1/2,3/4}(x)$ in the interval $-1 \leq x < 1$

Notice that if we select a sequence of reduced rationals $p/q$ tending to an irrational number $\rho$, so that $q$ tends to infinity, then the above intervals shrink to the point

$$\delta(\lambda, \rho) = (1 - \lambda) \left( 1 + \sum_{k=1}^{+\infty} \left( \left( (k+1)\rho \right) - \left( k\rho \right) \right) \lambda^k \right),$$

since the integer part function $x \mapsto \lfloor x \rfloor$ is continuous at non-integer $x$. Therefore, for each irrational number $\rho$ with $0 < \rho < 1$, there is exactly one value of $\delta$ for which $\rho = \rho_{\lambda, \delta}$ and this value is given by the series $\delta(\lambda, \rho)$.

Fix now an irrational number $\rho$ with $0 < \rho < 1$. Following the approach given in [7], define a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by the formula

$$\varphi(t) = \sum_{k\geq0} \lambda^k \left( \delta(\lambda, \rho) + (1 - \lambda)[t - (k+1)\rho] \right).$$

It has been established in [7] that the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function which has the following properties:

(C1) $\varphi(t + \rho) = F_{\lambda, \delta(\lambda, \rho)}(\varphi(t))$, $\forall t \in \mathbb{R}$,

(C2) $\varphi(t + 1) = \varphi(t) + 1$, and

(C3) $\varphi(0) = 0$.

Reducing modulo 1, this establishes that $f_{\lambda, \delta(\lambda, \rho)}$ and the circle rotation map $R_\rho : x \in I \mapsto x + \rho \mod 1$ are topologically conjugated on the limit set $C = \varphi(I) \subset I$.

Notice that (C2) and (C3) imply that $\lfloor \varphi(t) \rfloor = [t]$ for every $t \in \mathbb{R}$, from which (C1) easily follows. The property (C3) is equivalent to the formula (3) for $\delta(\lambda, \rho)$ since

$$\varphi(0) = \sum_{k\geq0} \lambda^k \left( \delta(\lambda, \rho) + (1 - \lambda)[-(k+1)\rho] \right) = \frac{\delta(\lambda, \rho)}{1 - \lambda} + (1 - \lambda) \sum_{k\geq0} \lambda^k[-(k+1)\rho],$$

$$= \frac{\delta(\lambda, \rho)}{1 - \lambda} - (1 - \lambda) \sum_{k\geq0} \lambda^k[(k+1)\rho] = \frac{\delta(\lambda, \rho)}{1 - \lambda} - 1 - \frac{1 - \lambda}{\lambda} \sum_{k\geq1} \lambda^k[k\rho].$$

We now link the preceding facts to the dynamics of $f_{\lambda, \delta(\lambda, \rho)}$, or equivalently to the iterates of the real function $F_{\lambda, \delta(\lambda, \rho)}$ by reducing modulo 1. For any $x_0 \in \mathbb{R}$, we define the orbit of
$x_0$ under $F_{\lambda,\delta(\lambda,\rho)}$ iteratively by

$$x_{k+1} = F_{\lambda,\delta(\lambda,\rho)}(x_k) = \lambda x_k + \delta(\lambda,\rho) + (1 - \lambda)[x_k], \text{ for every } k \geq 0.$$  

We then easily obtain by induction on $n$ the formula

$$x_n = \lambda^n x_0 + \sum_{k=0}^{n-1} \lambda^k \left( \delta(\lambda,\rho) + (1 - \lambda) [x_{n-k-1}] \right), \text{ for every } n \geq 0.$$  

It is proved in [7] that for each initial point $x_0$, there exists $t_0 \in \mathbb{R}$ such that

$$[x_k] = [t_0 + \rho k] \text{ for every } k \geq 0.$$  

Therefore

$$x_n = \lambda^n x_0 + \sum_{k=0}^{n-1} \lambda^k \left( \delta(\lambda,\rho) + (1 - \lambda) [t_0 + n\rho - (k + 1)\rho] \right) = \varphi(t_0 + n\rho) + O(\lambda^n),$$

as $n$ tends to infinity. Hence the $f_{\lambda,\delta(\lambda,\rho)}$-orbit of $x_0$ modulo 1 converges exponentially fast to the orbit of $\varphi(t_0)$ modulo 1 on $C$.

### 5.2. The rational rotation number case.

According to [10], any contracted rotation $f_{\lambda,\delta}$ with rational rotation number $\rho_{\lambda,\delta}$ has a unique periodic orbit in an extended meaning, allowing multivalues at the discontinuity point $0 \in \mathbb{R}/\mathbb{Z}$. Then, for every $x \in I$, the $\omega$-limit set

$$\omega(x) := \cap_{t \geq 0} \bigcup_{k \geq t} F^K_{\lambda,\delta}(x)$$

coincides with this periodic orbit.

Let $f : I \to I$ be a map of the unit interval which is continuous outside finitely many points. We say that $f$ is a piecewise contracting map, when there exists $0 < \lambda < 1$ such that for every open interval $J \subset I$ on which $f$ is continuous, the inequality $|f(x) - f(y)| \leq \lambda|x - y|$ holds for any $x, y \in J$ (see [16]). We view as well $f$ as a circle map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by identifying the sets $\mathbb{R}/\mathbb{Z}$ and $I$, thanks to the canonical bijection $I \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. We now prove that $f = f_{\lambda,\delta}$ has at most one periodic orbit, as a corollary of the more general statement

**Theorem 10.** Let $f$ be a piecewise contracting and orientation-preserving circle map. Assume that $f$ has a unique point of discontinuity on the circle $\mathbb{R}/\mathbb{Z}$ and that $f$ is right continuous at this point. Then $f$ has at most one periodic orbit.

**Proof.** We may assume without loss of generality that the discontinuity is the origin $0 \in \mathbb{R}/\mathbb{Z}$. This is the case for any contracted rotation $f_{\lambda,\delta}$ (see Figure 2). Then, there exists a lift $F$ of $f$ satisfying the three properties (i), (ii), (iii) appearing on page 11, which are verified by the function $F_{\lambda,\rho}$. Moreover, there exists $0 < \lambda < 1$ such that for each integer $n$, we have inequalities of the form

$$0 \leq F(y) - F(x) \leq \lambda(y - x) \text{ for every } n \leq x \leq y < n + 1. \quad (16)$$

Let $O_1$ and $O_2$ be two periodic orbits of $f$. For $i = 1, 2$, define $x_i$ as the smallest element of $O_i$ in $I$. Exchanging possibly $O_1$ and $O_2$, we may assume that $0 \leq x_1 \leq x_2 < 1$. We then claim that

$$[F^k(x_1)] = [F^k(x_2)] \quad \text{and} \quad 0 \leq F^k(x_2) - F^k(x_1) \leq \lambda^k(x_2 - x_1) \quad (17)$$
for every integer $k \geq 0$. The proof is performed by induction on $k$ and the assertion clearly holds for $k = 0$. We have

$$0 \leq F^{k+1}(x_2) - F^{k+1}(x_1) = F(F^k(x_2)) - F(F^k(x_1)) \leq \lambda(F^k(x_2) - F^k(x_1)) \leq \lambda^{k+1}(x_2 - x_1),$$

by (16) and (17). Assume on the contrary that

$$|F^{k+1}(x_2)| \geq |F^{k+1}(x_1)| + 1,$$

however $|F^k(x_2)| = |F^k(x_1)|$. We thus have

$$F^{k+1}(x_1) < |F^{k+1}(x_1)| + 1 \leq |F^{k+1}(x_2)|$$

and

$$F^{k+1}(x_2) - F^{k+1}(x_1) \leq \lambda^{k+1}(x_2 - x_1),$$

so that

$$f^{k+1}(x_2) = \{F^{k+1}(x_2)\} = F^{k+1}(x_2) - |F^{k+1}(x_2)| \leq \lambda^{k+1}(x_2 - x_1) \leq x_2$$

which contradicts the choice of $x_2$. The claim is proved.

Reducing modulo 1 and letting $k$ tend to infinity, we deduce from (17) by periodicity that $f^k(x_2) = f^k(x_1)$ for any $k \geq 0$. We have proved that $O_1 = O_2$.

$$\Box$$

Appendix: Liouville Numbers

A real number $\xi$ is a Liouville number, if for every integer number $k \geq 1$, there exist integers numbers $p_k$ and $q_k$ with $q_k \geq 2$ such that

$$0 < \left| \xi - \frac{p_k}{q_k} \right| < \frac{1}{q_k^k}.$$ 

Liouville’s inequality shows that $\xi$ is a transcendental number. The sequence $(q_k)_{k \geq 1}$ is thus unbounded. For completeness, we include the proof of the following lemma:

Lemma 11. Let $\xi$ be Liouville and $r$ and $s$ be nonzero rational numbers, then $r \xi + s$ is a Liouville number.

Proof. We write $r = \frac{a}{b}$ and $s = \frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$ with $b \geq 1$ and $d \geq 1$. We claim that $\frac{a}{b} \xi + \frac{c}{d}$ is a Liouville number. Let $m \geq 2$ be arbitrarily fixed. There exists $k = k(m)$ such that $k > m$ and $q_k \geq |a|b^{m-1}d^m$. We have

$$\left| \frac{a}{b} \xi + \frac{c}{d} - \frac{adp_k + bck_k}{bdq_k} \right| = \left| \frac{a}{b} \xi - \frac{p_k}{q_k} \right| < \frac{|a|}{b} \frac{1}{q_k} \leq \frac{|a|}{b} \frac{1}{|a|b^{m-1}d^m q_k^{k-1}} \leq \frac{1}{(bdq_k)^m}$$

which proves that $r \xi + s$ is a Liouville number. 

$\Box$

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REFERENCES

[1] B. Adamcewski and Y. Bugeaud. *Nombres réels de complexité sous-linéaire : mesures d’irrationalité et de transcendance*, J. reine angew. Math. 568 (2011), 65-98.

[2] A. S. Besicovitch and S. J. Taylor. *On the complementary intervals of a linear closed set of zero Lebesgue measure*, Journal of the London Math. Soc. 29 (1954), 449-459.

[3] P.E. Böhmer. *Über die Transzendenz gewisser dyadischer Brüche*, Math. Ann. 96 (1927), 367-377.

[4] J. Brémont. *Dynamics of injective quasi-contractions*, Ergodic. Th. and Dyn. Syst. 26 (2006), 19-44.

[5] Y. Bugeaud. *Dynamique de certaines applications contractantes, linéaires par morceaux, sur [0,1]*, C. R. Acad. Sci. de Paris 317 Série I (1993), 575-578.

[6] Y. Bugeaud and J.-P. Conze. *Calcul de la dynamique d’une classe de transformations linéaires contractantes mod 1 et arbre de Farey*, Acta Arithmetica LXXXVIII.3 (1999), 201-218.

[7] R. Coutinho. *Dinámica simbólica linear*, Ph. D. Thesis, Instituto Superior Técnico, Universidade Técnica de Lisboa, February 1999.

[8] E. J. Ding and P. C. Hemmer. *Exact treatment of mode locking for a piecewise linear map*, Journal of Statistical Physics, 46 (1987), 99-110.

[9] O. Feely and L. O. Chua. *The effect of Integrator Leak in Σ – Δ Modulation*, IEEE Transactions on Circuits and Systems, 38 (1991), 1293-1305.

[10] J.-M. Gambaudo and C. Tresser. *On the dynamics of quasi-contractions*, Bol. Coc. Bras. Mat., 19 (1988), 61-114.

[11] A. Lasota and M. C. Mackey. *Noise and statistical periodicity*, Physica 28D (1987), 143-154.

[12] J.H. Loxton and A.J. van der Poorten. *Arithmetic properties of certain functions in several variables II*, Bull. Austral. Math. Soc., 16 (1977), 15-47.

[13] J.H. Loxton and A.J. van der Poorten. *Transcendence and algebraic independence by a method of Mahler*, in *Transcendence Theory: Advances and applications*, ed. by A. Baker and D.W. Masser, Academic Press (1977), 211-226.

[14] J. Nagumo and S. Sato. *On a response characteristic of a mathematical neuron model*, Kybernetik 10(3) (1972), 155-164.

[15] K. Nishioka. *Mahler Functions and Transcendence*, Springer Lecture Notes in Mathematics, Vol. 1631.

[16] A. Nogueira and B. Pires. *Dynamics of piecewise contractions of the interval*, Ergodic Theory and Dynamical Systems, Volume 35, Issue 7 (2015), 2198-2215.

[17] F. Rhodes and C. Thompson. *Rotation numbers for monotone functions on the circle*, J. London Math. Soc. (2) 34 (1986), 360-368.

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