On consistency of the quantum-like representation algorithm

Peter Nyman
International Center for Mathematical Modelling in Physics and Cognitive Sciences
University of Växjö, S-35195, Sweden
E-mail: peter.nyman@vxu.se

June 11, 2009

Abstract
In this paper we continue to study so called “inverse Born’s rule problem”: to construct representation of probabilistic data of any origin by a complex probability amplitude which matches Born’s rule. The corresponding algorithm – quantum-like representation algorithm (QLRA) was recently proposed by A. Khrennikov [1]–[5]. Formally QLRA depends on the order of conditioning. For two observables \( a \) and \( b \), \( b|a \)- and \( a|b \)-conditional probabilities produce two representations, say in Hilbert spaces \( H_{b|a} \) and \( H_{a|b} \). In this paper we prove that under natural assumptions these two representations are unitary equivalent. This result proves consistency QLRA.

Keywords quantum-like representation algorithm, inverse Born’s rule problem, order of conditioning, unitary equivalence of representations.

1 Introduction
During last 80 years tremendous efforts were put to clarify inter-relation between classical and quantum probabilities, see, e.g., von Neumann [6] for the first detailed presentation of this problem and see, e.g., Gudder [7]–[9], Svozil [10], [11], Fine [12], Garola et al. [13], [15], Dvurecenskij and Pulmanova [17], Ballentine [16], O. Nánásiová et al [18], [19], Allahverdyan et al [20] for modern studies. We remark that during the last 30 years the main interest was attracted by Bell’s inequality, see, e.g., for detailed presentation. However, the basic rule of QM is Born’s rule. Therefore the study of its origin is not less (and may be even more) important than investigations on Bell’s inequality. In this paper we continue to study so

\footnote{The list of references is far from to be complete, see Khrennikov’s monograph [21] for the detailed list of references.}
called “inverse Born’s rule problem” as it was formulated by Khrennikov [1]–[5]:

**IBP** (inverse Born problem): To construct representation of probabilistic data by a complex probability amplitude which matches Born’s rule.

Solution of IBP provides a possibility to represent probabilistic data by “wave functions” and operate with this data by using linear algebra (as we do in conventional QM). In a special case (for a pair of dichotomous observables) this problem was solved in [1]–[5] with the help of so called quantum-like representation algorithm – QLRA. Formally the output of QLRA depends on the order of conditioning. For two observables \( a \) and \( b \) conditional probabilities produce two representations, say in Hilbert spaces \( H_{b|a} \) and \( H_{a|b} \). In this paper we prove that under natural assumptions these two representations are unitary equivalent. This result proves consistency of QLRA.

### 2 Inversion of Born’s Rule

We consider the simplest situation. There are given two dichotomous observables of any origin: \( a = \alpha_1, \alpha_2 \) and \( b = \beta_1, \beta_2 \). We set \( X_a = \{\alpha_1, \alpha_2\} \) and \( X_b = \{\beta_1, \beta_2\} \) – “spectra of observables”.

We assume that there is given the matrix of transition probabilities \( P_{b|a} = (p_{b|a}^{\beta\alpha}) \), where \( p_{b|a}^{\beta\alpha} \equiv P(b = \beta | a = \alpha) \) is the probability to obtain the result \( b = \beta \) under the condition that the result \( a = \alpha \) has been obtained. There are also given probabilities \( p_{a}^{\alpha} \equiv P(a = \alpha), \alpha \in X_a \), and \( p_{b}^{\beta} \equiv P(b = \beta), \beta \in X_b \). Probabilistic data \( C = \{p_{a}^{\alpha}, p_{b}^{\beta}\} \) are related to some experimental context (in physics preparation procedure).

IBP is to represent this data by a probability amplitude \( \psi \) (in the simplest case it is complex valued) such that Born’s rule holds for both observables:

\[
\begin{align*}
p_{\beta} = |\langle \psi | e_{\beta}^{b|a}\rangle|^2, & \quad p_{\alpha} = |\langle \psi | e_{\alpha}^{b|a}\rangle|^2, \\
\end{align*}
\]

where \( \{e_{\beta}^{b|a}\}_{\beta \in X_b} \) and \( \{e_{\alpha}^{b|a}\}_{\alpha \in X_a} \) are orthonormal bases for observables \( b \) and \( a \), respectively (so the observables are diagonal in respective bases).

In [1]–[5] the solution of IBP is given in the form of an algorithm which constructs a probability amplitude from data. Formally, the output of this algorithm depends on the order of conditioning. By starting with the matrix of transition probabilities \( P_{a|b} \), instead of \( P_{b|a} \), we construct another probability amplitude \( \psi_{a|b} \) (the amplitude in [1]–[5] should be denoted by \( \psi_{b|a} \)) and other bases, \( \{e_{\beta}^{a|b}\}_{\beta \in X_b} \) and \( \{e_{\alpha}^{a|b}\}_{\alpha \in X_a} \). We shall see that under natural assumptions these two representations are unitary equivalent.

### 3 QLRA

#### 3.1 \( H_{b|a} \)-conditioning

Suppose that the matrix of transition probabilities \( P_{b|a} \) is given. In [1]–[5] the following formula for interference of probabilities (generalizing the
classical formula of total probability was derived: $p^b_\beta = \sum_\alpha p^a_\alpha p^{b|\alpha}_\beta + 2\lambda_\beta \sqrt{\prod_\alpha p^a_\alpha p^{b|\alpha}_\beta}$, where the “coefficient of interference”

$$\lambda_\beta = \frac{p^b_\beta - \sum_\alpha p^a_\alpha p^{b|\alpha}_\beta}{2\sqrt{\prod_\alpha p^a_\alpha p^{b|\alpha}_\beta}}.$$  \hspace{1cm} (2)

We shall proceed under the conditions:

(1) $P^{b|a}$ is doubly stochastic.

(2) Probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ consist of strictly positive probabilities.

(3) Coefficients of interference $\lambda_\beta, \beta \in X_b$, are bounded by one: $|\lambda_\beta| \leq 1$.

Probabilistic data $C$ such that (3) holds is called trigonometric, because in this case we have the conventional formula of trigonometric interference\footnote{This formula can be easily derived in the conventional QM formalism, see, e.g., [21], by transition from the basis of eigenvectors for the $a$-observable to the basis of eigenvectors for the $b$-observables. We recall that in QM observables are given by self-adjoint operators. However, we proceed in the opposite way. We would like to produce a complex probability amplitudes and operator representation of the observables by using this formula.}

$$p^b_\beta = \sum_\alpha p^a_\alpha p^{b|\alpha}_\beta + 2\cos \theta_\beta \sqrt{\prod_\alpha p^a_\alpha p^{b|\alpha}_\beta},$$  \hspace{1cm} (3)

where $p^a_\alpha p^{b|\alpha}_\beta$ is given, see [1] – [5] for details, by $e_{\alpha_1} = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), e_{\alpha_2} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$. To solve IBP completely, we would like to have Born’s rule not only for the $b$-variable, but also for the $a$-variable: $p^{b|a}_\alpha = |\langle \psi^{b|a}_\alpha | \psi^{b|a}_\beta \rangle|^2, \alpha \in X_a$. Here the $a$-basis in the Hilbert space $H^{b|a}$ is given, see [1] – [5] for details, by $e_{\alpha} = \left( \begin{array}{c} \sqrt{P^{b|a}_{\alpha_1 \alpha}} \\ \sqrt{P^{b|a}_{\alpha_2 \alpha}} \end{array} \right), e_{\alpha_1} = \left( \begin{array}{c} \sqrt{P^{b|a}_{\alpha_1 \alpha_1}} \\ \sqrt{P^{b|a}_{\alpha_2 \alpha_1}} \end{array} \right), e_{\alpha_2} = \left( \begin{array}{c} \sqrt{P^{b|a}_{\alpha_1 \alpha_2}} \\ \sqrt{P^{b|a}_{\alpha_2 \alpha_2}} \end{array} \right)$. It is orthonormal, since $P^{b|a}$ is assumed to be doubly stochastic. In this basis the amplitude $\psi^{b|a}_\alpha$ is represented as $\psi^{b|a}_\alpha = \sqrt{p^{b|a}_\alpha} e_{\alpha_1} + e^{i\theta_1} \sqrt{p^{b|a}_{\alpha_2} e_{\alpha_2} e_{\alpha_2}}$.
We recall that in QM a pure state \( \Psi \) is defined as an equivalent class with respect to multipliers of the form \( c = e^{i\gamma} \). We shall use similar terminology. Each complex amplitude \( \psi^{b|a} \) produced by QLRA determines a quantum-like state (representing given probabilistic data) – the equivalence class \( \Psi^{b|a} \) determined by the representative \( \psi^{b|a} \).

### 3.2 \( H^{a|b} \)-conditioning

Here

\[
\psi^{a|b}_\alpha = \sqrt{p^{a|b}_{\alpha \beta_1}} e^{i\phi_\alpha} \sqrt{p^{a|b}_{\alpha \beta_2}}, \quad \alpha \in X_a .
\]

(7)

For any trigonometric probabilistic data \( C \), QLRA produces the complex amplitude \( \psi^{a|b} \) (the normalized vector in the two dimensional complex Hilbert space, say \( H^{a|b} \)):

\[
\psi^{a|b} = \psi^{a|b}_{\alpha_1} e^{a|b}_{\alpha_1} + \psi^{a|b}_{\alpha_2} e^{a|b}_{\alpha_2},
\]

(8)

where \( e^{a|b}_{\alpha_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( e^{a|b}_{\alpha_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Here the \( b \)-basis in the Hilbert space \( H^{a|b} \) is given by \( e^{a|b}_{\beta_1} = \begin{pmatrix} \sqrt{p^{a|b}_{\beta_1 \alpha_1}} \\ \sqrt{p^{a|b}_{\beta_1 \alpha_2}} \end{pmatrix} \), \( e^{a|b}_{\beta_2} = \begin{pmatrix} -\sqrt{p^{a|b}_{\beta_2 \alpha_1}} \\ -\sqrt{p^{a|b}_{\beta_2 \alpha_2}} \end{pmatrix} \).

In this basis the amplitude \( \psi^{a|b} \) is represented as

\[
\psi^{a|b} = \sqrt{p^{a|b}_{\alpha_1}} e^{a|b}_{\alpha_1} + e^{i\phi_1} \sqrt{p^{a|b}_{\alpha_2}} e^{a|b}_{\alpha_2}
\]

(9)

As in the case of \( H^{b|a} \)-representation, the quantum-like state (representing given probabilistic data) is defined as the equivalence class \( \Psi^{a|b} \) with the representative \( \psi^{a|b} \).

### 4 Unitary equivalence of \( b|a \)- and \( a|b \)-representations

Thus, as we have seen by selecting to types of conditioning, we represented the probabilistic data \( C = \{p^{a|b}_\alpha, p^{b|a}_\beta\} \) by two quantum-like states, \( \Psi^{b|a} \) and \( \Psi^{a|b} \). We are interested in consistency of these representations.

We remark that any linear operator \( W : H^{b|a} \to H^{a|b} \) induces the map of equivalence classes of the unit spheres with respect to multipliers \( c = e^{i\gamma} \). We define the unitary operator \( U^{a|b}_{b|a} : H^{b|a} \to H^{a|b} \) by \( U^{a|b}_{b|a}(e^{a|b}_\alpha) = e^{a|b}_\alpha \), \( \alpha \in X_a \). It induces the mentioned map of equivalent classes.

**Theorem.** The operator \( U^{a|b}_{b|a} \) maps \( \Psi^{b|a} \) into \( \Psi^{a|b} \) if and only if the following inter-relation of symmetry takes place for matrices of transition probabilities \( P^{b|a} \) and \( P^{a|b} \):

\[
p_{\beta \alpha} = p_{\alpha \beta},
\]

(10)

for all \( \alpha \) and \( \beta \) from spectra of observables \( a \) and \( b \).

**Proof.** Take the representative of \( \Psi^{b|a} \) given by (8). Then
\[ C_{\alpha\beta}^{a|b} = \sqrt{P_{a|b}^{\alpha|\beta} e^{i\theta_{\alpha|\beta}}} \]

Our aim is to show that this vector is equivalent to the vector \( \psi^{a|b} \) given by (10). By using \( H^{a|b} \) analogs of (2) and (3) for the coefficients of interference and its cos-expression we determine \( \cos \theta_{\alpha|\beta} \):

\[ p_{\alpha|\beta} = p_{\beta|\alpha} - p_{\beta|\alpha} + p_{\beta|\alpha} \cos \theta_{\alpha|\beta} \]

By using (2) and (3) we obtain \( \cos \theta_{\alpha|\beta} \):

\[ \cos \theta_{\alpha|\beta} = \frac{p_{\alpha|\beta} - p_{\beta|\alpha} + p_{\beta|\alpha} \cos \theta_{\alpha|\beta}}{2 \sqrt{p_{\alpha|\beta} p_{\beta|\alpha}}} \]

We also calculate

\[ \psi_{\alpha|\beta} \psi_{\alpha|\beta} = \sqrt{P_{\alpha|\beta}} \left( P_{\alpha|\beta} - P_{\beta|\alpha} + P_{\beta|\alpha} \cos \theta_{\alpha|\beta} \right) \]

where \( \psi_{\alpha|\beta} = \sqrt{P_{\alpha|\beta} p_{\beta|\alpha}} - e^{i\theta_{\alpha|\beta}} \sqrt{P_{\beta|\alpha} p_{\beta|\alpha}} \) is given by (10). We use that \( |\psi_{\alpha|\beta}|^2 = p_{\alpha|\beta} \Leftrightarrow \psi_{\alpha|\beta} = \sqrt{p_{\alpha|\beta}} (\cos \gamma_{\alpha|\beta} + i \sin \gamma_{\alpha|\beta}) \) where \( \gamma_{\alpha|\beta} = \arg \psi_{\alpha|\beta} \), \( j \in \{1, 2\} \) and this gives that

\[ \psi_{\alpha|\beta} \psi_{\alpha|\beta} = \sqrt{p_{\alpha|\beta} p_{\beta|\alpha}} (\cos \gamma_{\alpha} - \gamma_{\alpha} + i \sin \gamma_{\alpha}) \]

The real part of the equations (13) and (14) gives

\[ (\gamma_{\alpha} - \gamma_{\alpha}) = p_{\beta|\alpha} \sqrt{P_{\alpha|\beta} p_{\beta|\alpha}} - p_{\beta|\alpha} \sqrt{P_{\beta|\alpha} p_{\beta|\alpha}} \]

Moreover, since \( p_{\beta|\alpha} = 1 - p_{\alpha|\beta} \) and from the condition that \( P^{a|b} \) is double stochastic i.e. \( p_{\alpha|\beta} = p_{\beta|\alpha} = 1 - p_{\beta|\alpha} = 1 - p_{\beta|\alpha} \), we rewrite (15)

\[ \sqrt{p_{\alpha|\beta} p_{\beta|\alpha}} \cos (\gamma_{\alpha} - \gamma_{\alpha}) = \left( 2p_{\beta|\alpha} - 1 \right) \sqrt{P_{\alpha|\beta} (1 - P_{\alpha|\beta})} \]

Then by (2) and (3) we obtain \( \cos \theta_{\beta|\alpha} \):

\[ \cos \theta_{\beta|\alpha} = \frac{p_{\beta|\alpha} - P_{\alpha|\beta} P_{\beta|\alpha} - P_{\beta|\alpha} P_{\beta|\alpha}}{2 \sqrt{P_{\alpha|\beta} P_{\beta|\alpha} P_{\beta|\alpha} P_{\beta|\alpha}}} \]

Multiply (17) with \( 2 \sqrt{p_{\alpha|\beta} p_{\beta|\alpha}} \) and use again that \( p_{\beta|\alpha} = 1 - p_{\alpha|\beta} \) and \( P^{a|b} \) is double stochastic and

\[ 2 \sqrt{p_{\alpha|\beta} p_{\beta|\alpha}} \cos \theta_{\beta|\alpha} = \frac{p_{\alpha|\beta} - 1 + p_{\beta|\alpha} + P_{\alpha|\beta} P_{\beta|\alpha} - 2 p_{\beta|\alpha} P_{\beta|\alpha}}{\sqrt{P_{\alpha|\beta} P_{\beta|\alpha} P_{\beta|\alpha} P_{\beta|\alpha}}} \]
We will show that \( \cos (\gamma_{a_2} - \gamma_{a_1}) = \cos \theta_{b_1} \) or equivalent, we show that
\[
2 \sqrt{p_{a_1}^a p_{a_2}^a} \cos (\gamma_{a_2} - \gamma_{a_1}) = 2 \sqrt{p_{a_1}^a p_{a_2}^a} \cos \theta_{b_1}.
\] (19)
Multiply \( \sqrt{p_{a_1}^a p_{a_2}^a} \cos (\gamma_{a_2} - \gamma_{a_1}) \) by \( 2 \sqrt{p_{a_1}^{a|b} (1 - p_{a_1}^{a|b})} \) in the left-hand side (19) such that \( LHS = 2 \sqrt{p_{a_1}^{a|b} (1 - p_{a_1}^{a|b})} \sqrt{p_{a_1}^a p_{a_2}^a} \cos (\gamma_{a_2} - \gamma_{a_1}) \) and replace \( \cos \theta_{b_1} \) with \( \frac{p_{a_1}^{a|b} - p_{a_1}^{a|b} p_{a_1}^{a|b} - (1 - p_{a_1}^{a|b})(1 - p_{a_1}^{a|b})}{2 \sqrt{p_{a_1}^{a|b} p_{a_1}^{a|b} p_{a_2}^{a|b}}} \) in right-hand side
\[
LHS = 2 \left( 2p_{b_1}^b - 1 \right) p_{a_1}^{a|b} (1 - p_{a_1}^{a|b})
+ \left( p_{a_1}^a - p_{b_1}^{b|a} p_{a_1}^{a|b} - (1 - p_{b_1}^b)(1 - p_{a_1}^{a|b}) \right) \left( 1 - 2p_{a_1}^{a|b} \right).
\] (20)
From equation (15) and (20) must.
\[
\frac{p_{a_1}^a - 1 + p_{b_1}^b + p_{a_1}^{a|b} - 2p_{a_1}^{a|b} p_{a_1}^a}{\sqrt{p_{a_1}^{a|b} p_{b_1}^{b|a} p_{a_2}^{a|b}}} = \frac{p_{a_1}^a - 1 + p_{b_1}^b + p_{a_1}^{a|b} - 2p_{a_1}^{a|b} p_{a_1}^a}{\sqrt{p_{a_1}^{a|b} p_{b_1}^{b|a} p_{a_2}^{a|b}}}
\iff
p_{a_1}^{a|b} = p_{a_1}^{a|b}.
\]
Therefore must \( \cos (\gamma_{a_2} - \gamma_{a_1}) = \cos \theta_{b_1} \) iff \( P^{b|a} = P^{a|b} \). Let
\[
U^{a|b}_{b|a} = \left( \frac{\sqrt{p_{b_1}^{b|a} \sqrt{p_{b_1}^{b|a} \sqrt{p_{b_2}^{b|a}}}}}{\sqrt{p_{b_2}^{b|a}}} \right).
\] (22)
Then let us show that this vector is equivalent to the vector \( \psi^{a|b} \) given by (5).
\[
U^{a|b}_{b|a} = \sqrt{p_{a_1}^{a|b} \epsilon_{a_1}^{a|b} + e^{\theta_{b_1}} \sqrt{p_{a_2}^{a|b} \epsilon_{a_2}^{a|b}}}
= \sqrt{p_{a_1}^{a|b} \epsilon_{a_1}^{a|b} + e^{(\gamma_{a_2} - \gamma_{a_1})} \sqrt{p_{a_2}^{a|b} \epsilon_{a_2}^{a|b}}}
\] (23)
Then put $\psi_{a_j}^{p|b} = \sqrt{p_{a_j}} e^{i(\gamma_{a_j})_j}$, $j \in \{1, 2\}$ into (24)

$$
\psi_{a_j}^{p|b} = \sqrt{p_{a_1}} e^{i(\gamma_{a_1})} e_{a_1}^{a|b} + \sqrt{p_{a_2}} e^{i(\gamma_{a_2})} e_{a_2}^{a|b}
$$

The complex amplitudes $\psi_{a_j}^{a|b}$ and $U_{b|a}^{a|b} \psi_{b|a}^{b|a}$ differs only by the multiplicative factor $e^{i(\gamma_{a_1})}$. Hence, they belong to the same equivalent class of vectors on the unit sphere. Thus they are two representatives of the same quantum state $\psi_{b|a}^{b|a}$.

**Acknowledgements**

I am grateful to my supervisor Professor Andrei Khrennikov for many discussions on the formulations of quantum mechanics and for introducing me into this field of research. I am also very thankful to Guillaume Adenier for all the discussions we have had about the subject of quantum mechanics.

**References**

[1] Khrennikov, A. Yu.: The principle of supplementarity: A contextual probabilistic viewpoint to complementarity, the interference of probabilities, and the incompatibility of variables in quantum mechanics. *Found. Phys.*, 35, N. 10, 1655 - 1693 (2005).

[2] Khrennikov, A. Yu.: Interference in the classical probabilistic model and its representation in complex Hilbert space. Physica, E 29, 226-236 (2005).

[3] Khrennikov, A. Yu.: Linear and nonlinear analogues of the Schrödinger equation in the contextual approach to quantum mechanics. Dokl. Akad. Nauk 404, N. 1, 33-36 (2005); Doklady Mathematics, 72, N. 2, 791-794 (2005).

[4] Khrennikov, A. Yu.: Representation of the contextual statistical model by hyperbolic amplitudes, J. Math. Phys., 46, N. 6, 062111–062124 (2005).

[5] Khrennikov, A. Yu.: Schrödinger dynamics as the Hilbert space projection of a realistic contextual probabilistic dynamics. Europhys. Lett., 69 (5), 678-684 (2005).

[6] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press, Princeton, N.J., 1955.

[7] Gudder, S. P.: Special methods for a generalized probability theory. Trans. AMS 119, 428 (1965).

[8] Gudder, S. P.: Axiomatic quantum mechanics and generalized probability theory (Academic Press, New York, 1970).

[9] Gudder, S. P.: “An approach to quantum probability,” Quantum Prob. White Noise Anal. 13, 147 (2001).

[10] Svozil,K.: Quantum logic, Springer, Berlin, 1998.
[11] Svozil, K.: Randomness and undecidability in physics, World Scientific, Singapore, 1993.

[12] Fine, A.: Hidden variables, joint probabilities, and Bell inequalities, Phys. Rev. Lett., vol. 48, pp. 291–295, 1982.

[13] Garola, C. and Solombrino, L.: The theoretical apparatus of Semantic Realism: a new language for classical and quantum physics. Found. Phys. 26, 1121 (1996).

[14] Garola, C. and Solombrino, L.: Semantic Realism versus EPR-like paradoxes: the Furry, Bohm-Aharonov and Bell paradoxes. Found. Phys. 26, 1329 (1996).

[15] Garola, C.: A simple model for an objective interpretation of quantum mechanics. Found. Phys. 32, 1597 (1996).

[16] Ballentine, L. E.: Interpretations of probability and quantum theory, Q. Prob. White Noise Anal. 13, 71 (2001).

[17] Dvurecenskij A. and Pulmanova, O.: New trends in quantum structures. Kluwer Academic Publ., Dordrecht, 2000.

[18] Nánásiová, O.: Map for simultaneous measurements for a quantum logic. International Journal of Theoretical Physics 42, 1889-1903 (2003).

[19] Nánásiová, O. and Khrennikov, A. Yu.: Representation theorem of observables on a quantum system. International Journal of Theoretical Physics 45, 469 - 482 (2006).

[20] Allahverdyan, A., Khrennikov, A. Yu. and Nieuwenhuizen, Th. M.: Brownian entanglement, Phys. Rev. A, 71, 032102-1 – 032102-14 (2005).

[21] Khrennikov, A. Yu.: Interpretations of Probability. VSP International Science Publishers, Utrecht (1999); second addition (completed) De Gruyter, Berlin (2009).