The Complement of Polyhedral Product Spaces and the Dual Simplicial Complexes

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Abstract
In this paper, we define and prove basic properties of complement polyhedral product spaces, dual complexes and polyhedral join complexes. Then we compute the universal algebra of polyhedral join complexes under certain split conditions and the Alexander duality isomorphism on certain polyhedral product spaces.

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1 Introduction

The polyhedral product theory, especially the homotopy type of polyhedral product spaces, is developing rapidly nowadays. The first known polyhedral product space was the moment-angle complex introduced by Buchstaber and Panov [8] and was widely studied by mathematicians in the area of toric topology and geometry (see [1], [9], [12], [14], [15]). Later on, the homotopy types of polyhedral product spaces were studied by Grbić and Theriault [13], [14], [15], Beben and Grbić [7], Bahri, Bendersky, Cohen and Gitler [3], [4], [5] and many others ([6], [10], [11]). The cohomology ring of homology split polyhedral product spaces and the cohomology algebra over a field of polyhedral product spaces were computed in [17].

For a polyhedral product space $Z(K; X, A)$, is the complement space $M^c = (X_1 \times \cdots \times X_m) \setminus Z(K; X, A)$ a polyhedral product space? In Theorem 2.4, we show that $M^c = Z(K^\circ; X, A^c)$, where $K^\circ$ is the dual complex of $K$ relative to $[m]$ and $A^c = X \setminus A_k$. The moment-angle complex case of this theorem is obtained by Grujić and Welker [16].

Let $Z(K; Y, B)$, $(Y, B) = \{(Y_k, B_k)\}_{k=1}^m$ be the polyhedral product space defined as follows. For each $k$, $(Y_k, B_k)$ is a pair of polyhedral product spaces given by $(s_k = n_1 + \cdots + n_k)$

$$Y_k = Z(X_k; U_k, C_k), \quad B_k = Z(A_k; U_k, C_k), \quad (U_k, C_k) = \{(U_i, C_i)\}_{i=s_{k-1}+1}^{s_k},$$

where $(X_k, A_k)$ is a simplicial pair on $[n_k]$. In Theorem 2.9, we prove that $Z(K; Y, B)$ is also a polyhedral product space $Z(S(K; X, A); U, C)$, where $S(K; X, A)$ is defined in Definition 2.7. This theorem is a trivial extension of Ayzenberg’s Proposition 5.1 of [2], where $S(K; X, A)$ is denoted by $S^*_K(X, A)$ and given the name polyhedral join complex. When all $(X_k, A_k) = (\Delta^{[n_k]}, K_k)$, the complex $S(K; X, A)$ is denoted by $S(K; K_1, \cdots, K_m)$ which
is just the composition complex $K(K_1, \cdots, K_m)$ in Definition 4.5 of [2].

In Section 3, we compute the (co)homology group of polyhedral product inclusion complex $C_\ast(K; \emptyset)$ (defined in Definition 3.6) in Theorem 3.7. As an application, the reduced (co)homology group and the (right) total (co)homology group of polyhedral join complexes is computed in Theorem 3.9 and Theorem 3.11. In Example 3.10, we show that $S(K; L_1, \cdots, L_m)$ is a homological sphere if and only if $K$ is a homological sphere and each $L_k = \partial \Delta^{[m]}$. This result is a part of Ayzenberg’s Theorem 6.6 of [2], where homological can be replaced by simplicial.

In section 4, we compute the cohomology algebra of polyhedral product inclusion complex $C_\ast(K; \emptyset)$ in Theorem 4.7. As an application, we compute the cohomology algebra of $Z(K; Y, B)$ mentioned above and the (right) universal (normal, etc.) algebra of $S(K; X, A)$ ($S(K; L_1, \cdots, L_m)$) in Example 4.8 (4.9). In Theorem 5.6, we compute the Alexander duality isomorphism on the pair $(X_1 \times \cdots \times X_m, Z(K; X, A))$, where all $X_k$’s are orientable manifolds and all $A_k$’s are polyhedra.

2 Complement Spaces, Dual Complexes and Polyhedral Join Complexes

Conventions and Notations For a finite set $S$, $\Delta^S$ is the simplicial complex with only one maximal simplex $S$, i.e., it is the set of all subsets of $S$ including the empty set $\emptyset$. Define $\partial \Delta^S = \Delta^S \setminus \{S\}$. For $[m] = \{1, \cdots, m\}$, $\partial \Delta^{[m]} = \Delta^{[m]} \setminus \{m\}$. Specifically, define $\Delta^\emptyset = \{\emptyset\}$ and $\partial \Delta^\emptyset = \{\emptyset\}$. The void complex $\{\}$ with no simplex is inevitable in this paper.

For a simplicial complex $K$ on $[m]$ (ghost vertex $\{i\} \notin K$ is allowed) and $\sigma \subset [m]$ ($\sigma \notin K$ is allowed), the link of $\sigma$ with respect to $K$ is the simplicial
complex \( \text{link}_K \sigma = \{ \tau \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset \} \). This implies \( \text{link}_K \sigma = \{ \emptyset \} \) if \( \sigma \) is a maximal simplex of \( K \) and \( \text{link}_K \sigma = \{ \} \) if \( \sigma \notin K \). Specifically, if \( K = \{ \} \), then \( \text{link}_K \sigma = \{ \} \) for all \( \sigma \).

**Definition 2.1** For a simplicial complex \( K \) on \([m]\) and a sequence of topological (not CW-complex!) pairs \((X, A) = \{(X_k, A_k)\}_{k=1}^m\), the polyhedral product space \( \mathcal{Z}(K; X, A) \) is the subspace of \( X_1 \times \cdots \times X_m \) defined as follows. For a subset \( \tau \) of \([m]\), define
\[
D(\tau) = Y_1 \times \cdots \times Y_m, \quad Y_k = \begin{cases} X_k & \text{if } k \in \tau, \\ A_k & \text{if } k \notin \tau. \end{cases}
\]
Then \( \mathcal{Z}(K; X, A) = \bigcup_{\tau \in K} D(\tau) \). Empty space \( \emptyset \) is allowed in a topological pair and \( \emptyset \times X = \emptyset \) for all \( X \). Define \( \mathcal{Z}(\{ \} ; X, A) = \emptyset \).

Notice that \( D(\sigma) = \mathcal{Z}(\Delta^\sigma; X, A), D(\emptyset) = A_1 \times \cdots \times A_m = \mathcal{Z}(\{ \emptyset \}; X, A) \) and \( D([m]) = X_1 \times \cdots \times X_m = \mathcal{Z}(\Delta^{|m|}; X, A) \). But \( \emptyset = \mathcal{Z}(\{ \} ; X, A) \) has no corresponding \( D(-) \).

**Example 2.2** For \( \mathcal{Z}(K; X, A) \), let \( S = \{ k \mid A_k = \emptyset \} \). Then
\[
\mathcal{Z}(K; X, A) = \mathcal{Z}(\text{link}_K S; X', A') \times (\Pi_{k \in S} X_k),
\]
where \( (X', A') = \{(X_k, A_k)\}_{k \notin S} \) and link is as defined in conventions.

**Definition 2.3** Let \( K \) be a simplicial complex with vertex set a subset of \( S \neq \emptyset \). The dual of \( K \) relative to \( S \) is the simplicial complex
\[
K^\circ = \{ S \setminus \sigma \mid \sigma \subset S, \sigma \notin K \}.
\]
It is obvious that \( (K^\circ)^\circ = K \), \( (K_1 \cup K_2)^\circ = (K_1)^\circ \cap (K_2)^\circ \) and \( (K_1 \cap K_2)^\circ = (K_1)^\circ \cup (K_2)^\circ \). Specifically, \( (\Delta^S)^\circ = \{ \} \) and \( (\partial \Delta^S)^\circ = \{ \emptyset \} \).

**Theorem 2.4** For \( \mathcal{Z}(K; X, A) \), the complement space
\[
\mathcal{Z}(K; X, A)^c = (X_1 \times \cdots \times X_m) \setminus \mathcal{Z}(K; X, A) = \mathcal{Z}(K^\circ; X, A^c),
\]

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where \((X, A^c) = \{(X_k, A_k^c)\}_{k=1}^m\) with \(A_k^c = X_k \setminus A_k\) and \(K^o\) is the dual of \(K\) relative to \([m]\).

**Proof** For \(\sigma \subset [m]\) but \(\sigma \neq [m]\) (\(\sigma = \emptyset\) is allowed),

\[
\begin{align*}
(X_1 \times \cdots \times X_m) \setminus D(\sigma) \quad \text{(with space pair } (X_k, A_k)) \\
= \bigcup_{j \notin \sigma} X_1 \times \cdots \times (X_j \setminus A_j) \times \cdots \times X_m \\
= \bigcup_{j \in [m] \setminus \sigma} D([m] \setminus \{j\}) \quad \text{(with space pair } (X_k, A_k^c)) \\
= \mathcal{Z}((\Delta^o)^c; X, A^c)
\end{align*}
\]

So for \(K \neq \Delta^m\) or \(\{\}\), the above equality holds naturally. \(\square\)

**Example 2.5** Let \(\mathbb{F}\) be a field and \(V\) be a linear space over \(\mathbb{F}\) with base \(e_1, \cdots, e_m\). For a subset \(\sigma = \{i_1, \cdots, i_s\} \subset [m]\), denote by \(\mathbb{F}(\sigma)\) the subspace of \(V\) with base \(e_{i_1}, \cdots, e_{i_s}\). Then for \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\) and a simplicial complex \(K\) on \([m]\), we have

\[
V \setminus (\bigcup_{\sigma \in K} \mathbb{R}(\sigma)) = \mathbb{R}^m \setminus \mathcal{Z}(K; \mathbb{R}, \{0\}) = \mathcal{Z}(K^o; \mathbb{R}, \mathbb{R} \setminus \{0\}) \simeq \mathcal{Z}(K^o; D_1, S^0),
\]

\[
V \setminus (\bigcup_{\sigma \in K} \mathbb{C}(\sigma)) = \mathbb{C}^m \setminus \mathcal{Z}(K; \mathbb{C}, \{0\}) = \mathcal{Z}(K^o; \mathbb{C}, \mathbb{C} \setminus \{0\}) \simeq \mathcal{Z}(K^o; D_2, S^1).
\]

This example is in Lemma 2.4 of [16].

**Theorem 2.6** Let \(K\) and \(K^o\) be the dual of each other relative to \([m]\). The index set \(\mathcal{X}_m = \{ (\sigma, \omega) \mid \sigma, \omega \subset [m], \sigma \cap \omega = \emptyset \}\). For \((\sigma, \omega) \in \mathcal{X}_m\), define
simplicial complex $K_{\sigma,\omega} = \text{link}_K \sigma|_\omega = \{ \tau \subset \omega \mid \sigma \cup \tau \in K \}$ (so $K_{\sigma,\omega} = \{ \}$ if $\sigma \notin K$ or $K = \{ \}$). Then for any $(\sigma, \omega) \in \mathcal{P}_m$ such that $\omega \neq \emptyset$,

$$(K_{\sigma,\omega})^\circ = (K^\circ)_{\tilde{\sigma},\omega}, \quad \tilde{\sigma} = [m]\setminus (\sigma \cup \omega),$$

where $(K_{\sigma,\omega})^\circ$ is the dual of $K_{\sigma,\omega}$ relative to $\omega$.

**Proof** Suppose $\sigma \in K$. Then

$$(K^\circ)_{\tilde{\sigma},\omega} = \{ \eta \mid \eta \subset \omega, [m]\setminus (\tilde{\sigma} \cup \eta) = \sigma \cup (\omega \setminus \eta) \notin K \}$$

$$= \{ \omega \setminus \tau \mid \tau \subset \omega, \sigma \cup \tau \notin K \}$$

$$= (K_{\sigma,\omega})^\circ.$$

If $\sigma \notin K$, then $(K^\circ)_{\tilde{\sigma},\omega} = \Delta^\omega = (K_{\sigma,\omega})^\circ$. □

A sequence of simplicial pairs $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ in this paper means that the vertex set of $X_k$ is a subset of $[n_k]$ ($n_k > 0$) which is the subset

$$\{s_{k-1}+1, s_{k-1}+2, \cdots, s_{k-1}+n_k\} \quad (s_k = n_1+\cdots+n_k, \ s_0 = 0)$$

of $[n]$ with $n = n_1+\cdots+n_m$.

For simplicial complexes $Y_1, \cdots, Y_m$ such that the vertex set of $Y_k$ is a subset of $[n_k]$, the union simplicial complex is

$$Y_1 \ast \cdots \ast Y_m = \{ \sigma \subset [n] \mid \sigma \cap [n_k] \in Y_k \text{ for } k=1,\cdots,m \}.$$

**Definition 2.7** Let $K$ be a simplicial complex on $[m]$ and $(X, A)$ be as above. The *polyhedral join complex* $S(K; X, A)$ is the simplicial complex on $[n]$ defined as follows. For a subset $\tau \subset [m]$, define

$$S(\tau) = Y_1 \ast \cdots \ast Y_m, \quad Y_k = \left\{ \begin{array}{ll} X_k & \text{if } k \in \tau, \\ A_k & \text{if } k \notin \tau. \end{array} \right.$$ 

Then $S(K; X, A) = \bigcup_{\tau \in K} S(\tau)$. Void complex $\{ \}$ is allowed in a simplicial pair and $\{ \} \ast X = \{ \}$ for all $X$. Define $Z(\{ \}; X, A) = \{ \}$. 

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The name polyhedral join comes from Definition 4.2 of [2].

**Example 2.8** For $S(K; X, A)$, let $S(K; X, A) = \{ k | A_k = \{ \} \}$. Then

$$S(K; X, A) = S(\text{link}_K S; X', A') \ast (\ast_{k \in S} X_k),$$

where $(X', A') = \{(X_k, A_k)\}_{k \in S}$.

**Theorem 2.9** Let $Z(K; Y, B)$, $(Y, B) = \{(Y_k, B_k)\}_{k=1}^m$ be the polyhedral product space defined as follows. For each $k$, $(Y_k, B_k)$ is a pair of polyhedral product spaces given by

$$(Y_k, B_k) = (Z(X_k; U_k, C_k), Z(A_k; U_k, C_k)), \quad (U_k, C_k) = \{(U_i, C_i)\}_{i=s_k-1+1}^{s_k},$$

where $(X_k, A_k)$ is a simplicial pair on $[n_k]$. Then

$$Z(K; Y, B) = Z(S(K; X, A); U, C),$$

where $(U, C) = \{(U_k, C_k)\}_{k=1}^n$, $n = n_1 + \cdots + n_m$.

**Proof** If $K = \{ \}$ or $U_k = \emptyset$ for some $k$ (this implies $Y_l = \emptyset$ for some $l$), then $Z(K; Y, B) = Z(S(K; X, A); U, C) = \emptyset$. So we suppose $K \neq \{ \}$ and $U_k \neq \emptyset$ for all $k$ in the remaining part of proof.

We first prove the case $C_k \neq \emptyset$ for all $k$. If $X_k = \{ \}$ for some $k$, then $Y_k = \emptyset$ and $S(K; X, A) = \{ \}$. So $Z(K; Y, B) = Z(S(K; X, A); U, C) = \emptyset$. Suppose $X_k \neq \{ \}$ for all $k$. Let $S = \{ k | A_k = \{ \} \} = \{ k | B_k = \emptyset \}$. If $S \not\subseteq K$, then $\text{link}_K S = \{ \}$. From Example 2.2 and Example 2.8 we have $Z(K; Y, B) = Z(S(K; X, A); U, C) = \emptyset$. Suppose $S \in K$. Let $Z_k^\tau = Y_k$ if $k \in \tau$ and $Z_k^\tau = B_k$ if $k \not\in \tau$. For $\tau_k \subset [n_k]$, $W_t^\tau_k = U_k$ if $t \in \tau_k$ and $W_t^\tau_k = C_k$ for $t \not\in \tau_k$. 


if \( k \in [n_k] \setminus \tau_k \). Then

\[
\mathcal{Z}(K; Y, B) = \bigcup_{\tau \in K} Z_{1}^\tau \times \cdots \times Z_{m}^\tau
\]

\[
= \bigcup_{\tau_1, \ldots, \tau_m \in S(K; X, A)} (W_{1}^{\tau_1} \times \cdots \times W_{n_1}^{\tau_1}) \times \cdots \times (W_{s_{m-1}+1}^{\tau_m} \times \cdots \times W_{n}^{\tau_m})
\]

\[
= \mathcal{Z}(S(K; X, A); U, C)
\]

where \( \tau, \tau_1, \ldots, \tau_m \) are taken over all subsets such that \( \tau \in K \), \( \tau_k \in X_k \) if \( k \in \tau \) and \( \tau_k \in A_k \) if \( k \notin \tau \).

Now we prove the case \( \sigma = \{k \mid C_k = \emptyset\} \neq \emptyset \). Let \( \sigma_k = \sigma \cap [n_k] \). Then from Example 2.2 we have

\[
Y_k = Y'_k \times (\times k \in \sigma_k U_k), \quad Y'_k = \mathcal{Z}(\text{link}_{X_k} \sigma_k; U'_k, C'_k),
\]

\[
B_k = B'_k \times (\times k \in \sigma_k U_k), \quad B'_k = \mathcal{Z}(\text{link}_{A_k} \sigma_k; U'_k, C'_k),
\]

\[
\mathcal{Z}(K; Y, B) = \mathcal{Z}(K; Y', B') \times (\times k \in \sigma U_k),
\]

where \((U'_k, C'_k) = \{(U_k, C_k)\}_{k \in [n_k] \setminus \sigma_k}\) and \((Y'_k, B'_k) = \{(Y'_k, B'_k)\}_{k \notin \sigma}\). Then

\[
\mathcal{Z}(K; Y, B)
\]

\[
= \mathcal{Z}(K; Y', B') \times (\times k \in \sigma U_k)
\]

\[
= \mathcal{Z}(S(K; \text{link}_{(X, A)} \sigma); U', C') \times (\times k \in \sigma U_k)
\]

\[
= \mathcal{Z}(\text{link}_{S(K; (X, A)} \sigma; U', C') \times (\times k \in \sigma U_k) \quad \text{(by Theorem 2.10)}
\]

\[
= \mathcal{Z}(S(K; X, A); U, C),
\]

where \((U', C') = \{(U_k, C_k)\}_{k \notin \sigma}\). \(\Box\)

The above theorem is a trivial extension of Proposition 5.1 of [2]. With this theorem we see that to compute the cohomology algebra of \( \mathcal{Z}(K; Y, B) \), we have to compute the universal algebra of \( S(K; X, A) \), which is the central work of this paper.
Theorem 2.10 For the $S(K; X, A)$ in Definition 2.7 and $(\sigma, \omega) \in \mathcal{X}_n$ (the simplicial complex $(-)_{\sigma, \omega}$ is as defined in Theorem 2.6),

$$S(K; X, A)_{\sigma, \omega} = S(K; X_{\sigma, \omega}, A_{\sigma, \omega}),$$

where $(X_{\sigma, \omega}, A_{\sigma, \omega}) = \{(X_k)_{\sigma_k, \omega_k}, (A_k)_{\sigma_k, \omega_k}\}_{k=1}^m$, $\sigma_k = \sigma \cap [n_k]$, $\omega_k = \omega \cap [n_k]$.

For $\sigma \subset [n]$, $\sigma_k = \sigma \cap [n_k]$ (the links as defined in conventions),

$$\text{link}_{S(K; X, A)} \sigma = S(K; \text{link}_{X, A} \sigma),$$

where $\text{link}_{X, A} \sigma = \{(\text{link}_{X, k} \sigma_k, \text{link}_{A, k} \sigma_k)\}_{k=1}^m$. Precisely, if there is some $\sigma_k \notin X_k$, then $\sigma \notin S(K; X, A)$ and so $\text{link}_{S(K; X, A)} \sigma = \{\}$. Suppose $\sigma_k \in X_k$ for all $k$. Let $\hat{\sigma} = \{k \mid \sigma_k \notin A_k\}$, $\sigma' = \cup_{k \notin \hat{\sigma}} \sigma_k$, $n' = \Sigma_{k \in \hat{\sigma}} (n_k - |\sigma_k|)$. Then

$$\text{link}_{S(K; X, A)} \sigma = S(K; \text{link}_{X, A} \sigma' \cup \Delta^{n'}, \sigma) ,$$

where $\text{link}_{X, A} \sigma = \{(\text{link}_{X, k} \sigma_k, \text{link}_{A, k} \sigma_k)\}_{k \notin \hat{\sigma}}$.

Proof Let $Y_k^\tau = X_k$ if $k \in \tau$ and $Y_k^\tau = A_k$ if $k \notin \tau$ (\{\} is allowed). Then

$$S(K; X, A)_{\sigma, \omega} = \cup_{\tau \in K} (Y_1^\tau * \cdots * Y_m^\tau)_{\sigma, \omega}.$$

For a simplicial complex $L$ on $[l]$ and $\sigma \subset [l]$, $\text{link}_L \sigma = L_{[\sigma],[l]\setminus\sigma}$. So the second equality of the theorem is the special case of the first equality for $\omega = [n] \setminus \sigma$. The third equality holds by Example 2.8.

The dual of $S(K; X, A)$ relative to $[n]$ is in general not a polyhedral join complex. But the dual of the following type is.

Definition 2.11 The composition complex $S(K; L_1, \cdots, L_m)$ is the polyhedral join complex $S(K; X, A)$ such that each $(X_k, A_k) = (\Delta^{[n_k]}, L_k)$. 9
The name composition complex comes from Definition 4.5 of [2].

**Theorem 2.12** Let $S(K; L_1, \cdots, L_m)^\circ$ be the dual of $S(K; L_1, \cdots, L_m)$ relative to $[n]$. Then

$$S(K; L_1, \cdots, L_m)^\circ = S(K^\circ; L_1^\circ, \cdots, L_m^\circ),$$

where $K^\circ$ is the dual of $K$ relative to $[m]$ and $L_k^\circ$ is the dual of $L_k$ relative to $[n_k]$. So if $K$ and all $L_k$ are self dual ($X \cong X^\circ$ relative to its non-empty vertex set), then $S(K; L_1, \cdots, L_m)$ is self dual.

**Proof** For $\sigma \subset [m]$ but $\sigma \neq [m]$ ($\sigma = \emptyset$, $L_k = \Delta^{[n_k]}$ or $\{ \}$ are allowed),

$$S(\Delta^\sigma; L_1, \cdots, L_m)^\circ$$

$$= \{ [n] \setminus \tau \mid \tau \in \bigcup_{j \notin \sigma} \Delta^{n_1} \ast \cdots \ast (\Delta^{n_j} \setminus L_j) \ast \cdots \ast \Delta^{n_m} \}$$

$$= \bigcup_{j \notin \sigma} \Delta^{n_1} \ast \cdots \ast L_j^\circ \ast \cdots \ast \Delta^{n_m}$$

$$= S((\Delta^\sigma)^\circ; L_1^\circ, \cdots, L_m^\circ),$$

So for $K \neq [m]$ or $\{ \}$,

$$S(K; L_1, \cdots, L_m)^\circ$$

$$= (\bigcup_{\sigma \in K} S((\Delta^\sigma); L_1, \cdots, L_m))^\circ$$

$$= S(\bigcap_{\sigma \in K} (\Delta^\sigma)^\circ; L_1^\circ, \cdots, L_m^\circ)$$

$$= S(K^\circ; L_1^\circ, \cdots, L_m^\circ).$$

For $K = \Delta^{[m]}$ or $\{ \}$, the equality holds naturally. \qed

### 3 Homology and Cohomology Group

This is a paper following [17]. All the basic definitions such as indexed groups and (co)chain complexes, diagonal tensor product, etc., are as in [17].

In this section, we prove that the (co)homology group of the polyhedral product inclusion complex $C_*(K; \underline{U})$ is the diagonal tensor product of
the total (co)homology group of $K$ and the character group of the induced homomorphism $\theta^* (\theta^o)$ in Theorem 3.7.

**Conventions** In this paper, a group $A^\Lambda = \bigoplus_{\alpha \in \Lambda} A^\alpha$ indexed by $\Lambda$ is simply denoted by $A_*$ when there is no confusion. So is the (co)chain complex case. The diagonal tensor product $A^\Lambda \otimes B^\Lambda$ in [17] is simply denoted by $A^\Lambda \hat{\otimes} B^\Lambda$ in this paper (the index set $\Lambda$ can not be omitted in this case).

**Definition 3.1** Let $A_* = \bigoplus_{\alpha \in \Lambda} A^\alpha$, $B_* = \bigoplus_{\alpha \in \Lambda} B^\alpha$ be two groups indexed by the same set $\Lambda$. An *indexed group homomorphism* $f : A_* \to B_*$ is the direct sum $f = \bigoplus_{\alpha \in \Lambda} f_\alpha$ such that each $f_\alpha : A^\alpha \to B^\alpha$ is a graded group homomorphism. Define groups indexed by $\Lambda$ as follows.

$$\ker f = \bigoplus_{\alpha \in \Lambda} \ker f_\alpha, \quad \coker f = \bigoplus_{\alpha \in \Lambda} \coker f_\alpha,$$

$$\im f = \bigoplus_{\alpha \in \Lambda} \im f_\alpha, \quad \coim f = \bigoplus_{\alpha \in \Lambda} \coim f_\alpha.$$

For indexed group homomorphism $f = \bigoplus_{\alpha \in \Lambda} f_\alpha$ and $g = \bigoplus_{\beta \in \Gamma} g_\beta$, their tensor product $f \otimes g$ is naturally an indexed group homomorphism with $f \otimes g = \bigoplus_{(\alpha, \beta) \in \Lambda \times \Gamma} f_\alpha \otimes g_\beta$.

For indexed group homomorphism $f = \bigoplus_{\alpha \in \Lambda} f_\alpha$ and $g = \bigoplus_{\alpha \in \Lambda} g_\alpha$ indexed by the same set, their diagonal tensor product $f \hat{\otimes} g$ is the indexed group homomorphism $f \hat{\otimes} g = \bigoplus_{\alpha \in \Lambda} f_\alpha \otimes g_\alpha$.

Similarly, we have the definition of *indexed (co)chain homomorphism* by replacing the indexed groups in the above definition by indexed (co)chain complexes.

**Definition 3.2** An indexed group homomorphism $\theta : U_* \to V_*$ is called a *split homomorphism* if $\ker \theta$, $\coker \theta$ and $\im \theta$ are all free groups.

An indexed chain homomorphism $\vartheta : (C_*, d) \to (D_*, d)$ with induced homology group homomorphism $\theta : U_* = H_*(C_*) \to V_* = H_*(D_*)$ is called a *split inclusion* if $C_*$ is a chain subcomplex of the free complex $D_*$ and $\theta$ is a split...
homomorphism.

**Definition 3.3** Let \( \theta : U_* \to V_* \) be a split homomorphism with dual homomorphism \( \theta^o : V^* \to U^* \). The index set \( \mathcal{X} = \{(\emptyset,\emptyset), (\emptyset,\{1\}), (\{1\},\emptyset)\} \) and \( \mathcal{R} = \{(\emptyset,\emptyset), (\emptyset,\{1\})\} \subset \mathcal{X} \). \( \mathcal{S} \) may be taken to be either \( \mathcal{X} \) or \( \mathcal{R} \).

The indexed groups \( H_{s}^{\mathcal{S}}(\theta) = \bigoplus_{s \in \mathcal{S}} H^s(\theta) \) and its dual groups \( H_{s}^{\mathcal{S}}(\theta^o) = \bigoplus_{s \in \mathcal{S}} H^s(\theta^o) \) are given by

\[
H_s^s(\theta) = \begin{cases} 
\ker \theta & \text{if } s = ((1),\emptyset), \\
\text{im} \theta & \text{if } s = (\emptyset,\emptyset), \\
\text{coker} \theta & \text{if } s = (\{1\},\emptyset), \\
\end{cases}
\]

\[
H_s^s(\theta^o) = \begin{cases} 
\ker \theta^o & \text{if } s = (\{1\},\emptyset), \\
\text{im} \theta^o & \text{if } s = (\emptyset,\emptyset), \\
\text{coker} \theta^o & \text{if } s = (\emptyset,\{1\}), \\
\end{cases}
\]

\( H_{s}^{\mathcal{S}}(\theta) \) (\( H_{s}^{\mathcal{S}}(\theta^o) \)) is called the \( \text{(right for } \mathcal{R} \text{)} \) character group of \( \theta \) (\( \theta^o \)).

The indexed chain complexes \( (C_s^{\mathcal{S}}(\theta), d) = \bigoplus_{s \in \mathcal{S}} (C^s(\theta), d) \) and its dual cochain complexes \( (C_s^{\mathcal{S}}(\theta^o), \delta) = \bigoplus_{s \in \mathcal{S}} (C^s(\theta^o), \delta) \) are given by

\[
C_s^s(\theta) = \begin{cases} 
\ker \theta & \text{if } s = ((1),\emptyset), \\
\ker \theta \oplus \Sigma \ker \theta & \text{if } s = (\emptyset,\{1\}), \\
\text{im} \theta & \text{if } s = (\emptyset,\emptyset), \\
\end{cases}
\]

\[
C_s^s(\theta^o) = \begin{cases} 
\ker \theta^o & \text{if } s = ((1),\emptyset), \\
\ker \theta^o \oplus \Sigma \ker \theta^o & \text{if } s = (\emptyset,\{1\}), \\
\text{im} \theta^o & \text{if } s = (\emptyset,\emptyset), \\
\end{cases}
\]

where \( d \) is trivial on \( C_s^{0,\emptyset}(\theta) \) and \( C_s^{1,\emptyset}(\theta) \) and is the desuspension isomorphism on \( C_s^{0,\emptyset}(\theta) \). \( C_s^{\mathcal{S}}(\theta) \) (\( C_s^{\mathcal{S}}(\theta^o) \)) is called the \( \text{(right for } \mathcal{R} \text{)} \) character complex of \( \theta \) (\( \theta^o \)).

Let \( \theta = \bigoplus_{\alpha \in \Lambda} \theta_\alpha. \) Then \( H_{s}^{\mathcal{S}}(\theta) \) is also a group indexed by \( \Lambda \) and so denoted by \( H_{s}^{\mathcal{S}}\Lambda(\theta) = \bigoplus_{s \in \mathcal{S}, \alpha \in \Lambda} H^s_{s}(\theta) \) with

\[
H_{s}^{s,\alpha}(\theta) = \begin{cases} 
\text{coker} \theta_\alpha & \text{if } s = ((1),\emptyset), \\
\ker \theta_\alpha & \text{if } s = (\emptyset,\{1\}), \\
\text{im} \theta_\alpha & \text{if } s = (\emptyset,\emptyset), \\
\end{cases}
\]

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Other cases are similar.

**Theorem 3.4** For a split inclusion \( \vartheta: (C_*, d) \to (D_*, d) \) with induced homology homomorphism \( \theta: U_* \to V_* \), there are quotient chain homotopy equivalences \( q \) and \( q' \) satisfying the following commutative diagram

\[
\begin{array}{ccc}
(C_*, d) & \xrightarrow{q'} & (U_*, d) \\
\vartheta \downarrow & & \vartheta' \downarrow \\
(D_*, d) & \xrightarrow{q} & (C_*^X(\theta), d),
\end{array}
\]

where \( \vartheta' \) is the inclusion by identifying \( U_* = \ker \theta \oplus \coim \theta \) with \( \ker \theta \oplus \im \theta \subset C_*^X(\theta) \) (as in Definition 3.3).

There are also isomorphisms \( \phi \) and \( \phi' \) of chain complexes indexed by \( X \) satisfying the following commutative diagram

\[
\begin{array}{ccc}
(U_*, d) & \xrightarrow{\phi'} & S_*^X \hat{\otimes} H_*^X(\theta) \\
\vartheta' \downarrow & & i \hat{\otimes} 1 \downarrow \\
(C_*^X(\theta), d) & \xrightarrow{\phi} & (T_*^X \hat{\otimes} H_*^X(\theta), d)
\end{array}
\]

where \( S_*^X \) and \( T_*^X \) are as defined in \([17]\), 1 is the identity and \( i \) is the inclusion. Precisely, let \( \mathbb{Z}(x_1, \ldots, x_n) \) be the free abelian group generated by \( x_1, \ldots, x_n \), then \( (T_*^{X,(1)}, d) = (\mathbb{Z}(\beta, \gamma), d), (T_*^{(1),X}, d) = (\mathbb{Z}(\alpha), d) \), \( |\alpha| = |\beta|-1 = |\gamma| = |\eta| = 0 \), \( d\beta = \gamma \), \( S_*^X = \mathbb{Z}(\gamma, \eta) \).

If \( \theta \) is an epimorphism, then \( H_*^X(\theta) = H_*^X(\theta) = U_* \) by identifying \( \im \theta \) with \( \coim \theta \) and so all \( X \) is replaced by \( \mathcal{R} \).

**Proof** Take a representative \( a_i \) in \( C_* \) for every generator of \( \ker \theta \) and let \( \overline{a}_i \in D_* \) be any element such that \( \overline{a}_i a_i = 1 \). Take a representative \( b_j \) in \( C_* \) for every generator of \( \im \theta \). Take a representative \( c_k \) in \( D_* \) for every generator of \( \coker \theta \). So we may regard \( U_* \) as the chain subcomplex of \( C_* \) freely generated by all \( a_i \)’s and \( b_j \)’s and regard \( (C_*^X(\theta), d) \) as the chain subcomplex of \( D_* \) freely generated by all \( a_i \)’s, \( \overline{a}_i \)’s, \( b_j \)’s and \( c_k \)’s. Then we have the following
commutative diagram of short exact sequences of chain complexes

\[
\begin{array}{cccccc}
0 & \rightarrow & U_* & \overset{i}{\rightarrow} & C_* & \overset{j}{\rightarrow} & C_*/U_* & \rightarrow & 0 \\
\vartheta' & \downarrow & \vartheta & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & C_*^\mathbb{X}(\theta) & \overset{i}{\rightarrow} & D_* & \overset{j}{\rightarrow} & D_*/C_*^\mathbb{X}(\theta) & \rightarrow & 0.
\end{array}
\]

Since all the complexes are free, the two \(i\)'s have group homomorphism inverse. \(H_*\left(C_*/U_*\right) = 0\) and \(H_*\left(D_*/C_*^\mathbb{X}(\theta)\right) = 0\) imply that the inverse of \(i\)'s are complex homomorphisms. So we may take \(q, q'\) to be the inverse of \(i\)'s.

\(\phi\) is defined as shown in the following table.

| \(x \in \text{coker } \theta\) | \(\Sigma \ker \theta\) | \(\ker \theta\) | \(\im \theta\) |
|---|---|---|---|
| \(\phi(x) = \alpha \hat{\otimes} x\) | \(\beta \hat{\otimes} dx\) | \(\gamma \hat{\otimes} x\) | \(\eta \hat{\otimes} x\) |

\[\square\]

Compare the proof of the above theorem with that of Theorem 2.3 of [17]. The work of this and the next section is just to generalize all definitions for graded groups \(A_*\left(\mathbb{X}_*\right)\) in [17] to definitions for indexed groups \(A_* = \bigoplus_{\alpha \in \Lambda} A_*^\mathbb{X}\left(\mathbb{X}_*^\alpha\right)\). Then all the proofs in this paper are naturally obtained from [17] by replacing all the graded groups ((co)chain complexes) by indexed groups ((co)chain complexes). So we omit the proof when the analogue in [17] is given.

**Definition 3.5** For \(k = 1, \cdots, m\), let \(\vartheta_k: ((C_k)_*, d) \rightarrow ((D_k)_*, d)\) be a split inclusion with induced homology group homomorphism \(\theta_k: (U_k)_* \rightarrow (V_k)_*\). Denote \(\vartheta = \{\vartheta_k\}_{k=1}^m, \theta = \{\theta_k\}_{k=1}^m\) and their dual \(\vartheta^\circ = \{\vartheta_k^\circ\}_{k=1}^m, \theta^\circ = \{\theta_k^\circ\}_{k=1}^m\).

The index set \(\mathcal{J} = \mathcal{X}^\circ\) or \(\mathcal{A}\).

The indexed group \(H^\mathcal{J}_*(\vartheta)\) and its dual group \(H^\mathcal{J}_*(\theta^\circ)\) are given by

\[H^\mathcal{J}_*(\vartheta) = H_*^{\mathcal{J}}(\theta_1) \otimes \cdots \otimes H_*^{\mathcal{J}}(\theta_m), \quad H^\mathcal{J}_*(\theta^\circ) = H_*^{\mathcal{J}}(\theta_1^\circ) \otimes \cdots \otimes H_*^{\mathcal{J}}(\theta_m^\circ).\]

Since \(\mathcal{J}_m = \mathcal{J} \times \cdots \times \mathcal{J}\) (\(m\)-fold) by the 1-1 correspondence

\[(\sigma, \omega) \rightarrow (s_1, \cdots, s_m), \quad \sigma = \{k \mid s_k = (\{1\}, \emptyset)\}, \quad \omega = \{k \mid s_k = (\emptyset, \{1\})\}.\]
we may write $H_\mathcal{S}^*(\theta) = \oplus_{(\sigma, \omega) \in \mathcal{S}} H_{\sigma, \omega}^*(\theta)$, $H_{\mathcal{S}^*}^*(\theta^\circ) = \oplus_{(\sigma, \omega) \in \mathcal{S}} H_{\sigma, \omega}^*(\theta^\circ)$.

Then by definition,

$$H_{\sigma, \omega}^*(\theta) = H_1 \otimes \cdots \otimes H_m, \quad H_k = \begin{cases} \text{coker} \theta_k & \text{if } k \in \sigma, \\ \text{ker} \theta_k & \text{if } k \in \omega, \\ \text{im} \theta_k & \text{otherwise}, \end{cases}$$

$$H_{\sigma, \omega}^*(\theta^\circ) = H_1 \otimes \cdots \otimes H_m, \quad H_k = \begin{cases} \text{ker} \theta^\circ_k & \text{if } k \in \sigma, \\ \text{coker} \theta^\circ_k & \text{if } k \in \omega, \\ \text{im} \theta^\circ_k & \text{otherwise}. \end{cases}$$

$H_\mathcal{S}^*(\theta) (H_{\mathcal{S}^*}^*(\theta^\circ))$ is called the (right for $\mathcal{S}$) character group of $\theta (\theta^\circ)$.

The indexed chain complex $(C_{\mathcal{S}}^*(\theta), d)$ and its dual cochain complex $(C_{\mathcal{S}^*}^*(\theta^\circ), \delta)$ are given by

$$C_{\mathcal{S}}^*(\theta) = C_1^*(\theta) \otimes \cdots \otimes C_m^*(\theta), \quad C_{\mathcal{S}^*}^*(\theta^\circ) = C_1^*(\theta^\circ) \otimes \cdots \otimes C_m^*(\theta^\circ).$$

$C_{\mathcal{S}}^*(\theta) (C_{\mathcal{S}^*}^*(\theta^\circ))$ is called the (right for $\mathcal{S}$) character complex of $\theta (\theta^\circ)$.

**Definition 3.6** Let $K$ be a simplicial complex on $[m]$ and everything else be as in Definition 3.5. For $\mathcal{S} = \mathcal{X}$ or $\mathcal{R}$, $T_{\mathcal{S}}^*, S_{\mathcal{S}^*}$ are as in Theorem 3.4.

The total chain complex $(T_{\mathcal{S}}^*(K), d)$ of $K$ is the chain subcomplex of $(T_{\mathcal{S}}^* = T_1^* \otimes \cdots \otimes T_m^*, d)$ defined as follows. For a subset $\tau$ of $[m]$, define

$$(U_\tau, d) = (U_1 \otimes \cdots \otimes U_m, d), \quad (U_k, d) = \begin{cases} (T_1^*, d) & \text{if } k \in \tau, \\ S_{\mathcal{S}^*} & \text{if } k \not\in \tau. \end{cases}$$

Then $(T_{\mathcal{S}}^*(K), d) = (+_{\tau \in K} U_\tau, d)$. Define $(T_{\mathcal{S}}^*(\{\}), d) = 0$. The group $H_{\mathcal{S}}^*(K) = H_*(T_{\mathcal{S}}^*(K))$ is called the total homology group of $K$.

So the dual cochain complex $(T_{\mathcal{S}^*}^*(K), \delta)$ of $(T_{\mathcal{S}}^*(K), d)$ is a quotient complex of $(T_{\mathcal{S}}^*(K), d)$ and is called the total cochain complex of $K$. The group $H_{\mathcal{S}^*}^*(K) = H^*(T_{\mathcal{S}^*}^*(K))$ is called the total cohomology group of $K$. 

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The polyhedral product character complex \((C^\mathcal{X}_m(K; \hat{\theta}), d)\) is the subcomplex of \((C^\mathcal{X}_m(\theta), d)\) defined as follows. For a subset \(\tau\) of \([m]\), define
\[
(H_*(\tau), d) = (H_1 \otimes \cdots \otimes H_m, d), \quad (H_k, d) = \begin{cases} 
(C_*^{\mathcal{X}}(\theta_k), d) & \text{if } k \in \tau, \\
((U_k)_*), d & \text{if } k \notin \tau.
\end{cases}
\]
Then \((C^\mathcal{X}_m(K; \hat{\theta}), d) = (+_{\tau \in K} H_*(\tau), d)\). Define \((C^\mathcal{X}_m(\{\}; \theta), d) = 0\).

So the dual cochain complex \((C^*_{\mathcal{X}_m}(K; \hat{\theta}^\circ), \delta)\) of \((C^\mathcal{X}_m(K; \hat{\theta}), d)\) is a quotient complex of \((C^*_{\mathcal{X}_m}(\hat{\theta}^\circ), \delta)\).

Replace \(\mathcal{X}\) by \(\mathcal{I}\) in the above definitions, we get the right analogues (right total complex, right polyhedral product character complex, etc.).

The polyhedral product inclusion complex \((C_*(K; \bar{\theta}), d)\) is the subcomplex of \(((D_1)_* \otimes \cdots \otimes (D_m)_*), d)\) defined as follows. For a subset \(\tau\) of \([m]\), define
\[
(E_*(\tau), d) = (E_1 \otimes \cdots \otimes E_m, d), \quad (E_k, d) = \begin{cases} 
((D_k)_*), d & \text{if } k \in \tau, \\
((C_k)_*), d & \text{if } k \notin \tau.
\end{cases}
\]
Then \((C_*(K; \bar{\theta}), d) = (+_{\tau \in K} E_*(\tau), d)\). Define \((C_*(\{\}; \bar{\theta}), d) = 0\).

So the dual cochain complex \((C^*(K; \bar{\theta}^\circ), \delta)\) of \((C_*(K; \bar{\theta}), d)\) is a quotient complex of \(((D_1)^* \otimes \cdots \otimes (D_m)^*), \delta)\).

By Theorem 4.7 of [17], \(H^\sigma_{\mathcal{X}}(K) \cong \tilde{H}_{s-1}(K_{\sigma, \omega}), \ H^\sigma_{\mathcal{X}}(K) \cong \tilde{H}_{s-1}(K_{\sigma, \omega}), \)
where \(H^\mathcal{X}_m(K) = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} H^\sigma_{\mathcal{X}}(K), \ H^\mathcal{X}_m(K) = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} H^\sigma_{\mathcal{X}}(K)\).

**Theorem 3.7** For the \(K, \hat{\theta}\) and \(\bar{\theta}\) in Definition 3.5 and Definition 3.6, there is a quotient chain homotopy equivalence (\(\mathcal{X}_m\) neglected)
\[
\varphi_{(K; \hat{\theta})}: (C_*(K; \hat{\theta}), d) \xrightarrow{\cong} (C^\mathcal{X}_m(K; \hat{\theta}), d)
\]
and an isomorphism of chain complexes indexed by \(\mathcal{X}_m\)
\[
\phi_{(K; \theta)}: (C^\mathcal{X}_m(K; \hat{\theta}), d) \xrightarrow{\cong} (T^\mathcal{X}_m(K) \otimes \mathcal{H}^\mathcal{X}_m(\hat{\theta}), d).
\]
So we have (co)homology group isomorphisms
\[
H_*(C_*(K; \bar{\theta})) \cong H^\mathcal{X}_m(K) \otimes \mathcal{H}^\mathcal{X}_m(\hat{\theta}) = H^\mathcal{X}_m(K) \otimes (H^\mathcal{X}_*(\theta_1) \otimes \cdots \otimes H^\mathcal{X}_*(\theta_m)),
\]
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\[ H^*(C^*(K; \mathfrak{F})) \cong H^*_{x_m}(K) \otimes H^*_{x_m}(\mathfrak{F}) = H^*_{x_m}(K) \otimes (H^*_x(\theta_1) \otimes \cdots \otimes H^*_x(\theta_m)). \]

If each \( \theta_k \) is an epimorphism, then 
\[ H^*_x(\theta_k) = H^*_{x_k}(\theta_k) = (U_k)^* \]
and so all \( \mathcal{X} \) is replaced by \( \mathcal{R} \).

**Proof** Similar to that of Theorem 2.6 and Theorem 4.6 of [17]. \( \square \)

**Definition 3.8** A polyhedral join complex \( S(K; X, A) \) is **homology split** if the reduced simplicial homology homomorphism
\[ \iota_k: \tilde{H}_*(A_k) \to \tilde{H}_*(X_k) \]
induced by inclusion is split for \( k = 1, \ldots, m \).

A polyhedral join complex \( S(K; X, A) \) is **total homology split** if the reduced simplicial homology homomorphism
\[ \iota_{\sigma_k, \omega_k}: \tilde{H}_*(A_{k_{\sigma_k, \omega_k}}) \to \tilde{H}_*((X_k)_{\sigma_k, \omega_k}) (\sigma, \omega \text{ as in Theorem 2.6}) \]
induced by inclusion is split for all \( (\sigma_k, \omega_k) \in \mathcal{X}_m \).

**Theorem 3.9** For homology split \( S(K; X, A) \),
\[ \tilde{H}_{*-1}(S(K; X, A)) \cong H^*_{x_m}(K) \otimes H^*_{x_m}(X, A) \]

\[ = H^*_{x_m}(K) \otimes (H^*_x(X_1, A_1) \otimes \cdots \otimes H^*_x(X_m, A_m)), \]

\[ \tilde{H}^{-1}_{*-1}(S(K; X, A)) \cong H^*_{x_m}(K) \otimes H^*_{x_m}(X, A) \]

\[ = H^*_{x_m}(K) \otimes (H^*_x(X_1, A_1) \otimes \cdots \otimes H^*_x(X_m, A_m)), \]

where \( H^*_{x_m}(\cdot) = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} H^*_{\sigma, \omega}(\cdot) \), 
\( H^*_{x_m}(\cdot) = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} H^*_{\sigma, \omega}(\cdot) \) with
\[
H^*_x(X_k, A_k) = \begin{cases} 
\Sigma \text{coker } \iota_k & \text{if } s = ((1), \emptyset), \\
\Sigma \text{ker } \iota_k & \text{if } s = (\emptyset, \{1\}), \\
\Sigma \text{im } \iota_k & \text{if } s = (\emptyset, \emptyset), 
\end{cases}
\]

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$H^*_\sigma(X, A) = \begin{cases} 
\Sigma \ker \iota_k & \text{if } s = \langle \{1\}, \emptyset \rangle, \\
\Sigma \coker \iota_k & \text{if } s = \langle \emptyset, \{1\} \rangle, \\
\Sigma \text{im} \iota_k & \text{if } s = \langle \emptyset, \emptyset \rangle,
\end{cases}$

$H^{\sigma, \omega}_*(X, A) = H_1 \otimes \cdots \otimes H_m, \quad H_k = \begin{cases} 
\Sigma \coker \iota_k & \text{if } k \in \sigma, \\
\Sigma \ker \iota_k & \text{if } k \in \omega, \\
\Sigma \text{im} \iota_k & \text{otherwise,}
\end{cases}$

$H^{\sigma, \omega}_*(X, A) = H_1 \otimes \cdots \otimes H_m, \quad H_k = \begin{cases} 
\Sigma \ker \iota_k & \text{if } k \in \sigma, \\
\Sigma \coker \iota_k & \text{if } k \in \omega, \\
\Sigma \text{im} \iota_k & \text{otherwise,}
\end{cases}$

where $\iota_k$ is as in Definition 3.8 with dual $\iota_k^\circ$ and $\Sigma$ means suspension.

If each $\iota_k$ is an epimorphism, then all $\mathcal{R}$ is replaced by $\mathcal{R}$ and we have $H^{\mathcal{R}}_*(X, A_k) = H_*(A_k), H^{\mathcal{R}}_*(X, A_k) = H^*(A_k).

If the reduced simplicial (co)homology is taken over a field, then the conclusion holds for all polyhedral join complexes.

Proof A corollary of Theorem 3.7 by taking $\vartheta = \{\vartheta_k\}_{k=1}^m$ with split inclusion $\vartheta_k : (\Sigma \tilde{C}_*(A_k), d) \to (\Sigma \tilde{C}_*(X_k), d)$ the suspension reduced simplicial complex inclusion. Regard this graded group inclusion as an indexed chain homomorphism such that the index set has only one element. Then $(C_*(K; \vartheta), d) = (\Sigma \tilde{C}_*(S(K; X, A)), d)$ and $H^{\mathcal{R}}_*(\vartheta_k) = H^{\mathcal{R}}_*(X, A_k).$ □

Example 3.10 For $S(K; X, A) = S(K; L_1, \cdots, L_m)$ such that all $H_*(L_k)$ is free, each $\iota_k : \tilde{H}_*(L_k) \to \tilde{H}_*(\Delta^{[n_k]})$ is an epimorphism. By definition, $H^0_*(\Delta^{[n_k]}, L_k) = 0, H^0_*(\Delta^{[n_k]}, L_k) = \tilde{H}_{s-1}(L_k), H^0_*(\Delta^{[n_k]}, L_k) = 0$ if $\omega \neq [m].$ So

$H^{\mathcal{R}_m}_*(X, A) = H^{\mathcal{R}_m}_*(X, A) = \tilde{H}_{s-1}(L_1) \otimes \cdots \otimes \tilde{H}_{s-1}(L_m),$

$H^{\mathcal{R}_m}_*(K) \otimes H^{\mathcal{R}_m}_*(X, A) = \tilde{H}_{s-1}(K) \otimes \tilde{H}_{s-1}(L_1) \otimes \cdots \otimes \tilde{H}_{s-1}(L_m).$

So by Theorem 3.9,

$\tilde{H}_{s-1}(S(K; L_1, \cdots, L_m)) \cong \tilde{H}_{s-1}(K) \otimes \tilde{H}_{s-1}(L_1) \otimes \cdots \otimes \tilde{H}_{s-1}(L_m),$
\[ \widetilde{H}^{*-1}(S(K; L_1, \ldots, L_m)) \cong \widetilde{H}^{*-1}(K) \otimes \widetilde{H}^{*-1}(L_1) \cdots \otimes \widetilde{H}^{*-1}(L_m). \]

The above homology equality implies that \( S(K; L_1, \ldots, L_m) \) is homology spherical (\( \widetilde{H}_s(-) \cong \mathbb{Z} \)), so \( \{ \emptyset \} \) is homology spherical but \( \{ \} \) is not) if and only if \( K \) and all \( L_k \) are homology spherical. By Theorem 2.10, for \( \sigma \in \tilde{K} = S(K; L_1, \ldots, L_m) \), \( \sigma_k = \sigma \cap [n_k] \), \( \hat{\sigma} = \{ k \mid \sigma_k \notin L_k \} \), \( n' = \Sigma_{k \in \hat{\sigma}}(n_k - |\sigma_k|) \),

\[ \text{link}_K\sigma = S(\text{link}_K\hat{\sigma}; \text{link}_{L_i}\sigma_{i_1}, \ldots, \text{link}_{L_i}\sigma_{i_s}) \ast \Delta^{[n']}\), \( \{ i_1, \ldots, i_s \} = [m] \setminus \hat{\sigma}. \]

This implies that \( S(K; L_1, \ldots, L_m) \) is a homological sphere (\( L \) is a homological sphere if \( \text{link}_L\sigma \) is homology spherical for all \( \sigma \in L \)) if and only if \( K \) is a homological sphere and each \( L_k = \partial \Delta^{[n_k]} \). This result is a part of Theorem 6.6 of [2].

We have ring isomorphism \( H^*(S(K; X, A)) \cong H^*|S(K; X, A)| \), where \( | \cdot | \) means geometrical realization. So \( \widetilde{H}^*(S(K; X, A)) \) is a ring by adding a unit to it. This ring is not considered in this paper.

**Theorem 3.11** For a total homology split \( S(K, X, A) \), we have

\[
\begin{align*}
H^*_\mathcal{I}_n(S(K; X, A)) & \cong H^*_{\mathcal{I}_m}(K) \otimes (H^*_{\mathcal{I}_{m,\mathcal{I}_n}(X, A)}) \\
& \cong H^*_{\mathcal{I}_m}(K) \otimes (H^*_{\mathcal{I}_{m,\mathcal{I}_1}(X, A)} \otimes \cdots \otimes H^*_{\mathcal{I}_{m,\mathcal{I}_n}(X, A)}),
\end{align*}
\]

\[
H^*_\mathcal{I}_n(S(K; X, A))
\begin{align*}
& \cong H^*_{\mathcal{I}_m}(K) \otimes (H^*_{\mathcal{I}_{m,\mathcal{I}_1}(X, A)} \otimes \cdots \otimes H^*_{\mathcal{I}_{m,\mathcal{I}_n}(X, A)}),
\end{align*}
\]

where \( \mathcal{I}_n = \mathcal{I}_{1,\ldots,\mathcal{I}_n} \) by the 1-1 correspondence

\[
(\sigma, \omega) \leftrightarrow (\sigma_1, \omega_1, \ldots, \sigma_m, \omega_m), \; \sigma_k = \sigma \cap [n_k], \; \omega_k = \omega \cap [n_k].
\]

\( \mathcal{I} = \mathcal{I}^r \) or \( \mathcal{R} (\mathcal{I}_{n_1} = \mathcal{I}_{m,\mathcal{I}_1}, \mathcal{I}_{n_2} = \mathcal{R}_{n_2} \) is possible). Since \( \mathcal{I}_m = \mathcal{I} \times \cdots \times \mathcal{I} \) as in Definition 3.5, we have \( (\mathcal{I}_m; \mathcal{I}_n) = (\mathcal{I}; \mathcal{I}_{n_1}) \times \cdots \times (\mathcal{R}; \mathcal{I}_{n_m}) \) and so
for \((\hat{\sigma}, \hat{\omega}) \in \mathcal{R}_m, (\sigma, \omega) \in \mathcal{I}_n\), we have by definition

\[
H^\Lambda_{\chi}(\hat{\omega}) = \oplus_{\alpha, \beta \in \Gamma} H^\alpha_{\chi\beta}(\hat{\omega}), \quad H^\Lambda_{\chi}(\hat{\omega}) = \oplus_{\alpha, \beta \in \Gamma} H^\alpha_{\chi\beta}(\hat{\omega}),
\]

\[
H^s_{\sigma;\hat{\omega};\hat{w}}(X_k, A_k) = \begin{cases} 
\Sigma \text{coker } \iota_{\sigma;\omega} & \text{if } s = \{1\}; \\
\Sigma \ker \iota_{\sigma;\omega} & \text{if } s = \emptyset; \{1\}; \\
\Sigma \text{im } \iota_{\sigma;\omega} & \text{if } s = \emptyset,
\end{cases}
\]

\[
H^s_{\hat{\sigma};\hat{\omega};\hat{w}}(X, A) = H_1 \otimes \cdots \otimes H_m, \quad H_k = \begin{cases} 
\Sigma \text{coker } \iota_{\hat{\sigma};\hat{\omega}} & \text{if } k \in \hat{\sigma}; \\
\Sigma \ker \iota_{\hat{\sigma};\hat{\omega}} & \text{if } k \in \hat{\omega}; \\
\Sigma \text{im } \iota_{\hat{\sigma};\hat{\omega}} & \text{otherwise},
\end{cases}
\]

where \(\iota_{\sigma;\omega}\) is as in Definition 3.8 with dual \(\iota_{\sigma;\hat{\omega}}\) and \(\Sigma\) means suspension.

If each \(\iota_{\sigma;\omega}\) is an epimorphism, then all \(\mathcal{R}\) is replaced by \(\mathcal{R}\) and we have \(H^s_{\hat{\sigma};\hat{\omega};\hat{w}}(X_k, A_k) = H^s_{\hat{\omega};\hat{w}}(A_k), H^s_{\hat{\sigma};\hat{\omega};\hat{w}}(X_k, A_k) = H^s_{\hat{\omega};\hat{w}}(A_k)\).

If the (right) total (co)homology group is taken over a field, then the theorem holds for all polyhedral join complexes.

**Proof** A corollary of Theorem 3.7 by taking \(\mathcal{R} = \{\partial_k\}_{k=1}^m\) with split inclusion \(\partial_k: T^\partial_{\partial_k}(A_k) \rightarrow T^\partial_{\partial_k}(X_k)\) the (right) total chain complex inclusion. Then \((C_s(K; \mathcal{R}), d) = (T^\partial_{\partial_k}(S(K; \mathcal{X}; A)), d)\) and \(H^s_{\hat{\sigma};\hat{\omega}}(\theta_k) = H^s_{\hat{\sigma};\hat{\omega}}(X_k, A_k)\).

**Example 3.12** Apply Theorem 3.11 for \(S(K; \mathcal{X} A) = S(K; L_1, \cdots, L_m), L_k \neq \emptyset\) for \(k = 1, \cdots, m\). So either all \(H^s_{\hat{\sigma};\hat{\omega}}(L_k)\) are free or the (co)homology is taken over a field.
For \( \mathcal{S} = \mathcal{R} \), \( \theta_k : H^\mathcal{R}_{nk}(L_k) \to H^\mathcal{R}_{nk}(\Delta[n_k]) \cong \mathbb{Z} \) is an epimorphism. So
\[
H^\mathcal{S}_{nk}(S(K; L_1, \ldots, L_m)) \cong H^\mathcal{S}_{nk}(K) \otimes (H^\mathcal{S}_{nk}(L_1) \otimes \cdots \otimes H^\mathcal{S}_{nk}(L_m)),
\]
\[
H^\mathcal{S}_{nk}(S(K; L_1, \ldots, L_m)) \cong H^\mathcal{S}_{nk}(K) \otimes (H^\mathcal{S}_{nk}(L_1) \otimes \cdots \otimes H^\mathcal{S}_{nk}(L_m)).
\]

For \( \mathcal{S} = \mathcal{R} \), \( \theta_k : H^\mathcal{R}_{nk}(L_k) \to H^\mathcal{R}_{nk}(\Delta[n_k]) \) is not an epimorphism. By definition, \( H^\mathcal{R}_k(\Delta[n_k]_L) \cong H^\mathcal{R}_{nk}(L_k) \) for \( \omega_k \neq \emptyset \), \( H^\mathcal{R}_k(\emptyset \Delta[n_k]) \cong \mathbb{Z} \) for \( \sigma_k \in L_k \) and \( H^\mathcal{R}_k(\emptyset \Delta[n_k]) \cong \mathbb{Z} \) for \( \sigma_k \notin L_k \). So by identifying \( X \otimes \mathbb{Z} \otimes Y \) with \( X \otimes Y \), we have that for \( (\sigma, \omega) \in \mathcal{S}_n \),
\[
H^\mathcal{S}_k(\Delta_{nk}) \cong H^\mathcal{S}_{nk}(K) \otimes (H^\mathcal{S}_{nk}(L_1) \otimes \cdots \otimes H^\mathcal{S}_{nk}(L_m)),
\]
\[
H^\mathcal{S}_{nk}(S(K; L_1, \ldots, L_m)) \cong H^\mathcal{S}_{nk}(K) \otimes (H^\mathcal{S}_{nk}(L_1) \otimes \cdots \otimes H^\mathcal{S}_{nk}(L_m)).
\]

where \( \sigma_k = \sigma \cap [n_k], \omega_k = \omega \cap [n_k], \hat{\sigma} = \{ k \mid \sigma_k \notin L_k, \omega_k = \emptyset \}, \hat{\omega} = \{ k \mid \omega_k \neq \emptyset \}. \)

4 Universal Algebra

In this section, we prove that the cohomology algebra of \( C_*(K; \hat{\varrho}) \) is the diagonal tensor product of the total cohomology algebra of \( K \) induced by \( \hat{\varrho} \) and the character algebra of the dual \( \hat{\varrho}^* \) in Theorem 4.7. The (co)associativity is not required for a (co)algebra as in \[17] .

**Theorem 4.1** Let \( \varrho : (C_*, d) \to (D_*, d) \) be a split inclusion with induced homology homomorphism \( \theta : U_* \to V_* \) such that \( \varrho \) is also a coalgebra homomorphism \( \varrho : (C_*, \psi C) \to (D_*, \psi D) \) with induced homology coalgebra homomorphism \( \theta : (U_*, \psi U) \to (V_*, \psi V) \).

Then the group \( C^\varrho_*(\theta) \) in Definition 3.3 has a unique character coproduct \( \hat{\psi}(\varrho) \) satisfying the following three conditions.

i) \( \hat{\psi}(\varrho) \) makes the following diagram (q, q' and \( \varrho \) as in Theorem 3.4).
commutative except the homotopy commutative \((q \otimes q)\psi_D \simeq \hat{\psi}(\vartheta)q\).

\(\text{ii) } \hat{\psi}(\vartheta)\) is independent of the choice of \(\psi_C, \psi_D\) up to homotopy, i.e., if \(\psi_C, \psi_D\) are replaced by \(\psi'_C, \psi'_D\) such that \(\psi'_C \simeq \psi_C, \psi'_D \simeq \psi_D\) and we get \(\hat{\psi}'(\vartheta)\) for \(\psi'_C\) and \(\psi'_D\), then \(\hat{\psi}'(\vartheta) = \hat{\psi}(\vartheta)\).

\(\text{iii) Denote by } \alpha = \text{coker } \theta, \beta = \Sigma \text{ker } \theta, \gamma = \text{ker } \theta, \eta = \text{im } \theta. \text{ Then } \hat{\psi}(\vartheta)\) satisfies the following four conditions.

\begin{enumerate}
  \item \(\hat{\psi}(\vartheta)(\eta) \subset \eta \otimes \eta \oplus \gamma \otimes \eta \oplus \eta \otimes \gamma \oplus \gamma \otimes \gamma.\)
  \item \(\hat{\psi}(\vartheta)(\gamma) \subset \gamma \otimes \gamma \oplus \gamma \otimes \eta \oplus \eta \otimes \gamma.\)
  \item \(\hat{\psi}(\vartheta)(\beta) \subset (\beta \otimes \gamma \oplus \beta \otimes \eta \oplus \eta \otimes \beta) \oplus \beta \otimes \alpha \oplus \alpha \otimes \eta \oplus \eta \otimes \alpha \oplus \eta \otimes \eta.\)
  \item \(\hat{\psi}(\vartheta)(\alpha) \subset \alpha \otimes \alpha \oplus \alpha \otimes \eta \oplus \eta \otimes \alpha \oplus \eta \otimes \eta.\)
\end{enumerate}

\text{Proof Similar to that of Theorem 2.8 of [17].} \hfill \square

\textbf{Definition 4.2} For the \(\vartheta\) in Theorem 4.1, all the chain complexes in Theorem 3.4 are coalgebras defined as follows.

The \textit{character coalgebra complex} of \(\vartheta\) is \((C^X_*(\theta), \hat{\psi}(\vartheta))\).

If \(\theta\) is an epimorphism, then \(C^X_*(\theta) = C^X_*(\theta)\) and \((C^X_*(\theta), \hat{\psi}(\vartheta))\) is called the \textit{right character coalgebra complex} of \(\vartheta\).

The \textit{character coalgebra} of \(\vartheta\) is \((H^X_*(\theta), \hat{\psi}(\vartheta))\) with coproduct defined as follows.

\begin{enumerate}
  \item \(\hat{\psi}(\vartheta)(x) = \hat{\psi}(\vartheta)(x) = \psi_V(x)\) for all \(x \in \alpha = \text{coker } \theta.\)
  \item \(\hat{\psi}(\vartheta)(y) = \hat{\psi}(\vartheta)(y) = \psi_U(y)\) for all \(y \in \eta = \text{coim } \theta.\)
\end{enumerate}
(3) For \( z \in \gamma = \ker \theta \), denote by \( \overline{z} \) be the unique element in \( \beta \) such that \( d\overline{z} = z \). Suppose \( \hat{\psi}(\vartheta)(z) = \sum y_i \otimes z_i + \sum z'_j \otimes z''_j \) with \( z_i, z'_j, z''_j \in \gamma \) and \( y_i \in \eta \), then \( \hat{\psi}(\vartheta)(\overline{z}) = \sum (\text{sign} y_i) y_i \otimes z_i + \sum z'_j \otimes z''_j + \zeta \) with \( \zeta \in (\alpha \oplus \eta) \otimes (\alpha \oplus \eta) \).

Define \( \psi(\vartheta)(z) = \hat{\psi}(\vartheta)(z) + \zeta \). If \( \theta \) is an epimorphism, then \( (H^*_R(\theta), \psi(\vartheta)) \) is the right character coalgebra of \( \vartheta \). The group isomorphisms \( H^*_R(\theta) \cong V_* \) is in general not an algebra isomorphism.

For \( \mathcal{S} = \mathcal{X} \) or \( \mathcal{S} \), the dual algebra \( (H^*_X(\vartheta \circ \theta), \pi(\vartheta \circ \theta)) \) of \( (H^*_X(\theta), \psi(\vartheta)) \) is the (right) character algebra of \( \vartheta \).

The index coalgebra complex \( (T^*_\mathcal{X}, \hat{\psi}_\vartheta) \) of \( \vartheta \) is defined as follows. Let symbols \( x, x'_1, x''_2, \ldots \) be one of \( \alpha, \beta, \gamma, \eta \) that are both the generators of \( T^*_\mathcal{X} \) and the group summands of \( C^*_\mathcal{X}(\vartheta) \) in Theorem 4.1. If the group summand \( x \) satisfies \( \hat{\psi}_\vartheta(x) \subseteq \bigoplus \otimes x'_i \otimes x''_i \) such that no summand \( x'_i \otimes x''_i \) can be canceled, then the generator \( x \) satisfies \( \hat{\psi}_\vartheta(x) = \sum x'_i \otimes x''_i \).

If \( \theta \) is an epimorphism, then \( (T^*_R, \hat{\psi}_\vartheta) \) is the right index coalgebra complex of \( \vartheta \).

Notice that \( \hat{\psi}(\vartheta) \) is degree preserving but \( \psi(\vartheta) \) is in general not. When \( \psi(\vartheta) = \hat{\psi}(\vartheta) \) and so \( \psi(\vartheta) \) keeps degree, the coproduct is called normal, for example, (right) normal, (right) strictly normal.

Example 4.3 Take \( \vartheta: (T^*_\mathcal{X}m(L), \psi_C) \rightarrow (T^*_\mathcal{X}m(\Delta^m), \psi_D) \) in Theorem 4.1 to be the right total chain complex homomorphism induced by inclusion such that \( T^*_\mathcal{X}m(L) \) is free and the vertex set of \( L \) is \( [m] \). \( \psi_C, \psi_D \) are the right universal coproduct defined in Definition 7.4 of [17]. Then the induced homology coalgebra homomorphism \( \theta: H^*_\mathcal{X}m(L) \rightarrow H^*_\mathcal{X}m(\Delta^m) \cong \mathbb{Z} \) is an epimorphism. By definition, \( \eta \cong H^0,0_\mathcal{X}(L) \cong \mathbb{Z} \) with generator denoted by 1, \( \gamma \cong \bigoplus_{\omega \neq 0} H^0,\omega_\mathcal{X}(L), \alpha = 0 \). Since there is no ghost vertex, \( H^0,\omega_\mathcal{X}(L) = 0 \) for all
So \( \omega \neq \emptyset \). So \( \hat{\psi}(\varnothing)(1) = \mathbb{1} \). For \( x \in \gamma \), \( \hat{\psi}(\varnothing)(x) = \mathbb{1} + \Sigma x_i \otimes x''_i \) with \( x'_i, x''_i \in \gamma \). Since the degree of 1 is 0, \( \mathbb{1} \) can not be a summand of \( \hat{\psi}(\varnothing)(x) \).

So \( \hat{\psi}(\varnothing)(x) = \mathbb{1} + \mathbb{1} \otimes x + \Sigma x_i \otimes x''_i \). With these, we have

\[
\hat{\psi}(\varnothing)(\eta) \subset \eta \otimes \eta \quad (\eta = \mathbb{1}) \text{ and so } \hat{\psi}(\varnothing)(\eta) = \eta \otimes \eta \quad (\eta \text{ the generator of } T^\varnothing).
\]

\[
\hat{\psi}(\varnothing)(\gamma) \subset \gamma \otimes \gamma + \gamma \otimes \eta + \eta \otimes \gamma \text{ and so } \hat{\psi}(\varnothing)(\gamma) = \gamma \otimes \gamma + \gamma \otimes \eta + \eta \otimes \gamma.
\]

\[
\hat{\psi}(\varnothing)(\beta) \subset \beta \otimes \gamma + \beta \otimes \eta + \eta \otimes \beta \text{ and so } \hat{\psi}(\varnothing)(\beta) = \beta \otimes \gamma + \beta \otimes \eta + \eta \otimes \beta.
\]

So \( \hat{\psi}(\varnothing) \) is the right strictly normal coproduct in Definition 7.4 of [17] and we have the coalgebra isomorphism \((H^\varnothing_*(\varnothing), \psi(\varnothing)) = (H^\varnothing_*(L), \psi_U)\).

**Example 4.4** Take \( \varnothing: (T^\varnothing_*(L), \psi_C) \to (T^\varnothing_*(\Delta[m]), \psi_D) \) in Theorem 4.1 to be the total chain complex homomorphism induced by inclusion such that \( T^\varnothing_*(L) \) is free and the vertex set of \( L \) is \([m]\). \( \psi_C, \psi_D \) are the universal coproduct defined in Definition 6.1 of [17]. The induced homology coalgebra homomorphism \( \theta: H^\varnothing_*(L) \to H^\varnothing_*(\Delta[m]) \) is not an epimorphism. By definition, \( \eta \cong \mathbb{1} \oplus_{\sigma \in K} H^\varnothing_*(\eta^\sigma) \), where each \( H^\varnothing_*(\eta^\sigma) \cong \mathbb{1} \) with generator denoted by \( \eta^\sigma \). \( \alpha \cong \mathbb{1} \oplus_{\omega \neq \emptyset} H^\varnothing_*(\omega^\omega) \). Since there is no ghost vertex, \( H^\varnothing_0(L) = 0 \) for all \( \omega \neq \emptyset \). So \( \hat{\psi}(\varnothing)(\eta^\sigma) = \Sigma (\eta^\sigma \otimes \eta^\sigma) + (\alpha^\sigma \otimes \eta^\sigma) + (\eta^\sigma \otimes \alpha^\sigma) \), \( \hat{\psi}(\varnothing)(\eta^\sigma) = \Sigma (\eta^\sigma \otimes \eta^\sigma) \), where the sums are taken over all \( \sigma \cup \omega^\omega \subset \sigma \). For \( x \in \gamma \), \( \hat{\psi}(\varnothing)(x) = \Sigma_{\sigma' \subset \sigma} (\eta^\sigma \otimes x + x \otimes \eta^\sigma) + \Sigma x'_i \otimes x''_i \) with \( x'_i, x''_i \in \gamma \). Since the degree of all \( \eta^\sigma \) or \( \alpha^\sigma \) is 0, \( \xi^\sigma \otimes \xi^\omega \) \((\xi = \alpha \text{ or } \eta)\) can not be a summand of \( \hat{\psi}(\varnothing)(\sigma) \). So \( \hat{\psi}(\varnothing)(\sigma) = \Sigma_{\sigma' \subset \sigma} (\eta^\sigma \otimes x + x \otimes \eta^\sigma) + \Sigma x'_i \otimes x''_i \). With these, we have

\[
\hat{\psi}(\varnothing)(\eta) \subset \eta \otimes \eta \text{ and so } \hat{\psi}(\varnothing)(\eta) = \eta \otimes \eta.
\]

\[
\hat{\psi}(\varnothing)(\gamma) \subset \gamma \otimes \gamma + \gamma \otimes \eta + \eta \otimes \gamma \text{ and so } \hat{\psi}(\varnothing)(\gamma) = \gamma \otimes \gamma + \gamma \otimes \eta + \eta \otimes \gamma.
\]

\[
\hat{\psi}(\varnothing)(\beta) \subset \beta \otimes \gamma + \beta \otimes \eta + \eta \otimes \beta \text{ and so } \hat{\psi}(\varnothing)(\beta) = \beta \otimes \gamma + \beta \otimes \eta + \eta \otimes \beta.
\]

\[
\hat{\psi}(\varnothing)(\alpha) \subset (\alpha \otimes \eta) \otimes (\alpha \otimes \eta) \text{ and so } \hat{\psi}(\varnothing)(\alpha) = \alpha \otimes \alpha + \alpha \otimes \eta + \eta \otimes \alpha + \eta \otimes \eta.
\]

We call the above \( \hat{\psi}(\varnothing) \) the strictly normal coproduct of \( T^\varnothing_* \). The coproduct \( \hat{\psi}(\varnothing) = \hat{\psi}(\varnothing) \). \( H^\varnothing_*(L) \) is a subcoalgebra of \( H^\varnothing_*(\varnothing) = \mathbb{1} \oplus H^\varnothing_*(L) \). Dually,
we call \((H^*_X(\theta), \pi(\varphi))\) the augmented cohomology algebra of \(L\) relative to \([m]\) and denote it by \(\tilde{H}^*_X(L)\). Then \(\tilde{H}^*_X(L) = \alpha^\circ \oplus H^*_{X_m}(L)\) with \(\alpha^\circ\) the dual group of \(\alpha\) and \(\alpha^\circ\) is an ideal such that \(H^*_{X_m}(L) = \tilde{H}^*_X(L)/\alpha^\circ\).

Notice that coalgebras \((T^\mathcal{X}, \hat{\psi}_\varphi)\) and \((H^*_X(\theta), \psi(\varphi))\) in Example 3.3 and 3.4 remain unchanged if the (right) universal coproducts \(\psi_C, \psi_D\) are replaced by any other coproduct.

**Theorem 4.5** For the coalgebras in Definition 4.2, all the chain homomorphisms in Theorem 3.4 induce cohomology algebra isomorphisms.

**Proof** Similar to that of Theorem 6.4 of [17]. \(\square\)

**Definition 4.6** Let \(K, \vartheta, \theta\) be as in Definition 3.5 and Definition 3.6 such that each \(\vartheta_k : ((C_k)_*, \psi_{C_k}) \to ((D_k)_*, \psi_{D_k})\) and \(\theta_k : ((U_k)_*, \psi_{U_k}) \to ((V_k)_*, \psi_{V_k})\) satisfy the condition of Theorem 4.1. \(\mathcal{X} = \mathcal{X}\) or \(\mathcal{R}\). Then all the chain complexes in Theorem 3.7 are coalgebras defined as follows.

The polyhedral product inclusion complex \(C^*_X(K; \vartheta)\) is a subcoalgebra of \((D_1)_* \otimes \cdots \otimes (D_m)_*\). Its cohomology algebra product is denoted by \(\cup_{(K; \vartheta^\circ)}\).

The (right) polyhedral product character complex \(C^*_S(K; \vartheta)\) is a subcoalgebra of the (right) character coalgebra complex \(\mathcal{C^*_S}(\vartheta)\) = \((C^*_S(\vartheta_1) \otimes \cdots \otimes C^*_S(\vartheta_m), \hat{\psi}(\vartheta_1) \otimes \cdots \otimes \hat{\psi}(\vartheta_m))\).

The (right) character coalgebra \(\vartheta^\circ\) is \((H^*_S(\theta(\varphi)), \pi(\varphi)) = (H^*_S(\theta_1) \otimes \cdots \otimes H^*_S(\theta_m), \pi(\varphi_1) \otimes \cdots \otimes \pi(\varphi_m))\).

The (right) total chain complex \(T^*_S(K)\) of \(K\) is a subcoalgebra the (right) index coalgebra complex \((T^*_S, \hat{\psi}_\varphi) = (T^*_S \otimes \cdots \otimes T^*_S, \hat{\psi}_\varphi \otimes \cdots \otimes \hat{\psi}_\varphi)\) of \(\vartheta\) and is called the (right) total coalgebra complex of \(K\) induced by \(\vartheta\). Its cohomology algebra is called the (right) total cohomology algebra of \(K\) induced by \(\vartheta\) and is denoted by \((H^*_S(K), \cup_{(K; \vartheta^\circ)})\).
Theorem 4.7 For the coalgebras in Definition 4.6, all the chain homomorphisms in Theorem 3.7 induce cohomology algebra isomorphisms. So we have cohomology algebra isomorphism

\[(H^*(C^*(K; \mathfrak{g})), \cup_{(K; \mathfrak{g})}) \cong (H^*_\mathcal{X}_m(K) \otimes H^*_\mathcal{Y}_m(\mathfrak{g}), \cup_{(K; \mathfrak{g})} \otimes \pi(\mathfrak{g})).\]

Proof The group isomorphisms in Theorem 3.7 naturally induce cohomology algebra isomorphisms. \(\square\)

Example 4.8 Let everything be as in Theorem 2.9. We compute the cohomology algebra over a field of \(\mathcal{Z}(K; \mathcal{Y}, B).\) By Theorem 6.9 and 7.11 of [17], we have algebra isomorphisms

\[
(H^*(Y_k), \cup) \cong (H^*_\mathcal{Y}_n_k(X_k) \otimes H^*_\mathcal{Y}_n_k(U_k, C_k), \cup_{X_k} \hat{\otimes} \pi(U_k, C_k)),
\]

\[
(H^*(B_k), \cup) \cong (H^*_\mathcal{Y}_n_k(A_k) \otimes H^*_\mathcal{Y}_n_k(U_k, C_k), \cup_{A_k} \hat{\otimes} \pi(U_k, C_k)),
\]

where \(\mathcal{Y}_n_k = \mathcal{R}_n_k\) or \(\mathcal{R}_n_k, \cup_{X_k}\) and \(\cup_{A_k}\) are the (right) universal (or (right) normal, etc.) product appearing in the theorems induced by the coproduct \(\psi_{X_k}\) and \(\psi_{A_k}\). Take \(\vartheta_k: (T^*_\mathcal{Y}_n_k(A_k), \psi_{A_k}) \to (T^*_\mathcal{Y}_n_k(X_k), \psi_{X_k})\) to be the (right) total chain complex inclusion and apply Theorem 4.5 for \(\vartheta = \{\vartheta_k\}_{k=1}^m\). Then we have algebra isomorphisms

\[
(H^*(\mathcal{Z}(K; Y, B), \cup) \cong (H^*_\mathcal{Y}_n(S(K; X, A)) \otimes H^*_\mathcal{Y}_n(U, C), \cup \hat{\otimes} \pi(U, C)),
\]

\[
(H^*_\mathcal{Y}_n(S(K; X, A), \cup) \cong (H^*_\mathcal{Y}_m(K) \otimes H^*_\mathcal{Y}_m; \mathcal{X}, A), \cup_{(K; \mathfrak{g})} \otimes \pi(\mathfrak{g})),
\]

where \(\mathcal{Y}_n = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m\) (\(\mathcal{Y}_1 = \mathcal{X}_n_1, \mathcal{Y}_2 = \mathcal{R}_n_2\) is possible), i.e.,

\[
H^*_\mathcal{Y}_n(\mathcal{Z}(K; Y, B)) \cong H^*_\mathcal{Y}_m(K) \otimes H^*_\mathcal{Y}_m; \mathcal{X}_m(A) \hat{\otimes} H^*_\mathcal{Y}_m(U, C).
\]

When all \(\mathcal{Y}_n = \mathcal{X}_n_k\) and the coproduct \(\psi_{X_k}, \psi_{A_k}\) are the universal (or normal, etc.) coproduct, then \(H^*_\mathcal{Y}_n(S(K; X, A)), \cup_{(K; \mathfrak{g})}\) is just the universal (or normal, etc.) algebra of \(\mathcal{S}(K; X, A).\) The right algebra case also holds by replacing \(\mathcal{X}\) by \(\mathcal{R}\).
Example 4.9 Let $L_k$ satisfy the condition of Example 4.3 and 4.4 for $k = 1, \ldots, m$. Then we have algebra isomorphisms

$$H^*_{\mathfrak{X}_n}(S(K; L_1, \ldots, L_m)) \cong H^*_{\mathfrak{X}_m}(K) \widehat{\otimes} (H^*_{\mathfrak{X}_{n_1}}(L_1) \otimes \cdots \otimes H^*_{\mathfrak{X}_{n_m}}(L_m)),$$

$$H^*_{\mathfrak{Y}_n}(S(K; L_1, \ldots, L_m)) \cong H^*_{\mathfrak{Y}_m}(K) \widehat{\otimes} (\tilde{H}^*_{\mathfrak{Y}_{n_1}}(L_1) \otimes \cdots \otimes \tilde{H}^*_{\mathfrak{Y}_{n_m}}(L_m)),$$

where $H^*_{\mathfrak{X}_m}(K)$ is the (right) strictly normal algebra of $K$ induced by the (right) strictly normal coproduct of $T^*\mathfrak{X}$ in Example 4.4 (4.3) by Definition 4.6, $\tilde{H}^*_{\mathfrak{Y}_{n_k}}(L_k)$ is the augmented cohomology algebra of $L_k$ relative to $[n_k]$ defined in Example 4.4.

5 Duality Isomorphism

In this section, we compute the Alexander duality isomorphism on some special type of polyhedral product spaces.

Theorem 5.1 Let $(X, A) = \{(X_k, A_k)\}_{k=1}^m$ be a sequence of topological pairs satisfying the following conditions.

1) Each homology group homomorphism $i_k : H_*(A_k) \to H_*(X_k)$ induced by inclusion is a split homomorphism.

2) Each $X_k$ is a closed orientable manifold of dimension $r_k$.

3) Each $A_k$ is a proper compact polyhedron subspace of $X_k$.

Let $(X, A^c) = \{(X_k, A_k^c)\}_{k=1}^m$ with $A_k^c = X_k \setminus A_k$. Then for all $(\sigma, \omega) \in \mathcal{P}_m$, there are duality isomorphisms ($r = r_1 + \cdots + r_m$)

$$\gamma_{\sigma, \omega} : H^*_\sigma(\sigma, \omega)(X, A) \to H^*_{r-|\omega|-*}(X, A^c),$$

$$\gamma^0_{\sigma, \omega} : H^*_\sigma(\sigma, \omega)(X, A) \to H^*_\sigma(\sigma, \omega)(X, A^c),$$

where $\bar{\sigma} = [m] \setminus (\sigma \cup \omega), H^*_\sigma(-)$ and $H^*_\sigma(-)$ are as in Theorem 3.9.
If the (co)homology is taken over a field, then the conclusion holds for $(X, A)$ satisfying the following conditions.

1) Each $X_k$ is a closed manifold of dimension $r_k$ orientable with respect to the homology theory over the field.

2) Each $A_k$ is a proper compact polyhedron subspace of $X_k$.

Proof We have the following commutative diagram of exact sequences

$$
\begin{array}{c}
\cdots \rightarrow H_n(A_k) \xrightarrow{i_k} H_n(X_k) \xrightarrow{j_k} H_n(X_k, A_k) \xrightarrow{\partial_k} H_{n-1}(A_k) \rightarrow \cdots \\
\downarrow \alpha_k \quad \downarrow \gamma_k \quad \downarrow \beta_k \quad \downarrow \alpha_k \\
\cdots \rightarrow H'^{r-n}(X_k, A_k^c) \xrightarrow{q_k^c} H'^{r-n}(X_k) \xrightarrow{p_k^c} H'^{r-n}(A_k^c) \xrightarrow{\partial_k^c} H'^{r-n+1}(X_k, A_k^c) \rightarrow \cdots,
\end{array}
$$

where $\alpha_k, \beta_k$ are the Alexander duality isomorphisms and $\gamma_k$ is the Poncaré duality isomorphism. So we have the following group isomorphisms

$$(\partial_k^c)^{-1}\alpha_k : \ker i_k \xrightarrow{\cong} \Sigma \ker p_k^c,$$

$$\gamma_k : \im i_k \xrightarrow{\cong} \ker p_k^c,$$

$$p_k^c \gamma_k : \coker i_k \xrightarrow{\cong} \im p_k^c.$$

Define $\theta_k : H^*_X(X_k, A_k) \rightarrow H^*_X(X_k, A_k^c)$ to be the direct sum of the above three isomorphisms. Then $\theta_1 \otimes \cdots \otimes \theta_m = \bigoplus_{(\sigma, \omega) \in \mathcal{X}_m} \gamma_{\sigma, \omega}$. The degree correspondence of $\theta_k$ is $H_\sigma^{*, 0} \rightarrow H^{r-k-n}_{\sigma, 0} ; H_*^{1, 0} \rightarrow H^{r-n}_{1, 0} ; H_*^{0, 1} \rightarrow H^{r-n-1}_{0, 1}$.

Now we construct similar homomorphism $\tilde{\theta}_k$ for the proof of Theorem 5.6. We have group isomorphisms

$$\begin{align*}
\alpha_k : & \quad \ker i_k \xrightarrow{\cong} \im \partial_k^c \subset H^*(X_k, A_k^c), \\
\gamma_k : & \quad \im i_k \xrightarrow{\cong} \coim q_k^c \subset H^*(X_k, A_k^c), \\
p_k^c \gamma_k : & \quad \coker i_k \xrightarrow{\cong} \coim p_k^c \subset H^*(X_k).
\end{align*}$$

Define $\tilde{\theta}_k : H^*_X(X_k, A_k) \rightarrow \tilde{H}^*_X(X_k, A_k^c)$ to be the direct sum of the above three isomorphisms with $\tilde{H}^*_X(X_k, A_k)$ the direct sum of the right side groups.

Then $\tilde{\theta}_1 \otimes \cdots \otimes \tilde{\theta}_m = \bigoplus_{(\sigma, \omega) \in \mathcal{X}_m} \tilde{\gamma}_{\sigma, \omega}$. Since $\tilde{H}_\sigma^* \cong H^*_\sigma, \tilde{H}^*_{1, 0} \cong H^*_1, \tilde{H}^*_{0, 1} \cong H^*_{0, 1}$, we have $\tilde{\gamma}_{\sigma, \omega} = \gamma_{\sigma, \omega}$. \qed
Theorem 5.2 Let $K$ and $K^\circ$ be the dual of each other relative to $[m]$. Then for all $(\sigma, \omega) \in \mathcal{X}_m$, $\omega \neq \emptyset$, there are duality isomorphisms

$$
\gamma_{K, \sigma, \omega}^\circ: H^\circ_{\sigma, \omega}(K) = \tilde{H}^\circ_{\sigma, \omega}(K) \to H^\circ_{\sigma, \omega}(K) = \tilde{H}^\circ_{\sigma, \omega}(K) = \tilde{H}^\circ_{[|\omega|-s-2}(K),
$$

$$
\gamma_{K, \sigma, \omega}^\circ: H^\circ_{\sigma, \omega}(K) = \tilde{H}^\circ_{\sigma, \omega}(K) \to H^\circ_{\sigma, \omega}(K) = \tilde{H}^\circ_{[|\omega|-s-2}(K),
$$

where $\tilde{\sigma} = [m] \setminus (\sigma \cup \omega)$, $|\omega|$ is the cardinality of $\omega$.

Proof Let $(C_*(\Delta^\omega, K_{\sigma, \omega}), d)$ be the relative simplicial chain complex. Since $\tilde{H}_s(\Delta^\omega) = 0$, we have a boundary isomorphism

$$
\partial: H_*(\Delta^\omega, K_{\sigma, \omega}) \xrightarrow{\cong} \tilde{H}_{s-1}(K_{\sigma, \omega}) = H^\circ_{\sigma, \omega}(K).
$$

$C_s(\Delta^\omega, K_{\sigma, \omega})$ has a set of generators consisting of all non-simplices of $K_{\sigma, \omega}$, i.e., $K_{\sigma, \omega}^c = \{\eta \subset \omega | \eta \notin K_{\sigma, \omega}\}$ is a set of generators of $C_*(\Delta^\omega, K_{\sigma, \omega})$. So we may denote $(C_*(\Delta^\omega, K_{\sigma, \omega}), d)$ by $(C_*(K_{\sigma, \omega}^c), d)$, where $\eta \in K_{\sigma, \omega}^c$ has degree $|\eta|-1$ with $|\eta|$ the cardinality of $\eta$. The correspondence $\eta \to \omega \setminus \eta$ for all $\eta \in K_{\sigma, \omega}^c$ induces a dual complex isomorphism

$$
\psi: (C_*(K_{\sigma, \omega}^c), d) \to (\tilde{C}_*(K_{\sigma, \omega}^c), \delta).
$$

Since $(K_{\sigma, \omega}^c)^\circ = (K^\circ)_{\tilde{\sigma}, \omega}$, we have induced homology group isomorphism

$$
\tilde{\psi}: H_*(\Delta^\omega, K_{\sigma, \omega}) \to H^\circ_{\tilde{\sigma}, \omega}(K). \text{ Define } \gamma_{K, \sigma, \omega} = \tilde{\psi}\partial^{-1}.
$$

Notice that when $\omega = \emptyset$, for $\sigma \in K$, $\tilde{\sigma} = [m] \setminus \sigma \notin K^\circ$ and there is no isomorphism from $H^\circ_{\sigma, \emptyset}(K) = \mathbb{Z}$ to $H^\circ_{\emptyset, \emptyset}(K) = 0$; for $\sigma \notin K$, $\tilde{\sigma} = [m] \setminus \sigma \in K^\circ$ and there is no isomorphism from $H^\circ_{\sigma, \emptyset}(K) = 0$ to $H^\circ_{\emptyset, \emptyset}(K) = \mathbb{Z}$.

Example 5.3 For the $S(K; L_1, \ldots, L_m)$ and index sets $\sigma, \omega, \hat{\sigma}, \hat{\omega}, \sigma_k, \omega_k$ in Example 3.12, $\gamma_{S(K; L_1, \ldots, L_m), \sigma, \omega} = \gamma_{K, \hat{\sigma}, \hat{\omega}} \otimes (\otimes_{\omega_k \neq \emptyset} \gamma_{L_k, \sigma_k, \omega_k})$.

Definition 5.4 For homology split $M = \mathcal{Z}(K; X, A)$, let $i: H_*(M) \to H_*(X)$ and $i^*: H^*(\tilde{X}) \to H^*(M)$ be the singular (co)homology homomorphism induced by the inclusion map from $M$ to $\tilde{X} = X_1 \times \cdots \times X_m$. From the
long exact exact sequences
\[
\cdots \to H_n(M) \xrightarrow{1} H_n(\tilde{X}) \xrightarrow{j} H_n(\tilde{X}, M) \xrightarrow{\partial} H_{n-1}(M) \to \cdots
\]
\[
\cdots \to H^{n-1}(M) \xrightarrow{\partial^o} H^n(\tilde{X}, M) \xrightarrow{j^o} H^n(\tilde{X}) \xrightarrow{i^o} H^n(M) \to \cdots
\]
we define
\[
\hat{H}_*(M) = \text{coim } i, \overline{\Omega}_*(M) = \text{ker } i, \hat{H}_*(\tilde{X}, M) = \text{im } j, \overline{\Omega}_*(\tilde{X}, M) = \text{coker } j,
\]
\[
\hat{H}^*(M) = \text{im } i^o, \overline{\Omega}^*(M) = \text{coker } i^o, \hat{H}^*(\tilde{X}, M) = \text{coim } j^o, H^*(\tilde{X}, M) = \text{ker } j^o.
\]

**Theorem 5.5** For a homology split space \( M = Z(K; X, A) \), we have the following group decompositions
\[
H_*(M) = \hat{H}_*(M) \oplus \overline{\Omega}_*(M), \quad H_*(\tilde{X}, M) = \hat{H}_*(\tilde{X}, M) \oplus \overline{\Omega}_*(\tilde{X}, M),
\]
\[
H^*(M) = \hat{H}^*(M) \oplus \overline{\Omega}^*(M), \quad H^*(\tilde{X}, M) = \hat{H}^*(\tilde{X}, M) \oplus \overline{\Omega}^*(\tilde{X}, M)
\]
and direct sum group decompositions
\[
\overline{\Omega}_{s+1}(\tilde{X}, M) \cong \overline{\Omega}_s(M) \cong \bigoplus_{(\sigma, \omega) \in \mathcal{X}_m} H_{s, \omega}^\sigma(K) \otimes H_{s, \omega}^\sigma(X, A),
\]
\[
\overline{\Omega}^{s+1}(\tilde{X}, M) \cong \overline{\Omega}^s(M) \cong \bigoplus_{(\sigma, \omega) \in \mathcal{X}_m} H_{s, \omega}^\sigma(K) \otimes H_{s, \omega}^\sigma(X, A),
\]
\[
\hat{H}_s(M) \cong \bigoplus_{\sigma \in K} H_{s, \emptyset}^\sigma(X, A), \quad \hat{H}_s(\tilde{X}, M) \cong \bigoplus_{\sigma \notin K} H_{s, \emptyset}^\sigma(X, A),
\]
\[
\hat{H}^s(M) \cong \bigoplus_{\sigma \in K} H_{s, \emptyset}^\sigma(X, A), \quad \hat{H}^s(\tilde{X}, M) \cong \bigoplus_{\sigma \notin K} H_{s, \emptyset}^\sigma(X, A),
\]
where \( \mathcal{X}_m = \{ (\sigma, \omega) \in \mathcal{X}_m | \omega \neq \emptyset \} \).

The conclusion holds for all polyhedral product spaces if the (co)homology group is taken over a field.

**Proof** By definition, \( i = \bigoplus_{(\sigma, \omega) \in \mathcal{X}_m} i_{\sigma, \omega} \) with
\[
i_{\sigma, \omega} : H_{s, \omega}^\sigma(K) \otimes H_{s, \omega}^\sigma(X, A) \xrightarrow{i \otimes 1} H_{s, \omega}^\sigma(\Delta^{|m|}) \otimes H_{s, \omega}^\sigma(X, A),
\]
where \( i \) is induced by inclusion and \( 1 \) is the identity. So
\[
\hat{H}_s(M) = \bigoplus_{\sigma \in K} H_{s, \emptyset}^\sigma(K) \otimes H_{s, \emptyset}^\sigma(X, A) \cong \bigoplus_{\sigma \in K} H_{s, \emptyset}^\sigma(X, A)
\]
Theorem 5.6 For $M = \mathcal{Z}(K; \mathcal{X}, \mathcal{A})$ such that $(\mathcal{X}, \mathcal{A})$ satisfies the condition of Theorem 5.1, the Alexander duality isomorphisms

\[ \alpha: H_*(M) \to H^{r-*}(\tilde{X}, M^c), \quad \alpha^\circ: H^*(M) \to H_{r-*}(\tilde{X}, M^c) \]

have direct sum decomposition $\alpha = \hat{\alpha} \oplus \overline{\alpha}$, $\alpha^\circ = \hat{\alpha}^\circ \oplus \overline{\alpha}^\circ$, where

\[ \hat{\alpha}: \hat{H}_*(M) \to \hat{H}^{r-*}(\tilde{X}, M^c), \quad \overline{\alpha}: \overline{H}_*(M) \to \overline{H}^{r-*}(\tilde{X}, M^c) \cong \overline{H}^{r-*-1}(M^c), \]

\[ \hat{\alpha}^\circ: \hat{H}^*(M) \to \hat{H}_{r-*}(\tilde{X}, M^c), \quad \overline{\alpha}^\circ: \overline{H}^*(M) \to \overline{H}_{r-*}(\tilde{X}, M^c) \cong \overline{H}_{r-*-1}(M^c) \]

are as follows. Identify all the above groups with the direct sum groups in Theorem 5.5. Then

\[ \hat{\alpha} = \oplus_{\sigma \in K} \gamma_{\sigma, \emptyset}, \quad \overline{\alpha} = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} \gamma_{K, \sigma, \omega} \otimes \gamma_{\sigma, \omega}, \]

\[ \hat{\alpha}^\circ = \oplus_{\sigma \in K} \gamma_{\sigma, \emptyset}^\circ, \quad \overline{\alpha}^\circ = \oplus_{(\sigma, \omega) \in \mathcal{X}_m} \gamma_{K, \sigma, \omega}^\circ \otimes \gamma_{\sigma, \omega}^\circ, \]

where $\gamma_-, \gamma_-^\circ$ are as in Theorem 5.1 and Theorem 5.2.

Proof Denote by $\alpha = \alpha_M$, $\hat{\alpha} = \hat{\alpha}_M$, $\overline{\alpha} = \overline{\alpha}_M$. Then for $M = \mathcal{Z}(K; \mathcal{X}, \mathcal{A})$ and $N = \mathcal{Z}(L; \mathcal{X}, \mathcal{A})$, we have the following commutative diagrams of exact sequences

\[
\begin{array}{cccccccc}
\ldots & \to & H_k(M \cap N) & \to & H_k(M) \oplus H_k(N) & \to & H_k(M \cup N) & \to & \ldots \\
\alpha_{M \cap N} \downarrow & & \alpha_M \oplus \alpha_N \downarrow & & \alpha_{M \cup N} \downarrow & & & \\
\ldots & \to & H^{r-*}(\tilde{X}, (M \cap N)^c) & \to & H^{r-*}(\tilde{X}, M^c) \oplus H^{r-*}(\tilde{X}, N^c) & \to & H^{r-*}(\tilde{X}, (M \cup N)^c) & \to & \ldots
\end{array}
\]

\[ (1) \]

\[
\begin{array}{cccccccc}
0 & \to & H_k(M \cap N) & \to & H_k(M) \oplus H_k(N) & \to & H_k(M \cup N; \mathcal{X}, \mathcal{A}) & \to & 0 \\
\hat{\alpha}_{M \cap N} \downarrow & & \hat{\alpha}_M \oplus \hat{\alpha}_N \downarrow & & \hat{\alpha}_{M \cup N} \downarrow & & & \\
0 & \to & H^{r-*}(\tilde{X}, (M \cap N)^c) & \to & H^{r-*}(\tilde{X}, M^c) \oplus H^{r-*}(\tilde{X}, N^c) & \to & H^{r-*}(\tilde{X}, (M \cup N)^c) & \to & 0
\end{array}
\]

\[ (2) \]
For \((\sigma, \omega) \in \mathcal{F}_m\), \(A = H^{\sigma, \omega}_1(X, A)\), \(B = H^{\sigma, \omega}_2(X, A)\), \(\gamma_1 = \gamma_{K \cap L, \sigma, \omega}\), \(\gamma_2 = \gamma_{K, \sigma, \omega} \oplus \gamma_{L, \sigma, \omega}\), \(\gamma_3 = \gamma_{K \cup L, \sigma, \omega}\), we have the commutative diagram

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & H^{\sigma, \omega}_k(K \cap L) \otimes A & \rightarrow & (H^{\sigma, \omega}_k(K) \oplus H^{\sigma, \omega}_k(L)) \otimes A & \rightarrow & H^{\sigma, \omega}_k(K \cup L) \otimes A & \rightarrow & \cdots \\
\gamma_1 \otimes \gamma_{\sigma, \omega} \downarrow & & \gamma_2 \otimes \gamma_{\sigma, \omega} \downarrow & & \gamma_3 \otimes \gamma_{\sigma, \omega} \downarrow & & \\
\cdots & \rightarrow & H^{\sigma, \omega}[k^{-1}((K \cap L)^c) \otimes B & \rightarrow & (H^{\sigma, \omega}[k^{-1}(K) \cap H^{\sigma, \omega}[k^{-1}(L)]) \otimes B & \rightarrow & H^{\sigma, \omega}[k^{-1}((K \cup L)^c) \otimes B & \rightarrow & \cdots
\end{array}
\]

The direct sum of all the above diagrams is the following diagram.

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & \overline{H}_k(M \cap N) & \rightarrow & \overline{H}_k(M) \oplus \overline{H}_k(N) & \rightarrow & \overline{H}_k(M \cup N) & \rightarrow & \cdots \\
\overline{\pi}_{M \cap N} \downarrow & & \overline{\pi}_M \oplus \overline{\pi}_N \downarrow & & \overline{\pi}_{M \cup N} \downarrow & & \\
\cdots & \rightarrow & \overline{H}^{r-k}(\tilde{X}, (M \cap N)^c) & \rightarrow & \overline{H}^{r-k}(\tilde{X}, M^c) \oplus \overline{H}^{-k}(\tilde{X}, N^c) & \rightarrow & \overline{H}^{-k}(\tilde{X}, (M \cup N)^c) & \rightarrow & \cdots
\end{array}
\]

(1), (2) and (3) imply that if the theorem holds for \(M\) and \(N\) and \(M \cap N\), then it holds for \(M \cup N\). So by induction on the number of maximal simplices of \(K\), we only need prove the theorem for the special case that \(K\) has only one maximal simplex.

Now we prove the theorem for \(M = \mathcal{Z}(\Delta^S; X, A)\) with \(S \subset [m]\). Then

\[
M = Y_1 \times \cdots \times Y_m, \quad Y_k = \begin{cases} X_k & \text{if } k \in S, \\ A_k & \text{if } k \notin S. \end{cases}
\]

So \((\tilde{X}, M^c) = (X_1, Y_1^c) \times \cdots \times (X_m, Y_m^c)\).

Let \(\tilde{H}^*_\mathcal{F}(X_k, A_k), \tilde{\theta}_k, \gamma_{\sigma, \omega}, \tilde{\gamma}_{\sigma, \omega}\) be as in the proof of Theorem 5.1. Then we have the following commutative diagram

\[
\begin{array}{cccccccc}
H^*_k(M) & \rightarrow & H^*_k(\tilde{X}, M^c) \\
\| & & \| \\
H^*_k(Y_1) \otimes \cdots \otimes H^*_k(Y_m) & \rightarrow & H^*_k(\tilde{X}_1, Y_1^c) \otimes \cdots \otimes H^*_k(\tilde{X}_m, Y_m^c) \\
\| \otimes & & \| \\
\oplus_{\sigma \subseteq S, \omega \cap S = \emptyset} H^{\sigma, \omega}_2(X, A) & \rightarrow & \oplus_{\sigma \subseteq S, \omega \cap S = \emptyset} H_{\sigma, \omega}^{\tilde{\mathcal{F}}}(X, A^c) \\
\cap & & \cap \\
\oplus_{\sigma \subseteq S, \omega \cap S = \emptyset} H^{\sigma, \omega}_2(X_1, A_1) \otimes \cdots \otimes H^{\sigma, \omega}_2(X_m, A_m) & \rightarrow & \oplus_{\sigma \subseteq S, \omega \cap S = \emptyset} \tilde{H}_{\sigma, \omega}^{\tilde{\mathcal{F}}}(X, A^c) \\
\end{array}
\]

where \(\tilde{H}^*_\mathcal{F}(X_1, A_1^c) \otimes \cdots \otimes \tilde{H}^*_\mathcal{F}(X_m, A_m^c) = \oplus_{(\sigma, \omega) \in \mathcal{F}_m} \tilde{H}^*_{\sigma, \omega}(X, A^c)\).
For \( \sigma \subset S, \omega \cap S = \emptyset, \omega \neq \emptyset \), we have
\[
\begin{align*}
\gamma \Delta^\sigma_{\sigma, \omega} \otimes \gamma \sigma, \omega & \quad \mapsto \quad H_0^{\sigma, \omega}(\Delta^\sigma) \otimes H^{\sigma, \omega}_0(\underline{X}, \underline{A}) \\
& \quad \mapsto \quad H^{\sigma, \omega}_0(\underline{X}, \underline{A}) \quad (c \mathbb{F}^{r-s-1}(M^c))
\end{align*}
\]

So with the identification of the theorem, the third row of (4) is the direct sum \( \alpha_M = \bigoplus_{\sigma \subset S} \gamma_{\sigma, \omega} \) and \( \alpha_M = \hat{\alpha}_M \oplus \pi_M \).

Example 5.7 Regard \( S^{r+1} \) as one-point compactification of \( \mathbb{R}^{r+1} \). Then for \( q \leq r \), the standard space pair \((S^{r+1}, S^q)\) is given by
\[
S^q = \{(x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} \mid x_1^2 + \cdots + x_{q+1}^2 = 1, x_i = 0, \text{ if } i > q+1\}.
\]

Let \( M = Z_K \left( \begin{array}{ccc} r_1+1 & \cdots & r_m+1 \\
q_1 & \cdots & q_m \end{array} \right) = Z(K; X, A) \) be the polyhedral product space such that \((X_k, A_k) = (S^{r_k+1}, S^{q_k})\). Since \( S^{r-q} \) is a deformation retract of \( S^{r+1} \setminus S^q \), the complement space \( M^c = Z(K^\circ; X, A^c) \) is homotopy equivalent to \( Z_K \left( \begin{array}{ccc} r_1+1 & \cdots & r_m+1 \\
r_1-q_1 & \cdots & r_m-q_m \end{array} \right) \).

Since all \( H^\sigma_*(X, A) \cong \mathbb{Z} \), we may identify \( H^\sigma_*(K) \otimes H^\sigma_*(X, A) \) with \( \Sigma H^\sigma_*(K) \), where \( t = \Sigma_{k \in \sigma} (r_k+1) + \Sigma_{k \in \omega} q_k \). For \( \sigma \subset [m] \), let \( \mathbb{Z}_\sigma \) be the free group generated by \( \sigma \) with degree 0. Then
\[
\hat{H}_*(M) = \bigoplus_{\sigma \subset K} \Sigma_{k \in \sigma} (r_k+1) \mathbb{Z}_\sigma,
\]
\[
\overline{H}_*(M) = \bigoplus_{(\sigma, \omega) \in \mathcal{F}_m} \Sigma_{k \in \sigma} (r_k+1) + \Sigma_{k \in \omega} q_k H^\sigma_*(K).
\]

Dually, the cohomology of the complement space \( M^c \) is
\[
\hat{H}^*(M^c) = \bigoplus_{\sigma \subset K^\circ} \Sigma_{k \in \sigma} (r_k+1) \mathbb{Z}_\sigma,
\]
\[
\overline{H}^*(M^c) = \bigoplus_{(\sigma, \omega) \in \mathcal{F}_m} \Sigma_{k \in \sigma} (r_k+1) + \Sigma_{k \in \omega} (r_k-q_k) H^\sigma_*(K^\circ).
\]

In this case, the direct sum of \( \gamma_{K, \sigma, \omega} : H^\sigma_*(K) \to H^{\sigma, \omega}_{\sigma, \omega}(K^\circ) \) over all \((\sigma, \omega) \in \mathcal{F}_m \) (regardless of degree) is the isomorphism \( \overline{H}_*(M) \cong \overline{H}^{r-s-1}(M^c) \).
Specifically, $Z(K; S^{2n+1}, S^n) = Z_K \left( \frac{2n+1}{n} \ldots \frac{2n+1}{n} \right)$. Then we have

$$\overline{H}_*(Z(K; S^{2n+1}, S^n)) \cong \overline{H}^{(2n+1)m-s-1}(Z(K^n; S^{2n+1}, S^n)).$$

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