Superconducting Transition in Doped Antiferromagnet

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We investigate the superconducting transition in a doped antiferromagnet. Based on the phase string framework of the t-J model, an effective model describing the phase-coherence transition is obtained and is studied through duality transformation and renormalization group treatment. We show that such a topological transition is controlled by spin excitations, with the transition temperature determined by a characteristic spin excitation energy. The existence of an Ising-like long range order of staggered current loops is also discussed.

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One of puzzles in high-\(T_c\) cuprate superconductors is that the behavior of the energy gap in the quasiparticle channel is quite different from that of the superconducting transition temperature \(T_c\). In the underdoped regime, two scales show divergent doping dependence: the energy gap seems to increase as the hole concentration \(\delta\) is reduced, while \(T_c\) itself monotonically decreases [1]. The former is even present above \(T_c\). By contrast, in the BCS theory of superconductivity, the quasiparticle energy gap is simply proportional to \(T_c\) at \(T = 0\) and vanishes above \(T_c\). So \(T_c\) seems controlled by a rather different low-energy physics in the cuprate superconductors. For example, the well-known Uemura plot [2] shows the direct proportionality between \(T_c\) and the phase stiffness.

Emery and Kivelson have conjectured [3], for underdoped cuprates, \(T_c\) is decided by the phase coherence of the pairing order parameter. On the other hand, a quantitative feature has been recently revealed by inelastic neutron-scattering experiments. It was found [4] that \(T_c\) is correlated, in roughly a linear relation, with the characteristic spin energy scale \(E_g\) associated with a magnetic “resonance-like” peak [5], which decreases with the doping concentration.

How to establish the connection of \(T_c\) with either the phase coherence [3,4] or the spin resonance energy [5,6] is currently a hot topic in various modelling, but few attempt has been made to put both the phase coherence and spin resonance energy within a single framework in understanding the mechanism of superconducting transition in the cuprates. In this paper, we approach this issue by using a microscopic theory of doped antiferromagnet based on the \(t-J\) model. In such a description, one starts from the half-filling where the antiferromagnetism is well understood. The doping will then introduce the so-called phase string effect as doped holes pick up sequences of nontrivial signs from the spin background during their hopping [7]. Such a phase string effect at finite doping will destroy the long-range antiferromagnetic spin order, leading to a sharp “resonance-like” peak at a doping-dependent energy scale \(E_g \sim J\delta\) in the superconducting phase, characterized by the holon Bose condensation [10]. Due to the same phase string effect, spin excitations in the superconducting state induce strong phase frustrations on the holon concentration and eventually destroy the phase coherence of the latter at a finite temperature. We will demonstrate that \(T_c\) as the onset of the phase coherence temperature is indeed proportional to \(E_g\), which is not directly associated with the “energy gap” in the quasiparticle channel. As a by-product of the microscopic theory, we also show that there exists staggered current loops in the superconducting ground state, with a hidden broken \(Z_2\) symmetry.

We start from the \(t-J\) model. In the slave-particle representation one can take the aforementioned singular phase string effect into account by a decomposition [8]: \(c_{i\sigma} = h_i^\dagger b_{i\sigma} e^{i\Theta_{i\sigma}}\), where \(h_i^\dagger\) and \(b_{i\sigma}\) are bosonic holon and spinon operators, respectively. Using it, the local singularity of phase strings can be “gauged away” from the Hamiltonian and kept in the phase factor [9]. It is believed that the resulting nontrivial topological effect of phase strings is the key to understanding the physics of the two-dimensional (2D) \(t-J\) model. Based on the exact reformulation of the \(t-J\) model with explicitly incorporating the phase string effect, an effective Hamiltonian was obtained [10]:

\[
H_{\text{eff}} = H_h + H_s, \quad \text{in which the holon Hamiltonian is given by}
\]

\[
H_h = -t_h \sum_{<ij>} e^{i(A_{ij}^s + \phi_{ij})} h_i^\dagger h_j + \text{H.c.}
\]  

Here the phase string effect is precisely tracked by the lattice gauge field \(A_{ij}^s\) and \(\phi_{ij}\), satisfying \(\sum_s A_{ij}^s = \pi/4\sum_{\text{plaquettes}} \sum_{s} \sigma n_{i\sigma}^{h} \) and \(\sum_{s} \phi_{ij}^s = \pm \pi\) per plaquette, respectively. Note that \(n_{i\sigma}^{h}\) denotes the spinon number operator at site \(i\). So \(A_{ij}^s\) will mediate the main influence of the spinon degrees of freedom on the holon part in a form of gauge field.

In this framework, the superconducting phase is realized by the holon Bose condensation [10]. To study its phase transition, we shall assume that the amplitude of the holon condensation has been formed at some charac-
teristic temperature $T^* \geq T_c$ such that the holon operator can be written as $h_i = \sqrt{\rho_h} e^{i \vartheta_i} (\rho_h \simeq \delta)$. Then the holon Hamiltonian can be reexpressed as

$$H_h = -\rho_h g \sum_{(ij)} \cos[\theta_i - \theta_j - \varphi_{ij} - A_{ij}^s],$$

(2)

where $g = 2t_h$. In the spinon resonating-valence-bond (RVB) background, the paired spinons will not directly contribute to the lattice gauge field $A_{ij}^s$ according to its definition. So only thermally excited spinons will be seen by holons through $A_{ij}^s$ in (2). Without the lattice gauge field $A_{ij}^s$, the holon Bose condensation would occur as a conventional Kosterlitz-Thouless (KT) transition ($T_\text{KT}$ is the holon number operator) describes fictitious quanta.

$\text{So}$ condensed, $[9]$. In the superconducting phase as holons are Bose-condensed, $H_s$ has been studied in Ref. 9, where it has been estimated $[11]$ to be over $1000K$, using the parameters of the $t - J$ Hamiltonian, which is about one order of magnitude higher than observed in the cuprate superconductors. In the following, we will show how $A_{ij}^s$ can effectively bring down the transition temperature to a value determined by the spin characteristic energy, consistent with the experiment.

We note that the spinon part is governed by $H_s = -J_s \sum_{<ij>} e^{i \vartheta_A h_A^i h_A^j - \sigma} + H.c.$ At half-filling, without $e^{i \vartheta A h_A}$, $H_s$ reduces to the mean-field version of Schwinger-boson representation of the Heisenberg model $[10]$, which well captures the antiferromagnetic correlations there. Upon doping, $\sum_{\sigma} A_{ij}^h = 1/2 \sum_{\ell \in \sigma} n^h_{\ell} (n^h_{\ell})$ is the holon number operator describes fictitious quantized $\pi$ fluxoids bound to holons and seen by spinons. So $A_{ij}^h$ will represent the doping influence on the spinon part, a consequence again due to the phase string effect $[9]$. In the superconducting phase as holons are Bose-condensed, $H_s$ has been studied in Ref. 9, where it is shown that a resonant-like peak will emerge in the spin dynamic susceptibility at $(\pi, \pi)$ at a finite energy $E_g = 2E_s$ with $E_s$ denoting the single spinon excitation energy. If higher energies are neglected at low $T$, one has $H_s \approx E_g / 2 \sum_{\sigma} \gamma_{\sigma} \gamma_{\sigma}$, where $\gamma_{\sigma}$ is the Bogoliubov operator for spinon excitations $[10]$. Note that $A_{ij}^h$ only depends on the density of holons. So one expects that the peak is still present at $T_c < T < T^*$ as long as the amplitude of the holon condensation persists. To first order approximation, in the following we shall treat $E_g$ or $E_s$ as a $T$-independent quantity below $T^*$.

Now we can write down the partition function corresponding to the present system as follows:

$$Z = \sum_{\{n_{\sigma}\}} \int D\theta D\theta' A^s_{\mu} D\phi^0_{\mu} \times$$

$$\delta^{\beta} \sum_{\rho_h} e^{i \vartheta_A h_A^i h_A^j - \sigma} \times$$

$$\sum_{\sigma} \gamma_{\sigma} \gamma_{\sigma},$$

(3)

where $\beta = \rho_h g / T$, $\gamma_{\sigma} \equiv \gamma_{\sigma}^i \gamma_{\sigma}^j$, and the primes in $D' A^s_{\mu}$ and $D' \phi^0_{\mu}$ imply that $A^s_{\mu}$ and $\phi^0_{\mu}$ satisfy the constraints on $\sum_{\sigma} A^s_{\mu}$ and $\sum_{\sigma} \phi^0_{\mu}$ given above. For convenience, the subscript $\mu$ is introduced here to denote the link $ij$, and $\theta_i - \theta_j$ is replaced by $\Delta_{ij}(\theta(r))$.

First of all, let us discuss the role of $\phi^0_{\mu}(r)$. Note that the Hamiltonian (2) has the form of an extended xy model. Apart from the $U(1)$ symmetry, there is also an additional local $Z_2$ symmetry which corresponds to the invariance for a transformation $\sum_{\sigma} \phi^0_{\mu} = \pm \pi \rightarrow \mp \pi$ at each plaquette (one can realize this by changing a link phase by $2\pi$ within each plaquette). Like the xy model, one may reexpress the partition function (3) in the Coulomb gas representation through a standard duality transformation $[12]$. In this representation there are three different topological charges on the dual lattice site, corresponding to the vorticities of a supercurrent loop induced by the $\pi$ flux of $\phi^0_{\mu}(r)$, vortices bound to spinons through gauge field $A_{ij}^s$, and the conventional $2\pi$ vortices, respectively. All these topological charges are coupled with each other through long range logarithmic interactions. This lead to correlations among the topological charges. It is clear that topological charges with opposite vorticities have a tendency to pair at low temperature. Especially the pairing between the opposite sites of topological vortices related to $\phi^0_{\mu}$ means that local $Z_2$ symmetry can be broken at low temperature $[13]$. One can expect that apart from the topological transition, there is also a Ising-like long rang order of staggered current loop at low temperature $[14]$. Because of $\phi^0_{\mu}$, there are two degenerate ground states and, correspondingly, there are two low energy modes $[15]$. By constructing a Landau-Ginzburg-Wilson description, one can demonstrate that (3) may be decomposed into two coupled extended xy models. In this representation the topological transition is determined by the extended xy terms corresponding to each low-energy mode, while the $Z_2$ symmetry broken determined by the renormalized behavior of the coupling constant $h$ between the two modes where $h$ is renormalized to strong coupling, and the two modes are locked with relative phase $0$ or $\pi$ $[13]$. Since generally the $Z_2$ symmetry broken temperature is higher than the topological transition temperature $[17]$, in the following we mainly focus on the low-$T$ phase without further considering the fluctuation effects induced by $\phi^0_{\mu}(r)$. The main effect of the modes coupling due to $\phi^0_{\mu}(r)$ will be represented by the renormalization of $\beta$. One can introduce an effective $\beta' = \rho_h g' / T$ with $g'$ being replaced by $g$ to denote the effect. A further study on the $Z_2$ symmetry-broken and staggered current loop is to be given elsewhere.

The distinctive feature under this single mode approximation is the spinon-vortex introduced by $A^s_{\mu}$ with the corresponding vorticity given by $\gamma_{\sigma}^i \gamma_{\sigma}^j$. From the standard Villain approximation (3), and duality transformation $[12]$, we can then arrive at the following form
with this result we see that the $2\pi$ the density of doped hole:

$$m(r^*) = 0, \pm 1, \pm 2, \ldots,$$

represent the topological charges of the ordinary $2\pi$ vortex of the xy model, and $\pi(r)$ represents the topological charge of the $\pi$ vortices associated with spinon-vortices, where $r^*$ denotes a lattice site dual to $r$. From (4) we see that there are two types of vortex perturbations for the “spin wave” fixed line of the system: one is the vortex with vorticity $2\pi$ and the other is the vortex with vorticity $\pi$. According to Amit et. al [19], the perturbation of vortex with vorticity $n\pi$ has the scaling dimension $n^2/2$ near the critical point.

With this result we see that the $2\pi$ vortex perturbation is irrelevant compared to the $\pi$ vortex. It will not affect the critical behavior of the system. We can thus only focus on the effect of the $\pi$ vortices, associated with excited spinons, in the following study of low-temperature topological transition. The last term on the right hand side of (4) controls the fluctuations of $n(r^*)$. In fact, one can write down the relation:

$$n(r^*) \simeq \sum s, \sigma |w_s(r^*)|^2 \sigma n_{s, \sigma}^\gamma,$$  

in which $w_s(r)$ is the single-particle wave function for spinons at $E_s$ [4] and in the coherent-state representation [20] $w_s(r) \simeq c_s e^{-(r-r_s)^2/4l_0^2}$, with $c_s$ the normalized constant. In this representation the quantum numbers of the states are denoted by the position $r_s$ of their centers which form a von Neumann lattice with lattice constants $a_s = b_s \approx 2l_0$ which is a function of the density of doped hole: $l_0 = \frac{a}{\sqrt{\pi} \sigma}$. At low temperature $T \ll E_g$, the terms with $n_{s, \sigma}^\gamma = 0$ dominates, and the terms with $n_{s, \sigma}^\gamma = 1$ can be regarded as small corrections. Then

$$\sum_{n_{s, \sigma}^\gamma} \exp \{\pi i \sum_s n(r^*) - E_g/2T \sum n_{s, \sigma}^\gamma \} \simeq \exp \{y_b \sum \cos[\pi \sum_s |w_s(r^*)|^2 \phi(r^*)]\},$$

where $y_b = 1/\cosh(E_g/2T)$ ($y_b < 1$), in deducing it all the terms with order $O(y_b^n)$ ($n > 1$) have been omitted. It is easy to understand that $y_b$ measures how easy (difficult) to excite a spinon. Finally the continuum form of the partition function is obtained as follows (for convenience in the following we use $r$ instead of $r^*$ to denote the dual lattice site):

$$Z = \int D\phi e^{-\frac{1}{2} \int d^2r(\Delta_\phi \phi)^2 + \frac{\nu_s^4}{2l_0^2} \int d^2r \cos[\sqrt{\pi} \phi(r) x]}$$

$$\int d^2r |w_s(r)|^2 \phi(r) - y \int d^2r \beta \int d^2r \partial_s \phi(r^*)^2 \phi(r)^2,$$

where $a$ is the lattice constant. We have introduced a term $y \int d^2r \frac{\nu_s^4}{2l_0^2} \int d^2r \partial_s \phi(r^*)^2 \phi(r)^2$ into the partition function, which will be generated by renormalization group (RG) procedure discussed below with its initial value being zero. The partition function is now parametrized by three quantities: $\beta', y$, and $y_b$.

Compared to the conventional xy model, the partition function (6) looks more complicated. The physical origin of this complexity comes from the spinons, which are at centers of the $\pi$ vortices. Note that an excited spinon does a cyclotron motion (3), and in our coherent-state representation its distribution function $|w_s(r)|^2$ has an attenuation radius $l_0$. So the spinon-vortex has a finite vortex-core of radius $l_0$, which is reflected in (6) through $|w_s(r)|^2$. We shall treat the problem by means of the Wilson’s RG analysis [21].

In this RG procedure we first divide $\phi$ into the high energy part and low energy part, respectively: $\phi = \phi_f + h$, where $\phi_f(r) = \int_0^l \int_0^r d^2p/(2\pi)^2 \phi(r) \exp[ip \cdot \mathbf{x}]$, $\phi_f(r) = \int_0^l \int_0^r d^2p/(2\pi)^2 \phi(r) \exp[ip \cdot \mathbf{x}]$, $\Lambda = \Lambda_f - d\lambda$, and $\Lambda = 1/a$ is the momentum cut off; then the high energy part $h$ is averaged out by means of the cumulant expansion; finally we rescale the system so as to restore the original cutoff. Because of the presence of a characteristic length scale $2l_0$ of the vortex core, the RG analysis is separated into two steps; we first treat local physics within the length scale $2l_0$, then study the low energy and long wave-length physics beyond such a scale. After some algebra, following recursion relations can be obtained

$$dl_0 = -d_0 \frac{da}{a},$$  

$$dy_b = -\frac{\beta'}{4} (\Lambda_0^2) \frac{dy_b}{a},$$  

$$dy = c_1 \beta'^2 \frac{1}{(\Lambda_0^2)^2} \frac{dy}{a},$$

where $c_1 = 0.018\pi^2$. From (7)-(9) the effective parameters at the length scale $2l_0$ can be determined: $y_b = y_b(\Lambda_f) = \frac{\mu_y}{\cosh(E_g/2T)}$, $y = y(\Lambda_f) = 2c_1 \beta'^2 \frac{1}{\cosh(E_g/2T)}$, where $\Lambda_f = 1/(2l_0)$. With the length scale $2l_0$ being reduced to our “new” lattice constant $a$, the effective attenuation radius of $|w_s(r)|^2$ also “shrinks” from $l_0$ to $a/2$. So the effective range of $|w_s(r)|^2$ is within the unit plaquette of the new lattice now, which can be reasonably treated as the $\delta(r)$ function, such that $\frac{1}{2} \int d^2r |w_s(r)|^2 \phi(r) \approx \delta(r_s)$.

The kinetic term in (6) can be then rewritten as $-\frac{1}{2} (1 + 2y') \int d^2r \left[ \partial_r \phi(r) \right]^2$.

After the variable change $\phi \rightarrow \frac{1}{\sqrt{1 + 2y'}} \phi$, we finally obtain

$$Z = \int D\phi e^{-\frac{1}{2} \int d^2r(\Delta_\phi \phi)^2 + \frac{\nu_s^4}{2l_0^2} \int d^2r \cos[\sqrt{\pi} \phi(r) x]}$$

$$\int d^2r |w_s(r)|^2 \phi(r) - y \int d^2r \beta \int d^2r \partial_s \phi(r^*)^2 \phi(r)^2,$$

where $\beta'' = \beta'/(1 + 2y')$. Equation (10) is exactly the partition function of sine-Gordon model [20,22,24] with effective parameters $y_b'$ and $\beta''$. 

3
The dashed line represents the ordinary KT transition temperature (i.e., $T_c$), at which pairing between spinon-vortices and -antivortices dissolves, is determined by the following equation \[ (11): \]

\[
\pi/4 \beta'' - 2 = y'/2. \tag{11}
\]

Numerical results are shown in Fig. 1. $T_c$ as a function of $E_g$ is plotted at several doping concentrations in Fig. 1(a). It shows that $T_c$ increases monotonically with $E_g$ and saturates at larger $E_g$. According to Fig. 1(a), $T_c/g'$ linearly scales with $E_g/g'$ at small $E_g/g'$ where $T_c/E_g$ does not depend on $g'$, with the ratio ($\sim 1/4 - 1/5$) being only weakly doping dependent. So $T_c$ essentially is determined by the characteristic spin resonance-like energy $E_g$. For instance, plugging in the experimental value of $E_g = 41 \text{ meV}$ for the optimal doping YBCO compound \[ , \]

one estimates $T_c \sim 100 \text{ K}$, very close to the experimental value. Since $E_g$ as a function of $\delta$ has already been obtained in the same framework \[ , \]

as re-plotted in the inset of Figs. 1(b) and (c) (with $J_g = 0.1 \text{ eV}$ in Ref. \[ , \]

one can use such a calculated $E_g(\delta)$ to determine $T_c$ vs. $\delta$ as shown in Fig. 1(b) (solid curve). For comparison, the dashed line in Fig. 1(b) represents $T_{KT} \sim \pi \delta g'/2$ without including the spinon-vortices due to $A_{s'}$. Here we choose $g' \sim 2t_h = 0.2 \text{ eV}$, and find $T_{KT} \sim 510 \text{ K}$ at $\delta = 0.14$ while $T_c = 107 \text{ K}$. As noted before, $T_c$ is not sensitive to $g'$, but $T_{KT}$ does. So if we take $g' = 0.5 \text{ eV}$ at the same $\delta = 0.14$, $T_c$ increases to 164 $K$ but $T_{KT}$ reaches 1276 $K$. Therefore, $E_g$ effectively brings $T_c$ down to the right order of magnitude as the consequence that the spinon-vortices instead of the conventional $2\pi$ vortices control the superconducting phase coherence transition. In Fig. 1(c), $E_g(\delta) - T_c(\delta)$ is shown, which also are both qualitatively and quantitatively in good agreement with the experimental results \[ . \]

In conclusion, we have established a quantitative theory of superconducting transition based on an effective spin-charge separation description of the doped AF Mott insulator. The underlying physics is that the phase coherence transition is controlled by thermal spin excitations, which substantially reduce $T_c$ from $T_{KT}$ to a fraction of $E_g/k_B$. It resolves the long-standing issue of how $T_c$ can be quantitatively connected to the characteristic spin energy scale in a doped AF Mott insulator. The obtained $T_c - \delta$ and $E_g - T_c$ relations are in good agreement with those observed in the cuprate superconductors, lending a strong support for the experimental relevance of the spin-charge separation theory.

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The role of $A_{ij}$ is to introduce $\pi$ vortices associated with spinons. We note that the FFXY model without $A_{ij}$ has been also studied recently by Z. Nussinov in cond-mat/0107339.

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