L-PACKETS AND DEPTH FOR SL₂(K) WITH K A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2

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Abstract. Let \( G = \text{SL}_2(K) \) with \( K \) a local function field of characteristic 2. We review Artin-Schreier theory for the field \( K \), and show that this leads to a parametrization of certain \( L \)-packets in the smooth dual of \( G \). We relate this to a recent geometric conjecture. The \( L \)-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal \( L \)-packets of cardinality 4 are parametrised by biquadratic extensions. Each supercuspidal packet of cardinality 4 is accompanied by a singleton packet for \( \text{SL}_1(D) \). We compute the depths of the irreducible constituents of all these \( L \)-packets for \( \text{SL}_2(K) \) and its inner form \( \text{SL}_1(D) \).

1. Introduction

The special linear group \( \text{SL}_2 \) has been a mainstay of representation theory for at least 45 years, see \cite{GGPS}. In that book, the authors show how the unitary irreducible representations of \( \text{SL}_2(\mathbb{R}) \) and \( \text{SL}_2(\mathbb{Q}_p) \) can be woven together in the context of automorphic forms. This comes about in the following way. The classical notion of a cusp form \( f \) in the upper half plane leads first to the concept of a cusp form on the adele group of \( \text{GL}_2 \) over \( \mathbb{Q} \), and thence to the idea of an automorphic cuspidal representation \( \pi_f \) of the adele group of \( \text{GL}_2 \). We recall that the adele group of \( \text{GL}_2 \) is the restricted product of the local groups \( \text{GL}_2(\mathbb{Q}_p) \) where \( p \) is a place of \( \mathbb{Q} \). If \( p \) is infinite then \( \mathbb{Q}_p \) is the real field \( \mathbb{R} \); if \( p \) is finite then \( \mathbb{Q}_p \) is the \( p \)-adic field. The unitary representation \( \pi_f \) may be expressed as \( \otimes \pi_p \) with one local representation for each local group \( \text{GL}_2(\mathbb{Q}_p) \). It is this way that the unitary representation theory of groups such as \( \text{GL}_2(\mathbb{Q}_p) \) enters into the modern theory of automorphic forms.

Let \( X \) be a smooth projective curve over \( \mathbb{F}_q \). Denote by \( F \) the field \( \mathbb{F}_q(X) \) of rational functions on \( X \). For any closed point \( x \) of \( X \) we denote by \( F_x \) the completion of \( F \) at \( x \) and by \( \mathfrak{o}_x \) its ring of integers. If we choose a local coordinate \( t_x \) at \( x \) (i.e., a rational function on \( X \) which vanishes at \( x \) to order one), then we obtain isomorphisms \( F_x \simeq \mathbb{F}_{q_x}(t_x) \) and \( \mathfrak{o}_x \simeq \mathbb{F}_{q_x}[[t_x]] \), where \( \mathbb{F}_{q_x} \) is the residue field of \( x \); in general, it is a finite extension of \( \mathbb{F}_q \) containing \( q_x = q^\deg(x) \) elements. Thus, we now have a local function field attached to each point of \( X \).

With all this in the background, it seems natural to us to study the representation theory of \( \text{SL}_2(K) \) with \( K \) a local function field. The case when \( K \) has characteristic 2 has many special features – and we focus on this case in this article. A local function field \( K \) of characteristic 2 is of the form \( K = \mathbb{F}_q((t)) \), the field of Laurent series with coefficients in \( \mathbb{F}_q \), with \( q = 2^f \). This example is particularly interesting because there are countably many quadratic extensions of \( \mathbb{F}_q((t)) \).

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Artin-Schreier theory is a branch of Galois theory, and more specifically is a positive characteristic analogue of Kummer theory, for Galois extensions of degree equal to the characteristic $p$. Artin and Schreier (1927) introduced Artin-Schreier theory for extensions of prime degree $p$, and Witt (1936) generalized it to extensions of prime power degree $p^n$. If $K$ is a field of characteristic $p$, a prime number, any polynomial of the form

$$X^p - X + \alpha$$

for $\alpha \in K$, is called an Artin-Schreier polynomial. When $\alpha$ does not lie in the subset $\{y \in K \mid y = x^p - x \text{ for } x \in K\}$, this polynomial is irreducible in $K[X]$, and its splitting field over $K$ is a cyclic extension of $K$ of degree $p$. This follows since for any root $\beta$, the numbers $\beta + i$, for $1 \leq i \leq p$, form all the root – by Fermat’s little theorem – so the splitting field is $K(\beta)$. Conversely, any Galois extension of $K$ of degree $p$ equal to the characteristic of $K$ is the splitting field of an Artin-Schreier polynomial. This can be proved using additive counterparts of the methods involved in Kummer theory, such as Hilbert’s theorem 90 and additive Galois cohomology. These extensions are called Artin-Schreier extensions.

For the moment, let $F$ be a local nonarchimedean field with odd residual characteristic. The $L$-packets for $\text{SL}_2(F)$ are classified in the paper [LR] by Lansky-Raghuram. They comprise: the principal series $L$-packets $\xi_E = \{\pi_E^1, \pi_E^2\}$ where $E/F$ is a quadratic extension; the unramified supercuspidal $L$-packet of cardinality 4; and the supercuspidal $L$-packets of cardinality 2.

We now revert to the case of a local function field $K$ of characteristic 2. We consider $\text{SL}_2(K)$. Drawing on the accounts in [Da, Th1, Th2], we review Artin-Schreier theory, adapted to the local function field $K$, with special emphasis on the quadratic extensions of $K$.

The $L$-packets in the principal series of $\text{SL}_2(K)$ are parametrized by quadratic extensions, and the supercuspidal $L$-packets of cardinality 4 are parametrised by biquadratic extensions $L/K$. There are countably many such supercuspidal $L$-packets.

In this article, we do not consider supercuspidal $L$-packets of cardinality 2.

The concept of depth can be traced back to the concept of level of a character. Let $\chi$ be a non-trivial character of $K^\times$. The level of $\chi$ is the least integer $n \geq 0$ such that $\chi$ is trivial on the higher unit group $U_K^{n+1}$, see [BH, p.12]. The depth of a Langlands parameter $\phi$ is defined as follows. Let $r$ be a real number, $r \geq 0$, let $\text{Gal}(K_s/K)^r$ be the $r$-th ramification subgroup of the absolute Galois group of $K$. Then the depth of $\phi$ is the smallest number $d(\phi) \geq 0$ such that $\phi$ is trivial on $\text{Gal}(K_s/K)^r$ for all $r > d(\phi)$.

The depth $d(\pi)$ of an irreducible $\mathcal{G}$-representation $\pi$ was defined by Moy and Prasad [MolPr1, MolPr2] in terms of filtrations $P_{x,r}(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroups $P_x \subset \mathcal{G}$.

Let $\mathcal{G} = \text{SL}_2(K)$. Let $\text{Irr}(\mathcal{G})$ denote the smooth dual of $\mathcal{G}$. Thanks to a recent article [ABPS1], we have, for every Langlands parameter $\phi \in \Phi(\mathcal{G})$ with $L$-packet $\Pi_\phi(\mathcal{G}) \subset \text{Irr}(\mathcal{G})$

$$d(\phi) = d(\pi) \quad \text{for all } \pi \in \Pi_\phi(\mathcal{G}).$$

The equation (1) is a big help in the computation of the depth $d(\pi)$. To each biquadratic extension $L/K$, there is attached a Langlands parameter $\phi = \phi_{L/K}$, and an $L$-packet $\Pi_\phi$ of cardinality 4. The depth of the parameter $\phi_{L/K}$ depends on the extension $L/K$. More precisely, the numbers $d(\phi)$ depend on the breaks in the
upper ramification filtration of the Galois group
\[ \text{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \]
For certain extensions $L/K$ the allowed depths can be any odd number $1, 3, 5, 7, \ldots$. For the other extensions $L/K$, the allowed depths are $3, 5, 7, 9, \ldots$. Accordingly, the depth of each irreducible supercuspidal representation $\pi$ in the packet $\Pi_\phi$ is given by the formula
\[ d(\pi) = 2n + 1 \]
where $n = 0, 1, 2, 3, \ldots$ or $1, 2, 3, 4, \ldots$ depending on $L/K$. Let $D$ be a central division algebra of dimension 4 over $K$. The parameter $\phi$ is relevant for the inner form $\text{SL}_1(D)$, which admits singleton $L$-packets, and the depths are given by the formula (2).

This contrasts with the case of $\text{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a unique bi-quadratic extension $L/K$, and a unique tamely ramified parameter $\phi : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R})$ of depth zero.

We move on to consider the geometric conjecture in [ABPS]. Let $\mathcal{B}(G)$ denote the Bernstein spectrum of $G$, let $s \in \mathcal{B}(G)$, and let $T^s, W^s$ denote the complex torus, finite group, attached by Bernstein to $s$. For more details at this point, we refer the reader to [R]. The Bernstein decomposition provides us, inter alia, with the following data: a canonical disjoint union
\[ \text{Irr}(G) = \bigsqcup \text{Irr}(G)^s \]
and, for each $s \in \mathcal{B}(G)$, a finite-to-one surjective map
\[ \text{Irr}(G)^s \to T^s/W^s \]
on to the quotient variety $T^s/W^s$. The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a bijection
\[ \text{Irr}(G)^s \simeq T^s//W^s \]
where $T^s//W^s$ is the extended quotient of the torus $T^s$ by the finite group $W^s$. If the action of $W^s$ on $T^s$ is free, then the extended quotient is equal to the ordinary quotient $T^s/W^s$. If the action is not free, then the extended quotient is a finite disjoint union of quotient varieties, one of which is the ordinary quotient. The bijection (3) is subject to certain constraints, itemised in [ABPS].

In the case of $\text{SL}_2$, the torus $T^s$ is of dimension 1, and the finite group $W^s$ is either 1 or $\mathbb{Z}/2\mathbb{Z}$. So, in this context, the content of the conjecture is rather modest: but a proof is required, and such a proof is duly given in §7.

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2. Artin-Schreier theory

Let $K$ be a local field with positive characteristic $p$, containing the $n$-th roots of unity $\zeta_n$. The cyclic extensions of $K$ whose degree $n$ is coprime with $p$ are described by Kummer theory. It is well known that any cyclic extension $L/K$ of degree $n$, $(n, p) = 1$, is generated by a root $\alpha$ of an irreducible polynomial $x^n - a \in K[x]$. We fix an algebraic closure $\overline{K}$ of $K$ and a separable closure $K^s$ of $K$ in $\overline{K}$. If $\alpha \in K^s$
is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree $n$ and is called a Kummer extension of $K$.

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by $c h(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^n - x - a$. Essentially, every cyclic extension of $K$ with degree $p = c h(K)$ is generated by a root $\alpha$ of $x^p - x - a \in K[x]$.

Let $\varphi$ denote the Artin-Schreier endomorphism of the additive group $K^\times$:

$$\varphi: K^\times \to K^\times, \quad x \mapsto x^p - x.$$ 

Given $a \in K$ denote by $K((\varphi^{-1}(a))$ the extension $K(\alpha)$, where $\varphi(\alpha) = a$ and $\alpha \in K^\times$. We have the following characterization of finite cyclic Artin-Schreier extensions of degree $p$:

**Theorem 2.1.**  
(i) Given $a \in K$, either $\varphi(x) - a \in K[x]$ has one root in $K$ in which case it has all the $p$ roots are in $K$, or is irreducible.  
(ii) If $\varphi(x) - a \in K[x]$ is irreducible then $K((\varphi^{-1}(a))/K$ is a cyclic extension of degree $p$, with $\varphi^{-1}(a) \subset K^\times$.  
(iii) If $L/K$ be a finite cyclic extension of degree $p$, then $L = K((\varphi^{-1}(a))$, for some $a \in K$.

(See [Th1] p.34 for more details.)

We fix now some notation. $K$ is a local field with characteristic $p > 1$ with finite residue field $k$. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$.

Let $\mathfrak{o}$ be the ring of integers in $K$ and denote by $\mathfrak{p} \subset \mathfrak{o}$ the (unique) maximal ideal of $\mathfrak{o}$. This ideal is principal and any generator of $\mathfrak{p}$ is called a uniformizer. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi \mathfrak{o} \cong \varpi \mathbb{F}_q[[\varpi]]$.

A normalized valuation on $K$ will be denoted by $\nu$, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$. The group of units is denoted by $\mathfrak{o}^\times$.

### 2.1. The Artin-Schreier symbol

Let $L/K$ be a finite Galois extension. Let $N_{L/K}$ be the norm map and denote by $\text{Gal}(L/K)^{ab}$ the abelianization of $\text{Gal}(L/K)$. The reciprocity map is a group isomorphism

$$K^\times / N_{L/K} L^\times \cong \text{Gal}(L/K)^{ab}.$$  

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism $K^\times \to K^\times / N_{L/K} L^\times$

$$b \in K^\times \mapsto (b, L/K) \in \text{Gal}(L/K)^{ab}.$$  

From the Artin symbol we obtain a pairing

$$K \times K^\times \to \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha,$$

where $\varphi(\alpha) = a$, $\alpha \in K^\times$ and $L = K(\alpha)$.

**Definition 2.2.** Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is defined by

$$[a, b] = (b, L/K)(\alpha) - \alpha.$$
The Artin-Schreier symbol is a bilinear map satisfying the following properties, see \[Ne\] p.341:

\[
\begin{align*}
(7) \quad [a_1 + a_2, b] &= [a_1, b] + [a_2, b]; \\
(8) \quad [a, b_1 b_2] &= [a, b_1] + [a, b_2]; \\
(9) \quad [a, b] &= 0, \forall a \in K \iff b \in N_{L/K} L^\times, L = K(\alpha) \text{ and } \varphi(\alpha) = a; \\
(10) \quad [a, b] &= 0, \forall b \in K^\times \iff a \in \varphi(K).
\end{align*}
\]

2.2. The groups $K/\varphi(K)$ and $K^\times/K^{\times p}$. In this section we recall some properties of the groups $K/\varphi(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing \[\mathcal{F}\]. Dalawat \[Da2\, Da\] interprets $K/\varphi(K)$ and $K^\times/K^{\times p}$ as $\mathbb{F}_p$-spaces. This interpretation will be particularly useful in §4.

Consider the additive group $K$. By \[Da, Proposition 11\], the $\mathbb{F}_p$-space $K/\varphi(K)$ is countably infinite. Hence, $K/\varphi(K)$ is infinite as a group.

**Proposition 2.3.** $K/\varphi(K)$ is a discrete abelian torsion group.

**Proof.** The ring of integers decomposes as a (direct) sum

$$\mathfrak{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\varphi(\mathfrak{o}) = \varphi(\mathbb{F}_q) + \varphi(\mathfrak{p}).$$

The restriction $\varphi : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, see \[Da\, Lemma 8\]. Hence,

$$\varphi(\mathfrak{o}) = \varphi(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \varphi(K)$. It follows that $\varphi(K)$ is an open subgroup of $K$ and $K/\varphi(K)$ is discrete. Since $\varphi(K)$ is annihilated by $p$, $K/\varphi(K)$ is a torsion group. $\square$

Now we concentrate on the multiplicative group $K^\times$. For any $n > 0$, let $U_n$ be the kernel of the reduction map from $\mathfrak{o}^\times$ to $(\mathfrak{o}/\mathfrak{p}^n)^\times$. In particular, $U_1 = \ker(\mathfrak{o}^\times \to k^\times)$. The $U_n$ are $\mathbb{Z}_p$-modules, because they are commutative pro-$p$-groups. By \[Da2, Proposition 20\], the $\mathbb{Z}_p$-module $U_1$ is not finitely generated. As a consequence, $K^\times/K^{\times p}$ is infinite, see \[Da2, Corollary 21\]. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

**Proposition 2.4.** $K^\times/K^{\times p}$ is a profinite abelian $p$-torsion group.

**Proof.** There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{o}^\times$. The group of units is a direct product $\mathfrak{o}^\times \cong \mathbb{F}_q^\times \times U_1$, with $q = p^f$. By \[Iw, p.25\], the group $U_1$ is a direct product of countable many copies of the ring of $p$-adic integers

$$U_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots = \prod_{N} \mathbb{Z}_p.$$ 

Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the $p$-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{N} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

Now, $K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \ldots$$
\[= p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{n} p\mathbb{Z}_p.\]

Note that \(p\mathbb{Z}/(q-1)\mathbb{Z} = \mathbb{Z}/(q-1)\mathbb{Z}\), since \(p\) and \(q-1\) are coprime. Denote by \(z = \prod_n z_n\) an element of \(\prod_{n} \mathbb{Z}_p\), where \(z_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p\), for every \(n\).

The map \(\varphi : \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{n} \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_{n} \mathbb{Z}/p\mathbb{Z}\)

defined by \((x, y, z) \mapsto (x \mod p, \prod_n pr_0(z_n))\)

where \(pr_0(z_n) = a_{0,n}\) is the projection, is clearly a group homomorphism.

Now, \(\mathbb{Z}/p\mathbb{Z} \times \prod_{n} \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}\) is a topological group for the product topology, where each component \(\mathbb{Z}/p\mathbb{Z}\) has the discrete topology. It is compact, Hausdorff and totally disconnected. Therefore, \(\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}\) is a profinite group.

Since \(\text{ker}\varphi = p\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \prod_{n} p\mathbb{Z}_p\),

it follows that there is an isomorphism of topological groups

\[K^\times/K^{\times p} \cong \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z},\]

where \(K^\times/K^{\times p}\) is given the quotient topology. Therefore, \(K^\times/K^{\times p}\) is profinite.

From propositions 2.3 and 2.4, \(K/\wp(K)\) is a discrete abelian group and \(K/K^{\times p}\) is an abelian profinite group, both annihilated by \(p = ch(K)\). Therefore, Pontryagin duality coincides with \(\text{Hom}(\cdot, \mathbb{Z}/p\mathbb{Z})\) on both of these groups, see [Th2]. The pairing \(\langle \cdot, \cdot \rangle\) restricts to a pairing

\[(11) \quad [\cdot, \cdot] : K/\wp(K) \times K^\times/K^{\times p} \to \mathbb{Z}/p\mathbb{Z},\]

which we refer from now on to the Artin-Schreier pairing. It follows from \([9]\) and \([10]\), that the pairing is nondegenerate (see also [Th2 Proposition 3.1]). The next result shows that the pairing is perfect.

**Proposition 2.5.** The Artin-Schreier symbol induces isomorphisms of topological groups

\[K^\times/K^{\times p} \cong \text{Hom}(K/\wp(K), \mathbb{Z}/p\mathbb{Z}), bK^{\times p} \mapsto (a + \wp(K) \mapsto [a, b])\]

and

\[K/\wp(K) \cong \text{Hom}(K^\times/K^{\times p}, \mathbb{Z}/p\mathbb{Z}), a + \wp(K) \mapsto (bK^{\times p} \mapsto [a, b])\]

**Proof.** The result follows by taking \(n = 1\) in Proposition 5.1 of [Th2], and from the fact that Pontryagin duality for the groups \(K/\wp(K)\) and \(K^\times/K^{\times p}\) coincide with \(\text{Hom}(\cdot, \mathbb{Z}/p\mathbb{Z})\) duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. \(\square\)
Let $B$ be a subgroup of the additive group of $K$ with finite index such that $\varphi(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent $p$ is again a finite abelian Galois extension of exponent $p$. Therefore, the composite
\[
K_B = K(\varphi^{-1}(B)) = \prod_{a \in B} K(\varphi^{-1}(a))
\]
is a finite abelian Galois extension of exponent $p$. On the other hand, if $L/K$ is a finite abelian Galois extension of exponent $p$, then $L = K_B$ for some subgroup $\varphi(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent $p$, which we denote by $K_p = K(\varphi^{-1}(K))$. The extension $K_p/K$ is infinite and Galois. The corresponding Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^{x_p}$, see [Th2, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^{x^2}$ and will play a fundamental role in the sequel.

3. Quadratic characters

From now on we take $K$ to be a local function field with $ch(K) = 2$. Therefore, $K$ is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [Iw, p.25],
\[
U_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots = \prod_{N} \mathbb{Z}_2
\]
with countably infinite many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By proposition 2.4, there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group
\[
\mathbb{F}_q((\varpi))^\times/\mathbb{F}_q((\varpi))^\times^2 \cong \prod_{N} \mathbb{Z}/2\mathbb{Z} = G_2
\]
where $G_2$ is the Galois group of the maximal abelian extension of exponent 2. Since $G_2$ is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol
\[
[a, \cdot] : K^\times/K^{x^2} \rightarrow \mathbb{Z}/2\mathbb{Z}.
\]

Now, let $\varphi$ denote the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character
\[
(12) \quad \chi_a : K^\times \rightarrow \mathbb{C}^\times, \quad \chi_a = \varphi([a, \cdot])
\]

Proposition 2.5 shows that every quadratic character of $\mathbb{F}_q((\varpi))^\times$ arises in this way.

Example 3.1. The unramified quadratic extension of $K$ is $K(\varphi^{-1}(\varpi))$, see [Da] proposition 12. According to Dalawat, the group $K/\varphi(K)$ may be regarded as an $\mathbb{F}_2$-space and the image of $\varpi$ under the canonical surjection $K \rightarrow K/\varphi(K)$ is an $\mathbb{F}_2$-line, i.e., isomorphic to $\mathbb{F}_2$. Since $\varphi_q : p \rightarrow p$ is an isomorphism, the image of $p$ in $K/\varphi(K)$ is $\{0\}$, see lemma 8 in [Da]. Now, choose any $a_0 \in \varpi \setminus p$ such that the image of $a_0$ in $\varpi/p$ has nonzero trace in $\mathbb{F}_2$, see [Da] Proposition 9. The
quadratic character \( \chi_{a_0} = \varphi([a_0, \cdot]) \) associated with \( K(\varphi^{-1}(\mathfrak{p})) \) via class field theory is precisely the unramified character \( (n \mapsto (-1)^n) \) from above. Note that any other choice \( b_0 \in \mathfrak{o}\setminus\mathfrak{p} \), with \( a_0 \neq b_0 \), gives the same unique unramified character, since there is only one nontrivial coset \( a_0 + \varphi(K) \) for \( a_0 \in \mathfrak{o}\setminus\mathfrak{p} \).

Let \( G \) denote \( \text{SL}_2(K) \), let \( B \) be the standard Borel subgroup of \( G \), let \( T \) be the diagonal subgroup of \( G \). Then, \( \chi \) inflates to a character of \( B \). Denote by \( \pi(\chi) \) the (unitarily) induced representation \( \text{Ind}^G_B(\chi) \). The representation space \( V(\chi) \) of \( \pi(\chi) \) consists of locally constant complex valued functions \( f : G \rightarrow \mathbb{C} \) such that, for every \( a \in K^\times \), \( b \in K \) and \( g \in G \), we have

\[
 f\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = |a| \chi(a) f(g)
\]

The action of \( G \) on \( V(\chi) \) is by right translation. The representations \( (\pi(\chi), V(\chi)) \) are called (unitary) principal series of \( G = \text{SL}_2(K) \).

Let \( \chi \) be a quadratic character of \( K^\times \). The reducibility of the induced representation \( \text{Ind}^G_B(\chi) \) is well known in zero characteristic. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic \( p \).

**Theorem 3.2.** [Ca, Ca2] The representation \( \pi(\chi) = \text{Ind}^G_B(\chi) \) is reducible if, and only if, \( \chi \) is either \( |\cdot|^\pm \) or a nontrivial quadratic character of \( K^\times \).

For a proof see [Ca, Theorems 1.7, 1.9] and [Ca2] §9.

From now on, \( \chi \) will be a quadratic character. It is a classical result that the unitary principal series for \( \text{GL}_2 \) are irreducible. For a representation of \( \text{GL}_2 \) parabolically induced by \( 1 \otimes \chi \), Clifford theory tells us that the dimension of the intertwining algebra of its restriction to \( \text{SL}_2 \) is 2. This is exactly the induced representation of \( \text{SL}_2 \) by \( \chi \):

\[
 \text{Ind}^{	ext{GL}_2(K)}_B(1 \otimes \chi)|_{\text{SL}_2(K)} \cong \text{Ind}^{	ext{SL}_2(K)}_B(\chi)
\]

where \( B \) denotes the standard Borel subgroup of \( \text{GL}_2(K) \). This leads to reducibility of the induced representation \( \text{Ind}^G_B(\chi) \) into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

\[
 (13) \quad \pi(\chi) = \text{Ind}^G_B(\chi) = \pi(\chi)^+ \oplus \pi(\chi)^-
\]

define an \( L \)-packet \( \{ \pi(\chi)^+, \pi(\chi)^- \} \) for \( \text{SL}_2 \).

**4. Biquadratic extensions of \( \mathbb{F}_q((\varpi)) \)**

Quadratic extensions \( L/K \) are obtained by adjoining an \( \mathbb{F}_2 \)-line \( D \subset K/\varphi(K) \). Therefore, \( L = K(\varphi^{-1}(D)) = K(\alpha) \) where \( D = \text{span}\{a + \varphi(K)\} \), with \( \alpha^2 - \alpha = a \).

In particular, if \( a_0 \in \mathfrak{o}\setminus\mathfrak{p} \) such that the image of \( a_0 \) in \( \mathfrak{p}/\mathfrak{p} \) has nonzero trace in \( \mathbb{F}_2 \), the \( \mathbb{F}_2 \)-line \( V_0 = \text{span}\{a_0 + \varphi(K)\} \) contains all the cosets \( a_i + \varphi(K) \) where \( a_i \) is an integer and so \( K(\varphi^{-1}(a)) = K(\varphi^{-1}(V_0)) = K(a_0) \) where \( \alpha^2 - \alpha = a_0 \) gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering \( \mathbb{F}_2 \)-planes \( W = \text{span}\{a + \varphi(K), b + \varphi(K)\} \subset K/\varphi(K) \). Therefore, if \( a + \varphi(K) \) and \( b + \varphi(K) \) are \( \mathbb{F}_2 \)-linearly independent then \( K(\varphi^{-1}(W)) := K(\alpha, \beta) \) is biquadratic, where \( \alpha^2 - \alpha = a \) and \( \beta^2 - \beta = b \), \( \alpha, \beta \in K^\times \). Therefore, \( K(\alpha, \beta)/K \) is biquadratic if \( b - a \not\in \varphi(K) \).
A biquadratic extension containing the line $V_0$ is of the form $K(\alpha, \beta)/K$. There are countably many quadratic extensions $L_0/K$ containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions $L/K$ which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\alpha, \beta)/K$.

4.1. **Ramification.** The space $K/\wp(K)$ comes with a filtration

\begin{equation}
0 < V_0 < V_1 = V_2 < V_3 = V_4 < \ldots \subset K/\wp(K)
\end{equation}

where $V_0$ is the image of $\mathfrak{o}_K$ and $V_i \ (i > 0)$ is the image of $p^{-i}$ under the canonical surjection $K \to K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension $f$. The $\mathbb{F}_2$-dimension of $V_n$ is

\begin{equation}
dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f,
\end{equation}

for every $n \in \mathbb{N}$, where $\lceil x \rceil$ is the smallest integer bigger than $x$.

Let $L/K$ denote a Galois extension with Galois group $G$. For each $i \geq -1$ we define the $i$th-ramification subgroup of $G$ (in the lower numbering) to be:

\[ G_i = \{ \sigma \in G : \sigma(x) - x \in p_L^{i+1}, \forall x \in \mathfrak{o}_L \}. \]

An integer $t$ is a break for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the Hasse-Herbrand function $\psi = \psi_{L/K}$:

\[ G^u = G_{\psi(u)}. \]

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(0) = 0$.

Let $G_2 = Gal(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^x/K_{x2}$ (proposition 24), the pairing $K^x/K^x2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$ from (11) coincides with the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group $G_2$ comes equipped with a ramification filtration $(G^2_2)_{u \geq -1}$ in the upper numbering, see [Da, p.409]. For $u \geq 0$, we have an orthogonal relation [Da, Proposition 17]

\begin{equation}
(G^2_2)^\perp = p^{-|u|+1} \cdot V_{|u|-1}
\end{equation}

under the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [Da, Proposition 17], the positive breaks in the filtration $(G^v)_v$ occur precisely at integers prime to $p$. So, for $c(h(K)) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If $G$ is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^v)_v$ (see [Da], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.
Given a plane \( W \subset K/\wp(K) \), the filtration \((V_i)\) on \( K/\wp(K) \) induces a filtration \((W_i)\) on \( W \), where \( W_i = W \cap V_i \). There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

**Case 1:** \( W \) contains the line \( V_0 \), i.e. \( L_0 = K(\wp^{-1}(W)) \) contains the unramified quadratic extension \( K(\wp^{-1}(V_0)) = K(\alpha_0) \) of \( K \). The extension has residue degree \( f(L_0/K) = 2 \) and ramification index \( e(L_0/K) = 2 \). In this case, there is an integer \( t > 0 \), necessarily odd, such that the filtration \((W_i)\) looks like

\[
0 \subset W_0 = W_{t-1} \subset W_t = W
\]

By the orthogonality relation (16), the upper ramification filtration on \( G = Gal(L_0/K) \) looks like

\[
\{1\} = \ldots = G^{t+1} \subset G^t = \ldots = G^0 \subset G^{-1} = G
\]

Therefore, the upper ramification breaks occur at \(-1\) and \( t \).

The number of such \( W \) is equal to the number of planes in \( V_t \) containing the line \( V_0 \) but not contained in the subspace \( V_{t-1} \). This number can be computed and equals the number of biquadratic extensions of \( K \) containing the unramified quadratic extensions and with a pair of upper ramification breaks \((-1, t)\), \( t > 0 \) and odd. Here is an example.

**Example 4.1.** The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks \((-1, 1)\) is equal to the number of planes in an \( 1 + f \)-dimensional \( \mathbb{F}_2 \)-space, containing the line \( V_0 \). There are precisely

\[
1 + 2 + 2^2 + \ldots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1
\]

of such biquadratic extensions.

**Case 2.1:** \( W \) does not contain the line \( V_0 \) and the induced filtration on the plane \( W \) looks like

\[
0 = W_{t-1} \subset W_t = W
\]

for some integer \( t \), necessarily odd.

The number of such \( W \) is equal to the number of planes in \( V_t \) whose intersection with \( V_{t-1} \) is \( \{0\} \). Note that, there are no such planes when \( f = 1 \). So, for \( K = \mathbb{F}_2((\epsilon)) \), case 2.1 does not occur.

Suppose \( f > 1 \). By the orthogonality relation, the upper ramification ramification filtration on \( G = Gal(L/K) \) looks like

\[
\{1\} = \ldots = G^{t+1} \subset G^t = \ldots = G^{-1} = G
\]

Therefore, there is a single upper ramification break occurring at \( t > 0 \) and is necessarily odd.

For \( f = 1 \) there is no such biquadratic extension. For \( f > 1 \), the number of these biquadratic extensions equals the number of planes \( W \) contained in an \( \mathbb{F}_2 \)-space of dimension \( 1 + f \), \( t = 2i - 1 \), which are transverse to a given codimension-\( f \) \( \mathbb{F}_2 \)-space.

**Case 2.2:** \( W \) does not contain the line \( V_0 \) and the induced filtration on the plane \( W \) looks like

\[
0 = W_{t_1-1} \subset W_{t_1} = W_{t_2-1} \subset W_{t_2} = W
\]
for some integers \( t_1 \) and \( t_2 \), necessarily odd, with \( 0 < t_1 < t_2 \).

The orthogonality relation for this case implies that the upper ramification filtration on \( G = \text{Gal}(L/K) \) looks like
\[
\{1\} = \ldots = G^{t_2+1} \subset G^{t_2} = \ldots = G^{t_1+1} \subset G^{t_1} = \ldots = G
\]
The upper ramification breaks occur at odd integers \( t_1 \) and \( t_2 \).

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks \((t_1, t_2)\).

5. Langlands parameter

We have the following canonical homomorphism:
\[
W_K \rightarrow W_K^{ab} \simeq K^\times \rightarrow K^\times/K^{\times 2}.
\]
According to §2, we also have
\[
K^\times/K^{\times 2} \simeq \prod \mathbb{Z}/2\mathbb{Z}
\]
the product over countably many copies of \( \mathbb{Z}/2\mathbb{Z} \). Using the countable axiom of choice, we choose two copies of \( \mathbb{Z}/2\mathbb{Z} \). This creates a homomorphism
\[
W_K \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]
There are countably many such homomorphisms.

Following [We], denote by \( \alpha, \beta, \gamma \) the images in \( \text{PSL}_2(\mathbb{C}) \) of the elements
\[
z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]
in \( \text{SL}_2(\mathbb{C}) \).

Note that \( z_\alpha, z_\beta, z_\gamma \in \text{SU}_2(\mathbb{C}) \) so that \( \alpha, \beta, \gamma \in \text{PSU}_2(\mathbb{C}) = \text{SO}_3(\mathbb{R}) \).

Denote by \( J \) the group generated by \( \alpha, \beta, \gamma \):
\[
J := \{ \epsilon, \alpha, \beta, \gamma \} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]
The group \( J \) is unique up to conjugacy in \( G = \text{PSL}_2(\mathbb{C}) \).

The pre-image of \( J \) in \( \text{SL}_2(\mathbb{C}) \) is the group \( \{ \pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma \} \) and is isomorphic to the group \( \mathbb{U}_8 \) of unit quaternions \( \{ \pm 1, \pm i, \pm j, \pm k \} \).

The centralizer and normalizer of \( J \) are given by
\[
C_G(J) = J, \quad N_G(J) = O
\]
where \( O \simeq S_4 \) the symmetric group on 4 letters. The quotient \( O/J \simeq \text{GL}_2(\mathbb{Z}/2) \) is the full automorphism group of \( J \).

Each biquadratic extension \( L/K \) determines a Langlands parameter
\[
\phi : \text{Gal}(L/K) \rightarrow \text{SO}_3(\mathbb{R}) \subset \text{SO}_3(\mathbb{C})
\]
Define
\[
S_\phi = C_{\text{PSL}_2(\mathbb{C})}(\text{im } \phi)
\]
Then we have \( S_\phi = J \), since \( C_G(J) = J \), and whose conjugacy class depends only on \( L \), since \( O/J = \text{Aut}(J) \).

Define the new group
\[
S_\phi = C_{\text{SL}_2(\mathbb{C})}(\text{im } \phi)
\]
To align with the notation in [ABPS2], replace $\phi^\#$ in [ABPS] by $\phi$ in the present article. We have the short exact sequence

$$1 \to Z_\phi \to S_\phi \to S_\phi \to 1$$

with $Z_\phi = \mathbb{Z}/2\mathbb{Z}$.

Let $D$ be a central division algebra of dimension 4 over $K$, and let $\text{Nrd}$ denote the reduced norm on $D^\times$. Define

$$\text{SL}_1(D) = \{x \in D^\times : \text{Nrd}(x) = 1\}.$$ 

Then $\text{SL}_1(D)$ is an inner form of $\text{SL}_2(K)$. In the local Langlands correspondence [ABPS], the $L$-parameter $\phi$ is enhanced by elements $\rho \in \text{Irr}(S_\phi)$. Now the group $S_\phi \simeq U_8$ admits four characters $\rho_1, \rho_2, \rho_3, \rho_4$ and one irreducible representation $\rho_0$ of degree 2.

The parameter $\phi$ creates a big packet with five elements, which are allocated to $\text{SL}_2(K)$ or $\text{SL}_1(D)$ according to central characters. So $\phi$ assigns an $L$-packet $\Pi_\phi$ to $\text{SL}_2(K)$ with 4 elements, and a singleton packet to the inner form $\text{SL}_1(D)$. None of these packets contains the Steinberg representation of $\text{SL}_2(K)$ and so each $\Pi_\phi$ is a supercuspidal $L$-packet with 4 elements.

To be explicit: $\phi$ assigns to $\text{SL}_2(K)$ the supercuspidal packet

$$\{\pi(\phi, \rho_1), \pi(\phi, \rho_2), \pi(\phi, \rho_3), \pi(\phi, \rho_4)\}$$

and to $\text{SL}_1(D)$ the singleton packet

$$\{\pi(\phi, \rho_0)\}$$

and this phenomenon occurs countably many times.

Each supercuspidal packet of four elements is the JL-transfer of the singleton packet, in the following sense: the irreducible supercuspidal representation $\theta$ of $\text{GL}_2(K)$ which yields the 4-packet upon restriction to $\text{SL}_2(K)$ is the image in the JL-correspondence of the irreducible smooth representation $\psi$ of $\text{GL}_1(D)$ which yields two copies of $\pi(\phi, \rho_0)$ upon restriction to $\text{SL}_1(D)$:

$$\theta = \text{JL}(\psi).$$

Each parameter $\phi : W_K \to \text{PGL}_2(\mathbb{C})$ lifts to a Galois representation

$$\phi : W_K \to \text{GL}_2(\mathbb{C}).$$

This representation is triply imprimitive, as in [We]. Let $\Xi(\phi)$ be the group of characters $\chi$ of $W_K$ such that $\chi \otimes \phi \simeq \phi$. Then $\Xi(\phi)$ is non-cyclic of order 4.

6. Depth

Let $L/K$ be a biquadratic extension. We fix an algebraic closure $\overline{K}$ of $K$ such that $L \subset \overline{K}$. From the inclusion $L \subset \overline{K}$, there is a natural surjection

$$\pi_{L/K} : \text{Gal}(\overline{K}/K) \to \text{Gal}(L/K)$$

Let $K^{ur}$ be the maximal unramified extension of $K$ in $\overline{K}$ and let $K^{ab}$ be the maximal abelian extension of $K$ in $\overline{K}$. We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections.
In the above notation, we have $\pi_{L/K} = \pi_2 \circ \pi_1$.

Let

\[(\forall r) \pi_{L/K} I^{(r)} = \mathcal{J}^{(r)}\]

Proof. This follows immediately from the above diagram. Here, we identify $I^{(r)}$ with $\iota_1(I^{(r)})$ and $\mathcal{J}^{(r)}$ with $\iota_3(\mathcal{J}^{(r)})$.

\[\square\]

**Lemma 6.2.** Let $L/K$ be a biquadratic extension, let $\phi$ be the Langlands parameter $\bigl[17\bigr]$, $\phi = \alpha \circ \pi_{L/K}$ with $\alpha : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R})$. Then we have $d(\phi) = r - 1$ where $r$ is the least integer for which $\mathcal{J}^{(r)} = 1$.

Proof. The depth of a Langlands parameter $\phi$ is easy to define. For $r \in \mathbb{R} \geq 0$ let $\text{Gal}(F_s/F)^r$ be the $r$-th ramification subgroup of the absolute Galois group of $F$. Then the depth of $\phi$ is the smallest number $d(\phi) \geq 0$ such that $\phi$ is trivial on $\text{Gal}(F_s/F)^r$ for all $r > d(\phi)$.

Note that $\alpha$ is injective. Therefore

\[\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K})I^{(r)} = 1 \iff \alpha(\mathcal{J}^{(r)}) = 1 \iff \mathcal{J}^{(r)} = 1.\]

\[\square\]

For example, the parameter $\phi$ has depth zero if it is *tamely ramified*, i.e. the least integer $r$ for which $\mathcal{J}^{(r)} = 1$ is $r = 1$. The relative wild inertia group is 1, but the relative inertia group is not 1.

**Case 1:** There are two ramification breaks occurring at $-1$ and some odd integer $t > 0$:

\[\{1\} = \cdots \subset \mathcal{J}^{(t+1)} \subset \mathcal{J}^{(t)} = \cdots \mathcal{J}^{(0)} = \mathcal{J}_{L/K} \subset \text{Gal}(L/K), \quad d(\phi) = t\]
The allowed depths are 1, 3, 5, 7, ....

**Case 2.1:** One single ramification break occurs at some odd integer \( t > 0 \):

\[
\{1\} = \ldots = \mathcal{J}^{(t+1)} \subset \mathcal{J}^{(t)} = \ldots = \mathcal{J}^{(0)} = \mathcal{J}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t
\]

The allowed depths are 1, 3, 5, 7, ....

**Case 2.2:** There are two ramification breaks occurring at some odd integers \( t_1 < t_2 \):

\[
\{1\} = \ldots = \mathcal{J}^{(t_2+1)} \subset \mathcal{J}^{(t_2)} = \ldots = \mathcal{J}^{(t_1+1)} \subset \mathcal{J}^{(t_1)} = \ldots = \mathcal{J}^{(0)} = \mathcal{J}_{L/K} = \text{Gal}(L/K); \quad d(\varphi) = t_2
\]

The allowed depths are 3, 5, 7, 9, ....

(In the above, \( \mathcal{J}^{(0)} = \mathcal{J}_{L/K} \))

**Theorem 6.3.** Let \( L/K \) be a biquadratic extension, let \( \phi \) be the Langlands parameter \((77)\). For every \( \pi \in \Pi_{\phi}(\text{SL}_2(K)) \) and \( \pi \in \Pi_{\phi}(\text{SL}_1(D)) \) there is an equality of depths:

\[
d(\pi) = d(\varphi).
\]

The depth of each element in the \( L \)-packet \( \Pi_{\phi} \) is given by the largest break in the ramification of the Galois group \( \text{Gal}(L/K) \). The allowed depths are 1, 3, 5, 7, .... except in Case 2.2, when the allowed depths are 3, 5, 7, ....

**Proof.** This follows from Lemma \((6.2)\), the above computations, and Theorem 3.4 in [ABPS1]. \( \square \)

This contrasts with the case of \( \text{SL}_2(\mathbb{Q}_p) \) with \( p > 2 \). Here there is a unique biquadratic extension \( L/K \), and a unique tamely ramified parameter \( \phi : \text{Gal}(L/K) \to \text{SO}_3(\mathbb{R}) \) of depth zero.

**6.1. Quadratic extensions.** Let \( E/K \) be a quadratic extension. There are two kinds: the unramified one \( E_0 = K(\alpha_0) \) and countably many totally (and wildly) ramified \( E = K(\alpha) \).

**Theorem 6.4.** For the unramified principal series \( L \)-packet \( \{\pi_1^E, \pi_2^E\} \), we have

\[
d(\pi_1^E) = d(\pi_2^E) = -1.
\]

For the ramified principal series \( L \)-packet \( \{\pi_1^E, \pi_2^E\} \), we have

\[
d(\pi_1^E) = d(\pi_2^E) = n
\]

with \( n = 1, 2, 3, 4, \ldots \).

**Proof.** Case 1: \( E_0/K \) unramified. Then, \( f(E_0/K) = 2 \). In this case, we have \( G_0 = \{1\} \), and \( G_0 = G^0 = \mathcal{J}_{E_0/K} \). There is only one ramification break at \( t = 0 \) and the filtration of \( G = \text{Gal}(E_0/K) \) in the upper numbering is

\[
\{1\} = G^0 \subset G^{-1} = G = \mathbb{Z}/2\mathbb{Z}.
\]

The filtration on the relative inertia \( \mathcal{J}^{(t)} \) is

\[
\{1\} = \mathcal{J}_{L_0/K} \subset G = \mathbb{Z}/2\mathbb{Z}
\]

with only one break at \( t = 0 \). Negative depth, as expected.

Case 2: \( E/K \) is totally ramified. Then, \( e(E/K) = 2 \), which is divisible by the residue degree, so the extension is wildly ramified. In this case, there is one break
at some $t \geq 1$. This is because of wild ramification, since $G^1 = \{1\}$ if and only if
the extension is tamely ramified. The filtration of $G$ in the upper numbering is
\[ \{1\} = G^{t+1} \subset G^t = \ldots = G^0 = G = \mathbb{Z}/2\mathbb{Z} \]
The filtration on the relative inertia $\mathcal{I}^{(r)}$ is
\[ \{1\} = \mathcal{I}^{(t+1)} \subset \mathcal{I}^{(t)} = \ldots = G = \mathbb{Z}/2\mathbb{Z} \]
with only one break at $t \geq 1$. \qed

7. A commutative triangle

In this section we confirm part of the geometric conjecture in [ABPS] for $\text{SL}_2(F_q((\varpi)))$.

We begin by recalling the underlying ideas of the conjecture.

Let $\mathcal{G}$ be the group of $K$-points of a connected reductive group over a nonarchimedean local field $K$. The Bernstein decomposition provides us, inner alia, with the following data: a canonical disjoint union
\[ \text{Irr}(\mathcal{G}) = \bigsqcup \text{Irr}(\mathcal{G})^s \]
and, for each $s \in \mathcal{B}(\mathcal{G})$, a finite-to-one surjective map
\[ \text{Irr}(\mathcal{G})^s \rightarrow T^s/W^s \]
The geometric conjecture in [ABPS] amounts to a refinement of these statements. The refinement comprises the assertion that we have a bijection
\[ \text{Irr}(\mathcal{G})^s \simeq (T^s//W^s)_2 \]
where $(T^s//W^s)_2$ is the extended quotient of the second kind of the torus $T^s$ by the finite group $W^s$. This bijection is subject to certain constraints, itemised in [ABPS].

We proceed to define the extended quotient of the second kind. Let $W$ be a finite group and let $X$ be a complex affine algebraic variety. Suppose that $W$ is acting on $X$ as automorphisms of $X$. Define
\[ \bar{X}_2 := \{(x,\tau) : \tau \in \text{Irr}(W^s)\} \]
Then $W$ acts on $\bar{X}_2$:
\[ \alpha(x,\tau) = (\alpha \cdot x, \alpha_* \tau) \]

**Definition 7.1.** The extended quotient of the second kind is defined as
\[ (X//W)_2 := \bar{X}_2/W. \]

Thus the extended quotient of the second kind is the ordinary quotient for the action of $W$ on $\bar{X}_2$.

We recall that $(G,T)$ are the complex dual groups of $(\mathcal{G},\mathcal{T})$, so that $G = \text{PSL}_2(\mathbb{C})$. Let $W_K$ denote the Weil group of $K$. If $\phi$ is an $L$-parameter
\[ W_K \times \text{SL}_2(\mathbb{C}) \rightarrow G \]
then an enhanced Langlands parameter is a pair $(\phi,\rho)$ where $\phi$ is a parameter and $\rho \in \text{Irr}(S_\phi)$. 
Theorem 7.2. Let $G = \text{SL}_2(K)$ with $K = \mathbb{F}_q((\varpi))$. Let $s = [T, \chi]_G$ be a point in the Bernstein spectrum for the principal series of $G$. Let $\text{Irr}(G)^s$ be the corresponding Bernstein component in $\text{Irr}(G)$. Then there is a commutative triangle of natural bijections

\[
\begin{array}{c}
\text{Irr}(G)^s \\
\downarrow \\
\text{L}(G)^s
\end{array}
\]

where $\text{L}(G)^s$ denotes the equivalence classes of enhanced parameters attached to $s$.

Proof. We recall that $T^s = \{ \psi \chi : \psi \in \Psi(T) \}$ where $\Psi(T)$ is the group of all unramified quasicharacters of $T$. With $\lambda \in T^s$, we define the parameter $\phi(\lambda)$ as follows:

\[
\phi(\lambda) : W_K \times \text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}), \quad (w, \Phi_n K, Y) \mapsto \left( \begin{array}{cc} \lambda(w)^n & 0 \\ 0 & 1 \end{array} \right),
\]

where $A_\ast$ is the image in $\text{PSL}_2(\mathbb{C})$ of $A \in \text{SL}_2(\mathbb{C})$, $Y \in \text{SL}_2(\mathbb{C})$, $w \in I_K$ the inertia group, and $\Phi_K$ is a geometric Frobenius. Define, as in §3,

\[
\pi(\lambda) := \text{Ind}^G_B(\lambda).
\]

**Case 1.** $\lambda^2 \neq 1$. Send the pair $(\lambda, 1) \in T^s//W^s$ to $\pi(\lambda) \in \text{Irr}(G)^s$ (via the left slanted arrow) and to $\phi(\lambda) \in \text{L}(G)^s$ (via the right slanted arrow).

**Case 2.** Let $\lambda^2 = 1, \lambda \neq 1$. Let $\phi = \phi(\lambda)$. To compute $S_\phi$, let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

\[
C_G(\text{im} \phi) = T \cup wT
\]

So $\phi$ is a non-discrete parameter, and we have

\[
S_\phi \cong \mathbb{Z}/2\mathbb{Z}.
\]

We have two enhanced parameters, namely $(\phi, 1)$ and $(\phi, \epsilon)$ where $\epsilon$ is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$.

Since $\lambda^2 = 1$, there is a point of reducibility. We send

\[
(\lambda, 1) \mapsto \pi(\lambda)^+, \quad (\lambda, \epsilon) \mapsto \pi(\lambda)^-
\]

via the left slanted arrow, and

\[
(\lambda, 1) \mapsto (\phi(\lambda), 1), \quad (\lambda, \epsilon) \mapsto (\phi(\lambda), \epsilon)
\]

via the right slanted arrow. Note that this includes the case when $\lambda$ is the unramified quadratic character of $K^\times$.

**Case 3.** Let $\lambda = 1$. The principal parameter

\[
\phi_0 : W_K \times \text{SL}_2(\mathbb{C}) \to \text{SL}_2(\mathbb{C}) \to \text{PSL}(2, \mathbb{C})
\]

is a discrete parameter for which $S_{\phi_0} = 1$. In the local Langlands correspondence for $G$, the enhanced parameter $(\phi_0, 1)$ corresponds to the Steinberg representation $\text{St}$ of $\text{SL}_2(K)$. Note also that, when $\phi = \phi(1)$, we have $S_\phi = 1$. We send

\[
(1, 1) \mapsto \pi(1), \quad (1, \epsilon) \mapsto \text{St}
\]

via the left slanted arrow and

\[
(1, 1) \mapsto (\phi(1), 1), \quad (1, \epsilon) \mapsto (\phi_0, 1)
\]
via the right slanted arrow. This establishes that the geometric conjecture in \[ABPS\] is valid for \(\operatorname{Irr}(\mathcal{G})^s\).

Let \(L/K\) be a quadratic extension of \(K\). Let \(\lambda\) be the quadratic character which is trivial on \(N_{L/K}L^\times\). Then \(\lambda\) factors through \(\operatorname{Gal}(L/K) \simeq K^\times/N_{L/K}L^\times \simeq \mathbb{Z}/2\mathbb{Z}\) and \(\phi(\lambda)\) factors through \(\operatorname{Gal}(L/K) \times \operatorname{SL}_2(\mathbb{C})\). The parameters \(\phi(\lambda)\) serve as parameters for the \(L\)-packets in the principal series of \(\operatorname{SL}_2(K)\).

It follows from \(\S 3\) that, when \(K = \mathbb{F}_q((\pi))\), there are countably many \(L\)-packets in the principal series of \(\operatorname{SL}_2(K)\).

### 7.1. The tempered dual.

If we insist, in the definition of \(T^s\), that the unramified character \(\psi\) shall be unitary, then we obtain a copy \(T^s\) of the circle \(T\). We then obtain a compact version of the commutative triangle, in which the tempered dual \(\operatorname{Irr}_{\text{temp}}(\mathcal{G})^s\) determined by \(s\) occurs on the left, and the bounded enhanced parameters \(\mathfrak{L}^b(\mathcal{G})^s\) determined by \(s\) occur on the right. We now isolate the bijective map

\[
(T^s//W^s)_2 \to \operatorname{Irr}_{\text{temp}}(\mathcal{G})^s
\]

and restrict ourselves to the case where \(T^s\) contains two ramified quadratic characters. Let \(T := \{z \in \mathbb{C} : |z| = 1\}\), \(W := \mathbb{Z}/2\mathbb{Z}\). We then have \(T^s = T\), \(W^s = W\) and the generator of \(W\) acts on \(T\) sending \(z\) to \(z^{-1}\).

The left-hand-side and the right-hand-side of the map (21) each has its own natural topology, as we proceed to explain.

The topology on \((\mathbb{T}/W)_2\) comes about as follows. Let \(\operatorname{Prim}(C(\mathbb{T}) \rtimes W)\) denote the primitive ideal space of the noncommutative \(C^*\)-algebra \(C(\mathbb{T}) \rtimes W\). By the classical Mackey theory for semidirect products, we have a canonical bijection

\[
\operatorname{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}/W)_2.
\]

The primitive ideal space on the left-hand side of (22) admits the Jacobson topology. So the right-hand side of (22) acquires, by transport of structure, a compact non-Hausdorff topology. The following picture is intended to portray this topology.

The reduced \(C^*\)-algebra of \(\mathcal{G}\) is liminal, and its primitive ideal space is in canonical bijection with the tempered dual of \(\mathcal{G}\). Transporting the Jacobson topology on the primitive ideal space, we obtain a locally compact topology on the tempered dual of \(\mathcal{G}\), see [\text{Dix}^b\ 3.1.1, 4.4.1, 18.3.2]. This makes \(\operatorname{Irr}_{\text{temp}}(\mathcal{G})^s\) into a compact space, in the induced topology.

We conjecture that these two topologies make (21) into a homeomorphism. This is a strengthening of the geometric conjecture [\text{ABPS}]. In that case, the double-points in the picture arise precisely when the corresponding (parabolically) induced representation has two irreducible constituents. This conjecture is true for \(\operatorname{SL}_2(\mathbb{Q}_p)\) with \(p > 2\), see [\text{Pi} \ 3.1.2]. While in conjectural mode, we mention the following point: the standard Borel subgroup of \(\operatorname{SL}_2(K)\) admits countably many ramified quadratic characters and so, following the construction in [ChP], the geometric conjecture predicts that tetrahedra of reducibility will occur countably many times; however, the
$R$-group machinery is not, to our knowledge, available in positive characteristic, so this remains conjectural.

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