Quasipatterns in parametrically forced systems

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We examine two mechanisms that have been put forward to explain the selection of quasipatterns in single and multi-frequency forced Faraday wave experiments. Both mechanisms can be used to generate stable quasipatterns in a parametrically forced partial differential equation that shares some characteristics of the Faraday wave experiment. One mechanism, which is robust and works with single-frequency forcing, does not select a specific quasipattern: we find, for two different forcing strengths, a 12-fold quasipattern and the first known example of a spontaneously formed 14-fold quasipattern. The second mechanism, which requires more delicate tuning, can be used to select particular angles between wavevectors in the quasipattern.

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I. INTRODUCTION

The Faraday wave experiment consists of a horizontal layer of fluid that spontaneously develops a pattern of standing waves on its surface as it is driven by vertical oscillation with amplitude exceeding a critical value; see [1, 2] for surveys. Faraday wave experiments have repeatedly produced new patterns of behaviour that required new ideas for their explanation. An outstanding example of this was the discovery of quasipatterns in experiments with one frequency [3] and with two commensurate frequencies [4]. Quasipatterns do not have any translation symmetries, but their spatial Fourier transforms have 8, 10 or 12-fold rotational order. There is as yet no satisfactory mathematical treatment of quasipatterns [5].

Two mechanisms have been proposed for quasipattern formation, both building on ideas of Newell and Pomeau [6]. One applies to single frequency forcing [7] and has been tested experimentally [8]. Another was developed to explain the origin of the two length scales in superlattice patterns [9, 10] found in two-frequency experiments [11]. The ideas have not been tested quantitatively, but have been used qualitatively to control quasipattern [1, 12] and long-scale superlattice pattern [13] formation in two and three-frequency experiments.

With advances in computing power, we are able to go to larger domains, higher resolutions and longer integration times to obtain very clean examples of approximate quasipatterns, going further than previous numerical studies [14]. In addition, we report here the first example of a spontaneously formed 14-fold quasipattern. One issue, which we do not address here, is the distinction between a true quasipattern and an approximate quasipattern, as found in numerical experiments with periodic boundary conditions. We take the point of view that a system that produces an approximate quasipattern is a good place to look for true quasipatterns.

One aim of this paper is to demonstrate that both proposed mechanisms for quasipattern formation are viable. In order to claim convincingly that we understand the pattern selection process, we have designed a partial differential equation (PDE) and forcing functions that produce a priori the required patterns. The PDE has multi-frequency forcing and shares many of the characteristics of the real Faraday wave experiment, but has an easily controllable dispersion relation and simple nonlinear terms:

\[
\frac{\partial A}{\partial t} = (\mu + i\omega)A + (\alpha + i\beta)\nabla^2 A + (\gamma + i\delta)\nabla^4 A \\
+ Q_1A^2 + Q_2|A|^2 + C|A|^2A + i\text{Re}(A)f(t) \tag{1}
\]

where \(f(t)\) is a real-valued forcing function with period \(2\pi\), \(A(x, y, t)\) is a complex-valued function, \(\mu < 0\), \(\omega, \alpha, \beta, \gamma, \delta\) are real parameters, and \(Q_1, Q_2, C\) are complex parameters.

II. PATTERN SELECTION

Resonant triads play a key role in the understanding of pattern selection mechanisms. Consider a two (or more) frequency forcing function of the form

\[
f(t) = f_m \cos(nt + \phi_m) + f_n \cos(nt + \phi_n) + ..., \tag{2}
\]

where \(m\) and \(n\) are integers, \(f_m\) and \(f_n\) are amplitudes, and \(\phi_m\) and \(\phi_n\) are phases. We consider \(m\) to be the dominant driving frequency, and focus on a pair of waves, each with wavenumber \(k_m\) satisfying the linear dispersion relation \(\Omega(k_m) = m/2\). These waves have the correct natural frequency to be driven parametrically by the forcing \(f(t)\). We write the critical modes in traveling wave form \(z_1e^{ik_1x+int/2}\) and \(z_2e^{ik_2x+int/2}\). These waves will interact nonlinearly with waves \(z_3e^{ik_3x+ilt(k_3)t}\), where
\( \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \) and \( \Omega(k_3) \) is the frequency associated with \( k_3 \), provided that either (1) the same resonance condition is met with the temporal frequencies, i.e., \( \Omega(k_3) = \frac{2\pi}{T} + \frac{n\pi}{T} \), or (2) any mismatch \( \Delta = |\Omega(k_3) - \frac{2\pi}{T} - \frac{n\pi}{T}| \) \( \) in this temporal resonance condition can be compensated by the forcing \( f(t) \). The first case corresponds to the 1 : 2 resonance, which occurs even for single frequency forcing \( (f_n = 0) \), and the second applies, e.g., to two-frequency forcing with the third wave oscillating at the difference frequency: \( \Omega(k_3) = |m-n| \) and \( \Delta = n \). Note that in both cases, the temporal frequency \( \Omega(k_3) \) determines the angle \( \theta \) between the wave-vectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) via the dispersion relation \( (\text{figure 1a}) \), and therefore provides a possible selection mechanism for certain preferred angles appearing in the power spectrum associated with the pattern. Selecting an angle of \( 0^\circ \) \( (\text{figure 1b}) \) is a special case.

The nonlinear interactions of the modes can be understood by considering resonant triad equations describing standing wave patterns, which take the form

\[
\begin{align*}
\dot{z}_1 &= \lambda z_1 + q_1 z_2 z_3 + (a|z_1|^2 + b|z_2|^2)z_1 + \cdots \\
\dot{z}_2 &= \lambda z_2 + q_1 z_1 z_3 + (a|z_2|^2 + b|z_1|^2)z_2 + \cdots \\
\dot{z}_3 &= \lambda z_3 + q_3 z_1 z_2 + \cdots ,
\end{align*}
\]

where all coefficients are real, and the dot refers to timescales long compared to the forcing period. The quadratic coupling coefficients \( q_j \) are \( O(1) \) in the forcing in the 1 : 2 resonance case, and \( O(|f_n|) \) in the difference frequency case. For other angles \( \theta \) between the wavevectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \) we expect \( q_1 \approx q_2 \approx 0 \) because the temporal resonance condition for the trial of waves is not met. Here we are assuming that the \( z_3 \)-mode is damped when \( \lambda \) goes through zero \( (i.e., \lambda_3 < 0 \) in \( \mathbf{3} \)), so \( z_3 \) can be eliminated via center manifold reduction near the bifurcation point \( (z_3 \approx \frac{q_3 z_1 z_2}{|\lambda_3|}) \), resulting in the bifurcation problem

\[
\begin{align*}
\dot{z}_1 &= \lambda z_1 - (|z_1|^2 + B_\theta |z_2|^2)z_1 \\
\dot{z}_2 &= \lambda z_2 - (|z_2|^2 + B_\theta |z_1|^2)z_2 ,
\end{align*}
\]

where we have rescaled \( z_1 \) and \( z_2 \) by a factor of \( 1/\sqrt{|a|} \) and assumed that \( a < 0 \). Here \( B_\theta = b/a + \frac{q_3 \lambda_3}{a|\lambda_3|} \) includes the contribution from the slaved mode \( z_3 \), and depends on the angle \( \theta \) between the two wavevectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \).

The function \( B_\theta \) has important consequences for the stability of regular patterns. Within the context of \( \mathbf{4} \), stripes are stable if \( B_\theta > 1 \), while rhombs associated with a given angle \( \theta \) are preferred if \( |B_\theta| < 1 \). By judicious choice of forcing frequencies, we have some ability to control the magnitude of \( B_\theta \) over a range of angles \( \theta \) \( \mathbf{5} \), which allows the enhancement or suppression of certain combinations of wavevectors in the resulting patterns. Alternatively, if we choose forcing frequencies that select an angle of 0\(^{\circ}\), then this can lead to a large resonant contribution: \( a \) can become large \( \mathbf{6} \). This causes the rescaled cross-coupling coefficient \( B_\theta \) to be small over a broad range of \( \theta \) away from \( \theta = 0 \). (As \( \theta \rightarrow 0 \), it can be shown that \( B_\theta \rightarrow 2 \).)

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**FIG. 1:** (a) If the dispersion relation satisfies \( \Omega(2k_m) = 2\Omega(k_m) \), then two modes with wavenumber \( k_m \) and aligned wavevectors \( \mathbf{k}_1 = \mathbf{k}_2 \) (inner circle) resonate in space and time with a mode with \( \mathbf{k}_3 = 2\mathbf{k}_1 \) (outer circle). (b) With two-frequency forcing, consider two modes with wavevectors \( \mathbf{k}_1 \) and \( \mathbf{k}_2 \), with the same wavenumber \( k_m \), and with \( \Omega(\mathbf{k}_m) = m/2 \) (middle circle). The nonlinear combination of these two waves can, in the presence of forcing at frequency \( \nu \), interact with a mode with wavevector \( \mathbf{k}_3 \) (inner circle), provided \( \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \) and \( \Omega(k_3) = |m-n| \).

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**FIG. 2:** \( B_\theta \) for the two cases. (a) single frequency forcing with 1 : 2 resonance. The parameter values are \( \omega = \frac{1}{2} \), \( \beta = -\frac{1}{2} \), \( \delta = 0 \), \( \mu = -0.005 \), \( \alpha = 0.001 \), \( \gamma = 0 \), \( Q_1 = 3 + 4i \), \( Q_2 = -6 + 8i \), \( C = -1 + 10i \), \( m = 1 \), \( \phi_1 = 0 \) and \( f_1 = 0.024002 \). (b) multi-frequency \((4, 5, 8)\) forcing, with \( \omega = 0.633975 \), \( \beta = -1.366025 \), \( \delta = 0 \), \( \mu = -0.2 \), \( \alpha = -0.2 \), \( \gamma = -0.15 \), \( Q_1 = 1+i \), \( Q_2 = -2 + 2i \), \( C = -1 + 10i \), \( f_1 = 0.53437 \), \( f_5 = 0.76316 \), \( f_s = 1.49063 \), \( \phi_4 = 0 \), \( \phi_5 = 0 \) and \( \phi_s = 0 \). The + symbols are the result of a separate calculation.
III. RESULTS

We present parameter values that demonstrate that the two mechanisms are viable methods of predicting parameter values for stable approximate quasipatterns.

The dispersion relation of the PDE (1) is $\Omega(k) = \omega - \beta k^2 + \delta k^4$. With single-frequency forcing, we choose $m = 1$, and a spatial scale so that modes with $k = 1$ are driven subharmonically: $\Omega(1) = 0$. To have $1 : 2$ resonance in space and time, we impose $\Omega(2) = 1$, which leads to $\omega = \frac{1}{8} + 4\delta$ and $\beta = -\frac{1}{8} + 5\delta$. We choose $\delta = 0$, small values for the damping coefficients $\mu$, $\alpha$ and $\gamma$, and order one values for the nonlinear coefficients. We solve the linear stability problem to find the critical value of the amplitude $f_1$ in the forcing function, and use weakly nonlinear theory to calculate $B_\theta$ (figure 2a). This curve has $B_0 = 2$, but $B_\theta$ drops away sharply, and is close to zero for $\theta \geq 30^\circ$, for the reasons explained above. We use $B_\theta$ at $30^\circ$, $60^\circ$ and $90^\circ$ and find that, within the restrictions of a twelve-mode expansion, 12-fold quasipatterns are stable.

A numerical solution of the PDE (1) at $1.1$ times critical is shown in figure 3(a), in a square domain with periodic boundary conditions, of size $30 \times 30$ wavelengths, with $512^2$ Fourier modes (dealiased). The timestepping method was the fourth-order ETDRK4 [15], with $20$ timesteps per period of the forcing. The solution is an approximate quasipattern: the primary modes that make up the pattern are $(30,0)$ and $(26,15)$ and their reflections, in units of basic lattice vectors. These two wavevectors are $29.98^\circ$ apart, and differ in length by $0.05\%$. The amplitudes of the modes differ by $0.5\%$. The initial condition was not in any invariant subspace, and the PDE was integrated for $160000$ periods of the forcing. However, when we increase the forcing to $1.3$ times critical, we find that the $12$-fold quasipattern is unstable and is replaced (after a transient of $50000$ periods) by a $14$-fold quasipattern (figure 3b). In this case, the modes are $(30,0)$, $(27,13)$, $(19,23)$ and $(7,29)$, differing in length by $0.5\%$ and having angles within $1.5^\circ$ of $360^\circ/14$. The amplitudes differ by about $10\%$.

The second method of producing quasipatterns involves the weakly damped difference frequency mode, and is more selective, but also requires some fine-tuning of the parameters. In order to use triad interactions to encourage modes at $30^\circ$, we choose $m = 4$, $n = 5$ forcing, setting $\Omega(1) = 2$, and requiring that a wavenumber involved in $30^\circ$ mode interactions ($k^2 = 2 - \sqrt{3}$) correspond to the difference frequency $\Omega(k) = 1$. One solution is $\omega = 0.633975$, $\beta = -1.366025$ and $\delta = 0$. Twelve-fold quasipatterns also require modes at $90^\circ$ to be favoured, and for these choices of parameters, $\Omega(\sqrt{2})$ is $3.37$. Although this is not particularly close to $4$, we can use $1 : 2$ resonance (driving at frequency $8$) to control the $90^\circ$ interaction. The resulting $B_\theta$ curve (figure 2b) shows pronounced dips at $30^\circ$ and $90^\circ$ as required. Again, $B_{30}$, $B_{90}$ and $B_{90}$ are used to show that, within a $12$-amplitude cubic truncation, $12$-fold quasipatterns are stable, this time between $0.9995$ and $1.0095$ times critical. Squares are also stable above $1.0015$ times critical.

A numerical solution of the PDE (1) at $1.003$ times critical is shown in figure 3c, in a periodic domain $112 \times 112$ wavelengths (integrated using $1536^2$ Fourier modes). This solution was followed for over $10000$ forcing periods. The important wavevectors are $(112,0)$ and $(97,56)$, which are $29.9987^\circ$ apart and differ in length by $0.004\%$. The amplitudes of these modes differ by $1\%$.

IV. DISCUSSION

We have investigated two mechanisms for quasipattern formation for Faraday waves within a single PDE model of pattern formation via parametric forcing, and have demonstrated that both mechanisms are viable. One uses $1 : 2$ resonance in space and time to enhance the self-interaction coefficient $a$ and so suppress the cross-coupling coefficient $B_\theta$ for angles greater than about $30^\circ$, and leads to “turbulent crystals” [3]. Within this framework, there is little distinction between $8$, $10$, $12$ or $14$-fold quasipatterns, or indeed any other combination of modes, and there is no way of knowing in advance which pattern will be formed. The mechanism is robust (the patterns are found well above onset), and requires only a single frequency in the forcing. A dispersion relation that supports $1 : 2$ resonance in space and time is needed.

The existence of $14$-fold (and higher) quasipatterns has been suggested before [2, 7, 16], but we have presented here the first example of a spontaneously formed $14$-fold quasipattern that is a stable solution of a parametrically forced PDE. Examples where $14$-fold symmetry is imposed externally have been reported in optical experiments [17]. The Fourier spectra of $12$-fold and $14$-fold quasipatterns are very different [3], which may have consequences for their mathematical treatment.

The second mechanism uses three-wave interactions involving a damped mode associated with the difference of the two frequencies in the forcing to select a particular angle ($30^\circ$ in the example presented here). Using different primary frequencies, or altering the dispersion relation, allows other angles, or combinations of angles, to be selected. The advantage is that a forcing function can be designed to produce a particular pattern. On the other hand, the strongest control of $B_\theta$ occurs for parameters close to the bicritical point, which limits the range of validity of the weakly nonlinear theory used to compute stability. This issue will be pursued elsewhere.

It should be noted that the stability calculations done here are in the framework of a twelve-mode amplitude expansion truncated a cubic order. The fact that stable 12-fold quasipatterns are found where they are expected demonstrates that this approach provides useful information about the stability of the approximate quasipatterns, in spite of the concerns about small divisors raised by [3].

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