Spectrum of Laplacians for Graphs with Self-Loops

Behçet Açıkmeşe
Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin, USA

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Abstract

This note introduces a result on the location of eigenvalues, i.e., the spectrum, of the Laplacian for a family of undirected graphs with self-loops. We extend on the known results for the spectrum of undirected graphs without self-loops or multiple edges. For this purpose, we introduce a new concept of pseudo-connected graphs and apply a lifting of the graph with self-loops to a graph without self-loops, which is then used to quantify the spectrum of the Laplacian for the graph with self-loops.

1 Introduction

Graph theory has proven to be an extremely useful mathematical framework for many emerging engineering applications [1, 2, 3, 4, 5, 6]. This note introduces a result, Theorem 1, on the location of eigenvalues, i.e., the spectrum, of the Laplacian for a family of undirected graphs with self-loops. We extend on the known results for the spectrum of undirected graphs without self-loops or multiple edges [7, 8, 9]. For this purpose, we introduce a new concept of pseudo-connected graphs and apply a lifting of the graph with self-loops to a graph without self-loops, which is then used to quantify the spectrum of the Laplacian for the graph with self-loops.

The primary motivation for this result is to analyze interconnected control systems that emerge from new engineering applications such as control of multiple vehicles or spacecraft [10, 11, 12, 13, 2]. This result first appeared [14], where it proved key to
extend the stability proofs for observers [15, 16] to decentralized observers. We believe that the result can also be useful in designing decentralized control [17, 18, 19, 20, 21] and optimization algorithms.

**Notation:** The following is a partial list of notation (see the Appendix for the graph theoretic notation): \( \mathbb{R}^n \) is the \( n \) dimensional real vector space; \( \| \cdot \| \) is the vector 2-norm; \( I \) is the identity matrix and \( I_m \) is the identity matrix in \( \mathbb{R}^{m \times m} \); \( \mathbf{1}_m \) is a vector of ones in \( \mathbb{R}^m \); \( e_i \) is a vector with its \( i \)th entry +1 and the others zeros; \( \sigma(A) \) is the set of eigenvalues of \( A \); \( \sigma_+(S) \) are the positive eigenvalues of \( S = S^T \); \( \rho(A) \) is the spectral radius of \( A \).

**2 Main Result on Laplacians for Graphs with Self-Loops**

This section first summarizes some known facts about graph theory, and then introduces the main result of this note on graphs with self-loops. For graphs with self-loops, we will introduce the concept of pseudo connectedness, which is useful in our developments.

Let \( G = (V, E) \) represents a finite graph with a set of vertices \( V \) and edges \( E \) with \( (i, j) \in E \) denoting an edge between the vertices \( i \) and \( j \). \( L(G) \) is the Laplacian matrix for the graph \( G \); \( a(G) \) is the algebraic connectivity of the graph \( G \), which is the second smallest eigenvalue of \( L(G) \). \( E \) is the vertex-edge adjacency matrix. Each row of the vertex-edge adjacency matrix describes an edge between two vertices with entries corresponding to these vertices are +1 and −1 (it does not matter which entry is + or −) and the rest of the entries are zeros. Note that if the edge described by a row is a self-loop then there is only one non-zero entry with +1. Hence a row of \( E_{i,k} \), denoted by \( \pi \), defining an edge between \( p \)'th and \( q \)'th vertices of the graph has its \( j \)th entry of \( \pi_j \) as follows

\[
\pi_j = \begin{cases} 
1 & j = p \\
-1^{(j-p)} & j = q \\
0 & \text{otherwise}
\end{cases}
\]

\( \mathcal{A} \) is the adjacency matrix, and \( \mathcal{D} \) is the diagonal matrix of node in-degrees for \( G \), then the following gives a relationship to compute the Laplacian matrix

\[
L(G) = E^T E = \mathcal{D} - \mathcal{A}.
\]
The following relationships are well known in the literature \cite{7} and \cite{8} for a connected undirected graph $G$ with $N$ vertices and without any *self-loops or multiple edges*

\[
a(G) \geq 2(1 - \cos(\pi/N)) \quad (2)
\]

\[
2d(G) \geq \max(\sigma(L(G))), \quad (3)
\]

where $d(G)$ is the maximum in-degree of $G$. Indeed the inequality (3) is valid for any undirected graph without self-loops or multiple edges whether they are connected or not. Also, due to the connectedness of the graph, the minimum eigenvalue of the Laplacian matrix is 0 with algebraic multiplicity of 1 and the eigenvector of $1$. Next we characterize the location of the Laplacian eigenvalues for a connected undirected graph $G$ with self-loops. Having a self-loop does not change whether a graph is connected or not, that is, a graph with self-loops is connected if and only if the same graph with the self-loops removed is connected. Furthermore, we define the Laplacian of an undirected graph with at least one self-loop as

\[
L(G) = L(G^o) + \sum_{(i,i) \in E} e_i e_i^T \quad (4)
\]

where $G^o$ is the largest subgraph of $G$ with the self-loops removed, and

\[
L(G^o) = \sum_{(i,j) \in E, i \neq j} (e_i - e_j)(e_i - e_j)^T. \quad (5)
\]

The following definition introduces the concept of the pseudo-connected graphs, which is our fourth contribution.

**Definition 1** An undirected graph $G(V,E)$ without multiple edges is pseudo-connected if every vertex is connected to itself and/or to another vertex and if every connected subgraph of $G$ has at least one vertex with a self-loop.

Next we develop useful results on the eigenvalues of undirected graphs with self-loops, which are instrumental in the stability analysis of the decentralized observer. We refer to \cite{1} for a graph theoretic view of multi-agent networks.

**Lemma 1** The Laplacian of a pseudo-connected graph is positive definite.
Proof: A pseudo-connected graph can be partitioned into subgraphs that are connected with at least one self-loop in each subgraph. Note that some of these subgraphs can have a single vertex that has a self-loop. Clearly each subgraph with a single vertex and a self-loop has Laplacian 1. If we can also show that the connected subgraphs that have multiple vertices with at least one self-loop have positive definite Laplacians, then the Laplacian of the overall graph will also be positive definite. This will conclude the proof.

To do that we prove that a connected graph \( G \) with at least one self-loop has a positive definite Laplacian. Let \( G^0 \) be the connected graph formed by removing the self-loops from \( G \). Any vector \( v \neq 0 \), which can be expressed as \( v = w + \zeta 1 \) where \( w^T 1 = 0 \), and either or both \( w \neq 0 \) and \( \zeta \neq 0 \). Then, by using (5) \( \mathcal{L}(G) = \mathcal{L}(G^0) + Q_o \) where \( Q_o := \sum_{i=1}^q e_i e_i^T \) and \( q \) is the number of self-loops and having \( 1^T Q_o 1 = q \),

\[
v^T \mathcal{L}(G) v = w^T \mathcal{L}(G^0) w + w^T Q_o w + 2 \zeta w^T Q_o 1 + q \zeta^2 \geq 0.
\]

If \( w \neq 0 \), \( w^T \mathcal{L}(G^0) w > 0 \) (due to connectedness of \( G^0 \)), we have \( v^T \mathcal{L}(G) v > 0 \). Next, if \( w = 0 \) and \( \zeta \neq 0 \), then \( v^T \mathcal{L}(G) v = q \zeta^2 > 0 \). Consequently \( \mathcal{L}(G) = \mathcal{L}(G)^T > 0 \), where \( q \) is the number of self-loops.

Next we introduce the concept of lifted graph to characterize the eigenvalues of the Laplacian of a graph with self-loops.

Definition 2 Given an undirected graph \( G(E, V) \) with \( N \) vertices and with at least one self-loop, its lifted graph \( \hat{G}(\hat{E}, \hat{V}) \) is a graph with \( 2N + 1 \) vertices and with no self-loops such that (Figure 1): For every vertex \( i \) in \( G \) there are vertices \( i \) and \( i + N + 1 \) in \( \hat{G} \), \( i = 1, ..., N \), and also a middle vertex \( N + 1 \) with the following edges

\[
(i, j) \in E \Rightarrow (i, j) \in \hat{E} \text{ and } (i + N + 1, j + N + 1) \in \hat{E}
\]

\[
(i, i) \in E \Rightarrow (i, N + 1) \in \hat{E} \text{ and } (N + 1, i + N + 1) \in \hat{E}.
\]

The following theorem is the main result of this section on the eigenvalues of the Laplacians of pseudo-connected graphs.

Theorem 1 For a finite undirected graph, \( G \), with self-loops but without multiple-edges:

\[
\sigma (\mathcal{L}(G)) \subseteq \sigma \left( \mathcal{L}(\hat{G}) \right) \cap [0, 2d(G^0) + 1],
\]
where $G^o(V,E^o)$ is a subgraph of $G(V,E)$ where $E^o \subset E$ and $E^o$ contains all the edges of $E$ that are not self-loops. Particularly, if $G$ is a pseudo-connected graph, then

$$\sigma(L(G)) \subseteq \sigma_+\left(L(\hat{G})\right) \cap [0, 2d(G^o)+1].$$

(7)

**Proof:** Consider the edge-vertex adjacency matrix $E^o$ for $G^o$. We have the following relationship for the vertex adjacency matrices of $G$ and $\hat{G}$, $E$ and $\hat{E}$, in terms of $E^o$

$$\hat{E} = \begin{bmatrix} E^o & 0 & 0 \\ S & 1 & 0 \\ 0 & 1 & S \\ 0 & 0 & E^o \end{bmatrix}, \quad E = \begin{bmatrix} E^o \\ S \end{bmatrix}$$

where the matrix $S$ has entries of +1 or 0. This implies that

$$L(\hat{G}) = \begin{bmatrix} E^o^T E^o + S^T S & S^T \mathbf{1} & 0 \\ \mathbf{1}^T S & 2N & \mathbf{1}^T S \\ 0 & S^T \mathbf{1} & E^o^T E^o + S^T S \end{bmatrix}$$

and $L(G) = E^o^T E^o + S^T S$. Now suppose that $\psi \in \sigma(L(G))$ with the corresponding eigenvector $v$. Then

$$L(\hat{G}) \begin{bmatrix} v \\ 0 \\ -v \end{bmatrix} = \begin{bmatrix} L(G)v \\ 0 \\ -L(G)v \end{bmatrix} = \psi \begin{bmatrix} v \\ 0 \\ -v \end{bmatrix}.$$
Consequently $\psi \in \sigma \left( L(\hat{G}) \right)$ too. Next note that $0 \leq S^T S \leq I$, which implies that $L(G) \leq L(G^o) + I$. This implies that

$$\max(\sigma(L(G))) \leq \max(\sigma(L(G^o))) + 1 \leq 2d(G^o) + 1$$  \hspace{1cm} (8)

which follows from (3). This proves the relationship given by (6). Now by using Lemma 1 the relationship given by (7) directly follows from (6).

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