Error bounds for interpolation with piecewise exponential splines of order two and four

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Abstract

Explicit pointwise error bounds for the interpolation of a smooth function by piecewise exponential splines of order four are given. Estimates known for cubic splines are extended to a natural class of piecewise exponential splines which are appearing in the construction of multivariate polysplines. The error estimates are derived in an inductive way using error estimates for the interpolation of a smooth function by exponential splines of order two.

Key words: Exponential splines, Interpolation $L$-splines, Approximation rate, Error estimate

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Let $C^m[a,b]$ be the space of all $m$ times continuously differentiable functions $f:[a,b] \rightarrow \mathbb{C}$ on the closed interval $[a,b]$. A function $g:[t_1,t_n] \rightarrow \mathbb{C}$ is an exponential spline\footnote{We follow here the terminology in [32, p. 405]. In [21], [22] tensions splines with varying parameters are called exponential splines. This terminology seems to be misleading.} for the knots $t_1 < \ldots < t_n$ and $\Lambda = (\lambda_0,\ldots,\lambda_N) \in \mathbb{C}^{N+1}$ if $g \in C^{N-1}[t_1,t_n]$ and the restriction of $g$ to each open interval $(t_j,t_{j+1})$ is a solution of the differential equation $L_{(\lambda_0,\ldots,\lambda_N)}(g) = 0$ where

$$L_{(\lambda_0,\ldots,\lambda_N)} = \prod_{j=0}^{N} \left( \frac{d}{dx} - \lambda_j \right).$$

(1)

Exponential splines are used in many applications, e.g. in signal processing [10], in non-parametric regression [7], [29], in statistical modelling and smoothing of big data, see [8], [10], [11], and for approximating solutions of partial differential equations (see [24] and its references). The choice of the differential operator in (1) depends on the specific aspects of the underlying problem. The motivation for this paper came from the effort to provide error estimates for polysplines of
order 4, \cite{11}. Our main interest is related to the differential operator

\[ L(\xi,\xi, -\xi, -\xi) = \left( \frac{d^2}{dt^2} - \xi^2 \right) \]  

with \( \xi \in \mathbb{R} \) which arises naturally in the context of biharmonic functions, see \cite{17, 18} and \cite{19}. Polysplines on strips can be described by Fourier methods and the parameter \( \xi \) in (2) is equal to \( |y| \) where \( y \in \mathbb{R}^n \) is arbitrary, see \cite{11, 13, 14, 15, 16}. Hence it is crucial to have error estimates which are valid uniformly for all parameters \( \xi \in \mathbb{R} \).

Let us emphasize that \( L^\infty \)- and \( L^2 \)-error estimates for (more general) interpolation \( L \)-splines have been discussed by several authors, see \cite{31, 34, 32}. Our aim is to achieve exact control of the constants in the estimates which depend on the differential operator, in particular we want to obtain error estimates in the case of operator (2) uniformly for all \( \xi \in \mathbb{R} \).

For \( F \in C^1 [t_1, t_n] \) let us denote by \( I_4 (F) \) the exponential spline for the operator \( L(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \) satisfying the conditions

\[ I_4 (F) (t_j) = F (t_j) \text{ for } j = 1, ..., n, \]  
\[ \frac{d}{dt} I_4 (F) (t_1) = \frac{d}{dt} F (t_1) \text{ and } \frac{d}{dt} I_4 (F) (t_n) = \frac{d}{dt} F (t_n). \]  

We want to determine explicit constants \( C = C(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \) such that for any \( F \in C^4 [t_1, t_n] \) and any partition \( t_1 < ... < t_n \) the following error estimate

\[ |F (t) - I_4 (F) (t)| \leq C \cdot \Delta^4 \cdot \max_{x \in [t_1, t_n]} |L(\lambda_0, \lambda_1, \lambda_2, \lambda_3) F (x)| \]  

holds for \( t \in [t_1, t_n] \) where

\[ \Delta := \max_{j=1,...,n-1} |t_{j+1} - t_j|. \]  

In order to give a flavor of the results in this paper we state now one of the major results:

**Theorem 1** For \( t_1 < ... < t_n \) and \( F \in C^4 [t_1, t_n] \) let \( I_4 (F) \) be the exponential spline for the operator \( L(\xi,\xi, -\xi, -\xi) \) satisfying (3) and (4). Then

\[ \max_{t \in [t_1, t_n]} |F (t) - I_4 (F) (t)| \leq \Delta^4 \cdot \frac{5}{64} \max_{\theta \in [t_1, t_n]} |L(\xi,\xi, -\xi, -\xi) F (\theta)|. \]  

When we put \( \xi = 0 \) in (7) we obtain an error estimate for cubic splines. Note that then our result provides the estimate for cubic splines in \cite{3} p. 55 but with an additional factor 5/4. However, our estimate holds uniformly for all \( \xi \in \mathbb{R} \).

\[ ^2 \text{As already mentioned in } \cite{3}, \text{ Hall and Meyer have shown in } \cite{9} \text{ that the best constant } C \text{ in } (7) \text{ for cubic splines is } 5/384. \]
Our approach to the error estimates for exponential splines of order 4 is inspired by the elegant exposition of Carl de Boor in [3] of the error estimates for interpolation cubic splines. It is shown there that this error can be estimated in two steps by using two simpler error estimates: the pointwise error estimate for interpolation with continuous piecewise linear functions for the knots \( t_1 < \ldots < t_n \) (which is rather easy) and the pointwise error estimate of best \( L^2 \)-approximation by continuous piecewise linear functions for \( t_1 < \ldots < t_n \) (which is more delicate). For the latter estimate diagonal dominance of certain matrices is an important tool, and, according to [3, p. 35], this technique in spline analysis was first used in [36]. In the case of exponential splines we shall follow the same ideas and prove generalizations of every step. According to our knowledge this is a new contribution in the literature. In any case, the computational aspects are lot more challenging than in the polynomial case.

Let us now outline the structure and the main results of the paper: in section 2 we introduce at first the concept of generalized hat functions. In section 3 exponential splines of order 2 for the differential operator \( L(\lambda_0, \lambda_1) \) are discussed which interpolate a function \( f \) at the points \( t_1, \ldots, t_n \) (the analog of interpolating linear splines). This interpolation exponential spline of order 2 will be denoted by \( I_2(f) \). In section 4 we provide the following error estimate

\[
|f(t) - I_2(f)(t)| \leq \max_{j=1, \ldots, n-1} M_{\lambda_0, \lambda_1}^{t_j, t_{j+1}} \max_{\theta \in [t_j, t_{j+1}]} |L(\lambda_0, \lambda_1)f(\theta)|
\]

for all \( t \in [t_1, t_n] \). Here the constant \( M_{\lambda_0, \lambda_1}^{a, b} \) for the interval \([a, b] \) is defined by

\[
M_{\lambda_0, \lambda_1}^{a, b} = \max_{t \in [a, b]} \left| \Omega_{\lambda_0, \lambda_1, 0}^{a, b}(t) \right|
\]

and \( \Omega_{\lambda_0, \lambda_1, 0}^{a, b} \) is a solution of the differential equation \( L(\lambda_0, \lambda_1, 0)u = 0 \) such that \( \Omega_{\lambda_0, \lambda_1, 0}^{a, b}(a) = \Omega_{\lambda_0, \lambda_1, 0}^{a, b}(b) = 0 \) and \( L(\lambda_0, \lambda_1)\Omega_{\lambda_0, \lambda_1, 0}^{a, b} = -1 \).

In section 5 we shall compute the constant \( M_{\lambda_0, \lambda_1}^{a, b} \) for several cases. For the operator \( L_{(\xi, -\xi)} \) we shall prove the estimate

\[
M_{\xi, -\xi}^{a, b} \leq \frac{1}{8} (b - a)^2
\]

for all \( a < b \). It is surprising that in the case \( \lambda_0 \leq 0 \leq \lambda_1 \) the following estimate holds:

\[
M_{\lambda_0, \lambda_1}^{a, b} < \frac{1}{4} (b - a)^2.
\]

Simple examples show that for positive \( \lambda_0, \lambda_1 \) the numbers \( M_{\lambda_0, \lambda_1}^{a, b}/(b - a)^2 \) are not bounded.

Section 5 is devoted to an error estimate of the best \( L^2 \)-approximation to \( f \in C^2[a, b] \) by exponential splines of order 2 (for the operator \( L(\lambda_0, \lambda_1) \)) for the knots \( t_1 < \ldots < t_n \). Let us denote by \( \mathcal{H}_n \) the space of exponential splines of...
order 2 with respect to $L(\lambda_0, \lambda_1)$ and for the partition $t_1 < \ldots < t_n$. We denote by $P^{\mathcal{H}_n}(f)$ the best $L_2$-approximation to $f$ from the subspace $\mathcal{H}_n$. Following the pathway provided in \cite{3} (estimate of the best approximation by hat functions and estimate of solutions of tridiagonal matrices) one obtains explicit estimates for the operator norm $\|P^{\mathcal{H}_n}\|_{\text{op}}$ (defined with respect to the uniform norm), see Theorem 15. For the important case $\lambda_0 = \xi$ and $\lambda_1 = -\xi$ for $\xi \in \mathbb{R}$ we show that

$$\|P^{\mathcal{H}_n}\|_{\text{op}} \leq 4,$$

which is slightly larger than the constant in the polynomial case which is 3, see \cite[p. 34]{3}.

In section 6 we use the results of error estimates of exponential splines of order 2 for a short proof of the error estimate for exponential splines for differential operators of the type $L(\lambda_0, \lambda_1, -\lambda_0, -\lambda_1)$.

In the Appendix, section 7, we have compiled some results about exponential polynomials which are needed in the previous sections.

Most of the above results are valid for the so-called piecewise exponential splines. Let us recall that $g : [t_1, t_n] \rightarrow \mathbb{C}$ is a piecewise exponential spline for the knots $t_1 < \ldots < t_n$ and variable frequencies $(\lambda_{0,j}, \ldots, \lambda_{N,j})$ for $j = 1, ..., n - 1$, if $g \in C^{N-1}[t_1, t_n]$ and the restriction of $g$ to each interval $(t_j, t_{j+1})$ is a solution of the equation $L(\lambda_{0,j}, \ldots, \lambda_{N,j}) (g) = 0$ depending on $j = 1, ..., n - 1$. This concept was introduced by Späth in \cite{37} in the context of tension splines where different values of the tension parameter $\rho^2_j$ could be chosen for the intervals $(t_{j-1}, t_j)$. Recall that splines in tension are exponential splines for the differential operator

$$L(0,0,\rho,-\rho) = \frac{d^2}{dt^2} \left( \frac{d^2}{dt^2} - \rho^2 \right),$$

see also \cite{26, 27, 28}. The extension of the above results to piecewise L-splines does require only one additional burden – that for terminology and notation. Let us mention that some results in the literature about piecewise L-splines are erroneous (e.g. in \cite{25}) as pointed out recently by Z. Ayalon, N. Dyn and D. Levin in \cite{1}.

Finally let us introduce some standard notations. We denote the space of all exponential polynomials or L-polynomials with respect to the differential operator $L(\lambda_0, \ldots, \lambda_N)$ by

$$E(\lambda_0, \ldots, \lambda_N) = \{ f \in C^{N+1}(\mathbb{R}) : L(\lambda_0, \ldots, \lambda_N)f = 0 \}$$

The maximum norm for $f \in C[a,b]$ is defined by

$$\|f\|_{[a,b]} := \max_{t \in [a,b]} |f(t)|$$

and the distance of $f \in C[a,b]$ to a subspace $U$ of $C[a,b]$ is defined by

$$\text{dist}_{C[a,b]}(f,U) = \inf \left\{ \|f - g\|_{[a,b]} : g \in U \right\}.$$
1 Generalized hat functions

Hat functions, also called chapeau functions, are used in finite element methods. It is well known that hat functions provide a basis of the space of linear splines, cf. [3]. From the viewpoint of spline analysis hat functions are just linear splines with minimal support, or briefly, linear B-splines.

Now we shall generalize this concept, see Definition 2 below. When \( \varphi_j(t) = t \) for \( t \in \mathbb{R} \) and \( j = 1, \ldots, n-1 \), we obtain the definition of hat functions \( H_1, \ldots, H_n \) as given in [3, p. 32]. In many applications we take the same function \( \varphi = \varphi_j \) for \( j = 1, \ldots, n-1 \). Our definition allows us to develop error estimates for the case of piecewise exponential splines.

**Definition 2** Let \( \delta > 0 \), \( n \geq 3 \) and let \( \varphi_j : [-\delta, \delta] \to \mathbb{C} \) be continuous strictly increasing functions with \( \varphi_j(0) = 0 \) for \( j = 1, \ldots, n-1 \). Let the points \( t_1 < \ldots < t_n \) be given such that \( t_j - t_{j-1} \leq \delta \) for all \( j = 2, \ldots, n \). Define for \( j = 2, \ldots, n-1 \) the hat function \( H_j(t) \) with support in \([t_{j-1}, t_{j+1}]\) by

\[
H_j(t) = \begin{cases} \frac{\varphi_j(t-t_{j-1})}{\varphi_j(t_{j-1})} & \text{for } t \in [t_{j-1}, t_j] \\ \frac{\varphi_j(t-t_{j+1})}{\varphi_j(t_{j+1})} & \text{for } t \in [t_j, t_{j+1}] \end{cases}
\]

Further for \( j = 1 \) and \( j = n \) we define

\[
H_1(t) = \begin{cases} \frac{\varphi_1(t-t_2)}{\varphi_1(t_1-t_2)} & \text{for } t \in [t_1, t_2] \end{cases},
\]

\[
H_n(t) = \begin{cases} \frac{\varphi_{n-1}(t-t_{n-1})}{\varphi_{n-1}(t_{n-1}-t_n)} & \text{for } t \in [t_{n-1}, t_n] \end{cases},
\]

and zero elsewhere.

Obviously \( H_1, \ldots, H_n \) are non-negative and linearly independent. Given a function \( f \in C[t_1, t_n] \) we define the interpolant with respect to the hat functions \( H_1, \ldots, H_n \) as

\[
I_2(f) = \sum_{j=1}^{n} f(t_j) H_j.
\]

Note that \( I_2(f)(t_j) = f(t_j) \) for \( j = 1, \ldots, n \), so \( I_2(f) \) interpolates \( f \) at the points \( t_1, \ldots, t_n \).

**Proposition 3** Assume that \( \varphi_j \) are strictly increasing functions on \([-\delta, \delta]\) such that \( \varphi_j(0) = 0 \) for each \( j = 1, \ldots, n-1 \), and \( \max_{j=2,\ldots,n} (t_j - t_{j-1}) \leq \delta \). Then

\[
\sum_{j=1}^{n} |H_j(t)| \leq 2 \text{ and } |I_2(f)(t)| \leq 2 \|f\|_{[t_1, t_n]} \tag{9}
\]

for \( f \in C[t_1, t_n] \) and \( t \in [t_1, t_n] \). If \( U_n \) denotes the linear space generated by \( H_1, \ldots, H_n \), then

\[
dist_{C[a,b]}(f, U_n) \leq \|f - I_2(f)\|_{[t_1, t_n]} \leq 3 \cdot dist_{C[a,b]}(f, U_n) \tag{10}
\]
Taking the infimum over all $g$ is now obvious. Note that the first inequality in (10) is trivial. For Proposition 4 $\lambda$ always increasing on the real line even if $g$ments hold:

It is now easy to see that

$$0 \leq H_j(t) \leq 1$$

and $j = 1, \ldots, n$. Since each $H_j$ has support in $[t_{j-1}, t_j]$ the statement in (9) is now obvious. Note that the first inequality in (10) is trivial. For $g \in U_n$ we have $g = I_2(g)$ and

$$\|f - I_2(f)\|_{[t_1, t_n]} \leq \|f - g\|_{[t_1, t_n]} + \|I_2(g - f)\|_{[t_1, t_n]} \leq 3 \|f - g\|_{[t_1, t_n]}.$$  

Taking the infimum over all $g \in U_n$ gives the result. ■

2 Piecewise exponential splines of order 2 and generalized hat functions

For arbitrary complex numbers $\lambda_0, \lambda_1$ we define $\Phi(\lambda_0, \lambda_1)$ to be the unique exponential polynomial in $E(\lambda_0, \lambda_1)$ satisfying $\Phi(\lambda_0, \lambda_1) (0) = 0$ and $\Phi(\lambda_0, \lambda_1) (0) = 1$. We call $\Phi(\lambda_0, \lambda_1)$ the fundamental function for $(\lambda_0, \lambda_1)$, see the appendix for a general discussion. For $\lambda_0 \neq \lambda_1$ the simple formula

$$\Phi(\lambda_0, \lambda_1) (t) = \frac{e^{\lambda_1 t} - e^{\lambda_0 t}}{\lambda_1 - \lambda_0} = e^{(\lambda_0 + \lambda_1) t/2} \frac{e^{(\lambda_1 - \lambda_0) t/2} - e^{-(\lambda_1 - \lambda_0) t/2}}{\lambda_1 - \lambda_0}.$$  

holds. In the case that $\lambda_0 = \lambda_1 \neq 0$ we define $\Phi(\lambda_0, \lambda_0) (t) = e^{\lambda_0 t}$. In the case $\lambda_0 = \lambda_1 = 0$ we just have $\Phi(0,0) (t) = t$. In any of these three cases there exists an odd function $\psi(\lambda_0, \lambda_1)$ such that $\Phi(\lambda_0, \lambda_1) (t) = e^{(\lambda_0 + \lambda_1) t/2} \psi(\lambda_0, \lambda_1) (t)$, and this leads to the following useful formula:

$$\Phi(\lambda_0, \lambda_1) (-t) = -e^{-(\lambda_0 + \lambda_1) t} \Phi(\lambda_0, \lambda_1) (t).$$  

(11)

For simplicity we shall assume in this paper that $\lambda_0, \lambda_1$ are real numbers although it might be interesting for applications to include exponential polynomials with complex frequencies (see e.g. [40]): if $\lambda_0 = i\alpha$ and $\lambda_1 = -i\alpha$ for $\alpha > 0$ then

$$\Phi(i\alpha, -i\alpha) (t) = \frac{1}{\alpha} \sin \alpha t$$

is real-valued and strictly increasing on $[-\pi/2\alpha, \pi/2\alpha]$.

The following example on Figure 1 shows that the function $\Phi(\lambda_0, \lambda_1)$ is not always increasing on the real line even if $\lambda_0$ and $\lambda_1$ are real:

**Proposition 4** Assume that $\lambda_0, \lambda_1$ are real numbers. Then the following statements hold:

(i) If $\lambda_0 \leq 0 \leq \lambda_1$ then $\Phi(\lambda_0, \lambda_1)$ is strictly increasing on $\mathbb{R}$.

(ii) If $0 < \lambda_0 < \lambda_1$ then $\Phi(\lambda_0, \lambda_1)$ is strictly increasing on $[-\delta, \delta]$ with $\delta = \frac{\ln \lambda_1 - \ln \lambda_0}{\lambda_1 - \lambda_0}$. 


Proof. The case $\lambda_0 = \lambda_1$ leads in (i) to the polynomial case $\lambda_0 = \lambda_1 = 0$. For $\lambda_0 \leq 0 \leq \lambda_1$ with $\lambda_0 \neq \lambda_1$ the statement is obvious since
\[
\frac{d}{dt} \Phi_{(\lambda_0, \lambda_1)}(t) = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_0 e^{\lambda_0 t}}{\lambda_1 - \lambda_0} \geq 0.
\]
In the case (ii), $\Phi_{(\lambda_0, \lambda_1)}(t)$ has one critical point $t_0$, namely
\[
e^{(\lambda_1 - \lambda_0)t_0} = \frac{\lambda_0}{\lambda_1}, \text{ so } t_0 = \frac{\ln \lambda_0 - \ln \lambda_1}{\lambda_1 - \lambda_0}.
\]
The function $t \mapsto \Phi_{(\lambda_0, \lambda_1)}(t)$ is increasing for all $t > t_0$, and decreasing for all $t < t_0$. Note that $\Phi_{(\lambda_0, \lambda_1)}(t) \to 0$ for $t \to -\infty$. 

Throughout the whole paper we make the following assumptions and use of notations:

(i) Let $\delta > 0$. We assume that for the real numbers $\lambda_{0,j} \leq \lambda_{1,j}$ the functions
\[
\varphi_j := \Phi_{(\lambda_{0,j}, \lambda_{1,j})}, \quad j = 1, \ldots, n - 1,
\]
are increasing on $[-\delta, \delta]$.

(ii) For given $t_1 < \ldots < t_n$ with $|t_{j+1} - t_j| \leq \delta$, for $j = 1, \ldots, n - 1$ the corresponding hat functions are denoted by $H_1, \ldots, H_n$, and their linear span is denoted by $H_n$.

The following result is now obvious:

Proposition 5 The generalized hat functions $H_1, \ldots, H_n$ form a linear basis of the vector space $H_n$ of all piecewise exponential splines for the knots $t_1 < \ldots < t_n$ and the differential operators $L_{(\lambda_{0,j}, \lambda_{1,j})}$ on the interval $(t_j, t_{j+1})$ for $j = 1, \ldots, n - 1$. 

Figure 1: Graph of the fundamental function
The generalized hat functions $H_1, ..., H_n$ in Proposition 5 are piecewise exponential splines of order 2 with minimal compact support, or shorter, piecewise exponential B-splines of order 2. B-splines for special classes of exponential polynomials have been used by many authors, see e.g. [6, p. 197] or [42].

One basic feature of linear hat functions is the partition of unity, saying that the expression

$$U(t) := \sum_{j=1}^{n} H_j(t)$$

is equal to the constant function 1. Note that $U(t)$ is the piecewise exponential spline of order 2 interpolating the constant function 1, and in general this function is not constant. The following Figure 2 shows the three basis functions $H_1$, $H_2$, $H_3$ for the case $\lambda_1 = -\lambda_0$:

![Figure 2: $H_1, H_2, H_3$ for $\lambda_0 = -5$ and $\lambda_1 = 5$.](image)

On Figure 3 we see the sum $U(t)$ of the above basis functions $H_1$, $H_2$, $H_3$.

For positive frequencies $\lambda_0 = 0.2$ and $\lambda_1 = 2$ by Proposition 4 we know that $\Phi(\lambda_0, \lambda_1)$ is increasing on $[-1.2, 1.2]$ since $\frac{\ln 2 - \ln 0.2}{2 - 0.2} = 1.2792$. Figure 4 shows that in this case the sum of the hat functions (the upper curve) is not bounded by 1.

In [4] we have seen that $0 \leq U(t) \leq 2$. We show in the next proposition that this inequality can be improved when the frequencies $\lambda_{0,j}$ and $\lambda_{1,j}$ have different sign for each $j = 1, ..., n - 1$:

**Proposition 6** In addition to (i)–(ii) assume that $\lambda_{0,j} \leq 0 \leq \lambda_{1,j}$ for $j = 1, ..., n - 1$. Then the following inequality holds:

$$0 \leq \sum_{j=1}^{n} H_j(t) \leq 1.$$
Proof. It suffices to show that \( H_j(t) + H_{j+1}(t) \leq 1 \) for \( t \in [t_j, t_{j+1}] \) and for \( j = 1, \ldots, n - 1 \). For \( t \in [t_j, t_{j+1}] \) we have

\[
f(t) := H_j(t) + H_{j+1}(t) = \frac{\varphi_j(t - t_j) + \varphi_j(t - t_{j+1})}{\varphi_j(t_j - t_{j+1})}.
\]

It follows that \( f(t_j) = f(t_{j+1}) = 1 \) and \( f \neq 0 \). Note that \( f \) cannot have a local maximum and a local minimum since otherwise the derivative \( f' \) would have two different zeros which is a contradiction to the fact that \( f' \in E(\lambda_0, \lambda_1) \) has at most one zero, see Proposition 28 in the Appendix. If we show that \( f'(t_j) < 0 \) then \( f \) will have a local minimum, and not a local maximum, so \( f(t) \leq 1 \) for \( t \in [t_j, t_{j+1}] \), and the proof will be complete.

Writing \( \lambda_0 \) and \( \lambda_1 \) instead of \( \lambda_{0,j} \) and \( \lambda_{1,j} \), and \( t_j = a \) and \( t_{j+1} = b \), by our basic assumption (12) we have \( \varphi_j = \Phi(\lambda_{0,j}, \lambda_{1,j}) \), and after putting \( h := t_{j+1} - t_j \), we obtain the equality

\[
f'(a) = \frac{\Phi'(\lambda_0, \lambda_1)(-h)}{\Phi(\lambda_0, \lambda_1)(-h)} + \frac{1}{\Phi(\lambda_0, \lambda_1)(h)} + \frac{1}{\Phi(\lambda_0, \lambda_1)(h)} - e^{(\lambda_0 + \lambda_1)h} \frac{\Phi'(\lambda_0, \lambda_1)(-h)}{\Phi(\lambda_0, \lambda_1)(h)} - e^{(\lambda_0 + \lambda_1)h} \frac{\Phi'(\lambda_0, \lambda_1)(-h)}{\Phi(\lambda_0, \lambda_1)(h)}.
\]

we have used the fact that \( \Phi'(\lambda_0, \lambda_1)(0) = 1 \) and equation (11). We multiply the last equation by \( \Phi(\lambda_0, \lambda_1)(h) > 0 \). Then it suffices to show that for all \( h > 0 \)

\[
g(h) := 1 - e^{(\lambda_0 + \lambda_1)h} \Phi'(\lambda_0, \lambda_1)(-h) < 0.
\]

Consider the case \( \lambda_1 \neq \lambda_0 \), then

\[
g(h) = 1 - e^{(\lambda_0 + \lambda_1)h} \frac{\lambda_1 e^{\lambda_1 h} - \lambda_0 e^{\lambda_0 h}}{\lambda_1 - \lambda_0} = 1 - \frac{\lambda_1 e^{\lambda_1 h} - \lambda_0 e^{\lambda_0 h}}{\lambda_1 - \lambda_0}.
\]

It follows that \( g'(h) = \lambda_0 \lambda_1 \Phi(\lambda_0, \lambda_1)(h) \leq 0 \) (here we use that \( \lambda_0 \lambda_1 \leq 0 \)) and we conclude that \( g \) is decreasing. Since \( g(0) = 0 \) we conclude that \( g(h) \) is negative for \( h > 0 \), which ends the proof in the case \( \lambda_1 \neq \lambda_0 \).
In the case $\lambda_1 = \lambda_0$ we see that $\lambda_1 = \lambda_0 = 0$, which is the well known polynomial case.

In the case of two positive frequencies one may obtain surprising effects if we do not restrict the interval lengths of the partition $t_1 < \ldots < t_n$. We have seen that in this case the fundamental function $\Phi_{(\lambda_0, \lambda_1)}$ may not be increasing and a plot for the defining formula of a hat functions $H_j(t)$ looks like the graph on Figure 5.

However, if we restrict the size $\delta$ of the partition, the function $\Phi_{(\lambda_0, \lambda_1)}$ is increasing on $[-\delta, \delta]$, and this excludes such pathological behaviour of the hat function.
3 Error estimate for interpolation with piecewise exponential splines of order 2 in the uniform norm

For the approximation of \( f \in C^2 [t_1, t_n] \) by linear splines the following estimate is well known:

\[
\| f - I_2 (f) \|_{[t_1, t_n]} \leq \frac{1}{8} \max_{j=1, \ldots, n-1} |t_{j+1} - t_j|^2 \cdot \max_{\theta \in [t_1, t_n]} |f'' (\theta)| ,
\]

(13)

see e.g. [3, p. 31]. The estimate (13) is a simple consequence from two facts: (i) \( g (t) = f (t) - I_2 (f) (t) \) vanishes at \( t_j \) and \( t_{j+1} \), and (ii) the following inequality is valid for all \( g \in C^2 [a, b] \) with \( g (a) = g (b) = 0 \):

\[
|g (t)| \leq \frac{1}{8} |b-a|^2 \cdot \max_{\theta \in [a, b]} |g'' (\theta)| .
\]

(14)

We will find analogues of (14) in the context of \( L \)-splines where we will replace the second derivative in (14) by

\[
\max_{\theta \in [a, b]} \left| L_{(\lambda_0, \lambda_1)} f (\theta) \right| .
\]

We need now the following fact (see Remark 29 in the Appendix): if \( \lambda_0, \lambda_1 \) are real and \( a < b \) then there exists an exponential polynomial \( \Omega_{a,b,\lambda_0,\lambda_1,0} \) in \( E(\lambda_0, \lambda_1, 0) \) such that

\[
\Omega_{a,b,\lambda_0,\lambda_1,0} (a) = \Omega_{a,b,\lambda_0,\lambda_1,0} (b) = 0 \quad \text{and} \quad L_{(\lambda_0, \lambda_1)} \Omega_{a,b,\lambda_0,\lambda_1,0} = -1 .
\]

(15)

We will further exploit the fact that \( L_{(\lambda_0, \lambda_1)} \Omega_{a,b,\lambda_0,\lambda_1,0} \) is a constant function. In the polynomial case (i.e. \( \lambda_0 = \lambda_1 = 0 \)) we have

\[
\Omega_{a,0,0,0} (t) = \frac{1}{2} (t-a) (b-t) .
\]

If \( \lambda_0 \neq \lambda_1 \) are both non-zero one can define \( \Omega_{a,b,\lambda_0,\lambda_1,0} (t) \) by putting

\[
\Omega_{a,b,\lambda_0,\lambda_1,0} = \frac{\Phi_{(\lambda_0, \lambda_1)} (t-a) - e^{(\lambda_0 + \lambda_1)} (b-a) \Phi_{(\lambda_0, \lambda_1)} (t-b)}{\lambda_0 \lambda_1 \Phi_{(\lambda_0, \lambda_1)} (b-a)} - \frac{1}{\lambda_0 \lambda_1} .
\]

(16)

\[\text{Theorem 7} \quad \text{The smallest constant } C \text{ such that for all } f \in C^2 [a, b] \text{ with } f (a) = f (b) = 0 \text{ the following inequality holds}
\]

\[
|f (t)| \leq C \max_{\theta \in [a, b]} \left| L_{(\lambda_0, \lambda_1)} f (\theta) \right|
\]

(17)

for all \( t \in [a, b] \) is given by

\[
C = M_{\lambda_0, \lambda_1} := \max_{t \in [a, b]} \left| \Omega_{a,b,\lambda_0,\lambda_1,0} (t) \right| .
\]

(18)

Moreover \( \Omega_{a,b,\lambda_0,\lambda_1,0} \) is positive on \( (a, b) \).
Proof. Suppose that $C$ is the best constant such that (17) holds for all $f \in C^2[a,b]$ with $f(a) = f(b) = 0$. Inserting $f(t) = \Omega_{\lambda_0,\lambda_1,0}^{a,b}$ gives

$$|\Omega_{\lambda_0,\lambda_1,0}^{a,b}(t)| \leq C$$

since $L_{(\lambda_0,\lambda_1)}f = -1$. It follows that $M_{\lambda_0,\lambda_1}^{a,b} \leq C$. In order to show equality we recall that the Green function $G_{\lambda_0,\lambda_1}^{a,b}(t,\xi)$ of the differential operator $L_{(\lambda_0,\lambda_1)}$ for vanishing boundary conditions (see [2, p. 65]) is defined by

$$G_{\lambda_0,\lambda_1}^{a,b}(t,\xi) = \begin{cases} f(t) \frac{g(\xi)}{w(\xi)} & \text{for } t \in [a,\xi] \\ g(t) \frac{f(\xi)}{w(\xi)} & \text{for } t \in [\xi,b] \end{cases}$$

where $f(t) = \Phi_{(\lambda_0,\lambda_1)}(t - a)$ vanishes in $t = a$ and $g(t) = \Phi_{(\lambda_0,\lambda_1)}(t - b)$ vanishes in $t = b$, and $w(t) = f(t)g'(t) - f'(t)g(t)$ is the Wronski determinant. Note that $f(t) \geq 0$ for all $t \in (a,b)$ and $g(t) \leq 0$ for all $t \in (a,b)$. A calculation shows that

$$w(t) = -e^{(\lambda_0 + \lambda_1)(t-a)}\Phi_{(\lambda_0,\lambda_1)}(a-b) \geq 0 \text{ for all } t \in (a,b).$$

Hence $\xi \mapsto G_{\lambda_0,\lambda_1}^{a,b}(t,\xi) \leq 0$ for all $\xi \in (a,b)$. It is known that for any function $f \in C^2[a,b]$ with $f(a) = f(b) = 0$ the representation

$$f(t) = \int_a^b G_{\lambda_0,\lambda_1}^{a,b}(t,\xi) \cdot L_{(\lambda_0,\lambda_1)}(f)(\xi) \, d\xi$$

holds. Let us take $f(t) = \Omega_{\lambda_0,\lambda_1,0}^{a,b}(t)$ in (19). Since $L_{(\lambda_0,\lambda_1)}\Omega_{\lambda_0,\lambda_1,0}^{a,b} = -1$ we infer that

$$\Omega_{\lambda_0,\lambda_1,0}^{a,b}(t) = -\int_a^b G_{\lambda_0,\lambda_1}^{a,b}(t,\xi) \, d\xi. \quad (20)$$

It follows that $\Omega_{\lambda_0,\lambda_1,0}^{a,b}$ is positive on $(a,b)$ since $\xi \mapsto G_{\lambda_0,\lambda_1}^{a,b}(t,\xi) \leq 0$ for all $\xi \in (a,b)$. From (19) we see that

$$|f(t)| \leq \int_a^b |G_{\lambda_0,\lambda_1}^{a,b}(t,\xi)| \, d\xi \cdot \max_{\theta \in [a,b]} |L_{(\lambda_0,\lambda_1)}f(\theta)|$$

$$= \left|\Omega_{\lambda_0,\lambda_1,0}^{a,b}(t)\right| \cdot \max_{\theta \in [a,b]} |L_{(\lambda_0,\lambda_1)}f(\theta)|.$$ 

This implies that

$$\|f\|_{[a,b]} \leq \max_{t \in [a,b]} \left|\Omega_{\lambda_0,\lambda_1,0}^{a,b}(t)\right| \cdot \max_{\theta \in [a,b]} |L_{(\lambda_0,\lambda_1)}f(\theta)|$$

and therefore $C \leq M_{\lambda_0,\lambda_1}^{a,b}$. This ends the proof. ■

In the case of symmetric frequencies we can determine the best constant:
Theorem 8 For \((\lambda_0, \lambda_1) = (-\xi, \xi)\) the following identity holds

\[ M_{\xi,-\xi}^{a,b} = \max_{t \in [a,b]} \Omega_{\xi,-\xi,0}^{a,b} (t) = (b-a)^2 \cdot M^* (\xi (b-a)) \]  

(21)

where

\[ M^* (x) = \frac{\sinh x - 2 \sinh (x/2)}{x^2 \sinh x}, \]  

(22)

and the following estimate holds:

\[ \Omega_{\xi,-\xi,0}^{a,b} (t) \leq \frac{1}{2} (t-a)(b-t) \leq \frac{1}{8} (b-a)^2. \]  

(23)

Proof. It is easy to see that the following function

\[ f(t) := \frac{1}{\xi^2} \left( 1 - \frac{\sinh \xi (t-a)}{\sinh \xi (b-a)} + \frac{\sinh \xi (t-b)}{\sinh \xi (b-a)} \right), \]  

(24)

is equal to \(\Omega_{\xi,-\xi,0}^{a,b} (t)\) since it is an exponential polynomial in \(E(\xi,-\xi,0)\) which vanishes in \(t=a\) and \(t=b\) and \(L(\xi,-\xi) f(t) = -1\). Further we see that

\[ F(t) := \frac{1}{2} (t-a)(b-t) - f(t) \]

is a concave function since

\[ F''(t) = -1 + \frac{\sinh \xi (t-a)}{\sinh \xi (b-a)} - \frac{\sinh \xi (t-b)}{\sinh \xi (b-a)} = -\xi^2 \Omega_{\xi,-\xi}^{a,b} (t) < 0. \]

Since \(F(a) = F(b) = 0\) it follows that \(F(t) \geq 0\) for all \(t \in [a,b]\), which proves (24). It is easy to see that \(f\) has only one critical point, namely \(t_* = (a+b)/2\) which leads to the maximum of \(f\). Hence

\[ \max_{t \in [a,b]} f(t) = f(t_*) = \frac{1}{\xi^2} \left( 1 - \frac{2 \sinh \xi (b-a)/2}{\sinh \xi (b-a)} \right). \]

Now it is easy to derive formula (22). □

The following picture shows that \(\Omega_{\xi,-\xi,0}^{0,1}(1,-30,0)\), defined on the unit interval \([0,1]\), is not symmetric, while the Green function on the diagonal, \(G_{(1,-30)}^{0,1}\) is symmetric and strictly larger. Both are smaller or equal than \((1-t)^2(1-t)\) (but not \(\frac{1}{2}t(1-t)\), which is different from the polynomial case), as seen from Figure 6.

Next we give an upper bound for \(\Omega_{\xi_0,\lambda_0,\xi_1,0}^{a,b}(t)\):

Proposition 9 Let \(\lambda_0 \leq 0 \leq \lambda_1\). Then for all \(t \in [a,b]\)

\[ \Omega_{\xi_0,\lambda_0,\lambda_1,0}^{a,b} (t) \leq (b-a) \cdot \left| G_{\lambda_0,\lambda_1}^{a,b} (t,t) \right| \]
Proposition 4 shows that the function \(t \mapsto \Phi(-\lambda_0,-\lambda_1)(\xi-a)\) is increasing and positive on \((a,t)\), hence for for \(\xi \in [a,t]\) we obtain the inequality
\[
|G^{a,b}_{\lambda_0,\lambda_1}(t,\xi)| \leq |\Phi(\lambda_0,\lambda_1)(t-b)| \frac{\Phi(-\lambda_0,-\lambda_1)(t-a)}{\Phi(-\lambda_0,-\lambda_1)(b-a)} = |G^{a,b}_{\lambda_0,\lambda_1}(t,t)|
\]
where for proving the last identity we have used the identities \((25)\) and \((26)\) for \(\xi = t\).

When \(\xi \geq t\) then \(t \in [a,\xi]\) and the definition of the Green function shows that
\[
G^{a,b}_{\lambda_0,\lambda_1}(t,\xi) = f(t) \frac{g(\xi)}{w(\xi)} = \frac{\Phi(\lambda_0,\lambda_1)(t-a)}{-e^{(\lambda_0+\lambda_1)(\xi-a)}\Phi(\lambda_0,\lambda_1)(a-b)} \Phi(-\lambda_0,-\lambda_1)(\xi-b)
\]
\[
= \frac{\Phi(\lambda_0,\lambda_1)(t-a)}{-e^{(\lambda_0+\lambda_1)(\xi-b)}e^{(\lambda_0+\lambda_1)(b-a)}\Phi(\lambda_0,\lambda_1)(a-b)} \Phi(-\lambda_0,-\lambda_1)(\xi-b)
\]
\[
= \frac{\Phi(\lambda_0,\lambda_1)(t-a)}{e^{(\lambda_0+\lambda_1)(b-a)}\Phi(-\lambda_0,-\lambda_1)(b-a)} \Phi(-\lambda_0,-\lambda_1)(\xi-b).
\]

Since \(\lambda_0 \leq 0 \leq \lambda_1\), Proposition 4 shows that the function \(x \mapsto -\Phi(-\lambda_0,-\lambda_1)(x-b)\) is decreasing for all \(x\), and for \(\xi \leq t\) we infer
\[
-G^{a,b}_{\lambda_0,\lambda_1}(t,\xi) \leq \frac{\Phi(\lambda_0,\lambda_1)(t-a)}{e^{(\lambda_0+\lambda_1)(b-a)}\Phi(-\lambda_0,-\lambda_1)(b-a)} (-\Phi(-\lambda_0,-\lambda_1)(t-b)) = -G^{a,b}_{\lambda_0,\lambda_1}(t,t).
\]

Figure 6: \(\Omega^{0,1}_{(1,-30,0)}(t)\) is dash, and \(G^{0,1}_{1,-30}(t,t)\) is a solid line.
It follows that for fixed \( t \in (a, b) \) and for all \( \xi \in [a, b] \) the inequality
\[
|G_{\lambda_0, \lambda_1}^{a,b}(t, \xi)| \leq |G_{\lambda_0, \lambda_1}^{a,b}(t, t)|
\]
holds. Then the integral representation \([20]\) leads to the estimate
\[
\Omega_{\lambda_0, \lambda_1, 0}^{a,b}(t) \leq (b - a) \cdot |G_{\lambda_0, \lambda_1}^{a,b}(t, t)|.
\]
The proof is complete. ■

**Theorem 10** Assume that \( \lambda_0 \leq \lambda_1 \) are real numbers. Then
\[
(b - a) \cdot |G_{\lambda_0, \lambda_1}^{a,b}(t, t)| \leq \frac{1}{4} (b - a)^2.
\]

**Proof.** The definition of the Green function in the last proof shows that
\[
G_{\lambda_0, \lambda_1}^{a,b}(t, t) = \frac{\Phi_{(\lambda_0, \lambda_1)}(t - b) \Phi_{(-\lambda_0, -\lambda_1)}(t - a)}{\Phi_{(-\lambda_0, -\lambda_1)}(b - a)}.
\]

It is clear that \( F(t) := \Phi_{(\lambda_0, \lambda_1)}(t - b) \Phi_{(-\lambda_0, -\lambda_1)}(t - a) \) is an exponential polynomial in \( E(0, \lambda_1 - \lambda_0, \lambda_0 - \lambda_1) \). Further, \( F \) vanishes at \( t = a \) and \( t = b \). For every constant \( \lambda \) and function \( f(t) \) we define the differential operator \( D_{\lambda} f := df/dt - \lambda f \). We apply the product rule \( D_{(\lambda + \mu)} (ab) = D_{\lambda} a \cdot b + a D_{\mu} b \) and also formula \([59]\) (in the Appendix) to obtain the equalities
\[
D_{(-\lambda_0 + \lambda_1)}(F) = \Phi_{(-\lambda_1)}(t - a) \Phi_{(\lambda_0, \lambda_1)}(t - b) + \Phi_{(-\lambda_0, -\lambda_1)}(t - a) \Phi_{(\lambda_0)}(t - b)
\]
and
\[
D_{(-\lambda_0 - \lambda_1)} D_{(-\lambda_0 + \lambda_1)}(F) = \Phi_{(-\lambda_1)}(t - a) \Phi_{(\lambda_1)}(t - b) + \Phi_{(-\lambda_0)}(t - a) \Phi_{(\lambda_0)}(t - b)
\]
\[
= e^{-\lambda_1 (t-a)} e^{-\lambda_1 (t-b)} + e^{-\lambda_0 (t-a)} e^{-\lambda_0 (t-b)} = e^{\lambda_1 (a-b)} + e^{\lambda_0 (a-b)}.
\]

Using the uniqueness property of the function \( \Omega \), we obtain the following equality:
\[
G_{\lambda_0, \lambda_1}^{a,b}(t, t) = \frac{e^{\lambda_1 (a-b)} + e^{\lambda_0 (a-b)}}{e^{-\lambda_1 (b-a)} - e^{-\lambda_1 (b-a)}} (\lambda_1 - \lambda_0) \Omega_{(\lambda_1 - \lambda_0, -(\lambda_1 - \lambda_0), 0)}^{a,b}(t).
\]

Further with \( t = a - b \), we obtain
\[
\frac{e^{\lambda_1 t} + e^{\lambda_0 t}}{e^{-\lambda_1 t} - e^{-\lambda_0 t}} = \frac{e^{(-\lambda_1 - \lambda_0) t/2} e^{\lambda_1 t} + e^{\lambda_0 t}}{e^{(-\lambda_1 - \lambda_0) t/2} e^{-\lambda_1 t} - e^{-\lambda_0 t}} = \frac{e^{(\lambda_1 - \lambda_0) t/2} + e^{(-\lambda_1 - \lambda_0) t/2}}{e^{(\lambda_1 - \lambda_0) t/2} - e^{(-\lambda_1 - \lambda_0) t/2}}
\]
\[
= \frac{\cosh ((\lambda_1 - \lambda_0) t/2)}{\sinh ((\lambda_1 - \lambda_0) t/2)}.
\]

Since \( \cosh (x/2) \sinh (x/2) = \frac{1}{2} \sinh x \), it follows that
\[
(b - a) \cdot G_{(\lambda_0, \lambda_1)}^{a,b}(t, t) \leq -T ((\lambda_1 - \lambda_0) (b - a)) \cdot \Omega_{(\lambda_1 - \lambda_0, -(\lambda_1 - \lambda_0), 0)}^{a,b}(t)
\]
\[15\]
where we have put
\[
T(x) = \frac{x \cosh x/2}{\sinh (x/2)} = \frac{1}{2} \frac{x \sinh x}{\sinh^2 (x/2)}.
\]

Further, equation (21) and definition (22) lead to the estimate
\[
(b - a) \cdot \left| C_{G(a,b)}^{a,b} (t, t) \right| \leq (b - a)^2 \cdot T((\lambda_1 - \lambda_0)(b - a)) \cdot M^{*}((\lambda_1 - \lambda_0)(b - a))
\]
and
\[
T(x) M^{*} (x) = \frac{1}{2} \frac{x \sinh x}{\sinh^2 (x/2)} \leq 1.
\]

Finally, we apply the above estimates to piecewise exponential splines in the following result.

**Theorem 11** Let \( \lambda_{0,j} \leq \lambda_{1,j} \) be given for \( j = 1, \ldots, n - 1 \), and define \( \Phi_j = \Phi_{(\lambda_{0,j}, \lambda_{1,j})} \). Then the following estimate
\[
\| f - I_2 (f) \|_{[t_1, t_n]} \leq \max_{j=1,\ldots,n} \max_{j=1,\ldots,n-1} \max_{\theta \in [t_j, t_{j+1}]} | L(\lambda_{0,j}, \lambda_{1,j}) f (\theta) |
\]
holds for \( f \in C^2 [t_1, t_n] \) where \( I_2 (f) \) is piecewise exponential spline of order 2 interpolating \( f \).

**Proof.** For each \( t \in [t_j, t_{j+1}] \) we estimate the function \( f(t) - I_2 (f)(t) \). Since this function vanishes at \( t_j \) and \( t_{j+1} \) we can use (17) with \( C = M^{a,b}_{\lambda_{0}, \lambda_{1}} \) defined in (18).

Let us put \( \Delta := \max_{j=1,\ldots,n-1} | t_{j+1} - t_j | \). We have proved that \( M^{a,b}_{\xi,-\xi} \leq \frac{1}{8} (b - a)^2 \) for any real \( \xi \). For the piecewise exponential spline \( I_2 (f) \) with frequencies \( (\xi_j, -\xi_j) \) interpolating the function \( f \) at \( t_1 < \ldots < t_n \) we obtain then the estimate
\[
\| f - I_2 (f) \|_{[t_1, t_n]} \leq \Delta^2 \max_{j=1,\ldots,n-1} \max_{\theta \in [t_j, t_{j+1}]} | L(\xi_j, -\xi_j) f (\theta) |.
\]
In the case of \( \xi = 0 \) this reduces to the well-known classical estimate (13).

**4 Estimate for the best \( L^2 \)-approximation**

We denote by \( C(X) \) the space of all continuous complex-valued functions on a compact space \( X \). The maximum norm is denoted by \( \| f \|_X = \max_{x \in X} | f (x) | \), and we define the weighted inner product
\[
\langle f, g \rangle_w = \int_X f(x) g(x) w(x) \, dx
\]
(27)
where \( w(x) \) is a positive continuous function. We denote by \( \|f\|_w = \sqrt{\langle f, f \rangle_w} \) the induced norm.

For any finite dimensional subspace \( U_n \) of \( C(X) \) and \( f \in C(X) \) we define by \( P^{U_n}(f) \) the best \( L^2 \)-approximation from \( U_n \), so

\[
\|f - P^{U_n}(f)\|_w = \inf \{ \|f - h\|_w : h \in U_n \}.
\]

Then \( P^{U_n} : C(X) \to C(X) \) defined by \( f \mapsto P^{U_n}(f) \) is linear operator and a projection. The operator norm of \( P^{U_n} \) with respect to the uniform norm is defined as

\[
\|P^{U_n}\|_{\text{op}} = \sup_{f \in C(X)} \frac{\|P^{U_n}(f)\|_X}{\|f\|_X} \in [0, \infty].
\]

The operator norm \( \|P^{U_n}\|_{\text{op}} \) is a useful tool in approximation theory since for all \( f \in C(X) \)

\[
\text{dist}_{C(X)}(f, U_n) \leq \|f - P^{U_n}f\|_X \leq (1 + \|P^{U_n}\|) \text{dist}_{C(X)}(f, U_n) \tag{28}
\]

where we have defined \( \text{dist}_{C(X)}(f, U_n) = \inf_{g \in U_n} \|f - g\|_X \). This inequality is elementary: we can estimate for any \( g \in U_n, f \in C(X) \) and \( x \in X \)

\[
\|f(x) - P^{U_n}(f)(x)\| \leq \|f(x)-g(x)\| + \|P^{U_n}(g-f)(x)\|
\]

\[
\leq \|f-g\|_X + \|P^{U_n}\| \|f-g\|_X.
\]

using the fact that \( g = P^{U_n}g \) for any \( g \in U_n \).

**Definition 12** Assume that \( U_n \) is a subspace generated by linearly independent functions \( B_1, \ldots, B_n \) in \( C(X) \). Then we define a matrix \( S = (s_{i,j})_{i,j=1,\ldots,n} \) by

\[
s_{i,j} = \langle B_i, B_j \rangle_w = \int_X B_i(x) \overline{B_j(x)} w(x) \, dx.
\]

We say that \( S \) is tridiagonal, if \( s_{i,j} \) is zero whenever \( j \neq i, i+1, i-1 \), and we say that \( S \) is row diagonally dominant with dominance factor \( c \in (0, 1) \) if

\[
|s_{j-1,j}| + |s_{j,j+1}| \leq c|s_{j,j}| \text{ for } j = 1, \ldots, n
\]

with the convention that \( s_{0,1} = 0 \) and \( s_{n,n+1} = 0 \). Alternatively, we require \( |s_{1,2}| \leq cs_{11} \) and \( |s_{n-1,n}| \leq cs_{nn} \).

The following result is a generalization of well known techniques used in the error estimates of cubic splines, see e.g. \[3\] p. 34. We include the proof for convenience for the reader but emphasize that it follows closely the lines in \[3\] p. 34.

**Theorem 13** Let \( U_n \) be the subspace generated by linearly independent functions \( B_1, \ldots, B_n \) in \( C(X) \) and assume that the matrix \( S = (s_{i,j})_{i,j=1,\ldots,n} \) is tridiagonal and row diagonally dominant with dominance constant \( c \in (0, 1) \). Then

\[
\|P^{U_n}\|_{\text{op}} \leq \frac{\max_{x \in X} \sum_{j=1}^n |B_j(x)|}{1-c} \frac{\max_{j=1,\ldots,n} \int_X |B_j(y)| w(y) \, dy}{\int_X |B_j(y)|^2 w(y) \, dy} \tag{29}
\]
**Proof.** For \( g \in C(X) \) and the best approximant \( g^* := P_{U_n}(g) \) to \( U_n \) it is well known that \( g^* - g \) is orthogonal to \( U_n \), so

\[
\langle f, g^* - g \rangle_w = 0, \text{ hence } \langle f, g^* \rangle_w = \langle f, g \rangle_w \tag{30}
\]

for all \( f \in U_n \). We take \( f = B_i \) in (30) and write \( g^* = \alpha_1 B_1 + \cdots + \alpha_n B_n \). From (30) it follows that the coefficients \( \alpha_1, \ldots, \alpha_n \) satisfy the equations

\[
\sum_{j=1}^n \alpha_j s_{i,j} = \langle B_i, g \rangle_w := \beta_i
\]

for \( i = 1, \ldots, n \). Since the basis \( B_1, \ldots, B_n \) is tridiagonal we arrive after division by \( s_{ii} > 0 \) at

\[
\alpha_{i-1} s_{i-1,i} + \alpha_i + \alpha_{i+1} s_{i+1,i} = \frac{\beta_i}{s_{ii}}. \tag{31}
\]

Let \( j \) be the index such that \( \alpha_j = \max_{i=1, \ldots, n} |\alpha_i| \). Then equation (31) leads to

\[
|\alpha_j| \leq \frac{\beta_j}{s_{j,j}} + |\alpha_j| \frac{|s_{j-1,j}| + |s_{j,j+1}|}{s_{j,j}} \leq \frac{\beta_j}{s_{j,j}} + |\alpha_j| c.
\]

Thus, \((1-c) |\alpha_j| \leq \frac{1}{1-c} \frac{\beta_j}{s_{j,j}} - \frac{1}{1-c} \frac{\max_{i=1, \ldots, n} \beta_i}{s_{ii}}\)

It follows that

\[
|P_{U_n}(g)(x)| = |g^*(x)| \leq \sum_{j=1}^n |\alpha_j| |B_j(x)| \leq \frac{\sum_{j=1}^n |B_j(x)|}{1-c} \max_{i=1, \ldots, n} \frac{\beta_i}{s_{ii}}.
\]

The statement of the theorem follows now from the estimate

\[
\frac{\beta_j}{s_{j,j}} = \frac{|\langle B_j, g \rangle_w|}{\langle B_j, B_j \rangle_w} = \frac{\int_X |B_j(y)|^2 w(y) \, dy}{\int_X |B_j(y)|^2 w(y) \, dy} \leq \|g\|_X \frac{\int_X |B_j(y)||w(y)| \, dy}{\int_X |B_j(y)|^2 w(y) \, dy}.
\]

Assume now that \( \varphi_j : [-\delta, \delta] \to \mathbb{C} \) are continuous strictly increasing functions with \( \varphi_j(0) = 0 \) for \( j = 1, \ldots, n - 1 \), and define the hat functions \( H_1, \ldots, H_n \) for \( t_1 < \ldots < t_n \) as in Section 2. Let \( U_n \) be the linear span of \( H_1, \ldots, H_n \). We consider the inner product

\[
\langle f, g \rangle_p = \int_{t_1}^{t_n} f(t) g(t) e^{pt} \, dt. \tag{32}
\]
We introduce the following notations:

\[ A_\varphi (h) = \frac{\varphi (h)}{\varphi (-h)} \int_0^h \varphi (\tau - h) \varphi (\tau) e^{pt} dt, \quad B_\varphi (h) = \frac{\varphi (-h)}{\varphi (h)} \int_0^h \varphi (\tau - h) \varphi (\tau) e^{pt} dt \]

\[ C_\varphi (h) = \varphi (h) \int_0^h \varphi (t) e^{pt} dt, \quad D_\varphi (h) = \varphi (-h) \int_0^h \varphi (t) e^{pt} dt \]

**Proposition 14** Assume that \( \varphi_j : [-\delta, \delta] \to \mathbb{C} \) are continuous strictly increasing functions with \( \varphi_j (0) = 0 \) for \( j = 1, ..., n - 1 \), and let \( t_1 < ... < t_n \) with \( t_{j+1} - t_j \leq \delta \) for \( j = 1, ..., n - 1 \). Then the matrix \( S = (s_{ij})_{i,j=1,...,n} \) with \( s_{ij} = (H_i, H_j)_p \) is row diagonally dominant with constant \( c \in (0, 1) \) if

\[
\max_{j=1,...,n-1} \{ A_{\varphi_j} (h_j), B_{\varphi_j} (h_j) \} \leq c
\]

where \( h_j := t_{j+1} - t_j \). Further, the following estimate holds

\[
\| p^\mu_n \|_{op} \leq \max_{t \in [t_1, t_n]} \sum_{j=1}^n |H_j (t)| \frac{\max_{j=1,...,n-1} \{ C_{\varphi_j} (h_j), D_{\varphi_j} (h_j) \}}{1 - c}
\]

**Proof.** We shall apply Theorem [13] Let us define

\[
c_j = \int_{t_{j-1}}^{t_j} \left| \frac{\varphi_j-1 (t - t_j - 1)}{\varphi_j-1 (t_j - t_j - 1)} \right|^2 e^{pt} dt \quad \text{for } j = 2, ..., n,
\]

\[
d_j = \int_{t_{j-1}}^{t_j+1} \left| \frac{\varphi_j (t - t_j + 1)}{\varphi_j (t_j - t_j + 1)} \right|^2 e^{pt} dt \quad \text{for } j = 1, ..., n - 1.
\]

Since \( H_j \) has support in \([t_{j-1}, t_{j+1}]\) for \( j = 2, ..., n - 1 \) we obtain

\[
s_{j,j} = \int_{t_{j-1}}^{t_n} |H_j (t)|^2 e^{pt} dt = c_j + d_j
\]

for \( j = 2, ..., n - 1 \), and for \( j = 1 \) and \( j = n \) we have

\[
s_{1,1} = \int_{t_1}^{t_2} |H_1 (t)|^2 e^{pt} dt = d_1 \text{ and } s_{n,n} = \int_{t_{n-1}}^{t_n} |H_n (t)|^2 e^{pt} dt = c_n.
\]

Since the product function \( H_{j-1} H_j \) has support in \([t_{j-1}, t_j]\) we see that for \( j = 2, ..., n \)

\[
s_{j-1,j} = \int_{t_{j-1}}^{t_j} \frac{\varphi_{j-1} (t - t_j)}{\varphi_{j-1} (t_j - t_j)} \frac{\varphi_{j-1} (t - t_j - 1)}{\varphi_{j-1} (t_j - t_j - 1)} e^{pt} dt.
\]
Similarly, \( H_j H_{j+1} \) has support in \([t_j, t_{j+1}]\), and for \( j = 1, \ldots, n-1 \)

\[
s_{j,j+1} = \int_{t_j}^{t_{j+1}} \frac{\varphi_j(t+1) \varphi_j(t) e^{\mu t}}{\varphi_j(t) - t_{j+1})} dt.
\]

It follows that for \( j = 2, \ldots, n-1 \)

\[
\left| s_{j-1,j} \right| + 2 \left| s_{j,j+1} \right| \leq \max \left\{ \frac{|s_{j-1,j}|}{c_j}, \frac{|s_{j,j+1}|}{d_j} \right\}.  \quad (40)
\]

Hence, in order to apply Theorem 13, it suffices to require that

\[
\max \left\{ \frac{|s_{j,j+1}|}{c_j+1}, \frac{|s_{j,j+1}|}{d_j} \right\} \leq c \text{ for all } j = 1, \ldots, n-1.
\]

The last condition is satisfied since \( (37) \) holds and for \( j = 1, \ldots, n-1 \)

\[
\frac{s_{j,j+1}}{c_j+1} = \frac{\varphi_j(t_{j+1}+1) \varphi_j(t-j) e^{\mu t}}{\varphi_j(t) - t_{j+1})} \int_{t_j}^{t_{j+1}} (\varphi_j(t-j)) e^{\mu t} dt = A_{\varphi_j}(t_{j+1}+1) - t_j).
\]

Similarly, for \( j = 1, \ldots, n-1 \), we have \( \frac{s_{j,j+1}}{d_j} = B_{\varphi_j}(t_{j+1}+1) - t_j) \).

The inequality \( (38) \) is a simple consequence of \( (29) \) and the following claim:

\[
\frac{\int_{t_j}^{t_{j+1}} H_j(t) w(t) dt}{\int_{t_j}^{t_{j+1}} H_j(t)^2 w(t) dt} \leq \max \left\{ C_{\varphi_j}(t_{j+1}+1) - t_j, D_{\varphi_j}(t_{j+1}+1) - t_j) \right\}  \quad (41)
\]

for all \( j = 1, \ldots, n-1 \). Indeed, for \( j = 2, \ldots, n-1 \) we have

\[
\frac{\int_{t_j}^{t_{j+1}} H_j(t) e^{\mu t} dt}{\int_{t_j}^{t_{j+1}} H_j(t)^2 e^{\mu t} dt} = \frac{\sigma_{j-1,j} + \sigma_{j,j+1}}{c_j + d_j} \leq \max \left\{ \frac{\sigma_{j-1,j}}{c_j}, \frac{\sigma_{j,j+1}}{d_j} \right\}
\]

where we define for \( j = 2, \ldots, n-1 \)

\[
\sigma_{j-1,j} = \int_{t_{j-1}}^{t_j} \varphi_{j-1}(t-j) e^{\mu t} dt \quad \text{and} \quad \sigma_{j,j+1} = \int_{t_j}^{t_{j+1}} \varphi_j(t-j) e^{\mu t} dt.
\]

Now \( (41) \) follows from the fact that

\[
\frac{\sigma_{j,j+1}}{c_j + j} = \varphi_j(t_{j+1}+1) \int_{t_j}^{t_{j+1}} \varphi_j(t-j) e^{\mu t} dt \quad \text{and} \quad \frac{\sigma_{j,j+1}}{d_j} = D_{\varphi_j}(t_{j+1}+1) - t_j),
\]

and \( \frac{\sigma_{j,j+1}}{d_j} = D_{\varphi_j}(t_{j+1}+1) - t_j) \) for \( j = 1, \ldots, n-1 \).
5 Estimate for the best $L^2$-approximation by piecewise exponential splines of order 2

Let us recall the basic assumptions (i) and (ii) from Section 3, and add a new feature:

(i) Let $\delta > 0$. We assume that for the real numbers $\lambda_{0,j} \leq \lambda_{1,j}$ the functions

$$\varphi_j := \Phi(\lambda_{0,j}, \lambda_{1,j}), \ j = 1, \ldots, n-1,$$

are increasing on $[-\delta, \delta]$.

(ii) For given $t_1 < \ldots < t_n$ with $|t_{j+1} - t_j| \leq \delta$, for $j = 1, \ldots, n-1$ the corresponding hat functions are denoted by $H_1, \ldots, H_n$, and their linear span is denoted by $\mathcal{H}_n$.

(iii) For a real number $p$ we define on $C[t_1, t_n]$ the inner product

$$\langle f, g \rangle_p := \int_{t_1}^{t_n} f(t) g(t) e^{pt} dt.$$

The motivation for the introduction of a weight $e^{pt}$ in (iii) originates from consideration of polysplines on annuli where the weight depends on the dimension of the euclidean space.

Define the functions $T^{(p)}_{(\lambda_0, \lambda_1)} : \mathbb{R} \to \mathbb{R}$ and $S^{(p)}_{(\lambda_0, \lambda_1)} : \mathbb{R} \to \mathbb{R}$ by putting

$$T^{(p)}_{(\lambda_0, \lambda_1)} = \frac{1}{2} \Phi(\lambda_0, \lambda_1, -\lambda_0 - p - \lambda_1),$$
$$S^{(p)}_{(\lambda_0, \lambda_1)} = \frac{1}{2} \Phi(-\lambda_0, -\lambda_1, \lambda_0, -p - \lambda_0 - \lambda_1).$$

Here we recall that the function $\Phi(\lambda_0, \lambda_1, \ldots, \lambda_n)$ denotes the fundamental function, see formula (45) in the Appendix.

The following Theorem 15 is the main result of the present section which will be proved as a consequence of Proposition 14. After that we will prove a simple explicit estimate of the operator norm $\|P^{H_n}\|_{op}$ for sufficiently small grid sizes $\max |t_{j+1} - t_j|$. In the second part of the section we will give estimates of the functions defined in (42) and (43) for different configurations of the constants $\{\lambda_{0,j}, \lambda_{1,j}\}$ and give an explicit form of the estimate of Theorem 15.

**Theorem 15** In addition to (i)-(iii) assume that

$$c(\delta) := \sup_{|h| \leq \delta} \sup_{j=1, \ldots, n-1} \left| T^{(p)}_{(\lambda_0, \lambda_1)} (h) \right| < 1.$$  (44)

Then for any partition $t_1 < \ldots < t_n$ with $t_{j+1} - t_j \leq \delta$ the matrix $(s_{i,j})_{i,j=1, \ldots, n}$ defined by $s_{i,j} = \langle H_i, H_j \rangle_p$ is row diagonally dominant and

$$\|P^{H_n}\|_{op} \leq \max_{t \in [t_1, t_n]} \sum_{j=1}^{n} \left| H_j (t) \right| \frac{\sup_{|h| \leq \delta} \max_{j=1, \ldots, n-1} \left| S^{(p)}_{(\lambda_0, \lambda_1)} (h) \right|}{1 - c(\delta)}.$$  (45)
Proof. In view of Proposition 14 it suffices to show that $A_\varphi (h) = T^{(p)}_{(\lambda_0, \lambda_1)} (h)$ and $B_\varphi (h) = T^{(p)}_{(\lambda_0, \lambda_1)} (-h)$ for any $h > 0$, and $C_\varphi (h) = S^{(p)}_{(\lambda_0, \lambda_1)} (h)$ and $D_\varphi (h) = S^{(p)}_{(\lambda_0, \lambda_1)} (-h)$ for any $h > 0$ where $A_\varphi$, $B_\varphi$, $C_\varphi$ and $D_\varphi$ are defined in (33) to (36) for the function $\varphi = \Phi_{(\lambda_0, \lambda_1)}$.

In the sequel we often apply the following simple rules:

\[
\begin{align*}
\int_0^h \Phi_{(\lambda_0, \lambda_1)} (t - h) \Phi_{(\lambda_0, \lambda_1)} (t) e^{pt} dt &= - \int_0^h \Phi_{(-\lambda_0, -\lambda_1)} (h - t) \Phi_{(p, \lambda_0, p + \lambda_1)} (t) dt. \\
\Phi_{(\lambda_0, \ldots, \lambda_N)} (t) &= \Phi_{(\lambda_0, \ldots, \lambda_N)} (-t) \quad (46) \\
\Phi_{(\lambda_0, \ldots, \lambda_N)} (t) &= (-1)^N \Phi_{(-\lambda_0, \ldots, -\lambda_N)} (t) \quad (47) \\
\Phi_{(\lambda_0, \ldots, \lambda_N)} (pt) &= p^N \Phi_{(p, \lambda_0, \ldots, p + \lambda_N)} (t) \quad (48)
\end{align*}
\]

Then (47) and (46) shows that

\[
\int_0^h \Phi_{(\lambda_0, \lambda_1)} (t - h) \Phi_{(\lambda_0, \lambda_1)} (t) e^{pt} dt = - \int_0^h \Phi_{(-\lambda_0, -\lambda_1)} (h - t) \Phi_{(p, \lambda_0, p + \lambda_1)} (t) dt.
\]

Theorem 33 in the appendix (applied to the right hand side) shows that

\[
\int_0^h \Phi_{(\lambda_0, \lambda_1)} (t - h) \Phi_{(\lambda_0, \lambda_1)} (t) e^{pt} dt = - \Phi_{(p, \lambda_0, p + \lambda_1, -\lambda_0, -\lambda_1)} (h). \quad (49)
\]

Further we have seen in (11) that

\[
\frac{\Phi_{(\lambda_0, \lambda_1)} (h)}{\Phi_{(\lambda_0, \lambda_1)} (-h)} = \frac{1}{e^{-(\lambda_0 + \lambda_1)h}}. \quad (50)
\]

Now (49), and the identity (61) in the appendix show that

\[
A_\varphi (h) = \frac{\varphi (h) \int_0^h \varphi (\tau - h) \varphi (\tau) e^{\tau t} d\tau}{\varphi (-h) \int_0^h \varphi (\tau)^2 e^{\tau t} d\tau} = \frac{e^{-ph} \Phi_{(p, \lambda_0, p + \lambda_1, -\lambda_0, -\lambda_1)} (h)}{2e^{-(\lambda_0 + \lambda_1)h} \Phi_{(2\lambda_0 + 2\lambda_1, \lambda_0 + \lambda_1, -p)} (h)}. \quad (51)
\]

Apply the rule (46) for the nominator and the denominator, then

\[
A_\varphi (h) = \frac{\Phi_{(\lambda_0, \lambda_1, -\lambda_0, -\lambda_1, p + \lambda_0, -\lambda_0)} (h)}{2\Phi_{(\lambda_0, -\lambda_1, -\lambda_0, -\lambda_0, p + \lambda_0, -\lambda_1)} (h)} = T^{(p)}_{(\lambda_0, \lambda_1)} (h).
\]

Similarly, the definition of $B_\varphi (h)$ in (34) together with (49) and (62) shows that

\[
B_\varphi (h) = \frac{\Phi_{(p, \lambda_0, p + \lambda_1, -\lambda_0, -\lambda_1)} (h)}{2e^{(\lambda_0 + \lambda_1)h} \Phi_{(-2\lambda_0, -2\lambda_1, -\lambda_0 + \lambda_1, p)} (h)} = \frac{\Phi_{(p, \lambda_0, p + \lambda_1, -\lambda_0, -\lambda_1)} (h)}{2\Phi_{(\lambda_1, -\lambda_0, -\lambda_0, -\lambda_0, p + \lambda_0, + \lambda_1)} (h)}.
\]

where we used (46). Using (47) we see that $T^{(p)}_{(\lambda_0, \lambda_1)} (-h) = B (h)$ for $h > 0$.

Note that $\Phi_{(\lambda_0, \lambda_1)} (t) e^{pt} = \Phi_{(p, \lambda_0, p + \lambda_1)} (t)$ by (46), then we see using (46) and (61) in the appendix

\[
C_\varphi (h) = \Phi_{(\lambda_0, \lambda_1)} (h) \frac{\int_0^h \Phi_{(\lambda_0, \lambda_1)} (t) e^{pt} dt}{\int_0^h \Phi_{(\lambda_0, \lambda_1)} (t)^2 e^{pt} dt} = \frac{\Phi_{(\lambda_0, \lambda_1)} (h) \Phi_{(p, \lambda_0, p + \lambda_1, 0)} (h)}{2e^{pt} \Phi_{(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1, -p)} (h)}.
\]
Multiply the nominator and denominator with $e^{-(\lambda_0 + \lambda_1)h}$ and use (46) to the nominator and denominator:

$$C_\varphi(h) = \frac{1}{2} \frac{\Phi(-\lambda_0, -\lambda_1) (h) \Phi(\lambda_0, \lambda_1, -p) (h)}{\Phi(\lambda_0, \lambda_1, 1 - \lambda_0, 0, -p - \lambda_0 - \lambda_1) (h)} = S_{(\lambda_0, \lambda_1)}^{(p)} (h).$$

Further the identities (63) and (62) in the appendix and (47) show that

$$D_\varphi(h) = \Phi(\lambda_0, \lambda_1) (-h) \int_0^h \Phi(\lambda_0, \lambda_1) (t-h) e^{pt} dt \int_0^h \Phi(\lambda_0, \lambda_1) (t-h)^2 e^{pt} dt = \frac{\Phi(\lambda_0, \lambda_1) (-h) \Phi(-\lambda_0, -\lambda_1, p) (h)}{2\Phi(-2\lambda_0, -2\lambda_1, -\lambda_0 - \lambda_1, p) (h)}.$$}

Using similar arguments it follows that the last expression is equal to $S_{(\lambda_0, \lambda_1)}^{(p)} (-h)$.

**Corollary 16** Let $\lambda_{0,j}, \lambda_{1,j}$ be real and define $\varphi_j(t) = \Phi(\lambda_{0,j}, \lambda_{1,j}) (t)$ for $j = 1, \ldots, n - 1$, and $p$ be real number. Then for any $\eta > 0$ there exists $\delta > 0$ such that for any partition $t_1 < \ldots < t_n$ with $t_{j+1} - t_j \leq \delta$ the matrix $(s_{i,j})_{i,j=1,\ldots,n}$ defined by $s_{i,j} = \langle H_i, H_j \rangle_p$ is row diagonally dominant, and

$$\|P^{\mathcal{H}_n}\| \leq 6 + \eta.$$

**Proof.** Using the rule of L'Hôpital we see that for $h \to 0$

$$T_{(\lambda_0, \lambda_1)}^{(p)} (h) = \frac{\Phi(\lambda_0, \lambda_1, -p - \lambda_0 - \lambda_1) (h)}{2\Phi(\lambda_0 - \lambda_1, 1 - \lambda_0, 0, -p - \lambda_0 - \lambda_1) (h)} \to \frac{1}{2}.$$

Using Leibniz's rule and the fact that $\Phi(\lambda_0, \lambda_1) (0) = 0$ and $\Phi(\lambda_0, \lambda_1, -p) (0) = \Phi' (\lambda_0, \lambda_1, -p) (0) = 0$ we have

$$\lim_{t \to 0} \frac{d^3}{dt^3} \left( \Phi(\lambda_0, \lambda_1) (t) \Phi(\lambda_0, \lambda_1, -p) (t) \right) = 3\Phi' (\lambda_0, \lambda_1) (0) \Phi'' (\lambda_0, \lambda_1, -p) (0) = 3.$$}

Since $\Phi_{(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1, -p)}^{(3)} (0) = 1$ we conclude

$$\lim_{t \to 0} S_{(\lambda_0, \lambda_1)}^{(p)} (t) = \frac{3}{2}.$$}

Now we apply Theorem 15. Take $\delta > 0$ so small such that $\varphi_j(t)$ are increasing on $[-\delta, \delta]$. Then $\sum_{j=1}^n |H_j(t)| \leq 2$ by Proposition 3. By choosing $\delta > 0$ small enough we may assume that $|S_{(\lambda_0, \lambda_1)}^{(p)} (h)| \leq \varepsilon + 3/2$ and $c(\varepsilon) \leq \varepsilon + 1/2$ for all $0 \leq |h| \leq \delta$ and $j = 1, \ldots, n-1$ where $c(\varepsilon)$ is defined in (44). Then $1-c(\varepsilon) \geq 1/2 - \varepsilon$ and (45) shows that

$$\|P^{\mathcal{H}_n}\|_{\text{op}} \leq \frac{2}{\frac{1}{2} - \varepsilon} \left( \frac{3}{2} + \frac{1}{2} \varepsilon^2 \right).$$}

The right hand side of the last equation converges to 6 for $\varepsilon \to 0$, and now the claim is immediate. ■

The following lemma is well known and we include the simple proof for completeness. 

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Lemma 17 Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If $D_{\lambda} f (t) > 0$ for $t > a$ and $f (a) \geq 0$ then $f (t)$ is positive for all $t > a$. If $D_{\lambda} f (t) < 0$ for $t < a$ and $f (a) \leq 0$ then $f (t)$ is negative for all $t < a$.

Proof. Define $g (t) = e^{-\lambda t} f (t)$. Then $g' (t) = e^{-\lambda t} D_{\lambda} f (t) > 0$ for $t > a$. Hence $g$ is strictly increasing for $t > a$, and since $g (a) = e^{-\lambda a} f (a) \geq 0$ we infer that $g (t) > 0$. The second statement is proven in a similar way. ■

Theorem 18 Assume that $\lambda_0 \leq \lambda_1$ are real numbers. Then $S_{(\lambda_0, \lambda_1)}^{(0)} (t) \leq 2$ for all real numbers $t$.

Proof. We assume that $\lambda_0 < \lambda_1$, and at first we include the case of a real number $p$. In order to prove that $S_{(\lambda_0, \lambda_1)}^{(p)} (t) \leq C$ for all $t > 0$ it suffices to show

$$F_p (t) := 2 C \Phi_{(\lambda_0 - \lambda_1, 0, -p)} (t) - \Phi_{(-\lambda_0, 0)} (t) \Phi_{(\lambda_0, \lambda_1, -p)} (t)$$

is non-negative for $t > 0$. Using (46) we can rewrite

$$\Phi_{(-\lambda_0, 0)} (t) \Phi_{(\lambda_0, \lambda_1, -p)} (t) = \frac{e^{-\lambda_0 t} - e^{-\lambda_1 t}}{\lambda_1 - \lambda_0} \Phi_{(0, -\lambda_0, -\lambda_1 - p)} (t) = \frac{\Phi_{(0, -\lambda_1 - \lambda_0, -\lambda_1 - p)} (t) - \Phi_{(-\lambda_0, 0)} (t) \Phi_{(\lambda_0, \lambda_1, -p)} (t)}{\lambda_1 - \lambda_0}.$$ 

Using (59) in the appendix it follows that

$$F_p' = 2 C \Phi_{(\lambda_0 - \lambda_1, 0, -p)} - \frac{\Phi_{(\lambda_1 - \lambda_0, -\lambda_1 - p)} - \Phi_{(-\lambda_0, 0, -\lambda_1 - p)}}{\lambda_1 - \lambda_0}.$$ 

Next we consider

$$G_p := (\lambda_1 - \lambda_0) D_{(-p, -\lambda_0 - \lambda_1)} F_p' = 2 C (\lambda_1 - \lambda_0) \Phi_{(\lambda_0 - \lambda_1, 0, -\lambda_0)} - R$$

where we define

$$R = D_{(-p, -\lambda_0 - \lambda_1)} \Phi_{(\lambda_1 - \lambda_0, -\lambda_0 - p)} - D_{(-p, -\lambda_0 - \lambda_1)} \Phi_{(\lambda_0 - \lambda_1, -\lambda_1 - p)}.$$ 

Then

$$R = D_{(-p, -\lambda_0)} \Phi_{(\lambda_1 - \lambda_0, -\lambda_0 - p)} + \lambda_1 \Phi_{(\lambda_1 - \lambda_0, -\lambda_0 - p)}$$

$$- D_{(-p, -\lambda_1)} \Phi_{(\lambda_0 - \lambda_1, -\lambda_1 - p)} - \lambda_0 \Phi_{(\lambda_0 - \lambda_1, -\lambda_1 - p)}$$

$$= \Phi_{(\lambda_1 - \lambda_0)} - \Phi_{(\lambda_0 - \lambda_1)} + \lambda_1 \Phi_{(\lambda_1 - \lambda_0, -\lambda_0)} - \lambda_0 \Phi_{(\lambda_0 - \lambda_1, -\lambda_1)}.$$ 

Since $\Phi_{(\lambda_0 - \lambda_1, -\lambda_0)} (t) = \frac{e^{(\lambda_1 - \lambda_0) t} - e^{(\lambda_0 - \lambda_1) t}}{2 (\lambda_1 - \lambda_0)}$ we obtain the following formula:

$$G_p (t) = (C - 1) e^{(\lambda_1 - \lambda_0) t} - (C - 1) e^{(\lambda_0 - \lambda_1) t}$$

$$- \lambda_1 \Phi_{(\lambda_1 - \lambda_0, -\lambda_0 - p)} (t) + \lambda_0 \Phi_{(\lambda_0 - \lambda_1, -\lambda_1 - p)} (t).$$

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We specialize to the case $p = 0$, and obtain the equality:

$$G_0 (t) = (C - 1) e^{(\lambda_1 - \lambda_0) t} - (C - 1) e^{-(\lambda_1 - \lambda_0) t}$$

$$- \lambda_1 \frac{e^{(\lambda_1 - \lambda_0) t} - e^{-\lambda_0 t}}{\lambda_1} + \lambda_0 \frac{e^{(\lambda_0 - \lambda_1) t} - e^{-\lambda_1 t}}{\lambda_0}$$

$$= (C - 2) e^{(\lambda_1 - \lambda_0) t} - (C - 2) e^{-(\lambda_1 - \lambda_0) t} + e^{-\lambda_0 t} - e^{-\lambda_1 t}.$$

$$= (C - 2) (\lambda_1 - \lambda_0) \Phi_{(\lambda_1 - \lambda_0, \lambda_1 - \lambda_0)} + (\lambda_1 - \lambda_0) \Phi_{(-\lambda_0, -\lambda_1)} (t).$$

If we take $C = 2$ in this equation we see that $G_0 (t) = (\lambda_1 - \lambda_0) \Phi_{(-\lambda_0, -\lambda_1)} (t) > 0$ for all $t > 0$. Thus $G_0 (t)$ is positive for $t > 0$ and $G_0 (0) = 0$. Lemma 17 shows that $F_0 (t) \geq 0$ for all $t > 0$, hence $S^{(0)}_{(\lambda_0, \lambda_1)} (t) \leq 2$ for all $t > 0$.

For $t < 0$ we see that $G_0 (t) < 0$ and Lemma 17 shows that $F_0 (t) \leq 0$ for all $t < 0$. Further $F_0' (0) = 0$, so $F_0 (t) \leq 0$ for all $t < 0$. Since both $\Phi_{(\lambda_0 - \lambda_1, \lambda_1 - \lambda_0, 0, -p - \lambda_0 - \lambda_1)} (t)$ and $\Phi_{(-\lambda_0, -\lambda_1)} (t)$ are negative for $t < 0$ and $\Phi_{(\lambda_0, \lambda_1, -p)} (t) > 0$ for $t < 0$, it follows that $S^{(0)}_{(\lambda_0, \lambda_1)} (t) \leq 2$ for $t < 0$.

The case $\lambda_0 = \lambda_1$ follows from a continuity argument in the variables $\lambda_0, \lambda_1$.

**Theorem 19** Let $\xi_j$ be real numbers and define $\varphi_j (t) = \Phi_{(\xi_j, -\xi_j)} (t)$ for $j = 1, \ldots, n - 1$. Then the matrix $(s_{i,j})_{i,j=1,\ldots,n}$ defined by $s_{i,j} = \langle H_i, H_j \rangle_0$ is row diagonally dominant, and

$$\|P^{H_n}\|_{op} \leq 4.$$

**Proof.** By Proposition 6, $\sum_{j=1}^{n} |H_j (t)| \leq 1$, and Theorem 18 shows that $S^{(0)}_{(\lambda_0, \lambda_1)} (t) \leq 2$ for all real $t$. Hence according to Theorem 15

$$\|P^{H_n}\|_{op} \leq \frac{2}{1 - e}.$$

It suffices to show that for any real $\xi$ and real $t$

$$T^{(0)}_{(\xi, -\xi)} (t) = \frac{1}{2} \Phi_{(\xi, -\xi, -\xi, -\xi)} (t) \leq \frac{1}{2}.$$

It is easily verified that

$$\Phi_{(1, -1, 1, -1)} (t) = \frac{1}{2} (t \cosh t - \sinh t) \quad \text{and} \quad \Phi_{(2, -2, 0, 0)} (t) = \frac{1}{8} (\sinh (2t) - 2t).$$

Since $\Phi_{A \lambda N} (t) = \frac{1}{\pi} \Phi_{A \lambda N} (\xi t)$ it follows that

$$0 \leq \frac{\Phi_{(\xi, -\xi, -\xi, -\xi)} (t)}{\Phi_{(2\xi, -2\xi, 0, 0)}} (t) \leq \frac{\Phi_{(1, -1, 1, -1)} (\xi t)}{\Phi_{(2, -2, 0, 0)}} (\xi t) \quad \text{and} \quad \frac{4 (t \cosh t - \sinh t)}{\sinh (2t) - 2t} \leq 1.$$

The proof is complete. 

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Remark 20 In the polynomial case (i.e. $\xi = 0$) the estimate of the operator norm can be improved to $\| P_{H_n} \|_{op} \leq 3$ since in this case

$$S_{(0,0)}^{(0)}(h) = \frac{1}{2} \frac{\Phi_{(0,0)}(h) \Phi_{(0,0,0)}(h)}{\Phi_{(0,0,0,0)}(h)} = \frac{1}{2} \frac{h}{h^3} = \frac{3}{2}.$$ 

For tension splines one has the following result:

Theorem 21 For $(\lambda_0, \lambda_1) = (0, \rho)$ and $p = 0$ the matrix $(s_{i,j})_{i,j=1,\ldots,n}$ defined by $s_{i,j} = \langle H_i, H_j \rangle_0$ is row diagonally dominant, and

$$\| P_{H_n} \|_{op} < \infty.$$ 

Proof. The matrix $(s_{i,j})_{i,j=1,\ldots,n}$ is row diagonally dominant if

$$T_{(0,\rho)}^{(0)}(t) = \frac{1}{2} \frac{\Phi_{(0,\rho,0,-\rho)}(t)}{\Phi_{(-\rho,0,\rho)}(t)} = \frac{1}{2} \frac{\Phi_{(0,1,0,-1)}(pt)}{\Phi_{(-1,1,0,1)}(pt)}$$

is smaller than 1. Since $\Phi_{(0,1,0,-1)}(t) = \sinh t - t$ and

$$\Phi_{(-1,1,0,1)}(t) = \frac{1}{2} (e^t (t - 2) + \sinh t + 2)$$

one obtains that

$$\frac{\Phi_{(0,1,0,-1)}(t)}{\Phi_{(-1,1,0,1)}(t)} = \frac{2 (\sinh t - t)}{e^t (t - 2) + \sinh t + 2} = \frac{2 \left(\frac{1}{2} e^t - \frac{1}{2} - te^t\right)}{e^{2t} (t - 2) + \frac{1}{2} e^{2t} - \frac{1}{2} + 2e^t} \to 2$$

for $t \to -\infty$. ■

It is a natural question whether one has similar results for the general case $\lambda_0 < \lambda_1$. For positive frequencies $(\lambda_0, \lambda_1) = (2, 1)$ one obtains

$$T_{(2,1)}^{(0)}(t) = \frac{1}{2} \frac{\Phi_{(2,1,-2,-1)}(t)}{\Phi_{(-1,1,0,3)}(t)} = \frac{4 (\sinh t - \frac{1}{2} \sinh 2t)}{-e^{3t} + 3 \sinh t + 9 \cosh t - 9 + 1}$$

and the graph of $T_{(2,1)}^{(0)}(t)$ is provided by Figure 7.

Hence the constant $c(\delta)$ in unbounded in the variable $\delta$. In the case $(\lambda_0, \lambda_1) = (-2, 1)$ one obtains that

$$T_{(-2,1)}^{(0)}(t) = \frac{4}{3} \frac{\sinh t - \frac{1}{2} \sinh 2t}{-e^{-t} - \frac{1}{3} \sinh 3t + \frac{1}{5} \cosh 3t + \frac{8}{9}}$$

In this case the function $t \to T_{(-2,1)}^{(0)}(t)$ is not symmetric and it does not attain its maximum at $t = 0$ as seen by its graph provided on Figure 8.

The following theorem gives an upper bound for $T_{(\lambda_0, \lambda_1)}^{(0)}(t)$:
Figure 7: Graph of \( T^{(0)}_{(2,1)} \)

**Theorem 22** For \( \lambda_0 < 0 < \lambda_1 \) the following inequality holds for all real \( t \)

\[
T^{(0)}_{(\lambda_0, \lambda_1)}(t) = \frac{1}{2} \frac{\Phi_{(\lambda_0, \lambda_1, -\lambda_0, -\lambda_1)}(t)}{\Phi_{(\lambda_0, -\lambda_1, \lambda_1 - \lambda_0, 0, -\lambda_0 - \lambda_1)}(t)} \leq \max \left\{ \frac{2\lambda_1 - \lambda_0}{2\lambda_1 - 4\lambda_0}, \frac{\lambda_1 - 2\lambda_0}{4\lambda_1 - 2\lambda_0} \right\} < 1.
\]

\( (51) \)

**Proof.** According to Lemma 34 in the appendix, for any \( a < 0 \) the function

\[
F_{a,-a-1}(t) = 2C_{a,-a-1}\Phi_{(a-1, 1-a, 0, -a-1)}(t) - \Phi_{(a, 1-a, -1)}(t)
\]

is positive for \( t > 0 \) where

\[
C_{a,-a-1} = \max \left\{ \frac{1}{2}, \frac{3}{2(2-a)} : \frac{3 + 4a}{2(1 - 2a)(2 - a)} : \frac{a + 2}{2(1 - 2a)} \right\}.
\]

It is easy to see that

\[
C_{a,-a-1} \leq \max \left\{ \frac{1}{2}, \frac{2 - a}{2 - 4a} \right\}.
\]

Let us put \( a = \lambda_0/\lambda_1 < 0 \) and \( b = -a - 1 \). Then

\[
F_{a,-a-1}(\lambda_1 t) = 2C_{a,-a-1}\Phi_{a-1,1-a,0,-a-1}(\lambda_1 t) - \Phi_{a,1-a,-1}(\lambda_1 t)
\]

is positive and therefore

\[
\frac{\Phi_{(\lambda_0, \lambda_1, -\lambda_0, -\lambda_1)}(t)}{\Phi_{(\lambda_0, -\lambda_1, \lambda_1 - \lambda_0, 0, -\lambda_0 - \lambda_1)}(t)} = \frac{\Phi_{(a,-a,1,1-a,-1)}(\lambda_1 t)}{\Phi_{a-1,1-a,0,-a-1}(\lambda_1 t)} \leq 2C_{a,-a-1}
\]

Clearly this implies that \( T^{(0)}_{(\lambda_0, \lambda_1)}(t) \leq C_{a,-a-1} \) for all \( t > 0 \).
Now put $a = \lambda_1/\lambda_0 < 0$ and $b = -1 - a$. Then $F_{a,-a-1}(t)$ is positive for $t > 0$. Since $-\lambda_0 t > 0$ we infer that

$$F_{a,-a-1}(-\lambda_0 t) = 2C_{a,-a-1}\Phi_{(a-1,1-a,0,-a-1)}(-\lambda_0 t) - \Phi_{(a,1,-a,-1)}(-\lambda_0 t) \quad (52)$$

is positive for all $t > 0$. Hence

$$\Phi_{(\lambda_0,\lambda_1,-\lambda_0,-\lambda_1)}(-t) = \Phi_{(a,1,-a,-1)}(\lambda_0 (-t)) \leq 2C_{a,-a-1}$$

It follows that $T_{(-\lambda_0,\lambda_1)}^{(0)}(t) \leq C$ for all $t \in \mathbb{R}$ where

$$C \leq \max \left\{ C_{a,-a-1} : a = \frac{\lambda_0}{\lambda_1} \text{ or } a = \frac{\lambda_1}{\lambda_0} \right\}$$

Hence

$$C \leq \max \left\{ \frac{1}{2}, \frac{2 - \frac{\lambda_0}{\lambda_1}}{2 - 4\frac{\lambda_0}{\lambda_1}}, \frac{2 - \frac{\lambda_1}{\lambda_0}}{2 - 4\frac{\lambda_1}{\lambda_0}} \right\} = \max \left\{ \frac{2\lambda_1 - \lambda_0}{2\lambda_1 - 4\lambda_0}, \frac{\lambda_1 - 2\lambda_0}{4\lambda_1 - 2\lambda_0} \right\}.$$ 28

Now we specify the general result of Theorem 15.

**Theorem 23** Let $\lambda_{ij} < 0 < \lambda_{1,j}$ be real numbers and define $\varphi_j = \Phi_{(\lambda_{ij},\lambda_{1,j})}$ for $j = 1, \ldots, n-1$. Then the matrix $(s_{i,j})_{i,j=1,\ldots,n}$ defined by $s_{i,j} = \langle H_i, H_j \rangle_0$ is row diagonally dominant, and

$$\|P^H_n\|_{op} \leq 2 \max_{j=1,\ldots,n-1} \max \left\{ \frac{2\lambda_{1,j} - 4\lambda_{0,j}}{-3\lambda_{0,j}}, \frac{4\lambda_{1,j} - 2\lambda_{0,j}}{3\lambda_{1,j}} \right\}.$$
Proof. By Proposition 6 \[ \sum_{j=1}^{n} |H_j(t)| \leq 1, \] and Theorem 18 shows that \( S_{(\lambda_0, \lambda_1)}^{(0)}(t) \leq 2 \) for all real \( t \). Hence according to Theorem 15 \[ \|P^{H_n}\|_{op} \leq \frac{2}{1-c}. \]

Theorem 22 shows that
\[
|c| \leq \max_{j=1, \ldots, n-1} \max \left\{ \frac{2\lambda_{1,j} - \lambda_{0,j}}{2\lambda_{1,j} - 4\lambda_{0,j}}, \frac{\lambda_{1,j} - 2\lambda_{0,j}}{4\lambda_{1,j} - 2\lambda_{0,j}} \right\} < 1.
\]

It follows that for some \( j = 1, \ldots, n \), at least one of the following two inequalities holds:
\[
1 - c \geq 1 - \frac{2\lambda_{1,j} - \lambda_{0,j}}{2\lambda_{1,j} - 4\lambda_{0,j}} = \frac{-3\lambda_{0,j}}{2\lambda_{1,j} - 4\lambda_{0,j}}
\]
\[
1 - c \geq 1 - \frac{\lambda_{1,j} - 2\lambda_{0,j}}{4\lambda_{1,j} - 2\lambda_{0,j}} = \frac{3\lambda_{1,j}}{4\lambda_{1,j} - 2\lambda_{0,j}}
\]

It follows that
\[
\frac{1}{1-c} \leq \max_{j=1, \ldots, n-1} \max \left\{ \frac{2\lambda_{1,j} - 4\lambda_{0,j}}{-3\lambda_{0,j}}, \frac{4\lambda_{1,j} - 2\lambda_{0,j}}{3\lambda_{1,j}} \right\}.
\]

\[ \blacksquare \]

6 Error estimates for interpolation with exponential splines of order 4

Assume that \( f, g : (t_{j-1}, t_j) \to \mathbb{R} \) are differentiable and that \( f(t), g(t) \) have limits when \( t \to t_j, t_{j+1} \) for \( t \in (t_j, t_{j+1}) \), and let \( D_{\lambda} = f' - \lambda f \). Partial integration shows that
\[
\int_{t_j}^{t_{j+1}} D_{\lambda_{2,j}} f(t) \cdot g(t) \, dt = fg|_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}} f(t) \cdot D_{-\lambda_{2,j}} g(t) \, dt \quad (53)
\]
where \( f|_{t_j}^{t_{j+1}} \) is defined as the difference \( f(t_{j+1}) - f(t_j) \). Note that
\[
D_{-\lambda_{2,j}} (h(t)e^{pt}) = e^{pt} (h'(t) + ph(t)) + \lambda_{2,j} h(t)) = e^{pt} D_{-\lambda_{2,j}} h(t). \quad (54)
\]

Take now \( g = he^{pt} \) in formula (53) and use (54):
\[
\int_{t_j}^{t_{j+1}} D_{\lambda_{2,j}} f \cdot he^{pt} \, dt = he^{pt}|_{t_j}^{t_{j+1}} - \int_{t_j}^{t_{j+1}} f D_{-\lambda_{2,j}} h \cdot e^{pt} \, dt. \quad (55)
\]

Now replace \( f \) by \( D_{\lambda_{3,j}} f \), and we see that
\[
\int_{t_j}^{t_{j+1}} D_{\lambda_{2,j}} D_{\lambda_{3,j}} f \cdot he^{pt} \, dt = D_{\lambda_{3,j}} f \cdot he^{pt}|_{t_j}^{t_{j+1}}
\]
\[
- \int_{t_j}^{t_{j+1}} D_{\lambda_{3,j}} f \cdot D_{-\lambda_{2,j}} h \cdot e^{pt} \, dt.
\]
We apply (55) to the last summand with $\lambda_{3,j}$ instead of $\lambda_{2,j}$ and replace $h$ by $D_{-p-\lambda_{2,j}}h$, so we have the general identity
\[
\int_{t_j}^{t_{j+1}} D_{\lambda_{2,j}} D_{\lambda_{3,j}} f \cdot he^{pt} dt = D_{\lambda_{3,j}} f \cdot he^{pt} \bigg|_{t_j}^{t_{j+1}} \\
- f D_{-p-\lambda_{2,j}} h \cdot e^{pt} \bigg|_{t_j}^{t_{j+1}} + \int_{t_j}^{t_{j+1}} f D_{-p-\lambda_{3,j}} D_{-p-\lambda_{2,j}} h \cdot e^{pt} dt.
\]

**Proposition 24** Assume that for the real numbers $\lambda_{0,j}, \lambda_{1,j}, \lambda_{2,j}, \lambda_{3,j}$ for $j = 1, ..., n - 1$ there exists a real number $p$ such that for all $j = 1, ..., n - 1$
\[
\lambda_{0,j} = -p - \lambda_{2,j} \quad \text{and} \quad \lambda_{1,j} = -p - \lambda_{3,j}.
\]
(56)
Assume further that $F \in C^2 [t_1, t_n]$ vanishes on $t_1, ..., t_n$ and $F'(t_1) = 0$ and $F'(t_n) = 0$. Define $f \in C [t_1, t_n]$ by setting $f(t) = D_{\lambda_{2,j}} D_{\lambda_{3,j}} F(t)$ for $t \in [t_j, t_{j+1}]$ and $j = 1, ..., n - 1$. Then
\[
(f, h)_p = 0
\]
for any piecewise exponential spline $h$ with respect to $L(\lambda_{0,j}, \lambda_{1,j})$ for $j = 1, ..., n - 1$.

**Proof.** Since we assume that $F(t_j) = 0$ for $j = 1, ..., n$ we obtain
\[
(f, h)_p = \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} D_{\lambda_{2,j}} D_{\lambda_{3,j}} F \cdot he^{pt} dt = \sum_{j=1}^{n-1} D_{\lambda_{3,j}} f \cdot he^{pt} \bigg|_{t_j}^{t_{j+1}}
\]
since the expressions $F D_{-p-\lambda_{2,j}} h \cdot e^{pt} \bigg|_{t_j}^{t_{j+1}}$ clearly vanish and
\[
\int_{t_j}^{t_{j+1}} F(t) D_{-p-\lambda_{3,j}} D_{-p-\lambda_{2,j}} h(t) \cdot e^{pt} dt = 0
\]
since $h(t)$ is an exponential spline $h$ with respect to $L(\lambda_{0,j}, \lambda_{1,j})$ for $j = 1, ..., n - 1$, and condition (55) is satisfied. Since $D_{\lambda_{3,j}} F(t)$ and $h(t) e^{pt}$ is continuous we infer that the following sum is telescoping:
\[
\sum_{j=1}^{n-1} \left( D_{\lambda_{3,j}} f \cdot he^{pt} \bigg|_{t_j}^{t_{j+1}} \right) = D_{\lambda_{3,j}} F(t_n) h(t_n) e^{pt_n} - D_{\lambda_{3,j}} F(t_1) h(t_1) e^{pt_1} = 0
\]
since $F(t_n) = F'(t_n) = 0$ and $F(t_1) = F'(t_1) = 0$. □

**Theorem 25** Assume that there exists a real number $p$ such that $-p - \lambda_{2,j} = \lambda_{0,j}$ and $-p - \lambda_{3,j} = \lambda_{1,j}$ for $j = 1, ..., n$. Let $t_1 < ... < t_n$ and $F \in C^4 [t_1, t_n]$. Assume that $I_4(F)$ is a piecewise exponential spline with respect to $L(\lambda_{0,j}, \lambda_{1,j}, \lambda_{2,j}, \lambda_{3,j})$ for $j = 1, ..., n - 1$, interpolating $F$ at the points $t_1, ..., t_n$ and such that
\[
\frac{d}{dt} F(t_1) = \frac{d}{dt} I_4(F)(t_1) \quad \text{and} \quad \frac{d}{dt} F(t_n) = \frac{d}{dt} I_4(F)(t_n).
\]

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Then for any \( t \in [t_j, t_{j+1}] \) the estimate
\[
|F(t) - I_4(F)(t)| \leq C \cdot \max_{\zeta \in [t_j, t_{j+1}], j=1, \ldots, n-1} |L(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) F(\zeta)|
\]
holds, where
\[
C = \left( 1 + \| P^{H_n} \|_{op} \right) \max_{j=1, \ldots, n-1} M^{t_j, t_{j+1}}_{\lambda_2, \lambda_3, \lambda_4} \max_{j=1, \ldots, n-1} M^{t_j, t_{j+1}}_{\lambda_0, \lambda_1, \lambda_3}
\]
and \( \| P^{H_n} \|_{op} \) is given in \[ (5) \] with respect to the weight function \( e^{pt} \).

**Proof.** The function \( t \mapsto I_4(F)(t) \) for \( t \in [t_j, t_{j+1}] \) is an exponential polynomial in \( E(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \). Further \( g(t) := F(t) - I_4(F)(t) \) for \( t \in [t_j, t_{j+1}] \) vanishes in \( t_j \) and \( t_{j+1} \). Using inequality \[ (17) \] for the function \( g(t) \) and for the constants \( \lambda_2, \lambda_3, \lambda_4 \), we see that
\[
|F(t) - I_4(F)(t)| \leq M^{t_j, t_{j+1}}_{\lambda_2, \lambda_3, \lambda_4} \max_{\zeta \in [t_j, t_{j+1}]} |D_{\lambda_2, \lambda_3} \left[ F - I_4(F) \right](\zeta)|
\]
for \( t \in [t_j, t_{j+1}] \). We define \( f \in C[t_1, t_n] \) and \( f_0 \in C[t_1, t_n] \) by putting
\[
f(t) = D_{\lambda_2, \lambda_3} F(t) \quad \text{for } t \in [t_j, t_{j+1}]
\]
\[
f_0(t) = D_{\lambda_2, \lambda_3} (I_4(F))(t) \quad \text{for } t \in [t_j, t_{j+1}].
\]
From Proposition \[ (24) \] we infer that \( \langle f - f_0, h \rangle_p = 0 \) for any exponential spline \( h \) with respect to \( L(\lambda_0, \lambda_1, \lambda_2) \) for \( j = 1, \ldots, n - 1 \). Hence \( f_0 \) is the best \( L^2 \)-approximation to the function \( f \) from the subspace \( H_n \) with weight function \( e^{pt} \). Hence by inequality \[ (28) \] we obtain
\[
\max_{\zeta \in [t_j, t_{j+1}]} |f(\zeta) - f_0(\zeta)| \leq \left( 1 + \| P^{H_n} \|_{op} \right) \| f - I_2(f) \|_{[t_1, t_n]}.
\]
Further
\[
\| f - I_2(f) \|_{[t_1, t_n]} \leq \max_{j=1, \ldots, n} M^{t_j, t_{j+1}}_{\lambda_0, \lambda_1, \lambda_2} \max_{\zeta \in [t_j, t_{j+1}], j=1, \ldots, n-1} |L(\lambda_0, \lambda_1, \lambda_2) f(\zeta)|
\]
This ends the proof. \( \square \)

**Remark 26** Note that the last proof also provides an estimate for the derivatives of second order:
\[
\max_{t \in [t_j, t_{j+1}]} |D_{\lambda_2, \lambda_3} F(t)| \leq \left( 1 + \| P^{H_n} \|_{op} \right) \max_{x \in [t_j, t_{j+1}], j=1, \ldots, n-1} |L(\lambda_0, \lambda_1, \lambda_2, \lambda_3) F(x)|.
\]
Now we are able to derive our main result:
Theorem 27 Let $\xi_j$ for $j = 1, \ldots, n - 1$ real numbers and $t_1 < \ldots < t_n$. Let $F \in C^4 [t_1, t_n]$ and assume that $I_4 (F)$ is a piecewise exponential spline for the operators $L_{(\xi_j, -\xi_j, \xi_j, -\xi_j)}$, which interpolates $F$ at the points $t_1, \ldots, t_n$ and

$$\frac{d}{dt} F (t_1) = \frac{d}{dt} I_4 (F) (t_1) \quad \text{and} \quad \frac{d}{dt} F (t_n) = \frac{d}{dt} I_4 (F) (t_n).$$

Then for any $t \in [t_1, t_n]$ we have the estimate

$$|F (t) - I_4 (F) (t)| \leq \frac{\Delta^4}{16} \frac{5}{4} \max_{\xi \in [t_j, t_{j+1}], j = 1, \ldots, n-1} |L_{(\xi_j, -\xi_j, -\xi_j, \xi_j)} F (\xi)|.$$

Proof. Theorem 8 shows that $M_{\xi_j, -\xi_j}^j t_j \leq \frac{1}{8} |t_{j+1} - t_j|^2$, and Theorem 19 shows $\|P^{R_n}\| \leq 4$. Hence

$$C = (1 + \|P^{R_n}\|) \max_{j=1, \ldots, n-1} M_{\xi_j, -\xi_j}^j t_j \max_{j=1, \ldots, n} M_{\xi_j, -\xi_j}^j t_j \leq \frac{5}{64} |t_{j+1} - t_j|^4.$$

\[ \blacksquare \]

7 Appendix: Exponential polynomials

For given complex numbers $\lambda_0, \ldots, \lambda_N$ the elements of the space

$$E (\lambda_0, \ldots, \lambda_N) = \{ f \in C^{N+1} (\mathbb{R}) : L_{(\lambda_0, \ldots, \lambda_N)} f = 0 \}$$

are called exponential polynomials or $L$-polynomials, and $\lambda_0, \ldots, \lambda_N$ are also called exponents or frequencies. In the case of pairwise different $\lambda_j, j = 0, \ldots, N$, the space $E (\lambda_0, \ldots, \lambda_N)$ is the linear span of the functions

$$e^{\lambda_0 z}, e^{\lambda_1 z}, \ldots, e^{\lambda_N z}.$$

When $\lambda_j$ occurs $m_j$ times in $\Lambda_N = (\lambda_0, \ldots, \lambda_N)$, a basis of the space $E (\lambda_0, \ldots, \lambda_N)$ is given by the linearly independent functions $z^s e^{\lambda_j z}$ for $s = 0, 1, \ldots, m_j - 1$.

There exists a unique function $\Phi^{(N)} (\lambda_0, \ldots, \lambda_N)$ such that

$$\Phi^{(N)} (\lambda_0, \ldots, \lambda_N) (0) = \ldots = \Phi^{(N-1)} (\lambda_0, \ldots, \lambda_N) (0) = 0 \quad \text{and} \quad \Phi^{(N)} (\lambda_0, \ldots, \lambda_N) (0) = 1. \quad (57)$$

The function $\Phi^{(N-1)} (\lambda_0, \ldots, \lambda_N) (0)$ is called the fundamental function. An explicit definition is the formula

$$\Phi (\lambda_0, \ldots, \lambda_N) (t) = \frac{1}{2 \pi i} \int_{\Gamma_t} \frac{e^{tz}}{(z - \lambda_0) \cdots (z - \lambda_N)} dz, \quad (58)$$

where $\Gamma_t$ is the path in the complex plane defined by $\Gamma_t (t) = re^{it}, t \in [0, 2\pi]$, surrounding all the complex numbers $\lambda_0, \ldots, \lambda_N$, see [23]. The integral representation (58) implies the formula

$$\left( \frac{d}{dt} - \lambda_{N+1} \right) \Phi (\lambda_0, \ldots, \lambda_{N+1}) (t) = \Phi (\lambda_0, \ldots, \lambda_N) (t). \quad (59)$$
Lemma 31 For real \( t > 0 \) is an exponential polynomial in \( \Phi \).

The following result is a well-known, see e.g. [32], [11].

Proposition 28 If \( \lambda_0, ..., \lambda_N \) are real then the space \( E(\lambda_0, ..., \lambda_N) \) is an extended Chebyshev space on \( \mathbb{R} \), i.e. each non-zero function \( f \in E(\lambda_0, ..., \lambda_N) \) has at most \( N \) zeros (including multiplicities) on the real line.

Remark 29 It is a simple and well-known consequence of the above Proposition that for every choice of the numbers \( t_1 < ... < t_N \) and the data \( y_1, ..., y_N \) there exists a unique \( f \in E(\lambda_0, ..., \lambda_N) \) with \( f(t_j) = y_j \) for \( j = 1, ..., n \). Hence, for \( (\lambda_0, \lambda_1, 0) \in \mathbb{R}^3 \), numbers \( a < b \) and \( t_\ast \in (a, b) \) there exists \( f \in E(\lambda_0, \lambda_1, 0) \) such that \( f(a) = f(b) = 0 \) and \( f(t_\ast) = 1 \). This implies the existence of the function \( \Omega_{(\lambda_0, \lambda_1, 0)} \) in section [\[3\]].

The fundamental function \( \Phi(\lambda_0, ..., \lambda_N) \) has a zero of order \( N \), hence it follows that \( \Phi(\lambda_0, ..., \lambda_N) \) for all \( t \neq 0 \).

Proposition 30 If \( \lambda_0, ..., \lambda_N \) are real numbers then \( \Phi(\lambda_0, ..., \lambda_N) \) for all \( t > 0 \).

Lemma 31 For real \( \lambda_0, \lambda_1 \) the following identities hold:

\[
\int_0^h \Phi(t, \lambda_0, \lambda_1) (t) e^{pt} \, dt = 2e^{ph} \Phi(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1, -p) (h) \tag{60}
\]

\[
\int_0^h \Phi(t, \lambda_0, \lambda_1) (t - h) e^{pt} \, dt = -2\Phi(-2\lambda_0, -2\lambda_1, -\lambda_0 - \lambda_1, p) (h) \tag{61}
\]

\[
\int_0^h \Phi(t, \lambda_0, \lambda_1) (t - h) e^{pt} \, dt = -2\Phi(-\lambda_0, -\lambda_1, p) (h) \tag{62}
\]

Proof. For \( \lambda_0 \neq \lambda_1 \) real, the function

\[
f(t) := \Phi(t, \lambda_0, \lambda_1) (t)^2 = \frac{1}{(\lambda_1 - \lambda_0)^2} \left( e^{2\lambda_1 t} - 2e^{(\lambda_1 + \lambda_0)t} + e^{2\lambda_0 t} \right).
\]

is an exponential polynomial in \( E(2\lambda_1, 2\lambda_2, \lambda_1 + \lambda_0) \) such that \( f(0) = f'(0) = 0 \) and \( f''(0) = 2 \). Thus

\[
\Phi(t, \lambda_0, \lambda_1) (t)^2 = 2\Phi(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1) (t).
\]

This formula is also valid for \( \lambda_0 = \lambda_1 \) by a limit argument in \( (\lambda_0, \lambda_1) \). With [46] we see that

\[
\left| \Phi(t, \lambda_0, \lambda_1) (t)^2 e^{pt} = 2e^{pt} \Phi(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1) (t) = 2\Phi(2\lambda_0, 2\lambda_1, \lambda_0 + \lambda_1) (t)
\]

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Integration as in (60) shows that
\[
\int_0^h \left| \Phi_{(\lambda_0, \lambda_1)}(t) \right|^2 e^{pt} dt = 2 \Phi_{(p+2\lambda_0, p+2\lambda_1, p+\lambda_0+\lambda_1, 0)}(h)
\]
which with (46) gives (61).

Note that substitution \( \tau_1 = h - t \) yields
\[
I := \int_0^h \Phi_{(\lambda_0, \lambda_1)}(t-h)^2 e^{pt} dt = e^{ph} \int_0^h \Phi_{(\lambda_0, \lambda_1)}(-\tau)^2 e^{-pt} d\tau.
\]

By (47), \( \Phi_{(\lambda_0, \lambda_1)}(-\tau) = -\Phi_{(-\lambda_0, -\lambda_1)}(\tau) \). Using (61) for \( (-\lambda_0, -\lambda_1) \) and \(-p\) shows that
\[
I = e^{ph} 2 \Phi_{(-p-2\lambda_0, -p-2\lambda_1, -p-\lambda_0-\lambda_1, 0)}(h) = 2 \Phi_{(-2\lambda_0, -2\lambda_1, -\lambda_0-\lambda_1, p)}(h),
\]
which proves (62). The case (63) is similar and left to the reader. \( \blacksquare \)

Let \( g : [0, b] \to \mathbb{C} \) be differentiable and \( f : [0, b] \to \mathbb{C} \) continuous. Then it is well known that
\[
A(y) = \int_0^y f(t) g(y-t) dt
\]
is differentiable and
\[
\frac{d}{dy} \int_0^y f(t) g(y-t) dt = f(y) g(0) + \int_0^y f(t) \left( \frac{d}{dy} g \right)(y-t) dy.
\]
For the differential operator \( D_\lambda = \frac{d}{dy} - \lambda \) it is straightforward to verify that
\[
D_\lambda \int_0^y f(t) g(y-t) dt = f(y) g(0) + \int_0^y f(t) D_\lambda g(y-t) dy
\]
The next result follows by induction.

**Proposition 32** Let \( g : [0, b] \to \mathbb{C} \) be \( k+1 \) times continuously differentiable function with \( g^{(l)}(0) = 0 \) for \( l = 0, \ldots, k \) and assume that \( f : [0, b] \to \mathbb{C} \) is continuous. Then
\[
A(y) = \int_0^y f(t) g(y-t) dt
\]
is \( k+1 \) times differentiable and
\[
D_{\lambda_k} \cdots D_{\lambda_0} A(y) = \int_0^y f(t) (D_{\lambda_k} \cdots D_{\lambda_0} g)(y-t) dy.
\]

**Theorem 33** Let \( \lambda_0, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+m} \) be complex numbers. Then the following identity holds:
\[
\int_0^y \Phi_{(\lambda_0, \ldots, \lambda_n)}(t) \Phi_{(\lambda_{n+1}, \ldots, \lambda_{n+m})}(y-t) dt = \Phi_{(\lambda_0, \ldots, \lambda_{n+m})}(y). \tag{64}
\]
Proposition 32 (for \( k \)) suffices to show that for \( l > 0 \) is positive for all \( F \).

Let \( a, b \), \( \lambda \), and \( \lambda_n \) be fixed and \( \Phi \) be a polynomial with frequencies \( \lambda = 2 \) we obtain

\[
D_{\lambda_n+1} \cdots D_{\lambda_{n+m}} A (y) = \int_0^y f (t) D_{\lambda_n+1} \cdots D_{\lambda_{n+m}} \Phi (\lambda_{n+1} \cdots \lambda_{n+m}) (y-t) \, dt
\]

for each \( l = 2, \ldots, m \), and we conclude that \( A^{(k)} (0) = 0 \) for \( k = 0, \ldots, m-1 \). For \( l = 2 \) we obtain

\[
G (y) := D_{\lambda_n+1} \cdots D_{\lambda_{n+m}} A (y) = \int_0^y f (t) \Phi (\lambda_{n+1}) (y-t) \, dt.
\]

Proposition 32 applied to \( g_2 (y) = \Phi (\lambda_{n+1}) (y) \) shows that

\[
D_{\lambda_{n+1}} G (y) = f (y) g_2 (0) + \int_0^y f (t) D_{\lambda_{n+1}} g_2 (y-t) \, dy.
\]

The last integral vanishes since \( D_{\lambda_{n+1}} g_2 = 0 \). Since \( g_2 (0) = 1 \) and \( f (t) = \Phi (\lambda_0, \ldots, \lambda_n) (t) \) we obtain

\[
D_{\lambda_n + 1} \cdots D_{\lambda_{n+m}} A (y) = \Phi (\lambda_0, \ldots, \lambda_n) (y).
\]

Thus \( A^{(k)} (0) = 0 \) for all \( k = 0, \ldots, n+m \), and \( A^{(n+m)} (0) = 1 \). In order to show that \( A (y) = \Phi (\lambda_0, \ldots, \lambda_{n+m}) (y) \) it suffices to show that it an exponential polynomial with frequencies \( \lambda_0, \ldots, \lambda_{n+m} \), but this is clear since

\[
D_{\lambda_0} \cdots D_{\lambda_n} D_{\lambda_{n+1}} \cdots D_{\lambda_{n+m}} A (y) = D_{\lambda_0} \cdots D_{\lambda_n} \Phi (\lambda_0, \ldots, \lambda_n) (y) = 0.
\]

Lemma 34 Let \( a \) be negative number and \( b \) a real number. Then the function \( F_{(a,b)} \) defined by

\[
F_{(a,b)} (t) = 2C_{a,b} \Phi_{(a-1,1-a,0,b)} (t) - \Phi_{(a,1,-a,-1)} (t)
\]

is positive for all \( t > 0 \), where we have put

\[
C_{a,b} := \max \left\{ \frac{1}{2} \left( \frac{2 - a - b}{2} + \frac{2a^2 + (2a - 1)b + 2 - 3a}{2(1 - 2a)(2 - a)} \right), \frac{1 - b}{2(1 - 2a)} \right\}.
\]

Proof. In view of Lemma 17 and the fact that \( F_{(a,b)} (0) = F'_{(a,b)} (0) = 0 \) it suffices to show that for \( C = C_{a,b} \)

\[
G := D_b D_0 F_{(a,b)} = 2C \Phi_{(a-1,1-a)} - \Phi_{(a,1,-a,-1)}' + b \Phi_{(a,1,-a,-1)}'
\]

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is strictly positive for $t > 0$. If $a^2 \neq 1$ and $a \neq 0$ we know that

$$
\Phi_{(a,-a,1,-1)}(t) = \frac{1}{1-a^2} \left( \sinh t - \frac{1}{a} \sinh at \right)
$$

$$
\Phi'_{(a,-a,1,-1)}(t) = \frac{1}{1-a^2} (\cosh t - \cosh at)
$$

$$
\Phi''_{(a,-a,1,-1)}(t) = \frac{1}{1-a^2} (\sinh t - a \sinh at)
$$

$$
\Phi_{(a-1,1-a)}(t) = \frac{\sinh(1-a)t}{1-a} = \frac{1}{2} \frac{e^t e^{-at} - e^{-t} e^{at}}{(1-a)}.
$$

Hence

$$(1-a)G(t) = 2C \sinh(1-a)t - \frac{\sinh t - a \sinh at}{1+a} + \frac{b \cosh t - \cosh at}{1+a}.$$ Let us put $v = e^{-at}$ for $t \geq 0$. Then $v \geq 1$ since $a < 0$ and $v^{\frac{1}{a}} = (e^{-at})^{\frac{1}{a}} = e^t$. It follows that

$$2(1-a)G(t) = 2C \left( v^{1-\frac{1}{a}} - v^{-1+\frac{1}{a}} \right) - \frac{(v^{\frac{1}{a}} - v^{\frac{1}{a}})}{1+a} - a(v^{-1} - v) + \frac{b}{1+a} \left( \frac{v^{1-a}}{v^{\frac{1}{a}}} - v^{\frac{1}{a}} - v^{-1} - v \right)
$$

$$= 2C \left( v^{1-\frac{1}{a}} - v^{-1+\frac{1}{a}} \right) + \frac{b-1}{a+1} v^{\frac{1}{a}} + \frac{b+1}{a+1} v^{\frac{1}{a}} - \frac{b-a}{a+1} v^{-1} - \frac{b+a}{a+1} v^{-\frac{1}{a}}.
$$

Multiply this expression by $v^{1-\frac{1}{a}}$. Then it suffices to show that for all $v > 1$

$$F(v) := 2C v^{2-\frac{1}{a}} - 2C + \frac{b-1}{a+1} v^{1-\frac{1}{a}} + \frac{b+1}{a+1} v^{\frac{1}{a}} - \frac{b-a}{a+1} v^{-1} - \frac{b+a}{a+1} v^{-\frac{1}{a}}$$

is strictly positive. Note that $F(1) = 0$. By Lemma 17 it suffices to show that

$$F'(v) = 2C \left( 2 - \frac{2}{a} \right) v^{1-\frac{1}{a}} + \frac{(b-1)(1-\frac{2}{a}) v^{-\frac{2}{a}}}{a+1} + \frac{b+1}{a+1} + \frac{(b-a) v^{-1-\frac{1}{a}}}{a(a+1)}
$$

$$- \frac{(b+a)(2-\frac{1}{a}) v^{1-\frac{1}{a}}}{a+1}$$

is strictly positive. Note that

$$F'(1) = \frac{2}{a} (2C - 1) (a - 1) \geq 0$$

for any $a < 0$ since $C \geq \frac{1}{2}$. Again by Lemma 17 it suffices to show that

$$F''(v) = 2C \left( 2 - \frac{2}{a} \right) \left( 1 - \frac{2}{a} \right) v^{-\frac{2}{a}} + \frac{(b-1) (2 - \frac{2}{a}) v^{1-\frac{1}{a}}}{a+1} + \frac{b-a}{a+1} \left( 1 - \frac{1}{a} \right) (2 - \frac{1}{a}) v^{-\frac{1}{a}}
$$

$$+ \frac{b-a}{a+1} \left( -1 - \frac{1}{a} \right) v^{-\frac{1}{a}} - \frac{b+a}{a+1} \left( 1 - \frac{1}{a} \right) \left( 2 - \frac{1}{a} \right) v^{-\frac{1}{a}}$$

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is strictly positive. Multiply the expression with $v^{2 + \frac{1}{a}}$ and define

$$
\tilde{F}(v) = v^{2 + \frac{1}{a}} F''(v) = 2C \left(2 - \frac{2}{a}\right) \left(1 - \frac{2}{a}\right) v^{2 - \frac{1}{a}} - \frac{(b - 1)}{a + 1} \left(1 - \frac{2}{a}\right) v^{1 - \frac{1}{a}}
+ \frac{b - a}{a + 1} \left(-1 - \frac{1}{a}\right) \frac{1}{a} - \frac{b + a}{a + 1} \left(1 - \frac{1}{a}\right) \left(2 - \frac{1}{a}\right) v^2.
$$

Note that

$$
\tilde{F}(1) = -\frac{2}{a^2} (a - 1) (a + b + 4C - 2aC - 2).
$$

Since $a < 0$ we infer that $\tilde{F}(1) \geq 0$ when $a + b + 4C - 2aC - 2 \geq 0$. This is true since by our assumption

$$
C \geq \frac{2 - a - b}{2 (2 - a)}.
$$

By Lemma 17 it suffices to show that $\tilde{F}'(v)$ is positive for $v > 1$ where

$$
\tilde{F}'(v) = 4C \left(1 - \frac{1}{a}\right) \left(1 - \frac{2}{a}\right) \left(2 - \frac{1}{a}\right) v^{1 - \frac{1}{a}} - \frac{b - 1}{a + 1} \left(1 - \frac{2}{a}\right) \left(1 - \frac{1}{a}\right) v^{-\frac{1}{a}}
- \frac{b + a}{a + 1} \left(1 - \frac{1}{a}\right) \left(2 - \frac{1}{a}\right) v^{2v}.
$$

Divide by $v \geq 1$ and $1 - \frac{1}{a} \geq 0$, then it suffices to show that

$$
H(v) = \frac{\tilde{F}'(v)}{v \left(1 - \frac{1}{a}\right)}
= 4C \left(1 - \frac{2}{a}\right) \left(2 - \frac{1}{a}\right) v^{1 - \frac{1}{a}} - \frac{b - 1}{a + 1} \left(1 - \frac{2}{a}\right) \left(1 - \frac{1}{a}\right) v^{-\frac{1}{a}} - \frac{b + a}{a + 1} \left(2 - \frac{1}{a}\right) v^{2v}
$$

is strictly positive for $v > 1$. A computation shows that

$$
a^2 \frac{2}{2} H(1) = 2C (1 - 2a) (2 - a) + 3a + 2b - 2ab - 2a^2 - 2.
$$

We see that $H(1) \geq 0$ if we assume the inequality

$$
C \geq \frac{2a^2 + (2a - 1) b + 2 - 3a}{2 (1 - 2a) (2 - a)}.
$$

(66)

By Lemma 17 it suffices to show $H'$ is strictly positive for $v > 1$. This is seen from the following argument:

$$
H'(v) = 4C \left(1 - \frac{2}{a}\right) \left(2 - \frac{1}{a}\right) \left(-\frac{1}{a}\right) v^{-1 - \frac{1}{a}} - \frac{b - 1}{a + 1} \left(1 - \frac{2}{a}\right) \left(1 - \frac{1}{a}\right) v^{2 - \frac{1}{a}}
$$

is positive if for all $v > 1$

$$
Cv > \frac{b - 1}{a + 1} \frac{2}{a} \left(1 - \frac{2}{a}\right) \left(-\frac{1}{a}\right) = \frac{b - 1}{4a - 2} = \frac{1 - b}{2 - 4a}.
$$
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