RESTRICTING SLE(8/3) TO AN ANNULUS

ROBERT O. BAUER

Abstract. We study the probability that chordal SLE_{8/3} in the unit disk from \exp(\imath x) to 1 avoids the disk of radius \( q \) centered at zero. We find the initial/boundary-value problem satisfied by this probability as a function of \( x \) and \( a = \ln q \), and show that asymptotically as \( q \) tends to one this probability decays like \( \exp \left( -\frac{c x}{1 - q} \right) \) with \( c = \frac{5\pi}{8} \) for \( 0 < x \leq \pi \). We also give a representation of this probability as a multiplicative functional of a Legendre process.

1. Introduction

In this paper we study certain hitting probabilities for the chordal Schramm-Loewner evolution with parameter \( \kappa = 8/3 \) (SLE_{8/3}). We study this question for SLE_{8/3} because this process lies in the intersection of two important classes of conformally invariant measures.

On the one hand, we have chordal SLE: these are families of measures on non-self-crossing curves \( \gamma \), indexed by the simply connected domain \( D \) the curve \( \gamma \) lives in, and the endpoints \( z, w \) of \( \gamma \) on \( \partial D \). We can think of \( \gamma \) as a random interface separating two different materials on \( D \). If \( P_{D,z \rightarrow w} \) denotes the law of the curve \( \gamma \) in \( D \) from \( z \) to \( w \), then the family \( \{ P_{D,z \rightarrow w} \} \) is a Schramm-Loewner evolution if members of the family are related by

1. conformal invariance: if \( f \) is a conformal map from \( D \) to \( D' \) sending \( z, w \) to \( z', w' \), then \( f \circ P_{D,z \rightarrow w} = P_{D',z' \rightarrow w'} \);

2. domain Markovianity: if \( \gamma \) has law \( P_{D,z \rightarrow w} \), \( z' \) is an interior point of \( \gamma \), and we condition on the segment \( \gamma' \) of \( \gamma \) from \( z \) to \( z' \), then the remaining segment of \( \gamma \), from \( z' \) to \( w \), has law \( P_{D\setminus\gamma',z' \rightarrow w} \);

and if, for the particular case where \( D \) is the upper half-plane \( \mathbb{H} \), \( z = 0, w = \infty \), the law of \( \gamma \) is symmetric with respect to the imaginary axis. Suppose \( \{ P_{D,z \rightarrow w} \} \) is such a family. Using Löwner’s theory of slit mappings [21], Schramm showed that if \( t \in [0, \infty) \mapsto \gamma_t \in \mathbb{H} \) is correctly parameterized, \( \gamma_0 = 0, D_t \) denotes the unbounded component of \( \mathbb{H} \setminus \gamma(0, t) \), and \( g_t : D_t \rightarrow \mathbb{H} \) is conformal with ‘hydrodynamic’
normalization at infinity
\[ \lim_{z \to \infty} g_t(z) - z = 0, \]
then, under \( P_{\mathbb{H},0,\infty} \), \( g_t(\gamma_t) = \sqrt{\kappa} B_t \) for a standard 1-dimensional Brownian motion \( \{ B_t : t \geq 0 \} \) starting at zero and a constant \( \kappa \geq 0 \). \[26\].

On the other hand, we have restriction measures. These are again families of measures \( \{ P_{D,z,w} \} \) indexed by simply connected domains \( D \) and two boundary points \( z, w \), but this time describing random, closed, simply connected subsets (which we denote also by \( \gamma \)) of \( D \) such that \( \gamma \cap \partial D = \{ z, w \} \). For example, a simple curve in \( D \) from \( z \) to \( w \) is such a set. We dropped the \( \to \) in the notation as \( \gamma \) is a point-set without a ‘direction.’ A family \( \{ P_{D,z,w} \} \) is called a restriction measure if it is conformally invariant (as in (1) above), and satisfies the

(3) restriction property: if \( \gamma \) has distribution \( P_{D,z,w} \), \( D' \subset D \) and \( z, w \in \partial D' \), then conditional on \( \{ \gamma \subset D' \} \) the distribution of \( \gamma \) is \( P_{D',z,w} \).

In the statement of the restriction property it is understood that \( z \) and \( w \) are bounded away from the part of the boundary of \( D \) that does not belong to \( \partial D' \). An example of a restriction measure is provided by the ‘filling’ of a Brownian excursion in \( D \) from \( z \) to \( w \). Restriction is a powerful property. If \( \{ P_{D,z,w} \} \) denotes a restriction measure, and if \( D_2 \subset D_1 \subset D \) and \( z, w \in \partial D_2 \), then restriction implies in particular that

(1) \[ P_{D,z,w} \{ \gamma \subset D_2 \} = P_{D_1,z,w} \{ \gamma \subset D_2 \} P_{D,z,w} \{ \gamma \subset D_1 \}. \]

By conformal invariance it is enough to consider the case when \( D \) is the upper half-plane \( \mathbb{H} \), \( z = 0 \), and \( w = \infty \). That is, suppose that \( D_2 \subset D_1 \subset \mathbb{H} \) with \( 0, \infty \in \partial D_2 \). Denote \( \Phi_{1,2} : D_{1,2} \to \mathbb{H} \) the conformal map with normalization \( \lim_{z \to \infty} \Phi_{1,2}(z)/z = 1 \), \( \Phi_{1,2}(0) = 0 \). Then we can rewrite (1) as

(2) \[ P_{\mathbb{H},0,\infty} \{ \gamma \subset D_2 \} = P_{\mathbb{H},0,\infty} \{ \gamma \subset \Phi_1(D_2) \} P_{\mathbb{H},0,\infty} \{ \gamma \subset D_1 \}. \]

As we can identify a domain with the unique normalized conformal map from that domain to \( \mathbb{H} \), we may write \( F(\Phi_{1,2}) = P_{\mathbb{H},0,\infty} \{ \gamma \subset D_{1,2} \} \). In particular, (2) is equivalent to

(3) \[ F(\Phi_2) = F(\Phi_2 \circ \Phi_1^{-1}) \cdot F(\Phi_1), \]
that is, \( F \) is a homomorphism from the semigroup of conformal maps (with composition) to \( [0, \infty) \) (with multiplication). Lawler, Schramm, and Werner showed that this implies the remarkable result that there exists an \( \alpha > 0 \) such that

(4) \[ P_{\mathbb{H},0,\infty} \{ \gamma \in D \} = F(\Phi) = \Phi'(0)^\alpha, \]
where $D$ is a simply connected subdomain of $\mathbb{H}$ containing $0, \infty$ as boundary points, see [19]. If $\gamma$ is both, an SLE and a restriction measure, then

\[
P_{\mathbb{H}, 0 \to \infty} \{ \gamma \subset D | \gamma[0, t] \subset D \} = 1 \{ \gamma[0, t] \subset D \} P_{\mathbb{H}, W_t \to \infty} \{ \gamma \subset g_t(D) \}
\]

(5)

where $h_t$ is the normalized conformal map from $g_t(D)$ to $\mathbb{H}$, and $W_t = \sqrt{\kappa} B_t$. The first equality in (5) is on account of $\gamma$ being an SLE, the second a consequence of restriction. It follows that $h_t'(W_t)^\alpha$ is a martingale on $\{ \gamma[0, t] \subset D \}$. A calculation now shows that this implies $\kappa = 8/3$ and $\alpha = 5/8$, [19]. The self-avoiding random walk satisfies the discrete version of the restriction property and it is conjectured that the scaling limit of self-avoiding random walk is SLE$_{8/3}$, [20].

We now ask what happens if we restrict to domains $D \subset \mathbb{H}$ with ‘holes,’ i.e if $D$ is no longer simply connected. Then there is no homeomorphism from $D$ to $\mathbb{H}$. Even more, while connectivity classifies topological equivalence, it does not classify conformal equivalence. For example, two annuli are conformally equivalent if and only if the ratio of outer to inner radius of the former equals that of the latter. In other words, there is a conformal parameter, or modulus, which labels conformal equivalence classes of doubly connected domains, [1].

However, it is easy to extend restriction measures to multiply connected domains. Suppose $\{P_{D,z,w}\}$ is a restriction measure as above. If $D'$ is finitely connected and $z, w$ points on the same boundary component of $D'$, we define

\[
P_{D', z, w} = P_{D, z, w} \{ \cdot | \gamma \subset D' \},
\]

(6)

where $D \supset D'$ is simply connected and $z, w \in \partial D$. Restriction for simply connected domains implies that $P_{D', z, w}$ is independent of the choice of $D$, and an inclusion/exclusion argument of Beffara shows that then (4) holds for arbitrary finitely connected domains $D', D$ with $D' \subset D$, $z, w \in \partial D' \cap \partial D$, [6]. The identity (1) still holds in this more general context but (2) and (4) no longer make sense. Thus, while we can define restriction measures in multiply connected domains, we cannot calculate—or do not have a functional expression for—the probability that $\gamma$ hits a ‘hole.’ Finding a functional expression which generalizes (1) to multiply connected domains is the main motivation for this paper.

To begin, we decided to focus on the simplest case, just one hole, and address this case for the restriction measure which also is an SLE, making SLE-tools available. So suppose $\gamma$ is a chordal SLE$_{8/3}$ in the unit disk $U = \{ |z| < 1 \}$ from $e^{ix}$ to 1 and $A_q = \{ q < |z| < 1 \}$ an
annulus. Then

\[ P_{U,e^{ix}} \gamma \subset A_q \]

is a function \( F \) of \( x \) and \( a = \ln q \). In this paper we show that \( F \) is \( C^{1,2} \), find the initial/boundary-value problem to which this function is the solution, see Theorem 6.1 and show in Theorem 5.5 that asymptotically

\[
F(a, x) \approx \exp \left( -\frac{5\pi}{8} \cdot \frac{x}{1 - q} \right), \quad 0 \leq x \leq \pi,
\]

as \( q \searrow 1 \). Using this strong decay we obtain a stochastic representation for \( F(a, x) \) as

\[
\exp \left( \int_a^\sigma \left[ \frac{1}{12} - \sum_{n=1}^{\infty} \frac{2ne^{2nb}}{1 - e^{2nb}} (1 - \cos nY_b) \right] db \right), \sigma < 0
\]

in Theorem 5.6. Here \( Y \) is a Legendre process on \([0, 2\pi]\) starting at \( x \) at time \( a < 0 \) and \( \sigma \) is the first time \( Y \) hits the boundary. We give an alternative expression in terms of Jacobi’s \( \vartheta \)-function and Weierstrass’ \( \wp \)-function.

In [27], Werner also studies the asymptotics of a non-intersection probability in annuli as \( q \searrow 1 \), namely the probability, appropriately rescaled, that chordal SLE\(_{8/3}\) from “near 1” to 1 in the unit disk stays in the annulus \( A_q \) and goes the long way (around the hole), see [27, Lemma 18]. He finds that that probability decays like \( \exp(-5\pi^2/(4(1 - q))) \). This result can be guessed from (7) as follows. The probability that a chordal SLE\(_{8/3}\) from “near 1” to 1 goes around the disk of radius \( q \) centered at zero is, for \( q \) close to \( 1 \), approximately the same as the probability that a chordal SLE\(_{8/3}\) from 1 to \(-1\) goes around the disk of radius \( q \) via the upper half-plane, followed by an independent SLE\(_{8/3}\) from \(-1\) to 1, which goes around the disk of radius \( q \) via the lower half-plane. Thus the probability Werner calculates should behave asymptotically like the square of (7) for \( x = \pi \), which indeed is the case.

Concerning the behavior of \( F(a, x) \) as \( q \searrow 0 \) a brief analysis of the initial/boundary-value problem leads to the conjecture

\[
F(a, x) = 1 - cq^{2/3} \sin^2 x/2, \quad q \searrow 0,
\]

for some constant \( c \), see Proposition 6.2. We give evidence for this conjecture based on an analysis of the partial differential equation solved by \( F(a, x) \) in the last section. That \( 1 - F \) decays like \( q^{2/3} \) can actually be derived from the known Hausdorff dimension (i.e. \( 4/3 \)) of SLE\(_{8/3}\).
Our approach rests on the argument of Beffara alluded to above, see Lemma 4.1, and earlier work by Dubédat [9], as well as [4], [5], where the Loewner equation in multiply connected domains is discussed and explicit expressions for the change of the conformal parameters under Loewner evolution are given. Using Beffara’s argument, it is easy to see that if $D \subset A_q$ is doubly connected, $e^{ix}, 1 \in \partial D$, then

$$P_{A_q,e^{ix} \to 1}\{ \gamma \subset D \} = \frac{F(a',x')}{F(a,x)}[h'(e^{ix})h'(1)]^{5/8},$$

where $h$ is defined in terms of the unique conformal equivalence from $D$ to $A_q'$ which keeps 1 fixed, $e^{ix'}$ is the image of $e^{ix}$ under this equivalence, and $a' = \ln q'$. Equation (10) is the generalization of (4) for SLE$_{8/3}$.

In [9], Dubédat discusses questions similar to those we discuss here, although he considers SLE$_6$ and ‘locality.’ Zhan [29] constructs SLE$_2$ in an annulus as the scaling limit of loop-erased random walk, by adapting the approach taken by Schramm from simply connected domains to doubly connected domains. To do so, he exploits particular properties of the discrete walk. It is also clear from our calculations that $\kappa = 2$ is special in that some of the martingales mentioned below have a particularly simple form in this case. However, we will not pursue this here.

Restriction in multiply connected domains has also been discussed in [28], [10], and [17]. In particular, these authors find restriction (local) martingales similar to ours. Due to the greater generality, the expressions these authors find are less explicit and the asymptotics of these (local) martingales are not discussed. In the case of connectivity two we find here the asymptotics of the restriction martingale, leading to a stochastic functional representation of the intersection probability. We also give a proof that $F$ is smooth enough to apply Itô’s lemma, an issue that, to our knowledge, had not been addressed previously. The question of smoothness of the intersection probabilities had been raised by John Cardy. While it had been expected that the intersection probability would be given as the solution to a partial differential equation, we are the first to derive it for SLE$_{8/3}$. A similar equation has been derived in [2] in the percolation case ($\kappa = 6$), but the smoothness necessary to apply Itô’s formula in that context is not discussed. Finally, the limiting behavior of the intersection probability as the annulus becomes thinner and thinner is new, though it is clearly related to the estimate obtained in [27]. It fits with a recent calculation of Cardy using Coulomb gas methods, [8].
Going from ‘locality’ to ‘restriction’ in SLE-type calculations involves taking one more derivative, which leads to expressions which are considerably more expansive. For this reason we begin this paper by changing coordinates from the upper half-plane to a half-strip, where elliptic functions—the indispensable tool of function theory in annuli—have their simplest expression. In Section 3 we use elliptic functions to describe Loewner evolution in an annulus. In Section 4 we study the ‘conditional probability martingale’ derived from $F$ and use it to show that $F$ has enough smoothness to apply the Itô formula later in the paper. In Section 5 we obtain the asymptotic behavior for $F$ as $q \nearrow 1$ and the stochastic representation mentioned above. Finally, in Section 6 we apply Itô’s formula to derive the partial differential equation for $F$.

2. CHORDAL SLE IN A HALF-STRIP

Denote $B_t$ a standard 1-dimensional Brownian motion, $\kappa > 0$ a constant, and set $W_t = \sqrt{\kappa}B_t$. For $u$ in the upper half-plane $\mathbb{H}$ denote $g_t(u)$ the solution to the chordal Loewner equation at time $t$,$$
abla g_t(u) = 2/g_t(u) - W_t, \quad g_0(u) = u.$$
The solution exists up to a time $T_u = \sup\{t : \min_{s \leq t} |g_s(u) - W_s| > 0\}$, and if $K_t = \{u : T_u \leq t\}$, then $g_t$ is the conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with hydrodynamic normalization at infinity, $\lim_{z \to \infty} g_t(z) - z = 0$. The stochastic process of conformal maps $g_t$ is called chordal Schramm-Loewner evolution in $\mathbb{H}$ from $B_0$ to $\infty$ with parameter $\kappa$, see [18]. The random growing compact $K_t$ is generated by a curve $t \mapsto \gamma_t$ with $\gamma_0 = B_0$. If $\kappa \leq 4$, then $\gamma$ is simple, see [21]. We will sometimes write $\gamma$ for $\gamma[0, \infty)$.

The function

$$u = \cot(z/2) = i\frac{e^{iz} + 1}{e^{iz} - 1}$$

maps the halfstrip $HS \equiv \{z : 0 \leq \Re(z) \leq 2\pi, \Im(z) < 0\}$ onto the upper half-plane. We will use $u$ to denote the map as well as the variable for the image domain. The sides

$$(11) \quad \{iy : y < 0\}, \quad \{2\pi + iy : y < 0\}$$

of $HS$ are mapped to the slit $\{iy : y > 1\} \subset \mathbb{H}$ and the real interval $(0, 2\pi)$ in the $z$-plane corresponds to the real axis in the $u$-plane. Furthermore, the point $\infty$ in the (extended) $z$-plane corresponds to $i \in \mathbb{H}$ and the point $\infty$ in (the closure of) $\mathbb{H}$ has the pre-images $0, 2\pi \in HS$. If we identify the sides $(11)$, i.e. $iy \approx 2\pi + iy$, then $u = \cot z/2$ is
conformal from $HS$ onto $\mathbb{H}$. In the following we will always assume this identification for points in the $z$-plane. The inverse mapping is given by

$$z = \frac{1}{i} \ln \frac{u + i}{u - i},$$

and we recall the derivatives

$$u'(z) = -\frac{1}{2} \csc^2(z/2), \quad u''(z) = \frac{1}{2} \csc^2(z/2) \cot(z/2).$$

We define chordal $\text{SLE}_\kappa$ in $HS$ from $x \in (0, 2\pi)$ to 0 as the conformal image of $\text{SLE}_\kappa$ in $\mathbb{H}$ from $\cot x/2$ to $\infty$ under the mapping (12). This definition is natural in light of the characterization of $\text{SLE}$ as the unique family of measures on non-self-crossing curves which are conformally invariant, satisfy a Markovian-type property and a certain symmetry condition.

**Remark 2.1.** It follows from the Riemann mapping theorem that there is a one-parameter family of conformal maps from $HS$ onto $\mathbb{H}$ which send 0 to $\infty$ and $x$ to $\cot x/2$. Choosing a function other than $\cot z/2$ from this family would only result in a linear time-change for the $\text{SLE}$ measures. As we will not be interested in when a particular event occurs but rather if it occurs this is of no concern. In fact, we will change the time parameter when it simplifies our calculations.

If the process $X$ is defined by $X_t = u^{-1}(W_t)$, then

$$dX_t = -2\sqrt{\kappa} \sin^2(X_t/2) \, dB_t + 2\kappa \sin^4(X_t/2) \cot(X_t/2) \, dt.$$  

Under the random time-change $t \to s$ with $ds = 4 \sin^4(X_t/2) \, dt$, we get

$$dX_s = -\sqrt{\kappa} \, dB_s + \frac{\kappa}{2} \cot(X_s/2) \, ds.$$  

For this new time parameter, let $\tilde{g}_s = u^{-1} \circ g_s \circ u$. Then, for each $z \in HS$,

$$\partial_s \tilde{g}_s(z) = \Xi_1(\tilde{g}_s(z), X_s), \quad \tilde{g}_0(z) = z,$$

with

$$\Xi_1(z, x) = \frac{2u'(x)^2}{u'(z)[u(z) - u(x)]} = -\frac{\sin^2(z/2)}{\sin^4(x/2)[\cot(z/2) - \cot(x/2)]}.$$

Note that the vector field $\Xi_1(\cdot, x)$ has a pole with residue 2 at $x$. $\Xi$ is the variation kernel of the Riemann sphere expressed in the coordinate $u$, see [25]. The variation kernel is a reciprocal differential (holomorphic vector field) in $z$—this explains the $u'$-term in the denominator—and
a quadratic differential in $x$—which explains the $u'(x)^2$-term in the numerator.

**Remark 2.2.** The solution $X_s$ to the SDE (14) is a Bessel-like process on the interval $(0, 2\pi)$. At the boundary points it behaves like the 3-dimensional Bessel process, see [16]. In particular, with probability 1, $X_s$ never leaves $(0, 2\pi)$.

### 3. SLE viewed in an annulus

For a real number $a < 0$, $\cot(z/2)$ maps the rectangle

$R_a \equiv \{0 \leq \Re(z) \leq 2\pi, a < \Im(z) < 0\}$

onto $\mathbb{H} \setminus C_a$, where $C_a$ denotes the disk

$$\{u : \left| u - i \frac{1 + q^2}{1 - q^2} \right| \leq \frac{2q}{1 - q^2}, \quad q = e^a \}.$$

This doubly connected domain is conformally equivalent to the annulus

$$A_q \equiv \{ q < |z| < 1 \},$$

the image of $R_a$ under the map $z \mapsto v = \exp(-iz)$.

For $t > 0$, suppose that $\tilde{K}_s \equiv u^{-1}(K_s) \subset R_a$. Then $v(\tilde{K}_s) \subset A_q$ and the doubly connected domain $A_q \setminus v(\tilde{K}_s)$ is conformally equivalent to a unique annulus $A_{q'}$. If $a' = \ln q'$, then $a < a' < 0$. Furthermore, there is a unique conformal map $\tilde{h}_s : A_q \setminus v(\tilde{K}_s) \to A_{q'}$ with $\tilde{h}_s(1) = 1$, see [11]. Set $f_s = v^{-1} \circ \tilde{h}_s \circ v$. Then $f_s$ maps $R_a \setminus \tilde{K}_s$ onto $R_{a'}$, fixing $0, 2\pi$.

To describe the time evolution of $f_s$ we need to use elliptic functions. Denote $\zeta$ the Weierstrass $\zeta$-function with periods $2\pi$, $2ia$, i.e.

$$\zeta(z) = \zeta(z|a) = \frac{\eta}{\pi}z + \frac{1}{2} \cot(z/2) + 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin nz,$$

where

$$\eta = \pi \left( \frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \right),$$

see [12]. $\zeta$ is regular in the entire $z$-plane except for poles with residue 1 at the lattice points $2n\pi + 2mia$, $n, m \in \mathbb{Z}$. $\zeta$ is an odd function and $\zeta(\pi) = \eta$. For each $x \in (0, 2\pi)$, $a < 0$, define the vector field $\Xi_2(\cdot, x)$ by

$$\Xi_2(z, x) = \Xi_2(z, x|a) = 2 \left[ \zeta(z - x) - \frac{\eta}{\pi}z + \zeta(x) \right].$$

$\zeta$, $\eta$, and $\Xi_2$ all depend on $a$. We will use $a$ in the notation if any ambiguity as to the particular value of that parameter could arise.
Proposition 3.1. The vector field $\Xi_2(\cdot, x)$ (i) is regular except for poles with residue 2 at the points of the shifted lattice $\{2n\pi + x + 2mia : n, m \in \mathbb{Z}\}$, (ii) is periodic with period $2\pi$ (i.e. $\Xi_2(z, x) = \Xi_2(z + 2\pi, x)$), (iii) vanishes at $z = 0$, and (iv) has constant imaginary part $+i, -i$ on the lines $\{\Im(z) = a\}, \{\Im(z) = -a\}$, respectively.

Proof. Property (i) follows immediately from the properties of $\zeta$, and (ii), (iii) follow by inspection from (18). Next, if $\Im(z) = 0$, then

$$\Im(\Xi_2(z + ia, x)) = 1 - q^2 \frac{1}{1 - 2q \cos(z - x) + q^2} - 2 \sum_{n=1}^{\infty} q^n \cos n(z - x) = 1,$$

(19)

where the last equality follows from a well-known identity for Chebyshev polynomials, see [2]. Similarly, $\Im(\Xi_2(z - ia, x)) = -1$ if $\Im(z) = 0$. □

For chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to $\infty$, and $A < 0$, set

$$T_A = \inf\{s : K_s \cap C_A \neq \emptyset\}.$$

If $\kappa \leq 4$, then $\gamma$ is almost surely a simple curve and thus $K_t = \gamma[0, t]$. In particular, for $\kappa \leq 4$, $T_A = \infty$ if and only if $\gamma \cap C_A = \emptyset$. On $s < T_A$, let $a = a(s)$ be defined as the unique $a$ such that

$$h_s(\tilde{g}_s(R_A \setminus \tilde{K}_s)) = R_a.$$

Then $a(0) = A$ and $a(s) > a(t)$ if $s > t$ (for an integral expression for $a(s) - a(t)$ see [13]). Set

$$A^* = \lim_{s \to T_A} a(s).$$

Then $A^* \leq 0$ and $A^* = 0$ if and only if $T_A < \infty$. The last statement holds with probability 1 and is a consequence of the fact that a.s. $\gamma_s \to \infty$ as $s \to \infty$. We now change the time parameter from $s$ to $a$ and write $\gamma_a, X_a, \tilde{g}_a$, and $h_{A,a}$ for $\gamma_{s(a)}, X_{s(a)}, \tilde{g}_{s(a)}$, and $h_{s(a)}$. We include $A$ in the subscript of $h$ to keep note of the fact that the definition of $h$ depends on $A$ (or rather $R_A$). Then $\gamma[A, a] = \gamma[0, s]$.

Theorem 3.2. For $A \leq a < A^*$ we have

$$\partial_s a = h_{A,a}'(X_a)^2$$
Proof. Set \( f_{A,a} = h_{A,a} \circ \tilde{g}_a \). Then \( f_{A,a} \) is the unique conformal map from \( R_A \setminus \gamma[A,a] \) onto \( R_a \) with \( f_{A,a}(0) = 0 \). By [15],

\[
\partial_a f_{A,a}(z) = \Xi_2(f_{A,a}(z), Y_{A,a}(a))\]

where \( Y_{A,a} = h_{A,a}(X_a) \). Note that \( Y_{A,a} \) is the image of the tip of the slit \( \gamma[A,a] \) under \( f_{A,a} \), i.e \( Y_{A,a} = \lim_{z \to X_a} f_{A,a}(z) \). Also, it is clear from the mapping properties of \( f_{A,a} \) that the left-hand side of (21) is zero at \( z = 0 \) and has constant imaginary part 1 if \( \Im(z) = A \). Next, by the chain rule

\[
\partial_a h_{A,a}(z) = \partial_a f_{A,a}(\tilde{g}_a^{-1}(z)) + (f_{A,a})'(\tilde{g}_a^{-1}(z))\partial_a \tilde{g}_a^{-1}(z).
\]

Since \( \partial_a \tilde{g}_a^{-1}(z) = -(\tilde{g}_a^{-1})'(z)(\partial_a \tilde{g}_a)(\tilde{g}_a^{-1}(z)) \), we get from (15)

\[
\partial_a \tilde{g}_a^{-1}(z) = -(\tilde{g}_a^{-1})'(z)\Xi_1(z, X_a)\frac{\partial s}{\partial a}.
\]

Hence

\[
\partial_a h_{A,a}(z) = \Xi_2(h_{A,a}(z), h_{A,a}(X_a)|a) - \Xi_1(z, X_a)h'_{A,a}(z)\frac{\partial s}{\partial a},
\]

and

\[
\partial_s h_s(z) = \Xi_2(h_{A,a}(z), h_{A,a}(X_a)|a)\frac{\partial a}{\partial s} - \Xi_1(z, X_a)h''_{A,a}(z).
\]

To determine \( \partial a / \partial s \) we note that the domains \( \tilde{g}_s(R_A) \) change smoothly because \( \Xi_1(z, x) \) is smooth away from \( x \). The map \( h_s \) can be written explicitly in terms of domain functionals, namely the harmonic measures and their conjugates. By Hadamard’s formula for the variation of domain functionals under smooth boundary perturbations, see [25], it follows that \( \partial_s h_s(z) \) extends continuously to the boundary. In particular, the residues of the two terms on the right in (22) have to cancel. The residue of the first term is \( 2(\partial a / \partial s)/h'_{A,a}(X_a) \), the residue of the second \( 2h''_{A,a}(X_a) \). The theorem now follows. \( \square \)

We will now draw a number of conclusions from (20). To simplify notation, we will indicate differentiation with respect to \( a \) by ·, and suppress the subscripts \( a, A \) when convenient.

Corollary 3.3. On \([A, A^*]\) we have

\[
\hat{h}(X) = 2 \left[ \zeta(h(X)) - \frac{\eta}{\pi} h(X) \right] - 3 \frac{h''(X)}{h'(X)^2} - 3 \frac{\cot(X/2)}{h'(X)}
\]
and
\[ d(h(X)) = -\sqrt{\kappa} dB + 2 \left[ \zeta(h(X)) - \frac{\eta}{\pi} h(X) \right] da + \frac{\kappa - 6}{2} \left[ \frac{h''(X)}{h'(X)^2} + \frac{\cot(X/2)}{h'(X)} \right] da. \] (24)

Proof. Taking the limit \( z \to X_a \) in (20) gives (23). The calculation is done by Taylor expansion. By an appropriate version of Itô's lemma (23),
\[ d(h(X)) = \dot{h}(X) da + h'(X) dX + 1/2 h''(X) dX dX, \]
where \( dX dX \) is the differential of the quadratic variation. Also, by (14),
\[ dX_a = -\sqrt{\kappa} \frac{dB}{h'(X)} + \frac{\kappa \cot(X/2)}{2 h'(X)^2} da. \] (25)

Now (24) follows from (23). \( \square \)

Remark 3.4. A time change of the results (23) and (24) had previously been obtained in [9].

Denote \( \wp = -\zeta' \) the Weierstrass \( \wp \)-function,
\[ \wp(z) = \wp(z|a) = -\frac{\eta}{\pi} + \frac{1}{4} \csc^2(z/2) - 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos nz, \]
see [12]. Then it follows from (20) that
\[ \dot{h}'(z) = -2 \left[ \wp(h(z) - h(X)) + \frac{\eta}{\pi} \right] h'(z) - \frac{h''(z)}{h'(X)^2} \cdot \frac{\sin^3(z/2)}{\sin^3(X/2)} \csc \frac{z - X}{2} \]
\[ + \frac{h'(z)}{h'(x)^2 \sin^2(X/2)} \left[ \frac{1}{2} \csc^2 \frac{z - X}{2} - \frac{\cos(z/2)}{\sin(X/2)} \csc \frac{z - X}{2} \right]. \] (26)

In particular,
\[ \dot{h}'(0) = -2 \left[ \wp(h(X)) + \frac{\eta}{\pi} \right] h'(0), \] (27)
so that
\[ h'_{A,a}(0) = \exp \left( -2 \int_{A}^{0} \left[ \wp(h(X_b)) + \frac{\eta}{\pi} \right] db \right). \] (28)

Note that \( \eta \) in the integrand also depends on \( b \), the explicit form of the dependence being given in (17).
Corollary 3.5. We have
\[
\hat{h}'(X) = -\frac{2\eta}{\pi} h'(X) + \frac{2}{3h'(X)} - \frac{3\cot^2(X/2)}{2h'(X)} - 3\cot(X/2) \frac{h''(X)}{h'(X)^2} + \frac{h''(X)^2}{2h'(X)^3} - \frac{4\, h''(X)}{3\, h'(X)^2}
\]
and, for real \(\alpha\),
\[
\frac{d(h'(X)^\alpha)}{d\hat{h}'(X)^\alpha} = \left[ \frac{2}{3h'(X)^2} - \frac{3\cot^2(X/2)}{2h'(X)^2} + \frac{\kappa - 6}{2} \cot(X/2) \frac{h''(X)}{h'(X)^2} \right] d\alpha + \left[ \frac{1 + (\alpha - 1)\kappa}{2} \frac{h''(X)^2}{h'(X)^4} + \frac{\kappa - 8/3}{2} \frac{h''(X)}{h'(X)^3} - \frac{2\eta}{\pi} \right] d\alpha - \sqrt{\kappa} \frac{h''(X)}{h'(X)^2} dB.
\]
(29)

Proof. The first identity follows by taking the limit in (26), and then the second follows from Itô’s lemma, just as in the proof of Corollary 3.3. The calculation is tedious but straightforward and is omitted. \(\square\)

4. Conditional probabilities and restriction martingales

For a simply connected domain \(D\) and boundary points \(p, q\), we define chordal SLE in \(D\) from \(p\) to \(q\) by conformal invariance from chordal SLE in \(\mathbb{H}\) from 0 to \(\infty\). This is well defined up to a linear time-change. Denote \(P_{D,p\to q}\) the law of chordal SLE in \(D\) from \(p\) to \(q\), and \(E_{D,p\to q}\) expectation with respect to \(P_{D,p\to q}\). Then
\[
P_{HS,x\to 0}\{\gamma \subset R_A|\gamma[0,s]\} = P_{\mathbb{H},\cot x\to \infty}\{\gamma \cap C_A = \emptyset|\gamma[0,s]\}
= E_{\mathbb{H},\cot x\to \infty}\{1\{\gamma[0,s] \cap C_A = \emptyset\}1\{\gamma[0,s] \cap C_A = \emptyset\}|\gamma[0,s]\}
= 1\{s < T_A\}E_{\mathbb{H},\cot x\to \infty}\{1\{g_s(\gamma[s,\infty)) \cap g_s(C_A) = \emptyset\}|\gamma[0,s]\}
(30)
= 1\{t < T_A\}P_{\mathbb{H},W_s\to \infty}\{\gamma \cap g_s(C_A) = \emptyset\},
\]
where \(W\) is a time changed Brownian motion starting at \(\cot x\). We note that the last equality follows from basic properties of SLE. Now we need a result from [6].

Lemma 4.1. (Beffara) Let \(\kappa = 8/3\). If \(K\) and \(K'\) are compact subsets of \(\mathbb{H}\) such that \(\mathbb{H}\setminus K\) and \(\mathbb{H}\setminus K'\) are conformally equivalent, then
\[
P_{\mathbb{H},x\to \infty}\{\gamma \cap K = \emptyset\} = P_{\mathbb{H},\Phi(x)\to \infty}\{\gamma \cap K' = \emptyset\} [\Phi'(x)\Phi'(\infty)]^{5/8},
\]
where \(\Phi\) is a conformal map from \(\mathbb{H}\setminus K\) onto \(\mathbb{H}\setminus K'\) with \(\Phi(\infty) = \infty\) and \(\Phi'(\infty) = \lim_{z\to \infty} 1/\Phi'(z)\).
Theorem 4.2. If $F(A, x)$ denotes the probability that chordal SLE$_{8/3}$ in the halfstrip $HS$ from $x$ to $0$ stays in the rectangle $R_A$, then

$$F(a, h_{A,a}(X_a)) \left[ \frac{\sin^2(X_a/2)}{\sin^2(h_{A,a}(X_a)/2)} h'_{A,a}(X_a)h'_{A,a}(0) \right]^{5/8}$$

is a martingale on $[A, A^*]$.

Proof. It follows from (30) that $P_{\mathbb{H}, W_s \to \infty} \{ \gamma \cap g_s(C_A) = \emptyset \}$ is a martingale on $s < T_A$. Since

$$u \circ h_s \circ u^{-1}(g_s(C_A)) = C_a, \quad a = a(s),$$

it follows from Lemma 4.1 that

$$P_{\mathbb{H}, W_s \to \infty} \{ \gamma \cap g_s(C_A) = \emptyset \}$$

is a martingale on $s < T_A$. Since

$$P_{\mathbb{H}, W_s \to \infty} \{ \gamma \cap g_s(C_A) = \emptyset \} = 0,$$

we have

$$P_{\mathbb{H}, h_s \circ u^{-1}(W_s) \to \infty} \{ \gamma \cap C_a = \emptyset \} = P_{\mathbb{H}, h_s \circ u^{-1}(W_s) \to \infty} \{ \gamma \cap a \cap C_a = \emptyset \} \times [((u \circ h_s \circ u^{-1})(W_s))(u \circ h_s \circ u^{-1})(\infty)]^{5/8}.$$

Next,

$$(u \circ h \circ u^{-1})(w) = h'(u^{-1}(w)) \frac{\sin^2(u^{-1}(w)/2)}{\sin^2(h(u^{-1}(w))/2)}.$$

If $z = u^{-1}(w)$, then $w \to \infty$ implies $z \to 0$. As $\lim_{z \to 0} h(z) = 0$, we have

$$\lim_{z \to 0} \frac{\sin^2 z/2}{\sin^2(h(z)/2)} = \lim_{z \to 0} \left[ \frac{\sin z/2}{z} \cdot \frac{z}{h(z)} \cdot \frac{h(z)}{\sin(h(z)/2)} \right]^2 = h'(0)^{-2}.$$

Since $F(A, x) = P_{HS, X_a \to \infty} \{ \gamma \cap R_A \}$, the theorem now follows. \qed

The martingale in this theorem is a functional of the Markov process $X_a$ and the non-Markov process $h_{A,a}(X_a)$. Under an appropriate change of measure $h_{A,a}(X_a)$ becomes a Markov process $Y$. This change of measure also introduces a drift to the process in Theorem 4.2 and we have to multiply by a factor given by Girsanov’s formula to obtain a martingale under this new measure. The new martingale turns out to be a function of $Y$ times an exponential functional of $Y$. Our reason to change measure is that we are able to obtain the asymptotics of this new martingale in Theorem 5.6, while it was not clear to us how to carry out this step for the original martingale in Theorem 4.2.

To change $h_{A,a}(X_a)$ into a Markov process we will first remove the two drift terms in its Itô decomposition, see (21). We carry this out in two steps to better see how the constituent parts fit together. Finally, we perform a third change of measure, which transforms $Y$ from a multiple of a linear Brownian motion to a Bessel-type process on the interval $[0, 2\pi]$, a 0-dimensional Legendre process. This last step is
natural since it takes the geometry of our setup (i.e. the circle) into account, and, more importantly, leads to a multiplicative stochastic functional in the martingale replacing the martingale from Theorem 4.2 whose exponent is an integral with non-singular integrand.

**Proposition 4.3.** If $\kappa = 8/3$, $A < 0$, and

$$M_{A,a} = \left[ h'_{A,a}(X_a) \frac{\sin^2(X_a/2)}{\sin^2(X_A/2)} \exp \left( \int_A^a \frac{2\eta}{\pi} \, db \right) \right]^{5/8}, \quad A \leq a < A^*,$$

then $M$ is a martingale with $M_{A,A} = 1$ and

$$dM = -\frac{5}{8} \sqrt{8/3} M \left[ \frac{\cot(X/2)}{h'(X)} + \frac{h''(X)}{h'(X)^2} \right] dB.$$

**Proof.** We have

$$\frac{d \left[ h'(X) \sin^2(X/2) \right]^\alpha}{\alpha \left[ h'(X) \sin^2(X/2) \right]^\alpha} = -\sqrt{\kappa} \left[ \frac{\cot(X/2)}{h'(X)} + \frac{h''(X)}{h'(X)^2} \right] dB$$

$$- \frac{2\eta}{\pi} da + \frac{1}{2} \left( \frac{\kappa - 8/3}{4h'(X)^2} - \frac{h'''(X)}{2h'(X)^3} \right) da$$

$$+ \frac{\kappa(1 + 2\alpha) - 6}{2} \left[ \frac{\cot^2(X/2)}{2h'(X)^2} + \cot(X/2) \frac{h''(X)}{h'(X)^3} \right] da.$$

(31)

If $\kappa = 8/3$ and $\alpha = 5/8$ then all drift terms except for the first vanish. Since $M$ is also bounded for $a < 0$ the proposition follows. □

**Remark 4.4.** If $\kappa > 0$ is arbitrary and $\alpha = (6 - \kappa)/2\kappa$, then the drift term of $d \left[ h'(X) \sin^2(X/2) \right]^\alpha / \alpha \left[ h'(X) \sin^2(X/2) \right]^\alpha$ reduces to

$$-2\eta/\pi \, da + (\kappa - 8/3)[Sh(X) - 1/2]/2h'(X)^2 \, da,$$

where $Sh = h'''/h' - (3/2)(h''/h')^2$ is the Schwarzian derivative of $h$.

Denote $P$ the law of the underlying Brownian motion $B$, and denote $F_a$ the associated filtration after the time-change $t \to a$. Define the probability measure $Q$ by

$$\frac{dQ}{dP} |_{F_a} = M_{A,a}.$$
Corollary 4.5. Under the measure $Q$,

$$F(a, h_{A,a}(X_a)) \left[ \frac{h'_{A,a}(0)}{\sin^2(h_{A,a}(X_a)/2)} \exp \left( -\int_A^a \frac{2\eta}{\pi} \, db \right) \right]^{5/8}$$

is a martingale and $Y_{A,a} \equiv h_{A,a}(X_a)$ satisfies

$$dY = -\sqrt{8/3} \, dB + 2 \left[ \zeta(Y) - \eta \pi Y \right] \, da.$$ 

Proof. The two statements follow from Girsanov’s theorem, in conjunction with Theorem 4.2, Proposition 4.3, and (24). □

Let $\theta(x|a) = \vartheta_1(x/2\pi|2\pi)$, where $\vartheta_1$ is Jacobi’s theta function

$$\vartheta(x|a) = \vartheta_1(x/2\pi) = -i \sum_{n=-\infty}^{\infty} e^{i(x + n/2 + a(n+1/2)^2 + i\pi n)}.$$

Then

$$\frac{\partial}{\partial a} \vartheta(x|a) = -\frac{\partial^2}{\partial x^2} \vartheta(x|a),$$

and

$$\frac{\partial}{\partial x} \ln \vartheta(x|a) = \zeta(x) - \frac{\eta}{\pi} x, \quad \frac{\partial^2}{\partial x^2} \ln \vartheta(x|a) = -\wp(x) - \frac{\eta}{\pi},$$

see [12].

We note that if $A^* < 0$, then $A^*$ is the first time that $Y$, starting at $Y_A$ at time $A < 0$, hits $\{0, 2\pi\}$. If $A^* = 0$, then $Y$ does not hit $\{0, 2\pi\}$.

Proposition 4.6. If $\kappa = 8/3$, $A < 0$, and

$$N_{A,a} = \left[ \vartheta_1 \left( \frac{Y_{A,a}}{2\pi} \right) / \vartheta_1 \left( \frac{Y_{A,A}}{2\pi} \right) \right]^{-3/4} h_{A,a}^{-1/8}, \quad A \leq a < A^*,$$

then, under $Q$, $N$ is a martingale with $N_{A,A} = 1$ and

$$dN = \frac{3}{4} \sqrt{8/3N} \left[ \zeta(Y) - \eta \frac{Y}{\pi} \right] dB.$$ 

Proof. Denoting differentiation with respect to the spatial variable by $a'$ and using (32), we have

$$\frac{d}{\beta a'} \left[ \frac{\vartheta(Y/2\pi)}{\vartheta(Y/2\pi)^{\beta}} \right] = -\sqrt{\kappa \theta} \, dB$$

$$+ \left[ \frac{(\kappa/2 - 1)}{\theta} \frac{\theta''}{\theta} + \left( 2 + \frac{\kappa}{2} (\beta - 1) \right) \left( \frac{\theta'}{\theta} \right)^2 \right] da.$$ 

(34)
The term in brackets can be rewritten as
\[
\left(1 + \frac{\beta\kappa}{2}\right) \frac{\theta''}{\theta} \, da + \left[\left(1 - \beta\right)\frac{\kappa}{2} - 2\right] \left(\ln \theta\right)'' \, da.
\]
Thus for \(\kappa = 8/3\), \(\beta = -3/4\),
\[
d \left[\eta_1(Y/2\pi)^{-3/4}\right] = \frac{3}{4} \sqrt{\frac{8}{3}} \theta_1(Y/2\pi)^{-3/4} \frac{\theta'}{\theta} \, dB
\]
\[\quad - rac{1}{2} \theta_1(Y/2\pi)^{-3/4} \left(\ln \theta\right)'' \, da.
\]
(35)
The proposition now follows from (33) and (28).

\[\square\]

Define the probability measure \(R\) for \(a < A^*\) by
\[
\frac{dR}{dQ}\big|_{\cal F_a} = N_{A,a}.
\]

**Proposition 4.7.** If \(Y_{A,a} = h_{A,a}(X_a)\), then under the measure \(R\),
\[
F(a, Y_{A,a}) \frac{\partial_1(Y_{A,a}/2\pi)^{3/4}}{\sin^{5/4}(Y_{A,a}/2)} \exp \left(- \int_A^a \left[\varphi(Y_{A,b}) + \frac{9\eta}{4\pi} \right] \, db \right)
\]
is a martingale for \(a < A^*\) and \(Y_{A,a}\) satisfies
\[
dY = -\sqrt{\frac{8}{3}} \, dB.
\]

**Proof.** This is again a consequence of Gisanov’s theorem. \[\square\]

Finally, let
\[
\tilde{N}_{A,a} = \frac{\sin^{-1/2}(Y_{A,a}/2)}{\sin^{-1/2}(Y_{A,A}/2)} \exp \left[-\frac{1}{4} \int_A^a \csc^2(Y_{A,b}/2) \, db \right].
\]
It is an easy calculation that—under the measure \(R\)—\(\tilde{N}_{A,a}\) is a martingale on \(a < A^*\). If we define the measure \(\bar{R}\) by \(d\bar{R}/dR|_{\cal F_a} = \tilde{N}_{A,a}\), then we have the following

**Proposition 4.8.** Under the measure \(\bar{R}\) the process \(Y_{A,a}\) satisfies
\[
dY = -\sqrt{\frac{8}{3}} \, dB - 2/3 \cot Y/2 \, da,
\]
and
\[
\mathcal{M}_{A,a} \equiv F(a, Y_{A,a}) \exp \left[- \int_A^a \left(\varphi(Y_{A,b}) - \frac{1}{4} \csc^2(Y_{A,b}/2) \right) \, db \right]
\]
\[
\times \left[\prod_{n=1}^{\infty} \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n} \cos Y_{A,A} + Q^{4n}} \right]^{3/4}
\]
(36)
is a martingale for \(a < A^*\) with \(\mathcal{M}_{A,A} = F(A, Y_{A,A})\).
Proof. It follows from the infinite product representation of \( \vartheta_1 \), see [12], that

\[
\vartheta_1 \left( \frac{y}{2\pi} \right) \frac{\sin(y/2)}{\sin(y/2)} = q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos y + q^{4n}).
\]

Also,

\[
\exp \left[ \sum_{n=1}^{\infty} \int_{A}^{a} \frac{2nq^{2n}}{1 - q^{2n}} \, db \right] = \prod_{n=1}^{\infty} \frac{1 - Q^{2n}}{1 - q^{2n}},
\]

and

\[
\frac{1 - 2q^{2n} \cos x + q^{4n}}{1 - 2Q^{2n} \cos y + Q^{4n}} = \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n} \cos y + Q^{4n}} \cdot \frac{1 - 2q^{2n} \cos x + q^{4n}}{1 - 2q^{2n} + q^{4n}} \left( \frac{1 - q^{2n}}{1 - Q^{2n}} \right)^2.
\]

Now Girsanov’s theorem, Proposition [14], and the explicit expression for \( \varphi \) show that \( \mathcal{M} \) is a martingale.

\[\square\]

**Corollary 4.9.** For any \( A < a < 0 \), \( x \in [0, 2\pi] \),

\[
F(A, x) = \left( \prod_{n=1}^{\infty} \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n} \cos x + Q^{4n}} \right)^{3/4} \times \mathbb{E}[F(a, Y_{A,a}) \left( \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos Y_{A,a} + Q^{4n}}{1 - 2Q^{2n} + Q^{4n}} \right)^{3/4} \times \exp(- \int_{A}^{a} \left( \varphi(Y) - \frac{1}{4} \csc^2 Y/2 \right) \, db)]
\]

(38) (where \( Y_{A,A} = x \)), and \( F(a, x) \) is \( C^{1,2} \) as a function of \( a \) and \( x \).

**Proof.** First, (38) is a consequence of (36) and the optional sampling theorem. Next, \( x \mapsto F(a, x) \) is continuous because the chordal Loewner equation is continuous as a map from the space of continuous paths with the topology of uniform convergence on compacts (the input) to the space of conformal maps with the Carathéodory metric (as output). See [3], for a discussion. It then follows from the Feynman-Kac formula that the right-hand side of equation (38) is \( C^{1,2} \), see [14].

\[\square\]

5. **ASYMPTOTIC BEHAVIOR OF THE NON-INTERSECTION PROBABILITY**

The stochastic representation of the non-intersection probability

\[
(a, x) \in [\infty, 0] \times [0, 2\pi] \mapsto F(a, x) \equiv P_{U, \epsilon \tau \rightarrow 1}(\gamma \subset A_\epsilon)
\]
we obtain in this section rests on the asymptotics of \( F(a, x) \) as \( a \to 0 \).
In particular, this probability decays fast enough to control the limiting behavior of the martingale \( \mathcal{M} \) from Proposition 4.8.

For each \( q \in [0, 1) \) there exists a unique \( L = L(q) \in [0, 1) \) such that \( A_q \) and \( \mathbb{U} \setminus [-L, L] \) are conformally equivalent. As \( q \) increases to 1, \( L \) increases to 1 as well. Denote \( f \) the conformal equivalence, normalized by \( f(1) = 1 \). For \( x \in (0, \pi] \), let \( z_1 = e^{ix/2}, z_2 = e^{-ix/2} \). By symmetry, if \( w_{1,2} = f(z_{1,2}) \), then \( w_2 = \bar{w}_1 \).

In what follows we will mean by \( h(a) \asymp g(a) \) as \( a \to 0 \), that

\[
\lim_{a \to 0} \log h(a)/ \log g(a) = 1.
\]

**Lemma 5.1.** For \( x \in (0, \pi] \), we have

\[
1 - L \asymp e^\frac{x^2}{\pi}, \text{ and } |f'(z_1)| \asymp |1 - f(z_1)| \asymp e^\frac{x}{\pi}(\pi - x)
\]
as \( a \to 0 \).

**Proof.** From [22] Chap. VI, Sec. 3,

\[
f(z) = L \frac{2iK}{\pi} \log \frac{z}{q} + K; q^4
\]
where \( \text{sn}(z) \) is the analytic function for which \( \text{sn}'(0) = 1 \) and which maps the rectangle \( \{ z : -K < \Re z < K, 0 < \Im z < K' \} \) onto the upper half-plane in such a way that \( \text{sn}(\pm K) = \pm 1 \) and \( \text{sn}(\pm K' + iK') = \pm k^{-1} \). Furthermore, \( q^4 = \exp(-\pi K'/K) \), and \( L = \sqrt{k} \). It is classical that \( \text{sn}'(z) = [(1 - \text{sn}^2(z))(1 - k^2 \text{sn}^2(z))]^{1/2} \). Thus

(39) \[
f'(z) = (2iK/\pi z)(L^2 - f^2(z))(1 - L^2 f^2(z))^{1/2}.
\]

Define \( h, \tau \) by \( q^4 = h = e^{i\pi \tau} \), and set \( v = \frac{1}{\pi} \log \frac{z}{q} + \frac{1}{2} \). Then it follows from [12] II, 3., and using that texts notation, that

\[
L = \frac{\theta_2(0) \tau}{\theta_3(0) \tau}, \text{ and } f(z) = \frac{\theta_1(v \tau)}{\theta_0(v \tau)}.
\]

Using linear transformations of theta functions we may write

\[
\frac{\theta_2(0) \tau}{\theta_3(0) \tau} = \frac{\theta_0(0) - \frac{1}{\tau}}{\theta_3(0) - \frac{1}{\tau}}, \text{ and } \frac{\theta_1(v \tau)}{\theta_0(v \tau)} = \frac{\theta_1(\frac{v}{\tau} - \frac{1}{\tau})}{\theta_3(\frac{v}{\tau} - \frac{1}{\tau})}.
\]

Hence, if \( h' = \exp(-i\pi/\tau) \), and using the series representation of \( \theta_0 \) and \( \theta_3 \), we get

\[
L = \frac{1 + 2 \sum_{n=1}^{\infty}(-1)^n h'^n}{1 + 2 \sum_{n=1}^{\infty} h'^n} = 1 - 4h' + O((h')^2),
\]
which is the first statement of the lemma. For the second, we use the infinite product representation of \( \theta_1 \) and \( \theta_2 \), giving

\[
\frac{\theta_1(z) - \frac{1}{2}}{\theta_2(z) - \frac{1}{2}} = \frac{e^{2i\pi/\tau} - 1}{e^{2i\pi/\tau} + 1} \prod_{n=1}^{\infty} \frac{(1 - (h')^{2n} e^{2i\pi/\tau})(1 - (h')^{2n} e^{-2i\pi/\tau})}{(1 + (h')^{2n} e^{2i\pi/\tau})(1 + (h')^{2n} e^{-2i\pi/\tau})}.
\]

Since \( \exp(2i\pi/\tau) = i\exp(-4\pi a(\pi - x)) \), the infinite product is \( 1 + O(\exp(\pi^2/4a)) \), and

\[
\frac{e^{2i\pi/\tau} - 1}{e^{2i\pi/\tau} + 1} = 1 + 2ie^{\pi a(\pi - x)} + O(e^{\pi^2/4a}),
\]

as \( a \not\to 0 \). Using equation (39), the lemma now follows. \( \square \)

Recall that \( z_1 = e^{ix/2} \), \( w_1 = f(z_1) \), and set \( u = i(1 + w_1)/(1 - w_1) \). The following result can be derived from Lemma 5.2. However, we will give a direct argument.

**Lemma 5.2.** The probability \( P_{U, e^{ix}}(\gamma \subset A_q) \) is equal to

\[
P_{H, u\to u} \left( \gamma \cap I \left[ \frac{1 - L}{1 + L}, \frac{1 - L}{1 - L} \right] = \emptyset \right) \left| \frac{f'(z_1)(1 - z_1)}{1 - f'(z_1)} \right|^{5/4}.
\]

**Proof.** Denote \( B \) a simple curve connecting the inner and outer boundary of \( A_q \), so that \( B \) is bounded away from \( z_1 \) and \( z_2 \). Denote \( \phi \) a conformal map from \( A_q \) onto \( U \) such that \( \phi(z_{1,2}) = z_{1,2} \), and \( \psi \) a conformal map from \( f(A_q \setminus B) \) onto \( U \) such that \( \psi(w_{1,2}) = w_{1,2} \). Then, by conformal restriction,

\[
P_{U, z_1 \to z_2} (\gamma \subset A_q \setminus B) = |\phi'(z_1)|/|\phi'(z_2)|^{5/8},
\]

(40) \[
P_{U, w_1 \to w_2} (\gamma \subset f(A_q \setminus B)) = |\psi'(w_2)|/|\psi'(w_2)|^{5/8}.
\]

Since \( T = \phi \circ f \circ \psi^{-1} \) maps \( U \) onto \( U \) and sends \( w_{1,2} \) to \( z_{1,2} \), there is a pair \( w_0, z_0 \in \partial U \) such that \( T \) is the linear transformation given by

\[
\frac{T(w) - w_1}{T(w) - w_2} = \frac{w_0 - w_2}{w_0 - w_1} = \frac{z - z_1}{z - z_2} = \frac{z_0 - z_2}{z_0 - z_1}.
\]

A calculation gives

\[
T'(w_1)T'(w_2) = \left( \frac{z_1 - z_2}{w_1 - w_2} \right)^2,
\]

which together with \( |f'(z_1)| = |f'(z_2)| \) implies

(41) \[
P_{U, z_1 \to z_2} (\gamma \subset A_q \setminus B) = P_{U, w_1 \to w_2} (\gamma \subset f(A_q \setminus B)) \left| \frac{f'(z_1)(z_1 - z_2)}{w_1 - w_2} \right|^{5/4}.
\]
By an inclusion/exclusion argument, equation (41) also holds if $A_q \setminus B$ is replaced by $A_q$. Finally, by conformal invariance,

$$P_{\mathbb{H},u \to -u}(\gamma \cap i \left[ \frac{1-L}{1+L}, \frac{1+L}{1-L} \right] = \emptyset).$$

Note that because $x \in (0, \pi]$ we have $\arg z_1, \arg w_1 \in (0, \pi/2]$ and so $u \leq -1$. We will use the following lower and upper bounds:

$$P_{\mathbb{H},u \to -u}(\gamma \cap i(0, \frac{1+L}{1-L}] = \emptyset) + P_{\mathbb{H},u \to -u}(\gamma \cap i[\frac{1-L}{1+L}, \infty) = \emptyset)$$

and

$$P_{\mathbb{H},u \to -u}(\gamma \cap i[\frac{1-L}{1+L}, \infty) = \emptyset)$$

(42)

For $c \in \mathbb{R}, d > 0$, set

$$g_{c,d}(z) = \frac{|c|}{\sqrt{c^2 + d^2}} \sqrt{z^2 + d^2}.$$ 

Then $g_{c,d}$ maps $\mathbb{H} \setminus i(0, d]$ conformally onto $\mathbb{H}$ such that $g_{c,d}(\pm c) = \pm c$. Furthermore,

$$|g'_{c,d}(c)g'_{c,d}(-c)| = \frac{c^4}{(c^2 + d^2)^2},$$

and so by conformal restriction

$$P_{\mathbb{H},c \to -c}(\gamma \cap i[0, d] = \emptyset) = [c^2/(c^2 + d^2)]^{5/4}.$$

**Corollary 5.3.** We have

$$P_{\mathbb{H},u \to -u}(\gamma \cap i(0, \frac{1+L}{1-L}] = \emptyset) + P_{\mathbb{H},u \to -\frac{1}{u}}(\gamma \cap i(0, \frac{1+L}{1-L}] = \emptyset) \leq e^{5\pi a}$$

as $a \to 0$. 


Proof. By (44),
\[ P_{\mathbb{H},u \rightarrow -u} (\gamma \cap i(0, \frac{1 + L}{1 - L}) = \emptyset) = \left( \frac{u(1 - L)}{1 + L} \right)^{5/2} \left( 1 + \frac{(u(1 - L))^2}{(1 + L)^2} \right)^{-5/4}, \]
and from Lemma 5.1
\[ \left( \frac{u(1 - L)}{1 + L} \right)^{5/2} \left( 1 + \frac{(u(1 - L))^2}{(1 + L)^2} \right)^{-5/4} \asymp e^{\frac{5a}{aL}}. \]
Similarly,
\[ P_{\mathbb{H},\frac{1}{u} \rightarrow -\frac{1}{u}} (\gamma \cap i(0, \frac{1 + L}{1 - L}) = \emptyset) \asymp e^{\frac{5a^2}{a} + \frac{5\pi}{a}(\pi - x)}, \]
so that this term is negligible compared to the first if \(0 < x < \pi\), and of the same order if \(x = \pi\). \(\square\)

Lemma 5.4. We have
\[ P_{\mathbb{H},u \rightarrow -u} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}, \infty) \neq \emptyset \right) \asymp e^{2\pi/a}, \]
as a \(a \rightarrow 0\).

Proof. First,
\[ P_{\mathbb{H},u \rightarrow -u} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}, \infty) \neq \emptyset \right) \]
\[ = P_{\mathbb{H},u \rightarrow -u} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}) \neq \emptyset \right) + P_{\mathbb{H},\frac{1}{u} \rightarrow -\frac{1}{u}} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}) \neq \emptyset \right) \]
\[ - P_{\mathbb{H},u \rightarrow -u} \left( \gamma \cap i((0, \frac{1 - L}{1 + L}) \cup (\frac{1 + L}{1 - L}, \infty)) \neq \emptyset \right). \]
The last probability on the right equals
\[ P_{U, w_1 \rightarrow w_2} (\gamma \cap ((-1, -L] \cup [L, 1)) \neq \emptyset). \]
To calculate this probability, note that
\[ g_L(w) \equiv \frac{1 + w^2 - \sqrt{(1 + w^2)^2 - 4p^2w^2}}{2pw} \]
maps \(U \setminus ((-1, -L] \cup [L, 1))\) onto \(U\) if \(2p = (L + 1/L)\), see [13, Chapter 3]. Here, the square root is chosen so that \(g_L(i) = i\). Setting \(w = e^{i\varphi}\), this can be written
\[ g_L(w) = \begin{cases} \frac{1}{p} \cos \varphi + i \sqrt{1 - \frac{1}{p^2} \cos^2 \varphi}, & \text{if } \varphi \in (0, \pi/2]; \\ \frac{1}{p} \cos \varphi - i \sqrt{1 - \frac{1}{p^2} \cos^2 \varphi}, & \text{if } \varphi \in [-\pi/2, 0). \end{cases} \]
Then
\[ g'_L(w)g'_L(\bar{w}) = -\frac{\sin^2 \varphi}{p^2 - 1 + \sin^2 \varphi}. \]

Denote \( T \) a (fractional) linear transformation from \( \mathbb{U} \) onto \( \mathbb{U} \) sending \( g_L(w_{1,2}) \) onto \( w_{1,2} \). Then, as in the proof of Lemma 5.2,
\[ T'(g_L(w_1))T'(g_L(w_2)) = \frac{\sin^2 \varphi}{1 - \frac{1}{p^2} \cos^2 \varphi}, \]
where now \( \varphi = \arg w_1 \). Thus, by conformal restriction,
\[ (47) \quad P_{\mathbb{U}, w_1 \to w_2} (\gamma \cap ((-1, -L] \cup [L, 1)) \neq \emptyset) = 1 - \left[ \frac{p \sin^2 \varphi}{p^2 - 1 + \sin^2 \varphi} \right]^{5/4}. \]

Finally, from the definition of \( u \) and \( \varphi \) in terms of \( w_1 \), it follows that \( u = -\cot(\varphi/2) \) and so \( 4/\sin^2 \varphi = (u + 1/u)^2 \). A calculation now gives
\[ \frac{p^2 - 1 + \sin^2 \varphi}{\sin^2 \varphi} \]
\[ = 1 + \left( \frac{1 - L}{1 + L} \right)^2 \left( u^2 + \frac{1}{u^2} \right) + \frac{(1 - L)^4}{8(L + L^3)} \left[ 2 + \left( \frac{1 - L}{1 + L} \right)^2 \left( u^2 + \frac{1}{u^2} \right) \right]. \]

On the other hand, (44) implies
\[ (49) \quad P_{h, u \to -u} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}) \neq \emptyset \right) = 1 - \left( 1 + \left( \frac{1 - L}{1 + L} \right)^2 \frac{1}{u^2} \right)^{-5/4}. \]

and
\[ (50) \quad P_{h, -\frac{1}{u} \to \frac{1}{u}} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}) \neq \emptyset \right) = 1 - \left( 1 + \left( \frac{1 - L}{1 + L} \right)^2 u^2 \right)^{-5/4}. \]

Combining (49), (50), (47), and (45), we get
\[ P_{h, u \to -u} \left( \gamma \cap i(0, \frac{1 - L}{1 + L}) \neq \emptyset, \gamma \cap i(\frac{1 + L}{1 - L}, \infty) \neq \emptyset \right) \]
\[ = 1 - \left( 1 + \left( \frac{1 - L}{1 + L} \right)^2 \frac{1}{u^2} \right)^{-5/4} + 1 - \left( 1 + \left( \frac{1 - L}{1 + L} \right)^2 u^2 \right)^{-5/4} \]
\[ - 1 + \left( \frac{p^2 - 1 + \sin^2 \varphi}{p \sin^2 \varphi} \right)^{-5/4}. \]
Using (48), straightforward expansion of the right hand side of (51) shows it to be equal to

\[
\frac{5}{256}(1 - L)^4 + \frac{5}{128}(1 - L)^5 + (1 - L)^4 O(u^2(1 - L)^2). 
\]

□

From the upper and lower bounds (43), (42), Corollary 5.3 and Lemma 5.4 we get

**Theorem 5.5.** For every \( x \in (0, \pi] \) we have

\[
F(a, x) \asymp \exp \left( \frac{5\pi x}{8a} \right)
\]

as \( a \nearrow 0 \).

We now combine the previous result and Proposition 4.8 to obtain a stochastic representation of \( F(A, x) \).

**Theorem 5.6.** Under the measure \( \tilde{\mathcal{R}} \) we have

\[
\sup_{a < A^*} |\mathcal{M}_{A,a}| < \infty.
\]

Furthermore, if \( A^* = 0 \), then \( \lim_{a \nearrow A^*} \mathcal{M}_{A,a} = 0 \), while if \( A^* < 0 \) and \( Q^* = e^{A^*} \), then

\[
\lim_{a \nearrow A^*} \mathcal{M}_{A,a} = \left[ \prod_{n=1}^{\infty} \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n}\cos Y_{A,a} + Q^{4n}} \right]^{3/4} \times \exp \left[ - \int_{A}^{A^*} \left( \varphi(Y_{A,b}) - \frac{1}{4} \csc^2(Y_{A,b}/2) \right) db \right].
\]

(53)

Finally, if \( x = Y_{A,A} \), then

\[
F(A, x) = \left[ \prod_{n=1}^{\infty} \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n}\cos x + Q^{4n}} \right]^{3/4} \times \mathbb{E}^{\tilde{\mathcal{R}}} \left[ \exp \left[ - \int_{A}^{A^*} \left( \varphi(Y_{A,b}) - \frac{1}{4} \csc^2(Y_{A,b}/2) \right) db \right], A^* < 0 \right].
\]

(54)

**Proof.** That \( \mathcal{M} \) is a bounded martingale follows from the limiting behavior as \( a \nearrow A^* \), which we now establish. If \( A^* < 0 \), then \( Y_{A,A^*} = 0 \) and (53) follows directly from (36). On the other hand, if \( A^* = 0 \), then \( Y_{A,A^*} \neq 0 \) a.s. and it follows from Theorem 5.5 that \( F(a, Y_{A,a}) \) decays...
like $\exp(-cx/(1 - q))$ with $c = 5\pi/8$, and $x = \min\{Y_{A,A}, 2\pi - Y_{A,A}\}$.

We will now show that
\[
\prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos x + q^{4n}}{1 - 2q^{2n} + q^{4n}}^{3/4} \leq \exp\left[ \frac{1}{1 - q} \left( \frac{\pi^2}{8} - \frac{3}{8}[Li_2(e^{ix}) + Li_2(e^{-ix})]\right) \right],
\]
(55)

where $Li_2$ denotes the dilogarithm. Set $x_n = 1 - q^{2n}$, $n \geq 0$. Then
\[
x_n - x_{n-1} = (1 - x_n)(1 - q^2)/q^2,
\]
and by simple integral comparison,
\[
\left. \sum_{n=1}^{\infty} \ln(1 - q^{2n}) \right| = \frac{q^2}{1 - q^2} \sum_{n=1}^{\infty} \frac{\ln x_n}{x_n - x_n} (x_n - x_{n-1}) \\
\geq \frac{q^2}{1 - q^2} \int_0^1 \ln x \frac{dx}{1 - x} = -\frac{1}{1 - q} \cdot \frac{\pi^2 q^2}{6(1 + q)}.
\]

Thus
\[
-\frac{3}{2} \sum_{n=1}^{\infty} \ln(1 - q^{2n}) \leq \frac{1}{1 - q} \cdot \frac{\pi^2}{8}.
\]
(56)

Similarly, if we set $y_n = -q^{2n}$, $n \geq 0$, then $y_n - y_{n-1} = -y_n(1 - q^2)/q^2$ for $n \geq 1$, and so
\[
\left. \sum_{n=1}^{\infty} \ln(1 - 2q^{2n} \cos x + q^{4n}) \right| \\
= -\frac{q^2}{1 - q^2} \sum_{n=1}^{\infty} \frac{\ln(1 + 2y_n \cos x + y^2_n)}{y_n} (y_n - y_{n-1}) \\
\leq \frac{q^2}{1 - q^2} \int_0^1 \ln(1 - 2y \cos x + y^2) \frac{dy}{y} \\
= \frac{q^2}{1 - q^2} (-Li_2(e^{ix}) - Li_2(e^{-ix})).
\]
(57)

Now, (56) and (57) imply (55). It is elementary that
\[
5\pi x \geq \pi^2 - 3[Li_2(e^{ix}) + Li_2(e^{-ix})]
\]
for $x \in [0, \pi]$, with equality holding for $x = 0$. Thus $\mathcal{M}$ is a bounded martingale and (54) follows from the optional sampling theorem. □

**Remark 5.7.** Under $\tilde{R}$, $Y$ is a Legendre process whose boundary behavior is that of a 0-dimensional Bessel process, i.e. 0 and $2\pi$ are
absorbing, see [23]. It can also be interpreted as the driving function of a radial SLE($\kappa, \rho$). By (37),
\[
\prod_{n=1}^{\infty} \frac{1 - 2Q^{2n} + Q^{4n}}{1 - 2Q^{2n} \cos x + Q^{4n}}
\]
is the quotient of $y \mapsto \vartheta_1(y/2\pi)/\sin(y/2)$ evaluated at $y = 0$ and at $y = x$. Also,
\[
\exp \left[ - \int_{A}^{A^*} \left( \varphi(Y_{A,a}) - \frac{1}{4} \csc^2(Y_{A,a}/2) \right) da \right]
\]

\[
= \left( \frac{Q^*}{Q} \right)^{1/12} \exp \left[ - \int_{A}^{A^*} \frac{2nq^{2n}}{1 - q^{2n}} (1 - \cos nY_{A,a}) da \right].
\]

Remark 5.8. Obviously, $F(a, x) = \mathbb{E}^P[1, A^* < 0]$, where $P$ is the original SLE-measure under which $Y = h(X)$ is the non-Markov process satisfying equation [24]. Thus the price we incur for switching to a Markov process representation is an exponential functional. We note that this exponential functional can be given an interpretation using the Brownian loop soup.

6. The Partial differential equation

It follows from Corollary 4.9 that $F(a, x)$ is smooth enough in $(a, x)$ to apply Itô’s formula, and we have

**Theorem 6.1.** If $G(a, x) = F(a, x)\vartheta_1(x/2\pi)^{3/4} \sin^{-5/4}(x/2)$, then
\[
-\partial_a G = \frac{4}{3} G'' - \left( \varphi(x) + \frac{9\eta}{4\pi} \right) G.
\]

Furthermore, $F(a, x)$ is the unique solution to the evolution equation
\[
-\partial_a F = \frac{4}{3} F'' + \left[ 2\zeta(x) - \frac{2n}{\pi} x - \frac{5}{3} \cot \frac{x}{2} \right] F'
\]
\[
+ \left[ \frac{15}{16} \csc^2 \frac{x}{2} - \frac{5}{4} \left( \cot \frac{x}{2} \left[ \zeta(x) - \frac{\eta}{\pi} x \right] + \varphi(x) + \frac{2n}{\pi} + \frac{5}{12} \right) \right] F,
\]
for $(a, x) \in (-\infty, 0) \times (0, 2\pi)$, and with initial condition
\[
\lim_{a \to -\infty} F(a, x) = 1,
\]
and boundary condition
\[
F(a, 0) = F(a, 2\pi) = 1.
\]
Finally, the solution $F$ is symmetric, $F(a, x) = F(a, 2\pi - x)$. 

Proof. The partial differential equation for $G$ is a consequence of Theorem 4.7 and Itô's lemma. The equation for $F$ follows from the equation for $G$. Finally, that $F(a, 0) = 1$ is clear and it is also known, for example by considering the Hausdorff dimension of the SLE$_{8/3}$ curve, that $\lim_{a \to -\infty} F(a, x) = 1$.

We now briefly discuss the case $q \searrow 0$. As we could not find stronger convergence results for PDEs such as (60) in the literature we can only establish the rate in a weak sense, see Remark 6.4.

Using the formulas for $\zeta$, $\eta$, and $\wp$, we can write (60) as

$$-\partial_a F = \frac{4}{3} F'' + \left[ -\frac{2}{3} \cot(x/2) + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin nx \right] F' + \frac{5}{2} \sum_{n=2}^{\infty} \frac{q^{2n}}{1-q^{2n}} \left[ n(1 + \cos nx) - \cot(x/2) \sin nx \right] \cdot F.$$  

(61)

In particular, the coefficient of the zeroth-order term is nonsingular in $x$ and vanishes for $x = 0$. We note also that the summation in the zeroth-order term begins with $n = 2$ because $(1 + \cos x)/\sin x = \cot x/2$.

To guess the behavior of $F$ as $q \searrow 0$ we consider the PDE obtained by setting $q = 0$ in (61),

$$-\partial_a H = \frac{4}{3} H'' - \frac{2}{3} \cot x/2 \cdot H'.$$

(62)

Then (61) is a perturbation of (62) if $q$ is small. If we replace $H$ by $1 - H$, then $1 - H$ satisfies the same equation. We consider the mixed initial-boundary value problem for (62) where

$$\lim_{a \to -\infty} H(a, x) = 0, \text{ for } x \in (0, 2\pi), \text{ and } H(a, 0) = 0, \text{ for } a \in (-\infty, 0).$$

(63)

The solution should describe the asymptotic behavior of $P_{H, \cot x \to \infty} \{ \gamma \cap C_a \neq \emptyset \}$ as $a \to -\infty$.

Proposition 6.2. The solutions to the mixed initial-boundary value problem (62), (63) are given by

$$H(a, x) = cq^{2/3} \sin^2 x/2,$$

for an arbitrary positive constant $c$.

Proof. This follows easily from separation of variables.

Remark 6.3. The exponent $2/3$ is as expected. It is a special case of the “first moment estimate” given in [7], where it is shown that the Hausdorff dimension of SLE$_{8/3}$ is $4/3$. 


It is clear from the form of the equation (62) and the initial-boundary value conditions that multiplication of a solution by a constant gives another solution. For the full equation (61) this is not the case. The corresponding equation for $1 - F$ has the same initial and boundary value conditions as (63) but the equation is no longer homogeneous.

**Remark 6.4.** The Galerkin approximation, see [11], for (61) (or rather for the inhomogeneous equation satisfied by $1 - F$), using the orthonormal system $(1/\sqrt{\pi}) \sin((2k-1)x/2)$, $k = 1, 2, \ldots$, gives as first approximation to $1 - F$

$$\pi^{-1/2} q^{2/3} (1 - q^2)^{1/2} \prod_{n=2}^{\infty} (1 - q^{2n})^{5/4} \sin(x/2).$$

It is a weak solution of the equation for $1 - F$ when testing against the 1-dimensional space spanned by $w_1$. For larger subspaces the systems of ODEs the Galerkin approximation gives rise to, did not appear tractable to us.

**References**

[1] Ahlfors, Lars V., Complex analysis, 3rd ed., McGraw-Hill Book Co., New York, 1978.

[2] Andrews, George E. and Askey, Richard and Roy, Ranjan, Special functions, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, Cambridge, 1999.

[3] Bauer, Robert O., *Discrete Löwner evolution*, Annales de la Faculté des Sciences de Toulouse, Vol. XII, no. 4(2003). 433–451.

[4] Bauer, Robert O., Friedrich, Roland M., *On radial stochastic Loewner evolution in multiply connected domains*, J. Funct. Anal., 237 (2006), 565–588.

[5] Bauer, Robert O., Friedrich, Roland M., *On chordal and bilateral stochastic Loewner evolution in multiply connected domains*, to appear in Math. Zeitschrift. Available at www.arXiv.org, math.PR/0503178.

[6] Beffara, Vincent, Thesis, University of Paris, Orsay, 2003.

[7] Beffara, Vincent, *The dimension of SLE curves*, preprint.

[8] Cardy, John, *The O(n) model on the annulus*, preprint, math-ph/0604043.

[9] Dubédat, Julien, *Critical percolation in annuli and SLE_6*, Comm. Math. Phys., 245, 2004, no. 3, 627–637.

[10] Dubédat, Julien, *Commutation relations for SLE*, to appear, Comm. Pure Applied Math.

[11] Evans, Lawrence C., Partial differential equations, AMS, Providence, Rhode Island, 1999.

[12] Hurwitz, Adolf, Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen, 4th edition, Springer-Verlag, Berlin, 1964.

[13] Ivanov, V. I. and Trubetskov, M. K., *Handbook of conformal mapping with computer-aided visualization*, CRC Press, Boca Raton, Florida, 1995.
Karatzas, Ioannis and Shreve, Steven E., Brownian motion and stochastic calculus, Graduate Texts in Mathematics, 113, Second Ed., Springer-Verlag, New York, 1991.

Komatu, Yūsaku, *Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten*, Proc. Phys.-Math. Soc. Japan (3), 25, 1943, 1–42.

Lawler, Gregory F., Conformally invariant processes in the plane, Mathematical Surveys and Monographs, 114, American Mathematical Society, Providence, RI, 2005.

Lawler, Gregory F., *The Laplacian-b random walk and the Schramm-Loewner Evolution*, Illinois Journal of Mathematics, 50 (2006), 701–746.

Lawler, Gregory F. and Schramm, Oded and Werner, Wendelin, *Values of Brownian intersection exponents. I. Half-plane exponents*, Acta Math. 187, no. 2, 2001, pp. 237–273.

Lawler, Gregory and Schramm, Oded and Werner, Wendelin, *Conformal restriction: the chordal case*, J. Amer. Math. Soc., 16, 2003, no. 4, 917–955 (electronic).

Lawler, Gregory F. and Schramm, Oded and Werner, Wendelin, *On the scaling limit of planar self-avoiding walk*, in “Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2,” Proc. Sympos. Pure Math., vol. 72, 339–364, Amer. Math. Soc., Providence, RI, 2004.

Löwner, Karl, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann., 89, 1923, no. 1-2, 103–121.

Nehari, Zeev, Conformal mapping, McGraw-Hill, New York, 1952.

Rezvuz, Daniel and Yor, Marc, Continuous martingales and Brownian motion, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293, 3rd edition, Springer-Verlag, Berlin, 1999.

Rohde, Steffen and Schramm, Oded, *Basic properties of SLE*, Ann. of Math. (2), 161, no. 2, 2005, pp. 883–924.

Schiffer, Menahem and Spencer, Donald C., Functionals of finite Riemann surfaces, Princeton University Press, Princeton, N. J., 1954.

Schramm, Oded, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math., 118, 2000, 221–288.

Werner, Wendelin, *The conformally invariant measure on self-avoiding loops*, preprint, [math.PR/0511605]

Zhan, Dapeng, *Random Loewner chains in Riemann surfaces*, thesis, California Institute of Technology, 2004.

Zhan, Dapeng, *Stochastic Loewner evolution in doubly connected domains*, Probab. Theory Related Fields, 129, 2004, no. 3, 340–380.

Altgeld Hall, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: rbauer@math.uiuc.edu