On a multiplicative order of Gauss periods and related questions
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Abstract. We obtain explicit lower bounds on multiplicative order of elements that have more general form than finite field Gauss period. In a partial case of Gauss period this bound improves the previous bound of O.Ahmadi, I.E.Shparlinski and J.F.Voloch

1 Introduction

A problem of constructing primitive element for the given finite field is notoriously difficult in computational number theory. That is why one considers less restrictive question: to find an element with large order. It is sufficient in this case to obtain a lower bound of the order.

Such question in particular arises in concern with primality proving AKS algorithm. This unconditional, deterministic polynomial-time algorithm for proving primality of integer \(n\) was presented by M.Agrawal, N.Kayal and N.Saxena in [1].

The idea of AKS consists in the following: to choose small integer \(r\) with specific properties and to show that the set of elements \(\theta+a\), \(a=1,\ldots, \left\lfloor \sqrt{r \log n} \right\rfloor\) generates large enough subgroup in the multiplicative group of the finite field \(\mathbb{Z}_p[x]/h(x)\) where \(\theta\) is a coset of \(x\), \(p\) is a prime divisor of \(n\), \(h(x)\) is an irreducible divisor of \(r\)-th cyclotomic polynomial \(\Phi_r(x) = C_r(x) = x^{r-1}+x^{r-2}+\ldots+x+1\). Usually in AKS implementation \(r\) is prime and \(h(x) = \Phi_r(x)\).

The fastest known deterministic version of AKS runs in \((\log n)^{6+o(1)}\) time, the fastest randomized version – in \((\log n)^{4+o(1)}\) time [6]. If given in [1, p.791] conjecture were true, this would improve a complexity of AKS to \(O(\log n)^{3+o(1)}\). The conjecture means that the element \(\theta-1\) generates large enough subgroup.

We denote \(F_q\) finite field with \(q\) elements where \(q\) is a power of prime number \(p\). Let \(r=2s+1\) be a prime number coprime with \(q\). Let \(q\) be a primitive root modulo \(r\), that is a multiplicative order of \(q\) modulo \(r\) equals to \(r-1\).

Set \(F_q(\theta) = F_{q^{-1}} = F_q[x]/\Phi_r(x)\) where \(\theta = x (\mod \Phi_r(x))\), \(\theta\) generates an extension \(F_q(\theta) = F_{q^{-1}}\). It is clear that \(\theta' = 1\). The element \(\theta + \theta^{-1}\) is called a Gauss period of type \(((r-1)/2,2)\). It allows to construct normal base [2].

\(<u_1,\ldots, u_k>\) denotes a group generated by elements \(u_1,\ldots, u_k\). \(p_2(n)\) denotes the highest power of 2 dividing integer \(n\).

A partition of integer \(C\) consists of such non-negative integers \(u_1,\ldots, u_C\) that \(\sum_{j=1}^{C} ju_j = C\). \(U(C)\) denotes a number of partitions of \(C\). \(U(C,d)\) denotes a number of such partitions \(u_1,\ldots, u_C\) of \(C\).
for which \( u_1, \ldots, u_C \leq d \). \( Q(C,d) \) denotes a number of such partitions \( u_1, \ldots, u_C \) of \( C \) for which \( u_1, \ldots, u_C \neq 0 \mod d \).

It is shown in [2] that a multiplicative order of Gauss period \( \theta + \theta^{-1} \) which generates normal base over finite field is at least \( U((r-3)/2,p-1) \).

We generalize this result and show that for any integer \( e \), any integer \( f \) coprime with \( r \), any non-zero element \( a \) in the field \( F_q \) the multiplicative order of the element \( \theta^e(\theta^f + a) \) in the field \( F_q(\theta) \) is at least \( U(r-2,p-1) \). In particular, a multiplicative order of the element \( \theta + \theta^{-1} = \theta^{-1}(\theta^2 + 1) \) is at least \( U(r-2,p-1) \). This bound improves the previous bound of O.Ahmadi, I.E.Shparlinski and J.F.Voloch given in [2]. We show that if \( a \neq 1,-1 \) then a multiplicative order of element \( \theta^e(\theta^f + a) \) is least \( \lceil U((r-3)/2,p-1) \rceil^2/2 \). We also prove that an order of subgroup \( \langle \theta + \theta^{-1}, (a \theta + 1)(\theta + a)^{-1} \rangle \) is at least \( \lceil U((r-2,p-1) U((r-3)/2,p-1)) \rceil^2/2 \) and construct a generator of this cyclic subgroup.

We also give explicit lower bounds for the multiplicative order of the elements using results from [3,7].

A construction of large order elements in the case \( q \equiv 1 \mod r \) (\( r \) is not primitive modulo \( r \)) is given in [5]. Overview of some alternative constructions of large order elements in a finite field can be found in [4].

### 2 Multiplicative orders of finite field elements

**Theorem 1.** Let \( q \) be a power of prime number \( p \), \( r=2s+1 \) be a prime number coprime with \( q \), \( q \) be a primitive root modulo \( r \), \( \theta \) generates an extension \( F_q(\theta) = F_{q^{r-1}} \), \( e \) be any integer, \( f \) be any integer coprime with \( r \), \( a \) be any non-zero element in the finite field \( F_q \). Then

(a) element \( \theta^e(\theta^f + a) \) has a multiplicative order at least \( U(r-2,p-1) \)

(b) if \( a \neq 1,-1 \) then element \( \theta^e(\theta^f + a) \) has a multiplicative order at least \( \lceil U((r-3)/2,p-1) \rceil^2/2 \)

**Proof.** (a) Let us consider automorphism of the field \( F_q(\theta) \) which takes \( \theta \) to \( \theta^f \). Since the automorphism sends element \( \theta^e(\theta + a) \) to element \( \theta^e(\theta^f + a) \), where \( g \equiv ef^{-1} \mod r \), multiplicative orders of these elements coincide. Hence, it is sufficient to prove that element of the form \( \theta^e(\theta + a) \) has a multiplicative order at least \( U(r-2,p-1) \).

Since \( q \) is primitive modulo \( r \), then for each \( j=1,\ldots,r-2 \) exists such integer \( \alpha_j \) that \( q^{\alpha_j} \equiv j \mod r \). Then \( q^{\alpha_j} \)-th powers of element \( \theta^e(\theta + a) \) which are equal to

\[
\left(\theta^e(\theta + a)\right)^{\alpha_j} = \theta^{e\alpha_j} (\theta^{\alpha_j} + a) = \theta^{\alpha_j}(\theta^j + a)
\]
belong to group $\langle \theta^e (\theta + a) \rangle$. We consider the following products
\[
\prod_{j=1}^{r-2} [\theta^j (\theta^j + a)]^{u_j}, \text{ where } \sum_{j=1}^{r-2} j u_j = r - 2, 0 \leq u_1, \ldots, u_{r-2} \leq p - 1,
\]
which also belong to the group, and show that a number of such products equals to $U(r - 2, p - 1)$.

Let partitions $(u_1, \ldots, u_{r-2})$ and $(v_1, \ldots, v_{r-2})$ of integer $r-2$, where it part appears no more than $p-1$ times, be different, but the correspondent products coincide:
\[
\prod_{j=1}^{r-2} [\theta^j (\theta^j + a)]^{u_j} = \prod_{j=1}^{r-2} [\theta^j (\theta^j + a)]^{v_j}
\]
Then we have from (1):
\[
\theta^{\sum_{j=1}^{r-2} j u_j} \prod_{j=1}^{r-2} (\theta^j + a)^{u_j} = \theta^{\sum_{j=1}^{r-2} j v_j} \prod_{j=1}^{r-2} (\theta^j + a)^{v_j}
\]
\[
\prod_{j=1}^{r-2} (\theta^j + a)^{u_j} = \prod_{j=1}^{r-2} (\theta^j + a)^{v_j}
\]
(2)

Since the characteristic polynomial of $\theta$ is the polynomial $\Phi_r(x)$ we obtain
\[
\prod_{j=1}^{r-2} (x^j + a)^{u_j} = \prod_{j=1}^{r-2} (x^j + a)^{v_j}
\]
(3)

As there are polynomials of degree $r-2<\deg \Phi_r(x)$ from left and right side in equality (3) then these polynomials coincide as polynomials over $F_q$.

Let $k$ be the smallest integer for which $u_k \neq v_k$. Without loss of generality suppose $u_k > v_k$. After removing from (3) common factors we obtain
\[
(x^k + a)^{u_k - v_k} \prod_{j=1}^{r-2} (x^j + a)^{u_j} = \prod_{j=1}^{r-2} (x^j + a)^{v_j}
\]
(4)

Then there is the term $(u_k - v_k) a^{u_k - v_k - 1} a^{\sum_{j=1}^{k-2} u_j} x^k$ in the polynomial in the left side of (4) with minimal non-trivial power of $x$. Since $0 < u_k, v_k < p - 1$, $u_k \neq v_k$, $a \neq 0$, the term is non-zero. And all terms with non-trivial power of $x$ in the polynomial in the right side have a power higher than $k$ - a contradiction.

(b) Order of a multiplicative group of the field $F_q^{r-1}$ equals to $q^{r-1} - 1 = (q^{(r-1)/2} - 1)(q^{(r-1)/2} + 1)$. The factors $q^{(r-1)/2} - 1$ and $q^{(r-1)/2} + 1$ have the greatest common divisor 2 since their sum equals to $2q^{(r-1)/2}$.

Subgroup $H$ of the group generated by element $\theta^e (\theta + a)$ contains subgroup $H_1$ generated by element
\[
v = [\theta^e (\theta + a)]^{q^{r-1}/2 - 1} = \theta^{q^{r-1}/2} (\theta^{q^{r-1}/2} + a)(\theta + a)^{-1} = \theta^e (\theta^{q^{r-1}/2} + a)(\theta + a)^{-1} = \theta^{(r+1)} (a \theta + 1)(\theta + a)^{-1}
\]
and subgroup $H_2$ generated by element
\[ w = [\theta'(\theta+a)]^{q(r-1)/2+1} = \theta^{q(r-1)/2}(\theta^{r-1} + a)(\theta + a) = \theta^{q(r-1)/2} \] (note that since \( q \) is primitive modulo \( r \) and \( r \) is prime, then \( q^{-1} = 1 \mod r \), a \( q^{(r-1)/2} = -1 \mod r \)).

Element \([\theta'(\theta+a)]^{q(r-1)/2-1}\) has order that divides \( q^{(r-1)/2}+1 \), and element \([\theta'(\theta+a)]^{q(r-1)/2+1}\) has order that divides \( q^{(r-1)/2}-1 \).

Let us consider element \( z = \begin{cases} v^2w & \text{if } \rho_2(q^{-1/2} - 1) = 2 \\ vw^2 & \text{if } \rho_2(q^{-1/2} + 1) = 2 \end{cases} \)

If \( \rho_2(q^{-1/2} - 1) = 2 \) then orders of elements \( v^2 \) and \( w \) are coprime. If \( \rho_2(q^{-1/2} + 1) = 2 \) then orders of elements \( v \) and \( w^2 \) are coprime. In both cases an order of element \( z \) is a product of orders of \( v \) and \( w \) divided by 2.

Element \( w = \theta^{r+1}(a\theta+1)(\theta+a) \) generates subgroup \( H_2 \) of order at least \( U((r-3)/2,p-1) \).

Indeed, elements \( \theta^{-2(r+1)}(a\theta^2+1)(\theta^2+a) \), \( \ldots \), \( \theta^{-(r-3)/2(r+1)}(a\theta^{(r-3)/2}+1)(\theta^{(r-3)/2}+a) \) belong to subgroup \( H_2 \).

We consider products of the powers

\[
\prod_{j=1}^{(r-3)/2} [\theta^{-j}(a\theta^j + 1)(\theta^j + a)]^{uj}_j, \text{ where } \sum_{j=1}^{(r-3)/2} j u_j = (r-3)/2, 0 \leq u_1, \ldots, u_{r-2} \leq p-1, \]

which also belong to the subgroup, and show that a number of such products equals to \( U((r-3)/2, p-1) \).

Let partitions \((u_1, \ldots, u_{r-2})\) and \((v_1, \ldots, v_{r-2})\) of integer \((r-3)/2\), where it part appears no more than \( p-1 \) times, be different, but the correspondent products coincide:

\[
\prod_{j=1}^{(r-3)/2} [\theta^{-j}(a\theta^j + 1)(\theta^j + a)]^{uj}_j = \prod_{j=1}^{(r-3)/2} [\theta^{-j}(a\theta^j + 1)(\theta^j + a)]^{vj}_j \quad (5)
\]

Then we have from (5):

\[
\theta^{-(r+1) \sum_{j=1}^{(r-3)/2} j u_j (r-3)/2} \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)]^{uj}_j = \theta^{-(r+1) \sum_{j=1}^{(r-3)/2} j v_j (r-3)/2} \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)]^{vj}_j
\]

\[
\prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)]^{uj}_j = \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)]^{vj}_j \quad (6)
\]

Then we can write

\[
\prod_{j=1}^{(r-3)/2} [(ax^j + 1)(x^j + a)]^{uj}_j = \prod_{j=1}^{(r-3)/2} [(ax^j + 1)(x^j + a)]^{vj}_j \quad (7)
\]

There are polynomials of degree \( r-3 < \deg \Phi(x) \) from left and right side in equality (7).
Let $k$ be the smallest integer for which $u_k \neq v_k$. Without loss of generality suppose $u_k > v_k$.

After removing from (7) common factors we obtain

\[
[(ax^k + 1)(x^k + a)]^{(r-3)/2} \prod_{j=k+1}^{r-3/2} [(ax^j + 1)(x^j + a)]^{u_j} = \prod_{j=k+1}^{r-3/2} [(ax^j + 1)(x^j + a)]^{v_j}
\]

\[
[ax^{2k} + (a^2 + 1)x^k + a]^{(r-3)/2} \prod_{j=k+1}^{r-3/2} [(ax^j + 1)(x^j + a)]^{u_j} = \prod_{j=k+1}^{r-3/2} [(ax^j + 1)(x^j + a)]^{v_j}
\] (8)

Then there is the term $(u_k - v_k)(a^2 + 1)a^{u_k - v_k - 1}a^{r-3/2}x^k$ in the polynomial in the left side of (8) with minimal non-trivial power of $x$. Since $0 < u_k, v_k < p - 1$, $u_k \neq v_k$, $a \neq 0, 1, -1$, the term is non-zero.

And all terms with non-trivial power of $x$ in the polynomial in the right side have a power higher than $k$ - a contradiction.

Element $v = \theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}$ generates subgroup $H_1$ of order at least $U((r-3)/2, p-1)$.

Indeed, elements $\theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}, \ldots, \theta^{-(r-3)/2(r+1)}(a\theta^{(r-3)/2} + 1)(\theta^{(r-3)/2} + a)^{-1}$ belong to subgroup $H_1$.

We consider products of the powers

\[
\prod_{j=1}^{(r-3)/2} [\theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}]^{u_j}
\]

\[
\prod_{j=1}^{(r-3)/2} [\theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}]^{v_j}
\]

where $\sum_{j=1}^{(r-3)/2} j u_j = (r-3)/2$, $0 \leq u_1, \ldots, u_{r-2} \leq p - 1$, which also belong to the subgroup, and show that a number of such products equals to $U((r-3)/2, p-1)$.

Let partitions $(u_1, \ldots, u_{r-2})$ and $(v_1, \ldots, v_{r-2})$ of integer $(r-3)/2$, where it part appears no more than $p-1$ times, be different, but the correspondent products coincide:

\[
\prod_{j=1}^{(r-3)/2} [\theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}]^{u_j} = \prod_{j=1}^{(r-3)/2} [\theta^{-(r+1)}(a\theta + 1)(\theta + a)^{-1}]^{v_j}
\] (9)

Then we have from (9):

\[
\theta^{-(r+1)} \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)^{-1}]^{u_j} = \theta^{-(r+1)} \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)^{-1}]^{v_j}
\]

\[
\prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)^{-1}]^{u_j} = \prod_{j=1}^{(r-3)/2} [(a\theta^j + 1)(\theta^j + a)^{-1}]^{v_j}
\] (10)

Then we obtain

\[
\prod_{j=1}^{(r-3)/2} [(ax^j + 1)(x^j + a)^{-1}]^{u_j} = \prod_{j=1}^{(r-3)/2} [(ax^j + 1)(x^j + a)^{-1}]^{v_j}
\] (11)

There are polynomials of degree $r-3 < \deg \Phi_r(x)$ from left and right side in equality (11).
Let \( k \) be the smallest integer for which \( u_k \neq v_k \). Without loss of generality suppose \( u_k > v_k \). After removing from (11) common factors we obtain
\[
\prod \prod_{j=k+1}^{(r-3)/2} \frac{(x^j + 1)(x^j + a)^{y-j}}{u_k - x^j} = \prod \prod_{j=k+1}^{(r-3)/2} \frac{(x^j + 1)(x^j + a)^{y-j}}{v_k - x^j}.
\]

Let us denote absolute term for \( \prod \prod_{j=k+1}^{(r-3)/2} (x^j + 1)(x^j + a)^{y-j} \) by \( c \), and absolute term for \( \prod \prod_{j=k+1}^{(r-3)/2} (x^j + 1)(x^j + a)^{y-j} \) by \( d \). It is clear that \( c \neq 0 \), \( d \neq 0 \). Since absolute terms from left and from right in (12) must be equal, we have \( c = a^{u_k - v_k} d \). Since coefficients near \( x^k \) from left and from right in (12) must be equal, we have \( c(u_k - v_k)a = d(u_k - v_k)a^{u_k - v_k - 1} \). Then \( a^2 = 1 \) - a contradiction.

Hence, order of element \( \theta^r(\theta + a) \) is at least \( \lfloor U(r-3)/2, p-1) \rfloor^2/2 \).

Remark 1. Element \( \theta + \theta^{-1} \) belongs to the subfield \( F_{q^{(r-1)/2}} \) of the field \( F_q(\theta) = F_{q^{r-1}} \).

Corollary 2. Element \( \theta + \theta^{-1} \) has a multiplicative order at least \( U(r-2, p-1) \) and this order is a divisor of \( q^{(r-1)/2} - 1 \)

Proof. A fact that a multiplicative order of element \( \theta + \theta^{-1} \) is at least \( U(r-2, p-1) \) follows from the theorem 1, (a). Since
\[
(\theta + \theta^{-1})^{q^{(r-1)/2} - 1} = (\theta^{q^{(r-1)/2} + 1} + \theta^{-q^{(r-1)/2}})(\theta + \theta^{-1})^{-1} = (\theta^{-1} + \theta)(\theta + \theta^{-1})^{-1} = 1
\]
an order of element \( \theta + \theta^{-1} \) is a divisor of \( q^{(r-1)/2} - 1 \).

Corollary 3. If \( a \neq 1, -1 \) then element
\[
z = \begin{cases} 
(\theta + \theta^{-1})^2(a\theta + 1)(\theta + a)^{-1} & \text{if } \rho_2(q^{(r-1)/2} - 1) = 2 \\
(\theta + \theta^{-1})(a\theta + 1)(\theta + a)^{-1} & \text{if } \rho_2(q^{(r-1)/2} + 1) = 2 
\end{cases}
\]
has a multiplicative order at least \( \lfloor U(r-2, p-1) U((r-3)/2, p-1)/2 \rfloor \)

Proof. According to corollary 2 element \( \theta + \theta^{-1} \) has order that divides \( q^{(r-1)/2} - 1 \) and generates subgroup of order at least \( U(r-2, p-1) \), element \( (\theta + \theta^{-1})^2 \) has order that divides \( q^{(r-1)/2} - 1 \) and generates subgroup of order at least \( U(r-2, p-1)/2 \). According to proof of the theorem 1, (b) element \( (a\theta + 1)(\theta + a)^{-1} \) has order that divides \( q^{(r-1)/2} + 1 \) and generates subgroup of order at least \( U((r-3)/2, p-1) \), element \( [(a\theta + 1)(\theta + a)^{-1}]^2 \) has order that divides \( q^{(r-1)/2} + 1 \) and generates subgroup of order at least \( U((r-3)/2, p-1)/2 \).
\[
\text{If } \rho_2(q^{(r-1)/2}-1)=2 \text{ then orders of elements } (\theta + \theta^{-1})^2 \text{ and } (a\theta + 1)(\theta + a)^{-1} \text{ are coprime.}
\]
\[
\text{If } \rho_2(q^{(r-1)/2}+1)=2 \text{ then orders of elements } \theta + \theta^{-1} \text{ and } [(a\theta + 1)(\theta + a)^{-1}]^2 \text{ are coprime.}
\]

Both in the first and in the second case an order of element \( z \) is a product of orders of its factors.

### 3  Explicit lower bounds on multiplicative orders

We use some known estimates from [3,7] to derive explicit lower bounds on the multiplicative order of the elements \( \theta^r(\theta^r + a) \) and \( z \).

According to [3, corollary 1.3(Glaisher)] the following equality is true:

\[
U(n,d-1) = Q(n,d)
\]

We consider two different cases.

Case 1) \( r-3 \geq 2p^2 \), that is \( r \) is big comparatively to \( p \).

In this case the following corollary holds. Note that \( 2.5 \approx \pi \sqrt{2/3} \).

**Corollary 4.** Let \( r-3 \geq 2p^2 \), \( e \) be any integer, \( f \) be any integer coprime with \( r \), \( a \in F_q \) be any non-zero element. Then

(a) element \( \theta^r(\theta^r + a) \) in the extension field \( F_q(\theta) \) has a multiplicative order larger than

\[
\left( \frac{p(p-1)}{160(r-2)} \right)^{\sqrt{p}} \exp \left( 2.5 \sqrt{(1-\frac{1}{p})(r-2)} \right)
\]

(b) if \( a \neq 1,-1 \) then element \( \theta^r(\theta^r + a) \) has a multiplicative order larger than

\[
\frac{1}{2} \left( \frac{p(p-1)}{80(r-3)} \right)^{2\sqrt{p}} \exp \left( 2.5 \sqrt{\frac{2}{(1-\frac{1}{p})(r-3)}} \right)
\]

(c) if \( a \neq 1,-1 \) then element

\[
\rho_2(q^{(r-1)/2}-1) = 2 \quad \rho_2(q^{(r-1)/2}+1) = 2
\]

has a multiplicative order larger than

\[
\frac{1}{2} \left( \frac{p(p-1)}{80(r-3)} \right)^{\sqrt{p}} \exp \left( 2.5 \frac{\sqrt{2}}{2} \sqrt{(1-\frac{1}{p})(r-3)} \right)
\]

**Proof.**

(a) According to [7, theorem 5.1] the following inequality holds for \( n \geq d^2 \)

\[
Q(n,d) > \left( \frac{d(d-1)}{160n} \right)^{\sqrt{n}} \exp \left( 2.5 \sqrt{(1-\frac{1}{d})n} \right)
\]

\[
(14)
\]
According to theorem 1, equality (13) and inequality (14) we have for $r-2\geq p^2$:

$$ord(\theta^r(\theta^f + a)) \geq U(r-2, p-1) = Q(r-2, p) > \left(\frac{p(p-1)}{160(r-2)}\right)^{\sqrt{p}} \exp\left(2.5 \sqrt{1 - \frac{1}{p}(r-2)}\right)$$

(b) Analogous to proof of (a) using [7,theorem 5.1], theorem, equality (13) and inequality (14).
Note that if $r-3\geq 2p^2$ then $r-2\geq p^2$.

(c) Analogous to proof of (a) using [2,theorem 5.1], theorem, equality (13) and inequality (14).

Case 2) $r-2p$, that is $r$ is the same magnitude as $p$ or small comparatively to $p$.
In this case the following corollary holds.

**Corollary 5.** Let $r-2<r$, $e$ be any integer, $f$ be any integer coprime with $r$, $a \in F_q$ be any non-zero element. Then

(a) element $\theta^r(\theta^f + a)$ in the extension field $F_q(\theta)$ has a multiplicative order larger than

$$\exp\left(\frac{2.5\sqrt{r-2}}{13(r-2)}\right)$$

(b) if $a\neq 1,-1$ then element $\theta^r(\theta^f + a)$ has a multiplicative order larger than

$$\frac{2\exp\left(2.5\sqrt{2\sqrt{r-3}}\right)}{169(r-3)^2}$$

(c) if $a\neq 1,-1$ then element $z = \left[(\theta + \theta^{-1})^2(a\theta + 1)(\theta + a)^{-1}\right.\left.\text{if } \rho_2(q^{(r-1)/2} - 1) = 2\right]$

$$\left.\left[(\theta + \theta^{-1})[(a\theta + 1)(\theta + a)]^2\right.\text{if } \rho_2(q^{(r-1)/2} + 1) = 2\right]$$

has a multiplicative order larger than

$$\exp\left(\frac{2.5(1+\sqrt{2}/2)\sqrt{r-3}}{169(r-2)(r-3)}\right)$$

**Proof.**

(a) If $n<p$ then $U(n,p-1)=U(n)$. According to [7,theorem 4.2] the following inequality holds for $n<d$

$$U(n) > \frac{\exp(2.5\sqrt{n})}{13n} \quad (15)$$

According to theorem 1, equality 13 and inequality (15) we have for $r-2<p$:

$$ord(\theta^r(\theta^f + a)) \geq U(r-2, p-1) = U(r-2) > \frac{\exp(2.5\sqrt{r-2})}{13(r-2)}$$

(b) Analogous to proof of (a) using [7,theorem 4.2], theorem, equality (13) and inequality (15).
Note that if $r-2<p$ then $(r-3)/2<p$.

(c) Analogous to proof of (a) using [7,theorem 4.2], theorem, equality (13) and inequality (15).
Remark 2. Element $θ^r(θ^f+a)$ asymptotically has a multiplicative order larger than
\[ \exp(2.5\sqrt[r]{r}) = 12.18^{\sqrt[r]{r}}. \]

If $a≠1,-1$ then element $θ^r(θ^f+a)$ asymptotically has a multiplicative order larger than
\[ \exp(2.5\sqrt[2r]{r}) = 33.95^{\sqrt[2r]{r}}. \]

If $a≠1,-1$ then element
\[ z = \begin{cases} 
(\theta + \theta^{-1})^2(a \theta + 1)(\theta + a)^{-1} & \text{if } \rho_2(q^{(r-1)/2} - 1) = 2 \\
(\theta + \theta^{-1})[(a \theta + 1)(\theta + a)^{-1}]^2 & \text{if } \rho_2(q^{(r-1)/2} + 1) = 2 
\end{cases} \]
asymptotically has a multiplicative order larger than
\[ \exp\left(2.5(1 + \frac{\sqrt{2}}{2})\sqrt[r]{r}\right) = 70.1^{\sqrt[r]{r}} \]

4 Examples

Let us denote lower bounds for orders of elements $θ^r(θ^f+1)$, $θ^r(θ^f+a)$,
\[ z = \begin{cases} 
(\theta + \theta^{-1})^2(a \theta + 1)(\theta + a)^{-1} & \text{if } \rho_2(q^{(r-1)/2} - 1) = 2 \\
(\theta + \theta^{-1})[(a \theta + 1)(\theta + a)^{-1}]^2 & \text{if } \rho_2(q^{(r-1)/2} + 1) = 2 
\end{cases} \]
by $z_1$, $z_2$, $z_3$ respectively.
Logarithms of $|Fq^r|$, $1$ and of $z_1$, $z_2$, $z_3$ in examples 1-4 are given in the table.

Example 1
$q=p=5$, $r=257$ – prime number, element $5$ is primitive modulo $257$ and $Fq^r = F_{5256}$. Since $r-3≥2p^2$
we have case 1 in this example.

Example 2
$q=p=3$, $r=401$ – prime number, element $3$ is primitive modulo $401$ and $Fq^r = F_{3400}$. Since $r-3≥2p^2$
we have case 1 in this example.

Example 3
$q=p=11$, $r=1009$ – prime number, element $q =11$ is primitive modulo $r=1009$ and $Fq^r = F_{111008}$.
Since $r-3≥2p^2$ we have case 1 in this example.

Example 4
$q=p=107$, $r=97$ – prime number, element $q =107$ is primitive modulo $r=97$ and $Fq^r = F_{10796}$. Since
$r-2<p$ we have case 2 in this example.
Table

| Q | r  | $\log_2|F_{q+1}^*|$ | $\log_2z_1$ | $\log_2z_2$ | $\log_2z_3$ |
|---|----|------------------|-----------|-----------|-----------|
| 1 | 5  | 257              | 594.41    | 26.93     | 27.03     | 64.43     |
| 2 | 3  | 401              | 634       | 35.65     | 39.22     | 77.86     |
| 3 | 11 | 1009             | 3487.1    | 74.24     | 90.13     | 153.64    |
| 4 | 107| 97               | 647.18    | 24.71     | 28.71     | 38.89     |

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