FINITE MODEL PROPERTIES FOR RESIDUATED SEMIGROUPS

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Abstract. We have a quick look at various finite model properties for residuated semigroups. In particular, we solve Problem 19.17 from the monograph by Hirsch and Hodkinson [HH02].

1. Introduction

Residuated semigroups have been investigated on their own right, but also in connection with substructural logics, like the Lambek Calculus (LC) Lambek [La58], and relevance logics Anderson et al. [ABD92]. Indeed, relational semantics, i.e. models based on binary relations, have been proposed for these substructural logics, see van Benthem [vB91] and Routley and Meyer [RM73], and the algebraic settings of these relational semantics give rise to residuated semigroups (and their expansions to larger similarity types). For instance, representable residuated semigroups (see below) provide sound and complete semantics for LC, see Andréka and Mikulás [AM94]. The connection between (fragments of) relevance logics and (variants of) residuated semigroups have been investigated in e.g. Bimbó et al. [BDM09], Maddux [Ma10], Hirsch and Mikulás [HM11] and Mikulás [Mi09, Mi15].

In this note, we will concentrate on representable residuated semigroups with finite bases (see below) and show that they have smaller expressive power than representable semigroups in general. In particular, we show that there is a finite representable residuated semigroup that is not isomorphic to any representable residuated semigroup with a finite base, thus providing a solution to Problem 19.17 of Hirsch and Hodkinson [HH02].

Next we define (representable) residuated semigroups. Strictly speaking, these are not algebras but algebraic structures (because of the ordering \( \leq \)) but the reader should not have any problem with applying the usual algebraic techniques (like taking subalgebras) to algebraic structures.

Definition 1.1. A residuated semigroup \( A = (A, \leq, \circ, \setminus, /) \) is an algebra such that the following are satisfied:

- \( \leq \) is a reflexive ordering on \( A \),
- \( (A, \circ) \) is a semigroup,
- \( \circ \) is monotonic w.r.t. \( \leq \),
- \( \setminus, / \) are the right and left residuals of \( \circ \):

\[
y \leq x \setminus z \text{ iff } x \circ y \leq z \text{ iff } x \leq z / y
\]

for every \( x, y, z \) in \( A \).

We denote the class of residuated semigroups by \( \text{RS} \).

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Next we define subclasses of RS that are isomorphic to algebras of binary relation and the relation ≤ and operations ◦, \, / have “relational” interpretations.

**Definition 1.2.** A representable residuated semigroup \( A = (A, \leq, \circ, \setminus, /) \) is a subalgebra of some \((\wp(W), \subseteq, \circ_W, \setminus_W, /_W)\) where \( W \) is a transitive relation, \( \subseteq \) is the subset relation and the operations are interpreted as follows:

1. \( x \circ_W y = \{(u, v) \in W : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w\} \),
2. \( x \setminus_W y = \{(u, v) \in W : \text{for every } w, (w, u) \in x \text{ implies } (w, v) \in y\} \),
3. \( x /_W y = \{(u, v) \in W : \text{for every } w, (v, w) \in y \text{ implies } (u, w) \in x\} \),

see Figure 1. Let \( U \) be the smallest set such that \( W \subseteq U \times U \). We call \( U \) the base and \( W \) the unit of \( A \). We denote the class (of isomorphs) of representable residuated semigroups by \( \text{RRS} \).

Let \( \text{RRS}^{\square} \) be the “square” subclass of \( \text{RRS} \), where we additionally require that \( W = U \times U \).

The subscript \( W \) to the operations of representable algebras is to remind the reader that their meanings depend on the choice of the unit \( W \) of the algebra. In particular, if the unit \( W \) is an irreflexive relation, then \( (u, u) \notin x \setminus_W x \) for any element \( u \) in the base and algebra element \( x \), in stark contrast to \( (v, v) \in y \setminus_V y \) for any base element \( v \) and algebra element \( y \) for a reflexive unit \( V \).

**Definition 1.3.** Let \( K \) be a class of representable algebras (like \( \text{RRS} \) or \( \text{RRS}^{\square} \)).

We say that \( A \in K \) is representable on a finite base, or finitely representable, if the base \( U \) can be chosen to be finite in Definition 1.2. The class \( K \) has the finite representation property (frp) if every finite \( A \in K \) is finitely representable.

### 2. The main result

Next we show that \( \text{RRS} \) lacks the frp.

**Theorem 2.1.** There is a finite \( A \in \text{RRS} \) that cannot be represented on a finite base.

**Proof.** Let \( U = (\mathbb{Q} \times \{0\}) \cup (\mathbb{Q} \times \{1\}) \) and

\[
W = \{(q, 0), (r, 0) : q, r \in \mathbb{Q}, q < r\} \cup \{(q, 1), (r, 0) : q, r \in \mathbb{Q}, q \leq r\}
\]
Table 1. Table for composition

| $\cap_W$ | $a$ | $b$ | $\bot$ | $ba$ | $\top$ | $b'$ | $a'$ |
|----------|-----|-----|--------|------|-------|------|------|
| $a$      | $a$ | $ba$| $\bot$ | $ba'$| $ba$ | $a'$ | $a'$ |
| $b$      | $\bot$| $\bot$| $\bot$ | $\bot$| $\bot$| $\bot$| $\bot$|
| $ba$     | $\bot$| $\bot$| $\bot$ | $\bot$| $\bot$| $\bot$| $\bot$|
| $\top$   | $a$ | $ba$| $\bot$ | $ba'$| $ba$ | $a'$ | $a'$ |
| $b'$     | $\bot$| $\bot$| $\bot$ | $\bot$| $\bot$| $\bot$| $\bot$|
| $a'$     | $a$ | $ba$| $\bot$ | $ba'$| $ba$ | $a'$ | $a'$ |

Table 2. Tables for the residuals

| $\setminus_W$ | $a$ | $b$ | $\bot$ | $ba$ | $\top$ | $b'$ | $a'$ |
|----------------|-----|-----|--------|------|-------|------|------|
| $a$            | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $b$            | $b'$ | $b'$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $\bot$         | $b'$ | $b'$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $ba$           | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $\top$         | $b'$ | $b'$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $b'$           | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $a'$           | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |

| $/$ | $a$ | $b$ | $\bot$ | $ba$ | $\top$ | $b'$ | $a'$ |
|-----|-----|-----|--------|------|-------|------|------|
| $a$ | $a$ | $\bot$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $b$ | $b'$ | $b'$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $ba$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $\top$ | $b'$ | $b'$ | $\bot$ | $b'$ | $b'$ | $b'$ | $b'$ |
| $b'$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $a'$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |

where $\mathbb{Q}$ denotes the set of rational numbers. Define

$$a = \{(q,0), (r,0) : q, r \in \mathbb{Q}, q < r\},$$

(5)

$$b = \{(q,1), (q,0) : q \in \mathbb{Q}\}.$$ (6)

Let $\mathcal{A} = (A, \leq, \cap_W, \setminus_W, /_W)$ be the subalgebra of $(\wp(W), \subseteq, \cap_W, \setminus_W, /_W)$ generated by $a, b$. $\mathcal{A}$ has the following seven elements: bottom element $\emptyset$ (that we will denote by $\bot$), top element $W$ (that we will denote by $\top$), $a$, $b$, $b \setminus_W a = \{((q,1),(r,0)) : q,r \in \mathbb{Q}, q < r\}$, $a \cup b \setminus_W a$ (that we will denote by $a'$) and $b \cup b \setminus_W a$ (that we will denote by $b'$). We include the tables for the operations (where we abbreviate $b \cap_W a$ as $ba$).

Observe that

$$a = a \cap_W a$$ (7)

$$a \not\leq a \cap_W (b \setminus_W b) \cap_W a$$ (8)

since $a$ is a transitive and dense relation, and $b \setminus_W b = b'$ and $a \cap_W b' = \bot$ whence

$$a \cap_W (b \setminus_W b) \cap_W a = a \cap_W b' \cap_W a = \bot \cap_W a = \bot \not\geq a$$

in $\mathcal{A}$.

\footnote{Note that $\cup$ is not an operation of $\mathcal{A}$, but it makes sense to take the union of two elements of an algebra of binary relations.}
Next assume that $\mathcal{A}$ can be represented on a finite base. That is, we have a finite set $U'$ and a transitive relation $V \subseteq U' \times U'$ such that $\mathcal{A}$ is isomorphic to $\mathcal{B} = (B, \leq, \circ_V, \setminus_V, /_V) \subseteq (\wp(V), \subseteq, \circ_V, \setminus_V, /_V)$.

Let $(x, y) \in a$ in $\mathcal{B}$. We show that $(x, y) \in a \circ_V (b \setminus /_V b) \circ_V a$. Since $a$ is dense ($a \leq a \circ_V a$), we have $z \in U'$ (not necessarily distinct from $x, y$) such that $(x, z), (z, y) \in a$. Continuing in the same vein (using $(x, z), (z, y) \in a \leq a \circ_V a$, etc.) we can create an $a$-path of elements $(x = z_0, z_1, \ldots, z_i, \ldots, z_{n-1}, z_n = y)$ of unlimited length (i.e. each $(z_i, z_{i+1}) \in a$). Since $U'$ is finite, after finitely many iterations of the process of expanding the path we get that $z_i = z_j$ for two elements of the path. Since $a$ is transitive ($a \circ_V a \leq a$), we have $(x, z_i), (z_i, z_j), (z_j, y) \in a$.

Thus $(z_i, z_j) = (z_i, z_i) \in V$, whence $(z_i, z_i) \in b \setminus /_V b$. We get that $(x, y) \in a \circ_V (b \setminus /_V b) \circ_V a$, contradicting that $\mathcal{B}$ should satisfy $\mathfrak{S}$. □

3. Conclusions

The above proof shows that RRS does not have the finite base property (fbp) w.r.t. implications: the implication

$$(a \leq a \circ a \land a \circ a \leq a) \rightarrow a \leq a \circ b \setminus /_V b \circ a$$

is not RRS-valid but cannot be refuted in any RRS with finite base.

The solution to the following problem would answer the question of the finite relational model property (frmp) for the LC: whether non-derivable sequents of LC are refutable in relational models with finite bases.

**Problem 3.1.** Does RRS have the finite base property w.r.t. atomic formulas $t \leq s$ (where $t, s$ are $\{\circ, \setminus, /\}$-terms of variables).

We can ask the same problems for the square version $\text{RRS}^\square$ of RRS. These problems are related to that version of the LC that allows the empty word and sequents with empty antecedents [La61]. The corresponding RRS is a class with reflexive units $W$, thus every $x \setminus W x$ and $x / W x$ subsume the identity relation, see Andréka and Mikulás [AM94]. Thus the proof of Theorem 2.1 does not work in this case.

**Problem 3.2.** Does $\text{RRS}^\square$ have the finite representation and finite base properties?

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