Renormalization Group Method Applied to Kinetic Equations: roles of initial values and time

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Abstract

The so-called renormalization group (RG) method is applied to derive kinetic and transport equations from the respective microscopic equations. The derived equations include Boltzmann equation in classical mechanics, Fokker-Planck equation, a rate equation in a quantum field theoretical model. Utilizing the formulation of the RG method which elucidates the important role played by the choice of the initial conditions, the general structure and the underlying assumptions in the derivation of kinetic equations in the RG method is clarified. It is shown that the present formulation naturally leads to the choice for the initial value of the microscopic distribution function at arbitrary time $t_0$ to be on the averaged distribution function to be determined. The averaged distribution function may be thought as an integral constant of the solution of microscopic evolution equation; the RG equation gives the slow dynamics of the would-be initial constant, which is actually the kinetic equation governing the averaged distribution function. It is further shown that the averaging as given above gives rise to a coarse-graining of the time-derivative which is expressed with the initial time $t_0$, thereby leads to time-irreversible equations even from a time-reversible equation. It is shown that a further reduction of Boltzmann equation to fluid dynamical equations and the adiabatic elimination of fast variables in Fokker-Planck equation are also performed in a unified way in the present method.

1 Introduction

Statistical physics, especially of non-equilibrium phenomena may be said to be a collection of theories on how to reduce the dynamics of many-body systems to ones with fewer variables\cite{1}. BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy\cite{2} which is equivalent to Liouville equation hence time-reversible can be reduced to the time-irreversible Boltzmann equation\cite{3} given solely in terms of the single-particle distribution function for dilute gas systems\cite{4}. The derivation of Boltzmann equation by Bogoliubov\cite{4} shows that the dilute-gas dynamics as a \textit{dynamical system} with many-degrees of freedom has an \textit{attractive manifold}\cite{5} spanned by the one-particle distribution function, which is also an \textit{invariant manifold}\cite{5}. Boltzmann equation in turn can be further reduced to the hydrodynamic equation (Navier-Stokes equation) by a perturbation theory like Chapman-Enskog method\cite{6} or Bogoliubov’s method\cite{4, 7}. Langevin equation which may be time-irreversible can be reduced to the time-irreversible Fokker-Planck equation with a longer time scale than the scale in Langevin equation\cite{8}. Two basic ingredients are commonly seen in the reduction of dynamics, which are interrelated with but relatively independent of each other: (i) The reduced dynamics is characterized with a longer time scale
than that appearing in the original (microscopic) evolution equation. (ii) The reduced

dynamics is described by a time-irreversible equation even when the original microscopic
equation is time-reversible. The fundamental problem in the theory of deriving kinetic or
transport equations is to clarify the mechanism of and to implement the above two basic
ingredients.

It seems that the basic notions to implement the two basic ingredients (i) and (ii) are
given by (1) the coarse-grained time-derivative and (2) the choice of the initial conditions
in solving the microscopic equations, respectively:

(1) In an attempt to characterize hydro-dynamical processes microscopically, Mori pointed
out that time derivatives appearing in equations which define transport coefficients are
“the average” of time derivatives describing microscopic dynamics [9]. His definition of
the macroscopic derivative of an observable $F$ is

$$\frac{\delta}{\delta t} \langle F \rangle(t) \equiv \frac{1}{\tau} \{ \langle F \rangle(t + \tau) - \langle F \rangle(t) \} = \frac{1}{\tau} \int_0^\tau ds \frac{d}{ds} \langle F \rangle(t + s), \quad (1.1)$$

where $\tau$ is some time scale between microscopic (mean free) time and macroscopic (relaxation) time. An important point is that $\tau$ is finite. The idea of the coarse-graining of
time in kinetic and transport equations were first given by Kirkwood[10]; see also [11] for
a rigorous formulation.

(2) The importance of the choice of the initial condition in the derivation of kinetic equa-
tion is noticed and emphasized in the literature[3, 4, 12, 13, 14]. For instance, Kawasaki
clarifies in an excellent monograph[14] that the initial value of the microscopic distribution
function before averaging must be given solely in terms of the averaged distribution
to obtain a closed equation for the distribution function of macroscopic slow variables,
which is equivalent to the construction of an invariant manifold mentioned above. He also
clarifies that by this initial condition, the dominating class of states (“typical states”) are
selected which leads to an increase of the entropy, while exceptional states from which the
entropy would increase could be unstabilized to the dominating typical states by a mech-
anism producing chaotic behavior. Bogoliubov[4] also gave exactly the same scenario as
described above in his asymptotic theory for the reduction to Boltzmann equation; see
chap. 9 of [1].

In this paper, we apply the so-called renormalization group (RG) [15] method to derive
and reduce kinetic equations to a slower dynamics [16, 18, 19, 20]: We show that if the
RG method is properly formulated so that the essential role played by the choice of initial
conditions is manifest as is done in [19, 20], the general structure in the derivation of
kinetic equations in the RG method is exactly in accordance with the basic principles (1)
and (2) mentioned above.

The reduction-theoretical aspect of the RG method[16] and the improved formulation
given in [19] were reformulated mathematically with the notion of invariant manifolds fa-
miliar in the theory of dynamical systems [5] by Ei, Fujii and one of the present author[20].
In [20], it was shown that the perturbative RG method can be used to construct invariant
manifolds successively as the initial values of evolution equations using the Wilsonian
RG [13, 21, 22]; the would-be integral constants, which have one-to-one correspondence
with the initial values, in the unperturbed solution, constitute natural coordinates of the
invariant manifold. It was also shown that the RG equation determines the slow motion
of the would-be integral constants on the invariant manifold of the dynamical system, hence a reduction of evolution equation is achieved.

We shall show that the straightforward application of the RG method as formulated in [19, 20] naturally leads to the choice for the initial value of the microscopic distribution function at an arbitrary time $t_0$ to be on the averaged distribution, which is an implementation of (2) in the RG method, thereby leads to time-irreversible equations even from a time-reversible equation. The averaged distribution function may be thought as an integral constant of the solution of microscopic evolution equation. The RG equation gives the slow dynamics of the would-be initial constant, which is actually the kinetic equation governing the averaged distribution function. It will be further shown that the averaging as given above automatically gives rise to a coarse-graining of the time-derivative, which is expressed with the initial time $t_0$. This shows that the initial time $t_0$ has a macroscopic nature in contrast to the time $t$ appearing in the microscopic equations, which is an implementation of (1) in our method.

It should be noticed here that the RG equation has been already applied to kinetic equations both in classical and quantum regimes by others [23, 24]: In [23], it was ascertained that Boltzmann equation is a renormalization group equation on the basis of the work by M.S. Green [25] for the uniform system which shows that the perturbative solution for BBGKY hierarchy exhibit a secular term, and a sketch was given to derive Fokker-Planck equation from a simple Langevin equation noticing again an appearance of a secular term. On the basis of these two examples, they claimed that all other kinetic equations are also RG equations. Boyanovsky, de Vega and their collaborators [24] have derived a Boltzmann equations for quantum field theories including gauge theories in the prescription of Illinois group [16]. They noticed that the existence of a mesoscopic scale which is microscopically large but macroscopically small is essential for the applicability of the RG method to derive kinetic equations as well as the validity of the description in terms of a kinetic equation itself; we remark here that the existence of such mesoscopic scales may be considered as the so-called intermediate asymptotics [26], which was actually the basic observation for the application of the RG equation to asymptotic analysis of differential equations [16]. A model dealt in [23, 24] will be retreated in our formulation, and implicit assumptions in their treatment will be made explicit so that the roles of the initial conditions and the scale transformation of the time-derivative will become clear for the RG method to lead to kinetic or transport equations.

In section 2, we shall deal with Langevin equation and derive the Fokker-Planck equation as a typical problem of dynamical reduction leading to a kinetic equation in the RG method. We shall summarize the basic structure of the reduction given by the RG method. One will see that a similar definition to Eq.(1.1) of the macroscopic time-derivative naturally emerges in the RG method. In section 3, Boltzmann equation is derived from Liouville equation of the classical mechanics; we shall clarify the difference between the present method and the one by Bogoliubov. In section 4, we apply the RG method to derive a Boltzmann equation(rate equation) in quantum field theory in a way where it is transparent how the initial value are chosen and the coarse-grained time-derivative is introduced in the RG method. This section is a recapitulation of the work by Boyanovsky et al in our point of view. In section 5, the RG method as formulated in [20] is applied to obtain the fluid dynamical limit of Boltzmann equation. This is an example to reduce
a kinetic equation to a further slower dynamics, which appear quite often in statistical physics reflecting the hierarchy of the space-time of the nature. Another typical problem in this category is to derive Smoluchowski equation from Kramers equation. In section 6, we show that it is possible to develop a systematic theory based on the RG method for the adiabatic elimination of fast variables in Fokker-Planck equations. Section 7 is devoted to a summary and concluding remarks. In Appendix A, the calculational procedure sketched in \[23\] for deriving Fokker-Planck equation in the RG method is properly elaborated and worked out so that the essential role of the initial condition in the procedure is fully recognized; it is shown that the initial value to obtain the “stochastic distribution function” must be actually the averaged one for the RG equation to give Fokker-Planck equation in conformity with the observation by Kawasaki and others mentioned above.
2 From Langevin to Fokker-Planck equation

In this section, we apply the RG equation to derive the Fokker-Planck (FP) equation starting from the stochastic Liouville equation corresponding to Langevin equation. This derivation is thought to be a typical one for the reduction of evolution equations appearing in non-equilibrium physics. It will be clarified that the initial values of the stochastic distribution function at arbitrary time $t_0$ are naturally chosen to be on the averaged distribution function for the RG equation to derive the FP equation governing the averaged distribution function. We shall also notice that the time derivative in the RG equation which will be converted to the derivative in the FP equation is with respect to a macroscopic time, hence the coarse-graining of time is automatically built in in the present RG method.

2.1 Application of the RG method to a generic equation Langevin equation

We consider the following generic Langevin equation with $R_i (i = 1, 2, ..., n)$ being stochastic variables;

$$\frac{du}{dt} = h(u) + \hat{g}(u)R,$$  \hspace{1cm} (2.1)

where $u = (u_1, u_2, ..., u_n)$, $h = (h_1, h_2, ..., h_n)$, $\hat{g} a n$ times $n$ matrix and $(\hat{g}(u)R)_i = \sum_j g_{ij}R_j$. Notice that the noise enters multiplicatively. Here we assume without loss of generality that the noise has the vanishing average,

$$\langle R(t) \rangle = 0,$$  \hspace{1cm} (2.2)

where $\langle O(t) \rangle$ denotes the average of $O(t)$ with respect to the noise $R$. Let $f(u, t)$ be the distribution function with $R(t)$ given; the continuity equation reads

$$\frac{\partial f(u, t)}{\partial t} + \nabla_u \cdot vf(u, t) = 0,$$  \hspace{1cm} (2.3)

where $\nabla_u = \sum_i \partial/\partial u_i$ and $v = du/dt$ is the velocity of $u$, which is given in (2.1). Inserting (2.1) into (2.3), one has the Kubo’s stochastic Liouville equation,

$$\frac{\partial f}{\partial t} = -\nabla_u \cdot [(h + \hat{g}R)f].$$  \hspace{1cm} (2.4)

Whereas it is rather easy task to derive the FP equation in an exact way if the noise is Gaussian, it is formidable difficult if the noise is non-Gaussian. Although the present approach is based on the perturbation theory and of approximate nature, it will be found that the first order calculation suffices to derive the exact FP equation when the noise in Gaussian, and furthermore that the method is applicable even to non-Gaussian noises without difficulties.

The solution to (2.4) with the initial condition given at $t = t_0$ is formally given by

$$\tilde{f}(u, t; t_0) = T\exp\left[\int_{t_0}^{t} dsL(s)\right]f(u, t_0; t_0),$$  \hspace{1cm} (2.5)
where

\[ L(s) = -\nabla_u \cdot (h(u) + \hat{g}R(s)), \tag{2.6} \]

and \( T \) denotes the time ordering operator. The initial distribution \( \tilde{f}(u, t_0; t_0) \) will be specified later and found to play a significant role in the present method.

Now we are interested in the averaged distribution function \( \tilde{P}(u, t; t_0) \) which is defined as an average of \( f(u, t; t_0) \) with respect to the noise \( R \), i.e.,

\[ \tilde{P}(u, t; t_0) = \langle T \exp[\int_{t_0}^{t} dsL(s)]f(u, t_0) \rangle. \tag{2.7} \]

We take an interaction picture dividing the “Hamiltonian” \( L \) as follows;

\[ L = L_0 + L_1, \tag{2.8} \]

\[ L_0 = -\nabla_u \cdot h, \quad L_1 = -\nabla_u \hat{g}R. \tag{2.9} \]

We first define \( U_0(t) \) as the time-evolution operator governed by the unperturbed “Hamiltonian” \( L_0 \),

\[ U_0(t, t_0) = T \exp[\int_{t_0}^{t} dsL_0(s)], \tag{2.10} \]

which satisfies the evolution equation

\[ \frac{\partial}{\partial t} U_0(t, t_0) = L_0 U_0(t, t_0), \tag{2.11} \]

with the initial condition

\[ U_0(t_0, t_0) = 1. \tag{2.12} \]

Then to incorporate \( L_1 \), we define another microscopic distribution function \( \rho_1(u, t; t_0) \) by

\[ f(u, t; t_0) = U_0(t, t_0) \rho_1(u, t; t_0). \tag{2.13} \]

We remark that the initial values of \( f \) and \( \rho_1 \) at \( t = t_0 \) coincides with each other, which we take to be equal to the averaged distribution function \( P(u, t_0) \) at \( t = t_0 \) as depicted in Fig.1;

\[ \tilde{f}(u, t = t_0; t_0) = \rho_1(u, t = t_0; t_0) = P(u, t_0). \tag{2.14} \]

One will recognize that this choice of the initial condition is inevitable for the RG equation to be identified with Fokker-Planck equation. One can easily verify that \( \rho_1(u, t; t_0) \) is formally solved to be

\[ \rho_1(u, t; t_0) = T \exp[\int_{t_0}^{t} dsL_1(s; t_0)] \rho_1(u, t_0; t_0), \tag{2.15} \]
Distribution functions

\[ f(u, t) \]

\[ P(u, t_0) = \tilde{f}(u, t = t_0; t_0) \]

Figure 1: The initial value of \( \tilde{f}(u, t = t_0; t_0) = \rho_1(u, t = t_0; t_0) \) at \( t = t_0 \) is set on the averaged distribution function \( P(u, t_0) \), which will be found to obey Fokker-Planck equation as the RG equation.

where

\[ L_1(t; t_0) = U_0^{-1}(t, t_0)L_1(t)U_0(t, t_0), \quad (2.16) \]

is an “interaction Hamiltonian” in the “interaction picture”.

Thus we obtain the compact form of \( \tilde{P}(u, t; t_0) \) as follows,

\[ \tilde{P}(u, t; t_0) = \langle U_0(t, t_0)\rho_1(u, t; t_0) \rangle, \quad (2.17) \]

\[ = U_0(t, t_0)\langle T\exp[\int_{t_0}^{t} dsL_1(s; t_0)]P(u, t_0) \rangle, \quad (2.18) \]

\[ \equiv U_0(t, t_0)S(t; t_0)P(u, t_0), \quad (2.19) \]

where we have used the fact that \( \rho_1(u, t = t_0, t_0) = P(u, t_0) \) and the abbreviation

\[ S(t; t_0) \equiv \langle T\exp[\int_{t_0}^{t} dsL_1(s; t_0)] \rangle. \]

The computation may be performed in a perturbative way:

\[ S(t; t_0) = 1 + T\int_{t_0}^{t} ds\langle L_1(s) \rangle + \frac{1}{2}T\int_{t_0}^{t} ds_1\int_{t_0}^{t} ds_2\langle L_1(s_1)L_1(s_2) \rangle + \ldots \]

\[ = 1 + \frac{1}{2}T\int_{t_0}^{t} ds_1\int_{t_0}^{t} ds_2\Gamma(s_1, s_2) + \ldots \]

\[ \Gamma(s_1, s_2) \equiv \langle L_1(s_1)L_1(s_2) \rangle, \quad (2.21) \]

where we have put

\[ \Gamma(s_1, s_2) \equiv \langle L_1(s_1)L_1(s_2) \rangle. \]
If the noise is stationary, which we shall assume from now, \( \Gamma(s_1, s_2) \) will be a function of the difference \( s_1 - s_2 \); furthermore, owing to the time-reversible invariance of the microscopic law, \( \Gamma(s_1, s_2) \) becomes a function of the absolute value \( |s_1 - s_2| \), i.e., \( \Gamma(s_1, s_2) = \Gamma(|s_1 - s_2|) \).

Then one has for \( t > t_0 \),

\[
S(t; t_0) = 1 + (t - t_0)G(t - t_0) + \cdots,
\]

where we have put for \( t > 0 \)

\[
G(t) = \int_0^t ds\Gamma(s).
\]

If we stop at the second order approximation, we have

\[
\tilde{P}(u, t; t_0) = U(t; t_0)[1 + (t - t_0)G(t - t_0)]P(u, t_0).
\]

Notice the appearance of the secular term which indicates that the above formula is only valid for \( t \) around \( t_0 \).

Now we apply the RG equation to (2.24) which reads

\[
\frac{\partial}{\partial t} \tilde{P}(u, t; t_0) \bigg|_{t=t_0} = 0,
\]

which leads to

\[
\partial_t U_0(t, t_0) \big|_{t_0=t} P(u, t) + \partial_t P(u, t) - G(0)P(u, t) = 0,
\]

where use has been made that \( U_0(t_0, t_0) = 1, \forall t_0 \). Noticing that \( \partial_0 U_0(t, t_0) \big|_{t_0=t} = -L_0 = \nabla_u \cdot h \), we arrive at Fokker-Planck equation,

\[
\partial_t P(u, t) = -\nabla_u \cdot h P(u, t) + G(0)P(u, t).
\]

The concrete form of \( G(0) \) depends on the character of the noise \( R(t) \). This is one of the main results of this section.

To see that (2.25) is the desired equation, let us evaluate \( G(0) \) for a simple Gaussian noise given by

\[
\langle R_i(t)R_j(t') \rangle = 2\delta_{ij}D_i\delta(t - t').
\]

For this case, one has

\[
\Gamma(s) = U_0^{-1}\partial_i g_{ij}\partial_k g_{kl}2D_j\delta_{jl}\delta(s),
\]

where \( \partial_i = \partial/\partial u_i \). Then \( G(t) \) is evaluated as follows;

\[
G(t) \equiv \int_0^t ds\Gamma(s) = \frac{1}{2}U_0^{-1}\partial_i g_{ij}\partial_k g_{kl}2D_j\delta_{jl},
\]

Here we have used the identity \( \theta(0) = 1/2 \), in accordance with the Stratonovich scheme. Notice that \( G(t) \) in this case is independent of \( t \).

Inserting \( G(0) \) thus obtained into (2.25), one has the familiar form of Fokker-Planck equation for the multiplicative Gaussian noise,

\[
\partial_t P(u, t) = -\nabla_u \cdot h P(u, t) + D_j\partial_i g_{ij}\partial_k g_{kj}P(u, t).
\]

This shows that the initial distribution \( P(u, t_0) \) satisfies Fokker-Planck equation and justifies the identification of the initial distribution with the averaged one made in Eq. (2.14), as depicted in Fig. 1.
2.2 Discussion

Firstly, it is noteworthy that we have been naturally led to identify the initial values of the microscopic distribution function \( f(u, t_0, t_0) \) before averaging with the averaged value \( P(u, t_0) \) at an arbitrary initial time \( t = t_0 \). As mentioned in §1, the necessity to take such an initial condition to achieve reduction of evolution equation was advocated by Bogoliubov[1] and others[13, 14] including Boltzmann[3]. Secondly, this means that the nature of the initial time \( t_0 \) in the RG method is completely different from that of the time \( t \) in the stochastic equation (microscopic equation); \( t_0 \) represents the coarse-grained time describing the variation of the averaged quantity, and the derivative \( \partial / \partial t_0 \) in the RG equation is a macroscopic time-derivative. Again as mentioned in §1, this coarse-graining of time was also noted by others [10, 9, 11] in different approaches.

This automatic averaging and the appearance of the macroscopic time-derivative given in the RG method may be generically understood as a generalization of the scheme given in §2 of [20]: First discretize the variable \( u \rightarrow u_i \) and write as \( P(u, t) = X_i(t) \) and use a vector notation \( X = (X_1, X_2, \ldots) \). Thus the discretized stochastic Liouville equation with the initial value \( X(t_0) \) at an arbitrary time \( t_0 \) may be solved perturbatively, and the solution is denoted as \( \tilde{X}(t; t_0, X(t_0)) \), which satisfies the initial condition

\[
\tilde{X}(t_0; t_0, X(t_0)) = X(t_0).
\] (2.29)

We could solve the same equation with the initial condition given at a shifted initial time \( t = t_0 + \Delta t \);

\[
\tilde{X}(t_0 + \Delta t; t_0 + \Delta t, X(t_0 + \Delta t)) = X(t_0 + \Delta t).
\] (2.30)

We suppose that the time difference \( \Delta t \) is macroscopically small but microscopically so large that it may be taken as infinity. For the time \( t \) between \( t_0 \) and \( t_0 + \Delta t \), i.e., \( t_0 < t < t_0 + \Delta t \), the perturbation should be valid. If \( t \rightarrow t_0 \) and \( \Delta t \rightarrow \infty \) in the microscopic scale, we may anticipate that the system is relaxed to the averaged trajectory \( X(t) \) and have

\[
\tilde{X}(t; t_0 + \Delta t, X(t_0 + \Delta t)) \simeq \tilde{X}(t; t_0, X(t_0)),
\]

which implies that the macroscopic time derivative \( \delta / \delta t_0 \) vanishes,

\[
0 = \frac{\delta \tilde{X}}{\delta t_0} \equiv \frac{\tilde{X}(t; t_0 + \Delta t, X(t_0 + \Delta t)) - \tilde{X}(t; t_0, X(t_0))}{\Delta t_0},
\] (2.31)

\[
= \frac{\partial \tilde{X}}{\partial t_0} \bigg|_{t_0=t} + \frac{\partial \tilde{X}}{\partial X} \cdot \frac{dX}{dt_0}.
\] (2.32)

Notice that in the macroscopic scale, the equality \( t_0 \approx t \approx t_0 + \Delta t \) should be taken for granted. This is the RG equation underlying the derivation of Fokker-Planck equation and also other transport equations including kinetic equations as will be shown in the proceeding sections.

In the following sections, we shall show that the general structure elucidated in this section persists in other important examples of the reduction to kinetic equations.
3 From BBGKY hierarchy to Boltzmann equation

In this section, we apply the RG method to derive Boltzmann equation for a classical dilute gas starting from BBGKY hierarchy\[3\]. As is well known, Bogoliubov first derived Boltzmann equation from BBGKY hierarchy in his classic paper\[4\]. His derivation starts from an ansatz that the many particle distribution function depends on time only through the one-particle distribution function and uses a special perturbative expansion method. His approach is actually an application and generalization of the asymptotic theory by Krylov and Bogoliubov (KB) successful to non-linear oscillators\[7\]. Here we do not make that ansatz and start from the naive perturbation theory. We will see how the ansatz given by KB can be incorporated in the RG method. The importance of the initial condition again emerges, which is in accordance with the observation given in \[28\]. This implies that the appearance of a secular term\[23\] does not constitutes the final story for the derivation of Boltzmann equation.

3.1 Derivation of Boltzmann equation in the RG method

Consider a system of $N$ identical classical particles enclosed in a volume $V$. Following the notation of \[29\], we denote the $i$-th particle’s phase space coordinate by $x_i = (r_i, p_i)$. The Hamiltonian of the system is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} U(|r_i - r_j|).$$

We assume that the potential $U$ depends only on the relative distance of two particles and that its range $d$ is much shorter than the mean free path $l$. The $N$-particle distribution function $f_N(x_1, \cdots, x_N, t)$ is normalized as

$$\int f_N(x_1, \cdots, x_N, t) \frac{\prod_{i=1}^{N} dr_i dp_i}{N!} = 1.$$  \hspace{1cm} (3.2)

We define the $s$-particle distribution function by

$$f_s(x_1, \cdots, x_s, t) = \int f_N(x_1, \cdots, x_N, t) \frac{dx_{s+1} \cdots dx_N}{(N-s)!}.$$  \hspace{1cm} (3.3)

Then the normalization condition for $f_s$ becomes

$$\int f_s(x_1, \cdots, x_s, t) dx_1 \cdots dx_s = \frac{N!}{(N-s)!} \approx N^s,$$  \hspace{1cm} (3.4)

from which we see that $f_s$ is of $s$-th order in the particle density $n = \frac{N}{V}$. We assume that $n \ll 1$.

The kinetic equation for $f_s$ is obtained by integrating Liouville equation $\frac{d}{dt} f_N = 0$ over $x_{s+1}, \cdots, x_N$. Equations for $f_1$ and $f_2$ read

$$\frac{d}{dt} f_1(x_1, t) = (\frac{\partial}{\partial t} + iL_1') f_1(x_1, t) = - \int dx_2 L'_{12} f_2(x_1, x_2, t),$$  \hspace{1cm} (3.5)

$$\frac{d}{dt} f_2(x_1, x_2, t) = (\frac{\partial}{\partial t} + iL_{12}) f_2(x_1, x_2, t) = - \int dx_3 (iL'_{13} + iL'_{23}) f_3(x_1, x_2, x_3, t).$$  \hspace{1cm} (3.6)
where

\[ L_{12} = L_1^0 + L_2^0 + L'_{12}, \]

\[ L_i^0 = -i \frac{p_i}{m} \frac{\partial}{\partial r_i}, \quad L'_{ij} = i \frac{\partial U(r_i - r_j)}{\partial r_j} \cdot \left( \frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_i} \right). \]  

(3.7)

These are the first two equations of the BBGKY hierarchy which is a series of equations relating the evolution of \( f_s \) to \( f_{s+1} \). Our goal is to derive an equation (or equations) which captures the essence of the system’s dynamics described by BBGKY hierarchy. In the language of the theory of dynamical systems\(^{[3]}\), we wish to construct a low-dimensional invariant manifold in the (practically) infinite-dimensional functional space spanned by \( \{ f_s \} \) and derive the reduced equations of motion on it.

Whereas Liouville equation or, equivalently, BBGKY hierarchy describes microscopic collisions between particles in detail, what interests us is the macroscopic variation of the system caused by the accumulation of many collisions. More concretely, we wish to know the variation of the system over the space-time scale much longer than the collision radius and much shorter than the mean free path and the mean free time. Such scale is called the mesoscale. The derivatives appearing in (3.5) and (3.6) are, so to speak, microscopic derivatives, while those appearing in kinetic equations are macroscopic derivatives. We must take into account their difference when deriving kinetic equations.

Following\(^{[20]}\), suppose that we have found the solution to BBGKY hierarchy \( \{ f_s(t) \} \) up to an arbitrary time \( t_0 \). With the initial condition \( \{ f_s(x_0, t_0) \} \) we try to solve (3.5) and (3.6) by the perturbative expansion in the density (virial expansion) to obtain a solution \( \tilde{f}_s(x, t) \) around \( t \sim t_0 \). Recalling that \( f_s \) is of \( s \)-th order in the density, we expand as follows.

\[ \tilde{f}_1(x_1, t) = \tilde{f}_1^0(x_1, t) + \tilde{f}_1^1(x_1, t) + \tilde{f}_1^2(x_1, t) + \cdots, \]  

(3.8)

\[ \tilde{f}_2(x_1, x_2, t) = \tilde{f}_2^0(x_1, x_2, t) + \tilde{f}_2^1(x_1, x_2, t) + \cdots, \]  

(3.9)

\[ \tilde{f}_3(x_1, x_2, x_3, t) = \tilde{f}_3^0(x_1, x_2, x_3, t) + \cdots, \]  

(3.10)

where \( \tilde{f}_i^j(x_1, \cdots, x_i, t) \) is of \( (i+j) \)-th order in the density. Substituting the above expansion in (3.5) and (3.6), we get

\[ \frac{d}{dt} \tilde{f}_1^0(x_1, t) = 0, \]  

(3.11)

\[ \frac{d}{dt} \tilde{f}_2^0(x_1, x_2, t) = 0, \]  

(3.12)

\[ (\frac{\partial}{\partial t} + \frac{p_1}{m} \cdot \frac{\partial}{\partial r_1}) \tilde{f}_1^1(x_1, t) = \int dx_2 \frac{\partial}{\partial r_1} U(|r_1 - r_2|) \cdot \frac{\partial}{\partial p_1} \tilde{f}_2^0(x_1, x_2, t), \]  

(3.13)

where we have dropped terms which result in the surface integral. We also expand the initial condition

\[ f_1(x_1, t_0) = f_1^0(x_1, t_0) + f_1^1(x_1, t_0) + \cdots, \]  

\[ f_2(x_1, x_2, t_0) = f_2^0(x_1, x_2, t_0) + \cdots. \]  

(3.14)
Equation (3.11) and (3.12) are easily integrated:

$$f_1^0(x_1, t) = e^{-iL_0^1(t-t_0)}f_1^0(x_1, t_0),$$
$$f_2^0(x_1, x_2, t) = e^{-iL_{12}(t-t_0)}f_2^0(x_1, x_2, t_0) = f_2^0(x_{10}, x_{20}, t_0),$$

(3.15)

where

$$x_{i0}(x_1, x_2, t, t_0), \ i = 1, 2$$

(3.16)

are positions and momenta of the particles 1 and 2 at time $t_0$ under the influence of the 2-body Hamiltonian

$$H^{(2)}(x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + U(|r_1 - r_2|).$$

(3.17)

The initial values $f_1^0(x_1, t_0)$ and $f_2^0(x_1, x_2, t_0)$ may be considered as the integration constants of the lowest-order equation. In the RG method as formulated in [19, 20], the integration constants will constitute the coordinates of the zeroth invariant manifold [5]. The decisive step of the present approach is to choose the initial condition as follows

$$f_2^0(x_1, x_2, t_0) = f_1^0(x_1, t_0),$$

(3.18)

irrespective of the distance between $r_1$ and $r_2$. The underlying picture of this choice is that the system is so dilute that the two particles at an arbitrary time $t_0$ are most probably located at distance much longer than the collision radius $d$, so that the correlation of the two particles is negligible and $f_2$ can be set to the product of one-particle distribution functions. We remark that a probabilistic nature enters at this point [8].

The integration of (3.13) from $t_0$ to $t$ with $\frac{t}{v} \gg t - t_0$ ($v$ is the average velocity), which implies that $t - t_0$ is small in the macroscopic scale, gives

$$\bar{f}_1(x_1, t) = e^{-iL_0^1(t-t_0)}f_1^0(x_1, t_0)$$

$$+ \int_{t_0}^{t} dt' e^{-iL_0^1(t-t')} \int dx_2 \frac{\partial}{\partial r_1} U(|r_1 - r_2|) \cdot \frac{\partial}{\partial p_1} f_1^0(x_{10}, t_0) f_2^0(x_{20}', t_0,$$

(3.19)

where we have used (3.15) and (3.18), and $x_{10}'$ and $x_{20}'$ are given by (3.16) with the replacement $t \rightarrow t'$. We remark that the condition $\frac{t}{v} \gg t - t_0$ is also required for the expansion in the density to be valid [4]. In (3.19), only $x_2$ for $|r_1 - r_2| \leq d$ contributes to the integral. In this region, we can write

$$r_{i0}' \sim r_i - \frac{p_{i0}}{m} (t' - t_0).$$

(3.20)

for a microscopically large period $t' - t_0 \gg \frac{d}{v}$. Here we have neglected vectors whose magnitudes are of order $d$. Then the perturbative solution in the mesoscopic regime $\frac{t}{v} \gg t - t_0 \gg \frac{d}{v}$ is

$$\bar{f}_1(x_1, t) = \bar{f}_1^0(x_1, t) + \bar{f}_1^1(x_1, t)$$

$$= e^{-iL_{12}^1(t-t_0)}f_1^0(x_1, t_0) + \int_{t_0}^{t} dt' e^{-iL_0^1(t-t')} \int dx_2 \frac{\partial}{\partial r_1} U(|r_1 - r_2|)$$

$$\cdot \frac{\partial}{\partial p_1} f_1^0(r_1 - \frac{p_{10}}{m} (t' - t_0), p_{10}, t_0) f_2^0(r_2 - \frac{p_{20}}{m} (t' - t_0), p_{20}, t_0).$$

(3.21)
Note that $p_{i0} = p'_{i0}$: The magnitudes of $\frac{v}{t}$ and $\frac{d}{v}$ are of course different for different systems. For a dilute gas system, typical values are $10^{-8} \sim 10^{-9}$ and $10^{-12} \sim 10^{-13}$, respectively. The second term of the r.h.s. of (3.22) is the secular term. Indeed, it can be shown that in the spatially homogeneous case it is proportional to $t - t_0$.

Accordingly, we have chosen $f^1_1(t, t_0)$ to be zero following the prescription given in §2. The RG equation reads

$$\frac{\partial}{\partial t} f^0_1(x, t) \bigg|_{t=t_0} = 0,$$

(3.22)

$$\Rightarrow \quad \frac{\partial}{\partial t} f^0_1(x, t) + \frac{p_1}{m} \cdot \frac{\partial}{\partial r_1} f^0_1(x, t)$$

$$= \int dx_2 \frac{\partial}{\partial r_1} U(|r_1 - r_2|) \cdot \frac{\partial}{\partial p_1} f^0_1(r_1, \rho_0, t)f^0_1(r_1, \rho_{20}, t),$$

(3.23)

In (3.22) we have imposed that $t = t_0$ although the expression (3.22) is valid for $t - t_0 \gg \frac{d}{v}$. This manipulation can be justified by the same logic given in the last part in §2 and will appear also in the case of field theory discussed in the following section: The $t$-derivative is the microscopic derivative and the $t_0$-derivative is the macroscopic one. Through the RG equation, we can automatically go over to the mesoscopic physics from the microscopic physics. Thus the mesoscopic nature of Boltzmann equation is transparent in our approach. We have also replaced the argument $r_2$ of $f^0_1$ with $r_1$ in (3.23) because only $|r_1 - r_2| \leq d$ contributes to the integration.

(3.23) is the kinetic equation we have been seeking for. In the language of the RG method, it is the renormalization group equation describing the slow motion on the invariant manifold with the coordinate $f^0_1(x, t)$ [23]. To obtain the usual Boltzmann equation which contains the gain minus loss term, we have to manipulate the r.h.s. ignoring the spatial dependence. The result is

$$\frac{\partial}{\partial t} f^0_1(x, t) + \frac{p_1}{m} \cdot \frac{\partial}{\partial r_1} f^0_1(x, t)$$

$$= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi \int d\rho_2 v_{12} \{ f^0_1(r_1, \rho_1', t)f^0_1(r_1, \rho_2', t) - f^0_1(r_1, \rho_1, t)f^0_1(r_1, \rho_2, t) \},$$

(3.24)

where we have introduced the cylindrical coordinate pointing the direction of the relative velocity $v_{12} = (p_2 - p_1)/m$. Comparing (3.11) and (3.23), we see the change of the equation by including the lowest-order contribution of the collision.

### 3.2 Role of the initial condition

Bogoliubov [4] was the first to point out that Boltzmann equation represents the mesoscopic physics and derived it from Liouville equation. In his derivation, he assumed that the system can be described only in terms of the one-particle distribution function. That is, starting from an arbitrary initial condition, the system will rapidly reach the state in which $f_s (s \geq 2)$ are functionals of $f_1$.

$$f_s(x_1, \ldots, x_s, t) = f_s(x_1, \ldots, x_s; f_1(\cdot, t)) \quad s \geq 2.$$  

(3.25)
(\(f_s\) depends on time only through \(f_1\).) In fact, this assumption is a basis of any kinetic theory. The special expansion method based on this assumption leads to a special solution to BBGKY hierarchy. In the RG method, we started with a naive perturbative expansion without any knowledge about kinetic theory. Physics enters when we choose a special initial condition (3.13) and with this choice we can construct an invariant manifold spanned by the one-particle distribution function in the infinite-dimensional functional space, which was originally envisaged by Bogoliubov[4]; see also [28].
4 Derivation of kinetic equation in quantum field theory

In this section we apply the RG method to derive Boltzmann equation for quantum field theory. In fact, the application of the RG method to quantum field theory has been extensively done by Boyanovsky, de Vega and Wang [24] in the prescription given by Illinois group [16]. In the present section, using the self-interacting scalar field theory, one of the model Lagrangians which Boyanovsky et al treated, we shall apply the RG method as formulated in [19, 20] and elucidate how the choice of the initial values is essential to derive the time-irreversible equation and that the scale of the time in the RG equation is different from the microscopic time in the original equation. Although the formulation and the points to be emphasized are shifted from what is given in [24], we shall closely follow the calculation by Boyanovsky et al [24].

Consider the following Hamiltonian:

$$H = H'_0 + H'_{int},$$

$$H'_0 = \frac{1}{2} \int d^3x \left[ \pi^2 + (\nabla \phi)^2 + m_0^2 \phi^2 \right],$$

$$H'_{int} = \int d^3x \frac{\lambda}{4!} \phi^4.$$  

At finite temperature, any particle feels the mean field produced by the thermal excited particles which are present even in the classical regime; hence a change of the particle picture from the bare one to what we call the quasiparticle. Fortunately, in the case of the scalar theory the medium (thermal) effect only affects the mass of the particle. So it is convenient to rewrite the Hamiltonian so that the unperturbed one is written in terms of the quasiparticle operators as follows;

$$H = H_0 + H_{int},$$

$$H_0 = \frac{1}{2} \int d^3x \left[ \pi^2 + (\nabla \phi)^2 + m_{eff}^2 \phi^2 \right] = \int d^3k \omega_k [a(k) a(k)],$$

$$H_{int} = \int d^3x \left[ \frac{\lambda}{4!} \phi^4 + \frac{1}{2} \delta m^2 \phi^2 \right]$$

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \phi(k, t) e^{ik \cdot x}, \quad \phi(k, t) = \frac{1}{\sqrt{2\omega_k}} \left[ a(k, t) + a^\dagger(-k, t) \right],$$

$$\pi(x, t) = \int \frac{d^3k}{(2\pi)^{3/2}} \pi(k, t) e^{ik \cdot x}, \quad \pi(k, t) = i \frac{\omega_k}{2} \left[ a(k, t) + a^\dagger(-k, t) \right],$$

$$\omega_k = \sqrt{k^2 + m_{eff}^2}, \quad m_{eff}^2 + \delta m^2 = m_0^2.$$  

Here we have shifted the mass of the unperturbed Hamiltonian to the quasiparticle mass $m_{eff}$ which is to be determined self-consistently; in the equilibrium, the result is nothing but the effective mass to be given in the hard thermal loop resummation scheme [31, 24];

$$m_{eff}^2 = m_0^2 + \frac{\lambda}{2} \langle \phi^2(x) \rangle = m_0^2 + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1 + 2n_k}{2\omega_k}, \quad \omega_k = \sqrt{k^2 + m_{eff}^2}. $$  

(4.3)
We suppose that the distribution function \( n_k(t) \equiv n(t; \omega_k(t)) \) of the quasi-particles varies slowly. Here we have assumed that the system is spatially homogeneous, so the evolution equation for the distribution function to be obtained may be rather called the rate equation.

To describe the slow variation, we anticipate that \( n_k(t) \) is a macroscopic distribution function varying slowly like the distribution function \( P(u, t) \) governed by Fokker-Planck equation discussed in §2; the corresponding counterpart of the distribution function \( \tilde{P}(u, t; t_0) \) which includes rapid time-variations will be introduced shortly.

To solve the Heisenberg equation for \( a(k, t) \), we impose the following initial condition at \( t = t_0 \) where \( t_0 \) is arbitrary:

\[
\langle a^\dagger(k, t_0) a(k', t_0) \rangle = n_k(t_0) \delta(k - k'),
\]

which implies that the quasi-particles has the mass dependent on the initial time \( t_0 \) as follows,

\[
m^2_{\text{eff}}(t_0) = m_0^2 + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1 + 2n_k(t_0)}{2\omega_k(t_0)}, \quad \omega_k(t_0) = \sqrt{k^2 + m_{\text{eff}}^2(t_0)}.
\]

We define the microscopic distribution function around \( t \sim t_0 \) by

\[
\tilde{n}_k(t; t_0, [n(t_0)]) \delta(0) = \langle a^\dagger(k, t) a(k, t) \rangle,
\]

with the initial condition

\[
\tilde{n}_k(t_0; t_0, [n(t_0)]) = n_k(t_0).
\]

\( \tilde{n}_k(t; t_0, [n(t_0)]) \) is a counterpart of the microscopic distribution function \( \tilde{P}(x, v; t; t_0) \) in contrast to \( n_k(t_0) \). In the following, we shall suppress the third argument in \( \tilde{n}_k \) and write as \( \tilde{n}_k(t; t_0) \). The equation of motion for the distribution function \( \tilde{n}_k(t; t_0) \) is obtained by simply differentiating with respect to time and using the Heisenberg equation of motion. The expectation value can be evaluated perturbatively by the closed time path formalism \[32, 33, 34\]. To two loop order, the result is

\[
\frac{d}{dt} \tilde{n}_k(t; t_0) = \frac{\lambda^2}{3} \int_{-\infty}^{\infty} d\omega R(\omega, k; [n(t_0)]) \frac{\sin[(\omega - \omega_k)(t - t_0)]}{\pi(\omega - \omega_k)},
\]

\[
R(\omega, k; [n(t_0)]) = \frac{\pi}{2\omega_k} \int \frac{d^3q}{(2\pi)^3} \frac{(2\pi)^3 \delta^3(k - q_1 - q_2 - q_3)}{2\omega_{\bar{q}}},
\]

\[
\times \left\{ \delta(\omega + \omega_{q_1} + \omega_{q_2} + \omega_{q_3})N_1(t_0) + 3\delta(\omega + \omega_{q_1} + \omega_{q_2} - \omega_{q_3})N_2(t_0) + 3\delta(\omega + \omega_{q_1} - \omega_{q_2} + \omega_{q_3})N_3(t_0) + \delta(\omega - \omega_{q_1} - \omega_{q_2} - \omega_{q_3})N_4(t_0) \right\}.
\]
\[ N_1(t_0) = (1 + n_{k,1}(t_0))(1 + n_{q_1}(t_0))(1 + n_{q_2}(t_0))(1 + n_{q_3}(t_0)) - n_k(t_0)n_{q_1}(t_0)n_{q_2}(t_0)n_{q_3}(t_0), \]
\[ N_2(t_0) = (1 + n_{k,1}(t_0))(1 + n_{q_1}(t_0))(1 + n_{q_2}(t_0))n_{q_3}(t_0) - n_k(t_0)n_{q_1}(t_0)n_{q_2}(t_0)(1 + n_{q_3}(t_0)), \]
\[ N_3(t_0) = (1 + n_{k,1}(t_0)n_{q_1}(t_0)n_{q_2}(t_0)(1 + n_{q_3}(t_0)) - n_k(t_0)(1 + n_{q_1}(t_0))(1 + n_{q_2}(t_0))n_{q_3}(t_0), \]
\[ N_4(t_0) = (1 + n_{k,1}(t_0)n_{q_1}(t_0)n_{q_2}(t_0)n_{q_3}(t_0) - n_k(t_0)(1 + n_{q_1}(t_0))(1 + n_{q_2}(t_0))(1 + n_{q_3}(t_0)). \]

Notice that the distribution functions in r.h.s are all the initial ones. Integration over \( t \) can be easily performed

\[ \tilde{n}_k(t) = n_k(t_0) + \frac{\lambda^2}{3} \int_{-\infty}^{\infty} d\omega R(\omega, k; [n(t_0)]) \frac{1 - \cos[(\omega - \omega_k)(t - t_0)]}{\pi(\omega - \omega_k)^2}. \] (4.9)

The second term of r.h.s. can include a secular term. Indeed, owing to the limiting behavior familiar in deriving the Fermi’s golden rule

\[ \frac{1 - \cos[(\omega - k)(t - t_0)]}{\pi(\omega - k)^2} \xrightarrow{t-t_0 \to \infty} (t - t_0)\delta(\omega - k), \] (4.10)

(4.9) becomes for \( t - t_0 \to \infty \)

\[ \tilde{n}_k(t; t_0) \sim n_k(t_0) + \frac{\lambda^2}{3} (t - t_0)R(\omega_k, k; [n(t_0)]) + \text{non-secular terms}. \] (4.11)

Notice that the r.h.s. is composed of the macroscopic distribution function. The important point is that although we have taken the formal limit \( t - t_0 \to \infty \), (4.11) is still a local expression valid near \( t_0 \). By “near \( t_0 \)” we mean that \( t - t_0 \) is small in comparison with the macroscopic time scale which is in the same order as the relaxation time. Thus we are naturally led to suppose that the time variable \( t_0 \) is a macroscopic one. The limit \( t - t_0 \to \infty \) merely means that \( t - t_0 \) is large in comparison with the microscopic time scale which is in the same order as the inverse of the (thermal) mass of the particle. As was emphasized by Boyanovsky et al [24], the separation of these scales is justified in the weak coupling regime, and the existence of such an intermediate scale is essential for the validity of the kinetic equation like Boltzmann equation. A nice point in the RG method lies in the fact that we naturally get to change our standpoint to observe the system from the mesoscopic point of view, as noticed above.

Having made this change of the time scale, the RG equation

\[ \frac{d}{dt} \tilde{n}_k(t; t_0) \bigg|_{t=t_0} = 0 \] (4.12)

leads to the equation for the initial value \( n_k(t) \) as the macroscopic variable with the macroscopic time \( t \) as follows

\[ \frac{d}{dt} n_k(t) = \frac{\lambda^2}{3} R(\omega_k, k; [n(t)]) = \frac{\lambda^2}{3} \frac{\pi}{2\omega_k} \int \prod_i \frac{d^3q_i}{(2\pi)^3(2\pi)^3} (2\pi)^3 \delta^3(k - q_1 - q_2 - q_3) \times \left\{ \delta(\omega + \omega_{q_1} + \omega_{q_2} + \omega_{q_3})N_1(t) + 3\delta(\omega + \omega_{q_1} + \omega_{q_2} - \omega_{q_3})N_2(t) + 3\delta(\omega - \omega_{q_1} - \omega_{q_2} + \omega_{q_3})N_3(t) + \delta(\omega - \omega_{q_1} - \omega_{q_2} - \omega_{q_3})N_4(t) \right\}, \] (4.13)
which is nothing other than Boltzmann equation governing the slow variation of the distribution function. Thus the integration constant $n_k(t)$ as given by the initial value has become a dynamical variable whose dynamics (4.13) represents the reduced dynamics of the system, in accordance with the general property of the RG method elucidated by Ei et al. We should remark that Eq.(4.13) is supplemented with the time-dependent mean-field equation governing the time-evolution of $m_{\text{eff}}(t)$, which is now given by

$$m_{\text{eff}}^2(t) = m_0^2 + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1 + 2n_k(t)}{2\omega_k(t)}, \quad \omega_k(t) = \sqrt{k^2 + m_{\text{eff}}^2(t)}. \quad (4.14)$$

This condition leads to the r.h.s. of (4.8) which only contains the potential secular term and is equal to the time-dependent version of (4.3). Thus the quasiparticle defined by the hard thermal loop resummation scheme is the relevant slow dynamical variable and we must choose the initial condition to be the distribution function of it to obtain the rate equation. The importance of the choice of the initial condition is again recognized.
5 Fluid dynamical limit of Boltzmann equation

In this section, we apply the RG method formulated above to obtain the fluid dynamical limit of Boltzmann equation; in other words, we shall derive Euler and Navier-Stokes equations successively.

5.1 Basics of Boltzmann equation

Boltzmann equation is an evolution equation of the one-particle distribution function \( f(r, v, t) \) in the phase space, and reads

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} = I[f].
\]  (5.1)

Here the left-hand side describes the change due to the canonical equation of motion while the right-hand side the change due to collisions;

\[
I[f] = \int dv_1 \int dv' \int dv'_1 w(v v_1 | v' v'_1) \\
\times \left\{ f(r, v', t) f(r, v'_1, t) - f(r, v, t) f(r, v_1, t) \right\},
\]  (5.2)

which is called the collision integral. The transition probability \( w(v v_1 | v' v'_1) \) has the following symmetry due to the time-reversal invariance of the microscopic equation of motion;

\[
w(v v_1 | v' v'_1) = w(v' v'_1 | v v_1).
\]  (5.3)

Furthermore, the invariance under the particle-interchange implies the following equality;

\[
w(v v_1 | v' v'_1) = w(v_1 v | v'_1 v') = w(v'_1 v' | v v_1),
\]  (5.4)

where (5.3) has been used in the last equality.

Owing to the conservation of the particle number, the total momentum and the total kinetic energy during the collisions, the following equalities hold,

\[
\int dv I[f] = 0, \quad \int dv v I[f] = 0, \quad \int dv v^2 I[f] = 0.
\]  (5.5)

We say that the function \( \varphi(v) \) is a collision invariant if it satisfies the following equation;

\[
\int dv \varphi(v) I[f] = 0.
\]  (5.6)

For a collision invariant \( \varphi(v) \), we can define the density \( n_\varphi \) and the current \( j_\varphi \) as follows;

\[
n_\varphi = \int dv \varphi(v) f(r, v, t), \quad j_\varphi = \int dv v \varphi(v) f(r, v, t),
\]  (5.7)

which satisfy the continuity or balance equation;

\[
\partial_t n_\varphi + \nabla \cdot j_\varphi = 0.
\]  (5.8)
Thus we have formally the following fluid-dynamical equations as the balance equations for the conservation of the particle number, total momentum and kinetic energy;

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (5.9)
\]

\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_j u_i)}{\partial j} + \partial_j P_{ji} = 0, \quad (5.10)
\]

\[
\frac{\partial (\rho u_i^2 + e)}{\partial t} + \partial_j \left[ (\rho u_i^2 + e) u_i + Q_i \right] + \partial_i (P_{ij} u_j) = 0, \quad (5.11)
\]

respectively. Here, \(\rho, \mathbf{u}, e, P_{ij}, Q_i\) are the particle density, the stream velocity, the energy density, the pressure tensor and the heat flux, respectively and defined as follows;

\[
\rho(r, t) = m \int dv f(r, \mathbf{v}, t), \quad (5.12)
\]

\[
\rho(r, t) \mathbf{u}(r, t) = m \int dv v f(r, \mathbf{v}, t), \quad (5.13)
\]

\[
e(r, t) = \int dv \frac{m}{2} |v - \mathbf{u}|^2 f(r, \mathbf{v}, t), \quad (5.14)
\]

\[
P_{ij}(r, t) = \int dv m(v_i - u_i)(v_j - u_j) f(r, \mathbf{v}, t), \quad (5.15)
\]

\[
Q_i(r, t) = \int dv \frac{m}{2} |v - \mathbf{u}|^2 (v_i - u_i) f(r, \mathbf{v}, t). \quad (5.16)
\]

These equations are formal ones in the sense that the distribution function \(f\) is not yet solved: The solution obtained from Boltzmann equation will give the explicit forms of the internal energy, the transport coefficients and so on.

The \(H\) function is defined as follows:

\[
H(r, t) = \int dv f(r, \mathbf{v}, t)(\ln f(r, \mathbf{v}, t) - 1). \quad (5.17)
\]

For equilibrium states, the \(H\) function is equal to the entropy \(S\) with the sign changed. Defining the corresponding current by

\[
\mathbf{J}_H(r, t) = \int dv \mathbf{v} f(r, \mathbf{v}, t)(\ln f(r, \mathbf{v}, t) - 1), \quad (5.18)
\]

one has the balance equation;

\[
\frac{\partial H}{\partial t} + \nabla \cdot \mathbf{J}_H = \int dv f \ln f. \quad (5.19)
\]

This shows that when \(\ln f\) is a collision invariant, the \(H\) function is conserved. In fact, one can show that this condition is satisfied when \(f(r, \mathbf{v}, t)\) is a local equilibrium distribution function as given by (5.32) below.

### 5.2 Derivation of fluid dynamical equations in the RG method

To make it clear that the following discussion fits to the general formulation given in [20], we discretize the argument \(\mathbf{v}\) as \(\mathbf{v} \rightarrow \mathbf{v}_i\): Discriminating the arguments \((r, t)\) and \(\mathbf{v}_i\) in \(f(r, \mathbf{v}_i, t)\), we indicate \(\mathbf{v}_i\) as a subscript \(i\) for the distribution function;

\[
f(r, \mathbf{v}_i, t) = f_i(r, t) \equiv (f(r, t))_{i}. \quad (5.20)
\]
Then Boltzmann equation reads

\[
\frac{\partial f_i}{\partial t} = \dot{I}[f]_i - \mathbf{v}_i \cdot \frac{\partial f_i}{\partial \mathbf{r}}, \quad (5.21)
\]

where

\[
\dot{I}[f]_i = \sum_{j,k,l} w_{ijkl}(\mathbf{v}_i \cdot \mathbf{v}_j)(\mathbf{v}_k \cdot \mathbf{v}_l)(f_k f_l - f_i f_j)(\mathbf{r}, t). \quad (5.22)
\]

Now let us consider a situation where the fluid motion is slow with long wave-lengths so that

\[
\mathbf{v}_i \cdot \frac{\partial f_i}{\partial \mathbf{r}} = O(\epsilon), \quad (5.23)
\]

where \(\epsilon\) is a small quantity, \(|\epsilon| < 1\).

To take into account the smallness of \(\epsilon\) in the following calculations formally, let us introduce the scaled coordinate \(\bar{\mathbf{r}}\) defined by

\[
\bar{\mathbf{r}} = \epsilon \mathbf{r}, \quad \frac{\partial}{\partial \mathbf{r}} = \epsilon \frac{\partial}{\partial \bar{\mathbf{r}}}. \quad (5.24)
\]

Then (5.21) now reads

\[
\frac{\partial f_i}{\partial t} = \dot{I}[f]_i - \epsilon \mathbf{v}_i \cdot \frac{\partial f_i}{\partial \bar{\mathbf{r}}}, \quad (5.25)
\]

which has a form to which the perturbation theory given in \[20\] is naturally applicable.

In accordance with the general formulation given in \[20\], we first expand the solution as follows;

\[
f_i(\bar{\mathbf{r}}, t) = f_i^{(0)}(\bar{\mathbf{r}}, t) + \epsilon f_i^{(1)}(\bar{\mathbf{r}}, t) + \cdots.
\]

Let \(\tilde{f}_i(\bar{\mathbf{r}}, t; t_0)\) be a solution around \(t \sim t_0\) given by a perturbation theory with \(f_i(\bar{\mathbf{r}}, t_0)\) being the initial value at \(t = t_0\); \n
\[
\tilde{f}_i(\bar{\mathbf{r}}, t = t_0; t_0) = f_i(\bar{\mathbf{r}}, t_0). \quad (5.26)
\]

We expand \(\tilde{f}_i(\bar{\mathbf{r}}, t; t_0)\) as

\[
\tilde{f}_i(\bar{\mathbf{r}}, t; t_0) = \tilde{f}_i^{(0)}(\bar{\mathbf{r}}, t; t_0) + \epsilon \tilde{f}_i^{(1)}(\bar{\mathbf{r}}, t; t_0) + \cdots, \quad (5.27)
\]

and the respective initial condition is set up as follows,

\[
\tilde{f}_i^{(l)}(\bar{\mathbf{r}}, t = t_0; t_0) = f_i^{(l)}(\bar{\mathbf{r}}, t_0), \quad (l = 0, 1, 2, \ldots). \quad (5.28)
\]

The 0-th order equation reads

\[
\frac{\partial \tilde{f}_i^{(0)}}{\partial t} = (I[\tilde{f}^{(0)}])_i. \quad (5.29)
\]
Now we are interested in the slow motion which may be achieved asymptotically as \( t \to \infty \). Therefore we take the following stationary solution,

\[
\frac{\partial \tilde{f}^{(0)}}{\partial t} = 0,
\]

which is a fixed point of the equation satisfying

\[
(\hat{I}[\tilde{f}^{(0)}])_i = 0,
\]

for arbitrary \( \mathbf{r} \). Notice that (5.31) shows that the distribution function \( \tilde{f}^{(0)} \) is a function of collision invariants. Such a distribution function is a local equilibrium distribution function or Maxwellian;

\[
\tilde{f}^{(0)}(\mathbf{r}, t; t_0) = n(\mathbf{r}, t_0) \left( \frac{m}{2\pi k_B T(\mathbf{r}, t_0)} \right)^{3/2} \exp \left[ -\frac{m|\mathbf{v}_i - \mathbf{u}(\mathbf{r}, t_0)|^2}{2k_B T(\mathbf{r}, t_0)} \right].
\]

Here, the local density \( n \), local temperature \( T \), local flux \( \mathbf{u} \) are all dependent on the initial time \( t_0 \) and the space coordinate \( \mathbf{r} \) but independent of time \( t \).

The first-order equation reads

\[
\left( \frac{\partial}{\partial t} - A \right) \tilde{f}^{(1)} = -\mathbf{v}_i \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \mathbf{r}}.
\]

Here the linear operator \( A \) is defined by

\[
\left[ \hat{I}'[\tilde{f}^{(0)}] \tilde{f}^{(1)} \right]_i = \sum_{j,k} \frac{\partial \hat{I}}{\partial f_j} \left. \tilde{f}^{(0)} \right| \tilde{f} = \tilde{f}^{(0)} \tilde{f}^{(1)} \equiv (A \tilde{f}^{(1)})_i.
\]

The operator \( A \) satisfies the following; Let \( f_i = f^{(0)}_i \psi_i \). Then

\[
(A f)_i = f^{(0)}_i \sum_{j,k,l} \psi_j \psi_k \psi_l \psi_i \equiv f^{(0)}_i (A \psi)_i.
\]

Here \( A \) is an operation acting on \( \psi \); \( \psi_i = (\psi)_i \). Defining the inner product between \( \varphi \) and \( \psi \) by

\[
\langle \varphi, \psi \rangle = \int d\mathbf{v} \varphi \psi,
\]

one can show that \( A \) is self-adjoint;

\[
\langle \varphi, A \psi \rangle = \langle A \varphi, \psi \rangle.
\]

One can further show that the five invariants \( m, \mathbf{v}, \frac{m}{2} v^2 \) span the kernel of \( A \);

\[
\text{Ker}A = \{ m, \mathbf{v}, \frac{m}{2} v^2 \}.
\]

The other eigenvalues are found to be negative because one can show

\[
\langle \varphi, A \varphi \rangle \leq 0.
\]
We write the projection operator to the kernel as $P$ and define $Q = 1 - P$. Applying the general formulation in [20], one can readily obtain the first-order solution,

$$\tilde{f}^{(1)} = -(t - t_0)Pv \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \tilde{r}} + A^{-1}Qv \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \tilde{r}}. \quad (5.40)$$

The perturbative solution up to the $\epsilon$ order is found to be

$$\tilde{f}(\tilde{r}, t, t_0) = \tilde{f}^{(0)}(\tilde{r}, t, t_0) + \epsilon[-(t - t_0)Pv \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \tilde{r}} + A^{-1}Qv \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \tilde{r}}]. \quad (5.41)$$

Notice the appearance of a secular term.

If one stops the approximation and apply the RG equation, $\partial \tilde{f}/\partial t|_{t_0=t}=0$, one has

$$\frac{\partial \tilde{f}^{(0)}}{\partial t} + \epsilon P v \cdot \frac{\partial \tilde{f}^{(0)}}{\partial \tilde{r}} = 0. \quad (5.42)$$

This is a master equation from which equations governing the time evolution of $n(r, t)$, $u(r, t)$ and $T(r, t)$ in $f^{(0)}$, i.e., fluid dynamical equations: In fact, taking an inner product between $m/2$ and $mv^2/2$ with this equation, one has fluid dynamical equations as given in (5.9)-(5.11). The only difference from (5.9)-(5.11) lies in the fact that $f$ is now explicitly solved and the energy density $e$, the pressure tensor $P_{ij}$ and the heat flux $Q_i$ are given as follows;

$$e(r, t) = \int d^3v \frac{m}{2} |v - \tilde{u}|^2 f^{(0)}(r, v, t) = \frac{3}{2} k_B T(r, t), \quad (5.43)$$

$$P_{ij}(r, t) = \int d^3v (v_i - u_i)(v_j - u_j) f^{(0)}(r, v, t) = nk_B T(r, t) \delta_{ij} \equiv P(r, t) \delta_{ij}, \quad (5.44)$$

$$Q_i(r, t) = \int d^3v \frac{m}{2} |v - u|^2 (v_i - u_i) f^{(0)}(r, v, t) = 0. \quad (5.45)$$

We have defined the pressure $P$ using the equation of state for the ideal gas in the second line. There is no heat flux because the distribution function $f^{(0)}$ in the formulae is the one for the local equilibrium. Inserting these formulae into (5.9)-(5.11), we end up with a fluid dynamical equation without dissipation, i.e., the Euler equation in this approximation;

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0, \quad (5.46)$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j) + \frac{\partial P}{\partial x_i} = 0, \quad (5.47)$$

$$\frac{\partial}{\partial t} (\rho u_i^2 + e) + \frac{\partial}{\partial x_j} \left[ (\rho u_i^2 + e + P) u_j \right] = 0. \quad (5.48)$$

We notice that these equations have been obtained from the RG equation (5.42). It should be emphasized however that the distribution function obtained in the present approximation takes the form

$$f(r, v, t) = f^{(0)}(r, v, t) + A^{-1}Qv \cdot \frac{\partial f^{(0)}(r, v, t)}{\partial r}, \quad (5.49)$$
which incorporates as a perturbation a distortion from the local equilibrium distribution and gives rise to dissipations.

One can proceed to the second order approximation straightforwardly and obtain fluid dynamical equation with dissipations as the RG equation. The perturbation equation in the second order reads

\[
\left( \frac{\partial}{\partial t} - A \right) \tilde{f}^{(2)}_i = -\mathbf{v}_i \cdot \frac{\partial \tilde{f}^{(1)}_i}{\partial \mathbf{r}}.
\]

(5.50)

Here, we must make an important notice: We have actually used the linearized Boltzmann equation\[35\] neglecting the second-order term of \( \tilde{f}^{(1)} \) in the collision integral: It is known that the neglected term produces the so called Burnett terms which are absent in the usual Navier-Stokes equation \[30\].

Inserting the first order solution \( \tilde{f}^{(1)} \) into \( \tilde{f}^{(2)} \), we have for the second order solution

\[
f^{(2)}(\mathbf{r}, t; t_0) = -(t - t_0)\{A^{-1}Q \mathbf{v} \cdot \nabla P \mathbf{v} \cdot \nabla + P \mathbf{v} \cdot \nabla A^{-1}Q \mathbf{v} \cdot \nabla\} f^{(0)}(\mathbf{r}, t_0) - \{(A^{-1}Q \mathbf{v} \cdot \nabla)^2 - \frac{1}{2}(t - t_0)^2(P \mathbf{v} \cdot \nabla)^2\} f^{(0)}(\mathbf{r}, t_0).
\]

(5.51)

Adding up the all the solutions up to the second order and applying the RG equation \( \frac{df}{dt}|_{t_0=t} = 0 \), we have

\[
\frac{\partial f^{(0)}}{\partial t} + \epsilon\{P \mathbf{v} \cdot \nabla f^{(0)} + A^{-1}Q \mathbf{v} \cdot \nabla \frac{\partial f^{(0)}}{\partial t}\} + \epsilon^2\{A^{-1}Q \mathbf{v} \cdot \nabla P \mathbf{v} \cdot \nabla + P \mathbf{v} \cdot \nabla A^{-1}Q \mathbf{v} \cdot \nabla\} f^{(0)} = 0.
\]

(5.52)

Notice that the time derivative of the distribution function in the first order terms in \( \epsilon \) must be retained in this order, because the time derivative is at most of order \( \epsilon \). Applying the projection operators P and Q from the left, we have

\[
\frac{\partial f^{(0)}}{\partial t} + \epsilon P \mathbf{v} \cdot \nabla f^{(0)} + \epsilon^2 P \mathbf{v} \cdot \nabla A^{-1}Q \mathbf{v} \cdot \nabla f^{(0)} = 0,
\]

(5.53)

\[
\epsilon A^{-1}Q \mathbf{v} \cdot \nabla \{\frac{\partial f^{(0)}}{\partial t} + \epsilon P \mathbf{v} \cdot \nabla f^{(0)}\} = 0,
\]

(5.54)

respectively. Clearly, the second equation follows from the first one in this order. The third term in the first equation represents dissipations. Taking inner products between the first equation and the particle number, the velocity and the kinetic energy, one will obtain a fluid dynamical equation with dissipations included, i.e., Navier-Stokes equation\[35\].

In summary, we have shown that the fluid dynamical limit of the Boltzmann equation can be obtained neatly in the RG method as formulated in \[20\]. Such problems to reduce a kinetic equation to a further slower dynamics appear quite often, reflecting the hierarchy of the space-time of nature. A typical problem \[8, 36, 37, 38, 39, 40\] in this category is to derive Smoluchowski equation\[41\] from Kramers equation\[42\]. As is expected, it is possible to develop a systematic theory for the adiabatic elimination of fast variables in Fokker-Planck equations, which is described in the next section.
6 Adiabatic Elimination of Fast Variables in Fokker-Planck Equation

In this section, we shall show that the RG method as formulated in [20] gives a systematic way for the adiabatic elimination of fast variables appearing in Fokker-Planck equations [36, 3, 37, 38, 39]. A typical problem in this category is to reduce the so called Kramers equation [42] to Smoluchowski equation [41]. Although Brinkman [36] was the first for a serious attempt, reliable derivations was given rather lately [3, 37, 38, 39]; see [3, 40] for a review. Recently, it was shown that the RG method as formulated in [16] can be used to eliminate a fast variable in the Fokker-Planck equation [43]. We shall show that the RG method gives the results which have a clear correspondence with those given in other methods [3, 40]; thereby, we would say, our method gives a foundation for these methods.

6.1 A generic FP equation with a fast variable

Using a generic example, we shall present the method for adiabatic elimination of fast variables based on the RG equation.

The example is the following 2-dimensional Langevin equation with $\gamma$ being a large number,

$$\begin{align*}
\dot{x} &= h_x(x, y) + g_x(x, y) \Gamma_x(t), \\
\dot{y} &= \gamma h_y(x, y) + f(x, y) + \sqrt{\gamma} g_y(x, y) \Gamma_y(t),
\end{align*}$$

where $\Gamma_i(t)$ ($i = x, y$) are independent Gaussian noise satisfying

$$\langle \Gamma_i(t) \Gamma_j(t') \rangle = 2\delta_{ij} \delta(t - t').$$

(6.1)

This example is treated by Gardiner [3]. We follow Gardiner for the notations. We suppose that owing to the large friction $\gamma$, the variable $y$ is a fast variable. Our task is to eliminate the fast variable adiabatically and obtain the reduced dynamics for the slow variable $x$ only. Before doing this task, we first eliminate the most rapid variables, i.e., $\Gamma_i(t)$ ($i = x, y$) to obtain the dynamics written only in terms of $x$ and $y$. This is tantamount to transforming the Langevin equation to Fokker-Planck equation as was done in §2; it implies that what we are trying to is to further reduce a kinetic equation to a slower dynamics. The corresponding Fokker-Planck equation for the probability $W(x, y, t)$ reads

$$\frac{\partial W}{\partial t} = [\hat{L}_1 + \gamma \hat{L}_0] W,$$

(6.3)

where

$$\begin{align*}
\hat{L}_1 &= -(\frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} f(x, y)) + \frac{\partial^2}{\partial x^2} D_{xx}, \\
\hat{L}_0 &= -(\frac{\partial}{\partial y} D_x + \frac{\partial^2}{\partial y^2} D_{yy}),
\end{align*}$$

(6.4, 6.5)

with

$$D_i = h_i(x, y) + g_i(x, y) \frac{d}{dx_i} g_i(x, y), \quad D_{ii} = g_i^2(x, y), \quad (i = x, y).$$

(6.6)
To apply the perturbation theory, we introduce a scaled time $\tau$ by

$$\epsilon \tau = t,$$  \hspace{1cm} (6.7)

with $\epsilon = 1/\gamma < 1$ and write $\tau$ as $t$ for the moment. Our equation now reads

$$(\partial_t - \hat{L}_0)W = \epsilon \hat{L}_1 W,$$  \hspace{1cm} (6.8)

We denote the eigenvalues and the eigenfunctions of the unperturbed operator $\hat{L}_0(y, x)$ as follows;

$$\hat{L}_0 \varphi_n(y; x) = -\lambda_n(x) \varphi_n(y; x).$$  \hspace{1cm} (6.9)

Notice that $x$ in $\hat{L}_0(y; x)$ plays a role only as a parameter. The corresponding adjoint equation reads

$$\hat{L}_0^\dagger \varphi_n^\dagger(y; x) = -\lambda_n(x) \varphi_n^\dagger(y; x).$$  \hspace{1cm} (6.10)

We assume that $\lambda_n(x) \geq 0$ and that the 0-eigenvalue $\lambda_0 = 0$ is non-degenerate: The eigenfunction $\varphi_0(y; x)$ is proportional to the stationary distribution function, so we can write as $\varphi_0(y; x) = W_{st}(y; x)$. It can be shown that the eigenfunctions $\varphi_n(y; x)$ and $\varphi_n^\dagger(y; x)$ with $n \geq 1$ are written in terms of $\varphi_0(y; x)$ as

$$\varphi_n(y; x) = \sqrt{\varphi_0(y; x)} \psi_n(y; x), \quad \varphi_n^\dagger(y; x) = \psi_n(y; x)/\sqrt{\varphi_0(y; x)},$$  \hspace{1cm} (6.11)

where $\psi_n(y; x)$ is an eigenfunction of the Hermite operator

$$\hat{H}_0 = \frac{1}{\sqrt{\varphi_0(y; x)}} \hat{L}_0 \sqrt{\varphi_0(y; x)},$$  \hspace{1cm} (6.12)

with the eigenvalue $\lambda_n(x)$;

$$\hat{H}_0 \psi_n(y; x) = -\lambda_n \psi_n(y; x).$$  \hspace{1cm} (6.13)

Owing to the Hermiteness of $\hat{H}_0$, the orthonormality and completeness hold;

$$\int \psi_n(y; x) \psi_m(y; x) dy = \int \varphi_n^\dagger(y; x) \varphi_m(y; x) dy = \delta_{nm},$$  \hspace{1cm} (6.14)

$$\sum_{n=0}^{\infty} \psi_n(y; x) \psi_n(y'; x) = \sum_{n=0}^{\infty} \varphi_n^\dagger(y; x) \varphi_n(y'; x) = \delta(y - y').$$  \hspace{1cm} (6.15)

For later use, we define the projection operator $P$ to the kernel of $\hat{L}_0$ by

$$[P \varphi](y; x) = \varphi_0(y; x) \int \varphi_0^\dagger(y'; x) \varphi(y'; x) dy'.$$  \hspace{1cm} (6.16)

The projection operator to the orthogonal space to the kernel is denoted as $Q$,

$$Q = 1 - P = \sum_{n=1}^{\infty} \varphi_n(y; x) \int dy' \varphi_n^\dagger(y'; x).$$  \hspace{1cm} (6.17)
We remark that
\[ \varphi_0^*(y; x) = 1. \quad (6.18) \]

In the following, we assume that
\[ P\hat{L}_1 P = 0, \quad (6.19) \]

which can be always satisfied by redefinition of \( \hat{L}_0 \) and \( \hat{L}_1 \). In terms of the notions in quantum field theory, \( \hat{L}_1 \) is normal-ordered with respect to the vacuum \( |0\rangle \equiv \varphi_0; \langle 0|\hat{L}_1|0\rangle = 0. \)

Let us suppose that a solution is given in the perturbation series;
\[ W(t, x, y) = W_0(t, x, y) + \epsilon W_1(t, x, y) + \epsilon^2 W_2(t, x, y) + \ldots. \]

Then following the general scheme of the RG method as given in Appendix A of [20], we first construct the perturbed solution
\[ \tilde{W}(t, x, y; t_0) = \tilde{W}_0(t, x, y; t_0) + \epsilon \tilde{W}_1(t, x, y) + \epsilon^2 \tilde{W}_2(t, x, y; t_0) + \ldots \quad (6.20) \]

with the initial conditions given at arbitrary time \( t = t_0; \)
\[ \tilde{W}_i(t_0, x, y; t_0) = W_i(t_0, x, y). \quad (6.21) \]

Notice that this implies that
\[ W(t, x, y) = \tilde{W}(t, x, y; t). \quad (6.22) \]

Now the equations for \( \tilde{W} \ (n = 0, 1, 2, \ldots) \) read
\[ (\partial_t - \hat{L}_0)\tilde{W}_0 = 0, \]
\[ (\partial_t - \hat{L}_0)\tilde{W}_n = \hat{L}_1\tilde{W}_{n-1}, \quad (n = 1, 2, \ldots). \quad (6.23) \]

The solution to the lowest order equation around the initial time \( t = t_0 \) now reads
\[ \tilde{W}_0(t, x, y; t_0) = \sum_{n=0}^{\infty} C_n(x; t_0)\varphi_n(y; x)e^{-\lambda_n t}. \quad (6.24) \]

Notice that the coefficient function \( C_n(x; t_0) \) may depend on the initial time \( t_0 \) as well as \( x \). This solution implies that the initial value is taken as
\[ W(t_0, x, y) = \tilde{W}_0(t_0, x, y; t_0) = \sum_{n=0}^{\infty} C_n(x; t_0)\varphi_n(y; x)e^{-\lambda_n t_0}. \quad (6.25) \]

If we are only concerned with the asymptotic (long time) behavior as \( t \to +\infty \), however, we may keep only the stationary solution in the sum, namely,
\[ \tilde{W}_0(t, x, y; t_0) = C_0(x; t_0)\varphi_0(y; x) = W_0(t_0, x, y). \quad (6.26) \]

Notice the last equality, which gives the functional form for the true solution in the 0-th order, holds because of the stationarity of the solution. We remark also that
\[ P\tilde{W}_0(t, x, y; t_0) = \tilde{W}_0(t, x, y; t_0). \quad (6.27) \]
Let us solve Eq.(6.23) order by order. The solution to Eq.(6.23) is formally given by

\[ \tilde{W}_n(t, x, y; t_0) = \frac{1}{\partial_t - \hat{L}_0} \hat{L}_1 \tilde{W}_{n-1} = \frac{1}{\partial_t - \hat{L}_0} (P + Q) \hat{L}_1 \tilde{W}_{n-1}, \quad (6.28) \]

\[ \equiv X_n(t, x, y; t_0) + Y_n(t, x, y; t_0), \quad (6.29) \]

where

\[ X_n \equiv P\tilde{W}_n(t, x, y; t_0) = \frac{1}{\partial_t - \hat{L}_0} P\hat{L}_1 \tilde{W}_{n-1}, \]

\[ Y_n \equiv Q\tilde{W}_n(t, x, y; t_0) = \frac{1}{\partial_t - \hat{L}_0} Q\hat{L}_1 \tilde{W}_{n-1}, \quad (6.30) \]

with \( n = 1, 2, ... \). To obtain the special solutions to these inhomogeneous equations, one must impose an initial condition. We impose the initial condition at \( t = t_0 \) so that possible \( P \)-space components in the perturbed solutions is “renormalized away” to the unperturbed solution[19, 20]; namely, we shall construct special solutions which satisfy

\[ P\tilde{W}_n(t_0, x, y; t_0) = X_n(t_0, x, y; t_0) = 0. \quad (6.31) \]

We shall see that this condition is satisfied owing to the appearance of secular terms proportional to \((t - t_0)^n\) multiplied to the \( P \) operator.

The special solutions to the perturbative equations (6.23) with the initial condition (6.31) may be obtained in a mechanical way by the method given in Appendix A of[20]: For any constant vector \( U \) the following formulae hold[20];

\[ \frac{1}{\partial_t - \hat{L}_0} QU = \frac{1}{-\hat{L}_0} QU, \quad (6.32) \]

\[ \frac{1}{\partial_t - \hat{L}_0} (t - t_0)^n PU = \frac{1}{(n + 1)} (t - t_0)^{n+1} PU, \quad (6.33) \]

\[ \frac{1}{\partial_t - \hat{L}_0} (t - t_0)^n PU = \frac{1}{(n + 1)} (t - t_0)^{n+1} PU, \]

for \( n = 0, 1, 2,... \), especially for \( n = 0 \),

\[ \frac{1}{\partial_t - \hat{L}_0} PU = (t - t_0)PU, \quad (6.35) \]

and

\[ \frac{1}{\partial_t - \hat{L}_0} (t - t_0)Q\hat{L}_1 U = (t - t_0) \frac{1}{-\hat{L}_0} Q\hat{L}_1 U - \frac{1}{(-\hat{L}_0)^2} Q\hat{L}_1 U. \quad (6.36) \]

Here, notice that \([\hat{L}_0, Q] = 0\), so \( Q/\hat{L}_0 = \hat{L}_0^{-1} Q = Q\hat{L}_0^{-1} Q\). We remark that these special solutions are all compatible with the initial condition (6.31).

In the first order, one immediately sees that \( X_1 \) identically vanishes because

\[ P\hat{L}_1 \tilde{W}_0 = P\hat{L}_1 P\tilde{W}_0 = 0, \quad (6.37) \]
on account of (6.19) and (6.27). Thus we have for the first order solution

\[ \tilde{W}_1(t, x, y; t_0) = Y_1 = \frac{1}{-\hat{L}_0} Q \hat{L}_1 \tilde{W}_0. \]  \hspace{1cm} (6.38)

A couple of remarks are in order: (i) One may replace the operator \( Q \) with \( P + Q = 1 \) in this expression owing to Eq. (6.37). Although such simplification in the expressions may be done also for higher-order solutions, we shall retain the \( Q \) operator for definiteness, since the expression \((- \hat{L}_0)^{-1} \hat{L}_1 \) itself is ill-defined because of the zero eigenvalue of \( \hat{L}_0 \).

(ii) The solution (6.38) satisfies the initial condition (6.31) as

\[ \tilde{W}_1(t_0, x, y, t_0) = W_1(t_0, x, y) = 1 - \hat{L}_0 Q \hat{L}_1 \tilde{W}_0. \]  \hspace{1cm} (6.39)

The second order terms are evaluated as follows:

\[ X_2 = \frac{1}{\partial_t - \hat{L}_0} P \hat{L}_1 \frac{1}{-\hat{L}_0} Q \hat{L}_1 \tilde{W}_0, \]

\[ = (t - t_0) P \hat{L}_1 \frac{Q}{-\hat{L}_0} \hat{L}_1 \tilde{W}_0, \]  \hspace{1cm} (6.40)

where (6.35) has been used. Similarly, using (6.32) we have

\[ Y_2 = \frac{1}{\partial_t - \hat{L}_0} Q \hat{L}_1 \frac{1}{-\hat{L}_0} Q \hat{L}_1 \tilde{W}_0, \]

\[ = \left( \frac{Q}{-\hat{L}_0} \right)^2 \tilde{W}_0. \]  \hspace{1cm} (6.41)

Adding these terms, we have \( \tilde{W}_2(t, x, y, t_0) \). We remark that

\[ \tilde{W}_2(t_0, x, y, t_0) = W_2(t_0, x, y) = \frac{1}{\hat{L}_0} Q \hat{L}_1 \frac{1}{\hat{L}_0} Q \hat{L}_1 W_0(t_0, x, y), \]  \hspace{1cm} (6.42)

which satisfies the initial condition (6.31).

Let us proceed to the third order equation. On account of (6.19), \( P \hat{L}_1 X_2 = 0 \). So,

\[ X_3 = \frac{1}{\partial_t - \hat{L}_0} P \hat{L}_1 Y_2, \]

\[ = (t - t_0) P \hat{L}_1 \left( \frac{Q}{-\hat{L}_0} \right)^2 \tilde{W}_0, \]  \hspace{1cm} (6.43)

where (6.35) has been used in the last equality. Similarly using (6.32) and (6.36), we have

\[ Y_3 = [(t - t_0) \frac{Q}{-\hat{L}_0} \hat{L}_1 P \hat{L}_1 \frac{Q}{-\hat{L}_0} \hat{L}_1 \]

\[ - \frac{Q}{(-\hat{L}_0)^2} \hat{L}_1 P \hat{L}_1 \frac{Q}{-\hat{L}_0} \hat{L}_1 + \left( \frac{Q}{-\hat{L}_0} \hat{L}_1 \right)^3] \tilde{W}_0. \]  \hspace{1cm} (6.44)
Adding the above terms, we have for the third order solution
\[ \tilde{W}_3(t, x, y; t_0) = (t - t_0)\{P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2 + \frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 \}
\]
\[ - \frac{Q}{(-L_0)^2}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 + \left(\frac{Q}{-L_0}\hat{L}_1\right)^3\tilde{W}_0, \quad (6.45) \]
which satisfies the initial condition (6.31).

Then utilizing the formulae (6.38), \(X_4\) is evaluated to be
\[ X_4 = \frac{1}{2}(t - t_0)^2P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 + (t - t_0)\{-P \hat{L}_1\left(\frac{Q}{-L_0}\right)^2\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1
\]
\[ + P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^3\}]\tilde{W}_0. \quad (6.46) \]
Similarly with \(Y_4\), we obtain
\[ Y_4 = (t - t_0)\{\frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2 + \left(\frac{Q}{-L_0}\hat{L}_1\right)^2P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 \}
\]
\[ - \frac{Q}{(-L_0)^2}\hat{L}_1P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2 - \frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1
\]
\[ - \frac{Q}{-L_0}\hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 + \left(\frac{Q}{-L_0}\hat{L}_1\right)^4\tilde{W}_0. \quad (6.47) \]
Adding the two terms, we have
\[ \tilde{W}_4(t, x, y; t_0) = \frac{1}{2}(t - t_0)^2P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1
\]
\[ + (t - t_0)\{-P \hat{L}_1\left(\frac{Q}{-L_0}\right)^2\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 + P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^3
\]
\[ + \frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2 + \left(\frac{Q}{-L_0}\hat{L}_1\right)^2P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 \}
\[ - \frac{Q}{(-L_0)^2}\hat{L}_1P \hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2 - \frac{Q}{-L_0}\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1
\]
\[ - \frac{Q}{(-L_0)^2}\hat{L}_1\left(\frac{Q}{-L_0}\hat{L}_1\right)^2\hat{L}_1P \hat{L}_1\frac{Q}{-L_0}\hat{L}_1 + \left(\frac{Q}{-L_0}\hat{L}_1\right)^4\tilde{W}_0. \quad (6.48) \]
We remark that \(PW_4(t_0, x, y; t_0) = 0\).

If we stop at this order, the perturbed solution is given by the sum
\[ \tilde{W}(t, x, y; t_0) \simeq \sum_{i=0}^4 e^{\text{i}\tilde{W}_i(t, x, y; t_0)}. \quad (6.49) \]
This solution becomes invalid as \(t - t_0 \to \infty\) due to the secular terms.

Now the RG equation can now re-sum the secular terms: The RG equation
\[ \frac{d\tilde{W}}{dt_0}_{|_{t_0=t}} = 0 \quad (6.50) \]
gives for the reduced equations in the P- and Q-space
\[ \partial_t P W_0(t, x, y) = e^2 P \hat{L}_1 \left[ \sum_{n=1}^{3} e^{n-1} \left( \frac{Q}{-L_0} \hat{L}_1 \right)^n - e^2 \frac{Q}{-L_0} \hat{L}_1 P \hat{L}_1 \frac{Q}{-L_0} \hat{L}_1 \right] W_0(t, x, y) \] (6.51)

and
\[ \left\{ e \frac{Q}{-L_0} \hat{L}_1 + e^2 \left( \frac{Q}{-L_0} \hat{L}_1 \right)^2 \right\} \partial_t W_0 = \left\{ e^3 \frac{Q}{-L_0} \hat{L}_1 P \hat{L}_1 \frac{Q}{-L_0} \hat{L}_1 + e^4 \frac{Q}{-L_0} \hat{L}_1 P \hat{L}_1 \left( \frac{Q}{-L_0} \hat{L}_1 \right)^2 \right\} \]
\[ + e^4 \left( \frac{Q}{-L_0} \hat{L}_1 \right)^2 P \hat{L}_1 \frac{Q}{-L_0} \hat{L}_1 \} W_0, \] (6.52)

respectively. Here we have utilized the fact that \( \hat{W}_0(t, x, y; t_0 = t) = W_0(t, x, y) \) and Eq.(6.27). It is readily verified that Eq.(6.52) follows from Eq.(6.51). Eq.(6.51) is exactly the same as the one given by Gardiner[8] where the Laplace transformation was utilized. The fourth-order term in (6.51) may be rearranged as a wave-function renormalization in this order of approximation;

\[ (1 + e^2 P \hat{L}_1 \frac{Q}{(-L_0)^2} \hat{L}_1) \partial_t W_0(t, x, y) = e^2 P \hat{L}_1 \left[ \sum_{n=1}^{3} e^{n-1} \left( \frac{Q}{-L_0} \hat{L}_1 \right)^n \right] W_0(t, x, y). \] (6.53)

In fact, multiplying Eq.(6.52) by \(-e P \hat{L}_1 \frac{Q}{(-L_0)^2} \hat{L}_1 \) and summing the result with Eq.(6.51), one arrives at Eq.(6.53).

Eq.(6.51) or (6.53) is actually the equation for the amplitude \( C_0(x, t) \): Performing the projection \( P \) to the both sides, which is nothing but performing the integration \( \int dy \) of the both sides, one has after recovering the original time \( t \to t/\epsilon \) with \( \epsilon = 1/\gamma \),

\[ (1 + \gamma^{-2} \hat{L}^{(N)}(x)) \partial_t C_0(x, t) = \sum_{n=2}^{4} \gamma^{-(n-1)} \hat{L}_n(x) C_0(x, t), \] (6.54)

with

\[ \hat{L}^{(N)}(x) = \sum_{n=1}^{\infty} \hat{L}_{0,n} \frac{1}{\lambda_n^2(x)} \hat{L}_{n,0}, \] (6.55)

\[ \hat{L}_2(x) = \sum_{n=1}^{\infty} \hat{L}_{0,n} \frac{1}{\lambda_n(x)} \hat{L}_{n,0}, \] (6.56)

\[ \hat{L}_3(x) = \sum_{n,m=1}^{\infty} \hat{L}_{0,n} \frac{1}{\lambda_n(x)} \hat{L}_{n,m} \frac{1}{\lambda_m(x)} \hat{L}_{m,0}, \] (6.57)

\[ \hat{L}_4(x) = \sum_{n,m,l=1}^{\infty} \hat{L}_{0,n} \frac{1}{\lambda_n(x)} \hat{L}_{n,m} \frac{1}{\lambda_m(x)} \hat{L}_{m,l} \frac{1}{\lambda_l(x)} \hat{L}_{l,0}. \] (6.58)

Here \( \hat{L}_{n,m}(x) \) are the \( y \)-averaged operators with respect to \( x \), defined by

\[ \hat{L}_{n,m}(x) = \int dy \varphi_n^\dagger(y; x) \hat{L}_a(x, y) \varphi_m(y; x). \] (6.59)
6.2 Derivation of Smoluchowski equation from a Kramers equation

As an example, let us take the following simple case:

\[ h_x(x,y) = y, \quad g_x(x,y) = 0, \quad h_y(x,y) = -y, \]
\[ f(x,y) = F(x), \quad g_y(x,y) = \sqrt{T(x)}. \]  

(6.60)

This equation is a Langevin equation with a multiplicative noise. In this case, we have

\[ \hat{L}_0 = \partial_y y + T(x)\partial_y^2, \quad \hat{L}_1 = -y\partial_x - f(x)\partial_y. \]  

(6.61)

The eigenvalue problem for \( \hat{L}_0 \) is easily solved; the eigenfunctions belonging to the eigenvalue \( \lambda_n = n, (n = 0, 1, 2, \ldots) \) may be written

\[ \varphi_0(y; x) = e^{-y^2/2T(x)/\sqrt{2T(x)}}, \quad \varphi_{n\geq1}(y; x) = \frac{1}{2^n n!} H_n(y/\sqrt{2T(x)})\varphi_0(y; x), \]  

(6.62)

where \( H_n(x) \) is the Hermite polynomial in the \( n \)-th order. The conjugate eigenfunctions are found to be

\[ \varphi_n^\dagger(y; x) = H_n(y/\sqrt{2T(x)}), \quad \varphi_0^\dagger(y; x) = 1. \]  

(6.63)

Accordingly, the projection operator \( P \) is given by a simple integration,

\[ P = \varphi_0(y; x) \int dy. \]  

(6.64)

To obtain the reduced dynamics, we only have to calculate the \( y \)-averaged operators \( \hat{L}_{0,n}(x) \) and \( \hat{L}_{0,0}(x) \). First we notice that \( P\hat{L}_1 P = 0 \), hence \( \hat{L}_1 = \hat{L}_{0,0}(x) = 0 \). The non-vanishing term is evaluated to be

\[ \hat{L}_{0,n}(x) = \int dy \hat{L}_1 \varphi_n(y; x), \]
\[ = -\int dy (y\partial_x + f(x)\partial_y)\varphi_n(y; x), \]
\[ = -\delta_{1,n} \frac{\partial}{\partial x} \sqrt{\frac{T}{2}}. \]  

(6.65)

So we only have to calculate \( \hat{L}_{1,0} \):

\[ \hat{L}_{1,0}(x) = \int dy \varphi_1^\dagger(y; x) \hat{L}_1 \varphi_0(y; x), \]
\[ = -\int dy \varphi_1^\dagger(y; x)(y\partial_x + f(x)\partial_y)\varphi_0(y; x). \]  

(6.66)

A straightforward calculation gives

\[ (y\partial_x + f(x)\partial_y)\varphi_0(y; x) = \varphi_1(y; x)\sqrt{\frac{2}{T}}\left\{ T \frac{\partial}{\partial x} - f(x) + \frac{\partial T}{\partial x} \right\} \]
\[ + \varphi_3(y; x)\sqrt{\frac{72}{T}} \frac{\partial T}{\partial x}. \]  

(6.67)
hence

\[ \hat{L}_{1,0}(x) = -\sqrt{\frac{2}{T}} \{ T \frac{\partial}{\partial x} - f(x) + \frac{\partial T}{\partial x} \}. \quad (6.68) \]

Using the above results, one obtains in the second order approximation

\[ \hat{L}_2 = \hat{L}_{0,1}\hat{L}_{1,0}, \]
\[ = \frac{\partial}{\partial x} \left( T \frac{\partial}{\partial x} - f(x) + \frac{\partial T}{\partial x} \right). \quad (6.69) \]

Thus we end up with\[13\]

\[ \frac{\partial C_0(x, t)}{\partial t} = \gamma^{-1} \hat{L}_2 C_0(x, t), \]
\[ = \gamma^{-1} \frac{\partial}{\partial x} \left( T \frac{\partial}{\partial x} - f(x) + \frac{\partial T}{\partial x} \right) C_0(x, t). \quad (6.70) \]

One can proceed to higher orders. The third order correction is found to vanish owing to the parity conservation \( \hat{L}_{1,1} = \hat{L}_{1,3} = 0 \). So one must calculate the fourth order to have a correction to the second order result. Although it is straightforward to calculate \( \hat{L}_4 \) with \( x \)-dependent \( T(x) \), we shall, in the following, take a simple case where \( T \) is independent of \( x \). To obtain the fourth order correction, one needs to evaluate the following operators:

\[ \hat{L}_{1,m} = -\delta_{m,2} \sqrt{\frac{T}{2}} \frac{\partial}{\partial x}, \quad (6.71) \]
\[ \hat{L}_{2,l} = \sqrt{\frac{T}{2}} \{ \delta_{l,3} \frac{\partial}{\partial x} - \delta_{l,1} \{ \frac{\partial}{\partial x} - \frac{f(x)}{T} \} \}, \quad (6.72) \]
\[ \hat{L}_{3,0} = 0. \quad (6.73) \]

Thus one has

\[ \hat{L}_4 = \frac{d^2}{dx^2} \left( T \frac{\partial}{\partial x} - f \right)^2, \quad (6.74) \]

which is exactly the same as given in other methods\[8\].

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7 Summary and Concluding Remarks

In this paper, we have applied the so-called renormalization group (RG) method to derive and reduce kinetic equations; the equations treated include Boltzmann equation, the fluid dynamical equation, Fokker-Planck equation for classical dynamics and also the rate equation in quantum field theory. In contrast to previous works [23, 24], our main purpose was to elucidate the general structure of the reduction of the dynamical equations in the hierarchy of the evolution equations. We have noticed that the significance of the choice of the initial value on the attractive manifold which is also an invariant manifold [5] in deriving kinetic equations is fully recognized and emphasized by Bogoliubov [4], Lebowitz [13], Kubo [12] and Kawasaki [14], for instance. The notion of coarse-grained time derivative was also noticed by Mori [10] and others [10, 11]. Our point was that these basic ingredients naturally appear in the RG-theoretical derivation of kinetic equations when properly formulated so as to respect the role played by the initial condition as formulated in [19, 20].

We have also shown that the further reduction of kinetic equations can be performed in a unified manner in the the RG method as formulated in [20]. Such problems include obtaining the fluid dynamical limit of Boltzmann equation and deriving Smoluchowski equation from Kramers equation.

It is well known that the RG in quantum field theory (QFT) and statistical physics [15, 21, 22] works well as a powerful tool for obtaining the infrared effective theories with fewer degrees of freedom than in the original Lagrangian relevant in the high-energy region. This is a kind of the reduction of the dynamics [21]. So one could imagine that the RG may be applied to derive and reduce kinetic equations in a unified manner. Now we know that it is the case. Conversely, does the coarse graining of time as shown in the present work appear and play a role in the RG equations in QFT and statistical mechanics? In this respect, one may notice the resemblance of the exact RG equation like Wilsonian or Wegner-Houghton RG equation with Fokker-Planck equation; the quantum field theory with high-energy degrees of freedom remained corresponds to Langevin equation and the exact RG equation to Fokker-Planck. Such an analogy is not fully pursued yet [14].

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Appendix A  
Another calculational procedure for reduction of Langevin to Fokker-Planck equation

In this Appendix, we shall rederive Fokker-Planck equation for (2.1) starting from the stochastic Liouville equation in a more elementary way sketched in [23] for far simpler equation; we shall work out a detailed derivation to show how the identification of the initial condition is important for obtaining the averaged equation, thereby exhibit clearly the similarity of the discussion with the other problems considered in the text.

We first make a change of independent variables [23] for (2.4) \((t, u) \rightarrow (\tau, x)\) by

\[ \tau = t, \quad x = \epsilon \{ u - \int_{t_0}^t ds h(u(s)) \}, \]

where \(\epsilon\) is supposed to be small. Then Eq. (2.4) is converted to

\[ \frac{\partial f}{\partial \tau} = -\epsilon \left[ \nabla (\hat{g} R f) + \nabla \cdot h f \right], \]

where \(\nabla = \sum_i \partial / \partial x_i\). We now try to solve Eq.(A.2) around \(t \sim \forall t_0\) by the perturbation theory. We suppose that the initial distribution \(\bar{f}(u, t_0)\) is given at \(t = t_0\). The corresponding solution is written as \(\tilde{f}(u, t; t_0)\) and is expanded as

\[ \tilde{f} = \tilde{f}_0 + \epsilon \tilde{f}_1 + \epsilon^2 \tilde{f}_2 + \cdots. \]

The initial distribution is also expanded;

\[ \bar{f} = \bar{f}_0 + \epsilon \bar{f}_1 + \epsilon^2 \bar{f}_2 + \cdots. \]

The equations in the first few orders read

\[ \frac{\partial \tilde{f}_0}{\partial \tau} = 0, \]
\[ \frac{\partial \tilde{f}_1}{\partial \tau} = -[\nabla (\hat{g} R \tilde{f}_0) + \nabla \cdot h \tilde{f}_0], \]
\[ \frac{\partial \tilde{f}_2}{\partial \tau} = -[\nabla (\hat{g} R \tilde{f}_1) + \nabla \cdot h \tilde{f}_1]. \]

The first order solution is a stationary one;

\[ \tilde{f}_0(x, t; t_0) = \bar{f}_0(x, t_0), \]

where the initial distribution \(\tilde{f}_0(x, t_0)\) is not yet specified. Owing to the absence of explicit time-dependence of \(\tilde{f}_0\), the first and the second order equations can be readily solved;

\[ \tilde{f}_1(x, t; t_0) = -\int_{t_0}^t ds \nabla \cdot (\hat{g} R \bar{f}_0) - (t - t_0)(\nabla \cdot h) \bar{f}_0, \]
\[ \tilde{f}_2(x, t; t_0) = \int_{t_0}^t ds_1 \int_{t_0}^{s_1} ds_2 L_1(s_1)L_1(s_2) \bar{f}_0 + \frac{1}{2} \nabla \cdot h \nabla \cdot h \bar{f}_0(t - t_0)^2 + \text{terms linear in } R. \]
Here $L_1(s)$ is defined in Eq.(2.9) in the text. The averaged distribution function $\tilde{P}(\mathbf{x}, t; t_0)$ is now given by

$$
\tilde{P}(\mathbf{u}, t; t_0) = \langle \tilde{f}(\mathbf{u}, t; t_0) \rangle,
\tilde{f}_0(\mathbf{u}, t_0) - \epsilon(t - t_0)(\nabla \cdot \mathbf{h} \tilde{f}_0(\mathbf{u}, t_0) \\
+ \epsilon^2 \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \langle L_1(s_1)L_1(s_2) \rangle \tilde{f}_0(\mathbf{u}, t_0) \\
+ \frac{1}{2}(t - t_0)^2 \nabla \cdot \mathbf{h} \nabla \cdot \mathbf{h} \tilde{f}_0(\mathbf{u}, t_0). \tag{A.9}
$$

For steady noises, the correlation $\int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \langle L_1(s_1)L_1(s_2) \rangle$ can be further reduced as was done in §2:

$$
\int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \langle L_1(s_1)L_1(s_2) \rangle = (t - t_0)G(t - t_0). \tag{A.10}
$$

The RG equation $\partial \tilde{P}/\partial t_0|_{t_0=t} = 0$ gives

$$
\partial \tilde{f}_0(\mathbf{u}, t)/\partial t + \epsilon \nabla \cdot \mathbf{h} \tilde{f}_0 - \epsilon^2 G(0) \tilde{f}_0(\mathbf{u}, t_0) = 0. \tag{A.11}
$$

Recovering the original variables, we finally have

$$
\partial \tilde{f}_0/\partial t + \nabla(\cdot \mathbf{h} \tilde{f}_0) - G(0) \tilde{f}_0 = 0. \tag{A.12}
$$

This is the desired Fokker-Planck equation, provided that the initial distribution $\tilde{f}_0(\mathbf{u}, t_0)$ is identified with the averaged distribution function $P(\mathbf{u}, t)$. This means that the initial distribution function at an arbitrary time $t = t_0$ before averaging must coincide with the averaged distribution to be determined. The initial value may be considered as the integral constant in the unperturbed equation, which would move slowly being governed by the RG equation. In other words, the averaging is automatically made by the RG method.
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