Simple Mechanisms for Non-linear Agents

Yiding Feng∗ Jason D. Hartline† Yingkai Li‡

Abstract

We consider agents with non-linear preferences given by private values and private budgets. We quantify the extent to which posted pricing approximately optimizes welfare and revenue for a single agent. We give a reduction framework that extends the approximation of multi-agent pricing-based mechanisms from linear utility to non-linear utility. This reduction framework is broadly applicable as Alaei et al. (2012) have shown that mechanisms for linear agents can generally be interpreted as pricing-based mechanisms. We give example applications of the framework to oblivious posted pricing (e.g., Chawla et al., 2010), sequential posted pricing (e.g., Yan, 2011), and virtual surplus maximization (Myerson, 1981).

1 Introduction

Mechanism design for budgeted agents has been studied extensively in the literature as a canonical model of non-linearity that challenges classical methods for mechanism design. For selling a single item to a single-agent, if the agent has linear utility then posting a take-it-or-leave-it price is optimal. On the other hand, for an agent with both a private value and a private budget, posting a single price can be arbitrary bad against the optimal mechanism (Example B.1) and the optimal mechanism may need an exponential-size menu (Devanur and Weinberg, 2017). For selling to multiple agents, if the agents have linear utility then (relatively simple) virtual value based mechanisms are optimal and (even simpler) price posting mechanisms are approximately optimal. On the other hand, for an agent with non-linear utility the optimal mechanism requires solving a convex program subject to interim feasibility constraints (Alaei et al., 2012). This paper gives a general approach for reducing non-linear approximation mechanisms to linear approximation mechanisms.

Single-agent price-posting is a key building-block in mechanism design for a linear agents. Price posting is optimal for a single linear agent and a single item. For multiple linear agents, Chawla et al. (2010) introduce both (a) oblivious posted pricings and (b) sequential posted pricings and prove that these are good approximations in various environments. Yan (2011) formalized the connection between sequential posted pricings and correlation gap which governs their approximation. The approximation of oblivious posted pricings are often

∗Department of Computer Science, Northwestern University. Email: yidingfeng2021@u.northwestern.edu.
†Department of Computer Science, Northwestern University. Email: hartline@northwestern.edu.
‡Department of Computer Science, Northwestern University. Email: yingkai.li@u.northwestern.edu.
given by prophet inequalities, e.g., [Kleinberg and Weinberg, 2012, Feldman et al., 2016]. Moreover, the (c) optimal mechanism for linear agents can be viewed as reduction to single-agent price posting [Bulow and Roberts, 1989, Alaei et al., 2012].

As noted above for non-linear agents, mechanisms based on price-posting are not generally good even for single-agent mechanisms. This paper defines a notion of single-agent approximation by price-posting (see next paragraph) and shows that, for non-linear agents that satisfy this definition, approximately optimal multi-agent price-posting-based mechanisms can be derived from analogous mechanisms for linear agents. This reduction framework allows many known approximation mechanisms for linear agents to be lifted to non-linear agents environments. The approximation factors we obtain are the product of the single-agent approximation factor of posted-pricing and the approximation factor of the linear multi-agents mechanism. As examples of the reduction, we apply it to the simple linear-agent mechanisms (a), (b), and (c), above.

The single-agent price-posting approximation that governs our reduction is defined as follows. The literature on revenue optimal mechanism design for a single agent under ex ante constraint defines the so-called revenue curve (cf. Bulow and Roberts, 1989). Fixing any class of mechanisms and a single agent, the revenue curve is a mapping from an ex ante constraint \( q \in [0, 1] \) to the revenue of the optimal mechanism in the family that sells with the given ex ante probability \( q \). Specifically, the price-posting revenue curve is generated by fixing mechanism class to all price-posting mechanisms; and the optimal revenue curve is by allowing all possible mechanisms. In this paper we consider general objectives and general payoff curves that correspond to these objective. The posted-pricing approximation that governs our reduction is the closeness of the concave hull of the price-posting payoff curve and the optimal payoff curve.

**Main Result.** In Section 3, based on the definition of closeness between the price-posting payoff curve and the optimal payoff curve, we introduce a reduction framework to approximately reduce the analysis of the approximation bounds for simple price-posting-based mechanisms for agents with non-linear utility to agents with linear utilities. We instantiate this reduction framework with a set of constant-factor closeness results for agents with budgets and a number of constant-factor approximation results for pricing-based mechanisms for linear agents to obtain constant-factor approximation of pricing-based mechanisms for agents with budgets. Though we instantiate this reduction framework for agents with budgets, it can be applied to agents with non-linear utility generally.

For welfare-maximization (Section 4), we show constant closeness between the payoff curves without any assumption on the valuation or the budget distribution. For revenue-maximization (Section 5), we show constant-factor closeness between the payoff curves under certain assumptions on the valuation or the budget distribution. We also construct examples showing the necessity of our assumptions (Example B.1). Our single-agent analyses are summarized in Table 1 with their corresponding assumptions.

For non-linear agents that that are each \( \zeta \)-close, our framework reduces the multi-agent \( \zeta \gamma \)-approximation to linear agent \( \gamma \)-approximation. For order-oblivious posted pricing with

\footnote{1 In this paper, we consider non-linear agents whose utility function is a mapping from her private type and the outcome to her von Neumann-Morgenstern utility for the outcome.}
Table 1: Comparison of upper bounds for $\zeta$-closeness in this paper and approximate regularity studied in Feng et al. (2019) under various assumptions. * indicates tight ratio. † requires both assumptions on valuation and budget distribution. ‡ is under the assumption of regular valuation distribution and independent budget. △ is under the assumption of independent budget and budget exceeding expectation w.p. at least $1/\kappa$.

| Assumption | Public Budget Revenue | Private Budget Revenue | Welfare |
|------------|-----------------------|------------------------|---------|
| Regular Value, Independent Budget | $1^*$ | $(1 + 3\kappa - 1/\kappa)$ | 2 |
| $\zeta$-Closeness | $2^*$ | $3^*$ | $\infty^*$ |
| Approx-Regularity | $1^*$ | $\infty^*$ | $\sqrt{2(\kappa + 2)(\kappa + 1)^{\gamma}}$ | $\infty^*\Delta$ |

non-adaptive prices, when feasibility constraint is $k$-unit environment, $\gamma = 2$ (Chawla et al., 2010). For order-oblivious posted pricing with adaptive prices, when feasibility constraint is a matroid, $\gamma = 2$ (Kleinberg and Weinberg, 2012). For sequential posted priceings, when feasibility constraint is a matroid, $\gamma = e/(e - 1)$ and when the feasibility constraint is $k$-unit environment, $\gamma = 1/(1 - 1/\sqrt{2\pi})$ (cf. Yan, 2011). See Section 3.1 for detailed discussion.

Our construction of pricing-based mechanisms based on the optimal mechanisms for linear agents (Section 3.2) gives $\gamma$ equal to the gap between the ex ante relaxation and the optimal mechanism (for linear agents).

Our analyses of the closeness between the concave hull of the price-posting payoff curve and optimal payoff curve for agents with private budget are interesting independently of our reduction framework. The setting of our single-agent analysis with an ex ante constraint is equivalent to the mechanism design problem for a continuum of i.i.d. agents with unit-demand and limited supply. For example, Richter (2016) shows that posted pricing is optimal in the continuum model with regular and decreasing density value distribution and, critically, no unit-demand constraint.

Related Work. Our work builds on the Feng et al. (2019) study of the approximation of anonymous pricing for non-linear agents. For agents with linear utility, anonymous pricing is a constant approximation to the optimal revenue if agents have regular distributions, where the regularity is defined as the concavity of the price-posting revenue curve, which implies the equivalence between the price-posting revenue curve and the optimal revenue curve. Feng et al. (2019) define a notion of approximate regularity for agents with non-linear utility, which characterizes (a) the closeness between price-posting revenue curve and optimal revenue curve (i.e., how well posted pricing can approximate optimal mechanism for a single agent), and (b) the approximate concavity of price-posting revenue curve (i.e., the sensitivity of revenue against prices for a single agent). Then they introduce a reduction framework which extends the approximation bound of anonymous pricing for linear agent with regularity to non-linear agents with approximate regularity.

The main contributions of our results, relative to Feng et al. (2019), are the following four...
points. (i) We separate the definition of approximate regularity to closeness and approximate concavity. (The regularity assumption is unnecessary for non-anonymous pricing-based mechanisms to be good approximations.) (ii) We relax the definition of closeness by considering the closeness between the concave hull of the price-posting payoff curve and the ex ante payoff curve. Under this relaxed definition, we give tighter closeness bounds for revenue and agents with private budgets. (iii) We extend the reduction framework to general objectives besides revenue and give an analysis of welfare for agents with private budgets. (iv) We extend the reduction framework from single-item environments to downward-closed environments.

Next we further review the recent work on mechanism design for budgeted agents for single-item and multi-unit environments.

**Optimal mechanisms:** In single-item environments, Laffont and Robert (1996) and Maskin (2000) study the revenue-maximization and welfare-maximization problems for symmetric agents with public budgets. Boulatov and Severinov (2018) generalize their results to agents with i.i.d. values but asymmetric public budgets. Che and Gale (2000) consider the single agent problem with private budget and decreasing-marginal-revenue valuation distribution, and characterize the optimal mechanism by a differential equation. Devanur and Weinberg (2017) consider the single agent problem with private budget and an arbitrary valuation distribution, characterize the optimal mechanism by a linear program, and use an algorithmic approach to construct the solution. Pai and Vohra (2014) generalize the characterization of the optimal mechanism for symmetric agents with public budgets to symmetric agents with uniformly distributed private budgets. Richter (2016) shows that a price-posting mechanism is optimal for selling a divisible good to a continuum of agents with private budgets if their valuations are regular with decreasing density. For more general settings, no closed-form characterizations are known. However, the optimal mechanism can be solved by a polynomial-time solvable linear program over interim allocation rules (cf. Alaei et al., 2012; Che et al., 2013).

**Simple approximation mechanisms:** In single-item environment, Feng et al. (2019) study anonymous pricing and show constant approximations for budgeted agent with assumptions on the valuation and budget distributions. Abrams (2006) shows that posting the market clearing price (the price where demand meets supply) gives a two approximation to the revenue of the optimal mechanism for selling multiple units to a set of asymmetric non-unit-demand agents with public values and public budgets. For matroid environment, Chawla et al. (2011) show that a simple lottery mechanism is a constant approximation to the optimal pointwise individually rational mechanism for agents with MHR valuation and private budgets. Note that in our paper, the benchmark is the optimal mechanism under interim individually rationality, and the revenue gap between those two benchmark mechanisms may be unbounded. For general feasibility constraints, Alaei et al. (2013) approximately reduce the multi-agent problem to the single-agent ex ante optimization problem, and show, for example, that in the special case of the single item environment, sequentially running single agent ex ante optimal mechanisms gives an \( e/(e - 1) \)-approximation. However, the ex ante optimal mechanism for single budgeted agent is still complicated. For multiple items, Cheng et al. (2018) shows that selling items separately or as a bundle is approximately optimal for a single agent with additive valuation.
2 Preliminaries

In this paper, we consider general payoff maximization in downward-closed environment for agents with budgets. For example, both welfare maximization and revenue maximization are special cases of payoff maximization.

Agent Models. There is a set of agents $N$ where $|N| = n$. An agent’s utility model is defined as $(T, F, u)$ where $T$, $F$, and $u$ are the type space, distribution and utility function. The outcome for an agent is the distribution over the pair $(x, p)$, where allocation $x \in \{0, 1\}$ and payment $p \in \mathbb{R}_+$. The utility function of each player $u$ is a mapping from her private type and the outcome to her von Neumann-Morgenstern utility for the outcome.

A utility model of focus for the paper is the private-budget model: Each agent $i \in N$ has private value $v_i$ and private budget constraint $w_i$. We refer to the pair $(v_i, w_i)$ as the private type of the agent. The valuation $v_i$ for each agent $i$ is sampled from the marginal valuation distribution $F_i$ and her budget $w_i$ is sampled from the marginal budget distribution $G_i$. The pair $(v_i, w_i)$ is independent across different agents while we allow the value $v_i$ to be correlated with budget $w_i$ for each agent $i$. We also use $F_i$ and $G_i$ to denote the cumulative probability function for the valuation and budget of agent $i$. For each budgeted agent $i$, her utility given an outcome $(x_i, p_i)$ is $u_i = v_i x_i - p_i$ if the payment does not exceed her budget, i.e., $p_i \leq w_i$. Otherwise, her utility is $u_i = -\infty$. Note that this agent has linear utility if her budget is always infinite. In the following sections, we will drop the subscripts when we discuss the single agent problems.

Mechanisms. In this paper, we consider sealed-bid mechanisms: in a mechanism $\{(x_i, p_i)\}_{i \in N}$, agents simultaneously submit sealed bids $\{b_i\}_{i \in N}$ from their type spaces to the mechanism, and each agent $i$ gets allocation $x_i(\{b_i\}_{i \in N})$ with payment $p_i(\{b_i\}_{i \in N})$. The outcome of mechanisms is a distribution of the allocation payment pair $(x_i, p_i)$ for each agent $i$ where the allocation is a probability $x_i \in [0, 1]$ and the price is $p_i \in \mathbb{R}_+$. There is a downward-closed constraint $\mathcal{X} \subseteq \{0, 1\}^n$ on the set of feasible outcomes.

In this paper, we consider Bayesian incentive compatible (i.e. no agent can gain strictly higher expected utility than reporting her private type truthfully if all other agents are reporting their private types truthfully) and interim individual rational (i.e. the expected utility is non-negative for all agents and all private types if all agents are reporting their private types truthfully) mechanisms. For later discussion, we also define dominant strategy incentive compatible for a mechanism if no agent can gain strictly higher expected utility than reporting her private type truthfully, regardless of other agents’ report.

Payoff Curves. In this paragraph, we define the payoff curves, and introduce the revenue curves and welfare curves as special cases of the payoff curves. More specifically, we define the optimal payoff curves and price-posting payoff curves respectively.

Definition 2.1. Given ex ante constraint $q$, the optimal payoff curve $R(q)$ is a mapping from quantile $q$ to the optimal ex ante payoff for the single agent problem, i.e., the optimal payoff of the mechanism which in expectation sells the item with probability $q$. 
Fact 2.1. The optimal payoff curve is concave.

Fact 2.1 holds because the space of mechanisms is closed under convex combination. We also study mechanisms based on simple per-unit posted posting.

Definition 2.2. Posting per-unit price $p$ is offering a menu $\{(x, x \cdot p) : x \in [0,1]\}$ to the agent. A budgeted agent with value $v$ and budget $w$ given per-unit price $p$ will purchase the lottery $x = \min\{1, w/p\}$ if $v \geq p$, and purchase the lottery $x = 0$ otherwise.

Definition 2.3. The market clearing price $p^q$ for the ex ante constraint $q$ is the per-unit price such that the item is sold with probability $q$.

Definition 2.4. Given ex ante constraint $q$, the price-posting payoff curve $P(q)$ is a mapping from quantile $q$ to the optimal price-posting payoff for the single agent problem, i.e., the optimal payoff of the price posting mechanism which sells the item with probability $q$ in expectation over the type distribution and the probabilities of the selected lottery.

Price-posting payoff curves are not generally concave, we can iron it to get the concave hull of the price-posting payoff curves.

Definition 2.5. The ironed price-posting payoff curve $\bar{P}$ is the concave hull of the price-posting payoff curve $P$.

Next we review the relation between the optimal revenue curves and the concave hull of the price-posting revenue curves for agents with linear utilities.

Lemma 2.2 (Bulow and Roberts, 1989). The optimal revenue curve $R$ of a linear agent is equal to her ironed price-posting revenue curve $\bar{P}$.

A similar result holds for the welfare curve. Note that the price-posting welfare curve is always concave for agents with linear utility.

Lemma 2.3. The optimal welfare curve $R$ of a linear agent is equal to her price-posting welfare curve $P$, both are concave and $R = P = \bar{P}$.

In general, for agents with budgets, the optimal payoff (e.g., revenue or welfare) curves and the concave hull of the price-posting payoff curves are not equivalent, and the ex ante optimal mechanism is more complicated and extracts strictly higher payoff than the optimal price posting mechanism and randomizations over price posting mechanisms.

Ex Ante Relaxation. Next we provide the benchmark of our paper, the ex ante relaxation. For auctions with downward-closed feasibility constraints, any sequence of ex ante quantiles $\{q_i\}_{i \in N}$ is ex ante feasible with respect to constraint $\mathcal{X}$ if there exists a randomized, ex post feasible allocation such that the probability agent $i$ receives an item, i.e., marginal allocation probability for agent $i$, is exactly equal to $q_i$. We denote the set of ex ante feasible quantiles with respect to feasibility constraint $\mathcal{X}$ by $\text{EAF}(\mathcal{X})$. The optimal ex ante payoff given a specific collection of payoff curves $\{R_i\}_{i \in N}$ and feasibility constraint $\mathcal{X}$ is

$$\text{EAR}(\{R_i\}_{i \in N}, \mathcal{X}) = \max_{\{q_i\}_{i \in N} \subseteq \text{EAF}(\mathcal{X})} \sum_{i \in N} R_i(q_i).$$
3 Reduction Framework for pricing-based Mechanisms

In this section, we introduce a reduction framework that extends the approximation of any multi-agent mechanisms for agents with linear utilities to agents with non-linear utilities (e.g., budgeted). To establish this result, we first formally define the analogue of any mechanism from linear agents to non-linear agents (which we call pricing-based model-free mechanism, Definition 3.2), and provide an algorithmic approach (Definition 3.3) to construct such an analogue for deterministic dominant strategy incentive compatible mechanisms. Next, we show that the approximation guarantee of a mechanism for linear agents is generalized to the approximation guarantee of the mechanism’s analogue for non-linear agents.

Definition 3.1. A model-free mechanism \( \hat{M} \) is a mapping from the utility models \( \{(T_i, F_i, u_i)\}_{i \in N} \) of the agents to a well-defined mechanism \( M = \{(x_i, p_i)\}_{i \in N} \) for those agents, where \( x_i : \prod_{i \in N} T_i \to \Delta(\{0, 1\}) \) and \( p_i : \prod_{i \in N} T_i \to \Delta(\mathbb{R}_+) \) for each agent \( i \).

Definition 3.2. A model-free mechanism \( \hat{M} \) is pricing-based if given any utility model of the agents with the same pricing-posting payoff curves \( \{P_i\}_{i \in N} \), let mechanism \( M \) be the mechanism mapped from the model-free mechanism \( \hat{M} \), then

i. Identical payoff: the mechanism \( M \) has the same payoff, denoted \( \hat{M}(\{P_i\}_{i \in N}) \equiv M(\{P_i\}_{i \in N}) \);

ii. Identical feasibility: the mechanism \( M \) has the same distribution over outcomes.

Myerson (1981) shows that the payoff of any mechanism \( M_L \) for linear agents depends only on their pricing-posting payoff curves \( \{P_i\}_{i \in N} \).

Lemma 3.1 (Myerson, 1981). In any mechanism \( M_L = \{(x_i, p_i)\}_{i \in N} \) for linear agents with pricing-posting payoff curves \( \{P_i\}_{i \in N} \), the payoff is \( \sum_{i \in N} E[x_i(v_i(q)) \cdot P_i'(q)] \), where \( v_i(q) \equiv \sup\{v : F_i(v) = 1 - q\} \). Denote the corresponding payoff as \( M_L(\{P_i\}_{i \in N}) \).

For non-linear agents, however, mechanisms (e.g. revenue-optimal mechanism) are not uniquely pinned down by the pricing-posting payoff curves in general. Adapting the technique from Alaei et al. (2013), we introduce an algorithmic approach to construct pricing-based model-free mechanisms corresponding to any deterministic dominant strategy incentive compatible mechanism for linear agents.

Implementation of Model-free Mechanisms. In this part we formally show how to construct a model-free mechanism for agents with von Neumann-Morgenstern utilities given any deterministic dominant-strategy incentive compatible mechanism for linear agents. The construction is a simplification of a construction in Alaei et al. (2013) that is possible because we are considering price-posting mechanisms for the single-agent problems.

For any linear agent environment, a deterministic dominant-strategy incentive compatible mechanism \( M_L \) can be represented by a mapping from the quantiles of other agents to a threshold quantile for each agent. The agent wins when her quantile is below the threshold and loses when her quantile is above the threshold. Denote the function that maps the profile of other agent quantiles to a quantile threshold for agent \( i \) by \( q_{i \rightarrow M_L}(\{q_j\}_{j \in N \setminus \{i\}}) \).
For any non-linear agent model \((T,F,u)\), the single-agent pricing problem identifies the per-unit (market clearing) price \(p^\hat{q}\) to offer the agent for any ex ante allocation constraint \(\hat{q}\). Denote the allocation probability selected by an agent with type \(t\) as \(x^\hat{q}(t)\). A key property, which will be discussed after the definition, is that \(x^\hat{q}(t)\) is monotonic in \(\hat{q}\).

**Definition 3.3.** The model-free mechanism \(\hat{M}\) for deterministic dominant-strategy incentive compatible mechanism \(M_L\) and agents \(\{(T_i,F_i,u_i)\}_{i\in N}\) is

1. For each agent \(i\) with private type \(t_i\), map the type to a random quantile \(q_i\) according to the cumulative distribution \(H_i(q) = x^\hat{q}_i(t_i)\).
2. For each agent \(i\), calculate quantile threshold as \(\hat{q}_i = \hat{q}_i^{M_L}(\{q_j\}_{j\in N\setminus\{i\}})\).
3. For each agent \(i\), set payment as \(p^\hat{q}_i x^\hat{q}_i(t_i)\), and allocation \(x_i = 1\) if \(q_i < \hat{q}_i\) and \(x_i = 0\) otherwise.

**Definition 3.4.** An item is an ordinary good for agents if when offered a per-unit price for the item her demand is weakly decreasing in the price.

**Definition 3.5** (Alaei et al., 2013). An agent has monotone ex ante mechanisms if, for any private type, the probability she wins in the \(q\) ex ante mechanism is monotone non-decreasing in \(\hat{q}\).

Restricting to price-posting mechanisms, the monotone ex ante mechanisms property is satisfied for an ordinary good.

**Lemma 3.2.** For an ordinary good, agents have monotone ex ante mechanisms with respect to price-posting payoff curves.

**Proof.** For an ordinary good by definition, every agent’s expected allocation probability is weakly decreasing in the price. Thus, the per-unit price in each \(q\) ex ante mechanism (with respect to the price-posting payoff curve \(P\)) is weakly decreasing in \(q\). Now consider the \(q\) ex ante mechanism with respect to the ironed price-posting payoff curve \(\bar{P}\) for all quantile \(q\). The per-unit price is monotone (by the previous argument) on quantiles that are not in ironed intervals. Within an ironed interval, the mechanism is a mix over two end-points of non-ironed intervals which linearly interpolates between the end-points and is thus monotone.

We now show that the model-free mechanism constructed in Definition 3.3 is well defined, specifically, under the mild assumption that the agents utilities correspond to ordinary goods (Definition 3.4), the function \(G_i\) of Step 1 is monotonic and can be viewed as a distribution function. Finally, we show this model-free mechanism is pricing-based and guarantees the same payoff as the original mechanism for linear agents.

**Theorem 3.3.** The mechanism \(\hat{M}\) defined in Definition 3.3 from the mechanism \(M_L\) is a pricing-based model-free mechanism which is dominant strategy incentive compatible, individual rational and achieves the same payoff as the mechanism \(M_L\) for linear agents.
Proof. Since for each agent $i$, her type $t_i$ is drawn from $F_i$ and $q_i$ is drawn from $H_i$ condition on $t_i$, the (unconditional) distribution of $q_i$ is uniform on $[0, 1]$. Thus, from each agent $i$’s perspective, the other agents’ quantiles are distributed independently and uniformly on $[0, 1]$. This agent faces a distribution over ex ante posted pricing that is identical to the distribution of “critical quantiles” in the mechanism $\mathcal{M}_L$. Thus, the identical-payoff property of Definition 3.1 is satisfied. Since the distribution of $q_i$ is uniform on $[0, 1]$, the identical-feasibility property is satisfied by construction.

Remark. Alaei et al. (2013) give a model-free mechanism that corresponds to marginal payoff maximization with the single agent ex ante optimal mechanisms (rather than the ex ante price-posting mechanisms). The advantage of our implementation (i.e. Definition 3.3) with the ex ante price-posting mechanisms is two-fold. First, under a mild assumption (Definition 3.4) the price-posting payoff curves always satisfy the above monotonicity property and thus the reduction is simple. Second, the single-agent problems are simple and thus the whole mechanism is simple.

Reduction of approximation guarantee through $\zeta$-closeness. In this part, we define $\zeta$-closeness for a single agent with non-linear utility (Definition 3.6) which will later be used in our reduction framework. For linear agents, the ironed price-posting payoff curves equal the optimal payoff curves. For non-linear agents, however, the ironed price-posting payoff curves are not generally equivalent to the optimal payoff curves. The $\zeta$-closeness of an agent measures how close her ironed price-posting payoff curve is to her optimal payoff curve.

Definition 3.6. An agent’s ironed price-posting payoff curve $\bar{P}$ is $\zeta$-close to her optimal payoff curve $R$, if for all $q \in [0, 1]$, there exists a quantile $q^i \leq q$ such that $\bar{P}(q^i) \geq 1/\zeta \cdot R(q)$. Such an agent is $\zeta$-close.

Based on the definition of $\zeta$-closeness, we present the meta-theorem (Theorem 3.4): a reduction framework that extends the approximation of multi-agent mechanisms from linear utilities to non-linear utilities.

Theorem 3.4. For downward-closed feasibility constraint $\mathcal{X}$ and agents with price-posting payoff curves $\{P_i\}_{i \in N}$ and optimal payoff curves $\{R_i\}_{i \in N}$, if each agent is $\zeta$-close, and a linear agent mechanism $\mathcal{M}_L$ is a $\gamma$-approximation to the ex ante relaxation, i.e., $\mathcal{M}_L(\{P_i\}_{i \in N}) \geq 1/\gamma \cdot \text{EAR}(\{P_i\}_{i \in N}, \mathcal{X})$, then its corresponding pricing-based model-free mechanism $\hat{\mathcal{M}}$ is a $\gamma \zeta$-approximation to the ex ante relaxation for non-linear agents, i.e., $\hat{\mathcal{M}}(\{P_i\}_{i \in N}) \geq 1/\gamma \zeta \cdot \text{EAR}(\{R_i\}_{i \in N}, \mathcal{X})$.

Theorem 3.4 holds immediately from the definition of the pricing-based model-free mechanisms and the following lemma.

Lemma 3.5. For downward-closed feasibility constraint $\mathcal{X}$ and agents with ironed price-posting payoff curves $\{\bar{P}_i\}_{i \in N}$ and the optimal payoff curves $\{R_i\}_{i \in N}$, if each agent is $\zeta$-close, the ex ante relaxation on the ironed price-posting payoff curve is a $\zeta$-approximation to the ex ante relaxation on the optimal payoff curves, i.e., $\text{EAR}(\{\bar{P}_i\}_{i \in N}, \mathcal{X}) \geq 1/\zeta \cdot \text{EAR}(\{R_i\}_{i \in N}, \mathcal{X})$. 

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Proof. Let \( \{q_i\}_{i \in N} \in \text{EAF}(\mathcal{X}) \) be the profile of optimal ex ante quantiles for optimal payoff curves \( \{R_i\}_{i \in N} \). Since the ironed price-posting payoff curves \( \{\bar{P}_i\}_{i \in N} \) are \( \zeta \)-close to the optimal payoff curves \( \{R_i\}_{i \in N} \), there exists a sequence of quantiles \( \{\bar{q}_i\}_{i \in N} \) such that for any agent \( i \), \( \bar{q}_i \leq q_i \) and \( \bar{P}(\bar{q}_i) \geq 1/\zeta \cdot R(q_i) \). Since \( \mathcal{X} \) is downward-closed, \( \{\bar{q}_i\}_{i \in N} \) is also feasible for EAF(\( \mathcal{X} \)). Therefore,

\[
\text{EAR}(\{R_i\}_{i \in N}, \mathcal{X}) = \sum_{i \in N} R_i(q_i) \leq \zeta \cdot \sum_{i \in N} \bar{P}_i(\bar{q}_i) \leq \zeta \cdot \text{EAR}(\{\bar{P}_i\}_{i \in N}, \mathcal{X}).
\]

Optimal mechanisms for agents with private budget utility have been studied in the literature (e.g. Che and Gale (2000); Devanur and Weinberg (2017) for single-agent, Pai and Vohra (2014) for i.i.d. agents and Alaei et al. (2012) for non-i.i.d. agents). The characterization of these optimal mechanisms are complicated even for simple distributions (e.g. uniform value and uniform budget). However, with the reduction framework in Theorem 3.4, due to the closeness between price-posting payoff curve and optimal payoff curve, we can approximately extend simple mechanisms from linear agents to non-linear agents.

Example 3.7 (Uniform value, uniform budget, numerical). For a private-budgeted agent with value and budget both drawn uniformly from \([0, 1]\), her price-posting revenue curve is roughly 1.02-close to her optimal revenue curve. See Figure 1a and Appendix C for more details.

3.1 Posted Pricing Mechanism

In this subsection, we focus on posted pricing mechanisms, for which the construction of the corresponding pricing-based model-free mechanism is simpler than Definition 3.3. We apply closeness property between the ironed price-posting payoff curve and optimal payoff curve to obtain approximation results for agents with non-linear utility. In particular, we consider the following three families of mechanisms:

1. **sequential posted pricing (with non-adaptive prices)**: agents in sequence (specified by mechanisms) are offered take-it-or-leave-it non-adaptive prices.

2. **oblivious posted pricing with non-adaptive prices**: agents in sequence (unknown to mechanisms in advance) are offered take-it-or-leave-it non-adaptive prices.

3. **oblivious posted pricing with adaptive prices**: agents in sequence (unknown to mechanisms in advance) are offered take-it-or-leave-it adaptive prices.

All posted pricing mechanisms for linear agents can be converted into quantile space without loss of generality.

**Definition 3.8.** The model-free mechanism \( \hat{M} \) for posted pricing mechanism \( M_L \) and agents \( \{(T_i, F_i, u_i)\}_{i \in N} \) is

\[
\text{With the technique online contention resolution introduced by Feldman et al. (2016), it is known that for some environments (e.g. matroid) the same approximation bounds with adaptive prices can be guaranteed in oblivious posted pricing which posts non-adaptive prices but adaptively rejects agents.}
1. For each agent $i$, map the price $p_i$ posted to agent $i$ in mechanism $M_L$ to quantile $\hat{q}_i$, i.e., the probability that the item is sold to a linear agent with the same price-posting payoff curve.

2. For each agent $i$, post the market clearing price $\hat{p}_i$.

For agents with linear utilities, the approximations for posted pricing mechanisms against the optimal Bayesian mechanisms have been studied in Chawla et al. (2010); Yan (2011); Feldman et al. (2016), etc. Note that our reduction framework Theorem 3.4 requires a stronger guarantee (i.e. an approximation of $\gamma$ against the ex ante relaxation) for linear agents. For sequential posted pricing, we can directly apply the approximation results for linear agents given by correlation gap (cf. Agrawal et al., 2010; Yan, 2011) in our framework, since the benchmark for correlation gap is indeed the ex ante relaxation. However, for oblivious posted pricing with/without adaptive prices, approximation results for linear agents are given by prophet inequalities (Definition 3.10), where the standard benchmark is smaller than the ex ante relaxation. Therefore, a stronger definition (i.e. ex ante prophet inequality Definition 3.11) is required for the reduction to non-linear agents.

**Definition 3.9 (Bayesian online selection problem).** A gambler faces a series of $n$ games, one on each of $n$ days. Game $i$ has prize $v_i$ drawn independently from distribution $F_i$. There is a feasibility constraint $X$ including all subsets of prizes which the gambler can pick. The gambler knows the feasibility constraint and the prize distribution in advance. On day $i$ the gambler realizes the prize $v_i \sim F_i$ of game $i$ and must immediately make an irrevocable decision on whether to select this prize. The final set of prizes selected must satisfy feasibility constraint $X$. The gambler’s value is the total value of prizes selected.

**Definition 3.10 (prophet inequality).** In Bayesian online selection problem, prophet inequality is the ratio of the gambler with an online algorithm to a prophet who knows all prize realizations in advance and picks any feasible subset respecting the feasibility constraint $X$.

**Definition 3.11 (ex ante prophet inequality).** In Bayesian online selection problem, ex ante prophet inequality compares the gambler to the ex ante relaxation (a.k.a. the expected value of an ex ante prophet).

In fact, for certain feasibility constraints, the approximation guarantees in ex ante prophet inequality are the same as in the original prophet inequality. Lee and Singla (2018) proved that the same approximation bound (i.e. 2) holds for matroid environments. In Section 6 we introduce a meta approach (Proposition 6.2) to extend the results from prophet inequalities to ex ante prophet inequalities. Applying this meta approach to the analysis in Chawla et al. (2010) for $k$-unit environment and Kleinberg and Weinberg (2012) for matroid environment, we extend their results to ex ante prophet inequalities with basically the same argument.

In conclusion, the approximation guarantee $\gamma$ of sequential posted pricing for linear is given by correlation gap. For matroid environment, $\gamma$ is $6/(e-1)$; and for $k$-unit environment, $\gamma$ is $1/(1-1/\sqrt{2\pi})$ (Agrawal et al., 2010; Yan, 2011). For oblivious posted pricing, the approximation guarantee $\gamma$ is given by ex ante prophet inequality. For $k$-unit environment, $\gamma$ is 2 with non-adaptive prices (Section 6); for matching environment, $\gamma$ is 6.75 with non-adaptive prices (Chawla et al., 2010); and for matroid environment, $\gamma$ is 2 with adaptive prices (Appendix A).
3.2 Marginal Payoff Mechanism

The ex ante relaxation gives an upper bound on the optimal mechanism. The amount by which it is an upper bound depends on the feasibility constraint and the single-agent payoff curves. In the special case of linear agents, the gap between the ex ante relaxation and the optimal mechanism is precisely determined by the payoff curves and the feasibility constraint.

Definition 3.12. The ex ante gap for feasibility constraint $\mathcal{X}$ and optimal payoff curves $\{R_i\}_{i \in N}$ is the ratio between the ex ante relaxation $\text{EAR}(\{R_i\}_{i \in N}, \mathcal{X})$ and the payoff of the optimal mechanism for linear agents $\text{OPT}(\{R_i\}_{i \in N}, \mathcal{X})$.

By applying the marginal revenue mechanism of Bulow and Roberts (1989) in Definition 3.3, we obtain the model-free pricing-based marginal payoff mechanism. The implementation is simpler than Alaei et al. (2013) where the marginal revenue mechanism is implemented based on the ex ante optimal mechanisms.

Definition 3.13. The pricing-based model-free marginal payoff mechanism, denoted by $\text{MPM}_X$, is the pricing-based model-free mechanism (defined in Definition 3.3) that corresponds to the linear agent marginal revenue mechanism subject to feasibility constraint $\mathcal{X}$. Denote the payoff of $\text{MPM}_X$ for agents with price-posting payoff curves $\{P_i\}_{i \in N}$ as $\text{MPM}_X(\{P_i\}_{i \in N})$.

Theorem 3.6. For downward-closed feasibility constraint $\mathcal{X}$, given agents with the ironed price-posting payoff curves $\{ar{P}_i\}_{i \in N}$ and the optimal payoff curves $\{R_i\}_{i \in N}$, if each agent is $\zeta$-close, the worst case ratio between the the marginal payoff mechanism with respect to price-posting payoff curves and the ex ante relaxation on the optimal payoff curves is $\zeta \gamma$, i.e., $\text{MPM}_X(\{P_i\}_{i \in N}) \geq \frac{1}{\zeta \gamma} \cdot \text{EAR}(\{R_i\}_{i \in N}, \mathcal{X})$, where $\gamma$ is the ex ante gap for $\mathcal{X}$.

For matroid environments, applying correlation gap, the gap between optimal payoff and ex ante relaxation for linear agents is at most $\epsilon/(e-1)$. For $k$-unit environment, it is at most $1/(1-1/\sqrt{2\pi k})$. We note that these worst-case bounds against the ex ante relaxation are the same as those given by the sequential posted pricings that follow from the correlation gap (discussed previously). However, the advantage of using the marginal payoff mechanism over these sequential pricings is the same magnitude as the advantage of using the optimal mechanism over sequential posted pricing for linear agents. There can be significant improvement in payoff. We quantify the benefits of using marginal payoff mechanism for budgeted agents with uniform distributions in Figure 3. Moreover, for downward-closed environments, the gap between optimal payoff and ex ante relaxation for linear agents is at most $O(\log n)$ (Alaei et al., 2013).

4 Closeness for Welfare Maximization

We have introduced the welfare curve and the reduction framework for any payoff functions. In this section, we will focus on showing that for agents with budgets, the ironed price-posting welfare curve is close to the optimal welfare curve.
Figure 1: Figure 1a illustrates the comparison between the price-posting revenue curve (dashed line) and the ex ante revenue curve (solid line) for selling a single item to a private-budgeted agent with value and budget both drawn uniformly from $[0, 1]$. The $x$-axis is the ex ante probability and the $y$-axis is the expected revenue. The price-posting revenue curve for this uniform budgeted agent is 1.02-close to her ex ante revenue curve.

Figure 1b illustrates the comparison between approximation ratio of optimal oblivious posted pricing (grey line) and marginal payoff mechanism (black line) to the ex ante relaxation for selling a single item to i.i.d. private-budgeted agents with value and budget both drawn uniformly from $[0, 1]$. The $x$-axis is the number of agents and the $y$-axis is the approximation ratio. When there are 15 agents, the approximation ratio for oblivious posted pricing is 1.23 and the approximation ratio for marginal payoff mechanism is 1.11. Note that the revenue for optimal oblivious posted pricing is calculated by backward induction instead of applying the prices from correlation gap. See Appendix C for more details.

The ex ante optimal mechanism might be complicated and hard to characterize. However, as we show below, without any assumption on the valuation distribution or the budget distribution, posting the market clearing price guarantees a 2-approximation in welfare.

**Theorem 4.1.** For a single agent with private-budget utility and any ex ante constraint $q$, the welfare from market clearing is a 2-approximation to ex ante optimal welfare, i.e., the price-posting welfare curve is 2-close to the optimal welfare curve, which implies that the ironed price-posting welfare curve $\bar{P}$ is 2-close to the optimal welfare curve $R$, i.e., $\bar{P}(q) \geq \frac{1}{2} \cdot R(q)$ for any $q$.

The proof of Theorem 4.1 adapts the price decomposition technique from Feng et al. (2019) and extends it for welfare analysis.

Fix an arbitrary ex ante constraint $q$, denote EX as the $q$ ex ante welfare-optimal mechanism, and Payoff[EX] as its welfare. We want to decompose EX into two mechanisms EX† and EX‡ according to the market clearing price $p^q$ and bound the welfare from those two mechanisms separately. The decomposed mechanism may violate the incentive constraint for budgets, and we refer to this setting as the random-public-budget utility model. Note that the market clearing price is the same in both the private budget model and the random-public-budget utility model. Intuitively, mechanism EX† contains per-unit prices at most the market clearing price, while mechanism EX‡ contains per-unit prices at least the market clearing price. Both mechanisms EX† and EX‡ satisfy the ex ante con-
Lemma 4.2 (Feng et al., 2019). Any incentive compatible mechanism for a single agent with private-budget utility, and her behavior in the mechanisms.

Definition 4.1 (Feng et al., 2019). An allocation-payment function \( \tau : [0, 1] \rightarrow \mathbb{R}_+ \) is a mapping from the allocation \( x \) to the payment \( p \).

Lemma 4.2 (Feng et al., 2019). Any incentive compatible mechanism for a private budgeted agent is equivalent to providing a convex and non-decreasing allocation-payment function for each budget and letting the agent choose the utility maximization allocation and payment according to the allocation-payment functions.

Now we give the construction of \( \text{EX}^\dagger \) and \( \text{EX}^\ddagger \) by constructing their allocation-payment functions. For agent with budget \( w \), let \( \tau^* \) be the allocation-payment function in mechanism \( \text{EX} \), and \( x^*_w \) be the utility maximization allocation for a linear agent with value equal to the market clearing price \( p^\theta \), i.e., \( x^*_w = \max \{ x : \tau^*_w(x) \leq p^\theta \} \). For agents with budget \( w \), we define the allocation-payment functions \( \tau^\dagger_w \) and \( \tau^\ddagger_w \) for \( \text{EX}^\dagger \) and \( \text{EX}^\ddagger \) respectively below,

\[
\tau^\dagger_w(x) = \begin{cases} 
\tau^*_w(x) & \text{if } x \leq x^*_w, \\
\infty & \text{otherwise};
\end{cases}
\]

\[
\tau^\ddagger_w(x) = \begin{cases} 
\tau^*_w(x^*_w + x) - \tau^*_w(x^*_w) & \text{if } x \leq 1 - x^*_w, \\
\infty & \text{otherwise}.
\end{cases}
\]

By construction, for each type of the agent, the allocation from \( \text{EX} \) is upper bounded by the sum of the allocation from \( \text{EX}^\dagger \) and \( \text{EX}^\ddagger \), which implies that the welfare from \( \text{EX} \) is upper bounded by the sum of the welfare from \( \text{EX}^\dagger \) and \( \text{EX}^\ddagger \), and the requirements for the decomposition are satisfied.

As sketched above, we separately bound the welfare in \( \text{EX}^\dagger \) and \( \text{EX}^\ddagger \) by the welfare from posting the market clearing price.

Lemma 4.3. For a single agent with random-public-budget utility, independently distributed value and budget, and any ex ante constraint \( q \), the welfare from posting the market clearing price \( p^\theta \) is at least the welfare from \( \text{EX}^\dagger \), i.e., \( P(q) \geq \text{Payoff}[\text{EX}^\dagger] \).

Proof. Consider agent with type \((v, w)\) and agent with type \((v', w)\), where both value \( v \) and \( v' \) are higher than the market clearing price \( p^\theta \). Notice that the allocations for these two types are the same in \( \text{EX}^\dagger \) and in market clearing, since the per-unit price in both mechanisms is at most \( p^\theta \) which makes the mechanisms unable to distinguish these two types.

Let \( x^\dagger \) be the allocation rule in \( \text{EX}^\dagger \) and let \( x^\ddagger \) be the allocation rule in posting the market clearing price \( p^\theta \). For any value \( v \geq p^\theta \), and for any budget \( w \), the allocation for type \((v, w)\) is lower in \( \text{EX}^\dagger \) than in market clearing, i.e., \( x^\dagger(v, w) \leq x^\ddagger(v, w) \). Otherwise suppose the type \((v^*, w)\) has strictly higher allocation in \( \text{EX}^\dagger \) for some value \( v^* \geq p^\theta \), i.e, \( x^\dagger(v^*, w) > x^\ddagger(v^*, w) \). By the fact stated in previous paragraph, we have that for any budget \( w \) and any value \( v, v^* \geq p^\theta \), \( x^\ddagger(v, w) = x^\ddagger(v^*, w) \), \( x^\dagger(v, w) = x^\dagger(v^*, w) \), and the expected
allocation in EX\(^\dagger\) is

\[
E_{v,w}[x(v, w)] \geq \Pr[v \geq p^q] \cdot E_{v,w}[x^q(v, w) \mid v \geq p^q]
\]

\[
= \Pr[v \geq p^q] \cdot E_{v,w}[x^q(v, w) \mid v \geq p^q]
\]

\[
> \Pr[v \geq p^q] \cdot E_{v,w}[x^q(v^*, w) \mid v \geq p^q]
\]

\[
= \Pr[v \geq p^q] \cdot E_{v,w}[x^q(v, w) \mid v \geq p^q] = q,
\]

which implies EX\(^\dagger\) violates the ex ante constraint \(q\), a contradiction. Further, for any type \((v, w)\) with \(v \geq p^q\), \(x^q(v, w) \leq x^q(v, w)\) implies that the allocation in market clearing “first order stochastic dominantes” the allocation in EX\(^\dagger\), i.e., for any budget \(w\), and for any threshold \(v^\dagger\), the allocation from all types with value \(v \geq v^\dagger\) and budget \(w\) in market clearing is at least the allocation from those types in EX\(^\dagger\). Taking expectation over the valuation and the budget, the expected welfare from market clearing is at least the welfare from EX\(^\dagger\), i.e., \(P(q) \geq \text{Payoff}[\text{EX}\^\dagger]\).

**Lemma 4.4.** For a single agent with random-public-budget utility, independently distributed value and budget, and any ex ante constraint \(q\), the welfare from market clearing is at least the welfare from EX\(^\dagger\), i.e., \(P(q) \geq \text{Payoff}[\text{EX}\^\dagger]\).

**Proof.** In both EX\(^\dagger\) and market clearing, types with value lower than \(p^q\) will purchase nothing, so we only consider the types with value at least \(p^q\) in this proof. Consider any type \((v, w)\) where \(v \geq p^q\), its allocation in market clearing is at least its allocation in EX\(^\dagger\), because the per-unit price in EX\(^\dagger\) is higher. Thus, the welfare from market clearing is at least the welfare from EX\(^\dagger\), i.e., \(P(q) \geq \text{Payoff}[\text{EX}\^\dagger]\). \(\square\)

**Proof of Theorem 4.1.** Combining Lemma 4.3 and 4.4 for any quantile \(q\), we have

\[
R(q) = \text{Payoff}[\text{EX}] \leq \text{Payoff}[\text{EX}\^\dagger] + \text{Payoff}[\text{EX}\^\dagger] \leq 2P(q) \leq \max_{q' \leq q} 2P(q').
\]

5 Closeness for Revenue Maximization

In this section we analyze the closeness of ex ante and price-posting revenue curves. We show that approximate closeness is satisfied under weaker assumptions than those given by Feng et al. (2019). Since we improve the definition of closeness by comparing the ironed price-posting revenue curve with the optimal revenue curve. For simplicity, in this section, we use the notation \(\text{Payoff}_w[.]\) to denote the revenue given any mechanism if the budget of the agent is \(w\), and \(\text{Payoff}[.]\) to denote the revenue by taking expectation over the budget \(w\).

5.1 Public Budget

In this section, we consider the setting where agents have public budgets. For an agent with a public budget, Feng et al. (2019) show that the ironed price-posting revenue curve is 1-close to her optimal revenue curve if her valuation distribution is regular (Theorem 5.1). Here we show that for agents with general valuation distribution, the ironed price-posting revenue curve is 2-close to her optimal revenue curve (Theorem 5.3).
Figure 2: The thin solid line is the allocation rule for the optimal ex ante mechanism. The thick dashed line on the left side is the allocation of the decomposed mechanism with lower price, while the thick dashed line on the right side is the allocation of the decomposed mechanism with higher price.

**Theorem 5.1** (Feng et al., 2019). An agent with public budget and regular valuation distribution has the ironed price-posting revenue curve $\bar{P}$ that equals to (i.e. 1-close) her optimal revenue curve $R$.

For an agent with a general valuation distribution, closeness follows from a characterization of the ex ante optimal mechanism from Alaei et al. (2013).

**Lemma 5.2** (Alaei et al., 2013). For a single agent with public budget, the $q \in [0, 1]$ ex ante optimal mechanism has a menu with size at most two.

**Theorem 5.3.** An agent with public budget has the ironed price-posting revenue curve $\bar{P}$ that is 2-close to her optimal revenue curve $R$.

**Proof.** By Lemma 5.2, the allocation rule $x_q$ of the ex ante revenue maximization mechanism for the single agent with public budget has a menu of size at most two. We decompose its allocation into $x_L$ and $x_H$ as illustrated in Figure 2. Note that both allocation $x_L$ and $x_H$ are (randomized) price-posting allocation rules, and neither allocation violates the allocation constraint $q$. Thus,

$$R(q) = \text{Payoff}[x_q] = \text{Payoff}[x_L] + \text{Payoff}[x_H] \leq 2 \max_{q^p \leq q} \bar{P}(q^p).$$

\[ \square \]

### 5.2 Private Budget

In this section, we study the closeness of the ironed price-posting revenue curve and the optimal revenue curve for agents with private budget. For agents with linear utilities, those two curves are equivalent for any valuation distribution. However, for an agent with private budget, the gap between them can be unbounded. Specifically, according to Feng et al. (2019), when the budget distribution is correlated with the valuation distribution, posting prices is not a constant approximation to the optimal revenue for a single agent even with strong regularity assumption on the marginal valuation distribution and budget distribution. Therefore, in this section, we focus on the case when the budget distribution is independent with the
valuation distribution for each agent. Note that even with the independence assumption, without any further assumption on the valuation or the budget distribution, posting prices is not approximately optimal even for a single agent, see Example B.1 in the appendix. Therefore, we consider mild assumption on either the valuation distribution or the budget distribution and show the corresponding closeness property.

From [Feng et al., 2019], we know that regularity on the valuation distribution is sufficient to guarantee the closeness between the ironed price-posting revenue curves and the optimal revenue curve, without further assumption on the budget distribution.

Theorem 5.4 (Feng et al., 2019). A single agent with private-budget utility and regular valuation distribution has a ironed price-posting revenue curve \( \overline{P} \) that is \( 3 \)-close to her optimal revenue curve \( R \), if her value and budget are independently distributed.

We also consider the assumption that the budget exceeds its expectation with constant probability at least \( \frac{1}{\kappa} \). This assumption on budget distribution is also studied in Cheng et al. (2018) and Feng et al. (2019). Notice that a common distribution assumption, monotone hazard rate, is a special case of it with \( \kappa = e \) (cf. Barlow and Marshall, 1965).

Theorem 5.5. A single agent with private-budget utility has a price-posting revenue curve \( P \) that is \( (1 + 3\kappa - \frac{1}{\kappa}) \)-close to her optimal revenue curve \( R \), if her value and budget are independently distributed, and the probability the budget exceeds its expectation is \( \frac{1}{\kappa} \).

Theorem 5.5 implies that for this private-budget agent, her ironed price-posting revenue curve \( \overline{P} \) is also \( (1 + 3\kappa - \frac{1}{\kappa}) \)-close to her optimal revenue curve \( R \). Let \( w^* \) denote the expected budget of the agent. For any ex ante constraint \( q \), denote EX as the \( q \) ex ante optimal mechanism. We consider two cases whether the market clearing price \( p^q \) is larger than the expected budget \( w^* \). For the case where the market clearing price is at least the expected budget \( w^* \), we use Lemma 5.6 in Feng et al. (2019).

Lemma 5.6 (Feng et al., 2019). When the market clearing price \( p^q \) is at least the expected budget \( w^* \), \( \text{Payoff}[EX] \leq (2 + \kappa - \frac{1}{\kappa}) P(q) \).

Now we focus on the case where the market clearing price is smaller than the expected budget, i.e., \( p^q < w^* \). Our analysis here is similar to the analysis for welfare, i.e., the price decomposition technique. Consider the decomposition of EX into three mechanisms \( EX^\dagger, EX^\delta \) and \( EX^\ddagger \) such that mechanism \( EX^\dagger \) contains per-unit prices at most the market clearing price, mechanism \( EX^\ddagger \) contains per-unit prices at least the expected budget, while mechanism \( EX^\delta \) contains per-unit prices between the market clearing price and the expected budget. All mechanisms satisfy the ex ante constraint \( q \), and the sum of their welfare is upper bounded by the welfare of the original ex ante mechanism \( EX \), i.e., \( \text{Payoff}[EX] \leq \text{Payoff}[EX^\dagger] + \text{Payoff}[EX^\delta] + \text{Payoff}[EX^\ddagger] \).

We construct the allocation-payment functions \( \tau^\dagger_w, \tau^\ddagger_w \) and \( \tau^\delta_w \) for \( EX^\dagger \), \( EX^\ddagger \), and \( EX^\delta \) respectively. For each budget \( w \), let \( \tau_w \) be the allocation-payment function for types with budget \( w \) in mechanism \( EX \), and \( x^*_w \) be the utility maximization allocation for the agent with value and budget equal to the market clearing price \( p^q \), i.e., \( x^*_w = \arg\max\{x : \tau_w(x) \leq p^q\} \). Let \( x^*_w \) be the utility maximization allocation for the agent with value and budget equal to
the expected budget \( w^\ast \), i.e., \( x^\ast_w = \text{argmax}\{x : \tau^\ast_w(x) \leq w^\ast\} \). Then the allocation-payment functions \( \tau^\dagger_w \), \( \tau^\ast_w \) and \( \tau^\delta_w \) are defined respectively as follows,

\[
\tau^\dagger_w(x) = \begin{cases} 
\tau_w(x) & \text{if } x \leq x^\ast_w, \\
\infty & \text{otherwise;} 
\end{cases} \\
\tau^\ast_w(x) = \begin{cases} 
\tau_w(x^\ast_w + x) - \tau_w(x^\ast_w) & \text{if } x \leq x^\ast_w - x^\ast_w, \\
\infty & \text{otherwise;} 
\end{cases} \\
\tau^\delta_w(x) = \begin{cases} 
\tau_w(x^\ast_w + x) - \tau_w(x^\ast_w) & \text{if } x \leq 1 - x^\ast_w, \\
\infty & \text{otherwise.}
\end{cases}
\]

Lemma 5.7 [Feng et al., 2019]. When \( p^q \leq w^\ast \), \( \text{Payoff}^{\text{EX}^\dagger} \leq P(q) \) and there exists \( q^\dagger \in [0, q] \) such that \( \text{Payoff}^{\text{EX}^\dagger} \leq (1 + \kappa - 1/\kappa) \cdot P(q^\dagger) \).

Lemma 5.8. For a single agent with private-budget utility, independently distributed value and budget, when \( p^q \leq w^\ast \), there exists \( q^\dagger \leq q \) such that the ironed price-posting revenue from \( q^\dagger \) is a \((2\kappa - 1)\)-approximation to the revenue from \( \text{EX}^\delta \), i.e., \((2\kappa - 1)P(q^\dagger) \geq \text{Payoff}^{\text{EX}^\delta} \).

Proof. Let \( q^\dagger = \text{argmax}_{q \leq q} P(q') \). Suppose the support of the budget distribution is from \([w, \bar{w}]\). Let \( \bar{p} \) be the price larger than the market clearing price \( p^q \) and smaller than the expected budget \( w^\ast \) that maximizes revenue without the budget constraint. Consider the following calculation with justification below.

\[
\text{Payoff}^{\text{EX}^\delta} = \int_w^{w^\ast} \text{Payoff}_w^{\tau^\delta_w} dG(w) + \int_{w^\ast}^{\bar{w}} \text{Payoff}_w^{\tau^\delta_w} dG(w) \\
\leq (a) \int_w^{w^*} \text{Payoff}_w^{\tau^\delta_w} dG(w) + \int_{w^*}^{\bar{w}} \frac{w}{w^\ast} \text{Payoff}_w^{\tau^\delta_w} dG(w) \\
\leq (b) \int_w^{w^*} \text{Payoff}_w^{\tau^\delta_w} dG(w) + \int_{w^*}^{\bar{w}} \frac{w}{w^\ast} \text{Payoff}_w^{\tau^\delta_w} dG(w) \\
\leq (c) (2 - \frac{1}{\kappa}) \text{Payoff}_w^{\tau^\delta_w} \\tau^\delta_w \\
\leq (d) (2\kappa - 1) \text{Payoff}[\bar{p}] \leq (2\kappa - 1) P(q^\dagger).
\]

Inequality (a) holds because given the allocation payment function \( \tau^\delta_w \), the revenue only increases if we increase the budget to \( w^\ast \), i.e., \( \text{Payoff}_w^{\tau^\delta_w} \leq \text{Payoff}_w^{\tau^\delta_w} \) for any \( w \leq w^\ast \). Moreover, for any \( w > w^\ast \), given the allocation payment function \( \tau^\delta_w \), the revenue is either the same for budget \( w \) and \( w^\ast \), or the budget binds for agent with expected budget \( w^\ast \). Since the revenue from agent with budget \( w \) is at most \( w \), we know that \( \text{Payoff}_w^{\tau^\delta_w} \leq \text{Payoff}_w^{\tau^\delta_w} \). Note that for allocation payment rule \( \tau^\delta_w \), per-unit prices are larger than the market clearing price \( p^q \) and smaller than the expected budget \( w^\ast \), and budget does not bind for agents with budget \( w^\ast \). Therefore, by definition, the optimal per-unit price in this range is \( \bar{p} \), \( \text{Payoff}_w^{\tau^\delta_w} \leq \text{Payoff}_w^{\tau^\delta_w}(\bar{p}) \) and inequality (b) holds. Inequality (c) holds because \( \int_w^{w^*} dG(w) \leq 1 - 1/\kappa \) by the assumption that the probability the budget exceeds its expectation is at least \( \kappa \), and \( \int_{w^*}^{\bar{w}} w dG(w) \leq 1 \). Inequality (d) holds because \( \text{Payoff}_w^{\tau^\delta_w} \leq \kappa \cdot \text{Payoff}[\bar{p}] \) for any randomized prices \( \bar{p} \) according to [Cheng et al. (2018)]. Inequality (e) holds by the definition of the price-posting revenue curve \( P \) and quantile \( q^\dagger \), the fact that price \( \bar{p} \) is larger than the market clearing price \( p^q \). \( \square \)
Proof of Theorem 5.5. Let \( q^\dagger = \arg\max_{q \leq q} P(q') \). Combining Lemma 5.6, 5.7 and 5.8 we have

\[
\text{Payoff}[EX] \leq \text{Payoff}[EX^\dagger] + \text{Payoff}[EX^\dagger] + \text{Payoff}[EX^\dagger] \leq (1 + 3\kappa - 1/\kappa) P(q^\dagger). \]

\[\square\]

6 Ex Ante Prophet Inequality

Prophet inequality, which is closely related to oblivious posted pricing for agents with linear utility, mostly analyzes the approximation guarantee of online algorithms with respect to the optimal offline algorithm. To study the approximation of posted pricing for non-linear agents, our framework utilize results in \textit{ex ante prophet inequality}, a stronger version with respect to ex ante relaxation.

In this section, we provide meta approach, which shows that some prophet inequality results in the literature can be extended to be ex ante prophet inequalities. The main idea is as follows, (a) show the gambler with an online algorithm under product distribution can approximate the ex post prophet who can correlate realizations with the same marginals; (b) construct a correlated distribution with the same marginals such that the expected value of the ex post prophet given the correlated distribution is equivalent to the expected value of the ex ante prophet.

Lemma 6.1. The expected value of an ex ante prophet subject to ex ante feasible quantiles \( \text{EAF}(\mathcal{X}) \) is equal to the ex post prophet who can correlate prize realization (with the same marginals) and picks feasible subset subject to feasibility constraint \( \mathcal{X} \).

Proof. For any prize distributions \( \{F_i\}_{i \in N} \) and feasibility constraint \( \mathcal{X} \), lemma statement is equivalent to the existence of a correlated distribution \( \nu' \) such that

1. The marginal distribution of \( \nu'_i \) is from \( F_i \).
2. The expected value of the ex post prophet with this correlated distribution equals to the ex ante prophet with product distribution, i.e., \( \text{E}_\nu'[\text{Payoff}[\nu', \mathcal{X}]] = \text{EAF}(\nu) \).

The existence of the correlation distribution is guaranteed by the following construction. Given any distributions \( \{F_i\}_{i \in N} \) and feasibility constraint \( \mathcal{X} \), suppose \( q \) is the optimal ex ante probability profile and \( D \) is the distribution over feasible set which induces \( q \). To generate correlated distribution \( \nu' \), we first sample a set \( S \) under distribution \( D \), and for each agent \( i \), sample \( v_i \) from \( F_i \) conditional on \( v_i \geq v_i^q \) if \( i \in S \), and sample \( v_i \) from \( F_i \) conditional on \( v_i < v_i^q \) if \( i \notin S \), where \( v_i^q \) is the value such that \( \text{Pr}_{v \sim F_i}[v \geq v_i^q] = q_i \).

First note that the marginal distribution for agent \( i \) in correlated distribution \( \nu' \) is \( F_i \). By selecting the set \( S \) associate with each realized valuation profile \( \nu' \), which is feasible for the feasibility constraint \( \mathcal{X} \), we have \( \text{EAF}(\nu) \leq \text{E}_\nu'[\text{Payoff}[\nu', \mathcal{X}]] \). Since the marginal distribution for each agent \( i \) is \( F_i \), we have \( \text{EAF}(\nu) \geq \text{E}_\nu'[\text{Payoff}[\nu', \mathcal{X}]] \). Therefore, \( \text{EAF}(\nu) = \text{E}_\nu'[\text{Payoff}[\nu', \mathcal{X}]] \).

Proposition 6.2. For any prophet inequality bound which is obtained by comparing the gambler’s value to an upper bound on the prophet’s value and where the upper bound is invariant to correlation structure on prize distribution, the gambler’s algorithm obtains the same approximation bound for the ex ante prophet inequality.
We apply Proposition 6.2 to obtain ex ante prophet inequality under \( k \)-unit environment. For general matroid, we obtain the same ratio with adaptive prices and defer its proof to Appendix A.

**Theorem 6.3.** For \( k \)-unit environment \( \mathcal{X} \), and for linear agents with distributions \( \{F_i\}_{i \in N} \), there exists an anonymous threshold \( \theta \) such that

\[
\text{Payoff}[\theta, \{F_i\}_{i \in N}, \mathcal{X}] \geq \frac{1}{2} \cdot \text{EAR}(\{F_i\}_{i \in N}, \mathcal{X})
\]

where \( \text{Payoff}[\theta, \{F_i\}_{i \in N}, \mathcal{X}] \) is the expected welfare of threshold \( \theta \), and \( \text{EAR}(\{F_i\}_{i \in N}, \mathcal{X}) \) is the optimal ex ante relaxation.

**Proof.** By Chawla et al. (2010), for \( k \)-unit environments, there exists an anonymous threshold \( \theta \) such that

\[
2 \text{Payoff}[\theta, \{F_i\}_{i \in N}, \mathcal{X}] \geq (k \theta + \sum_{i \in N} E_{v_i}[(v_i - \theta)^+]) \geq E_v[\max_i v_i].
\]

Note that the upper bound of the optimal payoff \( k \theta + \sum_{i \in N} E_{v_i}[(v_i - \theta)^+] \) considered in Chawla et al. (2010) is invariant to correlation structure. Therefore, by applying Proposition 6.2, there exists an anonymous threshold \( \theta \) that achieves 2-approximation to the ex ante relaxation.

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Appendix

A  Ex Ante Matroid Prophet Inequality

In this section, we reduce the ex ante prophet inequality for matroid environment to the matroid prophet inequality analysis in [Kleinberg and Weinberg (2012)]. Our reduction does not require the matroid property.

Kleinberg and Weinberg (2012) consider algorithms with adaptive threshold $\theta = \{\theta_i\}_{i \in N}$ in prophet inequality for downward-closed environment: every prize $i$ is selected if and only if $v_i \geq \theta_i$. They define the property $\alpha$-balanced threshold and show that it implies $\alpha$-approximation against an (ex post) prophet. Finally, they design an algorithm with $2$-balanced threshold for matroid environment. In this section, we generalize the $\alpha$-balanced threshold (Definition A.1) with correlation and show that it implies $\alpha$-approximation against ex ante prophet (Lemma A.1). The proof follows from Kleinberg and Weinberg (2012)'s original argument, with the observation that independence among prizes is unnecessary. Notably the algorithm in Kleinberg and Weinberg (2012) constructs a threshold for matroid set system that satisfies the generalized $2$-balanced threshold property.

For this extension, we first generalize their notion of $\alpha$-balanced thresholds to correlated distributions on valuations. Let $v = \{v_i\}_{i \in N}$ and $v' = \{v'_i\}_{i \in N}$ be the valuation profiles drawn from the independent and correlated distribution, respectively. We assume $v$ and $v'$ are independent and have the same marginal distribution. Let $\theta = \{\theta_i\}_{i \in N}$ be the threshold in algorithm $A$ where $\theta_i$ is independent of $\{v_j\}_{j \neq i}$ and $v'$. Denote $A = \{i \in [n] : v_i \geq \theta_i\}$ as set of prizes chosen by threshold $\theta$. Denote $B$ as the feasible set in $X$ that maximizes the total value in the set for valuation profile $v'$. Let $C(A)$, $R(A)$ be a partition of $B$ such that (i) $A \cap R(A) = \emptyset$ and $A \cup R(A)$ is a basis for the matroid; and (ii) maximizes the total value of set $R(A)$. Next we formally define $\alpha$-balanced thresholds for the correlated benchmark.

**Definition A.1.** For a parameter $\alpha > 1$, an algorithm $A$ has $\alpha$-balanced threshold $\theta$ if for all valuation profile $v$,

$$\sum_{i \in A} \theta_i \geq \frac{1}{\alpha} \cdot E_{v'} \left[ \sum_{i \in C(A)} v'_i \right] \quad \text{and} \quad E_{v'} \left[ \sum_{i \in R(A)} \theta_i \right] \leq (1 - \frac{1}{\alpha}) \cdot E_{v'} \left[ \sum_{i \in R(A)} v'_i \right].$$

**Lemma A.1.** Any algorithm $A$ with $\alpha$-balanced threshold is an $\alpha$-approximation to the ex ante prophet.

---

3 In downward-closed environment, without loss of generality, an algorithm with adaptive threshold guarantees to output a feasible set of prizes selected, as it can set threshold $\theta_i = \infty$ to reject prize $i$.

4 Think $v$ as the valuation profile which an algorithm $A$ executes by the gambler, and $v'$ as the valuation profile which observed by ex ante prophet.
Proof. Invoking Proposition 6.2, it is sufficient to show

\[ E_v \left[ \sum_{i \in A} v_i \right] \geq \frac{1}{\alpha} \cdot E_{v'} \left[ \sum_{i \in B} v'_i \right]. \]

Since \( C(A) \) and \( R(A) \) is a partition of \( B \),

\[ E_{v'} \left[ \sum_{i \in B} v'_i \right] = E_{v'} \left[ \sum_{i \in C(A)} v'_i \right] + E_{v'} \left[ \sum_{i \in R(A)} v'_i \right]. \]

Let \( (\cdot)^+ = \max\{\cdot, 0\} \). We will derive the following three inequalities

1. \[ E_v \left[ \sum_{i \in A} \theta_i \right] \geq E_{v'} \left[ \sum_{i \in C(A)} v'_i \right] \] (1)
2. \[ E_v \left[ \sum_{i \in A} (v_i - \theta_i)^+ \right] \geq E_{v'} \left[ \sum_{i \in R(A)} (v'_i - \theta_i)^+ \right] \] (2)
3. \[ E_{v'} \left[ \sum_{i \in R(A)} (v'_i - \theta_i)^+ \right] \geq E_{v'} \left[ \sum_{i \in R(A)} v'_i \right]. \] (3)

Summing (1), (2) and (3) with the fact that \( \theta + (v_i - \theta_i)^+ = v_i \) for all \( i \in A \), finishes the proof.

Inequality (1) is satisfied by the definition of \( \alpha \)-balanced threshold. For inequality (2), notice that

\[ E_v \left[ \sum_{i \in A} (v_i - \theta_i)^+ \right] = (a) E_v \left[ \sum_{i \in N} (v_i - \theta_i)^+ \right] = (b) E_{v'} \left[ \sum_{i \in N} (v_i - \theta_i)^+ \right] \geq E_{v'} \left[ \sum_{i \in R(A)} (v'_i - \theta_i)^+ \right] \]

where (a) holds by definition of \( A \), i.e., prize \( i \) is selected in \( A \) if \( v_i \geq \theta_i \); and (b) holds since \( v, v' \) are independent with the same marginal and \( \theta_i \) is independent with \( v_i \) and \( v' \). For inequality (3), using the definition of \( \alpha \)-balance threshold, we have

\[ E_{v'} \left[ \sum_{i \in R(A)} v'_i \right] \leq E_{v'} \left[ \sum_{i \in R(A)} (\theta_i + (v'_i - \theta_i)^+) \right] \]

\[ \leq (1 - \frac{1}{\alpha}) \cdot E_{v'} \left[ \sum_{i \in R(A)} v'_i \right] + E_{v'} \left[ \sum_{i \in R(A)} (v'_i - \theta_i)^+ \right], \]

which implies that

\[ E_{v'} \left[ \sum_{i \in R(A)} (v'_i - \theta_i)^+ \right] \geq \frac{1}{\alpha} \cdot E_{v'} \left[ \sum_{i \in R(A)} v'_i \right]. \]

which concludes the proof. \square
For matroid environment, Kleinberg and Weinberg (2012) design an algorithm which has 2-balanced threshold for the correlated benchmark (Definition A.1).

Lemma A.2 (Kleinberg and Weinberg, 2012). For matroid environment $\mathcal{X}$, there exists a 2-balanced adaptive threshold

$$\theta_i = \frac{1}{2} E_{v_i'} \left[ \sum_{i \in C(A_{i-1} \cup \{i\})} v_i' - \sum_{i \in C(A_{i-1})} v_i' \right],$$

where $A_{i-1}$ is the set chosen till day $i - 1$.

Therefore, invoking Lemma A.1 and A.2 we have the following theorem.

Theorem A.3. For matroid environment $\mathcal{X}$, and for linear agents with distributions $\{F_i\}_{i \in \mathbb{N}}$, there exists a profile of adaptive threshold $\theta$ such that

$$\text{Payoff}[\theta, \{F_i\}_{i \in \mathbb{N}}, \mathcal{X}] \geq \frac{1}{2} \cdot \text{EAR}(\{F_i\}_{i \in \mathbb{N}}, \mathcal{X})$$

where $\text{Payoff}[\theta, \{F_i\}_{i \in \mathbb{N}}, \mathcal{X}]$ is the expected welfare of setting the threshold as $\theta_i$ when item $i$ arrives, and $\text{EAR}(\{F_i\}_{i \in \mathbb{N}}, \mathcal{X})$ is the optimal ex ante relaxation.

B. Necessity of assumptions for agents with private budget

If there is no assumption on the budget distribution and the valuation distribution, even if those distributions are independent from each other, for the single agent problem, price posting is not a constant approximation to the optimal revenue.

Example B.1. Consider the budget distribution is the discrete equal revenue distribution, i.e., $g(i) = 1/\pi \cdot i^2$, where $\pi = \pi^2/\alpha$. Let the quantile function of the valuation distribution be $q(i) = 1/\ln i$. The optimal price posting revenue is a constant. Next consider the pricing function $\tau(x) = \frac{1}{1-x}$. From this pricing function, the value $v_i$ corresponding to payment $i$ is $v_i = i^2$. Note that the revenue from this payment function is infinity, i.e.,

$$\text{Payoff}[\tau] \geq \lim_{m \to \infty} \sum_{i=1}^{m} (i \cdot q(v_i) \cdot g(i))$$

$$= \frac{1}{2\pi} \lim_{m \to \infty} \sum_{i=1}^{m} \frac{1}{i \cdot \ln i}$$

$$= \frac{1}{2\pi} \lim_{m \to \infty} \ln \ln m \to \infty.$$
Comparing to Feng et al. (2019), we propose a reduction framework for general payoff maximization (e.g., welfare maximization) using pricing-based mechanisms. Note that this reduction does not hold for anonymous pricing considered in Feng et al. (2019). The main reason is that anonymity is not maintained in the reduction framework. For example, for the welfare objective, prophet inequalities indicate that anonymous pricing achieves 2-approximation for linear agents, while we show that anonymous pricing is not a constant approximation for budgeted agents even when the valuations and the budgets for all agents are public information.

Example B.2. Consider there is a single item and 2 agents. Suppose $\epsilon < 1$ is a constant arbitrarily close to 0. Agent 1 has value $v_1 = \frac{1}{\epsilon^2}$ and budget $w_1 = 1$. Agent 2 has value $v_2 = \frac{1}{\epsilon}$ and budget $w_2 = \frac{1}{\epsilon}$. By allocating the item to agent 1, and the optimal welfare is $\frac{1}{\epsilon^2}$. For any anonymous price $p$, suppose agent 2 arrives first. If $p \leq \frac{1}{\epsilon}$, agent 2 gets the item and welfare is $\frac{1}{\epsilon}$. If $p > \frac{1}{\epsilon}$, since the budget of agent 1 is 1, she can purchase the lottery with allocation at most $\epsilon$, and the total welfare of anonymous pricing is at most $\frac{2}{\epsilon}$. Therefore, if $\epsilon \to 0$, the approximation ratio $\frac{1}{2\epsilon} \to \infty$.

C Numerical Result for Uniformly Distributed Private-budgeted Agents

In this section, we discuss the numerical results of the approximation ratios of revenue-maximization for i.i.d. private-budgeted agents with value and budget drawn uniformly from $[0,1]$ independently. This example and the optimal mechanisms have been studied in Che and Gale (2000) for a single agent and Pai and Vohra (2014) for multiple agents. For both scenarios, the optimal mechanisms are complicated. However, Figure 1a suggests that for a single agent, posting a single price is a good approximation to the optimal mechanism for all ex ante probability constraint; Figure 1b suggests that for multi-agents, simple pricing based mechanisms (i.e. oblivious posted pricing and marginal payoff maximization) achieve good approximation to the optimal mechanism. Next, we explain how the numerical results are computed.

First we focus on the single agent problem, i.e., the calculation of the price-posting revenue curve and ex ante revenue curve illustrated in Figure 1a. For the price-posting revenue curve, we directly compute the probability the item is sold and the corresponding revenue for any price $p$. Thus, we can have the closed-form characterization for the mapping from the ex ante allocation constraint to the optimal price-posting revenue. For the ex ante revenue curve, by approximating the continuous uniform distribution with a discretized uniform distribution, we can write this optimization problem as a finite dimensional linear program, which allows us to numerically evaluate the optimal ex ante revenue given any ex ante allocation constraint $q$. By evaluating the curve on quantiles $q \in \{0, 1/50, \ldots, 1\}$ with grid size $1/50$, we have the numerical figure for the ex ante revenue curve.

For the multi-agent problem, since both oblivious posted pricing and marginal payoff mechanism are pricing based mechanism, the revenues of both mechanisms for private-budgeted agents are equivalent to the revenues of both mechanisms for linear agents with the same price-posting revenue curve. By the above paragraph, we have the closed-form for the
price-posting revenue curve, which pins down the value distribution of such linear agents. First note that since agents are i.i.d., the revenue from oblivious posted pricing (OPP) is the same as sequential posted pricing (SPP). We compute the revenue for both OPP and SPP using dynamic programming (i.e. backward induction). For i.i.d. regular linear agents, the revenue of the marginal payoff mechanism is the same as the revenue of the second price auction with monopoly reserve, which can be solved analytically. Finally, we can numerical calculate the optimal ex ante relaxation using the ex ante revenue curve for a single agent, and evaluate the approximation ratio for both mechanisms when number of agents ranges from 1 to 15.