Simulation of Continuous Markov Processes in Discrete-Time for Signals Described by Stochastic Differential Equations

V M Artyushenko¹, V I Volovach² and V N Budilov²

¹Technological University, 42, Gagarin str., Korolev, 141070, Russia
²Volga Region State University of Service, 4, Gagarin str., Togliatty, 445017, Russia

E-mail: volovach.vi@mail.ru

Abstract. We considered the issues associated with methods for solving the problems of representation of continuous Markov processes in discrete-time by the example of radar signals, which are subject to the simultaneous effect of additive and multiplicative noise described by stochastic differential equations. We considered the algorithm, which allows us to carry out computer simulation of fading amplitude, phase and information parameter of the signal in discrete observation time using the example of a continuous frequency-modulated signal, the amplitude of which adheres to the Nakagami distribution that is typical for radar signals under simultaneous effect of additive and multiplicative noise. It is shown that with the help of computer simulation it is possible to create not only optimal algorithms, but also their corresponding optimal structural diagrams, which makes it possible to elevate the processing of radar signals to higher standards.

1. Introduction
The widespread introduction of modern digital systems and microprocessor technology requires constant development of the theory and technology of discrete (digital) processing of continuous signals. Such systems are most widely used in radar, radio navigation, communications, radio telemetry, information processing and transmission, that is, in those systems where the used computing devices receive and issue information about continuous processes in discrete time.

The aim of this work is to obtain algorithms for discrete-time simulation of continuous Markov processes described by stochastic differential equations.

As an example, we consider a frequency modulated signal in the presence of fading. As a rule, such a signal is observed during the processing of radar signals subject to the simultaneous effect of both additive and multiplicative noise [1-6 etc.].

2. The initial assumptions
Suppose the following fluctuation is available for observation

\[ y(t) = s(t, \lambda) + n(t), \]
where $s(t, \lambda)$ is a useful signal (narrow-band) with the vector information parameter $\lambda$ encoded in it, $n(t)$ is additive normal white noise with zero mean and delta correlation function $\langle n(t) \rangle = 0$, $\langle n(t_1)n(t_2) \rangle = 0.5N_0 \delta(t_2 - t_1)$. Here $N_0$ is known one-sided spectral density of white noise.

Note that the information vector parameter, in the general case, is described by the system of equations

$$\frac{d \lambda}{dt} = C \lambda(t) + n_0(t),$$

where $\lambda(t)$ is vector process, $C$ is square matrix, $n_0(t)$ is vector white Gaussian noise with mathematical expectation and correlation matrix

$$\mu[n_0(t_1)n_0^T(t_1)] = Q \delta(t_2 - t_1),$$

where $Q$ is a symmetric non-negative definite matrix, $T$ is the sign of the matrix transposition.

The general solution to the equation has the form

$$\lambda(t) = \Phi(t - t_0) \lambda(t_0) + \int_{t_0}^t \Phi(t - \tau) n_0 d\tau,$$

where $\Phi(t)$ is a transition matrix satisfying the equation $\frac{d\Phi}{dt} = C \Phi(t)$, with the initial condition $\Phi(0) = I$, where $I$ is the identity matrix.

### 3. Narrow-band useful signal description

Let a narrow-band useful signal with a vector information parameter $\lambda(t)$ encoded in it [7] has the form

$$s(t, \lambda) = U(t) \cos[\omega_0 + \Psi(t)],$$

where $U(t)$ and $\Psi(t)$ are random processes.

The first process $U(t) \geq 0$ takes into account the amplitude fading of the radio signal. The second one contains information about an information parameter $\lambda(t)$:

$$\Psi(t) = \varphi(t) + M \int_0^t \lambda(\tau) d\tau,$$

where $M_f$ is a constant frequency modulation factor, $\varphi(t)$ is random «wandering» phase of the radio signal, described by a stochastic differential equation

$$\frac{d \varphi}{dt} = n_\varphi(t).$$

Here $n_\varphi(t)$ is normal white noise («phase» noise caused by oscillator frequency instability) with zero mean and delta correlation function [8]:

$$\langle n_\varphi(t) \rangle = 0; \quad \langle n_\varphi(t_1)n_\varphi(t_2) \rangle = 0.5N_\varphi \delta(t_2 - t_1).$$

We will assume that the amplitude fading of the signal is described by the Nakagami distribution

$$W(U) = \left\{ \frac{2}{\Gamma(m)} \right\} \left( \frac{m}{\sigma_U^2} \right)^m U^{2m-1} \exp\left\{ -\frac{m}{\sigma_U^2} U^2 \right\},$$

where
where $\Gamma(.)$ is a gamma function, $m$ is a distribution parameter, $\sigma_U^2$ is an amplitude variance.

Suppose both processes $U(t)$ and $\Psi(t)$ are Markov and are described by the equations

\begin{align*}
\dot{\Psi} &= M \dot{\lambda} + n_\Psi(t); \\
\dot{\lambda} &= -\alpha \lambda + n_\lambda(t); \\
\dot{U} &= -\frac{1}{4} N_U^2 \Omega_U^2 \left( \frac{2m}{\sigma_U^2} U - \frac{2m-1}{U} \right) - \Omega_U n_U(t),
\end{align*}

where $(-\alpha \lambda) = \alpha(\lambda)$ is a drift coefficient for information parameter $\lambda(t)$, $\Omega_U$ is a parameter of the amplitude distribution, $n_\lambda(t)$ is normal white noise with known characteristics, zero mean and delta correlation function $\langle n_\lambda(t_1) n_\lambda(t_2) \rangle = 0.5 N \delta(t_1 - t_2)$, $n_\lambda(t)$ is normal white noise with one-sided spectral density $N_U$.

Here and below, the dot above denotes the derivatives with respect to time $t$.

Note that the drift coefficient for the signal amplitude $U(t)$ depends on $U$ nonlinearly:

$$
\alpha(U) = -\frac{1}{4} N_U^2 \Omega_U^2 \left( \frac{2m}{\sigma_U^2} U - \frac{2m-1}{U} \right).
$$

4. Modeling continuous Markov processes

Let us bring the system of differential equations (1)-(3) to a linear form. To convert the nonlinear differential equation (3) to a linear form, we use the expansion in Newton's series:

$$
\frac{1}{U} = \frac{1}{1+(U-1)} = 1 - (U-1) + (U-1)^2 - (U-1)^3 + (U-1)^4 - \ldots
$$

Let us denote by

$$
x = (U-1)^2 - (U-1)^3 + (U-1)^4 - \ldots
$$

and write $U^{-1} = 2 - U + x$.

The accuracy of calculating $x$ must satisfy the inequality $U - x = \Delta_U \leq \Delta$, where $\Delta_U$ is the error that occurs when expanding into Newton's series, $\Delta$ is the sampling step in the transition from continuous to discrete time.

Since the «final» value $x$ will be equal to some specific number, it is advisable for further calculations to designate $K = 2 + x$, where $K$ is a dimensionless coefficient.

Taking into account the designations made, we write down that $U^{-1} = K - U$.

Substituting this equality into equation (3), we obtain

$$
\dot{U} = -\frac{1}{4} N_U^2 \Omega_U^2 \left( \frac{2m}{\sigma_U^2} U - (2m-1)(K-U) \right) - \Omega_U n_U(t).
$$

Making the necessary transformations

$$
\frac{2m}{\sigma_U^2} U - (2m-1)(K-U) = U \frac{2m+\sigma_U^2(2m-1)}{\sigma_U^2} - K(2m-1),
$$

write down
Let us introduce new notation:

\[
\gamma = -\frac{1}{4} N_U^2 \Omega_U^2 \left( \frac{2m + \sigma_U^2 (2m-1)}{\sigma_U} \right), \quad n_u(t) = -\frac{1}{4} N_U^2 \Omega_U^2 (2m-1) - \Omega_U n_U(t).
\]

With this in mind, we write that \( \dot{U} = -\gamma U + n_u(t) \).

Then the system of equations (1)-(3) will take the form:

\[
\begin{aligned}
\dot{\Psi} &= M_f \lambda + n_q(t); \\
\dot{\lambda} &= -\alpha \lambda + n_l(t); \\
\dot{U} &= -\gamma U + n_u(t).
\end{aligned}
\]

Let us find the matrices \( C \) and \( Q \). For the differential system (4), the following record is valid

\[
C = \begin{bmatrix} 0 & M_f & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\gamma \end{bmatrix}.
\]

Knowing that \( n_q(t), n_l(t), n_u(t) \) are independent white Gaussian noise with zero mathematical expectation and one-sided spectral densities \( N_q = N_l = N_0, N_u \neq N_0 \), we write down

\[
Q = \begin{bmatrix} N_q & 0 & 0 \\ 0 & N_l & 0 \\ 0 & 0 & N_u \end{bmatrix} = \frac{N_0}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & N_u/N_0 \end{bmatrix}.
\]

Representing \( N_U = aN_0 \), where \( a \) is a dimensionless coefficient, matrix (5) will take the form:

\[
Q = \frac{N_0}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}.
\]

Let us define the transition matrix \( \Phi(t) \). Using the results of [9], we write \( \Phi(t) = \exp\{Ct\} \).

To calculate the matrix exponent, we use the Laplace transform. Suppose the matrix each element of which is a Laplace image of the corresponding element of the matrix \( \Phi(t) \) is denoted by \( \hat{\Phi}(p) = \alpha \{\Phi(t)\} \). Then we have \( (pI - C)\hat{\Phi}(p) = I \).

Hence, if the determinant \( \det(pI - C) \neq 0 \) get

\[
\hat{\Phi}(p) = (pI - C)^{-1}.
\]

We use this ratio to calculate the matrix exponent \( \hat{\Phi}(p) = \alpha^{-1}\{(pI - C)^{-1}\} \). Based on the above, we find
where \( p \) is a matrix of images, \( I \) is a unity matrix. As then, using expression (6), we define \((pI - C)^{-1}\). Let us denote \( B \).

According to \([10]\), we can write that

\[
B^{-1} = \frac{1}{\det B} \begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}^T,
\]

where \( \det B \) is a determinant of the matrix \( B \), \( B_{ik} \) is the algebraic complement of the element \( b_{ik} \) of the matrix \( B \) (it is understood as the minor \( M_{ik} \) of the element \( b_{ik} \) multiplied by \((-1)^{i+k}\), that is

\[
\text{det} B = (-1)^{i+k} M_{ik}, \quad (i = \overline{1,n}, k = \overline{1,n}).
\]

So, \( \det B = p(p+\alpha)(p+\gamma) \), then, after transformations, we can write the transposed matrix

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}^T = \begin{bmatrix}
(p+\alpha)(p+\gamma) & M_f & (p+\gamma) \\
0 & p(p+\gamma) & 0 \\
0 & 0 & p(p+\alpha)
\end{bmatrix}.
\]

Substituting the obtained expressions into (7), we obtain

\[
B^{-1} = \frac{1}{p(p+\alpha)(p+\gamma)} \begin{bmatrix}
(p+\alpha)(p+\gamma) & M_f & (p+\gamma) \\
0 & p(p+\gamma) & 0 \\
0 & 0 & p(p+\alpha)
\end{bmatrix}^{-1} \begin{bmatrix}
(M_f e^{-\alpha t}) & 0 \\
0 & e^{-\alpha t} & 0 \\
0 & 0 & e^{-\gamma t}
\end{bmatrix}.
\]

Using the inverse Laplace transform, we write

\[
\Phi(t) = \alpha^{-1} \{(pI - C)^{-1}\} = \alpha^{-1} \{B^{-1}\} = \begin{bmatrix}
1 & M_f e^{-\alpha t} & 0 \\
0 & e^{-\alpha t} & 0 \\
0 & 0 & e^{-\gamma t}
\end{bmatrix}.
\]

Let us pass from \( \Phi(t) \) \( \Phi(\Delta) = \exp\{\Delta\} \), that is, we will pass from continuous to discrete time. For this, the time interval \([0, T]\) is divided by \( n \) equidistant points \( t_k = h\Delta \), where \( \Delta = T/n \) is the sampling step in time, \( h = 0, 1, \ldots, n \). We choose a discretization step in time \( \Delta \) so that the inequality \( \alpha\Delta \ll 1 \). Then, using the approximate equality \( \exp\{-\alpha\Delta\} \approx 1 - \alpha\Delta \), you can write

\[
\Phi(\Delta) = \begin{bmatrix}
1 & M_f (1-\alpha\Delta) & 0 \\
0 & 1-\alpha\Delta & 0 \\
0 & 0 & 1-\gamma
\end{bmatrix}.
\]

Let us define the correlation matrix of discrete vector white noise \( D(\Delta) \). If we assume that \( t \to 0 \), then according to \([8]\), we get
The smaller the step \( \Delta \) the more accurate the approximate ratio. Proceeding from the fact that as any symmetric nonnegative definite matrix \( D(\Delta) \) can be represented as \( D = \Gamma \Gamma^T \), taking into account (8), we write that \( Q\Delta = \Gamma \Gamma^T \), where \( \Gamma \) is the lower triangular matrix (all elements above the main diagonal are equal to zero).

Let us define a triangular matrix \( G \). According to [10], we write:

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33}
\end{bmatrix}
= \begin{bmatrix}
q_{11} & 0 & 0 \\
q_{21} & q_{22} & 0 \\
q_{31} & q_{32} & q_{33}
\end{bmatrix}
= \Gamma \Gamma^T.
\]  

Recurrent procedure for calculating the elements of the lower and upper triangular matrices \( q_{ij} \) is following:

\[
q_{ij} = \frac{Q_{ij} - \sum_{k=1}^{i-1} q_{ik} q_{jk}}{\left( Q_{ij} - \sum_{k=1}^{i-1} q_{jk}^2 \right)^{0.5}},
\]  

\[
\sum_{k=1}^{0} q_{ik} q_{jk} = 0, \quad 1 \leq j \leq i \leq n.
\]

Using expression (9), we define the elements of the lower triangular matrix \( \Gamma \):

\[
q_{11} = (Q_{11})^{0.5} = 1; \quad q_{21} = \frac{Q_{21}}{(Q_{11})^{0.5}} = 0; \quad q_{31} = \frac{Q_{31}}{(Q_{11})^{0.5}} = 0;
\]

\[
q_{22} = \left( Q_{22} - \frac{Q_{21}^2}{Q_{11}} \right)^{0.5} = 1; \quad q_{32} = \frac{Q_{32} - \sum_{k=1}^{2} q_{1k} q_{2k}}{(Q_{32} - \sum_{k=1}^{2} q_{2k}^2)^{0.5}} = 0; \quad q_{33} = \left( Q_{33} - q_{31}^2 q_{32}^2 \right)^{0.5} = \left( \frac{N_U}{N_0} \right)^{0.5}.
\]

Taking into account expression (5), we finally write

\[
\Gamma = \left[ \frac{\Delta N_0}{2} \right]^{0.5} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \left( \frac{N_U}{N_0} \right)^{0.5}
\end{bmatrix}.
\]  

Thus, using equation (10), for the system of equations (4) we finally obtain

\[
\begin{bmatrix}
\Psi_{h} \\
\lambda_{h} \\
U_h
\end{bmatrix} = \begin{bmatrix}
1 & M_f (1 - \alpha \Delta) & 0 \\
0 & 1 - \alpha \Delta & 0 \\
0 & 0 & 1 - \gamma
\end{bmatrix} \begin{bmatrix}
\Psi_{h-1} \\
\lambda_{h-1} \\
U_{h-1}
\end{bmatrix} + \left[ \frac{\Delta N_0}{2} \right]^{0.5} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \left( \frac{N_U}{N_0} \right)^{0.5}
\end{bmatrix} n_{h-1}.
\]  

Relation (11) allows us to simulate continuous Markov processes \( \Psi, \lambda, \) \( U \) in discrete time.

Fig. 1 shows a shaping filter of a vector process \( \lambda_h = \{ \Psi_h, \lambda_h, U \} \), realizing the obtained algorithm (12). Process recovery scheme \( \lambda_h \) by discrete readings is shown in Fig. 2 (here ADC is analog-to-digital converter).
5. Conclusions
On the example of a continuous frequency-modulated signal, the amplitude of which obeys the Nakagami distribution, characteristic of radar signals subject to the simultaneous effect of additive and multiplicative noise, an algorithm is considered that allows computer modelling of signal parameters Ψ, λ and U in discrete observation time.

Structural diagrams of the filter forming the vector process are obtained $\lambda_h = \{\Psi, \lambda, U\}$. Structural diagrams of the filter forming the vector process are obtained.

With the help of computer modelling, it is possible not only to reproduce various situations associated with the processing of radar signals, but also to create both optimal algorithms and the corresponding optimal structural schemes, which allows raising the processing of radar signals to a qualitatively new level.

![Figure 1. Vector process shaping filter.](image1)

Scientific novelty lies in the discrete time representation of continuous Markov processes described by stochastic differential equations.

Practical significance includes the possibility of computer modeling of useful signal parameters, as well as the creation of optimal structural diagrams on its basis.

6. References
[1] Van Trees H L, Bell K and Tiany Z 2013 Detection Estimation and Modulation Theory, 2nd Edition, Part I, Detection, Estimation and Filtering Theory (London: Wiley & Sons, Inc.)
[2] Bentsman J 2016 Introduction to Signal Processing, Instrumentation, and Control An Integrative Approach University of Illinois at Urbana-Champaign, USA, 2016 (Urbana-Champaign: University of Illinois)
[3] Barkat M 2005 Signal Detection and Estimation (Norwood: Artech House)
[4] Palahina E and Palahin V 2016 Signal detection in additive-multiplicative non-Gaussian noise using higher order statistics 2016 Proc. 26th Int. Conf. Radioelektronika (Kosice, Slovakia)
[5] Artyushenko V M, Volovach V I and Shakurskiy M V 2016 The Demodulation Signal under the Influence of Additive and Multiplicative non-Gaussian Noise 2016 Proc. of IEEE East-West Design & Test Symposium (EWDTS’2016) (Yerevan, Armenia)
[6] Shevgunov T, Efimov E and Kirdyashkin V 2019 Scattering target identification based on radial basis function artificial neural networks in the presence of non-stationary noise PeriódicoTchêQuímica 16(33) 530-40

[7] Artyushenko V M and Volovach V I Nonlinear Estimation of Signal Parameters under the Influence of Narrowband Non-Gaussian Noise Optoelectronics, instrumentation and data processing 55(1) 66-3

[8] Deutsch R 2017 Nonlinear Transformations of Random Processes (New York: Dover Publications)

[9] Krishnan V 2016 Probability and Random Processes 2nd ed. (New York: Wiley)

[10] 1989 Matrix Theory and Applications (Phoenix, Arizona: American Mathematical Society)