Inertialess gyrating engines

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Abstract
A typical model for a gyrating engine consists of an inertial wheel powered by an energy source that generates an angle-dependent torque. Examples of such engines include a pendulum with an externally applied torque, Stirling engines, and the Brownian gyrating engine. Variations in the torque are averaged out by the inertia of the system to produce limit cycle oscillations. While torque generating mechanisms are also ubiquitous in the biological world, where they typically feed on chemical gradients, inertia is not a property that one naturally associates with such processes. In the present work, seeking ways to dispense of the need for inertial effects, we study an inertia-less concept where the combined effect of coupled torque-producing components averages out variations in the ambient potential and helps overcome dissipative forces to allow sustained operation for vanishingly small inertia. We exemplify this inertia-less concept through analysis of two of the aforementioned engines, the Stirling engine, and the Brownian gyrating engine. An analogous principle may be sought in biomolecular processes as well as in modern-day technological engines, where for the latter, the coupled torque-producing components reduce vibrations that stem from the variability of the generated torque.

Keywords: Stirling engine, Brownian gyrator, limit cycle oscillation, averaging

Significance Statement:
Certain mechanisms are capable of generating torque from temperature gradients and, by utilizing inertia, produce sustained operation as thermodynamic engines. The present paper studies the effect of coupling several such mechanisms together to produce a sustained torque so that inertia is no longer needed. It is envisioned that a similar principle might be at work in bio-molecular engines that draw energy from chemical gradients and where inertia is not typically a significant factor.

Introduction
The paradigm studied herein, referred to as a gyrating engine, is a system with a rotational degree of freedom characterized by an angle $\theta$ and driven by an external torque $T$ that depends on $\theta$, which, however, may not necessarily retain the same sign during a cycle. Specifically, the device obeys

$$\dot{\theta} = \omega, \quad I\ddot{\theta} = T(\theta) - \Gamma\omega,$$

(1)

where $I$ is the moment of inertia and $\Gamma$ is the friction coefficient. The term $-\Gamma\omega$ corresponds to external dissipation, though it can just as well represent torque proportional to angular velocity $\omega$ exchanged with an external subsystem acting as a load. This model captures the general principle behind a wide range of mechanisms that convert thermal/chemical energy to rotary motion, whether synthetic or natural, from steam-engines to biomolecular motors.

We focus on two different types of gyroting engines, a low-temperature-differential Stirling engine (1) that draws power from a temperature differential and a Brownian gyrating engine powered by Nyquist–Johnson thermal noise of two resistors kept at different temperature (2). The salient feature in embodiments of these devices is the inertia needed to average out fluctuations and ensure sustained operation. Analogous biomolecular mechanisms, however, seem to dispense of such a need for inertial effects (3–5). A cursory view of the workings of biomolecular engines reveals a many-fold symmetry of multiple torque-generating units at work. With this in mind, we study the coupling of multiple gyrating engines as a way to eliminate the need for inertia in sustained limit cycle oscillation.

The basic idea explored in this paper is based on the principle that a phase difference between coupled gyroting engines can average out the applied torque. Thereby, angular variations in torque and load can be matched via a suitable geometric arrangement. We present analysis that highlights similarities between the two paradigms, the Stirling and Brownian gyroting engines, as well as provides quantitative and qualitative features of such arrangements. Our interest is mainly in enabling sustained operation in the presence of sign-indefinite generated torque by individual engines, that is, in ensuring that the combined torque of multiple units retains its sign.
The same principle can be used to minimize the variance of the effective torque being applied. Indeed, the idea of coupling engines to reduce torque variations is not new. Multicylinder internal combustion engines reduce torsional vibrations (6,7). However, exploring this principle for inertia-less operation of gyrating engines is new and may help elucidate the functionality of certain biomolecular gyrating engines.

Specifically, there are three motor proteins that have been unambiguously identified as rotary engines, the F0/F1 ATP synthase and the bacterial flagellar motor (8); they are powered by chemical gradients with the flagellar and F0 motor tapping onto transmembrane ion-motive force while the workings of these 50-nm-scale motors, much remains to be understood (9). In regard to the mechanics, their geometry, that engages several torque-generating subunits (10,11) (up to 11 in flagellar motors, and often a three-fold symmetry in ATPases), leads inescapably to the conclusion that a principle such as the one studied herein must be at work.

In order for the engine to operate sustainably, the temperature difference must exceed a certain threshold, as noted in ref. (14); we also refer to ref. (15) for a detailed exposition of the coupling between the thermal gradient and the mechanics of the Stirling engine from a thermodynamic perspective.

Indeed, the underlying thermodynamics of the Stirling engine cycle have been thoroughly studied (16–20). However, models that include the gyrating dynamics of the engine are scarce. The simplified model that we adopt herein is based on the one developed in (1) that has two degrees of freedom, the flywheel angle \( \theta \), and its angular velocity \( \omega = \theta \). The equations of motion are those given in [1] with the torque given by

\[
T^\alpha(\theta) = s r (p(\theta) - p_0) \sin \theta, 
\]

where \( s \) is the section area of the power piston, \( r \) is the crank radius, \( p(\theta) \) is the pressure inside the cylinder, and \( p_0 \) is the external atmospheric pressure. The pressure \( p(\theta) \) is estimated using the ideal gas law scaled by a dimensionless parameter \( \zeta \) that accounts for the nonuniformity of temperature and pressure in the cylinder, and it is

\[
p(\theta) = \zeta \frac{nRT(\theta)}{V(\theta)}. 
\]

where \( n \) is the number of moles of gas in the cylinder and \( R \) is the molar gas constant. The effective temperature \( T(\theta) \) and the volume \( V(\theta) \) of the gas in the cylinder can be expressed as follows:

\[
T(\theta) = T_0 + \frac{\Delta T}{2} \sin(\theta), 
\]

\[
V(\theta) = V_0 + sr(1 - \cos \theta), 
\]

where \( T_0 = (T_{\text{top}} + T_{\text{btm}})/2 \) is the mean of the top and bottom temperatures, \( \alpha \) is a dimensionless coefficient that models the heat transfer, \( \Delta T = T_{\text{btm}} - T_{\text{top}} \) is the temperature difference, and \( V_0 \) is the volume at \( \theta = 0 \).

We remark that in the model proposed by (1), the temperature is more generally expressed as a function of both \( \theta \) and \( \omega \). Specifically, the temperature’s dependence on the angular position of the engine is delayed by a factor of \( r_0 \), with \( \sin(\theta - \omega r) \) replacing \( \sin(\theta) \) in (3). However, experimental evidence (1) suggests that \( r = 15 \times 10^{-3} \) s. Thus, in our analysis, we have adopted the simplifying assumption that \( \omega r = 0 \); numerical simulations confirm that for our purposes, the effect of the small delay \( r \) is indeed negligible.

### Brownian gyrating engine

The second example is that of a Brownian gyration-based engine that was recently introduced in (2). This consists of the coupling between an electrical system, known as the Brownian gyration (21), and a mechanical subsystem with an inertial wheel. Note that we distinguish between the Brownian gyration and the Brownian gyrating engine, that consists of coupling the Brownian gyration to the mechanical subsystem that mediates energy extraction.

The electrical embodiment of the Brownian gyration consists of three capacitors and two resistors (see Fig. 2, top), which are in contact with two heat baths at different temperatures giving rise to Johnson–Nyquist fluctuating currents at the two resistors. The temperature-induced amplitude imbalance in the fluctuating currents results in, on average, a circulating current (in a nonequi-
librium steady state) that effectively transfers heat between the two heat baths. This particular embodiment was introduced in ref. (22); equivalent realizations have been extensively studied, both theoretically (21, 23–25) and experimentally (23, 26, 27).

The mechanical subsystem, that together with the Brownian gyrator forms the Brownian gyration engine, includes dielectric padding in the three capacitors that can vary in its position through mechanical coupling to the rotating wheel as depicted at the bottom of Fig. 2. In this way, the angular position of the (inertial) wheel forces the dielectric material in and out of the (inertial) wheel forces the dielectric material in and out of the respective capacitors. This mechanical coupling renders the capacitor-matrix a function of the dynamic variable \( \theta \). In our analysis, the geometry of the linkages actuating the dielectric material has been chosen such that the capacitance matrix as a function of \( \theta \) is of the form

\[
C(\theta) = \begin{bmatrix}
C_1(\theta) + C_c(\theta) & -C_2(\theta) \\
-C_1(\theta) & C_2(\theta) + C_c(\theta)
\end{bmatrix}
\]

\[
= C_0 \begin{bmatrix}
2 + \beta g_2(\theta) & -1 - \beta \cos(\theta) \\
-1 - \beta \cos(\theta) & 2 + \beta g_2(\theta)
\end{bmatrix},
\]

where \( C_1, C_2, \) and \( C_c \), depicted in Fig. 2, are expressed in terms of a nominal capacitance \( C_0 \), and the \( \theta \)-functions \( g_1(\theta) = \cos(\theta + 2\pi/3) + \cos(\theta) \) and \( g_2(\theta) = \cos(\theta - 2\pi/3) + \cos(\theta) \) with \( 0 < \beta < 1 \). The mechanical part can provide inertia as well as a resistive torque (modeled as \( -R \gamma \)) that absorbs generated power.

As long as there is enough time-scale separation between the mechanical and the electrical subsystems, as shown in (2), the dynamics of the Brownian gyration engine obey [1] with

\[
T^\theta(\theta) = -\frac{1}{2} \text{Tr} \left[ k_B^{-1}(\theta) \Sigma(\theta) \right],
\]

where \( \text{Tr} \{ \cdot \} \) denotes the trace operation, and \( \Sigma(\theta) \) is the matrix covariance of the (Gaussian) state-vector \( \eta = [q_1(t), q_2(t)]^T \) of charges at the two capacitors \( C_1 \) and \( C_2 \), respectively. By virtue of the time-scale separation, the matrix covariance satisfies the algebraic Lyapunov equation

\[
-R^{-1}C^{-1}(\theta) \Sigma(\theta) - \Sigma(\theta)R^{-1}(\theta)R^{-1} + R^{-1}DD'R^{-1} = 0,
\]
figure), facilitating the downward sliding along the periodic potential, which is tilted due to the applied torque.

The situation with the Stirling and Brownian gyrating engines is analogous. The coupling of a number of engines, with a suitable phase difference between one another, averages out the "bumps" in the "corrugated" potential and enables sustained operation for a vanishingly small applied torque.

Results

We begin by highlighting the effect of coupling several damped pendula with an applied constant torque and a certain phase difference. Specifically, for this case, we consider two pendula coupled with a phase difference of \( \pi \) radians (see Fig. 3, bottom). The effective torque on the combined system is

\[
T^e_{2}\left(\theta\right) = \frac{1}{2}\left(T^p(\theta) + T^p(\theta + \pi)\right)
\]
effectively canceling the undulations of the potential; the \( \frac{1}{2} \) factor scales the power of the two engines so as that they can be compared to one engine. Thus, the effective torque remains constant, and thereby the overall potential driving the system of two engines has a constant tilt with no undulations. The system requires neither any inertia nor a minimum amount of actuation to achieve sustained continuous rotation. The cartoon shown in Fig. 3 helps exemplify the effect.

The underlying principle is readily seen to rely on ensuring the sign-definiteness of the effective torque. This is carried out via cancellation of respective terms between the Fourier series expansion of applied torques from contributing units. The sign-definiteness of the effective torque guarantees stable limit cycle oscillation (see the “Materials and methods” section for details.) Evidently, in more complicated examples, higher order harmonics are not immune and can likewise be eliminated or suppressed by coupling more engines as shown in the analysis that follows.

Inertialess Stirling engine

We consider the equidistant (in the \( \theta \) space) coupling of two and three Stirling engines, which generate combined torque

\[
T^e_{2}\left(\theta\right) = \frac{1}{2}\left(T^p(\theta) + T^p(\theta + \pi)\right)
\]
and

\[
T^e_{3}\left(\theta\right) = \frac{1}{2}\left(T^p(\theta) + T^p(\theta + 2\pi/3) + T^p(\theta + 4\pi/3)\right),
\]
for the two- and three-engine configurations, respectively. The resulting potential \( U \) is shown in the insert in Fig. 4 over two periods. It changes from a periodic sloped shape (in the case of one engine), to practically a sloped straight line already for two coupled engines, and more so for three. The main plot in Fig. 4 shows the averaged steady state angular velocity as a function of inertia. It is seen that, for this set of parameters, three engines dispense completely of the need for inertia, ensuring a limit cycle; with two engines, the need for inertia is already minimal.

Figure 5 illustrates how the averaged final angular velocity varies with the temperature difference \( \Delta T \) that powers the engine(s). Evidently, the coupling of multiple Stirling engines reduces the threshold temperature difference needed for continuous operation. When three engines are coupled, the threshold is virtually eliminated, guaranteeing the existence of a limit cycle for vanishingly small \( \Delta T \). In the “Materials and methods” section, we show that a sufficient condition for the torque \( T^e_{\alpha}\left(\theta\right) \) to be always positive is

\[
\frac{\alpha \Delta T}{4T_0} \gg \frac{\dot{\theta} T}{V_0} = \epsilon,
\]
for a typical value \( \epsilon \sim 10^{-3} \) in experimental settings of (1). Therein, we also prove that the torque being always positive is a sufficient condition for convergence to an asymptotically stable limit cycle. Then, the mean angular velocity of the wheel is, up to first order in \( \epsilon \),

\[
\langle \omega \rangle \approx \frac{\alpha \Delta T \gamma R}{4T} \epsilon = \omega_0^S,
\]
Fig. 4. Left: normalized averaged steady state angular velocity \( \langle \omega/\omega_0^S \rangle \) vs \( \log(2/\epsilon)^2 \) for one, two, and three coupled engines, with \( \Delta T = 10 \) K. Note that \( Z \) is normalized by \( Z_0 = 1/\omega_0^S \) and plotted in a logarithmic scale, where \( \omega_0^S \) is obtained from [4]. Similarly, the angular velocity is also normalized by \( \omega_0^S \). The case with \( \gamma = 15 \) ms is plotted in a dashed line and shows to what extent the assumption of the torque being \( \omega \)-independent holds. Right: effective potential along two cycles for one, two and three coupled engines.

Fig. 5. Averaged limit cycle angular velocity \( \langle \omega \rangle \) as a function of the temperature difference \( \Delta T \) for one, two, and three coupled Stirling engines (solid lines). The yellow-dashed line represents the case with \( \gamma = 15 \) ms and three coupled engines, and numerically shows to what extent our assumption of the torque being \( \omega \)-independent is valid. This agreement is highlighted in the blow-up of the figure. An estimation of the average angular velocity in the limit cycle, based on [4], has been marked by black “×”, showing a good agreement with the numerical results. The (flat) green line corresponds to a stable equilibrium present when the effective torque fails to be sign-definite.
in complete agreement with the numerical results (see the “Materials and methods” section for the derivation). Also, note that the dependence of the angular velocity on the temperature difference is linear, confirming the hypothesis first introduced by Kolin (29) and experimentally supported by Toyabe and Izumida, and Boutammachte and Norr (1, 30).

Inertialless Brownian gyrating engine

We now consider the coupling of two and three Brownian gyrating engines with combined torque

$$T^2_{\beta}(\theta) = \frac{1}{2} \left( T^{\beta}(\theta) + T^{\beta}(\theta + \pi) \right)$$

and

$$T^3_{\beta}(\theta) = \frac{1}{3} \left( T^{\beta}(\theta) + T^{\beta}(\theta + 2\pi/3) + T^{\beta}(\theta + 4\pi/3) \right),$$

respectively. The resulting potential $U$ is drawn over two periods in the insert of Fig. 6. It displays the same qualitative behavior as that of the Stirling engine’s potential. As we decrease the inertia, we observe that the the limit cycle is similarly maintained in the case of three engines for vanishingly small inertia (see Fig. 6).

Figure 7 displays the averaged angular velocity during operation as a linear function of the temperature difference $\Delta T := T_2 - T_1$ that powers the gyration, beyond a threshold that decreases with the number of coupled engines, as before. Similarly to the Stirling case, one can derive a sufficient condition for the existence of a limit cycle, namely

$$\frac{\sqrt{3}}{64} \frac{\Delta T}{T_0} \gg \beta,$$

where $T_0 = (T_1 + T_2)/2$. When this limit cycle is present, we can approximate the average angular velocity as

$$\langle \omega \rangle \approx \frac{\sqrt{3}k_B \Delta T}{64T} \beta^2 = \omega^B_0,$$

up to second-order terms in $\beta$.

Remarks on equalizing the torque

A main objective in coupling engines, in our exposition so far, has been the sustenance of inertialless operation. To this end, we sought to cancel harmonics by coupling engines with equal phase difference from one another (equidistantly). However, this is by no means the only metric that one may adopt for quantifying perfor-

mance. In particular, one may optimize the phase difference between engines as to maximize the minimal value of the torque along the cycle. Another possible metric for selecting phase differences is the variance of the torque, so as to limit vibrations. We highlight this point by considering the special case of two engines, to be coupled accordingly.

We discuss the case where we seek to minimize the variance of the effective torque, in coupling two engines. That is, we seek

$$\theta^B_{\text{opt}} = \arg\min_{\theta} \frac{1}{2\pi} \int_0^{2\pi} \left( (T(\theta) - \langle T(\theta) \rangle)^2 \right) d\theta,$$

where $\langle T(\theta) \rangle$ is the mean value of the applied torque over a cycle and $\theta^B_{\text{opt}}$ represents a phase difference between engines.

Clearly, $\theta^B_{\text{opt}} = 0$ maximizes the variance of the effective torque. As one may expect, $\theta^B_{\text{opt}} = \pi$ represents another potential extremum. However, whether it corresponds to a minimum, a maximum or an inflection point depends on the specific shape of the torque-profile as a function of $\theta$. For instance, for a Stirling engine (keeping terms up to second order in $\epsilon = \frac{V}{k_B T}$), we obtain that, as long as $b_1 < 4a_2$, $\theta^B_{\text{opt}} = \pi$ corresponds to a maximum, while the minimum is achieved for

$$\theta^B_{\text{opt}} = \arccos \left( -\frac{b_1}{4a_2} \right),$$

where $b_1 = \epsilon \left( 1 - \frac{\theta^\omega_{\text{opt}}}{\theta_{\text{opt}}} \right)$ and $a_2 = \epsilon \frac{\Delta T}{k_B T}$ (see the “Materials and methods” section). Otherwise, $\theta^B_{\text{opt}} = \pi$, a value that was confirmed by our numerical experiments. Intuitively, $\pi$ is the optimal solution when the odd harmonics in $T(\theta)$ dominate. The general case with a larger number of coupled engines can be worked out similarly.

Conclusions

The present paper details a proof-of-concept: the need for inertia to ensure limit cycle oscillations in gyrating engines can be dispensed of when a number of torque-generating subunits are coupled with a suitable phase difference from one another. When the effective torque produced by the combined contribution of sub-
units remains sign-definite over a cycle, the system operates in a
limit cycle making power available for external work. The under-
lying principle was demonstrated with two examples, a Stirling
engine and a Brownian gyrating engine.

It is postulated that a similar principle is at work in biomole-
cular engines, albeit in a significantly more complicated guise, given
the complexity of such engines. Indeed, in ref. (31), a model was
presented and partially tested to explain specific physical mecha-
nisms for torque generation in bacterial flagellar motors (RFMs).
In this, a number of torque generating units with a “wide and gently
slipping energy well” contribute in ways that are reminiscent
of the principle presented herein. Although the physics of torque
generation remain poorly understood, it was proposed in ref. (31)
that both electrostatic and steric forces are at work, with the lat-
ter generating a “push.” The resulting torque profile may likely
need more units to smooth out higher harmonics that may thus be present.
Understanding how ion-driven molecular machines work is of fundamental importance in cellular biology,
and thus the authors see likely that the principle discussed herein
may help explain the workings of multiple torque-generating sub-
units (10, 11) and, perhaps, even the necessity for a large number
(up to 11 in flagellar motors) of such units for the corresponding
torque-generating potential.

Materials and methods
In this section, we provide further technical insights and proofs
to the claims in the paper. We begin by showing that sign-
definiteness of the effective torque implies that a system obey-
ing [1] has indeed a unique asymptotically stable limit cycle.
We continue on by showing that for any θ-periodic torque pro-
file for which a certain continuity condition holds, a finite num-
er of engines always suffice to ensure sign-definiteness of the
torque, and thereby stable operation of the system of coupled en-
gines. We then specialize to the case of the Stirling and Brown-
nian gyration-based engines with a fixed number of units (three,
in particular), and we derive alternative sufficient conditions for
sign-definiteness of the effective torque as well as explicit expres-
sions for the average angular velocity. We finally expand on a point
raised in the “Remarks on equalizing the torque” section by work-
ing out in detail the phase difference θ0 between two coupled
engines that minimizes the variation of the effective torque. We
conclude by tabulating the values of parameters used in the numer-
cal simulations.

Sign-definiteness of torque implies a unique
stable limit cycle
Herein, we prove that if the torque T(θ) is strictly positive for all
values of θ, then a unique globally attractive limit cycle exists for
any (and hence for a vanishingly small) amount of inertia I. The
basis of the argument to establish existence of such a limit cycle
is the Poincaré–Bendixson theorem (32, page 391, Theorem 2.1; 33,
Theorem 9.0.6). This theorem states that a trajectory of a second-
order system, confined in a bounded two-dimensional region of
the phase space that contains no fixed points, is either a periodic
orbit itself or it converges (asymptotically) to one. The phase space
is a cylinder [0, 2π) × R, as is the case of the system in [1].

A fixed point of [1] requires that θ = 0 (from the first of the
two equations). But then, T(θ) − γω cannot vanish, since T(θ) > 0
for all θ, and hence [1] has no fixed points. We observe that any
(bounded) region D = [θ, ω] ∈ [0, 2π) × [−M, M)], for sufficiently
large M, is positively (in time) invariant. That is, any trajectory
that begins in D is confined within D for all times. Thus, by the
Poincaré–Bendixson theorem, there exists an asymptotically at-
tractive periodic orbit.

We now argue that the claimed periodic orbit is in fact unique,
i.e. it represents a globally attractive stable limit cycle. Starting
from a point [θ = 0, ω(0)] that lies on a period orbit, we integrate
\[ Iω = T(θ) − γω \] over the cycle θ ∈ [0, 2π]. The integral of the left
hand side is
\[ \int_{0}^{2π} Iωdθ = \int_{0}^{cT} T(θ) dθ = \frac{T}{2}(ω(0) - ω(0)) = 0, \]
where cT is the time-duration of a cycle. Integrating the right hand
side now gives
\[ \int_{0}^{2π} \omega dθ = \frac{2π(T)}{I}, \]
where (T) is a (fixed) constant that only depends on the shape of
the torque profile. Since trajectories do not cross, [6] can only be
satisfied by a unique periodic orbit.

Number of gyrating engines required to dispense of inertia
We consider gyrating engines obeying [1]. Following two different
approaches we show that the torque profile T(θ) satisfies
\[ |T(θ + Δ) − T(θ)| < L|Δ| \] for all θ, Δ, and with L < ∞ (i.e. it is Lip-
chitz) and provided the average torque over a cycle is not zero (and
which, without loss of generality, is assumed positive), there is an
integer m so that m equidistantly coupled engines ensure a glob-
ally attractive limit cycle. In other words, we establish that under
natural and mild conditions on the torque profile, a finite number
of coupled Stirling or Brownian gyrationing engines is always suffi-
cient to maintain a stable limit cycle for any set of parameters.

To establish the claim, we show that a finite number of coupled
equidistantly coupled engines is sufficient to ensure strictly positive torque for all values
of the angular position θ. Assuming that T(θ) is Lipschitz and
periodic, we consider the Fourier series expansion
\[ T(θ) = c_0 + \sum_{k=1}^{∞} a_k \cos(kθ) + b_k \sin(kθ). \] (7)
For m equidistantly coupled engines, the effective torque is
\[ T_m(θ) = \frac{1}{m} \sum_{k=0}^{m-1} T(θ - \frac{2π}{m} k) \]
\[ = c_0 + \frac{1}{m} \sum_{k=1}^{m} \sum_{ℓ=0}^{m-1} a_k \cos(k(θ - \frac{2π}{m} ℓ)) + b_k \sin(k(θ - \frac{2π}{m} ℓ)) \]
\[ = c_0 + \sum_{k=1}^{∞} a_k \cos(kmθ) + b_k \sin(kmθ) \]
\[ = \sum_{k=1}^{∞} c_{km} \cos(kmθ), \]
where \( c_k = \frac{1}{m}(a_k + ib_k) \) for \( k > 0 \) and \( c_0 = \frac{c}{2π} \) for \( k < 0 \). The third
equality follows from cancellation, due to phase difference, of all
terms with indices that are not multiples of m. Since T(θ) is Lipschitz,
the amplitude of the harmonics decays faster than \( k^{-1} \) and
the series \( |c_k|: k > 0 \) is summable, see e.g. refs. (32–34).
This thus exists an m such that \( \sum_{k=0}^{∞} |c_k| \) is summable, and for this m, \( T_m(θ) > 0 \) for all θ.

An alternative argument can be drawn as follows. Denote by L
the torque’s Lipschitz constant, i.e. \( L = \inf |T(θ + Δ) − T(θ)| < k|Δ| \) for all \( θ, Δ \in [0, 2π] \). Then, \( T_m(θ) \) is also Lipschitz with Lip-
shitz constant $\leq I$. It is also periodic with period $2\pi/m$ and average $c_0$, which we assume positive. Let $\theta_0$ be such that $T_m(\theta_0) = c_0$, which always exists since $T_m$ is continuous. Then, over a period $\theta \in [\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}]$, $T_m(\theta)$ takes values in the interval $[c_0 - \frac{c_0}{m}, c_0 + \frac{c_0}{m}]$. Thus, if we take $m = \lceil \frac{\pi}{c_0} \rceil$, that is, we take the smallest integer $m$ such that $m \geq \frac{\pi}{c_0}$, it follows that $T_m(\theta) > 0$ over the period, and hence for all $\theta$.

We note that the number $m = \lceil \frac{\pi}{c_0} \rceil$ of the needed engines is tight when $T(\theta)$ has the shape of a triangular wave with slope $I$ and period $2\pi$.

**Alternative analysis for the Stirling case**

We derive a condition for three coupled Stirling engines ($m = 3$) to suffice for sustained limit cycle operation. Let $\epsilon = \frac{\pi}{2}$ and consider the expansion of the dimensionless torque in terms of $\epsilon$, 

$$
\frac{T^3(\theta)}{\xi nR T_0} = \epsilon \left( \frac{1 + a \Delta T \sin(\theta)}{1 + \epsilon (1 - \cos \theta)} - \frac{p_0 V_0}{\xi nR T_0} \right) \sin \theta
$$

$$
= \epsilon \left( 1 + a \frac{\Delta T}{2T_0} \sin \theta - \frac{p_0 V_0}{\xi nR T_0} \right) \sin \theta + O(\epsilon^2)
$$

$$
= \epsilon \frac{a \Delta T}{4T_0} + \epsilon \left( 1 - \frac{p_0 V_0}{\xi nR T_0} \right) \sin \theta + \epsilon \frac{a \Delta T}{4T_0} \cos(2\theta) + O(\epsilon^2).
$$

Note that the two first harmonics vanish when coupling three Stirling engines, leaving only the constant term and higher order terms in $\epsilon$. Therefore, as long as 

$$
\epsilon \frac{a \Delta T}{4T_0} \gg \epsilon,
$$

three engines are enough to ensure that the torque is sign-definite. The resulting system will gyrate at approximately constant angular velocity

$$
\langle \omega \rangle \approx \frac{a \Delta T \xi nR}{4T_0} \epsilon.
$$

**Alternative analysis for the Brownian case**

In analogy with the Stirling engine, we expand the dimensionless torque for the Brownian gyrating engine in the dimensionless parameter $\beta$. This parameter controls the variation of the capacitance and, by expanding around zero, we assume that this variation is small. That is, we assume that our system is within the linear response regime. The expansion gives 

$$
\frac{T^3(\theta)}{k_0 T_0} = f_1(\theta) \beta + f_2(\theta) \beta^2 + O(\beta^3),
$$

where $f_1(\theta)$ depends on $\theta$ through terms linear in $\sin(\theta)$ and $\cos(\theta)$, while $f_2(\theta)$ contains second harmonics and a constant term. Terms independent of $\beta$ vanish, since for $\beta = 0$ energy cannot be extracted from the system. Therefore, up to second order in $\beta$, the only term that contributes to the average torque is $f_2(\theta) \beta^2$, whose average value over a cycle can be computed to be 

$$
\frac{1}{k_0 T_0} \int_0^{2\pi} T^3(\theta) d\theta = \frac{\beta^2}{2T_0} \int_0^{2\pi} f_2(\theta) d\theta = \frac{\sqrt{3} \Delta T}{64 T_0} \beta^2.
$$

Consequently, if two engines are coupled, the first-order term in $\beta$ vanishes, whereas, if three engines are coupled, the remaining terms are of third order or higher. Thus, provided the constant term dominates over higher order terms and a globally attractive limit cycle operation is present for the three coupled engines. In that case, the average angular velocity can be approximated by 

$$
\langle \omega \rangle \approx \frac{\sqrt{3}k_0 \Delta T}{64T_0} \beta^2.
$$

**Optimizing phase difference**

We now expand on the point raised in the “Remarks on equalizing the torque” section that phase differences between coupled engines may be optimized to minimize the variation of the effective torque. Doing so, for two coupled engines, amounts to solving the following optimization problem:

$$
\min_{\theta_0} \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{T(\theta + 2\theta_0)}{2} - \langle T \rangle \right)^2 d\theta.
$$

Due to the periodicity of $T$, the problem reduces to minimizing the integral of the product $T(\theta) T(\theta + \theta_0)$ over a cycle.

We bring in the Fourier series expansion [7], written for the terms in this product, and consider the partial derivative of the integral with respect to $\theta_0$ so as to obtain the first-order condition for optimality 

$$
-\sum_{k=1}^{\infty} k (a_k^2 + b_k^2) \sin(k \theta_0) = 0.
$$

We see that $\theta_0 = m \pi$, $m \in \mathbb{N}$ are solutions and thus potential extrema. Minimality hinges on the second derivative, which suffice to be strictly positive, i.e.

$$
-\sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) \cos(k \theta_0) > 0.
$$

It is clear that $\theta_0 = 0$ corresponds always to a maximum, while $\theta_0 = \pi$ may correspond to a maximum, a minimum, or be inconclusive, depending on the torque profile as a function of $\theta$. For instance, assuming $b_1$ and $a_2$ are the only nonzero terms in the Fourier expansion, as is the case of the Stirling engine (up to second-order approximation in $\epsilon$), $\theta_0 = \pi$ corresponds to a maximum as long as $b_1^2 < 4a_2$. In this case, there are two other extrema at

$$
\theta_0^{opt} = \pm \cos \left( \frac{b_1}{4a_2} \right),
$$

which are in fact minima. All in all, the optimal phase difference for two coupled Stirling engines is equal to $\theta_0^{opt} = \pi$ when

$$
\left( 1 - \frac{p_0 V_0}{\xi nR T_0} \right)^2 \geq \left( \frac{a \Delta T}{2T_0} \right)^2,
$$

and it is

$$
\theta_0^{opt} = \pm \cos \left( -4 \left( \frac{\xi nR T_0 - p_0 V_0}{\xi nR \Delta T} \right)^2 \right),
$$

otherwise. For the parameters used in this paper, it follows that the optimal phase is exactly $\pi$.

**Parameters used**

The parameters we have used in the different numerical experiments are specified in Table 1. Note that, for proper comparison, $\Gamma$ has been chosen such that $\log_{10}(\Gamma/T_0) = 2$ both for the Stirling and the Brownian gyrating engines in Figs. 5 and 7, respectively.
Table 1. Parameters used.

| Parameter          | Value   | Units   |
|--------------------|---------|---------|
| **Stirling engine problem** |         |         |
| \( s \)            | 71      | \( \text{mm}^2 \) |
| \( r \)            | 3.5     | \( \text{mm} \) |
| \( \zeta \)        | 0.94    |         |
| \( \rho_0 \)       | 101.3   | \( \text{kPa} \) |
| \( n \)            | 0.00185 | \( \text{mol} \) |
| \( R \)            | 8.314   | \( \text{J\,K}^{-1}\text{mol}^{-1} \) |
| \( T_{\text{mp}} \) | 297.15  | \( K \) |
| \( \alpha \)       | 0.17    |         |
| \( V_0 \)          | 44900   | \( \text{mm}^3 \) |
| \( I \)            | 10^{-1} to 10^{-6} (Fig. 4) | \( \text{kgm}^2\text{s}^{-1} \) |
| \( \Delta T \)      | 10 (Fig.4) | \( K \) |
| \( \Gamma \)       | 4.38 x 10^{-6} | \( \text{kgm}^2\text{s}^{-1} \) |

**Brownian gyrator problem**

| Parameter          | Value   | Units   |
|--------------------|---------|---------|
| \( C_a \)          | 2       | \( \text{mF} \) |
| \( \beta \)        | 0.1     |         |
| \( R_1, R_2 \)     | 1       | \( \Omega \) |
| \( T_1 \)          | 200     | \( K \) |
| \( k_0 \)          | 1.38 x 10^{-23} | \( \text{kgm}^2\text{s}^{-1}K^{-1} \) |
| \( I \)            | 10^{-12} to 10^{-19} (Fig. 6) | \( \text{kgm}^2\text{s}^{-1} \) |
| \( \Delta T \)      | 10 (Fig.6) | \( K \) |
| \( \Gamma \)       | 4.32 x 10^{-12} | \( \text{kgm}^2\text{s}^{-1} \) |

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**Authors’ Contributions**

J.V.S. and O.M.M. carried out the technical development of the work, O.M.M. oversaw the completion, all authors, J.V.S., O.M.M., A.T., Y.C., and T.T.G. contributed to the writing and development of ideas, and T.T.G. proposed the topic.

**Data Availability**

All study data are included in the article and/or Supplementary Material.

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