PERIODIC SOLUTIONS OF THE SINH-GORDON EQUATION AND INTEGRABLE SYSTEMS

MARKUS KNOPF

Abstract. We study the space of periodic solutions of the elliptic sinh-Gordon equation by means of spectral data consisting of a Riemann surface $Y$ and a divisor $D$. We show that the space $M^p_g$ of real periodic finite type solutions with fixed period $p$ can be considered as a completely integrable system $(M^p_g, \Omega, H_2)$ with a symplectic form $\Omega$ and a series of commuting Hamiltonians $(H_n)_{n \in \mathbb{N}}$. In particular we relate the gradients of these Hamiltonians to the Jacobi fields $(\omega_n)_{n \in \mathbb{N}_0}$ from the Pinkall-Sterling iteration. Moreover, a connection between the symplectic form $\Omega$ and Serre duality is established.

Contents

1. Introduction 1
2. Conformal CMC immersions into $\mathbb{S}^3$ 2
3. A formal diagonalization of the monodromy $M_\lambda$ 2
4. Polynomial Killing fields for finite type solutions 6
5. The associated spectral data 6
6. Isospectral and non-isospectral deformations 8
7. The phase space $(M^p_g, \Omega)$ 11
8. Polynomial Killing fields and integrals of motion 16
9. An inner product on $\Lambda^r sl_2(\mathbb{C})$ 23
10. The symplectic form $\Omega$ and Serre duality 27

1. INTRODUCTION

The elliptic sinh-Gordon equation is given by

$$\Delta u + 2 \sinh(2u) = 0, \quad (1.1)$$

where $\Delta$ is the Laplacian of $\mathbb{R}^2$ with respect to the Euclidean metric and $u : \mathbb{R}^2 \to \mathbb{R}$ is a twice partially differentiable function which we assume to be real.

The sinh-Gordon equation arises in the context of particular surfaces of constant mean curvature (CMC) since the function $u$ can be extracted from the conformal factor $e^{2u}$ of a conformally parameterized CMC surface. The study of CMC tori in 3-dimensional space forms was strongly influenced by algebro-geometric methods (as described in [2]) that led to a complete classification by Pinkall and Sterling [31] for CMC-tori in $\mathbb{R}^3$. Moreover, Bobenko [4] gave explicit formulas for CMC tori in $\mathbb{R}^3$, $S^3$ and $H^3$ in terms of theta-functions and introduced a
description of such tori by means of spectral data. We also refer the interested reader to [37]. Every CMC torus yields a doubly periodic solution $u : \mathbb{R}^2 \to \mathbb{R}$ of the sinh-Gordon equation. With the help of differential geometric considerations one can associate to every CMC torus a hyperelliptic Riemann surface $Y$, the so-called spectral curve, and a holomorphic line bundle $E$ on $Y$ (the so-called eigenline bundle) that is represented by a certain divisor $D$. Hitchin [21], and Pinkall and Sterling [31] independently proved that all doubly periodic solutions of the sinh-Gordon equation correspond to spectral curves of finite genus. We say that solutions of (2.2) that correspond to spectral curves of finite genus are of finite type.

In the present setting we will relax the condition on the periodicity and demand that $u$ is only simply periodic with a fixed period. After rotating the domain of definition we can assume that this period is real. This enables us to introduce simply periodic Cauchy data with fixed period $p \in \mathbb{R}$ consisting of a pair $(u, u_y) \in C^\infty(\mathbb{R}/p\mathbb{Z}) \times C^\infty(\mathbb{R}/p\mathbb{Z})$. Moreover, we demand that the corresponding solution $u$ of the sinh-Gordon equation is of finite type.

This paper summarizes the author’s PhD thesis [24]. Its main goal is to identify the sinh-Gordon equation (2.2) as a completely integrable system (compare with [15]) and illustrate its features in the finite type situation.

2. Conformal CMC immersions into $S^3$

2.1. The Lie groups $SL(2, \mathbb{C})$ and $SU(2)$. Let us consider the Lie group $SL(2, \mathbb{C}) := \{ A \in M_{2\times2}(\mathbb{C}) \mid \det(A) = 1 \}$. The Lie algebra $\mathfrak{s}_2(\mathbb{C}) := \{ B \in M_{2\times2}(\mathbb{C}) \mid \text{tr}(B) = 0 \}$ of $SL(2, \mathbb{C})$ is spanned by the matrices $\epsilon_+, \epsilon_- \in \mathfrak{s}_2(\mathbb{C})$ with

$$
\epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
$$

It will also be convenient to identify $S^3$ with the Lie group $SU(2) = \{ A \in M_{2\times2}(\mathbb{C}) \mid \det(A) = 1, \; A^\dagger = A^{-1} \}$. The Lie algebra of $SU(2)$ is denoted by $su(2)$ and a direct computation shows that $su(2) = \{ B \in M_{2\times2}(\mathbb{C}) \mid \text{tr}(B) = 0, \; B^\dagger = -B \} \simeq \mathbb{R}^3$.

2.2. Extended frames. We start with the following version of a result by Bobenko [5] (cf. [23], Theorem 1.1).

Theorem 2.1. Let $u : \mathbb{C} \to \mathbb{R}$ and $Q : \mathbb{C} \to \mathbb{C}$ be smooth functions and define

$$
\alpha_\lambda = \frac{1}{2} \begin{pmatrix} u_z dz - u_\bar{z} d\bar{z} & i\lambda^{-1} e^u dz + iQ e^{-u} d\bar{z} \\ iQ e^{-u} dz + i\lambda e^u d\bar{z} & -u_z dz + u_\bar{z} d\bar{z} \end{pmatrix}.
$$

Then $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ if and only if $Q$ is holomorphic, i.e. $Q_\bar{z} = 0$, and $u$ is a solution of the reduced Gauss equation

$$
2u_{\bar{z}z} + \frac{1}{2} (e^{2u} - Q\bar{Q} e^{-2u}) = 0.
$$

For any solution $u$ of the above equation and corresponding extended frame $F_\lambda$, and $\lambda_0, \lambda_1 \in S^1, \lambda_0 \neq \lambda_1$, i.e. $\lambda_k = e^{ik}$ the map defined by the Sym-Bobenko-formula

$$
f = F_{\lambda_1} F_{\lambda_0}^{-1}
$$

is a conformal immersion $f : \mathbb{C} \to SU(2) \simeq S^3$ with constant mean curvature

$$
H = i \frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1} = \cot(t_0 - t_1),
$$

where $t_k = -\lambda_k$.
conformal factor \( v = e^u / \sqrt{H^2 + 1} \), and Hopf differential \( \tilde{Q} dz^2 \) with \( \tilde{Q} = -\frac{i}{2} (\lambda_1^{-1} - \lambda_0^{-1}) Q \).

**Assumption 2.2.** Let us assume that the Hopf differential \( Q \) is constant with \( |Q| = 1 \).

2.3. **The monodromy.** The central object for the following considerations is the monodromy \( M_\lambda \) of a frame \( F_\lambda \).

**Definition 2.3.** Let \( F_\lambda \) be an extended frame and assume that \( \alpha_\lambda = F_\lambda^{-1} dF_\lambda \) has period \( p \in \mathbb{C} \), i.e. \( \alpha_\lambda(z + p) = \alpha_\lambda(z) \). Then the **monodromy** of the frame \( F_\lambda \) with respect to the period \( p \) is given by

\[
M_\lambda^p := F_\lambda(z + p) F_\lambda^{-1}(z).
\]

Note that we have \( dM_\lambda^p = 0 \) and thus \( M_\lambda^p \) does not depend on \( z \). Setting \( F_\lambda(0) = 1 \) we get

\[
M_\lambda := M_\lambda^p = F_\lambda(p) F_\lambda^{-1}(0) = F_\lambda(p).
\]

2.4. **Isometric normalization.** We can rotate the coordinate \( z \) by a map \( z \mapsto w(z) = e^{i\varphi}z \) in such a way that \( \Im(p) = 0 \). As a consequence the extended frame \( F_\lambda \) is multiplied with the matrix

\[
B_w = \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix}
\]

with \( \delta = e^{i\varphi} \in S^1 \). This corresponds to the isometric normalization described in [18], Remark 1.5, and the corresponding gauged \( \alpha_\lambda \) is of the form

\[
\alpha_\lambda = \frac{1}{2} \begin{pmatrix} u z dz - u \bar{z} d\bar{z} & i \lambda^{-1} e^u dz + i \gamma e^{-u} d\bar{z} \\ i \gamma e^{-u} dz + i \lambda e^u d\bar{z} & -u z dz + u \bar{z} d\bar{z} \end{pmatrix},
\]

where the constant \( \gamma \in S^1 \) is given by \( \gamma = \delta^{-1} \tilde{Q} = \tilde{\delta} Q \).

2.5. **The sinh-Gordon equation.** We can normalize the above parametrization with \( \delta = 1 \) and \( |\gamma| = 1 \) in (2.1) by choosing the appropriate value for \( Q \in S^1 \). Then we can consider the system

\[
dF_\lambda = F_\lambda \alpha_\lambda \quad \text{with} \quad F_\lambda(0) = 1.
\]

Since \( |\gamma| = 1 \), we see that the compatibility condition \( 2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0 \) from Theorem 2.1 holds if and only if

\[
2u z + \frac{1}{2}(e^{2u} - \gamma \gamma e^{-2u}) = 2u z + \sinh(2u) = 0.
\]

Thus the reduced Gauss equation turns into the **sinh-Gordon equation** in that situation. For the following we make an additional assumption.

**Assumption 2.4.** Let \( \gamma = \delta = 1 \). This yields

\[
\alpha_\lambda = \frac{1}{2} \begin{pmatrix} u z dz - u \bar{z} d\bar{z} & i \lambda^{-1} e^u dz + i e^{-u} d\bar{z} \\ i \lambda e^u d\bar{z} + i e^{-u} dz & -u z dz + u \bar{z} d\bar{z} \end{pmatrix}.
\]

If we evaluate \( \alpha_\lambda \) along the vector fields \( \partial \) and \( \partial_{\bar{z}} \) we obtain

\[
U_\lambda := \alpha_\lambda(\partial), \quad V_\lambda := \alpha_\lambda(\partial_{\bar{z}}).
\]

These matrices will be important for the upcoming considerations. In particular \( U_\lambda \) reads

\[
U_\lambda = \frac{1}{2} \begin{pmatrix} -iu_y & i \lambda^{-1} e^u + i e^{-u} \\ i \lambda e^u + i e^{-u} & iu_y \end{pmatrix}.
\]

**Remark 2.5.** Due to (2.3) we can identify the tuple \((u, u_y)\) with the matrix \( U_\lambda \).
3. A Formal Diagonalization of the Monodromy $M_\lambda$

We want to diagonalize the monodromy $M_\lambda$ and therefore need to diagonalize $\alpha_\lambda$. A diagonalization for the Schrödinger-operator is done in [33] based on a result from [17]. In order to adapt the techniques applied there we search for a $\lambda$-dependent periodic formal power series $\tilde{g}_\lambda(x)$ such that

$$\tilde{\beta}_\lambda = \tilde{g}_\lambda^{-1} \alpha_\lambda \tilde{g}_\lambda + \tilde{g}_\lambda^{-1} \frac{d}{dx} \tilde{g}_\lambda$$

is a diagonal matrix, i.e.

$$\tilde{\beta}_\lambda(x) = \left( \begin{array}{cc} \sum_m (\sqrt{\lambda})^m b_m(x) & 0 \\ 0 & -\sum_m (\sqrt{\lambda})^m b_m(x) \end{array} \right)$$

with $m \geq -1$. Since $F_\lambda(x) = \tilde{g}_\lambda(0) \tilde{G}_\lambda(x) \tilde{g}_\lambda(x)^{-1}$ (where $\tilde{G}_\lambda$ solves $\frac{d}{dx} \tilde{G}_\lambda(x) = \tilde{G}_\lambda(x) \tilde{\beta}_\lambda(x)$ with $\tilde{G}_\lambda(0) = 1$) we get

$$M_\lambda = F_\lambda(p) = \tilde{g}_\lambda(0) \tilde{G}_\lambda(p) \tilde{g}_\lambda(p)^{-1} = \tilde{g}_\lambda(0) \tilde{G}_\lambda(p) \tilde{g}_\lambda(0)^{-1}$$

and due to

$$\tilde{G}_\lambda(x) = \exp \left( \int_0^x \sum_m (\sqrt{\lambda})^m b_m(t) \, dt \right)$$

we obtain for the eigenvalue $\mu$ of $M_\lambda$

$$\mu = \exp \left( \int_0^p \sum_m (\sqrt{\lambda})^m b_m(t) \, dt \right) \quad \text{or equivalently} \quad \ln \mu = \sum_m (\sqrt{\lambda})^m \int_0^p b_m(t) \, dt.$$  

Let us start with the following lemma that is obtained by a direct calculation.

**Lemma 3.1.** By performing a gauge transformation with

$$g_\lambda(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{z}{2}} & 0 \\ 0 & \sqrt{\lambda} e^{-\frac{z}{2}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

the frame $F_\lambda(z)$ is transformed into $F_\lambda(z)g_\lambda(z)$ and the map $G_\lambda(z) := g_\lambda(0)^{-1} F_\lambda(z) g_\lambda(z)$ solves

$$dG_\lambda = G_\lambda \beta_\lambda \quad \text{with} \quad \beta_\lambda = g_\lambda^{-1} \alpha_\lambda g_\lambda + g_\lambda^{-1} d g_\lambda \quad \text{and} \quad G_\lambda(0) = 1.$$  

Evaluating the form $\beta_\lambda$ along the vector field $\frac{\partial}{\partial z}$ and setting $y = 0$ yields $\beta_\lambda \left( \frac{\partial}{\partial z} \right) = \frac{1}{\sqrt{\lambda}} \beta_{-1} + \beta_0 + \sqrt{\lambda} \beta_1$ with

$$\beta_{-1} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 0 & -u_z \\ -u_z & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} \frac{i}{2} \cosh(2u) & -\frac{i}{2} \sinh(2u) \\ \frac{i}{2} \sinh(2u) & \frac{i}{2} \cosh(2u) \end{pmatrix}.$$  

From the following theorem we obtain a periodic formal power series $\tilde{g}_\lambda(x)$ of the form $\tilde{g}_\lambda(x) = 1 + \sum_{m \geq 1} a_m(x) (\sqrt{\lambda})^m$ such that $\tilde{g}_\lambda(x) := g_\lambda(x) \tilde{g}_\lambda(x)$ (with $g_\lambda(x)$ defined as in Lemma 3.1) diagonalizes $\alpha_\lambda$ around $\lambda = 0$.

**Theorem 3.2.** Let $(u, u_y) \in C^\infty(\mathbb{R}/p\mathbb{Z}) \times C^\infty(\mathbb{R}/p\mathbb{Z})$. Then there exist two series

$$a_1(x), a_2(x), \ldots \in \text{span}\{\epsilon_+, \epsilon_-\} \quad \text{of periodic off-diagonal matrices and}$$

$$b_1(x), b_2(x), \ldots \in \text{span}\{\epsilon\} \quad \text{of periodic diagonal matrices, respectively}$$

such that

$$g_\lambda(x) = 1 + \sum_{m \geq 1} a_m(x) (\sqrt{\lambda})^m.$$
such that $a_{m+1}(x)$ and $b_m(x)$ are differential polynomials in $u$ and $u_y$ with derivatives of order $m$ at most and the following equality for formal power series holds asymptotically around $\lambda = 0$:

$$\beta_\lambda(x) \left( 1 + \sum_{m \geq 1} a_m(x)(\sqrt{\lambda})^m \right) + \sum_{m \geq 1} \frac{d}{dx} a_m(x)(\sqrt{\lambda})^m = \left( 1 + \sum_{m \geq 1} a_m(x)(\sqrt{\lambda})^m \right) \sum_{m \geq -1} b_m(x)(\sqrt{\lambda})^m. \tag{*}$$

Here $b_{-1}(x)$ and $b_0(x)$ are given by $b_{-1}(x) \equiv \beta_{-1} = \frac{i}{\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b_0(x) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

**Remark 3.3.** Since we only consider finite type solutions we can guarantee that the power series in (\ref{eq:*}) indeed are convergent, see [24, Theorem 4.35].

**Proof.** We start the iteration with $b_0(x) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and will inductively solve the given ansatz in all powers of $\sqrt{\lambda}$:

1. $(\sqrt{\lambda})^{-1}$: $\beta_{-1} = \beta_{-1}$. ✓
2. $(\sqrt{\lambda})^0$: $\beta_{-1}a_1(x) + \beta_0(x) = b_0(x) = 0$ and thus

$$a_1(x) = -\beta_{-1}^{-1} \beta_0(x) = \begin{pmatrix} 0 & -i\partial u \\ i\partial u & 0 \end{pmatrix}.$$  

3. $(\sqrt{\lambda})^1$: $\beta_{-1} a_2(x) + \beta_0(x) a_1(x) + \beta_1(x) + \frac{d}{dx} a_1(x) = b_1(x) + a_2(x) \beta_{-1} + a_1(x) b_0(x)$.

Rearranging terms and sorting with respect to diagonal (d) and off-diagonal (off) matrices we get two equations:

$$b_1(x) = \beta_0(x) a_1(x) + \beta_1(x) + \frac{d}{dx} a_1(x) = \begin{pmatrix} -i(\partial u)^2 + \frac{i}{2} \cosh(2u) & 0 \\ 0 & i(\partial u)^2 - \frac{i}{2} \cosh(2u) \end{pmatrix},$$

$$[\beta_{-1}, a_2(x)] = -\beta_{1, \text{off}}(x) - \frac{d}{dx} a_1(x).$$

In order to solve the second equation for $a_2(x)$ we make the following observation: set $a(x) = a_+(x) \epsilon_+ + a_-(x) \epsilon_-$. Since $[\epsilon, \epsilon_+] = 2i \epsilon_+$ and $[\epsilon, \epsilon_-] = -2i \epsilon_-$, we can define a linear map $\phi: \text{span}\{\epsilon_+, \epsilon_-\} \rightarrow \text{span}\{\epsilon_+, \epsilon_-\}$ by

$$\phi(a(x)) := [\beta_{-1}, a(x)] = [i\epsilon, a(x)] = i a_+(x) \epsilon_+ - i a_-(x) \epsilon_- \in \text{span}\{\epsilon_+, \epsilon_-\}.$$  

Obviously $\ker(\phi) = \{0\}$ and thus $\phi$ is an isomorphism. Therefore we can uniquely solve the equation $[\beta_{-1}, a_2(x)] = -\beta_{1, \text{off}}(x) - \frac{d}{dx} a_1(x)$ and obtain $a_2(x)$.

We now proceed inductively for $m \geq 2$ and assume that we already found $a_m(x)$ and $b_{m-1}(x)$. Consider the equation

$$\beta_{-1} a_{m+1}(x) + \beta_0(x) a_m(x) + \beta_1(x) a_{m-1}(x) + \frac{d}{dx} a_m(x) = b_m(x) + a_m(x) \beta_{-1} + \sum_{i=1}^{m} a_i(x) b_{m-i}(x)$$

for the power $(\sqrt{\lambda})^m$. Rearranging terms and after decomposition in the diagonal (d) and off-diagonal (off) part we get

$$b_m(x) = \beta_0(x) a_m(x) + \beta_{1, \text{off}}(x) a_{m-1}(x),$$
\[ [\beta_{-1}, a_{m+1}(x)] = -\beta_{1,1}(x) a_{m-1}(x) - \frac{d}{dx} a_m(x) + \sum_{i=1}^{m} a_i(x) b_{m-i}(x). \]

From the discussion above we see that these equations can uniquely be solved and one obtains \( a_{m+1}(x) \) and \( b_m(x) \). By induction one therefore obtains a unique formal solution of (2) with the desired properties. \( \square \)

With the help of Theorem 3.2 we can reproduce Proposition 3.6 presented in [23].

**Corollary 3.4.** The logarithm \( \ln \mu \) of the eigenvalue \( \mu \) of the monodromy \( M_\lambda \) has the following asymptotic expansion

\[
\ln \mu = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} (-i(\partial u)^2 + \frac{i}{2} \cosh(2u)) \, dt + O(\lambda) \quad \text{at } \lambda = 0.
\]

**Proof.** From Theorem 3.2 we know that at \( \lambda = 0 \) we have

\[
\ln \mu = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} b_1(t) \, dt + \sum_{m \geq 2} (\sqrt{\lambda})^m \int_{-\infty}^{\infty} b_m(t) \, dt
\]

\[
= \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} (-i(\partial u)^2 + \frac{i}{2} \cosh(2u)) \, dt + O(\lambda).
\]

\( \square \)

### 4. Polynomial Killing fields for finite type solutions

In the following we will consider the variable \( y \) as a flow parameter. If we define the Lax operator \( L_\lambda := \frac{d}{dy} + U_\lambda \), the zero-curvature condition can be rewritten as Lax equation via

\[
\frac{d}{dy} \frac{d}{dx} U_\lambda - \frac{d}{dx} V_\lambda - [U_\lambda, V_\lambda] = 0 \iff \frac{d}{dy} L_\lambda = [L_\lambda, V_\lambda].
\]

This corresponds to the sinh-Gordon flow and there exist analogous equations for the “higher” flows. In the finite type situation there exists a linear combination of these higher flows such that the corresponding Lax equation is stationary, i.e. there exists a map \( W_\lambda \) such that

\[
\frac{d}{dy} L_\lambda = [L_\lambda, W_\lambda] = \frac{d}{dx} W_\lambda + [U_\lambda, W_\lambda] \equiv 0.
\]

This leads to the following definition (cf. Definition 2.1 in [18]).

**Definition 4.1.** A pair \( C^\infty(\mathbb{R}/\mathbb{P}\mathbb{Z}) \times C^\infty(\mathbb{R}/\mathbb{P}\mathbb{Z}) \ni (u, u) \simeq U_\lambda(\cdot, 0) \) corresponding to a periodic solution of the sinh-Gordon equation is of **finite type** if there exists \( g \in \mathbb{N}_0 \) such that

\[
\Phi_\lambda(x) = \frac{\lambda^{-1}}{2} \begin{pmatrix} ie^u & 0 \\ 0 & e^u \end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} \omega_n e^{u \tau_n} & \omega_n e^{u \tau_n} \\ \omega_n e^{u \tau_n} & -\omega_n \end{pmatrix}
\]

is a solution of the Lax equation

\[
\frac{d}{dx} \Phi_\lambda = [\Phi_\lambda, U_\lambda(\cdot, 0)]
\]

for some periodic functions \( \omega_n, \tau_n, \sigma_n : \mathbb{R}/\mathbb{P}\mathbb{Z} \to \mathbb{C} \).

**Remark 4.2.** Let us justify why we can set \( y = 0 \) in the above definition:

- For a solution \( u \) of the sinh-Gordon equation (2.2) on a strip around the \( y = 0 \) axis, that is of finite type in the sense of Definition 2.1 in [18], we see that \((u(\cdot, 0), u(\cdot, 0)) \in C^\infty(\mathbb{R}/\mathbb{P}\mathbb{Z}) \times C^\infty(\mathbb{R}/\mathbb{P}\mathbb{Z}) \) is of finite type in the sense of Definition 2.1.
Let us omit the tilde in the following proposition.

Proposition 4.3 ([18], Proposition 2.2). Suppose $\Phi_\lambda$ is of the form
\[
\Phi_\lambda(z) = \frac{\lambda^{-1}}{2} \begin{pmatrix} 
0 & i e^u \\
0 & 0 
\end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} 
\omega_n & e^{u} \tau_n \\
e^{u} \sigma_n & -\omega_n 
\end{pmatrix}
\]
that is a solution of the Lax equation (according to Definition 2.1 in [18])
\[
d\Phi_\lambda = [\Phi_\lambda, \alpha] \iff \begin{cases} 
\frac{d}{d\tau} \Phi_\lambda(x,y) = \Phi_\lambda(x,y), U_\lambda(x,y) \\
\frac{d}{dy} \Phi_\lambda(x,y) = \Phi_\lambda(x,y), V_\lambda(x,y)
\end{cases}
\]
we obtain a map $\Phi_\lambda$ as in Definition [31] by setting
\[
\Phi_\lambda(x) := \Phi_\lambda(x, 0).
\]
Let us omit the tilde in the following proposition.

Proposition 4.3 ([18], Proposition 2.2). Suppose $\Phi_\lambda$ is of the form
\[
\Phi_\lambda(z) = \frac{\lambda^{-1}}{2} \begin{pmatrix} 
0 & i e^u \\
0 & 0 
\end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} 
\omega_n & e^{u} \tau_n \\
e^{u} \sigma_n & -\omega_n 
\end{pmatrix}
\]
for some $u : \mathbb{C} \to \mathbb{R}$, and that $\Phi_\lambda$ solves the Lax equation $d\Phi_\lambda = [\Phi_\lambda, \alpha]$. Then
(i) The function $u$ is a solution of the sinh-Gordon equation, i.e. $\Delta u + 2 \sinh(2u) = 0$.
(ii) The functions $\omega_n$ are solutions of the homog. Jacobi equation $\Delta \omega_n + 4 \cosh(2u)\omega_n = 0$.
(iii) The following iteration gives a formal solution of $d\Phi_\lambda = [\Phi_\lambda, \alpha]$. Let $\omega_n, \sigma_n, \tau_{n-1}$ with a solution $\omega_n$ of $\Delta \omega_n + 4 \cosh(2u)\omega_n = 0$ be given. Now solve the system
\[
\tau_{n, z} = i e^{-2u} \omega_n, \quad \tau_{n, \bar{z}} = 2i u_z \omega_{n, z} - i \omega_{n, \bar{z}},
\]
for $\tau_{n, z}$ and $\tau_{n, \bar{z}}$. Then define $\omega_{n+1}$ and $\sigma_{n+1}$ by
\[
\omega_{n+1} := -i \tau_{n, z} - 2i u_z \tau_n, \quad \sigma_{n+1} := e^{2u} \tau_n + 2i \omega_{n+1, \bar{z}}.
\]
(iv) Each $\tau_n$ is defined up to a complex constant $c_n$, so $\omega_{n+1}$ is defined up to $-2i c_n u_z$.
(v) $\omega_0 = u_z, \omega_{g-1} = cu_z$ for some $c \in \mathbb{C}$, and $\lambda^g \Phi_{1/\lambda}$ also solves $d\Phi_\lambda = [\Phi_\lambda, \alpha]$.

In [31] Pinkall-Sterling construct a series of solutions for the induction introduced in Proposition 4.3 (iii). From this Pinkall-Sterling iteration we obtain for the first terms of $\omega = \sum_{n \geq -1} \lambda^n \omega_n$
\[
\omega_{-1} = 0, \quad \omega_0 = u_z = \frac{1}{2} (u_x - i u_y), \quad \omega_1 = u_{zzz} - 2(u_z)^3,
\]
\[
\omega_2 = u_{zzzz} - 10u_{zzz}(u_z)^3 - 10(u_{zzz})^2 u_z + 6(u_z)^5, \ldots
\]

4.1. Potentials and polynomial Killing fields. We follow the exposition given in [18], Section 2. For $g \in \mathbb{N}_0$ consider the $3g + 1$-dimensional real vector space
\[
\Lambda^g_{-1} \mathfrak{s}l_2(\mathbb{C}) = \left\{ \xi_\lambda = \sum_{n=1}^{g} \lambda^n \tilde{\xi}_n \middle| \tilde{\xi}_{-1} \in i \mathbb{R} \epsilon_+, \tilde{\xi}_n = -\tilde{\xi}_{g-n} \in \mathfrak{s}l_2(\mathbb{C}) \text{ for } n = -1, \ldots, g \right\}
\]
and define an open subset of $\Lambda^g_{-1} \mathfrak{s}l_2(\mathbb{C})$ by
\[
\mathcal{P}_g := \{ \xi_\lambda \in \Lambda^g_{-1} \mathfrak{s}l_2(\mathbb{C}) \mid \tilde{\xi}_{-1} \in i \mathbb{R}^+ \epsilon_+, \text{ tr}(\tilde{\xi}_{-1} \xi_0) \neq 0 \}.
\]
Every \( \xi_\lambda \in \mathcal{P}_g \) satisfies the so-called **reality condition**
\[
\lambda^{g-1} \xi_{1/\lambda} = -\xi_\lambda.
\]

**Definition 4.4.** A **polynomial Killing field** is a map \( \zeta_\lambda : \mathbb{R} \to \mathcal{P}_g \) which solves
\[
\frac{d}{dx} \zeta_\lambda = [\zeta_\lambda, U_\lambda(\cdot, 0)] \quad \text{with} \quad \zeta_\lambda(0) = \xi_\lambda \in \mathcal{P}_g.
\]

For each initial value \( \xi_\lambda \in \mathcal{P}_g \), there exists a unique polynomial Killing field given by
\[
\zeta_\lambda(x) := F^{-1}_\lambda(x) \xi_\lambda F_\lambda(x)
\]
with \( \frac{d}{dx} F_\lambda(x) = F_\lambda(x) U_\lambda(x, 0) \), since there holds
\[
\frac{d}{dx} \zeta_\lambda = \frac{d}{dx} (F^{-1}_\lambda \xi_\lambda F_\lambda) = -F^{-1}_\lambda (\frac{d}{dx} F_\lambda) F^{-1}_\lambda \xi_\lambda F_\lambda + F^{-1}_\lambda \xi_\lambda (\frac{d}{dx} F_\lambda)
\]
\[
= -U_\lambda(\cdot, 0) F^{-1}_\lambda \xi_\lambda F_\lambda + F^{-1}_\lambda \xi_\lambda F_\lambda U_\lambda(\cdot, 0)
\]
\[
= [\zeta_\lambda, U_\lambda(\cdot, 0)].
\]

In order to obtain a polynomial Killing field \( \zeta_\lambda \) from a pair \((u, u_y)\) of finite type we set
\[
\zeta_\lambda(x) := \Phi_\lambda(x) - \lambda^{g-1} \Phi_{1/\lambda}(x) \quad \text{and} \quad \zeta_\lambda(0) =: \xi_\lambda = \Phi_\lambda(0) - \lambda^{g-1} \Phi_{1/\lambda}(0).
\]

**Remark 4.5.** These polynomial Killing fields \( \zeta_\lambda \) are periodic maps \( \zeta_\lambda : \mathbb{R}/p\mathbb{Z} \to \mathcal{P}_g \).

Suppose we have a polynomial Killing field
\[
\zeta_\lambda(x) = \begin{pmatrix} 0 & \beta_1(x) \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0(x) & \beta_0(x) \\ \gamma_0(x) & -\alpha_0(x) \end{pmatrix} \lambda^0 + \ldots + \begin{pmatrix} \alpha_g(x) & \beta_g(x) \\ \gamma_g(x) & -\alpha_g(x) \end{pmatrix} \lambda^g.
\]

Then one can assign a matrix-valued form \( U(\zeta_\lambda) \) to \( \zeta_\lambda \) defined by
\[
U(\zeta_\lambda) = \begin{pmatrix} \alpha_0(x) - \overline{\alpha_0}(x) & \lambda^{-1} \beta_1(x) - \overline{\gamma_0}(x) \\ -\lambda^{-1} \beta_1(x) + \overline{\gamma_0}(x) & -\alpha_0(x) + \overline{\alpha_0}(x) \end{pmatrix} dx.
\]

**Remark 4.6.** This shows that \((u, u_y) \simeq U_\lambda(\cdot, 0)\) is uniquely defined by \(\zeta_\lambda\).

5. **The associated spectral data**

In this section we want to establish a 1:1-correspondence between pairs \((u, u_y)\) that originate from solutions of the sinh-Gordon equation and the so-called spectral data \((Y(u, u_y), D(u, u_y))\) consisting of the spectral curve \(Y(u, u_y)\) and a divisor \(D(u, u_y)\) on \(Y(u, u_y)\).

5.1. **Spectral curve defined by** \(\xi_\lambda \in \mathcal{P}_g\). Let us introduce the definition of a spectral curve \(Y\) that results from a periodic polynomial Killing field \(\zeta_\lambda : \mathbb{R}/p\mathbb{Z} \to \mathcal{P}_g\).

**Definition 5.1.** Let \(Y^*\) be defined by
\[
Y^* = \{ (\lambda, v) \in \mathbb{C}^* \times \mathbb{C}^* \mid \det(v1 - \zeta_\lambda) = v^2 + \det(\xi_\lambda) = 0 \}
\]
and suppose that the polynomial \(a(\lambda) = -\lambda \det(\xi_\lambda)\) has \(2g\) pairwise distinct roots. By declaring \( \lambda = 0, \infty \) to be two additional branch points and setting \( v = \bar{v}\lambda \) one obtains that
\[
Y := \{ (\lambda, v) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid v^2 = \lambda a(\lambda) \}
\]
defines a compact hyperelliptic curve \(Y\) of genus \(g\), the **spectral curve**. The genus \(g\) of \(Y\) is called the **spectral genus**.

**Remark 5.2.** Note that the eigenvalue \(\bar{v}\) of \(\xi_\lambda\) is given by \(\bar{v} = \frac{1}{\lambda}\).
5.2. **Divisor.** The spectral curve $Y$ encodes the eigenvalues of $\xi_\lambda \in \mathbb{P}_g$. By considering the eigenvectors of $\xi_\lambda$ one arrives at the following lemma.

**Lemma 5.3.** On the spectral curve $Y$ there exist unique meromorphic maps $v(\lambda, \nu)$ and $w(\lambda, \nu)$ from $Y$ to $\mathbb{C}^2$ such that

(i) For all $(\lambda, \nu) \in Y^*$ the value of $v(\lambda, \nu)$ is an eigenvector of $\xi_\lambda$ with eigenvalue $\nu$ and $w(\lambda, \nu)$ is an eigenvector of $\xi_\lambda^1$ with eigenvalue $\nu$, i.e.

$$
\xi_\lambda v(\lambda, \nu) = \nu v(\lambda, \nu), \quad \xi_\lambda^1 w(\lambda, \nu) = \nu w(\lambda, \nu).
$$

(ii) The first component of $v(\lambda, \nu)$ and $w(\lambda, \nu)$ is equal to 1, i.e. $v(\lambda, \nu) = (1, v_2(\lambda, \nu))^t$ and $w(\lambda, \nu) = (1, w_2(\lambda, \nu))^t$ on $Y$.

The holomorphic map $v : Y \to \mathbb{CP}^1$ from Lemma 5.3 motivates the following definition.

**Definition 5.4.** Set the **divisor** $D(u, u_y)$ as

$$
D(u, u_y) = -(v(\lambda, \nu))
$$

and denote by $E(u, u_y)$ the holomorphic line bundle whose sheaf of holomorphic sections is given by $\mathcal{O}_{D(u, u_y)}$. Then $E(u, u_y)$ is called the **eigenline bundle**.

5.3. **Characterization of spectral curves.** Given a hyperelliptic Riemann surface $Y$ with branch points over $\lambda = 0 (y_0)$ and $\lambda = \infty (y_\infty)$ we can deduce conditions such that $Y$ is the spectral curve of a periodic finite type solution of the sinh-Gordon equation.

Let us recall the well-known characterization of such spectral curves (cf. [23], Section 1.2, in the case of immersed CMC tori in $\mathbb{S}^3$). Note that $M_\lambda$ and $\xi_\lambda$ commute, i.e. $[M_\lambda, \xi_\lambda] = 0$. Thus it is possible to diagonalize them simultaneously and $\mu$ and $\nu$ can be considered as two different functions on the same Riemann surface $Y$.

**Theorem 5.5 (23).** Let $Y$ be a hyperelliptic Riemann surface with branch points over $\lambda = 0 (y_0)$ and $\lambda = \infty (y_\infty)$. Then $Y$ is the spectral curve of a periodic real finite type solution of the sinh-Gordon equation if and only if the following three conditions hold:

(i) Besides the hyperelliptic involution $\sigma$ the Riemann surface $Y$ has two further anti-holomorphic involutions $\eta$ and $\rho = \eta \circ \sigma$. Moreover, $\eta$ has no fixed points and $\eta(y_0) = y_\infty$.

(ii) There exists a non-zero holomorphic function $\mu$ on $Y \setminus \{y_0, y_\infty\}$ that obeys

$$
\sigma^* \mu = \mu^{-1}, \quad \eta^* \mu = \mu, \quad \rho^* \mu = \mu^{-1}.
$$

(iii) The form $d \ln \mu$ is a meromorphic differential of the second kind with double poles at $y_0$ and $y_\infty$ only.

Following the terminology of [16][22][23], we will describe spectral curves of periodic real finite type solutions of the sinh-Gordon equation via hyperelliptic curves of the form

$$
\nu^2 = \lambda a(\lambda) = -\lambda^2 \det(\xi_\lambda) = (\lambda \tilde{\nu})^2.
$$

Here $\tilde{\nu}$ is the eigenvalue of $\xi_\lambda$ and $\lambda : Y \to \mathbb{CP}^1$ is chosen in a way such that $y_0$ and $y_\infty$ correspond to $\lambda = 0$ and $\lambda = \infty$ with

$$
\sigma^* \lambda = \lambda, \quad \eta^* \lambda = \lambda^{-1}, \quad \rho^* \lambda = \lambda^{-1}.
$$
Note that the function $\lambda : Y \to \mathbb{CP}^1$ is fixed only up to a Möbius transformation of the form $\lambda \mapsto e^{2i\phi}\lambda$. Moreover, $d\ln \mu$ is of the form

$$
d\ln \mu = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda},
$$

where $b$ is a polynomial of degree $g + 1$ with $\lambda^{g+1}b(\lambda^{-1}) = -b(\lambda)$.

**Definition 5.6.** The spectral curve data of a periodic real finite type solution of the sinh-Gordon equation is a pair $(a, b) \in \mathbb{C}^g|\lambda| \times \mathbb{C}^{g+1}|\lambda|$ such that

(i) $\lambda^{2g}a(\lambda^{-1}) = a(\lambda)$ and $\lambda^{-g}a(\lambda) \leq 0$ for all $\lambda \in \mathbb{S}^1$ and $|a(0)| = 1$.

(ii) On the hyperelliptic curve $\nu^2 = \lambda a(\lambda)$ there is a single-valued holomorphic function $\mu$ with essential singularities at $\lambda = 0$ and $\lambda = \infty$ with logarithmic differential

$$
d\ln \mu = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda}
$$

with $b(0) = i\sqrt{a(0)}\frac{p}{2}$ that transforms under the three involutions

$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu), \quad \rho : (\lambda, \nu) \mapsto (\lambda^{-1}, -\lambda^{-1-g}\nu), \quad \eta : (\lambda, \nu) \mapsto (\lambda^{-1}, \lambda^{-1-g}\nu)$

according to $\sigma^*\mu = \mu^{-1}$, $\rho^*\mu = \mu^{-1}$ and $\eta^*\mu = \eta^{-1}$.

**Remark 5.7.** The conditions (i) and (ii) from Definition 5.6 are equivalent to the following conditions (cf. Definition 5.10 in [18]):

(i) $\lambda^{2g}a(\lambda^{-1}) = a(\lambda)$ and $\lambda^{-g}a(\lambda) \leq 0$ for all $\lambda \in \mathbb{S}^1$ and $|a(0)| = 1$.

(ii) $\lambda^{g+1}b(\lambda^{-1}) = -b(\lambda)$ and $b(0) = i\sqrt{g(0)}\frac{p}{2}$.

(iii) $\int_{\alpha_i} b(\lambda) \frac{d\lambda}{\nu} = 0$ for all roots $\alpha_i$ of $a$.

(iv) The unique function $h : \tilde{Y} \to \mathbb{C}$, where $\tilde{Y} = Y \setminus \bigcup \gamma_i$ and $\gamma_i$ are closed cycles over the straight lines connecting $\alpha_i$ and $1/\alpha_i$, obeying $\sigma^*h = -h$ and $dh = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda}$, satisfies $h(\alpha_i) \in \pi i\mathbb{Z}$ for all roots $\alpha_i$ of $a$.

5.4. **The moduli space.** Since a Möbius transformation of the form $\lambda \mapsto e^{2i\phi}\lambda$ changes the spectral curve data $(a, b)$ but does not change the corresponding periodic solution of the sinh-Gordon equation we introduce the following definition.

**Definition 5.8.** For all $g \in \mathbb{N}_0$ let $\mathcal{M}_g(p)$ be the space of equivalence classes of spectral curve data $(a, b)$ from Definition 5.6 with respect to the action of $\lambda \mapsto e^{2i\phi}\lambda$ on $(a, b)$. $\mathcal{M}_g(p)$ is called the moduli space of spectral curve data for Cauchy data $(u, u_y)$ of periodic real finite type solutions of the sinh-Gordon equation.

Each pair of polynomials $(a, b) \in \mathcal{M}_g(p)$ represents a spectral curve $Y_{(a, b)}$ for Cauchy data $(u, u_y)$ of a periodic real finite type solution of the sinh-Gordon equation.

**Definition 5.9.** Let

$$
\mathcal{M}^1_g(p) := \{(a, b) \in \mathcal{M}_g(p) \mid a \text{ has } 2g \text{ pairwise distinct roots and } (a, b) \text{ have no common roots}\}
$$

be the moduli space of non-degenerated smooth spectral curve data for Cauchy data $(u, u_y)$ of periodic real finite type solutions of the sinh-Gordon equation.
The term “non-degenerated” in Definition 5.9 reflects the following fact (cf. [19], Section 9): If one considers deformations of spectral curve data \((a, b)\), the corresponding integral curves have possible singularities, if \(a\) and \(b\) have common roots. By excluding the case of common roots of \((a, b)\), one can avoid that situation and identify the space of such deformations with certain polynomials \(c \in \mathbb{C}^{g+1} [\lambda]\) (see Section 6.4).

**Remark 5.10.** By studying Cauchy data \((u, u_y)\) whose spectral curve \(Y(u, u_y)\) corresponds to \((a, b) \in M_1^1(p)\), we have the following benefits:

1. Since \((a, b) \in M_1^1(p)\) correspond to Cauchy data \((u, u_y)\) of finite type, we can avoid difficult functional analytic methods for the asymptotic analysis of the spectral curves \(Y\) at \(\lambda = 0\) and \(\lambda = \infty\).
2. Since \((a, b) \in M_1^1(p)\) have no common roots, we obtain non-singular smooth spectral curves \(Y\) and can apply the standard tools from complex analysis for their investigation.

Note, that these assumptions can be dropped in order to extend our results to the more general setting. This was done in [23] for the case of the non-linear Schrödinger operator, for example.

5.5. **Spectral data.** With all this terminology at hand let us introduce the spectral data associated to a periodic real finite type solution of the sinh-Gordon equation.

**Definition 5.11.** The spectral data of a periodic real finite type solution of the sinh-Gordon equation is a pair \((Y(u, u_y), D(u, u_y))\) such that \(Y(u, u_y)\) is a hyperelliptic Riemann surface of genus \(g\) that obeys the conditions from Theorem 5.5 and \(D(u, u_y)\) is a divisor of degree \(g + 1\) on \(Y(u, u_y)\) that obeys \(\eta(D) - D = (f)\) for a meromorphic \(f\) with \(f\eta^*f = -1\).

From the investigation of the so-called "Inverse Problem" it is well-known that the correspondence between \((u, u_y)\) and the spectral data is bijective, since \((Y, D)\) uniquely determine \(\xi_\lambda\) and \(\zeta_\lambda\) respectively (cf. [21]).

6. **Isospectral and non-isospectral deformations**

We will now consider deformations that correspond to variations of the divisor \(D\) or the spectral curve \(Y\) and call them isospectral and non-isospectral deformations, respectively.

6.1. **The projector \(P\).** We will use the meromorphic maps \(v: Y \to \mathbb{C}^2\) and \(w^t: Y \to \mathbb{C}^2\) from Lemma 5.3 to define a matrix-valued meromorphic function on \(Y\) by setting \(P := \frac{vw^t}{w^tv}\). Given a meromorphic map \(f\) on \(Y\) we also define

\[
P(f) := \frac{vw^t}{w^tv}.
\]

It turns out that \(P\) is a projector and that it is possible to extend the definition of the projector \(P\) to a projector \(P_x\) that is defined by

\[
P_x(f) := \frac{v(x)fw(x)^t}{w(x)^tv(x)} = F_{x}(x)P(f)F_{x}(x).
\]

Moreover, there holds \(\zeta_\lambda(x) = P_x\left(\frac{\lambda}{\lambda}\right) + \sigma^*P_x\left(\frac{\xi}{\lambda}\right)\).
6.2. General deformations of $M_\lambda$ and $U_\lambda$. In the next lemma we consider the situation of a general variation with isospectral and non-isospectral parts.

**Lemma 6.1.** Let $v_1, w_i^1$ be the eigenvectors for $\mu$ and $v_2, w_i^2$ the corresponding eigenvectors for $\frac{1}{p}$ of $M(\lambda)$. Then
\[
\delta M(\lambda)v_1 + M(\lambda)\delta v_1 = (\delta \mu)v_1 + \mu \delta v_1 \quad \text{and} \quad \delta M(\lambda)v_2 + M(\lambda)\delta v_2 = \delta \left(\frac{1}{p}\right)v_2 + \frac{1}{p}\delta v_2 \quad (*)
\]
if and only if
\[
\delta M(\lambda) = \left[ \sum_{i=1}^{2} \frac{(\delta v_i)w_i^1}{w_i^1v_i} M(\lambda) \right] + (P(\delta \mu) + \sigma^*P(\delta \mu)).
\]

**Proof.** A direct calculation shows
\[
\left( \sum_{i=1}^{2} \frac{(\delta v_i)w_i^1}{w_i^1v_i} M(\lambda) \right) v_1 = \left( \sum_{i=1}^{2} \frac{(\delta v_i)w_i^1}{w_i^1v_i} \right) \mu v_1 - M(\lambda)\delta v_1 + (\delta \mu)v_1 = \mu \delta v_1 - M(\lambda)\delta v_1 + (\delta \mu)v_1 \equiv \delta M(\lambda)v_1.
\]

An analogous calculation for $v_2$ gives
\[
\delta M(\lambda)v_2 = \left( \sum_{i=1}^{2} \frac{(\delta v_i)w_i^2}{w_i^2v_i} M(\lambda) \right) + (P(\delta \mu) + \sigma^*P(\delta \mu)) v_2
\]
and the claim is proved. \qed

The above considerations also apply for the equation $L_\lambda(x)v(x) = \left( \frac{d}{dx} + U_\lambda \right)v(x) = \frac{\ln p}{p} \cdot v(x)$ around $\lambda = 0$ and yield the following lemma.

**Lemma 6.2.** Let $v_1(x), w_i^1(x)$ be the eigenvectors for $\mu$ and $v_2(x), w_i^2(x)$ the corresponding eigenvectors for $\frac{1}{p}$ of $M_\lambda(x)$ and $M^1(x)$ respectively. Then
\[
\delta U_\lambda v_1(x) + L_\lambda(x)\delta v_1(x) = \left( \frac{\delta \ln \mu}{p} \right)v_1(x) + \frac{\ln \mu}{p} \delta v_1(x),
\]
\[
\delta U_\lambda v_2(x) + L_\lambda(x)\delta v_2(x) = -\left( \frac{\delta \ln \mu}{p} \right)v_2(x) - \frac{\ln \mu}{p} \delta v_2(x)
\]
around $\lambda = 0$ if and only if
\[
\delta U_\lambda = \left[ \frac{L_\lambda(x)}{\ln \mu} \cdot \sum_{i=1}^{2} \frac{(\delta v_i(x))w_i^1(x)}{w_i^1(x)v_i(x)} \right] + \left( P_\lambda \left( \frac{\delta \ln \mu}{p} \right) + \sigma^*P_\lambda \left( \frac{\delta \ln \mu}{p} \right) \right). \tag{6.1}
\]

**Proof.** Following the steps from the proof of Lemma 6.1 and keeping in mind that $\delta p = 0$ in our situation yields the claim. \qed

**Remark 6.3.** Equation (6.1) reflects the decomposition of the tangent space into isospectral and non-isospectral deformations.

6.3. Infinitesimal isospectral deformations of $\xi_\lambda$ and $U_\lambda$. We will now investigate infinitesimal isospectral deformations. Note that there also exist global isospectral deformations that are described in [18].
6.3.1. The real part of $H^1(Y, \mathcal{O})$. Let us begin with the investigation of $H^1(Y, \mathcal{O})$, the Lie algebra of the Jacobian $\text{Jac}(Y)$.

For this, consider disjoint open simply-connected neighborhoods $U_0, U_\infty$ of $y_0, y_\infty$. Setting $U := Y \setminus \{y_0, y_\infty\}$ we get a cover $U := \{U_0, U_\infty\}$ of $Y$. The only non-empty intersections of neighborhoods from $U$ are $U_0 \setminus \{y_0\}$ and $U_\infty \setminus \{y_\infty\}$. Since $U_0$ and $U_\infty$ are simply connected we have $H^1(U_0, \mathcal{O}) = 0 = H^1(U_\infty, \mathcal{O})$. Moreover, $H^1(U, \mathcal{O}) = 0$ since $U$ is a non-compact Riemann surface. This shows that $U$ is a Leray cover and therefore $H^1(Y, \mathcal{O}) = H^1(U, \mathcal{O})$, see [14, Theorem 12.8].

We will consider triples of the form $(f_0, 0, f_\infty)$ of meromorphic functions with respect to this cover $U$. Then the Cech-cohomology is induced by all pairs of the form $(f_0 - 0, f_\infty - 0) = (f_0, f_\infty)$.

**Lemma 6.4.** The equivalence classes $[h_i]$ of the $g$ tuples $h_i := (f_0^i, f_\infty^i)$ given by $h_i = (\nu \lambda^{-i}, -\nu \lambda^{-i})$ for $i = 1, \ldots, g$ are a basis of $H^1(Y, \mathcal{O})$. In particular $f_\infty = -f_0$.

**Proof.** We know that for $i = 1, \ldots, g$ the differentials $\omega_i = \frac{\lambda^{i-1} d\lambda}{\lambda^2}$ span a basis for $H^0(Y, \Omega)$ and that the pairing $\langle \cdot, \cdot \rangle : H^0(Y, \Omega) \times H^1(Y, \mathcal{O}) \to \mathbb{C}$ given by $(\omega, [h]) \mapsto \text{Res}(h \omega)$ is non-degenerate due to Serre duality [14, Theorem 17.9]. Therefore we can calculate the dual basis of $\omega_i$ with respect to this pairing and see

$$\langle \omega_i, [h_j] \rangle = \text{Res}_{\lambda=0} f_0^i \omega_i + \text{Res}_{\lambda=\infty} f_\infty^i \omega_i = \text{Res}_{\lambda=0} \lambda^{i-1-j} d\lambda - \text{Res}_{\lambda=\infty} \lambda^{i-1-j} d\lambda$$

$$= \text{Res}_{\lambda=0} \lambda^{i-j-1} d\lambda + \text{Res}_{\lambda=\infty} \lambda^{-i+j+1} \frac{d\lambda}{\lambda^2} = \text{Res}_{\lambda=0} (\lambda^{i-j-1} + \lambda^{-i+j+1}) d\lambda$$

$$= 2 \cdot \delta_{ij}.$$ 

This shows $\text{span}_\mathbb{C} \{[h_1], \ldots, [h_g]\} = H^1(Y, \mathcal{O})$ and concludes the proof. \hfill $\square$

**Definition 6.5.** Let $H^1_R(Y, \mathcal{O}) := \{[f] \in H^1(Y, \mathcal{O}) | \eta f = f\}$ be the real part of $H^1(Y, \mathcal{O})$ with respect to the involution $\eta$.

**Lemma 6.6.** An element $[f] = [(f_0, f_\infty)] \in H^1(Y, \mathcal{O})$ satisfies $\eta f = f$ if and only if $f_\infty = \eta f_0 = -\sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu$ satisfies $c_i = -c_{g-1-i}$ for $i = 0, \ldots, g-1$. In particular $\text{dim}_\mathbb{R} H^1_R(Y, \mathcal{O}) = g$. Any element $[f] = [(f_0, \eta f_0)] \in H^1_R(Y, \mathcal{O})$ can be represented by $f_0(\lambda, \nu)$ with

$$f_0(\lambda, \nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu$$

and $\bar{c}_i = -c_{g-1-i}$ for $i = 0, \ldots, g-1$.

**Proof.** The first part of the lemma is obvious. Now a direct calculation gives

$$-\eta^i \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu = -\sum_{i=0}^{g-1} \bar{c}_i \lambda^{i+1} \lambda^{-i-1} \nu = -\sum_{i=0}^{g-1} \bar{c}_i \lambda^{i-g} \nu \bar{c}_{g-1-i} = -\sum_{j=0}^{g-1} \bar{c}_{g-1-j} \lambda^{-j-1} \nu = \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu$$

if and only if $\bar{c}_i = -c_{g-1-i}$ for $i = 0, \ldots, g-1$. The subspace of elements $(c_0, \ldots, c_{g-1}) \in \mathbb{C}^g$ that obey these conditions is a real $g$-dimensional subspace. \hfill $\square$
6.3.2. The Krichever construction and the isospectral group action. The construction procedure for linear flows on Jac\(Y\) is due to Krichever \cite{Krichever80}. In \cite{McIntosh81} McIntosh describes the Krichever construction for finite type solutions of the sinh-Gordon equation. Let us consider the sequence

\[ 0 \rightarrow H^1(Y, \mathbb{Z}) \rightarrow H^1(Y, \mathcal{O}) \overset{\exp}{\rightarrow} H^1(Y, \mathcal{O}^*) \simeq \text{Pic}(Y) \overset{\text{deg}}{\rightarrow} H^2(Y, \mathbb{Z}) \rightarrow 0, \tag{6.2} \]

where Pic\(Y\) is the Picard variety of \(Y\).

**Definition 6.7.** Let \(\text{Pic}_d(Y)\) be the connected component of the Picard variety of \(Y\) of divisors of degree \(d\) and let

\[ \text{Pic}_d^\mathbb{R}(Y) := \{ D \in \text{Pic}_d(Y) \mid \eta(D) - D = (f) \text{ for a merom. } f \text{ with } f\eta \overline{f} = -1 \} \]

be the set of quaternionic divisors of degree \(d\) with respect to the involution \(\eta\).

Recall that Jac\(Y\) \(\simeq\) Pic\(0(Y)\). Pic\(d(Y)\) is also called the real part of Pic\(d(Y)\). If we consider \(H^2_0(Y, \mathcal{O})\) and restrict to the connected component Pic\(0^\mathbb{R}(Y)\) of Pic\(Y\), we get a map \(L : \mathbb{R}^g \simeq H^2_0(Y, \mathcal{O}) \rightarrow \text{Pic}_0^\mathbb{R}(Y)\)

\((t_0, \ldots, t_{g-1}) \mapsto L(t_0, \ldots, t_{g-1})\) from (6.2) and Lemma 6.6.

It is well known that the divisor \(D(u, u_g)\) from Definition 6.4 satisfies \(D(u, u_g) \in \text{Pic}_{g+1}^\mathbb{R}(Y)\) (see \cite{Segal85}). One can also show \cite[Theorem 5.17]{Segal85} that the action of the tensor product on holomorphic line bundles induces a continuous commutative and transitive group action of \(\mathbb{R}^g\) on \(\text{Pic}_{g+1}^\mathbb{R}(Y)\), which is denoted by

\[ \pi : \mathbb{R}^g \times \text{Pic}_{g+1}^\mathbb{R}(Y) \rightarrow \text{Pic}_{g+1}^\mathbb{R}(Y), \quad ((t_0, \ldots, t_{g-1}), E) \mapsto \pi(t_0, \ldots, t_{g-1})(E) \]

with

\[ \pi(t_0, \ldots, t_{g-1})(E) = E \otimes L(t_0, \ldots, t_{g-1}). \]

Here \(L(t_0, \ldots, t_{g-1})\) is the family in Pic\(g+1^\mathbb{R}(Y)\) which is obtained by applying Krichever’s construction procedure.

6.3.3. Loop groups and the Iwasawa decomposition. For real \(r \in (0, 1]\), denote the circle with radius \(r\) by \(S_r = \{ \lambda \in \mathbb{C} \mid |\lambda| = r\}\) and the open disk with boundary \(S_r\) by \(I_r = \{ \lambda \in \mathbb{C} \mid |\lambda| < r\}\). Moreover, the open annulus with boundaries \(S_r\) and \(S_{1/r}\) is given by \(A_r = \{ \lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}\) for \(r \in (0, 1]\). For \(r = 1\) we set \(A_1 := S^1\). The loop group \(\Lambda_r, SL(2, \mathbb{C})\) of \(SL(2, \mathbb{C})\) is the infinite dimensional Lie group of analytic maps from \(S_r\) to \(SL(2, \mathbb{C})\), i.e.

\[ \Lambda_r, SL(2, \mathbb{C}) = \mathcal{O}(S_r, SL(2, \mathbb{C})). \]

We will also need the following two subgroups of \(\Lambda_r, SL(2, \mathbb{C})\): First let

\[ \Lambda_r, SU(2) = \{ F \in \mathcal{O}(A_r, SL(2, \mathbb{C})) \mid F|_{S^1} \in SU(2) \}. \]

Thus we have

\[ F_\lambda \in \Lambda_r, SU(2) \iff F_{1/\lambda} = F_\lambda^{-1}. \]

The second subgroup is given by

\[ \Lambda^+_r, SL(2, \mathbb{C}) = \{ B \in \mathcal{O}(I_r \cup S_r, SL(2, \mathbb{C})) \mid B(0) = \begin{pmatrix} \rho & c \\ 0 & 1/\rho \end{pmatrix} \text{ for } \rho \in \mathbb{R}^+ \text{ and } c \in \mathbb{C} \}. \]

The normalization \(B(0) = B_0\) ensures that

\[ \Lambda_r, SU(2) \cap \Lambda^+_r, SL(2, \mathbb{C}) = \{ 1 \}. \]

Now we have the following important result due to Pressley-Segal \cite{PressleySegal84} that has been generalized by McIntosh \cite{McIntosh81}. 


Theorem 6.8. The multiplication $\Lambda_r SU(2) \times \Lambda^+ r SL(2, \mathbb{C}) \to \Lambda_r SL(2, \mathbb{C})$ is a surjective real analytic diffeomorphism. The unique splitting of an element $\phi_\lambda \in \Lambda_r SL(2, \mathbb{C})$ into

$$\phi_\lambda = F_\lambda B_\lambda$$

with $F_\lambda \in \Lambda_r SU(2)$ and $B_\lambda \in \Lambda^+ r SL(2, \mathbb{C})$ is called the r-Iwasawa decomposition of $\phi_\lambda$ or Iwasawa decomposition if $r = 1$.

Remark 6.9. The r-Iwasawa decomposition also holds on the Lie algebra level, i.e., $\Lambda_r \mathfrak{sl}_2(\mathbb{C}) = \Lambda_r \mathfrak{su}_2 \oplus \Lambda^+ r \mathfrak{sl}_2(\mathbb{C})$. This decomposition will play a very important role in the following.

In order to describe the infinitesimal isospectral deformations of $\xi_\lambda$ and $U_\lambda$, we follow the exposition of [1], IV.2.e. Let us transfer those methods to our situation.

Theorem 6.10. Let $f_0(\lambda, \nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu$ be the representative of $[(f_0, \eta^t f_0)] \in H^1_{\mathbb{R}}(Y, \mathcal{O})$ from Lemma 6.6 and let

$$A_{f_0} := P(f_0) + \sigma^* P(f_0) = \sum_{i=0}^{g-1} c_i \lambda^{-i}(P(\lambda^{-1} \nu) + \sigma^* P(\lambda^{-1} \nu)) = \sum_{i=0}^{g-1} c_i \lambda^{-i} \xi_\lambda$$

be the induced element in $\Lambda_r \mathfrak{sl}_2(\mathbb{C})$. Then the vector field of the isospectral action $\pi : \mathbb{R}^g \times \text{Pic}^{\mathbb{R}}_{g+1}(Y) \to \text{Pic}^{\mathbb{R}}_{g+1}(Y)$ at $E(u, u_y)$ takes the value

$$\dot{\xi}_\lambda = [A_{f_0}^+, \xi_\lambda] = -[\xi_\lambda, A_{f_0}].$$

Here $A_{f_0} = A_{f_0}^+ + A_{f_0}^-$ is the Lie algebra decomposition of the Iwasawa decomposition.

Proof. We write $\nu$ for $\tilde{v}$. Obviously there holds $A_{f_0} v = f_0 v$. If we write $\tilde{v} = e^{\beta_0(t)} v$ for local sections $\tilde{v}$ of $\mathcal{O}_D \otimes L(t_0, \ldots, t_{g-1})$ and $v$ of $\mathcal{O}_D$ with $\beta_0(t) = \sum_{i=0}^{g-1} c_i t_i \lambda^{-i-1} \nu$ we get

$$\delta \tilde{v} = \dot{\beta}_0(t) \tilde{v} + e^{\beta_0(t)} \delta v = f_0 \tilde{v} + e^{\beta_0(t)} \delta v$$

$$= A_{f_0} \tilde{v} + e^{\beta_0(t)} \delta v$$

$$= e^{\beta_0(t)} (A_{f_0} v + \delta v).$$

Moreover, $(f_0) \geq 0$ on $Y^*$. This shows that $A_{f_0} v + \delta v$ is a vector-valued section of $\mathcal{O}_D$ on $Y^*$ and defines a map $A_{f_0}^+$ such that

$$A_{f_0} v + \delta v = A_{f_0}^+ v$$

holds. Since $A_{f_0} v = f_0 v$ we also obtain the equations

$$\begin{cases}
\dot{\xi}_\lambda v = \nu v, \\
\xi_\lambda (A_{f_0}^+ v - \delta v) = \nu (A_{f_0}^+ v - \delta v).
\end{cases}$$

This implies

$$\xi_\lambda \delta v + [A_{f_0}^+, \xi_\lambda] v = \nu \delta v.$$ Differentiating the equation $\xi_\lambda v = \nu v$ we additionally obtain

$$\dot{\xi}_\lambda v + \xi_\lambda \delta v = \nu v + \nu \delta v = \nu \delta v.$$ Combining the last two equations yields

$$\dot{\xi}_\lambda v = [A_{f_0}^+, \xi_\lambda] v$$

and concludes the proof since this equation holds for a basis of eigenvectors. □
Remark 6.11. The decomposition of \( A_f \in \Lambda_r \mathfrak{sl}(2, \mathbb{C}) = \Lambda_r \mathfrak{su}_2 \oplus \Lambda^+_r \mathfrak{sl}_2(\mathbb{C}) \) yields \( A_{f_0} = A^+_f + A^-_f \) and therefore \( A_{f_0} v + \delta v = A^+_f v \) implies
\[
\delta v = -A^-_f v.
\]
In particular \( A_{f_0}^- \) is given by \( A_{f_0}^- = -\sum \frac{\partial \mu}{\partial v} \).

We want to extend Theorem 6.10 to obtain the value of the vector field induced by \( \pi : \mathbb{R}^g \times \text{Pic}^R_{g+1}(Y) \rightarrow \text{Pic}^R_{g+1}(Y) \) for \( U_\lambda \).

Theorem 6.12. Let \( f_0(\lambda, \nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i-1} \nu \) be the representative of \( [(f_0, \eta^x)] \in H^1_{\Theta}(Y, \mathcal{O}) \) from Lemma 6.8 and let
\[
A_{f_0}(x) := P_\nu(f_0) + \sigma^x P_\nu(f_0) = \sum_{i=0}^{g-1} c_i \lambda^{-i}(P_\nu(\lambda^{-i} \nu) + \sigma^x P_\nu(\lambda^{-i} \nu)) = \sum_{i=0}^{g-1} c_i \lambda^{-i} \zeta(x)
\]
be the induced map \( A_{f_0} : \mathbb{R} \rightarrow \Lambda_r \mathfrak{sl}_2(\mathbb{C}) \). Then the vector field of the isospectral action \( \pi : \mathbb{R}^g \times \text{Pic}^R_{g+1}(Y) \rightarrow \text{Pic}^R_{g+1}(Y) \) at \( E(u, u_y) \) takes the value
\[
\delta U_\lambda(x) = [A^+_f(x), L_\lambda(x)] = [L_\lambda(x), A^+_f(x)].
\]
Here \( A_{f_0}(x) = A^+_f(x) + A^-_f(x) \) is the Lie algebra decomposition of the Iwasawa decomposition.

Proof. Obviously \( A_{f_0}(x)v(x) = f_0v(x) \). In analogy to the proof of Theorem 6.10 we obtain a map \( A_f^+(x) \) such that
\[
A_{f_0}(x)v(x) + \delta v(x) = A^+_f(x)v(x)
\]
holds. Since \( A_{f_0}(x)v(x) = f_0v(x) \) we also obtain the equations
\[
\begin{cases}
L_\lambda(x)v(x) = \ln \frac{\mu}{p} v(x), \\
L_\lambda(x)(A^+_f(x)v(x) - \delta v(x)) = \ln \frac{\mu}{p} (A^+_f(x)v(x) - \delta v(x))
\end{cases}
\]
around \( \lambda = 0 \). This implies
\[
L_\lambda(x)\delta v(x) + [A^+_f(x), L_\lambda(x)]v(x) = \ln \frac{\mu}{p} \delta v(x).
\]
Differentiating the equation \( L_\lambda(x)v(x) = \ln \frac{\mu}{p} v(x) \) we additionally obtain
\[
\delta L_\lambda(x)v(x) + L_\lambda(x)\delta v(x) = \frac{\delta \ln \mu}{p} v(x) + \frac{\ln \mu}{p} \delta v(x) = \ln \frac{\mu}{p} \delta v(x).
\]
Combining the last two equations yields
\[
\delta L_\lambda(x)v(x) = \delta U_\lambda(x)v(x) = [A^+_f(x), L_\lambda(x)]v(x).
\]

6.4. Infinitesimal deformations of spectral curves.

Definition 6.13. Let \( \Sigma^g \) denote the space of smooth hyperelliptic Riemann surfaces \( Y \) of genus \( g \) with the properties described in Theorem 5.13 such that \( \text{d} \ln \mu \) has no roots at the branchpoints of \( Y \).
We will now investigate deformations of $\Sigma^p \simeq M^1_g(p)$. For this, following the expositions of [16,19,22], we derive vector fields on open subsets of $M^1_g(p)$ and parametrize the corresponding deformations by a parameter $t \in [0, \varepsilon)$. We consider deformations of $Y(u, u_y)$ that preserve the periods of $d \ln \mu$. We already know that
\[
\int_{a_i} d \ln \mu = 0 \quad \text{and} \quad \int_{b_i} d \ln \mu \in 2\pi i \mathbb{Z} \quad \text{for} \quad i = 1, \ldots, g.
\]
Considering the Taylor expansion of $\ln \mu$ with respect to $t$ we get
\[
\ln \mu(t) = \ln \mu(0) + t \partial_t \ln \mu(0) + O(t^2)
\]
and thus
\[
d\ln \mu(t) = d\ln \mu(0) + t d\partial_t \ln \mu(0) + O(t^2).
\]
Given a closed cycle $\gamma \in H_1(Y, \mathbb{Z})$ we have
\[
\int_{\gamma} d\ln \mu(t) = \int_{\gamma} d\ln \mu(0) + t \int_{\gamma} d\partial_t \ln \mu(0) + O(t^2)
\]
and therefore
\[
\left. \frac{d}{dt} \left( \int_{\gamma} d\ln \mu(t) \right) \right|_{t=0} = \int_{\gamma} d\partial_t \ln \mu(0) = 0.
\]
This shows that deformations resulting from the prescription of $\partial_t \ln \mu|_{t=0}$ at $t = 0$ preserve the periods of $d \ln \mu$ infinitesimally along the deformation and thus are isoperiodic. If we set $\lambda = 0$ we can consider $\ln \mu$ as a function of $\lambda$ and $t$ and get
\[
\ln \mu(\lambda, t) = \ln \mu(\lambda, 0) + t \partial_t \ln \mu(\lambda, 0) + O(t^2)
\]
as well as
\[
\partial_\lambda \ln \mu(\lambda, t) = \partial_\lambda \ln \mu(\lambda, 0) + t \partial^2_t \lambda \ln \mu(\lambda, 0) + O(t^2).
\]
On the compact spectral curve $Y$ the function $\partial_\lambda \ln \mu$ is given by
\[
\partial_\lambda \ln \mu = \frac{b(\lambda)}{\lambda^\nu}
\]
and the compatibility condition $\partial^2_\lambda \ln \mu|_{t=0} = \partial^2_\lambda \ln \mu|_{t=0}$ will lead to a deformation of the spectral data $(a, b)$ or equivalently to a deformation of the spectral curve $Y$ with its differential $d \ln \mu$. Therefore this deformation is non-isospectral. In the following we will investigate which conditions $\delta \ln \mu := (\partial_t \ln \mu)|_{t=0}$ has to obey in order obtain such a deformation.

**The Whitham deformation.** Following the ansatz given in [19] we consider the function $\ln \mu$ as a function of $\lambda$ and $t$ and write $\ln \mu$ locally as
\[
\ln \mu = \begin{cases} 
    f_{a_i}(\lambda)\sqrt{\lambda - \alpha_i + \pi i n_i} & \text{at} \quad \alpha_i \text{ of} \ a, \\
    f_0(\lambda)\lambda^{-1/2} + \pi i n_0 & \text{at} \quad \lambda = 0, \\
    f_\infty(\lambda)\lambda^{1/2} + \pi i n_\infty & \text{at} \quad \lambda = \infty.
\end{cases}
\]
Here we choose small neighborhoods around the branch points such that each neighborhood contains at most one branch point. Moreover, the functions $f_{a_i}, f_0, f_\infty$ do not vanish at the
corresponding branch points. If we write \( \dot{g} \) for \( (\partial_t g)|_{t=0} \) we get

\[
(\partial_t \ln \mu)|_{t=0} = \begin{cases} 
  f_\alpha(\lambda)\sqrt{\lambda - \alpha_i} & \text{at a zero } \alpha_i \text{ of } a, \\
  f_0(\lambda)\lambda^{-1/2} & \text{at } \lambda = 0, \\
  f_\infty(\lambda)\lambda^{1/2} & \text{at } \lambda = \infty.
\end{cases}
\]

Since the branches of \( \ln \mu \) differ from each other by an element in \( 2\pi i \mathbb{Z} \) we see that \( \delta \ln \mu = (\partial_t \ln \mu)|_{t=0} \) is a single-valued meromorphic function on \( Y \) with poles at the branch points of \( Y \), i.e. the poles of \( \delta \ln \mu \) are located at the zeros of \( a \) and at \( \lambda = 0 \) and \( \lambda = \infty \). Thus we have

\[
\delta \ln \mu = \frac{c(\lambda)}{\nu}
\]

with a polynomial \( c \) of degree at most \( g + 1 \). Since \( \eta^* \delta \ln \mu = \delta \ln \bar{\mu} \) and \( \eta^* \nu = \bar{\lambda}^{-g-1} \bar{\nu} \) the polynomial \( c \) obeys the reality condition

\[
\lambda^{g+1}c(\lambda^{-1}) = c(\lambda).
\]

(6.3)

Differentiating \( \nu^2 = \lambda a \) with respect to \( t \) we get \( 2\nu \dot{\nu} = \lambda \dot{a} \). The same computation for the derivative with respect to \( \lambda \) gives \( 2\nu' \nu = a + \lambda a' \). Now a direct calculation shows

\[
\begin{align*}
\partial^2_{t\lambda} \ln \mu|_{t=0} &= \partial_t \left( \frac{b}{\nu} \right) \bigg|_{t=0} = \frac{\dot{b} \nu - b \dot{\nu}}{\nu^2} = \frac{2ba - b\dot{a}}{2\nu^3}, \\
\partial^2_{\lambda\lambda} \ln \mu|_{t=0} &= \partial_\lambda \left( \frac{c}{\nu} \right) \bigg|_{t=0} = \frac{c' \nu - cv' + 2c' \nu' - 2cv \nu'}{2\nu^3} = \frac{2c' \lambda a - ca - c\lambda a'}{2\nu^3}.
\end{align*}
\]

The compatibility condition \( \partial^2_{t\lambda} \ln \mu|_{t=0} = \partial^2_{\lambda\lambda} \ln \mu|_{t=0} \) holds if and only if

\[
-2\dot{b}a + b\dot{a} = -2\lambda ac' + ac + \lambda a' c.
\]

(6.4)

Equation (6.4) is the so-called Whitham equation. Both sides of this equation are polynomials of degree at most \( 3g + 1 \) and therefore describe relations for \( 3g + 2 \) coefficients. If we choose a polynomial \( c \) that obeys the reality condition (6.3) we obtain a vector field on \( \mathcal{M}_g^1(p) \). Since \( (a, b) \in \mathcal{M}_g^1(p) \) have no common roots, the polynomials \( a, b, c \) in equation (6.4) uniquely define a tangent vector \( (\dot{a}, \dot{b}) \) (see [19], Section 9). In the following we will specify such polynomials \( c \) that lead to deformations which do not change the period \( p \) of \( (u, u_g) \) (cf. [19], Section 9).

Preserving the period \( p \) along the deformation. If we evaluate the compatibility equation (6.4) at \( \lambda = 0 \) we get

\[
-2\dot{b}(0)a(0) + b(0)\dot{a}(0) = a(0)c(0).
\]

Moreover,

\[
\dot{p} = 2 \frac{d}{dt} \left( \frac{b(0)}{i\sqrt{a(0)}} \right) \bigg|_{t=0} = \frac{-2\dot{b}(0)a(0) + b(0)\dot{a}(0)}{-i(a(0))^{3/2}} = i \frac{c(0)}{\sqrt{a(0)}}.
\]

This proves the following lemma.

**Lemma 6.14.** Vector fields on \( \mathcal{M}_g^1(p) \) that are induced by polynomials \( c \) obeying (6.3) preserve the period \( p \) of \( (u, u_g) \) if and only if \( c(0) = 0 \).

Let us take a closer look at the space of polynomials \( c \) that induce a Whitham deformation.
Lemma 6.15. For the coefficients of the polynomial \( c(\lambda) = \sum_{i=0}^{g+1} c_i \lambda^i \) obeying (6.3) there holds \( c_i = \bar{c}_{g+1-i} \) for \( i = 0, \ldots, g + 1 \).

Proposition 6.16. The space of polynomials \( c \) corresponding to deformations of spectral curve data \((a, b) \in M(p)\) (with fixed period \( p \)) is \( g \)-dimensional.

Proof. The space of polynomials \( c \) of degree at most \( g + 1 \) obeying the reality condition (6.3) is \((g + 2)\)-dimensional. From Lemma 6.14 we know that \( \dot{p} = 0 \) if and only if \( c(0) = 0 \). This yields the claim. \qed

6.5. \( M(p) \) is a smooth \( g \)-dimensional manifold. From Proposition 6.16 we know that the space of polynomials \( c \) corresponding to deformations of \( M(p) \) with fixed period \( p \) is \( g \)-dimensional. In the following we want to show that \( M(p) \) is a real \( g \)-dimensional manifold. Therefore, we follow the terminology introduced by Carberry and Schmidt in [10].

Let us recall the conditions that characterize a representative \((a, b) \in \mathbb{C}^g[\lambda] \times \mathbb{C}^{g+1}[\lambda]\) of an element in \( M(p) \):

(i) \( \lambda^{2g}a(\lambda^{-1}) = a(\lambda) \) and \( \lambda^{-g}a(\lambda) < 0 \) for all \( \lambda \in S^1 \) and \( |a(0)| = 1 \).

(ii) \( \lambda^{g+1}b(\lambda^{-1}) = -b(\lambda) \).

(iii) \( f_i(a, b) := \int_{\alpha_i}^{1/\alpha_i} \frac{b \, d\lambda}{\lambda} = 0 \) for the roots \( \alpha_i \) of \( a \) in the open unit disk \( \mathbb{D} \subset \mathbb{C} \).

(iv) The unique function \( h : \tilde{Y} \to \mathbb{C} \) with \( \sigma^*h = -h \) and \( dh = \frac{b \, d\lambda}{\nu} \) satisfies \( h(\alpha_i) \in \pi i \mathbb{Z} \) for all roots \( \alpha_i \) of \( a \).

Definition 6.17. Let \( H^g \) be the set of polynomials \( a \in \mathbb{C}^g[\lambda] \) that satisfy condition (i) and whose roots are pairwise distinct.

Every \( a \in H^g \) corresponds to a smooth spectral curve. Moreover, every \( a \in H^g \) is uniquely determined by its roots.

Definition 6.18. For every \( a \in H^g \) let the space \( \mathcal{B}_a \) be given by

\[ \mathcal{B}_a := \{ b \in \mathbb{C}^{g+1}[\lambda] \mid b \text{ satisfies conditions (ii) and (iii)} \} \]

Since (iii) imposes \( g \) linearly independent constraints on the \((g + 2)\)-dimensional space of polynomials \( b \in \mathbb{C}^{g+1}[\lambda] \) obeying the reality condition (ii) we get

Proposition 6.19 (cf. [10]). \( \dim \mathcal{B}_a = 2 \). In particular every \( b_0 \in \mathbb{C} \) uniquely determines an element \( b \in \mathcal{B}_a \) with \( b(0) = b_0 \).

Using the Implicit Function Theorem one obtains the following proposition.

Proposition 6.20. The set

\[ M := \{ (a, b) \in \mathbb{C}^g[\lambda] \times \mathbb{C}^{g+1}[\lambda] \mid a \in H^g, (a, b) \text{ have no common roots and b satisfies (ii)} \} \]

is an open subset of a \((3g + 2)\)-dimensional real vector space. Moreover, the set

\[ N := \{ (a, b) \in M \mid f_i(a, b) = 0 \text{ for } i = 1, \ldots, g \} \]

defines a real submanifold of \( M \) of dimension \( 2g + 2 \) that is parameterized by \((a, b(0))\). If \( b(0) = b_0 \) is fixed we get a real submanifold of dimension \( 2g \).

The results in [18][19] yield the following theorem (cf. Lemma 5.3 in [20]).
Theorem 6.21. For a fixed choice $n_1, \ldots, n_g \in \mathbb{Z}$ the map $h = (h_1, \ldots, h_g) : N \to (i\mathbb{R}/2\pi i\mathbb{Z})^g \simeq (S^1)^g$ with

$$h_j : N \to i\mathbb{R}/2\pi i\mathbb{Z}, \ (a, b) \mapsto h_j(a, b) := \ln \mu(\alpha_j) - \pi i n_j$$

is smooth and its differential $dh$ has full rank. In particular $M_1^g(p) = h^{-1}[0]$ defines a real submanifold of dimension $g$. Here we consider $b(0) = b_0$ as fixed, i.e. $\dim_{\mathbb{R}}(N) = 2g$.

Proof. Let us consider an integral curve $(a(t), b(t))$ for the vector field $X_c$ that corresponds to a Whitham deformation that is induced by a polynomial $c$ obeying the reality condition (6.3). Then there holds

$$h(a(t), b(t)) = \ln \mu(\alpha) \equiv \text{const.}$$

along $(a(t), b(t))$, where the $b_j$ are the $b$-cycles of $Y$. Taking the derivative yields

$$\left. \frac{d}{dt} f_{b_j}(a, b) \right|_{t=0} = \int_{b_j} \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda} = \ln \mu(\alpha_j) = \pi i n_j \in \pi i \mathbb{Z}$$

Morover, for the $a$-cycles $a_j$ we have

$$f_{a_j}(a, b) = \int_{a_j} \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda} = 0$$

and consequently

$$\left. \frac{d}{dt} f_{a_j}(a, b) \right|_{t=0} = \int_{a_j} \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda} = \ln \mu(\alpha_j) = 0$$

Since all integrals of $\partial_t(\partial_\lambda \ln \mu)|_{t=0}$ vanish, there exists a meromorphic function $\phi$ with

$$d\phi = \partial_t(\partial_\lambda \ln \mu)|_{t=0} d\lambda$$

Due to the Whitham equation (6.4) this function is given by $\phi = (\partial_t \ln \mu)|_{t=0} = \frac{\partial t}{\partial a_j}$. Thus the map in (6.5) is bijective. This shows $\dim(\ker dh) = g$ and consequently $\dim(\im dh) = g$ as well. Therefore $dh : \mathbb{R}^{2g} \to \mathbb{R}^g$ has full rank and the claim follows. \qed

7. The phase space $(M_1^g, \Omega)$

In the following we will define the phase space of our integrable system. We need some preparation and first recall the generalized Weierstrass representation [II]. Set

$$\Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C}) = \{ \xi_{\lambda} \in \mathcal{O}(C^*, \mathfrak{sl}_2(\mathbb{C})) \mid (\lambda \xi_{\lambda})_{\lambda=0} \in \mathbb{C}^* e_+ \}.$$ 

A potential is a holomorphic 1-form $\xi_{\lambda} dz$ on $C$ with $\xi_{\lambda} \in \Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C})$. Given such a potential one can solve the holomorphic ODE $d\phi_{\lambda} = \phi_{\lambda} \xi_{\lambda}$ to obtain a map $\phi_{\lambda} : C \to \Lambda_\nu SL(2, \mathbb{C})$. Then Theorem 6.8 yields an extended frame $F_{\lambda} : C \to \Lambda_{\nu} SU(2)$ via the $r$-Iwasawa decomposition

$$\phi_{\lambda} = F_{\lambda} B_{\lambda}.$$
It is proven in [11] that each extended frame can be obtained from a potential $\xi_\lambda dz$ by the Iwasawa decomposition. Note, that we have the inclusions

$$\mathcal{P}_g \subset \Lambda_\infty^1 \mathfrak{sl}_2(\mathbb{C}) \subset \Lambda_\infty^1 \mathfrak{sl}_2(\mathbb{C}).$$

An extended frame $F_\lambda : \mathbb{C} \to \Lambda_\tau SU(2)$ is of finite type, if there exists $g \in \mathbb{N}$ such that the corresponding potential $\xi_\lambda dz$ satisfies $\xi_\lambda \in \mathcal{P}_g \subset \Lambda_\infty^1 \mathfrak{sl}_2(\mathbb{C})$. We say that a polynomial Killing field has minimal degree if and only if it has neither roots nor poles in $\lambda \in \mathbb{C}^*$. We will need the following proposition that summarizes two results by Burstall-Pedit [8,9].

**Proposition 7.1** ([8, Proposition 4.5]). For an extended frame of finite type there exists a unique polynomial Killing field of minimal degree. There is a smooth 1:1 correspondence between the set of extended frames of finite type and the set of polynomial Killing fields without zeros.

Consider the map $A : \xi_\lambda \mapsto A(\xi_\lambda) := -\lambda \det \xi_\lambda$ (see [13]) and set $\mathcal{P}^1_g(p) := A^{-1}[\mathcal{M}_g^1(p)]$. Moreover, denote by $C^\infty_p := C^\infty(\mathbb{R}/p\mathbb{Z})$ the Frechet space of real infinitely differentiable functions of period $p \in \mathbb{R}^+$. The above discussion yields an injective map

$$\phi : \mathcal{P}_g^1(p) \subset \Lambda_\infty^1 \mathfrak{sl}_2(\mathbb{C}) \to \phi[\mathcal{P}_g^1(p)] \subset C^\infty_p \times C^\infty_p, \xi_\lambda \mapsto (u(\xi_\lambda), u_\bar{y}(\xi_\lambda)).$$

**Definition 7.2.** Let $M_g^p$ denote the space of $(u, u_\bar{y}) \in C^\infty_p \times C^\infty_p$ (with fixed period $p$) such that $(u, u_\bar{y})$ is of finite type in the sense of Def. [4.7], where $\Phi_\lambda$ is of fixed degree $g \in \mathbb{N}_0$, and $\xi_\lambda(0) \in \mathcal{P}_g^1(p)$ with $\xi_\lambda = \Phi_\lambda - \lambda^{-1}g \Phi_1/\lambda^3$, i.e. $M_g^p := \phi[\mathcal{P}_g^1(p)]$.

Now we are able to prove the following lemma.

**Lemma 7.3.** The map $\phi : \mathcal{P}_g^1(p) \to M_g^p, \xi_\lambda \mapsto (u(\xi_\lambda), u_\bar{y}(\xi_\lambda))$ is an embedding.

**Proof.** From the previous discussion we know that $\phi : \mathcal{P}_g^1(p) \to M_g^p$ is bijective. We show that $\phi^{-1} : M_g^p \to \mathcal{P}_g^1(p)$ is continuous. Assume that $g$ is the minimal degree for $\xi_\lambda \in \mathcal{P}_g^1(p)$ (see Proposition [7,1]). Then the Jacobi fields

$$(\omega_0, \partial_y \omega_0), \ldots, (\omega_{g-1}, \partial_y \omega_{g-1}) \in C^\infty(\mathbb{C}/p\mathbb{Z}) \times C^\infty(\mathbb{C}/p\mathbb{Z})$$

are linearly independent over $\mathbb{C}$ with all their derivatives up to order $2g + 1$. We will now show that they stay linearly independent if we restrict them to $\mathbb{R}$. For this, suppose that they are linearly dependent on $\mathbb{R}$ with all their derivatives up to order $2g + 1$. Since $u$ solves the elliptic sinh-Gordon equation with analytic coefficients $u$ is analytic on $\mathbb{C}$ [27]. Thus the $(\omega_i, \partial_y \omega_i)$ are analytic as well since they only depend on $u$ and its $k$-th derivatives with $k \leq 2g + 1 \leq 2g + 1$ (see [31], Proposition 3.1). Thus they stay linearly dependent on an open neighborhood and the subset $M \subset \mathbb{C}$ of points such that these functions are linearly dependent is open and closed. Therefore $M = \mathbb{C}$, a contradiction!

By considering all derivatives of $(u, u_\bar{y})$ up to order $2g + 1$ we get a small open neighborhood $U$ of $(u, u_\bar{y}) \in C^\infty(\mathbb{R}/p\mathbb{Z}) \times C^\infty(\mathbb{R}/p\mathbb{Z})$ such that the functions

$$(\tilde{\omega}_0, \partial_y \tilde{\omega}_0), \ldots, (\tilde{\omega}_{g-1}, \partial_y \tilde{\omega}_{g-1}) \in C^\infty(\mathbb{R}/p\mathbb{Z}) \times C^\infty(\mathbb{R}/p\mathbb{Z})$$

remain linearly independent for $(\tilde{u}, \tilde{u}_\bar{y}) \in U$. Given $(u, u_\bar{y}) \in U$ there exist numbers $a_0, \ldots, a_{g-1}$ such that the $g$ vectors

$$((\omega_0(a_j), \partial_y \omega_0(a_j)), \ldots, (\omega_{g-1}(a_j), \partial_y \omega_{g-1}(a_j)))^T$$
are linearly independent. Recall that $(\omega_g, \partial_g \omega_g) = \sum_{i=0}^{g-1} c_i(\omega_i, \partial_g \omega_i)$ in the finite type situation, which assures the existence of a polynomial Killing field. Inserting these $a_0, \ldots, a_{g-1}$ into the equation $(\omega_g, \partial_g \omega_g) = \sum_{i=0}^{g-1} c_i(\omega_i, \partial_g \omega_i)$ we obtain an invertible $g \times g$ matrix and can calculate the $c_i$. This shows that the coefficients $c_i$ continuously depend on $(u, u_y) \in U$. Thus for $(u, u_y) \in M_g^P$ and $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\|\xi_\lambda(u, u_y) - \xi_\lambda(\tilde{u}, \tilde{u}_g)\| < \epsilon$$

holds for all Cauchy data $(\tilde{u}, \tilde{u}_g) \in M_g^P$ with $\|(u, u_y) - (\tilde{u}, \tilde{u}_g)\| < \delta_\epsilon$, where the norm is given by the supremum of the first $2g + 1$ derivatives. □

Let us study the map

$$Y : M_g^P \to \Sigma_g^P \simeq \mathcal{M}_g^1(p), \ (u, u_y) \mapsto Y(u, u_y)$$

that appears in the diagram

$$\begin{array}{ccc}
\mathcal{P}_g^1(p) & \xrightarrow{A} & \Sigma_g^P \\
\phi \downarrow & & \downarrow Y \\
M_g^P & \xrightarrow{Y} & \Sigma_g^P
\end{array}$$

Proposition 7.4. The map $Y : M_g^P \to \Sigma_g^P \simeq \mathcal{M}_g^1(p), \ (u, u_y) \mapsto Y(u, u_y)$ is a principle bundle with fiber $\text{Iso}(Y(u, u_y)) \simeq \text{Pic}_{g+1}^\mathbb{Z}(Y(u, u_y)) \simeq (\mathbb{S}^1)^g$. In particular $M_g^P$ is a manifold of dimension $2g$.

Proof. Due to Theorem 6.21 the space $\mathcal{M}_g^1(p)$ is a smooth $g$-dimensional manifold. From Proposition 4.12 in [18] we know that the mapping

$$A : \mathcal{P}_g^1(p) \to \mathcal{M}_g^1(p), \ \xi_\lambda \mapsto -\lambda \det(\xi_\lambda)$$

is a principal fiber bundle with fiber $(\mathbb{S}^1)^g$ and thus $\mathcal{P}_g^1(p)$ is a manifold of dimension $2g$. Due to Lemma 7.3 the map $\phi : \mathcal{P}_g^1(p) \to M_g^P$ is an embedding and thus $M_g^P$ is a manifold of dimension $2g$ as well. □

Note, that the structure of such “finite-gap manifolds” is also investigated in [12] and [3][20].

7.1. Hamiltonian formalism. It will turn out that $M_g^P$ can be considered as a symplectic manifold with a certain symplectic form $\Omega$. To see this, we closely follow the exposition of [30] and consider the phase space of $(q, p) \in C^\infty_p \times C^\infty_p$ equipped with the symplectic form

$$\Omega((\delta q, \delta p), (\tilde{\delta} q, \tilde{\delta} p)) = \int_0^P (\delta q(x)\tilde{\delta} p(x) - \tilde{\delta} q(x)\delta p(x)) \ dx$$

and the Poisson bracket

$$\{f, g\} = \int_0^P \langle \nabla f, J \nabla g \rangle \ dx \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
and

\[ X(H) = J \nabla H. \]

Since \( X(H) \) is a vector field it defines a flow \( \Phi : O \subset M \times \mathbb{R} \to M \) such that \( \Phi((q_0, p_0), t) \) solves

\[ \frac{d}{dt} \Phi((q_0, p_0), t) = X(H)(\Phi((q_0, p_0), t)) \quad \text{with} \quad \Phi((q_0, p_0), 0) = (q_0, p_0). \]

In the following we will write \( (q(t), p(t))^t := \Phi((q_0, p_0), t) \) for integral curves of \( X(H) \) that start at \((q_0, p_0)\). A direct calculation shows

\[ \frac{d}{dt} f(q(t), p(t)) \bigg|_{t=0} = df(X(H)) = \int_0^p (\nabla f, J \nabla H) \, dx = \{ f, H \} \]

and we see again that \( f \) is an integral of motion if and only if \( f \) and \( H \) are in involution. Set \((q, p) = (u, u_y)\), where \( u \) is a solution of the sinh-Gordon equation, i.e.

\[ \Delta u + 2 \sinh(2u) = u_{xx} + u_{yy} + 2 \sinh(2u) = 0. \]

Setting \( t = y \) we can investigate the so-called sinh-Gordon flow that is expressed by

\[ \frac{d}{dy} \begin{pmatrix} u \\ u_y \end{pmatrix} = \begin{pmatrix} -u_y - 2 \sinh(2u) \\ 0 \end{pmatrix} = J \nabla H_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \]

with the Hamiltonian

\[ H_2(q, p) = \int_0^p \frac{1}{2} p^2 - \frac{1}{2} (q_x)^2 + \cosh(2q) \, dx = \int_0^p \frac{1}{2} (u_y)^2 - \frac{1}{2} (u_x)^2 + \cosh(2u) \, dx \]

and corresponding gradient

\[ \nabla H_2 = (q_{xx} + 2 \sinh(2q), p)^t = (u_{xx} + 2 \sinh(2u), u_y)^t. \]

**Remark 7.5.** Since we have a loop group splitting (the \( r \)-Iwasawa decomposition) in the finite type situation, all corresponding flows can be integrated. Thus the flow \((q(y), p(y))^t = (u(x, y), u_y(x, y))^t \) that corresponds to the sinh-Gordon flow is defined for all \( y \in \mathbb{R} \).

Due to Remark 7.5 \( q(y) = u(x, y) \) is a periodic solution of the sinh-Gordon equation with \( u(x + p, y) = u(x, y) \) for all \((x, y) \in \mathbb{R}^2\). The Hamiltonian \( H_2 \) is an integral of motion, another one is associated with the flow of translation (here we set \( t = x \)) induced by the functional

\[ H_1(q, p) = \int_0^p pq_x \, dx = \int_0^p u_y u_x \, dx \quad \text{with} \quad \frac{d}{dx} \begin{pmatrix} u \\ u_y \end{pmatrix} = \begin{pmatrix} u_x \\ u_{yx} \end{pmatrix} = J \nabla H_1. \]

8. **Polynomial Killing fields and integrals of motion**

Following [23], we will now describe how the functions \( \varphi((\lambda, \mu), z) := F_{-1}^{-1}(z) \nu(\lambda, \mu) \) and \( \psi((\lambda, \mu), z) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma^* \varphi((\lambda, \mu), z) \) can be used to describe the functions \( \omega_n, \sigma_n, \tau_n \) from the Pinkall-Sterling iteration.

**Proposition 8.1** ([23], Proposition 3.1). Define \( \omega := \psi_1 \varphi_1 - \psi_2 \varphi_2 \). Then

(i) The function \( h = \psi^t \varphi \) satisfies \( dh = 0 \).

(ii) The function \( \omega \) is in the kernel of the Jacobi operator and can be supplemented to a parametric Jacobi field with corresponding (up to complex constants)

\[ \tau = \frac{i \psi_2 \varphi_1}{e^{au}}, \quad \sigma = \frac{i \psi_1 \varphi_2}{e^{au}}. \]
Proposition 8.2 ([23], Proposition 3.3). Let $h = \psi^t \varphi$. Then the entries of

$$P(z) := \frac{\psi \varphi^t}{\psi^t \varphi} = \frac{1}{h} \begin{pmatrix} \psi_1 \varphi_1 & \psi_1 \varphi_2 \\ \psi_2 \varphi_1 & \psi_2 \varphi_2 \end{pmatrix}$$

have the asymptotic expansions at $\lambda = 0$

$$-\frac{i \omega}{2h} = \frac{1}{\sqrt{\lambda}} \sum_{n=1}^{\infty} \omega_n (-\lambda)^n, \quad -i \tau = \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \tau_n (-\lambda)^n, \quad -\frac{i \sigma}{h} = \frac{1}{\sqrt{\lambda}} \sum_{n=1}^{\infty} \sigma_n (-\lambda)^n.$$

Definition 8.3. Consider the asymptotic expansion

$$\ln \mu = \frac{1}{\sqrt{\lambda}} \frac{i \rho}{2} + \sqrt{\lambda} \sum_{n=0}^{\infty} c_n \lambda^n \text{ at } \lambda = 0$$

and set $H_{2n+1} := (-1)^{n+1} \Re(c_n)$ and $H_{2n+2} := (-1)^{n+1} \Im(c_n)$ for $n \geq 0$.

Remark 8.4. Since

$$\ln \mu = \frac{1}{\sqrt{\lambda}} \frac{i \rho}{2} + \sqrt{\lambda} \int_0^P (-i(\partial u)^2 + \frac{1}{4} \cosh(2u)) \, dt + O(\lambda)$$

at $\lambda = 0$ we see that the functions $H_1, H_2$ are given by

$$H_1 = \int_0^P \frac{1}{4} u_y u_x \, dx,$$

$$H_2 = -\int_0^P \frac{1}{4} (u_y)^2 - \frac{1}{4} (u_x)^2 + \frac{1}{2} \cosh(2u) \, dx.$$

These functions are proportional to the Hamiltonians that induce the flow of translation and the sinh-Gordon flow respectively.

We will now illustrate the link between the Pinkall-Sterling iteration from Proposition 4.3 and these functions $H_n$ (which we call Hamiltonians from now on) and show that the functions $H_n$ are pairwise in involution. Recall the formula

$$\frac{d}{dt} H((u, u_y) + t(\delta u, \delta u_y)) \bigg|_{t=0} = d H(u, u_y)(\delta u, \delta u_y) = \Omega(\nabla H(u, u_y), (\delta u, \delta u_y))$$

from the last section. First we need the following lemma.

Lemma 8.5. For the map $\ln \mu$ we have the variational formula

$$\frac{d}{dt} \ln \mu((u, u_y) + t(\delta u, \delta u_y)) \bigg|_{t=0} = \int_0^P \frac{1}{\psi^t \varphi} \delta U \varphi \, dx$$

with

$$\delta U_\lambda = \frac{1}{2} \begin{pmatrix} -i \delta u_y & i \lambda^{-1} e^u \delta u - i e^{-u} \delta u \\ i \lambda e^u \delta u - i e^{-u} \delta u & i \delta u_y \end{pmatrix}.$$

Proof. We follow the ansatz presented in [30], Section 6, and obtain for $F_\lambda(x)$ solving $\frac{d}{dx} F_\lambda = F_\lambda U_\lambda$ with $F_\lambda(0) = 1$ the variational equation

$$\frac{d}{dx} \frac{d}{dt} F_\lambda(\delta u, \delta u_y) \bigg|_{t=0} = \left( \frac{d}{dt} F_\lambda(\delta u, \delta u_y) \bigg|_{t=0} \right) U_\lambda + F_\lambda \delta U_\lambda$$

with

$$\left( \frac{d}{dt} F_\lambda(\delta u, \delta u_y) \bigg|_{t=0} \right)(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
and
\[ \delta U_\lambda = \frac{1}{2} \left( i \lambda e^u \delta u + i e^{-u} \delta u - i \delta u_y \right). \]

The solution of this differential equation is given by
\[ \left( \frac{d}{dt} F_\lambda(\delta u, \delta u_y) \right)_{t=0} = \left( \int_0^x F_\lambda(y) \delta U_\lambda(y) F_\lambda^{-1}(y) \, dy \right) F_\lambda(x) \]
and evaluating at \( x = p \) yields \( \frac{d}{dt} M_\lambda(\delta u, \delta u_y)_{t=0} = \left( \int_0^p F_\lambda(y) \delta U_\lambda(y) F_\lambda^{-1}(y) \, dy \right) M_\lambda. \) Due to Lemma 6.1, there holds
\[ \delta M_\lambda = \frac{d}{dt} M_\lambda(\delta u, \delta u_y)_{t=0} = \left[ \sum_{i=1}^{2} \left( \delta v_i \psi_i, \psi_i \delta U_\lambda \lambda \right) + (P(\delta \mu) + \sigma P(\delta \mu)). \]

If we multiply the last equation with \( \psi_i \) from the left and \( v_i \) from the right we get
\[ \mu \psi_i^\dagger \delta v_i - \mu \psi_i^\dagger \delta v_i + \delta \mu \psi_i^\dagger v_i = \delta \mu = \mu \int_0^p \frac{1}{\psi_2^2} \psi_1^\dagger \delta U_\lambda \rho \, d\tau \]
and therefore \( \frac{d}{dt} \ln \mu(\delta u, \delta u_y)_{t=0} = \int_0^p \frac{1}{\psi_2^2} \psi_1^\dagger \delta U_\lambda \rho \, d\tau \). This proves the claim. \( \Box \)

We now apply Lemma 8.3 and Corollary 2.1 to establish the link between solutions \( \omega_n \) of the homogeneous Jacobi equation from Proposition 4.3 and the Hamiltonians \( H_n \).

**Theorem 8.6.** For the series of Hamiltonians \((H_n)_{n \in \mathbb{N}}\) and solutions \((\omega_n)_{n \in \mathbb{N}}\) of the homogeneous Jacobi equation from the Pinkall-Sterling iteration, there holds
\[ \nabla H_{2n+1} = (\Re(\omega_n(0)), \Re(\partial_y \omega_n(0), 0)) \quad \text{and} \quad \nabla H_{2n+2} = (\Im(\omega_n(0)), \Im(\partial_y \omega_n(0), 0)). \]

**Proof.** Considering the result of Lemma 8.3, a direct calculation gives
\[
\frac{d}{dt} \ln \mu(\delta u, \delta u_y)_{t=0} = \int_0^p \frac{1}{\psi_2^2} \psi_1^\dagger \delta U_\lambda \rho \, d\tau
\]
and \( \nabla H_{2n+1} = (\Re(\omega_n(0)), \Re(\partial_y \omega_n(0), 0)) \) and \( \nabla H_{2n+2} = (\Im(\omega_n(0)), \Im(\partial_y \omega_n(0), 0)). \)

around $\lambda = 0$ due to Proposition 8.2. On the other hand we know from Corollary 3.4 that we have the following asymptotic expansion of $\ln \mu$ around $\lambda = 0$

$$
\ln \mu = \frac{1}{\sqrt{\lambda}} \mu + \sqrt{\lambda} \int_0^T \left( -i(\partial u)^2 + \frac{i}{2} \cosh(2u) \right) \, dt + O(\lambda)
$$

$$
= \frac{1}{\sqrt{\lambda}} \mu + \sqrt{\lambda} \sum_{n \geq 0} c_n \lambda^n
$$

$$
= \frac{1}{\sqrt{\lambda}} \mu + \sqrt{\lambda} \sum_{n \geq 0} (-1)^{n+1} H_{2n+1} \lambda^n + i \sqrt{\lambda} \sum_{n \geq 0} (-1)^n H_{2n+2} \lambda^n.
$$

Thus we get

$$
\frac{d}{dt} \ln \mu(\delta u, \delta u_y) \bigg|_{t=0} = \sqrt{\lambda} \sum_{n \geq 0} (-1)^{n+1} \Omega(\nabla H_{2n+1}, (\delta u, \delta u_y)) \lambda^n
$$

$$
+ i \sqrt{\lambda} \sum_{n \geq 0} (-1)^n \Omega(\nabla H_{2n+2}, (\delta u, \delta u_y)) \lambda^n
$$

and a comparison of the coefficients of the two power series yields the claim. \qed

9. An inner product on $\Lambda_\tau \mathfrak{sl}_2(\mathbb{C})$

We already introduced a differential operator $L_\lambda := \frac{d}{dt} + U_\lambda$ such that the sinh-Gordon flow can be expressed in commutator form, i.e.

$$
\frac{d}{dt} L_\lambda = \frac{d}{dt} U_\lambda = [L_\lambda, V_\lambda] = \frac{d}{dt} V_\lambda + [U_\lambda, V_\lambda].
$$

In the following we will translate the symplectic form $\Omega$ with respect to the identification $(u, u_y) \simeq U_\lambda$. First recall that the span of $\{\epsilon_+, \epsilon_-, \epsilon\}$ is $\mathfrak{sl}_2(\mathbb{C})$ and that the inner product

$$
\langle \cdot, \cdot \rangle : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{C}, \ (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle := \text{tr}(\alpha \cdot \beta)
$$

is non-degenerate. We will now extend the inner product $\langle \cdot, \cdot \rangle$ to a non-degenerate inner product $\langle \cdot, \cdot \rangle_\Lambda$ on $\Lambda_\tau \mathfrak{sl}_2(\mathbb{C}) = \Lambda_\tau \mathfrak{su}_2(\mathbb{C}) \oplus \Lambda_\tau \mathfrak{sl}_2(\mathbb{C})$.

**Lemma 9.1.** The map $\langle \cdot, \cdot \rangle_\Lambda : \Lambda_\tau \mathfrak{sl}_2(\mathbb{C}) \times \Lambda_\tau \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathbb{R}$ given by

$$
(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_\Lambda := \Re \left( \text{Res}_{\lambda = 0} \frac{d}{d\lambda} \text{tr}(\alpha \cdot \beta) \right)
$$

is bilinear and non-degenerate. Moreover, there holds

$$
\langle \cdot, \cdot \rangle_\Lambda |_{\Lambda_\tau \mathfrak{su}_2(\mathbb{C}) \times \Lambda_\tau \mathfrak{su}_2(\mathbb{C})} \equiv 0 \quad \text{and} \quad \langle \cdot, \cdot \rangle_\Lambda |_{\Lambda_\tau^+ \mathfrak{sl}_2(\mathbb{C}) \times \Lambda_\tau^+ \mathfrak{sl}_2(\mathbb{C})} \equiv 0,
$$

i.e. $\Lambda_\tau \mathfrak{su}_2(\mathbb{C})$ and $\Lambda_\tau^+ \mathfrak{sl}_2(\mathbb{C})$ are isotropic subspaces of $\Lambda_\tau \mathfrak{sl}_2(\mathbb{C})$ with respect to $\langle \cdot, \cdot \rangle_\Lambda$.

**Proof.** The bilinearity of $\langle \cdot, \cdot \rangle_\Lambda$ follows from the bilinearity of $\text{tr}(\cdot)$. Now consider a non-zero element $\xi = \sum_{\lambda \in \mathbb{I}} \lambda^j \xi_\lambda \in \Lambda_\tau \mathfrak{sl}_2(\mathbb{C})$ and pick out an index $j \in \mathbb{I}$ such that $\xi_j \neq 0$. Setting $\tilde{\xi} = i\lambda^{-j} \xi_j$ we obtain

$$
\langle \xi, \tilde{\xi} \rangle_\Lambda = \Re \left( \text{Res}_{\lambda = 0} \frac{d}{d\lambda} \text{tr}(\xi \cdot \tilde{\xi}) \right) = \text{tr}(\xi_j \cdot \tilde{\xi}_j) \in \mathbb{R}^+
$$

since $\xi_j \neq 0$. This shows that $\langle \cdot, \cdot \rangle_\Lambda$ is non-degenerate, i.e. the first part of the lemma.

We will now prove the second part of the lemma, namely that $\Lambda_\tau \mathfrak{su}_2(\mathbb{C})$ and $\Lambda_\tau^+ \mathfrak{sl}_2(\mathbb{C})$ are isotropic subspaces of $\Lambda_\tau \mathfrak{sl}_2(\mathbb{C})$ with respect to $\langle \cdot, \cdot \rangle_\Lambda$. 
First we consider $\alpha^+ = \alpha^+_\lambda = \sum_i \lambda^i \alpha^+_i$, $\bar{\alpha}^+ = \sum_i \bar{\lambda}^i \bar{\alpha}^+_i \in \Lambda_r \mathfrak{su}_2(\mathbb{C})$ with
\[
\alpha^+_\lambda = -\bar{\alpha}^+_{1/\bar{\lambda}} \quad \text{and} \quad \bar{\alpha}^+ = -\bar{\alpha}^+_{1/\bar{\lambda}}.
\]
Then one obtains
\[
\langle \alpha^+, \bar{\alpha}^+ \rangle_A = \Im \left( \operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} \tr(\alpha^+ \cdot \bar{\alpha}^+) \right) = \Im(\tr(\alpha^+_{-1} \cdot \bar{\alpha}^+_1 + \alpha^+_0 \cdot \bar{\alpha}^+_0 + \alpha^+_1 \cdot \bar{\alpha}^+_{-1})).
\]
A direct calculation gives
\[
\tr(\alpha^+_{-1} \bar{\alpha}^+_1 + \alpha^+_0 \bar{\alpha}^+_0 + \alpha^+_1 \bar{\alpha}^+_{-1}) = \tr((-\bar{\alpha}^+_{-1})^t(-\bar{\alpha}^+_1)^t + (-\bar{\alpha}^+_0)^t(-\bar{\alpha}^+_0)^t + (-\bar{\alpha}^+_1)^t(-\bar{\alpha}^+_{-1})^t)
\]
and thus $\tr(\alpha^+_{-1} \bar{\alpha}^+_1 + \alpha^+_0 \bar{\alpha}^+_0 + \alpha^+_1 \bar{\alpha}^+_{-1}) \in \mathbb{R}$. This shows
\[
\langle \alpha^+, \bar{\alpha}^+ \rangle_A = \Im(\tr(\alpha^+_{-1} \cdot \bar{\alpha}^+_1 + \alpha^+_0 \cdot \bar{\alpha}^+_0 + \alpha^+_1 \cdot \bar{\alpha}^+_{-1})) = 0.
\]
Now consider $\beta^- = \sum_{i \geq 0} \lambda^i \beta^i_-$, $\bar{\beta}^- = \sum_{i \geq 0} \bar{\lambda}^i \bar{\beta}^-_i \in \Lambda^+_r \mathfrak{sl}_2(\mathbb{C})$ where $\beta^-_0, \bar{\beta}^-_0$ are of the form
\[
\beta^-_0 = \begin{pmatrix} h_0 & e_0 \\ 0 & -h_0 \end{pmatrix}, \quad \bar{\beta}^-_0 = \begin{pmatrix} \bar{h}_0 & \bar{e}_0 \\ 0 & -\bar{h}_0 \end{pmatrix}
\]
with $h_0, \bar{h}_0 \in \mathbb{R}$ and $e_0, \bar{e}_0 \in \mathbb{C}$. Then one gets
\[
\langle \beta^-, \bar{\beta}^- \rangle_A = \Im \left( \operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} \tr(\beta^- \cdot \bar{\beta}^-) \right) = \Im(\tr(\beta^-_0 \cdot \bar{\beta}^-_0)) = \Im(2h_0 \bar{h}_0) = 0.
\]
This yields the second claim and concludes the proof. \hfill \Box

**Remark 9.2.** Recall the notion of Manin triples $(\mathfrak{g}, \mathfrak{p}, \mathfrak{q})$ that were introduced by Vladimir Drinfeld [13]. They consist of a Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric bilinear form, together with two isotropic subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{q}$ as a vector space. Thus Lemma [9.1] shows that $(\Lambda^+_r \mathfrak{sl}_2(\mathbb{C}), \Lambda^-_r \mathfrak{su}_2(\mathbb{C}), \Lambda^+_r \mathfrak{sl}_2(\mathbb{C}))$ is a Manin triple.

10. The symplectic form $\Omega$ and Serre duality

This section incorporates our previous results and establishes a connection between the symplectic form $\Omega$ and Serre Duality [14, Thm. 17.9]. Moreover, we will show that $(M^p, \Omega, H_2)$ is a completely integrable Hamiltonian system.

**Definition 10.1.** Let $H^0_\mathbb{R}(Y, \Omega) := \{ \omega \in H^0(Y, \Omega) \mid \eta \omega = -\omega \}$ be the real part of $H^0(Y, \Omega)$ with respect to the involution $\eta$.

Since $\eta$ is given by $(\lambda, \mu) \mapsto (1/\bar{\lambda}, \bar{\mu})$ we have
\[
\eta^* \delta \ln \mu = \delta \ln \mu \quad \text{and} \quad \eta^* \bar{\Delta} = -\frac{d\lambda}{\lambda}.
\]
Let us define the map $\omega : T(u, u_y)M^p \to H^0_\mathbb{R}(Y(u, u_y), \Omega)$ by
\[
(\delta u, \delta u_y) \mapsto \omega(\delta u, \delta u_y) := \delta \ln \mu(\delta u, \delta u_y) \frac{d\lambda}{\lambda}.
\]
Remark 10.2. Due to Theorem [6.21] we can identify the space $T_{Y(u,u_y)}\Sigma^p_y$ of infinitesimal non-isospectral (but iso-periodic) deformations of $Y(u,u_y)$ with the space $H^1_R(Y(u,u_y),\Omega)$ via the map $c \mapsto \omega(c) := \frac{\ln \mu}{p} = \delta \ln \mu$. Therefore $\omega$ can be identified with $dY$, the derivative of $Y : M_\Sigma^p \rightarrow \Sigma^p_y$. Due to Proposition [7.2] the map $Y : M_\Sigma^p \rightarrow \Sigma^p_y$ is a submersion. Thus the map $\omega : T_{Y(u,u_y)}\Sigma^p_y \rightarrow H^0_R(Y(u,u_y),\Omega)$ is surjective.

Definition 10.3. Let $L_{(u,u_y)} \subset T_{Y(u,u_y)}M_\Sigma^p$ be the kernel of the the map $\omega : T_{Y(u,u_y)}M_\Sigma^p \rightarrow H^0_R(Y(u,u_y),\Omega)$, i.e. $L_{(u,u_y)} := \ker(\omega)$.

Now we are able to formulate and prove the main result. The proof is based on the ideas and methods presented in the proof of [33], Theorem 7.5.

Theorem 10.4.
(i) There exists an isomorphism of vector spaces $d\Gamma_{(u,u_y)} : H^1_R(Y(u,u_y),\mathcal{O}) \rightarrow L_{(u,u_y)}$.
(ii) For all $[f] \in H^1_R(Y(u,u_y),\mathcal{O})$ and all $(\delta u, \delta u_y) \in T_{(u,u_y)}M_\Sigma^p$ the equation
\[ \Omega(d\Gamma_{(u,u_y)}([f]), (\delta u, \delta u_y)) = i \text{Res}([f] \omega(\delta u, \delta u_y)) \] (10.1)
holds. Here the right hand side is defined as in the Serre Duality Theorem [74].
(iii) $(M_\Sigma^p, Y, H_2)$ is a Hamiltonian system. In particular, $\Omega$ is non-degenerate on $M_\Sigma^p$.

Remark 10.5.
(i) From the Serre Duality Theorem [74], Thm. 17.9] we know that the pairing $\text{Res} : H^1(Y(u,u_y),\mathcal{O}) \times H^0(Y(u,u_y),\Omega) \rightarrow \mathbb{C}$ is non-degenerate.
(ii) $L_{(u,u_y)} \subset T_{(u,u_y)}M_\Sigma^p$ is a maximal isotropic subspace with respect to the symplectic form $\Omega : T_{(u,u_y)}M_\Sigma^p \times T_{(u,u_y)}M_\Sigma^p \rightarrow \mathbb{R}$, i.e. $L_{(u,u_y)}$ is Lagrangian.

Proof.
(i) Let $[f] = [(f_0, \eta^*f_0)] \in H^1_R(Y,\mathcal{O})$ be a cocycle with representative $f_0$ as defined in Lemma [6.6]. Then, defining $A_{f_0}(x) := P_x(f_0) + \sigma^*P_x(f_0)$, we get
\[ A_{f_0}(x) = \sum_{i=0}^{g-1} c_i \lambda^{-i}P_x(\lambda^{-1} \nu) + \sigma^*P_x(\lambda^{-1} \nu) = \sum_{i=0}^{g-1} c_i \lambda^{-i} \zeta_\lambda(x) \]
and also
\[ \delta U_\lambda(x) = [A_{f_0}^+(x), L_\lambda(x)] = [L_\lambda(x), A_{f_0}^-(x)] \]
due to Theorem [6.12]. Moreover,
\[ \delta v(x) = -A_{f_0}^-(x)v(x) \]
holds due to Remark [6.11] with $A_{f_0}^-(x) = -\sum \frac{\langle \delta v_i(x), w_i(x) \rangle}{w_i(x)v_i(x)}$. From Lemma [6.2] we know that in general
\[ \delta U_\lambda(x) = \left[ L_\lambda(x), -\sum_{i=1}^2 \frac{\langle \delta v_i(x), w_i(x) \rangle}{w_i(x)v_i(x)} \right] \]
\[ \quad + \left( P_x \frac{\delta \ln \mu}{p} + \sigma^*P_x \frac{\delta \ln \mu}{p} \right) \]
Since $\delta U_\lambda(x) = [L_\lambda(x), A_{f_0}^-(x)]$ we see that $\delta \ln \mu(\delta u_{f_0}, \delta u_y) = 0$ and consequently
\[ (\delta u_{f_0}, \delta u_y) \in \ker(\omega) \]
Thus we have an injective map $d\Gamma_{(u,u_y)} : H^1_R(Y(u,u_y),\mathcal{O}) \rightarrow L_{(u,u_y)}$. Due to Remark [6.2] we know that $\omega : T_{(u,u_y)}M_\Sigma^p \rightarrow H^0_R(Y(u,u_y),\Omega)$ is surjective. Since
Recall that $U^\lambda = \begin{bmatrix} e^{2i\delta u} & e^{2iu} \\ e^{-iu} & e^{-2i\delta u} \end{bmatrix}$ and consequently we have

\[
\delta U^\lambda = \frac{1}{2} \begin{bmatrix}
-i\delta u_y & i\lambda^{-1}e^u + ie^{-u} \\
-i\lambda^{-1}e^{-u} & -i\delta u_y
\end{bmatrix}.
\]

We can now use the above equations in order to obtain relations on the coefficients $B^{-}_0$ and $B^{-}_1$ of $B^{-} = \sum_{i\geq 0} \lambda^i B^{-}_i$ where $B^{-}_0$ is of the form

\[
B^{-}_0 = \begin{pmatrix} h_0 & e_0 \\ 0 & -h_0 \end{pmatrix}
\]
and $B^{-}_1$ is of the form $B^{-}_1 = \begin{pmatrix} h_1 & e_1 \\ f_1 & -h_1 \end{pmatrix}$. Since $\delta U^{-}_1 = [B^{-}_0, U^{-}_1]$ a direct calculation yields

\[
B^{-}_0 = \begin{pmatrix} \frac{1}{2} \delta u & e_0 \\ 0 & -\frac{1}{2} \delta u \end{pmatrix}.
\]

Moreover, the sum $[B^{-}_1, U^{-}_1] + [B^{-}_0, U_0]$ is given by

\[
\begin{pmatrix}
\frac{\lambda}{2} e_0 e^{-u} - \frac{1}{2} f_1 e^u & i h_1 e^u + i e_0 u_y - \frac{1}{2} e^{-u} \delta u \\
-\frac{1}{2} e^{-u} \delta u & -\frac{i}{2} e_0 e^{-u} + \frac{1}{2} f_1 e^u
\end{pmatrix}.
\]

For the diagonal entry of $[B^{-}_1, U^{-}_1] + [B^{-}_0, U_0]$ the $\oplus$-part is given by the imaginary part and therefore

\[
-\frac{i}{2} \delta u_y = \frac{i}{2} \Re(e_0) e^{-u} - \frac{i}{2} \Re(f_1) e^u.
\]

Thus we get

\[
\delta u_y = \Re(f_1) e^u - \Re(e_0) e^{-u}.
\]
For $B^{-}(x) := A_{f_{0}}^{-}(x)$ we obtain

$$\exists \left( \text{tr}(\delta U_{0} A_{f_{0},0}^{-}) + \text{tr}(\delta U_{-1} A_{f_{0},1}^{-}) \right) = -\delta u_{g} \Re(h_{0}) + \frac{i}{2} \delta u \left( \Re(f_{1}) e^{u} - \Re(c_{0}) e^{-u} \right)$$

Now a direct calculation gives

$$\frac{1}{2} \Omega(d\Gamma_{(u,u_{g})}([f]), (\delta u, \delta u_{g})) = \frac{1}{2} \int_{0}^{P} (\delta u_{f_{0}} \delta u_{g} - \delta u \delta u_{f_{0}}) dx$$

Setting $\hat{P}_{x}(\delta \ln \mu) := P_{x}(\delta \ln \mu) + \sigma^{*} P_{x}(\delta \ln \mu)$, we further obtain

$$\int_{0}^{P} \langle [L_{\lambda}(x), B^{-}(x)], A_{f_{0}}(x) \rangle_{A} dx$$

Recall, that $\text{tr}([B^{-}(x), L_{\lambda}(x)] \cdot A_{f_{0}}(x)) = \text{tr}(B^{-}(x) \cdot [L_{\lambda}(x), A_{f_{0}}(x)])$. Moreover, there holds $[L_{\lambda}(x), A_{f_{0}}(x)] = 0$ and we get

$$\int_{0}^{P} \langle B^{-}(x), [L_{\lambda}(x), A_{f_{0}}(x)] \rangle_{A} dx$$
Writing out the last equation yields
\[
\Omega(d\Gamma_{(u,u_y)}([f]), (\delta u, \delta u_y)) = \frac{2}{P} \int_0^P \Im \left( \text{Res}_{y_0} \frac{d\lambda}{\lambda} \text{tr} \left( A_{f_0}(x) \hat{P}_x (\delta \ln \mu) \right) \right) dx
\]
\[
= -2 \Im \left( \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) \right)
\]
\[
i \left( \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) - \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) \right)
\]
\[
i \left( \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) - \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) \right)
\]
\[
i \left( \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) + \text{Res}_{y_0} (f_0 \cdot \delta \ln \mu) \right)
\]
and thus
\[
\Omega(d\Gamma_{(u,u_y)}([f]), (\delta u, \delta u_y)) = i \text{Res}([f] \omega(\delta u, \delta u_y)).
\]

(iii) In order to prove (iii) we have to show that \( \Omega \) is non-degenerate on \( M^P_\mathbb{R} \). From equation (10.1) we know that
\[
\Omega((\delta u, \delta u_y), (\delta u, \delta u_y)) = 0 \text{ for } (\delta u, \delta u_y), (\delta u, \delta u_y) \in L_{(u,u_y)} = \ker(\omega).
\]
Moreover, \( \omega : T_{(u,u_y)}M^P_\mathbb{R} \to H^0_\mathbb{R}(Y(u,u_y), \Omega) \) is surjective since \( \text{dim } T_{(u,u_y)}M^P_\mathbb{R} = 2g \) and \( \text{dim } \ker(\omega) = g = \text{dim } H^0_\mathbb{R}(Y(u,u_y), \Omega). \) Thus we have
\[
T_{(u,u_y)}M^P_\mathbb{R}/\ker(\omega) \simeq H^0_\mathbb{R}(Y(u,u_y), \Omega)
\]
and there exists a basis \( \{ \delta a_1, \ldots, \delta a_g, \delta b_1, \ldots, \delta b_g \} \) of \( T_{(u,u_y)}M^P_\mathbb{R} \) such that
\[
\text{span} \{ \delta a_1, \ldots, \delta a_g \} = \ker(\omega) \text{ and } \omega[\text{span} \{ \delta b_1, \ldots, \delta b_g \}] = H^0_\mathbb{R}(Y(u,u_y), \Omega).
\]
Now \( L_{(u,u_y)} = \ker(\omega) \simeq H^1_\mathbb{R}(Y(u,u_y), \mathcal{O}) \) and since the pairing from Serre duality is non-degenerate we obtain with equation (10.1) (after choosing the appropriate basis)
\[
\Omega(\delta a_i, \delta b_j) = \delta_{ij} \text{ and } \Omega(\delta b_i, \delta a_j) = -\delta_{ij}.
\]
Summing up the matrix representation \( B_\Omega \) of \( \Omega \) on \( T_{(u,u_y)}M^P_\mathbb{R} \) has the form
\[
B_\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
and thus \( \Omega \) is of full rank. This shows (iii) and concludes the proof of Theorem 10.4. \( \square \)

References

[1] M. Audin. Hamiltonian systems and their integrability. Providence, RI: American Mathematical Society (AMS); Paris: Société Mathématique de France (SMF), 2008.
[2] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skij, A. R. Its, and V. B. Matveev. Algebro-geometric approach to nonlinear integrable equations. Berlin: Springer-Verlag, 1994.
[3] A. Bobenko and S. Kuksin. Small-amplitude solutions of the sine-Gordon equation on an interval under Dirichlet or Neumann boundary conditions. J. Nonlinear Sci., 5(3):207–232, 1995.
[4] A. I. Bobenko. All constant mean curvature tori in \( \mathbb{R}^3, S^3 \) and \( H^3 \) in terms of theta-functions. Math. Ann., 290(2):209–245, 1991.
[5] A. I. Bobenko. Constant mean curvature surfaces and integrable equations. Russ. Math. Surv., 46(4):1–45, 1991.
[6] A. I. Bobenko. Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. In Fordy, Allan P. (ed.) et al., Harmonic maps and integrable systems. Based on conference, held at Leeds, GB, May 1992. Braunschweig: Vieweg. Aspects Math. E23, 83-127, 1994.
[7] A. I. Bobenko. Exploring surfaces through methods from the theory of integrable systems: the Bonnet problem. In Surveys on geometry and integrable systems. Based on the conference on integrable systems in differential geometry, Tokyo, Japan, July 17–21, 2000, pages 1–53. Tokyo: Mathematical Society of Japan, 2008.

[8] F. Burstall and F. Pedit. Harmonic maps via Adler-Kostant-Symes theory. In Harmonic maps and integrable systems. Based on conference, held at Leeds, GB, May 1992, pages 221–272. Braunschweig: Vieweg, 1994.

[9] F. Burstall and F. Pedit. Dressing orbits of harmonic maps. Duke Math. J., 80(2):353–382, 1995.

[10] E. Carberry and M. U. Schmidt. The Closure of Spectral Data for Constant Mean Curvature Tori in $\mathbb{S}^3$. To appear in J. Reine Angew. Math., Feb. 2012.

[11] J. Dorfmeister, F. Pedit, and H. Wu. Weierstrass type representation of harmonic maps into symmetric spaces. Commun. Anal. Geom., 6(4):633–668, 1998.

[12] J. Dorfmeister and H. Wu. Constant mean curvature surfaces and loop groups. J. Reine Angew. Math., 440:43–76, 1993.

[13] V. Drinfel’d. Quantum groups. Proc. Int. Congr. Math., Berkeley/Calif. 1986, Vol. 1, 798-820, 1987.

[14] O. Forster. Lectures on Riemann surfaces. Graduate Texts in Mathematics, Vol. 81. New York - Heidelberg -Berlin: Springer-Verlag, VIII, 1981.

[15] A. Gerding, F. Pedit, and N. Schmitt. Constant mean curvature surfaces: an integrable systems perspective. In Harmonic maps and differential geometry. A harmonic map fest in honour of John C. Wood’s 60th birthday, Cagliari, Italy, September 7–10, 2009, pages 7–39. Providence, RI: American Mathematical Society (AMS), 2011.

[16] P. G. Grinevich and M. U. Schmidt. Period preserving nonisospectral flows and the moduli space of periodic solutions of soliton equations. Physica D, 87(1-4):73–98, 1995.

[17] G. Haak, M. Schmidt, and R. Schrader. Group theoretic formulation of the Segal-Wilson approach to integrable systems with applications. Rev. Math. Phys., 4(3):451–499, 1992.

[18] L. Hauswirth, M. Kilian, and M. U. Schmidt. Finite type minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. To appear in Illinois J. Math., Oct. 2012.

[19] L. Hauswirth, M. Kilian, and M. U. Schmidt. Properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. ArXiv e-print 1210.5953, Oct. 2012.

[20] L. Hauswirth, M. Kilian, and M. U. Schmidt. The geometry of embedded constant mean curvature tori in the 3-sphere via integrable systems. ArXiv e-print 1309.4278, Sept. 2013.

[21] M. Kilian and M. U. Schmidt. On the moduli of constant mean curvature cylinders of finite type in the 3-sphere. ArXiv e-print 0712.0108v2, Dec. 2007.

[22] M. Kilian and M. U. Schmidt. On infinitesimal deformations of CMC surfaces of finite type in the 3-sphere. In Riemann surfaces, harmonic maps and visualization. Proceedings of the 16th Osaka City University International Academic Symposium, Osaka, Japan, December 15–20, 2008, pages 231–248. Osaka: Osaka Municipal Universities Press, 2010.

[23] M. Knopf. Periodic solutions of the sinh-Gordon equation and integrable systems. PhD thesis, 2013.

[24] I. M. Krichever. Methods of algebraic geometry in the theory of nonlinear equations. Russ. Math. Surv., 32(6):185–213, 1977.

[25] S. Kuksin. Analysis of Hamiltonian PDE’s. Oxford: Oxford University Press, 2000.

[26] H. Lewy. Neuer Beweis des analytischen Charakters der Lösungen elliptischer Differentialgleichungen. Math. Ann., 101:699–619, 1929.

[27] I. McIntosh. Global solutions of the elliptic 2D periodic Toda lattice. Nonlinearity, 7(1):85–108, 1994.

[28] I. McIntosh. Harmonic tori and their spectral data. In Surveys on geometry and integrable systems. Based on the conference on integrable systems in differential geometry, Tokyo, Japan, July 17–21, 2000, pages 285–314. Tokyo: Mathematical Society of Japan, 2008.

[29] H. P. McKean. The sine-Gordon and sinh-Gordon equations on the circle. Commun. Pure Appl. Math., 34:197–257, 1981.

[30] U. Pinkall and I. Sterling. On the classification of constant mean curvature tori. Ann. Math. (2), 130(2):407–451, 1989.

[31] A. Pressley and G. Segal. Loop groups. Oxford (UK): Clarendon Press, repr. with corrections edition, 1988.
M. U. Schmidt. *Integrable systems and Riemann surfaces of infinite genus*. Memoirs of the American Mathematical Society Series, Number 581, 1996.

Institut für Mathematik, Universität Mannheim, 68131 Mannheim, Germany.

E-mail address: knopf@math.uni-mannheim.de