Canonical structure of renormalization group equations and separability of Hamiltonian systems

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Abstract. We investigate perturbed Hamiltonian systems with two degrees of freedom by renormalization group method, which derives a reduced equation called renormalization group equation (RGE) by handling secular terms. We found that RGE is not always a Hamiltonian system. The necessary and sufficient condition that RGE becomes a Hamiltonian system up to the second leading order of a small parameter is that the original system is separable by Cartesian coordinates. Moreover, RGE is integrable when it is a Hamiltonian system. These statements are partial generalizations of our previous paper.
1. Introduction

To understand the nature of non-linearity in dynamical systems, it is indispensable to investigate temporal evolutions of the system. Particularly, in Hamiltonian systems, slow relaxation of $1/f$ spectra is observed in systems with two [1, 2] and many [3, 4] degrees of freedom, and understanding its cause is an important subject of Hamiltonian dynamical systems. The slow relaxation is connected with self-similar hierarchical structure of phase space, which is observed in a 2-dimensional mapping [5]. The structure is also supposed in systems with many degrees of freedom with the aid of Kolmogorov-Arnold-Moser theorem [3] and Poincaré-Birkhoff theorem [7]. By assuming self-similar hierarchical structure and transition probabilities between levels of the hierarchy, some abstract models [8, 9, 10] are proposed to understand the slow relaxation. The abstract models catch universal properties of Hamiltonian systems, but they are not available to investigate details of individual systems owing to their assumptions. For instance, Poincaré recurrences $P(t)$ (integrated probability to return into a given region after a time larger than $t$) decays with $t$ as $P(t) \sim 1/t^p$, and the exponent $p$ takes different values in different 2-dimensional mappings [2]. One approach to grasp such individual properties is to numerically calculate equations of motion, and another approach is to analytically construct approximate solutions. In this paper we focus on the latter approach in nearly integrable systems with perturbation techniques.

Naive perturbation often yields secular terms due to resonances and they invalidate perturbative expansion. This problem of secular terms is dealt with singular perturbation techniques [11], e.g., averaging methods, multiple scale methods and matched asymptotic expansions. However, these techniques are not easy to use because we must select a suitable assumption about the structure of perturbation series. Recently, perturbative renormalization group method [12, 13] is proposed as a tool for global asymptotic analysis of the solution of differential equations. It unifies the methods listed above, and can treat many systems irrespective of their features. Renormalization group method omits fast motion and constructs reduced equations of motion from secular terms. The reduced equation is called renormalization group equation (RGE), which is geometrically regarded as an envelope equation.

Although RGE gives approximate but global solutions to original systems, we have one question: whether does RGE keep properties of original systems? We miss characteristic behaviors of orbits in phase space if RGE becomes a dissipative system which often has attractors and orbits exponentially converge to them. In contrast, Hamiltonian systems never have attractors and orbits eternally wander.

In the previous paper [14], we treated a Hamiltonian system with a homogeneous cubic or quartic potential function added as perturbation, and showed that if the RGE becomes a Hamiltonian system then the original Hamiltonian system is integrable. By
investigating properties of RGE, we can justify integrability of the original Hamiltonian system. The present paper aim to extend the result for general perturbation function $V(q_1, q_2)$. Moreover, we mention integrability and an integral of RGE.

The plan of this paper is as follows. In section 2 we review renormalization group method by using a simple example. The main theorem is shown in section 3, and it is proved in sections 4, 5 and 6. Section 7 is devoted to summary and discussions.

2. Renormalization Group Method

Here we review renormalization group method by using a simple system. Let us consider the system

$$H(q, p) = H_0(q, p) + \epsilon V(q),$$

where

$$H_0(q, p) = \frac{1}{2}(p^2 + q^2), \quad V(q) = \frac{1}{2}q^2,$$

and $\epsilon$ is a small parameter. The equation of motion is

$$\frac{d^2q}{dt^2} + q = -\epsilon q,$$

and the exact solution is

$$q = B_0 \cos(\sqrt{1 + \epsilon} \ t) + C_0 \sin(\sqrt{1 + \epsilon} \ t),$$

where $B_0$ and $C_0$ are constants of integration and determined by initial condition at the initial time $t = t_0$.

We perturbatively solve equation (3) by expanding $q$ as a series of positive powers of $\epsilon$:

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots.$$  \hspace{1cm} (5)

This naive expansion gives

$$q(t; t_0, B_0, C_0) = B_0 \cos t + C_0 \sin t + \frac{\epsilon}{2}(t - t_0)(C_0 \cos t - B_0 \sin t)$$

$$- \frac{\epsilon^2}{8} \left[(t - t_0)(C_0 \cos t - B_0 \sin t) + (t - t_0)^2(B_0 \cos t + C_0 \sin t)\right]$$

$$+ O(\epsilon^3).$$  \hspace{1cm} (6)

We ignored other homogeneous parts of $q_j \ (j \geq 1)$, which are kernel of the linear operator $L \equiv \frac{d^2}{dt^2} + 1$, since they can be included in $q_0$. This naive expansion breaks when $\epsilon(t - t_0) \geq 1$ because amplitude of $\epsilon q_1$ exceeds one of $q_0$. 

To remove the secular terms, we regard $B_0$ and $C_0$ as functions of the initial time $t_0$. The temporal evolutions of $B(t_0)$ and $C(t_0)$ are determined by the following equation\cite{12, 13}:

$$\left. \frac{\partial q}{\partial t_0} \right|_{t_0=t} = 0, \quad \forall t. \quad (7)$$

From the equation (7), we obtain the RGE

$$\frac{dB(t)}{dt} = \left( \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \right) C(t) + O(\epsilon^3),$$

$$\frac{dC(t)}{dt} = -\left( \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \right) B(t) + O(\epsilon^3), \quad (8)$$

where $B(t_0) = B_0$ and $C(t_0) = C_0$. The renormalized solution $q^{RG}$ is

$$q^{RG} = q(t; t, B(t), C(t)) = B_0 \cos \left( \left( 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \right) t \right) + C_0 \sin \left( \left( 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \right) t \right) + O(\epsilon^3), \quad (9)$$

and this solution is the same as the exact solution (4) up to $O(\epsilon^2)$.

Let us check whether the reduced equation (8) reflects properties of the original system (1), which are (i) symplectic properties, (ii) integrability and (iii) the form of an integral. In equation (8), $B$ and $C$ are canonical conjugate variables and the equation is governed by the Hamiltonian

$$H^{RG} = \frac{1}{2} \left( \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \right) (B^2 + C^2), \quad (10)$$

and canonical equations

$$\frac{dB(t)}{dt} = \frac{\partial H^{RG}}{\partial C},$$

$$\frac{dC(t)}{dt} = -\frac{\partial H^{RG}}{\partial B}. \quad (11)$$

Equation (8) is therefore integrable and an integral of the system is $B^2 + C^2$ which corresponds to $H_0$. As shown in section 4 these statements hold in any Hamiltonian systems with one degree of freedom whose integrable part is a harmonic oscillator. What happens in systems with two degrees of freedom where chaos may appear? This is the main question of this article and we show a part of the answer in the next section.

3. Main theorem

In this section we present the main theorem.
**Theorem 1** Let Hamiltonian be represented as follows:

$$H(q_1, q_2, p_1, p_2) = H_0(q_1, q_2, p_1, p_2) + \epsilon V(q_1, q_2),$$  \hspace{1cm} (12)

where

$$H_0(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2),$$  \hspace{1cm} (13)

$\epsilon$ is a small parameter and the potential $V(q_1, q_2)$ is an analytic function which has only even order terms of $q_1$ and $q_2$, for instance, $q_1^2 q_2^3$. We perturbatively solve the equation of motion by setting the 0-th order solution as

$$q_{j0}(t) = B_j \cos t + C_j \sin t. \quad (j = 1, 2) \hspace{1cm} (14)$$

Then the following two conditions (A) and (B) are equivalent.

(A) The renormalization group equation (RGE) of the system (12) is a Hamiltonian system up to the second order of $\epsilon$.

(B) The original system (12) is separable by Cartesian coordinates, that is, rotation of the coordinates $q_1$ and $q_2$. Moreover, when (A) (or (B)) is satisfied, the RGE is integrable and $(B_1^2 + C_1^2 + B_2^2 + C_2^2)/2$ is an integral which corresponds to $H_0$.

Remark 1: RGE for the system (12) is always a Hamiltonian system up to $O(\epsilon^4)$. Hamiltonian of the RGE is the same as one obtained by canonical perturbation theory \[13, 16\] and the Hamiltonian is time-average of $V(q_{10}, q_{20})$ as shown in subsection 4.2.

Remark 2: RGE is an equation for variables which are constants of integration introduced in the 0-th order solution, and hence the form of RGE changes when we use different form of the 0-th order solution, for instance,

$$q_{j0}(t) = \sqrt{2A_j} \cos(t + \theta_j).$$ \hspace{1cm} (15)

However, the theorem holds for new variables if they are canonical conjugate pair, since constructing RGE and changing constants of integration are commutable operations.

Remark 3: We excluded odd order terms of $q_1$ and $q_2$ from the perturbation $V(q_1, q_2)$. We must construct RGE up to $O(\epsilon^4)$ if $V(q_1, q_2)$ has odd order terms (c.f. \[14\]) and need longer calculation. But the extension is straightforward.

We prove this theorem 1 in sections 4, 5 and 6. Sections 4 and 5 are devoted to prove (A)$\Rightarrow$(B) and (B)$\Rightarrow$(A), respectively. Section 6 is for integrability and the form of an integral.
4. Proof of (A) \(\implies\) (B)

We first derive RGE up to \(O(\epsilon^2)\) for system (12), and we investigate conditions that RGE is a Hamiltonian system. Then we compare the conditions with separability conditions of system (12).

4.1. Renormalization Group Equation

Equation of motion in system (12) is
\[
\ddot{q}_1 + q_1 = -\epsilon V_1(q_1, q_2), \\
\ddot{q}_2 + q_2 = -\epsilon V_2(q_1, q_2),
\]
where the subscripts of \(V\) represent partial derivatives with the variables, namely, \(V_1 = \partial V/\partial q_1\) and \(V_{12} = \partial^2 V/\partial q_1 \partial q_2\). We expand \(q_1\) and \(q_2\) as power series of \(\epsilon\),
\[
q_1 = q_{10} + \epsilon q_{11} + \epsilon^2 q_{12} + \cdots, \\
q_2 = q_{20} + \epsilon q_{21} + \epsilon^2 q_{22} + \cdots.
\]
Equations of motion in each order of \(\epsilon\) are
\[
\ddot{q}_{j0} + q_{j0} = 0, \\
\ddot{q}_{j1} + q_{j1} = -V_j(q_0), \\
\ddot{q}_{j2} + q_{j2} = -\sum_{k=1}^{2} V_{jk}(q_0)q_{k1},
\]
where \(q_0 = (q_{10}, q_{20})\) and \(j = 1, 2\). Following an assumption described in theorem 1, we set the 0-th order solution as
\[
q_{j0} = B_j \cos t + C_j \sin t. \quad (j = 1, 2)
\]
Since \(V_j(q_0)\) is a periodic function of time with period \(2\pi\), we expand it by Fourier series:
\[
-V_j(q_0) = u(0)_{j} + \sum_{n=1}^{\infty} \left( u^{(n)}_{j} \cos nt + v^{(n)}_{j} \sin nt \right).
\]
Then solution to \(O(\epsilon^1)\) is
\[
q_{j1} = u^{(0)}_{j} + \frac{t - t_0}{2}(u^{(1)}_{j} \sin t - v^{(1)}_{j} \cos t) + \sum_{n=2}^{\infty} \frac{1}{1 - n^2}(u^{(n)}_{j} \cos nt + v^{(n)}_{j} \sin nt),
\]
where \(t_0\) is the initial time.

Let us proceed to solution to \(O(\epsilon^2)\). We also expand \(V_{jk}(q_0)\) by Fourier series,
\[
-V_{jk}(q_0) = u^{(0)}_{jk} + \sum_{n=1}^{\infty} (u^{(n)}_{jk} \cos nt + v^{(n)}_{jk} \sin nt).
\]
Terms in the right-hand-side of equation (20) which make secular terms in $q_{j2}$ are
\begin{align}
- \sum_{k=1}^{2} V_{jk}(\bar{\eta}_0)q_{k1} = G_j t \cos t + H_j t \sin t + K_j \cos t + J_j \sin t,
\end{align}
where, by using relations between Fourier components shown in Appendix A,
\begin{align}
G_j &= 2 \sum_{k=1}^{2} \left( \frac{\partial u^{(0)}}{\partial B_k} \frac{\partial^2 u^{(0)}}{\partial B_j \partial C_k} - \frac{\partial u^{(0)}}{\partial C_k} \frac{\partial^2 u^{(0)}}{\partial B_j \partial B_k} \right), \\
H_j &= 2 \sum_{k=1}^{2} \left( \frac{\partial u^{(0)}}{\partial B_k} \frac{\partial^2 u^{(0)}}{\partial C_j \partial C_k} - \frac{\partial u^{(0)}}{\partial C_k} \frac{\partial^2 u^{(0)}}{\partial C_j \partial B_k} \right), \\
K_j &= \frac{\partial}{\partial B_j} \left\{ (u_1^{(0)})^2 + (u_2^{(0)})^2 + \sum_{n=2}^{\infty} \frac{1}{1-n^2} \left[ (u_1^{(n)})^2 + (v_1^{(n)})^2 + (u_2^{(n)})^2 + (v_1^{(n)})^2 \right] \right\}, \\
J_j &= \frac{\partial}{\partial C_j} \left\{ (u_1^{(0)})^2 + (u_2^{(0)})^2 + \sum_{n=2}^{\infty} \frac{1}{1-n^2} \left[ (u_1^{(n)})^2 + (v_1^{(n)})^2 + (u_2^{(n)})^2 + (v_1^{(n)})^2 \right] \right\}.
\end{align}

Solution to $O(\epsilon^2)$ is written with these symbols as
\begin{align}
q_{j2} &= (t - t_0) \left[ \left( \frac{G_j}{4} - \frac{J_j}{2} \right) \cos t + \left( \frac{H_j}{4} + \frac{K_j}{2} \right) \sin t \right] \\
&\quad + (t - t_0)^2 \left[ -\frac{H_j}{4} \cos t + \frac{G_j}{4} \sin t \right] + \text{(non-secular terms)}.
\end{align}

From the secular terms of equations (23) and (30), we obtain RGE as
\begin{align}
\dot{B}_j &= -\frac{\epsilon}{2} v_j^{(1)} + \epsilon^2 \left( \frac{G_j}{4} - \frac{J_j}{2} \right) = \frac{\partial \Phi}{\partial C_j} + \frac{\epsilon^2}{4} G_j, \\
\dot{C}_j &= \frac{\epsilon}{2} u_j^{(1)} + \epsilon^2 \left( \frac{H_j}{4} + \frac{K_j}{2} \right) = -\frac{\partial \Phi}{\partial B_j} + \frac{\epsilon^2}{4} H_j,
\end{align}
where the potential $\Phi$ is defined by
\begin{align}
\Phi = -\epsilon u^{(0)} - \frac{\epsilon^2}{2} \left\{ (u_1^{(0)})^2 + (u_2^{(0)})^2 + \sum_{n=2}^{\infty} \frac{1}{1-n^2} \left[ (u_1^{(n)})^2 + (v_1^{(n)})^2 + (u_2^{(n)})^2 + (v_1^{(n)})^2 \right] \right\},
\end{align}
and $u^{(0)}$ is the constant component of Fourier series of $-V(\bar{\eta}_0)$
\begin{align}
u^{(0)} = -\frac{1}{2\pi} \int_{0}^{2\pi} V(q_{10}, q_{20}) \, dt.
\end{align}

4.2. Conditions that RGE is a Hamiltonian system

We show the conditions that the RGE (31) becomes a Hamiltonian system. The conditions are
\begin{align}
\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0,
\end{align}
where $\Delta_l (l = 1, 2, \cdots, 6)$ are defined by

$$
\Delta_1 \equiv \frac{\partial B_1}{\partial B_1} + \frac{\partial C_1}{\partial C_1}, \quad \Delta_2 \equiv \frac{\partial B_2}{\partial B_2} + \frac{\partial C_2}{\partial C_2}, \quad \Delta_3 \equiv \frac{\partial C_1}{\partial B_1} - \frac{\partial C_2}{\partial B_2}, \\
\Delta_4 \equiv \frac{\partial B_1}{\partial C_2} - \frac{\partial B_2}{\partial C_1}, \quad \Delta_5 \equiv \frac{\partial B_1}{\partial C_1} + \frac{\partial B_2}{\partial C_2}, \quad \Delta_6 \equiv \frac{\partial C_1}{\partial B_2} + \frac{\partial B_2}{\partial C_1}.
$$

(35)

The term written by $\Phi$ in equation (31) always satisfies conditions (34). Accordingly, whether RGE is a Hamiltonian system is determined by $G_j$ and $H_j$, which originates in secular terms of the 1-st order solution. From equations (26) and (27), the concrete forms of $\Delta_l$’s become

$$
\Delta_1 = \frac{\epsilon^2}{2} \sum_{k=1}^{2} \left( \frac{\partial u^{(0)}}{\partial B_k} \frac{\partial}{\partial C_k} - \frac{\partial u^{(0)}}{\partial C_k} \frac{\partial}{\partial B_k} \right) \left( \frac{\partial^2}{\partial B_k^2} + \frac{\partial^2}{\partial C_k^2} \right) u^{(0)}, \\
\Delta_2 = \frac{\epsilon^2}{2} \sum_{k=1}^{2} \left( \frac{\partial u^{(0)}}{\partial B_k} \frac{\partial}{\partial C_k} - \frac{\partial u^{(0)}}{\partial C_k} \frac{\partial}{\partial B_k} \right) \left( \frac{\partial^2}{\partial B_k^2} + \frac{\partial^2}{\partial C_k^2} \right) u^{(0)}, \\
\Delta_3 = \Delta_4 = \frac{\epsilon^2}{2} \left[ \frac{\partial^2 u^{(0)}}{\partial B_1^2} \left( \frac{\partial^2}{\partial C_1^2} - \frac{\partial^2}{\partial C_2^2} \right) u^{(0)} - \frac{\partial^2 u^{(0)}}{\partial C_1 C_2} \left( \frac{\partial^2}{\partial B_1^2} - \frac{\partial^2}{\partial B_2^2} \right) u^{(0)} \right], \\
\Delta_5 = \frac{\epsilon^2}{2} \left( \frac{\partial u^{(0)}}{\partial B_1} \frac{\partial}{\partial C_1} - \frac{\partial u^{(0)}}{\partial C_1} \frac{\partial}{\partial B_1} + \frac{\partial u^{(0)}}{\partial B_2} \frac{\partial}{\partial C_2} - \frac{\partial u^{(0)}}{\partial C_2} \frac{\partial}{\partial B_2} \right) \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} \right) u^{(0)}, \\
\Delta_6 = \frac{\epsilon^2}{2} \left[ \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} \right) u^{(0)} \right] \left( \frac{\partial^2}{\partial B_2^2} + \frac{\partial^2}{\partial C_2^2} \right) u^{(0)} - \frac{\partial^2 u^{(0)}}{\partial B_1 \partial C_2} \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} - \frac{\partial^2}{\partial B_2^2} - \frac{\partial^2}{\partial C_2^2} \right) u^{(0)} \right],
$$

(36)

where

$$
\Delta_5 = \Delta_5' + \Delta_6' , \\
\Delta_6 = \Delta_5' - \Delta_6' .
$$

(37)

Note that RGE is always a Hamiltonian system up to $O(\epsilon^1)$ since terms of $O(\epsilon^1)$ do not appear in $\Delta_l$’s. In this order, a Hamiltonian of the RGE becomes time-average of $V(q_0)$:

$$
H_1^{RG} = -u^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} dt \ V(q_0),
$$

(38)

and $H_1^{RG}$ is the same as the one obtained by canonical perturbation theory [13, 16].

4.3. Condition that the original system is separable

In this subsection we clarify the condition that the original system [12] is separable by rotation of the coordinates $(q_1, q_2)$. The integrable part $H_0$ is always separable by any
rotation, and hence we only have to consider the perturbation \( V \). In the next subsection, we compare the condition written by \( u(0) \) with equation (36).

**Theorem 2** The condition that the original system (12) becomes separable by rotation of the coordinates \((q_1, q_2)\) is

\[
c_1(V_{11} - V_{22}) + 2c_2V_{12} = 0. \quad (c_1^2 + c_2^2 = 1)
\]

(39)

In particular, when the potential \( V(q_1, q_2) \) has only even order terms of \( q_1 \) and \( q_2 \), the condition (39) is equivalent to each of the following three conditions:

\[
(I) \quad \left[ c_1 \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} - \frac{\partial^2}{\partial B_2^2} - \frac{\partial^2}{\partial C_2^2} \right) + 2c_2 \left( \frac{\partial^2}{\partial B_1 \partial B_2} + \frac{\partial^2}{\partial C_1 \partial C_2} \right) \right] u(0) = 0,
\]

(40)

\[
(II) \quad \left[ c_1 \left( \frac{\partial^2}{\partial B_1^2} - \frac{\partial^2}{\partial C_1^2} + \frac{\partial^2}{\partial B_2^2} + \frac{\partial^2}{\partial C_2^2} \right) + 2c_2 \left( \frac{\partial^2}{\partial B_1 \partial B_2} - \frac{\partial^2}{\partial C_1 \partial C_2} \right) \right] u(0) = 0,
\]

(41)

\[
(III) \quad \left[ c_1 \left( \frac{\partial^2}{\partial B_1 \partial C_1} - \frac{\partial^2}{\partial B_2 \partial C_2} \right) + c_2 \left( \frac{\partial^2}{\partial B_1 \partial C_2} + \frac{\partial^2}{\partial C_1 \partial B_2} \right) \right] u(0) = 0.
\]

(42)

**Corollary 3** From (I) and (II), the two conditions

\[
(I') \quad c_1 \left( \frac{\partial^2}{\partial B_1^2} - \frac{\partial^2}{\partial C_1^2} \right) + 2c_2 \left( \frac{\partial^2}{\partial B_1 \partial B_2} \right) u(0) = 0,
\]

(43)

\[
(II') \quad c_1 \left( \frac{\partial^2}{\partial C_1^2} - \frac{\partial^2}{\partial C_2^2} \right) + 2c_2 \left( \frac{\partial^2}{\partial C_1 \partial C_2} \right) u(0) = 0,
\]

(44)

are also equivalent to each of (I), (II) and (III).

Equation (39) is a special case of Darboux equation (17), which is used in the previous paper [14]. To prove theorem 2, we introduce the following lemma.

**Lemma 4** Let us assume that \( V(q_1, q_2) \) has only even order terms of \( q_1 \) and \( q_2 \) and \( V(q_1, q_2) \) is not constant. We define the function \( F(q_1, q_2) \) as

\[
F(q_1, q_2) \equiv c_1[V_{11}(q_1, q_2) - V_{22}(q_1, q_2)] + 2c_2V_{12}(q_1, q_2),
\]

(45)

then

\[
(a) \quad F(q_1, q_2) = 0 \quad \text{for } \forall q_1, \forall q_2
\]

(46)

\[
\Leftrightarrow (b) \quad \int_0^{2\pi} dt F(q_{10}, q_{20}) = 0 \quad \text{for } \forall B_1, \forall C_1, \forall B_2, \forall C_2
\]

(47)

\[
\Leftrightarrow (c) \quad \int_0^{2\pi} dt F(q_{10}, q_{20}) \cos 2t = 0 \quad \text{for } \forall B_1, \forall C_1, \forall B_2, \forall C_2
\]

(48)

\[
\Leftrightarrow (d) \quad \int_0^{2\pi} dt F(q_{10}, q_{20}) \sin 2t = 0 \quad \text{for } \forall B_1, \forall C_1, \forall B_2, \forall C_2
\]

(49)
Proof of Lemma 4:
The direction (a) \(\Rightarrow\) (b),(c),(d) is obvious, and hence we prove the reverse direction, that is, (b),(c),(d) \(\Rightarrow\) (a). We may set \(C_1 = C_2 = 0\) since (b),(c) and (d) are satisfied for any values of \(B_j\) and \(C_j\). Then \(F(q_{10}, q_{20})\) is expanded as

\[
F(q_{10}, q_{20}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{F^{(m,n)}(0,0)}{m!n!} B_1^m B_2^n \cos^{m+n} t,
\]

where

\[
F^{(m,n)}(0,0) = \frac{\partial^{m+n} F}{\partial q_1^m \partial q_2^n}(0,0).
\]

As a result, in case of (b),

\[
0 = \int_0^{2\pi} F(q_{10}, q_{20}) dt = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{F^{(m,n)}(0,0)}{m!n!} B_1^m B_2^n \times \begin{cases} 2\pi 2^{-m-n} C_{(m+n)/2} & (m + n : \text{even}) \\ 0 & (m + n : \text{odd}) \end{cases} (52)
\]

and \(F^{(m,n)}(0,0) = 0\) for even \(m + n\). Consequently, \(F(q_1, q_2) = 0\) because \(F(q_1, q_2)\) has only even order terms of \(q_1\) and \(q_2\). We can prove (c), (d) \(\Rightarrow\) (a) through similar ways. \(\blacksquare\)

Now let us prove theorem 2.

Proof of Theorem 2: Let us prove the first half of theorem 2. We rotate coordinates from \((q_1, q_2)\) to \((\tilde{q}_1, \tilde{q}_2)\) as

\[
q_1 = \tilde{q}_1 \cos \theta + \tilde{q}_2 \sin \theta,
q_2 = -\tilde{q}_1 \sin \theta + \tilde{q}_2 \cos \theta.
\]

The separability condition is that there exists a real value of \(\theta\) such that

\[
\frac{\partial^2 V}{\partial \tilde{q}_1 \partial \tilde{q}_2} = 0, \quad \text{for } \forall \tilde{q}_1, \forall \tilde{q}_2.
\]

This condition is equivalent to

\[
\left(V_{11} - V_{22}\right) \sin 2\theta + 2V_{12} \cos 2\theta = 0, \quad \text{for } \forall q_1, \forall q_2,
\]

and consequently, we have proved the first half of theorem 2.

From relations between Fourier components shown in Appendix A, the conditions (I),(II) and (III) are equivalent to

\[
(I) : \int_0^{2\pi} dt F(q_{10}, q_{20}) = 0,
\]

and

\[
(II) : \int_0^{2\pi} dt F(q_{10}, q_{20}) = 0,
\]

where

\[
F^{(m,n)}(0,0) = \frac{\partial^{m+n} F}{\partial q_1^m \partial q_2^n}(0,0).
\]
\[ (\Pi) : \int_0^{2\pi} dt F(q_{10}, q_{20}) \cos 2t = 0, \quad (57) \]
\[ (\Pi I) : \int_0^{2\pi} dt F(q_{10}, q_{20}) \cos t \sin t = \frac{1}{2} \int_0^{2\pi} dt F(q_{10}, q_{20}) \sin 2t = 0, \quad (58) \]
respectively. Then, from lemma 4, we conclude that theorem 2 has been proved. \(\Box\)

4.4. Final step of proof of \((A) \Rightarrow (B)\)

We prove \((A) \Rightarrow (B)\) of theorem 1 with the aid of theorem 2 and corollary 3. The conditions \((\Pi')\) and \((\Pi I')\) are rewritten in matrix form
\[
\begin{pmatrix}
\left( \frac{\partial^2}{\partial B_2 \partial B_1} - \frac{\partial^2}{\partial C_2 \partial C_1} \right) u^{(0)} \\
\left( \frac{\partial^2}{\partial C_2 \partial C_1} - \frac{\partial^2}{\partial C_2 \partial C_1} \right) u^{(0)}
\end{pmatrix}
\begin{pmatrix}
2 \frac{\partial^2}{\partial B_2 \partial B_1} u^{(0)} \\
2 \frac{\partial^2}{\partial C_2 \partial C_1} u^{(0)}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\quad (59)
\]
The constants \(c_1\) and \(c_2\) are not zero simultaneously, and the conditions \((\Pi')\) and \((\Pi I')\) must be degenerate accordingly. The degenerate condition is
\[
\det \begin{pmatrix}
\left( \frac{\partial^2}{\partial B_2 \partial B_1} - \frac{\partial^2}{\partial C_2 \partial C_1} \right) u^{(0)} \\
\left( \frac{\partial^2}{\partial C_2 \partial C_1} - \frac{\partial^2}{\partial C_2 \partial C_1} \right) u^{(0)}
\end{pmatrix}
\begin{pmatrix}
2 \frac{\partial^2}{\partial B_2 \partial B_1} u^{(0)} \\
2 \frac{\partial^2}{\partial C_2 \partial C_1} u^{(0)}
\end{pmatrix}
= 0
\]
\[
\iff \frac{\partial^2 u^{(0)}}{\partial B_1 \partial B_2} \left( \frac{\partial^2}{\partial C_2 \partial C_1} - \frac{\partial^2}{\partial C_2 \partial C_1} \right) u^{(0)} - \frac{\partial^2 u^{(0)}}{\partial C_1 \partial C_2} \left( \frac{\partial^2}{\partial B_2 \partial B_1} - \frac{\partial^2}{\partial B_2 \partial B_1} \right) u^{(0)} = 0,
\quad (60)
\]
and this separable condition is equivalent to \(\Delta_3 = \Delta_4 = 0\). We also find that the degenerate condition between \((\Pi)\) and \((\Pi I)\) is equivalent to \(\Delta_6 = 0\). Consequently,
\[
\text{RGE is a Hamiltonian system up to } O(\epsilon^2)
\iff \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 0
\implies \text{Potential function } V(q_1, q_2) \text{ is separable by Cartesian coordinates.}
\]

5. Proof of \((B) \implies (A)\)

We assume that the Hamiltonian system \((12)\) is separable by Cartesian coordinates. RGE for the Hamiltonian system is also separable since constructing RGE and rotating coordinates are commutable operations. As a result, what we have to prove is the following theorem.

Theorem 5 RGE up to \(O(\epsilon^2)\) for the system with one degree of freedom
\[
H(q_1, p_1) = \frac{1}{2}(p_1^2 + q_1^2) + \epsilon V(q_1)
\quad (61)
\]
is a Hamiltonian system. \(V\) is an analytic function of \(q_1\).
Proof of Theorem 5: RGE for system (61) is obtained from equation (31) by omitting terms with subscript 2:

\[
\begin{align*}
\dot{B}_1 &= \frac{\partial \Phi}{\partial C_1} + \epsilon^2 \frac{G_1}{4}, \\
\dot{C}_1 &= -\frac{\partial \Phi}{\partial B_1} + \epsilon^2 \frac{H_1}{4},
\end{align*}
\]

(62)

where

\[
\Phi = -\epsilon u^{(0)} - \frac{\epsilon^2}{2} \left\{ (u^{(0)}_1)^2 + \sum_{n=2}^{\infty} \frac{1}{1 - n^2} [(u^{(n)}_1)^2 + (v^{(n)}_1)^2] \right\},
\]

(63)

\[
G_1 = 2 \left( \frac{\partial u^{(0)}}{\partial B_1} \frac{\partial^2 u^{(0)}}{\partial B_1 \partial C_1} - \frac{\partial u^{(0)}}{\partial C_1} \frac{\partial^2 u^{(0)}}{\partial B_1^2} \right),
\]

(64)

\[
H_1 = 2 \left( \frac{\partial u^{(0)}}{\partial B_1} \frac{\partial^2 u^{(0)}}{\partial C_1^2} - \frac{\partial u^{(0)}}{\partial C_1} \frac{\partial^2 u^{(0)}}{\partial B_1 \partial C_1} \right).
\]

(65)

The condition that the RGE (62) becomes a Hamiltonian system is

\[
\Delta_1 \equiv \frac{\partial \dot{B}_1}{\partial B_1} + \frac{\partial \dot{C}_1}{\partial C_1} = 0,
\]

(66)

and the form of \( \Delta_1 \) is

\[
\frac{4}{\epsilon^2} \Delta_1 = \frac{\partial G_1}{\partial B_1} + \frac{\partial H_1}{\partial C_1} = 2 \left[ \frac{\partial u^{(0)}}{\partial B_1} \frac{\partial}{\partial C_1} \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} \right) u^{(0)} - \frac{\partial u^{(0)}}{\partial C_1} \frac{\partial}{\partial B_1} \left( \frac{\partial^2}{\partial B_1^2} + \frac{\partial^2}{\partial C_1^2} \right) u^{(0)} \right]
\]

\[
= \frac{\partial u^{(0)}}{\partial B_1} \frac{\partial u^{(0)}}{\partial C_1} - \frac{\partial u^{(0)}}{\partial C_1} \frac{\partial u^{(0)}}{\partial B_1}.
\]

(67)

We prove that the \( \Delta_1 \) is always zero by showing both \( u^{(0)} \) and \( u^{(0)}_{11} \) are functions of \( B_1^2 + C_1^2 \). From the definition of \( u^{(0)} \),

\[
\begin{align*}
u^{(0)} &= -\frac{1}{2\pi} \int_0^{2\pi} dt V(B_1 \cos t + C_1 \sin t) \\
&= -\frac{1}{2\pi} \int_0^{2\pi} dt V(\sqrt{B_1^2 + C_1^2} \cos(t - \delta)) \\
&= -\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{V^{(n)}(0)}{n!} (B_1^2 + C_1^2)^{n/2} \int_0^{2\pi} dt \cos^n(t - \delta) \\
&= -\sum_{n=0}^{\infty} \frac{V^{(2n)}(0)}{(2n)!} (B_1^2 + C_1^2)^n \times 2^{-2n} 2n C_n,
\end{align*}
\]

(68)

where \( \tan \delta = C_1/B_1 \) and \( V^{(k)} = \partial^k V/\partial q^k \). Consequently, \( u^{(0)} \) is a function of \( B_1^2 + C_1^2 \).

We can also prove that \( u^{(0)}_{11} \) is a function of \( B_1^2 + C_1^2 \) through the same way. \( \square \)
6. Integral of renormalization group equation

From the consideration of the system with one degrees of freedom, RGE with two degrees of freedom is always integrable when it is a Hamiltonian system since it is separable. Moreover, \( H_0 \) of system (12) is an integral of RGE.

**Theorem 6** For the RGE system with one degrees of freedom (62), the quantity \( B^2_1 + C^2_1 \) is an integral of the equation. For the RGE system with two degrees of freedom (31), if RGE is a Hamiltonian system then \( B^2_1 + C^2_1 + B^2_2 + C^2_2 \) is an integral of the equation.

**Proof of Theorem 6:** First we consider systems with one degree of freedom. We prove that the time derivative of \( (B^2_1 + C^2_1)/2 \) becomes zero:

\[
\frac{1}{2} \frac{d}{dt} (B^2_1 + C^2_1) = B_1 \dot{B}_1 + C_1 \dot{C}_1 = B_1 \frac{\partial \Phi}{\partial C_1} - C_1 \frac{\partial \Phi}{\partial B_1} + \frac{\epsilon^2}{4} (B_1 G + C_1 H) = 0. \tag{69}
\]

We prove equation (69) by showing that \( \Phi \) is a function of \( B^2_1 + C^2_1 \) and that \( B_1 G + C_1 H = 0 \).

We first consider the terms written by \( \Phi \). The concrete form of \( \Phi \) is

\[
\Phi = -\epsilon u^{(0)} - \frac{\epsilon^2}{2} \left\{ u^{(0)}_1 + \sum_{n=2}^{\infty} \frac{1}{1 - n^2} [u^{(n)}_1]^2 + (v^{(n)}_1)^2 \right\}. \tag{70}
\]

The first and the second terms \( u^{(0)}, v^{(0)}_1 \), are constant components of \( -V(q_0) \) and \( -V_1(q_0) \), respectively. They are therefore functions of \( B^2_1 + C^2_1 \) accordingly (c.f. equations (68)). From the definition of \( u^{(n)}_1 \),

\[
u^{(n)}_1 = \frac{1}{\pi} \int_0^{2\pi} dt V_1(B_1 \cos t + C_1 \sin t) \cos nt
= \frac{1}{\pi} \int_0^{2\pi} dt V_1(\sqrt{B^2_1 + C^2_1} \cos(t - \delta)) \cos nt
= \frac{1}{\pi} \int_0^{2\pi} dt \sum_{k=0}^{\infty} \frac{V_1^{(k)}(0)}{k!} (B_1^2 + C_1^2)^{k/2} \cos^k (t - \delta) \cos nt
= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{V_1^{(k+1)}(0)}{k!} (B_1^2 + C_1^2)^{k/2} (C_1^{(k)} \cos n\delta - S_1^{(k)} \sin n\delta), \tag{71}
\]

where

\[
C_1^{(k)} \equiv \int_0^{2\pi} dt \cos^k t \cos nt, \quad S_1^{(k)} \equiv \int_0^{2\pi} dt \cos^k t \sin nt. \tag{72}
\]

We also have

\[
v^{(n)}_1 = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{V_1^{(k+1)}(0)}{k!} (B_1^2 + C_1^2)^{k/2} (C_1^{(k)} \sin n\delta + S_1^{(k)} \cos n\delta), \tag{73}
\]
and

\[(u_1^{(n)})^2 + (v_1^{(n)})^2 = \frac{1}{\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{V^{(k+1)}(0)V^{(l+1)}(0)}{k! \ l!} \left( (B_1^2 + C_1^2)^{(k+l)/2} \left( C_n^{(k)}C_n^{(l)} + S_n^{(k)}S_n^{(l)} \right) \right). (74)\]

Consequently, \(\Phi\) is a function of \(B_1^2 + C_1^2\) since \(C_n^{(k)}C_n^{(l)} + S_n^{(k)}S_n^{(l)}\) is independent of \(B_1\) and \(C_1\).

Next we prove that \(B_1 G + C_1 H = 0\). From (64), (65) and the fact that \(u^{(0)}\) is a function of \(B_1^2 + C_1^2\),

\[G = -2C_1[(u^{(0)})']^2, \quad H = 2B_1[(u^{(0)})']^2, \quad (75)\]

where \((u^{(0)})'\) is derivative of \(u^{(0)}(B_1^2 + C_1^2)\), and hence \(B_1 G + C_1 H = 0\).

Finally we consider systems with two degrees of freedom. When RGE is a Hamiltonian system, both \(B_1^2 + C_1^2\) and \(B_2^2 + C_2^2\) are integrals of RGE in separated coordinate. Accordingly, \(B_1^2 + C_1^2 + B_2^2 + C_2^2\) is an integral in any Cartesian coordinates since it is invariant by rotation of the coordinates \((q_1, q_2)\) which induces rotation of two pairs of variables: \((B_1, B_2)\) and \((C_1, C_2)\).

Now we have completed proof of theorem 1. \(\square\)

7. Summary and discussions

We considered renormalization group equation (RGE) for perturbed Hamiltonian system with two degrees of freedom whose integrable part is two harmonic oscillators with the same angular frequency, and perturbation is a function of positions having only even order terms. RGE is not always Hamiltonian system and we presented the necessary and sufficient condition that RGE becomes Hamiltonian system as the theorem 1. The theorem states that RGE in Cartesian coordinates becomes a Hamiltonian system up to \(O(\epsilon^2)\) if and only if the original system is separable by rotation of the coordinates. When RGE is a Hamiltonian system, it has an integral which corresponds to the integral part of original system and RGE is integrable accordingly. The theorem also assert that RGE is a Hamiltonian system when the original system has one degree of freedom. RGE is always Hamiltonian system up to \(O(\epsilon^1)\) and its Hamiltonian is the same as one obtained by canonical perturbation theory.

The theorem excluded odd order terms from the perturbation potential function. We must construct RGE up to \(O(\epsilon^4)\) to extend the theorem to any analytic perturbation functions since odd order terms give information on the separability at \(O(\epsilon^4)\). But it is straightforward to extend our result to include odd order terms in potential function. We expect that the theorem holds for any perturbation from the result of homogeneous cubic perturbation case [14], which satisfies the theorem.
RGE is not a Hamiltonian system when the original system has two degrees of freedom and have no integral other than Hamiltonian of the system. On the other hand, RGE with one degree of freedom is always Hamiltonian system. We therefore expect that we can understand properties of chaotic orbits by investigating how symplectic properties break in the process of constructing RGE. Solutions to non-Hamiltonian RGE will also give some aspects of chaos.

Future works of this topic is as follows. We suppose that it is straightforward to extend the main theorem of this paper to systems with many degrees of freedom. Another extension is generalization of integrable part of original systems. We are particularly interested in the case that integrable part is non-linear and angular frequencies change as values of actions.

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Appendix A. Some relations between Fourier components

We show some useful relations between Fourier components of $V(q_0)$, $-V_j(q_0)$ and $-V_{jk}(q_0)$. From the definition, $u_1^{(1)}$ is represented by $u^{(0)}$ as

\[ u_1^{(1)} = \frac{1}{\pi} \int_0^{2\pi} dt V_1(q_{10}, q_{20}) \cos t \]
\[ = \frac{\partial}{\partial B_1} \left[ \frac{1}{\pi} \int_0^{2\pi} dt V(q_{10}, q_{20}) \right] \]
\[ = 2 \frac{\partial u^{(0)}}{\partial B_1}. \] (A1)

Similar relations hold for other components:

\[ u_j^{(1)} = 2 \frac{\partial u^{(0)}}{\partial B_j}, \quad v_j^{(1)} = 2 \frac{\partial u^{(0)}}{\partial C_j}, \] (A2)

and for $n \geq 2$,

\[ u_j^{(n-1)} = \frac{\partial u^{(n)}}{\partial B_j} + \frac{\partial v^{(n)}}{\partial C_j}, \quad u_j^{(n+1)} = \frac{\partial u^{(n)}}{\partial B_j} - \frac{\partial v^{(n)}}{\partial C_j}, \]
\[ v_j^{(n-1)} = -\frac{\partial u^{(n)}}{\partial B_j} - \frac{\partial v^{(n)}}{\partial C_j}, \quad v_j^{(n+1)} = \frac{\partial v^{(n)}}{\partial B_j} + \frac{\partial u^{(n)}}{\partial C_j}. \] (A3)

Some Fourier components of $V_{jk}(q_{10}, q_{20})$ are rewritten by $u^{(0)}$ as follows:

\[ 2u_{jk}^{(0)} = \frac{\partial u_k^{(1)}}{\partial B_j} + \frac{\partial v_k^{(1)}}{\partial C_j} = 2 \left( \frac{\partial^2}{\partial B_j \partial B_k} + \frac{\partial^2}{\partial C_j \partial C_k} \right) u^{(0)}, \]
\[ u_{jk}^{(2)} = \frac{\partial u_k^{(1)}}{\partial B_j} - \frac{\partial v_k^{(1)}}{\partial C_j} = 2 \left( \frac{\partial^2}{\partial B_j \partial B_k} - \frac{\partial^2}{\partial C_j \partial C_k} \right) u^{(0)}, \] (A4)
\[ v_{jk}^{(2)} = \frac{\partial v_k^{(1)}}{\partial B_j} + \frac{\partial u_k^{(1)}}{\partial C_j} = 2 \left( \frac{\partial^2}{\partial B_j \partial C_k} + \frac{\partial^2}{\partial C_j \partial B_k} \right) u^{(0)}. \]

From the definition of Fourier components, the following equation is always satisfied for any Fourier components:

\[ \frac{\partial^2}{\partial B_1 \partial C_2} X = \frac{\partial^2}{\partial C_1 \partial B_2} X, \] (A5)

where $X$ is an arbitrary Fourier component.
Appendix B. Hamiltonian of RGE in quartic perturbation

We show the explicit form of the Hamiltonian of RGE in quartic perturbation when RGE is Hamiltonian system. First we describe how we obtain the Hamiltonian from the RGE. Second the concrete form of the Hamiltonian is shown.

Appendix B.1. Method of constructing Hamiltonian of RGE

When RGE is a Hamiltonian system, it is written as

\[
\frac{dB_1}{dt} = f_1(B_1, C_1, B_2, C_2; \epsilon) = \frac{\partial H^{RG}}{\partial C_1},
\]

(B6)

\[
\frac{dC_1}{dt} = g_1(B_1, C_1, B_2, C_2; \epsilon) = -\frac{\partial H^{RG}}{\partial B_1},
\]

(B7)

\[
\frac{dB_2}{dt} = f_2(B_1, C_1, B_2, C_2; \epsilon) = \frac{\partial H^{RG}}{\partial C_2},
\]

(B8)

\[
\frac{dC_2}{dt} = g_2(B_1, C_1, B_2, C_2; \epsilon) = -\frac{\partial H^{RG}}{\partial B_2}.
\]

(B9)

By integrating equation (B6) with respect to \(C_1\), we have

\[
H^{RG} = H^{RG(1)} + M^{(1)}(B_1, B_2, C_2),
\]

(B10)

\[
H^{RG(1)} = \int f_1 dC_1,
\]

(B11)

where \(M^{(1)}\) is an arbitrary function of \(B_1, B_2\) and \(C_2\). To determine \(M_1\), we substitute equation (B10) to equation (B7) and we obtain

\[
\frac{\partial M^{(1)}}{\partial B_1} = \left( -\frac{\partial H^{RG(1)}}{\partial B_1} - g_1 \right)|_{C_1=0},
\]

(B12)

since the left-hand-side of equation (B12) does not depend on \(C_1\) when the RGE is a Hamiltonian system. By integrating equation (B12) with respect to \(B_1\), we get \(M^{(1)}\) as

\[
M^{(1)}(B_1, B_2, C_2) = H^{RG(2)} + M^{(2)}(B_2, C_2),
\]

(B13)

where

\[
H^{RG(2)} = \int \left( -\frac{\partial H^{RG(1)}}{\partial B_1} - g_1 \right)|_{C_1=0} dB_1.
\]

(B14)

Next, to determine \(M^{(2)}(B_2, C_2)\), we substitute equations (B11) and (B13) to equation (B8), then

\[
\frac{\partial M^{(2)}}{\partial C_2} = \left( -\frac{\partial}{\partial C_2} H^{RG(1)} + H^{RG(2)} + f_2 \right)|_{B_1=C_1=0},
\]

(B15)
By integrating equation (B15) with respect to \( C_2 \), we get \( M^{(2)} \) as

\[
M^{(2)}(B_2, C_2) = H^{\text{RG}(3)} + M^{(3)}(B_2),
\]

where

\[
H^{\text{RG}(3)} = \int \left( \frac{\partial}{\partial C_2} (H^{\text{RG}(1)} + H^{\text{RG}(2)}) + f_2 \right) \bigg|_{B_1=C_1=0} dC_2.
\]

Finally, to determine \( M^{(3)} \), we substitute equations (B10), (B13) and (B16) to equation (B8), then

\[
\frac{\partial M^{(3)}}{\partial B_2} = \left( -\frac{\partial}{\partial B_2} (H^{\text{RG}(1)} + H^{\text{RG}(2)} + H^{\text{RG}(3)}) - g_2 \right) \bigg|_{B_1=C_1=C_2=0}
\]

By integrating equation (B18) with respect to \( B_2 \), we get \( M^{(3)} \) as

\[
M^{(3)}(B_2) = H^{\text{RG}(4)},
\]

where

\[
H^{\text{RG}(4)} = \int \left( \frac{\partial}{\partial C_2} (H^{\text{RG}(1)} + H^{\text{RG}(2)} + H^{\text{RG}(3)}) - g_2 \right) \bigg|_{B_1=C_1=C_2=0} dB_2.
\]

Consequently, the Hamiltonian \( H^{\text{RG}} \) is

\[
H^{\text{RG}} = H^{\text{RG}(1)} + H^{\text{RG}(2)} + H^{\text{RG}(3)} + H^{\text{RG}(4)}.
\]

Appendix B.2. RGE Hamiltonian for quartic perturbation

We present explicit form of Hamiltonian of RGE for the system

\[
H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + \epsilon(\alpha_1 q_1^4 + \alpha_2 q_1^3 q_2 + \alpha_3 q_1^2 q_2^2 + \alpha_4 q_1 q_2^3 + \alpha_5 q_2^4).
\]

We know that RGE for this system has Hamiltonian \( H^{\text{RG}} \) if the coefficients \( \alpha_1, \cdots, \alpha_5 \) satisfy the following condition [14]:

\[
\begin{align*}
9\alpha_2^2 + 4\alpha_3^2 - 24\alpha_1\alpha_3 - 9\alpha_2\alpha_4 &= 0, \\
9\alpha_4^2 + 4\alpha_3^2 - 24\alpha_3\alpha_5 - 9\alpha_2\alpha_4 &= 0, \\
(\alpha_2 + \alpha_4)\alpha_3 - 6(\alpha_1\alpha_4 + \alpha_2\alpha_5) &= 0.
\end{align*}
\]

Then the \( H^{\text{RG}} \) is

\[
H^{\text{RG}} = \epsilon H^{\text{RG}(1)} + \epsilon^2 H^{\text{RG}(2)},
\]

where

\[
H^{\text{RG}(1)} = \frac{3}{8} \left\{ \alpha_1 (B_1^2 + C_1^2)^2 + \alpha_5 (B_2^2 + C_2^2)^2 \\
+ [\alpha_2 (B_1^2 + C_1^2) + \alpha_4 (B_2^2 + C_2^2)](B_1 B_2 + C_1 C_2) \right\}
+ \frac{1}{8} \alpha_3 (B_1^2 + C_1^2)(B_2^2 + C_2^2) + \frac{1}{4} \alpha_3 (B_1 B_2 + C_1 C_2)^2,
\]

\[
H^{\text{RG}(2)} = \frac{3}{8} \left\{ \alpha_1 (B_1^2 + C_1^2)^2 + \alpha_5 (B_2^2 + C_2^2)^2 \\
+ [\alpha_2 (B_1^2 + C_1^2) + \alpha_4 (B_2^2 + C_2^2)](B_1 B_2 + C_1 C_2) \right\}
+ \frac{1}{8} \alpha_3 (B_1^2 + C_1^2)(B_2^2 + C_2^2) + \frac{1}{4} \alpha_3 (B_1 B_2 + C_1 C_2)^2.
\]
and

\[
H_R^{RG} = -\frac{5}{32} \left[ \alpha_1^2 (B_1^2 + C_1^2)^3 + \alpha_2^2 (B_2^2 + C_2^2)^3 \right]
\]
\[
- \frac{5}{512} \left[ \alpha_2^2 (B_1^2 + C_1^2)^3 + \alpha_4^2 (B_2^2 + C_2^2)^3 \right]
\]
\[
- \frac{45}{512} \left[ \alpha_2^2 (B_1^2 + C_1^2)^2 (B_2^2 + C_2^2) + \alpha_4^2 (B_1^2 + C_1^2)(B_2^2 + C_2^2)^2 \right]
\]
\[
- \frac{5}{128} \alpha_3^2 (B_1^2 + C_1^2)(B_2^2 + C_2^2)(B_1^2 + B_2^2 + C_1^2 + C_2^2)
\]
\[
- \frac{15}{64} \alpha_1 \alpha_2 (B_1^2 + C_1^2)^2 + \alpha_4 \alpha_5 (B_2^2 + C_2^2)^2 \right] (B_1 B_2 + C_1 C_2)
\]
\[
- \frac{15}{64} \alpha_2 \alpha_4 (B_1 B_2 + C_1 C_2 + B_1 C_2 - B_2 C_1)
\]
\[
\times (B_1 B_2 + C_1 C_2 - B_1 C_2 + B_2 C_1)(B_1^2 + B_2^2 + C_1^2 + C_2^2)
\]
\[
- \frac{5}{32} \left[ \alpha_1 \alpha_3 (B_1^4 - C_1^4)(B_2^2 - C_2^2) + \alpha_3 \alpha_5 (B_1^2 - C_1^2)(B_2^4 - C_2^4) \right]
\]
\[
- \frac{5}{32} \left[ \alpha_1 \alpha_3 (B_1^2 + C_1^2) + \alpha_3 \alpha_5 (B_2^2 + C_2^2) \right] B_1 B_2 C_1 C_2
\]
\[
- \frac{5}{64} \left( \alpha_1 \alpha_4 + \alpha_2 \alpha_5 \right) (B_1 B_2 + C_1 C_2)^3
\]
\[
- \frac{5}{128} \left\{ \alpha_2 \alpha_3 \left[ (B_1^2 + C_1^2)^2 + 3(B_1^2 B_2^2 + C_1^2 C_2^2) \right.
\right.
\]
\[
+ 2(B_1 C_2 + B_2 C_1)^2 - 2B_1 B_2 C_1 C_2 \] + \alpha_3 \alpha_4 \left[ (B_2^2 + C_2^2)^2 + 3(B_1^2 B_2^2 + C_1^2 C_2^2) \right.
\right.
\]
\[
+ 2(B_1 C_2 + B_2 C_1)^2 - 2B_1 B_2 C_1 C_2 \] \right\} (B_1 B_2 + C_1 C_2). (B26)