Complex instruction set computing architecture for performing accurate quantum $Z$ rotations with less magic

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We present quantum protocols for executing arbitrarily accurate $\pi/2^k$ rotations of a qubit about its $Z$ axis. Reduced instruction set computing (RISC) architectures typically restrict the instruction set to stabilizer operations and a single non-stabilizer operation, such as preparation of a “magic” state from which $T = Z(\pi/4)$ gates can be teleported. Although the overhead required to distill high-fidelity copies of this magic state is high, the subsequent quantum compiling overhead to realize arbitrary $Z$ rotations in a RISC architecture can be much greater. We develop a complex instruction set computing (CISC) architecture whose instruction set includes stabilizer operations and preparation of magic states from which $Z(\pi/2^k)$ gates can be teleported, for $2 \leq k \leq k_{\text{max}}$. This results in a reduction in the resources required to achieve a desired gate accuracy for $Z$ rotations. The key to our construction is a family of shortened quantum Reed-Muller codes of length $2^{k+2} - 1$, whose magic-state distillation threshold shrinks with $k$ but is greater than $0.85\%$ for $k \leq 6$.

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I. INTRODUCTION

One of the biggest challenges in quantum information science is that quantum information is incredibly fragile. Even with great experimental care, decoherence can quickly corrupt key features such as superposition and entanglement. To circumvent the ravages of decoherence, one can consider alternative models of quantum computation, such as adiabatic quantum computation [1–3], which may offer direct physical immunity to certain classes of noise [4–14]. Another approach is to encode quantum information redundantly in an error-correcting code and process it fault-tolerantly to suppress the catastrophic propagation of errors [15, 16]. Somewhat miraculously, this latter approach works, and works arbitrarily well, when quantum computations are expressed as quantum circuits in which each elementary operation has a failure probability below a value known as the accuracy threshold [17–23]. Estimates for the accuracy threshold vary, and depend in part on the specifics of the fault-tolerant quantum computing protocol used. One of the more favorable estimates is $\approx 1\%$ for a protocol based on Kitaev’s surface codes [24–27]. An outstanding grand challenge in quantum information science is finding a way to marry fault-tolerance methods with intrinsically robust computational models to achieve fault tolerance with more achievable resource requirements [28–31].

One of the factors driving up the resource requirements in fault-tolerant quantum computing is the need to restrict the set of elementary operations in the “primitive” or “physical” instruction set to be finite. This is necessary because these instructions are presumed to be implementable only up to some maximal accuracy. One of the main jobs of a fault-tolerant quantum computing protocol is to define how one should sequence these primitive instructions together to synthesize arbitrarily accurate versions of each element of a universal “encoded” or “logical” instruction set, even when the primitive instructions themselves are faulty. Then, using these logical instructions, one can realize any quantum algorithm arbitrarily reliably, even in the face of decoherence and other sources of noise.

In a typical fault-tolerant quantum computing protocol, some logical instructions are “easy” to synthesize in that their error is solely a function of the errors in the primitive instructions from which they are composed. The accuracy of these logical instructions can be improved arbitrarily well by using arbitrarily good quantum codes. More quantitatively, the number of gates and qubits required to achieve approximation error $\epsilon$ for the “easy” instructions scales as $O(\log^\alpha(1/\epsilon))$, where $\alpha$ depends on the protocol, predominantly on the quantum code and classical decoding algorithm it uses. Standard techniques for realizing such gates include transversal action [22, 23] and code deformation [25, 26]. 2D topological codes using most-likely-error decoding can achieve $\alpha = 3$ [25, 26]; Pippenger has conjectured that it should be possible to lower $\alpha$ all the way to 1 [32].

Most protocols also have a set of logical instructions that are “hard” to synthesize, requiring additional methods and resources. The Eastin-Knill theorem, for example, guarantees that no protocol can realize a universal logical instruction set by transversal action alone [33]. A typical approach to synthesizing these hard logical instructions is to use the “magic state” approach, in which the “hard” instructions are state preparations that are distilled to high fidelity using the “easy” operations [34]. The number of ideal gates and qubits required to achieve approximation error $\epsilon$ in this approach scales as...
\(O(\log^2(1/\epsilon))\), where \(\beta\) depends on the magic-state distillation protocol. When the the resource costs for the “easy” gates are also considered, the combined overhead scales as \(O(\log^{\alpha+\beta}(1/\epsilon))\). In the well-studied Bravyi-Kitaev 15-to-1 distillation protocol [34], \(\beta = \log_2 15 \approx 2.47\). More recent constructions by Bravyi and Haah [35] and by Jones [36] achieve \(\beta = \log_2 3 \approx 1.58\). Bravyi and Haah conjecture that it should be possible to lower \(\beta\) all the way to 1 [39].

As an aside, it is worth mentioning that fault-tolerant quantum computing protocols based on some quantum codes have no “hard” logical instructions at all. For example, the 3D (and higher-dimensional) topological color codes have this feature [37, 38]. They cleverly circumvent the Eastin-Knill theorem by making (non-transversal!) quantum error correction be the process by which magic-states are prepared. A challenge to using these codes in practice is that implementing them without relying on long-distance quantum communication requires 3D spatial geometry, but many quantum technologies are naturally restricted to 1D or 2D. Even more challenging is that the only explicit 3D color code of which we are aware is the 15-qubit shortened quantum Reed-Muller code [37]. Concatenated schemes using the 15-qubit code would lead to a fault-tolerant scheme with only “easy” instructions, but concatenated schemes typically suffer significant performance losses when realized in a fixed spatial dimension. For example, the largest accuracy threshold of which we are aware for a concatenated-coding protocol in a semiregular 2D geometry is \(1.3 \times 10^{-5}\) [39].

Because of the additional overhead incurred in synthesizing “hard” logical instructions, research to date has focused on what one might term reduced instruction set computing, or RISC, architectures in which only a single “hard” logical instruction is added to an otherwise “easy” logical instruction set. However, while a RISC architecture minimizes the number of hard instructions in an instruction set, it does not necessarily minimize the number of hard instructions used in specific algorithms. For example, in order to compile the logical instructions into a sequence that approximates a quantum computation with error at most \(\epsilon\), one must use \(O(\log^\gamma(1/\epsilon))\) gates, where \(\gamma\) depends on the quantum compiling algorithm used. The overall cost of fault-tolerantly implementing a quantum computation is then \(O(\log^{\alpha+\beta+\gamma}(1/\epsilon))\). By increasing the size of the instruction set so that one has a complex instruction set computing, or CISC architecture, one can optimize both \(\beta\) and \(\gamma\) together rather than separately. When quantum compiling is optimized independently, \(\gamma\) can be no lower than 1 [40], a value recently achieved by an explicit Diophantine-equation-based algorithm by Selinger [41] and Kliuchnikov et al. [42]. For comparison’s sake, the more well-studied Dawson-Nielsen variant of the Solovay-Kitaev algorithm achieves \(\gamma = \log 5/\log(3/2) \approx 3.97\) [43].

To compare and contrast the RISC and CISC approaches more concretely without being encumbered by details of quantum error correcting codes and fault tolerance (which only contribute to \(\alpha\) and a delineation of which logical instructions are “easy” or “hard”—properties shared by both approaches), we abstract these details away and simply consider the straightforward problem of how to approximate \(\pi/2^k\) rotations of a qubit about its \(Z\) axis with a desired error at most \(\epsilon\) when we are given the ability to perform a proscribed set of “easy” instructions that are error-free and a proscribed set of “hard” instructions that have error at most \(\epsilon > \epsilon\). In this setting, it is clear that some kind of distillation of the hard instructions will be necessary to synthesize the \(Z\) rotations with lower error. \(Z(\pi/2^k)\) rotations are a natural candidate transformation to use to compare RISC and CISC approaches, because they arise in many quantum algorithms, for example those that make use of the quantum Fourier transform [44].

In Sec. II, we formulate the statement of the problem we are considering more precisely. In Sec. III, we review the standard RISC solution to this problem. In Sec. IV, we describe our CISC solution, and compare it to the RISC solution, demonstrating that for a regime of target \(\epsilon\) our solution offers a reduction in the number of resource states used to achieve this task. Sec. V concludes. Appendix A elaborates the shortened quantum Reed-Muller codes we use to effect our protocol, and Appendix B formulates a testable set of criteria one can use to check if a code admits \(Z(\pi/2^k)\) transversally.

### II. PROBLEM STATEMENT

Consider quantum \(Z\) rotations of the form

\[
Z_k := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2^k} \end{pmatrix} = e^{i\pi/2^{k+1}} R_z(\frac{\pi}{2^k}),
\]

for integers \(k \geq 0\). As a shorthand, we use \(Z\) to denote the Pauli operator \(Z_0\) and \(S\) and \(T\) to denote the rotations \(Z_1\) and \(Z_2\) respectively. We are interested in the scenario in which the \(Z_k\) gates are not available directly, but rather their action on \(|+\rangle\) states is, where \(|+\rangle := H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\) and \(H := (X + Z)/\sqrt{2}\). For concreteness, let \(Z_{k_{\text{max}}}\) denote the set of states of the form

\[
Z_k|+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + e^{i\pi/2^k} |1\rangle \right)
\]

for \(2 \leq k \leq k_{\text{max}}\).

In conjunction with the set \(S\) of stabilizer operations [45], the set \(Z_{k_{\text{max}}}\) can effect universal quantum computation, even when restricted to \(k_{\text{max}} \leq 2\) [44]. Here we restrict our attention to a certain (overcomplete) generating set for \(S\), namely the set consisting of the operations

\[
\{I, X, Y, Z, S, S_1, H\} \cup \{0\}, |+\rangle, M_Z, M_X\}
\]

and

\[
\{\Lambda(X^{q_1} \otimes \cdots \otimes X^{q_m}) \mid q_i \in \{0,1\}\},
\]

where
where $I$, $X$, $Y$, and $Z$ denote the Pauli operators, $M_X$ and $M_Z$ denote projective measurements in the $X$ and $Z$ bases (but which may be “destructive” in that they do not necessarily prepare $X$ or $Z$ eigenstates after the measurement), and $\Lambda(X^q)$ denotes the one-control, many-target controlled-NOT gate, where the number of targets $m$ is some efficiently computable number. The unitary gates in this generating set generate a subgroup of the stabilizer operations known as the Clifford operations [45], which are the set of operations that conjugate (tensor products of) Pauli operators to (tensor products of) Pauli operators.

These generators of $S$ are “easy” to perform at the logical level for the 4.8.8 2D color codes, motivating our choice [38]. The set is also almost “easy” for Kitaev’s 2D surface codes [24], except generating $S$ and $S^\dagger$ requires some constant startup costs that can be amortized [46]. Amazingly, as noted in the introduction, all elements from the set $S \cup Z_2$—a universal set—are “easy” to perform at the logical level for 3D color codes, but 3D geometries are required to realize error correction with these codes in a spatially local manner [38].

While errors in the “easy” operations can be suppressed arbitrarily close to zero by using arbitrarily large 2D topological codes, errors in the operations in $Z_{k_{\text{max}}}$ cannot, making these operations “hard” for these codes. The states in $Z_{k_{\text{max}}}$ can be “injected” into such codes at the logical level [26], but doing so also injects the errors in the state. In other words, if the states in $Z_{k_{\text{max}}}$ have errors that are at most $\epsilon$ (as measured by the trace distance [44]) as primitive instructions, then the injected states will have errors that are essentially the same when they become logical instructions, assuming the injection process itself adds errors at a low enough probability [82].

Motivated by these properties of 2D topological codes, we will fix the control model for our study to be the aforementioned generators of $S$ and $Z_{k_{\text{max}}}$, and the error model to be one in which the operations in $S$ are error-free but in which the $Z_{k_{\text{max}}}^+$ states in $Z_{k_{\text{max}}}$ each err by at most $\epsilon$, as measured by the trace distance. Notice that this control model makes no reference to codes or fault-tolerant quantum computing protocols. We have abstracted these away to focus on how to combine elementary operations in $S$ and $Z_{k_{\text{max}}}$ to achieve high-fidelity $Z$ rotations.

The question we address here is,

*How many resource states drawn from $Z_{k_{\text{max}}}$ does it take to approximate $Z_k$ with error at most $\epsilon'$ as a function of $k_{\text{max}}$, $k$, $\epsilon$, and $\epsilon'$?*

The values of $k$ we are interested in could be smaller than, equal to, or larger than $k_{\text{max}}$. However, since $Z_0$ and $Z_1$ are both in the error-free set $S$, we restrict our attention to $k \geq 2$.

### III. TRADITIONAL QUANTUM RISC ARCHITECTURE SOLUTION

The standard method for refining the accuracy of a $Z_k$ rotation is to synthesize it with what one might term a quantum reduced instruction set computing, or quantum RISC, architecture. The main idea is to only synthesize $T := Z_2$ gates to high accuracy and then rely on a quantum compiling algorithm to approximate $Z_k$ arbitrarily well with a quantum circuit over $T$ gates and adaptive stabilizer operations. The overall process can be broken into the three steps of quantum compiling, quantum gate teleportation, and magic-state distillation.

#### A. Protocol

1. **Quantum compiling**

The first step, *quantum compiling*, generates a classical description of an ideal quantum circuit that approximates $Z_k$ to accuracy $\epsilon_{qc}$ using $O(\log^3(1/\epsilon_{qc}))$ quantum operations drawn from some instruction set, for some small constant $\gamma$. While the error $\epsilon_{qc}$ can be measured in multiple ways, a wise choice is to measure $\epsilon_{qc}$ using the completely-bounded (“diamond”) trace distance [19, 47, 48] for reasons that we will explain later. Examples of quantum compiling algorithms include the Solovay-Kitaev algorithm [19, 40, 43, 44, 49–51], the Kitaev phase kickback algorithm [52–54], programmed ancilla algorithms [36, 55, 56], genetic algorithms [57], and even Diophantine-equation algorithms [41, 42]. When the accuracy demand is not great, it is sometimes even plausible to use algorithms which take exponential time to find very short approximation sequences [58–61]. As noted in the introduction, values for $\gamma$ range from 3.97 to 1.

Quantum compiling algorithms typically assume that the elements of the instruction set are error-free. If one implements the compiled circuit $Z_k^{(qc)}$ for $Z_k$ with operations that may be in error, the resulting approximation error will increase. To calculate the total error $\epsilon_k$ in this flawed circuit $\tilde{Z}_k^{(qc)}$, we use the fact that the diamond norm has many useful mathematical properties, including obeying the triangle inequality, the chaining inequality, and unitary invariance [62]. Using these, we can bound $\epsilon_k$ as

$$
\epsilon_k = d_0 \left( Z_{k}, \tilde{Z}_k^{(qc)} \right) \leq d_0 \left( Z_{k}, Z_k^{(qc)} \right) + d_0 \left( Z_k^{(qc)}, \tilde{Z}_k^{(qc)} \right) \leq \epsilon_{qc} + n_T \epsilon_T,
$$

where the compiled circuit uses $n_T$ $T$ gates, each with error at most $\epsilon_T$. To achieve the desired approximation
error of $\epsilon'$, it follows that sufficient conditions are
\[
\begin{align*}
\epsilon_{qc} & \leq C_{qc} \epsilon' \\
\epsilon_T & \leq C_T \epsilon'/n_T,
\end{align*}
\]
for positive constants constrained to obey
\[
C_T + C_{qc} \leq 1.
\]  

For comparison with our protocol introduced later, we chose the Diophantine equation-based compiling protocol presented by Selinger in Ref. [41]. This protocol saturates the asymptotic lower bound on the number of $T$ gates required to approximate a single-qubit gate, and for $Z$ rotations has a $T$ count of
\[
n_T (\epsilon_{qc}) \approx 11 + 4 \log_2 \left( \frac{1}{\epsilon_{qc}} \right),
\]
(11)

2. **Quantum gate teleportation**

The second step, *quantum gate teleportation*, replaces each $T$ gate in the quantum-compiled circuit by an adaptive stabilizer circuit that teleports the $T$ gate from the state $T|+\rangle$ to the desired qubit. An example of a teleportation circuit using $T|+\rangle$ is depicted in Fig. 1. The circuit is also correct if both $T$ operators are changed to $T^\dagger$; it is even correct if only one of the $T$ operators is changed to a $T^\dagger$ if the classical control is also changed to act on a 0 instead of a 1.

Each teleportation circuit requires the use of just a single $T|+\rangle$ resource state. The accuracy requirement set by $\epsilon_T$ will determine whether these are ‘bare’ $T|+\rangle$ states of accuracy $\epsilon$ or whether these states are the result of one or more rounds of distillation, described in the next section.

3. **Magic-state distillation**

The third step, *magic-state distillation*, generates $T|+\rangle$ or $T^\dagger|+\rangle$ states with accuracy $\epsilon_T$ from a much larger collection of states whose accuracy is only $\epsilon$. Reichardt showed that this is possible using an ideal (error-free) stabilizer circuit if and only if $\epsilon$ is less than the distillation threshold $\frac{2 - \sqrt{2}}{4} \approx 0.146$ [63]. When operations in the stabilizer circuit can err, the evaluation of the threshold is more complex, as studied by Jochym-O’Connor et al. [64].

There are multiple variations on how to implement magic-state distillation discussed in the literature [34, 35, 65–68]; a popular one is the 15-to-1 Bravyi-Kitaev protocol [34] based on the 15-qubit shortened quantum Reed-Muller code $QRM(1, 4)$. (See Appendix A for an explanation of this notation.)

To date, the best distillation scheme in terms of resource costs is a hybrid of the 15-to-1 Bravyi-Kitaev protocol [34], the 10-to-2 Meier-Eastin-Knill protocol [68], and the $(3k+8)$-to-$k$ family of protocols discovered by Bravyi and Haah [35]. Bravyi and Haah optimized combinations of these protocols to find the most efficient way of producing a state $T|+\rangle$ of target accuracy $\epsilon_T$ [35]. The optimization yields about a factor of two improvement over a scheme which utilizes only a combination of the 15-to-1 and the 10-to-2 protocols. We perform no such optimization over protocols when we compare to our own distillation protocols, because we already see a savings of more than an order of magnitude over these.

We chose to compare our protocol to resource costs incurred by the Selinger approximation protocol in conjunction with the Meier-Eastin-Knill (MEK) 10-to-2 protocol. For completeness we now provide a brief description of how the MEK protocol functions [68]. The goal is to prepare a target resource state, in our case $T|+\rangle$, with some desired accuracy $\epsilon_T$ given only faulty copies of the same state with error $\epsilon > \epsilon_T$. The simplest way to prepare such a state would be to measure an operator whose eigenstate is $T|+\rangle$, but given access to only Clifford operations this cannot be done. To circumvent this problem, more resource states of accuracy $\epsilon$ are consumed to perform the desired measurement. For the MEK protocol, the total number of resource states consumed per round is 10. Additionally, the measurement performed is a *logical* measurement on an encoded qubit (or qubits). This allows for the detection of errors during the measurement procedure and is responsible for the increased accuracy of the output resource states. If the desired logical measurement outcome has been observed, the syndrome for the code is measured and, conditioned on the syndrome being error free, running the decoding circuit leaves two resource states with error $O(\epsilon^2)$.

The code utilized by the MEK protocol is the $[4, 2, 2]$ quantum error-detecting code. The distilled states are the eigenstates of $H$, denoted $|H\rangle$, which are related to $T|+\rangle$ by a Clifford rotation as follows:

\[
|H\rangle = SH (T|+\rangle).
\]
(12)

The protocol proceeds as follows:

1. Encode two (‘‘twirled’’) copies of $|H\rangle$ in the $[4, 2, 2]$ code. “Twirling” is performed by the probabilistic process that applies either $I$ or $H$ to the state, each with probability $1/2$.

2. Perform a measurement of logical $H_1 H_2$, which for this code is the same as transversal $H$ up to a SWAP. This measurement uses eight additional $|H\rangle$ states, which can be inferred from the identities in Fig. 1 of Ref. [68].
3. If a $-1$ outcome is obtained for the measurement of $H_1 H_2$, start over. If a $+1$ outcome is obtained, measure the syndrome for the code.

4. If an error-free syndrome is reported, decode. Otherwise, start over.

5. After decoding there will be two higher fidelity $|H\rangle$ states with error $O(\epsilon^2)$.

Note that the syndrome measurements can be pushed through the decoding circuit, becoming single-qubit measurements after decoding is performed.

Counting the number of resource states required to produce $n_T$ states of accuracy $\epsilon_T$ is accomplished by numerically evaluating the recursive relationship

$$n_T(\ell) = 5n_T(\ell - 1)/a(\epsilon_T),$$

where $a(\epsilon)$ is the probability of the protocol accepting, given above Eq. (3) in Ref. [68], and $\epsilon$ is the accuracy after $\ell$ rounds of distillation, and the base of the recursion is $n_T(0) = 5/a(\epsilon)$. Intuitively, this just says that to produce one resource state of accuracy $O(\epsilon^2)$ requires on average $5/a(\epsilon)$ states of fidelity $\epsilon$. We use this, in conjunction with Eq. (3) in [68] to calculate how many resource states are required to achieve a target $\epsilon_T$.

### B. Resource analysis

As mentioned in the introduction, asymptotically the total number of operations required to approximate a $Z_k$ gate with error $\epsilon$ is $O(\log^{\alpha+\beta+\gamma}(1/\epsilon^2))$, where the exponents describe various overheads of the steps involved: fault-tolerant stabilizer operations ($\alpha$), magic-state distillation ($\beta$), and quantum compiling ($\gamma$). While a good starting point, asymptotic analysis like this fails to convey the great number of elementary operations needed to implement $Z_k$ gates, as it sweeps the (large!) constants under the rug. The explicit expression for the expected number of states used by the RISC approach to approximate $Z_k$ to error $\epsilon'$ using $T|+\rangle$ states whose error is $\epsilon$ is

$$n_{\text{states}}^{\text{RISC}}(Z_k, \epsilon', \epsilon) = \left[ 11 + 4 \log_2 \left( \frac{1}{C_{\text{qe}}\epsilon'} \right) \right] \times n_T^{\text{states}} \left( \frac{C_{T} \epsilon'}{n_T}, \epsilon \right),$$

where $n_T^{\text{states}}(C_{T} \epsilon'/n_T, \epsilon)$ is the number of $T|+\rangle$ states of error $\epsilon$ required to produce a $|+\rangle$ state of error $C_{T} \epsilon'/n_T$. The idea here is to first use the results of Ref. [41] to approximate $Z_k$ to accuracy $C_{\text{qe}}\epsilon'$, and then replace each $T$ gate in the compiled sequence with a teleportation circuit using a $T|+\rangle$ state of accuracy $C_{T} \epsilon'/n_T$.

To better appreciate the compiling resources needed, we consider the case when $C_{\text{qe}} = C_T = 1/2$, which balances the quality demands of quantum compiling and magic-state distillation. We give the $T|+\rangle$ state a generous error rate of $\epsilon = 10^{-4}$, which is well below the estimated threshold of $\approx 1\%$ for fault-tolerant quantum computation with surface codes [26, 27]. The number of states $n_{\text{states}}^{\text{RISC}}$ required to synthesize $Z_k$ with these parameters to various approximation levels are plotted in the dashed curve in Fig. 4. One appealing feature, especially for large values of $k$, is that the curve does not depend on $k$—the number of states needed is solely a function of the desired output precision.

### IV. QUANTUM CISC ARCHITECTURE SOLUTION

Now that we’ve described how to implement $Z_k$ rotations using a quantum RISC architecture, it’s natural to ask if extending the instruction set to a quantum complex instruction set computing architecture, or quantum CISC architecture, could provide any advantage in terms of a reduction in the required number of resource states. The point is that in any given quantum algorithm instance, one isn’t interested in applying arbitrary gates but rather a specific set of gates, say $Z_k$ gates up to some maximum value of $k$ in a quantum Fourier transform. Because of this, it may make more sense to just include those gates in the instruction set to begin with rather than compiling them from a more limited instruction set. Even if it is only feasible to include gates up to some value of $Z_{k_{\text{max}}}$, it is reasonable to expect that the length of the resulting compiled sequences will be shorter if an arbitrary gate is required.

#### A. Protocol

In our protocol we consider a programmed-ancilla CISC architecture, in which we pre-compile $Z_k|+\rangle$ states offline that can be used later to teleport the gate $Z_k$ on demand via the circuit in Fig. 2. While the teleportation may require a $Z_{k-1}$ gate for correction, iterating this process recursively is a negative binomial process that converges exponentially quickly—the expected number of $Z$ rotations for any $k$ is two: $Z_k$ on $|+\rangle$ and $Z_{k-1}$ after the measurement. To achieve error at most $\epsilon'$ on the teleported $Z_k$ gate, the $Z_k|+\rangle$ state and the $Z_{k-1}$ gate need to be performed with errors at most $C_1 \epsilon'$ and $C_2 \epsilon'$ respectively, where $C_1 + C_2 \leq 1$.

![FIG. 2: Magic-state circuit for teleporting the $Z_k$ gate.](image)

Our CISC approach is distinguished from previous programmed-ancilla approaches [36, 55, 56] in that we distill ancilla $Z_k|+\rangle$ states directly as instructions unto
themselves. This is a “top-down” approach in which some of the time auxiliary $Z_{k-1}|+\rangle$ states are needed, and even less of the time $Z_{k-2}|+\rangle$ states are needed, and so on, until we get to the point that very rarely do we need $T|+\rangle$ states. The previous approaches are “bottom-up” in that they always compile from $T|+\rangle$ states upwards until the $Z_k$ gate is performed; some of these schemes (notably the recent one by Duclos-Cianci and Svore [56]) reduce resources by including intermediate targets, but ultimately they all start from $T|+\rangle$ preparations at the lowest level. By starting from the top, we avoid the need to probe all the way to the bottom most of the time. As we will see, this results in savings in the number of operations needed to synthesize $Z_k$ gates.

The key to our construction is a family of shortened quantum Reed-Muller codes that are defined in Appendix A. The property of these codes that we harness here is that the $QRM(1, k+2)$ codes admit the logical $Z_k$ gate transversally, namely by applying $Z_k^\dagger$ to each qubit independently. We know this because these codes satisfy the conditions we derived in Appendix B. Because of this transversality property, we can use the $QRM(1, k+2)$ code to distill $Z_k^\dagger|+\rangle$ states using circuits that are essentially the same as the one used in Refs. [26, 69] to distill $Z_2|+\rangle$ states using the 15-qubit code, a circuit that is more compact than the one originally described by Bravyi and Kitaev [34]. Specifically, if we replace the encoding circuit for $QRM(1, 4)$ with the encoding circuit for $QRM(1, k+2)$ and replace each $T$ with $Z_k$, the circuit becomes a distillation circuit for $Z_k^\dagger|+\rangle$ states. Due to the numerical results in Ref. [64] that showed that magic states which are left untwirled can still be distilled, we also omit twirling our bare input states. As an example, we depict the distillation circuit for $Z_3^\dagger$ in Fig. 3; we derived the encoding circuit for $QRM(1, 5)$ in the figure using the methods outlined in Refs. [44, 70]. We defer a proof of why these codes have the transversality property to Appendix B and instead focus on how the protocol works here. We will note here, though, that our proof generalizes the “tri-orthogonality” condition that Bravyi and Haah used to establish the transversality of $T$ gates for their codes to a lemma in coding theory proved by Ward that we call Ward’s Divisibility Test [71, 72].

Using the $QRM(1, k+2)$ code to distill $Z_k|+\rangle$ states yields the following distillation polynomial:

$$
\epsilon_{\text{out}}(\epsilon) = \frac{1 - (1 - 2\epsilon)^{2^{k+1}-1} \left[ 2\epsilon(2^{k+2} - 1) + (1 - 2\epsilon)^{2^{k+1}} \right]}{2 \left[ 1 + (2^{k+2} - 1)(1 - 2\epsilon)^{2^{k+1}} \right]}
\approx \left( 1 - 3 \cdot 2^{k+1} + 2^{2k+3} \right) \left( \epsilon^3 / 3 \right) + O(\epsilon^5).
$$

(15)

Approximate values for the distillation threshold for various values of $k$ are listed in Table I; these are the same threshold values one would have obtained if one had used the code for distilling $Z_k|+\rangle$ to distill $Z_{k+1}|+\rangle$, but the improvement in accuracy in such a case would only be to $O(\epsilon)$ instead of $O(\epsilon^3)$ by generalizing the method of Reichardt [63]. Although the distillation threshold drops as $k$ increases, it is still larger than or comparable to the thresh-
old of $\approx 1\%$ for fault-tolerant quantum computation with surface codes [25–27] for values of $k$ less than or equal to 6, where it takes the value $\epsilon_k \approx 0.85\%$. This then sets a reasonable upper limit on the size of the complex instruction set one should consider for performing $Z_k$ gates in this way; going further would place greater fidelity demands on the elementary operations than fault-tolerance does.

To achieve $\epsilon_{\text{out}} \leq \epsilon'$, one must iterate the distillation circuit

$$\ell(\epsilon', \epsilon) = \left\lfloor \frac{\log \epsilon'}{\log \epsilon_{\text{out}}(\epsilon)} \right\rfloor$$

(17)
times. The expected number of repetitions per iteration needed to achieve distillation success is

$$E[t(\epsilon)] = \frac{2^{k+2}}{1 + (2^{k+2} - 1)(1 - 2\epsilon)^{2k+1}}.$$ (18)

Unlike in the RISC protocol, in which the corrective step in the teleportation circuit added no error, in our protocol each teleportation circuit may add error in its adaptive $Z_{k-1}$ gate. Therefore, we must implement the $Z_{k-1}$ gate with low error using our protocol recursively. We require that the error in the corrective $Z_{k-1}$ gate be at most the error in the $Z_k$ gates in Fig. 3. Due to the differences in the distillation polynomials for different values of $k$, it turns out that the error in the $Z_{k-1}$ gates for the corrective step is always less than the error in the $Z_k$ gates as long as both are being implemented by magic states that have been subjected to the same number of levels of distillation using our protocols.

### B. Resource analysis

Asymptotically, our CSC protocol achieves a value of $\beta = \beta_k := \log_3(2^{k+2} - 1)$ and $\gamma = 0$. The sum $\beta + \gamma$ is less than the sum of the 15-to-1 Bravyi-Kitaev magic-state distillation $\beta$ and the Dawson-Nielsen compiling $\gamma$ for $k \leq 9$. However, since the distillation threshold drops below 0.85% after $k = 6$, as argued earlier, it is probably wisest to stop at $k = 6$. Compared to the best values we know for $\beta$ ($\approx 1.58$ by Refs. [35, 36]) and $\gamma$ (1 by Ref. [42]), our CSC protocol would appear to be only superior for $k \leq 2$. However it is important to remember, as mentioned earlier, that arguing about asymptotics in this way can be very misleading as the constants involved can be huge. Indeed, asymptotically our protocol is inferior in that it requires many more resource states than the Selinger + MEK scheme. However, we find that for a fairly long range of values of $\epsilon'$ and $k$, our protocol performs better, not becoming worse until $\epsilon' \approx 10^{-19}$ for $k = 5$ and $k = 6$ and staying comparable or better for $k = 3$ and $k = 4$ to accuracies of $\epsilon' < 10^{-70}$. Due to the discrete jumps taken in the resource requirements of our protocol, the precise analysis becomes a bit subtle. The plot in Fig. 4 gives a better feel for when it is favorable to use our CSC protocol.

An important difference in accounting for the resource demands of our protocol as compared to the RISC solution is that, while we incur no overhead from quantum compiling, we do have a potentially more resource-intensive teleportation step. While in the RISC protocol the eventual use of a distilled magic state required only a possible Clifford correction in the teleportation procedure, in the CSC protocol we have to also account for the fact that when teleporting a $Z_k|\psi\rangle$ state it may be necessary to perform a $Z_{k-1}$ correction that is accurate to at least the same $\epsilon'$.

For the CSC architecture, we only allow ourselves access to $Z_k|\psi\rangle$ states of precision $\epsilon$ and the use of QRM-based distillation routines, even for $k = 2$. Because of this, we slightly over count the resources required by not optimizing over the best routine to produce a $Z_2|\psi\rangle$ state of a desired $\epsilon'$.

We produce our counts via the following recursive formula:

$$n_{\text{states}}(k, \ell) = \left(2^{k+2} - 1\right) n_{\text{states}}(k, \ell - 1) \left[ n_{\text{states}}(k, \ell - 1) + 1 \right] \cdot E[t(\epsilon)]$$

(19)

$$+ \frac{1}{2} n_{\text{states}}(k - 1, \ell - 1),$$

where the base of the recursion is given by

$$n_{\text{states}}(2, \ell) = E[t(\epsilon)]15^\ell.$$ (20)

The factor $E[t(\epsilon)]$, which accounts for the need to repeat the protocol if an improper measurement outcome is obtained, is very nearly 1 for the first level of distillation given bare states of accuracy $\epsilon = 10^{-4}$, and is even closer to 1 at higher levels when the input states are accurate to even higher precision. The leading $2^{k+2} - 1$ is due to the number of $Z_k|\psi\rangle$ states needed at each level $\ell$ of distillation. The first term in the square brackets accounts for the fact that distilling a new state at level $\ell$ requires states already distilled to level $\ell - 1$, while the second term accounts for the fact that each of these $Z_k|\psi\rangle$ states

| $k$ | $\epsilon_{\text{out}}/\epsilon'$ | $\epsilon_k$ |
|-----|-------------------------------|----------|
| 2   | 35                            | 14.15%   |
| 3   | 155                           | 6.94%    |
| 4   | 651                           | 3.44%    |
| 5   | 2667                          | 1.71%    |
| 6   | 10,795                        | 0.85%    |
| 7   | 43,435                        | 0.43%    |
| 8   | 174,251                       | 0.21%    |
| 9   | 698,027                       | 0.11%    |
| 10  | 2,794,155                     | 0.05%    |

TABLE I: Distillation polynomials (to most significant order) and distillation thresholds for distilling $Z_k^*|\psi\rangle$ states.
from level $\ell - 1$ are injected to our circuit via teleportation and on average half will require a $Z_{k-1}$ correction, also from distillation level $\ell - 1$. The final term counts the resources needed for the final teleportation step that consumes the distilled magic state. Here, half the time we will need to perform a $Z_{k-1}$ correction which must be distilled to the same level as the $Z_k$ gate being applied.

V. CONCLUSIONS

Fig. 4 shows the results of counting resource states for the various protocols we’ve described. Interpreting the results is subtle, with our protocols performing better when using only one or two rounds of distillation and losing out later as the asymptotics take over. As mentioned earlier, our protocols are asymptotically much worse that the current state of the art, but for accuracies of $\epsilon' > 10^{-10}$, or indeed as low as $10^{-70}$ for $k = 3$ or $k = 4$, the CISC solution outperforms the RISC solution. Some of the CISC protocols show an interesting reentrant behavior, becoming better than the RISC protocol as accuracy demands increase even though they started out using more states at lower accuracies. This is due to the large steps in accuracy when another level of distillation is used in our scheme.

![FIG. 4: Log of the number of resource states required to synthesize the quantum $Z(\pi/2^k)$ gate as a function of the log of the inverse of the desired precision $\epsilon'$ for the RISC architecture described in the text and our CISC architecture.

The difference between the architectures at low precision demand reflects the fact that when the hardware error rate is already below this demand (i.e., when $\epsilon < \epsilon'$), the only gates required by our quantum CISC architecture are those used to teleport the gate $Z_k$ from the state $Z_k|+\rangle$ to the target state $|\psi\rangle$. The RISC architecture doesn’t include the $Z_k$ gate for $k > 2$, so it must instead use a quantum compiling strategy to synthesize $Z_k$ from $T|+\rangle$ states.

Our CISC architecture does have some limitations. To begin, as can be seen in Fig. 4, as $k$ increases, even at fixed precision demand $\epsilon'$, the number of gates our CISC architecture uses increases. At any fixed $\epsilon'$, even those corresponding to very low accuracies, there will be some $k$ for which the RISC architecture uses fewer gates. However, a feature not apparent in this plot but apparent from Table I is that, even before this happens, the distillation threshold for our CISC architecture drops to a point below the accuracy threshold for fault-tolerant quantum computation. Using our CISC architecture beyond $k = 6$ would be foolhardy, as suddenly the distillation of encoded instructions and not the capacity of the underlying code would set the experimental hardware demands at the physical level. For this reason, we advocate using our CISC architecture up to $k = 6$, and then relying on an external quantum compiling algorithm (but with a larger base instruction set than a quantum RISC architecture would have) to synthesize $Z_k$ rotations for larger $k$ values.

We focused on synthesizing $Z_k$ rotations for two reasons. First, numerous quantum algorithms rely on the quantum Fourier transform, which in turn is naturally decomposed into Clifford operations and $Z_k$ rotations. We thought it was important to focus on synthesizing transformations that arise in actual algorithms rather than operations that occur only in the abstract. Second, and more significantly, we were able to find a code family, the shortened quantum Reed-Muller codes, that we could leverage to create distillation protocols for $Z_k$ rotations. The key enabling property these codes possess is code divisibility. With this insight, we generalized the “tri-orthogonality” condition of Bravyi and Haah [35] to a condition we call Ward’s Divisibility Test, which recognizes its analogous role in classical coding theory [71]. We haven’t sought codes beyond the shortened quantum Reed-Muller codes that pass Ward’s Divisibility Test for admitting a $Z_k$-distillation protocol. However, we present and prove the correctness of this test in Appendix B in the hopes that others will find it helpful in the quest to improve quantum CISC architectures.

One of the overall messages of our work is that it is not optimal to first optimize the number of gates used to synthesize a universal instruction set and then optimize the number of universal instructions needed to synthesize a gate of interest, in this case, a $Z_k$ gate. Instead, one can reap significant advantages by approaching this as a single optimization problem. The best conjectured asymptotic scaling when approached as two separate problems requires a number of gates that scales as $O(\log^2(1/\epsilon'))$. By approaching this as a single optimization problem, one may be able to achieve $O(\log(1/\epsilon'))$ for the combined process.

The resource tradeoff space for implementing quantum operations with finite discrete instruction sets is an area ripe for investigation. Beyond just minimizing the number of resource states required to approximate transformations of interest (our focus here), one might be interested in minimizing other metrics, such as the number of gates, the number of qubits used, the depth of the approximating quantum circuit, or the size of the ap-
proximating quantum circuit (which is its depth times the number of qubits). Depending on the task at hand, one instruction set may be more suitable than another. Investigations along these lines help us better understand the limits and capabilities of finite-instruction-set quantum information processing.

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Appendix A: Quantum Reed-Muller codes

One of the challenges in discussing quantum Reed-Muller codes is that there is not a unique definition of what a quantum Reed-Muller code is in the literature [34, 73–77]. Fortunately, there is at least a well-established definition for what a classical Reed-Muller code is. We state the definition for classical Reed-Muller codes below, confining our attention to binary codes. We refer the reader to standard texts for the definitions of supporting concepts such as Boolean monomials and $GF(2)$ [78].

Definition 1. The $r$th-order binary Reed-Muller code of length $2^m$, denoted $RM(r, m)$, is the linear code over $GF(2)$ whose generator matrix is composed of row vectors corresponding to the Boolean monomials over $GF(2)^2^m$ of degree at most $r$.

As an example, the generator matrix for the $RM(1, 4)$ code is

$$G = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}.$$  

From this definition, the codespace of binary Reed-Muller codes is just the space of Boolean polynomials over $GF(2)^2^m$ of degree at most $r$. It is a minor combinatorial exercise to work out that the code $RM(r, m)$ has rank $k = \sum_{i=0}^{r} \binom{m}{i}$ and code distance $d = 2^{m-r}$. In standard coding theory notation, we say that the code $RM(r, m)$ is an

$$[n, k, d] = \left[2^m, \sum_{i=0}^{r} \binom{m}{i}, 2^{m-r}\right].$$

It is straightforward to work out that the dual code to $RM(r, m)$ is $RM(m - r - 1, m)$. We use this to define a quantum Reed-Muller code as a CSS code composed of $RM(r, m)$ and its dual:

Definition 2. The $r$th-order quantum binary Reed-Muller code of length $2^m$, denoted $QRM(r, m)$, is the CSS code [79, 80] whose defining $X$ and $Z$ parity check matrices are the generator matrices for $RM(r, m)$ and its dual $RM(m - r - 1, m)$ respectively.

Notice that in this definition, somewhat confusingly, the quantum parity-check matrices are formed from classical generator matrices, not classical parity-check matrices.

We are most interested in the shortened quantum binary Reed-Muller codes, which we denote by $QRM(r, m)$. These codes are formed by shortening each of the binary Reed-Muller codes from which it is formed. The process of shortening first punctures a code by removing a bit on which only row of the generator matrix has support and then expurgates it by removing the row in the generator matrix that had support on that bit. For the Reed-Muller codes, this corresponds to removing the first row and last column of the generator matrix when presented in standard form, as in Eq. (A1). In essence, shortening a Reed-Muller code restricts the space of Boolean polynomials defining the code to those which have no constant term and which also satisfy $p(0) = 0$. An equivalent way of characterizing the shortened Reed-Muller code is as the even subcode of the punctured Reed-Muller code. The parameters of the resulting quantum code are $[2^m - 1, 1]$. Code parameters for small Reed-Muller codes, their duals, and their shortened quantum constructs are listed in Table II. Notice that the length of the code $n$ does not uniquely specify which shortened quantum Reed-Muller code one is referring to for $n > 15$.

Appendix B: Criteria for a code to admit transversal $Z(\pi/2^k)$ rotations

The shortened quantum Reed-Muller codes $QRM(1, k + 2)$ admit a transversal implementation of $Z_k$ by applying $Z_k^i$ to each qubit in the code independently. This result follows, e.g., from arguments made by Campbell et al. in Ref. [77]. Another way to see this is to note that these codes obey Theorem 1 below. We offer this alternative approach because it may be generalizable in a way that others could use to find more efficient codes that admit $Z_k$ transversally. It also relies on a lemma (Lemma 1) that naturally generalizes an otherwise unusual criterion of “tri-orthogonality” noted by Bravyi and Haah [35] for the $QRM(1, 4)$ code. We believe that this Lemma, which we call Ward’s Divisibility Test, makes better contact with the classical coding theory literature.
Table II: Parameters for (primal) Reed-Muller \(R(r,m)\) codes, their duals \(R(m-r-1,1)\), and their CSS-combined shortened quantum versions \(\overline{RM}(r,m)\) for small values. Shortened \(R(0,m)\) codes have no \(X\) generator, so the resulting quantum codes are just classical codes; they are referred to by \(\varnothing\) in the table.

| \(r, m\) | \(m-r-1, m\) | \(n, k, d\) primal | \(n, k, d\) dual | \(n, k\) |
|---------|--------------|-------------------|------------------|--------|
| (0,1)   | (0,1)        | [2,1,2]           | [2,1,2]          | \(\varnothing\) |
| (0,2)   | (1,2)        | [4,1,4]           | [4,3,2]          | \(\varnothing\) |
| (0,3)   | (2,3)        | [8,1,8]           | [8,7,2]          | \(\varnothing\) |
| (1,3)   | (1,3)        | [8,4,4]           | [8,4,4]          | [7,1] |
| (0,4)   | (3,4)        | [16,1,16]         | [16,15,2]        | \(\varnothing\) |
| (1,4)   | (2,4)        | [16,5,8]          | [16,11,4]        | [15,1] |
| (0,5)   | (4,5)        | [32,1,32]         | [32,31,2]        | \(\varnothing\) |
| (1,5)   | (3,5)        | [32,6,16]         | [32,26,4]        | [31,1] |
| (2,5)   | (2,5)        | [32,16,8]         | [32,16,8]        | [31,1] |
| (0,6)   | (5,6)        | [64,1,64]         | [64,63,2]        | \(\varnothing\) |
| (1,6)   | (4,6)        | [64,7,32]         | [64,57,4]        | [63,1] |
| (2,6)   | (3,6)        | [64,22,32]        | [64,42,8]        | [63,1] |

**Theorem 1.** A quantum \([n, 1]\) CSS code \([79, 80]\) with stabilizer generators defined by the parity check matrix \(H = diag(H^X, H^Z)\) via

\[
S^X_i := \bigotimes_{j=1}^n X^H_{ij}, \quad S^Z_i := \bigotimes_{j=1}^n Z^{H^Z_{ij}}, \tag{B1}
\]

where \(H^X\) has rows \(v_1, \ldots, v_{k+2}\), implements \((Z_k)^a\) transversally if

\[
\text{wt}(v_{\sigma(1)} \cdots v_{\sigma(j)}) \equiv 0 \mod 2^{k+2-j} \tag{B2}
\]

for all \(1 \leq j \leq k+2\) and all \(\sigma \in \Sigma_j\), and

\[n \equiv a \mod 2^{k+1}, \tag{B3}\]

where ‘\(\otimes\)’ denotes the tensor product, ‘\(\text{wt}\)’ denotes the Hamming weight of a binary vector, ‘\(\Sigma_j\)’ denotes the permutation group on \(j\) items, and \(v_1 \cdots v_j\) denotes the componentwise product of \(v_1, \ldots, v_j\).

When \(a\) in this Theorem is odd, \(\gcd(a, 2^{k+1}) = 1\), which means we can use an algorithm like the extended Euclidean algorithm \([81]\) to efficiently find numbers \(x\) and \(y\) such that \(ax + 2^{k+1}y = 1\). Iterating \((Z_k)^a\) \(x\) times results in a conditional phase of \(\pi(1 - 2^{k+1})/2^k \cong \pi/2^k\); in other words, \((Z_k)^{ax} \cong Z_k\) when \(a\) is odd.

Condition (B2) generalizes the tri-orthogonality condition of Bravyi and Haah \([35]\) into a kind of \((k+1)\)-orthogonality condition. More fundamentally, we want the classical linear code generated by \(H^X\) to be a code in which every codeword has a Hamming weight divisible by \(2^{k+1}\). Ward studied such divisible codes in depth and one of his results is that \(2^{k+1}\)-divisibility is testable by the condition of Eq. (B2) \([71]\). More explicitly, Ward’s Divisibility Test is captured by Lemma 1 below. (Ward’s result is actually more general; we use a version specialized to the binary case, as noted by Proposition 4.2 in Ref. \([72]\).)

**Lemma 1** (Ward’s Divisibility Test \([71]\)). The binary linear code with generator matrix \(H^X\) whose row vectors are \(v_1, \ldots, v_{k+2}\) is divisible by \(2^{k+1}\) if and only if

\[
2^{k+2-j}|\text{wt}(v_{\sigma(1)} \cdots v_{\sigma(j)}) \tag{B4}
\]

for all \(1 \leq j \leq k+1\) and all permutations \(\sigma \in \Sigma_j\).

While Ward’s Divisibility Test has the advantage of being an explicit algorithm for testing divisibility, it is not particularly efficient, as it takes a time that is exponential in \(k\) to execute. For codes with a high degree of structure, such as the shortened \(\overline{RM}(1, k+2)\) Reed-Muller codes, demonstrating \(2^{k+1}\)-divisibility is much simpler, as noted in Ref. \([72]\).

**Proof of Theorem 1.** By Ward’s Divisibility Test, every vector \(v\) in the rowspan \(\mathcal{L}\) of \(H^X\) has a Hamming weight divisible by \(2^{k+1}\). Since the logical \(\overline{0}\) for the code is \(\overline{0} := \sum_{v \in \mathcal{L}} |v\rangle\) (ignoring normalization), the action of transversal \(Z_k\) on \(\overline{0}\) is

\[
Z_k^\otimes n |\overline{0}\rangle = \sum_{\sigma \in \Sigma_j} Z^\otimes_n |v\rangle\tag{B5}
\]

\[
= \sum_{v \in \mathcal{L}} (e^{i\pi/2^k})^{n-|v|} |v\rangle\tag{B6}
\]

\[
= \sum_{v \in \mathcal{L}} |v\rangle\tag{B7}
\]

\[
= |\overline{0}\rangle\tag{B8}
\]

Similarly, using Eq. (B3), the action of transversal \(Z_k\) on (unnormalized) \(|\overline{1}\rangle = X|\overline{0}\rangle\) is

\[
Z_k^\otimes n |\overline{1}\rangle = Z^\otimes_n X|\overline{0}\rangle\tag{B9}
\]

\[
= \sum_{v \in \mathcal{L}} Z^\otimes_n |v \oplus 1\rangle\tag{B10}
\]

\[
= \sum_{v \in \mathcal{L}} (e^{i\pi/2^k})^{n-|v|} |v \oplus 1\rangle\tag{B11}
\]

\[
= \sum_{v \in \mathcal{L}} (e^{i\pi\sigma/2^k})^{n-|1\rangle} |v \oplus 1\rangle\tag{B12}
\]

\[
= e^{i\pi\sigma/2^k} |\overline{1}\rangle,\tag{B14}
\]

where \(1 := (1, \ldots, 1)\) denotes the all-ones vector, whose appearance comes from the fact that up to local qubit basis changes, \(X = X^\otimes_n\) for all CSS codes. These actions of \(Z_k^\otimes_n\) replicate \((Z_k)^a\) on the logical basis, and therefore \(Z_k\) implements \((Z_k)^a\) transversally.\(\square\)
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[80] How errors propagate in the injection process is an understudied problem in our opinion. However, we will not consider this issue here because we are abstracting away the details of quantum error correcting codes in our analysis.