COEXISTENCE EQUILIBRIA OF EVOLUTIONARY GAMES ON GRAPHS UNDER DETERMINISTIC IMITATION DYNAMICS

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Abstract. Cooperative behaviour is often accompanied by the incentives to defect, i.e., to reap the benefits of others’ efforts without own contribution. We provide evidence that cooperation and defection can coexist under very broad conditions in the framework of evolutionary games on graphs under deterministic imitation dynamics. Namely, we show that for all graphs there exist coexistence equilibria for certain game-theoretical parameters. Similarly, for all relevant game-theoretical parameters there exists a graph yielding coexistence equilibria. Our proofs are constructive and robust with respect to various utility functions which can be considered. Finally, we briefly discuss bounds for the number of coexistence equilibria.

1. Introduction. Cooperative behaviour in complex systems and natural networks is an exciting phenomenon occurring in groups of cells, animals [4, 6, 11] and, most importantly, human societies [1], social organizations and related networks [10]. Increased levels of cooperation can lead to advanced organizational structures. It has been suggested that cooperation is the third fundamental driving force of evolution besides mutation and natural selection [12]. Naturally, in most cases, cooperative actions are accompanied by the presence of defective ones (i.e., free-riding behaviour in which individuals collect the benefits of cooperation of others without contributing themselves) and both coexist in various forms [7, 9]. The goal of this paper is to formally show that in a simple framework of evolutionary games on graphs one can easily observe omnipresence of configurations in which cooperation and defection coexist.

In standard evolutionary game theory [8, 9], infinite homogeneous populations are considered and cooperation and defection coexist in the case of Stag hunt and...
Hawk and dove games (whereby the coexistence equilibrium is unstable in the former and stable in the latter case). In recent years, numerous studies of finite and heterogenous populations (modelled by evolutionary games on graphs) revealed that the introduction of spatial structure could extend the areas of coexistence of cooperation and defection to other social-dilemma games, especially prisoner’s dilemma [5, 7, 13, 14, 15, 16]. We contribute to this line of research and provide constructive proofs showing that, under deterministic imitation dynamics, for every social dilemma parameter cooperation and defection can coexist. Similarly, we show that for each graph/network there exist game-theoretical parameters such that cooperation and defection can coexist (we find these parameters in SH and FC parameter regions). Finally, in order to quantify the ubiquity of the states in which cooperation and defection coexist, we construct a specific class of graphs such that the number of coexistence equilibria grows exponentially with the number of vertices of underlying graphs.

The paper is organized as follows. Firstly, in Section 2 we introduce our formal model of an evolutionary game on a graph as well as the concept of coexistence equilibria. Next, in Section 3, we study the stationary boundaries of clusters of cooperators and defectors. Based on this knowledge, we are able to show in Section 4 that coexistence equilibria (or fixed points) exist for all social-dilemma game parameters as well as for all graphs. Finally, we estimate the possible number of coexistence equilibria on graphs in Section 5. In Section 6 we provide a robustness analysis for our constructions to show that they could be used also if another utility function is considered. We conclude with final remarks and open problems in Section 7.

2. Evolutionary games on graphs. We consider undirected graphs whose vertices represent players and the edges/links represent the interaction between the players. Each player can either cooperate $C$ or defect $D$, the set of possible states for each vertex is thus $S = \{D, C\} = \{0, 1\}$. We use the following notation for neighbourhoods on graphs. $N_1(i)$ denotes all vertices with distance 1 from vertex $i$ and $N_{\leq 1}(i)$ includes all vertices whose distance is at most one (similarly $N_2(i)$ denotes all vertices with distance 2 etc.).

In each time step each vertex (player) determines its utility $u$ from interactions with its neighbours. This utility is given by the underlying game-theoretical parameters:

|   | C | D |
|---|---|---|
| C | a | b |
| D | c | d |

Based on the values of its utility and the utilities of its neighbours it then chooses its next state (following a dynamical rule $\varphi$).

Putting those ideas together, we can formulate formally an evolutionary game on a graph as a dynamical system in the following way (see [5] for more details):

**Definition 2.1.** An *evolutionary game on a graph* is a quintuple $(G, p, u, T, \varphi)$, where

(a) $G = (V, E)$ is a connected graph,
(b) $p = (a, b, c, d)$ are game-theoretical parameters,
(c) $u : S^V \to \mathbb{R}^V$ is a utility function,
(d) $T : \mathbb{N}_0 \to 2^V$ is an update order,
(e) $\varphi : (\mathbb{N}_0)^2 \times S^V \to S^V$ is a (generally nonautonomous) dynamical system.
Remark 1. (a) We assume that the game-theoretical parameters \( p = (a, b, c, d) \) satisfy \( \min\{a, c\} > \max\{b, d\} \), i.e., we consider the so-called social-dilemmas which include Prisoner’s dilemma (PD, \( c > a > d > b \)), Stag hunt (SH, \( a > c > d > b \)), Hawk and dove (HD, \( c > a > b > d \)) and Full cooperation (FC, \( a > c > b > d \)), see Figure 1. Without loss of generality one could assume that \( a = 1 \) and \( d = 0 \), see [5, Remark 9] for details.

(b) There are two most common choices of utility functions, either the aggregate utility

\[
u^A_i(x) = a \sum_{j \in N_1(i)} x_i x_j + b \sum_{j \in N_1(i)} x_i (1 - x_j) + c \sum_{j \in N_1(i)} (1 - x_i) x_j + d \sum_{j \in N_1(i)} (1 - x_i)(1 - x_j),
\]

for \( x \in S^V \) or the mean utility

\[
u^M_i(x) = \frac{1}{|N_1(i)|} \nu^A_i(x).
\]

In the case of regular graphs, the dynamics is the same, but it differs for irregular graphs, see [5].

(c) Two major examples of update orders (for others see [5, Section 5]) are synchronous (\( T(t) = V \) for each \( t \in \mathbb{N}_0 \)), and sequential (vertices can be ordered
so that $T(t) = \{(t + 1) \mod n\}$. However, in this paper we deal with fixed points and our results apply to any update order.

(d) In this paper we use the (deterministic) imitation dynamics $\varphi^{ID}$, in which a vertex follows the strategy in its 1-neighbourhood which currently yields the highest utility. For other dynamics (birth-death, death-birth), see [5, Remark 4].

Mathematically, we define $\varphi^{ID}$ via its components $\varphi^{ID}_i : (N_0^2 \times SV) \rightarrow S$ by

$$\varphi^{ID}_i(t + 1, t, x) = \begin{cases} x_{\text{max}} & \text{if } i \in T(t), |A_i(x)| = 1 \text{ and } A_i(x) = \{x_{\text{max}}\}, \\ x_i & \text{otherwise}, \end{cases} \quad (3)$$

where $A_i(x)$ is the set of strategies in the neighbourhood of $x$ which yield the highest utility and is given by

$$A_i(x) = \{x_k : k \in \arg\max \{u_j(x) : j \in N_{\leq 1}(i)\}\}. \quad (4)$$

The cardinality of $A_i(x)$ is used to ensure that all vertices with the highest utility have the same state. If that is not the case, the vertex preserves its current state (in order to keep the dynamics deterministic).

In this paper we study states in which cooperation and defection coexist and which remain unchanged by the dynamics $\varphi$.

**Definition 2.2.** We say that a state $x \in SV$ is a coexistence equilibrium (coexistence fixed point) of the evolutionary game on a graph $(G, p, u, T, \varphi)$ if

(a) it is a fixed point, i.e., $\varphi(t + 1, t, x) = x$ for all $t \in N_0$,

(b) it is a coexistence state, i.e., $0 < \sum_{i \in V} x_i < |V|$. 

The following observation enables us to easily consider all update orders $T$ at once.

**Lemma 2.3.** Let $T : N_0 \rightarrow 2^V$ be the synchronous update order, i.e., $T(t) = V$ for all $t \in N_0$. If a state $x \in SV$ is a coexistence equilibrium of the evolutionary game on a graph $(G, p, u, T, \varphi)$ then it is a coexistence equilibrium of any evolutionary game $(G, p, u, \tilde{T}, \varphi)$, where $\tilde{T}$ is an arbitrary update order.

**Proof.** Indeed, if $\varphi(t + 1, t, x) = x$ in the synchronous case, all vertices $i \in V = T(t)$ preserve their strategy $x_i$. Thus, if $\tilde{T}(t) \subset V$ (only a subset of vertices is being updated) we have $\varphi(t + 1, t, x) = x$ as well. \[\square\]

**Remark 2.** (i) Consequently, we consider the synchronous update order throughout the paper and shorten the nonautonomous notation $\varphi(t + 1, t, x)$ and write autonomously $\varphi(x)$ instead. However, note that the dynamical properties like the stability of fixed points (which we do not study in this paper) need not be preserved when we move from the synchronous (autonomous) to the general (nonautonomous) update order, see examples in [5].

(ii) Following the idea of the proof, we could replace synchronous update order by any fair update order in Lemma 2.3. An update order $T : N_0 \rightarrow 2^V$ is fair if for each vertex $v \in V$ and each time $t_0 \in N_0$ there exists $t > t_0$ such that $v \in T(t)$ (i.e., $v$ is updated at time $t$). See [5, Definition 15] for more details.
3. Cluster boundaries. In a deterministic imitation dynamics $\varphi^{ID}$ which we introduced above, the vertices which only have neighbours with the same state never change their strategy. Consequently, we introduce clusters of cooperators and defectors and study their boundaries on which the change of strategies could occur, see Figure 2 for the illustration and note that the importance of clusters has been already pointed out [13]. For a given state $x \in S^V$ we introduce the sets of inner cooperators (abbreviated by IC, cooperators with only cooperative neighbours) and inner defectors (ID, defectors with only defective neighbours)

$$V_{IC}(x) := \{ i \in V : x_i = 1, \text{ and } x_j = 1 \text{ for all } j \in N_1(i) \},$$
$$V_{ID}(x) := \{ i \in V : x_i = 0, \text{ and } x_j = 0 \text{ for all } j \in N_1(i) \}.$$

In contrast, if a cooperator has at least one defective neighbour, we call it a boundary cooperator (BC). Similarly, boundary defectors (BD) have at least one cooperative neighbour. For a given state $x \in S^V$ we define the set of boundary cooperators by

$$V_{BC}(x) := \{ i \in V : x_i = 1, \text{ and there exists } j \in N_1(i) \text{ with } x_j = 0 \},$$

and the set of boundary defectors by

$$V_{BD}(x) := \{ i \in V : x_i = 0, \text{ and there exists } j \in N_1(i) \text{ with } x_j = 1 \}.$$

Obviously, for all $x \in S^V$ we have $V = V_{IC}(x) \cup V_{BC}(x) \cup V_{ID}(x) \cup V_{BD}(x)$.

This definition of cluster boundaries enables us to prove the following simple sufficient condition for a state $x$ to be a coexistence equilibrium. This statement is the cornerstone of our later constructions.

**Lemma 3.1.** Let $(G, p, u, T, \varphi^{ID})$ be an evolutionary game on a graph and let $x \in S^V$ be a coexistence state. If for each $i \in V_{BD}(x)$ and each $j \in V_{BC}(x) \cap N_1(i)$ there exists $v \in V_{IC}(x) \cap N_1(j)$ such that

$$u_v > u_i > u_j,$$

(5)

then $x$ is a coexistence equilibrium of $(G, p, u, T, \varphi^{ID})$.

**Proof.** We need to prove that neither boundary cooperators nor boundary defectors change states. Indeed, the fact that for all $i \in V_{BD}(x)$ and each $j \in V_{BC}(x) \cap N_1(i)$ we have $u_i > u_j$ implies that (see (3)-(4))

$$A_i(x) = \{0\} \text{ and consequently } \varphi^{ID}_i(x) = 0 \text{ (} = x_i\text{).}$$

Similarly, each $j \in V_{BC}(x)$ has a cooperative neighbour $v \in V_{IC}(x) \cap N_1(j)$ such that for all $i \in V_{BD}(x) \cap N_1(j)$ inequalities (5) hold, which implies that

$$A_j(x) = \{1\} \text{ and consequently } \varphi^{ID}_j(x) = 1 \text{ (} = x_j\text{).}$$

**Remark 3.** The inequalities (5) are not necessary (only sufficient) for $x$ to be a fixed point. For example, the boundary defector $i$ could have a cooperating neighbour $j \in V_{BC}(x)$ with a higher utility as long as it has a defective neighbour $k \in V_{BD}(x)$ such that

$$u_k > u_j > u_i.$$

4. Construction of coexistence equilibria. First, we show that for any given game-theoretical parameters $p = (a, b, c, d)$ there exists a graph such that the evolutionary game has a coexistence equilibrium.
Theorem 4.1. For each $p = (a, b, c, d)$ and any update order $\mathcal{T}$ there exists a connected graph $G$ such that the evolutionary game on a graph $(G, p, u^M, \mathcal{T}, \varphi^{ID})$ has a coexistence equilibrium.

Proof. We construct a graph $G$ and a state configuration $x \in S^V$ of the coexistence equilibrium at once. We have four vertex types (see Figure 2). In our construction, inner cooperators (IC) and inner defectors (ID) are always vertices of degree 1 (i.e., leaf vertices). On the other hand, $m$ boundary cooperators (BC) and $m$ boundary defectors (BD) form a complete bipartite graph $K_{m,m}$ so that each boundary cooperator is connected to all $m$ boundary defectors and vice versa. Moreover, each boundary defector has exactly $\ell$ neighbouring inner defectors and each boundary cooperator has exactly $n$ neighbouring inner cooperators (see Figure 2 for illustration of this construction). For given parameters $p = (a, b, c, d)$ the respective utilities are

$$
\begin{align*}
    u^M_{IC} &= a, \\
    u^M_{BC} &= \frac{na + mb}{n + m}, \\
    u^M_{BD} &= \frac{\ell d + mc}{\ell + m}, \\
    u^M_{ID} &= d.
\end{align*}
$$

The inequalities (5) hold if

$$
a > \frac{\ell d + mc}{\ell + m} > \frac{na + mb}{n + m}.
$$

Since $a > b$ we have that $a > \frac{na + mb}{n + m}$ for all $m > 0$. Moreover, the fact that $a > d$ and $b < c$ imply that we can find $\ell$ and $m$ such that (6) hold. $\Box$
So far, we showed that for any $p = (a, b, c, d)$ there exists a graph such that a coexistence equilibrium exists. Next, we prove that for any given graph we can find a suitable parameter vector such that the evolutionary game on this graph yields a coexistence equilibrium.

**Theorem 4.2.** For each connected graph $G$ and any update order $T$ there exists a parameter vector $p = (a, b, c, d)$ such that the evolutionary game on a graph $(G, p, u^M, T, \varphi^{ID})$ has a coexistence equilibrium.

**Proof.** Firstly, we construct a coexistence equilibrium for non-complete graphs, then for complete graphs.

1. If the graph is not complete, we can pick any vertex $v$ which is not connected to all the other vertices. We define the states of vertices $x \in S^V$ by

$$x_i = \begin{cases} 1 & i \in N_{\leq 1}(v), \\ 0 & i \notin N_{\leq 1}(v), \end{cases}$$

i.e., we define a cluster of cooperators (vertices which are at most at distance one from $v$) and a set of defectors (vertices whose distance from $v$ is at least 2, this set is not necessarily a connected subgraph of $G$). In other words, we have (see Figure 3):

$$V_{IC}(x) := \{v\},$$

$$V_{BC}(x) := \{j \in V : j \in N_1(v)\},$$

$$V_{BD}(x) := \{i \in V : i \in N_2(v)\}.$$ 

If we denote by $k_{\text{max}}$ the maximal degree of the graph $G$, the utilities of the relevant vertices satisfy

$$u^M_v = a,$$

$$u^M_j \leq \frac{(k_{\text{max}} - 1)a + b}{k_{\text{max}}},$$

$$u^M_i \leq c,$$
\[
\frac{c + (k_{\text{max}} - 1)d}{k_{\text{max}}},
\]
for all \( j \in V_{BC}(x) \) and \( i \in V_{BD}(x) \).

Consequently, the former inequality in (5) holds if we choose
\[
a > c,
\]
and the latter if we choose
\[
c > (k_{\text{max}} - 1)(a - d) + b.
\]

2. If the graph is complete, i.e. \( G = K_n \) and we pick arbitrary \( m \) vertices as cooperators and the remaining \( n - m \) as defectors, we have the following utilities for cooperators and defectors (note that \( V_{ID}(x) = V_{IC}(x) = \emptyset \) in this case)
\[
\begin{align*}
  u_{M}^c &= (m - 1)a + (n - m)b, \\
  u_{M}^d &= mc + (n - m - 1)d.
\end{align*}
\]

If \( u_C = u_D \), we have that \( \varphi_i(x) = x_i \). This situation occurs if
\[
(m - 1)a + (n - m)b = mc + (n - m - 1)d.
\]

Remark 4. Note that, with the exception of the complete graph, the coexistence equilibria are stable with respect to small perturbations of parameters (since the construction is based on inequalities rather than equalities as in the case of complete graphs).

5. Number of coexistence equilibria. In this section, we construct a special narrow class of graphs which shows that there can be an exponential number of coexistence equilibria.

Theorem 5.1. For each \( n \geq 6 \), there exists a connected graph \( G \) with \( n \) vertices such that the evolutionary game on a graph \( (G, p, u^M, T, \varphi^{ID}) \) has at least \( 2^\lfloor n/3 \rfloor \) coexistence equilibria for some parameter vector \( p = (a, b, c, d) \) and any update order \( T \).

Proof. Consider the undirected cycle \( G = C_n \) of length \( n \) with vertices numbered \( 0, \ldots, n - 1 \). Let \( X \) be the set of states \( x \) on \( C_n \) such that each cluster of defectors and cooperators has at least size 3, that is
\[
X = \{ x \in \{0, 1\}^n \mid W(x) \cap \{(1, 0, 1), (1, 0, 0, 1), (0, 1, 0), (0, 1, 1, 0)\} = \emptyset \}
\]
where we write
\[
W(x) = \{(x_j \text{ mod } n, x_{j+1} \text{ mod } n, \ldots, x_{j+k-1} \text{ mod } n) \mid j, k \in \{0, \ldots, n - 1\}\}.
\]

We calculate the following mean utilities
\[
\begin{align*}
  u_{j}^M &= \frac{a + b}{2} & \text{for all } j \in V_{BC}(x), \\
  u_{i}^M &= \frac{c + d}{2} & \text{for all } i \in V_{BD}(x).
\end{align*}
\]

Thus for \( a > \frac{c + d}{2} > \frac{a + b}{2} \) the inequalities (5) hold and we get a coexistence equilibrium. It remains to estimate the cardinality of \( X \). Since for \( n = 3k \) we have \( \{(1, 1, 1), (0, 0, 0)\}^k \subseteq X \), \( |X| > 2^k \) holds. Consequently, for general \( n \), we have \( |X| > 2^{\lfloor n/3 \rfloor} \). \( \square \)
Theorem 6.1. For each equilibrium (counterpart of Theorem 4.1) parameters there exists a graph such that the evolutionary game has a coexistence connected graph $u_6$.

 Aggregate utility function. in the case of stochastic evolutionary dynamics on graphs in [2, 3], influence of graph automorphisms and the problem of attainability has been studied bound. Also some of these equilibria cannot be attained from other states (the automorphism group of the cycle has size $2^n$). Hence $y$ is a fixed point, if

$$\begin{align*}
u_j^M &= \frac{(k+\ell)a + \ell b}{k + 2\ell} \\
u_i^M &= \frac{2(\ell c + kd)}{k + 2\ell}
\end{align*}$$

for all $j \in V_{BC}(x)$, for all $i \in V_{BD}(x)$.

Remark 5. The parameter range that we used in the proof is a subset of SH, HD and PD. We can get a parameter range intersecting all 4 scenarios by considering the lexicographic product $L = C_n[G]$ where $G$ is a $k$ regular graph with $\ell$ vertices (i.e., $k < \ell$). Thus $L$ has vertex set $\{1, \ldots, n\} \times V(G)$ and $((i_1,j_1),(i_2,j_2)) \in E(L)$ if $(i_1,i_2) \in E(C_n)$ or $i_1 = i_2, (j_1,j_2) \in E(G)$. Then $L$ is a $k + 2\ell$ regular graph. Each state on $x$ can be mapped to a state $f(x)$ with $f(x)_{(i,j)} := x_i$. Define $T = \{x \in \{0,1\}^n | W(x) \cap \{(0,0),(0,1,0),(0,1,1,0)\} = \emptyset\}$. Then for a state $y$ in $f(T)$ we have

$$\begin{align*}
u_j^M &= \frac{(k+\ell)a + \ell b}{k + 2\ell} \\
u_i^M &= \frac{2(\ell c + kd)}{k + 2\ell}
\end{align*}$$

for all $j \in V_{BC}(x)$, for all $i \in V_{BD}(x)$.

Remark 6. Considering equivalence classes of states under automorphisms of the graph can heavily reduce the number of states. In this sense, it is worth noting that our constructions which yield the exponential number of coexistence equilibria are based on symmetric graphs. Notice however that the number of coexistence equilibria which differ under automorphisms in our example is at least $2^{2n(2^n)}$ (the automorphism group of the cycle has size $2n$), so we still get an exponential lower bound. Also some of these equilibria cannot be attained from other states (the influence of graph automorphisms and the problem of attainability has been studied in the case of stochastic evolutionary dynamics on graphs in [2, 3]).

6. Aggregate utility function. All above results can relatively simply be formulated for the aggregate utility function $u^A$ given by (1). First, for any given parameters there exists a graph such that the evolutionary game has a coexistence equilibrium (counterpart of Theorem 4.1).

Theorem 6.1. For each $p = (a,b,c,d)$ and any update order $T$ there exists a connected graph $G$ such that the evolutionary game on a graph $(G,p,u^A,T,\varphi^{ID})$ has a coexistence equilibrium.

Proof. We only have to modify the construction in the proof of Theorem 4.1 slightly by adding a set of $k$ cooperators to every inner cooperator (see Figure 2). In that case, the required inequalities (6) turn into

$$(k+1)a > \ell d + mc > na + mb.$$

The former inequality will always be satisfied if $k$ is large enough (i.e., if we add sufficiently many cooperating neighbours to inner cooperators in Figure 2). The latter inequality will be satisfied if $m$ is large enough (since $b < c$), i.e., if each boundary cooperator will have enough boundary cooperators.

Also in the case of the aggregate utility function, for each graph we can find admissible parameters for which the evolutionary game on a graph has a coexistence equilibrium.
Theorem 6.2. For each connected graph $G$ and any update order $T$ there exists a parameter vector $p = (a, b, c, d)$ such that the evolutionary game on a graph $(G, p, u^A, T, \varphi^{ID})$ has a coexistence equilibrium.

Proof. In this case, the construction is the same as in the proof of Theorem 4.2, we only assume that $\max\{b, d\} \leq 0$ and $\min\{a, c\} > 0$. Considering the worst case scenarios we observe that a vertex $j \in V_{BC}(x)$ can be connected to every vertex from the cooperating cluster and must be connected to at least one boundary defector. A vertex $i \in V_{BD}(x)$ can be connected to all vertices from the cooperating cluster except the vertex $v$ (yielding the highest utility). On the other hand, the vertex $i$ could be connected to exactly one cooperator and to every other defecting vertex in the graph (yielding the lowest utility). Consequently,

\[
\begin{align*}
    u_i^A & \geq a, \\
    u_j^A & \leq (k_{\text{max}} - 1) a + b, \\
    u_i^A & \leq k_{\text{max}} c, \\
    u_i^A & \geq c + (k_{\text{max}} - 1) d.
\end{align*}
\]

These estimates imply that (5) hold if $a > k_{\text{max}} c$ and $c + (k_{\text{max}} - 1) d > (k_{\text{max}} - 1) a + b$. Thus, if we set $a > k_{\text{max}} c$ and $b \ll d$ both inequalities are satisfied. \hfill $\Box$

Finally, in Theorem 5.1 we showed that there is an exponential growth of the number of coexistence equilibria with respect to the number of vertices of the underlying graph. Since the construction in the proof was based on 2-regular graphs (on which the mean and aggregate utility functions $u^M$ and $u^A$ satisfy $u^A = 2 u^M$), we can straightforwardly claim the same statement for the aggregate utility function.

Theorem 6.3. For each $n \geq 6$, there exists a connected graph $G$ with $n$ vertices such that the evolutionary game on a graph $(G, p, u^A, T, \varphi^{ID})$ has at least $2^{\lfloor n/3 \rfloor}$ coexistence equilibria for some parameter vector $p = (a, b, c, d)$ and any update order $T$.

7. Final remarks. Evolutionary games on graphs have been studied in very complex settings under more general assumptions (large random nonconstant graphs, random updating, etc.). In this paper, we showed analytically that even in the deterministic settings the theory offers rich behaviour and yields coexistence equilibria for all graphs and for all game theoretical parameters. Consequently, we answered problems (A) and (D) which we posed in [5, Section 9]. Besides the other questions listed there we mention other issues related to coexistence equilibria worth investigation.

(A) Cluster dynamics: Our results are based on a simple observation, Lemma 3.1. This result could indicate that deeper analysis of dynamics on cluster boundaries could provide finer insight into the behaviour of evolutionary games on graphs. Specifically, since Lemma 3.1 is a sufficient condition for $x$ being a coexistence equilibrium can we, e.g., obtain a helpful necessary condition?

(B) Stability and attractivity: A natural and surprisingly nontrivial question concerns the stability of equilibria of evolutionary games on graphs. In [5, Definition 7] we defined stability via perturbation of a single vertex in a state $x$ and provided simple results for complete and $k$-regular graphs. However, it seems that this concept is very difficult to study on general graphs. Is there a sophisticated way to analyze stability of equilibria in this sense? Alternatively,
is there a better concept of stability/attractivity for evolutionary games on graphs?

(C) **Nonexistence:** Describe conditions under which there is no coexistence equilbrium for an evolutionary game on a graph.

(D) **Periodic coexistence:** In this paper we studied coexistence equilibria, i.e., fixed points. Note that we could ask similar questions related to coexistence cycles, in which we could observe periodic dynamics. Most importantly, given game-theoretical parameters \((a, b, c, d)\) (see Figure 2), can we find a graph \(G\) such that the evolutionary game on \(G\) yields a cycle (or even a cycle of given length)?

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