What are GT-shadows?

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Abstract

Let $B_4$ (resp. $PB_4$) be the braid group (resp. the pure braid group) on 4 strands and $NFI_{PB_4}(B_4)$ be the poset whose objects are finite index normal subgroups $N$ of $B_4$ that are contained in $PB_4$. In this paper, we introduce GT-shadows which may be thought of as “approximations” to elements of the profinite version $\widehat{GT}$ of the Grothendieck-Teichmueller group [7, Section 4]. We prove that GT-shadows form a groupoid whose objects are elements of $NFI_{PB_4}(B_4)$. We show that GT-shadows coming from elements of $\widehat{GT}$ satisfy various additional properties and we investigate these properties. We establish an explicit link between GT-shadows and the group $\widehat{GT}$ (see Theorem 3.8). We also present selected results of computer experiments on GT-shadows. In the appendix of this paper, we give a complete description of GT-shadows in the Abelian setting. We also prove that, in the Abelian setting, every GT-shadow comes from an element of $\widehat{GT}$. Objects very similar to GT-shadows were introduced in paper [14] by D. Harbater and L. Schneps. A variation of the concept of GT-shadows for the coarse version of $\widehat{GT}$ was studied in papers [12] and [13] by P. Guillot.

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1 Introduction

The absolute Galois group \( G_\mathbb{Q} \) of the field \( \mathbb{Q} \) of rational numbers and the Grothendieck-Teichmüller group \( \hat{\text{GT}} \) introduced by V. Drinfeld in \cite{7} are among the most mysterious objects in mathematics\cite{8,9,15,16,19,23,24,26}.

Using the outer action of \( G_\mathbb{Q} \) on the algebraic fundamental group of \( \mathbb{P}_\mathbb{Q} \setminus \{0, 1, \infty\} \), one can produce a natural group homomorphism

\[
G_\mathbb{Q} \to \hat{\text{GT}}
\]

and, due to Belyi’s theorem \cite{3}, this homomorphism is injective. Although both \( G_\mathbb{Q} \) and \( \hat{\text{GT}} \) are uncountable, it is very hard to produce explicit examples of elements in \( G_\mathbb{Q} \) and in \( \hat{\text{GT}} \). In particular, the famous question on surjectivity of \( \text{Gal}(\mathbb{P}_\mathbb{Q} \setminus \{0, 1, \infty\}) \) posed by Ihara at his ICM address \cite{16} is still open.

The group \( G_\mathbb{Q} \) can be studied by investigating finite degree field extensions of \( \mathbb{Q} \). In fact \( G_\mathbb{Q} \) coincides with the limit of the functor that sends a finite degree Galois extension \( K \) of \( \mathbb{Q} \) to the Galois group \( \text{Gal}(K/\mathbb{Q}) \). The goal of this paper is to propose a loose analog of such a functor for \( \hat{\text{GT}} \).

The most elegant definition of the group \( \hat{\text{GT}} \) involves (the profinite completion \( ̂\text{PaB} \) of) the operad \( \text{PaB} \) of parenthesized braids \cite{1}, \cite[Chapter 6]{9}, \cite{29}. \( \text{PaB} \) is an operad in the category of groupoids that is “assembled from” braid groups \( B_n \) for all \( n \geq 1 \). The objects of \( \text{PaB}(n) \) are words of the free magma generated by symbols 1, 2, \ldots, \( n \) in which each generator appears exactly once. For example, \( \text{PaB}(3) \) has exactly 12 objects: \((12)3, (21)3, (23)1, (32)1, (31)2, (13)2, 1(23), 2(13), 2(31), 3(21), 3(12), 1(32)\). For every \( n \geq 2 \) and every object \( \tau \) of \( \text{PaB} \), we have

\[
\text{Aut}_{\text{PaB}(n)}(\tau) = \text{PB}_n,
\]

where \( \text{PB}_n \) is the pure braid group on \( n \) strands.

As an operad in the category of groupoids, \( \text{PaB} \) is generated by these two morphisms:

\[
\beta := \begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
2
\end{array}
\end{array}
\]

\[
\alpha := \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
2
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
3
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
3
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bullet \\
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[1.2\]

\(^1\)This list of references is far from complete.
Moreover, any relation on $\beta$ and $\alpha$ in $\hat{\text{PaB}}$ is a consequence of the pentagon relation and the two hexagon relations (see (A.13), (A.14) and (A.15) in Appendix A.3). The hexagon relations come from two ways of connecting $(12)3$ to 3(12) and two ways of connecting 1(23) to (23)1 in $\text{PaB}(3)$. Similarly, the pentagon relation comes from two ways of connecting $((12)3)4$ to $1(2(34))$ in $\text{PaB}(4)$. For more details about the operad $\text{PaB}$ and its profinite completion $\hat{\text{PaB}}$, see Appendix A.

By definition, $\hat{\text{GT}}$ is the group $\text{Aut}(\hat{\text{PaB}})$ of (continuous) automorphisms$^2$ of the profinite completion $\hat{\text{PaB}}$ of $\text{PaB}$.

Since the morphisms $\beta$ and $\alpha$ from $\{1,2\}$ are topological generators of $\hat{\text{PaB}}$, every $\hat{T} \in \hat{\text{GT}}$ is uniquely determined by its values

$$\hat{T}(\beta) \in \text{Hom}_{\hat{\text{PaB}}}(\langle (1,2), (2,1) \rangle), \quad \hat{T}(\alpha) \in \text{Hom}_{\hat{\text{PaB}}}(\langle (1,2)3, 1(2,3) \rangle).$$

Moreover, since $\text{Aut}_{\hat{\text{PaB}}}(\langle 1,2,3 \rangle) = \hat{\text{PB}}_3$, $\text{Aut}_{\hat{\text{PaB}}}(\langle (1,2) \rangle) = \hat{\text{PB}}_2$ and $\hat{\text{PB}}_2 \cong \hat{\mathbb{Z}}$, the underlying set of $\hat{\text{GT}}$ can be identified with the subset of pairs $(\hat{m}, \hat{f}) \in \hat{\mathbb{Z}} \times \hat{\text{PB}}_3$ satisfying some relations and technical conditions.

Recall that $\hat{\text{PB}}_3$ is isomorphic to the direct product $\hat{\text{F}}_2 \times \hat{\mathbb{Z}}$ of the free group $\hat{\text{F}}_2$ on two generators and the infinite cyclic group. The $\hat{\text{F}}_2$-factor is generated by the two standard generators $x_{12}, x_{23}$ and the $\hat{\mathbb{Z}}$-factor is generated by the element $c := x_{23}x_{12}x_{13}$. In this paper, we tacitly identify $\hat{\text{F}}_2$ (resp. its profinite completion $\hat{\text{F}}_2$) with the subgroup $\langle x_{12}, x_{23} \rangle \leq \hat{\text{PB}}_3$ (resp. the topological closure of $\langle x_{12}, x_{23} \rangle$ in $\hat{\text{PB}}_3$). Occasionally, we denote the standard generators of $\hat{\text{F}}_2$ by $x$ and $y$.

One can show$^2$ (see, for example, Corollary 2.21 in Section 2 of this paper) that, for every $\hat{T} \in \hat{\text{GT}}$, the corresponding element $\hat{f} \in \hat{\text{PB}}_3$ belongs to the topological closure $([\hat{\text{F}}_2, \hat{\text{F}}_2])^{\text{cl}}$ of the commutator subgroup of $\hat{\text{F}}_2$.

\textbf{Remark 1.1} Due to Proposition 2.18 the restriction of every (continuous) automorphism $\hat{T} \in \text{Aut}(\hat{\text{PaB}})$ to $\hat{\text{F}}_2 \leq \hat{\text{PB}}_3 = \text{Aut}_{\hat{\text{PaB}}}(\langle 1,2,3 \rangle)$ gives us an automorphism of $\hat{\text{F}}_2$. In fact, many authors introduce $\hat{\text{GT}}$ as the subgroup of (continuous) automorphisms of $\hat{\text{F}}_2$ of the form

$$x \mapsto \hat{x}^\lambda, \quad y \mapsto \hat{f}^{-1}y^\lambda \hat{f},$$

where the pair $(\hat{\lambda}, \hat{f}) \in \hat{\mathbb{Z}}^\times \times ([\hat{\text{F}}_2, \hat{\text{F}}_2])^{\text{cl}}$ satisfies certain cocycle relations and the “invertibility condition.” Another equivalent definition of $\hat{\text{GT}}$ is based on the use of the outer automorphisms of the profinite completions of the pure mapping class groups. For more details about this definition, we refer the reader to [15, Main Theorem].

\textbf{Remark 1.2} It is known (see [20, Theorem 2]) that, for every $(\hat{m}, \hat{f}) \in \hat{\text{GT}}$, the element $\hat{f}$ satisfies further rather subtle properties. It would be interesting to investigate whether $\hat{\text{GT}}$-shadows satisfy consequences of these properties.

### 1.1 The link between $G_Q$ and $\hat{\text{GT}}$

For completeness, we briefly recall here the link between the absolute Galois group $G_Q$ of rationals and the Grothendieck-Teichmueller group $\hat{\text{GT}}$.

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$^2$We tacitly assume that our automorphisms act as identity on objects.

$^3$This statement can also be found in many introductory papers on $\hat{\text{GT}}$. 

3
Applying the basic theory of the algebraic fundamental group \[11\], \[28\] Section 5.6 to 
\[\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\},\]
we get an outer action of the absolute Galois group \( G_\mathbb{Q} \) on \( \hat{\mathbb{F}}_2 \). Using the fact that this action preserves the inertia subgroups, we can lift this outer action to an honest action of the form
\[g(x) = x^{\chi(g)}, \quad g(y) = \hat{f}_g(x, y)^{-1} y^{\chi(g)} \hat{f}_g(x, y), \quad g \in G_\mathbb{Q},\]  
(1.4)
where \( \chi : G_\mathbb{Q} \to \hat{\mathbb{Z}}^\times \) is the cyclotomic character and \( \hat{f}_g(x, y) \) is an element of \((\hat{\mathbb{F}}_2, \hat{\mathbb{F}}_2)^{cl}\) that depends only on \( g \).

It is known \[7\] Section 4], \[16\] Section 3], \[28\] Theorem 4.7.7], \[28\] Fact 4.7.8] that,
• \( \forall g \in G_\mathbb{Q} \), the pair \( ((\chi(g) - 1)/2, \hat{f}_g(x, y)) \in \hat{\mathbb{Z}} \times \hat{\mathbb{F}}_2 \) defines an element of \( \hat{\mathbb{G}}_T \);
• the assignment \( g \in G_\mathbb{Q} \mapsto ((\chi(g) - 1)/2, \hat{f}_g(x, y)) \in \hat{\mathbb{Z}} \times \hat{\mathbb{F}}_2 \) defines the group homomorphism \( (1.1) \).
• finally, using Belyi’s theorem \[3\], one can prove that the homomorphism \( (1.1) \) is injective.

For more details, we refer the reader to \[17\].

1.2 The groupoid \( G_{TSh} \) of \( G_T \)-shadows and its link to \( \hat{\mathbb{G}}_T \)
Let us denote by \( \text{PaB}^{\leq 4} \) the truncation of the operad \( \text{PaB} \) up to arity 4, i.e.
\[\text{PaB}^{\leq 4} := \text{PaB}(1) \sqcup \text{PaB}(2) \sqcup \text{PaB}(3) \sqcup \text{PaB}(4).\]

Moreover, let \( \text{NFI}_{\text{PB}_4}(B_4) \) be the poset of finite index normal subgroups \( N \triangleleft B_4 \) such that \( N \leq \text{PB}_4 \).

To every \( N \in \text{NFI}_{\text{PB}_4}(B_4) \), we assign an equivalence relation \( \sim_N \) on \( \text{PaB}^{\leq 4} \) that is compatible with the structure of the truncated operad and the composition of morphisms. For every \( N \in \text{NFI}_{\text{PB}_4}(B_4) \), the quotient
\[\text{PaB}^{\leq 4} / \sim_N\]
is a truncated operad in the category of finite groupoids.

In this paper, we introduce a groupoid \( G_{TSh} \) whose objects are elements of \( \text{NFI}_{\text{PB}_4}(B_4) \). Morphisms from \( N \) to \( N \) are isomorphisms of truncated operads
\[\text{PaB}^{\leq 4} / \sim_N \xrightarrow{\cong} \text{PaB}^{\leq 4} / \sim_N.\]  
(1.5)

We call these isomorphisms \( G_T \)-shadows.

Just as \( \text{PaB} \), the truncated operad \( \text{PaB}^{\leq 4} \) is generated by the braiding \( \beta \in \text{PaB}(2) \) and the associator \( \alpha \in \text{PaB}(3) \). Hence morphisms of \( G_{TSh} \) to \( N \in \text{NFI}_{\text{PB}_4}(B_4) \) are in bijection with pairs
\[(m + N_{\text{ord}} \mathbb{Z}, f_{\text{NPB}_3}) \in \mathbb{Z}/N_{\text{ord}} \mathbb{Z} \times \text{PB}_3/\text{NPB}_3,\]  
(1.6)
that satisfy appropriate versions of the hexagon relations, the pentagon relation and some technical conditions. Here, the integer \( N_{\text{ord}} \) and the (finite index) normal subgroup \( \text{NPB}_3 \triangleleft \text{PB}_3 \) are obtained from \( N \) via a precise procedure described in Subsection 2.2.
We denote by $\text{GT}(N)$ the set of such pairs $[(m,f)]$ and identify them with GT-shadows whose target is $N$. From now on, we denote by $[(m,f)]$ the GT-shadow represented by a pair $(m,f) \in \mathbb{Z} \times \text{PB}_3$.

A GT-shadow $[(m,f)] \in \text{GT}(N)$ is called genuine if there exists an element $\hat{T} \in \hat{\text{GT}}$ such that the diagram

$$
\begin{array}{ccc}
\text{PaB}^{\leq 4} & \xrightarrow{\hat{T}} & \text{PaB}^{\leq 4} \\
\downarrow & & \downarrow \\
\text{PaB}^{\leq 4} / \sim_N & \xrightarrow{\sim} & \text{PaB}^{\leq 4} / \sim_N,
\end{array}
$$

(1.7)

commutes. In (1.7), the lower horizontal arrow is the isomorphism corresponding to $[(m,f)]$ and the vertical arrows are the canonical projections. If such $\hat{T}$ does not exist, we say that the GT-shadow $[(m,f)]$ is fake.

In this paper, we show that genuine GT-shadows satisfy additional conditions. For example, every genuine GT-shadow in $\text{GT}(N)$ can be represented by a pair $(m,f)$ with

$$
f \in [F_2,F_2],
$$

(1.8)

where $[F_2,F_2]$ is the commutator subgroup of $F_2 \leq \text{PB}_3$.

A GT-shadow $[(m,f)]$ satisfying all these additional conditions (see Definition 2.19) is called charming. In this paper, we show that charming GT-shadows form a subgroupoid of $\text{GTSh}$ and we denote this subgroupoid by $\text{GTSh}^\heartsuit$.

The groupoid $\text{GTSh}^\heartsuit$ is highly disconnected. However, it is easy to see that, for every $N \in \text{NFI}_{\text{PB}_3}(B_4)$, the connected component $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ is a finite groupoid (see Proposition 3.1). In all examples we have considered so far (see [4] and Section 4 of this paper), $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ has at most two objects and, for many of examples of $N \in \text{NFI}_{\text{PB}_3}(B_4)$ the groupoid $\text{GTSh}_{\text{conn}}(N)$ has exactly one object (i.e. $\text{GT}(N)$ is a group). Such elements of $\text{NFI}_{\text{PB}_3}(B_4)$ play a special role and we call them isolated. We denote by $\text{NFI}_{\text{PB}_3,\text{isolated}}(B_4)$ the subposet of isolated elements of $\text{NFI}_{\text{PB}_3}(B_4)$.

In this paper, we show that the subposet $\text{NFI}_{\text{PB}_3,\text{isolated}}(B_4)$ is cofinal (i.e., for every $N \in \text{NFI}_{\text{PB}_3}(B_4)$, there exists $K \in \text{NFI}_{\text{PB}_3,\text{isolated}}(B_4)$ such that $K \leq N$). We show that the assignment $N \mapsto \text{GT}(N)$ upgrades to a functor $\mathcal{M}L\mathcal{C}$ from the poset $\text{NFI}_{\text{PB}_3,\text{isolated}}(B_4)$ to the category of finite groups and we prove that the limit of this functor is precisely the Grothendieck-Teichmüller group $\hat{\text{GT}}$ (see Theorem 3.8).

**Remark 1.3** Recall [15] that, omitting the pentagon relation from the definition of $\hat{\text{GT}}$, we get the coarse version $\hat{\text{GT}}_0$ of the Grothendieck-Teichmüller group. It is not hard to show that $\hat{\text{GT}}_0$ is the group of continuous automorphisms of the truncated operad $\hat{\text{PaB}}^{\leq 3}$ and $\hat{\text{GT}}$ is a subgroup of $\hat{\text{GT}}_0$. In papers [12] and [13], P. Guillot studies a variant of GT-shadows for this coarse version $\hat{\text{GT}}_0$ of the Grothendieck-Teichmüller group.

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4This name was suggested to the authors by David Harbater.

5It should be mentioned that, in the computer implementation [4], we only considered GT-shadows of the form $[(m,f)]$ with $f \in F_2 \leq \text{PB}_3$. 

5
1.3 Organization of the paper

In Section 2, we introduce the poset of compatible equivalence relations on the truncated operad \( \mathcal{P}_a \mathcal{B}^{\leq 4} \), and we show that \( \text{NFI}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) can be identified with the subposet of this poset. We introduce the concept of \( \text{GT} \)-pair and show that \( \text{GT} \)-pairs coming from elements of \( \hat{\text{GT}} \) satisfy certain conditions. This consideration motivates the concept of \( \text{GT} \)-shadow (see Definition 2.9). We prove that \( \text{GT} \)-shadows form a groupoid \( \text{GTSh} \): objects of this groupoid are elements of \( \text{NFI}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) and morphisms are \( \text{GT} \)-shadows.

In Section 2, we also investigate further conditions on \( \text{GT} \)-shadows coming from elements of \( \hat{\text{GT}} \), introduce charming \( \text{GT} \)-shadows and prove that charming \( \text{GT} \)-shadows form a sub-groupoid of \( \text{GTSh} \). In this section, we introduce a natural functor \( \text{Ch}_{\text{cyclo}} \) from \( \text{GTSh} \) to the category of finite cyclic groups. We call this functor the virtual cyclotomic character.

In Section 3, we introduce an important subposet \( \text{NFI}_{\text{isolated}}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) of \( \text{NFI}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) and construct a functor \( \mathcal{ML} \) from \( \text{NFI}_{\text{isolated}}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) to the category of finite groups. In this section, we prove that the limit of the functor \( \mathcal{ML} \) is precisely the Grothendieck-Teichmueller group \( \hat{\text{GT}} \).

In Section 4, we present selected results of computer experiments. We outline the basic information about 35 selected elements of \( \text{NFI}_{\mathcal{P}_a \mathcal{B}_4}(\mathcal{B}_4) \) and the corresponding connected components of the groupoid \( \text{GTSh} \). We say a few words about selected remarkable examples. Finally, we discuss two versions of the Furusho property (see Properties 4.2 and 4.3) and list selected open questions.

In Appendix A, we give a brief reminder of (pure) braid groups, the operad \( \mathcal{P}_a \mathcal{B} \) and its completion.

In Appendix B, we give a complete description of charming \( \text{GT} \)-shadows in the Abelian setting and we prove that, in this setting, every charming \( \text{GT} \)-shadow is genuine (see Theorem B.2).

1.4 Notational conventions

For a set \( X \) with an equivalence relation and \( a \in X \) we will denote by \([a]\) the equivalence class that contains the element \( a \). For a groupoid \( \mathcal{G} \), the notation \( \gamma \in \mathcal{G} \) means that \( \gamma \) is a morphism of this groupoid.

Every finite group is tacitly considered with the discrete topology. For a group \( G \), \( \hat{G} \) denotes the profinite completion \([25]\) of \( G \). The notation \([G, G]\) is reserved for the commutator subgroup of \( G \). For a normal subgroup \( H \triangleleft G \) of finite index, we denote by \( \text{NFI}_H(G) \) the poset of finite index normal subgroups \( N \) in \( G \) such that \( N \leq H \). Moreover, \( \text{NFI}(G) := \text{NFI}_G(G) \), i.e. \( \text{NFI}(G) \) is the poset of normal finite index subgroups of a group \( G \).

For a group \( G \) and elements \( K \leq N \) of the poset \( \text{NFI}(G) \), the notation \( \mathcal{P}_N \) (resp. \( \mathcal{P}_{K, N} \)) is reserved for the reduction homomorphism \( G \to G/N \) (resp. \( G/K \to G/N \)). The notation \( \hat{\mathcal{P}}_N \) is reserved for the canonical (continuous) homomorphism from \( \hat{G} \) to \( G/N \). Similar notation is used for the canonical functors to finite quotients of a groupoid.

The notation \( B_n \) (resp. \( \mathcal{P}_B_n \)) is reserved for the Artin braid group on \( n \) strands (resp. the pure braid group on \( n \) strands). \( S_n \) denotes the symmetric group on \( n \) letters. The standard generators of \( B_n \) are denoted by \( \sigma_1, \ldots, \sigma_{n-1} \) and the standard generators of \( \mathcal{P}_B_n \) are denoted by \( x_{ij} \) (for \( 1 \leq i < j \leq n \)). We will tacitly identify the free group \( F_2 \) on two generators with the subgroup \( \langle x_{12}, x_{23} \rangle \) of \( \mathcal{P}_B_3 \).
We will freely use the language of operads [6, Section 3], [9, Chapter 1], [21], [22], [27]. In this paper, we work with operads in the category of sets and in the category of (topological) groupoids. The category of topological groupoids is understood in the “strict sense.” For example, the associativity axioms for the elementary insertions\(^\odot\) (for operads in the category of groupoids) are satisfied “on the nose.”

For an integer \( q \geq 1 \), a \( q\)-truncated operad in the category of groupoids is a collection of groupoids \( \{G(n)\}_{1 \leq n \leq q} \) such that

- For every \( 1 \leq n \leq q \), the groupoid \( G(n) \) is equipped with an action of \( S_n \).
- For every triple of integers \( i, n, m \) such that \( 1 \leq i \leq n, n, m, n + m - 1 \leq q \) we have functors
  \[
  \odot_i : G(n) \times G(m) \to G(n + m - 1).
  \]  
(1.9)
- The axioms of the operad for \( \{G(n)\}_{1 \leq n \leq q} \) are satisfied in the cases where all the arities are \( \leq q \).

For every operad \( \mathcal{O} \) and every integer \( q \geq 1 \), the disjoint union \( \mathcal{O}^{\leq q} := \bigsqcup_{n=0}^{q} \mathcal{O}(n) \) is clearly a \( q\)-truncated operad. In this paper, we only consider \( 4\)-truncated operads. So we will simply call them truncated operads.

The operad \( \mathbb{P} B \) of parenthesized braids, its truncation \( \mathbb{P} B^{\leq 4} \) and its completion \( \widehat{\mathbb{P} B}^{\leq 4} \) play the central role in this paper. See Appendix A for more details.

Acknowledgement. We are thankful to Benjamin Collas, David Harbater, Julia Hartmann, Florian Pop, Leila Schneps, Dmitry Vaintrob and John Voight for useful discussions. V.A.D. is thankful to Pavol Severa for showing him the works of Pierre Guillot. V.A.D. discussed vague ideas of the construction presented in this paper during a walk near Zürich with Thomas Willwacher in October of 2016. V.A.D. is thankful to Thomas for leaving him with the question “What are GT-shadows?” and for giving the title to this paper! V.A.D. is especially thankful to Leila Schneps for her unbounded enthusiasm about this project and its possible continuations. V.A.D. is also thankful to Leila Schneps for her suggestion to modify the original definition of charming GT-shadows, her comments about the introduction, and her input concerning hypothetical versions of Furusho’s property. A.A.L. acknowledges both the Temple University Honors Program and the Undergraduate Research Program for their active support of undergraduate researchers. V.A.D. and K.Q.L. acknowledge a partial support from NSF grant DMS-1501001.

2 GT-pairs and GT-shadows

2.1 The poset of compatible equivalence relations on \( \mathbb{P} B^{\leq 4} \)

An equivalence relation \( \sim \) on the disjoint union of groupoids,\(^6\)

\[
\mathbb{P} B^{\leq 4} = \mathbb{P} B(1) \sqcup \mathbb{P} B(2) \sqcup \mathbb{P} B(3) \sqcup \mathbb{P} B(4)
\]

\(^6\)In the literature, elementary insertions are sometimes called partial compositions.

\(^7\)Recall that \( \mathbb{P} B(0) \) is the empty groupoid.
is an equivalence relation on the set of morphisms of $\mathsf{PaB}^{\leq 4}$ such that, if $\gamma \sim \tilde{\gamma}$, then the source (resp. the target) of $\gamma$ coincides with the source (resp. the target) of $\tilde{\gamma}$. In particular, $\gamma \sim \tilde{\gamma}$ implies that $\gamma$ and $\tilde{\gamma}$ have the same arity.

**Definition 2.1** An equivalence relation $\sim$ on $\mathsf{PaB}^{\leq 4}$ is called **compatible** if

- for every pair of composable morphisms $\gamma, \tilde{\gamma} \in \mathsf{PaB}(n)$ the equivalence class of the composition $\gamma \cdot \tilde{\gamma}$ depends only on the equivalence classes of $\gamma$ and $\tilde{\gamma}$;

- for every $\gamma, \tilde{\gamma} \in \mathsf{PaB}(n)$ and every $\theta \in S_n$
  \[ \gamma \sim \tilde{\gamma} \iff \theta(\gamma) \sim \theta(\tilde{\gamma}); \]

- for every tuple of integers $i, n, m$, $1 \leq i \leq n$, $n, m, n + m - 1 \leq 4$, and every $\gamma_1, \tilde{\gamma}_1 \in \mathsf{PaB}(n)$, $\gamma_2, \tilde{\gamma}_2 \in \mathsf{PaB}(m)$ we have
  \[ \gamma_1 \sim \tilde{\gamma}_1 \Rightarrow \gamma_1 \circ_i \gamma_2 \sim \gamma_1 \circ_i \tilde{\gamma}_2, \quad \gamma_2 \sim \tilde{\gamma}_2 \Rightarrow \gamma_1 \circ_i \gamma_2 \sim \gamma_1 \circ_i \tilde{\gamma}_2. \]

It is clear that, for every compatible equivalence relation $\sim$ on $\mathsf{PaB}^{\leq 4}$, the set

\[ \mathsf{PaB}^{\leq 4}/\sim \tag{2.1} \]

of equivalence classes of morphisms in $\mathsf{PaB}^{\leq 4}$ is a truncated operad in the category of groupoids. The set of objects of (2.1) coincides with the set of objects of $\mathsf{PaB}^{\leq 4}$. The action of symmetric groups and the elementary insertions are defined by the formulas

\[ \theta([\gamma]) := [\theta(\gamma)], \quad \theta \in S_n, \quad \gamma \in \mathsf{PaB}(n), \]

\[ [\gamma_1] \circ_i [\gamma_2] := [\gamma_1 \circ_i \gamma_2], \quad \gamma_1 \in \mathsf{PaB}(n), \quad \gamma_2 \in \mathsf{PaB}(m). \]

The conditions of Definition 2.1 guarantee that the composition of morphisms, the action of the symmetric groups on $\mathsf{PaB}(n)/\sim$ and the elementary operadic insertions are well defined. The axioms of the (truncated) operad follow directly from their counterparts for $\mathsf{PaB}^{\leq 4}$.

Compatibly equivalent relations on $\mathsf{PaB}^{\leq 4}$ form a poset with the following obvious partial order: we say that $\sim_1 \leq \sim_2$ if $\sim_1$ is finer than $\sim_2$, i.e.

\[ \gamma \sim_1 \tilde{\gamma} \Rightarrow \gamma \sim_2 \tilde{\gamma}. \]

It is clear that, for every pair of compatible equivalence relations $\sim_1, \sim_2$ on $\mathsf{PaB}^{\leq 4}$ such that $\sim_1 \leq \sim_2$, we have a natural onto morphism of truncated operads

\[ \mathcal{P}_{\sim_1, \sim_2} : \mathsf{PaB}^{\leq 4}/\sim_1 \rightarrow \mathsf{PaB}^{\leq 4}/\sim_2. \tag{2.2} \]

Moreover, the assignment $\sim \mapsto \mathsf{PaB}^{\leq 4}/\sim$ upgrades to a functor from the poset of compatible equivalence relations to the category of truncated operads.

For every compatible equivalence relation $\sim$ on $\mathsf{PaB}^{\leq 4}$, we denote by $\mathcal{P}_\sim$ the natural (onto) morphism of truncated operads:

\[ \mathcal{P}_\sim : \mathsf{PaB}^{\leq 4} \rightarrow \mathsf{PaB}^{\leq 4}/\sim. \tag{2.3} \]
2.2 From $\text{NFI}_{PB_4}(B_4)$ to the poset of compatible equivalence relations

In this paper, we mostly consider compatible equivalence relations for which the set of morphisms of (2.1) is finite and a large supply of such compatible equivalence relations come from elements of the poset $\text{NFI}_{PB_4}(B_4)$.

For $N \in \text{NFI}_{PB_4}(B_4)$, we set

$$N_{PB_3} := \varphi_{123}^{-1}(N) \cap \varphi_{123,4}^{-1}(N) \cap \varphi_{1,2,3,4}^{-1}(N) \cap \varphi_{23}^{-1}(N)$$  \hspace{1cm} (2.4)

and

$$N_{PB_2} := \varphi_{12}^{-1}(N_{PB_3}) \cap \varphi_{12,3}^{-1}(N_{PB_3}) \cap \varphi_{1,2,3}^{-1}(N_{PB_3}) \cap \varphi_{23}^{-1}(N_{PB_3}),$$  \hspace{1cm} (2.5)

where $\varphi_{123}$, $\varphi_{123,4}$, $\varphi_{1,2,3,4}$, $\varphi_{23}$ are the homomorphisms defined in (A.16) and $\varphi_{12}$, $\varphi_{12,3}$, $\varphi_{1,23}$, $\varphi_{23}$ are the homomorphisms defined in (A.17) (see also the explicit formulas in (A.18) and (A.19)).

We claim that

**Proposition 2.2** For every $N \in \text{NFI}_{PB_4}(B_4)$, the subgroup $N_{PB_3}$ (resp. $N_{PB_2}$) is an element of the poset $\text{NFI}_{PB_3}(B_3)$ (resp. the poset $\text{NFI}_{PB_2}(B_2)$).

**Proof.** Since every subgroup of $B_2$ is normal and $N_{PB_2}$ has a finite index in $PB_2$, the statement about $N_{PB_2}$ is obvious.

It is also easy to see that $N_{PB_3}$ is a subgroup of finite index in $PB_3$. So it suffices to prove that

$$g N_{PB_3} g^{-1} \subseteq N_{PB_3} \hspace{1cm} \forall g \in B_3.$$  \hspace{1cm} (2.6)

Let $h \in N_{PB_3}$ and $g \in B_3$. Then

$$\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = ou(m(g \cdot h \cdot g^{-1}) \circ_2 id_{12}).$$  \hspace{1cm} (2.7)

Using identity (A.11), we get

$$m(g \cdot h \cdot g^{-1}) = \theta(m(g)) \cdot \theta(\chi) \cdot m(g^{-1}),$$

where $\theta = \rho(g)$ and $\chi := m(h)$.

Therefore

$$m(g \cdot h \cdot g^{-1}) \circ_2 id_{12} = (\theta(m(g)) \circ_2 id_{12}) \cdot (\theta(\chi) \circ_2 id_{12}) \cdot (m(g^{-1}) \circ_2 id_{12}).$$  \hspace{1cm} (2.7)

Combining (2.6) with (2.7), we get

$$\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = ou(\theta(m(g)) \circ_2 id_{12}) \cdot ou(\theta(\chi) \circ_2 id_{12}) \cdot ou(m(g^{-1}) \circ_2 id_{12}).$$  \hspace{1cm} (2.8)

Since

$$ou(\theta(m(g)) \circ_2 id_{12}) \cdot ou(m(g^{-1}) \circ_2 id_{12}) = ou(\theta(m(g)) \circ_2 id_{12} \cdot m(g^{-1}) \circ_2 id_{12}) =$$

$$ou((\theta(m(g)) \cdot m(g^{-1})) \circ_2 id_{12}) = ou(m(g \cdot g^{-1}) \circ_2 id_{12}) = ou(id_{(1,23),4} = 1_{B_4},$$

the element $\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) \in B_4$ can be rewritten as

$$\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = \tilde{g} \cdot ou(\theta(\chi) \circ_2 id_{12}) \cdot \tilde{g}^{-1},$$
Thus it remains to prove that
\[ \text{ou}(\theta(x) \circ_2 \text{id}_{12}) \in \mathbb{N}. \] (2.9)

For this purpose, we consider the three possible cases: \( \theta(1) = 2 \), \( \theta(2) = 2 \) and \( \theta(3) = 2 \):

- If \( \theta(1) = 2 \) then \( \text{ou}(\theta(x) \circ_2 \text{id}_{12}) = \varphi_{12,3,4}(h) \) and (2.9) is a consequence of \( h \in \varphi_{12,3,4}^{-1}(\mathbb{N}) \).
- If \( \theta(2) = 2 \) then \( \text{ou}(\theta(x) \circ_2 \text{id}_{12}) = \varphi_{1,23,4}(h) \) and (2.9) is a consequence of \( h \in \varphi_{1,23,4}^{-1}(\mathbb{N}) \).
- If \( \theta(3) = 2 \) then \( \text{ou}(\theta(x) \circ_2 \text{id}_{12}) = \gamma_{1,2,34}(h) \) and (2.9) is a consequence of \( h \in \varphi_{1,2,34}^{-1}(\mathbb{N}) \).

We proved that the element \( ghg^{-1} \) belongs to \( \varphi_{1,23,4}^{-1}(\mathbb{N}) \subseteq \text{PB}_3 \). The proofs for the remaining 4 homomorphisms \( \varphi_{123}, \varphi_{12,3,4}, \varphi_{1,2,34} \) are similar and we omit them. \( \square \)

It is clear that \( N_{\text{PB}_2} = \langle x_{12}^N \rangle \), where \( N_{\text{ord}} \) is the index of \( N_{\text{PB}_2} \) in \( \text{PB}_2 \), i.e. \( N_{\text{ord}} \) is the least common multiple of orders of \( x_{12}N_{\text{PB}_2}, x_{23}N_{\text{PB}_2}, x_{12}x_{13}N_{\text{PB}_2} \) and \( x_{13}x_{23}N_{\text{PB}_2} \) in \( \text{PB}_3/N_{\text{PB}_2} \).

Using the identities \( x_{12}x_{13} = x_{23}^{-1}c, x_{13}x_{23} = x_{12}^{-1}c \) involving the generator \( c \) (see (A.5)) of the center of \( \text{PB}_3 \), it is easy to prove the following statement:

**Proposition 2.3** Let \( N_{\text{PB}_2} = \langle x_{12}^N \rangle \) be the subgroup of \( \text{PB}_2 \) defined in (2.5). Then \( N_{\text{ord}} \) coincides with

1. the least common multiple of orders of elements \( x_{12}N_{\text{PB}_3}, x_{23}N_{\text{PB}_3} \) and \( x_{12}x_{13}N_{\text{PB}_3} \);
2. the least common multiple of orders of elements \( x_{12}N_{\text{PB}_3}, x_{23}N_{\text{PB}_3} \) and \( x_{13}x_{23}N_{\text{PB}_3} \); and
3. the least common multiple of orders of elements \( x_{12}N_{\text{PB}_3}, x_{23}N_{\text{PB}_3} \) and \( cN_{\text{PB}_3} \)

\( \square \)

Given \( N \in N_{\text{Fl}_{\text{PB}_4}(B_4)} \) and the corresponding normal subgroups \( N_{\text{PB}_4} \) and \( N_{\text{PB}_2} \), we will now define an equivalence relation \( \sim_N \) on the set of morphisms in \( \text{PaB} \leq 1 \).

The groupoid \( \text{PaB}(1) \) has exactly one object and exactly one (identity) morphism. So \( \text{PaB}(1) \) has only one equivalence relation.

For two isomorphisms \( (2 \leq n \leq 4) \)
\[ \gamma, \tilde{\gamma} \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_2), \]
we declare that \( \gamma \sim_N \tilde{\gamma} \) if and only if
\[ \text{ou}(\gamma^{-1} \cdot \tilde{\gamma}) \in N_{\text{PB}_n}, \] (2.10)
where \( N_{\text{PB}_4} := N \). In other words, \( \gamma \sim_N \tilde{\gamma} \) if and only if
\[ \tilde{\gamma} = \gamma \cdot \eta, \]
where \( \text{ou}(\eta) \in N_{\text{PB}_n} \) and the source of \( \eta \) coincides with the target of \( \eta \).

We claim that
Proposition 2.4 For every \( N \in \text{NFI}_{PB_4}(B_4) \), \( \sim_N \) is a compatible equivalence relation on \( \text{PaB}^{\leq 4} \) in the sense of Definition 2.7. Moreover, the assignment

\[
N \mapsto \sim_N
\]

upgrades to a functor from the poset \( \text{NFI}_{PB_4}(B_4) \) to the poset of compatible equivalence relations on \( \text{PaB}^{\leq 4} \).

Proof. The first property of \( \sim_N \) follows from the fact that \( N_{PB_4} := N \) (resp. \( N_{PB_3}, N_{PB_2} \)) is normal in \( B_4 \) (resp. \( B_3, B_2 \)).

The second property of \( \sim_N \) follows from the obvious identity:

\[
\text{ou}(\gamma) = \text{ou}(\theta(\gamma)) \quad \forall \, \gamma \in \text{PaB}(n), \, \theta \in S_n.
\]

The proof of the last property is based on the observation that elementary operadic insertions for \( \text{PaB} \) can be expressed in terms of the operations \( \circ_i \text{id}_r, \text{id}_r \circ_i \text{id}_r \), and the composition of morphisms in the groupoids \( \text{PaB}(3) \) and \( \text{PaB}(4) \).

Let \( n \in \{2, 3\} \) and \( \eta \) be a morphism in \( \text{PaB}(n) \) whose target coincides with its source. In particular, \( \text{ou}(\eta) \in \text{PB}_n \). Let us prove that, if \( \text{ou}(\eta) \in \text{NFI}_{PB_n} \), then, for every \( \tau \in \text{Ob}(\text{PaB}(m)) \) with \( n + m - 1 \leq 4 \), we have

\[
\text{ou}(\eta \circ_i \text{id}_r) \in \text{PB}_{n+m-1}, \quad \forall \, 1 \leq i \leq n
\]

and

\[
\text{ou}(\text{id}_r \circ_i \eta) \in \text{PB}_{n+m-1}, \quad \forall \, 1 \leq i \leq m.
\]

Let \( h = \text{ou}(\eta) \). If \( m = 2 \) then there exists \( 1 \leq j \leq n \) (resp. \( j \in \{1, 2\} \)) such that

\[
\text{ou}(m(h) \circ_j \text{id}_{12}) = \text{ou}(\eta \circ_i \text{id}_r) \quad \text{(resp. } \text{ou}(\text{id}_{12}) \circ_j m(h) = \text{ou}(\text{id}_r \circ_i \eta)).
\]

Thus, if \( m = 2 \), (2.11) and (2.12) follow directly from the definitions of \( N_{PB_3}, N_{PB_2} \) and the definitions of the homomorphisms \( \varphi_{123}, \ldots, \varphi_{12}, \ldots \) (see (A.16) and (A.17)).

If \( m = 3 \), then there exist \( j, k \in \{1, 2\} \) such that

\[
\text{ou}(\eta \circ_i \text{id}_r) = \text{ou}((\eta \circ_j \text{id}_{12}) \circ_k \text{id}_{12}).
\]

For example, if \( \eta \in \text{Hom}_{\text{PaB}}((2, 1), (2, 1)) \), then \( \text{ou}(\eta \circ_2 \text{id}_{2(1,3)}) = \text{ou}((\eta \circ_3 \text{id}_{12}) \circ_3 \text{id}_{12}). \)

Thus (2.11) for \( m = 3 \) follows from (2.11) for \( m = 2 \). Similarly, (2.12) for \( m = 3 \) follows from (2.12) for \( m = 2 \).

We will now use (2.11) and (2.12) to prove the last property of \( \sim_N \).

Consider \( \gamma_1, \tilde{\gamma}_1 \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_2) \) and \( \gamma_2, \tilde{\gamma}_2 \in \text{Hom}_{\text{PaB}(m)}(\omega_1, \omega_2) \). First suppose \( \gamma_1 \sim_N \tilde{\gamma}_1 \), so \( \tilde{\gamma}_1 = \gamma_1 \cdot \eta \) for some \( \eta \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_1) \) such that \( \text{ou}(\eta) \in \text{NFI}_{PB_n} \). It follows that

\[
\tilde{\gamma}_1 \circ_i \gamma_2 = (\gamma_1 \cdot \eta) \circ_i (\gamma_2 \cdot \text{id}_{\omega_1}) = (\gamma_1 \circ_i \gamma_2) \cdot (\eta \circ_i \text{id}_{\omega_1})
\]

Due to (2.11), \( \text{ou}(\eta \circ_i \text{id}_{\omega_1}) \in \text{NFI}_{PB_{n+m-1}} \) and hence \( \tilde{\gamma}_1 \circ_i \gamma_2 \sim \gamma_1 \circ_i \gamma_2 \).

Now suppose \( \gamma_2 \sim_N \tilde{\gamma}_2 \), so \( \tilde{\gamma}_2 = \gamma_2 \cdot \eta' \) for some \( \eta' \in \text{Hom}_{\text{PaB}(m)}(\omega_1, \omega_1) \) such that \( \text{ou}(\eta') \in \text{NFI}_{PB_m} \). It follows that

\[
\gamma_1 \circ_i \tilde{\gamma}_2 = (\gamma_1 \cdot \text{id}_{\tau_1}) \circ_i (\gamma_2 \cdot \eta') = (\gamma_1 \circ_i \gamma_2) \cdot (\text{id}_{\tau_1} \circ_i \eta')
\]

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Due to (2.12), \( (\id_\tau \circ \eta') \in N_{PB_{n+m-1}} \) and hence \( \gamma_1 \circ_1 \gamma_2 \sim \gamma_1 \circ_2 \gamma_2 \).

This completes the proof of the fact that \( \sim_N \) is indeed a compatible equivalence relation on \( \PaB^{\leq 4} \).

It is clear that, if \( \tilde{N}, N \in NFI_{PB_4}(B_4) \) and \( \tilde{N} \leq N \), then \( \tilde{N}_{PB_3} \leq N_{PB_3} \) and \( \tilde{N}_{PB_2} \leq N_{PB_2} \).

Thus the assignment \( N \mapsto \sim_N \) upgrades to a functor from the poset \( NFI_{PB_4}(B_4) \) to the poset of compatible equivalence relations. \( \square \)

Later, we will need the following technical statement about \( NFI_{PB_4}(B_4) \):

**Proposition 2.5**

A) For every \( N \in NFI(PB_3) \), there exists \( K \in NFI_{PB_4}(B_4) \) satisfying the property

\[ K_{PB_3} \leq N. \]

B) For every \( N \in NFI(PB_2) \) there exists \( K \in NFI_{PB_4}(B_4) \) such that \( K_{PB_2} \leq N \).

**Proof.** Stronger versions of these statements are proved in Subsection 3.1 (see Proposition 3.9). So we omit the proof of this proposition. \( \square \)

### 2.3 The set of GT-pairs \( GT_{pr}(N) \)

Let \( N \in NFI_{PB_4}(B_4) \) and \( \sim_N \) be the corresponding compatible equivalence relation on \( \PaB^{\leq 4} \).

Let \( N_{PB_3} \) (resp. \( N_{PB_2} \)) be the corresponding normal subgroup of \( PB_3 \) (resp. \( PB_2 \)) and \( N_{ord} \) be the index of \( N_{PB_2} \) in \( PB_2 \).

Since the groupoid \( \PaB(0) \) is empty, Theorem A.1 implies that the truncated operad \( \PaB^{\leq 4} \) is generated by morphisms \( \alpha \) and \( \beta \) shown in figure 2.1

![Fig. 2.1: The isomorphisms \( \alpha \) and \( \beta \)](image)

Moreover any relation on \( \alpha \) and \( \beta \) in \( \PaB^{\leq 4} \) is a consequence of the pentagon relation

\[
\begin{array}{c}
(1(23))4 \xrightarrow{\alpha \circ_1 \id_{12}} \xrightarrow{\alpha \circ_2 \id_{12}} 1((23)4) \\
((12)3)4 \xrightarrow{\alpha \circ_1 \id_{12}} \xrightarrow{\alpha \circ_3 \id_{12}} 1(2(34)) \\
(12)(34) \xrightarrow{\alpha \circ_1 \id_{12}} \xrightarrow{\alpha \circ_3 \id_{12}} (12)(34)
\end{array}
\]

(2.13)
and the hexagon relations

\[
\begin{align*}
(12)3 \xrightarrow{\beta \circ_1 \text{id}_{12}, \alpha} 3(12) & \quad (1, 3, 2) \alpha \\
\downarrow_{\alpha} & \\
1(23) \xrightarrow{\text{id}_{12} \circ_2 \beta} 1(32) & \quad (2, 3) \alpha^{-1} \quad (2, 3) \circ_1 \beta \\
\end{align*}
\]

\[(2.14)\]

\[
\begin{align*}
1(23) \xrightarrow{\beta \circ_2 \text{id}_{12}, \alpha^{-1}} (23)1 & \quad (1, 2, 3) \alpha \\
\downarrow_{\alpha^{-1}} & \\
(12)3 \xrightarrow{\text{id}_{12} \circ_1 \beta} (21)3 & \quad (1, 2) \alpha \\
\end{align*}
\]

\[(2.15)\]

Thus morphisms of truncated operads

\[
T : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_N
\]

are in bijection with pairs

\[(m + N_{\text{ordZ}}, f_{\text{NPB}_3}) \in \mathbb{Z}/N_{\text{ordZ}} \times \text{PB}_3/\text{NPB}_3\]  

(2.16)

satisfying the relations

\[
s_1^n x_{12}^m f^{-1} s_2 x_{23}^m f_{\text{NPB}_3} = f^{-1} s_1 s_2 (x_{13} x_{23})^m N_{\text{NPB}_3},
\]

\[
f^{-1} s_2 x_{23}^m f s_1 x_{12}^m N_{\text{NPB}_3} = s_2 s_1 (x_{12} x_{13})^m f N_{\text{NPB}_3}
\]

(2.18), (2.19)

in B_3/N_{PB_3} and

\[
\varphi_{234}(f) \varphi_{1, 23, 4}(f) \varphi_{123}(f) N = \varphi_{1, 2, 34}(f) \varphi_{12, 3, 4}(f) N
\]

in PB_3/N.

More precisely, this bijection sends a pair (2.17) to the morphism of truncated operads

\[
T_{m, f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_N
\]

(2.16)

defined by the formulas:

\[
T_{m, f}(\alpha) := [\alpha \cdot m(f)], \quad T_{m, f}(\beta) := [\beta \cdot m(x_{12}^m)],
\]

(2.17)

where m is the map from B_n to PaB(n) defined in Appendix A.2.

This observation motivates our definition of a GT-pair:

**Definition 2.6** For \(N \in \text{NFI}_{\text{PB}_4}(B_4)\), the set \(\text{GT}_{\text{pr}}(N)\) consists of pairs

\[
(m + N_{\text{ordZ}}, f_{\text{NPB}_3}) \in \mathbb{Z}/N_{\text{ordZ}} \times \text{PB}_3/\text{NPB}_3
\]

satisfying (2.18), (2.19) and (2.20). Elements of \(\text{GT}_{\text{pr}}(N)\) are called **GT-pairs**.

We will represent GT-pairs by tuples \((m, f) \in \mathbb{Z} \times \text{PB}_3\). It is straightforward to see that, if relations (2.18), (2.19) and (2.20) are satisfied for a pair \((m, f)\), then they are also satisfied for \((m + qN_{\text{ord}}, fh)\), where \(q\) is an arbitrary integer and \(h\) is an arbitrary element in \(\text{NPB}_3\). A GT-pair in \(\mathbb{Z}/N_{\text{ordZ}} \times \text{PB}_3/\text{NPB}_3\) represented by a tuple \((m, f) \in \mathbb{Z} \times \text{PB}_3\) will be often denoted by

\[
[(m, f)].
\]
For a tuple \((m, f)\) representing a GT-pair in \(\text{GT}_{\text{pr}}(N)\), we denote by \(T_{m,f}\) the corresponding morphism of truncated operads:

\[
T_{m,f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_N.
\] (2.21)

It is clear that the assignment \(\circ\) from Appendix [A.2] induces the obvious map

\[
\text{PaB}(n)/\sim_N \rightarrow B_n/\text{NPB}_n,
\] (2.22)

for every \(2 \leq n \leq 4\) and, by abuse of notation, we will use the same symbol \(\circ\) for the map (2.22).

Using this map together with the \(m : B_n \rightarrow \text{PaB}(n)\) from Appendix [A.2] and morphism (2.21), we define group homomorphisms \(B_2 \rightarrow B_2/\text{NPB}_2\), \(B_3 \rightarrow B_3/\text{NPB}_3\), \(B_4 \rightarrow B_4/\text{NP}B_4\). Restricting these homomorphisms to \(\text{PB}_2\), \(\text{PB}_3\) and \(\text{PB}_4\), we get group homomorphisms \(\text{PB}_2 \rightarrow \text{PB}_2/\text{NPB}_2\), \(\text{PB}_3 \rightarrow \text{PB}_3/\text{NPB}_3\), \(\text{PB}_4 \rightarrow \text{PB}_4/\text{NP}B_4\), respectively. More precisely,

**Corollary 2.7** For every pair \((m + N_{\text{ord}}\mathbb{Z}, fN_{\text{NPB}_3}) \in \text{GT}_{\text{pr}}(N)\), and every \(2 \leq n \leq 4\), the assignment

\[
T^{\text{B}_n}_{m,f}(g) := \circ \circ T_{m,f} \circ m(g)
\] (2.23)

defines a group homomorphism from \(B_n \rightarrow B_n/\text{NPB}_n\). The restriction of \(T^{\text{PB}_n}_{m,f}\) to \(\text{PB}_n\) gives us a group homomorphism

\[
T^{\text{PB}_n}_{m,f}(g) : \text{PB}_n \rightarrow \text{PB}_n/\text{NPB}_n.
\] (2.24)

The action of \(T^{\text{B}_4}_{m,f}\) on the generators of \(B_4\) is given by the formulas:

\[
T^{\text{B}_4}_{m,f}(\sigma_1) := \sigma_1 x_{12}^m N, \quad T^{\text{B}_4}_{m,f}(\sigma_2) := \varphi_{123}(f)^{-1}(\sigma_2 x_{23}^m) \varphi_{123}(f) N,
\] (2.25)

\[
T^{\text{B}_4}_{m,f}(\sigma_3) := \varphi_{123,4}(f)^{-1}(\sigma_3 x_{34}^m) \varphi_{123,4}(f) N.
\]

The action of \(T^{\text{B}_3}_{m,f}\) on the generators of \(B_3\) are given by the formulas:

\[
T^{\text{B}_3}_{m,f}(\sigma_1) := \sigma_1 x_{12}^m N_{\text{NPB}_3}, \quad T^{\text{B}_3}_{m,f}(\sigma_2) := f^{-1}(\sigma_2 x_{23}^m) f N_{\text{NPB}_3}.
\] (2.26)

Finally, \(T^{\text{B}_2}_{m,f}\) sends \(\sigma_1\) to \(\sigma_1 x_{12}^m N_{\text{NPB}_2}\).

**Proof.** It is clear that, for every two composable morphisms \(\gamma_1, \gamma_2 \in \text{PaB}(n)/\sim_N\), we have

\[
\circ(\gamma_1 \cdot \gamma_2) = \circ(\gamma_1) \cdot \circ(\gamma_2).
\] (2.27)

Then using \((A.11)\) from Appendix [A.2] and the compatibility of \(T_{m,f}\) with the structure of
the truncated operad we get

\[ T_{m,f}^{B_n}(g_1 \cdot g_2) = \text{ou}\left(T_{m,f}(m(g_1 \cdot g_2))\right) \]

\[ = \text{ou}\left(T_{m,f}\left((\rho(g_2)^{-1}(m(g_1)) \cdot m(g_2))\right)\right) \]

\[ = \text{ou}\left(T_{m,f}\left((\rho(g_2)^{-1}(m(g_1))) \cdot T_{m,f}(m(g_2))\right)\right) \]

\[ = \text{ou}\left((\rho(g_2)^{-1}T_{m,f}(m(g_1))) \cdot T_{m,f}(m(g_2))\right) \]

\[ = \text{ou}\left((\rho(g_2)^{-1}T_{m,f}(m(g_1))) \cdot \text{ou}\left(T_{m,f}(m(g_2))\right)\right) \]

\[ = \text{ou}\left(T_{m,f}(m(g_1))\right) \cdot \text{ou}\left(T_{m,f}(m(g_2))\right) \]

\[ = T_{m,f}^{B_n}(g_1) \cdot T_{m,f}^{B_n}(g_2) \]

whence \(T_{m,f}^{B_n}\) is a group homomorphism.

The second statement of the corollary follows immediately from the fact \(m\) is a right inverse of \(\text{ou}\) and \(T_{m,f}\) acts trivially on objects of \(\text{PaB}\).

We will now prove (2.25). The easier cases of \(T_{m,f}^{B_3}\) and \(T_{m,f}^{B_2}\) are left for the reader.

For the generator \(\sigma_1\), we have:

\[ T_{m,f}^{B_4}(\sigma_1) = \text{ou}\left(T_{m,f}(m(\sigma_1))\right) \]

\[ = \text{ou}\left(T_{m,f}(\text{id}_{(12)3} \circ_1 \beta)\right) \]

\[ = \text{ou}\left(\text{id}_{(12)3} \circ_1 [\beta \cdot m(x_{12}^m)]\right) \]

\[ = \sigma_1 x_{12}^m N. \]

For the generator \(\sigma_2\), we have:

\[ T_{m,f}^{B_4}(\sigma_2) = \text{ou}\left(T_{m,f}(m(\sigma_2))\right) \]

\[ = \text{ou}\left(T_{m,f}\left((2,3)(\text{id}_{12} \circ_2 \alpha^{-1}) \cdot (\text{id}_{12} \circ_2 \beta) \cdot (\text{id}_{12} \circ_1 \alpha)\right)\right) \]

\[ = \text{ou}\left((2,3)(\text{id}_{12} \circ_1 [m(f^{-1}) \cdot \alpha^{-1}]) \cdot (\text{id}_{12} \circ_2 [\beta \cdot m(x_{12}^m)]) \cdot (\text{id}_{12} \circ_1 [\alpha \cdot m(f)])\right) \]

\[ = \varphi_{123}(f)^{-1}(\sigma_2 x_{23}^m) \varphi_{123}(f) N. \]

Finally, for the generator \(\sigma_3\), we have:

\[ T_{m,f}^{B_4}(\sigma_3) = \text{ou}\left(T_{m,f}(m(\sigma_3))\right) \]

\[ = \text{ou}\left(T_{m,f}\left((3,4)(\alpha^{-1} \circ_1 \text{id}_{12}) \cdot (\text{id}_{12} \circ_3 \beta) \cdot (\alpha \circ_1 \text{id}_{12})\right)\right) \]

\[ = \text{ou}\left((3,4)[m(f)^{-1} \cdot \alpha^{-1}] \circ_1 \text{id}_{12}) \cdot (\text{id}_{12} \circ_3 [\beta \cdot m(x_{12}^m)]) \cdot ([\alpha \cdot m(f)] \circ_1 \text{id}_{12})\right) \]

\[ = \varphi_{12,3,4}(f)^{-1}(\sigma_3 x_{34}^m) \varphi_{12,3,4}(f) N. \]

This completes the proof of Corollary 2.7. \(\square\)

Let us now use Corollary 2.7 to prove the following statement:
Corollary 2.8 Let $N \in NFI_{PB_4}(B_4)$, $[(m, f)] \in GT_{pr}(N)$ and $c$ be the generator of the center of $PB_3$ (see (A.5)). Then

$$T_{m,f}^{PB_3}(x_{12}) = x_{12}^{2m+1} N_{PB_3}, \quad T_{m,f}^{PB_3}(x_{23}) = f^{-1}x_{23}^{2m+1} f N_{PB_3},$$

$$T_{m,f}^{PB_3}(x_{13}) = x_{13}^{m} \sigma_1^{-1} f^{-1}x_{23}^{2m+1} f \sigma_1 x_{12}^{m} N_{PB_3}, \quad T_{m,f}^{PB_3}(c) = c^{2m+1} N_{PB_3}.$$

\[\text{Proof.}\] The first equation in (2.28) is a simple consequence of the first equation in (2.26).

Using the second equation in (2.26), we get

$$T_{m,f}^{PB_3}(x_{23}) = T_{m,f}^{PB_3}(\sigma_2^{2}) = (f^{-1}(\sigma_2 x_{23}^{m}))^{2} N_{PB_3} = f^{-1}(\sigma_2 x_{23}^{m}) f N_{PB_3} = f^{-1}x_{23}^{2m+1} f N_{PB_3}.$$ 

Thus the second equation in (2.28) is proved.

Using the definition of $x_{13} := \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_1^{-1} x_{23} \sigma_1$, the first equation in (2.29) and the second equation in (2.28), we get

$$T_{m,f}^{PB_3}(x_{13}) = T_{m,f}^{PB_3}(\sigma_1^{-1} x_{23} \sigma_1) = x_{12}^{-m} \sigma_1^{-1} f^{-1}x_{23}^{2m+1} f \sigma_1 x_{12}^{m} N_{PB_3}.$$ 

Thus the first equation in (2.29) is also satisfied.

To prove the second equation in (2.29), we use the formulas (2.18), (2.19), (2.26), and the identities $x_{13}x_{23} = x_{12}^{c}$, $x_{12}x_{13} = x_{23}^{c}$ extensively.

$$T_{m,f}^{PB_3}(c) = T_{m,f}^{PB_3}((\sigma_1 \sigma_2)^{3})$$

$$= (T_{m,f}^{PB_3}(\sigma_1 \sigma_2))^{3}$$

$$= \sigma_1 x_{12}^{m} f^{-1} \sigma_2 x_{23}^{m} f \sigma_1 x_{12}^{m} f^{-1} \sigma_2 x_{23}^{m} f \sigma_1 x_{12}^{m} f^{-1} \sigma_2 x_{23}^{m} f N_{PB_3}$$

$$= f^{-1} \sigma_1 x_{12}^{m} f^{-1} \sigma_2 x_{23}^{m} f \sigma_1 x_{12}^{m} f^{-1} \sigma_2 x_{23}^{m} f N_{PB_3}$$

$$= f^{-1} \sigma_1 x_{12}^{m} c \sigma_1 x_{12}^{m} \sigma_2 x_{23}^{m} f N_{PB_3}$$

$$= f^{-1} \sigma_1 x_{12}^{m} c \sigma_1 x_{12}^{m} \sigma_2 x_{23}^{m} f N_{PB_3}$$

$$= c^{2m+1} f^{-1}(\sigma_1 \sigma_2 \sigma_1 \sigma_2) f N_{PB_3}$$

$$= c^{2m+1} N_{PB_3}.$$

\[\square\]

2.4 GT-pairs coming from automorphisms of $\widehat{PB}$

Let $N \in NFI_{PB_4}(B_4)$ and $\sim_N$ be the corresponding compatible equivalence relation. Since the groupoids $PB(n)/\sim_N$ (for $1 \leq n \leq 4$) are finite, we have a canonical continuous onto morphism of truncated operads

$$\hat{\mathcal{P}}_N : \widehat{PB} \rightarrow \widehat{PB}/\sim_N.$$

Thus, given any continuous automorphism $\hat{T} : \widehat{PB} \rightarrow \widehat{PB}$, we can produce the morphism of truncated operads

$$T_N : PB^{\leq 4} \rightarrow PB^{\leq 4}/\sim_N$$

by setting

$$T_N := \hat{\mathcal{P}}_N \circ \hat{T} \circ \mathcal{I},$$

(2.31)
where $\mathcal{I}$ is the natural embedding of truncated operads

$$\mathcal{I} : \text{PaB}^{\leq 4} \to \widehat{\text{PaB}}^{\leq 4}. \tag{2.32}$$

In other words, for every continuous automorphism of $\widehat{\text{PaB}}$ and every $N \in \text{NFI}_{\text{PB}_4}(B_4)$, we get a GT-pair $[(m, f)]$ corresponding to $T_N$. In this situation, we say that the GT-pair $[(m, f)]$ comes from the automorphism $T$.

GT-pairs coming from automorphisms of $\widehat{\text{PaB}}$ satisfy additional properties. Indeed, since $\mathcal{I}(\text{PaB}^{\leq 4})$ is dense in $\widehat{\text{PaB}}^{\leq 4}$ and the morphism $P_N \circ \hat{T} : \widehat{\text{PaB}}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$ is continuous and onto, the morphism $T_N$ is also onto.

Thus, if a GT-pair $[(m, f)]$ comes from a (continuous) automorphism of $\widehat{\text{PaB}}$ then the group homomorphisms

$$T_{m,f}^{\text{PB}_4} : \text{PB}_4 \to \text{PB}_4/N, \tag{2.33}$$

$$T_{m,f}^{\text{PB}_3} : \text{PB}_3 \to \text{PB}_3/N_{\text{PB}_3}, \tag{2.34}$$

$$T_{m,f}^{\text{PB}_2} : \text{PB}_2 \to \text{PB}_2/N_{\text{PB}_2}, \tag{2.35}$$

are onto.

GT-pairs satisfying these properties are called GT-shadows. More precisely,

**Definition 2.9** Let $N$ be a finite index normal subgroup of $B_4$ such that $N \leq \text{PB}_4$. Furthermore, let $N_{\text{PB}_3}$, $N_{\text{PB}_2}$ be the corresponding normal subgroups of $B_3$ and $B_2$, respectively and let $N_{\text{ord}}$ be the index of $N_{\text{PB}_2}$ in $\text{PB}_2$. The set $\text{GT}(N)$ consists of GT-pairs $[(m, f)] \in \text{GT}_{pr}(N)$ for which group homomorphisms (2.33), (2.34), (2.35) are onto. Elements of $\text{GT}(N)$ are called GT-shadows.

It is easy to see that homomorphism (2.35) is onto if and only if

$$(2m + 1) + N_{\text{ord}}\mathbb{Z} \text{ is a unit in the ring } \mathbb{Z}/N_{\text{ord}}\mathbb{Z}. \tag{2.36}$$

We say that a GT-pair $[(m, f)]$ is friendly if $m$ satisfies condition (2.36).

Due to the following proposition, only homomorphisms (2.34) and (2.35) matter:

**Proposition 2.10** Let $N \in \text{NFI}_{\text{PB}_4}(B_4)$ and $[(m, f)] \in \text{GT}_{pr}(N)$. The following statements are equivalent:

1. $[(m, f)]$ is a GT-shadow;

2. group homomorphisms (2.34) and (2.35) are onto;

3. the map of truncated operads $T_{m,f} : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$ is onto.

**Proof.** The implication 1. $\Rightarrow$ 2. is obvious. It is also clear that, if $T_{m,f} : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_N$ is onto then group homomorphisms (2.33), (2.34), (2.35) are onto. Thus the implication 3. $\Rightarrow$ 1. is also obvious.

It remains to prove the implication 2. $\Rightarrow$ 3.
Since group homomorphism (2.35) is onto, there exists \( \gamma_2 \in \text{Hom}_{\mathcal{PB}_4}(12, 12) \) such that
\[
T_{m,f}(\gamma_2) = [m(x_{12}^{-m})].
\]
Therefore,
\[
T_{m,f}(\beta \cdot \gamma_2) = T_{m,f}(\beta) \cdot T_{m,f}(\gamma_2) = [\beta].
\]

Since homomorphism (2.35) is onto, there exists \( \gamma_3 \in \text{Hom}_{\mathcal{PB}_4}((12)3, (12)3) \) such that
\[
T_{m,f}(\gamma_3) = [m(f^{-1})].
\]
Therefore,
\[
T_{m,f}(\alpha \cdot \gamma_3) = T_{m,f}(\alpha) \cdot T_{m,f}(\gamma_3) = T_{m,f}(\alpha) \cdot [m(f^{-1})] = [\alpha].
\]

Since, as a truncated operad in the category of groupoids, \( \mathcal{PB}^{\leq 4} \) is generated by \( \beta \) and \( \alpha \), the truncated operad \( \mathcal{PB}^{\leq 4}/\sim_\mathcal{N} \) is generated by the equivalence classes \([\beta] \in \mathcal{PB}(2)/\sim_\mathcal{N} \) and \([\alpha] \in \mathcal{PB}(3)/\sim_\mathcal{N} \).

Using the fact that \([\beta] \) and \([\alpha] \) belong to the image of \( T_{m,f} \), we conclude that the morphism of truncated operads \( T_{m,f} \) is indeed onto.

Since the implication \( 2. \Rightarrow 3. \) is established, the proposition is proved. \( \square \)

### 2.5 The groupoid \( \mathcal{GTSh} \)

Let \( N \in \text{NFI}_{\mathcal{PB}_4}(B_4) \) and \([(m, f)] \in \mathcal{GT}(N) \). The morphism of truncated operads
\[
T_{m,f} : \mathcal{PB}^{\leq 4} \to \mathcal{PB}^{\leq 4}/\sim_\mathcal{N}
\]
gives us the obvious compatible equivalence relation \( \sim_\mathcal{s} \):
\[
\gamma_1 \sim_\mathcal{s} \gamma_2 \iff T_{m,f}(\gamma_1) = T_{m,f}(\gamma_2).
\]

#### Proposition 2.11

Let \( N \in \text{NFI}_{\mathcal{PB}_4}(B_4) \), \([(m, f)] \in \mathcal{GT}(N) \) and
\[
N^s := \ker(T_{m,f}^{\mathcal{PB}_4}) \subseteq \mathcal{PB}_4.
\]
Then \( N^s \in \text{NFI}_{\mathcal{PB}_4}(B_4) \) and the compatible equivalence relation \( \sim_\mathcal{s} \) coincides with \( \sim_{N^s} \).

**Proof.** To prove the first statement, we observe that, since \( N \in \mathcal{PB}_4 \), the standard homomorphism \( \rho : B_4 \to S_4 \) induces a group homomorphism \( \tilde{\rho} : B_4/N \to S_4 \). Furthermore, using equations (2.25), it is easy to see that the composition
\[
\psi := \tilde{\rho} \circ T_{m,f}^{\mathcal{PB}_4} : B_4 \to S_4
\]
coincides with \( \rho \). Thus \( N^s \) is the kernel of a group homomorphism \( T_{m,f}^{\mathcal{PB}_4} \) from \( B_4 \) to a finite group \( B_4/N \). Hence \( N^s \) is a finite index normal subgroup of \( B_4 \). Since we also have \( N^s \subseteq \mathcal{PB}_4 \), we conclude that \( N^s \in \text{NFI}_{\mathcal{PB}_4}(B_4) \).

Although the proof of the second statement is rather technical, the main idea is to show that group homomorphisms \( T_{m,f}^{\mathcal{PB}_4} \) (for \( n = 2, 3, 4 \)) are, in some sense, compatible with the homomorphisms \( \varphi_{123}, \varphi_{1234}, \varphi_{12345}, \varphi_{12345} \), \( \varphi_{1234}, \varphi_{12345} \), \( \varphi_{12345} \) (see equations (A.18) and (A.19)). This fact is deduced from the compatibility of \( T_{m,f} \) with the structures of truncated operads. Then the desired second statement of Proposition 2.11 is a simple consequence of this compatibility property of homomorphisms \( T_{m,f}^{\mathcal{PB}_4} \) (for \( n = 2, 3, 4 \)).
Let us consider $h \in \text{PB}_n$ (for $n \in \{2, 3\}$) and denote by $\tilde{h}$ any representative of the coset $T_{m,f}^{\text{PB}_n}(h)$ in $\text{PB}_n/\text{NPB}_n$. Our first goal is to prove that, for every

$$\varphi \in \begin{cases} 
\{\varphi_{123}, \varphi_{12,3,4}, \varphi_{12,3,4}, \varphi_{1,2,3,4}, \varphi_{234}\} & \text{if } n = 3, \\
\{\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}\} & \text{if } n = 2,
\end{cases}$$

there exists $g \in \text{PB}_{n+1}/\text{NPB}_{n+1}$ such that

$$g^{-1}T_{m,f}^{\text{PB}_{n+1}}(\varphi(h))g = \varphi(\tilde{h}) \text{NPB}_{n+1}.$$  \hspace{1cm} (2.38)

Indeed, let $n = 3$ and $\varphi = \varphi_{1,23,4}$. Setting $\eta := m(h)$ and using the compatibility of $T_{m,f}$ with operadic insertions and compositions we get

$$T_{m,f}(\eta) \circ_2 \text{id}_{12} = T_{m,f}(\eta \circ_2 \text{id}_{12}).$$  \hspace{1cm} (2.39)

Applying $\text{ou}$ to the left hand side of (2.39), we get

$$\text{ou}(T_{m,f}(\eta) \circ_2 \text{id}_{12}) = \varphi_{1,23,4}(\tilde{h}) \text{NPB}_4,$$  \hspace{1cm} (2.40)

where $\tilde{h}$ is an element of the coset $T_{m,f}^{\text{PB}_3}(h)$ in $\text{PB}_3/\text{NPB}_3$.

As for the right hand side of (2.39), we have

$$T_{m,f}(\eta \circ_2 \text{id}_{12}) = T_{m,f}(\alpha_{(1,2,3)4}^{(1(2,3)4)} \cdot m(\varphi_{1,23,4}(h)) \cdot \alpha_{(1,2,3)4}^{(1(2,3)4)} = T_{m,f}(\alpha_{(1,2,3)4}^{(1(2,3)4)} \cdot T_{m,f}(m(\varphi_{1,23,4}(h))) \cdot T_{m,f}(\alpha_{(1,2,3)4}^{(1(2,3)4)}).$$

Thus

$$\text{ou}(T_{m,f}(\eta \circ_2 \text{id}_{12})) = g^{-1}T_{m,f}^{\text{PB}_4}(\varphi_{1,23,4}(h))g,$$  \hspace{1cm} (2.41)

where $g = \text{ou}(T_{m,f}(\alpha_{(1,2,3)4}^{(1(2,3)4)}))$.

Combining (2.40) with (2.41), we conclude that (2.38) holds for $n = 3$ and $\varphi = \varphi_{1,23,4}$.

Let us now consider the case when $n = 2$ and $\varphi = \varphi_{12}$.

As above, setting $\eta := m(h)$ and using the compatibility of $T_{m,f}$ with operadic insertions and compositions we get

$$\text{id}_{12} \circ_1 T_{m,f}(\eta) = T_{m,f}(\text{id}_{12} \circ_1 \eta).$$  \hspace{1cm} (2.42)

Applying $\text{ou}$ to the left hand side of (2.42), we get

$$\text{ou}(\text{id}_{12} \circ_1 T_{m,f}(\eta)) = \varphi_{12}(\tilde{h}) \text{NPB}_3,$$  \hspace{1cm} (2.43)

where $\tilde{h}$ is an element of the coset $T_{m,f}^{\text{PB}_2}(h)$ in $\text{PB}_2/\text{NPB}_2$.

The right hand side of (2.42) can be rewritten as follows:

$$T_{m,f}(\text{id}_{12} \circ_1 \eta) = T_{m,f}(m(\varphi_{12}(h))).$$

Hence

$$\text{ou}(T_{m,f}(\text{id}_{12} \circ_1 \eta)) = \text{ou}(T_{m,f}(m(\varphi_{12}(h)))) = T_{m,f}^{\text{PB}_2}(\varphi_{12}(h)).$$  \hspace{1cm} (2.44)

Combining (2.43) with (2.44), we conclude that (2.38) holds for $n = 2$ and $\varphi = \varphi_{12}$ with

$g = \text{1}_{\text{PB}_3/\text{NPB}_3}$.

The proof of (2.38) for the remaining case proceeds in the similar way.
Let us now prove that, for every \( n \in \{2, 3, 4\} \),
\[
h \in \mathbb{N}_\text{PB}_n \Rightarrow m(h) \sim_s \text{id}_{(1,2).} \quad (2.45)
\]
where \( ((1,2)\ldots) \) denotes 12 (resp. (1,2)3, ((1,2)3)4) if \( n = 2 \) (resp. \( n = 3, n = 4 \)).

For \( n = 4 \), (2.45) is a straightforward consequence of the definition of \( \mathbb{N}^\text{s} \). So let \( n = 3 \) and \( \tilde{h} \) be an element of the coset \( T_{m,f}^\text{PB}_3(h) \) in \( \text{PB}_3/\text{NPB}_3 \).

Since \( T_{m,f}^\text{PB}_3(\varphi(h)) = 1 \) in \( \text{PB}_4/\text{NP} \) for every \( \varphi \in \{ \varphi_{123}, \varphi_{12,3,4}, \varphi_{1,2,3,4}, \varphi_{12,3,4} \} \), equation (2.38) implies that
\[
\tilde{h} \in \varphi_{123}^{-1}(\text{NP}_3) \cap \varphi_{12,3,4}^{-1}(\text{NP}_3) \cap \varphi_{1,2,3,4}^{-1}(\text{NP}_3) \cap \varphi_{12,3,4}^{-1}(\text{NP}_3) \cap \varphi_{12,3,4}^{-1}(\text{NP}_3).
\]

In other words, \( \tilde{h} \in \mathbb{N}_\text{PB}_3 \) and hence \( T_{m,f}^\text{PB}_3(h) = 1 \) in \( \text{PB}_3/\text{NPB}_3 \). Thus (2.45) holds for \( n = 3 \).

Let us now consider the case \( n = 2 \) and denote by \( \tilde{h} \) an element of the coset \( T_{m,f}^\text{PB}_2(h) \) in \( \text{PB}_2/\text{NPB}_2 \).

Since \( \varphi(h) \in \mathbb{N}_\text{PB}_2 \) for every \( \varphi \in \{ \varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3} \} \) and implication (2.45) is proved for \( n = 3 \), we conclude that
\[
T_{m,f}^\text{PB}_2(\varphi(h)) = 1 \quad \forall \varphi \in \{ \varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3} \}.
\]

Therefore, equation (2.38) implies that
\[
\tilde{h} \in \varphi_{12}^{-1}(\text{NP}_2) \cap \varphi_{12,3}^{-1}(\text{NP}_2) \cap \varphi_{1,2,3}^{-1}(\text{NP}_2) \cap \varphi_{12,3}^{-1}(\text{NP}_2) \cap \varphi_{12,3}^{-1}(\text{NP}_2).
\]

In other words, \( \tilde{h} \in \mathbb{N}_\text{PB}_2 \) and hence \( T_{m,f}^\text{PB}_2(h) = 1 \) in \( \text{PB}_2/\text{NPB}_2 \). Thus implication (2.45) holds for \( n = 2 \) as well.

Let us now prove that, for every \( n \in \{2, 3, 4\} \) and \( h \in \text{PB}_n \)
\[
m(h) \sim_s \text{id}_{(1,2).} \Rightarrow h \in \mathbb{N}^\text{s}_\text{PB}_n \quad (2.46)
\]

Again, for \( n = 4 \), (2.46) is a straightforward consequence of the definition of \( \mathbb{N}^\text{s} \). So let \( h \in \text{PB}_3 \).

Since \( m(h) \sim_s \text{id}_{(1,2)3} \), \( T_{m,f}^\text{PB}_3(h) \) is the identity element of \( \text{PB}_3/\text{NPB}_3 \). Hence, equation (2.38) implies that \( \varphi(h) \in \mathbb{N}^\text{s} \) for every \( \varphi \in \{ \varphi_{123}, \varphi_{12,3,4}, \varphi_{1,2,3,4}, \varphi_{12,3,4}, \varphi_{234} \} \) or equivalently \( h \in \mathbb{N}^\text{s}_\text{PB}_3 \).

Similarly, if \( h \in \text{PB}_2 \) and \( m(h) \sim_s \text{id}_{12} \) then \( T_{m,f}^\text{PB}_2(h) \) is the identity element of \( \text{PB}_2/\text{NPB}_2 \). Hence, equation (2.38) implies that \( T_{m,f}^\text{PB}_2(\varphi(h)) = 1 \) in \( \text{PB}_3/\text{NPB}_3 \) for every
\[
\varphi \in \{ \varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3} \},
\]
or equivalently
\[
m(\varphi(h)) \sim_s \text{id}_{(1,2)3} \quad \forall \varphi \in \{ \varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3} \}.
\]

Since implication (2.46) is already proved for \( n = 3 \), we conclude that
\[
\varphi(h) \in \mathbb{N}^\text{s}_\text{PB}_3 \quad \forall \varphi \in \{ \varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3} \}.
\]

Thus \( h \in \mathbb{N}^\text{s}_\text{PB}_2 \) and (2.46) is proved for \( n = 2 \).

Let \( n \in \{2, 3, 4\} \), \( \tau \in \text{Ob(PaB}(n)) \), \( \eta \in \text{Aut}_{\text{PaB}}(\tau) \) and \( h := \text{oU}(\eta) \in \text{PB}_n \). Our next goal is to prove that
\[
h \in \mathbb{N}^\text{s}_\text{PB}_n \Leftrightarrow \eta \sim_s \text{id}_{\tau} \quad (2.47)
\]
Since \( T_{m,f} \) is compatible with the action of the symmetric groups, we may assume, without loss of generality, that the underlying permutation of \( \tau \) is the identity permutation in \( S_n \).

Therefore
\[
\eta = \alpha^\tau_{((1,2)), m(h)} \alpha^\tau_{(1,2)}.
\] (2.48)

and hence \( T_{m,f}(\eta) = \text{id}_\tau \) if and only if \( T_{m,f}(m(h)) = \text{id}_{((1,2))} \) and the latter is equivalent to \( m(h) \sim_s \text{id}_{((1,2))} \).

Thus (2.47) is a consequence of implications (2.45) and (2.46).

Finally, let us use (2.47) to prove the statement of the proposition.

Let \( \gamma, \tilde{\gamma} \in \mathcal{P}_{\mathcal{A}B}(n) \) (with \( n \in \{2, 3, 4\} \)) and \( \tau \) be the source of both morphisms. Clearly, \( \gamma \sim_s \tilde{\gamma} \) if and only if \( \eta \sim_s \text{id}_\tau \), where \( \eta = \gamma^{-1} \cdot \tilde{\gamma} \).

Thus, due to (2.47), \( \gamma \sim_s \tilde{\gamma} \) if and only if 
\[
\text{ou}(\gamma^{-1} \cdot \tilde{\gamma}) \in N^s_{\mathcal{P}_{\mathcal{A}B}n}.
\]

Proposition 2.11 is proved. \( \square \)

Proposition 2.11 has the following important consequences:

**Corollary 2.12** For every GT-shadow \([(m, f)] \in \mathcal{G}(\mathcal{N}) \)

- \( |\mathcal{P}_{\mathcal{A}B} : N^s| = |\mathcal{P}_{\mathcal{A}B} : \mathcal{N}| \),
- \( |\mathcal{P}_{\mathcal{A}B} : N_{\mathcal{P}_{\mathcal{A}B}^3}| = |\mathcal{P}_{\mathcal{A}B} : \mathcal{N}_{\mathcal{P}_{\mathcal{A}B}^3}|, \) and
- \( N^s_{\text{ord}} = N_{\text{ord}} \) or equivalently \( N^s_{\mathcal{P}_{\mathcal{A}B}^2} = \mathcal{N}_{\mathcal{P}_{\mathcal{A}B}^2} \).

\( \square \)

**Corollary 2.13** For every GT-shadow \([(m, f)] \in \mathcal{G}(\mathcal{N}) \) the morphism of truncated operads \( T_{m,f} : \mathcal{P}_{\mathcal{A}B}^{\leq 4} \to \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}} \) factors as follows

\[
\begin{array}{ccc}
\mathcal{P}_{\mathcal{A}B}^{\leq 4} & \xrightarrow{\sim_{\mathcal{N}}} & \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}} \\
\mathcal{P}_{\mathcal{A}B}^{\leq 4} & \xrightarrow{T_{m,f}} & \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}} \\
\end{array}
\]

where \( \mathcal{P}_{\mathcal{A}B}^{\leq 4} \) is the canonical projection and \( T_{m,f}^{\text{isom}} \) is an isomorphism of truncated operads.

The assignment \([(m, f)] \mapsto T_{m,f}^{\text{isom}} \) gives us a bijection from the set
\[
\{(m, f) \in \mathcal{G}(\mathcal{N}) \mid N^s = \ker(T_{m,f}^{\text{P}_{\mathcal{A}B}^4})\}
\]
to the set \( \text{Isom}(\mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}}, \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}}) \) of isomorphisms of truncated operads (in the category of groupoids).

**Proof.** Due to Proposition 2.10 and the definition of the equivalence relation \( \sim_s \), we have the following commutative diagram of morphisms of truncated operads:

\[
\begin{array}{ccc}
\mathcal{P}_{\mathcal{A}B}^{\leq 4} & \xrightarrow{T_{m,f}} & \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}} \\
\mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_s & \xrightarrow{T_{m,f}^{\text{isom}}} & \mathcal{P}_{\mathcal{A}B}^{\leq 4} / \sim_{\mathcal{N}} \\
\end{array}
\]

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with \(T_{m,f}^{\text{isom}}\) being a bijection\(^8\) on the level of morphisms.

Thanks to Proposition \ref{prop:iso-comp}, the equivalence relation \(\sim_s\) coincides with \(\sim_N\). Hence \(T_{m,f}^{\text{isom}}\) is a morphism of truncated operads

\[
T_{m,f}^{\text{isom}} : \text{PaB}^{\leq 4}/\sim_N \overset{\cong}{\longrightarrow} \text{PaB}^{\leq 4}/\sim_N. \tag{2.49}
\]

Let us denote by \(S_{m,f}^{\text{isom}} : \text{PaB}^{\leq 4}/\sim_N \rightarrow \text{PaB}^{\leq 4}/\sim_N\), the inverse of \(T_{m,f}^{\text{isom}}\) (viewed as a map of morphisms) and show that \(S_{m,f}^{\text{isom}}\) is compatible with the composition of morphisms and with the operadic insertions.

As for the compatibility with operadic insertions, we have

\[
S_{m,f}^{\text{isom}}([\gamma_1] \circ_i [\gamma_2]) = S_{m,f}^{\text{isom}}((T_{m,f}^{\text{isom}}([\tilde{\gamma}_1])) \circ_i (T_{m,f}^{\text{isom}}([\tilde{\gamma}_2])))
\]

\[
= S_{m,f}^{\text{isom}}((T_{m,f}^{\text{isom}}([\tilde{\gamma}_1] \circ_i [\tilde{\gamma}_2]))) = [\tilde{\gamma}_1] \circ_i [\tilde{\gamma}_2] = S_{m,f}^{\text{isom}}([\gamma_1]) \circ_i S_{m,f}^{\text{isom}}([\gamma_2]),
\]

for any \([\gamma_1] \in \text{PaB}(n)/\sim_N\), \([\gamma_2] \in \text{PaB}(k)/\sim_N\) and \(k \leq 4\), \(1 \leq i \leq n \leq 4\).

The compatibility of \(S_{m,f}^{\text{isom}}\) with the composition of morphisms is proved in a similar fashion.

Let us now consider an isomorphism of truncated operads

\[
T_{m,f}^{\text{isom}} : \text{PaB}^{\leq 4}/\sim_N \overset{\cong}{\longrightarrow} \text{PaB}^{\leq 4}/\sim_N.
\]

Pre-composing \(T_{m,f}^{\text{isom}}\) with the canonical projection \(\mathcal{P}_N : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_N\) we get an onto morphism \(T := T_{m,f}^{\text{isom}} \circ \mathcal{P}_N\) of truncated operads. Since \(T\) is uniquely determined by a GT-shadow \([(m,f)] \in \text{GT}(N)\) such that \(\ker(T_{m,f}^{\text{PB}_4}) = N^g\), we conclude that the assignment \([(m,f)] \mapsto T_{m,f}^{\text{isom}}\) is indeed a bijection

\[
\{(m,f) \in \text{GT}(N) \mid N^g = \ker(T_{m,f}^{\text{PB}_4})\} \overset{\cong}{\longrightarrow} \text{Isom}(\text{PaB}^{\leq 4}/\sim_N, \text{PaB}^{\leq 4}/\sim_N). \tag{2.50}
\]

Corollary \ref{cor:iso-comp} is proved. \(\square\)

Let us now observe that the assignment

\[
\text{Hom}(\bar{N}, N) := \text{Isom}(\text{PaB}^{\leq 4}/\sim_N, \text{PaB}^{\leq 4}/\sim_N) \tag{2.51}
\]

upgrades the set \(N\mathcal{F}_{\text{PB}_4}(B)\) to a groupoid. The set of objects of this groupoid is \(N\mathcal{F}_{\text{PB}_4}(B)\) and the set of morphisms from \(\bar{N}\) to \(N\) is the set \(\text{Isom}(\text{PaB}^{\leq 4}/\sim_{\bar{N}}, \text{PaB}^{\leq 4}/\sim_N)\) of isomorphisms of truncated operads (in the category of groupoids). Morphisms of this groupoid are composed in the standard way.

The second statement of Corollary \ref{cor:iso-comp} allows us to tacitly identify \(\text{PB}_4\) with the set

\[
\{(m,f) \in \text{GT}(N) \mid \ker(T_{m,f}^{\text{PB}_4}) = \bar{N}\}.
\]

We will use the identification in the remainder of this paper and we denote by \(\text{GTSh}\) the resulting groupoid of GT-shadows.

The following proposition gives us an explicit formula for the composition of morphisms in \(\text{GTSh}\):
Proposition 2.14 Let $N^{(1)}, N^{(2)}$ and $N^{(3)}$ be elements of $NFI_{PB_4}(B_4)$ and
\[ [(m_1, f_1)] \in \text{Hom}_{GTSh}(N^{(1)}, N^{(2)}), \quad [(m_2, f_2)] \in \text{Hom}_{GTSh}(N^{(2)}, N^{(3)}). \]
Then their composition $[(m_2, f_2)] \circ [(m_1, f_1)]$ is represented by the pair $(m, f)$ where
\[ m := 2m_1m_2 + m_1 + m_2, \quad f N^{(3)}_{PB_3} := f_2 N^{(3)}_{PB_3} \cdot T_{m_2,f_2}(f_1). \] 

Proof. Let $[(m_2, f_2)] \in \text{GT}(N^{(3)})$ and $[(m_1, f_1)] \in \text{GT}(N^{(2)})$, where
\[ N^{(2)} := \ker(T_{m_2,f_2}^{PB_4}) \quad \text{and} \quad N^{(1)} := \ker(T_{m_1,f_1}^{PB_4}). \]
In other words, the $\text{GT}$-shadow $[(m_1, f_1)]$ (resp. $[(m_2, f_2)]$) is a morphism from $N^{(1)}$ to $N^{(2)}$ (resp. a morphism from $N^{(2)}$ to $N^{(3)}$) in $GTSh$.
By Corollary 2.13 we have the following diagram of morphisms of truncated operads
\[
\begin{array}{ccc}
\text{PaB}^{\leq 4}/ \sim_{N^{(1)}} & \xrightarrow{T_{m,f}} & \text{PaB}^{\leq 4}/ \sim_{N^{(2)}} \\
\downarrow & & \downarrow \\
\text{PaB}^{\leq 4}/ \sim_{N^{(3)}} & \xrightarrow{T_{\text{iso}}_{m_2,f_2}} & \text{PaB}^{\leq 4}/ \sim_{N^{(3)}},
\end{array}
\]
where the vertical arrow is the canonical projection.
Formula (2.52) is obtained by looking at the image of the associator $[\alpha] \in \text{PaB}^{\leq 4}/ \sim_{N^{(1)}}$ (resp. the braiding $[\beta] \in \text{PaB}^{\leq 4}/ \sim_{N^{(1)}}$) under $T_{\text{iso}}_{m_2,f_2} \circ T_{m_1,f_1}$. For $[\alpha]$, we have
\[
T_{m,f}(\alpha) = T_{\text{iso}}_{m_2,f_2}(T_{\text{iso}}_{m_1,f_1}[\alpha]) = T_{\text{iso}}_{m_2,f_2}(T_{m_1,f_1}(\alpha)) = T_{m_2,f_2}(\alpha \cdot m(f)) = T_{m_2,f_2}(\alpha) \cdot T_{m_2,f_2}(m(f)) = [\alpha \cdot m(f)],
\]
where $f$ is any representative of the coset $f_2 N^{(3)}_{PB_3} \cdot T_{m_2,f_2}(f_1)$ in $PB_3/N^{(3)}_{PB_3}$.
Similarly, computing $T_{m,f}(\beta)$, it is easy to see that $m \equiv 2m_1m_2 + m_1 + m_2 \mod N^{(3)}_{\text{ord}}. \Box$

Remark 2.15 Later we will see that it makes sense to focus only on $\text{GT}$-shadows that can be represented by pairs $(m, f)$ with
\[ f \in F_2 \leq PB_3. \] 
Let us call such $\text{GT}$-shadows practical.
Using (2.28) and (2.52), we get the following formula for the composition $[(m, f)] := [(m_2, f_2)] \circ [(m_1, f_1)]$ of practical $\text{GT}$-shadows $[(m_2, f_2)]$ and $[(m_1, f_1)]$:
\[ m := 2m_1m_2 + m_1 + m_2, \quad f(x, y) := f_2(x, y) f_1(x^{2m_2+1}, f_2(x, y)^{-1} y^{2m_2+1} f_2(x, y)). \] 
Due to this observation, practical $\text{GT}$-shadows form a subgroupoid of $\text{GTSh}$.

The authors do not know whether there exists $N \in NFI_{PB_4}(B_4)$ and an onto morphism of truncated operads $\text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/ \sim_N$ that cannot be represented by a pair $(m, f) \in \mathbb{Z} \times F_2$. 23
Let us observe that to every \( N \in NFI_{PB_4}(B_4) \) we assign the (finite) cyclic group

\[ \text{PB}_2/\langle x_{12}^{N_{\text{ord}}} \rangle \cong \mathbb{Z}/N_{\text{ord}}\mathbb{Z}, \]

where \( N_{\text{ord}} \) is the index of \( N_{PB_2} \) in \( \text{PB}_2 \). Moreover, if \( [(m, f)] \) is a morphism from \( N^s \) to \( N \) in the groupoid \( \text{GTSh} \), then both \( N \) and \( N^s \) correspond to the same quotient \( \text{PB}_2/\langle x_{12}^{N_{\text{ord}}} \rangle \) of \( \text{PB}_2 \).

Proposition 2.14 implies that the assignment \( N \mapsto \mathbb{Z}/N_{\text{ord}}\mathbb{Z} \) upgrades to a functor \( \text{Ch}_{\text{cyclot}} \) from \( \text{GTSh} \) to the category of finite cyclic groups. More precisely,

**Corollary 2.16** Let \( [(m, f)] \) be a morphism from \( N^{(1)} \) to \( N^{(2)} \) in the groupoid \( \text{GTSh} \). The assignments

\[ N \mapsto \text{PB}_2/N_{PB_2}, \quad [(m, f)] \mapsto \text{Ch}_{\text{cyclot}}(m, f) \in \text{Aut}(\text{PB}_2/N_{PB_2}^{(2)}), \]

(2.56)

\[ \text{Ch}_{\text{cyclot}}(m, f)(x_{12}^{N_{PB_2}^{(2)}}) := x_{12}^{2m+1}N_{PB_2}^{(2)} \]

define a functor \( \text{Ch}_{\text{cyclot}} \) from the groupoid \( \text{GTSh} \) to the category of finite cyclic groups.

**Proof.** Since, for every \( \text{GT} \) shadow \( [(m, f)] \), \( 2m + 1 \) represents an invertible element of the ring \( \mathbb{Z}/N_{\text{ord}}^{(2)}\mathbb{Z} \), \( \text{Ch}_{\text{cyclot}}(m, f) \) is clearly an automorphism of \( \text{PB}_2/N_{PB_2}^{(2)} = \text{PB}_2/N_{PB_2}^{(1)} \).

Thus it remains to show that \( \text{Ch}_{\text{cyclot}} \) is compatible with the composition of \( \text{GT} \)-shadows. For this purpose, we consider two composable \( \text{GT} \)-shadows: \( [(m_1, f_1)] \in \text{Hom}_{\text{GTSh}}(N^{(1)}, N^{(2)}) \) and \( [(m_2, f_2)] \in \text{Hom}_{\text{GTSh}}(N^{(2)}, N^{(3)}) \).

Since \( N^{(1)}, N^{(2)} \) and \( N^{(3)} \) belong to the same connected component of \( \text{GTSh} \), \( N_{PB_2}^{(1)} = N_{PB_2}^{(2)} = N_{PB_2}^{(3)} \) or equivalently \( N_{\text{ord}}^{(1)} = N_{\text{ord}}^{(2)} = N_{\text{ord}}^{(3)} \). So let us set \( N_{PB_2} := N_{PB_2}^{(1)} \) and \( N_{\text{ord}} := N_{\text{ord}}^{(1)} \).

Let \( [(m, f)] := [(m_2, f_2)] \circ [(m_1, f_1)] \).

Due to the first equation in (2.52), \( m \equiv 2m_1m_2 + m_1 + m_2 \mod N_{\text{ord}} \). Hence

\[ \text{Ch}_{\text{cyclot}}(m, f)(x_{12}N_{PB_2}) = x_{12}^{2(m_1m_2+m_1+m_2)+1}N_{PB_2} = x_{12}^{4m_1m_2+2m_1+2m_2+1}N_{PB_2} = x_{12}^{2m_1+1}(2m_2+1)N_{PB_2} = (x_{12}^{N_{PB_2}})^{2m_2+1} = \text{Ch}_{\text{cyclot}}(m_2, f_2) \circ \text{Ch}_{\text{cyclot}}(m_1, f_1)(x_{12}N_{PB_2}). \]

Thus \( \text{Ch}_{\text{cyclot}} \) is indeed a functor from \( \text{GTSh} \) to the category of finite cyclic groups. \( \square \)

We call the functor \( \text{Ch}_{\text{cyclot}} \) the **virtual cyclotomic character**. This name is justified by the following remark:

**Remark 2.17** Let \( N \in NFI_{PB_4}(B_4) \), \( g \in G_0 \) and \( [(m, f)] \) be the \( \text{GT} \)-shadow in \( \text{GT}(N) \) induced by the element in \( \widehat{\text{GT}} \) corresponding to \( g \) then

\[ \text{Ch}_{\text{cyclot}}(m, f)(x_{12}N_{PB_2}) = x_{12}^{\chi(g)N_{\text{ord}}N_{PB_2}}, \]

(2.57)

where \( \chi : G_0 \to \widehat{\mathbb{Z}}^\times \cong \text{Aut}(\widehat{\mathbb{Z}}) \) is the cyclotomic character and \( \chi(g)N_{\text{ord}} \) represents the image of \( \chi(g) \) in \( \text{Aut}(\mathbb{Z}/N_{\text{ord}}\mathbb{Z}) \cong (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times \). Equation (2.57) follows from the discussion in [28 Example 4.7.4] and [28 Remark 4.7.5]. See also [17 Proposition 1.6].
2.6 Charming GT-shadows

Recall that $\hat{PB}_3$ is isomorphic to $F_2 \times \mathbb{Z}$ where the $F_2$-factor is freely generated by $x_{12}$ and $x_{33}$ and the $\mathbb{Z}$-factor is generated by the central element $c$ given in (A.5). This implies that $\hat{PB}_3 \cong \hat{F}_2 \times \hat{\mathbb{Z}}$. Due to the following proposition, the action of $\hat{GT}$ on $\hat{PB}_3$ (viewed as the automorphism group of $(12)3$ in $\hat{PaB}$) respects this decomposition:

**Proposition 2.18** For every (continuous) automorphism $\hat{T}$ of $\hat{PaB}^{<4}$, its restriction to the subgroup $\hat{F}_2 \leq \hat{PB}_3$ gives us an automorphism of $\hat{F}_2$

$$\hat{T}|_{\hat{F}_2} : \hat{F}_2 \to \hat{F}_2$$

defined by the formulas

$$\hat{T}(x) := x^{2^n+1}, \quad \hat{T}(y) := \hat{f}^{-1}y^{2^n+1}\hat{f}. \quad (2.58)$$

The restriction of $\hat{T}$ to the central factor $\hat{\mathbb{Z}}$ of $\hat{PB}_3$ gives us the continuous automorphism of $\hat{\mathbb{Z}}$ defined by the formula

$$\hat{T}(c) := c^{2^n+1}. \quad (2.59)$$

**Proof.** Due to Proposition 2.5 the action of $\hat{T}$ on $\hat{PB}_3$ is determined by the group homomorphisms

$$T_{m,f}^{PB_3} : PB_3 \to PB_3/N_{PB_3}, \quad N \in NFI_{PB_4}(B_4)$$

corresponding to $\hat{GT}$-shadows $[(m, f)]$ that come from $\hat{T}$.

Combining this observation with equations (2.28) and the second equation in (2.29) and using the fact that $c$ is a central element of $PB_3$, we conclude that the restrictions of $\hat{T}$ to $\hat{F}_2$ and to $\hat{\mathbb{Z}}$ give us group homomorphisms

$$\hat{T}|_{\hat{F}_2} : \hat{F}_2 \to \hat{F}_2 \quad \text{and} \quad \hat{T}|_{\hat{\mathbb{Z}}} : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}, \quad (2.60)$$

respectively.

Since the restrictions of the inverse of $\hat{T}$ to $\hat{F}_2$ and to $\hat{\mathbb{Z}}$ give us inverses of the two homomorphisms in (2.60), respectively, the homomorphisms in (2.60) are indeed automorphisms. Explicit formulas (2.58) and (2.59) are consequences of equations (2.28) and the second equation in (2.29). □

If a $\hat{GT}$-shadow $[(m, f)]$ comes from an automorphism of $\hat{PaB}$ then it satisfies further conditions. The following definition is motivated by these conditions.

**Definition 2.19** Let $N \in NFI_{PB_4}(B_4)$. A $\hat{GT}$-shadow $[(m, f)] \in \hat{GT}(N)$ is called genuine if it comes from an automorphism of $\hat{PaB}$. Otherwise, $[(m, f)]$ is called fake. Furthermore, a $\hat{GT}$-shadow $[(m, f)] \in \hat{GT}(N)$ is called charming if

- the coset $fN_{PB_3}$ can be represented by $f_1 \in [F_2, F_2]$ and
- the group homomorphism

$$T_{m,f}^{F_2} := T_{m,f}^{PB_3}|_{F_2} : F_2 \to F_2/(N_{PB_3} \cap F_2) \quad (2.61)$$

is onto.

---

9In fact, some specialists like to define $\hat{GT}$ as a certain subgroup of continuous automorphisms of $\hat{F}_2$. 
Since the intersection $N_{PB_3} \cap F_2$ plays an important role, we will denote it by $N_{F_2}$.

$$N_{F_2} := N_{PB_3} \cap F_2.$$  \hfill (2.62)

Clearly, the kernel of the homomorphism $T_{m,f}^Z : F_2 \rightarrow F_2/N_{F_2}$ coincides with $N_{F_2}$ and $|F_2 : N_{F_2}^*| = |F_2 : N_{F_2}|$ for every charming GT-shadow $[(m, f)]$.

Let us prove that

**Proposition 2.20** Every genuine GT-shadow is charming.

**Proof.** Let $N \in NFl_{PB_3}(B_4)$ and $[(m, f)] \in GT(N)$ be a genuine GT-shadow. The element $f \in PB_3$ can be written uniquely as

$$f = g c^k,$$

where $g \in F_2, k \in \mathbb{Z}$ and $c$ is defined in (A.5).

Since $N_{PB_3}$ is a normal subgroup of finite index in $PB_3$, the subgroup $N_{F_2} := N_{PB_3} \cap F_2$ is normal in $F_2$ and it has a finite index in $F_2$. Similarly, the subgroup $N_Z := N_{PB_3} \cap \mathbb{Z}$ has a finite index in $Z$. Therefore, the subgroup $N_{F_2} \times N_Z$ is normal and it has finite index in $PB_3$.

Due to Proposition 2.20, there exists $K \in NFl_{PB_3}(B_4)$ such that $K_{PB_3}$ is contained in $N_{F_2} \times N_Z$. Since $[(m, f)]$ is a genuine GT-shadow, there exists $(m_1, f_1) \in \mathbb{Z} \times PB_3$ such that $(m_1, f_1)$ represents the same GT-shadow $[(m, f)]$ in $GT(N)$ and $[(m_1, f_1)] \in GT(K)$.

Thus, without loss of generality, we may assume that $m = m_1$ and $f = f_1$, i.e. $[(m, f)] \in GT(K)$.

Using relation (2.18), we have

$$\sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f K_{PB_3} = f^{-1} \sigma_2 x_{23}^m g(x_{12}, x_{23}) c^{-m+k} K_{PB_3}.$$

Next, using (A.5) and the fact that $c$ is a central element of $B_3$, we get that

$$x_{12}^m \sigma_2^{-1} \sigma_1^{-1} g(x_{12}, x_{23}) \sigma_1 x_{12}^m g(x_{12}, x_{23})^{-1} \sigma_2 x_{23}^m g(x_{12}, x_{23}) c^{-m+k} \in K_{PB_3}.$$

Using equations in (A.6) from Appendix A.1 we deduce that

$$x_{12}^m g(x_{23}^{-1} x_{12}^{-1} c, x_{12}) (x_{23}^{-1} x_{12}^{-1} c)^m g(x_{23}^{-1} x_{12}^{-1} c, x_{23})^{-1} x_{23}^m g(x_{12}, x_{23}) c^{-m+k} \in K_{PB_3}, \text{ or}$$

$$x_{12}^m g(x_{23}^{-1} x_{12}^{-1} c, x_{12}) (x_{23}^{-1} x_{12}^{-1} c)^m g(x_{23}^{-1} x_{12}^{-1} c, x_{23})^{-1} x_{23}^m g(x_{12}, x_{23}) c^k \in K_{PB_3}.$$

Since $K_{PB_3}$ is a subgroup of $N_{F_2} \times N_Z$, we have $c^k \in N_Z \subset N_{PB_3}$. Hence $f c^{-k} N_{PB_3} = g(x_{12}, x_{23}) N_{PB_3}$, and so the GT-shadow has a representative of the form $(m, f)$ where $f \in F_2$.

It remains to show that

- $[(m, f)]$ can be represented by a pair $(m_1, f_1)$ with $f_1 \in [F_2, F_2]$ and $f$.
- Homomorphism (2.61) is onto.

Since homomorphism (2.61) does not depend on the choice of the representative of the GT-shadow $[(m, f)]$, we first prove that this homomorphism is indeed onto.

Due to Proposition 2.18 we have the following commutative diagram:

$$
\begin{array}{ccc}
\widehat{F}_2 & \xrightarrow{T_{m,f}} & \widehat{F}_2 \\
\uparrow{i} & & \uparrow{\tilde{T}_{N_{F_2}}} \\
F_2 & \xrightarrow{T_{m,f}} & F_2/N_{F_2}
\end{array}
$$

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Since $F_2$ is dense in $\hat{F}_2$, we get that the composition $\hat{P}_{N_{F_2}} \circ \hat{T}|_{\hat{F}_2} \circ i$ is surjective whence we conclude $T_{m,f}$ is onto.

Let us now prove that $[(m, f)]$ can be represented by a pair $(m, \hat{f})$ with $\hat{f} \in [F_2, F_2]$. Let $q$ be the least common multiple of the orders of $x_{12}N_{F_2}$ and $x_{23}N_{F_2}$ in $F_2/N_{F_2}$ and $\psi_2 : PB_4 \to S_q$, $\psi_y : PB_4 \to S_q$ be the group homomorphisms defined by equations

$$\psi_x(x_{12}) := (1, 2, \ldots, q), \quad \psi_x(x_{23}) = \psi(x_{13}) = \psi(x_{14}) = \psi(x_{24}) = \psi(x_{34}) := id_{S_q}$$

and

$$\psi_y(x_{34}) := (1, 2, \ldots, q), \quad \psi_y(x_{12}) = \psi_y(x_{23}) = \psi(x_{13}) = \psi(y_{14}) = \psi(y_{24}) := id_{S_q},$$

respectively.

Let $K$ be an element of $NF_{PB_4}(B_4)$ such that

$$K \leq N \cap \ker(\psi_x) \cap \ker(\psi_y). \quad (2.63)$$

Since $[(m, f)]$ is a genuine GT-shadow, there exists a GT-shadow $[(m_1, f_1)] \in GT(K)$ such that $(m_1, f_1)$ is also a representative of $[(m, f)]$. We can assume, without loss of generality, that $f_1 \in F_2$.

Applying equation (2.20) to $f_1$ we see that

$$f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23}) \in K. \quad (2.64)$$

Inclusions (2.63) and (2.64) imply that

$$\psi_x(f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23})) = id_{S_q},$$

and

$$\psi_y(f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23})) = id_{S_q}.$$ 

Hence the sum $s_x$ of exponents of $x_{12}$ in $f_1$ and the sum $s_y$ of exponents of $x_{23}$ in $f_1$ are multiples of $q$, i.e. $x_{12}^{-s_x} \in N_{F_2}$ and $x_{23}^{-s_y} \in N_{F_2}$.

Thus $(m, f_1x_{12}^{-s_x}x_{23}^{-s_y})$ is yet another representative of the GT-shadow $[(m, f)]$ in $GT(N)$ and, by construction, $f_1x_{12}^{-s_x}x_{23}^{-s_y} \in [F_2, F_2]$.

The following statement can be found in many introductory (and “not so introductory”) papers on the Grothendieck-Teichmueller group $\hat{GT}$. Here, we deduce it from Proposition 2.20.

**Corollary 2.21** For every $(m, \hat{f}) \in \hat{GT}$, $\hat{f}$ belongs to the topological closure of commutator subgroup of $\hat{F}_2$.

**Proof.** It suffices to show that, for every $N \in NF_{PB_4}(F_2)$, the element $\hat{P}_N(\hat{f}) \in F_2/N$ can be represented by $f_1 \in [F_2, F_2]$. Let us observe that $N \times \langle c \rangle \in NF_{PB_3}$.

Due to Proposition 2.51 there exists $K \in NF_{PB_4}(B_4)$ such that $K_{PB_4} \leq N \times \langle c \rangle$. Clearly, $K_{F_2} \leq N$.

Since the pair $(\hat{P}_{K_{PB_4}}(m), \hat{P}_{K_{F_2}}(\hat{f}))$ is a charming GT-shadow in $GT(K)$, the element $\hat{P}_{K_{F_2}}(\hat{f}) \in F_2/K_{F_2}$ can be represented by $f_1 \in [F_2, F_2]$. Since $K_{F_2} \leq N$, the same element $f_1 \in [F_2, F_2]$ represents the coset $\hat{P}_N(\hat{f}) \in F_2/N$. \(\Box\)
Let us denote by $\text{GT}^\odot(N)$ the subset of all charming GT-shadows in $\text{GT}(N)$ and prove that $\text{GT}(N)$ can be safely replaced by $\text{GT}^\odot(N)$ in all the constructions of Section 2.5. More precisely,

**Proposition 2.22** The assignment

$$\text{Hom}_{\text{GTSh}}(\bar{N}, N) := \{[(m, f)] \in \text{GT}^\odot(N) \mid \bar{N} = \ker(T_{m,f}^{\text{PB}_3})\}, \quad \bar{N}, N \in \text{NFI}_{\text{PB}_3}(B_4)$$

upgrades the set $\text{NFI}_{\text{PB}_3}(B_4)$ to a groupoid.

**Proof.** Let $[(m_1, f_1)] \in \text{Hom}_{\text{GTSh}}(N^{(1)}, N^{(2)})$ and $[(m_2, f_2)] \in \text{Hom}_{\text{GTSh}}(N^{(2)}, N^{(3)})$. Since the GT-shadows $[(m_1, f_1)]$ and $[(m_2, f_2)]$ are charming, we may assume, without loss of generality, that $f_1, f_2 \in [F_2, F_2]$.

Due to Remark 2.13, the composition $[(m_2, f_2)] \circ [(m_1, f_1)]$ is represented by a pair $(m, f)$ with

$$f = f_2 f_1 (x^{2m_2+1}, f_2^{-1}(x, y) g^{2m_2+1} f_2(x, y)).$$

Since $f_1, f_2 \in [F_2, F_2]$, it is clear that $f$ also belongs to $[F_2, F_2]$.

Since $T_{m_2, f_2}^{F_2} : F_2 \to F_2/N^{(3)}$ is the composition of the onto homomorphism $T_{m_1, f_1}^{F_2} : F_2 \to F_2/N^{(2)}$ and the isomorphism $T_{m_2, f_2}^{F_2, \text{isom}} : F_2/N^{(2)} \to F_2/N^{(3)}$, the homomorphism $T_{m, f}^{F_2}$ is also onto.

We proved that the subset of charming GT-shadows is closed under composition.

To prove that the subset of charming GT-shadows is closed under taking inverses, we start with a charming GT-shadow $[(m, f)] \in \text{Hom}_{\text{GTSh}}(N^n, N)$ and assume that $f \in [F_2, F_2]$. Let $[(\bar{m}, \bar{f})] \in \text{Hom}_{\text{GTSh}}(N, N^n)$ be the inverse of $[(m, f)]$ in $\text{GTSh}$. In other words,

$$2m\bar{m} + m + \bar{m} \equiv 0 \mod N_{\text{ord}}$$

and

$$f T_{m,f}^{\text{PB}_3}(\bar{f}) = 1_{\text{PB}_3/N_{\text{PB}_3}}.$$  

(2.66)

Our goal is to show that the coset $\bar{f}N_{\text{PB}_3}$ can be represented by $g \in [F_2, F_2]$.

Since $f^{-1}$ belongs to $[F_2, F_2]$, we have

$$f^{-1} = [g_{11}, g_{12}][g_{21}, g_{22}] \cdots [g_{r1}, g_{r2}],$$

where each $g_{ij} \in F_2$ and $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$.

Since the homomorphism $T_{m, f}^{F_2} : F_2 \to F_2/N_{F_2}$ is onto, for every $g_{ij}$, there exists $\bar{g}_{ij} \in F_2$ such that $T_{m,f}^{F_2}(\bar{g}_{ij}) = g_i N_{F_2}$. Hence, for

$$g := [\bar{g}_{11}, \bar{g}_{12}][\bar{g}_{21}, \bar{g}_{22}] \cdots [\bar{g}_{r1}, \bar{g}_{r2}] \in [F_2, F_2]$$

we have $T_{m,f}^{\text{PB}_3}(g) = f^{-1} N_{\text{PB}_3}$ or equivalently

$$f T_{m,f}^{\text{PB}_3}(g) = 1_{\text{PB}_3/N_{\text{PB}_3}}.$$  

(2.67)

Combining (2.66) with (2.67) we conclude that the element $g^{-1} \bar{f}$ belongs to the kernel of $T_{m,f}^{\text{PB}_3} : \text{PB}_3 \to \text{PB}_3/N_{\text{PB}_3}$. Thus, due to Proposition 2.11, $g$ also represents the coset $\bar{f}N_{\text{PB}_3}$.

Since, by construction $g \in [F_2, F_2]$, the desired statement is proved.  

□
3 The Main Line functor $\mathcal{ML}$ and $\hat{\mathcal{GT}}$

In this section, we use (charming) $\mathcal{GT}$-shadows to construct a functor $\mathcal{ML}$ from a certain subposet of $\text{NFI}_{\text{PB}_4}(B_4)$ to the category of finite groups. We prove that the limit of the functor $\mathcal{ML}$ is isomorphic to the Grothendieck-Teichmüller group $\hat{\mathcal{GT}}$.

3.1 Connected components of $\text{GTSh}^\triangleright$, settled $\mathcal{GT}$-shadows and isolated elements of $\text{NFI}_{\text{PB}_4}(B_4)$

Since the set $\text{NFI}_{\text{PB}_4}(B_4)$ is infinite, so is the groupoid $\text{GTSh}^\triangleright$. Moreover, the groupoid $\text{GTSh}^\triangleright$ is highly disconnected. Indeed, if $\tilde{N}$ and $N$ are connected by a morphism in $\text{GTSh}^\triangleright$, then they must have the same index in $\text{PB}_4$.

For $N \in \text{NFI}_{\text{PB}_4}(B_4)$ we denote by $\text{GTSh}^\triangleright_{\text{conn}}(N)$ the connected component of $N$ in the groupoid $\text{GTSh}^\triangleright$. Clearly, an element $\tilde{N}$ of $\text{NFI}_{\text{PB}_4}(B_4)$ is an object of $\text{GTSh}^\triangleright_{\text{conn}}(N)$ if and only if there exists $[(m, f)] \in \text{GT}^\triangleright(N)$ such that

$$\tilde{N} = \ker(T_{m,f}^{\text{PB}_4}).$$

We call objects of the groupoid $\text{GTSh}^\triangleright_{\text{conn}}(N)$ conjugates of $N$.

Since $\text{GT}^\triangleright(N)$ is a finite set for every $N \in \text{NFI}_{\text{PB}_4}(B_4)$, it is easy to show that

**Proposition 3.1** For every $N \in \text{NFI}_{\text{PB}_4}(B_4)$, the (connected) groupoid $\text{GTSh}^\triangleright_{\text{conn}}(N)$ is finite.

To establish a more precise link between (charming) $\mathcal{GT}$-shadows and the group $\hat{\mathcal{GT}}$, we will be interested in a certain subposet of $\text{NFI}_{\text{PB}_4}(B_4)$. Let us start with the following definition:

**Definition 3.2** Let $N \in \text{NFI}_{\text{PB}_4}(B_4)$ and $[(m, f)] \in \text{GT}^\triangleright(N)$. A charming $\mathcal{GT}$-shadow $[(m, f)]$ is called settled if its source coincides with $N$, i.e. $\ker(T_{m,f}^{\text{PB}_4}) = N$. An element $N$ of the poset $\text{NFI}_{\text{PB}_4}(B_4)$ is called isolated if every $\mathcal{GT}$-shadow in $\text{GT}^\triangleright(N)$ is settled.

Clearly, a $\mathcal{GT}$-shadow $[(m, f)] \in \text{GT}^\triangleright(N)$ is settled if and only if $[(m, f)]$ is an automorphism of the object $N$ in the groupoid $\text{GTSh}^\triangleright$. Moreover, an element $N \in \text{NFI}_{\text{PB}_4}(B_4)$ is isolated if and only if the groupoid $\text{GTSh}^\triangleright_{\text{conn}}(N)$ has exactly one object. In this case, $\text{GT}^\triangleright(N)$ is the group of automorphisms of the object $N$ in the groupoid $\text{GTSh}^\triangleright$.

The following proposition gives us a simple way to produce many examples of isolated elements of $\text{NFI}_{\text{PB}_4}(B_4)$:

**Proposition 3.3** For every $N \in \text{NFI}_{\text{PB}_4}(B_4)$, the normal subgroup

$$N^2 := \bigcap_{K \in \text{Ob}(\text{GTSh}^\triangleright_{\text{conn}}(N))} K$$

(3.1)

is an isolated element of $\text{NFI}_{\text{PB}_4}(B_4)$.
Proof. Let $[(m, f)] \in \GTSh^{\circ}(N^\sharp)$ and $N^{\sharp,s}$ be the source of the corresponding morphism in $\GTSh^{\circ}$, i.e. $N^{\sharp,s} := \ker(T_{m,f}^{\PB_4})$.

Since $N^\sharp \leq K$, the same pair $(m, f) \in \mathbb{Z} \times F_2$ represents a $\GT$-shadow in $\GTSh^{\circ}(K)$. Moreover, the homomorphism from $\PB_4$ to $\PB_4/K$ corresponding to $[(m, f)] \in \GTSh^{\circ}(K)$ is the composition $\mathcal{P}_{N^\sharp,K} \circ T_{m,f}^{\PB_4}$ of $T_{m,f}^{\PB_4}$ with the canonical projection

$$\mathcal{P}_{N^\sharp,K} : \PB_4/N^\sharp \to \PB_4/K.$$ 

Let $h \in N^\sharp$, $\tilde{h} \in \PB_4$ be a representative of $T_{m,f}^{\PB_4}(h)$ and $K^\sharp$ be the source of the $\GT$-shadow $[(m, f)] \in \GTSh^{\circ}(K)$ (i.e. $K^\sharp := \ker(\mathcal{P}_{N^\sharp,K} \circ T_{m,f}^{\PB_4})$).

Since $N^\sharp \leq K^\sharp$, we have

$$\mathcal{P}_{N^\sharp,K}(T_{m,f}^{\PB_4}(h)) = 1_{\PB_4/K}. \quad (3.2)$$

Identity (3.2) implies that $\tilde{h} \in K$ for every $K \in \Ob(\GTSh_{\conn}(N))$ and hence $\tilde{h} \in N^\sharp$. Therefore $T_{m,f}^{\PB_4}(h) = 1_{\PB_4/N^\sharp}$ or equivalently $h \in N^{\sharp,s}$.

We proved that

$$N^\sharp \leq N^{\sharp,s}. \quad (3.3)$$

Since these subgroups have the same index in $\PB_4$, inclusion (3.3) implies that $N^{\sharp,s} = N^\sharp$.

Since we started with an arbitrary $\GT$-shadow in $\GTSh^{\circ}(N^\sharp)$, we proved that $N^\sharp$ is indeed an isolated element of $\NFI_{\PB_4}(B_4)$.

\[\square\]

Remark 3.4 In all examples we have considered so far (see Section 4 on selected results of computer experiments), $\GTSh_{\conn}(N)$ has at most two objects. Hence equation (3.1) gives us a practical way to produce examples of isolated elements of $\NFI_{\PB_4}(B_4)$.

Let us denote by

$$\NFI_{\PB_4}^{isolated}(B_4)$$

the subposet of isolated elements of $\NFI_{\PB_4}(B_4)$.

Since $N^\sharp \leq N$ for every $N \in \NFI_{\PB_4}(B_4)$, Proposition 3.3 implies that

Corollary 3.5 The subposet $\NFI_{\PB_4}^{isolated}(B_4)$ of $\NFI_{\PB_4}(B_4)$ is cofinal. In other words, for every $N \in \NFI_{\PB_4}(B_4)$, there exists $K \in \NFI_{\PB_4}^{isolated}(B_4)$ such that $K \leq N$. \[\square\]

Although, Corollary 3.5 implies that the poset $\NFI_{\PB_4}^{isolated}(B_4)$ is directed (it is a cofinal subposet of a directed poset), it is still useful to know that the intersection of two isolated elements of $\NFI_{\PB_4}(B_4)$ is an isolated element of $\NFI_{\PB_4}(B_4)$:

Proposition 3.6 For every $N^{(1)}, N^{(2)} \in \NFI_{\PB_4}^{isolated}(B_4)$,

$$N^{(1)} \cap N^{(2)}$$

is also an isolated element of $\NFI_{\PB_4}(B_4)$.

Proof. $K := N^{(1)} \cap N^{(2)}$ is clearly an element of $\NFI_{\PB_4}(B_4)$. So our goal is to prove that $K$ is isolated.

Let $[(m, f)] \in \GTSh^{\circ}(K)$ and $K^\sharp$ be the kernel of the homomorphism $T_{m,f}^{\PB_4} : \PB_4 \to \PB_4/K$. 

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Recall that $\mathcal{P}_{K,N^{(1)}}$ (resp. $\mathcal{P}_{K,N^{(2)}}$) is the canonical homomorphism from $PB_4/K$ to $PB_4/N^{(1)}$ (resp. to $PB_4/N^{(2)}$). Since $K \leq N^{(1)}$ and $K \leq N^{(2)}$, the pair $(m, f)$ also represents a $GT$-shadow in $GT^\triangledown(N^{(1)})$ and a $GT$-shadow in $GT^\triangledown(N^{(2)})$. Moreover, the compositions $\mathcal{P}_{K,N^{(1)}} \circ T_{m,f}^{PB_4}$ and $\mathcal{P}_{K,N^{(2)}} \circ T_{m,f}^{PB_4}$ are the homomorphisms $PB_4 \rightarrow PB_4/N^{(1)}$ and $PB_4 \rightarrow PB_4/N^{(2)}$ corresponding to these $GT$-shadows in $GT^\triangledown(N^{(1)})$ and $GT^\triangledown(N^{(2)})$, respectively.

Let us now consider $h \in K^e$. Since $T_{m,f}^{PB_4}(h) = 1_{PB_4/N^{(1)}}$, we have

$$\mathcal{P}_{K,N^{(1)}} \circ T_{m,f}^{PB_4}(h) = 1_{PB_4/N^{(1)}}, \quad \mathcal{P}_{K,N^{(2)}} \circ T_{m,f}^{PB_4}(h) = 1_{PB_4/N^{(2)}}. \quad (3.5)$$

Since $N^{(1)}$, $N^{(2)}$ are both isolated, identities (3.5) imply that $h \in N^{(1)}$ and $h \in N^{(2)}$. Hence $h \in K$.

Since we showed that $K^e \leq K$ and both subgroups have the same (finite) index in $PB_4$, we have the desired equality $K^e = K$. \hfill $\square$

Recall that, for every isolated element $N \in NFI_{PB_4}(B_4)$, the set $GT^\triangledown(N)$ is a finite group. More precisely, $GT^\triangledown(N)$ is the (finite) group of automorphisms of $N$ in the groupoid $GTSh^\triangledown$.

Let us denote this finite group by $ML(N)$ and prove that

**Proposition 3.7** The assignment

$$N \mapsto ML(N)$$

upgrades to a functor $ML$ from the poset $NFI_{PB_4}^{isolated}(B_4)$ to the category of finite groups.

**Proof.** Let $K \leq N$ be isolated elements of $NFI_{PB_4}(B_4)$. Our goal is to define a group homomorphism

$$ML_{K,N} : ML(K) \rightarrow ML(N) \quad (3.6)$$

and show that, for every triple of nested elements $N^{(1)} \leq N^{(2)} \leq N^{(3)}$ of $NFI_{PB_4}^{isolated}(B_4)$,

$$ML_{N^{(2)},N^{(3)}} \circ ML_{N^{(1)},N^{(2)}} = ML_{N^{(1)},N^{(3)}}. \quad (3.7)$$

For this proof, it is convenient to identify $GT$-shadows $[(m, f)] \in GT^\triangledown(K)$ with the corresponding onto morphisms $T_{m,f} : PaB^{\leq 4} \rightarrow PaB^{\leq 4} / \sim_K$ of truncated operads. So let $[(m, f)] \in GT^\triangledown(K)$ and $T_{m,f}$ be the corresponding morphism.

Recall that $\mathcal{P}_{K,N}$ denotes the canonical onto morphism of truncated operads

$$\mathcal{P}_{K,N} : PaB^{\leq 4} / \sim_K \rightarrow PaB^{\leq 4} / \sim_N.$$

Composing $\mathcal{P}_{K,N}$ with $T_{m,f}$ we get an onto morphism

$$\mathcal{P}_{K,N} \circ T_{m,f} : PaB^{\leq 4} \rightarrow PaB^{\leq 4} / \sim_N$$

and hence an element of $GT^\triangledown(N)$.

We set

$$ML_{K,N}(T_{m,f}) := \mathcal{P}_{K,N} \circ T_{m,f}. \quad (3.8)$$

To prove that $ML_{K,N}$ is a group homomorphism from $ML(K)$ to $ML(N)$, we recall that, since $K$ is isolated, every onto morphism of truncated operads $T : PaB^{\leq 4} \rightarrow PaB^{\leq 4} / \sim_K$
factors as follows

\[
\begin{array}{ccc}
\text{PaB}^{\leq 4} & \xrightarrow{\mathcal{P}_K} & \text{PaB}^{\leq 4} / \sim_K \\
\text{PaB}^{\leq 4} / \sim_K & \xrightarrow{T} & \text{PaB}^{\leq 4} / \sim_K \\
\end{array}
\]

Let us now show that, for every onto morphism of truncated operads \( T : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_K \), the diagram

\[
\begin{array}{ccc}
\text{PaB}^{\leq 4} & \xrightarrow{\mathcal{P}_K} & \text{PaB}^{\leq 4} / \sim_K \\
\text{PaB}^{\leq 4} / \sim_K & \xrightarrow{T} & \text{PaB}^{\leq 4} / \sim_K \\
\text{PaB}^{\leq 4} / \sim_K & \xrightarrow{\mathcal{P}_{K,N}} & \text{PaB}^{\leq 4} / \sim_N \\
\end{array}
\]

commutes.

Since the top triangle of (3.9) commutes by definition of \( T^{\text{isom}} \), we only need to prove the commutativity of the square. Let \( \gamma \in \text{PaB}^{\leq 4} / \sim_K \) (resp. \( \gamma \in \text{PaB}^{\leq 4} / \sim_N \)). Since \( T^{\text{isom}}([\gamma]_K) = T(\gamma) \), \( (\mathcal{P}_{K,N} \circ T)^{\text{isom}}([\gamma]_N) = \mathcal{P}_{K,N} \circ T(\gamma) \) and \( \mathcal{P}_{K,N}([\gamma]_K) = [\gamma]_N \), we have

\[
\mathcal{P}_{K,N} \circ T^{\text{isom}}([\gamma]_K) = \mathcal{P}_{K,N} \circ T(\gamma) = (\mathcal{P}_{K,N} \circ T)^{\text{isom}}([\gamma]_N) = (\mathcal{P}_{K,N} \circ T)^{\text{isom}} \circ \mathcal{P}_{K,N}([\gamma]_K).
\]

Thus (3.9) indeed commutes.

Now let \( T_1 \) and \( T_2 \) be onto morphisms (of truncated operads)

\[
T_1, T_2 : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_K.
\]

Since

\[
T_1^{\text{isom}} \circ T_2 : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_K
\]

is the composition of \( T_1 \) and \( T_2 \) in \( \text{GTSh}^\Diamond \) and

\[
(\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ (\mathcal{P}_{K,N} \circ T_2) : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N
\]

is the composition of \( \mathcal{P}_{K,N} \circ T_1 \) and \( \mathcal{P}_{K,N} \circ T_2 \) in \( \text{GTSh}^\Diamond \), our goal is to prove that

\[
\mathcal{P}_{K,N} \circ (T_1^{\text{isom}} \circ T_2) = (\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ (\mathcal{P}_{K,N} \circ T_2).
\]

(3.10)

Due to commutativity of (3.9) for \( T = T_1 \) we have

\[
\mathcal{P}_{K,N} \circ T_1^{\text{isom}} \circ T_2 = (\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ \mathcal{P}_{K,N} \circ T_2.
\]

Thus equation (3.10) indeed holds and we proved that \( \mathcal{M}_\mathcal{L}_{K,N} \) is a group homomorphism.

Let us now consider isolated elements \( N^{(1)} \leq N^{(2)} \leq N^{(3)} \) of \( \text{NFI}_{\text{PB}_4(B_4)} \). Since

\[
\mathcal{P}_{N^{(1)},N^{(3)}} = \mathcal{P}_{N^{(2)},N^{(3)}} \circ \mathcal{P}_{N^{(1)},N^{(2)}},
\]

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we have
\[ \mathcal{ML}_{N^{(2)},N^{(3)}} \circ \mathcal{ML}_{N^{(1)},N^{(2)}}(T_{m,f}) = P_{N^{(2)},N^{(3)}} \circ P_{N^{(1)},N^{(2)}} \circ T_{m,f} \]
\[ = P_{N^{(1)},N^{(3)}} \circ T_{m,f} = \mathcal{ML}_{N^{(1)},N^{(3)}}(T_{m,f}), \]
for every \([(m, f)] \in GT^\varnothing(N^{(1)}).
Thus the desired identity (3.7) holds and the proposition is proved. \(\square\)

We call the functor \(\mathcal{ML}\) the \textit{Main Line functor}.

In the next section, we will prove the following theorem:

\textbf{Theorem 3.8} \textit{The (profinite version) \(\hat{GT}\) of the Grothendieck-Teichmueller group is isomorphic to \(\lim(\mathcal{ML})\).}

3.2 Proof of Theorem 3.8

We will need the following auxiliary statements:

\textbf{Proposition 3.9}

A) For every \(N \in \text{NFI}(PB_3)\), there exists \(K \in \text{NFI}_{\text{isolated}}^{\text{PB}_4}(B_4)\) satisfying the property
\[ K_{PB_3} \leq N. \]

B) For every \(N \in \text{NFI}(PB_2)\) there exists \(K \in \text{NFI}_{\text{isolated}}^{\text{PB}_4}(B_4)\) such that \(K_{PB_2} \leq N.\)

\textbf{Proof.} Let \(N \in \text{NFI}(PB_3)\) and \(\psi\) be a group homomorphism from \(PB_3\) to \(S_n\) such that \(\ker(\psi) = N.\)

Using relations (A.3) on the generators of \(PB_4\), it is easy to show that the equations
\[ \tilde{\psi}(x_{12}) := \psi(x_{12}), \quad \tilde{\psi}(x_{23}) := \psi(x_{23}), \quad \tilde{\psi}(x_{13}) := \psi(x_{13}), \]
\[ \tilde{\psi}(x_{14}) = \tilde{\psi}(x_{24}) = \tilde{\psi}(x_{34}) := \text{id}_{S_n} \]
define a group homomorphism \(\tilde{\psi}: PB_4 \rightarrow S_n.\)

Moreover, the kernel of \(\tilde{\psi}\) satisfies the property
\[ \varphi_{123}^{-1}(\ker(\tilde{\psi})) = N. \]

Hence
\[ \varphi_{123}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{123,4}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{1,23,4}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{1,2,34}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{234}^{-1}(\ker(\tilde{\psi})) \leq N. \quad (3.11) \]

Let \(\tilde{N}\) be the normal subgroup of \(PB_4\) obtained by intersecting all normal subgroups of \(PB_4\) of index \(|PB_4 : \ker(\tilde{\psi})|\). Since \(\tilde{N}\) is a characteristic subgroup of \(PB_4\) of finite index (in \(PB_4\)), we have
\[ \tilde{N} \in \text{NFI}_{PB_4}(B_4). \]

\footnote{One of the authors of this paper is trying to live in the sequence of suburbs of Philadelphia called the Main Line. The functor \(\mathcal{ML}\) is named after this beautiful sequence of suburbs.}
Furthermore, due to Corollary 3.5 there exists an isolated element \( K \) of \( \text{NFI}_{\text{PB}_B} (B_4) \) satisfying the property \( K \leq \check{N} \). Combining \( K \leq \check{N} \) with \( \check{N} \leq \ker(\tilde{\psi}) \) and (3.11), we deduce that \( K_{\text{PB}_B} \leq N \).

Thus desired Statement A) is proved. Just as for Statement A), we start with a group homomorphism \( \kappa : \text{PB}_2 \to S_n \) whose kernel coincides with \( N \).

It is easy to see that the equations
\[
\tilde{\kappa}(x_{12}) := \kappa(x_{12}), \quad \tilde{\kappa}(x_{23}) := \kappa(x_{12})^{-1}, \quad \tilde{\kappa}(x_{13}) := \text{id}_{S_n}, \\
\tilde{\kappa}(x_{14}) = \tilde{\kappa}(x_{24}) = \tilde{\kappa}(x_{34}) := \text{id}_{S_n}
\]
define a group homomorphism \( \tilde{\kappa} : \text{PB}_4 \to S_n \).

The kernel of \( \tilde{\kappa} \) satisfies the property
\[
\varphi_{12}^{-1}(\varphi_{123}^{-1}(\ker(\tilde{\kappa}))) = N.
\] (3.12)

Let \( \check{N} \) be the normal subgroup of \( \text{PB}_4 \) obtained by intersecting all normal subgroups of \( \text{PB}_4 \) of index \( |\text{PB}_4 : \ker(\tilde{\kappa})| \). Since \( \check{N} \) is a characteristic subgroup of \( \text{PB}_4 \) of finite index (in \( \text{PB}_4 \)), we have
\[
\check{N} \in \text{NFI}_{\text{PB}_B} (B_4).
\]

As above, there exists an isolated element \( K \) of \( \text{NFI}_{\text{PB}_B} (B_4) \) satisfying the property \( K \leq \check{N} \). Combining \( K \leq \check{N} \) with \( \check{N} \leq \ker(\tilde{\kappa}) \) and (3.12), we deduce that \( K_{\text{PB}_2} \leq N \).

Thus Statement B) is also proved.\( \square \)

Proposition 3.9 allows us to produce a more practical description of \( \widetilde{\text{PaB}}^{\leq 4} \). To give this description, we note that the assignment \( K \mapsto \text{PaB}^{\leq 4}/K \) upgrades to a functor from the poset \( \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4) \) to the category of truncated operads in finite groupoids. Indeed, for every pair \( K_1 \leq K_2 \) of elements of \( \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4) \) we have the obvious morphism of truncated operads
\[
\mathcal{P}_{K_1,K_2} : \text{PaB}^{\leq 4}/ \sim_{K_1} \to \text{PaB}^{\leq 4}/ \sim_{K_2}.
\]

Moreover, for every triple \( K_1 \leq K_2 \leq K_3 \) of elements of \( \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4) \), we have \( \mathcal{P}_{K_2,K_3} \circ \mathcal{P}_{K_1,K_2} = \mathcal{P}_{K_1,K_3} \).

Let us denote by
\[
\widetilde{\text{PaB}}^{\leq 4}
\] (3.13)
the limit of this functor.

More concretely, \( \text{PaB}(n) \) consists of functions
\[
\gamma : \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4) \to \bigsqcup_{K \in \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4)} \text{PaB}(n)/ \sim_K
\]
satisfying these two conditions:

- for every \( K \in \text{NFI}_{\text{PB}_B}^{\text{isolated}} (B_4) \), \( \gamma(K) \in \text{PaB}(n)/ \sim_K \) and

34
for every pair $K_1 \leq K_2$ in $\text{NFI}_{PB_4}^{\text{isolated}}(B_4)$, $\mathcal{P}_{K_1,K_2}(\gamma(K_1)) = \gamma(K_2)$.

Since for every pair $K_1 \leq K_2$ of elements of $\text{NFI}_{PB_4}^{\text{isolated}}(B_4)$ we have

$$\mathcal{P}_{K_1,K_2} \circ \hat{\mathcal{P}}_{K_1} = \hat{\mathcal{P}}_{K_2},$$

the assignment

$$\Psi(\hat{\gamma})(K) := \hat{\mathcal{P}}_{K}(\hat{\gamma}), \quad \hat{\gamma} \in \hat{\text{PaB}}(n)$$

defines a morphism of truncated operads

$$\Psi : \hat{\text{PaB}} \to \hat{\text{PaB}}.$$  \hspace{1cm} (3.14)

Let us prove that

**Corollary 3.10** The morphism $\Psi$ in (3.14) is an isomorphism of truncated operads in the category of topological groupoids.

**Proof.** Since the compatibility with the structures of truncated operads and the composition of morphisms is obvious, it suffices to prove that $\Psi$ is a homeomorphism of topological spaces.

Let $\tau$, $\tau'$ be objects of $\text{PaB}(n)$ and $\hat{\gamma}_1, \hat{\gamma}_2 \in \text{Hom}_{\text{PaB}}(\tau, \tau')$ such that $\Psi(\hat{\gamma}_1) = \Psi(\hat{\gamma}_2)$ or equivalently, for every $K \in \text{NFI}_{PB_4}^{\text{isolated}}(B_4)$

$$\Psi(\hat{\gamma}_2^{-1} \cdot \hat{\gamma}_1)(K)$$

is the identity automorphism of $\tau$ in $\text{PaB}(n)/\sim_K$.

Thus, due to Proposition 3.9 the image of $\hat{\gamma}_2^{-1} \cdot \hat{\gamma}_1$ in $\text{PB}_n/N$ is the identity element for every $N \in \text{NFI}(\text{PB}_n)$. Therefore $\hat{\gamma}_2^{-1} \cdot \hat{\gamma}_1$ is the identity element of $\hat{\text{PB}}_n$ and hence

$$\hat{\gamma}_1 = \hat{\gamma}_2.$$

We proved that $\Psi$ is one-to-one.

Let $\gamma \in \hat{\text{PaB}}(n)$, $\tau$ and $\tau'$ be the source and the target of $\gamma$, respectively. Let $\lambda$ be any isomorphism from $\tau$ to $\tau'$ in $\text{PaB}(n)$. By abuse of notation, we will use symbol $\lambda$ for its obvious image in $\hat{\text{PaB}}(n)$ and in $\hat{\text{PaB}}(n)$.

Due to Proposition 3.9 there exists an element $\hat{h} \in \hat{\text{PB}}_n$ such that

$$\hat{\mathcal{P}}_{K}(\hat{h}) = (\lambda^{-1} \cdot \gamma)(K), \quad \forall \ K \in \text{NFI}_{PB_4}^{\text{isolated}}(B_4).$$  \hspace{1cm} (3.15)

Equation (3.15) implies that $\Psi(\lambda \cdot \hat{h}) = \gamma$. Thus we proved that $\Psi$ is onto.

Since, for every $K \in \text{NFI}_{PB_4}^{\text{isolated}}(B_4)$, the composition of $\Psi$ with the canonical projection

$$\hat{\text{PaB}} \to \text{PaB}/\sim_K$$

coincides with the continuous map

$$\hat{\mathcal{P}}_{K} : \hat{\text{PaB}} \to \text{PaB}/\sim_K,$$

we conclude that $\Psi$ is continuous.

Since $\Psi$ is a continuous bijection from a compact space $\hat{\text{PaB}}$ to a Hausdorff space $\hat{\text{PaB}}$, $\Psi$ is indeed a homeomorphism.  \hspace{1cm} $\square$
Due to Corollary 3.10, we can safely replace $\overline{\hat{\text{PaB}}}^{\leq 4}$ by $\overline{\text{PaB}}^{\leq 4}$ in all further considerations. We will also use the same symbol $\mathcal{I}$ (resp. $\hat{P}_K$ for $K \in \text{NFI}_{\text{PB}_4}^\text{isolated}(B_4)$) for the canonical embedding $\mathcal{I} : \text{PaB}^{\leq 4} \to \overline{\text{PaB}}^{\leq 4}$ and the canonical projection $\hat{P}_K : \text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_K$.

Recall that, for every $\hat{T} \in \hat{\mathcal{G}T}$ and $K \in \text{NFI}_{\text{PB}_4}^\text{isolated}(B_4)$, the formula $T_K := \hat{P}_K \circ \hat{T} \circ \mathcal{I}$ defines an onto morphism of truncated operads $\text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/K$. Since $K$ is an isolated element of $\text{NFI}_{\text{PB}_4}(B_4)$, Corollary 2.13 implies that the onto morphism $T_K$ factors as follows:

$$T_K = T_K^\text{isom} \circ P_K,$$  \hspace{1cm} (3.16)

where $T_K^\text{isom}$ is an isomorphism of truncated operads $T_K^\text{isom} : \text{PaB}^{\leq 4}/K \to \text{PaB}^{\leq 4}/K$ and $P_K$ is the canonical projection $\text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/K$.

We claim that

**Proposition 3.11** For every $\hat{T} \in \hat{\mathcal{G}T}$ and for every $K \in \text{NFI}_{\text{PB}_4}^\text{isolated}(B_4)$ the diagram

$$\begin{array}{ccc}
\overline{\text{PaB}}^{\leq 4} & \xrightarrow{\hat{T}} & \overline{\text{PaB}}^{\leq 4} \\
\hat{P}_K \downarrow & & \downarrow \hat{P}_K \\
\text{PaB}^{\leq 4}/\sim_K & \xrightarrow{T_K^\text{isom}} & \text{PaB}^{\leq 4}/\sim_K
\end{array}$$

commutes.

**Proof.** By definition of $T_K^\text{isom}$, $\hat{P}_K \circ \hat{T} \circ \mathcal{I}(\gamma) = T_K^\text{isom} \circ P_K(\gamma)$, for every $\gamma \in \text{PaB}^{\leq 4}$.

Hence

$$\hat{P}_K \circ \hat{T}(\mathcal{I}(\gamma)) = T_K^\text{isom} \circ \hat{P}_K(\mathcal{I}(\gamma)), \hspace{1cm} \forall \gamma \in \text{PaB}^{\leq 4}.$$  \hspace{1cm} (3.18)

Since the image $\mathcal{I}(\text{PaB}^{\leq 4})$ of $\text{PaB}^{\leq 4}$ in $\overline{\text{PaB}}^{\leq 4}$ is dense in $\overline{\text{PaB}}^{\leq 4}$ and the target $\text{PaB}^{\leq 4}/\sim_K$ of the compositions $\hat{P}_K \circ \hat{T}$ and $T_K^\text{isom} \circ P_K$ is Hausdorff, identity (3.18) implies that diagram (3.17) indeed commutes. \hspace{1cm} $\square$

**Proof of Theorem 3.8** Let $K, \tilde{K}$ be elements of $\text{NFI}_{\text{PB}_4}^\text{isolated}(B_4)$ such that $\tilde{K} \leq K$ and $P_{\tilde{K},K}$ be the canonical projection from $\text{PaB}^{\leq 4}/\sim_{\tilde{K}}$ to $\text{PaB}^{\leq 4}/\sim_K$. Furthermore, let $T_K$ and $T_{\tilde{K}}$ be onto morphisms from $\text{PaB}^{\leq 4}$ to $\text{PaB}^{\leq 4}/\sim_{\tilde{K}}$ and $\text{PaB}^{\leq 4}/\sim_K$, respectively, coming from $\hat{T} \in \hat{\mathcal{G}T}$.

Since $\hat{P}_K = P_{\tilde{K},K} \circ P_K$, the diagram

$$\begin{array}{ccc}
\text{PaB}^{\leq 4} & \xrightarrow{T_K} & \text{PaB}^{\leq 4}/\sim_{\tilde{K}} \\
\downarrow T_{\tilde{K}} & & \downarrow P_{\tilde{K},K} \\
\text{PaB}^{\leq 4}/\sim_K
\end{array}$$

commutes. Hence the assignment $\hat{T} \mapsto \{T_K\}_{K \in \text{NFI}_{\text{PB}_4}^\text{isolated}(B_4)}$ gives us a map from $\hat{\mathcal{G}T}$ to $\text{lim}(\mathcal{M}\mathcal{L})$

$$\hat{\mathcal{G}T} \to \text{lim}(\mathcal{M}\mathcal{L}).$$  \hspace{1cm} (3.19)
Let us show that the map (3.19) is a group homomorphism.

Indeed, let \( \hat{T}^{(1)}, \hat{T}^{(2)} \in \widehat{GT}, \hat{T} := \hat{T}^{(1)} \circ \hat{T}^{(2)} \) and \( K \in NFI_{\text{isolated}}^\text{isolated}(B_4) \). Using Proposition 3.11 we get

\[
\hat{P}_K \circ \hat{T} = \hat{P}_K \circ \hat{T}^{(1)} \circ \hat{T}^{(2)} = T_K^{(1), \text{isom}} \circ \hat{P}_K \circ \hat{T}^{(2)} = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}} \circ \hat{P}_K.
\]

On the other hand, \( \hat{P}_K \circ \hat{T} = T_K^{\text{isom}} \circ \hat{P}_K \) and hence

\[
T_K^{\text{isom}} \circ \hat{P}_K = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}} \circ \hat{P}_K.
\]

Since \( \hat{P}_K : \widehat{\text{PaB}}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_K \) is onto, identity (3.20) implies that

\[
T_K^{\text{isom}} = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}}.
\]

Thus the map (3.19) is indeed a group homomorphism.

Our next goal is to show that homomorphism (3.19) is one-to-one and onto.

To prove that (3.19) is one-to-one, we consider \( \hat{T} \in \widehat{GT} \) such that \( T_K \) coincides with the canonical projection

\[
\text{PaB}^{\leq 4} \to \text{PaB}^{\leq 4}/\sim_K
\]

for every \( K \in NFI_{\text{isolated}}^\text{isolated}(B_4) \).

Hence, for every \( \gamma \in \text{PaB}^{\leq 4} \), we have

\[
\hat{P}_K \circ \hat{T}(I(\gamma)) = \hat{P}_K \circ I(\gamma), \quad \forall \ K \in NFI_{\text{isolated}}^\text{isolated}(B_4).
\]

This means that the restriction of \( \hat{T} \) to the subset \( I(\text{PaB}^{\leq 4}) \subset \widehat{\text{PaB}}^{\leq 4} \) coincides with the restriction of the identity map \( \text{id} : \widehat{\text{PaB}}^{\leq 4} \to \text{PaB}^{\leq 4} \) to the subset \( I(\text{PaB}^{\leq 4}) \). Since the subset \( I(\text{PaB}^{\leq 4}) \) is dense in \( \widehat{\text{PaB}}^{\leq 4} \) and the space \( \widehat{\text{PaB}}^{\leq 4} \) is Hausdorff, we conclude that \( \hat{T} \) is the identity map \( \text{id} : \widehat{\text{PaB}}^{\leq 4} \to \text{PaB}^{\leq 4} \). Thus the injectivity of (3.19) is established.

Note that an element of \( \text{lim}(\mathcal{M}) \) is a family \( \{T_K^{\text{isom}}\}_{K \in NFI_{\text{isolated}}^\text{isolated}(B_4)} \) of isomorphisms of truncated operads

\[
T_K^{\text{isom}} : \text{PaB}^{\leq 4}/\sim_K \xrightarrow{\cong} \text{PaB}^{\leq 4}/\sim_K
\]

satisfying the following property: for every pair \( K \leq \hat{K} \) in \( NFI_{\text{isolated}}^\text{isolated}(B_4) \), the diagram

\[
\begin{array}{ccc}
\text{PaB}^{\leq 4}/\sim_K & \xrightarrow{T_K^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_K \\
\downarrow{\hat{T}_{k,\hat{K}}} & & \downarrow{\hat{T}_{k,\hat{K}}} \\
\text{PaB}^{\leq 4}/\sim_\hat{K} & \xrightarrow{T_{\hat{K}}^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_\hat{K}
\end{array}
\]

commutes.

Due to commutativity of (3.21), the formula

\[
\hat{T}(\gamma)(K) := T_K^{\text{isom}}(\gamma(K))
\]

(3.22)
defines a morphism of truncated operads in groupoids \( \hat{T} : \tilde{\mathbb{P}aB}^{\leq 4} \to \tilde{\mathbb{P}aB}^{\leq 4} \).

To prove that \( \hat{T} \) is continuous, we need to show that the composition
\[
\hat{P}_K \circ \hat{T} : \tilde{\mathbb{P}aB}^{\leq 4} \to \mathbb{P}aB^{\leq 4} / \sim_K
\]
is continuous for every \( K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(B_4) \).

By definition of \( \hat{T} \) and \( \hat{P}_K \)
\[
\hat{P}_K \circ \hat{T} = T^\text{isom}_K \circ \hat{P}_K
\]for every \( K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(B_4) \).

Since \( T^\text{isom}_K \) is an automorphism of the (finite) groupoid \( \mathbb{P}aB^{\leq 4} / \sim_K \) equipped with the discrete topology and \( \hat{P}_K \) is continuous, identity (3.23) implies that the composition \( \hat{P}_K \circ \hat{T} \) is indeed continuous.

Thus equation (3.22) defines a continuous endomorphism of the operad \( \tilde{\mathbb{P}aB}^{\leq 4} \).

To find the inverse of \( \hat{T} \), we denote by \( S^\text{isom}_K \) the inverse of \( T^\text{isom}_K \) for every \( K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(B_4) \).

Then it is easy to see that the formula
\[
\hat{S}(\gamma)(K) := S^\text{isom}_K(\gamma(K))
\]
defines the inverse of \( \hat{T} \).

The proof of surjectivity of (3.19) is complete.

Let us consider \( K, N \in \text{NFI}_{\text{PB}_4}(B_4) \) with \( K \leq N \) and a pair \((m, f)\) \( \in \mathbb{Z} \times F_2 \) that represents a GT-shadow in \( \text{GT}^\Diamond(K) \). Clearly, the same pair \((m, f)\) also represents a GT-shadow in \( \text{GT}^\Diamond(N) \). In other words, if \( K \leq N \), then we have a natural map
\[
\text{GT}^\Diamond(K) \to \text{GT}^\Diamond(N).
\]

It makes sense to consider this map even if neither \( K \) nor \( N \) are isolated.

**Definition 3.12** We say that a GT-shadow \([m, f] \in \text{GT}^\Diamond(N)\) survives into \( K \) if \([m, f] \) belongs to the image of the map (3.24). In other words, there exists \((m_1, f_1) \in \mathbb{Z} \times F_2\) such that \([(m_1, f_1)] \in \text{GT}^\Diamond(K), m_1 \cong m \mod N_{\text{ord}} \) and \( f_1 N_{F_2} = f N_{F_2} \).

The following statement is a straightforward consequence of Proposition 3.3 and Theorem 3.8.

**Corollary 3.13** Let \( N \in \text{NFI}_{\text{PB}_4}(B_4) \) and \([m, f] \in \text{GT}^\Diamond(N)\). The GT-shadow \([m, f] \) is genuine if and only if \([m, f] \) survives into \( K \) for every \( K \in \text{NFI}_{\text{PB}_4}(B_4) \) such that \( K \leq N \).

\( \square \)

### 4 Selected results of computer experiments

In the computer implementation [4], an element \( N \) of \( \text{NFI}_{\text{PB}_4}(B_4) \) is represented by a group homomorphism \( \psi \) from \( \text{PB}_4 \) to a symmetric group such that \( N = \ker(\psi) \). Each homomorphism \( \psi: \text{PB}_4 \to S_d \) is, in turn, represented by a tuple of permutations
\[
(g_{12}, g_{23}, g_{13}, g_{14}, g_{24}, g_{34}) \in (S_d)^6
\]
satisfying the relations of \( \text{PB}_4 \) (see (A.3)).

It should be mentioned that, in [4], we consider only practical \( \text{GT}-\)shadows (see Remark 2.15). In particular, throughout this section, \( \text{GT}(N) \) denotes the set of practical \( \text{GT}-\)shadows with the target \( N \). Clearly, every charming \( \text{GT}-\)shadow is practical.

Table I presents basic information about 35 selected elements

\[
N^{(0)}, \ N^{(1)}, \ldots, \ N^{(34)} \in \text{NFI}_{\text{PB}_4}(B_4).
\]

For every \( N^{(i)} \) in this list, the quotient \( F_2/N^{(i)}_{F_2} \) is non-Abelian. Table I also shows \( N^{(i)}_{\text{ord}} := |\text{PB}_2 : N^{(i)}_{\text{PB}_2}| \), the size of \( \text{GT}(N^{(i)}) \) (i.e. the total number of practical \( \text{GT}-\)shadows with the target \( N^{(i)} \)) and the size of \( \text{GT}^\circ(N^{(i)}) \). The last column indicates whether \( N^{(i)} \) is isolated or not.

For every non-isolated element \( N \) in the list (4.2), the connected component \( \text{GTSh}_{\text{conn}}^\circ(N) \) has exactly two objects. More precisely,

- \( N^{(4)} \) is a conjugate of \( N^{(3)} \) and \( N^{(3)} \cap N^{(4)} = N^{(14)} \);
- \( N^{(11)} \) is a conjugate of \( N^{(10)} \) and \( N^{(10)} \cap N^{(11)} = N^{(24)} \);
- \( N^{(17)} \) is a conjugate of \( N^{(16)} \) and \( N^{(16)} \cap N^{(17)} = N^{(30)} \);
- \( N^{(27)} \) is a conjugate of \( N^{(26)} \) and \( N^{(26)} \cap N^{(27)} = N^{(34)} \).

For \( N^{(31)} \), \( \text{GT}(N^{(31)}) \) has 588 elements. To find the size of \( \text{GT}(N^{(31)}) \), the computer had to look at \( \approx 9 \times 10^6 \) elements of the group \( F_2/N^{(31)}_{F_2} \). For the iMac with the processor 3.4 GHz, Intel Core i5, it took over 9 full days to complete this task.

For \( N^{(32)} \), \( \text{GT}(N^{(32)}) \) has 800 elements. To find the size of \( \text{GT}(N^{(32)}) \), the computer had to look at over \( 9 \times 10^6 \) elements of the group \( F_2/N^{(32)}_{F_2} \). For the iMac with the processor 3.4 GHz, Intel Core i5, it took almost 10 full days to complete this task.

**Remark 4.1** Recall that the definition of an isolated element of \( \text{NFI}_{\text{PB}_4}(B_4) \) (see Definition 3.2) is based on charming \( \text{GT}-\)shadows. In principal, it is possible that there exists an isolated element \( N \in \text{NFI}_{\text{PB}_4}(B_4) \) for which \( \text{GT}(N) \) has a non-settled element. We did not encounter such examples in our experiments.

### 4.1 Selected remarkable examples

For the 19-th example \( N^{(19)} \) in table I, the quotient \( F_2/N^{(19)}_{F_2} \) has order 7776 = \( 2^5 \cdot 3^5 \). Due to the similarity between this order and the historic year 1776, we decided to call the subgroup \( N^{(19)} \) the *Philadelphia subgroup* of \( \text{PB}_4 \). This subgroup is the kernel of the homomorphism from \( \text{PB}_4 \) to \( S_9 \) that sends the standard generators of \( \text{PB}_4 \) to the permutations

\[
\begin{align*}
g_{12} := & \ (1, 3, 2)(4, 6, 5), \quad g_{23} := \ (1, 4, 9)(2, 7, 6), \quad g_{13} := \ (1, 7, 5)(3, 6, 9), \\
g_{14} := & \ (2, 6, 7)(3, 8, 5), \quad g_{24} := \ (1, 8, 6)(3, 4, 7), \quad g_{34} := \ (1, 2, 3)(7, 9, 8),
\end{align*}
\]

(4.3)

respectively.

Since \( N^{(19)} \) is isolated, \( \text{GT}^\circ(N^{(19)}) \) is a group. We showed that \( \text{GT}^\circ(N^{(19)}) \) is isomorphic to the dihedral group \( D_6 = \langle r, s \mid r^6, s^2, rsrs \rangle \) of order 12. We also showed that the kernel of the restriction of the virtual cyclotomic character to \( \text{GT}^\circ(N^{(19)}) \) coincides with the cyclic subgroup \( \langle r \rangle \) of order 6.
is a group. This is what we showed about this group: Due to Proposition 3.3, the Mighty Dandy is an isolated element and hence $\text{GT}(N^{(34)})$ is a group. This is what we showed about this group:

| $i$ | $|PB_4 : N^{(i)}|$ | $|F_2 : N^{(i)}_{F_2}|$ | $|F_2/N^{(i)}_{F_2}|$ | $N^{(i)}_{\text{ord}}$ | $|\text{GT}(N^{(i)})|$ | $|\text{GT}^\oplus(N^{(i)})|$ | isolated? |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------|
| 0   | 8               | 16              | 2               | 4               | 4               | 4               | True     |
| 1   | 8               | 16              | 2               | 4               | 8               | 4               | True     |
| 2   | 12              | 36              | 4               | 3               | 18              | 6               | True     |
| 3   | 21              | 63              | 7               | 3               | 36              | 12              | False    |
| 4   | 21              | 63              | 7               | 3               | 36              | 12              | False    |
| 5   | 24              | 288             | 8               | 6               | 72              | 12              | True     |
| 6   | 24              | 144             | 4               | 6               | 72              | 12              | True     |
| 7   | 48              | 144             | 4               | 6               | 72              | 12              | True     |
| 8   | 60              | 1500            | 60              | 5               | 100             | 20              | True     |
| 9   | 60              | 900             | 4               | 15              | 360             | 24              | True     |
| 10  | 72              | 144             | 18              | 4               | 16              | 8               | False    |
| 11  | 72              | 144             | 18              | 4               | 16              | 8               | False    |
| 12  | 108             | 972             | 27              | 6               | 72              | 12              | True     |
| 13  | 120             | 6000            | 60              | 10              | 400             | 40              | True     |
| 14  | 147             | 441             | 49              | 3               | 216             | 72              | True     |
| 15  | 168             | 8232            | 168             | 7               | 294             | 42              | True     |
| 16  | 168             | 1344            | 168             | 4               | 64              | 32              | False    |
| 17  | 168             | 1344            | 168             | 4               | 64              | 32              | False    |
| 18  | 180             | 13500           | 60              | 15              | 600             | 40              | True     |
| 19  | 216             | 7776            | 216             | 6               | 72              | 12              | True     |
| 20  | 240             | 6000            | 60              | 10              | 400             | 40              | True     |
| 21  | 324             | 8748            | 108             | 9               | 486             | 54              | True     |
| 22  | 504             | 40824           | 504             | 9               | 486             | 54              | True     |
| 23  | 504             | 24696           | 504             | 7               | 294             | 42              | True     |
| 24  | 648             | 1296            | 162             | 4               | 32              | 16              | True     |
| 25  | 720             | 54000           | 240             | 15              | 1800            | 120             | True     |
| 26  | 1512            | 40824           | 504             | 9               | 486             | 54              | False    |
| 27  | 1512            | 40824           | 504             | 9               | 486             | 54              | False    |
| 28  | 2520            | 63000           | 2520            | 5               | 200             | 40              | True     |
| 29  | 2520            | 45300           | 2520            | 6               | 144             | 48              | True     |
| 30  | 28224           | 225792         | 28224           | 4               | 512             | 256             | True     |
| 31  | 181440          | 8890560        | 181440          | 7               | 588             | 84              | True     |
| 32  | 181440          | 9072000        | 181440          | 10              | 800             | 160             | True     |
| 33  | 181440          | 40824000       | 181440          | 15              | $\geq 1800$     | 120             | True     |
| 34  | 762048          | 20575296       | 254016          | 9               | $\geq 4374$     | 486              | True     |

Table 1: The basic information about selected 35 compatible equivalence relations

The last element $N^{(34)}$ in $\text{PB}_4$ has the biggest index $762,048 = 2^6 \cdot 3^5 \cdot 7^2$ in $\text{PB}_4$. This subgroup is the kernel of the homomorphism from $\text{PB}_4$ to $S_{18}$ that sends the standard generators of $\text{PB}_4$ to

\[
g_{12} := (1,3,5,7,9,2,4,6,8)(10,12,14,16,18,11,13,15,17), \]
\[
g_{22} := (1,3,7,8,2,4,9,6,5)(10,15,17,11,12,16,18,14,13) \]
\[
g_{13} := (1,3,5,7,9,2,4,6,8)(10,11,15,17,13,12,18,14,16) \]
\[
g_{14} := (1,3,7,8,2,4,9,6,5)(10,15,17,11,12,16,18,14,13) \]
\[
g_{24} := (1,7,6,2,4,8,9,3,5)(10,15,14,11,16,18,12,13,17) \]
\[
g_{34} := (1,3,5,7,9,2,4,6,8)(10,12,14,16,18,11,13,15,17) \]

respectively. We call this subgroup the Mighty Dandy.

Due to Proposition 3.3, the Mighty Dandy is an isolated element and hence $\text{GT}^\oplus(N^{(34)})$ is a group. This is what we showed about this group:

40
• GT\(\Diamond\)(N\(^{(34)}\)) has order 486 = 2 \cdot 3^5;
• the kernel Ker\(_{34}\) of the restriction of the virtual cyclotomic character to GT\(\Diamond\)(N\(^{(34)}\)) is an Abelian subgroup of order 81 = 3^4; in fact, Ker\(_{34}\) is isomorphic to \(\mathbb{Z}_9 \times \mathbb{Z}_9\);
• GT\(\Diamond\)(N\(^{(34)}\)) is isomorphic to the semi-direct product
\[
(\mathbb{Z}_2 \times \mathbb{Z}_3) \ltimes (\mathbb{Z}_9 \times \mathbb{Z}_9);
\]
• the Sylow 3-subgroup Syl of GT\(\Diamond\)(N\(^{(34)}\)) is a non-Abelian group of order 3^5 = 243; Syl is a normal subgroup of GT\(\Diamond\)(N\(^{(34)}\)) and it is isomorphic to \(\mathbb{Z}_3 \ltimes (\mathbb{Z}_9 \times \mathbb{Z}_9)\);

Although every element \(N\) in the list (4.2) has the property \(|F_2 : N_{F_2}| > |PB_4 : N|\), there are examples \(N \in NF_{PB_4}(B_4)\) for which \(|PB_4 : N|\) is significantly bigger than the index \(|F_2 : N_{F_2}|\).

One such example was suggested to us by Leila Schneps. Leila’s subgroup \(N^L\) of PB\(_4\) is the kernel of a homomorphism from PB\(_4\) to \(S_{130}\) and it can be retrieved from one of the storage files in [4]. Here is what we know about \(N^L\):
• the index of \(N^L\) in PB\(_4\) is \(2^{29} \cdot 3^{12} = 285315214344192\);
• the index of \(N^L_{PB_3}\) in PB\(_3\) is \(2^{12} \cdot 3^6 = 2985984\);
• the index of \(N^L_{F_2}\) in \(F_2\) is \(2^{10} \cdot 3^5 = 248832\);
• \(N^L_{\text{ord}} = 12\);
• the order of the commutator subgroup of \(F_2/N^L_{F_2}\) is \(2^6 \cdot 3^3 = 1728\);
• there are only 48 = 2^4 \cdot 3 charming GT-shadows for \(N^L\);
• \(N^L\) is an isolated element of \(NF_{PB_4}(B_4)\) and hence GT\(\Diamond\)(N\(^L\)) is a group.

We found that the group GT\(\Diamond\)(N\(^L\)) is isomorphic to the semi-direct product
\[
\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3),
\]
where the non-trivial element of \(\mathbb{Z}_2\) acts on
\[
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \{ a|a^2 \} \times \{ b|b^2 \} \times \{ c|c^2 \} \times \{ d|d^3 \} \tag{4.6}
\]
by the automorphism
\[
a \mapsto b, \quad b \mapsto a, \quad c \mapsto c, \quad d \mapsto d^{-1}.
\]

The restriction of the virtual cyclotomic character to GT\(\Diamond\)(N\(^L\)) gives us the group homomorphism
\[
\text{GT}^\Diamond(N^L) \to (\mathbb{Z}/12\mathbb{Z})^\times
\]
and the kernel of this homomorphism is the subgroup of (4.6) generated by \(ab, c\) and \(d\).
4.2 Is there a charming GT-shadow that is also fake?

Table 1 shows that the set $\text{GT}^\triangledown(N)$ of charming GT-shadows corresponding to a given $N \in \text{NFI}_{\text{PB}_4}(B_4)$ is typically a proper subset of $\text{GT}(N)$. For example, for the Philadelphia subgroup $N^{(19)}$, we have 72 GT-shadows and only 12 of them are charming.

Due to Proposition 2.20 every non-charming GT-shadow is fake. Thus, for a typical $N$ from our list of 35 elements of $\text{NFI}_{\text{PB}_4}(B_4)$, we have many examples of a fake GT-shadows.

For instance, $\text{GT}^\triangledown(N^{(19)})$ contains at least 60 fake GT-shadows.

It is more challenging to find examples of charming GT-shadows that are fake. At the time of writing, we did not find a single example of a charming GT-shadow that is also fake.

Here is what we did. In the list (4.2), there are exactly 24 pairs $(N^{(i)}, N^{(j)})$ with $i \neq j$ such that $N^{(j)} \leq N^{(i)}$.

For each such pair, we showed that every GT-shadow in $\text{GT}^\triangledown(N^{(i)})$ survives into $N^{(j)}$, i.e. the natural map $\text{GT}^\triangledown(N^{(j)}) \to \text{GT}^\triangledown(N^{(i)})$ is onto. We also looked at other selected examples of elements $K \leq N$ in $\text{NFI}_{\text{PB}_4}(B_4)$ in which $N$ belongs to the list (4.2) and $K$ is obtained by intersecting $N$ with another element of (4.2). In all examples we have considered so far, the natural map $\text{GT}^\triangledown(K) \to \text{GT}^\triangledown(N)$ is onto.

4.3 Versions of the Furusho property and selected open questions

Two versions of the Furusho property are motivated by a remarkable theorem which says roughly that, in the prounipotent setting, the pentagon relation implies the hexagon relations. For a precise statement, we refer the reader to [2, Theorem 3.1] and [10, Theorem 1].

We say that an element $N \in \text{NFI}_{\text{PB}_4}(B_4)$ satisfies the strong Furusho property if

Property 4.2 For every $fN_{F_2} \in F_2/N_{F_2}$ satisfying pentagon relation (2.20) modulo $N$, there exists $m \in \mathbb{Z}$ such that

- $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ and
- the pair $(m, f)$ satisfies hexagon relations (2.18), (2.19).

Furthermore, we say that an element $N \in \text{NFI}_{\text{PB}_4}(B_4)$ satisfies the weak Furusho property if

Property 4.3 For every $fN_{F_2} \in [F_2/N_{F_2}, F_2/N_{F_2}]$ satisfying pentagon relation (2.20) modulo $N$, there exists $m \in \mathbb{Z}$ such that

- $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ and
- the pair $(m, f)$ satisfies hexagon relations (2.18), (2.19).

Using [4], we showed that the following 11 elements of the list (4.2)

$$N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}, N^{(6)}, N^{(7)}, N^{(9)}, N^{(10)}, N^{(11)}, N^{(14)}, N^{(24)}$$

(4.7)

satisfy Property 4.2 and the remaining 24 elements of (4.2) do not satisfy Property 4.2.

For instance, for the Philadelphia subgroup $N^{(19)}$, $N_{\text{ord}}^{(19)} = 6$ and there are 216 elements $fN_{F_2}^{(19)}$ in $F_2/N_{F_2}^{(19)}$ that satisfy the pentagon relation modulo $N^{(19)}$. However, for only 36 of
these 216 elements, there exists \( m \in \{0, 1, \ldots, 5\} \) such that \( 2^m + 1 \) represents a unit in \( \mathbb{Z}/6\mathbb{Z} \) and the pair \((m, f)\) satisfies hexagon relations (2.18), (2.19) (modulo \( N_{PB_3}^{(19)} \)).

Using [4], we also showed that the following 13 elements of the list (4.2)

\[
N^{(0)}, N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}, N^{(5)}, N^{(6)}, N^{(7)}, N^{(9)}, N^{(10)}, N^{(11)}, N^{(14)}, N^{(24)}
\] (4.8)
satisfy Property 4.3 and the remaining 22 elements of (4.2) do not satisfy Property 4.3.

For instance, for the Mighty Dandy \( N^{(34)}, N^{ord}_{N^{(34)}} = 9 \) and there are 4096 elements in \([F_2/N_{F_2}^{(34)}, F_2/F_{F_2}^{(34)}]\) that satisfy the pentagon relation modulo \( N^{(34)} \). However, for only 243 of them, there exists \( m \in \{0, 1, \ldots, 8\} \) such that \( 2^m + 1 \) represents a unit in \( \mathbb{Z}/9\mathbb{Z} \) and the pair \((m, f)\) satisfies hexagon relations (2.18), (2.19) (modulo \( N_{PB_3}^{(34)} \)).

We conclude this section with selected open questions. Most of these questions are motivated by our experiments [4].

**Question 4.4** Let \( N \in NFL_{PB_4}(B_4) \) and \((m, f) \in \mathbb{Z} \times F_2\) be a pair satisfying (2.18), (2.19), (2.20) (relative to \( \sim_N \)). Recall that, due Proposition 2.10 if the group homomorphisms \( T_{PB_2}^{m,f} \) and \( T_{PB_3}^{m,f} \) are onto then so is the group homomorphism

\[
T_{PB_4}^{m,f} : PB_4 \to PB_4/N.
\]

Using [4], the authors could not find an example of a pair \((m, f) \in \mathbb{Z} \times F_2\) for which \( T_{PB_4}^{m,f} \) is onto but \( T_{PB_2}^{m,f} \) is not onto or \( T_{PB_3}^{m,f} \) is not onto. Can one prove that, if \( T_{PB_4}^{m,f} \) is onto, then so are the group homomorphisms \( T_{PB_2}^{m,f} \) and \( T_{PB_3}^{m,f} \)?

**Question 4.5** Is it possible to find an example of a non-isolated \( N \in NFL_{PB_4}(B_4) \) for which the connected component \( GTSh_{\text{conn}}^\heartsuit(N) \) has more than 2 objects? In other words, is it possible to find \( N \in NFL_{PB_4}(B_4) \) that has \( >2 \) distinct conjugates?

**Question 4.6** Is it possible to find \( K, N \in NFL_{PB_4}(B_4) \) such that \( K \leq N \) and the natural map

\[
GT^\heartsuit(K) \to GT^\heartsuit(N)
\]
is not onto? In other words, can one produce an example of a charming GT-shadow that is also fake?

**Question 4.7** Is it possible to find \( N \in NFL_{PB_4}(B_4) \) for which \( F_2/N_{F_2} \) is non-Abelian and we can identify all genuine GT-shadows in the set \( GT^\heartsuit(N) \)?

Note that, if \( F_2/N_{F_2} \) is Abelian, all charming GT-shadows can be described completely and they are all genuine. (See Theorem B.2 in Appendix B.)

### A The operad \( PB \) and its profinite completion

The operad \( PB \) of parenthesized braids is an operad in the category of groupoids and it was introduced by D. Tamarkin in [29].

In this appendix, we give a brief reminder of the operad \( PB \) and its profinite completion. For a more detailed exposition, we refer the reader to [9, Chapter 6].

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11For the iMac with the processor 3.4 GHz, Intel Core i5, it took over 52 hours to find all these elements.
12A very similar construction appeared in beautiful paper [1] by D. Bar-Natan.
A.1 The groups $B_n$ and $PB_n$

The Artin braid group $B_n$ on $n$ strands is, by definition, the fundamental group of the orbifold

$$\text{Conf}(n, \mathbb{C})/S_n,$$

where $\text{Conf}(n, \mathbb{C})$ denotes the configuration space of $n$ (labeled) points on $\mathbb{C}$: $\text{Conf}(n, \mathbb{C}) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$.

It is known [18, Chapter 1] that $B_n$ has the following presentation

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} \text{ if } |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \text{ for } 1 \leq i \leq n-2 \rangle,$$

where $\sigma_i$ is the element depicted in figure [A.1].

Fig. A.1: The generator $\sigma_i$

Recall that the pure braid group $PB_n$ on $n$ strands is the kernel of the standard group homomorphism $\rho : B_n \rightarrow S_n$. This homomorphism sends the generator $\sigma_i$ to the transposition $(i, i+1)$.

We denote by $x_{ij}$ (for $1 \leq i < j \leq n$) the following elements of $PB_n$

$$x_{ij} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$$

and recall [18, Section 1.3] that $PB_n$ has the following presentation:

$$PB_n \cong \langle \{x_{ij}\}_{1 \leq i < j \leq n} \mid \text{the relations} \rangle$$

with the relations

$$x_{rs}^{-1} x_{ij} x_{rs} = \begin{cases} x_{ij} & \text{if } s < i \text{ or } i < r < s < j, \\ x_{rj} x_{ij} x_{rj}^{-1} & \text{if } s = i, \\ x_{rj} x_{sj} x_{ij} x_{sj}^{-1} x_{rj}^{-1} & \text{if } r = i < s < j, \\ x_{rj} x_{sj} x_{rj}^{-1} x_{sj}^{-1} x_{ij} x_{sj} x_{rj} x_{sj}^{-1} x_{rj}^{-1} & \text{if } r < i < s < j. \end{cases}$$

(A.3)

For example, the standard generators of $PB_3$ are

$$x_{12} := \sigma_1^2, \quad x_{23} := \sigma_2^2, \quad x_{13} := \sigma_2 \sigma_1^2 \sigma_2^{-1}.$$  \hspace{1cm} (A.4)

The element

$$c := x_{23} x_{12} x_{13} = x_{12} x_{13} x_{23} = (\sigma_1 \sigma_2)^3 = (\sigma_2 \sigma_1)^3$$

has an infinite order; it generates the center of $PB_3$ and the center of $B_3$.

The elements $x_{12}$ and $x_{23}$ generate a free subgroup in $PB_3$. Thus $PB_3$ is isomorphic to $F_2 \times \mathbb{Z}$.

A direction calculation shows that

$$\sigma_1^{-1} x_{23} \sigma_1 = x_{13}, \quad \sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{12}^{-1} c, \quad \sigma_2^{-1} x_{13} \sigma_2 = x_{12}.$$  \hspace{1cm} (A.6)
A.2 The groupoid $\text{PaB}(n)$

Objects of $\text{PaB}(n)$ are parenthesizations of sequences $(\tau(1), \tau(2), \ldots, \tau(n))$ where $\tau$ is a permutation $S_n$. For example, $\text{PaB}(2)$ has exactly two objects $(12)$ and $(21)$ and $\text{PaB}(3)$ has 12 objects:

$$(12)3, (21)3, (23)1, (32)1, (31)2, (13)2, 1(23), 2(13), 2(31), 3(21), 3(12), 1(32).$$

To define morphisms in $\text{PaB}(n)$, we denote by $p$ the obvious projection from the set of objects of $\text{PaB}(n)$ onto $S_n$. For example,

$$p((23)1) := \left(\begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \end{array}\right).$$

For two objects $\tau_1, \tau_2$ of $\text{PaB}(n)$ we set

$$\text{Hom}_{\text{PaB}}(\tau_1, \tau_2) := \rho^{-1}(p(\tau_2)^{-1} \circ p(\tau_1)) \subset B_n,$$

where $\rho$ is the standard homomorphism $B_n$ to $S_n$.

For instance, $\text{Hom}_{\text{PaB}}(2(31), (31)2)$ consist of elements $g \in B_n$ such that

$$\rho(g) = \left(\begin{array}{c} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{array}\right).$$

An example of an isomorphism from $2(31)$ to $(31)2$ is shown in figure A.2

![Figure A.2: An example of an isomorphism from 2(31) to (31)2 in PaB(3)](image)

The composition of morphisms in $\text{PaB}(n)$ comes from the multiplication in $B_n$. For example, if $\eta$ is the element of $\text{Hom}_{\text{PaB}}(\tau_1, \tau_2)$ corresponding to $h \in B_n$ and $\gamma$ is the element of $\text{Hom}_{\text{PaB}}(\tau_2, \tau_3)$ corresponding to $g \in B_n$ then their composition $\gamma \cdot \eta$ is the element of $\text{Hom}_{\text{PaB}}(\tau_1, \tau_3)$ corresponding to $g \cdot h$. Note that we use $\cdot$ for the composition of morphisms in $\text{PaB}$ and the multiplication of elements in braid groups.

By definition of morphisms, we have a natural forgetful map

$$\text{ou} : \text{PaB}(n) \rightarrow B_n.$$  

This map assigns to a morphism $\gamma \in \text{PaB}(n)$ the corresponding element of the braid group $B_n$. Moreover, since the composition of morphisms in $\text{PaB}(n)$ comes from the multiplication in $B_n$, we have

$$\text{ou}(\gamma \cdot \eta) = \text{ou}(\gamma) \cdot \text{ou}(\eta)$$

for every pair $\gamma, \eta$ of composable morphisms.

The isomorphisms $\alpha \in \text{PaB}(3)$ and $\beta \in \text{PaB}(2)$ shown in figure A.3 play a very important role. We call $\beta$ the braiding and $\alpha$ the associator. Note that, although $\alpha$ corresponds to the identity element in $B_3$, it is not an identity morphism in $\text{PaB}(3)$ because $(12)3 \neq 1(23)$.  

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The symmetric group $S_n$ acts on $\text{Ob}(\mathbf{P}a\mathbf{B}(n))$ in the obvious way. Moreover, for every $\theta \in S_n$ and $\gamma \in \text{Hom}_{\mathbf{P}a\mathbf{B}(n)}(\tau_1, \tau_2)$, we denote by $\theta(\gamma)$ the morphism from $\theta(\tau_1)$ to $\theta(\tau_2)$ that corresponds to the same element of the braid group $B_n$, i.e.

$$ou(\theta(\gamma)) = ou(\gamma).$$ (A.9)

For example, if $\theta = (1, 2) \in S_3$ then

$$\theta(\alpha) = \begin{pmatrix} 2 & 1 \\ (2 & 1) \\ 3 \end{pmatrix}.$$ 

For our purposes, it is convenient to assign to every element $g \in B_n$ the corresponding morphism $m(g) \in \mathbf{P}a\mathbf{B}(n)$ from $(..(1, 2)3)\ldots n)$ to $(..(i_1,i_2)i_3)\ldots i_n)$, where $i_k := \rho(g)^{-1}(k)$. It is easy to see that the map

$$m : B_n \to \mathbf{P}a\mathbf{B}(n)$$ (A.10)

defined in this way is a right inverse of $ou$ (see (A.8)).

It is also easy to see that, for every pair $g_1, g_2 \in B_n$, we have

$$m(g_1 \cdot g_2) = \rho(g_2)^{-1}(m(g_1)) \cdot m(g_2).$$ (A.11)

For example, for $\sigma_1, \sigma_2 \in B_3$, $m(\sigma_1) = \text{id}_{12} \circ_1 \beta$ and

$$m(\sigma_2) := \begin{pmatrix} 1 & (1, 3) \\ (1 & 2) \\ 3 \end{pmatrix}.$$ 

The composition $m(\sigma_2) \cdot m(\sigma_1)$ is not defined because the source of $m(\sigma_2)$ does not coincide with the target of $m(\sigma_1)$. On the other hand, the source of $(1, 2)(m(\sigma_2))$ coincides with the target of $m(\sigma_1)$ and $(1, 2)(m(\sigma_2)) \cdot m(\sigma_1) = m(\sigma_2 \cdot \sigma_1)$.

### A.3 The operad structure on $\mathbf{P}a\mathbf{B}$

We already explained how the symmetric group $S_n$ acts on the groupoid $\mathbf{P}a\mathbf{B}(n)$. Furthermore, it is easy to see that $\{\text{Ob}(\mathbf{P}a\mathbf{B}(n))\}_{n \geq 1}$ is the underlying collection of the free operad
(in the category of sets) generated by the collection \( T \) with
\[
T(n) := \begin{cases} 
\{1, 2, 1\} & \text{if } n = 2, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Thus the functors
\[
\circ_i : \mathcal{P}^B(n) \times \mathcal{P}^B(m) \to \mathcal{P}^B(n + m - 1)
\]
act on the level of objects in the obvious way.

For example,
\[
(23)1 \circ_2 12 := ((23)4)1, \quad 21 \circ_1 (23)1 := 4((23)1), \quad 2(3(14)) \circ_3 1(32) := 2((3(54))(16)),
\]
where we use the gray color to indicate what happens with the inserted sequence. For instance, in the third example, \( 1(32) \mapsto (3(54)) \).

To define the action of the functor \( \circ_i \) on the level of morphisms, we proceed as follows: given \( \gamma \in \mathcal{P}^B(n) \), \( \tilde{\gamma} \in \mathcal{P}^B(m) \) and \( 1 \leq i \leq n \), we set \( g := \text{ou}(\gamma) \) and \( \tilde{g} := \text{ou}(\tilde{\gamma}) \); we compute the source and the target of \( \gamma \circ_i \tilde{\gamma} \) using the rules of operad \( \{\text{Ob}(\mathcal{P}^B(k))\} \). Finally, to get the element of \( \mathcal{B}_{n+m-1} \) corresponding to \( \gamma \circ_i \tilde{\gamma} \), we replace the strand of \( g \) that originates at the position labeled by \( i \) by a “thin” version of \( \tilde{g} \). For example,

For a more precise definition of operadic multiplications on \( \mathcal{P}^B \) we refer the reader to [9, Chapter 6].

The (iso)morphisms \( \alpha \) and \( \beta \) satisfy the following pentagon relation
\[
\begin{align*}
(1(23))4 \xrightarrow{id_{12} \circ_1 \alpha} & \quad ((12)3)4 \xrightarrow{\alpha_1 \circ_1 id_{12}} \quad (12)(34) \xrightarrow{\alpha_3 \circ id_{12}} \quad (1(23))4 \\
(1(23))4 \xrightarrow{id_{12} \circ_2 \alpha} & \quad 1((23)4) \xrightarrow{\alpha_2 \circ id_{12}} \quad 1(2(34)) \xrightarrow{\alpha_3 \circ id_{12}} \quad (1(23))4
\end{align*}
\]
and the two hexagon relations:
\[
\begin{align*}
(12)3 \xrightarrow{\beta \circ_1 id_{12}} & \quad 3(12) \xrightarrow{(1, 3, 2) \circ} \quad (31)2 \\
1(23) \xrightarrow{id_{12} \circ_2 \beta} & \quad 1(32) \xrightarrow{(2, 3)} \quad (13)2
\end{align*}
\]

(A.13)

(A.14)
\[
\begin{array}{cccc}
1(23) & \beta \circ \text{id}_{12} & (23)1 & (1, 2, 3) \alpha \\
\downarrow \alpha^{-1} & & \downarrow \hspace{1cm} (1, 2) \text{id}_{12} \circ \beta \\
(12)3 & \text{id}_{12} \circ \beta & (21)3 & (1, 2) \alpha \\
\end{array}
\]

It is known [9, Theorem 6.2.4] that\(^{13}\)

**Theorem A.1** As the operad in the category of groupoids, \(\text{PaB}\) is generated by morphisms \(\alpha\) and \(\beta\) shown in figure A.3. Moreover, any relation on \(\alpha\) and \(\beta\) in \(\text{PaB}\) is a consequence of (A.13), (A.14) and (A.15).

### A.4 The cosimplicial homomorphisms for pure braid groups in arities 2, 3, 4

The collection \(\{\text{PB}_n\}_{n \geq 1}\) of pure braid groups can be equipped with the structure of a cosimplicial group. For our purposes we will need the cofaces of this cosimplicial structure only in arities 2, 3 and 4.

Let \(\tau_1\) and \(\tau_2\) be objects of \(\text{PaB}(n)\) which differ only by parenthesizations, i.e. \(p(\tau_1) = p(\tau_2)\). For such objects, we denote by \(\alpha_{\tau_1}^{\tau_2}\) the isomorphism from \(\tau_1\) to \(\tau_2\) given by the identity element of \(B_n\). For example, the associator \(\alpha\) is precisely \(\alpha_{(123)}^{(1)(23)}\) and \(\alpha^{-1}\) is precisely \(\alpha_{(123)}^{(1)(23)}\).

Using the identity morphism \(\text{id}_{12} \in \text{PaB}(2)\), the maps \(\text{ou}, \text{m}\) (see (A.8), (A.10)) and the operadic insertions, we define the following maps from \(\text{PB}_3\) to \(\text{PB}_4\) and the maps from \(\text{PB}_2\) to \(\text{PB}_3\):

\[
\begin{align*}
\varphi_{123}(h) & := \text{ou}(\text{id}_{12} \circ_1 m(h)), & \varphi_{12,3,4}(h) & := \text{ou}(m(h) \circ_1 \text{id}_{12}), \\
\varphi_{1,2,34}(h) & := \text{ou}(m(h) \circ_2 \text{id}_{12}), & \varphi_{234}(h) & := \text{ou}(\text{id}_{12} \circ_2 m(h)), \\
\varphi_{12}(h) & := \text{ou}(\text{id}_{12} \circ_1 m(h)), & \varphi_{23}(h) & := \text{ou}(\text{id}_{12} \circ_2 m(h)), \\
\varphi_{12,3}(h) & := \text{ou}(m(h) \circ_1 \text{id}_{12}), & \varphi_{1,23}(h) & := \text{ou}(m(h) \circ_2 \text{id}_{12}).
\end{align*}
\]

We claim that

**Proposition A.2** The equations in (A.16) (resp. in (A.17)) define group homomorphisms from \(\text{PB}_3\) (resp. \(\text{PB}_2\)) to \(\text{PB}_4\) (resp. \(\text{PB}_3\)).

**Proof.** Let us consider the map \(\varphi_{1,2,34} : \text{PB}_3 \to \text{PB}_4\). For elements \(h, \tilde{h} \in \text{PB}_3\), we set

\[
\gamma := m(h), \quad \tilde{\gamma} := m(\tilde{h}).
\]

Since \(\text{PaB}\) is an operad in the category of groupoids, we have

\[
(\gamma \cdot \tilde{\gamma}) \circ_2 \text{id}_{12} = (\gamma \circ_2 \text{id}_{12}) \cdot (\tilde{\gamma} \circ_2 \text{id}_{12}).
\]

Hence

\[
\varphi_{1,2,34}(h) \cdot \varphi_{1,2,34}(\tilde{h}) = \text{ou}(\gamma \circ_2 \text{id}_{12}) \cdot \text{ou}(\tilde{\gamma} \circ_2 \text{id}_{12}) = \text{ou}
(\gamma \circ_2 \text{id}_{12}) \cdot (\tilde{\gamma} \circ_2 \text{id}_{12})).
\]

\(^{13}\)A very similar statement is proved in [9]. See Claim 2.6 in loc. cit. It goes without saying that Theorem A.1 can be thought of as a version of MacLane’s coherence theorem for braided monoidal categories.
\[
\varphi(\gamma \cdot \tilde{\gamma}) = \varphi_{1,2,3}(h \cdot \tilde{h}),
\]
where the last identity is a consequence of \(\gamma \cdot \tilde{\gamma} = m(h \cdot \tilde{h})\).

The proofs for the remaining 8 maps are very similar and we leave it to the reader. \(\square\)

Since all 9 maps in (A.16) and (A.17) are group homomorphisms, they are uniquely determined by their values on generators of \(PB_3\) and \(PB_4\), respectively. It is easy to see that

\[
\begin{align*}
\varphi_{123}(x_{12}) &= x_{12}, & \varphi_{123}(x_{23}) &= x_{23}, & \varphi_{123}(x_{13}) &= x_{13}, \\
\varphi_{234}(x_{12}) &= x_{23}, & \varphi_{234}(x_{23}) &= x_{34}, & \varphi_{234}(x_{13}) &= x_{24}, \\
\varphi_{123,4}(x_{12}) &= x_{13}x_{23}, & \varphi_{123,4}(x_{23}) &= x_{34}, & \varphi_{123,4}(x_{13}) &= x_{14}x_{24}, \\
\varphi_{1,23,4}(x_{12}) &= x_{12}x_{13}, & \varphi_{1,23,4}(x_{23}) &= x_{24}x_{34}, & \varphi_{1,23,4}(x_{13}) &= x_{14}, \\
\varphi_{1,2,34}(x_{12}) &= x_{12}, & \varphi_{1,2,34}(x_{23}) &= x_{23}x_{24}, & \varphi_{1,2,34}(x_{13}) &= x_{13}x_{14}.
\end{align*}
\]

(A.18)

(A.19)

\section{A.5 The profinite completion \(\widehat{\text{PaB}}\) of \(\text{PaB}\)}

Let \(G\) be a connected groupoid with finitely many objects and \(G\) be the group that represents the isomorphism class of \(\text{Aut}(a)\) for some object \(a\) of \(\mathcal{G}\). We tacitly assume that the group \(G\) is residually finite. Following [5], an equivalence relation \(\sim\) on \(\mathcal{G}\) is called compatible, if

1. \(\gamma_1 \sim \gamma_2 \Rightarrow\) the source (resp. the target) of \(\gamma_1\) coincides with the source (resp. the target) of \(\gamma_2\);
2. \(\gamma_1 \sim \gamma_2 \Rightarrow \gamma_1 \cdot \gamma \sim \gamma_2 \cdot \gamma\) and \(\tau \cdot \gamma_1 \sim \tau \cdot \gamma_2\) (if the compositions are defined);
3. the set \(\mathcal{G}/\sim\) of equivalence classes is finite.

It is clear that, for every compatible equivalence relation \(\sim\) on \(\mathcal{G}\), the quotient \(\mathcal{G}/\sim\) is naturally a finite groupoid (with the same set of objects).

Compatible equivalence relations on \(\mathcal{G}\) form a directed poset and the assignment \(\sim \mapsto \mathcal{G}/\sim\) gives us a functor from this poset to the category of finite groupoids. In [5], the profinite completion \(\mathcal{G}\) of the groupoid \(\mathcal{G}\) is defined as the limit of this functor.

In [5], it was also shown that compatible equivalence relations on \(\mathcal{G}\) are in bijection with finite index normal subgroups \(N\) of \(G\). This gives us the following “pedestrian” way of thinking about morphisms in \(\widehat{\mathcal{G}}(a, b)\): choose \(\lambda \in \mathcal{G}(a, b)\), then every morphism in \(\gamma \in \widehat{\mathcal{G}}(a, b)\) can be uniquely written as

\[
\gamma = \lambda \cdot h,
\]
where \(h \in \widehat{\mathcal{G}}\).

In [5], we also proved that the assignment \(\mathcal{G} \mapsto \widehat{\mathcal{G}}\) upgrades to a functor from the category of groupoids to the category of topological groupoids. Moreover, this is a symmetric monoidal functor.

Thus “putting hats” over \(\text{PaB}(n)\) for every \(n \geq 0\) gives us an operad \(\widehat{\text{PaB}}\) in the category of topological groupoids.

\footnote{\(\mathcal{G}(a, b)\) is non-empty because \(\mathcal{G}\) is connected.}
B Charming GT-shadows in the Abelian setting. Examples of genuine GT-shadows

Let us prove the following statement:

**Proposition B.1** For \(N \in \text{NF}_{PB_4}(B_4)\), the following conditions are equivalent:

a) the quotient group \(PB_4/N\) is Abelian;

b) the quotient group \(PB_3/N_{PB_3}\) is Abelian;

c) the quotient group \(F_2/N_{F_2}\) is Abelian.

**Proof.** Implications a) \(\Rightarrow\) b) and b) \(\Rightarrow\) c) are straightforward so we leave them to the reader.

Let us assume that the quotient group \(F_2/N_{F_2}\) is Abelian. Then the images of \(x_{12}\) and \(x_{23}\) in \(PB_3/N_{PB_3}\) commute. Furthermore, since the image of \(c\) in \(PB_3/N_{PB_3}\) is obviously in the center of \(PB_3/N_{PB_3}\) and \(PB_3 = \langle x_{12}, x_{23}, c \rangle\), we conclude that the quotient group \(PB_3/N_{PB_3}\) is also Abelian.

To show that the generators \(\bar{x}_{ij} := x_{ij}N \ (1 \leq i < j \leq 4)\) of \(PB_4/N\) commute with each other, we consider the group homomorphisms from \(PB_3\) to \(PB_4\) given by formulas (A.18).

Note that, for every homomorphism \(\phi : PB_3 \to PB_4\) in the set \(\{\phi_{234}, \phi_{123,4}, \phi_{1,2,3,4}, \phi_{1,2,34}, \phi_{234}\}\), we have \(N_{PB_3} \leq \phi^{-1}(N) \leq PB_3\). Therefore, since the quotient \(PB_3/N_{PB_3}\) is Abelian, the quotient \(PB_3/\phi^{-1}(N)\) is also Abelian.

Applying these observations to every \(\phi\) in (B.1), we deduce that

- the elements \(\bar{x}_{12}, \bar{x}_{23}, \bar{x}_{13}\) commute with each other;
- the elements \(\bar{x}_{23}, \bar{x}_{34}, \bar{x}_{24}\) commute with each other;
- the elements \(\bar{x}_{13}\bar{x}_{23}, \bar{x}_{34}\) and \(\bar{x}_{14}\bar{x}_{24}\) commute with each other;
- the elements \(\bar{x}_{12}, \bar{x}_{23}\bar{x}_{24}\) and \(\bar{x}_{13}\bar{x}_{14}\) commute with each other;
- the elements \(\bar{x}_{14}, \bar{x}_{12}\bar{x}_{13}\) and \(\bar{x}_{24}\bar{x}_{34}\) commute with each other.

Using these observations one can show that \([\bar{x}_{ij}, \bar{x}_{kl}] = 1_{PB_4/N}\) for every pair in the set
\[
\{(i, j), (k, l)\} \mid 1 \leq i < j \leq 4, 1 \leq k < l \leq 4, \} - \{(1, 2), (3, 4)\}, \{(1, 3), (2, 4)\}, \{(2, 3), (1, 4)\}.
\]

Luckily, due to (A.3), we have
\[
x_{12}x_{34} = x_{34}x_{12}, \quad x_{23}x_{14} = x_{14}x_{23}, \quad x_{13}^{-1}x_{24}x_{13} = [x_{14}, x_{34}]x_{24}[x_{14}, x_{34}]^{-1}.
\]

Thus all generators \(\bar{x}_{ij}\) of \(PB_4/N\) commute with each other. \(\square\)

If one of the three equivalent conditions of Proposition B.1 is satisfied then we say that we are in the Abelian setting.

We can now prove the following analog of the Kronecker-Weber theorem:
Theorem B.2 Let \( N \in \text{NF}_{PB_4}(B_3) \). If the quotient group \( PB_4/N \) is Abelian then

\[
\text{GT}^\circ(N) = \{(m + N_{\text{ord}}Z, \bar{1}) \mid 0 \leq m \leq N_{\text{ord}} - 1, \ \gcd(2m + 1, N_{\text{ord}}) = 1\}, \tag{B.2}
\]

where \( \bar{1} \) is the identity element of \( F_2/N_{F_2} \). Furthermore, every \( \text{GT} \)-shadow in (B.2) is genuine.

Proof. Since \( \bar{1} \) can be represented by the identity element of \( F_2 \), every element of the set

\[
X_N := \{(m + N_{\text{ord}}Z, \bar{1}) \mid 0 \leq m \leq N_{\text{ord}} - 1, \ \gcd(2m + 1, N_{\text{ord}}) = 1\} \tag{B.3}
\]
satisfies the pentagon relation (2.20).

For every element of \( X_N \), the hexagon relations (2.18) and (2.19) boil down to

\[
\sigma_1 x_1^m \sigma_2 x_2^m N_{PB_3} = \sigma_1 \sigma_2 (x_{13} x_{23})^m N_{PB_3}, \tag{B.4}
\]

and

\[
\sigma_2 x_2^m \sigma_1 x_1^m N_{PB_3} = \sigma_2 \sigma_1 (x_{12} x_{13})^m N_{PB_3}. \tag{B.5}
\]

Equation (B.4) follows easily from the identity

\[
\sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{13} x_{23}
\]

and the fact that the quotient \( PB_3/N_{PB_3} \) is Abelian.

Similarly, equation (B.5) follows easily from the identity

\[
\sigma_1^{-1} x_{23} \sigma_1 = x_{13}
\]

and the fact that the quotient \( PB_3/N_{PB_3} \) is Abelian.

We proved that every element of \( X_N \) is a \( \text{GT} \)-pair for \( N \). Moreover, since \( 2m + 1 \) represents a unit in the ring \( Z/N_{\text{ord}} Z \), every \( \text{GT} \)-pair in \( X_N \) is friendly, i.e. the group homomorphism \( T_{m,1}^{PB_2} : PB_2 \to PB_2/N_{PB_2} \) is onto.

Due to (2.28) and the second identity in (2.29), we have

\[
T_{m,1}^{PB_3}(x_{12}) = x_{12}^{2m+1} N_{PB_3}, \quad T_{m,1}^{PB_3}(x_{23}) = x_{23}^{2m+1} N_{PB_3}, \quad T_{m,1}^{PB_3}(c) = c^{2m+1} N_{PB_3}
\]

for every \( m \in Z \).

Since the orders of the elements \( x_{12} N_{PB_3}, x_{23} N_{PB_3} \) and \( c N_{PB_3} \) divide \( N_{\text{ord}} \) and \( 2m + 1 \) represents a unit in \( Z/N_{\text{ord}} Z \), all three cosets \( x_{12} N_{PB_3}, x_{23} N_{PB_3} \) and \( c N_{PB_3} \) belong to the image of \( T_{m,1}^{PB_3} \). Thus, due to Proposition 2.10 every element of \( X_N \) is a \( \text{GT} \)-shadow.

Furthermore, every \( \text{GT} \)-shadow in \( X_N \) is charming. The first condition of Definition 2.19 is clearly satisfied and the second one follows from the fact that \( 2m + 1 \) represents a unit in \( Z/N_{\text{ord}} Z \) and the orders of the elements \( x_{12} N_{F_2}, x_{23} N_{F_2} \) divide \( N_{\text{ord}} \).

Since the inclusion \( \text{GT}^\circ(N) \subseteq X_N \) is obvious, the first statement of Theorem B.2 is proved.

Let us now show that every \( \text{GT} \)-shadow in \( \text{GT}^\circ(N) \) is genuine.

Due to Remark 2.17 and the surjectivity of the cyclotomic character, we know that, for every \( \bar{\lambda} \in (Z/N_{\text{ord}} Z)^\times \), there should exist at least one genuine \( \text{GT} \)-shadow \( [(m, f)] \in \text{GT}^\circ(N) \) such that

\[
2 \bar{m} + \bar{1} = \bar{\lambda}. \tag{B.6}
\]

Let us assume that \( N_{\text{ord}} \) is odd. In this case \( \bar{2} \in (Z/N_{\text{ord}} Z)^\times \) and hence, for every fixed \( \bar{\lambda} \in (Z/N_{\text{ord}} Z)^\times \), equation (B.6) has exactly one solution \( \bar{m} \in Z/N_{\text{ord}} Z \).
Since, for every \( \tilde{\lambda} \in (\mathbb{Z}/\text{ord}\mathbb{Z})^\times \), we have exactly one GT-shadow \((\tilde{m}, 1)\) in \(\text{GT}^\vee(\mathbb{N})\) such that \(2\tilde{m} + 1 = \tilde{\lambda}\), the surjectivity of the cyclotomic character implies that every GT-shadow in \(\text{GT}^\vee(\mathbb{N})\) is genuine.

The case when \(\text{ord}\mathbb{N} = 2k\) (for \(k \geq 1\)) requires more work. In this case, equation (B.6) has exactly two solutions for every \(\tilde{\lambda} \in (\mathbb{Z}/2k\mathbb{Z})^\times\). More precisely, if \(2\tilde{m} + 1 = \tilde{\lambda}\) then the solution set for (B.6) is \(\{\tilde{m}, \tilde{m} + k\}\).

The proof of the desired statement about \(\text{GT}^\vee(\mathbb{N})\) is based on the fact that the integers \(2\tilde{m} + 1\) and \(2\tilde{m} + 2k + 1\) represent two distinct units in the ring \(\mathbb{Z}/4k\mathbb{Z}\).

Let \(K\) be an element of \(\text{NFI}_{P_B}(B_4)\) satisfying these three properties:

- \(K \leq \mathbb{N}\);
- \(P_B/\mathbb{N}\) is Abelian;
- \(4k\) divides \(K_0 := |P_B : K_PB_2|\).

One possible way to construct such \(K\) is to define a group homomorphism \(\psi : P_B \to S_{4k}\) by the formulas

\[
\psi(x_{ij}) := (1, 2, \ldots, 4k), \quad \forall \ 1 \leq i < j \leq 4
\]

and set \(K := \mathbb{N} \cap \ker(\psi)\).

Since the natural group homomorphism

\[
(\mathbb{Z}/K_0\mathbb{Z})^\times \to (\mathbb{Z}/4k\mathbb{Z})^\times
\]
is onto, there exist \(\tilde{\lambda}_1 \neq \tilde{\lambda}_2\) in \(\mathbb{Z}/K_0\mathbb{Z}\) whose images in \(\mathbb{Z}/4k\mathbb{Z}\)^\times are the two distinct units represented by \(2m + 1\) and \(2m + 2k + 1\), respectively.

Therefore there exist genuine GT-shadows \([,(m_1, 1)]\) and \([,(m_2, 1)]\) in \(\text{GT}^\vee(K)\) such that

\[
2m_1 + 1 \equiv \lambda_1 \text{ mod } K_0 \quad \text{and} \quad 2m_2 + 1 \equiv \lambda_2 \text{ mod } K_0.
\]

Consequently, \(m_1\) and \(m_2\) satisfy these congruences mod \(4k\):

\[
2m_1 + 1 \equiv 2m + 1 \text{ mod } 4k \quad \text{and} \quad 2m_2 + 1 \equiv 2m + 2k + 1 \text{ mod } 4k.
\]

Thus the images of the genuine GT-shadows \([,(m_1, 1)]\) and \([,(m_2, 1)]\) in \(\text{GT}(\mathbb{N})\) are \([,(m, 1)]\) and \([,(m + k, 1)]\).

\[\square\]

**Remark B.3** Note that, in the Abelian setting, every charming GT-shadow comes from an element of \(G\). The authors do not know whether there is a genuine GT-shadow (in the non-Abelian setting) that does not come from an element of \(G\). Of course, if such a GT-shadow exists then the homomorphism (1.1) is not onto\(^\text{15}\).

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\(^{15}\)Some mathematicians believe that, in modern mathematics, there are no tools for tackling this question.
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