SURFACE DRINFELD TORSORS I : HIGHER GENUS ASSOCIATORS

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ABSTRACT. We develop a higher genus version of Drinfeld associators by means of operad theory. We start by introducing a framed version of rational associators and Grothendieck–Teichmüller groups and show that their definition is independent of the framing data. Next, we define a framed version of the universal KZ connection and we use it to show that over the complex numbers, the rational framed Drinfeld torsor is not empty. Next, we concentrate on the higher genus version of this story. We define an operad module of framed parenthesized higher genus braidings in prounipotent groupoids and we define its chord diagram counterpart. We then use these operadic modules to operadically define higher genus associators and Grothendieck–Teichmüller groups, which again do not depend on the framing data. Finally, we compare our results in the genus 1 case with those appearing in the litterature.

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This article is the first of a series devoted to the study of surface analogs of the so-called rational Drinfeld torsor which consist, for each \( \mathbb{Q} \)-ring \( k \), on the (bi-)torsor \( \text{Ass}(k), \overline{\mathcal{GT}}(k), \text{GRT}(k) \) where \( \text{Ass}(k) \) is the set of Drinfeld \( k \)-associators acted upon by the \( k \)-prounipotent Grothendieck–Teichmüller group \( \overline{\mathcal{GT}}(k) \) and its graded version \( \text{GRT}(k) \). Different authors \([14, 16, 9, 15]\) already constructed analogs of Drinfeld torsors in the cyclotomic and elliptic cases and operadic descriptions of these torsors have become available recently in the mentioned cases \([19, 11, 12]\) and the twisted elliptic case \([11]\). In this article we will concentrate on the first part of this program. Namely, we develop the operadic construction of this torsor in the framed higher genus context. A second paper will be devoted to the study of Drinfeld torsors associated to orbit configuration spaces which are finite (possibly ramified) covers over framed configuration spaces of points on oriented surfaces.

One motivation for studying them is that Drinfeld torsors consist on a somehow “useful fiction” for finding interesting families of relations for analogs of multiple zeta values associated to a wide range of complex curves, through the study of algebraic relations satisfied by the monodromy of the flat universal KZB connection associated to the curve as such monodromy has proven to produce a \( \mathbb{C} \)-point in the set of Drinfeld associators associated to the curve in the already know cases.

Initially, Grothendieck–Teichmüller groups and associators were, in the genus 0, cyclotomic and genus 1 cases, constructed by using braided monoidal categories, braided modules categories and elliptic structures over braided monoidal categories respectively. Already in V. Drinfeld’s work, associators had an implicit operadic nature (made explicit in \([2]\)) which permits to define associators as formality isomorphisms between operads closely related to the little disks operad \( \mathbb{D}_2 \). More specifically, for \( k \) a \( \mathbb{Q} \)-ring, there are operads in \( k \)-prounipotent groupoids \( \overline{\mathcal{PaB}}(k) \), encapsulating the combinatorics of parenthesized braidings, and \( \text{GPaCD}(k) \), encapsulating the combinatorics of parenthesized chord diagrams. The former is obtained (roughly) by considering a parenthesized groupoid version of the pure braid group. The latter is obtained from the so-called Kohno-Drinfeld Lie \((k)\)-algebras \( \mathfrak{t}_n(k) \). In this scope, the \( k \)-prounipotent Grothendieck–Teichmüller group consists on the group of automorphisms of \( \overline{\mathcal{PaB}}(k) \) which are the identity on objects, the graded Grothendieck–Teichmüller group is the group of automorphisms of \( \text{GPaCD}(k) \) which are the identity on objects, and the set of \( k \)-associators consists on the set of isomorphisms \( \overline{\mathcal{PaB}}(k) \rightarrow \text{GPaCD}(k) \) of operads in \( k \)-prounipotent groupoids which are the identity on objects. It can be shown that these operadic point of view is compatible with the classic one, namely that there is a one-to-one correspondence between the operadic definition of these objects and the objects defined in the literature in terms of elements satisfying certain equations. Let us also mention that in \([19]\), B. Fresse developed very powerful tools to study rational homotopy theory of operads in order to understand, from a homotopical viewpoint, a deep relationship between operads and Grothendieck–Teichmüller groups which was first foreseen by M. Kontsevich in his work on deformation quantization process in mathematical physics.
Let us explain the general approach for constructing Drinfeld torsors in the framed higher genus context.

Let \( n \geq 1 \) and let \( M \) be a closed smooth manifold of dimension 2. Consider the configuration space of \( M \)

\[
\text{Conf}(M, n) = \{ \mathbf{x} = (x_1, \ldots, x_n) \in M^n; x_i \neq x_j \text{ if } i \neq j \}.
\]

The spaces \( \text{Conf}(M, n) \) are weakly equivalent to their Axelrod–Singer–Fulton–MacPherson (ASFM) compactification \( \overline{\text{Conf}}(M, n) \). These spaces are acted on by the symmetric group \( S_n \) by relabelling the marked points and the collection \( \overline{\text{Conf}}(M, -) := \{ \overline{\text{Conf}}(M, n) \}_{n \geq 0} \) is actually a \( S \)-module. When \( M \) is parallelizable, \( \overline{\text{Conf}}(M, -) \) forms a right \( C(\mathbb{R}^2, -) \)-module \( \overline{\text{Conf}}(M, -) \). Otherwise, in order to endow \( \overline{\text{Conf}}(M, -) \) with a well defined operadic structure, we need to introduce framed versions of the above considered configuration spaces. This consists on setting a choice of trivialization of the tangent bundle of \( M \) in order to specify in which direction we will insert the disks on \( M \) constructed by the ASFM compactification.

Let \( M \) be a Riemannian closed oriented compact 2-manifold and consider the bundle projection \( \pi_M: SO(M) \to M \), where \( SO(M) \) is the principal \( \text{GL}_2 \)-bundle of special orthogonal linear frames on \( M \). The framed configuration space \( \text{Conf}^f(M, n) \) of \( n \) distinct points in \( M \) is

\[
\text{Conf}^f(M, n) = \{ (\mathbf{x}, f_1, \ldots, f_n) \in \text{Conf}(M, n) \times SO(M)^n; f_i \in \pi^{-1}_M(x_i) \}.
\]

This is the same to define \( \text{Conf}^f(M, n) \) as the pullback of the diagram

\[
\begin{array}{ccc}
\text{SO}(M)^n & \longrightarrow & M^n \\
\downarrow & & \downarrow \\
\text{Conf}(M, n) & \longrightarrow & \text{Conf}^f(M, n)
\end{array}
\]

so \( \text{Conf}^f(M, n) \to \text{Conf}(M, n) \) is a principal \( SO(2)^n \)-bundle. If \( M \) is parallelizable, then \( \text{Conf}^f(M, n) \) is isomorphic to \( \text{Conf}(M, n) \times SO(2)^n \). For instance, this is the case when \( M = \mathbb{R}^2 \) (by considering its reduced version) or when \( M = \mathbb{T} \).

Let \( \overline{C}^f(\mathbb{R}^2, n) \) be the ASFM compactification of \( C^f(\mathbb{R}^2, n) = C(\mathbb{R}^2, n) \times SO(2)^n \). Now, if \( M \) is an oriented 2-manifold, then the collection of its framed ASFM compactifications forms a right \( \overline{C}^f(\mathbb{R}^2, -) \)-module denoted \( \overline{\text{Conf}}^f(M, -) \) where each space \( \overline{\text{Conf}}^f(M, n) \) is a principal \( SO(2)^n \)-bundle over \( \overline{\text{Conf}}(M, n) \).

In general, if \( M = \Sigma_g \) has genus \( g \), the \( S \)-module \( D^f_{2, g} \) of framed little 2-disks on \( \Sigma_g \) can be endowed with a well-defined operadic module structure over the framed little 2-disks operad \( D^f_2 \). In particular, if \( g = 1 \), as \( \mathbb{T} \) is pararellizable so each space \( D^f_{2, 1}(n) \) is isomorphic to \( D_{2, 1}(n) \times SO(2)^n \).

\[1\] In the case of non-oriented manifolds one can only consider the bundle projection \( O(M) \to M \).
Then we also have
\[
\begin{array}{c}
\xymatrix{
D_M^f(n) \ar[r] & \text{Conf}^f(M,n) \ar[d] & \text{Conf}^f(M,n) \\
D_M(n) \ar[u] & \text{Conf}(M,n) \ar[r] & \text{Conf}(M,n)
}
\end{array}
\]
where again the horizontal maps are $\mathfrak{S}_n$-equivariant homotopy equivalences.

If $M$ is parallelizable, then the semi-direct product in the below spaces becomes an usual product and we get a square of $\mathfrak{S}$-modules
\[
\begin{array}{c}
\xymatrix{
D_M^f \ar[r] & \text{Conf}^f(M,n)
}
\end{array}
\]

If $M$ is not parallelizable the second line of this square doesn’t enhance into an operadic module morphism but we still have a weak equivalence $\text{Conf}^f(M,n) \xrightarrow{\sim} D_M^f$ of modules over $\mathbb{C}^f(\mathbb{R}^2,-) \xrightarrow{\sim} D_2^f$.

**Plan of the paper.** After briefly recalling the categorical and operadical language we use to define rigourously rational Drinfeld torsors in Section 1, we introduce in Section 2 a full suboperad $\mathbf{PaB}^f \subset \pi_1(D^f_2)$ of framed parenthesized braidings. We do so by restricting the object sets of the groupoid so that $B(\mathbf{PaB}^f) \xrightarrow{\sim} B(\pi_1(D^f_2))$. We then use a framed version of the Kohno-Drinfeld Lie $k$-algebra and construct an operad $G\mathbf{PaCD}^f(k)$ of parenthesized framed (group-like) horizontal chord diagrams. These two operads will allow us to operadically define the bitorsor consisting of framed associators and framed Grothendieck–Teichmüller groups. After showing that such torsor is not empty over $\mathbb{C}$ by means of the monodromy of a framed version of the universal KZ connection, we show that this torsor is isomorphic to the unframed rational Drinfeld torsor. As an application of this, we relate associators and Grothendieck–Teichmüller groups to the rational homotopy theory of the framed little disks operad.

We then turn in Section 3 to the genus $g$ situation and we introduce a full submodule $\mathbf{PaB}^f_g \subset \pi_1(D^f_{2,g})$ of genus $g$ framed parenthesized braidings by restricting the object sets of the groupoid so that $B(\mathbf{PaB}^f_g) \xrightarrow{\sim} B(\pi_1(D^f_{2,g}))$ and we give a presentation of this operadic module by using the presentation of surface frame braids from [4]. We then construct a framed version $f^g_{g,n}(k)$ of the genus $g$ Kohno-Drinfeld Lie $k$-algebra contained in [17] and use it to define a $G\mathbf{PaCD}^f(k)$-module $G\mathbf{PaCD}^f_g(k)$ of genus $g$ parenthesized (group-like) framed chord diagrams.

The main result of this article is then the following

**Theorem (Theorem 3.17).** There is an isomorphism between the following two sets:

- the set $\text{Ass}^f_g(k)$ of couples $(F,G)$, where $F$ is an operad isomorphism $\mathbf{PaB}^f(k) \xrightarrow{\sim} G\mathbf{PaCD}^f(k)$ and $G$ is an isomorphism between the $\mathbf{PaB}^f(k)$-module $\mathbf{PaB}^f_g(k)$ and
the $\text{GPaCD}_f^I(k)$-module $\text{GPaCD}_f^I(k)$ which is the identity on objects and which is compatible with $F$;

- the set $\text{Ass}_g(k)$ consisting on tuples $(\mu, \phi, A_{1, 1}, \ldots, A_{g, 2})$ where $(\mu, \phi) \in \text{Ass}(k)$ and $A_{a, \pm} \in \exp(i_{g, 2}(k))$, for $a = 1, \ldots, g$, satisfying equations (34), (35), (36), (37) and (38).

Next, we operadically define genus $g$ (graded) Grothendieck–Teichmüller groups, extract from them descriptions la Drinfeld. We finish this section by making a conjecture on the existence of a genus $g \mathbb{C}$-associator by means of a yet-to-be-defined framed extension of the genus $g$ universal KZB connection which was constructed in [17].

Finally, in Section 4 we compare different genus 1 version of the Drinfeld torsor associated to configuration spaces of the 2-torus, depending on the framed/unframed and reduced/non-reduced versions of it.

It should be interesting to relate the Lie algebra of our genus $g$ graded Grothendieck–Teichmüller group to the higher genus Kashiwara–Vergne Lie algebra $\mathfrak{tv}(g, n+1)$ which is being studied in the recent work [1]. Finally, we should point out that the recent paper [13] makes a complementary construction of higher genus associators which intersects ours in the genus 0 case. It should be interesting to link their construction and ours via the study of higher genus version of the Arnold-Kohno isomorphism $\kappa_n : C^*_{\text{CE}}(t_n) \rightarrow H^*(\mathbb{C}(\mathbb{C}, n))$.

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1. Preliminaries on rational Drinfeld torsors

In this section we fix the categorical and operadical notation that will be used later on and which is used consistently in [11]. We also recall the definitions and results concerning the operadically defined rational Drinfeld torsor which is the triple consisting of the Grothendieck–Teichmüller groups, its graded version and the set rational Drinfeld associators. Let $k$ be a $\mathbb{Q}$-ring.

1.1. Pointed modules over an operad. Consider a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ having small colimits and such that $\otimes$ commutes with these. We make use of the following notation (we refer the reader to [11] for further details):

- Let $\mathcal{S}\text{-mod}$ be the category of $\mathcal{S}$-modules in $\mathcal{C}$, endowed with
  - the symmetric monoidal product $\otimes$ defined by
    $$(S \otimes T)(n) := \coprod_{p+q=n} (S(p) \otimes T(q))_{\mathcal{S}_p \times \mathcal{S}_q}.$$
where, for each group inclusion \( H \subset G \), \((-)^G_H \) is left adjoint to the restriction functor from the category of objects carrying a \( G \)-action to the category of objects carrying an \( H \)-action;

- the monoidal unit defined by
  \[
  1_\phi(n) := \begin{cases} 
  1 & \text{if } n = 0, \\
  \emptyset & \text{else}
  \end{cases}
  \]

• An operad in \( \mathcal{C} \) is a unital monoid in \((\mathcal{G}\text{-mod}, \circ, 1_\circ)\), where
  - \( \circ \) is the (non-symmetric) monoidal product \( \circ \) on \( \mathcal{G}\text{-mod} \) defined by
    \[
    (S \circ T)(n) := \prod_{k \geq 0} T(k) \otimes \bigotimes_k (S^{\otimes k}(n)) .
    \]
  - the monoidal unit \( 1_\circ(n) \) for \( \circ \) is given by
    \[
    1_\circ(n) := \begin{cases} 
    1 & \text{if } n = 1, \\
    \emptyset & \text{else}
    \end{cases}
    \]

The category of operads in \( \mathcal{C} \) will be denoted \( \text{Op}\mathcal{C} \).

• A module over an operad \( \mathcal{O} \) (in \( \mathcal{C} \)) is a left \( \mathcal{O} \)-module in \((\mathcal{S}\text{-mod}, \circ, 1_\circ)\). In particular, there are operations that decrease the arity and, in the case of modules, we have a distinguished morphism \( \mathcal{O} \to \mathcal{P} \) of \( \mathcal{G}\text{-mod} \).

• Let \( \mathcal{P} \to \mathcal{Q} \) be a morphism between operads in \( \mathcal{C} \), let \( \mathcal{M} \) be a module over \( \mathcal{P} \), and let \( \mathcal{N} \) be a module over \( \mathcal{Q} \). Operadic module morphisms \( \mathcal{M} \to \mathcal{N} \) are considered to lie in the category of \( \mathcal{P} \)-modules (via the restriction functor), and will simply be referred to as module morphisms.

• We write \( \text{OpR}\mathcal{C} \) for the category of pairs \((\mathcal{P}, \mathcal{M})\), where \( \mathcal{P} \) is an operad and \( \mathcal{M} \) is a right \( \mathcal{O} \)-module, in \( \mathcal{C} \). A morphism \((\mathcal{P}, \mathcal{M}) \to (\mathcal{Q}, \mathcal{N})\) in \( \text{OpR}\mathcal{C} \) is a pair \((f, g)\), where \( f : \mathcal{P} \to \mathcal{Q} \) is a morphism between operads and \( g : \mathcal{M} \to \mathcal{N} \) is a morphism of \( \mathcal{P} \)-modules.

• Let \( \mathcal{P}, \mathcal{Q} \) be two operads (resp. modules) in groupoids. If we are given a morphism \( f : \text{Ob}(\mathcal{P}) \to \text{Ob}(\mathcal{Q}) \) between the operads (resp. operad modules) of objects of \( \mathcal{P} \) and \( \mathcal{Q} \), then (following [19]), the fake pull-back operad (resp. operad module) \( f^* \mathcal{Q} \) is defined by
  - \( \text{Ob}(f^* \mathcal{Q}) := \text{Ob}(\mathcal{P}) \),
  - \( \text{Hom}_{(f^* \mathcal{Q})}(p, q) := \text{Hom}_{\mathcal{Q}}(f(p), f(q)) \).

• We denote by \( \text{CoAlg}_k \) the symmetric monoidal category of complete filtered topological coassociative cocommutative counital \( k \)-coalgebras, with monoidal product given by the completed tensor product \( \hat{\otimes}_k \) over \( k \).

• Let \( \text{Cat}(\text{CoAlg}_k) \) be the symmetric monoidal category of small \( \text{CoAlg}_k \)-enriched categories, with symmetric monoidal product \( \otimes \) given by
  - \( \text{Ob}(C \otimes C') := \text{Ob}(C) \times \text{Ob}(C') \),
  - \( \text{Hom}_{C \otimes C'}((c, c'), (d, d')) := \text{Hom}_C(c, d) \hat{\otimes}_k \text{Hom}_C(c', d') \).
Let $grLie_k$ be the category of positively graded finite dimensional Lie $k$-algebras, with symmetric monoidal structure given by the direct sum $\oplus$. There is a lax symmetric monoidal functor

$$\hat{U} : grLie_k \longrightarrow \mathbf{Cat}(\mathbf{CoAlg}_k)$$

sending a positively graded Lie algebra to the degree completion of its universal envelopping algebra, which is a complete filtered cocommutative Hopf algebra when viewed as a $\mathbf{CoAlg}_k$-enriched category with only one object.

There is a functor that goes from the category of surjective morphisms $G \to S$ with finitely generated kernel and with $S$ a finite group to the category of groupoids. It sends $\varphi : G \to S$ to the groupoid $G(\varphi)$ defined by $\text{Ob}(G(\varphi)) = S$ and, for $s, s' \in S$,

$$\text{Hom}_{G(\varphi)}(s, s') = \{ g \in G | \varphi g = s^{-1} s' \}$$

with multiplication of arrows in $G(\varphi)$ identical to the one in $G$.

Top will denote the category of topological spaces endowed with the cartesian product as symmetric monoidal product.

Consider the symmetric monoidal category $\mathbf{Grpd}$ of groupoids, with symmetric monoidal structure given by the cartesian product. There is a $k$-prounipotent completion functor

$$\mathcal{G} \mapsto \hat{G}(k) = G(\mathcal{G}(k))$$

for operads (resp. modules) in groupoids. Consider the symmetric monoidal category $\mathbf{Grpd}_k$ of $k$-prounipotent groupoids (being the image of the completion functor $\mathcal{G} \mapsto \hat{G}(k)$).

For $\mathcal{C}$ being $\mathbf{Grpd}$, $\mathbf{Grpd}_k$, or $\mathbf{Cat}(\mathbf{CoAlg}_k)$, the notation

$$\text{Aut}_{\mathcal{C}}^+ \quad \text{(resp. } \text{Iso}_{\mathcal{C}}^+)$$

refers to those automorphisms (resp. isomorphisms) which are the identity on objects. Likewise, in the case of operadic modules, the superscript “+” still indicates that we consider couples of morphisms that are both the identity on objects.

1.2. Rational Drinfeld torsors. Consider the inclusions of topological operads

$$\mathbf{Pa}(-) \subset \mathcal{C}(\mathbb{R}, -) \subset \mathcal{C}(\mathbb{C}, -)$$

where, for any finite set $I$,

- $\mathbf{Pa}(I)$ is the set of ordered maximal parenthesizations of $\underbrace{\cdot \cdot \cdot}_{|I| \text{ times}}$,

- $\mathcal{C}(\mathbb{R}, I)$ (resp. $\mathcal{C}(\mathbb{C}, I)$) is the Axelrod–Singer–Fulton–MacPherson (ASFM) compactification of the reduced configuration space of points indexed by $I$ in $\mathbb{R}$ (resp. in $\mathbb{C}$).

As a pointed operad in groupoids having $\mathbf{Pa}$ as operad of objects,

$$\mathbf{Pa}B := \pi_1(\mathcal{C}(\mathbb{C}, -), \mathbf{Pa})$$

is freely generated by
graded Lie Kohno-Drinfeld

We will denote $\tilde{\mathfrak{t}}$ the Lie algebra of Kohno-Drinfeld $\mathfrak{k}$ together with the following relations:

1. $\omega f$ is a quotient of $\mathfrak{t}$
2. $\mathfrak{grLie}$ equivalence ways:

We will depict the generator $x_i$ as a fundamental group of $\text{Conf}$ sending $t_{ij}$, $1 \leq i < j \leq n$, which satisfy a prescribed family of relations. We will depict the generator $x_i$ in the following two equivalent ways:

The holonomy Lie algebra of the configuration space $\text{Conf}(C, n)$ is isomorphic to the so-called Kohno-Drinfeld graded Lie $C$-algebra $t_n$ generated by $t_{ij}$, $1 \leq i < j \leq n$, with relations

(S) $t_{ij} = t_{ji}$,

(L) $|t_{ij}, t_{kl}| = 0$ if $\# \{i, j, k, l\} = 4$,

(4T) $|t_{ij}, t_{ik} + t_{jk}| = 0$ if $\# \{i, j, k\} = 3$.

The collection of Kohno-Drinfeld Lie $k$-algebras $t_n(k)$, defined likewise, is provided with the structure of an operad in the category $\mathfrak{grLie}_k$. The center of $t_3(k)$ is $c_3 = t_{12} + t_{13} + t_{23}$, the quotient of $t_3(k)$ by $c_3$ is the free Lie algebra $f_3(k)$. Along this paper we consider the inclusion $f_2(k) \subset t_3(k)$ sending $x$ to $t_{12}$ and $y$ to $t_{23}$.

We then consider the operad of chord diagrams $\mathcal{CD}(k) := \tilde{\mathcal{U}}(t(k))$ in $\mathcal{Cat}({\mathcal{CoAlg}_k})$. It has only one object in each arity. The terminal morphism of operads $\omega_1 : Pa = \text{Ob}(Pa(k)) \to \text{Ob}(\mathcal{CD}(k))$ allow us to consider the fake pull-back operad

$$Pa\mathcal{CD}(k) := \omega_1^* \mathcal{CD}(k)$$
of parenthesized chord diagrams. More explicitly, we have $\text{Ob}(\text{PaCD}(k)) = \text{Pa}$ and for all $p, q \in \text{PaCD}(k)(n)$, $\text{Mor}_{\text{PaCD}(k)(n)}(p, q) = \text{Mor}_{\text{CD}(k)(p, pt)}(\Upsilon(n, k))$. As is shown in [19, Theorem 10.3.4], the operad $\text{PaCD}(k)$ does not have a presentation in terms of generators and relations (as is the case for $\text{PaB}$) but has, nevertheless, a universal property with respect to generators $H^{1,2}, X^{1,2}$ and $a^{1,2,3}$ depicted as follows

These elements satisfy the following relations:

- $X^{2,1} = (X^{1,2})^{-1}$,
- $a^{1,2,3,4, a^{1,2,3}} = a^{1,2,3}a^{1,2,3}a^{2,3,4}$,
- $X^{12,3} = a^{1,2,3}X^{2,3}(a^{1,3,2})^{-1}X^{1,3,2}$,
- $H^{1,2} = X^{1,2}H^{2,1}(X^{1,2})^{-1}$,
- $H^{12,3} = a^{1,2,3}(H^{2,3} + X^{2,3}(a^{1,3,2})^{-1}H^{1,3}a^{1,3,2}X^{3,2})(a^{1,2,3})^{-1}$.

**Definition 1.1.** We call a rational Drinfeld torsor over $k$ the bi-torsor $(\text{GT}(k), \text{Ass}(k), \text{GRT}(k))$ where

\begin{align*}
(1) & \quad \text{GT}(k) = \text{Aut}^*_{\text{OpGrpd}_k}(\text{PaB}(k)), \\
(2) & \quad \text{Ass}(k) = \text{Iso}^*_{\text{OpGrpd}_k}(\text{PaB}(k), \text{PaCD}(k)), \\
(3) & \quad \text{GRT}(k) = \text{Aut}^*_{\text{OpGrpd}_k}(\text{PaCD}(k)).
\end{align*}

There is a bi-torsor isomorphism

\begin{align*}
\text{GT}(k), \text{Ass}(k), \text{GRT}(k) & \to (\text{GT}(k), \text{Ass}(k), \text{GRT}(k)),
\end{align*}

where

- $\text{Ass}(k)$ is the set of couples $(\mu, \varphi) \in k^* \times \exp(\hat{\text{F}}_2(k))$ such that
- $\varphi^{2,1,2} = (\varphi^{1,2,3})^{-1}$, in $\exp(\hat{i}_3(k))$,
- $\varphi^{1,2,3}e^{\mu t_{13}}/2, \varphi^{2,3,1}e^{\mu t_{12}}/2 \varphi^{1,2,3}e^{\mu t_{12}}/2 = e^{\mu (t_{12} + t_{13} + t_{23})}/2$, in $\exp(\hat{i}_3(k))$,
- $\varphi^{1,2,3,4}e^{\mu t_{12}}/2, \varphi^{2,3,4} = \varphi^{1,2,3,4}e^{\mu t_{12}}/2$, in $\exp(\hat{i}_3(k))$.

- $\text{GT}(k)$ is the group of pairs $(\lambda, f) \in k^* \times \hat{\text{F}}_2(k)$ which satisfy the following equations:
- $f(x, y) = f(y, x)^{-1}$, in $\hat{\text{F}}_2(k)$,
- $x f(x_1, x_2) x f(x_2, x_3) x f(x_3, x_1) = 1$, in $\hat{\text{F}}_2(k)$ (for all $x_1, x_2, x_3$),
- $f(x_{12}x_{23}, x_{34}) f(x_{12}, x_{23}) f(x_{12}x_{34}, x_{23}) = f(x_{12}, x_{23}) f(x_{12}x_{34}, x_{23})$, in $\hat{\text{F}}_2(k)$.

- $\text{GRT}(k) = \text{GRT}_1 \times k^*$ where $\text{GRT}_1$ is the group of elements $g \in \exp(\hat{\text{F}}_2(k))$ such that
- $g^{2,1,2} = g^{-1}$ and $g^{1,2,3}g^{2,3,1}g^{1,2,3} = 1$, in $\exp(\hat{i}_3(k))$,
- $t_{12} + \text{Ad}(g^{1,2,3})(t_{23}) + \text{Ad}(g^{2,3,1})(t_{13}) = t_{12} + t_{13} + t_{23}$, in $\hat{i}_3(k)$,
\[ g^{1,2,3}g^{1,23,4}g^{2,3,4}g^{1,2,3,4} = g^{12,3,4}g^{1,2,3,4} \text{ in } \exp(t_{i}(k)), \]

with multiplication law given by

\[(g_{1} * g_{2})(t_{12}, t_{23}) = g_{1}(t_{12}, \text{Ad}(g_{2}(t_{12}, t_{23}))(t_{23}))g_{2}(t_{12}, t_{23})\]

and where \( k^{s} \) acts on \( \text{GRT}_{1} \) by \( \lambda \cdot g(x,y) = g(\lambda x, \lambda y) \).

2. Operads associated to framed configuration spaces (framed associators)

2.1. The operad of parenthesized framed braidings.

2.1.1. Compactified framed configuration spaces of the plane. To any finite set \( I \) we associate framed configuration space

\[ \text{Conf}^{f}(\mathbb{C}, I) := \text{Conf}(\mathbb{C}, I) \times \text{SO}(2)^{\times I}. \]

We also consider its reduced version

\[ \text{C}^{f}(\mathbb{C}, I) := \text{Conf}^{f}(\mathbb{C}, I)/\text{S}_{I}. \]

The symmetric group \( \text{S}_{I} \) acts on \( \text{Conf}^{f}(\mathbb{C}, I) \) by relabelling the indices of the marked points and the map \( \text{Conf}^{f}(\mathbb{C}, [I]) := \text{Conf}^{f}(\mathbb{C}, I)/\text{S}_{I} \rightarrow \text{Conf}(\mathbb{C}, [I]) \) is a locally trivial bundle with fiber \( \text{SO}(2)^{I} \).

We then consider the ASFM compactification \( \overline{\text{C}}^{f}(\mathbb{C}, I) \) of the reduced framed configuration space \( \text{C}^{f}(\mathbb{C}, I) \). The boundary \( \partial \overline{\text{C}}^{f}(\mathbb{C}, I) = \overline{\text{C}}^{f}(\mathbb{C}, I) - \text{C}^{f}(\mathbb{C}, I) \) of \( \overline{\text{C}}^{f}(\mathbb{C}, I) \) is made of the following irreducible components: for any decomposition \( I = J_{1} \sqcup \cdots \sqcup J_{k} \) there is a component

\[ \partial_{J_{1}, \ldots, J_{k}} \overline{\text{C}}^{f}(\mathbb{C}, I) \cong \prod_{i=1}^{k} \overline{\text{C}}^{f}(\mathbb{C}, J_{i}) \times \overline{\text{C}}^{f}(\mathbb{C}, J_{i}). \]

The collection of spaces \( \overline{\text{C}}^{f}(\mathbb{C}, I) \) for all finite sets \( I \) assemble into an \( \mathcal{S} \)-module denoted \( \overline{\text{C}}^{f}(\mathbb{C}, -) \), and the inclusion of boundary components with respect to the direction of the frame data provides \( \overline{\text{C}}^{f}(\mathbb{C}, -) \) with the structure of an operad in topological spaces. This operad will be called the framed ASFM operad. It turns out to be weakly equivalent to the framed little 2-disks operad. Partial operadic composition morphisms can be pictured as follows:
2.1.2. The operad of framed parenthesized braidings. We have inclusions of topological operads

$$\mathbb{P}a \subset C(\mathbb{R}, -) \subset C^f(\mathbb{C}, -).$$

where the right inclusion is given by setting all framing elements pointing to the right. Then it makes sense to define the operad in groupoids of framed parenthesized braidings

$$\mathbb{P}aB^f = \pi_1(C^f(\mathbb{C}, -), \mathbb{P}a).$$

Example 2.1 (Description of $\mathbb{P}aB^f(1)$). Recall that $C^f(\mathbb{C}, 1) = \{(z_1, f_1)\} \approx S^1$. Besides the identity morphism in $\mathbb{P}aB^f(1)$, there is an element denoted $F^1 \in \text{End}_{\mathbb{P}aB^f(1)}(1)$ corresponding to the 360 clockwise twisting of the framing and which can be depicted as follows:

Two incarnations of $F^1$.

Example 2.2 (Description of $\mathbb{P}aB^f(2)$). Recall that $C(\mathbb{C}, 2) \approx S^1$. Then $C^f(\mathbb{C}, 2) \approx (S^1)^3$. We have two arrows $F^{1,2}$ and $F^{1,2}$ in $\mathbb{P}aB^f(2)$ going from $(12)$ to $(21)$ which can be depicted as follows:

The arrows $F^{1,2}$ (left) and $F^{1,2}$ (right).

Next, the image of $F^1$ in $\mathbb{P}aB^f(2)$ will be denoted $F^{12}$. It can be depicted as follows:

Two incarnations of $F^{12}$.

The element $F^{12}$ consists of a single ribbon, with a second a ribbon glued along its surface, being twisted 360 degrees and the blue strand is the transport of the glued ribbon lying in the surface of this ribbon.

Finally, we have arrows $R^{1,2}$ and $R^{1,2}$ in $\mathbb{P}aB^f(2)$ going from $(12)$ to $(21)$ which can be depicted as follows:
Remark 2.3. One can actually see that $\tilde{R}^{1,2}$, $F^{12}$ and $\tilde{F}^{1,2}$ are obtained from the arrows $F^{1,2}$ and $R^{1,2}$ via the following identities:

- $\tilde{R}^{1,2} = (R^{2,1})^{-1}$,
- $\tilde{F}^{1,2} = F^{12}(F^{1,2})^{-1}(R^{1,2}R^{2,1})^{-1}$,
- $F^{12} = F^{1,\varnothing} o_{1} I_{1}^{1,2} = F^{12,\varnothing}$.

Example 2.4 (Notable arrows in $\text{PaB}^{f}(3)$). We have an arrow $\Phi^{1,2,3}$ from (12)3 to 1(23) in $\text{PaB}^{f}(3)$. It can be depicted as follows:

The arrow $\Phi^{1,2,3}$ in $\text{PaB}^{f}(3)$.

Recall the definition of the operad $\text{CoB}$ of coloured braids from [19, Subsection 5.2.8] and of its framed version $\text{CoB}^{f}$ contained in [8]. As in the case of the operad $\text{PaB}$, the operad $\text{PaB}^{f}$ can be defined as the fake pullback of the framed version $\text{CoB}^{f}$ of $\text{CoB}$ along the operad map $\text{Pa} \rightarrow \mathcal{G}$ and we have a presentation of $\text{PaB}^{f}$ in terms of generators and relations. Namely, we have the following theorem which is an straightforward corollary of [8, Lemma 7.4].

Theorem 2.5. As a pointed operad in groupoids having $\text{Pa}$ as operad of objects, $\text{PaB}^{f}$ is freely generated by $F := F^{1,2} \in \text{PaB}^{f}(2)$, $R := R^{1,2} \in \text{PaB}^{f}(2)$ and $\Phi := \Phi^{1,2,3} \in \text{PaB}^{f}(3)$.
together with the following relations:

(R1) \( F^{\varnothing, 2} = \text{Id}_1 \) \((\text{in } \text{End}_{\text{PaB}^I}(1))\),

(R2) \( \Phi^{\varnothing, 2, 3} = \Phi^{\varnothing, \varnothing, 3} = \Phi^{1, 2, \varnothing} = \text{Id}_{1, 2} \) \((\text{in } \text{Hom}_{\text{PaB}^I}(12, 12))\),

(F) \( F^{1, 2} R^{1, 2} F^{2, 1} R^{2, 1} = F^{1, 2} \) \((\text{in } \text{Hom}_{\text{PaB}^I}(12, 12))\),

(H1) \( R^{1, 2} \Phi^{2, 1, 3} R^{1, 3} = \Phi^{1, 2, 3} R^{1, 23} \Phi^{2, 3, 1} \) \((\text{in } \text{Hom}_{\text{PaB}^I}(12))\),

(H2) \( R^{1, 2} \Phi^{2, 1, 3} R^{1, 3} = \Phi^{1, 2, 3} R^{1, 23} \Phi^{2, 3, 1} \) \((\text{in } \text{Hom}_{\text{PaB}^I}(3))\),

(P) \( \Phi^{12, 3, 4} \Phi^{1, 2, 3} \Phi^{1, 23, 4} \Phi^{2, 3, 4} \) \((\text{in } \text{Hom}_{\text{PaB}^I}(4))\).

Remark 2.6. Combining relations (R1) and (F) we obtain \( F^{1, \varnothing} = F^{1} \).

2.1.3. The non-symmetric operad \( \text{PB}^I \) of framed braidings. Let us now introduce two non-symmetric operads that will be of use in Lemma 3.9 and Theorem 3.6.

The collection \( \text{PB}^I := \{\text{PB}_n^I\}_{n \geq 0} \) can be endowed with the structure of a non-symmetric operad given by partial compositions

\[
\circ_i : \text{PB}_n^I \times \text{PB}_m^I \rightarrow \text{PB}_{n+m-1}^I
\]

\[
(b, b') \mapsto b \circ_i b'
\]

where \( b \circ_i b' \) is defined by replacing the \( i \)-labelled strand in \( b \) by the braid \( b' \) made very thin. Via the homotopy equivalence between framed little disks and framed configuration spaces we presented in the last section, one checks that the above operadic composition for \( \text{PB}^I \) is induced by that on \( D_2^I \).

The fundamental group of the unordered framed configuration space \( \text{Conf}^I(\mathbb{C}, [n]) \) was studied in [26] and is isomorphic to the framed (also called ribbon) braid group \( B_2^I \) generated by elements \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, f_1, f_2, \ldots, f_n \) together with relations

(B1) \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \), \quad \text{for all } 1 \leq i \leq n - 2,

(B2) \( (\sigma_i, \sigma_j) = 1 \), \quad \text{if } |i - j| > 1,

(FB1) \( f_i f_j = f_j f_i \), \quad \text{for all } 1 \leq i, j \leq n,

(FB2) \( \sigma_i f_j = f_{\sigma_n(j)} \sigma_i \), \quad \text{for all } 1 \leq i, j \leq n.

For convenience, we will rather think of the framed braid group \( B_n^I \) as a subgroup of \( B_{2n} \) with two generating elements \( \tau_i \) and \( f_i \) such that

\[
(1) \quad \tau_i = \sigma_{2i} \sigma_{2i-1} \sigma_{2i+1} \sigma_{2i},
\]

\[
(2) \quad f_i = \sigma_{2i-1}^{-1}
\]

Geometrically, if we denote \((z_1, z_2, \ldots, z_n, z_{n+1})\) a point in \( \text{Conf}^I(\mathbb{C}, [n]) \), then \( \tau_i \) exchanges \((z_1, z_i)\) and \((z_{i+1}, z_{2i+1})\) in clockwise direction and \( f_i \) makes a 360 degrees twist of \( z_i \) around \( z_1 \) in the clockwise direction.

The space \( \text{Conf}^I(\mathbb{C}, [n]) \) is an Eilenberg–Maclane space of type \( K(B_n^I, 1) \) and the group \( B_n^I \) is identified with the semidirect product \( \mathbb{Z}^n \rtimes B_n \) where the action of the braid group
B_n on \(\mathbb{Z}^n\) is given by \(\sigma(r_1, \ldots, r_n) = (r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(n)})\). An element of \(\mathcal{B}_n^f\) is written as \(f_1^{r_1}, f_2^{r_2}, \ldots, f_n^{r_n}, \alpha \in \mathcal{B}_n^f\) with \(\alpha \in \mathcal{B}_n\). The \(r_i\)'s are called framings. The group composition law in this notation is given by

\[
(f_1^{r_1}, f_2^{r_2}, \ldots, f_n^{r_n}, \alpha)(f_1^{s_1}, f_2^{s_2}, \ldots, f_n^{s_n}, \beta) = f_1^{r_1+s_1}, f_2^{r_2+s_2}, \ldots, f_n^{r_n+s_n}, \alpha \beta.
\]

Let \(n \geq 0\), and \(p\) be the object \(\ldots((12)3)\ldots)n\) of \(\mathcal{P}A\mathcal{B}^f(n)\). Then \(\text{Aut}_{\mathcal{P}A\mathcal{B}^f(n)}(p)\) identifies with the fundamental group \(\mathbb{P}B^f_n = \pi_1(\mathcal{C}(\mathbb{C}, n), p)\), which is isomorphic to the direct product \(\mathbb{Z}^n \times \mathbb{P}B^f_n\).

In the same way, one can construct a non-symmetric operad in groupoids \(\mathcal{B}^f\) in the following way:

- The objects of \(\mathcal{B}^f(n)\) are unnumbered maximal parenthesizations of length \(n\). In particular, this means that for every object \(p\) of \(\mathcal{P}a(n)\), there is a corresponding object \([p]\) in \(\mathcal{B}^f(n)\), and \([p] = [q]\) if \(p\) and \(q\) only differ by a permutation (but have the same underlying parenthesization).
- \(\mathcal{B}^f\) is freely generated by \(F := F^{**} \in \mathcal{B}^f(2)\), \(R := R^{**} \in \mathcal{B}^f(2)\) and \(\Phi := \Phi^{***} \in \mathcal{B}^f(3)\) together with relations (H1), (H2), (P) and the following relation:

\[
(F) \quad R^{**} R^{**} F^* F^* = F^{***} \quad \text{in} \ \text{End}_{\mathcal{B}^f(2)(**)},
\]

- \(\mathcal{B}^f\) is the image of \(\mathcal{P}a\mathcal{B}^f\) via the forgetful map \(\mathcal{O}p \to \mathcal{N}s\mathcal{O}p\) sending an operad to a non-symmetric operad.
- The operad of coloured framed braids is nothing but

\[
\mathbb{C}^{\mathcal{O}B} = \mathcal{G}(\mathcal{B}^f \to \mathbb{S}).
\]

It follows that there are group morphisms \(\mathcal{B}^f_n \to \text{Aut}_{\mathcal{B}(n)}(p) \to \mathbb{S}_n\), the left one being an isomorphism.

For example, arrows in \(\text{Aut}_{\mathcal{B}(3)}(**) \bullet\) can be depicted as follows (we neglect for simplicity the framing data):

\[
(5)
\]

We let the reader depict the generators \(F \in \mathcal{B}^f(2)\), \(R \in \mathcal{B}^f(2)\) and \(\Phi \in \mathcal{B}^f(3)\) accordingly.

2.2. Horizontal framed chord diagrams and rational framed associators.


2.2.1. The operad of framed chord diagrams. Let $t^f_n(k)$ denote the graded Lie algebra over $k$ generated by $t_{ij}$, $1 \leq i, j \leq n$ with relations

(FS) \quad t_{ij} = t_{ji}, \quad \text{for } 1 \leq i, j \leq n,

(FL) \quad [t_{ij}, t_{kl}] = 0, \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset,

(F4T) \quad [t_{ij}, t_{ik} + t_{jk}] = 0, \quad \text{if } \{i, j\} \cap \{k\} = \emptyset.

It is easy to see that we have a decomposition $t^f_n(k) = \bigoplus_{i=1}^{n} k t_{ii} \oplus t^f_n(k)$.

**Remark 2.7.** The above definition coincides with that appearing in [4], indeed it is isomorphic to the graded Lie algebra over $k$ generated by $t_{ij}$, $1 \leq i \neq j \leq n$ and $t_k$, $1 \leq k \leq n$, with relations (S), (L), (4T) and

(FL') \quad [t_i, t_j] = 0 \quad \text{if } 1 \leq i, j \leq n,

(FL'') \quad [t_i, t_{jk}] = 0 \quad \text{if } 1 \leq i, j, k \leq n.

The Lie algebra $t^f_n(k)$ is acted on by the symmetric group $S_n$, and one can show that the $S$-module in $grLie_k t^f_n(k) : = \{t^f_n(k)\}_{n \geq 0}$ is provided with the structure of an operad in $grLie_k$. Partial compositions are defined as follows: for $I, J$ a finite sets and $k \in I$,

\[
\circ_k : t^f_I(k) \otimes t^f_J(k) \quad \Rightarrow \quad t^f_{I \cup \{j\} - \{i\}}(k)
\]

\[
(t_{ij}, 0) \quad \Rightarrow \quad \begin{cases} 
  t_{ij} & \text{if } k \notin \{i, j\} \\
  \sum_{p \in J} t_{ip} & \text{if } k = i \\
  \sum_{p \in I} t_{pj} & \text{if } j = k
\end{cases}
\]

In particular, this has a translation into insertion-coproduct morphisms. We call $t^f(k)$ the operad of infinitesimal framed braids. We then consider the operad of framed chord diagrams $CD^f(k) : = \hat{U}(t^f(k))$ in $\text{Cat}(\text{CoAlg}_k)$.

**Remark 2.8.** Morphisms in $CD^f(k)(n)$ can be represented as linear combinations of diagrams of chords on $n$ vertical strands, where the chord diagram corresponding to $t_{ij}$ can be represented as in the unframed case, the chord corresponding to $t_{ii}$ as

and the composition is given by vertical concatenation of diagrams. Relations (S), (L) and (4T) can be described as in the in the unframed case and the remaining relations defining each
The operad \( \text{PaCD}^{f}(k) \) of parenthesized framed chord diagrams. As the operad \( \text{CD}^{f}(k) \) has only one object in each arity, we have an obvious terminal morphism of operads \( \omega_{1} : \text{Pa} = \text{Ob}(\text{Pa}(k)) \rightarrow \text{Ob}(\text{CD}^{f}(k)) \), and thus we can consider the operad

\[
\text{PaCD}^{f}(k) := \omega^{*}_{1} \text{CD}^{f}(k)
\]

in \( \text{Cat}(\text{CoAss}_{k}) \) of parenthesized framed chord diagrams. More explicitly we have:

- \( \text{Ob}(\text{PaCD}^{f}(k)) := \text{Pa} \),
- \( \text{Mor}_{\text{PaCD}^{f}(k)}(p, q) := \text{CD}^{f}(k) \).

**Example 2.9** (Notable arrows in \( \text{PaCD}^{f}(k) \)). We have the following arrow \( P^{1} \) in \( \text{PaCD}^{f}(k)(1) \)

\[
P^{1} = t_{11},
\]

as well as the following arrows in \( \text{PaCD}^{f}(k)(2) \) and \( \text{PaCD}^{f}(k)(3) \)

\[
P^{1,2} := t_{11}, \quad H^{1,2} := t_{12}, \quad X^{1,2} = 1, \quad \alpha^{1,2,3} = 1.
\]
Remark 2.10. The elements $a^{1,2,3}$, $X^{1,2}$, $H^{1,2}$ and $P^{1,2}$ are generators of $\text{PaCD}^f(k)$ and satisfy the following relations:

- $X^{2,1} = (X^{1,2})^{-1}$,
- $\tilde{P}^{1,2} P^{1,2} = P^{1,2}$,
- $P^{1,2} = \tilde{P}^{1,2}$ $\sigma_1$ Id$^{1,2}$,
- $a^{12,3,4} a^{1,2,3,4} = a^{1,2,3} a^{1,2,3,4} a^{2,3,4}$,
- $X^{12,3} = a^{1,2,3} X^{2,3}(a^{1,3,2})^{-1} X^{1,3} a^{3,1,2}$,
- $H^{1,2} = X^{1,2} H^{2,1}(X^{1,2})^{-1}$,
- $H^{12,3} = a^{1,2,3} H^{2,3}(a^{1,2,3})^{-1} + (X^{2,1})^{-1} a^{2,1,3} H^{1,3}(a^{2,1,3})^{-1} X^{2,1}$.

2.2.3. Rational framed associators.

Definition 2.11. A framed $k$-associator is an isomorphism between the operads $\text{PaB}^f(k)$ and $\text{GPaCD}^f(k)$ in $\text{Grpd}_k$ which is the identity on objects. We denote

$$\text{Ass}^f(k) := \text{Iso}^+_\text{op}(\text{PaB}^f(k), \text{GPaCD}^f(k))$$

the set of framed $k$-associators.

Proposition 2.12. There is a one-to-one correspondence between the set of framed $k$-associators $\text{Ass}^f(k)$ and the set $\text{Ass}(k)$ of $k$-associators.

Proof. A morphism $\tilde{H} : \text{PaB}^f(k) \rightarrow \text{GPaCD}^f(k)$ is uniquely determined by a morphism $H : \text{PaB}^f(k) \rightarrow \text{GPaCD}^f(k)$. Such a morphism is uniquely determined by two scalar parameters $\mu, \lambda \in k$ and $\varphi \in \exp(t_1)$ such that we have the following assignment in the morphism sets of the parenthesized chord diagram operad $\text{GPaCD}^f(k)$:

- $H(F^{1,2}) = e^{\lambda t_1} \cdot \text{Id}^{1,2}$,
- $H(R^{1,2}) = e^{\mu t_{12}} \cdot X^{1,2}$,
- $H(\Phi^{1,2,3}) = \varphi \cdot a^{1,2,3}$,

where $F^{1,2}$, $R^{1,2}$ and $\Phi^{1,2,3}$ are the generators of $\text{PaB}^f$. The triples $(\lambda, \mu, \varphi)$ then satisfy

- $H(F^{1,2}) = e^{\lambda t_1} \cdot \text{Id}^{1,2}$,
- $H(R^{1,2}) = e^{\mu t_{12}} \cdot X^{1,2}$,
- $H(\Phi^{1,2,3}) = \varphi \cdot a^{1,2,3}$,

where $F^{1,2}$, $R^{1,2}$ and $\Phi^{1,2,3}$ are the generators of $\text{PaB}^f$. The triples $(\lambda, \mu, \varphi)$ then satisfy

- $(\mu, \varphi) \in \text{Ass}(k)$,
- $e^{\lambda(t_1+t_2)+\mu t_{12}} = e^{\lambda(t_1+t_2)+\mu t_{12}}$.

From the last equation one can easily deduce (by using the map $t_2 \rightarrow t_3$) that $2\mu = \lambda$, which in turn implies that condition $e^{\lambda(t_1+t_2+2t_{12})} = e^{\lambda(t_1+t_2+2t_{12})}$ is trivially satisfied as the $t_i$ are central. This finishes the proof.

Theorem 2.13. The set $\text{Ass}^f(k)$ is non empty.

We will prove this statement in the following subsection by using the regularized monodromy of a framed version of the universal KZ connection.
2.2.4. The framed universal KZ connection. We use the conventions for principal bundles and monodromy actions from [10, Appendix A]. Define the framed universal KZ connection on the trivial $\exp(t_i^f)$-principal bundle over $\text{Conf}_\mathbb{C}(\mathbb{C}, n)$ as the connection given by the holomorphic 1-form

$$w_n^{f,KZ} := \sum_{1 \leq i \leq n} t_i d \log(\lambda_i) + \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij} \in \Omega^1(\text{Conf}_\mathbb{C}(\mathbb{C}, n), t_n^f),$$

which takes its values in $t_n^f$ and where the $\lambda_i \in \mathbb{C}^*$, for all $1 \leq i \leq n$, are the fiber coordinates.

**Proposition 2.14.** The connection $\nabla_n^{f,KZ} := d - w_n^{f,KZ}$ is flat.

**Proof.** Let $w_1 := \sum_{1 \leq i \leq n} t_i d \log(\lambda_i)$ and $w_2 := \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij}$. We want to show that $[w_1 + w_2, w_1 + w_2] = 0$. We have

$$[w_1 + w_2, w_1 + w_2] = [w_1, w_1] + [w_2, w_2] + [w_1, w_2] + [w_2, w_1] = 2[w_1, w_2]$$

since $[w_1, w_1] = 0$ because the relation (FT1), $[w_2, w_2] = 0$ because of flatness of the unframed KZ connection, and $[w_2, w_1] + [w_1, w_2] = 2[w_1, w_2]$. Next, because of relation (FT2), we have

$$[w_1, w_2] = [t_i d \log(\lambda_i), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] + \sum_{1 \leq i < j \leq n} [t_j d \log(\lambda_j), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}].$$

And finally,

$$\sum_{1 \leq i < j \leq n} [t_i d \log(\lambda_i), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] + \sum_{1 \leq i < j \leq n} [t_j d \log(\lambda_j), \frac{dz_i - dz_j}{z_i - z_j} t_{ij}] = 0.$$

This concludes the proof. \qed

In particular, we get morphism of splitting short exact sequences

\begin{equation}
\begin{array}{cccc}
1 & \longrightarrow & k^n & \longrightarrow & \mathcal{PB}_n^{f}(k) & \longrightarrow & \mathcal{PB}_n(k) & \longrightarrow & 1 \\
& & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^n & \longrightarrow & \exp(t_i^f(k)) & \longrightarrow & \exp(t_n^f(k)) & \longrightarrow & 1
\end{array}
\end{equation}

showing that $\mathcal{PB}_n^{f}(k) \rightarrow \exp(t_n^f(k))$ is a $k$-pro-unipotent group isomorphism. Similarly we get an isomorphism

$$\mathcal{PB}_n^{f}(k) \rightarrow \exp(t_n^f(k)) \cong \mathfrak{g}_n.$$

**Proof of Theorem 2.13.** Let $x \in \text{Conf}_\mathbb{C}(\mathbb{C}, n)$ and let $T_x^{f,KZ}$ be the parallel transport morphism associated to $\omega^{f,KZ}_{f,n}$. Then

$$T_x^{f,KZ}(\lambda_i) = e^{2i \pi t_i +} \in \exp(t_n^f)$$

so that $T_x^{f,KZ}(f_i) = (T_x^{f,KZ}(\sigma_i))^2$. \qed
2.3. The group $\text{GT}^f$ and homotopy theory of the framed little disks operad.

**Definition 2.15.** The framed Grothendieck–Teichmüller group is defined as the group

$$\text{GT}^f := \text{Aut}^*_\text{OpGrpd}(\text{PaB}^f)$$

of automorphisms of the operad in groupoids $\text{PaB}^f$ which are the identity of objects. One defines similarly the $k$-pro-unipotent version

$$\overline{\text{GT}}^f(k) := \text{Aut}^*_\text{OpGrpd}_k(\text{PaB}^f(k))$$

There are also pro-$\ell$ and profinite versions, denoted $\text{GT}^f_\ell$ and $\overline{\text{GT}}^f$ respectively, defined by replacing the $k$-pro-unipotent completion of $\text{PaB}$ by its pro-$\ell$ and profinite completions.

**Definition 2.16.** The graded framed Grothendieck–Teichmüller group is the group

$$\text{GRT}^f(k) := \text{Aut}^*_\text{OpGrpd}_k(\text{GPaCD}^f(k))$$

of automorphisms of $\text{GPaCD}^f(k)$ that are the identity on objects.

By [8, Lemma 7.7], there is a group isomorphism

$$\overline{\text{GT}}(k) \cong \overline{\text{GT}}^f(k) := \text{Aut}^*_\text{OpGrpd}_k(\text{PaB}^f(k))$$

and the fact that $t^f_n(k) = \bigoplus_{n=1}^\infty k_n \oplus t_n(k)$ gives us a further isomorphism

$$\text{GRT}(k) \cong \text{GRT}^f(k) := \text{Aut}^*_\text{OpGrpd}_k(\text{GPaCD}^f(k)).$$

The above results permit to extend the results in [20] in the following manner. Consider the diagram

$$\begin{array}{ccc}
D^f_2(n) & \xrightarrow{\cong} & \text{Conf}^f(\mathbb{R}^2, n) & \xleftarrow{\cong} & \overline{\text{C}}^f(\mathbb{R}^2, n) \\
\downarrow & & \downarrow & & \downarrow \\
D_2(n) & \xrightarrow{\cong} & \text{Conf}(\mathbb{R}^2, n) & \xleftarrow{\cong} & \overline{\text{C}}(\mathbb{R}^2, n)
\end{array}$$

where the horizontal arrows are $\mathfrak{S}_n$-equivariant homotopy equivalences and the vertical arrows are $\text{SO}(2)^n$-principal bundles. This diagram does not enhance into an operad map. Nevertheless, in [23], an operad morphism $\phi: \overline{\text{C}}(\mathbb{R}^2, -) \longrightarrow D_2$ was constructed and it is easy to verify that $\phi$ is equivariant for the action of $\text{SO}(2)$ on these two operads and by construction, the data of the framings are compatible with this map (since the rotation of a disk will preserve that disk). Thus, we can construct a square

$$\begin{array}{ccc}
D^f_2 & \xleftarrow{\cong} & \overline{\text{C}}^f(\mathbb{R}^2, -) \\
\downarrow & & \downarrow \\
D_2 & \xleftarrow{\cong} & \overline{\text{C}}(\mathbb{R}^2, -)
\end{array}$$

where the horizontal arrows are weak equivalences of operads in topological spaces (see [23] for details).
On the one hand, let $C_{\text{CE}}(t^n_f)$ be the Chevalley-Eilenberg cochain complex of $t^n_f$. It is a quasi-free commutative dg-algebra generated by the module $(t^n_f)\wedge$ in degree 1. Now $t^n_f$ is equipped with a weight grading such that each $t^n_{ij}, 1 \leq i, j \leq n$ is homogeneous of weight 1.

On the other hand, let $\Omega^*(\text{Conf}^f(C, n))$ be the de-Rham complex of $\text{Conf}^f(C, n)$. Now, as $\text{Conf}^f(C, n)$ is a trivial $\text{SO}(2)^n$-principal bundle over $\text{Conf}(C, n)$, then using the Kunneth formula and the cohomology of $S^1$, one can then show that the dg-algebra quasi-isomorphism $\kappa_n: C_{\text{CE}}(t_n) \to \text{H}^*(C(C, n))$ extends into a dg-algebra quasi-isomorphism

$$
\kappa_n^f: C_{\text{CE}}^*(t^n_f) \to \text{H}^*(C^f(C, n))
$$

Thus, as $\text{Ass}^f(Q) = \emptyset$, $LG, \text{H}^*(D^f_2)$ represents a rationalization of the framed little 2-disc operad $D^f_2$. Next, Fresse introduced an operadic replacement $A_2$ of Sullivan’s functor, showed that the the couple $(G_\bullet, A_2)$ is a Quillen pair and showed that if the components of an operad $\mathcal{O}$ in simplicial sets have a degree-wise finitely generated cohomology, then we have a weak equivalence $A_2(\mathcal{O})(n) = A(\mathcal{O})(n))$ for each arity $n$ so that the assignment $\mathcal{O} \to O^c_2 := LG_\bullet A(\mathcal{O})$ is equivalent to Sullivan’s rationalization of $\mathcal{O}(n)$ arity-wise.

Then equation (8) induces a rational weak equivalence

$$
(\text{D}^f_2)\wedge_Q \to LG_\bullet \text{H}^*(D^f_2),
$$

which in turn induces a weak-equivalence of Hopf dg-cooperads $A_2(D^f_2) \simeq \text{H}^*(D^f_2)$.

Let $Ho(\text{OpTop})$ be the homotopy category of the category of operads in topological spaces. We can then summarize the results of this section as follows.

**Theorem 2.17.** There is a torsor isomorphism

$$
(\tilde{\mathcal{GT}}^f(Q), \text{Ass}^f(Q), \mathcal{GRT}^f(Q)) \to (\tilde{\mathcal{GT}}(Q), \text{Ass}(Q), \mathcal{GRT}(Q))
$$
and the following maps are bijections

\[
\text{Ass}^f(\mathbb{Q}) \rightarrow \text{Iso}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}, LG^\bullet, H^*(D_2^f)),
\]

\[
\text{GRT}^f(\mathbb{Q}) \rightarrow \text{Aut}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}})
\]

**Proof.** The fact that the map (10) is a torsor isomorphism is a straightforward consequence of the fact that the set of complex associators is not empty, fact proven in Theorem 2.13, the fact that \((\text{GRT}^f(\mathbb{C}), \text{Ass}^f(\mathbb{C}), \text{GRT}^f(\mathbb{C}))\) has a natural torsor structure so that \(\text{Ass}^f(\mathbb{Q})\) is not empty and the fact that we have group isomorphisms \(\text{GRT}^f(\mathbb{C}) \rightarrow \text{GRT}(\mathbb{C})\) and \(\text{GRT}^f(\mathbb{C}) \rightarrow \text{GRT}(\mathbb{C})\).

The proof of the fact that (11) and (12) are bijections comes from the fact that the set \(\text{Ass}^f(\mathbb{Q})\) is not empty thus we have a chains of set morphisms

\[
\text{Ass}^f(\mathbb{Q}) \rightarrow \text{Ass}(\mathbb{Q}) \rightarrow \text{Iso}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}, LG^\bullet, H^*(D_2^f)) \rightarrow \text{Iso}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}, LG^\bullet, H^*(D_2^f))
\]

and

\[
\text{GRT}^f(\mathbb{Q}) \rightarrow \text{GRT}(\mathbb{Q}) \rightarrow \text{Aut}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}) \rightarrow \text{Aut}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}).
\]

The central morphisms are those constructed and proven to be isomorphisms by Fresse in [20], and the leftmost bijections are those coming from the isomorphism (10). The rightmost maps are constructed as sections of the restriction maps

\[
\text{Iso}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}, LG^\bullet, H^*(D_2^f)) \rightarrow \text{Iso}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}, LG^\bullet, H^*(D_2^f))
\]

\[
\text{Aut}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}}) \rightarrow \text{Aut}_{\text{Ho(OpTop)}}((D_2^f)^\wedge_{\mathbb{Q}})
\]

since \(\text{PaB}^f\) and \(t^f\) are quotients of \(\text{PaB}\) and \(t\). \(\square\)

3. Modules associated to framed configuration spaces (genus \(g\) associators)

3.1. The module of parenthesized genus \(g\) framed braidings.

3.1.1. Compactified configuration spaces of surfaces. Let \(g \geq 0\) and \(n > 0\) be two integers and consider a compact topological oriented surface \(\Sigma_g\) of genus \(g\).

The boundary \(\partial \text{Conf}^f(\Sigma_g, I) = \text{Conf}^f(\Sigma_g, I) - \text{Conf}^f(\Sigma_g, I)\) is made of the following irreducible components: for any decomposition \(I = J_1 \amalg \cdots \amalg J_k\) there is a component

\[
\partial J_1, \ldots, J_k \text{Conf}^f(\Sigma_g, I) \cong \prod_{i=1}^k \text{Conf}^f(\Sigma_g, J_i) \times \text{Conf}^f(\Sigma_g, k).
\]

The inclusion of boundary components with respect to the direction of the frame data provide \(\text{Conf}^f(\Sigma_g, -)\) with the structure of a module over the operad \(\text{Conf}^f(\Sigma, -)\) in topological spaces.

We can represent the action of \(\text{Conf}^f(\Sigma, -)\) on \(\text{Conf}^f(\Sigma_g, -)\) as follows (in the case \(g = 2\):
3.1.2. A presentation for surface framed braid groups. We recall that composition of paths are read from left to right. In particular the commutator of two elements \( A, B \), is \( (A, B) = ABA^{-1}B^{-1} \). For \( n \geq 1 \), we denote \( \text{PB}_{g,n}^f \) the fundamental group of \( \text{Conf}^f(\Sigma_g, n) \) and we define the framed braid group on \( \Sigma_g \) as the group \( B_{g,n}^f \) generated by

\[
X_1^a, Y_1^a, \ldots, X_g^a, Y_g^a, \tau_1, \ldots, \tau_{n-1}, f_1, \ldots, f_n,
\]

together with the following relations

- (T1) \( \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \), for all \( 1 \leq i \leq n - 2 \),
- (T2) \( (\tau_i, \tau_j) = 1 \), if \( |i - j| > 1 \),
- (FT1) \( \tau_i f_j = f_j \tau_i \), for all \( 1 \leq i, j \leq n \),
- (FT2) \( \tau_i f_j = f_j \tau_i \), for all \( j \neq i, i + 1 \)
- (FB3) \( \tau_i f_i = f_{i+1} \tau_i \), \( f_i \tau_i = \tau_i f_{i+1} \) for all \( 1 \leq i \leq n - 2 \),
- (FBG1) \( (X_1^a, \tau_i) = (Y_1^a, \tau_i) = 1 \), for all \( i = 2, \ldots, n-1 \), and \( 1 \leq a \leq g \),
- (FBG2) \( (X_a^a, X_2^a) = (Y_a^a, Y_2^a) = 1 \), for \( 1 \leq a \leq g \),
- (FBG3) \( (X_2^a, Y_1^a) = \tau_1^2 \), for \( 1 \leq a \leq g \),
- (FBG4) \( (X_1^a, X_2^b) = (Y_1^a, X_2^b) = (Y_1^a, Y_2^b) = 1 \) for \( 1 \leq b < a \leq g \),
- (FBG5) \( \prod_{a=1}^g ((X_1^a)^{-1}, Y_1^a) = \tau_1 \tau_{n-2} \tau_{n-2} \ldots \tau_1 f_1^{2(g-1)} \).

Here \( X_{i+1}^a = \tau_i X_i^a \tau_i \) and \( Y_{i+1}^a = \tau_i Y_i^a \tau_i \) for \( i = 1, \ldots, n - 1 \).

The corresponding geometric configuration of points and paths for the above presentation is the same that the one used in [4] which we now recall. Let \( B_{g,2n} \) be the fundamental group of \( \text{Conf}(\Sigma_g, [2n]) \) based at the point \( p = (p_1, \ldots, p_n) \) where the \( p_i \) are aligned in the right-most \( A \)-generating cycle of \( \Sigma_g \) (see the picture below). It is generated by paths \( \tilde{X}_1^g, \tilde{Y}_1^g, \tilde{X}_1^{a}, \tilde{Y}_1^{a}, \sigma_1, \ldots, \sigma_{2n-1} \) corresponding geometrically in particular to the following paths:
The geometric configuration of $B_{g,2n}$, for $n = 2$.

There is a morphism $B_{g,n} \rightarrow \mathcal{S}_n$ given by $\tilde{X}_i^a, \tilde{Y}_1^a \mapsto 1$, $\sigma_i \mapsto s_i := (i, i + 1)$. It is proved in [3] that the fundamental group $\pi_1(\text{Conf}(\Sigma_g, n))$ is isomorphic to the genus $g$ pure braid group $\text{PB}_{g,n}$ which is the kernel of this map and is generated by $\tilde{X}_i^a, \tilde{Y}_1^a$ $(1 \leq i \leq n, 1 \leq a \leq g)$, where $Z_{i+1}^a = \sigma_i Z_i^a \sigma_i$ for $Z$ any of the letters $X, Y$.

Then, $B_{g,n}^f$ is seen as a subgroup of $B_{g,2n}$ and its generators can be written in terms of the generators of $B_{g,2n}$:

$$X_i^a = \tilde{X}_i^a \tilde{X}_2^a \sigma_1^2, \quad Y_1^a = \tilde{Y}_1^a \tilde{Y}_2^a \sigma_1^2.$$ 

Let us assume that $g > 1$. In [4], the authors showed that $\text{PB}_{g,n}^f$ can be exhibited as a non-splitting central extension

$$1 \rightarrow \mathbb{Z}^n \rightarrow \text{PB}_{g,n}^f \xrightarrow{\beta_n} \text{PB}_{g,n} \rightarrow 1,$$

where $\beta_n$ is the morphism induced by the projection map $\text{Conf}^f(\Sigma_g, n) \rightarrow \text{Conf}(\Sigma_g, n)$ (i.e. $\beta_n$ consists in forgetting the framing). $\text{Conf}^f(\Sigma_g, n)$ is an Eilenberg–Maclane space of type $(\text{PB}_{g,n}^f, 1)$. This short exact sequence extends to the following non-split short exact sequence

$$1 \rightarrow \mathbb{Z}^n \rightarrow B_{g,n}^f \xrightarrow{\beta_n} B_{g,n} \rightarrow 1,$$

where again $\beta_n$ consists in forgetting the framing. $\text{Conf}^f(\Sigma_g, [n])$ is an Eilenberg–Maclane space of type $(B_{g,n}^f, 1)$.

**Definition 3.1.** Let $\text{CoB}_g^f$ the $\text{CoB}_g^f$-module in groupoids with $\mathcal{S}$-module of objects $\mathcal{S}$ and where, for $n \geq 1$, the morphisms of $\text{CoB}_g^f(n)$ consists of isotopy classes of genus $g$ framed braids (i.e. elements of the braid group $B_g^f(n)$) $\alpha$ together with a colouring bijection $i \mapsto \alpha_i$ between the index set $i \in \{1, \ldots, n\}$ which leaves the last strand uncoloured and the strands $\alpha_i \in \{\alpha_1, \ldots, \alpha_n\}$ of our braid $\alpha$ and the data of a special braid corresponding to the framing.

3.1.3. The $\text{PaB}_g^f$-module of parenthesized framed genus $g$ braids. Let us choose an embedding $S^1 \rightarrow \Sigma_g$. To any finite set $I$ we associate the ASFM compactification $\overline{\text{Conf}}(S^1, I)$ of the configuration space $\text{Conf}(S^1, I)$ of $S^1$. To any finite set $I$, we associate the framed ASFM compactification $\overline{\text{Conf}}^f(\Sigma_g, I)$ of $\text{Conf}^f(\Sigma_g, I)$. The inclusion of boundary components provide $\overline{\text{Conf}}(S^1, -)$ with the structure of a module over the operad $\overline{\text{C}}(\mathbb{R}, -)$ in $\text{Top}$.

We have inclusions of topological modules

$$\text{Pa} \subset \overline{\text{Conf}}(S^1, -) \subset \overline{\text{Conf}}^f(\Sigma_g, -),$$
over the topological operads
\[ \mathcal{P}_a \subset \mathcal{C}(\mathbb{R}, -) \subset \mathcal{C}^f(\mathbb{C}, -). \]

We then define
\[ \mathcal{P}_a \mathcal{B}^f := \pi_1(\conf^f(\Sigma_g, -), \mathcal{P}_a), \]
which is a \( \mathcal{P}_a \mathcal{B}^f \)-module in groupoids.

As all our modules are considered to be pointed, there is a map of \( \mathcal{S} \)-modules \( \mathcal{P}_a \mathcal{B}^f \to \mathcal{P}_a \mathcal{B}_g^f \) and we abusively denote \( R_1, 2, \tilde{R}_1, 2, \Phi_1, 2, 3 \) and \( F_1, 2 \) the images in \( \mathcal{P}_a \mathcal{B}_g^f \) of the corresponding arrows in \( \mathcal{P}_a \mathcal{B}^f \). Notice that in this case, \( \mathcal{P}_a \mathcal{B}_g^f(1) \) is not the trivial groupoid so the choice of the pointing here is not functorial (apart from the reduced elliptic case studied in [11]).

**Example 3.2** (Structure of \( \mathcal{P}_a \mathcal{B}_g^f(1) \)). As opposed to the unframed reduced genus 1 case studied in [11], we have non trivial arrows in arity 1. More precisely, we have \( 2g \) automorphisms, \( A_1^a \) and \( B_1^a \in \text{Aut}_{\mathcal{P}_a \mathcal{B}_g^f(1)} \), for all \( 1 \leq a \leq g \), corresponding to the inverse generating loops in \( \conf^f(\Sigma_g, 1) \). Here is a picture for \( A_1^1 \) and \( B_1^1 \) for \( g = 2 \):

![Diagram](image)

All other \( A_1^a \) and \( B_1^a \) are depicted in the same way. We will formally depict these arrows as *pic diagrams* pointing to the left, expressing the fact that the paths considered go in the opposite direction of the generating paths considered in \( \pi_1(\Sigma_g) \):

\[
\begin{array}{c}
\text{A}_a \\
1 \\
\text{B}_a \\
1
\end{array}
\]

**Remark 3.3.** One has to be careful with the above notation. Indeed, taking into account the framing data for, say, \( A_1^1 \), the above geometrical picture corresponds to the following
In other words, seeing $\text{Aut}_{\text{PaB}^f_g}(1) = \mathbb{PB}_{g,1}$ as a subgroup of $\mathbb{B}_{g,2}$, the element $A_1$ is identified with the composite $X_1^1 X_2^1 \sigma_1^2$.

**Example 3.4** (Structure of $\text{PaB}^f_g(2)$). We have $4g$ automorphisms, $A_1^{1,2}, \tilde{A}_1^{1,2}, B_1^{1,2}$ and $\tilde{B}_1^{1,2} \in \text{End}_{\text{PaB}^f_g(2)}(12)$, for all $1 \leq a \leq g$ corresponding, for the case of $A_1^{1,2}$ and $\tilde{B}_1^{1,2}$ to the following paths in $\text{Conf}^f(\Sigma_g, 2)$ (we neglect for simplicity the framing data):

We will represent the paths $A_1^{1,2}$ and $B_1^{1,2}$ as follows:

(15)

All other $A_a^{1,2}$ and $B_a^{1,2}$ are depicted along the same representation as that for $B_1^{1,2}$.

Moreover, $\tilde{A}_1^{1,2}$ and $\tilde{B}_1^{1,2}$ can also be depicted as follows

**Remark 3.5.** Doubling the braid $A_a \in \text{PaB}^f_g(1)$ amounts to taking $\circ_1(A_a, \text{id}_{12}) \in \text{PaB}^f_g(2)$, we get an arrow $A_a^{12}$ depicted as follows:

As both braids represent actually ribbon braids, then it is then a fact that
\[ A_a^{12}(A_a^{1,2})^{-1}(R^{1,2}R^{2,1})^{-1} = \begin{array}{c}
1 \\
\bullet \\
1 \\
\circ \\
2 \\
A_a \\
\end{array} \]

This means that, contrary to the reduced genus 1 case, \( A_a^{1,2} \) and \( \tilde{A}_a^{1,2} \) are not equal. Nevertheless, one can retrieve the latter arrow from the first one:

\[ \tilde{A}_a^{1,2} = (A_a^{12})^{-1}A_a^{1,2}R^{1,2}R^{2,1}. \]

The following theorem can be understood as a rephrasing of the MacLane-Joyal-Street coherence theorem for framed genus \( g \) \( D_2 \)-modules.

**Theorem 3.6.** As a \( \text{PaB}_g^f \)-module in groupoids having \( \text{Pa} \) as \( \text{Pa} \)-module of objects, \( \text{PaB}_g^f \) is freely generated by \( A_a^{1,2} \) and \( B_a^{1,2} \), for \( 1 \leq a \leq g \), in \( \text{Aut}_{\text{PaB}_g^f(2)}(12) \), together with the following relations, for all \( 1 \leq a \leq g \) and \( Z \) any of the letters \( A, B \):

\[
\begin{align*}
(R_g) & \quad Z_a^{(g,1)} = 1d^1, \quad \left( \text{in } \text{End}_{\text{PaB}_g^f(1)(1)} \right), \\
(D_g) & \quad Z_a^{(12g,1)} = \Phi^{1,2,3}Z_a^{1,23}R^{1,23}F^{2,3,1}Z_a^{2,31}R^{2,31}\Phi^{3,1,2}Z_a^{3,12}R^{3,12}, \\
(N_g) & \quad \text{lId}_{a}^{12,3} = (\Phi^{1,2,3}Z_a^{1,23}(\Phi^{1,2,3})^{-1}, R^{1,2}\Phi^{2,1,3}Z_a^{2,13}(\Phi^{2,1,3})^{-1}R^{2,1}), \quad \text{for } (1 \leq a < b \leq g), \\
(E_1_g) & \quad R^{1,2}R^{2,1} = (R^{1,2}\Phi^{2,1,3}A_a^{1,3}(\Phi^{2,1,3})^{-1}R^{2,1}, R^{1,2}\Phi^{2,1,3}B_a^{1,3}(\Phi^{2,1,3})^{-1}), \\
\end{align*}
\]

as relations holding in the automorphism group of \((12)3\) in \( \text{PaB}_g^f(3) \), and

\[
\begin{align*}
(E_2_g) & \quad R^{1,2}R^{2,1}(F^{1,2})^{2(g-1)} = \prod_{a=1}^{g}((A_a^{1,2})^{-1}, B_a^{1,2}), \\
\end{align*}
\]

as a relation holding in the automorphism group of \((12)3\) in \( \text{PaB}_g^f(2) \).

**Remark 3.7.** Some direct consequences of the above theorem:

- By removing the third strand in \((D_g)\) and using \( Z_a^{(g,1)} = 1d^1 \), one deduces that \( A_a^{1,1} = A_a^1 \) and \( B_a^{1,1} = B_a^1 \), where \( A_a^1, B_a^1 \) are the elements introduced in Example 3.2;
- \( \text{PaB}_g^f \) identifies with the fake pull-back \( \omega^\ast \text{CoB}_g^f \) of the \( \text{CoB}_g^f \)-module

\[
\text{CoB}_g^f := \mathcal{G}(B_g^f \rightarrow \mathcal{G}),
\]

along the forgetful functor \( \omega : \text{Pa} \rightarrow \mathcal{G} \).

- Relation \((N_g)\) can be understood as a naturality condition between couples of elements \((A_a, B_a)\) suggesting that \( \text{PaB}_g^f \) is a \((\text{PaB}_g^f \otimes \ldots \otimes \text{PaB}_g^f)\)-module in the category of right \( \text{PaB}_g^f \)-modules.

Before proving this theorem let us state a fact that will be useful later on.

**Lemma 3.8.** Relation \((D_g)\) is equivalent to

\[
\begin{align*}
Z_a^{12,3} = \Phi^{1,2,3}Z_a^{1,23}(\Phi^{1,2,3})^{-1}R^{1,2}\Phi^{2,1,3}Z_a^{2,13}(\Phi^{2,1,3})^{-1}R^{2,1}. \\
\end{align*}
\]
Proof of Lemma 3.8. On the one hand, as $Z_{a}^{1,23} = \text{Id}_1$, erasing the third strand both in relation (48) and in (17), implies
\[ Z_{a}^{1,2} R_{a}^{1,2} Z_{a}^{2,1} R_{a}^{2,1} = Z_{a}^{12}. \]

Then, by doubling the first braid, this is equivalent to
\[ (Z_{a}^{1,23})^{-1} = R_{a}^{3,12} Z_{a}^{12,3} R_{a}^{12,3}. \]

By using the above equation and the hexagon $R_{a}^{1,23} \Phi_{a}^{2,3,1} = (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} R_{a}^{1,3}$, then equation (48) reads
\[
Z_{a}^{(12)3} = \Phi_{a}^{1,2,3} Z_{a}^{1,23} R_{a}^{1,23} \Phi_{a}^{2,3,1} Z_{a}^{2,31} R_{a}^{2,31} \Phi_{a}^{3,1,2} Z_{a}^{3,12} R_{a}^{3,12} - \Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} R_{a}^{1,3} Z_{a}^{2,31} R_{a}^{2,31} \Phi_{a}^{3,1,2} Z_{a}^{3,12} R_{a}^{3,12} = \Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} R_{a}^{1,3} Z_{a}^{2,31} R_{a}^{2,31} \Phi_{a}^{3,1,2} (R_{a}^{12,3})^{-1} (Z_{a}^{12,3})^{-1} (Z_{a}^{12,3}).
\]

Next, as we have $R_{a}^{1,3} Z_{a}^{2,31} = Z_{a}^{2,13} R_{a}^{1,3}$ and $R_{a}^{1,3} R_{a}^{2,31} = R_{a}^{2,13} R_{a}^{1,3}$, equation (48) is equivalent to
\[
Z_{a}^{12,3} = \Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} R_{a}^{1,3} Z_{a}^{2,31} R_{a}^{2,31} \Phi_{a}^{3,1,2} (R_{a}^{12,3})^{-1}.
\]

Now, by applying the permutation (123) → (312), the second hexagon relation yields
\[
(R_{a}^{12,3})^{-1} = (\Phi_{a}^{3,1,2})^{-1} (R_{a}^{1,3})^{-1} \Phi_{a}^{1,3,2} (R_{a}^{2,3})^{-1} (\Phi_{a}^{1,2,3})^{-1}.
\]

Thus, equation (48) is equivalent to
\[
Z_{a}^{12,3} = \Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} Z_{a}^{2,13} R_{a}^{2,13} \Phi_{a}^{3,1,2} (R_{a}^{2,3})^{-1} (\Phi_{a}^{1,2,3})^{-1}.
\]

By applying the permutation (12) → (21), the first hexagon relation yields
\[
R_{a}^{2,13} = (\Phi_{a}^{2,1,3})^{-1} R_{a}^{2,1} \Phi_{a}^{1,2,3} R_{a}^{2,3} (\Phi_{a}^{1,3,2})^{-1}.
\]

Thus, equation (48) is equivalent to
\[
Z_{a}^{12,3} = \Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} Z_{a}^{2,13} (\Phi_{a}^{2,1,3})^{-1} R_{a}^{2,1}.
\]

□

Proof of Theorem 3.6. Let $Q$ be the $\text{PaB}^{f}$-module with the above presentation. We first show that there is a morphism of $\text{PaB}^{f}$-modules $Q \to \text{PaB}^{f}_{a}$. We have already seen that there are 2g automorphisms $A_{a}^{1,2}, B_{a}^{1,2}$ of (12) in $\text{PaB}^{f}_{a}(2)$ (see Example 3.4). We have to prove that they indeed satisfy the relations (R$_{a}$), (D$_{a}$), (N$_{a}$), (E1$_{a}$), and (E2$_{a}$).

Relation (R$_{a}$) is satisfied: This is straightforwardly satisfied as it corresponds topologically to removing the first brand to the paths $A_{a}^{1,2}, B_{a}^{1,2}$ that move the first strand, leaving the second strand untouched.

Relation (D$_{a}$) is satisfied: The decagon relation (D$_{a}$) can be depicted as follows (for simplicity we abusively neglect picturing the framing data):

\[
\Phi_{a}^{1,2,3} Z_{a}^{1,23} (\Phi_{a}^{1,2,3})^{-1} R_{a}^{1,2} \Phi_{a}^{2,1,3} Z_{a}^{2,13} (\Phi_{a}^{2,1,3})^{-1} R_{a}^{2,1}.
\]
It is satisfied in $\text{PaB}^f$, expressing the fact that when all (here, three) points with their associated framing data move along a generating loop on $\Sigma_g$ (in the opposite direction), this corresponds to the path in the framed configuration space of points on $\Sigma_g$ moving and twisting simultaneously the three points. Thus, the number of twists in the l.h.s. and r.h.s. of $(D_g)$ are equal and cancel out.

Relation $(N_g)$ is satisfied: This is straightforwardly satisfied as the braids corresponding to the l.h.s. and r.h.s. of the comutator are independent.

Relation $(E_{1g})$ is satisfied: One can interpret the path in the r.h.s. of $(E_{1g})$ as follows. Consider the path

$$K = (R^{1,2}_g \Phi^{2,1,3}_a A^{2,13}_a (\Phi^{2,1,3}_a)^{-1} R^{2,1}_g, \Phi^{1,2,3}_a B^{1,23}_a (\Phi^{1,2,3}_a)^{-1}) .$$

This can be depicted as follows:

By noticing that the the braid numbered by 1 passes above the one numbered by 2 first, one can see that $K$ is homotopic to the braid $R^{1,2} R^{2,1}$.

Relation $(E_{2g})$ is satisfied: Relation $(E_{2g})$ is more difficult to draw so we sketch the way to think of the right-hand-side. Align the points in a generating cycle of the genus $g$ surface (this means that they are in the boundary of the compactified framed configuration space). Then
If a point travels through a cycle, its corresponding framing will naturally start to spin (in clockwise direction) as one can see in the following picture, for $g = 2$ and for $g = 4$

If we consider a polygon with $4g$ sides corresponding to a genus $g$ surface, then for each marked point travelling through the generating cycles, the framing attached to that point will be twisted by an angle of $\pi - \frac{\pi}{g}$.

If we suppose that the marked points were chosen to be in the $A_1$-cycle of $\Sigma_g$, the right hand side of $(E_{2g})$ can be drawn as follows, for $g = 2$:

In conclusion, one can then easily see that if, for a 2-point configuration, the first point travels around all the generating cycles concerned in the right-hand-side of relation $(E_{2g})$, its corresponding framing data will make $2g \times \frac{(g-1)}{g} = 2(g-1)$ complete spins and the first point
will have done a complete loop around the second point. This is exactly the left-hand-side of
equation (E2).}

Thus, by the universal property of \( Q \), there is a morphism of \( \mathbb{P}aB^f \)-modules \( Q \to \mathbb{P}aB^f_g \),
which is the identity on objects. To show that this map is in fact an isomorphism, it suffices
to show that it is an isomorphism at the level of automorphism groups of objects arity-wise, as
all groupoids are connected. Let \( n \geq 0 \), and \( p \) be the object \( 1\overbrace{\cdots ((n-2)((n-1))}^{\cdots} \) of
\( Q(n) \) and \( \mathbb{P}aB^f_g(n) \). We want to show that the induced morphism

\[
\text{Aut}_{\mathbb{P}aB^f_g(n)}(p) \longrightarrow \text{Aut}_{\mathbb{P}aB^f_g(n)}(p) \cong \pi_1 \left( \text{Conf}^f(\Sigma_g, n), p \right)
\]

is an isomorphism.

On the one hand, as \( \text{Conf}^f(\Sigma_g, n) \) is a manifold with corners, we are allowed to move the
basepoint \( p \) to the point \( p_{\text{reg}} \) in which is based the fundamental group in subsection 3.1.2. We
then have an isomorphism of fundamental groups \( \pi_1(\text{Conf}^f(\Sigma_g, n), p) \cong \pi_1(\text{Conf}^f(\Sigma_g, n), p_{\text{reg}}) \).

On the other hand, one can construct a non-symetric module \( \tilde{Q} \) in groupoids over \( B^f \)
carrying an action of the (algebraic version of the) framed braid group \( B^f_{g,n} \) on \( \Sigma_g \) in the
following sense:

- for each \( n \geq 1 \), \( \tilde{Q}(n) \) is a groupoid with maximal parenthesizations of unnumbered
elements as objects. We will make abuse of notation on still numbering these elements
in order to count them.
- \( \tilde{Q} \) is freely generated by \( A_{g}^{1} := A_{g}^{1,2} \) and \( B_{g}^{1,2} := B_{g}^{1,2} \) in \( \tilde{Q}(2) \), for all \( 1 \leq i \leq g \), satisfying
relations \( (R_g) \), \( (D_g) \), \( (N_g) \), \( (E1_g) \) and \( (E2_g) \).

In the following lemma we show that there are group morphisms \( B_{g,n}^f \rightarrow \text{Aut}_{\tilde{Q}(n)}(p) \rightarrow \mathfrak{S}_n \),
the left one being unique.

**Lemma 3.9.** Let \( \tilde{Q} \) be the operadic \( B^f \)-module with unnumbered maximal parenthesizations
as objects and with generators \( A_i^{1,2} := A_i^{1} \) and \( B_i^{1,2} := B_i^{2} \), for all \( 1 \leq i \leq g \), in \( \tilde{Q}(2) \) satisfying
relations \( (E1_g) \) and \( (E2_g) \). Let \( p \) be the object in \( \tilde{Q}(n) \) given by the \( n \)-length rightmost maximal
parenthesization

\[
p := (\bullet(\bullet(\cdots ((\bullet)) \cdots)).
\]

Then there is a unique group isomorphism

\[
\phi_n : B_{g,n}^f \rightarrow \text{Aut}_{\tilde{Q}(n)}(p),
\]

such that, for \( \Phi : B_{g,n}^f \rightarrow \text{Aut}_{\tilde{Q}(n)}(p) \),

- \( X_{a}^{1} \mapsto A_{a}^{1,2} \), for all \( 1 \leq a \leq g \);
- \( Y_{1}^{1} \mapsto B_{1}^{1,2-n} \), for all \( 1 \leq a \leq g \);
- \( \tau_{i} \mapsto (\Phi)_{i}^{-1} F_{i,i+1} \Phi_{i} ; \) for all \( 1 \leq i \leq n - 1 \);
- \( f_{i} \mapsto (\Phi)_{i}^{-1} F_{i,i+1} \Phi_{i} ; \) for all \( 1 \leq i \leq n - 1 \);
- \( f_{n} \mapsto F_{n-1,n} \),
where $A^{1,2,\ldots,n} \in \text{Aut}_{\tilde{Q}(n)}(\tilde{p})$ is obtained from $A^{1,2}$, $F^{i,i+1}$ and $F^n$ are obtained from $F^{1,2}$ and $R^{i,i+1} \in \text{Aut}_{\tilde{Q}(n)}(\tilde{p})$ is obtained from $R^{1,2}$ by some finite sequences of arrows involving the associator and the operadic module morphisms since the parenthesizations are unmarked.

In particular, by applying a finite sequence of associators one can show that the above lemma remains true for all possible choices of base points $p \in \tilde{Q}(n)$.

**Proof.** For simplicity, we omit the associativity constraints. One can show by induction that the image of $X^{a}_{1} \overset{\tau_{i-1}}{=} \tau_{i-1} X^{a}_{i-1} \tau_{i-1}$ is obtained from $A_{1,2,\ldots,n}$ and $F_{i,i+1}$ are obtained from $F_{1,2}$ and $R_{i,i+1} \in \text{Aut}_{\tilde{Q}(n)}(\tilde{p})$ is obtained from $R_{1,2}$ by some finite sequences of arrows involving the associator and the operadic module morphisms since the parenthesizations are unmarked.

In particular, by applying a finite sequence of associators one can show that the above lemma remains true for all possible choices of base points $p \in \tilde{Q}(n)$.

Now, in Lemma 3.8, we prove that relation $(D_g)$ for $Z = A$ is equivalent to

$$A^{1,2,3}_a = \Phi^{1,2,3} A^{1,23}_a (\Phi^{1,2,3})^{-1} R^{1,2} \Phi^{2,1,3}_a A^{2,13}_a (\Phi^{2,1,3})^{-1} R^{2,1}.$$  

Again, by naturality since $\tilde{Q}$ is a $B^f$-module, we have $A^{1,2,3}_a R^{1,2} = R^{1,2} A^{21,3}_a$ so that replacing $A^{1,2,3}_a$ and $A^{21,3}_a$ by the r.h.s. of equation (19), this implies that

$$(A^{1,2,3}_a, R^{1,2} A^{21,3}_a R^{2,1}) = 1.$$  

Removing the third strand in this equation implies that relation $(X^a_1, X^a_2) = 1$ is preserved by $\phi_n$. The same reasoning applies to $(D_g)$ for $Z = B$ and implies that $(Y^a_1, Y^a_2) = 1$ is also preserved by $\phi_n$. In the same way, we obtain relation (FBG4) directly from $(N_g)$ since our
operadic modules are pointed so we can remove the third strand. Next, relation (FBG3) is satisfied as we can retrieve the third strand in (E1) to obtain the desired relation. Finally, relation (FBG5) is obtained directly from (E2). Thus, we have a group morphism. Let us show that it is bijective.

\( \phi_n \) is surjective: The fact that the map \( \phi_n \) is surjective is a consequence of the fact that all the defining relations in \( \tilde{Q}(n) \) come from the defining relations of \( B_{g,n}^f \) and the operadic module partial compositions.

\( \phi_n \) is injective: Let us now show the injectivity of this map. Let \( \bar{Q} \) be the operad module with same objects as \( \tilde{Q} \) and, for every object \( p \) of \( \bar{Q}(n) \), we define \( \text{Aut}_{\bar{Q}(n)}(p) := B_{g,n}^f \). Next we have a map \( \tilde{Q} \to \bar{Q} \) sending the generators \( A_1, A_2 \) to \( X_a^1 \) and \( B_1, B_2 \) to \( Y_a^1 \) in \( B_{g,2}^f \). Indeed, the fact that both modules are pointed and since the following relations hold in \( B_{g,2}^f \) and \( B_{g,3}^f \):

\[
\begin{align*}
(X_1^a \tau_1 \tau_2)^3 &= X_1^a_{123}, \\
(Y_1^a \tau_1 \tau_2)^3 &= Y_1^a_{123}, \\
(\tau_1 X_1^a \tau_1, Y_1^a) &= \tau_1^2, \\
(X_1^a, X_2^b) &= (X_1^a, Y_2^b) = (Y_1^a, X_2^b) = (Y_1^a, Y_2^b) = 1, \\
\prod_{a=1}^g ((X_1^a)^{-1}, Y_1^a) &= \tau_1^2 f_1^2(g-1),
\end{align*}
\]

we show that relations \((R_g), (D_g), (N_g), (E1_g)\) and \((E2_g)\) are preserved.

Then, as \( \text{PaB}^f \) acts on both of these operadic modules we conclude that there is a map \( \text{Aut}_{\bar{Q}(n)}(p) \to \text{Aut}_{\tilde{Q}(n)}(p) \). In order to prove the injectivity of \( \phi \), we are left to prove that the composite

\[
B_{g,n}^f \to \text{Aut}_{\bar{Q}(n)}(p) \to \text{Aut}_{\tilde{Q}(n)}(p)
\]

is the identity morphism, which is true by construction of both maps. \(\square\)

*End of the proof of Theorem 3.6.*

In the same way the collection \( \{\text{PB}_{g,n}^f\}_{n \geq 1} \) of pure genus \( g \) braids owns a non-symmetric module over the non-symmetric operad \( \text{PB}^f \), constructed in Section 2.1.3, denoted \( \text{PB}_{g}^f \).

Moreover, one the forgetful map \( \text{Op} \to \text{NsOp} \) between the category of operads in \( \mathcal{C} \) and the category of non-symmetric operads in \( \mathcal{C} \) induces a map \( Q \to \tilde{Q} \). Then, one has by construction of \( \tilde{Q} \) that \( \text{Aut}_{\bar{Q}(n)}(p) \) is the kernel of the map \( \text{Aut}_{\tilde{Q}(n)}([p]) \to \mathfrak{S}_n \). One can
actually show that we have a commuting diagram

\[
\begin{array}{ccccccc}
\text{PB}_{g,n}^f & \xrightarrow{\sim} & \text{Aut}_{\mathcal{Q}(n)}(p) & \longrightarrow & \pi_1 \left( \text{Conf}^f(\Sigma_g,n),p \right) & \longrightarrow & \pi_1 \left( \text{Conf}^f(\Sigma_g,n),p_{\text{reg}} \right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{B}_{g,n}^f & \xrightarrow{\sim} & \text{Aut}_{\mathcal{Q}(n)}([p]) & \longrightarrow & \pi_1 \left( \text{Conf}^f(\Sigma_g,n)/\mathcal{S}_n,[p] \right) & \longrightarrow & \pi_1 \left( \text{Conf}^f(\Sigma_g,n)/\mathcal{S}_n,[p_{\text{reg}}] \right) \\
\mathcal{S}_n & \longrightarrow & \mathcal{S}_n & \longrightarrow & \mathcal{S}_n & \longrightarrow & \mathcal{S}_n
\end{array}
\]

where all vertical sequences are short exact sequences. Thus, in order to show that the map \( \text{Aut}_{\mathcal{Q}(n)}(p) \rightarrow \pi_1 \left( \text{Conf}^f(\Sigma_g,n),p \right) \) is an isomorphism, we are left to show that

\[
\phi : \text{B}_{g,n}^f \longrightarrow \pi_1 \left( \text{Conf}^f(\Sigma_g,n)/\mathcal{S}_n,[p_{\text{reg}}] \right)
\]

is indeed an isomorphism. This is the case since the map \( \phi \) is exactly the isomorphism constructed in [4, Theorem 13]. This completes the proof of Theorem 3.6.

3.2. An alternative presentation of \( \text{PaB}_g^f \). In this subsection we exhibit an alternative presentation for the module \( \text{PaB}_g^f \). This will be then used in the particular case \( g = 1 \) to show that the set of genus 1 non-reduced associates over \( \mathbb{C} \) is not empty.

Let \( \text{PaB}_{g,f}^{\text{bis}} \) be the \( \text{PaB}^f \)-module having \( \text{Pa} \) as \( \text{Pa} \)-module of objects and freely generated by morphisms \( \tilde{A}_a^{1,2} \) and \( \tilde{B}_a^{1,2} \) in arity 2 with relations

\[
\begin{align*}
(20) \quad & \tilde{Z}_a^{1,0} = \text{Id}_1, \\
(21) \quad & \tilde{Z}_a^{12,3} \tilde{Z}_a^{12,3} = \Phi^{1,2,3} \tilde{Z}_a^{12,3} (\Phi^{1,2,3})^{-1} (R_a^{2,1})^{-1} \Phi^{2,1,3} \tilde{Z}_a^{2,13} (\Phi^{2,1,3})^{-1} (R_a^{1,2})^{-1}, \\
(22) \quad & \text{Id}^{1,2,3} = \left( \tilde{Z}_a^{12,3}, (\tilde{Z}_b^{12,3})^{-1} \Phi^{1,2,3} \tilde{Z}_b^{12,3} R_a^{2,3} R_a^{3,2} (\Phi^{1,2,3})^{-1} \right), \\
(23) \quad & \Phi^{1,2,3} R_a^{1,2,3} (\Phi^{1,2,3})^{-1} = \left( (\tilde{A}_a^{1,2,3})^{-1}, \tilde{B}_a^{1,2,3} \Phi^{1,2,3} (\tilde{B}_a^{1,2,3})^{-1} (\Phi^{1,2,3})^{-1} \right), \\
(24) \quad & R_a^{1,2} R_a^{2,1} (\Phi^{1,2})^{2g-1} = \prod_{a=1}^{g} (\tilde{A}_a^{1,2} (\tilde{B}_a^{1,2})^{-1}).
\end{align*}
\]

**Proposition 3.10.** As \( \text{PaB} \)-modules in groupoids having \( \text{Pa} \) as \( \text{Pa} \)-module of objects, \( \text{PaB}_g^f \) and \( \text{PaB}_{g,f}^{\text{bis}} \) are isomorphic.

**Proof.** In \( \text{PaB}_g^f \) there are morphisms \( \tilde{A}_a^{1,2} = R_a^{1,2} (\tilde{A}_a^{2,1})^{-1} (R_a^{2,1})^{-1}, \tilde{B}_a^{1,2} = R_a^{1,2} (\tilde{B}_a^{2,1})^{-1} (R_a^{2,1})^{-1} \). These correspond topologically to moving the point indexed by 2 in the direction of the generating cycles of \( \Sigma_g \). The fact that the assignment \( \text{Pa} \rightarrow \text{Pa} \), \( (\tilde{A}_a^{1,2}, \tilde{B}_a^{1,2}) \rightarrow (\tilde{A}_a^{1,2}, \tilde{B}_a^{1,2}) \) defines a morphism of \( \text{PaB} \)-modules \( \text{PaB}_{g}^{\text{bis}} \rightarrow \text{PaB}_1 \) is justified topologically by the fact that, in \( \text{PaB}_g^f \), the paths representing the l.h.s. and r.h.s. of relations (20), (21), (22), (23) and (24) are homotopic.

In order to prove that this morphism is an isomorphism, let us show that \( (R_g), (D_g), (N_g), (E_{1g}), (E_{2g}) \) are equivalent to (20), (21), (22), (23) and (24).
First of all, as we have \( \hat{Z}^{1,2}_a = R^{1,2}(Z^{1,2}_a)^{-1}(R^{1,2})^{-1} \), then one can readily see that (R_g) is equivalent to (20).

Second of all, let’s prove that (D_g) is equivalent to (21).

On the one hand, in Lemma 3.8 we showed that relation (D_g) is equivalent to

\[
Z^{1,2}_a = R^{1,2}(Z^{1,2}_a)^{-1}(R^{1,2})^{-1} \quad \text{(25)}
\]

On the other hand, doubling one of the strands, applying a permutation in the equality \( \hat{Z}^{1,2}_a = R^{1,2}(Z^{1,2}_a)^{-1}(R^{1,2})^{-1} \) and plugging the result into relation (21), one obtains

\[
(Z^{a}_{2})^{-1} R^{1,23}(Z^{a}_{23})^{-1}(R^{1,23})^{-1} = \Phi^{1,2,3} R^{1,23}(Z^{23}_a)^{-1}(R^{1,23})^{-1}(\Phi^{1,2,3})^{-1}
\]

The inverse of this equation reads

\[
R^{1,23}Z^{a}_{23}(R^{1,23})^{-1} Z^{a}_{23} = R^{1,2} \Phi^{2,1,3} R^{2,13}(Z^{23}_a)^{-1}(R^{2,13})^{-1}(\Phi^{1,2,3})^{-1},
\]

which is equivalent to

\[
\Phi^{1,2,3} R^{2,13} R^{23}_a Z^{a}_{23} (R^{1,23})^{-1}(\Phi^{1,2,3})^{-1},
\]

Now, using

\[
\begin{align*}
R^{1,23}(Z^{a}_{2})^{-1} R^{1,2} &= R^{1,2} R^{213}(Z^{a}_{23})^{-1}(R^{213})^{-1}, \\
(Z^{a}_{2})^{-1} &= R^{213}(Z^{a}_{23})^{-1} Z^{a}_{23} R^{213}, \\
Z^{a}_{23} &= (R^{1,23})^{-1}(Z^{a}_{23})^{-1}, \\
Z^{a}_{23} &= (R^{1,23})^{-1}(Z^{a}_{23})^{-1} Z^{a}_{23} (R^{1,23})^{-1}, \\
Z^{a}_{23} &= (R^{1,23})^{-1}(Z^{a}_{23})^{-1} Z^{a}_{23} (R^{1,23})^{-1},
\end{align*}
\]

we deduce

\[
Z^{a}_{23} = R^{1,2} R^{213} R^{1,23}(Z^{a}_{23})^{-1} Z^{a}_{23} (R^{1,23})^{-1}(\Phi^{1,2,3})^{-1}(R^{2,13})^{-1}(\Phi^{1,2,3})^{-1}.
\]

Now, the elements \( Z^{a}_{23} \) and \( Z^{a}_{23} \) can, after suitable permutations and conjugation with associators, be moved to the rightmost part of the r.h.s. of the above equation and cancel out. Using \( R^{1,2} R^{213} R^{231}(Z^{a}_{23})^{-1} = Z^{a}_{23} R^{1,2} R^{213} R^{231}, \) we obtain the following equation

\[
Z^{a}_{23} = \Phi^{1,2,3} R^{1,23} R^{213}(\Phi^{1,2,3})^{-1} Z^{a}_{23} (\Phi^{1,2,3})^{-1} (R^{1,23})^{-1}(\Phi^{1,2,3})^{-1}.
\]

Now by using \( R^{2,13} R^{1,23} Z^{a}_{23} = Z^{a}_{23} R^{213} R^{1,23} \) and \( \Phi^{1,2,3} (R^{213})^{-1}(\Phi^{1,2,3})^{-1} Z^{a}_{23} = Z^{a}_{23} (\Phi^{1,2,3})^{-1} (R^{213})^{-1}(\Phi^{1,2,3})^{-1} \), we now left to compute the expression

\[
R^{1,23} R^{1,23}(\Phi^{1,2,3})^{-1}(R^{1,23})^{-1}(\Phi^{1,2,3})^{-1}.
\]

By using suitable hexagon relations we obtain

\[
\begin{align*}
R^{1,23} R^{1,23} &= (\Phi^{1,2,3})^{-1} R^{1,23} (\Phi^{1,2,3})^{-1} R^{1,23}, \\
(R^{1,23})^{-1} &= (\Phi^{1,2,3})^{-1} R^{1,23} (\Phi^{1,2,3})^{-1} R^{1,23}.
\end{align*}
\]

\[
\Phi^{1,2,3} (R^{1,23})^{-1} (\Phi^{1,2,3})^{-1} R^{1,23} (\Phi^{1,2,3})^{-1} R^{1,23}.
\]
\[ \Phi^{1,2,3} R^{1,23} R^{23,1} (\Phi^{1,2,3})^{-1} = R^{1,2} \Phi^{2,1,3} R^{1,3} R^{3,1} (\Phi^{2,1,3})^{-1} R^{2,1}. \]

By using the above relations we deduce that (26) equals \((\Phi^{2,1,3})^{-1} R^{2,1} R^{1,2} R^{2,1}\), so that
\[ Z_{a}^{12,3} = \Phi^{1,2,3} Z_{a}^{123} (\Phi^{1,2,3})^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} Z_{a}^{213} (\Phi^{2,1,3})^{-1} R^{2,1} R^{1,2} R^{2,1}. \]

Now, in the automorphism group of (21),3, we have
\[ \Phi^{2,1,3} Z_{a}^{12,3} (\Phi^{2,1,3})^{-1} R^{2,1} R^{1,2} R^{2,1} = R^{2,1} R^{1,2} \Phi^{2,1,3} Z_{a}^{12,3} (\Phi^{2,1,3})^{-1} R^{2,1}, \]

In conclusion, we obtain
\[ Z_{a}^{12,3} = \Phi^{1,2,3} Z_{a}^{213} (\Phi^{1,2,3})^{-1} R^{1,2} \Phi^{2,1,3} Z_{a}^{213} (\Phi^{2,1,3})^{-1} R^{2,1}, \]

which is precisely equation (D_3).

Third of all, let us assume relations (20) and (21) and let us prove that (23) is equivalent to (E_1) and that (22) is equivalent to (N_2). Relation (21) for \( \tilde{Z} = \tilde{B} \) is equivalent to
\[ \Phi^{1,2,3} (\tilde{B}^{1,23})^{-1} (\Phi^{1,2,3})^{-1} \tilde{B}^{12,3} = (R^{2,1})^{-1} R^{2,1} \tilde{B}^{213} (\Phi^{2,1,3})^{-1} (R^{1,2})^{-1} (\tilde{B}^{213})^{-1}. \]

Thus, (23) is equivalent to
\[ \Phi R^{2,3} R^{1,2} \Phi^{-1} = (\tilde{A}_{a}^{12,3})^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} \tilde{B}_{a}^{213} (\Phi^{2,1,3})^{-1} (R^{1,2})^{-1} (\tilde{B}_{a}^{213})^{-1} \tilde{A}_{a}^{12,3} R^{1,2} \Phi^{2,1,3} (\tilde{B}_{a}^{213})^{-1} (\tilde{B}_{a}^{213})^{-1}. \]

Now, as \( \tilde{B}_{a}^{213} \) commutes with all elements in this equation, we simplify it and, as we have 
\[ (R^{1,2})^{-1} \tilde{A}_{a}^{12,3} = \tilde{A}_{a}^{213} (R^{1,2})^{-1}, \]
we deduce that (23) is equivalent to
\[ \Phi R^{2,3} R^{1,2} \Phi^{-1} = (\tilde{A}_{a}^{12,3})^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} \tilde{B}_{a}^{213} (\Phi^{2,1,3})^{-1} \tilde{A}_{a}^{213} \Phi^{2,1,3} (\tilde{B}_{a}^{213})^{-1} (\tilde{B}_{a}^{213})^{-1}. \]

which is equivalent to
\[ (\Phi^{2,1,3})^{-1} \tilde{A}_{a}^{213} R^{2,1} \Phi R^{2,3} R^{1,2} \Phi^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} = \tilde{B}_{a}^{213} (\Phi^{2,1,3})^{-1} \tilde{A}_{a}^{213} \Phi^{2,1,3} (\tilde{B}_{a}^{213})^{-1} (\tilde{B}_{a}^{213})^{-1}. \]

Now, by plugging \( \tilde{A}_{a}^{213} = R^{213} (A_{a}^{3,21})^{-1} (R^{213})^{-1} \) and \( \tilde{B}_{a}^{213} = R^{213} (B_{a}^{3,12})^{-1} (R^{213})^{-1} \) in the above equation and using
\[ \Phi^{2,1,3} = R^{213} (\Phi^{3,2,1})^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} (\Phi^{2,1,3})^{-1}, \]
\[ R^{2,1} \Phi^{2,3,1} R^{1,2} \Phi^{-1} = R^{2,1} \Phi^{2,1,3} (R^{2,1})^{-1} \Phi^{2,1,3} = R^{213} (\Phi^{3,2,1})^{-1} R^{1,2} \Phi^{2,1,3} (R^{1,3})^{-1}, \]
\[ B_{a}^{13,2} = (R^{213})^{-1} (B_{a}^{213})^{-1} (R^{13})^{-1} (R^{13})^{-1}, \]
we obtain
\[ R^{1,3} (\Phi^{2,3,1})^{-1} R^{2,3} \Phi^{3,2,1} (A_{a}^{3,21})^{-1} (\Phi^{3,2,1})^{-1} R^{3,2} \Phi^{2,3,1} (R^{1,3})^{-1} = R^{2,13} R^{13,2} B_{a}^{213} R^{1,3} (\Phi^{2,3,1})^{-1} \]
\[ R^{2,3} \Phi^{3,2,1} (A_{a}^{3,21})^{-1} (\Phi^{3,2,1})^{-1} R^{2,3} \Phi^{3,2,1} (R^{1,3})^{-1} B_{a}^{13,2} (R^{2,13})^{-1} (R^{2,13})^{-1}. \]

After suitable permutations of the indices and conjugations with associators, one can move the \( R^{213} R^{13,2} \) term in the r.h.s of the above equation to the rightmost part, where it cancels out.

We obtain, by performing the permutation (123) \( \Rightarrow (312) \) that
\[ R^{2,13} (\Phi^{1,2,3})^{-1} R^{13,2} \Phi^{2,1,3} (A_{a}^{3,21})^{-1} (\Phi^{2,1,3})^{-1} R^{2,13} \Phi^{2,1,3} (R^{3,2})^{-1} \]
\[ = B_{a}^{3,2} R^{3,2} (\Phi^{1,2,3})^{-1} R^{13,2} \Phi^{2,1,3} (A_{a}^{3,21})^{-1} (\Phi^{2,1,3})^{-1} (R^{1,2})^{-1} \Phi^{2,1,3} (R^{3,2})^{-1} (B_{a}^{3,2})^{-1}. \]

Now, using \( B_{a}^{3,2} R^{3,2} = R^{3,2} B_{a}^{3,2} \) this equation can be rewritten as
\[ R^{1,2} \Phi^{2,1,3} (A_{a}^{3,21})^{-1} (\Phi^{2,1,3})^{-1} R^{2,1}. \]
\[ \Phi^{1,2,3}B_0^{1,23}(\Phi^{1,2,3})^{-1}R^{1,2}\Phi^{2,1,3}(A_0^{2,13})^{-1}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\Phi^{1,2,3}(B_0^{1,23})^{-1}(\Phi^{1,2,3})^{-1}. \]

This is equivalent to
\[ R^{1,2}\Phi^{2,1,3}A_0^{2,13}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\Phi^{1,2,3}(B_0^{1,23})^{-1}(\Phi^{1,2,3})^{-1}R^{1,2}\Phi^{2,1,3} \]
\[ (A_0^{2,13})^{-1}(\Phi^{2,1,3})^{-1}R^{2,1}\Phi^{1,2,3}B_0^{1,23}(\Phi^{1,2,3})^{-1} = \text{Id}_{(12)3}. \]

Taking the inverse of this relation and multiplying it by \( R^{1,2}R^{2,1} \) in the rightmost part of each side we obtain
\[ R^{1,2}R^{2,1} = \Phi^{1,2,3}(B_0^{1,23})^{-1}\Phi^{1,2,3}(\Phi^{2,1,3})^{-1}(R^{1,2})^{-1}\Phi^{1,2,3}B_0^{1,23} \]
\[ (\Phi^{1,2,3})^{-1}R^{1,2}\Phi^{2,1,3}(A_0^{2,13})^{-1}(\Phi^{2,1,3})^{-1}R^{2,1}. \]

Finally, as \( \Phi^{1,2,3}(B_0^{1,23})^{-1}(\Phi^{1,2,3})^{-1} \) commutes with \( R^{1,2}R^{2,1} \) in the automorphism group of \((12)3\), we obtain
\[ R^{1,2}\Phi^{2,1,3}A_0^{2,13}(\Phi^{2,1,3})^{-1}R^{2,1}\Phi^{1,2,3}B_0^{1,23}(\Phi^{1,2,3})^{-1} \]
\[ (R^{1,2})^{-1}(\Phi^{2,1,3})^{-1}(A_0^{2,13})^{-1}\Phi^{2,1,3}(R^{2,1})^{-1}\Phi^{1,2,3}(B_0^{1,23})^{-1}(\Phi^{1,2,3})^{-1} = R^{1,2}R^{2,1}, \]
which is precisely equation \((E1_g)\). One can then eventually obtain the equivalence between equations \((22)\) and \((N_g)\) using the same procedure.

Last of all, after plugging \((A_0^{1,2})^{-1} = (R^{2,1})^{-1}A_0^{2,1}R^{2,1}\) and \(B_0^{1,2} = (R^{2,1})^{-1}(B_0^{2,1})^{-1}R^{2,1}\) into equation \((E2_g)\), we obtain
\[ R^{1,2}R^{2,1} = ((R^{2,1})^{-1}A_0^{2,1}R^{2,1}, (R^{2,1})^{-1}(B_0^{2,1})^{-1}R^{2,1}) \]
\[ = (R^{2,1})^{-1}(A_0^{2,1}, (B_0^{2,1})^{-1})R^{2,1} \]
\[ R^{2,1}R^{1,2} = (R^{1,2})^{-1}(A_0^{1,2}, (B_0^{1,2})^{-1})R^{1,2} \]
\[ R^{1,2}R^{2,1} = (A_0^{1,2}, (B_0^{1,2})^{-1}) \]
which is precisely equation \((24)\).

\[\square\]

**Remark 3.11.** One can also notice that for \( g = 1 \), removing the third strand in relation \((E1_g)\) implies relation \((24)\) and removing the first strand in relation \((23)\) implies relation \((E2_g)\).

### 3.3. Genus \( g \) Grothendieck–Teichmüller groups.

Let us finish this section by defining Grothendieck–Teichmüller groups in genus \( g \) operadically, and then making explicit descriptions of this groups.

**Definition 3.12.** The \((k\text{-}pronipotent\text{-}version\text{\ of\ }the\ }g\text{\ Genus\ }\text{Grothendieck–Teichmüller\ group\ }\) is defined as the group

\[
\overline{\mathcal{GT}}_g^f(k) := \text{Aut}_{\text{opRTGrpd}_k}(\overline{\mathcal{PB}}^f_k, \overline{\mathcal{PB}}^f_g(k))
\]

of couples \((F,G)\) where \( F \in \overline{\mathcal{GT}}_g^f(k) \) and \( G \) is an automorphism of the \( \overline{\mathcal{PB}}^f_k \text{-module} \) \( \overline{\mathcal{PB}}^f_g(k) \) which is the identity on objects and which is compatible with \( G \).

The presentation of \( \overline{\mathcal{PB}}^f_g \) then implies the following: each automorphism \( F \) of \( \overline{\mathcal{PB}}^f_g \) compatible with an automorphism \( G \) of \( \overline{\mathcal{PB}}^f_g \) is uniquely determined by \((\lambda,f) \in \overline{\mathcal{GT}}_g^f(k)\) such that
\[ F(R^{1,2}) = (R^{1,2} R^{2,1})^{\nu} R^{1,2}, \]
\[ F(\Phi^{1,2,3}) = f(x, y) \cdot \Phi^{1,2,3}, \]
\[ G(A^{1,2}_a) = g^{a,1}_a(X^a_1, X^a_2, Y^a_1, Y^a_2, f_1, f_2; 1 \leq a \leq g), \]
\[ G(B^{1,2}_a) = h^{a,1}_a(X^a_1, X^a_2, Y^a_1, Y^a_2, f_1, f_2; 1 \leq a \leq g), \]
where \( \nu = \frac{\lambda_1}{2} \) and \( g^{a,1}_a, h^{a,1}_a \) are elements of \( \overline{F}_{g,3}(k) \), for \( 1 \leq a \leq g \). These elements satisfy the following relations, induced by \((R_g), (D_g), (N_g), (E1_g)\) and \((E2_g)\):
\[
\begin{align*}
g^{a,1}_a(1) &= 1, \quad h^{a,1}_a(1) = 1, \\
(f(\tau^2_1, \tau^2_2) g^{a,1}_a(\tau_1 \tau^2_2 \tau_1)) &= g^{a(12)}_a \\
(f(\tau^2_1, \tau^2_2) h^{a,1}_a(\tau_1 \tau^2_2 \tau_1)) &= h^{a(12)}_a \\
u &= (u g^{a,2}_a, h^{a,1}_a) \\
1 &= (g^{a,1}_a, u h^{a,2}_a) = (h^{a,1}_a, u h^{a,2}_a) = (h^{a,2}_a, u g^{a,1}_a)
\end{align*}
\]
(Identities in \( \overline{F}_{g,3}(k) \)) where \( u = f(\tau^2_1, \tau^2_2)^{-1} 2^{\lambda_1} f(\tau^2_1, \tau^2_2) \) and
\[
\lambda^2 \lambda f_1^{2\lambda(x-1)} = \prod_{a=1}^g (g^{a,1}_a)^{-1} h^{a,1}_a.
\]
The image of the composition in \( \overline{G}_{g_n}(k) \) is given by
\[
(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1 \lambda_2, f_1(x^{\lambda_2}, f_2(x, y) y^{\lambda_2} f_2(x, y)^{-1}) f_2(x, y))
\]
and, for \( \nu = \frac{\lambda_1}{2} \), by
\[
g^{1,2}_a g^{1,2}_a \cdot g^{1,2}_a = g^{1,2}_a (g^{1,2}_1, \tau^1_1 \tau^1_2 \tau^1_1, \tau^1_2 \tau^1_1 \tau^1_2 \tau^1_1) = \sum \left( y^a_i, y^a_j \right) = \sum_{j \neq i} t_{ij} - 2(g-1)t_{ii},
\]
\[
(f_{g_n}) \sum \left( x^a_i, y^a_i \right) = 0 = \left( y^a_i, y^a_j \right).
\]
\[
(f_{g_n}) \sum \left( x^a_i, y^a_i \right) = 0 \quad \text{for all} \ i \neq j,
\]
\[
(f_{g_n}) \left( x^a_i, t_{ij} \right) = \left( y^a_i, t_{ij} \right) = 0 \quad \text{if} \ i, j \cap \{ k \} = \emptyset,
\]
\[
(f_{g_n}) \left( x^a_i + x^a_j, t_{ij} \right) = \left( y^a_i + y^a_j, t_{ij} \right) = 0 \quad \text{for all} \ i, j.
\]

3.4. Horizontal framed genus g chord diagrams and genus g associators.

3.4.1. The \( \text{CD}^f(k) \)-module of genus g framed chord diagrams. Let \( t^f_{g,n}(k) \) denote the graded Lie algebra over \( k \) generated by \( t_{ij}, 1 \leq i, j \leq n, x^a_i, y^a_i \) for \( 1 \leq i \leq n, 1 \leq a \leq g \) with relations \((FS), (FL), (F4T)\) and the following additional genus g relations
\[
(S_g) \quad [x^a_i, y^a_j] = \delta_{ab} t_{ij} \quad \text{for all} \ i \neq j,
\]
\[
(N_g) \quad [x^a_i, x^a_j] = 0 = [y^a_i, y^a_j] \quad \text{for all} \ i \neq j,
\]
\[
(FT_g) \quad \sum_{a=1}^g \left( x^a_i, y^a_i \right) = \sum_{j \neq i} t_{ij} - 2(g-1)t_{ii},
\]
\[
(FL_g) \quad [x^a_i, t_{ij}] = [y^a_i, t_{ij}] = 0 \quad \text{if} \ \{ i, j \} \cap \{ k \} = \emptyset,
\]
\[
(F4T_g) \quad [x^a_i + x^a_j, t_{ij}] = [y^a_i + y^a_j, t_{ij}] = 0 \quad \text{for all} \ i, j.
\]

The Lie algebra \( t^f_{g,n}(k) \) is acted on by the symmetric group \( \mathfrak{S}_n \). One can show that the \( \mathfrak{S} \)-module in \( grLie_k \)
\[
t^f_{g,n}(k) := \{ t^f_{g,n}(k) \}_{n \geq 0}
\]
is a \( t^f(k) \)-module in \( grLie_k \). Partial compositions are defined as follows: for \( I, J \) two finite sets and \( k \in I \),
We call $t^f_g(k)$ the module of infinitesimal genus $g$ framed braids.

We also define the $\mathbf{CD}^f(k)$-module $\mathbf{CD}^f_g(k) := \hat{U}(t^f_g(k))$ of genus $g$ framed chord diagrams whose morphisms can be pictured as chords on $n$ vertical strands with extra chords corresponding to the generators $x^a_i$ and $y^a_i$ as follows:

The relations introduced in the definition of $t^f_{g,n}$ can then be pictured as follows:

\[ \begin{align*}
\circ_k : & \quad t^f_{g,l}(k) \oplus t^f_{j}(k) \\
(0, t_{\alpha\beta}) & \quad \rightarrow \quad t^f_{g, \lambda \cup \{l\}}(k) \\
(t_{ij}, 0) & \quad \rightarrow \quad \begin{cases} 
 t_{ij} & \text{if } k \notin \{i, j\} \\
 \sum_{p \in J} t_{pj} & \text{if } k = i \\
 \sum_{p \in J} t_{ip} & \text{if } j = k 
\end{cases} \\
(x^a_i, 0) & \quad \rightarrow \quad \begin{cases} 
x^a_i & \text{if } k \neq i \\
 \sum_{p \in J} x^a_p & \text{if } k = i 
\end{cases} \\
(y^a_i, 0) & \quad \rightarrow \quad \begin{cases} 
y^a_i & \text{if } k \neq i \\
 \sum_{p \in J} y^a_p & \text{if } k = i 
\end{cases}
\end{align*} \]

\[ (S_g) \]

\[ (N_g) \]
3.4.2. The $\text{PaCD}^f(k)$-module of parenthesized framed genus $g$ chord diagrams. As in the framed genus 0 situation, the module of objects $\text{Ob}(\text{CD}^f_g(k))$ of $\text{CD}^f_g(k)$ is terminal. Thus, we have a morphism of modules $\omega_2 : \text{Pa} = \text{Ob}(\text{Pa}(k)) \to \text{Ob}(\text{CD}^f_g(k))$ over the morphism of operads $\omega_1$ from §2.2.2, and thus we can define the $\text{PaCD}^f(k)$-module

$$\text{PaCD}^f_g(k) := \omega_2^* \text{CD}^f_g(k),$$

in $\text{Cat}(\text{CoAss}_k)$, of so-called parenthesized genus $g$ framed chord diagrams. We have

- $\text{Ob}(\text{PaCD}^f_g(k)) := \text{Pa},$
- $\text{Mor}_{\text{PaCD}^f_g(k)(n)}(p,q) := \text{End}_{\text{CD}^f_g(k)(n)}(pt).$
Example 3.13 (Notable arrows in $\mathbf{PaCD}_g(k)$). We have the following arrows $X_a$, $Y_a$ in $\mathbf{PaCD}_g(k)(1)$

![Diagram of arrows](image)

and $X_a^{1,2}$, $Y_a^{1,2}$ in $\mathbf{PaCD}_g(k)(2)$

![Diagram of arrows](image)

Remark 3.14. One can write the elements $X_a^{12}$, $Y_a^{12}$, $\tilde{X}_a^{12}$ and $\tilde{Y}_a^{12}$ in terms of $X_a^{1,2}$ and $Y_a^{1,2}$ by means of the following relations:

- $\tilde{X}_a^{12} = X_a^{12} - X_a^{1,2}$, $\tilde{Y}_a^{12} = Y_a^{12} - Y_a^{1,2}$,
- $X_a^{12} = X_a^{12,0}$, $Y_a^{12} = Y_a^{12,0}$.

Remark 3.15. There is a map of $\mathcal{G}$-modules $\mathbf{PaCD}^f(k) \rightarrow \mathbf{PaCD}^f_g(k)$ and we abusively denote $p^{1,2}$, $X^{1,2}$, $H^{1,2}$ and $a^{1,2,3}$ the images in $\mathbf{PaCD}^f_g(k)$ of the corresponding arrows in $\mathbf{PaCD}^f(k)$. The elements $X_a^{1,2}$ and $Y_a^{1,2}$ are generators of the $\mathbf{PaCD}^f(k)$-module $\mathbf{PaCD}^f_g(k)$ and satisfy the following relations for all $1 \leq a \leq g$:

- $X_a^{2,1} = (X^{1,2})^{-1}X_a^{1,2}X_a^{1,2}$, $Y_a^{2,1} = (X^{1,2})^{-1}Y_a^{1,2}X_a^{1,2}$,
- $X_a^{2,0} = Y_a^{2,0} = 0$, $X_a^{1,0} = X_a$, $Y_a^{1,0} = Y_a$,
- $\tilde{X}_a^{12,3} = \tilde{a}^{1,2,3}X_a^{1,2}X_a^{1,2}(a^{1,2,3}X_a^{1,2})^{-1}X_a^{1,2}X_a^{1,2}$,
- $\tilde{Y}_a^{12,3} = \tilde{a}^{1,2,3}X_a^{1,2}X_a^{1,2}(a^{1,2,3})^{-1}X_a^{1,2}X_a^{1,2}$,
- $H_1^{1,2} = [a^{1,2,3}X_a^{1,2}X_a^{1,2}(a^{1,2,3})^{-1}X_a^{1,2}X_a^{1,2}]$,
- $H_1^{1,2} + (P^1)^2g^{-1} = \sum_{a=1}^g [X_a^{1,2}, X_a^{1,2}]$.

3.4.3. Genus $g$ associators.

Definition 3.16. A genus $g$ associator over $k$ is couple $(F,G)$ where $F \in \mathbf{Ass}^f_g(k)$ is a $k$-associator and $G$ is an isomorphism between the $\mathbf{PaB}^f(k)$-module $\mathbf{PaB}^f_g(k)$ and the $\mathbf{GPaCD}^f(k)$-module $\mathbf{GPaCD}^f_g(k)$ which is the identity on objects and which is compatible with $F$. We denote its set by

$$\mathbf{Ass}_g^f(k) := \Iso^*_{\mathrm{Op\mathfrak{R}}^\text{Grpd}_k} \left( ((\mathbf{PaB}^f(k), \mathbf{PaB}^f_g(k)), (\mathbf{GPaCD}^f(k), \mathbf{GPaCD}^f_g(k))) \right).$$
We have a morphism of short exact sequences

\[
\begin{array}{cclll}
1 & \longrightarrow & k' & \longrightarrow & \hat{PB}_{g,n}(k) \\
& & \downarrow & & \downarrow \\
1 & \longrightarrow & \exp(\hat{t}_{g,n}(k)) & \longrightarrow & \exp(t_{g,n}(k))
\end{array}
\]

where the right vertical arrow was constructed in [17]. This shows that the map \(\hat{PB}_{g,n}(k) \rightarrow \exp(\hat{t}_{g,n}(k))\) is a \(k\)-pro-unipotent group isomorphism. We will derive this result from the flatness of a connection defined over \(\text{Conf}^f(\Sigma_g,n)\) in a future work.

**Theorem 3.17.** There is a one-to-one correspondence between elements of \(\text{Ass}_g(k)\) and elements of the set \(\text{Ass}_g(k)\) consisting on tuples \((\mu, \varphi, A_{1,\pm}^{1,2}, \ldots, A_{g,\pm}^{1,2})\) where \((\mu, \varphi) \in \text{Ass}(k)\) and \(A_{a,\pm}^{1,2} \in \exp(\hat{t}_{g,\pm}^j)\), for \(a = 1, \ldots, g\), satisfying the following equations in \(\exp(\hat{t}_{g,1}^j)\):

\[
A_{a,\pm}^{1,2} = 1
\]

the following equations in \(\exp(\hat{t}_{g,2}^j)\):

\[
\alpha_{a,\pm}^{1,2,3} = A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}, \quad \text{where} \quad \alpha_{a,\pm}^{1,2,3} = \varphi^{1,2,3}A_{a,\pm}^{1,2,3}e^{\mu t_{12} + t_{13}}/2,
\]

\[
\epsilon^{\mu t_{12}} = (\varphi^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}),
\]

for all \(1 \leq a \leq g\)

\[
1 = (\varphi^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}A_{a,\pm}^{1,2,3}A_{g,\pm}^{1,2,3}),
\]

for all \(1 \leq b < a \leq g\) and the following equation in \(\exp(\hat{t}_{g,2}^j)\):

\[
\epsilon^{\mu t_{12} + (b-1)t_{13}} = \sum_{a=1}^{g} (A_{a,\pm}^{1,2})^{-1}, A_{a,\pm}^{1,2}.
\]

**Proof.** Let \(\hat{F}\) be a framed \(k\)-associator \(\text{PaB}^f(k) \rightarrow \text{GPaCD}^f(k)\) and let \(\hat{G}\) be an isomorphism

\[
\text{PaB}^f_g(k) \rightarrow \text{GPaCD}_g^f(k)
\]

of \((\text{PaB}^f(k), \text{GPaCD}^f(k))\)-modules which is the identity on objects and which is compatible with \(\hat{F}\). It corresponds to a unique morphism \(G : \text{PaB}_g \rightarrow \text{GPaCD}_g^f(k)\). From the presentation of \(\text{PaB}^f_g\), we know that \(G\) is uniquely determined by the images of \(A_{a,\pm}^{1,2}, B_{a}^{1,2} \in \text{Hom}_{\text{PaB}_g^f(k)(1)}(1)\) and \(A_{a,\pm}^{1,2}, B_{a}^{1,2} \in \text{Hom}_{\text{PaB}_g^f(k)(2)}(12)\), for all \(1 \leq a \leq g\) at the morphisms level. Thus, there are elements \(A_{a,\pm}^{1,2} \in \exp(\hat{t}_{g,2}^j)\), for \(a = 1, \ldots, g\), such that

- \(G(A_{a,\pm}^{1,2}) = A_{a,\pm}^{1,2}X_{a,\pm}^{1,2}\),
- \(G(B_{a}^{1,2}) = A_{a,\pm}^{1,2}Y_{a,\pm}^{1,2}\).

These elements must satisfy the following relations (34), (35), (36), (37) and (38), which are the images of relations \((R_g), (D_g), (N_g), (E_1g)\) and \((E_2g)\). 

\(\square\)
3.5. Graded genus \( g \) Grothendieck–Teichmüller groups.

**Definition 3.18.** The graded genus \( g \) Grothendieck–Teichmüller group is the group

\[
\text{GRT}_g(k) := \text{Aut}^+_{\text{OpR Grpd}_k}(\text{GPaCD}^I(k), \text{GPaCD}^\varphi_g(k))
\]

of automorphisms of the \( \text{GPaCD}^I(k) \)-module \( \text{GPaCD}^\varphi_g(k) \) which are the identity on objects.

Notice that there is an isomorphism

\[
\text{Aut}^+_{\text{OpR Cat}(\text{CoAlg}_k)}(\text{PaCD}^I(k), \text{PaCD}^\varphi_g(k)) \simeq \text{Aut}^+_{\text{OpR Grpd}_k}(\text{GPaCD}^I(k), \text{GPaCD}^\varphi_g(k)).
\]

To any element \((F, G)\) in \( \text{GRT}_g(k) \) one can associate tuples \((\mu, g, u_{1,1}^{1,2}, \ldots, u_{g,1}^{1,2})\), such that \((\mu, g) \in \text{GRT}(k)\) and

- \( G(X_{a}^{1,2}) = u_{1,a}^{1,2}, X_{a}^{1,2} \),
- \( G(Y_{a}^{1,2}) = u_{1,1}^{1,2}, Y_{a}^{1,2} \).

Here \( u_{1,1}^{1,2}, \ldots, u_{g,1}^{1,2} \in \tilde{\mathcal{U}}_{g,1}^f(k) \) satisfy, for \( 1 \leq a \leq g \),

\[
\begin{align*}
  u_{1,a}^{1,0} &= u_{a}^{1,1}, & u_{a,1}^{a,1} &= 1 \\
  \text{Ad}(g^{1,2,3})(u_{a}^{3}^{2,13}) + \text{Ad}(g^{2,1,3})(u_{a}^{2}^{3,12}) + (u_{a}^{2})^{3,12} &= x_{1}^{a,1} + x_{2}^{a,2} + x_{3}^{a,3}, \\
  [\text{Ad}(g^{1,2,3})(u_{a}^{2}^{1,23}), (u_{a}^{3})^{3,12}] &= 0, \\
  [\text{Ad}(g^{2,1,3})(u_{a}^{1}^{2,13}), \text{Ad}(g^{1,2,3})(u_{a}^{2}^{1,23})] &= 0, \\
  [\text{Ad}(g^{2,1,3})(u_{a}^{3}^{2,13}), \text{Ad}(g^{1,2,3})(u_{a}^{0}^{1,23})] &= t_{12},
\end{align*}
\]

as relations in \( \tilde{\mathcal{U}}_{g,1}^f(k) \) and

\[
\sum_{a=1}^{g} [u_{a}^{1}, u_{a}^{2}] = t_{12} + 2(g - 1)t_{1}.
\]

Let us denote by \( \text{GRT}_f^g(k) \) the set of such tuples. Set \((\mu, g, u_{1,1}^{1,2}, \ldots, u_{g,1}^{1,2}) := (\tilde{\mu}, \tilde{\gamma}, \tilde{u}_{1,1}^{1,2}, \ldots, \tilde{u}_{g,1}^{1,2})\), where \( \tilde{\mu}, \tilde{\gamma} \) are as in subsection 1.2 and, for all \( 1 \leq a \leq g \),

\[
\begin{align*}
  \tilde{u}_{a,1}^{1,2} := u_{a,1}^{1,2} &\left(\begin{array}{c}
  x_{1}, x_{2}, y_{1}, y_{2}; 1 \leq a \leq g \\
  x_{1}^{a}, x_{2}^{a}, y_{1}^{a}, y_{2}^{a}; 1 \leq a \leq g
\end{array}\right), \\
  \ldots, \tilde{u}_{g,1}^{1,2} &\left(\begin{array}{c}
  x_{1}, x_{2}, y_{1}, y_{2}; 1 \leq a \leq g \\
  x_{1}^{a}, x_{2}^{a}, y_{1}^{a}, y_{2}^{a}; 1 \leq a \leq g
\end{array}\right).
\end{align*}
\]

The group \( k^\times \) acts on \( \text{GRT}_f^g(k) \) by rescaling. We then set \( \text{GRT}_g(k) := \text{GRT}_f^g(k) \rtimes k^\times \).
3.5.1. The non framed case for chord diagrams. Let us consider \( g > 0 \) and \( n \geq 0 \) and define \( t_{g,n}(k) \) as the \( k \)-Lie algebra with generators \( x_i^a, y_i^a, t_{ij} \) for \( i \neq j \in [n], 1 \leq a \leq g \) satisfying relations (S), (L), (4T) and

\[
(S) \quad [x_i^a, y_j^b] = \delta_{ab} t_{ij} \quad \text{for all} \quad i \neq j,
\]

\[
(N) \quad [x_i^a, x_j^b] = 0 = [y_i^a, y_j^b] \quad \text{for all} \quad i \neq j,
\]

\[
(T) \quad \sum_{a=1}^{g} [x_i^a, y_i^a] = - \sum_{j \neq i} t_{ij},
\]

\[
(I) \quad [x_k^a, t_{ij}] = [y_k^a, t_{ij}] = 0 \quad \text{if} \quad \# \{i, j, k\} = 3,
\]

\[
(4T) \quad [y_i^a + y_j^a, t_{ij}] = [x_i^a + x_j^a, t_{ij}] = 0 \quad \text{for} \quad i \neq j.
\]

The Lie algebra \( t_{g,n}(k) \) is equipped with a grading given by \( \deg(x_i^a) = (1, 0), \deg(y_i^a) = (0, 1) \). The total degree defines a positive grading on \( t_{g,n}(k) \); we denote by \( t_{g,n}(k) \) the corresponding completion. If \( k = \mathbb{C} \), we will denote \( t_{g,n}(k) := t_{g,n} \).

The Lie algebra \( t_{g,n}(k) \) is acted on by the symmetric group \( \mathfrak{S}_n \), and one can show that the \( \mathfrak{S}_n \)-module in \( \text{grLie}_k \)

\[
t_g(k) := \{ t_{g,n}(k) \}_{n \geq 0}
\]

is a \( t(k) \)-module in \( \text{grLie}_k \).

The collection of the Lie algebras \( t_{g,n}(k) \), for \( n \geq 1 \) is provided with the structure of an \( t(k) \)-module over the operad \( t \) in \( (\text{positively} \ \text{graded} \ \text{finite} \ \text{dimensional}) \) Lie algebras over \( k \), denoted \( t_g(k) \). Partial compositions are defined as follows: for \( I \) a finite set and \( i \in I \),

\[
\alpha_k : \ t_{g,I}(k) \otimes t_I(k) \rightarrow t_{g,J \cup I - \{i\}}(k) \quad \text{with}
\]

\[
(t_{ij}, 0) \quad \rightarrow \quad \sum_{p \in J} t_{ip} \quad \text{if} \quad k = i
\]

\[
(x_i^a, 0) \quad \rightarrow \quad x_i^a \quad \text{if} \quad k \neq i
\]

\[
(y_i^a, 0) \quad \rightarrow \quad y_i^a \quad \text{if} \quad k \neq i
\]

Since we are in possession of operad modules \( \text{Pa}(k) \) and \( \text{CD}_g(k) \) in \( \text{Cat(} \text{CoAss}_k) \) and of an operad module morphism \( f : \text{Pa} \rightarrow \text{Ob(} \text{CD}_g(k)) \), we are ready to define the \( \text{PaCD}(k) \)-module

\[
\text{PaCD}_g(k) := f^* \text{CD}_g(k)
\]

in \( \text{Cat(} \text{CoAss}_k) \) of parenthesized genus \( g \) chord diagrams. We have \( \text{Ob(} \text{PaCD}_g(k)) := \text{Pa} \) and \( \text{Mor}_{\text{PaCD}_g(k)}(p, q) := U(\hat{t}_{g,n}(k)) \).
3.6. Towards the genus $g$ KZB associator. Recall the following result.

**Theorem 3.19.** (Bezrukavnikov, Enriquez) There is a monodromy morphism $\text{PB}_{g,n} \to \exp(t_{g,n})$ inducing an isomorphism of Lie algebras $\text{Lie}(\text{PB}_{g,n})^C \to t_{g,n}$.

Let us recall the construction from [17] of the universal genus $g$ KZB connection (defined over the configuration spaces). Endow the surface $\Sigma_g$ with a complex structure and denote $C$ the resulting smooth closed complex curve. For any $z \in C$, the fundamental group of $C$ based at $z$ is isomorphic to the group generated by $X^a, Y^a, 1 \leq a \leq g$, such that $\prod_{a=1}^{g}(X^a, Y^a) = 1$ and $\text{PB}_{g,n} := \pi_1(\text{Conf}(C, n), z)$ where $z := (z_1, \ldots, z_n) \in \text{Conf}(C, n)$.

Define a map $\rho : \text{PB}_{g,n} \to \exp(\hat{t}_{g,n})$ by means of the following composite

$$\text{PB}_{g,n} \to \pi_1(C^n, z) \to \pi_g \to F_g \to \exp(\hat{t}_g)^n,$$

where

- $F_g$ is the free group with generators $\gamma_a, 1 \leq a \leq g$,
- $\pi_g \to F_g$ is the composite $\pi_g \to \pi_g/N \to F_g$

where $\pi_g \to \pi_g/N$ is the quotient morphism, where $N$ is the normal subgroup generated by the $X^a, 1 \leq a \leq g$,
- $\pi_g/N \to F_g, Y^a \mapsto \gamma_a$ is the isomorphism induced from the presentation of $\pi_g/N$, where $F_g \to \exp(\hat{t}_g)$ is the assignment $\gamma_a \mapsto \exp(x_a)$.

The principal $\exp(t_{g,n})$-bundle with flat connection on $\text{Conf}(C, n)$ corresponding to $\rho_0$ is then $i^*(\mathcal{P}_n)$, where $i : \text{Conf}(C, n) \to C^n$ is the inclusion and

$$(\mathcal{P}_n \to C^n) = (\mathcal{P}_n^0 \to C)^n \times_{\exp(\hat{t}_{g,n})} \exp(\hat{t}_{g,n}),$$

where $(\mathcal{P}_n^0 \to C)$ is the principal $\exp(\hat{t}_g)$-bundle with flat connection corresponding to the above morphism $\pi_g \to F_g \to \exp(\hat{t}_g)$.

Denote the set of flat connections of degree 1 by

$$\mathcal{F}_1 = \{ \alpha \in \Omega^1(C^n - \text{Diag}, \mathcal{P}_n \times_{\text{ad}} \hat{t}_{g,n}[1]) | d\alpha = \alpha \wedge \alpha = 0 \}$$

and denote its subset of holomorphic flat connections by

$$\mathcal{F}_1^{\text{hol}} = \{ \alpha \in H^0(C^n, \Omega^1_{C,n} \otimes (\mathcal{P}_n \times_{\text{ad}} \hat{t}_{g,n}[1]) \times \text{Diag}) | d\alpha = \alpha \wedge \alpha = 0 \}$$

with $\text{Diag} = \sum_{i,j} \text{Diag}_{ij}$ and $\text{Diag}_{ij} \subset C^n$ is the diagonal corresponding to $z_i = z_j$. Then, Enriquez showed that there is an element $\alpha_{KZ} \in \mathcal{F}_1^{\text{hol}}$ given by

$$\alpha_{KZ}^{g,n} = \sum_{i=1}^{n} \alpha_i,$$

where $\alpha_i \in H^0(C, \mathcal{K}_C^{(i)} \otimes (\mathcal{P}_n \times_{\text{ad}} \hat{t}_{g,n}[1])((\sum_{j \neq i} \Delta_{ij}))$ expands as $\alpha_i \equiv \sum_{1 \leq a \leq g} \omega_a^{(i)} y_a$ modulo $\hat{t}_{g,n}[1, q]$.

As in [17], $\mathcal{K}_C^{(i)} = \mathcal{O}_{C}^{\delta_{i-1}} \otimes \mathcal{O}_{C}^{\delta_{n-i}}, \omega_a^{(i)} = 1^{\delta_{i-1}} \otimes \omega_a \otimes 1^{\delta_{n-i}}$, where $(\omega_a)_{1 \leq a \leq g}$ are the holomorphic differentials such that $f_{\mathcal{A}_a} \omega_b = \delta_{ab}$ and the images of $X^a$ and $Y^a$ under $\pi_g \to \pi_g^{ab} \sim H_1(C, \mathbb{Z})$ are $X^a$ and $Y^a$ respectively.
Consider integers \((g, n)\) in hyperbolic position (i.e. \(2 - 2g - n < 0\)) and let \(S\) be a genus \(g\) topological compact oriented surface, \(x_1, \ldots, x_n\) \(n\) marked points on it. Now let \(X\) be a Riemann surface modeled on \(S\) with genus \(g\) and \(n\) marked points. As \(X\) is hyperbolic, the Uniformisation Theorem says that \(X\) is isomorphic to a quotient \(\mathfrak{h}/\Gamma\) of the Poincaré half-plane \(\mathfrak{h}\) by a discrete subgroup \(\Gamma\) of \(\text{PSL}(2, \mathbb{R})\). Fix \(\tau \in \mathfrak{h}\) and consider a uniformization \(\Sigma_g\) of \(X\). This corresponds to a point \(\kappa\) in the moduli space \(\mathcal{M}_{g,n}\). Such a point can be described by \(3g + n - 3\) parameters. Enriquez chowed that, under this uniformization, the one form \(\alpha_{KZ}\) induces a flat connection

\[
\nabla_{\kappa}^{KZ} := d - \alpha_{\kappa}^{KZ}
\]

over \(\text{Conf}(\Sigma_{g,n})\). Now, the fundamental group \(\pi_1(\Sigma_{g,n}^\times, z_0)\) of \(\Sigma_{g,n}^\times := \Sigma_{g,n} - p_0\), where \(p_0\) is a puncture in the curve, is the nothing but the free group \(\text{F}\) over \(\text{Conf}\) by a discrete subgroup \(\Gamma\) of \(\text{PSL}(2, \mathbb{R})\). The fundamental group \(\Pi_1(\Sigma_{g,n}^\times, z_0)\) is isomorphic to a quotient \(\mathfrak{h}/\Gamma\) of the Poincaré half-plane \(\mathfrak{h}\) by a discrete subgroup \(\Gamma\) of \(\text{PSL}(2, \mathbb{R})\).

**Definition 3.20.** The non-framed genus \(g\) KZB associator is the tuple

\[
\epsilon_g(\kappa) := (A_1(\kappa), B_1(\kappa), \ldots, A_g(\kappa), B_g(\kappa))
\]

where

\[
A_a(\kappa) := T_{-v_0, v_0}^{g, KZB}(X_1^a)
\]

\[
B_a(\kappa) := T_{-v_0, v_0}^{g, KZB}(Y_1^a)
\]

where \(X_1^a\) and \(Y_1^a\), \(1 \leq a \leq g\) are the generating elements in \(\pi_1(\text{Conf}([\Sigma_{g,n}]))\).

We do not know what kind of monodromy relations these associators may have. In particular, if we want to relate them to our operadic definition of genus \(g\) associators we need to extend the universal KZB connection to its framed version. In this direction, we propose the following:

**Conjecture 3.21.** There is a flat universal framed KZB connection \(\nabla_{g,n}^{f, KZB}\) defined on the principal \(\exp(\hat{t}_{g,n})\)-bundle over \(\text{Conf}(C, n)\) constructed as above such that

- its pullback of \(\nabla_{g,n,\kappa}^{f, KZB}\) to the associated \(\exp(\hat{t}_{g,n})\)-bundle over \(C^n\) is

\[
\nabla_{g,n,\kappa}^{f, KZB} := d - \alpha_{g,n}^{f, KZB}
\]

where

\[
\alpha_{g,n}^{f, KZB} := \alpha_{g,n}^{KZB} + \sum_{1 \leq i \leq n} t_i d \log(\lambda_i);
\]

- the 1-form \(\alpha_{g,n}^{f, KZB}\) is \((\mathcal{C}^*)^n\)-basic and the induced connection on the \(\exp(\hat{t}_{g,n})\)-bundle over \(\text{Conf}(C, n)\) given above coincides with the universal genus \(g\) KZB connection defined by Enriquez.
Let $\kappa$ represent a point in the moduli space $M_{g,n}$. If conjecture 3.21 holds, then the monodromy of the connection $\nabla^{\text{KZB}}_{g,n,\kappa}$ induces a tuple

$$e_g(\kappa) := (A^g_1(\kappa), B^g_1(\kappa), \ldots, A^g_g(\kappa), B^g_g(\kappa))$$

where

$$A^g_a(\kappa) := T^{g,\text{KZB}}_{-\nu_a, \nu_a}(X^a_1)$$

$$B^g_a(\kappa) := T^{g,\text{KZB}}_{-\nu_a, \nu_a}(Y^a_1)$$

where $X^a_i$ and $Y^a_i$ are the inverse generating loops in $\pi_1(\text{Conf}_f(\Sigma, 2))$. Let $(2i\pi, \Phi^{\text{KZ}}_{g, \kappa})$ be the framed KZ associator coming from the framed universal KZ connection defined above.

**Conjecture 3.22.** The data $(2i\pi, \Phi^{\text{KZ}}_{g, \kappa}, e^g_f(\kappa))$, where

$$e^g_f(\kappa) = (A^g_1(\kappa), B^g_1(\kappa), \ldots, A^g_g(\kappa), B^g_g(\kappa))$$

is a genus $g$ $C$-associator.

### 4. Torsor comparisons in the elliptic case

#### 4.1. Four modules of genus 1 parenthesized braidings.

Since our base space $T$ is parallelizable and has a translation action, there are four variants of the module of parenthesized elliptic braids corresponding to the framed/unframed and the reduced/non-reduced situations. This subsection is devoted into comparing these four operadic modules.

On the one hand, the above subsection applied to $g = 1$ gives a $\text{PaB}^f$-module

$$\text{PaB}^f_1 := \pi_1(\text{Conf}^f(T, -), \text{Pa}).$$

As $\text{Conf}(T, -)$ is a module over $\overline{C}(\mathbb{C}, -)$, we obtain a $\text{PaB}$-module

$$\text{PaB}_1 := \pi_1(\overline{\text{Conf}}(T, -), \text{Pa}).$$

The operadic pointings are chosen to be the unit of of $\text{PaB}^f_1(1)$ and $\text{PaB}_1(1)$ respectively.

On the other hand, as constructed in [11], to any finite set $I$ we associate the ASFM compactification $\overline{C}(S^1, I)$ of the reduced configuration space $C(S^1, I) := \text{Conf}(S^1, I)/S^1$ of $S^1$. The inclusion of boundary components provide $\overline{C}(S^1, -)$ with the structure of a module over the operad $\overline{C}(\mathbb{R}, -)$ in $\text{Top}$.

Thus, we can construct a $\text{PaB}^f$-module

$$\text{PaB}^f_{\text{rel}} := \pi_1(\text{Conf}^f(T, -)/T, \text{Pa}),$$

and a $\text{PaB}$-module

$$\text{PaB}_{\text{rel}} := \pi_1(\overline{\text{Conf}}(T, -)/T, \text{Pa}).$$

Here, the action of $T$ on the configuration space is given by global translation of the marked points.

In [11], we showed that, as a $\text{PaB}$-module in groupoids having $\text{Pa}$ as $\text{Pa}$-module of objects, $\text{PaB}_{\text{rel}}$ is freely generated by morphisms two morphisms satisfying certain relations. For each $n \geq 1$ and each $p \in \text{PaB}_{\text{rel}}(n)$, the group $\text{Aut}_{\text{PaB}_{\text{rel}}(n)}(p)$ is isomorphic to the reduced pure
braid group $\mathbb{PB}_{1,n}$ with $n$ strands on the torus. In [11], we give a presentation of this group conjugated to the one we use in here (i.e. [4, Definition 5]).

As a $\mathbb{PaB}$-module in groupoids having $\mathbb{Pa}$ as $\mathbb{Pa}$-module of objects, $\mathbb{PaB}_1$ is isomorphic freely generated by $A_{1,2}$ and $B_{1,2}$ in arity 2, together with relations

\begin{align}
A_{1,2}^2 &= \text{Id}_{1,2}, \quad B_{1,2}^2 = \text{Id}_{1,1}, \\
\Phi_{1,2} A_{1,2} A_{2,3} R_{1,2} \Phi_{2,3} A_{2,3} A_{3,1} R_{2,3} \Phi_{3,1} A_{3,1} R_{3,1} &= A^{(1,2)}, \\
\Phi_{1,2} B_{1,2} B_{2,3} R_{1,2} \Phi_{2,3} B_{2,3} B_{3,1} R_{2,3} \Phi_{3,1} B_{3,1} R_{3,1} &= B^{(1,2)}, \\
R_{1,2}^{-1} R_{2,1} &= (R_{1,2} \Phi_{2,1,3} A_{2,1,3}^{-1} R_{2,1} \Phi_{1,2,3} B_{1,2,3}^{-1})^{-1}, \\
R_{1,2}^{-2} &= ((A_{1,2})^{-1}, B_{1,2}).
\end{align}

The quotient map $\mathbb{PB}_{1,n} \to \mathbb{FB}_{1,n} = \mathbb{PB}_{1,n} / (X_1 \ldots X_n, Y_1 \ldots Y_n)$ induces a unique $\mathbb{PaB}$-module morphism

$$F : \mathbb{PaB}_1 \to \mathbb{PaB}_{\text{eff}}$$

given by the identity on objects and

- $F(A_{1,2}) = (A_{1,2})^{-1}, \quad F(B_{1,2}) = (B_{1,2})^{-1}.$

Recall from [8] that we have a short exact sequence of operads in groupoids

$$1 \to \mathbb{PaB} \to \mathbb{PaB}^f \to \mathbb{Z} \to 1,$$

where $\mathbb{Z}$ is viewed as the operad in groupoids with a single object in each arity $n$ and $\mathbb{Z}^n$ as endomorphism of the object. One can also show that we have an isomorphism of operads $\mathbb{PaB}^f \cong \mathbb{PaB} \times \mathbb{Z}$ (see [27] for more details). Then the inclusion $\mathbb{PaB}_1 \to \mathbb{PaB}_1^f$ (which topologically sends the marked points to the same marked points with all framings attached to them aligned to the right on the real line) induces a short exact sequence

$$1 \to \mathbb{PaB}_1 \to \mathbb{PaB}_1^f \to \mathbb{Z} \to 1$$

of modules over (52). Then, we have an isomorphism

$$\mathbb{PaB}_1^f \cong \mathbb{PaB}_1 \times \mathbb{Z}.$$

This leads to an isomorphism of Drinfeld torsors between the framed and non-framed genus 1 non-reduced situations as we will see below.

Then, one can eventually obtain a further short exact sequence

$$1 \to \mathbb{PaB}_{\text{eff}} \to \mathbb{PaB}_{\text{eff}}^f \to \mathbb{Z} \to 1$$

of modules over (52). Here $\mathbb{Z}$ is viewed as the operad in groupoids with a single object in each arity $n$ and $\mathbb{Z}^n / \mathbb{Z}$ as endomorphism of the object, with diagonal action of $\mathbb{Z}$ on $\mathbb{Z}^n$. 


4.2. Reminders on elliptic associators. In the genus 1 case, the $\sum_i x_i$ and $\sum_i y_i$ are central in $t_{1,n}(k)$, and we also consider the quotient

$$t_{1,n}(k) = t_{1,n}(k)/(\sum_i x_i, \sum_i y_i).$$

In particular, $t_{1,2}(k)$ is equal to the free Lie $k$-algebra $f_2(k)$ on two generators $x = x_1$ and $y = y_2$. The Lie algebra $t_{1,n}$ is acted on by the symmetric group $\mathfrak{S}_n$, and one can show that the $\mathfrak{S}$-module in $grLie_k$

$$t_{\text{eff}}(k) := \{t_{1,n}(k)\}_{n \geq 0}$$

actually is a $t(k)$-module in $grLie_k$.

The same formula defines a $(k)$-module structure on $t_{\text{eff}}(k)$. We call $t_{\text{eff}}(k)$ the module of infinitesimal reduced elliptic braids and we define the $CD(k)$-module $CD_{\text{eff}}(k) := U(t_{\text{eff}}(k))$ of elliptic chord diagrams. As in the genus zero case, the module of objects $\text{Ob}(CD_{\text{eff}}(k))$ of $CD_{\text{eff}}(k)$ is terminal. Hence we have a morphism of modules $\omega_2: Pa = \text{Ob}(Pa(k)) \to \text{Ob}(CD_{\text{eff}}(k))$ over the morphism of operads $\omega_1$, and thus we can define the $PaCD(k)$-module

$$PaCD_{\text{eff}}(k) := \omega_2^*CD_{\text{eff}}(k),$$

in $\mathcal{C}(\mathfrak{C}, \mathfrak{A})$, of so-called parenthesized elliptic chord diagrams. There is a map of $\mathfrak{S}$-modules $PaCD(k) \to PaCD_{\text{eff}}(k)$ and we abusively denote $X^{1,2}$, $H^{1,2}$ and $a^{1,2,3}$ the images in $PaCD_{\text{eff}}(k)$ of the corresponding arrows in $PaCD(k)$. We have elements $X_{\text{eff}}^{1,2}$, $Y_{\text{eff}}^{1,2}$ in $PaCD_{\text{eff}}(k)(2)$ which are generators of the $PaCD(k)$-module $PaCD_{\text{eff}}(k)$ and satisfy a certain number of relations. The elliptic Drinfeld torsor over $k$ is the torsor $(\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k))$ defined by

$$\text{Ell}(k) := \text{Iso}^*_{\mathcal{C}(\mathfrak{G}, \mathfrak{R})}((\tilde{PaB}(k), \tilde{PaB}_{\text{eff}}(k)), (GPaCD(k), GPaCD_{\text{eff}}(k)))$$

$$\tilde{G}T_{\text{eff}}(k) := \text{Aut}^*_{\mathcal{C}(\mathfrak{G}, \mathfrak{R})}(\tilde{PaB}(k), \tilde{PaB}_{\text{eff}}(k))$$

$$GRT_{\text{eff}}(k) := \text{Aut}^*_{\mathcal{C}(\mathfrak{G}, \mathfrak{R})}(PaCD(k), PaCD_{\text{eff}}(k)).$$

There is a torsor isomorphism

$$(\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k)) \cong (\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k)),$$

where $(\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k))$ is the torsor constructed in [16, Definition 3.12, Definition 4.1, Subsection 5.2].

4.3. Torsor comparisons. Let $(\tilde{G}T_1(k), \tilde{\text{Ell}}(k), GRT_1(k))$ be the Drinfeld $k$-torsor associated to $PaB_1$ and $PaCD_1(k)$.

As we saw before, the genus $g$ Drinfeld torsor is independent of the framing data so there are obvious torsor isomorphisms.

$$(\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k)) \to (\tilde{G}T_1(k), \tilde{\text{Ell}}(k), GRT_1(k))$$

$$(\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k)) \to (\tilde{G}T_{\text{eff}}(k), \text{Ell}(k), GRT_{\text{eff}}(k)).$$

It remains to compare the reduced and non-reduced versions of the genus 1 Drinfeld torsor.
There is a one-to-one correspondence between elements of $\mathcal{Ell}(k)$ and elements of the set $\text{Ass}_1(k)$ consisting on tuples $(\mu, \varphi, A^{1,2}_k)$ where $(\mu, \varphi) \in \text{Ass}(k)$ and $A^{1,2}_k \in \exp(t_{1,2})$, satisfying the following equations in $\exp(t_{1,1}(k))$:

\begin{equation}
\alpha^{1,2,3} A^{1,23}_k = A^{(12)3}_k, \quad \text{where} \quad \alpha = \varphi^{1,2,3} A^{1,23}_k e^{\mu(t_{12}+t_{13})/2},
\end{equation}

\begin{equation}
e^{\mu t_{13}} = (e^{\mu t_{12}/2} \varphi^{2,1,3} A^{2,13}_+ (\varphi^{2,1,3})^{-1} e^{\mu t_{12}/2}, \varphi^{1,2,3} A^{1,23}_-(\varphi^{1,2,3})^{-1}),
\end{equation}

and the following equation in $\exp(t_{1,2}(k))$:

\begin{equation}
e^{\mu t_{12}} = \left((A^{1,2}_+)^{-1}, A^{1,2}_-\right).
\end{equation}

**Proof.** This is a straightforward application of Theorem 3.17 for $g = 1$ and forgetting about the framing. \hfill \square

**Proposition 4.1.** The set $\text{Ass}_1(k)$ is isomorphic to the set consisting on tuples $(\mu, \varphi, A^{1,2}_1, A^{1,2}_2)$ where $(\mu, \varphi) \in \text{Ass}(k)$ and $A^{1,2}_k \in \exp(t_{1,2})$, satisfying the following equations in $\exp(t_{1,1}(k))$:

\begin{equation}
\tilde{A}^{1,0}_k = 1,
\end{equation}

\begin{equation}
\tilde{A}^{1,2,3}_k \tilde{A}^{1,23}_k = \varphi^{1,2,3} \tilde{A}^{1,23}_+(\varphi^{1,2,3})^{-1} e^{-\mu t_{12}/2} \varphi^{2,1,3} \tilde{A}^{2,13}_+(\varphi^{2,1,3})^{-1} e^{-\mu t_{12}/2},
\end{equation}

\begin{equation}
((\tilde{A}^{1,23}_+)^{-1}, (\tilde{A}^{1,23}_-)^{-1}) = (\varphi e^{\mu t_{23}}, \varphi^{-1}),
\end{equation}

and the following equation in $\exp(t_{1,2}(k))$:

\begin{equation}
e^{\mu t_{12}} = \left(\tilde{A}^{1,2}_+ (\tilde{A}^{1,2}_-)^{-1}\right).
\end{equation}

**Proof.** By using proposition 3.10 applied to $g = 1$ and forgetting about the framing, one can show that $\text{PaB}_1$ has the following alternative presentation

\begin{equation}
\tilde{Z}^{1,0} = \text{Id}_1,
\end{equation}

\begin{equation}
\tilde{Z}^{1,23} \tilde{Z}^{1,23} = \Phi^{1,2,3} \tilde{Z}^{1,23} (\Phi^{1,2,3})^{-1} (R^{2,1})^{-1} \Phi^{2,1,3} \tilde{Z}^{2,13} (\Phi^{2,1,3})^{-1} (R^{1,2})^{-1},
\end{equation}

\begin{equation}
\Phi^{1,2,3} R^{2,3} R^{3,2} (\Phi^{1,2,3})^{-1} = ((\tilde{A}^{1,23}_+)^{-1}, \tilde{B}^{1,23}_+ (\tilde{B}^{1,23}_+)^{-1}) (\Phi^{1,2,3})^{-1},
\end{equation}

\begin{equation}
R^{1,2} R^{2,1} = (\tilde{A}^{1,2}_+ (\tilde{B}^{1,2}_+)^{-1}).
\end{equation}

Then the equivalence between equations (58)–(61) and equations (62)–(65) follow straightforwardly. \hfill \square

One can prove using the monodromy of the non-reduced version of the universal elliptic KZB connection contained in [9] that $\mathcal{Ell}(C)$ is not empty.

**Theorem 4.2.** The set $\mathcal{Ell}(C)$ is not empty.
Proof. The proof goes along the same lines as the one in [9] but since our conventions for the fundamental group generators and monodromy actions differ from it, we give the proof in full detail. Recall our conventions for monodromy actions from [10, Appendix A]. In [9], it was shown that there is a flat universal elliptic KZB connection over Conf$(E_{\tau}, n)$, where $\tau \in \mathfrak{h}$ and $E_{\tau}$ is a normalized elliptic curve extending to the non reduced moduli space of marked elliptic curves. For $\tau \in \mathfrak{h}$, let $U_{\tau,n} \subset \mathbb{C}^n - \text{Diag}_{\tau,n}$ be the open subset of all $z = (z_1, \ldots, z_n)$ of the form $z_i = a_i + \tau b_i$, where $0 < a_1 < \cdots < a_n < 1$ and $0 < b_n < \cdots < b_1 < 1$. If $z_0 \in U_{\tau,n}$, then it defines a point both in the ordered and unordered configuration spaces Conf$(E_{\tau}, n)$ and Conf$(E_{\tau}, [n])$.

As $U_{\tau,n}$ is simply connected, a solution of the elliptic KZB system in [9, Subsection 4.1] on this domain is then unique. Then there is a unique solution $F^{(n)}(z)$ with the prescribed expansion of [9, Subsection 4.1] to this system on $U_{\tau,n}$. The domains $H_n := \{ z \in \mathbb{C}^n | z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, 0 < a_1 < a_2 < \cdots < a_n < 1 \}$ and $D_n := \{ z \in \mathbb{C}^n | z_i = a_i + b_i \tau, a_i, b_i \in \mathbb{R}, 0 < b_n < \cdots < b_1 < 1 \}$ are also simply connected and invariant, and we denote by $F^H(z)$ and $F^V(z)$ the prolongations of $F^{(n)}(z)$ to these domains. Then we have solutions of the elliptic (non-reduced) KZB system on $H_n$ and $D_n$ given by $z \mapsto F^H(z + \sum_{j=1}^n \delta_j)$ and $z \mapsto e^{2\pi i (z_1 + \cdots + z_n)} F^V(z + \tau \sum_{j=1}^n \delta_j)$ respectively. We define $A^F_i, B^F_i \in \exp((1, n))$ by

$$F^H(z) = A^F_i F^H \left( z + \sum_{j=1}^n \delta_j \right),$$

$$F^V(z) = B^F_i e^{2\pi i (z_1 + \cdots + z_n)} F^V \left( z + \tau \sum_{j=1}^n \delta_j \right).$$

Let denote by $\gamma_n : B_{1,n} \to \exp((1, n)) \rtimes \mathfrak{g}_n$, the unique morphism induced by the solution $F^{(n)}(z)$ and, for all $1 \leq i \leq n$, denote $A_i$ for the class of the projection of the path $[0, 1] \ni t \mapsto (z_0 + t \sum_{j=1}^n \delta_j), B_i$ for the class of the projection of $[0, 1] \ni t \mapsto (z_0 + t \tau \sum_{j=1}^n \delta_j)$. The paths $A_1, B_1$ are generators of $B_{1,n}$.

Let us also denote $\tilde{A}_{KZB} := \gamma_1(A_1), \tilde{B}_{KZB} := \gamma_1(B_1)$, $\tilde{A}_{KZB}^{1,2} := \gamma_2(A_2), \tilde{B}_{KZB}^{1,2} := \gamma_2(B_2)$ and denote by $\varphi_{KZ}$ the KZ associator. It is then clear that

$$\tilde{Z}^{1,0} = 1, \quad \tilde{Z}^{0,1} = Z^1.$$

for $Z$ any of the letters $\tilde{A}_{KZB}, \tilde{B}_{KZB}$. Then, our presentation of $B_{1,n}$ implies that we have relation $A_3^{-1} A_2 = \sigma_1 A_2^{-1} \sigma_1 A_1$ and $B_3^{-1} B_2 = \sigma_1 B_2^{-1} \sigma_1 B_1$. Then the image by $\gamma_3$ of these relations yield

$$Z^{1,2} Z^{1,3} = \varphi_{KZ}^{1,2,3} Z^{1,23} (\varphi_{KZ}^{1,2})^{-1} e^{-\pi i t_{12}} \varphi_{KZ}^{2,1,3} Z^{2,13} (\varphi_{KZ}^{2,1})^{-1} e^{-\pi i t_{12}}$$

for $Z$ any of the letters $\tilde{A}_{KZB}, \tilde{B}_{KZB}$. Next, the image by $\gamma_3$ of $(A_3^{-1}, B_3^{-1} B_2) = P_{23}$ then gives

$$(\tilde{A}_{KZB}^{1,2})^{-1}, (\tilde{B}_{KZB}^{1,2})^{-1} \varphi_{KZ}^{1,2} \tilde{B}_{KZB}^{1,23} \varphi_{KZ} = \varphi_{KZ} e^{2\pi i s_{12}} (\varphi_{KZ})^{-1}.$$

Finally, the image by $\gamma_2$ of $(A_2, B_2^{-1}) = P_{12}$ then gives

$$e^{\mu_{12}} = (\tilde{A}_{KZB}^{1,2}, (\tilde{B}_{KZB}^{1,2})^{-1}).$$

By Proposition 4.1, we obtain that $(2\pi i, \varphi_{KZ}, \tilde{A}_{KZB}, \tilde{B}_{KZB})$, where $A_{KZB} = \gamma_3(\tilde{A}_1), B_{KZB} = \gamma_2(\tilde{B}_1)$ satisfy relations (58), (59), (60) and (61) so that it is in $\text{Ass}_1(\mathbb{C})$. ∎
We then get a surjective map of torsors
\[(\widehat{\operatorname{GT}}_1(k), \widetilde{\operatorname{Ell}}(k), \operatorname{GRT}_1(k)) \rightarrow (\widehat{\operatorname{GT}}_{\text{eff}}(k), \operatorname{Ell}(k), \operatorname{GRT}_{\text{eff}}(k)).\]

By functoriality and by the fact that the operadic pointing both in \(\mathbf{PaB}_1\) and \(\mathbf{PaB}_{\text{eff}}\) have been chosen to be the unit, we retrieve maps of torsors
\[(\widehat{\operatorname{GT}}(k), \operatorname{Ass}(k), \operatorname{GRT}(k)) \rightarrow (\widehat{\operatorname{GT}}_1(k), \widetilde{\operatorname{Ell}}(k), \operatorname{GRT}_1(k)) \rightarrow (\widehat{\operatorname{GT}}_{\text{eff}}(k), \operatorname{Ell}(k), \operatorname{GRT}_{\text{eff}}(k)).\]

The composite of these maps is an operadic version of the map first constructed in [16].

REFERENCES

[1] A. Alekseev, N. Kawazumi, Y. Kuno & F. Naef, The Goldman–Turaev Lie bialgebra and the Kashiwara–Vergne problem in higher genera, preprint available at https://arxiv.org/abs/1804.09566, (2018).

[2] D. Bar-Natan, On associators and the Grothendieck–Teichmüller Group I, Selecta Mathematica New Series 4 (1998) 183–212.

[3] P. Bellingeri, On presentations of surface braid groups J. Algebra 274, 2, (2004) 543-563.

[4] P. Bellingeri & S. Gervais, Surface framed braids Geometricae Dedicata 159, 1, (2012) 51–69.

[5] R. Bezrukavnikov, Koszul DG-algebras arising from configuration spaces, Geom. and Funct. Analysis, 4, 2 (1992), 119–135.

[6] J. Birman, On braid groups, Commun. Pure Appl. Math. 12 (1969), 41–79.

[7] J. Birman, Mapping class groups and their relationship to braid groups, Commun. Pure Appl. Math. 12 (1969), 213–38.

[8] P. Boavida de Brito, G. Horel & M. Robertson Operads of genus zero curves and the Grothendieck–Teichmüller group, Geometric Topology 23 (2019) 299-346.

[9] D. Calaque, B. Enriquez & P. Etingof, Universal KZB equations: the elliptic case, Algebra, Arithmetic and Geometry Vol I: in honor of Yu. I. Manin, Progress in Mathematics 269 (2010), 165–266.

[10] D. Calaque & M. Gonzalez, On the universal ellipsitomic KZB connection, https://arxiv.org/abs/1908.03887, (2019).

[11] D. Calaque & M. Gonzalez, Ellipsitomic associators, To appear soon.

[12] D. Calaque & M. Gonzalez, A moperadic approach to cyclotomic associators, https://arxiv.org/abs/2004.00572.

[13] R. Campos & N. Idrissi & T. Willwacher, Configuration spaces of surfaces, https://arxiv.org/abs/1911.12281, (2019).

[14] V. Drinfeld, On quasitriangular quasi-Hopf algebras and a group closely connected with \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), Leningrad Math. J. 1, 2 (1991), 829-860.

[15] B. Enriquez, Quasi-reflection algebras and cyclotomic associators, Selecta Mathematica (NS) 13 (2008), no. 3, 391–463.

[16] B. Enriquez Elliptic associators, Selecta Mathematica (NS), 20, 2 (2014), 491–584.

[17] B. Enriquez, Flat connections on configuration spaces and formality of braid groups of surfaces, Advances in Mathematics, 252 (2014) 204226.

[18] B. Fresse, Modules over operads and functors, Lecture Notes in Mathematics, Springer, 1967 (2009).

[19] B. Fresse, Homotopy of operads and Grothendieck–Teichmüller groups: Part 1: The algebraic theory and its topological background, Mathematical Surveys and Monographs, American Mathematical Society, 217 (2017); 532 pp.

[20] B. Fresse, Homotopy of operads and Grothendieck–Teichmüller groups: Part 2: The applications of (rational) homotopy theory methods, Mathematical Surveys and Monographs, American Mathematical Society, 217 (2017); 704 pp.

[21] W. Fulton & R. MacPherson, A compactification of configuration spaces, Annals of Mathematics, 139 (1994), 183-225.
[22] A. Grothendieck, *Esquisse d’un programme*, London Math. Soc. Lecture Note Ser., 242, Geometric Galois actions, 1, 5–48, Cambridge Univ. Press, Cambridge (1997).
[23] E. Hoefel, *Explicit Homotopy equivalences between some operads*, preprint available at https://arxiv.org/abs/1110.3116 (2011).
[24] G. Horel, *Profinite completion of operads and the Grothendieck–Teichmüller group*, Advances in Math. 321 (2017), pp. 326390.
[25] C. Kassel, *Quantum groups*, Springer Science & Business Media, 155 (2012).
[26] H. Ko & L. Smolinsky, *The framed braid group and 3-manifolds*, Proc. Amer. Math. Soc. 115 (1992) 541551.
[27] N. Wahl, *Ribbon braids and related operads*, PhD thesis, Oxford University, (2001).