The Brown-Henneaux’s central charge from the path-integral boundary condition

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Abstract

We derive Brown-Henneaux’s commutation relation and central charge in the framework of the path integral. If we use the leading part of the asymptotic symmetry to derive the Ward-Takahashi identity, we can see the central charge arises from the fact that the boundary condition of the path integral is not invariant under the transformation.
1 Introduction

In the (2+1) dimensional spacetime with a negative cosmological constant $\Lambda = -1/l^2$, Brown and Henneaux [1] have shown that the asymptotic symmetry of the asymptotically $AdS_3$ spacetime is the conformal group in 2 dimensions and this symmetry is canonically realized by the Poisson bracket algebra of the Hamiltonian generators with a central charge,

$$c = \frac{3l}{2G}.$$  \hspace{1cm} (1)

This central charge was also obtained in the Chern-Simons formulation of the (2+1) dimensional gravity [2] or in the context AdS/CFT correspondence [3]. Combining this central charge with the Cardy formula, Strominger [4] has suggested that the Bekenstein-Hawking entropy of the BTZ black hole [5] can be understood as the density of states of some conformal field theory. For further generalizations, see Refs. [6, 7].

In this paper, we consider to derive this central charge in the path integral formulation since it was originally obtained by the canonical formulation. In view of the equivalence of these two approaches to quantum theory, we must obtain the same result within the path integral. In the context of the path integral, the usual central charge is the quantum anomaly which is understood as the Jacobian factor of the path integral measure [8]. However, Brown-Henneaux’s central charge is classical one because it exists at the level of the Poisson bracket. Thus, we want to clarify the origin of this classical central charge in the formulation of path integral.
2 Transformations

We consider the asymptotically $AdS_3$ spacetime which is defined by the boundary condition [1],

\begin{align}
g_{tt} &= -\frac{r^2}{l^2} + \mathcal{O}(1), & g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}(1/r^4), & g_{\phi\phi} &= r^2 + \mathcal{O}(1), \\
g_{tr} &= \mathcal{O}(1/r^3), & g_{r\phi} &= \mathcal{O}(1/r^3), & g_{t\phi} &= \mathcal{O}(1).
\end{align}

(2)

The asymptotic symmetry of this spacetime becomes

\begin{align}
\xi^t &= lT(t, \phi) + \frac{l^3}{r^2} \bar{T}(t, \phi) + \mathcal{O}(1/r^4), \\
\xi^r &= rR(t, \phi) + \mathcal{O}(1/r), \\
\xi^\phi &= \Phi(t, \phi) + \frac{l^2}{r^2} \bar{\Phi}(t, \phi) + \mathcal{O}(1/r^4),
\end{align}

(3)

where they satisfy

\begin{align}
l \partial_t T(t, \phi) &= \partial_\phi \Phi(t, \phi) = -R(t, \phi), \\
l \partial_t \Phi(t, \phi) &= \partial_\phi T(t, \phi),
\end{align}

(4)

and

\begin{align}
\bar{T}(t, \phi) &= -\frac{1}{2} \partial_t R(t, \phi), \\
\bar{\Phi}(t, \phi) &= \frac{1}{2} \partial_\phi R(t, \phi).
\end{align}

(5)

This transformation preserves the above boundary condition (2) and is the conformal group in 2 dimensions.

We may consider another transformation which is the leading part of the asymptotic symmetry,

\begin{align}
\xi'^t &= lT(t, \phi), \\
\xi'^r &= rR(t, \phi), \\
\xi'^\phi &= \Phi(t, \phi),
\end{align}

(6)
where $T, R, \Phi$ again satisfy the above equations (4). Note that this transformation is not the asymptotic symmetry since it breaks the boundary conditions for $g_{tr}$ and $g_{r\phi}$. However, the charge of this transformation is the same as that of the asymptotic symmetry as we see later.

3 Action and Charge

If we assume that the boundary of the spacetime is only at infinity $r = r_* \to \infty$ whose unit normal vector is $u^a$, the action becomes [9]

$$S = \frac{1}{16\pi G} \int_M \sqrt{-g} \ (R - 2\Lambda) \ d^3x + \frac{1}{8\pi G} \int_{r = r_*} \sqrt{-\gamma} \Theta \ d^2x, \quad (7)$$

where $\gamma_{ab}$ is the induced metric on the boundary $r = r_*$ defined by $\gamma_{ab} = g_{ab} - u_a u_b$ and $\Theta^{ab}$ is the extrinsic curvature of the boundary defined by $\Theta^{ab} = \gamma^{ac} \nabla_c u^b$. The generic variation, namely $\delta g_{ab} \neq 0$ at $r = r_*$, of this action is [10, 11]

$$\delta S = -\frac{1}{16\pi G} \int_M \sqrt{-g} \tilde{G}^{ab} \delta g_{ab} \ d^3x - \frac{1}{16\pi G} \int_{r = r_*} \sqrt{-\gamma} \Pi^{ab} \delta \gamma_{ab} \ d^2x, \quad (8)$$

where $\tilde{G}^{ab} = R^{ab} - \frac{1}{2} g^{ab} R + \Lambda g^{ab}$ and $\Pi^{ab} = \Theta^{ab} - \Theta \gamma^{ab}$. By using this formula, we find that the change of the action under the transformation $\delta g_{ab} = \nabla_a \zeta_b + \nabla_b \zeta_a$ becomes

$$\delta_{\zeta} S = -\frac{1}{8\pi G} \int_M \sqrt{-g} \tilde{G}^{ab} \nabla_a \zeta_b \ d^3x$$

$$- \frac{1}{8\pi G} \int_{r = r_*} \sqrt{-\gamma} \left[ \Pi^{ab} \partial_a \tilde{\zeta}^b + \eta \left( \Theta^{ab} \Theta_{ab} - \Theta^2 \right) \right] d^2x, \quad (9)$$

where $\eta = \zeta^a u_a$ and $\tilde{\zeta}^a = \zeta^a - \eta u^a$ is the tangential part of $\zeta^a$ to the boundary $r = r_*$. Unfortunately, this would diverge in the limit of $r_* \to \infty$. Therefore, it is usual to subtract a functional of the boundary data $\gamma_{ab}$ from
the action \( [\Pi] \). We here choose so that

\[
\delta S = -\frac{1}{8\pi G} \int_M \sqrt{-g} \, \tilde{G}^{ab} \nabla_a \zeta_b \, d^3x
\]

\[-\frac{1}{8\pi G} \int_{r=r_*} \sqrt{-\gamma} \left[ \left( \Pi^a_{\ b} - \tilde{\Pi}^a_{\ b} \right) D_a \zeta^b
\right.
\]

\[+ \eta \left( \Theta^{ab} \Theta_{ab} - \Theta^2 - \tilde{\Theta}^{ab} \tilde{\Theta}_{ab} + \tilde{\Theta}^2 \right) \, d^2x, \tag{10}
\]

where the hats mean that they are evaluated by the \( M = J = 0 \) BTZ black hole rather than \( AdS_3 \) spacetime for a technical reason.

We can identify the charge of Brown and Henneaux as

\[
J[\xi] = -\frac{1}{8\pi G} \lim_{r \to \infty} \int_{r=r_*} d\phi \sqrt{\sigma} \left( \Pi^a_{\ b} - \tilde{\Pi}^a_{\ b} \right) \tilde{\xi}^b n_a, \tag{11}
\]

where \( n^a \) is the unit normal vector of the time slice and \( \sigma_{ab} \) is the induced metric on the boundary \( r = r_* \) of the time slice. To our knowledge, this alternative expression \([\Pi]\) for Brown-Henneaux’s charge has not been discussed before. It is easy to check that this charge is actually identical to Brown-Henneaux’s charge by expanding around the \( M = J = 0 \) BTZ black hole. See, however, Ref. \([\Pi]\) for a related definition of the global charge.

Note that \( J[\xi] = J[\xi'] \) for the transformations in Eqs. \((\Pi)\) and \((\Pi)\) since the non-leading terms does not contribute to the charge. After a straightforward calculation, we find that the change of this charge under the asymptotic symmetry becomes

\[
\delta_{\xi_2} J[\xi_1] = J\left[ [\xi_1, \xi_2] \right] + K[\xi_1, \xi_2] + \cdots, \tag{12}
\]

where \( K[\xi_1, \xi_2] \) is Brown-Henneaux’s central charge,

\[
K[\xi_1, \xi_2] = -\frac{1}{8\pi G} \int d\phi \left( T_1 \partial_\phi^3 + \Phi_1 l^3 \partial_\phi^3 \right) l\Phi_2, \tag{13}
\]

and ‘\( \cdots \)’ means the terms which vanish by using the equations of motion. By Fourier transformation, this becomes the usual central term with the central
charge ([4]). This provides an alternative derivation of Brown-Henneaux’s central charge which is simpler than the original one. On the other hand, an interesting aspect of the leading transformation ([3]), which motivated the present work, is that the change of the charge ([4]) gives rise to

\[
\delta \xi_2 J[\xi_1] = J[[\xi_1, \xi_2]] + \cdots,
\]

without any central charge. Since the remaining quantities which appear in the Ward-Takahashi identity are the same as those of the asymptotic symmetry, one might think that we can obtain the commutator without any central charge if we use this transformation. We thus want to understand the origin of the central charge by this transformation.

4 Commutation relation

We begin with the path integral

\[
\langle J[\xi_1] \rangle = \int_B d\mu J[\xi_1] e^{iS},
\]

where \(d\mu\) and \(B\) denote the measure and boundary condition of the path integral, respectively. To obtain the Ward-Takahashi identity, we perform the infinitesimal change of the integration variable corresponding to the leading transformation \(\xi_2\) rather than the asymptotic symmetry. (We assume that \(d\mu\) is invariant under this transformation since we want to calculate the classical central charge.) We then have the Ward-Takahashi identity,

\[
\langle \delta \xi_2 J[\xi_1] \rangle = -i \langle T^* J[\xi_1] \delta \xi_2 S \rangle - \Delta[\xi_1', \xi_2'],
\]

where

\[
\Delta[\xi_1', \xi_2'] \equiv \left( \int_{B+\delta \xi_2 B} - \int_B \right) d\mu J[\xi_1] e^{iS},
\]
and the boundary condition $B + \delta_{\xi_2}B$ denotes that the transformed metric $g_{ab} + \delta_{\xi_2}g_{ab}$ must satisfy the asymptotically AdS$_3$ condition (4). Note that we have an extra term $\Delta[\xi'_1, \xi'_2]$ because the leading transformation breaks the boundary condition of the path integral $B$. By using the asymptotically AdS$_3$ condition (2), we evaluate the first term of the right-hand side. Then, one finds that

$$
\langle T^* J[\xi_1] \delta_{\xi_2} S \rangle = -\langle T^* J[\xi_1] \int dt \frac{\partial}{\partial t} J[\xi_2] + \cdots \rangle
$$

$$
= -\int dt \frac{\partial}{\partial t} \langle T^* J[\xi_1] J[\xi_2] + \cdots \rangle
$$

$$
= -\int dt \frac{\partial}{\partial t} \langle T J[\xi_1] J[\xi_2] + \cdots \rangle
$$

$$
= \langle [J[\xi_1], J[\xi_2]] + \cdots \rangle, \tag{18}
$$

where “…” again means the terms which vanish by using the equations of motion. Here we used the standard Bjorken-Johnson-Low argument to convert the $T^*$-product to the canonical T-product. Combining with Eq. (14), we can obtain the anomalous commutator of two charges as

$$
\langle [J[\xi_1], J[\xi_2]] \rangle = \langle iJ[\xi_1, \xi_2] \rangle + i\Delta[\xi'_1, \xi'_2], \tag{19}
$$

by using the equations of motion.

Next, we want to evaluate the anomalous term $\Delta[\xi'_1, \xi'_2]$ which is defined by Eq. (17). By performing the infinitesimal change of the integration variable corresponding to the inverse transformation of $\xi'_2$ in the first integral and that of $\xi_2$ in the second integral of Eq. (17), the integrals become

$$
\int_{B+\delta_{\xi_2}B} d\mu J[\xi_1] e^{iS} = \int_{B} d\mu \left( J[\xi_1] - iJ[\xi_1] \delta_{\xi_2} S - \delta_{\xi_2} J[\xi_1] \right) e^{iS},
$$

$$
\int_{B} d\mu J[\xi_1] e^{iS} = \int_{B} d\mu \left( J[\xi_1] - iJ[\xi_1] \delta_{\xi_2} S - \delta_{\xi_2} J[\xi_1] \right) e^{iS}. \tag{20}
$$
Note that the boundary condition of both of the path integral become the same. Since $\delta_{\xi_2} S = \delta_{\xi_2'} S$ but $\delta_{\xi_2} J[\xi_1] \neq \delta_{\xi_2'} J[\xi_1]$ as Eqs. (12) and (14) show, one can obtain that

$$\Delta[\xi_1', \xi_2'] = \langle \delta_{\xi_2} J[\xi_1] - \delta_{\xi_2'} J[\xi_1] \rangle = \langle K[\xi_1, \xi_2] \rangle,$$

(21)

by using the equations of motion. Thus, we finally obtain

$$\langle [J[\xi_1'], J[\xi_2']] \rangle = \langle iJ[\xi_1', \xi_2'] \rangle + iK[\xi_1, \xi_2],$$

(22)

which is consistent with the result of Brown and Henneaux.

5 Discussion

We have reproduced Brown-Henneaux’s commutator and central charge in the frame of the path integral. By using the leading transformation (3) of the asymptotic symmetry to derive the Ward-Takahashi identity, we have shown that Brown-Henneaux’s central charge arises from the path integral boundary condition. That is, the central charge arises from the fact that the boundary condition of the path integral is not invariant under the transformation. This is in contrast to the usual quantum case, where the anomaly arises from the fact that the measure of the path integral is not invariant under the relevant transformation. Other classical central charges, such as in $N = 2$ supersymmetric theory [12], may also be understood as above in the path integral formalism.

Of course, we can derive the above commutator by using the asymptotic symmetry itself. Then, the central charge arises from the transformation law of the charge. However, the present analysis suggests the possibility that the classical central charge may arise in more general theories if the boundary
condition of the path integral is non-trivial. A path integral generally provides a more transparent framework to study various topological properties such as related to the black hole. In view of this, it is important to understand the origin of the central charge in path integral. It is also gratifying that one can derive the fundamental result in a variety of ways.

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References

[1] J. D. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986).
[2] M. Bañados, Phys. Rev. D52, 5816 (1995); M. Bañados, T. Brotz, and M. E. Ortiz, Nucl. Phys. B545, 340 (1999).
[3] M. Henningson and K. Skenderis, JHEP 9807, 023 (1998); S. Hyun, W. T. Kim, and J. Lee, Phys. Rev. D59, 084020 (1999); V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999).
[4] A. Strominger, JHEP 9802, 009 (1998).
[5] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D48, 1506 (1993).
[6] S. Carlip, Phys. Rev. Lett. 82, 2828 (1999); Class. Quantum Grav. 16, 3327 (1999).
[7] M. Natsuume, T. Okamura, and M. Sato, Phys. Rev. D61, 104005 (2000).
[8] K. Fujikawa, Phys. Lett. B188, 115 (1987).

[9] G. W. Gibbons and S. W. Hawking, Phys. Rev. D15, 2752 (1977); S. W. Hawking, in General Relativity, an Einstein Centenary Survey, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[10] J. W. York, Found. Phys. 16, 249 (1986).

[11] J. D. Brown and J. W. York, Phys. Rev. D47, 1407 (1993).

[12] K. Fujikawa and K. Okuyama, Nucl. Phys. B521, 401 (1998).