BAND WIDTH ESTIMATES OF CMC INITIAL DATA SETS

XIAOXIANG CHAI AND XUEYUAN WAN

Abstract. We generalize a band width estimate of Gromov to CMC initial data sets. We give three independent proofs: via the stability of a hypersurface with prescribed null expansion, via a perturbation of the spacetime harmonic function and via the Dirac operator.

1. Introduction

An initial data set \((M, g, k)\) is an \(n\)-dimensional Riemannian manifold \((M, g)\) immersed as a spacelike hypersurface in an \((n+1)\)-dimensional Lorentzian spacetime \(S^{n,1}\) where \(k(X, Y) = \langle \nabla_X e_0, Y \rangle\) is the second fundamental form with respect to a choosed unit timelike normal \(e_0\). We call \(M\) a constant mean curvature (or in short CMC) initial data set if \(M\) is of constant mean curvature in \(S^{n,1}\), that is, \(\text{tr}_g k\) is a constant. The Einstein tensor \(G\) of \(S^{n,1}\) when restricted to \(M\) gives the constraint equation

\[ 2\mu := G_{00} = R_g - |k|^2_g + (\text{tr}_g k)^2, \]

and

\[ J_i := G_{0i} = (\text{div}_g k - d(\text{tr}_g k))_i, \]

where \(R_g\) is the scalar curvature of \((M, g)\), \(\mu\) and \(J\) are respectively called energy density and current density.

Given a two-sided hypersurface \(\Sigma\) in \(M\), we denote by \(p\) the second fundamental form computed with respect to the choosed normal \(\nu\). Further, we denote \(H = \text{div}_\Sigma \nu\) the mean curvature. The quantity

\[ \theta^\pm = \pm H + \text{tr}_\Sigma k \]

is called outward (inward) null expansion of the hypersurface \(\Sigma\). A hypersurface with vanishing null expansion \(\theta^+ (\theta^-)\) is called a marginally outer (inner) trapped hypersurface or in short MOTS (MITS). We will be concerned with only the outward null expansion and we use the shorthand \(\theta = \theta^+\).

Let \(M = T^{n-1} \times [-1, 1], \partial_\pm M = T^{n-1} \times \{\pm 1\}\), \(\theta_\pm\) be the expansion at \(\partial_\pm M\) computed with respect to the outward normal \(\nu_+\) and \(\theta_-\) be the

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null expansion computed with respect to the inward normal $\nu_-$. It is easy to see that this choice of normal vector field is consistent in the sense that $\langle \nu_\pm, \frac{\partial}{\partial t} \rangle > 0$ at both $\partial \pm M$. Here $t$ is the coordinate representing the segment $[-1, +1]$. We denote the expansion at $\partial \pm M$ computed in this way by $\theta_\pm$.

The width of $M$ is defined to be the distance between two boundaries $\partial \pm M$, which is denoted by $\text{width}(M, g)$.

Our main result is a width estimate for a CMC initial data set $(M, g, k)$. Before stating the main result, we define some quantities we frequently use.

Let $\sigma, \lambda$ be two real constants, define $\eta = \eta(t) = \eta_0(t)$ to be the solution to the ordinary differential equation

$$\sigma + \frac{n}{n-1} \eta^2 - 2\eta \lambda + 2\eta' = 0, \quad \eta' < 0.$$  

The solution of (1.1) is given in the Appendix. The explicit form of $\eta$ depends on the sign of $\sigma - \frac{n-1}{n} \lambda^2$, see respectively (5.2), (5.3) and (5.4). Define the interval $[r_-, r_+]$ by setting

$$r_\pm = \pm \frac{(n-1)\pi}{n} \left[ \frac{n-1}{n} (\sigma - \frac{n-1}{n} \lambda^2) \right]^{-\frac{1}{2}}$$

if $\sigma > \frac{n-1}{n} \lambda^2$ and otherwise fixing $r_\pm$ with the property $0 \notin [r_-, r_+]$.

Now we are ready to state the theorem.

**Theorem 1.1.** Let $M = \mathbb{T}^{n-1} \times [-1, 1]$, assume on $(M, g, k)$ that $\text{tr}_g k = \lambda$, $\mu - |J| \geq \frac{1}{2} \sigma$ (1.2) and the null expansions at the boundaries $\partial_\pm M$ satisfy

$$\theta_- \leq \eta(t_-), \theta_+ \geq \eta(t_+),$$

for some $t_\pm$ with $r_- < t_- < t_+ < r_+$. Then

$$\text{width}(M, g) \leq t_+ - t_-.$$  

**Remark 1.2.** In fact, using the method of the Dirac operator, we can prove Theorem 1.1 when $M$ is a spin band of infinite vertical $\hat{A}$-area or a $K\bar{O}$-band, see Theorems 4.14, 4.18.

Now we briefly discuss the history and previous results related to the Theorem 1.1. Gromov [Gro18] established via torical symmetrization the width estimate for the case $k = 0$, $\sigma = n(n-1)$ with the metric in Example 1.3 as a rigid band in the sense that (1.2), (1.3) and (1.4) are equalities.

**Example 1.3 ([Gro18]).** The metric $g = dt^2 + \cos(\frac{n t}{2})^\frac{4}{n} \tau$ where $\tau$ is the standard metric on the torus $\mathbb{T}^{n-1}$ has scalar curvature $R_g = n(n-1)$. The mean curvature of each $t$-level set is $H = -(n-1) \tan(\frac{\pi t}{2})$ and satisfies

$$n(n-1) + \frac{n}{n-1} H^2 + 2H' = 0.$$  

Zhu [Zhu21] established an optimal band width estimate for manifold with constant positive Ricci curvature lower bound via $\mu$-bubble. Zhu’s method was further applied by [Rä21] to prove the band width estimate...
under scalar curvature lower bound. The spinorial proof was developed by [Zei20a, CZ21b].

There are other two examples of rigid bands. The case \( k \equiv 0, \sigma < 0 \) is also due to Gromov.

**Example 1.4** ([Gro19]). The metric \( dt^2 + \sinh(\frac{nt}{2})^4 \tau \) on \( (0, \infty) \times \mathbb{T}^{n-1} \) has scalar curvature \(-n(n-1)\) and each \( t \)-level set has mean curvature \( H = (n - 1) \coth(\frac{nt}{2}) \) satisfying

\[
-n(n-1) + \frac{n}{n-1} H^2 + 2H' = 0.
\]

The metric is a rigid band with \( k = 0 \), \( \sigma = -2n(n-1) \).

**Example 1.5.** The metric \( g = dt^2 + (\frac{nt}{2})^4 \tau \) on \( (0, \infty) \times \mathbb{T}^{n-1} \) is of zero scalar curvature, and each \( t \)-level set has mean curvature \( H = \frac{2(n-1)}{nt} \) and satisfies

\[
R_g + \frac{n}{n-1} H^2 + 2H' = 0
\]

is a rigid band with \( k \equiv 0 \) and \( \sigma = 0 \).

The article is organized as follows:

In Section 2, we generalize the \( \mu \)-bubble to the spacetime settings, which we call a hypersurface of prescribed null expansion (see Definition 2.1) and use it to give a proof of Theorem 1.1 in dimensions less than eight. We also discuss some easy generalizations of Theorem 1.1 where the initial data set is not CMC. In Section 3, we restrict to dimension three only and use a perturbation of spacetime harmonic functions to give a simple proof. We can show rigidity in this case. In Section 4, we give the proof via the Dirac operator.

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## 2. Proof via Hypersurfaces of Prescribed Null Expansion

Gromov [Gro18, Gro21] studied stable \( \mu \)-bubbles in the study of scalar curvature, which leads to many interesting results in positive scalar curvature other than band width estimates. For example, positive mass theorem for asymptotically flat manifolds with arbitrary ends [LUY21], nonexistence of metrics with positive scalar curvature [CL20] on aspherical manifolds, see also [Gro21] for a survey of more results.

A \( \mu \)-bubble is just a hypersurface of prescribed mean curvature, and the use of \( \mu \)-bubble in scalar curvature problems dates back to minimal surface techniques used by Schoen-Yau settled the positive mass conjecture [SY79b] and [SY79a]. The analog of minimal surfaces in initial data sets is a marginally outer trapped surface (see [AMS05]).
A MOTS arises boundaries of blow up sets of Jang equation [SY81]. The existence of MOTS was proven by [AM09, Eic09]. It was applied to establish the spacetime positive mass theorem [EHLS16]. Motivated by the $\mu$-bubble, we propose the notion of a hypersurface of prescribed null expansion.

**Definition 2.1.** Given a smooth function $p$ on $(M, g, k)$, a hypersurface $\Sigma$ is called a hypersurface of prescribed null expansion if

$$\theta = p$$

along $\Sigma$. We say that $\Sigma$ is stable if there exists a vector field $X = \varphi \nu$ with nonzero $\varphi \geq 0$ such that the variation of $\theta - p$ is nonnegative along $X$, that is,

$$\delta_X(\theta - p) \geq 0. \quad (2.2)$$

The definition is motivated by both the stability of MOTS [AMS05] and the stability of the $\mu$-bubble. The case $p \equiv 0$ gives the definition of MOTS and the case $k \equiv 0$ recovers surfaces of prescribed mean curvature or in Gromov’s terminology a $\mu$-bubble. We can write down the stability (2.2) explicitly by calculating the variation of $\theta - p$.

**Lemma 2.2.** If $\Sigma$ is a stable hypersurface of prescribed null expansion $p$ in the initial data set $(M, g, k)$, then there exists a nonzero function $\varphi \geq 0$ such that $L \varphi \geq 0$, where $L$ is given by

$$L \varphi = -\Delta_\Sigma \varphi + 2\langle W, \nabla_\Sigma \varphi \rangle + (\text{div} W - W + Q)\varphi$$

$$- \frac{1}{2}(p^2 - 2p \text{tr} g k + 2\nu(p))\varphi,$$

where $W$ is the vector field tangential to $\Sigma$ which is dual to $k(\nu, \cdot)$,

$$Q = \frac{1}{2}R_\Sigma - \mu - J(\nu) - \frac{1}{2}|\chi|^2.$$

Here $\chi = h + k$ is called the null second fundamental form or the shear tensor. We call $L$ the stability operator.

**Proof.** From Definition 2.1, there exists a nonzero vector field $\varphi \nu$ such that $\delta_{\varphi \nu}(\theta - p) \geq 0$ with $\varphi \geq 0$. It suffices to prove

$$\delta_{\varphi \nu}(\theta - p) = L \varphi. \quad (2.4)$$

Now we calculate $\delta_{\varphi \nu}(\theta - p)$. The variation of $\delta_X \theta$ was already done in [AEM10, (10)]. They derived that

$$\delta_X \theta = -\Delta_\Sigma \varphi + 2\langle W, \nabla_\Sigma \varphi \rangle + (\text{div} W - W + Q)\varphi - \frac{1}{2}\theta(\theta - 2 \text{tr} g k).$$

With $X(p) = \varphi \langle \nabla p, \nu \rangle$ and $\theta = p$, we obtain $\delta_X(\theta - p) = L \varphi$ where $L \varphi$ is given by (2.3). $\square$

From Definition 2.1, a hypersurface $\Sigma$ of prescribed null expansion is stable if and only if the principal eigenvalue $\lambda_1$ of $L$ is nonnegative. The operator $L$ is not necessarily self-adjoint; hence its eigenvalues can have complex values. But it does have a real eigenvalue $\lambda_1$ and with least real part among all eigenvalues of $L$ by Krein-Rutman theorem (see [AMS05]). The eigenvalue
\( \lambda_1 \) is called the principal eigenvalue and its eigenfunction can be chosen strictly positive.

Let \( \chi^0 \) be the trace free part of the null second fundamental form, then

\[
|\chi|^2 = |\chi|^2 + \frac{1}{n-1} (\tr g \chi)^2 = |\chi|^2 + \frac{1}{n-1} p^2
\]

by (2.1). So the stability operator can be written in the following form:

\[
L \phi = -\Delta_{\Sigma} \phi + 2\langle W, \nabla_{\Sigma} \phi \rangle + (\div W - W) \phi
\]

\[
+ (\frac{1}{2} R_{\Sigma} - \frac{1}{2} |\chi|^2) \phi - [\mu + J(\nu) + \frac{1}{2} (\frac{n}{n-1} p^2 - 2p \tr g k + 2\nu(p)) \phi
\]

Because of the above and \([GS06]\), we introduce the relaxed energy dominant energy condition.

**Definition 2.3.** We say that an initial data set \((M, g, k)\) satisfies the relaxed dominant energy condition if there is a smooth function \(p\) defined on \(M\) such that

\[
(2.8) \quad \mu - |J| + \frac{1}{2} (\frac{n}{n-1} p^2 - 2p \tr g k - 2|\nabla p|) \geq 0.
\]

We have the following classifications of the topology of a stable hypersurfaces of prescribed null expansion.

**Theorem 2.4.** Assume that \(\Sigma\) is a stable hypersurface of prescribed null expansion \(p\) in an initial data set \((M, g, k)\) satisfying (2.8), then \(\Sigma\) is of positive Yamabe type unless \(\Sigma\) is Ricci flat, \(\chi^0\) vanishes and

\[
(2.9) \quad \mu + J(\nu) + \frac{1}{2} (\frac{n}{n-1} p^2 - 2p \tr g k + 2\nu(p)) \]

vanishes along \(\Sigma\).

**Proof.** The proof proceeds as \([GS06]\) using the relaxed dominant energy condition (2.8) and the stability (2.2). We refer the readers to \([GS06]\) for the details. \(\square\)

**Remark 2.5.** In fact, let \(\tilde{k} = k - \frac{1}{n-1} pg\), then a hypersurface \(\Sigma\) of prescribed null expansion is a MOTS in the new initial data set \((M, g, \tilde{k})\). The stability operator (2.3), (2.7), the energy condition (2.8), Theorem 2.4 and the existence of hypersurface with prescribed null expansion can be readily obtained by transferring to \((M, g, \tilde{k})\). We will use this fact about the existence in proving Theorem 1.1 via constructing a hypersurface of prescribed null expansion. See \([LLU22, \text{Section 6}]\).

**Lemma 2.6.** Let \((M, g, k)\) be as in Theorem 1.1, if the width of \(M\) is greater than \(\tau_+ - \tau_-\), then there exists a smooth function \(p\) such that

\[
(2.10) \quad \mu - |J| + \frac{1}{2} (\frac{n}{n-1} p^2 - 2p \tr g k - 2|\nabla p|) > 0
\]

in \(M\) and

\[
(2.11) \quad \theta_- - p(\partial_\tau - \partial_- M) < 0 \quad \text{and} \quad \theta_+ - p(\partial_\tau + \partial_+ M) > 0.
\]
Proof. By the assumption on the width of $M$, we fix a small $\varepsilon > 0$ such that $\text{width}(M,g) > t_+ - t_- + 2\varepsilon$. We use a lemma due to Zhu [Zhu21, Lemma 4.1] to get a surjective smooth function

$$\phi : (M, g) \to [t_- - \varepsilon, t_+ + \varepsilon]$$

with $\text{Lip}(\phi) < 1$ and such that $\phi^{-1}(t_- - \varepsilon) = \partial_- M$ and $\phi^{-1}(t_+ + \varepsilon) = \partial_+ M$. We define

$$p(x) = \eta \circ \phi(x), \quad x \in M$$

where $\eta = \eta(t)$ is the function satisfying the ordinary differential equation (5.1) and $\eta'(t) < 0$. Then

$$\frac{n}{n-1} p^2 - 2p \text{tr}_g k - 2|\nabla p| \geq \frac{n}{n-1} p^2 - 2p \text{tr}_g k - 2|\eta'| \text{Lip}(\phi)$$

$$= -\sigma - 2\eta' \circ \phi + 2|\eta' \circ \phi| \text{Lip}(\phi)$$

$$= -\sigma - 2\eta' \circ \phi(1 - \text{Lip}(\phi))$$

$$> -\sigma,$$

where the last inequality follows from $\eta' < 0$. So (2.10) is true because of (1.2). Furthermore, since the function $\eta$ is monotonically decreasing, we have the strict expansion bound (2.11) of the boundaries from the bounds (1.3).

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 when $\dim M \leq 7$. We argue by contradiction, assume that there exists a small $\varepsilon > 0$ such that $\text{width}(M,g) > t_+ - t_- + 2\varepsilon$. We can invoke Lemma 2.6 to construct an $p$ satisfying (2.10) and (2.11).

By the bounds (2.11), we can apply Eichmair [Eic09] (see Remark 2.5) to find a closed hypersurface $\Sigma$ of prescribed null expansion $p$ which is homologous to $\partial_\pm M$ and stable. Note that also $\partial_\pm M \cap \Sigma = \emptyset$. Because of (2.10) in Lemma 2.6, we can apply Theorem 2.4 to conclude that $\Sigma$ must admit a metric of positive scalar curvature. However, this contradicts the construction of $\Sigma$ which says that $\Sigma$ is topologically $\mathbb{T}^{n-1}$.

Remark 2.7. In fact, Theorems 1.1 can be generalized to the cases with a general warped product. See [Rä21]. The proofs are similar, and we leave the details to the readers. A general version is also available via the spinor methods [CZ21b].

We are not able to show a rigidity. We wish to apply [EGM21, Theorem 1.2] for the initial data set $(M, g, k - \frac{1}{n-1} \eta \circ \phi g)$ with $\phi$ constructed later in Lemma 3.3. However, $\eta \circ \phi$ is only Lipschitz. The paper [EGM21] used [AM09, Theorem 5.1] in which they required that $\eta \circ \phi$ can be differentiated at least twice.

The rigidity is true for $k = \frac{1}{n-1} \lambda g$ via spinorial techniques [CZ21b] because the dominant energy scalar $\mu - |J| \geq \frac{1}{2} \sigma$ is just a scalar curvature bound, the
bounds (1.3) are just bounds on the mean curvatures of \( \partial \pm M \) and everything reduces to the cases by [Gro18], [CZ21b] and [Rä21].

We mention how to generalize of the Theorems 1.1 when \( \text{tr}_g \kappa \) is not constant. Let \( \lambda = \inf_M \text{tr}_g \kappa \) and \( \Lambda = \sup_M \text{tr}_g \kappa \). For example, we require that \( \eta(t_{\pm}) > 0 \), then

\[
(2.13) \quad -2\eta \text{tr}_g \kappa \geq -2\eta \Lambda.
\]

We get a solution \( \eta \) of (1.1) by replacing \( \lambda \) by \( \Lambda \). Then we can state a slight generalization of Theorem 1.1. In the following theorem, we require that \( \sigma > \frac{n-1}{n} \Lambda^2 \).

**Theorem 2.8.** Given real constants \( \Lambda, \sigma \) with \( \sigma > \frac{n-1}{n} \Lambda^2 \), define

\[
\eta(t) = \frac{n-1}{n} \Lambda - \sqrt{\frac{n-1}{n} (\sigma - \frac{n-1}{n} \Lambda^2)} \tan \left( \frac{n}{2(n-1)} \sqrt{\frac{n-1}{n} (\sigma - \frac{n-1}{n} \Lambda^2)} t \right).
\]

Let \( M = T^{n-1} \times [-1, 1] \), assume on \( (M, g, k) \) that \( \text{tr}_g \kappa \leq \Lambda \),

\[
\mu - |J| \geq \frac{1}{2} \sigma
\]

and the null expansions at the boundaries \( \partial \pm M \) satisfy

\[
\theta_- \leq \eta(t_-), \quad \theta_+ \geq \eta(t_+),
\]

where \( t_\pm \) are two numbers with \( \eta(t_\pm) > 0 \). Then

\[
\text{width}(M, g) \leq t_+ - t_-.
\]

### 3. Proof via a Perturbation of the Spacetime Harmonic Function

Stern [Ste19] initiated an approach using a harmonic map to prove that there does not exist a metric of positive scalar curvature on \( T^3 \). The approach has been generalized to the spacetime by [BKKS22, HKK20] to show a spacetime positive mass theorem. See [BHK+21] more related results. We assume that \( M = T^2 \times [-1, 1] \) in this section. We recall the integral inequality.

**Lemma 3.1 ([HKK20]).** If

\[
(3.1) \quad \Delta_g u + \text{tr}_g \kappa |\nabla u| = 0
\]

on \( M \), then \( u \in C^{2, \alpha} \cap W^{3,p} \) and satisfies

\[
(3.2) \quad \int_{\partial \pm M} \pm \partial_{\nu \pm} |\nabla u| + k(\nabla u, \pm \nu) \geq \int_{\bar{u}}^u \int_{\Sigma_s} \frac{1}{2} |\nabla^2 u - k_{ij} |\nabla u| |^2 + \mu + J(\frac{\nabla u}{|\nabla u|}) - K_{\Sigma_s},
\]

where \( \Sigma_s \) are level sets of \( u \), \( K_{\Sigma_s} \) is its Gauss curvature, \( \bar{u} = \sup_M u \) and \( u = \inf_M u \).
Proof of Theorem 1.1 in dimension 3. Let \( p \) be the function constructed in Lemma 2.6, \( k = k - \frac{1}{2} pg \) and \( u \) be a solution of

\[
\Delta_g u + \operatorname{tr}_g \tilde{k} |\nabla u| = \Delta_g u + (\operatorname{tr}_g k - \frac{3}{2} p)|\nabla u| = 0 \text{ in } M
\]

and \( u = \pm 1 \) on \( \partial_{\pm} M \). Replacing the \( k, \mu, J \) by \( \tilde{k}, \tilde{\mu}, \tilde{J} \) in (3.2), by an easy calculation,

\[
\tilde{\mu} = \mu + \frac{1}{2}(\frac{3}{2} p^2 - 2p \operatorname{tr}_g k), \quad \tilde{J} = J + \nabla p.
\]

So

\[
\tilde{\mu} + J(\nabla u) = \mu + J(\nabla u) + \frac{1}{2} \left( \frac{3}{2} p^2 - 2p \operatorname{tr}_g k + 2(\nabla p, \nabla \nabla u) \right)
\]

\[
\geq \mu - |J| + \frac{1}{2} \left( \frac{3}{2} p^2 - 2p \operatorname{tr}_g k - 2|\nabla p| \right) > 0,
\]

by (2.10). Each regular level set of \( u \) in (3.3) is copies of topological torus, applying Gauss-Bonnet theorem on each regular level set, we have the right hand side of (3.2) is strictly positive.

We study now the boundary terms in (3.2). On \( \partial_{+} M, Du \) is nonzero and \( \frac{Du}{|Du|} \) points outward by the strong maximum principle, so \( \nabla u = |\nabla u| \nu_+ \).

Using (3.1) and the decomposition of the Laplacian \( \Delta_g \),

\[
( - \operatorname{tr}_g k + \frac{3}{2} p)|\nabla u| = \Delta_g u = H_+ (\nabla u, \nu_+) + (\nabla^2 u)(\nu_+, \nu_+).
\]

So

\[
\partial_{\nu_+} |\nabla u| + \tilde{k} (\nabla u, \nu_+)
\]

\[
= \frac{1}{|\nabla u|} (\nabla^2 u)(\nabla u, \nu_+) + |\nabla u| k(\nu_+, \nu_+) - \frac{1}{2} p|\nabla u|
\]

\[
= (\nabla^2 u)(\nu_+, \nu_+) + |\nabla u| k(\nu_+, \nu_+) - \frac{1}{2} p|\nabla u|
\]

\[
= - H_+ |\nabla u, \nu_+| + ( - \operatorname{tr}_g k + \frac{3}{2} p)|\nabla u| + |\nabla u| k(\nu_+, \nu_+) - \frac{1}{2} p|\nabla u|
\]

\[
= - H_+ |\nabla u| - \operatorname{tr}_{\nu_+} k|\nabla u| + p|\nabla u|
\]

\[
= ( - \theta_+ + p)|\nabla u|.
\]

Similarly on \( \partial_{-} M, Du = |\nabla u| \nu_- \) and

\[
\partial_{\nu_-} |\nabla u| + \tilde{k} (\nabla u, \nu_-) = - (\theta_- - p)|\nabla u|.
\]

So the left hand side of (3.2)

\[
\int_{\partial_{\pm} M} \pm \partial_{\nu_{\pm}} |\nabla u| + k(\nabla u, \pm \nu_{\pm}) = \int_{\partial_{\pm} M} \mp (\theta_{\pm} - p)|\nabla u| < 0
\]

is negative by (2.11). We have a contradiction. So we have the width of \((M, g, k)\) is less than \( t_+ - t_- \). \( \square \)

Remark 3.2. The band width estimate via the spacetime harmonic function of the case \( k = 0 \) is due to the work (in preparation) of S. Hirsch, D. Kazaras, M. Khuri and Y. Zhang.
Now we turn to study the rigidity.

**Lemma 3.3.** If \( \text{width}(M,g) = t_+ - t_- \), then there exists a function \( \phi : M \to [t_-, t_+] \) such that \( \phi(\partial \pm M) = t_\pm \) with \( \text{Lip}(\phi) \leq 1 \).

**Proof.** Let 
\[
\phi(x) = \min\{t_- + \text{dist}(x, \partial_- M), t_+\}.
\]
It is easy to see that \( \text{Lip}(\phi) \leq 1 \) and \( \phi(\partial_- M) = t_- \). Since \( \text{width}(M,g) = t_+ - t_- \), so the distance of any \( x \in \partial_+ M \) to the boundary \( \partial_- M \) is greater or equal to \( t_+ - t_- \). Therefore
\[
t_- + \text{dist}(x, \partial_- M) \geq t_- + t_+ - t_- = t_+,
\]
implying that \( \phi(\partial_+ M) = t_+ \).

**Remark 3.4.** We can also apply the Arzela-Ascoli lemma to a family of smooth functions constructed in [Zhu21, Lemma 4.1] to obtain a \( \phi \) with the same properties. It is not as explicit.

With the help of Lemma 3.3, we have a rigidity result.

**Theorem 3.5.** Assume that \( (M^3, g, k) \) satisfies the conditions in Theorem 1.1 and moreover \( \text{width}(M,g) = t_+ - t_- \). Then \( \phi \) constructed in Lemma 3.3 is
\[
\phi(x) = \text{dist}(x, \partial_- M) + t_-.
\]

**Proof.** Let \( p = \eta \circ \phi \), then the (2.8) and the bounds (1.3) on the null expansion of the boundaries are satisfied. Using the previous proof of Theorem 1.1 with a solution of (3.3) (which does not depend on these curvature bounds), we have the following integral inequality,
\[
0 \geq \int_{\partial \pm M} (\theta_\pm - p) |\nabla u|
\geq \int_{-1}^1 \int_{\Sigma_{\pm t}} \left( \frac{1}{2} |\nabla^2 u - k| \frac{\nabla u}{|\nabla u|} + \frac{1}{2} |p| \right)^2 + \mu + J(\nabla u, \frac{\nabla u}{|\nabla u|}) - K_{\Sigma_{\pm t}}
+ \int_{-1}^1 \int_{\Sigma_{\pm t}} \frac{1}{2} (3p^2 - 2p \text{tr} g \cdot k + 2(\nabla p, \frac{\nabla u}{|\nabla u|}))
\geq 0,
\]
by Gauss-Bonnet theorem on the level set \( \Sigma_{\pm t} \). Hence we get \( \theta_\pm = p = \eta(t_\pm) \) on \( \partial \pm M \), \( \mu + J(\nabla u, \frac{\nabla u}{|\nabla u|}) = \frac{1}{2} \sigma \) and
\[
\frac{3}{2} p^2 - 2p \text{tr} g \cdot k + 2(\nabla p, \frac{\nabla u}{|\nabla u|}) = -\sigma.
\]
Since \( \eta \) satisfies (5.1), we deduce that
\[
\eta' \circ \phi(\nabla \phi, \frac{\nabla u}{|\nabla u|}) = (\nabla p, \frac{\nabla u}{|\nabla u|}) = \eta' \circ \phi.
\]
By \( \eta' < 0 \), we have that \( \nabla \phi \) is parallel to \( \nabla u \) and \( |\nabla \phi| = 1 \) proving that \( \phi \) is a distance function, and because \( \phi(\partial_- M) = t_- \), we have \( \phi = t_- + \text{dist}(\cdot, \partial_- M) \).
Actually, we can assert that each level set of \( u \) coincide with a level set of \( \phi \) and each level set is a flat, stable torus of prescribed null expansion \( \eta \circ \phi \). For this, we refer the readers to the work of Tsang [Tsa21] on the Gromov dihedral rigidity of initial data sets.

4. Proof via the Dirac operator

In this section, following the method in [CZ21b], we give a proof of Theorem 1.1 via the Dirac operator.

4.1. Callias operator. In this subsection, we review basics of Callias operators; one can refer to [CZ21b].

The following definition of the Dirac bundle can be found in [LM89, Definition 5.2].

**Definition 4.1 (Dirac bundle).** A \((\mathbb{Z}_2\text{-graded})\) Dirac bundle over \( M \) is a Hermitian vector bundle \( S \to M \) with a metric connection \( \nabla : C^\infty(M, S) \to C^\infty(M, T^*M \otimes S) \) (endowed with a parallel and orthogonal \( \mathbb{Z}_2 \)-grading \( S = S^+ \oplus S^- \)) and a parallel bundle map \( c : T^*M \to \text{End}(S) \), called Clifford multiplication, such that \( c(\omega) \) is anti-self-adjoint (and odd), and \( c(\omega)^2 = -|\omega|^2 \) for all \( \omega \in T^*M \).

To define the Callias operator, we also need the following definitions of relative Dirac bundle and admissible potential, see [CZ21b, Definitions 2.2 and 3.1].

**Definition 4.2 (Relative Dirac bundle).** Let \( K \subset M^\circ \) be compact subset in the interior. A relative Dirac bundle with support \( K \) is a \( \mathbb{Z}_2 \)-graded Dirac bundle \( S \to M \) together with an odd, self-adjoint, parallel bundle involution \( \sigma \in C^\infty(M \setminus K, \text{End}(S)) \) satisfying \( c(\omega)\sigma = -\sigma c(\omega) \) for every \( \omega \in T^*M \mid_{M \setminus K} \) and such that \( \sigma \) admits a smooth extension to a bundle map on an open neighborhood of \( M \setminus K \).

**Definition 4.3 (Admissible potential).** A Lipschitz function \( \psi : M \to \mathbb{R} \) is called an admissible potential if \( \psi = 0 \) on \( K \) and there exists a compact set \( K \subset L \subset M \) such that \( \psi \) is equal to a nonzero constant on each component of \( M \setminus L \).

Let \( \psi : M \to \mathbb{R} \) be an admissible potential and \( S \to M \) be a relative Dirac bundle. Then we can define the Callias operator as follows, see [CZ21b, (3.1)].

**Definition 4.4 (Callias operator).** The Callias operator is defined as

\[
B_\psi := D + \psi \sigma,
\]

where \( \psi \) is an admissible potential, and \( D \) is the Dirac operator associated to the metric connection \( \nabla \).
Example 4.5 ([CZ21b, Example 2.6]). Let $(M, g)$ be an $n$-dimensional Riemannian spin band and let $S_M \to M$ be the associated complex spinor bundle endowed with the connection induced by the Levi-Civita connection. Let $E \to M$ be a Hermitian bundle equipped with a metric connection. Then $S := (S_M \otimes E) \oplus (S_M \otimes E)$ is a $\mathbb{Z}_2$-graded Dirac bundle with Clifford multiplication 

$$c := \begin{pmatrix} 0 & c_S \otimes \text{id}_E \\ c_S \otimes \text{id}_E & 0 \end{pmatrix}$$

Moreover, $S$ turns into a relative Dirac bundle with the involution

$$\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

globally defined on $M$. The Dirac operator on $S$ is given by

$$D = \begin{pmatrix} 0 & \hat{D}_E \\ \hat{D}_E & 0 \end{pmatrix}$$

where $\hat{D}_E : C^\infty(M, S_M \otimes E) \to C^\infty(M, S_M \otimes E)$ is the spinor Dirac operator on $(M, g)$ twist with the bundle $E$. The curvature term is

$$R = \text{scal}_g + R_E,$$

where $R_E = \sum_{i<j} c(e^i)c(e^j)(\text{id}\otimes R^E)$.

4.2. Spectral estimates. Let $(M, g, k)$ be an spin initial data set. Recall that

$$2\mu = R_g - |k|^2 + (\text{tr}g)^2, \quad J_i = (\text{div}g k - d(\text{tr}g))i.$$  

Let $E \to M$ be a Hermitian vector bundle with a metric connection, and let $\nabla$ be the induced metric connection on the Dirac bundle $S := (S_M \otimes E) \oplus (S_M \otimes E)$. According to Example 4.5, $S$ turns into a relative Dirac bundle with involution $\sigma$. The associated Dirac operator is given by

$$D = c(e^i)\nabla_{e^i}.$$ 

Now we define a new connection on $S$ by

$$\tilde{\nabla}_{e^i} = \nabla_{e^i} - \frac{1}{2}k_{ij}c(e^j)\sigma.$$ 

It is a connection on $S$ since $\frac{1}{2}k_{ij}c(e^j)\sigma \in A^1(M, \text{End}(S))$. Set

$$\tilde{D} = c(e^i)\tilde{\nabla}_{e^i}.$$ 

By a direct calculation, one has

**Proposition 4.6.** For any $u \in \Gamma(S)$, we have the following

(i) $\tilde{\nabla}_{e^i}u = \nabla_{e^i}u + \frac{1}{2}k_{ij}c(e^j)\sigma u$, $\tilde{D}u = Du + \frac{\text{tr}g}{2} \sigma u$,

(ii) $\nabla^*_u u = -\nabla_{e^i}u$, $\nabla^*_u u = -\nabla_{e^i}u - k_{ij}c(e^j)\sigma(u)$, $D^*u = Du$, $\tilde{D}^*u = Du$;

(iii) $D^2u = \nabla^* \nabla u + R u, \tilde{D}^2u = \tilde{\nabla}^* \tilde{\nabla} u + \tilde{R} u$, $\tilde{R} = \frac{1}{2}(\mu_E - J_i c(e^i)\sigma)$.

where $R = R_g + R_E$ and $\mu_E = \mu + 2R_E$. 


Remark 4.7. The proof of Proposition 4.6 is the same as the identities for the Dirac-Witten operator because the endomorphism $c(e^0)$ is precisely an involution in a relative Dirac bundle. One can refer to [HZ03, Section III] for the related calculations of the Dirac-Witten operators.

The Green’s formula for the Dirac operator $\tilde{D}$ is given by

$$\int_M \langle \tilde{D} u, v \rangle = \int_M \langle u, \tilde{D} v \rangle + \int_{\partial M} \langle u, c(\nu^b) v \rangle,$$

where $u, v \in \mathcal{C}_c^\infty(M, S)$. Here $\nu$ denotes the normal vector field pointing inward. For the connection $\nabla$, we have the following Green’s formula

$$\int_M \langle u, \nabla^* \nabla u \rangle = \int_M \langle \nabla u, \nabla u \rangle + \int_{\partial M} \langle u, \nabla_\nu u \rangle.$$

Hence

$$\int_M |\tilde{D} u|^2 = \int_M \langle u, \tilde{D}^2 u \rangle + \int_{\partial M} \langle u, c(\nu^b) \tilde{D} u \rangle$$

$$= \int_M \langle u, \tilde{\nabla}^* \tilde{\nabla} u \rangle + \int_M \langle u, \tilde{\nabla} u \rangle + \int_{\partial M} \langle u, c(\nu^b) \tilde{D} u \rangle$$

$$= \int_M |\tilde{\nabla} u|^2 + \int_M \langle u, \tilde{\nabla} u \rangle + \int_{\partial M} \langle u, c(\nu^b) \tilde{D} u + \nabla_\nu u \rangle.$$

The boundary Dirac operator is defined as follows

$$\mathcal{A} := \sum_{i=1}^{n-1} c^\partial(e^i) \nabla^\partial_{e_i},$$

where $e_n = -\nu$ and $c^\partial(e^i) = c(e^i)c(\nu^b)$ and $\nabla^\partial_{e_i} = \nabla_{e_i} + \frac{1}{2} c^\partial(\nabla_{e_i} \nu^b)$, see e.g. [CZ21b, Section 2]. Denote by $p$ the second fundamental form on $\partial M$, then $\operatorname{tr}_{\partial M} h = \sum_{i=1}^{n-1} \langle e_i, \nabla_{e_i}(-\nu) \rangle$. Then

$$\mathcal{A} = \frac{1}{2} \operatorname{tr}_{\partial M} h - c(\nu^b) D - \nabla_\nu$$

$$= \frac{1}{2} \operatorname{tr}_{\partial M} h - c(\nu^b)(\tilde{D} - \frac{\operatorname{tr}_{\partial M} k}{2} \sigma) - (\nabla_\nu + \frac{1}{2} c^\partial(\nabla_{e_i} \nu^b))$$

$$= \frac{1}{2} \operatorname{tr}_{\partial M} h + \frac{\operatorname{tr}_{\partial M} k}{2} c(\nu^b) \sigma - \frac{1}{2} k_{av} c(\nu^b) \sigma - \frac{1}{2} k_{av} c(e^a) \sigma - c(\nu^b) \tilde{D} + \nabla_\nu$$

$$= \frac{1}{2} \operatorname{tr}_{\partial M} h + \frac{\operatorname{tr}_{\partial M} k}{2} c(\nu^b) \sigma - \frac{1}{2} k_{av} c(e^a) \sigma - c(\nu^b) \tilde{D} + \nabla_\nu$$

which follows that

$$\int_M |\tilde{D} u|^2 = \int_M |\tilde{\nabla} u|^2 + \int_M \langle u, \tilde{\nabla} u \rangle$$

$$+ \int_{\partial M} \left( u, \left( \frac{1}{2} \operatorname{tr}_{\partial M} h + \frac{\operatorname{tr}_{\partial M} k}{2} c(\nu^b) \sigma - \frac{1}{2} k_{av} c(e^a) \sigma - A \right) u \right).$$
The Penrose operator of $\tilde{\nabla}$ is defined as
\[
\tilde{\mathcal{P}}_\xi u = \tilde{\nabla}_\xi u + \frac{1}{n} c(\xi^a) \tilde{D} u.
\]
The Friedrich inequality is then
\[
|\tilde{\nabla} u|^2 - \frac{1}{n} |\tilde{D} u|^2 = |\tilde{\mathcal{P}} u|^2 \geq 0,
\]
which follows that
\[
\int_M |\tilde{D} u|^2 = \frac{n}{n-1} \int_M |\tilde{\mathcal{P}} u|^2 + \frac{n}{n-1} \int_M \langle u, \tilde{\mathcal{R}} u \rangle
\]
\[
+ \frac{n}{n-1} \int_{\partial M} \left\langle u, \left( \frac{1}{2} \text{tr}_{\partial M} h + \frac{\text{tr}_{\partial M} k}{2} c(\nu^b) \sigma - \frac{1}{2} k_{\alpha\kappa\nu} c(e^a)\sigma - A \right) u \right\rangle.
\]
Let $\psi : M \to \mathbb{R}$ be a Lipschitz function such that $2\psi + \text{tr}_g k$ is an admissible potential, then
\[
\mathcal{B}_\psi = \tilde{D} + \psi \sigma = D + \left( \frac{1}{2} \text{tr}_g k + \psi \right) \sigma
\]
is a Callias operator. For any $u \in C^\infty_c(M, \mathbb{S})$, one has
\[
(4.2) \quad \int_M |\mathcal{B}_\psi u|^2 = \int_M (|\tilde{D} u|^2 + |\psi|^2 |u|^2)
\]
\[
+ \int_M \left\langle u, (c(\text{d}\psi)\sigma + \psi(\text{tr}_g k)) u \right\rangle + \int_{\partial M} \left\langle u, \psi(c(\nu^b)\sigma(u) \right\rangle.
\]
Thus
\[
\int_M |\mathcal{B}_\psi u|^2 = \frac{n}{n-1} \int_M |\tilde{\mathcal{P}} u|^2 + \frac{n}{n-1} \int_M \langle u, \tilde{\mathcal{R}} u \rangle
\]
\[
+ \frac{n}{n-1} \int_{\partial M} \left\langle u, \left( \frac{1}{2} \text{tr}_{\partial M} h + \frac{\text{tr}_{\partial M} k}{2} c(\nu^b) \sigma - \frac{1}{2} k_{\alpha\kappa\nu} c(e^a)\sigma - A \right) u \right\rangle
\]
\[
+ \int_M |\psi|^2 |u|^2 + \left\langle u, (c(\text{d}\psi)\sigma + \psi(\text{tr}_g k)) u \right\rangle + \int_{\partial M} \langle u, \psi(c(\nu^b)\sigma(u) \rangle.
\]
Using the fact $|\tilde{\mathcal{P}} u| \geq 0$ and $\langle u, c(\text{d}\psi)\sigma u \rangle \leq |\text{d}\psi||u|^2$, so
\[
\int_M |\mathcal{B}_\psi u|^2 \geq \int_M (|\psi|^2 - |\text{d}\psi| + \psi(\text{tr}_g k)) |u|^2 + \frac{n}{n-1} \int_M \langle u, \tilde{\mathcal{R}} u \rangle
\]
\[
+ \frac{n}{n-1} \int_{\partial M} \left\langle u, \left( \frac{1}{2} \text{tr}_{\partial M} h + \frac{\text{tr}_{\partial M} k}{2} + \frac{n-1}{n} \psi \right) c(\nu^b) \sigma - \frac{1}{2} k_{\alpha\kappa\nu} c(e^a)\sigma - A \right) u \right\rangle.
\]
Let $s : \partial M \to \{ \pm 1 \}$ be a choice of signs, and we consider the following boundary condition
\[
\{ u \in C^\infty_c(M, \mathbb{S}) | \chi(u|_{\partial M}) = u|_{\partial M} \},
\]
where $\chi := sc(\nu^b)\sigma$ is the boundary chirality. Then $\chi^2 = \text{Id}$ and
\[
\chi A = -\chi A, \quad \chi(c(e^a)\sigma) = -(c(e^a)\sigma)\chi.
\]
Hence
\[
\langle u, Au \rangle = 0 = \langle u, c(e^a)\sigma u \rangle.
\]
Under this boundary condition, one has
\[
\int_M |\mathcal{B}_\psi u|^2 \geq \int_M (|\psi|^2 - |d\psi| + \psi \text{tr}_g k)|u|^2 + \frac{n}{n-1} \int_M \left\langle u, \tilde{\mathcal{R}} u \right\rangle \\
+ \frac{n}{n-1} \int_{\partial M} \left( u, \left( \frac{1}{2} \text{tr}_{\partial M} h + \left( \frac{\text{tr}_{\partial M} k}{2} + \frac{n-1}{n} \psi \right) c(\nu^\flat) \right) u \right).
\]

Since
\[
\left\langle u, \tilde{\mathcal{R}} u \right\rangle \geq \frac{1}{2} (\mu - |J|)|u|^2 - \|R^E\|_\infty |u|^2,
\]
where \(\|R^E\|_\infty := \sup_{\|u\|=1} \{ \langle R^E u, u \rangle \} \), so
\[
\int_M |\mathcal{B}_\psi u|^2 \geq \frac{n}{n-1} \int_M \left( \mu - |J| + \frac{1}{2} \left( \frac{n}{n-1} \tilde{\psi}^2 - 2\tilde{\psi} \text{tr}_g k - 2|d\tilde{\psi}| \right) - 2\|R^E\|_\infty \right) |u|^2 \\
+ \frac{n}{n-1} \int_{\partial M} \left( \left( \frac{\text{tr}_{\partial M} h + \left( \frac{\text{tr}_{\partial M} k}{2} + \frac{n-1}{n} \psi \right) s \right) u \right).
\]

Now we set \(\tilde{\psi} = -\frac{2(n-1)}{n}\psi\) and \(\theta_{\pm} = \pm \text{tr}_{\partial M} h + \text{tr}_{\partial M} k\) and \(s = \pm 1\) on \(\partial \pm M\), thus
(4.3)
\[
\int_M |\mathcal{B}_\psi u|^2 \geq \frac{n}{2(n-1)} \int_M \left( \mu - |J| + \frac{1}{2} \left( \frac{n}{n-1} \tilde{\psi}^2 - 2\tilde{\psi} \text{tr}_g k + 2|d\tilde{\psi}| - 2\|R^E\|_\infty \right) \right) |u|^2 \\
+ \frac{n}{2(n-1)} \int_{\partial_+ M} (\theta_+ - \tilde{\psi}) |u|^2 + \frac{n}{2(n-1)} \int_{\partial_- M} (\theta_- + \tilde{\psi}) |u|^2.
\]

**Remark 4.8.** For \(k = 0\), (4.3) is reduced to [CZ21b, Theorem 4.3 and (4.2)].

Now we denote
\[
H^1_s(M,S) := \{ u \in H^1_{loc}(M,S) \cap L^2(M,S) | \tilde{D} u \in L^2(M,S) , \chi(u|_{\partial M}) = u|_{\partial M} \}.
\]

Then we get

**Theorem 4.9.** Let \((M,g,k)\) be a CMC initial data set satisfying the conditions in Theorem 1.1. For any \(\epsilon > 0\), if there exists a Hermitian bundle \(E\) such that \(\|R^E\|_\infty < \epsilon\) and \(H^1_s(M,S) \cap \ker \mathcal{B}_\psi \neq \{0\}\), then
\[
\text{width}(M,g) \leq t_+ - t_-.
\]

**Proof.** By Lemma 2.6, if \(\text{width}(M,g) > t_+ - t_-\), we can find \(\tilde{\psi}\) such that
\[
\mu - |J| + \frac{1}{2} \left( \frac{n}{n-1} \tilde{\psi}^2 - 2\tilde{\psi} \text{tr}_g k + 2|d\tilde{\psi}| - 2\|R^E\|_\infty \right) > \epsilon \geq 2\|R^E\|_\infty
\]
and
\[
\theta_- - \tilde{\psi} < 0 \quad \text{on} \quad \partial_- M
\]
\[
\theta_+ - \tilde{\psi} > 0 \quad \text{on} \quad \partial_+ M
\]
which follows that
\[ 0 = \int_M |B_\psi u|^2 > 0, \]
contradiction. The proof is complete. \(\square\)

Remark 4.10. We can choose a suitable \(\tilde{\psi}\) such that Lemma 2.6 holds and
\[-\frac{n}{n-1}\tilde{\psi} + \lambda \text{ is an admissible potential.} \]
Note that
\[ 2\psi + \text{tr}_g (k) = -\frac{n}{n-1}\tilde{\psi} + \lambda = -\frac{n}{n-1}\eta \circ \phi + \lambda, \]
where \(\phi : (M, g) \to [t_-, t_+ + \epsilon]\) and \(\text{width}(M, g) > t_+ - t_- + 2\epsilon, \phi^{-1}(t_- - \epsilon) = \partial_- M\) and \(\phi^{-1}(t_- + \epsilon) = \partial_+ M\), see Lemma 2.6 and \(\eta' < 0\). The Lipschitz function \(\phi\) can be chosen such that \(\phi\) is constant near \(\partial_{\pm} M\). In fact, let

\[ \phi(p) = \begin{cases} t_- - \epsilon & p \in (M \setminus M')_- \\ \phi' & p \in M' \\ t_+ + \epsilon & p \in (M \setminus M')_+. \end{cases} \]

Then \(\phi : (M, g) \to [t_- - \epsilon, t_+ + \epsilon]\) is a Lipschitz function such that \(\partial_- M \subset \phi^{-1}(t_- - \epsilon)\) and \(\partial_+ M \subset \phi^{-1}(t_- + \epsilon)\), which is constant near \(\partial_{\pm} M\). By taking \((M \setminus M')_{\pm}\) small enough, so \(\phi\) satisfies Lemma 2.6.

Therefore, \(2\psi + \text{tr}_g (k)\) is a nonzero constant for some small \(\epsilon > 0\). Hence \(B_\psi = \tilde{D} + \psi \sigma\) is a Callias operator.

Remark 4.11. Inspired by the rigidity theorem [CZ21b, Theorem 8.3], we assume that \((M, g, k)\) satisfies the following conditions:

(i) \(\mu - |J| - 2\|R^E\|_\infty \geq \frac{1}{2}\sigma\);
(ii) \(x : M \to [t_-, t_+]\) is a smooth width function for some \(t_-, t_+ \in \mathbb{R}\);
(iii) \(\theta_+ \geq \eta(t_+)\) and \(\theta_- \leq \eta(t_-)\);
(iv) \(\ker B_{\psi,s} \neq \{0\}\).
For any $u \in \ker B_{\psi,s}$ with $u \neq 0$, we have
\[
0 = \int_M |B_{\psi}u|^2 \\
\geq \frac{n}{n-1} \int_M |\tilde{P}u|^2 \\
+ \frac{n}{2(n-1)} \int_M \langle -\tilde{\psi}' \rangle \langle u, (c(dx)\sigma + 1)u \rangle \\
+ \frac{n}{2(n-1)} \int_M \left( \mu - |J| - 2 \|R^E\|_\infty - \frac{1}{2} \sigma \right) |u|^2 \\
+ \frac{n}{2(n-1)} \int_M \left( \frac{1}{2} \sigma + \frac{n}{2(n-1)} \tilde{\psi}^2 + \tilde{\psi}' - \tilde{\psi}\tr_g k \right) |u|^2 \\
+ \frac{n}{2(n-1)} \int_{\partial^+ M} (\theta_+ - \tilde{\psi}) |u|^2 + \frac{n}{2(n-1)} \int_{\partial^- M} (\theta_- + \tilde{\psi}) |u|^2 \\
\geq 0
\]
by the above assumptions. We obtain
\[
\tilde{P}_\xi u = \tilde{\nabla}_\xi u + \frac{1}{n} c(\xi^b) \tilde{D}u = 0
\]
on $M$ and
\[
c(dx)\sigma u = -u
\]
almost everywhere. Since $B_{\psi}u = 0$, so
\[
(\tilde{\nabla}_\xi - \frac{\psi}{n} c(\xi^b))\sigma u = 0.
\]
Note that the boundary elliptic regularity implies that $u$ is smooth. If $u$ vanishes at some point $p_0 \in M$, for any point $p_1 \in M$, let $\gamma = \gamma(t)$ be a smooth path in $M$ connected $p_0$ and $p_1$, then
\[
(\tilde{\nabla}_{\gamma(t)} - \frac{\psi}{n} c(\gamma'(t))^b)\sigma u = 0.
\]
which is a linear ordinary differential equation, and has the unique solution $u \equiv 0$ along $\gamma(t)$. Thus $u \equiv 0$ on $M$ since $M$ is connected, which contradicts to $u \neq 0$. Hence $|u| > 0$. The argument to deduce $|u| > 0$ also can be found in the proof of [CZ21a, Lemma 2.6]. Thus
\[
\mu - |J| - 2 \|R^E\|_\infty = \frac{1}{2} \sigma, \quad \theta_\pm = \eta(t_\pm).
\]

4.3. **Band width estimates.** In this subsection, we give some examples of bands satisfying the assumptions of Theorem 4.9.

4.3.1. **Infinite vertical $\hat{A}$-area.** The following definition of infinite vertical $\hat{A}$-area can be found in [CZ21b, Definition 7.3].
Definition 4.12 (Infinite vertical $\tilde{A}$-area). A band $M$ is said to have infinite vertical $\tilde{A}$-area, if for every $\varepsilon > 0$, there exists a Hermitian vector bundle $E \to M$ such that $\|R^E\|_\infty < \varepsilon$ and such that we have
$$\int_{\partial_- M} \tilde{A}(\partial_- M) \wedge \text{ch}(E|_{\partial_- M}) \neq 0.$$

Remark 4.13. If $X$ is a closed spin manifold (even dimensional) with infinite $\tilde{A}$-area, then $M = X \times [-1, 1]$ has infinite vertical $\tilde{A}$-area. An important class of examples consists of even-dimensional compactly enlargeable manifolds, for example, $X = \mathbb{T}^{2m}$. On the other hand, if $M$ is a $\tilde{A}$-ovtorical band, then $M$ also has infinite vertical $\tilde{A}$-area.

From Theorem 4.9, we obtain

Theorem 4.14. Let $(M, g, k)$ be a CMC initial data set satisfying the conditions in Theorem 1.1. If $(M, g)$ is a spin band of infinite vertical $\tilde{A}$-area, then
$$\text{width}(M, g) \leq t_+ - t_-.$$

Proof. From [CZ21b, Corollary 3.10], one has
$$\text{ind}(B_{\psi,s}) = \text{ind}(\mathcal{D}_{\partial_- M, E|_{\partial_- M}}) = \int_{\partial_- M} \tilde{A}(\partial_- M) \wedge \text{ch}(E|_{\partial_- M}) \neq 0,$$
which follows that
$$\ker B_{\psi,s} = H^1_s(M, S) \cap \ker B_{\psi} \neq \{0\}.$$

Remark 4.15. For $k = 0$ and $\sigma = n(n-1)$, Theorem 4.14 is exactly [CZ21b, Theorem 7.6].

4.3.2. $\mathcal{K}O$-bands. The following definition of $\mathcal{K}O$ band can be found in [Zei20b, Page 9].

Definition 4.16 ($\mathcal{K}O$-band). A band $M \in \mathcal{K}O$ is called a $\mathcal{K}O$ band if $M$ is spin and admits a flat bundle $E \to M$ of finitely generated projective Hilbert-$A$-modules for some unital Real $C^*$-algebra $A$ such that the twisted Dirac operator on $\partial \pm M$ has non-vanishing index $\text{ind}(\mathcal{D}_{\partial_- M, E|_{\partial_- M}}) \neq 0 \in K\text{O}_{n-1}(A)$.

Remark 4.17. In particular, all overtorical bands and $\tilde{A}$-ovtorical bands are $\mathcal{K}O$ bands, see [Zei20b, Proposition 5.2]. If $M = X \times [-1, 1]$ where $X$ is a closed spin manifold with nonvanishing Rosenberg index $\alpha(X)$, then $M$ is a $\mathcal{K}O$-band, see [Zei20a, Section 3].

Let $(M, g)$ be an $n$-dimensional Riemannian spin manifold. Let $A$ be a Real unital $C^*$-algebra and let $(E, \nabla^E)$ be a bundle of finitely generated projective Hilbert $A$-modules endowed with a metric connection. Then one can also define the generalized Callias-type operator $B_{\psi,s}$, see [Cec20, Section...
Similarly, we can obtain the estimate (4.3), and Theorem 4.9 holds for this flat bundle $E$.

**Theorem 4.18.** Let $(M, g, k)$ be a CMC initial data set satisfying the conditions in Theorem 1.1. If $M$ is a KO-band, then
\[
\text{width}(M, g) \leq t_+ - t_-.
\]

*Proof.* If $M$ is a KO-band, then there exists a flat bundle $E \to M$ such that
\[
\text{ind}(\mathcal{D}_{\partial_+M,E|\partial_+M}) \neq 0.
\]
Hence
\[
\text{ind}(\mathcal{B}_{\psi,s}) = \text{ind}(\mathcal{D}_{\partial_+M,E|\partial_+M}) \neq 0.
\]
By Theorem 4.9, the proof is complete. \qed

In particular, we obtain a proof of Theorem 1.1 via the Dirac operator for bands $T^{n-1} \times [-1, 1]$ of all dimensions.

### 5. Appendix

In this section, we will discuss the solutions $\eta = \eta(t)$ of the ordinary differential equation
\[
\sigma + \frac{n-1}{n-1} \eta^2 - 2\eta \lambda + 2\eta' = 0, \quad \eta'(t) < 0.
\]
Let $p(x) = \frac{n-1}{n-1} x^2 - 2\lambda x + \sigma$, the solution to (5.1) depends on the sign of $\sigma - \frac{n-1}{n} \lambda^2$. In what follows, $c$ is a constant. We divide into three cases:

**Case 1:** when $\sigma = \frac{n-1}{n} \lambda^2$, $p$ has only one root. The ordinary differential equation (5.1) reduces to
\[
\frac{n-1}{n-1} (\eta - \frac{n-1}{n} \lambda)^2 + 2\eta' = 0.
\]
The solution of (5.1) is
\[
\eta(t) = \frac{n-1}{n} \lambda + \frac{2(n-1)}{n(t-c)}
\]
on \{t > c\} or \{t < c\}.

**Case 2:** when $\sigma > \frac{n-1}{n} \lambda^2$, it is easy to see that
\[
p(x) = \frac{n-1}{n-1} (x - \frac{n-1}{n} \lambda)^2 + (\sigma - \frac{n-1}{n} \lambda^2)
\]
has no real root. We solve that
\[
\eta(t) = \frac{n-1}{n} \lambda - \frac{n-1}{n} \left(\sigma - \frac{n-1}{n} \lambda^2\right) \tan \left(\frac{n}{2(n-1)} \sqrt{n-1}(\sigma - \frac{n-1}{n} \lambda^2)(t - c)\right)
\]
with $t$ satisfying
\[
\left|\frac{n}{2(n-1)} \sqrt{n-1}(\sigma - \frac{n-1}{n} \lambda^2)(t - c)\right| < \frac{\pi}{2}.
\]

**Case 3:** when $\sigma < \frac{n-1}{n} \lambda^2$, $p$ has two real roots $x_{\pm}$. Denote
\[
a = \sqrt{\frac{n-1}{n}(\frac{n-1}{n} \lambda^2 - \sigma)}.
\]
Then
\[ x_\pm = \frac{n-1}{n} \lambda \pm a. \]
So (5.1) turns into
\[ \frac{n}{n-1} (\eta - \frac{n-1}{n} \lambda + a) (\eta - \frac{n-1}{n} \lambda - a) = -2 \eta' = -2(\eta - \frac{n-1}{n} \lambda)'. \]
We get the solution
\[ \eta(t) = \frac{n-1}{n} \lambda + a \coth\left( \frac{n}{2(n-1)} a(t - c) \right) \]
which is
(5.4)
\[ \eta(t) = \frac{n-1}{n} \lambda + \sqrt{\frac{n-1}{n} \left( \frac{n-1}{n} \lambda^2 - \sigma \right)} \coth\left( \frac{n}{2(n-1)} \sqrt{\frac{n-1}{n} \left( \frac{n-1}{n} \lambda^2 - \sigma \right)} (t - c) \right) \]
with \( t > c \) or \( t < c \).

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