Homomesy in products of two chains

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Abstract

Many invertible actions \( \tau \) on a set \( S \) of combinatorial objects, along with a natural statistic \( f \) on \( S \), exhibit the following property which we herein coin homomesy: the average of \( f \) over each \( \tau \)-orbit in \( S \) is the same as the average of \( f \) over the whole set \( S \). This phenomenon was first noticed by Panyushev in 2007 in the context of the rowmotion map acting on the set of antichains of a root poset; Armstrong, Stump, and Thomas proved Panyushev’s conjecture in 2011. We describe a theoretical framework for results of this kind that seems to apply more broadly, giving examples in a variety of contexts. These include linear actions on vector spaces, sandpile dynamics, Suter’s action on certain subposets of Young’s Lattice, and the rowmotion and promotion operators acting on certain posets. We give a detailed description of the latter situation for products of two chains.

Keywords: antichains, homomesy, orbit, order ideals, Panyushev complementation, poset, product of chains, promotion, rowmotion, sandpile, Suter’s symmetry toggle group.

1 Introduction

We begin with the definition of our main unifying concept, and supporting nomenclature.

Definition 1. Given a set \( S \), an invertible map \( \tau \) from \( S \) to itself such that each \( \tau \)-orbit is finite, and a function (or “statistic”) \( f : S \to K \) taking values in some field \( K \) of characteristic zero, we say the triple \((S, \tau, f)\) exhibits homomesy iff there exists a constant \( c \in K \) such that for every \( \tau \)-orbit \( O \subset S \)

\[
\frac{1}{\#O} \sum_{x \in O} f(x) = c. \tag{1}
\]

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In this situation we say that the function \( f : S \to K \) is **homomesic**\(^1\) under the action of \( \tau \) on \( S \), or more specifically **\( c \)-mesic**.

Typically our statistic \( f \) will be a natural measure (e.g., cardinality) and our set \( S \) will be a **finite** collection of combinatorial objects (e.g., order ideals in a poset), in which case homomesy can be restated equivalently as all orbit-averages being equal to the global average:

\[
\frac{1}{\#O} \sum_{x \in O} f(x) = \frac{1}{\#S} \sum_{x \in S} f(x).
\] (2)

We will also apply the term homomesy more broadly to include the case that the statistic \( f \) takes values in a vector space over a field of characteristic 0 (as in sections 2.4 and 2.6).

We have found many instances of (2) where the actions \( \tau \) and the statistics \( f \) are natural ones. Many (but far from all) situations that support examples of homomesy also support examples of the cyclic sieving phenomenon of Reiner, Stanton, and White \([RSW04]\) and more exploration of the linkages and differences is certainly in order. At the stated level of generality this notion appears to be new, but specific instances can be found in earlier literature. In particular, in 2007, Panyushev \([Pan08]\) conjectured and in 2011, Armstrong, Stump, and Thomas \([AST11]\) proved that if \( S \) is the set of antichains in the root poset of a finite Weyl group, \( \Phi \) is the operation variously called the Brouwer-Schrijver map \([BS74]\), the Fon-der-Flaass map \([Fon93, CF95]\), the reverse map \([Pan08]\), Panyushev complementation \([AST11]\), and rowmotion \([SW12]\), and \( f(A) \) is the cardinality of the antichain \( A \), then \((S, \Phi, f)\) satisfies (2).

Examples of homomesy are given starting with Section 2. These include the following triples \((S, \tau, f)\), each of which will be explained in more detail in what follows.

1. Reversal of permutations with the statistic that counts inversions;
2. Cyclic rotation of bitstrings with the \( \{0, 1\} \)-function that indicates whether a string satisfies the ballot condition;
3. Cyclic rotation of bitstrings with the statistic that counts the number of (multiset) inversions in the string;
4. Rowmotion acting on the set of order ideals of \([a] \times [b]\) with the cardinality statistic;
5. Rowmotion acting on the set of antichains of \([a] \times [b]\) with the cardinality statistic;
6. Promotion (in the sense of \([SW12]\)) acting on the set of order ideals of \([a] \times [b]\) with the cardinality statistic;
7. Suter’s map on Young diagrams with a weighted cardinality statistic.

Our main results for this paper involve studying the action of this rowmotion operator and also the (Striker-Williams) promotion operator on the poset \( P = [a] \times [b] \), the product of two chains. (Here and throughout this article we use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\)

\(^1\)Greek for “same middle”
and the associated $n$-element poset.) We show that the statistic $f := \# A$, the size of
the antichain, is homomesic with respect to the promotion operator, and that the statistic
$f = \# I(A)$, the size of the corresponding order ideal, is homomesic with respect to both the
promotion and rowmotion operators.

Although these results are of intrinsic interest, we think the main contribution of the
dpaper is its identification of homomesy as a phenomenon that (as we expect future articles
to show) occurs quite widely. Within any linear space of functions on $S$, the functions that
are homomesic under $\tau$, like the functions that are invariant under $\tau$, form a subspace. There
is a loose sense in which the notions of invariance and homomesy (or, more strictly speaking,
$0$-mesy) are complementary; an extremely clean case of this complementarity is outlined
in subsection 2.4 and a related complementarity (in the context of continuous rather than
discrete orbits) is sketched in subsection 2.5. This article gives a general overview of the
broader picture as well as a few specific examples done in more detail for the operators of
promotion and rowmotion on the poset $[a] \times [b]$.

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2 Examples of Homomesy

Here we give a variety of examples of homomesy in combinatorics, the first two of which
long predate the general notion of homomesy; we also give non-combinatorial examples that
establish links with different branches of mathematics. For examples of homomesy associated
with piecewise-linear maps and birational maps, see [EP13].

2.1 Inversions in permutations

Let $S$ be the set of permutations of $1, 2, \ldots, n$, let $\tau$ send $(\pi_1, \ldots, \pi_n)$ to its reversal $(\pi_n, \ldots, \pi_1)$,
and let $f(\pi)$ be the number of inversions in $\pi$. Since $\tau^2$ is the identity, and since $f(\pi) +
f(\tau(\pi)) = n(n - 1)/2$, $f$ is $c$-mesic under the action of $\tau$, where $c = n(n - 1)/4$.

2.2 Ballot theorems

Let $S$ be the set of strings $(s_1, s_2, \ldots, s_n)$ of length $n = a + b$, consisting of $a$ terms equal
to $-1$ and $b$ terms equal to $+1$, with $a, b \geq 1$; we think of each such string as an order for
counting $n$ ballots in a two-way election, $a$ of which are for candidate A and $b$ of which are for candidate B. If $a < b$, then candidate B will be deemed the winner once all $a + b$ ballots have been counted, and we ask for the probability that at every stage in the counting of the ballots, candidate B is in the lead. This probability is the same as the expected value of $f(s)$, where $f(s)$ is 1 if $s_1 + \cdots + s_i > 0$ for all $1 \leq i \leq n$ and is 0 otherwise, and where $s$ is chosen uniformly at random from $S$. Bertrand’s Theorem states that this probability is $(b - a)/(a + b)$.

We can recast the famous “cycle lemma” proof of Bertrand’s Theorem, due to Dvoretzky and Motzkin [DM47], in our framework as follows: Let $\tau := C_L : S \to S$ be the leftward cyclic shift operator that sends $(s_1, s_2, s_3, \ldots, s_n)$ to $(s_2, s_3, \ldots, s_n, s_1)$. Then over any orbit $O$ one has

$$\sum_{s \in O} f(s) = b - a$$

(see [R07] for details), implying

$$\frac{1}{\#O} \sum_{s \in O} f(s) = \frac{b - a}{a + b} = \frac{1}{\#S} \sum_{s \in S} f(s).$$

Dvoretzky and Motzkin used this method to prove a more general version of Bertrand’s Theorem due to Barbier [B1887]: if $b > ra$ for some positive integer $r$, then the probability that throughout the counting candidate B always has more than $r$ times as many votes as candidate A is $(b - ra)/(a + b)$.

### 2.3 Inversions in two-element multiset permutations

Let $S$ be the same set of words as in the previous subsection, with $f(s) := \text{inv}(s) := \#\{i < j : s_i > s_j\}$. For fixed $i < j$, the number of $s$ in $S$ with $s_i > s_j$ (i.e., with $s_i = 1$ and $s_j = 0$) is $\binom{n - 2}{a - 1}$, so the probability that an $s$ chosen uniformly at random from $S$ satisfies $s_i < s_j$ is $\binom{n - 2}{a - 1}/\binom{n}{a} = \frac{ab}{n(n-1)}$, so by additivity of expectation, the expected value of $\text{inv}(s)$ is

$$\sum_{i < j} \frac{ab}{n(n-1)} = \frac{n(n-1)}{2} \frac{ab}{n(n-1)} = ab/2.$$

**Proposition 2.** Let $\tau$ be the left-shift $C_L$ on $S$ and $f(s) := \text{inv}(s)$ as above. Then over each orbit $O$ we have

$$\frac{1}{\#O} \sum_{s \in O} f(s) = \frac{ab}{2} = \frac{1}{\#S} \sum_{s \in S} f(s).$$

In other words, the inversion statistic is homomesic under the action of cyclically rotating bitstrings.

One way to prove Proposition 2 is to write

$$f(s) = \sum_{i < j} (1 + s_i)(1 - s_j)/4 = \frac{1}{4} \sum_{i < j} (1 + s_i - s_j - s_is_j).$$

For all $s$, $\sum_{i < j} 1$ is $\frac{n(n-1)}{2}$ and $\sum_{i < j} s_is_j$ is

$$\left( \frac{a(a - 1)}{2} + \frac{b(b - 1)}{2} \right) (+1) + (ab) (-1) = \frac{n(n - 1)}{2} - 2ab.$$
so
\[ f(s) = \frac{1}{4} \left( 2ab + \sum_{i<j} (s_i - s_j) \right). \]

Since the average value of \( \sum_{i<j} (s_i - s_j) \) over each orbit is 0, the average value of \( f \) over each orbit is \( ab/2 \).

Proposition 2 it also follows from one of our results from Section 3. More specifically, we have a bijection between order-ideals in the poset \( P = [a] \times [b] \) and strings consisting of \( a \) 1's and \( b \) -1's. Then promotion on \( J(P) \) is equivariant with leftward cyclic shift on strings, and the cardinality of an order ideal is equal to the number of inversions in the associated string. Theorem 14 then yields the claimed result.

In the specific case \( a = b = 2 \), the six-element set \( S \) decomposes into two orbits, shown in Figure 1. (Here we recode the elements of \( S \) as ordinary bit-strings, representing +1 and -1 by 1 and 0, respectively.) As frequently happens, not all orbits are the same size; however, one could also view the orbit of size 2 as a multiset orbit of size 4, cycling through the same set of elements twice. This perspective, where we view all orbits as multiset orbits of the same size, facilitates the discussion of certain comparisons.

### 2.4 Linear actions on vector spaces

Let \( V \) be a (not necessarily finite-dimensional) vector space over a field \( K \) of characteristic zero, and define \( f(v) = v \) (that is, our “statistic” is just the identity function). Let \( T : V \to V \) be a linear map such that \( T^n = I \) (the identity map on \( V \)) for some fixed \( n \geq 1 \). Say \( v \) is invariant under \( T \) if \( Tv = v \) and 0-mesic under \( T \) if \( (v + Tv + \cdots + T^{n-1}v)/n = 0 \). Every \( v \in V \) can be written uniquely as the sum of an invariant vector \( \hat{v} \) and a 0-mesic vector \( \hat{v} \). (One suggestive way of paraphrasing the above is: Every element of the kernel of \( I - T \) can be written uniquely as the sum of an element of the kernel of \( I - T \) and an element of the kernel of \( I + T + T^2 + \cdots + T^{n-1} \).) For, one can check that \( v = \hat{v} + \hat{v} \) is such a decomposition, with \( \hat{v} = (v + Tv + \cdots + T^{n-1}v)/n \) and \( \hat{v} = v - \hat{v} \), and no other such decomposition is possible because that would yield a nonzero vector that is both invariant and 0-mesic, which does not exist. In representation-theoretic terms, we are applying symmetrization to \( v \) to extract from it the invariant component \( \hat{v} \)

\begin{align*}
\ldots & \xrightarrow{f} 0011 \xrightarrow{f} 0110 \xrightarrow{f} 1100 \xrightarrow{f} 1001 \xrightarrow{f} \ldots \\
0 & \quad 2 \quad 4 \quad 2 \\
\ldots & \xrightarrow{f} 1010 \xrightarrow{f} 0101 \xrightarrow{f} \ldots \\
3 & \quad 1
\end{align*}

Figure 1: The two orbits of the action of the cyclic shift on binary strings consisting of two 0’s and two 1’s. The average value of the inversion statistic is \((0 + 2 + 4 + 2)/4 = 2\) on the orbit of size 4 and \((3 + 1)/2 = 2\) on the orbit of size 2.
associated with the trivial representation of the cyclic group, and the homomesic (0-mesic) component ˆv consists of everything else.

This picture relates more directly to our earlier definition if we use the dual space V* of linear functionals on V as the set of statistics on V. As a concrete example, let V = \( \mathbb{R}^n \) and let \( T \) be the cyclic shift of coordinates sending \((x_1, x_2, \ldots, x_n) \) to \((x_n, x_1, \ldots, x_{n-1})\).

The T-invariant functionals form a 1-dimensional subspace of V* spanned by the functional \((x_1, x_2, \ldots, x_n) \mapsto x_1 + x_2 + \ldots + x_n\), while the 0-mesic functionals form an \((n-1)\)-dimensional subspace of V* spanned by the n-1 functionals \((x_1, x_2, \ldots, x_n) \mapsto x_i - x_{i+1}\) (for \(1 \leq i \leq n-1\)).

2.5 A circle action

Let \( S \) be the set of (real-valued) functions \( f(t) \) satisfying the differential equation \( f''(t) + f(t) = 0 \), that is, the set of functions of the form \( f(t) = A \cos t + B \sin t \). Then we have \( f'' = -f \), \( f''' = -f' \), \( f'''' = f \), etc., so that every polynomial function of \( f, f', f'', f''', \ldots \), can be written as a polynomial in \( f \) and \( f' \). Given an element \( p \) of the ring \( \mathbb{R}[x, y] \), we will say \( p \) is invariant if \( \frac{d}{dt}p(f(t), f'(t)) = 0 \) for all \( f \) in \( S \), and c-mesic if \( \frac{1}{2 \pi} \int_0^{2 \pi} f(x(t), y(t)) \) = c for all \( f \) in \( S \). For example, \( x^2 + y^2 \) is invariant (one can think of this quantity as the total energy of a harmonic oscillator, with invariance corresponding to conservation of energy) and the \( x \) and \( y \) are 0-mesic (one can think of these as the mean displacement and mean velocity of a harmonic oscillator). We can give a basis for \( \mathbb{R}[x, y] \) (viewed as a vector space over \( \mathbb{R} \)) consisting of the constant 1 and the powers of \( x^2 + y^2 \) (which jointly span the “invariant” subspace of \( \mathbb{R}[x, y] \)) along with the functions \( x, y, xy, x^2 - y^2, x^3, x^2 y, xy^2, y^3 \), etc. (which jointly span the 0-mesic subspace of \( \mathbb{R}[x, y] \)).

**Proposition 3.** Let \( V_n \) be the \((n+1)\)-dimensional vector subspace of \( \mathbb{R}[x, y] \) spanned by the monomials \( x^a y^b \) with \( a + b = n \). When \( n \) is odd, all of \( V_n \) is 0-mesic. When \( n \) is even, \( V_n \) can be written as the direct sum of an \( n \)-dimensional subspace of homomesic functions and a 1-dimensional subspace of functions that are invariant under time-evolution.

**Proof.** Define \( \int_{S_1} p(x, y) \frac{1}{2 \pi} \int_0^{2 \pi} p(\cos t, \sin t) \) dt. Consider the monomial \( x^a y^b \) with \( a + b = n \). If \( a \) (resp. \( b \)) is odd, the involution \((x, y) \mapsto (-x, y) \) (resp. \((x, -y)\)) shows that \( \int_{S_1} x^a y^b \) = 0. If \( a \) and \( b \) are both even, then \( \int_{S_1} x^a y^b \) is some positive number \( c_{a,b} \). Now let \( a, b \) vary subject to \( a + b = n \). If \( n \) is odd, then \( x^a y^b \) is 0-mesic for all \( a, b \) with \( a + b = n \) (since at least one of \( a, b \) is odd), so all of \( V_n \) is 0-mesic. If \( n \) is even, then for \( a, b \) even, \((1/c_{a,b})x^a y^b - (1/c_{n,0})x^n y^0 \) is 0-mesic, and these functions span an \((n/2)\)-dimensional space; adding in the 0-mesic functions \( x^a y^b \) with \( a, b \) odd we get an \((n-1)\)-dimensional space of homomesies, complementing the 1-dimensional space of invariants spanned by \((x^2 + y^2)^{n/2}\). \( \square \)

Greg Kuperberg points out that another term one might give to the 0-mesic subspace of \( \mathbb{R}[x, y] \) in this context is the coinvariant kernel, since this subspace is the mutual kernel of all invariant linear maps from \( \mathbb{R}[x, y] \) to \( \mathbb{R} \). Here, as in the preceding section, we get a clean complementarity between invariance and homomesy. That is, every element in \( \mathbb{R}[x, y] \) can be written uniquely as the sum of an invariant element and a 0-mesic element.
2.6 Sandpile dynamics

Let \( G \) be a finite directed graph with vertex set \( V \). For \( v \in V \) let \( \text{outdeg}(v) \) be the number of directed edges emanating from \( v \), and for \( v, w \in V \) let \( \text{deg}(v, w) \) be the number of directed edges from \( v \) to \( w \) (which we will permit to be larger than 1, even when \( v = w \)). Define the \textit{combinatorial Laplacian} of \( G \) as the matrix \( \Delta \) (with rows and columns indexed by the vertices of \( V \)) whose \( v, \text{th} \) entry is \( \text{outdeg}(v) - \text{deg}(v, v) \) and whose \( v, \text{th} \) entry for \( v \neq w \) is \( -\text{deg}(v, w) \). Specify a \textit{global sink} \( t \) with the property that for all \( v \in V \) there is a forward path from \( v \) to \( t \), let \( V^- = V \setminus \{t\} \), and let \( \Delta' \) (the reduced Laplacian) be the matrix \( \Delta \) with the row and column associated with \( t \) removed. By the Matrix-Tree theorem, \( \Delta' \) is nonsingular. A \textit{sandpile configuration} on \( G \) (with sink at \( t \)) is a function \( \sigma \) from \( V^- \) to the nonnegative integers. (For more background on sandpiles, see Holroyd, Levine, Mészáros, Peres, Propp, & Wilson [HLMPPW08].) We say \( \sigma \) is \textit{stable} if \( \sigma(v) < \text{outdeg}(v) \) for all \( v \in V^- \). For any sandpile-configuration \( \sigma \), Dhar’s least-action principle for sandpile dynamics (see Levine & Propp [LP10]) tells us that the set of nonnegative-integer-valued functions \( u \) on \( V^- \) such that \( \sigma - \Delta' u \) is stable has a minimal element \( \phi = \phi(\sigma) \) in the natural (pointwise) ordering; we call \( \phi \) the \textit{firing vector} for \( \sigma \) and we call \( \sigma - \Delta' \phi \) the \textit{stabilization} of \( \sigma \), denoted by \( \sigma^o \). If we choose a source vertex \( s \in V^- \), then we can define an action on sandpile configurations via \( \tau(\sigma) = (\sigma + 1_s)^o \), where \( 1_v \) denotes the function that takes the value 1 at \( v \) and 0 elsewhere. Say that \( \sigma \) is \textit{recurrent} (relative to \( s \)) if \( \tau^m(\sigma) = \sigma \) for some \( m > 0 \). (This notion of recurrence is slightly weaker than that of [HLMPPW08]; they are equivalent when every vertex is reachable by a path from \( s \).) Then \( \tau \) restricts to an invertible map from the set of recurrent sandpile configurations to itself. Let \( f(\sigma) = \phi(\sigma + 1_s) \). Since \( \tau(\sigma) = \sigma + 1_s - \Delta' f(\sigma) \) we have \( \tau(\sigma) - \sigma = 1_s - \Delta' f(\sigma) \); if we average this relation over all \( \sigma \) in a particular \( \tau \)-orbit, the left side telescopes, giving \( 0 = 1_s - \Delta' \bar{f} \), where \( \bar{f} \) denotes the average of \( f \) over the orbit. Hence:

\textbf{Proposition 4.} Under the action of \( \tau \) on recurrent sandpile configurations described above, the function \( f : \sigma \mapsto \phi(\sigma + 1_s) \) is homomesic, and its orbit-average is the function \( f^* \) on \( V^- \) such that \( \Delta' f^* = 1_s \) (unique because \( \Delta' \) is nonsingular).

\textbf{Example 5.} Figure 2 shows an example of the \( \tau \)-orbits for the case where \( G \) is the bidirected cycle graph with vertices 1, 2, 3, and 4, with a directed edge from \( i \) to \( j \) iff \( i - j = \pm 1 \mod 4 \); here the discrete Laplacian is

\[
\Delta = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2 \\
\end{pmatrix}.
\]

Let the source be \( s = 2 \) and global sink be \( t = 4 \). The sandpile configuration \( \sigma \) is represented by the triple \((\sigma(1), \sigma(2), \sigma(3))\). The four recurrent configurations \( \sigma \) are \((1, 0, 1), (1, 1, 1), (0, 1, 1), \) and \((1, 1, 0)\), and the respective firing vectors \( f(\sigma) \) are \((0, 0, 0), (1, 2, 1), (0, 1, 1), \) and \((1, 1, 0)\). The average value of the firing vector statistic \( f \) is \( f^* = (\frac{1}{2}, 1, \frac{1}{2}) \) on each orbit. Treating \( f^* \) as a column vector and multiplying on the left by \( \Delta' \) gives the column vector
... $\tau \rightarrow (1,0,1) \quad \tau \rightarrow (1,1,1) \quad \tau \rightarrow \ldots$

\[ \begin{array}{ccc}
  f & \downarrow & f \\
  (0,0,0) & \downarrow & (1,2,1) \\
\end{array} \]

... $\tau \rightarrow (0,1,1) \quad \tau \rightarrow (1,1,0) \quad \tau \rightarrow \ldots$

\[ \begin{array}{ccc}
  f & \downarrow & f \\
  (0,1,1) & \downarrow & (1,1,0) \\
\end{array} \]

Figure 2: The two orbits in the action of the sandpile map $\tau$ on recurrent configurations on
the cycle graph of size 4, with source at 2 and sink at 4. There are two orbits, each of size
2, and the average of $f$ along each orbit is $\left( \frac{1}{2}, 1, \frac{1}{2} \right)$.

$(0,1,0) = 1_s$:

\[
\begin{pmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
  1/2 \\
  1 \\
  1/2
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix}.
\]

We should mention that in this situation all orbits are of the same cardinality. This
is a consequence of the fact that the set of recurrent chip configurations can be given the
structure of a finite group (the “sandpile group” of $G$). For, given any finite abelian group $G$
and any element $h \in G$, the action of $h$ on $G$ by multiplication has orbits that are precisely
the cosets of $G/H$, where $H$ is the subgroup of $G$ generated by $h$, and all these cosets have
size $|H|$.

Similar instances of homomesy were known for a variant of sandpile dynamics called
rotor-router dynamics; see Holroyd-Propp [HP10]. It was such instances of homomesy that
led the second author to seek instances of the phenomenon in other, better-studied areas of
combinatorics.

2.7 Suter’s map

This subsection needs to be written.

3 Promotion and rowmotion in products of two chains

For a finite poset $P$, we let $J(P)$ denote the set of order ideals (or down-sets) of $P$, $F(P)$
denote the set of (order) filters (or up-sets) of $P$, and $A(P)$ be the set of antichains of $P$.
(For standard definitions and notation about posets and ideals, see Stanley [S11].) There is
a bijection $J(P) \leftrightarrow A(P)$ given by taking the maximal elements of $I \in J(P)$ or conversely
by taking the order ideal generated by an antichain $A \in A(P)$. Similarly, there is a bijection
$F(P) \leftrightarrow A(P)$. Composing these with the complementation bijection between $J(P)$ and
$F(P)$ leads to an interesting map that has been studied in several contexts [BS74, Fon93,
CF95, Pan08, AST11, SW12], namely $\Phi_A := A(P) \rightarrow J(P) \rightarrow F(P) \rightarrow A(P)$ and the
companion map $\Phi_J := J(P) \rightarrow F(P) \rightarrow A(P) \rightarrow J(P)$, where the subscript indicates
whether we consider the map to be operating on antichains or order ideals. We often drop
the subscript and just write $\Phi$ when context makes clear which is meant. Following Striker
and Williams [SW12] we call this map rowmotion.

Let $[a] \times [b]$ denote the poset that is a product of chains of lengths $a$ and $b$. Figure 5
shows an orbit of the action of $\Phi_J$ starting from the ideal generated by the antichain $\{(2, 1)\}$. Note that the elements of $[4] \times [2]$ here are represented by the squares rather than the points in the picture, with covering relations represented by shared edges. One can also view this as an orbit of $\Phi_A$ if one just considers the maximal elements in each shaded order ideal.

This section contains our main specific results, namely that the following triples exhibit
homomesy:

$$(J([a] \times [b]), \Phi_J, \#I) ; \quad (A([a] \times [b]), \Phi_A, \#A) ; \quad \text{and} \quad (J([a] \times [b]), \partial_J, \#I).$$

Here $\partial_J$ is the promotion operator to be defined in the next subsection, and $\#I$ (resp. $\#A$) denotes the statistic on $J(P)$ (resp. $A(P)$) that is the cardinality of the order ideal $I$ (resp. the antichain $A$). All maps operate on the left (e.g., we write $\partial_JI$, not $I\partial_J$).

3.1 Background on the toggle group

Several of our examples arise from the toggle group of a finite poset (first explicitly defined in [SW12]; see also [CF95, Sta09, SW12]). We review some basic facts and provide some pointers to relevant literature.

Definition 6. Given $x \in P$, we define the toggle operation $\sigma_x : J(P) \to J(P)$ (“toggling at $x$”) via

$$\sigma_x(I) = \begin{cases} I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P); \\ I & \text{otherwise}, \end{cases}$$

where $A \triangle B$ denotes $A \setminus B \cup B \setminus A$.

Proposition 7 ([CF95]). (a) For every $x \in P$, $\sigma_x$ is an involution, i.e., $\sigma_x^2 = 1$.
(b) For every $x, y \in P$ where neither $x$ covers $y$ nor $y$ covers $x$, the toggles commute, i.e.,
$\sigma_x\sigma_y = \sigma_y\sigma_x$.

Proposition 8 ([CF95]). Let $x_1, x_2, \ldots, x_n$ be any linear extension (i.e., any order-preserving listing of the elements) of a poset $P$ with $n$ elements. Then the composite map $\sigma_{x_1}\sigma_{x_2} \cdots \sigma_{x_n}$ coincides with the rowmotion operator $\Phi_J$.

Corollary 9 ([SW12], Cor. 4.9). Let $P$ be a graded poset of rank $r$, and set $T_k := \prod_{|x|=k} \sigma_x$, the product of all the toggles of elements of fixed rank $k$. (This is well-defined by Proposition 7.) Then the composition $T_1T_2 \cdots T_r$ coincides with $\Phi_J$, i.e., rowmotion is the same as toggling by ranks from top to bottom.

We focus on the case $P = [a] \times [b]$, whose elements we write as $(k, \ell)$; the Hasse diagram of $P$ consists of the points $(i, j) := (\ell - k, \ell + k - 2) \ (1 \leq k \leq a, \ 1 \leq \ell \leq b)$ so that the poset-elements $(k, \ell) = (1, 1), (a, 1), (1, b), \text{and} (a, b)$ are respectively the bottom, left, right, and top elements of the Hasse diagram.
Definition 10. In this situation, we call the sets with constant \( j \) ranks (in accordance with standard poset terminology), sets with constant \( i \) files, sets with constant \( j - i \) positive fibers, and sets with constant \( j + i \) negative fibers. (The words “positive” and “negative” indicate the slopes of the lines on which the fibers lie in the Hasse diagram.) More specifically, the element \((k, \ell) \in [a] \times [b]\) belongs to rank \( k + \ell - 2 \), to file \( \ell - k \), to positive fiber \( k \), and to negative fiber \( \ell \).

To each order ideal \( I \in J([a] \times [b]) \) we associate a lattice path of length \( a + b \) joining the points \((-a, a)\) and \((b, b)\) in the plane, where each step is of type \((i, j) \rightarrow (i + 1, j + 1)\) or of type \((i, j) \rightarrow (i + 1, j - 1)\), as follows. Given \( 1 \leq k \leq a \) and \( 1 \leq \ell \leq b \), represent \((k, \ell) \in [a] \times [b]\) by the square centered at \((\ell - k, \ell + k - 1)\) with vertices \((\ell - k, \ell + k - 2), (\ell - k, \ell + k), (\ell - k - 1, \ell + k - 1), \) and \((\ell - k + 1, \ell + k - 1)\). Then the squares representing the elements of the order ideal \( I \) form a “Russian-style” Young diagram whose upper border is a path joining some point on the line of slope \(-1\) to some point on the line of slope \(+1\). Adding extra edges of slope \(-1\) at the left and extra edges of slope \(+1\) at the right, we get a path joining \((-a, a)\) to \((b, b)\). See Figures 3, 4, and 5 for several examples of this correspondence.

Definition 11. We can think of this path as the graph of a (real) piecewise-linear function \( h_I : [-a, b] \rightarrow [0, a + b] \); we call this function (or its restriction to \([-a, b] \cap \mathbb{Z}\)) the height function representation of the ideal \( I \). To this height function we can in turn associate a word consisting of \( a \) \(-1\)'s and \( b \) \(+1\)'s, whose \( i \)th term (for \( 1 \leq i \leq a + b \)) is \( h_I(i - a) - h_I(i - a - 1) = \pm 1 \); we call this the sign-word associated with the order ideal \( I \).

Note that the sign-word simply lists the slopes of the segments making up the path, and that either the sign-word or the height-function encodes all the information required to determine the order ideal.

Proposition 12. Let \( I \in J([a] \times [b]) \) correspond with height function \( h_I : [-a, b] \rightarrow \mathbb{R} \). Then

\[
\sum_{k=-a}^{b} h_I(k) = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + 2\# I.
\]

So to prove that the cardinality of \( I \) is homomesic, it suffices to prove that the function \( h_I(-a) + h_I(-a+1) + \cdots + h_I(b) \) is homomesic (where our combinatorial dynamical system acts on height functions \( h \) via its action on order ideals \( I \)).

3.2 Promotion in products of two chains

In general a ranked poset \( P \) may not have an embedding in \( \mathbb{Z} \times \mathbb{Z} \) that allows files to be defined; when they are, however, then all toggles corresponding to elements within the same file commute by Proposition 7, so their product is a well defined operation on \( J(P) \). This allows one to define an operation on \( J(P) \) by successively toggling all the files from left to right, in analogy to Corollary 9.

Theorem 13 (Striker-Williams [SW12 § 6.1]). Let \( x_1, x_2, \ldots, x_n \) be any enumeration of the elements \((k, \ell)\) of the poset \([a] \times [b]\) arranged in order of increasing \( \ell - k \). Then the action on \( J(P) \) given by \( \partial := \sigma_{x_n} \circ \sigma_{x_{n-1}} \circ \cdots \circ \sigma_{x_1} \) viewed as acting on the paths (or binary strings representing them) is just a leftward cyclic shift.
Figure 3: One promotion orbit in $J([3] \times [2])$

Striker and Williams call this well-defined composition $\partial$ promotion (since it is related to Schützenberger’s notion of promotion on linear extensions of posets). They show that it is conjugate to rowmotion in the toggle group, obtaining a much simpler bijection to prove Panyushev’s conjecture in Type A, and generalizing an equivariant bijection for $[a] \times [b]$ of Stanley [Sta09, remark after Thm 2.5]. This definition and their results apply more generally to the class they define of $rc$-posets, whose elements fit neatly into “rows” and “columns” (which we call here “ranks” and “files”). As with $\Phi$, we can think of $\partial$ as operating either on $J(P)$ or $A(P)$, adding subscripts $\partial_{J}$ or $\partial_{A}$ if necessary. Since the cyclic left-shift has period $a+b$, so does $\partial$.

**Theorem 14.** The cardinality statistic is homomesic under the action of promotion $\partial_{J}$ on $J([a] \times [b])$, with all averages equal to $ab/2$.

**Proof.** To show that $\#I$ is homomesic, by Proposition [12] it suffices to show that $h_{I}(k)$ is homomesic for all $-a \leq k \leq b$. Note that here we are thinking of $I$ as varying over $J(P)$, and $h_{I}$ as being a function-valued function on $J(P)$.

We can write $h_{I}(k)$ as the telescoping sum $h_{I}(-a) + (h_{I}(-a+1) - h_{I}(-a)) + (h_{I}(-a+2) - h_{I}(-a+1)) + \cdots + (h_{I}(k) - h_{I}(k-1))$; to show that $h_{I}(k)$ is homomesic for all $k$, it will be enough to show that all the increments $h_{I}(k) - h_{I}(k-1)$ are homomesic. Note that these increments are precisely the terms of the sign-word of $I$. Create a square array with $a+b$ rows and $a+b$ columns, where the rows are the sign-words of $I$ and its successive images under the action of $\partial$; each row is just the cyclic left-shift of the row before. Since each row contains $a-1$’s and $b+1$’s, the same is true of each column. Thus, for all $k$, the average value
Figure 4: The other promotion orbit in $J([3] \times [2])$

of the $k$th terms of the sign-words of $I$, $\partial I$, $\partial^2 I$, $\ldots$, $\partial^a+b-1 I$ is $(b-a)/(b+a)$. This shows that the increments are homomesic, as required, which suffices to prove the theorem.

Our proof actually shows the more refined result that the restricted cardinality functions $#(I \cap S)$ where $S$ is any file of $[a] \times [b]$ are homomesic with respect to the action of $\partial_I$.

The next example shows that the cardinality of the antichain $A_I$ associated with the order ideal $I$ is not homomesic under the action of promotion $\partial$.

**Example 15.** Consider the two promotion orbits of $\partial_A$ shown in Figures 3 and 4. Although the statistic $#I$ is homomesic, giving an average of 3 in both cases, the statistic $#A$ averages to $\frac{1}{5}(0+1+1+1+1) = \frac{4}{5}$ in the first orbit and to $\frac{1}{5}(1+2+2+1+2) = \frac{8}{5}$ in the second.

### 3.3 Rowmotion in products of two chains

Unlike promotion, the rowmotion operator turns out to exhibit homomesy with respect to both the statistic that counts the size of an order ideal and the statistic that counts the size of an antichain.

#### 3.3.1 Rowmotion on order ideals in $J([a] \times [b])$

We can describe rowmotion nicely in terms of the sign-word. We define blocks within the sign-word as occurrences of the subword $-1,+1$ (that is, a $-1$ followed immediately by a $+1$). Once we have found all the blocks, we identify all the gaps between the blocks, where a gap is bounded by two consecutive blocks, or between the beginning of the word
and the first block, or between the last block and the end of the word. (In the case where there are no blocks at all, the entire word is considered a gap.) To apply rowmotion to a sign-word, reverse all the blocks and all the gaps. For example, consider the binary word $-1, +1, +1, -1, -1, +1, +1$. To apply rowmotion to it, we first divide it into blocks and gaps as $-1, +1, | +1, -1, -1, | -1, +1, | +1$, and then reverse each block and gap in place, obtaining $+1, -1, | -1, -1, +1, | +1, -1, | +1$, or (dropping the dividers) $+1, -1, -1, +1, +1, -1, +1$. Note that the dividers correspond to the red dots in Figure 5, so one can visualize $\Phi_J$ as reversing (180° rotation of) each lattice-path segment that corresponds to a block or a gap in the sign-word. (See animations within talk slides at http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf.)

It turns out that all we really need to know for purposes of proving homomesy is that the sign-word for $I$ has $-1, +1$ in a pair of adjacent positions if and only if the sign-word for $\Phi I$ has $+1, -1$ in the same two positions. This can be seen directly for $J([a] \times [b])$ from the description of $\Phi_J$ given at the start of Section 3 (See also Figure 5). This situation occurs if and only if the antichain $A(\Phi(I))$ contains an element in the associated file of $[a] \times [b]$.

Theorem 16. The cardinality statistic is homomesic under the action of rowmotion $\Phi_J$ on $J([a] \times [b])$, with all averages equal to $ab/2$.

Proof. As in the previous section, to prove that $\# I$ is homomesic under rowmotion, it suffices to prove that all the increments $h_I(k) - h_I(k-1)$ are homomesic. A result of Fonder-Flaass [Fon93, Theorem 2], states that the size of any $\Phi$-orbit in $[a] \times [b]$ is a divisor of $a + b$, so this is equivalent to showing that for all $k$, the $k$th element of the sign-word of $\sum_{m=0}^{a+b-1} \Phi^m I$ under the action of rowmotion is independent of $I$. 

Figure 5: A rowmotion orbit in $J([4] \times [2])$
Create a square array with $a+b$ rows and $a+b$ columns, where the rows are the sign-words of $I$ and its successive images under the action of $\Phi$. Consider any two consecutive columns of the array, and the width-2 subarray they form. There are just four possible combinations of values in a row of the subarray: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, and $(-1, -1)$. However, we have just remarked that a row is of type $(-1, +1)$ if and only if the next row is of type $(+1, -1)$ (where we consider the row after the bottom row to be the top row). Hence the number of rows of type $(-1, +1)$ equals the number of rows of type $(+1, -1)$. It follows that any two consecutive column-sums of the full array are equal, since other row types contribute the same value to each column sum. That is, within the original square array, every two consecutive columns have the same column-sum. This common value of the column-sum must be $1/(a+b)$ times the grand total of the values of the square array. But since each row contains $a$ $-1$’s and $b$ $+1$’s, each row-sum is $b-a$, so the grand total is $(a+b)(b-a)$, and each column-sum is $b-a$. Since this is independent of which rowmotion orbit we are in, we have proved homomesy for elements of the sign-word of $I$ as $I$ varies over $J([a] \times [b])$, and this gives us the desired result about $\#I$, just as in the proof of Theorem 14.

3.3.2 Rowmotion on antichains in $\mathcal{A}([a] \times [b])$

In his survey article on promotion and evacuation, Stanley [Sta09, remark after Thm 2.5] gave a concrete equivariant bijection between rowmotion $\Phi_A$ acting on antichains in $\mathcal{A}([a] \times [b])$ and cyclic rotation of certain binary strings. Armstrong (private communication) gave a variant description that clarified the correspondence, which he learned from Thomas and which we use in what follows.

**Definition 17.** Fix $a$, $b$, and $n = a+b$. Call sets in $[a] \times [b]$ of the form $\{(k, \ell) : \ell \in [b]\}$ (with $k$ fixed) **rows** and sets of the form $\{(k, \ell) : k \in [a]\}$ (with $\ell$ fixed) **columns**. Define the
Stanley-Thomas word $w(A)$ of an antichain $A$ in $[a] \times [b]$ to be $w_1 w_2 \cdots w_{a+b} \in \{-1, +1\}^{a+b}$ with

$$w_i := \begin{cases} +1, & \text{if } A \text{ has an element in row } i \text{ (for } i \in [a]) \text{ or} \\ -1 & \text{A has NO element in column } i \text{ (for } a+1 \leq i \leq n); \end{cases}$$

**Example 18.** As illustrated in Figure 6, let $A = \{(1,5), (5,3), (6,2)\}$. By definition, the Stanley-Thomas word $w(A)$ should have +1 in entries 1, 5, and 6 (rows where $A$ appears) and in entries 8 and 11 (columns where $A$ does not appear, with indices shifted by 7 = $a$). Indeed one sees that $w(A) = +1, -1, -1, +1, +1, -1, | +1, -1, -1, +1, -1$ (where the divider between $a$ and $a+1$ is just for ease of reading). Note that applying rowmotion gives $A' = \Phi(A) = \{(2,4), (6,3), (7,1)\}$ with Stanley-Thomas word $w(A') = -1, +1, -1, -1, -1, -1, -1, -1, +1$ = $C_R w(A)$, the rightward cyclic shift of $w(A)$.

**Proposition 19** (Stanley-Thomas). The correspondence $A \leftrightarrow w_A$ is a bijection from $\mathcal{A}([a] \times [b])$ to binary words $w \in \{-1, +1\}^{a+b}$ with exactly $a$ $-1$’s and $b$ $+1$’s. Furthermore, this bijection is equivariant with respect to the actions of rowmotion $\Phi_A$ and rightward cyclic shift $C_R$.

Note that the classical result that $\Phi_A^{a+b}$ is the identity map follows immediately.

**Theorem 20.** The cardinality statistic is homomesic under the action of rowmotion $\Phi_J$ on $\mathcal{A}([a] \times [b])$, with all averages equal to $ab/(a+b)$.

**Proof.** It suffices to prove a more refined claim, namely, that if $S$ is any row or column of $[a] \times [b]$, the cardinality of $A \cap S$ is homomesic under the action of rowmotion on $A$. By the previous result, rowmotion corresponds to cyclic shift of the Stanley-Thomas word, and the entries in the Stanley-Thomas word tell us which fibers (rows or columns) contain an element of $A$ and which do not. Specifically, for $1 \leq k \leq a$, if $S$ is the $k$th row, then $A$ intersects $S$ iff the $k$th symbol of the Stanley-Thomas word is a $+1$. Since the Stanley-Thomas word contains $a$ $-1$’s and $b$ $+1$’s, the multiset orbit of $A$ of size $a+b$ has exactly $b$ elements that are antichains that intersect $S$. That is, the sum of $\#(A \cap S)$ over the multiset orbit of size $a+b$ is exactly $b$, for each of the $a$ rows of $[a] \times [b]$. Summing over all the rows, we see that the sum of $\#A$ over the multiset orbit is $ab$. Hence $\#A$ is homomesic with average $ab/(a+b)$.

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