Conformal matter in warped backgrounds

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Abstract

Warped backgrounds in five dimensional models can provide solutions to various hierarchy problems in particle physics if the standard model matter is associated with the zero modes of bulk fields with nontrivial profiles along the extra dimension. We investigate the case of a conformally coupled massless scalar. This field possesses a zero mode whose action density is exponentially enhanced along the transverse direction in a warped background, i.e. near the negative tension brane in the RS setting. At the zero mode level effective 5D composites of this scalar and its derivatives, which are suppressed by powers of the fundamental gravitational scale, yield 4D interactions of electroweak strength. This suggests a certain importance of conformally coupled scalar fields for extra-dimensional model building.
1 Introduction

The last few years have witnessed an explosion of interest in field theoretic models involving extra dimensions of spatial signature. For a large part this rapid development was stimulated by the work of Ref. [1] where the presence of a small number of compact extra dimensions, accessible only to gravity, was proposed as a mechanism for lowering the $4 + n$-dimensional fundamental scale relative to the 4-dimensional Planck scale. In this scenario, the large hierarchy between the electroweak and the Planck scale can be re-expressed as a large hierarchy between the inverse compactification radius and the electroweak scale, and the latter is identified with the higher dimensional gravitational scale. Another important impetus to extradimensional model building was launched by the orbifold construction of Randall and Sundrum (RS) [2] where the possibility of a purely geometrical solution to the gauge hierarchy problem was demonstrated. The key observation in [2] is that a conversion of the 5D Planck scale to the 4D electroweak scale occurs if a single compact extra dimension is sufficiently strongly warped due to the presence of a bulk cosmological constant and if the relevant matter is localized on a brane with negative tension which is positioned at the minimum of the warp factor.

For both extradimensional approaches to be viable and plausible there is the need to (i) stabilize the extra dimension(s) and (ii) localize the low-energy Standard Model matter fields on 4D hypersurfaces in a dynamical fashion. Condition (i) is dictated by the necessity to be in agreement with the observed 4D late time cosmology [4]. There have been many proposals for a stabilization of the set-up in [2] which by itself does not generate a potential for the radion field. General criteria for the stabilization of RS-like scenarios were worked out recently in Ref. [5]. The most popular scenario uses a minimally coupled bulk scalar field which is subject to brane potentials [6]. Operating in the gravitational background of the RS solution, in this scenario a radion potential with an acceptable minimum is generated by a nontrivial bulk profile of the scalar field. Stabilization of extra dimension(s) due to the Casimir force of conformally coupled bulk matter was investigated in Refs. [8].

In the framework of field theory in Minkowski space localization mechanisms for low-energy matter fields are known for a long time [9]. In a realization of (ii) for the warped case the weak/Planck hierarchy would appear due to the zero mode property of a dynamically localized profile in a gravitational background which is more fundamental than the framework that simply assumes localization [2].

The challenge in the RS framework is to localize the zero modes of bulk matter gravitationally. A considerable amount of work has been devoted to address this question in the past [3]. It was found that the gravitational localization of the massless fermion fields is possible while for the gauge fields this mechanism does not work.

In case of the gravitational localization of fermions, the transverse action density behaves as $a^{-1}$ where $a = a(y)$ denotes the warp factor [3]. This scaling corresponds to the strong enhancement near the negative tension brane and can be interpreted
as localization of the zero mode on the ”visible” brane in the RS set-up. In case of
the minimally coupled bulk scalar field, its zero mode has a constant profile along
the extra dimension which implies that the transverse action density is proportional
to $\sim a^2$. Thus, the zero mode is effectively localized on the ”hidden” brane. A
nontrivial $y$ profile for the zero mode of a minimally coupled scalar requires the
introduction of extremely fine-tuned sources on the branes (see section 2). This
problem does not arise in the fermionic case since the bulk equation of motion is
first order.

One may wonder whether a localization of fermionic and scalar low-energy matter
on different branes is an artefact of the minimal coupling of the latter to gravity.
The main purpose of this Letter is to point out that this is indeed the case. We
investigate the profiles of the scalar fields in curved 5D backgrounds characterized
by an arbitrary scale factor $a(y)$. We compare the minimally and the conformally
coupled cases. In the former case we show that nontrivial profiles for a zero mode
require the introduction of sources on the branes. De-tuning the sources would result
in a mass of the would-be zero mode of the minimally coupled scalar. If the energy
scale associated with the inverse size of the extra dimension is large, like in the RS
model, such a mode would not be useful for low-energy phenomenology.

We then address the case of the conformally coupled scalar. In this case we show
how one can construct a nontrivial zero mode profile which (i) automatically solves
the associated scalar field equation and scales as $\sim a^{-3/2}$, (ii) does not require a
fine-tuning of brane sources to ensures the masslessness condition , and (iii) in the
case of a large warping exhibits a localization of the transverse action density on
the negative tension brane. In a next step we investigate how interactions between
conformally coupled scalars in the bulk reduce to the interaction between the zero
modes in the effective 4D theory. We argue that effective terms in the bulk potential,
which are suppressed by the inverse powers of the Planck scale, in the 4D theory
are enhanced due to the $\sim a^{-3/2}$ scaling and become electroweak size interactions.

We believe that these features are worth being advocated in view of an ample
use of conformally coupled scalars in extra-dimensional model building. Moreover,
these scalars can be thought of as being composites of gauge and/or fermionic funda-
mentals that when minimally are also conformally coupled. Prime examples for
such composites are the Goldstone bosons arising from the spontaneous breakdown
of a continuous, global symmetry.

2 Minimally coupled scalar

Throughout this paper we assume that the geometry in the transverse $y$ direction
is given by a scale factor $a = a(y)$,

$$ds^2 = a^2(y)\eta_{\mu\nu}x^\mu x^\nu + dy^2$$  \hspace{1cm} (1)

where $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$. We assume that the extra dimension is compact and
has two special points at $y = 0$ and $y = L$ according to the standard $S_1/Z_2$ orbifold
hypothesis. Moreover, we assume that \(a(y)\) and a massless, noninteracting minimally coupled scalar \(\phi\) in the bulk are even under the orbifold reflection \(y \rightarrow -y\). 3-branes and sources \(V_{0,L}\) for \(\phi\) on these branes can be placed at the fixed points \(y = 0, L\). For most of what follows the precise form of the given scale factor \(a(y)\) is irrelevant.

We consider the following action for a minimally coupled massless bulk scalar field \(\phi\) in the background (1),

\[
S = - \int d^4x \int dy \sqrt{-g} \left( \frac{1}{2} \partial_M \phi \partial^M \phi + V_0(\phi) \delta(y) + V_L(\phi) \delta(y-L) \right).
\]

(2)

We are interested in the lowest mass mode in a KK decomposition of \(\phi\). If the mass of the lowest mode is zero or small compared to other scales in the problem, the corresponding equation of motion for the \(y\) dependence \(^1\) in the bulk reads

\[
1/a^4 \partial_y (a^4(y) \partial_y \phi(y)) = 0.
\]

(3)

The field \(\phi(y)\) is continuous at the boundaries and subject to the two boundary conditions,

\[
\left[ \frac{d \phi}{d y} \right] (y = 0) = \frac{dV_0}{d \phi}, \quad \left[ \frac{d \phi}{d y} \right] (y = L) = -\frac{dV_L}{d \phi}.
\]

(4)

The square bracket in (4) denotes the discontinuity of its argument and is defined as

\[
[f](y) \equiv \lim_{\varepsilon \to 0} (f(y + \varepsilon) - f(y - \varepsilon)).
\]

Eqs. (4) are usually called jump conditions. The general solution for the bulk equation (3) is (7),

\[
\phi = B + A \left| \int_0^y \frac{dy}{a^4(y)} \right|.
\]

(5)

Both \(A\) and \(B\) are smooth functions of the 4D coordinates \(\{x^\mu\}\). A nontrivial \(y\) dependence of the zero mode, corresponding to \(A \neq 0\), leads to nonvanishing jumps on the branes, \(\partial_y \phi(y = 0, L) = 2A/a^4(y = 0, L)\), and thus needs brane sources according to (4).

A linear combination of \(A(x)\) and \(B(x)\) is a massless mode in 4D if it satisfies the four dimensional equation of motion \(\partial_\mu \partial^\mu (B + cA) = 0\) and allows for an arbitrary rescaling \(B + cA \rightarrow \text{const} \times (B + cA)\). However, for generic brane potentials \(V_0, L\) the two boundary conditions (4) determine \(A(x)\) and \(B(x)\) algebraically which means that their magnitudes are fixed and thus no \(x\) dependence is allowed for. Therefore, none of the solutions with a nontrivial profile can represent an effective scalar field with zero mass in 4D. A trivial alternative is possible, \(A = V_0 = V_L = 0\) (3). This leaves \(B(x)\) unconstrained, and thus it represents a true zero mode. In the RS background the action density for this mode is enhanced near the positive tension brane and therefore it is not suitable for model building purposes. Another possibility is

\(^1\)In the following we denote the zero mode as well as the full field by \(\phi\).
to introduce an explicit correlation between the two sources \( V_{0,L} \) and the scale factor \( a(y = 0, L) \) such that the two boundary conditions on \( A \) and \( B \) degenerate into a single one. This, however, is an extreme fine-tuning.

### 3 Conformally coupled scalar

**Field equation and solution**

We now turn to the case of a conformally coupled bulk scalar in \( D \) dimensions. The corresponding action reads

\[
S = - \int d^Dx \sqrt{-g} \left( \frac{1}{2} \partial_M \phi \partial^M \phi + \frac{1}{2} \xi_D R_D \phi^2 \right),
\]

where \( \xi_D \) is given by \( \xi_D \equiv \frac{1}{4} \left( \frac{D-2}{D-1} \right) \). One may also add brane-localized sources for the \( \phi \) field as in Eq. (2). In conformal coordinates, \( ds^2 = a^2(z) \eta_{MN} dx^M dx^N \), a solution to the scalar field equation in the bulk,

\[
\phi^{;M}_{;M} - \xi R \phi = 0,
\]

can be obtained by rescaling a solution of the equation of motion in a flat background

\[
\partial_M \partial^M \tilde{\phi} = 0
\]

as \( \phi = a^{(2-D)/2}(z) \tilde{\phi} \). Here \( z \) refers to the \( D \)-th coordinate in the conformal frame. The most general solution \( \tilde{\phi} = \tilde{\phi}(z) \) to Eq. (8) is

\[
\tilde{\phi}(z) = Az + B.
\]

A finite transformation from RS to conformal coordinates is given as

\[
z(y) = \int_0^y \frac{dy}{a(z(y))}.
\]

Even orbifold symmetry, \( a(z(y)) = a(z(-y)) \), implies even orbifold symmetry of

\[
\phi = a^{(2-D)/2}(z(y)) \left\{ A \left| \int_0^y \frac{dy}{a(z(y))} \right| + B \right\}.
\]

Again, \( A \) and \( B \) are smooth functions of the 4D coordinates \( \{x^\mu\} \) and satisfy \( \partial_\mu \partial^\mu A = \partial_\mu \partial^\mu B = 0 \). The field \( \phi(z(y), \{x^\mu\}) \) in Eq. (11) is a solution of (10) in the RS frame. Note the similarity between (5) and (11) for \( D = 5 \). However, in contrast to the solution (5) both contributions in (11) have a nontrivial \( y \) dependence because of an overall \( a^{-3/2} \) factor.
Jump conditions

For the discussion of the jump conditions we retreat to \( D = 5 \). Since both terms in Eq. (11) have a \( y \) dependence the jump conditions are expected to severely constrain the bulk solution (11). In RS coordinates we derive the jump conditions for \( \phi \) on the branes from the scalar field equation that is modified due to the presence of the term \( \sim \xi_5 \phi^2 R_5 \) in the action:

\[
\frac{1}{a^4} \frac{\partial_y \phi}{\partial_y} \left( a^4 \frac{\partial_y \phi}{\partial_y} \right) = \xi_5 R_5 \phi \,.
\]

(12)

The jump conditions become

\[
\frac{[\partial_y \phi](y = 0, L)}{\phi(y = 0, L)} = -8\xi_5 \frac{[\partial_y a](y = 0, L)}{a(y = 0, L)}.
\]

(13)

The occurrence of \([\partial_y a](y = 0, L)\) on the RHS of (13) originates from a term proportional to \( \partial_y^2 a \) in \( R_5 \). Inserting Eq. (11) for \( D = 5 \) into Eq. (13) and considering the jump at \( y = 0 \), we obtain

\[
(8\xi_5 - 3/2) \frac{[\partial_y a](y = 0)}{a(y = 0)} = \frac{1}{a(y = 0)} \frac{A}{B}.
\]

(14)

The LHS of (14) vanishes since \( 8\xi_5 = 3/2 \) and consequently we must set \( A = 0 \). For the remaining part

\[
\phi = a^{-3/2}(y) B
\]

(15)

the jump conditions (13) are identically satisfied at \( y = 0, L \). Unlike the minimally coupled case the nontrivial profile (15) solves the scalar sector in the background \( a(y) \) exactly without the need to introduce fine-tuned potentials on the branes.

4 Zero mode effective 4D Lagrangian

We now derive a 4D effective action for the associated canonically normalized 4D scalar field \( \chi \). The field \( \chi(\{x^\mu\}) \) is defined as

\[
\phi = \frac{B(x)}{a^{3/2}(y)} = \frac{Z^{1/2} \chi(x)}{a^{3/2}(y)}.
\]

(16)

Integrating the bulk action density in Eq. (6) over \( y \) in RS coordinates, we obtain

\[
S = - \int_0^L dy \frac{a^4 Z}{a^2 a^3} \int d^4x \sqrt{-g_4} \partial_\mu \chi \partial^\mu \chi.
\]

(17)

No potential terms are generated\(^2\) for \( \chi \) since there is an exact cancellation between \( R \phi^2 \) and \( (\partial_y \phi)^2 \) terms in the integral over \( y \). From (17) we read off an expression

\(^2\)The situation is different if gravitational back reaction is considered. In the absence of a genuine stabilization mechanism, which enforces the static geometry assumed in this paper, the scale factor \( a \) would depend on time due to the stress-energy arising from the scalar sector.
for $Z$ in terms of the integral over the inverse of the scale factor:

$$\frac{1}{Z} = 2 \int_0^L \frac{dy}{a(y)}. \quad (18)$$

For warped geometries with $a(y = 0) \gg a(y = L)$, the integral $18$ is saturated around $a(y = L)$. In particular, using the RS relation to solve the gauge hierarchy problem,

$$a(y = 0) = 1 \quad \text{and} \quad a(y = L) \sim M_W/M_{Pl}, \quad (19)$$

one immediately concludes that

$$Z = \left( 2 \int_0^L \frac{dy}{a(y)} \right)^{-1} \sim \kappa a(y = L) \sim M_W, \quad (20)$$

where $\kappa \sim M_{Pl}$ denotes the value of the coefficient in the RS exponential and $M_W$ denotes the weak scale. It is worth emphasizing that the precise functional form of the scale factor $a(y)$ is not important for an estimate of $Z$. For example, the cosh-like solutions [11] with the hierarchy [19] generate a similar value for $Z$.

On the level of zero modes we may now break perturbatively the conformal symmetry of the scalar sector by adding higher dimensional composites to the 5D Lagrangian in Eq. (6). Using the estimate of the $Z$-factor (20) in the RS framework, we then may investigate the effective 4D scaling of these operators. Let us first look at nonDerivative couplings. Terms in the bulk action, that are suppressed by inverse powers of the UV cutoff parameter $M_{Pl}$, are

$$S^{(5)}_n = -\int d^4x \int dy a^4 \phi^n M_{Pl}^{5-\frac{2n}{3}}, \quad (21)$$

where $n > 3$. The action (21) can be generalized to the case of several conformally coupled scalar fields. When reduced to the zero mode level and after integrating over $y$ Eq. (21) becomes

$$S^{(4)}_n = -\int d^4x \chi^n Z^\frac{n}{2} M_{Pl}^{4-\frac{3n}{2}} a^{4-\frac{3n}{2}}(y = L) \sim -\int d^4x \chi^n M_W^{4-n}. \quad (22)$$

Eq. (22) shows that at the zero mode level Planck scale suppressed operators in 5D induce interactions in the effective 4D theory that are of electroweak strength. For example, a quartic term $\phi^4$ in the bulk, being suppressed by the first power of $M_{Pl}$, gives rise to a quartic interaction of the zero modes of strength $\sim 1$. It is also instructive to look at the bilinear term $M^2 \phi^2$ where $M$ is some mass parameter. Its scaling is different from (22) because the integral over $y$ is saturated near $a_0$. The requirement that the mass term for the zero mode arising from this operator should be of order $M^2 \chi^2$ implies that $M \sim \sqrt{M_{Pl} M_W}$.

A similar picture arises for the $M_{Pl}$-suppressed operators that contain derivatives such as

$$M_{Pl}^{5-5n} \left( \partial_M \phi \partial^M \phi \right)^n. \quad (23)$$
These operators are relevant for the interaction of Goldstone bosons. For exponential warping and on the zero mode level terms $\sim (\partial_y \phi \partial^y \phi)^n$ can be treated as polynomial interactions $\sim M_{Pl}^2 \phi^{2n}$. It is then easy to see that the 5D operator (23) induces a 4D chain of operators of the form

$$\sim - \int d^4x M_W^{4-4r-2s}(\partial_\mu \chi \partial^\mu \chi)^r \chi^{2s}, \quad (r + s = n).$$

(24)

Thus, pure derivative couplings in 5D generate weak-scale suppressed mixed derivative and polynomial operators.

5 Conclusions

We have shown that in a gravitational background given by a single warp factor $a = a(y)$ and in the setting of Ref. [2] a conformally coupled bulk scalar field has a zero mode with a non-trivial $y$-profile, $\phi \sim a^{-3/2}$. This leads to localization of this zero mode at the negative tension brane. The behaviour is similar to what has been observed for the minimally coupled fermion fields ($\psi \sim a^{-2}$). We show that such zero modes can be used in models that have an RS type of solution to the gauge hierarchy problem. In particular, we demonstrate that the effective 5D interactions of the zero modes, which are suppressed by powers of the Planck scale, correspond to the effective 4D interactions of electroweak strength. Similar results should also hold for fermions. The missing piece then is, of course, a plausible localization mechanism for gauge fields in the RS context.

Apart from the gauge hierarchy problem there are proposals to use nontrivial bulk profiles for the resolution of the fermion flavor hierarchy [12] and various cosmological problems [13], etc. We have shown that such a nontrivial scalar bulk profile exists ($B = \text{const} \neq 0$ in (15)) in the conformally coupled case as an exact solution to the scalar field equation in a warped background.

The conformal coupling of the scalar sector to an AdS background breaks AdS SUSY explicitly. As far as grand unification is concerned, it has been advocated recently that a high-scale non-SUSY realization in a compactified AdS background is possible due to a logarithmic running of the gauge coupling even beyond the lowest KK excitation energy [14].

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