NONLINEAR INSTABILITY IN GRAVITATIONAL EULER-POISSON SYSTEM FOR $\gamma = \frac{6}{5}$

JUHI JANG

Abstract. The dynamics of gaseous stars can be described by the Euler-Poisson system. Inspired by Rein’s stability result for $\gamma > \frac{4}{3}$, we prove the nonlinear instability of steady states for the adiabatic exponent $\gamma = \frac{6}{5}$ in spherically symmetric and isentropic motion.

1. Introduction

The motion of gaseous stars can be described by the Euler-Poisson equations:

\begin{align}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p &= -\rho \nabla \Phi,
\end{align}

where $\Delta \Phi = 4\pi \rho$

where $t \geq 0$, $x \in \mathbb{R}^3$, $\rho$ is the density, $u \in \mathbb{R}^3$ the velocity, $p$ the pressure of the gas, and $\Phi$ the potential function of the self-gravitational force. We consider the isentropic motion i.e. $p = A\rho^\gamma (1 < \gamma < 2)$, where $A$ is an entropy constant and $\gamma$ is an adiabatic exponent. In our case $A$ will be normalized as $\frac{2\pi}{9}$ and $\gamma$ will be chosen as $\frac{6}{5}$.

For the spherically symmetric motion the above equations where $\rho > 0$ can be written as following:

\begin{align}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{2}{r} \rho u &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{4\pi}{r^2} \int_0^r \rho s^2 ds &= 0.
\end{align}

First consider the stationary solutions $(\rho_0(r), 0)$ of (1.4) and (1.5). They satisfy

\begin{equation}
\frac{d\rho}{dr} + \frac{4\pi \rho}{r^2} \int_0^r \rho s^2 ds = 0.
\end{equation}

This ordinary differential equation has been well studied. One interesting question relevant to our context can be given like this: Given the mass $M > 0$, how many solutions are there for (1.6) with $\int_{\mathbb{R}^3} \rho dx = M(\rho) = M$? For our purpose we summarize the answer according to the range of $\gamma$. See $\textit{}$ or $\textit{}$ for details. If $\frac{6}{5} < \gamma < 2$ and any $M > 0$, there exists at least one compactly supported stationary solution.
\( \rho \) such that \( M(\rho) = M \). For \( 4/3 < \gamma < 2 \), every stationary solution is compactly supported and unique. If \( \gamma = \frac{6}{5} \) and any \( M > 0 \), there is a unique ground-state solution (not compactly supported) \( \rho \). The solution can be expressed in terms of Lane-Emden function and moreover it can be written explicitly as \( \rho_0(r) = \frac{1}{(1 + r^2)^{\frac{2}{5}}} \) up to scaling in \( r \) and constant multiplication. On the other hand, if \( 1 < \gamma < \frac{6}{5} \), there is no stationary solution with finite total mass.

The stability question has been a great interest and it has been conjectured by astrophysicists that stationary solutions for \( \gamma < \frac{4}{3} \) are unstable; indeed one can easily check that when \( \gamma \in (1, \frac{4}{3}) \) steady states are not minimizers of the energy functional

\[
E = E(\rho, u) = \int (\frac{1}{2} \rho |u|^2 + \frac{1}{\gamma - 1} \rho) dx - \frac{1}{2} \int \int \frac{\rho(x)\rho(y)}{|x - y|} dxdy
\]

by constructing a scaling invariant family of steady states, and this indicates the possibility of certain kind of instability. So far only partial results are known in this direction. The linear stability of the above stationary solutions was studied in \[7\] in Lagrangian formulation. It was shown that any stationary solution is stable when \( \gamma \in (\frac{4}{3}, 2) \) and unstable when \( \gamma \in (1, \frac{4}{3}) \). In accordance with the linear stability, a nonlinear stability for \( \gamma > \frac{4}{3} \) was established recently in \[8\] by Rein using the variational approach based on the fact the steady states are minimizers of the energy functional \( E \) defined in (1.7). See \[9\] for a great overview of mathematical results on the nonlinear stability problems. For \( \gamma = \frac{4}{3} \), the energy of a steady state is zero and any small perturbation can make the energy positive and cause part of the system go off to infinity, which implies an instability of such a state. This kind of instability was investigated in \[4\]. However, the method they employed is not applicable when \( \gamma < \frac{4}{3} \). The stability question for \( \frac{6}{5} \leq \gamma < \frac{4}{3} \) under the physical consideration waits for a satisfactory answer.

Our main result in this paper concerns a fully nonlinear, dynamical instability of the steady profile for \( \gamma = \frac{6}{5} \):

\[
\rho_0(r) = \frac{1}{(1 + r^2)^{\frac{2}{5}}} \quad \text{and} \quad u_0(r) = 0
\]

We shall show that this steady profile is unstable under an appropriate energy-like measurement \( \mathcal{E}_l \) which will be precisely defined later in Section 5. Here \( l \leq 0 \) represents the strength of weights. As \( l \) gets smaller, \( \mathcal{E}_l \) is equipped with the stronger weight. Indeed, \( \mathcal{E}_0 \) corresponds to the positive part of the real energy \( E \). First we rewrite the Euler-Poisson system (1.4) and (1.5) for \( \gamma = \frac{6}{5} \) by letting \( \rho = \rho_0 + \sigma \):

\[
\sigma_t + \frac{1}{r^2}(r^2 \rho_0 u)_r + \frac{1}{r^2}(r^2 \sigma u)_r = 0
\]

\[
u_t + uu_r + \frac{4\pi}{15}(\rho_0 + \sigma)^{-\frac{2}{3}}(\rho_0 + \sigma)_r + \frac{4\pi}{r^2} \int_0^r (\rho_0 + \sigma)s^2 ds = 0
\]

If we define \( \mathcal{E}_0(t) \) and \( T^\delta \) by

\[
\mathcal{E}_0(t) = \int_{\mathbb{R}^3} (\rho_0 + \sigma)u^2 + \frac{4\pi}{15}(\rho_0 + \sigma)^{-\frac{2}{3}}\sigma^2 dx \quad \text{and} \quad T^\delta = \frac{1}{\sqrt{\mu_0}} \ln \frac{\theta}{\delta} \quad \text{where} \sqrt{\mu_0} \text{ is the sharp linear growth rate,}
\]
the main theorem can be stated as follows:

**Theorem 1.1.** Let $l \leq -3$ be fixed. There exist $\theta > 0$ and $C > 0$ such that for any small $\delta > 0$, there exists a family of solutions $\nu^\delta(t) = (\sigma^\delta(t), u^\delta(t))$ of (1.9) and (1.10) such that

\[
\sqrt{E_l(0)} \leq C\delta, \quad \text{but} \quad \sup_{0 \leq t \leq T^\delta} \sqrt{E_0(t)} \geq \theta.
\]

We remark that the escape time $T^\delta$ is determined by the exponential growth rate of the linearized system (2.1) and (2.2) and the instability occurs before the possible blowup of the smooth solutions. The local existence of regular solutions for $\gamma = \frac{5}{3}$ including (1.8) was shown in [5] and [1] independently. Gamblin uses the paradifferential calculus and Bezard uses the Gagliardo-Nirenberg inequality and Littlewood-Paley theory. In this paper we take their existence results for granted without proving again.

Rein’s work in [8] implies that the steady states for $\gamma > \frac{4}{3} > \frac{5}{3}$ are stable under the energy functional $E$ as minimizers. By contrast with those cases, our steady profile for $\gamma = \frac{5}{3}$ does not minimize $E$ and that motivated us to investigate an instability under the energy-like measurement $E^0$. The proof of Theorem 1.1 is based on the bootstrap argument from the linear instability to the nonlinear dynamical model. In general, passing a linearized instability to a nonlinear instability needs much effort in the PDE context: the spectrum of the linear part is fairly complicated and the unboundedness of the nonlinear part usually yields a loss in derivatives. In order to get around these difficulties, the careful analysis on the linearized system which controls the sharp exponential growth rate of its solutions is necessary. In addition, the energy estimates on the whole system and the interplay with the linear analysis can close the argument. See [6] for its original method.

The main difficulty in this paper is to derive Proposition 6.1, a key estimate for the bootstrap argument. There are two important ingredients: the first idea is to find the form of

\[
\frac{d}{dt} E \leq \eta E + (E)^2 + \text{(quadratic but lower derivative terms)}
\]

where $\eta$ is smaller than the sharp growth rate $\sqrt{\mu_0}$, but it turns out that (1.12) is not enough to close the energy estimate; so we introduce the weighted energy $E_l$ by the utilization of a family of symmetrizers as weights. New undesirable quadratic terms come out during the weighted energy estimates, but they turn out to have weaker weights. As for weighted cubic terms, we introduce the weighted Gagliardo-Nirenberg inequality (6.19). Here is the modified version of (1.12):

\[
\frac{d}{dt} E_l \leq \eta E_l + (E_l)^2 + \text{(quadratic but lower derivative terms)}
\]

The potential part $\nabla \Phi$ not only has the smoothing effect to the whole system but also behaves nicely with respect to weights. Eventually, in cooperating with weights, the chain type estimates by using (1.13) complete the bootstrap argument as well as the proof of Proposition 6.1.
Another technical difficulty lies in that both linearized and full system do not comply submissively with spatial derivatives. Our argument in both linear and nonlinear parts heavily depends on the estimates on pure temporal derivative terms. As for spatial and mixed derivative terms we use the equations directly to control them in terms of temporal derivative terms. Furthermore, when one uses the polar coordinates, we seem to end up with simpler one dimensional flow but the singularity at the origin comes into play. For clear and precise understanding of the largest growing mode, we take the polar coordinates in the linear analysis while the rectangular ones are used for the nonlinear analysis to take care of spatial and mixed derivative terms.

Other cases of $\gamma$, $\frac{4}{5} < \gamma < \frac{2}{3}$ attain more physical, interesting feature involving the vacuum boundary. For those cases, steady states satisfying (1.6) are compactly supported and even the local existence of the Euler-Poisson system including those stationary solutions has not been completed yet. At this moment the above argument does not seem to apply directly to such $\gamma$'s. But we believe the method developed in this paper can make a contribution to show the instability for those $\gamma$'s. We will leave them for future study.

The paper proceeds as follows. The first half of this article is devoted to develop the linear theory: finding the largest growing mode of the linearized Euler-Poisson system and deriving some regularity of it. While the linear instability was studied in [2], it was done in Lagrangian formulation and it does not give the precise growth rate. For our purpose we shall demonstrate the explicit linear instability analysis in Eulerian coordinates. In Section 2, we formulate a variational problem to find the biggest eigenvalue $\sqrt{\mu_0}$ and corresponding eigenfunction. In the subsequent section, the fast decay property of the largest growing mode is derived. In Section 4, we show that $\sqrt{\mu_0}$ dominates exponential growth rate of any solutions for the linearized system. In the other half the nonlinear analysis is carried out. In Section 5, the weighted instant energy and total energy are introduced and it will be shown that the total energy is bounded by the instant energy containing only temporal derivative terms. We perform the weighted energy estimates in Section 6 to get the precise estimate of (1.13). Finally in Section 7, the bootstrap argument and Theorem 1.1 will be proven.

### 2. Existence of the Largest Growing Mode

Firstly, we study the linearized Euler-Poisson equations for the spherically symmetric case. We assume $\rho > 0$ which is our concern. Letting $\rho = \rho_0 + \sigma$, we linearize (1.4) and (1.5) around a given steady state $(\rho_0, 0)$ and get the linearized Euler-Poisson equations in terms of $\sigma$ and $u$.

\begin{align}
\frac{\partial \sigma}{\partial t} + \frac{\partial \rho_0}{\partial r} u + \rho_0 \frac{\partial u}{\partial r} + \frac{2}{r} \rho_0 u &= 0 \tag{2.1} \\
\frac{\partial u}{\partial t} + A(\gamma \rho_0^{-\gamma} \frac{\partial \sigma}{\partial r} + \gamma (\gamma - 2) \rho_0^{-\gamma} \frac{\partial \rho_0}{\partial r} \sigma) + \frac{4\pi}{r^2} \int_0^r \sigma s^2 ds &= 0. \tag{2.2}
\end{align}
To find a growing mode for (2.1) and (2.2), let \( \sigma = e^{\lambda t} \phi(r) \) and \( u = e^{\lambda t} \psi(r) \). Then (2.1) becomes

\[
(2.3) \quad \lambda \phi + \frac{\partial \rho_0}{\partial r} \psi + \rho_0 \frac{\partial \psi}{\partial r} + \frac{2}{r} \rho_0 \psi = 0.
\]

Since \( \frac{\partial \rho_0}{\partial r} \psi + \rho_0 \frac{\partial \psi}{\partial r} + \frac{2}{r} \rho_0 \psi = \frac{1}{\gamma}(r^2 \rho_0 \psi)_r \), (2.3) gives a simple relation between \( \phi \) and \( \psi \):

\[
(2.4) \quad \psi(r) = -\frac{\lambda}{r^2 \rho_0} \int_0^r \phi(s)s^2 ds.
\]

Similarly, (2.2) becomes

\[
(2.5) \quad \lambda \psi + A[\gamma \rho_0^{\gamma - 2} \frac{\partial \phi}{\partial r} + \gamma (\gamma - 2) \rho_0^{\gamma - 3} \frac{\partial \rho_0}{\partial r} \phi] + \frac{4\pi r^2}{\gamma} \int_0^r \phi(s)^2 ds = 0.
\]

Multiplying (2.5) by \( \frac{A}{\gamma} \) and using (2.3), we get

\[
\frac{\lambda^2 \psi}{A} = \gamma \rho_0^{\gamma - 2} (-\lambda \phi)' + (\gamma (\gamma - 2) \rho_0^{\gamma - 3} \rho_0' (-\lambda \phi)) + \frac{4\pi \rho_0}{A} (-\frac{\lambda}{r^2 \rho_0} \int_0^r \phi(s)ds)
\]

\[
= \gamma \rho_0^{\gamma - 2} [\rho_0'' \psi + 2 \rho_0' \psi' + \rho_0 \psi'' - 2 \rho_0 \psi' \frac{2 \rho_0 \psi'}{r^2} + \frac{2 \rho_0 \psi'}{r} + 2 \rho_0 \psi'] + \gamma (\gamma - 2) \rho_0^{\gamma - 3} \rho_0' \frac{\rho_0' \psi + \rho_0 \psi' + 2 \rho_0 \psi}{A} \psi
\]

\[
= \gamma \rho_0^{\gamma - 1} \psi' + \gamma (\gamma - 2) \rho_0^{\gamma - 2} [\rho_0' \psi + \frac{2 \gamma}{r} \rho_0^{\gamma - 1}] \psi' + [\gamma \rho_0^{\gamma - 2} \rho_0' - \frac{2 \gamma}{r} \rho_0^{\gamma - 1}] \psi + \gamma \rho_0^{\gamma - 2} \rho_0' + \frac{4\pi \rho_0}{A} \psi,
\]

where \( \rho = \frac{d}{dr} \) and this is the 2nd order ordinary differential equation. For the further simplification recall that \( \rho_0 \) satisfies (1.6): \( A \gamma \rho_0^{\gamma - 2} \rho_0 + \frac{4\pi r^2}{\gamma} \int_0^r \rho_0 s^2 ds = 0 \). Compute \( \frac{1}{\gamma^2} \frac{d}{dr} [r^2 \cdot (1.6)] \), and then we get the following relation:

\[
\gamma \rho_0^{\gamma - 2} \rho_0' + \frac{2 \gamma}{r} \rho_0^{\gamma - 2} \rho_0 + \gamma (\gamma - 2) \rho_0^{\gamma - 3} (\rho_0')^2 + \frac{4\pi \rho_0}{A} = 0.
\]

In turn we have

\[
(2.6) \quad \frac{\lambda^2 \psi}{A} = \gamma \rho_0^{\gamma - 1} \psi'' + \gamma (\gamma - 2) \rho_0^{\gamma - 2} \rho_0 + \gamma (\gamma - 2) \rho_0^{\gamma - 3} (\rho_0')^2 + \frac{4\pi \rho_0}{A} = 0.
\]

Multiply (2.6) by \( \frac{\partial \rho_0}{\gamma} \) to obtain the following:

\[
\lambda^2 \frac{\rho_0 r^2}{A} \psi = r^2 \rho_0 \psi'' + [\gamma r^2 \rho_0^{\gamma - 1} \rho_0 + 2 \gamma r \rho_0^{\gamma - 1}] \psi' + [2(\gamma - 2) \rho_0^{\gamma - 1} \rho_0' - 2 \rho_0^\gamma] \psi
\]

\[
(2.7) \quad \frac{\partial \rho_0}{\gamma} \psi'' = (r^2 \rho_0')' \psi' + 2 [\frac{\gamma - 2}{\gamma} r (\rho_0')' - \rho_0^\gamma] \psi
\]

Denote the RHS of (2.7) by \( L \psi \):

\[
(2.8) \quad L \psi \equiv (r^2 \rho_0')' + 2 [\frac{\gamma - 2}{\gamma} r (\rho_0')' - \rho_0^\gamma] \psi
\]
Note that the linear operator $L$ is self-adjoint and hence $\lambda^2$ is real.

**Lemma 2.1.** Suppose $\chi$ and $\omega$ ($\omega > 0$) satisfy $L\chi = \omega^2 \frac{\partial \sigma}{\partial \theta} \chi$. Define $\varphi = -\frac{1}{\omega^2}(r^2 \rho_0 \chi)'$. We assume $\chi$ and $\varphi$ are well defined admissible functions in a suitable sense which will be clarified later on. Then $\sigma = e^{\pm \omega t} \varphi$ and $u = e^{\pm \omega t} \chi$ are a solution pair of the linearized equations (2.1) and (2.2).

Proof. This is obvious by the definition of $\chi$ and $\varphi$. One can keep track of the derivation of $L$ to see it. $\Box$

Lemma 2.1. tells us that $(e^{\pm \omega t} \varphi, e^{\pm \omega t} \chi)$ satisfying all the assumptions is a growing mode for the linearized Euler-Poisson equations. Next we show such a growing mode actually exists when $\gamma = \frac{6}{5}$. This can be done by looking at the eigenvalue problem of the operator $L$ due to Lemma 2.1. In other words, we only need to find $\psi$ and $\lambda$ ($\lambda > 0$) such that $L\psi = \lambda^2 \frac{\partial \sigma}{\partial \theta} \psi$, where $L\psi$ is defined in (2.8).

From now on we fix $\gamma = \frac{6}{5}$ and $A = \frac{2\pi}{9}$ and corresponding

$$\rho_0(r) = \frac{1}{(1 + r^2)^{\frac{5}{2}}}.$$

The starting equation is $L\psi = \mu \frac{\partial \sigma}{\partial \theta} \psi$ on $(0, \infty)$, where $\mu = \lambda^2$. It is well known that the largest eigenvalue $\mu_0$ is given by a variational formula:

$$\mu_0 = \sup \{ \frac{Q(\psi)}{I(\psi)} : Q(\psi) < \infty, I(\psi) < \infty \}$$

where

$$Q(\psi) = (L\psi, \psi) = -\int_0^\infty r^2 \rho_0^2 (\psi')^2 dr + 2 \int_0^\infty \left( \gamma - 2 \frac{\gamma}{\gamma} r(\rho_0')^2 - \rho_0' \right) \psi^2 dr$$

$$= -\int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{5}{2}}} (\psi')^2 dr + 2 \int_0^\infty \frac{3r^2 - 1}{(1 + r^2)^{\frac{5}{2}}} (\psi)^2 dr$$

and

$$I(\psi) = \left( \frac{\rho_0 r^2}{A^2 \gamma} \psi, \psi \right) = \int_0^\infty \frac{\rho_0 r^2}{A^2 \psi^2 dr} = \frac{15}{4\pi} \int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{5}{2}}} (\psi)^2 dr.$$

Hence once the above formula attains the sup, the largest eigenvalue $\mu_0$ and corresponding eigenfunction $\psi_0$ of the linear operator $L$ gives a largest growing mode of the linearized equations. In order to carry it out, first define a norm for any $\psi \in C_c^\infty(0, \infty)$,

$$\| \psi \|^2 = \int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{5}{2}}} (\psi')^2 dr + \int_0^\infty \frac{2}{(1 + r^2)^4} \psi^2 dr + \frac{15}{4\pi} \int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{5}{2}}} \psi^2 dr.$$

Let $H = C_c^\infty(0, \infty)$ in the above norm. It is clear that $\frac{Q(\psi)}{I(\psi)} = \frac{Q(\psi)}{I(\psi)}$ for any nonzero constant $c$. Thus the variational problem can be rephrased as to find a maximum $\mu_0$ of $Q(\psi)$ on $H$ under the normalization condition $I(\psi) = 1$.

**Proposition 2.2.** There exists a $\psi_0 \in H$ such that $I(\psi_0) = 1$ and $Q(\psi_0) = \mu_0$ ($\mu_0 > 0$) i.e. the sup $\mu_0$ is attained on $H$. 
Proof: First we claim $\mu_0 > 0$. Consider $\psi = \sqrt{r}$.

$$Q(\sqrt{r}) = \frac{-J_0^\infty (r^2 - (1+r^2)^3)\frac{1}{2\sqrt{r}} dr + 2J_0^\infty \frac{3r^2}{(1+r^2)^3} \sqrt{r}^2 dr}{\frac{15\pi}{4\pi} J_0^\infty \frac{r^2}{(1+r^2)^3} \sqrt{r}^2 dr}$$

Since $\mu_0 \geq \frac{Q(\sqrt{r})}{I(\sqrt{r})}$ and $I(\sqrt{r}) > 0$, it is enough to show $Q(\sqrt{r}) > 0$.

$$Q(\sqrt{r}) = (-\frac{1}{4} + 6) \int_0^\infty \frac{r}{(1+r^2)^3} dr - 8 \int_0^\infty \frac{r}{(1+r^2)^4} dr$$

$$= \frac{23\pi}{4} \left[ \frac{1}{2(1+r^2)^2} \right]_0^\infty - [\frac{1}{3(1+r^2)^3}]_0^\infty$$

$$= \frac{23}{8} - \frac{8}{3} > 0$$

And note that the positive part of $Q$ is uniformly bounded by $I$ because $\frac{3r^2}{(1+r^2)^3} = O(\frac{1}{r^4})$ and $\frac{r^2}{(1+r^2)^3} = O(\frac{1}{r^3})$ for sufficiently large $r$. This implies $\mu_0$ is finite. To show $\mu_0$ is attained on $H$ let $\{\psi_n\}$ be a maximizing sequence i.e.

$$Q(\psi_n) \nearrow \mu_0 \text{ as } n \to \infty \text{ and } I(\psi_n) = 1 \text{ for all } n.$$ 

Let $\psi_0$ be its weak limit. Then by the lower semicontinuity of weak convergence, we have

$$\lim inf \int \frac{r^2}{(1+r^2)^3} (\psi'_n)^2 dr \geq \int \frac{r^2}{(1+r^2)^3} (\psi'_0)^2 dr,$$

$$\lim inf \int \frac{1}{(1+r^2)^4} \psi'_n^2 dr \geq \int \frac{1}{(1+r^2)^4} \psi'_0^2 dr,$$

$$\lim inf \int \frac{r^2}{(1+r^2)^2} \psi_n^2 dr \geq \int \frac{r^2}{(1+r^2)^2} \psi_0^2 dr.$$

Claim 1. (Compactness of the positive part) There exists a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ such that

$$\int \frac{r^2}{(1+r^2)^4} \psi_{n_k}^2 dr \to \int \frac{r^2}{(1+r^2)^4} \psi_0^2 dr \text{ as } n_k \to \infty.$$ 

Claim 2. $I(\psi_0) = 1$.

Since Claim 1 immediately implies $Q(\psi_0) = \mu_0$, the conclusion follows from Claim 1 and Claim 2. It remains to prove Claim 1 and Claim 2. Claim 2 follows from a simple scaling argument. Suppose $\frac{15\pi}{4\pi} \int \frac{r^2}{(1+r^2)^2} \psi^2_0 dr = \alpha^2 < 1$. Then

$$I(\frac{\psi_0}{\alpha}) = 1 \text{ and } Q(\frac{\psi_0}{\alpha}) = \frac{1}{\alpha^2} \mu_0 > \mu_0$$

which is a contradiction to the definition of $\mu_0$. To prove Claim 1, first observe that

$$\int_0^\infty \frac{r^2}{(1+r^2)^4} \psi_n^2 dr \leq \frac{1}{(1+R^2)^2} \int_R^\infty \frac{r^2}{(1+r^2)^2} \psi_n^2 dr \leq \frac{4\pi}{15(1+R^2)^2}.$$
Fix $R > 0$. On the finite interval $(0, R)$, since $\int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi^2 dr \sim \|\psi\|_{L^2(B_R(0))}^2$, $\int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi^2 dr \sim \|\psi\|_{L^2(B_R(0))}^2$, and $\int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi'' dr \sim \|\psi''\|_{L^2(B_R(0))}$, where $B_R(0)$ is a ball with radius $R$ in $\mathbb{R}^3$, we can apply the Rellich-Kondrachov Compactness theorem, which says $H^1(B_R(0))$ is compactly embedded in $L^q(B_R(0))$ for each $1 \leq q < 6$. So there exists a subsequence $\{\psi_{n_k}\}$ such that

$$
\int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_{n_k}^2 dr \longrightarrow \int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_0^2 dr, \text{ as } n_k \longrightarrow \infty.
$$

For any given $\epsilon > 0$, choose $R > 0$ large enough so that $\frac{4\pi}{15(1 + R^2)^{\frac{3}{2}}} < \epsilon$.

$$
\int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_{n_k}^2 dr = \int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_{n_k}^2 dr + \int_R^\infty \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_{n_k}^2 dr < \int_0^R \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_{n_k}^2 dr + \epsilon
$$

Now take the limit of $n_k \longrightarrow \infty$. Since $\epsilon > 0$ is arbitrary, this finishes the proof of Proposition 2.1. \(\Box\)

Finally, let us make sure $\psi_0$ is in fact an eigenfunction corresponding to $\mu_0$. Consider a perturbation $\psi_0 + \epsilon \eta$ around $\psi_0$. Since $\mu_0 = Q(\psi_0)$,

$$
\frac{Q(\psi_0 + \epsilon \eta)}{I(\psi_0 + \epsilon \eta)} \leq Q(\psi_0) = \mu_0
$$

for all sufficiently small $\epsilon$ and all admissible function $\eta$. And hence we have

$$
Q(\psi_0 + \epsilon \eta) \leq Q(\psi_0) I(\psi_0 + \epsilon \eta) + 2\epsilon (L\psi_0, \eta) + \epsilon^2 Q(\eta) \leq \mu_0 I(\eta)
$$

Here $(\cdot, \cdot)$ is the standard inner product in $L^2(0, \infty)$. Note $I(\psi_0) = 1$ and $Q(\eta) \leq \mu_0 I(\eta)$. Hence in order for the above inequality to hold for all $\epsilon$ and $\eta$, the coefficient of $\epsilon$ should vanish, i.e. $(L\psi_0, \eta) = \mu_0 \frac{d^2}{dr^2} \psi_0 \eta$ for all $\eta$. Thus, $L\psi_0 = \mu_0 \frac{d^2}{dr^2} \psi_0$ and therefore $\sqrt{\mu_0}$ and $\psi_0$ give a largest growing mode for the linearized Euler-Poisson equations.

3. The Regularity of the Largest Growing Mode

Recall $\psi_0$ satisfies the following 2nd linear ordinary differential equation:

$$
\frac{15\mu_0}{4\pi} \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_0 = \left( \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_0' \right)' + 2 \frac{3r^2 - 1}{(1 + r^2)^{\frac{3}{2}}} \psi_0
$$

$$
= \frac{r^2}{(1 + r^2)^{\frac{3}{2}}} \psi_0'' + \frac{2r - 4r^3}{(1 + r^2)^{\frac{3}{2}}} \psi_0' + 2 \frac{3r^2 - 1}{(1 + r^2)^{\frac{3}{2}}} \psi_0
$$

The interior regularity easily follows from the elliptic theory of the 2nd order differential equations: since each coefficient of $\psi_0''$, $\psi_0'$ and $\psi_0$ in (3.1) is in $C^\infty(\epsilon, R)$
where $\epsilon$ is a small enough positive fixed number and $R$ is a large enough fixed number, $\psi_0$ is also $C^\infty$ on $(\epsilon, R)$. In the following two subsections we investigate the behavior of $\psi_0$ when $r$ is either very small or very large.

3.1. The behavior of $\psi_0$ near the origin. In this first subsection we show $\psi_0$ is analytic near the origin. This rather surprising property easily follows from the classical theorem by Frobenius from the ODE theory. For reference, see [2]. Before we prove the analyticity, it will be shown that the maximizing property implies the boundedness of $\psi_0$ at the origin.

Lemma 3.1. There exists a decreasing sequence \( \{\epsilon_k\} \) with \( \epsilon_k \searrow 0 \) such that \( \psi_0(\epsilon_k)\psi_0'(\epsilon_k) \geq 0 \) for each \( k \).

Proof. Consider $\psi_1 = \Theta\psi_0$, where $\Theta$ is a lipschitz cutoff function defined by

$$
\Theta(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq \epsilon, \\
\epsilon - r & \text{if } \epsilon \leq r \leq 2\epsilon, \\
\epsilon & \text{if } r \geq 2\epsilon.
\end{cases}
$$

Here, $\epsilon$ is a sufficiently small positive number to be clarified. Then clearly $\psi_1$ is in the admissible set $\mathcal{H}$. First note that $I(\psi_1) \leq I(\epsilon\psi_0)$ since $\psi_1^2 \leq \epsilon^2\psi_0^2$. Next let us look at the difference between $Q(\psi_1)$ and $Q(\epsilon\psi_0)$. Since $\psi_1$ has the same value as $\epsilon\psi_0$ on $r \geq 2\epsilon$ and $\psi_1^2 = \psi_0^21_{(\epsilon, 2\epsilon)} + 2(r - \epsilon)\psi_0\psi_0'1_{(\epsilon, 2\epsilon)} + (r - \epsilon)^2\psi_0^21_{(\epsilon, 2\epsilon)} + \epsilon^2\psi_0^21_{(2\epsilon, \infty)}$, $Q(\psi_1) - Q(\epsilon\psi_0)$ is

\[
\begin{align*}
&- \int_\epsilon^{2\epsilon} \frac{r^2}{(1 + r^2)^3}\psi_0^2 + (r - \epsilon)^2\psi_0'^2 + 2(r - \epsilon)\psi_0\psi_0' dr + 2 \int_\epsilon^{2\epsilon} \frac{3r^2 - 1}{(1 + r^2)^3}(r - \epsilon)^2\psi_0'^2 dr \\
&+ \int_{2\epsilon}^{\infty} \frac{r^2}{(1 + r^2)^3}\epsilon^2\psi_0'^2 dr - 2 \int_{\epsilon}^{2\epsilon} \frac{3r^2 - 1}{(1 + r^2)^3}\epsilon^2\psi_0'^2 dr.
\end{align*}
\]

After collecting similar terms, (3.2) can be written as following:

\[
\begin{align*}
&\int_0^{\epsilon} \frac{r^2}{(1 + r^2)^3}\epsilon^2\psi_0'^2 dr + \int_\epsilon^{2\epsilon} \frac{r^2}{(1 + r^2)^3}r(2\epsilon - r)\psi_0'^2 dr - \int_\epsilon^{2\epsilon} \frac{2r^2(2\epsilon - r)}{(1 + r^2)^3}\psi_0\psi_0' dr \\
&+ \int_{\epsilon}^{2\epsilon} \frac{2 - 6\epsilon^2}{(1 + r^2)^4}\epsilon^2\psi_0'^2 dr + \int_\epsilon^{2\epsilon} \frac{5r^4 - 12\epsilon r^3 - 3r^2 + 4\epsilon}{(1 + r^2)^4}\epsilon^2\psi_0'^2 dr.
\end{align*}
\]

Call the first three terms of (3.3) (I) and the last two terms (II). Now suppose the lemma were false. Then for any small enough $\epsilon > 0$, we may assume $\psi_0\psi_0' < 0$ on $(0, 2\epsilon)$. It is obvious to see that (I) is positive. And because $\psi_0^2$ is decreasing on $(0, 2\epsilon)$,

$$
\int_\epsilon^{2\epsilon} \frac{2 - 6\epsilon^2}{(1 + r^2)^4}\epsilon^2\psi_0'^2 dr > \int_\epsilon^{2\epsilon} \frac{2 - 6\epsilon^2}{(1 + r^2)^4}\epsilon^2\psi_0'^2 dr
$$
and hence

\[
(II) > \int_{\epsilon}^{2\epsilon} \frac{2 - 6r^2}{(1 + r^2)^4} \psi_0^2 dr + \int_{\epsilon}^{2\epsilon} \frac{5r^4 - 12\epsilon r^3 - 3r^2 + 4r \epsilon}{(1 + r^2)^4} \psi_0^2 dr
\]

\[
= \int_{\epsilon}^{2\epsilon} \frac{5r^4 - 12\epsilon r^3 - 6\epsilon^2 r^2 + \epsilon^2}{(1 + r^2)^4} \psi_0^2 dr + \int_{\epsilon}^{2\epsilon} \frac{2 + 4\epsilon r - 3r^2}{(1 + r^2)^4} \psi_0^2 dr
\]

\[
= (III) + (IV).
\]

(III) \geq 0 since \(5r^4 - 12\epsilon r^3 - 6\epsilon^2 r^2 + \epsilon^2 \geq 0\) on \((\epsilon, 2\epsilon)\) for sufficiently small \(\epsilon > 0\).

Claim. (IV) \geq 0.

Claim implies \(Q(\psi_1) - Q(\epsilon \psi_0) = (I) + (II) > 0\). This leads to

\[
\frac{Q(\psi_1)}{I(\psi_1)} > \frac{Q(\epsilon \psi_0)}{I(\epsilon \psi_0)} = \frac{Q(\psi_0)}{I(\psi_0)} = \mu_0.
\]

which is a contradiction to the definition of \(\mu_0\). Therefore, to finish the proof it suffices to verify Claim. Let \(\alpha\) be a zero of \(\epsilon^2 + 4\epsilon r - 3r^2 = 0\) with \(\epsilon < \alpha < 2\epsilon\). Then

\[
\epsilon^2 + 4\epsilon r - 3r^2 = \begin{cases} 
\geq 0 & \text{if } \epsilon \leq r \leq \alpha, \\
\leq 0 & \text{if } \alpha \leq r \leq 2\epsilon.
\end{cases}
\]

The decreasing assumption on \(\psi_0^2\) will be again used:

\[
(IV) \geq \min_{[\epsilon, \alpha]} \frac{\rho_0^2(r)}{(1 + r^2)^4} \int_{\epsilon}^{\alpha} \epsilon^2 + 4\epsilon r - 3r^2 dr + \max_{[\alpha, 2\epsilon]} \frac{\rho_0^2(r)}{(1 + r^2)^4} \int_{\alpha}^{2\epsilon} \epsilon^2 + 4\epsilon r - 3r^2 dr
\]

\[
= \frac{\rho_0^2(\alpha)}{(1 + \alpha^2)^4} \int_{\epsilon}^{2\epsilon} \epsilon^2 + 4\epsilon r - 3r^2 dr
\]

\[
= 0.
\]

The first equality holds because \(\frac{\rho_0^2(r)}{(1 + r^2)^4}\) is decreasing and the second one is simply due to the fact \(\int_{\epsilon}^{2\epsilon} \epsilon^2 + 4\epsilon r - 3r^2 dr = 0\). This completes the argument.\(\square\)

The Frobenius theorem with Lemma 3.1 gives rise to the following analytic property of \(\psi_0\) around the origin.

**Lemma 3.2.** \(\psi_0\) is analytic at \(r = 0\) and moreover \(\psi_0(r) = ar + o(r^2)\) around the origin where \(a\) is a constant.

Proof. In order to employ the Frobenius theorem, we need to show the equation (3.1) has a regular singular point at \(r = 0\). To check this out in the context of [2] (p. 215), we rewrite (3.1) in the following form

\[
\psi_0'' + \frac{2 - 4r^2}{r(1 + r^2)} \psi_0' + \left\{ \frac{2}{r^2} \frac{3r^2 - 1}{1 + r^2} - \frac{15\mu_0}{4\pi} (1 + r^2)^{\frac{\alpha}{2}} \right\} \psi_0 = 0.
\]

Let \(P(r)\) and \(Q(r)\) be coefficients of \(\psi_0'\) and \(\psi_0\) respectively in the above. Then it is clear that \(rP(r)\) and \(r^2Q(r)\) are analytic at \(r = 0\), which means \(r = 0\) is a regular singular point of (3.1). Let \(p_0, q_0\) be the zeroth order term of \(rP(r)\) and \(r^2Q(r)\) respectively. It is easy to check \(p_0 = 2\) and \(q_0 = -2\). The indicial equation \(r(r-1) + p_0r + q_0 = 0\) has two roots \(r_1 = 1\) or \(r_2 = -2\). Hence by the Frobenius theorem \(\psi_0\) has a power series representation of either \(y_1(r) = r \sum_{n=0}^{\infty} a_n r^n\) or
\[ y_2(r) = ay_1(r) \ln r + r^{-2} \sum_{n=0}^{\infty} b_n r^n. \] However, \( y_2(r) \) is impossible by Lemma 3.1 and therefore \( \psi_0 = r \sum_{n=0}^{\infty} a_n r^n \) is obtained. \( \square \)

3.2. **Asymptotic behavior of \( \psi_0 \).** In this subsection we first observe \( \psi_0^2(r) \) is nonincreasing near the infinity owing to the maximizing property and then the fast decay of \( \psi_0 \) in an appropriate sense will be shown by using a standard bootstrap argument.

**Lemma 3.3.** There exists an increasing sequence \( \{R_k\} \) with \( R_k \to \infty \) such that \( \psi_0(R_k)\psi_0'(R_k) \leq 0 \) for each \( k \).

**Proof.** Consider \( \psi_1 = \Theta \psi_0 \), where \( \Theta \) is a lipschitz cutoff function defined by

\[
\Theta(r) = \begin{cases} 
1 & \text{if } 0 \leq r \leq R, \\
R + 1 - r & \text{if } R \leq r \leq R + 1, \\
0 & \text{if } r \geq R + 1.
\end{cases}
\]

Here, \( R \) is a sufficiently large number to be determined. Note that \( (\psi_1')^2 = (\psi_0')^2 \Theta_1^{(0,R)} + (R + 1 - r)^2(\psi_0')^2 \Theta_1^{(R,R+1)} + \psi_0^2(\psi_0')^2 \Theta_1^{(R,R+1)} - 2(R + 1 - r)\psi_0\psi_0' \Theta_1^{(R,R+1)} \). Then clearly \( \psi_1 \in \mathcal{H} \). Let us compute \( I(\psi_1) \) and \( Q(\psi_1) \). Recall \( I(\psi_0) = 1 \) and \( Q(\psi_0) = \mu_0 \).

\[
I(\psi_1) = I(\psi_0) - 15 R \int_0^{\infty} \frac{r^2}{(1 + r^2)^2} \psi_0^2 dr + 15 \frac{4\pi}{R} \int_0^{R+1} \frac{r^2}{(1 + r^2)^2} \psi_0^2 dr
\]

\[
= 1 - \frac{15 R}{4\pi} \left[ \int_0^{R+1} \frac{r^2}{(1 + r^2)} \psi_0^2 dr \right]
\]

\[
\geq 1 - C_R
\]

\[
Q(\psi_1) = Q(\psi_0) - \left[ \int_0^{\infty} \frac{r^2}{(1 + r^2)^3} \psi_0^2 dr + 2 \int_0^{\infty} \frac{3r^2 - 1}{(1 + r^2)^4} \psi_0^2 dr \right]
\]

\[
+ \left( R + 1 - r \right)^2 \frac{r^2}{(1 + r^2)^3} \psi_0^2 dr + \left( R + 1 - r \right)^2 \frac{r^2}{(1 + r^2)^3} \psi_0^2 dr
\]

\[
- 2 \int_0^{R+1} \frac{r^2}{(1 + r^2)^3} \psi_0 \psi_0' dr - 2 \int_0^{R+1} \frac{3r^2 - 1}{(1 + r^2)^4} \psi_0^2 dr
\]

\[
\geq \mu_0 - \left[ \int_0^{R+1} \frac{r^2}{(1 + r^2)^3} \psi_0 \psi_0' dr + \int_0^{R+1} \frac{r^2}{(1 + r^2)^3} \psi_0^2 dr 
\]

\[
+ 2 \int_0^{R+1} \frac{3r^2 - 1}{(1 + r^2)^4} \psi_0^2 dr \right]
\]

\[
\geq \mu_0 - 2 \int_0^{R+1} \frac{r^2}{(1 + r^2)^3} \psi_0 \psi_0' dr - D_R
\]

Now suppose the proposition were false. Then there exists large enough \( R_0 \) such that if \( R > R_0 \), then \( -\psi_0 \psi_0' > 0 \) on \( (R, R + 1) \). On the other hand, \( D_R \leq \frac{\mu_0}{15} C_R \) for
$R > R_0 > 0$, where $K$ is a fixed constant. So we get the following inequalities.

$$
\frac{Q(\psi_1)}{I(\psi_1)} \geq \frac{\mu_0 + 2 \int_R R^{n+1} (R + 1 - r) \frac{r^2}{(1 + r^2)} \psi_0 \psi_0' dr - D_R}{1 - C_R}
$$

Choosing sufficiently large $R > R_0 > 0$ such that $\mu_0 \geq \frac{K}{R}$, we get $\frac{Q(\psi_1)}{I(\psi_1)} > \mu_0$ which contradicts the definition of $\mu_0$. Therefore the lemma follows. □

Lemma 3.3 with the integration by parts leads to the next proposition that shows the fast decay of $\psi_0$ in a suitable sense. It will play a key role when we prove the main theorem in Section 7 in a sense that the initial data with different weights are not essentially different.

**Proposition 3.4.** \(\int_0^\infty \frac{r^{n+2}}{(1 + r^2)^2} \psi_0^2 dr\) and \(\int_0^\infty \frac{r^{n+2}}{(1 + r^2)^2} \psi_0'^2 dr\) are bounded for each nonnegative integer $n$.

Proof. Let $R_k$ be given as in Lemma 3.3. Multiply (3.1) by $r^n \psi_0$ and integrate over $(0, R_k)$. Then we get

$$
\frac{15\mu_0}{4\pi} \int_0^{R_k} \frac{r^{n+2}}{(1 + r^2)^2} \psi_0'^2 dr = \int_0^{R_k} r^n (\frac{r^2}{(1 + r^2)^3} \psi_0')' \psi_0 dr + 2 \int_0^{R_k} r^n \frac{3r^2 - 1}{(1 + r^2)^3} \psi_0'^2 dr
$$

The first term on the RHS of (3.4) can be rewritten as following using the integration by parts,

$$
\frac{R_k^{n+2}}{(1 + R_k^2)^2} \psi_0'(R_k) \psi_0(R_k) - \int_0^{R_k} (n r^{n-1} \psi_0 + r^n \psi_0') \frac{r^2}{(1 + r^2)^3} \psi_0 dr
$$

$$
= \frac{R_k^{n+2}}{(1 + R_k^2)^3} \psi_0'(R_k) \psi_0(R_k) - \int_0^{R_k} \frac{n r^{n+1}}{(1 + r^2)^3} (r^2 \psi_0')' + (\psi_0')^2 dr
$$

$$
= \frac{R_k^{n+2}}{(1 + R_k^2)^3} \psi_0'(R_k) \psi_0(R_k) - \frac{n}{2} \int_0^{R_k} \frac{R_k^{n+1}}{(1 + R_k^2)^3} \psi_0'^2(R_k) dr - \frac{n}{2} \int_0^{R_k} \frac{R_k^{n+2}}{(1 + r^2)^3} (\psi_0')^2 dr
$$

$$
+ n \int_0^{R_k} \frac{(n - 5) r^{n+2} + (n + 1) r^n}{(1 + r^2)^4} \psi_0'^2 dr
$$

Plugging this into (3.4), we get the following relation.

$$
\frac{15\mu_0}{4\pi} \int_0^{R_k} \frac{r^{n+2}}{(1 + r^2)^2} \psi_0'^2 dr + \int_0^{R_k} \frac{r^{n+2}}{(1 + r^2)^3} (\psi_0')^2 dr
$$

$$
+ \frac{R_k^{n+1}}{(1 + R_k^2)^3} \int_0^{R_k} \frac{n}{2} \psi_0'^2(R_k) - R_k \psi_0'(R_k) \psi_0(R_k)
$$

$$
= n \int_0^{R_k} \frac{(n - 5) r^{n+2} + (n + 1) r^n}{(1 + r^2)^4} \psi_0'^2 dr + 2 \int_0^{R_k} \frac{r^n}{(1 + r^2)^3} \psi_0'^2 dr + 2 \int_0^{R_k} \frac{3r^2 - 1}{(1 + r^2)^3} \psi_0'^2 dr
$$

Observe that for each $R_k$, each term of the LHS of (3.5) is nonnegative because of the previous lemma and the small $r$ parts of the RHS of (3.5) are finite due to the behavior of $\psi_0$ near the origin. First, when $n = 0, 1, 2, 3$, all terms
in the RHS of (3.5) are finite for any $R_k$ since they are uniformly bounded by
\[ \frac{15}{4\pi} \int_0^\infty \frac{r^2}{(1+r^2)^{\frac{5}{2}}} \psi_0^2 dr = 1. \]
As taking the limit of $R_k$, we know each term of the LHS converges as $R_k \to \infty$. In particular, $\int_0^\infty \frac{r^{n+2}}{(1+r^2)^{\frac{5}{2}}} \psi_0^2 dr$ and $\int_0^\infty \frac{r^{n+2}}{(1+r^2)^{\frac{3}{2}}} (\psi_0^2)' dr$ are bounded for $n = 0, 1, 2, 3$. The standard induction on $n$ with (3.4) gives the desired result for general $n$. □

4. The Linear Growth Rate

In this section, we show that $\sqrt{\mu_0}$ is the dominating exponential growth for the linearized Euler-Poisson equations. The linearized Euler-Poisson equations (2.1) and (2.2) for $\gamma = \frac{6}{5}$ are

\[ \sigma_t + \frac{1}{r^2} (r^2 \rho_0 u)_r = 0 \quad \text{(4.1)} \]

\[ u_t + \frac{4\pi}{15} (\rho_0^{-\frac{1}{5}} \sigma)_r + \frac{4\pi}{r^2} \int_0^r \sigma s^2 ds = 0 \quad \text{(4.2)} \]

Multiply (4.2) by $\frac{15}{4\pi} r^2 \rho_0$, take the $t$ derivative and use (4.1) to get rid of $\sigma$.

\[ \frac{15}{4\pi} r^2 \rho_0 u_{tt} = -r^2 \rho_0 (\rho_0^{-\frac{1}{5}} \sigma)_{rr} - 15 \rho_0 \int_0^r \sigma s^2 ds \]

\[ = r^2 \rho_0 \left( \frac{1}{r^2} \rho_0^{-\frac{1}{5}} (r^2 \rho_0 u)_r \right)_r + 15 r^2 \rho_0^2 u \]

\[ = r^2 \rho_0^{\frac{3}{5}} u_{rr} + \left\{ r^2 \rho_0 (\rho_0^{\frac{1}{5}})_{rr} + \rho_0^{\frac{1}{5}} (r^2 \rho_0)_r \right\} u_r \]

\[ + \left\{ r^2 \rho_0 \left( \frac{1}{r^2} \rho_0^{-\frac{1}{5}} (r^2 \rho_0)_{rr} \right) + 15 r^2 \rho_0^2 \right\} u \]

After putting back $\rho_0 = \frac{1}{(1+r^2)^{\frac{5}{2}}}$ in the above, we get an equivalent 2nd order equation for $u \equiv \Psi$:

\[ \frac{15}{4\pi} \frac{r^2}{(1+r^2)^{\frac{5}{2}}} \Psi_{tt} = \left( \frac{r^2}{(1+r^2)^3} \Psi_r \right)_r + 2 \left( \frac{3r^2 - 1}{(1+r^2)^4} \right) \Psi \quad \text{(4.3)} \]

Here we use $\Psi$ instead of $u$ in order to distinguish the linear analysis from the nonlinear one. In this way of writing it is also easy to compare (4.3) with (2.8). Note that $\Psi$ in (4.3) is a function of both $t$ and $r$ while $\psi$ in (2.8) is a function of only $r$. Denote the RHS of (4.3) by $L_0 \Psi$. $L_0$ is basically same as $L$ in Section 2.

Define the following quantities.

\[ W_0(r) \equiv \frac{15}{4\pi} \frac{r^2}{(1+r^2)^{\frac{5}{2}}} \rho_0 \]

\[ \|f\|_W^2 \equiv (W, f) = \int_0^\infty W f^2 dr \text{ where } W \text{ is a given weight function} \]

\[ P_f(t) \equiv \int_0^\infty \frac{r^2}{(1+r^2)^{\frac{3}{2}}} f^2(t) dr + 2 \int_0^\infty \frac{1}{(1+r^2)^{\frac{3}{2}}} f^2(t) dr \]
Lemma 4.1. For every solution $\Psi$ to (4.3) there exists $C_{\mu_0}, C_{\mu_0, \alpha} > 0$ such that

(1) $\|\Psi(t)\|_{W_0}, \|\Psi_t(t)\|_{W_0} \leq C_{\mu_0} e^{\sqrt{\mu_0}} (\|\Psi_t(0)\|_{W_0} + \|\Psi(0)\|_{W_0} + \sqrt{P_{\Psi}(0)})$.

(2) For any $\alpha \geq 1$,

$$\|\partial_t^{\alpha+1}\Psi(t)\|_{W_0} \leq C_{\mu_0, \alpha} e^{\sqrt{\mu_0}} (\|\Psi_t(0)\|_{W_0} + \|\Psi(0)\|_{W_0} + \sqrt{P_{\Psi}(0)}) + C_{\mu_0, \alpha} \sum_{i=1}^{\alpha} (\|\partial_t^{i+1}\Psi(0)\|_{W_0} + \sqrt{P_{\partial_t^i\Psi}(0)}).$$

Proof. Take the inner product of (4.3) with $\Psi$. Then we get

$$(W_0 \Psi_{tt}, \Psi_t) = (L_0 \Psi, \Psi_t)$$

$\iff \frac{d}{dt} (W_0 \Psi_t, \Psi_t) = \frac{d}{dt} (L_0 \Psi, \Psi).$$

The above equivalence comes from the self-adjointness of $L_0$. Next integrate the above with respect to $t$ to get

(4.4) $(W_0 \Psi_t(t), \Psi_t(t)) = (L_0 \Psi(t), \Psi(t)) + (W_0 \Psi_t(0), \Psi_t(0)) - (L_0 \Psi(0), \Psi(0)).$

Since $(L_0 \Psi(t), \Psi(t)) \leq \mu_0 (W_0 \Psi(t), \Psi(t))$ for all $t$ and $-(L_0 \Psi(0), \Psi(0)) \leq P_{\Psi}(0)$, from (4.4) we get

(4.5) $\|\Psi_t(t)\|_{W_0}^2 \leq \mu_0 \|\Psi(t)\|_{W_0}^2 + \|\Psi_t(0)\|_{W_0}^2 + P_{\Psi}(0).$

Since $\|\Psi(t)\|_{W} \leq \int_0^t \|\Psi_t(\tau)\|_W d\tau + \|\Psi(0)\|_{W}$, plugging this into (4.5), we get

$$\|\Psi_t(t)\|_{W_0} \leq \sqrt{\mu_0} \int_0^t \|\Psi_t(\tau)\|_{W_0} d\tau + C (\|\Psi_t(0)\|_{W_0} + \|\Psi(0)\|_{W_0} + \sqrt{P_{\Psi}(0)}).$$

By Gronwall's inequality, we obtain

$$\|\Psi_t(t)\|_{W_0} \leq C e^{\sqrt{\mu_0}} (\|\Psi_t(0)\|_{W_0} + \|\Psi(0)\|_{W_0} + \sqrt{P_{\Psi}(0)})$$

and in success

$$\|\Psi(t)\|_{W_0} \leq C e^{\sqrt{\mu_0}} (\|\Psi_t(0)\|_{W_0} + \|\Psi(0)\|_{W_0} + \sqrt{P_{\Psi}(0)}).$$

Notice that $C$ only depends on $\mu_0$. For higher derivatives, take $\partial_t^\alpha$ of (4.3): $W_0(r) \partial_t^\alpha \Psi_{tt} = L_0 \partial_t^\alpha \Psi$. Take the inner product of this with $\partial_t^\alpha \Psi$ to get

$$\|\partial_t^{\alpha+1}\Psi(t)\|_{W_0}^2 = (W_0 \partial_t^{\alpha+1}\Psi(t), \partial_t^{\alpha+1}\Psi(t))$$

$$= (L_0 \partial_t^\alpha \Psi(t), \partial_t^\alpha \Psi(t)) + (W_0 \partial_t^{\alpha+1}\Psi_t(0), \partial_t^{\alpha+1}\Psi_t(0)) - (L_0 \partial_t^\alpha \Psi(0), \partial_t^\alpha \Psi(0))$$

$$\leq \mu_0 \|\partial_t^\alpha \Psi(t)\|_{W_0}^2 + \|\partial_t^{\alpha+1}\Psi(0)\|_{W_0}^2 + P_{\partial_t^\alpha \Psi}(0).$$

Thus (2) easily follows. $\square$
Next we show that the energy estimates with Lemma 4.1 lead to the same exponential growth rate on the $\sigma$ satisfying (4.1) and (4.2). To avoid the confusion with the nonlinear analysis, we use $\Phi$ instead of $\sigma$.

Define a weight function for $\Phi$ by

$$V_0(r) \equiv r^2 - \frac{\hat{4}}{\hat{3}} = r^2(1 + r^2)^2.$$  

Notice that $(V_0, W_0)$ is chosen not randomly but to be a symmetrizer of (4.1) and (4.2) that makes the energy estimate work. $\|\Phi\|_{V_0} + \|\Psi\|_{W_0}$ resembles the real energy (1.7).

**Lemma 4.2.** There are constants $C_1, C_2 > 0$ such that

$$\|\partial_t^\alpha \Psi(t)\|_{V_0} \leq C_1 \left\{ \|\partial_t^{\alpha-1} \Psi(t)\|_{W_0} + \|\partial_t^\alpha \Psi(0)\|_{V_0} + \|\partial_t^\alpha \Psi(0)\|_{W_0} \right\} \text{ for any } \alpha \geq 1$$

and $\|\Phi(t)\|_{V_0} \leq C_2 e^{\sqrt{\rho_0}} \left\{ \|\Phi(0)\|_{V_0} + \|\Psi(0)\|_{W_0} + \|\Psi(t)\|_{W_0} + \|\Psi(0)\| W_0 + \sqrt{\rho_0}(\Phi(0)) \right\}$.

**Proof.** Fix $\alpha \geq 1$. Compute the following equation:

$$\int_0^\infty \left[ \rho_0^{-\frac{\hat{4}}{\hat{3}}} \partial_t^\alpha \Phi \cdot \partial_t^{\alpha-1} (4.1) + \frac{15}{\hat{4}\pi} \rho_0 \partial_t^\alpha \Psi \cdot \partial_t^\alpha (4.2) \right] r^2 dr = 0.$$

The choice of weight functions $\rho_0^{-\frac{\hat{4}}{\hat{3}}}$ and $\frac{15}{\hat{4}\pi} \rho_0$ above yields a nice cancellation after integrating by parts, i.e. $\int_0^\infty \rho_0^{-\frac{\hat{4}}{\hat{3}}} \partial_t^\alpha \Phi \cdot (r^2 \rho_0 \partial_t^\alpha \Psi) dr + \frac{15}{\hat{4}\pi} \rho_0 \partial_t^\alpha \Psi \cdot (\rho_0^{-\frac{\hat{4}}{\hat{3}}} \partial_t^\alpha \Phi, dr = 0$

and it results in

$$(4.6) \quad \frac{d}{dt} \int_0^\infty \left[ \rho_0^{-\frac{\hat{4}}{\hat{3}}} (\partial_t^\alpha \Phi)^2 + \frac{15}{\hat{4}\pi} \rho_0 (\partial_t^\alpha \Psi)^2 \right] r^2 dr + 15 \int_0^\infty \rho_0 \partial_t^\alpha \Psi \left( \int_0^r \partial_t^\alpha \Phi s^2 ds \right) dr = 0.$$

On the other hand, by (4.1), we get $\int_0^\infty \partial_t^\alpha \Phi s^2 ds = -r^2 \rho_0 \partial_t^{\alpha-1} \Psi$ and hence (4.6) can be rewritten as

$$(4.7) \quad \frac{d}{dt} \int_0^\infty \rho_0^{-\frac{\hat{4}}{\hat{3}}} (\partial_t^\alpha \Phi)^2 r^2 dr + \int_0^\infty \frac{15}{\hat{4}\pi} \rho_0 (\partial_t^\alpha \Psi)^2 r^2 dr = \frac{15 d}{2} \int_0^\infty \rho_0 \partial_t^{\alpha-1} \Psi \cdot r^2 dr.$$

As taking $\int_0^\infty$ of (4.7), the desired result is obtained. Note that we have used $\rho_0^2 \leq \rho_0$. As for $\alpha = 0$, utilize $\|\Phi(t)\|_{V_0} \leq \int_0^\infty \|\Phi(\tau)\|_{V_0} d\tau + \|\Phi(0)\|_{V_0}$. □

The next two lemmas show that $\rho_0$ also determines the exponential growth rate even with strong weights. Lemma 4.3 will play a crucial role in the proof of Theorem 1.1. Main idea of proofs is to utilize the linear operator $L$. The results only contain $\Psi$ estimates and the estimates on $\Phi$ can be derived similarly.

**Lemma 4.3.** For any $\alpha \geq 0$, there exists $C_{\rho_0} > 0$ such that

$$\int_0^\infty \frac{r^2}{(1 + r^2)^{\frac{\hat{4}}{\hat{3}}}} (\partial_t^\alpha \Psi)^2 dr + 2 \int_0^\infty \frac{1}{(1 + r^2)^\frac{\hat{4}}{\hat{3}}} (\partial_t^\alpha \Psi)^2 dr \leq C_{\rho_0} e^{\sqrt{\rho_0}} I_\alpha,$$

where $I_\alpha = \sum_{i=0}^{\alpha+2} \|\partial_t^i \Psi(0)\|_{W_0} + \sum_{i=0}^{\alpha+1} P_{\partial_t^i} \Psi(0)$ is given initial data, for every solution $\Psi$ to (4.3).
Proof. Multiply $\partial_t^\alpha (\ref{eq:4.3})$ by $\partial_t^\alpha \Psi$ and integrate to get
\begin{equation}
\int_0^\infty W_0 \partial_t^{\alpha+2} \partial_t^\alpha \Psi dr = -\int_0^\infty \frac{r^2}{(1+r^2)^3} (\partial_t^\alpha \Psi_r)^2 dr + 2 \int_0^\infty \frac{3r^2 - 1}{(1+r^2)^4} (\partial_t^\alpha \Psi)^2 dr.
\end{equation}
If we move two negative terms in the RHS of (4.8) into the LHS, we obtain
\begin{align*}
\int_0^\infty & \frac{r^2}{(1+r^2)^3} (\partial_t^\alpha \Psi_r)^2 dr + 2 \int_0^\infty \frac{1}{(1+r^2)^4} (\partial_t^\alpha \Psi)^2 dr \\
& \phantom{=} - \int_0^\infty W_0 \partial_t^{\alpha+2} \partial_t^\alpha \Psi dr \\
& \leq 6 \int_0^\infty \frac{r^2}{(1+r^2)^4} (\partial_t^\alpha \Psi)^2 dr + \frac{1}{2} (\|\partial_t^{\alpha+2} \Psi(t)\|_{W_0}^2 + \|\partial_t^\alpha \Psi(t)\|_{W_0}^2) \\
& \leq C (\|\partial_t^{\alpha+2} \Psi(t)\|_{W_0}^2 + \|\partial_t^\alpha \Psi(t)\|_{W_0}^2)
\end{align*}
But the last quantity is bounded by $Ce^{2\sqrt{m} I_\alpha}$ by Lemma 4.1 and the conclusion follows. \square

For the next lemma define the weight functions $W_l$ for $l \leq 0$ by
\begin{equation*}
W_l(r) \equiv (1+r^2)^{-\frac{l+1}{2}} W_0(r) = \frac{15}{4\pi} r^2 r_0^l.
\end{equation*}
We remark that $W_l$ is a linear version of symmetrizers $S_l$ defined in Section 5.

Lemma 4.4. For any $\alpha \geq 0$, there exists $C_\mu > 0$ such that
\begin{equation*}
\|\partial_t^\alpha \Psi(t)\|_{W_0} \leq C_\mu e^{\sqrt{m} I_\alpha},
\end{equation*}
where $I_\alpha$ is defined in Lemma 4.3, for every solution $\Psi$ to (4.3).

Proof. Let us only consider $\alpha = 0, 1$. Other cases can be treated in the similar way. Letting $L_k = (1+r^2)^{\frac{k}{2}} L_0$, we get $(W_k \Psi_m, \Psi_l) = (L_k \Psi, \Psi_l)$. But because $L_k$ is not self-adjoint any more, we do not have a simple equivalent expression as in Lemma 4.1. Instead, we try to find a relation between $L_k$ and $L_0$. The following identities are needed and they are obtained by definitions and straightforward computations.

\begin{align*}
(L_k \Psi, \Psi) &= \int_0^\infty (1+r^2)^{\frac{k}{2}} \Psi \frac{r^2}{(1+r^2)^3} \psi_r dr + 2 \int_0^\infty (1+r^2)^{\frac{k}{2}} \frac{3r^2 - 1}{(1+r^2)^4} \Psi^2 dr \\
(L_0(\Psi(1+r^2)^{\frac{k}{2}}), \Psi(1+r^2)^{\frac{k}{2}}) &= -\int_0^\infty \frac{r^2}{(1+r^2)^3} \{ (1+r^2)^{\frac{k}{2}} \psi_r^2 + kr(1+r^2)^{\frac{k-1}{2}} \psi r \\
&\quad + \frac{k^2}{4} (1+r^2)^{\frac{k}{2}-2} \psi^2 \} dr + 2 \int_0^\infty \frac{3r^2 - 1}{(1+r^2)^4} (1+r^2)^{\frac{k}{2}} \psi^2 dr
\end{align*}
The last two identities imply the following:
\begin{equation*}
(L_k \Psi, \Psi) = (L_0(\Psi(1+r^2)^{\frac{k}{2}}), \Psi(1+r^2)^{\frac{k}{2}}) + \frac{k^2}{4} \int_0^\infty \frac{r^4}{(1+r^2)^3} (\Psi(1+r^2)^{\frac{k}{2}})^2 dr.
\end{equation*}
Define $Q_k$ and $I_k$ similar to $Q$ and $I$ in Section 2 as following

$$Q_k(\Psi) \equiv (L_0(\Psi(1 + r^2)^{\frac{3}{2}}), \Psi(1 + r^2)^{\frac{3}{2}}) - \beta \int_0^\infty \frac{r^4}{(1 + r^2)^\frac{7}{2}} (\Psi(1 + r^2)^{\frac{5}{2}})^2 dr,$$

$$I_k(\Psi) \equiv I(\Psi(1 + r^2)^{\frac{3}{2}}) = (W_k(\Psi), \Psi).$$

where $\beta$ is a small positive constant. Then by doing the same variational analysis in Section 2, one can show that there exists $\mu_k > 0$ such that $\mu_k$ is the maximum of a functional $\frac{Q_k(\Psi)}{I_k(\Psi)}$ and hence $Q_k(\Psi) \leq \mu_k I_k(\Psi)$.

Claim. $\mu_k < \mu_0$.

To see the Claim, pick a $\Psi_1$ such that $Q_k(\Psi_1) = \mu_k$ and $I_k(\Psi_1) = 1$. By the definition of $Q_k$,

$$Q_k(\Psi_1) = (L_0(\Psi_1(1 + r^2)^{\frac{3}{2}}), \Psi_1(1 + r^2)^{\frac{3}{2}}) - \beta \int_0^\infty \frac{r^4}{(1 + r^2)^\frac{7}{2}} (\Psi_1(1 + r^2)^{\frac{5}{2}})^2 dr.$$

Since $(L_0(\Psi_1(1 + r^2)^{\frac{3}{2}}), \Psi_1(1 + r^2)^{\frac{3}{2}}) \leq \mu_0 I(\Psi_1(1 + r^2)^{\frac{3}{2}}) = \mu_0 I_k(\Psi_1) = \mu_0$,

$$Q_k(\Psi_1) = \mu_k \leq \mu_0 - \beta \int_0^\infty \frac{r^4}{(1 + r^2)^\frac{7}{2}} (\Psi_1(1 + r^2)^{\frac{5}{2}})^2 dr < \mu_0.$$

Thus,

$$Q_k(\Psi) = Q_k(\Psi_1) + (\beta + \frac{k^2}{4}) \int_0^\infty \frac{r^4}{(1 + r^2)^\frac{7}{2}} (\Psi(1 + r^2)^{\frac{5}{2}})^2 dr$$

$$\leq \mu_k (W_k(\Psi, \Psi)) + (\beta + \frac{k^2}{4}) \int_0^\infty \frac{r^4}{(1 + r^2)^\frac{7}{2} - \frac{k}{2}} \Psi^2 dr$$

(4.9)

Now we are ready to go back to $(W_k(\Psi_t, \Psi_t)) = (L_k(\Psi, \Psi_t))$.

$$L_k(\Psi, \Psi_t) = \int_0^\infty (\frac{r^2}{1 + r^2})^3 \Psi_r r (1 + r^2)^{\frac{3}{2}} \Psi_t dr + 2 \int_0^\infty \frac{3r^2 - 1}{1 + r^2} (1 + r^2)^{\frac{3}{2}} \Psi \Psi_t dr$$

$$= - \int_0^\infty \frac{r^2}{1 + r^2} \Psi_r \psi r \psi r dr - k \int_0^\infty \frac{r^3}{1 + r^2} \Psi_r \psi r \psi r dr + 2 \int_0^\infty \frac{3r^2 - 1}{1 + r^2} \Psi \Psi_t dr$$

$$= \frac{1}{2} \frac{d}{dt} (W_k(\Psi, \Psi_t)) - k \int_0^\infty \frac{r^3}{1 + r^2} \Psi \Psi_t dr - k \int_0^\infty \frac{r^3}{1 + r^2} \Psi \Psi_t dr$$

Hence,

$$\frac{1}{2} \frac{d}{dt} (W_k(\Psi, \Psi_t)) = \frac{1}{2} \frac{d}{dt} (L_k(\Psi, \Psi) - k \int_0^\infty \frac{r^3}{1 + r^2} \Psi \Psi_t dr$$

(4.10)

Let $k = 1$. By Lemma 4.1 and 4.3, using the Cauchy-Schwarz inequality,

$$- \int_0^\infty \frac{r^3}{(1 + r^2)^\frac{3}{2}} \Psi \Psi_t dr - \int_0^\infty \frac{r^3}{(1 + r^2)^\frac{3}{2}} \Psi \Psi_t dr \leq C e^{2\sqrt{\mu_0}t} I_0$$

for a constant $C$ and initial data $I_0$. Rewriting the above (4.10) when $k = 1$, get

$$\frac{1}{2} \frac{d}{dt} (W_1(\Psi, \Psi_t)) \leq \frac{1}{2} \frac{d}{dt} (L_1(\Psi, \Psi)) + C e^{2\sqrt{\mu_0}t} I_0.$$
Taking the integral with respect to \( t \) and using (4.9), we have
\[
(W_1 \Psi_t, \Psi_t) \leq (L_1 \Psi, \Psi) + Ce^{2\sqrt{\mu_0}t}I_0
\]
\[
\leq \mu_1 (W_1 \Psi, \Psi) + (\beta + \frac{1}{4}) \int_0^\infty \frac{r^4}{(1 + r^2)^2} \Psi^2 dr + Ce^{2\sqrt{\mu_0}t}I_0
\]

Using Lemma 4.1 again, we obtain
\[
(4.11) (W_1 \Psi_t, \Psi_t) \leq \mu_1 (W_1 \Psi, \Psi) + Ce^{2\sqrt{\mu_0}t}I_0.
\]

Since \( \|\Psi(t)\|_{W_1} \leq \int_0^t \|\Psi_\tau(\tau)\|_{W_1} d\tau + \|\Psi(0)\|_{W_1} \), combining this with (4.11), get
\[
\|\Psi(t)\|_{W_1} \leq \sqrt{\mu_1} \int_0^t \|\Psi(\tau)\|_{W_1} d\tau + Ce^{\sqrt{\mu_0}t} \sqrt{I_0}.
\]

Since we have \( \sqrt{\mu_1} < \sqrt{\mu_0} \), Gronwall inequality gives \( \|\Psi(t)\|_{W_1} \leq Ce^{\sqrt{\mu_0}t} \sqrt{I_0} \) as well as \( \|\Psi_t(t)\|_{W_1} \leq Ce^{\sqrt{\mu_0}t} \sqrt{I_0} \). The standard induction on \( k \) with (4.10) claims the desired result for all \( k \).

5. Weighted Instant Energy and Weighted Total Energy

In this section, by the utilization of symmetrizers of the Euler-Poisson system, we introduce suitable measurements of perturbations \( \sigma \) and \( u \) of steady states resembling weighted Sobolev norms: the weighted instant energy \( E_l \) and the weighted total energy \( \tilde{E}_l \), where \( l \) is an index associated to weights. The symmetrizers will play the same role as weights in the linear analysis for small solutions. Then the total energy is shown to be bounded by the instant energy under a certain smallness assumption in using the equations directly, which makes it sufficient to play only with the instant energy. The weighted energy estimates for the instant energy will be carried out in the next section. Before going any farther we remark that it is convenient to work on rectangular coordinates rather than polar coordinates because one can avoid the singularity of the origin coming from the spherical symmetry.

We are interested in sufficiently small solutions \( \sigma, u \) satisfying the neutrality condition
\[
\int_{\mathbb{R}^3} \sigma dx = 0.
\]
\( \sigma \) is assumed to be relatively smaller than \( \rho_0 \), in particular, we assume
\[
\frac{9}{10} \rho_0 \leq \rho_0 + \sigma \leq \frac{11}{10} \rho_0.
\]

For such small solutions the Euler-Poisson system (1.1), (1.2) and (1.3) can be rewritten in the rectangular coordinates as the following:

\[
(5.2) \quad \sigma_t + (\rho_0 + \sigma) \nabla \cdot u + \nabla (\rho_0 + \sigma) \cdot u = 0
\]

\[
(5.3) \quad u_t + (u \cdot \nabla) u + \frac{4\pi}{15} (\rho_0 + \sigma) - \frac{4}{5} \nabla \sigma - \{ \frac{16\pi}{75} \rho_0 \frac{-2}{7} \sigma + h(\sigma, \rho_0) \} \nabla \rho_0 + \nabla \phi = 0
\]

\[
(5.4) \quad \Delta \phi = 4\pi \sigma
\]
where \( h(\sigma, \rho_0) = -\frac{4\pi}{15} (\rho_0 + \sigma)^{-\frac{4}{5}} - \frac{4}{5} \rho_0^{-\frac{2}{5}} \) represents higher order terms. 

\( u \) takes the vector form, i.e. \( u(x, t) = u(r, t) \frac{x}{|x|} \), where \( x \in \mathbb{R}^3 \), \( r = |x| \) and we denote each component of \( u \) by \( u^k \). Note \( \nabla \times u = 0 \).

Now let us consider the symmetrizers \( S_l \) where \( l \in \mathbb{R} \) for the Euler-Poisson system.

\[
S_l = \left( \begin{array}{cccc}
\frac{4\pi}{15} (\rho_0 + \sigma)^{-\frac{4}{5} + l} & 0 & 0 & 0 \\
0 & (\rho_0 + \sigma)^{1 + l} & 0 & 0 \\
0 & 0 & (\rho_0 + \sigma)^{1 + l} & 0 \\
0 & 0 & 0 & (\rho_0 + \sigma)^{1 + l}
\end{array} \right)
\]

Define the instant energy \( E_l(t) \) and the total energy \( \tilde{E}_l(t) \) by

\[
E_l(t) = \sum_{j=0}^{3} \int_{\mathbb{R}^3} S_l (\partial_t^l \sigma, \partial_t^l u^1, \partial_t^l u^2, \partial_t^l u^3) : (\partial_t^l \sigma, \partial_t^l u^1, \partial_t^l u^2, \partial_t^l u^3) dx
\]

\[
= \sum_{j=0}^{3} \int_{\mathbb{R}^3} \frac{4\pi}{15} (\rho_0 + \sigma)^{-\frac{4}{5} + l} (\partial_t^l \sigma)^2 + (\rho_0 + \sigma)^{1 + l} |\partial_t^l u|^2 dx
\]

\[
\equiv \sum_{j=0}^{3} E_l^j
\]

\[
\tilde{E}_l(t) = \sum_{j=0}^{3} \sum_{i=0}^{j} \int_{\mathbb{R}^3} \frac{4\pi}{15} (\rho_0 + \sigma)^{-\frac{4}{5} + l + \frac{i}{j}} |\partial_t^l \sigma|^2 + (\rho_0 + \sigma)^{1 + l + \frac{i}{j}} |\partial_t^l u|^2 dx
\]

\[
\equiv \sum_{j=0}^{3} \sum_{i=0}^{j} \tilde{E}_l^{j,i}
\]

Here \( \partial_x \) represents any spatial first derivatives. Note that \( E_0^0(t) \) is a part of the real energy \( 2E \) of \((\rho, u)\) defined in (1.7). The case \( l = 0 \), however, is not enough for our purpose because we cannot close the energy estimate at the step \( l = 0 \). Being convinced by Lemma 4.4, we try different \( l \)'s as well. As \( l < 0 \) gets smaller, the weights become stronger due to the behavior of \( \rho_0 \). Unfortunately, as \( l \) varies, new quadratic terms come out while performing the energy estimates. This phenomenon seems undesirable but it turns out that they are equipped with weaker weights. And that opens another door.

Observe that the weights of mixed derivative terms in \( \tilde{E}_l \) are different, in fact a little better, from the ones of temporal derivative terms. This is not a coincidence but rather a nature of the system; the same feature can be seen in the linear analysis. See Lemma 4.3. \( \tilde{E}_l \) contains all the spatial and mixed derivatives of \( \sigma, u \) and it is easy to see that

\[
E_l = \sum_{j=0}^{3} E_l^{j,0} \leq \tilde{E}_l.
\]

Now we want to show the converse, in other words, \( \tilde{E}_l \) is also bounded by \( E_l \) under a certain smallness assumption:

\[
(5.5) \quad \tilde{E}_{-\frac{4}{5}} + \tilde{E}_{-\frac{2}{5}} + \tilde{E}_{-\frac{4}{5}} \leq \theta_1
\]
where $\theta_1$ is a sufficiently small constant. In order to appreciate the utilization of such an assumption, first we prove the next lemma.

Notation. $\int dx$ represents $\int_{\mathbb{R}^3} dx$ and when dealing with line integrals $\int dr$, each end value will be specified.

**Lemma 5.1.** Suppose (5.5) holds for $0 \leq t \leq T$. Then there exists a constant $C > 0$ such that for each $0 \leq t \leq T$,

\begin{equation}
\sup_{x \in \mathbb{R}^3} |\frac{\sigma}{\rho_0 + \sigma}| + |\frac{\sigma_t}{\rho_0 + \sigma}| + \left| \frac{\nabla \sigma}{(\rho_0 + \sigma)^{\frac{1}{2}}} \right| + \left| \frac{u}{(\rho_0 + \sigma)^{\frac{1}{2}}} \right| + |u_t| + |\nabla u| \leq C \sqrt{\theta_1}.
\end{equation}

In particular, the assumption (5.1) is justified.

Proof. The above smallness assumption (5.5) together with the Sobolev imbedding theorem yields the result. To see how it works, let us apply the Sobolev imbedding theorem to $|\frac{\sigma}{\rho_0 + \sigma}|$.

\begin{equation}
\sup_{x \in \mathbb{R}^3} |\frac{\sigma}{\rho_0 + \sigma}|^2 \leq C \left( |\frac{\sigma}{\rho_0 + \sigma}|^2 \right)_{L^2(\mathbb{R}^3)}
\end{equation}

\begin{align*}
&\leq C \left\{ \int |\frac{\sigma}{\rho_0 + \sigma}|^2 dx + \int |\frac{\partial_\sigma \sigma}{\rho_0 + \sigma}|^2 dx + \int |\frac{\partial_\sigma^2 \sigma}{\rho_0 + \sigma}|^2 dx \\
&+ \int |\frac{\partial_x (\rho_0 + \sigma)}{\rho_0 + \sigma}|^2 dx + \int |\frac{\partial_x \sigma \cdot \partial_x (\rho_0 + \sigma)}{\rho_0 + \sigma}|^2 dx \\
&+ \int |\frac{\partial_x^2 (\rho_0 + \sigma)}{\rho_0 + \sigma}|^2 dx + \int \left\{ |\frac{\sigma}{\rho_0 + \sigma}||\frac{\partial_x \sigma \cdot \partial_x (\rho_0 + \sigma)}{\rho_0 + \sigma}|^2 \right\} dx \right\}
\end{align*}

Recall the behavior of $\rho_0$, namely $\rho_0(r) = O(r^{-5})$ for large $r$. Hence $|\frac{\partial_\sigma \sigma}{\rho_0 + \sigma}|$ and $|\frac{\partial_\sigma^2 \sigma}{\rho_0 + \sigma}|$ are uniformly bounded. Thus (5.7) becomes

\begin{align}
\sup_{x \in \mathbb{R}^3} |\frac{\sigma}{\rho_0 + \sigma}|^2 &\leq C \left\{ \mathcal{E}_{-\frac{1}{20}}^{0.0} + \mathcal{E}_{-\frac{1}{20}}^{-1.1} + \mathcal{E}_{-\frac{1}{20}}^{-2.2} \right\} \\
&+ \sup_{x \in \mathbb{R}^3} \left| \frac{\sigma}{\rho_0 + \sigma} \right|^2 \left( \mathcal{E}_{-\frac{1}{20}}^{0.0} + \mathcal{E}_{-\frac{1}{20}}^{-1.1} + \mathcal{E}_{-\frac{1}{20}}^{-2.2} \right) \\
&+ \sup_{x \in \mathbb{R}^3} \left| \frac{\sigma}{\rho_0 + \sigma} \right|^2 \sup_{x \in \mathbb{R}^3} \left| \frac{\partial_x \sigma}{(\rho_0 + \sigma)^{\frac{1}{2}}} \right|^2 \mathcal{E}_{-\frac{1}{4}}^{-0.1} \\
&\leq C(\theta_1 + \sup_{x \in \mathbb{R}^3} \left| \frac{\sigma}{\rho_0 + \sigma} \right|^2 \theta_1 + \sup_{x \in \mathbb{R}^3} \left| \frac{\sigma}{\rho_0 + \sigma} \right|^2 \sup_{x \in \mathbb{R}^3} \left| \frac{\partial_x \sigma}{(\rho_0 + \sigma)^{\frac{1}{2}}} \right|^2 \theta_1).
\end{align}

We can get similar estimates to (5.8) for other terms in (5.6). Call the LHS of (5.6) $S$. Consequently, we obtain the following:

\begin{equation}
S^2 \leq \theta_1 + C S^2 \theta_1 + C S^4 \theta_1.
\end{equation}

Since $\theta_1$ is small enough, (5.9) immediately implies the lemma.\[\square\]

Since $\tilde{\xi}_i^{1,0} = \xi_i^{1,0}$, we only need to show that $\tilde{\xi}_i^{1,1} \leq C \xi_1^{1,1}$ where $1 \leq i \leq j \leq 3$ is bounded by $CS\xi_1$; the precise statement is given in the following lemma. Main idea for accomplishing the goal is to estimate the spatial and mixed derivative terms directly from the equations in terms of temporal derivative terms.
Lemma 5.2. Suppose (5.5) holds for $0 \leq t \leq T$. Let $l \leq 0$. Then there exists $C > 0$ such that for each $1 \leq i \leq j \leq 3$ and for $0 \leq t \leq T$,

$$\mathcal{E}_{t}^{i,j}(t) \leq C \sum_{k=1}^{j} \mathcal{E}_{t}^{k}(t) + C \sum_{k=0}^{j-1} \mathcal{E}_{t}^{k+\frac{1}{2}}.$$  

Proof. Let us start with one spatial derivative terms corresponding to $i = 1$. No temporal derivative terms i.e. when $i = 1, j = 1$ are treated carefully since it is instructive and other cases can be easily shown from it. Various case of $i$ for a fixed $i$ will be done in turn. Then we move onto other cases of $i$. First of all, solve (5.2) and (5.3) for $\nabla \sigma$ and $\nabla \cdot u$ to get

$$\nabla \sigma = -\frac{15}{4\pi} (\rho_0 + \sigma)^{\frac{3}{2}} \left( u_t - \frac{16\pi}{75} \rho_0^{\frac{2}{3}} \nabla \rho_0 \sigma + \nabla \phi + (u \cdot \nabla) u - h \nabla \rho_0 \right)$$ (5.10)

$$\nabla \cdot u = -\frac{1}{\rho_0 + \sigma} \left( \sigma_t + \nabla \rho_0 \cdot u + \nabla \sigma \cdot u \right).$$ (5.11)

Notice that the estimate on $\nabla \cdot u$ is enough for $\partial_x u$ since $\nabla \times u = 0$. In order to get a right weight of $\mathcal{E}_{t}^{1,1}$ for $\sigma$ and $u$, multiply (5.10) and (5.11) by $(\rho_0 + \sigma)^{-\frac{3}{2} + \frac{t}{2}}$ and $(\rho_0 + \sigma)^{\frac{3}{2} - \frac{t}{2}}$ respectively, square and integrate them over $\mathbb{R}^3$.

$$\int (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\nabla \sigma|^2 dx + \int (\rho_0 + \sigma)^{\frac{3}{2} - t} |\nabla \cdot u|^2 dx$$ (5.12)

$$\leq C \int (\rho_0 + \sigma)^{1+t} \left( |u_t|^2 + (\rho_0 - \frac{2}{3}) |\nabla \rho_0|^2 \sigma^2 + |\nabla \phi|^2 + |(u \cdot \nabla) u|^2 + |\nabla \rho_0|^2 h^2 \right) dx$$

$$+ C \int (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\sigma_t|^2 + |\nabla \rho_0 \cdot u|^2 + |\nabla \sigma \cdot u|^2 | dx$$

$$\leq C \int (\rho_0 + \sigma)^{1+t} |u_t|^2 + (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\sigma_t|^2 dx$$

$$+ C \int (\rho_0 + \sigma)^{1+t} |\rho_0^{\frac{2}{3}} \nabla \rho_0|^2 \sigma^2 + (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\nabla \rho_0 \cdot u|^2 dx$$

$$+ C \int (\rho_0 + \sigma)^{1+t} |(u \cdot \nabla) u|^2 + (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\nabla \sigma \cdot u|^2 | dx$$

$$+ C \int (\rho_0 + \sigma)^{1+t} |\nabla \rho_0|^2 h^2 dx + C \int (\rho_0 + \sigma)^{1+t} |\nabla \phi|^2 dx$$

We rearranged terms according to the order of algebraic degree. The key difficulty lies in the potential part and so it will be done last. The first integral is bounded by $C \mathcal{E}_{t}^{1}$ by the definition. Since $|\rho_0^{\frac{2}{3}} \nabla \rho_0| = 5r(1 + r^2) \leq C(\rho_0 + \sigma)^{-\frac{3}{2}}$ and similarly $|\nabla \rho_0| \leq C(\rho_0 + \sigma)^{\frac{3}{2}}$,

$$\int (\rho_0 + \sigma)^{1+t} |\rho_0^{\frac{2}{3}} \nabla \rho_0|^2 \sigma^2 dx + \int (\rho_0 + \sigma)^{-\frac{3}{2} + t} |\nabla \rho_0 \cdot u|^2 dx$$ (5.13)

$$\leq C \int (\rho_0 + \sigma)^{-\frac{3}{2} + t} \sigma^2 dx + C \int (\rho_0 + \sigma)^{\frac{3}{2} - t} |u|^2 dx$$

$$\leq C \mathcal{E}_{t}^{0}.$$
For higher order terms, we use the smallness assumption to get the desired estimates: Lemma 5.1 is used.

\[
\int (\rho_0 + \sigma)^{1+l}|(u \cdot \nabla)u|^2 + (\rho_0 + \sigma)^{-\frac{1}{2}+l} |\nabla \sigma \cdot u|^2 \, dx
\]

\[
(5.14)
\]

\[
\leq \int \left| \frac{u}{(\rho_0 + \sigma)^{\frac{3}{4}}} \right|^2 \{ (\rho_0 + \sigma)^{\frac{5}{4}+l} |\nabla \cdot u| + (\rho_0 + \sigma)^{-\frac{1}{4}+l} |\nabla \sigma|^2 \} \, dx
\]

\[
\leq C\theta_1 \int (\rho_0 + \sigma)^{\frac{5}{4}+l} |\nabla \cdot u|^2 + (\rho_0 + \sigma)^{-\frac{1}{4}+l} |\nabla \sigma|^2 \, dx
\]

Hence the higher order terms in the third integral of the RHS in (5.12) can be absorbed into its LHS, since \( \theta_1 \) is sufficiently small. Because \( h \) is a higher order term depending on only \( \rho_0 \) and \( \sigma \), after applying Lemma 5.1 to the fourth integral in (5.2), we get

\[
(5.15) \quad \int (\rho_0 + \sigma)^{1+l} |\nabla \rho_0|^2 \, dx \leq C\theta_1 \mathcal{C}_0^0 \frac{1}{l+\frac{3}{8}}.
\]

In order to handle the potential part, we write the poisson equation (5.4) for the spherically symmetric case in polar coordinates as

\[
\phi_{rr} + \frac{2}{r} \phi_r = 4\pi \sigma \quad \text{where} \quad \phi_r = \frac{4\pi}{r} \int_0^r \sigma s^2 \, ds.
\]

Recall the \( L^p \) estimates \( \| \partial^2 \phi \|_{L^2(\mathbb{R}^3)} \leq C \| \sigma \|_{L^2(\mathbb{R}^3)} \) and in the spherically symmetric case it implies that since

\[
\sum_{i,j=1}^3 (\partial_i \partial_j \phi)^2 = \phi_{rr}^2 + \frac{2}{r^2} \phi_r^2,
\]

we have

\[
\| \frac{1}{r} \phi_r \|_{L^2(\mathbb{R}^3)} \leq C \| \sigma \|_{L^2(\mathbb{R}^3)}.
\]

We consider two cases: \( l \geq -\frac{3}{8} \) and \( l < -\frac{3}{8} \). In the first case, \( (\rho_0 + \sigma)^{1+l} \sim \frac{1}{(1 + r^2)^{-\frac{5(l+1)}{2}}} \) is uniformly bounded, and therefore we get

\[
(5.16) \quad \int (\rho_0 + \sigma)^{1+l} |\nabla \phi|^2 \, dx = 4\pi \int_0^1 (\rho_0 + \sigma)^{1+l} \phi_r^2 r^2 \, dr \leq C \int_0^1 \left( \frac{\phi_r}{r} \right)^2 r^2 \, dr \leq C \int \sigma^2 \, dx.
\]

The \( L^p \) estimate has been used at the last inequality. When \( l < -\frac{3}{8} \), we divide the integral into two parts.

\[
(5.17) \quad \int (\rho_0 + \sigma)^{1+l} |\nabla \phi|^2 \, dx = 4\pi \int_0^1 (\rho_0 + \sigma)^{1+l} \phi_r^2 r^2 \, dr
\]

\[
= 4\pi \int_0^1 (\rho_0 + \sigma)^{1+l} \phi_r^2 r^2 \, dr + 4\pi \int_1^\infty (\rho_0 + \sigma)^{1+l} \phi_r^2 r^2 \, dr
\]

\[
\equiv (I) + (II)
\]

In the unit ball we can do the same as in (5.16), since the weight function is bounded.

\[
(5.18) \quad (I) \leq C \int_0^1 \left( \frac{\phi_r}{r} \right)^2 r^2 \, dr \leq C \int_0^1 \sigma^2 r^2 \, dr \leq C \int (\rho_0 + \sigma)^m \sigma^2 \, dx, \text{ for any } m.
\]
For the second term \((II)\), we use the neutrality condition \(\int_{\mathbb{R}^3} \sigma s^2 ds = 0\) which is equivalent to \(\int_0^\infty \sigma s^2 ds = -\int_r^\infty \sigma s^2 ds\) in polar coordinates.

\[(5.19)\]

\[
(II) \leq C \int_1^\infty (1 + r^2)^{-\frac{5(1+i+l)}{2}} \left( \frac{1}{r^2} \right) \int_r^\infty \sigma s^2 ds^2 r^2 dr
\]

\[
= C \int_1^\infty \frac{1}{r^2(1 + r^2)} (1 + r^2)^{-\frac{2}{5}(\frac{1}{2} + l)} \int_r^\infty \sigma s^2 ds^2 dr
\]

\[
\leq C \int_1^\infty \frac{1}{r^2(1 + r^2)} \left\{ \int_r^\infty (1 + s^2)^{-\frac{1}{2}(\frac{1}{2} + l)} \sigma^2 s^2 ds \right\}^2 dr
\]

\[
\leq C \int_1^\infty \frac{1}{r^2(1 + r^2)} \left[ \int_r^\infty (1 + s^2)^{-\frac{1}{2}(\frac{1}{2} + l)} \sigma^2 s^2 ds \right]^2 dr
\]

\[
\leq C \left\{ \int_1^\infty \frac{1}{r^2(1 + r^2)} \left[ \int_r^\infty (1 + s^2)^{-\frac{1}{2}(\frac{1}{2} + l)} \sigma^2 s^2 ds \right] dr \right\}
\]

\[
\leq C \int_1^\infty (\rho_0 + \sigma)^{1+l} \sigma^2 dx
\]

From (5.16), (5.18) and (5.19) we conclude that for any \(l \leq 0\),

\[(5.20)\]

\[
\int (\rho_0 + \sigma)^{1+l} |\nabla \phi|^2 dx \leq C \xi_t^{0+l}.
\]

Thus, from (5.12), (5.13), (5.14), (5.15) and (5.20), we obtain

\[
\tilde{\xi}_{t}^{1,1}(t) \leq C \xi_t^1 + C \xi_t^{0,\frac{1}{2}}.
\]

Next we focus on one spatial, one temporal derivative terms, namely the case \(i = 1, j = 2\). Take \(\partial_t\) of (5.10) and (5.11):

\[(5.21)\]

\[
\nabla \sigma_t = -\frac{3}{\pi \rho_0 + \sigma} \frac{\sigma_t}{\rho_0 + \sigma} \{ u_t - \frac{16\pi}{l_0} \rho_0 - \frac{2}{5} \nabla \rho_0 \sigma + \nabla \phi + (u \cdot \nabla) u - h \nabla \rho_0 \}
\]

\[
- \frac{15}{4\pi} (\rho_0 + \sigma)^{\frac{3}{2}} \{ u_{tt} - \frac{16\pi}{l_0} \rho_0 - \frac{2}{5} \nabla \rho_0 \sigma_t + \nabla \phi_t + (u_t \cdot \nabla) u_t + (u \cdot \nabla) u_t - h_t \nabla \rho_0 \}
\]

\[
\nabla \cdot u_t = \frac{1}{\rho_0 + \sigma} \{ \sigma_t + \nabla \rho_0 \cdot u + \nabla \sigma \cdot u \}
\]

\[
- \frac{1}{\rho_0 + \sigma} \{ \sigma_{tt} + \nabla \rho_0 \cdot u_t + \nabla \sigma_t \cdot u + \nabla \sigma \cdot u_t \}
\]

The first part of (5.21) and (5.22) is bounded by \(\frac{\sigma_t}{\rho_0 + \sigma} \nabla \sigma|\) and \(\frac{\sigma_t}{\rho_0 + \sigma} \nabla \cdot u|\) that have been already estimated at the previous step. Note that \(\frac{\sigma_t}{\rho_0 + \sigma}\) is small enough and it does not cause any trouble. The other part has the same structure as before and therefore we can do the same: multiply the same weights used in the previous case, square and integrate. The potential term is easily taken care of, since the dynamics of \(\nabla \phi_t\) gets simpler and better in the sense that

\[
\nabla \phi_t = 4\pi \nabla \Delta^{-1} \sigma_t = -4\pi \nabla \Delta^{-1} \nabla \cdot (\rho_0 + \sigma)u = -4\pi (\rho_0 + \sigma)u.
\]
After higher order terms being absorbed we obtain
\[
\tilde{E}_t^{2,1} \leq C(\mathcal{E}^2 + \mathcal{E}^1) + C(\mathcal{E}^1 + \mathcal{E}^0 + \mathcal{E}^0).
\]

Considering \(\partial_t(5.21)\) and \(\partial_t(5.22)\), each term in the RHS either has been estimated or can be dealt with in the same manner as the previous cases, and therefore the estimates on \(\partial_t^2 \partial_t^2 \mathcal{E}^1\) and \(\partial_t^2 \partial_t^2 u\) follows:
\[
\tilde{E}_t^{3,1} \leq C(\mathcal{E}^3 + \mathcal{E}^2 + \mathcal{E}^1) + C(\mathcal{E}^1 + \mathcal{E}^0 + \mathcal{E}^0 + \mathcal{E}^0).
\]

Now we move onto two spatial derivative terms, the case \(i = 2\). The only but important difference is to use a different weight to close the estimate. It explains why the total energy is designed with having different weights according to the number of spatial derivatives. Compute \(\partial_x(5.10)\) and \(\partial_x(5.11)\) to get
\[
(5.23)\quad \nabla \partial_x \sigma = -3 \frac{\partial_x (\rho_0 + \sigma)}{\pi (\rho_0 + \sigma)^2} \{u_0 - \frac{16}{T} \rho_0^0 \nabla \rho_0 \sigma + \nabla \phi + (u \cdot \nabla) u - h \nabla \rho_0 \}
- \frac{15}{4\pi} (\rho_0 + \sigma)^{\frac{2}{3}} \{\partial_x u_0 - \frac{16}{T} \rho_0^0 \nabla \partial_x \sigma + \nabla \partial_x \phi + (\partial_x u \cdot \nabla) u + (u \cdot \nabla) \partial_x u
- \partial_x h \nabla \rho_0 - \frac{16}{T} \partial_x (\rho_0^0 \nabla \rho_0) \sigma - h \nabla \partial_x \rho_0 \}
\]
\[
\nabla \cdot \partial_x u = \frac{\partial_x (\rho_0 + \sigma)}{(\rho_0 + \sigma)^2} \{\sigma_t + \nabla \rho_0 \cdot u + \nabla \sigma \cdot u \}
- \frac{1}{\rho_0 + \sigma} \{\partial_x \sigma_t + \nabla \rho_0 \cdot \partial_x u + \nabla \partial_x \sigma \cdot u + \nabla \sigma \cdot \partial_x u + \nabla \partial_x \rho_0 \cdot u \}.
\]

In order to get right exponents \(-\frac{2}{3} + l, \frac{2}{3} + l\) of \(\nabla \partial_x \sigma\) and \(\nabla \cdot \partial_x u\) in \(\tilde{E}_t^{2,2}\) we multiply \((5.23)\) and \((5.24)\) by \((\rho_0 + \sigma)^{-\frac{1}{2} + \frac{l}{3}}\) and \((\rho_0 + \sigma)^{-\frac{2}{3} + \frac{l}{3}}\) respectively, and square them. Notice that our chosen weight functions are of polynomial type due to the behavior of \(\rho_0\): \((\rho_0 + \sigma) \sim \rho_0 = O(r^{-5})\). Thus as one takes the spatial derivative, one gets \(|\partial_x \rho_0| = O(r^{-6}) \sim (\rho_0 + \sigma)^{\frac{2}{3}}\). So we get the following:
\[
(5.25)\quad \int (\rho_0 + \sigma)^{-\frac{2}{3} + l} |\nabla \partial_x \sigma|^2 dx + \int (\rho_0 + \sigma)^{\frac{2}{3} + l} |\nabla \partial_x u|^2 dx
\leq C \int ((\rho_0 + \sigma)^{-\frac{1}{2} + l} + \frac{\partial_x \sigma}{(\rho_0 + \sigma)^{\frac{1}{2}}})^2 (\rho_0 + \sigma^{1+ l})(|u_t|^2 + |\rho_0^{-\frac{2}{3}} \nabla \rho_0|^2 \sigma^2 + |\nabla \phi|^2)
+ \{|u \cdot \nabla| u|^2 + |\nabla \rho_0|^2 h^2\} dx
+ C \int ((\rho_0 + \sigma)^{-\frac{1}{2} + l} + \frac{\partial_x \sigma}{(\rho_0 + \sigma)^{\frac{1}{2}}})^2 (\partial_x \sigma)^2 + |\nabla \partial_x \rho_0|^2 + |(\partial_x u \cdot \nabla) u|^2
+ |(u \cdot \nabla) \partial_x u|^2 + |\nabla \rho_0|^2 |\partial_x h|^2 + |\partial_x (\rho_0^{-\frac{2}{3}} \nabla \rho_0)|^2 \sigma^2 + |\nabla \partial_x \rho_0|^2 h^2\} dx
+ C \int ((\rho_0 + \sigma)^{-\frac{1}{2} + l} (\partial_x \sigma)^2 + |\nabla \partial_x u|^2 + |\nabla \partial_x \sigma|^2 + |\nabla \sigma \partial_x u|^2 + |\nabla \partial_x u|^2 + |\nabla \partial_x \rho_0|^2 |u|^2) dx
\]
The first and second integrals are exactly same as the RHS of (5.12). The ones in the third and fourth integrals except the potential term have been already estimated at the previous steps since each of them contains only one spatial derivative with the right exponent of the corresponding weight. The potential part does not produce any further difficulty, indeed it behaves better both in weights and in derivatives. Notice that \( \triangle \partial_x \phi = 4\pi \partial_x \sigma \) and \( \int_{\mathbb{R}^3} \partial_x \sigma dx = 0 \). Thus we can do the same as we did in (5.17), (5.18), (5.19) and we get the following estimate similar to (5.20):

\[
(5.26) \quad \int (\rho_0 + \sigma)^{\frac{5}{4} + l} |\nabla \partial_x \phi|^2 dx \leq C \varepsilon_0^{2.5}
\]

Consequently, we get the desired estimates for \( \partial_x^2 \sigma \) and \( \partial_x^2 u \):

\[
\tilde{E}_l^{2.2} \leq C(\varepsilon_l^2 + \varepsilon_l^1) + C(\varepsilon_{l+1}^1 + \varepsilon_{l+1}^0)\]

Consider \( \partial_t (5.23) \) and \( \partial_t (5.24) \). Taking \( \partial_t \) does not destroy the structure of equations. As going along the same track, the desired result on \( \partial_t \partial_x^2 \sigma \) and \( \partial_t \partial_x^2 u \) is easily obtained.

Lastly, for three full spatial derivative terms, namely the case \( i = 3, j = 3 \), compute \( \partial_x (5.23) \) and \( \partial_x (5.24) \). Since we are dealing with one more spatial derivative, we have to modify the weights again. Multiply them by \((\rho_0 + \sigma)^{-\frac{10}{4} + \frac{1}{2}} \) and \((\rho_0 + \sigma)^{\frac{7}{4} + \frac{1}{2}} \), square and integrate them. Then most terms have been already treated before. As for the potential part, noting that \( \triangle \partial_x^2 \phi = 4\pi \partial_x^2 \sigma \) and \( \int_{\mathbb{R}^3} \partial_x^2 \sigma dx = 0 \), we get the similar estimate to (5.26):

\[
(5.26) \quad \int (\rho_0 + \sigma)^{\frac{7}{4} + l} |\nabla \partial_x^2 \phi|^2 dx \leq C \varepsilon_l^{2.5}
\]

At last this finishes the lemma. \( \square \)

Lemma 5.2 shows that any spatial and mixed derivative terms can be estimated in terms of time derivative terms with suitable weights, i.e. \( \tilde{\varepsilon}_l \) and \( \tilde{\varepsilon}_l \) are more or less equivalent measurements. Now we take time derivatives which do not destroy the structure of the system much and do the energy estimates.

6. Weighted Nonlinear Energy Estimates

In this section we perform the nonlinear energy estimates with the utilization of a family of symmetrizers of the system. Energy estimates with weights on \( \partial_x^j u \) for \( 0 \leq j \leq 3 \) are carried out to derive the following key estimate so as to build the bootstrap argument which will be discussed in the next section. Throughout this section, (5.1) and (5.5) are assumed.

**Proposition 6.1.** Suppose \( \tilde{\varepsilon}_{-\frac{5}{3}}^1 + \tilde{\varepsilon}_{-\frac{5}{3}}^2 + \tilde{\varepsilon}_{-\frac{5}{3}}^3 \leq \theta_1 \) for \( 0 \leq t \leq T \) where \( \theta_1 \ll 1 \) is sufficiently small.

1. Let \( l = 0 \). Then, for any fixed small \( \eta > 0 \), there exist \( C, C_\eta > 0 \) such that

\[
\frac{1}{2} \frac{d}{dt} \varepsilon_0 \leq (C \sqrt{\theta_1} + \eta) \varepsilon_0 + C_\eta (\varepsilon_0)^{\frac{3}{2}} (\varepsilon_3)^{\frac{3}{2}} + C_\eta (\varepsilon_0^0 + \varepsilon_0^1 + \varepsilon_0^2).
\]
(2) Let $l < 0$. Then, for any fixed small $\eta > 0$, there exist $C, C_\eta > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_l \leq (C\sqrt{\theta_1 + \eta})\mathcal{E}_l + C\mathcal{E}_{l+\frac{5}{3}} + C_\eta(\mathcal{E}_l)\frac{\partial}{\partial t} + C_\eta(\mathcal{E}_l^0 + \mathcal{E}_l^1 + \mathcal{E}_l^2).$$

In particular, if $l \leq -\frac{3}{2}$, then we have

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_l \leq (C\sqrt{\theta_1 + \eta})\mathcal{E}_l + C\mathcal{E}_{l+\frac{5}{3}} + C_\eta(\mathcal{E}_l^0 + \mathcal{E}_l^1 + \mathcal{E}_l^2).$$

Proposition 6.1 will be proven by a series of lemmas in which we will derive the estimate on $\mathcal{E}_l^j$ for each $j$; each lemma has its own significance and we will need all of them to prove the bootstrap argument. Let us start with the simplest case $j = 0$: the zeroth order estimate.

**Lemma 6.2.** $(\mathcal{E}_l^0)$ Let $l \leq 0$. There exists a constant $C > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_l^0 \leq C\sqrt{\theta_1}\mathcal{E}_l^0 + C\mathcal{E}_{l+\frac{5}{3}}.$$

Proof. Consider

$$0 = \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{2}{5} + l} \sigma \, dx + \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{2}{5} + l} \nabla \cdot (\rho_0 + \sigma) u \sigma \, dx + \int (\rho_0 + \sigma)^{1+l} u_\tau \cdot dx + \int (\rho_0 + \sigma)^{1+l} (u \cdot \nabla) u \cdot \sigma \, dx + \int (\rho_0 + \sigma)^{1+l} \nabla \phi \cdot u \sigma \, dx$$

$$+ \int (\rho_0 + \sigma)^{1+l}[(\rho_0 + \sigma)\frac{\partial}{\partial t} u + \frac{4}{5}\rho_0^{-\frac{2}{5}} \nabla \rho_0 \sigma + h(\sigma, \rho_0)] \cdot u \sigma \, dx$$

(6.1)

We compute the first three terms in turn; call them $(I), (II)$ and $(III)$.

$$(I) = \frac{1}{2} \frac{d}{dt} \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{2}{5} + l} \sigma^2 \sigma^2 \sigma^2 \, dx - \frac{14\pi}{2} \frac{15}{5} (-\frac{4}{5} + l) \int (\rho_0 + \sigma)^{-\frac{2}{5} + l} \sigma \sigma \, dx$$

(II) = $- \frac{4\pi}{15} (-\frac{4}{5} + l) \int (\rho_0 + \sigma)^{-\frac{2}{5} + l} \nabla \rho_0 + \sigma \cdot [(\rho_0 + \sigma) u] \sigma \, dx$

(6.2)

$$(II) = \frac{1}{2} \frac{d}{dt} \int (\rho_0 + \sigma)^{1+l} |u|^2 \sigma \, dx + \frac{1}{2}(1 + l) \int (\rho_0 + \sigma)^{1+l} |u|^2 \sigma \, dx$$

$$(III) = \frac{1}{2} \frac{d}{dt} \int (\rho_0 + \sigma)^{1+l} |u|^2 \sigma \, dx - \frac{1}{2}(1 + l) \int (\rho_0 + \sigma)^{1+l} |u|^2 \sigma \, dx$$
After cancellation (6.1) becomes

\[(6.3) \quad \frac{1}{2} \frac{d}{dt} \left[ 4\pi \int \frac{(\rho_0 + \sigma)^{-\frac{4}{5} + l} \sigma^2 dx + \int (\rho_0 + \sigma)^{1+l} u^2 dx} {\rho_0 + \sigma} \right] \]

\[= \frac{4\pi}{15} \left( \frac{4}{5} + l \right) \int \frac{(\rho_0 + \sigma)^{-\frac{4}{5} + l} \sigma}{\rho_0 + \sigma} \sigma^2 dx + \frac{1}{2} (1 + l) \int (\rho_0 + \sigma)^{1+l} \frac{\sigma}{\rho_0 + \sigma} u^2 dx \]

\[= \int (\rho_0 + \sigma)^{1+l} (u \cdot \nabla) u \cdot udx - \int (\rho_0 + \sigma)^{1+l} \nabla \phi \cdot udx \]

\[+ \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1+l} \left( \frac{4}{5} + l \right) (\rho_0 + \sigma)^{-\frac{4}{5} + l} \nabla (\rho_0 + \sigma) \sigma + \frac{4}{5} \rho_0^\frac{2}{5} \nabla \rho_0 \sigma - h(\sigma, \rho_0) \cdot udx \]

Next we estimate the potential part. By the Cauchy-Schwartz inequality and (5.20), we have

\[(6.4) \quad \int (\rho_0 + \sigma)^{1+l} \nabla \phi \cdot udx \leq \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx + \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx \]

\[\leq CE_0^{l+\frac{2}{3}} \]

The last term in (6.3) is rewritten as

\[(6.5) \quad \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1+l} \left[ l_{\rho_0}^\frac{2}{5} \nabla \rho_0 \sigma + \tilde{h}(\sigma, \rho_0) \right] \cdot udx \]

where \(\tilde{h}\) is higher order term including \(\sigma\) and \(\nabla \sigma\). Recall \(\rho_0^{-\frac{2}{5}} \rho_0' = -5r(1 + r^2) \sim (\rho_0 + \sigma)^{-\frac{4}{5}}\). When \(l = 0\), we only have cubic terms left in the above and hence we are done. The quadratic term when \(l \neq 0\) can be treated as the following:

\[(6.6) \quad \int (\rho_0 + \sigma)^{1+l} \left[ l_{\rho_0}^{-\frac{2}{5}} \nabla \rho_0 \sigma \right] \cdot udx \]

\[\leq \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx + \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx \]

\[\leq \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx + \frac{1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\nabla \phi}{\rho_0 + \sigma} |u|^2 dx \]

\[\leq CE_0^{l+\frac{2}{3}} \]

Apply Lemma 5.1 to the first three integrals in the RHS of (6.3). With (6.4) and (6.6) the wanted result follows.\(\square\)

For higher order terms, the spirit of details is same as before but we have extra terms to deal with. While doing higher derivatives, the necessity of the cooperation with mixed and spatial estimates occurs.

Let us compute

\[\int S_i \partial_i \left( \frac{5.2}{5.3} \right) \cdot \partial_i \left( \frac{\sigma}{u} \right) dx = 0 : \]
(6.7)
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{4\pi}{15} \int (\rho_0 + \sigma)^{\frac{5}{2} + t} (\partial_t^l \sigma)^2 \, dx + \int (\rho_0 + \sigma)^{1 + t} |\partial_t^l u|^2 \, dx \right\} \\
= -\frac{4\pi}{15} \frac{1}{2} \int (\rho_0 + \sigma)^{\frac{5}{2} + t} \frac{\sigma_t}{\rho_0 + \sigma} (\partial_t^l \sigma)^2 \, dx \\
+ \frac{l + 1}{2} \int (\rho_0 + \sigma)^{1 + t} \frac{\sigma_i}{\rho_0 + \sigma} |\partial_t^l u|^2 \, dx \\
- \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{5}{2} + t} \partial_t^l \nabla \cdot [(\rho_0 + \sigma)u] \partial_t^l \sigma \, dx \\
- \int (\rho_0 + \sigma)^{1 + t} \partial_t^l [(u \cdot \nabla)u] \cdot \partial_t^l u \, dx \\
- \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1 + t} \{ \partial_t^l [(\rho_0 + \sigma)^{-\frac{5}{2}} \nabla \sigma] - \frac{4}{\sigma} \rho_0^{-\frac{5}{2}} \nabla \rho_0 \partial_t^l \sigma + \partial_t^l h(\sigma, \rho_0) \} \cdot \partial_t^l u \, dx \\
- \int (\rho_0 + \sigma)^{1 + t} \nabla \partial_t^l \phi \cdot \partial_t^l u \, dx
\]

For computational convenience, we separate \(\partial_t^l\) terms from lower derivative terms in the RHS of (6.7). Some terms contain unfavorably \((j + 1)^{th}\) derivative terms. The worst terms seem to come from the third and fifth integrals: 
\[-\frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{5}{2} + t} \nabla \cdot [(\rho_0 + \sigma)\partial_t^l u] \partial_t^l \sigma \, dx \text{ and } -\frac{4\pi}{15} \int (\rho_0 + \sigma)^{1 + t} [(\rho_0 + \sigma)^{-\frac{5}{2}} \nabla \partial_t^l \sigma - \frac{4}{\sigma} \rho_0^{-\frac{5}{2}} \nabla \rho_0 \partial_t^l \sigma] \cdot \partial_t^l u \, dx.\]

Use the integration by parts to get some nice cancellation:

(6.8)
\[
-\frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{5}{2} + t} \nabla \cdot [(\rho_0 + \sigma)\partial_t^l u] \partial_t^l \sigma \, dx \\
-\frac{4\pi}{15} \int (\rho_0 + \sigma)^{1 + t} [(\rho_0 + \sigma)^{-\frac{5}{2}} \nabla \partial_t^l \sigma - \frac{4}{\sigma} \rho_0^{-\frac{5}{2}} \nabla \rho_0 \partial_t^l \sigma] \cdot \partial_t^l u \, dx \\
= \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{5}{2} + t} \nabla \cdot [(\rho_0 + \sigma)\partial_t^l u] \partial_t^l \sigma \, dx \\
-\frac{4\pi}{15} \int (\rho_0 + \sigma)^{1 + t} [(\rho_0 + \sigma)^{-\frac{5}{2}} \nabla \partial_t^l \sigma - \frac{4}{\sigma} \rho_0^{-\frac{5}{2}} \nabla \rho_0 \partial_t^l \sigma] \cdot \partial_t^l u \, dx \\
= \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1 + t} [(\rho_0 + \sigma)^{-\frac{5}{2}} \nabla \rho_0 \partial_t^l \sigma + \tilde{h}(\sigma, \nabla \sigma, \rho, \partial_t^l \sigma)] \cdot \partial_t^l u \, dx
\]
where \( \tilde{h}(\sigma, \nabla \sigma, \rho, \partial_t \sigma) = (-\frac{4}{5} + l)\{(\rho_0 + \sigma)^{-\frac{4}{5}} \nabla \sigma + [(\rho_0 + \sigma)^{-\frac{4}{5}} - \rho_0^{-\frac{4}{5}}] \nabla \rho_0\} \partial_t \sigma \).

Notice that we have the above quadratic term only when \( l \neq 0 \). Taking into account (6.8) and grouping by similarity, we rewrite (6.7) as following:

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{4}{5} + l} (\partial_t \sigma)^2 dx + \int (\rho_0 + \sigma)^{1+l} |\partial_t u|^2 dx \right)
= \frac{1}{2} \frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{4}{5} + l} \frac{\sigma_1}{\rho_0 + \sigma} (\partial_t \sigma)^2 dx
+ \frac{l + 1}{2} \int (\rho_0 + \sigma)^{1+l} \frac{\sigma_1}{\rho_0 + \sigma} (\partial_t \sigma)^2 dx
+ \left( -\frac{4\pi}{15} \int (\rho_0 + \sigma)^{1+l} \left[ \partial_t^l \sigma \nabla \cdot u + \nabla \partial_t^l \sigma \cdot u \right] \partial_t \sigma dx \right.
- \int (\rho_0 + \sigma)^{1+l} \left[ \partial_t^l u \cdot \nabla \right] \partial_t \sigma dx
\]

\[
+ \left( \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1+l} \left[ \rho_0 - \frac{4}{5} \rho_0 \partial_t \sigma + \tilde{h}(\sigma, \nabla \sigma, \rho, \partial_t \sigma) + \partial_t \tilde{h}(\sigma, \rho_0) \right] \cdot \partial_t^l u dx \right)
+ \left( - \int (\rho_0 + \sigma)^{1+l} \nabla \partial_t^l \phi \cdot \partial_t \sigma \right)
+ \left( -\frac{4\pi}{15} \int (\rho_0 + \sigma)^{-\frac{4}{5} + l} \sum_{i=1}^{j-1} \nabla \cdot \left[ \partial_t^{j-i} (\rho_0 + \sigma) \partial_t^i u \right] \partial_t \sigma dx \right.
- \int (\rho_0 + \sigma)^{1+l} \sum_{i=1}^{j-1} \left[ \partial_t^{j-i} u \partial_t^i \sigma \right] \cdot \partial_t \sigma dx
\]

\[
\left. - \frac{4\pi}{15} \int (\rho_0 + \sigma)^{1+l} \sum_{i=1}^{j-1} \left[ \partial_t^{j-i} (\rho_0 + \sigma)^{-\frac{4}{5}} \partial_t^i \nabla \sigma \right] \cdot \partial_t^l u dx \right) = (I) + (II) + (III) + (IV) + (V)
\]

Note that (V) is alive only for \( j = 2, \) or 3. First three groups have the exactly same structure as (6.3). Each term can be easily estimated as in Lemma 6.2. For (I), by using Lemma 5.2, we get immediately

\[
(I) \leq C \sqrt{t_i^j} E_i^j \text{ for all } j.
\]
Other \((j+1)^{th}\) derivative terms are in \((II)\). After integrating by parts, the third and fourth integrals become

\[
-\frac{4\pi}{15} \int (\rho_0 + \sigma) \frac{4}{\pi^{1/2}} [\nabla \cdot \nabla \sigma \cdot u + \nabla \partial_t^j \sigma \cdot u] dx \\
= -\frac{2\pi}{15} \int (\rho_0 + \sigma) \frac{4}{\pi^{1/2}} (\partial_t^j \sigma) dx
\]

\[
+ \frac{2\pi}{15} \left(-\frac{4}{5} + l\right) \int (\rho_0 + \sigma) \frac{4}{\pi^{1/2}} \nabla (\rho_0 + \sigma) \cdot \frac{u}{(\rho_0 + \sigma)^{3/2}} (\partial_t^j \sigma)^2 dx,
\]

\[(6.11)\]

\[
- \int (\rho_0 + \sigma)^{1+l}[\partial_t^j u \cdot \nabla u + (u \cdot \nabla) \partial_t^j u] \partial_t^j u dx
\]

\[
= - \int (\rho_0 + \sigma)^{1+l}(\partial_t^j u \cdot \nabla u) u \cdot \partial_t^j u dx + \frac{1}{2} \int (\rho_0 + \sigma)^{1+l}(\nabla \cdot u) |\partial_t^j u|^2 dx
\]

\[
+ \frac{1+l}{2} \int (\rho_0 + \sigma)^{1+l} \nabla (\rho_0 + \sigma) \cdot \frac{u}{(\rho_0 + \sigma)^{3/2}} |\partial_t^j u|^2 dx
\]

Therefore \((6.11)\) with Lemma 5.1 gives rise to:

\[(6.12)\]

\[
(II) \leq C \sqrt{\theta_1} \xi_1^j \text{ for all } j.
\]

\((III)\) is similar to \((6.5)\) in the zeroth estimate and so it can be treated in the same way. If we do the same as in \((6.6)\) and use Lemma 5.1, we get the following: for each \(j\),

\[(6.13)\]

\[
(III) \leq C \sqrt{\theta_1} \xi_1^j \text{ when } l = 0, \text{ and } \\
(III) \leq C \xi_1^j + C \sqrt{\theta_1} \xi_1^j \text{ when } l \neq 0.
\]

The potential part \((IV)\) and cubic terms \((V)\) are somewhat new, complex and they’d rather be done separately according to different \(j\)’s. Before we split the cases, from the dynamics of \(\nabla \partial_t^j \phi : \nabla \partial_t^j \phi = -4\pi \partial_t^{j-1}[(\rho_0 + \sigma)u]\), we reduce \((IV)\) to the following:

\[(6.14)\]

\[
(IV) = 4\pi \int (\rho_0 + \sigma)^{1+l} \partial_t^j u \cdot \partial_t^{j-1}[(\rho_0 + \sigma)u] dx
\]

Now let \(j = 1\). Here is the first order \(\partial_t\) estimate.

**Lemma 6.3.** \((\xi_1^1)\) For any small \(\eta > 0\), there exist constants \(C, C_\eta > 0\) such that

\[
\frac{1}{2} \frac{d}{dt} \xi_1^1 \leq (C \sqrt{\theta_1} + \eta) \xi_1^1 + C_\eta \xi_0^2 \text{ for } l = 0,
\]

\[
\frac{1}{2} \frac{d}{dt} \xi_1^1 \leq (C \sqrt{\theta_1} + \eta) \xi_1^1 + C \xi_1^1 + C_\eta \xi_0^2 \text{ for } l < 0.
\]

Proof. Since \((V)\) has no effect on \(j = 1\), it is sufficient to take care of \((IV)\). The potential part is shown to be even better in terms of derivatives as we can expect
in (6.14). By the Cauchy-Schwartz inequality, we get

\[
(IV) = 4\pi \int (\rho_0 + \sigma)^{2+l} \partial_t u \cdot u dx
\]

\[
\leq \eta \int (\rho_0 + \sigma)^{1+l} |\partial_t u|^2 dx + C_\eta \int (\rho_0 + \sigma)^{3+l} |u|^2 dx
\]

\[
\leq \eta E^1_l + C_\eta E^0_{l+2}, \text{ for any small } \eta > 0.
\]

Thus (6.10), (6.12), (6.13) and (6.15) give the desired result. \(\square\)

The second order \(\partial_t^2\) estimate can be done in the same spirit. We have extra cubic terms to deal with from (V). Lemma 5.2 plays an important role. Let \(j = 2\).

**Lemma 6.4.** \((\mathcal{E}^2_l)\) For any small \(\eta > 0\), there exist constants \(C, C_\eta > 0\) such that

\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}^2_0 \leq (C \sqrt{\theta_1 + \eta}) \mathcal{E}^2_0 + C_\eta \sum_{i=0}^{1} \mathcal{E}^i_0 \text{ for } l = 0,
\]

\[
\frac{1}{2} \frac{d}{dt} \mathcal{E}^2_l \leq (C \sqrt{\theta_1 + \eta}) \mathcal{E}^2_l + C \mathcal{E}^2_{l+1} + C_\eta \sum_{i=0}^{1} \mathcal{E}^i_l \text{ for } l < 0.
\]

Proof. The potential part (6.14) can be computed like (6.15). By Lemma 5.1,

\[
(IV) = 4\pi \int (\rho_0 + \sigma)^{2+l} \partial_t^2 u \cdot \partial_t u + \frac{\sigma_t}{\rho_0 + \sigma} u |u| dx
\]

\[
\leq \eta \mathcal{E}^2_l + C_\eta (\mathcal{E}^1_{l+2} + \theta_1 \mathcal{E}^0_{l+2}), \text{ for any small } \eta > 0.
\]

Terms in (V) for \(j = 2\) are at least cubic including mixed derivatives. We can take the sup for the lowest, first derivative term and then we end up with manageable quadratic terms. In order to see how it works, we illustrate the estimate on the first term in (V):

\[
\int (\rho_0 + \sigma)^{-\frac{4}{3} + l} \nabla \cdot (\partial_t \sigma \partial_t u) \partial_t^2 \sigma dx
\]

\[
= \int (\rho_0 + \sigma)^{\frac{4}{3} + l} \frac{\partial \sigma}{(\rho_0 + \sigma)} \nabla \cdot \partial_t u \partial_t \sigma dx + \int (\rho_0 + \sigma)^{-\frac{4}{3} + l} \frac{\partial_t u}{(\rho_0 + \sigma)} \cdot \nabla \partial_t \sigma \partial_t^2 \sigma dx
\]

\[
\leq C \sqrt{\theta_1} \left\{ \int (\rho_0 + \sigma)^{\frac{4}{3} + l} |\nabla \cdot \partial_t u|^2 dx + \int (\rho_0 + \sigma)^{-\frac{4}{3} + l} (\partial_t^2 \sigma)^2 dx \right\}
\]

\[
+ C \sqrt{\theta_1} \left\{ \int (\rho_0 + \sigma)^{-\frac{4}{3} + l} |\nabla \partial_t \sigma|^2 dx + \int (\rho_0 + \sigma)^{-\frac{4}{3} + l} (\partial_t^2 \sigma)^2 dx \right\}
\]

\[
\leq C \sqrt{\theta_1} (\mathcal{E}^1_{l+1} + \mathcal{E}^2_l)
\]

We have used the Cauchy-Schwartz inequality at the first inequality. Notice the changes of the exponents in weights. We can do the same to other terms in (V). Ultimately, applying Lemma 5.2, we get the following:

\[
(V) \leq C \sqrt{\theta_1} \mathcal{E}^2_l + C \sum_{k=0}^{1} \mathcal{E}^k_{l+\frac{2}{3}}
\]
Note that $\mathcal{E}_{t+k} \leq C\mathcal{E}_t$ for $k > 0$. Thus (6.10), (6.12), (6.13), (6.16) and (6.18) give the desired result. □

To finish the proof of Proposition 6.1, only $j = 3$ i.e. $\partial^3_l$ case is left. The difficulty is to handle the weighted new cubic terms in $(V)$ of which each factor is at least second derivative of $\sigma$ and $u$ and hence we cannot utilize Lemma 5.1 directly. To overcome it, we introduce the weighted Gagliard-Nirenberg inequality. This job is done in the next lemma. One can see that $(IV)$ and other terms in $(V)$ for $j = 3$ can be treated similarly as in (6.16) and (6.17). Therefore, the following lemma finally establishes Proposition 6.1.

**Lemma 6.5.** Let $l \leq 0$. For any small fixed $\eta > 0$, there exists $C_\eta > 0$ such that

$$\int (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot u \partial^3_l \sigma| dx, \quad \int (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot u^2 \partial^2_l \sigma| dx$$

$$\int (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot \partial^3_l u| dx, \quad \int (\rho_0 + \sigma)^{1+l} |\partial^2_l u \cdot \partial_t \partial^3_l u| dx$$

are bounded by $\eta \mathcal{E}_l^3 + C_\eta (\mathcal{E}_l^3) \frac{1}{2} (\mathcal{E}_{l+(3-2l)}^2)^{\frac{1}{4}} + C_\eta (\mathcal{E}_{l+(3-2l)}^2) \frac{1}{2} (\mathcal{E}_{l+(3-2l)}^2)^{\frac{1}{4}}$. In particular, if $l \leq -\frac{3}{2}$, they are bounded by $\eta \mathcal{E}_l + C_\eta (\mathcal{E}_l)^2$.

Proof. First we split each term in $\partial^3_l$ term and $\partial^2_l$ term by the Cauchy-Schwartz inequality. In order to take care of $L^4$ norm of $\partial^2_l$ terms we use the Gagliard-Nirenberg inequality $\|f\|_{L^4(\mathbb{R}^3)} \leq \frac{1}{\lambda} \|\nabla f\|_{L^2(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}$. Since the inequality complies well with the localization and the weight functions are nice, using a partition of unity, in our case we get the weighted version of the Gagliard-Nirenberg inequality

$$\int_0^\infty w_k f^4 r^2 dr \leq C (\int_0^\infty w_k^\beta |\nabla f|^2 r^2 dr)^{\frac{1}{2}} (\int_0^\infty w_k^\alpha f^2 r^2 dr)^{\frac{\gamma}{2}}$$

$$+ C (\int_0^\infty w_k^\beta f^2 r^2 dr)^{\frac{1}{2}} (\int_0^\infty w_k^\alpha f^2 r^2 dr)^{\frac{\gamma}{2}}$$

where $w_k = (1 + r^2)^{\frac{\beta}{2}} \sim (\rho_0 + \sigma)^{1 - \frac{\gamma}{2}}$ and $\frac{\gamma}{2} \alpha + 1 = 1$ and $\frac{\gamma}{2} \beta = 1$. Its proof is given at the end of the argument. Only the first and the last term are treated in this proof. The other cases can be estimated in the same way.

$$\int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot u \partial^3_l \sigma| r^2 dr$$

$$\leq \eta \int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma|^2 r^2 dr + C_\eta \int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot u|^2 r^2 dr$$

$$\int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma \partial_t \cdot u|^2 r^2 dr$$

$$\leq \frac{1}{2} \int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial^2_l \sigma|^4 r^2 dr + \frac{1}{2} \int_0^\infty (\rho_0 + \sigma)^{-\frac{\alpha}{4} + l} |\partial_t \partial^3_l u|^4 r^2 dr$$
Here it comes:

Note that

\[ \int_{0}^{\infty} (\rho_0 + \sigma)^{-\frac{2}{3} + l} (\partial_t^2 \sigma)^4 r^2 dr \]

\[ \leq C \int_{0}^{\infty} (\rho_0 + \sigma)^{-\frac{2}{3} + l} |\partial_t^2 \nabla \sigma|^2 r^2 dr \int_{0}^{\infty} (\rho_0 + \sigma)^{-\frac{2}{3} - l} (\partial_t^2 \sigma)^2 r^2 dr + \]

\[ + C \int_{0}^{\infty} (\rho_0 + \sigma)^{-\frac{2}{3} + l} (\partial_t^2 \sigma)^2 r^2 dr \int_{0}^{\infty} (\rho_0 + \sigma)^{-\frac{2}{3} - l} (\partial_t^2 \sigma)^2 r^2 dr \]

\[ \leq C(\mathcal{E}_i^2)^\frac{1}{2}(\mathcal{E}_{i+1}^{(-3, -2)}) + C(\mathcal{E}_i^2)^\frac{1}{2}(\mathcal{E}_{i+1}^{(-3, -2)}) \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{2}{3} + l} |\partial_t \nabla u|^4 r^2 dr \]

Here is the verification of each exponent: \(-\frac{14}{3} + l = \frac{2}{5}(-\frac{3}{5} + l) + \frac{1}{5}(-\frac{10}{3} - l)\) and \(-\frac{10}{5} - l = -\frac{4}{5} + l + (-3 - 2l)\). Note that \(-\frac{10}{5} - l \geq -\frac{4}{5} + l\) for \(l \leq -\frac{4}{5}\). Now let us look at the last term. We go through the similar computation as in (6.20) and (6.21).

\[ \int_{0}^{\infty} (\rho_0 + \sigma)^{1 + l} |\partial_t^2 u \cdot \nabla \partial_t u \partial_t^3 u|^2 r^2 dr \leq \eta \int_{0}^{\infty} (\rho_0 + \sigma)^{1 + l} |\partial_t^3 u|^2 r^2 dr + \]

\[ + C \eta \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} + l} |\partial_t^3 u|^4 r^2 dr + \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} + l} |\nabla \partial_t u|^4 r^2 dr \]

\[ \leq C \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} + l} |\partial_t^3 u|^2 r^2 dr \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} - l} |\partial_t^3 u|^2 r^2 dr + \]

\[ + C \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} + l} |\partial_t^3 u|^2 r^2 dr \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} - l} |\partial_t^3 u|^2 r^2 dr \]

\[ \leq C(\mathcal{E}_i^2)^\frac{1}{2}(\mathcal{E}_{i+1}^{(-3, -2)}) + C(\mathcal{E}_i^2)^\frac{1}{2}(\mathcal{E}_{i+1}^{(-3, -2)}) \int_{0}^{\infty} (\rho_0 + \sigma)^{\frac{5}{2} + l} |\nabla \partial_t u|^4 r^2 dr \]

Observe that \(\frac{5}{2} + l = \frac{3}{5}(\frac{5}{2} + l) + \frac{1}{5}(-2 - l)\). The only missing part is the proof of the weighted Gagliardi-Nirenberg inequality. Here it comes:
Proof of (6.19): We choose a partition of unity \( \{ \varphi_n \}_{n \geq 0} \) as following:

\[
\varphi_0(r) = \begin{cases} 
1 - r & \text{if } 0 \leq r \leq 1 \\
0 & \text{if } r \geq 1 
\end{cases}
\]

\[
\varphi_n(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq \frac{n-1}{2} \\
\frac{r - \frac{n-1}{2}}{\frac{1}{2}} & \text{if } \frac{n-1}{2} \leq r \leq \frac{n+1}{2} \\
\frac{n+2 - r}{\frac{1}{2}} & \text{if } \frac{n+1}{2} \leq r \leq \frac{n+2}{2} \\
0 & \text{if } r \geq \frac{n+2}{2} 
\end{cases}
\]

It is easy to check \( 0 \leq \varphi_n \leq 1 \), \( \text{supp } \varphi_n = [\frac{n-1}{2}, \frac{n+2}{2}] \) (supp \( \varphi_0 = [0, 1] \)), \( \sum_{n=0}^{\infty} \varphi_n(r) = 1 \) for all \( r \geq 0 \), and \( |\varphi'_n(r)| \leq 1 \) a.e.

\[
\int_0^\infty (1 + r^2)^k f^4 r^2 dr = \sum_{n=0}^{\infty} \int_0^\infty (1 + r^2)^k f^4 r^2 dr
\]

(6.24)

\[
\leq C \sum_{n=0}^{\infty} \int_0^\infty \varphi_n^4(r)(1 + r^2)^k f^4 r^2 dr
\]

We only consider \( k \geq 0 \). Other cases can be proven in the same manner. Note that \( (1 + r^2)^k \leq (1 + (\frac{n+2}{2})^2)^k \) on \( \text{supp } \varphi_n \). First we localize the half real line according to the partition of unity. Since weights are monotonic, they can be localized as well. And then we apply the Gagliardo-Nirenberg inequality.

\[
(9.1) \leq C \sum_{n=0}^{\infty} (1 + (\frac{n+2}{2})^2)^k \int_0^\infty \varphi_n^4 f^4 r^2 dr
\]

\[
\leq C \sum_{n=0}^{\infty} (1 + (\frac{n+2}{2})^2)^k \left( \int_0^\infty [\varphi_n f] r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty \varphi_n^2 f^2 r^2 dr \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{n=0}^{\infty} (1 + (\frac{n+2}{2})^2)^k \left( \int_0^\infty [\varphi_n f'] r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty \varphi_n^2 f^2 r^2 dr \right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{n=0}^{\infty} (1 + (\frac{n+2}{2})^2)^k \left( \int_0^\infty \frac{1}{\alpha} f^2 r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty f^2 r^2 dr \right)^{\frac{1}{2}}
\]

\[
+ C \sum_{n=0}^{\infty} (1 + (\frac{n+2}{2})^2)^k \left( \int_0^\infty \frac{1}{\beta} f^2 r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty f^2 r^2 dr \right)^{\frac{1}{2}}
\]

\[
\leq C \left( \int_0^\infty (1 + r^2)^{\alpha} f^2 r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty (1 + r^2)^{\beta} f^2 r^2 dr \right)^{\frac{1}{2}}
\]

\[
+ C \left( \int_0^\infty (1 + r^2)^{\alpha'} f^2 r^2 dr \right)^{\frac{3}{2}} \left( \int_0^\infty (1 + r^2)^{\beta'} f^2 r^2 dr \right)^{\frac{1}{2}}
\]

In the above \( C \) is a generic constant. Note that \( \frac{3}{2} \alpha + \frac{1}{2} \beta = \frac{3}{2} \alpha' + \frac{1}{2} \beta' = k. \)
7. Nonlinear Instability

Now we are ready to prove the bootstrap argument. The proof completely depends on the estimates in Section 6 and the Gronwall inequality.

**Proposition 7.1.** Let \( \nu(t) = (\nu_{u(t)}) \) be a solution of the Euler-Poisson system (1.9) and (1.10). Let \( \text{Re} \leq -3 \) be given. Assume that

\[
\sqrt{\tilde{E}_t}(0) \leq C_0 \delta \text{ and } \sqrt{E_0}(t) \leq C_0 \delta e^{\sqrt{\tilde{E}_t}t} \text{ for } 0 \leq t \leq T.
\]

Then there exist constants \( C_5, \theta_0 > 0 \) such that if \( 0 \leq t \leq \min\{T, T^\delta\} \), then

\[
\sqrt{\tilde{E}_t}(t) \leq C_5 \delta e^{\sqrt{\tilde{E}_t}t} \leq C_5 \theta_0,
\]

where \( T^\delta = \frac{1}{\sqrt{\nu_0}} \ln \frac{\theta_0}{\delta} \).

Proof. In Proposition 6.1, Lemma 6.2, 6.3 and 6.4, choose \( \theta_1, \eta \) small enough so that \( C \sqrt{\theta_1} + \eta \leq \frac{\sqrt{\nu_0}}{2} \). Therefore there exist constants \( C_1 \geq 0, C_2, C_3 > 0 \) such that for \( l \leq 0 \),

\[
\begin{align*}
(a_l) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l^0 \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l^0 + C_2 E_l^0, \\
(b_l) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l^1 \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l^1 + C_1 \tilde{E}_l^1 \eta + C_2 E_l^0, \\
(c_l) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l^2 \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l^2 + C_1 \tilde{E}_l^1 \eta + C_2 (E_l^0 + \tilde{E}_l^1), \\
(d_l) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l + C_1 \tilde{E}_l \eta + C_2 (E_l^0 + \tilde{E}_l^1) \tilde{E}_l^2 + C_3 \tilde{E}_l \eta^2 (\tilde{E}_l \eta) \tilde{E}_l^2.
\end{align*}
\]

Note that \( C_1 = 0 \) when \( l = 0 \). Define \( T\ast \) by

\[
T\ast = \sup\{t : \tilde{E}_l(t) \leq \min\{\theta_1, \frac{\sqrt{\mu_0}}{4C_3}\}, s \in [0, t], l \ast \leq l \leq 0\}.
\]

Here \( \theta_1 \) is a small constant coming from the smallness assumption to guarantee that the nonlinear energy estimates work.

Let \( 0 \leq t \leq \min\{T, T\ast\} \). Then since \( \sqrt{\tilde{E}_l^0}(t) \leq C_0 \delta e^{\sqrt{\tilde{E}_l^0}t} \) by the hypothesis, from \( (a_l) \), we get

\[
(a_l) \implies \sqrt{\tilde{E}_l^0}(t) \leq C_0' \delta e^{\sqrt{\tilde{E}_l^0}t} \text{ for all } l \leq 0
\]

by the standard Gronwall inequality. We use \( C_0' \) as a generic constant. Consider the following diagram:

\[
\begin{align*}
(b_0) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l^0 \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l^0 + C_2 E_l^0 \implies \sqrt{\tilde{E}_l^0}(t) \leq C_0' \delta e^{\sqrt{\tilde{E}_l^0}t} \\
(b_{-k}) \quad & \frac{1}{2} \frac{d}{dt} \tilde{E}_l^1 \eta \leq \frac{\sqrt{\mu_0}}{2} \tilde{E}_l^1 \eta + C_1 \tilde{E}_l^1 + C_2 E_l^0 \implies \sqrt{\tilde{E}_l^1 \eta}(t) \leq C_0' \delta e^{\sqrt{\tilde{E}_l^1 \eta}t}
\end{align*}
\]

Likewise, for all \( k \geq 0 \), we have

\[
(b_{-k}) = \sqrt{\tilde{E}_l^1 \eta}(t) \leq C_0' \delta e^{\sqrt{\tilde{E}_l^1 \eta}t}.
\]

By the same bootstrap argument with \( (c_{-k}) \) where \( k \geq 0 \), we have

\[
\sqrt{\tilde{E}_l \eta^2 \tilde{E}_l^2}(t) \leq C_0' \delta e^{\sqrt{\tilde{E}_l \eta^2 \tilde{E}_l^2}t}.
\]
By the behavior of $\psi_T\delta$ small but fixed positive constant independent of $\theta$, we know Gronwall inequality to get the following Euler-Poisson system (2.1) and (2.2) obtained in Section 2. Normalize $\nu_0$ such that $(\nu_0)^2 < \min\{\theta_1, \frac{\sqrt{\mu}}{4C_3}\}$. We consider following two cases:

(i) $T^\delta \leq \min\{T, T^*\}$; in this case, the conclusion follows without any extra work.

(ii) $T^\delta > \min\{T, T^*\}$; then $T \leq T^* < T^\delta$. If this is true, then again the conclusion is trivial. We show this is the only possibility. If not, we have $T^* < T < T^\delta$. Letting $t = T^*$, from (7.2) and the definition of $T^\delta$, we get

$$\tilde{E}_l(t) \leq C^2_2 \delta^2 e^{2\sqrt{\mu} t},$$

for any $l \leq 0$ and some constant $C_4$. And in success by Lemma 5.2, for any $l \leq 0$, we also have

$$\tilde{E}_l(t) \leq C^2_2 \delta^2 e^{2\sqrt{\mu} t},$$

for $0 \leq t \leq \min\{T, T^*\}$.

Now choose $\theta_0$ such that $(C_5 \theta_0)^2 < \min\{\theta_1, \frac{\sqrt{\mu}}{4C_3}\}$. We consider following two cases:

(i) $T^\delta \leq \min\{T, T^*\}$: in this case, the conclusion follows without any extra work.

(ii) $T^\delta > \min\{T, T^*\}$: then $T \leq T^* < T^\delta$. If this is true, then again the conclusion is trivial. We show this is the only possibility. If not, we have $T^* < T < T^\delta$. Letting $t = T^*$, from (7.2) and the definition of $T^\delta$, we get

$$\tilde{E}_l(t) \leq C^2_2 \delta^2 e^{2\sqrt{\mu} t^*} < (C_5 \delta \exp\sqrt{\mu} t^*)^2 = (C_5 \theta_0)^2.$$

But this is impossible by the choice of $\theta_0$ since it would contradict the definition of $T^*$. This establishes the proposition.$\square$

From now on we regard $\delta > 0$ as an arbitrary small parameter and $\theta$ as a small but fixed positive constant independent of $\delta$. Recall that $T^\delta$ is defined by $\theta = \delta \exp\sqrt{\mu} t^\delta$ or equivalently $T^\delta = \frac{1}{\sqrt{\mu_0}} \ln \frac{\theta}{\delta}$.

**Proof of Theorem 1.1.** Let $\nu_0 = (\nu_0)$ be a growing mode for the linearized Euler-Poisson system (2.1) and (2.2) obtained in Section 2. Normalize $\nu_0$ such that

$$||\nu_0||^2 = \frac{16}{15} \pi^2 (||\phi_0||^2_{W_0} + ||\psi_0||^2_{W_0}) = 1.$$

By the behavior of $\psi_0$ in Proposition 3.4 and the relation (2.4) between $\phi_0$ and $\psi_0$, we know $|\delta \nu_0|$ is relatively small compared to $\rho_0$: $\rho_0 + \delta \rho_0 \sim \gamma \rho_0$ where $\gamma$ is close to 1, and $\int_0^\infty \phi_0 r^2 dr = 0$. We may assume $\frac{15}{2} \rho_0 < \rho_0 + \delta \rho_0 < \frac{2}{15} \rho_0$. Let

$$4\pi \int_0^\infty \frac{4\pi}{15} (\rho_0 + \delta \rho_0)^{-\frac{2}{3} + 1} \phi_0^2 r^2 dr + 4\pi \int_0^\infty (\rho_0 + \delta \rho_0)^{1 + 2}\psi_0^2 r^2 dr = a^2 < \infty.$$

Now solve the Euler-Poisson system with a family of initial data $\nu|_{t=0} = \delta \nu_0$. The continuity equation gives rise to $\int_0^\infty \sigma \delta r^2 dr = 0$. Denote the corresponding
\begin{align*}
\mathcal{E}_t\text{-solution by } \nu(t) \equiv \nu^\delta(t) = (\sigma^\delta(\tau(t)), u^\delta(\tau(t))). \text{ It can be written as }
\nu(t) = \delta \sqrt{\rho_0} \nu_0 + \int_0^t \mathcal{L}(t - \tau) N(\tau) d\tau,
\end{align*}
where \( \mathcal{L} \) is the solution operator for the linearized Euler-Poisson system and \( N \) is nonlinear part.

\[ N = \left( uu_r + \frac{4\pi}{15} \left[ -\frac{2}{5} \rho_0^{\frac{4}{5}} \sigma_r + (\rho_0 + \sigma) \right] h \right) \]

where \( h = (\rho_0 + \sigma)^{-\frac{4}{5}} - \rho_0^{-\frac{4}{5}} + \frac{4}{5} \rho_0^{-\frac{2}{5}} \sigma \).

Define \( T \) by

\[ T = \sup \{ s : \text{ for } 0 \leq t \leq s, \sqrt{\mathcal{E}_0} \leq \max \{ 3, \alpha \} \delta \sqrt{\rho_0} \}. \]

Then by Proposition 7.1, there exist \( C_1 \) and \( \theta_0 > 0 \) such that for \( 0 \leq t \leq \min \{ T, T^\delta \} \),

\[ \sqrt{\mathcal{E}_t(t)} \leq \sqrt{\mathcal{E}_t} \leq C_1 \delta \sqrt{\rho_0} \leq C_1 \theta_0. \]

Note for sufficiently small \( \theta_0 \), by Lemma 5.1, it means that \( \left| \frac{\sigma^\delta}{\rho_0 + \sigma} \right| \ll 1 \) i.e. \( \rho_0 + \sigma^\delta \) behaves like \( \rho_0 \). So we can find a small constant \( \beta, 0 \leq \beta \leq \frac{1}{2} \) such that

\[ (1 - \beta)^2 \rho_0^{-\frac{4}{5}} \leq (\rho_0 + \sigma^\delta)^{-\frac{4}{5}} \leq (1 + \beta)^2 \rho_0^{-\frac{4}{5}} \]

(7.3)

To emphasize which functions we deal with, we denote \( \mathcal{E}_0^0 \) being plugged \( f = (f_1, f_2) \) in but its weight part unchanged by \( \| f \|_Y^2 \):

\[ \| f \|_Y^2 = \int_0^\infty \int_0^\infty \left( (\rho_0 + \sigma^\delta)^{-\frac{4}{5}} (f_1)^2 r^2 dr + 4 \int_0^\infty (\rho_0 + \sigma^\delta) (f_2)^2 r^2 dr \right) \]

In this notation, by (7.3), for \( 0 \leq t \leq \min \{ T, T^\delta \} \),

\[ \| \delta \sqrt{\rho_0} \nu_0 \|_Y \geq (1 - \beta) \delta \sqrt{\rho_0} \| \nu_0 \|_0 = (1 - \beta) \delta \sqrt{\rho_0}. \]

For the nonlinear parts, from the linearized estimates in the Lemma 4.1, 4.2 and 4.3, for \( 0 \leq t \leq \min \{ T, T^\delta \} \), we have

(7.5)

\[ \| \nu^\delta(t) - \delta \sqrt{\rho_0} \nu_0 \|_Y = \int_0^t \mathcal{L}(t - \tau) N(\tau) d\tau \]

\[ \leq C \int_0^t e^{\sqrt{\rho_0}(t-\tau)} \| N(\tau) \|_Y + \| \partial_t N(\tau) \|_Y + \| \partial_{tt} N(\tau) \|_Y d\tau \]

\[ \leq C \int_0^t \left\{ \frac{\sigma}{\rho_0 + \sigma} + \frac{\sigma_t}{\rho_0 + \sigma} + \frac{\nabla \sigma}{(\rho_0 + \sigma)^{\frac{4}{5}}} \right\} d\tau \]

\[ \leq C \int_0^t e^{\sqrt{\rho_0}(t-\tau)} (\delta \sqrt{\rho_0}) (\delta e^{\sqrt{\rho_0}}) d\tau \]

\[ \leq C_2 (\delta \sqrt{\rho_0})^2 \]
where $C_2$ is a constant. At the second inequality we have used Lemma 5.2. The next inequality follows from Proposition 7.1 and Lemma 5.1.

Now if necessary, fix $\theta_0$ sufficiently small such that $C_2\theta_0 \leq \frac{1-\beta}{2}$.

Claim. $T^\delta \leq T$.

Proof. If not i.e. $T^\delta > T$, by (7.4) and (7.5)

$$
\|\nu^\delta\|_Y(T) \leq \|\delta e^{\sqrt{m_0}t}\nu_0\|_Y(T) + \|\nu^\delta - \delta e^{\sqrt{m_0}t}\nu_0\|_Y(T)
\leq (1 + \beta)\delta e^{\sqrt{m_0}T}\|\nu_0\|_0 + C_2\theta_0\delta e^{\sqrt{m_0}T}
\leq \frac{3 + \beta}{2}\delta e^{\sqrt{m_0}T} < 2\delta e^{\sqrt{m_0}T}
$$

which would contradict the definition of $T$.

Once we have $T^\delta \leq T$, again by (7.4) and (7.5),

$$
\sqrt{\epsilon_0(T^\delta)} \geq (1 - \beta)\delta e^{\sqrt{m_0}T^\delta} - \frac{1 - \beta}{2}\delta e^{\sqrt{m_0}T^\delta} = \frac{1 - \beta}{2}\theta_0 > 0.
$$

Set $\theta = \frac{1-\beta}{2}\theta_0$. This finishes the proof of the theorem. □

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Department of Mathematics, Brown University, Providence, RI 02912, USA
E-mail address: juhijang@math.brown.edu