On a probabilistic derivation of the basic particle statistics (Bose-Einstein, Fermi-Dirac, canonical, grand-canonical, intermediate) and related distributions

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Abstract

Combining intuitive probabilistic assumptions with the basic laws of classical thermodynamics, using the latter to express probabilistic parameters in terms of the thermodynamic quantities, we get a simple unified derivation of the fundamental ensembles of statistical physics avoiding any limiting procedures, quantum hypothesis and even statistical entropy maximization. This point of view leads also to some related classes of correlated particle statistics.

Key words: Bose-Einstein and Fermi-Dirac distributions, canonical ensemble, grand-canonical ensemble, intermediate statistics, correlated statistics.

1 Introduction

The Bose-Einstein (BE) and Fermi-Dirac (FD) statistics are key concepts in modern physics, which remarkably start penetrating even in social sciences (see e.g. [1]). There are many approaches to the derivation of the basic particle distributions: canonical (or Boltzmann), grand canonical, BE and FD. For instance, the canonical ensemble can be derived via equilibria with some hypothetical external reservoirs or from the principle of maximal entropy (see e.g. [11]). It can also be derived from the microcanonical ensemble by passing to the limit of projections to a single state of an ensemble of identical particles, as the number of particles tends to infinity (see e.g. [10], [17]). BE and FD distributions can be derived from the grand canonical ensemble (which in turns is obtained via entropy optimization or via external reservoirs) or via entropy optimization for certain energy level packing models (see e.g. [11]). We can refer to [8] for the modern presentation of the original derivations due to Planck and Bose. For the specific setting of the black body radiation the historical background from the modern perspective can be found in [2], see also a discussion in [16]. Since the usual derivation of BE statistics includes some law of large number limit, the pre-limit (number of particle dependent) versions were introduced

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in [15]. Usually the BE distributions are associated with quantum behavior, and the canonical Boltzmann distribution is obtained as their classical limit.

Papers [5] and [6] present pure probabilistic derivations of the basic ensembles and the review of various approaches to such derivations (including a curious idea of Brillouin assigning particles positive or negative volume to derive Fermi-Dirac or Bose-Einstein statistics respectively). These papers exploit the conditional probabilities arising from adding and taking away particles from an ensemble and derive the distributions postulating certain properties of such probabilities.

In the present paper we follow a similar methodology, though searching for the most direct postulate and combining it with the basic laws of classical thermodynamics (coincidence of intensive variables for systems in equilibria) in order to express probabilistic parameters in terms of the basic thermodynamic quantities. Our conditioning postulate is close in spirit to Johnson’s ‘sufficientness postulate’. However, the latter is given in terms of a Markov chain that ‘creates’ new particles (as stressed in [5]), and we employ a different point of view dealing with a fixed finite collection of particles. As a result we get a unified and very elementary derivation of all basic distributions (including even more exotic intermediate statistics) avoiding any limiting procedures, entropy maximization or quantum hypothesis. Additionally this point of view leads to the derivation of more general classes of particle distributions with correlated statistics.

In Section 2 we introduce our conditioning postulate revealing a specific feature of the geometric distribution that relates it to particle statistics. In the following sections we derive all basic statistics as consequences of this feature and the classical laws of thermodynamics. Finally we discuss some extensions showing, in particular, a remarkable robustness of our conditioning postulate which leads to some interesting class of correlated statistics.

2 Geometric distribution

Suppose one particle can be in one of $k$ states. The state space of the system of many particles (in its statistical description) consists of vectors $n = (n_1, \cdots, n_k)$ of $k$ non-negative integers, where $n_j$ denotes the number of particles in the state $j$. Adding a particle of type $j$ to such $n$ produces the new state

$$n + e_j = (n_1, \cdots, n_{j-1}, n_j + 1, n_{j+1}, \cdots, n_k),$$

where $e_j$ is the unit coordinate vector (with $j$th coordinate 1 and other coordinates vanishing).

Let us denote by $n^+ = (n_1, \cdots, n_k)^+$ the event that there are at least $n_j$ particles in the state $j$ for each $j = 1, \cdots, k$ (we found $n_j$ particles, but there can be more), that is

$$(n_1, \cdots, n_k)^+ = \bigcup_{m_1, \cdots, m_k: m_j \geq n_j} (m_1, \cdots, m_k).$$

Our ‘conditioning postulate’ is as follows: the conditional probabilities

$$q_j = \mathbf{P}((n + e_j)^+|n^+)$$

$$= \mathbf{P}((n_1, \cdots, n_{j-1}, n_j + 1, n_{j+1}, \cdots, n_k)^+|(n_1, \cdots, n_{j-1}, n_j, n_{j+1}, \cdots, n_k)^+),$$

depend only on the type $j$ of a particle and not on the state $n$. This postulate is a kind of no memory property (it can be also interpreted as some no-interaction axiom).
Proposition 2.1. Condition (1) with some \( q_j \in (0, 1) \) is equivalent to the condition

\[
P(n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_k) = q_j P(n_1, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k),
\]

that is, the ratio \( P(n + e_j)/P(n) = q_j \) depends only on the type of particles. Each of conditions (1) and (2) is equivalent to the formula

\[
P(n_1, \ldots, n_k) = \prod_{j=1}^{k}[q_j^{n_j}(1 - q_j)],
\]

that is \((n_1, \ldots, n_k)\) is a random vector of independent geometrical distributions.

Proof. (2) \(\Rightarrow\) (3): Assume (2) holds with some \( q_j \in (0, 1). \) Denoting by \( P_0 \) the probability of the vacuum state (the state without particles) we find directly that

\[
P(n_1, \ldots, n_k) = P_0 \prod_{j=1}^{k} q_j^{n_j}.
\]

By the normalization condition for probabilities (4),

\[
1 = P_0 \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \prod_{j=1}^{k} q_j^{n_j} = \frac{P_0}{(1 - q_1) \cdots (1 - q_k)},
\]

implying (3).

(3) \(\Rightarrow\) (1): It follows from (3) that

\[
P((n_1, \ldots, n_k)^+) = \sum_{m_1=n_1}^{\infty} \cdots \sum_{m_k=n_k}^{\infty} \prod_{j=1}^{k} q_j^{m_j} \frac{q_j^{m_j}}{1 - q_j} = \prod_{j=1}^{k} q_j^{n_j},
\]

implying (1).

(1) \(\Rightarrow\) (3). Condition (1) implies (3) and hence the independence of all \( n_j. \) For \( k = 1, \)

(3) follows directly.

By the independence, the average number of particles in a state \( j \) is independent of other particles and equals the expectation of the corresponding geometric random variable

\[
N_j = \mathbb{E}n_j = (1 - q_j) \sum_{n_j=0}^{\infty} n_j q_j^{n_j} = \frac{q_j}{1 - q_j} = \frac{1}{q_j^{-1} - 1}.
\]

Inverting this formula we see that the values of \( q_j \) can be uniquely identified from the average number of particles in each state:

\[
q_j = \frac{1}{N_j^{-1} + 1}.
\]

From physics one expects

\[
q_j = e^{\beta(\mu - \epsilon_j)}, \quad \beta = 1/k_B T,
\]

where \( \mu \) is the chemical potential, \( \epsilon_j \) is the energy of the \( j \)-th type of particle, \( k_B \) is the Boltzmann constant, and \( T \) is the temperature.
where $\mu$ is the chemical potential, $T$ temperature, in which case (6) concretizes to

$$E_{n_j} = \frac{1}{\exp\{\beta(\epsilon_j - \mu)\} - 1},$$

(9)

which is the Bose-Einstein distribution.

In the next section we derive (8)-(9) from (3) and basic thermodynamics.

Remark 1. There are several well known and insightful ways to characterize geometric random vector (3). For instance, it can be derived from the entropy maximization as the distribution on the collections $\{n_1, \ldots, n_k\}$ maximizing the entropy under the constraints of given $E_{n_j}$. Alternatively it arises as the monkey-typing process of Mandelbrot and Miller [14], which can be recast in terms of particle accumulation. It is also an invariant distribution for the Markov chain on the collections $\{n_1, \ldots, n_k\}$ that moves $n_j$ to $n_j+1$ or $n_j-1$ (the latter only when $n_j \neq 0$) with given probabilities $r^j_+$ and $r^j_-$, in which case $q_j = r^j_+ / r^j_-$ (used already in [4], see also [9]). Yet another way arises from packing randomly $k$ energy levels with indistinguishable particles, each $j$th level having given number $L_j$ of states, so that given numbers $N_j$ of particles go to the $L_j$ states of the $j$th level (with all possible distribution equally probable). In this way the probabilities $P_n$ to have $n$ particles in any given state of any $j$th level can be described by the Yule-Simon growth process and become geometric in the limit $N_j \to \infty$, $L_j \to \infty$ (see discussion and references in [7] and [18]). The limit in this scheme can be taken differently. Namely, let $L_j = a_j L$ with fixed $a_j$ and let $L$ and $N = N_1 + \cdots + N_k$ tend to infinity. As was noted in [12] the limiting distribution depends on whether the ratio $L/N$ tends to infinity, to a finite number or to zero, in which cases the limiting distribution is exponential (Gibbs canonical), Bose-Einstein or power law (Pareto type) respectively.

### 3 Bose-Einstein distribution

We shall now identify the expression for $q_j$ in terms of the classical variables of thermodynamics, that is, obtain (8). Of course some properties of thermodynamic variables have to be taken into account for such a derivation. We shall use the principle that intensive variables, like temperature and chemical potential, coincide for systems in equilibrium. This principle can be taken for granted as an empirical fact or derived from the classical principle of increasing entropy (that is, from the second law of thermodynamics).

System distributed by (3) can be looked at as $k$ systems in equilibrium, each one characterized by its number of particles $N_j$, and having common temperature $T$ and chemical potential $\mu$. Assume $\epsilon_j$ is the energy in state $j$.

Writing the fundamental equations for each system in terms of the grand potentials

$$\Phi_j = E_j - S_j T - \mu N_j,$$

$j = 1, \ldots, k$, the natural variables are $\mu, T$, and thus each $q_j$ must be a function of $\mu$ and $T$. By (6), the energy of the subsystem containing the states $j$ is

$$E_j = \epsilon_j N_j = \epsilon_j q_j / (1 - q_j).$$

Consequently,

$$\Phi_j = \left(\frac{\epsilon_j - \mu}{1 - q_j}\right) q_j - k_B T \left[ - \ln(1 - q_j) - \frac{q_j}{1 - q_j} \ln q_j \right],$$
where \( S_j = -\ln(1 - q_j) - \frac{q_j}{1 - q_j} \ln q_j \) is the entropy of the geometric distribution with the parameter \( q_j \). Since \( \frac{\partial \Phi_j}{\partial \mu} = -N_j \) (by the definition of \( \Phi \) as the Legendre transform of the energy \( E = E(S, N) \)) and \( N_j = q_j/(1 - q_j) \),

\[
\frac{\partial \Phi_j}{\partial \mu} = -N_j + \frac{\partial \Phi_j}{\partial q_j} \frac{\partial q_j}{\partial \mu} = -N_j.
\]

Similarly, \( \frac{\partial \Phi_j}{\partial T} = -S_j \), so that

\[
\frac{\partial \Phi_j}{\partial T} = -S_j + \frac{\partial \Phi_j}{\partial q_j} \frac{\partial q_j}{\partial T} = -S_j.
\]

As we cannot have both \( \frac{\partial q_j}{\partial \mu} = 0 \) and \( \frac{\partial q_j}{\partial T} = 0 \) (otherwise \( q_j \) is a constant independent of thermodynamics, so that the corresponding state \( j \) can be considered as some irrelevant background) it follows that

\[
\frac{\partial \Phi_j}{\partial q_j} = \frac{\epsilon_j - \mu}{(1 - q_j)^2} + k_B T \frac{\ln q_j}{(1 - q_j)^2} = 0,
\]

which implies \( \Phi \) by expressing \( q_j \) as the subject.

### 4 Canonical and grand canonical ensembles

Assuming \( \Phi \), what is the conditional probability of having a particle in state \( j \) given that there is only one particle in the system? It equals

\[
P(j|\text{one particle}) = \frac{P_0 q_j}{\sum_{k} P_0 q_k}.
\]

Under \( \Phi \) it implies

\[
P(j|\text{one particle}) = Z^{-1} e^{-\beta \epsilon_j}, \quad Z = \sum_{l} e^{-\beta \epsilon_l}, \quad (10)
\]

that is, the standard canonical ensemble.

The grand canonical ensemble for bosons is just distribution \( \Phi \) with \( q_j \) from \( \Phi \). Conditioning on the total number of particles, that is, taking the conditional probability

\[
P(n_1, \ldots, n_k | N) = P(n_1, \ldots, n_k | \text{number of particles is } N),
\]

yields the canonical ensemble for \( N \) particles. By \( \Phi \),

\[
P(n_1, \ldots, n_k | N) = \frac{P(n_1, \ldots, n_k)}{P(\text{number of particles is } N)} = \frac{P_0 \prod e^{-\beta(\epsilon_j - \mu)n_j}}{P(\text{number of particles is } N)}
\]

that is

\[
P(n_1, \ldots, n_k | N) = Z_{ge}^{-1}(N) \prod e^{-\beta(\epsilon_j - \mu)n_j}, \quad (11)
\]

with

\[
Z_{ge}(N) = \frac{P(\text{number of particles is } N)}{P_0} = \sum_{n_1=1}^{N} \cdots \prod e^{-\beta(\epsilon_j - \mu)n_j}, \quad (12)
\]
the grand canonical partition function reduced to \( N \) particle states. One can calculate this function by induction yielding

\[
Z_{gc}(N) = \sum_{j=1}^{k} q_j^{N+k-1} \prod_{m \neq j} (q_j - q_m)^{-1},
\]

(13)
in the case of different \( q_j \) from (8).

5 Fermi-Dirac and intermediate statistics

Similarly to the discussion above and keeping the main assumption (2), we can analyze the situation with the exclusion principle, that is, when the particles cannot occupy the same state, so that the vector-states \((n_1, \cdots, n_k)\) can have coordinates zero or one only. In this case (4) remains true, but only for this kind of vectors, and the normalization condition yields

\[
1 = P_0 \sum_{n_1, n_2, \cdots, n_k=0}^{1} \prod_{j=1}^{k} q_j^{n_j} = P_0 (1 + q_1) \cdots (1 + q_k),
\]

so that

\[
P_0 = \prod_{j=1}^{k} (1 + q_j)^{-1}, \quad P(n_1, \cdots, n_k) = \prod_{j=1}^{k} \frac{q_j^{n_j}}{1 + q_j}.
\]

(14)

Therefore the random vector \((n_1, \cdots, n_k)\) is an independent collection of \( k \) Bernoulli random variables, each taking values 0 or 1 with the probabilities \( 1/(1 + q_j), q_j/(1 + q_j) \).

The average number of particles in state \( j \) is thus the expectation of the \( j \)th Bernoulli random variable and equals

\[
E n_j = \frac{q_j}{1 + q_j} = \frac{1}{q_j^{-1} + 1}.
\]

(15)

If \( q_j \) are given by (8), (15) turns to

\[
E n_j = \frac{1}{\exp\{\beta(\epsilon_j - \mu)\} + 1},
\]

(16)

which is the Fermi-Dirac (FD) distribution.

Instead of (5), for FD statistics one has

\[
P((n_1, \cdots, n_k)^+) = \prod_{j: n_j=1} \frac{q_j}{1 + q_j},
\]

as this is just the probability that all levels are occupied.

In a more general situation, the number of particles in each state can be bound by some number \( K \geq 1 \). The corresponding intermediate statistics was initially suggested by G. Gentile Jr (see [3] for a review). For instance, if one aims at using Bose-Einstein statistics for molecules, their number is bounded (by the total number of molecules), and
the intermediate statistics with bounded occupation numbers may be more realistic, than their $K \to \infty$ limit.

Under assumption (2) and limiting the occupation numbers by a constant $K$, we get the normalization condition in the form

$$1 = P_0 \sum_{n_1, n_2, \ldots, n_k=0}^{K} \prod_{j=1}^{k} q_j^{n_j} = P_0 \prod_{m=1}^{k} \frac{1 - q_m^{K+1}}{1 - q_m},$$

so that

$$P(n_1, \ldots, n_k) = \prod_{j=1}^{k} \left( \frac{q_j^{n_j}(1 - q_j)}{1 - q_j^{K+1}} \right). \quad (17)$$

Thus we obtain the intermediate statistics (sometimes also called parastatistics) for the average number of particles in state $j$ (Gentile’s formula) as another corollary of postulate (1):

$$E_{n_j} = \frac{1 - q_j}{1 - q_j^{K+1}} \sum_{n=1}^{K} nq_j^n = \frac{q_j[1 + Kq_j^{K+1} - (1 + K)q_j^K]}{(1 - q_j)(1 - q_j^{K+1})}. \quad (18)$$

We refer to [13] for some recent applications of this statistics.

6 Generalized Bose-Einstein distribution and canonical ensemble for magnetic systems

The Bose-Einstein distribution (9) was derived from the geometric distribution for the simplest system characterized only by the temperature and the chemical properties of the energy levels. In general, different states of a system $\{1, \ldots, k\}$ can be characterized by other local extensive variables, not only the energy $E$. Let us denote them $U = (U_1, \ldots, U_m)$ and their normalized values (per particle) in $j$th state by $u^j = (u^j_1, \ldots, u^j_m)$. Denoting the dual intensive variables $\nu = (\nu_1, \ldots, \nu_m)$ we can write the thermodynamic potential of the system with the basic variables $T, \nu$ as

$$\Phi = E - ST - (\nu, U) = E - ST - \sum_{l=1}^{m} \nu_l U_l,$$

and the corresponding thermodynamic potentials for subsystems combining particles in states $j$ as

$$\Phi_j = \epsilon_j N_j - S_j T - (\nu, u^j)N_j = \frac{(\epsilon_j - (\nu, u^j))q_j}{1 - q_j} - k_B T[- \ln(1 - q_j) - \frac{q_j}{1 - q_j} \ln q_j].$$

Since $\partial \Phi_j/\partial \nu = -u_j N_j$ and $\partial \Phi_j/\partial T = -S_j$ (by the definition of the thermodynamic potential as the Legendre transform of the energy $E = E(S, U)$), it follows that

$$\frac{\partial \Phi_j}{\partial \nu} = -u_j N_j + \frac{\partial \Phi_j}{\partial q_j} \frac{\partial q_j}{\partial \nu} = -u_j N_j,$$

and

$$\frac{\partial \Phi_j}{\partial T} = -S_j + \frac{\partial \Phi_j}{\partial q_j} \frac{\partial q_j}{\partial T} = -S_j.$$
As previously, we cannot have both \( \partial q_j / \partial \nu = 0 \) and \( \partial q_j / \partial T = 0 \), it follows that

\[
\frac{\partial \Phi_j}{\partial q_j} = \frac{\epsilon_j - (\nu, u_j)}{(1 - q_j)^2} + k_B T \ln \frac{q_j}{(1 - q_j)^2} = 0,
\]

which implies the following extension of (8):

\[
q_j = e^{\beta((\nu, u_j) - \epsilon_j)}.
\]

Formula (8) is obtained from (19) if \( m = 1 \), \( u_j = 1 \) and \( \mu = \nu \).

The probability of a particle to be in \( j \)th state conditioned on having only one particle becomes now

\[
P(j|\text{one particle}) = Z^{-1} e^{-\beta(\epsilon_j - (\nu, u_j))}, \quad Z = \sum_{l} e^{-\beta(\epsilon_l - (\nu, u_l))},
\]

extending (10) and yielding the general version of the canonical ensemble.

For instance, for the simplest magnetic system specified by a finite number of sites \( \{1, \cdots, L\} \), each of which can have a spin \( \sigma \) chosen from a fixed subset of a vector space (in the simplest case \( \sigma = \pm 1 \)), a state is a configuration \( \Sigma \), that is an assignment of \( \sigma_l \), the values of \( \sigma \) at each site \( l \). A configuration \( \Sigma \) is characterized by its energy \( E(\Sigma) \) (some given function) and the magnetization \( M(\Sigma) = \sum \sigma_l \). The canonical ensemble for such magnetic system subject to an external magnetic field \( H \) is the distribution on the configurations given by the formula

\[
P(\Sigma) = Z^{-1} e^{-\beta(E(\Sigma) - HM(\Sigma))}, \quad Z = \sum_{\Sigma} e^{-\beta(E(\Sigma) - HM(\Sigma))},
\]

(see e.g. [11]), which is seen to be given by (20) with the index \( j \) counting sites replaced by \( \Sigma \), \( \nu = H \) and \( u_j \) denoted by \( M(\Sigma) \).

Another example is the so-called pressure ensemble for gases obtained by choosing \( \nu \) to be the pressure.

### 7 Further links, extensions and exercises

1. From (13) one can find the number of particles in state \( i \) conditioned on the total number \( N \):

\[
E(n_i|n_1 + \cdots + n_k = N) = Z_{g_c}^{-1}(N) \sum_{n_1 + \cdots + n_k = N} n_i \prod_{j=1}^{k} q_j^{n_j} \frac{q_j^{N+1} - q_i^{N+1} - (N+1)q_i^N q_j}{(q_j - q_i)^2 \prod_{m \neq i,j} (q_j - q_m)}.
\]

(22)

This formula is seen to be close to Gentile’s intermediate distribution (18). In fact, (18) and (22) refer to the number of particles under slightly different constraints.

What will be the limit of (22), when \( N \to \infty \)? Suppose

\[
q_i > \max_{j \neq i} q_j.
\]
Then one can check (to perform calculations it is handy to start with \( k = 2 \)) that
\[
\lim_{N \to \infty} E(n_i|n_1 + \cdots + n_k = N)/N = 1.
\]
The main point is this exact 1 on the r.h.s., which means that almost all particles will eventually settle on the level \( i \) of the lowest energy. One can get even more precise result. Namely,
\[
\lim_{N \to \infty} E(n_j|n_1 + \cdots + n_k = N) = \frac{q_j}{q_i - q_j} = \frac{1}{e^{\beta(\epsilon_i - \epsilon_j)} - 1}; \quad j \neq i,
\]
that is, in the limit \( N \to \infty \), other levels contain only finite number of particles, which are distributed according to the BE statistics on \( k - 1 \) levels with the chemical potential coinciding with the lowest energy level. This is a performance of the general effect of the Bose-Einstein condensation.

2. If in distributions (3) or (4), all \( q_j \) are close to each other, so that one can write \( q_j = p + \epsilon_j \) with small \( \epsilon_j \), then, in the first order of approximation, (4) becomes
\[
P(n_1, \cdots, n_k) = P_0p^N(1 + \frac{1}{p} \sum_j \epsilon_j n_j),
\]
with \( N = \sum n_j \), that is, the r.h.s. is bilinear with respect to the occupation numbers and transition rates. Such bilinear form is used by the authors of [1] in their psychological experiments with 11 animals.

3. As was mentioned, condition (1) is reminiscent to Johnson’s ‘sufficiency postulate’ (see [19] for its full discussion) stating that in the Markov process creating new particles the probability to create a particle of type \( i \) depends only on the number of existing particles of this type. This is different from (1) and leads one to a different distribution. In particular, if this probability of creation depends only on the type of a particle, the resulting probability of the occupation numbers \( n = (n_1, \cdots, n_k) \) becomes \( N! \prod p_j^{n_j}/n_j! \) (see [5]), which differs by the multinomial coefficient from the multivariate geometric. We refer to [20] for further extensions related to the Johnson-Carnap continuum of inductive methods.

4. Let us now discuss a rather amazing robustness of our basic postulate (1) and some correlated statistics arising from its extension.

**Proposition 7.1.** Assume that
\[
P((n + e_j)^+|n^+) = q(j, \frac{n_j}{n_1 + \cdots + n_k}), \quad n = (n_1, \cdots, n_k),
\]
that is, unlike our initial postulate, the conditional probabilities on the l.h.s. of this equation are allowed to depend not only on \( j \), but also on the fraction of \( j \)th particle in the state \( n \). If \( k > 2 \) it follows that
\[
q(j, \frac{m}{l}) = q(j, 1) = q_j
\]
for all \( m/l \neq 0 \), so that the deviation from \( q_j \) not depending on the fraction of \( j \)th particles in the state \( n \) can actually manifest itself only in the choice of \( q_{0j} = q(j, 0) \). For the unconditional probabilities one gets the formula
\[
P((n_1, \cdots, n_k)^+) = \omega \prod_{j \in I}[q_{0j} q_{j}^{n_j-1}], \quad n_1 + \cdots + n_k > 0,
\]
\[ P(n_1, \cdots, n_k) = \prod_{j \in I} (1 - q_j) \prod_{j \notin I} (1 - q_{0j}) P((n_1, \cdots, n_k)^+), \quad n_1 + \cdots + n_k > 0, \tag{26} \]

where \( I = \{ j : n_j \neq 0 \} \), and

\[ P(0, \cdots, 0) = 1 - \omega [1 - \prod_{j=1}^{k} (1 - q_{0j})]. \tag{27} \]

Here \( q_j \in (0, 1], q_{0j} \in (0, 1], \omega > 0 \) are arbitrary constants subject to the constraint \( P(0, \cdots, 0) \geq 0 \), that is

\[ \omega [1 - \prod_{j=1}^{k} (1 - q_{0j})] \leq 1. \tag{28} \]

In particular, \( \omega \leq 1 \) if \( q_{0j} = 1 \) for at least one index \( j \).

The proof of this theorem is an insightful exercise based on the exploitation of the consistency equations:

\[ P(n^+) = P((n - e_j)^+) q(j, n_j - 1 \over n_1 + \cdots + n_k), \quad j = 1, \cdots, k. \tag{29} \]

If \( \omega = 1 \) in (25), the vector \( n = (n_1, \cdots, n_k) \) is seen to have independent coordinates, which are represented by just slight extensions of the geometric distributions. However, if \( \omega \neq 1 \), the coordinates of vector \( n \) become dependent:

\[ E_{n_j} = \frac{\omega q_{0j}}{1 - q_j}, \quad E(n_i n_j) = \frac{1}{\omega} E_{n_i} E_{n_j}, \quad Cov(n_i, n_j) = (\frac{1}{\omega} - 1) E_{n_i} E_{n_j}. \tag{30} \]

Moreover, (25) is sensitive to the number of remaining types: under the condition that \( n_j = 0 \), the coefficient \( \omega \) for the remaining particles turns to \( \omega (1 - q_{0j})/(1 - \omega q_{0j}) \).

The distribution of each coordinate \( n_j \) is given by

\[ \omega q_{0j} q_j^{n_j - 1} (1 - q_j), \quad n_j \neq 0, \]

and the entropy of this distribution is found to be

\[ S_j = S_{Ber}(\omega q_{0j}) + \omega q_{0j} S_{Geom}(q_j), \]

where \( S_{Ber}(a) = -a \ln a - (1 - a) \ln(1 - a) \) and \( S_{Geom}(a) \) denote the entropies of the Bernoulli and geometric random variables with a parameter \( a \). This allows one to find the difference between the entropy of the vector \( n \) and the sum of the entropies \( S_j \) of \( n_j \). This difference vanishes if \( \omega = 1 \) (as it should be for independent coordinates), and otherwise it represents the nontrivial entropy of mixing of particles lying on different energy levels.

One can also check that for any \( k > 1 \) the distribution (26), (27) can be obtained as the maximum entropy distribution on \( \{0, 1, \cdots\}^k \) subject to given expectations of the number of particles in each state, the probabilities for each level to be nonempty and the probability of vacuum, that is, by \( 2k + 1 \) parameters, which can be fixed by the choice of \( q_j, q_{0j}, \omega \).

It seems that for \( k = 2 \) there are other distributions satisfying (24), but it is not at all clear (at least for the author), how they look like.
As shows already [23], the general form of the grand canonical distribution [3] is preserved under various conditioning and limiting procedures. To support this claim one can also check, for instance, that under [3], the distribution $P((n_1, n_2)|n_3 = n_1 + n_2)$ has the same form with the parameters $k = 2$, $q'_1 = q_1q_3$, $q'_2 = q_2q_3$, and the distribution $P((n_1, n_2)|n_1 = n_2)$ again the same form with the parameters $k = 1$, $q' = q_1q_2$. However, examples of distribution (26), (27) can be also obtained from the standard grand canonical distribution (3) by an appropriate conditioning, for instance, by conditioning on the absence of vacuum. In fact, under (3),

$$P((n_1, \cdots, n_k)|n_1 + \cdots + n_k > 0) = \frac{\prod_{j=1}^k [q_{nj}^n(1 - q_j)]}{1 - \prod_j (1 - q_j)},$$

(31)

which is (26), (27) with $q_{0j} = q_j$ and $\omega = [1 - \prod_j (1 - q_j)]^{-1}$.

5. A continuous variable version of axiom (1) is the following condition on the random vector $\tau = (\tau_1, \cdots, \tau_k)$ with non-negative coordinates:

$$\frac{\partial}{\partial s}|_{s=0} P(\tau_j > t_j + s| \tau_l > t_l \forall l) = q_j.$$

(32)

This is easily seen to imply that $\tau_j$ are independent exponential random variables. The analog of (24) is the condition

$$\frac{\partial}{\partial s}|_{s=0} P(\tau_j > t_j + s| \tau_l > t_l \forall l) = q_j \left(\frac{t_j}{t_1 + \cdots + t_k}\right).$$

(33)

In analogy with Proposition 7.1 one can show that if the vector has absolutely continuous distribution and $k > 2$, then all $q_j$ on the r.h.s. of (33) must be constant, that is, (32) holds. Some analogs of more general distributions (26) can be obtained assuming the discontinuity of the distribution of $\tau$ on the boundary of its range. For continuous random variables $\tau_j$ more natural interpretation is in terms of time to default (finances) or survival time (engineering). Looking at distributions (26), (27) as natural discretizations of continuous random vectors satisfying (33) can lead to a performance of the BE distributions for estimating the rates of defaults or survivals.

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