Improved Optimistic Algorithm For The Multinomial Logit Contextual Bandit

December 1, 2020

Abstract

We consider a dynamic assortment selection problem where the goal is to offer a sequence of assortments of cardinality at most $K$, out of $N$ items, to minimize the expected cumulative regret (loss of revenue). The feedback is given by a multinomial logit (MNL) choice model. This sequential decision making problem is studied under the MNL contextual bandit framework. The existing algorithms for MNL contextual bandit have frequentist regret guarantees as $O(\kappa \sqrt{T})$, where $\kappa$ is an instance dependent constant. $\kappa$ could be arbitrarily large, e.g. exponentially dependent on the model parameters, causing the existing regret guarantees to be substantially loose. We propose an optimistic algorithm with a carefully designed exploration bonus term and show that it enjoys $O(\sqrt{T})$ regret. In our bounds, the $\kappa$ factor only affects the poly-log term and not the leading term of the regret bounds.

1 INTRODUCTION

Linear contextual bandit [Dani et al., 2008] [Rusmevichientong and Tsitsiklis, 2010] [Abbasi-Yadkori et al., 2011] [Chu et al., 2011] [Abeille et al., 2017] [Hao et al., 2020] is an extension of the standard Multi-armed bandit (Bubeck and Cesa-Bianchi, 2012) framework. It caters to the exploitation-exploration dilemma in situations where the actions space is large or the contextual (side) information models some useful attributes of the environment. Generalized linear bandits (GLB) (Filippi et al., 2010) [Li et al., 2017] provide a natural extension to the linear contextual bandits by allowing controlled non-linearity in the reward function. In this work we focus on the Contextual multinomial logit (MNL) bandit, which is a generalization of logistic bandit (a variant of GLB) and holds significant commercial interests.

Here, we consider a dynamic assortment selection problem [Caro and Gallien, 2007] [Rusmevichientong et al., 2010] [Sauré and Zeevi, 2013] [Agrawal et al., 2017, 2019] with contextual (side) information [Oh and Iyengar, 2019a,b]. At each round of interaction, the learner aims to offer an assortment of size at most $K$ out of $N$ available items to an user with the goal of maximizing revenue. The feedback is the specific choice made by the user. The user chooses at most one item from the assortment. We assume that the user choice follows a multinomial logit (MNL) model. This framework has application in numerous domains such as recommendation systems, online retail, product placements in brick-mortar stores, news and social media feed, etc. Here, the utility of each item in the assortment is given by linear function of $d$-dimensional feature vectors. Our model is fairly general, the contexts can be composed of the combined information of both the item and the user and are allowed to change over time. For example, in case of online retail, the user information may refer to user’s purchase history, age group, preferences etc, while the item information may refer to price profile, popularity, ratings etc of the product.

On a comparative note, in the logistic bandit setting [Li et al., 2017] [Faury et al., 2020], the reward is modeled by a sigmoid function. In Filippi et al. (2010), the authors introduced GLM-UCB which is a generic optimistic algorithm for generalized linear models (and hence also applies to the logistic case) and achieves

\footnote{The reward function is called link function in the terminology of Generalized linear models (GLM).}
Figure 1: Illustration of $\kappa$ parameter (logistic case, multinomial logit case closely follows): A representative plot of derivative of the reward function. The x-axis represent the linear function $x^\top \theta$ and the y-axis is proportional to $1/\kappa$. $\kappa$ is small only in the narrow region around 0 and grows arbitrarily large depending on the problem instance ($x^\top \theta$ values).

$\tilde{O}\left(\kappa d\sqrt{T}\right)$ frequentist regret. Li et al. (2017) proposed SupCB-GLM and proved $\tilde{O}\left(\kappa \sqrt{d \log(K)T}\right)$ regret for it. For MNL bandits, Oh and Iyengar (2019a) gave a $\tilde{O}\left(\kappa \sqrt{dT}\right)$ regret achieving algorithm. In these works (also our own), $\kappa$ is an instance dependent parameter which characterizes the non-linearity of the problem. For example, in case of logistic rewards, $\kappa$ is inversely proportion to the derivative of the sigmoid function and could become arbitrarily large, even exponential for simple real-life examples. See Figure 1 for an illustration and Section 2.6 for a practical example.

**Contributions.** We propose a new optimistic algorithm, CB-UCB(MNL) for the study of contextual multinomial logit bandits. We prove that CB-UCB(MNL) enjoys $\tilde{O}\left(d\sqrt{T}\right)$ with the parameter $\kappa$ only affecting terms with logarithmic dependence on $T$. The generalization of logistic bandit (assortment cardinality = 1) to the MNL contextual is non-trivial since the MNL model cannot be expressed in form a generalized linear model (Chen et al., 2018). Also, in the MNL contextual bandit, the feedback (item chosen by the user) is a function of the entire assortment, this makes the regret analysis complicated. Our results significantly enhance the utility of contextual MNL bandit model for practical dynamic assortment selection problems.

CB-UCB(MNL) follows the standard template of the upper bound confidence learning strategies (Auer, 2002; Chu et al., 2011) with a delicate alteration to the form of exploration bonus. We make use of Bernstein-style concentration for self-normalized martingales which was recently proposed in Faury et al. (2020) in the context of logistic bandits to define our confidence interval. We also leverage self-concordance (Bach, 2010) like relation for the multinomial logit reward function (Zhang and Lin, 2015) which helps us to limit the effect of $\kappa$ on the final regret bounds.

2 PRELIMINARIES

2.1 Notations

For a vector $x \in \mathbb{R}^d$, $x^\top$ denotes the transpose. Given by positive definite matrix $M \in \mathbb{R}^{d \times d}$, the induced norm is given by $\|x\|_M = \sqrt{x^\top Mx}$. For two symmetric matrices $M_1$ and $M_2$, $M_1 \succeq M_2$ means that $M_1 - M_2$ is positive semi-definite. For any positive integer $n$, $[n] := \{1, 2, 3, \cdots, n\}$. $I_d$ denotes an identity matrix of dimension $d \times d$. The platform (i.e. the learner) is denoted by the pronouns she/her/hers.

---

$\tilde{O}\left(\kappa^2\right)$ order burn-in period

Both Oh and Iyengar (2019a) and Li et al. (2017) have slightly different definition of $\kappa$, however it leads to similar issues. Further both analyses require $\tilde{O}\left(\kappa^2\right)$ order burn-in period.
2.2 Model Setting

Rewards model. We consider the MNL contextual bandit problem (Oh and Iyengar 2019b). At every round \( t \), the platform (learner) is presented with set \( N \) of distinct items, indexed by \( i \in [N] \) and their attribute vectors (contexts): \( \{ x_{t,i} \}_{i=1}^{N} \) such that \( \forall \ i \in [N], \ x_{t,i} \in \mathbb{R}^{d} \), where \( N = |N| \) is the cardinality of set \( N \). The platform then selects an assortment \( Q_{t} \subset N \) and the interacting user (environment) offers the reward \( r_{t,i} \) to the platform. The assortment has the maximum cardinality of \( K \), i.e. \( |Q_{t}| \leq K \). The platform’s decision is based on the entire history of interaction. The history is represented by the filtration set \( \mathcal{F}_{t} := \mathcal{F}_{0}, \sigma \left( \{ x_{s,i} \}_{i=1}^{N}, Q_{s} \}_{s=1}^{t-1} \right) \). where \( \mathcal{F}_{0} \) is any prior information available to the platform. The interaction lasts for \( t = 1, 2, \cdots, T \) rounds. Conditioned on \( \mathcal{F}_{t} \), the reward \( r_{t,i} \) is a binary vector such that \( r_{t,i} \in \{0,1\}^{Q_{t}} \) and follows a multinomial distribution. Specifically, the probability that \( r_{t,i} = 1, \forall i \in Q_{t} \) is given by the softmax function:

\[
\mathbb{P} \left( r_{t,i} = 1 | Q_{t}, \mathcal{F}_{t} \right) = \mu_{i}^{ML} (Q_{t}, \theta_{*}) := \frac{\exp \left( x_{t,i}^{\top} \theta_{*} \right)}{1 + \sum_{j \in Q_{t}, \exp \left( x_{t,j}^{\top} \theta_{*} \right)}}, \quad (1)
\]

where \( \theta_{*} \) is an unknown time-invariant parameter. 1 is denominator accounts for the case when \( r_{t,i} = 0, \forall i \in Q_{t} \). Also, the expected revenue due to the assortment \( Q_{t} \) is given by:

\[
\mu^{ML} (Q_{t}, \theta_{*}) := \sum_{i \in Q_{t}} \mu_{i}^{ML} (Q_{t}, \theta_{*}). \quad (2)
\]

Also, \( \{ x_{t,i} \} \) may vary adversarially in each round in our model, unlike in Li et al. (2017); Oh and Iyengar (2019a), where the attribute vectors are assumed to be drawn from an unknown i.i.d. distribution. When \( K = 1 \), the above model reduces to the case of logistic bandit.

As choice model. Eq (1) can also be considered in the light of the platform-user setting where the platform presents an assortment of items to an user and the user selects atmost one item from this assortment. In this interpretation, the probability of choosing an item \( i \) is given by \( \mu_{i}^{ML} (Q_{t}, \theta_{*}) \). Likewise, the probability of the user not selecting any item is given by: \( 1/ \left( 1 + \sum_{j \in Q_{t}, \exp \left( x_{t,j}^{\top} \theta_{*} \right)} \right) \). Clearly, the platform attempts to offer such assortment such that the propensity of the user to make a successful selection is high.

Regret. The platform does not know the value of \( \theta_{*} \). Our learning algorithm CB-UCB(MNL) (see Algorithm 1) sequentially makes the assortment selection decisions, \( Q_{1}, Q_{2}, \cdots, Q_{T} \) so that the cumulative expected revenue \( \sum_{t=1}^{T} \mu^{ML} (Q_{t}, \theta_{*}) \) is high. The performance is quantified by pseudo-regret, which is the gap between the expected revenue generated by the algorithm and that of the optimal assortment in hindsight. The learning goal is to minimize the cumulative pseudo-regret up to time \( T \), defined as:

\[
R_{T} := \sum_{t=1}^{T} \left[ \mu^{ML} (Q_{t}^{*}, \theta_{*}) - \mu^{ML} (Q_{t}, \theta_{*}) \right], \quad (3)
\]

where \( Q_{t}^{*} \) is the offline optimal assortment at round \( t \) under full information of \( \theta_{*} \), defined as:

\[
Q_{t}^{*} := \arg \max_{Q \subset N} \mu^{ML} (Q, \theta_{*}).
\]

As in the case of contextual linear bandits (Abbasi-Yadkori et al. 2011; Chu et al. 2011), the emphasis here is to make good sequential decisions while tracking the the true parameter \( \theta_{*} \) with a close estimate \( \hat{\theta}_{t} \) (see

\[3 \sigma (\{ \cdot \}) \text{ denotes the } \sigma \text{-algebra set over the sequence } \{ \cdot \}.\]
Section 2.4. Our algorithm (like others) does not necessarily “improves” the estimate at each round. However it ensures that estimate lies within a confidence interval of the true parameter and aggregate prediction error for all $T$ rounds is bounded.

Our model is fairly general, the contextual information $x_{t,i}$ can be used to model combined information of the item $i$ in the set $N$ and the user at round $t$. Suppose the user at round $t$ is represented by $v_t$ and the item $i$ has attribute vector as $w_{t,i}$, then $x_{t,i} = \text{vec}(v_t w_{t,i}^\top)$ (vectorized outer product of $v_t$ and $w_{t,i}$). We assume that the platform knows the interaction horizon $T$. Additional notations: $X_Q$, denotes the design matrix whose columns are the attribute vectors $(x_{t,i})$ of the items in the assortment $Q_t$. Also, we now denote $\mu^ML(Q_t, \theta_*) as $\mu^ML(X_{Q_t}^\top, \theta_*)$ to signify that $\mu^ML(Q_t, \theta_*) : \mathbb{R}^{|Q_t|} \rightarrow \mathbb{R}$.

2.3 Assumptions

Following Filippi et al. (2010); Li et al. (2017); Oh and Iyengar (2019b); Faury et al. (2020), we introduce the subsequent assumptions on the problem structure.

Assumption 1 (Bounded parameters). $\theta_* \in \Theta$, where $\Theta$ is a compact subset of $\mathbb{R}^d$. $S := \max_{\theta \in \Theta} \|\theta\|_2$ is known to the learner. Further, $\|x_{t,i}\|_2 \leq 1$ for all values of $t$ and $i$.

This assumption simplifies analysis and removes scaling constants from the equations. It is common to the contextual bandit literature.

Assumption 2. There exists $\kappa > 0$ such that for every item $i \in Q_t$ and for any $Q_t \subset N$ and all rounds $t$:

$$\inf_{Q_t \subset N, \theta \in \mathbb{R}^d} \mu^ML_i \left( X_{Q_t}^\top \theta \right) \left( 1 - \mu^ML_i \left( X_{Q_t}^\top \theta \right) \right) \geq \frac{1}{\kappa}. $$

Note that $\mu^ML_i \left( X_{Q_t}^\top \theta \right) \left( 1 - \mu^ML_i \left( X_{Q_t}^\top \theta \right) \right)$ denotes the derivative of the softmax function along the $i$th direction. This assumption comes as a necessity from the likelihood theory (Lehmann and Casella 2006) as it ensures that the fisher matrix for $\theta_*$ estimation is invertible for all possible input instances. We refer to Oh and Iyengar (2019b) for a detailed discussion in this regard. We denote $L$ and $M$ as the upper bounds on the fist and second derivative of the the softmax function along any component respectively. We have $L, M \leq 1$ (Gao and Pavel 2017) for all problem instances.

2.4 Maximum Likelihood Estimate

CB-UCB(MNL), described in Algorithm 1, uses regularized maximum likelihood estimator to compute an estimate $\theta_\text{est}$ of $\theta_*$. Recall that $r_t = \{r_{t,1}, r_{t,2}, \cdots, r_{t,|Q_t|}\} \in \{0,1\}^{|Q_t|}$ follows multinomial distribution. Therefore, the regularized log-likelihood (negative cross entropy loss) function under parameter $\theta$ could be written as:

$$L^\lambda_i(\theta) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} r_{s,i} \log \left( \mu^ML_i(X_{Q_s}^\top \theta) \right) - \frac{\lambda}{2} \|\theta\|_2^2, \quad (4)$$

$L^\lambda_i(\theta)$ is concave in $\theta$ for $\lambda > 0$, the maximum likelihood estimator is given by calculating the critical point of $L^\lambda_i(\theta)$. The gradient $\nabla_{\theta} L^\lambda_i(\theta)$ is given by:

$$\nabla_{\theta} L^\lambda_i(\theta) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \left[ \mu^ML_i(X_{Q_s}^\top \theta) - r_{t,i} \right] x_{s,i} + \lambda \theta \quad (5)$$

Setting $\nabla_{\theta} L^\lambda_i(\theta) = 0$, we get $\hat{\theta}_i$ as the solution of Eq (5) as:

$$\sum_{s=1}^{t-1} \sum_{i \in Q_s} \left[ \mu^ML_i(X_{Q_s}^\top \hat{\theta}_i) - r_{t,i} \right] x_{s,i} + \lambda \hat{\theta}_i = 0. \quad (6)$$
For analysis we also define

\[ \begin{align*}
  g_t(\theta) & := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^{ML}_i(\mathbf{X}_{Q_s}^\top \theta)x_{s,i} + \lambda \theta, \\
  g_t(\hat{\theta}_t) & := \sum_{s=1}^{t-1} \sum_{i \in Q_s} r_{t,i}x_{s,i}.
\end{align*} \]

Suitably chosen regularization parameter \( \lambda (\lambda > 1) \) makes CBA-UCB(MNL) burn-in period free, in contrast to some previous works, e.g. \cite{filippi2010}. Moreover, we also show that \( \lambda \) plays an important role in bounding regret (see Corollary 2).

### 2.5 Confidence Set

Recently in \cite{faury2020}, the authors proposed a new Bernstein-like tail inequality for self-normalized vectorial martingales (see Appendix A.1), using which we derive a confidence set on \( \theta_* \) that would be later used to calibrate exploration bonuses for CBA-UCB(MNL) in Algorithm 1. The confidence set is presented as:

\[ C_t(\delta) := \left\{ \theta \in \Theta : \left\| g_t(\theta) - g_t(\hat{\theta}_t) \right\|_{H_{t-1}^{-1}(\theta)} \leq \gamma_t(\delta) \right\}, \]

where

\[ H_{t-1}(\theta_1) := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \dot{\mu}_i^{ML}(\mathbf{X}_{Q_s}^\top \theta_1)x_{s,i}x_{s,i}^\top + \lambda \mathbf{I}_d, \]

and

\[ \gamma_t(\delta) := \frac{\sqrt{\lambda}}{2} + \frac{2}{\sqrt{\lambda}} \log \left( \frac{(\lambda + LKt/d)^{d/2}}{\delta} \right) \]

\[ + \frac{2d}{\sqrt{\lambda}} \log(2). \]

\( \dot{\mu}_i^{ML}(\cdot) \) is the partial derivative of \( \mu_i^{ML} \) in the direction of the \( i \)-th component of the assortment. The value of \( \gamma_t(\delta) \) is an outcome of the concentration result of \cite{faury2020}. As a consequence of this concentration, we have \( \theta_* \in C_t(\delta) \) with probability at least \( 1 - \delta \).

The Bernstein-like concentration inequality used here is similar to Theorem 1 of \cite{abbasi2011} with the difference that we take into account local variance information (hence local curvature information of the reward function) in defining \( H_{t-1} \). Above discussion is formalized in Appendix A.1.

For later sections we also define the following norm inducing design matrix based on the all the contexts observed till time \( t - 1 \):

\[ V_{t-1} := \sum_{s=1}^{t-1} \sum_{i \in Q_s} x_{s,i}x_{s,i}^\top + \lambda \mathbf{I}_d. \]

### 2.6 On The Parameter \( \kappa \)

Assumptions similar to Assumption 2 have been made in the previous generalized linear bandit literature \cite{filippi2010,li2017,oh2019b} and the quantity \( \kappa \) features in their regret guarantees as a multiplicative factor of the primary term (as \( \tilde{O}(\kappa \sqrt{T}) \)). In this section, we illustrate that this dependence grows exponentially for practical scenarios. For the sake of making an intuitive argument consider \( K = 1 \), i.e. reward model reduces to the logistic case, and we represent \( x_{t,i} \) by just \( x_t \). Set \( z_t = x_t^\top \theta \) with \( z_t \in \mathbb{R}^d \). From Assumption 2 we have:

\[ \sup_{z_t} \left( \exp(-z_t/2) + \exp(z_t/2) \right) \leq \kappa. \]
Without any further constraints on \( z_t \), \( \kappa \) can be exponentially large. In the MNL model, \( \kappa \) is very large if there are few times in the assortment that have very high expected rewards (or there exists few items with very low rewards). In the context of platform-user setting, this relation implies that \( \kappa \) is very large if few items out of many in an assortment are more likely to be chosen by the user (or there exists few items which have very low probability of being chosen by the user). For example, the click through rate (CTR) of web advertisements (also for web search results) fall sharply after few positions, items listed on the second page have probability of being clicked \( \approx 10^{-3} \), causing \( \kappa \approx 10^{-3} \). This implies that the present algorithms have substantial gap in their regret bounds for simple practical scenarios. Our main result of dropping \( \kappa \) dependence from leading term of the regret bounds (see Section 4) makes the MNL contextual bandit framework useful for practical use-cases.

3 ALGORITHM

At each round \( t \), the attribute parameters (contexts) \( \{x_{t,1}, x_{t,2}, \cdots, x_{t,N}\} \) are made available to the algorithm CB-UCB(MNL). The algorithm calculates the estimate of the true parameter \( \theta_* \) as \( \hat{\theta}_t \) according to Eq (6). If \( \hat{\theta}_t \in \Theta \), then \( \theta_t^{\text{est}} \) is set as \( \hat{\theta}_t \). If \( \hat{\theta}_t \notin \Theta \), then the algorithm calculates \( \theta_t^{\text{est}} \) by projecting it on the set \( \Theta \), as:

\[
\theta_t^{\text{est}} = \arg\min_{\theta \in \Theta} \left\| g_t(\theta) - g_t(\hat{\theta}_t) \right\|_{H_{\theta_t}^{-1}(\theta)}. 
\]  

(12)

The choice of norm in this projection step ensures that \( \theta_t^{\text{est}} \) satisfies a slightly inflated version of confidence set of Eq (5) when \( \theta_* \in C_t(\delta) \) as :

\[
C_t(\delta) := \left\{ \theta \in \Theta, \left\| g_t(\theta^{\text{est}}) - g_t(\hat{\theta}_t) \right\|_{H_{\theta_t}^{-1}(\theta)} \leq 2\gamma_t(\delta) \right\}.
\]

For a simple exposition of our results, we skip the difference between \( C_t(\delta) \) and \( C_t(\hat{\delta}) \) in the main text and correctly account for the constant factor in the technical appendix.

Remark 1. It is easy to check if \( \hat{\theta}_t \in \Theta \) and this step can be done online and hence the computationally expensive projection step of Eq (12) need not be carried out. This projection step is similar to one in the previous literature \( \text{(Faury et al., 2020, Oh and Iyengar, 2019a, Filippi et al., 2010)} \) and is approximately evaluated \( \text{(Mairal, 2013)} \).

CB-UCB(MNL) is an optimistic (upper confidence bound) algorithm such that action (choice of assortment set \( Q_t \)) based on the estimate \( \theta_t^{\text{est}} \) of \( \theta_* \) is given by:

\[
Q_t = \arg\max_{A_t \in A} \mu_t^{\text{ML}}(X_t, \theta_t^{\text{est}}) + \left( \Psi_t(A_t, \theta_t^{\text{est}}) \right)_t
\]

(13)

The exploration bonus given by \( \Psi_t(Q_t, \theta) \)

\[
:= (2 + 4S) \gamma_t(\delta) \sum_{i \in Q_t} \mu_t^{\text{ML}}(X_{Q_t}, \theta) ||x_t,i||_{H_{\theta_t}^{-1}(\theta)}^2 + \kappa(4 + 8S)M\gamma_t(\delta)^2 \sum_{i \in Q_t} ||x_t,i||_{V_{\theta_t}^{-1}}^2.
\]

(14)

The choice of the exploration is determined by the upper bound on the prediction error of \( \theta \) at \( X_{Q_t} \), defined as:

\[
\Delta_{\text{pred}}(X_{Q_t}, \theta) := \left| \mu^{\text{ML}}(X_{Q_t}, \theta_t^{\text{est}}) - \mu^{\text{ML}}(X_{Q_t}, \theta) \right|.
\]

\( \Delta_{\text{pred}}(X_{Q_t}, \theta) \) represents the difference in perceived rewards due to the inaccuracy in the estimation of the parameter \( \theta_* \). Setting the exploration bonus to be upper-bound on the prediction error ensures that the regret at each round is within constant factors of the bonus term. The implication is that \( R_T \leq 2 \sum_{t=1}^T \Psi_t(Q_t, \theta_t^{\text{est}}) \).

This relation is proved in Lemma 3.

\[\text{This optimization step is typical to optimistic algorithms, as optimistic algorithms need to search over action space}\]
Algorithm 1: CB-UCB(MNL)

1 Input: regularization parameter: $\lambda$, $N$ distinct items
2 for $t \geq 1$ do
3 Given: Set $\{x_{t,1}, x_{t,2}, \cdots, x_{t,N}\}$ of parameters for each item at $t$.
4 Estimate $\hat{\theta}_t$ according to Eq (6);
5 if $\hat{\theta}_t \in \Theta$ then
6 Set $\theta_{t}^{\text{est}} = \hat{\theta}_t$
7 end
8 else
9 Compute $\theta_{t}^{\text{est}}$ by projecting $\hat{\theta}_t$ according to Eq (12)
10 end
11 Construct the set $\mathcal{A}$ of all possible assortments of $N$ with cardinality up to $K$.
12 Play $Q_t = \arg\max_{A_t \in \mathcal{A}} \mu_{ML}(X_{A_t}^T \theta_{t}^{\text{est}} + \Psi_t(A_t, \theta_{t}^{\text{est}}))$.
13 Observe rewards $r_t$
14 end

4 MAIN RESULTS

In this section, we present regret upper bounds for CB-UCB(MNL) in form of Theorem 1.

Theorem 1. With probability at least $1 - \delta$:

$$R_T \leq C_1 \gamma_t(\delta) \sqrt{2dT \log \left(1 + \frac{LK T}{d \lambda} \right)} + C_2 \kappa \gamma_t(\delta)^2 d \log \left(1 + \frac{KT}{d \lambda} \right),$$

where the constants are given as $C_1 = (4 + 8S)$, $C_2 = 4(4 + 8S)^{3/2}M$ and $\gamma_t(\delta)$ is given by Eq (10).

The formal proof is deferred to the technical Appendix, here we outline major steps leading this result. First, we show that the cumulative regret is upper bounded by sum of upper bounds on the prediction error. In Section 4.1 we arrive at an expression for prediction error upper bound. Then in Section 4.2 we give the outline for calculating the $T$ rounds summation of the prediction error. Finally, in Section 4.3 we describe the final steps leading to the statement of Theorem 1. The order dependence on the model parameters is made explicit by the following corollary.

Corollary 2. Setting the regularization parameter $\lambda = d \log (KT)$ where $K$ is the maximum cardinality of the assortments to be selected, makes $\gamma(\delta) = O \left( d^{3/2} \log^{1/2} (KT) \right)$. The regret upper bounds follow:

$$R_T = O \left( d \sqrt{T \log (KT)} + \kappa d^2 \log^2 (KT) \right).$$

Through Lemma 3 we establish that as long as the exploration bonus dominates the maximum prediction error at every round, the regret is upper bounded by a constant times summation of exploration bonus till the horizon $T$.

Lemma 3 (Prediction error as regret). For any $T \geq 1$, assortment set $\mathcal{A}$ and $\theta \in C_t(\delta)$ if the exploration bonus $\Psi_t(\mathcal{A}, \theta) \geq \Delta^{\text{pred}}(X_\mathcal{A}, \theta)$ then the regret of CB-UCB(MNL) is upper bounded as:

$$R_T \leq 2 \sum_{t=1}^{T} \Psi_t(Q_t, \theta_t),$$

7
where \( \theta_t \) denotes estimate of the true parameter \( \theta^* \) and \( X_{Q_t} \) refers to the design matrix composed of the attributes of the items in the assortment chosen by the algorithm at time \( t \).

**Proof.** From the definition of regret (see Eq (3)) we write:

\[
R_T = \sum_{t=1}^{T} \mu_{ML}^{T}(X_{Q_t}^\top \theta^*) - \mu_{ML}^{T}(X_{Q_t}^\top \theta_t)
\]

\[
= \sum_{t=1}^{T} \left( \mu_{ML}^{T}(X_{Q_t}^\top \theta^*) - \mu_{ML}^{T}(X_{Q_t}^\top \theta_t) \right)
\]

\[
+ \sum_{t=1}^{T} \left( \mu_{ML}^{T}(X_{Q_t}^\top \theta_t) - \mu_{ML}^{T}(X_{Q_t}^\top \theta^*) \right)
\]

\[
\leq \sum_{t=1}^{T} \Delta_{pred}(X_{Q_t}^\top \theta_t) + \sum_{t=1}^{T} \left( \mu_{ML}^{T}(X_{Q_t}^\top \theta_t) - \mu_{ML}^{T}(X_{Q_t}^\top \theta^*) \right)
\]

\[
\leq \sum_{t=1}^{T} \Delta_{pred}(X_{Q_t}^\top \theta_t) + \sum_{t=1}^{T} \Delta_{pred}(X_{Q_t}^\top \theta_t).
\]

From Eq (13) we have:

\[
\mu_{ML}^{T}(X_{Q_t}^\top \theta_t) + \Psi_t(Q_t, \theta_t) \geq \mu_{ML}^{T}(X_{Q_t}^\top \theta_t) + \Psi_t(Q_t, \theta_t),
\]

and

\[
\mu_{ML}^{T}(X_{Q_t}^\top \theta_t) - \mu_{ML}^{T}(X_{Q_t}^\top \theta^*) \leq \Psi_t(Q_t, \theta_t) - \Psi_t(Q_t, \theta_t).
\]

Therefore we have:

\[
R_T \leq \sum_{t=1}^{T} \Delta_{pred}(X_{Q_t}^\top \theta_t) + \sum_{t=1}^{T} \Psi_t(Q_t, \theta_t)
\]

\[
+ \sum_{t=1}^{T} \Delta_{pred}(X_{Q_t}^\top \theta_t) - \sum_{t=1}^{T} \Psi_t(Q_t, \theta_t).
\]

(15)

Applying \( \Psi_t(Q_t, \theta_i) \geq \Delta_{pred}(X_{Q_t}, \theta_i) \) on Eq (15), we obtain the Lemma statement.

Lemma 3 justifies using prediction error upper bound as a candidate choice for the exploration bonus in CB-UCB(MNL) (see Eq (13)). We now focus on upper bounding the prediction error. Lemma 4 gives the expression for an upper bound on the prediction error.

### 4.1 Bounds on prediction error

**Lemma 4.** Let \( \theta_i \) be the projected estimate, as in Eq (12) of the true parameter \( \theta^* \), then:

\[
\Delta_{pred}(X_{Q_t}^\top \theta_t)
\]

\[
\leq (2 + 4S) \gamma_t(\delta) \sum_{i \in Q_t} \mu_{ML}^i(X_{Q_t}^\top \theta_t) \|x_{t,i}\|_{H^{(\gamma_t)}_{i}^{-1}(\theta_i)}
\]

\[
+ \kappa(4 + 8S) \gamma_t(\delta)^2 \sum_{i \in Q_t} \|x_{t,i}\|^2_{V_{i}^{-1}},
\]

where \( V_{i}^{-1} \) is given by Eq (11).
The detailed proof is provided in the technical appendix. Here we develop the main ideas leading to this result. In the previous works [Filippi et al., 2010; Li et al., 2017; Oh and Iyengar, 2019a], the first step was to upper bound the prediction error by Lipschitz constant of the softmax (or sigmoid for the logistic bandit case) function, as:

$$|\mu^{ML}(X_Q^T\theta_s) - \mu^{ML}(X_Q^T\theta_t)| \leq L|X_Q^T(\theta_s - \theta_t)|. \quad (16)$$

By making the prediction error linear, we loose the local information carried by $\theta_t$. Instead, we construct a local information preserving norm to aid in the analysis. First, we introduce the notation:

$$\alpha_t(X_Q^T, \theta_t, \theta_s)x_{i,s}^T(\theta_s - \theta_t) := \mu^{ML}(X_Q^T\theta_s) - \mu^{ML}(X_Q^T\theta_t).$$

Therefore, we have:

$$\sum_{i \in Q_t} \alpha_t(X_Q^T, \theta_t, \theta_s)x_{i,s}^T(\theta_s - \theta_t) = \mu^{ML}(X_Q^T\theta_s) - \mu^{ML}(X_Q^T\theta_t). \quad (17)$$

From Eq (7a), we obtain:

$$g(\theta_s) - g(\theta_t) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \alpha_i(X_Q^T, \theta_t, \theta_s)x_{i,s}^T(\theta_s - \theta_t) + \lambda(\theta_s - \theta_t)$$

$$= Gm_t(\theta_t, \theta_s)(\theta_s - \theta_t),$$

where,

$$Gm_t(\theta_t, \theta_s) := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \alpha_i(X_Q^T, \theta_t, \theta_s)x_{i,s}^T + \lambda I_d.$$

Leveraging Assumption 2, $Gm_t(\theta_t, \theta_s)$ is a positive definite matrix for $\lambda > 0$. Therefore we write:

$$||\theta_s - \theta_t||_{Gm_t(\theta_t, \theta_s)} = ||g(\theta_s) - g(\theta_t)||_{Gm_t^{-1}(\theta_t, \theta_s)}. \quad (18)$$

Using Eq (18) with Cauchy-Schwarz inequality, helps to simplify the prediction error as:

$$\Delta^{\text{pred}}(X_Q^T, \theta_t) \leq \sum_{i \in Q_t} \alpha_i(X_Q^T, \theta_t, \theta_s)||x_{i,s}^T||_{Gm_t^{-1}}||g_t(\theta_s) - g_t(\theta_t)||_{Gm_t}.$$ 

It is not straight-forward to bound $||g(\theta_s) - g(\theta_t)||_{Gm_t^{-1}(\theta_t, \theta_s)}$, the previous literature [Filippi et al., 2010; Oh and Iyengar, 2019a] have utilized $Gm_t^{-1}(\theta_t, \theta_s) \succeq \kappa^{-1}V_{t}$, thereby incurring loose regret bounds.

We derive self-concordance style relations for the multinomial logit function. We show $|\tilde{\mu}_t^{ML}(\cdot)| \leq \mu_t^{ML}(\cdot)$ (see Appendix A.3), where $\tilde{\mu}_t^{ML}(\cdot)$ is the double derivative along the $i_{th}$ component. This allows us to relate $Gm_t^{-1}(\theta_t, \theta_s)$ and $Hm_t^{-1}(\theta_t)$ (or $Hm_t^{-1}(\theta_s)$) in the form of following lemma.

**Lemma 5.** For all $\theta_1, \theta_2 \in \Theta$ such that $S := \max_{\theta \in \Theta}||\theta||_2$, the following inequalities hold:

$$Gm_t(\theta_1, \theta_2) \succeq (1 + 2S)^{-1}Hm_t(\theta_1)$$

$$Gm_t(\theta_1, \theta_2) \succeq (1 + 2S)^{-1}Hm_t(\theta_2)$$

9
Lemma 5 provides an important relation between $\mathbf{G}_t$ and $\mathbf{H}_t$, as this combined with the definition of the confidence set, as in Eq (8), gives an handle on $\|g(\theta_*) - g(\theta_t)\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)}$, which we upper bound as below.

Lemma 6. For $\theta_t = \theta_t^{\text{est}}$ as calculated by the algorithm CB-UCB(MNL) and $S = \max_{\theta \in \Theta} \|\theta\|_2$, we have the following relation:

$$\|g(\theta_t) - g(\theta_*)\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)} \leq 2\sqrt{1 + 2S\gamma_t(\delta)}.$$ 

Proofs of Lemma 5 and 6 have been deferred to Appendix A.3.

Now, we use Lemma 6 in Eq (43) to get:

$$\Delta^{\text{pred}}(\mathbf{X}_t, \theta_t) \leq 2\sqrt{1 + 2S\gamma_t(\delta)} \sum_{\mathbf{i} \in \mathbf{Q}_t} \left| \alpha_i(\mathbf{X}_t, \theta_t, \theta_*) \|x_{t,i}\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)} \right|.$$ 

The quantity $\alpha_i(\mathbf{X}_t, \theta_t, \theta_*)$ as described in the Eq (17) is simplified by first order Taylor expansion. Thereby, we get:

$$\Delta^{\text{pred}}(\mathbf{X}_t, \theta_t) \leq 2\sqrt{1 + 2S\gamma_t(\delta)} \sum_{\mathbf{i} \in \mathbf{Q}_t} \mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta_t) \|x_{t,i}\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)} + (4 + 8S)\gamma_t(\delta)^2 \sum_{\mathbf{i} \in \mathbf{Q}_t} \|x_{t,i}\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)}^2.$$ 

Applying Lemma 5, we have $\|x_{t,i}\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)} \leq \sqrt{1 + 2S}\|x_{t,i}\|_{\mathbf{H}_t^{-1}(\theta_*)}$. Also, we have $\sum_{\mathbf{i} \in \mathbf{Q}_t} \|x_{t,i}\|_{\mathbf{G}_t^{-1}(\theta_*, \theta_t)}^2 \leq \kappa \sum_{\mathbf{i} \in \mathbf{Q}_t} \|x_{t,i}\|_{\mathbf{V}_t^{-1}}^2$ from Assumption 2. Thus, we arrive at the statement of Lemma 6.

We set the exploration bonus of CB-UCB(MNL) as the upper bound of the prediction error as given by Lemma 6 and invoke Lemma 3 to find an upper bound on regret. This requires calculating $T$ round summation of terms $\sum_{\mathbf{i} \in \mathbf{Q}_t} \mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta_t) \|x_{t,i}\|_{\mathbf{H}_t^{-1}(\theta_*)}$ and $\kappa \sum_{\mathbf{i} \in \mathbf{Q}_t} \|x_{t,i}\|_{\mathbf{V}_t^{-1}}^2$, which we discuss in Section 4.2. We also note that $\gamma_t(\delta)$, defined in Eq (10) is a logarithmically (i.e. very slowly) increasing quantity and hence for rounds $t \leq T$, it is upper bounded by $\gamma_T(\delta)$.

4.2 Sum of prediction error

In the term $\sum_{\mathbf{i} \in \mathbf{Q}_t} \mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta_t) \|x_{t,i}\|_{\mathbf{H}_t^{-1}(\theta_*)}$, $\|x_{t,i}\|_{\mathbf{H}_t^{-1}(\theta_*)}$ depends on the sequence $\{\mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta_t)^{-1}\}_{t,i}$. Without any additional work, one may conclude that $\mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta_t)^{-1}$ may be arbitrarily large (as discussed in Section 2.6). We make a key observation that we only need to account for $\theta \in \{C_t(\delta) \cap \Theta\}$, since the projection step in Eq (12) ensures that $\theta_t \in C_t(\delta)$. In this spirit, we first define:

$$\mu_i^{\text{ML}}(\mathbf{X}_t) := \inf_{\theta \in C_t(\delta)} \mu_i^{\text{ML}}(\mathbf{X}_t^\top, \theta).$$

and then lower bound $\mathbf{H}_t(\theta_t)$ (for $\theta_t = \theta_t^{\text{est}}$) as:

$$\mathbf{H}_t(\theta_t) = \sum_{s=1}^{t-1} \sum_{\mathbf{i} \in \mathbf{Q}_s} \mu_i^{\text{ML}}(\mathbf{X}_s^\top, \theta_s) x_{s,i}^\top x_{s,i} \mathbf{I}_d + \lambda \mathbf{I}_d \geq \sum_{s=1}^{t-1} \sum_{\mathbf{i} \in \mathbf{Q}_s} \inf_{\theta \in C_t(\delta)} \mu_i^{\text{ML}}(\mathbf{X}_s^\top, \theta) x_{s,i}^\top x_{s,i} \mathbf{I}_d + \lambda \mathbf{I}_d \geq \sum_{s=1}^{t-1} \sum_{\mathbf{i} \in \mathbf{Q}_s} \mu_i^{\text{ML}}(\mathbf{X}_s) x_{s,i}^\top x_{s,i} \mathbf{I}_d + \lambda \mathbf{I}_d.$$ 

(22)
Since $\hat{\mu}_i^{ML}(\cdot)$ is positive for all $i \in [K]$, we define a norm inducing positive semi-definite matrix as:

$$J_m := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \hat{\mu}_i^{ML}(X_{Q_s}) x_{s,i} x_{s,i}^T + \lambda I_d.$$  

It also follows from Eq (22) that:

$$\|x\|_{H_l^{-1}(\theta)} \leq \|x\|_{J_m^{-1}}. \quad (23)$$

$J_m$ is alternatively expressed as:

$$J_m := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \tilde{x}_{s,i} \tilde{x}_{s,i}^T + \lambda I_d, \quad (24)$$

where $\tilde{x}_{s,i} = \sqrt{\hat{\mu}_i^{ML}(X_{Q_s})} x_{s,i}$. This representation of $J_m$ is crucial for calculating the $T$ rounds summation of the prediction error. For a counterfactual scenario where the learner receives context vectors as $\tilde{x}_{s,i}$ instead of $x_{s,i}$, we utilize results for summation of self-normalized vectors from Abbasi-Yadkori et al. (2011).

We now apply first order Taylor expansion on the first term of Eq (21) and introduce the definition of $J_m$ to obtain the statement of Lemma 7.

**Lemma 7.**

$$\Delta^{\text{pred}}(X_{Q_t}, \theta_t) \leq (2 + 4S) \gamma_t(\delta) \sum_{i \in Q_t} ||\tilde{x}_{t,i}||_{J_m^{-1}}$$

$$+ \kappa (4 + 8S)^{3/2} M \gamma_t(\delta)^2 \sum_{i \in Q_t} ||x_{t,i}||^2_{V_m^{-1}}, \quad (25)$$

where $\tilde{x}_{t,i} = \sqrt{\hat{\mu}_i^{ML}(X_{Q_t})} x_{t,i}$ with $\hat{\mu}_i^{ML}(X_{Q_t}) = \inf_{\theta \in C_t(\delta)} \hat{\mu}_i^{ML}(X_{Q_t}^\top \theta)$.

The proof is provided in the technical appendix.

### 4.3 Regret Calculation

The final step is to calculate $T$ round summation of the prediction error as given in Eq (25). The following two lemmas give the upper bounds on the self-normalized vector summations.

**Lemma 8.**

$$\sum_{t=1}^T \min \left\{ \sum_{i \in Q_t} ||\tilde{x}_{t,i}||_{J_m^{-1}}^2, 1 \right\} \leq 2d \log \left( 1 + \frac{LKT}{d\lambda} \right).$$

**Lemma 9.**

$$\sum_{t=1}^T \min \left\{ \sum_{i \in Q_t} ||x_{t,i}||_{V_m^{-1}}^2, 1 \right\} \leq 2d \log \left( 1 + \frac{KT}{d\lambda} \right).$$

While the proofs of Lemmas 8 and 9 are in the technical appendix, here we describe how we use them. Since we assume that the maximum reward at any round is bounded by 1, there we write:

$$R_T \leq \sum_{t=1}^T \min \left\{ 2 \Delta^{\text{pred}}(X_{Q_t}, \theta_t), 1 \right\}. \quad (26)$$
Using Cauchy-Schwarz inequality we write:

\[
\sum_{t=1}^{T} \min_{\mathcal{Q}_t} \left\{ \sum_{i \in \mathcal{Q}_t} \|x_{t,i}\| J_{m_t-1}, 1 \right\} \leq \sqrt{T} \sum_{i \in \mathcal{Q}_t} \min_{\mathcal{Q}_t} \left\{ \sum_{i \in \mathcal{Q}_t} \|x_{t,i}\|^2 J_{m_t-1}, 1 \right\} .
\]  

(27)

Now we substitute the prediction error upper bound from Eq (25) in Eq (26) using Eq (27). Hence, we retrieve the upper bounds of Theorem 1.

5 DISCUSSION

In this work, we proposed an optimistic algorithm for learning under the MNL contextual bandit framework. Using techniques from Faury et al. (2020), we developed an improved technical analysis to deal with non-linear nature of the reward function. As a result, the leading term in our regret bound do not suffer from the problem dependent parameter \( \kappa \). This contribution is significant as \( \kappa \) can be very large. For example, for \( \kappa = O(\sqrt{T}) \), then results of Oh and Iyengar (2019a,b) suffers \( \tilde{O}(T) \) regret, while our algorithm continues to enjoy \( \tilde{O}(\sqrt{T}) \).

Our result is still \( O\left(\sqrt{d}\right) \) away from the mini-max lower bound (Chu et al., 2011) for linear contextual bandit. In case of logistic bandits Li et al. (2017) makes i.i.d. assumption on the contexts in order to bridge the gap (however with a \( \kappa \) factor). Working in a restrictive setting like theirs may help to improve our results. Oh and Iyengar (2019b) gave a Thompson sampling (TS) based learning strategy for MNL contextual bandit. Thompson sampling approaches may not have to search the entire action to take decision like optimistic algorithms (like ours) do. Results in Oh and Iyengar (2019b) however suffer from the prohibitive scaling of the problem dependent parameter \( \kappa \). Using our analysis in a TS based learning strategy could bring in together the best of both worlds.

References

Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems, pages 2312–2320, 2011.

Marc Abeille, Alessandro Lazaric, et al. Linear thompson sampling revisited. Electronic Journal of Statistics, 11(2):5165–5197, 2017.

Shipra Agrawal, Vashist Avadhanula, Vineet Goyal, and Assaf Zeevi. Thompson sampling for the mnl-bandit. In Conference on Learning Theory, pages 76–78, 2017.

Shipra Agrawal, Vashist Avadhanula, Vineet Goyal, and Assaf Zeevi. Mnl-bandit: A dynamic learning approach to assortment selection. Operations Research, 67(5):1453–1485, 2019.

Peter Auer. Using confidence bounds for exploitation-exploration trade-offs. Journal of Machine Learning Research, 3(Nov):397–422, 2002.

Francis Bach. Self-concordant analysis for logistic regression. Electronic Journal of Statistics, 4:384–414, 2010.

S. Bubeck and N. Cesa-Bianchi. Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems. Foundations and Trends in Machine Learning. Now Publishers, 2012. ISBN 9781601986269.
Felipe Caro and Jérémie Gallien. Dynamic assortment with demand learning for seasonal consumer goods. Management Science, 53(2):276–292, 2007.

Xi Chen, Yining Wang, and Yuan Zhou. Dynamic assortment optimization with changing contextual information. arXiv preprint arXiv:1810.13069, 2018.

Wei Chu, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandits with linear payoff functions. In Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics, pages 208–214, 2011.

Varsha Dani, Thomas P Hayes, and Sham M Kakade. Stochastic linear optimization under bandit feedback. In Conference on Learning Theory, 2008.

Louis Faury, Marc Abeille, Clément Calauzènes, and Olivier Fercoq. Improved optimistic algorithms for logistic bandits. arXiv preprint arXiv:2002.07530, 2020.

Sarah Filippi, Olivier Cappe, Aurélien Garivier, and Csaba Szepesvári. Parametric bandits: The generalized linear case. In Advances in Neural Information Processing Systems, pages 586–594, 2010.

Bolin Gao and Lacra Pavel. On the properties of the softmax function with application in game theory and reinforcement learning. arXiv preprint arXiv:1704.00805, 2017.

Botao Hao, Tor Lattimore, and Csaba Szepesvari. Adaptive exploration in linear contextual bandit. volume 108 of Proceedings of Machine Learning Research, pages 3536–3545. PMLR, 2020.

Tor Lattimore and Csaba Szepesvári. Bandit algorithms. Cambridge University Press, 2020.

Erich L Lehmann and George Casella. Theory of point estimation. Springer Science & Business Media, 2006.

Lihong Li, Yu Lu, and Dengyong Zhou. Provably optimal algorithms for generalized linear contextual bandits. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 2071–2080, 2017.

Julien Mairal. Stochastic majorization-minimization algorithms for large-scale optimization. In Advances in Neural Information Processing Systems, pages 2283–2291, 2013.

Min-hwan Oh and Garud Iyengar. Multinomial logit contextual bandits. 2019a.

Min-hwan Oh and Garud Iyengar. Thompson sampling for multinomial logit contextual bandits. In Advances in Neural Information Processing Systems, pages 3151–3161, 2019b.

Paat Rusmevichientong and John N Tsitsiklis. Linearly parameterized bandits. Mathematics of Operations Research, 35(2):395–411, 2010.

Paat Rusmevichientong, Zuo-Jun Max Shen, and David B Shmoys. Dynamic assortment optimization with a multinomial logit choice model and capacity constraint. Operations research, 58(6):1666–1680, 2010.

Denis Sauré and Assaf Zeevi. Optimal dynamic assortment planning with demand learning. Manufacturing & Service Operations Management, 15(3):387–404, 2013.

Yuchen Zhang and Xiao Lin. Disco: Distributed optimization for self-concordant empirical loss. In International conference on machine learning, pages 362–370, 2015.
A APPENDIX

A.1 Confidence Set

In this section, we justify the design of confidence set defined in Eq (8). This particular choice is based on the following concentration inequality for self-normalized vectorial martingales.

**Theorem 10.** Appears as Theorem 1 in Faury et al. (2020) Let \( \{F_t\}_{t=1}^{\infty} \) be a filtration. Let \( \{x_t\}_{t=1}^{\infty} \) be a stochastic process in \( \mathcal{B}_2(d) \) such that \( x_t \) is \( F_t \) measurable. Let \( \{\varepsilon_t\}_{t=1}^{\infty} \) be a martingale difference sequence such that \( \varepsilon_{t+1} \) is \( F_{t+1} \) measurable. Furthermore, assume that conditionally on \( F_t \) we have \( |\varepsilon_t| \leq 1 \) almost surely, and note \( \sigma_{t}^{2} := \mathbb{E}[\varepsilon_{t+1}^{2}|F_{t}] \). Let \( \lambda > 0 \) and for any \( t \geq 1 \) define:

\[
H_t := \sum_{s=1}^{t-1} \sigma_s^{2} x_s x_s^T + \lambda I_d, \quad S_t := \sum_{s=1}^{t-1} \varepsilon_{s+1} x_s.
\]

Then for any \( \delta \in (0,1] \):

\[
P\left( \exists t \geq 1, \|S_t\|_{H_t^{-1}} \geq \frac{\sqrt{\lambda}}{2} + \frac{2}{\sqrt{\lambda}} \log \left( \frac{\det (H_t)^{\frac{1}{2}} \lambda^{-\frac{1}{2}}}{\delta} \right) + \frac{2}{\sqrt{\lambda}} d \log (2) \right) \leq \delta.
\]

Using Theorem 10 we show that the following holds with high probability.

**Lemma 11** (confidence bounds for multinomial logistic rewards). With \( \hat{\theta}_t \) as the regularized maximum log-likelihood estimate as defined in Eq (8), the following follows with probability at least \( 1 - \delta \):

\[
\forall t \geq 1, \quad \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_s) \right\|_{H_t^{-1}} \leq \gamma_t(\delta),
\]

where \( H_t \) is defined in Eq (7a) and (7b).

**Proof.** \( \hat{\theta}_t \) is the maximizer of the regularized log-likelihood:

\[
\mathcal{L}_t^\lambda(\theta) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} r_{s,i} \log \left( \mu^ML(x_{Q_s}^T \theta) \right) - \frac{\lambda}{2} \|\theta\|_2^2,
\]

where \( \mu^ML(x_{Q_s}^T \theta) \) is given by Eq (1) as \( \frac{e^{x_{Q_s}^T \theta}}{1 + \sum_{j \in Q_s} e^{x_{Q_s}^T \theta}} \). Solving for \( \nabla \theta \mathcal{L}_t^\lambda = 0 \), we obtain:

\[
\sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^ML(x_{Q_s}^T \theta) x_{s,i} + \lambda \hat{\theta}_t = \sum_{s=1}^{t-1} \sum_{i \in Q_s} r_{s,i} x_{s,i}
\]

This result, combined with the definition of \( g_t(\theta^*_s) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^ML(x_{Q_s}^T \theta^*_s) x_{s,i} + \lambda \theta^*_s \) yields:

\[
g_t(\hat{\theta}_t) - g_t(\theta^*_s) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \varepsilon_{s,i} x_{s,i} - \lambda \theta^*_s
\]

where we denoted \( \varepsilon_{s,i} := r_{s,i} - \mu^ML(x_{Q_s}^T \theta^*_s) \) for all \( s \geq 1 \) and \( i \in [K] \) and \( S_{t,K} := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \varepsilon_{s,i} x_{s,i} \) for all \( t \geq 1 \). For any \( \lambda \geq 1 \), from the definition of \( H_t \) it follows that \( H_t^{-1}(\theta^*_s) \leq I_d \). Hence,
Also from Remark 2, we have:

\[ \|g_t(\hat{\theta}_t) - g_t(\theta_*)\|_{H^{-1}_m(\theta_*)} \leq \|S_t\|_{H^{-1}_m(\theta_*)} + \sqrt{t} \]

Later in the proof of Theorem 21 we present our choice of \( \lambda \) which always ensures \( \lambda \geq 1 \).

\[ \left\| g_t(\hat{\theta}_t) - g_t(\theta_*) \right\|_{H^{-1}_m(\theta_*)} \leq \|S_t\|_{H^{-1}_m(\theta_*)} + \sqrt{t} \lambda S \]  

(28)

Conditioned on the filtration set \( F_t,i \) (see Section 2.2 to review the definition of the filtration set), \( \varepsilon_{s,i} \) is a martingale difference is bounded by 1 as we assume the maximum reward that is accrued at any round is upper bounded by 1. To apply Theorem 10 we calculate for all \( s \geq 1 \):

\[ \mathbb{E} [\varepsilon^2_{s,i} | F_t] = \mathbb{E} \left[ r_{s,i} - \mu^{ML}_i (X_{Q,s}^\top \theta_*) \right]^2 | F_t \] = \( \mathbb{V} [r_{s,i} | F_t] = \mu^{ML}_i (X_{Q,s}^\top \theta_*) \left( 1 - \mu^{ML}_i (X_{Q,s}^\top \theta_*) \right) \).  

(29)

Also from Remark 2 we have:

\[ \mu^{ML}_i (X_{Q,s}^\top \theta_*) = \mu^{ML}_i (X_{Q,s}^\top \theta_*) \left( 1 - \mu^{ML}_i (X_{Q,s}^\top \theta_*) \right) \]

Therefore setting \( H_i(\theta_*) \) as \( H_m(\theta_*) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^{ML}_i (X_{Q,s}^\top \theta_*) x_{s,i} x_{s,i}^\top + \lambda I_d \) and \( S_t \) as \( S_t \), we invoke an instance of Theorem 10 to obtain:

\[ 1 - \delta \leq \mathbb{P} \left( \forall t \geq 1, \|S_t\|_{H^{-1}_m(\theta_*)} \leq \frac{\sqrt{\lambda}}{2} + \frac{2}{\sqrt{\lambda}} \log \left( \frac{\det(H_m(\theta_*))^{1/2} \lambda^{-d/2}}{\delta} \right) + \frac{2d}{\sqrt{\lambda}} \log(2) \right) \]  

(30)

We simplify \( \det(H_m(\theta_*)) \), using the fact that the multinomial logistic function is \( L \)-Lipschitz (see Assumption 2):

\[ \det(H_m(\theta_*)) = \det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^{ML}_i (X_{Q,s}^\top \theta_*) x_{s,i} x_{s,i}^\top + \lambda I_d \right) \leq L^d \det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} x_{s,i} x_{s,i}^\top + \frac{\lambda}{L} I_d \right) \]

Further, using Lemma 24 and using \( \|x_{s,i}\|_2 \leq 1 \) we write:

\[ L^d \det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} x_{s,i} x_{s,i}^\top + \frac{\lambda}{L} I_d \right) \leq \left( \lambda + \frac{LKt}{d} \right)^d \]

This we simplify Eq (30) as:

\[ 1 - \delta \leq \mathbb{P} \left( \forall t \geq 1, \|S_t\|_{H^{-1}_m(\theta_*)} \leq \frac{\sqrt{\lambda}}{2} + \frac{2}{\sqrt{\lambda}} \log \left( \frac{(\lambda + LKt/d)^{d/2} \lambda^{-d/2}}{\delta} \right) + \frac{2d}{\sqrt{\lambda}} \log(2) \right) \]

\[ \leq \mathbb{P} \left( \forall t \geq 1, \|S_t\|_{H^{-1}_m(\theta_*)} \leq \frac{\sqrt{\lambda}}{2} + \frac{2}{\sqrt{\lambda}} \log \left( \frac{(1 + \frac{LKt}{d})^{d/2}}{\delta} \right) + \frac{2d}{\sqrt{\lambda}} \log(2) \right) \]

\[ = \mathbb{P} \left( \forall t \geq 1, \|S_t\|_{H^{-1}_m(\theta_*)} \leq \gamma_\ell(\delta) - \sqrt{\lambda} S \right) \]  

(31)

Combining Eq (28) and Eq (31) yields:

\[ \mathbb{P} \left( \forall t \geq 1, \left\| g_t(\hat{\theta}_t) - g_t(\theta_*) \right\|_{H^{-1}_m(\theta_*)} \leq \gamma_\ell(\delta) \right) \geq \mathbb{P} \left( \forall t \geq 1, \|S_t\|_{H^{-1}_m(\theta_*)} + \sqrt{\lambda} S \leq \gamma_\ell(\delta) \right) \geq 1 - \delta \]

This completes the proof.
Lemma 11 implies that $\theta_*$ lies in the confidence set of Eq (8) with probability at least $1 - \delta$. Further from Eq (12), $\theta_t^{\text{est}}$ satisfies:

$$\theta_t^{\text{est}} = \arg\min_{\theta \in \Theta} \left\| g_t(\theta) - g_t(\hat{\theta}) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}(\theta)}.$$  

Thus using triangle inequality, we write:

$$\left\| g_t(\hat{\theta}_t) - g_t(\theta_t^{\text{est}}) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}} \leq \left\| g_t(\hat{\theta}_t) - g_t(\theta_*) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}} + \left\| g_t(\theta_*) - g_t(\theta_t^{\text{est}}) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}} \leq 2 \left\| g_t(\hat{\theta}_t) - g_t(\theta_*) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}}.$$  

This ensures:

$$C_t' (\delta) := \left\{ \theta \in \Theta, \left\| g_t(\theta_t^{\text{est}}) - g_t(\hat{\theta}_t) \right\|_{\mathbf{H}_{\mathbf{m}}^{-1}(\theta)} \leq 2 \gamma_t (\delta) \right\}.$$  

It is insightful to compare Theorem 10 with Theorem 1 of Abbasi-Yadkori et al. (2011). The later is re-stated below:

**Theorem 12.** Let $\{\mathcal{F}_t\}_{t=0}^\infty$ be a filtration. Let $\{\eta_t\}_{t=1}^\infty$ be a real-valued stochastic process such that $\eta_t$ is $\mathcal{F}_t$-measurable and $\eta_t$ is conditionally $R$-sub-Gaussian for some $R \geq 0$, i.e

$$\forall \lambda \in \mathbb{R}, \quad \mathbb{E} \left[ \exp (\lambda \eta_t) \mid \mathcal{F}_{t-1} \right] \leq \exp \left( \frac{\lambda^2 R^2}{2} \right).$$

Let $\{x_t\}_{t=1}^\infty$ be an $\mathbb{R}^d$-valued stochastic process such that $X_t$ is $\mathcal{F}_{t-1}$-measurable. Assume $\mathbf{V}$ is a $d \times d$ positive definite matrix. For any $t \geq 0$, define:

$$\mathbf{V}_t = \mathbf{V} + \sum_{s=1}^{t} x_s x_s^\top, \quad S_t = \sum_{s=1}^{t} \eta_s x_s.$$  

Then, for any $\delta > 0$, with probability at least $1 - \delta$, for all $t \geq 0$,

$$\left\| S_t \right\|_{\mathbf{V}_t^{-1}} \leq 2R \log \left( \frac{\det (\mathbf{V})^{-1/2} \det (\mathbf{V})^{-1/2}}{\delta} \right).$$

Theorem 12 makes an uniform sub-Gaussian assumption and unlike Theorem 10 does not take into account local variance information.

### A.2 Local Information Preserving Norm

Deviating from the previous analyses as in Filippi et al. (2010), Li et al. (2017), we describe norm which preserves the local information. The matrix $\mathbf{X}_{\mathcal{Q}_s}$ is the design matrix composed of the contexts $x_{s,1}, x_{s,2}, \ldots, x_{s,K}$ received at time step $s$ as its columns. The expected reward due to the $i\text{th}$ item in the assortment is given by:

$$\mu_{i}^{\text{ML}}(\mathbf{X}_{\mathcal{Q}_s}^\top, \theta) = \frac{e^{\mathbf{x}_{s,i}^{\top} \theta}}{1 + \sum_{j \in \mathcal{Q}_s} e^{\mathbf{x}_{s,j}^{\top} \theta}}.$$  

Further, we consider the following integral:

$$\int_{0}^{1} \mu_{i}^{\text{ML}} \left( \nu \mathbf{X}_{\mathcal{Q}_s}^\top, \mathbf{1} - \nu \right) \left( \mathbf{X}_{\mathcal{Q}_s}^\top, \theta_i \right) \cdot d\nu = \int_{x_{s,i}^{\top}, \theta_2}^{x_{s,i}^{\top}, \theta_1} \frac{1}{x_{s,i}(\theta_2 - \theta_1)} \mu_{i}^{\text{ML}}(t_i) \cdot dt_i, \quad (32)$$
where $\dot{\mu}_i^{ML}$ is the partial derivative of $\mu_i^{ML}$ in the direction of the $i$th component and $\int_{x_i, \theta_i} x_i \cdot t \cdot dt_i$ represents integration of $\dot{\mu}_i^{ML}(\cdot)$ with respect to the coordinate $t_i$ (hence the limits of the integration only consider change in the coordinate $t_i$). For notation purposes which would become clear later, we define:

$$\alpha_i(X_{Q_i}, \theta_1, \theta_2) x_{s,i}^T(\theta_2 - \theta_1) := \dot{\mu}_i^{ML}(X_{Q_i}^T \theta_2) - \mu_i^{ML}(X_{Q_i}^T \theta_1)$$

$$= \frac{e^{x_{s,i}^T \theta_2}}{1 + \sum_{j \in Q_i} e^{x_{s,j}^T \theta_2}} - \frac{e^{x_{s,i}^T \theta_1}}{1 + \sum_{j \in Q_i} e^{x_{s,j}^T \theta_1}} = \int_{x_{s,i}^T \theta_1} \dot{\mu}_i^{ML}(t_i) \cdot dt_i,$$

where the second step is due to Fundamental Theorem of Calculus. We have exploited the two ways to view the multinomial logit function: sum of individual probabilities and a vector valued function. We write:

$$\sum_{i \in Q_i} \alpha_i(X_{Q_i}, \theta_1, \theta_2) x_{s,i}^T(\theta_2 - \theta_1) = \sum_{i \in Q_i} \int_{0}^{1} \dot{\mu}_i^{ML} \left( \nu X_{Q_i}^T \theta_2 + (1 - \nu) X_{Q_i}^T \theta_1 \right) \cdot d\nu$$

We also have:

$$\mu_i^{ML}(X_{Q_i}^T \theta_1) - \mu_i^{ML}(X_{Q_i}^T \theta_2) = \sum_{i = 1}^{K} \alpha_i(X_{Q_i}, \theta_2, \theta_1) x_{s,i}^T(\theta_1 - \theta_2).$$

It follows that:

$$g(\theta_1) - g(\theta_2) = \sum_{s = 1}^{l-1} \sum_{i \in Q_s} \left( \frac{e^{x_{s,i}^T \theta_1}}{1 + \sum_{j \in Q_s} e^{x_{s,j}^T \theta_1}} - \frac{e^{x_{s,i}^T \theta_2}}{1 + \sum_{j \in Q_s} e^{x_{s,j}^T \theta_2}} \right) x_{s,i} + \lambda(\theta_1 - \theta_2)$$

$$= \sum_{s = 1}^{l-1} \sum_{i \in Q_s} \alpha_i(X_{Q_s}, \theta_2, \theta_1) x_{s,i} x_{s}^T(\theta_1 - \theta_2) + \lambda(\theta_1 - \theta_2)$$

$$= G_{\theta_1, \theta_2}(\theta_1 - \theta_2),$$

where $G_{\theta_1, \theta_2} := \sum_{s = 1}^{l-1} \sum_{i \in Q_s} \alpha_i(X_{Q_s}, \theta_1, \theta_2) x_{s,i} x_{s}^T + \lambda I_d$. Since $\alpha(X_{Q_s}, \theta_1, \theta_2) \geq \frac{1}{r}$ (from Assumption 2), therefore $G_{\theta_1, \theta_2} \geq O_{d \times d}$. Hence we get:

$$\|\theta_1 - \theta_2\|_{G_{\theta_1, \theta_2}^{-1}} = \|g(\theta_1) - g(\theta_2)\|_{G_{\theta_1, \theta_2}^{-1}}.$$

### A.3 Self-Concordance Style Relations for Multinomial Logistic Function

**Lemma 13.** For an assortment $Q_\theta$ and $\theta_1, \theta_2 \in \Theta$, the following holds:

$$\sum_{i \in Q_s} \alpha_i(X_{Q_s}, \theta_2, \theta_1) = \sum_{i \in Q_s} \int_{\nu = 0}^{1} \dot{\mu}_i^{ML} \left( \nu X_{Q_s}^T \theta_2 + (1 - \nu) X_{Q_s}^T \theta_1 \right) \cdot d\nu \geq \sum_{i \in Q_s} \dot{\mu}_i^{ML}(X_{Q_s}^T \theta_1) \left( 1 + |x_{s,i}^T \theta_1 - x_{s,i}^T \theta_2| \right)^{-1}$$

**Proof.** We write:

$$\int_{\nu = 0}^{1} \dot{\mu}_i^{ML} \left( \nu X_{Q_s}^T \theta_2 + (1 - \nu) X_{Q_s}^T \theta_1 \right) \cdot d\nu = \sum_{i \in Q_s} \int_{x_{s,i}^T \theta_1}^{x_{s,i}^T \theta_2} \frac{1}{x_{s,i}(\theta_2 - \theta_1)} \dot{\mu}_i^{ML}(t_i) \cdot dt_i,$$

where $\int_{x_{s,i}^T \theta_1}^{x_{s,i}^T \theta_2} \dot{\mu}_i^{ML}(t_i) \cdot dt_i$ represents integration of $\dot{\mu}_i^{ML}(\cdot)$ with respect to the coordinate $t_i$ (hence the limits of the integration only consider change in the coordinate $t_i$). For some $z > z_1 \in \mathbb{R}$, consider:

$$\int_{z_1}^{z} \frac{d}{dt_i} \log \left( \dot{\mu}_i^{ML}(t_i) \right) \cdot dt_i = \int_{z_1}^{z} \frac{\nabla^2 \mu_i^{ML}(t_i)}{\dot{\mu}_i^{ML}(t_i)} dt_i,$$
where \( \nabla^2 \mu_{i,i}^{ML}(\cdot) \) is the double derivative of \( \mu^{ML}(\cdot) \). Using Lemma 22 we have \( -1 \leq \frac{\nabla^2 \mu_{i,i}^{ML}(\cdot)}{\mu_{i,i}^{ML}(\cdot)} \leq 1 \). Thus we get:

\[
-(z - z_1) \leq \int_{z_1}^{z} \frac{d}{dt_i} \log \left( \mu_i^{ML}(t_i) \right) dt_i \leq (z - z_1)
\]

Using Fundamental Theorem of Calculus, we get:

\[
-(z - z_1) \leq \log \left( \frac{\mu_i^{ML}(z)}{\mu_i^{ML}(z_1)} \right) \leq (z - z_1)
\]

\[
\therefore \mu_i^{ML}(z_1) \exp(-(z - z_1)) \leq \mu_i^{ML}(z) \leq \mu_i^{ML}(z_1) \exp(z - z_1)
\]

(38)

Using Eq (38) and for \( z_2 \geq z_1 \in \mathbb{R} \), and for all \( i \in [K] \) and we have:

\[
\mu_i^{ML}(z_1) \frac{1 - \exp(-(z_1 - z_2))}{z_2 - z_1} \leq \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \mu_i^{ML}(t_i) dt_i \leq \mu_i^{ML}(z_1) \frac{\exp(z_1 - z_2) - 1}{z_2 - z_1}.
\]

(39)

Reversing the role of \( z_1 \) and \( z_2 \), such that \( z_2 \leq z_1 \) then again by using Eq (38) we write:

\[
\mu_i^{ML}(z_1) \frac{\exp(-(z_1 - z_2)) - 1}{z_2 - z_1} \leq \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \mu_i^{ML}(t_i) dt_i \leq \mu_i^{ML}(z_1) \frac{\exp(z_1 - z_2) - 1}{z_2 - z_1}.
\]

(40)

Combining Eq (39) and (40) and for all \( i \in [K] \) we get:

\[
\mu_i^{ML}(z_1) \frac{1 - \exp(-(z_1 - z_2))}{|z_1 - z_2|} \leq \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \mu_i^{ML}(t_i) dt_i.
\]

(41)

If \( x \geq 0 \), then \( e^{-x} \leq (1 + x)^{-1} \), and therefore \( (1 - e^{-x})/x \geq (1 + x)^{-1} \). Thus we lower bound the left hand side of Eq (41) as:

\[
\mu_i^{ML}(z_1) \frac{(1 + |z_1 - z_2|)^{-1}}{|z_1 - z_2|} \leq \mu_i^{ML}(z_1) \frac{1 - \exp(-(z_1 - z_2))}{|z_1 - z_2|} \leq \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} \mu_i^{ML}(t_i) dt_i.
\]

Using above with \( z_2 = x_{s,i}^{\top} \theta_2 \) and \( z_1 = x_{s,i}^{\top} \theta_1 \) in Eq (37) gives:

\[
\sum_{i \in Q_x} \int_{\nu=0}^{1} \mu_i^{ML} \left( \nu X_{Q_x}^{\top} \theta_2 + (1 - \nu) X_{Q_x}^{\top} \theta_1 \right) \cdot d\nu = \sum_{i \in Q_x} \int_{x_{s,i}^{\top} \theta_1}^{x_{s,i}^{\top} \theta_2} \frac{1}{x_{s,i}^{\top} (\theta_2 - \theta_1)} \mu_i^{ML}(t_i) \cdot dt_i \geq \sum_{i \in Q_x} \mu_i^{ML} \left( X_{Q_x}^{\top} \theta_1 \right) \left( 1 + |x_{s,i}^{\top} \theta_1 - x_{s,i}^{\top} \theta_2 | \right)^{-1}
\]

\[
\]

Lemma 14. For all \( \theta_1, \theta_2 \in \Theta \) such that \( S := \max_{\theta \in \Theta} \| \theta \|_2 \) (Assumption [4]), the following inequalities hold:

\[
\text{Gm}_i(\theta_1, \theta_2) \succeq (1 + 2S)^{-1} \text{Hm}_i(\theta_1)
\]

\[
\text{Gm}_i(\theta_1, \theta_2) \succeq (1 + 2S)^{-1} \text{Hm}_i(\theta_2)
\]

18
Proof. From Lemma 13, we have:

\[
\sum_{i \in \mathcal{Q}_s} \alpha_i \left( |x_{s,i}^T \theta_1 - x_{s,i}^T \theta_2| \right)^{-1} \mu_i^{ML}(X_{Q_s}^T \theta_1) \\
\geq \sum_{i \in \mathcal{Q}_s} \left( 1 + \|x_{s,i}\|^2 \right)^{-1} \mu_i^{ML}(X_{Q_s}^T \theta_1) \\
\geq (1 + 2S)^{-1} \mu_i^{ML}(X_{Q_s}^T \theta_1)
\]

\[(\theta_1, \theta_2 \in \Theta, \|x_{s,i}\|_2 \leq 1)\]

Now we write \(\text{Gm}_t(\theta_1, \theta_2)\) as:

\[
\text{Gm}_t(\theta_1, \theta_2) = \sum_{s=1}^{t-1} \sum_{i \in \mathcal{Q}_s} \alpha_i (X_{Q_s}, \theta_2, \theta_1) x_{s,i} x_{s,i}^T + \lambda I_d
\]

\[
\geq (1 + 2S)^{-1} \sum_{s=1}^{t-1} \sum_{i \in \mathcal{Q}_s} \mu_i^{ML}(X_{Q_s}^T \theta_1) x_{s,i} x_{s,i}^T + \lambda I_d
\]

\[
= (1 + 2S)^{-1} \left( \sum_{s=1}^{t-1} \sum_{i \in \mathcal{Q}_s} \mu_i^{ML}(X_{Q_s}^T \theta_1) x_{s,i} x_{s,i}^T + (1 + 2S)\lambda I_d \right)
\]

\[
\geq (1 + 2S)^{-1} \left( \sum_{s=1}^{t-1} \sum_{i \in \mathcal{Q}_s} \mu_i^{ML}(X_{Q_s}^T \theta_1) x_{s,i} x_{s,i}^T + \lambda I_d \right)
\]

\[
= (1 + 2S)^{-1} \text{Hm}_t(\theta_1).
\]

Since, \(\theta_1\) and \(\theta_2\) have symmetric roles in the definition of \(\alpha_i (X_{Q_s}, \theta_2, \theta_1)\), we also obtain the second relation by a change of variable directly. □

Lemma 14 combined with the projection step of Eq (12) gives following result

Lemma 15. For \(\theta = \theta_{t}^{\text{ext}}\) as calculated by Eq (12), we have the following relation with probability at least \(1 - \delta\):

\[
\|g(\theta) - g(\theta_{t}^{\text{ext}})\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})} \leq 2\sqrt{1 + 2S}\gamma_t(\delta).
\]

Proof. From triangle inequality, we write :

\[
\|g(\theta_{t}^{\text{ext}}) - g(\theta)\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})} \leq \|g(\theta_{t}^{\text{ext}}) - g(\hat{\theta}_t)\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})} + \|g(\hat{\theta}_t) - g(\theta)\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})},
\]

where \(\hat{\theta}_t\) is the MLE estimate. Further Lemma 14 gives:

\[
\|g(\theta_{t}^{\text{ext}}) - g(\theta)\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})} \leq \sqrt{1 + 2S}\|g(\theta_{t}^{\text{ext}}) - g(\hat{\theta}_t)\|_{\text{Hm}_t^{-1}(\theta)} + \sqrt{1 + 2S}\|g(\hat{\theta}_t) - g(\theta)\|_{\text{Hm}_t^{-1}(\theta)}.
\]

since \(\theta\) is the minimizer of \(\|g(\theta) - g(\hat{\theta}_t)\|_{\text{Hm}_t^{-1}(\theta)}\), therefore we write:

\[
\|g(\theta_{t}^{\text{ext}}) - g(\theta)\|_{\text{Gm}_t^{-1}(\theta, \theta_{t}^{\text{ext}})} \leq 2\sqrt{1 + 2S}\|g(\theta_{t}^{\text{ext}}) - g(\hat{\theta}_t)\|_{\text{Hm}_t^{-1}(\theta)}.
\]

Finally, the proof follows from Lemma 11 as:

\[
\|g(\theta_{t}^{\text{ext}}) - g(\theta)\|_{\text{Hm}_t^{-1}(\theta)} \leq \gamma_t(\delta).
\]

\[
19
\]
A.4 Bounds on prediction error

Lemma 16. For an assortment \( Q_s \) and \( \theta = \theta^{\text{est}}_t \) as given by Eq (12), the following holds with probability at least \( 1 - \delta \):

\[
\alpha_i(\mathbf{X}_{Q_s}, \theta, \theta) \leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) + 2\sqrt{1 + 2SM\gamma_t(\delta)} \|x_{s,i}\|_{\mathbf{G}_{\gamma_t^{-1}(\theta, \theta)}}.
\]

Proof. Consider the multinomial logit function:

\[
\alpha_i(\mathbf{X}_{Q_s}, \theta, \theta)x_{s,i}^\top(\theta_s - \theta) = \frac{e_{x_{s,i}}^\top \theta}{1 + \sum_{j \in Q_s} e_{x_{s,j}}^\top \theta} - \frac{e_{x_{s,i}}^\top \theta}{1 + \sum_{j \in Q_s} e_{x_{s,j}}^\top \theta}.
\]  (42)

We use second-order Taylor expansion for each component of the multinomial logit function at \( a_i \). Consider for all \( i \in [K] \):

\[
f_i(r_i) = \frac{e_{r_i}}{1 + e_{r_i}} \sum_{j \in Q_s, j \neq i} e_{r_j}
\leq f(a_i) + f'(a_i)(r_i - a_i) + \frac{f''(a_i)(r_i - a_i)^2}{2}.
\]  (43)

In Eq (43), we substitute: \( f(\cdot) \rightarrow \mu_i^{\text{ML}}, r_i \rightarrow x_{s,i}^\top \theta, \) and \( a_i \rightarrow x_{s,i}^\top \theta \). Thus we re-write Eq (42) as:

\[
\alpha_i(\mathbf{X}_{Q_s}, \theta, \theta)x_{s,i}^\top(\theta_s - \theta) \leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) (x_{s,i}^\top(\theta_s - \theta)) + \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) (x_{s,i}^\top(\theta_s - \theta))^2,
\]

\[
\therefore \quad \alpha_i(\mathbf{X}_{Q_s}, \theta, \theta) \leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) (x_{s,i}^\top(\theta_s - \theta)) + \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) |x_{s,i}^\top(\theta_s - \theta)|
\leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) + M|x_{s,i}^\top(\theta_s - \theta)|,
\]

where we upper bound \( \hat{\mu}^{\text{ML}}_i \) by \( M \). An application of Cauchy-Schwarz gives us:

\[
|x_{s,i}^\top(\theta_s - \theta)| \leq \|x_{s,i}\|_{\mathbf{G}_{\gamma_t^{-1}(\theta, \theta)}} \|\theta_s - \theta\|_{\mathbf{G}_{\gamma_t}(\theta, \theta)}
= \|x_{s,i}\|_{\mathbf{G}_{\gamma_t^{-1}(\theta, \theta)}} \|g_s(\theta_s) - g_s(\theta)\|_{\mathbf{G}_{\gamma_t}(\theta, \theta)} \quad \text{(From Eq (36))}
\]  (44)

Upon Combining the last two equations we get:

\[
\alpha_i(\mathbf{X}_{Q_s}, \theta, \theta) \leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) + \|x_{s,i}\|_{\mathbf{G}_{\gamma_t^{-1}(\theta, \theta)}} \|g_s(\theta_s) - g_s(\theta)\|_{\mathbf{G}_{\gamma_t}(\theta, \theta)}.
\]

From Lemma 15 we get:

\[
\alpha_i(\mathbf{X}_{Q_s}, \theta, \theta) \leq \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_s}^\top \theta \right) + 2\sqrt{1 + 2SM\gamma_t(\delta)} \|x_{s,i}\|_{\mathbf{G}_{\gamma_t^{-1}(\theta, \theta)}}.
\]

Lemma 17. For an assortment \( Q_t \) and \( \theta = \theta^{\text{est}}_t \) as given by Eq (12), the following holds with probability at least \( 1 - \delta \):

\[
\Delta^{\text{pred}}(\mathbf{X}_{Q_t}, \theta) \leq (2 + 4S)\gamma_t(\delta) \sum_{i \in Q_t} \hat{\mu}^{\text{ML}}_i \left( \mathbf{X}_{Q_t}^\top \theta \right) \|x_{t,i}\|_{\mathbf{H}_{\gamma_t^{-1}(\theta)}} + \kappa(4 + 8S)M\gamma_t(\delta) \sum_{i \in Q_t} \|x_{t,i}\|_{\mathbf{V}_{\gamma_t^{-1}}}^2
\]
Proof.
\[ \Delta_{\text{pred}}(X_{Q_t}, \theta) = \left| \mu_{\text{ML}}(X_{Q_t}, \theta^*) - \mu_{\text{ML}}(X_{Q_t}, \theta) \right| \]
\[ = \left| \sum_{i \in Q_t} \alpha_i(X_{Q_t}, \theta, \theta^*) x_{t,i}(\theta_1 - \theta_2) \right| \]
\[ \leq \left| \sum_{i \in Q_t} \alpha_i(X_{Q_t}, \theta, \theta^*) \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} \|g_i(\theta^*) - g_i(\theta)\|_{\Gamma_{\mu_i}(\theta, \theta^*)} \right| \]
(From Eq \((35)\))
\[ \leq 2\sqrt{1 + 2S_{\gamma}(\delta)} \sum_{i \in Q_t} \alpha_i(X_{Q_t}, \theta, \theta^*) \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} \]
(Cauchy-Schwarz inequality and Eq \((36)\))
\[ \leq 2\sqrt{1 + 2S_{\gamma}(\delta)} \sum_{i \in Q_t} \left( \mu_i^{\text{ML}}(X_{Q_t}, \theta) \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} + 2\sqrt{1 + 2SM_{\gamma}(\delta)} \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}^2 \right) \]
(From Lemma \((16)\))
\[ \] (45)

Upon re-arranging the terms we get:
\[ \Delta_{\text{pred}}(X_{Q_t}, \theta) \leq 2\sqrt{1 + 2S_{\gamma}(\delta)} \sum_{i \in Q_t} \mu_i^{\text{ML}}(X_{Q_t}, \theta) \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} + (4 + 8S)M_{\gamma}(\delta)^2 \sum_{i \in Q_t} \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}^2 \]
\[ \leq (2 + 4S)_{\gamma}(\delta) \sum_{i \in Q_t} \mu_i^{\text{ML}}(X_{Q_t}, \theta) \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} + \kappa(4 + 8S)M_{\gamma}(\delta)^2 \sum_{i \in Q_t} \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}^2 , \]
where we use \(\Gamma_{\mu_i^{-1}(\theta, \theta^*)} \geq \kappa^{-1}\Gamma_{\mu_i} \) from Assumption \(2\).

Lemma 18. For an assortment \(Q_t\) and \(\theta = \theta_t^{\text{est}}\) as given by Eq \((12)\), the following holds with probability at least \(1 - \delta\):
\[ \Delta_{\text{pred}}(X_{Q_t}, \theta) \leq (2 + 4S)_{\gamma}(\delta) \sum_{i \in Q_t} \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}} + \kappa(4 + 8S)_{\gamma}(\delta)^2 \sum_{i \in Q_t} \|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}^2 , \]
where \(\hat{x}_{t,i} = \sqrt{\mu_i^{\text{ML}}(X_{Q_t})} x_{t,i}\) with \(\mu_i^{\text{ML}}(X_{Q_t}) := \inf_{\theta \in C_{t}(\delta)} \mu_i^{\text{ML}}(X_{Q_t}, \theta)\).

Proof. We first study the quantity \(\|x_{t,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}\). In \(\Gamma_{\mu_i}(\theta) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu_i^{\text{ML}}(X_{Q_s}, \theta) x_{s,i}x_{s,i}^\top + \lambda I_d\), without any additional constraints on \(\theta \in \Theta\), \(\mu_i^{\text{ML}}(X_{Q_s}, \theta)\) could be very small, potentially inflating \(\sum_{i \in Q_s} \|x_{s,i}\|_{\Gamma_{\mu_i^{-1}(\theta, \theta^*)}}\). We observe that we only need to account for \(\theta \in \{C_t(\delta) \cap \Theta\}\), since the projection step in Eq \((12)\) ensures that \(\theta_t^{\text{est}} \in C_t(\delta)\). In this spirit, we first define:
\[ \hat{\mu}_i^{\text{ML}}(X_{Q_s}) : = \inf_{\theta \in C_t(\delta)} \mu_i^{\text{ML}}(X_{Q_s}, \theta), \]
and then lower bound \(\Gamma_{\mu_i}(\theta)\) as:
\[ \Gamma_{\mu_i}(\theta) = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \hat{\mu}_i^{\text{ML}}(X_{Q_s}, \theta) x_{s,i}x_{s,i}^\top + \lambda I_d \]
\[ \geq \sum_{s=1}^{t-1} \sum_{i \in Q_s} \inf_{\theta \in C_t(\delta)} \mu_i^{\text{ML}}(X_{Q_s}, \theta) x_{s,i}x_{s,i}^\top + \lambda I_d \]
\[ = \sum_{s=1}^{t-1} \sum_{i \in Q_s} \hat{\mu}_i^{\text{ML}}(X_{Q_s}) x_{s,i}x_{s,i}^\top + \lambda I_d \]
(46)
Since $\mu_{i,ML}^-(\cdot)$ is positive for all $i \in [K]$, there we define a norm inducing positive semi-definite matrix as:

$$J_m := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu_{i,ML}^-(X_{Q_s}^\top x_{s,i}x_{s,i}^\top + \lambda I_d).$$

It also follows from Eq (46) that:

$$\|x\|_{H_m^{-1}(\theta)} \leq \|x\|_{J_m^{-1}}. \tag{47}$$

We define $\theta_{t,i} := \arg\min_{\theta \in C_t(\delta)} \mu_{i,ML}^-(X_{Q_i}^\top \theta^0)$. Now, consider the first order-Taylor expansion of $\mu_{i,ML}^-(X_{Q_i}^\top \theta_t)$ for all $i \in [K]$:

$$\mu_{i,ML}^-(X_{Q_i}^\top \theta_t) - \mu_{i,ML}^-(X_{Q_i}^\top \theta^0) \leq \mu_{i,ML}^-(X_{Q_i}^\top \theta_t) + M \|x_{t,i}\|_{\text{Gm}_{t,i}^{-1}(\theta_t, \theta^0)} \|x_{t,i}\|_{\text{Gm}_{t,i}^{-1}(\theta_t, \theta^0)} \tag{48}$$

Using Eq (46) we simplify, the first term of the prediction error upper bound (see Lemma 17) as:

$$\mu_{i,ML}^-(X_{Q_i}^\top \theta_t) \|x_{t,i}\|_{\text{Hm}_{t,i}^{-1}(\theta)} \leq \mu_{i,ML}^-(X_{Q_i}^\top \theta_t) \|x_{t,i}\|_{\text{Hm}_{t,i}^{-1}(\theta)} + 2\sqrt{\lambda} M \sqrt{1 + 2\gamma_t(\delta)} \|x_{t,i}\|_{\text{V}_{\gamma_t}^{-1}} \|x_{t,i}\|_{\text{Hm}_{t,i}^{-1}(\theta)} \tag{49}$$

where $\tilde{x}_{t,i} = \sqrt{\mu_{i,ML}^-(X_{Q_i}^\top x_{t,i})} x_{t,i}$.

Using Lemma 17 we get:

$$\Delta_{\text{pred}}(X_{Q_i}, \theta) \leq (2 + 4S) \gamma_t(\delta) \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|_{\text{Jm}_{t,i}^{-1}(\theta)} + \kappa(4 + 8S) \gamma_t(\delta)^2 \sum_{i \in Q_t} \|x_{t,i}\|_{\text{V}_{\gamma_t}^{-1}}^2$$

A.5 Regret calculation

The following two lemmas give the upper bounds on the self-normalized vector summations.

Lemma 19.

$$\sum_{t=1}^T \min_{i \in Q_t} \left\{ \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|_{\text{Jm}_{t,i}^{-1}(\theta)}^2, 1 \right\} \leq 2d \log \left( 1 + \frac{LKT}{d\lambda} \right).$$
Proof. The proof follows by a direct application of Lemma 23 and 24 as:

\[
\sum_{t=1}^{T} \min \left\{ \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2, 1 \right\} \leq 2 \log \left( \frac{\det(J_{1}^{\top}+1)}{\lambda^d} \right) \quad \text{(From Lemma 23)}
\]

\[
= 2 \log \left( \frac{\det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu_i^{ML} (X_{Q_s}) \tilde{x}_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right)
\]

\[
\leq 2 \log \left( \frac{\det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} L \tilde{x}_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right) \quad \text{(Upper bound by Lipschitz constant)}
\]

\[
\leq 2 \log \left( \frac{L^d \det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} \tilde{x}_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right)
\]

\[
\leq 2 \log \left( \frac{\det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} L \tilde{x}_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right)
\]

\[
\leq 2d \log \left( 1 + \frac{KT}{d\lambda} \right). \quad \text{(From Lemma 24)}
\]

Similar to Lemma 19, we prove the following.

Lemma 20.

\[
\sum_{t=1}^{T} \min \left\{ \sum_{i \in Q_t} \|x_{t,i}\|^2, 1 \right\} \leq 2d \log \left( 1 + \frac{KT}{d\lambda} \right).
\]

Proof:

\[
\sum_{t=1}^{T} \min \left\{ \sum_{i \in Q_t} \|x_{t,i}\|^2, 1 \right\} \leq 2 \log \left( \frac{\det(V_{1}^{\top}+1)}{\lambda^d} \right) \quad \text{(From Lemma 23 set } \mu_i^{ML}(\cdot) = 1)\]

\[
= 2 \log \left( \frac{\det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} x_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right)
\]

\[
\leq 2 \log \left( \frac{\det \left( \sum_{s=1}^{t-1} \sum_{i \in Q_s} L x_{t,i}^{T} + \lambda I_d \right)}{\lambda^d} \right)
\]

\[
\leq 2d \log \left( 1 + \frac{KT}{d\lambda} \right). \quad \text{(From Lemma 24)}
\]

\[
\Box
\]

Theorem 21. With probability at least \( 1 - \delta \):

\[
R_T \leq C_1 \gamma_T(\delta) \sqrt{2d \log \left( 1 + \frac{KT}{d\lambda} \right) T} + C_2 \kappa \gamma_T(\delta)^2 d \log \left( 1 + \frac{KT}{d\lambda} \right),
\]

where the constants are given as \( C_1 = (4 + 8S) \), \( C_2 = 4(4 + 8S)^{\gamma} M \) and \( \gamma_T(\delta) \) is given by Eq (10).
In the algorithm CB-UCB(MNL) we fix the exploration bonus as $2\Delta_{\text{pred}}(X^T_{Q_t}, \theta_t^\ast)$. For the interaction at round $t$. Lemma[3] suggests that the cumulative regret is given by:

$$R_T \leq \sum_{t=1}^{T} \min \left\{ 2\Delta_{\text{pred}} \left( X_{Q_t}, \theta_t^\ast \right), 1 \right\}$$

$$\leq \sum_{t=1}^{T} 2 \min \left\{ (2 + 4S) \gamma_t(\delta) \sum_{i \in Q_t} \| x_{t,i} \| J_{m_{t-1}}^{-1} + \kappa(4 + 8S)^{3/2} M \gamma_t(\delta)^2 \sum_{i \in Q_t} \| x_{t,i} \| V_{m_{t-1}}^{-1}, 1 \right\}$$

(From Lemma[18]

$$\leq (4 + 8S) \gamma_t(\delta) \sqrt{T} \sum_{t=1}^{T} \min \left\{ \sum_{i \in Q_t} \| x_{t,i} \| J_{m_{t-1}}^{-1}, 1 \right\} + 2(4 + 8S)^{3/2} \kappa M \gamma_t(\delta)^2 \sum_{t=1}^{T} \min \left\{ \sum_{i \in Q_t} \| x_{t,i} \| V_{m_{t-1}}^{-1}, 1 \right\}$$

(Using Cauchy-Schwarz inequality)

$$\leq (4 + 8S) \gamma_t(\delta) \sqrt{2d \log \left( 1 + \frac{LKT}{d\lambda} \right)} T + 4(4 + 8S)^{3/2} \kappa M \gamma_t(\delta)^2 d \log \left( 1 + \frac{KT}{d\lambda} \right).$$

(From Lemma[19] and [20]

For a choice of $\lambda = d \log(KT) \gamma_t(\delta) = O \left( d^{1/2} \log^{1/2}(KT) \right)$.

### A.6 Technical Lemmas

**Remark 2** (Derivatives for MNL choice function). *For the multinomial logit choice function, where the expected reward due to item $i$ of the assortment $S_t$ is modeled as:

$$f_i(S_t, r) = \frac{e^{r_i}}{1 + e^{r_i} + \sum_{j \in S_t, j \neq i} e^{r_j}}$$

the partial derivative with respect to the expected reward of $i$th item is given as:

$$\frac{\partial f_i}{\partial r_i} = f_i(S_t, r) \left( 1 - f_i(S_t, r) \right)$$

and the double derivative as:

$$\frac{\partial^2 f_i}{\partial r_i^2} = f_i(S_t, r) \left( 1 - f_i(S_t, r) \right) \left( 1 - 2f_i(S_t, r) \right).$$

**Lemma 22** (Self-Concordance like relation for MNL). *For the multinomial logit choice function, where the expected reward due to item $i$ of the assortment $S_t$ is modeled as:

$$f_i(S_t, r) = \frac{e^{r_i}}{1 + e^{r_i} + \sum_{j \in S_t, j \neq i} e^{r_j}}$$

the following relation holds:

$$\left| \frac{\partial^2 f_i}{\partial r_i^2} \right| \leq \frac{\partial f_i}{\partial r_i}$$
Proof. The proof directly follows from Remark \ref{Remark1} and the observation $|1 - 2f_i(S_t, r)| \leq 1$ for all items, \(i\) in the assortment choice.

\begin{lemma}[Generalized elliptical potential] Let \(\{X_Q\} \) be a sequence in \(\mathbb{R}^{d \times K}\) such that for each \(s\), \(X_{Q_s}\) has columns as \(\{x_{s,1}, x_{s,2}, \ldots, x_{s,K}\}\) where \(\|x_{s,i}\|_2 \leq w_i \in \mathbb{R}^d\) for all \(s \geq 1\) and \(i \in [K]\). Also, let \(\lambda\) be a non-negative scalar. For \(t \geq 1\), define \(J_{M_t} := \sum_{s=1}^{t-1} \sum_{i \in Q_s} \mu^{ML}_{d} (X_{Q_s}) x_{s,i}x_{s,i}^T + \lambda I_d\) where \(\mu^{ML}(X_{Q_s})\) is strictly positive for all \(i \in [K]\). Then the following inequality holds:

\[
\sum_{t=1}^{T} \min_{i \in Q_t} \left\{ \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2_{J_{M_t}^{-1}}, 1 \right\} \leq 2 \log \left( \frac{\det(J_{M_{T+1}})}{\lambda^d} \right)
\]

with \(\tilde{x}_{t,i} = \sqrt{\mu^{ML}(X_{Q_s})} x_{s,i}\).

Proof. By the definition of \(J_{M_t}\):

\[
\det(J_{M_{t+1}}) = \det(J_{M_t} + \sum_{i \in Q_t} \tilde{x}_{t,i}\tilde{x}_{t,i}^T) = \det(J_{M_t}) \det(I_d + J_{M_t}^{-1/2} \sum_{i \in Q_t} \tilde{x}_{t,i}\tilde{x}_{t,i}^T J_{M_t}^{-1/2}) = \det(J_{M_t}) \left( 1 + \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2_{J_{M_t}^{-1}} \right).
\]

Taking log from both sides and summing from \(t = 1\) to \(T\):

\[
\sum_{t=1}^{T} \log \left( 1 + \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2_{J_{M_t}^{-1}} \right) = \sum_{t=1}^{T} \log \left( \frac{\det(J_{M_{t+1}})}{\det(J_{M_t})} \right)
\]

\[
= \sum_{t=1}^{T} \log \left( \frac{\det(J_{M_{t+1}})}{\det(J_{M_t})} \right) 
= \log \left( \frac{\det(J_{M_{T+1}})}{\det(I_d)} \right) 
= \log \left( \frac{\det(J_{M_{T+1}})}{\lambda^d} \right). \quad \text{(By a telescopic sum cancellation)}
\]

(51)

For any \(a\) such that \(0 \leq a \leq 1\), it follows that \(a \leq 2 \log(1 + a)\). Therefore, we write:

\[
\sum_{t=1}^{T} \min_{i \in Q_t} \left\{ \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2_{J_{M_t}^{-1}}, 1 \right\} \leq 2 \sum_{t=1}^{T} \log \left( 1 + \sum_{i \in Q_t} \|\tilde{x}_{t,i}\|^2_{J_{M_t}^{-1}} \right)
\]

\[
= 2 \log \left( \frac{\det(J_{M_{T+1}})}{\lambda^d} \right). \quad \text{(From Eq. (51))}
\]

\begin{lemma}[Determinant-Trace inequality, see Lemma 10 in \cite{Abbasi-Yadkori11}], Let \(\{x_s\}_{s=1}^\infty\) a sequence in \(\mathbb{R}^d\) such that \(\|x_s\|_2 \leq X\) for all \(s \in \mathbb{N}\), and let \(\lambda\) be a non-negative scalar. For \(t \geq 1\) define \(V_t := \sum_{s=1}^{t-1} x_s x_s^T + \lambda I_d\). The following inequality holds:

\[
\det(V_{t+1}) \leq \left( \lambda + tX^2/d \right)^d.
\]

\end{lemma}