Abstract

In this paper, we study elimination of imaginaries in henselian valued fields of equicharacteristic zero and residue field algebraically closed. The results are sensitive to the complexity of the value group. We focus first on the case where the ordered abelian group has finite spines, and then prove a better result for the dp-minimal case. In previous work the author proved that an ordered abelian with finite spines weakly eliminates imaginaries once one adds sorts for the quotient groups $\Gamma/\Delta$ for each definable convex subgroup $\Delta$, and sorts for the quotient groups $\Gamma/(\Delta + l\Gamma)$ where $\Delta$ is a definable convex subgroup and $l \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the quotient sorts. Jahnke, Simon, and Walsberg (J. Symb. Log. 82 (2017) 151–165) characterized $dp$-minimal ordered abelian groups as those without singular primes, that is, for every prime number $p$ one has $[\Gamma : p\Gamma] < \infty$.

We prove the following two theorems:

**Theorem.** Let $K$ be a henselian valued field of equicharacteristic zero with residue field algebraically closed and value group of finite spines. Then $K$ admits weak elimination of imaginaries once one adds codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, and the quotient sorts for the value group.

**Theorem.** Let $K$ be a henselian valued field of equicharacteristic zero, with residue field algebraically closed and whose value group is $dp$-minimal. Then $K$ eliminates imaginaries once one adds codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, the quotient sorts for
the value group and constants to distinguish the elements of each of the finite groups $\Gamma/\ell\Gamma$, where $\ell \in \mathbb{N}_{>0}$.

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INTRODUCTION

Valued fields are among the fundamental structures in number theory, algebraic geometry, asymptotic analysis and nonstandard analysis. Their model theory has been a major topic of study during the last century, it was initiated in [29] by Robinson’s model completeness result of ACVF, the theory of algebraically closed non-trivially valued fields. Many other results were obtained after his work, however precisely because of the complexity of valued fields, much of the work centers on quantifier elimination and basic properties of formulae. Quantifier elimination results give a classification and complete description of definable sets, and they have been used as a central tool in many applications of model theory.

In [27], Poizat introduced the idea of classifying the larger class of interpretable sets, which are quotients of definable sets by definable equivalence relations, such as quotient groups in algebraically closed fields.

For instance, let $M$ be a structure. Given $D \subseteq \mathcal{M}^k$ be some definable set and $E$ some definable equivalence relation over $D$. For any $a \in D$, the equivalence class $e = a/E$ is said to be an imaginary element and $D/E$ is an interpretable set in $M$. Instead we refer to tuples of elements in $D$ to be real elements in $M$.

Shelah formalized the well-known so-called eq construction that makes every interpretable set definable. However, Shelah’s construction escapes the classification of interpretable sets, and in the eq construction such sets are far to be well-understood. Poizat’s main goal was formalizing the idea of giving a complete classification of the interpretable sets, by introducing the concept of elimination of imaginaries.

A structure $M$ eliminates imaginaries if given $X$ an $\emptyset$-definable set and $E$ an $\emptyset$-definable equivalence relation on $X$ the class $e = a/E$ is interdefinable with a real tuple, that is, there is a tuple $b$ of elements in $M$ such that $e \in \text{dcl}^e(b)$ and $b \in \text{dcl}^e(e)$.

There is a stronger notion of elimination of imaginaries that we call as uniform elimination of imaginaries. A structure $M$ uniformly eliminates imaginaries if for any $\emptyset$-definable set $X \subseteq M^n$ and a $\emptyset$-definable equivalence relation $E(x, y)$ on $X$ we can find an $\emptyset$-definable function $f : X \to M^k$ such that for any $x, y \in X$ the relation $E(x, y)$ holds if and only if $f(x) = f(y)$.

For example, Poizat proved that in any algebraically closed field or a differentially closed field of characteristic zero any interpretable set is definably isomorphic to a definable set, meaning that the categories of definable sets and interpretable sets are equivalent. Elimination of imaginaries is saying that the class of definable sets is closed by taking definable quotients, and we say that algebraically closed fields and differentially closed fields of characteristic zero eliminate imaginaries. The property of elimination of imaginaries later became an essential feature in model-theoretic applications, such as diophantine geometry and algebraic dynamics and made available the use of more sophisticated tools like stability, that take place in the category of interpretable sets.

The study of elimination of imaginaries in henselian valued fields was initiated by Haskell, Hrushovski, and Macpherson in [12], where they proved that ACVF eliminates imaginaries down to the well-known geometric sorts. These sorts consist of the field $K$ and, for all $n \in \mathbb{N}_{>0}$, the space $S_n := GL_n(K)/GL_n(\mathcal{O})$ of free rank $n$ submodules of $K^n$, where $\mathcal{O}$ denotes the valuation ring, and of the spaces $T_n := \bigcup_{S \subseteq S_n} S/\mathcal{M}s$ where $\mathcal{M} \subseteq \mathcal{O}$ is the unique maximal ideal. Their work set up the ground-field to develop a tame geometry theory for ACVF and significant applications arose afterward. In [20], Hrushovski and Rideau-Kikuchi gave a complete characterization of definable abelian groups in ACVF, and further work of Hrushovski and Loeser studies connections with the Berkovich theory of rigid analytic spaces.
Other results of elimination of imaginaries down to the geometric sorts have been obtained for other henselian valued fields: real closed valued fields [25], separably closed valued fields of finite imperfection degree [14], enrichments of ACVF in [28], and p-adic fields and their ultraproducts [19]. The former result allowed to uniformize and extend Denef’s result on the rationality of certain zeta functions.

The starting point in this project relies on the Ax–Kochen–Ershov theorem, which states that the first-order theory of a henselian valued field of equicharacteristic zero or unramified mixed characteristic with perfect residue field is completely determined by the first-order theory of its value group and its residue field. A natural principle follows from this theorem: model theoretic questions about the valued field itself can be understood by reducing them to its residue field, its value group and their interaction in the field.

A fruitful application of this principle has been achieved to describe the class of definable sets. For example, in [26] Pas proved field quantifier elimination relative to the residue field and the value group once angular component maps are added in the equicharacteristic zero case. Further studies of Basarab and Kuhlmann show a quantifier elimination relative to the RV sorts (see [3, 24], respectively).

The current results in the field on elimination of imaginaries in henselian valued fields are all obtained for particular instances, while the more general approach of obtaining a relative statement for broader classes of henselian valued fields is still an open question in the field. The precise formulation of elimination of imaginaries for henselian valued fields in an Ax–Kochen/Ershov style can be phrased in the following way: can we find a family of sorts $\mathcal{C}$ such that for any henselian valued field that is finitely ramified one obtains weak elimination of imaginaries in the sorts $K \cup \mathcal{C} \cup k^{eq} \cup \Gamma^{eq}$, where $k^{eq}$ and $\Gamma^{eq}$ are Shelah’s eq expansion of the residue field and the value group (respectively).

The question of whether a henselian valued field eliminates imaginaries in a given language is of course subject to the complexity of its value group and its residue field, because both are interpretable structures in the valued field itself. Following the Ax–Kochen–Ershov style principle, it seems natural to first attempt to solve this question by looking at the problem in two orthogonal directions: one by making the residue field as docile as possible and studying which troubles would the value group bring into the picture, or by making the value group tame and understanding the difficulties that the residue field would contribute to the problem.

Hils and Rideau-Kikuchi in [15] proved that under the assumption of having a definably complete value group and requiring that the residue field eliminates the $\exists^\omega$ quantifier, then any definable set admits a code once the geometric sorts and the linear sorts are added to the language. Any definably complete ordered abelian group is either divisible or a $\mathbb{Z}$-group (i.e., a model of Presburger Arithmetic).

This paper addresses the first approach in the setting of henselian valued fields of equicharacteristic zero. We suppose the residue field to be algebraically closed and we obtain results that are sensitive to the complexity of the value group. We first analyze the case where the value group has finite spines. An ordered abelian group with finite spines weakly eliminates imaginaries once we add sorts for the quotient groups $\Gamma/\Delta$ for each definable convex subgroup $\Delta$, and sorts for the quotient groups $\Gamma/(\Delta + l\Gamma)$ where $\Delta$ is a definable convex subgroup and $l \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the quotient sorts. The first result that we obtain is:

**Theorem 1.1.** Let $K$ be a valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines. Then $K$ admits weak elimination of imaginaries once we add codes
for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, and the quotient sorts for the value group (This is Theorem 5.12).

Later, we prove a better result for the dp-minimal case.

**Theorem 1.2.** Let $K$ be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then $K$ eliminates imaginaries once we add codes for all the definable $\mathcal{O}$-submodules of $K^n$ for each $n \in \mathbb{N}$, the quotient sorts for the value group and constants to distinguish the finite quotients $\Gamma/\ell \Gamma$ for each $\ell \in \mathbb{N}_{\geq 2}$ (This corresponds to Theorem 6.49).

This document is organized as follows.

- **Section 2:** We introduce the required background, including quantifier elimination statements, the state of the model theory of ordered abelian groups and some results about valued vector spaces.
- **Section 3:** We study definable $\mathcal{O}$-modules of $K^n$.
- **Section 4:** We start by presenting Hrushovski’s abstract criterion to eliminate imaginaries. We introduce the stabilizer sorts, where the $\mathcal{O}$-submodules of $K^n$ can be coded.
- **Section 5:** We prove that each of the conditions of Hrushovski’s criterion hold. This is the density of definable types in definable sets in 1-variable $X \subseteq K$ and finding canonical basis for definable types in the stabilizer sorts and $\Gamma^{\text{val}}$. We conclude this section proving the weak elimination of imaginaries of any henselian valued field of equicharacteristic zero, residue field algebraically closed and value group with finite spines down to the stabilizer sorts.
- **Section 6:** We show a complete elimination of imaginaries statement when the value group is $dp$-minimal. We prove that any finite set of tuples in the stabilizer sorts can be coded.

## 2 | PRELIMINARIES

### 2.1 | Quantifier elimination for valued fields of equicharacteristic zero and residue field algebraically closed

In this section, we recall several results relevant for our statement. In particular, we state a quantifier elimination relative to the value group in the canonical three sorted language $\mathcal{L}_{\text{val}}$ for the class of valued fields of equicharacteristic zero and residue field algebraically closed.

#### 2.1.1 | The three-sorted language $\mathcal{L}_{\text{val}}$

We consider valued fields as three sorted structures $(K, k, \Gamma)$. The first two sorts are equipped with the language of fields $\mathcal{L}_{\text{fields}} = \{0, 1, +, \cdot, (\cdot)^{-1}, -\}$, we refer to the first one as the main field sort while we call the second one as the residue field sort. The third sort is supplied with the language of ordered abelian groups $\mathcal{L}_{\text{OAG}} = \{0, <, +, -\}$, and we refer to it as the value group sort. We also add constants $\infty$ to the second sort and the third sort. We introduce a function symbol $v : K \to \Gamma \cup \{\infty\}$, interpreted as the valuation and, we add a map $\text{res} : K \to k \cup \{\infty\}$, where $\text{res} : \mathcal{O} \to k$ is interpreted as a surjective homomorphism of rings, while for any element $x \in K \setminus \mathcal{O}$ we have $\text{res}(x) = \infty$. We denote this language as $\mathcal{L}_{\text{val}}$. 
2.1.2 | The extension theorem

Let $\mathcal{K} = (K, k, \Gamma)$ be a valued field and $\mathcal{O}$ its valuation ring. A triple $\mathcal{E} = (E, k_E, \Gamma_E)$ is a substructure if $E$ is a subfield of $K$, $k_E$ is a subfield of $k$, $\Gamma_E$ is a subgroup of $\Gamma$, $v(E^\times) \subseteq \Gamma_E$ and $\text{res}(O_E) \subseteq k_E$ where $O_E = O \cap E$.

**Definition 2.1.** Let $\mathcal{K}_1 = (K_1, k_1, \Gamma_1)$ and $\mathcal{K}_2 = (K_2, k_2, \Gamma_2)$ be valued fields of equicharacteristic zero with residue field algebraically closed.

Let $\mathcal{E} = (E, k_E, \Gamma_E)$ be a substructure of $\mathcal{K}_1$ a triple $(f, f_r, f_v) = \mathcal{E} \to \mathcal{K}_2$ is said to be an admissible embedding if it is a $\mathcal{L}_{\text{val}}$ isomorphism and $f_v = \Gamma_E \to \Gamma_2$ is a partial elementary map between $\Gamma_1$ and $\Gamma_2$, that is, for every $\mathcal{L}_{OAC}$ formula $\phi(x_1, \ldots, x_n)$ and tuple $e_1, \ldots, e_n \in \Gamma_E$ we have that

$$\Gamma_1 \models \phi(e_1, \ldots, e_n)$$

if and only if $\Gamma_2 \models \phi(f_v(e_1), \ldots, f_v(e_n))$.

Let $\kappa = \max\{|k_E|, |\Gamma_E|\}$. If $\mathcal{K}_2$ is $\kappa^+$-saturated we say that $(f, f_r, f_v)$ is an admissible map with small domain.

**Theorem 2.2** (The Extension theorem). The theory of henselian valued fields of equicharacteristic zero with residue field algebraically closed admits quantifier elimination relative to the value group in the language $\mathcal{L}_{\text{val}}$. This is, given $\mathcal{K}_1 = (K_1, k_1, \Gamma_1)$ and $\mathcal{K}_2 = (K_2, k_2, \Gamma_2)$ henselian valued fields of equicharacteristic zero with residue field algebraically closed, a substructure $\mathcal{E}$ of $\mathcal{K}_1$ and $(f, f_r, f_v) : \mathcal{E} \to \mathcal{K}_2$ and admissible map with small domain, for any $b \in K_1$ there is an admissible map $\hat{f}$ extending $f$ whose domain contains $b$.

**Proof.** This is straightforward using the standard techniques to obtain elimination of field quantifiers already present in the area. We refer the reader, for example, to [33, Theorem 5.21]. The unique step that requires the presence of an angular component map is when for a subfield $E \subseteq K_1$ we want to add an element $\gamma$ to $v(E^\times)$ and there is some prime number $p$ such that $p\gamma \in v(E^\times)$. For this, take $a \in E$ and $c \in K_1$ be such that $v(a) = p\gamma$ and $v(c) = \gamma$. We first aim to find $b_1 \in K_1$ that is a root of the polynomial $Q(x) \in O_E[x]$, where $Q(x) = x^p - \frac{a}{c^p}$. Let $d = \text{res}(\frac{a}{c^p})$, because $k_1$ is algebraically closed there is some $z \in k_1$ such that $z^p - d = 0$. Let $\alpha \in K_1$ be such that $\text{res}(\alpha) = z$, then $v(Q(\alpha)) > 0$ while $v(Q'(\alpha)) = 0$, because $p \neq \text{char}(k_1)$. Indeed, $Q'(x) = px^{p-1}$ and $z \neq 0$ because $d \neq 0$.

By henselianity, we can find $b_1 \in K_1$ that is a root of $Q(x)$. Then $x_1 = (b_1 \cdot c)$ is a $p$th-root of $a$.

The following is an immediate consequence of relative quantifier elimination.

**Corollary 2.3.** The residue field and the value group are both purely stably embedded and orthogonal to each other.

2.2 | Some results on the model theory of ordered abelian groups

In this subsection, we summarize many interesting results about the model theory of ordered abelian groups. We start by recalling the following folklore fact.
**Fact 2.4.** Let \((\Gamma, \leq, +, 0)\) be a nontrivial ordered abelian group. Then the topology induced by the order in \(\Gamma\) is discrete if and only if \(\Gamma\) has a minimum positive element. In this case we say that \(\Gamma\) is discrete, otherwise we say that it is dense.

The following notions were isolated in the 1960s by Robinson and Zakon in [30] to understand some model complete extensions of the theory of ordered abelian groups.

**Definition 2.5.** Let \(\Gamma\) be an ordered abelian group and \(n \in \mathbb{N}_{\geq 2}\).

1. Let \(\gamma \in \Gamma\). We say that \(\gamma\) is \(n\)-divisible if there is some \(\beta \in \Gamma\) such that \(\gamma = n\beta\).
2. We say that \(\Gamma\) is \(n\)-divisible if every element \(\gamma \in \Gamma\) is \(n\)-divisible.
3. \(\Gamma\) is said to be \(n\)-regular if any interval with at least \(n\) points contains an \(n\)-divisible element.

**Definition 2.6.** An ordered abelian group \(\Gamma\) is said to be regular if it is \(n\)-regular for all \(n \in \mathbb{N}\).

Robinson and Zakon in their seminal paper [30] completely characterized the possible completions of the theory of regular groups, obtained by extending the first-order theory of ordered abelian groups with axioms asserting that for each \(n \in \mathbb{N}\) if an interval contains at least \(n\) elements then it contains an \(n\)-divisible element. The following is [30, Theorem 4.7].

**Theorem 2.7.** The possible completions of the theory of regular groups that are not trivial are:

1. the theory of discrete regular groups, and
2. the completions of the theory of dense regular groups \(T_\chi\) where

\[\chi =: \text{Primes} \rightarrow \mathbb{N} \cup \{\infty\},\]

is a function specifying the index \(\chi(p) = [\Gamma : p\Gamma]\).

Robinson and Zakon proved as well that each of these completions is the theory of some archimedean group. In particular, any discrete regular group is elementarily equivalent to \((\mathbb{Z}, \leq, +, 0)\).

The following definitions were introduced by Schmitt in [31].

**Definition 2.8.** We fix an ordered abelian group \(\Gamma\) and \(n \in \mathbb{N}_{\geq 2}\). Let \(\gamma \in \Gamma\). We define:

- \(A(\gamma)\) = the largest convex subgroup of \(\Gamma\) not containing \(\gamma\);
- \(B(\gamma)\) = the smallest convex subgroup of \(\Gamma\) containing \(\gamma\);
- \(C(\gamma) = B(\gamma)/A(\gamma)\);
- \(A_n(\gamma)\) = the smallest convex subgroup \(C\) of \(\Gamma\) such that \(B(\gamma)/C\) is \(n\)-regular;
- \(B_n(\gamma)\) = the largest convex subgroup \(C\) of \(\Gamma\) such that \(C/A_n(\gamma)\) is \(n\)-regular.

In [31, chapter 2], Schmitt shows that the groups \(A_n(\gamma)\) and \(B_n(\gamma)\) are definable in the language of ordered abelian groups \(L_{OAG} = \{+, -, \leq, 0\}\) by a first-order formula using only the parameter \(\gamma\).

We recall that the set of convex subgroups of an ordered abelian group is totally ordered by inclusion.
Definition 2.9. Let $\Gamma$ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, we define the $n$-regular rank to be the order type of:

$$\langle \{A_n(\gamma) \mid \gamma \in \Gamma \setminus \{0\}\}, \subseteq \rangle.$$ 

The $n$-regular rank of an ordered abelian group $\Gamma$ is a linear order, and when it is finite we can identify it with its cardinal. In [9], Farré emphasizes that we can characterize the $n$-regular rank without mentioning the subgroups $A_n(\gamma)$. The following is [9, Remark 2.2].

Definition 2.10. Let $\Gamma$ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, then:

1. $\Gamma$ has $n$-regular rank equal to 0 if and only if $\Gamma = \{0\}$,
2. $\Gamma$ has $n$-regular rank equal to 1 if and only if $\Gamma$ is $n$-regular and not trivial,
3. $\Gamma$ has $n$-regular rank equal to $m$ if there are $\Delta_0, \ldots, \Delta_m$ convex subgroups of $\Gamma$, such that:
   - $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_m = \Gamma$,
   - for each $0 \leq i < m$, the quotient group $\Delta_{i+1}/\Delta_i$ is $n$-regular,
   - the quotient group $\Delta_{i+1}/\Delta_i$ is not $n$-divisible for $0 < i < m$.

In this case, we define $RJ_n(\Gamma) = \{\Delta_0, \ldots, \Delta_{m-1}\}$. The elements of this set are called the $n$-regular jumps.

Definition 2.11. Let $\Gamma$ be an ordered abelian group. We say that it is poly-regular if it is elementarily equivalent to a subgroup of the lexicographically ordered group $(\mathbb{R}^n, +, \leq_{\text{lex}}, 0)$.

In [4], Belegradek studied poly-regular groups and proved that an ordered abelian group is poly-regular if and only if it has finitely many proper definable convex subgroups, and all the proper definable subgroups are definable over the empty set. In [35, Theorem 2.9], Weispfenning obtained quantifier elimination for the class of poly-regular groups in the language of ordered abelian groups extended with predicates to distinguish the subgroups $\Delta + \ell \Gamma$ where $\Delta$ is a convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$.

Definition 2.12. Let $\Gamma$ be an ordered abelian group. We say that it has bounded regular rank if it has finite $n$-regular rank for each $n \in \mathbb{N}_{\geq 2}$. For notation, we will use $RJ(\Gamma) = \bigcup_{n \in \mathbb{N}_{\geq 2}} RJ_n(\Gamma)$.

The class of ordered abelian groups of bounded regular rank extends the class of poly-regular groups and regular groups. The terminology of bounded regular rank becomes clear with the following Proposition (item 3).

Proposition 2.13. Let $\Gamma$ be an ordered abelian group. The following are all equivalent.

1. $\Gamma$ has finite $p$-regular rank for each prime number $p$.
2. $\Gamma$ has finite $n$-regular rank for each natural number $n \geq 2$.
3. There is some cardinal $\kappa$ such that for any $H \equiv \Gamma$, $|RJ(H)| \leq \kappa$.
4. For any $H \equiv \Gamma$, any definable convex subgroup of $H$ has a definition without parameters.
5. There is some cardinal $\kappa$ such that for any $H \equiv \Gamma$, $H$ has at most $\kappa$-definable convex subgroups.

Moreover, in this case $RJ(\Gamma)$ is the collection of all proper definable convex subgroups of $\Gamma$ and all are definable without parameters. In particular, there are only countably many definable convex subgroups.
2.2.1 Quantifier elimination and the quotient sorts

In [6], Cluckers and Halupczok introduced a language $\mathcal{L}_{qe}$ to obtain quantifier elimination for ordered abelian groups relative to the auxiliary sorts $S_n$, $T_n$, and $T_n^+$, whose precise description can be found in [6, Definition 1.5]. This language is similar in spirit to the one introduced by Schmitt in [31], but has lately been preferred by the community as it is more in line with the many-sorted language of Shelah’s imaginary expansion $\mathcal{M}^{eq}$. Schmitt does not distinguish between the sorts $S_n$, $T_n$ and $T_n^+$. Instead for each $n \in \mathbb{N}$ he works with a single sort $Sp_n(\Gamma)$ called the $n$-spine of $\Gamma$, whose description can be found in [11, section 2]. In [6, section 1.5], it is explained how the auxiliary sorts of Cluckers and Halupczok are related to the $n$-spines $Sp_n(\Gamma)$ of Schmitt. In [9, section 2], it is shown that an ordered abelian group $\Gamma$ has bounded regular rank if and only if all the $n$-spines are finite, and $Sp_n(\Gamma) = RJ_n(\Gamma)$. In this case, we define the regular rank of $\Gamma$ as the cardinal $|RJ(\Gamma)|$, which is either finite or $\aleph_0$. Instead of saying that $\Gamma$ is an ordered abelian group with finite spines, we prefer to use the classical terminology of bounded regular rank, as it emphasizes the relevance of the $n$-regular jumps and the role of the divisibilities to describe the definable convex subgroups.

We define the Presburger Language $\mathcal{L}_{\text{Pres}} = \{0, 1, +, −, <, (P_m)_{m \in \mathbb{N}_{\geq 2}}\}$. Given an ordered abelian group $\Gamma$, we naturally see it as a $\mathcal{L}_{\text{Pres}}$-structure. The symbols $\{0, +, −, <\}$ take their obvious interpretation. If $\Gamma$ is discrete, the constant symbol $1$ is interpreted as the least positive element of $\Gamma$, and by $0$ otherwise. For each $m \in \mathbb{N}_{\geq 2}$, the symbol $P_m$ is a unary predicate interpreted as $m \Gamma$.

**Definition 2.14** (The language $\mathcal{L}_b$). Let $\Gamma$ be an ordered abelian group with bounded regular rank, we view $\Gamma$ as a multi-sorted structure where:

1. we add a sort for the ordered abelian group $\Gamma$, and we equip it with a copy of the language $\mathcal{L}_{\text{Pres}}$ extended with predicates to distinguish each of the convex subgroups $\Delta \in RJ(\Gamma)$; we refer to this sort as the main sort;
2. we add a sort for each of the ordered abelian groups $\Gamma/\Delta$, equipped with a copy of the language $\mathcal{L}_{\text{Pres}}^{\Delta} = \{0^\Delta, 1^\Delta, +^\Delta, −^\Delta, <^\Delta, (P^\Delta_m)_{m \in \mathbb{N}_{\geq 2}}\}$; we add as well a map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, interpreted as the natural projection map.

**Remark 2.15.** To keep the notation as simple and clear as possible, for each $\Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$ and $\beta \in \Gamma/\Delta$ we will write $\beta \in n(\Gamma/\Delta)$ instead of $P^\Delta_n(\beta)$.

The following statement is a direct consequence of [1, Proposition 3.14].

**Theorem 2.16.** Let $\Gamma$ be an ordered abelian group with bounded regular rank. Then $\Gamma$ admits quantifier elimination in the language $\mathcal{L}_b$.

We will consider an extension of this language that we will denote as $\mathcal{L}_{\text{bq}}$, where for each natural number $n \geq 2$ and $\Delta \in RJ(\Gamma)$ we add a sort for the quotient group $\Gamma/(\Delta + n\Gamma)$ and a map $\pi^n_\Delta : \Gamma \to \Gamma/(\Delta + n\Gamma)$. We will refer to the sorts in the language $\mathcal{L}_{\text{bq}}$ as quotient sorts.

The following is [34, Theorem 5.1].
Theorem 2.17. Let $\Gamma$ be an ordered abelian group with bounded regular rank. Then $\Gamma$ admits weak elimination of imaginaries in the language $\mathcal{L}_{bq}$, that is, once one adds all the quotient sorts.

Definable end-segments in ordered abelian groups with bounded regular rank

Definition 2.18.

1. A nonempty set $S \subseteq \Gamma$ is said to be an end-segment if for any $x \in S$ and $y \in \Gamma$, $x < y$ we have that $y \in S$.
2. Let $n \in \mathbb{N}, \Delta \in RJ(\Gamma), \beta \in \Gamma \cup \{-\infty\}$ and $\square \in \{\geq, >\}$. The set:

$$S^\Delta_n(\beta) := \{\eta \in \Gamma \mid n\eta + \Delta \square \beta + \Delta\}$$

is an end-segment of $\Gamma$. We call any of the end-segments of this form as divisibility end-segments.
3. Let $S \subseteq \Gamma$ be a definable end-segment and $\Delta \in RJ(\Gamma)$. We consider the projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, and we write $S_\Delta$ to denote $\rho_\Delta(S)$. This is a definable end-segment of $\Gamma/\Delta$.
4. Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ an end-segment. We say that $S$ is $\Delta$-decomposable if it is a union of $\Delta$-cosets.
5. We denote as $\Delta_S$ the stabilizer of $S$, that is, $\Delta_S := \{\eta \in \Gamma \mid \eta + S = S\}$.

Definition 2.19. Let $\Gamma$ be an ordered abelian group. Let $S, S' \subseteq \Gamma$ be definable end-segments. We say that $S$ is a translate of $S'$ if there some $\beta \in \Gamma$ such that $S = \beta + S'$. Given a family $S$ of definable end-segments we say that $S$ is complete if every definable end-segment is a translate of some $S' \in S$.

Fact 2.20. Let $\Gamma$ be an ordered abelian group with bounded regular rank. Let $\beta, \gamma \in \Gamma, \Delta \in RJ(\Gamma)$ and $n \in \mathbb{N}_{\geq 2}$. If $\beta - \gamma \in \Delta + n\Gamma$ then $S^\Delta_n(\gamma)$ is a translate of $S^\Delta_n(\beta)$.

The following is [34, Proposition 3.3].

Proposition 2.21. Let $\Gamma$ be an ordered abelian group of bounded regular rank. Any definable end-segment is a divisibility end-segment.

Remark 2.22. Let $\Gamma$ be an ordered abelian group and $\Delta$ be an $\emptyset$-definable convex subgroup. Any complete set of representatives in $\Gamma$ modulo $k\Gamma$ for $k \in \mathbb{N}$ is also a complete set of representative of $\Gamma$ modulo $\Delta + k\Gamma$. Moreover, there is an $\emptyset$-definable surjective function $f : \Gamma/k\Gamma \to \Gamma/(\Delta + k\Gamma)$.

Corollary 2.23. Let $\Gamma$ be an ordered abelian group with bounded regular rank. For each $n \in \mathbb{N}_{\geq 2}$ let $C_n$ be a complete set of representatives of the cosets $n\Gamma$ in $\Gamma$. Define $S^\Delta_n := \{S^\Delta_n(\beta) \mid \beta \in C_n\}$. Then $S = \bigcup_{\Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}} S^\Delta_n$ is a complete family.

Proof. It is an immediate consequence of Proposition 2.21, Fact 2.20, and Remark 2.22. \qed

The following is [34, Fact 4.1].

Fact 2.24. Let $S \subseteq \Gamma$ be a definable end-segment. Then $\Delta_S$ is a definable convex subgroup of $\Gamma$, therefore $\Delta_S \in RJ(\Gamma)$. Furthermore, $\Delta_S$ is the largest convex subgroup of $\Gamma$ such that $S$ is $\Delta$-decomposable.
Definition 2.25. Let $T$ be a complete $\mathcal{L}$-first-order theory. Let $\mathcal{C}$ be the monster model and $A \subseteq \mathcal{C}$ be a set of parameters. Let

$$\Sigma \subseteq \{ \psi(x; a) \mid \psi(x; y) \text{ is a } \mathcal{L}\text{-formula and } a \in A \},$$

and suppose that $\Sigma$ is closed under boolean combinations. Let $b \in \mathcal{C}$ we define the $\Sigma$-type of $b$:

$$tp_{\Sigma}(b/A) = \{ \psi(x; a) \in \Sigma \mid \mathcal{C} \models \psi(b; a) \}.$$ 

To easier the notation, we usually refer to this type simply as a partial type.

We say that $tp_{\Sigma}(b/A)$ is definable over $A_0 \subseteq A$, if for every formula $\psi(x, y) \in \Sigma$ there is some $\mathcal{L}(A_0)$ formula $d\psi(y)$ such that:

$$\psi(x, a) \text{ if and only if } \mathcal{C} \models \psi(a).$$

Definition 2.26. Let $S \subseteq \Gamma$ be a definable end-segment. Let

$$\Sigma_{S_{\text{gen}}}^S(x) := \{ x \in S \} \cup \{ x \notin B \mid B \subseteq S \text{ and } B \text{ is a definable end-segment } \}.$$ 

We refer to this partial type as the generic type in $S$. This partial type is $\forall S \exists$-definable.

2.2.2 The dp-minimal case

In 1984, the classification of the model theoretic complexity of ordered abelian groups was initiated by Gurevich and Schmitt, who proved that no ordered abelian group has the independence property. During the last years finer classifications have been achieved, in particular dp-minimal ordered abelian groups have been characterized in [21].

Definition 2.27. Let $\Gamma$ be an ordered abelian group and let $p$ be a prime number. We say that $p$ is a singular prime if $[\Gamma : p\Gamma] = \infty$.

The following result corresponds to [21, Proposition 5.1].

Proposition 2.28. Let $\Gamma$ be an ordered abelian group, the following conditions are equivalent:

1. $\Gamma$ does not have singular primes,
2. $\Gamma$ is $dp$-minimal.

Definition 2.29 (The language $\mathcal{L}_{dp}$). Let $\Gamma$ be a dp-minimal ordered abelian group. We consider the language extension $\mathcal{L}_{dp}$ of $\mathcal{L}_{bg}$ [see Definition 2.14] where for each $n \in \mathbb{N}_{\geq 2}$ we add a set of constant symbols for the elements of the finite group $\Gamma/n\Gamma$.

The following is [34, Corollary 5.2].

Corollary 2.30. Let $\Gamma$ be a $dp$-minimal ordered abelian group. Then $\Gamma$ admits elimination of imaginaries in the language $\mathcal{L}_{dp}$.
The following will be a very useful fact.

**Fact 2.31.** Let $\Gamma$ be a dp-minimal ordered abelian group and let $S \subseteq \Gamma$ be a definable end-segment. Then any complete type $q(x)$ extending $\Sigma_S^{\text{gen}}(x)$ is $\Gamma S$-$\text{definable}$.  

**Proof.** Let $\Sigma_S^{\text{gen}}(x)$ be the generic type of $S$ and $q(x)$ be any complete extension. $\Sigma_S^{\text{gen}}(x)$ is $\Gamma S$-$\text{definable}$, and by Theorem 2.16 $q(x)$ is completely determined by the quantifier free formulae. It is sufficient to verify that for each $\Delta \in RJ(\Gamma)$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ the set:

$$Z = \{ \beta \in \Gamma \mid (\rho_\Delta(x) - \rho_\Delta(\beta) + k \Delta \in n(\Gamma/\Delta)) \in q(x) \}$$

is $\Gamma S$-$\text{definable}$. First, we note that there is a canonical one to one correspondence

$$g : = (\Gamma/\Delta)/n(\Gamma/\Delta) \rightarrow \Gamma/(\Delta + n\Gamma).$$

Let $c = g(k\Delta + n(\Gamma/\Delta)) \in \text{dcl}^e(\emptyset)$. Let $\mu \in \Gamma/n\Gamma$ be such that $\pi_n(a) = \mu$, where $a \models q(x)$. Let $f$ be the $\emptyset$-definable function given by Remark 2.22. Then $\beta \in Z$ if and only if $\models \pi^n_\Delta(\beta) = f(\mu) + c$, and $f(\mu) + c \in \text{dcl}^e(\emptyset)$. \qed

We conclude this subsection with the following remark, that simplifies the presentation of a complete family in the dp-minimal case.

**Remark 2.32.** Let $\Gamma$ be a dp-minimal ordered abelian group. For each $n \in \mathbb{N}_{\geq 2}$, let $\Omega_n$ be a finite set of constants in $\Gamma$ to distinguish representatives for each of the cosets of $n\Gamma$ in $\Gamma$.

Let $S^n_\Delta : = \{ S^n_\Delta(d) \mid d \in \Omega^n \}$. The set $S_{dp} = \bigcup_{\Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}} S^n_\Delta$ is a complete family whose elements are all definable over $\emptyset$.

### 2.3 Henselian valued fields of equicharacteristic zero with residue field algebraically closed and value group of bounded regular rank

The main goal of this section is to describe the $1$-definable subsets $X \subseteq K$, where $K$ is a henselian valued field with residue field algebraically closed and with value group of bounded regular rank.

#### 2.3.1 The language $\mathcal{L}$

Let $(K, \nu)$ be a henselian valued field of equicharacteristic zero, whose residue field is algebraically closed and whose value group is of bounded regular rank. We will view this valued field as an $\mathcal{L}$-structure, where $\mathcal{L}$ is the language extending $\mathcal{L}_{\text{val}}$ in which the value group sort is equipped with the language $\mathcal{L}_b$ described in Definition 2.14. Let $T$ be the complete $\mathcal{L}$-$\text{first-order}$ theory of $(K, \nu)$. (In particular, we are fixing a complete theory for the value group.)

**Corollary 2.33.** The first-order theory $T$ admits quantifier elimination in the language $\mathcal{L}$.

**Proof.** This is a direct consequence of Theorems 2.2 and 2.16. \qed
2.3.2  Description of definable sets in 1-variable

In this subsection, we give a description of the definable subsets in 1-variable $X \subseteq K$, where $K \models T$. We denote as $\mathcal{O}$ its valuation ring.

**Definition 2.34.** Let $(K, \mathcal{O})$ be a henselian valued field of equicharacteristic zero and let $\Gamma$ be its value group. Let $\Delta$ be a convex subgroup of $\Gamma$ then the map:

$$v_\Delta : \begin{cases} 
K & \to \Gamma / \Delta \\
x & \mapsto v(x) + \Delta,
\end{cases}$$

is a henselian valuation on $K$ and it is commonly called as the **coarsened valuation induced by $\Delta$**. Note that $v_\Delta = \rho_\Delta \circ v$.

The following is a folklore fact.

**Fact 2.35.** There is a one-to-one correspondence between the $\mathcal{O}$-submodules of $K$ and the end-segments of $\Gamma$. Given $M \subseteq K$ an $\mathcal{O}$-submodule, we have that $S_M := \{v(x) \mid x \in M\}$ is an end-segment of $\Gamma$. We refer to $S_M$ as the end-segment induced by $M$. And given an end-segment $S \subseteq \Gamma$, the set $M_S := \{x \in K \mid v(x) \in S\}$ is an $\mathcal{O}$-submodule of $K$.

**Definition 2.36.**

1. Let $M$ and $N$ be $\mathcal{O}$-submodules of $K$, we say that $M$ is a **scaling** of $N$ if there is some $b \in K^\times$ such that $M = bN$.
2. A family $\mathcal{F}$ of definable $\mathcal{O}$-submodules of $K$ is said to be **complete** if any definable submodule $M \subseteq K$ is a scaling of some $\mathcal{O}$-submodule $N \in \mathcal{F}$.

**Fact 2.37.** Let $\mathcal{F} = \{M_S \mid S \in \mathcal{S}\}$, where $\mathcal{S}$ is the complete family of definable end-segments described in Corollary 2.23. Then $\mathcal{F}$ is a complete family of $\mathcal{O}$-submodules of $K$, which contains $\{0\}$ and $K$.

**Definition 2.38.**

1. Let $w : K \to \Gamma_w$ be a valuation, $\gamma \in \Gamma_w$ and $a \in K$. The **closed ball of radius $\gamma$** centered at $a$ according to the valuation $w$ is the set of the form

$$\bar{B}_\gamma(a) = \{x \in K \mid \gamma \leq w(x - a)\}.$$ 

And the **open ball of radius $\gamma$** centered at $a$ according to the valuation $w$ is the set of the form

$$B_\gamma(a) = \{x \in K \mid \gamma < w(x - a)\}.$$ 

2. A **Swiss cheese according to the valuation $w$** is a set of the form

$$A \setminus (B_1 \cup \cdots \cup B_n),$$

where $A$ is either $K$, a point or a ball and for each $i \leq n$, $B_i \subseteq A$ and the $B_i$ is a ball or a point according to the original valuation $w : K \to \Gamma_w$. 


(3) A 1-torsor of $K$ is a set of the form $a + bI$ where $a, b \in K$ and $I \in \mathcal{P}$.

(4) A generalized Swiss cheese is either a singleton element in the field $\{a\}$ or a set of the form $A \setminus (B_1 \cup \cdots \cup B_n)$ where $A$ is a 1-torsor for each $i \leq n$, $B_i \subseteq A$ and the $B_i$ is either a 1-torsor or a singleton element $\{b_i\}$ of the field.

(5) A basic positive congruence formula in the valued field is a formula of the form $zv_\Delta(x - a) - \beta + k \in n(\Gamma/\Delta)$, where $k, z \in \mathbb{Z}$, $a \in K$, $\beta \in \Gamma/\Delta$, $n \in \mathbb{N}_{\geq 2}$ and $k_\Delta = k \cdot 1^\Delta$, where $1^\Delta$ is the minimum positive element of $\Gamma/\Delta$ if it exists.

(6) A basic negative congruence formula in the valued field is a formula of the form $zv_\Delta(x - a) - \beta + k \notin n(\Gamma/\Delta)$, where $k, z \in \mathbb{Z}$, $a \in K$, $\beta \in \Gamma/\Delta$, $n \in \mathbb{N}_{\geq 2}$ and $k_\Delta = k \cdot 1^\Delta$, where $1^\Delta$ is the minimum positive element of $\Gamma/\Delta$ if it exists.

(7) A basic congruence formula in the valued field is either a basic positive congruence formula in the valued field or a basic negative congruence formula in the valued field.

(8) A finite congruence restriction in the valued field is a finite conjunction of basic congruence formulae in the valued field.

(9) A nice set is a set of the form $S \cap C$ where $S$ is a generalized Swiss cheese and $C$ is the set defined by a finite congruence restriction in the valued field.

To describe completely the definable subsets of $K$, we will need the following lemmas, which permit us to reduce the valuation of a polynomial into the valuation of linear factors of the form $v(x - a)$. We recall a definition and some results present in [10] that will be useful for this purpose.

**Definition 2.39.** Let $(K, w)$ be a henselian valued field, $\alpha \in K$ and $S$ a Swiss cheese. Let $p(x) \in K[x]$, we define:

$$m(p, \alpha, S) := \max\{i \leq d \mid \exists x \in S \forall j \leq d \left( w(a_i(x - \alpha)^j) \leq w(a_j(x - \alpha)^j) \right)\},$$

where the $a_i$ are the coefficients of the expansion of $p$ around $\alpha$, that is, $p(x) = \sum_{i=0}^{d} a_i(x - \alpha)^i$.

Thus, $m(p, \alpha, S)$ is the highest order term in $p$ centered at $\alpha$ that can have minimal valuation (among the other terms of $p$) in $S$.

The following is [10, Proposition 3.4].

**Proposition 2.40.** Let $(K, w)$ be a henselian valued field of characteristic zero. Let $p(x) \in K[x]$ and $S$ a Swiss cheese in $K$. Then there are (disjoint) sub-Swiss cheeses $T_1, \ldots, T_n \subseteq S$ and $\alpha_1, \ldots, \alpha_n \in K$ such that $S = \bigcup_{1 \leq i \leq n} T_i$, where for all $x \in T_i$, $w(p(x)) = w(a_{im_i}(x - \alpha_i)^{m_i})$, where $p(x) = \sum_{n=0}^{d} a_{in}(x - \alpha_i)^n$ and $m_i = m(p, \alpha_i, T_i)$. Furthermore, $\alpha_1, \ldots, \alpha_k$ can be taken algebraic over the subfield of $K$ generated by the coefficients of $p(x)$.

Though the preceding proposition is stated for a single polynomial, the same result will hold for any finite number of polynomials $\Sigma$. We refer the reader for details about this fact to [10] (see comments after [10, Proposition 3.4]).

**Fact 2.41.** Let $(K, w)$ be a henselian valued field of equicharacteristic zero, and $Q_1(x), Q_2(x) \in K[x]$ be two polynomials in a single variable. Let $R = \{x \in K \mid Q_2(x) = 0\}$. There is a finite union of Swiss
cheeses \( K = \bigcup_{i \leq k} T_i \), coefficients \( \varepsilon_i \in K \), elements \( \gamma_i \in \Gamma \) and integers \( z_i \in \mathbb{Z} \) such that for any \( x \in T_i \setminus R \):

\[
w(Q_1(x)) - w(Q_2(x)) = \gamma_i + z_i w(x - \varepsilon_i).
\]

**Proof.** The statement is a straightforward computation after applying Proposition 2.40, and it is left to the reader. \( \square \)

**Proposition 2.42.** Let \( K \models T \), for each \( \Delta \in RJ(\Gamma) \) let \( v_\Delta : K \to \Gamma/\Delta \) be the coarsened valuation induced by \( \Delta \). Let \( Q_1(x), Q_2(x) \in K[x] \) and \( R = \{ x \in K \mid Q_1(x) = 0 \text{ or } Q_2(x) = 0 \} \). Let \( X \subseteq K \setminus R \) be the set defined by a formula of the form:

\[
\gamma \leq \Delta v_\Delta(Q_1(x)) - v_\Delta(Q_2(x)) \text{ or } v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n(\Gamma/\Delta);
\]

where \( \gamma \in \Gamma/\Delta \) and \( n \in \mathbb{N} \). Then \( X \) is a finite union of nice sets.

**Proof.** First we observe that a Swiss cheese with respect to the coarsened valuation \( v_\Delta \) is a generalized Swiss cheese with respect to \( v \). The statement follows by a straightforward computation after applying Fact 2.41, and it is left to the reader. \( \square \)

We conclude this section by characterizing the definable sets in 1-variable.

**Theorem 2.43.** Let \( K \models T \) and \( X \subseteq K \) be a definable set. Then \( X \) is a finite union of nice sets.

**Proof.** By Corollary 2.33, \( X \) is a Boolean combination of sets defined by formulae of the form

\[
\gamma \leq \Delta v_\Delta(Q_1(x)) - v_\Delta(Q_2(x)) \text{ or } v_\Delta\left(\frac{Q_1(x)}{Q_2(x)}\right) - \gamma \in n(\Gamma/\Delta), \text{ where } \Delta \in RJ(\Gamma), \gamma \in \Gamma/\Delta \text{ and } n \in \mathbb{N}_{\geq 2}.
\]

By Proposition 2.42, each of these formulae defines a finite union of nice sets. Because the intersection of two generalized Swiss cheeses is again a generalized Swiss cheese and the complement of a generalized Swiss cheese is a finite union of generalized Swiss cheeses the statement follows. \( \square \)

### 2.4 \( \mathcal{O} \)-modules and homomorphisms in maximal valued fields

In this section, we recall some results about modules over maximally complete valued fields. We follow ideas of Kaplansky in [23] to characterize the \( \mathcal{O} \)-submodules of finite-dimensional \( K \)-vector spaces.

**Definition 2.44.**

1. Let \( K \) be a valued field and \( \mathcal{O} \) its valuation ring. We say that \( K \) is maximal, if whenever \( \alpha_r \in K \) and (integral or fractional) ideals \( I_r \) are such that the congruences \( x - \alpha_r \in I_r \) are pairwise consistent, then there exists in \( K \) a simultaneous solution of all the congruences.

2. Let \( K \) be a valued field and \( M \subseteq K^n \) be an \( \mathcal{O} \)-module. We say that \( M \) is maximal if whenever ideals \( I_r \subseteq \mathcal{O} \) and elements \( s_r \in M \) are such that \( x - s_r \in I_r M \) is pairwise consistent in \( M \), then there exists in \( M \) a simultaneous solution of all the congruences.
(3) Let $N$ be an $\mathcal{O}$-submodule. Let $\alpha \in \mathcal{O}$ and fix an element $x \in N$. Say that $x$ is $\alpha$-divisible in $N$ if there is some $n \in N$ such that $x = \alpha n$.

We start by recalling a very useful fact.

**Fact 2.45.** Let $K$ be a henselian valued field of equicharacteristic zero, then there is an elementary extension $K < K'$ that is maximal.

**Proof.** Let $K$ be a henselian valued field of equicharacteristic zero, let $T$ be its $L_{\text{val}}$-complete first-order theory and $\mathbb{C}$ the monster model of $T$. By [33, Lemma 4.30], there is some maximal immediate extension of $K \subseteq F \subseteq \mathbb{C}$. By [33, Theorem 7.12], $K < F$. \qed

The following is [23, Lemma 5].

**Lemma 2.46.** Let $K$ be a maximal valued field, then any (integral or fractional) ideal $I$ of $\mathcal{O}$ is maximal as an $\mathcal{O}$-submodule of $K$. Moreover, any finite direct sum of maximal $\mathcal{O}$-modules is also maximal.

**Fact 2.47.** Let $N \subseteq K$ be a nonzero $\mathcal{O}$-submodule. Let $n \in N \setminus \{0\}$ then $N = nI$ where $I$ is a copy of $K$ or a fractional ideal of $\mathcal{O}$.

**Definition 2.48.** Let $K$ be a field and $n \in \mathbb{N}_{\geq 1}$, we say that a set $\{a_1, \ldots, a_n\}$ is an upper triangular basis of the vector space $K^n$ if it is a $K$-linearly independent set and the matrix $[a_1, \ldots, a_n]$ is upper triangular.

**Theorem 2.49.** Let $K$ be a maximal valued field and $n \in \mathbb{N}_{\geq 1}$. Let $N \subseteq K^n$ be an $\mathcal{O}$-submodule (not necessarily definable). Then $N$ is maximal, and one can find an upper triangular basis $\{a_1, \ldots, a_n\}$ of $K^n$ such that $N = \{a_1 x_1 + \cdots + a_n x_n \mid x_i \in I_i\}$, where each $I_i$ is either a copy of $K$ or a fractional ideal of $\mathcal{O}$. In this case we say that $[a_1, \ldots, a_n]$ is a representation matrix for the module $N$.

**Proof.** We proceed by induction on $n$, the base case is given by Fact 2.47 and Lemma 2.46. For the inductive step, let $\pi : K^{n+1} \to K$ be the projection into the last coordinate and let $M = \pi(N)$. We consider the exact sequence of $\mathcal{O}$-modules $0 \to N \cap (K^n \times \{0\}) \to N \to M \to 0$.

By induction, $N \cap (K^n \times \{0\})$ is maximal and of the required form. And there is an upper triangular basis $\{a_1, \ldots, a_n\}$ of $K^n \times \{0\}$ such that $[a_1, \ldots, a_n]$ is a representation matrix for $N \cap (K^n \times \{0\})$. If $M = \{0\}$ we are all set, so we may take $m \in M$ such that $m \neq 0$ and $M = mI$ for some fractional ideal $I$ of $\mathcal{O}$ or $I = K$.

**Claim 2.49.1.** There is some element $x \in N$ such that $\pi(x) = m$ and for any $\alpha \in \mathcal{O}$, if $m$ is $\alpha$-divisible in $M$ then $x$ is $\alpha$-divisible in $N$.

**Proof.** Let $J = \{\alpha \in \mathcal{O} \mid m$ is $\alpha$-divisible in $M\}$. For each $\alpha \in J$, let $m_\alpha \in M$ be such that $m = \alpha m_\alpha$ and take $n_\alpha \in \pi^{-1}(m_\alpha) \cap N$. Fix an element $y \in N$ satisfying $\pi(y) = m$ and let $s_\alpha = y - \alpha n_\alpha \in N \cap (K^n \times \{0\})$.

Consider $S = \{x - s_\alpha \in \alpha N \cap (K^n \times \{0\}) \mid \alpha \in J\}$ this is system of congruences in $N \cap (K^n \times \{0\})$. We will argue that it is pairwise consistent. Let $\alpha, \beta \in \mathcal{O}$, then either $\frac{\alpha}{\beta} \in \mathcal{O}$ or $\frac{\beta}{\alpha} \in \mathcal{O}$ (or...
both). Without loss of generality, assume that $\frac{\alpha}{\beta} \in \mathcal{O}$, then:

$$s_\alpha - s_\beta = (y - \alpha n_\alpha) - (y - \beta n_\beta) = \beta n_\beta - \alpha n_\alpha = \beta \left( n_\beta - \frac{\alpha}{\beta} n_\alpha \right) \in N \cap (K^n \times \{0\}).$$

Thus, $s_\alpha$ is a solution to the system \( \{ x - s_\alpha \in \alpha N \cap (K^n \times \{0\}) \} \cup \{ x - s_\beta \in \beta N \cap (K^n \times \{0\}) \} \). By maximality of $N \cap (K^n \times \{0\})$, we can find an element $z \in N \cap (K^n \times \{0\})$ such that $z$ is a simultaneous solution to the whole system of congruences in $S$. Let $x = y - z \in N$, then $x$ satisfies the requirements. Indeed, for each $\alpha \in J$, we had chosen $z - s_\alpha \in \alpha N \cap (K^n \times \{0\})$, so $z = s_\alpha + \alpha w$ for some $w \in N \cap (K^n \times \{0\})$. Thus, $x = y - z = y - s_\alpha - \alpha w = y - (y - \alpha n_\alpha) - \alpha w = \alpha(n_\alpha - w) \in \alpha N$, as desired. \( \square \)

Let $s : M \to N$ be the map sending an element $\alpha m$ to $\alpha x$, where $\alpha \in K$. As $N$ is a torsion-free module, $s$ is well-defined. One can easily verify that $s$ is a homomorphism such that $\pi \circ s = id_M$. Thus, $N$ is the direct sum of $N \cap (K^n \times \{0\})$ and $s(M)$, so it is maximal by Lemma 2.46. Moreover, $[a_1, \ldots, a_n, x]$ is a representation matrix for $N$, as required. \( \square \)

**Proposition 2.50.** Let $K$ be a maximal valued field. Let $M, N \subseteq K$ be $\mathcal{O}$-submodules. For any $\mathcal{O}$-homomorphism $h : M \to K/N$ there is some $a \in K$ such that for any $x \in M$, $h(x) = ax + N$.

**Proof.** By Fact 2.47, $M = bI$ where $I$ is a copy of $K$ or a fractional ideal of $\mathcal{O}$. It is sufficient to prove the statement for $b = 1$. Let $S_j = \{v(y) \mid y \in I\}$ be the end-segment induced by $I$. Let $\{\gamma_\alpha \mid \alpha \in \kappa\}$ be a co-initial strictly decreasing sequence in $S_j$, where $\kappa$ is a regular cardinal. Choose an element $x_\alpha \in K$ such that $v(x_\alpha) = \gamma_\alpha$, then for each $\alpha < \beta < \kappa, x_\beta \mathcal{O} \subseteq x_\alpha \mathcal{O}$ and $I = \bigcup_{\alpha \in \kappa} x_\alpha \mathcal{O}$.

**Claim 2.50.1.** For each $\alpha \in \kappa$, there is an element $a_\alpha \in K$ such that for all $x \in x_\alpha \mathcal{O}$ we have $h(x) = a_\alpha x + N$.

For each $\alpha$ choose an element $y_\alpha$ such that $h(x_\alpha) = y_\alpha + N$ and let $a_\alpha = x_\alpha^{-1} y_\alpha$. Fix an element $x \in x_\alpha \mathcal{O}$, then:

$$h(x) = h(x_\alpha (x_\alpha^{-1} x)) = (x_\alpha^{-1} x) \cdot h(x_\alpha) = (x_\alpha^{-1} x) \cdot (a_\alpha x_\alpha + N) = a_\alpha x + N.$$

**Claim 2.50.2.** Given $\beta < \alpha < \kappa$, then $a_\alpha - a_\beta \in x_\beta^{-1} N$.

Note that $x_\beta \in x_\beta \mathcal{O} \subseteq x_\alpha \mathcal{O}$, by Claim 2.50.1 we have $h(x_\beta) = a_\alpha x_\beta + N = a_\beta x_\beta + N$, then $(a_\alpha - a_\beta)x_\beta \in N$. Hence, $(a_\alpha - a_\beta) \in x_\beta^{-1} N$.

**Claim 2.50.3.** Without loss of generality, we may assume that for any $\alpha < \kappa$ there is some $\alpha < \alpha' < \kappa$ such that for any $\alpha' \leq \alpha'' < \kappa$ $a_\alpha - a_\alpha'' \notin x_\alpha^{-1} N$. 


Suppose the statement is false. Then there is some \( \alpha \) such that for any \( \alpha < \alpha' \) we can find \( \alpha' \leq \alpha'' \) such that \( a_\alpha - a_{\alpha''} \in x_{\alpha''}^{-1}N \). Define:

\[
h^* : \begin{cases} 
I \to K/N \\
x \to a_\alpha x + N.
\end{cases}
\]

We will show that for any \( x \in I \), \( h(x) = h^*(x) \). Fix an element \( x \in I \), as \( < \gamma_\alpha \mid \alpha \in \kappa > \) is co-initial and decreasing in \( S_I \) we can find an element \( \alpha' > \alpha \) such that \( v(x) > \gamma_{\alpha'} \), so \( x \in x_{\alpha'} \cap \subseteq x_{\alpha''} \cap \). Then

\[
(a_\alpha - a_{\alpha'})x = (a_\alpha - a_{\alpha''})x + (a_{\alpha''} - a_{\alpha'})x \\
\in x_{\alpha'}^{-1}xN \subseteq N \\
\in x_{\alpha''}^{-1}xN \subseteq N
\]

We conclude that \( (a_\alpha - a_{\alpha'})x \in N \). By Claim 2.50.1, we have \( h(x) = a_{\alpha'} x + N \), thus \( h^*(x) = h(x) \) and \( h^* \) witnesses the conclusion of the statement.

By Claim 2.50.3, we can find a function \( g : \kappa \to \kappa \) such that for any \( \alpha \in \kappa \), we have that for all \( \alpha < g(\alpha) \leq \delta < \kappa \),

\[
a_\delta - a_\alpha \not\in x_\delta^{-1}N.
\]

Claim 2.50.4. There is pseudoconvergent sub-sequence \( < b_\alpha \mid \alpha \in \lambda > \) of \( < a_\alpha \mid \alpha \in \kappa > \) and a strictly increasing and cofinal function \( f : \lambda \to \kappa \) such that for all \( \alpha \in \lambda \) one has \( b_\alpha = a_{f(\alpha)} \). As \( \kappa \) is regular \( \lambda = \kappa \).

Proof. By transfinite recursion, we build a sequence \( < b_\alpha \mid \alpha < \lambda > \) and a strictly increasing function \( f : \lambda \to \kappa \) satisfying the following conditions.

1. For each \( \alpha < \lambda \), \( b_\alpha = a_{f(\alpha)} \).
2. For each \( \alpha < \eta < \lambda \), \( f(\alpha) < f(\eta) \).
3. For each \( \eta < \alpha < \lambda \), \( g(f(\eta)) \leq f(\alpha) \).
4. The sequence \( < b_\alpha \mid \alpha < \lambda > \) is pseudoconvergent. This is, for each \( \alpha_1 < \alpha_2 < \alpha_3 < \lambda \),

\[
v(a_{\alpha_3} - a_{\alpha_2}) > v(a_{\alpha_2} - a_{\alpha_1}).
\]

Set \( b_0 = a_0 \) and \( f(0) = 0 \). Suppose \( < b_\eta \mid \eta < \mu > \) and \( f \upharpoonright _\mu \) have been defined. Let \( \mu^* = \sup\{f(\eta) \mid \eta < \mu\} \), we may assume that \( \mu^* < \kappa \), otherwise we can take \( \mu^* = \lambda \) and we are done. By regularity of \( \kappa \),

\[
\varepsilon = \sup\{g(f(\eta)) \mid \eta < \mu\} < \kappa.
\]

Set \( f(\mu) = \varepsilon \) and \( b_\mu = a_\varepsilon \).

We continue arguing that the four conditions are still satisfied. The first and third condition follow immediately by construction. For (2), note that for any \( \alpha < \mu \), \( f(\alpha) < g(f(\alpha)) \leq \varepsilon = f(\mu) \). It is only left to verify that the sequence is pseudoconvergent, so fix \( \alpha_1 < \alpha_2 < \mu \). Combining Claim 2.50.2 with the fact that \( f(\alpha_2) < f(\mu) \), we have that

\[
b_{\alpha_2} - b_\mu = a_{f(\alpha_2)} - a_{f(\mu)} \in x_{f(\alpha_2)}^{-1}N.
\]
Because $\alpha_1 < \alpha_2$, by the induction hypothesis, $g(f(\alpha_1)) \leq f(\alpha_2)$. Hence,

$$b_{\alpha_1} - b_{\alpha_2} = a_{f(\alpha_1)} - a_{f(\alpha_2)} \notin x^{-1}_{f(\alpha_2)} N.$$  

We conclude that $v(b_{\alpha} - b_{\alpha_2}) > v(b_{\alpha_2} - b_{\alpha_1})$, as required. \hfill \Box

As $K$ is maximal there is some $\alpha \in K$ that is a pseudolimit of $< b_{\alpha} \mid \alpha \in \lambda >$. We aim to prove that $h(x) = ax + N$ for $x \in I$. Fix an element $x \in I$. We can find some $\alpha \in \lambda$ such that $x \in x_{f(\alpha)} \Theta \subseteq I$. By Claim 2.50.1, $h(x) = a_{f(\alpha)} x + N$, hence it is sufficient to prove that $(a - a_{f(\alpha)}) x \in N$. Because $x \in x_{f(\alpha)} \Theta$ it is enough to show that

$$(a - a_{f(\alpha)}) = (a - b_{\alpha}) \in x_{-1} f(\alpha) N.$$  

Let $\alpha < \beta < \kappa$, by Claim 2.50.2 $(b_{\beta} - b_{\alpha}) = (a_{f(\beta)} - a_{f(\alpha)}) \in x_{f(\alpha)}^{-1} N$. Also, $v(a - a_{f(\alpha)}) = v(a_{f(\beta)} - a_{f(\alpha)})$ thus $(a - a_{f(\alpha)}) = u(a_{f(\beta)} - a_{f(\alpha)})$ for some $u \in \Theta^x$, thus $(a - a_{f(\alpha)}) \in x_{f(\alpha)}^{-1} N$, as desired. \hfill \Box

## 2.5 Valued vector spaces

We introduce valued vector spaces and some facts that will be required throughout this paper. An avid and curious reader can consult [2, section 2.3] for a more exhaustive presentation. Throughout this section, we fix $(K, \Gamma, v)$ a valued field and $V$ a $K$-vector space.

**Definition 2.51.** A tuple $(V, \Gamma(V), val, +)$ is a valued vector space structure if:

1. $\Gamma(V)$ is a linear order,
2. there is an action $+: \Gamma(K) \times \Gamma(V) \to \Gamma(V)$ that is strictly order preserving in each coordinate,
3. $val: V \to \Gamma(V)$ is a map such that for all $v, w \in V$ and $\alpha \in K$ we have:
   - $val(v + w) \geq \min\{val(w), val(v)\}$,
   - $val(\alpha v) = \alpha \cdot val(v)$.

The following fact is [22, Remark 1.2].

**Fact 2.52.** Let $V$ be a finite-dimensional valued vector space over $K$, then the action of $\Gamma(K)$ on $\Gamma(V)$ has finitely many orbits. Furthermore, $|\Gamma(V)/\Gamma(K)| \leq \dim_{\mathbb{K}}(V)$.

**Definition 2.53.** Let $(V, \Gamma(V), val, +)$ be a valued vector space.

1. Let $\alpha \in V$ and $\gamma \in \Gamma(V)$. A ball in $V$ is a set of the form:

$$\text{Ball}_\gamma(a) = \{x \in V \mid val(x - a) \geq \gamma\} \text{ or } Ball_\alpha(a) = \{x \in V \mid val(x - a) > \gamma\}.$$  

2. We say that $(V, \Gamma(V), val, +)$ is maximal if every nested family of balls in $V$ has nonempty intersection.

**Definition 2.54.** Let $(V, \Gamma(V), val, +)$ be a valued vector space and let $W$ be a subspace of $V$. Then $(W, \Gamma(W), val, +)$ is also a valued vector space, where $\Gamma(W) = \{val(w) \mid w \in W\}$. 

We say that:

1. **W is maximal in V** if every family of nested balls

   \[ \{ \text{Ball}_\alpha(x_\alpha) \mid \alpha \in S \} \text{, where } S \subseteq \Gamma(W) \text{ and for each } \alpha \in S \ x_\alpha \in W \]

   that has nonempty intersection in \( V \) has nonempty intersection in \( W \);

2. **\( W \leq V \) has the optimal approximation property** if for any \( v \in V \setminus W \) the set \( \{ \text{val}(v - w) \mid w \in W \} \) attains a maximum.

The following is a folklore fact.

**Fact 2.55.** Let \((V, \Gamma(V), \text{val}, +)\) be a valued vector space, and \(W\) a subspace of \(V\) the following statements are equivalent.

1. **\( W \) is maximal in \( V \).**
2. **\( W \) has the optimal approximation property in \( V \).**

Additionally, if \( W \) is maximal then it is maximal in \( V \).

We conclude this subsection with the definition of separated basis.

**Definition 2.56.** Let \((V, \Gamma(V), \text{val}, +)\) be a valued vector space. Assume that \( V \) is a \( K \)-vector space of dimension \( n \). A basis \( \{v_1, \ldots, v_n\} \subseteq V \) is a separated basis if for any \( \alpha_1, \ldots, \alpha_n \in K \) we have that:

\[
\text{val} \left( \sum_{i \leq n} \alpha_i v_i \right) = \min \{ \text{val}(\alpha_i v_i) \mid i \leq n \}.
\]

### 3 | DEFINABLE MODULES

In this section, we study definable \( \mathcal{O} \)-submodules in henselian valued fields of equicharacteristic zero.

**Corollary 3.1.** Let \((F, v)\) be a henselian valued field of equicharacteristic zero and \( N \) be a definable \( \mathcal{O} \)-submodule of \( F^n \). Then \( N \) is definably isomorphic to a direct sum of copies of \( F \) or fractional ideals of \( \mathcal{O} \). Moreover, if \( N \cong \bigoplus_{i<n} I_i \) there is some upper triangular basis \( \{a_1, \ldots, a_n\} \) of \( F^n \) such that \( [a_1, \ldots, a_n] \) is a representation matrix of \( N \).

**Proof.** By Fact 2.45, we can find \( F' \) an elementary extension of \( F \) that is maximal, so we can apply Theorem 2.49. As the statement that we are trying to show is first-order expressible, it must hold as well in \( F \). ⊠

**Corollary 3.2.** Let \((F, v)\) be a henselian valued field of equicharacteristic zero and let \( N, M \subseteq F \) be a definable \( \mathcal{O} \)-submodules. Then for any definable \( \mathcal{O} \)-homomorphism \( h : M \to F/N \). Then there is some \( b \in F \) such that for any \( y \in M \) one has \( h(y) = by + N \).

**Proof.** By Fact 2.45, we can find an elementary extension \( F < F' \) that is maximal. The statement follows by applying Proposition 2.50, because it is first-order expressible. ⊠
3.1  Definable modules in valued fields of equicharacteristic zero with residue field algebraically closed and value group with bounded regular rank

Let \((K, v)\) be a henselian valued field of equicharacteristic zero with residue field algebraically closed and value group of bounded regular rank. Let \(\mathcal{O}\) be its valuation ring and \(T\) be the complete \(\mathcal{L}\)-first-order theory of \((K, v)\). In this section, we study the definable \(\mathcal{O}\)-modules and torsors. Let \(\mathcal{F}\) be the complete family of \(\mathcal{O}\)-submodules of \(K\) described in Fact 2.37. From now on, we fix 
\[I = \mathcal{F} \setminus \{0, K\}.\]

Remark 3.3. If \(K \models T\), then \(N \cong \bigoplus_{i \leq n} I_i\), where each \(I_i \in I \cup \{0, K\} = \mathcal{F}\). This follows because \(\mathcal{F}\) is a complete family of \(\mathcal{O}\)-modules of \(K\), that contains \(K\) and 0.

Definition 3.4. Let \(K \models T\). A definable torsor \(U\) is a coset in \(K^n\) of a definable \(\mathcal{O}\)-submodule of \(K^n\), if \(n = 1\) we say that \(U\) is a 1-torsor. Let \(U\) be a definable 1-torsor, we say that \(U\) is:

1. closed if it is a translate of a submodule of \(K\) of the form \(a\mathcal{O}\);
2. it is open if it is either \(K\) or a translate of a submodule of the form \(aI\) for some \(a \in K\), where \(I \in I\) and \(I \neq \mathcal{O}\).

Definition 3.5. Let \((I_1, \ldots, I_n) \in I^n\) be a fixed tuple.

1. An \(\mathcal{O}\)-module \(M \subseteq K^n\) is of type \((I_1, \ldots, I_n)\) if there is a matrix \(A \in B_n(K)\), \(A \cdot M = I_1 \oplus I_2 \oplus \cdots \oplus I_n\).
2. An \(\mathcal{O}\)-module \(M \subseteq K^n\) of type \((\mathcal{O}, \ldots, \mathcal{O})\) is said to be an \(\mathcal{O}\)-lattice of rank \(n\).
3. A torsor \(Z\) is of type \((I_1, \ldots, I_n)\), if \(Z = d + M\) where \(M \subseteq K^n\) is an \(\mathcal{O}\)-submodule of \(K^n\) of type \((I_1, \ldots, I_n)\).

Proposition 3.6. Let \(N \subseteq K^n\) be a \(\mathcal{O}\)-module of type \((I_1, \ldots, I_n)\) and \(\tilde{d} \in K^n\) such that \(Z = \tilde{d} + N\). Let \(\tilde{b} = \begin{bmatrix} d \\ 1 \end{bmatrix} \in K^{n+1}\). Define the \(\mathcal{O}\)-submodule of \(K^{n+1}\)

\[L_{\tilde{d}} := N_2 + \tilde{b} \mathcal{O} = \left\{ \begin{bmatrix} n + \tilde{d}r \\ r \end{bmatrix} | r \in \mathcal{O}, n \in N \right\},\]

where \(N_2 = N \times \{0\}\). \(L_{\tilde{d}}\) is a \(\mathcal{O}\)-submodule of \(K^{n+1}\) of type \((I_1, \ldots, I_n, \mathcal{O})\). For any \(\tilde{d}, \tilde{d}' \in Z\) one has \(L_{\tilde{d}} = L_{\tilde{d}'}\), so we simply denote \(L = L_{\tilde{d}}\). The codes \(^rZ^n\) and \(^rL^n\) are interdefinable.

Proof. By a standard computation, one can verify that the definition of \(L_{\tilde{d}}\) is independent of the choice of \(\tilde{d}\), that is, if \(\tilde{d} - \tilde{d}' \in N\) then \(L_{\tilde{d}} = L_{\tilde{d}'}\). Thus, we simply write \(L\). We aim to show that \(L\) and \(Z\) are interdefinable. It is clear that \(^rL^n \in dcl^{eq}(^rZ^n)\), while \(^rZ^n \in dcl^{eq}(^rL^n)\) because \(Z = \pi_{<n}(L \cap (K^n \times \{1\}))\) where \(\pi_{<n} : K^{n+1} \to K^n\) is the projection into the first \(n\)-coordinates. \(\square\)

3.2  Definable 1-\(\mathcal{O}\)-modules

In this subsection, we study the quotient modules of 1-dimensional modules.
Notation 3.7. Let $M \subseteq K$ be a definable $\mathcal{O}$-module. We denote by $S_M := \{v(x) \mid x \in M\}$ the end-segment induced by $M$. We recall as well that we write $I$ to denote the complete family of $\mathcal{O}$-submodules of $K$ previously fixed.

Definition 3.8. A definable 1-$\mathcal{O}$-module is an $\mathcal{O}$-module that is definably isomorphic to a quotient of a definable $\mathcal{O}$-submodule of $K$ by another, that is, something of the form $aI/bJ$ where $a, b \in K$ and $I, J \in I \cup \{0, K\}$.

The following operation between $\mathcal{O}$-modules will be particularly useful in our setting.

Definition 3.9. Let $N, M$ be $\mathcal{O}$-submodules of $K$, we define the colon module $Col(N : M) = \{x \in K \mid xM \subseteq N\}$. It is a well-known fact from Commutative Algebra that $Col(N : M)$ is also an $\mathcal{O}$-module.

Lemma 3.10. Let $K \models T$. Let $A$ be a 1-definable $\mathcal{O}$-module. Suppose that $A = A_1/A_2$, where $A_2 \leq A_1$ are $\mathcal{O}$-submodules of $K$. Then the $\mathcal{O}$-module $\text{Hom}_\mathcal{O}(A, A)$ is definably isomorphic to the 1-definable $\mathcal{O}$-module

$$(Col(A_1 : A_1) \cap Col(A_2 : A_2))/Col(A_2 : A_1).$$

Proof. By Fact 2.45, without loss of generality we may assume $K$ to be maximal, because the statement is first-order expressible. Let

$$B = \{f : A_1 \to A_1/A_2 \mid f \text{ is a homomorphism and } A_2 \subseteq \ker(f)\}.$$ 

$B$ is canonically in one-to-one correspondence with $\text{Hom}_\mathcal{O}(A, A)$. By Corollary 3.2, for every homomorphism $f \in B$ there is some $b_f \in K$ satisfying that for any $x \in A_1$, $f(x) = b_f x + A_2$ and we say that $b_f$ is a linear representation of $f$.

Claim 3.10.1. Let $f \in B$. If $b_f$ is a linear representation of $f$, then $b_f \in Col(A_1 : A_1) \cap Col(A_2 : A_2)$.

Proof. First we verify that $b_f \in Col(A_1 : A_1)$. Let $x \in A_1$, by hypothesis $f(x) = b_f x + A_2 \in A_1/A_2$. Then there is some $y \in A_1$ such that $b_f x + A_2 = y + A_2$ and therefore $b_f x - y \in A_2 \subseteq A_1$. Consequently, $b_f x \in y + A_1 = A_1$, and as $x$ is an arbitrary element we conclude that $b_f \in Col(A_1 : A_1)$. We check now that $b_f \in Col(A_2 : A_2)$, and we fix an element $x \in A_2$. By hypothesis, $b_f x + A_2 = A_2$ so $b_f x \in A_2$, and as $x \in A_2$ is an arbitrary element we conclude that $b_f \in Col(A_2 : A_2)$.

Claim 3.10.2. Let $f \in B$ if $b_f, b_f'$ are linear representations of $f$, then $b_f - b_f' \in Col(A_2 : A_1)$

Proof. Let $x \in A_1$, by hypothesis $f(x) = b_f x + A_2 = b_f' x + A_2$, so $(b_f - b_f') x \in A_2$. Because $x$ is arbitrary in $A_1$ we have that $(b_f - b_f') \in Col(A_2 : A_1)$.

We consider the map $\phi : B \to (Col(A_1 : A_1) \cap Col(A_2 : A_2))/Col(A_2 : A_1)$ that sends an $\mathcal{O}$-homomorphism $f$ to the coset $b_f + Col(A_2 : A_1)$. By Claim 3.10.2, such map is well-defined.
By a standard computation $\phi$ is an injective $\mathcal{O}$-homomorphism. To show that $\phi$ is surjective, let

$$b \in \text{Col}(A_1 : A_1) \cap \text{Col}(A_2 : A_2),$$

and consider $f_b : A_1 \to A_1/A_2$, the map that sends the element $x$ to $bx + A_2$. Because $b \in \text{Col}(A_2 : A_2)$, for any $x \in A_2$ we have that $bx \in A_2$ thus $A_2 \subseteq \ker(f_b)$. Consequently, $f_b \in B$ and $\phi(f_b) = b + \text{Col}(A_2 : A_1)$.

**Lemma 3.11.** Let $n \in \mathbb{N}_{>2}$ and $M \subseteq K^n$ be an $\mathcal{O}$-module.

1. Let $\pi^{n-1} : K^n \to K^{n-1}$ be the projection into the first $(n-1)$-coordinates and $B_{n-1} = \pi^{n-1}(M)$. Take $A_1 \subseteq K$ be the $\mathcal{O}$-module such that $\ker(\pi^{n-1}) = M \cap (\{0\}^{n-1} \times K) = (\{0\}^{n-1} \times A_1)$.

2. Let $\pi_n : K^n \to K$ be the projection into the last coordinate and $B_1 = \pi_n(M)$. Let $A_{n-1} \subseteq K^{n-1}$ be the $\mathcal{O}$-module such that $\ker(\pi_n) = M \cap (K^{n-1} \times \{0\}) = (A_{n-1} \times \{0\})$.

Then $A_{n-1} \leq B_{n-1}$ and both lie in $K^{n-1}$, and $A_1 \leq B_1$ and both lie in $K$. The map $\phi : B_{n-1} \to B_1/A_1$ given by $b \mapsto a + A_1$ where $(b, a) \in M$, is a well-defined homomorphism of $\mathcal{O}$-modules whose kernel is $A_{n-1}$. In particular, $B_{n-1}/A_{n-1} \cong B_1/A_1$. Furthermore if $M$ is definable, $\phi$ is also definable.

**Proof.** Let $\bar{m} \in A_{n-1}$, then $(\bar{m}, 0) \in M$ thus $\pi^{n-1}(\bar{m}, 0) = \bar{m} \in B_{n-1}$. We conclude that $A_{n-1}$ is a submodule of $B_{n-1}$. Likewise $A_1 \leq B_1$. For the second part of the statement, it is a straightforward computation to verify that the map $\phi : B_{n-1} \to B_1/A_1$ (defined as in the statement), is a well-defined surjective homomorphism of $\mathcal{O}$-modules whose kernel is $A_{n-1}$. Finally, the definability of $\phi$ follows immediately by the definability of $M$. □

## 4 | THE STABILIZER SORTS

### 4.1 | An abstract criterion to eliminate imaginaries

We start by recalling Hurshovski’s criterion, The following is [18, Lemma 1.17].

**Theorem 4.1.** Let $T$ be a first-order theory with home sort $K$ (meaning that $\mathfrak{M}_{eq} = dcl_{eq}(K)$). Let $\mathcal{G}$ be some collection of sorts. If the following conditions all hold, then $T$ has weak elimination of imaginaries in the sorts $\mathcal{G}$.

1. Density of definable types: for every nonempty definable set $X \subseteq K$ there is an $acl_{eq}(r_X^n)$-definable type in $X$.

2. Coding definable types: every definable type in $K^n$ has a code in $\mathcal{G}$ (possibly infinite). This is, if $p$ is any (global) definable type in $K^n$, then the set $r_p^n$ of codes of the definitions of $p$ is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$.

**Proof.** A very detailed proof can be found in [22, Theorem 6.3]. The first part of the proof shows weak elimination of imaginaries as it is shown that for any imaginary element $e$ we can find a tuple $a \in \mathcal{G}$ such that $e \in dcl_{eq}(a)$ and $a \in acl_{eq}(e)$. □
We start by describing the sorts that are required to be added to apply this criterion and show that any valued field of equicharacteristic zero, with residue field algebraically closed and value group of bounded regular rank admits weak elimination of imaginaries.

**Definition 4.2.** For each \( n \in \mathbb{N} \), let \( \{e_1, ..., e_n\} \) be the standard basis of \( K^n \) and \( (I_1, ..., I_n) \in \mathcal{I}^n \), that is, \( I_i \not\in \{0, K\} \).

1. Let \( C_{(I_1, ..., I_n)} = e_1 I_1 + e_2 I_2 + \cdots + e_n I_n \), we refer to this module as the **canonical** \( \mathcal{O} \)-submodule of \( K^n \) of type \( (I_1, ..., I_n) \).
2. We denote as \( B_n(K) \) the multiplicative group of \( n \times n \)-upper triangular and invertible matrices.
3. We define the subgroup \( Stab(I_1, ..., I_n) = \{A \in B_n(K) \mid AC(I_1, ..., I_n) = C(I_1, ..., I_n)\} \).
4. Let \( \Lambda(I_1, ..., I_n) := \{M \mid M \subseteq K^n \text{ is an } \mathcal{O} \text{-module of type } (I_1, ..., I_n)\} \).
5. Let \( U_n \subseteq (K^n)^n \) be the set of \( n \)-tuples \( (\bar{b}_1, ..., \bar{b}_n) \), such that \( B = [\bar{b}_1, ..., \bar{b}_n] \) is an invertible upper triangular matrix. We define the equivalence relation \( E(I_1, ..., I_n) \) on \( U_n \) as:
   \[
   E(I_1, ..., I_n)(\bar{a}_1, ..., \bar{a}_n; \bar{b}_1, ..., \bar{b}_n) \text{ holds if and only if} \]
   \[
   (\bar{a}_1, ..., \bar{a}_n) \text{ and } (\bar{b}_1, ..., \bar{b}_n) \text{ generate the same } \mathcal{O} \text{-module of type } (I_1, ..., I_n), \]
   that is,
   \[
   \left\{ \sum_{1 \leq i \leq n} x_i \bar{a}_i \mid x_i \in I_i \right\} = \left\{ \sum_{1 \leq i \leq n} x_i \bar{b}_i \mid x_i \in I_i \right\}.
   \]
6. We denote as \( \bar{\rho}(I_1, ..., I_n) \) the canonical projection map:
   \[
   \bar{\rho}(I_1, ..., I_n) : \left\{ \begin{array}{c} U_n \\ (\bar{a}_1, ..., \bar{a}_n) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} U_n \big/ E(I_1, ..., I_n) \\ [(\bar{a}_1, ..., \bar{a}_n)]_{E(I_1, ..., I_n)} \end{array} \right\}.
   \]

**Remark 4.3.**

(1) Given any definable \( \mathcal{O} \)-submodule \( M \subseteq K^n \), it is of some type \( (I_1, ..., I_n) \) as long as \( M \) is not contained in a proper \( K \)-subvector space and does not contain any nontrivial \( K \)-subvector space of \( K^n \) (as \( 0, K \not\in \mathcal{I} \)).

(2) The set \( \{^r M^n \mid M \in \Lambda(I_1, ..., I_n)\} \) can be canonically identified with \( B_n(K)/Stab(I_1, ..., I_n) \). By Corollary 3.1 given any \( \mathcal{O} \)-module \( M \) of type \( (I_1, ..., I_n) \) we can find an upper triangular basis \( \{\bar{a}_1, ..., \bar{a}_n\} \) of \( K^n \) such that \( [a_1, ..., a_n] \) is a matrix representation of \( M \). The code \( ^r M^n \) is interdefinable with the coset \( [a_1, ..., a_n]Stab(I_1, ..., I_n) \).

(3) Fix some \( n \in \mathbb{N}_{\geq 2} \) and let \( (I_1, ..., I_n) \) be a fixed tuple. The sort \( B_n(K)/Stab(I_1, ..., I_n) \) is in definable bijection with the equivalence classes of \( U_n \big/ E(I_1, ..., I_n) \). In fact, we can consider the \( \emptyset \)-definable map:
   \[
   f : \left\{ \begin{array}{c} U_n \big/ E(I_1, ..., I_n) \\ [(\bar{a}_1, ..., \bar{a}_n)]_{E(I_1, ..., I_n)} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} B_n(K)/Stab(I_1, ..., I_n) \\ [\bar{a}_1, ..., \bar{a}_n]Stab(I_1, ..., I_n) \end{array} \right\}.
   \]
   We denote as \( \rho(I_1, ..., I_n) : U_n \rightarrow B_n(K)/Stab(I_1, ..., I_n) \) the composition maps \( \rho(I_1, ..., I_n) = f \circ \bar{\rho}(I_1, ..., I_n) \).
**Definition 4.4** (The stabilizer sorts). We consider the language $\mathcal{L}_G$ extending the three sorted language $\mathcal{L}$ (defined in Subsection 2.3.1), where:

1. we equipped the value group with the multi-sorted language $\mathcal{L}_{bq}$ introduced in Subsection 2.2.1;
2. for each $n \in \mathbb{N}$, we consider the parameterized family of sorts $B_n(K)/Stab(I_1,\ldots,I_n)$ and maps
   \[ \rho(I_1,\ldots,I_n) : U_n \to B_n(K)/Stab(I_1,\ldots,I_n), \]
   where $(I_1,\ldots,I_n) \in I^n$.

We refer to the sorts in the language $\mathcal{L}_G$ as the **stabilizer sorts**. We denote as $\mathcal{G}$ their union, that is,

\[
K \cup k \cup \Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\} \cup \{\Gamma/(\Delta + n\Gamma) \mid \Delta \in RJ(\Gamma), n \in \mathbb{N}_{\geq 2}\} \\
\cup \{B_n(K)/Stab(I_1,\ldots,I_n) \mid n \in \mathbb{N}, (I_1,\ldots,I_n) \in I^n\}.
\]

**Remark 4.5.** The geometric sorts for the case of $ACVF$ are a particular instance of the stabilizer sorts. Let $S_n$ denotes the set of $\mathcal{O}$-lattices of $K^n$ of rank $n$, these are simply the $\mathcal{O}$-modules of type $(\mathcal{O},\ldots,\mathcal{O})$. For each $\Lambda \in S_n$, let $res(\Lambda) = \Lambda \otimes_{\mathcal{O}} k = \Lambda/\mathcal{M}\Lambda$, which is a $k$-vector space of dimension $n$.

Let $T_n = \bigcup_{\Lambda \in S_n} res(\Lambda) = \{(\Lambda, x) \mid \Lambda \in S_n, x \in res(\Lambda)\}$. Each of these torsors is considered in the stabilizer sorts as the code of an $\mathcal{O}$ module of type $(\mathcal{M},\ldots,\mathcal{M},\mathcal{O})$, because any torsor of the form $a + \mathcal{M}\Lambda$ for some $\Lambda \in S_n$ can be identified with an $\mathcal{O}$-module of type $(\mathcal{M},\ldots,\mathcal{M},\mathcal{O})$ (see Proposition 3.6).

### 4.2 An explicit description of the stabilizer sorts

In this subsection, we state an explicit description of the subgroups $Stab(I_1,\ldots,I_n)$.

**Notation 4.6.** For each $\Delta \in RJ(\Gamma)$, we denote as $\mathcal{O}_\Delta$ the valuation ring of $K$ of the coarsened valuation $v_\Delta : K^\times \to \Gamma/\Delta$ induced by $\Delta$.

**Fact 4.7.** Let $I \in I$ and let $S_I = \{v(x) \mid x \in I\}$. Then $Stab(I) = \mathcal{O}_{S_I}^\times = \{x \in K \mid v(x) \in \Delta_{S_I}\}$.

**Proof.** This is an immediate consequence of Fact 2.24. \qed

**Proposition 4.8.** Let $n \in \mathbb{N}$, and $(I_1,\ldots,I_n) \in I^n$. Then

\[
Stab(I_1,\ldots,I_n) = \{(a_{i,j})_{1 \leq i,j \leq n} \in B_n(K) \mid a_{ii} \in \mathcal{O}_{\Delta_{S_{I_i}}}^\times \land a_{i,j} \in Col(I_i,I_j) \text{ for each } 1 \leq i < j \leq n \}.
\]

**Proof.** This is a straightforward computation and it is left to the reader. \qed
5 | WEAK ELIMINATION OF IMAGINARIES FOR HENSELIAN VALUED FIELD WITH VALUE GROUP OF BOUNDED REGULAR RANK

Let \((K, v)\) be a henselian valued field of equicharacteristic zero, with residue field algebraically closed and value group of bounded regular rank. Let \(T\) be its complete \(\mathcal{L}_v\)-first-order theory and \(\mathcal{M}\) its monster model. In this section, we show that both conditions required by Hrushovski’s criterion to obtain weak elimination of imaginaries down to the stabilizer sorts hold.

5.1 | Density of definable types

In this subsection, we prove density of definable types for 1-definable sets \(X \subseteq K(\mathcal{M})\) (i.e., \(X\) is a definable subset in 1-variable of the main sort). There are two ways to tackle this problem. One can either use the quantifier elimination (see Corollary 2.33) and obtain a canonical decomposition of \(X\) into nice sets \(T_i \in \text{acl}^q(\Gamma X^\sim)\). Then one needs to build a global type \(p(x) \vdash x \in T_i\) that is \(\text{acl}^q(\Gamma T_i^\sim)\)-definable. This approach was successfully achieved for the case of ACVF, and it was initiated by Holly in [16], who proves a canonical decomposition for definable sets in ACVF and RCVF. Her work essentially gives a way to code one-definable sets in the main field down to the geometric sorts. It is worth pointing out, that finding a canonical decomposition is often a detailed technical work. Instead of following this strategy, we follow a different approach that exploits the power of generic types (which are definable partial types) combined with the very nice quantifier elimination that we have in this setting (Corollary 2.33). This approach is very close to ideas of Rideau-Kikuchi and Hils, for example, in [15, section 3] and [28, section 9].

Definition 5.1. Let \(U \subseteq K(\mathcal{M})\) be a definable 1-torsor, let

\[
\Sigma^\text{gen}_U(x) = \{x \in U\} \cup \{x \notin B \mid B \subsetneq U \mid B\text{ is a proper subtorsof }U\}.
\]

This is a \(\Gamma U^\sim\)-definable partial type, and we refer to it as the generic type of the torsor \(U\).

Proposition 5.2. Let \(U \subseteq K(\mathcal{M})\) be a definable closed 1-torsor. Then there is a unique complete global type \(p(x)\) extending \(\Sigma^\text{gen}_U(x)\), that is, \(\Sigma^\text{gen}_U(x) \subseteq p(x)\). Moreover, \(p(x)\) is \(\Gamma U^\sim\)-definable.

Proof. Let \(a \in U\) and \(Y_a = \{v(x - a) \mid x \in U\} \subseteq \Gamma\). As \(U\) is a closed \(\Theta\)-module then \(Y_a\) has a minimum element \(\gamma\). For any other element \(b \in U\), we have that \(\gamma = \min(Y_a) = \min(Y_b)\), thus \(\gamma \in \text{dcl}^q(\Gamma U^\sim)\).

By quantifier elimination (see Corollary 2.33), it is sufficient to show that \(\Sigma^\text{gen}_U(x)\) determines also the congruence and coset formulæ. Let \(c\) be a realization of \(\Sigma^\text{gen}_U(x)\). For any \(a \in U(\mathcal{M})\) we have that \(v(c - a) = \gamma\). Let \(p(x) = \text{tp}(c/\mathcal{M}), A \in R\mathcal{J}(\Gamma), \epsilon \in \mathbb{N}\) and \(\beta \in \Gamma\). If \(a \in U(\mathcal{M})\) then:

\[
\models v_\Delta(c - a) = \rho_\Delta(\beta) + k^\Delta \text{ if and only if } \models \phi^k_\Delta(\beta) := \gamma = \rho_\Delta(\beta) + k^\Delta, \text{ and}
\]

\[
\models v_\Delta(c - a) - \rho_\Delta(\beta) + k^\Delta \in \epsilon(\Gamma/\Delta) \text{ if and only if } \models \psi^k_\Delta(\beta) := \gamma - \rho_\Delta(\beta) + k^\Delta \in \epsilon(\Gamma/\Delta).
\]

We observe that \(\psi^k_\Delta(\beta)\) and \(\phi^k_\Delta(\beta)\) are \(\mathcal{L}(\text{dcl}^q(\Gamma U^\sim))\)-formulae, and their definition is completely independent from the choice of \(c\).
If \( a \notin U(\mathfrak{M}) \), then for any \( b \in U(\mathfrak{M}) \) we have that \( v(c - a) = v(b - a) \). Therefore,

\[
\models v_\Delta(c - a) = \rho_\Delta(\beta) + k^\Delta \text{ if and only if }
\]

\[
\models v_\Delta^k(a, \beta) := \exists b \in U(v_\Delta(b - a) = \rho_\Delta(\beta) + k^\Delta ), \text{ and }
\]

\[
\models v_\Delta(c - a) - \rho_\Delta(\beta) + k^\Delta \in \mathcal{E}(\Gamma/\Delta) \text{ if and only if }
\]

\[
\models \eta_\Delta^k(a, \beta) := \exists b \in U(v_\Delta(b - a) - \rho_\Delta(\beta) + k^\Delta \in \mathcal{E}(\Gamma/\Delta)).
\]

Both formulae \( \varepsilon_\Delta^k(a, \beta) \) and \( \eta_\Delta^k(a, \beta) \) are \( \mathcal{L}(U^\gamma) \)-definable and completely independent from the choice of \( c \).

We conclude that \( p(x) \) is a \( rU^\gamma \)-definable type. Furthermore, for any possible realization \( c \models \Sigma g \) \( e_n U(x) \) we obtain the same scheme of definition. Hence, there is a unique extension \( p(x) \) of \( \Sigma g_{\mathfrak{M}}(x) \).

\[\Box\]

**Definition 5.3.** Let \( X \subseteq K(\mathfrak{M}) \). We say that \( X \) is bounded if there is some \( a \in \mathfrak{M} \), such that \( X \subseteq a^\circ \). Otherwise we say that it is unbounded.

The following proposition is a useful fact that will simplify our proof of density of definable types. It follows from hensel-minimality, a tame geometric notion developed by Cluckers, Halupczok, and Rideau-Kikuchi in [7] for henselian valued fields of equicharacteristic zero. However, in our context one can prove the result by hand and we include details to keep the document self-contained.

**Proposition 5.4.** Let \( X \subseteq K(\mathfrak{M}) \) be a bounded definable set. Then there is a finite set \( D \subseteq K(\mathfrak{M}) \) such that for any torsor \( U \) such that \( a \not\in U \) for all \( a \in D \), either \( U \cap X = \emptyset \) or \( U \subseteq X \).

**Proof.** We first show the result for nice sets.

**Claim 5.4.1.** Let \( T = S \cap C \) be a bounded nice set where \( S \) is a generalized Swiss cheese and \( C \) is a set defined by a finite congruence restriction. Then there is a finite set \( D \subseteq K(\mathfrak{M}) \) such that for any torsor \( U \) such that for all \( a \in D \) \( a \not\in U \), either \( U \cap T = \emptyset \) or \( U \subseteq T \).

**Proof.** If \( T \) is finite the statement follows immediately, so we assume that \( T \) is infinite. As \( T \) is a bounded nice set \( T = S \cap C \), we may assume that \( S = A \setminus (B_{i_1} \cup \cdots \cup B_{i_n}) \) and \( A \neq K \). Without loss of generality, we suppose that whenever \( l \neq k \) then \( B_{i_k} \cap B_{i_l} = \emptyset \). For each \( 1 \leq l \leq n \) take \( d_l \in B_{i_l}(\mathfrak{M}) \) and \( d_0 \in A \setminus (B_{i_1} \cup \cdots \cup B_{i_n}) \). Let \( D_0 = \{d_l \mid 0 \leq l \leq n\} \). We say that an element \( e \in K(\mathfrak{M}) \) is mentioned in \( C \) if a basic formula of the form

\[
zv_\Delta(x - e) - \rho_\Delta(\beta) - k^\Delta \in (\Gamma/\Delta)/(n\Gamma/\Delta) \text{ or }
\]

\[
zv_\Delta(x - e) - \rho_\Delta(\beta) - k^\Delta \notin (\Gamma/\Delta)/(n\Gamma/\Delta)
\]

is one of the formulae that occur in the finite congruence restriction that defines \( C \). Let

\[
E = \{e \in K(\mathfrak{M}) \mid e \text{ is mentioned in } C\}.
\]

And let \( D = D_0 \cup E \). Let \( U \) be a definable torsor such that for all \( d \in D \), \( d \not\in U \).
We may assume that $U \subseteq A$, because otherwise $U \cap T = \emptyset$.

For each $l \leq n$ because $d_l \in B_i \setminus U$ then either $U \not\subseteq B_i$ and hence $U \cap T_i = \emptyset$, or $U \cap B_i = \emptyset$.

Then we may assume that for each $l \leq n$, $U \cap B_i = \emptyset$. Let $e \in E$, as $e \not\in U$ then there is some $\delta_e \in \Gamma$ such that for any $x, x' \in U$ one has that $v(x - e) = v(x' - e) = \delta_e$.

Then if $\{\delta_e \mid e \in E\}$ satisfy the required congruence restrictions imposed in $C$, then $U \subseteq T$, and hence $U \cap T_i = \emptyset$, or $U \cap B_i = \emptyset$.

Then we may assume that for each $l \leq n$, $U \cap B_i = \emptyset$. Let $e \in \mathcal{E}$, as $e \not\in U$ then there is some $\delta_e \in \Gamma$ such that for any $x, x' \in U$ one has that $v(x - e) = v(x' - e) = \delta_e$.

Then if $\{\delta_e \mid e \in \mathcal{E}\}$ satisfy the required congruence restrictions imposed in $C$, then $U \subseteq T$, and $U \cap T = \emptyset$, as required.

We continue arguing for any definable bounded set $X$. By quantifier elimination $X$ is a finite union of nice sets, that is, $X = \bigcup_{i \leq n} T_i$ where $T_i = S_i \cap C_i$, $S_i$ is a generalized Swiss cheese and $C_i$ is a definable set defined by a finite congruence restriction. By Claim 5.4.1 for each $i \leq n$ there is a finite set $D_i \subseteq K(\mathfrak{M})$ such that for any torsor $U$ such that for all $a \in D_i a \not\in U$ one has $U \cap T_i = \emptyset$ or $U \subseteq T_i$.

Let $D = \bigcup_{i \leq n} D_i$ and let $U$ be a torsor that does not contain any point $a \in D$, then if $U \cap X \neq \emptyset$ there is some $i \leq n$ such that $U \cap T_i \neq \emptyset$ so $U \subseteq T_i \subseteq X$.

\begin{theorem}
For every nonempty definable set $X \subseteq K(\mathfrak{M})$, there is an acl\textsuperscript{eq}($\Gamma X$)-definable global type $p(x) \vdash x \in X$.
\end{theorem}

\begin{proof}
Let $X \subseteq K(\mathfrak{M})$ be a 1-definable set. We first argue that without loss of generality $X$ is bounded and $X \subseteq \Theta$. We proceed by following cases.

- If $X' = X \cap \Theta$ is nonempty, then we may replace $X$ by $X' \subseteq X$ and $\Gamma X' \subseteq \dcf\textsuperscript{eq}($\Gamma X$). Then given a global type $p(x) \vdash x \in X' \subseteq X$ that is acl\textsuperscript{eq}($\Gamma X$) definable is also acl\textsuperscript{eq}($\Gamma X$)-definable.

- If $X \cap \Theta = \emptyset$, then we may replace $X$ by $X'' = \{a^{-1} \mid a \in X\} \subseteq \Theta$, and note that $\Gamma X'' \subseteq \dcf\textsuperscript{eq}($\Gamma X$). Given an acl\textsuperscript{eq}($\Gamma X''$)-definable global type $q(x) \vdash x \in X''$ and $a \models q(x)$, one can take $\tp(a^{-1}/\mathfrak{M}) \vdash x \in X$ that is acl\textsuperscript{eq}($\Gamma X''$)-definable, so it is acl\textsuperscript{eq}($\Gamma X$)-definable.

Therefore, we assume that $X$ is bounded and $X \subseteq \Theta$.

\begin{claim}
There is a proper 1-torsor $U$ such that $\Gamma U \subseteq \acl\textsuperscript{eq}($\Gamma X$)$ and the partial type:

$\Sigma_U^{\text{gen}}(x) \cup \{x \in X\}$ is consistent.
\end{claim}

\begin{proof}
Let $B$ be the family of balls $B$ (closed or open) such that $B \cap X \neq \emptyset$. We say that $B_1 \sim B_2$ if and only if $B_1 \cap X = B_2 \cap X$. This is a $\Gamma X$-definable equivalence relation on $B$.

Let $\pi : B \to B/\sim$ be the natural $\Gamma X$-definable map sending a ball to its class $[B]_\sim$. For each class $\mu \in B/\sim$ the set $U_\mu = \bigcap_{B \in B, \pi(B) = \mu} B$ is a $\mu$-definable 1-torsor. Moreover, for any $B \in B$, $B \cap X = U_\mu \cap X$ if and only if $\pi(B) = \mu$. In particular, if $B \subseteq U_\mu$, then $\pi(B) = \mu$.

$B/\sim$ admits a partial $\Gamma X$-definable order, defined as:

$\mu_1 \triangleleft \mu_2$ if and only $B_1 \cap X \subset B_2 \cap X$ where $\pi(B_1) = \mu_1$ and $\pi(B_2) = \mu_2$.

There is a maximal element $\mu_0$ in $(B/\sim, \triangleleft)$. This maximal element is obtained by taking the class of a ball $B_0$ such that $B_0 \cap X = X$. Because $X$ is bounded, then such ball $B_0$ exists. Note that $\mu_0 \in \acl\textsuperscript{eq}($\Gamma X$)$.
For each \( \mu \in B/\sim \), we write \( P(\mu) \) to denote the set of immediate predecessors of \( \mu \) (if they exist). This is

\[
P(\mu) := \{ \beta \in B/\sim \mid \beta \vartriangleleft \mu \text{ and } \neg \exists z (\beta \vartriangleleft z \vartriangleleft \mu) \}.
\]

If \( \Sigma^\text{gen}_{U_\mu}(x) \cup \{ x \in X \} \) is inconsistent then \( P(\mu) \) is finite and has size at least 2.

Indeed, if \( \Sigma^\text{gen}_{U_\mu}(x) \cup \{ x \in X \} \) is inconsistent, by compactness there are finitely many proper subtorsors of \( U_\mu, V_1, \ldots, V_l \) such that:

\[
U_\mu \cap X \subseteq \bigcup_{i \leq l} V_i.
\]

Without loss of generality, we may assume that such subtorsors are pairwise disjoint and for each \( i \leq l \) we have that \( V_i \cap X \neq \emptyset \). (If the intersection with \( X \) is empty one can simply remove them from the list, and if \( V_i \subseteq V_j \) we can reduce the set to the maximal subtorsors, that is, we can remove \( V_j \) from the list.)

As every two disjoint subtorsors of \( U_\mu \) can be separated by disjoint balls, we can find a finite set of pairwise disjoint balls \( B_1, \ldots, B_k \), such that for any \( i \leq k \), one has \( B_i \subseteq U_\mu, B_i \cap X \neq \emptyset \) and

\[
U_\mu \cap X \subseteq \bigcup_{i \leq k} B_i.
\]

Let \( \beta_i = \pi(B_i) \vartriangleleft \mu \). Then \( P(\mu) = \{ \beta_i \mid i \leq k \} \subseteq \text{acl}^\text{eq}(\Gamma X^\gamma, \mu) \).

We now start looking for a 1-torsor \( U \in \text{acl}^\text{eq}(\Gamma X^\gamma) \) such that \( \Sigma^\text{gen}_{U}(x) \cup \{ x \in X \} \) is consistent.

Let \( \mu_0 \in \text{acl}^\text{eq}(\Gamma X^\gamma) \) be the maximal element of \( (B/\sim, \vartriangleleft) \), if \( \Sigma^\text{gen}_{U_{\mu_0}}(x) \cup \{ x \in X \} \) is consistent, the torsor \( U_{\mu_0} \) satisfies the required conditions. We may assume that \( \Sigma^\text{gen}_{U_{\mu_0}}(x) \cup \{ x \in X \} \) is inconsistent, thus it has finitely many predecessors \( P(\mu_0) \subseteq \text{acl}^\text{eq}(\Gamma X^\gamma) \). For each \( \beta \in P(\mu_0) \) exactly one of the following cases hold:

1. \( \Sigma^\text{gen}_{U_{\beta}}(x) \cup \{ x \in X \} \) is consistent, then the torsor \( U_{\beta} \) satisfies the required conditions of the claim; or
2. \( \Sigma^\text{gen}_{U_{\beta}}(x) \cup \{ x \in X \} \) is inconsistent, and \( \beta \) has finitely many predecessors \( P(\beta) \subseteq \text{acl}^\text{eq}(\Gamma X^\gamma, \beta) \subseteq \text{acl}^\text{eq}(\Gamma X^\gamma) \).

By iterating this process for each of the predecessors, we build a discrete tree \( T \subseteq B/\sim \) of finite ramification.
By Proposition 5.4, there is a finite set $D \subseteq K(\mathfrak{M})$ such that for any torsor $U$ such that for all $a \in D$, $a \not\in U$, one has $U \cap X = \emptyset$ or $U \subseteq X$. If the tree $T$ is infinite, there must be some $\mu \in T$ such that $U_\mu$ does not contain any point in $D$. Then $U_\mu \subseteq X$, so $U_\mu = U_\mu \cap X$ and $\Sigma_{U_\mu}^{gen}(x) \cup \{x \in X\}$ must be consistent. ☐

**Claim 5.5.2.** There is a complete acl$^q(\langle X \rangle)$-definable global type $p(x)$ extending $\Sigma_{U}^{gen}(x) \cup \{x \in X\}$, where $U$ is the 1-torsor given by Claim 5.5.1.

**Proof.** Let $U$ be the torsor obtained in Claim 5.5.1. If $U$ is a closed 1-torsor, we let $c$ be a realization of $\Sigma_{U}^{gen}(x) \cup \{x \in X\}$. By Proposition 5.2, the type $p(x) = \tp(c/\mathfrak{M}) \vdash x \in X$ is $\langle X \rangle$-definable. The statement follows as $\langle X \rangle \in \text{acl}^q(\langle X \rangle)$.

We may assume that $U$ is an open torsor. We observe that for any realization $c \models \Sigma_{U}^{gen}(x)$ given $a \neq a' \in U(\mathfrak{M})$ we have $v(c-a) = v(c-a')$.

We recall our notation for the value group. For each convex subgroup $\Delta \in R_J(\Gamma)$, we add a sort $\Gamma/\Delta$ and a projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$. And for each $\Delta \in R_J(\Gamma)$ and $\ell \in \mathbb{N}_{\geq 2}$ we add a sort for the quotient $\Gamma/(\Delta + \ell \Gamma)$ and the projection map $\pi^\ell_{\Delta}$.

Let $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}_{\geq 2}$ be a fixed bijection. We build an increasing sequence of partial consistent types $(\Sigma_k(x) \mid k \in \mathbb{N})$ by induction.

- **Stage 0:** Let $\Sigma_0(x) := \Sigma_{U}^{gen}(x) \cup \{x \in X\}$.

- **Stage $k + 1$:** Let $\phi(k) = (n, \ell)$. At this stage we decide the congruence modulo $\Delta_n + \ell \Gamma$. To simplify the notation we will assume that $\ell \geq 2$, otherwise the argument will follow in a similar manner (instead of working with the quotient group $\Gamma/(\Delta_n + \ell \Gamma)$ and the projection map $\pi^\ell_{\Delta}$ we argue with $\Gamma/\Delta_n$ and the projection map $\rho_{\Delta_n}$).

Let

$$\Lambda_k(x) := \Sigma_k(x) \cup \{v_{\Delta_n}(x-a) - \rho_{\Delta_n}(\beta) \not\in \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma(\mathfrak{M})\}.$$  

If the partial type $\Lambda_k(x)$ is consistent, then we set $\Sigma_{k+1}(x) = \Lambda_k(x)$. Otherwise, let

$$A_k = \{\mu \in \Gamma/(\Delta_n + \ell \Gamma)(\mathfrak{M}) \mid \Sigma_k(x) \cup \{\pi^\ell_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\} \text{ is consistent}\}.$$  

$A_k$ is a finite set. We fix $\mu \in A_k$ and we define:

$$\Sigma_{k+1}(x) := \Sigma_k(x) \cup \{\pi^\ell_{\Delta_n}(v(x-a)) = \mu \mid a \in U(\mathfrak{M})\}.$$  

Let $J = \{k \in \mathbb{N}_{\geq 1} \mid \Lambda_k(x) \text{ is inconsistent}\}$.

**Claim 5.5.3.** For all $k \in \mathbb{N}$ we have that for any automorphism $\sigma \in \text{Aut}(\mathfrak{M} / \text{acl}^q(\langle X \rangle))$, $\sigma(\Sigma_k(x)) = \Sigma_k(x)$ and if $k \in J$ then $\sigma(A_k) = A_k$. In particular, $A_k \subseteq \text{acl}^q(\langle X \rangle)$ for all $k \in J$.

**Proof.** We proceed by induction, for the base case $k = 0$ the statement follows because $\langle U \rangle \in \text{acl}^q(\langle X \rangle)$. We assume that for any $\sigma \in \text{Aut}(\mathfrak{M} / \text{acl}^q(\langle X \rangle))$ we have that $\sigma(\Sigma_k(x)) = \Sigma_k(x)$. We fix $\tau \in \text{Aut}(\mathfrak{M} / \text{acl}^q(\langle X \rangle))$ and we aim to show that $\tau(\Sigma_{k+1}(x)) = (\Sigma_{k+1}(x))$. If $\Lambda_k(x)$ is consistent, then:

$$\begin{aligned}
\tau(\Sigma_{k+1}(x)) &= \tau\left(\Sigma_k(x) \cup \{v_{\Delta_n}(x-a) - \rho_{\Delta_n}(\beta) \not\in \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma(\mathfrak{M})\}\right) \\
&= \Sigma_k(x) \cup \{v_{\Delta_n}(x-\tau(a)) - \rho_{\Delta_n}(\tau(\beta)) \not\in \ell(\Gamma/\Delta_n) \mid a \in U(\mathfrak{M}), \beta \in \Gamma(\mathfrak{M})\} = \Sigma_{k+1}(x).
\end{aligned}$$
If $A_k(x)$ is inconsistent then $k \in J$. And we first argue that $\tau(A_k) = A_k$. By definition of $A_k$, if $\mu \in A_k$ then

$$
\Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - a)) = \mu \mid a \in \mathcal{M}\}
$$

is consistent,

because $\tau$ is an isomorphism,

$$
\tau(\Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - a)) = \mu \mid a \in \mathcal{M}\})
= \Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - \tau(a))) = \tau(\mu) \mid a \in \mathcal{M}\}
= \Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - a)) = \tau(\mu) \mid a \in \mathcal{M}\}
$$

hence $\tau(\mu) \in A_k$. We conclude that $\tau(A_k) = A_k$, and because $\tau$ is an arbitrary element in $\text{Aut}(\mathcal{M} / \text{acl}^{eq}((\forall X))$ we conclude that $A_k \subseteq \text{acl}^{eq}(\text{acl}^{eq}(\forall X)) = \text{acl}^{eq}(\forall X)$. In particular, for any $\mu \in A_k$, $\tau(\mu) = \mu$. Consequently,

$$
\tau(\Sigma_{k+1}(x)) = \Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - \tau(a))) = \mu \mid a \in \mathcal{M}\}
= \Sigma_k(x) \cup \{\pi_{A_k}^\epsilon (v(x - a)) = \mu \mid a \in \mathcal{M}\} = \Sigma_{k+1}(x),
$$

as required. □

Let

$$
\Sigma_\infty(x) := \bigcup_{k \in \mathbb{N}} \Sigma_k(x).
$$

By construction, this is a consistent $\text{acl}^{eq}(\forall X)$-definable partial type and

$$
\Sigma_\infty(x) \vdash x \in X.
$$

By quantifier elimination (Corollary 2.33), $\Sigma_\infty(x)$ determines a complete global type $p(x) \vdash x \in X$. This type $p(x)$ is $\text{acl}^{eq}(\forall X)$-definable as $\Sigma_\infty(x)$ is. □

This completes the proof for density of definable types.

5.2 Coding definable types

In this subsection, we prove that any definable type can be coded in the stabilizer sorts $G$. Let $x = (x_1, \ldots, x_k)$ be a tuple of variables in the main field sort. By quantifier elimination any definable type $p(x)$ over a model $K$ is completely determined by boolean combinations of formulae of the form:

(1) $Q_1(x) = 0$,
(2) $v_\Delta(Q_1(x)) \leq v_\Delta(Q_2(x))$,
(3) $v_\Delta(Q_1(x)/Q_2(x)) - k_\Delta \in n(\Gamma/\Delta)$,
(4) $v_\Delta(Q_1(x)/Q_2(x)) = k_\Delta$. 
where \( Q_1(x), Q_2(x), \in K[X_1, \ldots, X_k], n \in \mathbb{N}_{\geq 2}, \Delta \in RJ(\Gamma), k \in \mathbb{Z}, \) and \( k_\Delta = k \cdot 1_\Delta \) where \( 1_\Delta \) is the minimal element of \( \Gamma / \Delta \) if it exists. We will approximate such a type by considering for each \( l \in \mathbb{N} \) the definable vector space \( D_l/I_l \), where \( D_l \) is the set of polynomials of degree at most \( l \) and \( I_l \) is the subspace of \( D_l \) of polynomials \( Q(x) \) such that \( Q(x) = 0 \) is a formula in \( p(x) \). The formulae of the second kind, essentially give \( D_l/I_l \) a valued vector space structure with all the coarsened valuations, while the formulae of the third and fourth kind simply impose some binary relations in the linear order of the valued vector space. This philosophy reduces the problem of coding definable types into finding a way to code the possible valuations that could be induced over some power of \( K \) while taking care of the further structure on the linear order of the valued vector space.

The following is [22, Lemma 3.3].

**Fact 5.6.** Let \( K \) be any field. Let \( V \) be a subspace of \( K^n \) then \( V \) can be coded by a tuple of \( K \), and \( V \) and \( K^n/V \) have a \( ^\Gamma V^n \)-definable basis.

We start by coding the \( \Theta \)-submodules of \( K^n \).

**Lemma 5.7.** Let \( K \models T \) and \( M \subseteq K^n \) be a definable \( \Theta \)-submodule. Then the code \( ^\Gamma M^n \) can be coded in the stabilizer sorts.

**Proof.** Let \( V^+ \) be the span of \( M \) and \( V^- \) the maximal \( K \)-subspace of \( K^n \) contained in \( M \). By Fact 5.6, the subspaces \( V^+ \) and \( V^- \) can be coded by a tuple \( c \) in \( K \), and the quotient vector space \( V^+/V^- \) admits a \( c \)-definable basis. Hence, \( V^+/V^- \) can be identified over \( c \) with some power \( K^m \). And \( ^\Gamma M^n \) is interdefinable over \( c \) with the code of the image \( M/V^- \) in \( K^m \). But this image is an \( \Theta \)-submodule of \( K^m \) of some type \((I_1, \ldots, I_m) \in I^m \) so it admits a code in \( B_m(K)/Stab(I_1, \ldots, I_m) \). So \( M \) admits a code in the stabilizer sorts, as required. □

**Definition 5.8 (Valued relation).** Let \( K \models T \) and \( \Gamma \) be its value group. Let \( V \) be some finite-dimensional \( K \)-vector space and \( R \subseteq V \times V \) be a definable subset that defines a total pre-order. We say that \( R \) is a valued relation if there is an interpretable valued vector space structure \((V, \Gamma(V), val, +)\) in \( K \) such that \((v, w) \in R \) if and only if \( val(v) \leq val(w) \).

Given a relation \( R \subseteq V \times V \) that defines a total pre-order satisfying that:

- for all \( v, w \in V \) \((v, v + w) \in R \) or \((w, v + w) \in R \),
- for all \( v \in V \) \((v, v) \in R \),
- for all \( v, w \in V \) and \( \alpha \in K \), if \((v, w) \in R \) then \((\alpha v, \alpha w) \in R \).

We can define an equivalence relation \( E_R \) on \( V \) as \( E_R(v, w) \leftrightarrow (v, w) \in R \land (w, v) \in R \).

The set \( \Gamma(V) = V/E_R \) is therefore interpretable in \( K \) and we call it the linear order induced by \( R \). Let \( val : V \to \Gamma(V) \) be the canonical projection map that sends each vector to its class. We can naturally define an action of \( \Gamma(K) \) on \( \Gamma(V) \) as:

\[
\Gamma(K) \times \Gamma(V) \to \Gamma(V) \\
(a, [v]_{E_R}) \mapsto [av]_{E_R}, \text{ where } a \in K \text{ satisfying } v(a) = a.
\]

This is a well-defined map by the third condition imposed on \( R \). The structure \((V, \Gamma(V), val, +)\) is an interpretable valued vector space structure over \( V \) and we refer to it as the valued vector space structure induced by \( R \).
Lemma 5.9. Let $R \subseteq K^n \times K^n$ be a binary relation inducing a valued vector space structure $(K^n, \Gamma(K^n), \text{val}, +)$ on $K^n$. Then we can find a basis $\{v_1, \ldots, v_n\}$ of $K^n$ such that:

1. it is a separated basis for val, this is given any set of coefficients $\lambda_1, \ldots, \lambda_n \in K$,
$$\text{val}\left(\sum_{i \leq n} \lambda_i v_i\right) = \min\{\text{val}(\lambda_i) + \text{val}(v_i) \mid i \leq n\};$$

2. for each $i \leq n$, $\gamma_i = \text{val}(v_i) \in \text{dcl}\text{eq}(\Gamma(R^n))$.

Proof. Because the statement we are proving is first-order expressible, by Fact 2.45 we may assume that $K$ is maximal. We proceed by induction on $n$. For the base case, note that $K = \text{span}_K\{1\}$ then $\gamma = \text{val}(1) \in \text{dcl}\text{eq}(\Gamma(R^n))$. We assume the statement for $n$ and we want to prove it for $n + 1$. Let $W = K^n \times \{0\}$, $\text{val}_W = \text{val}^W$, $\Gamma(W) = \{\text{val}(w) \mid w \in W\}$, and $R_W = R \cap (W \times W)$. Then $(W, \Gamma(W), \text{val}_W, +)$ is a valued vector space structure on $W$ and $\Gamma(R_W) \subseteq \text{dcl}\text{eq}(\Gamma(R^n))$. The subspace $W$ admits an $\emptyset$-definable basis, so it can be canonically identified with $K^n$. By the induction hypothesis we can find $\{w_1, \ldots, w_n\}$ a separated basis of $W$ such that $\text{val}_W(w_i) \in \text{dcl}\text{eq}(\Gamma(R_W)) \subseteq \text{dcl}\text{eq}(\Gamma(R^n))$. As $W$ is finite-dimensional it is maximal by Lemma 2.46. By Fact 2.55, $W$ has the optimal approximation property in $K^{n+1}$. We can therefore define the valuation over the quotient space $K^{n+1}/W$ as follows:

$$\text{val}_{K^{n+1}/W} : \left\{(K^{n+1}/W) \rightarrow \Gamma(K^{n+1}) \mid v + W \mapsto \max\{\text{val}(v + w_0) \mid w_0 \in W\}\right\}.$$  

Define $R_{K^{n+1}/W} = \{(w_1 + W, w_2 + W) \mid \text{val}_{K^{n+1}/W}(w_1 + W) \leq \text{val}_{K^{n+1}/W}(w_2 + W)\}$, which is a valued relation over the quotient space $K^{n+1}/W$. As $K^{n+1}/W = K^{n+1}/(K^n \times \{0\})$ is definably isomorphic over $\emptyset$ to $K$, we can find a nonzero coset $v + W$ such that $\text{val}_{K^{n+1}/W}(v + W) \in \text{dcl}\text{eq}(\Gamma(R_{K^{n+1}/W})) \subseteq \text{dcl}\text{eq}(\Gamma(R^n))$. Let $w^* \in W$ be a vector where the maximum of $\{\text{val}_{K^{n+1}/W}(v + w) \mid w \in W\}$ is attained, that is, $\text{val}_{K^{n+1}/W}(v + W) = \text{val}(v + w^*)$. It is sufficient to show that $\{w_1, \ldots, w_n, v + w^*\}$ is a separated basis for $K^{n+1}$.

Let $\alpha \in K$, we show that for any $w \in W$ $\text{val}(v + w^* + \alpha w) = \min\{\text{val}(v + w^*), \text{val}(\alpha w)\}$. If $\text{val}(v + w^*) = \text{val}(\alpha w)$ then $\text{val}(v + w^* + \alpha w) = \min\{\text{val}(v + w^*), \text{val}(\alpha w)\}$. So let’s assume that $\gamma = \text{val}(v + w^*) = \text{val}(\alpha w)$.

Theorem 5.10. Let $K$ be a model of $T$ and $\Gamma$ its value group. Let $R$ be a definable valued relation on $K^n$ and $(K^n, \Gamma(K^n), \text{val}, +)$ be the valued vector space structure induced by $R$. Then $\Gamma(R^n)$ is interdefinable with a tuple of elements in the stabilizer sorts and there is an $\Gamma(R^n)$-definable bijection between $\Gamma(K^n)$ and finitely many disjoint copies of $\Gamma$ (all contained in $\Gamma^s$, where $s$ is the number of $\Gamma$-orbits over $\Gamma(K^n)$).

Proof. As the statement that we are trying to prove is first-order expressible, without loss of generality we may assume that $K$ is maximal. Let $R$ be a valued relation over $K^n$ and let $\Gamma(K^n), \text{val}, +)$ be the valued vector space structure induced by $R$. By Lemma 5.9, we can find a separated basis $\{v_1, \ldots, v_n\}$ of $K^n$, such that for each $i \leq n$, $\text{val}(v_i) \in \text{dcl}\text{eq}(\Gamma(R^n))$. Let $\{\gamma_1, \ldots, \gamma_s\} \subseteq$
\{\text{val}(v_i) \mid i \leq n\} be a complete set of representatives of the orbits of \(\Gamma\) over the linear order \(\Gamma(K^n)\), this is:

\[
\Gamma(K^n) = \bigcup_{i \leq s} \Gamma + \gamma_i.
\]

For each \(i \leq s\), we define \(B_i := \{x \in K^n \mid \text{val}(x) \geq \gamma_i\}\). Each \(B_i\) is an \(O\)-submodule of \(K^n\), so by Lemma 5.7, \(\Gamma B_i\) is interdefinable with a tuple in the stabilizer sorts. The valued vector space structure over \(K^n\) is completely determined by the closed balls containing 0, and each of these ones is of the form \(\alpha B_i\) for some \(\alpha \in K\) and \(i \leq s\). Thus, the code \(\Gamma^R\) is interdefinable with the tuple \((\Gamma B_1, \ldots, \Gamma B_s)\). We conclude that \(\Gamma^R\) can be coded in the stabilizer sorts.

For the second part of the statement, consider the map:

\[
f : \bigcup_{i \leq s} \Gamma + \gamma_i \to \Gamma^s
\]

\[
\alpha + \gamma_i 
\xrightarrow{\text{ith coordinate}} (0, \ldots, 0, \alpha, 0, \ldots, 0).
\]

As \(\{\gamma_1, \ldots, \gamma_s\} \subseteq dcl^q(\Gamma^R)\) this is a \(\Gamma^R\)-definable bijection between \(\Gamma(K^n)\) and finitely many disjoint copies of \(\Gamma\), contained in \(\Gamma^s\). □

**Theorem 5.11.** Let \(p(x)\) be a definable global type in \(\mathfrak{M}^n\). Then \(p(x)\) can be coded in \(\mathcal{C} \cup \Gamma^e\).

**Proof.** Let \(p(x)\) be a definable global type, and let \(K\) be a small model where \(p(x)\) is defined. Let \(q(x) = p(x) \restriction_K\) it is sufficient to code \(q(x)\).

For each \(\ell \in \mathbb{N}\) let \(D_\ell\) be the space of polynomials in \(K[X_1, \ldots, X_n]\) of degree less or equal than \(\ell\). This is a finite-dimensional \(K\)-vector space with an \(\emptyset\)-definable basis. Let \(I_\ell := \{Q(\bar{x}) \in D_\ell \mid Q(\bar{x}) = 0 \in q(\bar{x})\}\), this is a subspace of \(D_\ell\). Let \(R_\ell := \{(Q_1(\bar{x}), Q_2(\bar{x})) \in D_\ell \times D_\ell \mid v(Q_1(\bar{x})) \leq v(Q_2(\bar{x})) \in q(\bar{x})\}\), this relation induces a valued vector space structure on the quotient space \(V_\ell = D_\ell / I_\ell\). Let \((V_\ell, \Gamma(V_\ell), v_{\ell}, +_\ell)\) be the valued vector space structure induced by \(R_\ell\) over \(V_\ell\).

For each \(\Delta \in RJ(\Gamma)\) and \(k \in \mathbb{Z}\), a formula of the form \(v_{\Delta}(Q_1(\bar{x})) = v_{\Delta}(Q_2(\bar{x})) + k_{\Delta}\) determines a definable relation \(\varphi_{\ell, \Delta}^k \subseteq \Gamma(V_\ell)^2\), defined as:

\[
(v_{\ell}(Q_1(\bar{x})), v_{\ell}(Q_2(\bar{x}))) \in \varphi_{\ell, \Delta}^k \text{ if and only if } v_{\Delta}(Q_1(\bar{x})) = v_{\Delta}(Q_2(\bar{x})) + k_{\Delta} \in q(\bar{x}).
\]

Similarly, for each \(\Delta \in RJ(\Gamma), k \in \mathbb{Z}\) and \(n \in \mathbb{N}_{\geq 2}\) we consider the definable binary relation \(\psi_{\ell, \Delta}^{nk} \subseteq \Gamma(V_\ell)^2\) determined as:

\[
(v_{\ell}(Q_1(\bar{x})), v_{\ell}(Q_2(\bar{x}))) \in \psi_{\ell, \Delta}^{nk} \text{ if and only if } v_{\Delta}(Q_1(\bar{x})) - v_{\Delta}(Q_2(\bar{x})) + k_{\Delta} \in n(\Gamma/\Delta) \in q(\bar{x}).
\]

Likewise, for each \(\Delta \in RJ(\Gamma)\) and \(k \in \mathbb{Z}\) we consider the definable binary relations \(\theta_{\ell, \Delta}^{k} \subseteq \Gamma(V_\ell)^2\) defined as:

\[
(v_{\ell}(Q_1(\bar{x})), v_{\ell}(Q_2(\bar{x}))) \in \theta_{\ell, \Delta}^k \text{ if and only if } v_{\Delta}(Q_1(\bar{x})) < v_{\Delta}(Q_2(\bar{x})) + k_{\Delta} \in q(\bar{x}).
\]

Let

\[
S_{\ell} = \{\varphi_{\ell, \Delta}^k \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\} \cup \{\psi_{\ell, \Delta}^{nk} \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}, n \in \mathbb{N}_{\geq 2}\} \cup \{\theta_{\ell, \Delta}^{k} \mid \Delta \in RJ(\Gamma), k \in \mathbb{Z}\}
\]
We denote as $\mathcal{V}_\ell = (V_\ell, \Gamma(V_\ell), \text{val}_\ell, +, S_\ell)$ the valued vector space on $V_\ell$ with the enriched structure over the linear order $\Gamma(V_\ell)$. By quantifier elimination (see Corollary 2.33), the type $q(x)$ is completely determined by boolean combinations of formulae of the form:

- $Q_1(x) = 0$,
- $v_\Delta(Q_1(x)) < v_\Delta(Q_2(x))$,
- $v_\Delta(Q_1(x) - Q_2(x)) - k_\Delta \in n(\Gamma/\Delta)$,
- $v_\Delta(Q_1(x) - Q_2(x)) = k_\Delta$,

where $Q_1(x), Q_2(x), \in \mathcal{K}[X_1, \ldots, X_k]$, $n \in \mathbb{N}_{\geq 2}$, $\Delta \in \mathcal{R}_J(\Gamma)$, $k \in \mathbb{Z}$ and $k_\Delta = k \cdot 1_\Delta$ where $1_\Delta$ is the minimum positive element of $\Gamma/\Delta$ if it exists. Hence, the type $p(x)$ is entirely determined (and determines completely) by the sequence of valued vector spaces $(\mathcal{V}_\ell | \ell \in \mathbb{N})$ with enriched structure over the linear order.

By Fact 5.6 for each $\ell \in \mathbb{N}$ we can find codes $\lceil I^\ell \rceil$ in the home sort for the $I^\ell$s. After naming these codes, each quotient space $V_\ell = D/I_\ell$ has a definable basis, so it can be definably identified with some power of $K$. Therefore, without loss of generality we may assume that the underlying set of the valued vector space with enriched structure $V_\ell$ is some power of $K$. By Theorem 5.10, the relation $R_\ell$ admits a code $\lceil R_\ell \rceil$ in the stabilizer sorts. Moreover, there is a $\lceil R_\ell \rceil$ definable bijection $f:\Gamma(V_\ell) \to \Gamma^s$, where $s \in \mathbb{N}_{\geq 2}$ is the number of $\Gamma$-orbits over $\Gamma(V_\ell)$.

In particular, for each $\Delta \in \mathcal{R}(\Gamma)$, $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ the definable relations $\phi_k^\ell$, $\psi_k^{\ell,n}$ and $\theta_k^\ell$ are interdefinable over $\lceil R \rceil$ with $f(\phi_k^\ell)$, $f(\psi_k^{\ell,n})$ and $f(\theta_k^\ell)$, all subsets of $\Gamma^{2s}$. Consequently, the type $q(x)$ can be coded in the sorts $\mathcal{G} \cup \Gamma^{eq}$, as every definable subset $D$ in some power of $\Gamma$ admits a code in $\Gamma^{eq}$.

**Theorem 5.12.** Let $K$ be a valued field of equi-characteristic zero, residue field algebraically closed and value group of bounded regular rank. Then $K$ admits weak elimination of imaginaries in the language $\mathcal{L}_G$, where the stabilizer sorts are added.

**Proof.** By Theorem 4.1, $K$ admits weak elimination of imaginaries down to the sorts $\mathcal{G} \cup \Gamma^{eq}$, where $\mathcal{G}$ are the stabilizer sorts. Hrushovski’s criterion requires us to verify the following two conditions:

1. the density of definable types, this is Theorem 5.5, and
2. the coding of definable types, this is Theorem 5.11.

By Corollary 2.3, the value group $\Gamma$ is stably embedded. By Theorem 2.17, the ordered abelian group with bounded regular rank $\Gamma$ admits weak elimination of imaginaries once one adds the quotient sorts,

$$\{\Gamma/\Delta | \Delta \in \mathcal{R}(\Gamma)\} \cup \{\Gamma/(\Delta + \ell \Gamma) | \Delta \in \mathcal{R}(\Gamma), \ell \in \mathbb{N}_{\geq 2}\}.$$

We conclude that $K$ admits weak elimination of imaginaries down to the stabilizer sorts $\mathcal{G}$.

## 6 Elimination of Imaginaries for Henselian Valued Field with dp-Minimal Value Group

Let $(K, v)$ be a henselian valued field of equi-characteristic zero, residue field algebraically closed and dp-minimal value group. We see $K$ as a multi-sorted structure in the language $\hat{\mathcal{L}}$ extending the
language $\mathcal{L}_C$ (described in Definition 4.4), where the value group is equipped with the language $\mathcal{L}_{dp}$ described in Subsection 6, that is, for each convex subgroup $\Delta$ we add the quotient group $\Gamma/\Delta$ equipped with a copy of the language of Pressburger arithmetic and a projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$. And we also add sorts for the finite quotients $\Gamma/(\Delta + \ell \Gamma)$ where $\Delta$ is a convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$ and constants for the finite groups $\Gamma/\ell \Gamma$ where $\ell \in \mathbb{N}_{\geq 2}$. Let $\mathcal{P}$ be the complete family of $O$-submodules of $K$ described in Fact 2.37. From now on we fix a complete family $I = \mathcal{P} \setminus \{0, K\}$.

Remark 6.1. If we work with the complete family of $O$-submodules induced by the complete family of end-segments given by Remark 2.32, then each of the $\mathcal{O}$-modules in $I$ is definable over the empty set. In this setting, we are adding a finite set of constants $\Omega_n$ in $\Gamma$ choosing representatives of $n\Gamma$ in $\Gamma$ for each $n \in \mathbb{N}$. The results we obtain in this section will hold in the same manner if we work with this language instead.

Our main goal is the following theorem.

**Theorem 6.2.** Let $K$ be a henselian valued field of equicharacteristic zero, residue field algebraically closed and $dp$-minimal value group. Then $K$ eliminates imaginaries in the language $\hat{\mathcal{L}}$, where the stabilizer sorts are added.

**Definition 6.3.** We say that a multi-sorted first-order theory $T$ codes finite sets if for every model $M \models T$, and every finite subset $S \subseteq M$, the code $\Gamma S^\gamma$ is interdefinable with a tuple of elements in $M$.

The following is a folklore fact (see, e.g., [27]).

**Fact 6.4.** Let $T$ be a complete multi-sorted theory. If $T$ has weak elimination of imaginaries and codes finite sets then $T$ eliminates imaginaries.

In view of Theorem 5.12 and Fact 6.4, it is only left to show that any finite set can be coded in $\mathcal{G}$.

**Definition 6.5.**

1. An equivalence relation $E$ on a set $X$ is said to be proper if it has at least two different equivalence classes. It is said to be trivial if for any $x, y \in X$ we have $E(x, y)$ if and only if $x = y$.
2. A finite set $F$ is primitive over $A$ if there is no proper nontrivial $(\Gamma F^\gamma \cup A)$-definable equivalence relation on $F$. If $F$ is primitive over $\emptyset$ we just say that it is primitive.

To code finite sets, we need numerous smaller results. This section is organized as follows.

1. Subsection 6.1: We analyze the stable and stably embedded multi-sorted structure $VS_{k,C}$, consisting of the $k$-vector spaces $\text{red}(s)$, where $s$ is some $O$-lattice definable over $C$, an arbitrary imaginary set of parameters. This structure has elimination of imaginaries by results of Hrushovski in [17].
2. Subsection 6.2: We introduce the notion of germ of a definable function $f$ over a definable type $p$. We prove that germs can be coded in the stabilizer sorts.
(3) Subsection 6.3: We show that the code of any $O$-submodule $M \subseteq K^n$ is interdefinable with the code of its projection to the last coordinate and the germ of the function describing each of the fibers. We prove the same statement for torsors.

(4) Subsection 6.4: We prove several results on coding finite sets in the 1-dimensional case, for example, if $F$ is a primitive finite set of 1-torsors then it can be coded in $G$.

(5) Subsection 6.5: We carry a simultaneous induction to prove that any finite set $F \subseteq G$ can be coded in the stabilizer sorts, and any definable function $f : F \rightarrow G$ admits a code in the stabilizer sorts.

(6) Subsection 6.6: We state the result on full elimination of imaginaries down to the stabilizer sorts.

### 6.1 The multi-sorted structure of $k$-vector spaces

By Corollary 2.3, the residue field $k$ is stably embedded and it is a strongly minimal structure, because it is a pure algebraically closed field. This enables us to construct, over any imaginary base set of parameters $C$, a part of the structure that naturally inherits stability-theoretic properties from the residue field. Given a $O$-lattice $s \subseteq K^n$ we have that $\text{red}(s) = s/Ms$ is a $k$-vector space.

**Definition 6.6.** For any imaginary set of parameters $C$, we let $VS_{k,C}$ be the many-sorted structure whose sorts are the $k$ vector spaces $\text{red}(s)$ where $s \subseteq K^n$ is an $O$-lattice of rank $n$ definable over $C$. Each sort $\text{red}(s)$ is equipped with its $k$-vector space structure. In addition, $VS_{k,C}$ has any $C$-definable relation on products of the sorts.

**Definition 6.7.** A definable set $D$ is said to be internal to the residue field if there is a finite set of parameters $F \subseteq G$ such that $D \subseteq \text{dcl}^q(kF)$.

Each of the structures $\text{red}(s)$ is internal to the residue field, and the parameters needed to witness the internality lie in $\text{red}(s)$, so in particular each of the $k$-vector spaces $\text{red}(s)$ is stably embedded. The entire multi-sorted structure $VS_{k,C}$ is also stably embedded and stable, and in this subsection we will prove that it eliminates imaginaries.

**Notation 6.8.** We recall that given an $O$-submodule $M$ of $K$, we write $S_M$ to denote the end-segment induced by $M$, that is, $\{v(x) \mid x \in M\}$.

We recall some definitions from [17] to show that $VS_{k,C}$ eliminates imaginaries.

**Definition 6.9.** Let $t$ be a theory of fields (possibly with additional structure). A $t$-linear structure $\mathcal{A}$ is a structure with a sort $k$ for a model of $t$, and additional sorts $(V_i \mid i \in I)$ denoting finite-dimensional vector spaces. Each $V_i$ has (at least) a $k$-vector space structure, and $\dim V_i < \infty$. We assume that:

1. $k$ is stably embedded,
2. the induced structure on $k$ is precisely given by $t$,
3. the $V_i$ are closed under tensor products and duals.

Moreover, we say it is flagged if for any finite-dimensional vector space $V$ there is a filtration $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ by subspaces, with $\dim V_i = i$ and $V_i$ is one of the distinguished sorts.
The following is [17, Lemma 5.2].

**Lemma 6.10.** If \( k \) is an algebraically closed field and \( A \) is a flagged \( k \)-linear structure, then \( A \) admits elimination of imaginaries.

**Notation 6.11.** Let \( A \) be an \( \mathcal{O} \)-module. Let \( MA = \{ xa \mid x \in M, a \in A \} \) we denote as \( \text{red}(A) \) the quotient \( \mathcal{O} \)-module \( A/MA \).

We observe that \( \text{red}(A) = A/MA \) is canonically isomorphic to \( A \otimes_{\mathcal{O}} k \).

**Fact 6.12.** Let \( A \subseteq K^n \) and \( B \subseteq K^m \) be \( \mathcal{O} \)-lattices. Then \( \text{red}(A) \otimes_k \text{red}(B) \) can be canonically identified with \( \text{red}(A \otimes_{\mathcal{O}} B) \).

**Proof.** This is a straightforward computation and it is left to the reader. \[ \square \]

**Remark 6.13.** Given \( A \subseteq K^n \) and \( B \subseteq K^m \) \( \mathcal{O} \)-lattices, there is some \( \mathcal{O} \)-lattice \( C \subseteq K^{mn} \) such that \( A \otimes_{\mathcal{O}} B \) is canonically identified with \( C \). This isomorphism induces as well a one to one correspondence between \( \text{red}(A \otimes_{\mathcal{O}} B) \) and \( \text{red}(C) \).

**Proof.** Given \( K^n \) and \( K^m \) two vector spaces, the tensor product \( K^n \otimes K^m \) is a \( K \) vector space whose basis is \( \{ e_i \otimes e_j \mid i \leq n, j \leq m \} \) and it is canonically identified with \( K^{nm} \), via a linear map \( \phi \) that extends the bijection between the basis sending \( e_i \otimes e_j \) to \( e_{ij} \). Given \( A \subseteq K^n \) and \( B \subseteq K^m \) \( \mathcal{O} \)-lattices, then \( A \otimes_{\mathcal{O}} B \) is an \( \mathcal{O} \)-lattice of \( K^n \otimes K^m \) and we denote as \( C = \phi(A \otimes_{\mathcal{O}} B) \). This map induces as well an identification between \( \text{red}(A \otimes_{\mathcal{O}} B) \) and \( \text{red}(C) \) such that the following map commutes:

\[
\begin{array}{ccc}
A \otimes_{\mathcal{O}} B & \xrightarrow{\phi} & C \\
\downarrow \text{red} & & \downarrow \text{red} \\
\text{red}(A \otimes_{\mathcal{O}} B) & \xrightarrow{i} & \text{red}(C)
\end{array}
\]

**Fact 6.14.** Let \( A \subseteq K^n \) and \( B \subseteq K^m \) be \( \mathcal{O} \)-lattices. Then there is an isomorphism

\[ \phi : \text{red}(Hom_{\mathcal{O}}(A, B)) \to \text{Hom}_k(\text{red}(A), \text{red}(B)), \]

where for any \( f \in Hom_{\mathcal{O}}(A, B) \) and \( a \in A \):

\[ \phi(f + MA \text{Hom}_{\mathcal{O}}(A, B)) : \begin{cases} \text{red}(A) \to \text{red}(B) \\ a + MA \mapsto f(a) + MB. \end{cases} \]

**Proof.** This is a straightforward computation and it is left to the reader. \[ \square \]

**Remark 6.15.** Given an \( \mathcal{O} \)-lattice \( A \subseteq K^n \), then \( Hom_{\mathcal{O}}(A, \mathcal{O}) \) can be canonically identified with some \( \mathcal{O} \)-lattice \( C \) of \( K^n \). So there is a correspondence between \( \text{red}(Hom_{\mathcal{O}}(A, \mathcal{O})) \) and \( \text{red}(C) \).
Proof. Let $A$ be an $\mathcal{O}$-lattice of $K^n$. By linear algebra $K^n$ can be identified with its dual space $(K^n)^*$. Let 

$$A^* = \{ T \in (K^n)^* \mid \text{for all } a \in A, T(a) \in \mathcal{O} \}.$$ 

$A^*$ is canonically identified with $\text{Hom}_\mathcal{O}(A, \mathcal{O})$ via the map that sends a transformation $T$ to $T \upharpoonright_A$. Also $A^*$ is isomorphic to some $\mathcal{O}$-lattice $C$ of $K^n$, as there is a canonical isomorphism between $K^n$ and its dual space. More explicitly, 

$$C = \{ z \in K^n \mid \text{for all } a \in A, \sum_{i=1}^n z_ia_i \in \mathcal{O} \}.$$ 

So, we have a definable $\mathcal{O}$-isomorphism $\phi$ between $\text{Hom}_\mathcal{O}(A, \mathcal{O})$ and $C$, and this correspondence induces an identification $\hat{\phi}$ between $\text{red}(\text{Hom}_\mathcal{O}(A, \mathcal{O}))$ and $\text{red}(C)$ making the following diagram commute:

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{O}(A, \mathcal{O}) & \xrightarrow{\phi} & C \\
\downarrow{\text{red}} & & \downarrow{\text{red}} \\
\text{red}(\text{Hom}_\mathcal{O}(A, \mathcal{O})) & \xrightarrow{\hat{\phi}} & \text{red}(C)
\end{array}
\]

Remark 6.16. Let $A \subseteq K^n$ be an $\mathcal{O}$-lattice. There is a sequence of $\mathcal{O}$-lattices $< A_i | i \leq n >$ such that $< \text{red}(A_i) | i \leq n >$ is a flag of $\text{red}(A)$ and for each $i \leq n$, $\Gamma A_i^\gamma \in \text{dcl}^{eq}(\Gamma A^\gamma)$.

Proof. We proceed by induction on $n$, the base case is trivial. Let $A \subseteq K^{n+1}$, and $\pi_{n+1} : K^{n+1} \to K$ be the projection into the last coordinate. Let $B \subseteq K^n$ be the $\mathcal{O}$-lattice such that 

$$\ker(\pi_{n+1}) = B \times \{0\} = A \cap (K^n \times \{0\}).$$

We observe that $\Gamma B^\gamma$, $\Gamma \pi_{n+1}(A)^\gamma \in \text{dcl}^{eq}(\Gamma A^\gamma)$. By Corollary 3.1, $B$ is a direct summand of $A$, so we have the exact splitting sequence 

$$0 \to B \to A \to \pi_{n+1}(A) \to 0.$$

Consequently, 

$$0 \to MB \to MA \to M\pi_{n+1}(A) \to 0$$

and 

$$0 \to \text{red}(B) \to \text{red}(A) \to \text{red}(\pi_{n+1}(A)) \to 0$$

are exact sequences that split. By the induction hypothesis, there is a sequence $\{0\} \subseteq A_1 \subseteq \cdots \subseteq A_n = B$ such that $< \text{red}(A_i) | i \leq n >$ is a flag of $\text{red}(B)$, $\dim(\text{red}(A_i)) = i$ and $\Gamma A_i^\gamma \in \text{dcl}^{eq}(\Gamma B^\gamma) \subseteq \text{dcl}^{eq}(\Gamma A^\gamma)$. Let $A_{n+1} = A$, the sequence $< A_i | i \leq n + 1 >$ satisfies the required conditions. \hfill \Box

Theorem 6.17. Let $C \subseteq K^{eq}$, then $VS_{k,C}$ has elimination of imaginaries.

Proof. The sorts $\text{red}(s)$ where $s$ is a $\mathcal{O}$-lattice of $K^n$ and $\text{dcl}^{eq}(C)$-definable form the multi-sorted structure $VS_{k,C}$. Each $\text{red}(s)$ carries a $k$-vector space structure. $VS_{k,C}$ is closed under tensor prod-
uct by Remark 6.13 and Fact 6.12. It is closed under duals by Remark 6.15 and Fact 6.14. By Remark 6.16, each sort $\text{red}(s)$ where $s$ is an $O$-lattice admits a complete filtration by $C$-definable vector spaces. Therefore, $\text{VS}_{k,C}$ is a flagged $k$-linear structure, so the statement is an immediate consequence of 6.10.

6.2 | Germs of functions

In this subsection, we show how to code the germ of a definable function $f$ over a definable type $p(x)$ in the stabilizer sorts.

**Definition 6.18.** Let $T$ be a complete first-order theory and $M \models T$. Let $B \subseteq M$ and $p$ be a $B$-definable type whose solution set is $P$. Let $f$ be an $M$-definable function whose domain contains $P$. Suppose that $f = f_c$ is defined by the formula $\phi(x, y, c)$ (so $f_c(x) = y$). We say that $f_c$ and $f_{c'}$ have the same germ on $P$ if the formula $f_c(x) = f_{c'}(x)$ lies in $p$. By the definability of $p$ the equivalence relation $E_{\phi}(c, c')$ that states $f_c$ and $f_{c'}$ have the same germ on $P$ is definable over $B$. The germ of $f_c$ on $P$ is defined to be the class of $c$ under the equivalence relation $E_{\phi}(y, z)$, which is an element in $M^{eq}$. We write $\text{germ}(f, p)$ to denote the code for this equivalence class.

**Definition 6.19.** Let $p$ be a global type definable over $B$ and let $C$ be a set of parameters. We say that a realization $a$ of $p$ is sufficiently generic over $BC$ if $a \models p \upharpoonright BC$.

We start proving some results that will be required to show how to code the germs of a definable function $f$ over a definable type $p$ in the stabilizer sorts.

Let $U \subseteq K$ be a 1-torsor, we recall Definition 5.1, where we defined the $\lceil U \rceil$-definable partial type.

$$\Sigma^\text{gen}_U(x) = \{x \in U\} \cup \{x \notin B \mid B \subset U \text{ is a proper subtorsor of } U\}.$$ 

We refer to this type as the generic type of $U$.

When considering complete extensions of $\Sigma^\text{gen}_U(x)$ one finds an important distinction between the closed and the open case. In Proposition 5.2, we proved that whenever $U$ is a closed 1-torsor then $\Sigma^\text{gen}_U(x)$ admits a unique complete extension. The open case inherits a higher level of complexity, as there are many complete extensions of the type. However, all of those extensions are still $\lceil U \rceil$-definable.

**Proposition 6.20.** Let $U$ be an open 1-torsor, then any completion of the generic type of $U$ is $\lceil U \rceil$-definable.

**Proof.** Let $c \models \Sigma^\text{gen}_U(x)$, it is sufficient to prove that $p(x) = \text{tp}(c / M)$ is $\lceil U \rceil$-definable. We note that by quantifier elimination, the completions of the partial type $\Sigma^\text{gen}_U(x)$ are determined by $\text{tp}_\Gamma(v(x - a) / \Gamma(M))$ where $a \in U(M)$.

First, we observe that for any $a, a' \in U(M)$ we have that $v(c - a) = v(c - a')$, because $c$ realizes the generic type of $U$. Let $\Delta \in R\Gamma, \ell' \in \mathbb{N}, \beta \in \Gamma, k \in \mathbb{Z}$. In particular,

$$(v^\Delta(x - a) - \rho^\Delta(\beta) + k^\Delta) \in \ell'(\Gamma/\Delta)) \in p(x) \text{ if and only if } (v^\Delta(x - a') - \rho^\Delta(\beta) + k^\Delta \in \ell'(\Gamma/\Delta)) \in p(x).$$
Pick some element $\delta \in \Gamma$ such that $\rho_\Delta(\delta) = k^\Delta$, and let $\mu = \pi_\Delta^\ell(v(c - a) + \delta) \in \text{dcl}^{eq}(\emptyset)$. Then, for any $a \in U(\mathfrak{M})$ we have:

$$(v_\Delta(x - a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta)) \in p(x) \text{ if and only if } \models \pi_\Delta^\ell(\beta) = \mu.$$ 

If $a \notin U(\mathfrak{M})$, then $v(c - a) = v(b - a)$ for any $b \in U(\mathfrak{M})$. Hence,

$$(v_\Delta(x - a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta)) \in p(x) \text{ if and only if } \\
\models \exists b \in U(v_\Delta(b - a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta)).$$

Let

$$\psi(a, \beta) := (a \in U \land \pi_\Delta^\ell(\beta) = \mu) \lor (a \notin U \land \phi(a, \beta)).$$

Hence,

$$(v_\Delta(x - a) - \rho_\Delta(\beta) + k^\Delta \in \ell(\Gamma/\Delta)) \in p(x) \text{ if and only if } \models \psi(a, \beta).$$

Note that $\forall \psi(x, z)^\mathfrak{M} \in \text{dcl}^{eq}(\forall U^\mathfrak{M})$.

We continue showing a $\text{dcl}^{eq}(\forall U^\mathfrak{M})$-definable scheme for the coset formulae. Let $a \in U(\mathfrak{M})$ and consider the definable end-segment $S_a = \{v(x - a) \mid x \in U\}$. For any $a \neq a'$, $S_a = S_{a'}$, thus we write $S$ to denote this set. Note that $\forall S^\mathfrak{M} \in \text{dcl}^{eq}(\forall U^\mathfrak{M})$ and let $\Delta \in RJ(\Gamma)$. We recall that we denote by $S_\Delta$ the set $\rho_\Delta(S)$, which is a definable end-segment in $\Gamma/\Delta$. If $S_\Delta$ has a minimum element $\gamma \in \text{dcl}^{eq}(S_\Delta) \subseteq \text{dcl}^{eq}(\forall S^\mathfrak{M})$ then for any $a \in U(\mathfrak{M})$ we have:

$$(v_\Delta(x - a) = \rho_\Delta(\beta) + k^\Delta) \in p(x) \text{ if and only if } \models \rho_\Delta(\beta) + k^\Delta = \gamma.$$ 

If $S_{\Delta}$ does not have a minimum, then for any $a \in U(\mathfrak{M})$,

$$(v_\Delta(x - a) = \rho_\Delta(\beta) + k^\Delta) \in p(x) \text{ if and only if } \models \beta \neq \beta.$$ 

Finally, for $a \notin U(\mathfrak{M})$ we have that $v(c - a) = v(b - a)$ for any $b \in U(\mathfrak{M})$, therefore for $\Delta \in RJ(\Gamma)$ and $k \in \mathbb{Z}$ we have that:

$$(v_\Delta(x - a) = \rho_\Delta(\beta) + k^\Delta) \in p(x) \text{ if and only if } \models \exists b \in U(v_\Delta(b - a) = \rho_\Delta(\beta) + k^\Delta).$$

Consequently, for each quantifier free formula $\phi(x, y)$, we have shown the existence of a formula $d\phi(y)$ such that $\forall d\phi(y)^\mathfrak{M} \in \text{dcl}^{eq}(\forall U^\mathfrak{M})$ and $\phi(x, b) \in p(x)$ if and only if $\models d\phi(b)$. By quantifier elimination (see Corollary 2.33), the type $p(x)$ is completely determined by the quantifier free formulae, we conclude that $p(x)$ is $\forall U^\mathfrak{M}$-definable. \hfill $\square$

**Remark 6.21.** The completions of the global partial type $\Sigma_\mathfrak{M}^{gen}(x)$ are determined by the type $tp_\Gamma(v(x - a)/\Gamma(\mathfrak{M}))$ where $a \in U(\mathfrak{M})$. 

Corollary 6.22. Let $U$ be a definable 1-torsor, then each completion $p(x)$ of $\sum_U^{\text{gen}}(x)$ is \( \Gamma \)-definable.

Proof. This follows immediately by combining Propositions 5.2 and 6.20.

Proposition 6.23. Let $M \subseteq \mathcal{M}$ be a definable \( \mathcal{O} \)-module and let $p(x)$ be a global type containing the generic type of $M$. Then $p(x)$ is stabilized additively by $M(\mathcal{M})$, that is, if $c$ is a realization of $p(x)$ and $a \in M(\mathcal{M})$ then $a + c$ is a realization of $p(x)$.

Proof. Let $c$ be a realization of the type $p(x)$, $a \in M(\mathcal{M})$ and $d = c + a$. As $\sum_M^{\text{gen}}(x) \subseteq p(x)$, $c \in M$ and $c \notin U$ for any proper subtorsor $U \subseteq M$. First we argue that $d \models \sum_M^{\text{gen}}(x)$. Because $M$ is an $\mathcal{O}$-submodule of $K$ (in particular closed under addition) we have $d \in M$. And if there is a subtorsor $U \varsubsetneq M$ such that $d \in U$, then $c \in -a + U \subseteq M$ contradicting that $c \models \sum_M^{\text{gen}}(x)$. For any $\Delta \in RJ(\Gamma)$, element $z \in M(\mathcal{M})$ and realization $b \models \sum_M^{\text{gen}}(x)$ we have $v_\Delta(z - b) = v_\Delta(b)$. Thus, for any $n \in \mathbb{N}$ and $\beta \in \Gamma/\Delta$:

$$v_\Delta(b - z) - \beta \in n(\Gamma/\Delta) \text{ if and only if } v_\Delta(b) - \beta \in n(\Gamma/\Delta).$$

We conclude that $d$ and $c$ must satisfy the same congruence and coset formulae, because $c$ is a realization of the generic type of $M$, $a \in M(\mathcal{M})$ and $v_\Delta(d) = v_\Delta(c + a) = v_\Delta(c - (-a)) = v_\Delta(c)$.

Corollary 6.24. Let $M \subseteq \mathcal{M}$ be a definable $\mathcal{O}$-module. Let $p(x)$ be a global type containing the generic type of $M$. Let $a \in M(\mathcal{M})$, then $a$ is the difference of two realizations of $p(x)$, that is, we can find $c, d \models p(x)$ such that $a = c - d$.

Proof. Let $c$ be a realization of $p(x)$ and fix $a \in M(\mathcal{M})$. By Proposition 6.23, $d = c - a$ is also a realization of $p(x)$. The statement now follows because $a = c - d$.

Fact 6.25. Let $L$ be an $\mathcal{O}$-module and $\Delta$ be a definable convex subgroup of $\Gamma$. Let $b \in K$ and $R = b \mathcal{O}_\Delta^x$. We define the set:

$$L + R = \{l + r \mid l \in L, r \in R\}.$$

Then one of the three cases hold: $L + R = L$, $L + R = R$ or $L + R = \mathcal{O}_\Delta$. Moreover, there is global type $p(x) \models x \in L + R$ which is $\mathcal{R}L + \mathcal{R}^\Gamma$-definable.

Proof. This is a straightforward computation and we leave details to the reader.

Proposition 6.26. Let $(I_1, \ldots, I_n) \in \mathcal{T}^n$, for every $\mathcal{O}$-module $M$ of type $(I_1, \ldots, I_n)$ we can find a type $p_M(\bar{x}_1, \ldots, \bar{x}_n) \in S_{n\times n}(K)$ such that:

1. $p_M(\bar{x})$ is definable over $\mathcal{R}M^n$,
2. a realization of $p_M(\bar{x})$ is a matrix representation of $M$. This is if $(\bar{d}_1, \ldots, \bar{d}_n) \models p_M(\bar{x})$ then $[\bar{d}_1, \ldots, \bar{d}_n]$ is a representation matrix for $M$.

Proof. We start by recalling the notation introduced in Definition 4.2. Let $U \subseteq K^{n\times n}$ the set of $n$-tuples $(\bar{b}_1, \ldots, \bar{b}_n)$ such that $[\bar{b}_1, \ldots, \bar{b}_n]$ is an upper triangular matrix and invertible. And let $f :$
\[ U \to B_n(K)/\text{Stab}(I_1,\ldots,I_n) \] the \( \emptyset \)-definable map, that sends the tuple \((\bar{b}_1,\ldots,\bar{b}_n)\) to the corresponding coset \([\bar{b}_1,\ldots,\bar{b}_n]/\text{Stab}(I_1,\ldots,I_n)\).

The code \( \mathcal{M}^\sim \) is interdefinable with the class \( A/\text{Stab}(I_1,\ldots,I_n) \) of the quotient \( B_n(K)/\text{Stab}(I_1,\ldots,I_n) \), where \( A \) is a (any) representation matrix for \( M \).

Let \( X = f^{-1}(A/\text{Stab}(I_1,\ldots,I_n)) \), this is the set of representation matrices of the \( \mathcal{O} \)-module \( M \) and it is \( \mathcal{M}^\sim \)-definable.

By Proposition 4.8, we have:

\[
\text{Stab}(I_1,\ldots,I_n) = \{(b_{ij})_{1 \leq i, j \leq n} \in B_n(K) \mid b_{ii} \in \mathcal{O}^\times_{\Delta S_i} \land b_{ij} \in \text{Col}(I_i, I_j) \text{ for each } 1 \leq i < j \leq n \}.
\]

This allows us to find a more explicit description of \( X \).

\[
X = A/\text{Stab}(I_1,\ldots,I_n) = \left[ \begin{array}{cccc}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \ldots & \alpha_{1n} \\
0 & \alpha_{22} & \alpha_{23} & \ldots & \alpha_{2n} \\
0 & 0 & \alpha_{33} & \ldots & \alpha_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \alpha_{nn}
\end{array} \right] \cdot \left[ \begin{array}{cccc}
\mathcal{O}^\times_{\Delta S_1} & \text{Col}(I_1, I_2) & \text{Col}(I_1, I_3) & \ldots & \text{Col}(I_1, I_n) \\
0 & \mathcal{O}^\times_{\Delta S_2} & \text{Col}(I_2, I_3) & \ldots & \text{Col}(I_2, I_n) \\
0 & 0 & \mathcal{O}^\times_{\Delta S_3} & \ldots & \text{Col}(I_3, I_n) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \mathcal{O}^\times_{\Delta S_n}
\end{array} \right]
\]

We use matrix notation to simplify the presentation of the argument. We write \( X_{ij} \) to denote the projection of \( X \) to the \((i,j)\)-coordinate. More explicitly,

\[
X_{ij} = \begin{cases} 
\{0\} & \text{if } i > j, \\
\alpha_{ii}\text{Col}(I_i, I_j) + \cdots + \alpha_{ij-1}\text{Col}(I_{j-1}, I_j) + \alpha_{ij}\mathcal{O}^\times_{\Delta S_i} & \text{if } i < j, \\
\alpha_{ii}\text{Col}(I_i, I_i) & \text{if } i = j.
\end{cases}
\]

Note that the set \( X_{ij} \) is \( \mathcal{M}^\sim \)-definable, one can verify by a straightforward computation that its definition does not depend of the choice of the representation matrix \( A \) for \( M \).

Moreover, if \( i < j \), then \( X_{ij} = L + R \) where \( L = \alpha_{ii}\text{Col}(I_i, I_j) + \cdots + \alpha_{ij-1}\text{Col}(I_{j-1}, I_j) \) is a \( \mathcal{O} \)-submodule of \( K \) while \( R = \alpha_{ij}\mathcal{O}^\times_{\Delta S_i} \). By Fact 6.25, there is a complete global type \( p_{ij}(x) \vdash x \in X_{ij} \) which is \( \mathcal{M}^\sim \)-definable, and therefore \( \mathcal{M}^\sim \)-definable. Likewise, \( X_{ii} = 0 + R \) where \( R = \alpha_{ii}\mathcal{O}^\times_{\Delta S_i} \) and by Fact 6.25 there is a \( \mathcal{M}^\sim \)-definable global type \( p_{ii}(x) \vdash x \in X_{ii} \), in particular it is \( \mathcal{M}^\sim \)-definable.

We consider the set \( J = \{(i, j) \mid 1 \leq i, j \leq n\} \), and we equip it with a linear order defined as:

\[
(i, j) < (i', j') \text{ if and only if } j < j' \text{ or } j = j' \land i' < i.
\]

Fix an enumeration of \( J = \{v_1,\ldots,v_{n^2}\} \) such that \( v_1 < v_2 < \cdots < v_{n^2} \). Hence, for each \( 1 \leq m \leq n^2 \) let:

\[
p_{v_m}(x) = \begin{cases} 
tp(0) & \text{if } v_m = (i, j) \text{ where } 1 \leq j < i \leq n, \\
p_{ij}(x) & \text{if } v_m = (i, j) \text{ for some } 1 \leq i \leq j \leq n.
\end{cases}
\]
Let $p_M(x) = p_{v_{n^2}} \otimes \cdots \otimes p_{v_1}$, which is $\Gamma(M)$-definable type, because it is the product of $\Gamma(M)$-definable types. Given any realization $(\tilde{d}_1, ..., \tilde{d}_n) \models p_M(x)$ then $[\tilde{d}_1, ..., \tilde{d}_n]$ is a representation matrix for $M$, because by construction $[\tilde{d}_1, ..., \tilde{d}_n] \in A\text{Stab}(I_1, ..., I_n)$.

**Theorem 6.27.** Let $X$ be a definable subset of $K^n$ and let $p(x) \vdash x \in X$ be a global type definable over $\Gamma X$. Let $f = X \to G$ be a definable function. Then the $p$-germ of $f$ is coded in $G$ over $\Gamma X$.

**Proof.** We first assume that $f : X \to B_n(K)/\text{Stab}(I_1, ..., I_n)$. Let

$$B = \text{dcl}_G(\text{germ}(f, p), \Gamma X) = \text{dcl}^G(\text{germ}(f, p), \Gamma X) \cap G.$$ 

By Theorem 5.11, the type $p(x)$ is $B$-definable. Suppose that $f$ is $c$-definable, and let $q = \text{tp}(c/B)$ and $Q$ be its set of realizations. Fix some $c' \in Q$. We denote by $f'$ the function obtained by replacing the parameter $c$ by $c'$ in the formula defining $f$. Let $M$ be a small model containing $Bc_c$.

**Step 1:** For any realization $a \models p(x) \upharpoonright M$ we have $f(a) = f'(a)$.

Let $a$ be a realization of $p(x) \upharpoonright M$. Let $u_{f(a)}(y)$ be the definable type over $\Gamma f(a)$ given by Proposition 6.26. Given any realization $d = (\tilde{d}_1, ..., \tilde{d}_n) \models u_{f(a)}(y), [\tilde{d}_1, ..., \tilde{d}_n]$ is a representation matrix for the module $f(a)$. In particular, $f(a) = p(I_1, ..., I_n)(\tilde{d}_1, ..., \tilde{d}_n) \in \text{dcl}^G(d)$. Let $d$ be a realization of $u_{f(a)}(y) \upharpoonright M$ and let $r(x, y) = \text{tp}(a, d/M)$, then the type $r(x, y)$ is $B$-definable, and therefore $B$-invariant.

Because $\text{tp}(c/B) = \text{tp}(c'/B)$ we can find an automorphism $\sigma \in \text{Aut}(M/B)$ sending $c$ to $c'$. Then $u_{f'(a)}(y) = \sigma(u_{f(a)}(y))$, which is a definable type over $\Gamma f'(a)$. Let $d'$ be a realization of $u_{f'(a)}(y) \upharpoonright M$. Let $r'(x, y) = \text{tp}(a, d'/M)$, then $\sigma(r(x, y)) = r'(x, y) = r(x, y)$ by the $B$-invariance of $r(x, y)$, so $\text{tp}(a, d/M) = \text{tp}(a, d'/M)$. As $f(a) \in \text{dcl}^G(d)$ and $f'(a) \in \text{dcl}^G(d')$, we must have that $\text{tp}(a, f(a)/M) = \text{tp}(a, f'(a)/M)$ and as $f$ and $f'$ are both definable over $M$ this implies that $f(a) = f'(a)$.

**Step 2:** The germ$(f, p)$ is coded in the stabilizer sorts $G$ over $\Gamma X$.

First, note that for any $a \models p(x) \upharpoonright Bc_c$ it is the case that $f(a) = f'(a)$. In fact, by Step 1 $f(x) = f'(x) \in \text{tp}(a/M)$ and $f(x) = f'(x)$ is a formula in $\text{tp}(a/Bc_c)$. Then $f$ and $f'$ both have the same $p$-germ. As $p(x)$ is definable over $B = \text{dcl}_G(B)$ the equivalence relation $E$ stating that $f$ and $f'$ have both the same $p$-germ is $B$-definable. As for any realization $a \models p(x) \upharpoonright Bc_c$ it is the case that $f(a) = f'(a)$, the class $E(x, c)$ is $B$-invariant, therefore germ$(f, p)$ is definable over $B = \text{dcl}_G(B)$.

We continue arguing that the statement for $f = X \to B_n(K)/\text{Stab}(I_1, ..., I_n)$ is sufficient to conclude the entire result. For each $\Delta \in R_J(\Gamma)$, there is a canonical isomorphism $\Gamma/\Delta \cong K^\times/\text{O}_{\Delta}^\times$ where $\text{O}_{\Delta}$ is the valuation ring of the coarsened valuation $v_{\Delta}$ induced by $\Delta$. The functions whose image lie in $\Gamma/\Delta$ are being considered in the previous case, because $\text{Stab}(\text{O}_{\Delta}) = \text{O}_{\Delta}^\times$. By Proposition 3.6 any definable function $f = X \to k = \text{O}/\text{M}$ can be seen as a function whose image lies in $B_2(K)/\text{Stab}_{(M, O)}$.

It is only left to consider the case where the target set is $K$. The proof follows in a very similar manner as the case for $f : X \to B_n(K)/\text{Stab}(I_1, ..., I_n)$. Let $a \models p(x)$. We let $a \models p(x)$ and let $r(x, y) : = \text{tp}(a, f(a)/M)$, this is a $B$-definable type by Theorem 5.11, in particular $B$-invariant. Likewise, $r'(x, y) : = \text{tp}(a, f'(a)/M)$ is $B$-invariant, thus $\text{tp}(a, f(a)/M) = \text{tp}(a, f'(a)/M)$. As $f$ and $f'$ are both definable over $M$, this implies that $f(a) = f'(a)$. The rest of the proof follows exactly as in the second step. □
6.3 Some useful lemmas

In this subsection, we prove several lemmas that will be required to code finite sets.

Notation 6.28. Let $U \subseteq K$ be a $1$-torsor, $U = a + bI$ where $I \in I$. We write $p_U(x)$ to denote some $U^\gamma$-definable type centered in $U$ extending the generic type of $U \Sigma^\gen_U(x)$. If $U$ is closed such type is unique (see Proposition 5.2) and for the open case there are several choices for this type, but all of them are $U^\gamma$-definable by Proposition 6.20.

Lemma 6.29. Let $F = \{B_1, ..., B_n\}$ be a primitive finite set of $1$-torsors. Let $W = \{x_1, ..., x_n \mid x_i \in B_i\}$, and $W^* = \{\{x_1, ..., x_n\} \mid x_1, ..., x_n \in W\}$ Then there is a $W^*^\gamma$-definable type $q$ concentrated on $W^*$. Furthermore, given $b^*$ a realization of $q$ sufficiently generic over a set of parameters $C$ if we take $B$ the finite set coded by $b^*$, then if $b \in B$ is the element that belongs to $B_i$ then $b_i$ is a sufficiently generic realization of some type $p_{B_i}(x)$, which is $B_i^\gamma$-definable and extends the generic type of $B_i$. Finally, the types $p_{B_i}(x)$ are compatible under the action of $Aut(\mathfrak{M}/\Gamma F^\gamma)$ meaning that if $\sigma \in Aut(\mathfrak{M}/\Gamma F^\gamma)$ and $\sigma(B_i) = B_j$ then $\sigma(p_{B_i}(x)) = p_{B_j}(x)$.

Proof. We focus first on the construction of the type $q$, and later we show that it satisfies the required conditions. Suppose that each 1-torsor $B_i = c_i + b_iI_i$ for some $I_i \in \mathfrak{V}$. By transitivity all the balls are of the same type $I \in \mathfrak{V}$ and for all $1 \leq i, j \leq n$ we have that $v(b_i) = v(b_j)$. Hence, we may assume that each $B_i$ is of the form $c_i + bI$ for some fixed $c_i, b \in K$ and $I \in I$.

We argue by following cases.

(1) Case 1: All the 1-torsors $B_i$ are closed.

For each $i \leq n$, let $p_{B_i}(x)$ be the unique $B_i^\gamma$-definable type given by Proposition 5.2. Define $r(x_1, ..., x_n) = p_{B_1}(x_1) \otimes ... \otimes p_{B_n}(x_n)$, this is a $(B_1^\gamma, ..., B_n^\gamma)$-definable type. Let $(a_1, ..., a_n) \models r(x_1, ..., x_n)$ and let $q = tp(r(a_1, ..., a_n)^\gamma/\mathfrak{M})$ its symmetrized version. This type is well-defined independently of the choice of the order, because each type $p_{B_i}(x)$ is generically stable, thus it commutes with any definable type by [32, Proposition 2.33]. The type $q$ is $W^*^\gamma$-definable and centered at $W^*$.

(2) Case 2: All the 1-torsors $B_i$ are open, that is, $I \in I \backslash \{\emptyset\}$.

Let $S_{bI} = v(b) + S_I = \{v(b) + v(x) \mid x \in I\}$, this is a definable end-segment of $\Gamma$ with no minimal element. Let $r(y)$ be some fixed completion of the partial generic type $\Sigma^\gen_{S_{bI}}(y)$. By Fact 2.31, $r(y)$ is $S_{bI}^\gamma$-definable. Fix elements $a = \{a_1, ..., a_n\} \in W(\mathfrak{M})$ and $\delta \models r(y)$, we define $C(a, \delta) = \{C_1(a), ..., C_n(a)\}$, where each $C_i(a)$ is the closed ball around $a_i$ of radius $\delta$. For each $i \leq n$ we take $p_{C_i(a)}(x)$ the unique extension of the generic type of $C_i(a)$ given by Proposition 5.2, this type is $C_i(a)^\gamma$-definable.

Let $q^a_\delta$ be the symmetrized generic type of $C_1(a) \times ... \times C_n(a)$, that is, we take $tp(r(b_1, ..., b_n)^\gamma/\mathfrak{M}\delta)$ where $(b_1, ..., b_n)$ is a realization of the generically stable type $p_{C_i(a)} \otimes ... \otimes p_{C_n(a)}$. Let $q^a$ be the definable global type satisfying that $d \models q^a$ if and only if there is some $\delta \models r(y)$ and $d \models q^a_\delta$.

Claim 6.29.1. The type $q^a$ does not depend on the choice of $a$.

Proof. Let $a' = \{a'_1, ..., a'_n\} \in W(\mathfrak{M})$ and $\delta \models r(y)$. For each $i \leq n$, $a_i, a'_i \in B_i$ meaning that $a_i - a'_i \in bI$, that is, $v(a_i - a'_i) \in S_{bI} = v(b) + S_I$ and note that $v(a_i - a'_i) \in \Gamma(\mathfrak{M})$. 

By construction, $\delta \in S_\beta$ and $\delta < v(a_i - a'_i)$, thus the closed ball of radius $\delta$ concentrated on $a_i$ is the same closed ball of radius $\delta$ concentrated on $a'_i$. As the set of closed balls $C(a, \delta) = C(a', \delta)$, we must have that $q^a_\delta = q'^a_\delta$, and as this holds for any $\delta \models r(\gamma)$ we conclude that $q^a$ does not depend on the choice of $a$ and we simply denote it as $q$.

This type $q$ is $\mathcal{W}^\gamma$-definable and it is centered in $W^\gamma$. This finalizes the construction of the type $q$ that we are looking for.

We continue checking that the type $q$ that we have constructed satisfies the other properties that we want. In both cases, by construction, if $b^*$ is a sufficiently generic realization of $q$ over $C$ and $B$ is the finite set coded by $b^*$ if we take $b_i$ the unique element of $B$ that lies on $B_i$ then $b_i$ realizes the generic type $\Sigma^\text{gen}_i(x)$. By Corollary 6.22, the type $tp(b_i/\mathfrak{M})$ is $\mathcal{W}^\gamma$-definable.

It is only left to show that the types $p_{B_i}(x)$ are compatible under the action of $Aut(\mathfrak{M}/\mathcal{W})$, that is, given $\sigma \in Aut(\mathfrak{M}/\mathcal{W})$, if $\sigma(B_i) = B_j$ then $\sigma(p_{B_i}(x)) = p_{B_j}(x)$. We proceed by cases.

If the torsors are closed, then the types $p_{B_i}(x)$ are all compatible under the action of $Aut(\mathfrak{M}/\mathcal{W})$ as there is a unique complete extension of the generic type of $B_i$, this is guaranteed by Proposition 5.2.

We now work the details for the open case. By construction, for all $k \leq n$ the type $p_{B_k}(x)$ that we are fixing is the unique extension of the generic type of the closed ball $C_\delta(a_k)$ where $a_k$ is some (any) element of $B_k(\mathfrak{M})$ and $\delta \models r(\gamma)$.

By quantifier elimination (i.e., Corollary 2.33), for any $a, a' \in B_1(\mathfrak{M})$ and $\delta, \delta' \models r(\gamma)$ over $\mathfrak{M}$, we have:

$$p_{C_\delta(a)}(x) |_{\mathfrak{M}} = p_{C_{\delta'}(a')}(x) |_{\mathfrak{M}},$$

(Note that all the congruence and coset formulae are decided in $r(\gamma)$, and for any $a, a' \in B_1(\mathfrak{M})$ we have that $v(x - a) = v(x - a') \in p_{B_i}(x)$).

Fix $\sigma \in Aut(\mathfrak{M}/\mathcal{W})$ and assume that $\sigma(B_i) = B_j$. Then then $\sigma(b_i) \in B_j$ and because the type $r(\gamma)$ is $\mathcal{W}^\gamma$-definable, it is $\mathcal{W}^\gamma$-invariant so one has $\sigma(r(\gamma)) = r(\gamma)$. It follows that $\sigma(p_{B_i}(x)) = p_{B_j}(x)$, as desired.

**Notation 6.30.** Let $M \subseteq K^n$ be a definable $\mathcal{O}$-module and let $Z = \tilde{d} + M$ be a torsor. Let $\pi_n = K^n \rightarrow K$ be the projection into the last coordinate. Consider the function that describes the fiber in $Z$ of each element at the projection, this is $h_Z(x) = \{y \in K^{n-1} \mid (y, x) \in Z\}$.

**Fact 6.31.** Let $M$ be a $\mathcal{O}$-submodule of $K^n$. Then for any $x, y \in \pi_n(M)$ we have that

$$h_M(x) + h_M(y) = h_M(x + y).$$

Furthermore, if $Z = \bar{d} + M \in K^n/M$ is a torsor, then for any $d_1, d_2 \in \pi_n(Z)$ we have that $d_1 - d_2 \in \pi_n(M)$ and:

$$h_M(d_1 - d_2) = h_Z(d_1) - h_Z(d_2).$$

**Proof.** This is a straightforward computation and it is left to the reader. □
Lemma 6.32. Let $n \geq 2$ be a natural number and $M \subseteq K^n$ be a definable $\mathcal{O}$-submodule. Then $\tau M^n$ is interdefinable with $(\tau \pi_n(M)^\gamma, \text{germ}(h_M, p_{\pi_n(M)}))$, where $p_{\pi_n(M)}$ is any complete extension of the generic type of $\pi_n(M)$.

Proof. Let $M_1$ and $M_2$ be $\mathcal{O}$-modules of the same type. Suppose that $A = \pi_n(M_1) = \pi_n(M_2)$, and $\text{germ}(h_{M_1}, p_A) = \text{germ}(h_{M_2}, p_A)$. We must show that $M_1$ and $M_2$ are the same $\mathcal{O}$-module.

Let $c$ be a realization of the type $p_A(\cdot)$ sufficiently generic over $\lceil M_1 \rceil \lceil M_2 \rceil$. Fix an element $y \in A$ and let $d = c - y$. By Proposition 6.23, $d$ is a realization of $p_A(\cdot)$ sufficiently generic over $\lceil M_1 \rceil \lceil M_2 \rceil$, and $y = c - d$. As $\text{germ}(h_{M_1}, p_A) = \text{germ}(h_{M_2}, p_A)$, we have that $h_{M_1}(c) = h_{M_2}(c)$ and $h_{M_1}(d) = h_{M_2}(d)$. By Fact 6.31, $h_{M_1}(y) = h_{M_1}(c) - h_{M_1}(d) = h_{M_2}(c) - h_{M_2}(d) = h_{M_2}(y)$. Consequently, $M_1 = M_2$ as desired. □

Corollary 6.33. Let $n \geq 2$ be a natural number and $N \subseteq K^n$ be a definable $\mathcal{O}$-submodule. Let $Z = b + N$ be a torsor, then $\tau Z^n$ is interdefinable with $(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)}))$, where $p_{\pi_n(Z)}$ is any global type containing the generic type of $\pi_n(Z)$.

Proof. We first show that $\lceil Z \rceil$ is interdefinable with $(\tau \pi_n(Z)^\gamma, \lceil N \rceil, \text{germ}(h_Z, p_{\pi_n(Z)}))$. Let $Z_1 = b_1 + N$ and $Z_2 = b_2 + N$ torsors, and suppose that $A = \pi_n(Z_1) = \pi_n(Z_2)$ and $\text{germ}(h_{Z_1}, p_A) = \text{germ}(h_{Z_2}, p_A)$. Let $c$ be a realization of the type $p_A(\cdot)$ sufficiently generic over $\tau Z_1^n \tau Z_2^n$, and then $h_{Z_1}(c) = h_{Z_2}(c)$. If $Z_1 \neq Z_2$, then they must be disjoint because they are different cosets of $N$. But if $h_{Z_1}(c) = h_{Z_2}(c)$ then $Z_1 \cap Z_2 \neq \emptyset$, so $Z_1 = Z_2$.

We continue showing that $N$ is definable over $(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)}))$. We will find a global type $p_{\pi_n(N)}(\cdot)$ extending the generic type of $\pi_n(N)$ such that

$$(\tau \pi_n(N)^\gamma, \text{germ}(h_N, p_{\pi_n(N)})) \in \text{dcl}^eq(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)})).$$

By Lemma 6.32, this guarantees that $\tau N^n \in \text{dcl}^eq(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)}))$.

First, let $y' \in \pi_n(Z)$, then $\pi_n(N) = \{y - y' \mid y \in \pi_n(Z)\}$. As this definition is independent from the choice of $y'$, we have $\tau \pi_n(N)^\gamma \in \text{dcl}^eq(\tau \pi_n(Z)^\gamma)$.

Claim 6.33.1. Let $q(x_2, x_1) = p_{\pi_n(Z)}(x_2) \otimes p_{\pi_n(Z)}(x_1)$ and $(d_2, d_1) \models q(x_2, x_1)$.

Then $\Sigma^\text{gen}_{\pi_n(Z)}(\cdot) \subseteq \text{tp}(d_2 - d_1/\mathcal{M})$.

Proof. We proceed by contradiction and we assume the existence of some proper $\mathcal{M}$-definable subtorser $B \subseteq \pi_n(N)$ such that $d_2 - d_1 \in B$. Then $d_2 \in d_1 + B \subseteq \pi_n(Z)$, and $d_1 + B$ is a proper $\mathcal{M}d_1$-definable torsor of $\pi_n(Z)$, but this contradicts that $d_2$ is a sufficiently generic realization of $\Sigma^\text{gen}_{\pi_n(Z)}(\cdot)$ over $\mathcal{M}d_1$. □

Let $p_{\pi_n(N)}(\cdot) = \text{tp}(d_2 - d_1/\mathcal{M})$, we observe that such type is independent of the choices of $d_1$ and $d_2$, as the congruence and coset formulae are completely determined in the type $q(x_2, x_1)$.

It is only left to show that $\text{germ}(h_N, p_{\pi_n(N)}) \in \text{dcl}^eq(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)}))$.

Let $\sigma \in \text{Aut}(\mathcal{M}/(\tau \pi_n(Z)^\gamma, \text{germ}(h_Z, p_{\pi_n(Z)})))$, we will show that $\text{h}_N(x) = \text{h}_Z(\sigma(x)) \in p_{\pi_n(N)}(x)$. Because $p_{\pi_n(N)}(x)$ is $\tau \pi_n(N)^\gamma$-definable then $\sigma(p_{\pi_n(N)}(x)) = p_{\pi_n(N)}(x)$. As $\sigma(\text{germ}(h_Z, p_{\pi_n(Z)})) = \text{germ}(h_Z, p_{\pi_n(Z)})$, then $h_Z(x) = h_{\sigma(Z)}(x) \in p_{\pi_n(Z)}$. 


As \( \sigma(N) = N \) then \( \sigma(\Gamma N) = \Gamma N \). Let \( C = \{ \Gamma Z, \sigma(Z) \setminus \Gamma N \} \). In particular, if \( d_1 \models p_{\pi_n(Z)} \restriction C \) and \( d_2 \models p_{\pi_n(Z)} \restriction C \), then \( h_Z(d_1) = h_{\sigma(Z)}(d_1) \) and \( h_Z(d_2) = h_{\sigma(Z)}(d_2) \). By Fact 6.31,

\[
h_N(d_2 - d_1) = h_Z(d_2) - h_Z(d_1) = h_{\sigma(Z)}(d_2) - h_{\sigma(Z)}(d_1) = h_{\sigma(Z)}(d_2 - d_1) = h_N(d_2 - d_1).
\]

Consequently, \( \sigma(\text{germ}(h_N, p_{\pi_n}(N))) = \text{germ}(h_N, p_{\pi_n}(N)) \). Because \( \sigma \) is arbitrary, we conclude that

\[
\text{germ}(h_N, p_{\pi_n}(N)) \in \text{dcl}^q(\Gamma \pi_n(Z) \setminus \text{germ}(h_Z, p_{\pi_n}(Z))),
\]
as required. \( \square \)

### 6.4 Some coding lemmas

**Notation 6.34.** Let \( N \) be a \( \Theta \)-submodule of \( K^n \), then:

\[
\mathcal{M}N = \{ x_n \in K^n \mid x \in \mathcal{M}, n \in N \},
\]
is a \( \Theta \)-submodule of \( N \). Let \( U \in K^n / N \) be a torsor, we define an equivalence relation \( E \) on \( U \) as:

\[
E(d_1, d_2) \text{ if and only if } d_1 - d_2 \in \mathcal{M}N.
\]

We write \( \text{red}(U) \) to denote the quotient \( U / E \).

**Lemma 6.35.** Let \( A \) be a definable \( \Theta \)-lattice in \( K^n \) and \( U \in K^n / A \) be a torsor. Let \( B \) be the \( \Theta \)-lattice in \( K^{n+1} \) that is interdefinable with \( U \) (given by Proposition 3.6). Then there is a \( \Gamma U \setminus \Theta \)-definable injection:

\[
f = \begin{cases} 
\text{red}(U) & \rightarrow \text{red}(B) \\
 b + MA & \mapsto (b, 0) + MB.
\end{cases}
\]

**Proof.** We recall how the construction of \( B \) was achieved. Given any \( \bar{d} \in U \), we can represent \( B = A_2 + \begin{bmatrix} d \\ 0 \end{bmatrix} \Theta \), where \( A_2 = A \times \{ 0 \} \). This definition is independent from the choice of \( \bar{d} \). We consider the \( \Gamma U \setminus \Theta \)-definable injection \( \phi = U \rightarrow B \) that sends each element \( \bar{b} \) to \( \begin{bmatrix} b \\ 1 \end{bmatrix} \). The interpretable sets \( \text{red}(U) \) and \( \text{red}(B) = B / MB \) are both \( \Gamma U \setminus \Theta \)-definable. It follows by a standard computation that for any \( b, b' \in U \), \( b - b' \in MA \) if and only if \( \begin{bmatrix} b - b' \\ 1 \end{bmatrix} \in MB \). This shows that the map \( f \) is a \( \Gamma U \setminus \Theta \)-definable injection. \( \square \)

**Lemma 6.36.** Let \( F \) be a primitive finite set of 1-torsors, then \( F \) can be coded in \( \mathcal{G} \).

**Proof.** If \( |F| = 0 \) or \( |F| = 1 \), the statement follows clearly. So, we may assume that \( |F| > 1 \). By primitivity all the torsors in \( F \) are translates of the same \( \Theta \)-submodule of \( K \). We can find \( b \in K \) and \( I \in I \) such that for any \( t \in F \) there is some \( a_t \in K \) satisfying \( t = a_t + bI \). Moreover, there is some \( \delta \not\in \nu(b) + S_I \) such that for any two different torsors \( t, t' \in F \) if \( x \in t \) and \( y \in t' \) then \( \nu(x - y) = \delta \). Let \( T = \bigcup_{t \in F} t \). We define

\[
J_F = \{ Q(x) \in K[x] \mid Q(x) \text{ has degree at most } |F| \text{ and for all } x \in T, \\
\nu(Q(x)) \in \nu(b) + (|F| - 1)\delta + S_I \}.
\]
Step 1: $\mathcal{J} F^{-1}$ is interdefinable with $\mathcal{F}^{-1}$.

Observe that $J_F$ is definable over $\mathcal{F}^{-1}$, because $v(b)$, $\mathcal{F}^{-1}$, $\delta$ lie in $\text{dcl}^{eq}(\mathcal{F}^{-1})$. Hence, it is sufficient to prove that we can recover $F$ from $J_F$. For this we will show that given a monic polynomial $Q(x) \in K[x]$ with exactly $|F|$-different roots in $K$ each of multiplicity one, we have that $Q(x) \in J_F$ if and only if $Q(x)$ satisfies the following condition:

Condition: Let $\{\beta_1, \ldots, \beta_{|F|}\} \subseteq K$ be the set of all the roots of $Q(x)$ (note that all of them are different). For each $1 \leq i \leq |F|$ there is some $t \in F$ such that $\beta_i \in t$. And all the roots of $Q(x)$ lie in different torsors, that is, if $i \neq j$ and $t, t' \in F$ are such that $\beta_i \in t$ and $\beta_j \in t'$ then $t \neq t'$.

We first show that a monic polynomial $Q(x)$ with exactly $|F|$-different roots in $K$ each of multiplicity one satisfying the condition above belongs to $J_F$. Let $R = \{\beta_1, \ldots, \beta_{|F|}\} \subseteq K$ be the set of all the (different) roots of $Q(x)$. Let $x \in T$, then there is some $t \in F$ such that $x \in t$. Let $\beta_i$ be the root of $Q(x)$ that belongs to $t$, then $x, \beta_i \in t$, so $v(x - \beta_i) \in v(b) + S_I$. For any other index $j \neq i$ let $t' \in F$ be such that $\beta_j \in t'$. Because $t \neq t'$ then $v(x - \beta_j) = \delta$. Summarizing we have:

$$v(Q(x)) = v\left(\prod_{k \leq |F|} (x - \beta_k)\right) = v(x - \beta_i) + \sum_{j \neq i} v(x - \beta_j) \in v(b) + (|F| - 1)\delta + S_I.$$  

Consequently, $Q(x) \in J_F$.

For the converse, let $Q(x) \in J_F$ be a monic polynomial with exactly $|F|$-different roots $R = \{\beta_1, \ldots, \beta_{|F|}\} \subseteq K$. We show that $Q(x)$ satisfies the condition, that is, each root belongs to some torsor $t \in F$ and any two different roots belong to different torsors of $F$.

Claim 6.36.1. Given any torsor $t \in F$, there is a unique root $\beta \in R$ such that for all elements $x \in t$, $v(x - \beta) > \delta$.

Proof. Let $t \in F$ be a fixed torsor. We first show the existence of some root $\beta \in R$ such that for any $x \in t$, we have $v(x - \beta) > \delta$. We argue by contradiction, so let $t \in F$ and assume that there is no root $\beta \in R$ such that $v(x - \beta) > \delta$ for all $x \in t$. Then for each element $x \in t$ we have:

$$v(Q(x)) = v\left(\prod_{i \leq |F|} (x - \beta_i)\right) = \sum_{i \leq |F|} v(x - \beta_i) \leq |F|\delta.$$  

In this case $Q(x) \notin J_F$, because $|F|\delta \notin v(b) + (|F| - 1)\delta + S_I$ as $\delta \notin v(b) + S_I$. This concludes the proof for existence.

For uniqueness, let $\{t_1, \ldots, t_{|F|}\}$ be some fixed enumeration of $F$.

For each $i \leq |F|$, let $\beta_i \in R$ be such that for all $x \in t_i$ we have $v(x - \beta_i) > \delta$. We first argue that for any $i \neq j$, we must have that $\beta_i \neq \beta_j$. Suppose by contradiction that $\beta_i = \beta_j = \beta$, and let $x \in t_i$ and $y \in t_j$, then:

$$\delta = v(x - y) = v((x - \beta) + (\beta - y)) \geq \min\{v(x - \beta), v(y - \beta)\} > \delta.$$  

The uniqueness now follows because $|F| = |R|$.
By Claim 6.36.1, we can fix enumerations \{t_i \mid i \leq |F|\} of \(F\) and \{\beta_i \mid i \leq |F|\} of \(R\) such that for any \(x \in t_i\), \(v(x - \beta_j) > \delta\). We note that if \(j \neq i\), then for any \(x \in t_i\) we have that \(v(x - \beta_j) = \delta\). In fact, fix some \(y \in t_j\), because \(v(y - \beta_j) > \delta\) we have:

\[
v(x - \beta_j) = v((x - y) + (y - \beta_j)) = \min\{v(x - y), v(y - \beta_j)\} = \delta.
\]

**Claim 6.36.2.** For each \(i \leq |F|\), we have that \(\beta_i \in t_i\).

**Proof.** Fix some \(i \leq |F|\). For any \(x \in t_i\):

\[
v(Q(x)) = v\left(\prod_{k \leq |F|} v(x - \beta_k)\right) = v(x - \beta_i) + \sum_{j \neq i} v(x - \beta_j) = v(x - \beta_i) + (|F| - 1)\delta.
\]

Because \(Q(x) \in J_F\), we must have that \(v(x - \beta_i) \in v(b) + S_i\). Thus, \(\beta_i \in t_i\).

**Step 2:** \(F\) admits a code in the geometric sorts.

By the first step, \(F\) is interdefinable with \(J_F\). The latter one is a definable \(\Gamma\)-module, so by Lemma 5.7 it admits a code in the stabilizer sorts \(\mathcal{G}\).

**Lemma 6.37.** Let \(F\) be a primitive finite set of 1-torsors such that \(|F| > 1\). There is an \(\Gamma F^n\)-definable \(\mathcal{O}\)-lattice \(s \subseteq K^2\) and an \(\Gamma F^n\)-definable injective map \(g = F \to VS_{k, \Gamma s^n}\).

**Proof.** Let \(F\) be a primitive finite set of 1-torsors. By primitivity, there is some \(d \in K\) and \(I \in I\) such that for any \(t \in F\) there is some \(a_t \in K\) satisfying \(t = a_t + d I\). Moreover, there is some \(\delta \in \Gamma \backslash (v(d) + S_I)\) such that for any pair of different torsors \(t, t' \in F\), and \(x \in t, y \in t'\) we have \(v(x - y) = \delta\). Let \(T = \bigcup_{t \in F} t\), and take elements \(c \in T\) and \(b \in K\) such that \(v(b) = \delta\). Let \(U = c + b \mathcal{O}\). Then \(U\) is the smallest closed 1-torsor that contains all the elements of \(F\). Note that \(U\) is definable over \(\Gamma F^n\). Let \(h\) be the map sending each element of \(F\) to the unique class that contains it in \(\text{red}(U)\). By construction, such a map must be injective and it is \(\Gamma F^n\)-definable. Let \(s\) be the \(\mathcal{O}\)-lattice in \(K^2\), whose code is interdefinable with \(\Gamma U^n\) (given by Proposition 3.6). By Lemma 6.35, there is a \(\Gamma s^n\)-definable injection \(f : \text{red}(U) \to \text{red}(s)\). Let \(g = f \circ h\), the composition map \(g = F \to VS_{k, \Gamma s^n}\) satisfies the required conditions.

**Lemma 6.38.** Let \(F\) be a finite set of 1-torsors and let \(f : F \to \mathcal{G}\) be a definable function. Suppose that \(F\) is primitive over \(\Gamma F^n\), then:

1. For any set of parameters \(C\) if \(f(F) \subseteq VS_{k, C}\) then \(f\) is coded in \(\mathcal{G}\) over \(C\),
2. if \(f(F) \subseteq K\) then \(f\) is coded in \(\mathcal{G}\),
3. if \(f(F)\) is a finite set of 1-torsors (by primitivity all have the same type \(I \in I\)). Then \(f\) is coded in \(\mathcal{G}\).

**Proof.** In all the three cases, we may assume that \(|F| > 1\), otherwise the statement follows clearly. By primitivity of \(F\) over \(\Gamma F^n\), \(f\) is either constant or injective. If it is constant and equal to \(c\), the tuple \((\Gamma F^n, c)\) is a code for \(f\). By Lemma 6.36, \(\Gamma F^n\) admits a code in \(\mathcal{G}\), so \((\Gamma F^n, c)\) is interdefinable
with a tuple in the stabilizer sorts. In the following arguments, we assume that $f$ is an injective function and that $|F| \geq 2$.

(1) By Lemma 6.36, $\Gamma F^n \in \mathcal{G}$. Let $s$ be the $\mathcal{O}$-lattice of $K^2$ and $g : F \to VS_{k,\mathcal{C}r_s \gamma}$ the injective map given by Lemma 6.37. Both $s$ and $g$ are $\Gamma F^n$-definable. Let $F^* = g(F) \subseteq VS_{k,\mathcal{C}r_s \gamma}$, the map $f \circ g^{-1} : F^* \to VS_{k,\mathcal{C}r_s \gamma}$ can be coded in $\mathcal{G}$ over $C$ by Theorem 6.17. Hence, the tuple $(f \circ g^{-1}, \Gamma F^n)$ is a code of $f$ over $C$, because $g$ is a $\Gamma F^n$-definable bijection and $(f \circ g^{-1}, \Gamma F^n)$ is interdefinable over $C$ with a tuple of elements in $\mathcal{G}$.

(2) Let $D = f(F) \subseteq K$, this is a finite set in the main field so it can be coded by a tuple $d$ of elements in $K$. Because $F$ is primitive over $\Gamma f^n$, then $D$ is a primitive set. Thus, there is some $\delta \in \Gamma$ such that for any pair of different elements $x, y \in D$ $\nu(x - y) = \delta$. Let $b \in K$ be such that $\nu(b) = \delta$, take $x \in D$ and let $U = x + b\mathcal{O}$, this is the smallest closed 1-torsor containing $D$. The elements of $D$ all lie in different classes of $\text{red}(U)$ and let $g : D \to \text{red}(U)$ be the definable map sending each element $x \in D$ to the unique element in $\text{red}(U)$ that contains $x$. Both $U$ and $g$ are $\Gamma D^n$-definable, and therefore $d$-definable. By Proposition 3.6, there is an $\mathcal{O}$-lattice $s \subseteq K^2$, whose code is interdefinable with $\Gamma U^n$. Let $h : \text{red}(U) \to \text{red}(s)$ the $\Gamma U^n$-definable injective map given by Lemma 6.35. Both $U$ and $h$ are $d$-definable. By (1) of this statement, the function $h \circ g \circ f : F \to VS_{k,\mathcal{C}r_s \gamma}$ can be coded in $\mathcal{G}$. As $f$ and $h \circ g \circ f$ are interdefinable over $d$, the statement follows.

(3) Let $D = f(F)$ then $D$ must be a primitive set of 1-torsors because $F$ is primitive over $\Gamma f^n$ (in particular, there are $I \in I$ and $b \in K$ and elements $a_i \in K$ such that for each $t \in f(F)$ we have $t = a_t + bI$). By Lemma 6.36, we may assume $\Gamma D^n$ is a tuple in the stabilizer sorts. Let $s \subseteq K^2$ and $g : D \to \text{red}(s) \subseteq VS_{k,\mathcal{C}r_s \gamma}$ the injective map given by Lemma 6.37. Both $s$ and $g$ are $\Gamma D^n$-definable. By part (1) of this statement, the composition $g \circ f$ can be coded in $\mathcal{G}$, and as $g$ is a $\Gamma D^n$-definable bijection the tuple $(g \circ f^n, \Gamma D^n)$ is interdefinable with $\Gamma f^n$. \qed

6.5 | Coding of finite sets of tuples in the stabilizer sorts

We start by recalling some terminology from previous sections for sake of clarity.

**Notation 6.39.** Let $M \subseteq K^n$ be an $\mathcal{O}$-module, and $(I_1, \ldots, I_n) \in I$ be such that $M \cong \bigoplus_{i \in \mathbb{N}} I_i$. For any torsor $Z = \tilde{d} + M \in K^n/M$ we say that $Z$ is of type $(I_1, \ldots, I_n)$ and it has complexity $n$. We denote by $\pi_n : K^n \to K$ the projection to the last coordinate and for a torsor $Z = \tilde{d} + M \in K^n/M$ we write as $A_Z = \pi_n(Z)$. We recall as well the notation introduced in Notation 6.30 for the function that describes the fiber in $Z$ of each element at the projection, this is $h_Z(x) = \{y \in K^{n-1} \mid (y, x) \in Z\}$.

**Definition 6.40.** Let $F$ be a finite set of torsors, the complexity of $F$ is the maximum complexity of the torsors $t \in F$.

The following is a very useful fact that we will use repeatedly.

**Fact 6.41.** Let $F$ be a finite set of torsors, then there is a finite set $F' \subseteq \mathcal{G}$ and a bijection $f : F \cong F'$ which is $\Gamma F^n$-definable and $\Gamma F'^n$-definable. In particular, $\Gamma F^n$ and $\Gamma F'^n$ are interdefinable.

Moreover, any definable function $f : F \to P$, where $P$ is a finite set of torsors or $P \subseteq \mathcal{G}$, is interdefinable with a function $g : F' \to \mathcal{G}$, where $F' \subseteq \mathcal{G}$.
Proof. The statement follows immediately by Proposition 3.6.

The main goal of this section is the following theorem.

**Theorem 6.42.** For every \( m \in \mathbb{N}_{\geq 1} \), the following hold.

- \( I_m \): For every \( r > 0 \) and finite set \( F \subseteq G \) of size \( m \) then \( rF \upharpoonright \uparrow \) is interdefinable with a tuple of elements in \( G \).
- \( II_m \): For every \( F \subseteq G \) of size \( m \) and \( f : F \to G \) a definable function, then \( rF \upharpoonright \uparrow \) is interdefinable with a tuple in \( G \).

We will prove this statement by induction on \( m \), we note that for \( m = 1 \) the statements \( I_m \) and \( II_m \) follow trivially. We now assume that \( I_k \) and \( II_k \) hold for each \( k \leq m \) and we want to show \( I_{m+1} \) and \( II_{m+1} \). To keep the steps of the proof easier to follow we break the proof into some smaller steps. We write each step as a proposition to make the document more readable, as we will be carrying multiple inductions.

**Proposition 6.43.** Let \( F \) be a finite set of torsors of size at most \( m + 1 \), then \( rF \upharpoonright \uparrow \) is interdefinable with a tuple of elements in \( G \).

Furthermore, any definable function \( f : S \to F \), where:

- \( S \) is a finite set of 1-torsors of size at most \( (m + 1) \),
- \( F \) is a finite set of at most \( m + 1 \) torsors,

can be coded in \( G \).

Proof. We will start by proving the following statements by a simultaneous induction on \( n \).

- \( A_n \): Any set \( F \) of torsors of size at most \( m + 1 \) of complexity at most \( n \) can be coded in \( G \).
- \( B_n \): Every definable function \( f : S \to F \), where \( S \) is a finite set of 1-torsors of size at most \( (m + 1) \) and \( F \) is a finite set of size at most \( (m + 1) \) torsors of complexity at most \( n \) can be coded in \( G \).

We observe first that we may assume in \( A_n \) that \( F \) is a primitive set of size \( m + 1 \). If \( |F| \leq m \), the statement follows immediately by Fact 6.41 combined with \( I_k \) for each \( k \leq m \). So, we may assume that \( F \) has \( m + 1 \) elements. If \( F \) is not primitive, then we can find a nontrivial equivalence \( E \) relation definable over \( rF \upharpoonright \uparrow \), and let \( C_1, ..., C_l \) be the equivalence classes. For each \( i \leq l \), \( |C_i| \leq m \), by Fact 6.41 and because \( I_k \) holds for each \( k \leq m \) \( rF \upharpoonright \uparrow \) is interdefinable with a tuple \( c_i \) of elements in \( G \). Because \( l \leq m \) and \( I_l \) holds, we can find a code \( c \) in the stabilizer sorts of the set \( \{c_1, ..., c_l\} \). The code \( rF \upharpoonright \uparrow \) is interdefinable with \( c \in G \).

Likewise, for \( B_n \) we may assume that \( S \) is primitive over \( rF \upharpoonright \uparrow \). Otherwise, there is a \( (rF \upharpoonright \uparrow \cup rS) \)-definable nontrivial equivalence relation \( E \) on \( S \) and let \( C_1, ..., C_l \) be the equivalence classes of this relation. For each \( i \leq l \), \( |C_i| \leq m \) and let \( f_i = f \upharpoonright C_i \). By Fact 6.41, for each \( i \leq l \) \( rF \upharpoonright \uparrow \) is interdefinable with a map \( g_i : S_i \to G \) where \( S_i \subseteq G \) and \( |S_i| \leq m \). Because \( II_k \) holds for each \( k \leq m \), \( f_i \) admits a code \( c_i \) in \( G \). Because \( I_k \) holds, we can find a code \( c \) for the finite set \( \{c_1, ..., c_l\} \). The codes \( rF \upharpoonright \uparrow \) and \( c \) are interdefinable.

We continue arguing for the base case \( n = 1 \). The statement \( A_1 \) holds by Lemma 6.36, while \( B_1 \) is given by (3) of Lemma 6.38. We now assume that \( A_n \) and \( B_n \) hold and we prove \( A_{n+1} \) and \( B_{n+1} \).
First we prove that $A_{n+1}$ holds. Let $F$ be a primitive finite set of torsors of size $m + 1$. By primitivity all the torsors in $F$ are of the same type. For each $Z \in F$ we write $A_Z$ to denote the projection of $Z$ into the last coordinate. By primitivity of $F$ the projections to the last coordinate are either all equal or all different. We argue by following cases.

(1) **Case 1:** All the projections are equal, that is, $A = A_Z$ for all $Z \in F$.

**Proof.** For each $x \in A$, the set of fibers $F_x = \{ h_Z(x) | Z \in F \}$ is a finite set of torsors of size at most $m + 1$ of complexity at most $n$. By the induction hypothesis $A_{n}$, $F_x$ admits a code in the stabilizer sorts. By compactness, we can uniformize such codes, and we can define the function $g : A \to \mathcal{G}$ by sending the element $x$ to the code $\{ h_Z(x) | Z \in F \}$. This is a $\mathcal{G}$-definable function. Let $p_A(x)$ be a global type extending the generic type of $A$, it is $\mathcal{G}$-definable by Corollary 6.22. By Theorem 6.27, the germ of $g$ over $p_A$ can be coded in $\mathcal{G}$ over $\{ A \}$. By Corollary 6.33 for any $Z \in F$ the code $\{ h_Z, p_A \}$ is interdefinable with $\{ A, \{ \text{germ}(h_Z, p_A) \} | Z \in F \}$. 

**Claim 6.43.1.** $\text{germ}(g, p_A)$ is interdefinable with the code $\{ \text{germ}(h_Z, p_A) | Z \in F \}$ over $\{ A \}$.

**Proof.** We first prove that $\text{germ}(g, p_A) \in \text{dcl}^{\mathcal{G}}(\{ A \}, \{ \text{germ}(h_Z, p_A) | Z \in F \})$.

Let $\sigma \in \text{Aut}(\mathfrak{M}/\{ \text{germ}(h_Z, p_A) | Z \in F \}, \mathcal{G})$, we want to show that $\sigma(\text{germ}(g, p_A)) = \text{germ}(\sigma(g), p_A) = \text{germ}(g, p_A)$. Let $B$ the set of all the parameters required to define all the objects that have been mentioned so far. It is therefore sufficient to argue that for any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we have $\sigma(g)(c) = g(c)$, where $\sigma(g) : A \to \mathcal{G}$ is the function given by sending the element $x$ to the code $\{ h_\sigma(Z)(x) | Z \in F \}$. Note that

$$\sigma(\{ \text{germ}(h_Z, p_A) | Z \in F \}) = \{ \text{germ}(h_\sigma(Z), p_A) | Z \in F \} = \{ \text{germ}(h_Z, p_A) | Z \in F \},$$

because $\sigma(\{ \text{germ}(h_Z, p_A) | Z \in F \}) = \{ \text{germ}(h_Z, p_A) | Z \in F \}$. As a result, for any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we must have that $\{ h_Z(c) | Z \in F \} = \{ h_\sigma(Z)(c) | Z \in F \}$ so $g(c) = \sigma(g)(c)$, as desired.

For the converse, let $\sigma \in \text{Aut}(\mathfrak{M}/\{ A \}, \text{germ}(g, p_A))$ we want to show that $\sigma(\{ \text{germ}(h_Z, p_A) | Z \in F \}) = \{ \text{germ}(h_Z, p_A) | Z \in F \}$. Let $c$ be a realization of $p_A(x)$ sufficiently generic over $B$ by hypothesis $g(c) = \sigma(g)(c)$. Then:

$$g(c) = \{ h_Z(c) | Z \in F \} = \{ h_\sigma(Z)(c) | Z \in F \} = \sigma(g)(c).$$

Therefore, for each $Z \in F$ there is some $Z' \in F$ such that $h_Z(c) = h_\sigma(Z')(c)$ and this implies that $\text{germ}(h_Z, p_A) = \text{germ}(h_\sigma(Z'), p_A)$. Thus,

$$\sigma(\{ \text{germ}(h_Z, p_A) | Z \in F \}) = \{ \text{germ}(h_\sigma(Z), p_A) | Z \in F \} = \{ \text{germ}(h_Z, p_A) | Z \in F \}.$$  

We conclude that $\sigma(\{ \text{germ}(h_Z, p_A) | Z \in F \}) = \{ \text{germ}(h_Z, p_A) | Z \in F \}$, as desired. □

Consequently, $F$ is coded by the tuple $(\{ A \}, \text{germ}(g, p_A))$ which is a sequence of elements in $\mathcal{G}$. □

(2) **Case 2:** All the projections are different, that is, $A_Z \neq A_{Z'}$ for all $Z \neq Z' \in F$. 

ELIMINATION OF IMAGINARIES IN $\mathbb{C}((\Gamma))$

**FIGURE 1** Graphical representation of the function $l_*$. 

**Proof.** To simplify the notation, fix some enumeration of the projections $S = \{A_Z \mid Z \in F\}$ say $\{A_1, ..., A_n\}$. Let $W = \{\{x_1, ..., x_n\} \mid x_i \in A_i\}$, such set is independent from the choice of the enumeration.

Each set $\{x_1, ..., x_n\} \in W$ admits a code in the home sort $K$, because fields uniformly code finite sets. We denote by $W^* = \{\{x_1, ..., x_n\}^\tau | \{x_1, ..., x_n\} \in W\}$, the set of all these codes.

For each $x^* \in W^*$, we define the function $f_{x^*} : S \to K$ that sends $A_Z \mapsto x_Z$, where $x_Z$ is the element in the set coded by $x^*$ that belongs to $A_Z$. Let $l_{x^*}$ be the function given by sending $A_Z \mapsto \llbracket h_Z(f_{x^*}(A_Z)) \rrbracket^\tau$.

This map sends the projection $A_Z$ to the code of the fiber in the module $Z$ at the point $x_Z$, which is the unique point in the set coded by $x^*$ that belongs to $A_Z$ (see Figure 1).

**Claim 6.43.2.** For each $x^* \in W^*$, the functions $f_{x^*}$ and $l_{x^*}$ can be coded in $G$.

**Proof.** We argue first for the function $f_{x^*}$. If $S$ is not primitive over $\llbracket f_{x^*} \rrbracket^\tau$ then there is a nontrivial equivalence relation $E$ definable over $(S \cup \llbracket f_{x^*} \rrbracket^\tau)$ and let $C_1, ..., C_l$ be the equivalence classes of $E$. For each $i \leq l$, $|C_i| \leq m$ and let $f_{x^*}^i = f_{x^*}^i \upharpoonright C_i : C_i \to K$. For each $i \leq l$ $\llbracket f_{x^*}^i \rrbracket^\tau$ is interdefinable with a tuple $c_i$ of elements in $G$, this follows by combining Fact 6.41 and $II_k$ for each $k \leq m$. Because $I_1$ holds, the set $\{c_1, ..., c_l\}$ admits a code $c$ in the stabilizer sorts. Then $\llbracket f_{x^*} \rrbracket^\tau$ and $c$ are interdefinable.

For the function $l_{x^*}$, the statement follows immediately by the induction hypothesis $B_n$, as the complexity of the fibers must be at most $n$, that is, $\{h_Z(f_{x^*}(A_Z)) \mid Z \in F\}$ is a finite set of at most $m + 1$ torsors of complexity at most $n$.

By compactness, we can uniformize all such codes, so we can define the function $g : W^* \to G$ by sending $x^* \mapsto (\llbracket f_{x^*} \rrbracket^\tau, \llbracket l_{x^*} \rrbracket^\tau)$.

By Lemma 6.29, there is some $\llbracket W^* \rrbracket^\tau$-definable type $q(x^*) \vdash x^* \in W^*$. The second part of Lemma 6.29 also guarantees that given $d^*$ a generic realization of $q$ over a set of parameters $B$, if we take $Y$ the set coded by $d^*$ and $b$ is the element in $Y$ that belongs to $A_Z$ then $b$ is a sufficiently generic realization over $B$ of some type $p_{A_Z}(x)$ which is $\llbracket A_Z \rrbracket^\tau$-definable and extends the generic type of $A_Z$. We recall as well that the types $p_{A_Z}(x)$ given by Lemma 6.29 are all compatible under the action of $\text{Aut}(\mathcal{M} / \llbracket F \rrbracket^\tau)$, this is for any $\sigma \in \text{Aut}(\mathcal{M} / \llbracket F \rrbracket^\tau)$ if $\sigma(Z) = Z'$ then $\sigma(p_{A_Z}(x)) = p_{A_{Z'}(x)}$. By Theorem 6.27 the germ of $g$ over $q$ can be coded in the stabilizer sorts $G$ over $\llbracket W^* \rrbracket^\tau \in dcl^{eq}(\llbracket S \rrbracket^\tau)$. Because $A_1$ holds, we may assume $\llbracket S \rrbracket^\tau \in G$.

**Claim 6.43.3.** The tuple $(\text{ker}(g, q), \llbracket S \rrbracket^\tau) \in G$ is interdefinable with $\llbracket F \rrbracket^\tau$. 

Proof. It is clear that \((\text{germ}(g, q), \gamma_S) \in \text{dcl}^d(\gamma_F)\). For the converse, let \(\sigma \in \text{Aut}(\mathfrak{M}/\text{germ}(g, q), \gamma_S)\) we want to show that \(\sigma(F) = F\). By Corollary 6.33 the code of each torsor \(Z \in F\) is interdefinable with the pair \((A_Z, \text{germ}(h_Z, p_{A_Z}))\).

Hence, it is sufficient to argue that:

\[
\sigma(\{(A_Z, \text{germ}(h_Z, p_{A_Z})) \mid Z \in F\}) = \{(A_Z, \text{germ}(h_Z, p_{A_Z})) \mid Z \in F\}.
\]

We have that \(\sigma(\gamma_F^\ast) = \gamma_F^\ast\) because \(\sigma(S) = S\). Therefore, \(\sigma(\text{germ}(g, q)) = \text{germ}(\sigma(g), q) = \text{germ}(g, q)\). Let \(B\) be the set of parameters required to define all the objects that have been mentioned so far. For any realization \(d^*\) of the type \(q\) sufficiently generic over \(B\) we have \(g(d^*) = \sigma(g)(d^*)\), where \(\sigma(g)\) is the function sending an element \(x^*\) in \(W^*\) to the tuple \((\sigma(f)_{x^*}, \sigma(l)_{x^*})\) (These are the functions described as above but working with \(\sigma(F)\) instead of \(F\)).

Consequently, \((\gamma f_{d^*})^\gamma = \gamma (\sigma(f)_{d^*})^\gamma, \gamma (\sigma(l)_{d^*})^\gamma\). Let \(D\) be the set of elements coded by \(d^*\).

The function \(f_{d^*} : S \to D \subseteq K\) is the function that sends the projection into the last coordinate \(A_Z = \pi(Z)\) to the unique point in \(D\) that belongs to \(A_Z\), we write such element as \(d_Z\) to simplify the notation.

Likewise, the function \(\sigma(f)_{d^*} : S \to D \subseteq K\) is the function that sends the projection to the last coordinate \(A_{\sigma(Z)} = \pi(\sigma(Z))\) to the unique point in \(D\) that belongs to \(A_{\sigma(Z)}\), to simplify the notation we denote such element as \(d_{\sigma(Z)}\).

Because \(\gamma f_{d^*} = \gamma (\sigma(f)_{d^*})\), the two functions are the same.

Then, for any \(Z \in F\) there is a unique torsor \(Z' \in F\) such that:

\[
(A_{\sigma(Z)}, d_{\sigma(Z)}) = (A_{Z'}, d_{Z'}).
\]

Thus, for every \(Z \in F\) there is a unique \(Z' \in F\) such that

\[
A_{\sigma(Z)} = A_{Z'} \quad \text{and} \quad d_{\sigma(Z)} = d_{Z'}.
\] (6.1)

Recall that for each \(\hat{Z} \in F\), \(d_{\hat{Z}}\) is a realization of some \(\gamma A_{\hat{Z}}\)-definable complete extension \(p_{A_{\hat{Z}}(x)}\) of the generic type of \(A_{\hat{Z}}\). Thus, \(\sigma(p_{A_{\hat{Z}}(x)}) = p_{A_{\sigma(\hat{Z})}(x)}\) is a \(\gamma A_{\sigma(\hat{Z})}\)-definable complete type extending the generic type of \(A_{\sigma(\hat{Z})}\).

Fix an element \(Z \in F\) and let \(Z'\) be the unique torsor in \(F\) such that \(A_{\sigma(Z)} = A_{Z'}\) and \(d_{\sigma(Z)} = d_{Z'}\).

Thus, \(d_{\sigma(Z)}\) is a realization of the \(\gamma A_{\sigma(Z)}\)-definable type \(p_{A_{\sigma(Z)}}(x)\). Because \(d_{Z'} = d_{\sigma(Z)}\) is a realization of the \(\gamma A_{Z'}\)-definable type \(p_{A_{Z'}}(x)\), then \(p_{A_{Z'}}(x) = p_{A_{\sigma(Z)}}(x)\).

We conclude that for any \(Z \in F\) there is a unique \(Z' \in F\) such that:

\[
\sigma(p_{A_{Z}}(x)) = p_{A_{\sigma(Z)}}(x) = p_{A_{Z'}}(x)
\] (6.3)

and \(d_{Z'} \models p_{A_{Z'}}(x)\). (6.4)

On the other hand, because \(\gamma l_{d^*}^\gamma = \gamma (\sigma(l)_{d^*})^\gamma\), the two functions are the same.
The function $l_{d^*}: S \to G$ sends the torsor $A_Z$ to the fiber in $Z$ at the point $d_Z$, that is, to $h_Z(d_Z)$, while $\sigma(l)_{d^*}: S \to G$ sends the torsor $A_{\sigma(Z)}$ to the fiber in $\sigma(Z)$ at the point $d_{\sigma(Z)}$, that is, to $h_Z(d_{\sigma(Z)})$.

Because the functions are the same, for each $Z \in F$ there is a unique $Z' \in F$ such that:

\[ (A_{\sigma(Z)}, h_{\sigma(Z)}(d_{\sigma(Z)})) = (A_{Z'}, h_{Z'}(d_{Z'})). \] (6.5)

Fix some $Z \in F$ and let $Z' \in F$ be such that 6.5 holds. By uniqueness of 6.1, $d_{\sigma(Z)} = d_{Z'}$.

By 6.5, $h_{\sigma(Z)}(d_{Z'}) = h_{Z'}(d_{Z'})$. Finally, by 6.3

\[ \sigma(\text{germ}(p_{A_Z}, h_Z)) = (\text{germ}(p_{A_{\sigma(Z)}}, h_{\sigma(Z)})) = \text{germ}(p_{A_{Z'}}, h_{Z'}). \]

Consequently,

\[ \sigma\left(\{(\gamma A_Z, \text{germ}(h_Z, p_{A_Z})) \mid Z \in F\}\right) \]
\[ = \{(\gamma A_{\sigma(Z)}, \text{germ}(h_{\sigma(Z)}, p_{A_Z})) \mid Z \in F\} \]
\[ = \{(\gamma A_{Z'}, \text{germ}(h_{Z'}, p_{A_{Z'}})) \mid Z' \in F\}, \]

as desired. \[ \square \]

This finalizes the proof for Case 2. \[ \square \]

Consequently $A_{n+1}$ holds. We prove $B_{n+1}$, that is, every definable function $f: S \to F$ where $S$ is a finite set of at most $m + 1$-torsors and $F$ is a finite set of torsors of complexity at most $n + 1$ can be coded in $G$. We recall that without loss of generality we may assume that $S$ is primitive over $\gamma f^\gamma$, so $F$ is also a primitive set. By primitivity $f$ is either constant or injective, if it is constant equal to $c$ then $\gamma f^\gamma$ is interdefinable with $(\gamma S^\gamma, c)$. By Proposition 3.6 and Lemma 6.36 this tuple is interdefinable with a tuple in $G$. Thus, we may assume that $f$ is an injective function. By primitivity of $F$, all the torsors in $F$ are of the same type so they have the same complexity. We may assume that all are of complexity exactly $n + 1$, otherwise the statement follows immediately by $B_n$. By primitivity of $F$, the projections to the last coordinate are either all equal or all different. We proceed again by cases.

(1) **Case 1: The projections are all equal, that is, there is a torsor $A$ such that $A = A_Z$ for all $Z \in F$.**

**Proof.** We fix $p_{A}(x)$ be some global type extending the generic type of $A$, it is $\gamma A^\gamma$-definable by Corollary 6.22. Let $f: S \to F$ be a definable injective map. For each $x \in A$ we define the function $g_x: S \to G$ by sending $t \mapsto \gamma h_{f(t)}(x)^\gamma$.

This is the function that sends each torsor $t \in S$ to the fiber at $x$ of the torsor $f(t) \in F$ (see Figure 2).

By the induction hypothesis $B_n$ for each $x \in A$, the function $g_x$ can be coded in $G$ because its range is of lower complexity. By compactness we can uniformize such codes, so we can define the function $r: A \to G$ by sending $x \mapsto \gamma g_x^\gamma$.

By Theorem 6.27, the germ of $r$ over $p_{A}(x)$ can be coded in $G$ over $\gamma A^\gamma$. By Lemma 6.36, the set $S$ admits a code $\gamma S^\gamma$ in the stabilizer sorts.
Claim 6.43.4. The code $\mathcal{R} f^\gamma$ is interdefinable with $(\mathcal{R} A^\gamma, \mathcal{R} S^\gamma, \text{germ}(r, p_A))$, and the latter is a sequence of elements in $\mathcal{G}$.

Proof. It is clear that $(\mathcal{R} A^\gamma, \mathcal{R} S^\gamma, \text{germ}(r, p_A)) \in \text{dcl}^q(\mathcal{R} f^\gamma)$.

We want to show that

$$\mathcal{R} f^\gamma \in \text{dcl}^q(\mathcal{R} A^\gamma, \mathcal{R} S^\gamma, \text{germ}(r, p_A)).$$

Let $\sigma \in \text{Aut}(\mathcal{M}/\mathcal{R} A^\gamma, \mathcal{R} S^\gamma, \text{germ}(r, p_A))$. By Corollary 6.33 for each torsor $Z \in F = \{f(t) \mid t \in S\}$, the code $\mathcal{R} Z^\gamma$ is being identified with the tuple $(\mathcal{R} A^\gamma, \text{germ}(h_Z, p_A))$. Thus, the function $f$ is interdefinable over $\mathcal{R} A^\gamma$ with the function: $f' : S \to \mathcal{G}$ that sends $t \mapsto \text{germ}(h_{f(t)}, p_A)$. So, it is sufficient to argue that $\sigma(\mathcal{R} f^\gamma) = \mathcal{R} f'^\gamma$. Let $B$ be the set of parameters required to define all the objects that have been mentioned so far. For any realization $c$ of $p_A(x)$ sufficiently generic over $B$ we have that $r(c) = \sigma(r)(c)$, because $\text{germ}(r, p_A) = \sigma(\text{germ}(r, p_A)) = \text{germ}(\sigma(r), p_A)$. By definition, $r(c) = \mathcal{R} g_c^\gamma$ and $\sigma(r)(c) = \mathcal{R} (\sigma(g_c))^\gamma$, where $\sigma(g_c) : S \to \mathcal{G}$ is the function that sends $t \mapsto \mathcal{R} h_{\sigma(f)(t)}(c)$. The later implies that $\text{germ}(\mathcal{R} h_{f(t)}, p_A) = \sigma(\mathcal{R} h_{f(t)}, p_A)$, meaning that $\sigma$ is acting as a bijection among the elements in the graph of $f'$. Therefore, $\sigma(\mathcal{R} f'^\gamma) = \mathcal{R} f'^\gamma$, as desired.

This completes the proof for the first case.

(2) Case 2: All the projections are different, that is, $A_Z \neq A_{Z'}$ for all $Z \neq Z' \in F$.

Proof. Let $f : S \to F$ be a definable injective function where $S$ is a finite set of 1-torsors primitive over $\mathcal{R} f^\gamma$. Let $\pi_{n+1} : K^{n+1} \to K$ be the projection into the last coordinate. We consider the definable function that sends each torsor $t \in S$ to the code of the projection into the last coordinate of the torsor $f(t) \in F$, more explicitly:

$$\pi_{n+1} \circ f : \begin{cases} S & \to \mathcal{G} \\ t & \mapsto \mathcal{R} \pi_{n+1}(f(t))^\gamma. \end{cases}$$

By Lemma 6.38(3), $\pi_{n+1} \circ f$ can be coded in $\mathcal{G}$, and by $A_{n+1}$ the finite set $F$ is coded by a tuple in $\mathcal{G}$. It is sufficient to show the following claim:
Claim 6.43.5. The code $\gamma f^\gamma$ is interdefinable with the tuple $(\gamma_{n+1} \circ f^\gamma, \gamma F^\gamma)$, which is a tuple in the stabilizer sorts.

Proof. Clearly, $(\gamma_{n+1} \circ f^\gamma, \gamma F^\gamma) \in \text{dcl}^\text{eq}(\gamma f^\gamma)$. Note that $\gamma S^\gamma \in \text{dcl}^\text{eq}(\gamma_{n+1} \circ f^\gamma)$ because $S$ is the domain of the given function. We can describe the function $f : S \to F$ by sending $t \mapsto \gamma Z_t^\gamma$, where $Z_t$ is the unique torsor in $F$ such that $\gamma_{n+1}(Z_t)^\gamma = (\gamma_{n+1} \circ f)(t)$, we conclude that $\gamma f^\gamma \in \text{dcl}^\text{eq}(\gamma_{n+1} \circ f^\gamma, \gamma F^\gamma)$. Consequently, $f$ is coded in $\mathcal{G}$ by the tuple $(\gamma_{n+1} \circ f^\gamma, \gamma F^\gamma)$.

This finalizes the proof for the second case.

Consequently, $A_n$ and $B_n$ hold for all $n \in \mathbb{N}$. The statement follows.

We continue arguing that $I m+1$ holds for $r = 1$.

Proposition 6.44. Let $F \subseteq \mathcal{G}$ be a finite set of size $m + 1$ then $F$ admits a code in $\mathcal{G}$.

Proof. If $F$ is not primitive, we show that $\gamma F^\gamma$ can be coded in $\mathcal{G}$, by using Fact 6.41 and the induction hypothesis $I_k$ for $k \leq m$. We may assume that $F$ is a primitive set, so all the elements of $F$ lie in the same sort. If $F$ is either contained in the main field or the residue field, then $F$ is coded by a tuple of elements in the same field, because fields code uniformly finite sets. If $F \subseteq \Gamma / \Delta$ for some $\Delta \in RJ(\Gamma)$ the statement follows as there is a definable order over the elements of $F$. If $F \subseteq B_n(K) / \text{Stab}(I_1, \ldots, I_n)$ for some $n \geq 2$, by Proposition 6.43 $F$ admits a code in $\mathcal{G}$. (Indeed, $\varnothing$-modules are in particular torsors).

We continue showing that $II m+1$ holds, we first prove the following statement.

Proposition 6.45. Let $F$ be a finite set of torsors of size $m + 1$ and $f : F \to P$ be a definable bijection, where $P$ is a finite set of torsors. Suppose that $F$ is primitive over $\gamma f^\gamma$, then $\gamma f^\gamma$ is interdefinable with a tuple of elements in $\mathcal{G}$.

Proof. We proceed by induction on the complexity of the torsors in $F$. The base case follows directly by Proposition 6.43. We assume the statement for any set of torsors $F$ with complexity $n$ and we prove it for complexity $n + 1$. By primitivity all the projections into the last coordinate are either equal or all distinct. For each torsor $Z \in F$, we denote as $A_Z$ the projection of $Z$ into the last coordinate. We argue by following cases.

(1) Case 1: All the projections are equal and let $A = A_Z$ for all $Z \in F$.

For each $x \in A$, let $I_x = \{h_Z(x) | Z \in F\}$ that describes the set of fibers at $x$. We define $B = \{x \in A | |I_x| = |F|\}$ that is a $\gamma F^\gamma$-definable set. For each $y \in B$ we consider the map $g_y : I_y \to P$ defined by sending $h_Z(y) \mapsto f(Z)$, which is the function that sends each fiber to the image of the torsor under $f$. By the induction hypothesis we can find a code $\gamma g_y^\gamma$ in $\mathcal{G}$, and by compactness we can uniformize such codes. Therefore, we can define the function: $r : B \to \mathcal{G}$ by sending $y \mapsto \gamma g_y^\gamma$.

Let $p_A(x)$ be a global complete type containing the generic type of $A$, it is $\gamma A^\gamma$-definable by Corollary 6.22. If we fix a realization of the generic type $c$ of $p_A(x)$ sufficiently generic over $\{\gamma Z^\gamma | Z \in F\}$, and $Z \neq Z' \in F$ then the fibers $h_Z(c)$ and $h_{Z'}(c)$ must be different by Corol-
lary 6.33. Hence, $p_A(x) \vdash x \in B$. By Theorem 6.27 the germ of $r$ over $p_A(x)$ can be coded in $\mathcal{G}$ over $^rA^n$.

**Claim 6.45.1.** The code $^rF^n$ is interdefinable with $(\text{germ}(r, p_A), ^rF^n)$ which is a tuple in the stabilizer sorts $\mathcal{G}$. (By Proposition 6.43 $^rF^n \in \mathcal{G}$, while germ$(r, p_A) \in \mathcal{G}$ by Theorem 6.27.)

**Proof.** Clearly, $(\text{germ}(r, p_A), ^rF^n) \in \text{dcl}^q(\mathcal{G})$. We will argue that for any automorphism $\sigma \in \text{Aut}(\mathcal{M}/^rF^n, \text{germ}(r, p_A))$ we have $\sigma(^rF^n) = ^rF^n$. As each torsor $Z \in F$ is being identified with the tuple $(^rA^n, \text{germ}(h_Z, p_A))$, and $^rA^n \in \text{dcl}^q(^rF^n)$ then it is sufficient to argue that:

$$\sigma([(h_Z, p_A, f(Z)) \mid Z \in F]) = [(h_Z, p_A, f(Z)) \mid Z \in F].$$

For any $Z \in F$ there is a unique torsor $Z' \in F$ such that $\sigma(Z') = Z$, because $\sigma(^rF^n) = ^rF^n$. Let $D$ be the set of parameters required to define all the objects that have been mentioned so far. For any realization $c$ of the type $p_A(x)$ sufficiently generic over $D$ we have $r(c) = \sigma(r(c))$, because $\sigma(\text{germ}(r, p_A)) = \text{germ}(\sigma(r), p_A)$. Consequently $r(c) = g_c = \sigma(g_c) = \sigma(r)(c)$. In particular, $h_{\sigma(Z')}(c) = h_Z(c)$ which implies that $\text{germ}(h_{\sigma(Z')}), p_A) = \text{germ}(h_Z, p_A)$. In addition, $\sigma(f(\sigma(Z'))) = \sigma(g_c(h_{\sigma(Z')}(c))) = g_c(h_Z(c)) = f(Z)$. Therefore,

$$\sigma([(h_Z, p_A, f(Z)) \mid Z \in F]) = [(h_Z, p_A, f(Z)) \mid Z \in F],$$

as desired. □

(2) **Case 2:** All the projections are different, that is, for all $Z \neq Z' \in F$ we have $A_Z \neq A_{Z'}$.

By Proposition 6.43, we can find a code in the stabilizer sorts for $F$, and $^rF^n \in \text{dcl}^q(^rF^n)$ as it is the domain of this function. Let $S = \{A_Z \mid Z \in F\}$ and define the function $g : S \to F$ by sending $A_Z \mapsto Z$, where $Z$ is the unique torsor in $F$ satisfying that $\pi_{n+1}(Z) = A_Z$. Clearly $g$ is a $^rF^n$-definable bijection. We consider the map $f \circ g : S \to P$ that sends $A_Z \mapsto f(Z)$. By Proposition 6.43, the function $f \circ g$ admits a code in the stabilizer sorts.

**Claim 6.45.2.** The code $^rF^n$ is interdefinable with the tuple $(^rF \circ g^n, ^rF^n)$ which is a tuple in the stabilizer sorts.

**Proof.** It is clear that $(^rF \circ g^n, ^rF^n) \in \text{dcl}^q(^rF^n)$. For the converse note that $S$ is definable from $^rF \circ g^n$ because it is its domain. As $F$ is given, we can define the function $\pi : F \to S$ that sends $Z \mapsto A_Z$. This is the map that sends each torsor to its projection into the last coordinate. We observe that $f = (f \circ g) \circ \pi$, in fact $f(Z) = (f \circ g)(A_Z)$. So $^rF^n \in \text{dcl}^q(^rF \circ g^n, ^rF^n)$. □

This completes the proof of the proposition. □

We are now ready to show that $II_{m+1}$ holds.

**Proposition 6.46.** For every $F \subseteq \mathcal{G}$ finite set of size $m + 1$ and definable function $f : F \to \mathcal{G}$, the code $^rF^n$ is interdefinable with a tuple of elements in $\mathcal{G}$.

**Proof.** Without loss of generality, we may assume that $F$ is primitive over $^rF^n$. Otherwise, there is a $(^rF^n \cup ^rF^n)$-definable equivalence relation on $F$ and we let $C_1, \ldots, C_l$ be the equivalence classes. For each $i \leq l$ we have $|C_i| \leq m$ and let $f_i = f \upharpoonright C_i$. By the induction hypothesis, for each $k \leq m$ the
Proposition 6.44. Summarizing, we may assume that over that position Let be the map given by Lemma 6.35. By Proposition 6.45, the composition and that property ELIMINATION OF IMAGINARIES IN \( \mathbb{C}((\Gamma)) \) respectively) \( f(\mathcal{F}) \subseteq \mathcal{K} \) lies in the stabilizer sorts and is interdefinable with the code of \( f \). A similar argument applies if \( F \subseteq \Gamma / (\Delta + \ell \Gamma) \) where \( \Delta \in R J(\Gamma) \) and \( \ell \in \mathbb{N}_{\geq 2} \) because \( \Gamma / (\Delta + \ell \Gamma) \subseteq dcl^{\mathcal{G}}(\emptyset) \).

Because \( rF^\gamma \) and \( rF^{-1} \) are interdefinable, a similar argument proves that \( f \) can be coded in \( \mathcal{G} \) if \( f(F) \) is contained in \( \Gamma / \Delta \) or \( \Gamma / (\Delta + \ell \Gamma) \) for some \( \Delta \in R J(\Gamma) \) and \( \ell \in \mathbb{N}_{\geq 2} \).

If \( F \) is contained in the residue field, then \( F \) is interdefinable with the code of a finite set of 1-torsors of type \( \mathcal{M} \). If \( F \subseteq B_n(\mathcal{K}) / \text{Stab}(I_1, \ldots, I_n) \), the statement follows by Proposition 6.45. If \( f(F) \subseteq \mathcal{K} \) or \( f(F) \) is contained in the residue field then by Lemma 6.38 (part (2) and (1), respectively) \( f \) can be coded in \( \mathcal{G} \).

If \( F \subseteq B_n(\mathcal{K}) / \text{Stab}(I_1, \ldots, I_n) \), for some \( n \geq 2 \), and \( f(F) \) is contained in one of the sorts of the form \( B_n(\mathcal{K}) / \text{Stab}(I'_1, \ldots, I'_n) \) the statement follows by Proposition 6.45, because \( \mathcal{O} \)-modules are torsors. Likewise, if \( f(F) \) is contained in the residue field, by Proposition 6.45 \( f \) can be coded in \( \mathcal{G} \), because each element in the residue field is a 1-torsor of type \( \mathcal{M} \). Let’s assume that \( f(F) \subseteq \mathcal{K} \), because fields code finite sets we can find a tuple \( d \) of elements in the field that code the finite set \( f(F) \). Let \( U \) be the smallest closed torsor that contains all the elements of \( f(F) \), this is a \( d \)-definable set. Let \( g \) be the function that sends each element \( x \in f(F) \) to the unique class of \( \text{red}(U) \) that contains such element. Let \( s \) be the \( \mathcal{O} \)-lattice whose code is interdefinable with \( rU^\gamma \), and let \( h = \text{red}(U) \to \text{red}(s) \) be the map given by Lemma 6.35. By Proposition 6.45, the composition \( h \circ g \circ f = F \to \text{red}(s) \) can be coded in the stabilizer sorts \( \mathcal{G} \). Let \( D = h \circ g(f(F)) \subseteq \text{red}(s) \). Because \( h \circ g = f(F) \to D \) is a \( d \)-definable bijection, then \( f \) is interdefinable with the tuple \( (d, r(h \circ g \circ f)^\gamma) \) which is a sequence of elements in \( \mathcal{G} \).

It is therefore left to consider the case where \( F \subseteq \mathcal{K} \). Because fields code uniformly finite sets, we may assume that \( f(F) \) is contained in the residue field or a sort of the form \( B_n(\mathcal{K}) / \text{Stab}(I_1, \ldots, I_n) \) and that \( rF^\gamma \) is a tuple of elements in the main field. Let \( U \) be the smallest closed torsor that contains all the elements of \( F \), this is a \( rF^\gamma \)-definable set. Let \( g \) be the function that sends each element \( x \in F \) to the unique class of \( \text{red}(U) \) that contains such element. Let \( s \) be the \( \mathcal{O} \)-lattice whose code is interdefinable with \( rU^\gamma \), and let \( h = \text{red}(U) \to \text{red}(s) \) be the map given by Lemma 6.35. Let \( D = h \circ g(F) \), which is an \( rF^\gamma \)-definable finite subset of \( \text{red}(s) \). By Proposition 6.45, the composition \( f \circ g^{-1} \circ h^{-1} = D \to \mathcal{G} \) can be coded in the stabilizer sorts \( \mathcal{G} \). Because \( h \circ g = F \to D \) is a \( rF^\gamma \)-definable bijection, then \( f \) is interdefinable with the tuple \( (rF^\gamma, rF^\gamma \circ g^{-1} \circ h^{-1}) \) which is a sequence of elements in \( \mathcal{G} \).

Finally, we conclude proving that \( I_{m+1} \) holds for \( r > 0 \).

**Proposition 6.47.** For any \( r > 0 \) let \( F \subseteq \mathcal{G} \) be a finite set of size \( m+1 \). Then \( F \) can be coded in \( \mathcal{G} \).

**Proof.** Let \( r > 0 \) and \( F \) be a finite set of \( \mathcal{G} \) of size \( m+1 \). Suppose that \( F \) is not primitive, that means that we can find a nontrivial equivalence \( E \) relation definable over \( rF^\gamma \), and let \( C_1, \ldots, C_l \) be such classes. For each \( i \leq l \), \( \lvert C_i \rvert \leq m \), because \( I_k \) holds for each \( k \leq m \) we can find a code \( c_i \in \mathcal{G} \). As
l ≤ m by \( I_l \) holds, we can find a code \( c \) in the stabilizer sorts of the set \( \{c_1, \ldots, c_l\} \), because \( l < m + 1 \). The code \( \tau F \) is interdefinable with \( c \).

We assume that \( F \) is a primitive set. Let \( \pi_i = \mathcal{G} \to \mathcal{G} \) be the projection into the \( i \)th coordinate. By primitivity of \( F \) each projection \( \pi_i \) is either constant or injective. As \( |F| > 1 \) there must be an index \( 1 ≤ i_0 ≤ r \) such that \( \pi_{i_0} \) is injective and \( F_0 = \pi_{i_0}(F) \) is a primitive finite subset of \( \mathcal{G} \). By Proposition 6.44, we can find a code \( \tau F_0 \) in \( \mathcal{G} \). For each other index \( i \neq i_0 \), by Proposition 6.46 we have that \( \pi_i \circ \pi_{i_0}^{-1} = F_0 \to \mathcal{G} \) can be coded in the stabilizer sorts. Then \( \tau F \) is interdefinable with the tuple \((\tau F_0, (\tau \pi_i \circ \pi_{i_0}^{-1})_{i \neq i_0})\) which is a tuple in the stabilizer sorts, as required.

This completes the induction on the cardinality of the set \( F \). Because \( I_m \) holds for each \( m \in \mathbb{N} \), we can conclude with the following statement.

**Theorem 6.48.** Let \( r > 0 \) and \( F \subseteq \mathcal{G} \), then \( \tau F \) is interdefinable with a tuple of elements in \( \mathcal{G} \).

### 6.6 | Putting everything together

We conclude this section with our main theorem.

**Theorem 6.49.** Let \( K \) be a henselian valued field of equicharacteristic zero, residue field algebraically closed and dp-minimal value group. Then \( K \) eliminates imaginaries in the language \( \hat{\mathcal{L}} \), where the stabilizer sorts are added.

*Proof.* By Theorem 5.12, \( K \) has weak elimination of imaginaries down to the stabilizer sorts. By Fact 6.4, it is sufficient to show that finite sets can be coded, this is guaranteed by Theorem 6.48.

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