INFINITELY MANY POSITIVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$

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Abstract. We consider the following nonlinear problem in $\mathbb{R}^N$

$$-\Delta u + V(|y|)u = u^p, \quad u > 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)$$  \hspace{1cm} (0.1)

where $V(r)$ is a positive function, $1 < p < \frac{N+2}{N-2}$. We show that if $V(r)$ has the following expansion: There are constants $a > 0$, $m > 1$, $\theta > 0$, and $V_0 > 0$, such that

$$V(r) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad \text{as } r \to +\infty,$$

then (0.1) has infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

1. Introduction

Standing waves for the following nonlinear Schrödinger equation in $\mathbb{R}^N$:

$$-i\frac{\partial \psi}{\partial t} = \Delta \psi - \tilde{V}(y) \psi + |\psi|^{p-1} \psi,$$  \hspace{1cm} (1.1)

where $p > 1$, are solutions of the form $\psi(t, y) = \exp(i\lambda t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that $\psi$ satisfies (1.1) if and only if $u$ solves the nonlinear elliptic problem

$$-\Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y| \to +\infty} u(y) = 0,$$  \hspace{1cm} (1.2)

where $V(y) = \tilde{V}(y) + \lambda$. In the rest of this paper, we will assume that $V$ is bounded, and $V(y) \geq V_0 > 0$.

A problem which is similar to (1.2) is the following scalar field equation:

$$-\Delta u + u = Q(y)u^p, \quad u > 0, \quad \lim_{|y| \to +\infty} u(y) = 0,$$  \hspace{1cm} (1.3)

where $Q(y)$ is bounded, and $Q(y) \geq Q_0 > 0$.

If

$$\inf_{y \in \mathbb{R}^N} V(y) < \lim_{|x| \to \infty} V(x), \quad \text{or} \quad \sup_{y \in \mathbb{R}^N} Q(y) > \lim_{|x| \to \infty} Q(x),$$  \hspace{1cm} (1.4)

then, using the concentration compactness principle [24, 25], one can show that (1.2) and (1.3) have a least energy solution. See for example [17, 24, 25, 28]. But if (1.4) does not hold, (1.2) (or (1.3)) may not have least energy solution. So, one needs to find solution with higher energy level. For results on this aspect, the readers can refer to [4, 5, 6]. Note
that the energy of the solutions in \[4, 6\] is less than twice of the first level at which the Palais-Smale condition fails.

Recently, Cerami, Devillanova and Solimini \[9\] showed that the following problem

\[-\Delta u + V(y)u = |u|^{p-1}u, \quad \lim_{|y| \to +\infty} u(y) = 0,\]

has infinitely many sign-changing solutions if \(V(y)\) tends to its limit at infinity from below with a suitably rate. Except \[10\], where \(Q(y)\) is periodic, there is no result on the multiplicity of positive solutions for (1.2) (or (1.3)).

On the other hand, if we consider the following singularly perturbed problem:

\[-\varepsilon^2 \Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y| \to +\infty} u(y) = 0, \quad \text{(1.5)}\]

or

\[-\varepsilon^2 \Delta u + u = Q(y)u^p, \quad u > 0, \quad \lim_{|y| \to +\infty} u(y) = 0, \quad \text{(1.6)}\]

where \(\varepsilon > 0\) is a small parameter, then the number of the critical points of \(V(y)\) (or \(Q(y)\)) (see for example \[11, 7, 8, 13, 16, 18, 27, 29\]), the type of the critical points of \(V(y)\) (or \(Q(y)\)) (see for example \[12, 21, 26\], and the topology of the level set of \(V(y)\) (or \(Q(y)\)) \[2, 3, 11, 19\], can affect the number of the solutions for (1.5) (or (1.6)). But for the singularly perturbed problems (1.5) and (1.6), the parameter \(\varepsilon\) will tend to zero as the number of the solutions tends to infinity. So, all these results do not give any multiplicity result for (1.2) (or (1.3)).

In this paper, we assume that \(V(y)\) is radial. That is, \(V(y) = V(|y|)\). Thus, we consider the following problem

\[-\Delta u + V(|y|)u = u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad \text{(1.7)}\]

where \(1 < p < \frac{N+2}{N-2}\) if \(N \geq 3\), \(1 < p < +\infty\) if \(N = 2\). We assume

\[\lim_{|y| \to +\infty} V(|y|) = V_0 > 0.\]

Note that if \(V(r)\) is non-decreasing, by [20], any solution of (1.7) is radial.

The aim of this paper is to obtain infinitely many non-radial positive solutions for (1.7) under an assumption for \(V(r)\) near the infinity. We assume that \(V(r) > 0\) satisfies the following condition:

(V): There is are constants \(a > 0\), \(m > 1\), \(\theta > 0\), and \(V_0 > 0\), such that

\[V(r) = V_0 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad \text{(1.8)}\]

as \(r \to +\infty\). (Without loss of generality, we may assume that \(V_0 = 1\).)

Our main result in this paper can be stated as follows:

**Theorem 1.1.** If \(V(r)\) satisfies (V), then problem (1.7) has infinitely many non-radial positive solutions.
Remark 1.2. To obtain the result in Theorem 1.1 (1.8) can not be changed to

\[ V(r) = V_0 - \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad (1.9) \]

In fact, it is easy to find a function \( V(r) \), satisfying \( V'(r) \geq 0 \) and (1.9). So, for this \( V \), all the solutions must be radial.

Remark 1.3. The radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinate system,

1. \( V(y) = V((y', y''), y_3, \ldots, y_N) \), where \( y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \),
2. \( V(y) = V_0 + \frac{a}{|y|^m} + O\left(\frac{1}{|y|^{m+\theta}}\right) \) as \( |y| \to +\infty \), where \( a > 0, m > 1, \theta > 0, \) and \( V_0 > 0 \) are some constants.

We believe that Theorem 1.1 is still true for non-radial potential \( V(y) \). So, we make the following conjecture: 

**Conjecture:** Problem (1.2) has infinitely many solutions, if

\[ V(y) = V_0 + \frac{a}{|y|^m} + O\left(\frac{1}{|y|^{m+\theta}}\right), \]

as \( |y| \to +\infty \), where \( V_0 > 0, a > 0, m > 0 \) and \( \theta > 0 \) are some constants.

Remark 1.4. Using the same argument, we can prove that if

\[ Q(r) = Q_0 - \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \]

where \( Q_0 > 0, a > 0, m > 1 \) and \( \theta > 0 \) are some constants, then

\[ -\Delta u + u = Q(|y|)u^p, \quad u > 0, \quad u \in H^1(\mathbb{R}^N), \]

has infinitely many positive non-radial solutions.

Remark 1.5. If \( V(r) \) tends to \( V_0 \) from below, i.e. \( V \) satisfies

\[ V(r) = V_0 - \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \quad (1.10) \]

as \( r \to +\infty \), then we can use similar method to prove the existence of infinitely many sign-changing solutions. This recovers the result in [9], at least when \( V \) is radially symmetric.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1. We will construct solutions with large number of bumps near the infinity. Since we assume

\[ \lim_{|y| \to +\infty} V(|y|) = 1, \]

we will use the solution of

\[ \begin{cases} -\Delta u + u = u^p, & u > 0, \quad \text{in } \mathbb{R}^N, \\
 u(y) \to 0, & \text{as } |y| \to \infty, \end{cases} \quad (1.11) \]
to build up the approximate solutions for (1.7). It is well-known that (1.11) has a unique solution \( U \), satisfying 
\[ U(y) = U(|y|), \quad U' < 0. \]

Let 
\[ x_j = (r \cos \frac{2(j - 1)\pi}{k}, r \sin \frac{2(j - 1)\pi}{k}, 0), \quad j = 1, \ldots, k, \]
where 0 is the zero vector in \( \mathbb{R}^{N-2} \), \( r \in [r_0 k \ln k, r_1 k \ln k] \) for some \( r_1 > r_0 > 0 \).

Set \( y = (y', y'') \), \( y' \in \mathbb{R}^2 \), \( y'' \in \mathbb{R}^{N-2} \). Define 
\[ H_s = \{ u : u \in H^1(\mathbb{R}^N), u \text{ is even in } y_h, h = 2, \ldots, N, \]
\[ u(r \cos \theta, r \sin \theta, y'') = u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'') \}. \]

Let 
\[ W_r(y) = \sum_{j=1}^{k} U_{x_j}(y), \]
where \( U_{x_j}(y) = U(y - x_j) \).

Theorem 1.1 is a direct consequence of the following result:

**Theorem 1.6.** Suppose that \( V(r) \) satisfies (V). Then there is an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), (1.7) has a solution \( u_k \) of the form 
\[ u_k = W_{r_k}(y) + \omega_k, \]
where \( \omega_k \in H_s \), \( r_k \in [r_0 k \ln k, r_1 k \ln k] \) and as \( k \to +\infty \), 
\[ \int_{\mathbb{R}^N} (|D\omega_k|^2 + \omega_k^2) \to 0. \]

We will use the techniques in the singularly perturbed elliptic problems to prove Theorem 1.6. For singularly perturbed problems (similar to (1.5)), a small parameter is present either in the front of the \( \Delta \) or in the nonlinearity. Here for our problem (1.7), there is no parameter to use. However we use the **loss of compactness** to build up solutions. More precisely, because of the domain \( \mathbb{R}^N \) we can use \( k \), the **number of the bumps** of the solutions, as the parameter in the construction of spike solutions for (1.7). This seems to be a new idea. This is partly motivated by recent paper of Lin-Ni-Wei [23] where they constructed multiple spikes to a singularly perturbed problem. There they allowed the number of spikes to depend on the small parameter.

This paper is organized as follows. In section 2, we will carry out the reduction. Then, we will study the reduced finite dimensional problem and prove Theorem 1.6. We will leave all the technical calculations in the appendix.

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2. Proof the main result

Let
\[ Z_j = \frac{\partial U_{x_j}}{\partial r}, \quad j = 1, \ldots, k, \]
where \( x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0). \) In this paper, we always assume
\[ r \in S_k =: [\left(\frac{m}{2\pi} - \beta k \ln k, \frac{m}{2\pi} + \beta k \ln k\right)], \quad (2.1) \]
where \( m \) is the constant in the expansion for \( V \), and \( \beta > 0 \) is a small constant.

Define
\[ E = \{ v : v \in H_s, \int_{\mathbb{R}^N} U_{x_j}^{p-1} Z_j v = 0, \quad j = 1, \ldots, k \} \]
The norm of \( H^1(\mathbb{R}^N) \) is defined as follows:
\[ \|v\| = \sqrt{\langle v, v \rangle}, \quad v \in H^1(\mathbb{R}^N), \]
where
\[ \langle v_1, v_2 \rangle = \int_{\mathbb{R}^N} (Dv_1 Dv_2 + V(|y|)v_1 v_2). \]

It is easy to check that
\[ \int_{\mathbb{R}^N} (Dv_1 Dv_2 + V(|y|)v_1 v_2 - pW^{p-1}_r v_1 v_2), \quad v_1, v_2 \in E, \]
is a bounded bilinear functional in \( E \). Thus, there is a bounded linear operator \( L \) from \( E \) to \( E \), such that
\[ \langle Lv_1, v_2 \rangle = \int_{\mathbb{R}^N} (Dv_1 Dv_2 + V(|y|)v_1 v_2 - pW^{p-1}_r v_1 v_2), \quad v_1, v_2 \in E. \]

The next lemma shows that \( L \) is invertible in \( E \).

**Lemma 2.1.** There is a constant \( \rho > 0 \), independent of \( k \), such that for any \( r \in S_k \),
\[ \|Lv\| \geq \rho \|v\|, \quad v \in E. \]

**Proof.** We argue by contradiction. Suppose that there are \( n \to +\infty \), \( r_k \in S_k \), and \( v_k \in E \), with
\[ \|Lv_k\| = o(1) \|v_k\|. \]
Then
\[ \langle Lv_k, \varphi \rangle = o(1) \|v_k\| \|\varphi\|, \quad \forall \varphi \in E. \quad (2.2) \]
We may assume that \( \|v_k\|^2 = k \).

Let
\[ \Omega_j = \{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \}. \]

By symmetry, we see from (2.2),

\[ \int_{\Omega_1} (Dv_k D\varphi + V(|y|) v_k \varphi - p W_r^{p-1} v_k \varphi) = \frac{1}{k} \langle Lv_k, \varphi \rangle = o\left(\frac{1}{\sqrt{k}}\right) \| \varphi \|, \quad \forall \varphi \in E. \tag{2.3} \]

In particular,

\[ \int_{\Omega_1} (|Dv_k|^2 + V(|y|) v_k^2 - p W_r^{p-1} v_k^2) = o(1), \tag{2.4} \]

and

\[ \int_{\Omega_1} (|Dv_k|^2 + V(|y|) v_k^2) = 1. \tag{2.4} \]

Let \( \bar{v}_k(y) = v_k(y - x_1) \). Then for any \( R > 0 \), since \( |x_2 - x_1| = r \sin \frac{\pi}{k} \geq \frac{m}{4} \ln k \), we see that \( B_R(x_1) \subset \Omega_1 \). As a result, from (2.4), we find that for any \( R > 0 \),

\[ \int_{B_R(0)} (|D\bar{v}_k|^2 + V(|y|) \bar{v}_k^2) \leq 1. \]

So, we may assume that there is a \( v \in H^1(\mathbb{R}^N) \), such that as \( k \to +\infty \),

\[ \bar{v}_k \to v, \quad \text{weakly in } H^1_{loc}(\mathbb{R}^N), \]

and

\[ \bar{v}_k \to v, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N). \]

Since \( \bar{v}_k \) is even in \( y_h, h = 2, \cdots, N \), it is easy to see that \( v \) is even in \( y_h, h = 2, \cdots, N \). On the other hand, from

\[ \int_{\mathbb{R}^N} U_{x_1}^{p-1} Z_1 v_k = 0, \]

we obtain

\[ \int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \bar{v}_k = 0. \]

So, \( v \) satisfies

\[ \int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} v = 0. \tag{2.5} \]

Now, we claim that \( v \) satisfies

\[ -\Delta v + v - p U^{p-1} v = 0, \quad \text{in } \mathbb{R}^N. \tag{2.6} \]
Define
\[ \tilde{E} = \{ \varphi : \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} U^{p-1} \frac{\partial U}{\partial x_1} \varphi = 0 \}. \]

For any \( R > 0 \), let \( \varphi \in C_0^\infty(B_R(0)) \cap \tilde{E} \) be any function, satisfying that \( \varphi \) is even in \( y_h, h = 2, \cdots, N \). Then \( \varphi_k(y) = \varphi(y - x_1) \in C_0^\infty(B_R(x_1)) \). Inserting \( \varphi_k \) into (2.3), using Lemma A.1, we find
\[ \int_{\mathbb{R}^N} (DvD\varphi + v\varphi - pU^{p-1}v\varphi) = 0. \] (2.7)

On the other hand, since \( v \) is even in \( y_h, h = 2, \cdots, N \), (2.7) holds for any function \( \varphi \in C_0^\infty(B_R(0)) \cap \tilde{E} \). By the density of \( C_0^\infty(\mathbb{R}^N) \) in \( H^1(\mathbb{R}^N) \), it is easy to show that
\[ \int_{\mathbb{R}^N} (DvD\varphi + v\varphi - pU^{p-1}v\varphi) = 0, \ \forall \ \varphi \in \tilde{E}. \] (2.8)

But (2.8) holds for \( \varphi = \frac{\partial U}{\partial x_1} \). Thus (2.8) is true for any \( \varphi \in H^1(\mathbb{R}^N) \). So, we have proved (2.6).

Since \( U \) is non-degenerate, we see that \( v = c \frac{\partial U}{\partial x_1} \) because \( v \) is even in \( y_h, h = 2, \cdots, N \). From (2.5), we find
\[ v = 0. \]

As a result,
\[ \int_{B_R(x_1)} v_k^2 = o(1), \ \forall \ R > 0. \]

On the other hand, it follows from Lemma A.1 that for any small \( \eta > 0 \), there is a constant \( C > 0 \), such that
\[ W_{\tilde{r}_k}(y) \leq Ce^{-(1-\eta)|y-x_1|}, \ y \in \Omega_1. \] (2.9)

Thus,
\[ o(1) = \int_{\Omega_1} (|Dv_k|^2 + V(|y|)v_k^2 - pW_{\tilde{r}_k}^{p-1}v_k^2) \]
\[ = \int_{\Omega_1} (|Dv_k|^2 + V(|y|)v_k^2) + O(e^{-(1-\eta)(p-1)R}) \int_{\Omega_1} v_k^2 \]
\[ \geq \frac{1}{2} \int_{\Omega_1} (|Dv_k|^2 + V(|y|)v_k^2) + o(1). \]

This is a contradiction to (2.4). \( \square \)
Define
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|Du|^2 + V(|y|)u^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}). \]
Let
\[ J(\phi) = I(W_r + \phi), \quad \phi \in E. \]

We have

**Proposition 2.2.** There is an integer \( k_0 > 0 \), such that for each \( k \geq k_0 \), there is a \( C^1 \) map from \( S_k \) to \( H_s \): \( \phi = \phi(r), \ r = |x_1| \), satisfying \( \phi \in E \), and
\[ \langle \frac{\partial J(\phi)}{\partial \phi}, \varphi \rangle = 0, \quad \forall \varphi \in E. \]
Moreover, there is a small \( \sigma > 0 \), such that
\[ \|\phi\| \leq \frac{C}{k^{\frac{n-1}{2}+\sigma}}. \] (2.10)

**Proof.** Expand \( J(\phi) \) as follows:
\[ J(\phi) = J(0) + l(\phi) + \frac{1}{2} \langle L\phi, \phi \rangle + R(\phi), \quad \phi \in E, \]
where
\[ l(\phi) = \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(|y|) - 1)U_{x_i}\phi + \int_{\mathbb{R}^N} (W^p - \sum_{i=1}^{k} U_{x_i}^p)\phi, \]
\( L \) is the bounded linear map from \( E \) to \( E \) in Lemma 2.1 and
\[ R(\phi) = \frac{1}{p+1} \int_{\mathbb{R}^N} (|W + \phi|^{p+1} - W^{p+1} - (p+1)W^{p}\phi - \frac{1}{2}(p+1)pW^{p-1}\phi^2). \]
Since \( l(\phi) \) is a bounded linear functional in \( E \), we know that there is an \( l_k \in E \), such that
\[ l(\phi) = \langle l_k, \phi \rangle. \]
Thus, finding a critical point for \( J(\phi) \) is equivalent to solving
\[ l_k + L\phi + R'(\phi) = 0. \] (2.11)
By Lemma 2.1, \( L \) is invertible. Thus, (2.11) can be rewritten as
\[ \phi = A(\phi) =: -L^{-1}l_k - L^{-1}R'(\phi). \]
Let
\[ S = \{ \phi : \phi \in E, \|\phi\| \leq \frac{1}{k^{\frac{n-1}{2}}} \}. \]
If $p \leq 2$, then it is easy to check that
\[ \|R'(\phi)\| \leq C\|\phi\|^p. \]
So, from Lemma 2.3 below,
\[ \|A(\phi)\| \leq C\|l_k\| + C\|\phi\|^p \leq \frac{C}{k^{m-1}+\sigma} + \frac{C}{k^{\frac{m(m-1)}{2}}} \leq \frac{1}{k^{\frac{m-1}{2}}}. \] (2.12)
Thus, $A$ maps $S$ into $S$ if $p \leq 2$.

On the other hand, if $p \leq 2$, then
\[ \|R''(\phi)\| \leq C\|\phi\|^{p-1}. \]
Thus,
\[ \|A(\phi_1) - A(\phi_2)\| = \|L^{-1}R'(\phi_1) - L^{-1}R'(\phi_2)\| \leq C\|\phi_1\|^{p-1} + C\|\phi_2\|^{p-1} \leq \frac{1}{2}\|\phi_1 - \phi_2\|. \]
So, we have proved that if $p \leq 2$, $A$ is a contraction map. Therefore, we have proved that if $p \leq 2$, $A$ is a contraction map from $S$ to $S$. So, the result follows from the contraction mapping theorem.

It remains to deal with the case $p > 2$.

Suppose that $p > 2$. Since
\[ \left|\left< R'(\phi), \xi \right> \right| \leq C\int_{\mathbb{R}^N} W_r^{p-2} |\phi|^2 |\xi| \leq C\left(\int_{\mathbb{R}^N} \left( W_r^{p-2} |\phi|^2 \right)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}} \|\xi\|. \]
we find
\[ \|R'(\phi)\| \leq C\left(\int_{\mathbb{R}^N} \left( W_r^{p-2} |\phi|^2 \right)^{\frac{p+1}{p}} \right)^{\frac{p}{p+1}}. \]
On the other hand, it follows from Lemma A.1 that $W_r$ is bounded. Since $2 < \frac{2(p+1)}{p} < p + 1$, we obtain
\[ \|R'(\phi)\| \leq C\left(\int_{\mathbb{R}^N} |\phi|^{\frac{2(p+1)}{p}} \right)^{\frac{p}{p+1}} \leq \|\phi\|^2. \]
For the estimate of $\|R''(\phi)\|$, we have
\[ \left| R''(\phi)(\xi, \eta) \right| \leq C\int_{\mathbb{R}^N} W_r^{p-2} |\phi||\xi||\eta| \leq C\int_{\mathbb{R}^N} |\phi||\xi||\eta| \leq C\left(\int_{\mathbb{R}^N} |\phi|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^N} |\xi|^3 \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^N} |\eta|^3 \right)^{\frac{1}{3}} \leq C\|\phi\|\|\xi\|\|\eta\|, \]
since $2 < 3 < p + 1$. So
\[ \|R''(\phi)\| \leq C\|\phi\|. \]
Thus,

\[ \| R'(\phi) \| \leq \frac{C}{k^{m-1}} \leq \frac{C}{k^{\frac{m-1}{2}} + \sigma}. \]

As a result,

\[ \| A(\phi) \| \leq C\| l_k \| + C\| R'(\phi) \| \leq \frac{C}{k^{\frac{m-1}{2}} + \sigma} \leq \frac{1}{k^{\frac{m-1}{2}}}. \] (2.13)

Thus, \( A \) maps \( S \) to \( S \).

On the other hand,

\[ \| R''(\phi) \| \leq C\| \phi \| \leq \frac{C}{k^{\frac{m-1}{2}}} , \]

which implies that \( A \) is a contraction map. So, we have proved that if \( p > 2 \), then \( A \) is a contraction map from \( S \) to \( S \). And the result follows from the contraction mapping theorem.

Finally, (2.10) follows from (2.12) and (2.13). \( \square \)

**Lemma 2.3.** There is a small \( \sigma > 0 \), such that

\[ \| l_k \| \leq \frac{C}{k^{\frac{m-1}{2}} + \sigma}. \]

**Proof.** By the symmetry of the problem,

\[ \sum_{i=1}^{k} \int_{\mathbb{R}^N} (V(|y|) - 1)U_{x_i}\phi = k \int_{\mathbb{R}^N} (V(|y|) - 1)U_{x_1}\phi \]

\[ = k \int_{\mathbb{R}^N} (V(|y-x_1|) - 1)U\phi(y - x_1) \leq kO\left( \frac{1}{r^m} \right)\| \phi \| \] (2.14)

because \( m > 1 \).

On the other hand, for any \( y \in \Omega_1 \),

\[ U_{x_i}^p \leq U_{x_1}^{p-1}U_{x_i}. \]

Thus,
\[
\left| \int_{\mathbb{R}^n} (W^p - \sum_{i=1}^k U_{x_i}^p) \phi \right| = k \left| \int_{\Omega} (W^p - \sum_{i=1}^k U_{x_i}^p) \phi \right|
\]
\[
\leq Ck \int_{\Omega} U_{x_1}^{p-1} \sum_{j=2}^k U_{x_j} |\phi| \leq Ck \sum_{j=2}^k e^{-\frac{p-\tau}{2} |x_j-x_1|} \left( \int_{\Omega} |\phi|^{p+1} \right)^{\frac{1}{p+1}}
\]
\[
\leq k^{\frac{p}{p+1}} \sum_{j=2}^k e^{-\frac{p-\tau}{2} |x_j-x_1|} \|\phi\|,
\]

where $\tau > 0$ is any small fixed constant.

From the definition of $S_k$ in (2.1), we see that for any $r \in S_k$
\[
\sum_{j=2}^k e^{-\frac{p-\tau}{2} |x_j-x_1|} \leq C e^{-\frac{p-\tau}{2} \frac{p}{k} r} \leq \frac{C}{k^{\frac{p}{2}} (m-\beta)}.
\]

Since
\[
\frac{pm}{2} - \frac{p}{p+1} > \frac{m-1}{2}, \quad \forall \ m > 1,
\]
we obtain from (2.15) that
\[
\left| \int_{\mathbb{R}^n} (W^p - \sum_{i=1}^k U_{x_i}^p) \phi \right| \leq \frac{C}{k^{\frac{p}{2}} + \sigma} \|\phi\|.
\]

The result follows from (2.14) and (2.16).

We are ready to prove Theorem 1.6. Let $\phi = \phi(r)$ be the map obtained in Proposition 2.2. Define
\[
F(r) = I(W_r + \phi), \quad \forall \ r \in S_k.
\]
It is well known that if $r$ is a critical point of $F(r)$, then $W_r + \phi$ is a solution of (1.7). (See [21] or [23].)

Proof of Theorem 1.6. It follows from Propositions 2.2 and A.3 that
\[
F(r) = I(W) + I(\phi) + \frac{1}{2} \langle L\phi, \phi \rangle + R(\phi)
\]
\[
= I(W) + O(\|l_k\| \|\phi\| + \|\phi\|^2)
\]
\[
=k \left( A + \frac{B_1}{r^m} - B_2 e^{-\frac{2\tau}{r}} + O\left( \frac{1}{k^{m+\sigma}} \right) \right)
\]

Consider
\[
\text{max}\{ F(r) : r \in S_k \}.
\] (2.17)

For the definition of \( S_k \), see (2.1). Since the function
\[
\frac{B_1}{r^m} - B_2 e^{-\frac{2\pi r}{k}}
\]
has a maximum point
\[
\bar{r}_k = \left( \frac{m}{2\pi} + o(1) \right) k \ln k,
\]
which is an interior point of \( S_k \), it is easy to check that (2.17) is achieved by some \( r_k \), which is in the interior of \( S_k \). Thus, \( r_k \) is a critical point of \( F(r) \). As a result
\[
W_{r_k} + \phi(r_k)
\]
is a solution of (1.7).

\[\square\]

**Appendix A. Energy Expansion**

In this section, we will give the energy expansion for the approximate solutions. Recall
\[
x_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,
\]
\[
\Omega_j = \{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \rangle \geq \cos \frac{\pi}{k} \}, \quad j = 1, \ldots, k,
\]
and
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|Du|^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}.
\]

Firstly, we have the following basic estimate:

**Lemma A.1.** For any \( y \in \Omega_1 \), and \( \eta \in (0, 1] \), there is a constant \( C > 0 \), such that
\[
\sum_{j=2}^{k} U_{x_j}(y) \leq C e^{-\eta|x_1|} \bar{r} e^{-(1-\eta)|y-x_1|}.
\]

**Proof.** For any \( y \in \Omega_1 \), we have \( |y-x_j| \geq |y-x_1| \).

If \( |y-x_1| \geq 2|x_j-x_1| \), then for any \( y \in \Omega_1 \),
\[
U_{x_j}(y) \leq C e^{-|y-x_j|} \leq C e^{-|y-x_1|} = C e^{-\eta|y-x_1|} e^{-(1-\eta)|y-x_1|} \leq C e^{-2\eta|x_j-x_1|} e^{-(1-\eta)|y-x_1|}.
\]

If \( |y-x_1| \leq 2|x_j-x_1| \), then
\[ |y - y_j| \geq |x_j - x_1| - |y - x_1| \geq \frac{1}{2}|x_j - x_1|. \]

So for any \( y \in \Omega_1 \),

\[ U_{x_j}(y) \leq C e^{-\eta|y - x_j|} e^{-(1-\eta)|y - x_1|} \leq C e^{-\frac{1}{2}\eta|x_j - x_1|} e^{-(1-\eta)|y - x_1|}. \]

Thus,

\[
\sum_{j=2}^{k} U_{x_j}(y) \leq C e^{-(1-\eta)|y - x_1|} \sum_{j=2}^{k} e^{-\frac{1}{2}\eta|x_j - x_1|} \\
\leq C e^{-(1-\eta)|y - x_1|} \sum_{j=2}^{k} e^{-\eta|x_1| \sin \frac{\pi j}{k}} \leq C_1 e^{-\eta|x_1| \frac{\pi}{k}} e^{-(1-\eta)|y - x_1|}. \]

\[ \square \]

In this appendix, we denote \( r = |x_1| \), and we always assume that

\[ r \in S_k, \]

where \( S_k \) is defined in (2.1).

**Proposition A.2.** We have

\[ I(U_{x_1}) = A + \frac{B_1}{r^m} + O\left(\frac{1}{k^{m+\theta}}\right), \]

where

\[ A = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} U^{p+1}, \quad B_1 = \frac{a}{2} \int_{\mathbb{R}^N} U^2. \]

**Proof.** We have

\[
I(U_{x_1}) = \frac{1}{2} \int_{\mathbb{R}^N} (|DU|^2 + U^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1} + \frac{1}{2} \int_{\mathbb{R}^N} (V(|y|) - 1) U_{x_1}^2 \]
\[
= A + \frac{1}{2} \int_{\mathbb{R}^N} (V(|y - x_1|) - 1) U^2. \tag{A.1} \]

On the other hand, for any small \( \tau > 0 \), using (V),

\[
\int_{\mathbb{R}^N} (V(|y - x_1|) - 1) U^2 = \int_{B_{\frac{1}{2\tau}}(0)} (V(|y - x_1|) - 1) U^2 + O(e^{-(1-\tau)r}) \]
\[
= \int_{B_{\frac{1}{2\tau}}(0)} \left(\frac{a}{|y - x_1|^m} + O\left(\frac{1}{|y - x_1|^{m+\theta}}\right)\right) U^2 + O(e^{-(1-\tau)r}). \tag{A.2} \]
But for any \( \alpha > 0 \),

\[
\frac{1}{|y - x_1|^{\alpha}} = \frac{1}{|x_1|^{\alpha}} \left( 1 + O \left( \frac{|y|}{|x_1|} \right) \right), \quad y \in B_{\frac{1}{2}|x_1|}(0).
\]

Thus,

\[
\int_{B_{\frac{1}{2r}(0)} \mid \frac{|y - x_1|}{r} \mid^{\alpha} U^2 = \frac{1}{|x_1|^{\alpha}} \int_{\mathbb{R}^N} U^2 + O \left( \frac{1}{|x_1|^{\alpha+1}} + e^{-\frac{(1-r)|x_1|}{\alpha}} \right). \tag{A.3}
\]

Inserting (A.3) into (A.2), we obtain

\[
\int_{\mathbb{R}^N} (V(|y - x_1|) - 1) U^2 = \frac{B_1}{r^m} + O \left( \frac{1}{k^{m+\theta} \ln^{m+\theta} k} \right). \tag{A.4}
\]

Thus, the result follows from (A.1) and (A.4).

\[\square\]

**Proposition A.3.** There is a small constant \( \sigma > 0 \), such that

\[
I(W_r) = k \left( A + \frac{B_1}{r^m} - B_2 e^{-\frac{2\pi r}{k}} + O \left( \frac{1}{k^{m+\theta}} \right) \right),
\]

where \( A \) and \( B_1 \) are the constants in Proposition A.2, and \( B_2 > 0 \) is a positive constant.

**Proof.** Using the symmetry,

\[
\int_{\mathbb{R}^N} (|DW_r|^2 + W_r^2) = \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\mathbb{R}^N} U_{x_j} U_{x_i} \tag{A.5}
\]

\[= k \int_{\mathbb{R}^N} U_{x_1}^{p+1} + k \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x_1}^{p} U_{x_i}.
\]

Recall

\[
\Omega_j = \{ y = (y',y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left( \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \right) \geq \cos \frac{\pi}{k} \}, \quad j = 1, \ldots, k.
\]

It follows from Lemma A.1 that

\[
\int_{\mathbb{R}^N} (V(|y|) - 1)W_r^2 = k \int_{\Omega_1} (V(|y|) - 1)W_r^2
\]

\[= k \int_{\Omega_1} (V(|y|) - 1) \left( U_{x_1} + O \left( e^{-\frac{1}{2}|x_1|^2 \pi e^{-\frac{1}{2}|y-x_1|}} \right) \right)^2 \tag{A.6}
\]

\[= k \int_{\Omega_1} (V(|y|) - 1)U_{x_1}^2 + kO \left( \int_{\Omega_1} |V(|y|)| - 1 |e^{-\frac{1}{2}|x_1|^2 \pi e^{-|y-x_1|}} \right)
\]

\[= k \left( \frac{B_1}{|x_1|^m} + O \left( \frac{1}{k^{m+\theta}} \right) \right).
\]
Suppose that $p \leq 3$. Then, for any $y \in \Omega_1$,

$$W^p_{r} = U^p_{x_1} + (p + 1)U^p_{x_1}\sum_{j=2}^{k} U_{x_j} + O\left(U^p_{x_1}\left(\sum_{j=2}^{k} U_{x_j}\right)^{\frac{p+1}{2}}\right).$$

Using Lemma A.1, we have

$$\int_{\mathbb{R}^N} W^p_{r} = k\int_{\Omega_1} W^p_{r} = k\int_{\Omega_1} U^p_{x_1} + (p + 1)\sum_{i=2}^{k} U^p_{x_i} U_{x_i} + O\left(\sum_{i=2}^{k} e^{-\frac{m-1}{2} |x_i - x|} + e^{-|x_i - x_1|}\right). \quad (A.7)$$

But for any $r \in S_k$,

$$\sum_{j=2}^{k} e^{-|x_j - x_1|} \leq C e^{-\frac{d}{k_m}} \leq \frac{C}{k^{m-\beta}}.$$ 

So, we obtain that for $p \in (1, 3]$, if $\beta > 0$ is small enough,

$$\int_{\mathbb{R}^N} W^p_{r} = k\int_{\Omega_1} W^p_{r} = k\int_{\Omega_1} U^p_{x_1} + (p + 1)\sum_{i=2}^{k} U^p_{x_i} U_{x_i} + O\left(\frac{1}{k^{m+\sigma}}\right). \quad (A.8)$$

Combining (A.5), (A.6), (A.7) and (A.8), we are led to
\[ I(W_r) = k \left( A + \frac{B_1}{r^m} - \frac{1}{2} \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x_i}^p U_{x_i} + O\left( \frac{1}{k^{m+\sigma}} \right) \right). \]

But there are constants \( \sigma > 0 \), \( \tilde{B}_2 > 0 \) and \( B'_2 > 0 \), such that

\[
\sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x_i}^p U_{x_i} = \tilde{B}_2 \sum_{i=2}^{k} e^{-|x_i-x_1|} + O\left( \sum_{i=2}^{k} e^{-|x_i-x_1|} \right)
= B'_2 e^{-\frac{2|x_1|}{k}} + O\left( e^{-\frac{2(1+\sigma)|x_1|}{k}} \right).
\]

So the result follows. \( \square \)

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