ENDOMORPHISMS OF ABELIAN VARIETIES, CYCLOTOMIC EXTENSIONS AND LIE ALGEBRAS

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Abstract. We prove an analogue of the Tate conjecture on homomorphisms of abelian varieties over infinite cyclotomic extensions of finitely generated fields of characteristic zero.

1. Introduction

The aim of this note is to extend Faltings’ results [4, 5] concerning the Tate conjecture on homomorphisms of abelian varieties [15, 16] over finitely generated fields $K$ of characteristic zero to their infinite cyclotomic extensions $K(\ell) = K(\mu_\infty)$. The possibility of such generalization (in the case of number fields $K$) was stated (without a detailed proof) in [19, §6, Subsect. 2, pp. 91–92]; it was pointed out there that this result follows from the theorem of Faltings combined with technique developed in [17] and a theorem of F.A. Bogomolov about homotheties [1, 2]. Our main result is the following assertion. (Here $\text{Gal}(E)$ stands for the absolute Galois group of $E$ while $T_\ell(X)$ and $T_\ell(Y)$ are the Tate modules of abelian varieties of $X$ and $Y$ respectively.)

Theorem 1.1. Suppose that $K$ is a field that is finitely generated over $\mathbb{Q}$ and $\ell$ is a prime. Let us put $E = K(\ell)$. If $X$ and $Y$ are abelian varieties over $E$ then the natural embedding of $\mathbb{Z}_\ell$-modules

$$\text{Hom}_E(X, Y) \otimes \mathbb{Z}_\ell \hookrightarrow \text{Hom}_{\text{Gal}(E)}(T_\ell(X), T_\ell(Y))$$

is bijective.

Remark 1.2. Replacing $K$ by its suitable finite (sub) extension (of $E$), we may in the course of the proof of Theorem 1.1 assume that both $X$ and $Y$ are defined over $K$.

Remark 1.3. A.N. Parshin [19, §6, Subsect. 2, pp. 91–92] conjectured that the following analogue of the Mordell conjecture holds true: if $K$ is a number field and $C$ is an absolutely irreducible smooth projective curve over $E = K(\ell)$ then the set $C(E)$ of its $E$-rational points is finite if the genus of $C$ is greater than 1. Theorem 1.1 has arisen from attempts to understand which parts of Faltings’ proof of the Mordell conjecture [4] can be extended to the case of $K(\ell)$.

The paper is organized as follows. In Section 2 we discuss the $\ell$-adic Lie algebras arising from Tate modules of abelian varieties and their centralizers. In Section 3 we deal with analogues of the Tate conjecture on homomorphisms over arbitrary fields. In Section 4 we prove the main result.

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2. Tate modules, $\ell$-adic Lie groups and Lie algebras

Let $K$ be a field, $\bar{K}$ its algebraic closure and $\text{Gal}(K) = \text{Aut}(\bar{K}/K)$ its absolute Galois group. If $m$ is a positive integer that is not divisible by $\text{char}(K)$ then we write $\mu_m$ for the cyclic order $m$ multiplicative subgroup of $m$th roots of unity in $K$ and $K(\mu_m)$ for the corresponding cyclotomic extension of $K$. If $\ell$ is a prime different from $\text{char}(K)$ then we write $K(\ell)$ for the “infinite” cyclotomic extension

$$E = K(\mu_{\ell^\infty}) = \bigcup_{i=1}^{\infty} K(\mu_{\ell^i}).$$

It is well known that the compact Galois group $\text{Gal}(K(\ell)/K)$ is canonically isomorphic to a closed subgroup of $\mathbb{Z}_\ell^*$. We write

$$\chi_\ell : \text{Gal}(K) \to \mathbb{Z}_\ell^*$$

for the corresponding cyclotomic character; its kernel coincides with $\text{Gal}(K(\ell)) = \text{Aut}(\bar{K}/K(\ell))$.

We write $\mathbb{Z}_\ell(1)$ for the projective limit of the groups $\mu_{\ell^i}$ where the transition map $\mu_{\ell^i} \to \mu_{\ell^{i+1}}$ is raising to $\ell$th power. It is well known that $\mathbb{Z}_\ell(1)$ is a free $\mathbb{Z}_\ell$-module of rank 1 provided with the natural structure of a Galois module while the defining homomorphism

$$\text{Gal}(K) \to \text{Aut}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell^*$$

coincides with $\chi_\ell$. Let us consider the 1-dimensional $\mathbb{Q}_\ell$-vector space

$$\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

which carries the structure of a Galois module provided by the same character

$$\chi_\ell : \text{Gal}(K) \to \mathbb{Z}_\ell^* \subset \mathbb{Q}_\ell^* = \text{Aut}_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell(1)).$$

Let $A$ be an abelian variety over $K$ and $\text{End}_K(A)$ the ring of its $K$-endomorphisms. If $X$ and $Y$ are abelian varieties over $K$ then we write $\text{Hom}_K(X,Y)$ for the group of $K$-homomorphisms from $X$ to $Y$. If $m$ is as above then we write $A_m$ for the kernel of multiplication by $m$ in $A(\bar{K})$. The subgroup $A_m$ is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\dim(A)$ [6] and a Galois submodule of $A(\bar{K})$.

We write $T_\ell(A)$ for the $\ell$-adic Tate module of $A$, which is the projective limit of the groups $A_{\ell^i}$ while the transition map $A_{\ell^i} \to A_{\ell^{i+1}}$ is multiplication by $\ell$ [5]. It is well-known that $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of rank $2\dim(A)$, the natural map $T_\ell(A) \to A_{\ell^i}$ gives rise to the isomorphisms

$$T_\ell(A)/\ell^i = A_{\ell^i}$$

and the Galois actions on $A_{\ell^i}$ give rise to the natural continuous homomorphism ($\ell$-adic representation)

$$\rho_\ell, A = \rho_\ell, A, K : \text{Gal}(K) \to \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)),$$

which provides $T_\ell(A)$ with the natural structure of a $\text{Gal}(K)$-module [10]. The image

$$G_{\ell, A, K} = \rho_\ell, A(\text{Gal}(K)) \subset \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(A))$$

is a compact $\ell$-adic Lie group [10]: clearly, $G_{\ell, A, K} \subset 1 + \ell^i \text{End}_{\mathbb{Z}_\ell}(T_\ell(A))$ if and only if $\text{Gal}(K)$ acts trivially on $T_\ell(A)/\ell^i = A_{\ell^i}$, i.e., $A_{\ell^i} \subset A(K)$. 
Clearly, Gal(K(ℓ)) = Aut(\(\tilde{K}/K(\ell)\)) is a compact normal subgroup of Gal(K). We write \(G^1_{\ell,A,K}\) for its image \(\rho_{\ell,A}(\text{Gal}(K(\ell)))\), which is a compact normal Lie subgroup of \(G_{\ell,A,K}\). By definition,

\[G^1_{\ell,A,K} = G_{\ell,A,K}(\ell)\].

Let us consider the 2dim(A)-dimensional \(\mathbb{Q}_\ell\)-vector space

\[V(\ell) = T(\ell) \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell\]

One may view \(T(\ell)\) as the \(\mathbb{Z}_\ell\)-lattice in \(V(\ell)\). We have

\[G_{\ell,A,K} \subset \text{Aut}_{\mathbb{Z}_\ell}(T(\ell)) \subset \text{Aut}_{\mathbb{Q}_\ell}(V(\ell))\]

which provides \(V(\ell)\) with the natural structure of a Gal(K)-module.

Notice that the Lie algebra of the compact \(\ell\)-adic Lie group \(\text{Aut}_{\mathbb{Z}_\ell}(T(\ell))\) coincides with \(\text{End}_{\mathbb{Q}_\ell}(V(\ell))\). The Lie algebra \(\mathfrak{g}_{\ell,A}\) of \(G_{\ell,A,K}\) is a \(\mathbb{Q}_\ell\)-linear Lie subalgebra of \(\text{End}_{\mathbb{Q}_\ell}(V(\ell))\). The Lie algebra \(\mathfrak{g}^0_{\ell,A}\) of \(G^1_{\ell,A,K}\) is an ideal in \(\mathfrak{g}_{\ell,A}\). It is known \([10]\) that the Lie algebras \(\mathfrak{g}_{\ell,A}\) and \(\mathfrak{g}^0_{\ell,A}\) will not change if we replace \(K\) by its finite algebraic extension.

Let \(\text{Id}\) be the identity map on \(V(\ell)\) and let

\[\text{tr} : \text{End}_{\mathbb{Q}_\ell}(V(\ell)) \rightarrow \mathbb{Q}_\ell\]

be the trace map. Let

\[\text{det} : \text{Aut}_{\mathbb{Q}_\ell}(V(\ell)) \rightarrow \mathbb{Q}_\ell^*\]

be the determinant map. We write \(\mathfrak{sl}(V(\ell))\) for the Lie subalgebra of traceless linear operators in \(\text{End}_{\mathbb{Q}_\ell}(V(\ell))\).

Let \(\text{End}_{\mathfrak{g}_{\ell,A}}(V(\ell))\) be the the centralizer of \(\mathfrak{g}_{\ell,A}\) in \(\text{End}_{\mathbb{Q}_\ell}(V(\ell))\) and \(\text{End}_{\mathfrak{g}^0_{\ell,A}}(V(\ell))\) be the the centralizer of \(\mathfrak{g}^0_{\ell,A}\) in \(\text{End}_{\mathbb{Q}_\ell}(V(\ell))\). Clearly,

\[\mathbb{Q}_\ell\text{Id} \subset \text{End}_{\mathfrak{g}_{\ell,A}}(V(\ell)) \subset \text{End}_{\mathfrak{g}^0_{\ell,A}}(V(\ell)) \subset \text{End}_{\mathbb{Q}_\ell}(V(\ell))\]

**Remark 2.1.** Since \(\mathfrak{g}_{\ell,A}\) is the Lie algebra of \(G_{\ell,A,K}\),

\[\text{End}_{\mathfrak{g}_{\ell,A}}(V(\ell)) \supset \text{End}_{G_{\ell,A,K}}(V(\ell)) = \text{End}_{\text{Gal}(K)}(V(\ell))\]

Since \(\mathfrak{g}^0_{\ell,A}\) is the Lie algebra of \(G^1_{\ell,A,K}\),

\[\text{End}_{\mathfrak{g}^0_{\ell,A}}(V(\ell)) \supset \text{End}_{G^1_{\ell,A,K}}(V(\ell)) = \text{End}_{\text{Gal}(K(\ell))}(V(\ell)) = \text{End}_{\text{Gal}(E)}(V(\ell))\]

In the following two propositions we assume that \(A\) has positive dimension.

**Proposition 2.2.**

1. \(\mathfrak{g}^0_{\ell,A} = \mathfrak{g}_{\ell,A} \cap \mathfrak{sl}(V(\ell)) \subset \mathfrak{sl}(V(\ell))\).

2. Suppose that \(\mathfrak{g}_{\ell,A}\) contains the homotheties \(\mathbb{Q}_\ell\text{Id}\). Then

\[\mathfrak{g}_{\ell,A} = \mathbb{Q}_\ell\text{Id} \oplus \mathfrak{g}^0_{\ell,A}\].

In particular, the centralizers \(\text{End}_{\mathfrak{g}_{\ell,A}}(V(\ell))\) and \(\text{End}_{\mathfrak{g}^0_{\ell,A}}(V(\ell))\) do coincide.

**Proof.** It is known \([5]\) Sect. 1.3] that a choice of a polarization on \(A\) gives rise to a nondegenerate alternating bilinear form

\[e_\ell : V(\ell) \times V(\ell) \rightarrow \mathbb{Q}_\ell(1)\]

such that

\[e_\ell(\rho_{\ell,A}(\sigma)(x), \rho_{\ell,A}(\sigma)(y)) = \chi_\ell(\sigma) \cdot e_\ell(x, y) \quad \forall x, y \in V(\ell); \sigma \in \text{Gal}(K)\].
By fixing a (non-canonical) isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$Q_\ell(1) \cong Q_\ell,$$

we may assume that the alternating form $e_\ell$ takes on values in $Q_\ell$. We obtain that $G_{\ell,A,K}$ lies in the corresponding group of symplectic similitudes

$$G_{\ell,A,K} \subset \text{Gp}(V_\ell(A), e_\ell) = \{ s \in \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \mid \exists c \in Q_\ell^* \text{ such that } e_\ell(sx, sy) = c \cdot e_\ell(x, y) \ \forall x, y \in V_\ell(A) \} \subset \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(A))$$

and $\chi_\ell$ coincides with the composition of

$$\rho_{\ell,A} : \text{Gal}(K) \to G_{\ell,A,K}$$

and

$$G_{\ell,A,K} \subset \text{Gp}(V_\ell(A), e_\ell) \to Q_\ell^*$$

where the scalar $c(s)$ is defined by

$$e_\ell(sx, sy) = c(s) \cdot e_\ell(x, y) \ \forall x, y \in V_\ell(A); s \in \text{Gp}(V_\ell(A), e_\ell).$$

Clearly,

$$c : \text{Gp}(V_\ell(A), e_\ell) \to Q_\ell^* \text{, } s \mapsto c(s)$$

is a homomorphism of $\ell$-adic Lie groups and

$$c(s)^{\dim(A)} = \det(s) \ \forall s \in \text{Gp}(V_\ell(A), e_\ell)$$

(recall that $V_\ell(A)$ is a $2\dim(A)$-dimensional $\mathbb{Q}_\ell$-vector space). It is also clear that $G^0_{\ell,A,K}$ coincides with the kernel of the homomorphism of $\ell$-adic Lie groups

$$c : G_{\ell,A,K} \subset \text{Gp}(V_\ell(A), e_\ell) \to Q_\ell^*,$$

and therefore $\mathfrak{g}^0_{\ell,A}$ coincides with the kernel of the corresponding tangent map of Lie algebras

$$\mathfrak{g}_{\ell,A} \to Q_\ell.$$

On the other hand, one may easily check that the tangent map is

$$\frac{1}{\dim(A)} \text{tr} : \mathfrak{g}_{\ell,A} \to Q_\ell$$

(because tr is the tangent map to det.) This implies that

$$\mathfrak{g}^0_{\ell,A} = \mathfrak{g}_{\ell,A} \cap \mathfrak{sl}(V_\ell(A)).$$

This proves the first assertion of Proposition. In order to prove the second assertion, notice that $\mathbb{Q}_\ell \text{Id} \cap \mathfrak{sl}(V_\ell(A)) = \{0\}$ and therefore $\mathfrak{g}_{\ell,A}$ contains $\mathbb{Q}_\ell \text{Id} \oplus \mathfrak{g}^0_{\ell,A}$. On the other hand, since $\mathfrak{g}^0_{\ell,A}$ is the kernel of $\mathfrak{g}_{\ell,A} \to Q_\ell$, its codimension in $\mathfrak{g}_{\ell,A}$ is (at most) 1. This implies that $\mathfrak{g}_{\ell,A} = \mathbb{Q}_\ell \text{Id} \oplus \mathfrak{g}^0_{\ell,A}$. □

**Proposition 2.3.** There exists a finite separable algebraic field extension $K_0/K$ that enjoys the following properties.

If $K'/K_0$ is a finite separable algebraic field extension then

$$\text{End}_{\text{Gal}(K')}(V_\ell(A)) = \text{End}_{\text{B}_{\ell,A}}(V_\ell(A)).$$
Proof. (Compare with [9] Prop. 1 and its proof.) Let us choose open neighborhoods $V$ of 0 in $\mathfrak{g}_{\ell,A}$ and $U$ of $\text{Id}$ in $G_{\ell,A,K}$ such that the $\ell$-adic exponential map $\exp$ and logarithm map $\log$ establish mutually inverse $\mathbb{Q}_{\ell}$-analytic isomorphisms between $V$ and $U$.

Let $G_0$ be an open subgroup of $G_{\ell,A,K}$ that lies in $U$. (The existence of such $G_0$ follows from Corollary 2 in [11] Part II, Ch. 4, Sect. 8, p. 117.)

Then $V_0 = \log(G_0)$ is an open subset of $\mathfrak{g}_{\ell,A}$ that contains 0 and $G_0 = \exp(V_0)$. Clearly, $G_0$ has finite index in $G_{\ell,A,K}$ and

$$\text{End}_{G_0}(V_0(A)) = \text{End}_{\mathfrak{g}_{\ell,A}}(V_0(A)) = \text{End}_{\mathfrak{g}_{\ell,A}}(V_0(A)).$$

The preimage of $G_0$ in $G_{\ell,A,K}$ is an open subgroup of finite index and therefore coincides with $\text{Gal}(K_0)$ for a certain finite separable algebraic field extension $K_0$ of $K$. It follows that

$$\text{End}_{\text{Gal}(K_0)}(V_0(A)) = \text{End}_{G_0}(V_0(A)) = \text{End}_{\mathfrak{g}_{\ell,A}}(V_0(A)).$$

If $K'/K_0$ is a finite separable algebraic field extension then $\text{Gal}(K')$ is a compact subgroup of finite index in $\text{Gal}(K_0)$ and its image $G' = \rho_{\ell,A}(\text{Gal}(K'))$ is a closed subgroup of finite index in $G_0$ and therefore is open in $G_0$ and therefore is also open in $G_{\ell,A,K}$. As above, $V' = \log(G')$ is an open subset of $\mathfrak{g}_{\ell,A}$ that contains 0 and $G' = \exp(V')$ and

$$\text{End}_{\text{Gal}(K')}(V_0(A)) = \text{End}_{G'}(V_0(A)) = \text{End}_{V'}(V_0(A)) = \text{End}_{\mathfrak{g}_{\ell,A}}(V_0(A)).$$

$\square$

3. Homomorphisms of abelian varieties

Throughout this Section, $X, Y, Z, A$ are abelian varieties over $K$ and $\ell$ is a prime different from char($K$). We write $\text{Hom}_K(X, Y)$ for the (finitely generated free) commutative group of $K$-homomorphisms from $X$ to $Y$. If $X = Y = A$ then $\text{Hom}_K(X, Y)$ coincides with the ring $\text{End}_K(A)$ of $K$-endomorphisms of $A$.

There is a natural embedding of $\mathbb{Z}_\ell$-modules

$$i_{X,Y,K} : \text{Hom}_K(X, Y) \otimes \mathbb{Z}_\ell \hookrightarrow \text{Hom}_{\text{Gal}(K)}(T_\ell(X), T_\ell(Y)),$$

whose cokernel is torsion free [9] [16].

Remark 3.1. Notice that if $L/K$ is a finite or infinite Galois extension and

$$i_{X,Y,L} : \text{Hom}_L(X, Y) \otimes \mathbb{Z}_\ell \hookrightarrow \text{Hom}_{\text{Gal}(L)}(T_\ell(X), T_\ell(Y))$$

is bijective then $i_{X,Y,K}$ is also bijective. This assertion follows easily from the following obvious description of $\text{Gal}(L/K)$-invariants

$$\text{Hom}_K(X, Y) = \{\text{Hom}_L(X, Y)\}^{\text{Gal}(L/K)},$$

$$\text{Hom}_{\text{Gal}(K)}(T_\ell(X), T_\ell(Y)) = \{\text{Hom}_{\text{Gal}(L)}(T_\ell(X), T_\ell(Y))\}^{\text{Gal}(L/K)}$$

and the $\text{Gal}(L/K)$-equivariance of $i_{X,Y,L}$.

Extending $i_{X,Y,K}$ by $\mathbb{Q}_{\ell}$-linearity, we obtain the natural embedding of $\mathbb{Q}_{\ell}$-vector spaces

$$\tilde{i}_{X,Y,K} : \text{Hom}_K(X, Y) \otimes \mathbb{Q}_{\ell} \hookrightarrow \text{Hom}_{\text{Gal}(K)}(V_\ell(X), V_\ell(Y)),$$

see [16] Section 1, displayed formula (2) on p. 135.

The following observations are due to J. Tate [16] Sect. 1, Lemma 1 and Lemma 3 and its proof on p. 135.
Lemma 3.2 (of Tate).

1. The map $i_{X,Y,K}$ is bijective if and only if $\tilde{i}_{X,Y,K}$ is bijective.

2. Let us put $Z = X \times Y$. If the embedding

$$\tilde{i}_{Z,Z,K} : \text{End}_K(Z) \otimes \mathbb{Q}_\ell \hookrightarrow \text{End}_{\text{Gal}(K)}(V_\ell(Z))$$

is bijective then

$$\tilde{i}_{X,Y,K} : \text{Hom}_K(X,Y) \otimes \mathbb{Q}_\ell \hookrightarrow \text{Hom}_{\text{Gal}(K)}(V_\ell(X), V_\ell(Y))$$

is also bijective.

Tate [15, 16] conjectured and G. Faltings [4, 5] proved that this embedding is actually a bijection when $K$ is finitely generated over the field $\mathbb{Q}$ of rational numbers.

Theorem 3.3. Suppose that $K$ is field and $\ell$ is a prime that is different from $\text{char}(K)$. Let us put $E = K(\ell)$. Suppose that $A$ is an abelian variety of positive dimension over $K$ such that for all finite separable algebraic field extensions $K'/K$ the embedding

$$\tilde{i}_{A,A,K'} : \text{End}_{K'}(A) \otimes \mathbb{Q}_\ell \hookrightarrow \text{End}_{\text{Gal}(K')}(V_\ell(A))$$

is bijective. If $g_{E,A}$ contains the homotheties $\mathbb{Q}_\ell \text{Id}$ then the injective maps

$$\tilde{i}_{A,A,E} : \text{End}_E(A) \otimes \mathbb{Q}_\ell \hookrightarrow \text{End}_{\text{Gal}(E)}(V_\ell(A))$$

and

$$i_{A,A,E} : \text{End}_E(A) \otimes \mathbb{Z}_\ell \hookrightarrow \text{End}_{\text{Gal}(E)}(T_\ell(A))$$

are bijective.

Proof. Let us consider the field $K_2 = K(A_{\ell^2})$ of definition of all points of $A_{\ell^2}$ and put $E_2 = K_2(\ell)$. Clearly, $K_2/K$ and $E_2/E$ are finite Galois extensions. Applying Remark 3.1 (to $E_2/E$ instead of $L/K$) and Lemma 3.2.1 to $X = Y = A$, we observe that in the course of the proof we may (and will) assume that

$$K = K_2 = K(A_{\ell^2}),$$

i.e., $A_{\ell^2} \subset A(K)$. Since $\ell^2 \geq 4$, it follows from a result of A. Silverberg [14] that all $\bar{K}$-endomorphisms of $A$ are defined over $K$. In particular,

$$\text{End}_K(A) = \text{End}_E(A) = \text{End}_{\bar{K}}(A).$$

Using Proposition 2.3 we may replace $K$ by its finite separable algebraic extension in such a way that

$$\text{End}_{\text{Gal}(K')}(V_\ell(A)) = \text{End}_{g_{E,A}}(V_\ell(A))$$

for all finite separable algebraic field extensions $K'$ of $K$.

Let $K_0/K$ satisfies the conclusion of Proposition 2.3. Replacing $K_0$ by its normal closure over $K$, we may and will assume that $K_0/K$ is a finite Galois extension. Let us put $E_0 = K_0(\ell)$. Clearly, $E_0/E$ is a finite Galois extension and

$$\text{End}_E(A) = \text{End}_K(A) = \text{End}_{K_0}(A) = \text{End}_{E_0}(A).$$

By the assumption of Theorem 3.3

$$\text{End}_{\text{Gal}(K_0)}(V_\ell(A)) = \text{End}_{K_0}(A) \otimes \mathbb{Q}_\ell.$$

By Proposition 2.3

$$\text{End}_{\text{Gal}(K_0)}(V_\ell(A)) = \text{End}_{g_{E,A}}(V_\ell(A)).$$
This implies that
\[ \text{End}_{K_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{g_\ell,A}(V_\ell(A)). \]

By Proposition 2.2
\[ \text{End}_{g_\ell,A}(V_\ell(A)) = \text{End}_{E_0}(V_\ell(A)). \]

This implies that
\[ \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{K_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{g_\ell,A}(V_\ell(A)). \]

By Remark 2.1 applied to \( K_0 \) and \( E_0 \) (instead of \( K \) and \( E \)),
\[ \text{End}_{g_\ell,A}(V_\ell(A)) \supset \text{End}_{\text{Gal}(E_0)}(V_\ell(A)). \]

So, we get
\[ \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{g_\ell,A}(V_\ell(A)) \supset \text{End}_{\text{Gal}(E_0)}(V_\ell(A)) \supset \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell, \]

which implies that
\[ \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{g_\ell,A}(V_\ell(A)) = \text{End}_{\text{Gal}(E_0)}(V_\ell(A)) = \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell. \]

In particular,
\[ \text{End}_{E_0}(A) \otimes \mathbb{Q}_\ell = \text{End}_{\text{Gal}(E_0)}(V_\ell(A)). \]

Now Lemma 3.2.1 applied to \( X = Y = A \) and to \( E_0 \) (instead of \( K \)) implies that
\[ \text{End}_{E_0}(A) \otimes \mathbb{Z}_\ell = \text{End}_{\text{Gal}(E_0)}(T_\ell(A)). \]

It follows from Remark 3.1 (applied to \( E_0/E \) instead of \( L/K \)) that
\[ \text{End}_{E}(A) \otimes \mathbb{Z}_\ell = \text{End}_{\text{Gal}(E)}(T_\ell(A)). \]

Again, Lemma 3.2.1 tells us that
\[ \text{End}_{E}(A) \otimes \mathbb{Q}_\ell = \text{End}_{\text{Gal}(E)}(V_\ell(A)). \]

Lemma 3.4 (of Clifford). Let \( G \) be a group and \( H \) its normal subgroup. Let \( W \) be a vector space of finite positive dimension over a field \( k \). Let \( \rho : G \to \text{Aut}_k(W) \) be a semisimple (completely reducible) linear representation of \( G \). Then the corresponding \( H \)-module \( W \) is also semisimple.

Proof. Let us split \( W \) into a finite direct sum \( W = \oplus W_i \) of simple \( G \)-modules \( W_i \).

By Theorem (49.2) of [3], the corresponding \( H \)-modules \( W_i \) are semisimple. This implies that the \( H \)-module \( W \) is a direct sum of semisimple \( H \)-modules \( W_i \)'s and therefore is also semisimple.

Proposition 3.5. Let \( L/K \) be a finite or infinite Galois extension of \( K \). If the \( \text{Gal}(K) \)-module \( V_\ell(A) \) is semisimple then the \( \text{Gal}(L) \)-module \( V_\ell(A) \) is semisimple.

Proof. Since \( L/K \) is Galois, the subgroup \( \text{Gal}(L) \) of \( \text{Gal}(K) \) is normal. Now the result follows from Lemma 3.4. □
4. Homotheties, centralizers and semisimplicity

**Theorem 4.1** (of Bogomolov). Suppose that $K$ is a field that is finitely generated over $\mathbb{Q}$ and $\ell$ is a prime. Let $A$ be an abelian variety of positive dimension over $K$. Then $\mathfrak{g}_{\ell,A}$ contains the homotheties $\mathbb{Q}_\ell \text{Id}$.

**Proof.** When $K$ is a number field, this assertion was proven by Bogomolov [1, 2]. The case of arbitrary finitely generated $K$ is also known [13, Sect. 1, p. 2] and follows from the number field case with the help of Serre’s variant of the Hilbert irreducibility theorem for infinite extensions ([13, Sect. 1], [12, Sect. 10.6], [7, Prop. 1.3 on pp. 163–164]). Indeed, there exist a number field $F$, an abelian variety $B$ over $F$ with $\dim(A) = \dim(B)$ and an isomorphism of $\mathbb{Z}_\ell$-modules $u : T_{\ell}(A) \cong T_{\ell}(B)$ such that $u^{-1} G_{\ell,B,F} u = G_{\ell,A,K}$. Extending $u$ by $\mathbb{Q}_\ell$-linearity, we obtain the isomorphism of $\mathbb{Q}_\ell$-vector spaces $V_{\ell}(A) \cong V_{\ell}(B)$, which we still denote by $u$.

Clearly, $u^{-1} g_{\ell,B} u = g_{\ell,A}$. Since $F$ is a number field, $\mathfrak{g}_{\ell,B}$ contains all the homotheties, which implies that $\mathfrak{g}_{\ell,A}$ also contains all the homotheties. This ends the proof. □

**Proof of Theorem 1.1.** In light of Remark 1.2, we may and will assume that $X$ and $Y$ are defined over $K$. Let us put $A = X \times Y$. Since $K$ is finitely generated over $\mathbb{Q}$, every finite algebraic extension $K'$ of $K$ is also finitely generated over $\mathbb{Q}$. By the theorem of Faltings [4, 5], the injection

$$\tilde{i}_{A,A,K'} : \text{End}_{K'}(A) \otimes \mathbb{Q}_\ell \to \text{End}_{\text{Gal}(K')} (V_{\ell}(A))$$

is bijective. Thanks to Theorem 4.1 we know that $\mathfrak{g}_{\ell,A}$ contains the homotheties $\mathbb{Q}_\ell \text{Id}$. Now the desired result follows from Theorem 3.3 combined with Lemma 3.2.

**Corollary 4.2.** Suppose that $K$ is a field that is finitely generated over $\mathbb{Q}$ and $\ell$ is a prime. Let $E_1$ be a field extension of $K$ that lies in $K(\ell)$. If $X$ and $Y$ are abelian varieties over $E_1$ then the natural embedding of $\mathbb{Z}_\ell$-modules

$$\text{Hom}_{E_1}(X,Y) \otimes \mathbb{Z}_\ell \to \text{Hom}_{\text{Gal}(E_1)} (T_{\ell}(X), T_{\ell}(Y))$$

is bijective.

**Proof.** Since $K(\ell)/E_1$ is a Galois extension, the result follows from Theorem 1.1 combined with Remark 3.1 applied to $K(\ell)/E_1$.

□

**Theorem 4.3.** Suppose that $K$ is a field that is finitely generated over $\mathbb{Q}$ and $\ell$ is a prime. Let $L/K$ be a finite or infinite Galois extension. (E.g., $L = K(\ell)$.) Let $A$ be an abelian variety of positive dimension over $K$. Then the $\text{Gal}(L)$-module $V_{\ell}(A)$ is semisimple.

**Proof.** Faltings [4, 5] proved that the $\text{Gal}(K)$-module $V_{\ell}(A)$ is semisimple. This also covers the case when $L/K$ is a finite (Galois) extension. The case of infinite $L/K$ follows from Faltings’ result combined with Proposition 3.3.

□

**Theorem 4.4.** Suppose that $K$ is a field that is finitely generated over $\mathbb{Q}$. Let $L/K$ be a finite or infinite Galois extension. Let $A$ be an abelian variety of positive dimension over $K$. Then the $\text{Gal}(L)$-module $A_{\ell}$ is semisimple for all but finitely many primes $\ell$.
Proof. For all but finitely many primes \( \ell \) the \( \text{Gal}(K) \)-module \( A_\ell \) is semisimple. Indeed, when \( K \) is a number field, this assertion is contained in Corollary 5.4.3 on p. 316 of [15] (the proof is based on results of Faltings [4]). The same proof works over arbitrary fields that are finitely generated over \( \mathbb{Q} \), provided one replaces the reference to Prop. 3.1 of [4] by the reference to the corollary on p. 211 of [5]. Since \( L/K \) is Galois, \( \text{Gal}(L) \) is a normal subgroup of \( \text{Gal}(K) \). Now the desired result follows from Lemma 3.4 applied to \( k = \mathbb{F}_\ell, W = A_\ell \) and \( G = \text{Gal}(K), H = \text{Gal}(L) \).

\[ \square \]

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