On the parameter $\mu_{21}$ of a complete bipartite graph

A.M. Khachatryan$^1$, R.R. Kamalian$^2$

$^1$Ijevan Branch of Yerevan State University, e-mail: khachatryanarpine@gmail.com
$^2$The Institute for Informatics and Automation Problems of NAS RA, e-mail: rrkamalian@yahoo.com

Abstract

A proper edge $t$-coloring of a graph $G$ is a coloring of edges of $G$ with colors $1, 2, \ldots, t$ such that all colors are used, and no two adjacent edges receive the same color. The set of colors of edges incident with a vertex $x$ is called a spectrum of $x$. An arbitrary nonempty subset of consecutive integers is called an interval.

Suppose that all edges of a graph $G$ are colored in the game of Alice and Bob with asymmetric distribution of roles. Alice determines the number $t$ of colors in the future proper edge coloring of $G$ and aspires to minimize the number of vertices with an interval spectrum in it. Bob colors edges of $G$ with $t$ colors and aspires to maximize that number. $\mu_{21}(G)$ is equal to the number of vertices of $G$ with an interval spectrum at the finish of the game on the supposition that both players choose their best strategies.

In this paper, for arbitrary positive integers $m$ and $n$, the exact value of the parameter $\mu_{21}(K_{m,n})$ is found.

Keywords: proper edge coloring, interval spectrum, game, complete bipartite graph.

We consider finite, undirected, connected graphs without loops and multiple edges containing at least one edge. For any graph $G$, we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of $G$, respectively. For any $x \in V(G)$, $d_G(x)$ denotes the degree of the vertex $x$ in $G$. For a graph $G$, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of vertices in $G$, respectively. For a graph $G$, and for any $V_0 \subseteq V(G)$, we denote by $G[V_0]$ the subgraph of the graph $G$ induced by the subset $V_0$ of its vertices. We denote by $C_4$ a simple cycle with four vertices.

An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element $p$ and the maximum element $q$ is denoted by $[p, q]$.

A function $\varphi : E(G) \to [1, t]$ is called a proper edge $t$-coloring of a graph $G$, if all colors are used, and for any adjacent edges $e_1 \in E(G), e_2 \in E(G)$, $\varphi(e_1) \neq \varphi(e_2)$.

The minimum value of $t$ for which there exists a proper edge $t$-coloring of a graph $G$ is denoted by $\chi'(G)$ [2].

For any graph $G$, and for any $t \in [\chi'(G), |E(G)|]$, we denote by $\alpha(G, t)$ the set of all proper edge $t$-colorings of $G$.

Let us also define a set $\alpha(G)$ of all proper edge colorings of a graph $G$:

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)} |E(G)| \alpha(G, t).$$

If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set $\{\varphi(e)/e \in E(G), e \text{ is incident with } x\}$ is called a spectrum of the vertex $x$ of the graph $G$ at the coloring $\varphi$ and is denoted by $S_G(x, \varphi)$; if $S_G(x, \varphi)$ is an interval,
we say that $\varphi$ is interval in $x$. If $G$ is a graph, $\varphi \in \alpha(G)$, $R \subseteq V(G)$, then we say, that $\varphi$ is interval on $R$ iff for $\forall x \in R$, $\varphi$ is interval in $x$. We say, that a subset $R$ of vertices of a graph $G$ has an $i$-property iff there exists $\varphi \in \alpha(G)$ interval on $R$. If $G$ is a graph, and a subset $R$ of its vertices has an $i$-property, we denote by $w_R(G)$ and $W_R(G)$ (omitting the index in these notations in a peculiar case with $R = V(G)$) the minimum and the maximum value of $t$, respectively, for which $\exists \varphi \in \alpha(G, t)$ interval on $R$. If $G$ is a graph, $\varphi \in \alpha(G)$, then set $V_{\text{int}}(G, \varphi) \equiv \{x \in V(G) / S_G(x, \varphi) \text{ is an interval}\}$ and $f_G(\varphi) \equiv |V_{\text{int}}(G, \varphi)|$. A proper edge coloring $\varphi \in \alpha(G)$ is called an interval edge coloring $[3, 4, 5]$ of the graph $G$ iff $f_G(\varphi) = |V(G)|$. The set of all graphs having an interval edge coloring is denoted by $\mathfrak{M}$.

For a graph $G$, and for any $t \in [\chi'(G), |E(G)|]$, we set $[6]$:

$$
\mu_1(G, t) \equiv \min_{\varphi \in \alpha(G, t)} f_G(\varphi), \quad \mu_2(G, t) \equiv \max_{\varphi \in \alpha(G, t)} f_G(\varphi).
$$

For any graph $G$, we set $[6]$:

$$
\mu_{11}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G, t), \quad \mu_{12}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G, t),
$$

$$
\mu_{21}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G, t), \quad \mu_{22}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G, t).
$$

Clearly, the parameters $\mu_{11}$, $\mu_{12}$, $\mu_{21}$ and $\mu_{22}$ are correctly defined for an arbitrary graph. Their exact values are found for simple paths, simple cycles and some outerplanar graphs $[7]$, Möbius ladders $[6]$, complete graphs $[8]$, complete bipartite graphs $[9][10]$, prisms $[11]$ and $n$-dimensional cubes $[11][12]$. The exact values of $\mu_{11}$ and $\mu_{22}$ for trees are found in $[13]$. The exact value of $\mu_{12}$ for an arbitrary tree is found in $[14]$.

In addition to the definitions given above, let us note that exact values of the parameters $\mu_{12}$ and $\mu_{21}$ have certain game interpretations. Suppose that all edges of a graph $G$ are colored in the game of Alice and Bob with asymmetric distribution of roles. Alice determines the number $t$ of colors in the future coloring $\varphi$ of the graph $G$, satisfying the condition $t \in [\chi'(G), |E(G)|]$, Bob colors edges of $G$ with $t$ colors.

When Alice aspires to maximize, Bob aspires to minimize the value of the function $f_G(\varphi)$, and both of them choose their best strategies, then at the finish of the game exactly $\mu_{12}(G)$ vertices of $G$ will receive an interval spectrum.

When Alice aspires to minimize, Bob aspires to maximize the value of the function $f_G(\varphi)$, and both of them choose their best strategies, then at the finish of the game exactly $\mu_{21}(G)$ vertices of $G$ will receive an interval spectrum.

In this paper, for arbitrary positive integers $m$ and $n$, we determine the exact value of $\mu_{21}$ for the complete bipartite graph $K_{m,n}$.

For $m \geq n$, let $K_{m,n}$ be a complete bipartite graph with a bipartition $(X,Y)$, where $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$.

Clearly, for any positive integers $m$ and $n$, $\chi'(K_{m,n}) = \Delta(K_{m,n}) = m$, $|E(K_{m,n})| = mn$.

First we recall some known results.

**Theorem 1.** $[3][4][5]$ If $R$ is the set of all vertices of an arbitrary part of a bipartite graph $G$, then:

1) $R$ has an $i$-property,

2) $W_R(G) = |E(G)|$,

3) for any $t \in [w_R(G), W_R(G)]$, there exists $\varphi_t \in \alpha(G, t)$ interval on $R$.

**Theorem 2.** $[15]$ For arbitrary positive integers $m$ and $n$, $w_Y(K_{m,n}) = n \cdot \left\lceil \frac{m}{n} \right\rceil$.

**Theorem 3.** $[5][10]$ For arbitrary positive integers $m$ and $n$, $K_{m,n} \in \mathfrak{M}$, $w(K_{m,n}) = m + n - \gcd(m, n)$, $W(K_{m,n}) = m + n - 1$; moreover, for any $t \in [w(K_{m,n}), W(K_{m,n})]$, there exists $\varphi_t \in \alpha(K_{m,n}, t)$ with $V_{\text{int}}(K_{m,n}, \varphi_t) = V(K_{m,n})$. 
Corollary 1. For arbitrary positive integers \( m \) and \( n \), the inequality \( \max\{m,n\} \leq \min\{m,n\} \cdot \left\lfloor \frac{\max\{m,n\}}{\min\{m,n\}} \right\rfloor \leq m + n - \gcd(m,n) \leq m + n - 1 \) is true.

Theorem 4. If \( G \) is a graph with \( \delta(G) \geq 2, \varphi \in \alpha(G,|E(G)|), V_{int}(G,\varphi) \neq \emptyset \), then \( G[V_{int}(G,\varphi)] \) is a forest each connected component of which is a simple path.

If \( G \) is a graph with \( \chi'(G) = \Delta(G), t \in [\Delta(G),|E(G)|], \xi \in \alpha(G,t) \), then for any \( j \in [1,t] \), we denote by \( E(G,\xi,j) \) the set of all edges of \( G \) colored by the color \( j \) at the coloring \( \xi \). The coloring \( \xi \) is called a harmonic \( t \)-coloring of the graph \( G \), if for any \( i \in [1,\Delta(G)] \), the set

\[
\bigcup_{1 \leq j \leq t, j \equiv i(\mod(\Delta(G)))} E(G,\xi,j)
\]

is a matching in \( G \).

Suppose that \( G \) is a graph with \( \chi'(G) = \Delta(G), t \in [\Delta(G),|E(G)|], \xi \) is a harmonic \( t \)-coloring of \( G \). Let us define a sequence \( \xi^*, \xi^*_1, \ldots, \xi^*_{\chi'(G)} \) of proper edge colorings of the graph \( G \).

Set \( \xi_0^* \equiv \xi \).

Case 1. \( t = \chi'(G) \). The sequence mentioned above is already constructed.

Case 2. \( t \in [\chi'(G)+1,|E(G)|] \). Suppose that \( j \in [1,t-\chi'(G)] \), and the proper edge colorings \( \xi^*_0, \ldots, \xi^*_{t-1} \) of the graph \( G \) are already constructed. Let us define \( \xi^*_j \). For an arbitrary \( e \in E(G) \), set:

\[
\xi^*_j(e) = \begin{cases} 
\xi^*_{j-1}(e), & \text{if } \xi^*_{j-1}(e) \neq \max(\{\xi^*_{j-1}(e)/e \in E(G)\}) \\
\xi^*_{j-1}(e) - \Delta(G), & \text{if } \xi^*_{j-1}(e) = \max(\{\xi^*_{j-1}(e)/e \in E(G)\})
\end{cases}
\]

Remark 1. Suppose that \( G \) is a graph with \( \chi'(G) = \Delta(G), t \in [\Delta(G),|E(G)|], \xi \) is a harmonic \( t \)-coloring of \( G \). All proper edge colorings \( \xi^*_0, \xi^*_1, \ldots, \xi^*_{\chi'(G)-1} \) of the graph \( G \) are already constructed. Then, for any \( j \in [1,\chi'(G)] \), \( \xi_j^* \) is a harmonic \( (t-j) \)-coloring of the graph \( G \).

Remark 2. Suppose that \( G \) is a graph with \( \chi'(G) = \Delta(G), t \in [1+\chi'(G),|E(G)|], \xi \) is a harmonic \( t \)-coloring of \( G \). It is not difficult to see, that for any \( j \in [1,\chi'(G)] \), \( \xi_j^* \) is a harmonic \( (t-j) \)-coloring of the graph \( G \).

Lemma 1. If integers \( m \) and \( n \) satisfy either conditions \( m \geq 3, n = 2 \), or the inequality \( m \geq n \geq 3 \), then \( \mu_G(K_{m,n}, mn) \leq m \).

Proof. Assume the contrary. Then there exists \( \varphi_0 \in \alpha(K_{m,n}, mn) \) with \( f_{K_{m,n}}(\varphi_0) = m + k \), where \( k \in [1,n] \). Clearly, \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = m + k - |V_{int}(K_{m,n},\varphi_0) \cap X| \).

Case 1. \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = 0 \). In this case we obtain a contradiction \( m < m + k = |V_{int}(K_{m,n},\varphi_0) \cap Y| \leq |Y| = m. \)

Case 2. \( |V_{int}(K_{m,n},\varphi_0) \cap X| = 1 \). In this case \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = m + k - 1 \geq m \). From this inequality we obtain that \( \Delta(K_{m,n}[V_{int}(K_{m,n},\varphi_0)]) \geq 3 \), but it is impossible because of Theorem 4.

Case 3. \( 2 \leq |V_{int}(K_{m,n},\varphi_0) \cap Y| \leq n \). In this case \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = m + k - |V_{int}(K_{m,n},\varphi_0) \cap X| \geq m - n + k \).

Clearly, if at least one of the inequalities \( m - n \geq 1 \) and \( k \geq 2 \) is true, we obtain the inequality \( |V_{int}(K_{m,n},\varphi_0) \cap Y| \geq 2 \), which contradicts Theorem 4.

Therefore, without loss of generality, we can assume, that \( m = n, k = 1 \), \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = m + 1 - |V_{int}(K_{m,n},\varphi_0) \cap X|, 1 \leq |V_{int}(K_{m,n},\varphi_0) \cap Y| \leq m - 1 \).

Let us notice that the inequality \( |V_{int}(K_{m,n},\varphi_0) \cap Y| \geq 2 \) is incompatible with the inequality \( |V_{int}(K_{m,n},\varphi_0) \cap X| \geq 2 \) because of Theorem 4, therefore \( |V_{int}(K_{m,n},\varphi_0) \cap Y| = 1 \).

An assumption \( |V_{int}(K_{m,n},\varphi_0) \cap X| \geq 3 \) implies the inequality \( \Delta(K_{m,n}[V_{int}(K_{m,n},\varphi_0)]) \geq 3 \), which contradicts Theorem 4.
An assumption $|V_{int}(K_{m,n}, \varphi_0) \cap X| = 2$ implies the equality $f_{K_{m,n}}(\varphi_0) = 3$, which is incompatible with the equality $f_{K_{m,n}}(\varphi_0) = m + k$ because of $m \geq 3$ and $k = 1$.

The Lemma is proved.

**Lemma 2.** If integers $m$ and $n$ satisfy either conditions $m \geq 3$, $n = 2$, or the inequality $m \geq n \geq 3$, then $\mu_2(K_{m,n}, mn) = m$.

**Proof.** It follows from Theorem [4] that there exists $\bar{\varphi} \in \alpha(K_{m,n}, mn)$ interval on $Y$. It means that $\mu_2(K_{m,n}, mn) \geq m$. From Lemma [1] we have $\mu_2(K_{m,n}, mn) \leq m$.

The Lemma is proved.

**Theorem 5.** For arbitrary positive integers $m$ and $n$, where $m \geq n$,

$$\mu_{21}(K_{m,n}) = \begin{cases} 
   m + 1, & \text{if } n = 1 \text{ or } m = n = 2 \\
   m & \text{otherwise.}
\end{cases}$$

**Proof.**

**Case 1.** $n = 1$. In this case $\chi'(K_{m,1}) = \Delta(K_{m,1}) = |E(K_{m,1})| = m$. It means that for all $\varphi \in \alpha(K_{m,1}, m)$, $f_{K_{m,1}}(\varphi) = m + 1$. Hence, $\mu_2(K_{m,1}, m) = m + 1$, $\mu_{21}(K_{m,1}) = m + 1$.

**Case 2.** $m = n = 2$. Clearly, $K_{2,2} \cong C_4$, and the theorem follows from the results of [7].

**Case 3.** $m \geq 3$, $n = 2$ or $m \geq n \geq 3$.

From Lemma [2] we have $\mu_2(K_{m,n}, mn) = m$. Let us show that for any $t \in [m, mn]$, the inequality $\mu_2(K_{m,n}, t) \geq m$ holds.

From Theorems [1] and [2] it follows that for any $t \in [n \cdot \lceil \frac{m}{n} \rceil, mn]$, there exists $\varphi_t \in \alpha(K_{m,n}, t)$ interval on $Y$ with $f_{K_{m,n}}(\varphi_t) \geq m$. It means that for any $t \in [n \cdot \lceil \frac{m}{n} \rceil, mn]$, the inequality $\mu_2(K_{m,n}, t) \geq m$ is true.

Now let us show that for any $t \in [m, m + n - 1]$, the inequality $\mu_2(K_{m,n}, t) \geq m$ is also true.

Let us define [5, 16] a proper edge $(m + n - 1)$-coloring $\xi$ of the graph $K_{m,n}$. For any integers $m$ and $n$, satisfying the inequalities $1 \leq i \leq n$, $1 \leq j \leq m$, set $\xi((x_i, y_j)) \equiv i + j - 1$. It is easy to see that $\xi$ is a harmonic $(m + n - 1)$-coloring of $K_{m,n}$ with $f_{K_{m,n}}(\xi) = m + n$. Let us consider the sequence $\xi_0^*, \xi_1^*, \ldots, \xi_{n-1}^*$ of proper edge colorings of $K_{m,n}$. Taking into account Remarks [1]–[3] it is not difficult to notice, that for any $j \in [1, n - 1]$, $f_{K_{m,n}}(\xi_j^*) = m + n - j$. Consequently, for any $j \in [1, n - 1]$, $f_{K_{m,n}}(\xi_j^*) \geq m + 1$.

It means that for any $t \in [m, m + n - 1]$, the inequality $\mu_2(K_{m,n}, t) \geq m$ is true indeed.

Now, taking into account Corollary [1] we can conclude that for any $t \in [m, mn]$, the inequality $\mu_2(K_{m,n}, t) \geq m$ is proved. From Lemma [2] we obtain $\mu_{21}(K_{m,n}) = m$.

The Theorem is proved.

**Corollary 2.** For any positive integers $m$ and $n$, where $m \geq n$, the inequality $m \leq \mu_{21}(K_{m,n}) \leq m + 1$ holds.

**References**

[1] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, New Jersey, 1996.

[2] V.G. Vizing, *The chromatic index of a multigraph*, Kibernetika 3 (1965), pp. 29–39.

[3] A.S. Asratian, R.R. Kamalian, *Interval colorings of edges of a multigraph*, Appl. Math. 5 (1987), Yerevan State University, pp. 25–34.

[4] A.S. Asratian, R.R. Kamalian, *Investigation of interval edge-colorings of graphs*, Journal of Combinatorial Theory. Series B 62 (1994), no.1, pp. 34–43.

[5] R.R. Kamalian, *Interval Edge Colorings of Graphs*, Doctoral dissertation, the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of USSR, Novosibirsk, 1990.
[6] N.N. Davtyan, R.R. Kamalian, *On boundaries of extremums of the number of vertices with an interval spectrum among the set of proper edge colorings of "Möbius ladders" with t colors under variation of t*, Proc. of the 3rd Ann. Sci. Conf. (December 5–10, 2008) of the RAU, Yerevan, 2009, pp. 81–84.

[7] N.N. Davtyan, R.R. Kamalian, *On properties of the number of vertices with an interval spectrum in proper edge colorings of some graphs*, the Herald of the RAU, №2, Yerevan, 2009, pp. 33–42.

[8] A.M. Khachatryan, *On boundaries of extremums of the number of vertices with an interval spectrum among the set of proper edge colorings of complete graphs with t colors under variation of t*, Proc. of the 5th Ann. Sci. Conf. (December 6–10, 2010) of the RAU, Yerevan, 2011, pp. 268–272.

[9] A.M. Khachatryan, *On the parameters µ₁₁, µ₁₂ and µ₂₂ of complete bipartite graphs*, the Herald of the RAU, №1, Yerevan, 2011, pp. 76–83.

[10] R.R. Kamalian, A.M. Khachatryan, *On the sharp value of the parameter µ₂₁ of complete bipartite graphs*, the Herald of the RAU, №2, Yerevan, 2011, pp. 19–27.

[11] R.R. Kamalian, A.M. Khachatryan, *On properties of a number of vertices with an interval spectrum among the set of proper edge colorings of some regular graphs*, Proc. of the 6th Ann. Sci. Conf. (December 5–9, 2011) of the RAU, Yerevan, 2012, to appear.

[12] A.M. Khachatryan, R.R. Kamalian, *On the µ-parameters of the graph of the n-dimensional cube*, Book of abstracts of the International Mathematical Conference devoted to the 70 year anniversary of Professor Vladimir Kirichenko, June 13-19 (2012), Mykolaiv, Ukraine, to appear.

[13] N.N. Davtyan, *On the least and the greatest possible numbers of vertices with an interval spectrum on the set of proper edge colorings of a tree*, Math. Problems of Computer Science, Vol. 32, Yerevan, 2009, pp. 107–111.

[14] N.N. Davtyan, R.R. Kamalian, *On the parameter µ₁₂ of a tree*, Proc. of the 4th Ann. Sci. Conf. (November 30 – December 4, 2009) of the RAU, Yerevan, 2010, pp. 149–151.

[15] R.R. Kamalian, *On one-sided interval colorings of bipartite graphs*, the Herald of the RAU, №2, Yerevan, 2010, pp. 3–11.

[16] R.R. Kamalian, *Interval colorings of complete bipartite graphs and trees*, Preprint of the Computing Centre of the Academy of Sciences of Armenia, 1989, 11p.

[17] N.N. Davtyan, A.M. Khachatryan, R.R. Kamalian, *On a subgraph induced at a labeling of a graph by the subset of vertices with an interval spectrum*, Book of abstracts of the 8th International Algebraic Conference in Ukraine. July 5–12(2011), Lugansk, pp. 61–62.