Interaction of a Bose–Einstein condensate and a superconductor via eddy currents

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Abstract. We study center-of-mass oscillations of a dipolar Bose–Einstein condensate in the vicinity of a superconducting surface. We show that the magnetic field of the magnetic dipoles induces eddy currents in the superconductor, which act back on the Bose–Einstein condensate. This leads to a shift of its oscillation frequency and to an anharmonic coupling of the Bose–Einstein condensate with the superconductor. The anharmonicity creates a coupling to one of the collective modes of the condensate that can be resonantly enhanced if the parameters of the condensate are chosen properly. This provides a new physical mechanism to couple a Bose–Einstein condensate and a superconductor, which becomes significant for $^{52}$Cr, $^{168}$Er or $^{164}$Dy condensates in superconducting microtraps.
1. Introduction

In the last few years, hybrid quantum systems have come into the focus of research in the context of quantum information processing [1–4]. They are able to combine the strengths of qubits based on solid-state devices, which can be better controlled, and qubits based on atomic systems, which promise longer coherence times. Such a hybrid can consist of a superconductor coupled to a Bose–Einstein condensate (BEC). To achieve a controlled coupling between a superconductor and BEC, it is necessary to understand the interaction between these two systems.

Coupled quantum systems based on BECs and solid-state devices have been suggested theoretically [5–8]. While the influence of a solid on a BEC is sizeable and has been well studied in the past, normally the influence of a BEC on a solid is very weak due to the low density of the BEC. In recent years, BECs have been condensed in superconducting microtraps [9–13], allowing the close approach of a BEC to a superconducting surface [14–17]. The use of a superconducting microtrap as opposed to a metallic one allows for a significantly longer lifetime of the atomic cloud in the vicinity of the surface [14, 18–20], and therefore also longer coherence time. It also promises a successful coupling of these two macroscopic quantum phenomena [6–8, 21–26].

In the last decade, there has also been intensive research, theoretical as well as experimental, in the field of dipolar BECs. The high magnetic dipole moment of the atoms leads to a long-ranged anisotropic interaction between the atoms in a BEC. This interaction is responsible for a number of interesting phenomena observable in dipolar BECs [27]. For some time $^{52}$Cr [28] was the only experimentally realized dipolar BEC. But most recently the condensation of $^{168}$Er and $^{164}$Dy [29, 30] has also been achieved.

Here we will study a dipolar BEC in the vicinity of a superconductor. We suggest that the mutual interaction of a dipolar BEC with a superconductor can become sizeable due to the large magnetic dipole moment of the atoms. Specifically, center-of-mass oscillations of the dipolar condensate within its trap create eddy currents in the superconductor surface. These eddy currents, in turn, shift the oscillation frequency of the condensate. Their anharmonicity creates a coupling of the center-of-mass oscillations of the condensate with one of its collective modes. We show that this anharmonic coupling can be resonantly enhanced, allowing for a sizeable interaction of the BEC with the solid and a new mechanism for coupling a BEC with a superconductor.
Figure 1. The dipole at \( z = 0 \) interacts with the mirror dipoles distributed with \( n_{1D}(z) \) along the z-axis. The interaction sign with the dipoles in the red region is negative and with the dipoles beyond the red region is positive. The BEC has an optimal length when all dipoles are in the red region.

2. Interaction between the superconducting surface and the Bose–Einstein condensate

Consider a weakly interacting dipolar BEC of \( N \) atoms at temperature \( T = 0 \) confined by an external harmonic trapping potential \( V_T(r) \). The trapping potential could be generated by the magnetic field of a superconducting atom chip or a laser field, for example. The spins are all aligned along \( \hat{e}_z \) by an external magnetic field (see figure 1). The interaction \( U(r, r') \) between two atoms consists of two contributions. One contribution is the isotropic contact interaction

\[
U_s(r, r') = g_s \delta(\mathbf{r} - \mathbf{r}') = \frac{4 \pi \hbar^2}{M} \delta^3(\mathbf{r} - \mathbf{r}'),
\]

where \( g_s = \frac{4 \pi \hbar^2}{M} \delta^3(\mathbf{r} - \mathbf{r}') \) gives the strength of the interaction and is determined by the mass \( M \) and the s-wave scattering length \( a_s \). The other contribution is the long ranged magnetic dipole–dipole interaction

\[
U_{md}(r, r') = -g_D \frac{3(z - z')^2}{|r - r'|^5} - \frac{1}{|r - r'|^3},
\]

where \( g_D = \mu_0 m^2 \), with \( m \) being the magnetic dipole moment. The time evolution of the BEC is given by the time-dependent Gross–Pitaevskii equation (GPE) \([31, 32]\)

\[
i\hbar \frac{\partial}{\partial t} \psi(r; t) = \left[ -\frac{\hbar^2}{2M} \nabla^2 + V_T(r) + (N - 1) \int_{\mathbb{R}^3} d\mathbf{r}' U(r, r') \left| \psi(r'; t) \right|^2 \right] \psi(r; t).\]

For simplicity, we will model the superconductor by a superconducting half space, which is a valid approximation when the BEC is sufficiently close to a plane superconductor surface. The presence of a superconducting half space modifies the magnetic field distribution of a nearby magnetic dipole due to the currents induced in the superconductor. As long as the distance between the magnetic dipole and the surface of the superconductor (in our case \( \sim 10 \mu \text{m} \)) is larger than the magnetic penetration depth of the superconductor (\( \sim 100 \text{nm for Nb} \)) and as long as the oscillation frequency of the dipole motion (in our case \( \sim 1 \text{Hz–1 kHz} \)) is smaller than the gap frequency (\( \sim 100 \text{GHz for Nb} \)), the superconductor acts as a perfect magnetic mirror. This means that at the surface of a superconductor, the normal component of a magnetic induction field has to vanish, \( \mathbf{B} \cdot \hat{n} = 0 \). The field distribution of the magnetic dipole close to a superconductor can thus be found by introducing a mirror dipole in the superconductor and adding up the field of the dipole and the mirror dipole. The mirror dipole emulates the effect of the induced eddy currents. In this way the magnetic interaction between a dipolar BEC and
a superconductor can be described by an additional external potential felt by the atoms in the BEC due to the mirror BEC:

\[ U_{SC}(r) = \int_{\mathbb{R}^3} d'r' n(r') U_{md}(r, r') . \]  

Here, \( n(r') \) is the density distribution of the mirror BEC. Note that this potential depends on the number of atoms in the BEC in contrast to other single-particle potentials such as the Casimir–Polder force, for example.

The BEC ground state is the stationary solution of (2), which will be determined numerically below. Before we do that, let us discuss first a useful approximation for the potential of the mirror. With a sufficiently large number of atoms in the BEC, the kinetic energy can be neglected, which leads to the Thomas–Fermi (TF) approximation [31, 32]. Within this approximation, an analytical expression for the density distribution \( N |\psi(r)|^2 \) of a BEC can be given. In a harmonic potential, the density distribution has an ellipsoidal shape,

\[ n_{TF}(r) = n_0 \left( 1 - \frac{x^2}{\lambda_x^2} - \frac{y^2}{\lambda_y^2} - \frac{z^2}{\lambda_z^2} \right) \quad \text{for } r \in \mathbb{D}_{TF}, \]

with \( \mathbb{D}_{TF} = \left\{ r \in \mathbb{R}^3 \left| \frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2} + \frac{z^2}{\lambda_z^2} \leq 1 \right\} \) and \( n_{TF}(r) = 0 \) for \( r \notin \mathbb{D}_{TF} \). Here \( n_0 \) is the central density and \( \lambda_x, \lambda_y \) and \( \lambda_z \) are the semi-axes of the ellipsoid. In the case where only contact interaction is present, it is easy to see that \( n_{TF}(r) \) is of the form (4). However, this is not so obvious for the case of a dipolar BEC. As has been discussed by Eberlein et al [33], the BEC density distribution remains ellipsoidal in shape also in the presence of the dipole–dipole interaction with only the semi-axes being modified. They have also shown that the BEC may become unstable if the dipole–dipole interaction becomes too large. The dimensionless parameter \( \varepsilon_D = \frac{\gamma_D}{\varepsilon_0} \) provides a measure of the strength of the dipole–dipole interaction compared with the strength of the contact interaction. In the region \(-1/2 < \varepsilon_D < 1\), the ground state is stable, while beyond this region it may or may not be stable depending on the trap geometries. In the following, we will only consider values of \( \varepsilon_D \) in the stable region. In the case where no dipole–dipole interaction is present (\( \varepsilon_D = 0 \)), the semi-axes are given by

\[ \lambda^{(0)}_a = \sqrt{\frac{2\mu^{(0)}}{M \omega_a^2}}, \quad a \in \{ x, y, z \} \]  

with the chemical potential \( \mu^{(0)} = g_s n_0^{(0)} \) and \( n_0^{(0)} \) being the central density of a non-dipolar BEC fixed by the normalization condition \( \int \mathbb{d}r n_{TF} = N \) and given by [31, 32]

\[ n_0^{(0)} = \frac{15}{8\pi} \frac{N}{\lambda_x^{(0)} \lambda_y^{(0)} \lambda_z^{(0)}} . \]  

For a dipolar BEC, these quantities need to be determined numerically; see for example [27, 33, 34]. Using \( n(r) = n_{TF}(r) \); integral (3) cannot be solved analytically. However, if the distance \( x_d \) between the BEC and the superconductor is large enough, i.e. \( \lambda_x, \lambda_y \ll x_d \), the problem can be further simplified. Density distribution (4) can be integrated over \( x \) and \( y \), yielding

\[ n_{1D}(z) = \frac{15}{16} \frac{N}{\lambda_z} \left( 1 - \frac{z^2}{\lambda_z^2} \right)^2 . \]
for \(|z| \leq \lambda_z\) and \(n_{1D}(z) = 0\) elsewhere. \(n_{1D}(z)\) is the so-called column density \([36]\) and represents an effective one-dimensional (1D) density distribution of the mirror BEC. Note that \(n_{1D}(z)\) is a good approximation for the three-dimensional mirror BEC even for \(\lambda_z \gg x_d\). Using \(n_{1D}(z)\), the interaction potential (3) is reduced to a 1D integral along the axis of the column density of the mirror BEC. The potential generated by the mirror BEC at a position \(r = (x, y = 0, z)^T\) can be written as

\[
U_{SC} (x, z) = -\frac{g_D}{4\pi} \int_{-\lambda_z}^{\lambda_z} dz' n_{1D}(z') \left( \frac{3 (z-z')^2}{(x^2 + (z-z')^2)^{5/2}} - \frac{1}{(x^2 + (z-z')^2)^{3/2}} \right). \tag{8}
\]

Here, we have evaluated \(U_{SC}\) only in the plane \(y = 0\), since the column density of the BEC is located in this plane and we are interested in oscillations in the \(x\)-direction (see the next section). The analytical solution of this integral is straightforward. The discussed model is depicted in figure 1.

3. Center-of-mass frequency shift

In this section we will discuss the change of the center-of-mass frequency of the dipolar BEC due to the presence of the superconductor. The external trapping potential \(V_T (r)\) provides a certain oscillation frequency that is modified by the interaction with the mirror BEC. In particular, we are interested in the frequency of the center-of-mass motion perpendicular to the surface. The frequency shift due to the superconductor is related to the curvature in the \(x\)-direction generated by \(U_{SC}\). By calculating the curvature, we have to take into account that the motion of the BEC also leads to motion of the mirror BEC. When the BEC moves toward the superconductor, so does the mirror BEC. This means that we have to take the derivative with respect to the distance to the superconductor rather than the distance to the mirror BEC. Using expression (8) as the interaction potential, we have to take the derivative with respect to \(x/2\) and the curvature change along the \(z\)-axis of the BEC reads

\[
g (z; x_d) = 4 \frac{\partial^2}{\partial x^2} U_{SC} (x, z) \bigg|_{x=2x_d}.
\]

For small-amplitude oscillations, the center-of-mass frequency \(\omega'_x\) perpendicular to the surface is determined by \([36]\)

\[
\omega'^2_x = \omega^2_x + \frac{1}{M} \frac{1}{N} \int_{-\lambda_z}^{\lambda_z} dz n_{1D}(z) g (z; x_d), \tag{9}
\]

where \(n_{1D}(z)\) represents the column density of the BEC and \(\omega_x\) is the frequency of the harmonic trapping potential \(V_T (r)\). If the frequency change is small, we have \(\omega'^2_x - \omega^2_x = (\omega'_x - \omega_x)(\omega'_x + \omega_x) \approx (\omega'_x - \omega_x) 2\omega_x\) and with that the relative frequency shift can be written as

\[
\gamma = \frac{\omega'_x - \omega_x}{\omega_x} = \frac{1}{2} \frac{1}{N} \frac{1}{2M} \int_{-\lambda_z}^{\lambda_z} dz n_{1D}(z) g (z; x_d). \tag{10}
\]

Using the semi-axes of a non-dipolar BEC \(\lambda_a = \lambda_a^{(0)}\), we find \(\gamma \propto \varepsilon_D\). The change of the semi-axes of the ellipsoidal BEC due to the dipole–dipole interaction only appears as a higher-order correction. The integral in expression (10) is best evaluated numerically. Although the atoms in the BEC do not experience an individual frequency shift, we can still consider a single atom
Figure 2. Frequency shift as a function of the number of atoms for different models and interaction strengths $\varepsilon_D = 0.15$ and 0.5. The curves labeled ‘non-dipolar TF’ represent calculations based on a TF density distribution for a BEC without dipole–dipole interaction. The curves labeled ‘dipolar TF’ are based on a TF density distribution that takes the dipole–dipole interaction into account. In both cases the frequency shift was calculated using equation (10). The points labeled ‘num’ are the results of a numerical time evolution of the GPE using the effective potential (13). The parameters for the curves are $\lambda(0)_{x} = 7 \, \mu m$, $x_d = 14 \, \mu m$ and $n_0(0) = 2.5 \times 10^{13} \, \text{cm}^{-3}$. The semi-axis in the z-direction varies from $\lambda(0)_{z} \approx 7 \, \mu m$ at $N = 15\,000$ and $\nu \approx 1$ to $\lambda(0)_{z} \approx 85 \, \mu m$ at $N = 175\,000$ and $\nu \approx 12$.

at the center of the BEC interacting with the mirror BEC in order to obtain an analytical order of magnitude estimate for the frequency shift. In this case, we have $\gamma = \frac{g(z = 0; x_d)}{2\omega^2 M}$. The sign of the dipole–dipole interaction depends on the relative position of the interacting dipoles; it can be attractive or repulsive. Considering a single dipole at $z = 0$, the strongest frequency shift is obtained when the interaction sign is the same with all the dipoles in the mirror BEC (see figure 1). Then, all contributions to the interaction integral (8) add up constructively. Depending on the distance $x_d$, there is an optimal length of the BEC, which reads for the semi-axis $\lambda_{z} = \sqrt{2} x_d$. Using $\lambda_{z}^{(0)} = \sqrt{2} x_d$, we find analytically

$$\gamma_{\text{max}} = \frac{g(z = 0; x_d)}{2\omega^2 M} \approx 0.11\varepsilon_D \left( \frac{\lambda_{z}^{(0)}}{x_d} \right)^4,$$

which represents a rule of thumb for the magnitude of the maximal frequency shift. As an example, for $^{52}\text{Cr}$ with $\varepsilon_D \approx 0.15$ [35] assuming a distance of $x_d = 2\lambda_{x}^{(0)}$ we find $\gamma_{\text{max}} \approx 10^{-3}$. A frequency shift of this magnitude is well within experimental resolution. A precision of $10^{-5}$ was demonstrated in an experiment where the Casimir–Polder force was measured via the frequency shift of a BEC [37].

In figure 2, we show the frequency shift (10) as a function of the number of atoms $N$ in the BEC. For these calculations we assumed a distance of $x_d = 14 \, \mu m$. Experimental findings [14, 24] and theoretical analysis [15, 16] both suggest that using a superconducting microtrap such distances can be achieved. The frequency shift is presented for two different
values of the dipole–dipole interaction parameter using three different models. First, we calculated the frequency shift (10) for a non-dipolar BEC, meaning that we used the semi-axes $\lambda_{\alpha}^{(0)}$ according to (5). So the dipole–dipole interaction is only taken into account in the interaction between the BEC and its mirror. In figure 2, this model is labeled by ‘non-dipolar TF’ in the plot legend. In this case the ratio of the trapping frequencies $v = \omega_x/\omega_z$ is given by the inverse ratio of the semi-axes $v = \lambda_z^{(0)}/\lambda_x^{(0)}$ according to (5). The larger the value of $v$, the more elongated the BEC in the $z$-direction. If the semi-axes $\lambda_x^{(0)}$ and $\lambda_y^{(0)}$ and the central density $n_0^{(0)}$ are kept constant, the trap ratio $v$ becomes proportional to the number of atoms, which is easy to see using relation (6):

$$v = \frac{\lambda_z^{(0)}}{\lambda_x^{(0)}} = \frac{15}{8\pi n_0^{(0)}} \frac{N}{\left(\lambda_x^{(0)}\right)^2 \lambda_y^{(0)}}. \tag{12}$$

Varying the number of atoms in this way is equivalent to changing the length of the BEC. Experimentally this could be achieved by adjusting the trap frequency $\omega_z$ according to the number of atoms $N$, such that relation (12) remains satisfied. Figure 2 shows that $\gamma$ has a maximum. This maximum appears at an optimal length of the BEC as has been discussed above. Increasing the number of atoms above the optimal number leads to a smaller frequency shift, because contributions from the edges of the mirror BEC with opposite sign compensate for contributions from the central region. In the limit $N \to \infty$, the frequency shift approaches 0. In order to detect the eddy current effect experimentally, we suggest using the frequency $\tilde{\omega}_x$ of a long BEC as the reference frequency. If the BEC is long enough, the frequency shift due to the eddy current effect is smaller than experimentally detectable. However, $\tilde{\omega}_x$ would still include possible shifts due to other effects, like, for example, the Casimir–Polder force, which do not depend on $N$. With $\tilde{\omega}_x$ as a reference, the frequency shift $\omega'_x - \tilde{\omega}_x$ can be measured as a function of $N$. Since other surface forces do not have this characteristic dependence on the number of atoms, the curve is a fingerprint for the eddy current effect.

We also calculated the frequency shift using a dipolar BEC. In figure 2, these results are labeled as ‘dipolar TF’. Here we have used the same parameters as in the other calculation, meaning that the trap frequencies remain the same as well as the distance to the surface. However, this time we used the correct dipolar semi-axis $\lambda_z$ instead of $\lambda_z^{(0)}$ in the density distribution. While the trap ratio $v$ is still proportional to $N$, it is no longer given exactly to be the ratio $\lambda_z/\lambda_x$. Also the central density $n_0$ slightly changes while varying $N$. Figure 2 shows that the effect of the modified semi-axes is negligible for $\varepsilon_D = 0.15$ but somewhat changes the result for $\varepsilon_D = 0.5$. However, the main features of the curve are preserved.

The points labeled ‘num’ in the plot legend of figure 2 are the results of numerical calculations. We obtained these results by solving the time-dependent GPE (2) numerically in three spatial dimensions using a time-splitting spectral method [38]. We only considered $U_s(r, r')$ in the GPE for the numerical calculations and will discuss the effect of $U_{\text{md}}(r, r')$ below. As the potential in the GPE, we used the following effective potential:

$$V_{\text{eff}}(r) = \frac{M}{2} \left[ \alpha_0^2 (1 + f(z; x_d))^2 x^2 + \alpha_y^2 y^2 + \alpha_z^2 z^2 \right], \tag{13}$$

where the function $f(z; x_d)$ describes the relative curvature change of the potential in the $x$-direction due to $U_{\text{SC}}$ and is defined by

$$f(z; x_d) = \frac{1}{2M\omega_x^2} g(z; x_d). \tag{14}$$
The effective potential \((13)\) is a good approximation if \(f(z; x_d) \ll 1\). We first determine the ground state of the GPE numerically. To excite the center-of-mass oscillation, we then shift the potential by a distance \(x_s\) in the \(x\)-direction and calculate the time evolution. In every time step \(t_i\) we calculate the \(x\)-coordinate of the center-of-mass

\[
\langle x(t_i) \rangle = \int \mathbf{r} \cdot |\psi(\mathbf{r}; t_i)|^2, 
\]

where \(\psi(\mathbf{r}; t_i)\) is the numerically determined solution of the GPE at that particular time step. After the time evolution is completed, we perform a Fourier analysis of the data to obtain the oscillation frequency. The results for the center-of-mass frequency are presented in figure 2 and labeled 'num'. We can see a very nice agreement with the results obtained for \(\gamma\) using \((10)\). The reason why it does not agree with the results for the dipolar BEC is because we did not take into account the modified semi-axes when we calculated \(V_{\text{eff}}(\mathbf{r})\). Since we neglected the dipole–dipole interaction in the GPE, neglecting it in \(f(z; x_d)\) is consistent. Again the dipole–dipole interaction is only taken into account between the BEC and its mirror. The numerically obtained results are expected to follow the curves labeled 'dipolar TF' in figure 2 if we include the dipole–dipole interaction in the GPE and consider it in \(f(z; x_d)\).

4. Coupling of the center-of-mass motion with the breather mode

Next we want to discuss the aforementioned coupling of a collective mode with the center-of-mass motion due to the eddy current effect. Center-of-mass motions can be excited by a sudden shift \(x_s\) of the trap minimum. In a harmonic potential, the center-of-mass motion does not excite collective shape fluctuations of the BEC and the shape of the BEC will remain constant during motion. However, in the vicinity of the superconductor the effective potential is no longer purely harmonic. The interaction with the mirror BEC generates additional anharmonic terms to the harmonic trapping potential. The excitation of collective modes due to terms like \(x^3\), \(x^4\) etc has been discussed previously [39–42]. The lowest order anharmonic term in \((13)\) is of the form \(x^2z^2\). Transforming into the center-of-mass system, we have \(x(t) \propto \sin(\omega' x t)\). The anharmonic term thus generates a time-dependent change of the trap frequency in the \(z\)-direction of the form \(\Delta \omega'z(t) \propto \sin^2(\omega' x t)\). This excites monopole–quadrupole modes of the BEC [43] with frequency \(2\omega'_x\). We thus expect to see a resonance if one of the monopole–quadrupole modes happens to have the frequency \(2\omega'_x\). The best candidate for this is the so-called breather mode.

In figure 3, we show the breather mode frequency \(\Omega_B\) of a BEC calculated within the TF approximation [34, 44]. The mode frequency is shown as a function of the aspect ratio \(\omega_x/\omega_z\) of the trapping frequencies. In a spherical trap with \(\omega_x = \omega_y = \omega_z\), the mode frequency is \(\Omega_B = \sqrt{2}\omega_z\). In a uniaxial elongated trap with \(\omega_x = \omega_y > \omega_z\), the mode frequency approaches \(2\omega_x\) (red line). In a triaxial elongated trap with \(\omega_x > \omega_y > \omega_z\), the mode frequency crosses \(2\omega_x\) (dashed blue line). In the latter case, we expect to see a strong resonant coupling of the breather mode and the center-of-mass motion at the crossing point.

To study the excitation of the breather mode, we again numerically solve the time-dependent GPE using potential \((13)\). However, this time we analyze the fluctuations of the width of the wave function

\[
\sigma_x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}, \tag{15}
\]

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Figure 3. Breather mode frequency $\Omega_B$ versus $\omega_x/\omega_z$ for different $\varepsilon_D$ and $\omega_y/\omega_x$ values: red solid line $\varepsilon_D = 0$ and $\omega_y/\omega_x = 1$; red dotted line $\varepsilon_D = 0.15$ and $\omega_y/\omega_x = 1$; blue dashed line $\varepsilon_D = 0$ and $\omega_y/\omega_x = 0.99$; blue dot-dashed line $\varepsilon_D = 0.15$ and $\omega_y/\omega_x = 0.99$.

Figure 4. Numerical results for the excitation spectrum of $\Delta_1\sigma_x$ for different trap ratios $\omega_x/\omega_z$. The trap ratios range from $\omega_x/\omega_z = 1$ to 4 in steps of 0.05. The colors represent the amplitudes on a logarithmic scale. The mode frequencies range from $\Omega/\omega_x = 0$ to 2.5. There are two modes visible, which correspond to collective modes of the BEC. The third structure at $\approx 2\omega_x$ corresponds to twice the oscillation frequency of the center of mass, which excites the collective modes. Parameters: $\omega_y/\omega_x = 0.99; \varepsilon_D = 0.15; \text{length of time evolution } t = 500 T_x$, with $T_x = \frac{2\pi}{\omega_x}$.

After calculating the time evolution, a Fourier analysis of $\sigma_x$ provides the frequency spectrum. The peaks in the frequency spectrum correspond to collective modes of the BEC. We have identified the peaks in the spectrum by comparing them with the results from TF calculations.

In figure 4, we present the frequency spectra of $\Delta_1\sigma_x = \sigma_x(t) - \sigma_x(0)$ in a slightly triaxial trap for different $\omega_x/\omega_z$ values. The yellow line starting at $\approx 2.3 \omega_x$ represents the breather mode frequency. For increasing trap ratio $\omega_x/\omega_z$, it approaches another yellow line at $\approx 2\omega_x$. This second line represents the double oscillation frequency of the center-of-mass motion. A resonance can be observed where the two lines meet (red). There is also a third yellow line.
visible in the spectrum. It belongs to another collective mode of the BEC, which is also excited due to the anharmonicity of the trap. However, this excitation is very weak compared to the resonance that occurs for the breather mode.

In figure 5, an enlarged region of the excitation spectrum of $\Delta \sigma_x$ near the resonance is shown. While the spectra are plotted on a linear scale, the color map on the bottom is plotted on a logarithmic scale. There are two peaks visible in the spectra shown: one corresponding to the breather mode frequency and the other representing twice the center-of-mass frequency ($\approx 2.0012 \omega_x$). One can clearly see the resonance peak at the crossing point just as we have already conjectured.

As shown in figure 3, the breather mode for a BEC with $\epsilon_D > 0$ has a different frequency than a BEC with $\epsilon_D = 0$. Since we have neglected $U_{md}(r, r')$ in the GPE, this effect is not included in our numerical results. Figure 3 shows that the crossing point of the breather mode and the double oscillation frequency is shifted towards smaller values of $\omega_x/\omega_z$ (dot-dashed blue line). Therefore, we also expect the resonance peak to appear at a smaller value of $\omega_x/\omega_z$.

5. Discussion and conclusion

We have studied a new physical mechanism to couple a BEC and a superconductor. The mechanism rests on the interaction of the magnetic dipole moments of a dipolar BEC with the superconductor. Center-of-mass oscillations of such a dipolar BEC create eddy currents in the superconductor, which act back on the BEC. We have demonstrated that this eddy current effect leads to a frequency change and can resonantly excite a collective mode of the BEC.

The frequency change of the center-of-mass motion of the BEC has a characteristic dependence on the number of atoms in the BEC, which can serve as an experimental fingerprint to distinguish this effect from other effects that may change the oscillation frequency. We have
also shown that the resonant excitation of the collective mode of the BEC becomes possible if the parameters of the external trapping potential are tuned properly. Both effects become significant in $^{52}\text{Cr}$, $^{168}\text{Er}$ or $^{164}\text{Dy}$ BECs.

In principle, there should be similar effects using a thermal cloud instead of a BEC. However, it will be much harder to observe those effects, because the density in a thermal cloud is much smaller. Also, such a system would not represent a coupled quantum system.

The eddy current effect we described here requires a trap to be formed a few ten micrometers away from a superconductor surface. For superconducting microtraps, it has been pointed out that the Meissner effect of the superconductor modifies the magnetic field distribution in its vicinity in such a way that a magnetic trap cannot be formed anymore if the trap position is brought too close to the superconducting surface [11]. In [15, 16], this effect has been studied theoretically for a finite conductor with rectangular cross-section and the limits for such traps have been discussed. It was shown that a trap distance of less than 10 $\mu$m can be achieved for a niobium conductor. A recent experiment [24] has demonstrated a distance of 14 $\mu$m from a superconducting strip.

In our calculations we assumed a superconducting half space. For this approximation to be appropriate, a finite superconducting strip with a rectangular cross-section should fulfill the following requirements:

- **Thickness**: As the eddy currents are flowing at the surface of the superconductor within the magnetic penetration depth $\lambda$, the thickness of the superconductor should be $2\lambda$ or more, which for niobium is $\sim$200 nm. These are typical film thicknesses used in thin film technology.

- **Width**: The width of the strip should be larger than both the width of the BEC and its distance from the superconductor. In the recent experiment of Bernon et al [24], it was shown that this requirement can be met.

- **Length**: The length should be larger than both the length of the BEC and its distance from the superconductor. Such an axial confinement could be made by a Z-shaped trap, for example, as in [24].

If some of these requirements cannot be met in an experiment, it is clear that the eddy current effect we describe will be reduced by a geometrical factor. This factor depends on the solid angle under which the BEC sees the superconductor surface.

In this work, we have only considered small-amplitude oscillations. For large amplitudes, we expect to see corrections to the results presented here. On the other hand, larger amplitudes should increase the discussed effects since the potential becomes more anharmonic. For example, the potential can no longer be assumed to be symmetric in the $x$-direction.

So far experiments on superconducting microtraps for BECs have only been done with $^{87}\text{Rb}$ atoms, which have a comparatively small magnetic dipole moment. Our calculations show that it is beneficial to study strongly dipolar BECs in superconducting microtraps. In such a system, a mutual coupling of a BEC and a superconductor via eddy currents becomes possible.

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