The influence of difference in the surface properties on the axisymmetric oscillations of an oblate drop

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Abstract. The forced axisymmetric oscillations of an oblate fluid drop are investigated. A drop is cylindrical in equilibrium, surrounded by another liquid and bounded axially by two parallel solid plates. These plates have different surfaces. Hocking’s boundary conditions hold on the contact line: the velocity of the contact line motion is proportional to the deviation of the contact angle from its equilibrium value. The Hocking’s parameter (so-called wetting parameter) is the proportionality coefficient in this case and it is different for each plate. The vibration force is parallel to the symmetry axis of the drop. The solution of the boundary value problem is found using Fourier series of Laplace operator eigen functions.

1. Introduction

A motion of triple contact line is one of the most important and well-known in the multiphase hydrodynamics [1-5]. For periodic or quasiperiodic motion the most frequently used condition for contact line velocity is the one applied by Hocking [6] for investigation of standing waves between two vertical walls. This condition assumes a linear relationship between the velocity of the contact line motion and the contact angle

\[ \frac{\partial \zeta}{\partial t} = \lambda \tilde{k} \cdot \nabla \zeta, \]

where \( \zeta \) is the deviation of the interface from the equilibrium position, \( \tilde{k} \) is the external normal to the solid surface, \( \lambda \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter) having the dimension of the velocity. There are two important limit of the boundary condition (1): (a) \( \zeta = 0 \) – the fixed contact line (pinned-end edge condition), (b) \( \tilde{k} \cdot \nabla \zeta = 0 \) – the constant contact angle. The Hocking parameter was constant in all papers in which this condition was used [7-19]. Note, that the surfaces of plates are the same in all the above papers.

In the present article, we consider the axisymmetric oscillations of sandwiched fluid drop which surrounded by other ideal liquid. We assume that Hocking parameter is unique for each plate. The free and forced axisymmetric oscillations of cylindrical drop for case of homogeneous identical plates were investigated in [9,15,16] and for case of inhomogeneous identical plates [19], eigen frequency were studied for case of different plates in [20].

2. Problem formulation

Following [9,15], consider the oscillations of an incompressible fluid drop of density \( \rho^i \) and kinematic viscosity \( \nu^i \) surrounded by other fluid of density \( \rho^e \) and kinematic viscosity \( \nu^e \). The
system is bounded by two parallel solid surfaces which are separated by a distance \( h^* \) (see figure 1). The vessel is closed at infinity. In equilibrium the drop has circle cylindrical form with radius \( R_0^* \). The equilibrium contact angle \( \theta_0 \) is equal to \( 0.5\pi \). The system is affected by a vibration field with amplitude \( A^* \) and frequency \( \omega^* \). The vibration force is directed perpendicularly the symmetry axis of the drop. We assume that the frequency is high to neglect the dissipative effects caused by acoustic radiation and viscous dissipation, i.e. \( \omega R_0^* \ll c \) and \( \delta = (v^*/\omega) \ll R_0^* \), where \( c \) is the sound velocity, \( \delta \) is the viscous boundary-layer thickness. The amplitude of external force is considered small in the sense that \( A^* \ll R_0^* \).

![Figure 1. Problem geometry.](image)

We work in cylindrical coordinates \( r^* \), \( \alpha \) and \( z^* \) because of the problem symmetry. Let the lateral surface of the drop be described by \( r^* = r_0^* + \zeta^*(\alpha, z^*, t^*) \), where \( \zeta^* \) is the surface deviation from equilibrium. The liquid motion is irrotational in the accepted approximations, which makes it convenient to introduce the velocity potential. Thus, the dynamics of the liquid is described by the Bernoulli and Laplace equations. We use the following quantities as the measurement units: \( R_0^* \) for length, \( h^* \) for height, \( \rho_0^* \sigma_0^* \) for time, \( \rho_0^* \sigma_0^* \) for velocity, \( \rho_0^* \sigma_0^* \) for velocity potential, \( \rho_0^* \sigma_0^* \) for pressure, \( \sigma_0^* \) for surface deviation. The amplitude of oscillations is considered small \( \varepsilon = A^*/R_0^* \ll 1 \), which allows us to linearize the governing equations and simplify the boundary conditions. Thus, the dimensionless linear boundary value problem is determined by

\[
\Delta \varphi_j = 0, \quad p_j = -\rho_j \left( \frac{\partial \varphi_j}{\partial t} - \omega^2 z e^{j\omega t} \right), \quad j = i, e, \quad (2)
\]

\[
r = 1: \left[ \frac{\partial \varphi_j}{\partial r} \right] = 0, \quad [p] = \zeta + b^2 \frac{\partial^2 \zeta}{\partial \alpha^2} + b^2 \frac{\partial^2 \zeta}{\partial z^2}, \quad \frac{\partial \zeta}{\partial t} = \frac{\partial \varphi}{\partial r}, \quad (3)
\]

\[
z = \pm 1/2: \frac{\partial \varphi_j}{\partial z} = 0, \quad (4)
\]

\[
r = 1, \quad z = \pm 1/2: \frac{\partial \zeta}{\partial t} = \pm \lambda_{ub} \frac{\partial \zeta}{\partial z}, \quad (5)
\]

where \( p \) is pressure, \( \varphi \) is velocity potential, \( \lambda_s \) and \( \lambda_b \) are the Hocking parameter of the “top” \( (z = 0.5) \) and “bottom” \( (z = -0.5) \) substrate, respectively. In what follows, the quantities with subscript \( i \) refer to the drop, and those with subscript \( e \) to the surrounding liquid. Boundary value problem (2)–(5) contains five dimensionless parameters: aspect ratio \( b = R_0^*/h^* \), the Hocking
constant (wetting parameter) – \( \lambda = \Lambda \lambda \left( \rho_e \rho_i \right)^{-1/2} \), the frequency – \( \omega = \omega \left( \rho_e \right)^{-1/2} \), the density of the external liquid – \( \rho_e = \rho_e \left( \rho_e + \rho_i \right)^{-1} \), the density of the drop liquid – \( \rho_i = \rho_i \left( \rho_e + \rho_i \right)^{-1} \), \( \rho_e + \rho_i = 1 \).

3. Forced oscillations

Consider forced linear oscillations of the drop. The external force excites only axisymmetric oscillations of the drop in contrast to the case of inhomogeneous plates [19]. However, the entire spectrum of modes is excited (both even and odd), in contrast to the case of identical homogeneous plates [15,16] in which only odd modes existed. Representing the fields of the velocity potentials and surface deviation as

\[
\varphi_i(r,z,t) = \text{Re} \left( i \omega \sum_{k=0}^{\infty} (a_{ik} R_{ik}^i(r) \sin((2k+1)\pi z) + a_{ik}^* R_{ik}^i(r) \cos(2\pi kz)) e^{i \omega t} \right),
\]

\[
\varphi_e(r,z,t) = \text{Re} \left( i \omega \sum_{k=0}^{\infty} (b_{ik} R_{ik}^e(r) \sin((2k+1)\pi z) + b_{ik}^* R_{ik}^e(r) \cos(2\pi kz)) e^{i \omega t} \right),
\]

\[
\zeta(z,t) = \text{Re} \left( \sum_{k=0}^{\infty} \left( c_{ik} \sin((2k+1)\pi z) + c_{ik}^* \cos(2\pi kz) \right) + d_{i} \sin \left( \frac{z}{b} \right) + d_{j} \cos \left( \frac{z}{b} \right) e^{i \omega t} \right),
\]

where \( R_{ik}^i(r) = I_0 \left( (2k+1)\pi br \right), R_{2k}^e(r) = \text{const}, R_{ik}^e(r) = \text{I}_0 \left( 2k\pi br \right), R_{ik}^e(r) = K_0 \left( (2k+1)\pi br \right), R_{2k}^e(r) = \text{const}, R_{ik}^e(r) = K_0 \left( 2k\pi br \right) \), \( I_0 \) and \( K_0 \) are the modified Bessel functions of the 0-th order. Substituting solutions (6)—(8) into (2)—(5), we obtain the expressions for the unknown amplitudes \( a_{ik}, b_{ik}, c_{ik}, d_i, \) and \( d_j \). These expressions are equivalent to the similar solutions obtained in [15] for \( \lambda_u = \lambda_b = \lambda \).

\[\begin{array}{cccc}
(a) & (b) & (c) & (d) \\
(e) & (f) & (g) & (h)
\end{array}\]

**Figure 2.** The accuracy of representation of the drop surface (a-d) and the contact angle (d-f) vs the frequency \( \omega \) \( (b = 1, \rho_i = 0.7) \), \( \lambda_u = 1 \), \( \lambda_b = 0.1 \) – solid line, \( \lambda_b = 1 \) – dashed, \( \lambda_b = 10 \) – dotted.

For convenience, as a maximum deviation of the drop surface from the equilibrium position, we prescribe on the “upper” plate \( z = 0.5 - \zeta_u = \max \left( \zeta \left( 0.5, 0 \right) \right) \), on the “bottom” plate \( z = -0.5 - \zeta_u = \max \left( \zeta \left( -0.5, 0 \right) \right) \), in the centre of the layer \( z = 0 - \zeta_u = \max \left( \zeta \left( 0, 0 \right) \right) \) and a “quarter” position
\[ z = 0.25 - \zeta_q = \max(\zeta_q(0.25,0)) \]; the values of the internal contact angle \( \gamma \) on the “upper” plate are \( \gamma_u \), at the “bottom” plate – \( \gamma_b \); the deviation from the equilibrium contact angle on the “upper” plate is \( \delta_z = \max(\gamma_u - 0.5\pi) \) and on the “bottom” plate – \( \delta_b = \max(\gamma_b - 0.5\pi) \).

**Figure 2.** Shows the oscillation amplitude of the drop surface and the deviation of the contact angle as a function of the frequency of the uniform electric field for several values of the Hocking parameters \( \lambda_u \) and \( \lambda_b \).

**Figure 3.** Evolution of the drop surface shape (a,e,i), the shape of the contact line (b,c,f,g,j,k) and the contact angles (d,h,l). \( T = 2\pi\omega^{-1} \) is the oscillation period (\( b = 1 \), \( \rho_i = 0.7 \), \( \varepsilon = 0.1 \), \( \lambda_u = 1 \)), (a-d) \( \omega = 10 \), \( \lambda_u = 0.1 \), (e-h) \( \omega = 25 \), \( \lambda_b = 1 \), (i-l) \( \omega = 10 \), \( \lambda_b = 10 \), (a-c, e-g,i-k) \( t = 0 \) – solid line, \( t = 0.125T \) – dashed, \( t = 0.25T \) – dotted, \( t = 0.375T \) – dash-dotted.

Figure 2 shows the oscillation amplitude of the drop surface and the deviation of the contact angle as a function of the frequency of the uniform electric field for several values of the Hocking parameters \( \lambda_u \) and \( \lambda_b \). The amplitudes of the surface oscillations and the contact angle reach maximum values in a linear resonance. It is also seen from the graphs that the values of the resonant frequencies decrease with an increase of \( \lambda_u \) or \( \lambda_b \). Despite weak dissipation at small values of the parameter \( \lambda_b \), the amplitude of contact line oscillations at \( z = 0.5 \) is greater than it is at \( z = -0.5 \) (figure 2a,d). The contact angle varies in a wide range (figure 2e, f). It is important to note that if at least one of the parameters \( \lambda_u \) or \( \lambda_b \) is finite, the amplitude of the surface oscillations is always finite. Consequently, dissipation is determined by the largest damping parameter. Note that the curves have a
resonant shape in the limiting case $\lambda \rightarrow 0$ (figure 2b). The amplitude of the contact line oscillations limits to infinity for $\lambda \rightarrow \infty$. Results for similar plates can be seen in [15,16] more details.

Also, for clarity, figure 2 shows the case of equality of the Hocking parameters $\lambda_u = \lambda_b = 1$. In this situation, the external force excites only odd spatial modes, so that there is no deviation of the drop surface in the center of the layer (figure 2c). Recall that for finite values of the parameter $\lambda$, a dissipation is maximum during the movement of the contact line, and therefore, the plots do not display pronounced resonance peaks (figure 2b,c). The motion of a drop does not depend on the parameter $\lambda$ at certain frequencies $\omega$: the contact line is like a fixed contact line for any values of $\lambda$ (figure 2b,c). The values of such “anti-resonant” frequencies are determined from the solution (6)—(8). Figure 3 shows the profile of the lateral surface (figure 3a,e,i) and the contact line (figure 3b,c,f,g,j,k) and changes in the internal contact angle (figure 3d,h,l) for several values $\omega$ and $\lambda$ at different moments of the oscillation period. The shape of the drop surface depends on the frequency of the vibrational field. For example, in figure 3, most of the vibration energy at a given frequency $\omega = 10$ (see figure 2a) is transmitted to the lowest spatial mode at $\lambda_u = 1$, $\lambda_b = 0.1$ (figure 2a) and resonance frequency $\omega = 7$ is at $\lambda_u = 1$, $\lambda_b = 10$. Also case of “anti-resonant” frequency $\omega = 25$ ($\lambda_u = 1$, $\lambda_b = 1$) is shown on figure 3e-h.

4. Conclusion
The behaviour of cylindrical drop between solid plates has been considered taking into account the dynamics of the contact angle under axisymmetric vibrations. The solid plates have different Hocking’s parameters. The used boundary condition on the contact line leads to damping of oscillations. In addition, the oscillations of different parts of the fluid are shifted in phase relative to each other, which leads to the appearance of travelling surface waves.

In the study of forced vibrations, resonance phenomena were detected. It is shown that dissipation on the contact line leads to a limitation of the maximum amplitude of oscillations in the resonance, as well as to a shift of the resonance frequency. At finite values of the parameter $\lambda$, due to dissipation as the contact line moves, the amplitude of oscillations remains limited.

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