Coupled-channels Faddeev AGS calculation of $K^{-}ppn$ and $K^{-}ppp$

quasi-bound states

S. Marri* and S. Z. Kalantari

Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

(Dated: November 29, 2016)

Abstract

Using separable $K\bar{N} - \pi\Sigma$ potentials in the Faddeev equations, we calculated the binding energies and widths of the $K^{-}pp$, $K^{-}ppn$ and $K^{-}ppp$ quasi-bound states on the basis of three- and four-body Alt-Grassberger-Sandhas equations in the momentum representation. One- and two-pole version of $K\bar{N} - \pi\Sigma$ interaction are considered and the dependence of the resulting few-body energy on the two-body $K\bar{N} - \pi\Sigma$ potential was investigated. The $s$-wave $[3+1]$ and $[2+2]$ sub-amplitudes are obtained by using the Hilbert-Schmidt expansion procedure for the integral kernels. As a result, we found a four-body resonance of the $K^{-}ppn$ and $K^{-}ppp$ quasi-bound states with a binding energy in the range $B_{K^{-}ppn} \sim 55 - 70$ and $B_{K^{-}ppp} \sim 90 - 100$ MeV, respectively. The calculations yielded full width of $\Gamma_{K^{-}ppn} \sim 16 - 20$ and $\Gamma_{K^{-}ppp} \sim 7 - 20$ MeV.

PACS numbers:

*s.marri@ph.iut.ac.ir*
I. INTRODUCTION

The dynamics of antikaon interacting with nucleons and nuclei is one of the current challenging problems in strangeness nuclear physics. The $\bar{K}N$ interaction at low energy is strongly attractive and generates the $\Lambda(1405)$ resonance (abbreviated as $\Lambda^*$) as a quasi-bound state embedded in the $\pi\Sigma$ continuum below the $\bar{K}N$ threshold. Thus, one expects unusual, interesting phenomena to be observed when the antikaon is injected or stopped in nuclei. Theoretical interests in $\bar{K}$-nuclear bound states were triggered by the works of Akaishi and Yamazaki (A-Y) looking for $\bar{K}$ bound states in several few-body systems [1–4], which were predicted to be not only deeply bound but also unusually shrunk. In addition to the lightest possible antikaon-nucleus system, $K^-pp$, a series of proton-rich $K^-$ bound systems were predicted [2], which can be called kaonic nuclear clusters (“KNC”). The proton and neutron distributions in KNC’s were studied extensively using antisymmetrized molecular dynamics (AMD) method by Dote et al. [3, 4]. Subsequently, theoretical studies of KNC’s, especially of $K^-pp$, were developed by using different models and methods to solve the three-body system [5–11]. These calculations have shown essentially that the $K^-pp$ system is bound below the break-up threshold in agreement with A-Y’s original prediction [2], though some differences between different predictions remain. Very recently, Maeda et al. [12] has carried out Faddeev and Faddeev-Yakubowsky calculations for the three and four body systems, $\bar{K}NN$, $\bar{K}NNN$, $\bar{K}\bar{K}N$ and $\bar{K}\bar{K}NN$, with varied elementary potentials, overviewing their binding energies, densities and shapes.

It was found and emphasized in refs. [13, 14] that the essential ingredient in KNC’s is the $\Lambda^* = K^-p$. The strong binding force in KNC’s originates not only from the direct $\bar{K}N$ interaction, but also from the exchange integral arising from the ”Platz-Wechsel” (place-exchange) effect a la Heitler-London type mechanism [15] for hydrogen molecular bonding. This multi-body attraction was named “super-strong nuclear force” [13].

Parallel to the theoretical activities, experimental searches for KNC’s have been carried out, but so far, most of the trials are not conclusive. The FINUDA group at DAPHNE first reported a $K^-pp$-like peak in the invariant-mass spectrum of $\Lambda - p$ that were emitted in $K^-$ capture by light targets [16], but this result was poor in statistics, and moreover, its interpretation of the observed spectrum in terms of a single Lorentzian peak without background component to yield a binding energy of $B_K = 115 \pm 7$ MeV and a width of $\Gamma = 67 \pm 14$ MeV was questioned [17].

In 2007 a theoretical study of the structure of $K^-pp$ and its formation in the $d(\pi^+, K^+)$ reaction
and in the \( p + p \rightarrow K^+ + K^- pp \) reaction was performed \cite{14}. The former method followed a well-known hypernuclear formation, but the formation probability of \( K^- pp \) was calculated to be about 1 % as much as the quasi-free background component. With such a pessimistic prediction and the non-availability of a suitable beam line and detection system no experimental trial had been challenged until a recent J-PARC E27 experiment \cite{18}. Concerning the other method using the \( p + p \) reaction, a very exotic formation mechanism was theoretically revealed in contrast to the conventional pessimistic expectation. In such a high-energy collision a large momentum around 1.6 GeV/c is transferred to the formed system, and thus, the sticking of \( K^- \) to the involved nucleus should be enormously small. On the contrary to the pessimistic view, the calculated cross section for \( K^- pp \) was found to be as large as the free production of \( \Lambda^* = \Lambda(1405) \). The reason for this surprising paradoxical consequence is that the formed state \( K^- pp \) is a condensed object in which \( \Lambda^* \) and \( p \) are bound with high internal momenta, which can be populated by high-energy short-range collisions of \( p + p \). The produced \( \Lambda^* \) is in the short proximity of the participating proton in the collision. A small working group (M. Maggiora, K. Suzuki, P. Kienle and T. Yamazaki) was formed to examine this surprising hypothesis using large amounts of existing exclusive data of \( p + p \rightarrow p + \Lambda + K^+ \) reactions, taken by the DISTO collaboration at Saturne of Saclay. In the conventional view, where the \( K^- pp \) is not dense, no such reaction will take place. Only when the \( K^- pp \) were unusually dense, a peak comparable to the free emission of \( \Lambda^* \) would be observed. In 2010 the DISTO group published the discovery of a gigantic peak in \cite{19} using the data at the incident energy of \( T_p = 2.85 \text{ GeV} \). Its mass was found to be \( M_X = 2267 \pm 2(\text{stat}) \pm 5(\text{syst}) \text{ MeV}/c^2 \), and a binding energy of \( B_X = 105 \text{ MeV} \) and a width of \( \Gamma_X = 118 \pm 8(\text{stat}) \pm 10(\text{syst}) \text{ MeV} \) were deduced. Recently, another report on the same reaction, but with an incident energy of 2.5 GeV was reported by the same group \cite{20}. The observed absence of the peak X at the \( T_p = 2.5 \text{ GeV} \) was interpreted as being due to the incident proton energy too low to produce the \( \Lambda^* \) doorway. More recently, the HADES group at GSI reported absence of X at the incident energy of 3.5 GeV \cite{21}. This was interpreted to be due to the too high incident energy, which made the collision dynamics to sit outside the favorite Dalitz zone of double resonance that was realized at \( T_p \).

We believe it to be vitally important to extend the theoretical and experimental search to four-body KNC’s. In the present study, we solve the Alt-Grassberger-Sandhas (AGS) equations for \( \bar{K}NN \) and \( \bar{K}NNN \) with an early phenomenological model of \( \bar{K}N \) interaction by applying our approach based on the coupled-channel AGS equations developed in \cite{6,22}.
This paper is composed as follows. In sect. II, we first give a brief recapitulation of the three-body equations and then present the formula corresponding to the four-body equations. The inputs for the AGS system of equations are given in sect. III. A discussion of the results can be found in section IV. Finally, we summarize our conclusions in sect. V.

II. FORMULATION OF THE PROBLEM

A. Three-body AGS equations

In the present work, we employ the three- and four-body Faddeev equations in momentum space, using the Alt-Grassberger-Sandhas form \cite{23}. Three-body Faddeev equations \cite{6} in the AGS form are given by

\[ K_{ij,I}^{\alpha \beta} = \delta_{\alpha \beta} M_{ij,I}^{\alpha \beta} + \sum_{k,I}^{} M_{i}^{\alpha} \tau_{k}^{\alpha \gamma} K_{ij,k}^{\gamma \beta}, \tag{1} \]

where the operator \( K_{ij,I}^{\alpha \beta} \) is the transition amplitude between channels \( \alpha \) and \( \beta \), the operator \( M_{ij,I}^{\alpha \beta} \) is the corresponding Born term and \( \tau_{i}^{\alpha \beta} \) is the two-body t-matrix embedded in three-body system. Here, the Faddeev partition indices \( i, j = 1, 2, 3 \) denote simultaneously a spectator particle and an interacting pair while the particle indices \( \alpha, \beta = 1, 2, 3 \) denote the three-body channels. We use these Faddeev equations to solve the \( \bar{K}NN - \pi \Sigma N \) three-body system. Depending on the two nucleon spin and isospin, we should treat the \( K^- pp \) or \( K^- d \) systems. The calculation scheme, which formally allows an exact solution, is based on the separable approximation of the appropriate integral kernels. The separable approximation of the kernel of the Faddeev integral equation permits one to represent the dynamical equations in terms of particle exchange diagrams \cite{22}. The key ingredient of the quasi-particle method \cite{24, 25} is the separable representation of the off-shell scattering amplitudes for the two- and three-body systems. We have to introduce also the separable representation for the three-body amplitudes and driving terms, which will be necessary to find the pole position of \( \bar{K}NN \) system. For this purpose we apply the Hilbert-Schmidt expansion (HSE) method

\[ M_{ij,I}^{\alpha}(p, p', \epsilon) = - \sum_{n=1}^{N_{r}} \lambda_{n}(\epsilon) u_{n;i,I}^{\alpha}(p, \epsilon) u_{n;j,I}^{\alpha}(p', \epsilon), \tag{2} \]

where the form factors \( u_{n;i,I}^{\alpha}(p, \epsilon) \) are taken as the eigenfunctions of the kernel of eq. (1), with the eigenvalues \( \lambda_{n}(\epsilon) \).
The separable form of the Faddeev transition amplitudes is given by

\[ K^{\alpha\beta}_{ij,I_j}(p, p', \epsilon) = \sum_{n=1}^{N} u^{\alpha}_{n;i,I_i}(p, \epsilon) \zeta_n(\epsilon) u^{\beta}_{n;j,I_j}(p', \epsilon), \]  

(3)

where the functions \( \zeta_n(\epsilon) \) obey the equation

\[ \zeta_n(\epsilon) = \lambda_n(\epsilon)/(\lambda_n(\epsilon) - 1). \]  

(4)

Then using the separable approximation for the Faddeev amplitudes and driving terms in (1), the Faddeev equations take the form

\[ u^{\alpha}_{n;i,I_i} = \frac{1}{\lambda_n} \sum_{k=1}^{3} \sum_{\gamma=1}^{3} \sum_{I_k} M^{\alpha}_{ik,I_k} T^{\gamma}_{k,I_k} u^{\gamma}_{n;k,I_k}. \]  

(5)

The AGS equation of (5) is a Fredholm type integral equation. To find the resonance energy of the three-body system using these equations, we should transform the integral equations into algebraic ones and then search for a complex energy at which the first eigenvalue of the kernel matrix becomes equal to one. Before we proceed to solve the AGS equations for both \((\bar{K}NN)_{s=0,1}\) systems, the operators involving two identical baryons should be antisymmetric. The baryon spins do not enter explicitly in the three-body equations because the total spin \(s\) remains unchanged in the process. In the \(K^{-}d\) case, the spin component is symmetric, then all operators in isospin base should be antisymmetric. In the case of \(K^{-}pp\) the spin component is antisymmetric. Thus, all operators in isospin base should be symmetric.

B. The four-body \(\bar{K}NNN\) equations

In four-body \(\bar{K}NNN\) system, there is three identical nucleons, therefore, the four-body equations for \(\bar{K}NNN\) system are reduced to three sets of integral equations. As it is shown in fig. 1, the whole dynamics is described in terms of the Faddeev amplitudes, which connect the three channels characterized by the following partitions

\[ \alpha = \{1, 2, 3\} = \{\bar{K}(NNN), N(\bar{K}NN), (\bar{K}N)(NN)\}. \]  

(6)

We need all possible amplitudes connecting the initial state, consisting of the \(3N\) bound state \((^3\text{He})\) and a free kaon, with all three channels listed in (6) via particle or two-body quasi-particle
FIG. 1. The four different rearrangement channels of the $\bar{K}NNN$ four-body system including the K- and H-type diagrams are represented. Antisymmetrization of three $N$’s is to be made within each channel.

exchange. The four-body Faddeev amplitudes obey a set of three coupled integral equations, whose structure is represented by the following matrix equation

$$
\begin{pmatrix}
A_{11} \\
A_{21} \\
A_{31}
\end{pmatrix}
= 
\begin{pmatrix}
0 & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & 0
\end{pmatrix}
\begin{pmatrix}
\zeta_1 & 0 & 0 \\
0 & \zeta_2 & 0 \\
0 & 0 & \zeta_3
\end{pmatrix}
\begin{pmatrix}
A_{11} \\
A_{21} \\
A_{31}
\end{pmatrix}.
$$

(7)

Here, we take into account only the dominant s-wave part of the interaction in the two-body subsystems and thus in the three- and four-particle states. Therefore, in all expressions, we drop the index $L = 0$. The explicit analytical form of the transition amplitudes between the channel states, taking into account the spin and isospin degrees of freedom, are given by

$$
A_{II',nn'}^{\alpha\beta,ss'} = R_{II',nn'}^{\alpha\beta,ss'} + \sum_{\gamma=1}^{3} \sum_{n''} R_{II',nn''}^{\alpha\gamma,ss''} \zeta_{n''}^{\gamma} A_{II',nn''}^{\gamma\beta,s''s'},
$$

(8)

where the operators $A_{II',nn'}^{\alpha\beta,ss'}$ are the four-body Faddeev amplitudes, $\zeta_{n''}^{\gamma}$-functions are represented by eq. (4) and the operators $R_{sI',sI}$ are driving terms, which describe the effective particle-exchange potential realized by the exchanged particle between the quasi-particles in the channels $\alpha$ and $\beta$, which can be written as

$$
R_{II',nn'}^{\alpha\beta,ss'}(p, p', E) = \frac{\Omega_{II'}^{ss'}}{2} \int_{-1}^{+1} d(\hat{p} \cdot \hat{p}') u_{n',I}(q', \epsilon_{\beta} - \frac{p'^2}{2M_{\beta}}) \times \tau(z) u_{n,I}(q, \epsilon_{\alpha} - \frac{p^2}{2M_{\alpha}}).
$$

(9)

Here, the symbols $\Omega_{II'}^{ss'}$ are the spin and isospin Clebsch-Gordan coefficients, the functions $u_{n,I}^{\alpha,s}$ are the form factors that generated by the separable representation of the sub-amplitudes appearing
in the channels (6) and $z$ is given as $z = E - \frac{\vec{p}^2}{2M_\beta} - \frac{\vec{p}'^2}{2M_\alpha} - \frac{\vec{p} \cdot \vec{p}'}{m}$. The energy $\epsilon_\alpha$ is the subsystem energy in channel $\alpha$. The momenta $\vec{q}(\vec{p}, \vec{p}')$ and $\vec{q}'(\vec{p}, \vec{p}')$ are given in terms of $\vec{p}$ and $\vec{p}'$. We use the relations

$$
\begin{align*}
\vec{q} &= \vec{p}' + \frac{M_\alpha}{m}\vec{p}, \\
\vec{q}' &= \vec{p} + \frac{M_\alpha'}{m}\vec{p}'.
\end{align*}
$$

where $m$ is exchanged particle or quasi-particle mass and the reduced masses $M_\alpha$ and $M_\alpha'$ in the channel $\alpha$ of the [3+1] subsystem are defined by

$$
\begin{align*}
M_\alpha &= m_i^\alpha (m_j^\alpha + m_k^\alpha + m_l^\alpha)/(m_i^\alpha + m_j^\alpha + m_k^\alpha + m_l^\alpha), \\
M_\alpha' &= m_j^\alpha (m_k^\alpha + m_i^\alpha)/(m_j^\alpha + m_k^\alpha + m_i^\alpha),
\end{align*}
$$

and in the case of the [2+2] subsystem are given by

$$
\begin{align*}
M_\alpha &= (m_i^\alpha + m_j^\alpha)(m_k^\alpha + m_l^\alpha)/(m_i^\alpha + m_j^\alpha + m_k^\alpha + m_l^\alpha), \\
M_\alpha' &= m_i^\alpha m_j^\alpha/(m_i^\alpha + m_j^\alpha).
\end{align*}
$$

The meaning of the driving terms $R_{II', ss'}^{\alpha\beta, ss'}$, is explained schematically by the diagrammatic representation in fig. 2. By cyclic permutation of the nucleons, one can obtain various relations between the different driving terms $R_{II', ss'}^{\alpha\beta, ss'}$. For example, by applying a combination of a cyclic permutation within an antisymmetrized $NN$-state, one obtains for the transition $2 \rightarrow 3$ the relation

$$
\mathcal{R}_{23} = \mathcal{R}_{13}^{23} + 2\mathcal{R}_{23}^{23},
$$

where the coefficient 2 in the term $\mathcal{R}_{23}^{23}$ comes from the identity of the nucleons.

Before we proceed to solve the four-body equations, we also need as input the equations describing two independent pairs of interacting particles ($\bar{K}N + (NN)$). The corresponding equations read in our case

$$
\begin{align*}
\mathcal{Y}_{ss'I}^{s'I} = & \mathcal{W}_{ss'I}^{ss'I} + \mathcal{W}_{ss'I}^{ss'I} \mathcal{Y}_{NN,NN}^{s'I, s'I}, \\
\mathcal{Y}_{s'I}^{ss'I} = & \mathcal{W}_{s'I}^{s'I} \mathcal{Y}_{NN,NN}^{s'I, s'I}.
\end{align*}
$$

Here, the operators $\mathcal{Y}_{s'I}^{s'I}$ are the Faddeev amplitudes which describe two independent pairs of interacting particles and the operators $\mathcal{W}_{s'I}^{s'I}$ are the effective potentials. A graphical representation of the system (14) is shown in fig. 3. Analogously to the treatment in the previous subsection,
FIG. 2. Diagrammatic representation of the potentials $\mathcal{R}^{\alpha\beta}$ in the separable approximation. The blue dashed line corresponds to the $\bar{K}$ and the black solid lines correspond to the nucleon. The symbols $u_\alpha$ will define the initial and final state of the system.

The separable form of the amplitude can easily be found

$$\mathcal{Y}^{sI,s'I'}_{i,j}(p,p',\epsilon) = \sum_{n=1}^{N_r} u^{sI}_{n;i}(p,\epsilon) \zeta_n(\epsilon) u^{s'I'}_{n;j}(p',\epsilon),$$

where the functions $u^{sI}_{n;i}$ are the eigenfunctions of the kernel of eq. (14).

$$u^{sI}_{n;i} = \frac{1}{\lambda_n} \sum_{j=\bar{K},NN} \mathcal{W}^{sI,s'I'}_{i,j} s'_{i,j} u^{s'I'}_{n;j}.$$  (16)

The conversion of the four-body equations to a numerically manageable form is yielded by expanding the two- and three-body Faddeev amplitudes in eqs. (1) and (14) into separable series of finite rank $N_r$. For to make a separable representation for these subsystem amplitudes, one can use the energy dependent pole expansion (EDPE) [26] or the Hilbert-Schmidt expansion [25]. The desired approach in this work is the Hilbert-Schmidt expansion (HSE). The inputs for the driving terms of equation (9) are two-body t-matrices, embedded in the four-body Hilbert space and the form factors, which are defined in eqs. (5) and (16). Before we proceed to solve the AGS equations (8), we should antisymmetrize the basic amplitudes with respect to the exchange of the nucleons for which we follow mainly the work of [22].

III. TWO-BODY INTERACTIONS

All two-body interactions are taken in $s$-wave and separable form. Thus, in the case of separable two-body potential we have

$$V_{\alpha\beta}(p_\alpha, p_\beta) = \lambda_{\alpha\beta} g_\alpha(p_\alpha) g_\beta(p_\beta).$$  (17)
Here, $\alpha$ and $\beta$ enumerate two-body channels and $p_\alpha$ is the c.m. momentum in the corresponding channel. The two-body $t$-matrices that serve as input for the three- and four-body problem are all taken in the separable form for a given partial wave

$$T_{\alpha\beta}(p_\alpha, p_\beta, E) = g_\alpha(p_\alpha)\tau_{\alpha\beta}(E)g_\beta(p_\beta),$$

(18)

where $E$ is the total energy, $\lambda_{\alpha\beta}$ are the coupling strength parameters of the interaction and the form factors are defined by $g_\alpha(p_\alpha)$.

The $\bar{K}N$ interaction, which is the most important interaction for the $\bar{K}NN$ and $\bar{K}NNN$ systems, is usually described either by pure phenomenological or by chirally motivated potentials. In our Faddeev calculations, we use two different effective interactions for the coupled-channel $\bar{K}N - \pi\Sigma$ interaction that, having a one- and two-pole structure of the $\Lambda(1405)$ resonance. The potentials that we use here for the $\bar{K}N$ interaction are given in ref. [27]. The parameters of the coupled-channel $\bar{K}N - \pi\Sigma$ potential were fitted to reproduce all existing experimental data on the low-energy $\bar{K}N$ system and the fitting was performed by using physical masses in $\bar{K}N$ and $\pi\Sigma$ channels with the inclusion of the Coulomb interaction.

The $s$-wave $\Sigma N$ interaction in the $I = 1/2$ isospin state is coupled with $\Lambda N$ channel, therefore, we used an optical potential for $\Sigma N$ interaction in this isospin state and a real potential for $I = 3/2$ channel. The parameters chosen for the $\Sigma N$ interaction were those given in ref. [28]. In this calculation, we use the spin independent version of $\Sigma N$ interaction.

In our three- and four-body study for singlet and triplet $NN$ interaction, we choose a poten-
tial of PEST type [29], which is a separablization of the Paris potential. The coupling strength parameter was set to \( \lambda = -1 \) and the form factors are defined by

\[
g^{NN}_{s,I}(p) = \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{6} \frac{c^{NN}_{n,I}}{p^2 + (\beta^{NN}_{n,I})^2},
\]

(19)

where the constants \( c^{NN}_{n,I} \) and \( \beta^{NN}_{n,I} \) are listed in ref. [29]. PEST potential is equivalent to the Paris potential for energies up to \( E_{lab} \sim 50 \text{ MeV} \). It reproduces the deuteron binding energy \( E_{B.E} = 2.2249 \text{ MeV} \), as well as the singlet and triplet \( NN \) scattering lengths, \( a(^1S_0) = 17.534 \text{ fm} \) and \( a(^3S_1) = 5.422 \text{ fm} \), respectively. The \(^3\text{He} \) binding energy, calculated with PEST potential is 9.7 MeV while the experimental value is 8.54 MeV.

IV. RESULTS AND DISCUSSIONS

Because \([\bar{K}NN]_{I=1/2,J^\pi=0^-}\) is the most important subsystem of the four-body \(\bar{K}NNN\) system, in fig. 4 we demonstrated how well a finite sum (2) may represent the exact amplitude. Thus, we calculated the ratio of the Schmidt norm for

\[
\Delta = \frac{||\vartheta_N||}{||\vartheta||},
\]

(20)

of the operators

\[
\vartheta = \mathcal{M}(\bar{K}N)_{I=0}N-(\bar{K}N)_{I=0}N,
\]

\[
\vartheta_N = \mathcal{M}(\bar{K}N)_{I=0}N-(\bar{K}N)_{I=0}N - \mathcal{M}^{N_r}_{(\bar{K}N)_{I=0}N-(\bar{K}N)_{I=0}N},
\]

(21)

where \( \mathcal{M}^{N_r}_{(\bar{K}N)_{I=0}N-(\bar{K}N)_{I=0}N} \) is given by the sum (2) containing only the first \( N_r \) terms. One can see that the rate of convergence is not very effective, but appears to be sufficient for the practical calculation.
As a starting three- and four-body calculation, we calculated the binding energies and widths of $K^- pp$ and $K^- ppn$ quasi-bound states using a one-channel complex $\bar{K}N$ potential [6]. During these calculations, we considered the $\bar{K}N$ potentials with the parameters $\lambda^{I, \text{Complex}}_{\bar{K}N, \bar{K}N}$ and $\beta_I$, which reproduce $M_\Lambda = 1405.1$ MeV, $\Gamma_\Lambda = 50$ MeV and the $K^- p$ scattering length, for which we used as a guideline the SIDDHARTA measured value: $a_{K^- p}^{\text{SIDD}} = (-0.65 + i 0.81)$ fm [30]. In table I, our results for the binding energy of the $K^- pp$ and $K^- ppn$ related to these data, and using $\beta_I = 3.5$ fm$^{-1}$, are represented. In table I we performed a calculation for the one-channel $\bar{K}NN$ system using a one-channel complex $\bar{K}N$ potential. For these data, we found a quasi-bound state for $K^- pp$ and $K^- ppn$ below the threshold.

TABLE I. The binding energies and widths of the quasi-bound state of the $K^- pp$ and $K^- ppn$ systems for one-channel complex potential.

| $a_{K^- p}$ (fm) | $E_{K^- pp}$ (MeV) | $E_{K^- ppn}$ (MeV) |
|------------------|-------------------|-------------------|
| -0.65+i0.81 [30] | -49.4-i43.5       | -60.2-i42.2       |

In the following we present the results for the binding energy of the $\bar{K}N$ and $\bar{K}NN$ in table II for the one- and two-pole version of $\bar{K}N - \pi \Sigma$ interaction. The binding energies for $\bar{K}N$ in table II are just a bit different from those given in the original ref. [27]. The reason is that the above calculations were performed with averaged masses and without Coulomb interaction while
TABLE III. The sensitivity of the binding energies and widths of the quasi-bound state of the $K^-ppn$ system to the number of terms $N_r$ in eqs. (2) and (15). $E_{K^-ppn}^{SIDD, One-pole}$ and $E_{K^-ppn}^{SIDD, Two-pole}$ correspond to the one- and two-pole version of the $\bar{K}N$ interaction, respectively. The real part of the pole $E_{K^-ppn}$ (in MeV) is measured from the $\bar{K}NNN$ threshold.

| $N_r$ | $E_{K^-ppn}^{SIDD, One-pole}$ | $E_{K^-ppn}^{SIDD, Two-pole}$ |
|------|-----------------|-----------------|
| 2    | -69.6-i10.5     | -56.7-i8.6      |
| 4    | -69.0-i11.1     | -56.2-i8.8      |
| 6    | -68.8-i11.0     | -55.9-i8.8      |

the fitting to the experimental data was performed with physical masses and Coulomb interaction. At the beginning, we solved eq. (5) with neglecting the $\Sigma N$ and $\pi N$ interactions. Thus, only $\bar{K}N$ and $NN$ t-matrices enter the equations. Therefore, we constructed the exact optical $\bar{K}N(-\pi \Sigma)$ potential, which is an approximation for the full coupled-channel interaction. The binding energies are calculated with respect to the $\bar{K}NN$ threshold. In the third column of table II the binding energy and width of the full coupled-channel calculation of the $\bar{K}NN - \pi \Sigma N$ by taking the $\Sigma N$ interaction into account are presented. One can see that the one-channel AGS calculation with exact optical $\bar{K}N(-\pi \Sigma)$ potential gives a good approximation to the full coupled-channel calculations. This result was expected because the exact optical potential provides exactly the same elastic $\bar{K}N - \bar{K}N$ amplitude as the coupled-channel model of interaction, see ref. [28].

TABLE II. The sensitivity of the binding energies and widths of the quasi-bound state of the $K^-pp$ systems to the $\bar{K}N$, $\Sigma N$ interactions. $E^0$ stands for no $\Sigma N$ interaction, while in calculating the $E^1$, $\Sigma N$ interaction is on. The real part of the pole $E_{K^-pp}$ is measured from the $\bar{K}NN$ threshold.

| $E_{K^-pp}$ (MeV) | $E_{K^-pp}^{(0)}$ (MeV) | $E_{K^-pp}^{(1)}$ (MeV) |
|-------------------|-----------------|-----------------|
| $V_{One-pole}$    | 1428.1-i46.6    | -48.7-i34.3     |
| $V_{Two-pole}$    | 1418.1-i56.9    | -45.4-i24.4     |
|                   | 1382.0-i104.2   | -47.1-i25.0     |

In table III we presented our results for the $K^-ppn$ quasi-bound state obtained by keeping a finite number of terms $N_r$, in the Hilbert-Schmidt expansion of the amplitudes (3) and (15). In this table, the rate of convergence of $K^-ppn$ binding energy is investigated and one can see that
the choice \( N_r = 4 \) provides rather satisfactory accuracy. In the four-body calculation we have neglected any \( \Sigma N - \Lambda N \) and \( \pi N \) interactions. The inclusion of these interactions would increase the number of channels in the four-body equations which would lead to much more complex formalism. As mentioned in the previous paragraph, the one-channel AGS calculation with exact optical \( \bar{K}N \) potential, giving exactly the same \( \bar{K}N - \bar{K}N \) amplitude as the corresponding coupled-channel potential, turns out to be a good approximation. Therefore, one can safely assume that \( \Sigma N - \Lambda N \) and \( \pi N \) interactions in the \( \pi\Sigma NN \) channel can not change the binding energy of the \( \bar{K}NNN - \pi\Sigma NN \) system more than a few MeV. Using the exact optical \( \bar{K}N \) potential, our two-channel four-body calculation with coupled-channel \( \bar{K}N - \pi\Sigma \) potential will be equivalent to the one-channel four-body calculation.

The binding energies and widths of the quasi-bound state of the \( K^-pp, K^-ppm \) and \( K^-ppp \) systems have been calculated and presented in table IV. We calculated \( K^-ppm \) and \( K^-ppp \) quasi-bound state positions by keeping four terms in the Hilbert-Schmidt expansion of the amplitudes (3) and (15).

Very recently, some few-body calculations are performed on \( K^-ppn \) by the variational method [31, 32] and the Faddeev approach [12]. The investigation of the \( \bar{K}NNN \) in ref. [31] uses the effective \( \bar{K}N \) interaction derived from chiral low energy theorem, a quasi-bound state was found with a binding energy 30 MeV and a width 30 MeV below the threshold energy of the \( \bar{K}NNN \) state. A similar conclusion was drawn using the Faddeev equation by Maeda et al. using a one-channel real potential [12]. The obtained binding energies for \( K^-ppn \) was about 69 MeV below threshold energy. The obtained binding energies of the \( \bar{K}NNN \) quasi-bound state in ref. [32] for A-Y and HW potentials are \( \sim 65 \) and \( \sim 18 \) MeV and the corresponding widths are \( \sim 74-80 \) and \( \sim 27-31 \), respectively. The comparison our results for \( K^-ppn \) obtained for PEST \( NN \) interaction and the coupled-channel \( \bar{K}N - \pi\Sigma \) interaction with calculations in ref. [12] within the Faddeev method for rank-two \( NN \) interaction and one-channel real \( \bar{K}N \) interaction shows that they are in the same range. However, this is in contrast to the chiral low energy potential, which is constructed to generate a bound state with a binding energy \( \sim 30 \) MeV.
TABLE IV. Pole positions (in MeV) of the quasi-bound states in the $K^{-}pp$, $K^{-}ppn$ and $K^{-}ppp$. The Faddeev AGS calculations performed with the phenomenological potentials from ref. [27]. The potentials $V^{SIDD}_{\text{One-pole}}$ and $V^{SIDD}_{\text{Two-pole}}$ are $\bar{K}N - \pi\Sigma$ potentials, which produce the one- and two-pole structure of the $\Lambda(1405)$ resonance, respectively. The binding energies (real part of the pole) are measured from the thresholds.

|                | $E_{K^{-}pp}$ | $E_{K^{-}ppn}$ | $E_{K^{-}ppp}$ |
|----------------|---------------|----------------|---------------|
| $V^{SIDD}_{\text{One-pole}}$ | -48.7-i34.3   | -68.8-i11.0    | -99.6-i10.5   |
| $V^{SIDD}_{\text{Two-pole}}$  | -45.4-i24.4   | -55.9-i18.8    | -87.8-i3.5    |

V. CONCLUSION

Starting from Faddeev AGS equations and using different versions of the $\bar{K}N - \pi\Sigma$ potentials, which produce the one- and two-pole structure of the $\Lambda(1405)$ resonance and separable expressions for the [3+1] and [2+2] subsystems. We employed the HSE method to reduce the problem to a set of single-variable integral equations. We solved the three- and four-body Faddeev equations, searching for $K^{-}pp$, $K^{-}ppn$ and $K^{-}ppp$ quasi-bound states. We studied the dependence of the pole energy on different models of $\bar{K}N - \pi\Sigma$ interaction. It was shown that a one-channel complex $\bar{K}N$ potential gives much broader three- and four-body quasi-bound state than the exact optical potential. The calculations yielded binding energy $B_{K^{-}pp} \sim 45-55$, $B_{K^{-}ppn} \sim 55-70$ and $B_{K^{-}ppp} \sim 90-100$ MeV for $K^{-}pp$, $K^{-}ppn$ and $K^{-}ppp$, respectively. The obtained widths for these systems are $\Gamma_{K^{-}pp} \sim 50-75$, $\Gamma_{K^{-}ppn} = 16 - 20$ and $\Gamma_{K^{-}ppp} = 7 - 20$ MeV. However, a similar calculation should be performed for the standard energy-dependent $\bar{K}N$ input potential, too. The quasi-bound states resulting from the energy-dependent potentials happen to be shallower, this is due to the energy dependence of the interaction. The energy-dependent potential will provide a weaker $\bar{K}N$ attraction for lower energies than the energy independent potential under consideration in this work. A definitive study of the $K^{-}pp$ quasi-bound state could be performed through fully exclusive formation reaction, such as the in-flight $^3\text{He}(K^{-}, N)$ reaction. This was performed at J-PARC [33]. As a next step, we will develop the four-body Faddeev AGS equations to make a practical calculation of the cross section of kaon-induced strange-dibaryon production reaction. In the present study, we have calculated $K^{-}ppn$ and $K^{-}ppp$ quasi-bound state positions.
using the HSE method to find the separable expressions for the [3+1] and [2+2] subsystems. There is another separable expansion method for the [3+1] and [2+2] subsystems, this method is called the energy-dependent pole expansion (EDPE) method and the form factors have an energy dependence [26]. To study which one of these methods (HSE and EDPE) has a better convergence rate, one can perform a similar calculation using the EDPE method.

The authors thank A. Fix for helpful comments and discussions. One of the authors (S. Marri) is thankful to Prof. T. Yamazaki for his fruitful discussions. The authors gratefully acknowledge the Sheikh Bahaei National High Performance Computing Center (SBNHPCC) for providing computing facilities and time. SBNHPCC is supported by the scientific and technological department of presidential office and Isfahan University of Technology (IUT).

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