ON POLYHEDRAL CONTROL SYNTHESIS FOR DYNAMICAL DISCRETE-TIME SYSTEMS UNDER UNCERTAINTIES AND STATE CONSTRAINTS

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ABSTRACT. We deal with a problem of target control synthesis for dynamical bilinear discrete-time systems under uncertainties (which describe disturbances, perturbations or unmodelled dynamics) and state constraints. Namely we consider systems with controls that appear not only additively in the right hand sides of the system equations but also in the coefficients of the system. We assume that there are uncertainties of a set-membership kind when we know only the bounding sets of the unknown terms. We presume that we have uncertain terms of two kinds, namely, a parallelootope-bounded additive uncertain term and interval-bounded uncertainties in the coefficients. Moreover the systems are considered under constraints on the state (“under viability constraints”). We continue to develop the method of control synthesis using polyhedral (parallelootope-valued) solvability tubes. The technique for calculation of the mentioned polyhedral tubes by the recurrent relations is presented. Control strategies, which can be constructed on the base of the polyhedral solvability tubes, are proposed. Illustrative examples are considered.

1. Introduction. We deal with a problem of target control synthesis for dynamical discrete-time systems with a bilinear structure under uncertainties and state constraints (SC). We use a deterministic model of uncertainty with set-membership description of the uncertain items (which describe disturbances, perturbations or unmodelled dynamics) when they are taken to be unknown but bounded with given bounds and we have no statistical information whatever [23, 24, 25]. The set-membership model of uncertainty is appropriate for many applied problems. The problems in dynamics and control with such model of uncertainty are described through set-valued functions known as trajectory tubes and are to be treated through set-valued analysis. The problem of construction of the mentioned trajectory tubes that describe the dynamics of reachable sets, solvability sets, informational domains [24, 25] may be called one of the fundamental problems of the mathematical control theory. Note that there are many different problem statements in the theory of dynamical systems concerning investigations of an influence of perturbations, and there were obtained deep and surprising results, in particular, in KAM theory (see, for example, [28, 12]).

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There are known approaches for solving the problems of terminal target feedback control for differential systems based on construction of so-called solvability tubes (in other terms, maximal stable bridges, Krasovskii’s bridges, backward reachable tubes) and there are known descriptions of trajectory tubes for differential systems through multivalued integrals and through evolution equations of several types (in particular, through funnel equations, through the evolution of support functions or through level sets of appropriate Hamilton-Jacobi-Bellman equations) [24, 25] (here and below, we note, as examples, only some references from numerous publications; see also references therein). SC further complicate the problem [13, 25] and lead to so-called viability tubes. A close target control problem, which is connected with constructing an appropriate set of initial states (so-called capture basin), was considered in viability theory [3, 6]. Since practical construction of the trajectory tubes (in particular, the solvability tubes, reachable tubes, viable trajectory tubes) as well as of capture basins and other so-called “kernels” from viability theory may be cumbersome, various numerical methods have been developed for approximation of the set-valued solutions and for numerical solution of the above-mentioned evolution equations and multivalued integrals including methods based on approximations of sets either by arbitrary polytopes with a large number of vertices or by unions of a large number of points [4, 5, 6, 7, 29, 34, 36]. Such methods are devised to obtain approximations as accurate as possible. But they can require much calculation, especially for large-dimensional systems. Other techniques are based on estimates of sets by domains of some fixed shape such as ellipsoids and parallelepipeds, including boxes aligned with coordinate axes as in interval analysis [7, 8, 10, 14, 15, 17, 18, 19, 20, 21, 24, 25, 26, 35]. The main advantage of the last techniques is that they enable to obtain approximate/particular solutions using relatively simple tools. More accurate approximations may be obtained by using the whole families (varieties) of such simple estimates (as suggested by A.B. Kurzhanski) [8, 15, 17, 18, 19, 20, 21, 24, 25, 26, 35]. The methods of interval analysis which use subpavings (unions of non-overlapping boxes) [14] serve the same purpose, but may require much computations and memory for large-dimensional systems.

As for solving the feedback target control problems for differential and discrete-time systems, constructive computation schemes using ellipsoidal techniques were proposed [8, 24, 25, 26, 35] and then expanded to the polyhedral techniques that use polyhedral (parallelotope-valued) solvability tubes [15, 17, 19, 20, 21, 24, 25, 26, 35] (this had required the development of a quite different techniques). In [30, 31], the polyhedral technique from [19, Sec. 3] was applied for constructing a real-time control of aircraft take-off in windshear using a sequential linearization of highly nonlinear point mass aircraft model equations around appropriate ascending paths. Also in [30], the example with the linear differential game known as “Boy and Crocodile” is considered, where the mentioned parallelotope technique is compared with a grid method which computes the maximal solvability tube almost exactly if the grid is sufficiently fine.

An important for investigations class of dynamical systems is produced by systems with a bilinear structure, where matrices of coefficients may contain not only uncertainties but also controls. Such models may be useful in many applied areas from physics and engineering to biology, ecology, socioeconomics [32, 33], [22, Ref. 23]. Different techniques to solving some control and stabilization problems for bilinear systems, including ellipsoidal and interval ones, can be found, for example in [1, 2, 5, 11, 25, 32, 33] and in references therein.
Here we develop the polyhedral method of control synthesis for bilinear discrete-time systems on a given fixed time interval. We consider systems with controls that appear not only additively in the right hand sides of the system equations but also in the coefficients of the system. We assume that there are uncertainties of two kinds, namely, the parallelepiped-bounded additive uncertain term and interval-bounded uncertainties in the coefficients. Note that the systems with controls (or/and uncertainties) in the system matrix are of bilinear type and have properties of nonlinear systems (in particular reachable sets and solvability sets of such systems can be non-convex). Moreover we consider the systems under constraints on the state ("under viability constraints"). To solve the problem of target control synthesis under these conditions we expand the techniques from [17, 19, 20, 21, 22] (some additional comments about these works will be given below). The technique for calculation of the polyhedral tubes by the recurrent relations is presented. Corresponding control strategies are described. Results of computer simulations in the illustrative examples are presented confirming the operability of the proposed method.

We use the following notation: $\mathbb{R}^n$ is the $n$-dimensional vector space; $(x, y) = x^\top y$ is the scalar product for $x, y \in \mathbb{R}^n$; $\top$ is the transposition symbol; $\|x\|_2 = (x^\top x)^{1/2}$, $\|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i|$ are the vector norms for $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$; $e^i = (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ is the unit vector oriented along the axis $x_i$ (the unit stands at position $i$); $e = (1, 1, \ldots, 1)^\top$; $\mathbb{R}^{n \times m}$ is the space of real $n \times m$-matrices $A = \{a^i_j\} = \{a^j_i\}$ with elements $a^i_j$ and columns $a^j$ (the upper index numbers the columns and the lower index numbers the components of vectors); $I$ is the identity matrix; $0$ is the zero matrix (vector); $E = \{e^i\}$ is the matrix with all elements equal to the unity: $e^i = 1$; $\text{Abs} A = \{\|a^i_j\|\}$ for $A = \{a^i_j\} \in \mathbb{R}^{n \times m}$; $\text{diag} \pi$, $\text{diag} \{\pi_i\}$ are the diagonal matrix $A$ with $a^i_j = \pi_i$, where $\pi_i$ are the components of the vector $\pi$; det $A$ is the determinant of $A \in \mathbb{R}^{n \times n}$; $A \ast B = \{a^i_j b^i_j\} \in \mathbb{R}^{n \times n}$ is the Hadamard product of $n \times n$-matrices $A = \{a^i_j\}$ and $B = \{b^i_j\}$ (elementwise product); int $\mathcal{X}$ is the set of interior points of the set $\mathcal{X} \subset \mathbb{R}^n$; $\text{Pr}_{[a, \bar{a}]}(z)$ is a projection of the real $z$ on the segment $[a, \bar{a}] \subset \mathbb{R}^1$, namely it is equal to $a$, $z$, $\bar{a}$ for $z < a$, $a \leq z \leq \bar{a}$, $z > \bar{a}$ respectively; the notation $k = 1, \ldots, N$ is used instead of $k = 1, 2, \ldots, N$ for brevity.

2. Problem formulation. Consider the controlled system ($x \in \mathbb{R}^n$ is the state):

$$
\begin{align*}
   x[k] &= (A[k] + V[k] + U[k]) x[k-1] + B[k] u[k] + C[k] v[k], \quad k = 1, \ldots, N, \\
   x[N] &\in \mathcal{M},
\end{align*}
$$

(1)

with a given terminal target set $\mathcal{M}$. Here $A[k] \in \mathbb{R}^{n \times n}$, $B[k] \in \mathbb{R}^{n \times n_u}$, $C[k] \in \mathbb{R}^{n \times n_v}$, are given matrices; $U[k] \in \mathbb{R}^{n \times n}$ and $u[k] \in \mathbb{R}^{n_u}$ serve as controls that satisfy the following constraints with given bounding sets:

$$
   u[k] \in \mathcal{R}[k] \subset \mathbb{R}^{n_u}, \quad k = 1, \ldots, N,  
$$

(2)

$$
   U[k] \in \mathcal{U}[k] = \{U \in \mathbb{R}^{n \times n} \mid \text{Abs} (U - \hat{U}[k]) \leq \hat{U}[k]\}, \quad k = 1, \ldots, N;  
$$

(3)

$v[k] \in \mathbb{R}^{n_v}$ (unknown but bounded disturbances) and $V[k] \in \mathbb{R}^{n \times n}$ (matrix uncertainties) are subjected to given set-valued constraints:

$$
   v[k] \in \mathcal{Q}[k] \subset \mathbb{R}^{n_v}, \quad k = 1, \ldots, N,  
$$

(4)

$$
   V[k] \in \mathcal{V}[k] = \{V \in \mathbb{R}^{n \times n} \mid \text{Abs} (V - \hat{V}[k]) \leq \hat{V}[k]\}, \quad k = 1, \ldots, N.  
$$

(5)

The functions $v[\cdot]$ and $V[\cdot]$ satisfying (4) and (5) are called admissible. Matrix and vector inequalities ($\leq, <, \geq, >$) here and below are understood component-wise.
Below we consider the following cases: (A) \textit{without uncertainty}, when \( v \) and \( V \) are given functions, i.e., \( \bar{Q} \equiv 0 \), \( \bar{V} \equiv 0 \); (B) \textit{under uncertainty} including two subcases: (B.i) \textit{only additive uncertainty} (\( \bar{V} \equiv 0 \)); (B.ii) \textit{also matrix uncertainty} (\( \bar{V} \not\equiv 0 \)).

The system is considered under the state constraints (SC)

\[
\bar{x[k]} \in \mathcal{W}[k] \subseteq \mathbb{R}^n, \quad k = 0, \ldots, N-1. \tag{6}
\]

We presume the sets \( \mathcal{R}[k], \mathcal{Q}[k], \mathcal{M} \), \( \mathcal{Y}[k] \) to be given and accept the following.

**Assumption 1.** The set \( \mathcal{M} \) is a nondegenerate parallelepiped \( \mathcal{M} = \mathcal{P}(p_1, p_1, \pi_1) = \mathcal{P}[p_1, \bar{P}_1] \) (det \( \bar{P}_1 \not\equiv 0 \)); the sets \( \mathcal{R}[k] \) and \( \mathcal{Q}[k] \) are parallelotopes \( \mathcal{R}[k] = \mathcal{P}[r[k], \bar{R}[k]], \mathcal{Q}[k] = \mathcal{P}[q[k], \bar{Q}[k]] \); \( \mathcal{Y}[k] \) are either zones \( \mathcal{Y}[k] = \bigcap_{i=1}^{m} \Sigma^i[k] \), \( \Sigma^i[k] = \Sigma(c_i[k], s^i[k], \sigma_i[k]) = \{ x \mid (x, s^i[k]) - c_i[k] \leq \sigma_i[k] \} \), or \( \mathcal{Y}[k] = \mathbb{R}^n \) (if \( SC \) are not imposed at time \( k \)); all matrices \( D[k] = A[k] + \bar{V}[k] + \bar{U}[k] \) are nonsingular.

By a parallelepiped \( \mathcal{P}(p, P, \pi) \subseteq \mathbb{R}^n \) we mean a set such that \( \mathcal{P} = \mathcal{P}(p, P, \pi) = \{ x \in \mathbb{R}^n \mid x = p + P \text{diag} \pi \xi, \|\xi\|_\infty \leq 1 \} \), where \( p \in \mathbb{R}^n \); \( P = \{ p^i \} \in \mathbb{R}^{n \times n} \) is a nonsingular matrix (det \( P \not\equiv 0 \)) such that \( \|p^i\|_2 = 1; \pi \in \mathbb{R}^n, \pi \geq 0; \) the condition \( \|p^i\|_2 = 1 \) may be omitted to simplify formulas. It may be said that \( p \) determines the center of the parallelepiped, \( P \) is the orientation matrix, \( p^i \) are the “directions”, and \( \pi_i \) are the values of its “semi-axes”. We call a parallelepiped \textit{nondegenerate} if all \( \pi_i > 0 \).

By a \textit{paralleloctope} \( \mathcal{P}(p, \bar{P}) \subseteq \mathbb{R}^n \) we mean a set \( \mathcal{P} = \mathcal{P}(p, \bar{P}) = \{ x \mid x = p + \bar{P} \xi, \|\xi\|_\infty \leq 1 \} \), where \( p \in \mathbb{R}^n \) and \( \bar{P} = \{ \bar{p}^i \} \in \mathbb{R}^{n \times m} \), \( m \leq n \). We call a paralleloctope \( \mathcal{P} \) \textit{nondegenerate} if \( m = n \) and det \( \bar{P} \not\equiv 0 \).

By a \textit{zone} (or \( m \)-zone) \( \mathcal{S} = \mathcal{S}(c, S, \sigma, \pi) \subseteq \mathbb{R}^n \) we mean an intersection of \( m \leq n \) strips \( \Sigma^i \); \( \mathcal{S} = \mathcal{S}(c, S, \sigma, \pi) = \bigcap_{i=1}^{m} \Sigma^i \); \( \Sigma^i = \Sigma(c^i, s^i, \sigma^i) = \{ x \mid (x, s^i) - c^i \leq \sigma^i \} \), where \( c \in \mathbb{R}^m \), \( S = \{ s^i \} \subseteq \mathbb{R}^{n \times m} \), vectors \( s^i \) are linearly independent; \( \sigma \in \mathbb{R}^m \), \( \sigma \geq 0 \).

Each parallelepiped \( \mathcal{P}(p, P, \pi) \) is a paralleloctope \( \mathcal{P}(p, \bar{P}) \) with \( \bar{P} = P \text{diag} \pi \).

Each nondegenerate paralleloctope is a parallelepiped with \( P = P \text{diag} \{ \|p^i\|_2^{-1} \} \), \( \pi_i = \|p^i\|_2 \) or, in a different way, with \( P = P \), \( \pi = \pi \), where \( \pi = (1, \ldots, 1)^{T} \). Each parallelepiped is a zone, and vice versa if \( m = n \).

The following Problem 1, which is similar to ones from [24, 25, 26, 35], was investigated earlier [17, 26, 35].

**Problem 1.** Let \( \bar{U}[k] \equiv 0 \). For any \( i, 0 \leq i \leq N-1 \), find a solvability set \( \mathcal{W}[i] \) and a control strategy \( u = u[k, x] \) with \( u[k, x] \in \mathcal{R}[k] \) such that each solution \( x[k] \) to

\[
x[k] = (A[k] + V[k])x[k-1] + B[k]u[k, x[k-1]] + C[k]v[k], \quad k = i+1, \ldots, N,
\]

that start from any \( x[i] \in \mathcal{W}[i] \) would reach the target set (\( x[N] \in \mathcal{M} \) and satisfy \( SC \) (6) whatever are admissible functions \( v[\cdot] \), \( V[\cdot] \) subjected to (4), (5)).

Similarly to [24, 25], we say that the multivalued function \( \mathcal{W}[k] \), \( k = 0, \ldots, N \), is a \textit{solvability tube} \( \mathcal{W}[\cdot] \).

The solution to Problem 1 for cases (A), (B.i) (i.e., without matrix uncertainty) is known (see [17, 20], and see also [35] for the case (A)). It contains recurrent relations for \( \mathcal{W}[\cdot] \), which involve operations with sets such as \( Minkowskis\’ \text{ sum} (\mathcal{X}^1 + \mathcal{X}^2 = \{ y \mid y = x^1 + x^2, x^1 \in \mathcal{X}^1 \}), Minkowskis\’ \text{ difference} (\mathcal{X}^1 - \mathcal{X}^2 = \{ y \mid y + \mathcal{X}^1 \subseteq \mathcal{X}^2 \}), \text{ affine transformation}, \) and \textit{intersection of sets}. Thus exact construction of \( \mathcal{W}[\cdot] \) by the mentioned relations can be very cumbersome. Even more difficulties arise for the cases with uncertainties/controls in matrices. Therefore, the ellipsoidal methods for solving Problem 1 were elaborated (see [35] for case (A) under SC and [26] for
cases (A), (B,i) without SC). Polyhedral techniques were also proposed [17], which use parallelepiped-valued estimates for the solvability tubes.

We call \( \mathcal{P}^- (\mathcal{P}^+) \) an internal (external) estimate for \( \mathcal{Q} \subseteq \mathbb{R}^n \) if \( \mathcal{P}^- \subseteq \mathcal{Q} \) (\( \mathcal{P}^+ \supseteq \mathcal{Q} \)).

In [17], for cases (A) and (B,i), the families of external \( \mathcal{P}^+[\cdot] \) and internal \( \mathcal{P}^-[\cdot] \) parallelepiped-valued and parallelepiped-valued (shorter, polyhedral) estimates for \( \mathcal{W}[\cdot] \) were introduced (\( \mathcal{P}^- [k] \subseteq \mathcal{W}[k] \subseteq \mathcal{P}^+[k], k = 1, \ldots, N \)), and control strategies \( u[k,x] \) were proposed, which may be constructed using \( \mathcal{P}^- [k] \) by solving systems of linear inequalities. If the initial point \( x[0] = x_0 \) is out of at least one of the external estimates \( \mathcal{P}^+[0] \), then there is no guarantee that it can be steered to the terminal set for any disturbances, and if it belongs to one of the internal estimates \( \mathcal{P}^-[0] \), then it can reach \( \mathcal{M} \) using the mentioned control strategy.

Now let us consider Problem 2, which concerns all cases (A) – (B,ii) under SC, where the controls appear both additively and in the system matrix.

**Problem 2.** For system (1)-(6), find a polyhedral solvability tube \( \mathcal{P}^-[\cdot] \) that satisfies \( \mathcal{P}^-[N] = \mathcal{M} \) and \( \mathcal{P}^-[k] \subseteq \mathcal{Y}[k], k = 0, \ldots, N-1 \), and find a corresponding feedback control strategies \( u = u[k,x] \) and \( U = U[k,x] \) such that \( u[k,x] \in \mathcal{R}[k], U[k,x] \in \mathcal{U}[k], k = 1, \ldots, N, \) and each solution \( x[\cdot] \) to

\[
x[k] = (A[k]+V[k]+U[k,x[k-1]])x[k-1]+B[k]u[k,x[k-1]]+C[k]v[k], \quad k = 1, \ldots, N,
\]

with \( x[0] = x_0 \in \mathcal{P}^-[0] \) would satisfy \( x[k] \in \mathcal{P}^-[k], k = 1, \ldots, N, \) whatever are admissible \( v[\cdot] \) and \( V[\cdot] \). Moreover, introduce a family of such tubes \( \mathcal{P}^-[\cdot] \).

Recall that in [19, 21] for cases (A), (B,i), (B,ii) without SC the polyhedral techniques were proposed for synthesis of controls which appear either additively or in the system matrix; such controls can be constructed by explicit formulas. For systems with \( U \equiv 0 \), this technique was expanded in [20] to systems with SC and also the technique from [17] was expanded for the case (B,ii) (the case under matrix uncertainties). It appears that both mentioned techniques provide one the same families of tubes \( \mathcal{P}^-[\cdot] \), while control strategies, generally speaking, are different.

To solve Problem 2 for the general case we expand the techniques from [19, 20, 21, 22]. As a result we can construct controls \( u \) and \( U \) by explicit formulas and obtain more rich family of the tubes \( \mathcal{P}^-[\cdot] \) than it could be obtained using constructions from [18] as it was done in [19, 22]. Note that expanding the technique from [17] for the general case with both controls \( u \) and \( U \) is an open question yet.

3. **Solving Problem 2.** To construct polyhedral solvability tubes we need some primary internal polyhedral estimates for sets. Recall the way of constructing for \( \mathcal{Q} \subseteq \mathbb{R}^{n \times n} \) internal estimates \( \mathcal{P}^-_{v,V}(\mathcal{Q}) \) with arbitrary fixed center \( v \in \operatorname{int} \mathcal{Q} \) and orientation matrix \( V = \{v^j\} \in \mathbb{R}^{n \times n} \) for the case when \( \mathcal{Q} \) is a bounded polytope with \( \operatorname{int} \mathcal{Q} \neq \emptyset \) given as the intersection of \( \mathcal{Y} \geq n+1 \) strips: \( \mathcal{Q} = \bigcap_{j=1}^{\mathcal{Y}} \Sigma^j, \Sigma^j = \Sigma(c_j, s^j, \sigma_j) = \{x | |(x,s^j)-c_j| \leq \sigma_j \} \) (see [20] for details). Let \( A = \{a^j_i\} = \{a^j\} \in M^{n \times \mathcal{Y}}, b \in \mathbb{R}^{\mathcal{Y}}, \) and vectors \( v^0, \nu^* \in \mathbb{R}^n \) be determined by the formulas:

\[
a^j_i = |(v^i, s^j)|, \quad i = 1, \ldots, n, \quad j = 1, \ldots, \mathcal{Y},
\]

\[
b_j = \min \{\sigma_j + c_j - (v, s^j), \sigma_j - c_j + (v, s^j)\}, \quad j = 1, \ldots, \mathcal{Y} ;
\]

\[
u^0 = (1/n) \min \{b_j/a^j_i | j = 1, \ldots, \mathcal{Y}, a^j_i \neq 0\}, \quad i = 1, \ldots, n,
\]

\[
\nu^* = \gamma \nu^0, \quad \gamma = \min \{|b_j/(a^j_i v^0) | j = 1, \ldots, \mathcal{Y}, (a^j_i v^0) \neq 0\}.
\]

Then \( a^j_i \neq 0 \) (\( j = 1, \ldots, \mathcal{Y} \), \( A \geq 0, b > 0 \), we have \( \nu^* > 0 \), and the parallelepiped \( \mathcal{P}^-_{v,V}(\mathcal{Q}) = \mathcal{P}(v, V; \nu^*) \) is an internal estimate for \( \mathcal{Q} \): \( \mathcal{P}^-_{v,V}(\mathcal{Q}) \subseteq \mathcal{Q} [20] \).
Here \( v \) and \( V \) serve as parameters, which specify a parametric family of estimates.

It was assumed above that the point \( v \in \text{int} \ Q \) is known. It is not difficult to find such point by some explicit formulas for some types of sets. In particular, let \( Q = P \cap S \), i.e., \( Q \) is the intersection of the parallelepiped \( P \) with the zone \( S = \bigcap_{i=1}^{m} \Sigma^i \). If \( S \) consists of the unique strip \( S = \Sigma^1 \), then such point can be found by some explicit formulas [20, Ref. 10], otherwise it can be found successively via \( m \) steps starting from \( P \), where at each step an internal estimate is constructed for the intersection of a current parallelepiped with the strip \( \Sigma^i \). It should be mentioned that, generally speaking, such successive procedure can produce rather rough estimates for \( m > 1 \).

An alternative attractive ways for calculating \( v \) (when \( V \) is fixed) is to find \( v \in \text{Argmax} \{ \{ P_{v,V} (Q) \} | v \in Q \} \) (for example, using the Nelder-Mead simplex method [27]).

Now let us consider Problem 2. Let us introduce the parametric family of tubes \( P^- [\cdot] \) that satisfy the following system of recurrent relations:

\[
P^- [k] = \begin{cases} P^0 [\cdot] & \text{if } P^0 [\cdot] \subseteq Y [k], \\ P^0 [\cdot], P^0 [\cdot] \cap Y [k] & \text{otherwise}, \end{cases} \tag{8}
\]

where \( P^0 [\cdot] = \{ \rho, \phi \} \) satisfies the following relations (\( k = N, \ldots, 1 \)):

\[
p^0 [k-1] = D[k]^{-1} (p^0 [k - B[k] \gamma[k] - C[k] q[k]), \quad D[k] = A[k] + \bar{U}[k] + \bar{V}[k], \tag{9}
\]

\[
\bar{P}^0 [k-1] = H[k], \quad \bar{P}^0 [k-1] = \max \{ \text{Abs} \ (\bar{P}^0 [k-1] - (\text{Abs} \ P)e) \}, \tag{10}
\]

\[
\Phi[k, P] = \Phi[k, P; \Omega[k], L[k]] = \frac{(\bar{U}[k] + \bar{V}[k]) \text{diag} (\eta[k, P] \Pi(L[k]) \text{diag} (\kappa[P])}{\text{Abs} \ (\bar{P}^0 [k-1] - (\text{Abs} \ P)e)}, \tag{11}
\]

\[
\nu[k, P] = \max \{ \text{Abs} \ (\bar{P}^0 [k-1] - (\text{Abs} \ P)e) \}, \quad \nu[k, P] = \max \{ \text{Abs} \ (\bar{P}^0 [k-1] - (\text{Abs} \ P)e) \}, \tag{12}
\]

\[
\kappa_i (P) = (\text{e}^T) (\text{Abs} \ (\text{Abs} P)e)^{-1}, \quad i = 1, \ldots, n, \tag{13}
\]

Here \( \ast \) denotes the operation of the Hadamard product of matrices; \( \Pi(L) \) denotes the matrix determined by a permutation \( L = \{ i_1, \ldots, i_n \} \) of numbers \( \{ 1, \ldots, n \} \), i.e. is obtained by the corresponding transposition of columns of the unit matrix: \( \Pi(L) = \{ e^{i_j} \}; \) \( \text{Abs} \ (\text{Abs} P)e \) is the set of all vertices of the unit cube \( C = P(0, 1, e) \); the operation of maximum is understood component-wise.

Formulas (8), (11), (12) contain several parameters, namely matrices \( \Gamma[k] \in \mathbb{R}^{n \times n}, \ \Omega[k] \in \mathbb{R}^{n \times n}, \ P^0 [k] \in \mathbb{R}^{n \times n} \), vectors \( p^0 [\cdot] \in \mathbb{R}^n \), and also permutations \( L[k] = \{ i_1[k], \ldots, i_n[k] \} \). We permit arbitrary permutations \( L[k] \), assume that other parameters satisfy the following conditions for all \( k = N, \ldots, 1 \):

\[
\Gamma[k] \in G = \{ \Gamma = \{ \gamma_i^j \} \in \mathbb{R}^{n \times n} | \| \Gamma \| \leq 1 \}, \quad \| \Gamma \| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\gamma_i^j|, \tag{14}
\]

\[
\Omega[k] \in C, \quad C = \{ \Omega = \{ \omega^i_j \} \in \mathbb{R}^{n \times n} | |\omega^i_j| \leq 1, \quad i, j = 1, \ldots, n \}, \tag{15}
\]

\[
\text{det} \ P^-[k] \neq 0, \quad P^- [k] \in \text{int} (P^0 [k] \cap Y [k]), \tag{16}
\]

and call such parameters \textit{admissible}. Thus we have the parametric family of the polyhedral tubes, where the parameters are the functions \( \Gamma[\cdot], \Omega[\cdot], L[\cdot], P^- [\cdot], p^- [\cdot] \).

We see that the centers of parallelotopes \( P^0 [k] \) are determined by system of explicit recurrent relations (9), (13) while the matrices \( P^0 [k] \) are determined by
the fulfilment of constraints (2) and (3) for all $x$ in (17). According to relations e following relations:

\begin{align*}
&\text{Proof. First of all we concretize elements } U \\
&\text{or diagonal matrices of the form } \lambda \text{ where } \lambda = \text{diag } \lambda, \text{ and } \phi_k = \text{min} \{1, (||\xi||_\infty)^{-1}\}, \text{ or diagonal matrices of the form } \\
&\lambda = \text{diag } \lambda, \quad \phi_k = \text{min} \{1, (||\xi||)^{-1}\}, \quad i = 1, \ldots, n, \\
&u_i = u_i + \left\{ \begin{array}{ll}
\text{Pr}_{[-u_i],[u_i]}(x_i P_i - p_{i-1}[k-1]) x_i^{-1} \Phi[k, P_{i-1}[k-1]] e_i \\
0 \text{ otherwise.}
\end{array} \right.
\end{align*}

Here, the index $k$ in control strategies $u[k, x]$ and $U[k, x]$ indicates that they are used in system (7) at the $k$th step. The dependence on $k-1$ (through paralleotope $P_{i-1}[k-1]$) is not indicated to simplify the notation. The similar remark is true for the notation for $\eta$ and $\beta$ in (12).

**Theorem 3.1.** Let $\Gamma[]$, $\Omega[]$, $L[]$, $P[]$, $P[]$, $p[]$, $\eta[]$ be arbitrary admissible parameters and the system (8)-(13) has a solution $(\rho^0[], \bar{\rho}^0[], P[], p[], \eta[])$ that satisfy the following relations:

\begin{align*}
e - \beta[k, \bar{\rho}^0[k-1]] - \gamma[k] \geq 0, \quad k = N, \ldots, 1, \\
\det \bar{\rho}^0[k] \neq 0, \det P[k] \neq 0, \text{ int } (P[k] \cap \mathcal{X}) \neq \emptyset, \quad k = N-1, \ldots, 0.
\end{align*}

Then the tube $P[] = \mathcal{P}[p[], P[]]$ and the above control strategies $u[]$ and $U[]$ give a particular solution to Problem 2.

**Proof.** First of all we concretize elements $\varphi_i^j$ of the matrix $\Phi = \{\varphi^j\}$ that appear in (17). According to relations $e^j = \Pi(L)e^j$ and orthogonality of matrices $\Pi(L)$, we obtain $\Pi(L)^T e^j = e^j$ and

\begin{equation}
\varphi_i^j = e^T \Phi e^j = e^T (\bar{U} * \Omega) \text{ diag } \eta \Pi^T e^j, \quad \phi_i = \bar{U} \eta \Pi^T e^j, \quad \phi_i \phi_j^T = \bar{U} \eta \Pi^T e^j \eta_j \Pi^T e^j.
\end{equation}

(here and below arguments of functions are omitted for brevity).

Due to projection type operations, formulas for controls $u[k, x]$ and $U[k, x]$ ensure the fulfillment of constraints (2) and (3) for all $x \in \mathbb{R}^n$, $k = 1, \ldots, N$.

Let us verify that if $\det \bar{\rho}^0[k-1] \neq 0$ and $P[k-1] \subseteq P^0[k-1]$, then $U[k, x]$ from (17) act on $x \in P[k-1]$ according to the following rule:

\begin{equation}
U[k, x] x = \bar{U} [k] x - \Phi[k, \bar{\rho}^0[k-1]] (x - p^0[k-1]).
\end{equation}
Indeed, it follows from the inclusions $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^0-[k-1]$ that there exist $\zeta[k-1]$ and $\zeta^0[k-1]$ such that

$$x = p^-[k-1] + \bar{P}^-[k-1]\zeta[k-1] = p^0-[k-1] + \bar{P}^0-[k-1]\zeta^0[k-1],$$

$$||\zeta[k-1]||_{\infty} \leq 1, \quad ||\zeta^0[k-1]||_{\infty} \leq 1.$$ (20)

Let $\bar{U}[k, x]$ be constructed by (17), where the projection operations are omitted. Then it can be verified using estimates similar to ones from [21, Proof of Theorem 2] that $\bar{U}[k, x]$ satisfy (3) and consequently $\bar{U}[k, x] = \bar{U}[k, x]$ for $x \in \mathcal{P}^-[k-1]$.

In order to prove (19) for $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^0-[k-1]$ it is sufficient to verify that if $x_j = 0$, then $\varphi^j \equiv 0$, because then we will have, using (17) and $\bar{U}[k, x] = \bar{U}[k, x]$, that $(U - \bar{U})x = -\sum_{j}x_j \varphi^j(x_j) = -\sum_{j}x_j \varphi^j(x_j) = -\sum_{j}x_j \varphi^j(x_j) = -\sum_{j}x_j \varphi^j(x_j)$. The situation with $x_j = 0$ and $\varphi^j \neq 0$ is impossible because if $\varphi^j \neq 0$, then (18) gives $\eta_j = 0$, and according to (20) and (12), we can obtain the contradiction: $|x_j| \geq |\varphi^j[0-[k-1]] - (|\varphi^j|^2 P^0-[k-1] \zeta^0[k-1]|)^\perp \bar{P}^0-[k-1]e| \geq |\varphi^j[0-[k-1]] - (|\varphi^j|^2 P^0-[k-1] \zeta^0[k-1]|)^\perp \bar{P}^0-[k-1]e|$. Thus we have (19) indeed.

Let $x[j]$ be the solution of (7) that corresponds to $x[0] = x_0 \in \mathcal{P}^0[-k, 0]$, to the control strategies $U = U[k, x]$ from (17) and $\hat{u} = \hat{u}[k, x]$ from (14), (15) or (14), (16), and to arbitrary admissible $v[-1]$ and $V[-1] + \Delta V[-1]$ (i.e., $v[k] = q[k] + Q[k] \chi[k]$, $||\chi[k]||_{\infty} \leq 1; Abs(\Delta V[k]) \leq V[k]$ for all $k$). Let us represent vectors $x[k]$ in the form $x[k] = p^-[k]+\bar{P}^-[k]\zeta[k]$. We prove by induction that if $x = x[k-1] \in \mathcal{P}^-[k-1]$, then $||\zeta[k]||_{\infty} \leq 1, i.e., x[k] \in \mathcal{P}^-[k]$. It follows from (7), (9), (20), and (4) that

$$\zeta[k] = \bar{P}^-[k]^{-1}(x[k] - p^-[k])$$

$$= \bar{P}^-[k]^{-1}(D[k] + U[k, x] - \bar{U}[k] + \Delta V[k]) + B[k]u[k, x] + C[k]v[k]$$

$$- (D[k]p^0-[k-1] + B[k]r[k] + C[k]q[k]))$$

$$= \bar{P}^-[k]^{-1}(D[k]\bar{P}^0-[k-1] \zeta^0[k-1] + U[k, x] - \bar{U}[k]) + B[k]u[k, x] - r[k]) + c[k, x],$$

where $c[k, x] = \bar{P}^-[k]^{-1}\Delta V[k] x + \bar{P}^-[k]^{-1}C[k]Q[k]\chi[k]$. Note that due to $||\chi[k]||_{\infty} \leq 1$ we have for $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^0-[k-1]$ that both pairs of formulas (14), (15) and (14), (16) give one and the same equality $u[k, x] = r[k] + R[k]\Gamma[k]\bar{P}^0-[k-1](x - p^0-[k-1])$. Taking into account (19), (20), (10), (11), we can conclude that

$$\zeta[k] = \bar{P}^-[k]^{-1}(D[k]\bar{P}^0-[k-1] \zeta^0[k-1] - \Phi[k, \bar{P}^0-[k-1]](x - p^0-[k-1])$$

$$+ B[k]R[k]\Gamma[k]\bar{P}^0-[k-1]^{-1}(x - p^0-[k-1])) + c[k, x]$$

$$= \bar{P}^-[k]^{-1}((D[k] - \Phi[k, \bar{P}^0-[k-1]])\bar{P}^0-[k-1] + B[k]R[k]\Gamma[k]\bar{P}^0-[k-1] + c[k, x]$$

$$= \text{diag}(e - \beta[k, \bar{P}^0-[k-1]] - \gamma[k]) \zeta^0[k-1] + c[k, x].$$

Using the relations $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^0-[k-1]$ and arguing similarly to the end of the proof of [20, Theorem 5.1], we obtain $||\zeta[k]||_{\infty} \leq e, i.e., \zeta[k] \in \mathcal{P}^-[k]$ indeed.

In [19], the technique was proposed for solving Problem 2 with $u \equiv 0$ without SC using a parametric family of polyhedral tubes with one parameter $J[\cdot]$. The following Corollary 1 from Theorem 3.1 (where $L^*$ serves as an analogue of $J$) provides the extension of the mentioned technique for the case of systems under SC and also shows that Theorem 3.1 generally gives more rich family of tubes $\mathcal{P}^-[\cdot]$.

**Corollary 1.** The analogue of Theorem 3.1 is true if we consider a family of the tubes $\mathcal{P}^-[\cdot]$ depending on three parameters $L^*[\cdot]$, $P^-[\cdot]$, and $p^-[\cdot]$, where $L^*[\cdot] = \mathcal{P}^-[\cdot]$. 

\[ \{l_1^*[k], \ldots, l_n^*[k] \} \] are permutations such that \( \Pi(L^*[k]) = \Pi(L[k])^\top \), and put \( \Omega[k] = \Pi(L^*[k])^\top \). In this case matrices \( \Phi \) from (12) become diagonal of the following form

\[ \Phi[k, P] = \text{diag} \alpha[k, P], \quad \alpha_i[k, P] = \alpha_i[k, P; L^*[k]] = \hat{u}_i^*[k] \eta_i^*[k, P] \kappa_i(P). \]

**Remark 1.** For systems without matrix controls (\( U[\cdot] = 0 \)) and without matrix uncertainties (cases (A), (B,i)), matrices \( \bar{P}^0[\cdot] = \bar{V}[\cdot] \) are determined by the explicit formulas, and parameters \( \Omega[\cdot] \) and \( L[\cdot] \) disappear.

**Remark 2.** Let the discrete-time system be obtained by the Euler approximations\(^1\) of some differential equations: \( A[k]=I+h_N A(t_{k-1}), \) \( \bar{U}[k] = h_N \bar{U}(t_{k-1}), \) \( \bar{U}[k] = h_N \bar{U}(t_{k-1}), \ldots, B[k]=h_N B(t_{k-1}), \) \( R[k]=R(t_{k-1}), \) \( t_k = kh_N \in [0, \theta], \) \( h_N = \theta N^{-1}. \) Let, for a fixed \( k, \) det \( \bar{P}^-[k] \neq 0 \) and the time step \( h_N \) be sufficient small. Then

(I) The operator \( H[k, P] \) from (11) is contractive, and therefore the equation \( \bar{P} = \bar{P}^0[\cdot] \) has a unique solution \( \bar{P} = \bar{P}^0[-k-1], \) which can be found by the simple iteration \( \bar{P}^{l+1} = H[k, P^l], \) \( l = 0, 1, \ldots, \) starting from \( \bar{P}^0 = \bar{P}^-[k] \). Also, we have \( e^{-\beta[k, P^0[-k-1]] - \gamma[k]} > 0. \) But certainly we can not derive from here the existence of nonsingular matrices \( \bar{P}^-[k] \) for all \( k = N, \ldots, 1. \)

(II) The parameters \( \Gamma[k] \) and \( \Omega[k] \) may be constructed by some special formulas using arguments of “local” volume optimization similarly to [15, 21].

**Remark 3.** Let us mention some heuristic ways to choose parameters \( \Omega \) and \( L: \)

(i) “by hand”, specify some admissible functions \( \Omega \) and \( L \) (for example, constant);

(ii) specify a function \( L \) and set \( \Omega \equiv \Pi(L); \)

(iii) specify a function \( \Omega \) (for example, \( \Omega \equiv E \)) and find values \( L[k] \) for each \( k \) by solving (using a finite search) optimization problems of the type \( f(L, \Omega) \rightarrow \max_L, \) where \( f = \log|\bar{P}^0[-k-1]|; \)

(iv) put \( \Omega = \Omega(L) = \Pi(L) \) and find \( L \) (and, consequently, \( \Omega \)) by solving a problem of the type \( f(L, \Omega(L)) \rightarrow \max_L \) similarly to the above case (iii).

**Remark 4.** With an unsuccessful choice of admissible parameters in formulas (8)-(13), it is not excluded a case when we can obtain the empty set \( \bar{P}^-[k] \) for some \( k \) and therefore we can not construct the solution of Problem 2 using such parameters.

We would like to emphasize once more a relatively simplicity of the proposed method. It may be efficient (in the sense of a number of operations and of memory consumption) for systems with not only small dimensions. Namely, the controls can be calculated by explicit formulas on the base of a polyhedral solvability tube. Cross-sections of such tubes are parallelotopes in \( \mathbb{R}^n \) whose centers and shape are determined by only \( n + n^2 \) numbers in contrast to maximal solvability tubes whose cross-sections are polytopes of general form, where the number of vertices and faces may increase greatly from each time step \( k \) to the next one, in particular due to the operation of the Minkowsky sum\(^2\). A whole number of examples shows that the method of simple iteration can be successfully applied for solving nonlinear matrix equations (10) not only for systems of the important type described in Remark 2, where a small number of iterations can be required (see, for examples, Sec. 4 below), but also for other types of systems (see, for example, [22, Example 1]). As for comparing a quality of estimates produced by ellipsoidal and polyhedral techniques

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\(^1\) Convergence questions are not considered in this paper; in this connection the paper [9] may be mentioned which analyzes the Euler approximations in a set-valued framework.

\(^2\) Some considerations of efficiency of calculating parallelotope-valued reachable rubes are given in [16, Sec. 9]; see also [30] about efficiency of the method for the case without SC on short time intervals.
in the sense of “size” (volume), there are examples, where polyhedral estimates can be better, but opposite examples exist too (see, in particular, [18, ref. 11]).

4. Examples. Some results of computer simulations under different assumptions and types of controls can be found in [19, 20, 22], including a couple of examples that have motivations arising from applied areas and deal with a system which may be interpreted as a hypothetical control ecological system [22, Example 1] and with a system which may be interpreted as a variant of the Richardson model known in political science [22, Example 2].

Here we consider an example of polyhedral control synthesis that illustrates Theorem 3.1 and demonstrates the new family of polyhedral solvability tubes described by relations (8)-(13). The considered system is obtained by the Euler approximations of a differential one considered on a time interval [0, \theta].

Let \( A \equiv I, \bar{U} \equiv \tau \begin{bmatrix} 0 & 0.5 \\ -4 & 0 \end{bmatrix}, \bar{U} \equiv 0 \) or \( \bar{U} \equiv \tau \begin{bmatrix} 1 & 0.5 \\ 1 & 0 \end{bmatrix}, \bar{V} \equiv 0, \bar{V} \equiv 0 \) or \( \bar{V} \equiv \tau \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, B \equiv \tau (0, 1)^T, \mathcal{R} \equiv \{0\} \) or \( \mathcal{R} \equiv \mathcal{P}(0, I, 1), C \equiv \tau (1, 0)^T, \mathcal{Q} \equiv \{0\} \) or \( \mathcal{Q} \equiv \mathcal{P}(0, I, 0, 1), \mathcal{M} \equiv \mathcal{P}((1, 0)^T, I, (0.1, 0.1)^T), \tau = \theta/N, \theta = 2, N=200 \). We use the following values of parameters of the tubes: \( \Gamma[\cdot] \) are constructed using arguments of “local” volume optimization [15]; \( P^{-}[k] = P^{0-}[k], k = 1, \ldots, N; p^{-}[k] \) are constructed by the Nelder-Mead method to maximize \( \text{vol} P_{p^{-}[k], P^{-}[k]}(P^{0-}[k] \cap \mathcal{Y}[k]) \) under above \( P^{-}[k] \). Eight pairs of \( \Omega[\cdot] \) and \( L[\cdot] \) were used, namely, 7 following constant values of \( \Omega[\cdot] \) were taken: \( \Omega \equiv E, \Omega \equiv \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \Omega \equiv \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}, \Omega \equiv \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \), and corresponding values of \( L[\cdot] \) were constructed by Remark 3 (iii), and the eighth pair of \( \Omega[\cdot] \) and \( L[\cdot] \) was found by Remark 3 (iv) (in figures, the corresponding \( P^{-}[0] \) are shown by bold lines). Three initial points \( x_0 = (-1.4, 1)^T, x_0 = (-1, 1)^T, x_0 = (-0.45, 2)^T \) were considered.

In Figures 1–4 we present results for three cases: case (A) without SC, case (B,ii) (under both additive and matrix uncertainties, where we put \( r[\cdot] \) similar to [15] and \( V[\cdot] \equiv \bar{V}[\cdot] + \bar{V}[\cdot] \)) without SC, and analogues case (B,ii;SC) under SC \( |x_2| \leq 2.2 \). We show state constraints by dash-dot lines. In three parts of Figure 1, which

![Figure 1](image-url)
corresponds to case (A), we present several paralleloptopes $P^{-}[0]$ (i.e. polyhedral solvability sets at time $k = 0$) and controlled trajectories for 3 different cases when we have only control $u$, or only control $U$, or else both controls $u$ and $U$. As expected, the union of polyhedral solvability sets for the third case is larger.
Figures 2–4 are obtained by using both $u$ and $U$. At the top of Figure 2, we show several paralleloptopes $P^{-}[0]$ and trajectories, which were constructed for cases (A), (B,ii), and (B,ii;SC) respectively. At the bottom of Figure 2, we once again present trajectories for $x_0 = (-0.45, 2)^T$ and also several cross-sections $P^{-}[k]$ of the used polyhedral solvability tubes to show the dynamics of cross-sections; the points of trajectories at the corresponding times are marked. As expected, the polyhedral solvability sets for the cases with uncertainties and SC turn out to be smaller than for the first one. Note that if $x_0 \notin \bigcup P^{-}[0]$, then we use the tube $P^{-}[\cdot]$ with the nearest $P^{-}[0]$, but there is no guarantee that the trajectory can be steered into $M$ without violating SC (if any) by the corresponding control under any disturbances. But we obtain that if points $x_0$ belong to some of $P^{-}[0]$, then the obtained controlled trajectories together with the corresponding tubes $P^{-}[\cdot]$ satisfy SC (if any) and reach the target set, which agrees with Theorem 3.1. Figures 2 (bottom), 3 and 4 correspond to $x_0 = (-0.45, 2)^T$.

It worth to be mentioned that the method of simple iteration was used for solving nonlinear equations (10), and it was sufficient a small number of iterations (approximately from 5 to 10 for case (B,ii;SC)).

5. Conclusion. The problem of feedback terminal target control for bilinear discrete-time uncertain systems under state constraints is considered. Polyhedral control synthesis using polyhedral (paralleloptope-valued) solvability tubes is elaborated. This technique provides guaranteed results under set-bounded uncertainties which appear in additive and matrix terms in right hand sides of the system equations. The method allows to calculate guaranteed polyhedral solvability tubes in advance (similarly to constructing maximal solvability tubes) and then to construct control strategies by explicit formulas using these tubes. Although the proposed polyhedral solvability tubes may turn out to be smaller than maximal solvability tubes, they are useful, especially on not too long time intervals, because we can rather easily calculate them, while it is hard to calculate maximal solvability tubes.

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