Finite Element Approximation and Analysis of Viscoelastic Wave Propagation with Internal Variable Formulations

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Abstract

We consider linear scalar wave equations with a hereditary integral term of the kind used to model viscoelastic solids. The kernel in this Volterra integral is a sum of decaying exponentials (The so-called Maxwell, or Zener model) and this allows the introduction of one of two types of families of internal variables, each of which evolve according to an ordinary differential equation (ODE). There is one such ODE for each decaying exponential, and the introduction of these ODEs means that the Volterra integral can be removed from the governing equation. The two types of internal variable are distinguished by whether the unknown appears in the Volterra integral, or whether its time derivative appears; we call the resulting problems the displacement and velocity forms. We define fully discrete formulations for each of these forms by using continuous Galerkin finite element approximations in space and an implicit ‘Crank-Nicolson’ type of finite difference method in time. We prove stability and \textit{a priori} bounds, and (using the FEniCS environment, https://fenicsproject.org/) give some numerical results. These bounds do not require Grönwall’s inequality and so can be regarded to be of high quality, allowing confidence in long time integration without an \textit{a priori} exponential build up of error. As far as we are aware this is the first time that these two formulations have been described together with accompanying proofs of such high quality stability and error bounds. The extension of the results to vector-valued viscoelasticity problems is straightforward and summarised at the end. The numerical results are reproducible by acquiring the python sources from https://github.com/Yongseok7717, or by running a custom built docker container (instructions are given).

Keywords: viscoelasticity, finite element method, internal variables, \textit{a priori} estimates

1. Introduction

Materials that exhibit both elastic and viscous response to imposed load and/or deformation are called viscoelastic. Typical examples of such solid materials are amorphous polymers, soft biotissue, metals at high temperatures and even concrete [1]. The mathematical description of the dynamic response of these materials uses a momentum balance law to relate external forces, \( f \), to acceleration, \( \ddot{u} \), and stress divergence, \( \nabla \cdot \sigma \). A faithful mathematical model of this physical set-up would require the introduction of a vector-valued partial differential equation as in elastodynamics (see e.g. [2, 3]) but we restrict ourselves here to a scalar analogue to keep the exposition as simple as possible (but we use the terminology of solid mechanics). So, with

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that in mind, once boundary data for displacement, $u$, and/or traction, $g$, together with initial
displacement and velocity data are specified, the physical problem is exemplified by the following
mathematical model: find $u: [0, T] \times \Omega \rightarrow \mathbb{R}$ such that

\begin{align*}
\rho \ddot{u} - \nabla \cdot \sigma &= f \quad \text{in} \quad (0, T) \times \Omega, \\
u &= 0 \quad \text{on} \quad [0, T] \times \Gamma_D, \\
\sigma \cdot n &= g_N \quad \text{on} \quad [0, T] \times \Gamma_N, \\
\dot{u} &= u_0 \quad \text{on} \quad \{0\} \times \Omega, \\
\ddot{u} &= \dot{u}_0 \quad \text{on} \quad \{0\} \times \Omega,
\end{align*}

where $\sigma$ is the stress; $\Omega$ is an open bounded polytopic domain in $\mathbb{R}^d$ with constant mass density
$\rho$; $\Gamma_D$ and $\Gamma_N$ are the ‘Dirichlet’ and ‘Neumann’ boundaries and $T > 0$ is a final time. As
usual $\Gamma_D$ and $\Gamma_N$ are disjoint and we will assume that the surface measure of $\Gamma_D$ is strictly
positive. Note that we use overdots to denote time differentiation: $\dot{u} := \frac{\partial u}{\partial t}$ and $\ddot{u} := \frac{\partial^2 u}{\partial t^2}$. In
classical continuum mechanics, the physical model is defined with a displacement vector so the
stress is a second order tensor and is defined by a constitutive relationship with the strain tensor.
Hence, in general, the linear viscoelastic dynamic equation is a vector-valued PDE of which the
above is a scalar analogue. However, (1.1) is not only a scalar analogue but also represents the
mathematical model of viscoelastic materials subjected to antiplane shear response. Antiplane
strain, for instance, allows us to reduce the second order tensor to a vector so that the viscoelastic
antiplane model in 3D can be dealt with by the scalar wave problem in 2D (see e.g. [4, 5, 6]).

The viscoelasticity literature contains a large number of ‘rheologically’ (spring and dashpot)
based phenomenological models (e.g. the Maxwell, Voigt, Kelvin-Voigt, generalised Maxwell,
..., models — see [2, 7] for more details) as well as models based on the fractional calculus —
or ‘power law’ models, see [8]. The spring and dashpot models are the ones of interest here because they give rise to constitutive laws that can be described by Volterra kernels of sums of
decaying exponentials. This, in turn, makes them much better suited to numerical approximation
than the fractional calculus models. We will return to this point below but for now we recall
these constitutive laws in the theory of linear viscoelasticity from [2]. For example, we have the
following equivalent constitutive hereditary laws

\begin{align*}
\sigma(t) &= D\varphi(0) \nabla u(t) - \int_0^t D\varphi_s(t - s) \nabla u(s) ds, \\
\sigma(t) &= D\varphi(t) \nabla u(0) + \int_0^t D\varphi(t - s) \nabla \dot{u}(s) ds,
\end{align*}

where $D$ is a positive constant, $\varphi(t)$ is a stress relaxation function and $\varphi_s(t - s) := \frac{\partial}{\partial s} \varphi(t - s)$. It is easy to see that the two constitutive equations are equivalent by integration by parts.
Now we can complete our model problem by defining the stress relaxation function $\varphi(t)$ and
then substituting for $\sigma$. The generalised Maxwell model produces the following stress relaxation
function (see e.g. [8]),

\begin{equation}
\varphi(t) = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q},
\end{equation}

with $N_\varphi \in \mathbb{N}$, $\varphi(0) = 1$, positive coefficients $\{\varphi_q\}_{q=1}^{N_\varphi}$ and $\{\tau_q\}_{q=1}^{N_\varphi}$.

The form (1.8) will permit us to introduce a way to deal with the Volterra integrals that
avoids any reference to the past ‘history’ of the solution. In this approach the stress $\sigma$ is defined
by using ‘hidden’ or internal variables (e.g. [2, 9]) in place of the Volterra integral that evolve
following an ordinary differential equation. Each of the above forms gives rise to different internal variables and so will be considered separately to give two formulations: the displacement form and the velocity form. For each of these formulations we will approximate the solution and the internal variables using the standard continuous Galerkin Finite Element Method (CGFEM) to discretize in space, and a second order implicit Crank-Nicolson finite difference method for the time discretisation.

The plan of the paper is as follows. In Section 2 we introduce the displacement forms of the internal variables in Subsection 2.1 and the velocity form in Subsection 2.2. The fully discrete approximations are then given in Section 3 where in Subsections 3.1 (displacement form) and 3.2 (velocity form) where we prove stability and a priori error estimates. Grönwall’s lemma is not used for these proofs and so the constants in these bounds do not grow exponentially with time. We can therefore have confidence in these schemes for long-time integration. The proofs require some long and technical calculations and details, and so where it aids presentation of the main ideas and specific arguments we sometimes omit the details. In Section 4 we use the FEniCS environment (see https://fenicsproject.org/) to give the results of some numerical experiments, and explain how the software can be acquired and the results reproduced. We finish in Section 5 with some general comments.

We introduce and use some standard notations so that \( L^p(\Omega) \), \( H^s(\Omega) \) and \( W^{s,p}(\Omega) \) denote the usual Lebesgue, Hilbert and Sobolev spaces. For any Banach space \( X \), \( \| \cdot \|_X \) is the \( X \) norm. For example, \( \| \cdot \|_{L^2(\Omega)} \) is the \( L^2 \) norm induced by the \( L^2 \) inner product which we denote by \( (\cdot, \cdot) \) for the entire domain but for \( S \subset \overline{\Omega} \), \( (\cdot, \cdot)_{L^2(S)} \) is the \( L^2 \) inner product over \( S \). In the case of time dependent functions, we expand this notation such that if \( f \in L^p(0,T;X) \) for some Banach space \( X \), we define

\[
\| f \|_{L^p(0,t_0;X)} = \left( \int_0^{t_0} \| f(t) \|^p_X \, dt \right)^{1/p}
\]

for \( t_0 \leq T \) and \( 1 \leq p < \infty \). When \( p = \infty \), we shall use ‘essential supremum’ norm where \( \| f \|_{L^\infty(0,t_0;X)} = \sup_{0 \leq t \leq t_0} \| f(t) \|_X \). We also use the same notation for vector valued functions in Section 5. For use later we recall the trace inequality,

\[
\| v \|_{L^2(\partial \Omega)} \leq C \| v \|_{H^1(\Omega)}, \text{ for any } v \in H^1(\Omega),
\]

where \( C \) is a positive constant depending only on \( \Omega \) and its boundary.

### 2. Weak formulations

Our first step is to define the test space \( V \),

\[
V = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \right\}.
\]

Then, multiplying (1.1) by \( v \in V \), integrating by parts and using the boundary data gives, in a standard way, that

\[
(\rho \ddot{u}(t), v) + (\sigma(t), \nabla v) = F_d(t; v)
\]

for all \( v \in V \) where the linear form \( F_d(\cdot) \) is defined by

\[
F_d(t; v) = (f(t), v) + (g_N(t), v)_{L^2(\Gamma_N)}.
\]

We now need to substitute for the stress using either the displacement or velocity forms.
2.1. Displacement form

Recalling (1.6) and (1.8) we write

\[ \sigma(t) = D \nabla \left( u(t) - \sum_{q=1}^{N_{\varphi}} \psi_q(t) \right), \quad (2.2) \]

with constant \( D > 0 \) and where, for \( 1 \leq q \leq N_{\varphi} \), the internal variables are defined by

\[ \psi_q(t) := \frac{\varphi_q}{\tau_q} \int_0^t e^{-(t-s)/\tau_q} u(s) \, ds, \quad (2.3) \]

where \( N_{\varphi} \in \mathbb{N} \), \( \tau_q \) and \( \varphi_q \) are positive and we recall the normalization \( \varphi(0) = 1 \). The following ODE’s for the internal variables follow easily,

\[ \dot{\psi}_q(t) = \frac{\varphi_q}{\tau_q} u(t) - \frac{1}{\tau_q} \psi_q(t) \quad \text{for} \quad q = 1, 2, \ldots, N_{\varphi}, \quad (2.4) \]

with \( \psi_q(0) = 0 \). Our weak formulation (2.1) can now be written as

\[ \begin{aligned}
(\rho \ddot{u}(t), v) + a(u(t), v) - \sum_{q=1}^{N_{\varphi}} a(\psi_q(t), v) &= F_d(t; v) \quad \forall v \in V.
\end{aligned} \quad (2.5) \]

In this the symmetric bilinear form \( a(\cdot, \cdot) \) is defined by \( a(w, v) = (D \nabla w, \nabla v) \). It follows from our assumption on \( \Gamma_D \), with Friedrichs and Poincare type inequalities that this bilinear form is coercive on \( V \), \cite{10}, and so we may define \((V, a(\cdot, \cdot))\) to be an inner product space equivalent to \((H^1(\Omega), (\cdot, \cdot)_{H^1(\Omega)})\) and define the energy norm \( \|v\|_V := \sqrt{a(v, v)} \) for all \( v \in V \). We then have the norm equivalence \( \kappa \|v\|_{H^1(\Omega)}^2 \leq \|v\|_V^2 \leq D \|v\|_{H^1(\Omega)}^2 \) for a positive constant \( \kappa \).

We use the energy inner product to enforce the internal variable ODE and then arrive at the following weak problem.

**P1** Find \([0, T] \to V \) maps \( u \) and \( \{\psi_q\}_{q=1}^{N_{\varphi}} \) such that

\[ \begin{aligned}
(\rho \ddot{u}(t), v) + a(u(t), v) - \sum_{q=1}^{N_{\varphi}} a(\psi_q(t), v) &= F_d(t; v) \quad \forall v \in V, \\
\tau_q a(\dot{\psi}_q(t), v) + a(\psi_q(t), v) &= \varphi_q a(u(t), v) \quad \forall v \in V, \quad q = 1, \ldots, N_{\varphi}
\end{aligned} \quad (2.6) \]

with \( u(0) = u_0, \dot{u}(0) = w_0 \) and \( \psi_q(0) = 0, \forall q \in \{1, \ldots, N_{\varphi}\} \).

2.2. Velocity form

On the other hand, using (1.7) and (1.8) with the velocity form of internal variable given by

\[ \zeta_q(t) = \int_0^t \varphi_q e^{-(t-s)/\tau_q} \dot{u}(s) \, ds, \]

for each \( q = 1, \ldots, N_{\varphi} \), we have

\[ \zeta_q(t) + \frac{1}{\tau_q} \zeta_q(t) = \varphi_q \dot{u}(t) \quad \text{for} \quad q = 1, 2, \ldots, N_{\varphi}, \quad (2.8) \]
with $\zeta_q(0) = 0$. Noticing that $\psi_q(t) = \varphi_q u(t) - \varphi_q e^{-t/\tau} u_0 - \zeta_q(t)$ (integrate by parts) and recalling that $\varphi(0) = 1$ we can observe that

$$u(t) = \sum_{q=1}^{N_q} \frac{\psi_q(t)}{\tau_q} = \varphi_0 u(t) + \sum_{q=1}^{N_q} \left( \varphi_q e^{-t/\tau} u_0 + \zeta_q(t) \right)$$

and, consequently, we can rewrite (2.2) with the above result so that (2.1) gives the weak problem for the velocity form with the auxiliary weak form from (2.8).

(P2) Find $(0, T] \rightarrow V$ maps $u$ and $\{\zeta_q\}_{q=1}^{N_q}$ such that

$$(\rho \ddot{u}(t), v) + \varphi_0 a(u(t), v) + \sum_{q=1}^{N_q} a(\zeta_q(t), v) = F_v(t; v) \quad \forall v \in V,$$

$$\tau_q a(\dot{\zeta}_q(t), v) + a(\zeta_q(t), v) = \tau_q \varphi_q a(\dot{u}(t), v) \quad \forall v \in V, \quad q = 1, \ldots, N_q,$$

(2.9)

with $u(0) = u_0$, $\dot{u}(0) = w_0$, $\zeta_q(0) = 0$, $\forall q$ and $F_v(t; v) = F_d(t; v) - \sum_{q=1}^{N_q} \varphi_q e^{-t/\tau} a(u_0, v)$. \vspace{5mm}

3. Fully discrete formulation

Let $V^h \subset V$ be a conforming finite element space built with continuous piecewise Lagrange basis functions with respect to an underlying quasi-uniform mesh. We write $t_n = n\Delta t$ with $\Delta t = T/N$ for $N \in \mathbb{N}$, and denote the fully discrete approximations to $u$ and $\dot{u}$ by $u(t_n) = u^n \approx Z^n_h \in V^h$ and $\dot{u}(t_n) = \dot{u}^n \approx W^n_h \in V^h$. Furthermore, we will use the following approximations,

$$\frac{\ddot{u}(t_{n+1}) + \ddot{u}(t_n)}{2} \approx \frac{W^{n+1}_h - W^n_h}{\Delta t} \quad \text{and} \quad \frac{u(t_{n+1}) + u(t_n)}{2} \approx \frac{Z^{n+1}_h + Z^n_h}{2},$$

and will impose the relation

$$\frac{W^{n+1}_h + W^n_h}{2} = \frac{Z^{n+1}_h - Z^n_h}{\Delta t},$$

(3.1)

in the fully discrete schemes that follow.

Remark: Let us recall Grönwall’s inequality from [11, 12]. The discrete Grönwall’s inequality is given in [12] as follows. If for $n \in \mathbb{N}$ and non-negative sequences $(a_n), (b_n)$ and $(g_n)$ we have

$$a_n \leq b_n + \sum_{i=0}^{n-1} g_i a_i,$$

then

$$a_n \leq b_n + \sum_{i=0}^{n-1} g_i b_i \exp \left( \sum_{j=i}^{n-1} g_j \right).$$

Grönwall inequality in both continuous and discrete versions is commonly used in a priori estimates for time dependent problems (see e.g. [3, 13, 14]). This is a convenient result but can often be pessimistic in that the constants appearing in the final bound are exponentially large in the final time $T$. In our proofs below we take care to avoid using this argument and obtain sharper constants.
3.1. Displacement form

Our fully discrete formulation for (P1) is:

(P1) Find \( Z^n_h, W^n, \Psi^n_h, \Psi^n_{hN}, \Psi^n_{hNq} \in V^h \) for \( n = 0, \ldots, N \) such that (3.1) holds along with:

\[
\left( \frac{W^{n+1}_h - W^n_h}{\Delta t}, v \right) + a \left( \frac{Z^{n+1}_h + Z^n_h}{2}, v \right) - \sum_{q=1}^{N_q} \left( \frac{\Psi^{n+1}_{hq} + \Psi^n_{hq}}{2}, v \right) = F_d(t_{n+1}; v) + F_d(t_n; v),
\]

\[
\tau_q a \left( \frac{\Psi^{n+1}_{hq} - \Psi^n_{hq}}{\Delta t}, v \right) + a \left( \frac{\Psi^{n+1}_{hq} + \Psi^n_{hq}}{2}, v \right) = \varphi_q a \left( \frac{Z^{n+1}_h - Z^n_h}{2}, v \right) \quad \text{for each } q,
\]

\[
a(\Psi^n_{hq}, v) = a(u_0, v),
\]

\[
W^n_h, v = (u_0, v),
\]

\[
\Psi^n_{hq} = 0 \quad \text{for each } q,
\]

each for all \( v \in V^h \).

It follows immediately that \( \| Z^n_0 \|_V \leq \| u_0 \|_V \) and and \( \| W^n_0 \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} \), and we have the following stability estimate.

**Theorem 3.1.** Suppose \( f \in C(0,T; L^2(\Omega)) \), \( g_N \in H^1(0,T; L^2(\Gamma_N)) \cap C(0,T; L^2(\Gamma_N)) \) and \( u_0 \in H^1(\Omega) \). Then (P1) has a unique solution. Moreover, there exists a positive constant \( C \) depending on \( \Omega, \partial \Omega \) and sets \( \{ \varphi_q \}_{q=0}^{N_q} \) and \( \{ \tau_q \}_{q=1}^{N_q} \), but independent of numerical solutions, \( h, \Delta t \) and \( T \) such that

\[
\frac{\rho}{2} \max_{0 \leq n \leq N} \| W^n_h \|_{L^2(\Omega)}^2 + \sum_{q=1}^{N_q} \frac{2\tau_q}{\Delta t^2\varphi_q} \| \Psi^{n+1}_{hq} - \Psi^n_{hq} \|_V^2
\]

\[
\leq C T^2 \left( \| w_0 \|_{L^2(\Omega)}^2 + \| u_0 \|_V^2 + \| f \|_{L^\infty(0,T; L^2(\Omega))}^2 + \| g_N \|_{L^2(0,T; L^2(\Gamma_N))}^2
\]

\[
+ \| g_N \|_{L^\infty(0,T; L^2(\Gamma_N))}^2 \right).
\]

**Proof.** The existence and uniqueness follows from the stability bounds, so we only have to establish that. Let \( m \in \mathbb{N} \) such that \( 1 \leq m \leq N \). Put \( v = W^{n+1}_h + W^n_h \) for \( 0 \leq n \leq m - 1 \) into (3.2) and \( v = \Psi^{n+1}_{hq} - \Psi^n_{hq} \) into (3.3) for each \( q \), respectively, add results and take the summation from \( n = 0 \) to \( n = m - 1 \). We obtain

\[
\rho \| W^m_h \|_{L^2(\Omega)}^2 + \| Z^m_h \|_V^2 + \sum_{q=1}^{N_q} \frac{1}{\varphi_q} \| \Psi^{m}_{hq} \|_V^2 + \sum_{q=1}^{N_q} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t^2\varphi_q} \| \Psi^{n+1}_{hq} - \Psi^n_{hq} \|_V^2
\]

\[
= \| W^0_0 \|_{L^2(\Omega)}^2 + \| Z^n_0 \|_V^2 + \sum_{n=0}^{m-1} \left( F_d(t_{n+1}; W^{n+1}_h + W^n_h) + F_d(t_n; W^{n+1}_h + W^n_h) \right)
\]

\[
+ \sum_{q=1}^{N_q} 2a(\Psi^{m}_{hq}, \Psi^n_{hq}),
\]
after noting that
\[
\frac{\Delta t}{2} a \left( \Psi_{bq}^{n+1} + \Psi_{bq}^{n} W_{h}^{n+1} + W_{h}^{n} \right) + a \left( \Psi_{bq}^{n+1} - \Psi_{bq}^{n}, Z_h^{n+1} + Z_h^{n} \right) \\
= 2a \left( \Psi_{bq}^{n+1}, Z_h^{n+1} \right) - 2a \left( \Psi_{bq}^{n}, Z_h^{n} \right) .
\]

First of all we will consider \( \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_{n}; W_{h}^{n+1} + W_{h}^{n})) \). By the definition and (3.1)
\[
\frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_{n}; W_{h}^{n+1} + W_{h}^{n})) \\
= \frac{\Delta t}{2} \sum_{n=0}^{m-1} (f(t_{n+1}) + f(t_n), W_{h}^{n+1} + W_{h}^{n}) \\
+ \sum_{n=0}^{m-1} (g_N(t_{n+1}) + g_N(t_n), Z_h^{n+1} - Z_h^{n})_{L_2(\Gamma_N)} ,
\]
Summing by parts yields
\[
\sum_{n=0}^{m-1} (g_N(t_{n+1}) + g_N(t_n), Z_h^{n+1} - Z_h^{n})_{L_2(\Gamma_N)} = 2 (g_N(t_m), Z_h^{m})_{L_2(\Gamma_N)} - 2 (g_N(t_0), Z_h^{0})_{L_2(\Gamma_N)} \\
- \sum_{n=0}^{m-1} (g_N(t_{n+1}) - g_N(t_n), Z_h^{n+1} + Z_h^{n})_{L_2(\Gamma_N)}.
\]
Here, since \( g_N \) is continuous and differentiable in time, we can obtain
\[
g_N(t_{n+1}) - g_N(t_n) = \int_{t_n}^{t_{n+1}} \dot{g}_N(s)ds,
\]
and then we have by Leibniz’s integral rule,
\[
\sum_{n=0}^{m-1} (g_N(t_{n+1}) - g_N(t_n), Z_h^{n+1} + Z_h^{n})_{L_2(\Gamma_N)} = \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} (\dot{g}_N(s), Z_h^{n+1} + Z_h^{n})_{L_2(\Gamma_N)} ds.
\]
Hence we have
\[
\frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_{n}; W_{h}^{n+1} + W_{h}^{n})) \\
= \frac{\Delta t}{2} \sum_{n=0}^{m-1} (f(t_{n+1}) + f(t_n), W_{h}^{n+1} + W_{h}^{n}) + 2 (g_N(t_m), Z_h^{m})_{L_2(\Gamma_N)} - 2 (g_N(t_0), Z_h^{0})_{L_2(\Gamma_N)} \\
- \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} (\dot{g}_N(s), Z_h^{n+1} + Z_h^{n})_{L_2(\Gamma_N)} ds.
\]
By applying the Cauchy-Schwarz inequality and the trace estimate from (1.9) we get,
\[
\left| \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_{n}; W_{h}^{n+1} + W_{h}^{n})) \right|
\]
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\[ \frac{\Delta t}{2} \sum_{n=0}^{m-1} |f(t_{n+1}) + f(t_n)|_{L^2(\Omega)} \| W_{h}^{n+1} + W_{h}^{n} \|_{L^2(\Omega)} + 2C \| g_{N}(t_m) \|_{L^2(\Gamma_N)} \| Z_{h}^{m} \|_{V} \]

\[ + 2C \| g_{N}(t_0) \|_{L^2(\Gamma_N)} \| Z_{h}^{0} \|_{V} + C \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} |\dot{g}_{N}(s)|_{L^2(\Gamma_N)} \| Z_{h}^{n+1} + Z_{h}^{n} \|_{V} \, ds, \]

for a positive constant $C$. Young’s inequality and the triangle inequality then give,

\[ \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_n; W_{h}^{n+1} + W_{h}^{n})) \]

\[ \leq \frac{2\Delta t}{\epsilon_a} \sum_{n=0}^{m} \| f(t_n) \|_{L^2(\Omega)}^2 + 2\Delta t \epsilon_a \max_{0 \leq n \leq N} \| W_{h}^{n} \|_{L^2(\Omega)}^2 \]

\[ + \frac{C}{\epsilon_b} \| g_{N}(t_n) \|_{L^2(\Gamma_N)}^2 + C \epsilon_b \max_{0 \leq n \leq N} \| Z_{h}^{n} \|_{V}^2 + C \| Z_{h}^{0} \|_{V}^2 \]

\[ + \frac{C}{\epsilon_c} \int_{t_n}^{t_{n+1}} |\dot{g}_{N}(s)|_{L^2(\Gamma_N)}^2 \, ds + 2C \Delta t \epsilon_c \sum_{n=0}^{m} \| Z_{h}^{n} \|_{V}^2, \quad (3.7) \]

for positive $\epsilon_a$, $\epsilon_b$, and $\epsilon_c$. It follows that

\[ \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d(t_{n+1}; W_{h}^{n+1} + W_{h}^{n}) + F_d(t_n; W_{h}^{n+1} + W_{h}^{n})) \]

\[ \leq \frac{2\Delta t}{\epsilon_a} \sum_{n=0}^{N} \| f(t_n) \|_{L^2(\Omega)}^2 + 2(T + \Delta t) \epsilon_a \max_{0 \leq n \leq N} \| W_{h}^{n} \|_{L^2(\Omega)}^2 \]

\[ + \left( \frac{C}{\epsilon_b} + 1 \right) \max_{0 \leq n \leq N} \| g_{N}(t_n) \|_{L^2(\Gamma_N)}^2 + C \epsilon_b \max_{0 \leq n \leq N} \| Z_{h}^{n} \|_{V}^2 + C \| Z_{h}^{0} \|_{V}^2 \]

\[ + \frac{C}{\epsilon_c} \| g_{N} \|_{L^2(0,T; L^2(\Gamma_N))}^2 + 2C(T + \Delta t) \epsilon_c \max_{0 \leq n \leq N} \| Z_{h}^{n} \|_{V}^2. \quad (3.8) \]

Secondly, we get from the Cauchy-Schwarz and Young’s inequalities that

\[ \sum_{q=1}^{N_{\tilde{\Psi}}} 2\alpha(Z_{h}^{n}, \Psi_{h}^{m}) \leq \sum_{q=1}^{N_{\tilde{\Psi}}} \epsilon_q \| Z_{h}^{m} \|_{V}^2 + \sum_{q=1}^{N_{\tilde{\Psi}}} \frac{1}{\epsilon_q} \| \Psi_{h}^{m} \|_{V}^2, \]

for any $\epsilon_q > 0$ for each $q$. If we take $\epsilon_q = \varphi_q = \varphi_0/(2N_{\tilde{\Psi}}) > 0$ for each $q$, we can obtain

\[ \rho \| W_{h}^{m} \|_{L^2(\Omega)}^2 + \frac{\varphi_0}{2} \| Z_{h}^{m} \|_{V}^2 + \sum_{q=1}^{N_{\tilde{\Psi}}} \frac{\varphi_0}{2 \varphi_q^2 \varphi_0 + \varphi_0 \varphi_q} \| \Psi_{h}^{m} \|_{V}^2 + \sum_{q=1}^{N_{\tilde{\Psi}}} \frac{2\gamma_q}{\Delta t \varphi_q} \| \Psi_{h}^{n-1} - \Psi_{h}^{n} \|_{V}^2 \]

\[ \leq \| W_{h}^{0} \|_{L^2(\Omega)}^2 + (1 + C) \| Z_{h}^{0} \|_{V}^2 + \frac{2\Delta t}{\epsilon_a} \sum_{n=0}^{N} \| f(t_n) \|_{L^2(\Omega)}^2 + 2(T + \Delta t) \epsilon_a \max_{0 \leq n \leq N} \| W_{h}^{n} \|_{L^2(\Omega)}^2 \]

\[ + \left( \frac{C}{\epsilon_b} + 1 \right) \max_{0 \leq n \leq N} \| g_{N}(t_n) \|_{L^2(\Gamma_N)}^2 + C \epsilon_b \max_{0 \leq n \leq N} \| Z_{h}^{n} \|_{V}^2 \]

\[ + \frac{C}{\epsilon_c} \| \dot{g}_{N} \|_{L^2(0,T; L^2(\Gamma_N))}^2 + 2C(T + \Delta t) \epsilon_c \max_{0 \leq n \leq N} \| Z_{h}^{n} \|_{V}^2, \]

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since, recalling that $\varphi(0) = 1$, we have
\[
1 - \sum_{q=1}^{N_v} \epsilon_q = \frac{\varphi_0}{2} > 0 \text{ and } \frac{1}{\varphi_q} = \frac{\varphi_0}{2\frac{\varphi^2}{q} N_{\varphi} + \varphi_q \varphi_0} > 0, \forall q \in \{1, \ldots, N_v\}.
\]
Noting that $a_n + b_n + c_n \leq C$ for all $n$ implies $\max_{0 \leq n \leq N} a_n + \max_{0 \leq n \leq N} b_n + c_n \leq 3C$ the result above gives
\[
\rho \max_{0 \leq n \leq N} \|W^0_h\|_{L^2(\Omega)} + \frac{\varphi_0}{2} \max_{0 \leq n \leq N} \|Z^0_h\|_{V}^2 + \sum_{q=1}^{N_v} \frac{\varphi_0}{2\frac{\varphi^2}{q} N_{\varphi} + \varphi_q \varphi_0} \|\psi_q\|_{V}^2
\]
\[
\leq 3 \left( \|W^0_h\|_{L^2(\Omega)}^2 + (1 + C) \|Z^0_h\|_{V}^2 + \frac{2\Delta t}{\epsilon_a} \sum_{n=0}^{N_v} \|f(t_n)\|_{L^2(\Omega)}^2 \right)
\]
\[
+ 2(T + \Delta t)\epsilon_a \max_{0 \leq n \leq N} \|W^0_h\|_{L^2(\Omega)}^2 + \left( \frac{C + 1}{\epsilon_b} \right) \max_{0 \leq n \leq N} \|g_N(t_n)\|_{L^2(\Gamma_N)}^2
\]
\[
+ C \max_{0 \leq n \leq N} \|Z^0_h\|_{V}^2 + \frac{C}{\epsilon_c} \max_{0 \leq n \leq N} \|g_N(t_n)\|_{L^2(0,T;L^2(\Gamma_N))}^2 + 2C(T + \Delta t)\epsilon_c \max_{0 \leq n \leq N} \|Z^0_h\|_{V}^2
\]
Choosing $\epsilon_a = \nu/(12(T + \Delta t))$, $\epsilon_b = \varphi_0/(24C)$ and $\epsilon_c = \varphi_0/(48C(T + \Delta t))$ and recalling that $\|W^0_h\|_{L^2(\Omega)} \leq \|w_0\|_{L^2(\Omega)}$ and $\|Z^0_h\|_{V} \leq \|u_0\|_{V}$, we conclude that there is a positive constant $C$ such that
\[
\frac{\rho}{2} \max_{0 \leq n \leq N} \|W^0_h\|_{L^2(\Omega)}^2 + \frac{\varphi_0}{4} \max_{0 \leq n \leq N} \|Z^0_h\|_{V}^2 + \sum_{q=1}^{N_v} \frac{\varphi_0}{2\frac{\varphi^2}{q} N_{\varphi} + \varphi_q \varphi_0} \|\psi_q\|_{V}^2
\]
\[
\leq CT^2 \left( \|w_0\|_{L^2(\Omega)}^2 + \|u_0\|_{V}^2 + \Delta t \sum_{n=0}^{N_v} \|f(t_n)\|_{L^2(\Omega)}^2 + \|g_n\|_{L^2(0,T;L^2(\Gamma_N))}^2 + \max_{0 \leq n \leq N} \|g_N(t_n)\|_{L^2(\Gamma_N)}^2 \right),
\]
for $0 \leq m \leq N - 1$. Note that the constant bound $C$ is independent of $h$, $\Delta t$, $T$ and solutions since $T + \Delta t \leq 2T$, but depends on trace inequality constant and coefficients of density and internal variables. Since $f$ and $g$ are continuous and bounded, we have
\[
\Delta t \sum_{n=0}^{N_v} \|f(t_n)\|_{L^2(\Omega)}^2 \leq 2T \|f\|_{L^\infty(0,T;L^2(\Omega))} \text{ and } \max_{0 \leq n \leq N} \|g_N(t_n)\|_{L^2(\Gamma_N)}^2 \leq \|g_N\|_{L^\infty(0,T;L^2(\Gamma_N))}^2.
\]
Recalling that $m$ is arbitrary then completes the proof. \[\square\]

**Note** Theorem 3.1 is an example of how the $\epsilon^T$ dependence of the constant that would arise from Grönnwall’s inequality can be avoided for these viscoelasticity problems. The results that follow are of a similar form.

We now turn to error bounds and begin by defining the elliptic projection $R : V \mapsto V^h$ (e.g. [15]) for $w \in V$ by
\[
a(Rw, v) = a(w, v), \forall v \in V^h.
\]
Elliptic regularity is required for optimal $f$ if $w \in V$. The elliptic projection gives Galerkin orthogonality such that for any $w \in V$
\[ a(w - Rw, v) = 0, \ \forall v \in V^h. \]

According to [15], we use the elliptic approximation estimates as follows.

**Lemma 3.1.** (see details in [15, pg 731-732] and [16])

If $w \in H^{s} (\Omega) \subset V$ and $V^h$ is a finite element space of degree $s_1$ in $V$,
\[ \|w - Rw\|_{L_2(\Omega)} \leq C|w|_{H^r(\Omega)}h^{r-1} \text{ and } \|w - Rw\|_V \leq C|w|_{H^r(\Omega)}h^{r-1} \]

where $r := (s_1 + 1, s_2)$ for some positive $C$ and with the spatial mesh size $h$. Furthermore, if there is elliptic regularity, we have $\|w - Rw\|_{L_2(\Omega)} \leq C|w|_{H^r(\Omega)}h^r$.

For details, we refer to [16, Chapter 5.5]. When the domain is neither smooth nor convex, it fails to get higher order with respect to $L^2$ on the domain is an essential condition of elliptic regularity. Moreover, with $\Gamma_N = \emptyset$, elliptic regularity holds for smooth $\Omega$ and convex polytope $\Omega$ e.g. see [17, 18, 16].

**Remark** Elliptic regularity is required for optimal $L_2(\Omega)$ error estimates. The general case for elliptic regularity estimates is given by
\[ \|v\|_{W^p_2(\Omega)} \leq \|\Delta v\|_{L_2(\Omega)}, \quad 1 < p < \mu, \]
where $\mu$ depends on $\partial\Omega$ e.g. see [16, 17, 18]. Therefore, if we assume that elliptic regularity is satisfied, it holds that
\[ |v|_{H^r(\Omega)} \leq \|\Delta v\|_{L_2(\Omega)}, \]
then we have $\|w - Rw\|_{L_2(\Omega)} \leq C|w|_{H^r(\Omega)}h^r$.

Let
\[ \theta := u - Ru, \quad \chi^n := Z_h^n - Ru^n, \quad \omega^n := W_h^n - Ru^n, \]
\[ \vartheta_q := \psi_q - R\psi_q, \quad \varsigma^n_q := \Psi^n_q - R\psi^n_q, \]
for each $q$. Additionally, we define
\[ \epsilon^n_h := u^n - Z_h^n = \theta^n - \chi^n, \quad \text{and } \tilde{\epsilon}^n_h := \tilde{u}^n - W_h^n = \tilde{\theta}^n - \omega^n, \]
and also $\psi^n - \Psi^n_q = \vartheta^n_q - \varsigma^n_q$ for each $q$.

**Lemma 3.2.** Suppose $u \in H^1(0, T; H^r(\Omega)) \cap C^1(0, T; H^{r+2}(\Omega))$. Then
\[
\max_{0 \leq k \leq N} \|\omega^k\|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \|\chi^k\|_V + \sum_{q=1}^{N_x} \max_{0 \leq k \leq N} \|\varsigma^k_q\|_V + \sqrt{\frac{\Delta t}{N_x}} \sum_{n=0}^{N-1} \sum_{q=1}^{N_x} \frac{\tau_q}{\varphi_q} \left\| \frac{\varsigma^{n+1}_q - \varsigma^n_q}{\Delta t} \right\|_V^2 \\
\leq CT \|u\|_{H^1(0, T; H^{r+2}(\Omega))} (h^{r-1} + \Delta t^2).
\]

If we also assume elliptic regularity,
\[
\max_{0 \leq k \leq N} \|\omega^k\|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \|\chi^k\|_V + \sum_{q=1}^{N_x} \max_{0 \leq k \leq N} \|\varsigma^k_q\|_V + \sqrt{\frac{\Delta t}{N_x}} \sum_{n=0}^{N-1} \sum_{q=1}^{N_x} \frac{\tau_q}{\varphi_q} \left\| \frac{\varsigma^{n+1}_q - \varsigma^n_q}{\Delta t} \right\|_V^2 \\
\leq CT \|u\|_{H^1(0, T; H^{r+2}(\Omega))} (h^{r} + \Delta t^2).
\]

Here, a positive constant $C$ depends on $\Omega$, $\partial\Omega$ and other coefficients such as $\rho$, $\{\varphi_q\}_{q=0}^{N_x}$ and $\{\tau_q\}_{q=0}^{N_x}$ but is independent of $h$, $\Delta t$, $T$, and the numerical solution.
Proof. Subtracting average (2.6) at $t_{n+1}$ and $t_n$ from (3.2) gives
\[
\left( \frac{\rho}{2} (\bar{u}^{n+1} + \bar{u}^n) - \frac{\rho}{\Delta t} (W^{n+1}_h - W^n_h), v \right) + \frac{1}{2} a \left( (u^{n+1} + u^n) - (Z^{n+1}_h + Z^n_h), v \right) - \frac{1}{2} \sum_{q=1}^{N_q} a \left( (\psi^{n+1}_q + \psi^n_q) - (\Psi^{n+1}_{hq} + \Psi^n_{hq}), v \right) = 0,
\]
for any $v \in V^h$. Using Galerkin orthogonality, we can rewrite this as
\[
\frac{\rho}{\Delta t} \left( \varpi^{n+1} - \varpi^n, v \right) + \frac{1}{2} a \left( \chi^{n+1} + \chi^n, v \right) - \frac{1}{2} \sum_{q=1}^{N_q} a \left( \varsigma^{n+1}_q + \varsigma^n_q, v \right) = \frac{\rho}{\Delta t} \left( \dot{\varpi}^{n+1} - \dot{\varpi}^n, v \right) + \rho (\mathcal{E}^n_1, v),
\]
for $0 \leq n \leq N - 1$, where
\[
\mathcal{E}_1(t) := \frac{\dot{u}(t + \Delta t) + \ddot{u}(t)}{2} - \frac{\dot{u}(t + \Delta t) - \ddot{u}(t)}{\Delta t}.
\]
Note that by (3.1)
\[
\frac{\chi^{n+1} - \chi^n}{\Delta t} = \frac{Z^{n+1} - Z^n}{\Delta t} - \frac{R u^{n+1} - R u^n}{\Delta t} = \frac{\varpi^{n+1} + \varpi^n}{2} - \mathcal{E}^n_2 - \mathcal{E}^n_3,
\]
where
\[
\mathcal{E}_2(t) := \frac{\dot{\varpi}(t + \Delta t) + \ddot{\varpi}(t)}{2} - \frac{\varpi^{n+1} - \varpi^n}{\Delta t}, \quad \mathcal{E}_3(t) := \frac{u(t + \Delta t) - u(t)}{\Delta t} = \frac{\dot{u}(t + \Delta t) + \ddot{u}(t)}{2},
\]
and so choosing $v = \frac{\chi^{n+1} - \chi^n}{\Delta t}$ in (3.11) and using (3.12), we can derive
\[
\frac{\rho}{2 \Delta t} \left( \| \varpi^{n+1} \|_{L^2(\Omega)}^2 - \| \varpi^n \|_{L^2(\Omega)}^2 \right) + \frac{1}{2 \Delta t} \left( \| \chi^{n+1} \|_{V}^2 - \| \chi^n \|_{V}^2 \right) - \frac{1}{2} \sum_{q=1}^{N_q} a \left( \varsigma^{n+1}_q + \varsigma^n_q, \frac{\chi^{n+1} - \chi^n}{\Delta t} \right)
\]
\[
= \frac{\rho}{2 \Delta t} \left( \dot{\varpi}^{n+1} - \dot{\varpi}^n, \varpi^{n+1} + \varpi^n \right) + \frac{\rho}{2} (\mathcal{E}^n_1, \varpi^{n+1} + \varpi^n) - \rho (\mathcal{E}^n_1, \mathcal{E}^n_2) - \rho (\mathcal{E}^n_1, \mathcal{E}^n_3) + \frac{\rho}{\Delta t} \left( \sigma^{n+1} - \sigma^n, \mathcal{E}^n_2 \right) + \frac{\rho}{\Delta t} \left( \sigma^{n+1} - \sigma^n, \mathcal{E}^n_3 \right) + \frac{\rho}{\Delta t} \left( \mathcal{E}^n_1, \mathcal{E}^n_3 \right).
\]
Summing this over $n = 0, \ldots, m - 1$, where $m \leq N$, we get
\[
\frac{\rho}{2 \Delta t} \left( \| \varpi^m \|_{L^2(\Omega)}^2 - \| \varpi^0 \|_{L^2(\Omega)}^2 \right) + \frac{1}{2 \Delta t} \left( \| \chi^m \|_{V}^2 - \| \chi^0 \|_{V}^2 \right) - \frac{1}{2 \Delta t} \sum_{n=0}^{m-1} \sum_{q=1}^{N_q} a \left( \varsigma^{n+1}_q + \varsigma^n_q, \chi^{n+1} - \chi^n \right)
\]
\[
= \frac{\rho}{2 \Delta t} \left( \| \varpi^0 \|_{L^2(\Omega)}^2 - \| \varpi^m \|_{L^2(\Omega)}^2 \right) + \frac{1}{2 \Delta t} \left( \| \chi^0 \|_{V}^2 - \| \chi^m \|_{V}^2 \right) + \frac{\rho}{2 \Delta t} \sum_{n=0}^{m-1} \left( \dot{\varpi}^{n+1} - \dot{\varpi}^n, \varpi^{n+1} + \varpi^n \right) + \frac{\rho}{\Delta t} \left( \sigma^{n+1} - \sigma^n, \mathcal{E}^n_2 \right) + \frac{\rho}{\Delta t} \left( \sigma^{n+1} - \sigma^n, \mathcal{E}^n_3 \right) + \frac{\rho}{\Delta t} \left( \mathcal{E}^n_1, \mathcal{E}^n_3 \right).
\]
In a similar way, let us consider the difference of (2.7) and (3.3) for each

In addition, we will apply summation by parts such that, since

\[
\frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} \left( \dot{\varphi}^{n+1} - \dot{\varphi}^n, E^m_2 \right) + \frac{\varphi_q}{2} \sum_{n=0}^{m-1} \left( E^m_1, \varpi^{n+1} + \varpi^n \right) - \rho \sum_{n=0}^{m-1} \left( E_1^n, \varpi^n \right) - \rho \sum_{n=0}^{m-1} \left( E_3^n, \varpi^n \right) \\
+ \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} \left( \varpi^{n+1} - \varpi^n, E_2^n \right) + \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} \left( \varpi^{n+1} - \varpi^n, E_3^n \right).
\]

(3.14)

In (3.15) and summing over \( n = 0, \ldots, m - 1 \) then we have

\[
\frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} \| \varpi^{n+1} - \varpi^n \|_V^2 + \frac{1}{2 \Delta t} \left( \| m^n \|_V^2 - \| q_0^n \|_V^2 \right) - \frac{\varphi_q}{2 \Delta t} \sum_{n=0}^{m-1} a \left( \chi^{n+1} + \chi^n, \varpi^n + \varpi_q^n - \varpi_q^n \right) \\
= \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} a \left( E^n_q, \varpi^{n+1} - \varpi^n \right),
\]

where for each \( q \)

\[
E_q(t) := \frac{\dot{\psi}_q(t + \Delta t) + \dot{\psi}_q(t)}{\Delta t} - \psi_q(t + \Delta t) - \psi_q(t).
\]

In addition, we will apply summation by parts such that, since \( \psi_q^0 = 0 \),

\[
\sum_{n=0}^{m-1} a \left( \chi^{n+1} + \chi^n, \varpi^{n+1} - \varpi^n \right) = 2a \left( m^n, m_q^n \right) - \sum_{n=0}^{m-1} a \left( \chi^{n+1} - \chi^n, \varpi^n + \varpi_q^n \right),
\]

and

\[
\sum_{n=0}^{m-1} a \left( E^n_q, \varpi^{n+1} - \varpi^n \right) = a \left( E^{m-1}_q, m_q^n \right) - \sum_{n=0}^{m-2} a \left( E^{n+1}_q - E^n_q, \varpi^n \right),
\]

then it implies

\[
\frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} a \left( \chi^{n+1} - \chi^n, \varpi^{n+1} + \varpi_q^n \right) + \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} \| \varpi^{n+1} - \varpi^n \|_V^2 \\
- \frac{1}{2 \Delta t} \| m^n \|_V^2 + \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-1} a \left( E^{m-1}_q, m_q^n \right) - \frac{\varphi_q}{\Delta t} \sum_{n=0}^{m-2} a \left( E^{n+1}_q - E^n_q, \varpi^n \right)
\]

(3.16)

since \( \psi_q^0 = 0, \forall q \). Using (3.16), multiplying (3.14) by \( \Delta t \) leads us to obtain

\[
\frac{\rho}{2} \| \varpi^m \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \chi^m \|_V^2 + \frac{1}{2} \sum_{q=1}^{N_q} \frac{1}{\varphi_q} \| m_q^m \|_V^2 + \Delta t \sum_{n=0}^{m-1} \frac{\varphi_q}{\Delta t} \left( \varpi^{n+1} - \varpi^n \right) \| \varpi^{n+1} - \varpi^n \|_V^2
\]

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\[= \frac{\rho}{2} \|\varpi^0\|^2_{L^2(\Omega)} + \frac{1}{2} \|\chi^0\|^2_V + \frac{\rho}{2} \sum_{n=0}^{m-1} \left( \frac{\dot{\varpi}^{n+1} - \dot{\varpi}^n}{\Delta t} \varpi^{n+1} + \varpi^n \right) \]

\[- \rho \sum_{n=0}^{m-1} \left( \frac{\dot{\varpi}^{n+1} - \dot{\varpi}^n}{\Delta t} \varpi^n + \varpi^n \right) - \rho \Delta t \sum_{n=0}^{m-1} (E_1^n, E_2^n) - \rho \Delta t \sum_{n=0}^{m-1} (E_1^n, E_3^n) \]

\[+ \rho \sum_{n=0}^{m-1} \left( \frac{\varpi^{n+1} - \varpi^n}{\Delta t} + \rho \sum_{n=0}^{m-1} (\varpi^{n+1} - \varpi^n, E_3^n) + \sum_{q=1}^{N_q} a(\chi, \varsigma_q) \right) \]

\[+ \sum_{q=1}^{N_q} \tau_q a(E_q^{m-1}, \varsigma_q^m) - \sum_{n=0}^{m-2} \sum_{q=1}^{N_q} \tau_q a(E_q^{n+1} - E_q^n, \varsigma_q^{n+1}) \]. \tag{3.17}

Note that elliptic projection and initial conditions imply \(\|\varpi^0\|^2_{L^2(\Omega)} \leq \|\dot{\varpi}^0\|^2_{L^2(\Omega)}\) and \(\|\chi^0\|^2_V = 0\).

On account of error estimates by Crank-Nicolson method, we can derive error bounds of \(\Delta t^2\) for \(E_1^n, E_2^n, E_3^n, E_q^n\) and \(\dot{E}_q^n\). More precisely, using the following argument such that

\[\frac{\dot{w}(t_{n+1}) + \dot{w}(t_n)}{2} - \frac{w(t_{n+1}) - w(t_n)}{\Delta t} = \frac{1}{2\Delta t} \int_{t_n}^{t_{n+1}} w^{(3)}(t)(t_{n+1} - t)(t - t_n)dt,\]

where \(w^{(3)}\) is the third time derivative of \(w\) and \(w^{(3)} \in L_2(t_n, t_{n+1}; L_2(\Omega))\), Cauchy-Schwarz inequality implies that

\[\left\|\frac{\dot{w}(t_{n+1}) + \dot{w}(t_n)}{2} - \frac{w(t_{n+1}) - w(t_n)}{\Delta t}\right\|_{L^2(\Omega)}^2 \leq \frac{\Delta t^3}{4} \|w^{(3)}\|_{L^2(t_n, t_{n+1}; L^2(\Omega))}.\]

We can also have similar bounds with respect to energy norm such that

\[\left\|\frac{\dot{w}(t_{n+1}) + \dot{w}(t_n)}{2} - \frac{w(t_{n+1}) - w(t_n)}{\Delta t}\right\|_V^2 \leq C\Delta t^3 \|w^{(3)}\|_{L^2(t_n, t_{n+1}; H^1(\Omega))},\]

for some positive \(C\), where \(w^{(3)} \in L_2(t_n, t_{n+1}; V)\). Therefore, since we suppose \(u \in H^4(0, T; H^4(\Omega))\), we can take the advantage of second order finite difference scheme at the end. For example,

\[\sum_{n=0}^{m-1} (E_1^n, E_2^n) \leq \frac{\Delta t}{2} \sum_{n=0}^{m-1} \|E_1^n\|^2_{L^2(\Omega)} + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \|E_2^n\|^2_{L^2(\Omega)} \]

\[\leq \frac{\Delta t}{2} \sum_{n=0}^{N_q-1} \|E_1^n\|^2_{L^2(\Omega)} + \frac{\Delta t}{2} \sum_{n=0}^{N_q-1} \|E_2^n\|^2_{L^2(\Omega)} \]

\[\leq \frac{\Delta t^4}{8} \|u^{(4)}\|^2_{L^2(0, T; L^2(\Omega))} + \frac{\Delta t^4}{8} \|\theta^{(4)}\|^2_{L^2(0, T; L^2(\Omega))}.\]

On the other hand, Lemma 3.1 allows us to have \(L_2\) estimates on the spatial domain for \(\theta(t)\) and its time derivatives. Moreover, we note that regularity of internal variables follows that of the solution. In this manner, we can observe the bounds of terms of \(E_1^n, E_2^n, E_3^n, E_q^n\) and \(\dot{E}_q^n\).
Now, using the same techniques in the proof of Theorem 3.1 such as Cauchy-Schwarz inequalities, Young’s inequalities with proper coefficients, norm axioms, summation by parts and the property of maximum onto (3.17), we can derive

\[
\max_{0 \leq k \leq N} \left\| \tilde{\varphi}^k \right\|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \left\| \chi^k \right\|_V + \sum_{0 \leq k \leq N} \max_{0 \leq k \leq N} \left\| \chi^k \right\|_V + \sqrt{\left( \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \left( \frac{\varsigma_{q+1} - \varsigma_q}{\Delta t} \right)^2 \right)}_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2).
\]

Furthermore if elliptic regularity is provided which means that (3.10) holds,

\[
\max_{0 \leq k \leq N} \left\| \tilde{\varphi}^k \right\|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \left\| \chi^k \right\|_V + \sum_{0 \leq k \leq N} \max_{0 \leq k \leq N} \left\| \chi^k \right\|_V + \sqrt{\left( \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \left( \frac{\varsigma_{q+1} - \varsigma_q}{\Delta t} \right)^2 \right)}_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r} + \Delta t^2).
\]

In the end, we can observe that a positive constant C is independent of T, h, \( \Delta t \), and the numerical solution. However it depends on the space domain, and other coefficients.

We would not expand all procedures but the main idea is given and it follows the almost same way to prove Theorem 3.1.

**Remark** As in the stability analysis, we have taken into account maximum with respect to discrete time rather than using Grönwall’s inequality in Lemma 3.2. Hence Lemma 3.2 enables us to have the following error estimates even for large T.

**Theorem 3.2.** Suppose \( u \in H^4(0,T;H^2(\Omega)) \cap C^1(0,T;H^2(\Omega)) \). Then we have

\[
\max_{0 \leq k \leq N} \left\| \hat{u}(t_k) - W^k_h \right\|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \left\| \hat{u}(t_k) - Z^k_h \right\|_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2)
\]

where C is a positive constant independent of T, h, \( \Delta t \), the numerical solution but depends on \( \Omega, \partial \Omega, \rho \) and other coefficients of internal variables. With elliptic regularity, it is also observed that

\[
\max_{0 \leq k \leq N} \left\| \hat{u}(t_k) - W^k_h \right\|_{L^2(\Omega)} \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2)
\]

for some positive C.

**Proof.** From Lemma 3.2, it is provided that

\[
\max_{0 \leq k \leq N} \left\| \tilde{\varphi}^k \right\|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \left\| \chi^k \right\|_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2)
\]

for some positive C. Combining with Lemma 3.1, we have for any 0 \( \leq n \leq N \)

\[
\| e^h_n \|_V = \| \vartheta^n - \chi^n \|_V \leq \| \vartheta^n \|_V + \| \chi^n \|_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2)
\]

for some positive C. In this same sense, we can derive

\[
\| \tilde{e}^h_n \|_{L^2(\Omega)} \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2).
\]

Since n is arbitrary, it is also true that

\[
\max_{0 \leq k \leq N} \| \tilde{e}^k \|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \| e^k_h \|_V \leq CT \| u \|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2).
\]
Furthermore, elliptic regularity allows us to have higher order in $L_2(\Omega)$ norm thus,
\[
\max_{0 \leq k \leq N} \| e_h^k \|_{L_2(\Omega)} \leq \max_{0 \leq k \leq N} \| e_h^k \|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \| \theta_h^k \|_{L_2(\Omega)} \leq CT \| u \|_{H^t(0,T;H^s(\Omega))}(h^r + \Delta t^2).
\]

By the norm equivalence between and energy norm and $H_1(\Omega)$ norm, it is easily seen that
\[
\| e_h^n \|_{L_2(\Omega)} \leq \| e_h^n \|_{H^t(\Omega)} \leq \frac{1}{\sqrt{K}} \| e_h^n \|_{V}, \quad \text{for any } n.
\]

**Corollary 3.1.** Under same conditions as Theorem 3.2, if elliptic regularity holds, then
\[
\max_{0 \leq k \leq N} \| u(t_k) - Z_h^n \|_{L_2(\Omega)} \leq CT \| u \|_{H^t(0,T;H^s(\Omega))}(h^r + \Delta t^2).
\]

**Proof.** In a similar way with the proof of Theorem 3.2,
\[
\| u(t) - Z_h^n \|_{L_2(\Omega)} = \| e_h^n \|_{L_2(\Omega)} \leq \| \theta_h^n \|_{L_2(\Omega)} + \| \chi_h^n \|_{L_2(\Omega)},
\]
for any $0 \leq n \leq N$. Note that $\| \chi_h^n \|_{L_2(\Omega)} \leq \| \chi_h^n \|_{H^t(\Omega)} \leq \frac{1}{\sqrt{K}} \| \chi_h^n \|_{V}$ by coercivity. Hence we have
\[
\| u(t) - Z_h^n \|_{L_2(\Omega)} \leq \| \theta_h^n \|_{L_2(\Omega)} + \| \chi_h^n \|_{L_2(\Omega)} \leq \| \theta_h^n \|_{L_2(\Omega)} + \frac{1}{\sqrt{K}} \| \chi_h^n \|_{V}.
\]
Since $n$ is arbitrary and Lemmas 3.1 and 3.2 yield
\[
\| \theta_h^n \|_{L_2(\Omega)} \leq C|u|^n_{H^t(\Omega)} h^r \quad \text{and} \quad \| \chi_h^n \|_{V} \leq CT \| u \|_{H^t(0,T;H^s(\Omega))}(h^r + \Delta t^2),
\]
we can obtain
\[
\max_{0 \leq k \leq N} \| u(t_k) - Z_h^n \|_{L_2(\Omega)} \leq CT \| u \|_{H^t(0,T;H^s(\Omega))}(h^r + \Delta t^2).
\]

\[
\square
\]

### 3.2. Velocity form

Recall the variational formulations of the velocity form (2.9)-(2.10). The fully discrete formulation for (P2) is introduced with Crank-Nicolson method as

\[
(P2)^h \text{ Find } Z_h^n, W_h^n, S_{h1}^n, S_{h2}^n, \ldots, S_{hN_v}^n \in V^h \text{ for } n = 0, \ldots, N \text{ such that}
\]

\[
\left( \frac{W_{h}^{n+1} - W_{h}^{n}}{\Delta t}, v \right) + \varphi_0 a \left( \frac{Z_{h}^{n+1} + Z_{h}^{n}}{2}, v \right) + \sum_{q=1}^{N_v} a \left( \frac{S_{hq}^{n+1} + S_{hq}^{n}}{2}, v \right) = \frac{1}{2} \left( F_v(t_{n+1}; v) + F_v(t_n; v) \right),
\]

\[
\tau_q a \left( \frac{S_{hq}^{n+1} - S_{hq}^{n}}{\Delta t}, v \right) + a \left( \frac{S_{hq}^{n+1} + S_{hq}^{n}}{2}, v \right) = \tau_q \varphi_q a \left( \frac{W_{h}^{n+1} + W_{h}^{n}}{2}, v \right) \quad \text{for each } q,
\]

\[
a(Z_h^n, v) = a(u_0, v), \quad \text{(3.20)}
\]

\[
(W_h^n, v) = (u_0, v), \quad \text{(3.21)}
\]

\[
S_{hq}^n = 0, \quad \text{for each } q. \quad \text{(3.22)}
\]

for any $v \in V^h$ where (3.1) holds.

We first show a stability estimates. The proof is similar to the displacement form but there are obvious differences.
Theorem 3.3. Assume that $f \in C(0,T;L_2(\Omega))$, $g_N \in H^1(0,T;L_2(\Gamma_N)) \cap C(0,T;L_2(\Gamma_N))$ and $u_0 \in H^1(\Omega)$. Then, $(P2)^h$ has a unique solution and there exists a positive constant $C$ depending on $\Omega$, $\partial \Omega$ and sets $\{\nu_q\}_{q=0}^{N_\nu}$ and $\{\tau_q\}_{q=1}^{N_\nu}$, but independent of numerical solutions, $h$, $\Delta t$ and $T$ such that

$$
\max_{0 \leq k \leq N} \|w^k\|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \|v^k\|_{V'} + \sum_{q=1}^{N_\nu} \max_{0 \leq n \leq N} \|S^n_{hq}\|_V + \sum_{q=1}^{N_\nu} \sum_{n=0}^{N_N-1} \Delta t \left\| S^n_{hq} + S^n_{hq}\right\|_V
\leq CT^2 \left( \|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_\infty(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} \right).
$$

Proof. Once we show a stability bound for the velocity form, it implies that the linear system to (3.18)-(3.22) can be solved uniquely.

With $v = W_{h}^{n+1} + W_{h}^n$ in (3.18) and $v = S_{h}^{n+1} + S_{h}^n$ in (3.19) for each $q$, we obtain for $1 \leq m \leq N$

$$
\rho \|W_{h}^{m}\|^2_{L_2(\Omega)} + \varphi_0 \|Z_{h}^{m}\|^2_V + \sum_{q=1}^{N_\nu} \frac{1}{\tau_q} \|\dot{S}_{h}^{m}\|_V + \sum_{q=1}^{N_\nu} \sum_{n=0}^{m-1} \frac{\Delta t}{2\tau_q \delta_q} \left\| S^n_{hq} + S^n_{hq}\right\|_V
\leq \rho \|W_{h}^{0}\|^2_{L_2(\Omega)} + \varphi_0 \|Z_{h}^{0}\|^2_V + \sum_{n=0}^{m-1} \frac{\Delta t}{2} \left( F_v \left( t_{n+1}; W_{h}^{n+1} + W_{h}^n \right) + F_v \left( t_n; W_{h}^{n+1} + W_{h}^n \right) \right). \quad (3.23)
$$

As shown in Theorem 1.1, particularly (3.8), we use the same arguments such as Cauchy–Schwarz inequalities, Young’s inequalities, summations by parts, (1.9), (3.1) and take the maximum to obtain a bound for the linear form of velocity form. The only difference is that $F_v$ contains $-\sum_{q=1}^{N_\nu} \varphi_0 e^{-t/\tau_q} a(u_0,v)$ more. However, we can easily show its boundedness by the fact, $\varphi_0(0) = 1$ and $0 < e^{-t/\tau_q} \leq 1$ for $0 \leq t, \forall q$. To be specific,

$$
-\sum_{q=1}^{N_\nu} \varphi_0 e^{-t/\tau_q} a(u_0,v) \leq \sum_{q=1}^{N_\nu} \varphi_0 e^{-t/\tau_q} \|u_0\|_V \|v\|_V
\leq \sum_{q=1}^{N_\nu} \varphi_0 \|u_0\|_V \|v\|_V \leq \|u_0\|_V \|v\|_V.
$$

Thus, we can complete our proof in a similar way as for Theorem 3.1. \hfill \Box

In a similar way with Lemma 3.2 and Theorem 3.2, error estimates for $(P2)^h$ can be derived. First of all, let us introduce the following notations $\nu_q := \zeta_q - R\zeta_q$ and $T^n_q := S^n_{hq} - R\zeta^n_q$, for $q = 1, \ldots, N_\nu$.

Lemma 3.3. Suppose $u \in H^k(0,T;H^{r\nu}(\Omega)) \cap C^1(0,T;H^{r\nu}(\Omega))$. Then there exists a positive constant $C$ such that

$$
\max_{0 \leq k \leq N} \|u^k\|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \|v^k\|_{V'} + \sum_{q=1}^{N_\nu} \max_{0 \leq k \leq N} \|\nu^n_k\|_V + \sum_{n=0}^{N_N-1} \sum_{q=1}^{N_\nu} \left( \|T^n_k + T^n_k\|^2_{V'} \right)^{1/2}
$$

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\[ \leq C T \|u\|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2). \]

Furthermore, if we assume elliptic regularity, we have

\[
\max_{0 \leq k \leq N} \|\pi_k\|_{L_2(\Omega)} + \max_{0 \leq k \leq N} \| \chi_k \|_V + \sum_{q=1}^{N_q} \max_{0 \leq k \leq N} \| T^k_q \|_V + \sqrt{\Delta t} \sum_{q=1}^{N_q} \| Y^q_n + Y^q_n \|_V^2 \\
\leq C T \|u\|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2).
\]

C is independent of h, \( \Delta t \), T and the numerical solution but depends on \( \rho \), \( \Omega \), \( \partial \Omega \) and other internal variable coefficients.

**Proof.** The proof will follow the similar steps in Lemma 3.2. By subtraction of (2.9) and (3.18) with \( v \in V_h \) and subtraction of average of (2.10) between \( t_{n+1} \) and \( t_n \) into (3.19) with

\[ v = \frac{\tau^{n+1} + \tau^n}{2} \text{ for each } q, \]

we have for any \( 0 \leq n \leq N - 1 \)

\[
\frac{\rho}{2} \left( \frac{\| \tau^{n+1} \|_{L_2(\Omega)}^2 - \| \tau^n \|_{L_2(\Omega)}^2}{2} \right) + \frac{\varphi_0}{2} \left( \frac{\| \chi^{n+1} \|_V^2 - \| \chi^n \|_V^2}{2} \right) \\
+ \frac{1}{2} \sum_{q=1}^{N_q} \frac{1}{\varphi_2} \left( \| \tau^{n+1}_q \|_V^2 - \| \tau^n_q \|_V^2 \right) + \Delta t \sum_{q=1}^{N_q} \| \tau^{n+1}_q + \tau^n_q \|_V^2 \\
\leq \frac{\rho}{2} \left( \| \tau^{n+1} - \tau^n \|_V^2 + \| \tau^n \|_V^2 \right) + \frac{\rho}{2} \Delta t \left( E^n_1, \tau^{n+1} + \tau^n \right) \\
- \rho \left( \tau^{n+1} - \tau^n, E^n_3 \right) - \rho \Delta t \left( E^n_2, E^n_3 \right) - \frac{\Delta t}{2} \sum_{q=1}^{N_q} \frac{1}{\varphi_2} a \left( E^n_q, \tau^{n+1}_q + \tau^n_q \right) - \frac{\Delta t}{2} \sum_{q=1}^{N_q} \frac{1}{\varphi_2} a \left( E^n_3, \tau^{n+1}_q + \tau^n_q \right) \tag{3.24}
\]

where

\[
E_1(t) := \frac{\tilde{u}(t + \Delta t) + \tilde{u}(t)}{2} - \frac{\tilde{u}(t + \Delta t) - \tilde{u}(t)}{\Delta t}, \quad E_2(t) := \frac{\tilde{\theta}(t + \Delta t) + \tilde{\theta}(t)}{2} - \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t}, \]

\[
E_3(t) := \frac{u(t + \Delta t) - u(t)}{\Delta t} - \frac{\tilde{u}(t + \Delta t) + \tilde{u}(t)}{2}, \quad E_4(t) := \frac{\tilde{\zeta}(t + \Delta t) + \tilde{\zeta}(t)}{2} - \frac{\zeta(t + \Delta t) - \zeta(t)}{\Delta t},
\]

for each \( q \). Thus, following the same way as Lemma 3.2, for example use of Cauchy-Schwarz inequalities, Young’s inequalities, summation by parts, Lemma 3.1, and Crank-Nicolson method, allows us to obtain our claim. \( \square \)

**Theorem 3.4.** Suppose \( u \in H^1(0,T;H^2(\Omega)) \cap C^1(0,T;H^2(\Omega)) \). Then we have

\[
\max_{0 \leq k \leq N} \| u(t_k) - Z_k^h \|_V + \max_{0 \leq k \leq N} \| \tilde{u}(t_k) - W_k^h \|_{L_2(\Omega)} \leq C T \|u\|_{H^4(0,T;H^2(\Omega))}(h^{r+1} + \Delta t^2)
\]

for some positive \( C \) such that is independent of \( h \), \( \Delta t \), and solutions but depends on \( \rho \), \( \Omega \), \( \partial \Omega \), \( T \) and other internal variable coefficients. In addition,

\[
\max_{0 \leq k \leq N} \| \tilde{u}(t_k) - W_k^h \|_{L_2(\Omega)} \leq C T \|u\|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2)
\]

if elliptic regularity is assumed.
Proof. By Lemma 3.3, we have
\[
\max_{0 \leq k \leq N} \|\varpi^k\|_{L^2(\Omega)} + \max_{0 \leq k \leq N} \|\chi^k\|_V \leq CT\|u\|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2)
\]
for some positive \(C\). Hence, as seen in Theorem 3.2, use of triangle inequalities and Lemmas 3.1 and 3.3 gives
\[
\max_{0 \leq k \leq N} \|\hat{e}^k_h\|_V + \max_{0 \leq k \leq N} \|\tilde{e}^k_h\|_{L^2(\Omega)} \leq CT\|u\|_{H^4(0,T;H^2(\Omega))}(h^{r-1} + \Delta t^2).
\]
Also, if elliptic regularity holds
\[
\max_{0 \leq k \leq N} \|\tilde{e}^k_h\|_{L^2(\Omega)} \leq CT\|u\|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2).
\]
Thus our claim is shown. Note that \(C\) is independent of \(T\), mesh sizes and solutions but depends on the domain, its boundary and the coefficient of density and internal variables.

Corollary 3.2. Under same conditions as Theorem 3.4, if elliptic regularity estimates hold, then
\[
\max_{0 \leq k \leq N} \|u(t_k) - Z^k_h\|_{L^2(\Omega)} \leq CT\|u\|_{H^4(0,T;H^2(\Omega))}(h^r + \Delta t^2),
\]
where a positive constant \(C\) is independent of \(T\), mesh sizes and solutions but depends on domains and constant coefficients.

Proof. The proof is parallel to Corollary 3.1 but instead of using the result from Theorem 3.2, Theorem 3.4 is applied here.

4. Numerical experiments

In this section we give some evidence that the convergence rates given in the theorems above are realised in practice, at least for model problems for which an exact solution can be generated. The tabulated results in this section can be reproduced by the python scripts (using FEniCS, https://fenicsproject.org/) at https://github.com/Yongseok7717 or by pulling and running a custom docker container as follows (at a bash prompt):

```bash
docker pull variationalform/fem:yjcg1
docker run -ti variationalform/fem:yjcg1
cd ./2019-11-26-codes/mainTable/
./main_Table.sh
```

The run may take around 30 minutes or longer depending on the host machine.

Let the exact solution to (1.1)-(1.5) be
\[
u(x, y, t) = e^{-t} \sin(xy) \in C^\infty(0, T; C^\infty(\Omega))
\]
where \(\Omega\) is the unit square and \(T = 1\). Hence \(\Omega\) satisfies the condition for elliptic regularity since \(\Omega\) is a convex polygonal domain [18, Chapter 4.3] so elliptic regularity estimates are satisfied. The Dirichlet boundary condition is given by
\[
u = 0 \text{ if } x = 0 \text{ or } y = 0, \forall t.
\]
While we set $\varphi_0 = 0.5$, $\varphi_1 = 0.1$, $\varphi_2 = 0.4$, $\tau_1 = 0.5$, $\tau_2 = 1.5$, and we suppose $\rho = 1$ and $D = 1$, internal variables for $q = 1, 2$, the source term $f$ and the Neumann boundary condition $g_N$ are governed by primal problem. Then our exact solution satisfies all conditions for stability and error bounds. The numerical simulations are implemented in FEniCS.

Recalling the error estimates, regardless of form of internal variables, $\|e_h^N\|_V$, $\|e_h^N\|_{L_2(\Omega)}$ and $\|e_h^N\|_{L_2(\Omega)}$ become $O(h^{s_1} + \Delta t^2)$, $O(h^{s_1+1} + \Delta t^2)$ and $O(h^{s_1+1} + \Delta t^2)$ respectively, since $s_2 = \infty$.

In other words, the convergence rate with respect to time is fixed at second order but the spatial convergence order depends on the degree of polynomials $s_1$. Let us consider $\Delta t \approx h$ then the errors should be

$$
\|e_h^N\|_V = O(h^{\min(s_1, 2)}), \quad \|e_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^{\min(s_1+1, 2)}).
$$

In the computational work, the numerical convergent rate $d_c$ can be estimated by

$$
d_c = \frac{\log(\text{error of } h_1) - \log(\text{error of } h_2)}{\log(h_1) - \log(h_2)}.
$$

For example, the rates of convergence are illustrated as gradient of lines in Figure 1. With linear polynomial basis, the energy errors have first order accuracy but the $L_2$ errors show optimal second order. On the other hand, in case of quadratic polynomial basis, we can observe only second order rates since the time-error convergence order is fixed at 2.

![Figure 1: Numerical convergent order: linear (left) and quadratic (right) polynomial basis](image)

To see higher order in space for the quadratic case, a significantly small $\Delta t$ allows us to observe that

$$
\|e_h^N\|_V = O(h^2) \quad \text{and} \quad \|e_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^3)
$$

on Table 1. In the same sense, once the spatial error is negligible, the convergence rate of time can be observed. Table 2 presents optimal results of Crank-Nicolson method regardless of type of internal variables, norms, and errors. With combining all results, we have

$$
\|e_h^N\|_V = O(h^2 + \Delta t^2) \quad \text{and} \quad \|e_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^3 + \Delta t^2),
$$

for $s_1 = 2$. 

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| $h$ | Displacement form | | Velocity form |
|-----|-------------------|------------------|
|     | $\|e_h^N\|_V$ | $\|\tilde{e}_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_V$ | $\|\tilde{e}_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_{L^2(\Omega)}$
| 1/4 | 2.2557E-3 | 8.1101E-5 | 6.9417E-5 | 2.2557E-3 | 8.1098E-5 | 6.9419E-5
| 1/8 | 6.0301E-4 | 1.0491E-5 | 9.2260E-6 | 6.0301E-4 | 1.0489E-5 | 9.2266E-6
| 1/16 | 1.5566E-4 | 1.2803E-6 | 1.1954E-6 | 1.5566E-4 | 1.2794E-6 | 1.1957E-6
| 1/32 | 3.9526E-5 | 1.6460E-7 | 1.5240E-7 | 3.9526E-5 | 1.6270E-7 | 1.5226E-7
| rate | 1.93 | 2.99 | 2.93 | 1.93 | 2.99 | 2.93

Table 1: Fixed time step size errors when $s_1 = 2$ and $\Delta t = 1/1200$

| $\Delta t$ | Displacement form | | Velocity form |
|-------|-------------------|------------------|
|       | $\|e_h^N\|_V$ | $\|\tilde{e}_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_V$ | $\|\tilde{e}_h^N\|_{L^2(\Omega)}$ | $\|e_h^N\|_{L^2(\Omega)}$
| 1/8  | 6.0705E-04 | 8.5271E-04 | 2.4904E-04 | 3.6453E-04 | 6.8608E-04 | 1.4780E-04
| 1/16 | 1.5316E-04 | 2.1327E-04 | 6.3192E-05 | 9.2174E-05 | 1.7163E-04 | 3.7643E-05
| 1/32 | 3.8373E-05 | 5.3325E-05 | 1.5856E-05 | 2.3105E-05 | 4.2915E-05 | 9.4542E-06
| 1/64 | 9.5993E-06 | 1.3332E-05 | 3.9677E-06 | 5.7818E-06 | 1.0729E-05 | 2.3663E-06
| rate | 1.99 | 2.00 | 1.99 | 1.99 | 2.00 | 1.98

Table 2: Fixed spatial mesh size errors when $s_1 = 2$ and $h = 1/512$

In summary, the displacement form and the velocity form provide appropriate numerical results as following the error estimate theorems. To be described in details, the energy estimates are given by $O(h_{\text{min}}(s_1+1, s_2) - 1 + \Delta t^2)$ and $L^2$ estimates with elliptic regularity follow $O(h_{\text{min}}(s_1+1, s_2) + \Delta t^2)$.

5. Conclusions

Our two fully discrete formulations show optimal energy error estimates as well as $L^2$ error estimates theoretically and experimentally with fixed second order accuracy in time. We can also expand and elevate the scalar problem (antiplane structure) to a vector-valued viscoelastic problem, straight forwardly. Instead of antiplane shear strain, we can define the strain tensor by Cauchy infinitesimal tensor $\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ for $i, j = 1, \ldots, d$ and $d = 2$ or $3$ where $u$ is a displacement vector. When we replace $D \rightarrow \underline{D}$ and $\nabla \rightarrow \underline{\varepsilon}$ where $\underline{D}$ is a symmetric positive definite fourth order tensor, the strain tensor $\underline{\sigma}$ is given in the same way with the scalar analogue. In addition, we have a symmetric bilinear form of vector valued problem such that $a(w, v) = (D\underline{\varepsilon}(w), \underline{\varepsilon}(v))$ for $w, v \in V$ where $V = \{ v \in H^1(\Omega)^d \mid v(x) = 0$ on $\Gamma_D \}$. Note that according to Korn’s inequality (e.g. [19, 20, 21, 16]), the bilinear form is coercive. Also, by the Cauchy-Schwarz inequality the bilinear form is continuous. Therefore variational problems and fully discrete formulations can be derived by introducing internal variables. As following the almost same way in the scalar analogue, the proof can be shown and the elliptic estimates can be considered as elastic problems hence we may use the results in [22, 16, 23]. In the end, we can obtain the same results as in the scalar case with respect to stability and error analysis.

In conclusion, a scalar/vector analogue of the linear viscoelastic problem can be formulated with two type of internal variables and the fully discrete formulations govern by CGFEM and Crank-Nicolson method provide stability and $a\, priori$ bounds.
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