Second-Order Coding Rate of Quasi-Static Rayleigh-Product MIMO Channels

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Abstract

With the development of innovative applications that require high reliability and low latency, ultra-reliable and low latency communications become critical for wireless networks. In this paper, the second-order coding rate of coherent quasi-static Rayleigh-product MIMO channels is investigated. We consider the coding rate within $O\left(\frac{1}{\sqrt{nV}}\right)$ of the capacity, where $M$ and $n$ denote the number of transmit antennas and the block-length, respectively, and derive the closed-form upper and lower bounds for the optimal average error probability. This analysis is achieved by setting up a central limit theorem (CLT) for the mutual information density (MID) with the assumption that the block-length, number of the scatterers, and number of the antennas go to infinity with the same pace. To obtain more physical insights, the high and low signal-to-noise ratio (SNR) approximations for the upper and lower bounds are also given. One interesting observation is that rank-deficiency degrades the performance of MIMO systems with finite blocklength (FBL) and the fundamental limits of the Rayleigh-product channel approach those of the Rayleigh case when the number of scatterers approaches infinity. Finally, the fitness of the CLT and the gap between the derived bounds and the performance of practical coding schemes are illustrated by simulations.

Index Terms

Finite blocklength (FBL), ultra-reliable and low-latency communications (URLLC), multiple-input multiple output (MIMO), packet error probability, random matrix theory (RMT).

I. INTRODUCTION

Ultra-reliable and low-latency communications (URLLC) are considered as one of the key enabling technologies of the fifth/sixth generation (5G/6G) wireless communication systems [1] to meet the stringent latency and reliability requirements of innovative applications, e.g., industrial automation, autonomous vehicles, and virtual/augmented reality (VR/AR) [2], [3]. However, the classical Shannon’s capacity [4] is too optimistic and can not be utilized to characterize the fundamental limits of systems with finite blocklength (FBL).

FBL Analysis: There are three fundamental metrics for the performance analysis with FBL, i.e., the packet error rate, the blocklength, and the coding rate. Characterizing the trade-off among the three metrics is of great challenges. In [4], Shannon showed that when the blocklength goes to infinity, the packet error probability vanishes and derived the maximum achievable coding rate. Then in [5], the conventional Shannon’s coding rate was refined to show that the maximal channel coding rate can be represented by a normal approximation

$$\log M(n, \varepsilon) = nC - \sqrt{nV} Q^{-1}(\varepsilon) + O(\log(n)),$$

where $M(n, \varepsilon)$ is the cardinality of a codebook which has blocklength $n$ and can be decoded with block error probability less or equal to $\varepsilon$, $C$ is the channel capacity, $V$ is the channel dispersion, determined by the channel characteristics, and $Q^{-1}(\cdot)$ denotes the inverse $Q$-function. Taking channel dynamics into account, the coding rate of the single-input single-output (SISO) fading channel was evaluated in [6] by analyzing the channel dispersion.

There have been engaged results works on the FBL performance of MIMO systems. With the coherent setting, the FBL capacity of independent and identically distributed (i.i.d.) isotropic MIMO fading channels with receiver channel state information (CSIR) was investigated in [10] by analyzing the channel dispersion. The length of the packet is usually shorter than the channel coherence time [7], [8], so it is reasonable to consider a quasi-static channel model. The maximal achievable rate of the quasi-static fading channel was investigated for given $n$, $\varepsilon$, and a per-codeword power constraint with the non-coherent setting for single-input multiple-output (SIMO) [9] and multi-input multi-output (MIMO) [8] systems. Given isotropic codebooks (with the exact energy constraint) and CSIR, the authors of [11] derived the closed-form expressions for the throughput, error probability, and the minimum number of antennas required for given error probability/throughput. In [12], by assuming that the numbers of transmit and receive antennas go to infinity with the same pace, the close-form expressions of the mean and variance for the channel dispersion of i.i.d. MIMO Rayleigh channels with CSIR were given.

The evaluation of the error probability in fading channels also attracted many interests. The closed-form maximal coding rate for additive white Gaussian noise (AWGN) channel was investigated and the bounds for the optimal error probability were given in [13]. In [14], the closed-form average packet error probability of a single antenna system over cascaded Nakagami-m channel with general correlation was given by Mellin transform. In [15], the closed-form upper and lower bounds for the
optimal average packet error probability of the MIMO quasi-static Rayleigh fading channels were given by utilizing random matrix theory (RMT). In [16], the Gallager bound for i.i.d. Rayleigh MIMO channels, which is an upper bound for the packet error probability, was investigated by RMT.

**Double-Scattering Channel:** It is well known that the capacity of MIMO channels with infinite blocklength severely degrades due to the lack of scatterers or the keyhole effects [17]. However, the FBL analysis for rank-deficient channels is not yet available in the literature. Many works, e.g., [11], [12], considered a rich scattering environment, e.g., the full-rank independent Rayleigh MIMO channels, which, as evident in [18], is incapable of characterizing the reduced-rank behavior of MIMO systems due to the lack of scatterers around the transceivers. Although the rank of the channel was considered in [10], the i.i.d. isotropic channel model is incapable of characterizing the double-scattering or keyhole effect.

Gesbert et al. proposed the double-scattering channel in [19], where both the spatial correlation and the rank deficiency were considered. Due to its practical significance and generality, there have been many works concerning the fundamental limits of the double-scattering MIMO channel including the ergodic capacity [17], [20], outage probability [21], and the diversity-multiplexing tradeoff [22], [23]. When the antenna elements and the scatterers are sufficiently separated, the spatial correlations can be ignored such that the double-scattering model degenerates to the Rayleigh-product channel, which is modeled as the product of two i.i.d. Gaussian random matrices. In [24], the authors investigated the three metrics (outage probability, symbol error rate, and ergodic capacity) over Rayleigh-product channels by finite-dimension RMT. By a similar approach, the three metrics were investigated considering the co-channel interference in [25] and the achievable sum-rate of MIMO systems over Rayleigh-product channels with linear receivers was investigated in [26].

The above analysis in the finite dimension regime involves complex integrals which can only be evaluated numerically. To tackle this issue, the asymptotic RMT was adopted to obtain strikingly simple forms of results. Furthermore, the high-dimensionality of future wireless systems, e.g., massive MIMO, favors the asymptotic results. In [27], the authors investigated the outage performance of Rayleigh-product channels and gave a closed-form expression for the outage probability when the transmitter and the receiver have equal number of antennas. The outage performance of the more general double-scattering channel was investigated in [21], [23], where the authors gave a closed-form expression for the outage probability of the double-scattering MIMO channel and investigated the impact of the number of scatterers. However, the outage probability is no longer applicable to characterize the reliability in the FBL regime.

**Our Contributions:** To the best of the authors’ knowledge, the FBL analysis over Rayleigh-product MIMO channels remains unavailable, especially regarding the impact of the number of scatterers on the fundamental limits. The characterization of the optimal average error probability turns out to be a difficult RMT problem due to the product structure of the channel, which will be the focus of this paper. The relation of this paper with the related works is summarized in Table[I].

In this paper, we will investigate the optimal average packet error probability of the coherent quasi-static Rayleigh-product MIMO channels for rates within $O(\frac{1}{\sqrt{Mn}})$ of the capacity, which is referred to as the second-order coding rate [29]. Here, $M$ and $n$ represent the number of transmit antennas and the blocklength, respectively, such that $Mn$ is equal to the number of symbols in each codeword. Inspired by the information spectrum approach in [13], [15], we first show that the upper and lower bounds for the optimal average error probability of a channel with given inputs are characterized by the mutual information density (MID). Then, we set up a central limit theorem (CLT) for the MID by RMT and derive the closed-form upper and lower bounds for the optimal average error probability. The main contributions of this paper are summarized as follows:

1. Assuming that the number of the antennas, the number of the scatterers, and the blocklength go to infinity with the same pace, we set up a CLT for the MID over Rayleigh-product channels for any codes with the exact energy constraint. Specifically, we show that the characteristic function of the MID converges to that of the Gaussian distribution by using the Gaussian tools (the integration by parts formula and Nash-Poincaré inequality) and give the closed-form expressions for the mean and variance.

2. Based on the CLT, we derive the closed-form upper and lower bounds for the optimal average error probability and the second-order outage probability. To evaluate the impact the number of scatterers, we give the high and low SNR approximations for the upper and lower bounds. It is shown that when the number of scatterers goes to infinity, the results in this paper resort to those of the Rayleigh case. On the other hand, when the blocklength approaches infinity, the second-order outage probability converges to the outage probability, i.e., [23] in Table[I]. Furthermore, our results can also degenerate to the Shannon analysis over the Rayleigh channel, i.e., [28] in Table[I] if both the number of the scatterers and the blocklength go to infinity.

3. Numerical results indicate that the gap between the upper and lower bounds is small for practical SNR region and the slope of the bounds for the optimal average packet error probability matches well with the packet error probability of the LDPC codes with maximal likelihood (ML) demodulator. We also compare the derived results with those of the Rayleigh channel by

| Channel Type                  | Shannon Analysis | FBL Analysis |
|------------------------------|------------------|--------------|
| Single-Rayleigh Channel      | [28]             | [13], [19]   |
| Rayleigh-product Channel     | [21], [23], [22] | This work    |

| TABLE I  
| SUMMARY OF RELATED WORKS. |
simulations, which show that the packet error probability of the Rayleigh-product channel is higher than that of the Rayleigh channel given the same rate, and the gap decreases as the rank of the Rayleigh-product channel increases, which agrees with the theoretical results.

**Paper Organization:** In Section II we introduce the system model and formulate the problem. In Section III we derive a CLT for the MID, based on which we give the closed-form expression for the upper and lower bounds of the average error probability. In Section IV we provide the simulation results by adopting LDPC codes. Section V concludes the paper.

**Notations:** Bold, upper case letters and bold, lower case letters represent matrices and vectors, respectively. $P(\cdot)$ represents the probability operator and $E_x$ denotes the expectation of $x$. $\mathbb{C}^N$ and $\mathbb{C}^{M\times N}$ denote the $N$-dimensional vector space and the $M$-by-$N$ matrix space, respectively. $A^H$ represents the conjugate transpose of $A$, and the $(i,j)$-th entry of $A$ is denoted by $[A]_{i,j}$ or $A_{ij}$. $\cdot^*$ represents the conjugate of a complex number. $|A|$ denotes the spectral norm of $A$. $\text{Tr} A$ refers to the trace of $A$. $I_N$ denotes the $N$ by $N$ identity matrix. The cumulative distribution function (CDF) of the standard normal distribution is denoted by $\Phi(x)$. $\mathcal{A}(\cdot)$ denotes the centered random variable and $\text{Cov}(x,y) = E_{x,y}[xy] - E_x E_y$ represents the covariance between $x$ and $y$. $\mathcal{D}_{N} \xrightarrow{D} \mathcal{D}_{N'}$ denotes the convergence in distribution. $a \downarrow b$ represents that $a$ approaches $b$ from the right. $\mathcal{P}(x)$ denotes a polynomial with positive coefficients. $\text{supp}(\cdot)$ denotes the support operator. Given a set $\mathcal{S}$, $\mathcal{P}(\mathcal{S})$ denote the set of probability measures with support of a subset of $\mathcal{S}$.

II. SYSTEM MODEL AND PROBLEM FORMULATION

In this paper, we consider a MIMO system with $N$ receive antennas and $M$ transmit antennas. The received signal $y_t \in \mathbb{C}^N$ (output of the channel) is given by

$$y_t = H x_t + \sigma w_t, \quad t = 1, 2, \ldots, n. \quad (1)$$

where $x_t \in \mathbb{C}^M$ represents the transmit signal (input of the channel) and $w_t \in \mathbb{C}^N$, whose entries follow $\mathcal{CN}(0,1)$, denotes the AWGN at time $t$. $n$ is the blocklength (the number of the channel uses required to transmit a codeword) and $\sigma^2$ is the noise power. $H \in \mathbb{C}^{N\times M}$ represents the Rayleigh-product channel matrix, given by

$$H = Z Y, \quad (2)$$

where the entries of $Z \in \mathbb{C}^{N\times L}$ and $Y \in \mathbb{C}^{L\times M}$ follow $\mathcal{CN}(0, \frac{1}{L})$ and $\mathcal{CN}(0, \frac{1}{M})$, respectively, with $L$ denoting the number of scatterers. We consider the quasi-static channel where $H$ does not change in $n$ channel uses. The transmitter has the statistical knowledge of $H$ while the receiver has perfect CSI. We introduce the following notations: $X^{(n)} = (x_1, x_2, \ldots, x_n)$, $Y^{(n)} = (y_1, y_2, \ldots, y_n)$, and $W^{(n)} = (w_1, w_2, \ldots, w_n)$. There are two types of constraints on the channel inputs, i.e., the maximal power constraint and the exact energy constraint (sphere constraint), which are given by

$$\mathcal{S}^n = \{X^{(n)} \in \mathbb{C}^{M\times n} | \frac{\text{Tr} X^{(n)} H X^{(n)^H}}{M n} \leq 1\}, \quad (3)$$

$$\mathcal{S}^n_e = \{X^{(n)} \in \mathbb{C}^{M\times n} | \frac{\text{Tr} X^{(n)} H X^{(n)^H}}{M n} = 1\}. \quad (4)$$

Obviously, the inputs satisfying the exact energy constraint is a subset of those following the maximal power constraint. In this paper, we will focus on the optimal average error probability with the maximal power constraint, which will be shown to be bounded by that with the exact energy constraint in Lemma I. In the following, we first give the definitions of the metrics concerned in this paper.

A. Average Error Probability

**The encoder mapping:** A $(P_e(n), G_n)$-code for the model in (1), which satisfies the maximal power constraint in (3), can be represented by the following mapping $f$,

$$f : G_n \rightarrow \mathbb{C}^{M\times n}. \quad (5)$$

Here the transmitted symbols are denoted by $X^{(n)}_m = f(m) \in \mathcal{S}^n$ with $m$ uniformly distributed in $G_n = \{1, 2, \ldots, G_n\}$. $C_n$ is the codebook, i.e., $\{f(1), f(2), \ldots, f(G_n)\}$.

**The decoder mapping:** The decoder mapping from the channel output $Y^{(n)}$ to the message can be represented by

$$g : \mathbb{C}^{N\times n} \rightarrow G_n \cup \{e\}. \quad (6)$$

The mapping $g$ gives the decision of $\hat{m} = g(Y^{(n)})$, where $Y^{(n)} = H f(m) + \sigma W^{(n)}$ denotes the received block. The decoder picks the transmitted message $m$ if it is correctly decoded otherwise an error $e$ occurs. Since $m$ is assumed to be uniformly distributed, the average error probability for a code $C_n$ with blocklength $n$, encoder $f$, decoder $g$ and the input $G_n$ is given by

$$P_e(n)(C_n) = \mathbb{P}(\hat{m} \neq m), \quad (7)$$
where the evaluation involves the randomness of $H$, $W^{(n)}$, and $m \in G_n$. Given a rate $R$, the optimal average error probability is given by

$$P_e^{(n)}(R) = \inf_{C_n: \supp(C_n) \subseteq S^n} \left\{ P_e^{(n)}(C_n) \left| \frac{1}{nM} \log(|C_n|) \geq R \right. \right\}.$$ 

Note $R$ is the per-antenna rate of each channel symbol.

Unfortunately, it is very difficult to obtain the exact expression of the optimal average error probability for any $M$, $L$, $N$, and $R$. Thus, we consider the rate $R$ within $O(\sqrt{nM})$ of the ergodic capacity for large $n$, which is referred to as the second-order coding rate [5, 13]. In particular, we will consider the optimal average error probability with respect to a perturbation $r$ around the ergodic capacity (in the FBL regime). To handle the difficulty caused by the randomness of Rayleigh-product MIMO channels, we will back off from the infinity by assuming that $n$, $M$, $N$, $L$ go to infinity with the same pace to obtain the closed-form evaluation for the optimal average error probability. This asymptotic regime has been widely used in evaluating the performance of large MIMO systems [21, 27, 28] and the strikingly simple expressions for the asymptotic performance have also been validated to be accurate even for the low-dimensional systems. In particular, the results in this paper will be derived with the following assumption.

**Assumption A.** (Asymptotic Regime) $0 < \lim_{M \to \infty} \inf_{n \to \infty} \frac{M}{n} \leq \frac{M}{n} \leq \lim_{M \to \infty} \sup_{n \to \infty} \frac{M}{n} < \infty$, $0 < \lim_{M \to \infty} \inf_{n \to \infty} \frac{M}{n} \leq \frac{M}{n} \leq \lim_{M \to \infty} \sup_{n \to \infty} \frac{M}{n} < \infty$.

**Assumption A** assumes that $M$, $N$, $L$, and $n$ go to infinity with the same pace and the asymptotic regime is widely used in the large system analysis [20, 23, 30]. We denote the ratios $\eta = \frac{N}{M}$, $\rho = \frac{M}{L}$, $\kappa = \frac{N}{L}$. $n \sqrt{\frac{M}{L}} \to \infty$ represents the asymptotic regime where $n$, $N$, $M$, and $L$ grow to infinity with the fixed ratios $\rho$, $\eta$, and $\kappa$.

### B. The Optimal Average Error Probability with Respect to the Second-Order Coding Rate

1) **Second-Order Error Probability:** Given a second-order coding rate $r$, the optimal average error probability for the system with $M$ transmit antennas and blocklength $n$ is given by [13]

$$P_e(r|\rho, \eta, \kappa) = \inf_{C_n: \supp(C_n) \subseteq S^n} \left\{ \limsup_{n \to \infty} \frac{1}{nM} \log(|C_n|) \left| \frac{1}{nM} \log(|C_n|) - EC(\sigma^2) \right| \geq r \right\},$$

where $C(\sigma^2) = \frac{1}{M} \log\det(I_N + \frac{1}{\sigma^2} HH^H)$ denotes the per antenna capacity. From (8) and (9), we can observe that for $r = O(1)$, the rate $R = EC(\sigma^2) + \frac{r}{\sqrt{nM}}$ is a $O(\sqrt{nM})$ perturbation around $EC(\sigma^2)$.

2) **Second-Order Outage Probability:** With the second-order coding rate $r$, the second-order outage probability is given by [15]

$$P_{out}(r|\rho, \eta, \kappa) = \inf_{C_n: \supp(C_n) \subseteq S^n} \left\{ \limsup_{n \to \infty} P_e(C_n) \left| \liminf_{n \to \infty} M\left(\frac{1}{nM} \log(|C_n|) - EC(\sigma^2)\right) \geq r \right. \right\}.$$ 

We can observe that the relation between the error probability and outage probability for the second-order coding rate $r$ is $P_{out}(r|\rho, \eta, \kappa) = P_e(\sqrt{r}|\rho, \eta, \kappa)$.

In this paper, we will give the closed-form upper and lower bounds for the optimal average error probability (outage probability).

### C. Preliminary Results

We first revise the first-order result regarding the MI of Rayleigh-product channels, i.e., the closed-form expression for the ergodic rate and then present the bounds for the error probability with given $H$ and $X^{(n)}$.

**Theorem 1.** (First-order result regarding the MI of Rayleigh-product channels) Given Assumption A and the channel $H$ in [2], the per antenna mutual information is given by

$$C(\sigma^2) = \frac{1}{M} \log\det(I_N + \frac{1}{\sigma^2} HH^H)$$

and there holds true that [20, Theorem 2]

$$C(\sigma^2) \overset{a.s.}{\rightarrow} C(\sigma^2),$$

and [27, Proposition 1]

$$EC(\sigma^2) \overset{\mathbb{P}}{\rightarrow} \bar{C}(\sigma^2) + O\left(\frac{1}{M^2}\right),$$

where $\bar{C}(\sigma^2)$ is given by

$$\bar{C}(\sigma^2) = -\frac{\log(\sigma^2)}{\kappa} - \left(\frac{\eta - 1}{\kappa}\right) \log\left(1 - \frac{\omega}{\eta(1 + \omega)}\right) + \log(1 + \omega) - \frac{2\omega}{1 + \omega} + \frac{\log(\eta)}{\kappa}.$$
Here $\omega$ is the root of the following cubic equation

$$P(\sigma^2) = \omega^3 + (2\sigma^2 + \eta \kappa - \kappa - \eta + 1) \frac{\omega^2}{\sigma^2} + (1 + \eta \kappa - \frac{2\eta}{\sigma^2} + \frac{1}{\sigma^2}) \omega - \frac{\eta}{\sigma^2} = 0,$$

which satisfies $\omega > 0$ and $\eta + (\eta - 1)\omega > 0$.

By Theorem 1 we can replace $\mathbb{E}C(\sigma^2)$ by $\overline{C}(\sigma^2)$ in (9) and (10) since $\sqrt{Mn(\mathbb{E}C(\sigma^2) - \overline{C}(\sigma^2))} = O(\frac{1}{\sqrt{n}})$, which results from $\mathbb{E}C(\sigma^2) - \overline{C}(\sigma^2) = O(\frac{1}{n})$. The convergence rate $O(\frac{1}{\sqrt{n}})$ is proved in [21] for the cases where $Z$ and $Y$ are both Gaussian matrices, and the convergence rate may not be valid for non-Gaussian matrices. The analysis of non-Gaussian matrices indicates that $\sqrt{Mn(\mathbb{E}C(\sigma^2) - \overline{C}(\sigma^2))} = O(1)$ for single-hop channels in [31], [32] when the entry of the channel matrix has a non-zero pseudo-variance or fourth-order cumulant, which is referred to as the bias. The bias also exists for the two-hop channel if the entries of $Z$ and $Y$ have non-zero pseudo-variance or fourth-order cumulant.

Next, we will introduce the bounds obtained by the information spectrum approach, which relies on the MID.

**Lemma 1. (Bounds for the optimal average error probability) [13] Eq. (77) and Eq. (89)** The optimal average error rate can be bounded by

$$\mathbb{F}(r|\rho, \eta, \kappa) \leq \mathbb{P}(r|\rho, \eta, \kappa) \leq \mathbb{G}(r|\rho, \eta, \kappa),$$

where

$$\mathbb{G}(r|\rho, \eta, \kappa) = \lim_{\delta \downarrow 0} \lim_{\zeta \downarrow 0} \sup_{N, d, n, m} \mathbb{P}(\sqrt{Mn}(I_{N,L,M}^{(n)} - \overline{C}(\sigma^2)) \leq r + \zeta),$$

and

$$\mathbb{F}(r|\rho, \eta, \kappa) = \inf_{\delta \downarrow 0} \lim_{\zeta \downarrow 0} \lim_{\delta \downarrow 0} \sup_{N, d, n, m} \mathbb{P}(\sqrt{Mn}(I_{N,L,M}^{(n+1)} - \overline{C}(\sigma^2)) \leq r - \zeta),$$

with $I_{N,L,M}^{(n)}$ representing the MID given in (16) at the top of this page. Here (18a) is induced by the input $X^{(n)} \in \mathbb{C}^{M \times n}$

$$\overline{X}^{(n)} \left(\frac{1}{Mn} \text{Tr}(\overline{X}^{(n)} \overline{X}^{(n), H})\right)^{-\frac{1}{2}},$$

where $\overline{X}^{(n)} \in \mathbb{C}^{M \times n}$ is an i.i.d. Gaussian matrix.

**Remark 1.** Note that the inf operation is taken over $\{\mathbb{P}(X^{(n+1)} \in \mathbb{P}(S^{(n+1)})) \} \in \mathbb{F}(r|\rho, \eta, \kappa)$. This result differs from the adaptation from the maximal power constraint to the equal power constraint by introducing an auxiliary symbol [3 Lemma 39]. Lemma 7 is important due to two reasons. First, it converts the evaluation with constraint $S$ to that on the sphere coding (exact energy constraint) $S_\omega$ in [4]. Second, it indicates that, to characterize $\mathbb{P}(r|\rho, \eta, \kappa)$, we need to investigate the distribution of the MID.

Based on the above results, we will investigate the asymptotic distribution of the MID defined in (16) with the exact energy constraint $S_\omega$.

### III. Main Results

In this section, we will characterize the distribution of the MID and then derive the closed-form upper/lower bounds for the optimal average error probability. To clearly present the main results, we introduce the following notations:

$$\delta = \frac{1}{\sigma^2} \left(\frac{N}{L} - \frac{M\omega}{L(1 + \omega)}\right),$$

$$\overline{\omega} = \frac{1}{1 + \omega}.$$

The derivatives of $\delta$, $\omega$, $\overline{\omega}$, and $\overline{C}(\sigma^2)$ with respect to $\sigma^2$ are given by

$$\delta' = \frac{d\delta}{d\sigma^2} = -\frac{\delta}{\Delta_{\sigma^2}},$$

$$\omega' = \frac{\omega\delta'}{\delta(1 + \delta\overline{\omega})},$$

$$\overline{\omega'} = -\frac{\omega\overline{\omega}^2 \delta'}{\delta(1 + \delta\overline{\omega})}.$$
where
\[ \Delta_n^2 = \sigma^2 + \frac{M \omega x^2}{L \delta (1 + \delta x^2)}. \]  
(21)
The above results can be obtained by taking derivative on both sides of \[ (15) \] and the proof is given in Appendix A.

The asymptotic distribution of the MID is given in the following theorem.

**Theorem 2.** (CLT of the MID) Given Assumption A and a sequence of \( X^{(n)} \), the distribution of the MID converges to a Gaussian distribution, i.e.,
\[ \sqrt{\frac{Mn}{V_n}} \left( I_{K,L,M}^{(n)} - \bar{C}(\sigma^2) \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \]
where \( V_n \) is given by
\[ V_n = -\rho \log(\Xi) + \eta + \frac{\sigma^4 \delta^l}{\kappa} + \frac{\rho \kappa \omega x^l}{M} \text{Tr} A_n^2 \left( \frac{\omega x^2 (1 + \delta x^2)}{1 + \delta x^2} - \frac{\omega x}{\delta (1 + \delta x^2)} \right), \]
with
\[ A_n = I_M - \frac{1}{n} X^{(n)}(X^{(n)})^H, \]
\[ \Xi = \frac{(1 + \delta x^2)(\sigma^2 + \frac{\omega x}{\delta (1 + \delta x^2)})}{\eta \kappa (1 + \delta x^2)}. \]
(25)

Proof. The proof of Theorem 2 is given in Appendix C. \( \square \)

**Remark 2.** (Degeneration to the Rayleigh case) When \( \kappa \to 0 \) with fixed \( \eta \) and \( \rho \) (denoted as \( \kappa \to \eta, \rho \to 0 \)), equation \[ (15) \] becomes
\[ \omega^2 + (1 - \frac{\eta}{\sigma^2} + \frac{1}{\sigma^2}) \omega - \frac{\eta}{\sigma^2} = 0, \]
whose solution \( \omega_\infty \) is \( \omega_\infty = \delta_0 = \frac{\eta - 2 \sigma^2 - \sqrt{(1 - \eta + \sigma^2)^2 + 4 \sigma^2}}{2 \sigma^2} \) so that \( \omega \xrightarrow{\kappa \to 0} \omega_\infty = \delta_0 \) and \( \omega' \xrightarrow{\kappa \to 0} \omega'_\infty = \delta'_0 \). We can further obtain from \[ (19) \] that
\[ \frac{L \delta_\infty}{M} = \frac{1}{\sigma^2} (\eta - \frac{\omega_\infty}{1 + \omega_\infty}) \xrightarrow{(a)} \omega_\infty, \]
where (a) follows from \[ (20) \].

We can rewrite \( \bar{C}(\sigma^2) \) as
\[ \bar{C}(\sigma^2) = \eta \log(1 + \frac{\kappa \omega x}{\sigma^2 \delta}) + \frac{1}{\kappa} \log(1 + \delta x) + \log(1 + \omega) - 2 \omega \xrightarrow{\kappa \to 0} \eta \log(1 + \frac{1}{\sigma^2 (1 + \delta_0)}) \]
\[ + \log(1 + \delta_0) - \frac{\delta_0^2}{1 + \delta_0}. \]

For the variance, we have
\[ \Xi \xrightarrow{\kappa \to 0} \frac{\delta_0 (\sigma^2 + \frac{1}{(1 + \delta_0)^2})}{\eta} = 1 - \frac{\delta_0^2}{\eta (1 + \delta_0)^2}, \]
\[ \frac{\kappa \omega x (1 + \delta x)}{1 + \delta x^2} - \frac{\kappa \omega x}{\delta (1 + \delta x^2)} \xrightarrow{\kappa \to 0} 0 - \delta_0, \]
so that
\[ V_n \xrightarrow{\kappa \to 0} -\rho \log(1 - \frac{\delta_0^2}{\eta (1 + \delta_0)^2}) + \eta + \sigma^4 \delta_0^4 - \frac{\rho \delta_0^4 \text{Tr} A_n^2}{M (1 + \delta_0)^2}. \]

The RHS of \[ (28) \] and \[ (31) \] are identical to \[ (15) \] Eqs. (12) and (20), respectively, which indicates that Theorem 2 is equivalent to that for the Rayleigh channel when \( \kappa \to 0 \).

Theorem 2 gives the asymptotic distribution of the MID given the sphere channel input \( X^{(n)} \). According to Lemma 1 we can give the approximations for the upper and lower bounds of the optimal average error probability by the following theorem.

**Theorem 3.** (Bounds for the optimal average error probability) The optimal average error probability \( \mathbb{P}_e(r | \rho, \eta, \kappa) \) for the second-order coding rate over Rayleigh-product fading channels is bounded by
\[ \mathbb{P}_e(r | \rho, \eta, \kappa) \geq \begin{cases} \Phi(-r \sqrt{V_n}), & r \leq 0, \\ \frac{1}{2}, & r > 0, \end{cases} \]
\[ (32) \]
where
\[ V_- = -\rho \log(\Xi) + \eta + \frac{\sigma^4 \delta'}{\kappa}, \]
\[ V_+ = V_- + \kappa \omega' \left[ \frac{\omega^2 (1 + \delta^2)}{1 + \delta^2} - \frac{\omega \omega'}{\delta (1 + \delta^2)} \right]. \]

**Proof.** **Lower bound:** By Theorem 2, we can obtain that
\[ \mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+1}}} \leq z \right) \geq \Phi\left( \frac{r - \varsigma}{\sqrt{V_{n+1}}} \right), \]
where \( V_{n+1} = \frac{n}{n+1} (V_- - \log(\Xi) + \frac{\rho + \Theta}{M} \sqrt{\frac{A_{\omega+1}}{M}}), \) with \( \Theta = \kappa \omega'(\frac{\omega^2 (1 + \delta^2)}{1 + \delta^2} - \frac{\omega \omega'}{\delta (1 + \delta^2)}) \). Since \( -\log(\Xi) + \Theta \to 0 \) by Slutsky’s lemma [33], we have
\[ \mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+1}}} \leq z \right) \geq \Phi\left( \frac{r - \varsigma}{\sqrt{V_{n+1}}} \right), \]
where \( V_{n+} = V_- + \frac{\rho + \Theta}{M} \sqrt{\frac{A_{\omega+1}}{M}}. \) Therefore, we can obtain
\[ \mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+}}} \leq r - \varsigma \right) = \mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+}}} \leq \frac{r - \varsigma}{\sqrt{V_{n+}}} \right) \]
\[ \geq \left\{ \begin{array}{ll}
\mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+}}} \leq \frac{r - \varsigma}{\sqrt{V_{n+}}} \right), & r \leq 0, \\
\mathbb{P}\left( \frac{\sqrt{M} n (I_{N,L,M}^{(n+1)} - \mathcal{C}(\sigma^2))}{\sqrt{V_{n+}}} \leq 0, & r > 0, \end{array} \right. \]
for the sequence \( c_n \downarrow 0 \). The inequality in step (a) holds true since \( V_{n+} \geq V_- > 0 \) and the case for \( r > 0 \) follows from the fact that \( \frac{\sqrt{A_{\omega+1}}}{M} = O(M) \) so that \( V_{n+} \to \infty \). The lower bound can be obtained by taking the limit \( \downarrow 0 \).

**Upper bound:** By the upper bound in [18a], we provide an exact implementation of \( \mathbf{X}^{(n)} \), which is constructed by the normalized Gaussian codebook, i.e.,
\[ \mathbf{X}^{(n)} = \frac{\mathbf{G}}{\sqrt{\frac{\text{Tr}(\mathbf{G}^H \mathbf{G})}{M_n}}} \]
where \( \mathbf{G} \in \mathbb{C}^{M \times n} \) is a Gaussian random matrix with i.i.d. entries. This indicates that
\[ \frac{\text{Tr} \mathbf{A}_\omega^2}{M} = 1 - \frac{2 \text{Tr} \mathbf{X}^{(n)} \mathbf{X}^{(n)^H}}{M_n} + \frac{\text{Tr} \mathbf{X}^{(n)} \mathbf{X}^{(n)^H}}{M_n^2} \xrightarrow{\text{as}} \rho^{-1}. \]
In this case, the variance in \([23]\) becomes \( V_+ \).

**Remark 3.** Theorem 3 depicts the optimal average error probability for a region of the coding rate close to the ergodic capacity, i.e., \( \frac{1}{n} \log(\| C_n \|) = \mathcal{C}(\sigma^2) + \frac{\rho + \Theta}{M} \). In [3], [13], the corresponding results over the AWGN channel depend only on the noise level \( \sigma^2 \) while the results in the Rayleigh fading channel depend on \( \sigma^2, \eta, \) and \( \rho. \) Theorem 3 shows the impact of the channel rank on the optimal average error probability by introducing \( \kappa. \) Although the loose lower bound \( \frac{1}{2} \) for \( r > 0 \) may not be accurate, the region of interest, i.e., \( r < 0, \) is accurate as the upper bound and lower bound are close to each other. Furthermore, for the bounds \( \Phi(\frac{r - \varsigma}{\sqrt{V_{n+}}} \) and \( \Phi(\frac{r - \varsigma}{\sqrt{V_{n+}}} \) with different blocklengths, Theorem 3 indicates that there is a crossing point at the error rate \( \frac{1}{2} \) and the corresponding SNR can be determined by setting \( r = \sqrt{M_n (R - \mathcal{C}(\sigma^2))} \). Note that the crossing point does not point on the blocklength \( n. \) The above analysis will be validated in Section IV.

**Remark 4.** With the relation \( \mathbb{P}_{out}(r|\rho, \eta, \kappa) = \mathbb{P}_c(\sqrt{pr}|\rho, \eta, \kappa), \) we can bound the second-order outage probability as
\[ \mathbb{P}_{out}(r|\rho, \eta, \kappa) \leq \Phi\left( \frac{r}{\sqrt{\rho}} \right), \]
and
\[ \mathbb{P}_{out}(r|\rho, \eta, \kappa) \geq \left\{ \begin{array}{ll}
\Phi\left( \frac{r}{\sqrt{\rho}} \right), & r \leq 0, \\
\frac{1}{2}, & r > 0, \end{array} \right. \]
The outage probability with infinite blocklength can be obtained by letting $\rho \to \infty$ while keeping $\eta$ and $\kappa$, i.e., $P_{\text{out}}(r) \xrightarrow{\eta, \kappa, \rho \to \infty} \Phi\left(\frac{-\log(\Xi)}{\sqrt{-\log(\Xi)}}\right)$, which is equivalent to the result in [22], [23] by setting the correlation matrices $R = I_N$, $S = I_L$, and $T = I_M$.

In [22], [23], [27], the blocklength is assumed to be infinitely large and then the limit is taken with respect to $N$, $L$, and $M$. In this work, we change the asymptotic regime such that the blocklength is increasing at the same pace with $N$, $L$, and $M$. In this case, the impact of the FBL is reflected by the terms after $-\log(\Xi)$ in (34), i.e., $\frac{1}{\rho}(\eta + \frac{\sigma^2}{\kappa})$ in $V_+$ and $\frac{1}{\rho}(\eta + \frac{\sigma^2}{\kappa} + \frac{\omega^2(1 + \frac{\delta^2}{\kappa})}{1 + \omega^2(1 + \frac{\delta^2}{\kappa})})$ in $V_-$.

**Proposition 1.** (Bounds with equal number of transceiver antennas) When the transceivers have the same number of antennas, i.e., $M = N$ ($\eta = 1$), the optimal average error probability is bounded with

$$V_+ = \rho W(\omega) + 1 + X(\omega),$$

$$V_- = V_+ + Y'(\omega),$$

where $\omega$ is the positive solution of the equation

$$\omega^3 + 2\omega^2 + (1 + \frac{\kappa}{\sigma^2} - \frac{1}{\sigma^2})\omega - \frac{1}{\sigma^2} = 0,$$

satisfying $\omega > 0$ and $1 + (1 - \kappa)\omega > 0$. $W(\omega)$, $X(\omega)$ and $Y(\omega)$ are given by

$$W(\omega) = \log\left(\frac{(1 + \omega)^2}{1 + 2\sigma^2 + 2\sigma^2\omega^2}\right),$$

$$X(\omega) = -\frac{(1 + \omega^2(1 + \omega))}{(1 + \omega)(1 + 2\omega^2 + 2\omega^2)}\omega^2\left(1 + \omega^2 + \omega + 1 + \kappa\right) + \frac{(1 + (1 - \kappa)\omega)^2}{\kappa^2((1 - \kappa)\omega^2 + 2\omega + 1)^2}. $$(44)

$$Y(\omega) = \frac{\kappa}{(1 + \omega)^3}\left(\sigma^2\left(1 + \omega^2\right)^2 + (1 - \kappa)\omega + 1 + \kappa\right) + \frac{(1 + (1 - \kappa)\omega)^2}{\kappa^2((1 - \kappa)\omega^2 + 2\omega + 1)^2}. $$(45)

**Proof.** Letting $z = \sigma^2$ and $M = N$ in the first line of (19), we have $\delta = \frac{1}{\rho}(\eta + \frac{\sigma^2}{\kappa}) = \frac{z}{\omega}$. Also, we have the following results

$$\Delta_{\sigma^2} = \omega^2 \frac{z + \omega \omega}{1 + \frac{\omega}{z}} = \omega^2 (1 + \omega)^3 + (1 - \kappa)\omega + 1 + \kappa = \frac{\omega^2(z + \omega^3) + (1 + \kappa)}{z(1 + \omega)^3 + (1 - \kappa)\omega + 1 + \kappa}. $$ (46)

Then, by (20), we can obtain

$$\delta' = \frac{\omega^2}{\Delta_{\sigma^2}} = \frac{\omega^2(z + \omega^3) + (1 + \kappa)}{z(1 + \omega)^3 + (1 - \kappa)\omega + 1 + \kappa}.$$

Let $X(\omega) = \frac{z^2}{\kappa}$ and $Y(\omega) = \kappa\sigma^2\left(\frac{\omega^2(1 + \frac{\delta^2}{\kappa})}{1 + \omega^2(1 + \frac{\delta^2}{\kappa})}\right)$, we can complete the proof by substituting the above results into (34). \qed

**Remark 5.** By letting $\rho \to \infty$ while keeping $\eta$ and $\kappa$, the upper bound and lower bound in (22) converge to the limiting outage probability, which can be given as $\Phi\left(\frac{-\log(\Xi)}{\sqrt{-\log(\Xi)}}\right)$. The outage probability is identical to the outage probability with infinite blocklength in [22] Proposition 2 and [27] Proposition 3, when setting the rate threshold $R = M\overline{C}(\sigma^2) + r$.

**Proposition 2.** (High SNR approximation) The high SNR approximations for $V_+$ and $V_-$ are given as

$$V_+ \approx \begin{cases} -\rho \log\left((1 - \kappa)(1 - \frac{1}{\eta})\right) + 1, & \text{when } \kappa < 1 \land \eta > 1, \\ -\rho \log\left((1 - \eta)(1 - \kappa)\right) + \eta, & \text{when } \eta < 1 \land \eta \kappa < 1, \\ -\rho \log\left((1 - \kappa)(1 - \frac{1}{\eta \kappa})\right) + \frac{1}{\kappa}, & \text{when } (\kappa > 1 \land \eta > 1) \lor (\eta \kappa > 1 \land \eta < 1). \end{cases} $$ (48)
\[
V_+ \approx \begin{cases}
- \rho \log((1 - \kappa)(1 - \frac{1}{\eta})) + 1, & \text{when } \kappa < 1 \wedge \eta > 1, \\
- \rho \log((1 - \eta)(1 - \eta \kappa)) + \eta(2 - \eta), & \text{when } \eta < 1 \wedge \eta \kappa < 1, \\
- \rho \log((1 - \frac{1}{\kappa})(1 - \frac{1}{\eta \kappa})) + \frac{1}{\kappa} + \frac{(\kappa - 1)}{\kappa^2}, & \text{when } (\kappa > 1 \wedge \eta > 1) \vee (\eta \kappa > 1 \wedge \eta < 1). 
\end{cases}
\] (49)

\[\text{Proof.} \text{ Define } \rho = \sigma^{-2} \text{ to represent the SNR. We can first obtain the high-SNR approximation for } \omega \text{ by analyzing the dominating terms of the Eq. (15). As proved in [21] Appendix I, when } N > M \text{ and } L > M \text{ (} \eta > 1 \text{ and } \kappa < 1), \omega = O(\rho) \text{ and the dominating term of } \omega^3 \text{ should be compensated by that of } (\eta - 1)(\kappa - 1)\omega^2 \rho. \text{ Therefore, we have } \omega = -(\eta - 1)(\kappa - 1)\omega^2 \rho + O(1). \text{ All other cases can be analyzed similarly. Therefore, we have}
\]

Case 1: \(N > M \text{ and } L > M\). In this case, we have \(\omega = -(\eta - 1)(\kappa - 1)\rho + O(1), \delta = (\eta \kappa - \kappa)\rho + O(1), \delta' = -(\eta \kappa - \kappa)\rho^2 + O(\rho), \text{ and } \omega' = -(\eta - 1)(1 - \kappa)\rho^2 + O(\rho), \text{ so that } V_+ = -\rho \log((1 - \kappa)(1 - \eta)) + 1 + o(1) \text{ and } V_+ = V_+ + o(1).

When \(M > N\), \(\omega\) should be \(O(1)\), due to the constraint \((\eta - 1)\omega + \eta > 0\). The dominating term is \([(\eta - 1)\kappa - 1)\omega^2 + (\eta - 2\eta + 1)\omega - \eta]\rho \text{ and } \omega \text{ can be obtained by letting the coefficient of } \rho \text{ be zero so that } \omega \in \{1 - \frac{\eta}{\eta - 1}\}.

Case 2: \(M > N \text{ and } L > N\). In this case, \(\omega = \frac{1}{\kappa - 1} + o(1)\) is not feasible since \((\eta - 1)\omega + \eta = \frac{\eta(1 - \eta)}{\kappa - 1} < 0\). Therefore, we have the approximations \(\omega = -\frac{\eta}{1 - \eta^3} + o(1), \delta = (1 - \eta)^{-1}(\frac{1}{\eta^2} - 1) + o(1), \delta' = -\frac{\eta^3(1 - \eta^2)}{(1 - \eta^3)(1 - \eta^2)} + o(1), \omega' = \frac{\eta^2}{1 - \eta^3} \delta' = \frac{\eta(1 - \eta)}{1 - \eta} + o(1) \text{ and}
\]

\[
\omega^4 \left( \frac{2}{\eta^2} \right) \left( \frac{\delta}{1 + \delta \omega^2} \right) - \frac{\kappa \omega' \omega}{\eta^2} + \frac{\eta (1 - \eta)}{(1 - \eta^2)(1 - \eta^3)} + o(1)
\]

so that \(V_- = -\rho \log((1 - \eta)(1 - \eta \kappa)) + \eta + o(1) \text{ and } V_+ = V_+ + \eta(1 - \eta) + o(1).

Case 3: \(N > M \text{ and } M > L\). In this case, we have the approximations \(\omega = \frac{1}{\kappa - 1} + o(1), \delta = (\eta \kappa - 1)\rho + O(1), \delta' = -(\eta \kappa - 1)\rho^2 + O(\rho), \text{ and } \omega' = O(1), \text{ so that } V_- = -\rho \log((1 - \kappa)(1 - \frac{1}{\eta \kappa})) + 1 + o(1) \text{ and } V_+ = V_+ + (\frac{\eta}{\kappa - 1}) + o(1).

Case 4: \(M > N \text{ and } N > L\). In this case, by the analysis before Case 2, we have \(\omega = O(1). \text{ If } \omega = \frac{\eta}{\kappa - 1}, \delta = (1 - \eta)^{-1}(\frac{1}{\kappa - 1}) + o(1) \text{ and the result for this case coincides with that of Case 3} \). \[\square\]

Remark 6. For the first two cases, the approximations of the variances will converge to those for the Rayleigh channel in [15] Remark 4] by letting \(\kappa \frac{\eta^2}{\kappa^2} \to 0\), and we can conclude that the variances of the MID in the Rayleigh-product channel are larger than those of the Rayleigh channel. In the first case, the gap between the upper bound and lower bound vanishes. The third case can not degenerate to the Rayleigh case, and illustrates the impact of the low rank in the high SNR regime. Here the edge cases (\(M = L \text{ or } M = N \text{ or } L = N\) ) are not discussed as the variance will increase with \(\frac{1}{\sigma^2}\) to infinity. The analysis for \(N = M \text{ can be derived from [21] Propositions 1].}

Proposition 3. (Low SNR approximation) At the low SNR region when \(\sigma^2 \to \infty\), the following parameters will converge to zero

\[
V_+ = 2(1 + \eta \kappa)\eta \sigma^{-2} + O(\sigma^{-4})
\]

\[
V_- = 2(1 + \eta \kappa)\eta \sigma^{-2} + O(\sigma^{-4})
\] (51)

Proof. Setting \(z = \sigma^2\), we have \(\delta = \eta \kappa z^{-1} + O(z^{-2}) \text{ and } \omega = \eta z^{-1} + O(z^{-2}). \text{ By further analysis of the dominating terms in (15), we can obtain}
\]

\[
\omega = \eta z^{-1} - \eta(1 + \eta \kappa)^{-1} + O(z^{-3}),
\]

\[
\delta = \eta \kappa z^{-1} - \eta \kappa z^{-2} + O(z^{-3}),
\]

\[
\delta' = -\eta \kappa z^{-2} + 2(1 + \eta \kappa)\eta \kappa z^{-3} + O(z^{-4}).
\] (52)

(51) can be obtained by noticing that \(\log(\Xi) = O(z^{-4}). \) \[\square\]

By Proposition 3 we can obtain \(\frac{V_+ - V_-}{V_+} = O(\sigma^{-2})\), which indicates the the gap between the upper and lower bounds is small at the low SNR region. Compared with the Rayleigh channel, the term \(2 \eta^2 \kappa \sigma^{-2}\) is introduced by the two-hop structure and shows the impact of the number of the scatterers \(L\). The term will vanish when \(\kappa \frac{\eta^2}{\kappa^2} \to 0\) so that \(V_+ \text{ and } V_- \text{ become } 2\eta \sigma^{-2}, \text{ which is the same as that of the Rayleigh channel.} \)
In this section, the analytical results derived in this paper will be verified by numerical simulations.

In Fig. 1, we compare the error bounds with different $L$. The parameters are set as $N = 16$, $M = 8$, $L = \{8, 16, 32\}$, $n = 36$, and $R = \log(2)$. The second-order coding rate is $r = \frac{R - C(\sigma^2)}{\sqrt{Mn}}$. We compare the theoretical bounds of the Rayleigh-product channel with those of the Rayleigh channel. It can be observed that the optimal average error probability of the Rayleigh-product channel is worse than that of the Rayleigh channel. Furthermore, as $L$ increases, the bounds for the Rayleigh product channel approach those of the Rayleigh channel.

To better illustrate the impact of $n$, the second-order outage probability with different $L$ is illustrated in Fig. 2 with the second-order coding rate $r = \frac{-C(\sigma^2)}{\sqrt{Mn}}$. The SNR is set to be 10 dB. The limiting outage probability derived in [21], [23] with infinity blocklength is plotted by the magenta dot line. It can be observed that as the blocklength $n$ goes to infinity, the second-order outage probability approaches the outage probability, which coincides with the analysis in Remark 4. For the case with short blocklength, the limiting outage probability is overly optimistic.

**IV. Numerical Results**

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Next, we compare the theoretical bounds with the performance of practical coding schemes by following the simulation setting for Rayleigh channel in [15], where the WiMAX standard with a LDPC code is adopted. Specifically, the system parameters are set as $M = 8$, $N = 16$, $L = 24$, and the inputs are generated uniformly. The coding scheme we adopt is the $1/2$ LDPC codes, which were used in the WiMAX standard [35]. A bit interleaved coded modulation (BICM) scheme with a random interleaver is used before modulation. The QPSK modulation is employed at the transmitter and the coding rate is set as $R = \log(2)$. At the receiver, the received signal is demodulated by the ML demodulator [36], [37],

$$L(s_i|\mathbf{r}, \mathbf{H}) = \log \frac{\sum_{\mathbf{c} \in C_1^{(i)}} p(\mathbf{r}|\mathbf{c}, \mathbf{H})}{\sum_{\mathbf{c} \in C_0^{(i)}} p(\mathbf{r}|\mathbf{c}, \mathbf{H})},$$

(53)

where $L(s_i|\mathbf{r}, \mathbf{H})$ represents the log likelihood ratio of $i$-th bit $s_i \in \{0,1\}$ and $C_1^{(i)} = \{c|c_i = 1, c \in C\}$ denotes the set of the codewords whose $i$-th digit is 1. Similarly, $C_0^{(i)}$ represents the set of the codewords whose $i$-th digit is 0. $p(\mathbf{r}|\mathbf{c}, \mathbf{H})$ is the conditional PDF of the received signal $\mathbf{r}$. The output of the demodulator is then decoded by the soft-decision LDPC decoder. The length of the LDPC codes are $l \in \{567,2304\}$ bits, which correspond to the blocklengths $n = \frac{l}{2M} \in \{36,144\}$. The packet error is compared with the theoretical bounds in Theorem 3 for different SNRs $(\frac{1}{\sigma^2})$. The second-order coding rate is $r = \frac{R-C(\sigma^2)}{\sqrt{Mn}}$. From Fig. 3 it can be observed that the upper bound and lower bound are nearly overlapped, which validates the tightness of the bounds. The theoretical bounds are nearly parallel with the error probability of the LDPC codes and the gap is around $2$ dB in the SNR range considered. Another observation is that there is a cross point at the error probability $0.5$ for both theoretical bounds and the simulation curves, which coincides with the analysis in Remark 3. Similar phenomenon can also be observed from the low rank case ($L = 4$) shown in Fig. 4.

V. CONCLUSION

In this paper, the optimal average packet error probability of MIMO systems over the quasi-static Rayleigh-product fading channel was investigated by the information-spectrum approach and RMT, where the results were derived in the asymptotic regime by assuming that the number of antennas and the blocklength go to infinity with the same pace. Considering the coding rate within a $O(\frac{1}{\sqrt{Mn}})$ perturbation of the asymptotic capacity, we first set up a CLT for the MID by the Gaussian tools and gave the closed-form expressions for the mean and the variance, which were then used to derive the upper and lower bounds for the optimal average packet error probability. We also provided the high and low SNR approximations for the upper and lower bounds. The results degenerated to previous FBL results over the Rayleigh channel when the number of scatterers approaches infinity, and the infinite-blocklength results in the Rayleigh-product channel when the blocklength approaches infinity. Besides the asymptotic results, the effect of the rank deficiency was also analyzed. Furthermore, we validated the fitness of the CLT and illustrated the gap between the derived bounds and the practical LDPC coding.
APPENDIX A
PROOF OF (20)

Proof. By taking derivative with respect to \( \sigma^2 \) on both sides of (15), we have

\[
3\omega^2 \omega' + 4\omega \omega' + (\eta - 1)(\kappa - 1) \left( \frac{2\omega \omega' - \omega^2}{\sigma^2} \right) + \omega' + (\eta \kappa - 2\eta + 1) \left( \frac{\omega' - \omega}{\sigma^2} \right) + \frac{\eta}{\sigma^4} = 0.
\]

By solving \( \omega' \) in (54), we have

\[
\omega' = \frac{(\eta - 1)(\kappa - 1)\frac{\omega^2}{\sigma^2} + (\eta \kappa - 2\eta + 1)\frac{\omega}{\sigma^2} - \frac{\eta}{\sigma^4}}{\sigma^2 + 3\sigma^2 \omega^2 + 4\sigma^2 \omega + 2(\eta - 1)(\kappa - 1)\omega + (\eta \kappa - 2\eta + 1)} := \frac{A}{B}.
\]

By (15), we can rewrite \( A \) and \( B \) as

\[
A = -\omega(1 + \omega)^2,
\]

and

\[
B = \sigma^2(1 + \omega)^2 + 2\sigma^2(\omega(1 + \omega) + 2(\eta - 1)(\kappa - 1)\omega + \eta \kappa - 2\eta + 1
\]

\[
= \sigma^2(1 + \omega)^2 + 2(\eta + (\eta - \kappa)\omega) + \eta \kappa - 2\eta + 1
\]

\[
= \sigma^2(1 + \omega)^2 + (\eta \kappa - \frac{\kappa \omega}{1 + \omega}) - \frac{\kappa \omega}{1 + \omega} + 1
\]

\[
= \sigma^2(1 + \omega)^2 + \sigma^2 \delta + \kappa \omega \left( \frac{1}{\delta} + \frac{\omega}{1 + \omega} - \frac{\omega}{1 + \omega} \right) = \sigma^2(1 + \omega)^2 + \sigma^2 \delta + \frac{\kappa \omega}{\delta} = \frac{\Delta \sigma^2}{\omega^2},
\]

where step (a) in (57) follows from \( \omega = \frac{\mathcal{L}}{M(\frac{1}{\delta} + \sigma^2)} \). Therefore, we can obtain

\[
\omega' = \frac{A\sigma^2}{B\sigma^2} = -\frac{\omega}{(1 + \delta \omega^2)\Delta \sigma^2}.
\]

By (19) and (58), we have

\[
\delta' = -\frac{\delta}{\sigma^2} - \frac{\kappa \sigma^2 \omega'}{\sigma^2} = -\frac{\delta}{\Delta \sigma^2}
\]

which concludes (20a) so that (20b) follows immediately. (20c) can be obtained by

\[
\omega' = -\omega^2 \omega' = -\frac{\omega^2 \delta'}{\delta(1 + \delta \omega^2)}.
\]

By [30 Eq. (90)], we have \( \frac{\partial C(\sigma^2) + \eta \log(\sigma^2)}{\partial \sigma^2} = \frac{\delta}{\kappa} \) to conclude (20d). \qed
APPENDIX B

MATHEMATICAL TOOLS AND USEFUL RESULTS

In this paper, we mainly use the Gaussian tools for the proof, which consists of the Nash-Poincaré Inequality and the Integration by Parts Formula.

1. Nash-Poincaré Inequality. Denote $x = [x_1, \ldots, x_N]^T$ as a complex Gaussian random vector satisfying $E x = 0$, $E x x^T = 0$, and $E x x^H = \Omega$. $f = f(x, x^*)$ is a $C^1$ complex function such that both itself and its derivatives are polynomially bounded. Then, the variance of $f$ satisfies the following inequality,

$$\text{Var}[f(x, x^*)] \leq E[\nabla_x f(x, x^*)^T \Omega \nabla_x f(x, x^*)] + E[\nabla_x f(x, x^*)^H \Omega \nabla_x f(x, x^*)], \quad (61)$$

where $\nabla_x f(x) = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}]^T$ and $\nabla_x f(x) = [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}]^T$. $\Omega$ is referred to as the Nash-Poincaré inequality, which gives an upper bound for functional Gaussian random variables and is widely utilized in the error estimation for the expectation of Gaussian matrices. By utilizing this inequality, it was shown that the approximation error of the deterministic approximation for the MI of MIMO channels is $O(\frac{1}{N^2})$ for both the Rayleigh and double-scattering channels [21, 28].

2. Integration by Parts Formula. The formula is given by

$$E[x_i f(x, x^*)] = \sum_{m=1}^{N} \Omega_{i,m} E[\frac{\partial f(x, x^*)}{\partial x^*_m}]. \quad (62)$$

If $\Omega = I_N$, (62) can be simplified as

$$E[x_i f(x, x^*)] = E[\frac{\partial f(x, x^*)}{\partial x^*_m}] \quad (63)$$

By this formula, the expectation for the product of a Gaussian random variable and a functional Gaussian random variable is converted to the expectation for the derivative of the functional Gaussian random variable.

**Lemma 2.** Given Assumption A and defining $Q(z) = (zI_N + HH^H)^{-1}$, there holds true that for finite $x, z > 0$, $m \geq 0$ and matrix $B$.

$$\text{Var}(\frac{\text{Tr} B Q(x) ZZ^H Q(z)^m M}{M}) \leq \frac{K_m \text{Tr} BB^H (1 + 2m z^{-2} + 2x^{-2})}{M^3 z^{2m} z^2}, \quad (64)$$

$$\text{Var}(\frac{\text{Tr} Q(x) ZZ^H Q(z)^m HBH^H M}{M}) \leq \frac{K_m \text{Tr} BB^H (3 + 2x^{-2} + 2m z^{-2})}{M^3 z^{2m} z^2}, \quad (65)$$

$$\text{Var}(\frac{\text{Tr} Q(x) Q(z)^m HBH^H M}{M}) \leq \frac{K_m \text{Tr} BB^H (2 + 2x^{-2} + 2m z^{-2})}{M^3 z^{2m} z^2}, \quad (66)$$

$$\text{Var}(\frac{\text{Tr} Q(z) ZZ^H (Q(z)HBH^H)^m M}{M}) \leq \frac{K_m \text{Tr} (BB^H)^m (2m + 1 + 2x^{-2} + 2m z^{-2})}{M^3 z^{2m} z^{2m}}, \quad (67)$$

$$\text{Var}(\frac{(\text{Tr} Q(z) ZZ^H)^m Q(z)HBH^H M}{M}) \leq \frac{K_m \text{Tr}(BB^H)^m (m + 2 + 2x^{-2} + 2m z^{-2})}{M^3 z^{2m} z^{2m}}. \quad (68)$$

**Proof.** The proof is similar to the proof of [21 Proposition 2] and omitted here. \hfill \Box

**Lemma 3.** ( [27] Lemma 4) Given Assumption A, there holds true that for finite $x, z > 0$,

$$\frac{E \text{Tr} Q(z)}{L} = \delta_z + O(\frac{\mathcal{P}(\frac{1}{z^2 M^2})}{z^2 M^2}),$$

$$\frac{E \text{Tr} Q(z) ZZ^H}{M} = \omega_z + O(\frac{\mathcal{P}(\frac{1}{z^2 M^2})}{z^2 M^2}), \quad (69)$$

$$\frac{\text{Tr} Q(z) HH^H}{M} = \omega_z \omega_z + O(\frac{\mathcal{P}(\frac{1}{z^2 M^2})}{z^2 M^2}),$$

where $\omega_z$ is the positive solution of

$$\omega^3 + (2z + \eta \kappa - \kappa - \eta + 1) \omega^2 + (1 + \frac{\eta \kappa}{z} - \frac{2\eta}{z} + \frac{1}{z}) \omega - \frac{\eta}{z} = 0, \quad (70)$$

such that $\eta + (\eta - 1)\omega_z > 0$. $\delta_z$ and $\omega_z$ can be obtained by (79). In particular, when $z = \sigma^2$, we have $\omega = \omega_z$, $\delta = \delta_z$, and $\omega z = \omega_z$.

**Remark 7.** Lemma 3 indicates that the approximation error of the quantities in Raleigh-product channels is $O(M^{-2})$, which is of the same order as that for the Rayleigh channel [28].
In this paper, we will use \((\delta, \omega, \overline{\omega})\) and \((\delta_z, \omega_z, \overline{\omega}_z)\) to denote the parameters generated by the polynomial \(P(\sigma^2)\) in \((15)\) and \(P(z)\) in \((70)\), respectively.

**Lemma 4.** Given Assumption A, \(Q = (\sigma^2I_N + HH^H)^{-1}\), and \(Q(z) = (zN + HH^H)^{-1}\), the following computation results hold true

\[
\frac{\omega}{\delta(1 + \delta_z\overline{\omega}_z)} = \frac{\omega_z}{\delta_z(1 + \delta\overline{\omega}_z)},
\]

\[
\mathbb{E} \text{Tr} \frac{QQ(z)}{L} = \frac{\delta}{\delta(1 + \delta \overline{\omega}_z)} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right),
\]

\[
\mathbb{E} \text{Tr} \frac{QZZ^HQ(z)}{M} = \frac{\omega_z}{\delta(1 + \delta \overline{\omega}_z)} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right),
\]

\[
\mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M} = \frac{\omega_z}{\delta(1 + \delta \overline{\omega}_z)} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right),
\]

\[
\mathbb{E} \text{Tr} \frac{QZZ^HQ(z)HH^H}{M} = \frac{\delta\omega^2\overline{\omega}_z}{1 + \delta \overline{\omega}_z} + \frac{M\omega_z^2\overline{\omega}_z}{L\delta_z(1 + \delta \overline{\omega}_z)^2} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right),
\]

\[
\mathbb{E} \text{Tr} \frac{QZZ^HQ(z)ZZ^H}{M} = \frac{M\omega_z(1 + \delta \overline{\omega}_z)}{L(1 + \delta \overline{\omega}_z)} + \frac{M\omega_z}{L\delta_z(1 + \delta \overline{\omega}_z)} + O\left(\frac{P(\frac{1}{z})}{M^2z}\right),
\]

where

\[
\Delta_{\sigma^2}(z) = \sigma^2 + \frac{M\omega_z^2\overline{\omega}_z}{L\delta_z(1 + \delta \overline{\omega}_z)}.
\]

**Proof.** By the definitions of \(\delta, \omega, \text{ and } \overline{\omega}\), we have

\[
\frac{\omega}{\delta(1 + \delta \overline{\omega}_z)} = \frac{\omega_z}{\delta_z(1 + \delta \overline{\omega}_z)} = \frac{\omega}{\delta(1 + \delta \overline{\omega}_z)} = \frac{\omega_z}{\delta_z(1 + \delta \overline{\omega}_z)} = \frac{\omega}{\delta(1 + \delta \overline{\omega}_z)},
\]

which concludes \((71)\).

Next, we will prove \((72)\) by first evaluating \(\mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M}\). With the integration by parts formula \(\mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} = \mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M} - \mathbb{E} \text{Tr} \frac{Q(z)ZZ^H}{M} \text{Tr} \frac{Q(z)QHH^H}{M}\), we have

\[
\mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M} = \frac{M}{1} \sum_{i,j} \mathbb{E} Y_{ij}^* |Z|^{2} (Q(z)QH_{ij}^*) = \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} - \mathbb{E} \text{Tr} \frac{Q(z)ZZ^H}{M} \text{Tr} \frac{Q(z)QHH^H}{M}
\]

\[
- \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} \text{Tr} \frac{QHH^H}{M} = \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} - \mathbb{E} \text{Tr} \frac{Q(z)ZZ^H}{M} \text{Tr} \frac{Q(z)QHH^H}{M}
\]

\[
- \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} \text{Tr} \frac{QHH^H}{M} = \mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M}
\]

where \(\varepsilon_{H,1} = -\frac{1}{M^2} \text{Cov}(\text{Tr} Q(z)ZZ^H, \text{Tr} Q(z)QHH^H)\) and \(\varepsilon_{H,2} = -\frac{1}{M^2} \text{Cov}(\text{Tr} Q(z)QZZ^H, \text{Tr} Q QHH^H)\). By using the Cauchy-Schwarz inequality and the variance control in \((64)\) and \((66)\) of Lemma 2, \(\varepsilon_{H,1}\) and \(\varepsilon_{H,2}\) can be bounded by

\[
|\varepsilon_{H,1}| \leq \frac{1}{M^2} \text{Var}^2 (\text{Tr} Q(z)ZZ^H) \text{Var}^2 (\text{Tr} Q(z)QHH^H) = O\left(\frac{P(z^{-1})}{M^2z}\right),
\]

\[
|\varepsilon_{H,2}| \leq \frac{1}{M^2} \text{Var}^2 (\text{Tr} Q(z)QZZ^H) \text{Var}^2 (\text{Tr} Q QHH^H) = O\left(\frac{P(z^{-1})}{M^2z}\right),
\]

where \(\mathcal{P}(\cdot)\) denotes a polynomial with positive coefficients. With the results of \((69)\) in Lemma 3, \((79)\) can be further written as

\[
\frac{\mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M}}{M} = \left(1 - \omega\overline{\omega}_z\right) \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} - \omega_z \mathbb{E} \text{Tr} \frac{Q(z)QHH^H}{M} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right)
\]

\[
= \omega \mathbb{E} \text{Tr} \frac{Q(z)QZZ^H}{M} + O\left(\frac{P(\frac{1}{z})}{zM^2}\right),
\]
where step (b) follows by moving $\frac{1}{M}\text{Tr}(Q(z)Q^{HH})$ to the LHS of the first line in \(81\) to solve $\frac{1}{M}\text{Tr}(Q(z)Q^{HH})$. Then we turn to evaluate $\frac{1}{M}\text{Tr}(Q(z)Q^{ZH})$. By the integration by parts formula \(62\), we have

$$
\frac{1}{M}\text{Tr}(Q(z)Q^{ZH}) = \frac{1}{M} \sum_{i,j} \text{E} [z_{j,i}^* | Q(z)Q^{ZH}]_{j,i} = \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M} 
$$

$$
= \frac{1}{M} \left( \frac{\text{E} \text{Tr}(Q(z)QQ^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M} \right) = \frac{1}{M} \left( \frac{\text{E} \text{Tr}(Q(z)QQ^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M} \right) 
$$

$$
= \frac{1}{M} \left( \frac{\text{E} \text{Tr}(Q(z)QQ^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M} \right) 
$$

where step (a) in \(82\) follows from \(69\) in Lemma \(8\) and the variance control in \(80\). By substituting \(81\) into \(82\) to replace $\frac{1}{M}\text{Tr}(Q(z)Q^{ZH})$ and solving $\frac{1}{M}\text{Tr}(Q(z)Q^{ZH})$, we have

$$
\frac{1}{M}\text{Tr}(Q(z)Q^{ZH}) = \left( \frac{1}{L} - \frac{\sigma^2}{L} \right) \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{L} + O(\frac{P(\frac{1}{z})}{z^2 M^2}).
$$

By using the resolvent identity $\sigma^2 Q + Q^{HH} = I_N$ to replace $Q^{HH}$ in the second last term of \(82\) and combining \(82\) with \(83\), we can obtain the approximation of $\frac{1}{M}\text{Tr}(Q(z)Q^{ZH})$.

$$
\frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{L} = \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{L} + \frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{L} + O(\frac{P(\frac{1}{z})}{z^2 M^2})
$$

where $\Delta_{z^2}(z) = \frac{\sigma^2}{L} + \frac{M \omega z}{L \delta_{z}(1 + \delta_{z} \omega z)}$ and step (a) in \(84\) follows from \(71\). This concludes the proof of \(72\). \(73\) can be obtained by \(74\). With \(74\), \(75\) can be then obtained by \(71\).

Next, we turn to evaluate $\frac{\text{E} \text{Tr}(Q(z)Q^{ZH})}{M}$ in \(75\). It follows from the resolvent identity

$$
\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} = \omega z - \sigma^2 \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} + O(\frac{P(\frac{1}{z})}{z^2 M^2}) = \omega z - \frac{\sigma^2}{M^2 z} \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} + O(\frac{P(\frac{1}{z})}{z^2 M^2})
$$

where step (a) in \(85\) follows from the definition of $\Delta_{z^2}(z)$ in \(77\) and step (b) follows from \(71\). This concludes the proof of \(75\). Next, we will evaluate $\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M}$ in \(76\). Notice that $\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M}$ can be written by integration by parts formula \(62\) and the variance control as

$$
\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} = \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} + O(\frac{P(\frac{1}{z})}{z^2 M^2})
$$

where step (a) in \(86\) follows from the definition of $\Delta_{z^2}(z)$ in \(77\) and step (b) follows from \(71\). This concludes the proof of \(75\). Next, we will evaluate $\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M}$ in \(76\). Notice that $\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M}$ can be written by integration by parts formula \(62\) and the variance control as

$$
\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} = \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} + O(\frac{P(\frac{1}{z})}{z^2 M^2})
$$

Similarly, we have

$$
\frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} = \delta \omega z + \left( \frac{\sigma^2}{M^2 z} \right) \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} - \frac{\text{E} \text{Tr}(Q^{ZH}Q^{ZH})}{M} + O(\frac{P(\frac{1}{z})}{z^2 M^2})
$$

where step (a) in \(87\) follows from the definition of $\Delta_{z^2}(z)$ in \(77\) and step (b) follows from \(71\). This concludes the proof of \(75\).
where \((u)\) is obtained by substituting \(E \text{Tr} QZ \bar{Z} H Q(z) HH^H\) using (86) and then solving \(E \text{Tr} QZ \bar{Z} H Q(z) ZZ^H\). By noticing that \(M \omega (1 + \delta)^{-2} \approx \omega\), we have
\[
E \text{Tr} QZ \bar{Z} H Q(z) ZZ^H = \frac{M \omega \omega (1 + \delta \omega)}{L (1 + \delta \omega)} + \frac{M \omega \omega}{L \omega (1 + \delta \omega)} + \frac{E \text{Tr} QZ \bar{Z} H Q(z)}{M} + \mathcal{O} \left( \frac{1}{M^2} \right),
\]
which concludes (76).

\[\square\]

\section*{Appendix C: Proof of Theorem 2}

In this proof, we will utilize the Gaussian tools introduced in Appendix B, which fully exploit the Gaussianity of the matrices \(W, Y, Z\), to evaluate the expectation of the functional quantities with respect to Gaussian random matrices. We prove the CLT by evaluating the characteristic function of the MID and showing that the characteristic function converges to that of the Gaussian distribution. This approach has been used in the second-order analysis of the MIMO channels [21], [28], [38]. Furthermore, the closed-form expressions of the mean and variance for the MID will be computed in the proof.

Given \(W, Y, Z\) over \(M, L, M\), we have
\[
\gamma_n W, Y, Z = \sqrt{nM} I_n, L, M
\]
and \(\Phi_n W, Y, Z(u) = e^{i\nu_n W, Y, Z}\), the characteristic function of the MID \(\Psi_n W, Y, Z(u)\) is given by
\[
\Psi_n W, Y, Z(u) = E \Phi_n W, Y, Z,
\]
where \(I_n, L, M\) is given in (16). The evaluation of \(\Psi_n W, Y, Z(u)\) is difficult due to the exponential structure. To overcome the difficulty, we first investigate its derivative with respect to \(u\), i.e.,
\[
\frac{\partial \Psi_n W, Y, Z(u)}{\partial u} = E \frac{\partial \Phi_n W, Y, Z(u)}{\partial u}.
\]
Then by evaluating the RHS of (20) and taking integral with respect to \(u\), we will find \(V_n\) and \(\tau_n\) such that
\[
E e^{-\frac{W, Y, Z}{\tau_n}} = \mathcal{N}(\frac{\gamma_n W, Y, Z}{\tau_n}, \frac{1}{\tau_n^2}),
\]
from which we can conclude the asymptotic Gaussianity of the MID. The evaluation is very challenging since there are three random matrices \(W, Y, Z\) involved. To overcome the difficulty, we consider splitting the evaluation of \(\Psi_n W, Y, Z(u)\) into two steps. In the first step, we will take the expectation with respect to \(W\). Then in the second step, we will take the expectation with respect to \(Y\) and \(Z\) jointly. Finally, we will determine \(V_n\) and \(\tau_n\) to conclude (21). Therefore, the proof includes three steps. In the following, we will use \(X\) to denote the codeword \(X^{(n)} \in \mathbb{C}^{N \times n}\) and
\[
A = A_n = I_n - \frac{X^{(n)}(X^{(n)})^H}{n}.
\]

\section*{A. Step 1: Expectation over \(W\)}

In this step, we will provide an approximation for \(\frac{\partial \Phi_n W, Y, Z(u)}{\partial u}\), which only relies on \(Y\) and \(Z\) by taking the expectation over \(W\). The approximation is given by the following Lemma.

\begin{lemma}
Given Assumption A, the derivative of the characteristic function \(\Psi_n W, Y, Z(u)\) can be approximated by
\[
\frac{\partial \Psi_n W, Y, Z(u)}{\partial u} = E(j \mu_n Y, Z - \frac{u^2}{2} Y, Z) \Phi_n Y, Z + \mathcal{O}(\frac{u^2 P(u)}{M^2}),
\]
where
\[
\mu_n Y, Z = \sqrt{n} \log (I_n + H H^H) - \frac{n}{\sqrt{M n}} \text{Tr} Q H A H^H,
\]
\[
\nu_n Y, Z = \frac{n}{M n} \text{Tr}(Q H H^H)^2 + \frac{2 \sigma_n^2}{M n} \text{Tr} Q^2 H X X^H H^H,
\]
\[
\beta_n Y, Z = \frac{n}{\sqrt{n^3 M^3}} \text{Tr}(Q H H^H)^3 + \frac{3 \sigma_n^2}{\sqrt{n^3 M^3}} \text{Tr} Q^2 H H^H Q H X X^H H^H H^H,
\]
with \(n\) denoting the number of columns of \(X\). Furthermore, there holds true that
\[
\Psi_n W, Y, Z(u) = E \Phi_n Y, Z + \mathcal{O}(\frac{u^4}{M^2}).
\]
\end{lemma}
\[ \mathbb{E} \operatorname{Tr} Q(z)H H^H \Phi_n^Y Z = \mathbb{E} Y_{ji}^*[Z^H Q(z)H]_{ji} \Phi_n^Y Z = \mathbb{E}[\operatorname{Tr} Q(z)Z Z^H H] - \frac{\operatorname{Tr} Q(z)Z Z^H}{M} \operatorname{Tr} Q(z)H H^H \]
\[ + \frac{\mu \sqrt{M}}{\sqrt{\nu M}} \operatorname{Tr} Q Z Z^H Q(z)H H^H - \frac{\mu}{M \nu n} \operatorname{Tr} Z Z^H Q(z)H A H^H Q + \frac{\mu}{M \nu n} \operatorname{Tr} Z Z^H Q(z)H H^H Q H A H^H Q \]
\[ - \sum_{i,j} \frac{\mu}{2 M} \frac{\partial \nu Y Z}{\partial Y_{ji}} [Z H Q(z)H]_{ji} + \sum_{i,j} \frac{\mu}{3 M} \frac{\partial \nu Y Z}{\partial Y_{ji}} [Z H Q(z)H]_{ji} \Phi_n^Y Z \]
\[ + \mathbb{E} \operatorname{Tr} Q(z)Z Z^H H \Phi_n^Y Z (1 - \frac{\mathbb{E} \operatorname{Tr} Q(z)H H^H}{M}) - \frac{\mathbb{E} \operatorname{Tr} Q(z)Z Z^H}{M} \mathbb{E} \operatorname{Tr} Q(z)H H^H \Phi_n^Y Z \]
\[ + \frac{\mu \sqrt{M}}{\sqrt{\nu M}} \operatorname{Tr} Q Z Z^H Q(z)H H^H \Phi_n^Y Z + \frac{\mu}{M \nu n} \operatorname{Tr} Z Z^H Q(z)H Y^H \Phi_n^Y Z + \frac{\mu}{M \nu n} \operatorname{Tr} Z Z^H Q(z)H H^H \Phi_n^Y Z \]
\[ + \mathbb{E} \operatorname{Tr} Q(z)Z Z^H \Phi_n^Y Z + \mathbb{E} \mathbb{E} \mathbb{E} Z^H \Phi_n^Y Z + \mathbb{E} \mathbb{E} \mathbb{E} Z^H \Phi_n^Y Z + \mathbb{E} \mathbb{E} \mathbb{E} Z^H \Phi_n^Y Z + O\left( \frac{\mathbb{P}(\frac{1}{\nu})}{2M} \right). \] (100)

**Proof.** The proof is similar to Step 1 in [15 Appendix D.D, Eq. (221) to (240)] since the error control for the Rayleigh-product channel has a similar behavior as that of the Rayleigh channel as discussed in Remark [2] We omit the proof here. \( \square \)

By Lemma [5] we can discard the dependence on \( W \) and turn to evaluate \( \mathbb{E} \left( \mu \mu_n^Y Z - \frac{\mu^2}{2} \nu_n^Y Z \right) \Phi_n^Y Z \) by taking the expectation with respect to \( Y \) and \( Z \).

**B. Step 2:** Evaluate \( \mathbb{E} \left( \mu \mu_n^Y Z - \frac{\mu^2}{2} \nu_n^Y Z \right) \Phi_n^Y Z \)

\( \mathbb{E} \left( \mu \mu_n^Y Z - \frac{\mu^2}{2} \nu_n^Y Z \right) \Phi_n^Y Z \) can be written as

\[ \mathbb{E} \left( \mu \mu_n^Y Z - \frac{\mu^2}{2} \nu_n^Y Z \right) c_n^Y Z = \mathbb{E} \mu \mu_n^Y Z \Phi_n^Y Z - \frac{\mu^2}{2} \nu_n^Y Z \Phi_n^Y Z = U_1 + U_2, \] (98)

1) **The evaluation of \( U_1 \):** \( U_1 \) can be rewritten as [31 Eq. (4)]

\[ U_1 = (\mu \sqrt{n \int_{\pi^2} ^{\infty} N \Phi_n^Y Z z - \mathbb{E} \operatorname{Tr} Q(z) \Phi_n^Y Z d z}) \]
\[ - \frac{n}{\sqrt{M n}} \mathbb{E} \operatorname{Tr} Q H A H^H \Phi_n^Y Z = U_{1,1} + U_{1,2}. \] (99)

In the following, we will evaluate \( U_1 \) and \( U_2 \), respectively. We first evaluate \( \mathbb{E} \operatorname{Tr} Q(z) \Phi_n^Y Z \) in \( U_{1,1} \), which can be converted to the evaluation of \( \mathbb{E} \operatorname{Tr} Q(z)H H^H \Phi_n^Y Z \) by the resolvent identity \( I_N = z \Phi(z) + Q(z)H H^H \). By the integration by parts formula in [62], we have the evaluation in [100] at the top of the next page. Step (a) in [100] follows from

\[ \mathbb{E} a b \Phi_n^Y Z = \mathbb{E} a b \Phi_n^Y Z + \mathbb{E} a b \Phi_n^Y Z \]
\[ = \mathbb{E} a b \Phi_n^Y Z + \mathbb{E} a b \Phi_n^Y Z - \mathbb{E} a b \Phi_n^Y Z + \mathbb{E} a b \Phi_n^Y Z + O\left( \frac{\mathbb{P}(\frac{1}{\nu})}{2M} \right), \] (101)

where \( a = \frac{\mathbb{E} \operatorname{Tr} Q(z)Z Z^H}{M}, b = \mathbb{E} \operatorname{Tr} Q(z)H H^H \). The term \( O\left( \frac{\mathbb{P}(\frac{1}{\nu})}{2M} \right) \) follows from [65] and [66] of Lemma [2] in Appendix [5] with

\[ |\mathbb{E} a b \Phi_n^Y Z b| \leq \text{Var}^*(a) \text{Var}^*(b) = O\left( \frac{\mathbb{P}(\frac{1}{\nu})}{2M} \right). \] (102)

Step (b) in [100] follows from [69] in Lemma [5]. Similarly, by Lemma [2] we can obtain

\[ \mathbb{E} \epsilon_1 \Phi_n^Y Z = \mathbb{E} \epsilon_1 \Phi_n^Y Z + \text{Cov}(\epsilon_1, \Phi_n^Y Z), \] (103)

and \( \text{Cov}(\epsilon_1, \Phi_n^Y Z) \leq \text{Var}^*(\epsilon_1) \text{Var}^*(\Phi_n^Y Z) = O\left( \frac{\mathbb{P}(\frac{1}{\nu})}{2M} \right) \). By the integration by parts formula, we have

\[ \mathbb{E} \epsilon_1 = - \frac{\mu n}{\sqrt{M n}} \operatorname{Tr} Z Z^H Q(z)H A H^H Q \]
\[ - \mathbb{E} \operatorname{Tr} Q Z Z^H Q(z) \frac{\operatorname{Tr} Q(z)H H^H}{M} - \mathbb{E} \operatorname{Tr} Q Z Z^H \frac{\operatorname{Tr} Q(z)H H^H}{M} \]
\[ = E_1 + E_2 + E_3. \] (104)
Noticing that \( \text{Tr} A = \text{Tr}(I_M - \frac{Xn^m}{n}) = 0 \), we have \( E_1 = 0 \). By (67) and \( \text{Tr} A^2 = O(M^2) \), we have

\[
|E_2| = \frac{u_n}{M^4 n} \left| \frac{\text{Tr} QZZH^H Q(z) \text{Tr} Q(z) HAH^H}{M} + \text{Cov}(\text{Tr} QZZH^H Q(z), \frac{\text{Tr} Q(z) HAH^H}{M}) \right|
\leq \frac{u_n}{M^3 |z|^2} \sqrt{\text{Tr} A^2} = O\left(\frac{1}{Mz^2}\right).
\]

and similarly we have

\[
|E_3| \leq \frac{u_n}{M^3 |z|^2} \sqrt{\text{Tr} A^2} = O\left(\frac{1}{Mz^2}\right).
\]

Similar analysis can be performed to obtain \( E_\epsilon \Phi_n^{YZ} = O\left(\frac{1}{Mz^2}\right) \) and \( E_\epsilon \Phi_n^{YZ} + O\left(\frac{1}{Mz^2}\right) \) in the last line of (100). Now we turn to evaluate \( E_\epsilon \Phi_n^{YZ} \), which is performed as

\[
\mathbb{E}_{\epsilon}\Phi_n^{YZ} = -\mathbb{E} \sum_{i,j} \frac{u_2}{2M} \frac{\partial_n Y, Z}{\partial Y_{j,i}} [Z^H Q(z)]H_{j,i} = -\frac{u_2n}{M} \mathbb{E} \text{Tr} HHH^H QHH^H QZZ^H Q(z)
\]

\[
- \frac{1}{M} \frac{\text{Tr} HHH^H QHH^H QZZ^H Q(z)}{M} + \frac{\sigma^2}{2M} \frac{\text{Tr} X^H H^H Q^2 ZZZ^H Q(z)}{M} + \frac{\sigma^2}{2M} \frac{\text{Tr} XX^H H^H Q^2 ZZZ^H Q(z)}{M}
\]

\[
= \frac{u_2n}{M} [E_{3,1} + E_{3,2} + E_{3,3} + E_{3,4} + E_{3,5}].
\]

By the matrix inequality \( \text{Tr} CB \leq \|C\| \text{Tr} B \), where \( B \) is semi-definite positive, we have

\[
\left| \frac{n^2 E_{3,1}}{M} \right| \leq \frac{n}{M} \mathbb{E}\|Z\|^2 \|Y\|^4 \leq \frac{2n}{M} \frac{K N}{M \sigma^4 z} = O\left(\frac{1}{Mz^2}\right).
\]

Similarly, we can show that \( E_{3,1} = 2,...,5 \) are all \( O\left(\frac{1}{Mz^2}\right) \) terms. By a similar approach, we can also show that \( E_\epsilon \Phi_n^{YZ} = \mathbb{E} \sum_{i,j} \frac{u_3}{3M} \frac{\partial_n Y, Z}{\partial Y_{j,i}} [Z^H Q(z)]H_{j,i} = O\left(\frac{1}{Mz^2}\right) \). Now we turn to evaluate \( \mathbb{E} \text{Tr} Q(z) ZZ^H \Phi_n^{YZ} \) in the second last line of (100).

By the integration by parts formula, we have

\[
\mathbb{E} \text{Tr} Q(z) ZZ^H \Phi_n^{YZ} = \sum_{i,j} Z_{j,i}^H \sum Q(z) [Q(z)]_{j,i} \Phi_n^{YZ}
\]

\[
= \mathbb{E} \text{Tr} Q(z) \frac{1}{M} \text{Tr} HHH^H + \frac{u_4}{M} \frac{\text{Tr} Q(z) HHH^H Q(z)}{M} - \frac{u_4}{M} \frac{\text{Tr} Q(z) HHH^H Q(z)}{M}
\]

\[
+ \frac{u_4}{M} \frac{\text{Tr} Q(z) HHH^H QHH^H Q(z)}{M} - \frac{u_4}{M} \frac{\text{Tr} Q(z) HHH^H QHH^H Q(z)}{M}
\]

\[
= \frac{u_4}{M} \mathbb{E} \text{Tr} Q(z) ZZ^H \Phi_n^{YZ} + O\left(\frac{1}{Mz^2}\right).
\]

By substituting (100) into (110) to replace \( \mathbb{E} \text{Tr} Q(z) HHH^H QHH^H Q(z) \), we can solve \( \mathbb{E} \text{Tr} Q(z) ZZ^H \Phi_n^{YZ} \) to obtain

\[
\frac{u_4}{M} \mathbb{E} \text{Tr} Q(z) ZZ^H \Phi_n^{YZ} = \frac{M \delta_z \omega_z - M \delta_z \omega_z^2}{1 + \delta_z \omega_z} \mathbb{E} \Phi_n^{YZ} + \frac{1 - M \delta_z \omega_z^2}{1 + \delta_z \omega_z} \mathbb{E} \text{Tr} Q(z) \Phi_n^{YZ}
\]

\[
- \frac{u_4}{M} \frac{\delta_z \omega_z}{M \sigma^4 z^2} \mathbb{E} \text{Tr} Q(z) ZZ^H Q(z) HH^H \Phi_n^{YZ} + \frac{u_4}{M} \frac{\delta_z \omega_z}{M \sigma^4 z^2} \mathbb{E} \text{Tr} Q(z) HHH^H QHH^H Q(z) \Phi_n^{YZ} + O\left(\frac{1}{Mz^2}\right).
\]

Noticing that \( 1 - \frac{M \delta_z \omega_z^2}{L \delta_z \omega_z^2} = \frac{M \omega_z}{L \delta_z \omega_z^2} \) and plugging (111) into (100), we can obtain

\[
\frac{u_4}{M} \frac{\delta_z \omega_z}{M \sigma^4 z^2} \mathbb{E} \text{Tr} Q(z) ZZ^H Q(z) HH^H \Phi_n^{YZ} - \frac{u_4}{M \omega_z} \frac{\delta_z \omega_z}{M \sigma^4 z^2} \mathbb{E} \text{Tr} Q(z) HH^H Q(z) \Phi_n^{YZ} + O\left(\frac{1}{Mz^2}\right).
\]
Now we turn to evaluate $U_{1,1}$ in [99]. By the integration by parts formula, we can obtain (116) at the top of this page. According to Lemma 2 and $\text{Tr} A^2 = O(M^2)$, we can obtain

$$\text{Var}(\frac{1}{M} \text{Tr} Q\text{H}AH^H) = O(\frac{\text{Tr} A^2}{M^3}) = O(\frac{1}{M}),$$

and

$$\text{Var}(\frac{1}{M} \text{Tr} QZZ^HQ\text{H}AH^H) = O(\frac{1}{M}),$$

to derive

$$\frac{1}{M} \text{E Tr } Q\text{H}AH^H = \frac{1}{M^2} \text{E Tr } QZZ^H \text{Tr } A - \frac{1}{M^2} \text{E Tr } QZZ^H \text{Tr } Q\text{H}AH^H = - \frac{\text{Cov}(\frac{1}{M} \text{Tr } Q\text{H}AH^H, \frac{1}{M} \text{Tr } QZZ^H)}{1 + \text{E Tr } QZZ^H} = O(\frac{\sqrt{\text{Var} A^2}}{M^2}) = O(\frac{1}{M^2}).$$
Similarly, we have
\[ \frac{1}{M} \mathbb{E} \text{Tr} QZZ^H QHAH^H = O\left( \frac{1}{M^{\frac{3}{2}}} \right), \] (120)
so that the third term in (116) is \( O\left( \frac{1}{\sqrt{M}} \right) \). Therefore, we can obtain
\[ \left| \frac{1}{M} \mathbb{E} g_n y^i z^j \right| \leq \frac{1}{M} \text{Var}^* a \text{Var}^* b = O\left( \frac{\sqrt{\text{Tr} A^2}}{M^{\frac{3}{2}}} \right), \] (121)
where \( a = \text{Tr} QHAH^H \) and \( b = \text{Tr} QZZ^H \). Considering the technique used in (101), we can evaluate the second term in (116) as
\[ -\frac{1}{M} \mathbb{E} \text{Tr} QZZ^H \text{Tr} QHAH^H \Phi_n^{y,z} = \frac{\mathbb{E} \text{Tr} QZZ^H}{M} \Phi_n^{y,z} - \omega \mathbb{E} \text{Tr} QHAH^H \Phi_n^{y,z} + O\left( \frac{1}{\sqrt{M}} \right), \] (122)
where \( \frac{\mathbb{E} \text{Tr} QZZ^H}{M} \). Similarly, we have
\[ |\mathbb{E} w^2 \partial_\nu y^{i,j} \frac{\partial}{\partial y} | \left| \frac{Z^H QHAH^H \Phi_n^{y,z}}{M} \right| = O\left( \frac{a^2}{M^2} + 1 \right), \] (123)
and
\[ |\mathbb{E} w^2 \partial_\nu y^{i,j} \frac{\partial}{\partial y} | \left| \frac{Z^H QHAH^H \Phi_n^{y,z}}{M} \right| = O\left( \frac{a^2}{M^2} + 1 \right). \] (124)

Therefore, by noticing \( \text{Tr} A = 0 \), moving \( -\omega \mathbb{E} \text{Tr} QHAH^H \Phi_n^{y,z} \) in (122) to the LHS of (116), and solving \( \mathbb{E} \text{Tr} QHAH^H \Phi_n^{y,z} \), (116) can be rewritten as
\[ \mathbb{E} \text{Tr} QHAH^H \Phi_n^{y,z} \]
\[ = \frac{\sqrt{\omega n}}{M} \sum_{i,j} y^{i,j} \left| Z^H QHAH^H QZZ^H QHAH^H \Phi_n^{y,z} \right| \]
\[ = \mathbb{E} \left\{ \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \right\} \]
\[ = \frac{\sqrt{\omega n}}{M} (X_1 - X_2) \Phi_n^{y,z} + O\left( \frac{\text{Tr} A^2}{M^{\frac{3}{2}}} \right). \] (125)

Now we will compute \( X_1 - X_2 \). By the integration by parts formula and the variance control in (124), we have
\[ X_1 = \mathbb{E} \left\{ \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \right\} \]
\[ = \mathbb{E} \left\{ \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \right\} \] (126)
\[ = \mathbb{E} \left\{ \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \frac{\text{Tr} QZZ^H}{M} \right\} \] (127)
By noticing that \( \text{Tr} A = 0 \), we can obtain that \( A_2 = 0 \). \( A_3 \) can be evaluated as
\[ A_3 = -\omega X_1 + O\left( \frac{\text{Tr} A^2}{M^{\frac{3}{2}}} \right), \] (128)
according to the bound of the variance in (124) and the evaluation of (69) in Lemma 3. By \( \text{Var}(a) = (\mathbb{E} a)^2 + \text{Var}(a) \) and (118), \( A_4 \) can be evaluated as
\[ A_4 = -\omega X_1 + O\left( \frac{1}{M} \right) = O\left( \frac{1}{M} \right), \] (129)
where $a = \frac{\text{Tr} QZ^H Q H H^H}{M}$ and the last equality follows from $\text{E}a = O\left(\frac{1}{M^2}\right)$, which can be derived by a similar approach as (119). Define $c = \frac{\text{Tr} QZ^H Q H H^H}{M}$ and $d = \frac{\text{Tr} Q H H}{M}$. By (68) in Lemma 2 we have $\text{Var}(c) = O\left(\frac{\text{Tr} A^2}{M}\right) = O\left(\frac{1}{M}\right)$ and $\text{Var}(d) = O\left(\frac{\text{Tr} A^2}{M}\right) = O\left(\frac{1}{M}\right)$. Therefore, $A_5$ can be handled by

$$A_5 = -\text{E}c\text{Ed} + \text{Cov}(c, d) = O\left(\frac{1}{M^2}\right) + O\left(\frac{1}{M}\right),$$

(130)

where $\text{E}c$ and $\text{Ed}$ can be evaluated by the same approach as (119). By moving $\omega X_1$ from $A_3$ to the LHS of (126) and using the evaluation of $A_1$ in (127), we can solve $X_1$

$$X_1 = \omega A_1 + O\left(\frac{\text{Tr} A^2}{M}\right) = \omega \omega \text{E} \text{Tr} QZ^H Q H H^H + O\left(\frac{\text{Tr} A^2}{M^2}\right).$$

(131)

Then we will evaluate $X_2$. By the integration by parts formula (62), we have

$$X_2 = \text{E} \frac{1}{M} \sum_{i,j} Y_{i,j}^* |Z^H QZ^H Q H H^H|_{i,j} = \text{E} \left\{ \frac{\text{Tr} A^2 \text{Tr} QZ^H Q H H^H}{M} \right\}$$

$$= \text{Tr} QZ^H Q H H^H - \text{Tr} QZ^H Q H H^H \frac{\text{Tr} QZ^H Q H H^H}{M} - \omega \text{E} \text{Tr} QZ^H Q H H^H + O\left(\frac{\text{Tr} A^2}{M^2}\right)$$

(132)

$$= \frac{\sigma^2}{M^2} \text{Tr} A^2 \text{Tr} QZ^H Q H H^H + O\left(\frac{\text{Tr} A^2}{M^2}\right),$$

where step (a) in (132) is obtained by the variance control in (124) and the evaluations in (69). Based on (131) and (132), we have

$$- \frac{n}{\sqrt{Mn}} (\omega X_1 - \omega X_2) = \frac{n^2 \sigma^4}{M^2} \text{Tr} A^2 \text{Tr} QZ^H Q H H^H + O\left(\frac{1}{M}\right)$$

(133)

$$= \frac{n^2 \sigma^4}{M^2} \text{Tr} A^2 \left(1 + \frac{\delta^2}{M^2} \omega^2 \right) + \frac{\omega \text{E} \text{Tr} Q^2 Z Z^H}{M^2} + O\left(\frac{1}{M}\right),$$

where step (a) in (133) follows from (76). By (128), we can obtain that

$$U_{1,2} = \frac{\rho \text{Tr} A^2 \left(\frac{1 + \frac{\delta^2}{M^2} \omega^2}{\sqrt{Mn}} - \frac{\omega \text{E} \text{Tr} Q^2 Z Z^H}{M} \right)}{M^2}$$

(134)

By substituting (115) and (133) into (99), we complete the evaluation of $U_1$ in (98).

2) The evaluation of $U_2$: By the resolvent identity, we have

$$\text{E} \frac{1}{M} \text{Tr} Q H H^H Q H H^H + \frac{2 \sigma^2}{Mn} \text{Tr} Q^2 H \frac{n}{M} H^H | \Phi_n^Y Z$$

$$= \text{E} \frac{1}{M} \text{Tr} Q H H^H Q H H^H + \frac{2 \sigma^2}{Mn} \text{Tr} Q^2 H \frac{n}{M} H^H | \Phi_n^Y Z$$

(135)

$$= \left(\frac{\rho \text{Tr} A^2 \left(\frac{1 + \frac{\delta^2}{M^2} \omega^2}{\sqrt{Mn}} - \frac{\omega \text{E} \text{Tr} Q^2 Z Z^H}{M} \right)}{M^2}\right)^2$$

from which we can obtain the evaluation of $U_2$. By far, we have completed the evaluation of (98).

C. Step 3: Convergence of $\text{E} \frac{w^Y Z}{\sqrt{n} \nu_n^Y}$

By the evaluations in Appendix C-B, we have the following approximation

$$\frac{\partial \psi^\alpha Y Z(u)}{\partial u} = \left(\frac{\nu_n^Y Z(u)}{Mn} - \frac{\omega^2}{2} \nu_n^Y Z(u)\right) \Phi_n^Y Z(u) + O\left(\frac{1}{M^2 u^2}\right)$$

(136)

$$= \left(\frac{\nu_n^Y Z(u)}{Mn} - \frac{\omega^2}{2} \nu_n^Y Z(u)\right) \Phi_n^Y Z(u) + O\left(\frac{1}{M^2 u^2}\right)$$

$$= \left(\frac{\nu_n^Y Z(u)}{Mn} - \frac{\omega^2}{2} \nu_n^Y Z(u)\right) \Phi_n^Y Z(u) + O\left(\frac{1}{M^2 u^2}\right) + \varepsilon^\alpha(u)$$

$$= \left(\frac{\nu_n^Y Z(u)}{Mn} - \frac{\omega^2}{2} \nu_n^Y Z(u)\right) \Phi_n^Y Z(u) + O\left(\frac{1}{M^2 u^2}\right) + \varepsilon^\alpha(u),$$
where $E^A(u) = \mathcal{O}\left(\frac{P(u)(\sqrt{\tau\mathbb{M}}^2 + u\sqrt{\mathbb{M}}^2)}{M^2}\right)$. Step (a) follows from (97) in Lemma 5. Solving $\Psi^{W,Y,Z}(u)$ from the differential equation in (136), we can obtain

$$
\psi_{W,Y,Z}(u) = e^{ju\sqrt{\mathbb{M}C}(\sigma^2)} \cdot \mathcal{O}\left(\frac{2u}{\sqrt{\mathbb{M}}^2}\right)
\times \left(1 + \int_0^u e^{-j\sqrt{\mathbb{M}C}(\sigma^2)} \cdot \mathcal{O}\left(\frac{2u}{\sqrt{\mathbb{M}}^2}\right) dv\right).
$$

(137)

Therefore, denoting $\gamma_n = \sqrt{nMC}(\sigma^2)$, the characteristic function of the normalized version of $\gamma_n$ can be written as

$$
\psi_{\text{norm}}(u) = e^{ju\sqrt{nM}C} - \mathcal{O}\left(\frac{2u}{\sqrt{\mathbb{M}}^2}\right)
\times \left(1 + \int_0^u e^{-j\sqrt{nM}C} - \mathcal{O}\left(\frac{2u}{\sqrt{\mathbb{M}}^2}\right) dv\right).
$$

(138)

where step (a) in (138) follows from

$$
\mathcal{O}\left(\frac{u\mathbb{M}^A(u)}{\sqrt{\mathbb{M}^A(u)}}\right) = \mathcal{O}\left(\frac{\mathbb{M}^A(u)}{\sqrt{\mathbb{M}^A(u)}}\right) = \mathcal{O}\left(\frac{\mathbb{M}^A(u)}{\sqrt{\mathbb{M}^A(u)}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\mathbb{M}}}\right),
$$

(139)

and $\tau = \mathcal{O}(1)$ is the coefficient of $\frac{\tau\mathbb{M}^2}{M}$ in (23). By (138) and Lévy’s continuity theorem, there holds true that

$$
\frac{\gamma_n - \gamma_n}{\sqrt{\mathbb{M}^A(u)}} \overset{\mathbb{P}}{\to} \mathcal{N}(0,1),
$$

(140)

which concludes the proof of Theorem 2.

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