EXCLUDING LONG PATHS

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Abstract. Ding (1992) proved that for each integer $m \geq 0$, and every infinite sequence of finite simple graphs $G_1, G_2, \ldots$, if none of these graphs contains a path of length $m$ as a subgraph, then there are indices $i < j$ such that $G_i$ is isomorphic to an induced subgraph of $G_j$. We generalise this result to infinite graphs, possibly with parallel edges and loops.

Keywords. tree-decomposition, tree-width, tree-diameter, well-quasi-ordering, better-quasi-ordering.

1. Introduction and main results

All graphs in this paper are undirected. Unless stated otherwise, a graph may be finite or infinite, and may contain parallel edges and loops. Let $m \geq 0$ be an integer. We use $P_m$ to denote a path with $m$ edges (and $m + 1$ vertices).

Robertson and Seymour [11] proved that the finite graphs are well-quasi-ordered by the minor relation. Thomas [12] found an example showing that the infinite graphs are not well-quasi-ordered by the minor relation. Later, Thomas [13] proved that the finite or infinite graphs without a given finite planar graph as a minor are well-quasi-ordered (furthermore, better-quasi-ordered) by the minor relation.

A Robertson chain of length $m$ is the graph obtained by duplicating each edge of $P_m$. Robertson conjectured in 1980’s that the finite graphs without a Robertson chain of length $m$ as a topological minor are well-quasi-ordered by the topological minor relation. This conjecture was proved by Liu [6].

By considering the type of a finite simple graph, Ding [1] proved that, for each integer $m \geq 0$, the finite simple graphs without $P_m$ as a subgraph are well-quasi-ordered by the induced subgraph relation. Another proof, based on the tree-depth, was given by Nešetřil and Ossona de Mendez [9]. We generalise Ding’s theorem to infinite graphs, possibly with parallel edges and loops.

_bin Jia gratefully acknowledges scholarships provided by The University of Melbourne._
Theorem 1.1. Given a finite graph $H$, the graphs (respectively, of bounded multiplicity) without $H$ as a subgraph are better-quasi-ordered by the (respectively, induced) subgraph relation if and only if $H$ is a disjoint union of paths.

Let $t \geq 1$ be an integer. A $t$-dipole is a graph with two vertices and $t$ edges between them. Clearly, a $t$-dipole does not contain $P_2$ as a subgraph. Further, the set of $t$-dipoles, for $t = 1, 2, \ldots$, is well-quasi-ordered by the subgraph relation, but not by the induced subgraph relation.

Our method in dealing with the graphs without $P_m$ as a subgraph is different from the methods of Ding [1] and Nešetřil and Ossona de Mendez [9]. Instead of studying the type or the tree-depth of a graph $G$, we prove Theorem 1.1 by investigating the tree-decompositions of $G$. A key step is to show that, if $G$ does not contain $P_m$ as a subgraph, then $G$ admits a tree-decomposition which attains the minimum width such that the diameter of the tree for the tree-decomposition is bounded by a function of $m$. Then we prove Theorem 1.1 by induction on the diameter.

2. Terminology

This section presents some necessary definitions and basic results about tree-decompositions and quasi-orderings of graphs.

A binary relation on a set is a quasi-ordering if it is reflexive and transitive. A quasi-ordering $\leq$ on a set $Q$ is a well-quasi-ordering if for every infinite sequence $q_1, q_2, \ldots$ of $Q$, there are indices $i < j$ such that $q_i \leq q_j$. And if this is the case, then $q_i$ and $q_j$ are called a good pair, and $Q$ is well-quasi-ordered by $\leq$.

Let $G$ be a hypergraph, $T$ be a tree, and $V := \{V_v \mid v \in V(T)\}$ be a set cover of $V(G)$ indexed by $v \in V(T)$. The pair $(T, V)$ is called a tree-decomposition of $G$ if the following two conditions are satisfied:

- for each hyperedge $e$ of $G$, there exists some $V \in V$ containing all the vertices of $G$ incident to $e$;
- for every path $[v_0, \ldots, v_i, \ldots, v_m]$ of $T$, we have $V_{v_0} \cap V_{v_m} \subseteq V_{v_i}$.

The width $tw(T, V)$ of $(T, V)$ is $\sup\{|V| - 1 \mid V \in V\}$. The tree-width $tw(G)$ of $G$ is the minimum width of a tree-decomposition of $G$. The tree-diameter $tdi(G)$ of $G$ is the minimum diameter of $T$ over the tree-decompositions $(T, V)$ of $G$ such that $tw(T, V) = tw(G)$.

Let $[v_0, e_1, v_1, \ldots, e_{m-1}, e_m, v_m]$ be a path of $T$. Denote by $V_T(v_0, v_m)$ the set of minimal sets, up to the subset relation, among $V_{v_i}$ and $V_{e_j} := V_{v_{j-1}} \cap V_{v_j}$ for $i \in \{0, 1, \ldots, m\}$ and $j \in [m] := \{1, 2, \ldots, m\}$. We identify repeated sets in $V_T(v_0, v_m)$.

A tree-decomposition $(T, V)$ of $G$ is said to be linked if

- for every pair of nodes $u$ and $v$ of $T$, and subsets $U$ of $V_u$ and $V$ of $V_v$ such that $|U| = |V| := k$, either $G$ contains $k$ disjoint paths from $U$ to $V$, or there exists some $W \in V_T(u, v)$ such that $|W| < k$. 
Kruskal’s theorem [4] states that finite trees are well-quasi-ordered by the topological minor relation. Nash-Williams [7] generalised this theorem and proved that infinite trees are better-quasi-ordered by the same relation. Let \( A \) be the set of all finite ascending sequences of nonnegative integers. For \( A, B \in A \), write \( A <_A B \) if \( A \) is a strict initial subsequence of some \( C \in A \), and by deleting the first term of \( C \), we obtain \( B \). Let \( B \) be an infinite subset of \( A \), and \( \bigcup B \) be the set of nonnegative integers appearing in some sequence of \( B \). \( B \) is called a block if it contains an initial subsequence of every infinite increasing sequence of \( \bigcup B \). For \( A, B \in A \), write \( A < A B \) if \( A \) is a strict initial subsequence of some \( C \in A \), and by deleting the first term of \( C \), we obtain \( B \). Let \( B \) be an infinite subset of \( A \), and \( \bigcup B \) be the set of nonnegative integers appearing in some sequence of \( B \). Let \( Q \) be a set with a quasi-ordering \( \leq_Q \). A \( Q \)-pattern is a function from a block \( B \) into \( Q \). A \( Q \)-pattern \( \varphi \) is good if there exist \( A, B \in B \subseteq A \) such that \( A <_A B \) and \( \varphi(A) \leq_Q \varphi(B) \). \( Q \) is said to be better-quasi-ordered by \( \leq_Q \) if every \( Q \)-pattern is good. For example, the set of nonnegative integers is better-quasi-ordered by the natural ordering. It follows from the definitions that a better-quasi-ordering is a well-quasi-ordering. And \( Q \) is better-quasi-ordered if and only if each subset of \( Q \) is better-quasi-ordered.

For an integer \( j \geq 1 \), define a quasi-ordering on \( Q^j \) as follows: \( (q_1, \ldots, q_j) \leq_{Q^j} (q'_1, \ldots, q'_j) \) if \( q_i \leq_Q q'_i \) for every \( i \in [j] \). The following lemma follows from the Galvin-Prikry theorem [2] (see also [14, (3.11)] and [5, Lemma 3]).

**Lemma 2.1.** Let \( k \geq 1 \) be an integer, and \( Q = \bigcup_{i=1}^k Q_i \) be a quasi-ordered set. Then the following statements are equivalent:

1. \( Q \) is better-quasi-ordered.
2. \( Q_i \) is better-quasi-ordered for every \( i \in [k] \).
3. \( Q^j \) is better-quasi-ordered for every integer \( j \geq 1 \).

Define a quasi-ordering \( \leq \) on the powerset of \( Q \) as follows. For \( S_1, S_2 \subseteq Q \), write \( S_1 \leq S_2 \) if there is an injection \( \varphi \) from \( S_1 \) to \( S_2 \) such that \( q \leq_Q \varphi(q) \) for every \( q \in S_1 \).

Let \( Q \) be a set with a quasi-ordering \( \leq_Q \). Let \( S \) be a set of sequences whose elements are from \( Q \). For \( S_1 := (q_1, q_2, \ldots) \in S \) and \( S_2 \in S \), we say \( S_1 \leq_S S_2 \) if there is a subsequence \( S_3 := (p_1, p_2, \ldots) \) of \( S_2 \) such that \( S_1 \) and \( S_3 \) have the same length, and that \( q_i \leq_Q p_i \) for every index \( i \) used in \( S_1 \). The following results are due to Nash-Williams [8].

**Lemma 2.2** ([8]). Every finite quasi-ordered set is better-quasi-ordered. And each better-quasi-ordering is a well-quasi-ordering. Moreover, a quasi-ordered set \( Q \) is better-quasi-ordered if and only if the powerset of \( Q \) is better-quasi-ordered if and only if every set of sequences whose elements are from \( Q \) is better-quasi-ordered.

### 3. Graphs without \( P_m \) as a subgraph

In this section, we show that a graph without a given finite path as a subgraph has bounded tree-diameter. We achieve this by modifying a given tree-decomposition \((T, \mathcal{V})\) of \( G \) such that \( \text{diam}(T) \) is reduced but \( \text{tw}(T, \mathcal{V}) \) remain unchanged.
Lemma 3.1. Let \( m \geq 3 \) be an integer, and \((T, \mathcal{V})\) be a tree-decomposition of a graph \( G \), and \([v_0, \ldots, v_m]\) be a path of \( T \). Let \( U \subseteq V(G) \). Then \( U = V_{v_0} \cap V_{v_1} = V_{v_{m-1}} \cap V_{v_m} \) if and only if \( U = V_{v_i} \cap V_{v_0} = V_{v_i} \cap V_{v_m} \) for all \( i \in [m-1] \).

Proof. \((\Leftarrow)\) Let \( i = 1 \), we have \( U = V_{v_0} \cap V_{v_1} \). Let \( i = m - 1 \), we have \( U = V_{v_{m-1}} \cap V_{v_m} \). 

\((\Rightarrow)\) Since \( U = V_{v_0} \cap V_{v_1} = V_{v_0} \cap V_{v_m} \), we have \( U = V_{v_0} \cap V_{v_t} \cap V_{v_{m-1}} \cap V_{v_m} \subseteq V_{v_0} \cap V_{v_m} \). Since \( 1 \leq i \leq m - 1 \), by the definition of a tree-decomposition, \( V_{v_0} \cap V_{v_i} \subseteq V_{v_i} \), and \( V_{v_0} \cap V_{v_{m-1}} \subseteq V_{v_{m-1}} \). So \( V_{v_0} \cap V_{v_m} \subseteq V_{v_0} \cap V_{v_i} \subseteq V_{v_0} \cap V_{v_{m-1}} \cap V_{v_m} \). Thus \( U = V_{v_0} \cap V_{v_i} = V_{v_0} \cap V_{v_{m-1}} \cap V_{v_m} \). Symmetrically, \( U = V_{v_t} \cap V_{v_m} \), and hence \( U = V_{v_0} \cap V_{v_i} = V_{v_t} \cap V_{v_m} \).

We list the operation that can be used to reduce \( \text{diam}(T) \) for \((T, \mathcal{V})\).

Operation 3.2. Let \((T, \mathcal{V})\) be a tree-decomposition of a finite graph \( G \) such that \( T \) is a finite tree. Let \( U \subseteq V(G) \) such that \( E_U := \{ e \in E(T) \mid \ V_e = U \} \) is not empty. Let \( T_U \) be the minimal subtree of \( T \) containing \( E_U \), and \( u \) be a center of \( T_U \). For each \( e \in E_U \) with end vertices \( v, w \in V(T) \setminus \{ u \} \) such that \( u \) is closer to \( v \) than to \( w \), delete \( e \) and add an extra edge between \( w \) and \( u \).

Let \( T' \) be obtained from \( T \) by applying Operation 3.2 to a subset \( U \) of \( V(G) \). Let \( E' := E(T') \setminus E(T) \). In Figure 1 \( \text{diam}(T) = 6 \), and the bold edges represent the edges of \( E' \). During the operation, the bold edge incident to \( u \) does not change. Other bold edges are deleted. The curve edges in \( T' \) represent the edges in \( E_U \). Note that \( \text{diam}(T') = 4 \), less than the diameter of \( T \).
Figure 1. Reducing the diameter of the tree for a tree-decomposition

Our next lemma is useful in proving that \((T', \mathcal{V})\) is a tree-decomposition of \(G\).

**Lemma 3.3.** Let \((T, \mathcal{V})\) be a tree-decomposition of a finite graph \(G\) such that \(T\) is a finite tree. Then \(T'\) is a finite tree. And for every pair of \(x, y \in V(T) = V(T')\), we have \(\mathcal{V}_T(x, y) = \mathcal{V}_{T'}(x, y)\).

**Proof.** If \(E_U = \emptyset\) or \(\text{diam}(T_U) \leq 2\), then \(T' = T\), and the lemma follows trivially. Now assume that \(\text{diam}(T_U) \geq 3\). By Operation \([3.2]\) we have that \(T'\) is connected, \(V(T') = V(T)\) and \(|E(T')| = |E(T)|\). So \(T'\) is a finite tree.

Let \(P\) and \(Q\) be paths from \(x\) to \(y\) in \(T\) and \(T'\) respectively. By Operation \([3.2]\) we have that \(E(P) \cap E_U = \emptyset\) if and only if \(E(Q) \cap E_{T'} = \emptyset\). And if this is the case, then \(P = Q\) and there is nothing to show. Now suppose that \(E(P) \cap E_U \neq \emptyset\).

For each \(f \in E(P) \cup E(Q)\), there are three cases: First, \(f \in E_U \cup E_{T'}\) and \(V_f = U\). Second, \(V_f \neq U\), and there are two edges \(e, e' \in E_U\) such that \(f\) is on the path of \(T\) between \(e\) and \(e'\). In this case, since \((\mathcal{V}, T)\) is a tree-decomposition of \(G\), we have that \(U = V_e \cap V_{e'} \subset V_f\).

Last, \(f \notin E_U \cup E_{T'}\) and \(f\) is not between two edges of \(E_U\) in \(T\). In this situation, assume for a contradiction that \(f\) is in a cycle of \(T \cup T'\). By Operation \([3.2]\) \(f\) is between \(u\) and an edge \(e \in E_U\) in \(T\). Note that \(u\) is a center of \(T_U\). So there is another edge \(e' \in E_U\) such that \(u\) is on the path of \(T\) from \(e\) to \(e'\), a contradiction. Thus \(f\) is a bridge of \(T \cup T'\). So \(f \in E(P) \cap E(Q)\).

By the analysis above, both \(\mathcal{V}_T(x, y)\) and \(\mathcal{V}_{T'}(x, y)\) are

\[
\min\{U, V_f \mid f \in E(P) \cap E(Q)\}.
\]

A tree-decomposition \((T, \mathcal{V})\) is **short** if for every pair of different \(e, f \in E(T)\), if \(V_e = V_f\), then \(e\) and \(f\) are incident in \(T\). Let \(T^*\) be obtained from \(T\) by applying Operation \([3.2]\) to each \(U \subseteq V(G)\). In the following, we verify that \((T^*, \mathcal{V})\) is a short tree-decomposition.
Lemma 3.4. Let \((T, \mathcal{V})\) be a tree-decomposition of a finite graph \(G\) such that \(T\) is a finite tree. Then all the following statements hold:

1. \(T^*\) is a finite tree such that \(\text{diam}(T^*) \leq \text{diam}(T)\).
2. \((T^*, \mathcal{V})\) is a tree-decomposition of \(G\) such that \(\text{tw}(T^*, \mathcal{V}) = \text{tw}(T, \mathcal{V})\).
3. \((T^*, \mathcal{V})\) is linked if and only if \((T, \mathcal{V})\) is linked.
4. For \(e, f \in E(T^*)\), if \(V_e = V_f\), then \(e\) and \(f\) are incident in \(T^*\).

Proof. (1) follows from Operation 3.2 and Lemma 3.3.

For (2), let \(x, y \in V(T^*) = V(T)\). Since \((T, \mathcal{V})\) is a tree-decomposition, \(V_x \cap V_y\) is a subset of every set in \(\mathcal{V}_T(x, y)\). By Lemma 3.3, \(V_x \cap V_y\) is a subset of every set in \(\mathcal{V}_{T^*}(x, y)\). Let \(z \in V(T)\) be on the path of \(T\) from \(x\) to \(y\). By the definition of \(\mathcal{V}_{T^*}(x, y)\), there exists some \(V \in \mathcal{V}_{T^*}(x, y)\) such that \(V \subseteq V_z\). So \(V_x \cap V_y \subseteq V_z\) and hence \((T^*, \mathcal{V})\) is a tree-decomposition of \(G\). Operation 3.2 does not change a set in \(\mathcal{V}\), so \(\text{tw}(T^*, \mathcal{V}) = \text{tw}(T, \mathcal{V})\).

By Lemma 3.3, for every pair of \(x, y \in V(T)\), we have \(\mathcal{V}_T(x, y) = \mathcal{V}_{T^*}(x, y)\). So (3) follows from the definition of a linked tree-decomposition.

(4) follows from Operation 3.2.

Introduced by Kríž and Thomas [3], an \(M\)-closure of a simple graph \(G\) is a triple \((T, \mathcal{V}, X)\), where \(X\) is a chordal graph without a complete subgraph of order \(\text{tw}(G) + 2\), \(V(G) = V(X)\), \(E(G) \subseteq E(X)\), and \((T, \mathcal{V})\) is a linked tree-decomposition of \(X\) such that each part induces a maximal complete subgraph of \(X\). An \(M\)-closure is short if the tree-decomposition is short.

Lemma 3.5. Every graph \(G\) of finite tree-width, with or without loops, admits a short linked tree-decomposition of width \(\text{tw}(G)\).

Proof. It is enough to consider the case that \(G\) is a simple graph. By Kríž and Thomas [3, (2.3)], every finite simple graph has an \(M\)-closure \((T, \mathcal{V}, X)\). By Lemma 3.4, \((T^*, \mathcal{V})\) is a short \(M\)-closure. In [3, (2.4)], replacing ‘an \(M\)-closure’ with ‘a short \(M\)-closure’ causes no conflict. So \(G\) has a short \(M\)-closure. The rest of the lemma follows from a discussion similar with [3, (2.2)].

Let \(s \geq 0\) be an integer. For \(i = 1, 2, \ldots\), let \(T_i\) be a tree with a center \(v_i\) and of diameter at most \(s\). Let \(T\) be obtained from these trees by adding an edge from \(v_1\) to each of \(v_2, v_3, \ldots\). Then \(\text{diam}(T) \leq 2\lfloor \frac{s}{2} \rfloor + 2 \leq s + 3\). Thus we have the following observation.

Observation 3.6. Let \(G\) be a graph, and \(s\) be the maximum tree-diameter of a connected component of \(G\). Then \(\text{tdi}(G) \leq s + 3\).

We now show that graphs without a given path as a subgraph have bounded tree-diameter.

Lemma 3.7. Let \(G\) be a graph without \(P_m\) as a subgraph. Then \(G\) admits a linked tree-decomposition \((T, \mathcal{V})\) such that \(\text{tw}(T, \mathcal{V}) = \text{tw}(G) \leq m - 1\), and \(\text{diam}(T) \leq 2(m^2 - m + 2)^m + 1\). And if \(G\) is connected, then \(\text{diam}(T) \leq 2(m^2 - m + 2)^m - 2\).
Proof. Let \( X \) be a finite subgraph of \( G \). Suppose for a contradiction that \( \text{tw}(X) \geq m \). Then by Robertson and Seymour [10], \( X \) contains a path of length \( m \), a contradiction. So \( \text{tw}(X) \leq m - 1 \). By a compactness theorem for the notion of tree-width [13, 15], we have that \( \text{tw}(G) \leq m - 1 \).

For the tree-diameter, by Observation [3.6] we only need to consider the case that \( G \) is nonnull and connected. By Lemma [3.3] \( G \) admits a short linked tree-decomposition \((T, \mathcal{V})\) of width \( \text{tw}(G) \). Let \( p := \text{tw}(G) + 1 \in [m] \).

Let \( P := [v_0, e_1, \ldots, e_s, v_s] \) be a path of length \( s \geq 1 \) in \( T \). We say \( P \) is \( t \)-rotund, where \( t \in [s] \), if there exists some \( k \in [p] \) and a sequence \( 1 \leq i_1 < \ldots < i_t \leq s \) such that \( V_{e_{i_1}}, \ldots, V_{e_{i_t}} \) are pairwise distinct, \( |V_{e_{i_j}}| = k \) for all \( j \in [t] \), and \( |V_{e_j}| \geq k \) for all \( j \) such that \( i_1 \leq j \leq i_t \). Let \( s^* \in [s] \) be the maximum number of edges of \( P \) corresponding to pairwise different subsets of \( V(G) \).

Claim. \( s \leq 2s^* \). If \( s \geq 2s^* + 1 \), then there are \( 1 \leq j_1 < j_2 < j_3 \leq s \) such that \( V_{e_{j_1}} = V_{e_{j_2}} = V_{e_{j_3}} \), contradicting the shortness of \((T, \mathcal{V})\).

Claim. If \( P \) is not \( t \)-rotund, then \( s^* \leq tp - 1 \). To see this, let \( s_k \) be the maximum number of edges of \( P \) corresponding to pairwise different subsets with \( k \) vertices of \( V(G) \). Since \( P \) is not \( t \)-rotund, we have that \( s_1 \leq t - 1 \). More generally, for \( k \geq 2 \), we have \( s_k \leq (s_1 + \ldots + s_{k-1} + 1)(t-1) \). By induction on \( k \) we have that \( s_k \leq tk^{-1}(t-1) \) for each \( k \in [p] \). So \( s^* = s_1 + \ldots + s_p \leq tp - 1 \).

Claim. If \( P \) is \( t \)-rotund, then \( t \leq p(m - 1) + 1 \). To prove this, recall that \((T, \mathcal{V})\) is a linked tree-decomposition. So there are \( k \) disjoint paths in \( G \) with at least \( |\bigcup_{j=1}^j V_{e_{i_j}}| \geq k + t - 1 \) vertices. Since \( G \) does not contain \( P_m \) as a subgraph, each of these \( k \) paths contains at most \( m \) vertices. So \( k + t - 1 \leq km \). As a consequence, \( t \leq k(m - 1) + 1 \leq p(m - 1) + 1 \).

Now let \( t \) be the maximum integer such that \( P \) is \( t \)-rotund. By the third claim, \( t \leq p(m - 1) + 1 \). Since \( P \) is not \((t+1)\)-rotund, by the second claim, \( s^* \leq (t+1)p-1 \). Thus \( s \leq 2s^* \leq 2[(t+1)p - 1] \leq 2[p(m - 1) + 2]p - 2 \leq 2(m^2 - m + 2)m - 2 \). 

4. Better-quasi-ordering

This section shows some better-quasi-ordering results for graphs without a given path as a subgraph.

A rooted hypergraph is a hypergraph \( G \) with a special designated subset \( r(G) \) of \( V(G) \). Note that \( r(G) \) can be empty. Let \( Q \) be a set with a quasi-ordering \( \leq Q \). A \( Q \)-labeled rooted hypergraph is a rooted hypergraph \( G \) with a mapping \( \sigma : E(G) \mapsto Q \). The lemma below says that graphs with finitely many vertices (respectively, of bounded multiplicity) are better-quasi-ordered by the (respectively, induced) subgraph relation.

Lemma 4.1. Let \( Q \) be a better-quasi-ordered set, and \( \mathcal{G} \) be a sequence of \( Q \)-labeled rooted hypergraphs (respectively, of bounded multiplicity) whose vertex sets are the subsets of \([p]\), where \( p \geq 1 \) is an integer. For \( X, Y \in \mathcal{G} \), denote by \( X \subseteq Y \) (respectively, \( X \leq Y \)) that \( r(X) = r(Y) \), and there is an isomorphism \( \varphi \) from
X to a (respectively, an induced) subgraph of Y such that for all \( i \in V(X) \) and \( e \in E(X) \), we have that \( \varphi(i) = i \) and \( \sigma(e) \leq Q \sigma(\varphi(e)) \). Then \( G \) is better-quasi-ordered by \( \subseteq \) (respectively, \( \leq \)).

**Proof.** There are \( \sum_{i=0}^{p} (\binom{p}{i}) 2^i = 3^p \) choices for vertex sets and roots. So by Lemma 2.1 it is safe to assume that all \( G \in \mathcal{G} \) have the same vertex set, say \([p] \), and the same root.

Then each \( G \) can be seen as a sequence of length \( 2^p - 1 \), indexed by the nonempty subsets of \([p] \). And for each nonempty \( V \subseteq [p] \), the term of the sequence indexed by \( V \) is the collection of elements of \( Q \) that are used to label the hyperedges \( e \) of \( G \) such that the set of end vertices of \( e \) is \( V \). By Lemmas 2.1 and 2.2 \( G \) is better-quasi-ordered by \( \subseteq \).

Now let \( \mu \) be an upper bound of the multiplicities. There are \( (\mu + 1)^{2^p - 1} \) unequal hypergraphs of vertex set \([p] \). By Lemma 2.1 we can assume that all these rooted hypergraphs are equal. In this situation, each \( G \in \mathcal{G} \) is a sequence of length \( 2^p - 1 \), indexed by the nonempty subsets of \([p] \). And for each nonempty \( V \subseteq [p] \), the term of the sequence indexed by \( V \) is the collection of elements of \( Q \) that are used to label the hyperedges \( e \) of \( G \) such that the set of end vertices of \( e \) is \( V \). Moreover, the length of the collection is bounded by \( \mu \) and is determined by \( V \). By Lemmas 2.1 and 2.2 \( G \) is better-quasi-ordered by \( \leq \). \( \blacksquare \)

In the following, we show that, for a better-quasi-ordered set \( Q \), the \( Q \)-labeled hypergraphs of bounded (respectively, multiplicity) tree-width and tree-diameter are better-quasi-ordered by the (respectively, induced) subgraph relation.

**Lemma 4.2.** Let \( p, s \geq 0 \) be integers, \( Q \) be a better-quasi-ordered set, \( \mathcal{G} \) be the set of quintuples \( \mathbf{G} := (G, T, V, r, V_G) \), where \( G \) is a \( Q \)-labeled hypergraph (respectively, of bounded multiplicity) with a tree-decomposition \( (T, \mathcal{V}) \) of width at most \( p - 1 \), \( T \) is a rooted tree of root \( r \) and height at most \( s \), and \( V_G \subseteq V_r \). Let \( \lambda : V(G) \mapsto [p] \) be a colouring such that for each \( v \in V(T) \), every pair of different vertices of \( V_r \) are assigned different colours. For \( X, Y \in \mathcal{G} \), denote by \( X \subseteq Y \) (respectively, \( X \leq Y \)) that there exists an isomorphism \( \varphi \) from \( X \) to a subgraph (respectively, an induced subgraph) of \( Y \) such that \( \varphi(V_X) = V_Y \), and that for each \( x \in V(X) \) and \( e \in E(X) \), \( \lambda(x) = \lambda(\varphi(x)) \) and \( \sigma(e) \leq Q \sigma(\varphi(e)) \). Then \( \mathcal{G} \) is better-quasi-ordered by \( \subseteq \) (respectively, \( \leq \)).

**Proof.** Let \( \mathcal{G}_s \) be the set of \( \mathbf{G} \in \mathcal{G} \) of which the height of \( T \) is exactly \( s \). By Lemma 2.1 it is enough to prove the lemma for \( \mathcal{G}_s \). The case of \( s = 0 \) is ensured by Lemma 4.1. Inductively assume it holds for some \( s - 1 \geq 0 \). By Lemma 2.2 the powerset \( \mathcal{M}_{s-1} \) of \( \mathcal{G}_{s-1} \) is better-quasi-ordered.

Denote by \( N_T(r) \) the neighborhood of \( r \) in \( T \). For each \( u \in N_T(r) \), let \( T_u \) be the connected component of \( T - r \) containing \( u \), and \( G_{T_u} \) be the subgraph of \( G \) induced by the vertex set \( \bigcup_{w \in V(T_u)} V_w \). Let \( \mathcal{V}_{T_u} := \{ V_w | w \in V(T_u) \} \), and \( G_{T_u} := (G_{T_u}, T_u, \mathcal{V}_{T_u}, u, V_r \cap V_u) \). Then \( G_{T_u} \in \mathcal{G}_{s-1} \). Let \( G_r \) be the subgraph of \( G \) induced by \( V_r \). Then \( G_r := (G_r, r, V_r, r, V_G) \in \mathcal{G}_0 \). Clearly, \( G \mapsto G_r \times \{ G_{T_u} | u \in N_T(r) \} \).
$N_T(r)$ is an order-preserving bijection from $\mathcal{G}_s$ to $\mathcal{G}_0 \times \mathcal{M}_{s-1}$. By Lemma 2.1, $\mathcal{G}_s$ is better-quasi-ordered since $\mathcal{G}_0$ and $\mathcal{M}_{s-1}$ are better-quasi-ordered.

We end this paper by proving that graphs (respectively, of bounded multiplicity) without a given path as a subgraph are better-quasi-ordered by the (respectively, induced) subgraph relation.

**Proof of Theorem 1.1.** $(\Leftarrow)$ follows from Lemmas 3.7 and 4.2.

$(\Rightarrow)$ Let $\mathcal{G}$ be the set of graphs without $H$ as a subgraph, quasi-ordered by the subgraph or induced subgraph relation. Suppose for a contradiction that $H$ is not a union of paths. Then $H$ contains either a cycle or a vertex of degree at least 3. For $i \geq 1$, let $C_i$ be the cycle of $|V(H)| + i$ vertices. Then $C_1, C_2, \ldots$ is a sequence without a good pair with respect to the subgraph or induced subgraph relation. So $\mathcal{G}$ is not well-quasi-ordered, not say better-quasi-ordered, a contradiction. Thus $H$ is a union of paths.

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