RIBBON COBORDISMS AS A PARTIAL ORDER

MARIUS HUBER

Abstract. We show that the notion of ribbon rational homology cobordism yields a partial order on the set of aspherical 3-manifolds, thus supporting a conjecture formulated by Daemi, Lidman, Vela-Vick and Wong. Our proof is built on Agol’s recent proof of the fact that ribbon concordance yields a partial order on the set of knots in the 3-sphere.

1. Introduction

In a recent preprint [Ago], Agol famously proved that ribbon concordance defines a partial order on the set of knots in $S^3$, thus resolving a long-standing conjecture [Gor81, Conjecture 1.1] due to Gordon. Agol’s proof relies on relating maps between knot groups to maps between the representation varieties of those groups. Before Agol’s result, work by Zemke [Zem19] sparked renewed interest in the study of ribbon concordances (see e.g. [LZ19], [JMZ20] and [Sar20]), which was eventually put into a broader framework in an article by Daemi, Lidman, Vela-Vick and Wong [DLVVW22] in which they defined and studied ribbon rational homology cobordisms.

Recall that, given closed, oriented 3-manifolds $Y_1$ and $Y_2$, a rational homology cobordism from $Y_1$ to $Y_2$ is an oriented 4-manifold $W$ with oriented boundary $\partial W = -Y_1 \sqcup Y_2$ such that inclusion of $Y_i$ into $W$ induces an isomorphism $H_\ast(Y_i; \mathbb{Q}) \cong H_\ast(W; \mathbb{Q})$, $i = 1, 2$. Such a cobordism $W$ from $Y_1$ to $Y_2$ is said to be ribbon, if $W$ admits a handle decomposition relative to $Y_1 \times I$ that consists of just 1- and 2-handles, where $I = [0, 1]$ (for the corresponding notion for 3-manifolds with boundary, see “Conventions and notation” at the end of this section). In the following, we refer to ribbon rational homology cobordisms simply as ribbon cobordisms.

This terminology stems from the fact that the exterior of a ribbon concordance $C \subset S^3 \times I$ from a knot $K_1$ to $K_2$ admits a natural handle decomposition which makes $(S^3 \times I) \setminus \nu C$ into a ribbon cobordism from $S^3 \setminus \nu K_1$ to $S^3 \setminus \nu K_2$ in the above sense. Using this point of view, Daemi, Lidman, Vela-Vick and Wong formulated a conjecture analogous to that of Gordon.

Conjecture 1.1 (Daemi-Lidman-Vela-Vick-Wong [DLVVW22, Conjecture 1.1]). The preorder on the set of homeomorphism classes of closed, connected, oriented 3-manifolds given by ribbon cobordisms is a partial order.

As speculated by Agol [Ago], it is natural to wonder whether the techniques used to prove Gordon’s Conjecture could be used to make progress in proving Conjecture 1.1. The purpose of the present note is to do exactly that, and to show that Conjecture 1.1 holds for the class of aspherical 3-manifolds (recall that a 3-manifold $Y$ is aspherical if $\pi_k(Y) = 0$ for all $k \geq 2$, or, equivalently, if $Y$ is irreducible and has infinite fundamental group). Combining this with previous work of the author [Hub21] on ribbon cobordisms between lens spaces, we obtain the following. Note that we do not require the manifolds involved to be closed.
Theorem 1.2. Let \( Y_1 \) and \( Y_2 \) be compact, oriented, 3-manifolds, possibly with boundary, such that there exists a ribbon cobordism \( W_i \) from \( Y_i \) to \( Y_j \), \( \{i, j\} = \{1, 2\} \). If \( Y_i \) is either aspherical or a lens space, \( i = 1, 2 \), then \( Y_1 \cong Y_2 \).

The above result is a rather direct consequence of the following two more technical results. To put these into context, recall that if \( Y_1 \) and \( Y_2 \) are compact 3-manifolds and if \( W \) is a ribbon cobordism from \( Y_1 \) to \( Y_2 \), one has the following diagram of maps induced by inclusion (\cite[Theorem 1.14]{DLVVW22} and \cite[Lemma 3.1]{Gor81}):

\[
\pi_1(Y_1) \quad \longrightarrow \quad \pi_1(W) \quad \longleftarrow \quad \pi_1(Y_2)
\]

Hence, if \( Y \) is a 3-manifold with finite fundamental group, and if \( W \) is a ribbon cobordism from \( Y \) to itself, then the inclusion of either boundary component into \( W \) induces an isomorphism of fundamental groups. Our main technical result states that this remains true if \( Y \) has infinite fundamental group, provided that \( Y \) is aspherical.

Theorem 1.3. Let \( Y \) be a compact, oriented, aspherical 3-manifold, possibly with boundary, and suppose that \( W \) is a ribbon cobordism from \( Y_1 \) to \( Y_2 \), where \( Y_i \cong Y \), \( i = 1, 2 \). Then the inclusion of \( Y_i \) into \( W \) induces an isomorphism \( \pi_1(Y_i) \cong \pi_1(W) \), \( i = 1, 2 \).

As a consequence, we obtain the following result concerning pairs of 3-manifolds with the property that there exists a ribbon cobordism in either direction.

Theorem 1.4. Let \( Y_1, Y_2 \) be compact, oriented, aspherical 3-manifolds, \( i = 1, 2 \), possibly with boundary, and suppose that there exists a ribbon cobordism \( W_i \) from \( Y_i \) to \( Y_j \), \( \{i, j\} = \{1, 2\} \). Then the inclusion of \( Y_i \) into \( W_j \) induces an isomorphism \( \pi_1(Y_i) \cong \pi_1(W_j) \), \( i, j = 1, 2 \). In particular, there exists an orientation-preserving homotopy equivalence \( f: (Y_i, \partial Y_i) \to (Y_j, \partial Y_j) \), \( \{i, j\} = \{1, 2\} \).

We conclude this introduction by pointing out that during the time of writing of the present note, essentially the same result was independently found by Friedl, Misev and Zentner \cite{FMZ}.

Conventions and notation. Given manifolds \( Y \) and \( Y' \), \( Y \cong Y' \) means that \( Y \) and \( Y' \) are related by an orientation-preserving homeomorphism. By a handle decomposition of a 4-dimensional cobordism \( W \) from \( Y \) to \( Y' \), we mean that \( W \) is built from \( Y \times [0, 1] \) by attaching 1-, 2- and 3-handles, where the attaching region of each handle is supported in \( \text{int}(Y) \times \{1\} \), or in the boundary of previously attached handles. In particular, if \( \partial Y' \) is non-empty, the attaching regions of the handles of \( W \) avoid \( \partial Y \times \{1\} \subset Y \times \{1\} \), and \( \partial Y \cong \partial Y' \).

Acknowledgments. I would like to thank to my advisor, Josh Greene, for the many helpful discussions we had along the course of this project.

2. Proofs of results

The following lemma is used in the algebro-geometric portion of the proof of Theorem 1.3, which, in turn, is virtually the same as the proof of \cite[Theorem 1.2]{Ago}. We provide a proof of the lemma for completeness, but also to highlight the use of residual finiteness of fundamental groups of 3-manifolds.

Lemma 2.1. Suppose \( \Gamma \) is a residually finite group, and let \( \gamma \in \Gamma \setminus \{1\} \). Then there exists \( n > 0 \) and a homomorphism \( \rho: \Gamma \to SO(n) \) such that \( \rho(\gamma) \neq 1 \).
Proof. We first show that any finite group embeds into $\text{SO}(n)$ for some $n > 0$. For this, recall that the symmetric group on $n$ elements $S_n$ is generated by the $n-1$ transpositions $\tau_{i,i+1} = (i, i+1)$, $i = 1, \ldots, n-1$. For $i = 1, \ldots, n-1$, define $\phi_{(i,i+1)}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi_{(i,i+1)}(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

One can check that $\phi_{(i,i+1)} \in \text{SO}(n+1)$ for all $i = 1, \ldots, n-1$, and hence the above assignment defines an embedding of $S_n$ into $\text{SO}(n+1)$. Since any finite group embeds into $S_n$ for some $n > 0$, it follows that the same holds with $S_n$ replaced by $\text{SO}(n)$.

Now, let $\Gamma$ be residually finite and $\gamma \in \Gamma$ non-trivial. By definition of residual finiteness, there exists a finite group $G$ and a surjection $\rho: \Gamma \to G$ such that $\rho(\gamma) \neq 1$. Postcomposing $\rho$ with an embedding of $G$ into $\text{SO}(n)$, for some $n > 0$, yields the claim. \hfill $\square$

Proof of Theorem 1.3. Suppose that $W$ is a ribbon cobordism as in the statement of Theorem 1.3. By [DLVVW22] Theorem 1.14, we have that

$$\pi_1(Y_1) \overset{i_1}{\longrightarrow} \pi_1(W) \overset{i_2}{\longleftarrow} \pi_1(Y_2),$$

where $i_k$ is the map induced by inclusion $i_k: Y_k \to W$, $k = 1, 2$, and $\pi_1(Y_1) \cong \pi_1(Y_2)$. As in [Ago], for $n > 0$ and a manifold $X$, let $R_n(X) = R_n(\pi_1(X))$ denote the representation variety of $\pi_1(X)$ to $\text{SO}(n)$. By [DLVVW22] Proposition 2.1, we then have that

$$R_N(Y_1) \overset{r_1}{\longleftarrow} R_N(W) \overset{r_2}{\longrightarrow} R_N(Y_2),$$

where $r_k$ is the restriction map, $k = 1, 2$. As shown in [Ago], $r_1$ is obtained by projection of $R_N(W)$ onto the subspace spanned by the coordinates corresponding to $\pi_1(Y_1)$ (regarded as a subgroup of $\pi_1(W)$) and hence is a polynomial map, and, moreover, $R_N(Y_1)$ and $R_N(Y_2)$ are related by a polynomial isomorphism. Precomposing this isomorphism with $r_1$, one obtains a surjective polynomial map $\phi: R_N(W) \to R_N(Y_2)$. By the argument given in [Ago], $r_2$, in fact, embeds $R_N(W)$ into $R_N(Y_2)$ as a real algebraic subset. This allows one to show that $i_2: \pi_1(Y_2) \to \pi_1(W)$ is injective as follows. Given $\gamma \in \pi_1(Y_2) \setminus \{1\}$, one can, using residual finiteness of 3-manifold groups (which follows from [Thu82] Theorem 3.3 and Geometrization) and Lemma 2.1 find $n > 0$ and a representation $\rho \in R_N(Y_2)$ with the property that $\rho(\gamma) \neq 1$. By the above, $R_N(W) \subset R_N(Y_2)$ is an algebraic subset that admits a surjective polynomial map to $R_N(Y_2)$ (namely, $\phi$). Thus, [Ago] Lemma A.2 implies that $R_N(W) = R_N(Y_2)$, and it follows that the representation $\rho$ is the restriction of some representation $\rho' \in R_N(W)$. Hence, $\rho'(i_2(\gamma)) = (r_2(\rho'))(\gamma) = \rho(\gamma) \neq 1$, which implies that $i_2(\gamma)$ is non-trivial. It follows that $i_2$ is injective and hence an isomorphism.

It remains to show that $i_1$ is an isomorphism. To show this, we adapt an argument used in the proof of [DLVVW22] Proposition 9.2. Set $\overline{W} = -W$, so that $\overline{W}$ is a rational homology cobordism from $Y_2$ to $Y_1$ which is built from $Y_2 \times I$ by attaching 2- and 3-handles. By what we have shown so far, the map $\pi_1(Y_2) \to \pi_1(\overline{W})$ induced by inclusion is an isomorphism, which implies that each of the 2-handles of $\overline{W}$ is attached to $Y_2 \times I$ along a null-homotopic curve in $Y_2 \times \{1\}$, and hence each attaching curve bounds an immersed disk in $Y_2 \times \{1\}$. Let $\overline{W}_2$ denote the space obtained by attaching just the 2-handles of $\overline{W}$ to $Y_2 \times I$. Define a map $\rho_2: \overline{W}_2 \to Y_2$ as follows. First, shrink each of the 2-handles to its core, then map each core to the disk in $Y_2 \times \{1\}$ bounded by its attaching curve via a map that is the identity on the attaching curve itself, and, finally, apply the obvious deformation retraction of $Y_2 \times I$ onto $Y_2 = Y_2 \times \{0\}$. The obstruction to extending $\rho_2$ over the 3-handles of $\overline{W}$ lies in $H^3(\overline{W}, \overline{W}_2; \pi_2(Y_2))$ (see e.g. [Hat02] Proposition 4.72). Since we assumed $Y_2$ to be aspherical, this group vanishes, and $\rho_2$ extends to a retraction $\rho: \overline{W} \to Y_2$. Indeed, by definition of a ribbon cobordism between manifolds with boundary,
Proof of Theorem 1.4. Let \( W \) be the composition of the cobordisms \( W_1 \) and \( W_2 \), i.e. \( W = W_1 \cup_{Y_2} W_2 \), so that \( W \) is a ribbon cobordism from \( Y_1 \) to \( Y_2 \). Letting \( h_1: \pi_1(W) \to \pi_1(W) \) denote the map induced by inclusion of \( W_1 \) into \( W \), \( i = 1, 2 \), and using [DLVVW22, Theorem 1.14], we obtain the following diagram of maps induced by inclusion on the level of fundamental groups.

\[
\begin{array}{ccccccccc}
\pi_1(Y_1) & \xrightarrow{i_1^1} & \pi_1(W_1) & \xleftarrow{i_1^2} & \pi_1(Y_2) & \xrightarrow{i_2^1} & \pi_1(W_2) & \xleftarrow{i_2^2} & \pi_1(Y_1) \\
\cong & h_1 & & & h_2 & & & & \cong
\end{array}
\]  

(2.3)

The fact that the maps \( h_1 \circ i_1^1 \) and \( h_2 \circ i_2^2 \) are isomorphisms follows from Theorem 1.3. This immediately implies that \( i_2^1 \) is injective, and hence an isomorphism. Switching the roles of \( W_1 \) and \( W_2 \), we see that \( i_2^2 \) is an isomorphism as well. Moreover, using the fact that \( W_2 \) is a ribbon cobordism, it follows by an argument similar to the one used to show injectivity of \( i_1^1 \) and \( i_2^2 \) (see e.g. the proof of [DLVVW22 Proposition 2.1]) that \( h_1 \) is injective. Note that, for that argument to apply, we need the fact that \( \pi_1(W_1) \) is residually finite; but \( \pi_1(W_1) \cong \pi_1(Y_2) \) via \( i_2^2 \) and \( \pi_1(Y_2) \), being the fundamental group of a compact 3-manifold, is residually finite. Now, since \( h_1 \circ i_1^1 \) is an isomorphism, \( h_1 \) is also surjective and hence an isomorphism, which implies that \( i_1^1 \) is an isomorphism, too. Switching the roles of \( W_1 \) and \( W_2 \), we see that \( h_2 \) and \( i_2^1 \) are isomorphisms as well. Hence all maps in (2.3) are isomorphisms.

It remains to prove the existence of the claimed homotopy equivalences. To that end, note that, because all horizontal maps in (2.3) are isomorphisms, we can apply the argument from the last paragraph of the proof of Theorem 1.3 to \( W_1 \) to obtain an orientation-preserving degree one map \( f : [Y_1, \partial Y_1] \to [Y_2, \partial Y_2] \) that induces an isomorphism of fundamental groups. Since we assumed \( Y_1 \) and \( Y_2 \) to be aspherical, it follows that \( f \) induces an isomorphism on all homotopy groups and hence is a homotopy equivalence by Whitehead's theorem (see e.g. [Hat02, Theorem 4.5]). Moreover, by construction of the retraction \( \rho \) from the proof of Theorem 1.3 we have that \( f(\partial Y_1) \subset \partial Y_2 \) and, indeed, that \( f \) fixes \( \partial Y_1 \) pointwise. It follows that \( f : (Y_1, \partial Y_1) \to (Y_2, \partial Y_2) \) is a homotopy equivalence, as desired. The above argument applied to \( W = W_2 \) yields a homotopy equivalence going in the other direction.

We are now in a position to prove our main result.

Proof of Theorem 1.2. Observe that, by (2.3), \( Y_1 \) is a lens space iff \( Y_2 \) is. Assume first that \( Y_1 \) and \( Y_2 \) are closed. If both \( Y_1 \) and \( Y_2 \) are lens spaces, the claim is a straightforward consequence of [Hub21, Theorem 1.2], so we may assume that...
both $Y_1$ and $Y_2$ are aspherical. By Theorem 1.4, there exists an orientation-preserving homotopy equivalence $f: Y_1 \to Y_2$. Note that $\pi_1(Y_2)$ is not just infinite, but torsion-free by (C.2) and (C.3), and hence the Borel Conjecture in dimension three implies that $f$ is homotopic to a homeomorphism. This homeomorphism must be orientation-preserving, because this property is preserved under homotopy, and it follows that $Y_1 \cong Y_2$.

It remains to address the case where $Y_1$, and hence also $Y_2$, has non-empty boundary. By what we assumed, it follows that both $Y_1$ and $Y_2$ are aspherical. By Theorem 1.4, there exists an orientation-preserving homotopy equivalence $f: (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$. Since $Y_i$ has non-empty boundary, and hence is Haken, $i = 1, 2$, $f$ is homotopic to a homeomorphism from $Y_1$ to $Y_2$ by Corollary 6.5. As before, this homeomorphism must be orientation-preserving, because $f$ was, and it follows that $Y_1 \cong Y_2$.

\[\square\]

References

[AFW15] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. 3-manifold groups. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[Ago] Ian Agol. Ribbon concordance of knots is a partial order. (preprint: arXiv:2201.03626 [math.GT]).

[DLVVW22] Aliakbar Daemi, Tye Lidman, David Shea Vela-Vick, and C.-M. Michael Wong. Ribbon homology cobordisms. Adv. Math., 408(part B):Paper No. 108580, 68, 2022.

[FMZ] Stefan Friedl, Filip Misev, and Raphael Zentner. Rational homology ribbon cobordism is a partial order. (preprint: arXiv:2204.10730v2 [math.GT]).

[Gor81] C. McA. Gordon. Ribbon concordance of knots in the 3-sphere. Math. Ann., 257(2):157–170, 1981.

[Hat02] Allen Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002.

[Hub21] Marius Huber. Ribbon cobordisms between lens spaces. Pacific J. Math., 315(1):111–128, 2021.

[JMZ20] András Juhász, Maggie Miller, and Ian Zemke. Knot cobordisms, bridge index, and torsion in Floer homology. J. Topol., 13(4):1701–1724, 2020.

[KL09] M. Kreck and W. Lück. Topological rigidity for non-aspherical manifolds. Pure Appl. Math. Q., 5(3, Special Issue: In honor of Friedrich Hirzebruch. Part 2):873–914, 2009.

[LZ19] Adam Simon Levine and Ian Zemke. Khovanov homology and ribbon concordances. Bull. Lond. Math. Soc., 51(6):1099–1103, 2019.

[Ron92] Yong Wu Rong. Degree one maps between geometric 3-manifolds. Trans. Amer. Math. Soc., 332(1):411–436, 1992.

[Sar20] Sucharit Sarkar. Ribbon distance and Khovanov homology. Algebr. Geom. Topol., 20(2):1041–1058, 2020.

[Thu82] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. Bull. Amer. Math. Soc. (N.S.), 6(3):357–381, 1982.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. Ann. of Math. (2), 87:56–88, 1968.

[Zem19] Ian Zemke. Knot Floer homology obstructs ribbon concordance. Ann. of Math. (2), 190(3):931–947, 2019.

Department of Mathematics, Boston College, Chestnut Hill, MA 02467

Email address: marius.huber@bc.edu