ON THE MULTIPLE $q$-GENOCCHI AND EULER NUMBERS

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Abstract. The purpose of this paper is to present a systemic study of some families of multiple $q$-Genocchi and Euler numbers by using multivariate $q$-Volkenborn integral (= $p$-adic $q$-integral) on $\mathbb{Z}_p$. From the studies of those $q$-Genocchi numbers and polynomials of higher order we derive some interesting identities related to $q$-Genocchi numbers and polynomials of higher order.

§1. Introduction

Let $p$ be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ and let $q$ be regarded as either a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume $|1 - q|_p < p^{-\frac{1}{p-1}}$, which implies that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Here, $| \cdot |_p$ is the $p$-adic absolute value in $\mathbb{C}_p$ with $|p|_p = \frac{1}{p}$. The $q$-basic natural number are defined by $[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}$, ($n \in \mathbb{N}$), and $q$-factorial are also defined as $[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$. In this paper we use the notation of Gaussian binomial coefficient as follows:

$$ \binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q} . $$

Note that $\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{n!}$. The Gaussian coefficient satisfies the following recursion formula:

$$ \binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q , \text{ cf. [12].} $$
From the recursion formula we derive
\[
\binom{n}{k}_q = \sum_{d_0 + \cdots + d_k = n-k, d_i \in \mathbb{N}} q^{d_1+2d_2+\cdots+kd_k}, \text{ cf.}[1, 2, 12, 13, 14].
\]

The \(q\)-binomial formulae are known as
\[
(b; q)_n = \prod_{i=1}^{n} (1 - bq^{i-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k} (-1)^k b^k,
\]
and
\[
\frac{1}{(b; q)_n} = \prod_{i=1}^{n} (1 - bq^{i-1})^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q b^k, \text{ cf.}[12].
\]

In this paper we use the notation
\[
[x]_q = \frac{1 - q^x}{1 - q}, \text{ and } [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]

Hence, \(\lim_{q \to 1} [x]_q = 1\), for any \(x\) with \(|x|_p \leq 1\) in the present \(p\)-adic case, cf.[1-18].

For \(d\) a fixed positive integer with \((p, d) = 1\), let
\[
X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,
\]
\[
X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
\]
\[
a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^n}\},
\]
where \(a \in \mathbb{Z}\) lies in \(0 \leq a < dp^N\). In [9], we note that
\[
\mu_q(a + dp^N\mathbb{Z}_p) = (1 + q) \left(\frac{-1}{1 + q^{dp^N}}\right) = \left(\frac{-q}{dp^N} \right)^a,
\]
is distribution on \(X\) for \(q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\). We say that \(f\) is a uniformly
differentiable function at a point \(a \in \mathbb{Z}_p\) and denote this property by \(f \in UD(\mathbb{Z}_p)\),
if the difference quotients \(F_f(x, y) = \frac{f(x) - f(y)}{x - y}\) have a limit \(l = f'(a)\) as \((x, y) \to (a, a)\). For \(f \in UD(\mathbb{Z}_p)\), this distribution yields an integral as follows:
\[
I_{-q} = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_{-q}} \sum_{x=0}^{dp^{N-1}} f(x)(-q)^x,
\]

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which has a sense as we see readily that the limit is convergent (see [9, 10, 14, 15]). Let \( q = 1 \). Then we have the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) as follows:

\[
I_{-1} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ cf.}[3, 6, 7, 8, 9, 13, 14].
\]

For any positive integer \( N \), we set

\[
\mu_q(a + lp^N \mathbb{Z}_p) = q^a [lp^N]_q, \text{ cf. [5, 9, 15, 16, 17, 18]},
\]

and this can be extended to a distribution on \( X \). This distribution yields \( p \)-adic bosonic \( q \)-integral as follows (see [12, 17, 18]):

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_{X} f(x) d\mu_q(x),
\]

where \( f \in UD(\mathbb{Z}_p) = \) the space of uniformly differentiable function on \( \mathbb{Z}_p \) with values in \( \mathbb{C}_p \).

In view of notation, \( I_{-1} \) can be written symbolically as \( I_{-1}(f) = \lim_{q \to -1} I_q(f) \), cf.[9]. For \( n \in \mathbb{N} \), let \( f_n(x) = f(x+n) \). Then we have

\[
(4) \quad q^n I_{-q}(f_n) = (-1)^n I_{-q}(f) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^l f(l), \text{ see [9]}. \]

For any complex number \( z \), it is well known that the familiar Euler polynomials \( E_n(z) \) are defined by means of the following generating function:

\[
F(z, t) = \frac{2}{e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \text{ for } |t| < \pi, \text{ cf.[13,14]}. \]

We note that, by substituting \( z = 0 \), \( E_n(0) = E_n \) is the familiar \( n \)-th Euler number defined by

\[
F(t) = F(0, t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf.[12]}. \]

The Genocchi numbers \( G_n \) are defined by the generating function

\[
\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi). \]

It satisfies \( G_1 = 1, G_3 = G_5 = \cdots = G_{2k+1} = 0 \), and even coefficients are given by

\[
G_n = 2(1 - 2^n) B_n = 2n E_{2n-1}(0),
\]
where $B_n$ are Bernoulli numbers and $E_n(x)$ are Euler polynomials. By meaning of the generalization of $E_n$, Frobenius-Euler numbers and polynomials are also defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad \text{and} \quad \frac{1-u}{e^t-u} e^{xt} = \sum_{n=0}^{\infty} H_n(u, x) \frac{t^n}{n!}, \quad \text{for} \quad u \in \mathbb{C}, \text{cf.}[12, 14].$$

Over five decades ago, Carlitz [1, 2] defined $q$-extension of Frobenius-Euler numbers and polynomials and proved properties analogous to those satisfied $H_n(u)$ and $H_n(u, x)$. In previous my paper [6, 7, 8] the author defined the $q$-extension of ordinary Euler and polynomials and proved properties analogous to those satisfied $E_n$ and $E_n(x)$. In [6] author also constructed the $q$-Euler numbers and polynomials of higher order and gave some interesting formulae related to Euler numbers and polynomials of higher order. The purpose of this paper is to present a systemic study of some families of multiple $q$-Genocchi and Euler numbers by using multivariate $q$-Volkenborn integral (= $p$-adic $q$-integral) on $\mathbb{Z}_p$. From the studies of these $q$-Genocchi numbers and polynomials of higher order we derive some interesting identities related to $q$-Genocchi numbers and polynomials of higher order.

§2. Preliminaries / $q$-Euler polynomials

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{r-1}}$. Let $f_1(x)$ be translation with $f_1(x) = f(x+1)$. From (4) we can derive

$$I_{-1}(f_1) = I_{-1}(f) + 2f(0).$$

If we take $f(x) = e^{(x+y)t}$, then we have Euler polynomials from the integral equation of $I_{-1}(f)$ as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = e^{xt} \frac{2}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n(x) t^n}{n!}.\,$$

That is,

$$\int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = E_n, \quad \text{and} \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x).$$

Now we consider the following multivariate $p$-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$(5) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left( \frac{2}{e^t+1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!},$$

where $E_n^{(r)}(x)$ are the Euler polynomials of order $r$. 

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From (5) we note that

\[
(6) \quad \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x).
\]

In view of (6) we can define the q-extension of Euler polynomials of higher order. For \( h \in \mathbb{Z}, \ k \in \mathbb{N}, \) let us consider the extended higher order q-Euler polynomials as follows:

\[
E_{m,q}^{(h,k)}(x) = \int_{Z_p} \cdots \int_{Z_p} [x_1 + \cdots + x_k + x]^m q^{\sum_{j=1}^{k} x_j (h-j)} d\mu_q(x_1) \cdots d\mu_q(x_r), \text{ see [6].}
\]

From this definition we can derive

\[
(7) \quad E_{m,q}^{(h,k)}(x) = [2]^k q \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{x_j} \frac{1}{(-q^{j+h}; q^{-1})_k}, \text{ see [6].}
\]

It is easy to see that

\[
(-q^{j+k-1}; q^{-1})_k = \prod_{l=0}^{k-1} (1 + q^l q^j) = (-q^j; q)_k
\]

In the special case \( h = k - 1, \) we can easily see that

\[
E_{m,q}^{(k-1,k)}(x) = [2]^k q \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{x_j} \frac{1}{(-q^{j+k-1}; q^{-1})_k}
\]

\[
= [2]^k q \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{x_j} \frac{1}{(-q^j; q)_k}
\]

\[
(8) \quad = [2]^k q \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{x_j} \sum_{n=0}^{\infty} \binom{k+n-1}{n} (-q^j)_n
\]

Let \( F^k(t, x) = \sum_{n=0}^{\infty} E_{n,q}^{(k-1,k)}(x) \) be the generating function of \( E_{n,q}^{(k-1,k)}(x). \) From (8) we note that

\[
F^k_q(t, x) = \sum_{m=0}^{\infty} E_{m,q}^{(k-1,k)}(x) \frac{t^m}{m!} = [2]^k q \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{k+n-1}{n} (-1)^n [n + x]_q^m \frac{t^m}{m!}
\]

\[
= [2]^k q \sum_{n=0}^{\infty} \binom{k+n-1}{n} (-1)^n [n + x]_q^m.
\]
Remark. For $w \in \mathbb{C}_p$ with $|1 - w| < 1$, we have

$$I_{-1}(w^x e^{tx}) = \int_{\mathbb{Z}_p} w^x e^{tx} d\mu_{-1}(x) = \frac{2}{we^t + 1} = \sum_{n=0}^{\infty} E_n(w) \frac{t^n}{n!},$$ see [3, 9, 11, 12, 15].

Thus, we note that $\int_{\mathbb{Z}_p} w^x x^e d\mu_{-1}(x) = E_n(w)$ and $E_n(w) = \frac{2}{w+1} H_n(-w^{-1})$, where $H_n(-w^{-1})$ are Frobenius-Euler numbers.

In the previous paper [11], the $q$-extension of $E_n(w)$ (= twisted $q$-Euler numbers) are studied as follows:

$$I_{-q}(w^x e^{[x]q}) = \int_{\mathbb{Z}_p} w^x e^{[x]q} d\mu_{-q}(x) = \sum_{n=0}^{\infty} E_{n,q}(w) \frac{t^n}{n!}.$$}

From (9) we note that

$$E_{n,q}(w) = \int_{\mathbb{Z}_p} w^x [x]^n_q d\mu_{-q}(x) = \frac{[2;q]_n}{(1-q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{1 + q^{2j+1}w},$$ see [11].

By the exactly same method of Eq.(7), we can also derive the multiple twisted $q$-Euler numbers as follows:

$$E_{m,q}^{(h,k)}(w, x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{\sum_{j=1}^{k} x_j} [x_1 + \cdots + x_k + x]^m_q \sum_{j=1}^{k} (h-j) x_j d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).$$

From (10) we can easily derive

$$E_{m,q}^{(h,k)}(w, x) = \frac{[2;k]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-q^x)^l \frac{1}{(-wq^{h+l}; q^{-1})}. $$

For $h = k - 1$, we have

$$E_{m,q}^{(k-1,k)}(w, x) = \frac{[2;k]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-q^x)^l \frac{1}{(-wq^k; q^{-k})}$$

$$= [2;q]_k \sum_{n=0}^{\infty} \binom{n+k-1}{n}_q (-w)^n [n+x]^m_q.$$}

Let $F_{q}^{k}(w, x) = \sum_{m=0}^{\infty} E_{m,q}^{(k-1,k)}(x) \frac{t^m}{m!}$. From (11), we can easily derive

$$F_{q}^{k}(w, x) = [2;k]_q \sum_{n=0}^{\infty} \binom{n+k-1}{n}_q (-w)^n [n+x]^{m}_q.$$
Remark. When we consider those $q$-Euler numbers and polynomials in complex number field, we assume that $q \in \mathbb{C}$ with $|q| < 1$.

§3. Genocchi and $q$-Genocchi numbers

From (4) we note that

$$t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \quad (12)$$

Thus, we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n + 1!}. \quad (13)$$

By (13) we easily see that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n + 1}, \text{ and } \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(x) d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n + 1},$$

where $G_n(x)$ are Genocchi polynomials (see [8]).

From the multivariate $p$-adic fermionic integral on $\mathbb{Z}_p$ we can also derive the Genocchi numbers of order $r$ as follows:

$$t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1 + \cdots + x_r)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left( \frac{2t}{e^t + 1} \right)^r = \sum_{n=0}^{\infty} G_n^{(r)} \frac{t^n}{n!}, \quad r \in \mathbb{N}, \quad (14)$$

where $G_n^{(r)}$ are the Genocchi numbers of order $r$.

From (14) we note that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{(r + n)_r t^{n+r}}{(n + r)!} = \sum_{n=0}^{\infty} G_n^{(r)} \frac{t^n}{n!}, \quad (15)$$

where $(x)_r = x(x-1) \cdots (x-r+1)$. By (14) and (15), we easily see that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \frac{1}{r! \binom{n+r}{r}} G_n^{(r)}(x), \text{ where } n \in \mathbb{N} \cup \{0\}, \quad (16)$$

and $G_0^{(r)} = G_1^{(r)} = \cdots = G_{r-1}^{(r)} = 0$. Thus, we obtain the following theorem.
Theorem 1. For $n \in \mathbb{N} \cup \{0\}$, $r \in \mathbb{N}$, we have

$$G_{n+r}^{(r)} = (n + r)_r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \binom{n + r}{r} r! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r),$$

where $(x)_r = x(x - 1) \cdots (x - r + 1)$.

Recently we constructed the $q$-extension of Genocchi numbers as follows:

$$t \int_{\mathbb{Z}_p} e^{[x]q} t d\mu_{-q}(x) = \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]q} t = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \text{ see } [8].$$

Thus, we note that

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) = G_{n,q} = n \frac{[2]_q}{(1 - q)^{n-1}} \sum_{l=0}^{n-1} \binom{n - 1}{l} \frac{(-1)^l}{1 + q^{l+1}}, \text{ see } [8].$$

In view of (14) we can define the $q$-extension of Genocchi numbers of higher order. For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, let us consider the extended higher order $q$-Genocchi numbers as follows:

$$\sum_{n=0}^{\infty} G_{n,q}^{(h,k)} \frac{t^n}{n!} = k \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1 + \cdots + x_k]q} q^{\sum_{j=1}^{h} x_j (h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^{h} x_j (h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \frac{t^{n+k}}{n!}$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^{h} x_j (h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \frac{(n+k)!}{(n+k)} \frac{t^{n+k}}{n!}.$$}

Thus, we have

$$G_{n+k,q}^{(h,k)} = k! \binom{n + k}{k} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n q^{\sum_{j=1}^{h} x_j (h-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= k! \binom{n + k}{k} \frac{[2]_q^k}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{l+1}(q-1)^k},$$

and

$$G_{0,q}^{(h,k)} = G_{1,q}^{(h,k)} = \cdots = G_{k-1,q}^{(h,k)} = 0.$$
If we take $h = k - 1$, then we have
\[
G_{n+k,q}^{(k-1,k)} = k! \left( \begin{array}{c} n+k \\ k \end{array} \right) \frac{[2]^k_q}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(-q^l; q)_k} \\
= k! \left( \begin{array}{c} n+k \\ k \end{array} \right) \frac{[2]^k_q}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{m=0}^\infty \binom{m+k-1}{m} (-q)^m q^m \\
= k! \left( \begin{array}{c} n+k \\ k \end{array} \right) [2]^k_q \sum_{m=0}^\infty \binom{m+k-1}{m} (-1)^m [m]^n_q.
\]

Therefore we obtain the following theorem.

**Theorem 2.** For $h \in \mathbb{Z}$, $k \in \mathbb{N}$, we have
\[
G_{n+k,q}^{(h,k)} = k! \left( \begin{array}{c} n+k \\ k \end{array} \right) \frac{[2]^k_q}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(-q^{h+l}; q^{-1})_k},
\]
and
\[
G_{n+k,q}^{(k-1,k)} = k! \left( \begin{array}{c} n+k \\ k \end{array} \right) [2]^k_q \sum_{m=0}^\infty \binom{m+k-1}{m} (-1)^m [m]^n_q.
\]

Let
\[
h_q^k(t) = \sum_{n=0}^\infty G_{n,q}^{(k-1,k)} \frac{t^n}{n!} = \sum_{n=0}^\infty G_{n+k,q}^{(k-1,k)} \frac{t^{n+k}}{(n+k)!},
\]
because $G_{0,q}^{(k-1,k)} = \cdots = G_{k-1,q}^{(k-1,k)} = 0$. By (18) and Theorem 2, we see that
\[
h_q^k(t) = \sum_{n=0}^\infty G_{n,q}^{(k-1,k)} \frac{t^n}{n!} = [2]^k_q \sum_{n=0}^\infty \binom{m+k-1}{m} (-q)^m q^m \\
= [2]^k_q \sum_{m=0}^\infty \binom{m+k-1}{m} (-1)^m [m]^n q^m.
\]

Remark. For $w \in \mathbb{C}_p$ with $|1-w|_p < 1$, we can also define $w$-Genocchi numbers (= twisted Genocchi numbers) as follows:
\[
t \int_{\mathbb{Z}_p} w^x e^{xt} d\mu_{-1}(x) = \frac{2t}{we^t + 1} = \sum_{n=0}^\infty G_{n,w} \frac{t^n}{n!}, \text{ cf.}[3, 11, 13].
\]
From this we note that $\lim_{w \to 1} G_{n,w} = G_n$. The $q$-extension of $G_{n,w}$ can be also defined by
\[
t \int_{\mathbb{Z}_p} w^x e^{[x]_q t} d\mu_{-q}(x) = \sum_{n=0}^\infty G_{n,q,w} \frac{t^n}{n!}, \text{ cf.} [3, 8, 11, 13].
\]
By (19) we easily see that
\[
G_{n,q,w} = n \frac{[2]_q}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{1+ql+1} \text{, cf.}[11].
\]

From this we also note that \(\lim_{w \to 1} G_{n,q,w} = G_{n,q}\).

Now we consider the extended \((q, w)\)-Genocchi numbers by using multivariate \(p\)-adic fermionic integral on \(\mathbb{Z}_p\). For \(h \in \mathbb{Z}, k \in \mathbb{N}, w \in \mathbb{C}_p\) with \(|1-w|_p < 1\), we define \(G_{n,k}^{(h,k)}\) as follows:

\[
G_{n+k,q,w}^{(h,k)} = k! \left( \begin{array}{c} n+k \\ k \end{array} \right) \frac{[2]_q^k}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{(-wq^{h+l}; q^{-1})_k},
\]

and

\[
G_{n+k,q,w}^{(k-1,k)} = k! \left( \begin{array}{c} n+k \\ k \end{array} \right) \frac{[2]_q^k}{(1-q)^n} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-w)^m [n]_q^m.
\]

Let \(h_{q,w}^{(k)}(t) = \sum_{n=0}^{\infty} G_{n,q,w}^{(k-1,k)} \frac{t^n}{n!}\). Then we have

\[
h_{q,w}^{(k)}(t) = \sum_{n=0}^{\infty} G_{n+k,q,w}^{(k-1,k)} \frac{t^{n+k}}{(n+k)!} = \frac{[2]_q^k}{(n+k)!} \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-w)^m [n]_q^m t^m.
\]

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