SINGULARITIES IN POSITIVE CHARACTERISTIC, STRATIFICATION AND SIMPLIFICATION OF THE SINGULAR LOCUS

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Abstract. We introduce an upper semi-continuous function that stratifies the highest multiplicity locus of a hypersurface in arbitrary characteristic. The blow-up along the maximum stratum defined by this function leads to a form of simplification of the singularities, also known as a reduction to the monomial case.

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Introduction

Resolution of singularities is a classical and central problem in algebraic geometry, that was proved by Hironaka in the mid sixties (cf. [17]) for varieties over fields of characteristic zero. This proof was existential, and several constructive (algorithmic) proofs of resolution of singularities have been published since the late eighties ([6], [9], [14], [23], [26], [27], [31]).

There are some results on low dimensional varieties over arbitrary fields due to Abhyankar (see also [21]), but the general question of resolution still remains open today. Another important contribution is de Jong’s work on alterations, which provides a weaker statement, but it is strong enough for some applications.

In the next paragraphs we describe some of the main ideas of the proof of algorithmic resolution in characteristic zero, paying special attention to the part of the argument that fails in positive characteristic. After this exposition, we explain the conclusions drawn by this paper.

Suppose that $X$ is a variety over a field of characteristic zero. An algorithmic desingularization of $X$ can be achieved in two steps, say A and B. In Step A a suitable sequence of monoidal transformations on smooth centers is defined so as to produce a simplification of the singularities of $X$. This is accomplished by using an inductive argument. Once this process is finished it is said that $X$ is within the monomial case. In Step B, the monomial case is treated: a combinatorial argument leads to a resolution of singularities of $X$.

**Step A: Simplification of singularities.** The goal here is to define a stratification of any scheme $X$ by means of an upper-semi-continuous function $\Gamma_X : X \to (\Lambda, \leq)$, where $(\Lambda, \leq)$ is a fixed totally ordered set, such that:

1. The maximum value of $\Gamma_X$, is achieved on a smooth closed subscheme, $\text{Max}\Gamma_X$, and it describes the worst singularities of $X$.
2. The blow-up of $X$ along $\text{Max}\Gamma_X$, $\pi_1 : X_1 \to X$, works to improve the singularities of $X$ in the following sense: There is an upper semi-continuous function $\Gamma_{X_1} : X_1 \to (\Lambda, \leq)$ such that:
   
   (a) If $x_1 \in X_1 \setminus \pi_1^{-1}(\text{Max}\Gamma_X) \simeq X \setminus \text{Max}\Gamma_X$ maps to $x \in X$, then $\Gamma_X(x) = \Gamma_{X_1}(x_1)$.
   (b) The maximum value of $\Gamma_X$, $\text{Max}\Gamma_X$, drops, i.e., $\text{Max}\Gamma_{X_1} < \text{Max}\Gamma_X$. 


After a finite number of monoidal transformations defined by $\Gamma$, which can have the following form:

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n,$$

a *simplification* of the singularities of $X$ is obtained. This is usually referred to as a reduction to the *monomial case*, and its meaning will be explored below. The question now is how to define the functions $\Gamma_{X_i} : X_i \rightarrow (\Lambda, \leq)$, for $i = 0, 1, \ldots, n - 1$.

Suppose that $X$ is embedded in a smooth $d$-dimensional scheme $V^{(d)}$. A first approximation to $\Gamma_X$ is to consider the order of the ideal of definition of $X$, $\mathcal{I}^{(d)} \subset \mathcal{O}_{V^{(d)}}$, at the closed points $x \in V^{(d)}$. Notice that this defines an upper-semicontinuous function say $\Gamma^{(d)} : X \rightarrow \mathbb{Z}_{\geq 0}$.

However this function is *too coarse* since in general it does not satisfy properties (1) and (2b). This becomes clear if we assume, for instance, that $X$ is a hypersurface: then the maximum order of $\mathcal{I}^{(X)}$ is achieved in the set of points with maximum multiplicity of $X$, which may not be a smooth subscheme of $V^{(d)}$.

An important point in characteristic zero is how the previous function $\Gamma^{(d)}$ can be *refined*, thanks to the existence of *hypersurfaces of maximal contact*: the closed set $\text{Max} \Gamma^{(d)}$ is locally contained in a smooth $(d - 1)$-dimensional scheme $V^{(d-1)}$, and it can be described by means of an ideal $\mathcal{I}^{(d-1)} \subset \mathcal{O}_{V^{(d-1)}}$. Then a new function $\Gamma^{(d-1)} : \text{Max} \Gamma^{(d)} \subset V^{(d-1)} \rightarrow \mathbb{Z}_{\geq 0}$ is defined in the restriction of $\text{Max} \Gamma^{(d)}$ to $V^{(d-1)}$, now using the order at closed points of the ideal $\mathcal{I}^{(d-1)}$. The construction of an upper-semicontinuous function $\Gamma : X \rightarrow (\Lambda, \leq)$ satisfying all properties (1) and (2) from Step A is made by collecting the information from $\Gamma^{(d)}, \Gamma^{(d-1)}, \ldots$ and employing an inductive argument.

**Step B: The monomial case.** Once Step A is accomplished as a composition of a finite number of monoidal transformations, $V^{(d)} \leftarrow V_n^{(d)}$, it can be assumed that, locally, the restriction of the *worst singularities* of $X_n \subset V_n^{(d)}$ to a (suitable chosen closed smooth-$(d - e)$-dimensional) subscheme $V_n^{(d-e)}$ of $V_n^{(d)}$ can be defined in terms of an ideal of a divisor with normal crossing support. Here $e \geq 1$, and the ideal of a divisor with normal crossings is called a monomial ideal. In this case (monomial case) it is relatively easy to enlarge sequence (1) to resolve the singularities of $X$.

**Hypersurfaces of maximal contact and the problems in positive characteristic.** Hypersurfaces of maximal contact play a central role in constructive resolution in characteristic zero. This topic is related to Abhyankar’s notion of Tschirnhausen transform (or Tschirnhausen substitution): given the equation of a singular embedded hypersurface, Tschirnhausen provides the equation of a smooth hypersurface, in the ambient space, that locally contains the highest multiplicity locus of the singular hypersurface (maximal contact). Moreover, this
containment is preserved by monoidal transformations with centers included in the locus with the highest multiplicity.

It is the work of J. Giraud where hypersurfaces of maximal contact arise by means of techniques that involve differential operators on smooth schemes, in a first attempt to address the problem of embedded desingularization in arbitrary characteristic (cf. [13]). This approach, which uses differential operators, played a central role in the development of algorithmic resolution of singularities in characteristic zero.

However in positive characteristic hypersurfaces of maximal contact may not exist (see for instance [15] and [25]), and for this reason the argument explained in Step A cannot be extended to this setting.

The aim of this paper. In this paper we show that the stratification of Step A is characteristic free. In other words, the stratifying functions $\Gamma_X : X \rightarrow (\Lambda, \leq)$, with the prescribed properties, can be defined in any characteristic.

This extension is made possible by the introduction of two tools: the characteristic free techniques introduced in [28] that avoid maximal contact; and the function defined by Main Theorem 10.1 (see Definition 10.2). Thus Main Theorem 10.1 provides a characteristic free form of induction; and this shows that Step A can be achieved over arbitrary characteristic (see also [29]).

About Step B. When the characteristic is zero, in the monomial case it is possible to define a stratification that lowers the multiplicity by blowing up at maximum stratum. The extension of Step A to arbitrary characteristic, i.e., the reduction to the monomial case, opens a door to new invariants. Over fields of positive characteristic, hypersurfaces whose highest multiplicity locus is in the “monomial case” turn out to have very particular properties. The treatment of this specific case, which we hope to address in the future, would imply resolution of singularities over arbitrary fields (see [3]).

Other approaches to resolution in arbitrary characteristic. The form of induction treated in this paper is different from that used in [21], and also different from those used in the Kawanoue program ([22] and [24]), and in [32]. All these approaches rely strongly on techniques of differential operators, but differ in their approach to induction. In these works restriction to smooth hypersurfaces of maximal contact is replaced by a notion of restriction to singular hypersurfaces, which are also, in some generalized sense, of maximal contact. Some questions concerning an approach to stratification of singularities in positive characteristic have also been addressed in [9]. There are other invariants for singularities in positive characteristic, not treated here, also related to the problem of embedded resolution of singularities, studied in works of Cossart, Hauser, and Moh. We include an example to illustrate the effect of Step A on a particular singularity. We chose here one of Hauser’s kangaroo points...
(cf. [16]); these are singularities where specific pathologies of positive characteristic arise (see Example 13.5).

Elimination: a strategy for overcoming the failure of maximal contact in positive characteristic

Maximal contact vs. elimination. The concepts of hypersurfaces of maximal contact and restriction to hypersurfaces of maximal contact are replaced by the notion of transversal projections and elimination algebras (respectively) in [28]. This allows us to use induction in any characteristic; here we illustrate this procedure.

Let \( k \) be a field, and assume that \( A \) is a smooth \( k \)-algebra and that \( X \) is a hypersurface in \( \text{Spec}(A[Z]) \) defined by
\[
f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_n \in A[Z].
\]
Let \( \Upsilon_n \) be the set of \( n \)-fold points of \( X \) (i.e., the points of multiplicity \( n = \deg f(Z) \)), and let \( B = A[Z]/\langle f(Z) \rangle \). Then the natural projection
\[
\beta : \text{Spec}(B) \to \text{Spec}(A)
\]
is a finite morphism, and Zariski’s multiplicity formula for projections ensures that the map induces a bijection between \( \Upsilon_n \) and \( \beta(\Upsilon_n) \) (see 7.1 for more details).

Now in this setting:

i. A suitable \( A[Z] \)- algebra, \( G \), is associated to \( f(Z) \) whose singular locus, \( \text{Sing } G \), is \( \Upsilon_n \) (we refer to Definition 3.6 for the notion of singular locus of a Rees algebra).

ii. An elimination algebra is associated to \( G \), say \( R_G \), which is now an algebra over the ring \( A \) (independent of the variable \( Z \)), and it has the property that \( \beta(\Upsilon_n) = \beta(\text{Sing } G) \subset \text{Sing } R_G \).

In this approach the highest multiplicity locus of \( X \) is projected bijectively on the smooth scheme \( \text{Spec}(A) \). This would be a substitute to the idea of restricting the highest multiplicity locus of \( X \) to a hypersurface of maximal contact.

Parallel to the arguments given in Step A, the ambient space \( V^{(d)} \) here is \( \text{Spec}(A[Z]) \) and the ideal \( I^{(d)} \) is substituted by \( G \). Then the restriction to the hypersurface of maximal contact \( V^{(d-1)} \) is replaced by the projection to \( \text{Spec}(A) \) and the information encoded by \( I^{(d-1)} \) is now encoded by the Rees algebra \( R_G \), over the ring \( A \), so it is independent of the variable \( Z \).

Now we proceed in the same manner as in characteristic zero. Consider a small variation of the order function:
\[
\Gamma^{(d)} : \begin{array}{ccc}
\text{Sing } G & \to \quad \mathbb{Q}_{\geq 0} \\
x & \to & \text{ord}_x G
\end{array}
\]
(see 3.9 for the definition) which, in general, turns out to be too coarse to satisfy properties (1) and (2b) as stated in Step A. Then we refine this variation by considering the order function

$$\Gamma^{(d-1)}: \text{Max} \Gamma^{(d)} \to \mathbb{Q}_{\geq 0} \xrightarrow{x \to \text{ord}_{x_1} \mathcal{R}_G},$$

where $$x_1 = \beta(x)$$ and we use the fact that $$\beta(\text{Max} \Gamma^{(d)}) \subset \text{Sing} \mathcal{R}_G$$, and then the argument proceeds by induction.

In this paper we show that this procedure can be iterated and that the functions we construct are independent of the choice of the projection. Therefore this part of the inductive argument used in characteristic zero is characteristic free: by induction on the dimension we can construct an upper-semi-continuous function to some ordered set, $$\Gamma: X \to (\Lambda, \geq)$$, that stratifies $$\mathcal{Y}_n$$ in smooth strata (see Main Theorem 10.1 and Theorem 13.1).

The idea of projecting the maximum multiplicity locus on a smooth subscheme is in the line of the arguments of Jung’s procedure for resolving hypersurface singularities by simplifying the discriminant.

The remainder of this paper is organized in five parts. Part 1 contains a brief exposition of the main ideas behind algorithmic resolution of singularities over fields of characteristic zero. As indicated above, algorithmic resolution is achieved in two steps A and B: a reduction to the monomial case, and then a treatment of the monomial case. We indicate below the conditions required to extend step A to arbitrary fields.

The problem of resolution over arbitrary fields will be formulated in terms of Rees algebras, so Part 2 is devoted to recalling some notions of the theory. In Sections 3 and 4 we present a brief introduction. Special attention will be paid to Rees algebras enriched with the action of differential operators. We will see how these objects provide a suitable framework to define our invariants. In Section 5 we discuss transformations of Rees algebras, and Hironaka’s notion of weak equivalence. The least number of variables needed to express the initial form of a hypersurface at a singular point is a central invariant in the theory. In fact, these are the variables that can be eliminated from the problem. This is what we call the $$\tau$$-invariant and its study, addressed in Section 6, will play a central role in the construction of our stratifying function.

Part 8 is dedicated to presenting elimination algebras and to reviewing some of their properties: Section 7 contains a detailed study of universal elimination algebras, and the specialization to usual elimination algebras via change of base rings. In Section 8 we study conditions in which elimination can be defined, with special attention on the fact that these conditions are open.
Part 4 contains the main results: In Section 9 we study the behavior of elimination algebras under monoidal transformations; Main Theorem 10.1 is stated in Section 10 and the proof is given in Section 11. Theorem 10.1 makes it possible to construct the upper-semicontinuous functions introduced in Definition 10.2, and Sections 12 and 13 are devoted to the stratification that results from this function.

Finally in Part 5 we explain how to use our results to achieve the monomial case. We refer to [29] for full details.

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Part 1. Algorithmic resolution of singularities over fields of characteristic zero

Here we briefly present the main ideas underlying the algorithmic resolution of singularities in characteristic zero. This is done here, as in [26] and [27], in terms of pairs and basic objects. We conclude this first part with an example to illustrate how the algorithm works (cf. Example 1.3). For more details we refer the reader to the introductory presentation in [11].

We will show how the language of Rees algebras, which is required for this new approach in arbitrary characteristic, parallels the one of pairs (see 3.10, 10.3 and 10.4).

1. The language of pairs and basic objects

Pairs are instruments that help us codify a suitable closed subset of a variety, and provide information regarding the singularities contained in it. They also provide a suitable language and appropriate tools to prove resolution of singularities using induction on the dimension of the ambient space.

However, once we start the process of resolution, exceptional divisors appear, and we need to keep track on this information too. The information provided by pairs and the ambient space where they exist, as well as the set of exceptional divisors that the resolution process produces, are codified in terms of basic objects.

Suppose that $X$ is a hypersurface embedded in some smooth $d$-dimensional space $V$. If our goal is to resolve the singularities of $X$, then we will start by paying attention to the worst singularities of $X$, namely, those points where the multiplicity of $X$ is the highest, say $b$. We will see that the natural pair associated to this closed set is $(\mathcal{I}(X), b)$. If there are no exceptional divisors to take care of, the basic object we will be interested in is $(V, (\mathcal{I}(X), b), \{\emptyset\})$. 
1.1. Pairs. A pair $(J, b)$ on a smooth scheme $V$ is defined by a non-zero sheaf of ideals $J \subset \mathcal{O}_V$ and a positive integer $b$. The singular locus of a pair $(J, b)$ consists of the set of points in $V$ where $J$ has order at least $b$, i.e.,

$$\text{Sing} \ (J, b) := \{ x \in V | \nu_x(J) \geq b \},$$

where $\nu_x$ denotes the order function at the local regular ring $\mathcal{O}_{V,x}$. The set $\text{Sing} \ (J, b)$ is closed in $V$.

As indicated above, it is typical to take $J$ as the sheaf of ideals defining a hypersurface $X \subset V$, and $b$ as the maximum of the multiplicities at points of $X$.

Hironaka defines the function:

$$\text{ord}_{(J, b)} : \text{Sing} \ (J, b) \to \mathbb{Q} \geq 1$$

$$x \to \text{ord}_{(J, b)}(x) = \frac{\nu_x(J)}{b}.$$ 

If $J$ is the defining ideal of a hypersurface $X \subset V$ as before, we can think that the worst singularities of $X$ are located at the points where the maximum value of Hironaka’s function is achieved. Thus, our aim is to lower the maximum value of this function by defining a suitable sequence of blow-ups at smooth centers in an effort to improve the singularities of a strict transform of $X$.

1.2. Basic objects and resolution. Since our resolution problem has been codified in terms of pairs, the next step is to understand how pairs transform under blow-ups. Given a pair $(J, b)$, a smooth closed subscheme $Y \subset V$ is said to be permissible if $Y \subset \text{Sing} \ (J, b)$. If $V \xleftarrow{\pi} V_1 \supset H = \pi^{-1}(Y)$ denotes the monoidal transformation at a permissible center $Y$ then the total transform of $J$ in $\mathcal{O}_{V_1}$, $J\mathcal{O}_{V_1}$, can be expressed as a product,

$$J\mathcal{O}_{V_1} = I(H)^b J_1$$

for a uniquely defined $J_1$ in $\mathcal{O}_{V_1}$. The new couple $(J_1, b)$ is called the transform of $(J, b)$, say:

$$V \xleftarrow{\pi} V_1, \quad (J, b) \xrightarrow{\pi} (J_1, b)$$ 

(3)

However, some geometric conditions have to be imposed in order to define a sequence of transformations of a pair. Every monoidal transformation introduces an exceptional divisor and we require that these divisors have normal crossings. To keep track of this additional information we define a couple $(V, E)$ to be a smooth scheme $V$ together with a set of smooth hypersurfaces $E = \{H_1, \ldots, H_r\}$ so that their union has normal crossings. If $Y$ is closed and smooth in $V$, and has normal crossings with $E$ (i.e., with the union of hypersurfaces of $E$), then we define a transform of the couple, say

$$(V, E) \xleftarrow{\pi} (V_1, E_1),$$

where $V \xleftarrow{\pi} V_1$ is the blow-up at $Y$; and $E_1 = \{H_1, \ldots, H_r, H_{r+1}\}$, where $H_{r+1}$ is the exceptional locus, and each $H_i$ denotes again the strict transform of $H_i$ in $V_1$, for $1 \leq i \leq r$. 

We finally define a basic object to be a couple \((V, E = \{H_1, \ldots, H_r\})\) together with a pair \((J, b)\), and we denote it by 
\[(V, (J, b), E).\]

We say that \((V, (J, b), E)\) is a \textit{d-dimensional basic object} if the dimension of \(V\) is \(d\). If a smooth center \(Y\) defines a transformation of \((V, E)\), and in addition \(Y \subset \text{Sing}(J, b)\), then a transform of the couple \((J, b)\), say \((J_1, b)\), is defined as above. In this case we say that 
\[ (V, (J, b), E) \leftarrow (V_1, (J_1, b), E_1) \]
is a transformation of the basic object. So we will ask permissible centers to satisfy this normal crossing condition.

A sequence of permissible transformations is denoted by 
\[(V, (J, b), E) \leftarrow (V_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (V_s, (J_s, b), E_s));\]
and such sequence is said to be a resolution of the basic object if \(\text{Sing}(J_s, b) = \emptyset\).

**Example 1.3.** A resolution of \((V, (J, b), E) = (V, (\mathcal{I}(X), b), \{\emptyset\})\), with \(b\) the maximum multiplicity of a hypersurface \(X\), lowers the maximum multiplicity of the strict transform of \(X\) in \(V_s\).

**Example 1.4.** The monomial case. Let \((V, (J, b), E)\) be a basic object with \(E = \{H_1, \ldots, H_l\}\). Notice that if \(J = \mathcal{I}(H_1)^{a_1} \cdots \mathcal{I}(H_l)^{a_l}\) with \(a_i \in \mathbb{N}\) for \(i = 1, \ldots, l\), then it is relatively easy to find a resolution of \((V, (J, b), E)\), which can be achieved using a combinatorial argument.

2. Algorithmic resolution of basic objects

Given a basic object \((V, (J, b), E)\), algorithms for resolving singularities provide a resolution as in (4), where the choice of the centers of the monoidal transformations is given as the “worst stratum” which is defined by a suitable upper-semi-continuous function.

We distinguish two steps in algorithmic resolution:

- **Step A**: Reduction to the monomial case. In this step a sequence of permissible transformations is defined to simplify the structure of the basic object.

- **Step B**: Treatment of the monomial case. This step involves the resolution of a basic object that is supposed to be within the monomial case.

Step A is accomplished by both defining a suitable upper-semi-continuous function constructed from the so called satellite functions, and using an inductive argument. In Step B the monomial case is treated using an upper-semi-continuous function of combinatorial nature.

In the following paragraphs we sketch how to accomplish Step A; see [I] for more details on this matter and a treatment of Step B.
Step A: Satellite functions

2.1. The first satellite function. Assume that $V$ is a $d$-dimensional smooth scheme, consider a sequence of transformations of basic objects which is not necessarily a resolution,

$$ (V_0, (J_0, b), E_0) = (V, (J, b), E) \leftarrow (V_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (V_s, (J_s, b), E_s), $$

and let $\{H_{r+1}, \ldots, H_{r+s}\}(\subset E_s)$ denote the exceptional hypersurfaces introduced by the sequence of blow-ups. We may assume, for simplicity, that these hypersurfaces are irreducible.

Then, there is a well defined factorization of the sheaf of ideals $J_s \subset \mathcal{O}_{V_s}$:

$$ J_s = I(H_{r+1})^{b_{s,r+1}} I(H_{r+2})^{b_{s,r+2}} \cdots I(H_{r+s})^{b_{s,r+s}} \cdot \mathcal{J}_s $$

such that $\mathcal{J}_s$ does not vanish along $H_{r+i}$ for $0 \leq i \leq s$.

Define $w\text{-ord}^{(d)}_{(J_s, b)}$ (or simply $w\text{-ord}^{(d)}_s$):

$$ w\text{-ord}^{(d)}_s : \text{Sing} (J_s, b) \rightarrow \mathbb{Q} $$

$$ x \rightarrow w\text{-ord}^{(d)}_s (x) = \frac{\nu_x(\mathcal{J}_s)}{b_i} $$

where $\nu_x(\mathcal{J}_s)$ denotes the order of $\mathcal{J}_s$ at $\mathcal{O}_{V_s,x}$. This function has the following properties:

1) It is upper semi-continuous. In particular the set of points where the maximum value, $\max w\text{-ord}^{(d)}_s$, is achieved is closed. This set is denoted by $\text{Max } w\text{-ord}^{(d)}_s$.

2) For any index $i \leq s$, there is an expression

$$ J_i = I(H_{r+1})^{b_{i,r+1}} \cdots I(H_{r+i})^{b_{i,r+i}} \cdot \mathcal{J}_i, $$

and hence the function $w\text{-ord}^{(d)}_i : \text{Sing} (J_i, b) \rightarrow \mathbb{Q}$ can also be defined.

3) If each transformation of basic objects $(V_i, (J_i, b_i), E_i) \leftarrow (V_{i+1}, (J_{i+1}, b_{i+1}) E_{i+1})$ in (5) is defined with center $Y_i \subset \text{Max } w\text{-ord}^{(d)}_i$, then

$$ \max w\text{-ord}^{(d)}_i \geq \max w\text{-ord}^{(d)}_1 \geq \cdots \geq \max w\text{-ord}^{(d)}_s. $$

Note that for $i = 0$ the function $w\text{-ord}^{(d)}_i$ coincides with the function $\text{ord}_{(J,b)} : \text{Sing} (J, b) \rightarrow \mathbb{Q}$ in (2); these functions differ for indices $i \geq 1$, and they are called satellite functions of the function introduced in (2). These functions represent small variations of the original function, and satisfy the inequalities stated in (8).

Observe that $\max w\text{-ord}^{(d)}_s = 0$ when

$$ J_s = I(H_{r+1})^{b_{s,r+1}} I(H_{r+2})^{b_{s,r+2}} \cdots I(H_{r+s})^{b_{s,r+s}}, $$

i.e., when $\mathcal{J}_s = \mathcal{O}_{V_s}$ in (5); in this case we say that $(V_s, (J_s, b), E_s)$ is in the monomial case. In the monomial case it is easy to enlarge sequence (5) to obtain a resolution. Therefore the functions $w\text{-ord}^{(d)}_i : \text{Sing} (J_i, b) \rightarrow \mathbb{Q}$ measure how far $J_i$ is from being locally monomial.
2.2. The second satellite functions. Consider a sequence of permissible transformations of $d$-dimensional basic objects,
\begin{equation}
(V, (J, b), E) \leftarrow (V_1, (J_1, b), E_1) \leftarrow \cdots \leftarrow (V_s, (J_s, b), E_s),
\end{equation}
with
\[
\max \text{w-ord}^{(d)} \geq \max \text{w-ord}_{1}^{(d)} \geq \ldots \max \text{w-ord}_{s}^{(d)}.
\]
Then if $\max \text{w-ord}_{s}^{(d)} > 0$, the function $t^{(d)}_s$ is defined in the following way: let $s_0 \leq s$ be the smallest index such that
\[
\max \text{w-ord}_{s_0}^{(d)} = \max \text{w-ord}_{s_0+1}^{(d)} = \ldots = \max \text{w-ord}_{s}^{(d)},
\]
and set
\[
E_s = E_s^+ \sqcup E_s^- \quad \text{where } E_s^- \text{ are the strict transforms of the hypersurfaces in } E_{s_0}.
\]
Then we can define:
\begin{equation}
t^{(d)}_s : \text{Sing} (J_s, b) \longrightarrow \mathbb{Q} \times \mathbb{N} \\
x \quad \mapsto (\text{w-ord}^{(d)}_s(x), n^{(d)}_s(x))
\end{equation}
where
\[
n^{(d)}_s(x) = \# \{ H_i \in E_s^- : x \in H_i \}
\]
and $\mathbb{Q} \times \mathbb{N}$ is lexicographically ordered. The function $t^{(d)}_s$ is upper-semicontinuous, and it is designed to ensure the normal crossing condition of the permissible centers with the smooth hypersurfaces in $E_s$.

**Remark 2.3.** Given a sequence of permissible transformations of basic objects, notice that the functions $t^{(d)}_i$ depend on Hinoraka’s order function (see (2) in 1.1), which is why they are called satellite functions.

**Step A: Induction and maximal contact**

In general satellite functions are too coarse to provide, just by themselves, an upper-semicontinuous function that lead to resolution, or even to the monomial case. For instance, it can be easily seen that for $(A^3_k, ((z^2 + (x^2 - y^3)^2), 2), E^{(3)} = \{\emptyset\})$ the maximum of $t^{(3)}$ is not smooth so it does not define a permissible center. Induction is thus used to solve this problem: the information provided by the satellite functions is refined using an inductive argument as explained in the next paragraphs.

**2.4. Simple basic objects and induction.** Let $V$ be a smooth scheme. A pair $(J, b)$ is said to be *simple* if $\text{ord}_{(J,b)} : \text{Sing} (J, b) \rightarrow \mathbb{Q}$ is the constant function 1; namely when the order of $J$ is exactly $b$ at the local ring $O_{V,x}$ for any $x \in \text{Sing} (J, b)$. A basic object $(V, (J, b), E)$ is said to be a *simple basic object* when $(J, b)$ is simple.

In the case of characteristic zero, the resolution of simple basic objects can be defined if we assume, by induction, the resolution of basic objects on lower dimensional ambient spaces,
which is guaranteed by the notion of *maximal contact*: a $d$-dimensional simple basic object $(V^{(d)}, (J, b), E^{(d)})$ can be restricted, locally, to a smooth hypersurface, defining a $(d - 1)$-dimensional basic object on this smooth lower dimensional space, $(V^{(d-1)}, (J', b'), E^{(d-1)})$. Furthermore, the link between the original basic object and the restricted one is sufficiently strong so that a resolution of the latter induces a resolution of the former, since there are commutative diagrams of transformations and restrictions:

$$(V^{(d)}, (J, b), E^{(d)}) \leftarrow (V_1^{(d)}, (J_1, b), E_1^{(d)}) \leftarrow \cdots \leftarrow (V_s^{(d)}, (J_s, b), E_s^{(d)})$$

$$(V^{(d-1)}, (J', b'), E^{(d-1)}) \leftarrow (V_1^{(d-1)}, (J_1', b'), E_1^{(d-1)}) \leftarrow \cdots \leftarrow (V_s^{(d-1)}, (J_s', b'), E_s^{(d-1)}).$$

In other words, simple basic objects can be resolved by induction: a resolution of simple basic objects in lower dimension can also be defined in terms of satellite functions leading to the reduction to the monomial case (and resolution in the monomial case) in the lower dimension.

**2.5. The non-simple case.** One property of the first satellite function is that it is naturally attached to a simple basic object. In fact, given a non-necessarily simple basic object, $(V^{(d)}, (J, b), E^{(d)})$, there is a simple basic object attached to it, $(V^{(d)}, (\tilde{J}, \tilde{J}), E^{(d)})$ such that its singular locus is the closed set where $w$-ord$_d$ takes the biggest value for $(V^{(d)}, (J, b), E^{(d)})$: max w-ord$_d$, and so that a resolution of $(V^{(d)}, (\tilde{J}, \tilde{J}), E^{(d)})$ induces a sequence of permissible transformations of $(V^{(d)}, (J, b), E^{(d)})$ henceforth lowering max w-ord$_d$. As a consequence, by successively resolving the simple basic objects attached to the functions w-ord$_i^{(d)}$ we produce a sequence where

$$\text{max w-ord}_d^{(d)} \geq \text{max w-ord}_1^{(d)} \geq \cdots \geq \text{max w-ord}_s^{(d)} = 0;$$

which implies that $J_s$ is monomial.

**2.6. Step A: technical problems.** There are three main sub-steps in step A: the local restriction to hypersurfaces of maximal contact, commutative diagrams of restrictions and resolutions, and the association of simple basic objects to non-simple basic objects.

- **Maximal contact.** For a fixed simple basic object there may be different choices of hypersurfaces of maximal contact. However it can be shown they all lead to the same resolution. This result is the main outcome of the so called Hironaka’s trick; an alternative and enlightening proof of this result is given by J. Wlodarczyk (see [31]).

- **Commutative diagrams of restrictions and permissible transformations.** Step A is accomplished by an inductive argument; for this argument to hold it is necessary that restrictions and permissible transformations commute as indicated in [2.4].

- **Association of simple basic objects to non-simple basic objects.** For a fixed basic object there may be different choices of simple basic objects that can be associated to it. It can be shown that all different choices lead to the same resolution.

**2.7. Summarizing.** Given a basic object $(V, (J, b), E)$, an upper-semi-continuous function is defined by means of the satellite functions in dimension $d$ and lower dimensions: if $x \in$
Sing\((J, b)\), then the upper-continuous-function associates to it a set of values \(t^{(d-j)}\) where \(j = 0, \ldots, r\) for some \(r < d\), each of them identifying \(x\) with its image under successive restrictions to hypersurfaces of maximal contact, and being interpreted as contained in the singular locus of some lower dimensional basic object. Then a resolution (or the monomial case) is achieved by blowing up the centers defined by these functions.

**Example 2.8.** To find a resolution of singularities of \(X := \{z^2 + (x^2 - y^3)^2 = 0\} \subset \mathbb{A}^3_k\), we start by finding a resolution of the basic object \((\mathbb{A}^3_k, (\mathcal{I}(X), 2), E^{(3)} = \{\emptyset\})\). Since

\[
\text{Sing} (\mathcal{I}(X), 2) = \{z = 0, x^2 - y^3 = 0\} = C,
\]

we can take \(\{z = 0\} \simeq \mathbb{A}^2_k\) as a hypersurface of maximal contact, and we associate to the original 3-dimensional basic object a 2-dimensional one,

\[
(\mathbb{A}^2_k, ((x^2 - y^3)^2), 2), E^{(2)} = \{\emptyset\}).
\]

Since this basic object is not simple, we attach to Max \(w\)-ord\(^{(2)}\) a simple basic object (s.b.o.), \((\mathbb{A}^2_k, (x^2 - y^3), 2, \{\emptyset\})\), and then find another hypersurface of maximal contact, \(\{x = 0\} \simeq \mathbb{A}^1_k\), and a 1-dimensional basic object:

\[
(\mathbb{A}^3_k, ((x^2 + (x^2 - y^3)^2), 2), \{\emptyset\})
\]

\[
\begin{array}{ccc}
\text{Restriction} & \downarrow & \text{Restriction} \\
(\mathbb{A}^2_k, ((x^2 - y^3)^2), 2), \{\emptyset\}) & \leftrightarrow & (\mathbb{A}^2_k, (x^2 - y^3), 2), \{\emptyset\}) \\
(\mathbb{A}^1_k, (y^3), 2), \{\emptyset\}) & \leftrightarrow & (\mathbb{A}^1_k, (y^3), 3), \{\emptyset\})
\end{array}
\]

The information can be interpreted in the following way:

- A resolution of \((\mathbb{A}^3_k, ((z^2 + (x^2 - y^3)^2), 2), \{\emptyset\})\) can be found by finding a resolution of \((\mathbb{A}^2_k, ((x^2 - y^3)^2), 2), \{\emptyset\})\).

- Lowering the maximum order of \((x^2 - y^3)^2\) in \(\mathbb{A}^2_k\) (which is achieved in \((0, 0)\)) is equivalent to finding a resolution of \((\mathbb{A}^2_k, (x^2 - y^3), 2), \{\emptyset\})\).

- A resolution of \((\mathbb{A}^2_k, (x^2 - y^3), 2), \{\emptyset\})\) can be found by resolving \((\mathbb{A}^1_k, (y^3), 2), \{\emptyset\})\).

- The maximum order of \(y^3\) in \(\mathbb{A}^1_k\) is forced to drop by resolving \((\mathbb{A}^1_k, (y^3), 3), \{\emptyset\})\).

By collecting the information provided by the order function in different dimensions, the following upper semi-continuous function is defined:

\[
\Gamma_X(p) = \begin{cases} 
((1, 0), (2, 0), (\frac{3}{2}, 0)) & \text{if } p = (0, 0, 0) \\
((1, 0), (1, 0), (\infty, 0)) & \text{if } p \in C \setminus (0, 0, 0).
\end{cases}
\]
Its maximum value indicates the first center that has to be blown up: \((0, 0, 0)\). On the other hand, the \(\infty\)-coordinate is attached to the points that are contained in a component of codimension one of the singular locus of a simple basic object: It can be shown that these components are always smooth, and therefore natural centers where to blow-up.

After a blow-up at the origin the maximum of the w-order in the second level, \(\max w\text{-ord}^{(2)}_1\), has dropped, so the function \(n_1^{(2)}\) plays a role in counting old exceptional divisors. The sequence defined by the algorithm takes the following form:

**Starting point:** 3-dimensional basic object

| Couples: | \((\mathbb{A}^3_k, \{\emptyset\})\) | \((V_1^{(3)}, \{H_1\})\) | \((V_2^{(3)}, \{H_1, H_2\})\) |
|----------|-----------------------------------|--------------------------|--------------------------|
| Pairs:   | \((z^2 + (x^2 - y^3)^2, 2)\) | \((z_1^2 + y_1^2(x_1^2 - y_1^2)^2, 2)\) | \((z_2^2 + y_2^2x_2^2(1 - y_2^2)^2, 2)\) |

**Restricting:** 2-dimensional basic object

| Couples: | \((\mathbb{A}^2_k, \{\emptyset\})\) | \((V_1^{(2)}, \{\overline{H}_1\})\) | \((V_2^{(2)}, \{\overline{H}_1, \overline{H}_2\})\) |
|----------|-----------------------------------|--------------------------|--------------------------|
| Pairs:   | \(((x^2 - y^3)^2, 2)\) | \((y_1^2(x_1^2 - y_1^2), 2)\) | \((y_2^2x_2^2(1 - y_2^2)^2, 2)\) |

After two permissible transformations the Step A of resolution has been accomplished since we have reached the monomial case in dimension 2. Notice that a resolution of \((A^2_k, (x^2 - y^3)^2), E^{(2)} = \{\emptyset\})\) induces a resolution of the original 3-dimensional basic object.

2.9. **About this paper.** The purpose of this paper is to show that Step A of the resolution process is characteristic free, i.e., this part of the algorithmic resolution can be performed in any characteristic if we replace the restriction to hypersurfaces of maximal contact with a different form of induction that requires projections. This new approach is formulated in terms of Rees algebras: our statements are made in terms of Rees algebras instead of pairs. There is a dictionary that translates between pairs and Rees algebras (see Section 3.10), and their role in resolution problems will be explained in and Section 10. When translating the algorithmic resolution to this new setting we encounter the corresponding technical problems as described in 2.6:

- **Projections to smooth schemes.** For a fixed simple Rees algebra there may be numerous suitable projections. In Main Theorem 10.1 we show that the w-order functions defined after projecting are independent of the choice of the projections. This leads to Definition 10.2 and consequently satellite functions can also be defined in our context.

- **Commutative diagrams of projections and permissible transformations.** This is addressed in Section 9.

- **Association of simple Rees algebras to non-simple Rees algebras.** For a fixed Rees algebra there may be different choices of simple Rees algebras that can be associated to it. In Section 12 we show that all of the different choices lead to the same invariants.
In summary, we show that the so called “reduction to the monomial case” is possible in positive characteristic, where, as indicated above, the reduction to the monomial case means that the monomial case arises in some lower dimension via induction. We hope to say more about the monomial case in the future (see [5]).

Part 2. Rees algebras

3. Rees algebras

We begin by introducing Rees algebras in the context in which they will be used throughout this paper. Special attention should be paid to Example 3.7 where Rees algebras are studied in the typical situations that we are interested in.

Definition 3.1. Let $B$ be a Noetherian ring, and let $\{I_n\}_{n \geq 0}$ be a sequence of ideals in $B$ satisfying the following conditions:

1. $I_0 = B$;
2. $I_k \cdot I_l \subset I_{k+l}$.

Then the graded subring $G = \oplus_{n \geq 0} I_n W^n$ of the polynomial ring $B[W]$ is said to be a Rees algebra if it is a finitely generated $B$-algebra.

Remark 3.2. A Rees algebra can be described by giving a finite set of generators

\[ \{f_{n_1} W^{n_1}, \ldots, f_{n_s} W^{n_s}\} \]

with $f_{n_i} \in B$ for $i = 1, \ldots, s$. An element $g \in I_n$ will be of the form $g = F_n(f_{n_1}, \ldots, f_{n_s})$ for some weighted homogeneous polynomial in $s$-variables $F_n(Y_1, \ldots, Y_s)$ where $Y_i$ has weight $n_i$ for $i = 1, \ldots, s$.

Example 3.3. A canonical example of a Rees algebra is the Rees ring of an ideal: fix an ideal $J \subset B$, and let $G = \oplus_n J^n W^n$. In fact, a Rees algebra is not very far away from being the Rees ring of an ideal in a sense that we make precise in the following lines. Let $G = \bigoplus_{n \geq 0} I_n W^n \subset B[W]$ be the Rees algebra generated by $\{f_{n_1} W^{n_1}, \ldots, f_{n_s} W^{n_s}\}$ with $f_i \in B$, and let $\bar{N}$ be a common multiple of all integers $n_i$, $i = 1, \ldots, s$. Then

\[ \bigoplus_{k \geq 0} I^k_{\bar{N}} W^{kn} \subset \bigoplus_{n \geq 0} I_n W^n \]

is a finite extension of Rees algebras (cf. [30, 2.3]). So, up to integral closure, a Rees algebra can be thought of as the Rees ring of a suitable ideal (see [28]).

Remark 3.4. Given a Rees algebra $G = \oplus_{n \geq 0} I_n W^n$ another can be defined by setting

\[ I'_n = \sum_{r \geq n} I_r, \]

and letting $L = \oplus_{n \geq 0} I'_n W^n$. Then $L$ is contained in the integral closure of $G$ (cf. [28, Remark 2.2 (2)]), and has the additional property that $I'_k \supset I'_s$ if $s \geq k$. So, up to integral closure it can always be assumed that a Rees algebra fulfills this condition.
3.5. Rees algebras on schemes. Let $V$ be a scheme and let $\{I_n\}_{n \geq 0}$ be a sequence of sheaves of ideals in $\mathcal{O}_V$ with $I_0 = \mathcal{O}_V$ and such that $I_k \cdot I_l \subseteq I_{k+l}$ for all non-negative integers $k,l$. The graded subsheaf of algebras $G = \oplus_{n \geq 0} I_n W^n$ of $\mathcal{O}_V[W]$ is said to be a sheaf of Rees algebras if there is an affine open cover $\{U_i\}$ of $V$, such that $G(U_i) \subseteq \mathcal{O}_V(U_i)[W]$ is a Rees $\mathcal{O}_V(U_i)$-algebra per Definition 3.1.

3.6. The singular locus of a Rees algebra. Let $V$ be a non-singular scheme and let $G = \oplus_n I_n W^n$ be a sheaf of Rees algebras. Let $\nu_x(J)$ denote the order of an ideal $J$ in the regular local ring $\mathcal{O}_{V,x}$. The singular locus of $G$, denoted by $\text{Sing } G$, is the closed set of all points $x \in V$ such that $\nu_x(I_n) \geq n$ for all non-negative integers $n$, i.e.,

$$\text{Sing } G = \bigcap_n \{x \in V : \nu_x(I_n) \geq n, \text{ for all } n \in \mathbb{Z}_{\geq 0}\}.$$

Example 3.7. Let $\langle f \rangle \subseteq \mathcal{O}_V$ be the ideal of an affine hypersurface $H$ in an affine smooth scheme $V$. Also let $b$ be a non-negative integer, and let $G$ be the Rees algebra generated by $f$ in degree $b$. Then $\text{Sing } G$ is the closed set of points of multiplicity at least $b$ of $H$ (this may be empty). The same holds if $J \subseteq \mathcal{O}_V$ is a sheaf of ideals, $b$ is a non-negative integer and $G$ is the Rees algebra generated by $J$ in degree $b$: then the singular locus of $G$ consists of the points of $V$ where the order of $J$ is at least $b$ (which may be empty).

3.8. Singular locus and integral closure. The singular locus of a Rees algebra is defined up to integral closure. In other words: If $G_1, G_2 \subseteq \mathcal{O}_V[W]$ have the same integral closure in $\mathcal{O}_V[W]$, then $\text{Sing } G_1 = \text{Sing } G_2$ (see [30, Proposition 4.4 (1)]).

3.9. The order of a Rees algebra at a point. [10, 6.3] Let $x \in \text{Sing } G = \oplus_{n \geq 0} I_n W^n$, and let $f W^n \in I_n W^n$. Then set

$$\text{ord}_x(f) = \frac{\nu_x(f)}{n} \in \mathbb{Q},$$

where $\nu_x(f)$ denotes the order of $f$ in the regular local ring $\mathcal{O}_{V,x}$. Notice that $\text{ord}_x(f) \geq 1$ since $x \in \text{Sing } G$. Now define

$$\text{ord}_x G = \inf\{\text{ord}_x(f) : f W^n \in I_n W^n, n \geq 1\}.$$

If $G$ is generated by $\{f_{n_1} W^{n_1}, \ldots, f_{n_m} W^{n_m}\}$ then

$$\text{ord}_x G = \min\{\text{ord}_x(f_{n_i}) : i = 1, \ldots, m\},$$

and therefore, if $x \in \text{Sing } G$ then $\text{ord}_x G$ is a rational number that is greater or equal to one. Furthermore if $N$ is a common multiple of all $n_i$, then

$$\text{ord}_x G = \frac{\nu_x(I_N)}{N}.$$

If $G_1, G_2 \subseteq \mathcal{O}_V[W]$ have the same integral closure, then $\text{ord}_x G_1 = \text{ord}_x G_2$ at any point $x \in \text{Sing } G_1 = \text{Sing } G_2$ (cf. [10, Proposition 6.4]).
3.10. Rees algebras vs. pairs. The notion of Rees algebra is essentially equivalent to Hironaka’s notion of pair. We assign to a pair \((J, b)\) over a smooth scheme \(V\) the Rees algebra:

\[
G_{(J,b)} = \mathcal{O}_V[J^b W^b],
\]

which is a graded subalgebra in \(\mathcal{O}_V [W]\). It turns out that every Rees algebra over \(V\) is a finite extension of \(G_{(J,b)}\) for a suitable pair \((J, b)\) ([29, Proposition 2.9]).

Observe that for \(G_{(J,b)} = \mathcal{O}_V[J^b W^b]\) there is an equality of closed sets

\[
\text{Sing}(G_{(J,b)}) = \text{Sing}(J, b),
\]

and also of functions

\[
\text{ord}_{G_{(J,b)}} = \text{ord}_{(J,b)},
\]

where the left-hand side is that defined in 3.9.

4. Differential Rees algebras

As indicated in the previous section (see Example 3.7) we are particularly interested in the multiplicity of embedded hypersurfaces. For this purpose, we will use a class of Rees algebras that are, in a sense, compatible with differential operators. This point will be clarified in 4.3.

Let \(V\) be a smooth scheme over a field \(k\). Then, for any non-negative integer \(s\), the sheaf of \(k\)-differential operators of order \(s\), \(\text{Diff}^s_k\), is a coherent sheaf locally free over \(V\). If \(s = 0\), the sheaf \(\text{Diff}^0_k\) can be naturally identified with \(\mathcal{O}_V\) and for each \(s \geq 0\) there are natural inclusions \(\text{Diff}^s_k \subset \text{Diff}^{s+1}_k\).

**Definition 4.1.** A Rees algebra \(G = \bigoplus_n I_n W^n\) is said to be a differential algebra, or a differential algebra relative to \(k\) if the following conditions hold:

i. For all non-negative integers \(n\) there is an inclusion \(I_n \supset I_{n+1}\).

ii. There is an affine open covering of \(V\), \(\{U_i\}\), such that for any \(D \in \text{Diff}^r_k(U_i)\) and any \(h \in I_n(U_i)\) we have that \(D(h) \in I_{n-r}(U_i)\) provided that \(n \geq r\).

4.2. The differential algebra generated by a Rees algebra. Let \(G\) be a Rees algebra on a smooth scheme \(V\) over a field \(k\). There is a natural way to construct a differential algebra containing \(G\) with the property of being the smallest differential algebra containing it (see [30, Theorem 3.4]). This Rees algebra will be denoted by \(\text{Diff}(G)\). In particular if \(G\) is locally generated on an affine open set \(U\) by \(\{f_{n_1} W^{n_1}, \ldots, f_{n_s} W^s\}\), then in [28] it is shown that \(\text{Diff}(G(U))\) is generated by

\[
\{D(f_{n_i}) W^{n_i-r} : D \in \text{Diff}^r_k, 0 \leq r < n_i, n_i \leq n_i, i = 1, \ldots, s\}.
\]

4.3. Differential algebras and singular locus. On a smooth scheme \(V\), of finite type over a field \(k\), the sheaves of differentials \(\text{Diff}^r_k\) for different values of \(r\) allow us to study the order of a sheaf of ideals. Similarly, differential algebras are the right structures for studying
the singular locus of a Rees algebra. More precisely, given a Rees algebra $G = \bigoplus_n I_n W^n$ on $V$,

$$\text{Sing } G = \cap_{r \geq 0} V(\text{Diff}_{k}^{-1}(I_r)),$$

(see [30, Definition 4.2]). This definition coincides with the one given in Definition 3.6 (see [30, Proposition 4.4]). In fact if $\text{Diff}(G)$ is the differential algebra generated by a Rees algebra $G$ then

$$\text{Sing } G = \text{Sing } \text{Diff}(G);$$

also if $x \in \text{Sing } G = \text{Sing } \text{Diff}(G)$ then

$$\text{ord}_x G = \text{ord}_x \text{Diff}(G)$$

(cf. [10, Proposition 6.4]). Furthermore, if $G$ is a differential algebra, then $\text{Sing } G = V(I_r)$ for any positive integer $r$ (see [30, Proposition 4.4]).

4.4. Differential algebras and integral closure. In many problems concerning resolution of singularities it is natural to consider ideals up to integral closure. For instance two ideals with the same integral closure have the same embedded principalizations (Log-resolutions). In the use of differential algebras as a tool to understand singularities, we need to consider algebras up to integral closure, so we need to understand how integral closure relates to differential algebras. This issue is treated in [30, Section 6] where it is proven that if $G_1 \subset G_2$ is a finite extension of differential algebras on a smooth scheme $V$ over a field $k$, then $\text{Diff}(G_1) \subset \text{Diff}(G_2)$ is also a finite extension. In other words, if $G_1$ is equal to $G_2$ up to integral closure, then so are $\text{Diff}(G_1)$ and $\text{Diff}(G_2)$.

Relative Differential Algebras

4.5. Let $\phi : V^{(d)} \rightarrow V^{(e)}$ be a smooth morphism of smooth schemes of dimensions $d$ and $e$ respectively. Then, for any non-negative integer $s$, the sheaf of relative differential operators of order $s$, $\text{Diff}^s(V^{(d)}/V^{(e)})$, is locally free over $V^{(d)}$.

**Definition 4.6.** Let $\phi : V^{(d)} \rightarrow V^{(e)}$ be a smooth morphism of smooth schemes of dimensions $d$ and $e$ respectively. A Rees algebra $G = \bigoplus_n I_n W^n \subset O_{V^{(d)}}[W]$ is said to be a $\phi$-relative differential algebra or simply a $\phi$-differential algebra if:

i. For all non-negative integers $n$ there is an inclusion $I_n \supset I_{n+1}$.

ii. There is an affine open covering $\{U_i\}$ of $V^{(d)}$ such that for any $D \in \text{Diff}^s(V^{(d)}/V^{(e)})(U_i)$ and any $h \in I_n(U_i)$ we have that $D(h) \in I_{n-s}(U_i)$ provided that $n \geq s$.

Relative differential algebras will play a central role in our arguments due to their relation to a form of elimination that we shall discuss in the next sections. The case of relative dimension one, $V^{(d)} \rightarrow V^{(d-1)}$, is of particular interest.

5. Rees algebras, permissible transformations and weak equivalence

In the previous section we attached to a Rees algebra a closed set, its singular locus, and a function along this closed set. We indicated that both, the closed set and the function, are the same for two algebras with the same integral closure. In our study of resolution we
consider Rees algebras up to integral closure, and all of the invariants we consider are the same for algebras with the same integral closure.

The purpose of this section is to introduce the concept of weak equivalence; two Rees algebras with the same integral closure will be weakly equivalent. To this end, we consider three kinds of transformations of Rees algebras: monoidal transformations, restrictions to open sets and products of smooth schemes with affine spaces. These will be used to define the equivalence relation. If two Rees algebras are equivalent according to this relation, then they will have the same resolution (this concept will be defined in the next sections). In fact, within an equivalence class of Rees algebras there is a natural procedure to choose one up to integral closure (see Theorem 5.8). This equivalence relation will play a role in Section 12.

5.1. Monoidal transformations. Let \( G = \bigoplus_n J_n W^n \subset O_V[W] \) be a Rees algebra. A monoidal transformation with center \( Y \subset V, V \leftarrow V' \), is said to be permissible if \( Y \subset \text{Sing} G \) is a smooth closed subscheme. If \( H \subset V' \) is the exceptional divisor, then for each \( n \in \mathbb{N} \),

\[ J_n O_{V'} = \mathcal{I}(H)^n J'_n \]

for some sheaf of ideals \( J'_n \subset O_{V'} \). Then the weak transform of \( G \) is defined as

\[ G' := \bigoplus_n J'_n W^n. \]

The next proposition gives a local description of the weak transform of a Rees algebra \( G \) after a permissible monoidal transformation.

**Proposition 5.2.** [10, Proposition 1.6] Let \( G = J_n W^n \) be a Rees algebra on a smooth scheme \( V \) over a field \( k \), and let \( V \leftarrow V' \) be a permissible transformation. If \( G \) is generated by \( \{g_{n_1} W^{n_1}, \ldots, g_{n_s} W^{n_s}\} \) then \( G' \) is generated by \( \{g'_{n_1} W^{n_1}, \ldots, g'_{n_s} W^{n_s}\} \), where \( g'_{n_i} \) denotes the weak transform of \( g_{n_i} \) for \( i = 1, \ldots, s \).

5.3. Integral closure, differential operators and weak transforms. [10, 4.1] Let \( G_1 \subset G_2 \subset G_3 \) be an inclusion of Rees algebras, such that \( G_3 \) is the differential algebra spanned by \( G_1 \), and let \( V \leftarrow V' \) be a permissible monoidal transformation with center \( Y \subset \text{Sing} G_1 \). Then:

(i) There is an inclusion of weak transforms

\[ G'_1 \subset G'_2 \subset G'_3. \]

(ii) The three algebras \( G'_1 \subset G'_2 \subset G'_3 \) span the same differential algebra.

(iii) If \( G_1 \subset G_2 \) is a finite extension, then \( G'_1 \subset G'_2 \) is a finite extension as well.

A notion of equivalence for Rees algebras

5.4. If \( G = \bigoplus I_k W^k \) is a differential \( O_V \)-algebra and \( V'' \rightarrow V \) is a smooth morphism, then the natural extension \( G'' = \bigoplus I_k O_{V''} W^k \) is also a differential algebra (cf. 30, Proposition
5.1]). Moreover, if \( \phi : T \to V \) is a morphism of smooth schemes then \( \phi^*(G) \) is a differential algebra on \( T \) and \( \text{Sing } \phi^*(G) = \phi^{-1}(\text{Sing } G) \) (cf. [30, Theorem 5.4]).

There are two types of smooth morphisms that we are specially interested in:

i. If \( U \subset V \) is an open subset, then the restriction of \( G \) to \( U \) is a Rees algebra, and if \( G \) is a differential algebra, so is its restriction.

ii. If \( \phi : T = V \times \mathbb{A}^k \to V \) is the projection, then the pull back \( \phi^*G \) is a Rees algebra. Moreover, if \( G \) is a differential algebra, then so is \( \phi^*G \).

**Definition 5.5.** Let \( G \) be a Rees algebra. A morphism \( V' \to V \) is a permissible transformation if it is either a permissible monoidal transformation as in Definition 5.1, or a smooth morphism as described in 5.4 (i) or (ii).

We shall consider a smooth scheme \( V \) together with a set \( E \) of smooth hypersurfaces having normal crossings, so we present our data as \((V, G, E)\), which we call a basic object; and we express a transformation as:

\[
(V, G, E) \xleftarrow{\phi} (U, G_U, E_U)
\]

which we call a pull-back, which essentially is, as above, a restriction to an open set or a restriction followed by multiplication by an affine space.

**Definition 5.6.** A local sequence of transformations of basic objects takes the following form:

\[
(V, G, E) \xleftarrow{\phi_1} (V_1', G_{1,1}, E_1') \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_s} (V_s', G_{s,s}, E_s'),
\]

where \((V, G, E) \xleftarrow{\phi} (V_1', G_{1,1}, E_1')\), and each \((V_i', G_i, E_i) \xleftarrow{\phi_i} (V_{i+1}', G_{i+1}, E_{i+1})\), is a pull-back, or a pull-back followed by a permissible monoidal transformation defined with a center \( Y_i \subset \text{Sing } (G_i) \) having normal crossings with the hypersurfaces in \( E_i \). In this last case \( E_{i+1} \) consists of the strict transforms of the hypersurfaces in \( E_i \) together with the exceptional hypersurface introduced by the monoidal transformation.

**Definition 5.7.** Two Rees algebras \( G_i, \ i = 1, 2 \), or two basic objects \((V, G_i, E), \ i = 1, 2\), are said to be weakly equivalent if: \( \text{Sing } (G_1) = \text{Sing } (G_2) \), and if any local sequence of transformations of one of them, say,

\[
(V', G_i, E') \xleftarrow{\phi_{i,j}} (V_{i,j}', G_{i,j}, E_{i,j}') \xleftarrow{\phi_{i,j+1}} \cdots \xleftarrow{\phi_{i,s}} (V_s', G_{i,s}, E_s'),
\]

defines a local sequence of transformation of the other, and \( \text{Sing } (G_{1,j}) = \text{Sing } (G_{2,j}) \) for \( 0 \leq j \leq s \).

The following Theorem is derived from the cited result of Hironaka. This fact, and many applications of it, are studied in [12].

**Theorem 5.8.** [20, p. 119] If \( G_1 \) and \( G_2 \) have the same integral closure then they are weakly equivalent. If \( \text{Diff}(G) \) is the differential algebra generated by \( G \) in 4.2, then \( G \) and \( \text{Diff}(G) \) are weakly equivalent. Moreover \( G_1 \) and \( G_2 \) are weakly equivalent if and only if \( \text{Diff}(G_1) \) and \( \text{Diff}(G_2) \) have the same integral closure.
Theorem 5.9. [20, p. 101] If $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly equivalent, then $\text{ord}_x \mathcal{G}_1 = \text{ord}_x \mathcal{G}_2$ for each $x \in \text{Sing} \mathcal{G}_1 = \text{Sing} \mathcal{G}_2$.

6. Simple points and tangent cones

Let $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ be a Rees algebra on a $d$-dimensional smooth scheme $V$ over a field $k$. We present here the notion of $\tau$-invariant at a singular point $x \in \text{Sing} \mathcal{G}$, together with a subspace of co-dimension $\tau$ in the tangent space of the point. Despite the fact that this invariant is defined at the tangent space of the point, it provides local information on the singularity: $\tau$ is a bound on the local co-dimension of the singular locus (see Theorem 6.5), and it is also the number of variables which are to be “eliminated”, via elimination algebras, as we shall see in the following sections (see 8.11).

Definition 6.1. A point $x \in \text{Sing} \mathcal{G}$ is simple if for some $k \geq 1$ the order of $I_k$ in $x$, $\nu_x(I_k)$, is $k$, (i.e., if $\text{ord}_x \mathcal{G} = 1$).

Simple points play a central role in elimination theory (see Section 8).

6.2. The tangent cone. [28, 4.2] Let $x \in \text{Sing} \mathcal{G}$ be a closed point. Consider the graded algebra associated to the closed point’s maximal ideal $m_x$, $\text{Gr}_{m_x}(\mathcal{O}_{V,x})$, which is isomorphic to a polynomial ring in $d$-variables. This is the coordinate ring associated to the tangent space of $V$ at $x$, namely $\text{Spec}(\text{Gr}_{m_x}(\mathcal{O}_{V,x})) = T_{V,x}$. The initial ideal or tangent ideal of $\mathcal{G}$ at $x$, $\text{In}_x(\mathcal{G})$, is the ideal of $\text{Gr}_{m_x}(\mathcal{O}_{V,x})$ generated by the elements $\text{In}_x(I_n)$ for all $n \geq 1$. Observe that $\text{In}_x(\mathcal{G})$ is zero unless $\text{ord}_x \mathcal{G} = 1$. The zero set of the tangent ideal in $\text{Spec}(\text{Gr}_{m_x}(\mathcal{O}_{V,x}))$ is the tangent cone of $\mathcal{G}$ at $x$, $\mathcal{C}_G, x$.

Tangent ideals have the following properties:

(i) The tangent ideal $\text{In}_x \mathcal{G}$ is zero unless $x \in \text{Sing} \mathcal{G}$ is a simple point.

(ii) If $\mathcal{G}$ is a differential algebra and if $k'$ (the residue field at $x$) is a field of characteristic zero, then $\text{In}_x \mathcal{G}$ is generated by linear forms. If $k'$ is a field of positive characteristic $p$, then there is a sequence $e_0 < e_1 < \cdots < e_r$ in $\mathbb{Z} \geq 0$, and $\text{In}_x \mathcal{G}$ is generated by elements of the form

$$l_1, \ldots, l_{s_0}, l_{s_0 + 1}, \ldots, l_{s_1}, \ldots, l_{s_r - 1}, \ldots, l_{s_r}$$

where $l_1, \ldots, l_{s_0}$ is a linear combination of powers $Z_i^{p^{e_0}}$; if $t \geq 0$,

$$l_{s_t + 1}, \ldots, l_{s_{t+1}}$$

is a linear combination of powers $Z_i^{p^t}$, and we require that $s_r$ homogeneous elements in (15) form a regular sequence at $\text{Gr}_{m_x}(\mathcal{O}_{V,x})$.

So $\langle l_1, \ldots, l_{s_r} \rangle$ define a subscheme of co-dimension $s_r$ in $T_{V,x}$. If $k'$ is a perfect field the radical of this ideal is spanned by linear forms, defining a subspace of co-dimension $s_r$ in $T_{V,x}$. 
The integer $s_r$ is said to be the $\tau$-invariant of the singularity and we will denote it by $\tau_{G,x}$. If $p^{e_0}$ is the smallest power of $p$ in (15) then the order of $I_n$ in $O_{V,x}$ is $n$ if and only if $n$ is a multiple of $p^{e_0}$.

From the algebraic point of view, $\tau_{G,x}$ indicates the minimum number of variables needed to describe $I_nG_x$. From the geometric point of view, $\tau_{G,x}$ is the co-dimension of the largest linear subspace $L_{G,x} \subset C_{G,x}$ such that $u + v \in L_{G,x}$ for all $u \in C_{G,x}$ and all $v \in L_{G,x}$.

(iii) If $G$ is a differential algebra then:

$$L_{G,x} = C_{G,x}.$$

(iv) For any Rees algebra $G$, the inclusion $G \subset \text{Diff}(G)$ defines an inclusion $C_{\text{Diff}(G),x} \subset C_{G,x}$, and:

$$C_{\text{Diff}(G),x} = L_{G,x}.$$

(v) If $Y \subset \text{Sing} G$ is a permissible center, then $T_{Y,x} \subset T_{V,x}$, is a linear subspace, and furthermore $T_{Y,x} \subset L_{G,x}$ for all $x \in Y \subset \text{Sing} G$. In particular $\tau_{G,x}$ bounds the local co-dimension of the regular scheme $Y$ in $V$, i.e., $\text{co-dim}_x Y \geq \tau_{G,x}$.

The following Theorem is due to Hironaka:

**Theorem 6.3.** If $G_1$ and $G_2$ are weakly equivalent, then for each $x \in \text{Sing} G_1 = \text{Sing} G_2$ there is an equality between their $\tau$-invariants, i.e., $\tau_{G_1,x} = \tau_{G_2,x}$.

**Definition 6.4.** A Rees algebra $G$ is said to be of co-dimensional type $\geq e$ if $\tau_{G,x} \geq e$ for all $x \in \text{Sing} G$.

**Theorem 6.5.** Let $x \in \text{Sing} G$. Then $\text{co-dim}_x \text{Sing} G \geq \tau_{G,x}$, and if equality holds then $\text{Sing} G$ is smooth locally at $x$.

We shall prove that locally at $x$, $\text{Sing} G$ is included in a complete intersection scheme of co-dimension $\tau_{G,x}$ (see Corollary 11.9), which proves the first assertion. The second claim will be addressed in Lemma 13.2 and Remark 13.3.

**Part 3. Elimination**

7. **Elimination via Universal Invariants**

As indicated in the Introduction and specially in Part 1 the resolution of singularities of a hypersurface over a field of characteristic zero can be achieved in two Steps A and B. In Step A, a suitable stratification of the locus of maximum multiplicity is constructed using an inductive argument; we briefly explained how the notion of maximal contact plays a role in this stratification.

We will introduce a new approach, which is a reformulation of maximal contact in the case of characteristic zero. As a result of this reformulation, *universal elimination algebras* can be defined. We will use these algebras in our inductive arguments.
This section is organized as follows: In 7.1 we discuss the motivation for using elimination (see Example 7.2 for more details); universal elimination algebras are defined in 7.3 and their relation to differential operators is described in 7.4; finally, in Theorem 7.6 we explain how these universal invariants specialize to the particular case that we are interested in.

7.1. The motivation. Assume that $S$ is a regular ring containing a field $k$. Let $f(Z) = Z^n + a_1Z^{n-1} + \ldots + a_n \in S[Z]$ and denote by $\Upsilon_n$ the set of points in $\{f(Z) = 0\}$ with multiplicity $n$. The natural inclusion $S \subset S[Z]$ induces a smooth morphism $\beta$, and a finite restriction $\beta\downarrow\text{Spec}(S)\to\text{Spec}(S)[/latex].

\begin{equation}
\text{Spec}(S[Z]/\langle f(Z) \rangle) \hookrightarrow \text{Spec}(S[Z])
\end{equation}

Our goal is to find equations in the coefficients of $f(Z)$ that describe the image in $\text{Spec}(S)$ of $\Upsilon_n$. The elimination algebra of $f(Z)$ will be the $k$-subalgebra of $S$ generated by these elements.

Notice that $B = S[Z]/\langle f(Z) \rangle$ is a free $S$-module of rank $n$. Let $Q$ be a prime ideal in $B$ dominating $S$ at a prime $P$. Under these conditions, Zariski’s projection formula for multiplicities ensures that the multiplicity of $B_Q$ is at most $n$, and if this multiplicity is $n$ then $Q$ is the unique prime in $B$ which dominates $P$, and $B_Q$ and $A_P$ have the same residue field (see [33, Corollary 1, p. 299]). The morphism $\overline{\beta}$ is said to be purely ramified over a point $x \in \text{Spec}(S)$ if the geometric fiber over $x$ is a unique point.

So the multiplicity formula shows that $\Upsilon_n$ is contained in the set of points where $\overline{\beta}$ is purely ramified.

Example 7.2. Suppose $n = 2$, and let $f(Z) = Z^2 + a_1Z + a_2$ with $a_1, a_2 \in S$. In this case the discriminant—namely $a_1^2 - 4a_2 \in S$—describes the image under $\overline{\beta}$ of the purely ramified locus in $\text{Spec}(S)$. Notice that $a_1^2 - 4a_2 \in S$ is a weighted homogeneous polynomial of degree 2 provided that we assign weight one to $a_1$ and weight two to $a_2$. It is not hard to check that if the characteristic of $S$ is not 2, then the closed subset of the regular scheme $\text{Spec}(S)$ where the discriminant has order at least two, is exactly the image of the two-fold points via $\text{Spec}(S[Z]/\langle f(Z) \rangle) \to \text{Spec}(S)$.

Using the language of Rees algebras, our datum is the $S[Z]$-algebra generated by $f(Z)$ in degree two, say $\mathcal{G} = S[Z][fW^2](\subset S[Z][W])$. Therefore its singular locus is the set of 2-fold points of $\{f(Z) = 0\}$. As we shall see, in this case the elimination algebra associated to $\mathcal{G}$ is the Rees algebra over $S$ generated by $a_1^2 - 4a_2$ in degree 2, say $\mathcal{R}_\mathcal{G} = S[(a_1^2 - 4a_2)W^2]$, and hence, its singular locus is the image of the two-fold points of $\{f(Z) = 0\}$.
However, this process fails if the characteristic is two, a situation that requires some attention. This problem can be remedied by extending $G$ to a differential algebra, i.e., adding to our original datum the result of applying all differential operators to $f(Z)$. This forces us to extend the notion of elimination algebra to the case of several polynomials since typically a differential algebra will have more than one generator. When a Rees algebra is differential, then its singular locus can be identified with the singular locus of its elimination algebra, once a smooth projection is fixed. These ideas will be explored further, specially in Section 8 (see 8.7).

7.3. The universal elimination algebra. Let $k$ be a field. Consider the polynomial ring in $n$ variables $k[Y_1, \ldots, Y_n]$, and the universal polynomial of degree $n$,

$$F_n(Z) = (Z - Y_1) \cdots (Z - Y_n) = Z^n - s_{n,1}Z^{n-1} + \ldots + (-1)^n s_{n,n} \in k[Y_1, \ldots, Y_n, Z],$$

where for $i = 1, \ldots, n$, $s_{n,i} \in k[Y_1, \ldots, Y_n, Z]$ denotes the $i$-th symmetric polynomial in $n$ variables.

Observe that the diagram

$$
\begin{array}{ccc}
\text{Spec } (k[s_{n,1}, \ldots, s_{n,n}][Z]/(F_n(Z))) & \leftarrow & \text{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]) \\
\alpha \downarrow & & \downarrow \alpha \\
\text{Spec}(k[s_{n,1}, \ldots, s_{n,n}]) & \\
\end{array}
$$

(17)

illustrates the universal situation we are interested in, and that (16) is a specialization of this case where:

$$\Theta : k[s_{n,1}, \ldots, s_{n,n}] \longrightarrow S$$

$$(-1)^i s_{n,i} \longrightarrow a_i.$$  

(18)

In the following lines we consider the universal case. Our goal is to find equations in the coefficients of the polynomial $F_n$ that describe the image of the $n$-fold points of $F_n = 0$. We begin by looking for equations in the coefficients that describe the purely ramified locus of the morphism; we reproduce arguments from [28, Section 1].

First notice that the group of permutations of $n$ elements, $S_n$, acts linearly on $k[Y_1, \ldots, Y_n]$ and that the subring of invariants is

$$k[Y_1, \ldots, Y_n]^{S_n} = k[s_{n,1}, \ldots, s_{n,n}].$$

Set $T = k[s_{n,1}, \ldots, s_{n,n}]$ and observe that $T \subset k[Y_1, \ldots, Y_n]$ is an inclusion of graded rings since the action of $S_n$ in $k[Y_1, \ldots, Y_n]$ is linear (i.e., it preserves the grading).

In the setting of (16) the purely ramified locus does not vary under changes of variable, and we consider all possible changes of the form

$$uZ - \alpha,$$

with $\alpha, u \in S$ and $u$ invertible. So, in finding equations in $S$ describing $\overline{\beta}(Y_n)$ we have to look for equations in the coefficients of $f$ that are invariant under changes as in (19).
We consider first changes of the form $Z - \alpha$. In the universal case, these changes of variable can be expressed as

$$F_n(Z + T) = (Z - (Y_1 - T)) \cdots (Z - (Y_n - T)) \in k[Y_1 - T, \ldots, Y_n - T]^{S_n}[Z].$$

The group $S_n$ also acts linearly on $k[Y_i - Y_j]_{1 \leq i, j \leq n}$ defining a graded subring

$$k[Y_i - Y_j]_{1 \leq i, j \leq n}^{S_n} \subset T = k[Y_1, \ldots, Y_n]^{S_n}.$$

These are functions on the coefficients of the universal polynomial, and hence on the coefficients of any monic polynomial of degree $n$, which are clearly invariant by any change of the form $Z - \alpha$. It will also be shown below how to further profit from the inclusion $k[Y_1 - Y_2, \ldots, Y_1 - Y_n]^{S_n} \subset k[Y_1 - T, \ldots, Y_n - T]^{S_n}$.

Let $U$ be the $k$-subalgebra of $T[Z]$ generated by $F_n$ in degree $n$, say $k[F_n]$. We define

$$k[H_{n_1}, \ldots, H_{n_r}] := k[Y_i - Y_j]_{1 \leq i, j \leq n}^{S_n}$$

and refer to it as the universal elimination algebra $R_U$ associated to $U$. Note that for $i = 1, \ldots, r$, each $H_{n_i}$ is a homogeneous polynomial in degree $n$, and it is also a weighted homogeneous polynomial in $s_{n,1}, \ldots, s_{n,n}$ where $s_{n,i}$ is homogeneous of degree $i$ in the variables $Y_1, \ldots, Y_n$ for $i = 1, \ldots, n$. For instance, in Example 7.2 the elimination algebra is generated by the discriminant in degree two.

The grading of $R_U$ will lead to the definition of ideals. For each positive index $N$, the homogeneous polynomials of degree $N$ form a finite dimensional vector space over $k$. Polynomials in this vector space are weighted homogeneous on the coefficients $s_{n,1}, \ldots, s_{n,n}$, and span an ideal, say $I_N \subset T$. In this way, for any specific monic polynomial of degree $n$, say $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_n \in S[Z]$ over a $k$-algebra $S$, an ideal $J_N$ is spanned by these weighted homogeneous equations on the coefficients $a_i$. These ideals $J_N(\subset S)$, defined for each positive index $N$, will be invariant under any change of variables in (19). Note also that for any two positive integers $N$, $M$: $J_N \cdot J_M \subset J_{N+M}$.

**Differential operators**

7.4. Now our purpose is to get a better understanding of the information encoded in the universal elimination algebra. Given a diagram as in (17), we want to study how the universal elimination algebra can be used to describe the image under $\alpha$ of the purely ramified locus, and in turn, of the set of $n$-fold points of $\{F_n = 0\} \subset \text{Spec} (K[s_{n,1}, \ldots, s_{n,n}][Z])$.

To understand how to reach this goal, we cite the following Lemma from [28], which relates the multiple roots of a polynomial to the vanishing of its derivatives:
Lemma 7.5. \[28\] Lemma 1.3 Let $K$ be an algebraically closed field and let $f(Z) \in K[Z]$ be a polynomial of degree $n$. Then the following are equivalent:

i. $\Delta^k f(z)$ is nilpotent in $K[Z]/(f(z))$ for $0 \leq k < n$.

ii. $f(Z) = (Z - \alpha)^n$ for some $\alpha \in K$.

We start by introducing differential operators in the universal case: let $T, Z$ be variables, and let $k[Y_1, \ldots, Y_n, Z, T]$ be the polynomial ring in $n+2$ variables. Consider the $k[Y_1, \ldots, Y_n]$-morphism:

$$
\text{Tay}: \ k[Y_1, \ldots, Y_n][Z] \longrightarrow \ k[Y_1, \ldots, Y_n][Z, T] \\
Z \longrightarrow Z + T.
$$

For each polynomial $G(Z) \in k[Y_1, \ldots, Y_n][Z]$,

$$
\text{Tay}(G(Z)) = \sum_{k \geq 0} G_k(Z)T^k,
$$

and for each index $k$, we can define the operators:

$$
\Delta^k: \ k[Y_1, \ldots, Y_n][Z] \longrightarrow k[Y_1, \ldots, Y_n][Z] \\
G(Z) \longrightarrow \Delta^k(G(Z)) := G_k(Z).
$$

For $k \geq 0$ the $\Delta^k$ are particular differential operators of degree $k$, relative to the inclusion $k[Y_1, \ldots, Y_n] \subset k[Y_1, \ldots, Y_n][Z]$.

Now consider the universal monic polynomial of degree $n$:

$$
F_n(Z) = (Z - Y_1) \cdots (Z - Y_n) = Z^n - s_{n,1}Z^{n-1} + \cdots + (-1)^n s_{n,n} \in k[Y_1, \ldots, Y_n, Z].
$$

Observe that

$$
\text{Tay}(F_n(Z)) = F_n(Z + T) = (Z + T - Y_1) \cdots (Z + T - Y_n) = (T - (-Z + Y_1)) \cdots (T - (-Z + Y_n)),
$$

and that the coefficients of this polynomial in the variable $T$ are precisely the symmetric polynomials in the variables $Z - Y_1, \ldots, Z - Y_n$, i.e.,

$$
\Delta^k(F_n(Z)) = (-1)^{n-k}s_{n,n-k}(-Z + Y_1, \ldots, -Z + Y_n),
$$

for $k = 1, \ldots, n - 1$. In this setting, the action of $S_n$ in $k[Y_1, \ldots, Y, Z]$ can be considered as a permutation of $Y_1, \ldots, Y_n$ that fixes $Z$. Hence,

$$
k[Z - Y_1, \ldots, Z - Y_n]^{S_n} = k[F_n(Z), \{\Delta^k(F_n(Z))\}_{k=1,\ldots,n-1}].
$$

Let us stress here that $\Delta^k(F_n(Z))$ is homogeneous of degree $n - k$ for $k = 1, \ldots, n - 1$, so $k[F_n(Z), \{\Delta^k(F_n(Z))\}_{k=1,\ldots,n-1}]$ is a graded subring in $k[Y_1, \ldots, Y, Z]$.

Now, since $Y_i - Y_j = (Z - Y_j) - (Z - Y_i)$, we have that

$$
k[Y_i - Y_j]_{1 \leq i, j \leq n} \subset k[Z - Y_1, \ldots, Z - Y_n].$$
Hence there is an inclusion of graded algebras

$$k[H_{n_1}, \ldots, H_{n_r}] = k[Y_i - Y_j]_{1 \leq i, j \leq n}$$

$$\subset k[Z - Y_1, \ldots, Z - Y_n]^{s_n} = k[F_n(Z), \{\Delta^k(F_n(Z))\}_{k=1,\ldots,n-1}],$$

and therefore for $i = 1, \ldots, r$ each $H_{n_i}$ is also weighted homogeneous in the variables

$$\{F_n(Z), \{\Delta^k(F_n(Z))\}_{k=1,\ldots,n-1}\}.$$

To conclude, it can be shown that $k[s_{n,1}, \ldots, s_{n,n}][Z]/(f(Z)) = k[s_{n,1}, \ldots, s_{n,n}][Y_1]$, and setting $Z = Y_1$ in $k[F_n(Z), \{\Delta^k(F_n(Z))\}_{k=1,\ldots,n-1}]:$

$$k[H_{n_1}, \ldots, H_{n_r}] = k[Y_i - Y_j]_{1 \leq i, j \leq n}$$

$$\subset k[F_n(Y_1), \{\Delta^k(F_n(Y_1))\}_{k=1,\ldots,n-1}] \subset k[s_{n,1}, \ldots, s_{n,n}][Y_1],$$

have the same integral closure. This result, in combination with Lemma 7.5, give the following Theorem:

**Theorem 7.6.** [28, Theorem 1.16] Let $S$ be a $k$-algebra, let $f(Z) = Z^n + a_1Z^{n-1} + \ldots + a_{n-1}Z + a_n \in S[Z]$ and consider a commutative diagram

$$\text{Spec}(S[Z]/(f(Z))) \rightarrow \text{Spec}(S[Z])$$

$$\text{Spec}(S[Z]) \Downarrow \beta$$

$$\text{Spec}(S).$$

as described in [16]. Let $\mathcal{G}$ be the Rees algebra generated by $f(Z)$ in degree $n$, and let $\mathcal{T}_n$ denote the set of $n$-fold points of $\{f(Z) = 0\} \subset \text{Spec}(S[Z])$, i.e., $\mathcal{T}_n = \text{Sing} \mathcal{G}$. Define the specialization morphism,

$$\mathcal{T} = k[s_{n,1}, \ldots, s_{n,n}] \rightarrow S$$

$$s_{n,i} \rightarrow (-1)^ia_i$$

which gives rise to the elimination algebra associated to $\mathcal{G}$,

$$\mathcal{R}_\mathcal{G} = S[H_{m_j}(a_1, \ldots, a_n)W^m_j, j = 1, \ldots, r] \subset S[W],$$

where $m_j$ denotes the degree of the weighted homogeneous polynomial $H_{m_j}(s_{n,1}, \ldots, s_{n,n})$.

Then:

i) The closed set $V(H_{m_j}(a_1, \ldots, a_n); j = 1, \ldots, r) \subset \text{Spec}(S)$ is the image of the set of points where $\beta$ is purely ramified.

ii) If $S$ is regular, then

$$\beta(\mathcal{T}_n) = \beta(\text{Sing} \mathcal{G}) \subset \text{Sing} \mathcal{R}_\mathcal{G}.$$ 

If in addition, the characteristic of $S$ is zero, then the inclusion in (28) is an equality.

### 7.7. Elimination algebras in the general case.

Elimination algebras can also be defined for Rees algebras with more than one generator. In particular, if we consider the differential
algebra generated by $\mathcal{G}$, namely $\text{Diff}(\mathcal{G})$ (see 4.2), then Theorem 7.6 can be qualitatively improved, since in this case:

$$\beta(\Upsilon_n) = \beta(\text{Sing Diff}(\mathcal{G})) = \text{Sing } \mathcal{R}_{\text{Diff}(\mathcal{G})}$$

in any characteristic (see [28, Corollary 4.12], or 8.7 below). We refer to [28 1.23-1.40] for more details on the construction of the elimination universal algebra associated to more than one polynomial. Also, we refer to 8.8 and 8.7 where we indicate how elimination algebras can be computed. In Example 13.5 we provide a concrete example.

8. A local projection and the elimination algebra

Once the universal case has been treated in the previous section, we are ready to study the case that we are interested in: how to define elimination in terms of algebras on smooth schemes. Let $V = V^{(d)}$ be a $d$-dimensional smooth scheme of finite type over a field $k$. Let $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} W_n$ be a sheaf of Rees algebras and let $x \in \text{Sing } \mathcal{G}$ be a simple point not contained in any component of co-dimension one of $\text{Sing } \mathcal{G}$ (see Definition 6.1). In the following we describe how to construct:

- A suitable local projection (in an étale neighborhood of $x$),
  $$\beta_{d,d-1} : V^{(d)} \to V^{(d-1)},$$
  with $\beta_{d,d-1}(x) = x_1$. In this case $\beta_{d,d-1}^* : \mathcal{O}_{V^{(d-1)},x_1} \to \mathcal{O}_{V^{(d)},x}$ is an inclusion.
- An elimination algebra
  $$\mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \subset \mathcal{O}_{V^{(d-1)}}[W]$$
  in a suitable neighborhood of $x_1$.

So we start with an algebra $\mathcal{G} \subset \mathcal{O}_{V^{(d)}}[W]$ and define $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \subset \mathcal{O}_{V^{(d-1)}}[W]$. Although $\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}$ depends on the projection $\beta_{d,d-1}$, it will satisfy some nice properties, and our main invariants will derive from them (see 8.7).

**Definition 8.1.** Let $\mathcal{G}$ be a Rees algebra on a smooth $d$-dimensional scheme $V^{(d)}$ over a field $k$, and let $x \in \text{Sing } \mathcal{G}$ be a simple point. We say that a local projection to a smooth $(d-1)$-dimensional scheme, $V^{(d-1)}$,

$$\beta_{d,d-1} : V^{(d)} \to V^{(d-1)}$$  
  $x \to x_1$

is $\mathcal{G}$-admissible locally at $x$ if the following conditions hold:

(i) The closed point $x$ is not contained in any component of co-dimension one of $\text{Sing } \mathcal{G}$.
(ii) The Rees algebra $\mathcal{G}$ is a $\beta_{d,d-1}$-relative differential algebra (see Definition 4.6).
(iii) Transversality: $\ker d\beta_{d,d-1} \cap \mathcal{C}_{\mathcal{G},x} = \{0\} \subset T_{V,x}$.

Now we will explain the role of conditions (ii) and (iii) in Definition 8.1.
8.2. Condition (ii) in Definition 8.1: Relative differential algebras. Notice that if $G = \oplus_{n \in \mathbb{N}} I_n W^n$ is an absolute differential algebra, then it is also a relative differential algebra for any smooth morphism $\beta_{d,d-1} : V^{(d)} \to V^{(d-1)}$ of schemes over a field $k$ defined in a neighborhood of $x \in \text{Sing } G$. A key point in our development is the study of properties of relative differential algebras, and their stability by monoidal transformations (see Section 9); whereas transforms of absolute differential algebras are not absolute differential.

As indicated before, absolute differential algebras give rise to relative differential algebras for arbitrary smooth maps (see 8.7, specially properties i, ii and iii for the relevance of working with differential and relative differential algebras).

8.3. Condition (iii) in Definition 8.1: Local projections and transversality. Almost any smooth local projection, or more generally, almost any smooth morphism defined locally, in a neighborhood of a simple point in the singular locus of a Rees algebra, will fulfill the condition in Definition 8.1 (iii). In 8.5 we show that this condition is open: it holds at any singular point in a neighborhood of $x$. First we will explain the meaning in terms of local rings, and then we describe a procedure to construct a smooth morphism satisfying this geometric condition at $x$:

Suppose that a local projection to a $(d-1)$-dimensional regular scheme,

$$\beta_{d,d-1} : V^{(d)} \to V^{(d-1)}$$

is defined. A regular system of parameters $\{y_1, \ldots, y_{d-1}\} \subset \mathcal{O}_{V^{(d-1)}},x_1$ extends to parameters $\{y_1, \ldots, y_{d-1}, y_d\} \subset \mathcal{O}_{V^{(d)}},x$. Condition (iii) in Definition 8.1 holds if and only if $\{\text{In}_x y_1 = 0, \ldots, \text{In}_x y_{d-1} = 0\} \subset \mathcal{T}_{V,x}$ is not contained in the tangent cone of $G$ at $x$, $C_{G,x}$.

This also shows how to produce local projections that fulfill Condition (iii): Let $G = \oplus_{n \in \mathbb{N}} I_n W^n$ be a Rees algebra, and let $x \in \text{Sing } G$ be a simple closed point. The graded ideal $\text{In}_x G$ defines the subscheme $C_{G,x}$ of $\mathcal{T}_{V,x}$ (in fact, recall that if $G$ is a differential algebra, then $C_{G,x} = L_{G,x}$, see 6.2(iii)). Now select a regular system of parameters $\{y_1, \ldots, y_{d-1}, y_d\} \subset \mathcal{O}_{V^{(d)}},x$ such that $\{\text{In}_x y_1 = 0, \ldots, \text{In}_x y_{d-1} = 0\} \subset \mathcal{T}_{V,x}$ is not contained in $C_{G,x}$. Note that there is a natural injective map from the ring of polynomials in $(d-1)$-variables with coefficients in $k$ into $\mathcal{O}_{V^{(d)},x}$, and localizing we get an inclusion of regular local rings,

$$k[Y_1, \ldots, Y_{d-1}]/(Y_1, \ldots, Y_{d-1}) \longrightarrow \mathcal{O}_{V^{(d)},x}$$

This is one way to produce a local projection as (30), to a $(d-1)$-dimensional regular scheme, satisfying condition (iii) in Definition 8.1.

8.4. Transversality and Zariski’s multiplicity formula for projections. With the same notation as in 8.3, fix a local projection as performed in (30). Now our goal is to study the image of $\text{Sing } G$ under the morphism $\beta_{d,d-1}$ in a neighborhood of $x$. We will show that if $Y$ is a smooth center in $\text{Sing } G$ containing $x$, then $Y$ and $\beta_{d,d-1}(Y)$ are isomorphic; in particular both are smooth.
Since $x \in \text{Sing } G$ is a simple point, there is some $n \in \mathbb{Z}_{>0}$ and an element $f \in I_n$ of order exactly $n$ at $x$. Therefore,

\begin{equation}
\text{Sing } G \subset \{ n \text{-fold-points of } f = 0 \} \subset V((f))
\end{equation}

and $\mathcal{C}_{G,x} \subset V(\text{In}_xf)$. Let $\{y_1, \ldots, y_{d-1}\}$ be a regular system of parameters in $\mathcal{O}_{V(d-1),x_1}$. Since $\beta_{d,d-1} : V(d) \rightarrow V(d-1)$ is smooth, $\{y_1, \ldots, y_{d-1}\}$ can be extended to a regular system of parameters $\{y_1, \ldots, y_{d-1}, Z\}$ in $\mathcal{O}_{V(d),x}$.

The condition of transversality imposed in Definition 8.1 (iii) ensures that $f \in I_n$ can be chosen so that $V(\text{In}_xf)$ and $\{\text{In}_xy_1 = 0, \ldots, \text{In}_xy_{d-1} = 0\}$ intersect only at the origin of the vector space $\mathbb{T}_{V,x}$. This last condition can be reformulated by saying that $\text{In}_xf \in \text{Gr}_{m_z}(\mathcal{O}_{V(d),x})$ is a homogeneous polynomial of degree $n$ in the variables $\{\text{In}_xy_1, \ldots, \text{In}_xy_{d-1}, \text{In}_xZ\}$, in which the monomial $(\text{In}_xZ)^n$ appears with non-zero coefficient.

Since Weierstrass Preparation Theorem holds in an étale neighborhood of $\mathcal{O}_{V(d-1),x}$, we may replace $\mathcal{O}_{V(d-1),x_1}$ and $\mathcal{O}_{V(d),x}$ in (8.3) by suitable étale neighborhoods if needed, and thus assume that there is a regular system of parameters, $\{y_1, \ldots, y_{d-1}\} \in \mathcal{O}_{V(d-1),x_1}$ that extends to a regular system of parameters in $\mathcal{O}_{V(d),x}$, $\{y_1, \ldots, y_{d-1}, z\}$ so that $f = z^n + a_1z^{n-1} + \ldots + a_{n-1}z + a_n$ with $a_i \in \langle y_1, \ldots, y_{d-1} \rangle$. Setting $z$ as $Z$,

\begin{equation}
f(Z) = Z^n + a_1Z^{n-1} + \ldots + a_{n-1}Z + a_n \in \mathcal{O}_{V(d-1),x_1}[Z].
\end{equation}

The map

\begin{equation}
\mathcal{O}_{V(d-1),x_1} \rightarrow \mathcal{O}_{V(d-1),x_1}[Z]/\langle f(Z) \rangle
\end{equation}

is a finite morphism of local rings which induces a finite projection

\[ \beta : V(f) \rightarrow V(d-1), \]

mapping $x \in V(f)$ to $x_1$. Also, notice that the extension of the maximal ideal $m_{x_1} \subset \mathcal{O}_{V(d-1),x_1}$ to $\mathcal{O}_{V(d),x_1}[Z]/\langle f(Z) \rangle$ is a reduction of the maximal ideal $M = (\overline{y}_1, \ldots, \overline{y}_{d-1}, \overline{Z}) \subset \mathcal{O}_{V(d-1),x_1}[Z]/\langle f(Z) \rangle$.

Let $V_n(f)$ be the closed set of $n$-fold points of the hypersurface $V(f)$ in $V(d)$. The map $\beta : V(f) \rightarrow V(d-1)$ is defined in a neighborhood of $x$ and it is the restriction of the smooth morphism $\beta_{d,d-1} : V(d) \rightarrow V(d-1)$. Then by Zariski’s multiplicity formula for projections (see 7.1):

(A) The projection $\beta_{d,d-1}$ induces a bijection between $V_n(f)$ and its image $\beta(V_n(f))$.

(B) For any irreducible scheme $Y \subset V_n(f)$, the finite map

\begin{equation}
\beta : Y \rightarrow \beta(Y)
\end{equation}

is birational.
Therefore, by (31) and (A), there is a bijection between $\beta_{d,d-1}(\text{Sing } G)$ and $\text{Sing } G$, and from (B) it follows that if $Y \subset \text{Sing } G$ is an irreducible subscheme, then $\beta : Y \rightarrow \beta(Y)$ is a finite birational map. Moreover, (A) also ensures that $\beta : Y \rightarrow \beta(Y)$ defines a bijection of the underlying topological spaces.

Assume, in addition, that $x \in Y \subset \text{Sing } G$ and that $Y$ is a regular center. Then $x$ is the unique point of $Y$ mapping to $\beta(x) = x_1 \in \beta(Y)$, and we claim now that $(Y, x)$ is étale over $(\beta(Y), x_1)$. This together with the previous properties would show that $\beta(Y)$ is regular at $x_1$, and that the finite birational map $\beta : Y \rightarrow \beta(Y)$ is in fact an isomorphism in an open neighborhood of $x$.

We will argue geometrically to prove that $\beta : Y \rightarrow \beta(Y)$ is étale at $x$. The smooth morphism $\beta_{d,d-1} : V^{(d)} \rightarrow V^{(d-1)}$ induces a linear map of tangent spaces, $d\beta_{d,d-1} : \mathbb{T}_{V^{(d)},x} \rightarrow \mathbb{T}_{V^{(d-1)},x_1}$. The claim is that $\ker d\beta_{d,d-1} \cap \mathbb{T}_{Y,x} = \{0\} \subset \mathbb{T}_{V^{(d)},x}$. This follows from our choice of $f \in I_n$ and the transversality condition in (8.1) (iii). In fact $\mathbb{T}_{Y,x} \subset V(\text{In}_x f)$ and $\ker d\beta_{d,d-1} = \{\text{In}_x y_1, \ldots, \text{In}_x y_{d-1}\}$ intersect a the origin of the vector space $\mathbb{T}_{V^{(d)},x}$. This proves that $Y$ and $\beta(Y)$ are isomorphic in a suitable neighborhood of $x$.

The previous discussion also shows that there is a change of variable of the form $Z' = Z - a$ in $\mathcal{O}_{V^{(d-1)},x_1}[Z]$, for a suitable $a \in \mathcal{O}_{V^{(d-1)},x_1}$, such that $I(Y)_x = \langle Z', v_1, \ldots, v_s \rangle$, where $\{v_1, \ldots, v_s\}$ is part of a regular system of parameters at $\mathcal{O}_{V^{(d-1)},x_1}$ (see also the proof of Theorem [9.1]). In fact, if $\overline{Z}$ denotes the restriction of $Z$ to $Y$, then there is an element $a \in \mathcal{O}_{V^{(d-1)},x_1}$ which restricts to the same function on $\mathcal{O}_{\beta(Y),x_1} = \mathcal{O}_{Y,x}$. So $Z - a$ will vanish along $Y$, and the claim follows from this fact.

**Remark 8.5.** The local projection constructed in (33) and the arguments and results described in (34) are also valid in an open neighborhood of $x$ in $\text{Sing } G$. To show this it is enough to prove that the transversality condition on $f$ holds in an open neighborhood of $x$; notice that the condition of transversality on $f$ can also be expressed in terms of differential operators that are relative to the smooth morphism $\beta_{d,d-1}$. Furthermore note that the operators $\Delta^k$ in (31) are also defined as relative differential operators, say:

$$
\Delta^k : \mathcal{O}_{V^{(d)},x} \rightarrow \mathcal{O}_{V^{(d)},x}
$$

where the inclusion $k[Y_1, \ldots, Y_n] \subset k[Y_1, \ldots, Y_n][Z]$ is replaced by the inclusion of regular rings $\mathcal{O}_{V^{(d-1)},x_1} \subset \mathcal{O}_{V^{(d)},x}$, and the role of $Z$ is that of the last parameter $y_d$ in $\mathcal{O}_{V^{(d)},x}$.

One can check that the condition of transversality imposed on $f \in I_n$ at the closed point $x \in \text{Sing } G$ can also be formulated by requiring that $\Delta^n(f)$ be a unit at the regular ring $\mathcal{O}_{V^{(n)},x}$, or, formally, that:

$$
\Delta^n(f)(x) \neq 0,
$$

(36)
which also shows that if the geometric condition in Definition 8.1 (iii) holds at \( x \), it also holds for all singular points in an open neighborhood of \( x \).

**Remark 8.6.** Fix a polynomial ring \( S[Z] \). A morphism,

\[
Tay: S[Z] \longrightarrow S[Z,T]; \quad Z \rightarrow Z + T,
\]

and operators \( \Delta^k: S[Z] \rightarrow S[Z] \) are defined by setting

\[
Tay(G(Z)) = \sum_{k \geq 0} \Delta^k(G(Z))T^k.
\]

Each \( \Delta^k \) is a differential operator of order \( k \) over the ring \( S \), and furthermore, for each positive integer \( N \), \( \{\Delta^k, k = 0, 1, \ldots N\} \) is a basis of \( Diff^N(S[Z]/S) \), the free \( S \)-module of \( S \)-differential operators of order \( N \).

Consider a finite number of monic polynomials, say

\[
f_i(Z) = Z^{n_i} + a_i^1 Z^{n_i-1} + \ldots + a_i^{n_i}, \quad i = 1, \ldots, r,
\]

and define a subalgebra of \( S[Z][W] \) of the form

\[
S[Z]\{f_i(Z)W^{n_i}, i = 1, \ldots, r\}.
\]

In general this Rees algebra will not be compatible with \( S \)-differential operators in the sense of Definition 4.6 (ii). However in \([30] \), Theorem 2.9] it is shown that there is a smallest extension of this algebra to one having this property, and such extension is

\[
S[Z]\{f_i(Z)W^{n_i}, \{\Delta^k(f_i(Z))W^{n_i-k}\}_{k=1,\ldots,n_i-1}\}_{i=1,\ldots,r}.
\]

But this extension is, in turn, the pull-back of an algebra in the universal setting in \([26] \), by a suitable morphism on \( S[W] \) (see Theorem 7.6 and 7.7).

**8.7. The elimination algebra \( R_{G_{\beta, d-1}} \).** Let \( V(d) \) be a \( d \)-dimensional smooth scheme over a field \( k \), let \( G = I_nW^n \subset O_{V(d)} \) be a Rees algebra, and assume that \( x \in Sing G \) is a simple closed point not contained in any component of co-dimension one of \( Sing G \). Construct a smooth morphism \( \beta_{d,d-1}: V(d) \rightarrow V(d-1) \) to some smooth \( (d-1) \)-dimensional scheme transversal to \( G \) in a neighborhood of \( x \) (this can be done, for instance, following the arguments given in \([8, 3] \). If in addition \( G \) is a \( \beta_{d,d-1} \)-relative differential algebra (i.e., if \( \beta_{d,d-1}: V(d) \rightarrow V(d-1) \) is locally \( G \)-admissible at \( x \)) then an **elimination algebra**

\[
R_{G_{\beta,d-1}} \subset O_{V(d-1),x_1}[W]
\]

can be defined (see \([28, 1.25, Definitions 1.42 and 4.10] \)). To do so, first choose a positive integer \( n \), and an element \( f \in I_n \) of order \( n \) at \( O_{V(d),x_1} \), and then produce a monic polynomial \( f(Z) \in I_n \) as in \([32]\) in a suitable \( \text{étale} \) neighborhood of \( x \). Then, it can be checked that, up to integral closure, we may assume that \( G \) is as in \([37]\), for \( S = O_{V(d-1),x_1} \), and suitable monic polynomials \( f_i(Z), i = 1, \ldots, r \). In particular, \( G \) is locally (and up to integral closure) the pull-back of the universal algebra so we define \( R_{G, \beta, d-1} \subset O_{V(d-1),x_1}[W] \) following the procedures indicated in Theorem 7.6 and 7.7.
This elimination algebra depends on the projection $\beta_{d,d-1}$ but by construction it does not depend on the election of $f$ once the projection is fixed, and it satisfies the following conditions:

i. The inclusion $\beta_{d,d-1}^* : O_{V^{(d-1)},x_1} \rightarrow O_{V^{(d)},x}$ induces an inclusion of Rees algebras $R_{G,\beta_{d,d-1}} \subset G$ (this follows now from \[26\]; see also \[28\] Theorem 4.13).

ii. If $G$ is a differential algebra, then $R_{G,\beta_{d,d-1}}$ is a differential algebra.

iii. There is an inclusion of closed subsets

\[ \beta_{d,d-1}(\text{Sing } G) \subset \text{Sing } R_{G,\beta_{d,d-1}} \]

and equality holds if $G$ is a differential Rees algebra (cf. \[28\] Corollary 4.12).

iv. The order of $R_{G,\beta_{d,d-1}}$ at $x_1$ does not depend on the projection, in other words, $\text{ord}_{x_1} R_{G,\beta_{d,d-1}}$ is independent of $\beta_{d,d-1}$ (see \[28\] Theorem 5.5).

**8.8. Another description of $R_{G,\beta_{d,d-1}}$.** Assume that locally, in a neighborhood of a simple point $x$, the Rees algebra $G$ is generated by

\[ \{f_n W^{n_1}, \ldots, f_n W^{n_s}\} \]

With the same notation as in \[8.7\] we can choose an element $fW^n$, assuming that $f$ has order $n$ at the local ring of the point. We can also assume that after multiplying by a unit: $f = F(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n \in O_{V^{(d-1)},x_1}[Z]$.

Note that multiplying by an element $f_n W^{n_s}$ induces an endomorphism

\[ L_{f_n} : (O_{V^{(d-1)},x_1}[Z]/\langle F(Z) \rangle)[W] \rightarrow (O_{V^{(d-1)},x_1}[Z]/\langle F(Z) \rangle)[W]. \]

Since $O_{V^{(d-1)},x_1}[Z]/\langle F(Z) \rangle[W]$ is a free $O_{V^{(d-1)},x_1}[W]$-module of rank $n$, each endomorphism $L_{f_n}$ has a characteristic polynomial or degree $n$,

(38)

\[ T^n + g_{1,n_1} T^{n-1} + \ldots + g_{n,n_1} \]

where $g_{j,n_1} \in O_{V^{(d-1)},x_1}[W]$ and the elimination algebra $R_{G,\beta_{d,d-1}}$ is generated by these coefficients up to integral closure (see \[28\] Corollary 4.12 and Example \[13.3\] for a computation in a concrete example). Furthermore, in a suitable neighborhood of $x$, and up to integral closure:

(39)

\[ G = O_{V}[\Delta^e(F(Z))W^{n-e}, 0 \leq e \leq n - 1] \odot \beta_{d,d-1}^*(R_{G,\beta_{d,d-1}}), \]

where the right hand side is the smallest subalgebra in $O_{V^{(d)}}[W]$ containing both algebras (cf.\[4\], or \[5\]).

**8.9. Elimination algebras and integral closure.** An important property of this form of elimination is its link with integral closure of graded algebras. Using the same notation as in \[8.7\] consider the following diagram:

\[
\begin{array}{ccc}
O_{V^{(d)},x_1}[W] & \xrightarrow{\gamma^*} & O_{V^{(d),x}}/\langle f_n \rangle[W] \\
\beta_{d,d-1}^* & \uparrow & \simeq O_{V^{(d-1),x_1}[Z]/\langle F(Z) \rangle}[W] \\
O_{V^{(d-1),x_1}[W]}
\end{array}
\]
where $\gamma^*$ denotes the natural restriction. Then the image of $R_{G,\beta_{d,d-1}}$ in $O_{V(d-1),x_1}[Z]/\langle F(Z)\rangle[W]$ is contained in $\gamma^*(G)$, and they both have the same integral closure (see [28, Theorem 4.11]). This Theorem also proves that if an inclusion of Rees algebras $G \subset G'$ is finite, then $R_{G,\beta_{d,d-1}} \subset R_{G',\beta_{d,d-1}}$ is finite.

8.10. Notation. In what follows, given a Rees algebra $G = G^{(d)}$ on a $d$-dimensional smooth scheme $V^{(d)}$ of finite type over a field $k$, we will refer to an elimination algebra as $R_{G,\beta_{d,d-1}}$ if we need to emphasize the projection, or just as $G^{(d-1)} \subset O_{V(d-1)}[W]$ if the choice of the projection is not relevant in the discussion.

8.11. Elimination algebras and the $\tau$-invariant. The equality expressed in (39) is an equality up to integral closure (both algebras have the same integral closure). A consequence of Theorem 8.3 is that the $\tau$-invariant at a point is well defined up to integral closure.

In [4] it is proven that if $G$ is a differential algebra, then $\tau_{R_{G,\beta_{d,d-1}}(x_1)} = \tau_{G,x} - 1$. In summary, this proof shows that (39) holds for any $n$ and any $f_n = F(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n \in O_{V(d-1),x_1}[Z]$, with the only condition that $f_n \in I_n$ have order $n$ at the local ring $O_{V(d-1),x}$; finally, for the case of differential algebras we may choose $f_n$ so that the initial form defines a linear subspace of co-dimension one in $T_{V(d),x}$.

So, in general, if $G$ is of co-dimensional type $\geq e \geq 1$ in a neighborhood of $x$ (i.e., if $\tau_{G,x} \geq e$ in $U \subset \text{Sing } G$) then we can expect to iterate the arguments in 8.3 $e$-times, and a sequence of local projections can be defined:

$$
\begin{align*}
V^{(d)} & \xrightarrow{\beta_{d,d-1}} V^{(d-1)} \rightarrow \ldots \rightarrow V^{(d-e)} \\
x = x_0 & \rightarrow x_1 \rightarrow \ldots \rightarrow x_e,
\end{align*}
$$

which by composition induces a local projection from $V^{(d)}$ to some $(d-e)$-dimensional smooth space $V^{(d-e)}$. In this way, by iteration, we can define elimination algebras

$$
G^{(d-1)} \subset O_{V(d-1)}[W], \ldots, G^{(d-e)} \subset O_{V(d-e)}[W]
$$

if for each $i = 1, \ldots, e$, the projection

$$
\beta_{d-(i-1),d-i} : V^{(d-(i-1))} \rightarrow V^{(d-i)}
$$

is $G^{(d-(i-1))}$-admissible locally at $x_{i-1}$. By [28, Corollary 4.12], there is an inclusion of closed subsets

$$
\beta_{d-(i-1),d-i} : \text{Sing } G^{(d-(i-1))} \subset \text{Sing } G^{(d-i)},
$$

which is an equality when $G^{(d-(i-1))}$ is a differential algebra for $i = 1, \ldots, e$.

Definition 8.12. Let $G$ be a Rees algebra on a smooth $d$-dimensional scheme $V^{(d)}$ over a field $k$ and let $x \in \text{Sing } G$ be a simple point with $\tau_{G^{(d)},x} \geq e$. We will say that a local projection to a smooth $(d-e)$-dimensional scheme over $k$

$$
\beta_{d,d-e} : V^{(d)} \rightarrow V^{(d-e)}
$$

$x \rightarrow x_e$
is locally $\mathcal{G}$-admissible at $x$ if it factorizes as a sequence of local $\mathcal{G}^{(d-i)}$-admissible projections as in Definition 8.1:

$$
(V^{(d)}, x_0 = x) \xrightarrow{\beta_{d,d-1}} (V^{(d-1)}, x_1) \to \ldots \xrightarrow{\beta_{d-(e-1),d-e}} (V^{(d-e)}, x_e)
$$

where for $i = 1, \ldots, e$, each $\mathcal{G}^{(d-i)} \subset \mathcal{O}_{V^{(d-i)}, x_i}[W]$ is the elimination algebra of $\mathcal{G}^{(d-(i-1))} \subset \mathcal{O}_{V^{(d-(i-1))}, x_{i-1}}[W]$, and $\beta_{d-(i-1),d-1}(x_{i-1}) = x_i$.

9. Elimination Algebras and Permissible Monoidal Transformations

The purpose of this section is to study the behavior of admissible projections under permissible monoidal transformations (see Definition 8.1), a result that will play a key role in the inductive construction of the function in Theorem 10.1.

**Theorem 9.1.** Let $\mathcal{G}^{(d)}$ be a Rees algebra on a smooth $d$-dimensional scheme $V^{(d)}$ and let $x \in \text{Sing } \mathcal{G}^{(d)}$ be a simple point (i.e., $\tau_{\mathcal{G}^{(d)}, x} \geq 1$). Suppose that a local $\mathcal{G}^{(d)}$-admissible projection is given, defining an elimination algebra:

$$
\beta_{d,d-1} : (V^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1)
$$

Let $Y \subset \text{Sing } \mathcal{G}^{(d)}$ be a permissible center. Then, locally in a neighborhood of $x$:

(i) The closed set $\beta_{d,d-1}(Y) \subset \text{Sing } \mathcal{G}^{(d-1)} \subset V^{(d-1)}$ is a permissible center for $\mathcal{G}^{(d-1)}$.

(ii) Given the monoidal transformations on $V^{(d)}$ and $V^{(d-1)}$ with centers $Y$ and $\beta_{d,d-1}(Y)$ respectively, there is a projection $\beta'_{d,d-1}$ defined in a suitable open set, and a commutative diagram of projections, and weak transforms:

$$
\begin{array}{ccc}
(V^{(d)}, x) & \xrightarrow{\pi^{(d)}} & U \subset V^{(d)'} \\
\downarrow{\beta_{d,d-1}} & & \downarrow{\beta'_{d,d-1}} \\
(V^{(d-1)}, x_1) & \xleftarrow{\pi^{(d-1)}} & V^{(d-1)'}
\end{array}
$$

Furthermore, if $x' \in \text{Sing } \mathcal{G}^{(d)'} \neq \emptyset$ maps to $x$, then,

(a) The projection $V^{(d)'} \to V^{(d-1)'}$ is $\mathcal{G}^{(d)'}$-admissible locally at $x'$ (see Definition 8.1). In particular $\mathcal{G}^{(d)'}$ is a $\beta'_{d,d-1}$-relative-differential algebra, defining an elimination algebra $\mathcal{G}^{(d-1)}$.

(b) Let $x'_1 = \beta'_{d,d-1}(x')$. Locally in an open neighborhood of $x'_1$, there is a natural inclusion $\mathcal{G}^{(d-1)'} \subset \mathcal{G}^{(d-1)}$ which is an equality up to integral closure.

To prove the Theorem we need to study the behavior of elimination algebras in the universal case, which is the purpose of the next lines. The proof of the Theorem is given in 9.3.
9.2. Monoidal transformations and weak transforms. Our goal is to understand:

i. How universal elimination algebras behave under permissible monoidal transformations.

ii. The behavior of differential algebras under permissible monoidal transformations.

Both points will be treated in the universal case.

Let \( F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_n) \) be the universal monic polynomial of degree \( n \), and let \( V \) be a new variable. We would like to make sense of the expression

\[
\left( \frac{1}{V} \right)^n F_n(Z) \in k[Y_1, \ldots, Y_n][V, V^{-1}][Z]
\]

which corresponds to the notion of weak transform of a degree-\( n \) element in a Rees algebra.

Notice that

\[
\left( \frac{1}{V} \right)^n F_n(Z) = \left( \frac{Z}{V} - \frac{Y_1}{V} \right) \cdot \left( \frac{Z}{V} - \frac{Y_2}{V} \right) \cdots \left( \frac{Z}{V} - \frac{Y_n}{V} \right) \in k[Y_1, \ldots, Y_n][V, V^{-1}][Z].
\]

Set

\[
F'_n \left( \frac{Z}{V} \right) := \left( \frac{Z}{V} - \frac{Y_1}{V} \right) \cdot \left( \frac{Z}{V} - \frac{Y_2}{V} \right) \cdots \left( \frac{Z}{V} - \frac{Y_n}{V} \right).
\]

Then \( F'_n \left( \frac{Z}{V} \right) = \frac{1}{V^n} F_n(Z) \) is a monic polynomial in the ring \( k \left[ \frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V} \right] \).\( \frac{Z}{V} \).

Let \( \Delta^k_1 \) be a differential operator on \( k \left[ \frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V} \right] \) relative to \( k \left[ \frac{Y_1}{V}, \frac{Y_2}{V}, \ldots, \frac{Y_n}{V} \right] \) for some \( k < n \). Then by (22),

\[
\Delta^k_1 \left( F'_n \left( \frac{Z}{V} \right) \right) = \frac{1}{V^{n-k}} \cdot \Delta^k(F_n(Z)).
\]

Now let \( S_n \) act on \( k[Y_1, \ldots, Y_n][V, V^{-1}][Z] \) by permuting the variables \( Y_i \) and fixing both \( V \) and \( Z \). In this way \( S_n \) acts also by permutation on the variables \( \frac{Y_i}{V} \) for \( i = 1, \ldots, n \) and fixes \( \frac{Z}{V} \). Therefore

\[
k \left[ \frac{Y_i}{V} - \frac{Y_j}{V} \right]^{S_n} = k[H'_{n_1}, \ldots, H'_{n_r}],
\]

where

\[
H'_{n_i} = \frac{1}{V_{n_i}} \cdot H_{n_i}
\]

for \( H_{n_i} \) as in (20).

9.3. Proof of Theorem 9.1. (i) There is an inclusion \( \beta_{d,d-1}(\text{Sing } G) \subset \text{Sing } G^{(d-1)} \) (see 8.7 (iii)), mapping \( Y \) to \( \beta(Y) \), which is an isomorphism (see 8.4). Since \( Y \) and \( \beta(Y) \) are isomorphic, then \( \beta(Y) \) is a regular permissible center for \( G^{(d-1)} \).
(ii) We claim that the monoidal transformations of \( V^{(d)} \) and \( V^{(d-1)} \) with centers \( Y \) and \( \beta_{d,d-1}(Y) \) respectively, produce, in a suitable open set \( U \subset V^{(d)'} \), a commutative diagram of projections:

\[
\begin{array}{ccc}
(V^{(d)}, x) & \xrightarrow{\pi^{(d)}} & (U \subset V^{(d)'}, x') \\
\downarrow \beta_{d,d-1} & \circ & \downarrow \beta'_{d,d-1} \\
(V^{(d-1)}, x_1) & \xrightarrow{\pi^{(d-1)}} & (V^{(d-1)'}), x_1'.
\end{array}
\]

Set \( \mathcal{G} = \oplus I_n W^n \subset \mathcal{O}_{V^{(d)}}[W] \) and \( \mathcal{G}' = \oplus I'_n W^n \subset \mathcal{O}_{V^{(d)'}[W]} \). To prove (ii) we argue as in (i). First, choose an integer \( n \) and select an element \( f \in I_n \) of order \( n \) transversal to \( \beta_{d,d-1} \) at \( x \), and consider its weak transform, \( f' \in I'_n \). The hypothesis of transversality on \( f \) can be reformulated by saying that the relative differential operator of order \( n \), \( \Delta^n \in \text{Diff}^n \mathcal{O}_{V^{(d)}/V^{(d-1)}} \), is such that \( \Delta^n f \) is a unit in a neighborhood of \( x \) (see [8.5] specially [36]).

We claim that the law of transformation of relative differentials in (40) specializes to show that there is a relative differential operator \( \Delta^n \in \text{Diff}^n \mathcal{O}_{V^{(d)}/V^{(d-1)}} \) such that \( \Delta^n(f') \) is a unit in a neighborhood of \( x' \) and therefore \( x' \) is a simple point (i.e., \( \tau_{\mathcal{G}^{(d)'},x'} \geq 1 \)).

We also claim that the law of transformation in (40) already shows that \( \mathcal{G}^{(d)'} \) is a \( \beta'_{d,d-1} \)-relative differential algebra locally at \( x' \).

To clarify these points note first that we may assume that, after multiplying by a unit, \( f \) is monic of degree \( n \), i.e., \( f = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n \in \mathcal{O}_{V^{(d-1)},x_1}[Z] \) in an étale neighborhood of the point \( x \). According to (i), the center \( Y \) maps isomorphically to \( \beta(Y) \); in particular the class (restriction) on \( Y \) of any element of \( \mathcal{O}_{V^{(d)},x} \) is also the class of an element of \( \mathcal{O}_{V^{(d-1)},x_1} \). Thus after a suitable change of variable of the form \( Z - \alpha, \alpha \in \mathcal{O}_{V^{(d-1)},x_1} \), we may assume that \( Z \) vanishes identically along \( Y \), and that \( I(Y) = \langle Z, y_1, \ldots, y_s \rangle \), where \( \{y_1, \ldots, y_s\} \) is part of a regular system of parameters in \( \mathcal{O}_{V^{(d-1)},x_1} \) and each coefficient \( a_i \) has order \( \geq i \) along the regular center \( Y \). The closed set \( \text{Sing } \mathcal{G}^{(d)} \) is included in the closed set of \( n \)-fold points of the hypersurface \( V(\langle f \rangle) \), and \( \text{Sing } \mathcal{G}^{(d)'} \) is included in the closed set of \( n \)-fold points of \( V(\langle f' \rangle) \).

Consider the open set \( U \subset V^{(d)'} \) which is the union of the charts \( \text{Spec} \left( \mathcal{O}_{V^{(d)}} \left[ Z, \frac{y_1}{y_j}, \ldots, \frac{y_s}{y_j} \right] \right) \) for \( j = 1, \ldots, s \). The inclusions \( \mathcal{O}_{V^{(d-1)}} \left[ \frac{y_1}{y_j}, \ldots, \frac{y_s}{y_j} \right] \subset \mathcal{O}_{V^{(d)}} \left[ Z, \frac{y_1}{y_j}, \ldots, \frac{y_s}{y_j} \right] \) define \( \beta'_{d,d-1} : U \to V^{(d-1)'} \), as above. The point \( x' \in \text{Sing } \mathcal{G}^{(d)'} \) is included the \( n \)-fold points of \( f' \), and

\[
f' = \frac{Z^n}{y_j} + \frac{a_1 Z^{n-1}}{y_j y_j^{n-1}} + \ldots + \frac{a_{n-1} Z}{y_j y_j^{n-1}} + \frac{a_n}{y_j^n} \in \mathcal{O}_{V^{(d-1)'},x'_1} \left[ \frac{Z}{y_j} \right].
\]
Since $x'$ is a point of multiplicity $n$ of $f' = 0$, the residue fields of $\mathcal{O}_{V(d')', x'}$ and of $\mathcal{O}_{V(d-1)', x_1'}$ are the same. So there is an element $a \in \mathcal{O}_{V(d-1)', x_1'}$ such that $Z_1 = \frac{Z}{y} - a$ vanishes at $x'$, and $f'$ is a monic polynomial of degree $n$ in $\mathcal{O}_{V(d-1)', x_1'}[Z_1]$. Moreover if $\{z_1, \ldots, z_{d-1}\}$ is a regular system of parameters at $\mathcal{O}_{V(d-1)', x_1'}$, then $\{z_1, \ldots, z_{d-1}, Z_1\}$ is a regular system of parameters at $\mathcal{O}_{V(d')', x'}$. Now the local arguments in 8.3 and 8.7 can be repeated for $
abla'_{d,d-1}(V(d')', x') \rightarrow (V(d-1)', x_1')$, to prove the statement in (b). More precisely notice that:

- Up to integral closure $\mathcal{G}(d)$ is generated by monic polynomials in the setting of (37), as indicated in 8.7.
- The weak transforms of the local generators of $\mathcal{G}(d)$ generate $\mathcal{G}(d)'$ (see Proposition 5.2).
- The closed point $x'$ is contained in $V((f')) \subset V(d')$.
- We claim now that the weak transform of $f$ in $\mathcal{G}(d)'$, $f'$, can be used to define the elimination algebra in a neighborhood of $x'$ as in 8.3. In Theorem 7.6 and 7.7 it is given an explicit description of the elimination algebra as a specialization of the universal algebra elimination algebra. It follows from (10) that $\mathcal{G}'$ is a relative differential algebra. Also from (11) that up to integral closure, $\mathcal{G}'(d-1) = \mathcal{G}(d-1)$.

Part 4. Main Theorem and inductive invariants

10. Main Theorem 10.1

In this section we discuss resolutions of Rees algebras (see 10.3), where the main invariant is the function $\text{ord}_x \bar{G}$, defined by $\text{ord}_x \bar{G}$ in 3.9 for each $x \in \text{Sing} \bar{G}$. When the characteristic is zero and $\text{ord}_x \bar{G} = 1$ there is a smooth hypersurface of maximal contact at $x$, and a new Rees algebra $\bar{G}$ is defined along this smooth hypersurface. In particular a new value $\text{ord}_x \bar{G}$ can be defined. It is then shown that this value is an invariant; in other words, independent of the choice of the hypersurface of maximal contact. This result is the main outcome of the so called Hironaka trick; an alternative and enlightening proof of this result is due to J. Wlodarczyk (see [31]).

In this work hypersurfaces of maximal contact are replaced by suitable projections, and we begin this section by formulating this result in this setting, in Main Theorem 10.1. We then recall briefly how resolution of Rees algebras can be achieved by induction (see 10.3 and 10.4) parallelling the ideas given in Part 1.

Let $\mathcal{G}(d)$ be a Rees algebra on a smooth $d$-dimensional scheme $V(d)$ over a field $k$, let $x \in \text{Sing} \mathcal{G}(d)$ be a closed point with $\tau_{\mathcal{G}(d),x} \geq e$. Assume that $x$ is not contained in any component of co-dimension $e$ of $\text{Sing} \mathcal{G}$, and that there are two different admissible projections.
to a \((d - e)\)-dimensional smooth space \([8, 12]\).

\[
\beta_{1, d, d - e}: V^{(d)} \rightarrow V_1^{(d - e)} \quad \beta_{2, d, d - e}: V^{(d)} \rightarrow V_2^{(d - e)}
\]

\(x \rightarrow x_{e, 1}\)

Then the question is to compare \(\text{ord}_{x_{e, 1}} G_1^{(d - e)}\) and \(\text{ord}_{x_{e, 2}} G_2^{(d - e)}\).

In [28, Theorem 5.5] it is shown that if \(e \geq 1\) then

\[
\text{ord}_{x_{1, 1}} G_1^{(d - 1)} = \text{ord}_{x_{1, 2}} G_2^{(d - 1)}.
\]

In this section we generalize this result, which leads to Definition 10.2. More precisely we prove the following theorem:

**Theorem 10.1 (Main Theorem).** Let \(V^{(d)}\) be a \(d\)-dimensional scheme smooth over a field \(k\), let \(G^{(d)} \subset \mathcal{O}_{V^{(d)}} [W]\) be a differential algebra, let \(x \in \text{Sing} G^{(d)}\) be a simple closed point, and let \(m \leq \tau_{G, x}\). Consider two different \(G^{(d)}\)-admissible local projections to some \((d - m)\)-dimensional smooth schemes with their corresponding elimination algebras:

\[
\begin{align*}
\beta_{1, d, d - m}: (V^{(d)}, x) &\rightarrow (V_1^{(d - m)}, x_{m, 1}) \\
\beta_{2, d, d - m}: (V^{(d)}, x) &\rightarrow (V_2^{(d - m)}, x_{m, 2})
\end{align*}
\]

Then:

\[
\text{ord}_{x_{m, 1}} G_1^{(d - m)} = \text{ord}_{x_{m, 2}} G_2^{(d - m)}.
\]

Moreover, if \(V^{(d)} \leftarrow V^{(d)'}\) is a composition of permissible monoidal transformations, \(x' \in \text{Sing} G^{(d)'}\) a closed point dominating \(x\), and

\[
\begin{array}{ccc}
(V^{(d)}, x) & \cong & (U \subset V^{(d)'}, x') \\
G^{(d)} & \cong & G^{(d)'} \\
\beta_{1, d, d - m}: (V^{(d)}, x) & \rightarrow & (V_1^{(d - m)}, x_{m, 1}) \\
\beta_{2, d, d - m}: (V^{(d)}, x) & \rightarrow & (V_2^{(d - m)}, x_{m, 2}) \\
G^{(d)}_{j^{(d - m)}} & \cong & G^{(d - m)'}_{j^{(d - m)'}},
\end{array}
\]

is the corresponding commutative diagram of elimination algebras and admissible projections for \(j = 1, 2\), then

\[
\text{ord}_{x_{m, j}} G_1^{(d - m)'} = \text{ord}_{x_{m, j}} G_2^{(d - m)'}.
\]

The Theorem provides the following upper semi-continuous functions:

**Definition 10.2.** (i) Let \(V^{(d)}\) be a \(d\)-dimensional scheme smooth over a field \(k\), let \(G^{(d)} \subset \mathcal{O}_{V^{(d)}} [W]\) be a differential algebra, let \(x \in \text{Sing} G^{(d)}\) be a simple closed point, and let \(m \leq \tau_{G, x}\). Then, in a neighborhood of \(x\), we define the function

\[
\begin{align*}
\text{ord}_{G^{(d)}}^{(d - m)}: \text{Sing} G^{(d)} &\rightarrow \mathbb{Q} \\
z &\rightarrow \text{ord}_{z, m} G^{(d - m)}
\end{align*}
\]
where $G^{(d-m)}$ is an elimination algebra defined by an arbitrary $G^{(d)}$-admissible local projection to some $(d - m)$-dimensional smooth scheme, $\beta_{d,d-m} : V^{(d)} \to V^{(d-m)}$, and $z_m = \beta_{d,d-m}(z)$ (notice that the function is well defined since it does not depend on the projection by Theorem 10.1).

(ii) Let $G^{(d)} \subset \mathcal{O}_{V^{(d)}}[W]$ be a differential algebra as in (i), let $\beta_{d,d-m} : V^{(d)} \to V^{(d-m)}$ be any $G^{(d)}$-admissible local projection in a neighborhood of a simple point $x \in \text{Sing } G^{(d)}$, and let $V^{(d)} \leftarrow V^{(d)'}$ be a composition of permissible monoidal transformations. Let $x' \in \text{Sing } G^{(d)'}$ be a closed point dominating $x$, and consider the corresponding commutative diagram of elimination algebras and admissible projections as in Theorem 9.1,

$$
\begin{array}{ccc}
(V^{(d)}, x) & \rightarrow & U \subset V^{(d)'} \\
G^{(d)} & \downarrow \beta_{d,d-m} & G^{(d)'} \\
\downarrow \beta_{d,d-m} & \circ & \downarrow \beta_{d,d-m}' \\
(V^{(d-m)}, x_m) & \leftarrow & V^{(d-m)'} \\
G^{(d-m)} & \leftarrow & G^{(d-m)'}.
\end{array}
$$

Then in a neighborhood of $x'$ the function

$$
\text{ord}^{(d-m)'}_{G^{(d)'}} : \text{Sing } G^{(d)'} \to \mathbb{Q}
$$

$$
z' \to \text{ord}_{z'_m} G^{(d-m)'}
$$

with $z'_m = \beta_{d,d-m}'(z')$ is well define since by Theorem 10.1 it is independent of the projection.

The proof of Theorem 10.1 will be presented to the next section. In the rest of this section we indicate some variations of the function in 10.2 that lead to the so-called reduction to the monomial case treated in the coming sections (see Part 4).

Theorem 10.1 is stated for a simple Rees algebra on a $d$-dimensional smooth scheme $V^{(d)}$; and we have to indicate why simple Rees algebras arise in resolution problems. Recall that there is a dictionary between Rees algebras and pairs as indicated in 3.10; in 2.4 the notion of simple pairs was introduced, which are analogous of simple Rees algebras. Moreover, the whole Section 2 was dedicated to showing how resolution of simple pairs leads to the so-called monomial case.

10.3. Resolution of Rees algebras. [29, 5.10] As pointed out in 3.10 there is a strong link between $(J, b)$ and the Rees algebra $G = G_{(J,b)}$ (see [12]). So a sequence of transformations of pairs and basic objects as in (3) defines a sequence of transformations or Rees algebras:

$$
(V, G, E) \leftarrow (V_1, G_1, E_1) \leftarrow \cdots \leftarrow (V_s, G_s, E_s).
$$

It follows from our notion of transformation of Rees algebras that each $G_i = G_{(J_i,b)}$, so

$$
\text{Sing } (G_i) = \text{Sing } (J_i, b).
$$
Furthermore, if $d$ denotes the dimension of $V$, then the functions
\[(44) \quad w-\text{ord}_{G_i}^{(d)} : \text{Sing}(G_i) \to \mathbb{Q}\]
are defined with the same properties as in the case of pairs (see 2.1).

We say that a sequence of transformations,
\[(45) \quad (V, G, E) \leftarrow (V_1, G_1, E_1) \leftarrow \cdots \leftarrow (V_s, G_s, E_s),\]
is a resolution of $(V, G, E)$ (or a resolution of $G$ if $E$ is empty), if $\text{Sing}(G_s) = \emptyset$.

10.4. The monomial case. [29, 6.11] Let $V^{(d)}$ be a $d$-dimensional scheme smooth over a field $k$, let $G \subset \mathcal{O}_{V^{(d)}}[W]$ be a differential algebra of co-dimensional type $\geq m$ (6.4). In this case the function
\[\text{ord}_{G}^{(d-m)} : \text{Sing} G \to \mathbb{Q}\]
is described in Definition 10.2. The discussion on basic objects and its resolution, which was presented in Section 2, also extends to this context and satellite functions $w-\text{ord}_{G}^{(d-m)}$ are defined, with the property that $w-\text{ord}_{G}^{(d-m)} = \text{ord}_{G}^{(d-m)}$, and if
\[(46) \quad (V^{(d)}, G, E) \leftarrow (V_1^{(d)}, G_1, E_1) \leftarrow \cdots \leftarrow (V_s^{(d)}, G_s, E_s),\]
is a sequence of monoidal transformations with center $Y_i \subset \text{Max} w-\text{ord}_{G}^{(d-m)}$, then
\[\max w-\text{ord}_{G}^{(d-m)} \geq \max w-\text{ord}_{G_1}^{(d-m)} \geq \cdots \geq \max w-\text{ord}_{G_s}^{(d-m)} .\]
When this holds we say that $(V_s, G_s, E_s)$ is in the monomial case if $\max w-\text{ord}_{G_s}^{(d-m)} = 0$ (here $m$ could be zero).

11. Proof of Theorem 10.1

The strategy. In this section we address the proof of Theorem 10.1. Recall our starting point: we assume the existence of two locally $G^{(d)}$-admissible projections to $(d - m)$-smooth dimensional schemes (8.12),
\[
\begin{array}{c}
(V^{(d)}, x) \\
G^{(d)}
\end{array} 
\begin{array}{c}
(V_1^{(d-m)}, x_{m,1}) \\
G_1^{(d-m)}
\end{array} 
\begin{array}{c}
(V_2^{(d-m)}, x_{m,2}) \\
G_2^{(d-m)}
\end{array}
\]
and using the hypothesis of Theorem 10.1 we want to show that $\text{ord}_{x_{m,i}} G_1^{(d-m)} = \text{ord}_{x_{m,i}} G_2^{(d-m)}$. To this end, as indicated in the following proposition, it will be enough to find a suitable local ring $(B, m)$ and suitable maps
\[\mathcal{O}_{V_i^{(d-m)}, x_{m,i}} \to B\]
so that the images of $G_i^{(d-m)}$ in $B[W]$ under these maps have the same integral closure for $i = 1, 2$. 
Proposition 11.1. [28] Lemma 5.7 and Corollary 5.8] Let \((B, m)\) be a local ring, let 
\((S_1, m_1), (S_2, m_2) \subset (B, m)\)
be two local regular rings, let \(\mathcal{H} = \oplus I_i W^k \subset B[W]\) be a Rees algebra and let 
\[\mathcal{H}_1 = \oplus J_{1,k} W^k \subset S_1[W]\quad \text{and} \quad \mathcal{H}_2 = \oplus J_{2,k} W^k \subset S_2[W]\]
be Rees algebras with inclusions 
\(\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{H}\).

Assume that for \(i = 1, 2\):

(i) The inclusions \(S_i \subset B\) are finite and flat extensions of local rings.
(ii) The ideals \(m_i B \subset B\) are reductions of \(m\).
(iii) The inclusions \(\mathcal{H}_i = \oplus J_{i,k} W^k \subset \mathcal{H} = \oplus I_k W^k\) are both finite.

Then
\[\text{ord}_{S_1}(\mathcal{H}_1) = \text{ord}_{S_2}(\mathcal{H}_2)\].

The basic idea of the proof of Theorem 10.1 is that under its assumptions we can find a suitable sequence of elements
\(f_1 W^{n_1}, \ldots, f_m W^{n_m} \in \mathcal{G}^{(d)}\)
so that the hypotheses of Proposition 11.1 hold for:
- \(B = \mathcal{O}_{V^{(d)},x}/\langle f_1, \ldots, f_m \rangle\);
- \(\mathcal{H}\) the image of \(\mathcal{G}^{(d)}\) in \(B[W]\) under the natural quotient map;
- \(S_1 = \mathcal{O}_{V^{(d),x},m,1}, S_2 = \mathcal{O}_{V^{(d),x},m,2}\);
- \(\mathcal{H}_1 = \mathcal{G}^{(d)-m}_1, \mathcal{H}_2 = \mathcal{G}^{(d)-m}_2\).

11.2. Idea of the proof of Theorem 10.1. Observe that there are two statements in Theorem 10.1: the first is a result about differential algebras, while the second part is the corresponding statement for the weak transform of a differential algebra after a finite sequence of monoidal transformations.

The first part of the Theorem will be proven in two steps: 1 and 2. In step 1 we will show that differential algebras contain sequences of elements with special properties. This will be used in step 2 to accomplish the first part of Theorem 10.1.

Similarly, the proof of the second part of the Theorem will be shown in two steps: 1’, and 2’. In step 1’ we will prove that, after a finite sequence of monoidal permissible transformations, the weak transform of a differential algebra contains a sequence of elements with special properties. This will be used in step 2’, where the second part of the Theorem 10.1 will be given.

Idea of the proof of the first part of Theorem 10.1
Step 1. Assume that \( V^{(d)} \) is a \( d \)-dimensional scheme smooth over a field \( k \), and that \( G^{(d)} \subset O_{V^{(d)}}[W] \) is a differential algebra. Let \( x \in \text{Sing} \ G^{(d)} \) be a simple closed point, and let \( m \leq \tau_{G,x} \). Suppose that

\[
(V^{(d)}, x) \to (V^{(d-m)}, x_m)
\]

is a \( G^{(d)} \)-admissible projection locally at \( x \). Under these assumptions we will show that there is factorization of (48) into local admissible projections, together with elimination algebras and elements

\[
(V^{(d)}, x) \to (V^{(d-1)}, x_1) \to \ldots \to (V^{(d-(m-1))}, x_{m-1}) \to (V^{(d-m)}, x_m)
\]

where \( f_1, \ldots, f_m \in G^{(d)} \) via the inclusions

\[
G^{(d-(m-1))} \subset \ldots \subset G^{(d-1)} \subset G^{(d)},
\]

and where each \( f_i \) is transversal to

\[
(V^{(d-(i-1))}, x_{i-1}) \to (V^{(d-i)}, x_i),
\]

for \( i = 1, \ldots, m \) (here we take \( x_0 = x \)). Set \( B = O_{V^{(d)}(x)} / \langle f_1, \ldots, f_m \rangle \) and let \( m_B \) be its maximal ideal.

Step 2. Under the assumptions of step 1, suppose that, in some neighborhood of \( x \), an arbitrary \( G^{(d)} \)-admissible local projection to some \( (d-m) \)-dimensional smooth scheme, and an elimination algebra are given:

\[
\beta_{1,d,d-m} : (V^{(d)}, x) \to (V_{1}^{(d-m)}, x_{m,1})
\]

Consider the local ring \( O_{V_1^{(d-m)}, x_{m,1}, m_{x_{m,1}}} \). Then we will show that there is an inclusion of local rings,

\[
O_{V_1^{(d-m)}, x_{m,1}} \subset B = O_{V^{(d)}(x)} / \langle f_1, \ldots, f_m \rangle,
\]

with the following properties:

(i) The inclusion is finite and flat;
(ii) The ideal \( m_{x_1}B \) is a reduction of \( m_B \);
(iii) The Rees algebras

\[
G_1^{(d-m)} \subset \overline{G^{(d)}} \subset B[W]
\]

have the same integral closure in \( B[W] \) (here \( \overline{G^{(d)}} \) denotes the image of \( G^{(d)} \) in \( B[W] \)).

Since (50) is an arbitrary admissible projection and \( B \) has been fixed in step 1, the first part of Theorem 10.1 will follow from Proposition 11.1.

Idea of the proof of the second part of Theorem 10.1
Step 1'. Fix $V^{(d)}$, $\mathcal{G}^{(d)} \subset \mathcal{O}_{V^{(d)}}[W]$, $x \in \text{Sing } \mathcal{G}^{(d)}$, $m \leq \tau_{\mathcal{G},x}$ and an admissible projection as in step 1:

\begin{equation}
(V^{(d)}, x) \rightarrow (V^{(d-m)}, x_m)
\end{equation}

together with the factorization given in (49), and the elements $f_1, \ldots, f_m \in \mathcal{G}^{(d)}$ with the properties stated in step 1.

Let $V^{(d)} \leftarrow V^{(d)'}$ be a composition of permissible monoidal transformations mapping $x'$ to $x$. Then by Theorem 9.1 sequence (19) can be lifted to a sequence of local admissible projections for the weak transform of $\mathcal{G}^{(d)}, \mathcal{G}^{(d)'}$, inducing a commutative diagram of permissible transformations, local admissible projections and elimination algebras,

\begin{equation}
\begin{array}{c}
(V^{(d')}, x' = x'_0) \rightarrow (V^{(d-1)'} , x'_1) \rightarrow \ldots \rightarrow (V^{(d-(m-1)')}, x'_{m-1}) \rightarrow (V^{(d-m')}, x'_m) \\
\mathcal{G}^{(d')} \mathcal{G}^{(d-1)'} \ldots \mathcal{G}^{(d-(m-1)')} \mathcal{G}^{(d-m')}
\end{array}
\end{equation}

where $x'_i$ maps to $x_i$ for $i = 0, \ldots, m$. Notice that then the strict transforms of $f_1, \ldots, f_m$ in $\mathcal{O}_{V^{(d)'}, f'_1, \ldots, f'_m}$, are in $\mathcal{G}^{(d)'}$, and that moreover,

\begin{equation}
f'_1 \in \mathcal{G}^{(d)'} f'_2 \in \mathcal{G}^{(d-1)'} \ldots f'_{m-1} \in \mathcal{G}^{(d-(m-1)')}.
\end{equation}

We will show that each $f'_i$ is transversal to

\begin{equation}
(V^{(d-(i-1)'}, x'_{i-1}) \rightarrow (V^{(d-i)'}, x'_i),
\end{equation}

for $i = 1, \ldots, m$ (here we take $x'_0 = x'$). Set $B' = \mathcal{O}_{V^{(d)'}, x'/\langle f'_1, \ldots, f'_m \rangle}$ and let $m_{B'}$ be its maximal ideal.

Step 2'. Under the assumptions of step 1', assume that in some neighborhood of $x$ an arbitrary $\mathcal{G}^{(d)}$-admissible local projection to some $(d - m)$-dimensional smooth scheme is given:

\begin{equation}
\beta_{1_{d,d-m}} : (V^{(d)}, x) \rightarrow (V^{(d-m)}, x_{m,1})
\end{equation}

\mathcal{G}^{(d)} \mathcal{G}^{(d-m)}.
Then the composition of monoidal permissible transformations $V^{(d)} \leftarrow V^{(d)'}$ from step 1’ induces a composition of permissible transformations and elimination algebras in (55),

\[
\begin{array}{ccc}
(V^{(d)}, x) & \leftarrow & (U \subset V^{(d)'} , x') \\
G^{(d)} & \leftarrow & G^{(d)'} \\
(V^{(d)}_{1}, x_{m,1}) & \leftarrow & (V^{(d)'}_{1}, x'_{m,1}) \\
G^{(d)}_{1} & \leftarrow & G^{(d)'}_{1}
\end{array}
\]

Consider the local ring $(O_{V^{(d)'}_{1}, x'_{m,1}}, m_{x'_{m,1}})$. Then we will show that there is an inclusion of local rings:

\[
O_{V^{(d)'}_{1}, x'_{m,1}} \subset B' = O_{V^{(d)'}, x'/\langle f_{1}', \ldots , f_{m}' \rangle}
\]

that is finite and flat, that the ideal $m_{x'_{m,1}} B$ is a reduction of $m_{B'}$ and that

\[
G^{(d)'}_{1} \subset \overline{G^{(d)'}} \subset B'[W]
\]

have the same integral closure in $B'[W]$ (here $\overline{G^{(d)'}$ denotes the image of $G^{(d)'}$ in $B'[W]$). Since $B'$ has been fixed in step 1’, the second part of Theorem 10.1 will follow from Proposition 11.1.

**About steps 1 and 1’**

The main difficulty in the proof of Theorem 10.1 is the accomplishment of steps 1 and 1’. More precisely, and with the same notation as above, given a differential algebra, a suitable ring $B$ is constructed and fixed in step 1. Then, in step 2, we have to show that for any admissible projection there is an inclusion as in (51) that satisfies properties (i), (ii) and (iii). Moreover, it is not immediate, either, that this situation can be carried out after a finite sequence of permissible transformations. Thus, the key of the proof is to find a suitable sequence of elements as in step 1 and step 1’. Most part of this section will be devoted to proving the existence of these particular sequences of elements for a differential algebra. First, we have to introduce some definitions and prove auxiliary results:

- Given a Rees-algebra $\mathcal{G}$ and a simple closed point $x \in \text{Sing} \, \mathcal{G}$, we introduce the notion of $\tau_{\mathcal{G},x}$-sequence (see Definition 11.3). We will see that the existence of such sequences is guaranteed when $\mathcal{G}$ is a differential algebra.

- However it is not clear that $\tau_{\mathcal{G},x}$-sequences behave well under permissible monoidal transformations, so they are not suitable for proving Theorem 10.1. This problem is overcome by introducing $\mathcal{G}$-nested sequences (see Definition 11.6). Nested sequences have some interesting properties as listed in 11.7. In particular, they behave well under permissible transformations.
It is worth pointing out that, while the notion of $\tau$-sequence is intrinsic to $G$, the concept of nested sequence is relative to a particular smooth projection and a suitable factorization of it, as in \[49\].

- The existence of nested sequences is not obvious: in Proposition \[11.8\] and in Corollary \[11.9\] we show how to construct $G$-nested sequences starting from a $\tau_{G,x}$-sequence.

Once the existence of nested sequences is established, the proof Theorem \[10.1\] will follow from their properties. The proof of Theorem \[10.1\] is stated in \[11.10\].

**About $\tau$-sequences**

**Definition 11.3.** Let $G = \oplus_n I_n W^n$ be a Rees algebra in a $d$-dimensional smooth scheme $V$ over a field $k$, let $x \in \text{Sing } G$ be a simple point, and let $k'$ be the residue field at $x$. We will say that a set of homogeneous elements $f_1 W^{n_1}, \ldots, f_s W^{n_s} \in G$ is a $\tau_{G,x}$-sequence of length $s$ if for $j = 1, \ldots, s$:

i. $n_j = p^{e_j}$;

ii. $\text{In}_x f_j \in \text{Gr}_{O_{V,x}} \simeq k'[Z_1, \ldots, Z_d]$ is a $k'$-linear combination of $Z_1^{p^{e_j}}, \ldots, Z_d^{p^{e_j}}$ for some $e_j \in \mathbb{N}$;

iii. The class of $\text{In}_x f_j$ is a regular element at the graded ring $\text{Gr}_{O_{V,x}}/\langle \text{In}_x f_i : i \neq j \rangle$.

By definition, if $f_1 W^{n_1}, \ldots, f_s W^{n_s} \in G$ is a $\tau_{G,x}$-sequence of length $s$, then $s \leq \tau_{G,x}$. A $\tau_{G,x}$-sequence $f_1 W^{n_1}, \ldots, f_s W^{n_s} \in G$ is said to be a maximal-$\tau_{G,x}$-sequence if $\tau_{G,x} = s$.

**11.4. On the conditions of Definition 11.3.** Let $f_1 W^{n_1}, \ldots, f_s W^{n_s} \in G$ be a $\tau_{G,x}$-sequence. If char $k = 0$ then condition (ii) says that $\text{In}_x f_1, \ldots, \text{In}_x f_s \in \text{Gr}_{O_{V,x}}$ are linear forms, while condition (iii) means that they are linearly independent. If char $k = p > 0$, then, up to a change of the base field, it can be assumed that $\text{In}_x f_j \in \text{Gr}_{O_{V,x}}$ is some $p^{e_j}$-th power of a linear form for $j = 1, \ldots, s$. Condition (iii) indicates that these linear forms are independent (see \[6.2\]). Notice that if $f_1 W^{n_1}, f_2 W^{n_2}, \ldots, f_s W^{n_s}$ is a $\tau$-sequence, then so is $(f_1)^p(W^{n_1})^p, f_2 W^{n_2}, \ldots, f_s W^{n_s}$. In particular it can always be assumed that $n_1 = \ldots = n_s$.

**Remark 11.5.** When $G^{(d)}$ is a differential algebra, then there is a maximal $\tau_{G^{(d)},x}$-sequence for each simple point $x \in \text{Sing } G^{(d)}$ (see \[6.2\]). However if $G^{(d)} \rightarrow G^{(d)'}$, then $(V^{(d)}, x) \leftarrow (V^{(d)'}, x')$ is a permissible monoidal transformation it is, in general, not true that the strict transforms of a $\tau_{G^{(d)},x}$-sequence form a $\tau_{G^{(d)'},x}$-sequence.

**About nested sequences**
Definition 11.6. Let $G^{(d)}$ be a Rees algebra, and let $x \in \text{Sing} G^{(d)}$ be a simple point with $\tau_{G^{(d)},x} \geq s$. Suppose that there is a $G^{(d)}$-admissible projection to some $(d-s)$-dimensional smooth scheme in a neighborhood of $x$,

$$(V^{(d)}, x) \to (V^{(d-s)}, x_s),$$

and a factorization into admissible projections

$$\begin{align*}
(V^{(d)}, x) & \xrightarrow{\beta_{d,d-1}} \ldots \xrightarrow{\beta_{d-(s-1),d-s}} (V^{(d-s)}, x_{s-1}), & (V^{(d-s)}, x_s) \\
G^{(d)} & \xrightarrow{f_1} \ldots \xrightarrow{f_{s}} G^{(d-s)},
\end{align*}$$

A set of homogeneous elements $f_1^{(d)}W_{n_1}, f_2^{(d-1)}W_{n_2}, \ldots, f_s^{(d-(s-1))}W_{n_s} \in G^{(d)}$ is said to be a $G^{(d)}$-nested sequence relative to sequence (56) if

$$(V^{(d)}, x = x_0) \xrightarrow{\beta_{d,d-1}} \ldots \xrightarrow{\beta_{d-(s-1),d-s}} (V^{(d-s)}, x_{s-1}),$$

and $f_i^{(d-(i-1))}$ is transversal to $\beta_{d-(i-1),d-i}$ for $i = 1, \ldots, s$ (see 8.3 and 8.4 for the notion of transversality and its role in constructing admissible smooth projections).

11.7. Some facts about nested sequences. Assume that

$$f_1^{(d)}W_{n_1}, f_2^{(d-1)}W_{n_2}, \ldots, f_s^{(d-(s-1))}W_{n_s} \in G^{(d)}$$

is a $G^{(d)}$-nested-sequence in a neighborhood of $x$ as in Definition 11.6 relative to a sequence as in (56). Then:

1. Nested sequences define complete intersections. In other words, the quotient

$$\mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle$$

is a complete intersection.

To see this, notice that since $f_i^{(d-(i-1))} \in G^{(d-(i-1))}$ is transversal to $\beta_{d-(i-1),d-i} : V^{(d-(i-1))} \to V^{(d-i)}$, for each $i = 1, \ldots, s$ the local ring homomorphism

$$\mathcal{O}_{V^{(d-i)},x_i} \to \mathcal{O}_{V^{(d-(i-1)),x_{i-1}}}/\langle f_i^{(d-(i-1))} \rangle$$

is finite and flat (up to an étale change of base, see 8.4). As a consequence,

$$\mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle$$

is a finite free $\mathcal{O}_{V^{(d-s)},x_s}$-module. Hence, the quotient is Cohen-Macaulay, and moreover, a complete intersection.

2. Nested sequences and reductions. If $m_{x_s}$ denotes the maximal ideal in $\mathcal{O}_{V^{(d-s)},x_s}$ then

$$m_{x_s} \mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle$$
is a reduction of the maximal ideal in \( O_{V^{(d)},x} / \langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle \) (see 3.4 specially the arguments involving formula (32)).

3. Nested sequences lift to nested sequences after permissible monoidal transformations. Let \( V^{(d)} \leftarrow V^{(d)'} \) be a permissible monoidal transformation, let \( G^{(d)'} \) be the weak transform of \( G^{(d)} \) in \( V^{(d)'} \), and let \( x_0' \in \text{Sing} \ G^{(d)'} \) be a closed point dominating \( x_0 \). Then the weak transforms of \( f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \) in \( V^{(d)'} \), which we denote by \( f_1^{(d)'}, f_2^{(d-1)'} \), \ldots, \( f_s^{(d-(s-1))'} \), form a \( G^{(d)'} \)-nested sequence relative to the transform of sequence (56) (see Theorem 9.1 and its proof). Therefore the quotient

\[
O_{V^{(d)'}}, x_0' / \langle f_1^{(d)'}, f_2^{(d-1)'}, \ldots, f_s^{(d-(s-1))'} \rangle
\]

defines a complete intersection, and hence it is flat over \( O_{V^{(d-s)'}}, x_0', s \). If \( I(E) \subset O_{V^{(d)'}}, x_0' \) denotes the ideal sheaf of the exceptional divisor, then the strict transform of the ideal

\[
\langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle O_{V^{(d)'}}, x_0'
\]

in \( O_{V^{(d)'}}, x_0' \) is

\[
\bigcup_{n \geq 0} \left( \langle f_1^{(d)}, f_2^{(d-1)}, \ldots, f_s^{(d-(s-1))} \rangle O_{V^{(d)'}}, x_0' : I(E)^n \right) = \langle f_1^{(d)'}, f_2^{(d-1)'}, \ldots, f_s^{(d-(s-1))'} \rangle \subset O_{V^{(d)'}}, x_0'
\]
since

\[
O_{V^{(d)'}}, x_0' / \langle f_1^{(d)'}, f_2^{(d-1)'} \rangle / \langle f_1^{(d)'}, f_2^{(d-1)'}, \ldots, f_s^{(d-(s-1))'} \rangle
\]
is flat over \( O_{V^{(d-m)'}}, x_0', m \).

4. Nested sequences and integral closure. There is a diagram

\[
\begin{array}{ccc}
G^{(d)} & \xrightarrow{\gamma_0} & O_{V^{(d)},x}[W] \\
G^{(d-s)} & \xrightarrow{\gamma_s} & O_{V^{(d-s)},x}[W] \\
& \downarrow \gamma_0 & \\
& O_{V^{(d)},x} / \langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle [W] & \\
\end{array}
\]

where \( \gamma_s \) is a finite map, \( \gamma_0 \) is the natural surjection, and

\[
\gamma_s^*(G^{(d-s)}) \subset \gamma_0^*(G^{(d)})
\]

is a finite extension of graded algebras in \( O_{V^{(d)},x} / \langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle [W] \).

To see this, notice that by property (1), for each \( i = 1, \ldots, s \), the map

\[
O_{V^{(d-i)},x_i} \longrightarrow O_{V^{(d-i)},x_i} / \langle f_1^{(d)}, \ldots, f_i^{(d-(i-1))} \rangle
\]
is finite, and factorizes as

\[
O_{V^{(d-i)},x_i} \xrightarrow{\eta_{d-i,d-(i-1)}} O_{V^{(d-(i-1)),x_{i-1}}} / \langle f_i^{(d-(i-1))} \rangle \xrightarrow{\delta_{d-(i-1)}} O_{V^{(d)},x} / \langle f_1^{(d)}, \ldots, f_i^{(d-(i-1))} \rangle.
\]
For $i = 1, \ldots, s$ consider the diagram

$$
\begin{align*}
\mathcal{G}(d-(i-1)) &\subset \mathcal{O}_{V(d-(i-1)),x_{i-1}}[W] \\
\mathcal{G}(d-i) &\subset \mathcal{O}_{V(d-i),x_i}[W]
\end{align*}
$$

where $\eta_{d-i,d-(i-1)}$ is finite, $\alpha_{d-(i-1)}^*$ is surjective and $f_{i}^{(d-(i-1))} \in \mathcal{G}(d-(i-1))$ is transversal to

$$
\beta_{d-(i-1),d-i} : (V(d-(i-1)), x_{i-1}) \to (V(d-i), x_i).
$$

According to Theorem [28, 4.11] (see also [8,7]), the inclusion

$$
\eta_{d-i,d-(i-1)}^* \mathcal{G}(d-i) \subset \alpha_{d-(i-1)}^*(\mathcal{G}(d-(i-1)))
$$

is a finite extension of graded algebras. Therefore,

$$
\delta_{d-(i-1)}^*(\eta_{d-i,d-(i-1)}^*(\mathcal{G}(d-i))) \subset \delta_{d-(i-1)}^*(\alpha_{d-(i-1)}^*(\mathcal{G}(d-(i-1))))
$$

is a finite extension of graded algebras in $\mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle[W]$.

Since the map

$$
\mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle \to \mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle,
$$

is surjective, the resulting maps

$$
\gamma_{i-1}^* : \mathcal{O}_{V(d-(i-1)),x_{i-1}} \to \mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle,
\gamma_i^* : \mathcal{O}_{V(d-i),x_i} \to \mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle
$$

are a composition of finite and surjective maps. Hence

$$
\gamma_i^*(\mathcal{G}(d-i)) \subset \gamma_{i-1}^*(\mathcal{G}(d-(i-1))) \subset \mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle[W]
$$

have the same integral closure.

Using an inductive argument we conclude that there is a sequence of inclusions of Rees algebras

$$
\gamma_s^*(\mathcal{G}(d-s)) \subset \ldots \subset \gamma_i^*(\mathcal{G}(d-1)) \subset \gamma_0^*(\mathcal{G}(d)) \subset \mathcal{O}_{V(d),x}/\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle[W],
$$

all having the same integral closure.

**The existence of nested sequences for differential algebras**
In the following we consider a differential algebra on a smooth scheme together with an admissible local projection. Our goal is to show that there is a \( \tau \)-sequence which is, in addition, a nested sequence for this given admissible projection. This will be settled in Corollary 11.9 where we also describe a procedure for constructing a sequence which is simultaneously a nested and a \( \tau \)-sequence. In this procedure we will start from an arbitrary \( \tau \)-sequence at a singular point.

**Proposition 11.8.** Let \( \mathcal{G} = \oplus_n I_n W^n \) be a differential algebra on a \( d \)-dimensional smooth scheme \( V^{(d)} \) over a field \( k \). Let \( x \in \text{Sing} \mathcal{G} \subset V^{(d)} \) be a simple closed point and let

\[
f_1^{(d)} W^{n_1}, \ldots, f_s^{(d)} W^{n_s}
\]

be a maximal \( \tau_{\mathcal{G},x} \)-sequence of length \( s \geq 2 \). Fix a \( \mathcal{G} \)-admissible projection in a neighborhood of \( x \), \( \beta_{d,d-1} : V^{(d)} \to V^{(d-1)} \) (see Definition 8.7). Then:

A) For some index \( i \), \( 1 \leq i \leq s \), \( f_i^{(d)} \) is transversal to \( \beta_{d,d-1} \).

B) Set \( i = 1 \) as in (A) (after reordering the sequence if needed) and construct an elimination algebra \( \mathcal{R}_{\mathcal{G},\beta_{d,d-1}} \) as described in 8.7. Then:

i. There is a \( \tau_{\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}} \)-sequence of length \( (s-1) \), \( f_2^{(d-1)} W^{l_2}, \ldots, f_s^{(d-1)} W^{l_s} \in R_{\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}} \).

ii. The previous \( \tau_{\mathcal{R}_{\mathcal{G},\beta_{d,d-1}}} \)-sequence can be constructed so that

\[
\langle f_2^{(d-1)}, \ldots, f_s^{(d-1)} \rangle \subset \langle f_1^{(d)}, \ldots, f_s^{(d)} \rangle \subset \mathcal{O}_{V^{(d)},x}
\]

via the inclusion \( \mathcal{O}_{V^{(d-1)},x_1} \subset \mathcal{O}_{V^{(d)},x} \).

**Proof:** A) Our hypotheses are that \( \mathcal{G} \) is a differential algebra and \( f_1^{(d)} W^{n_1}, \ldots, f_s^{(d)} W^{n_s} \) is a maximal \( \tau_{\mathcal{G},x} \)-sequence of length \( s \geq 2 \) at \( x \in \text{Sing} \mathcal{G} \subset V^{(d)} \). Recall that each \( n_i = p^{e_i} \), that each \( f_i \) has order \( p^{e_i} \) at \( \mathcal{O}_{V^{(d)},x} \), and that \( \text{In}_x(f_i) \in \text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x}) \) is homogeneous of degree \( p^{e_i} \) and a \( p^{e_i} \)-th power of a linear form. In addition, the tangent cone defined by \( \mathcal{G} \) at \( \mathbb{T}_{V^{(d)},x} = \text{Spec} (\text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x})) \) is the closed set defined by the ideal \( \langle \text{In}_x(f_1), \ldots, \text{In}_x(f_s) \rangle \). Since we are assuming that the conditions in Definition 8.7 hold, there must be an index \( i \) for which \( f_i \) is transversal to \( \beta_{d,d-1} : V^{(d)} \to V^{(d-1)} \).

B) By (A) we can assume that \( f_1 \) is transversal to \( \beta_{d,d-1} \) (here a reordering of the \( \tau \)-sequence may be needed). Suppose that \( \text{In}_x(f_1) = Y_1^{p^{e_1}} \) for some linear form \( Y_1 \in \text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x}) \). Let \( \{z_2, \ldots, z_d\} \) be a regular system of parameters in \( \mathcal{O}_{V^{(d-1)},x_1} \). Choose \( y_1 \) to be an element of order one at \( \mathcal{O}_{V^{(d)},x} \), so that \( \text{In}_x(y_1) = Y_1 \in \text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x}) \). Then \( \{y_1, z_2, \ldots, z_d\} \) is a regular system of parameters in \( \mathcal{O}_{V^{(d)},x} \), and \( \text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x}) \) is a polynomial ring in variables \( \{Y_1, Z_2, \ldots, Z_d\} \), where \( Z_i = \text{In}_x(z_i) \) for \( i = 2, \ldots, d \).

Recall that the \( \tau \)-sequence \( f_1^{(d)} W^{n_1}, \ldots, f_s^{(d)} W^{n_s} \) is defined with \( n_i = p^{e_i} \), which can be chosen so that \( e_1 = e_2 = \cdots = e_s = e \). Let \( k' \) denote the residue field of \( \mathcal{O}_{V^{(d)},x} \). Then:

- \( \text{Gr}_{m_x}(\mathcal{O}_{V^{(d)},x}) = k'[Y_1, Z_2, \ldots, Z_d] \).
- In\(_e(f_1^{(d)}) = Y^e_1,
- For \(j = 2, \ldots, s\), In\(_e(f_j^{(d)}) = \lambda_j Y_1^{p^e} + (L_j)^{p^e}\), for some \(\lambda_j \in k'\) and \(L_j\) a linear form in \(k'[Z_2, \ldots, Z_d]\);
- The linear forms \(\{L_j, j = 2, \ldots s\}\) are independent in \(Gr_{m_1}((O_{V(d-1),x_1}) = k'[Z_2, \ldots, Z_d]\).

Assume, for simplicity, that \(k' = k\) (by finite extension of base field), set \(f_1^{(d)} = f_1^{(d)}\) and let \(f_j^{(d)} = \lambda_j f_1^{(d)} - f_j^{(d)}\) for \(j = 2, \ldots, s\). Notice that \(\{f_1^{(d)}W^{p^e}, \ldots, f_s^{(d)}W^{p^e}\}\) is a \(\tau_{G,x}\)-sequence, that \(\langle f_1^{(d)}, \ldots, f_s^{(d)}\rangle = \langle f_1^{(d)}, \ldots, f_s^{(d)}\rangle\), and that a regular system of parameters \(\{v_2, \ldots, v_d\}\) can be chosen in \(O_{V(d-1),x_1}\) so that:

a) The set \(\{y_1, v_2, \ldots, v_d\}\) is a regular system of parameters in \(O_{V(d),x}\). In particular \(Gr_{m_2}((O_{V(d),x}) = k'[Y_1, V_2, \ldots, V_d]\), and \(V_i = In_x(v_i)\) for \(i = 2, \ldots, d\).

b) \(In_x(f_1^i) = Y_1^{p^e}\), and \(In_x(f_j^i) = V_j^{p^e}\), for \(i = 2, \ldots, s\).

Under these assumptions, part B i) of the Proposition was proven in [28, 5.12]. We briefly sketch the argument here: The setting now is that \(\{f_1^{(d)}W^{n_1}, \ldots, f_s^{(d)}W^{n_s}\}\) is a \(\tau_{G,x}\)-sequence of length \(s\), all \(n_i = p^e\), and there is a regular system of parameters \(\{y_2, \ldots, y_d\} \subset O_{V(d-1),x_1}\), which extends to \(\{y_1, y_2, \ldots, y_d\} \subset O_{V(d),x}\) and \(In_x f_i^{(d)} = In_x y_i^{p^e} \in Gr_{m_2}((O_{V(d),x})\) for \(i = 1, \ldots, s\).

We assume that \(f_1\) is a monic polynomial of degree \(p^e\) in \(y_1\) and coefficients in \(O_{V(d-1),x_1}\), so \(O_{V(d),x}/\langle f_1^{(d)}\rangle\) is a free \(O_{V(d-1),x_1}\)-module of rank \(p^e\).

For each \(i = 2, \ldots, s\), let \(\overline{f_i^{(d)}}\) be the image of \(f_i^{(d)}\) in \(O_{V(d),x}/\langle f_1^{(d)}\rangle\). Multiplying by \(\overline{f_i^{(d)}}\) induces a map of free \(O_{V(d-1),x_1}\)-modules:
\[
\Gamma_{\overline{f_i^{(d)}}}: O_{V(d),x}/\langle f_1^{(d)}\rangle \to O_{V(d),x}/\langle f_1^{(d)}\rangle,
\]
and similarly, multiplying by \(\overline{f_i^{(d)}}W^{n_i}\) defines a map of free \(O_{V(d-1),x_1}[W]\)-modules:
\[
\Gamma_{\overline{f_i^{(d)}}W^{n_i}}: O_{V(d),x}/\langle f_1^{(d)}W\rangle \to O_{V(d),x}/\langle f_1^{(d)}W\rangle.
\]
Let \(p_i(t)\) be the characteristic polynomial of \(\Gamma_{\overline{f_i^{(d)}}W^{n_i}}\), let \(g_iW^{t_i} \in O_{V(d-1),x_1}[W]\) be the determinant (i.e., \(g_iW^{t_i} = p_i(0)\)), and note that \(g_i\) is the determinant of \(\Gamma_{\overline{f_i^{(d)}}}: O_{V(d),x}/\langle f_1^{(d)}\rangle \to O_{V(d),x}/\langle f_1^{(d)}\rangle\).

Under these conditions, it can be shown that \(g_i\) has order \(p^{2e}\) in \(O_{V(d-1),x_1}\), that \(g_iW^{p^{2e}} \in R_{G,\beta_{d-1},1}\), and that \(In_x g_i = In_x (y_i)^{p^{2e}}\) for \(i = 2, \ldots, s\), where, as indicated before, \(\{y_2, \ldots, y_d\}\) is a regular system of parameters in \(O_{V(d-1),x_1}\) (see [28, 5.12] for more details on this proof).
To prove \( \text{B ii) } \), observe that the composition
\[
\mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)} \rangle \xrightarrow{\Gamma_{f_i^{(d)}}} \mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_i^{(d)} \rangle \to \mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_i^{(d)} \rangle,
\]
(where the last row is just the natural quotient morphism), maps the image of \( \Gamma_{f_i^{(d)}} \) to zero. Since \( g_i \) is the determinant of the first, any sufficiently high power of \( g_i \) is zero in \( \mathcal{O}_{V^{(d)},x}/\langle f_1^{(d)}, f_i^{(d)} \rangle \). In particular, for \( e' \) large enough, \( g_i^{e'} \in \langle f_1^{(d)}, f_i^{(d)} \rangle \), for \( i = 2, \ldots, s \).

Finally define \( f_i^{(d-1)} = g_i^{e'} \) and \( l_i = p^{2e'}\) for\( i = 2, \ldots, s \). So:
\[
f_2^{(d-1)} W^{l_2}, \ldots, f_s^{(d-1)} W^{l_s} \in \mathcal{R}_{G, \beta_{d,d-1}}
\]
is a \( \tau \)-sequence, and
\[
\langle f_2^{(d-1)}, \ldots, f_s^{(d-1)} \rangle \subset \langle f_1^{(d)}, \ldots, f_s^{(d)} \rangle \subset \mathcal{O}_{V^{(d)},x}.
\]
\[\square\]

**Corollary 11.9.** Let \( G^{(d)} = \oplus_n \mathbb{N} W^n \) be a differential algebra over a \( d \)-dimensional smooth scheme \( V^{(d)} \) over a field \( k \). Let \( x \in \text{Sing} G^{(d)} \) be a simple point, and let \( f_1^{(d)} W_n^1, \ldots, f_s^{(d)} W_n^s \in G^{(d)} \) be a maximal \( \tau_{G^{(d)},x} \)-sequence of length \( s \). Consider a \( G^{(d)} \)-admissible local projection to a \( (d-s) \)-dimensional scheme,
\[
\beta_{d,d-s} : (V^{(d)},x) \xrightarrow{G^{(d)}} (V^{(d-s)},x_s)
\]
and a factorization of \([52]\) as a sequence of \( G^{(d-i)} \)-admissible projections,
\[
(V^{(d)},x) \xrightarrow{G^{(d)}} \cdots \xrightarrow{G^{(d)}} (V^{(d-(s-1)},x_{s-1}) \xrightarrow{G^{(d-s)}} (V^{(d-s)},x_s)
\]
Then, after reordering \( f_1^{(d)} W_n^1, \ldots, f_s^{(d)} W_n^s \) if needed, for each \( i = 1, \ldots, s - 1 \):

\[\text{i. There is a } \tau_{G^{(d-i)},x_i} \text{-sequence of length } (s - i), \]
\[f_{i+1}^{(d-i)} W_{l_{i+1}}, \ldots, f_s^{(d-i)} W_{l_s} \in G^{(d-i)} \subset \mathcal{O}_{V^{(d-i)},x_i}[W].\]

\[\text{ii. There is an inclusion of ideals}
\]
\[\langle f_1^{(d-i)}, \ldots, f_s^{(d-i)} \rangle \subset \langle f_1^{(d-(i-1))}, f_{i+1}^{(d-(i-1))}, \ldots, f_s^{(d-(i-1))} \rangle \subset \mathcal{O}_{V^{(d-(i-1)),x_i-1}}
\]
via the inclusion \( \mathcal{O}_{V^{(d-(i-1)),x_i-1}} \to \mathcal{O}_{V^{(d-(i-1)),x_{i-1}}}; \)

\[\text{iii. Moreover there is a } G^{(d)} \text{-nested-sequence of length } m \text{ in a neighborhood of } x,
\]
\[f_1^{(d)} W_{l_1}, \ldots, f_s^{(d-(s-1))} W_{l_s}
\]
relative to sequence \([52]\), that is also a \( \tau_{G^{(d)},x} \)-sequence, and with
\[\langle f_1^{(d)}, \ldots, f_s^{(d-(s-1))} \rangle \subset \langle f_1^{(d)}, \ldots, f_s^{(d)} \rangle \subset \mathcal{O}_{V^{(d)},x}
\]
via the inclusions \( \mathcal{O}_{V^{(d-i)},x_i} \to \mathcal{O}_{V^{(d)},x} \) for \( i = 1, \ldots, s - 1 \).
Proof: After relabelling \( f_1^{(d)}, \ldots, f_s^{(d)} \in G^{(d)} \), we may assume that \( f_1^{(d)} \) is transversal to \( \beta_{d,d-1} \). Now the corollary follows from Proposition 11.8 and an inductive argument since the elimination algebra of a differential algebra is also a differential algebra. □

Proof of Theorem 10.1

11.10. The first part of the theorem will be proven in two steps.

Step 1. Assuming that \( G^{(d)} \) is a differential algebra we are going to show that there is a \( \tau \)-sequence which is also nested for some sequence of local admissible projections.

Since \( G^{(d)} \) is a differential algebra by Corollary 11.9 we can assume that there is a \( \tau_{G^{(d)},x} \)-sequence of length \( m \),

\[
(61) \quad f_1^{(d)} W^{r_1}, \ldots, f_m^{(d)} W^{r_m} \in G^{(d)}
\]

that is also \( G^{(d)} \)-nested relative to some sequence of \( G^{(d)} \)-local admissible projections

\[
(62) \quad G^{(d)} \rightarrow G_0^{(d-1)} \rightarrow \cdots \rightarrow G_0^{(d-m)}.
\]

Hence the map of local rings

\[
\mathcal{O}_{V_0^{(d-m)},x_0,m} \rightarrow \mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)})
\]

is finite and flat and therefore the quotient

\[
\mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)})
\]

is a \( (d-m) \)-dimensional Cohen-Macaulay ring (see 11.7). Let \( B = \mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)}) \), and denote by \( m_B \) its maximal ideal.

Step 2. Suppose that we are given an arbitrary \( G^{(d)} \)-admissible projection to some \( (d-m) \)-smooth scheme (8.12), and an elimination algebra

\[
(63) \quad \beta_{d,d-m} : V^{(d)} \rightarrow V^{(d-m)}
\]

then:

(a) Notice that there is a natural map

\[
\mathcal{O}_{V^{(d-m)},x_m} \rightarrow B = \mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)}).
\]

(b) We claim that the images of \( G^{(d-m)} \) and \( G^{(d)} \) in \( B[W] = \mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)})[W] \) have the same integral closure. Since the local admissible projection (63) is arbitrary, and the sequence \( f_1^{(d)}, \ldots, f_m^{(d)} \) and hence \( B = \mathcal{O}_{V^{(d)},x}/(f_1^{(d)}, \ldots, f_m^{(d)}) \) are fixed, the first part of Theorem 10.1 follows from Proposition 11.1.
The claim in (b) can be accomplished by finding a nested sequence relative to some factorization of (63) into locally admissible projections. This nested sequence will be constructed using the \( \tau \)-sequence found in Step 1.

So, consider any \( G(d) \)-admissible projection to some \((d - m)\)-smooth scheme with its corresponding elimination algebra,

\[
\beta_{d,d-m} : (V^{(d)}, x) \rightarrow (V^{(d-m)}, x_m)
\]

and construct a \( G(d) \)-nested sequence

\[
f_1^{(d)} W^{l_1}, \ldots, f_m^{(d-(m-1))} W^{l_m} \in G(d),
\]

in a neighborhood of \( x \), relative to some factorization of (64) using the \( \tau_{G,x} \)-sequence from step 1,

\[
f_1^{(d)} W^{n_1}, \ldots, f_m^{(d)} W^{n_m},
\]

as in Corollary 11.9 (ii). Notice that by construction,

\[
\langle f_1^{(d)}, \ldots, f_m^{(d-(m-1))} \rangle \subset \langle f_1^{(d)}, \ldots, f_m^{(d)} \rangle \subset O_{V^{(d)},x}.
\]

According to properties (1) and (2) in 11.7

\[
O_{V^{(d-m)},x_m} \rightarrow O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d-(m-1))} \rangle
\]

is a finite flat local map of \((d - m)\)-dimensional Cohen-Macaulay rings (here an étale change of base maybe needed) and by (66) the map

\[
O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d-(m-1))} \rangle \rightarrow O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d)} \rangle
\]

is surjective. Therefore

\[
O_{V^{(d-m)},x_m} \rightarrow B = O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d)} \rangle
\]

is flat.

Let \( m_{x_m} \) be the maximal ideal in \( O_{V^{(d-m)},x_m} \), and let \( m_x \) be the maximal ideal in \( O_{V^{(d)},x} \). Then by 11.7 (2),

\[
m_{x_m} O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d-(m-1))} \rangle
\]

is a reduction of

\[
m_x O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d-(m-1))} \rangle.
\]

Therefore by (66)

\[
m_{x_m} O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d)} \rangle
\]

is a reduction of the maximal ideal \( m_B \) of \( B \),

\[
m_x O_{V^{(d)},x}/\langle f_1^{(d)}, \ldots, f_m^{(d)} \rangle.
\]
Finally, consider the diagram:

\[ G^{(d)} \subset O_{V(d), x}[W] \]

\[ G^{(d-m)} \subset O_{V(d-m), x'_m}[W] \xrightarrow{\gamma^*_m} O_{V^{(d)}, x}/\langle f^{(d)}_1, \ldots, f^{(d-(m-1))}_m \rangle[W] \]

According to 11.7 (4) \( \gamma^*_m(G^{(d-m)}) \subset \gamma^*(G^{(d)}) \) is a finite extension of graded algebras. Thus by (66), their images in \( B \) still have the same integral closure.

Since (64) was an arbitrary \( G^{(d-m)} \)-admissible projection, by Proposition 11.1 \( \text{ord}_{x'_m} G^{(d-m)} \) is independent on the choice of the projection. This proves the first part of the Theorem.

The second part of the Theorem will be accomplished in two steps.

**Step 1′.** Fix the \( \tau_{G^{(d)}, x} \)-sequence found in (61) which is also a \( G^{(d)} \)-nested sequence relative to (62). Now suppose that \( V^{(d)} \leftarrow V^{(d)}' \) is a composition of permissible monoidal transformations, and that \( x' \in \text{Sing} G^{(d)'} \) is a closed point dominating \( x \). By Theorem 9.1 there is a commutative diagram of permissible monoidal transformations and admissible projections. Observe that the weak transforms of \( f^{(d)}_1, \ldots, f^{(d-m)}_m \in G^{(d)} \) in \( V^{(d)}' \), say \( f^{(d)'}_1, \ldots, f^{(d)'}_m \), form a \( G^{(d)'} \)-nested sequence in a neighborhood of \( x' \) (see 11.7). Let \( B' = O_{V^{(d)'}, x'}/\langle f^{(d)'}_1, \ldots, f^{(d)'}_m \rangle \) and let \( m_{B'} \) denote its maximal ideal.

**Step 2′.** Fix an arbitrary \( G^{(d)} \)-admissible local projection as in (64) and consider the composition of permissible monoidal transformations from step 1′, \( V^{(d)} \leftarrow V^{(d)'} \). Again by Theorem 9.1 there is a commutative diagram of elimination algebras and admissible projections:

\[ (V^{(d)}, x) \leftarrow (U \subset V^{(d)'}, x') \]

\[ (V^{(d)-m}, x_m) \leftarrow (V^{(d)'-m}, x'_m) \]

Recall, as observed in step 1′, that the weak transforms in \( V^{(d)'} \) of the \( G \)-nested sequence \( f^{(d)}_1, \ldots, f^{(d-m-(m-1))}_m \) given in (65), say \( f^{(d)'}_1, \ldots, f^{(d-m-(m-1))'}_m \), form a \( G^{(d)'} \)-nested sequence in a neighborhood of \( x' \) relative to some factorization of the \( G^{(d)'} \)-admissible local projection (67). Also there is an inclusion ideals

\[ \langle f^{(d)'}_1, \ldots, f^{(d-m-(m-1))'}_m \rangle \subset \langle f^{(d)'}_1, \ldots, f^{(d)'}_m \rangle, \]
Now the proof follows from a similar argument as the one given in step 2, using \((B', m_{B'})\) instead of \((B, m_B)\) (see also Theorem 9.1 and its proof).

\section{The non-simple case}

Let \(G^{(d)}\) be a Rees algebra and let \(x \in \text{Sing } G^{(d)}\) be a simple point with \(\tau_{G^{(d)}, x} \geq m\). Under the assumptions of Theorem 10.1 there are well defined upper-semi-continuous functions in a neighborhood of \(x\):

\[
\text{ord}^{(d-i)} : \text{Sing } G^{(d)} \rightarrow \mathbb{Q},
\]

\[
z \rightarrow \text{ord}^{(d-i)}_z G^{(d)} = \text{ord}_z G^{(d-i)}
\]

where \(z_i := \beta_{d,d-i}(z)\), and \(\beta_{d,d-i} : V^{(d)} \rightarrow V^{(d-i)}\) is a \(G^{(d-i)}\)-admissible local projection on to some \((d-i)\)-dimensional smooth scheme \(V^{(d-i)}\), for \(i = 0, 1, \ldots, m\) (8.12). Since \(\tau_{G^{(d)}, x} \geq m\),

\[
\text{ord}^{(d-i)}_x G^{(d)} = \ldots = \text{ord}^{(d-(m-1))} G^{(d)} = 1.
\]

In the following we denote by \(\text{max-ord}^{(d-i)}\) the maximum value of the function \(\text{ord}^{(d-i)}\) and we will use \(\text{Max-ord}^{(d-i)}\) to denote the the closed set

\[
\{ z \in \text{Sing } G^{(d)} : \text{ord}^{(d-i)}_z G^{(d)} = \text{max-ord}^{(d-i)} G^{(d)} \}.
\]

Now suppose that \(\tau_{G^{(d)}, x} \geq m\). Fix a \(G^{(d)}\)-admissible projection to some \((d-m)\)-dimensional smooth scheme, \(\beta_{d-m} : V^{(d)} \rightarrow V^{(d-m)}\), and let \(x_m := \beta_{(d-m)}(x)\). If \(m = d\), then \(\text{Sing } G^{(d)} = \{ x \}\) in a neighborhood of \(x\), and a resolution of \(G^{(d)}\) is achieved by blowing up this point (see 10.3).

On the other hand, if \(m < d\), and \(x_m \in \text{Sing } G^{(d-m)}\) is not a simple point contained in a component of co-dimension one of \(\text{Sing } G^{(d-m)}\) (see Lemma 13.2 and Remark 13.3), then it would be interesting to, somehow, “enlarge \(G^{(d)}\) to a larger Rees algebra \(\tilde{G}^{(d)} \supset G^{(d)}\) so that \(\text{Sing } \tilde{G}^{(d)} = \text{Max ord}^{(d-m)} G^{(d)}\) and \(\tau_{\tilde{G}^{(d)}, x} \geq m + 1\) (i.e., \(x_m \in \text{Sing } \tilde{G}^{(d-m)}\) will be a simple point). In this case, a stratification of \(\text{Sing } \tilde{G}^{(d)}\) will induce a stratification of \(\text{Max ord}^{(d-m)} \subset \text{Sing } G^{(d)}\) by descending induction on the value of \(\tau\).

The purpose of this section is to show how this enlargement can be done in full generality, the main result is the formulation of Theorems 12.9 and 12.10 in which the main properties of \(\tilde{G}^{(d)}\) are discussed. It is at this point where the notions of weak equivalence introduced in 5.7 appear in full strength. In fact, these two theorems show that \(\tilde{G}^{(d)}\) can be chosen so as to be well defined up to integral closure (see also Remark 12.8).

We begin by recalling the notion of twisted algebras introduced in [29].
Definition 12.1. Let $G = \bigoplus_{n \geq 0} J_n W^n$ be a Rees algebra on a smooth $d$-dimensional scheme $V$ and let $\omega$ be a positive rational number. The twisted algebra $G(\omega)$ is defined as

$$G(\omega) = \bigoplus_{n \geq 0} J_n^{\omega} W^n$$

where it is assumed that $J_n^{\omega} = 0$ if $\frac{n}{\omega}$ is not an integer.

As indicated in [3.10] our notion of Rees algebra is closely related to Hironaka’s notion of pair. A pair $(J, b)$ on a smooth scheme $V$ is defined by a non-zero sheaf of ideals $J \subset O_V$ and a positive integer $b$. We assign to a pair $(J, b)$ over $V$ the Rees algebra, say:

$$G(J, b) = O_V[J^b W^b],$$

which is a graded subalgebra in $O_V[W]$. It turns out that every Rees algebra over $V$ is a finite extension of $G(J, b)$ for a suitable pair $(J, b)$ (cf. [29, Proposition 2.9]).

Proposition 12.2. The twisted algebra of Definition 12.1 satisfies the following properties:

(i) If $G = G_{(J,b)}$ and if $w$ is a positive rational number with $bw \in \mathbb{Z}$ then $G(w) = G_{(J,wb)}$.

(ii) If $G_1$ and $G_2$ have the same integral closure, then so do $G_1(\omega)$ and $G_2(\omega)$.

(iii) $G(\omega)$ is a Rees algebra and $\omega \cdot \text{ord}_x G(\omega) = \text{ord}_x G$. In particular if $\omega = \text{ord}_x G$ then $\text{ord}_x G(\omega) = 1$.

(iv) If $\omega = \max\text{-ord}^{(d)} G$ then $G(\omega)$ is simple and $\text{Sing} G(\omega) = \text{Max}\text{-ord}^{(d)} G$.

For the proof we refer the reader to [10, Propositions 6.4, 6.5, 6.7 and Corollary 6.7].

Remark 12.3. Let $G$ be a Rees algebra, and let $\omega$ be the maximum of the function

$$\text{ord} : \text{Sing} G \rightarrow \mathbb{Q}.$$ 

If $x \in \text{Sing} G$, then $G = G(\omega)$ at $x$ if and only if $x$ is a simple point. If $x$ is not a simple point for $G$, then $\tau_{G,x} = 0$, but then $x \in \text{Sing} G(\omega)$ is a simple point, so in particular $\tau_{G(\omega),x} \geq 1$.

Definition 12.4. Let $G$ be a Rees algebra in a smooth scheme $V$, and let $x \in \text{Sing} G$. Let $\omega = \text{ord}_x G$. If $x$ is not a simple point, i.e., if $\omega > 1$, then, define $\tilde{G} = \text{Diff}(G(\omega))$ [4.2].

Remark 12.5. Using the same notation as in the previous definition, notice that $\text{Sing} \tilde{G} = \{z \in V : \text{ord}_z \tilde{G} = \omega\}$ in a neighborhood of $x$.

Definition 12.6. Given two algebras over $V$, for instance $G_1$ and $G_2$, set $G_1 \odot G_2$ as the smallest subalgebra of $O_V[W]$ containing both (as in [39]). Let $U$ be an affine open set in $V$. If the restriction of $G_1$ to $U$ is $O_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}]$, and that of $G_2$ is $O_V(U)[f_{s+1} W^{n_{s+1}}, \ldots, f_t W^{n_t}]$, then the restriction of $G_1 \odot G_2$ to $U$ is

$$O_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}, f_{s+1} W^{n_{s+1}}, \ldots, f_t W^{n_t}].$$

One can check that:

(1) $\text{Sing} (G_1 \odot G_2) = \text{Sing} (G_1) \cap \text{Sing} (G_2)$. In particular, if $V \leftarrow V'$ is a permissible transformation for $G_1 \odot G_2$, then it is also a permissible transformation for $G_1$ and for $G_2$. 
(2) If \( V \leftarrow V' \) is a permissible transformation for \( \mathcal{G}_1 \odot \mathcal{G}_2 \), and if \((\mathcal{G}_1 \odot \mathcal{G}_2)'\), \( \mathcal{G}_1' \), and \( \mathcal{G}_2' \) denote the transforms in \( V' \), then:
\[
(\mathcal{G}_1 \odot \mathcal{G}_2)' = \mathcal{G}_1' \odot \mathcal{G}_2'.
\]

**Theorem 12.7.** Let \( \mathcal{G} = \bigoplus_n I_n W^n \) be a differential algebra defined on a \( d \)-dimensional smooth scheme \( V^{(d)} \) over a field \( k \), and let \( x \in \text{Sing} \mathcal{G} \) be a simple point with \( \tau_{\mathcal{G},x} \geq 1 \). Assume that \( x \) is not contained in any component of co-dimension one of \( \text{Sing} \mathcal{G} \) and let
\[
\omega := \text{max-ord}^{(d-1)} \mathcal{G} > 1.
\]
Fix two \( \mathcal{G} \)-admissible local projections to some \((d-1)\)-dimensional smooth schemes, and consider the corresponding elimination algebras and twisted algebras as in Definition 12.4,
\[
\beta_1 : (V^{(d)}, x) \rightarrow (V_1^{(d-1)}, x_{1,1}) \quad \beta_2 : (V^{(d)}, x) \rightarrow (V_2^{(d-1)}, x_{1,2})
\]
\[
\mathcal{R}_{\mathcal{G},\beta_1} \subset \mathcal{R}_{\mathcal{G},\beta_1}(\omega) \quad \mathcal{R}_{\mathcal{G},\beta_2} \subset \mathcal{R}_{\mathcal{G},\beta_2}(\omega).
\]
Then
\[
\tilde{\mathcal{G}}_1 = \mathcal{G} \odot \mathcal{R}_{\mathcal{G},\beta_1}(\omega) \quad \text{and} \quad \tilde{\mathcal{G}}_2 = \mathcal{G} \odot \mathcal{R}_{\mathcal{G},\beta_2}(\omega)
\]
are weakly equivalent as in Definition 5.7.

**Remark 12.8.** Since \( \tilde{\mathcal{G}}_1 \) and \( \tilde{\mathcal{G}}_2 \) are weakly equivalent, Theorem 5.8 says that there is a canonical way to associate to \( \mathcal{G} \) a differential algebra, up to integral closure, \( \tilde{\mathcal{G}} \subset \mathcal{O}_V[W] \), such that
\[
\text{Max-ord}^{(d-1)} \mathcal{G} = \text{Sing} \tilde{\mathcal{G}} \quad \text{and} \quad \tau_{\tilde{\mathcal{G}},x} \geq 2
\]
for all \( x \in \text{Max-ord}^{(d-1)} \mathcal{G} \) in some neighborhood of \( x \).

**Proof of Theorem 12.7.** We have to show that \( \tilde{\mathcal{G}}_1 \) and \( \tilde{\mathcal{G}}_2 \) define the same singular locus under any permissible transformation in the sense of Definition 5.5. This is straightforward for permissible transformations as in 5.4 (i) and (ii), so we are only left with the case of permissible monoidal transformations.

Let \( V \leftarrow V' \) be a permissible monoidal transformation with center
\[
Y \subset \text{Max-ord}^{(d-1)} \mathcal{G} = \text{Sing} \tilde{\mathcal{G}}_1 = \text{Sing} \tilde{\mathcal{G}}_2,
\]
and let \( x' \in V' \) be a closed point that dominates \( x \). Then by Theorem 9.1 there is a commutative diagram of algebras and elimination algebras in a suitable open set of \( V' \):
\[
\begin{array}{ccc}
(V, x) & \leftarrow & (V', x') \\
\mathcal{G} & \subset & \tilde{\mathcal{G}}_i \\
\beta_i & & \beta_i' \\
(V_i^{(d-1)}, x_{1,i}) & \leftarrow & (V_i'^{(d-1)}, x_{1,i}') \\
\mathcal{R}_{\mathcal{G},\beta_i} & \subset & \mathcal{R}_{\mathcal{G},\beta_i}(\omega) \quad \mathcal{R}_{\mathcal{G},\beta_i}' \subset \mathcal{R}_{\mathcal{G},\beta_i}(\omega)'
\end{array}
\]
for \( i = 1, 2 \).

Now, on the one hand by Theorem 10.1,

\[
\beta'_i(\text{Max-w-ord}^{(d-1)} \mathcal{G}') = \text{Max-w-ord} \mathcal{R}_{\mathcal{G}', \beta_i}
\]

in a neighborhood of \( x'_i \) (see 2.1 for the definition of w-ord\(^{(d-i)}\)). On the other hand,

\[
\tilde{\mathcal{G}}'_i = (\mathcal{G} \circ \mathcal{R}_{\mathcal{G}, \beta_i}(\omega))' = \mathcal{G}' \circ \mathcal{R}_{\mathcal{G}, \beta_i}(\omega)',
\]

and \( \text{Max-w-ord}^{(d-1)} \mathcal{G}' = \text{Sing} \tilde{\mathcal{G}}_i \) for \( i = 1, 2 \). Therefore

\[
\text{Sing} \tilde{\mathcal{G}}'_1 = \text{Sing} \tilde{\mathcal{G}}'_2
\]

in a neighborhood of \( x' \). □

**Theorem 12.9.** Let \( \mathcal{G}^{(d)} \) be a differential algebra on a \( d \)-dimensional smooth scheme over a field \( k \), let \( x \in \text{Sing} \mathcal{G}^{(d)} \) be a simple point and assume that \( \tau_{\mathcal{G}^{(d)}, x} = m \geq 1 \). Then there is a differential algebra \( \tilde{\mathcal{G}}^{(d)} \) containing \( \mathcal{G}^{(d)} \) with the following properties:

(i) \( \tau_{\tilde{\mathcal{G}}^{(d)}, x} \geq m + 1 \).

(ii) Locally at \( x \) there is an equality of closed sets

\[
\text{Sing} \tilde{\mathcal{G}}^{(d)} = \text{Max ord}^{(d-m)} \mathcal{G}^{(d)}.
\]

(iii) The differential algebra \( \tilde{\mathcal{G}}^{(d)} \) is unique up to weak equivalence. Furthermore, this differential algebra is unique up to integral closures of algebras (see Remark 12.8).

**Proof:** Consider a \( \mathcal{G}^{(d)} \)-admissible local projection to some \((d - m)\)-smooth dimensional scheme,

\[
(V^{(d)}, x) \to (V^{(d-m)}, x_m),
\]

and a factorization as in the diagram,

\[
\begin{array}{c}
\mathcal{G}^{(d)} \subset \mathcal{O}_{V^{(d)}}[W] \\
\downarrow \beta_{d,d-(m-1)}^{*} \\
\mathcal{G}^{(d-(m-1))} \subset \mathcal{O}_{V^{(d-(m-1))}}[W] \\
\downarrow \beta_{d-(m-1),d-m}^{*} \\
\mathcal{G}^{(d-m)} \subset \mathcal{O}_{V^{(d-m)}}[W].
\end{array}
\]

By Theorem \([12.7]\) there is a differential algebra, \( \tilde{\mathcal{G}}^{(d-(m-1))} \), containing \( \mathcal{G}^{(d-(m-1))} \) with the following properties:

(a) Its \( \tau \)-invariant at \( x_{m-1} = \beta_{d,d-(m-1)}(x) \) is larger than that of \( \mathcal{G}^{(d-(m-1))} \), i.e.,

\[
\tau_{\tilde{\mathcal{G}}^{(d-(m-1))}, x_{m-1}} \geq \tau_{\mathcal{G}^{(d-(m-1))}, x_{m-1}} + 1,
\]

and therefore \( x_{m-1} \in \text{Sing} \tilde{\mathcal{G}}^{(d-(m-1))} \) is a simple point.

(b) By construction \( \tilde{\mathcal{G}}^{(d-(m-1))} \) is unique up to weak equivalence.
(c) There is an equality of closed sets (using the identification between singular loci),

\[ \text{Sing } \tilde{G}^{(d-(m-1))} = \text{Max-ord } G^{(d-m)} = \text{Max-ord}^{(d-m)} G^{(d)} \]

in a neighborhood of \( x \).

Now set

\[ \tilde{G}^{(d)} = G^{(d)} \odot \beta^*_d,d-(m-1) \tilde{G}^{(d-(m-1))}. \]

Next we check that this algebra satisfies the properties stated in the Theorem:

(i) By construction

\[ \tau_{\tilde{G}^{(d)},x} = \tau_{\tilde{G}^{(d-(m-1))},x_{m-1}} + (m-1) \geq \tau_{G^{(d-(m-1))},x_{m-1}} + 1 + (m-1) \geq m + 1. \]

(ii) Notice that via the natural identification of the singular loci and from (69), locally, in a neighborhood of \( x \),

\[ \text{Sing } \tilde{G}^{(d)} = \text{Sing } \tilde{G}^{(d-(m-1))} = \text{Sing } \tilde{G}^{(d-m)} = \text{Max-ord } G^{(d-m)} = \text{Max-ord}^{(d-m)} G^{(d)}. \]

(iii) The argument to show this part is similar to the proof of Theorem 12.7 since by Theorem 10.1 \( \text{ord}^{(d-m)} \) does not depend on the choice of the \( G^{(d)} \)-admissible projections and therefore \( \text{Max-ord}^{(d-m)} G^{(d)} = \text{Max-ord} G^{(d-m)} \) in a neighborhood of \( x \). \( \square \)

Finally, we state a similar result for permissible transformations of differential algebras.

**Theorem 12.10.** Let \( G^{(d)} \) be a differential algebra on a \( d \)-dimensional smooth scheme of finite type over a field \( k \), let \( x \in \text{Sing } G^{(d)} \) be a simple point and assume that \( \tau_{G^{(d)},x} = m \geq 1 \). Let

\[ V^{(d)} \leftarrow V^{(d)'} \]

be a composition of permissible monoidal transformations, let \( G^{(d)'} \) be the weak transform of \( G^{(d)} \) and let \( x' \in \text{Sing } G^{(d)'} \) be a closed point that dominates \( x \). Then there exists an algebra, \( \tilde{G}^{(d)'} \) containing \( G^{(d)'} \) with the following properties:

(i) \( \tau_{\tilde{G}^{(d)'}},x' \geq m + 1 \).

(ii) Locally at \( x' \) there is an equality of closed sets

\[ \text{Sing } \tilde{G}^{(d)'} = \text{Max ord}^{(d-m)} G^{(d)'} \]

(iii) The algebra \( \tilde{G}^{(d)'} \) is unique up to integral closure of algebras.

**Proof:** Similar to the proof of Theorem 12.9 since, as in that case, by Theorem 10.1, the functions \( \text{ord}^{(d-m)} \) are well defined for \( G^{(d)'} \) in a neighborhood of \( x' \). \( \square \)

### 13. Stratification of the singular locus by smooth strata

The purpose of this section is to prove the following theorem:
**Theorem 13.1.** Let \( \mathcal{G}^{(d)} \) be a differential algebra on a smooth \( d \)-dimensional scheme \( V^{(d)} \) over a field \( k \). Let \( \mathbb{Q}^* = \mathbb{Q} \cup \{ \infty \} \) and let

\[
I_d = \mathbb{Q}^* \times \mathbb{Q}^* \times \ldots \times \mathbb{Q}^* \quad \text{\( d \)-times}
\]

ordered lexicographically. Then there is an upper-semi-continuous function,

\[
\gamma_{\mathcal{G}^{(d)}} : \text{Sing} \mathcal{G}^{(d)} \to I_d
\]

such that:

(i) The level sets of \( \gamma_{\mathcal{G}^{(d)}} \) stratify \( \text{Sing} \mathcal{G}^{(d)} \) in smooth locally closed strata.

(ii) If \( k \) is a field of characteristic zero then \( \gamma_{\mathcal{G}^{(d)}} \) coincides with the resolution function used for resolution of singularities in characteristic zero.

The proof of the Theorem is given in 13.4. First we need the following lemma.

**Lemma 13.2.** Let \( \mathcal{G} \) be a Rees algebra on a smooth \( d \)-dimensional scheme \( V \) over a field \( k \), and assume that \( x \in \text{Sing} \mathcal{G} \) is a simple point. If \( x \) is contained in a component of co-dimension one, \( Y \), of \( \text{Sing} \mathcal{G} \), then \( Y \) is smooth.

**Proof:** We may assume that, up to integral closure \( \mathcal{G} = \bigoplus_n J^n W^{k+n} \) for some sheaf of ideals \( J \subset \mathcal{O}_V \). The hypothesis of the lemma asserts that \( Y \subset \text{Sing} \mathcal{G} \). Since \( \mathcal{G} \) is simple at \( x \), the hypothesis means that locally in a suitable neighborhood of \( x \), \( Y \subset V(J) \) where \( V(J) \) denotes the closed set determined by \( J \). By restricting to a smaller neighborhood \( U \) of \( x \) if needed, we may assume that \( I(Y) = \langle f \rangle \), for some reduced element \( f \in \mathcal{O}_V(U) \) and that \( J = \langle f^s \rangle \) for some positive integer \( s \). Since \( x \) is a simple point and \( Y \cap U \subset \text{Sing} \mathcal{G} \cap U \), \( f \) has to be smooth at \( x \). \( \square \)

**Remark 13.3.** As a consequence of the previous lemma an inductive argument, using elimination algebras, shows that if \( x \in \text{Sing} \mathcal{G} \) is contained in a component of co-dimension \( \tau_{\mathcal{G},x} \), then the component is smooth in a neighborhood of \( x \).

13.4. **Proof of Theorem 13.1** We start by defining the function \( \gamma_{\mathcal{G}^{(d)}} \). Let \( x \in \text{Sing} \mathcal{G}^{(d)} \). To associate a value to \( \gamma_{\mathcal{G}^{(d)}} \) at \( x \), we will argue by induction on the dimension of \( V^{(d)} \).

Suppose that \( V^{(1)} \) is a one-dimensional smooth scheme over a field \( k \), that \( \mathcal{G}^{(1)} \) is a non-zero differential algebra and that \( x \in \text{Sing} \mathcal{G}^{(1)} \) is a closed point. Set

\[
\gamma_{\mathcal{G}^{(1)}}(x) = (\text{ord}_{x}^{(1)} \mathcal{G}^{(1)}).
\]

Suppose that the function \( \gamma \) can be defined for any differential algebra \( \mathcal{G}^{(n)} \) on a \( n \)-dimensional smooth scheme over a field \( k \), \( V^{(n)} \), with \( n < d \). We will show that then the function can be defined for any non-zero differential algebra \( \mathcal{G}^{(d)} \) on a \( d \)-dimensional smooth scheme over a field \( k \), \( V^{(d)} \).
• First assume that \( x \in \text{Sing} \, \mathcal{G}^{(d)} \) is a simple closed point. We now distinguish between two cases:

**Case 1.** If \( x \) is contained in a component of co-dimension one, \( Y \), of \( \text{Sing} \, \mathcal{G}^{(d)} \), then set

\[
\gamma_{\mathcal{G}^{(d)}}(x) = \left( \text{ord}_x^{(d)} \mathcal{G}^{(d)}, \infty, \ldots, \infty \right) = \left( 1, \infty, \ldots, \infty \right) \text{ d-1-times}.
\]

Note that by Lemma 13.2 the closed subscheme \( Y \) is smooth locally at \( x \).

**Case 2.** If \( x \) is not contained in any component of co-dimension one of \( \text{Sing} \, \mathcal{G}^{(d)} \) then construct a \( \mathcal{G}^{(d)} \)-admissible local projection to a \( (d-1) \)-dimensional scheme \( V^{(d-1)} \), and an elimination algebra as in 8.3 and 8.7,

\[
\beta_{d,d-1} : (V^{(d)}, x) \rightarrow (V^{(d-1)}, x_1) \quad \mathcal{G}^{(d)} \rightarrow \mathcal{G}^{(d-1)}.
\]

By the induction hypothesis, \( \gamma_{\mathcal{G}^{(d-1)}}(x_1) \) is defined. Now set

\[
\gamma_{\mathcal{G}^{(d)}}(x) = \left( \text{ord}_x^{(d)} \mathcal{G}^{(d)}, \gamma_{\mathcal{G}^{(d-1)}}(x_1) \right).
\]

• If \( x \in \text{Sing} \, \mathcal{G} \) is not a simple point, then let \( \tilde{\mathcal{G}}^{(d)} \) be the twisted algebra as in Definition 12.3 with \( \omega = \text{ord}_x \mathcal{G} \). Then \( x \in \text{Sing} \, \tilde{\mathcal{G}}^{(d)} \) is a simple point and cases 1 and 2 can be applied to \( \tilde{\mathcal{G}}^{(d)} \). Now define

\[
\gamma_{\tilde{\mathcal{G}}^{(d)}}(x) = \left( \text{ord}_x^{(d)} \mathcal{G}^{(d)}, \gamma_{\tilde{\mathcal{G}}^{(d-1)}}(x_1) \right),
\]

where \( \gamma_{\tilde{\mathcal{G}}^{(d-1)}}(x) \) are the last \( (d-1) \)-coordinates of the function \( \gamma_{\tilde{\mathcal{G}}^{(d)}}(x) \).

We will see next that \( \gamma_{\mathcal{G}^{(d)}} \) is upper-semicontinuous and that it stratifies \( \text{Sing} \, \mathcal{G}^{(d)} \) in smooth strata.

The fact that this function takes only a finite number of values follows by induction. Thus it only remains to show that for any value \((a_1, a_2, \ldots, a_d) \in (\mathbb{Q}^*)^d\), the set \( \{x \in V^{(d)} : \gamma_{\mathcal{G}}(x) \geq (a_1, a_2, \ldots, a_d)\} \) is (locally) closed and smooth (if it is non-empty). Observe that it is enough to prove this fact in the case when \((a_1, a_2, \ldots, a_d)\) is the maximum value achieved by the function. As in the previous discussion we will use induction on the dimension of \( V^{(d)} \).

First suppose that \( V^{(1)} \) is a one-dimensional-scheme and let \( a_1 \in \mathbb{Q}^* \) be any value such that \( \{z \in V^{(1)} : \gamma_{\mathcal{G}}(z) \geq a_1\} \) is non-empty. Since \( \mathcal{G}^{(1)} \neq 0 \), \( \{z \in V^{(1)} : \sigma_{\mathcal{G}}(z) \geq a_1\} \) consists of a finite number of closed points which is clearly a smooth closed subscheme of \( V^{(1)} \).

Assume now that part (i) of the theorem holds for differential algebras in any \( n \)-dimensional smooth scheme \( V^{(n)} \) of finite type over a field \( k \) with \( n < d \). We will show that it also holds for differential algebras over a \( d \)-dimensional scheme \( V^{(d)} \) of finite type over a field \( k \).
Let \((a_1, \ldots, a_d) \in \mathbb{Q}^* \times \cdots \times \mathbb{Q}^*\) be the maximum value of \(\gamma_{G^{(d)}}(z)\), and let \(x \in \text{Sing } G^{(d)}\) with \(\gamma_{G^{(d)}}(x) = (a_1, \ldots, a_d)\). We will prove that there is an open subset \(U^{(d)} \subseteq V^{(d)}\) containing \(x\) such that \(U^{(d)} \cap \{z \in V^{(d)} : \gamma_{G^{(d)}}(z) = (a_1, \ldots, a_d)\}\) is closed and smooth. We distinguish three cases.

**Case 1.** If \(a_2 = \ldots = a_d = \infty\) then \(x\) is contained in a component of co-dimension one of \(\text{Sing } G^{(d)}\). In this case by Lemma \[3.2\] (applied to \(G^{(d)}\) if \(x\) is a simple point, or to some twisting, \(\tilde{G}^{(d)}\) of \(G^{(d)}\) otherwise), there is an open neighborhood \(U^{(d)}\) of \(x\) satisfying the required property.

**Case 2.** If \(a_2 \neq \infty\) and \(x\) is a simple point consider a \(G^{(d)}\)-admissible local projection and an elimination algebra as in \[8.3\] and \[8.7\] in an open neighborhood \(U^{(d)}\) of \(x\):

\[
(G^{(d)}, x) \longrightarrow (V^{(d-1)}, x_1)
\]

Notice that then \(\{z \in U^{(d)} : \text{ord}_{G^{(d)}}(z) = a_1\}\) can be identified with \(\text{Sing } G^{(d-1)}\) in some open neighborhood \(U^{(d-1)}\) containing \(x_1\). Therefore via this identification

\[
\{z \in U^{(d)} : \gamma_{G^{(d)}}(z) = (a_1, \ldots, a_d)\} =
\]

\[
= \text{Sing } G^{(d-1)} \cap \{z \in U^{(d-1)} : \gamma_{G^{(d-1)}}(z) = (a_2, \ldots, a_d)\} =
\]

\[
= \{z \in H^{(d-1)} : \gamma_{G^{(d-1)}}(z) = (a_2, \ldots, a_d)\}
\]

for some open subset \(H^{(d-1)} \subseteq V^{(d-1)}\). Restricting \(U^{(d-1)}\) if necessary we may assume that \((a_2, \ldots, a_d)\) is actually the maximum of \(\gamma_{G^{(d-1)}}\). According to our inductive hypothesis there is an open neighborhood of \(x_1\) where \(\{z \in U^{(d-1)} : \gamma_{G^{(d-1)}}(z) = (a_2, \ldots, a_d)\}\) is locally closed and smooth. Again, via the identification \(\text{Sing } G^{(d)} \cap U^{(d)}\) with \(\text{Sing } G^{(d-1)} \cap U^{(d-1)}\) we conclude that there is an open neighborhood of \(x\) where the stratum \(\{z \in V^{(d)} : \gamma_{G^{(d)}}(z) = (a_1, \ldots, a_d)\}\) is closed and smooth.

**Case 3.** If \(a_2 \neq \infty\) and \(x\) is not a simple point, then, in a suitable neighborhood of \(x\), replace \(G^{(d)}\) by \(\tilde{G}^{(d)}\) as in Theorem \[12.9\]. By restricting to a smaller neighborhood if needed, it can be assumed that in addition, \(\{z : \text{ord}_{G^{(d)}}(z) = a_1\} = \text{Sing } \tilde{G}^{(d)}\). Now the argument in Case 2 can be applied to \(\tilde{G}^{(d)}\).

**Part 5. Epilogue and example**

Let \(G^{(d)}\) be a Rees algebra on a smooth scheme \(d\)-dimensional smooth scheme \(V^{(d)}\) over a field \(k\). The study of a stratification on \(\text{Sing } G^{(d)}\) achieved by means of an upper-semicontinuous function is one example of an application of Main Theorem \[10.1\]. However, this stratification is mainly interesting due to the following fact: as in Part II (via the dictionary between Rees algebras and pairs) similar satellite functions can be defined thanks to Theorems \[9.1, 10.1\] and...
In this way an upper-semi-continuous function is constructed whose maximum value determines permissible centers, and the blow-up along these centers produces a simplification of the singularities. To be precise, once we blow-up at the smooth center defined by the function (on the worst points), a new upper-semi-continuous function is defined, which provides a new stratification and a new closed and smooth stratum of worst singularities. We then consider the blow-up at such center and so on.

A number of exceptional hypersurfaces arise in this process of monoidal transforms, and it is important that these hypersurfaces have normal crossings. So we have to define a procedure so that the maximum stratum (center of the monoidal transform) have normal crossings with the exceptional hypersurfaces introduced in the previous steps. Here is where the second satellite functions play an important role (see 2.2).

On the other hand, the notion of co-dimensional type in Definition 6.4 provides a natural form of induction used in resolution problems:

- Observe first that when $G^{(d)}$ is of co-dimensional type $d$, then $\text{Sing}(G^{(d)})$ is a zero dimensional closed set, and a resolution is achieved by blowing up these closed points.

- When $G^{(d)}$ is of co-dimensional type $\geq m$, then by Theorem 12.9 a new Rees algebra $\tilde{G}^{(d)}$ of co-dimensional type $\geq m + 1$ can be attached to $G^{(d)}$. Theorem 12.10 says that this Rees algebra is determined up to integral closure of algebras. This is what allows us to define the upper-semi-continuous functions after successive monoidal transforms, and it also leads to the reduction to the monomial case. In fact, if we assume by induction an algorithm of resolution for algebras of co-dimensional type $\geq m + 1$, then a sequence

$$(V^{(d)}, G^{(d)}, E^{(d)}) \leftarrow (V^{(1)}_1, G^{(1)}_1, E^{(1)}_1) \leftarrow \cdots \leftarrow (V^{(d-m)}_s, G^{(d-m)}_s, E^{(d-m)}_s)$$

$\max \text{ord}^{(d-m)}_{G^{(d)}} \geq \max \text{w-ord}^{(d-m)}_{G^{(d)}_1} \geq \cdots \geq \max \text{w-ord}^{(d-m)}_{G^{(d)}_s}$,

can be defined so that $\max \text{w-ord}^{(d-m)}_{G^{(d)}_s} = 0$ (see 29, Corollary 6.15). In other words, the sequence of transformations can be defined so that $(V^{(d)}_s, G^{(d)}_s, E^{(d)}_s)$ is in the monomial case as described in 10.3.

So Theorem 12.10 provides our form of induction. However, to be precise, the invariant dealt with in that Theorem is essentially that in (44) of 10.3 (or say, the first satellite function in equation (7) of 2.1). As indicated above, after the first monoidal transformation, the upper-semicontinuous function that defines the reduction to the monomial case makes use of the
second satellite function as in \( \text{(11)} \). The reader can find the formulation of Theorem \( \text{[12,10]} \) in terms of the second satellite function and other technical aspects in \( \text{[29]} \).

For the case of characteristic zero it is simple to extend sequence \( \text{(70)} \) (i.e., the monomial case) to a resolution of \((V^{(d)}, G^{(d)}, E^{(d)})\) (see Steps A and B in Section \( \text{[2]} \)). When the characteristic is positive, the containment \( \beta_3(\text{Sing } G^{(d)}) \subset \text{Sing } G_s^{(d-m)} \) may be strict, and then a resolution of \((V_s^{(d-m)}, G_s^{(d-m)}, E_s^{(d-m)})\) may not lift to a resolution of \((V^{(d)}, G^{(d)}, E^{(d)})\) (see \( \text{[5]} \) for concrete examples). However the condition of \((V^{(d-m)}, G^{(d-m)}, E^{(d-m)})\) being monomial in positive characteristic (i.e. the elimination algebra being monomial) opens the way to new invariants, as those treated in \( \text{[5]} \). We hope to be able to address the monomial case in arbitrary characteristic in the future.

We conclude this section with an example to illustrate the computation of an elimination algebra and its use in stratification.

**Example 13.5.** Assume that \( k \) is a field of characteristic 2, let \( V^{(3)} \) be the affine 3-dimensional space \( \text{Spec}(k[X, Y, Z]) \), and let

\[
S := \{ f = Z^2 + (Y^7 + YX^4) \in k[X, Y, Z] = 0 \}.
\]

Clearly, the maximum order at points of \( S \) is two. This maximum is achieved at the points of the curve \( \{ Z = 0, Y^3 + X^2 = 0 \} \), but this is not a smooth closed subscheme, and henceforth we are forced to look for other invariants that refine the order function.

Let \( \mathcal{G} \) be the differential algebra generated by \( f \) in degree two:

\[
\mathcal{G} = \mathcal{O}_{V^{(3)}}[Z^2 + (Y^7 + YX^4)W^2, (Y^3 + X^2)^2W].
\]

Notice that \( \text{Max-ord}_{\mathcal{G}}^{(3)} = 1 \), and that

\[
\text{Sing } \mathcal{G} = \text{Max-ord}_{\mathcal{G}}^{(3)} = \{ Z = 0, Y^3 + X^2 = 0 \}.
\]

Again, observe that the function \( \text{ord}_{\mathcal{G}}^{(3)} \) is too coarse: its singular locus is not even smooth.

Let \( V^{(2)} = \text{Spec}(k[X, Y]) \). We choose the \( \mathcal{G} \)-admissible projection

\[
\beta_{3,2} : V^{(3)} \rightarrow V^{(2)}
\]

and compute the corresponding elimination algebra, \( \mathcal{R}_\mathcal{G} \):

\[
\mathcal{R}_\mathcal{G} = \mathcal{O}_{V^{(2)}}[(Y^3 + X^2)^2W].
\]

Now \( \text{Max-ord}_{\mathcal{R}_\mathcal{G}}^{(2)} = \text{Max-ord}_{\mathcal{G}}^{(2)} = 4 \), and \( \text{Max-ord}_{\mathcal{G}}^{(2)} = \{(0, 0, 0)\} \). The procedure now involves associating a simple differential algebra to \( \text{Max-ord}_{\mathcal{G}}^{(2)} \) and then projecting onto some smooth scheme of dimension 1.
The first monoidal transformation is the blow-up at the origin, $V^{(3)} \leftarrow V^{(3)}_1$, which induces a blow-up at $V^{(2)}$ with center $\{(0,0)\}$, $V^{(2)} \leftarrow V^{(2)}_1$. Recall that by Theorem 9.1, there are commutative diagrams of monoidal transformations, restrictions and elimination algebras.

Consider the affine charts $U^{(3)}_{1,Y} = \text{Spec } (k[\frac{X}{Y}, Y, Z]) \subset V^{(3)}_1$, and $U^{(2)}_{1,Y} = \text{Spec } (k[\frac{X}{Y}, Y]) \subset V^{(2)}_1$. To simplify notation, set again $X = \frac{X}{Y}$ and $Z = \frac{Z}{Y}$:

$$V^{(3)} \xrightarrow{\pi_1} V^{(3)}_1 \cup U^{(3)}_{1,Y}$$

$$f_1 = Z^2 + Y^3 \cdot (Y + X^2)^2.$$

In $U^{(3)}_{1,Y}$ consider the weak transform of $G$, $G_1 = \mathcal{O}_{U^{(3)}_{1,Y}}[\{Z^2 + Y^3 \cdot (Y + X^2)^2\}W^2, Y^3(Y + X^2)^2W]$, and the weak transform of $\mathcal{R}_G$ in $U^{(2)}_{1,Y}$, $\mathcal{R}_{G_1} = \mathcal{O}_{U^{(2)}_{1,Y}}[Y^3(Y + X^2)^2W]$.

Notice that $\text{Max-ord}_{G_1}^{(3)} = 1$, and that $\text{Max-ord}_{G_1}^{(2)} = \text{Max-w-ord}_{\mathcal{R}_{G_1}} = 2$, so this invariant has dropped, and hence the second satellite function plays a role counting exceptional divisors (see 2.2). Now the same procedure that works for algorithmic resolution in characteristic zero applies here (see Section 1), and after two more blow-ups at closed points (the centers that are determined using the upper semi-continuous functions derived from Theorem 10.1), the monomial case is achieved.

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