Superhamiltonian formalism for $2D \ N = 1, 2$ theories

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Abstract

We show how to formulate 2-dimensional supersymmetric $N = 1, 2$ theories, both massive and conformal, within a manifestly supersymmetric hamiltonian framework, via the introduction of a (super)-Poisson brackets structure defined on superfields. In this approach, as distinct from the previously known superfield hamiltonian formulations, the dynamics is not separated into two unrelated $2D$ light-cone superspaces, but is recovered by specifying boundary conditions at a given “super-time” coordinate. So the approach proposed provides a natural generalization of canonical hamiltonian formalism. One of its interesting features is that the physical and auxiliary fields equations appear on equal footing as the Hamilton ones.
1 Introduction

The most convenient and efficient way to deal with supersymmetric theories is a manifestly supersymmetric approach based on the concept of superfields. This fact has been recognized since the early works on supersymmetry, and manifestly supersymmetric superfield lagrangian formalisms are of common use (see [1] and references therein). On the other hand, the hamiltonian methods of treating field theories, though being advantageous in a number of aspects (e.g., in what concerns the quantization), received little attention in the framework of the superfield approach.

A version of the superfield hamiltonian formalism for supersymmetric 2D theories is already available (see, e.g., [2], [3]). However, it has been formulated for the two separated 1-dimensional dynamics in the two unrelated 2D light-cone superspaces and so seems to be of immediate use mainly in superconformally-invariant integrable theories which admit (sometimes after a field redefinition) on-shell separation into independent left and right movers. Examples of that kind are supplied by super Liouville and super Toda theories.

In this paper we introduce another superfield Poisson brackets structure which directly generalizes the canonical (not the light-cone) bosonic Poisson structure and is applicable not only to superconformally-invariant $N = 1$ and $N = 2$, 2D models, but also to their massive deformations, such as the super sine-Gordon and sinh-Gordon models. We show that the dynamics of $N = 1$, 2 supersymmetric 2D theories can be readily recovered within the superhamiltonian framework based on such a super-Poisson brackets structure and that it takes a form very similar to the dynamics of bosonic 2D theories in the standard hamiltonian formulation.

The dynamics of our theory is recovered in terms of boundary conditions which are naturally expressed by specifying the values of the superfields at a given “supertime” coordinate, instead of specifying their values on two separated light-cone supersurfaces.

Our framework seems therefore the natural set-up for studying properties of integrable models like the exchange algebra (this was indeed the original motivation for our work [4, 5]).

Though we could formulate our 2D super-Poisson brackets formalism in a complete generality, we will specialize here to the three simplest non-trivial integrable field-theoretical examples, namely, the $N = 1$ (massive) super sinh-Gordon theory [6], the $N = 1$ superconformal affine Liouville theory (super-CAL) [5], and finally the $N = 2$ super sine-Gordon theory [7].

We decided to treat here the case of supersymmetric $N = 1, 2$ sin-Gordon theories instead of the super Liouville ones in order to stress the fact that our formalism can be applied to massive models as well. The super-CAL theory has been chosen as an example of superconformal theory which involves in particular a pair of superfields having momenta conjugate to each other.

The generalization of this formulation to more complicated theories is completely straightforward.
2 \( N = 1 \) superhamiltonian framework

Our starting point in this Section will be the algebra of \( N = 1, 2D \) supersymmetry. Its nonvanishing structure relations in the standard light-cone notation and in the realization via the \( N = 1, 2D \) superspace coordinates \((z^{++}, z^{--}, \theta^{+}, \theta^{-})\) read

\[
(Q_+)^2 = P_{++}, \quad (Q_-)^2 = P_{--}.
\]

(1)

Here

\[
Q_{\pm} = i \frac{\partial}{\partial \theta^{\pm}} + \theta^{\mp} \partial_{\pm \pm}, \quad P_{\pm \pm} = i \partial_{\pm \pm}.
\]

We will also need the algebra of \( N = 1 \) spinor covariant derivatives

\[
D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i \theta^{\mp} \partial_{\pm \mp},
\]

(2)

\[
(D_+)^2 = i \partial_{++}, \quad (D_-)^2 = i \partial_{--}, \quad \{D_+, D_-\} = 0.
\]

(3)

As the first step in developing a hamiltonian formalism one should choose which of the \( 2D \) coordinates will be identified with the evolution parameter, the time. One possibility is to take as such a coordinate \( z^{++} \) or \( z^{--} \); these options amount to the light-cone hamiltonian formalism. Two other possibilities lead to the canonical \( 2D \) hamiltonian approach a superfield extension of which we are going to construct. They correspond to the following two alternative identifications of the time-like and spatial \( 2D \) coordinates

\[
x = \frac{1}{2}(z^{++} + z^{--})
\]

\[
t = \frac{1}{2}(z^{++} - z^{--})
\]

(4)

or

\[
t = \frac{1}{2}(z^{++} + z^{--})
\]

\[
x = \frac{1}{2}(z^{++} - z^{--})
\]

(5)

In what follows we will keep to the former definition as it has been used in ref. [5] which was just a first incitement for undertaking the present investigation. We will comment on the second, more customary option in the end of this section.

Along with introducing the coordinates \( t \) and \( x \) as in eq. (4), we also pass to the new “rotated” fermionic coordinates

\[
\theta^1 = \frac{1}{2}(\theta^+ + \theta^-)
\]

\[
\theta^0 = \frac{1}{2}(\theta^+ - \theta^-).
\]

(6)

Defining the “rotated” spinor derivatives by

\[
D_0 = D_+ - D_- \quad D_1 = D_+ + D_-,
\]

\[
D_0 = \frac{\partial}{\partial \theta^0} + i(\theta^1 \partial_t + \theta^0 \partial_x)
\]

\[
D_1 = \frac{\partial}{\partial \theta^1} + i(\theta^1 \partial_x + \theta^0 \partial_t),
\]

(7)
one can rewrite the algebra (3) in the following equivalent form

\[ D_0^2 = D_1^2 = i\partial_x , \quad \{D_0, D_1\} = 2i\partial_t \] (8)

Note the useful relation

\[ D_+ D_- = \frac{i}{2}\partial_t - \frac{1}{2}D_1D_0. \] (9)

We also present the transformation laws of the coordinates \( t, x, \theta^1, \theta^0 \) under \( N = 1 \) supersymmetry:

\[ \delta \theta^1 = \epsilon^1 , \quad \delta \theta^0 = \epsilon^0 , \quad \delta x = -i(\epsilon^1 \theta^1 + \epsilon^0 \theta^0) , \quad \delta t = -i(\epsilon^1 \theta^0 + \epsilon^0 \theta^1) . \] (10)

Now we possess all the necessary matter to approach our task of constructing a superfield version of the hamiltonian formalism. As a sample model we take the super sinh-Gordon theory which is introduced via the action

\[ S \equiv \int d^2zd^2\theta L = \int d^2zd^2\theta \{D_+ \Phi D_- \Phi + i\alpha e^\Phi + i\beta e^{-\Phi}\} , \] (11)

where \( \Phi \) is a bosonic superfield. Without loss of generality the real parameter \( \alpha \) can be set equal to unity through a shift of the superfield \( \Phi \), while \( \beta \) is a mass parameter which measures the breaking of conformal invariance. The super Liouville theory is recovered by letting \( \beta \to 0 \).

The equation of motion is given by

\[ D_+ D_- \Phi = \frac{i}{2}(e^\Phi - \beta e^{-\Phi}) , \] (12)

or, with making use of (9),

\[ \dot{\Phi} = -iD_1D_0\Phi + (e^\Phi - \beta e^{-\Phi}) . \] (13)

By applying the derivative \( D_0 \) on both sides of the latter equation, one gets the important consequence

\[ D_0\dot{\Phi} = D_1\Phi' - D_0\Phi(e^\Phi + \beta e^{-\Phi}) . \] (14)

Here we denoted, as usual, \( \dot{\Phi} \equiv \partial_t\Phi, \Phi' \equiv \partial_x\Phi \).

Now we wish to show that eqs.(13), (14) can be re-derived as the first-order Hamilton equations in the framework of the appropriate superhamiltonian formalism.

To this end, let us first rewrite the action (11) so as to explicitly single out \( \dot{\Phi} \). This can be achieved by taking one of spinor derivatives, say \( D_1 \), off the integration measure in \( S \) and throwing it on the integrand

\[ S = \int d^2zd^2\theta L \] (15)

\[ L = \frac{1}{2}\dot{\Phi}D_0\Phi - \frac{1}{4}i\Phi' D_1\Phi - \frac{1}{4}D_0\Phi D_1D_0\Phi - \frac{i}{2}D_0\Phi(e^\Phi - \beta e^{-\Phi}) . \] (16)

\[ ^1\text{We define Berezin integrals as} \]

\[ \int d^2zd^2\theta \equiv \int d^2xdD_- = -\frac{1}{2} \int d^2zD_1D_0 , \int d^2zd^2\theta^1 \equiv \int d^2zD_1. \]
All the $\theta^0$-dependent terms in the lagrangian $\mathcal{L}$ are reduced to $x,t$-derivatives and so do not contribute to the integral in (13), though the involved superfields can still be regarded as given on the whole $N = 1$ superspace $(t, x, \theta^0, \theta^1)$. In other words, without loss of anything we may treat the superfield argument $\theta^0$ in $\mathcal{L}$ as some cyclic Grassmann variable which can be fixed at any “value” we wish (in particular, it can be put equal to zero). Note that in the action (15) there remains only one manifest supersymmetry

$$\delta \theta^1 = \epsilon^1, \quad \delta x = -i\epsilon^1 \theta^1$$

(17)

(the time coordinate can be made inert under $\epsilon^1$-supertranslations by the redefinition $t \rightarrow \hat{t} = t + i\theta^1 \theta^0$). Supertranslations with the parameter $\epsilon^0$ now cannot be realized as pure coordinate transformations; they mix $\Phi$ with $D_0 \Phi$ and $\dot{\Phi}$.

Next natural step is to define, in the standard way, the superfield momentum $\Pi_\Phi$ canonically conjugate to $\Phi$

$$\Pi_\Phi = \frac{\delta S}{\delta \dot{\Phi}} = \frac{\partial L}{\partial \dot{\Phi}} = \frac{i}{2} D_0 \Phi .$$

(18)

Surprisingly, the conjugate momentum turns out to be fermionic. This is of course a consequence of the fact that the superfield lagrangian $\mathcal{L}$ is a fermionic object in the present case.

Now, again following the text-book prescriptions, we introduce the superhamiltonian

$$H = \int dxd\theta^1 \{ \Pi_\Phi \cdot \dot{\Phi} - \mathcal{L} \} =$$

$$= \int dxd\theta^1 \{ \frac{i}{4} \Phi' D_1 \Phi - \Pi_\Phi D_1 \Pi_\Phi + \Pi_\Phi (e^\Phi - \beta e^{-\Phi}) \} .$$

(19)

Integration in (19) over the superplane $\{ x, \theta^1 \}$ is quite natural because the latter is the minimal superextension of the line $\{ x \}$ closed under $x$ translations and “$N = 1/2$” supersymmetry (17).

Using the equation of motion (13) and its corollary (14) one easily proves the conservation laws

$$D_0 H = \dot{H} = 0 ,$$

(20)

which state that on shell $H$ does not depend on the coordinates $\theta^0$ and $t$. Then it is natural to join these coordinates into a “supertime” $T \equiv (t, \theta^0)$ and, respectively, to identify the remaining set of coordinates as the “superspatial” coordinate $X \equiv (x, \theta^1)$. Thus, instead of the Poisson brackets at equal time appearing on the component level, in the superfield hamiltonian formalism we are led to define the Poisson brackets at equal supertime. Note that the superhamiltonian density (the integrand in (19)) includes explicit $\theta^0$ and the derivative $\partial_t$ which enter via the spinor derivative $D_1$. In order to avoid their appearance and so to be able to integrate by part with respect to $D_1$ one should redefine $t$ as follows

$$t \rightarrow \hat{t} = t - i\theta^1 \theta^0 .$$

(21)

In this basis

$$D_1 = \frac{\partial}{\partial \theta^1} + i\theta^1 \partial_x .$$
In accord with the last remark we define the super-Poisson brackets at equal supertime \( \tilde{T} \equiv (\tilde{t}, \theta^0) \). The only non-vanishing bracket is

\[
\{ \Pi_\Phi(X, \tilde{T}), \Phi(X', \tilde{T}') \}_{\tilde{T} = \tilde{T}'} = \Delta(X, X') \equiv \delta(x-x')(\theta^1 - \theta^{1'}) .
\] (22)

Here \( \Delta(X, X') \) is the supersymmetric delta function on the superplane \( \{X\} \):

\[
\int d^2 X \Delta(X, Y) F(X) = F(Y) , \quad d^2 X \equiv dx d\theta^1 ,
\]

with \( F(X) \) being an arbitrary "\( N = 1/2 \)" superfield. Note that the equality of supertimes, \( \tilde{T} = \tilde{T}' \), amounts to the following relations between the original coordinates

\[
\theta^0 = \theta^{0'} , \quad t - t' - i(\theta^1 - \theta^{1'}) \theta^0 = 0 .
\] (23)

These relations are invariant both under the \( \epsilon^0 \) and \( \epsilon^1 \) supertranslations (10), which justifies our choice of "super-simultaneity" in (22).

Now it is an easy exercise to check that the Hamilton equations pertinent to the superhamiltonian (19) and the super-Poisson structure (22)

\[
\dot{\Phi} = \{ H, \Phi \} \\
\dot{\Pi}_\Phi = \{ H, \Pi_\Phi \}
\] (24)

just reproduce the equations of motion (13), (14).

The hamiltonian (19) satisfying the super-conservation laws (20) and the super-Poisson structure (22) are the basic ingredients of the superhamiltonian formalism for \( N = 1, 2D \) theories. It is straightforward to check that after performing the integration over \( d\theta^1 \) and elimination of the auxiliary field in (19) the latter becomes the standard component hamiltonian, while (22) gives rise to the conventional Poisson brackets for the physical component fields present in the \( \theta \) expansions of \( \Phi, \Pi \). These basic structures are not specific just for the \( N = 1 \) sine-Gordon model: they can be equally defined for any other \( N = 1, 2D \) theory and shown to possess similar properties.

To demonstrate this, let us apply the previous construction to the \( N = 1 \) superconformal affine theory of ref. [5].

The action

\[
S = \int d^2 z d^2 \theta L \equiv \int d^2 z d\theta^1 \mathcal{L}
\]

is now given by

\[
S = \int d^2 z d^2 \theta \{ \frac{1}{4} D_0 \Phi D_1 \Phi + \frac{1}{2} D_0 \Sigma D_1 \Lambda + \frac{1}{2} D_0 \Lambda D_1 \Sigma + i\gamma (e^{\Phi} + e^{\Lambda - \Phi}) \} ,
\] (25)

with \( \Phi, \Sigma, \Lambda \) being independent real \( N = 1 \) superfields. The equations of motion are

\[
\dot{\Phi} = -iD_1 D_0 \Phi + \gamma (e^{\Phi} - e^{\Lambda - \Phi}) \\
\dot{\Lambda} = -iD_1 D_0 \Lambda \\
\dot{\Sigma} = -iD_1 D_0 \Sigma + \gamma e^{\Lambda - \Phi}
\] (26)
As before, we integrate in $S$ over $d\theta^0$ and define the conjugate momenta as follows

$$
\Pi_\Phi = \frac{\delta S}{\delta \dot{\Phi}} = \frac{1}{2} D_0 \Phi ,
\Pi_\Sigma = \frac{\delta S}{\delta \dot{\Sigma}} = \frac{1}{2} D_0 \Lambda ,
\Pi_\Lambda = \frac{\delta S}{\delta \dot{\Lambda}} = \frac{1}{2} D_0 \Sigma .
$$

(27)

Denoting with $\Psi$ a generic superfield, the super-Poisson brackets are introduced, at equal supertime $\tilde{T}$, through

$$
\{\Pi_\Phi(X,T),\Psi(X',T')\}_{\tilde{T}=\tilde{T}'} = \Delta(X,X')
$$

(28)

The superhamiltonian in this case is given by

$$
H = \int dx d\theta^1 \{ \Pi_\Phi \cdot \dot{\Phi} + \Pi_\Sigma \cdot \dot{\Sigma} + \Pi_\Lambda \cdot \dot{\Lambda} - L \}
$$

$$
= \int dx d\theta^1 \{ \frac{1}{4} i \Phi' D_1 \Phi + \frac{1}{4} \Sigma' D_1 \Lambda + \frac{1}{4} \Lambda' D_1 \Sigma + \gamma \Pi_\Phi e^\Phi + \gamma (\Pi_\Sigma - \Pi_\Phi) e^{\Lambda - \Phi}
$$

$$
- \Pi_\Phi D_1 \Pi_\Phi - \Pi_\Sigma D_1 \Pi_\Lambda - \Pi_\Lambda D_1 \Pi_\Sigma \} .
$$

(29)

The equations of motion are reproduced by the Hamilton system

$$
\dot{\Psi} = \{ H, \Psi \}
$$

$$
\dot{\Pi}_\Psi = \{ H, \Pi_\Psi \} .
$$

(30)

It is easy to check that the conservation laws (20) are valid in this case too.

Let us comment now on the alternative definition of $t$ and $x$ as in eq.(5). For this choice one cannot construct an analogous superhamiltonian formulation, though the component one in terms of physical fields certainly exists. The reason is clear from the anticommutation relations (8) (or the relations between the supertranslation generators which are basically the same). Under the choice we kept to until now there exists a $N = 1/2$ subalgebra $(Q_1, P_x)$ in $N = 1$ SUSY algebra, extending the $x$ translation generator $P_x$. Just due to this fact one can define a "spatial" superplane $(\theta_1, x)$ which is closed under this $N = 1/2$ SUSY and plays a crucial role in the above construction. The $N = 1/2$ SUSY is the only manifest supersymmetry of our superhamiltonian formalism. On the other hand, the choice of $t,x$ as in eq. (5) amounts to the interchange of $\partial_t$ and $\partial_x$ in eq. (8) and one can easily see that in this case there is no any kind of $N = 1/2$ subalgebra extending $P_x$. Therefore one cannot single out a closed subspace in $N = 1, 2D$ superspace, so as it would involve only $x$ in its bosonic sector.

However, the choice (5) is distinguished by the positivity arguments: indeed, in this case

$$
P_t = (Q_+)^2 + (Q_-)^2
$$

and the energy is strictly positive as a consequence of the $N = 1, 2D$ SUSY algebra. There is no such a remarkable property for the choice (4). Nevertheless, it is an easy effort to see that both choices are physically equivalent. Let us start with the component off-shell form of the action (11), assuming that the choice (4) has been done and the superfield $\Phi$ has the following $\theta$ expansion

$$
\Phi(x,t,\theta^+,\theta^-) = \phi(x,t) + \theta^+ \psi_+(x,t) + \theta^- \psi_-(x,t) + \frac{1}{2} \theta^+ \theta^- F(x,t) .
$$

(31)
Then one can check that the formal substitutions

\[ \psi_- \Rightarrow i\psi_-, \ F \Rightarrow iF, \ \alpha \Rightarrow i\alpha, \ \beta \Rightarrow i\beta \]  

(32)

take this component action just into the form pertinent to the choice (5) and vice versa. Thus we can use our superhamiltonian formalism at all stages of computations and at the final stage, in order to ensure the positiveness of energy, accomplish the substitutions (32). Note that in terms of superfields the replacements (32) correspond to invoking another type of reality condition for \( \Phi \)

\[ \Phi^\dagger(z, \theta^+, \theta^-) = \Phi(z, \theta^+, -\theta^-), \]

which means that \( \Phi \) is assumed to be real with respect to the new involution defined as the product of ordinary complex conjugation and reflection of \( \theta_- \). Note that the latter reflection is an obvious automorphism of the \( N = 1 \) superalgebra (1). Just owing to this property, the alternative definition of reality is still compatible with the \( N = 1, 2D \) supersymmetry.

Let us conclude this section by mentioning an application of our super-Poisson bracket formalism: it can be used to analyze the integrability property of the \( N = 1 \) super sinh-Gordon and superCAL theory: let a supersymmetric Lax pair (see ref.[5]) be defined through the positions

\[ L_+ = D_+ \Psi + e^{ad\Psi} \mathcal{E}_+, \quad L_- = -D_- \Psi + e^{-ad\Psi} \mathcal{E}_- \]  

(33)

where \( \Psi \) is a superfield taking value in the Cartan subalgebra of \( osp(2|2)^{(2)} \) (respectively \( osp(2|2)^{(2)} \)) and \( \mathcal{E}_+ \), \( \mathcal{E}_- \) are the sum of odd positive and negative simple root vectors of the same superalgebra. The zero-curvature condition for the above Lax pair is equivalent to the equations of motion derived in the superhamiltonian framework. The integrability property of such theories are made manifest by the appearance of the classical \( r \)-matrices \( r^\pm \) (they are defined in terms of the superalgebra generators and their explicit expression is given in [5]) through the relation for \( L = L_+ + L_- \):

\[ \{L_1(X, T), L_2(Y, T)\} = [r^\pm, L_1(X, T) + L_2(Y, T)]\delta(X, Y) \]  

(34)

where \( L_1 \equiv L \otimes 1, L_2 \equiv 1 \otimes L \). The l.h.s. is referred to the super-Poisson bracket. Notice in particular the presence of the (super-)conjugate momentum \( \Pi_\Psi \equiv (D_+ - D_-)\Psi \). In the super-Poisson bracket formalism the above equality (34) is directly proven, with no need of performing lengthy computation at the level of component fields.

3 The \( N = 2 \) superhamiltonian framework

In this section we extend the construction of the super-Poisson brackets and superhamiltonian formalism to the \( N = 2 \) case, taking as an illustration the \( N = 2 \) super sine-Gordon theory.

The two-dimensional \( N = 2 \) superspace is parametrized by \( z^{\pm\pm} \) and the complex fermionic coordinates \( \theta^{\pm} \) (and their conjugate \( \bar{\theta}^{\pm} \)). We will sometimes use the notation
\[ Z = (z^{+}, z^{-}, \theta^{+}, \theta^{-}, \vec{\theta}^{+}, \vec{\theta}^{-}) \]. The \( N = 1 \) superspace is recovered by letting \( \theta^{\pm} = \vec{\theta}^{\pm} \).

The \( N = 2 \) spinor derivatives \( D_{\pm}, \bar{D}_{\pm} \) are defined as:

\[
D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\vec{\theta}^{\pm} \partial_{\pm}
\]

\[
\bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm} \partial_{\pm}.
\] (35)

The only non-vanishing bracket between them is given by

\[
\{D_{\pm}, \bar{D}_{\pm}\} = 2i\partial_{\pm}.
\]

In particular we have

\[
D_{\pm}^2 = \bar{D}_{\pm}^2 = 0.
\]

Just as in the previous section, it is convenient to introduce the “rotated” spinor derivatives \( D_{0,1}, \bar{D}_{0,1} \) defined as

\[
D_{0} = D_{+} - D_{-} = \frac{\partial}{\partial \theta^{0}} - i\vec{\theta}^{1} \partial_{t} - i\vec{\theta}^{0} \partial_{x}
\]

\[
D_{1} = D_{+} + D_{-} = \frac{\partial}{\partial \theta^{1}} - i\vec{\theta}^{0} \partial_{t} - i\vec{\theta}^{1} \partial_{x}
\]

\[
\bar{D}_{0} = -\frac{\partial}{\partial \bar{\theta}^{0}} + i\theta^{1} \partial_{t} + i\theta^{0} \partial_{x}
\]

\[
\bar{D}_{1} = -\frac{\partial}{\partial \bar{\theta}^{1}} + i\theta^{0} \partial_{t} + i\theta^{1} \partial_{x}.
\] (36)

They satisfy the following anticommutation relations

\[
\{D_{0}, \bar{D}_{0}\} = \{D_{1}, \bar{D}_{1}\} = 2i\partial_{x}
\]

\[
\{D_{0}, \bar{D}_{1}\} = \{D_{1}, \bar{D}_{0}\} = 2i\partial_{t}
\]

\[
\{D_{i}, D_{j}\} = \{\bar{D}_{i}, \bar{D}_{j}\} = 0 \quad (i, j = 0, 1).
\] (37)

An \( N = 2 \) chiral superfield \( \Phi \) is introduced by the constraint

\[
D_{\pm} \Phi = 0,
\] (38)

while its conjugate satisfies

\[
\bar{D}_{\pm} \Phi^{\dagger} = 0
\] (39)

and so it is an anti-chiral \( N = 2 \) superfield:

\[
\Phi \equiv \Phi(z^{++}, z^{--}, \theta^{+}, \theta^{-}, \vec{\theta}^{+}, \vec{\theta}^{-}), \quad \Phi^{\dagger} \equiv \Phi^{\dagger}(z^{++}, z^{--}, \theta^{+}, \theta^{-}),
\] (40)
\[
\begin{align*}
\dot{z}^{\pm\pm}_R &= z^{\pm\pm} + i\theta^{\pm\pm}\overline{\theta}^{\pm\pm}, \quad \dot{z}^{\pm\pm}_L &= z^{\pm\pm} - i\theta^{\pm\pm}\overline{\theta}^{\pm\pm}.
\end{align*}
\] (41)

In what follows, the parametrizations of \(N = 2\) superspace using the bosonic coordinates \(z^{\pm\pm}_L\) and \(z^{\pm\pm}_R\) will be referred to, respectively, as the \(Z_L\) and \(Z_R\) ones.

The action \(S\) of the super sinh-Gordon theory reads \(7\):

\[
S = \int d^2 z d^4\theta \{\Phi\Phi^\dagger\} + \int d^2 z_R d^2\theta \{e^\Phi + \beta e^{-\Phi}\} + \int d^2 z_L d^2\theta \{e^{\Phi^\dagger} + \beta e^{-\Phi^\dagger}\} \quad (42)
\]

where we defined the \(N = 2\) superspace integration measures by

\[
\begin{align*}
\int d^2 z d^4\theta &= \int d^2 z_L d^2\theta \Omega_0 \Omega_1 = \int d^2 z_R d^2\theta \Omega_1 \Omega_0 \\
\int d^2 z L d^2\theta &= \int d^2 z_L D_0 D_1, \quad \int d^2 z_R d^2\theta = \int d^2 z_R \overline{\Omega}_0 \overline{\Omega}_1.
\end{align*}
\] (43)

For what follows it will be convenient to rewrite the action as

\[
\begin{align*}
S &= S_L + S_R \\
&= \frac{1}{2} \int d^2 z_L d^2\theta \{\Omega_0 \Omega_1 \Phi \Phi^\dagger + 2(e^{\Phi^\dagger} + \beta e^{-\Phi^\dagger})
\]
\[
+ \frac{1}{2} \int d^2 z_R d^2\theta \{\Phi \cdot D_1 D_0 \Phi^\dagger + 2(e^{\Phi} + \beta e^{-\Phi})\}. \quad (44)
\end{align*}
\]

The equations of motion following from \(\{42\}\) are

\[
\begin{align*}
D_1 D_0 \Phi^\dagger &= -e^{\Phi} + \beta e^{-\Phi} \\
\overline{\Omega}_0 \overline{\Omega}_1 \Phi^\dagger &= -e^{\Phi^\dagger} + \beta e^{-\Phi^\dagger} \quad (45)
\end{align*}
\]

Acting on them by spinor derivatives one gets the important consequences

\[
\begin{align*}
D_0 \Phi^\dagger - D_1 \Phi^\dagger &= \frac{i}{2} \overline{\Omega}_1 \Phi \left( e^{\Phi} + \beta e^{-\Phi} \right) \\
D_1 \Phi^\dagger - D_0 \Phi^\dagger &= \frac{i}{2} \overline{\Omega}_0 \Phi \left( e^{\Phi} + \beta e^{-\Phi} \right) \quad (46)
\end{align*}
\]

\[
\begin{align*}
\overline{\Omega}_1 \Phi - \overline{\Omega}_0 \Phi &= \frac{i}{2} D_1 \Phi^\dagger \left( e^{\Phi^\dagger} + \beta e^{-\Phi^\dagger} \right) \\
\overline{\Omega}_1 \Phi^\dagger - \overline{\Omega}_0 \Phi^\dagger &= \frac{i}{2} D_0 \Phi^\dagger \left( e^{\Phi^\dagger} + \beta e^{-\Phi^\dagger} \right) \quad (47)
\end{align*}
\]

Note that in the present case, as it follows from the anticommutation relations \(\{37\}\), one cannot cast the equations of motion \(\{45\}\) in the form analogous to \(\{13\}\). However, using \(\{37\}\) and the chirality conditions \(\{38\}\) and \(\{39\}\), one finds the algebraic identities

\[
\Phi = -\frac{i}{2} D_1 \overline{\Omega}_0 \Phi, \quad \Phi^\dagger = -\frac{i}{2} \overline{\Omega}_1 D_0 \Phi^\dagger \quad (48)
\]

which in the \(N = 2\) superhamiltonian formalism replace the dynamical relation \(\{13\}\).

After integration in both chiral parts of the action over, respectively, \(d\theta^0\), \(d\overline{\theta}^0\) one gets

\[
\begin{align*}
S &= \int d^2 z_L d\theta^1 \mathcal{L}_L + \int d^2 z_R d\overline{\theta}^1 \mathcal{L}_R \\
&= \int d^2 z_L d\theta^1 \{i(\overline{\Omega}_1 \Phi)^\dagger \Phi + i(\overline{\Omega}_0 \Phi) \Phi^\dagger - \frac{1}{2} \overline{\Omega}_1 (\overline{\Omega}_0 \Phi)(D_0 \Phi^\dagger) + D_0 \Phi^\dagger (e^{\Phi^\dagger} - \beta e^{-\Phi^\dagger})\}
\]
\[
- \int d^2 z_R d\overline{\theta}^1 \{i(D_1 \Phi)^\dagger \Phi + i(D_0 \Phi^\dagger) \Phi^\dagger - \frac{1}{2} D_1 (D_0 \Phi^\dagger)(\overline{\Omega}_0 \Phi) - \overline{\Omega}_0 \Phi(e^{\Phi} - \beta e^{-\Phi})\}. \quad (49)
\]
The super-conjugate momenta are defined to be

\[
\Pi_{\Phi^\dagger} = \frac{\delta S}{\delta \dot{\Phi}^\dagger} = 2i \overline{D}_0 \Phi \\
\Pi_\Phi = \frac{\delta S}{\delta \Phi} = -2i D_0 \Phi^\dagger = (\Pi_{\Phi^\dagger})^\dagger.
\] (50)

Notice that, when computing, e.g., \(\Pi_{\Phi^\dagger}\), one should rewrite the kinetic part of \(S\) in the basis where \(\dot{\Phi}^\dagger\) is unconstrained (\(z_L, \theta^1\) basis). Then both parts of the action give the same contribution, which explains the factor 2 in (50).

The \(N = 2\) super-Poisson brackets are defined at equal left-chiral or right-chiral super-times \(T_L \equiv (t_L, \theta^0)\), \(T_R \equiv (t_R, \overline{\theta}^1)\). With \(X_L \equiv (x_L, \theta^1)\), \(X_R \equiv (x_R, \overline{\theta}^1)\), the only non-vanishing brackets are

\[
\{\Pi_\Phi(Z_R), \Phi(Z_R')\}_{T_R = T_R'} = \Delta(X_R, X'_R) \\
\{\Pi_{\Phi^\dagger}(Z_L), \Phi^\dagger(Z'_L)\}_{T_L = T_L'} = \overline{\Delta}(X_L, X'_L)
\] (51)

where \(\Delta(X_R, Y_R)\), \(\overline{\Delta}(X_L, Y_L)\) are the chiral delta-functions on the conjugate “superspatial” planes \(x_R, \overline{\theta}^1\) and \(x_L, \theta^1\). Note that both sides of (51) identically vanish under \(D_0 (\overline{D}_0)\), \(D_1 (\overline{D}_1)\) acting on the second superargument and under \(D_0 (\overline{D}_0)\) acting on the first one. This follows from the chirality of \(\Phi\) and the definition of conjugate momenta. At the same time, \(D_1 (\overline{D}_1)\), while acting on the first superargument in the l.h.s. of (51), do not yield zero automatically, though annihilate the r.h.s. So it is a nontrivial consequence of the Poisson structure (49) that \(D_1 \Pi_\Phi (\overline{D}_1 \Pi_{\Phi^\dagger})\) commute with \(\Phi (\Phi^\dagger)\) at equal super-time, or, in other words, that the \(\theta^1 (\overline{\theta}^1)\) dependence of \(\Pi_\Phi \ (\Pi_{\Phi^\dagger})\) in (51) can be consistently neglected. This property is evidently compatible with the superfield equations of motion (43) (and, actually, with the equations corresponding to an arbitrary choice of the chiral superfields potential in (40)). It is straightforward to check that on shell the super-Poisson brackets (51) yield the Poisson brackets obtained from the theory formulated in terms of component fields.

As the next crucial step in constructing \(N = 2\) superhamiltonian formalism, we define the \(N = 2\) superhamiltonian by

\[
H = \int dx_L d\theta^1 \mathcal{H}_L + \int dx_R d\overline{\theta}^1 \mathcal{H}_R \equiv H_L + H_R
\] (52)

with

\[
\mathcal{H}_L \equiv \Pi_{\Phi^\dagger} \dot{\Phi}^\dagger - \mathcal{L}_L = -i (\overline{D}_1 \Phi)^\dagger \Phi^\dagger + \frac{1}{8} (\Pi_{\Phi^\dagger} \overline{D}_1 \Pi_\Phi + \overline{D}_1 \Pi_{\Phi^\dagger} \Pi_\Phi) - \frac{1}{2} \Pi_\Phi (e^{\Phi^\dagger} - e^{-\Phi^\dagger})
\] (53)

\[
\mathcal{H}_R \equiv \Pi_\Phi \dot{\Phi} - \mathcal{L}_R = i (D_1 \Phi^\dagger)^\dagger \Phi^\dagger - \frac{1}{8} (\Pi_\Phi D_1 \Pi_{\Phi^\dagger} + D_1 \Pi_{\Phi^\dagger} \Pi_\Phi) + \frac{1}{2} \Pi_{\Phi^\dagger} (e^{\Phi} - e^{-\Phi}).
\] (54)

In the process of obtaining the expressions (53), (54) we have eliminated \(\dot{\Phi}, \dot{\Phi}^\dagger\) with the help of the kinematical relations (18).

It is remarkable that \(H_L\) and \(H_R\) separately satisfy the following conservation laws

\[
\overline{D}_0 H_L = \overline{D}_1 H_L = D_0 H_L = \dot{H}_L = 0
\] (55)

and

\[
D_0 H_R = D_1 H_R = \overline{D}_0 H_R = \dot{H}_R = 0.
\] (56)
In both sets first laws are fulfilled kinematically, without assuming any dynamics for the involved superfields, while the remaining ones are genuine conservation laws: they are consequences of the equations of motion (43). It is an easy exercise to see that they are compatible with the algebra of $N = 2$ spinor derivatives (37).

It is interesting that $H_L$ and $H_R$ can be interpreted as the time-translation operators, respectively for $\Phi^\dagger$ and $\Phi$: it is straightforward to check that the pairs of the Hamilton equations

\[
\dot{\Phi} = 2\{H, \Phi\} \\
\dot{\Pi}_\Phi = 2\{H, \Pi_\Phi\}
\]

and

\[
\dot{\Phi} = 2\{H, \Phi\} \\
\dot{\Pi}_\Phi = 2\{H, \Pi_\Phi\}
\]

immediately yield, with respect to the super Poisson brackets (51), the identities (48) and the second equations from the conjugate pairs (46), (47). In fact, the whole system (46), (47) is reproduced, since all the spinor and ordinary derivatives of the first equations vanish as a consequence of the second ones and then the validity of the first equations is clear by 2D Lorentz covariance.

Actually, after passing in $H_R$ to the left- and in $H_L$ to the right-chiral bases one may evaluate the Poisson brackets of $H_L$ with $\Phi, \Pi_\Phi$ and $H_R$ with $\Phi^\dagger, \Pi_{\Phi^\dagger}$ and check that these brackets are such that the same evolution equations (57), (58) can be equivalently obtained as those with respect to the full superhamiltonian (52)

\[
\dot{\Phi} = \{H, \Phi\} \\
\dot{\Phi} = \{H, \Phi^\dagger\} \\
\dot{\Pi}_\Phi = \{H, \Pi_\Phi\} \\
\dot{\Pi}_{\Phi^\dagger} = \{H, \Pi_{\Phi^\dagger}\}
\]

The fact that the dynamics is splitted into two conjugate hamiltonians leading to two conjugate sets of equations of motion is a peculiar feature of the second supersymmetry: indeed, $N = 2$ supersymmetric integrable models can be formulated through two conjugate Lax pairs (see [8]).

Let us point out that, in contrast with the $N = 1$ case, in the $N = 2$ superhamiltonian approach one does not immediately reproduce the original Lagrangian superfield equations of motion (13), but merely their consequences (46), (47). The latter system actually is not fully equivalent to (13): these equations are restored from it up to an arbitrary complex integration constant (of dimension of mass) in their r.h.s., respectively $c$ and $c^\dagger$, which amounts to adding the linear terms $-c\Phi$ and $-c^\dagger\Phi^\dagger$ to the superpotentials in the Lagrangians (12), (14):

\[
L_L \Rightarrow L_L - c^\dagger\Phi^\dagger, \quad L_R \Rightarrow L_R - c\Phi.
\]

Once this constant is included in the Lagrangian, it will appear as well in the superhamiltonians (52) and (53), (54):

\[
\mathcal{H}_L \Rightarrow \mathcal{H}_L - \frac{1}{4i}c^\dagger\Phi, \quad \mathcal{H}_R \Rightarrow \mathcal{H}_R - \frac{1}{4i}c\Phi^\dagger.
\]
Just these modified hamiltonians now satisfy the conservation laws (53), (54): while checking the validity of the latter, one needs to make use of the original superfield equations of motion, not only of their consequences (13), (17). Thus we conclude that there exists a family of $N=2$ superhamiltonians parametrized by a complex parameter $c$, such that all these hamiltonians yield the same dynamical equations (16), (17) and the identities (18) as the associated Hamilton system and reproduce the original set of equations up to a freedom related to this parameter. For the time being the origin and meaning of this ambiguity is not quite clear for us. Note that such a deformation of the $N=2$ super Liouville action has been proposed in ref. [9] to avoid difficulties with infra-red divergences. The resulting more general theory was called the “$N=2$ Morse - Liouville” one.

4 Concluding Remarks

In this paper we have defined a superhamiltonian formalism for $N=1, 2$ 2D theories which naturally extends the standard hamiltonian formulation. Its main advantage compared to the standard canonical hamiltonian treatment of such theories lies in its manifest supersymmetry: e.g., the full supersymmetric set of the field equations including those for auxiliary fields arise as the Hamilton ones.

Our framework allows to introduce Poisson structures and perform computations with them at a superfield level; therefore we expect it to find natural applications in the integrable supersymmetric models and in any kind of supersymmetric theory where there is a need to investigate algebraic structures based on Poisson brackets.

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Appendix: Component form of superhamiltonian

In this appendix we present the superhamiltonian (19) for the $N=1$ sinh-Gordon model in terms of component fields.

The superfields $\Phi, \Pi_\Phi$ have the following component expansions

\[
\Phi = \phi + \theta^0 \psi_0 + \theta^1 \psi_1 + \theta^0 \theta^1 F \\
\Pi_\Phi = \frac{i}{2}(\psi_0 + i\theta^0 \phi' + \theta^1(F + i\dot{\phi}) + \theta^0 \theta^1(i\dot{\psi}_1' - i\dot{\psi}_0)) \tag{A.1}
\]

where $\psi_{0,1}$ are fermionic fields, $\phi, F$ bosonic ones ($F$ auxiliary).

The superfield equation of motion (12) amounts to the following set of equations for the component fields

\[
F = -i(e^\phi - \beta e^{-\phi}) \\
\dot{\psi}_0 - \psi_1' = -\psi_0(e^\phi + \beta e^{-\phi})
\]
\[ \begin{align*} 
\dot{\psi}_1 - \psi_0' &= \psi_1(e^\phi + \beta e^{-\phi}) \\
\dot{\phi} - \phi' &= iF(e^\phi + \beta e^{-\phi}) - i\psi_0\psi_1(e^\phi - \beta e^{-\phi}). 
\end{align*} \] (A.2)

The Hamiltonian density \( \mathcal{H} \) can be straightforwardly computed from eq.(19). After doing the \( \theta^1 \) integration there, one is left with an integral over the spatial coordinate \( x \). Though the integrand in it bears a dependence on \( \theta^0 \), the coefficient before \( \theta^0 \) turns out to be a total \( x \)-derivative and gives no contribution, in agreement with the first of the conservation laws (20). So the Hamiltonian \( \mathcal{H} \) is just given by

\[ \begin{align*} 
\mathcal{H} &= \frac{1}{4} \int dx ((F + i\dot{\phi})^2 - \phi'\phi' + i\psi_1\psi_1 + i\psi_0\psi_0 + \\
&\quad + 2i(F + i\dot{\phi})(e^\phi - \beta e^{-\phi}) - 2i\psi_0\psi_1(e^\phi + \beta e^{-\phi})). 
\end{align*} \] (A.3)

Notice that the fields \( F, \dot{\phi} \) enter into this Hamiltonian in the combination \( F + i\dot{\phi} \). Only after employing the algebraic equation for the auxiliary field \( F \), eq.(A.3) reduces to the standard Hamiltonian constructed directly in the component fields formalism.

To find out the significance of this deviation from the standard Hamiltonian approach, let us look at the component content of the super-Poisson brackets (22). The ordinary Poisson brackets at equal time implied by (22) for the component fields fall into the three categories: (i) The standard ones matching with those constructed in the component fields canonical formalism; (ii) The relations involving the auxiliary field \( F \); (iii) Some additional constraints which involve, e.g., \( \dot{F}, \ddot{\phi} \) and the validity of which should be checked with making use of the previous type relations and the equations of motion. For self-consistency of the whole set of these Poisson brackets it is essential that the "super-simultaneity" is defined as in eq. (23).

In the first category we have the following relations (for brevity we suppress the argument \( t \) of fields)

\[ \begin{align*} 
\{ \dot{\phi}(x), \phi(x') \}_- &= -2\delta(x - x') \\
\{ \psi_0(x), \psi_1(x') \}_+ &= -2i\delta(x - x') \\
\{ \phi(x), \phi(x') \}_- &= \{ \phi(x), \psi_{0,1}(x') \}_- = 0 \\
\{ \psi_0(x), \psi_0(x') \}_+ &= \{ \psi_1(x), \psi_1(x') \}_+ = 0 
\end{align*} \] (A.4)

The second category involves the relations

\[ \begin{align*} 
\{ F(x), \phi(x') \}_- &= \{ F(x), \psi_{0,1}(x') \}_- = 0 \\
\{ F(x) + i\dot{\phi}(x), F(x') + i\dot{\phi}(x') \}_- &= 0. 
\end{align*} \] (A.5)

An example of the constraints of the third category is supplied by

\[ \{ \dot{\psi}_0(x), \psi_0(x') \}_+ = 2i\partial_x \delta(x - x'). \] (A.6)

Now it is easy to see that the whole set of the component equations (A.2) amounts to the set of the Hamilton equations

\[ \dot{B} = \{ H, B \} \]
with, respectively, $B = \phi, \psi_0, \psi_1, F + i\dot{\phi}$. So only the first and second category of the component Poisson brackets are really of need to derive the equations of motion\footnote{Actually, for this derivation we merely need to know the Poisson brackets just between the fields $\phi, \psi_0, \psi_1, F + i\dot{\phi}$, but not, e.g., between $F$ and $F$, $F$ and $\phi$ separately.}. The remarkable difference from the standard component Hamiltonian formalism lies in the fact that the whole set of field equations including the equation for the auxiliary field $F$ follows as the Hamilton equations with respect to the Hamiltonian (A.3). This comes about just because (A.3) differs from the standard component Hamiltonian by terms which vanish upon exploiting the equation for $F$.

All these features are independent of the specific form of the chosen superpotential and therefore are valid for a more general class of theories than those explicitly treated in the present paper.

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