HAIRY BLACK HOLES IN STRING THEORY

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ABSTRACT: Solutions of bosonic string theory are constructed which correspond to four-dimensional black holes with axionic quantum hair. The basic building blocks are the renormalization group flows of the $CP^1$ model with a $\theta$ term and the $SU(1,1)/U(1)$ WZW coset conformal field theory. However the solutions are also found to have negative energy excitations, and are accordingly expected to decay to the vacuum.

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The classical endpoint of gravitational collapse is expected to be a simple object: a stationary black hole characterized solely by its mass, charge and angular momentum. This expectation is strengthened by the famous “no-hair” theorems [nhr]. These theorems assume a specific field content, but they generally suggest that the stationary, stable black hole configurations are labeled only by conserved quantities associated with local gauge symmetries.

Quantum mechanics leads to a further balding of black holes. For example, Hawking radiation will cause a charged black hole to radiate away mass until it reaches the extremal (mass = charge) limit. Thus one less quantum number is required to characterize the stable endpoint.

Several years ago it was suggested that quantum mechanics might lead to hair growth as well as hair loss [ghr]. In theories with axion strings, an arbitrary phase can be associated with the process of lassoing a black hole with an axion string. This phase arises from the axion-string interaction Lagrangian

\[ S_I = T \int_{\Sigma} B. \]  

The integral extends over the string worldsheet \( \Sigma \), \( T \) is the string tension and \( B \) is the axion two-form potential. If \( B \) is given by the closed but not exact two-form obeying \( \int_{S^2} B = \theta L^2 \) (where \( \theta \) is dimensionless and \( L \) is the appropriate distance scale) for any two sphere surrounding the horizon, it follows that a string which lassos the black holes picks up a phase \( \theta TL^2/\hbar \). \( \theta \) is classically unobservable, but may be measured quantum mechanically in Aharonov-Bohm type interference experiments. It is thus a new quantum number, or “quantum hair” associated to a quantum black hole. Generalizations of this idea involving discrete \( Z_N \) gauge symmetries were discussed in [kw].

Quantum hair has no perturbative effect on the Hawking radiation rate, so large hairy black holes will evaporate as usual. However, for small black holes, non-perturbative effects
could be important and one might suspect the existence of hairy extremal black holes[ghr]. These objects would be stabilized against Hawking radiation by quantum hair, just as charged extremal black holes are stabilized by their classical electromagnetic hair.

Evidence for such extremal objects was found (in the context of discrete gauge hair) in the elegant analysis of Coleman, Preskill and Wilczek [cpw]. A Euclidean instanton which can be described as a virtual string lassoing the black holes was shown to slow down the Hawking radiation rate, suggesting that it might actually turn off at a critical value of the mass. However, unlike electromagnetic charge, axion (or discrete) charge is periodic and cannot be made arbitrarily large. Hairy extremal black holes are therefore typically planckian objects, and their existence can not ordinarily be determined from low-energy semiclassical gravity. A complete quantum theory of gravity - including all higher dimension operators - is required.

In this paper we will use bosonic string theory to demonstrate the existence of hairy extremal black holes. In string theory, non-zero $\theta$ leads to a deformation of the classical solutions (which vanishes exponentially at large distances) and some exact classical solutions will be found as conformal field theories. The general solution will be qualitatively described in terms of two-dimensional renormalization group flows. We will find a tachyonic excitation in the spectrum, so these objects are unstable—this is in addition to the usual tachyonic instability of string theory. We briefly discuss the possibility of generalization to the superstring.

Classical solutions of bosonic string theory are provided by conformally invariant two-dimensional sigma models. The sigma model corresponding to a four-dimensional hairy black hole with line element

$$ds^2 = -N^2(\rho)dt^2 + d\rho^2 + R^2(\rho)d^2\Omega,$$

(lmt)
and axion hair

\[ B = \frac{\theta \epsilon \alpha'}{2}, \quad \text{(bte)} \]

and dilaton field \( \Phi(\rho) \) may be written*

\[
S_\sigma = \frac{1}{\pi \alpha'} \int d^2 \sigma \left( - N^2(\rho) \partial_+ t \partial_- t + \partial_+ \rho \partial_- \rho + \alpha' R_{+-} \Phi(\rho) \right. \\
\left. + R^2(\rho) G_{\mu \nu} \partial_+ X^\mu \partial_- X^\nu + \frac{\alpha' \theta}{2} \epsilon_{\mu \nu} \partial_+ X^\mu \partial_- X^\nu \right) \quad \text{(smn)}
\]

where \( G_{\mu \nu} \) (\( \epsilon_{\mu \nu} \)) \( \mu, \nu = 1, 2 \) is the unit metric (volume form) on the two-sphere, \( \alpha' = \hbar/2\pi T \) and \( \int d^2 \sigma \sqrt{g} R = 4 \int d^2 \sigma R_{+-} = 8\pi \) on \( S^2 \). We wish to find \( c = 4 \) conformal field theories of the form (smn). The additional \( c = 22 \) CFT required for \( c = 26 \) will be suppressed.

**Fig. 1:** Shown is a sketch of the renormalization group flows for the CP1 model. An infrared fixed point lies at \( \theta = \pi, R = R_C \).

*Note that, following convention, the \( \theta \) term in this action differs from (sss) by a factor of \( \hbar \).
Building blocks in our constructions are sigma models on $S^2$ - or $CP_1$ models - with $\theta$ terms, corresponding to the last two terms in (smn) with constant $R$. These have been the object of extensive investigations. The renormalization group flows in the $\theta$, $R$ planes are depicted in figure 1. The renormalization group flows have a fixed point at $\theta = \pi$, $R = R_c$ corresponding to a $c = 1$ CFT, a free scalar at its self-dual $SU(2)$-invariant radius. This was first conjectured by Haldane [hal], who argued that the critical behavior of the spin-$s$ antiferromagnetic chain was governed by the $CP_1$ sigma model, at $\theta = 0$ for integer spin and at $\theta = \pi$ for half-integer. The critical behavior for $s = \frac{1}{2}$ was already known, by bosonization and Bethe ansatz, to be given by the scalar at the self-dual radius. Subsequent work ([af] and references therein; see also [fr] for a review) confirmed this conjecture and filled out the phase diagram. The $CP_1$ sigma model with $\theta$ term has been applied previously to string theory black holes by Kogan [kogan], with a different interpretation of the renormalization group flow.

A $c = 4$ CFT can be obtained (following previous constructions [gs,gps]) by simply taking the tensor product with an $SU(1,1)/U(1)$ level $k = 8$, $c = 3$ coset model. By representing this coset model as a sigma model, it was identified [wtt] as a two-dimensional black hole. The full sigma model action with four-dimensional target space is, to leading order in $\alpha'$

$$S_\sigma = \frac{1}{\pi\alpha'} \int d^2\sigma \left[ -\tanh^2(\rho/\sqrt{6}\alpha') \partial_+ t \partial_- t + \partial_+ \rho \partial_- \rho 
+ \mathcal{R}_{+} \left( -\ln \cosh(\rho/\sqrt{6}\alpha') + \Phi_h \right) + (R_c^2 G_{\mu\nu} + \frac{\pi \alpha'}{2} \epsilon_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu \right],$$

(wwz)

where $\Phi_h$ is an arbitrary constant. (wwz) corresponds to a special type of four-dimensional hairy black hole. There is no asymptotically flat region; the two-spheres have radius $R_c$ for all $\rho$ and $\theta$ is restricted to equal $\pi$.

The one free parameter is the constant $\Phi_h$, the value of the dilaton at the horizon $\rho = 0$, which is related to the ADM mass of the black hole[wtt]. At the quantum level,
Hawking radiation presumably drives the mass to zero. In this limit the horizon moves off to infinity and the action is simply (after a coordinate transformation)

\[
S_\sigma = \frac{1}{\pi \alpha'} \int d^2 \sigma \left[ - \partial_+ t \partial_- t + \partial_+ \rho \partial_- \rho - \sqrt{\frac{\alpha'}{6}} R_{++} - \left( R_c^2 G_{\mu \nu} + \frac{\pi \alpha'}{2} \epsilon_{\mu \nu} \right) \partial_+ X^\mu \partial_- X^\nu \right] .
\]  
\[(xth)\]

This represents an extremal black hole with quantum hair.

It is of interest to find hairy extremal black holes which are asymptotically flat and thus might exist in our universe. These can be described starting from the renormalization group flows in figure 1. Consider a point \( \theta = \pi \), with \( R \) just above the fixed point \( R_c \). For small \( R - R_c \), the corresponding sigma model is nearly conformally invariant and has a small beta function

\[
\mu \partial_\mu \ln R(\mu) = \gamma (R - R_c)^2 .
\]  
\[(bta)\]

where \( \gamma > 0 \). The approach to the fixed point is along the perturbation \( j_\bar{z} \) and so is marginal as indicated by the quadratic beta function \([af]\). A conformal field theory to order \( R - R_c \) can be constructed by dressing the action with the \( \rho \) field

\[
S \sim \frac{1}{\pi \alpha'} \int d^2 \sigma \left( - \partial_+ t \partial_- t + \partial_+ \rho \partial_- \rho - \sqrt{\frac{\alpha'}{6}} R_{++} - \{ \rho + O(\rho^{-2}) \} \right) \\
\left( R_c^2 - 2R_c/\gamma \rho + O(\rho^{-2}) \right) G_{\mu \nu} + \frac{\pi \alpha'}{2} \epsilon_{\mu \nu} \partial_+ X^\mu \partial_- X^\nu \right]
\]  
\[(aan)\]

near \( \rho = -\infty \). Presumably it is possible to correct the action (aan) to obtain a CFT in a power series in \( R - R_c = -1/\gamma \rho \) in a neighborhood of the fixed point. As \( \rho \to -\infty \), \( R \) approaches \( R_c \), and the geometry approaches the previously discussed extremal black hole. As \( \rho \) increases, the radius \( R \) increases, and the black hole “throat” begins to open up. When \( R - R_c \) is of order one the radius rapidly increases and one enters the mouth region. At this point the expansion parameter used in constructing CFT’s by dressing renormalization group flows breaks down, and we have no quantitative tools to analyze the theory. However, if one assumes that \( R \) passes through the mouth region to larger
values, the theory can be analyzed in an expansion in $1/R$ on the other side of this region.
To leading order in this expansion, conformal invariance and $c = 4$ implies that $R \propto \rho$ and
$\Phi$ is constant. Thus it is plausible to assume that the theory ties on to an asymptotically
flat geometry.

Other renormalization group trajectories can similarly be used to construct extremal
black holes. For $\theta$ near but not equal to $\pi$, there is a long throat region produced by $R$
ingering near the fixed point. In this region a construction of the type (aan) yields an
approximate CFT. However for $\theta \neq \pi$, $R$ eventually becomes small and the approximations
break down. At the $R = 0$ infrared fixed point all excitations are infinitely massive and
a spacetime interpretation of the theory is no longer possible [gps]. If $\theta$ is not near $\pi$, a
throat region never forms, and one immediately descends into the vicinity of the infrared
fixed point.

As indicated in figure 1, the perturbation in the $\theta$ direction away from $\theta = \pi$ is relevant.
It corresponds to the $(j, \tilde{j}) = (\frac{1}{2}, \frac{1}{2})$ primary, of weight $(\frac{1}{4}, \frac{1}{4})$. Expanding around a linear
dilaton background, the mass-squared (including a term $(\nabla \Phi)^2$ from the linear dilaton),
is tachyonic, $m^2 = -17/6\alpha'$. The extremal object is therefore classically unstable, as are
the near-extremal objects with long throats. Presumably they will decay by emission of a
radial axion gradient, to flat $R^3$ with $\theta = 0$. The would-be hair, like a bad toupee, slips
off.

The picture of the $\theta = \pi$ extremal black hole is similar to that found for magnetically
charged extremal black holes [ghs,gps] which, for large charge, can be analyzed perturba-
tively. There is an asymptotically flat region, and a mouth connecting on to a semi-infinite
throat region. These also share with the present construction the feature that the time
coordinate is a free field and plays only a spectator role in the construction. The solutions
with $\theta$ neither 0 nor $\pi$ resemble the $Q = \pm 1$ solutions of [gps] in the degeneration to a
massive field theory at the origin.
A qualitative - but not quantitative - picture of the structure of hairy black holes can be developed in a mini-superspace type approximation. The renormalization group flows in figure 1 originate entirely from non-perturbative worldsheet instantons which wrap around the horizon, since those are the only configurations sensitive to $\theta$. The spacetime effective action which incorporates these instanton effects is non-local. However in an $S$-wave approximation in which all configurations are required to be spherically symmetric, a world sheet instanton is represented by a point in the two-dimensional $\rho, t$ plane. Summing over spherically symmetric world sheet instantons is then equivalent to summing over ordinary instantons in the two-dimensional effective theory. The effects of such instantons are reproduced by adding to the action the operator whose effects mimics that of the instanton†. The result is

$$S = \frac{2\pi}{\kappa^2} \int d^2x \sqrt{-g} e^{-2\Phi} \left[ R^2 R^{(2)} + 2(\nabla R)^2 + 2 + 4R^2(\nabla \Phi)^2 - 2\nabla^2 R^2 - \frac{\alpha'^2}{4R^2}(\nabla \theta)^2 + Ce^{-2R^2/\alpha' \cos \theta} \right]$$

where $x = (\rho, t)$. $C$ is a positive determinant. The first six terms are obtained by spherical reduction of the four-dimensional string action with the ansatz

$$ds^2 = g_{ab}(x)dx^a dx^b + R^2(x) d^2\Omega,$$

$$B = \frac{\alpha'}{2} \theta(x) \epsilon.$$  

The last term in (ssp) reproduces the effects of string instantons. One can think about this in two ways. The first is the usual string sigma model point of view, where the string wrapping the black hole is a world-sheet instanton, and the last term in the action represents the contribution of these instantons to the beta function. The fact that the dilute instanton approximation suggests a $\theta = \pi$ fixed point of the $CP1$ model was noted in [lev]. Alternately we can think of them - as in [cpw] - just like ordinary spacetime

† In the case at hand the single instanton action is infrared divergent. However, as explained in [claw], this does not mean that instanton effects can not be summarized by a local operator. Rather, a single insertion of the appropriate operator must reproduce the infrared divergence.
instantons involving solitonic strings wrapping around a black hole**. This should be
valid at large $R$ because - in the spirit of [daha] - the low-energy effective field theory does
not know if the core of the string contains a fundamental string or resembles a smooth
soliton.

The equations of motion following from (ssp) have a solution with constant $R$ and
constant $\theta = \pi$. However it is only suggestive since (ssp) can not be trusted when $R$ is
small and the instantons are not dilute. This approximation also misses the fact that the
approach to the fixed point is marginal.

The reduced action (ssp) can be thought of as describing spherical four-dimensional
dilaton gravity coupled to a scalar field $\theta(x)$ with a field dependent potential that vanishes
asymptotically. Non-extremal hairy black hole solutions can be constructed by solving the
radial equations with boundary conditions imposed at the horizons. Since the equations
degenerate at the horizon, there are (after gauge fixing) only three independent initial
data which may be taken to be the horizon values $\theta_h$, $\Phi_h$ and $R_h$ of $\theta$, $\Phi$ and $R$. $R_h$
directly determines the horizon area, while $\Phi_h$ determines the asymptotic value of $\Phi$. If
$\theta$ were an ordinary massive field there would be two possible asymptotic solutions: one
which grows and one which decays exponentially. Unless one takes $\theta_h = 0$, there would be
some admixture of the growing solution and the spacetime would not be asymptotically
flat. This is in accord with the no hair theorems for scalar fields. On the other hand if $\theta$
were exactly massless the asymptotic solutions go as a constant plus $1/R$, and there is no
danger of destroying asymptotic flatness. In fact the solution will have $\theta = \theta_h$ everywhere.
The action (ssp) is somewhere between the massive and massless case. Because the mass
vanishes asymptotically, there are no growing modes and no fear of destroying asymptotic
flatness. On the other hand since the potential is non-zero $\theta$ will not be constant if

** In [cpw] the effects of string instantons are - in contrast to the present case - non-
perturbative in $\hbar$ because they keep $T = \hbar/2\pi\alpha'$ (rather than $\alpha'$) fixed as $\hbar \to 0$. 

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θ_h \neq 0, \pi. In general the phase measured in string interference will be given by the asymptotic value of \( \theta(x) \) which is non-trivially related to \( \theta_h \).

It is natural to attempt a similar construction for the superstring. In this case, however, \( \theta \) can be eliminated by a chiral fermion rotation so there is no analog of figure 1. The analogous effect for black holes with discrete gauge hair was discussed in [gh]. However for a collection of neutral black holes characterized by different values of \( \theta \), there is no globally defined chiral rotation which eliminates all the \( \theta \)s. Thus we do expect quantum hair to arise, although the conformal field theoretic methods described herein are inadequate to describe it.

In closing we wish to discuss the observability of the axion hair in this solution. There is a limit, albeit artificial, in which it can be measured. This is the limit of string tree level, where we are doing conformal field theory in a fixed background. This limit is partly classical and partly quantum mechanical. The background fields do not fluctuate, but the propagation of test strings is quantum mechanical; for example, the vertex operators satisfy wave equations. A spherical world-sheet can loop the black hole, so interference effects from \( \theta \) will appear in tree-level amplitudes. Note also that because the background is classical the field satisfies \(< \Delta B >^2 = < (\Delta B)^2 >\). The left-hand side here is of two-instanton order. The right-hand side has both one- and two- instanton contributions, but the first enters at string loop order and so is suppressed in the limit discussed here.

Quantum effects make the axion hair difficult to measure for several reasons. In general \( \theta \) eigenstates are not energy eigenstates, and so the \( \theta \)-mode of the axion field will rapidly fluctuate (we believe this is equivalent to the arguments of [cpw]). In the present case this is exacerbated by the presence of the tachyonic mode.

In conclusion, under closer inspection axion hair turns out to be a toupee in bosonic string theory: it does not provide a new quantum label for stable, extremal black holes.
It remains a logical and interesting possibility that genuine quantum hair could exist in other contexts, such as superstring theories or discrete gauge symmetries.

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