Certain hyperbolic regular polygonal tiles are isoperimetric

Jack Hirsch · Kevin Li · Jackson Petty · Christopher Xue

Received: 12 October 2020 / Accepted: 27 January 2021 / Published online: 20 February 2021
© The Author(s), under exclusive licence to Springer Nature B.V. part of Springer Nature 2021

Abstract
The hexagon is the least-perimeter tile in the Euclidean plane. On hyperbolic surfaces, the isoperimetric problem differs for every given area. Cox conjectured that a regular \( k \)-gonal tile with 120-degree angles is isoperimetric for its area. We prove his conjecture and more.

Keywords Isoperimetric · Hyperbolic geometry · Tiling · Closed hyperbolic surfaces

Mathematics Subject Classification 52C20 · 51M09

Contents

1 Introduction ............................................... 65
   Methods .................................................. 66
2 Definitions ................................................ 67
3 Hyperbolic geometry ....................................... 67
4 Monohedral tilings of closed hyperbolic surfaces .............. 70
5 Regular polygonal tiles are isoperimetric ....................... 71
References .................................................. 77

1 Introduction

In 2001, Hales [10] proved that regular hexagons provide a least-perimeter tiling with possibly distinct or disconnected equal-area regions of the plane, and furthermore that no other such tiling of a flat torus does as well (Fig. 1). Efforts to generalize this result to hyperbolic surfaces have been unsuccessful (see Sect. 4). For monohedral tilings (by a single prototile) of a closed
Hales [10] proved that regular hexagons provide the least-perimeter equal-area tiling of the plane. Unlike Hales’s deep proof, our result does not require computers.

Moreover, our main result more generally treats multihedral tilings of closed hyperbolic surfaces with tiles of varying areas averaging $A_k = (k - 6)\pi/3$. For integer $k$, this is the area of $R_k$, the regular $k$-gon with $120^\circ$ angles. Let

$$P_k = 2k \cosh^{-1} \left( \frac{\cos(\pi/k)}{\sqrt{3}/2} \right).$$

Again for integer $k$, this is just the perimeter of $R_k$. Theorem 5.7 proves that the maximum perimeter of any tile in a given tiling must be at least $P_k$. It extends to all real $k > 6$, and in the case of integer $k$, if the maximum perimeter equals $P_k$, every tile must be equivalent to $R_k$.

**Theorem 5.7** For real $k > 6$, consider a curvilinear polygonal tiling of a closed hyperbolic surface with $N$ tiles of average area $A_k$, each with perimeter at most $P_k$. Then $k$ is an integer and every tile is equivalent to $R_k$.

**Methods**

To prove $R_k$ is the optimal tile of an appropriate closed hyperbolic surface, Proposition 3.7 first verifies that among $n$-gons (not necessarily tiles) of given area, the regular one minimizes perimeter. We seem to provide the first complete proof in the literature of this folk theorem, including Lemma 3.6 that the least-perimeter triangle of given area is isosceles. It follows easily that $R_k$ has less perimeter than all other $n$-gonal tiles for $n \leq k$. For $n > k$, Lemma 4.3, using the Gauss–Bonnet theorem, shows that in an $n$-gonal tiling, there are on average at most $k$ vertices of degree 3 or more per tile.

The main difficulty concerns nonconvex tiles with many sides. Cutting corners saves perimeter, but the resulting shape does not necessarily tile. Proposition 5.3 shows that the convex hulls of each tile’s vertices of degree at least 3 cover the surface, albeit with polygons generally of unequal areas and variable number of sides. By a new concavity Lemma 5.4, $k$-gons would enclose more area with the same perimeter, exhibiting a $k$-gon better than the regular $k$-gon, a contradiction.

Hales [10] remarks that, Fejes Tóth., who proved the honeycomb conjecture for convex tiles [14], predicted considerable difficulties for general tiles [13, p. 183] and said that the conjecture had resisted all attempts at proving it [15]. Removing the convexity hypothesis is the major advance of Hales’s work and of ours, although we focus on polygonal monohedral tilings. It remains an open question whether a hyperbolic multihedral tiling with areas $A_k$ could have less average perimeter than the regular polygon $R_k$ of area $A_k$ and angles $2\pi/3$. 
2 Definitions

Definition 2.1 (Tiling) Let $M$ be a closed Riemannian surface. A tiling of $M$ is an embedded multigraph on $M$ with no vertices of degree 0 or 1. A tiling is polygonal if

1. every edge is a geodesic;
2. every face is an open topological disk.

The oriented boundary of a face of a polygonal tiling is called a polygon. A tiling is monohedral if all faces are congruent. By default, faces of a tiling are not necessarily congruent, but in this case we call tilings multihedral for emphasis. We sometimes consider curvilinear polygonal tilings, relaxing condition (1).

Remark By definition our tilings are edge-to-edge. When tiling a closed surface with a tile, one copy might be edge-to-edge with itself. An example is tiling a hyperbolic two-holed torus with a single octagon. All eight vertices join at one point, and each edge coincides with another edge. A second example is tiling a one-holed torus by tiling the square fundamental region with thin vertical rectangles. Each rectangle is edge-to-edge with itself at top and bottom, and the two vertices of a vertical edge coincide.

All polygonal tilings are connected multigraphs as a consequence of (2).

Definition 2.2 (Equivalence) Two polygons $Q$ and $Q'$ are equivalent $Q \sim Q'$ if they are congruent after the removal of all vertices of measure $\pi$.

Remark We can’t in general define away vertices of measure $\pi$; a vertex in a tiling could, for example, have angles $\pi, \pi/2, \pi/2$.

Definition 2.3 (Convex Hull) Let $R$ be a polygonal region on a closed hyperbolic surface $M$.

The convex hull $H(R)$ is taken in the hyperbolic plane (with the minimal number of vertices). The convex hull of an $n$-gonal region $R$ is a $k$-gonal region for some $k \leq n$. The convex hull has no less area and no more perimeter.

Remark (Existence) By standard compactness arguments, there is a perimeter-minimizing tiling for prescribed areas summing to the area of the surface, except that polygons may bump up against themselves and each other, possibly with angles of measure 0 and $2\pi$, in the limit. We think that no such bumping occurs, but we have no proof.

3 Hyperbolic geometry

We begin with basic results of hyperbolic geometry. Proposition 3.7 seems to provide the first complete proof of the folk theorem that the regular hyperbolic $n$-gon has least perimeter among all $n$-gons of fixed area, based on the fact that the best triangle of given base and area is isosceles (Lemma 3.6). A key ingredient is the hyperbolic Heron’s formula (Proposition 3.5). Corollary 3.10 proves that the regular $k$-gon is optimal among polygons with $k$ or fewer sides.

Proposition 3.1 By the Gauss–Bonnet formula, an $n$-gon in the hyperbolic plane with interior angles $\theta_1, \ldots, \theta_n$ has area $(n-2)\pi - \sum \theta_i$. In particular, a regular $n$-gon with interior angle $\theta$ has area given by

$$A(n, \theta) = (n-2)\pi - n\theta.$$ (3.1.1)
**Proposition 3.2** (Law of Cosines) If $\ell$ is the length of the side opposing angle $\theta_3$ in a triangle with interior angles $\theta_i$, then

$$\cos \theta_3 = \sin \theta_1 \sin \theta_2 \cosh \ell - \cos \theta_1 \cos \theta_2.$$  

In particular, for right triangle $\triangle ABC$ with legs $a, b$,

$$\cosh (a) = \cos (\angle A) / \sin (\angle B).$$

**Proposition 3.3** A regular $n$-gon with interior angle $\theta$ has perimeter given by

$$P(n, \theta) = 2n \cosh^{-1} \left( \frac{\cos (\pi/n)}{\sin (\theta/2)} \right).$$ (3.3.1)

**Proof** Connect the center of the regular $n$-gon to each of its vertices to form $n$ isosceles triangles. Bisect each triangle into two right triangles by connecting the center of the polygon to the midpoint of each side of the polygon. Each triangle has interior angles $\pi/2, \pi/n$, and $\theta/2$. By Proposition 3.2, the length of the leg on the polygonal side of each of the $2n$ right triangles is $\cosh^{-1}(\cos (\pi/n) / \sin (\theta/2))$. \qed

**Definition 3.4** For real $k > 6$, let $A_k = A(k, 2\pi/3) = (k - 6)\pi/3$ and $P_k = P(k, 2\pi/3)$, extending the area and perimeter of the regular $k$-gon $R_k$ with angles $2\pi/3$ to real values of $k$. Note that $A_k$ and $P_k$ increase from 0 to $\infty$ as $k$ increases from 6 to $\infty$.

The hyperbolic version of Heron’s formula gives the areas of hyperbolic triangles in terms of their side lengths.

**Proposition 3.5** (Heron’s formula [16]) For a triangle in $\mathbb{H}^2$ with sides $x, y, z$, the area $A$ satisfies

$$\tan^2 \frac{A}{2} = \frac{1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2 \cosh x \cosh y \cosh z}{(1 + \cosh x + \cosh y + \cosh z)^2}.$$  

Carroll et al. [2] provide the following simple proof that among hyperbolic $k$-gons of given area, the regular one minimizes perimeter. The previously published proof by Bezdek [1] used without proof the nontrivial fact (Lemma 3.6) that for given base and area, an isosceles triangle minimizes perimeter. Carroll et al. (Proposition 2.5) deduced this fact from Heron’s formula, though their statement of Heron’s formula was not quite right. In 2016, in discussions with Steve Openshaw, Colin Adams—unaware of the Carroll et al. proof—produced a longer geometric proof (private communication).

**Lemma 3.6** For fixed base and perimeter, the isosceles triangle uniquely maximizes area in $\mathbb{H}^2$.

**Proof** Consider a triangle with side lengths $x, y, z$. By Proposition 3.5,

$$\tan^2 \frac{A}{2} = \frac{1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2 \cosh x \cosh y \cosh z}{(1 + \cosh x + \cosh y + \cosh z)^2},$$

where $A$ is area. Fixing the base $z$,

$$\tan^2 \frac{A}{2} = \frac{a - \cosh^2 x - \cosh^2 y + 2m \cosh x \cosh y}{(b + \cosh x + \cosh y)^2}$$ (3.6.1)
for constants $a$, $b$, and $m = \cosh(z)$. Fix $x + y = 2c$, thereby fixing perimeter. It is possible to simultaneously maximize the numerator and minimize the denominator (which are both positive). The numerator is maximized by maximizing

$$F(x) = 2m \cosh x \cosh(2c - x) - \cosh^2 x - \cosh^2(2c - x).$$

A short computation and simplification makes the critical equation

$$F'(x) = 4(\cosh(2c) - m) \sinh(c - x) \cosh(c - x) = 0.$$

Observe $0 < \cosh(c - x)$. Also, by the triangle inequality, $z < 2c$ so $m < \cosh(2c)$. Thus $\sinh(c - x) = 0$, which means $x = c$. This critical point is the unique global maximum as the derivative is positive for $0 < x < c$ and negative for $c < x < 2c$.

To minimize the denominator of Eq. (3.6.1), set $x = c = y$. Therefore Eq. (3.6.1), and thus area, is uniquely maximized for $x = y$, that is, when the triangle is isosceles.

\[ \square \]

**Proposition 3.7** [2, Proposition 2.5] In the hyperbolic plane, the regular $n$-gon $Q_n$ has less perimeter than any other $n$-gon $Q$ of the same area.

**Proof** First we show that the optimal $n$-gon $Q$ must be convex and equilateral. For fixed perimeter $P$, an area-maximizing $Q$, as the convex hull of $n$ points, exists by a standard compactness argument. If it has fewer than $n$ vertices, place extra vertices on one of the sides.

By Lemma 3.6, any two adjacent sides must be of equal length, ignoring the extra vertices. Now add one of the extra vertices. Repeating the argument with a segment bounded by the vertex and the following adjacent edge shows that there are no extra vertices. Therefore $Q$ is a convex equilateral $n$-gon.

Finally, assume $Q$ is not regular. Inscribe the regular $n$-gon $Q_n$ with the same edge lengths in a circle. Adding the little region between each edge of $Q_n$ and the circle to each edge of $Q$ would yield another region with the same perimeter as the circle and at least as much area, a contradiction.

\[ \square \]

The following monotonicity result is generalized to noninteger $n$ in Lemma 5.4.

**Proposition 3.8** The perimeter of a regular $n$-gon for a fixed area is decreasing as a function of $n$.

**Proof** Let $Q_n$ and $Q_{n+1}$ be the regular polygons of a fixed area with $n$ and $n + 1$ sides. Let $Q'_{n+1}$ be an $(n + 1)$-gon formed by adding a vertex of measure $\pi$ to $Q_n$. By Proposition 3.7,

$$P(Q_{n+1}) < P(Q'_{n+1}) = P(Q_n).$$

\[ \square \]

**Remark** As expected, the perimeter of a regular $n$-gon of area $A$ is increasing as a function of $A$, for $0 < A < (n - 2)\pi$. Indeed, by Propositions 3.1 and 3.3, the perimeter of the $n$-gon is

$$2n \cosh^{-1}\left(\frac{\cos \pi/n}{\sin(((n - 2)\pi - A)/2n)}\right),$$

and it is increasing because $\cosh^{-1}$ and sine are increasing over $(0, \infty)$ and $(0, \pi/2)$, respectively.
Corollary 3.9  The regular $k$-gon has less perimeter than any other $n$-gon of equal or greater area for $3 \leq n \leq k$.

Proof  The corollary follows immediately from Propositions 3.7 and 3.8.

Corollary 3.10  Tile a closed hyperbolic surface by polygons of equal area with $k$ or fewer sides. Then each of those tiles has perimeter at least that of the regular $k$-gon of the same area.

Proof  The corollary follows immediately from Corollary 3.9.

4 Monohedral tilings of closed hyperbolic surfaces

In 2005, Cox [4,5] and subsequently Šešum [12] proposed generalizing Hales’s hexagonal isoperimetric inequality to prove that a tiling by regular $k$-gons $R_k$ with $120^\circ$ angles ($k \geq 7$) minimizes perimeter among all (possibly multihedral) tilings of an appropriate closed hyperbolic surface. Carroll et al. [2] showed that their proposed polygonal isoperimetric inequality fails for $k > 66$. Corollary 5.8 independently proves $R_k$ optimal for monohedral tilings. Although Corollary 5.8 applies even if the regular polygon does not tile, Proposition 4.1 shows there are many closed hyperbolic surfaces which it does tile. It is possible for many-sided polygons to tile, but Proposition 4.4 shows that as $n$ increases, $n$-gonal tiles necessarily have many concave angles. Corollary 4.5 deduces that the regular polygon has less perimeter than any other convex polygonal tile.

Remark  By Gauss–Bonnet, the regular $k$-gon $R_k$ of area $A_k = (k-6)\pi/3$ ($k \geq 7$) has interior angles of $2\pi/3$ (Sect. 3). It therefore tiles $H^2$, as well as many closed hyperbolic surfaces (Proposition 4.1). For area not a multiple of $\pi/3$, there is no conjectured isoperimetric tile.

That every other tile of the same area has more perimeter than $R_k$ was known in the special case that the surface has area $A_k$, so that a single tile covers the whole surface. Choe [3, p. 653] proved the existence of such an isoperimetric single tile and shows that it is a polygon with $120^\circ$ interior angles. For example, the isoperimetric single tile on a flat torus is a $120^\circ$-angle hexagon (not a parallelogram) and always has at least the perimeter of the regular hexagon. On a closed hyperbolic surface of genus $g$, the isoperimetric single tile $T$ is a $120^\circ$-angle $(12g - 6)$-gon and always has at least the perimeter of the regular $(12g - 6)$-gon.

Proposition 4.1  For $k \geq 7$, there exist infinitely many closed hyperbolic surfaces tiled by the regular $k$-gon $R_k$ of area $A_k = (k-6)\pi/3$ and angles $2\pi/3$.

Proof  These surfaces are provided by work of Edmonds et al. [7, Main theorem] on torsion-free subgroups of Fuchsian groups and tessellations (see also [8,9]). Their work yields torsion-free subgroups $S$ of arbitrarily large finite index of the triangle group $(2, 3, k)$. This triangle group is the orientation-preserving symmetry group of the hyperbolic triangle of angles $\pi/2$, $\pi/3$, and $\pi/k$. Each quotient of $H^2$ by such a subgroup $S$ is a closed hyperbolic surface tiled by these triangles, which can be joined in groups of $2k$ to form a tiling by the regular $k$-gon of area $(k-6)\pi/3$ and hence angles $2\pi/3$ (by Gauss–Bonnet).

Example 4.2  The Klein Quartic Curve in $CP^2$ [11] is the set of complex solutions to the homogeneous equation

$$u^3 v + v^3 w + w^3 u = 0.$$  

The curve is a hyperbolic 3-holed torus. It is famously tiled by 24 regular heptagons.
The following results are instrumental in eliminating competing \( n \)-gons of large \( n \).

**Lemma 4.3** Consider a tiling of a closed hyperbolic surface by curvilinear polygons \( Q_i \) of average area \( A_k = (k - 6)\pi/3 \) for some real \( k > 6 \). Then each polygon has on average at most \( k \) vertices of degree at least 3, with equality if and only if every vertex has degree two or three.

**Proof** A tile with \( n \) edges and \( v \) vertices of degree at least 3 contributes to the tiling 1 face, \( n/2 \) edges, and at most \((n - v)/2 + v/3 \) vertices, with equality precisely if no vertices have degree greater than 3. Therefore it adds at most \( 1 - v/6 \) to the Euler characteristic \( F - E + V \). The Gauss–Bonnet theorem says that

\[
\int G = 2\pi(F - E + V).
\]

Hence the average contributions per tile satisfy

\[-A_k = -(k - 6)\pi/3 \leq 2\pi(1 - \bar{v}/6).
\]

Therefore \( \bar{v} \leq k \), with equality if and only if no vertices have degree more than 3. \( \square \)

**Proposition 4.4** Let \( Q \) be an \( n \)-gon of arbitrary area \( A_k = (k - 6)\pi/3 \) (real \( k > 6 \)) with \( \ell_1 \) (interior) angles of measure \( \pi \) and \( \ell_2 \) of measure greater than \( \pi \). If \( Q \) tiles a closed hyperbolic surface \( M \), then \( \ell_1 + 2\ell_2 \geq n - k \). Equality holds for a tiling (and therefore every tiling) if and only if every vertex is of degree two or three, and every concave angle has degree two.

**Proof** Take any tiling of \( M \) by \( Q \). Each vertex of degree two in the tiling has either two angles of measure \( \pi \) or exactly one angle of measure greater than \( \pi \). By Lemma 4.3,

\[
\ell_1 + 2\ell_2 \geq n - k,
\]

with equality precisely when every vertex has degree two or three, and every concave angle has degree 2. \( \square \)

The following corollary proves Corollary 5.8 among convex polygonal tiles.

**Corollary 4.5** The regular \( k \)-gon \( R_k \) has less perimeter than any non-equivalent convex polygonal tile of area \( A_k = (k - 6)\pi/3 \).

**Proof** Let \( Q \) be a convex \( n \)-gonal tile of area \( A_k \). By Proposition 4.4, \( Q \) contains at least \( n - k \) angles of measure \( \pi \). Hence \( Q \) is equivalent to a polygon with at most \( k \) sides. Unless \( Q \) is equivalent to \( R_k \), \( Q \) has strictly more perimeter by Corollary 3.9. \( \square \)

**Remark** Although it is easy to show that an isoperimetric curvilinear triangular tile must actually be polygonal by straightening the edges, an extension to all curvilinear \( k \)-gons remains conjectural because straightening one edge of a tile might cause it to intersect another part of the tile.

**5 Regular polygonal tiles are isoperimetric**

Our main result, Theorem 5.7, proves that regular \( k \)-gons \( R_k \) of area \( A_k = (k - 6)\pi/3 \) (with 120° angles and perimeter \( P_k \)) are optimal, even when they don’t tile. It provides
similar estimates for interpolated areas. It also allows for multihedral tilings, showing that the maximum perimeter of such tiles is greater than or equal to \( P_k \).

The main difficulty concerns nonconvex tiles with many sides. Cutting corners saves perimeter, but the resulting shape does not necessarily tile. Proposition 5.3 shows that the collection of convex hulls of each tile’s vertices of degree at least 3 covers the surface, although generally with polygons of unequal areas and variable number of sides. Fortunately, by Gauss–Bonnet, the average number of sides is at most \( k \) (Proposition 5.3). By a new concavity Lemma 5.4, the \( k \)-gons enclose more area with the same perimeter, exhibiting a \( k \)-gon better than \( R_k \), a contradiction.

To ensure that the convex hulls of the high-degree vertices cover, we start with straightening and flattening processes for curvy edges and degree-2 vertices.

**Definition 5.1 (Flattening)** Consider a polygonal chain \( ABC \) in \( H^2 \). To flatten vertex \( B \) is to replace \( ABC \) with the geodesic \( AC \). For a hyperbolic surface, flattening is done in the universal cover \( H^2 \).

**Lemma 5.2** Consider immersed curvilinear polygons \( P \) and \( Q \) in a hyperbolic surface that share either a vertex \( V \) and the incident edges or an edge. Replacing the edge with a geodesic or flattening \( V \) in the covering \( H^2 \) yields immersed curvilinear polygons whose union contains \( P \) and \( Q \).

**Proof** Let \( A \) and \( B \) be the adjacent vertices of \( V \). Let \( R \) be the region enclosed by the new geodesic and the edges it replaced. Note that the union of of the resulting polygons is simply \( P \cup Q \cup R \). \( \square \)

**Proposition 5.3** Let \( M \) be a closed hyperbolic surface tiled by curvilinear polygons \( Q_i \) of average area \( A_k = (k - 6)\pi/3 \) for real \( k > 6 \). Let \( Q_i^* \) be the convex hull of the vertices of degree three or higher of \( Q_i \). Then \( \{Q_i^*\} \) covers \( M \) and the average number of sides is less than or equal to \( k \).

**Proof** By Lemma 5.2, straightening edges and flattening all degree-2 vertices yields a covering by immersed polygons, each covered by the corresponding \( Q_i^* \). Hence \( \{Q_i^*\} \) covers \( M \). By Lemma 4.3, the average number of sides is less than or equal to \( k \). \( \square \)

**Remark** For fixed \( n \), every tile in a tiling by curvilinear \( n \)-gons of a connected closed surface, other than a sphere or \( \mathbb{R}P^2 \), has at least two vertices of degree at least 3. Indeed, suppose a tile has fewer than two vertices of degree at least 3. Such a tile must share all edges with itself or another tile (and actually has no vertices of degree at least 3). Since the surface is connected, there are no other tiles, and the surface is a sphere or \( \mathbb{R}P^2 \).

**Remark** Figure 2 illustrates an unbounded example in which the convex hulls of each tile’s vertices of degree at least three do not cover the surface.

The concavity of the following area function for fixed perimeter is a crucial ingredient in the proof of the main result, Theorem 5.7.

**Lemma 5.4** The area of the regular \( n \)-gon with perimeter \( P \) is given by

\[
A(n) = \pi(n - 2) - 2n \sin^{-1}(\cos \alpha \operatorname{sech} \beta)
\]

where \( \alpha = \pi/n \) and \( \beta = P/2n \). The function \( A(n) \) is strictly increasing and strictly concave on \([2, \infty)\). We extend \( A(n) \) continuously to be identically 0 on the interval \([0, 2]\).
Fig. 2  A tiling of the Euclidean plane by polygons of equal area, in which all vertices of degree three or more (here marked by dots) are collinear. The convex hull of these vertices is just a line, and certainly does not cover the plane.

**Remark**  For nonintegral $n$, Eq. (5.4.1) still holds when Eqs. (3.1.1) and (3.3.1) for $A(n, \theta)$ and $P(n, \theta)$ hold.

**Proof**  By Proposition 3.3, the perimeter of a regular $n$-gon with interior angle $\theta$ is given by

$$P(n, \theta) = 2n \cosh^{-1}\left(\frac{\cos(\pi/n)}{\sin(\theta/2)}\right),$$

which increases from 0 to $\infty$ as $\theta$ decreases from $(n - 2)\pi/n$ to 0. Solve for $\theta$ in the range $0 < \theta < \pi$ to find

$$\theta = 2 \sin^{-1}(\cos \alpha \sech \beta).$$

Equation (5.4.1) now follows from Proposition 3.1 (3.1.1). To prove that $A(n)$ is strictly concave, remove a trivially negative factor from the second derivative $A''(n)$ and simplify, reducing the problem to showing that

$$P^2 \tanh^2 \beta + (4\pi^2 - P^2) \sech^2 \beta + P^2 \cos^2 \alpha \sech^4 \beta + 4\pi P \tan \alpha \tanh \beta - 4\pi^2$$

is positive for $n > 2$ and $P > 0$. Substituting $T = \tan \alpha$ and $H = \tanh \beta$, rearranging terms, and simplifying give that it is sufficient to prove

$$P \left(H^2 + T^2\right) - \sqrt{1 + T^2} \cdot (PT - 2\pi H)$$

is positive. Since it vanishes at $P = 0$, it suffices to show that the derivative with respect to $P$,

$$\tanh^2(\alpha \beta) + \tan^2(\alpha) + 2\alpha \beta \tanh(\alpha \beta) \sech^2(\alpha \beta) - \sec(\alpha) \left(\tan(\alpha) - \alpha \sech^2(\alpha \beta)\right),$$

is positive for $0 < \alpha < \pi/2$ and $\beta > 0$. Substituting $c = \tanh^2(\alpha \beta)$ and simplifying reduces to showing that

$$c + \frac{\alpha}{\cos \alpha}(1 - c) > \frac{\sin \alpha}{1 + \sin \alpha}$$

for $0 < c < 1$ and $0 < \alpha < \pi/2$, which holds trivially.

Finally, strict monotonicity of $A(n)$ follows from strict concavity, since $A(n)$ remains positive for $n > 2$. \qed

The following lemma and corollary are needed in the proof of the main Theorem 5.7 to handle the interval $[0, 2)$ not covered by Lemma 5.4.

**Lemma 5.5**  Fix real $k > 6$. Consider $A(n)$ with fixed perimeter $P_k$. Then

$$A(k) < 2A\left(\frac{k}{2}\right).$$
Proof Let $\gamma = \cos(\pi/k)$, so $\sqrt{3}/2 < \gamma < 1$. By Eq. (3.3.1),
\[
\cosh\left(\frac{P}{2k}\right) = \frac{\cos(\pi/k)}{\sin(\pi/3)} = \frac{2\gamma}{\sqrt{3}}.
\]
By Eq. (3.1.1) and the double angle identities,
\[
A(k) = (k - 2)\pi - \frac{2k\pi}{3},
\]
\[
2A\left(\frac{k}{2}\right) = (k - 4)\pi - 2k\sin^{-1}\left(\frac{2\gamma^2 - 1}{8\gamma^2/3 - 1}\right).
\]
Algebraic manipulation shows the desired inequality is
\[
\sin^{-1}\left(\frac{2\gamma^2 - 1}{8\gamma^2/3 - 1}\right) < \frac{\pi}{3} - \frac{\pi}{k}.
\]
Both sides lie in the interval $[-\pi/2, \pi/2]$, over which sine is increasing. Thus it is equivalent to show
\[
\left(\frac{2\gamma^2 - 1}{8\gamma^2/3 - 1}\right) < \sin\left(\frac{\pi}{3} - \frac{\pi}{k}\right) = \frac{1}{2}\left(\gamma\sqrt{3} - \sqrt{1 - \gamma^2}\right).
\]
Equality is attained at $\gamma = \cos(\pi/6) = \sqrt{3}/2$, and the inequality is trivial at $\gamma = 1$. It thus suffices to show equality is never attained in $(\sqrt{3}/2, 1)$; there are many ways to do so, one of which we use here. After rearrangement, equality holds only at the roots of the equation
\[
\left(2(2\gamma^2 - 1) - \gamma\sqrt{3}(8\alpha^2/3 - 1)\right)^2 = \left((8\gamma^2/3 - 1)\cdot\sqrt{1 - \gamma^2}\right)^2,
\]
and so only at the roots of the sixth degree polynomial
\[
256\gamma^6 - 192\sqrt{3}\gamma^5 - 112\gamma^4 + 168\sqrt{3}\gamma^3 - 60\gamma^2 - 36\sqrt{3}\gamma + 27.
\]
The first through sixth derivatives of this polynomial, evaluated at $\gamma = \sqrt{3}/2$, are all positive:

\[
6\sqrt{3}, \; 384, \; 2554\sqrt{3}, \; 31,872, \; 69,120\sqrt{3}, \; 184,320.
\]
Since the sixth derivative is constant, they remain positive. Hence, equality is never attained in $(\sqrt{3}/2, 1)$, and so the desired strict inequality for $k > 6$ follows.

Corollary 5.6 Fix real $k > 6$. Consider $A(n)$ with fixed perimeter $P_k$. For all real $n \geq k$,
\[
A(n) < 2A\left(\frac{n}{2}\right).
\]
Proof By Lemma 5.5,
\[
A(k) < 2A\left(\frac{k}{2}\right).
\]
Since $A$ is strictly concave on $[2, \infty) \supset [k/2, \infty)$ and is strictly increasing,
\[
A(n) = A(k) + (A(n) - A(k)) < 2A\left(\frac{k}{2}\right) + 2\left(A\left(\frac{n}{2}\right) - A\left(\frac{k}{2}\right)\right) = 2A\left(\frac{n}{2}\right).
\]
\[\square\]
Recall that $A_k$ and $P_k$ are the area and perimeter of the regular polygon $R_k$ with $120^\circ$ angles, extended formulaically to all real $k > 6$ and increasing in $k$ (Definition 3.4). Our main theorem shows that as $k$ ranges from 6 to $\infty$ and the average area $A_k$ ranges from 0 to $\infty$, some tile must have perimeter at least $P_k$, with equality only if $k$ is an integer and every tile is equivalent to the regular $k$-gon $R_k$.

**Theorem 5.7** For real $k > 6$, consider a curvilinear polygonal tiling of a closed hyperbolic surface with $N$ tiles of average area $A_k$, each with perimeter at most $P_k$. Then $k$ is an integer and every tile is equivalent to $R_k$.

**Proof** By Proposition 5.3, the collection of convex hulls $Q_i^*$ of the vertices with degree at least 3 on each tile covers $M$, and of course $P(Q_i^*) \leq P(Q_i) \leq P_k$ by assumption. Since the $Q_i^*$ cover,

$$\frac{1}{N} \sum \text{Area}(Q_i^*) \geq A_k. \quad (5.7.1)$$

By Proposition 5.3, the number of sides $n_i$ of $Q_i^*$ satisfy

$$\frac{1}{N} \sum n_i \leq k.$$

The areas can be estimated in terms of $A(n)$ for $P_k$ as

$$\sum \text{Area}(Q_i^*) \leq \sum A(n_i) \leq N \cdot A\left(\frac{\sum n_i}{N}\right) \leq N \cdot A(k) = N \cdot A_k. \quad (5.7.2)$$

The first inequality follows from Proposition 3.7 and the remark after Proposition 3.8. The second inequality follows from the concavity of $A(n)$ for $n \geq 2$ (Lemma 5.4) and Jensen’s inequality. If any of the $n_i$ are 0 or 1, choose some $n_i > k$, and use Corollary 5.6 first to replace $0 + A(n_i)$ with $2A(n_i/2)$. If you run out of large enough $n_i$, the next inequality holds already. The third inequality follows from the fact that $A(n)$ is strictly increasing (again Lemma 5.4). The final equality holds by the definition of $A(n)$ for $P_k$.

By Eq. 5.7.1, equality must hold in every inequality. By the strict concavity of $A(n)$, equality in the second inequality implies that every $n_i = k$, which must therefore be an integer. Equality in the first inequality implies that every $Q_i^*$ has area $A$. By Proposition 3.7, $Q_i^*$ is the regular $k$-gon $R_k$ of area $A_k$. Finally

$$P(Q_i) \geq P(Q_i^*) = P_k,$$

and equality implies that $Q_i \sim R_k$.

Theorem 5.7 immediately implies the following corollary on monohedral tilings.

**Corollary 5.8** (Monohedral tilings) For $k \geq 7$, any non-equivalent tile of area $A_k = (k - 6)\pi/3$ of a closed hyperbolic surface has more perimeter than the regular $k$-gon $R_k$ (whether or not $R_k$ tiles).

**Remark** It remains an open question whether Corollary 5.8 extends to the hyperbolic plane, where matching discrepancies might be pushed off to infinity. Similarly considering large regions does not work, because truncation effects are too large.

The following proposition shows that in some sense the area of the regular hexagon $R_k$ increases more rapidly than the perimeter as the number of sides increases.
**Proposition 5.9** (Perimeter ratio) For real $k > 6$, $P_k/A_k$ is a (strictly) decreasing function of $k$.

**Proof** By Propositions 3.1 and 3.3, in terms of $x = \pi/6 - \pi/k$,

$$\frac{\pi^2 A_k}{9 P_k} = \frac{x}{\cosh^{-1} \left( \frac{\cos(\pi/6-x)}{\sin(\pi/3)} \right)}.$$

It suffices to show that the right hand side is strictly increasing in $x$ for $0 < x < 1$. By Wolfram Alpha, its derivative is given by

$$\frac{x \sin \left( \frac{\pi}{3} - x \right)}{\sin(\pi/3) \cos^{-1} \left( \frac{\cos \left( \frac{\pi}{6} - x \right)}{\sin(\pi/3)} \right)} + \frac{1}{\cosh^{-1} \left( \frac{\cos \left( \frac{\pi}{6} - x \right)}{\sin(\pi/3)} \right)},$$

which is positive for $0 < x < 1$.

Hence, $P_k/A_k$ is strictly decreasing for $k > 6$. \hfill \square

The following corollary shows in particular that reducing the area per tile of a monohedral tiling increases the total perimeter.

**Corollary 5.10** (Total perimeter) A tiling of a closed hyperbolic surface by $R_k$ has less total perimeter than any nonequivalent tiling by polygons of equal perimeter $P$ and average area $A_m \leq A_k$.

**Proof** Let $A$ denote the total area of the surface. By Theorem 5.7 and Proposition 5.9, the competing total perimeter is greater than or equal to the total perimeter of the $R_k$ tiling:

$$P \frac{A}{A_m} \geq P_m \frac{A}{A_m} \geq P_k \frac{A}{A_k},$$

with equality only if all the tiles are equivalent to $R_k$. \hfill \square

Our methods more easily yield the following weak version of Hales’s hexagonal honeycomb theorem [10]. For details, see Proposition 10.5 of Di Giosia et al. [6].

**Proposition 5.11** (Euclidean Hexagons) Consider a curvilinear polygonal tiling of a flat torus with tiles of average area $A$. Then some tile has at least as much perimeter as the regular hexagon $R_6$ of area $A$, with equality only if every tile is equivalent to $R_6$.

**Acknowledgements** This work is a product of the 2019 Summer Undergraduate Mathematics Research program at Yale (SUMRY) under the guidance of Frank Morgan of Williams College. The authors greatly thank Morgan for his help and insight over the many weeks spent researching and writing this paper. We thank the Young Mathematicians Conference (YMC) and Yale for supporting our trip to present at the 2019 YMC in Columbus, Ohio.

**Funding** The research was supported by 2019 Summer Undergraduate Mathematics Research program at Yale (SUMRY).

**Availability of data and materials** The manuscript has no data.

**Compliance with ethical standards**

**Conflict of interest** All authors declare that they have no conflict of interest.

**Code availability** The manuscript has no code.
References

1. Bezdek, K.: Ein elementarer Beweis für die isoperimetrische Ungleichung in der Euklidischen und hyperbolischen Ebene. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 27, 107–112 (1984)
2. Carroll, C., Jacob, A., Quinn, C., Walters, R.: On generalizing the Honeycomb theorem to compact hyperbolic manifolds and the sphere. Report of SMALL Geometry Group’06, Williams College (Aug. 2006). https://sites.williams.edu/Morgan/ongeneralizing-the-honeycomb-theorem-to-compact-hyperbolic-manifolds-and-the-sphere/
3. Choe, J.: On the existence and regularity of fundamental domains with least boundary area. J. Differ. Geom. 29(3), 623–663 (1989). https://doi.org/10.4310/jdg/1214443065
4. Cox, C.: The Honeycomb problem on hyperbolic surfaces (Aug. 2005). https://sites.williams.edu/Morgan/the-honeycomb-problem-on-hyperbolic-surfaces/
5. Cox, C.: The Honeycomb problem on hyperbolic surfaces (July 2011). https://sites.williams.edu/Morgan/the-honeycomb-problem-on-hyperbolic-surfaces/
6. Di Giosia, L., Habib, J., Hirsch, J., Kenigsberg, L., Li, K., Pittman, D., Petty, J., Xue, C., Zhu, W.: Optimal monohedral tilings of hyperbolic surfaces. arXiv e-prints arXiv:1911.04476 [math.MG] (Nov. 2019)
7. Edmonds, A.L., Ewing, J.H., Kulkarni, R.S.: Regular tessellations of surfaces and (p, q, 2)-triangle groups. Ann. Math. 116(1), 113–132 (1982)
8. Edmonds, A.L., Ewing, J.H., Kulkarni, R.S.: Torsion free subgroups of Fuchsian groups and tessellations of surfaces. Bull. Am. Math. Soc. 6(3), 456–458 (1982). https://doi.org/10.1090/S0273-0979-1982-15014-5
9. Edmonds, A.L., Ewing, J.H., Kulkarni, R.S.: Torsion free subgroups of Fuchsian groups and tessellations of surfaces. Invent. Math. 69(3), 331–346 (1982). https://doi.org/10.1007/BF01389358
10. Hales, T.C.: The Honeycomb conjecture. Discrete Comput. Geom. 25(1), 1–22 (2001). https://doi.org/10.1007/s004540010071
11. Klein, F.: Ueber die Transformation siebenter Ordnung der elliptischen Functionen. Math. Ann. 4, 428–471 (1878). https://gdz.sub.uni-goettingen.de/download/pdf/PPN235181684_0014/PPN235181684_0014.pdf, Trans. by Silvio Levy as “On the order-seven transformation of elliptic functions.” In: The Eightfold Way: The Beauty of Klein’s Quartic Curve. Vol. 35 (1998). Chap. 9, pp. 287–331. http://library.msri.org/books/Book35/files/klein.pdf
12. Šešum, V.: The Honeycomb Problem on Hyperbolic Surfaces. Undergraduate Thesis. Williams College (May 2006). https://sites.williams.edu/Morgan/the-honeycombproblem-on-hyperbolic-surfaces-by-vojislav-sesum/
13. Tóth, L.F.: Regular Figures. Vol. 48. International Series of Monographs on Pure and Applied Mathematics. The Macmillan Company, New York (1964)
14. Tóth, L.F.: Über das kürzeste Kurvennetz das eine Kugeloberfläche in flächengleiche konvexe Teil zerlegt. Mat. Term. tud. Ertesítő 62, 349–354 (1943)
15. Tóth, L.F.: What the bees know and what they do not know. Bull. Am. Math. Soc. 70(4), 468–481 (1964)
16. University of Glasgow. Hyperbolic Area—Heron’s Formula. https://www.maths.gla.ac.uk/wws/cabripages/hyperbolic/harea2.html

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.