Oblique correction in a walking lattice theory

Thomas DeGrand

Department of Physics, University of Colorado, Boulder, CO 80309, USA

Abstract

I compute the difference of vector and axial vector current correlators in the weak coupling phase of (lattice-regulated) SU(3) gauge theory with two flavors of symmetric-representation dynamical fermions. This is a walking theory at the bare parameter values chosen for the simulation. Otherwise, it is not a conventional technicolor candidate. The correlator difference shows scaling behavior in the fermion mass, and vanishes in the fermion zero mass limit. Consequences for the phenomenology of similar systems which might be candidates for beyond Standard Model physics are discussed. I check my methodology against ordinary QCD, by computing the Gasser-Leutwyler coefficient $\mathcal{T}_{10}$ and the charged - neutral pion mass difference from an approximate parameterization of the correlator.
I. INTRODUCTION AND BACKGROUND

We hope to discover new physics at the Large Hadron Collider. But even if a direct observation of physics beyond the Standard Model is difficult, it still may be possible to detect new physics through its effect on Standard Model processes, via precision electroweak measurements. One classic observable is the S-parameter of Peskin and Takeuchi[1]. In the simple system I will analyze, it is related to the difference between the momentum space vector and axial current correlators at zero momentum

\[ \Pi_{\mu\nu}(q) = \int d^4q \exp(\text{i}qx) \langle J^L_\mu(x)J^R_\nu(0) \rangle \]

\[ \equiv (q^2\delta_{\mu\nu} - q_\mu q_\nu)\Pi_{LR}^T(q^2) + q_\mu q_\nu \Pi_{LR}^L(q^2). \]

The S-parameter is proportional to the limiting value of \( d(q^2\Pi_{LR}^T(q^2))/dq^2 \) at small \( q^2 \), after Goldstone boson effects are subtracted.

Recently, many groups have begun to use lattice methods to study candidate beyond-Standard Model systems which replace the fundamental Higgs field by new, nonperturbative physics [2]. These models may be realizations of technicolor (for reviews, see Ref. [3]), or they may correspond to “hidden sector”[4] or “unparticle” systems [5] where the new physics is approximately conformal. To decide whether any particular model is viable, it is necessary to compute observables. This paper describes the measurement of \( \Pi_{LR}^T(q^2) \) for one candidate theory, \( SU(3) \) gauge fields coupled to \( N_f = 2 \) flavors of fermions in the symmetric (sextet) representation. Before lattice simulations began, this model was one of many candidate theories for walking technicolor [6–9]. Previous lattice simulations of this system include Refs. [10–15]. In this system, \( \Pi_{\mu\nu}(q) \) involves the correlation function of two fermionic (\( \psi \)) bilinears. In my conventions \( J^L_\mu = \bar{\psi}\gamma_\mu(1-\gamma_5)/2\psi \) and \( J^R_\mu = \bar{\psi}\gamma_\mu(1+\gamma_5)/2\psi \).

The specific lattice calculation is done at a set of bare parameters at which the gauge coupling runs slowly or “walks”: the theory is approximately conformal in the zero fermion mass limit. Tuning the fermion mass away from zero explicitly breaks conformal symmetry. What happens to \( \Pi_{LR}^T(q^2) \) in this situation is, as far as I know, unexplored. Unfortunately, this system is not a candidate for conventional technicolor. At the parameter value where I did the simulations, it shows no sign of spontaneous chiral symmetry breaking.

There is some discussion in the literature of \( \Pi_{LR}^T(q^2) \) for technicolor candidates which have slowly running or “walking” couplings, or which are conformal [16]. In these systems, conformal symmetry is is usually broken, either explicitly or through the coupling of the new physics sector to the Standard Model. Here, the new physics at very short distance which generates a fermion mass, is a bare fermion mass in the lattice action. The question is then, what happens to \( \Pi_{LR}^T(q^2) \) as the mass is tuned to zero. The expectation is that \( \Pi_{LR}^T(q^2) \) falls to zero in that limit. A second expectation often seen in the Beyond Standard Model literature is that \( \Pi_{LR}^L \) can be computed by saturation by a few light resonances in the appropriate channels, by tuning their masses and couplings. I will test these expectations.

By “slowly running” I mean that a suitably defined running coupling constant shows small variation, over the range of length scales accessible to a lattice simulation in finite volume at a fixed value of its bare couplings. In Ref. [13], this is done using a lattice version of the background field method, in which the system size \( L \) represents the scale at which a running coupling \( g^2(L) \) is measured. (It is called the Schrödinger functional method. Part of the extensive literature of the Schrödinger functional include Refs. [17–20].) It was observed
that the coupling ran more slowly than perturbation theory predicted. At one loop, this is
\[
\frac{1}{g^2(sL)} = \frac{2b_1}{16\pi^2} \log sL + \text{constant},
\]  
(2)

where \( b_1 = 13/3 \) for this theory. For a scale factor \( s = 2 \), this is a change in \( 1/g^2(sL) \) of about 0.038, compared to the measured value at this simulation’s bare coupling of \( 1/g^2(L) \) of about 0.38.

This is not what happens in ordinary QCD. With two flavors of fundamental representation fermions, \( b_1 = 29/3 \) and in perturbation theory the change in the inverse coupling over a scale factor of two is about 0.084. In practice, at coupling values used in standard QCD simulations, the change is much greater. (Ref. [20] compares their Schrödinger functional coupling to perturbation theory.) In fact, the whole framework of high precision lattice QCD simulation is built on the assumption that one can do a simulation in which the coupling is perturbative at the shortest available distances, and the system will become nonperturbative at the longest ones. This can be observed in (for example) a heavy quark potential behaving as \( V(r) \sim 1/r \) at small \( r \) and \( \sigma r \) for large \( r \). The ever bigger lattices used in state of the art simulations are present both to push the simulation volume (measured in centimeters\(^4\)) ever larger, while simultaneously shrinking the lattice spacing ever smaller.

“Eliminating lattice artifacts” is a coded phrase for writing down a theory at the cutoff scale which is continuum QCD, up to small and calculable corrections, which implies that it is weakly interacting there.

The slow running of the coupling for this model means that on any lattice size accessible for numerical simulations, the zero bare quark mass system is conformal for all practical purposes. What does control the correlation length (inverse of a mass) is the fermion mass. The scaling of the correlation length with fermion mass is given by a critical exponent \( \gamma_m \) which is related to the anomalous dimension \( \gamma_m \) of the operator \( \bar{\psi}\psi \), defined through the running of the fermion mass
\[
\mu \frac{\partial m(\mu)}{\partial \mu} = -\gamma_m(g^2(m(\mu)), \quad \text{(3)}
\]

The quantity \( \gamma_m \) is the interesting parameter for technicolor dynamics. Technicolor scenarios, conventional or not, are said to prefer to have \( \gamma_m \sim 1 \) to simultaneously generate phenomenologically interesting fermion masses while suppressing flavor changing neutral currents. (Compare the recent discussion in Ref. [21].) Our theory is not very desirable from a phenomenological point of view: two studies[12, 13] show that \( \gamma_m \) is small, about 0.35 at the parameter values of the simulation.

Finally, to complete the list of undesirable features of the system, as one tunes the bare gauge coupling larger and larger, it undergoes a first order transition into a confining phase where the axial Ward identity quark mass never vanishes. It is not known whether this transition is a lattice artifact, or not. For the purpose of this paper, the resolution of this question is only of indirect importance. At its bare coupling value, the lattice system has a slowly running coupling constant, no chiral symmetry breaking, and no confinement. Let us just regard it as a template for some exotic new physics scenario, and see what it produces for an electroweak observable: what is \( \Pi^T_{LR}(q^2) \)?

Readers should note: most lattice calculations are about numbers and precision. This calculation has neither. Rather, it is a qualitative study of a theory which does not resemble QCD. To try to discover what it does resemble, I (mostly) compute \( \Pi^T_{LR} \) rather than the \( S \)
parameter, because at the bare couplings where I did the simulation, there is no $S$ parameter. The true electroweak observable $S$ involves subtracting the contribution of the Higgs field – or of the particles which replace the Higgs field – from $\Pi_{LR}^T$. There is no evidence of Higgs-like dynamics in the weak coupling phase of this theory.

The realization of Eq. 1 in a lattice simulation involves a long stream of annoying technical problems which must be overcome. Fortunately, there are already two lattice calculations of the $S$ parameter in QCD, Refs. [22] and [23], plus earlier lattice work on current-current correlators [24, 25] which provide pretty explicit directions to follow. The problems to be addressed are:

- Lattice artifacts in Eq. 1 in a lattice calculation
  \[ \Pi_{\mu\nu}(q) = P_T^{\mu\nu}(q)\Pi_T(q) + P_L^{\mu\nu}(q)\Pi_L(q) + \ldots \]  
  where $P_T(q)$ and $P_L(q)$ are lattice analogs of the transverse and longitudinal projectors and the dots represents additional momentum-dependent terms, proportional to higher powers of products of $q$.

- If the lattice currents are not conserved, there is an additional quadratic divergence $\delta_{\mu\nu}Q/a^2$ in $\Pi_{\mu\nu}(q)$.

- If the lattice currents are not local, there are additional contact terms in $\Pi_{\mu\nu}(q)$.

- There is a lattice-to-continuum regularization factor for each current. If it is different for the vector and axial vector currents, the analysis of a quantity depending on their difference becomes more fraught.

If we are only interested in the difference of vector and axial vector currents, the use of valence fermions which encode exact chiral symmetry (overlap or domain wall fermions) alleviates the second, third, and fourth of these problems.

The next section describes technical details of the simulations. Readers uninterested in them should jump to the following two sections, for qualitative comparisons of $\Pi_{LR}^T$ from ordinary QCD and the walking theory.

II. NUMERICAL TECHNIQUES AND BACKGROUND

I will make qualitative comparisons of lattice data from simulations of ordinary QCD and of $SU(3)$ gauge theory coupled to two flavors of fermions in the symmetric (sextet) representation. In all these studies, the valence Dirac operator is taken to be the overlap operator [26, 27]. Details of the particular implementation of the action are described in Refs. [28–32]; suffice it to say that the massless overlap operator is defined as $D = R_0(1 + d(-R_0)/\sqrt{d^\dagger(-R_0)d(-R_0)})$ where $d(m) = d + m$ for some lattice approximation (“kernel”) $d$ to the massless continuum Dirac operator. The only new ingredient is the application to symmetric-representation fermions, already described in Ref. [12]. Eigenvalues of the squared Hermitian Dirac operator $D^\dagger D$ are computed using the “Primme” package of McCombs and Stathopoulos [33] and are used to precondition the calculation of propagators.

The lattice analog of Eq. 1 is computed using the difference of improved currents, that is, the vector current is

\[ V_{\mu}^{12} = \bar{q}_1\gamma_\mu(1 - \frac{aD}{2R_0})q_2. \]
The axial current is identical, apart from the substitution of $\gamma_\mu \gamma_5$ for $\gamma_\mu$. These currents are not conserved, and so the correlator of each current is quadratically divergent. However, because they are related by a Ward identity, the quadratic divergence cancels in the vector-axial difference. In addition, both currents have the same lattice-to-continuum regulator renormalization factor (Z-factor). The currents are local enough that $\Pi_{\mu\nu}$ does not have contact terms.

In practice, the correlator is computed using point currents and the “shifted” propagator

$$\hat{D}^{-1}(m_q) = \frac{1}{1 - m_q/(2R_0)}(D^{-1}(m_q) - \frac{1}{2R_0}).$$

(6)

In free field theory and to all orders in perturbation theory $\Pi^{LR} = 0$ at $m_q = 0$, basically because $\{\gamma_5, \hat{D}^{-1}\} = 0$.

This gives a sensible and useful $\Pi_{\mu\nu}^{LR}$ – a correlator of local currents with a common Z factor, and with the decomposition of Eq. 1 for small momenta. I now have to deal with the lattice artifacts in the decomposition of Eq. 4. Following Ref. [22], I observe that they are small. This is quantified via the observable

$$\Delta_J(q) = \sum_{\mu\nu} \bar{q}_\mu \bar{q}_\nu \left( \frac{1}{q^2} - \frac{\bar{q}_\nu}{\sum_\lambda \bar{q}_\lambda^2} \right) \Pi_{\mu\nu}(q).$$

(7)

In a slight variation on the method of Ref [22], I define the appropriate variable as $\bar{q}_\mu = (2/a) \sin q_\mu a/2$. (Of course, $q_\mu = (2\pi/L)n_\mu$ if the $\mu$ direction exhibits periodic boundary conditions and its length is $L$.) A $\Pi_{\mu\nu}$ which is a superposition of pure longitudinal and transverse terms (the continuum decomposition of Eq. 1 with $q$’s replaced by $\bar{q}$’s) will give $\Delta = 0$. When I come to analysis I will show that $\Delta$ is very small compared to $\Pi_{\mu\nu}^{LR}$.

Next, I have to perform the decomposition of $\Pi_{\mu\nu}$ into $\Pi_L$ and $\Pi_T$. An easy way to do this is to assume that

$$\Pi_{\mu\nu}(q) = P_{\mu\nu}^T(q)\Pi_T(q) + P_{\mu\nu}^L(q)\Pi_L(q)$$

(8)

where

$$P_{\mu\nu}^T(q) = \bar{q}^2 \delta_{\mu\nu} - \bar{q}_\mu \bar{q}_\nu$$

(9)

and

$$P_{\mu\nu}^L(q) = \bar{q}_\mu \bar{q}_\nu$$

(10)

are the transverse and longitudinal projectors. Next I form the chi-squared function, individual momentum mode by momentum mode, taking the sixteen ($\mu\nu$) correlators as the quantities to be fit,

$$\chi^2(q) = \sum_{\mu\nu} \left( \Pi_{\mu\nu}(q) - P_{\mu\nu}^T(q)\Pi_T(q) - P_{\mu\nu}^L(q)\Pi_L(q) \right)^2,$$

(11)

and treating $\Pi_T$ and $\Pi_L$ as fit parameters. Because $P_T$ and $P_L$ are projectors, $\text{Tr} P_T P_L = 0$, this minimization reduces to simple definitions of weighted averages,

$$\Pi_T(q) = \frac{\sum_{\mu\nu} P_{\mu\nu}^T(q)\Pi_{\mu\nu}(q)}{3(q^2)^2}$$

$$\Pi_L(q) = \frac{\sum_{\mu\nu} P_{\mu\nu}^L(q)\Pi_{\mu\nu}(q)}{(q^2)^2}.$$
Of course, all lattice data from the same set of configurations are highly correlated. The uncertainties in the pictures which follow are computed by folding Eq. 12 into a single elimination jackknife: I delete a lattice from my data set, compute $\Pi_T$, $\Pi_L$, and $\Delta$, and then present a jackknife average and uncertainty.

III. ILLUSTRATIONS FROM QCD

The remainder of the paper is a qualitative comparison of $\Pi_T^{LR}$ from ordinary QCD and from sextet QCD. I begin with ordinary QCD.

I have two data sets. The first is a quenched set of 20 lattices computed using the overlap operator for valence quarks on a background of Wilson gauge action configurations at a gauge coupling $\beta = 5.9$. The lattice size is $16^4$ sites. The lattice spacing is about 0.13 fm from the rho mass or 0.11 fm from the Sommer parameter. This data set can be compared to an extensively-analyzed companion used in a calculation of $B_K$, the kaon B-parameter [34]. (The parameters of the kernel action used in this study differ by about one per cent from those of the $B_K$ project, but that is not going to matter given the qualitative nature of my presentation.)

The second data set is a smaller volume ($12^4$ sites) set of simulations with $N_f = 2$ flavors of dynamical overlap fermions. The data set had three quark masses, $am_q = 0.03, 0.05, 0.10$, at a lattice spacing of roughly 0.14 fm from the Sommer parameter. These lattices have been used in several small projects [35] to date. They are not really big enough for reliable spectroscopy measurements.

I cannot extract an $S$ parameter from these results (though I tried). The reason is that to get $S$ from $\Pi_T^{LR}$ requires doing a fit of the data to a chiral perturbation theory calculation and removing the single pseudoscalar propagator contribution. $S$ is then related to one of the Gasser-Leutwyler $O(p^4)$ coefficients. Doing this requires going to very low $q^2$, so that chiral perturbation theory is in its domain of applicability. This in turn requires with present techniques making the lattice volume large. (For example, Ref. [22] used a $16^3 \times 32$ lattice.) My $16^4$ lattice is large but it is a quenched data set and I am unaware of the appropriate quenched chiral perturbation theory calculation. And of course, the quenched approximation is obsolete as a potential high-precision venue. For this study, I do not need $S$.

Figs. 1 and 2 show the $\Delta$ parameter and $\Pi_T^{LR}$ from the quenched data set. In keeping with past practice [22, 23], I have plotted $q^2\Pi_T^{LR}$. $\Delta$ seems satisfactorily small compared to $\Pi_T^{LR}$. Unsurprisingly, $\Pi_T^{LR}$ qualitatively resembles other published results.

The superimposed curves in Fig. 2 show phenomenological parameterizations of the data, where $\Pi_T^{LR}$ is saturated with a set of resonances. Introducing two quark flavor labels to characterize the currents, the pseudoscalar decay constant $f_\pi$ is defined through

$$\langle 0|\bar{u}\gamma_0\gamma_5d|\pi \rangle = m_\pi f_\pi$$

(13)

(so $f_\pi \sim 132$ MeV) while the vector meson decay constant of state $V$ is defined as

$$\langle 0|\bar{u}\gamma_i d|V \rangle = m_V^2 f_V \epsilon_i$$

(14)

and the axial vector meson decay constant of state $A$ is

$$\langle 0|\bar{u}\gamma_i\gamma_5 d|A \rangle = m_A^2 f_A \epsilon_i.$$  

(15)
FIG. 1: $q^2 \Delta^{V-A}$ vs $q^2$ and $q^2$ from quenched overlap fermions at $\beta = 5.9$. Valence masses are (octagons), $am_q = 0.10$ (squares), $am_q = 0.05$ (diamonds), $am_q = 0.035$. (crosses) $am_q = 0.025$, and (fancy diamonds) $am_q = 0.02$.

FIG. 2: $q^2 \Pi^{V-A}$ vs $q^2$ from quenched overlap fermions at $\beta = 5.9$. As in Fig. 1, valence masses are shown by octagons for $am_q = 0.10$, squares for $am_q = 0.05$, diamonds for $am_q = 0.035$, crosses for $am_q = 0.025$, and fancy diamonds for $am_q = 0.02$. The lines are the model parameterizations.
$\epsilon_i$ is a unit polarization vector. The transverse current correlator is then

$$\Pi_{LR}^T(q^2) = \sum_V f_V^2 V - \sum_A f_A^2 A - \frac{f_\pi^2}{q^2}.$$  \hspace{1cm} (16)

In a theory with spontaneous chiral symmetry breaking, the couplings are constrained by the first

$$\sum_V f_V^2 V - \sum_A f_A^2 A - f_\pi^2 = 0$$  \hspace{1cm} (17)

and second

$$\sum_V f_V^2 V - \sum_A f_A^2 A = 0$$  \hspace{1cm} (18)

Weinberg sum rules [36]. When the two sum rules are satisfied, $\Pi_{LR}^T(q^2)$ falls off asymptotically in $q$ as $1/q^6$.

A common phenomenological approach is to saturate the correlator by the lowest states in the vector and axial channels, the pion, rho and $a_1$ meson [37]. This involves fixing five parameters, the three couplings $f_\pi$, $f_\rho$, and $f_{a_1}$ and two masses $m_\rho$ and $m_{a_1}$. In principle, all of these can be measured in lattice simulations. In practice, $f_{a_1}$ and $m_{a_1}$ involve difficult measurements because the signal to noise ratio in a mesonic correlator (whose mass is $M$) scales like $\exp((M - 2m_\pi)t)$.

I attempted to extract all five quantities from spectroscopic fits to the quenched data. I could do this only for the heavier masses in the data set. Results are shown in Table II. Using them in the correlator does not reproduce either the data or the Weinberg sum rules. (Perhaps this result is neither surprising nor controversial.) However, I do want to
TABLE I: Measured lattice parameters (in lattice units; everything is scaled by a factor of the lattice spacing $a$ and the overall $Z$ factor is left out) from quenched $16^4$ simulations, used in constructing the parameterization of the transverse correlator. The last four columns are fit results from forcing the spectral function to obey the first and second Weinberg sum rules, saturated by the rho and $a_1$ mesons.

| $m_q$ | $f_\pi$ | $m_\rho$ | $f_\rho$ | $m_{a_1}$ | $f_{a_1}$ | fit $m_{a_1}$ | fit $f_{a_1}$ | $L_{10} \times 10^3$ | $a^2\Delta m_{\pi}^2 \times 10^3$ |
|-------|---------|---------|---------|---------|---------|---------|---------|---------------|------------------|
| 0.020 | 0.085(1)| 0.55(2) | 0.28(2) |         |         | 0.657(15) | 0.197(3) | 5.1(6)       | 0.62(4)          |
| 0.025 | 0.089(1)| 0.56(2) | 0.27(4) |         |         | 0.684(14) | 0.185(9) | 5.3(6)       | 0.66(5)          |
| 0.035 | 0.095(1)| 0.57(1) | 0.27(3) |         |         | 0.723(8)  | 0.168(6) | 5.6(7)       | 0.71(3)          |
| 0.050 | 0.100(1)| 0.60(1) | 0.26(2) | 0.93(5) | 0.182(3)| 0.780(8)  | 0.154(5) | 5.5(8)       | 0.81(3)          |
| 0.100 | 0.116(5)| 0.689(4)| 0.23(5) | 1.00(3) | 0.170(2)| 0.995(10)| 0.113(4) | 5.3(6)       | 1.16(2)          |

TABLE II: Measured lattice parameters from dynamical $12^4$ simulations, used in constructing the parameterization of the transverse correlator. The last four columns are fit results from forcing the spectral function to obey the first and second Weinberg sum rules, saturated by the rho and $a_1$ mesons.

| $m_q$ | $f_\pi$ | $m_\rho$ | $f_\rho$ | fit $m_{a_1}$ | fit $f_{a_1}$ | $L_{10} \times 10^3$ | $a^2\Delta m_{\pi}^2 \times 10^3$ |
|-------|---------|---------|---------|---------|---------|---------------|------------------|
| 0.03  | 0.092(3)| 0.57(2) | 0.22(2) | 0.84(10) | 0.10(3) | 4.8(4)       | 0.81(12)          |
| 0.05  | 0.135(5)| 0.64(2) | 0.25(2) | 1.19(26) | 0.07(4) | 7.1(7)       | 1.24(23)          |
| 0.10  | 0.138(6)| 0.80(1) | 0.23(2) | 1.20(15) | 0.10(3) | 5.3(5)       | 1.64(18)          |

compare the QCD results to those from the walking theory. Accordingly, one approach to parameterizing the data consists of taking the values of $f_\pi$, $m_\rho$, and $f_\rho$ from fits to lattice data, and determining $f_{a_1}$ and $m_{a_1}$ by forcing a solution to the to Weinberg sum rules.

The result of this calculation is shown as the curves in Fig. 2. In this figure, the bursts at $\bar{q} = 0$ are just the values of $f_\pi^2$. Fig. 3 shows $\bar{q}^2 \Pi_{T}^{-A}$ on a log-log scale to expose the approach to scaling at large $\bar{q}$. The data qualitatively resembles the analytic parameterization. The Weinberg sum rules are statements about the asymptotic behavior of the current correlators. Lattice QCD correlators presumably attempt to encode these statements until they are overwhelmed by lattice artifacts.

Next, we turn to figures from the $12^4$ volume dynamical overlap: Fig. 4 shows the $\Delta$ observable. Fig. 5 shows $\bar{q}^2 \Pi_{T}^{-A}$ vs $\bar{q}^2$. Again, the superimposed curves are from Eq. 16, taking fit values of $f_\pi$, $f_\rho$ and $m_\rho$, and determining $f_{a_1}$ and $m_{a_1}$ by saturating the two Weinberg sum rules. Fig. 6 reproduces Fig. 5 on a log-log scale to show the Weinberg sum rules at work. As a qualitative parameterization of the data, this procedure works well.

I cannot resist using the pseudoscalar and vector decay constants, plus the vector meson mass, to compute the Gasser-Leutwyler parameter $L_{10}$ and the charged - neutral pion mass difference $\Delta m_{\pi}^2$ via the Das, Guralnik, Mathur, Low, Young sum rule \[38\]. In the “lowest mass dominance” approximation to the spectral function, $L_{10} = (f_\rho^2 - f_{a_1}^2)/8 = f_{a_1}^2(R^2 - 1)$ and

$$\Delta m_{\pi}^2 = \frac{3\alpha}{4\pi} \left(\frac{1}{m_\rho^2 - m_{a_1}^2}\right) \log \frac{m_{a_1}^2}{m_\rho^2} = \frac{3\alpha}{4\pi} \frac{m_\rho^2}{R - 1} \log R$$

(19)

where $R = 1 - (f_\pi/(m_\rho f_\rho))^2$ is given to remind the reader, what are the real independent variables. The results are shown in the two tables and in Fig. 7. A fit to a linear dependence
FIG. 4: $q^2\Delta V^A$ vs $q^2$ vs $q^2$ from dynamical $N_f = 2$ simulations. Valence masses are (octagons) $am_q = 0.10$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.03$.

FIG. 5: $q^2\Pi_T^A$ vs $q^2$ from dynamical $N_f = 2$ simulations. Valence masses are (octagons) $am_q = 0.10$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.03$. The lines are the model parameterizations.
on the quark mass gives $\bar{T}_{10} = 5.1(6) \times 10^{-3}$ and $5.3(5) \times 10^{-3}$ for the dynamical and quenched data sets. This is in good agreement with large scale simulation results of Refs. [22, 23] and with phenomenological estimates [39].

Fitting the dynamical data set’s $\Delta m^2_{\pi}$ in the same way gives $\Delta m^2_{\pi} = 0.49(18) \times 10^{-3}$ or (with $a = 0.14$ fm) 980(360) MeV$^2$. The quenched data has $\Delta m^2_{\pi} = 0.48(3) \times 10^{-3}$ for 1540(100) MeV$^2$ or 1100(70) MeV$^2$, taking $a = 0.11$ or 0.13 fm. These results agree nicely with experiment, 1261 MeV$^2$. Readers should note that while these are lattice results, they are indirect and depend on the additional assumption that the spectral functions can be saturated by three input parameters plus the two Weinberg sum rules.

The summary of this section is that both quenched and dynamical lattice QCD possesses a $q^2 \Pi^{V-A}_{T}(q^2)$ which at small $q$ remains finite as the quark mass vanishes (it reduces to $-f^2_{\pi}$). At large $q$ it vanishes roughly in accordance with the Weinberg sum rules. A simple three-parameter function using information from the lowest excitations in the axial and vector channels seems to reproduce the data reasonably well and produce phenomenologically sensible results.

IV. $\Pi^{LR}$ FROM SEXTET FERMIONS

I repeat the calculation of $q^2 \Pi^{V-A}_{T}(q^2)$ using sextet representation overlap fermions on a background of dynamical simulations of $N_f = 2$ flavors of clover fermions in the sextet representation and SU(3) gauge fields. The dynamical simulations were performed using the Wilson gauge action at a coupling $\beta = 5.2$ and a hopping parameter $\kappa = 0.1285$. This corresponds to an AWI (axial Ward identity) quark mass of $am^a_q = 0.044$. I used five valence masses, $am_q = 0.100, 0.050, 0.035, 0.025$ and 0.020. The lattice volume is $16^4$ sites.
FIG. 7: Panel (a) shows lattice data for $\overline{T}_{10}$, and panel (b), for $\Delta m_{a}^{2}$ from the analysis described in the text. Squares show quenched data and crosses are for $N_f = 2$ simulations.

At this set of simulation parameters, the system seems to be deconfined (the string tension is immeasurably small) and chiral symmetry is restored (observed through the parity doubling of states).

Fig. 8 shows the $\Delta$ observable. It seems to be acceptably small. Fig. 9 shows $\overline{q}^{2}\Pi_{T}^{V-A}$ vs $\overline{q}^{2}$. Fig. 10 shows $-\overline{q}^{2}\Pi_{T}^{V-A}$ vs $\overline{q}^{2}$, replotted with logarithmic axes. These figures show very different behavior from the case of QCD.

Notice that in the large-$q^{2}$ limit, $\Pi_{T}^{LR}$ still falls to zero. This is consistent with the Weinberg sum rules, and is expected for any theory with an ultraviolet-attractive fixed point at $g = 0 \ [40]$.

Just as in the case of QCD, it is possible to measure spectroscopy in the pseudoscalar, vector, and axial vector channels. It is also possible to measure the three decay constants. In fact, it is easier to extract the axial vector meson properties here, than it is in an ordinary QCD simulation. This is because the system is chirally restored and the pion is not appreciably lighter than the other states. (The same effect occurs in the high temperature phase of QCD.) These parameters are shown in Table III. The obvious feature they show is chiral symmetry restoration: note the near degeneracy of vector and axial vector masses (called $m_{\rho}$ and $m_{a_{1}}$ in hadronic analogy), which becomes more pronounced as the fermion mass vanishes) and of the vector and axial vector couplings.

A quick check with a calculator shows that the three lowest states with their couplings do not saturate either Weinberg sum rule. So I play the same game that I did for the QCD data sets, determining $m_{a_{1}}$ and $f_{a_{1}}$ by forcing the two sum rules to saturate. The result of this exercise is shown in Fig. 10 (for the $m_{q} = 0.100$ data set, only). This fails completely.

I was somewhat surprised by this result, since lowest mass dominance is a venerable ingredient of strongly coupled beyond - Standard Model phenomenology (compare Ref. [1], or for a recent review, Ref. [41]), even sometimes of conformal beyond - Standard Model phenomenology (compare Refs. [42, 43]. However, note that I am looking at a system in which the breaking of (near) conformal symmetry is by a small mass, and I am looking at
FIG. 8: $q^2 \Delta^{V-A}$ vs $q^2$ from valence sextet overlap fermions on background sextet dynamical clover simulations at $\beta = 5.2$, $\kappa = 0.1285$. Valence masses are (octagons) $am_q = 0.10$, (bursts) $am_q = 0.075$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.035$. (crosses) $am_q = 0.020$.

FIG. 9: $q^2 \Pi_T^{V-A}$ vs $q^2$ from valence sextet overlap fermions on background sextet dynamical clover simulations at $\beta = 5.2$, $\kappa = 0.1285$. Valence masses are (octagons) $am_q = 0.10$, (bursts) $am_q = 0.075$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.035$. (crosses) $am_q = 0.020$. 
FIG. 10: As in Fig. 9, but a log-log scale. The line shows an attempted parameterization of the data, forcing the two Weinberg sum rules to be saturated by the three lowest resonances.

\[
\begin{array}{c|cccccc}
\hline
m_q & f_\pi & m_\rho & f_\rho & m_{a_1} & f_{a_1} \\
\hline
0.020 & 0.00050(4) & 0.33(1) & 0.0046(2) & 0.38(1) & 0.0044(2) \\
0.035 & 0.00080(6) & 0.37(1) & 0.0045(2) & 0.42(2) & 0.0043(1) \\
0.050 & 0.00102(8) & 0.37(1) & 0.0044(2) & 0.47(2) & 0.0042(2) \\
0.075 & 0.00131(9) & 0.43(1) & 0.0041(1) & 0.54(2) & 0.0037(10) \\
0.100 & 0.00151(10) & 0.48(1) & 0.0037(1) & 0.61(2) & 0.0037(2) \\
\hline
\end{array}
\]

TABLE III: Measured lattice parameters from valence overlap fermions from sextet QCD simulations.

\(m^2/q^2\) small. Often, the phenomenology is concerned with the other limit, \(m^2/q^2\) large.

I convert \(\Pi_{V-A}^V\) into an “effective” \(S\)-parameter by numerically differentiating it,

\[
S_{\text{eff}}(q^2) = 16\pi \frac{\Delta (\tilde{q}^2 \Pi_{V-A}^V)}{\Delta \tilde{q}^2} \tag{20}
\]

(For the resonance model, Eq. 16 this would give \(S_{\text{eff}} = 16\pi (f_V^2 - f_A^2)\).) This I do in the most naive way, sorting the data into increasing \(\tilde{q}^2\) and just taking successive differences. This rapidly becomes noisy at bigger \(\tilde{q}^2\), so I arbitrarily kept the ten smallest values, \(\tilde{q} < 1.24\), for each \(m_q\). I plot the result in Fig. 11. The collapse of the data to zero with the fermion mass is obvious. Presumably, a true “new physics” prediction would multiply \(S_{\text{eff}}\) by an overall scale representing the coupling constant of the dynamics to the W or Z boson.

It is a worthwhile exercise to look for scaling behavior in the data. To do this, I replot the data in Fig. 12. The data seems to fall on a scaling curve in terms of the dimensionless
FIG. 11: Derivative of the sextet QCD correlator $S_{\text{eff}} = 16\pi\Delta(\bar{q}^2\Pi^V_{1T}-A)/\Delta\bar{q}^2$. Valence masses are (octagons) $am_q = 0.10$, (bursts) $am_q = 0.075$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.035$. (crosses) $am_q = 0.020$. 

ratio $\bar{q}^2/m^2$. In free field theory, $S_{\text{eff}} = (48/\pi^2)(m^2/\bar{q}^2)$, which seems to be a reasonable parameterization of the data. However, motivated by various didactic reviews [41] of the expected algebraic scaling behavior of conformal theories, I also tried a little fit to

$$S_{\text{eff}} = a\left(\frac{m^2}{\bar{q}^2}\right)^p.$$  \hspace{2cm} (21)

The line is the result, $p = 0.953(3)$. The uncorrelated chi-squared is 924 for 38 degrees of freedom, but I should point out that the data is in fact strongly correlated.

Again, I remark that this is not the usual case studied in the technicolor literature, where the fermion mass is often taken to be large compared to $q$. Presumably to apply this result to some hidden sector beyond - Standard Model phenomenology, one might take $q^2 = m^2_Z$. (Or for another alternative, see Ref. [45].)

I believe that there are lessons to be drawn from this exercise. First, because the system only has explicit chiral symmetry breaking at nonzero fermion mass, $\Pi^V_{1T}(q^2)$ does vanish at zero fermion mass. $\Pi^V_{1T}(q^2)$ falls to zero for any value of the mass, at any nonzero $q$.

Second, a simple parameterization of $\Pi^V_{1T}(q^2)$ in terms of the (measured) properties of the lowest resonances fails, even though away from $m_q = 0$ the theory is not conformal; it is a theory of resonances. Presumably, one must just include more of the excited state spectrum in the sum. $S_{\text{eff}}$ shows power law scaling in terms of $m^2/\bar{q}^2$. 

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FIG. 12: Derivative of the sextet QCD correlator $S_{eff} = 16\pi \Delta(\bar{q}^2 \Pi^{V-A}_T)/\Delta q^2$ plotted as a function of the scaling combination $\bar{q}^2/m^2$. Valence masses are (octagons) $am_q = 0.10$, (bursts) $am_q = 0.075$, (squares) $am_q = 0.05$, (diamonds) $am_q = 0.035$, (crosses) $am_q = 0.020$. The line is the straight power law fit, Eq. 21.

V. CONCLUSIONS

This paper was a qualitative investigation of a precision electroweak observable for a lattice model of gauge fields and two flavors of symmetric representation fermions. Before lattice simulations began, this model was one of many candidate theories for walking technicolor [6–9]. At the bare parameters where I did my simulations, it does not present the appearance of being a conventional walking technicolor model. However, it does show conformal behavior “for all practical purposes,” that is, at the parameters of the simulation the scale dependent gauge coupling runs very slowly [13].

I only asked simple questions, because the literature for these models is relatively sparse. However, I discovered (as expected) that $\Pi^{V-A}_T(q^2)$ vanished for all $q$ as the quark fermion mass vanished, and that it vanished at large $q$ for all values of the fermion mass. The effective S-parameter is a scaling function of $m^2/q^2$. I was surprised to discover that a phenomenologically popular parameterization of $\Pi^{V-A}_T(q^2)$ in terms of the properties of the lowest resonances did not reproduce the data.

It might be, that all models are just different, and what I found for $\Pi^{V-A}_T(q^2)$ is not relevant to any other system. However, there are other would-be technicolor candidates in the literature, which share some features with this model for regions of their bare parameter spaces. $\Pi^{T}_{LR}$ in “minimal walking technicolor” (gauge group $SU(2)$ with a doublet of adjoint representation fermions [46]) is an interesting target for a lattice study.

And of course, phenomenologists should expect more calculations of $\Pi^{V-A}_T(q^2)$, done by groups on models which are (hopefully) more attractive technicolor candidates – perhaps ones which are not so conformal as this one.
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