On the Lebesgue constants of Fourier-Laplace series by Riesz Means

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Abstract. An asymptotic formula for the Lesbegue constant of the Riesz means of Fourier-Laplace series on the sphere obtained in this paper.

1. INTRODUCTION

Let us define $\sigma_n^\alpha f(x)$ the Cesaro means of order $\alpha$ of the partial sums of Fourier-Laplace series on unite sphere $S^N$ as

$$\sigma_n^\alpha f(x) = \int_{S^N} \Theta^\alpha(x, y, n)f(y) \, d\sigma(y),$$

where the kernel

$$\Theta^\alpha(x, y, n) = \sum_{k=0}^n \frac{A_{n-k}^\alpha}{A_n^\alpha} \sum_{j=1}^{a_k} Z_k(x, y),$$

$$Z_k(x, y) = \sum_{j=1}^{a_k} Y_j^{(k)}(x)Y_j^{(k)}(y), \quad Y_j^{(k)}(y) \text{ are spherical harmonics and } A_n^\alpha = \binom{n+\alpha}{\alpha}.$$

Investigations on the behaviour of the Cesaro means $\sigma_n^\alpha f(x)$ can be found in works [4] - [5] and [11] - [16]. The different aspects of the convergence and summability can be also found in the book [18]. Since $\sigma_n^\alpha f(x)$ is an integral operator the precise estimation of its kernel $\Theta^\alpha(x, y, n)$ is essential for the study. First estimations of the kernel $\Theta^\alpha(x, y, n)$ obtained by Gronwall [6] for the case of Legendre polynomials and Kogbetliantz [8] for the Gegenbauer polynomials.

The Lebesgue constant is $L_1$ norm of the kernel above. Note, that estimations of the Lebesgue constants of the Cesaro means studied by Khocholava [8], Akhobadze [2] and Macharashvili [10]. The Lebesgue constants of multiple Fourier series studied in [1] and [9].

This article focuses on Lebesque constants related to Fourier-Laplace series of the Laplace-Beltrami operator:

$$L_n^\alpha = \int_{S^N} |\Theta^\alpha(x, y, n)| d\sigma(y).$$

(1.1)
2. MAIN RESULT.
In the present paper we consider the Riesz means instead the Cesaro means of the partial sums of Fourier-Laplace series. The Riesz means of the partial sums will also be an integral operator and its kernel can be represented by

\[ \Theta^\alpha(x, y, n) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha \Gamma \left(k + \frac{N-1}{2}\right) \Gamma(k + N - 1) \frac{\nu + 1}{k^\frac{N+1}{2}} P_{k}^\nu (\cos \gamma), \]

where \( P_k^\nu(t) \) denote the Gegenbauer polynomials as follows:

\[ P_k^\nu(t) = \frac{(-2)^k \Gamma(k + \nu) \Gamma(k + 2\nu)}{\Gamma(2(k + \nu))} (1 - t^2)^{-(\nu - \frac{1}{2})} \frac{d^k}{dt^k} \left[(1 - t^2)^{k+\nu-\frac{1}{2}}\right]. \]

By this representation it is evident that \( \Theta^\alpha(x, y, n) \) depends only on the spherical distance between \( x \) and \( y \) hence, allows the Riesz means of the spectral function to be written as \( \Theta^\alpha(x, y, n) = \Theta^\alpha_n(\cos \gamma) \). The Riesz means of the kernel is studied in the works [3] and [17].

The main goal of the paper is to obtain the estimation of the Lebesque constant ( \( L_1 \) norm of the kernel). Let us use the same notation \( \mathcal{L}^\alpha_n \) for the Lebesque constant as in (1.1). Then following theorem is valid.

**Theorem 2.1.** The Lebesgue constants for Fourier-Laplace series have the following estimations

\[ C' n^{\frac{N-1}{2} - \alpha} < \mathcal{L}^\alpha_n < C'' n^{\frac{N-1}{2} - \alpha}, \quad \alpha < \frac{N-1}{2}, \]

\[ C' \ln n < \mathcal{L}^\alpha_n < C'' \ln n, \quad \alpha = \frac{N-1}{2}, \]

\[ 0 < \mathcal{L}^\alpha_n < C, \quad \alpha > \frac{N-1}{2}. \]

3. PROOF OF MAIN RESULT.
To estimate \( \mathcal{L}^\alpha_n \), we first denote (1.1) as follows,

\[ \mathcal{L}^\alpha_n = C \int_0^\pi |\Theta^\alpha_n(\cos \gamma)| \sin^{N-1} \gamma d\gamma \] (3.1)

Let us divide the integral on the right hand side of (3.1) into three parts as follows

\[ \mathcal{L}^\alpha_n = C \int_0^{\frac{\pi}{N+1}} |\Theta^\alpha_n(\cos \gamma)| \sin^{N-1} \gamma d\gamma + C \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} |\Theta^\alpha_n(\cos \gamma)| \sin^{N-1} \gamma d\gamma \]

\[ + C \int_{\frac{\pi}{2}}^{\pi} |\Theta^\alpha_n(\cos \gamma)| \sin^{N-1} \gamma d\gamma \] (3.2)

\[ = I_1 + I_2 + I_3. \]
3.1. Estimation from above

If $0 < \gamma_0 \leq \gamma \leq \pi$ and $0 \leq \gamma \leq \pi$, the kernel $\Theta_n^\alpha(\cos \gamma)$ can be easily estimated by

$$|\Theta_n^\alpha(\cos \gamma)| \leq C_4 n^{N-\frac{\alpha}{2}} \quad \text{and} \quad |\Theta_n^\alpha(\cos \gamma)| \leq C_5 n^N.$$ 

For the Riesz means of the spectral function $\Theta_n^\alpha(\cos \gamma)$ in $0 \leq \gamma \leq \pi$ we have the following estimation

$$|\Theta_n^\alpha(\cos \gamma)| < C n^N, \quad 0 \leq \gamma \leq \pi.$$ 

The first integrand $I_1$ can be estimated as follow

$$I_1 < C n^N \int_0^{\frac{\pi}{n+\frac{\alpha}{2}}} \sin^{N-1} \gamma \, d\gamma \leq C n^N \int_0^{\frac{\pi}{n+\frac{\alpha}{2}}} \gamma^{N-1} \, d\gamma = \frac{C n^N (\pi/2)^N}{(n+1)^N(N)} \leq \frac{C (\pi/2)^N}{N} = O(1). \quad (3.3)$$

Similarly, for the third term $I_3$ one has

$$I_3 < C n^N \int_{\pi-I}^{\pi} \sin^{N-1} \gamma \, d\gamma \leq C n^N \int_{\pi-I}^{\pi} \sin^{N-1}(\pi - \gamma) \, d\gamma.$$

Applying the change of variables $\omega = \pi - \gamma$ the last estimation gives

$$I_3 < C n^N \int_0^{\frac{\pi}{n+\frac{\alpha}{2}}} \sin^{N-1} \omega \, d\omega = O(1). \quad (3.4)$$

To estimate $I_2$ we need the following estimation of the kernel $\Theta_n^\alpha(\cos \gamma)$ (refer [17])

$$\Theta_n^\alpha(\cos \gamma) = (N-1)n^{\frac{N-1}{2}} \sin \left( n + \frac{N-1}{2} + \frac{n+1}{2} \right) \gamma - \frac{\pi}{2} \left( \frac{N-1}{2} + \alpha \right) + \frac{\pi n^\alpha (\gamma(n+1))^{\frac{N-1}{2}-\alpha-1}}{(2 \sin \frac{\pi}{2})^{2+2}} \left( 2 \sin \frac{\pi}{2} \right)^{\frac{N-1}{2}+1} \quad (3.5)$$

By (3.5), the integrand $I_2$ is bounded by the sum of the following integrals

$$I_2 < C \int_{\pi-I}^{\pi} \frac{n^{\frac{N-1}{2}-\alpha}}{(2 \sin \frac{\pi}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma + C \int_{\pi-I}^{\pi} \frac{n^{\frac{N-1}{2}-\alpha}}{(2 \sin \frac{\pi}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma$$

$$+ C \int_{\pi-I}^{\pi} \frac{\eta_n^\alpha(\gamma)}{(2 \sin \frac{\pi}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma = I_2^1 + I_2^2 + I_2^3. \quad (3.6)$$

We first estimate $I_2^1$:

$$I_2^1 = C \int_{\pi-I}^{\pi} \frac{n^{\frac{N-1}{2}-\alpha}}{(2 \sin \frac{\pi}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma = C \int_{\pi-I}^{\pi} \frac{n^{\frac{N-1}{2}-\alpha}}{(2 \sin \frac{\pi}{2})^{1+\alpha}} \sin^{N-1} \gamma \, d\gamma$$

$$= C n^{\frac{N-1}{2}-\alpha} \int_{\pi-I}^{\pi} \sin^{N-1} \gamma \, d\gamma \leq C n^{\frac{N-1}{2}-\alpha} \int_{\pi-I}^{\pi} \gamma^{N-1} \, d\gamma.$$
Consider the cases: \( \alpha < \frac{N-1}{2} \), \( \alpha = \frac{N-1}{2} \) and \( \alpha > \frac{N-1}{2} \).

If \( \alpha < \frac{N-1}{2} \), then

\[
I'_2 \leq \frac{C}{N-\frac{1}{2}-\alpha} \left[ \left( \pi - \frac{\pi}{2(n+1)} \right)^{N-1-\alpha} - \left( \frac{\pi}{2(n+1)} \right)^{N-1-\alpha} \right] n^{N-1-\alpha} \leq Cn^{\frac{N-1}{2} - \alpha}. \tag{3.7}
\]

If \( \alpha = \frac{N-1}{2} \),

\[
I'_2 \leq C \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \frac{1}{\gamma} d\gamma = C \ln \left[ (n+1) \left( 2 - \frac{1}{n+1} \right) \right] = C \ln(2n+1) \leq C \ln n. \tag{3.8}
\]

If \( \alpha > \frac{N-1}{2} \),

\[
I'_2 \leq Cn^{\frac{N-1}{2} - \alpha} \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \gamma^{N-1-\alpha-1} d\gamma \leq \frac{Cn^{\frac{N-1}{2} - \alpha}}{(\alpha - \frac{N-1}{2})(n+1)^{\frac{N-1}{2} - \alpha}} \left( \frac{\pi}{2} \right)^{\frac{N-1}{2} - \alpha} \leq C_\alpha. \tag{3.9}
\]

Hence, from (3.7), (3.8) and (3.9) we obtain

\[
I'_2 \leq \begin{cases} 
Cn^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\
C \ln n, & \alpha = \frac{N-1}{2}, \\
C_\alpha, & \alpha > \frac{N-1}{2}.
\end{cases} \tag{3.10}
\]

The term \( I''_2 \) is estimated by:

\[
I''_2 = C \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \frac{(n+1)^{\frac{N-3}{2} - \alpha} |\eta_n(\gamma)|}{(\sin \gamma)^{1/2+1} (\sin \frac{\gamma}{2})^{\alpha+1}} \sin^{N-1} \gamma d\gamma \leq Cn^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \frac{(\sin \gamma)^{\frac{N-3}{2}}}{(\sin \frac{\gamma}{2})^{\alpha+1}} d\gamma \leq Cn^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \gamma^{\frac{N-5}{2} - \alpha} d\gamma.
\]

Let us consider the following cases:

If \( \alpha < \frac{N-3}{2} \),

\[
I''_2 \leq Cn^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \gamma^{\frac{N-5}{2} - \alpha} d\gamma = Cn^{\frac{N-3}{2} - \alpha} \left[ \left( \pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} - \left( \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} \right] \leq Cn^{\frac{N-3}{2} - \alpha}. \tag{3.11}
\]

If \( \alpha = \frac{N-3}{2} \),

\[
I''_2 \leq Cn^{\frac{N-3}{2} - \alpha} \int_{\frac{\pi}{N+1}}^{\frac{\pi}{2}} \frac{1}{\gamma} d\gamma \leq C \ln n. \tag{3.12}
\]
If \( \alpha > \frac{N-3}{2} \)

\[
I_2'' \leq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{n+1} \frac{n}{2}}^{\frac{1}{n+1} \frac{n}{2}} \gamma^{\frac{N-5}{2} - \alpha} \, d\gamma = C n^{\frac{N-3}{2} - \alpha} \left[ \left( \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} - \left( \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} \right]
\]

\[
\leq \frac{C}{\alpha - \frac{N+1}{2}} \left( \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2} - \alpha} n^{\frac{N-3}{2} - \alpha} \leq C.
\]

(3.13)

By (3.11), (3.12) and (3.13), \( I_2 \) is estimated by

\[
I_2' \leq \begin{cases} 
C n^{\frac{N-3}{2} - \alpha}, & \alpha < \frac{N-3}{2}, \\
C \ln n, & \alpha = \frac{N-3}{2}, \\
C, & \alpha > \frac{N-3}{2}.
\end{cases}
\]

(3.14)

Finally, to estimate \( I_2 \) from above, \( I_2''' \) is estimated as follows:

\[
I_2''' = \frac{C}{n+1} \int_{\frac{1}{n+1} \frac{n}{2}}^{\frac{1}{n+1} \frac{n}{2}} \left| \varepsilon_n(\gamma) \right| \sin^{N-1} \gamma \, d\gamma \leq \frac{C}{n+1} \int_{\frac{1}{n+1} \frac{n}{2}}^{\frac{1}{n+1} \frac{n}{2}} \frac{1}{\gamma^2} \, d\gamma \leq \frac{C}{n+1} \frac{\pi(n+1)}{2} \leq C,
\]

(3.15)

Then from (3.6) taking into the note (3.10), (3.14) and (3.15) we obtain

\[
I_2 \leq \begin{cases} 
C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\
C \ln n, & \alpha = \frac{N-1}{2}, \\
C, & \alpha > \frac{N-1}{2}.
\end{cases}
\]

(3.16)

Consequently, using (3.3), (3.4) and (3.16) the Lebesgue constant \( L_n^\alpha \) (see (3.2)) is estimated as follows

\[
L_n^\alpha \leq \begin{cases} 
C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\
C \ln n, & \alpha = \frac{N-1}{2}, \\
C, & \alpha > \frac{N-1}{2}.
\end{cases}
\]

(3.17)
3.2. Estimation from below

The next step is to obtain the lower bound of the Lebesgue constant. We proceed by first estimating \( I_2' \) from below:

\[
I_2' = C \int_{\frac{1}{N+1}}^{0} n^{\frac{N-1}{2} - \alpha} (N - 1) \frac{\sin \left[ (n + \frac{N-1}{2} + \frac{\alpha+1}{2}) \gamma - \frac{\pi}{2} \left( \frac{N-1}{2} + \alpha \right) \right]}{(2 \sin \gamma)^{\frac{N-1}{2} - \alpha}} \sin^{N-1} \gamma \, d\gamma
\]

\[
= C \int_{\frac{1}{N+1}}^{0} n^{\frac{N-3}{2} - \alpha} (N - 1) \frac{\sin \left[ (n + \frac{N-1}{2} + \frac{\alpha+1}{2}) \gamma - \frac{\pi}{2} \left( \frac{N-1}{2} + \alpha \right) \right]}{\sin^{\frac{N-3}{2}} \gamma} \left( \cos \gamma \right)^{\frac{N-1}{2}} \, d\gamma
\]

\[
\geq C n^{\frac{N-3}{2} - \alpha} \int_{\frac{1}{N+1}}^{0} n^{\frac{N-3}{2} - \alpha} (N - 1) \frac{\sin \left[ (n + \frac{N-1}{2} + \frac{\alpha+1}{2}) \gamma - \frac{\pi}{2} \left( \frac{N-1}{2} + \alpha \right) \right]}{\gamma^{\alpha - \frac{N-3}{2}}} \, d\gamma.
\]

A new variable \( \omega = (n + \frac{N}{2} + \frac{\alpha}{2}) \gamma - \frac{\pi(N-1+2\alpha)}{4} \) is introduced and substituted into the chain of inequalities:

\[
I_2' \geq C n^{\frac{N-1}{2} - \alpha} \left( n + \frac{N-1}{2} + \frac{\alpha+1}{2} \right)^{\frac{3\pi}{2(n+1)}} \int_{\frac{(n+N-1+2\alpha)}{2}}^{\frac{(n+N+1+2\alpha)}{2}} \frac{\sin \omega}{\left( \omega + \frac{\pi(N-1+2\alpha)}{4} \right)^{\alpha - \frac{N-3}{2}}} \, d\omega
\]

\[
\geq C n^{\frac{N-1}{2} - \alpha} \left( n + \frac{N-1}{2} + \frac{\alpha+1}{2} \right)^{\frac{3\pi}{4} - \frac{\pi(N-1+2\alpha)}{4}} \int_{n\pi/2}^{\pi} \frac{\sin \omega \, d\omega}{\left( \omega + \frac{\pi(N-1+2\alpha)}{4} \right)^{\alpha - \frac{N-3}{2}}}.\]

Since

\[
\left( n + \frac{N-1}{2} + \frac{\alpha+1}{2} \right) = \frac{\pi}{2(n+1)} - \frac{\pi(N-1+2\alpha)}{4} \leq \pi,
\]

and

\[
\left( n + \frac{N-1}{2} + \frac{\alpha+1}{2} \right) = \frac{3\pi}{4} - \frac{\pi(N-1+2\alpha)}{4} \geq n\pi/2,
\]

we can now estimate \( I_2' \) as

\[
I_2' \geq C \int_{n\pi/2}^{\pi} \frac{\sin \omega \, d\omega}{\left( \omega + \frac{\pi(N-1+2\alpha)}{4} \right)^{\alpha - \frac{N-3}{2}}} = C \sum_{\tau=1}^{[\tau]} \int_{\tau\pi}^{(\tau+1)\pi} \frac{\sin \omega \, d\beta}{\left( \omega + \frac{\pi(N-1+2\alpha)}{4} \right)^{\alpha - \frac{N-3}{2}}}.
\]

Applying the change of variable \( \beta = t + \tau\pi \), we obtain

\[
I_2' \geq C \sum_{\tau=1}^{[\tau]} \int_{0}^{\pi} \frac{\sin(t + \tau\pi)) \, dt}{\left( t + \tau\pi + \frac{\pi(N-1+2\alpha)}{4} \right)^{\alpha - \frac{N-3}{2}}} \geq C \sum_{\tau=1}^{[\tau]} \int_{0}^{\pi} \frac{\sin \tau \, dt}{\tau^{\alpha - \frac{N-3}{2}}} = C \sum_{\tau=1}^{[\tau]} \frac{1}{\tau^{\alpha - \frac{N-3}{2}}}.
\]
Once more, the following 3 cases are considered:

If $\alpha < \frac{N-1}{2}$ then

$$I'_2 \geq C \int_1^{[\frac{n}{2}]^{-1}} t^{\frac{N-1}{2} - \alpha - 1} dt = \frac{C}{N-1 - \alpha} \left( \left( \left[ \frac{n}{2} \right] - 1 \right)^{\frac{N-1}{2} - \alpha} - 1 \right) \geq C n^{\frac{N-1}{2} - \alpha}. \quad (3.18)$$

If $\alpha = \frac{N-1}{2}$ then

$$I'_2 \geq C \int_1^{[\frac{n}{2}]^{-1}} \frac{dt}{t} = C \ln \left( \left[ \frac{n}{2} \right] - 1 \right) \geq C \ln n. \quad (3.19)$$

If $\alpha > \frac{N-1}{2}$ then

$$I'_2 \geq C \int_1^{[\frac{n}{2}]^{-1}} t^{\frac{N-1}{2} - \alpha - 1} dt = \frac{C}{\alpha - \frac{N-1}{2}} \left\{ 1 - \left( \left[ \frac{n}{2} \right] - 1 \right)^{\frac{N-1}{2} - \alpha} \right\} \geq C. \quad (3.20)$$

From (3.18), (3.19) and (3.20), we have

$$I'_2 \geq \begin{cases} C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.21)$$

Applying the reverse triangle inequality, $|a - b| \geq |a| - |b|$ with the Riesz mean of the spectral function from (3.5), gives the following

$$I_2 = C \int_{\frac{n}{2(n+1)}}^{\frac{n}{n+1}} |\Theta^1_1(\cos \gamma) + \Theta^2_2(\cos \gamma) + \Theta^3_3(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma$$

$$\geq C \int_{\frac{n}{2(n+1)}}^{\frac{n}{n+1}} |\Theta^1_1(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma - C \int_{\frac{n}{2(n+1)}}^{\frac{n}{n+1}} |\Theta^2_2(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma$$

$$- C \int_{\frac{n}{2(n+1)}}^{\frac{n}{n+1}} |\Theta^3_3(\cos \gamma)| \sin^{N-1} \gamma \, d\gamma$$

$$= I''_2 - I'''_2 - I''''_2. \quad (3.22)$$

Given the inequalities (3.21), (3.14) and (3.15) of (3.22), $I_2$ is bounded as follows

$$I_2 \geq \begin{cases} C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ 0, & \alpha > \frac{N-1}{2}. \end{cases} \quad (3.23)$$
and by manner of (3.2), $L_n^\alpha \geq I_2$. Consequently, gives the following estimates

$$L_n^\alpha \geq \begin{cases} 
C n^{\frac{N-1}{2} - \alpha}, & \alpha < \frac{N-1}{2}, \\
C \ln n, & \alpha = \frac{N-1}{2}, \\
0, & \alpha > \frac{N-1}{2}.
\end{cases}$$

Combination of the estimated upper bound of the Lebesgue constant in (3.17) and lower bound in (3.24), provides estimate of the Lebesgue constant, $L_n^\alpha$ in form of

$$C' n^{\frac{N-1}{2} - \alpha} < L_n^\alpha < C'' n^{\frac{N-1}{2} - \alpha}, \quad \alpha < \frac{N-1}{2},$$

$$C' \ln n < L_n^\alpha < C'' \ln n, \quad \alpha = \frac{N-1}{2},$$

$$0 < L_n^\alpha < C, \quad \alpha > \frac{N-1}{2}.$$

Theorem 2.1 is proved.

4. ACKNOWLEDGEMENT.

This paper has been supported by the Intenational Islamic University Malaysia under the post-doctoral scheme, Fundamental Research Grant Shceme, Grant Number: FRGS 14-142-0383.

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