On weakly SS-permutable subgroups on finite groups

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ON WEAKLY SS-PERMITABLE SUBGROUPS OF FINITE GROUPS

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Abstract. Suppose that $G$ is a finite group and $H$ is a subgroup of $G$. We say that: (1) $H$ is ss-permutable in $G$ if there is a subgroup $B$ of $G$ such that $G = HB$ and $H$ permutes with every Sylow subgroup of $B$; (2) $H$ is weakly ss-permutable in $G$ if there are a subnormal subgroup $T$ of $G$ and an ss-permutable subgroup $H_{ss}$ of $G$ contained in $H$ such that $G = HT$ and $H \cap T \leq H_{ss}$. We investigate the influence of weakly ss-permutable subgroups on the $p$-nilpotency and $p$-supersolvability of finite groups.

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1. INTRODUCTION

All groups considered in this paper are finite. A subgroup $H$ of a group $G$ is said to be $s$-permutable (or $s$-quasinormal) in $G$ if $H$ permutes with every Sylow subgroups of $G$ [5]. In 2008, Shirong Li, etc. [7], introduced the concept of ss-permutability (or ss-quasinormality ) which is a generalization of $s$-permutability. A subgroup $H$ of a group $G$ is called ss-permutable in $G$ if there is a subgroup $B$ of $G$ such that $G = HB$ and $H$ permutes with every Sylow subgroup of $B$. Li investigated the influence of ss-quasinormality of some subgroups on the structure of finite groups. More recently, Xuanli He, etc. [3], introduced the following concept, which covers both ss-permutability and Skiba’s weakly s-permutability [10] (A subgroup $H$ of a group $G$ is called weakly s-permutable in $G$ if there is a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{ss}$, where $H_{ss}$ is the maximal s-permutable subgroup of $G$ contained in $H$).

Definition 1. Let $H$ be a subgroup of $G$. $H$ is called weakly ss-permutable in $G$ if there is a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{ss}$, where $H_{ss}$ is an ss-permutable subgroup of $G$ contained in $H$.

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In [3], Xuanli He studied the influence of weakly ss-permutable subgroups on the supersolvability of groups. In the present paper we characterize $p$-nilpotency and $p$-supersolvability of finite groups with the assumption that some maximal subgroups or 2-maximal subgroups are weakly ss-permutable.

2. Preliminaries

**Lemma 1** ([3], Lemma 2.2). Let $U$ be a weakly ss-permutable subgroup of a group $G$ and $N$ a normal subgroup of $G$. Then

(a) If $U \leq H \leq G$, then $U$ is weakly ss-permutable in $H$.

(b) Suppose that $U$ is a $p$-group for some prime $p$. If $N \leq U$, then $U/N$ is weakly ss-permutable in $G/N$.

(c) Suppose $U$ is a $p$-group for some prime $p$ and $N$ is a $p'$-subgroup, then $UN/N$ is weakly ss-permutable in $G/N$.

(d) Suppose $U$ is a $p$-group for some prime $p$ and $U$ is not ss-permutable in $G$. Then $G$ has a normal subgroup $M$ such that $|G : M| = p$ and $G = UM$.

(e) If $U \leq O_p(G)$ for some prime $p$, then $U$ is weakly s-permutable in $G$.

**Lemma 2.** Let $p$ be a prime dividing the order of a group $G$ and $P$ a Sylow $p$-subgroup of $G$. If $N_G(P)$ is $p$-nilpotent and $P$ is abelian, then $G$ is $p$-nilpotent.

**Proof.** Since $N_G(P)$ is $p$-nilpotent, $N_G(P) = P \times H$, where $H$ is the normal $p$-complement of $N_G(P)$. Since $P$ is abelian and $[P, H] = 1$, we see that $C_G(P) = P \times H = N_G(P)$. Hence $G$ is $p$-nilpotent. □

**Lemma 3** ([1], A, 1.2). Let $U, V,$ and $W$ be subgroups of a group $G$. Then the following statements are equivalent:

1. $U \cap VW = (U \cap V)(U \cap W)$.
2. $UV \cap UW = U(V \cap W)$.

**Lemma 4** ([4], VI, 4.10). Assume that $A$ and $B$ are two subgroups of a group $G$ and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either $A$ or $B$ is contained in a nontrivial normal subgroup of $G$.

**Lemma 5** ([13], Lemma 3.16). Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let $P$ be a normal $p$-subgroup of a group $G$ such that $G/P \in \mathcal{F}$. If $G$ is $A_4$-free and $|P| \leq p^2$, then $G \in \mathcal{F}$.

**Lemma 6** ([6], Lemma 2.6). Let $H$ be a solvable normal subgroup of a group $G(H \neq 1)$. If every minimal normal subgroup of $G$ which is contained in $H$ is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of $H$ is the direct product of minimal normal subgroups of $G$ which are contained in $H$.

**Lemma 7** ([8], Lemma 2.2). If $P$ is a $s$-permutable $p$-subgroup of $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$. 
Lemma 8 ([12], Lemma 2.8), Let $M$ be a maximal subgroup of $G$ and $P$ a normal $p$-subgroup of $G$ such that $G = PM$, where $p$ is a prime. Then $P \cap M$ is a normal subgroup of $G$.

3. Results

Theorem 1. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$. If $N_G(P)$ is $p$-nilpotent and every maximal subgroups of $P$ is weakly $s\bar{s}$-permutable in $G$, then $G$ is $p$-nilpotent.

Proof. It is easy to see that the theorem holds when $p = 2$ by [3, Theorem 3.1], so it suffices to prove the theorem for the case of odd prime. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $G$ is not a non-abelian simple group.

By Lemma 2, $p^3||P|$ and so there exists a non-identity maximal subgroup $P_1$ of $P$. By the hypothesis, $P_1$ is weakly $s\bar{s}$-permutable in $G$. Then there are a subnormal subgroup $T$ of $G$ and an $s\bar{s}$-permutable subgroup $P_1$ of $G$ contained in $P_1$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{s\bar{s}}$. Suppose $G$ is simple, then $T = G$ and so $P_1 = (P_1)_{s\bar{s}}$ is $s\bar{s}$-permutable in $G$. By [7, Lemma 2.5] and Lemma 4, $G$ has a nontrivial normal subgroup, a contradiction.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. By Lemma 1, it is easy to see that every maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ is weakly $s\bar{s}$-permutable in $G/O_{p'}(G)$. Since $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is $p$-nilpotent, $G/O_{p'}(G)$ satisfies all the hypotheses of our theorem. The minimality of $G$ yields that $G/O_{p'}(G)$ is $p$-nilpotent, and so $G$ is $p$-nilpotent, a contradiction.

(3) If $M$ is a proper subgroup of $G$ with $P \leq M < G$, then $M$ is $p$-nilpotent.

It is clear to see $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is $p$-nilpotent. Applying Lemma 1, we immediately see that $M$ satisfies the hypotheses of our theorem. Now, by the minimality of $G$, $M$ is $p$-nilpotent.

(4) $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $p$-nilpotent. Moreover $\Phi(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. We shall prove that $G/N$ satisfies the hypothesis of the theorem. Since $P$ is a Sylow $p$-subgroup of $G$, $PN/N$ is a Sylow $p$-subgroup of $G/N$. If $|PN/N| \leq p^2$, then $G/N$ is $p$-nilpotent by Lemma 2. So we suppose $|PN/N| \geq p^3$. Let $M_1/N$ be a maximal subgroup of $PN/N$. Then $M_1 = N(M_1 \cap P)$. Let $P_1 = M_1 \cap P$. It follows that $P_1 \cap N = M_1 \cap P \cap N = P \cap N$ is a Sylow $p$-subgroup of $N$. Since $p = |PN/N|$:
$M_1/N = |PN : (M_1 \cap P)N| = |P : M_1 \cap P| = |P : P_1|$. $P_1$ is a maximal subgroup of $P$. By the hypothesis, $P_1$ is weakly ss-permutable in $G$, then there are a subnormal subgroup $T$ of $G$ and an ss-permutable subgroup $(P_1)_{ss}$ of $G$ contained in $P_1$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. So $G/N = P_1N/N \cdot TN/N = M_1/N \cdot TN/N$. Since $|\{N : P_1 \cap N, |N : T \cap N|\} = 1$, $(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap (P_1T)$. By Lemma 3, $(P_1N) \cap (TN/N) = (P_1 \cap T)N$. It follows that $(P_1N/N) \cap (TN/N) = (P_1 \cap T)N/N = (P_1 \cap T)N/N \leq (P_1)_{ss}N/N$. Since $(P_1)_{ss}N/N$ is ss-permutable in $G/N$ by [7, Lemma 2.1], $M_1/N$ is weakly ss-permutable in $G/N$. Since $N_{G/N}(PN/N) = N_G(P)N/N$ is $p$-nilpotent, we have that $G/N$ satisfies the hypothesis of the theorem. The choice of $G$ yields that $G/N$ is $p$-nilpotent. Consequently the uniqueness of $N$ and the fact that $\Phi(G) = 1$ are obvious.

(5) $G = PQ$ is solvable, where $Q$ is a Sylow $q$-subgroup of $G$ with $p \neq q$.

Since $G$ is not $p$-nilpotent, by a result of Thompson [11, Corollary], there exists a characteristic subgroup $H$ of $P$ such that $N_G(H)$ is not $p$-nilpotent. If $N_G(H) \neq G$, we must have $N_G(H)$ is $p$-nilpotent by step (3), a contradiction. We obtain $N_G(H) = G$. This leads to $O_p(G) \neq 1$. By step (4), $G/O_p(G)$ is $p$-nilpotent and therefore $G$ is $p$-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow $q$-subgroup of $Q$ such that $G_1 = PQ$ is a subgroup of $G$ [2, Theorem 6.3.5]. Invoking our claim (3) above, $G_1$ is $p$-nilpotent if $G_1 < G$. This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ [9, Theorem 9.3.1], a contradiction. Thus, we have proved that $G = PQ$ is solvable.

(6) The final contradiction.

By step (4), there exists a maximal subgroup $M$ of $G$ such that $G = MN$ and $M \cap N = 1$. Since $N$ is an elementary abelian $p$-group, $N \leq C_G(N)$ and $C_G(N) \cap M \leq G$. By the uniqueness of $N$, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence $N = O_p(G) = C_G(N)$. Obviously $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we take a maximal subgroup $P_1$ of $P$ such that $P \cap M \leq P_1$. By our hypotheses, $P_1$ is weakly ss-permutable in $G$, then there are a subnormal subgroup $T$ of $G$ and an ss-permutable subgroup $(P_1)_{ss}$ of $G$ contained in $P_1$ such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ss}$. Since $|G : T|$ is a power of $p$ and $T \vartriangleleft G$, $O_p(G) \leq T$. From the fact that $N$ is the unique minimal normal subgroup of $G$, we have $N \leq O_p(G) \leq T$. Hence $N \cap P_1 = N \cap (P_1)_{ss}$. By [7, Lemma 2.5], $(P_1)_{ss}G_q = G_q(P_1)_{ss}$ for any Sylow $q$-subgroup $G_q$ of $G$, where $q \neq p$. Since $N \cap P_1 = N \cap (P_1)_{ss} = N \cap (P_1)_{ss}G_q$, we have that $N \cap P_1$ is normalized by $G_q$. Obviously, $N \cap P_1 \leq P$. Therefore, $N \cap P_1$ is normal in $G$. The minimality of $N$ implies that $N \cap P_1 = 1$ or $N \cap P_1 = N$. If $N \cap P_1 = N$, then $N \leq P_1$ and so $P = NP_1 = P_1$, a contradiction. Hence we have $N \cap P_1 = 1$. Since $|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p$, $P_1 \cap N$ is a maximal subgroup of $N$.阴


Therefore \(|N| = p\), and so \(\text{Aut}(N)\) is a cyclic group of order \(p - 1\). If \(q > p\), then \(NQ\) is \(p\)-nilpotent and therefore \(Q \leq C_G(N) = N\), a contradiction. On the other hand, if \(q < p\), then, since \(N = C_G(N)\), we see that \(M \cong G/N = N_G(N)/C_G(N)\) is isomorphic to a subgroup of \(\text{Aut}(N)\) and therefore \(M\), and in particular \(Q\), is cyclic. Since \(Q\) is a cyclic group and \(q < p\), we know that \(G\) is \(q\)-nilpotent and therefore \(P\) is normal in \(G\). Hence \(N_G(P) = G\) is \(p\)-nilpotent, a contradiction. \(\square\)

**Corollary 1.** Let \(p\) be a prime dividing the order of a group \(G\) and \(H\) a normal subgroup of \(G\) such that \(G/H\) is \(p\)-nilpotent. If \(N_G(P)\) is \(p\)-nilpotent and there exists a Sylow \(p\)-subgroup \(P\) of \(H\) such that every maximal subgroup of \(P\) is weakly \(ss\)-permutable in \(G\), then \(G\) is \(p\)-nilpotent.

**Proof.** It is clear that \(N_H(P)\) is \(p\)-nilpotent and that every maximal subgroup of \(P\) is weakly \(ss\)-permutable in \(H\). By Theorem 1, \(H\) is \(p\)-nilpotent. Now let \(H_{p'}\) be the normal Hall \(p'\)-subgroup of \(H\). Then \(H_{p'}\) is normal in \(G\). If \(H_{p'} \neq 1\), then we consider \(G/H_{p'}\). It is easy to see that \(G/H_{p'}\) satisfies all the hypotheses of our corollary for the normal subgroup \(H/H_{p'}\) of \(G/H_{p'}\) by Lemma 1. Now by induction, we see that \(G/H_{p'}\) is \(p\)-nilpotent and so \(G\) is \(p\)-nilpotent. Hence we may assume \(H_{p'} = 1\) and therefore \(H = P\) is a \(p\)-group. In this case, by our hypotheses, \(N_G(P) = G\) is \(p\)-nilpotent. \(\square\)

**Theorem 2.** Let \(p\) be the smallest prime dividing the order of a group \(G\) and \(P\) a Sylow \(p\)-subgroup of \(G\). If \(G\) is \(A_4\)-free and every \(2\)-maximal subgroup of \(P\) is weakly \(ss\)-permutable in \(G\), then \(G\) is \(p\)-nilpotent.

**Proof.** Let \(P_2\) be a \(2\)-maximal subgroup of \(P\). By our hypotheses, \(P_2\) is weakly \(ss\)-permutable in \(G\), then there are a subnormal subgroup \(T\) of \(G\) and an \(ss\)-permutable subgroup \((P_2)_s\) of \(G\) contained in \(P_2\) such that \(G = P_2T\) and \(P_2 \cap T \leq (P_2)_s\). If \(P_2\) is not \(ss\)-permutable in \(G\), then \(G\) has a normal subgroup \(M\) such that \(|G: M| = p\) by Lemma 1(d). It follows that every maximal subgroup of \(P \cap M\) is weakly \(ss\)-permutable in \(M\) by Lemma 1(1). Hence we have that \(M\) is \(p\)-nilpotent by [3, Theorem 3.1]. It is easy to see that \(G\) is \(p\)-nilpotent. Now we may assume that every \(2\)-maximal subgroup of \(P\) is \(ss\)-permutable in \(G\). By [7, Theorem 1.7], we get that \(G\) is \(p\)-nilpotent too. \(\square\)

**Corollary 2.** Suppose that every \(2\)-maximal subgroup of any Sylow subgroup of a group \(G\) is weakly \(ss\)-permutable in \(G\). If \(G\) is \(A_4\)-free, then \(G\) is a Sylow tower group of supersolvable type.

**Proof.** Let \(p\) be the smallest prime dividing \(|G|\) and \(P\) a Sylow \(p\)-subgroup of \(G\). Then every \(2\)-maximal subgroup of \(P\) is weakly \(ss\)-permutable in \(G\). By Theorem 2, \(G\) is \(p\)-nilpotent. Let \(U\) be the normal \(p\)-complement of \(G\). By Lemma 1, \(U\) satisfies the hypothesis of the Corollary. It follows by induction that \(U\), and hence \(G\) is a Sylow tower group of supersolvable type. \(\square\)
Corollary 3. Let $p$ be the smallest prime dividing the order of a group $G$ and $G$ is $A_4$-free. Assume that $H$ is a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent. If there exists a Sylow $p$-subgroup $P$ of $H$ such that every 2-maximal subgroup of $P$ is weakly $ss$-permutable in $G$, then $G$ is $p$-nilpotent.

Proof. By Lemma 1, every 2-maximal subgroup of $P$ is weakly $ss$-permutable in $H$. By Theorem 2, $H$ is $p$-nilpotent. Now, let $H_p'$ be the normal $p$-complement of $H$. Then $H_p' \leq G$. By using the arguments as in the proof of Corollary 1, we may assume that $H_p' = 1$ and $H = P$ is a $p$-group. Since $G/H$ is $p$-nilpotent, let $K/H$ be the normal $p$-complement of $G/H$. By Schur-Zassenhaus’s theorem, there exists a Hall $p'$-subgroup $K_{p'}$ of $K$ such that $K = HK_{p'}$. By Theorem 2, $K$ is $p$-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal $p$-complement of $G$. □

Corollary 4. Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is 2-nilpotent. If there exists a Sylow 2-subgroup $P$ of $H$ such that every 2-maximal subgroup of $P$ is weakly $ss$-permutable in $G$ and $3 \nmid |G|$, then $G$ is 2-nilpotent.

Corollary 5. Let $G$ be a group of odd order and $H$ a normal subgroup of $G$ such that $G/H$ is 3-nilpotent. If there exists a Sylow 3-subgroup $P$ of $H$ such that every 2-maximal subgroup of $P$ is weakly $ss$-permutable in $G$, then $G$ is 3-nilpotent.

Theorem 3. Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and $G$ is $A_4$-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup $H$ of $G$ such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of $H$ is weakly $ss$-permutable in $G$.

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let $G$ be a counterexample of minimal order. By Lemma 1, every 2-maximal subgroup of any Sylow subgroup of $H$ is weakly $ss$-permutable in $H$. By Corollary 2, $H$ is a Sylow tower group of supersolvable type. Let $p$ be the maximal prime divisor of $|H|$ and let $P$ be a Sylow $p$-subgroup of $H$. Then $P$ must be a normal subgroup of $G$ and every 2-maximal subgroup of $P$ is weakly $ss$-permutable in $G$. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of $H/P$ are weakly $ss$-permutable in $G/P$ by Lemma 1 and $G/P$ is $A_4$-free. By the minimality of $G$, we have $G/P \in \mathcal{F}$. Let $N$ be a minimal normal subgroup of $G$ contained in $P$.

1. $P = N$.

If $N < P$, then $(G/N)/(P/N) \cong G/P \in \mathcal{F}$. We will show that $G/N \in \mathcal{F}$. If $|P/N| \leq p^2$, then $G/N \in \mathcal{F}$ by Lemma 5. If $|P/N| > p^2$, then every 2-maximal subgroup of $P/N$ is weakly $ss$-permutable in $G/N$ by Lemma 1. By the minimality of $G$, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, $N$ is the unique minimal normal subgroup of $G$ contained in $P$ and $N \not\subseteq \Phi(G)$. By Lemma 6, it follows that $P = F(P) = N$. 

(2) The final contradiction.

Since $N \trianglelefteq G$, we may take a 2-maximal $N_2$ of $N$ such that $N_2 \trianglelefteq G_p$, where $G_p$ is a Sylow $p$-subgroup of $G$. By the hypothesis, $N_2$ is weakly ss-permutable in $G$. Then there are a subnormal subgroup $T$ of $G$ and an ss-permutable subgroup $(N_2)_{ss}$ of $G$ contained in $N_2$ such that $G = N_2T$ and $N_2 \cap T \leq (N_2)_{ss}$. Thus $G = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in $G$ and $N$ is minimal normal in $G$, $N \cap T = N$. It follows that $T = G$ and so $N_2 = (N_2)_{ss}$ is ss-permutable in $G$. By [7, Lemma 2.2], $N_2$ is $s$-permutable in $G$. By Lemma 7, $O^p(G) \leq N_G(N_2)$. Thus $N_2 \leq G_pO^p(G) = G$. It follows that $N_2 = 1$ and so $|N| = p^2$. By Lemma 5, $G \in F$, a contradiction. \square

**Theorem 4.** Let $p$ be a prime, $G$ be a $p$-solvable group. If there exists a Sylow $p$-subgroup $P$ of $G$ such that every maximal subgroup of $P$ is weakly ss-permutable in $G$, then $G$ is $p$-supersolvable.

**Proof.** Suppose that the theorem is false and let $G$ be a counterexample of minimal order.

(1) $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $p$-supersolvable.

Let $N$ be a minimal normal subgroup of $G$. Since $P$ is the Sylow $p$-subgroup of $G$, $PN/N$ is the Sylow $p$-subgroup of $G/N$. Let $M/N$ be a maximal subgroup of $PN/N$, then $M = (M \cap P)N$. Let $P_1 = M \cap P$. Obviously, $P_1$ is the maximal subgroup of $P$. Since $G$ is $p$-solvable, $N$ is an elementary abelian $p$-group or $p'$-group. If $N$ is a $p'$-group, then $M/N = P_1N/N$. If $N$ is a $p$-group, then $M/N = P_1/N$. By hypothesis, $P_1$ is weakly ss-permutable in $G$ and so $M/N$ is weakly ss-permutable in $G/N$ by Lemma 1. Hence $G/N$ satisfies all the hypotheses of our theorem. The minimal choice of $G$ implies that $G/N$ is $p$-supersolvable. Clearly, $N$ is the unique minimal normal subgroup of $G$ as the class of $p$-supersolvable group is a formation.

(2) $O_p'(G) = 1$.

If $O_p'(G) \neq 1$, then $G/O_p'(G) \cong (G/N)/(O_p'(G)/N)$ is $p$-supersolvable by step (1) and so $G$ is $p$-supersolvable, a contradiction.

(3) The final contradiction.

Since $G$ is $p$-solvable, $N$ is an elementary abelian $p$-group by step (2). If $N$ is contained in all maximal subgroups of $G$, then $N \leq \Phi(G)$ and so $G$ is $p$-supersolvable, a contradiction. Hence there exists a maximal subgroup $M$ of $G$ such that $G = NM$ and $N \cap M = 1$. Applying Lemma 8, we have $O_p(G) \cap M \leq G$, so that $O_p(G) \cap M = 1$ and $N = O_p(G)$. By using the arguments as in the proof of Theorem 1, we have $|N| = p$ and so $G$ is $p$-supersolvable. \square
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