Q-FUNCTIONS AND BOUNDARY TRIPLETS 
OF NONNEGATIVE OPERATORS

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Dedicated to Lev Aronovich Sakhnovich in the occasion of his 80-th birthday

Abstract.
Operator-valued $Q$-functions for special pairs of nonnegative selfadjoint extensions of nonnegative not necessarily densely defined operators are defined and their analytical properties are studied. It is shown that the Kreĭn-Ovcharenko statement announced in [37] is valid only for $Q$-functions of densely defined symmetric operators with finite deficiency indices. A general class of boundary triplets for a densely defined nonnegative operator is constructed such that the corresponding Weyl functions are of Kreĭn-Ovcharenko type.

1. Introduction

Notations. We use the symbols $\text{dom } T$, $\text{ran } T$, $\text{ker } T$ for the domain, the range, and the null-subspace of a linear operator $T$. The closures of $\text{dom } T$, $\text{ran } T$ are denoted by $\overline{\text{dom } T}$, $\overline{\text{ran } T}$, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by $I$ and sometimes by $I_{\mathbb{H}}$. If $\mathfrak{L}$ is a subspace, i.e., a closed linear subset of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T|\mathcal{N}$ means the restriction of a linear operator $T$ on the set $\mathcal{N} \subset \text{dom } T$. The resolvent set of $T$ is denoted by $\rho(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathfrak{L}(\mathfrak{H}, \mathfrak{K})$ and the Banach algebra $\mathfrak{L}(\mathfrak{H}, \mathfrak{H})$ by $\mathfrak{L}(\mathfrak{H})$. A linear operator $A$ in a Hilbert space is called nonnegative if $(Af, f) \geq 0$ for all $f \in \text{dom } A$.

Let $\mathfrak{H}$ be a separable complex Hilbert space and let $S$ be a closed symmetric operator with equal deficiency indices in $\mathfrak{H}$. We do not suppose that $S$ is densely defined. As it is well known the Kreĭn’s resolvent formula for canonical and generalized resolvents plays crucial role in the spectral theory of selfadjoint extensions and its numerous applications. The essential part of this formula is the $Q$-function of $S$. Denote by $\mathfrak{N}_z$ the defect subspace of $S$, i.e.,

$$\mathfrak{N}_z = \mathfrak{H} \ominus \text{ran } (S - zI).$$

or, equivalently, $\mathfrak{N}_z = \text{ker}(S^* - zI)$. Choose a selfadjoint extension $\widetilde{S}$ of $S$. The following definitions can be found in M. Kreĭn and H. Langer papers [32], [33], [34] for a densely defined $S$ and in the Langer and Textorius paper [38] for the general case of a symmetric linear relation $S$.

Definition 1.1. Let $\mathfrak{H}$ be a Hilbert space whose dimension is equal to the deficiency number of $S$. The function

$$\rho(\widetilde{S}) \ni z \mapsto \Gamma(z) \in \mathfrak{L}(\mathfrak{H}, \mathfrak{H})$$

is called the $\gamma$-field, corresponding to $\widetilde{S}$ if

(1) the operator $\Gamma(z)$ isomorphically maps $\mathfrak{H}$ onto $\mathfrak{N}_z$ for all $z \in \rho(\widetilde{S})$, 
(2) for every \( z, \zeta \in \rho(\tilde{S}) \) the identity
\[
\Gamma(z) = \Gamma(\zeta) + (z - \zeta)(\tilde{S} - zI)^{-1}\Gamma(\zeta)
\]
holds.

**Definition 1.2.** Let \( \Gamma(z) \) be a \( \gamma \)-field corresponding to \( \tilde{S} \). An operator-valued function \( Q(z) \in L(H) \) with the property
\[
Q(z) - Q^*(\zeta) = (z - \zeta)\Gamma^*(\zeta)\Gamma(z), \quad z, \zeta \in \rho(\tilde{S})
\]
is called the \( Q \)-function of \( S \) corresponding to the \( \gamma \)-field \( \Gamma(z) \).

The \( \gamma \)-field corresponding to \( \tilde{S} \) can be constructed as follows: fix \( \zeta_0 \in \rho(\tilde{S}) \) and let \( \Gamma_{\zeta_0} \in L(H, \tilde{S}) \) be a bijection from \( H \) onto \( \mathfrak{N}_{\zeta_0} \). Then clearly the function
\[
\Gamma(z) = (\tilde{S} - \zeta_0 I)(\tilde{S} - zI)^{-1}\Gamma_{\zeta_0} = \Gamma_{\zeta_0} + (z - \zeta_0)(\tilde{S} - zI)^{-1}\Gamma_{\zeta_0}, \quad z \in \rho(\tilde{S})
\]
is a \( \gamma \)-field corresponding to \( \tilde{S} \). It follows from Definition 1.2 that
\[
Q(z) = C - i\text{Im} \zeta_0 \Gamma^*_{\zeta_0} \Gamma_{\zeta_0} + (z - \zeta_0)\Gamma^*_{\zeta_0} \Gamma_{\zeta_0},
\]
where \( C = \text{Re} Q(\zeta_0) \in L(H) \) is a selfadjoint operator. Thus, the \( Q \)-function is defined uniquely up to a bounded selfadjoint term in \( H \) and it is a Herglotz-Nevanlinna function. Moreover, for every \( z, \text{Im} z \neq 0, -i\text{Im} z \) \((Q(z) - Q^*(z))\) is positive definite. Hence, \(-Q^{-1}(z), \text{Im} z \neq 0, \) is a Herglotz-Nevanlinna function, too. Definition 1.2 combined with (1.1) gives the following representation for \( Q \):
\[
Q(z) = C - i\text{Im} \zeta_0 \Gamma^*_{\zeta_0} \Gamma_{\zeta_0} + (z - \zeta_0)\Gamma^*_{\zeta_0} \Gamma_{\zeta_0} \left( \Gamma_{\zeta_0} + (z - \zeta_0)(\tilde{S} - zI)^{-1}\Gamma_{\zeta_0} \right).
\]

One of the main results of the Krein–Langer–Textorius theory of \( Q \)-functions is the following statement: If \( Q \)-functions of two simple closed densely defined symmetric operators \( S_1 \) and \( S_2 \) coincide, then the operators \( S_1 \) and \( S_2 \) are unitarily equivalent.

This result remains valid if condition (1) in Definition 1.1 is replaced with a little bit weaker one: \( \Gamma(z) \) is one-to-one and has dense range in \( \mathfrak{N}_z \) at least for one (and then for all) \( z \) [24].

M. Krein and I. Ovcharenko in their papers [36] and [37] defined special \( Q \)-functions for a densely defined closed nonnegative operator \( S \) in the Hilbert space \( \tilde{S} \) with disjoint Friedrichs and Krein extensions \( S_F \) and \( S_K \) [31] \((\text{dom } S_F \cap \text{dom } S_K = \text{dom } S)\). Let \( H \) be a Hilbert space with \( \dim H = \text{dim } S \). Let \( a \geq 0 \) and let
\[
C_a := 2a \left( (S_K + aI)^{-1} - (S_F + aI)^{-1} \right), \quad C := C_1 = B_M - B_\mu,
\]
where \( B_M = (I - S_K)(I + S_K)^{-1}, \ B_\mu = (I - S_F)(I + S_F)^{-1}. \) Define the operator-valued functions \( \gamma_F(\lambda) \) and \( \gamma_K(\lambda) \)
\[
\mathbb{C} \setminus \mathbb{R}_+ \ni \lambda \mapsto L(H, \tilde{S}),
\]
as follows
\begin{enumerate}
\item \( \text{ran } \gamma_F(\lambda) = \text{ran } \gamma_K(\lambda) = \mathfrak{N}_\lambda \) for each \( \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \), where \( \mathfrak{N}_\lambda := \ker(S^* - \lambda I), \)
\item \( \gamma_F(\lambda) - \gamma_F(z) = (\lambda - z)(S_F - \lambda I)^{-1}\gamma_F(z), \ \gamma_K(\lambda) - \gamma_K(z) = (\lambda - z)(S_K - \lambda I)^{-1}\gamma_K(z), \)
\item \( \text{ran } \gamma_F(-a) = \text{ran } \gamma_K(-a) = C_a^{1/2} \) for each \( a > 0. \)
\end{enumerate}
The $L(H)$-valued functions $Q_F(\lambda)$ and $Q_K(\lambda)$ are defined as follows:

1) $Q_F(\lambda) = Q_F^0(z) = (\lambda - z)\gamma^F(z)\gamma^F(\lambda)$, $\lambda, z \in \mathbb{C} \setminus \mathbb{R}_+$,

2) $\lim_{x \to 0} Q_F(x) = 0$,

3) $Q_K(\lambda) = Q_K^0(z) = (\lambda - z)\gamma^K(z)\gamma^K(\lambda)$, $\lambda, z \in \mathbb{C} \setminus \mathbb{R}_+$,

4) $\lim_{x \to -\infty} Q_K(x) = 0$.

For example, one can take $H = \mathfrak{N} := \ker(S^* + I)$ and

\[
\begin{align*}
\gamma^F_0(\lambda) &:= (I + (\lambda + 1)(S_F - \lambda I)^{-1}) C^{1/2} | \mathfrak{N}, \\
\gamma^K_0(\lambda) &:= (I + (\lambda + 1)(S_K - \lambda I)^{-1}) C^{1/2} | \mathfrak{N}.
\end{align*}
\]

Then

\[
\begin{align*}
Q_F^0(\lambda) &= -2I_S + (\lambda + 1)C^{1/2} (I + (\lambda + 1)(S_F - \lambda I)^{-1}) C^{1/2} | \mathfrak{N}, \\
Q_K^0(\lambda) &= 2I_S + (\lambda + 1)C^{1/2} (I + (\lambda + 1)(S_K - \lambda I)^{-1}) C^{1/2} | \mathfrak{N}.
\end{align*}
\]

The following statement is formulated without proof in [37]. Let $Q$ be an $L(H)$-valued function holomorphic on $\mathbb{C} \setminus [0, \infty)$. Then $Q$ is the $Q_K$-function ($Q_F$-function) of a densely defined closed nonnegative operator if and only if the following conditions hold true:

1) $Q^{-1}(\lambda) \in L(H)$ for each $\lambda \in \mathbb{C} \setminus [0, \infty)$;

2) $\lim_{x \to 0} (Q(x)g, g) = \infty$ for each $g \neq 0$ \hspace{0.5cm} (2') $\lim_{x \to 0} Q(x) = 0$;

3) $\lim_{x \to -\infty} Q(x) = 0$ \hspace{0.5cm} (3') $\lim_{x \to -\infty} (Q(x)g, g) = -\infty$ for each $g \neq 0$;

4) $\lim_{x \to -\infty} (xQ(x)g, g) = -\infty$ for each $g \neq 0$ \hspace{0.5cm} (4') $\lim_{x \to -\infty} x^{-1} Q(x) = 0$.

In this paper it is shown that this statement holds true only for the case $\dim H < \infty$. More precisely, given an arbitrary closed not necessarily densely defined nonnegative symmetric operator $S$ with infinite defect numbers and disjoint nonnegative selfadjoint (operator) extensions (the case $\dom S = \{0\}$ is possible), we construct special pairs $\{\tilde{S}_0, \tilde{S}_1\}$ of disjoint $(\tilde{S}_0 \cap \tilde{S}_1 = S)$ nonnegative selfadjoint extensions different from the pair $\{S_F, S_K\}$ and define the corresponding $Q$-functions $\tilde{Q}_0$ and $\tilde{Q}_1$ of Krein-Ovcharenko type, i.e., possessing properties mentioned in the above statement. Furthermore, for the case of a densely defined nonnegative operator $S$ we construct a new general class of positive (generalized) boundary triplets. This class of boundary triplets extends the notions of (ordinary and generalized) basic boundary triplets as well as the earlier notions of positive boundary triplets appearing in [1] [7] [9] [17] [22] [29]. A key assumption used in the construction is the existence of a pair $\{\tilde{S}_0, \tilde{S}_1\}$ of nonnegative selfadjoint extensions of $S$ which are disjoint, i.e. $\dom \tilde{S}_1 \cap \dom \tilde{S}_0 = \dom S$, and whose associated closed forms satisfy the inclusion

\[
\tilde{S}_0[\cdot, \cdot] \subset \tilde{S}_1[\cdot, \cdot].
\]

With some further condition of the pair $\{\tilde{S}_0, \tilde{S}_1\}$ this class of boundary triplets is specialized to a class of boundary triplets leading to realization results for the classes of $Q$-functions of Krein-Ovcharenko type as introduced above.

In this paper we proceed on the base of the dual situation related to a non-densely defined Hermitian contraction $B$ and its selfadjoint contractive extensions. Recall that so-called $Q_\mu$- and $Q_M$-functions were introduced and studied in [34]. These functions are associated with the extremal extensions $B_\mu$ and $B_M$ of $B$ which are fundamental concepts going back to [31]. In [10] the $Q$-functions of Krein-Ovcharenko type, formally similar to $Q_\mu$- and $Q_M$-functions, were considered and therein analogous counterexamples to the statements of Theorem 2.2 in [35] were given.
In the last section of this paper boundary triplet technique plays a central role; the basic notions and some fundamental results related to the boundary triplets, their Weyl functions, boundary relations and their Weyl families for the adjoint of a symmetric linear relation can be found in [16, 17, 18, 19, 22, 23].

2. Basic Preliminaries

2.1. Closed nonnegative forms and nonnegative selfadjoint relations. Let $\mathfrak{h} = \mathfrak{h}[\cdot, \cdot]$ be a nonnegative form in the Hilbert space $\mathfrak{H}$ with domain $\text{dom} \mathfrak{h}$. The notation $\mathfrak{h}[h]$ will be used to denote $\mathfrak{h}[h, h]$, $h \in \text{dom} \mathfrak{h}$. The form $\mathfrak{h}$ is closed if

$$h_n \to h, \quad \mathfrak{h}[h_n - h_m] \to 0, \quad h_n \in \text{dom} \mathfrak{h}, \quad h \in \mathfrak{H}, \quad m, n \to \infty,$$

imply that $h \in \text{dom} \mathfrak{h}$ and $\mathfrak{h}[h_n - h] \to 0$. The form $\mathfrak{h}$ is closable if

$$h_n \to 0, \quad \mathfrak{h}[h_n - h_m] \to 0, \quad h_n \in \text{dom} \mathfrak{h} \quad \Rightarrow \quad \mathfrak{h}[h_n] \to 0.$$ 

The form $\mathfrak{h}$ is closable if and only if it has a closed extension, and in this case the closure of the form is the smallest closed extension of $\mathfrak{h}$. The inequality $\mathfrak{h}_1 \geq \mathfrak{h}_2$ for semibounded forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$ is defined by

$$\text{dom} \mathfrak{h}_1 \subset \text{dom} \mathfrak{h}_2, \quad \mathfrak{h}_1[h] \geq \mathfrak{h}_2[h], \quad h \in \text{dom} \mathfrak{h}_1.$$ 

In particular, $\mathfrak{h}_1 \subset \mathfrak{h}_2$ implies $\mathfrak{h}_1 \geq \mathfrak{h}_2$. If the forms $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are closable, the inequality $\mathfrak{h}_1 \geq \mathfrak{h}_2$ is preserved by their closures.

There is a one-to-one correspondence between all closed nonnegative forms $\mathfrak{h}$ and all nonnegative selfadjoint relations $H$ in $\mathfrak{H}$ via $\text{dom} H \subset \text{dom} \mathfrak{h}$ and

$$\mathfrak{h}[h, k] = (H_s h, k), \quad h \in \text{dom} H, \quad k \in \text{dom} \mathfrak{h};$$

here $H_s$ stands for the nonnegative selfadjoint operator part of $H$. In what follows the form corresponding to $H$ is shortly denoted by $H[\cdot, \cdot]$. Recall that a selfadjoint relation $H$ admits an orthogonal decomposition $H = H_s \oplus (\{0\} \times \text{mul} H)$, where $H_s$ is the selfadjoint operator part $H_s = PH$ acting on $\overline{\text{dom} H} = \mathfrak{H} \oplus \text{mul} H$ and $P$ stands for the orthogonal projection onto $\overline{\text{dom} H}$. The functional calculus for a selfadjoint relation can be defined on $\mathbb{R} \cup \{\infty\}$ by interpreting $\text{mul} H$ as an eigenspace at $\infty$; in particular, one defines $H^\frac{1}{2} = H_s^\frac{1}{2} \oplus (\{0\} \times \text{mul} H)$. The one-to-one correspondence in (2.2) can also be expressed as follows

$$H[h, k] = (h', k), \quad \{h, h', k\} \in H, \quad k \in \text{dom} \mathfrak{h},$$

since $(h', k) = (h', Pf) = (H_s h, k)$. The one-to-one correspondence can be made more explicit via the second representation theorem:

$$H[h, k] = (H_s^{\frac{1}{2}} h, H_s^{\frac{1}{2}} k), \quad h, k \in \text{dom} H[\cdot, \cdot] = \text{dom} H_s^{\frac{1}{2}}.$$ 

The formulas (2.2), (2.3) are analogs of Kato’s representation theorems for, in general, nondensely defined closed semi-bounded forms in [28, Section VI]; see e.g. [40, 9, 26].

Let $H_1$ and $H_2$ be nonnegative selfadjoint relations in $\mathfrak{H}$, then $H_1$ and $H_2$ are said to satisfy the inequality $H_1 \geq H_2$ if

$$\text{dom} H_1^{\frac{1}{2}, s} \subset \text{dom} H_2^{\frac{1}{2}, s} \quad \text{and} \quad \|H_1^{\frac{1}{2}, s} h\| \geq \|H_2^{\frac{1}{2}, s} h\|, \quad h \in \text{dom} H_1^{\frac{1}{2}, s}.$$ 

This means that the closed nonnegative forms $H_1[\cdot, \cdot]$ and $H_2[\cdot, \cdot]$ generated by $H_1$ and $H_2$ satisfy the inequality $H_1[\cdot, \cdot] \geq H_2[\cdot, \cdot]$; see (2.1), (2.3).

Given a form $\mathfrak{h}_1$ one can generate a class of forms by means of bounded operators.
Lemma 2.1. Let $\mathfrak{h}$ be a nonnegative form with $\text{dom}\, \mathfrak{h} \subset \mathfrak{S}$ and let $C$ be a bounded operator in $\mathfrak{S}$. Then

$$\mathfrak{h}^C[h, k] = \mathfrak{h}[Ch, Ck]$$

is also a nonnegative form. Moreover, if $\mathfrak{h}$ is closed or closable the same is true for $\mathfrak{h}^C$.

Proof. It is clear that $\mathfrak{h}^C$ defines a nonnegative form in $\mathfrak{S}$ whose domain is the preimage $C^{-1}(\text{ran} \, C \cap \text{dom} \, \mathfrak{h})$, so that $\text{dom} \, \mathfrak{h}^C$ can be even a zero subspace. Now assume that $\mathfrak{h}$ is closed and let $h_n \in \text{dom} \, \mathfrak{h}^C$ with $h_n \to h$ and $\mathfrak{h}^C[h_n - h_m] = \mathfrak{h}[Ch_n - Ch_m] \to 0$. Since $C$ is bounded (continuous) $Ch_n \to Ch$ and by closability of $\mathfrak{h}$ one concludes that $Ch \in \text{dom} \, \mathfrak{h}$ and $\mathfrak{h}[Ch_n - Ch] \to 0$. Consequently, $h \in \text{dom} \, \mathfrak{h}^C$ and $\mathfrak{h}^C[h_n - h] \to 0$ and thus $\mathfrak{h}^C$ is closed. Similarly one proves that $\mathfrak{h}^C$ is closable whenever $\mathfrak{h}$ is closable. \hfill \Box

The next result gives various characterizations for the inequality $H_1 \geq H_2$; it can be viewed as an extension of Douglas factorization in the present situation of linear relations, cf. [20].

Proposition 2.2. Let $H_1$ and $H_2$ be nonnegative selfadjoint relations in $\mathfrak{S}$. Then the following statements are equivalent:

(i) $H_1 \geq H_2$;
(ii) there exists a contraction $C \in \mathcal{L}(\mathcal{H})$ with $\text{ran} \, C \subset \overline{\text{dom} \, H_2}$ and $\ker H_1 \subset \ker C$ such that

$$CH_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}} \quad (\iff \quad H_2^{\frac{1}{2}} \subset H_1^{\frac{1}{2}})$$

in fact with these conditions $C$ is uniquely determined and it satisfies also the following inclusions

$$CH_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}}; \quad \text{ran} \, C^* \subset \overline{\text{dom} \, H_1 \ominus \ker H_1}; \quad \text{mul} \, H_2 \ominus \ker H_2 \subset \ker C^*;$$

(iii) there exists a contraction $C \in [\mathcal{H}]$ with $\text{ran} \, C \subset \overline{\text{dom} \, H_2}$ and $\ker H_1 \subset \ker C$ such that

$$(P_1 H_2^{\frac{1}{2}} h, P_1 H_2^{\frac{1}{2}} k) = H_{1,s}[C^* h, C^* k], \quad h, k \in \text{dom} \, H_2^{\frac{1}{2}};$$

where the form $[H_{1,s}^0]$ is as defined in Lemma 2.1 (see also (2.3));
(iv) there exists a contraction $C_1 \in [\mathfrak{S}]$ with $\text{ran} \, C_1 \subset \overline{\text{dom} \, H_2}$ such that

$$(H_1 + I)^{-\frac{1}{2}} = (H_2 + I)^{-\frac{1}{2}} C_1,$$

where $C_1$ is uniquely determined and $\ker C_1 = \text{mul} \, H_1$;
(v) for some nonnegative contraction $M$, $0 \leq M \leq I$, with $\text{ran} \, M \subset \overline{\text{dom} \, H_2}$ one has

$$(H_1 + I)^{-1} = (H_2 + I)^{-\frac{1}{2}} M (H_2 + I)^{-\frac{1}{2}};$$

(vi) the inequality $(H_1 + I)^{-1} \leq (H_2 + I)^{-1}$ holds.

Proof. Since $H_j$ is selfadjoint it admits an orthogonal decomposition $H_j = H_{s,j} \oplus \{0\} \times \text{mul} \, H_j$, where $H_{s,j} = P_j H_j$ is the selfadjoint operator part acting on $\overline{\text{dom} \, H_j} = \mathfrak{S} \ominus \text{mul} \, H_j$ and $P_j$ stands for the orthogonal projection onto $\overline{\text{dom} \, H_j}$; $j = 1, 2$.

(i) $\Rightarrow$ (ii) Let $f \in \overline{\text{dom} \, H_1^{\frac{1}{2}}}$ and define $C_0$ by setting $C_0 H_{1,s}^{\frac{1}{2}} f = H_{2,s}^{\frac{1}{2}} f$. Then the inequality in (2.4) shows that $\|C_0 H_{1,s}^{\frac{1}{2}} f\| = \|H_{2,s}^{\frac{1}{2}} f\| \leq \|H_{1,s}^{\frac{1}{2}} f\|$, and hence $C_0$ is a well-defined and contractive operator, which can be continued to a contraction from the closed subspace $\text{ran} \, H_{1,s}$ into the closed subspace $\overline{\text{ran} \, H_2,s}$. By extending $C_0$ to $\mathfrak{S} \ominus \overline{\text{ran} \, H_{1,s}}$ as a zero operator
gives a contractive operator $C \in [\mathcal{H}]$ with $\text{ran} \ C \subset \overline{\text{dom}} \ H_2 \ominus \ker \ H_2$ and $\text{mul} \ H_1 \oplus \ker \ H_1 \subset \ker \ C$. The last two inclusion are equivalent to the inclusions stated for $\ker C^*$ and $\text{ran} \ C^*$ in (ii). Moreover, by construction $CH_1^{\frac{1}{2}} = C_0 H_1^{\frac{1}{2}} \subset H_2^{\frac{3}{2}}$ and $CH_1^{\frac{3}{2}} = CP_1 H_1^{\frac{1}{2}} = CH_1^{\frac{3}{2}}$, so that $CH_1^{\frac{1}{2}} \subset H_2^{\frac{3}{2}} \subset H_2^{\frac{1}{2}}$. By boundedness of $C$, $CH_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}}$ is equivalent to $H_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}}$.

Finally it is shown that the conditions $\text{ran} \ C \subset \overline{\text{dom}} \ H_2$, $\ker \ H_1 \subset \ker \ C$, and $CH_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}}$ determine $C$ uniquely. From the first and third condition one obtains

$$CH_1^{\frac{1}{2}} = P_2 CH_1^{\frac{1}{2}} \subset P_2 H_2^{\frac{1}{2}} = H_2^{\frac{1}{2}},$$

and this implies that $\text{mul} \ H_1 \subset \ker \ C$. Hence $CH_1^{\frac{1}{2}} = CP_1 H_1^{\frac{1}{2}} = CH_1^{\frac{1}{2}} \subset H_2^{\frac{3}{2}}$, and now the condition $\ker \ H_1 \subset \ker \ C$ implies that $C$ restricted to the subspace $\overline{\text{dom}} \ H_1 \ominus \ker \ H_1$ is uniquely determined by the condition $CH_1^{\frac{1}{2}} \subset H_2^{\frac{3}{2}}$. It coincides with the closure of $C_0$ on $\overline{\text{dom}} \ H_1 \ominus \ker \ H_1$ and is a zero operator on the orthogonal complement $\text{mul} \ H_1 \ominus \ker \ H_1$.

(ii) $\Rightarrow$ (i) This implication is obtained directly by applying the definition in (2.4).

(ii) $\iff$ (iii) If $C$ is as in (ii) then $H_2^{\frac{1}{2}} \subset H_2^{\frac{3}{2}} C^*$ implies that $P_1 H_2^{\frac{1}{2}} \subset P_1 H_2^{\frac{1}{2}} C^*$ and in view of $\overline{\text{dom}} H_1 \subset \overline{\text{dom}} H_2$ this leads to $P_1 H_2^{\frac{1}{2}} \subset H_2^{\frac{1}{2}} C^*$ and (2.5).

Conversely, if (2.5) holds then $P_1 H_2^{\frac{1}{2}} \subset H_2^{\frac{1}{2}} C^* \subset H_1^{\frac{1}{2}} C^*$ and taking adjoints in $\mathfrak{F}$ it is easy to check that

$$H_2^{\frac{1}{2}} P_1 = (H_2^{\frac{1}{2}})^* P_1 = (P_1 H_2^{\frac{1}{2}})^* \subset (H_1^{\frac{1}{2}} C^*)^* \subset CH_1^{\frac{1}{2}},$$

which implies that $CH_1^{\frac{1}{2}} \subset H_2^{\frac{1}{2}}$.

(i) $\iff$ (vi) Recall that (i) is equivalent to $H_1^{-1} \leq H_2^{-1}$ and hence also to $H_1 + I \geq H_2 + I$ and $(I + H_1)^{-1} \leq (I + H_2)^{-1}$.

(ii), (vi) $\Rightarrow$ (iv) Apply (ii) to the inequality $(I + H_1)^{-1} \leq (I + H_2)^{-1}$ with $C_1 = C^*$; here the second inclusion from (ii) holds as an equality $(H_1 + I)^{-\frac{1}{2}} = (H_2 + I)^{-\frac{1}{2}} C_1$ due to boundedness. Moreover, $\text{ran} \ C_1 \subset \mathfrak{F} \ominus \ker ((H_2 + I)^{-\frac{1}{2}}) = \overline{\text{dom}} \ H_2$ clearly implies that $\ker C_1 = \ker ((H_1 + I)^{-\frac{1}{2}}) = \text{mul} \ H_1$.

(vi) $\Rightarrow$ (v) Write $(H_1 + I)^{-1} = (H_2 + I)^{-\frac{1}{2}} C_1 ((H_2 + I)^{-\frac{1}{2}} C_1)^* = (H_2 + I)^{-\frac{1}{2}} C_1 C_1^* (H_2 + I)^{-\frac{1}{2}}$ and take $M = C_1 C_1^*$.

(v) $\Rightarrow$ (vi) This is clear. \hfill $\Box$

Observe that if $\overline{\text{dom}} \ H_1 = \overline{\text{dom}} \ H_2$, then (2.5) can be expressed using the forms corresponding to $H_1$ and $H_2$ in the following simpler form:

$$H_2[\cdot, \cdot] \subset H_1^{\mathcal{C}}[\cdot, \cdot].$$

Notice also that for any fixed $t > 0$ the conditions (iii) and (v) can be also replaced by the equivalent conditions $(H_1 + t)^{-\frac{1}{2}} = (H_2 + t)^{-\frac{1}{2}} C_t$, $\|C_t\| \leq 1$, and $(H_1 + I)^{-1} \leq (H_2 + I)^{-1}$, respectively; see [26, Lemma 3.2].

To an arbitrary nonnegative l.r. $S$ in $\mathfrak{F}$ one can associate the following Cayley transform

$$(2.6) \quad S \mapsto B = \mathcal{C}(S) = -I + 2(I + S)^{-1} = \{\{f + f', f - f'\}, \{f, f'\} \in S\};$$

if $S$ is an operator then (2.6) can be rewritten in the form $\mathcal{C}(S) = (I-S)(I+S)^{-1}$. The Cayley transform (2.6) establishes a one-to-one correspondence between all nonnegative symmetric
(selfadjoint) relations $S$ and all (graphs of) Hermitian (selfadjoint, respectively) contractions $B$ with inverse transform

$$B \mapsto S = C(B) = (I - B)(I + B)^{-1} = \{(I + B)h, (I - B)h : h \in \mathcal{H}\}.$$  

For the proof of the next statement, see [10].

**Lemma 2.3.** Let $\tilde{S}$ be a nonnegative selfadjoint relation and let $\tilde{B} = C(\tilde{S})$ be its Cayley transform. Then

$$\mathcal{D}[\tilde{S}] = \text{ran} \ (I + \tilde{B})^{1/2};$$  
$$\tilde{S}[u, v] = -(u, v) + 2 \left((I + \tilde{B})^{(-1/2)}u, (I + \tilde{B})^{(-1/2)}v\right), \ u, v \in \mathcal{D}[\tilde{S}];$$  
$$\mathcal{D}[\tilde{S}^{-1}] = \text{ran} \ (I - \tilde{B})^{1/2};$$  
$$\tilde{S}^{-1}[f, g] = -(f, g) + 2 \left((I - \tilde{B})^{(-1/2)}f, (I - \tilde{B})^{(-1/2)}g\right), \ f, g \in \mathcal{D}[\tilde{S}^{-1}].$$

If $\tilde{S}$ is a nonnegative selfadjoint relation, then the form domain $\mathcal{D}[\tilde{S}]$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{\tilde{S}} := \tilde{S}[f, g] + \langle f, g \rangle.$$  

Observe, that if $\tilde{B} = C(\tilde{S})$ then Lemma 2.3 shows that

$$\langle f, g \rangle_{\tilde{S}} = 2 \left((I + \tilde{B})^{(-1/2)}f, (I + \tilde{B})^{(-1/2)}g\right), \ f, g \in \mathcal{D}[\tilde{S}] = \text{ran} \ (I + \tilde{B})^{1/2}.$$  

**2.2. Krein shorted operators.** For every nonnegative bounded operator $S$ in the Hilbert space $\mathcal{H}$ and every subspace $K \subset \mathcal{H}$ M.G. Krein [31] defined the operator $S_K$ by the relation

$$S_K = \max \{Z \in L(\mathcal{H}) : 0 \leq Z \leq S, \text{ran} \ Z \subset K\}.$$  

The equivalent definition

$$\langle S_K f, f \rangle = \inf_{\varphi \in K^\perp} \{(S(f + \varphi), f + \varphi)\}, \ f \in \mathcal{H}.$$  

Here $K^\perp := \mathcal{H} \ominus K$. The properties of $S_K$, were studied by M. Krein and by other authors (see [8] and references therein). $S_K$ is called the shorted operator (see [11, 2]). It is proved in [31] that $S_K$ takes the form

$$S_K = S^{1/2} P_\Omega S^{1/2},$$  

where $P_\Omega$ is the orthogonal projection in $\mathcal{H}$ onto the subspace

$$\Omega = \{f \in \text{ran} \ S : S^{1/2} f \in K\} = \text{ran} \ S \ominus S^{1/2} K^\perp.$$  

Moreover [31],

$$\text{ran} S^{1/2}_K = \text{ran} S^{1/2} P_\Omega = \text{ran} S^{1/2} \cap K.$$  

It follows that

$$S_K = 0 \iff \text{ran} S^{1/2} \cap K = \{0\}.$$  

A bounded selfadjoint operator $S$ in $\mathcal{H}$ has the block-matrix form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix} : \begin{array}{c} K \\ K^\perp \end{array} \rightarrow \begin{array}{c} K \\ K^\perp \end{array}.$$
It is well known (see [35]) that the operator $S$ is nonnegative if and only if

$$S_{22} \geq 0, \quad \mathsf{ran} \ S_{12}^* \subset \mathsf{ran} \ S_{22}^{1/2}, \quad S_{11} \geq \left(S_{22}^{-1/2} S_{12}^* \right)^* \left(S_{22}^{-1/2} S_{12}^* \right).$$

and the operator $S_K$ is given by the block matrix

$$S_K = \begin{pmatrix} S_{11} - \left(S_{22}^{-1/2} S_{12}^* \right)^* \left(S_{22}^{-1/2} S_{12}^* \right) & 0 \\ 0 & 0 \end{pmatrix},$$

where $S^{-1/2}_{22}$ is the Moore-Penrose pseudo-inverse. If $S^{-1}_{22} \in \mathcal{L}(\mathcal{K}^\perp)$ then

$$S_K = \begin{pmatrix} S_{11} - S_{12} S^{-1}_{22} S_{12}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

and the operator $S_{11} - S_{12} S^{-1}_{22} S_{12}^*$ is called the Schur complement of the matrix $S$. From (2.11) it follows that

$$S_K = 0 \iff \mathsf{ran} \ S_{12}^* \subset \mathsf{ran} \ S_{22}^{1/2} \quad \text{and} \quad S_{11} = \left(S_{22}^{-1/2} S_{12}^* \right)^* \left(S_{22}^{-1/2} S_{12}^* \right).$$

2.3. Selfadjoint contractive extensions of a nondensely defined Hermitian contraction. Let $B$ be a Hermitian contraction in $\mathcal{H}$ defined on the subspace $\mathcal{H}_0$, i.e., $(Bf, g) = (f, Bg)$ for all $f, g \in \mathcal{H}_0$ and $\|B\| \leq 1$. Set $\mathcal{N} = \mathcal{H} \ominus \mathcal{H}_0$. A description of all selfadjoint contractive (sc-)extensions of $B$ in $\mathcal{H}$ was given by M.G. Kreǐn [31]. In fact, he showed that all sc-extensions of $B$ form an operator interval $[B_\mu, B_M]$, where the extensions $B_\mu$ and $B_M$ can be characterized by

$$\sup_{\varphi \in \text{dom} \ B} \frac{|(B \varphi, h)|^2}{||\varphi||^2 - ||B \varphi||^2} = \infty$$

for all $h \in \mathcal{N} \setminus \{0\}$.

A description of the operator interval $[B_\mu, B_M]$ is given by the following equality (cf. [31], [35]):

$$\tilde{B} = (B_M + B_\mu)/2 + (B_M - B_\mu)^{1/2} \tilde{Z} (B_M - B_\mu)^{1/2}/2,$$

where $\tilde{Z}$ is a sc-operator in the subspace $\overline{\mathsf{ran}} (B_M - B_\mu) \subseteq \mathcal{N}$. It follows from (2.12), for instance, that for every sc-extension $\tilde{B}$ of $B$ the following identities hold:

$$(I - \tilde{B})|_{\mathcal{N}} = B_M - \tilde{B}, \quad (I + \tilde{B})|_{\mathcal{N}} = \tilde{B} - B_\mu,$$

cf. [31]. Hence, according to (2.10)

$$\begin{align*}
\mathsf{ran} \ (I - \tilde{B})^{1/2} \cap \mathcal{N} &= \mathsf{ran} \ (B_M - \tilde{B})^{1/2}, \\
\mathsf{ran} \ (I + \tilde{B})^{1/2} \cap \mathcal{N} &= \mathsf{ran} \ (\tilde{B} - B_\mu)^{1/2}.
\end{align*}$$
2.4. Nonnegative linear relations and their nonnegative selfadjoint extensions. Let $S$ be a nonnegative l.r. in $\mathfrak{H}$. Recall the definition of the Friedrichs extension $S_F$ of $S$ (see [28] for the case of densely defined $S$ and [40] for nonnegative l.r. case): $S_F$ is the unique selfadjoint relation associated with the closure of the form $S(\varphi,\psi) = (\varphi',\psi)$, $\{\varphi,\varphi'\} \in S$, $\psi \in \text{dom } S$:

$$S_F[\cdot,\cdot] = S[\cdot,\cdot] := \text{clos } S(\cdot,\cdot).$$

Consider the Cayley transform $B = C(S)$ of $S$ in (2.6). Then $B$ is a Hermitian contraction in $\mathfrak{N}$ and the formulas

$$\tilde{B} = -I + 2(I + \tilde{S})^{-1}, \quad \tilde{S} = (I - \tilde{B})(I + \tilde{B})^{-1}$$

establish a one-to-one correspondence between $sc$-extensions $\tilde{B}$ of $B$ and nonnegative selfadjoint extensions $\tilde{S}$ of $S$. In his famous paper [31] M.G. Krein proved, with $S$ being densely defined in $\mathfrak{H}$, that the Cayley transform of the left endpoint $B_\mu$ of the operator interval $[B_\mu, B_M]$ coincides with the Friedrichs extension $S_F$ of $S$, i.e.,

$$S_F = (I - B_\mu)(I + B_\mu)^{-1}.$$ 

This equality remains valid when $S$ is a l.r.; see [6, 15, 25]. Notice that $\mathcal{D}[S] = \mathcal{D}[S_F]$. In addition

1. if $S$ is a densely defined operator, then $S_F$ is characterized by

$$\text{dom } S_F = \text{dom } S^* \cap \mathcal{D}[S];$$

2. if $S$ is a nondensely defined operator, then

$$S_F = \{ \{f, S_0 f + h\} : f \in \text{dom } S_0, h \in \mathfrak{N} \}$$

where $S_0$ stands for the Friedrichs extension of the nonnegative operator $S_0 := P_{\mathfrak{N}_0} S$ having a dense domain in $\mathfrak{N}_0 = \text{dom } S$.

Let $z \in \mathbb{C} \setminus \mathbb{R}_+$ and let $\mathfrak{M}_z = \mathfrak{N} \oplus \text{ran } (S^* - zI)$ be the defect subspace of $S$ at $z$. Recall that

$$\mathcal{D}[S] \cap \mathfrak{M}_z = \{0\}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+;$$

see e.g. [31, 6]. The Cayley transform $S_K := (I - B_M)(I + B_M)^{-1}$ of the right endpoint possesses the following property (see [3] for the operator case and [15] for the case of l.r.):

$$S_K = ((S^{-1})_F)^{-1}.$$ 

It is a consequence of Proposition 2.2 and the formula (2.6) that if the Hermitian contractions $\tilde{B}_1$ and $\tilde{B}_2$ satisfy the inequality $\tilde{B}_1 \leq \tilde{B}_2$, then equivalently their Cayley transforms $\tilde{S}_1 = C(\tilde{B}_1)$ and $\tilde{S}_2 = C(\tilde{B}_2)$ satisfy the reverse inequality $\tilde{S}_1 \geq \tilde{S}_2$. It follows that the linear relations $S_F$ and $S_K$ are the maximal and minimal (in the sense of quadratic forms, see (2.4)) among all nonnegative selfadjoint extensions, i.e., if $\tilde{S}$ is a nonnegative selfadjoint extension of $S$, then

1. $\mathcal{D}[S] \subset \mathcal{D}[\tilde{S}] \subset \mathcal{D}[S_K],$

2. $S[\varphi] \geq \tilde{S}[\varphi]$ for all $\varphi \in \mathcal{D}[S]$ and $\tilde{S}[u] \geq S_K[u]$ for all $u \in \mathcal{D}[\tilde{S}]$.

These inclusions and inequalities were originally established by M.G. Krein in [31] for a densely defined $S$ and in [15] for a l.r. $S$. The minimality property of $S_K$ is obtained by Ando and Nishio in [3] for nondensely defined operator $S$. 

Q-FUNCTIONS OF NONNEGATIVE OPERATORS
The minimal nonnegative selfadjoint extension $S_K$ we will call the *Krein-von-Neumann extension* of $S$. Recall that $S$ admits a unique nonnegative selfadjoint extension, i.e. $S_K = S_F$, if and only if for at least for one (and then for all) $z \in \mathbb{C} \setminus \mathbb{R}_+$ the following condition is fulfilled:

$$\sup_{\varphi \in \text{dom}(S)} \frac{|(S\varphi, \varphi)|^2}{(S\varphi, \varphi)} = \infty \quad \text{for every} \quad \varphi \in \mathcal{N}_z \setminus \{0\}.$$ 

The domain $\mathcal{D}[S_K]$ and $S_K[u]$ can be characterized as follows [3], [4]:

$$\mathcal{D}[S_K] = \left\{ u \in \mathcal{H} : \sup_{\varphi \in \text{dom}(S)} \frac{|(S\varphi, u)|^2}{(S\varphi, \varphi)} < \infty \right\},$$

$$S_K[u] = \sup_{\varphi \in \text{dom}(S)} \frac{|(S\varphi, u)|^2}{(S\varphi, \varphi)}, \; u \in \mathcal{D}[S_K].$$

Observe that the form $S_F[\cdot, \cdot]$ is the closed restriction of the form $S_K[\cdot, \cdot]$ and the form $S_K^{-1}[\cdot, \cdot]$ is the closed restriction of the form $S_F^{-1}[\cdot, \cdot]$. Besides (see [2])

$$\inf_{\varphi \in \mathcal{D}[S_F]} S_K[f - \varphi] = 0 \quad \text{for all} \quad f \in \mathcal{D}[S_K],$$

$$\inf_{\varphi \in \mathcal{D}[S_F^{-1}]} S_F^{-1}[g - \psi] = 0 \quad \text{for all} \quad g \in \mathcal{D}[S_F^{-1}].$$

### 3. Special pairs of nonnegative selfadjoint linear relations and corresponding pairs of selfadjoint contractions

Let $\mathcal{A}$ and $\mathcal{B}$ be bounded selfadjoint operators which are nonnegative and satisfy the inequality $\mathcal{A} \leq \mathcal{B}$. In this case Proposition 2.2 yields the following equivalences (see also 10 and the references therein):

(i) $\mathcal{A} \leq \mathcal{B};$
(ii) $\mathcal{A} = \mathcal{B}^{1/2} \mathcal{Z} \mathcal{B}^{1/2}$, where $\mathcal{Z}$ is a nonnegative selfadjoint contraction in $\overline{\text{ran} \mathcal{B}}$;
(iii) $\text{ran} \mathcal{A}^{1/2} = \mathcal{B}^{1/2} \text{ran} \mathcal{Z}^{1/2}$, where $\mathcal{Z}$ is as in (ii).

Observe that if $0 \leq \mathcal{Z} \leq I$ then the block operator

$$P = \begin{pmatrix} \mathcal{Z} & (\mathcal{Z} - \mathcal{Z}^2)^{1/2} \\ (\mathcal{Z} - \mathcal{Z}^2)^{1/2} & I - \mathcal{Z} \end{pmatrix}$$

satisfies $P^* = P = P^2$. In particular, this shows that $\mathcal{Z}$ itself is an orthogonal projection precisely when $\text{ran} \mathcal{Z}^{1/2} \cap \text{ran} (I - \mathcal{Z})^{1/2} = \text{ran} (\mathcal{Z} - \mathcal{Z}^2)^{1/2} = \{0\}$. Since

$$\text{ran} \mathcal{A}^{1/2} \cap \text{ran} (\mathcal{B} - \mathcal{A})^{1/2} = \text{ran} \mathcal{B}^{1/2} \mathcal{Z}^{1/2} \cap \text{ran} \mathcal{B}^{1/2}(I - \mathcal{Z})^{1/2} = \text{ran} \mathcal{B}^{1/2}(\mathcal{Z} - \mathcal{Z}^2)^{1/2},$$

one concludes the following equivalence for $\mathcal{Z}$ in (ii) and (iii):

\begin{equation}
\mathcal{Z} = \mathcal{Z}^2 \quad \iff \quad \text{ran} \mathcal{A}^{1/2} \cap \text{ran} (\mathcal{B} - \mathcal{A})^{1/2} = \{0\}.
\end{equation}

Recall that for closed nonnegative forms $\mathfrak{h}_1 \subset \mathfrak{h}_2$ implies $\mathfrak{h}_1 \geq \mathfrak{h}_2$. The next proposition gives some necessary and sufficient conditions for the inclusion $\mathfrak{h}_1 \subset \mathfrak{h}_2$ to hold by means of the Cayley transforms of representing selfadjoint relations, and hence, can be seen as a further specification of Proposition 2.2.

**Proposition 3.1.** Let $\tilde{S}_0$ and $\tilde{S}_1$ be two nonnegative selfadjoint linear relations and let

$$\text{graph} \tilde{B}_k = \mathcal{C}(\tilde{S}_k) = \left\{ \{f + f', f - f'\}, \; \{f, f'\} \in \tilde{S}_k \right\} = -I + 2(I + \tilde{S}_k)^{-1}, \; k = 0, 1,$$
be their Cayley transforms. Suppose that \( \tilde{S}_1 \leq \tilde{S}_0 \) or, equivalently, that \( \tilde{B}_0 \leq \tilde{B}_1 \). Then the following conditions are equivalent:

(i) the form \( \tilde{S}_0[\cdot, \cdot] \) is a closed restriction of the form \( \tilde{S}_1[\cdot, \cdot] \);
(ii) the following equality holds

\[ \mathcal{D}[\tilde{S}_1] \cap \tilde{S}_1 \mathcal{D}[\tilde{S}_0] = \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2}; \]

(iii) the following equality holds

\[ I + \tilde{B}_0 = (I + \tilde{B}_1)^{1/2} \Pi (I + \tilde{B}_1)^{1/2}, \]

where \( \Pi \) is orthogonal projection acting in \( \text{ran} (I + \tilde{B}_1) \);

(iv) the following equality holds

\[ \text{ran} (I + \tilde{B}_0)^{1/2} \cap \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}. \]

**Proof.** The operators \( \tilde{B}_0 \) and \( \tilde{B}_1 \) are selfadjoint contractions in \( \mathcal{H} \) and Lemma 2.3 shows that (cf. (2.3))

\[ \tilde{S}_k[(I + \tilde{B}_k)^{1/2}f] + ||(I + \tilde{B}_k)^{1/2}f||^2 = 2||f||^2, \quad f \in \text{ran} (I + \tilde{B}_k), \quad k = 0, 1. \]

By Proposition 2.2 the inequality \( I + \tilde{B}_0 \leq I + \tilde{B}_1 \) is equivalent to the existence of a contraction \( W : \text{ran} (I + \tilde{B}_0) \to \text{ran} (I + \tilde{B}_1) \) (ker \( W = \{0\} \) in \( \mathcal{D}[\tilde{S}_0] \)) such that

\[ (I + \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} W, \]

in fact, \( W \) is given by

\[ W = (I + \tilde{B}_1)^{(-1/2)} (I + \tilde{B}_0)^{1/2} : \text{ran} (I + \tilde{B}_0) \to \text{ran} (I + \tilde{B}_1). \]

The identity (3.3) implies that

\[ I + \tilde{B}_0 = (I + \tilde{B}_1)^{1/2} W^* (I + \tilde{B}_1)^{1/2}, \quad \tilde{B}_1 - \tilde{B}_0 = (I + \tilde{B}_1)^{1/2} (I - W W^*) (I + \tilde{B}_1)^{1/2}. \]

In particular,

\[ \text{ran} (I + \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} \text{ran} W, \quad \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} \text{ran} D_{W^*}, \]

where \( D_{W^*} = (I - W W^*)^{1/2} \).

(i) \( \Rightarrow \) (iii) Suppose that the form \( \tilde{S}_0[\cdot, \cdot] \) is a closed restriction of the form \( \tilde{S}_1[\cdot, \cdot] \). Then it follows from (3.2) that \( ||Wf||^2 = ||f||^2 \), i.e., \( W \) is isometric and consequently \( \Pi := WW^* \) appearing in (3.3) is the orthogonal projection onto the closed subspace \( \text{ran} W \subset \text{ran} (I + \tilde{B}_1) \).

(iii) \( \Rightarrow \) (ii) It is clear from (3.5) that \( \mathcal{D}[\tilde{S}_0] = \text{ran} (I + \tilde{B}_0)^{1/2} \subset \text{ran} (I + \tilde{B}_1)^{1/2} = \mathcal{D}[\tilde{S}_1] \); cf. (3.6). Now suppose that \( v \in \mathcal{D}[\tilde{S}_1] \cap \tilde{S}_1 \mathcal{D}[\tilde{S}_0] \), i.e., that \( \tilde{S}_1[u, v] + (u, v) = 0 \) for all \( u \in \mathcal{D}[\tilde{S}_0] \); see (2.8). By Lemma 2.3 and (2.3) this can be rewritten as

\[ \left( (I + \tilde{B}_1)^{(-1/2)} (I + \tilde{B}_0)^{1/2} h, (I + \tilde{B}_1)^{(-1/2)} v \right) = 0, \quad h \in \mathcal{H}, \]

which in view of (3.4) is equivalent to \( \tilde{W}^* (I + \tilde{B}_1)^{(-1/2)} v = 0 \). This shows that

\[ \mathcal{D}[\tilde{S}_1] \cap \tilde{S}_1 \mathcal{D}[\tilde{S}_0] = (I + \tilde{B}_1)^{1/2} \text{ker} W^* \]

On the other hand, the identity in (iii) implies that

\[ \tilde{B}_1 - \tilde{B}_0 = (I + \tilde{B}_1)^{1/2} P (I + \tilde{B}_1)^{1/2}, \]
where \( P \) is the orthogonal projection from \( \text{ran} (I + \tilde{B}_1) \) onto \( \text{ran} (I + \tilde{B}_1) \cup \text{ran} W = \ker W^* \), where \( W^* \) acts on \( \text{ran} (I + \tilde{B}_1) \). Therefore,

\[
(3.9) \quad \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} \ker W^* = D[\tilde{S}_1] \cup_{\tilde{S}_1} D[\tilde{S}_0].
\]

(ii) \( \Rightarrow \) (i) Suppose that \( D[\tilde{S}_1] \cup_{\tilde{S}_1} D[\tilde{S}_0] = \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} \). According to \( (3.6) \) one has \( \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} \text{ran} D_{W^*} \) which combined with \( (3.7) \) leads to

\[
(3.10) \quad \text{ran} D_{W^*} = \ker W^*.
\]

By the commutation relation \( W^* D_{W^*} = D_W W^* \) the identity \( (3.10) \) gives \( D_W W^* = 0 \) and this implies that the restriction \( W| \text{ran} W^* \) is isometric. However, \( \text{ran} W^* = \text{ran} (I + \tilde{B}_1)^{1/2} \) and, thus, \( W \) is isometric on \( \text{ran} (I + \tilde{B}_1)^{1/2} \). Now \( (3.3) \) and \( (3.2) \) imply that \( \tilde{S}_1[u] = \tilde{S}_0[u] \) for all \( u \in D[\tilde{S}_0] = \text{ran} (I + \tilde{B}_0)^{1/2} \), i.e. the form \( \tilde{S}_0[\cdot, \cdot] \) is a closed restriction of the form \( \tilde{S}_1[\cdot, \cdot] \).

Finally, the equivalence of (iii) and (iv) is obtained directly from \( (3.1) \). \( \square \)

Observe that if the equivalent conditions in Proposition \( 3.1 \) are satisfied, then it follows from \( (2.9) \) and \( (3.8) \) that

\[
(3.11) \quad \| (\tilde{B}_1 - \tilde{B}_0)^{1/2} g \|_{\tilde{S}_1}^2 = 2 \| P g \|^2, \quad g \in \text{ran} (I + \tilde{B}_1).
\]

The next theorem will play an important role in the considerations that follow; for this purpose we first state and prove the following further result.

**Lemma 3.2.** Let \( \tilde{S}_0 \) and \( \tilde{S}_1 \) be two nonnegative selfadjoint linear relations such that \( \tilde{S}_1 \leq \tilde{S}_0 \) and let their Cayley transforms \( \tilde{B}_0 \) and \( \tilde{B}_1 \) be connected by \( (I + \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} W \), where \( W \) is as defined in \( (3.1) \). Then the associated forms satisfy the approximation property

\[
(3.12) \quad \inf \left\{ \tilde{S}_1[u - \varphi], \varphi \in D[\tilde{S}_0] \right\} = 0 \quad \text{for all} \quad u \in D[\tilde{S}_1]
\]

if and only if

\[
(3.13) \quad \text{ran} (I - \tilde{B}_1)^{1/2} \cap (I + \tilde{B}_1)^{1/2} \ker W^* = \{0\},
\]

or, equivalently, \( \text{ran} \tilde{S}_1^{1/2} = D[\tilde{S}_1^{-1}] \) satisfies

\[
(3.14) \quad D[\tilde{S}_1^{-1}] \cap \left( D[\tilde{S}_1] \cup_{\tilde{S}_1} D[\tilde{S}_0] \right) = \{0\}.
\]

**Proof.** First assume that \( (3.12) \) is satisfied. By means of Lemma \( 2.3 \) this condition can be rewritten as follows

\[
\inf \left\{ \| (I - \tilde{B}_1)^{1/2} f - (I - \tilde{B}_1)^{1/2} W g \|^2, \quad g \in \text{ran} (I + \tilde{B}_0) \right\} = 0
\]

for all \( f \in \text{ran} (I + \tilde{B}_1)^{1/2} \). Since \( \text{ran} (I - \tilde{B}_1^2)^{1/2} = \text{ran} (I - \tilde{B}_1)^{1/2} \cap \text{ran} (I + \tilde{B}_1)^{1/2} \) and moreover \( \text{ran} (I - \tilde{B}_1^2)^{1/2} = (\ker (I - \tilde{B}_1^2)^{1/2})^\perp = \text{ran} (I - \tilde{B}_1)^{1/2} \cap \text{ran} (I + \tilde{B}_1)^{1/2} \) the previous condition is equivalent to

\[
\inf \left\{ \| h - (I - \tilde{B}_1)^{1/2} W g \|^2, \quad g \in \text{ran} (I + \tilde{B}_0) \right\} = 0 \quad \text{for all} \quad h \in \text{ran} (I - \tilde{B}_1^2)^{1/2}.
\]
This means that the orthogonal complement $\Omega^\perp$ in $\text{ran} \ (I - \tilde{B}_1^2)^{1/2}$ of the linear manifold
\[
\Omega := \left\{ (I - \tilde{B}_1)^{1/2} W g, \ g \in \text{ran} \ (I + \tilde{B}_0) \right\}
\]
is equal to zero. However,
\[
\Omega^\perp = \left\{ \varphi \in \text{ran} \ (I - \tilde{B}_1^2)^{1/2} : (I - \tilde{B}_1)^{1/2} \varphi \in \ker W^* \right\}
\]
and since $\ker W^* \subset \text{ran} \ (I + \tilde{B}_1)$ one concludes that the condition $\Omega^\perp = \{0\}$ is equivalent to $\text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \ker W^* = \{0\}$. It remains to prove that this last condition is equivalent to the condition in (3.13). To see this first assume that $\text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \ker W^* = \{0\}$ and let $g \in \text{ran} \ (I - \tilde{B}_1)^{1/2} \cap (I + \tilde{B}_1)^{1/2} \ker W^*$. Then $g \in \text{ran} \ (I - \tilde{B}_1^2)^{1/2}$ and hence
\[
g = (I - \tilde{B}_1^2)^{1/2} u = (I + \tilde{B}_1)^{1/2} w \text{ for some } u \in \text{ran} \ (I - \tilde{B}_1^2) \text{ and } w \in \ker W^* \subset \text{ran} \ (I + \tilde{B}_1).
\]
This implies that
\[
(I + \tilde{B}_1)^{1/2} [(I - \tilde{B}_1)^{1/2} u - w] = 0
\]
and since $(I - \tilde{B}_1)^{1/2} u - w \in \text{ran} \ (I + \tilde{B}_1)$ one concludes that $(I - \tilde{B}_1)^{1/2} u = w$, which by the assumption $\text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \ker W^* = \{0\}$ implies that $(I - \tilde{B}_1)^{1/2} u = w = 0$. Therefore, also $g = 0$ and thus (3.13) follows. To prove the converse assume that (3.13) is satisfied and suppose that $g \in \text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \ker W^*$. Then $g \in \text{ran} \ (I + \tilde{B}_1)$ and clearly $(I + \tilde{B}_1)^{1/2} g \in \text{ran} \ (I - \tilde{B}_1)^{1/2} \cap (I + \tilde{B}_1)^{1/2} \ker W^*$ from which one concludes that $(I + \tilde{B}_1)^{1/2} g = 0$ and hence also $g = 0$. This proves that (3.12) and (3.13) are equivalent.

The equivalence of (3.13) and (3.14) is obtained by using Lemma 2.3 which shows that $\text{ran} \ (I - \tilde{B})^{1/2} = \mathcal{D}[\tilde{S}^{-1}]$, and the formula $\mathcal{D}[\tilde{S}_1] \cap \mathcal{D}[\tilde{S}_0] = (I + \tilde{B}_1)^{1/2} \ker W^*$ in (3.7). □

**Theorem 3.3.** Let $\tilde{S}_0$ and $\tilde{S}_1$ be two nonnegative selfadjoint relations. Then the following conditions are equivalent:

(i) the form $\tilde{S}_0[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_1[\cdot, \cdot]$ and
\[
\inf \left\{ \tilde{S}_1[u - \varphi], \ \varphi \in \mathcal{D}[\tilde{S}_0] \right\} = 0 \quad \text{for all } u \in \mathcal{D}[\tilde{S}_1];
\]

(ii) the form $\tilde{S}_1^{-1}[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_0^{-1}[\cdot, \cdot]$ and
\[
\inf \left\{ \tilde{S}_0^{-1}[v - \psi], \ \psi \in \mathcal{D}[\tilde{S}_1^{-1}] \right\} = 0 \quad \text{for all } v \in \mathcal{D}[\tilde{S}_0^{-1}];
\]

(iii) the form $\tilde{S}_0[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_1[\cdot, \cdot]$ and the form $\tilde{S}_1^{-1}[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_0^{-1}[\cdot, \cdot]$;

(iv) the Cayley transforms
\[
\text{graph } \tilde{B}_k = \mathcal{C}(\tilde{S}_k) = \left\{ (f + f', f - f'), \ \{f, f'\} \in \tilde{S}_k \right\}, \ \text{for } k = 0, 1
\]
satisfy the conditions
\[
\text{ran} \ (I + \tilde{B}_0)^{1/2} \cap \text{ran} \ (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \text{ran} \ (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}.
\]

**Proof.** By Proposition 3.13 the statement that the form $\tilde{S}_0[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_1[\cdot, \cdot]$ is equivalent to the equality $\text{ran} \ (I + \tilde{B}_0)^{1/2} \cap \text{ran} \ (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}$. Similarly by applying inverses, cf. (2.7), it can be seen that the statement that the form $\tilde{S}_1^{-1}[\cdot, \cdot]$ is a closed restriction of the form $\tilde{S}_0^{-1}[\cdot, \cdot]$ is equivalent to the equality
\[
\text{ran} \ (I - \tilde{B}_1)^{1/2} \cap \text{ran} \ (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}.
\]
This proves the equivalence (iii) ⇔ (iv).

(i) ⇒ (iv) By Lemma 3.2 the condition \( \inf \{ \tilde{S}_1[u - \varphi], \varphi \in D[\tilde{S}_0] \} = 0 \) for all \( u \in D[\tilde{S}_1] \) is equivalent to (3.13). On the other hand, since \( \tilde{S}_0[,\cdot] \) is a closed restriction of the form \( \tilde{S}_1[,\cdot] \) it follows from Proposition 3.1 that
\[
(3.15) \quad \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = (I + \tilde{B}_1)^{1/2} \ker^* W,
\]
see (3.9). Combining (3.15) with (3.13) gives \( \text{ran}(I - \tilde{B}_1)^{1/2} \cap \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\} \).

(ii) ⇒ (iv) The proof is similar to the proof of the previous implication (apply inverses).

(iv) ⇒ (i), (ii) Assume that (iv) holds. Again it follows from Proposition 3.1 that (3.15) is equivalent to (2.14) is equivalent to (3.14) and, similarly, (3.16) is equivalent to (3.13).

This means that the second condition in (iv) coincides with the condition (3.13) in Lemma 3.2 and, therefore, the approximation property (3.12) in (i) is satisfied. Moreover, by Proposition 3.1 the first property in (i) is equivalent to the first condition in (iv). Hence (iv) implies (i) and likewise one can derive (ii) from (iv). 

Remark 3.4. If \( \tilde{B}_0 \) and \( \tilde{B}_1 \) (\( \tilde{B}_0 \leq \tilde{B}_1 \)) are sc-extensions of a Hermitian contraction, then it follows from (2.11) that the condition
\[
\text{ran}(I + \tilde{B}_0)^{1/2} \cap \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}
\]
is equivalent to
\[
\text{ran}(\tilde{B}_0 - B\mu)^{1/2} \cap \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\},
\]
and, similarly,
\[
\text{ran}(I - \tilde{B}_1)^{1/2} \cap \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}
\]
is equivalent to
\[
\text{ran}(B_M - \tilde{B}_1)^{1/2} \cap \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}.
\]

Remark 3.5. If \( F \) and \( G \) are bounded nonnegative selfadjoint operators, then the parallel sum \( F : G \) can be defined (21). The conditions \( F : G = 0 \) and \( \text{ran} F^{1/2} \cap \text{ran} G^{1/2} = \{0\} \) are equivalent.

The following theorem has been established in (10).

Theorem 3.6. Let \( S \) be a nonnegative symmetric linear relation. The pair \( \{\tilde{S}_0, \tilde{S}_1\} \) of nonnegative selfadjoint linear relations satisfies the conditions
\[
(3.16) \quad \left\{ \begin{array}{l} \tilde{S}_0 \cap \tilde{S}_1 = S, \\
\text{the sesquilinear form} \tilde{S}_0[,\cdot] \text{ is a closed restriction of the form} \tilde{S}_1[,\cdot], \\
\text{the sesquilinear form} \tilde{S}_1^{-1}[,\cdot] \text{ is a closed restriction of the form} \tilde{S}_0^{-1}[,\cdot]
\end{array} \right.
\]
if and only if the pair \( \{\tilde{B}_0, \tilde{B}_1\} \) of selfadjoint contractions satisfies conditions
\[
(3.17) \quad \tilde{B}_0 \leq \tilde{B}_1, \quad \ker(\tilde{B}_1 - \tilde{B}_0) = \text{dom} B, \\
\text{ran}(\tilde{B}_1 - \tilde{B}_0^{1/2}) \cap \text{ran}(\tilde{B}_0 - B\mu)^{1/2} = \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} \cap \text{ran}(B_M - \tilde{B}_1)^{1/2} = \{0\},
\]
where \( B = C(S), \tilde{B}_k = C(\tilde{S}_k), k = 0, 1 \).

It is also shown in (10) that if the defect numbers of \( S \) are finite, then the pair \( \{S_F, S_K\} \) of nonnegative selfadjoint extensions of \( S \) is the unique pair satisfying the conditions (3.16) and that if the defect numbers are infinite, then there exist pairs \( \{\tilde{S}_0, \tilde{S}_1\} \) different from \( \{S_F, S_K\} \) with the properties (3.16).
4. Special pairs of selfadjoint extensions

Let \( B \) be a Hermitian contraction in \( \mathcal{H} \) with \( \text{dom} \, B = \mathcal{H}_0 \subset \mathcal{H} \) and let \( \mathfrak{N} = \mathcal{H} \ominus \mathcal{H}_0 \). In what follows it is assumed that \( \ker(B_M - B_\mu) = \text{dom} \, B \). It is clear that the pair \( \{B_\mu, B_M\} \) determined by extreme extensions of the operator interval \( [B_\mu, B_M] \) satisfies all the conditions in (3.17). According to [10] there exists a pair \( \{\tilde{B}_0, \tilde{B}_1\} \), which is different from the pair \( \{B_\mu, B_M\} \) and satisfies the conditions (3.17) if and only if \( \text{dim} \, \mathfrak{N} = \infty \). We repeat here the construction from [10], since it is essential also for the present paper.

4.1. Construction of special pairs of nonnegative selfadjoint contractions. Let \( \mathcal{H} \) be an infinite-dimensional separable Hilbert space and let \( \mathfrak{K} \) be an infinite-dimensional subspace of \( \mathcal{H} \) with an infinite-dimensional orthogonal complement \( \mathfrak{K}^\perp \). Then \( \mathfrak{K}^\perp \) can be identified with \( \mathfrak{K} \) and one can write \( \mathcal{H} \) as a direct sum \( \mathcal{H} = \mathfrak{K} \oplus \mathfrak{K}^\perp \).

It is well known that there exist unbounded selfadjoint operators on infinite dimensional Hilbert spaces \( \mathcal{H} \), whose (dense) domains have a trivial intersection; see [39], [21], concrete examples are given in [14], [30]. Consequently, there exist bounded nonnegative operators \( F \) and \( G \) in \( \mathfrak{K} \), such that \( \text{ran} \, F = \text{ran} \, G = \mathfrak{K} \) and \( \text{ran} \, F \cap \text{ran} \, G = \{0\} \).

Without loss of generality one can assume that \( \|F\| < 1 \). Then \( F \) is contractive and \( \ker(I - F^2) = \{0\} \).

Define
\[
\mathcal{X} = \begin{pmatrix} F^2 & 0 \\ 0 & I - F^2 \end{pmatrix}, \quad \mathfrak{M} = \left\{ \begin{pmatrix} Gh \\ h \end{pmatrix} : h \in \mathfrak{K} \right\}.
\]

Then \( \mathcal{X} = \mathcal{X}^* \) is a nonnegative contraction in \( \mathcal{H} \) with \( \ker \mathcal{X} = \{0\} \) and \( \mathfrak{M} \) is a closed linear subspace of \( \mathcal{H} \) such that \( \text{ran} \, \mathcal{X}^{1/2} \cap \mathfrak{M} = \{0\} \).

To see this assume that \( v \in \text{ran} \, \mathcal{X}^{1/2} \cap \mathfrak{M} \). Then for some \( h, x, y \in \mathfrak{K} \) one has
\[
v = \begin{pmatrix} Gh \\ h \end{pmatrix} = \begin{pmatrix} Fx \\ (I - F^2)^{1/2} y \end{pmatrix}.
\]

Since \( \text{ran} \, F \cap \text{ran} \, G = \{0\} \), (4.1) implies that \( Fx = Gh = 0 \). Due to \( \ker F = \ker G = \{0\} \) one obtains \( x = 0 \), \( h = 0 \). Consequently \( v = 0 \), and this proves the claim.

Next observe that
\[
I - \mathcal{X} = \begin{pmatrix} I - F^2 & 0 \\ 0 & F^2 \end{pmatrix}, \quad \mathfrak{M}^\perp = \left\{ \begin{pmatrix} k \\ -Gk \end{pmatrix} : k \in \mathfrak{K} \right\}.
\]

Clearly, \( \ker(I - \mathcal{X}) = \{0\} \) and a similar argument as above shows that
\[
\text{ran} \, (I - \mathcal{X})^{1/2} \cap \mathfrak{M}^\perp = \{0\}.
\]

Now consider
\[
\mathcal{Y} := \mathcal{X} + (I - \mathcal{X})^{1/2} \mathfrak{M} (I - \mathcal{X})^{1/2}.
\]

By definition \( \mathcal{X} \leq \mathcal{Y} \leq I \) and
\[
\mathcal{Y} = \mathcal{X} + (I - \mathcal{X})^{1/2} \mathfrak{M} (I - \mathcal{X})^{1/2}.
\]

In particular, \( \ker \mathcal{Y} = \{0\} \) and it follows from (3.17) that
\[
\text{ran} \, (I - \mathcal{Y})^{1/2} \cap \text{ran} \, (\mathcal{Y} - \mathcal{X})^{1/2} = \{0\}.
\]
Moreover, by factoring $\mathcal{Y} = \mathcal{Y}_0 \mathcal{Y}_0^*$ with the row operator $\mathcal{Y}_0 = (\mathcal{X}^{1/2}; (I - \mathcal{X})^{1/2} P_{\Omega})$ and using similar arguments as in (3.1) one concludes that

\begin{equation}
\text{ran } \mathcal{X}^{1/2} \cap \text{ran } (\mathcal{Y} - \mathcal{X})^{1/2} = \{0\}.
\end{equation}

Notice that due to $\ker(I - \mathcal{X}) = \{0\}$ the condition (4.2) is equivalent to

\[ \ker(\mathcal{Y} - \mathcal{X}) = \{0\}. \]

It is also worth to mention that (use e.g. (4.4))

\begin{equation}
\text{ran } \mathcal{X} \cap \text{ran } \mathcal{Y} = \{0\}.
\end{equation}

4.2. Construction of special pairs of selfadjoint contractions and selfadjoint contractive extensions. Next introduce the selfadjoint contractions $\bar{Z}_0$ and $\bar{Z}_1$ by

\begin{equation}
\bar{Z}_0 := 2 \mathcal{X} - I, \quad \bar{Z}_1 := 2 \mathcal{Y} - I.
\end{equation}

Then $\bar{Z}_0 \leq \bar{Z}_1$ and in view of (4.3) and (4.4) one has

\[ \text{ran } (I - \bar{Z}_1)^{1/2} \cap \text{ran } (\bar{Z}_1 - \bar{Z}_0)^{1/2} = \{0\}, \quad \text{ran } (I + \bar{Z}_0)^{1/2} \cap \text{ran } (\bar{Z}_1 - \bar{Z}_0)^{1/2} = \{0\}. \]

Additionally, by the construction one has

\[ \ker(I + \bar{Z}_0) = \{0\}, \quad \ker(I - \bar{Z}_0) = \{0\}, \quad \ker(\bar{Z}_1 - \bar{Z}_0) = \{0\}, \]

and hence also $\ker(I + \bar{Z}_1) = \{0\}$.

Now we are ready to make the construction of a pair $\{\bar{B}_0, \bar{B}_1\}$ of contractions with the desired properties.

**Corollary 4.1.** Let $B$ be a Hermitian contraction in $\mathfrak{H}$ with $\text{dom } B = \mathcal{H}_0$, let $\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{H}_0$, and assume that

\[ \dim \mathfrak{M} = \infty, \quad \ker(B_M - B_{\mu}) = \text{dom } B = \mathcal{H}_0. \]

Then there exists a pair $\{\bar{B}_0, \bar{B}_1\}$ of sc-extensions of $B$ with the properties (3.17) which differs from the pair $\{B_{\mu}, B_M\}$.

**Proof.** Let $\bar{Z}_0$ and $\bar{Z}_1$ be a pair of selfadjoint contractions in $\mathfrak{M}$ as constructed in (4.6) and define a pair of sc-extensions of $B$ by means of (2.13):

\[ \bar{B}_k = (B_M + B_{\mu})/2 + (B_M - B_{\mu})^{1/2} \bar{Z}_k (B_M - B_{\mu})^{1/2}/2, \quad k = 0, 1. \]

Then the pair $\{\bar{B}_0, \bar{B}_1\}$ satisfies all the conditions in (3.17) and, since clearly $\bar{Z}_0 \neq -I_\mathfrak{M}$ and $\bar{Z}_1 \neq I_\mathfrak{M}$, one concludes that $\bar{B}_0 \neq B_{\mu}$ and $\bar{B}_1 \neq B_M$. \hfill \Box

4.3. Construction of special pairs of nonnegative selfadjoint operators and nonnegative selfadjoint extensions.

**Corollary 4.2.** There exist pairs $\{\bar{S}_0, \bar{S}_1\}$ of unbounded nonnegative selfadjoint operators in $\mathfrak{H}$ such that

\begin{enumerate}
\item $\text{dom } \bar{S}_0 \cap \text{dom } \bar{S}_1 = \{0\}$,
\item $\text{dom } \bar{S}_0^{1/2} \subset \text{dom } \bar{S}_1^{1/2}$ and the form $\bar{S}_0[\cdot, \cdot]$ is the closed restriction of the form $\bar{S}_1[\cdot, \cdot]$,
\item $\inf_{\varphi \in \text{dom } \bar{S}_0^{1/2}} \bar{S}_1[f - \varphi] = 0$ for all $f \in \text{dom } \bar{S}_1^{1/2}$.
\end{enumerate}
Then for a nonnegative symmetric operator conditions (3.16) by means of Cayley transforms. For simplicity the next result is formulated from (4.6), (4.3), (4.4), Proposition 3.1, and Theorem 3.3 we get that the form
\[ \langle \tilde{S}_0[\varphi, \psi], \tilde{S}_0[1/2] \rangle = (I - \mathcal{X})^{1/2} \mathcal{X}^{-1/2} \varphi, (I - \mathcal{X})^{1/2} \mathcal{X}^{-1/2} \psi, \]
\[ \varphi, \psi \in D[\tilde{S}_0] = \text{dom} \mathcal{X}^{-1/2} = \text{ran} \mathcal{X}^{1/2}, \]
is a closed restriction of the form
\[ \tilde{S}_1[u, v] = (I - \mathcal{Y})^{1/2} \mathcal{Y}^{-1/2} u, (I - \mathcal{Y})^{1/2} \mathcal{Y}^{-1/2} v, \]
\[ u, v \in D[\tilde{S}_1] = \text{dom} \mathcal{Y}^{-1/2} = \text{ran} \mathcal{Y}^{1/2} \]
and
\[ \inf_{\varphi \in \text{dom} \tilde{S}_0^{1/2}} ||\tilde{S}_1^{1/2}(f - \varphi)||^2 = 0 \]
for all \( f \in \text{dom} \tilde{S}_1^{1/2}. \)

Let \( S \) be a nonnegative symmetric linear relation. It follows from Theorems 3.3 and 3.6 that one can construct pairs \( \{\tilde{S}_0, \tilde{S}_1\} \) of nonnegative selfadjoint extensions of \( S \) satisfying the conditions (3.16) by means of Cayley transforms. For simplicity the next result is formulated for a nonnegative symmetric operator \( S \) along the lines in Theorem 3.3.

**Corollary 4.3.** Let \( S \) be a closed nonnegative symmetric, not necessary densely defined, operator in the Hilbert space \( \mathcal{H} \) and assume that \( S \) admits disjoint nonnegative selfadjoint (operator) extensions. Then there exists a pair \( \{\tilde{S}_0, \tilde{S}_1\} \) of nonnegative selfadjoint extensions of \( S \) such that

\[
(4.7) \quad \begin{cases}
\tilde{S}_0 \cap \tilde{S}_1 = S, \\
\text{the sesquilinear form } \tilde{S}_0[\cdot, \cdot] \text{ is a closed restriction of the form } \tilde{S}_1[\cdot, \cdot], \\
\inf \left\{ \tilde{S}_1[u - \varphi], \varphi \in D[\tilde{S}_0] \right\} = 0 \quad \text{for all } u \in D[\tilde{S}_1].
\end{cases}
\]

Moreover, if \( n_+(S) = \infty \) then the pair \( \{\tilde{S}_0, \tilde{S}_1\} \) differs in general from the pair \( \{S_F, S_K\} \).

**Proof.** The Cayley transform \( B = \mathcal{C}(S) = -I + 2(I + S)^{-1} \) of \( S \) is a nondensely defined Hermitian contraction with \( \ker(I + B) = \{0\} \). The disjointness assumption implies that \( S_F \cap S_K = S \), i.e., \( S_F \) and \( S_K \) are also disjoint nonnegative extensions of \( S \). Therefore their Cayley transforms \( B_\mu = \mathcal{C}(S_F) \) and \( B_M = \mathcal{C}(S_K) \) satisfy the equality \( \ker(B_M - B_\mu) = \text{dom} \mathcal{B} \). Now it is clear that the pair \( \{B_\mu, B_M\} \) satisfies all the conditions in (3.17). Moreover, if \( n_+(S) = \infty \) then \( \dim \mathfrak{K} = \infty \) and hence by Corollary 4.1 there are also other pairs \( \{\tilde{B}_0, \tilde{B}_1\} \) of \( sc \)-extensions of \( B \) satisfying the properties (3.17). Finally, it follows from Theorems 3.3 and 3.6 that

\[ \tilde{S}_k = (I - \tilde{B}_k)(I + \tilde{B}_k)^{-1}, \quad k = 0, 1, \]
are nonnegative selfadjoint extensions of \( S \) satisfying the properties in (4.7). \( \square \)
5. \( Q \)-functions of Hermitian contraction corresponding to the special pairs of selfadjoint contractive extensions

The following classes of \( Q \)-functions of a nondensely defined Hermitian contraction \( B \) with \( \text{dom} B = \mathcal{H}_0 \subset \mathcal{H} \), associated to the pair \( \{ \tilde{B}_0, \tilde{B}_1 \} \) of \( sc \)-extensions of \( B \) in \( \mathcal{H} \) which satisfy the conditions in (3.17), were introduced and studied in \([10]\):

\[
(5.1) \quad \tilde{Q}_0(\lambda) = \left[ (\tilde{B}_1 - \tilde{B}_0)^{1/2}(\tilde{B}_0 - \lambda I)^{-1}(\tilde{B}_1 - \tilde{B}_0)^{1/2} + I \right] | \mathcal{M},
\]

\[
(5.2) \quad \tilde{Q}_1(\lambda) = \left[ (\tilde{B}_1 - \tilde{B}_0)^{1/2}(\tilde{B}_1 - \lambda I)^{-1}(\tilde{B}_1 - \tilde{B}_0)^{1/2} - I \right] | \mathcal{M}, \quad \lambda \in \text{Ext} \left[ -1, 1 \right]
\]

These functions belong to the Herglotz-Nevanlinna class. It is easy to verify that

\[
\tilde{Q}_0(\lambda)\tilde{Q}_1(\lambda) = \tilde{Q}_1(\lambda)\tilde{Q}_0(\lambda) = -I_\mathcal{M}, \quad \lambda \in \text{Ext} \left[ -1, 1 \right].
\]

Moreover, the function \( \tilde{Q}_0 \) possesses the properties

\[
\lim_{\lambda \to \infty} \tilde{Q}_0(\lambda) = I_\mathcal{M},
\]

\[
\lim_{\lambda \downarrow 1} \tilde{Q}_0(\lambda) = 0, \quad \lim_{\lambda \uparrow 1} (\tilde{Q}_0(\lambda)h, h) = +\infty, \quad h \in \mathcal{M} \setminus \{0\}
\]

while for \( \tilde{Q}_1 \) one has

\[
\lim_{\lambda \to \infty} \tilde{Q}_1(\lambda) = -I_\mathcal{M},
\]

\[
\lim_{\lambda \downarrow 1} \tilde{Q}_1(\lambda) = 0, \quad \lim_{\lambda \uparrow 1} (\tilde{Q}_1(\lambda)h, h) = -\infty, \quad h \in \mathcal{M} \setminus \{0\}.
\]

Observe that from (5.1), (5.2), and (5.3) follow implications

\[
\lambda \in (-\infty, -1) \Rightarrow \tilde{Q}_0(\lambda) > 0, \quad \tilde{Q}_1(\lambda) < 0,
\]

\[
\lambda \in (1, +\infty) \Rightarrow \tilde{Q}_1(\lambda) < 0, \quad \tilde{Q}_0(\lambda) > 0.
\]

For the pair \( \{ B_\mu, B_M \} \) the corresponding \( Q \)-functions, called the \( Q_\mu \) and \( Q_M \)-functions, were originally defined and investigated by Krein and Ovcharenko in \([35]\). It is stated in \([35]\) that if the function \( \tilde{Q}_0 \) (\( \tilde{Q}_1 \)) possesses the properties in (5.4), then there exists a nondensely defined Hermitian contraction \( B \) such that \( \ker(B_M - B_\mu) = \text{dom} B \) and \( \tilde{Q}_0 \) (respect., \( \tilde{Q}_1 \)) coincides with \( Q_\mu \) (respect., with \( Q_M \)). However, this statements appears to be true only in the case that \( \dim \mathcal{M} < \infty \).

The class of Herglotz-Nevanlinna functions holomorphic in \( \mathbb{C} \setminus [-1, 1] \) and satisfying conditions (5.4) (respect., (5.5)) is denoted in \([10]\) by \( \mathcal{S}_\mu(\mathcal{M}) \) (respect., by \( \mathcal{S}_M(\mathcal{M}) \)). Thus the function \( \tilde{Q}_0 \) defined by (5.1) belongs to the class \( \mathcal{S}_\mu(\mathcal{M}) \), while the function \( \tilde{Q}_1(\lambda) = \tilde{Q}_0^{-1}(\lambda) \) belongs to the class \( \mathcal{S}_M(\mathcal{M}) \). The next theorem, which contains a proper characterization for the conditions stated by Krein and Ovcharenko in \([35]\), has been established in \([10]\).

**Theorem 5.1.** Assume that \( \tilde{Q} \in \mathcal{S}_\mu(\mathcal{M}) \). Then there exist a Hilbert space \( \mathcal{H} \) containing \( \mathcal{M} \) as a subspace, a Hermitian contraction \( B \) in \( \mathcal{H} \) defined on \( \text{dom} B = \mathcal{H} \ominus \mathcal{M} \), and a pair \( \{ \tilde{B}_0, \tilde{B}_1 \} \) of \( sc \)-extensions of \( B \), satisfying (3.17) such that

\[
Q(\lambda) = \left[ (\tilde{B}_1 - \tilde{B}_0)^{1/2}(\tilde{B}_0 - \lambda I)^{-1}(\tilde{B}_1 - \tilde{B}_0)^{1/2} + I \right] | \mathcal{M},
\]

\[
-Q^{-1}(\lambda) = \left[ (\tilde{B}_1 - \tilde{B}_0)^{1/2}(\tilde{B}_1 - \lambda I)^{-1}(\tilde{B}_1 - \tilde{B}_0)^{1/2} - I \right] | \mathcal{M}, \quad \lambda \in \text{Ext} \left[ -1, 1 \right].
\]
If \( \dim \mathcal{N} < \infty \), then necessary
\[
\begin{align*}
\widetilde{B}_0 &= B_\mu, \\
\widetilde{B}_1 &= B_M.
\end{align*}
\]

It is emphasized that in the case \( \dim \mathcal{N} = \infty \) there exist pairs different from \( \{B_\mu, B_M\} \) satisfying (3.17) and their corresponding \( Q \)-functions given by (5.1) and (5.2) also satisfy (5.4) and (5.5), giving a contradiction to the above mentioned result in [35] in the infinite dimensional case \( \dim \mathcal{N} = \infty \).

Recall from [35] that two Hermitian operators \( B \) and \( B' \) defined on the subspaces \( \text{dom} \, B \) and \( \text{dom} \, B' \) of the Hilbert spaces \( \mathfrak{H} = \text{dom} \, B \oplus \mathcal{N} \) and \( \mathfrak{H}' = \text{dom} \, B' \oplus \mathcal{N} \), respectively, are said to be \( \mathcal{N} \)-unitaly equivalent [11], [12], if there is a unitary operator \( U \) from \( \mathfrak{H} \) onto \( \mathfrak{H}' \), such that
\[
U \upharpoonright \mathcal{N} = I_{\mathcal{N}}, \quad U(\text{dom} \, B) = \text{dom} \, B', \quad UB = B'U.
\]
Moreover, \( B \) in \( \mathfrak{H} \) is said to be \( \text{simple} \) if there is no nontrivial subspace invariant under \( B \). An equivalent condition due to M.G. Kre˘ın and I.E. Ovcharenko [35, Lemma 2.1] for Hermitian contraction \( B \) is that the subspace \( \mathcal{N} = \mathfrak{H} \ominus \text{dom} \, B \) is generating for some (equivalently for every) selfadjoint extension \( \widetilde{B} \) of \( B \):

\[
\mathfrak{H} = \text{span} \{ \widetilde{B}^{\mu} \mathcal{N} : n = 0, 1, \ldots \} = \text{span} \{ (\widetilde{B} - \lambda I)^{-1} \mathcal{N} : |\lambda| > 1 \}.
\]

In [35] it is shown that the simple part of the Hermitian contraction \( B \) is uniquely determined by its \( Q_\mu \) (\( Q_M \))-function up to unitary equivalence. An analogous statement holds for functions belonging to the classes \( \mathcal{G}_\mu(\mathcal{N}) \) and \( \mathcal{G}_M(\mathcal{N}) \). Moreover, the following generalization of this result for the pair \( \{\widetilde{B}_0, \widetilde{B}_1\} \) of \text{sc}-extensions of \( B \) is also true.

**Proposition 5.2.** Let \( B \) and \( B' \) be simple Hermitian contractions in \( \mathfrak{H} = \text{dom} \, B \oplus \mathcal{N} \) and \( \mathfrak{H}' = \text{dom} \, B' \oplus \mathcal{N} \), respectively, and let \( \widetilde{Q}_0(\lambda) \) and \( \widetilde{Q}'_0(\lambda) \) be defined via (5.1) \( (\widetilde{Q}_1(\lambda) \) and \( \widetilde{Q}'_1(\lambda) \) be defined via (5.2) \) with the pair \( \{\widetilde{B}_0, \widetilde{B}_1\} \) and \( \{\widetilde{B}_0', \widetilde{B}_1'\} \), respectively. If \( \widetilde{Q}_0(\lambda) \) and \( \widetilde{Q}'_0(\lambda) \) are equal, then \( B \) and \( B' \) and the pairs \( \{\widetilde{B}_0, \widetilde{B}_1\} \) and \( \{\widetilde{B}_0', \widetilde{B}_1'\} \) are unitarily equivalent with the same unitary operator \( U \).

We also recall another statement which concerns the compressed resolvent
\[
Q_{\widetilde{B}}(\lambda) := P_{\mathcal{N}}(\widetilde{B} - \lambda I)^{-1} \upharpoonright \mathcal{N}
\]
associated to a selfadjoint contraction \( \widetilde{B} \) and which can also be found from [11].

**Proposition 5.3.** Let \( \widetilde{B} \) be a selfadjoint contraction in the Hilbert space \( \mathfrak{H} \), and let \( \mathcal{N} \subseteq \mathfrak{H} \). Suppose that \( \widetilde{B} \) is \( \mathcal{N} \)-minimal, i.e. \( \mathfrak{H} = \text{span} \{ (\widetilde{B} - \lambda I)^{-1} \mathcal{N} : |\lambda| > 1 \} \). Then the following conditions are equivalent:

(i) \( \mathcal{N} = \mathfrak{H} \);

(ii) the operator-valued function \( Q_{\widetilde{B}}^{-1}(\lambda) + \lambda I \) is constant.

Since
\[
\left[ (\widetilde{B}_1 - \widetilde{B}_0)^{1/2}(\widetilde{B}_0 - \lambda I)^{-1}(\widetilde{B}_1 - \widetilde{B}_0)^{1/2} + I \right] \upharpoonright \mathcal{N}
= \left[ (\widetilde{B}_1 - \widetilde{B}_0)^{1/2}Q_{\widetilde{B}_0}(\lambda)(\widetilde{B}_1 - \widetilde{B}_0)^{1/2} + I \right] \upharpoonright \mathcal{N},
\]
one can apply Proposition 5.3 and see that it is possible that \( \text{dom} \, B = \{0\} \) in Theorem 5.1. The example of such a situation is provided by the pair of operators \( \{\widetilde{Z}_0, \widetilde{Z}_1\} \) in \( \mathfrak{H} \).
constructed in Subsection 4.2 and the corresponding functions satisfy
\[
\begin{align*}
\bar{Q}_0(\lambda) &= (\bar{Z}_1 - \bar{Z}_0)^{1/2}(\bar{Z}_0 - \lambda I)^{-1}(\bar{Z}_1 - \bar{Z}_0)^{1/2} + I_0 \in \mathcal{S}_e(\mathcal{H}), \\
\bar{Q}_1(\lambda) &= -\bar{Q}_0^{-1}(\lambda) = (\bar{Z}_1 - \bar{Z}_0)^{1/2}(\bar{Z}_1 - \lambda I)^{-1}(\bar{Z}_1 - \bar{Z}_0)^{1/2} - I_0 \in \mathcal{S}_M(\mathcal{H}).
\end{align*}
\]

As an addition to [10] the following statement will now be proved.

**Theorem 5.4.** Let \( B \) be a Hermitian contraction in \( \mathcal{H} \) with \( \text{dom} \, B = \mathcal{H}_0 \subset \mathcal{H} \). Suppose \( \ker(I+B) = \{0\} \). Let the pair \( \{\tilde{B}_0, \tilde{B}_1\} \) of sc-extensions of \( B \) satisfy the equivalent conditions in Proposition 3.1. Then the following conditions are equivalent:

(i) \( s - \lim_{\lambda \to -1} (\lambda + 1) \bar{Q}_0(\lambda) = 0 \);

(ii) \( \lim_{\lambda \uparrow -1} \frac{(\bar{Q}_1(\lambda)f, f)}{1 + \lambda} = -\infty, \, f \in \mathfrak{N} \setminus \{0\} \);

(iii) \( \ker(I + \tilde{B}_0) = \{0\} \).

**Proof.** Using (5.11) together with the following well-known relations for a nonnegative self-adjoint operator \( G \)
\[
\lim_{y \uparrow 0} y(G - yI)^{-1}f = \begin{cases} 
0, & \text{if } f \in \text{ran } G \\
-f, & \text{if } f \in \ker G
\end{cases},
\]
and the identity \( \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \mathfrak{N} \) we get that
\[
(i) \iff \mathfrak{N} \subseteq \text{ran} (I + \tilde{B}_0).
\]

On the other hand, using the equivalence \( \mathfrak{N} \subseteq \text{ran} (I + \tilde{B}_0) \iff \text{dom } B \supseteq \ker(I + \tilde{B}_0) \), the condition \( \ker(I + B) = \{0\} \), and the fact that \( \tilde{B}_0 \) is a sc-extension of \( B \), we have
\[
(i) \iff (iii).
\]

Due to the equality
\[
\bar{Q}_1(\lambda) = -\bar{Q}_0^{-1}(\lambda), \quad \lambda \in \text{Ext} [-1, 1],
\]
we get with \( \lambda < -1 \)
\[
||f||^2 = \left( \bar{Q}_0(\lambda)f, -\bar{Q}_1(\lambda) \right) \leq \sqrt{\left( \bar{Q}_0(\lambda)f, f \right)} \sqrt{\left( \bar{Q}_0(\lambda)\bar{Q}_1(\lambda)f, \bar{Q}_1(\lambda)f \right)}
\]
\[
= \sqrt{\left( \bar{Q}_0(\lambda)f, f \right) \sqrt{-\left( f, \bar{Q}_1(\lambda)f \right)}}.
\]

It follows that
\[
-f, \bar{Q}_1(\lambda)f \geq \frac{||f||^4}{\bar{Q}_0(\lambda)f, f}, \quad \lambda < -1.
\]

Hence \( (i) \Rightarrow (ii) \).

Next suppose that \( (ii) \) holds true. Since \( \tilde{B}_1 - \tilde{B}_0 = (I + \tilde{B}_1)^{1/2}P(I + \tilde{B}_1)^{1/2} \), where \( P \) is an orthogonal projection (see Proposition 3.1 (3.8)), we get that
\[
(5.7) \quad (\tilde{B}_1 - \tilde{B}_0)^{1/2}f = (I + \tilde{B}_1)^{1/2} \sqrt{f}, \quad f \in \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2} = \mathfrak{N}
\]
where $\mathcal{V}$ is an isometry from $\mathcal{H}(= \text{ran}(\tilde{B}_1 - \tilde{B}_0)^{1/2})$ into $\text{ran}(I + \tilde{B}_1)$. With $\lambda < -1$ one obtains

$$\left(\tilde{Q}_1(\lambda)f, f\right) = \left((\tilde{B}_1 - \lambda I)^{-1}(I + \tilde{B}_1)\mathcal{V}f, \mathcal{V}f\right) - \|f\|^2$$

$$= -(1 + \lambda)\left((\tilde{B}_1 - \lambda I)^{-1}\mathcal{V}f, \mathcal{V}f\right), \quad f \in \mathcal{H}.$$

Therefore

$$\left(\frac{\tilde{Q}_1(\lambda)f, f}{1 + \lambda}\right) = -\|((\tilde{B}_1 - \lambda I)^{-1/2}\mathcal{V}f\|^2.$$

One concludes that

$$(ii) \iff \text{ran} \mathcal{V} \cap \text{ran} (I + \tilde{B}_1)^{1/2} = \{0\}.$$ 

From the definition of the isometry $\mathcal{V}$ in (5.7) we have

$$\text{ran} \mathcal{V} \cap \text{ran} (I + \tilde{B}_1)^{1/2} = \{0\} \iff \text{ran} (I + \tilde{B}_1) \cap \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2} = \{0\}.$$ 

With $g \in \ker(I + \tilde{B}_0)$ the equality

$$I + \tilde{B}_1 = I + \tilde{B}_0 + (\tilde{B}_1 - \tilde{B}_0)$$

yields the identity $(I + \tilde{B}_1)g = (\tilde{B}_1 - \tilde{B}_0)g$. Thus, (ii) $\Rightarrow$ (iii). The proof is complete. $\square$

6. $Q$-FUNCTIONS OF A NONNEGATIVE SYMMETRIC OPERATOR CORRESPONDING TO THE SPECIAL PAIRS OF NONNEGATIVE SELFADJOINT EXTENSIONS

Let $S$ be a closed nonnegative symmetric operator, which is in general nondensely defined. It is assumed that $S$ admits disjoint nonnegative selfadjoint operator extensions. In the case of nondensely defined $S$ this yields, in particular, that $S_K$ is an operator (i.e. it has no multi-valued part).

Let the linear fractional transformation $B$ of $S$ be defined by

$$B := (I - S)(I + S)^{-1}.$$ 

Since $S_F \cap S_K = S$, we get $\ker(B_M - B_\mu) = \text{dom} B$. Consider two nonnegative selfadjoint operator extensions $\tilde{S}_0$ and $\tilde{S}_1$ of $S$ given by

$$\tilde{S}_k = (I - \tilde{B}_k)(I + \tilde{B}_k)^{-1}, \quad k = 0, 1,$$

where the pair of $sc$-extensions $\{\tilde{B}_0, \tilde{B}_1\}$ satisfies the condition (3.17). Notice that

$$\tilde{B}_1 - \tilde{B}_0 = 2(\tilde{S}_1 + I) - (\tilde{S}_0 + I)^{-1}$$

Next introduce the so-called $\gamma$-fields by the formulas

$$\mathbb{C} \setminus \mathbb{R}_+ \ni \lambda \mapsto \gamma_0(\lambda) := \left(I + (\lambda + 1)(\tilde{S}_0 - \lambda I)^{-1}\right)(\tilde{B}_1 - \tilde{B}_0)^{1/2} | \mathcal{N} \in \mathcal{L}(\mathcal{H}, \tilde{H}),$$

$$\mathbb{C} \setminus \mathbb{R}_+ \ni \lambda \mapsto \gamma_1(\lambda) := \left(I + (\lambda + 1)(\tilde{S}_1 - \lambda I)^{-1}\right)(\tilde{B}_1 - \tilde{B}_0)^{1/2} | \mathcal{N} \in \mathcal{L}(\mathcal{H}, \tilde{H}).$$

Then define

$$\tilde{Q}_0(\lambda) = -I_{\mathcal{H}} + \frac{\lambda + 1}{2}(\tilde{B}_1 - \tilde{B}_0)^{1/2}\left(I + (\lambda + 1)(\tilde{S}_0 - \lambda I)^{-1}\right)(\tilde{B}_1 - \tilde{B}_0)^{1/2} | \mathcal{N}, \lambda \in \mathbb{C} \setminus \mathbb{R}_+, (6.1)$$

$$\tilde{Q}_1(\lambda) = I_{\mathcal{H}} + \frac{\lambda + 1}{2}(\tilde{B}_1 - \tilde{B}_0)^{1/2}\left(I + (\lambda + 1)(\tilde{S}_1 - \lambda I)^{-1}\right)(\tilde{B}_1 - \tilde{B}_0)^{1/2} | \mathcal{N}, \lambda \in \mathbb{C} \setminus \mathbb{R}_+, (6.2)$$
If $\tilde{B} = (I - \tilde{S})(I + \tilde{S})^{-1}$ is the linear fractional transformation of a nonnegative selfadjoint operator, then its resolvent can be expressed in the form

\[(6.3) \quad (\tilde{B} - \mu I)^{-1} = -\frac{1}{1 + \mu} \left( I + \frac{2}{1 + \mu} \left( \tilde{S} - \frac{1 - \mu}{1 + \mu} I \right)^{-1} \right). \]

It follows that

\[(6.4) \quad \tilde{Q}_0(\lambda) = -\left( I_{\mathfrak{m}} + (\tilde{B}_1 - \tilde{B}_0)^{1/2} \left( \tilde{B}_0 - \frac{1 - \mu}{1 + \mu} I_{\mathfrak{m}} \right)^{-1} (\tilde{B}_1 - \tilde{B}_0)^{1/2} \right) \upharpoonright \mathfrak{m} = -\tilde{Q}_0 \left( \frac{1 - \mu}{1 + \lambda} \right), \]

\[\tilde{Q}_1(\lambda) = -\left( -I_{\mathfrak{m}} + (\tilde{B}_1 - \tilde{B}_0)^{1/2} \left( \tilde{B}_1 - \frac{1 - \mu}{1 + \mu} I_{\mathfrak{m}} \right)^{-1} (\tilde{B}_1 - \tilde{B}_0)^{1/2} \right) \upharpoonright \mathfrak{m} = -\tilde{Q}_1 \left( \frac{1 - \mu}{1 + \lambda} \right), \]

where the functions $\tilde{Q}_0$ and $\tilde{Q}_1$ are given by (5.1) and (5.2) with $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. From (6.4) and (5.6) it follows that

$\lambda \in (-\infty, 0) \Rightarrow \tilde{Q}_0(\lambda) < 0$, $\tilde{Q}_1(\lambda) > 0$.

**Definition 6.1.** Let $\mathcal{H}$ be a separable Hilbert space. Then denote by $\mathcal{G}_F(\mathcal{H})$ the class of Herglotz-Nevanlinna $\mathcal{L}(\mathcal{H})$-valued functions $\mathcal{M}(\lambda)$ holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$ and possessing the properties

1. $\mathcal{M}^{-1}(\lambda) \in \mathcal{L}(\mathcal{H})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,
2. $s - \lim_{x \to 0} \mathcal{M}(x) = 0$,
3. $\lim_{x \downarrow -\infty} (\mathcal{M}(x)g, g)_{\mathcal{H}} = -\infty$ for each $g \in \mathcal{H} \setminus \{0\}$,
4. $s - \lim_{x \downarrow -\infty} x^{-1} \mathcal{M}(x) = 0$.

**Definition 6.2.** Let $\mathcal{H}$ be a separable Hilbert space. Then denote by $\mathcal{G}_K(\mathcal{H})$ the class of Herglotz-Nevanlinna $\mathcal{L}(\mathcal{H})$-valued functions $\mathcal{N}(\lambda)$ holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$ and possessing the properties

1. $\mathcal{N}^{-1}(\lambda) \in \mathcal{L}(\mathcal{H})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,
2. $\lim_{x \uparrow 0} (\mathcal{N}(x)g, g)_{\mathcal{H}} = +\infty$ for each $g \in \mathcal{H} \setminus \{0\}$,
3. $s - \lim_{x \uparrow -\infty} \mathcal{N}(x) = 0$,
4. $\lim_{x \downarrow -\infty} x(\mathcal{N}(x)g, g)_{\mathcal{H}} = -\infty$ for each $g \in \mathcal{H} \setminus \{0\}$.

Clearly, the class $\mathcal{G}_F(\mathcal{H})$ is a subset of the inverse Stieltjes class and $\mathcal{G}_K(\mathcal{H})$ is subset of the Stieltjes class of $\mathcal{L}(\mathcal{H})$-valued functions [27].

**Theorem 6.3.** The function $\tilde{Q}_0$ belongs to the class $\mathcal{G}_F(\mathfrak{m})$, while the function $\tilde{Q}_1$ belongs to the class $\mathcal{G}_K(\mathfrak{m})$ and

$\tilde{Q}_0(\lambda)\tilde{Q}_1(\lambda) = \tilde{Q}_1(\lambda)\tilde{Q}_0(\lambda) = -I_{\mathfrak{m}}$

for each $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$.

**Proof.** The statements follow from (5.3), (5.4), (5.5), Theorem 5.4 and (6.4). \qed

**Theorem 6.4.** Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{M} \in \mathcal{G}_F(\mathcal{H})$ ($\mathcal{N} \in \mathcal{G}_K(\mathcal{H})$). Then there exists a Hilbert space $\mathfrak{N}$, containing $\mathcal{H}$ as a subspace, a closed simple nonnegative
possibly nondensely defined operator $S$ in $\mathcal{H}$, and a pair $\{\tilde{S}_0, \tilde{S}_1\}$ of nonnegative selfadjoint operator extensions of $S$, satisfying (4.7) and such that

$$\mathcal{M}(\lambda) = Y^* \tilde{Q}_0(\lambda)Y \quad (\mathcal{N}(\lambda) = Y^* \tilde{Q}_1(\lambda)Y), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+,$$

where $Y \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is an isomorphism and $\tilde{Q}_0 (\tilde{Q}_1)$ is given by (6.1) (6.2). If $\dim \mathcal{H} < \infty$ or $\dim \mathcal{H} = \infty$ but $\text{Im} \mathcal{M}(i) \cap \text{Im} \mathcal{N}(i)$ is positive definite, then $S$ is densely defined and the equalities $\tilde{S}_0 = S_F$, $\tilde{S}_1 = S_K$ hold true.

Proof. We will prove the statement for $\mathcal{M} \in \mathfrak{S}_F(\mathcal{H})$. Since the function $\mathcal{M}$ belongs to the inverse Stieltjes class, the operator $-\mathcal{M}(-1)$ is positive definite. Let $Y = (-\mathcal{M}(-1))^{1/2}$ and define

$$\tilde{Q}_0(\lambda) = Y^{-1} \mathcal{M}(\lambda)Y^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+,$$

$$\tilde{Q}_0(z) = -\tilde{Q}_0 \left( \frac{1 - z}{1 + z} \right), \quad z \in \mathbb{C} \setminus [-1, 1].$$

Due to $\mathcal{M} \in \mathfrak{S}_F(\mathcal{H})$ the function $\tilde{Q}_0$ belongs to the class $\mathfrak{S}_\mu(\mathcal{H})$ and, moreover,

$$s - \lim_{x \uparrow -1} (x + 1)\tilde{Q}_0(x) = 0.$$

By [10] Theorem 5.1 there exists a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ as a subspace, a simple Hermitian contraction $B$ defined on $\text{dom} B = \tilde{\mathcal{H}} \ominus \mathcal{H}$ with the property $\ker(B_M - B_\mu) = \text{dom} B$, and a pair $\{\tilde{B}_0, \tilde{B}_1\}$ of $sc$-extensions, satisfying (5.17) such that

$$\tilde{Q}_0(z) = \left[ (\tilde{B}_1 - \tilde{B}_0)^{1/2} (\tilde{B}_0 - zI)^{-1} (\tilde{B}_1 - \tilde{B}_0)^{1/2} + I \right] \upharpoonright \mathcal{H}, \quad z \in \mathbb{C} \setminus [-1, 1].$$

From Theorem 5.4 it follows that $\ker(I + \tilde{B}_0) = \{0\}$.

Now define

$$S = (I - B)(I + B)^{-1}.$$

Then $S$ is a closed nonnegative operator, possibly nondensely defined, and the pair $\{\tilde{S}_0, \tilde{S}_1\}$ of its nonnegative selfadjoint (operator) extensions defined by

$$\tilde{S}_k = (I - \tilde{B}_k)(I + \tilde{B}_k)^{-1}, \quad k = 0, 1,$$

satisfies conditions (4.7). Finally, (6.3) implies that the function

$$-I_{\tilde{\mathcal{H}}} + \frac{\lambda + 1}{2}(\tilde{B}_1 - \tilde{B}_0)^{1/2} \left( I + (\lambda + 1)(\tilde{S}_0 - \lambda I)^{-1} \right) (\tilde{B}_1 - \tilde{B}_0)^{1/2} \upharpoonright \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+,$$

coincides with $\tilde{Q}_0$. Thus, $\mathcal{M}(\lambda) = Y \tilde{Q}_0(\lambda)Y$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$.

Let $\dim \mathcal{H} < \infty$. Then $B$ is Hermitian contraction with finite equal deficiency indices. In this case the pair $\{\tilde{B}_0, \tilde{B}_1\}$ necessarily coincides with the pair $\{B_\mu, B_M\}$. Moreover, $\ker(I + B_F) = \{0\}$, so that the operator $S$ is densely defined, and the equalities $\tilde{S}_0 = S_F$ and $\tilde{S}_1 = S_K$ follow.

It is clear that $\text{Im} \mathcal{M}(i) = -Y \text{Im} Q_0(-i)Y$. If $\text{Im} \mathcal{M}(i)$ has a bounded inverse, then according to [10] Corollary 6.3 one has $\tilde{B}_0 = B_\mu$, $\tilde{B}_1 = B_M$, and $\text{ran}(B_M - B_\mu) = \mathcal{H}$, and since $\ker(I + S_F) = \{0\}$, one concludes again that the operator $S$ is densely defined and that $\tilde{S}_0 = S_F$ and $\tilde{S}_1 = S_K$. \hfill \Box
Thus if $\mathcal{H}$ is finite dimensional and $\mathcal{M} \in \mathcal{G}_F(\mathcal{H})$, then there exists a closed densely defined nonnegative operator $S$ with finite deficiency indices such that $\mathcal{M}$ is the $Q_F$-function of $S$ and $-\mathcal{M}^{-1}$ is the $Q_K$-function of the same $S$.

If $\dim \mathcal{H} = \infty$, then it is possible that $\text{dom} \ S = \{0\}$. Actually, one can take the pair $\{\tilde{S}_0, \tilde{S}_1\}$ in $\mathcal{F}$ as given in Corollary 4.12 and define the corresponding function

$$\tilde{Q}_0(\lambda) = -I + \frac{\lambda + 1}{2} (\tilde{Z}_1 - \tilde{Z}_0)^{1/2} \left( I + (\lambda + 1)(\tilde{S}_0 - \lambda I)^{-1} \right) (\tilde{Z}_1 - \tilde{Z}_0)^{1/2}, \ \lambda \in \mathbb{C} \setminus \mathbb{R}_+.$$ 

This function belongs to the class $\mathcal{G}_F(\mathcal{F})$ and $-\tilde{Q}_0^{-1}(\lambda) = \tilde{Q}_1(\lambda) \in \mathcal{G}_K(\mathcal{F})$, where

$$\tilde{Q}_1(\lambda) = I + \frac{\lambda + 1}{2} (\tilde{Z}_1 - \tilde{Z}_0)^{1/2} \left( I + (\lambda + 1)(\tilde{S}_1 - \lambda I)^{-1} \right) (\tilde{Z}_1 - \tilde{Z}_0)^{1/2}, \ \lambda \in \mathbb{C} \setminus \mathbb{R}_+.$$ 

7. Special boundary pairs, positive boundary triplets and their Weyl functions

In this section pairs of nonnegative selfadjoint extensions of a nonnegative symmetric operator and the associated $Q$-functions are investigated further by constructing specific classes of (generalized) boundary triplets and boundary pairs suitable for nonnegative operators. In particular, some new realization results for the classes of $Q$-functions introduced in the previous sections are obtained, a most appealing one concerns the class $\mathcal{G}_F(\mathcal{H})$ (see Definition 6.1) which is established in Theorems 7.13, 7.17 below.

7.1. Ordinary, generalized and positive boundary triplets.

**Definition 7.1.** [13], [29], [22], [23]. Let $S$ be a closed densely defined symmetric operator with equal defect numbers in $\mathcal{F}$. Let $\mathcal{H}$ be some Hilbert space and let $\Gamma_0$ and $\Gamma_1$ be linear mappings of $\text{dom} \ S^*$ into $\mathcal{H}$. A triplet $\{H, \Gamma_0, \Gamma_1\}$ is called a space of boundary values (s.b.v.) or an ordinary boundary triplet for $S^*$ if

a) for all $x, y \in \text{dom} \ S^*$ the Green's identity

$$(S^*x, y) - (x, S^*y) = (\Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, \Gamma_1 y)_{\mathcal{H}}, \ x, y \in \text{dom} \ S^*,$$

holds;

b) the mapping

$$\text{dom} \ S^* \ni x \mapsto \Gamma x = \{\Gamma_0 x, \Gamma_1 x\} \in \mathcal{H} \times \mathcal{H}$$

is surjective.

Denote $\mathcal{F}_+ := \text{dom} \ S^*$. When equipped with the inner product

$$(u, v)_+ := (u, v) + (S^*u, S^*v),$$

$\mathcal{F}_+$ becomes a Hilbert space. It follows from Definition 7.1 that $\Gamma_0, \Gamma_1 \in \mathcal{L}(\mathcal{F}_+, \mathcal{H})$, and $\ker \Gamma_k \supset \text{dom} \ S, \ k = 1, 2$, and, moreover, that the operators

$$\tilde{S}_0 = S^* \uparrow \ker \Gamma_0, \ \tilde{S}_1 = S^* \uparrow \ker \Gamma_1$$

are selfadjoint extensions of $S$ which in addition are transversal:

$$\text{dom} \ S^* = \text{dom} \tilde{S}_0 + \text{dom} \tilde{S}_1.$$ 

The function $M(\lambda)$ defined by

$$M(\lambda)(\Gamma_0 x_\lambda) = \Gamma_1 x_\lambda, \ x_\lambda \in \mathfrak{N}_\lambda,$$
where $\mathfrak{N}_0$ stands for the defect subspace of $S$ at $\lambda$, is called the Weyl function of the boundary triplet \cite{10}. With the corresponding $\gamma$-field given by 

$$\gamma(\lambda) := (\Gamma_0 | \mathfrak{N}_\lambda)^{-1}$$

the definition of the Weyl function can be rewritten in the form $M(\lambda) = \Gamma_1 \gamma(\lambda)$.

If the operators $\Gamma_0$ and $\Gamma_1$ are defined only on a linear manifold $\mathcal{L}$ which is dense in $\mathcal{H}_+$, are closable w.r.t. norms $|| \cdot ||_+ \text{ and } || \cdot ||_{\mathcal{H}}$, the Green’s identity (7.1) is valid for $x, y \in \mathcal{L}$, the mapping $\Gamma_0 : \mathcal{L} \to \mathcal{H}$ is surjective, and the operator $\tilde{\mathcal{S}}_0 := S^* \upharpoonright \ker \Gamma_0$ is selfadjoint, then \{\mathcal{H}, \Gamma_0, \Gamma_1\} is said to be a generalized boundary triplet; see \cite{17}.

**Definition 7.2.** Let $S$ be a densely defined closed positive definite symmetric operator in $\mathcal{H}$ and let $\tilde{\mathcal{S}}_0$ be a positive definite selfadjoint extension of $S$. An ordinary boundary triplet \{\mathcal{H}, \Gamma_0, \Gamma_1\} for $S^*$ is called a positive boundary triplet corresponding to the decomposition

$$\text{dom} \, S^* = \text{dom} \, \tilde{\mathcal{S}}_0 \hat{+} \ker S^*$$

if

$$(S^* f, g) = (\tilde{\mathcal{S}}_0 \mathcal{P}_0 f, g) + (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}}, \quad f, g \in \text{dom} \, S^*,$$

where $\mathcal{P}_0$ is the projector from $\tilde{\mathcal{S}}_+ = \text{dom} \, S^*$ onto $\text{dom} \, \tilde{\mathcal{S}}_0 = \ker \Gamma_0 \parallel \ker S^*$.\[
\]

By definition $\ker \Gamma_0 = \text{dom} \, \tilde{\mathcal{S}}_0$ and, moreover,

$$\ker \Gamma_1 = \ker S^+ \ker S^* = \ker S_K.$$\[
\]

Definition\cite{7} has been proposed by A.N. Kochubei \cite{29} (see also \cite{22}). To cover the general case of a nonnegative symmetric operator $S$ the following definition was suggested in \cite{4}:

**Definition 7.3.** Let $S$ be a densely defined closed nonnegative symmetric operator in $\mathcal{H}$. An ordinary boundary triplet \{\mathcal{H}, \Gamma_0, \Gamma_1\} for $S^*$ is called positive if the quadratic form

$$\omega(f, f) := (S^* f, f) - (\Gamma_1 f, \Gamma_0 f)_{\mathcal{H}}, \quad f \in \text{dom} \, S^*$$

is nonnegative.

It follows from Definition\cite{7,3} that if \{\mathcal{H}, \Gamma_0, \Gamma_1\} is a positive boundary triplet, then $\tilde{\mathcal{S}}_0$ and $\tilde{\mathcal{S}}_1$ are two mutually transversal nonnegative selfadjoint extensions of $S$ such that $\tilde{\mathcal{S}}_1 \leq \tilde{\mathcal{S}}_0$. Moreover, it is proved in \cite{4} that positive boundary triplets exist if and only if the Friedrichs and Kreǐn extensions are transversal. An ordinary boundary triplet for a densely defined closed nonnegative operator $S$, which satisfies the equalities

$$\ker \Gamma_0 = S_F \quad \text{and} \quad \ker \Gamma_1 = S_K,$$

is called basic; see \cite{4, 9}. The following theorem has been established in \cite{4}:

**Theorem 7.4.** Let \{\mathcal{H}, \Gamma_0^{(0)}, \Gamma_1^{(0)}\} be a basic boundary triplet. Then an ordinary boundary triplet \{\mathcal{H}', \Gamma_0', \Gamma_1'\} is positive if and only if the following equalities hold

$$\Gamma_0' = W \left( (I_{\mathcal{H}} + BC)\Gamma_0^{(0)} - B\Gamma_1^{(0)} \right),$$

$$\Gamma_1' = W^{-1} \left( -CT\Gamma_0^{(0)} + \Gamma_1^{(0)} \right)$$

for some bounded nonnegative selfadjoint operators $B$ and $C$ in $\mathcal{H}$ and a linear homeomorphism $W \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$.

Notice that in \cite{17} and \cite{7} generalized basic boundary triplets are constructed. In the next section a more general class of generalized positive boundary triplets is constructed.
7.2. Special boundary pairs and corresponding positive boundary triplets.

7.2.1. The linear manifold $\mathcal{L}$. In the rest of this section we assume that

(a) $S$ is a densely defined nonnegative symmetric operator in $\mathcal{H}$,
(b) $\widetilde{S}_0$ and $\widetilde{S}_1$ are two nonnegative selfadjoint extensions of $S$, such that $\text{dom} \widetilde{S}_1 \cap \text{dom} \widetilde{S}_0 = \text{dom} S$,
(c) the form $\widetilde{S}_0[\cdot, \cdot]$ is a closed restriction of the form $\widetilde{S}_1[\cdot, \cdot]$

Define the linear manifold $\mathcal{L}$ by the equality

\begin{equation}
\mathcal{L} := \text{dom} \widetilde{S}_0 + (\mathcal{D}[\widetilde{S}_1] \ominus \mathcal{S}_1) \mathcal{D}[\widetilde{S}_0].
\end{equation} 

Let $\mathfrak{N}_z$ be the defect subspace of $S$ at $z$ and denote

\begin{equation}
\mathfrak{N}_z := \mathfrak{N}_z \cap \mathcal{L}, \ z \in \text{Ext} [0, \infty).
\end{equation} 

Since $\mathcal{D}[\widetilde{S}_1] \ominus \mathcal{S}_1 \mathcal{D}[\widetilde{S}_0] \subset \mathfrak{N}_{-1}$, see Proposition 3.1, it is clear that

\begin{equation}
\mathcal{D}[\widetilde{S}_1] \ominus \mathcal{S}_1 \mathcal{D}[\widetilde{S}_0] = \mathfrak{N}_{-1}.
\end{equation} 

Consequently,

\begin{equation}
\mathfrak{N}_z = (\widetilde{S}_0 + I)(\widetilde{S}_0 - zI)^{-1}\mathfrak{N}_{-1} = \left(I + (z + 1)(\widetilde{S}_0 - zI)^{-1}\right)\mathfrak{N}_{-1},
\end{equation} 

and one has the decompositions

\begin{equation}
\mathcal{L} = \text{dom} \widetilde{S}_0 + \mathfrak{N}_z, \ \mathcal{D}[\widetilde{S}_1] = \mathcal{D}[\widetilde{S}_0] + \mathfrak{N}_z, \ z \in \text{Ext} [0, \infty).
\end{equation} 

In particular, with $z, \xi \in \text{Ext} [0, \infty)$ the subspaces in (7.4) are connected by

\begin{equation}
\mathfrak{N}_z = (\widetilde{S}_0 - \xi I)(\widetilde{S}_0 - zI)^{-1}\mathfrak{N}_\xi = \left(I + (z - \xi)(\widetilde{S}_0 - zI)^{-1}\right)\mathfrak{N}_\xi.
\end{equation} 

Lemma 7.5. Let $S$ and $\{\widetilde{S}_0, \widetilde{S}_1\}$ satisfy conditions (a), (b), (c). Then $\mathcal{L}$ defined in (7.3) satisfies

\begin{equation}
\mathcal{L} = \text{dom} \widetilde{S}_1 + (\mathcal{D}[\widetilde{S}_1] \ominus \mathcal{S}_1) \mathcal{D}[\widetilde{S}_0]
\end{equation} 

and

\begin{equation}
\text{dom} \widetilde{S}_0 + \text{dom} \widetilde{S}_1 \subset \mathcal{L} \subset \mathcal{D}[\widetilde{S}_1] \cap \text{dom} S^*.
\end{equation} 

Proof. According to Proposition 3.1 one has $\mathcal{D}[\widetilde{S}_1] \ominus \mathcal{S}_1 \mathcal{D}[\widetilde{S}_0] = \text{ran} (\widetilde{B}_1 - \widetilde{B}_0)^{1/2}$ and since $\text{dom} \widetilde{S}_k = \text{ran} (I + \widetilde{B}_k), \ k = 0, 1$, we have

\begin{align*}
(I + \widetilde{B}_1 - \widetilde{B}_0) f = (I + \widetilde{B}_0) f + (\widetilde{B}_1 - \widetilde{B}_0) f \in \text{dom} \widetilde{S}_0 + \text{ran} (\widetilde{B}_1 - \widetilde{B}_0)^{1/2}, \\
(I + \widetilde{B}_0) f = (I + \widetilde{B}_1) f - (\widetilde{B}_1 - \widetilde{B}_0) f \in \text{dom} \widetilde{S}_1 + \text{ran} (\widetilde{B}_1 - \widetilde{B}_0)^{1/2}.
\end{align*}

These identities combined with (7.3) lead to the sum representation in (7.7) and since $S$ is densely defined and $\widetilde{S}_1$ is nonnegative, the sum in (7.7) is direct.

The last two inclusions in the lemma are clear from (7.5) and (7.7). \qed

If $\widetilde{S}_0 = S_F$, $\widetilde{S}_1 = S_K$ then $\mathcal{L} = \mathcal{D}[S_K] \cap \text{dom} S^*$. Moreover, in the case of transversality one has automatically $\mathcal{L} = \text{dom} S^* = \text{dom} \widetilde{S}_0 + \text{dom} \widetilde{S}_1$. 

Proposition 7.6. Under the assumptions in Lemma 7.5 the sesquilinear form
\[ \text{dom } \tilde{\eta} = \mathcal{L}, \quad \tilde{\eta}[u, v] := \tilde{S}_1[u, v], \quad u, v \in \mathcal{L} \]
is closed in the Hilbert space \( \mathfrak{H}_+ \).

Proof. Let \( \{u_n\} \) be a sequence from \( \mathcal{L} \) such that
1. \( \lim_{n \to \infty} u_n = u \) in \( \mathfrak{H}_+ \),
2. \( \lim_{m,n \to \infty} \tilde{S}_1[u_n - u_m] = 0 \).

Due to (7.3) one can write \( u_n = f_n + (\tilde{B}_1 - \tilde{B}_0)^{1/2}g_n, \quad n \in \mathbb{N} \), where \( f_n \in \text{dom } \tilde{S}_0 \) and \( g_n \in \mathfrak{N} = \text{ran } (\tilde{B}_1 - \tilde{B}_0)^{1/2} \), which in view of (3.11) leads to
\[
\tilde{S}_1[u_n - u_m] + ||u_n - u_m||^2 \\
= \tilde{S}_1[f_n - f_m] + ||f_n - f_m||^2 + ||(\tilde{B}_1 - \tilde{B}_0)^{1/2}(g_n - g_m)||^2_{\tilde{S}_1} \\
= \tilde{S}_1[f_n - f_m] + ||f_n - f_m||^2 + 2||g_n - g_m||^2.
\]

Hence the sequences \( \{f_n\} \) and \( \{g_n\} \) converge in \( \mathfrak{H} \). Let \( g := \lim_{n \to \infty} g_n \). Then \( g \in \mathfrak{N} \) and
\[
\lim_{n \to \infty} (\tilde{B}_1 - \tilde{B}_0)^{1/2}g_n = (\tilde{B}_1 - \tilde{B}_0)^{1/2}g.
\]
It follows from (7.5) that
\[ S^*u_n = \tilde{S}_0f_n - (\tilde{B}_1 - \tilde{B}_0)^{1/2}g_n, \quad n \in \mathbb{N}, \]
and hence the sequence \( \{u_n\} \) converges in \( \mathfrak{H}_+ \). Consequently, \( \{f_n\} \) converges in \( \mathfrak{H}_+ \). Put
\[ f := \lim_{n \to \infty} f_n \] in the Hilbert space \( \mathfrak{H}_+ \).

Then \( f \in \text{dom } \tilde{S}_0 \) and
\[ u = f + (\tilde{B}_1 - \tilde{B}_0)^{1/2}g. \]
Thus the vector \( u \) belongs to \( \mathcal{L} \). Since the form \( \tilde{S}_0[\cdot, \cdot] \) is the closed restriction of the form \( \tilde{S}_1[\cdot, \cdot] \) we get that
\[
\lim_{n \to \infty} \tilde{S}_1[f - f_n] = \lim_{n \to \infty} \tilde{S}_0[f - f_n] = 0.
\]
Therefore,
\[ \tilde{S}_1[u - u_n] + ||u - u_n||^2 = \tilde{S}_0[f - f_n] + ||f - f_n||^2 + ||(\tilde{B}_1 - \tilde{B}_0)^{1/2}(g - g_n)||^2 \to 0, \quad n \to \infty, \]
and this completes the proof. \( \square \)

It follows from Proposition 7.6 that the linear manifold \( \mathcal{L} \) is a Hilbert space with respect to the inner product (cf. (7.2))
\[
(u, v)_{\tilde{\eta}} := \tilde{S}_1[u, v] + (u, v)_+.
\]

(7.8)

Lemma 7.7. The identity
\[ \tilde{S}_1[f, \varphi] = (S^*f, \varphi) \]
is satisfied for all \( f \in \mathcal{L} \) and all \( \varphi \in \mathcal{D}[\tilde{S}_0] \).
Proof. Let \( f = \psi + g, \psi \in \text{dom } \tilde{S}_0, g \in \mathcal{D}[\tilde{S}_1] \ominus \tilde{S}_1 \mathcal{D}[\tilde{S}_0] \). According to (7.9) \( \mathcal{D}[\tilde{S}_1] \ominus \tilde{S}_1 \mathcal{D}[\tilde{S}_0] \subset \mathcal{H}_{-1} \), so that \( S^* g = -g \) and, therefore, \( S^* f = \tilde{S}_0 \psi - g \). On the other hand,
\[
\tilde{S}_1 [f, \varphi] = \tilde{S}_1 [\psi, \varphi] + \tilde{S}_1 [g, \varphi] = \tilde{S}_0 [\psi, \varphi] - (g, \varphi) = (\tilde{S}_0 \psi - g, \varphi),
\]
where the second identity follows from (2.8). This completes the proof. \( \square \)

7.2.2. Boundary pairs and \( \gamma \)-fields.

Definition 7.8. The pair \( \{ \mathcal{H}, \Gamma_0 \} \) is called a boundary pair for \( \{ \tilde{S}_0, \tilde{S}_1 \} \) if \( \mathcal{H} \) is a Hilbert space, \( \Gamma_0 \) is a continuous linear operator from the Hilbert space \( \mathcal{D}[\tilde{S}_1] \) into \( \mathcal{H} \), and
\[
\ker \Gamma_0 = \mathcal{D}[\tilde{S}_0], \text{ ran } \Gamma_0 = \mathcal{H}.
\]
Due to (7.6) and the equality \( \ker \Gamma_0 = \mathcal{D}[\tilde{S}_0] \) the mapping \( \Gamma_0 : \tilde{\mathcal{H}}_z \rightarrow \mathcal{H} \) is a bijection, the inverse operator
\[
(7.9) \quad \Gamma_0(z) := \left( \Gamma_0|_{\tilde{\mathcal{H}}_z} \right)^{-1}
\]
belongs to \( \mathbf{L}(\mathcal{H}, \mathcal{D}[\tilde{S}_1]) \cap \mathbf{L}(\mathcal{H}, \tilde{\mathcal{S}}) \). Since \( ||\varphi_z||_z^2 = (1 + |z|^2)||\varphi_z||^2 \) for all \( \varphi_z \in \mathcal{H}_z \), the operator \( \Gamma_0(z) \) is continuous from \( \mathcal{H} \) into \( \mathcal{L} \) with respect to the inner product (7.8).

Definition 7.9. Let \( \{ \mathcal{H}, \Gamma_0 \} \) be a boundary pair for \( \{ \tilde{S}_0, \tilde{S}_1 \} \). The operator valued function \( \Gamma_0(z) \) defined by (7.9) is called the \( \Gamma_0 \)-field.

Since \( \ker \Gamma_0 = \mathcal{D}[\tilde{S}_0] \) and ran \( \Gamma_0 = \mathcal{H} \), one obtains the following equality:
\[
(7.10) \quad \Gamma_0(z) = \Gamma_0(\xi) + (z - \xi)(\tilde{S}_0 - zI)^{-1}\Gamma_0(\xi), \quad z, \xi \in \text{Ext } [0, \infty).
\]
Therefore, the \( \Gamma_0 \)-field is a holomorphic function in \( \text{Ext } [0, \infty) \) and ran \( \Gamma_0(z) = \tilde{\mathcal{H}}_z \). In addition,
\[
s - \lim_{x \downarrow -\infty} \Gamma_0(x) = 0.
\]
Observe that the operator \( \Gamma_0|_{\mathcal{L}} \) is closed in \( \tilde{\mathcal{S}}_+ \). To see this let \( \{ u_n \} \subset \mathcal{L} \) be a sequence such that
\[
u_n \rightarrow u \text{ in } \mathcal{H}_+, \quad \Gamma_0 u_n \rightarrow e \text{ in } \mathcal{H} \quad \text{when } n \rightarrow \infty.
\]
Due to (7.6) and (7.9)
\[
\{ f_n \} \subset \text{dom } \tilde{S}_0, \quad \{ e_n \} \subset \mathcal{H}.
\]
Since \( e_n = \Gamma_0 e_n \), \( n \in \mathbb{N} \), the sequence \( \{ e_n \} \) converges in \( \mathcal{H} \) to the vector \( e \). Therefore the sequence \( \{ \Gamma_0(1) e_n \} \) converges to \( \Gamma_0(1) e \) in \( \mathcal{H}_{-1} \) in the Hilbert space \( \mathcal{D}[\tilde{S}_1] \). Hence
\[
\lim_{n \rightarrow \infty} \Gamma_0(-1) e_n = \Gamma_0 e \in \tilde{\mathcal{S}}_+. \quad \text{It follows that the sequence } \{ f_n \} \text{ converges in } \tilde{\mathcal{S}}_+ \text{ to some vector } f \in \text{dom } \tilde{S}_0 \text{ and, thus, } u = f + \Gamma_0(1) e \in \mathcal{L}, \quad e = \Gamma_0 u \text{, i.e.}, \quad \Gamma_0|_{\mathcal{L}} \text{ is closed in } \mathcal{S}_+.
\]
Define the \( \mathbf{L}(\mathcal{H}) \)-valued function \( W(z, \xi) \) by
\[
(7.11) \quad (W(z, \xi) h, e)_\mathcal{H} := \tilde{S}_1 [\Gamma_0(z) h, \Gamma_0(\xi) e], \quad h, e \in \mathcal{H}.
\]
Clearly, \( W(z, \xi) \) is holomorphic in \( z \), anti-holomorphic in \( \xi \), and, in addition, it is a positive definite kernel.

Let \( \Gamma_0^*(z) \in \mathbf{L}(\mathcal{H}, \mathcal{H}) \) be the adjoint of the operator \( \Gamma_0(z) \in \mathbf{L}(\mathcal{H}, H) \).
Lemma 7.10. The function
\[
z\Gamma^*_0(\xi)\Gamma_0(z) - W(z, \xi)
\]
does not depend on \(\xi\).

Proof. By definition one has
\[
z (\Gamma_0(z)h, \Gamma_0(\xi)e) - (W(z, \xi)h, e)_H = (S^*\Gamma_0(z)h, \Gamma_0(\xi)e) - \tilde{S}_1[\Gamma_0(z)h, \Gamma_0(\xi)e].
\]
Now by adding and subtracting the term \(\Gamma_0(-1)e\) in the right side of the previous formula and taking into account that \(\Gamma_0(\xi)e - \Gamma_0(-1)e \in \text{dom} \tilde{S}_0\), the assertion follows from Lemma 7.7.

7.2.3. Boundary triplets and Weyl functions.

Theorem 7.11. Let \(\{\mathcal{H}, \Gamma_0\}\) be a boundary pair for \(\{\tilde{S}_0, \tilde{S}_1\}\). Then there exists a unique linear operator \(\Gamma_1 : \mathcal{L} \to \mathcal{H}\) such that
\[
\tilde{S}_1[u, v] = (S^*u, v) - (\Gamma_1 u, \Gamma_0 v)_H \quad \text{for all } u \in \mathcal{L} \quad \text{and all } v \in \mathcal{D}[\tilde{S}_1].
\]
The operator \(\Gamma_1\) is bounded from the Hilbert space \(\mathcal{L}\), equipped with the inner product (7.8), to the Hilbert space \(\mathcal{H}\). Moreover,
\[
\ker \Gamma_1 = \text{dom} \tilde{S}_1, \quad \text{ran} \Gamma_1 = \mathcal{H}.
\]

Proof. Decompose \(v = \varphi + g\), where \(\varphi \in \mathcal{D}[\tilde{S}_0], g \in \mathcal{D}[\tilde{S}_1] \ominus \mathcal{D}[\tilde{S}_0]\). Then Lemma 7.7 implies that
\[
\tilde{S}_1[u, v] - (S^*u, v) = \tilde{S}_1[u, \varphi] + \tilde{S}_1[u, g] - (S^*u, \varphi) - (S^*u, g) = \tilde{S}_1[u, g] - (S^*u, g).
\]
By Lemma 7.5 the vector \(u \in \mathcal{L}\) can be represented in the form \(u = h + \psi\), where \(h \in \text{dom} \tilde{S}_1\) and \(\psi \in \mathcal{D}[\tilde{S}_1] \ominus \mathcal{D}[\tilde{S}_0]\). This yields the equality
\[
\tilde{S}_1[u, v] - (S^*u, v) = \tilde{S}_1[h + \psi, v] - (\tilde{S}_1 h - \psi, v) = \tilde{S}_1[h, v] + (\psi, g) = (\psi, g)_{\tilde{S}_1}.
\]
Therefore, for all \(v \in \mathcal{D}[\tilde{S}_1]\) one has
\[
|\tilde{S}_1[u, v] - (S^*u, v)| = |(\psi, g)_{\tilde{S}_1}| \leq \|\psi\|_{\tilde{S}_1} ||g||_{\tilde{S}_1} \leq C \|\psi\|_{\tilde{S}_1} ||\Gamma_0 v||_H,
\]
i.e. \(\tilde{S}_1[u, v] - (S^*u, v)\) is a continuous linear functional w.r.t. \(\Gamma_0 v\) on \(\mathcal{H}\). It follows that there exists a linear operator \(\Gamma_1 : \mathcal{L} \to \mathcal{H}\) such that \(\tilde{S}_1[u, v] - (S^*u, v) = -(\Gamma_1 u, \Gamma_0 v)_H\) for all \(u \in \mathcal{L}\) and all \(v \in \mathcal{D}[\tilde{S}_1]\).

Now with \(u, v \in \mathcal{L}\) one obtains (see (2.8), (7.8))
\[
|(\Gamma_1 u, \Gamma_0 v)_H| = |(S^*u, v) - \tilde{S}_1[u, v]| \leq \sqrt{\tilde{S}_1[u]} \tilde{S}_1[v] + ||S^*u|| ||v|| \leq 2 ||u||_H ||v||_{\tilde{S}_1}.
\]
This implies that
\[
||\Gamma_1 u||_H \leq \tilde{C} ||u||_H, \quad u \in \mathcal{L},
\]
i.e., \(\Gamma_1 : \mathcal{L} \to \mathcal{H}\) is bounded.

The equality \(\ker \Gamma_1 = \text{dom} \tilde{S}_1\) follows directly from (7.12). In view of (7.13) one has
\[
-(\Gamma_1 \psi, \Gamma_0 g)_H = (\psi, g)_{\tilde{S}_1}, \quad \text{for all } \psi, g \in \mathcal{D}[\tilde{S}_1] \ominus \tilde{S}_1 \mathcal{D}[\tilde{S}_0].
\]
Since \(\ker \Gamma_0 = \mathcal{D}[\tilde{S}_0]\) and \(\Gamma_0 \left(\mathcal{D}[\tilde{S}_1] \ominus \tilde{S}_1 \mathcal{D}[\tilde{S}_0]\right) = \mathcal{H}\), it follows that \(\text{ran} \Gamma_1 = \mathcal{H}\). To see that \(\Gamma_1\) is surjective assume the converse. Then by Lemma 7.5 there exists a normalized sequence
$\{g_n\} \subset \mathcal{D}[\tilde{S}_1] \ominus \tilde{S}_1 \mathcal{D}[\tilde{S}_0]$ with $\|g_n\|_\eta = 1$ such that $\Gamma_1 g_n \to 0$, as $n \to \infty$. Now boundedness of $\Gamma_0$ implies that

$- (\Gamma_1 g_n, \Gamma_0 g_n) = \|g_n\|_{\tilde{S}_1}^2 \to 0$.

However, here $g_n \in \mathcal{H}_{-1}$ and hence the norms $\|g_n\|_{\tilde{S}_1}$ and $\|g_n\|_\eta$ are equivalent (see (7.8)), so that $\|g_n\|_\eta \to 0$; a contradiction. Therefore, $\text{ran} \Gamma_1 = \mathcal{H}$. □

**Definition 7.12.** Let $\{\mathcal{H}, \Gamma_0\}$ be a boundary pair for $\{\tilde{S}_0, \tilde{S}_1\}$ and let $\Gamma_1 : \mathcal{L} \to \mathcal{H}$ be as in (7.12). Then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the pair $\{\tilde{S}_0, \tilde{S}_1\}$.

Observe that the Green’s identity

$$(S^* u, v) - (u, S^* v) = (\Gamma_1 u, \Gamma_0)_{\mathcal{H}} - (\Gamma_0 u, \Gamma_1 v)_{\mathcal{H}}, \quad u, v \in \mathcal{L},$$

is satisfied. Due to (7.12) the boundary triplet introduced in Definition 7.12 is a generalization of the notion of an ordinary positive boundary triplet (see Definitions 7.2 and 7.3). Moreover, since $\text{ran} \Gamma_0 = \mathcal{H}$ and $\tilde{S}_0 := S^* | \ker \Gamma_0$ is a selfadjoint extension of $S$, this is a generalized boundary triplet for $S^*$ in the sense of [17].

The main result in this section connects the boundary triplet in Definition 7.12 to the study of boundary relations in 19.

**Theorem 7.13.** Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the pair $\{\tilde{S}_0, \tilde{S}_1\}$ as in Definition 7.12. Then the operator $\tilde{A}$ defined by

$$(7.15) \quad \tilde{A} \left( \frac{u}{\Gamma_0 u} \right) = \left( \begin{array}{c} S^* u \\ -\Gamma_1 u \end{array} \right), \quad u \in \mathcal{L}.$$ 

is a nonnegative selfadjoint extension of $S$ acting in the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$. Moreover,

$$(7.16) \quad \mathcal{D}[\tilde{A}] = \left\{ \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right), \quad v \in \mathcal{D}[\tilde{S}_1] \right\}, \quad \tilde{A} \left[ \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) \right] = \tilde{S}_1[v],$$

and $\text{dom} \tilde{A}^{1/2} \cap \mathcal{H} = \{0\}$ holds. If, in addition, the pair $\{\tilde{S}_0, \tilde{S}_1\}$ satisfies the properties (3.16), then

$$(7.17) \quad \inf_{u \in \text{dom} \tilde{S}_0} \tilde{A} \left[ \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) - \left( \begin{array}{c} u \\ 0 \end{array} \right) \right] = 0 \quad \text{for all} \quad v \in \mathcal{D}[\tilde{S}_1]$$

and, moreover,

$$\text{ran} \tilde{A}^{1/2} \cap \mathcal{H} = \{0\}.$$

**Proof.** It follows from (7.15) and (7.12) that

$$(7.18) \quad \tilde{A} \left( \frac{u}{\Gamma_0 u} \right), \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) = (S^* u, v) - (\Gamma_1 u, \Gamma_0 v) = \tilde{S}_1[u, v] \geq 0, \quad u, v \in \mathcal{L}.$$ 

Therefore, the operator $\tilde{A}$ is nonnegative and clearly $S \subset \tilde{A}$. Observe that

$$(7.19) \quad \text{graph} \tilde{A} \cap (\tilde{\mathcal{H}} \oplus \{0\})^2 = \text{graph} S.$$

Next it will be proved that $\mathcal{R}(\tilde{A} + I_{\tilde{S}_1}) = \tilde{\mathcal{H}}$. Given the vectors $h \in \tilde{\mathcal{H}}$ and $\varphi \in \mathcal{H}$ it is shown that the system of equations

$$\begin{align*}
S^* u + u &= h \\
-\Gamma_1 u + \Gamma_0 u &= \varphi
\end{align*}$$
has a unique solution \( u \in \mathcal{L} \). According to (7.6) the vector \( u \in \mathcal{L} \) has the decomposition

\[ u = u_1 + f_{-1}, \]

where \( u_1 \in \text{dom} \tilde{S}_0 \) and \( f_{-1} \in \tilde{\mathcal{H}}_{-1} = \mathcal{D}[\tilde{S}_1] \cap \tilde{\mathcal{D}}[\tilde{S}_0] \). Then \( S^* u + u = \tilde{S}_0 u_1 + u_1 \).

Since \( \tilde{S}_0 \) is a nonnegative selfadjoint operator, one obtains \( u_1 = (\tilde{S}_0 + I_\delta)^{-1} h \). Then

\[ \varphi = -\Gamma_1 u + \Gamma_0 u = -\Gamma_1 u_1 - \Gamma_1 f_{-1} + \Gamma_0 f_{-1}, \]

i.e.,

\[ -\Gamma_1 f_{-1} + \Gamma_0 f_{-1} = \varphi + \Gamma_1 u_1. \]

It follows from (7.9) and (7.14) that for all \( g \in \tilde{\mathcal{H}}_{-1} \),

\[ (-\Gamma_1 g + \Gamma_0 g, \Gamma_0 g)_{\mathcal{H}} = (-\Gamma_1 \Gamma_0(-1) \Gamma_0 g + \Gamma_0 g, \Gamma_0 g)_{\mathcal{H}} \geq ||\Gamma_0 g||^2_{\mathcal{H}}, \]

and hence the operator \( -\Gamma_1 \Gamma_0(-1) + I_{\mathcal{H}} \) is bounded and positive definite on \( \mathcal{H} \). It follows that the equation \( -\Gamma_1 f_{-1} + \Gamma_0 f_{-1} = \varphi + \Gamma_1 u_1 \) has a unique solution \( f_{-1} \in \tilde{\mathcal{H}}_{-1} \). Thus, \( \mathcal{R}(\tilde{A} + I_{\tilde{\mathcal{H}}}) = \tilde{\mathcal{H}} \). This shows that the operator \( \tilde{A} \) is selfadjoint and nonnegative in \( \tilde{\mathcal{H}} \).

Since the form \( \tilde{S}_1[u, v] \) is closed in \( \tilde{\mathcal{H}} \), the form

\[ \tilde{\tau} \left[ \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right), \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) \right] := \tilde{S}_1[u, v], \ u, v \in \mathcal{D}[\tilde{S}_1], \]

is closed in \( \tilde{\mathcal{H}} \) and by (7.18) the selfadjoint operator \( \tilde{A} \) is associated with \( \tilde{\tau} \) according to the first representation theorem in [28]. This proves (7.16).

The form \( \tilde{S}_0[\cdot, \cdot] \) is a closed restriction of the form \( \tilde{S}_1[\cdot, \cdot] \) with \( \text{dom} \tilde{S}_0 \) being a core of \( \mathcal{D}[\tilde{S}_0] \). Therefore, under the conditions (3.16) for \( \{\tilde{S}_0, \tilde{S}_1\} \), the formula (7.17) is obtained from Theorem 3.3.

It follows from (7.16) that

\[ \left( \begin{array}{c} 0 \\ h \end{array} \right) \notin \text{dom} \tilde{A}^{1/2}, \ h \neq 0. \]

Next assume that

\[ \left( \begin{array}{c} 0 \\ h \end{array} \right) \in \text{ran} \tilde{A}^{1/2}. \]

Then

\[ \left| \left( \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right), \left( \begin{array}{c} 0 \\ h \end{array} \right) \right) \right|^2 \leq C \left( \left( \begin{array}{c} S^* u \\ -\Gamma_1 u \end{array} \right), \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right) \right) \]

for all \( u \in \mathcal{L} \) and some \( C > 0 \). Thus

\[ |(\Gamma_0 u, h)_{\mathcal{H}}|^2 \leq C \tilde{S}_1[u], \ u \in \mathcal{L}. \]

Replacing \( u \) by \( u - \varphi \), where \( \varphi \in \text{dom} \tilde{S}_0 \), and noting that \( \Gamma_0 \varphi = 0 \), one obtains

\[ |(\Gamma_0 u, h)_{\mathcal{H}}|^2 \leq C \tilde{S}_1[u - \varphi], \ u \in \mathcal{L}, \ \varphi \in \text{dom} \tilde{S}_0. \]

Furthermore, since

\[ \inf \left\{ \tilde{S}_1[u - \varphi], \ \varphi \in \mathcal{D}[\tilde{S}_0] \right\} = 0 \quad \text{for all} \quad u \in \mathcal{D}[\tilde{S}_1]; \]

and \( \text{dom} \tilde{S}_0 \) is a core of \( \mathcal{D}[\tilde{S}_0] \), one concludes that

\[ (\Gamma_0 u, h)_{\mathcal{H}} = 0 \quad \text{for all} \quad u \in \mathcal{L}. \]

Now the identity \( \Gamma_0 \mathcal{L} = \mathcal{H} \) implies that \( h = 0 \), i.e. \( \text{ran} \tilde{A}^{1/2} \cap \mathcal{H} = \{0\} \). The proof is complete. \( \square \)
Taking into account the definition of a boundary relation and results established in [19] we arrive at the following statement.

Remark 7.14. In the theory of boundary relations [19] the operator $A$ is called the main transform of the mapping $\Gamma := (\Gamma_0, \Gamma_1)$. The selfadjointness of $A$ together with (7.19) means that $\{H, \Gamma\}$ is a boundary relation for $S^*$. 

The next statement is a converse to Theorem 7.13.

Theorem 7.15. Let $S$ be a densely defined symmetric operator in $\mathcal{H}$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a generalized boundary triplet for $S^*$ (in the sense of [17]) with $\ker \Gamma_i = \tilde{S}_i$, $i = 1, 2$, and such that

1. the main transform $\tilde{A}$ in (7.15) is a nonnegative selfadjoint operator,
2. the closed form associated with $\tilde{A}$ is given by
   $$\mathcal{D}[\tilde{A}] = \left\{ \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) : v \in \mathcal{D}[\tilde{S}_1] \right\}, \quad \tilde{A} \left[ \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) \right] = \tilde{S}_1[v],$$
   where $\Gamma_0$ is a linear operator acting from $\mathcal{D}[\tilde{S}_1]$ into $\mathcal{H}$ and extends the mapping $\Gamma_0$, and where $\tilde{S}_1[u, v]$ stands for the closure of the form $(\tilde{S}_1 u, v)$, $u, v \in \text{dom} \tilde{S}_1$ (see (2.15)).

Then $\{\tilde{S}_0, \tilde{S}_1\}$ is a pair of nonnegative selfadjoint extensions of $S$ which satisfies the conditions (a), (b), (c).

Proof. It is first shown that $\tilde{S}_1$ is a selfadjoint operator. Since $\text{dom} \tilde{S}_1 = \ker \Gamma_1$ it is clear from (7.15) that $\tilde{S}_1$ is a nonnegative extension of $S$. Let $\tilde{S}_{1F}$ be the Friedrichs extension of $\tilde{S}_1$ and let $u \in \text{dom} \tilde{S}_{1F}$; cf. (2.15). Then by the first representation theorem [28] the equality

$$(\tilde{S}_{1F} u, v) = \tilde{S}_1[u, v]$$

is valid for all $v \in \mathcal{D}[\tilde{S}_1]$. Since $\tilde{S}_{1F} \supset \tilde{S}_1 \supset S$, we get $\tilde{S}_{1F} \subset S^*$ and $\text{dom} \tilde{S}_{1F} \subset \mathcal{D}[\tilde{S}_1]$. Thus,

$$(S^* u, v) = \tilde{S}_1[u, v] = \tilde{A} \left[ \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right), \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) \right]$$

for all $v \in \mathcal{D}[\tilde{S}_1]$. On the other hand

$$(S^* u, v) = \left( \left( \begin{array}{c} S^* u \\ 0 \end{array} \right), \left( \begin{array}{c} v \\ \Gamma_0 v \end{array} \right) \right)_{\tilde{S}_1}.$$ 

Making use the first representation theorem again, we get

$$u \in \text{dom} \tilde{S}_{1F} \Rightarrow \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right) \in \text{dom} \tilde{A}, \quad \tilde{A} \left( \begin{array}{c} u \\ \Gamma_0 u \end{array} \right) = \left( \begin{array}{c} S^* u \\ 0 \end{array} \right).$$

Now definition (7.15) of the main transform $\tilde{A}$ yields the equality $\Gamma_1 u = 0$. This means that $u \in \ker \Gamma_1 = \text{dom} \tilde{S}_1$. Therefore, $\tilde{S}_{1F} = \tilde{S}_1$ and thus $\tilde{S}_1$ is selfadjoint.

It is clear from (7.13) and (7.16) that the equality $(\tilde{S}_0 u, v) = \tilde{S}_1[u, v]$ holds for all $u, v \in \text{dom} \tilde{S}_0$. Consequently, $\tilde{S}_0$ is nonnegative and the closed form corresponding to $\tilde{S}_0$ is a restriction of the closed form $\tilde{S}_1[\cdot, \cdot]$. Therefore, the pair $\{\tilde{S}_0, \tilde{S}_1\}$ satisfies all the conditions in (a), (b), (c).
**Definition 7.16.** Let \( \{\mathcal{H}, \Gamma_0\} \) be a boundary pair for \( \{\tilde{S}_0, \tilde{S}_1\} \), let \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be the corresponding boundary triplet, and let \( \Gamma_0(z) \) be as in (7.9). The operator valued function

\[
M(z) := \Gamma_1 \Gamma_0(z), \quad z \in \text{Ext } [0, \infty),
\]

is called the Weyl function.

An application of (7.12) and (7.11) shows that

\[
(W(z, \xi) h, e)_{\mathcal{H}} = z (\Gamma_0(z) h, \Gamma_0(\xi) e) - (M(z) h, e)_{\mathcal{H}}, \quad h, e \in \mathcal{H}.
\]

Hence,

\[
(7.20) \quad -M(z) = W(z, \xi) = z \Gamma_0^*(\xi) \Gamma_0(z).
\]

Since \( W^*(z, \xi) = W(\xi, z) \), this implies that

\[
-M^*(\xi) = W(z, \xi) = \bar{\xi} \Gamma_0^*(\xi) \Gamma_0(z).
\]

Therefore,

\[
(7.21) \quad \frac{M(z) - M^*(\xi)}{z - \bar{\xi}} = \Gamma_0^*(\xi) \Gamma_0(z),
\]

and

\[
W(z, \xi) = \frac{\bar{\xi}M(z) - zM^*(\xi)}{z - \bar{\xi}}.
\]

Next another expression for \( M(z) \) is derived by means of the Cayley transforms \( \tilde{B}_k = (I - \tilde{S}_k)(I + \tilde{S}_k)^{-1}, \ k = 0, 1. \)

Since \( \text{ran} \Gamma_0(-1) = \mathcal{D}[\tilde{S}_1] \cap \mathcal{D}[\tilde{S}_0] \), Proposition 3.1 shows that

\[
\text{ran} \Gamma_0(-1) = \text{ran} (\tilde{B}_1 - \tilde{B}_0)^{1/2}.
\]

Therefore, there exists a continuous linear isomorphism \( X_0 \) from \( \mathcal{H} \) onto the subspace \( \mathfrak{N}_{-1} = \mathfrak{N} \) such that

\[
\Gamma_0(-1) = (\tilde{B}_1 - \tilde{B}_0)^{1/2} X_0.
\]

**Theorem 7.17.** The Weyl function \( M(z) \) of the boundary triplet \( \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) takes the form

\[
M(z) = -2X_0^* \left( I + (\tilde{B}_1 - \tilde{B}_0)^{1/2} \left( \tilde{B}_0 - \frac{1-z}{1+z} I \right)^{-1} (\tilde{B}_1 - \tilde{B}_0)^{1/2} \right) X_0,
\]

\[
= -2X_0^* \tilde{Q}_0 \left( \frac{1-z}{1+z} \right) X_0, \quad z \in \text{Ext } [0, \infty),
\]

where \( \tilde{Q}_0 \) is defined by (5.1). If, in particular, the pair \( \{\tilde{S}_0, \tilde{S}_1\} \) satisfies the properties (4.7), then \( M(z) \) belongs to the class \( \mathcal{G}_F(\mathcal{H}) \).

**Proof.** From (7.11) and (7.20) one obtains for all \( h \in \mathcal{H}, \)

\[
-(M(-1)h, h)_{\mathcal{H}} = (W(-1, -1)h, h)_{\mathcal{H}} + ||\Gamma_0(-1)h||^2 = ||\Gamma_0(-1)h||_{\mathfrak{S}_1}^2.
\]

Now the definition of \( X_0 \) and (3.11) imply

\[
-(M(-1)h, h)_{\mathcal{H}} = ||(\tilde{B}_1 - \tilde{B}_0)^{1/2} X_0 h||_{\mathfrak{S}_1}^2 = 2||X_0 h||^2,
\]

which leads to

\[
M(-1) = -2X_0^* X_0.
\]
According to (7.21) one has \( M(z) = M(-1) + (z + 1)\Gamma_0(-1)\Gamma_0(z) \) and using (7.10) and (6.3) one obtains
\[
M(z) = M(-1) + (z + 1)\Gamma_0^*(z+1)\Gamma_0(z)
\]
\[
= -2X_0^*X_0 - 2X_0^*(\hat{B}_1 - \hat{B}_0)^{1/2}(\hat{B}_0 - \frac{1-z}{1+z}I)^{-1}(\hat{B}_1 - \hat{B}_0)^{1/2}X_0
\]
\[
= -2X_0^*\tilde{Q}_0\left(\frac{1-z}{1+z}\right)X_0.
\]
Finally, since \( X_0 \) is a linear isomorphism (homeomorphism) it follows from Theorem 6.3 that the function \( M(z) \) together with the function \( \tilde{Q}_0(z) = -2\tilde{Q}_0\left(\frac{1-z}{1+z}\right) \) belongs to the class \( \mathcal{G}_\mathcal{P}(\mathcal{H}) \) of Herglotz-Nevanlinna functions.

By means of (5.4) and (5.3) it is seen that \( M^{-1}(z) \in \mathcal{L}(\mathcal{H}) \) for all \( z \in \text{Ext}[0, \infty) \) and
\[
M^{-1}(z) = -\frac{1}{2}X_0^{-1}\tilde{Q}_1\left(\frac{1-z}{1+z}\right)X_0^{-1},
\]
where \( \tilde{Q}_1 \) is defined in (5.2).

In conclusion we mention one more general relation for the Weyl function \( M(z) \). Let \( \tilde{A} \) be defined by (7.15). Then
\[
P_\mathcal{H}(\tilde{A} - zI)^{-1}\mid \mathcal{H} = -(M(z) + zI)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\]
Indeed, since
\[
(\tilde{A} - zI)\begin{pmatrix} u \\ \Gamma_0u \end{pmatrix} = \begin{pmatrix} S^*u - zu \\ -\Gamma_1u - z\Gamma_0u \end{pmatrix},
\]
the equality
\[
(\tilde{A} - zI)\begin{pmatrix} u \\ \Gamma_0u \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix}
\]
holds if \( u \in \mathfrak{N}_z \cap \text{dom} \tilde{A} = \mathfrak{M}_z \). Hence \( u = \Gamma_0(z)e \) for certain \( e \in \mathcal{H} \). Then
\[
-\Gamma_1u - z\Gamma_0u = -(M(z) + zI)e = h.
\]
Hence \( P_\mathcal{H}(\tilde{A} - zI)^{-1}h = e = -(M(z) + zI)^{-1}h \).

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Q-FUNCTIONS OF NONNEGATIVE OPERATORS

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