Soft charges from the geometry of field space

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Infinite sets of asymptotic soft-charges were recently shown to be related to new symmetries of the S-matrix, spurring a large amount of research on this and related questions. Notwithstanding, the raison-d’être of these soft-charges rests on less firm ground, insofar as their known derivations through generalized Noether procedures are not fully gauge invariant: these derivations rely on incomplete gauge fixations that leave only the new symmetries unfixed. An a priori reason for leaving out yet more general symmetries is missing, motivating further enlargements of the symmetries put forward in the literature. In this article, we propose that a geometrical framework anchored in the space of field configurations gives a rationale for the leading order soft charges in gauge theories. Our framework also explains why this infinite enhancement of the symmetry group is a property of asymptotic infinity and should not be expected to hold within finite regions, where at most a finite number of physical charges—corresponding to the reducibility parameters of the quasi-local field configuration—is singled out. Finally, our formalism suggests a new proposal for the origin of magnetic-type charges at asymptotic infinity.

The work of Strominger and collaborators on the “soft-triangle” ([1] and references therein) unveiled fascinating and unexpected relations among (i) Weinberg’s soft theorems and their generalizations and (ii) the so-called (nonlinear) memory effects [5–7], and (ii) an enlarged group of asymptotic symmetries for the S-matrix, in both gauge theories (and gravity). Strong evidence for these enlarged symmetries has been gathered by “reverse-engineering” their charges from the soft theorems. However, while descriptions in the asymptotic phase space exist (e.g. [1, 8–11]), a derivation of these charges from first principles is still missing. In particular, the relationship between what is “gauge” and what is “symmetry” is most often clouded by the employment of gauge fixings that happen to leave only the sought after symmetry parameters unfixed. Moreover, to interpret the soft modes as Goldstone modes [1], the broken symmetries must be global and not gauge in nature. Hence the questions: can the distinction between gauge and global symmetries be made a priori? Does this distinction lead to the correct charges?\textsuperscript{1} Early on, the suspicion emerged that these questions are related to the interplay of gauge symmetry with the (infinitely far) boundaries\textsuperscript{2} of the spacelike Cauchy surfaces and of \( \mathcal{I}^\pm \). This intuition suggests in turn that new symmetries and charges should emerge even at finite (fiducial) boundaries ([14] and e.g. [15, Sect.1]). Here, on the other hand, we employ a framework [16–18] that (i) is manifestly gauge invariant at boundaries, (ii) provides a criterion to distinguish gauge from symmetry, (iii) can be applied to infinite Cauchy surfaces as well as to finite bounded regions, (iv) associates nonvanishing charges to symmetries but not to gauge, (v) engenders an infinite dimensional group of symmetries only at \( \mathcal{I} \), and (vi) associates to these symmetries charges that reproduce the leading (electric [10]) soft theorems. Items (iii), (v) and (vi) are entirely new results. As a byproduct we will also uncover a quasi-local analogue of the radiative degrees of freedom [19] as well as a new proposal for the origin of magnetic-type charges at asymptotic boundaries. The techniques employed are geometric in nature, although the investigated geometry is that of field space. We will focus on Yang-Mills (YM) theories in 4 spacetime dimensions. In appendix A, we comment on generalizations in the presence of massive charged matter and to higher dimensions.

The strategy we follow proceeds in two steps. First, we exploit the fiducial fibre bundle structure and the kinematic supermetric on the configuration space of the YM theory to construct geometrically a gauge-invariant quasi-local symplectic potential. Through this gauge-invariant symplectic potential, we can successfully parse physical symmetries from gauge transformations, and devise a gauge-invariant Noether procedure which assigns nonvanishing charges to the former only. On (portions of) the finite-time Cauchy surfaces, our gauge-invariant symplectic form carries only the quasi-local generalization of the radiative mode, and leaves aside the constrained Coloumbic components of the gauge field which are fully determined by the Gauss constraint. Only a finite number of gauge-invariant global symmetries and charges is nonvanishing. However, far into the future, close to future null infinity, our construction unveils a symmetry enhancement whose associated leading order charges coincide with the leading order (electric) soft charges. At asymptotic null infinity, two circumstances are fundamental: the induced metric on the hypersurface becomes (conformally) null and the gauge field is assumed to admit an expansion in inverse powers of the radial coordinate\textsuperscript{3}. The latter circumstance, related to

\textsuperscript{1}This assumption might interfere with the gauge-fixing: e.g. to fix radial gauge \( A_r = 0 \), and thus \( A^{(1)}_r = 0 \), a \( \ln(r) \)-gauge transformation is needed which undermines the expansion itself.
the peeling property, is a foundational component of the asymptotic-infinity formulation of isolated systems (see e.g. [20]). No gauge-fixing, or restriction on the gauge parameters, apart from their asymptotic expansion, is ever required. The origin of the new soft charges is not, as it is sometimes suggested (e.g. [15, Sect.1]), to be found in the generic breaking of gauge symmetry at the boundary of the Cauchy surface. On the contrary, ensuring manifest gauge-invariance both in the bulk and at the boundary is a cornerstone of the present approach.

I. CONFIGURATION SPACE GEOMETRY

Let \{\Sigma_t\}_t be a Cauchy foliation of a (3+1)-dimensional spacetime \(M \cong \Sigma_t \times \mathbb{R}\). For notational simplicity, we restrict our analysis to foliations of unit lapse \(N = 1\) (cf. appendix B). However, we keep the shift \(\beta_t\) nontrivial, since it will be relevant in the retarded-time coordinates needed for the asymptotic limit. Our focus is the field-space vector associated to an infinitesimal gauge transformation \(\xi \in \text{Lie}(G)\). With a mild abuse of notation,

\[
\xi^A = \delta \xi = \Delta \xi = \delta \xi + [A, \xi] \in T_A \Phi.
\]

Then, the defining properties of \(\varpi\) can be expressed as

\[
\begin{align*}
\varpi(\xi^A) &= \xi \\
\varpi(\delta \xi) &= \delta \xi + [\varpi, \xi]
\end{align*}
\]

where \(\varpi\) is the field-space Lie-derivative and \(\delta\) the (fiducial) field-space de Rham differential. To avoid confusion between field-space and spacetime quantities, we reserved double-struck symbols for field-space. In the second equation, the bracket \([\cdot, \cdot]\) is the Lie bracket in \(\text{Lie}(G)\). Hence, this equation states that \(\varpi\) transforms covariantly when transported along the gauge orbits in \(\Phi\) (if \(\xi\) is a field-independent gauge transformation, then \(\delta \xi \equiv 0\), see [16–18]). The first condition states that \(\varpi\) is essentially a projector on the tangent space of the gauge orbit, \(V_A = \text{Span}(\xi^A) \subset T_A \Phi\), and therefore its kernel defines the physical, or “horizontal”, directions \(H_A\). Given a generic variation \(\delta A\), its horizontal component is

\[
\delta_H A = \delta A - \varpi(\delta A) = \delta A - D\varpi(\delta A) \in H_A.
\]

Contrary to \(\delta A\), \(\delta_H A\) transforms covariantly even under field-dependent gauge transformations \(\delta g \neq 0\), i.e. \(\delta_H (A^g) = g^{-1}(\delta H A) g\). Notice that although a (local) gauge fixing \(\sigma : \Phi / G \hookrightarrow \Phi\) defines a unique \(\varpi\) such that \(\text{Im}T\sigma \subset H\), the converse is not true: first, because the horizontal distribution \(H\) might not be Frobenious-integrable (if \(\varpi = \delta \varpi + \varpi^\perp \neq 0\)), and second, because \(\varpi\) is defined along the entire gauge orbit and thus cannot select any one section of \(\Phi\). At this point it is important to notice that \(\varpi\) is not uniquely defined, since the algebraic split \(T\Phi = V \perp H\) is not canonical. This leads us to the next ingredient.

b. The Singer-DeWitt connection If \(\Phi\) is equipped with a supermetric which is constant in the gauge directions \(L_{\xi_t} G = 0\), the orthogonal decomposition \(T\Phi = V \perp H\) defines a functional connection \(\varpi \perp [17, 18, 22–27]\), through the condition that \(\delta_H A\) (5) is orthogonal to all vertical vectors \(\xi^A\) (3),

\[
0 \equiv G(\delta \xi, \delta A - D\varpi(\delta A)) \quad \forall \xi.
\]

This readily leads to the elliptic boundary value problem

\[
\begin{align*}
D^2 \varpi &= D^v \xi \quad \text{in } R \\
D_s \varpi &= \delta A \quad \text{at } \partial R
\end{align*}
\]

4 More precisely: \(\xi^A = \int_R \text{dvol} (x) (\Delta^2)^{1/2}(x) \frac{\delta A}{\sqrt{\text{dvol}(x)}} \in T_A \Phi\).
where the \( s \) subindex stands for the contraction with the spacelike normal \( s^i \) and \( D_i \) is the gauge covariant generalization of the Levi-Civita (LC) derivative on \( \Sigma \). We will refer to the above connection as the Singer-DeWitt (SdW) connection. The boundary condition crucially follows from the fact that gauge transformations have not been trivialized at \( \partial R \). In electromagnetism on a flat hypersurface, the above is a Poisson equation with Neumann boundary conditions. Its solutions are unique up to a constant. This remark is the seed of some fundamental considerations that we postpone to the next section on symmetry and charges.

We conclude by stressing the fact that (7) imposes boundary conditions on \( \varpi \), and not on \( \delta A \), which is free to take any value at \( \partial R \) or elsewhere. In particular, no boundary condition is imposed on the gauge field.

II. QUASI-LOCAL DEGREES OF FREEDOM

a. SdW-horizontal symplectic potential The physical relevance of the SdW connection descends from the fact that the supermetric \( \mathcal{G} \) features in the kinetic term of the configuration-space Lagrangian \( L = T - U \), i.e.

\[
T = \frac{1}{2} \mathcal{G}(E, E) = \frac{1}{2} \mathcal{G}(\partial_t A, \partial_t A) + \ldots,
\]

where we introduced the electric field

\[
E_i = F_{ni}[A] = \dot{A}_i - D_i A_t
\]

and the symbol

\[
\dot{A}_i = \partial_t A_i - \beta^k F_{ki}
\]

(recall that \( \beta_i \) is the shift of the foliation \( \{ \Sigma_t \}_t \)). In this “3+1” decomposition, \( A_t \) is a scalar on \( \Sigma_t \). Inspection of the YM action shows that \( A_t \) is a Lagrange multiplier for the Gauss constraint,

\[
C_G = D^i E_i = D^i \dot{A}_i - D^2 A_t \approx 0.
\]

Also, from these equations, it follows that the symplectic potential of a YM theory, when written in configuration space variables (i.e. in terms of the velocity fields, instead of the momenta) and restricted to \( R \), reads

\[
\theta = \int_R g^{ij} E_i \delta A_j = \mathcal{G}(E, \delta A).
\]

Now, the definition of the horizontal variations \( \delta H A \) allows us to introduce the horizontal (pre)symplectic potential \( \theta_H \) by simply replacing \( \delta A \) with its horizontal counterpart:

\[
\theta_H(A, \delta A) = \mathcal{G}(E, \delta H A).
\]

Notice that in these formulas, we are interpreting the electric field \( E \) as a vector field in \( \Phi \). This is possible because – forgetting \( D_i A_t \) for now – \( E_i \) is essentially equal to \( \partial_t A_i \) (albeit corrected for a nontrivial shift of \( \{ \Sigma_t \}_t \)), and \( \partial_t A_i \) can be interpreted as the tangent “velocity vector” to the history of \( A_i \) in configuration space. With this interpretation, \( \Pi = \mathcal{G}(E, \cdot) \in T^* \Phi \) is the momentum of \( A_t \). Decomposing \( E \), understood as a vector, into its horizontal and vertical components, we find

\[
E_i = \dot{A}_i - D_i \varpi(A) - D_i \varphi = \dot{A}_i^H - D_i \varphi,
\]

for some \( \varphi \in C^\infty(\Sigma, \text{Lie}(G)) \) that transforms gauge-covariantly: \( \delta_G \varphi = [\varphi, \xi] \). This decomposition corresponds to writing \( A_t \) as

\[
A_t = \varphi + \varpi(\dot{A}),
\]

which is particularly convenient, since \( \varpi(\dot{A}) \), under a gauge transformation \( \xi \), can be checked to transform as \( \delta_G \varpi(\dot{A}) = \mathcal{L}_\xi \varpi(\dot{A}) = [\varpi(\dot{A}), \xi] + \partial_t \xi \). This uses (4) and the fact that \( \dot{A} \) is a vector on \( \Phi \) and thus \( \mathcal{L}_\xi \dot{A} = [\xi^t, \dot{A}] \cdot T \Phi \) is a Lie bracket between vectors. The formula holds also for field-dependent gauge transformations, provided \( \partial_t \) is interpreted as a total derivative \( d/dt = \partial_t + \int (\partial_t \xi) \mathcal{L}_\xi \) (for details see [19, Appendix A]). Therefore, (14) automatically takes care of the nontrivial transformation properties that the Lagrange multipliers must satisfy to have full consistency under time-dependent gauge transformations, the inhomogeneous piece being taken care of by \( \varpi(A) \), which depends only on the configuration variables.

These considerations hold for any choice of \( \varpi \). But, for the SdW connection, \( \varpi = \varpi_\perp \) and \( \varphi = \varphi_\perp \), there is more: this is the only functional connection such that (i) the Gauss constraint \( C_G \) depends only on \( \varphi = \varphi_\perp \), and (ii) \( \varphi = \varphi_\perp \) drops from the horizontal symplectic potential:

\[
C_G = D^i E_i = -D^2 \varphi_\perp \approx 0
\]

\[
\theta_\perp = \mathcal{G}(\dot{A}_\perp, \delta_\perp A) = \int_R g^{ij} \dot{A}_\perp \delta_\perp A j
\]

the two facts being closely related: they mirror the two steps of symplectic reduction, where one first solves the constraint and then removes the “conjugate” degree of freedom. The first equation is easy to verify by recalling that SdW-horizontal vectors \( \delta A = \delta_\perp A \) are by definition in the kernel of \( \varpi_\perp \) and therefore annihilate the rhs of (7); while the second equation follows from (12) and the definition of the SdW connection \( \varpi_\perp \) as the one for which horizontality is given by \( \mathcal{G} \)-orthogonality to the fibres \( \pi^{-1}(\{ A \}) \subset \Phi \).

b. Quasi-local degrees of freedom To recapitulate, we used the kinetic supermetric \( \mathcal{G} \) to define a preferred functional connection \( \varpi = \varpi_\perp \), named after Singer and DeWitt (SdW), according to the slogan “horizontality from fibre orthogonality”. The preferred status of \( \varpi_\perp \) results from the fact that the SdW connection is the only functional connection for which the non-dynamical component of \( E \), i.e. the one fixed by the Gauss constraint, drops from \( \theta_H = \theta_\perp \). In other words, \( \varpi_\perp \) is the only choice of functional connection such that the corresponding gauge-invariant (horizontal) symplectic potential \( \theta_H = \theta_\perp \) contains only the quasi-local analogue of the
radiative degrees of freedom, with no Coulombic contribution left. This is possible because of the role the kinetic
supermetric \( G \) plays in the (usual) symplectic form \( \theta \) (when expressed in configuration space).

Notice that the quasi-local radiative degrees of freedom, i.e. the SdW-horizontal perturbations \( \delta_\bot A \) are non-
locally built from a generic perturbation \( \delta A \), since this requires solving (7). Thanks to the presence of boundary
conditions in (7), the nonlocality is limited to the region \( R \) and no further information from the rest of \( \Sigma \) is re-
quired. Finally, none of these considerations is affected by the presence of matter.

E.g., for \( G = SU(N) \), a scalar \( \psi \in C^\infty(\Sigma, V) \), with \( V \cong \mathbb{R}^N \), the fundamental representation of \( G \), can be
incorporated as follows:

\[
C_G = D^i E_i - 4\pi \rho \approx 0
\]

\[
\theta_\bot = \int_R \left( g^{ij} \dot{A}_i \partial_\bot A_j + 2\pi (\bar{\psi}^\bot \partial_\bot \psi + \partial_\bot \bar{\psi}^\bot) \right)
\]

where the charge density \( \rho \) is defined by \( \text{Tr}(\rho \xi) = \frac{1}{4}(\bar{\psi}\xi\psi - \bar{\psi}\psi) \) for all \( \xi \in \text{Lie}(G) \), and where the
matter horizontal vectors read \( \dot{\delta}_\bot \psi = \delta \psi + \partial_\bot \psi \). Crucially, \( \partial_\bot \psi \) is still defined purely in terms of \( \delta A \) according to
(7). In electromagnetism, \( \delta_\bot \psi \) and \( \delta_\bot A_i \), can be interpreted in terms of quasi-local generalization of the vari-
ations of a Dirac-dressed scalar field and of transverse photons, respectively. However, these interpretations are not
fully satisfactory. They carry the same benefits and drawbacks of an interpretation of \( \partial_\bot \psi \) as a gauge-fixing: al-
though it might provide an intuition, it captures only a limited aspect of the formalism and therefore is, in the
end, inadequate [18].

### III. SYMMETRIES AND CHARGES

As remarked above, in flat space electromagnetism, the defining equation of the SdW connection (7) admits
unique solutions only up to constant offsets. This is an important observation, since \( \chi = cte \) corresponds to ele-
ments in \( \text{Lie}(G) \) such that \( \delta_\chi A = 0 \). This generalizes to non-Abelian YM theories as follows: at \( A \in \Phi \), the
SdW connection is defined up to reducibility parameters of \( A \), i.e. up to a \( \chi \in \text{Lie}(G) \) such that \( \delta_\chi A = 0 \). Re-
ducibility parameters are the analogue of Killing vector fields in general relativity, and exist only at peculiar, re-
ducible, configurations in \( \Phi \). The geometrical reason for a kernel in the above equations is that \( \Phi \) fails to be a
bena-fide principal fibre bundle, because certain fibres (i.e. gauge orbits) degenerate, making \( \Phi/G \) a stratified
manifold. The lower the stratum of \( \Phi/G \) to which \( [A] \) belongs, the more symmetric \( [A] \) is [18].

On a compact and boundary-less hypersurface \( \Sigma \), the Hamiltonian generator of the gauge transformation \( \xi \) is
\( Q[\xi] = \theta(\xi^2) = \theta(A, \psi; \delta_\chi A, \delta_\chi \psi) \) and vanishes on-shell of the Gauss constraint, \( Q[\xi] \approx 0 \). This means that
gauge transformations have trivial charges. In presence of boundaries, however, \( Q[\xi] \equiv q[\xi] \) for \( q[\xi] = \int_{\partial R} \text{Tr}(\xi E_r) \neq 0 \). This poses a puzzle: do gauge transforma-
tion at the boundary of a fiducial region \( R \subset \Sigma \) suddenly turn into physical symmetries carrying nontrivial charges? This sounds implausible, unless (possibly gauge breaking) boundary conditions are imposed at \( \partial R \). This puzzle is resolved if one makes use of the “horizontal symplectic geometry” [16–18]. Indeed, the contraction of a vertical vector \( \xi^\bot \) with the horizontal form \( \theta_\bot \) identically vanishes,

\[
Q_\bot[\xi] = \theta_\bot(\xi^\bot) \equiv 0,
\]

unless \( \xi = \chi \) is a reducibility parameter of \( A \), i.e. \( D\chi = 0 \), in which case the generator coincides with the physical charge
density of the matter distribution [17, 18]:

\[
Q_\bot[\chi] = \theta_\bot(\chi^\bot) = -\int_R 4\pi \text{Tr}(\chi \rho) \approx Q[\chi].
\]

In contrast to the other equations in this section, (20) depends on the use of the SdW-connection since, in this
case, \( \partial_\bot(\chi^\bot) = 0 \). This means that a reducible vertical transformation \( \chi^\bot \) is also horizontal, i.e. “physical”,
with respect to the SdW connection [16–18]. Therefore, we see that another property of the SdW connection is to
select the physically meaningful charges associated to actual global symmetries of the gauge field configura-
tion. Notice that, in YM theory, if a continuous family of reducibility parameters \( \{\chi(t)\}_t \) exists along the on-shell motion \( \gamma = \{A_i(t)\}_t \in \Phi \), then \( \chi(t) \) is a reducibility par-
ter of the (on-shell) spacetime connection \( A_i(t) \). To
these spacetime reducibility parameters, one associates proper dynamically conserved charges, since then the
spacetime divergence \( D_{\mu}(J^\rho) \) vanishes on-shell (see also [20, 28]). Therefore, our configuration space reducibility
parameters are natural candidates for dynamically conserved charges (and in the instantaneous configurations
space \( \Phi \) this is the best one can do). In the next section, we apply this construction to fields on \( \Sigma_{t=+\infty} = \mathcal{I}^+ \).
This construction will feature fundamentally new proper-
ties with respect to the finite-time hypersurface case.

### IV. ASYMPTOTIC INFINITY

Consider Minkowski space in the retarded coordinates \((t, u, y^4)\), with lapse \( N = 1 \) and shift \( \beta_\mu dx^\mu = -du \),

\[
ds^2 = -2dtdu + du^2 + (t - u)^2 \gamma_{AB} dy^A dy^B,
\]

and consider a family of regions \( R_t \subset \Sigma_t \) defined by \( u \in [u_t, u] \). In the late time limit, \( t \to \infty \): \( \Sigma_t \to \mathcal{I}^+ \),
\( \gamma_{AB} \) becomes the (conformal) metric on the celestial sphere \( S \simeq S_2 \), and the gauge field is assumed to ap-
proach the vacuum configuration \( A^{(0)} \) at a certain rate. In particular, the assumption is made that in a neigh-


bourhood of $\mathcal{S}^+$ the field admits an expansion\(^5\) in powers of $t$ (which at leading order, and at fixed $u$, coincides with Penrose’s conformal factor):

$$A_i(x,t) = A_i^{(0)} + \frac{1}{2} A_i^{(1)} + \frac{1}{12} A_i^{(2)} + \ldots$$

(22)

Hence, at late time, configuration space $\Phi$ has an extra structure that gauge transformations must also respect. Therefore, they also come layered in inverse powers of $t$:

$$\xi = \xi^{(0)} + \frac{1}{2} \xi^{(1)} + \frac{1}{12} \xi^{(2)} + \ldots$$

(23)

Notice that we are not gauge-fixing the vacuum $A_0^{(0)}$, nor any of the subleading orders of $A$ (at least to the extent to which they admit the above expansion, see footnote 3). Once again, we will leave it to the configuration space geometry to distinguish gauge from physical symmetries. Observe that, in the asymptotic limit, (7) becomes

$$\begin{cases}
(D_u^{(0)})^2 \varpi^{(0)} = D_u^{(0)} \delta A_i^{(0)} + O(\frac{1}{t}) & \text{in } R_t \to \infty \\
(D_u^{(0)})^2 \varpi^{(0)} = \delta A_i^{(0)} + O(\frac{1}{t}) & \text{at } \partial R_t \to \infty
\end{cases}$$

(24)

Thus, using the fact that physical symmetries $\chi^2$ are given by transformations which are simultaneously vertical and horizontal, i.e. in the kernel of (24), we immediately find that – thanks to the degenerate nature of the limit metric $g_{ij}(t)$ and to the layering of $A \in \Phi_{t \to \infty}$ and $\xi \in \text{Lie}(G)_{t \to \infty} –$ the leading order $\chi^{(0)}$ does not need to be a reducibility parameter of $A_i^{(0)}$, as was the case for finite-boundaries: it suffices that $\chi^{(0)}$ is covariantly constant in retarded time:

$$D_u^{(0)} \chi^{(0)} = 0.$$  

(25)

Since $A_i^{(0)}$ is pure gauge (this is not needed in an Abelian theory), this equation gives us a celestial-sphere-worth of asymptotic symmetries for all configurations in $\Phi_{t \to \infty}$, rather than just a finite number only at the reducible configurations. Physically, this is the consequence of points on the celestial sphere being infinitely separated from each other. To compute the asymptotic charge of the physical symmetries $\chi^{(0)}$ through the first equality in (20), we must first define $\theta_\perp$ on $\Phi_{t \to \infty}$, verify its finiteness and hence evaluate it at $\delta A = \chi^{(0)} t$. Observe that even for $t \to \infty$, $\theta_\perp$ can be written as

$$\theta_\perp = \mathcal{G}(E, \delta A) = \mathcal{G}(A, \delta A),$$

(26)

if $A_t$ is assumed to admit an expansion in $t^{-1}$. This follows from the general equality, $\mathcal{G}(E + D\phi, \delta A) = \mathcal{G}(E, \delta A)$ for all $\phi$ admitting an expansion in $t^{-1}$, due to the defining relation of $\varpi^{(0)}$ to $\mathcal{G}$. To proceed, we expand $A_t = \partial_t A_t + F_{ui}$, obtaining

$$\begin{cases}
A_u = \frac{1}{2} \sigma + O(\frac{1}{t}) \\
A_A = F_{uA}^{(0)} + O(\frac{1}{t})
\end{cases}$$

(27)

where $\sigma = -A_u^{(1)} = F_{uA}^{(2)}$ is the charge aspect, i.e. the asymptotic electric field generated by a charge in the spacetime’s bulk. Notice that $F_{uA}^{(0)} = \partial_u A_A - \mathcal{D}_A A_u^{(0)}$, where $\mathcal{D}_A$ is the gauge-covariant LC derivative of $\gamma_{AB}$ on the celestial sphere $S$, and $A_u^{(0)}$ is a scalar on $S$. Hence, combining this and (26), and using $A_u^{(0)} = 0 = A^{(1)}_u$, we get

$$\theta_\perp = \int_{\mathcal{S}^+} \text{Tr} \left( \sigma \delta_\perp A_u^{(0)} + \gamma^{AB} F_{uA}^{(0)} \theta_\perp A_B^{(0)} \right) + O(\frac{1}{t}),$$

(28)

where we abbreviated $\int_{\mathcal{S}^+} = \int_{u_i} u_{(\Sigma \to \infty)} \sqrt{-\gamma} d^{D-1}y$. Finally, defining the charge as in the first equality of (20) and using (25), we obtain

$$Q_\perp (\chi^{(0)}) = \theta_\perp (\chi^{(0)} t) = - \int_{\mathcal{S}^+} \text{Tr} \left( \chi^{(0)} \mathcal{D} F_{uA}^{(0)} \right) + O(\frac{1}{t}),$$

(29)

These charges are readily recognized as matching Strominger’s “new asymptotic charges”, or “soft charges” [1, 29]. The term “soft” is due to the fact that, since $\chi^{(0)}$ is constant in time (25), the charge only involves the zero-mode (in retarded time) of the momentum conjugate to the gauge-invariant photon field $\delta_\perp A_A^{(0)}$.

$$N_A = \int d u F_{uA}^{(0)}.$$  

(30)

Using the Gauss law and performing the integration in $d u$, the soft charge expression can also be written as

$$Q_\perp (\chi^{(0)}) \approx \int \text{Tr} \left( \chi^{(0)} (\sigma (u_i) - \sigma (u)) \right)$$

(31)

Therefore, both the asymptotic symplectic potential (28) and the soft charges involve, from a spacetime perspective, Coulombic information for the electric field. This might be puzzling, since we stated that $\theta_\perp$ does not contain this information on any $\Sigma_t$. The explanation resides in the term $\beta^k F_{kA}$ correcting the naive expression of $E_i$ for the presence of a nonvanishing shift $\beta_i$: although at finite $t$ this term simply adds some magnetic field to the canonical momentum, asymptotically it gets boosted so much that it captures the retarded-time dynamics of $A_A^{(0)}$ as well as – via the Gauss law – the bulk’s Coulombic information. Another surprising feature of this framework lies in the fact that at finite times nontrivial charges require a nonvanishing charge density $\rho$ (20), while on $\mathcal{S}$ no matter charge density is required (29).

V. TOWARDS MAGNETIC SOFT CHARGES

In [10], the relevance of magnetic, rather than electric, soft charges was identified. The relation of these charges to gauge transformations raises some puzzles,
since electro-magnetic duality is explicitly broken -- even in vacuum -- by the introduction of an electromagnetic gauge potential $A_\mu$, subject to gauge transformations. This is of course related to the fact that since the magnetic charge density identically vanishes, one turns the would-be source equation for the magnetic field into an algebraic identity (Bianchi). In the present construction, this state of affairs is highlighted by the fact that only the (total) electric charge arises as a horizontal charge on a finite-time hypersurface $\Sigma_t$. Therefore, the proposed field-space analysis is consistent on finite-time hypersurfaces. Again, the question arises, whether it is possible for it to recover magnetic charges asymptotically.

Here, although we will not attempt to perform a complete analysis that could fully recover the results of [9, 10] (but see also [30] for a challenging example), we will still put forward the following observation: asymptotically magnetic charges can arise from considering foliations $\Sigma_t \to \mathcal{F}$ more general than the one considered in the previous section.

E.g. consider a rotating foliation in the axial direction, $\phi \rightarrow \phi + \tau^{-1}$, that comes at leading order in $\tau^{-1}$ with a shift $\beta_\phi dx^\phi = -d\nu - \tau d\phi + O(\tau^{-2})$ (recall that $\tau^{-1}$ is our conformal parameter). This shift corresponds to a frame with a finite rotational velocity at infinity. This more general shift introduces in the expression of the electric field-space analysis is consistent on finite-time hypersurfaces.

In $D + 1 = 4$, this term leads to a new contribution to the soft charge equal to

$$
\bar{Q}[\chi^{(0)}] = \int \mathcal{B} \Tr \left( B\tau \epsilon_{\phi A} \mathcal{D}_A \chi^{(0)} \right)
$$

where $\mathcal{B} = \frac{1}{2} \int d\nu e^{AB} F^{(0)}_{AB}$ is the (retarded-time) zero-mode of the leading order magnetic flux across the celestial sphere.

This type of soft charge shares properties with the magnetic charges entering Low’s subleading soft-theorem [9, 10, 31]. Once again, we did not prove that the present framework is capable of naturally recovering the subleading charges: this would require considerably more work and likely some new ideas. However, we did show that magnetic charges do not have to arise necessarily from an electromagnetic duality, but in certain circumstance can be a consequence of a “covariantization” of the choice of foliation approaching $\mathcal{F}$.

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Appendix A: Generalizations

a. Massive matter Generalization to the presence of massive matter requires to resolve future timelike infinity by the introduction of a spacelike future hyperboloid. For this, one defines $\Sigma_\tau$ to be given by the union $\mathcal{H}_\tau^+ \cup \mathcal{F}^+$, where $\mathcal{H}_\tau^+ = \{ x \in \mathbb{R}^4 | t^2 - r^2 = r^2, t \geq r \}$ which meets $\mathcal{F}^+$ at $t = \infty$ [9]. Now, solutions of the horizontality conditions on $\Sigma_\tau$ are given by $\chi^{(0)} = \chi^{(0)}(x^A)$ on $\mathcal{F}$ (as above) and on $\mathcal{H}_\tau^+$ by the solution $\lambda(\chi^{(0)})$ of the boundary value problem

$$
\begin{align}
D_\mathcal{H}^2 \lambda &= 0 & \text{on } \mathcal{H}_\tau^+ \\
\lambda &= \chi^{(0)} & \text{at } \partial \mathcal{H}_\tau^+ = \mathcal{H}_\tau^+ \cap \mathcal{F}^+ 
\end{align}
$$

where $D_\mathcal{H}^2$ is the gauge-covariant Laplacian on $\mathcal{H}_\tau^+$. This follows from the fact that the only boundary of $\Sigma_\tau$ is the past corner of $\mathcal{F}^+$. The above condition reproduce what used in [9] to generalize the soft charges in the presence of charged massive particles. Their result made use of the Lorentz gauge.

b. Higher dimensions $D + 1 > 4$ In dimension $D + 1 > 4$, the $t \rightarrow \infty$ limit of the symplectic potential (28) gives a quantity that a priori diverges as $t^{D-3}$. In [32], it is shown how these divergences can be reabsorbed on-shell into the spacetime and field-space cohomological ambiguities of $\theta$. The paper deals with electromagnetism (Abelian YM), but work in progress by the authors shows that the same renormalization procedure is possible in general relativity. There, the conformal factor was chosen to be the more standard inverse radial coordinate $r^{-1}$, rather than Minkowski time, therefore some details of the presentation might undergo slight changes. Nonetheless, since the difference in the two choices can be reabsorbed into a retarded-time dependence of $\gamma_{AB}$ ($\gamma_{AB}$ there), the general renormalizability argument of [32] will not be compromised, nor we expect the final results to be much different. Then, thanks to the detailed analysis of [32], we expect the (renormalized) soft charges computed from the limiting renormalized horizontal symplectic potential and $\chi^{(0)}$ as in (25) and (29) to coincide with those deduced by Strominger and collaborators from the soft theorems [33]. All the quoted results hold in even spacetime dimensions larger than $D + 1 \geq 6$. Electromagnetism and gravity do not admit a conformal compactification in odd spacetime dimensions in presence of radiation (see [34] and [32] for a bulk and $\mathcal{F}$ perspective, respectively).
Appendix B: Lapse $N \neq 1$

Here we report the generalization of the relevant formulas for the case of lapse $N \neq 1$. The kinetic term of the Lagrangian, $L = T - U$, is $T = \frac{1}{2} G(NE, NE) = \frac{1}{2} \mathcal{G}(\partial_t A, \partial_t A) + \ldots$ for the following kinetic supermetric

$$G(\delta_t A, \delta_t A) = \int_R \text{dvol} \, N^{-1} g^{ij} \text{Tr}(\delta_t A_i \delta_t A_j) \quad (B1)$$

and electric field

$$E_i = F_{ni}[A] = \frac{1}{N} (\dot{A}_i - D_t A_i). \quad (B2)$$

As a consequence of the lapse appearing in the equation above, both the Gauss constraint and the SdW boundary value problem have to be slightly modified according to

$$NC_G = D_i \dot{A}_i - a_i \dot{A}_i - D^2 A_i + a_i^* D_i A_i \approx 0 \quad (B3)$$

and

$$\begin{cases} D^2 \varpi_\perp - a_i^* D_i \varpi_\perp = D^2 \delta A_i - a_i^* \delta A_i & \text{in } R \\ D_i \varpi_\perp = \delta A_i & \end{cases} \quad (B4)$$

where $a_i = D_i \ln N$ is the acceleration of Eulerian observers of $\{\Sigma_t\}_t$. Similarly, the (pre)symplectic potential, horizontal or not, are also modified:

$$\theta = \int_R g^{ij} \text{Tr}(E_i \delta \varpi_\perp) = G(NE, \delta A) \quad (B5)$$

$$\theta_H = G(NE, \delta H A) = G(N \dot{A}_\perp, \delta \varpi_\perp) \quad (B6)$$

and the momentum of $\delta A$ is $\Pi = G(NE, \cdot)$. 

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