Aeroelastic stability of a shell supported by a cylinder with a hole of linearly varying radius

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Abstract. The aeroelastic stability of a shell supported by an elastic cylinder of varying thickness with the outer surface exposed to supersonic gas flow is studied. The supporting cylinder is considered as an elastic foundation, whose bedding value is determined from the equations of the three-dimensional theory of elasticity. The solution is obtained by the Bubnov-Galerkin method in the form of a trigonometric series in the axial coordinate. The characteristic equation is approximated by the Lagrange polynomial, whose stability is investigated with the use of the Routh-Hurwitz criterion. The dependences of the critical flow velocity on the length of the cylinder section with the tapered hole are shown for various values of the axial force, the damping decrement, and the shell thickness.

1. Introduction
The problem of flutter and dynamic stability of composite shells supported by an elastic cylinder was studied in a number of papers [1–10]. However, the flutter of shells supported by a cylinder with a hole of varying radius remains unexplored.

2. Formulation of the problem
In this paper, we investigate the stability of a cylindrical shell made of composite material supported by an elastic cylinder. The cylinder has a coaxial hole consisting of a tapered and a cylindrical section. The shell is loaded with axial forces and external supersonic gas flow. The ends of the shell are simply supported. The problem geometry is shown in figure 1.

The motion of the shell is described by the equation of the orthotropic shell theory. The elastic cylinder is considered as a Winkler foundation, whose bedding value is determined from the solution of the equations of the three-dimensional elasticity theory. The inertial properties of the cylinder are taken into account by the use of the Rayleigh method [11].

We introduce a cylindrical coordinate system nondimensionalized by the radius of the shell.
Then, the equation of motion of the shell can be written as [12]

\[
\begin{align*}
\left\{ a_3 \nabla^8 + a_2 \frac{\partial^4}{\partial \alpha^4} + \nabla^4 \left( a_5 \frac{\partial^2}{\partial \alpha^2} + a_8 \frac{\partial}{\partial \alpha} + a_9 \frac{\partial}{\partial t} \right) + \\
+ \nabla^4 B \sum_{j=1}^{2} \left( K_j + a_{7j} \frac{\partial^2}{\partial \alpha^2} \right) [\sigma_0 (\alpha - \alpha_{j-1}) - \sigma_0 (\alpha - \alpha_j)] \right\} w = 0, \tag{1}
\end{align*}
\]

where \( \nabla^8, \nabla^4 \) are differential operators of the form

\[
\begin{align*}
\nabla^8 &= a_1 \frac{\partial^8}{\partial \alpha^8} + [a_4 + a_6 (2a_1 - \nu_\beta)] \frac{\partial^8}{\partial \alpha^6 \partial \beta^2} + \\
&+ \left\{ a_2 \left( \frac{\partial^2}{\partial \beta^2} + 1 \right)^2 + [2a_6 (a_4 - a_6 \nu_\beta) + a_2] \frac{\partial^4}{\partial \alpha^4} \right\} \frac{\partial^4}{\partial \alpha^4} + \\
&+ a_4 \left[ (a_4 - a_6 \nu_\beta) \left( \frac{\partial^2}{\partial \beta^2} + 1 \right)^2 + 2a_1 a_6 \frac{\partial^4}{\partial \beta^4} \right] \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + a_2 a_4 \left( \frac{\partial^2}{\partial \beta^2} + 1 \right)^2 \frac{\partial^4}{\partial \beta^4};
\end{align*}
\]

\[
\begin{align*}
\nabla^4 &= a_1 \frac{\partial^4}{\partial \alpha^4} + (a_4 - a_6 \nu_\beta) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + a_2 \frac{\partial^4}{\partial \beta^4};
\end{align*}
\]

\[
\begin{align*}
a_1 &= \frac{G_{\alpha \beta} (1 - \nu_\alpha \nu_\beta)}{E_\alpha}; \quad a_2 = a_1 a_4; \quad a_3 = \frac{h^2}{12 R^2}; \quad a_4 = \frac{E_\beta}{E_\alpha}; \quad a_5 = \frac{B T_\alpha}{2 \pi R^3}; \quad a_6 = 2a_1 + \nu_\beta; \\
a_7 &= h \left( \frac{\alpha}{\alpha_1} \mu + \rho \right); \quad a_{72} = h (\mu + \rho); \quad a_8 = \frac{B \sigma_0 p_0 M}{R}; \quad a_9 = B \left( \frac{\sigma_0 p_0}{C_0} + \varepsilon \right); \\
B &= \frac{R^2 (1 - \nu_\alpha \nu_\beta)}{E_\alpha h}; \quad K_1 = \frac{\alpha}{\alpha_1} K_0; \quad K_2 = K_0; \quad \mu = \frac{R^2 - R_0^2}{6 R h} \rho_\varepsilon; \\
C_0 &= \sqrt{\frac{\sigma_0 p_0}{\rho_0}}; \quad M = \frac{V}{C_0}; \quad \varepsilon = \frac{\delta \rho h}{\pi R} \sqrt{\frac{E_\alpha}{\rho}};
\end{align*}
\]
\(\alpha\) and \(\beta\) denote the axial and circumferential nondimensional coordinates; \(w\) the radial shell displacement; \(R\) and \(h\) the radius and the thickness of the shell; \(R_0\) the radius of the cylindrical part of the hole; \(E_0\) and \(E_{\beta}\) the axial and circumferential moduli of elasticity; \(G_{\alpha\beta}\) the transverse shear modulus; \(\nu_\alpha\) and \(\nu_\beta\) Poisson’s ratios; \(K_0\) the bedding value of the elastic cylinder section with the cylindrical hole; \(\rho\) and \(\rho_c\) the shell and the cylinder density; \(\delta\) the damping decrement; \(z_0, p_0\) and \(p_0\) the polytropic index, pressure, and speed of sound in the freestream; \(M\) the Mach number; \(V\) the air velocity; \(\alpha_1\) the length of the tapered part of the hole.

### 3. Solution of the equation of motion

A solution of equation (1) is assumed of the form

\[
w = \cos n\beta \sum_{m=1}^{\infty} A_m \sin \gamma_m \alpha e^{\omega t},
\]

where \(\gamma_m = m\pi/\alpha_0; \alpha_0 = L/R; L\) is the length of the shell; \(m\) the number of axial half-waves; \(n\) the number of circumferential full waves; \(\omega\) the complex frequency.

Substituting (2) into (1) and using Galerkin’s procedure, we obtain an infinite system of algebraic equations

\[
(a_{10} \omega^2 + a_9 \omega + \psi_k) A_k + \sum_{m=1}^{\infty} f_{mk} A_m = 0, \quad (k = 1, 2, 3, \ldots),
\]

where \(a_{10} = B\rho; \quad a_{11}^{(m)} = B (K_{0m} + h\mu \omega^2); \quad \psi_k = (a_3 \nabla_k^8 + a_2 \gamma_k^4) / \nabla_k^4 - a_5 \gamma_k^2; \quad f_{mk} = a_{11}^{(m)} \left[ \frac{1}{\alpha_0} H_1^{(mk)} + H_2^{(mk)} \right] + a_8 H_3^{(mk)},
\]

\[
H_1^{(mk)} = \begin{cases} \frac{a_1 \alpha - \alpha_0 \gamma_k}{\alpha_0 \mu}; & \text{when } m = k_0; \\ \frac{1}{\alpha_0 \mu} (\alpha_1 \gamma_m \sin \theta_{mk} \alpha_1 + \cos \theta_{mk} \alpha_1 - 1) - \frac{1}{\alpha_0 \mu} (\alpha_1 \gamma_m \sin \varphi_{mk} \alpha_1 + \cos \varphi_{mk} \alpha_1 - 1), & \text{when } m \neq k; \end{cases}
\]

\[
H_2^{(mk)} = \begin{cases} \frac{\alpha_0 - \alpha_1 \gamma_k}{\alpha_0 \mu}; & \text{when } m = k_0; \\ \frac{1}{\alpha_0 \mu} \left( \frac{1}{\alpha \mu} \sin \varphi_{mk} \alpha - \frac{1}{\theta_{mk}} \sin \theta_{mk} \alpha \alpha_1 \right), & \text{when } m \neq k; \end{cases}
\]

\[
H_3^{(mk)} = \begin{cases} \frac{4 \alpha_0}{\alpha_0 \mu^2 - \alpha_1 \gamma_k^2}, & \text{if } m \pm k_0 \text{ is odd;} \\ 0, & \text{if } m \pm k \text{ is even;} \end{cases}
\]

\[
\varphi_{mk} = \pi (m + k) / \alpha_0; \quad \theta_{mk} = \pi (m - k) / \alpha_0; \quad \nabla_k^4 = a_1 \gamma_k^4 + (a_4 - a_6 \nu_\beta) \gamma_k^2 n^2 + a_2 n^4;
\]

\[
\nabla_k^8 = a_1 \gamma_k^8 + [a_4 + a_6 (2 \alpha_1 - \nu_\beta)] \gamma_k^6 n^2 + \left\{ a_2 (1 - n^2)^2 + [2a_6 (a_4 - a_6 \nu_\beta) + a_2] n^4 \right\} \gamma_k^4 + a_4 \left( [a_4 - a_6 \nu_\beta] (1 - n^2)^2 + 2a_1 a_6 n^4 \right) \gamma_k^2 n^2 + a_2 a_4 (1 - n^2)^2 n^4.
\]

For the chosen model of the elastic foundation, the bedding value can be expressed as [12]

\[
K_{0m} = \frac{2 \mu_0 \Delta}{RQ}; \quad Q = \sum_{r=1}^{6} F_r D_{qr}; \quad F_1 = -\frac{n^2}{\gamma_m} I_n(\gamma_m); \quad F_3 = \frac{q_1}{2} \gamma_m \left( \frac{n^2}{\gamma_m^2} + 1 \right) I_n(\gamma_m); \quad F_5 = -I_n(\gamma_m),
\]
where $\Delta$ and $D_{qr}$ are the determinant and a cofactor, respectively, of the matrix $a_{ij}$:

$$a_{11} = \frac{n^2}{x_m} I_n(x_m); \quad a_{13} = -I'_n(x_m) - q_1 \left( \frac{n^2}{x_m^2} + 1 \right) x_m I_n(x_m); \quad a_{15} = 2I'_n(x_m);$$

$$a_{21} = \left( \frac{n^2}{x_m^2} + \frac{1}{2} \right) I_n(x_m) - \frac{1}{x_m} I'_n(x_m); \quad a_{23} = \frac{q_1}{2} \left[ \frac{1}{x_m} I'_n(x_m) - \left( \frac{n^2}{x_m^2} + 1 \right) I_n(x_m) \right];$$

$$a_{25} = \frac{1}{x_m} \left[ I'_n(x_m) - \frac{1}{x_m} I_n(x_m) \right]; \quad a_{31} = \frac{n^2}{x_m} \left[ \frac{1}{x_m} I_n(x_m) - I'_n(x_m) \right];$$

$$a_{33} = \frac{q_1}{2} \left[ \left( \frac{n^2}{x_m^2} + 1 \right) x_m I'_n(x_m) - \left( \frac{n^2}{x_m^2} - q_2 \right) I_n(x_m) \right]; \quad a_{35} = \frac{1}{x_m} I'_n(x_m) - \left( \frac{n^2}{x_m^2} + 1 \right) I_n(x_m);$$

$$a_{41} = \frac{n^2}{\gamma_m} I_n(\gamma_m); \quad a_{43} = -I'_n(\gamma_m) - q_1 \left( \frac{n^2}{\gamma_m^2} + 1 \right) \gamma_m I_n(\gamma_m); \quad a_{45} = 2I'_n(\gamma_m);$$

$$a_{51} = \left( \frac{n^2}{\gamma_m^2} + \frac{1}{2} \right) I_n(\gamma_m) - \frac{1}{\gamma_m} I'_n(\gamma_m); \quad a_{53} = \frac{q_1}{2} \left[ \frac{1}{\gamma_m} I'_n(\gamma_m) - \left( \frac{n^2}{\gamma_m^2} + 1 \right) I_n(\gamma_m) \right];$$

$$a_{55} = \frac{1}{\gamma_m} \left[ I'_n(\gamma_m) - \frac{1}{\gamma_m} I_n(\gamma_m) \right]; \quad a_{61} = \frac{n^2}{\gamma_m^2} \left[ \frac{1}{\gamma_m} I_n(\gamma_m) - I'_n(\gamma_m) \right];$$

$$a_{63} = \frac{q_1}{2} \left[ \left( \frac{n^2}{\gamma_m^2} + 1 \right) \gamma_m I'_n(\gamma_m) - \left( \frac{n^2}{\gamma_m^2} - q_2 \right) I_n(\gamma_m) \right]; \quad a_{65} = \frac{1}{\gamma_m} I'_n(\gamma_m) - \left( \frac{n^2}{\gamma_m^2} + 1 \right) I_n(\gamma_m);$$

$$q_1 = \frac{\lambda + \mu_0}{\lambda + 2\mu_0}; \quad q_2 = \frac{\mu_0}{\lambda + \mu_0}; \quad \mu_0 = \frac{E_0}{2(1 + \nu_0)}; \quad \lambda = \frac{E_0\nu_0}{(1 + \nu_0)(1 - 2\nu_0)}; \quad x_m = z_0\gamma_m; \quad z_0 = \frac{R_0}{R},$$

where $E_0$ and $\nu_0$ are the elastic modulus and Poisson’s ratio of the cylinder material; $I_n(x)$ is the modified Bessel function of the first kind of order $n$; here the prime denotes the derivative with respect to the respective argument. To obtain even columns of the matrix $a_{ij}$ and $F_i$ with even indices, $I_n$ in the previous odd elements should be replaced with the modified Bessel function of the second kind $K_n$ of the same argument.

4. Stability analysis

Reducing the system (3) and setting the determinant of its matrix equal to zero, we obtain the characteristic equation. However, it is quite difficult to investigate this equation directly. It is more convenient to analyze its stability using the method described in reference [13].

Using the Lagrange polynomial approximation, we express the left-hand side of the characteristic equation $P(\omega) = 0$ as a polynomial in $\omega$

$$P(\omega) = \sum_{j=0}^{2r} b_j \omega^{2r-j} = 0,$$

where $b_j$ are unknown constant coefficients.

Calculating $P(\omega_i)$ for a number of $\omega_i$, we find the values of $b_j$ from the system of equations

$$\sum_{j=0}^{2r} b_j \omega_i^{2r-j} = P(\omega_i), \quad (i = 0, 1, \ldots, 2r).$$
The unperturbed state of the shell is stable if the real parts of all the complex frequencies are negative \( \text{Re} \omega_i < 0 \). A necessary and sufficient condition for that is the positivity of the Hurwitz determinants of the polynomial (4) [14]

\[
D_j(M) = \begin{vmatrix}
    b_1 & b_0 & 0 & 0 & \cdots & 0 & 0 \\
    b_3 & b_2 & b_1 & b_0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & b_j & b_{j-1} & b_{j-2} \\
    0 & \cdots & 0 & 0 & b_j & b_j \\
\end{vmatrix} > 0, \quad (j = 0, 1, \ldots, 2r).
\]

As the Mach number \( M \) increases, this inequality will cease to hold when either \( b_{2r} \) or \( D_{2r-1}(M) \) becomes zero. The former case corresponds to divergence and the latter to flutter of the shell.

5. Numerical example

As an example, two shells of different thicknesses were considered. The parameters of the shells are as follows:

\[
L/R = 6; \quad E_\alpha/E_0 = 1.5 \times 10^4; \quad E_\beta/E_0 = 2.3 \times 10^4; \quad G_{\alpha\beta}/E_0 = 2.5 \times 10^4;
\nu_\alpha/E_0 = 0.15; \quad \nu_\beta/E_0 = 0.23; \quad \nu_0 = 0.49; \quad \alpha_0 = 1.4; \quad p_0/E_0 = 0.03; \quad (\rho; \rho_c)C_0^2/E = 109.
\]

In figure 2 the critical Mach number is plotted vs. the nondimensional length of the tapered part of the hole for various values of the axial force and the damping decrement. Here \( T_b \) is the axial buckling force. Figure 3 shows these dependences for the shell with a different thickness-to-radius ratio.

6. Conclusions

We have proposed a mathematical model of the behavior of an orthotropic cylindrical shell supported by an elastic isotropic cylinder of linearly varying thickness in a supersonic gas flow. A characteristic equation is obtained, which, with the use of the Routh-Hurwitz asymptotic stability criterion, allows us to determine the dependence of the critical flow velocity on a number of design parameters. The effect of the cylinder thickness gradient on the occurrence of flutter has been shown.
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References
[1] Solomonov Yu S et al 2012 Proc. of Moscow Inst. of Therm. Tech. 12(1) 5–13
[2] Bakulin V N, Volkov E N and Nedbai A Ya 2015 Flutter of a sandwich cylindrical shell supported with annular ribs and loaded with axial forces Doklady Phys. 60(8) 360–363
[3] Volkov E N, Danilkin E V and Nedbai A Ya 2015 Proc. of Moscow Inst. of Therm. Tech. 15(1) 54–64
[4] Bakulin V N, Volkov E N and Nedbai A Ya 2016 Dynamic stability of a cylindrical shell reinforced by longitudinal ribs and a hollow cylinder under the action of axial forces J. Engineering Phys. and Thermophysics 89(3) 747–753
[5] Bakulin V N, Danilkin E V and Nedbai A Ya 2018 Dynamic stability of a cylindrical shell stiffened with a cylinder and longitudinal diaphragms at external pressure J. Engineering Phys. and Thermophysics 91(2) 537–543
[6] Korbut B A and Nagornyi Yu I 1972 Dynamics and Strength of Machines vol 15 (Khar’kov: Khar’kov State Univ.) pp 70–77
[7] Bakulin V N, Volkov E N and Simonov A I 2017 Dynamic stability of a cylindrical shell under alternating axial external pressure Russian Aeronautics 60(4) 508–513
[8] Bakulin V N, Bokov M A and Nedbai A Ya 2018 Aeroelastic stability of a cylindrical composite shell at a bilateral flow Mech. Comp. Materials 53(6) 801–808
[9] Bakulin V N, Konopelchev M A and Nedbai A Ya 2018 Flutter of a laminated cantilever cylindrical shell with a ring-stiffened edge Russian Aeronautics 61(4) 517–523
[10] Bakulin V N, Nedbai A Ya and Shepeleva I O 2019 Dynamic stability of orthotropic cylindrical shell of piecewise constant thickness under the action of external pulsating pressure Izv. VUZ. Aviatsionnaya Tekhnika 2 19–25
[11] Timoshenko S 1937 Vibration problems in engineering (New York: D. Van Nostrand Company, Inc.)
[12] Solomonov Yu S et al 2014 Applied Problems of the Mechanics of Composite Cylindrical Shells (Moscow: Fizmatlit)
[13] Moskvin V G 1973 Theory of Shells and Plates: Proc. of the 8th All-Union Conf. (Rostov-on-Don, September 16–22, 1971) (Moscow: Nauka) pp 527–531
[14] Chetaev N G 1961 The Stability of Motion (New York: Pergamon Press)