Volume Distance to Hypersurfaces:  
Asymptotic Behavior of its Hessian

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Abstract. The volume distance from a point \( p \) to a convex hypersurface \( M \subset \mathbb{R}^{N+1} \) is defined as the minimum \((N + 1)\)-volume of a region bounded by \( M \) and a hyperplane \( H \) through the point. This function is differentiable in a neighborhood of \( M \) and if we restrict its hessian to the minimizing hyperplane \( H(p) \) we obtain, after normalization, a symmetric bi-linear form \( Q \).

In this paper, we prove that \( Q \) converges to the affine Blaschke metric when we approximate the hypersurface along a curve whose points are centroids of parallel sections. We also show that the rate of this convergence is given by a bilinear form associated with the shape operator of \( M \). These convergence results provide a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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1. Introduction

Consider a strictly convex hypersurface \( M \subset \mathbb{R}^{N+1} \), a point \( p \) in the convex side of \( M \) and \( n \in S^N \). Denote by \( U(n, p) \) the region bounded by \( M \) and a hyperplane \( H(n, p) \) orthogonal to \( n \) through \( p \), with \( n \) pointing outwards the region, and by \( V(n, p) \) its volume. The volume distance \( v(p) \) of \( p \) to \( M \) is defined as the minimum of \( V(n, p) \), \( n \in S^N \).

The volume distance is an important object in computer vision which has been extensively studied in the planar case \( n = 1 \) ([1]) and was also considered in the case \( n = 2 \) ([4]). For \( n = 1 \), the hessian of the volume distance was studied in ([2], [3]), where it is shown that its determinant equals \(-1\). This property is not extended to higher dimensions. Nevertheless, we

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prove in this paper some asymptotic properties of the hessian of the volume distance in arbitrary dimensions.

A pair \((n, p)\) is called \textit{minimizing} when \(n\) is the minimum of \(V(n, p)\) with \(p\) fixed. A minimizing pair necessarily satisfies
\[
\frac{\partial V}{\partial n}(n, p) = 0. \tag{1.1}
\]
It is proved in \cite{5} that if \((n, p)\) satisfies \(1.1\), then \(p\) is the centroid of \(R(n, p)\).

In order to obtain \(n = n(p)\) implicitly defined by \(1.1\), the second derivative of \(V\) with respect to \(n\) must be non-degenerate. A formula for this second derivative can also be found in \cite{5}. From this formula, one concludes that the second derivative is positive definite in a half-neighborhood of \(M\), i.e., the part of a neighborhood of \(M\) contained in its convex side. Based on this, we verify that there exists a half-neighborhood \(D\) of \(M\) such that, for any \(p \in D\), there exists a unique \(n(p)\) that minimizes the map \(n \to V(n(p), p)\). Moreover, the map \(p \to n(p)\) is smooth and consequently \(v(p) = V(n(p), p)\) is also smooth.

For \(p \in D\), let
\[
Q(p) = \frac{1}{b(p)} \frac{\partial^2 V}{\partial n^2}(n(p), p), \tag{1.2}
\]
where \(b(p)\) denotes the \(N\)-dimensional volume of the region \(R(p) \subset H(p)\) bounded by \(M\). By making some calculations, we show that, for \(p \in D\),
\[
-\frac{1}{b(p)} D^2 v(p) \big|_{H(p)} = Q^{-1}(p) \tag{1.3}
\]
where \(D^2 v(p) \big|_{H(p)}\) means the restriction of \(D^2 v(p)\) to \(H(p)\).

This paper is concerned with the asymptotic behavior of the quadratic form \(Q\). In order to motivate a bit more this study, we remark that this quadratic form is an important tool in the study of \textit{floating bodies}. When \(M\) is the boundary of a convex body \(K\), one can define its floating body \(K_\delta\), for \(\delta > 0\), by the property that each support hyperplane of \(K_\delta\) cuts \(K\) in a region of volume \(\delta\). For smooth strictly convex bodies and \(\delta\) sufficiently small, the convex bodies exist and its boundary is a smooth surface (see \cite{5}). In \cite{6}, the quadratic form \(Q\) was a key ingredient in proving that \(K_\delta\) is well defined for every \(0 < \delta \leq \frac{1}{2} \text{vol}(K)\) if and only if \(K\) is symmetric with respect to a point. Also in \cite{9}, \(Q\) appears as a tool in proving that a convex body with a sequence of homothetic floating bodies must be an ellipsoid.

For \(q \in M\), denote by \(T_q M = H(n(q), q)\) the tangent plane to \(M\) at \(q\) and, for \(t > 0\), define \(\gamma_q(t)\) as the centroid of the region \(R(n(q), q + t\xi(q))\), where \(\xi(q)\) is the affine normal to \(M\) at \(q\). We shall consider two symmetric bilinear forms defined on \(T_q M\): the Blaschke metric \(h\) which is positive definite and \(h_S\) defined as \(h_S(X, Y) = h(X, SY)\), where \(S\) is the shape operator. By identifying \(H(\gamma_q(t))\) with \(T_q M\), the normalized hessian \(Q(\gamma_q(t))\) can also be seen as a symmetric bilinear form in \(T_q M\). The main result of the paper says that
\[
Q(\gamma_q(t)) = h(q) + th_S(q) + O(t^2),
\]
where $O(t^k)$ indicates a quantity such that $\lim_{t \to 0} \frac{O(t^k)}{t} = 0$, for any $\epsilon > 0$. This result can be regarded as a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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2. Hessian of the volume distance

2.1. Notation

Consider a strictly convex hypersurface $M \subset \mathbb{R}^{N+1}$, possibly with a non-empty boundary $\partial M$. Denote by $H(n, p) \subset \mathbb{R}^{N+1}$ the hyperplane passing through $p \in \mathbb{R}^{N+1}$ with normal $n \in S^N$. For $p \in \mathbb{R}^{N+1}$, denote by $E(p) \subset S^N$ the set of unitary vectors $n$ whose corresponding hyperplane $H(n, p)$ intersects $M - \partial M$ transversally at a closed hypersurface $\Gamma(n, p) \subset H(n, p)$ bounding a region $R(n, p) \subset H(n, p)$ containing $p$ in its interior and such that the region $U(n, p)$ bounded by $R(n, p)$ and $M$, with $n$ pointing outwards, has finite volume $V(n, p)$ (see figure 1). Denote by $D_1 \subset \mathbb{R}^{N+1}$ the set of $p \in \mathbb{R}^{N+1}$ such that $E(p) \neq \emptyset$ and the infimum $\inf\{V(n, p) \mid n \in E(p)\}$ is attained at $E(p)$. When $n \in E(p)$ attains this minimum, we call the pair $(n, p)$ minimizing and $v(p) = V(n, p)$ the volume distance to $M$. We remark that if $M$ is a closed hypersurface enclosing a convex region, then the domain $D_1$ of the volume distance is all the enclosed region.

![Figure 1. The section $R(n, p)$ and the enclosed region $U(n, p)$.](image)

For $q \in M$, denote by $\xi(q)$ the affine normal vector pointing to the convex side of $M$. Along this paper, we shall call a half-neighborhood of $M$ any set of the form

$$\{q + t \xi(q) \mid q \in M, 0 \leq t < T(q)\},$$

where $T(q) > 0$ is some smooth function of $q$.

Close to a pair $(n_0, p_0)$, consider cartesian coordinates $(x, z) \in \mathbb{R}^N \times I$, $I = (-\epsilon, \epsilon)$ such that $p_0 = (0, 0)$ and $n_0 = (0, 1)$. To describe the hypersurface
M in a neighborhood of $H(n_0, p_0)$, consider cylindrical coordinates $(r, \eta, z)$, where $x = r\eta, \eta \in S^{N-1}, r > 0$. Then $M$ is described by $r = r(\eta, z)$, for some smooth function $r$ (see figure 2). We write
\[ r(\eta, z) = r(\eta, 0) + r_z(\eta, 0)z + O(z^2), \tag{2.1} \]
for $z$ close to 0.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The curve $r = r(\eta, z)$ with fixed $\eta \in S^N$.}
\end{figure}

\section{2.2. Smoothness of the volume distance $v$ in a half-neighborhood of $M$}

The derivative $\frac{\partial V}{\partial n}(n, p_0)$ can be regarded as a linear functional on $T_n S^N$, which can be identified with $H(n, p_0)$. The proof of next proposition can be found in [5], p. 166.

\textbf{Proposition 2.1.} Denote by $p(n, p)$ the center of gravity of $R(n, p)$ and by $b(n, p)$ the $N$-dimensional volume of the region $R(n, p)$. Then
\[ \frac{\partial V}{\partial n}(n, p) = -b(n, p) \left( p(n, p) - p \right). \tag{2.2} \]
Thus, a pair $(n, p)$ is critical if and only if $p(n, p) = p$.

The second derivative $\frac{\partial^2 V}{\partial n^2}(n, p)$ can be seen as a linear operator of $T_n S^N$. Next proposition, whose proof can be found in [5], p. 168, describes this linear operator in the above defined cylindrical coordinates.

\textbf{Proposition 2.2.} Denote $M_N$ the symmetric positive definite $N \times N$ matrix $\eta \cdot \eta^t$, where $\eta$ is a column vector and $\eta^t$ its transpose. We have that
\[ \frac{\partial^2 V}{\partial n^2}(n_0, p_0) = \int_{S^{N-1}} r^{N+1}(\eta, 0)r_z(\eta, 0)M_N d\eta. \tag{2.3} \]

If $r_z(\eta) > 0$, for any $\eta \in S^{n-1}$, then formula (2.3) implies $\frac{\partial^2 V}{\partial n^2}(n_0, p_0)$ is positive definite. Based on this, we can prove the following proposition:
Proposition 2.3. There exists a half-neighborhood $D \subset D_1$ of $M$ such that for any $p \in D$ there exists a smooth function $n = n(p)$ such that the pair $(n(p), p)$ is minimizing and $\frac{\partial^2 V}{\partial n^2}(n(p), p)$ is positive definite.

Proof. Given $q \in M$ consider a neighborhood $W$ of $q$ in $M$ with the following property: for any pair $(n, p)$ such that $\Gamma(n, p) \subset W$, $r_z(n, p)$ is strictly positive. For $p$ fixed, denote by $E_1(p) = \{n \in S^{N-1} | \Gamma(n, p) \subset W\}$.

There is a half-neighborhood $U(q)$ of $q$ such that for any $p \in U(q)$, there exists a minimizing $n(p) \in E_1(p)$ and any minimizing pair $n(p)$ must be in $E_1(p)$. Since $r_z(n, p)$ is strictly positive, the map $n \in E_1(p) \rightarrow V(n, p)$ is convex, so the minimizer $n(p)$ is unique. Considering $D = \cup_{q \in M-\partial M} U(q)$, we complete the proof of the proposition. □

2.3. Derivatives of the volume distance

Consider $D$ the half-neighborhood of $M$ given by proposition 2.3 and let $p \in D$. Recall that $v(p) = V(n(p), p).$ (2.4)

Lemma 2.4. We have that $\frac{\partial V}{\partial p}(n, p) = b(n, p)n.$ (2.5)

As a consequence, $Dv(p) = b(n(p), p)n(p).$ (2.6)

Proof. Since $p \rightarrow V(n, p)$ is constant along the hyperplane $H(n, p)$, we conclude that $\frac{\partial V}{\partial p}(n, p)$ is parallel to $n$. Also, for $t$ small, $V(n, p + tn) - V(n, p) = tb(n, p) + O(t^2)$, and thus the first formula is proved. Now differentiating (2.4) we obtain (2.6). □

Proposition 2.5. The normalized hessian of $v$ is exactly $Q^{-1}$, i.e., $\frac{1}{b(p)} D^2 v(p)\big|_{H(p)} = Q^{-1}.$

Proof. Differentiating (2.6) with respect to $p$ and using that $n$ is orthogonal to $H(p)$, we obtain $D^2 v(p)\big|_{H(p)} = b(p) \frac{dn}{dp}\big|_{H(p)}.$

On the other hand, if we differentiate (1.1) with respect to $p$ we obtain $\frac{\partial^2 V}{\partial n^2}(n, p) \frac{dn}{dp} + \frac{\partial^2 V}{\partial n \partial p} = 0.$

Now, from (2.5), $\frac{\partial^2 V}{\partial n \partial p} = b(p)I + \frac{\partial b}{\partial n} n.$
We conclude that
\[
\frac{dn}{dp}igg|_{H(p)} = -b(p) \left[ \frac{\partial^2 V}{\partial n^2(n,p)} \right]^{-1},
\]
thus proving the proposition.

\[\Box\]

3. Convergence to the Blaschke metric

For \(q \in M\), consider the centroid \(\gamma_q(t), \ t > 0\) of the region \(R(n(q), q + t\xi(q))\), where \(n(q)\) is orthogonal to \(T_qM\) and \(\xi(q)\) is the affine normal vector at \(q\). Then \(Q(\gamma_q(t))\) is a symmetric bilinear form defined in \(H(\gamma_t(q))\), which can be identified with \(T_qM\). The aim of this section is to prove the following theorem:

**Theorem 3.1.** For \(q \in M\),
\[
Q(\gamma_q(t)) = h(q) + O(t), \tag{3.1}
\]
and so \(Q(\gamma_q(t))\) is converging to \(h(q)\) when \(t\) goes to 0.

By applying a suitable affine transformation, we may assume that \(q = (0,0)\), the tangent plane \(T_qM\) is \(z = 0\) and the affine normal at \(q\) is \((0,1)\). Then, close to \(q\), the surface \(M\) is defined by an equation of the form
\[
z = \frac{r^2}{2} + O(r^3). \tag{3.2}
\]
where \(O(r^k)\) may depend on \(\eta\) but satisfies \(\lim_{r \to 0} \frac{O(r^k)}{r^k} = 0\), for any \(\epsilon > 0\). In this coordinates \(h(q) = I\) and \(\xi(q) = (0,1)\). Thus we can choose \(t = z\) and write \(\gamma_q(z) = (\overline{x}(z), z)\).

The following lemma is the main tool for proving theorem 3.1:

**Lemma 3.2.** Define
\[
Q_1(z) = \frac{1}{b(z)} \int_{S^{N-1}} r^{N+1} \left( \eta, z \right) r_z(\eta, z) M_N(\eta) d\eta, \tag{3.3}
\]
where \(b(z)\) denotes the \(N\)-volume of the section parallel to the hyperplane \(z = 0\) at height \(z\). Then
\[
Q_1(z) = I + O(z).
\]

We now show how theorem 3.1 follows from lemma 3.2. Since \(\xi(q)\) is tangent to the centroid line ([8], p.52), we have that \(\overline{x}(z) = O(z^2)\). Now from equations (1.2) and (2.3) we conclude that \(Q(\gamma_q(z))\) is \(O(z^2)\)-close to \(Q_1(z)\). Hence lemma 3.2 implies that
\[
Q(\gamma_q(z)) = I + O(z),
\]
thus proving theorem 3.1.

It remains then to prove lemma 3.2.
Proof. Since \( \lim_{r \to 0} \frac{r}{\sqrt{2z^{1/2}}} = 1 \), we can write
\[
    r(\eta, z) = \sqrt{2z^{1/2} + O(z^{3/2})}. \tag{3.4}
\]
Straightforward calculations from (3.4) show that
\[
    \frac{r_N}{N^{2N/2}} = \frac{1}{N}z^{N/2} + O(z^{N/2+1}).
\]
Differentiating \( \frac{r_{N+2}}{N+2} \) with respect to \( z \) leads to
\[
    \frac{r_{N+1}r_z}{2N/2} = z^{N/2} + O(z^{N/2+1}).
\]
The integral of \( \eta_i\eta_j \) over \( S^{N-1} \) is equal to \( \frac{\lambda}{N} \delta_{ij} \), where \( \lambda = \lambda(N) \) is the Lebesgue measure of \( S^{N-1} \) and \( \delta_{ij} = 1 \), if \( i = j \), and 0, if \( i \neq j \). Thus the integral \( L(i, j) \) of \( r_{N+1}r_z\eta_i\eta_j \) satisfies
\[
    \frac{L(i, j)}{2N/2} = \frac{\lambda \delta_{ij}}{N} z^{N/2} + O(z^{N/2+1}).
\]
Also, calculating \( b(z) \) as the integral of \( r_N/N \) over \( S^{N-1} \) we obtain
\[
    \frac{b(z)}{2N/2} = \frac{\lambda}{N}z^{N/2} + O(z^{N/2+1}).
\]
Thus
\[
    2^{N/2}b(z)^{-1} = \frac{N}{\lambda}z^{-N/2} + O(z^{-N/2+1}).
\]
and so \( Q(z)(i, j) = b(z)^{-1}L(i, j) = \delta_{ij} + O(z) \). \( \square \)

4. Convergence to the shape operator

Along this section, we shall use the notation of [7]: let \( f : M \subset \mathbb{R}^N \to \mathbb{R}^{N+1} \) be the inclusion map and denote by \( \xi \) its normal vector field pointing to the convex part of \( M \). For \( X, Y \in \mathcal{X}(U) \), we write
\[
    D_Xf_\ast(Y) = f_\ast(\nabla_X Y) + h(X, Y)\xi \\
    D_X\xi = -f_\ast(SX),
\]
where \( \nabla \) denotes the Blaschke connection, \( h \) is the positive definite Blaschke metric and \( S \) is the shape operator. Denote by \( \nu : M \to \mathbb{R}_{N+1} \) the corresponding co-normal immersion.

Close to the hypersurface \( M \), we write \( p = \gamma_q(t), q \in M, t \in [0, T) \), where \( \gamma_q(t) \) is the centroid of the section through \( q + t\xi(q) \) parallel to \( T_qM \). Then \( p \) is not necessarily on the normal line \( q + t\xi(q) \), but we can write
\[
    p = q + t\xi(q) + Z, \tag{4.1}
\]
for some \( Z = Z(q, t) \in T_qM \), with \( Z = O(t^2) \) (see [8], p.52). Differentiating (4.1) with respect to \( t \) gives
\[
    \frac{\partial p}{\partial t} = \xi(q) + Z_t, \tag{4.2}
\]
for some $Z_t \in T_q M$, with $Z_t = O(t)$. We conclude that
$$v_t(p) = Dv(p) \cdot (\xi(q) + Z_t) = Dv(p) \cdot \xi(q),$$
where for the last equality we have used the orthogonality of $Dv(p)$ and $H(p)$
(see equation (2.6)). We have thus proved the following lemma:

**Lemma 4.1.** The derivative of $v$ is given by
$$Dv(p) = v_t(p) \nu(q),$$ (4.3)
where $\nu(q)$ is the co-normal vector at $q \in M$ and $v_t(p) = \frac{d}{dt} v(\gamma_q(t)).$

**Lemma 4.2.** For any $X \in T_q M$,
$$\lim_{t \to 0} \frac{1}{v_t} \cdot D^2 v(X, \xi) = 0.$$

**Proof.** Differentiate equation (4.3) with respect to $t$ and use (4.2) to obtain
$$D^2 v(\xi(q) + Z_t) = v_{tt} \nu(q).$$
Thus, for any $X \in T_q M$,
$$D^2 v(\xi(q) + Z_t, X) = 0.$$
So $D^2 v(X, \xi) = -D^2 v(X, Z_t)$ and hence
$$\frac{1}{v_t} \cdot D^2 v(X, \xi) = Q(\gamma_q(t))(X, Z_t).$$
By corollary 3.1, $Q(\gamma_q(t))$ is converging to $h$ and since $Z_t = O(t)$, we conclude
that this last expression converges to 0, thus proving the lemma. \qed

**Theorem 4.3.** The rate of convergence of the bi-linear form $Q(\gamma_q(t))$ to $h(q)$
is $h_S(q)$, i.e.,
$$\lim_{t \to 0} \frac{Q(\gamma_q(t))(X, Y) - h(q)(X, Y)}{t} = h_S(q)(X, Y).$$
for any $q \in M, X, Y \in T_q M$.

**Proof.** Observe first that if we differentiate (4.1) in the direction $X \in T_q M$, we obtain
$$D_X(p) = (I - tS)X + \nabla_X Z + h(X, Z)\xi(q),$$ (4.4)
with $\nabla_X Z = O(t^2)$ and $h(X, Z) = O(t^2)$. Then differentiate equation (4.3)
in the direction of $X \in T_q M$ to obtain
$$D^2 v(D_X(p)) = v_t \nu_X(q) + X(v_t)\nu(q).$$
Thus, for $Y \in T_q M$,
$$D^2 v(D_X(p), Y) = v_t \nu_X(q)(Y) = -v_t h(X, Y)$$
(see [7], p.57, for the last equality). Expanding this equation using (4.4) and dividing by $v_t$ we obtain
$$Q(\gamma_q(t))(1 - tS)X, Y) - h(X, Y) = -Q(\gamma_q(t))(\nabla_X Z, Y) + h(X, Z)\frac{D^2 v(\xi,Y)}{v_t}.$$
Now, from lemma 4.2 and theorem 3.1, we conclude that
\[
\lim_{t \to 0} \frac{Q(\gamma_q(t))(X, Y) - h(X, Y)}{t} = h(SX, Y),
\]
thus proving the theorem. \(\square\)

Example. Consider the surface \(M \subset \mathbb{R}^3\) described by the equation
\[
z = \frac{1}{2} (x^2 + y^2) + \frac{c}{6} (x^3 - 3xy^2) + \frac{1}{24} (a_{40}x^4 + 4a_{31}x^3y + 6a_{22}x^2y^2 + 4a_{13}xy^3 + a_{04}y^4).
\]
For this surface \(\xi(0, 0) = (0, 0, 1)\) and we write
\[
z = \frac{r^2}{2} + \frac{r^3}{6} P_3(\theta) + \frac{r^4}{24} P_4(\theta),
\]
where \(\eta = (\cos(\theta), \sin(\theta))\),
\[
P_3(\theta) = c (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = c \cos(3\theta)
\]
and
\[
P_4(\theta) = a_{40} \cos^4 \theta + 4a_{31} \cos^3 \theta \sin \theta + 6a_{22} \cos^2 \theta \sin^2 \theta + 4a_{13} \cos \theta \sin^3 \theta + a_{04} \sin^4 \theta.
\]
It is not difficult to show that, in a neighborhood of \((0, 0)\), the inverse function \(r = r(z)\) satisfies
\[
r(\theta, z) = \sqrt{2} z^{1/2} - \frac{P_3(\theta)}{3} z + \frac{5P_3^2(\theta) - 3P_4(\theta)}{18\sqrt{2}} z^{3/2} + O(z^2).
\]
From this equation, long but straightforward calculations show that \(Q(z) = I + zA + O(z^2)\), where
\[
A = \begin{bmatrix}
\frac{c^2}{2} & -\frac{1}{4}(a_{40} + a_{22}) & -\frac{1}{4}(a_{31} + a_{13}) \\
-\frac{1}{4}(a_{31} + a_{13}) & \frac{c^2}{2} & -\frac{1}{4}(a_{22} + a_{04})
\end{bmatrix}.
\]
On the other hand, we can calculate the shape operator of \(M\) at the origin following [7], p.47. In this way we verify that \(h_S = -A\), in accordance with theorem 4.3.

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