TWO-MATCHING COMPLEXES

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ABSTRACT. A two-matching complex is a simplicial complex which captures the relationship between two-matchings of a graph. In this paper, we will use discrete Morse Theory and the Matching Tree Algorithm to prove homotopical results. We will consider a class of graphs for which the homotopy type of the 2-matching complex transforms from a sphere to a point with the addition of leaves. We end the paper by defining \(k\)-matching sequences and looking at the 1- and 2-matching complexes of wheel graphs and perfect caterpillar graphs.

1. Introduction

Matchings and matching complexes are objects that have been well studied, for example [1] [8] [12] [14]. Matching complexes were introduced in the 70’s through work done by Brown and Quillen as a way to study the structure of subgroups and provide interesting connections to several areas in mathematics. A matching complex of a graph \(G\), denoted \(M_1(G)\), is a simplicial complex with vertices given by edges of \(G\) and faces given by matchings of \(G\), where a matching is a subset of edges \(H \subseteq E(G)\) such that any vertex \(v \in V(H)\) has degree at most 1. Some matching complexes that have been studied in detail are the full matching complex \(M_1(K_n)\), where \(K_n\) is the complete graph on \(n\) vertices, and the chessboard complex \(M_1(K_{m,n})\), where \(K_{m,n}\) is the complete bipartite graph with block size \(m\) and \(n\). Results about \(M_1(K_n)\) and \(M_1(K_{m,n})\) include connectivity bounds and rational homology. For a general survey on matching complexes see [13]. The homotopy type of matching complexes is a bit more mysterious. The homotopy type of matching complexes for paths and cycles [9], for forests [11], and for the \(k\)-matching complexes. Jonsson defines the bounded degree complex \(BD_n^\lambda(G)\) with \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) to be the complex of subgraphs of a graph \(G\) with \(n\) vertices such that the degree of vertex \(x_i\) is at most \(\lambda_i\), which is a natural generalization of matching complexes. When \(\lambda = (d, \ldots, d)\) we write \(BD_n^d(G) := BD_n^{d,d,\ldots,d}(G)\). The bounded degree complex \(BD_n^1(K_n)\) is the matching complex on complete graphs, that is \(M_1(K_n)\). For \(d \geq 2\), \(BD_n^d(G)\) is the \(d\)-matching complex on \(G\) with \(0\)-simplices given by edges in \(G\) and faces by \(d\)-matchings in \(G\), where a \(d\)-matching is a subset of edges \(H \subseteq E(G)\) such that any vertex \(v \in V(H)\) has degree at most \(d\). Bounded degree complexes are generalizations of matching complexes that involve relaxing the incidence conditions on the vertices. Such a generalization may provide insight into the complexity of matching complexes. For example, in Section 3 we use bounded degree complexes to inductively study \(k\)-matching complexes. Jonsson primarily focuses on the connectivity of \(BD_n^\lambda(K_n)\) considering the outcome for graphs with and without loops. For a further survey of bounded degree complexes see [13].

In Section 3, we connect our results to these connectivity results. The focus of this paper will be the topology of \(M_2(G) := BD_2^2(G)\) which we call the 2-matching complex of \(G\). Since a matching of \(G\) is also a 2-matching of \(G\), the matching complex of \(G\) is a subcomplex of the 2-matching complex of \(G\), with \(M_1(G) \subset M_2(G)\).
1.1. **Our contributions.** In this paper, we will use discrete Morse Theory and the Matching Tree Algorithm to prove homotopical results. In Section 2, we provide the necessary combinatorial and topological background for these techniques. In Section 3, we take a preliminary look at 2-matching complexes and consider a class of graphs for which the homotopy type of the 2-matching complex is contractible. Then, we look at graphs whose homotopy type of the 2-matching complex changes from a sphere to a point with the addition of leaves. We end this section with a constructive algorithm to maximize the number of additional leaves that can be added to a certain family of graphs without changing the homotopy type of $M_2(G)$. In Section 4, we define $k$-matching sequences and look at wheel graphs as a first example. We conclude with perfect caterpillar graphs and future directions.

## 2. Background

### Definition 2.1.** An (abstract) simplicial complex $\Delta$ on a set $X$ is a collection of subsets of $X$ such that

1. $\emptyset \in \Delta$
2. If $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of a simplicial complex are called faces and an $n$–simplex is the collection of all subsets of $[n]$. A subcomplex $\Delta' \subseteq \Delta$ of a complex $\Delta$ is a subcollection of $\Delta$ which satisfies (i) and (ii). For simplicial complexes $\Delta$ and $\Delta'$, the topological join is $\Delta \ast \Delta' = \{ \sigma \cup \sigma' : \sigma \in \Delta, \sigma' \in \Delta' \}$. A simplicial complex $\Delta$ is said to be a cone with cone point $\{x\} \in \Delta$ if for every face $\sigma \in \Delta$ we have $\sigma \cup \{x\} \in \Delta$, that is the simplicial complex $\Delta' \ast x$ for some $\Delta'$. Note that every cone is contractible. The suspension of a space $\Delta$ is denoted $\Sigma(\Delta)$ and is the join of $\Delta$ with two discrete points.

We will let $G$ be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the degree of $v$, $\deg(v)$ is the number of edges incident to $v$. If $V(G) \cap V(H) = \{x\}$, the wedge sum $G \vee H$ of $G$ with $H$ over $x$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let $\{u, v\} \in E(G)$ and $w$ a new vertex not in $V(G)$. The subdivision of $\{u, v\} \in E(G)$ is obtained by deleting $\{u, v\}$ and adding $w$ to $V(G)$ and $\{u, w\}, \{w, v\}$ to $E(G)$. A vertex $v \in V(G)$ is a leaf if its neighborhood contains exactly one vertex. For a graph $G$ with $v \in V(G)$, attaching a leaf to $v$ in $G$ refers to the process of adding a new vertex $w$ to $V(G)$ and $\{v, w\}$ to $E(G)$. Given a graph $G$ with two leaves $u, v$ and edges $\{v_1, u\}$ and $\{v_2, v\}$, define $G_{(u,v)}$ to be the graph obtained by identifying $u$ and $v$, labeled $uv$. That is $E(G_{(u,v)}) = E(G) \setminus \{\{v_1, u\}, \{v_2, v\}\} \cup \{\{v_1, uv\}, \{v_2, uv\}\}$ and $V(G_{(u,v)}) = V(G) \setminus \{u, v\} \cup \{uv\}$.

### Definition 2.2.** For a graph $G = (V(G), E(G))$ with max degree three, the clawed graph of $G$, denoted $C(G)$ or $CG$ is the graph obtained by subdividing every $e \in E(G)$ and attaching (possibly empty) set of leaves to every $v \in V(G)$ so that $\deg(v) = 3$ for all $v$. The graph $G$ is called the core of $C(G)$. See Example 2.3 for an example.

If $|E(G)|$ and $|V_{\leq 2}(G)|$ denote the number of edges and the number of vertices with degree less than or equal to 2 in a graph $G$ respectively, and $L$ is the number of leaves of $G$, the process of clawing $G$ introduces $|E(G)| + |V_{\leq 2}(G)| + L$ new vertices and $|V_{\leq 2}(G)| + L$ new edges.

### Example 2.3.** Clawing a graph: (A) Begin with a graph $G$, (B) Subdivide each edge (depicted with open circles), (C) attach a set of leaves to each vertex of $G$ so that $\deg(v) = 3$ for all $v \in V(G)$. We say graph $G$ is the core graph of $C(G)$ or $CG$ is the clawed graph with respect to $G$, $|E(G)| = 4 = |V_{\leq 2}(G)|$ and $L = 3$ so the total number of vertices added is 11 and the total number of new (leaf) edges is 7.

### Definition 2.4.** For an edge set $H \subseteq E(G)$, let $V(H)$ denote the set of vertices supported by $H$. That is, $V(H) := \bigcup_{e \in H} V(e)$. An induced claw unit of a graph is a $K_{1,3}$ subgraph with 1 vertex of degree 3 in $G$ and 3 vertices of degree less than or equal to 2 in $G$ (Figure 1).
We will be interested in deleting an induced claw unit in a graph. To do so, we consider an induced claw unit $c$ to be defined by the unique degree 3 vertex, call it $v$. We abuse notation and use $G \setminus c$ to denote the vertex deleted subgraph of $G \setminus \{v\}$, the graph obtained by deleting $v$ and all the edges incident to it.

**Definition 2.5.** A 2-matching of a graph $G$ is a subset of edges $H \subseteq E(G)$ such that any vertex $v \in V(H)$ has degree at most 2.

**Definition 2.6.** A 2-matching complex of a graph $G$, denoted $M_2(G)$ is a simplicial complex with vertices given by edges of $G$ and faces given by 2-matchings of $G$.

**Example 2.7.** See Figure 2 consisting of the graph $G$, its matching complex $M_1(G)$, and its 2-matching complex $M_2(G)$. The 2-matching complex of $G$ consists of 5 maximal faces. Namely, (1) $\{a,c,d\}$, (2) $\{a,c,e\}$, (3) $\{a,b,d,e\}$, (4) $\{b,c,d\}$, (5) $\{b,c,e\}$. These maximal faces form a simplicial complex that is homotopy equivalent to $S^2$, a 2-sphere. Notice that $M_1(G) \subseteq M_2(G)$.

2.1. **Discrete Morse Theory.** Robin Forman developed Discrete Morse Theory as a tool to study the homotopy type of simplicial complexes [5]. The underlying idea of the theory is to pair faces in a simplicial complex to give rise to a sequence of collapses that yields a homotopy equivalent cell complex.

**Definition 2.8.** A partial matching in a poset $P$ is a partial matching in the underlying graph of the Hasse diagram of $P$, i.e., it is a subset $M \subseteq P \times P$ such that
• 

- $(a, b) \in M$ implies $b > a$; i.e. $a < b$ and no $c$ satisfies $a < c < b$.
- each $a \in P$ belongs to at most one element in $M$.

When $(a, b) \in M$, we write $a = d(b)$ and $b = u(a)$.

A partial matching on $P$ is called acyclic if there does not exist a cycle

$$a_1 < u(a_1) > a_2 < u(a_2) > \cdots < u(a_m) > a_1$$

with $m \geq 2$ and all $a_i \in P$ being distinct.

Given an acyclic partial matching $M$ on a poset $P$, an element $c$ is critical if it is unmatched. If none of the critical cells can be further paired $M$ is called complete. If every element is matched by $M$, $M$ is called perfect.

The main theorem of discrete Morse theory as given in [9, Theorem 11.13] is

**Theorem 2.9.** Let $\Delta$ be a polyhedral cell complex and let $M$ be an acyclic matching on the face poset of $\Delta$. Let $c_i$ denote the number of critical $i$-dimensional cells of $\Delta$. The space $\Delta$ is homotopy equivalent to a cell complex $\Delta_c$ with $c_i$ cells of dimension $i$ for each $i \geq 0$, plus a single 0-dimensional cell in the case where the emptyset is paired in the matching.

A common way to obtain an acyclic matching is to toggle on an element $x$ in the vertex set of a face poset $P$.

**Definition 2.10.** Let $P$ be the face poset of a simplicial complex $\Delta$ and $Q \subseteq P$ a subposet. For $x$ an element in the vertex set of $Q$, toggling on an element $x$ is a partial matching that pairs subsets $a \in Q$, $x \not\in a$ with $a \cup \{x\}$, whenever possible.

It is often useful to create acyclic partial matchings on different sections of the face poset of a simplicial complex and then combine them to form a larger acyclic partial matching on the entire poset. This process is detailed in the following theorem known as the Cluster Lemma in [8] and the Patchwork Theorem in [9].

**Theorem 2.11.** Assume that $\varphi : P \rightarrow Q$ is an order-preserving map. For any collection of acyclic matchings on the subposets $\varphi^{-1}(q)$ for $q \in Q$, the union of these matchings is itself an acyclic matching on $P$.

The following theorem shows there is an intimate relationship between linear extensions and acyclic matchings [9].

**Theorem 2.12 (Kozlov, Theorem 11.2).** A partial matching on a poset $P$ is acyclic if and only if there exists a linear extension of $\mathcal{L}$ of $P$ such that $x$ and $u(x)$ follow consecutively.
Since \( x \) and \( u(x) \) follow consecutively in the linear extension, when we refer to these elements in the linear extension we will use the notation \( (x, u(x)) \) and consider them as a pair of consecutive elements in the poset.

**Lemma 2.13.** Toggling provides an acyclic partial matching.

**Proof.** To see this suppose we toggle on the element \( x \). Start with an element \( a_1 \in P \) such that \( x \not\in a_1 \), \( x \in u(a_1) \). Any element \( a_2 < u(a_1) \) with \( a_2 \neq a_1 \) contains \( x \) since \((a_1, u(a_1)) \in M\). Hence, there is no element \( u(a_2) \), and a cycle cannot be created. \( \square \)

Additionally, using the patchwork theorem, we see that performing repeated toggling yields an acyclic matching.

**Lemma 2.14.** Let \( P \) be a poset with vertex set \( T \) and \( \{x_1, x_2, ..., x_n\} \) with \( x_i \in T \) be a sequence of toggling elements of \( P \). Repeatedly toggling on \( x_1 \), then \( x_2 \) and so on, yields an acyclic matching on \( P \).

**Proof.** Let \( Q \) be a poset with elements \( \mathcal{R} \) and \( Y_i \) for \( i \in [n] \) with relations given by \( Y_i < Y_{i+1} \) for all \( i \in [n-1] \) and \( Y_n < \mathcal{R} \). Recursively define \( D_i := \{\alpha \in P|\{x_i\} \in \alpha \text{ or } \alpha \cup \{x_i\} \in P \text{ and } \alpha \not\in D_j \text{ for } j \leq i-1\} \). Define \( \varphi : P \to Q \) by \( \varphi^{-1}(Y_i) := D_1 \cup \{x_i\}, \varphi^{-1}(Y_i) := D_i \), for \( 2 \leq i \leq n \), and the remaining elements to \( \mathcal{R} \). The map \( \varphi \) is well-defined and order-preserving. On \( \varphi^{-1}(Y_i) \) toggle on \( x_i \), which is an acyclic matching by Lemma 2.13. The union of which forms an acyclic matching on \( P \) by Theorem 2.11. \( \square \)

We will use discrete Morse theory to determine the homotopy type of clawed graphs. We observe now that induced claw units in graphs behave nicely with 2-matching complexes.

**Proposition 2.15.** Let \( c \in G \) be an induced claw unit with edge set \( E(c) = \{x, y, z\} \). The following sets are in bijection with each other:

(i) The set of 2-matchings of \( G \setminus c \),
(ii) The set of 2-matchings containing \( \{y, z\} \), and
(iii) The set of 2-matchings containing \( x \) and not \( y \) or \( z \).

**Proof.** For any 2-matching \( m \) in \( G \setminus c \), both \( m \cup \{x\} \) (not containing \( y \) or \( z \)) and \( m \cup \{y, z\} \) are 2-matchings in \( G \). Notice that \( x \) and \( \{y, z\} \) cannot be in a 2-matching together since they all meet at a degree three vertex. \( \square \)

**Example 2.16.** Consider the graph in Figure 3. There is exactly 1 induced claw unit, call it \( c \), given by the edge set \( \{x, y, z\} \). The set \( \{e\} \) is the only 2-matchings of \( G \setminus c \). Notice that the 2-matchings containing \( \{y, z\} \) consists of exactly \( \{e, y, z\} \) and 2-matchings containing \( x \) and not \( y \) or \( z \) consists of exactly \( \{e, x\} \).

![Figure 3](image-url)

We turn our attention to a general connectivity result of \( M_2(G) \) for any graph \( G \). Since \( \mathcal{F}(M_2(G)) \) the face poset of a 2-matching complex of \( G \) has vertex set consisting of faces of \( M_2(G) \) with an order relation of containment, for \( a, b \in \mathcal{F}(M_2(G)) \), \( a < b \) if \( b = a \cup e \) for some \( e \in E(G) \). In
relation to Figure 3 suppose we define a partial matching of \( F(M_2(G)) \) by toggling on \( x \), where \( x \in E(G) \). Then, the matchings remaining after toggling are exactly those that contain \( \{y, z\} \) and therefore are in bijection with 2-matchings of \( G - c \) by Proposition \( 2.15 \). Hence, if you have two induced claw units \( c_1 \) and \( c_2 \) in \( G \), the choice of toggle edge in \( c_1 \) and \( c_2 \) and the order in which one toggles is irrelevant.

**Lemma 2.17.** Let \( G \) be a simple, finite graph and \( C = \{c_1, ..., c_n\} \) be a collection of induced claw units in \( G \) with \( E(c_i) := \{x_i, y_i, z_i\} \) for each \( c_i \). Then the connectivity of \( M_2(G) \) is at least \( 2|C| - 2 \). Further, if we fix the toggle edge in each \( c_i \), say \( x_i \), then every critical cell remaining after toggling on all of the \( x_i \)’s will consist of \( \{y_i, z_i\} \) for all \( i \), regardless of order.

**Proof.** Let \( P := F(M_2(G)) \) be the face poset of the 2-matching complex of \( G \). We define a partial (discrete Morse) matching on \( P \) by (arbitrarily) fixing \( x_i \) as the toggle edge for each \( c_i \). Our claim is that for any permutation \( \pi \in \mathfrak{S}_n \), the unmatched subposet that remains after toggling on \( x_{\pi(1)} \), \( x_{\pi(2)} \), \( x_{\pi(3)} \), ..., \( x_{\pi(n)} \) is the upper-order ideal \( P_{\geq \{y_{\pi(1)}, z_{\pi(1)}, y_{\pi(2)}, z_{\pi(2)}, ..., y_{\pi(n)}, z_{\pi(n)}\}} \). Since permutations can be generated by a sequence of transpositions, it suffices to consider the unmatched subposet obtained from toggling \( x_1 \), then \( x_2 \) and the unmatched subposet obtained from toggling \( x_2 \) then \( x_1 \).

Suppose first that we toggle on \( x_1 \). The edge \( x_1 \in E(G) \) forms a 2-matching with all 2-matchings of \( G \) that do not contain both \( y_1 \) and \( z_1 \) so the unmatched cells of \( P \) are precisely the elements containing both \( y_1 \) and \( z_1 \). That is, the unmatched subposet that remains is \( P_{\geq \{y_1, z_1\}} \). Now, toggling on \( x_2 \) matches all of the 2-matchings of \( G \) that contain \( y_1, z_1 \), but do not contain \( y_2, z_2 \). All elements \( a \in P_{\geq \{y_1, z_1\}} \) with \( \{x_2\} \in a \) will be paired with \( b := a \setminus \{x_2\} \) through toggling on \( x_2 \) and all elements \( b \) are in \( P_{\geq \{y_1, z_1\}} \) since \( \{x_1\} \subseteq b \). Notice that all matchings in \( P_{\geq \{y_1, z_1\}} \) are in bijection with 2-matchings in \( G - c_1 \) by Proposition \( 2.15 \) and \( c_2 \in G - c_1 \).

Hence, the unmatched subposet that remains after toggling on \( x_1 \) then \( x_2 \) is precisely \( P_{\geq \{y_1, z_1, y_2, z_2\}} \).

An analogous argument shows that the same upper order ideal remains after toggling first on \( x_2 \) and then \( x_1 \). By induction, we get that the unmatched subposet that remains after toggling on \( x_{\pi(1)} \), \( x_{\pi(2)} \), \( x_{\pi(3)} \), ..., \( x_{\pi(n)} \) is the upper-order ideal \( P_{\geq \{y_{\pi(1)}, z_{\pi(1)}, y_{\pi(2)}, z_{\pi(2)}, ..., y_{\pi(n)}, z_{\pi(n)}\}} \). Two elements from each induced claw unit contribute to \( \{y_{\pi(1)}, \bar{y}_{\pi(1)}, y_{\pi(2)}, \bar{y}_{\pi(2)}, ..., y_{\pi(n)}, \bar{y}_{\pi(n)}\} \) and an acyclic matching has been produced such that all unmatched sets are of size at least \( 2|C| \) as follows. We defines the the connectivity of \( M_2(G) \) is at least \( 2|C| - 2 \).

**Definition 2.18.** Let \( G \) be any graph. A claw-induced partial matching is an acyclic partial matching on \( F(M_2(G)) \) obtained by toggling on elements in the vertex set of \( F(M_2(G)) \) corresponding to edges in induced claw units of \( G \), whenever possible.

### 2.2. Matching Tree Algorithm (MTA)

In [2], the authors detail the Matching Tree Algorithm which provides an acyclic discrete Morse matching on the face poset of an independence complex of a graph \( G \). An independence complex \( \text{Ind}(G) \) of a graph \( G \) is a simplicial complex in which the vertices are given by vertices of \( G \) and faces are given by independent sets of vertices. The matching complex of a graph \( G \) is equal to the independence complex of the line graph of \( G \) where the vertices of the line graph are the edges of the graph and two vertices are adjacent if the corresponding edges are incident in the graph. In Section 3 we will use the Matching Tree Algorithm to find the homotopy type of the 1-matching complex of a wheel graph, by looking at the independence complex of the line graph.

Let \( G \) be a simple graph with vertex set \( V = V(G) \). Bousquet-Mélou, Linusson, and Nevo motivate the MTA with the following algorithm. Let \( \Sigma \) denote the independence complex of \( G \). Take a vertex \( p \in V \) and denote \( N(p) \) the set of its neighbors. Define \( \Delta = \{I \in \Sigma : I \in N(p) = \emptyset\} \). For \( I \in \Delta \) and \( p \notin I \), the set of pairs \((I, I \cup \{p\})\) form a perfect matching of \( \Delta \) and hence a matching of \( \Sigma \). The vertex \( p \) is called a **pivot**.
Notice that the unmatched elements of $\Sigma$ are those containing at least one element of $N(p)$. Choose an unmatched vertex and continue the process as many times as possible. This algorithm will give rise to a rooted tree, called a matching tree of $\Sigma$, whose nodes represent sets of unmatched elements. Some of the nodes are reduced to the empty set, and all others are of the form

$$\Sigma(A, B) = \{ I \in \Sigma : A \subset I \text{ and } B \cap I = \emptyset \},$$

where

$$A \cap B = \emptyset \text{ and } N(A) := \bigcup_{a \in A} N(a) \subset B.$$

The root of the tree is $\Sigma(\emptyset, \emptyset)$, which is equal to the set of all the independent sets of $G$. As we traverse the tree the sets $\Sigma(A, B)$ will become smaller and the leaves of the tree will have cardinality 0 or 1.

The following presentation of the Matching Tree Algorithm follows [7]. Begin with the root node $\Sigma(\emptyset, \emptyset)$ and at each node $\Sigma(A, B)$ where $A \cup B \neq V$ apply the following procedure:

1. If there is a vertex $v \in V \setminus (A \cup B)$ such that $N(v) \setminus (A \cup B) = \emptyset$, then $v$ is called a free vertex. Give $\Sigma(A, B)$ a single child labeled $\emptyset$.

2. Otherwise, if there is a vertex $v \in V \setminus (A \cup B)$ such that $N(v) \setminus (A \cup B)$ is a single vertex $w$, then $v$ is called a pivot and $w$ a matching vertex. Give $\Sigma(A, B)$ a single child labeled $\Sigma(A \cup \{w\}, B \cup N(w))$, which we call the right child, and $\Sigma(A, B \cup \{v\})$, which we call the left child.

**Remark 2.19.** Step (3) is motivated by the observation that if $v$ has at least two neighbors, say $w$ and $w'$ then some of the unmatched sets $I$ contain $w$, and some others don’t, but if they do not contain $w$ than they must contain $w'$.

The following theorem is the main theorem for the Matching Tree Algorithm, which is due to Bousquet-Mélou, Linusson, and Nevo [2], but is stated as it appears in Braun and Hough [3].

**Theorem 2.20.** A matching tree for $G$ yields an acyclic partial matching on the face poset of $\text{Ind}(G)$ whose critical cells are given by the non-empty sets $\Sigma(A, B)$ labeling non-root leaves of the matching tree. In particular, for each set $\Sigma(A, B)$, the set $A$ yields a critical cell in $\text{Ind}(G)$.

Thus far, we have provided combinatorial tools for determining the homotopy type of simplicial complexes. It is also possible to use more topological methods to approach homotopy type. This approach requires inductively determining the homotopy type of complexes of interest and appropriately “gluing” these spaces over a common subspace. For a more detailed discussion see [6] Section 4.G] and [9] Section 15.2. The following lemma follows from [6] Proposition 4G.1, where $X \vee Y$ is considered as a homotopy colimit.

**Lemma 2.21.** Let $X$ and $Y$ be two spaces such that $X \simeq_f X'$ and $Y \simeq_g Y'$, then $X \vee Y \simeq X' \vee Y'$.

3. **Contractibility in 2-matching complexes**

We begin this section by exploring graph properties that force contractible 2-matching complexes.

**Observation 3.1.** If $G$ and $H$ are two graphs with leaf vertices $v_1 \in V(G)$ and $v_2 \in V(H)$, it is immediate for $G \vee_{v_1 \sim v_2} H$ we have

$$M_2(G \vee_{v_1 \sim v_2} H) = M_2(G) \ast M_2(H),$$

where $\ast$ denotes the topological join.

**Proposition 3.2.** If $v_1, v_2 \in V(G)$ are two leaf vertices of a graph $G$, then $M_2(G) = M_2(G_{(v_1, v_2)})$. 

Proof. Let \( G \) be a graph with \( x, y \) two leaf nodes and \( m \in M_2(G) \) be a 2-matching. A 2-matching \( H \subseteq E(G) \) of a graph \( G \) consists of vertices \( v \in V(H) \) with degree at most 2. Identifying two leaf vertices of \( G \) does not affect the 2-matching since the identified vertex has degree 2. So \( m \) is also a 2-matching for \( G_(x,y) \), the graph which identifies \( x \) and \( y \).

\[ \square \]

Theorem 3.3. Let \( G \) be a graph with \( e = \{x, y\} \in E(G) \) such that \( \deg(x) \leq 2 \) and \( \deg(y) \leq 2 \). Then \( M_2(G) \) is contractible.

Proof. Since both endpoints of \( e \) have degree at most 2, \( e \) may be included in any 2-matching of \( G \setminus e \) and \( M_2(G \setminus e) \). Hence, \( M_2(G) \) is a cone and therefore contractible.

We observed in Theorem 3.3 that graphs that contain an edge with endpoints of degree less than or equal to 2 form a large class of graphs that have contractible 2-matching complexes. We will now explore 2-matching complexes that are close to but not contractible. In particular, we turn our attention to clawed graphs. We begin by considering clawed paths of even length. In the following proposition, we use the well-known fact that for two spheres \( S^m \) and \( S^n \), \( S^m \ast S^n \simeq S^{m+n+1} \).

Proposition 3.4. For \( n \geq 0 \), let \( CP_n \) be a clawed path with respect to a path of length \( n \). Then, \( M_2(CP_n) \simeq S^{2n+1} \).

Proof. Since \( P_0 \) consists of one vertex and no edges, we have \( CP_0 = K_{3,1} \). See Figure 4. It follows that \( M_2(CP_0) \simeq S^1 \). Consider now a clawed path of length 1, \( CP_1 \) consists of two copies of \( K_{3,1} \) intersecting at one vertex. By Observation 3.1 we have \( M_2(CP_1) = M_2(CP_0 \vee CP_0) = M_2(CP_0) \ast M_2(CP_0) = S^1 \ast S^1 \simeq S^{1+1+1} = S^3 \). Continuing inductively, we have \( M_2(CP_n) = M_2(CP_{n-1} \vee CP_0) \simeq S^{2(n-1)+1} \ast S^1 \simeq S^{2n-2+3} = S^{2n+1} \).

Corollary 3.5. \( M_2(CP_{n-1}) \simeq M_2(CC_n) \simeq S^{2n-1} \).

Proof. The result follows from Proposition 3.2, see Figure 5.

In the next proposition, we see that, even further, the 2-matching complex for a clawed cycle shares its homotopy type with the 2-matching complex of a fully whiskered cycle.

Definition 3.6. A fully whiskered graph \( W(G) \) is a graph in which a leaf is attached to every vertex of the graph \( G \).
Figure 5. On the left graph $CP_2$, the clawed path of length 2 and on the right $CC_3$, the clawed 3-cycle obtained by identifying the endpoints of $CP_2$. The core 3-cycle is shown with dashed lines.

Proposition 3.7. Let $WC_m$ denote a fully whiskered $2m$-cycle graph for $m \geq 3$. $M_2(WC_m) \simeq S^{2m-1}$.

Proof. Label the edges of the cycle by $1, 2, \ldots, 2m$ and each leaf edge by $x_{i,i+1}$ for $i \in [2m-1]$, and $x_{1,2m}$, where the index corresponds to the incident edges in the cycle as in Figure 6. Let the edge set $c_i := \{x_{i,i+1}, i, i+1\}$ for each $i \in \{1, 3, 5, \ldots, 2m-1\}$ denote an induced claw of $WC_m$. Then the collection $\mathcal{C} = \{c_1, c_3, \ldots, c_{2m-1}\}$ defines a family of $m$ induced claw units that are edge disjoint. If this were not the case, then one edge $j \in E(WC_m)$ would be an edge in two claws, but by the labeling system this would mean that $j = j + 1$ which is a contradiction to the edge labels on the cycle. Following the proof of Lemma 2.17 for each $i \in \{1, 3, 5, \ldots, 2m-1\}$ let $x_{i,i+1}$ be the toggle edge in the discrete Morse matching on the face poset of $M_2(WC_m)$. By Lemma 2.17 we know the connectivity of $M_2(WC_m)$ is at least $2(m-2)$. Further, every unmatched cell contains $\{1, 2, \ldots, 2m\}$, that is all of the edges in the even cycle. All of the edges in the cycle forms a maximal two matching of $WC_m$ and hence $\{1, 2, \ldots, 2m\}$ is the only critical cell of the discrete Morse matching and $M_2(WC_m) \simeq S^{2m-1}$. \hfill $\square$

Corollary 3.8. $M_2(WC_n) \simeq M_2(CC_n)$ for $n \geq 3$.

In Proposition 3.7 we considered fully whiskered $2m$-cycle graphs because we are interested in aligning this result with clawed path graphs, but there is no reason why we could not apply the same reasoning for fully whiskered odd-cycle graphs.

Theorem 3.9. Let $WC_n^d$ denote a fully whiskered $n$-cycle graph for odd $n$. Then, $M_2(WC_n^d) \simeq S^{n-1}$.

Proof. Using the same claw-induced partial matching as in the proof of Proposition 3.7 for all $i \in \{1, 3, \ldots, n-2\}$, the remaining unmatched cells must contain $\{1, 2, \ldots, n-1\}$. These cells form an upper order ideal in the partially matched face poset of $M_2(WC_n^d)$ and include precisely $\{x_{n,1}, 1, 2, \ldots, n-1, n\}$, $\{x_{n,1}, 1, 2, \ldots, n-1\}$, $\{1, 2, \ldots, n-1, x_{n-1,n}\}$, and $\{1, 2, \ldots, n-1\}$. Performing a final toggle on the edge $x_{n-1,n}$, we obtain one critical cell, $\{1, 2, \ldots, n\}$ and hence $M_2(WC_n^d) \simeq S^{n-1}$. \hfill $\square$

We saw in Corollary 3.8 that $M_2(CC_n) \simeq M_2(WC_n) \simeq S^{2n-1}$ and it is no coincidence that $CC_n$ is a subgraph of $WC_n$. The next lemma shows that there are certain degree two vertices such that attaching a leaf does not affect the homotopy type of the 2-matching complex. We call such vertices attaching sites.

Lemma 3.10. Let $CG := C(G)$ be a clawed graph with vertex set $V(CG)$, edge set $E(CG)$, and $v \in V(CG)$ a degree two vertex with $e_1, e_2 \in E(CG)$ the two incident edges to $v$. Define a complete
claw-induced partial matching on $P$, the face poset of $M_2(CG)$. Then both edges $e_1$ and $e_2$ are in a critical cell if and only if attaching a leaf to $v$ does not change the homotopy type. Further, if at least one edge, $\{e_1, e_2\}$ is not in any critical cell obtained from the complete claw-induced partial matching of $P$, the 2-matching complex of $CG$ with a leaf attached to $v$ is contractible.

**Proof.** Since $CG$ is a clawed graph and $\deg(v) = 2$, $v$ is the intersection of two claws $c_1$ and $c_2$. For each claw, one of the edges is a toggle edge and two are in critical cells. If $e_1$ and $e_2$ are in some critical cell; then they are in all critical cells since this would mean that one of the other edges in $c_1$ and $c_2$ are toggled on. Attaching a leaf $w$ to $v$ does not give rise to any additional cells since this would imply that $e_1, e_2$, and the edge $\{v, w\}$ are all in a 2-matching together, but this is not possible because they are all incident a common vertex.

Suppose now that no critical cell contains both $e_1$ and $e_2$ (but perhaps contains one). Then attaching a leaf $w$ to $v$ gives rise to several new critical cells, under the same matching $M$. For each critical cell $X$ in the claw-induced partial matching on $P$, $X \cup \{w, v\}$ is a critical cell in the claw-induced partial matching on $F(M_2(CG \cup \{w, v\}))$. Therefore, every critical cell can be further matched by toggling on $\{w, v\}$ and $M_2(CG \cup \{w, v\})$ is contractible. \hfill $\square$

**Theorem 3.11.** For a clawed graph $CG$, $M_2(CG) \simeq S^{2n-1}$ where $n = |E(CG)|$.

**Proof.** The clawed graph $CG$ consists of a collection of claws that have pairwise intersection of at most 1 vertex, that is a collection of $\frac{1}{3}n$ induced claw units. Each claw in this claw decomposition of $G$ gives rise to one toggle edge and two edges in the critical cell. By Lemma 2.17 the connectivity of $M_2(CG)$ is at least $\frac{2}{3}n - 2$. Further the complete claw-induced partial matching will consist of one critical cell consisting of two edges of each claw which defines a maximal matching on the clawed graph $CG$. Since a graph has $\frac{1}{3}n$ claws and two of every one belongs in the critical cell, the critical cell has size $\frac{2}{3}n$ and $M_2(CG) \simeq S^{2n-1}$. \hfill $\square$

We can relate these findings back to [8, Theorem 12.5] which gives a general connectivity bound for these complexes. For a real number $\nu$, a family of sets $\Delta$ is $AM(\nu)$ if $\Delta$ admits an acyclic matching such that all unmatched sets are of dimension $\lceil \nu \rceil$. For $\lambda = (\lambda_1, ..., \lambda_n)$ define $|\lambda| = \sum_{i=1}^{n} \lambda_i$. For a sequence $\mu = (\mu_1, ..., \mu_n), n \geq 1$, define

$$\alpha(n, \mu) = \min \{ \alpha : BD_\alpha^n \text{ is } AM(\frac{|\lambda| - \alpha}{2} - 1) \}.$$
Theorem 3.12. (Thm 12.5, [8]) Let $G$ be a graph on the vertex set $V$. Let $\{U_1, ..., U_t\}$ be a clique partition of $G$ and let $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mu = (\mu_1, ..., \mu_n)$ be sequences of nonnegative integers such that $\lambda_i \leq \mu_i$ for all $i$. Then $BD^\lambda_n(G)$ is $[\nu] - 1$ connected, where

$$\nu = \frac{|\lambda|}{2} - \frac{1}{2} \sum_{j=1}^{t} (\alpha(U_j, \mu_j) - 1)$$

Proposition 3.13. Theorem 3.12 is an example where Theorem 3.13 is not sharp.

Proof. To show this we need to choose a clique partition. By construction of the clawed graphs, the best we can do is choosing a partition of 2- and 1-cliques. Let $\lambda = (2, 2, ..., 2) = \mu$. By [8, Lemma 12.6], all values of $\alpha$ are 2 and any $\mu$ with $\lambda_i < \mu_i$ for $i = 1, 2$ would give rise to larger $\alpha$ values. So, the lower bound on connectivity is given by $\nu = \frac{|\lambda|}{2} - \frac{1}{2} \sum_{j=1}^{t} 2 - 1$. Let $T$ denote the number of claws in $G$. Since $|\lambda| = 2|V(G)|$ and $t = T + (|v| - 2T) = \frac{|E|}{3} + (|V| - 2\frac{|E|}{3})$, $\nu$ simplifies to $|v| - (\frac{|E|}{3} + |v| - 2\frac{|E|}{3}) - 1 = \frac{|E|}{3} - 1$. From Theorem 3.11 the actual dimension of the 2-matching complex is $\frac{|E|}{3} + 1$, greater than the lower bound obtained from Theorem 3.12. \qed

4. Clawed Non-separable Graphs

Suppose we have a graph with potential attaching sites, i.e. vertices of degree 2. It is natural to ask, which of these degree 2 vertices are actually attaching sites with respect to some matching. In addition, once we start attaching leaves, how many can we attach before the 2-matching complex becomes contractible? To analyze these questions, we will focus our attention on clawed non-separable graphs. Our overall goal of this section will be to maximize the number of attaching sites in a clawed graph by pairing toggle edges in the graph.

Definition 4.1. A non-separable, i.e. 2-connected, graph is a connected graph in which the removal of any one vertex results in a connected graph.

Non-separable graphs can be classified through the following construction [4, Proposition 3.1.1]:

1. Begin with a graph $G := n$-cycle
2. Choose two vertices of $G$, say $v_1$ and $v_2$.
3. Attach the two endpoints of a path to $v_1$ and $v_2$ respectively.
4. Set $G$ to be this new graph and return to (2).

Using this construction we can define a clawed non-separable graph.

Definition 4.2. A clawed non-separable graph is a graph obtained through the following construction.

1. Begin with $G$ a clawed $n$-cycle, that is $G := C(C_n)$.
2. Choose two leaves of $G$, say $v_1$ and $v_2$.
3. For each endpoint $x$ in a path $P$, let one of the leaves attached to $x$ be an endpoint of the clawed path, $CP$. Attach the two endpoints of a clawed path to $v_1$ and $v_2$ respectively.
4. Set $G$ to be this new graph and return to (2).

We can use the construction of clawed non-separable graphs to get an understanding of the relationship between the number of claws in a clawed non-separable graph and the number of leaves. This will eventually lead us to finding an upper bound for the number of attaching sites in such a graph. Recall that an attaching site is a degree two vertex such that attaching a leaf does not affect the homotopy type of the resulting 2-matching complex.

Proposition 4.3. Let $T$ be the number of claws in a clawed non-separable graph and $L$ the number of leaves. Then $T$ and $L$ have the same parity modulo 2.
Figure 7.

Proof. It is clear that for the clawed graph of a non-separable $n$-cycle the parity of $T$ and $L$ is the same. Then, by construction two leaves are chosen, changing the number of leaves but keeping the parity the same. For each additional claw we add another leaf and the parity remains the same. □

A consequence of this proposition is that there is an even number of possible toggle edges that are not in induced claw units that contain a leaf. Our strategy for obtaining an upper bound for the maximum number of attaching sites will be to pair the toggle edges such that two paired toggle edges are incident to each other.

Theorem 4.4. Let $C(H)$ be the clawed graph of a non-separable graph $H$ such that $C(H)$ has $T$ claws. Then, the upper bound for the maximum number of leaves that can be added before changing the homotopy type of the 2-matching complex of a clawed non-separable graph is $T$.

Proof. The total number of possible attaching sites is given by $3T - L$ since each claw has three vertices with degree less than three, we need to remove the number of leaves since the degree is one, and then divide by two since all remaining vertices are the intersection of two claws. Now to find the maximum number of attaching sites we subtract away the minimum number of vertices that have at least one edge that is toggled on.

There is one toggled edge per claw and for any claw that has a leaf we can choose the leaf as the toggle edge, which will maximize the number of attaching sites since no additional leaf can be added to either endpoint of a leaf edge. The most ideal matching pairs the toggled edges, so minimally we have $T - L$ vertices that cannot be sites.

Hence, we have a maximum of $\frac{3T - L}{2} - \frac{T - L}{2} = \frac{2T}{2} = T$ attaching sites. □

The strategy in the proof of Theorem 4.4 was to pair toggle edges as a way to maximize the number of attaching sites. We provide two examples (Figures 7 and 8) in which the toggle edges are depicted with a solid line and the edges in the critical cell are depicted as double lines. In Figure 7, we have an example of a clawed non-separable graph together with a partial matching which attains the maximum number of attaching sites, namely 5.

It is not always the case that we can achieve the upper bound for the number of attaching sites for clawed non-separable graphs. In Figure 8 we see that after toggling on the leaf edges and doing our best to pair the inner toggle edges we are still left with two independent induced claw units that are surrounded by edges that are already in the critical cell. No matter which edge we choose in either of these induced claw units as the toggle edge, we will decrease the total number of possible attaching sites and thereby the number of possible attaching sites is less than the maximum.

We end this section with a constructible algorithm to obtain a maximal number of attaching sites in a clawed non-separable graph.

This constructible algorithm to obtain a maximal number of attaching sites prioritizes using leaf edges as toggle edges followed by pairing non-leaf toggle edges. Using Lemma 2.17 we may arbitrarily choose one of the three edges in each of our claws without changing the homotopy type
generated by the claw-induced partial matching. At each step we are bringing together as many of the toggle edges as possible to attain the maximal number of attaching sites. Figure 9 provides an example.

1. Begin with a clawed \( n \)-cycle and a claw decomposition \( C = \{c_1, \ldots, c_n\} \). Choose all the leaves as toggle edges such that all edges in the cycle are in the critical cell.

2. Choose two claws, \( c_i \) and \( c_j \) to attach the next clawed path. Notice that \( c_i \) and \( c_j \) are induced claw units that contain a leaf, which we call leaf-claws. Change the matching on these two leaf-claws so that:
   
   (a) For each of the chosen leaf claws \( c_i \) and \( c_j \): If the leaf claw is incident to a previously chosen or currently chosen leaf claw change the matching to match the toggle edges of these two leaf-claws, prioritizing the leaf claws incident to only one previously chosen leaf-claws. In doing so the number of attaching sites will either remain the same or increase.

3. For the new clawed path, let all of the leaves be the toggle edges.

4. Return to (2).

This algorithm returns the maximum number of attaching sites. Consider taking a claw-induced matching on a clawed non-separable graph. If it was possible to increase the number of attaching sites of by modifying this matching, one of two scenarios may be present:

(i) there exists a leaf-claw such that the toggle edge is not the leaf edge, or

(ii) there exists a pair of incident claws such that neither one has a toggle edge that is already incident to another toggle edge.

Through this algorithm, all leaves are toggle edges so (i) is not present. Notice that if (ii) appeared in this construction it would arise from step (2) of the algorithm when we add a new clawed path, but during that step we are re-orienting so that whenever possible toggle edges are incident to each other.

5. \( k \)-matching sequences

We now turn our attention to the relationship between 1-matchings and 2-matchings. Define a \( k \)-matching sequence as the sequence \((M_1(G), M_2(G), M_3(G), \ldots, M_n(G))\), up to homotopy, for \( 1 \leq k \leq n \) and where \( M_n(G) \) is a contractible space. The \( n \)-matching complex \( M_n(G) \) is a cone, hence contractible precisely when there is an edge \( e \in E(G) \) with both endpoints having max degree \( n \). In this section we will look at the \( k \)-matching sequence for wheel graphs.

Let \( W_n \) be a wheel graph on \( n \) vertices, that is a graph formed by connecting every vertex of a \( n - 1 \) cycle to a single universal vertex. Label the edges of the cycle with \( c_0, \ldots, c_{n-2} \) and inner edges by \( \ell_0, \ell_1, \ldots, \ell_{n-2} \), where \( \ell \) is used to symbolize “leg” edges, such that \( c_i \) shares a vertex with \( \ell_{i-1} \) and \( \ell_i \) modulo \( n - 1 \). See Figure 10.

We will determine the homotopy type of the 1-matching complex and 2-matching complex of wheel graphs. In the proof of Theorem 5.2 we will first focus on the “legs” or spokes of the wheel graphs.
Figure 9. On the left most picture we start with a clawed cycle. Choosing two points, \( v_1 \) and \( v_2 \) we attach a clawed path of length 2. Since the chosen two leaf claws are incident, we pair the toggle edges of each. Then we choose two more vertices, \( v_1 \) and \( v_2 \) and continue. In this step there is one claw unit that is incident to two leaf claws and the other claw unit is incident to one.

Figure 10. \( W_5 \) and the labeling used in Theorems 5.2 and 5.4.

and then on the outer cycle. In [10], Kozlov proves the following proposition which will come in handy.

**Proposition 5.1 (Kozlov, [10] Proposition 5.2).** For \( n \geq 1 \), let \( C_n \) denote the cycle of length \( n \). The homotopy type of the independence complex of the cycle graph is

\[
\text{Ind}(C_n) \simeq \begin{cases} 
S^{\nu_n} \lor S^{\nu_n} & n \equiv 0 \text{ mod } 3 \\
S^{\nu_n} & n \not\equiv 0 \text{ mod } 3.
\end{cases}
\]

where \( \nu_n = \lceil \frac{n-4}{3} \rceil \).

**Theorem 5.2.** Let \( W_n \) be a wheel graph on \( n \) vertices. Then, for \( k \in \mathbb{N} \), the homotopy type of \( M_1(W_n) \) is given by:

\[
M_1(W_n) \simeq \begin{cases} 
S^{\nu_n} \lor S^{\nu_n} & n \equiv 1 \text{ mod } 3 \\
\lor S^{\nu_n} & n \equiv 2 \text{ mod } 3 \\
\lor S^{\nu_n} & n \equiv 0 \text{ mod } 3
\end{cases}
\]

where \( \nu_n = \lceil \frac{n-4}{3} \rceil \).

**Proof.** The strategy of this proof will be to define a Matching Tree Algorithm on the line graph of \( W_n \), see Figure 11. The line graph of \( W_n \), denoted \( L(W_n) \) is given by a complete graph on \( n-1 \) vertices, labeled \( \ell_0, \ldots, \ell_{n-2} \) and an \( (n-1) \)-cycle graph \( c_0, \ldots, c_{n-2} \) with the additional edges \( \{c_j, \ell_{j-1}\} \) and \( \{c_j, \ell_j\} \) where \( j \) is calculated modulo \( n-1 \). We derive the homotopy type of \( M_1(W_n) \) by defining an acyclic (discrete Morse) matching on the face poset of the independence complex of \( L(W_n) \) using the Matching Tree Algorithm.
Let $P$ denote the face poset of $Ind(L(W_n))$. To begin we start with a tentative pivot $\ell_0$ which gives rise to two children $\Sigma(\emptyset, \ell_0)$ and $\Sigma(\ell_0, \ell_1, \ldots, \ell_{n-2}, c_0, c_1)$. We first address the right child $\Sigma(\ell_0; \ell_1, \ldots, \ell_{n-2}, c_0, c_1)$. The elements of $V \setminus (A \cup B)$ are $c_2, c_3, \ldots, c_{n-2}$. Since $c_2$ has exactly 1 neighbor in $V \setminus (A \cup B)$, use $c_2$ as a pivot leading to 1 child $\Sigma(\ell_0, c_2; \ell_1, \ldots, \ell_{n-2}, c_0, c_1, c_2, c_4)$ where $c_3$ is the matching vertex. Continue in this fashion consecutively choosing the pivot $c_f(2), c_f(3), \ldots, c_f(k)$ where $3k < n - 1$ and $f(i) = j + 3(i) \mod n - 1$, with $j$ the index on the tentative pivot of this branch, namely the index of $\ell_j$.

Notice $\frac{n-1}{3}$ is the number of groups of 3 that we can break the $(n - 1)$ cycle into, where each group consists of 1 pivot and 2 neighbors of that pivot. Hence, for $n \equiv 0 \mod 3$ and $n \equiv 2 \mod 3$ $\frac{n-1}{3}$ is not a whole number meaning that all vertices in the outer cycle are either in $A$ or $B$ at the time we reach $c_f(k)$. Therefore $\ell_0, c_f(1), c_f(2), \ldots, c_f(k)$ is the single critical cell of this branch.

When $n \equiv 1 \mod 3$, $\frac{n-1}{3}$ is a whole number and we have a group of 3 left over when we reach $c_f(k)$, 2 of which are already in $A$. Therefore, we have an isolated vertex and an empty leaf results, i.e. there are no critical cells of this branch.

Now, turning our attention to $\Sigma(\emptyset, \ell_0)$ we iterate this process using $\ell_1$ as our tentative vertex. Due to the symmetry of $L(W_n)$, each branch beginning with $\Sigma(\ell_j, N(\ell_j))$ will either result in an empty leaf or a single critical cell as described above. The general structure of our matching tree can be seen in Figure 12.

Once all vertices of the complete graph have been chosen as tentative vertices, we are left with one child $\Sigma(\emptyset, \ell_0, \ldots, \ell_{n-2})$ and $V \setminus (A \cup B)$ consists of only vertices on the outside cycle. When $n \equiv 0 \mod 3$, and $m = n - 1 \equiv 2 \mod 3$, Proposition 5.1 states there exists a matching tree with one critical cell of size $\nu_n + 1$. Additionally, from each of the other branches we have critical cells of size $\nu_n + 1$. By Theorem 2.11 the homotopy type is a wedge of spheres, $M_1(W_n) \simeq \bigvee_n S^{\nu_n}$ when $n \equiv 0 \mod 3$.

When $n \equiv 1 \mod 3$, and $m = n - 1 \equiv 0 \mod 3$, each of the branches resulting from vertices of the complete graph are empty. Hence, $M_1(W_n) \simeq M_1(C_m) = Ind(C_m) \simeq S^{\nu_n}$.

Finally, when $n \equiv 2 \mod 3$, and $m = n - 1 \equiv 1 \mod 3$, a subtle shift occurs. Notice that $\nu_n = \nu_n + 1$ when $m = n - 1$ so Proposition 5.1 says we have one critical cell of size $\nu_n$ and each of the $n - 1$ branches gives rise to a critical cell of size $\nu_n + 1$. We now argue that we can further match the cells $\alpha := \{\ell_{n-2}, c_f(1), \ldots, c_f(k)\}$ and $\beta := \{c_f(0), \ldots, c_f(k)\}$. We do so by showing that there exists a linear extension with $u(\beta) = \alpha$, which by Theorem 2.12 gives us that there is an acyclic matching with $\alpha$ and $\beta$ paired, as desired.

First note that $\{\ell_{n-2}, c_f(1), \ldots, c_f(k)\}$ is a facet in the independence complex of $L(W_n)$ for $n \equiv 3 \mod 3$ which means it is a maximal element of the face poset. Since $\beta < \alpha \in P$, $\beta$ is a coatom.

We claim for any pair $(x, u(x))$ for which $\beta <_P x$ or $\beta <_P u(x)$ (i.e. $\beta <_L (x, u(x))$, $\alpha$ is incomparable to $x$ and to $u(x)$. If $\beta <_P x <_P u(x)$, then $\alpha$ is incomparable to $x$ and incomparable
to \( u(x) \) since \( \beta \prec \alpha \) and \( \alpha \) is maximal. Suppose \( \beta \) is incomparable to \( x \) and \( \beta \prec_P u(x) \). Since \( \beta \prec \alpha \), \( \alpha \) is incomparable to \( u(x) \). Since \( \beta \) is incomparable to \( x \), \( \beta \prec u(x) \), and \( x \prec u(x) \) it must be that \( \beta \cup x \subseteq u(x) \). In addition, \( \alpha \) and \( \beta \) differ by 1 element and if \( x \prec \alpha \) this would mean \( \alpha = \beta \cup x \) which is a contradiction to the incomparability of \( u(x) \).

This means that any pair \((x, u(x))\) in \( \mathcal{L} \) such that \( \beta \prec (x, u(x)) \) can be moved above \( \alpha \). The only concern is if there exists elements \((y, u(y))\) such that \((y, u(y)) \prec_L (x, u(x)) \) and \((y, u(y)) \succ_L (x, u(x)) \) but this is not possible as this means \((y, u(y)) \succ_L (x, u(x)) \) and we have seen \((y, u(y)) \) is incomparable to \( \alpha \).

Finally, we note that for any pair \((y, u(y))\) such that \((y, u(y)) \prec_L \alpha \), we have seen \( \beta \not\prec_L (y, u(y)) \) and therefore it is either the case that \((y, u(y)) \) is incomparable to \( \beta \) or \((y, u(y)) \prec_L \beta \).

Hence, we can rearrange \( \mathcal{L} \) so that \( u(\beta) = \alpha \) which implies pairing \( \alpha \) and \( \beta \) forms an acyclic matching. It follows from Theorem 2.9 that this homotopy type for \( M_1(W_n) \simeq \bigvee_{n-2} S^n \). □

The next theorem show that for \( n \geq 6 \), \( M_2(W_n) \) is contractible. We need the following lemma [8, Lemma 4.3]:

**Lemma 5.3.** Let \( \Delta_0 \) and \( \Delta_1 \) be disjoint families of subsets of a finite set such that \( \tau \not\subseteq \sigma \) if \( \sigma \in \Delta_0 \) and \( \tau \in \Delta_1 \). If \( \mathcal{M}_i \) is an acyclic matching on \( \Delta_i \) for \( i = 0, 1 \) then \( \mathcal{M}_0 \cup \mathcal{M}_1 \) is an acyclic matching on \( \Delta_0 \cup \Delta_1 \).

**Theorem 5.4.** Let \( W_n \) be a wheel graph on \( n \) vertices. Then, for \( k \in \mathbb{N} \), the homotopy type of \( M_2(W_n) \) is given by:

\[
M_2(W_n) \simeq \begin{cases} 
S^2 \vee S^2 \vee S^2 & n = 4 \\
S^3 \vee S^3 & n = 5 \\
pt & n \geq 6.
\end{cases}
\]

**Proof.** Let \( P_n \) be the face poset of \( M_2(W_n) \). See figure 10 for an example of the labeling of \( W_n \). Our strategy will be to define acyclic matchings on subposets of \( P_n \) and then apply Theorem 2.11. Define \( Q_n \) to be a poset on the elements \( \{c_0, c_2, R\} \) given by the relations \( c_0 \prec c_2 \prec R \). The
sets \( \Gamma_n \) are in bold to differentiate them from vertices of \( W_n \). Now, we define the poset map \( \Gamma_n : P_n \to Q_n \) by defining the preimage \( \Gamma_n^{-1}(\alpha) \) for each \( \alpha \in Q_n \).

- For \( n = 4 \) let \( \Gamma_n^{-1}(R) := \{ \{c_1, c_2, \ell_2\}, \{c_2, \ell_2, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_2, \ell, \ell_2\} \} \).
- For \( n \geq 5 \) let \( \Gamma_n^{-1}(R) := \{ m \in M_2(W_n) | \{c_1, \ell_0, \ell_1\} \subseteq m \text{ or } \{c_1, \ell_0, \ell_3, \ell_2\} \subseteq m \text{ or } \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\} \subseteq m \} \).
- \( \Gamma_n^{-1}(c_2) := \{ m \in M_2(W_n) | \{c_1, \ell_0\} \subseteq m \text{ or } \{c_{n-2}, \ell_{n-2}\} \subseteq m \} \setminus \Gamma_n^{-1}(R) \).
- \( \Gamma_n^{-1}(c_0) := \{ m \in M_2(W_n) | \{c_0\} \subseteq m \text{ or } m \cup \{c_0\} \in M_2(W_n) \} \).

Since every 2-matching of \( W_n \) either contains \( c_0, \{c_1, \ell_0\} \), or \( \{c_{n-2}, \ell_{n-2}\} \), elements of \( P_n \) have been assigned an image under \( \Gamma_n \) and, by definition, \( \Gamma_n \) is order-preserving poset map. For the preimages \( \Gamma_n^{-1}(c_0) \) and \( \Gamma_n^{-1}(c_1) \) perform a toggle on \( c_0 \) and \( c_1 \), respectively. That is, for each \( \sigma \in \Gamma_n^{-1}(\alpha) \) that does not contain \( \alpha \), pair \( \sigma \) with \( \sigma \cup \{ \alpha \} \). By Lemma 2.13, these matchings are acyclic. In addition, both of these toggles result in a perfect (discrete Morse) matching. Notice that what remains are the elements of \( \Gamma_n^{-1}(R) \) which is a set of disjoint subposets for \( n \geq 5 \) where each of the sets \( \{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\} \), and \( \{c_{n-2}, \ell_{n-2}, c_3, \ell_2\} \) are the minimal vertices of the respective subposets. Since the (poset) join between any two of these elements would contain more than two leg edges, which is not possible in a 2-matching, these posets are pairwise disjoint.

**Claim:** Each subposet either consists of 1 element or is associated to a contractible subcomplex for \( n \geq 4 \).

Recall that any subset of edges in a disjoint union of paths forms a 2-matching. Each of the sets \( \{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\} \), and \( \{c_{n-2}, \ell_{n-2}, c_3, \ell_2\} \) contains two leg edges and two cycle edges. Hence, the possible edges that we union with any of these elements to form a 2-matching form a disjoint union of paths when \( n \geq 6 \). When \( n \geq 7 \), toggling on \( c_4 \) will pair away all of the remaining cells since \( c_4 \) can be in any 2-matching containing the sets \( \{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_{n-2}, \ell_{n-2}, c_1, \ell_1\} \), and \( \{c_{n-2}, \ell_{n-2}, c_3, \ell_2\} \). For \( n = 6 \), toggles can be made with \( c_1, c_3 \) and \( c_4 \). Therefore, by Lemma 5.3, \( M_2(W_6) \simeq \text{pt} \) when \( n \geq 6 \).

When \( n = 5 \), \( \Gamma_n^{-1} = \{ \{c_1, \ell_1, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_1, \ell_0, c_3, \ell_2\}, \{c_1, \ell_0, c_3, \ell_3\}, \{c_1, \ell_1, c_3, \ell_3\} \} \). Toggling on \( c_1 \) and \( c_3 \) leaves 2 critical 3-cells, namely \( \{c_1, \ell_0, \ell_2, c_3\}, \{c_1, \ell_1, \ell_3, c_3\} \). Hence, \( M_2(W_5) \simeq S^3 \vee S^3 \). When \( n = 4 \), \( \Gamma_n^{-1}(R) := \{ \{c_1, \ell_2, c_2\}, \{c_2, \ell_2\}, \{c_2, \ell_2, \ell_0\}, \{c_1, \ell_0, \ell_1\}, \{c_2, \ell_1, \ell_2\} \} \) and toggling on \( c_1 \) leaves 3 critical 2-cells \( \{c_1, \ell_0, \ell_1\}, \{c_2, \ell_1, \ell_2\} \), and \( \{c_2, \ell_2, \ell_0\} \). Hence, \( M_2(W_4) = S^2 \vee S^2 \vee S^3 \).

Since \( M_3(W_n) \simeq \text{pt} \), we have the \( k \)-matching sequence of \( W_4 \) is \((S^2 \vee S^2; S^2 \vee S^2 \vee S^2; \text{pt}) \) and for \( W_5 \) is \( (\vee S^2; S^3 \vee S^3; \text{pt}) \) where \( n = \lceil \frac{n-4}{3} \rceil \).

### 6. Caterpillar graphs

A **caterpillar graph** is a tree in which every vertex is on a central path or only one edge away from the path. A perfect \( m \)-caterpillar of length \( n \), denoted \( G_n \) is a caterpillar graph with \( m \) legs at each vertex on the central path of \( n \) vertices (see Figure 13). We conclude the paper with a derivation of 2-matching complexes of perfect \( m \)-caterpillar graphs.

![Figure 13. A perfect m-caterpillar or length n.](image)

In [7], Jelić Milutinović et. al. calculate the homotopy type of \( M_1(G_n) \) using topological techniques.
Theorem 6.1. [7] Theorem 5.4] For \( m \geq 2 \), let \( G_n \) be a perfect \( m \)-caterpillar graph of length \( n \geq 1 \). Then the homotopy type of \( M(G_n) \) is given by:

\[
M(G_n) \simeq \begin{cases} 
\bigvee_{t=0}^{k} S^{k-1+t} & \text{if } n = 2k \\
\bigvee_{t=0}^{k} S^{k+t} & \text{if } n = 2k + 1
\end{cases}
\]

where \( \alpha_t = \binom{k+t}{k} (m-1)^{2t} \) and \( \beta_t = \binom{k+1+t}{k-t} (m-1)^{2t+1} \).

As we will now see the homotopy type of \( M_2(G_n) \) is also a wedge of spheres.

Definition 6.2. Let \( G_n \) be a perfect \( m \)-caterpillar of length \( n \) with the right most edge along the central path \( e = \{x_0, x_1\} \). Define \( BD(G_n) \) as the simplicial complex whose vertices are given by edges in \( G_n \) and faces are given by subgraphs \( H \) of \( G_n \) such that the \( \deg(x_1) \leq 1 \) and the degree of any other vertex is at most 2 in \( H \).

In order to obtain the 2-matching complex of \( G_n \), we will inductively use the bounded degree complex \( BD(G_{n-1}) \) to build up to \( M_2(G_n) \). Namely, our progression will be:

\[
M_2(G_{n-1}) \to BD(G_n) \to M_2(G_n) \to BD(G_{n+1}) \to M_2(G_{n+1}) \to \ldots.
\]

Notice that the only difference between \( BD(G_n) \) and \( M_2(G_n) \) is the possible degree of the last vertex on the central path. This will allow us to build an inductive argument on \( m \)-perfect caterpillar graphs.

Lemma 6.3. \( BD(G_n) \cong \Sigma_m(M_2(G_{n-1})) \vee \Sigma(BD(G_{n-1})) \)

Proof. Let \( m \) denote the number of legs off of each vertex along the central path as seen in Figure 13. For a bounded degree complex \( BD(G_n) \), let \( e = \{x_0, x_1\} \) be the right most edge along the central path and consider subgraphs \( H \) such that \( \deg(x_1) \leq 1 \) in \( H \). We can decompose these bounded degree subgraphs into those that contain \( e \) and those that do not. Namely, if we exclude \( e \), the bounded degree graphs are given by \( M_2(G_{n-1}) \ast M_1(St_m) \) where \( St_m \) is a star graph on \( m \) edges, and if we include \( e \) the bounded degree subgraphs are given by \( e \ast BD(G_{n-1}) \). These two complexes share \( BD(G_{n-1}) \) as a common subcomplex and hence

\[
BD(G_n) \cong M_2(G_{n-1}) \ast M_1(St_m) \bigcup_{BD(G_{n-1})} e \ast BD(G_{n-1}).
\]

Since \( e \ast BD(G_{n-1}) \) is a contractible space we get

\[
BD(G_n) \cong M_2(G_{n-1}) \ast M_1(St_m) \bigcup_{BD(G_{n-1})} e \ast BD(G_{n-1}) / e \ast BD(G_{n-1}) \cong \Sigma_m(M_2(G_{n-1}))/BD(G_{n-1}),
\]

where \( \Sigma_m(X) \) is \( X \) join a set of \( m \) discrete points. Since \( BD(G_{n-1}) \subseteq M_2(G_{n-1}) \) we see that \( BD(G_{n-1}) \) is contractible in \( \Sigma_m(M_2(G_{n-1})) \). Hence,

\[
BD(G_n) \simeq \Sigma_m(M_2(G_{n-1})) \vee \Sigma(BD(G_{n-1})).
\]

\[ \square \]

Lemma 6.4. \( M_2(G_n) \cong M_2(G_{n-1}) \ast M_2(St_m) \vee \Sigma_m(BD(G_{n-1})) \)

Proof. Let \( m \) be the number of legs off each vertex of the central path as seen in Figure 13. For the 2-matching complex \( M_2(G_n) \), let \( e = \{x_0, x_1\} \) be the right most edge along the central path and consider 2-matchings of \( G_n \). Following the argument analogously to Lemma 6.3 we can decompose these 2-matchings into those that contain \( e \) and those that do not. Hence,

\[
M_2(G_n) \cong M_2(G_{n-1}) \ast M_2(St_m) \bigcup_{BD(G_{n-1}) \ast M_1(St_m)} e \ast BD(G_{n-1}) \ast M_1(St_m).
\]
Using equations 2 and 3, we see that
\[ M_2(G_n) \cong M_2(G_{n-1}) * M_2(S_{t=0}) / BD(G_{n-1}) * M_1(S_{t=0}). \]
Further, since \( BD(G_{n-1}) \subseteq M_2(G_{n-1}) \) and \( M_1(S_{t=0}) \subseteq M_2(S_{t=0}) \) we get that \( BD(G_{n-1}) * M_1(S_{t=0}) \subseteq M_2(G_{n-1}) * M_2(S_{t=0}) \) is contractible and \( M_2(G_n) \cong M_2(G_{n-1}) * M_2(S_{t=0}) \vee \Sigma(\Sigma_m(BD(G_{n-1}))) \).

**Theorem 6.5.** Let \( G_n \) denote a perfect \( m \)-caterpillar graph of length \( n \). Then,
\[ (i) \] the homotopy type of \( BD(G_n) \) and \( M_2(G_n) \) are wedges of spheres of varying dimensions for all \( n \geq 1 \).
\[ (ii) \] the total number of spheres in \( BD(G_{i+1}) \) and \( M_2(G_{i+1}) \) is given by the coefficient of \( t^i \) in the series
\[
\sum_{i \geq 0} A_i t^i = \sum_{j \geq 0} B_j t^j = \frac{x}{1 - (1 + y)t - (x^2 - y)t^2}
\]
where \( x = (m - 1) \) and \( y = \binom{m-1}{2} \), and
\[ (iii) \] \( M_2(G_i) \cong \bigvee_{j \geq 0} S^{i+j} \) where \( \beta_{i,j} \) the number of spheres of dimension \( i+j \) is the coefficient of \( r^i t^j \) in \( B(r,t,x,y) = \sum_{i,j \geq 0} b_{i,j} r^i t^j = \frac{x}{1 - r t - (x^2 - y)r t^2} \) where \( x = (m - 1) \) and \( y = \binom{m-1}{2} \).

**Proof.** (i) Since \( BD(G_1) = M_1(S_{t=0}) \cong \bigvee_{(m-1)} S^0 \) and \( M_2(G_1) = M_2(S_{t=0}) \cong \bigvee_{(m-1)} S^1 \), (i) follows from Lemmas 6.3, 6.4 and 2.21.

(ii) Let \( A_i \) denote the total number of spheres in the homotopy type of \( BD(G_{i+1}) \) and \( B_i \) be the total number of spheres in the homotopy type of \( M_2(G_{i+1}) \). From Lemmas 6.3, 6.4 and 2.21 we know \( A_0 = x := (m - 1), B_0 = y := \binom{m-1}{2}, \) and \( A, B \) follow the recursions:
\[ (2) \quad A_i = A_{i-1} + xB_{i-1} \]
\[ (3) \quad B_i = xA_{i-1} + yB_{i-1}. \]
Using equations 2 and 3 we see that \( A_i = (1 + y)A_{i-1} + (x^2 - y)A_{i-2} \). Let \( A(t) = \sum_{i \geq 0} A_i t^i. \)

Multiplying by \( (1 - (1 + y)t - (x^2 - y)t^2) \) and solving we obtain
\[ A(t) = \frac{x}{1 - (1 + y)t - (x^2 - y)t^2}. \]
The argument for \( B(t) = \sum_{j \geq 0} B_j t^j \) is analogous.

(iii) Let \( \alpha_{i,j} \) be the total number of spheres of dimension \( j \) in \( BD(G_{i+1}) \) and \( \beta_{i,j} \) the total number of spheres of dimension \( j \) in \( M_2(G_{i+1}) \). Using that \( BD(G_1) \cong \bigvee_{(m-1)} S^0 \) and \( M_2(G_1) \cong \bigvee_{(m-1)} S^1 \), and Lemmas 6.3, 6.4 and 2.21 we obtain the following initial conditions
\[ \alpha_{0,0} = x := (m - 1) \]
\[ \beta_{0,1} = y := \binom{m-1}{2} \]
\[ \alpha_{0,j} = 0 \quad \text{for} \quad j \geq 1 \]
\[ \beta_{0,j} = 0 \quad \text{for} \quad j \geq 2 \]
\[ \alpha_{i,0} = 0 \quad \text{for} \quad i \geq 1 \]
\[ \beta_{i,0} = 0 \quad \text{for} \quad i \geq 0 \]
Additionally, \( \alpha_{i,j} \) and \( \beta_{i,j} \) follow the recursions
\[ (4) \quad \alpha_{i,j} = \alpha_{i-1,j-1} + x(\beta_{i-1,j-1}) \]
\[ (5) \quad \beta_{i,j} = x\alpha_{i-1,j-2} + y\beta_{i-1,j} - 2. \]
Using equations 4 and 5 we can see that
\[ \beta_{i,j} = \beta_{i-1,j-1} + (x^2 - y)(\beta_{i-2,j-3} + y(\beta_{i-2,j-3}). \]
Let \( B(r, t, x, y) = \sum_{i,j \geq 0} b_{i,j} r^i t^j \) and multiply by \( 1 - rt - (x^2 - y)r^2 t^3 - yrt^2 \). When we solve and use the initial conditions we find that
\[ B(r, t, x, y) = \frac{x}{1 - rt - (x^2 - y)r^2 t^3 - yrt^2} \]
and the result follows from substituting \( (m-1) \) for \( x \) and \( \binom{m-1}{2} \) for \( y \).

**Remark 6.6.** From Theorem 6.5 (iii), notice that the number of spheres in each dimension is given by a polynomial in \( x \) and \( y \). If we set \( x = y = 1 \), we can see that the number of terms in the sum given by the coefficient of \( r^i t^j \) is a binomial coefficient:
\[ B(r, t, 1, 1) = \frac{1}{1 - rt(1 + t)} = \sum_{k \geq 0} r^k t^k (1 + t)^k \]
and the coefficient of \( [r^i t^j] \) = \( \binom{i}{j} \).

7. **Future directions.**

The original motivation for this project was to study 1-matching complexes through the lens of \( k \)-matching complexes for \( k \geq 2 \). We end with a few open questions. Our exploration of 2-matching complexes led to observations about the flexibility of the homotopy type and how the homotopy type of clawed non-separable graphs change (or doesn’t change) as new leaves are added. One avenue to explore with this problem involves understanding the interaction between clawed non-separable graphs and additional leaves.

**Question 7.1.** Ranging over all clawed non-separable graphs, what is the average maximum number of leaves that can be added without affecting the homotopy type of the resulting 2-matching complex?

We have already seen that there are some graphs in which the maximum can be obtained and other graphs in which there is an obstruction to doing so. It would be interesting to know if clawed non-separable graphs tend to have structural properties that obstruct obtaining the maximum and can we expect the maximum number of leaves to be evenly distributed over all such graphs.

We can also ask about properties of graphs more generally.

**Question 7.2.** Given a graph, how can we determine when leaves can be attached without affecting the resulting homotopy type of the 2-matching complex?

In Section 5, we defined the \( k \)-matching complex of a graph and explored two examples, wheel graphs and perfect caterpillar graphs. Theorems 6.1 and 6.5 show that the homotopy type of \( M_1(G_n) \) and \( M_2(G_n) \) are both wedges of spheres with combinatorial structure. A future direction of this work would be to further understand the \( k \)-matching complex of perfect caterpillar graphs and caterpillar graphs in general.

**Conjecture 7.3.** The \( k \)-matching complex of caterpillar graphs are homotopy equivalent to a wedge of spheres.

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