QUASI-LOCAL MASS AND THE EXISTENCE OF HORIZONS

YUGUANG SHI\textsuperscript{1} AND LUEN-FAI TAM\textsuperscript{2}

Abstract. In this paper, we obtain lower bounds for the Brown-York quasilocal mass and the Bartnik quasilocal mass for compact three manifolds with smooth boundaries. As a consequence, we derive sufficient conditions for the existence of horizons for a certain class of compact manifolds with boundary and some asymptotically flat complete manifolds. The method is based on analyzing Hawking mass and inverse mean curvature flow.

1. Introduction

In this work, we will discuss the relations between different kinds of quasi-local mass and use them to derive sufficient conditions for the existence of horizons in the time symmetric case. In the time symmetric case, a horizon is defined to be a compact minimal surface which is the boundary of some open sets (see §3 for more precise definition).

In 1972, Thorne made the following conjecture, which later became known as the hoop conjecture (see [8]): Black holes with horizons form when and only when a mass $M$ gets compacted into a region whose circumference in every direction is $C \leq 4\pi M$. The conjecture is loosely formulated. Several concepts such as mass, circumference etc. are not clearly defined. Hence, this conjecture allows many different precise interpretations.

In 1983, Schoen and Yau in [20], and later Yau in [24] define a kind of radius of a bounded region, and derive an upper bound for this radius for a region in spacetime without apparent horizons in terms of the lower bound of mass density. By this and the study of the obstruction to the existence of regular solution to the Jang equation, they obtain some important results for the existence of black holes in the spirit of the hoop conjecture.

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In the time symmetric case, the method does not work because the Jang equation always has a solution. For this case, we will use the inverse mean curvature flow of Huisken and Ilmanen [12]. We will define a quantity \( m(\Omega) \) on a compact three manifold \( \Omega \) with boundary which involves Hawking mass of some subsets of \( \Omega \) and some other geometric quantities of \( \Omega \), see definition in (2.19). Then we compare this with the Brown-York mass \( m_{BY}(\partial \Omega) \) of \( \partial \Omega \) of a simply connected compact manifold with nonnegative scalar curvature and with connected smooth boundary which has positive Gauss curvature and positive mean curvature with respect to the outward normal. It is well-known that \( \partial \Omega \) can be isometrically embedded in \( \mathbb{R}^3 \). The first result is as follows: If \( \Omega \) contain no horizons, then \( m_{BY}(\partial \Omega) \geq m(\Omega) \). From this one can prove that if \( m(\Omega) \geq 2R \) where \( R \) is the radius of the smallest circumscribed ball of \( \partial \Omega \) in \( \mathbb{R}^3 \), then \( \Omega \) must contains a horizon. In particular, if \( m(\Omega) \geq 2 \text{diam}(\partial \Omega) \), then \( \Omega \) contains a horizon. Examples satisfying these conditions are given.

We will prove that if \( \Omega \) contains a round sphere \( \partial E \) in the sense of [7] such that its Hawking mass is larger than \( m_{BY}(\partial \Omega) \), then \( \Omega \) also contains a horizon. However, it is unclear if one can find a round sphere satisfying the condition.

It turns out that the quasi-local mass \( m_B(\Omega) \) (see the definition in §3) introduced by Bartnik for a compact manifold \( (\Omega, g) \) with smooth boundary with an admissible extension is also bounded below by \( m(\Omega) \). It was observed by Walter Simon (see [3]) that if \( (\Omega, g) \) has an admissible extension and suppose \( (\Omega, g) \) is isometrically embedded in an asymptotically flat and complete manifold \( M \) with nonnegative scalar curvature so that the ADM mass \( m_{ADM}(M) \) of \( M \) is less than \( m_B(\Omega) \), then \( M \) must contains a horizon. Our lower bound for \( m_B(\Omega) \) implies that if \( m_{ADM}(M) < m(\Omega) \), then \( M \) must contain a horizon.

We should emphasis that the quantity \( m(\Omega) \) defined in (2.19) is nontrivial, in the sense that \( m(\Omega) \geq 0 \) and is zero only if it is locally flat. One can prove that it is actually a domain in \( \mathbb{R}^3 \) in some cases.

The basic outline of the paper is as follows. In Section 1, we first construct examples to motivate the definition of \( m(\Omega) \). Then we prove the positivity of \( m(\Omega) \) (see Theorem 1.1).

In Section 2, we compare \( m(\Omega), m_{BY}(\Omega) \) to give sufficient conditions for the existence of horizons for compact manifold with boundary as mentioned above. We will give another lower bound for the Brown-York mass. In particular, we prove that the Hawking mass of the boundary is dominated by the Brown-York mass of the domain.

In Section 3, we prove that \( m(\Omega) \) is bounded above by the Bartnik mass \( m_B(\Omega) \) for \( \Omega \) which has an admissible extension. We will
also discuss some properties of Bartnik quasi-local mass including the conjecture of Bartnik that $m_B(\Omega)$ is realized by a static admissible extension, see [3] for more details of the conjecture.

The main analytical tool in this paper is the inverse mean curvature flow which has been studied by Huisken and Ilmanen [12]. We obtain our results by studying the obstruction for the monotonicity of Hawking mass under this flow.

2. Hawking mass of subsets of a domain

In this section, we will introduce a quantity involving Hawking mass of some subsets of a compact three manifold $(\Omega, g)$ with smooth boundary which will be used to give a condition for the existence of stable minimal spheres on a compact manifold with boundary. All Riemannian manifolds in this work are assumed to be oriented and connected with dimension three.

To motivate the definition, let us construct some examples.

**Proposition 2.1.** There exist asymptotically flat metrics $g_1, g_2$ with nonnegative scalar curvature on $\mathbb{R}^3$ such that $g_1 = g_2$ outside some compact set, $(\mathbb{R}^3, g_1)$ contains a stable minimal sphere but $(\mathbb{R}^3, g_2)$ does not contain any compact minimal surfaces.

Recall that an asymptotically flat (AF) three manifold $(M, g)$ with one end is a complete manifold with nonnegative scalar curvature which is in $L^1(M)$ such that for some compact set $K$ of $M$, $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates in $\mathbb{R}^3$, the metric $g$ satisfies:

\begin{equation}
(2.1) \quad g_{ij} = \delta_{ij} + b_{ij}
\end{equation}

with

\begin{equation}
(2.2) \quad ||b_{ij}|| + r||\partial b_{ij}|| + r^2||\partial \partial b_{ij}|| = O(r^{-1})
\end{equation}

where $r$ and $\partial$ denote the Euclidean distance and standard derivative operator on $\mathbb{R}^3$.

Recall also that the ADM mass of $M$ is defined as

\begin{equation}
(2.3) \quad m_{ADM}(M) = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ii,j} - g_{ij,i})\nu^j d\sigma_r
\end{equation}

where $S_r$ is the Euclidean sphere, $\nu$ is the outward unit normal of $S_r$ in $\mathbb{R}^3$ and the derivatives are taken with respect to the Euclidean metric.
Proof of Proposition 2.1. We first construct metric $g_2$ which contains no compact minimal surface such that $g_2$ is Schwarzschild near infinity. Let $m > 0$ be any positive constant, $\rho_0 > m$, and let $\rho_1 > \rho_0$. Fix a smooth nonincreasing function $h$ on $(0, \infty)$ such that $h(r) = 0$ for any $r \in (0, \rho_0)$, $h(r) = -\frac{m^2}{2}$ for any $r \in (\rho_1, \infty)$. Let $c_0 = 1 + \frac{m}{2\rho_1}$. Define a function $u$ on $\mathbb{R}^3$ as:

\[ u(x) = c_0 + \int_{\rho_1}^{r} \frac{h(\tau)}{\tau^2} d\tau. \]  

for $|x| = r$. If $|x| = r \leq \rho_0$, then

\[ u(x) = c_0 - \int_{\rho_0}^{\rho_1} \frac{h(\tau)}{\tau^2} d\tau > 0 \]

because $h \leq 0$. If $|x| = r \in (\rho_0, \rho_1)$, then

\[ u(x) = 1 + \frac{m}{2\rho_1} - \int_{r}^{\rho_1} \frac{h(\tau)}{\tau^2} d\tau > 0. \]

If $|x| = r \geq \rho_1$, then $u(x) = 1 + \frac{m}{2r}$. Hence $u$ is smooth and positive. Moreover,

\[ \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = \frac{h'}{r^2} \leq 0. \]

Let $g_2 = u^4 g_0$, where $g_0$ is the Euclidean metric. Then $(\mathbb{R}^3, g_2)$ is an AF manifold which is Schwarzschild near infinity.

To prove that $(\mathbb{R}^3, g_2)$ contains no compact minimal surfaces, it is sufficient to show that each Euclidean sphere with center at the origin has positive mean curvature in $g_2$. Let $H$ be the mean curvature of the Euclidean sphere $S_r = \{|x| = r\}$ with respect to $g_2$. Then

\[ H = \frac{1}{u^2} \left( \frac{2}{r} + \frac{4}{u} \frac{\partial u}{\partial r} \right) = \frac{2}{ru^3} \left( u + 2r \frac{\partial u}{\partial r} \right). \]
If \( r \leq \rho_0 \), then \( u' = \frac{\partial u}{\partial r} = 0 \). Since that \( u > 0 \) is positive, we have \( H > 0 \). If \( r \geq \rho_0 \), then

\[
2ru' + u = \frac{2h(r)}{r} + 1 + \frac{m}{2\rho_1} + \int_{\rho_1}^{r} \frac{h(\tau)}{\tau^2} d\tau
\]

\[
= \frac{2h(r)}{r} + 1 + \frac{m}{2\rho_1} + h(\xi) \left( \frac{1}{\rho_1} - \frac{1}{r} \right)
\]

\[
\geq \frac{2h(r)}{r} + 1 + \frac{m}{2\rho_1} + h(r) \left( \frac{1}{\rho_1} - \frac{1}{r} \right)
\]

(2.6)

\[
= 1 + \frac{m}{2\rho_1} + h(r) \left( \frac{1}{\rho_1} + \frac{1}{r} \right)
\]

\[
= 1 + \frac{m}{2\rho_1} - \frac{m}{2} \left( \frac{1}{\rho_1} + \frac{1}{r} \right)
\]

\[
\geq 1 - \frac{m}{2\rho_0}
\]

\[
> 0
\]

for some \( \xi \) between \( r \) and \( \rho_1 \), here we have used the fact that \( h \) is nonincreasing, \( h \geq -\frac{m}{2} \) and \( r \geq \rho_0 > m \). Hence we also have \( H > 0 \).

In [17], Miao has constructed a scalar flat AF metric \( g_1 \) on \( \mathbb{R}^3 \) which contains a stable minimal sphere and is Schwarzschild at infinity. By rescaling, we see that the proposition is true. However, because of later application, we will construct \( g_1 \) directly using similar method as in the construction of \( g_2 \).

Basically, we glue the standard sphere to the Schwarzschild manifold. Under the stereographic projection, the metric on the sphere minus a point is:

(2.7) \[
ds^2_{S^2} = \frac{1}{(1 + \frac{m}{4r^2})^2} (d\rho^2 + \rho^2 d\sigma^2)
\]

where \( d\rho^2 + \rho^2 d\sigma^2 \) is the standard Euclidean metric. On the other hand the Schwarzschild metric is given by

(2.8) \[
ds^2_{Sch} = (1 + \frac{m}{2r})^4 (dr^2 + r^2 d\sigma^2),
\]

defined on \( \mathbb{R}^3 \) minus the origin, where \( m > 0 \) is a constant. We need to rescale the metric so that the compact minimal surface \( \{ r = 1/2m \} \) is near \( \rho = \infty \) in the metric (2.7). Namely, let \( r = \epsilon \rho \), then

(2.9) \[
ds^2_{Sch} = \epsilon^2 (1 + \frac{m}{2\epsilon \rho})^4 (d\rho^2 + \rho^2 d\sigma^2),
\]
For \( \rho_0 > 0 \), define

\[
(2.10) \quad k(\rho) = \begin{cases} 
-\frac{\epsilon}{4}(1 + \frac{1}{4}\rho^2)^{-\frac{3}{2}}, & \rho \leq \rho_0; \\
-\frac{m}{2}\rho^{-2}, & \rho \geq 2\rho_0.
\end{cases}
\]

We want to find \( \epsilon > 0 \) and \( \rho_0 \) so that \( k \) can be defined to be nonincreasing. Let \( \epsilon = m^2/64, \rho_0 = 1/(4m) \). Then

\[
k(\rho_0) = -\frac{32m^2}{(1 + 64m^2)^{\frac{3}{2}}}.
\]

\[
k(2\rho_0) = -64m^2.
\]

Hence there exists \( m_0 > 0 \) such that \( k(\rho_0) > k(2\rho_0) \) for all \( 0 < m < m_0 \). For such \( m \) we can define \( k \) satisfying (2.10), and is smooth and nonincreasing. Next define

\[
(2.11) \quad u_m(x) = b_0 + \int_0^\rho k(\tau)d\tau
\]

if \( |x| = \rho \), where \( b_0 \) is chosen such that \( u_m(x) = \sqrt{\epsilon}(1 + m/(2\epsilon\rho)) \) for \( \rho \geq 2\rho_0 \). More precisely,

\[
b_0 = \frac{m}{4\sqrt{\epsilon}\rho_0} + \sqrt{\epsilon} - \int_0^{2\rho_0} k(\tau)d\tau
\]

\[
= m\left(\frac{65}{8} - \frac{8}{\sqrt{1 + 64m^2}}\right) + 1 - \int_0^{2\rho_0} k(\tau)d\tau
\]

Note that

\[
0 \geq \int_0^{2\rho_0} k(\tau)d\tau \geq -16m
\]

Hence

\[
(2.13) \quad u_m(x) = \begin{cases} 
-\frac{m}{8}\left(-\frac{8}{\sqrt{1 + 64m^2}}\right) - \int_0^{2\rho_0} k(\tau)d\tau + \left(1 + \frac{1}{4}\rho^2\right)^{-\frac{3}{2}}, & \rho \leq \rho_0; \\
m\left(\frac{65}{8} - \frac{8}{\sqrt{1 + 64m^2}}\right) + 1 - \int_0^\rho k(\tau)d\tau, & \rho_0 \leq \rho \leq 2\rho_0; \\
\sqrt{\epsilon}(1 + \frac{m}{2\rho}), & \rho \geq 2\rho_0,
\end{cases}
\]

for \( |x| = \rho \). Hence choosing a smaller \( m_0 > 0 \), we have \( u_m(x) > 0 \) for all \( x \) if \( 0 < m < m_0 \). As before, since \( k < 0 \) and \( k' \leq 0 \), we have \( \Delta u_m \leq 0 \). The metric \( ds_m^2 = u_m^4(d\rho^2 + \rho^2d\sigma^2) \) is an AF metric with nonnegative scalar curvature such that on \( \rho \geq 2\rho_0 \), the metric is

\[
e^2(1 + \frac{m}{2\epsilon\rho})^4(d\rho^2 + \rho^2d\sigma^2) = (1 + \frac{m}{2r})^4(dr + r^2d\sigma^2)
\]
which is Schwarzschild. Moreover, there is a minimal sphere at
\[ \rho = \frac{m}{2\epsilon} = \frac{32}{m} > 2\rho_0. \]
Hence \( \mathbb{R}^3 \) with the metric \( ds_m^2 \) has a horizon and is Schwarzschild at infinity. \( \square \)

From the proposition, in order to find a sufficient condition for the existence of compact minimal surfaces, we need to know information in the interior of the domain. This motivates us to introduce the following quantity using Hawking mass of some compact surfaces inside a domain.

Let \( E \) be an open set in a Riemannian manifold with compact \( C^1 \) boundary, then one can define the mean curvature \( H \) of \( \partial E \) in the weak sense, see [12]. Recall that the Hawking mass \( m_H(\partial E) \) of \( \partial E \) is defined as:

\[
(2.15) \quad m_H(\partial E) = \sqrt{\frac{|\partial E|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\partial E} H^2 \right),
\]
where \( |\partial E| \) is the area of \( \partial E \). In our convention, the standard unit sphere in \( \mathbb{R}^3 \) has mean curvature 2 with respect to the unit outward normal.

Let us recall the idea of minimizing hull introduced in [12]. Let \( (\Omega, g) \) be a Riemannian manifold.

**Definition 2.2.** Let \( E \) be a set in \( \Omega \) with locally finite perimeter. \( E \) is said to be a minimizing hull in \( \Omega \) if \( |\partial^* E \cap K| \leq |\partial^* F \cap K| \) for any set \( F \) with locally finite perimeter such that \( F \supset E \) and \( F \setminus E \Subset \subset \Omega \) and for any compact set \( K \subset \Omega \). Here \( \partial^* E \) and \( \partial^* F \) are the reduced boundaries of \( E \) and \( F \) respectively. \( E \) is said to be strictly minimizing hull if equality (for all \( K \)) implies \( E \cap \Omega = F \cap \Omega \) a.e.

Suppose \( E \) is an open set of \( \Omega \) such that there is a strictly minimizing hull in \( \Omega \) containing \( E \), then define \( E' \) to be the intersection of all strictly minimizing hulls containing \( E \). Note that \( E' \) consists of the Lebesque points of the intersection by definition. \( E' \) is called the strictly minimizing hull of \( E \). Let \( E \subset \subset \Omega \) be an open set. Suppose \( \Omega \) is compact with smooth boundary which has positive mean curvature with respect to the outward normal, then \( E' \) exists and \( E' \subset \subset \Omega \).

Let \( \Omega_1 \subset \subset \Omega_2 \subset \Omega \) such that \( \Omega_1 \) and \( \Omega_2 \) have smooth boundaries. We need the following lemma from [15].

**Lemma 2.3.** [Meeks-Yau] With the above notations and let \( d \) be the distance between \( \Omega_1 \) and \( \partial \Omega_2 \). Let \( \iota \) be the infinmum of the injectivity
radius of points in \( \{ x \mid d(x, \partial \Omega_2) > \frac{d}{4} \} \). Let \( K > 0 \) be the upper bound of the curvature of \( \Omega_2 \). Suppose \( N \) is a minimal surface and \( x \in N \) with \( d(x, \partial N) = \frac{d}{2} \), so that \( d(x, \partial N) \geq \frac{d}{2} \), then

\[
|N \cap B_x(r)| \geq CK^2 \int_0^r \tau^{-1} \sin^2 K\tau \, d\tau
\]

where \( r = \min \{ \frac{d}{2}, \ell \} \). Here \( C \) is a positive absolute constant.

For such \( \Omega_1, \Omega_2 \), let

\[
\alpha_{\Omega_1;\Omega_2}^2 = \min \left\{ \frac{CK^{-2} \int_0^r \tau^{-1} \sin^2 K\tau \, d\tau}{|\partial \Omega_1|}, 1 \right\}
\]

Let \( \mathcal{F}_{\Omega_2} \) be the family of precompact connected minimizing hulls with \( C^2 \) boundary in \( \Omega_2 \). Define

\[
m(\Omega_1;\Omega_2) = \sup_{E \subset \mathcal{F}_{\Omega_2}, E \subset \Omega_1} m_H(E).
\]

Define

\[
m(\Omega) = \sup \alpha_{\Omega_1;\Omega_2} m(\Omega_1;\Omega_2)
\]

where the supremum is taken over all \( \Omega_1 \subset \subset \Omega_2 \subset \Omega \) with smooth boundaries. Here and below \( \Omega_1 \) is always assumed to be nonempty.

In general, the Hawking mass of a compact surface may be negative. However, one can prove that \( m(\Omega) \geq 0 \).

**Theorem 2.4.** Let \( (\Omega, g) \) be a compact manifold with smooth boundary. Then \( \mathcal{F}_{\Omega_2} \neq \emptyset \) for any \( \Omega_2 \subset \subset \Omega \) and \( m(\Omega) \geq 0 \). If \( \Omega \) can be embedded in \( \mathbb{R}^3 \), then \( m(\Omega) = 0 \). On the other hand, suppose \( \Omega \) has nonnegative scalar curvature and \( m(\Omega) = 0 \), then \( \Omega \) is locally flat. In particular, if \( \Omega \) is simply connected, then \( \Omega \) is a domain in \( \mathbb{R}^3 \).

**Proof.** Since \( \Omega \) has smooth boundary, by taking a collar of \( \partial \Omega \), we can embed \( \Omega \) in a compact manifold \( \Omega_3 \) with smooth boundary. Taking a double of \( \Omega_3 \), we may assume that \( \Omega \) is isometrically embedded in a compact manifold \( \Omega_4 \) without boundary. Take a small geodesic disk in \( \Omega_4 \setminus \overline{\Omega} \) and glue it to the exterior of some compact set of \( \mathbb{R}^3 \), we may assume that \( \Omega \) is embedded in a complete noncompact manifold \( M \) with only one end and such that near infinity of \( M \) is isometric to the exterior of a compact set in \( \mathbb{R}^3 \).

Suppose \( \Omega_2 \subset \Omega \). Take a point \( x_0 \in \Omega_2 \) and let \( 3r > 0 \) be such that \( B_{x_0}(3r) \subset \Omega_2 \), \( 3r \) is less than the injectivity radius of \( x_0 \) and \( \partial B_{x_0}(r) \) has positive mean curvature for all \( 0 < \rho < 3r \). We claim that there exists \( 0 < \rho_0 < r \) such that \( B_{x_0}(\rho) \) is a strictly minimizing hull in \( \Omega_2 \) for all \( 0 < \rho < \rho_0 \). Let \( 0 < \rho < r \) and let \( F \) be the strictly minimizing
hull of $B_{r_0}(\rho)$ in $M$. $F$ exists because $M$ is Euclidean near infinity. Moreover, there is $R > 0$ independent of $\rho$ such that $F \subset B_p(R)$ where $p \in M$ is a fixed point. Suppose $F \cap (M \setminus B_{r_0}(2r)) \neq \emptyset$. Then there is a point $x \in \partial F$ and $x \in B_p(2R) \setminus B_{r_0}(2r)$. Moreover, $B_x(r) \cap B_{r_0}(r) = \emptyset$. Since $\partial F \setminus B_{r_0}(r)$ is minimal surface, see [12], by Lemma 1.1 we have $|\partial F| \geq c > 0$ for some constant $c > 0$ independent of $\rho$. Now choose $r > \epsilon_0 > 0$ such that $|\partial B_{r_0}(\rho)| < c$ for all $0 < \rho < \epsilon_0$. Then for $0 < \rho < \epsilon_0$, the strictly minimizing hull of $B_{r_0}(\rho)$ must be a subset of $B_{r_0}(3r)$. However, since $\partial B_{r_0}(s)$ has positive mean curvature for all $0 < s < 3r$, we conclude that $F = \partial B_{r_0}(\rho)$. This proves the claim. In particular, $B_{r_0}(\rho) \in F_{\Omega_2}$ for all $0 < \rho < \epsilon_0$ and so $F_{\Omega_2} \neq \emptyset$.

Let $H_\rho$ be the mean curvature of $\partial B_{r_0}(\rho)$, then it is easy to see that $H_\rho = \frac{2}{\rho} + O(1)$ as $\rho \to 0$ and

$$\lim_{\rho \to 0} \frac{|\partial B_{r_0}(\rho)|}{4\pi \rho^2} = 1.$$ 

Hence we have $\lim_{\rho \to 0} m_H(\partial B_{r_0}(\rho)) = 0$. From this it is easy to see that $m(\Omega_1; \Omega_2) \geq 0$ for all $\Omega_1 \subset \subset \Omega_2 \subset \Omega$, and so $m(\Omega) \geq 0$.

It is well known that $m_H(\Sigma) \leq 0$ for any compact surface in $\mathbb{R}^3$, see [23]. Hence if $\Omega$ is a domain in $\mathbb{R}^3$, then $m(\Omega) = 0$.

To prove the last two assertions, we first observe that by the definition of $m(\Omega)$, $m(\Omega') \leq m(\Omega)$ if $\Omega' \subset \Omega$. Now suppose $m(\Omega) = 0$, then for any $x_0 \in \Omega$, we have $m(B_{r_0}(\rho)) = 0$ for all $\rho$ small enough. We want to prove $\Omega$ is flat near $x_0$ using the idea in [12]. Suppose it is not flat near $x_0$, let $M$ be as above. Let $\rho_0 > 0$ be such that (i) $\rho_0 < r$ where $3r$ is the injectivity radius of $x_0$; (ii) $\partial B_{r_0}(\rho)$ has positive mean curvature for all $0 < \rho < 3r$; (iii) for all $0 < \rho < \rho_0$, $m(B_{r_0}(\rho)) = 0$ and $B_{r_0}(\rho)$ is a strictly minimizing hull in $M$. Moreover by the proof above we may assume that for any open set $E \subset B_{r_0}(\rho_0)$ with $C^2$ boundary, then the strictly minimizing hull $E'$ of $E$ in $M$ satisfies $E' \subset B_{r_0}(\rho_0)$.

Let $\rho_0 > \rho > 0$ and let $E_t^t, 0 \leq t < \infty$ be the weak solution of inverse mean curvature flow with initial condition $B_{r_0}(\rho)$ given by a locally Lipschitz proper function $u^\rho$. Such a solution exists by [12]. Moreover, for $x \in B_{r_0}(3r)$

$$(2.20) \quad |\nabla u^\rho(x)| \leq C_1 d^{-1}(x)$$

for some constant $C_1$ independent of $\rho$ and $d(x)$ is the distance between $x$ and $x_0$. By [12] p.421-422], we can find $c_i \to \infty, \rho_i \to 0$ such that $N_{i+c_i}$ converges for a.e. $t \in (-\infty, \infty)$ in $C^1$ to $N_i$ which is a solution of inverse mean curvature flow in $M \setminus \{x_0\}$ given by a locally Lipschitz proper function $u$. Here $N^\rho_{t+c_i} = \partial E^\rho_{t+c_i} = \partial \{u^\rho < t + c_i\}$
and \( N_t = \partial E_t = \partial \{ u < t \} \). Moreover, \( N_t \) is nearly equal to \( \partial B_{x_0} (e^{t \frac{r}{2}}) \) as \( t \to -\infty \). Since \( N_{t+\varepsilon_i}^{\rho_i} \) is connected for all \( i, N_t \) is also connected.

Let \( b > -\infty \) be such that \( E_{t+\varepsilon_i}^{\rho_i} \subset B_{x_0} (\rho_0) \) for all \( t \leq b \) and for all \( i \). Then as in \[12\] p.427, one can show that \( m_H (N_t) > 0 \) for all \( t < b \). Let us outline the idea. Let \( t < b \) be such that \( N_{t+\varepsilon_i}^{\rho_i} \) converges to \( N_t = \partial E_t \) in \( C^1 \). Observe that \( m_H (N_{t+\varepsilon_i}^{\rho_i}) \geq m_H (\partial B_{x_0} (\rho)) \geq -\delta_0 \), here \( \delta_0 \) is a constant independent of \( i \) by the monotonicity formula \[12\] Theorem 5.8] because \( \Omega \) has nonnegative scalar curvature. Since \( |N_{t+\varepsilon_i}^{\rho_i}| \) also converges to \( |N_t| \), we have

\[
(2.21) \quad \int_{N_{t+\varepsilon_i}^{\rho_i}} H^2 \leq C_2
\]

for some \( C_2 \) independent of \( i \). Since \( N_{t+\varepsilon_i}^{\rho_i} \) converges to \( N_t \) in \( C^1 \), the topology of \( N_{t_0+\varepsilon_i}^{\rho_i} \) is the same as that \( N_{t_0} \) provided \( i \) is large enough. By \[2.21\], \[12\] Lemma 5.5] and its proof, we have

\[
(2.22) \quad \int_{N_{t+\varepsilon_i}^{\rho_i}} |A|^2 \leq C_3
\]

for some constant \( C_3 \) for all \( i \) large enough, where \( A \) is the second fundamental form of \( N_{t+\varepsilon_i}^{\rho_i} \). By lower semicontinuity

\[
(2.23) \quad \int_{N_t} |A|^2 < \infty.
\]

Hence the monotonicity formula \[12\] Theorem 5.8] for \( N_t \) holds. Note that by upper semicontinuity \( m_H (N_t) \geq 0 \) for such \( t \). Hence if for such a \( t, m_H (N_t) = 0 \), then one can prove that \( m_H (N_s) = 0 \) for a.e. \( s < t \). Since there is no compact minimal surfaces in \( B_{x_0} (3r) \), one can argue as in \[12\] that \( \Omega \) is flat near \( x_0 \). This is a contradiction.

Now choose \( t_0 < b \) such that \( N_{t_0+\varepsilon_i}^{\rho_i} \) converge in \( C^1 \) to \( N_{t_0} \). Note that even though \( m_H (N_{t_0}) > 0 \) and \( E_{t_0} \) is a minimizing hull by \[12\], \( N_{t_0} \) may not have \( C^2 \) boundary. In order to show that \( m(B_{x_0} (\rho)) > 0 \) for \( \rho \) small, we need some approximation.

As before, we have

\[
\int_{N_{t_0}} |A|^2 < \infty.
\]

By (5.19) in \[12\], we can find an open set \( F \subset B_{x_0} (\rho_0) \) with smooth boundary such that \( m_H (\partial F) > 0, \partial F \) is connected. Let \( F' \) be the
strictly minimizing hull of $F$. Then $F' \subset B_{x_0}(\rho_0)$ and

$$m_H(\partial F') = \sqrt{\frac{|\partial F'|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\partial F'} H^2\right)$$

(2.24)

$$\geq \sqrt{\frac{|\partial F'|}{|\partial F|}} m_H(\partial F) > 0$$

because the mean curvature of $\partial F'$ is zero on $\partial F' \setminus \partial F$ and the mean curvature of $\partial F'$ is equal to the mean curvature of $\partial F$ a.e. on $\partial F' \cap \partial F$.

Since $B_{x_0}(3r)$ is foliated by geodesic sphere with positive mean curvature, $\partial F$ cannot be a minimal surface. Since $F$ is connected, $F'$ is connected. Since $\partial F'$ is $C^{1,1}$, by Lemma 5.6 in [12] we can find $G \subset B_{x_0}(\rho_0)$ with smooth boundary such that $G$ is a strictly minimizing hull in $M$ and $m_H(G) > 0$. $G$ can be chosen to be connected because $F'$ is connected. Therefore $G \in \mathcal{F}_{B_{x_0}(\rho_0)}$, we have $m(B_{x_0}(\rho_0)) > 0$. This is a contradiction because $m(B_{x_0}(\rho_0)) = 0$. Hence $\Omega$ is locally flat. If $\Omega$ is simply connected, then $\Omega$ is a domain in $\mathbb{R}^3$, see [22, p.42-44].

3. Sufficient conditions on the existence of compact minimal surfaces

Let $(\Omega, g)$ be a compact orientable three manifold with smooth boundary which consists of finitely many components and with nonnegative scalar curvature. In this section, we always assume that the boundary of $\Omega$ has positive Gauss curvature and positive mean curvature with respect to the outward normal. Hence each boundary component is diffeomorphic to a sphere. We want to obtain a sufficient condition on the existence of outermost horizon which is defined as in [4] as follows:

Let $\Sigma_1, \ldots, \Sigma_k$ be the components of $\partial \Omega$. For a fixed $i$, by adding points to each component except $\Sigma_i$, we obtain a topological space $\Omega_i$. Let $\mathcal{S}_i$ be the collection of surfaces which are smooth boundaries of precompact open sets in $\Omega_i$ containing those points. The boundary $\partial E$ of a set $E$ with $\partial E \in \mathcal{S}_i$ is called outer minimizing if $|\partial E| \leq |\partial F|$ for all $F \supset E$. A horizon (relative to $\Sigma_i$) is a surface $\partial E \in \mathcal{S}_i$ with zero mean curvature. A horizon is said to be outermost if it is not enclosed by another horizon, see [1] p.185. Suppose $\partial E$ is a horizon. Let $E'$ be its strictly minimizing hull in $\Omega_i$, which exists because the mean curvature of $\Sigma_i$ is positive. Then $\partial E'$ is also a horizon.

Let $(\Omega, g)$ and $\Sigma_i$ as before. By Weyl embedding theorem, see [19], each $\Sigma_i$ can be isometrically embedded in $\mathbb{R}^3$ and the embedding is
unique up to an isometry of $\mathbb{R}^3$. Let $H$ be the mean curvature of $\Sigma_i$ with respect to the outward normal and let $H_0$ be the mean curvature of $\Sigma_i$ when embedded in $\mathbb{R}^3$. The Brown-York mass [5, 6] of $\Sigma_i$ is defined to be

$$m_{BY}(\Sigma_i) = \frac{1}{8\pi} \int_{\Sigma_i} (H_0 - H) d\sigma$$

where $d\sigma$ is the volume element on $\Sigma_i$ induced by $g$.

In this section we will give some lower bounds of the Brown-York mass in terms of the Hawking mass of subsets of $\Omega$. We will also give a sufficient condition on of existence of horizons in $\Omega$ using the Brown-York mass and $m(\Omega)$ defined in the previous section. The first lower bound for the Brown-York mass is the following:

**Theorem 3.1.** Let $(\Omega, g)$ be a compact three manifold with connected smooth boundary and with nonnegative scalar curvature. Assume that $\Omega$ is simply connected and suppose $\partial \Omega$ has positive Gauss curvature and positive mean curvature with respect to the outward normal. Then

$$m_{BY}(\partial \Omega) \geq m_H(\partial E).$$

for any connected minimizing hull $E$ in $\Omega$ where $E \subset \subset \Omega$ with $C^{1,1}$ boundary. Moreover, equality holds for some minimizing hull $E$ with the above properties if and if $\Omega$ is a standard ball in $\mathbb{R}^3$ and $E$ is a standard ball in $\Omega$. In particular, $m_{BY}(\partial \Omega) \geq m_H(\partial \Omega)$ and equality holds if and only if $\Omega$ is a standard ball in $\mathbb{R}^3$.

**Proof.** Let $E$ be a minimizing hull in $\Omega$ with $C^{1,1}$ boundary such that $E \subset \subset \Omega$. We want to prove that $m_{BY}(\partial \Omega) \geq m_H(\partial E)$. Since $m_{BY}(\partial \Omega) \geq 0$, it is sufficient to prove the case when $m_H(\partial E) > 0$.

Isometrically embed $\Sigma = \partial \Omega$ in $\mathbb{R}^3$. As in [21], one can glue $\Omega$ to the exterior of $\partial \Omega$ in $\mathbb{R}^3$ to form a manifold $(M, h)$ such that the metric $h$ satisfies:

(i) $h|_{\Omega} = g$;
(ii) $h_{\partial M \setminus \Omega}$ is smooth up the boundary;
(iii) the scalar curvature of $h$ in $M \setminus \Omega$ is zero;
(iv) $h$ is Lipshitz near $\partial \Omega$;
(v) the mean curvatures of $\partial \Omega$ with respect to the outward normal are the same for metrics inside and outside $\Omega$;
(vi) $h$ is asymptotically flat;
(vii) if $\Sigma_r$ is the surface consisting points in the exterior to $\Sigma$ in $\mathbb{R}^3$ and with Euclidean distance $r$ from $\Sigma$, then $\Sigma_r$ has positive mean curvature with respect to $h$, which is bounded from below by $C/(1 + r)$ for some $C > 0$.  


Given $\epsilon > 0$, by (i)-(vi) and \[10\], there is a smooth AF metric $h_\epsilon$ on $M$ with nonnegative scalar curvature such that

$$ (1 - \epsilon) h_\epsilon(v, v) \leq h(v, v) \leq (1 + \epsilon) h_\epsilon(v, v) $$

for all tangent vector $v$ on $M$, and such that

$$ \lim_{\epsilon \to 0} m_{ADM}(h_\epsilon) = m_{ADM}(h) $$

where $m_{ADM}(h_\epsilon)$ and $m_{ADM}(h)$ are the ADM masses of $(M, h_\epsilon)$ and $(M, h)$ respectively.

Let $\theta > 0$ be given. We can find a connected open set $F \supset E$ with smooth boundary such that $F \subset \subset \Omega$,

$$ |\partial E| - \theta \leq |\partial F| \leq |\partial E| + \theta; \ m_H(\partial E) \leq m_H(\partial F) + \theta; \text{ and } m_H(\partial F) > 0. $$

For any $\epsilon > 0$, let $F'$ be the strictly minimizing hull of $F$ in $(M, h_\epsilon)$. $F'$ exists, precompact in $M$ and has $C^{1,1}$ boundary. Since $F$ is connected, $F'$ is connected. Moreover $M$ is simply connected because $\Omega$ is simply connected and $\partial \Omega$ is homeomorphic to the sphere. Hence $\partial F'$ is connected.

Choose $\delta > 0$ be small enough, such that the boundary of $\Omega_t$ for all $0 < t \leq \delta$ has positive mean curvature in $g$, where $\Omega_t = \{ x \in \Omega | d_g(x, \partial \Omega) > t \}$. Here $d_g$ is the distance with respect to $g$. There is $0 < t_0 < \delta$ such that $F \subset \subset \Omega_{t_0}$. Suppose $F' \setminus \Omega_{t_0} \neq \emptyset$. Since $M \setminus \Omega_\delta$ is foliated by compact surfaces of positive mean curvature with respect to $h$, if we denote the unit outward normal vector in $h$ of the surfaces by $\nu$, we have

$$ 0 < \int_{F' \setminus \Omega_{t_0}} div_h \nu dV_h $$

$$ = \int_{\partial F' \cap (M \setminus \Omega_{t_0})} h(\nu, \mu) - [F' \cap \partial \Omega_{t_0}]_g $$

where $\mu$ is the unit outward normal of $\partial F'$ in $h$. Here $div_h$ is the divergence with respect to $h$. Hence we have

$$ |\partial F' \cap (M \setminus \Omega_{t_0})|_h > [F' \cap \partial \Omega_{t_0}]_g $$

Now $F' \cap \Omega_{t_0} \supset E$ is a set of finite perimeter and $E$ is a minimizing hull in $(\Omega, g)$, we have

$$ |\partial (F' \cap \Omega_{t_0})|_g \geq |\partial E|_g. $$
From (3.7) and (3.8) we conclude that
\[ |\partial F'\cap (M \setminus \Omega_t)|_h + |\partial F' \cap \Omega_t|_h > |\partial F'\cap \partial \Omega_t|_g + |\partial F' \cap \Omega_t|_g = |\partial (F' \cap \Omega_t)|_g. \] (3.9)

Moreover the mean curvature of \( \partial F' \) in \( \partial \Omega \) is zero on \( \partial F' \) and the mean curvature of \( \partial F' \) and \( \partial F \) are equal a.e. in \( \partial F' \cap \partial F \). Using the inverse mean curvature flow with initial data \( F' \), we obtain by the Penrose inequality in [12] we have
\[ m_{ADM}(h) \geq \sqrt{\frac{|\partial F'|_h}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\partial F'} H^2 d\sigma_h \right) \]
\[ \geq \sqrt{\frac{|\partial F'|_h}{|\partial F'|_g}} \cdot \sqrt{\frac{|\partial F'|_g}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\partial F} H^2 d\sigma_h \right) \]
\[ = \sqrt{\frac{|\partial F'|_h}{|\partial F'|_g}} m_H(\partial F) \]
\[ \geq \sqrt{(1 - \epsilon) \frac{|\partial E'|_h}{|\partial E'|_g} m_H(\partial E) - \theta} \sqrt{\frac{|\partial F'|_h}{|\partial F'|_g}} \]
\[ \geq \sqrt{(1 - \epsilon) \frac{|\partial E|^2}{|\partial F|^2} m_H(\partial E) - \theta(1 + \epsilon)^{\frac{1}{2}}} \]
\[ \geq \sqrt{(1 - \epsilon) \frac{|\partial E|^2}{|\partial E|^2 + \theta} m_H(\partial E) - \theta(1 + \epsilon)^{\frac{1}{2}}} \]
(3.10)

Here, we have use the assumption \( m_H(\partial E) \geq 0 \) in the fifth inequality above, and \( d\sigma_h \) and \( d\sigma_g \) are the area elements on surfaces induced by \( h \) and \( g \) respectively and we have used (3.5) and (3.9). Let \( \epsilon \to 0 \), and then let \( \theta \to 0 \), we have \( m_{ADM}(h) \geq m_H(\partial E) \). By [21], we have
\[ m_{BY}(\partial \Omega) \geq m_{ADM}(h) \geq m_H(\partial E) \] (3.11)

If \( m_{BY}(\partial \Omega) = m_H(\partial E) \) for some \( E \) with the properties in the theorem, then \( m_{BY}(\partial \Omega) = m_{ADM}(h) \) by (3.11). Hence \( \Omega \) is a domain in \( \mathbb{R}^3 \) by [21] and \( m_H(\partial \Omega) = m_{BY}(\partial \Omega) = 0 \) by [23], \( \Omega \) is a standard ball in \( \mathbb{R}^3 \). Since \( m_H(\partial E) = 0 \), \( E \) is also a standard ball in \( \mathbb{R}^3 \).
Conversely, if $\Omega$ is a domain in $\mathbb{R}^3$, since each sphere has zero Hawking mass, and a ball which is small enough is a minimizing hull, we have $m_{BY}(\Omega) = m_H(\partial E) = 0$ for some small ball.

To prove the last statement, we observe that $\Omega_t$ is a minimizing hull in $\Omega$ if $t > 0$ is small. Hence $m_{BY}(\partial \Omega) \geq \lim_{t \to 0} m_H(\partial \Omega_t) = m_H(\partial \Omega)$. In fact, from the proof above, $m_{BY}(\partial \Omega) > \lim_{t \to 0} m_H(\partial \Omega_t)$ unless $\Omega$ is a domain in $\mathbb{R}^3$. Hence by a result of Willmore [23], $m_{BY}(\partial \Omega) = m_H(\partial \Omega)$ implies that $\partial \Omega$ is a standard sphere so $\Omega$ is a standard ball in $\mathbb{R}^3$. $\square$

In the above theorem, we do not assume that $(\Omega, g)$ contains no horizons. In order to obtain a sufficient condition that $(\Omega, g)$ contains a horizon, we need another estimate of the Brown-York mass. Let $m(\Omega)$ as defined in (2.19). We have the following:

**Theorem 3.2.** Let $(\Omega, g)$ be a compact manifold with connected smooth boundary and with nonnegative scalar curvature. Assume that $\Omega$ is simply connected and suppose $\partial \Omega$ has positive Gauss curvature and positive mean curvature with respect to the outward normal. Suppose $(\Omega, g)$ has no horizons. Then $m(\Omega) \leq m_{BY}(\partial \Omega)$. Equality holds if and only if $\Omega$ is a domain in $\mathbb{R}^3$.

**Remark 3.3.** If $\partial \Omega$ has more than one components, then it is easy to see that $\Omega$ will have an outermost minimizing horizon with respect to any boundary component because $\partial \Omega$ has positive mean curvature. If $\partial \Omega$ is connected and $\Omega$ is not simply connected, then $\Omega$ contains a minimal sphere or real projective space by [14].

**Proof of Theorem 3.2.** Assume that $\Omega$ has no horizons. Suppose $\Omega$ is not diffeomorphic to an open 3-ball in $\mathbb{R}^3$, then $\Omega$ must contains a minimal sphere $S$ or projective plane [14]. Since $\Omega$ is simply connected, $S$ must be a minimal sphere and is the boundary of some precompact open set $E$, see [11] p.107]. So $S$ is a horizon. This is a contradiction. Hence $\Omega$ is diffeomorphic to an open 3-ball in $\mathbb{R}^3$.

Let $\Omega_1 \subset \subset \Omega_2 \subset \subset \Omega$ with smooth boundaries. Let $E$ be a connected precompact minimizing hull in $\Omega_2$ with $C^2$ boundary such that $E \subset \Omega_1$. We want to prove that $m_{BY}(\partial \Omega) \geq \alpha_{\Omega_1,\Omega_2} m_H(\partial E)$. Again, we may assume that $m_H(\partial E) \geq 0$.

Let $E'$ be the strictly minimizing hull of $E$ in $\Omega$. $E'$ exists because $\partial \Omega$ has positive mean curvature. Moreover, $E'$ has $C^{1,1}$ boundary because $E$ has $C^2$ boundary. Note that $E'$ and $\partial E'$ are connected because $\Omega$ is
diffeomorphic to the unit ball in $\mathbb{R}^3$. By Theorem 3.1
\[
m_{\text{BY}}(\partial \Omega) \geq m_H(\partial \Omega')
\] (3.12)
\[
\geq \sqrt{\frac{|\partial \Omega'|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\partial \Omega} H^2 \right)
\]
because $\partial \Omega' \setminus \partial \Omega$ is minimal and the mean curvatures of $\partial \Omega$ and $\partial \Omega'$ are equal a.e. on their common part. Suppose $E' \subset \subset \Omega$, then $|\partial \Omega'| \geq |\partial \Omega|$ and we have $m_{\text{BY}}(\partial \Omega) \geq m_H(\partial \Omega)$.

Suppose $E'$ is not a precompact set in $\Omega$, then $\partial \Omega' \cap (M \setminus \Omega) \neq \emptyset$. On the other hand, if $\partial \Omega' \cap \partial \Omega = \emptyset$ then $\partial \Omega'$ would be a horizon contradicting the assumption that $\Omega$ contains no horizons. Hence
\[
\partial \Omega' \cap \Omega_1 \supset \partial \Omega' \cap \partial \Omega \neq \emptyset
\]
and there is a point $x \in \partial \Omega'$ and $d(x, \partial \Omega_2) = \frac{4}{2}$, where $d$ is the distance between $\Omega_1$ and $\partial \Omega_2$. By Lemma 2.3 we have
\[
|\partial \Omega'| \geq \alpha_{\Omega_1; \Omega_2}|\partial \Omega_1| \geq \alpha_{\Omega_1; \Omega_2}|\partial \Omega|
\] (3.13)
because $E$ is a minimizing hull in $\Omega_2$. Hence by (3.12), (3.13) we have
\[
m_{\text{BY}}(\partial \Omega) \geq \alpha_{\Omega_1; \Omega_2} m_H(\partial \Omega).
\] (3.14)
This completes the proof of the first part. If equality holds, then from the proof above, for any $\epsilon > 0$ we can find a strictly minimizing hull $E'$ in $\Omega$ with $C^{1,1}$ boundary such that $m_{\text{BY}}(\partial \Omega) \leq m_H(\partial \Omega') + \epsilon$. From the proof of Theorem 3.1 one can conclude that $\Omega$ must be a domain in $\mathbb{R}^3$. \hfill \Box

Let $\Omega$ be as in the theorem. Isometrically embed $\partial \Omega$ in $\mathbb{R}^3$. Let $R$ be the radius of the smallest circumscribed ball of $\partial \Omega$ in $\mathbb{R}^3$. We have the following:

**Corollary 3.4.** Let $(\Omega, g)$ be a compact manifold with nonnegative scalar curvature and with connected boundary which has positive mean curvature and positive Gauss curvature. Suppose $\Omega$ is simply connected and suppose $m(\Omega) \geq m_{\text{BY}}(\partial \Omega)$, then there is a horizon in $\Omega$ unless $\Omega$ is a domain in $\mathbb{R}^3$. Hence if $m(\Omega) \geq 2R$, then $\Omega$ contains a horizon. In particular, if $m(\Omega) \geq 2\text{diam}(\partial \Omega)$ then $\Omega$ contains a horizon. Here $\text{diam}(\partial \Omega)$ is the diameter of $\partial \Omega$ with respect to the metric induced by $g$.

**Proof.** The first statement is just a restatement of Theorem 3.2. Since
\[
m_{\text{BY}}(\Omega) \leq \frac{1}{8\pi} \int_{\partial \Omega} H_0 \leq 2R < 2\text{diam}(\partial \Omega)
\] (3.15)
by the Minkowski integral formula [13, p.136], the fact that $\partial \Omega$ has positive Gauss curvature and the Gauss-Bonnet formula. □

There are examples so that $m(\Omega) > m_{BY}(\Omega)$. Consider the metrics $(\mathbb{R}^3, ds^2_m = u^4_m(d\rho^2 + \rho^2 d\sigma^2))$ defined in the proof of Proposition 2.4, where $d\rho^2 + \rho^2 d\sigma^2$ is the Euclidean metric. Since as $m \to 0$, $ds^2_m$ converges uniformly on the Euclidean ball $B_4 = \{ x \in \mathbb{R}^3 | |x| < 4 \}$ to the standard metric on the unit sphere, we see that for sufficiently small $m > 0$, the boundary of $B_\tau$ is mean convex for all $0 < \tau < 2$ with respect to $ds^2_m$. $B_1$ is a minimizing hull in $B_2$. Moreover, there is $\delta > 0$ such that the Hawking mass $m_H(\partial B_1)$ of $\partial B_1$ with respect to $ds^2_m$ is at least $\delta$, provided $m > 0$ is small enough. Hence for all $m > 0$ small enough, $\alpha_{\Omega_1,\Omega_2} m(B_1; B_2) \geq \delta$. Now consider the domain $\Omega = B_\rho$, where $\rho = 8/m > 2\rho_0$. It is easy to see that $\partial \Omega$ has positive mean curvature and positive Gauss curvature with diameter $d \leq Cm$, here $C$ is a universal constant. Hence

$$m(\Omega) \geq \alpha_{\Omega_1,\Omega_2} m(B_1; B_2) \geq \delta > 2d \geq m_{BY}(\Omega)$$

provided $m$ is small enough.

There is another sufficient condition for the existence of horizons in terms of the Hawking mass of other kind of surfaces. Suppose $E \subset \subset \Omega$ is a bounded subset in $\Omega$ such that $E$ and connected and the volume of $E$ is equal $v_0$. Using the terminology in [8], $\partial E$ is called a round sphere if $\partial E$ is smooth and connected and has least area among all subsets of $\Omega$ with locally finite perimeter with volume $v_0$. The volume of a set of locally finite perimeter is simply the measure of the set with respect to the volume element of $\Omega$. If the scalar curvature of $\Omega$ is nonnegative, and $\partial E$ is a round sphere, then the Hawking mass $m_H(\partial E)$ of $\partial E$ is nonnegative by [8]. We have the following:

**Theorem 3.5.** Let $(\Omega, g)$ be a compact Riemannian three manifold with smooth and connected boundary and with nonnegative scalar curvature. Assume that $\Omega$ is simply connected and suppose that both the Gauss curvature and mean the curvature of the boundary (with respect to outward normal) is positive. If there is a round sphere $\partial E$ in $\Omega$ such that

$$m_H(\partial E) > m_{BY}(\partial \Omega),$$

then there is a horizon in $\Omega$.

First, we prove the following lemma:

**Lemma 3.6.** Let $(\Omega, g)$ be a compact Riemannian three manifold with smooth boundary and with nonnegative scalar curvature such that $\partial \Omega$
has positive mean curvature. Suppose \( E \subset \subset \Omega \) such that \( \partial E \) is a round sphere. Then either \( \Omega \) contains a horizon or \( E \) is a minimizing hull in \( \Omega \).

**Proof.** Suppose \( E \) is not minimizing hull in \( \Omega \). Since \( \partial \Omega \) has positive mean curvature, there exists a strictly minimizing hull \( E' \) containing \( E \) with \( E' \subset \subset \Omega \). Since \( E \) is not a minimizing hull, \( |\partial E'| < |\partial E| \). Let \( v_0 \) be the volume of \( E \), then \( vol(E') > v_0 \) because \( E \) is a round sphere. Choose \( \epsilon_0 > 0 \) small enough so that \( \partial \Omega_\epsilon \) is smooth and has positive mean curvature for all \( 0 < \epsilon < 2\epsilon_0 \), where \( \Omega_\epsilon = \{ x \in \Omega | d(x, \partial \Omega) > \epsilon \} \).

Moreover, \( \epsilon_0 \) is chosen so that \( E' \subset \subset \Omega_\epsilon \). Let \( \Phi \) be the family of precompact subsets \( F \) with locally finite perimeter of \( \Omega_\epsilon \) with the properties that \( vol(F) > v_0 \), and \( |\partial F| < \frac{1}{2}(|\partial E'| + |\partial E|) \). \( \Phi \) is not empty because \( E' \in \Phi \). Let

\[
(3.16) \quad v_1 = \inf_{F \in \Phi} vol(F).
\]

Note that \( v_1 \geq v_0 \). We claim that \( v_1 > v_0 \). In fact, let \( F_i \in \Phi \) be such that \( \lim_{i \to \infty} vol(F_i) = v_1 \). By compactness theorem for sets of locally finite perimeter, we may assume that the characteristic functions of \( F_i \) converges in \( L^1(\Omega) \) to a characteristic function of a subset \( F \) of \( \Omega_{\epsilon_0} \) with locally finite perimeter such that \( vol(F) = v_1 \). By lower semicontinuity, \( |\partial F| \leq \frac{1}{2}(|\partial E'| + |\partial E|) < |\partial E| \).

(For simplicity, we use \( |\partial F| \) to denote the area of the reduced boundary of \( F \)). Hence

\[
(3.17) \quad v_1 = vol(F) > v_0 \quad \text{by the definition of round spheres.}
\]

Let

\[
(3.18) \quad a = \inf_{F \in \Phi} |\partial F|.
\]

Choose a sequence \( F_i \in \Phi \) such that \( \lim_{i \to \infty} |\partial F_i| = a \). Since \( v_1 > v_0 \) and \( |\partial F_i| < \frac{1}{2}(|\partial E'| + |\partial E|) \), by the approximation result of sets of locally finite perimeter, we may choose \( F_i \) such that \( \partial F_i \) is smooth.

Choose \( 2\epsilon_0 > \epsilon_1 > \epsilon_0 \) such that \( vol(\Omega_{\epsilon_0} \setminus \Omega_{\epsilon_1}) \leq \frac{1}{2}(v_1 - v_0) \).

Consider the set \( \tilde{F}_i = F_i \cap \Omega_{\epsilon_1} \), then \( |\partial \tilde{F}_i| \leq |\partial F_i| < \frac{1}{2}(|\partial E'| + |\partial E|) \) since \( \partial \Omega_\epsilon \) has positive mean curvature for all \( 0 < \epsilon < 2\epsilon_0 \).

Moreover,

\[
(3.18) \quad vol(\tilde{F}_i) \geq vol(F_i) - vol(\Omega_{\epsilon_0} \setminus \Omega_{\epsilon_1}) \geq \frac{1}{2}(v_1 + v_0) > v_0
\]

by (3.11). Hence \( \tilde{F}_i \in \Phi \) and \( \lim_{i \to \infty} |\partial \tilde{F}_i| = a \). Passing to a subsequence if necessary, we may assume that the characteristic functions of \( \tilde{F}_i \) converges in \( L^1(\Omega) \) to a characteristic function of a subset \( \tilde{F} \) of \( \Omega_{\epsilon_1} \), with locally finite perimeter. Moreover, \( vol(\tilde{F}) \geq \frac{1}{2}(v_1 + v_0) > v_0 \) by (3.18) and \( |\partial \tilde{F}| = a \). Since \( E' \in \Phi \), \( a \leq |\partial E'| < \frac{1}{2}(|\partial E'| + |\partial E|) \).

Hence \( \tilde{F} \in \Phi \).
Let \( B_x(r) \subset \subset \Omega_\epsilon \) with \( \text{vol}(B_x(r)) < \frac{1}{2}(v_1 - v_0) \). Suppose \( F \) is any set of locally finite perimeter such that the symmetric difference \( F \Delta \tilde{F} \subset \subset B_x(r) \), then \( F \subset \subset \Omega_\epsilon \)

\[
\text{vol}(F) \geq \text{vol}(\tilde{F}) - \text{vol}(B_x(r)) \geq \frac{1}{2}(v_1 + v_0) > v_0
\]

by (3.16). If \( |\partial F| < |\partial \tilde{F}| \) then \( F \in \Phi \) because \( |\partial \tilde{F}| < \frac{1}{2}(|\partial E'| + |\partial E|) \). This is impossible by the construction of \( \tilde{F} \). Hence one can see that \( \tilde{F} \) is minimizing in any geodesic ball inside \( \Omega_\epsilon \), with radius small enough and \( \tilde{F} \) is a compact minimal surface. That is to say \( \Omega \) contains a horizon.

Proof of Theorem 3.5. Let \( \partial E \) be a round sphere. \( E \) is not a minimizing hull in \( \Omega \) by Theorem 3.1 and the assumption that \( \text{m}_H(\partial E) > \text{m}_{BY}(\partial \Omega) \). By Lemma 3.6 we see that \( \Omega \) contains a horizon. This completes the proof of the theorem.

4. On the Bartnik quasi-local mass

In this section, we will discuss relation between \( m(\Omega) \) and the quasi-local mass introduced by Bartnik \[1\]. We will also discuss some properties of the Bartnik quasi-local mass. We use the definition of the Bartnik quasi-local mass \( m_B(\Omega) \) in \[2\] of a compact manifold \((\Omega, g)\) with smooth boundary and with nonnegative scalar curvature.

Let \( \mathcal{P}M_o \) be the set of complete noncompact AF manifolds with nonnegative scalar curvature which admit no minimal two spheres or projective planes. For \( M \in \mathcal{P}M_o \), \( \Omega \subset \subset M \) means that \( \Omega \) is isometrically embedded in \( M \). Then the Bartnik quasi-local mass \( m_B(\Omega) \) of \( \Omega \) is defined as:

\[
m_B(\Omega) = \inf_{M \in \mathcal{P}M_o} \{ m_{ADM}(M) | \Omega \subset \subset M \}
\]

If \( \Omega \subset \subset M \in \mathcal{P}M_o \), then \( M \) is said to be an admissible extension of \( \Omega \). We should remark that for \( M \in \mathcal{P}M_o \), \( M \) is topologically \( \mathbb{R}^3 \), by \[14\], see also \[9\].

We have the following lower bound of \( m_B(\Omega) \).

**Theorem 4.1.** Suppose \( \Omega \) has an admissible extension. Then

\[
m_B(\Omega) \geq m(\Omega)
\]

**Proof.** The proof is exactly the same as the proof of Theorem 3.1. □

It was proved in \[12\] that if \( m_B(\Omega) = 0 \), then \( \Omega \) is locally flat. Hence it is a domain in \( \mathbb{R}^3 \) if \( \partial \Omega \) is topologically a sphere. If the boundary of \( \Omega \) has several components, then one can show that if \( m_B(\Omega) = 0 \),
then Ω is still a domain in \( \mathbb{R}^3 \) provided that each component of \( \partial \Omega \) is topologically a sphere. This follows from the following lemma.

**Lemma 4.2.** Let \((Ω, g)\) be a compact three manifold with boundary with nonnegative scalar curvature such that Ω has an admissible extension. Suppose \(m_B(Ω) = 0\). Let \(Σ_1, \ldots, Σ_k\) be the boundary components of Ω. Then Ω can be extended to a manifold \(\tilde{Ω}\) with compact closure with nonnegative scalar curvature such that \(\tilde{Ω}\) has only one boundary component which is equal to \(Σ_i\) for some \(i\) and \(m_B(\tilde{Ω}) = 0\).

**Proof.** Let \(M_l\) be a minimizing sequence of admissible extensions of Ω for \(m_B(Ω)\). Then each \(M_l\) is diffeomorphic to \(\mathbb{R}^3\). Each \(Σ_j\) separates \(\mathbb{R}^3\) into two components. Hence by taking a subsequence if necessary, we may assume that Ω is in the interior of \(Σ_1\), say, for all \(l\). Let \(\tilde{Ω}\) be the interior of \(Σ_1\) in \(M_1\). Note that \(\tilde{Ω}\) is diffeomorphic to the unit ball in \(\mathbb{R}^3\). Then \(Ω \subset \tilde{Ω}\), the scalar curvature of \(\tilde{Ω}\) is nonnegative, and \(\partial \tilde{Ω} = Σ_1\). We want to prove that \(m_B(\tilde{Ω}) = 0\).

Let \(\tilde{M}_l\) be the manifold obtained in the following way: \(\tilde{M}_l = M_l\) on \(M_l \setminus\) interior of \(Σ_1\) and replace Ω by \(\tilde{Ω}\) in the interior of \(Σ_1\), so that in a neighborhood of \(Σ_1\), \(M_l = \tilde{M}_l\). We want to prove that if \(l\) is large, \(\tilde{M}_l\) will not contain any stable minimal spheres.

Choose a \(\epsilon > 0\) be small enough, such that \(\{x ∈ Ω | d(x, Σ_1) = \epsilon\}\) is also topologically a sphere and so that if \(U = \{x ∈ Ω | d(x, Σ_1) < \epsilon\}\), then \(\overline{U} \cap Σ_j = \emptyset\) for all \(j \neq 1\). Let \(N\) be an outermost horizon of \(M_l\). If \(N \cap (\tilde{Ω} \setminus U) = \emptyset\), then \(N \subset M_l\). This is impossible by the fact that \(M_l\) is admissible and the fact that each component of \(N\) is a stable minimal sphere, see [12, 4]. Hence \(N \cap (\tilde{Ω} \setminus U) \neq \emptyset\) and there is a constant \(c > 0\) independent of \(i\), such that \(|N| ≥ c > 0\) by Lemma 2.3. On the other hand,

\[
\lim_{l \to \infty} m_{ADM}(\tilde{M}_l) = \lim_{l \to \infty} m_{ADM}(M_l) = 0.
\]

Hence by the Penrose inequality in [12, 4], no such \(N\) exists if \(l\) is large enough.

Note that \(\tilde{M}_l\) is homeomorphic to \(\mathbb{R}^3\). Hence \(\tilde{M}_l\) contains no minimal projective plane. If \(\tilde{M}_l\) contains a minimal sphere \(S\), then strictly minimizing hull of the interior of \(S\) is an outermost horizon. Hence no such \(S\) exists. We conclude that \(\tilde{M}_l\) is an admissible extension of \(\tilde{Ω}\) and so \(m_B(\tilde{Ω}) = 0\). \(\square\)

From this, we obtain the following:

**Proposition 4.3.** Let \((Ω, g)\) be as in the lemma. Suppose \(m_B(Ω) = 0\) and each \(Σ_i\) is topologically a sphere, then Ω is a domain in \(\mathbb{R}^3\).
In case the boundary of \( \partial \Omega \) is not a sphere, it is still unknown if \( m_B(\Omega) = 0 \) will imply \( \Omega \) is a domain in \( \mathbb{R}^3 \). In the following, using Theorem [4.1] we are going to construct a locally flat compact manifold which either has no admissible extension or its Bartnik mass is positive.

**Example 4.4.** Consider the cylinder in \( \mathbb{R}^3 \):

\[
C = \{ (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq r^2, 0 \leq z \leq l \}.
\]

Let \( T \) be the solid torus by identifying \( (x, y, 0) \) and \( (x, y, l) \) in \( C \). If \( l \) is small enough depending only on \( r \), then the Hawking mass of \( T \) is positive. Moreover, if \( T \) has an admissible extension, then the Bartnik mass is also positive.

To prove that \( m_H(T) > 0 \), it is easy to see that the mean curvature \( H \) of \( \partial C \) is \( \frac{1}{r} \), and

\[
\int_{\partial C} H^2 = \frac{2\pi l}{r},
\]

Suppose \( l < 8r \), hence

\[
\int_{\partial C} H^2 < 16\pi.
\]

We have \( m_H(T) > 0 \).

Now suppose that \( T \) has an admissible extension, we may conclude that \( m_B(T) > 0 \) because of the following consequence of Theorem [4.1]:

**Proposition 4.5.** Suppose \( (\Omega, g) \) has an admissible extension so that \( \partial \Omega \) is connected. Suppose \( \partial \Omega \) has positive mean curvature and suppose \( m_H(\partial \Omega) > 0 \). Then \( m_B(\Omega) > 0 \).

**Proof.** Choose \( \epsilon_0 > 0 \) be small enough such that \( \partial \Omega_{\epsilon} \) is smooth with positive mean curvature for all \( 0 < \epsilon < \epsilon_0 \), where

\[
\Omega_{\epsilon} = \{ x \in \Omega | d(x, \partial \Omega) > \epsilon \}.
\]

Moreover, by choosing \( \epsilon_0 \) small enough, we may also assume that \( m_H(\partial \Omega_{\epsilon}) > 0 \) for all \( 0 < \epsilon < \epsilon_0 \). Then each \( \Omega_{\epsilon} \) is a minimizing hull in \( \Omega \) because \( \Omega \setminus \Omega_{\epsilon_0} \) is foliated by positive mean curvature surfaces. Hence \( m(\Omega_{\epsilon}; \Omega) > 0 \) provided \( \epsilon > 0 \) is small enough. In particular, \( m(\Omega) > 0 \).

By Theorem [4.1] the proposition follows.

Even though it is unclear if the solid torus \( T \) in Example 4.4 has an admissible extension. One can prove that \( T \) has no smooth static extension. Namely, there is no AF manifold \( (M, g) \) with one end satisfying the following:

(a) \( g \) is smooth.
(b) \( T \) is isometrically embedded in \( M \).
(c) There is a smooth function $u$ defined on $M \setminus \text{interior of } T$ such that $u \to 1$ at infinity, $\text{Ric} = u^{-1} \nabla^2 u$ and $\Delta u = 0$. In fact, if $(M, g)$ is such an extension, then by [18], $u = 1 - \frac{m}{r} + O(r^{-2})$ where $r$ is the Euclidean distance in a coordinate system near infinity in the definition of AF manifold and $m$ is the ADM mass of $M$. Moreover, let $\Sigma = \partial T$, then on $\Sigma$ (see [18]):

$$\Delta_\Sigma u + H \frac{\partial u}{\partial \nu} = 0$$

where $\Delta_\Sigma$ is the Laplacian on $\Sigma$, $\nu$ is the unit outward normal and $H > 0$ is the mean curvature of $\Sigma$. Here we have used the fact that $\Sigma$ is flat, $H$ is constant and $T$ is locally flat. Hence we have

$$0 = \int_{M \setminus T} \Delta u = 4\pi m - \int_\Sigma \frac{\partial u}{\partial \nu} = 4\pi m + H^{-1} \int_\Sigma \Delta_\Sigma u = 4\pi m.$$

Hence $M$ must be $\mathbb{R}^3$. This is impossible by the uniqueness of embedding.

**References**

1. Bartnik, R., *New definition of quasilocal mass*, Phys. Rev. Lett. 62, (1989), 2346–2348.
2. Bartnik, R., *Energy in general relativity*, Tsing Hua lectures on geometry and analysis, (Hsinchu, 1990–1991), Internat. Press, Cambridge, MA, 1997, 5–27.
3. Bartnik, R., *Mass and 3-metrics of non-negative scalar curvature*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 231–240, Higher Ed. Press, Beijing, 2002.
4. Bray, H. L. *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. 59 (2001), 177–267.
5. Brown, J. D. and York, J. W., *Quasilocal energy in general relativity in Mathematical aspects of classical field theory* (Seattle, WA, 1991), Contemp. Math. 132, Amer. Math. Soc., Providence, RI, (1992), 129-142.
6. Brown, J. D. and York, J. W., *Quasilocal energy and conserved charges derived from the gravitational action*, Phys. Rev. D (3), textbf 47 (no. 4), (1993), 1407-1419.
7. Christodoulou, D. and Yau, S. T., *Some remarks on quasi-local mass*. Contemporary mathematics, Mathematics and General Relativity J. isenberg (ed.)(1986)9-14.
8. Flanagan, E., *Hoop conjecture for black-hole horizon formation*, Phys. Rev.D 44, 2409(1991).
9. Galloway, G. J., *On the topology of black holes*, Comm. Math. Phys. 151 (1993), 53–66.
10. Giusti, E., *Minimal surfaces and functions of bounded variation*, Notes on pure mathematics 10, Department of Pure Mathematics, 1977.
11. Hirsch, M. W., *Differential topology* Springer-Verlag, New York, 1976.
12. Huisken, G. and Ilmanen, T., *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. 59 (2001), 353–437.
13. Klingenberg, W. *A course in differential geometry*, Graduate Texts in Mathematics, Vol. 51. Springer-Verlag, New York-Heidelberg, 1978.
14. Meeks, W. H., Simon, L. and Yau, S.-T., *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. 116 (1982), 621–659.
15. Meeks, W. H., and Yau, S.-T., *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. 112 (1980), 441–484.
16. Miao, P., *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys. 6 (2002), no. 6, 1163–1182 (2003).
17. Miao, P., *Asymptotically flat and scalar flat metrics on $\mathbb{R}^3$ admitting a horizon*, Proc. Amer. Math. Soc. 132 (2004), 217–222 (electronic).
18. Miao, P., *A remark on boundary effects in static vacuum initial data sets*, Classical Quantum Gravity 22 (2005), no. 11, L53–L59.
19. Nirenberg, L., *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. 6, (1953). 337–394.
20. R. Schoen and S.-T. Yau *The existence of a black hole due to condensation of matter* Comm. Math. Phys. 90 (1983), 575-579.
21. Shi, Y.-G., Tam, L.-F., *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. 62 (2002), 79–125.
22. Spivak, M., *A Comprehensive introduction to differential geometry*, v.3, Publish or Perish, 1970-75.
23. Willmore, T. J., *Total curvature in Riemannian geometry*, Chichester: E. Horwood, 1982.
24. Yau, S.-T., *Geometry of three manifolds and existence of black hole due to boundary effect* Adv. Theor. Math. Phys. 5(2001)

**Key Laboratory of Pure and Applied Mathematics, School of Mathematics Science, Peking University, Beijing, 100871, P.R. China.**

*E-mail address: ygshi@math.pku.edu.cn*

**Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.**

*E-mail address: lftam@math.cuhk.edu.hk*