COREGULARITY OF FANO VARIETIES

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Abstract. The regularity of a Fano variety, denoted by \( \text{reg}(X) \), is the largest dimension of the dual complex of a log Calabi–Yau structure on \( X \). The coregularity is defined to be

\[
\text{coreg}(X) := \dim X - \text{reg}(X) - 1.
\]

The coregularity is the complementary dimension of the regularity. We expect that the coregularity of a Fano variety governs, to a large extent, the geometry of \( X \). In this note, we review the history of Fano varieties, give some examples, survey some important theorems, introduce the coregularity, and propose several problems regarding this invariant of Fano varieties.

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1. Preliminaries

In this note, we introduce and study the coregularity of Fano varieties. In this section, we present some basic objects that will be used to introduce the invariant: canonical line bundle, Fano manifolds, Fano varieties, Calabi–Yau pairs, theory of complements, and dual complexes.

1.1. Canonical line bundle. One of the main aims of algebraic geometry is to classify smooth projective varieties. Given a \( n \)-dimensional smooth projective variety \( X \subset \mathbb{P}^N \), we can define its tangent bundle \( T_X \). Dualizing the tangent bundle, we obtain the cotangent bundle \( \Omega_X := T_X^\ast \). In some sense, the tangent bundle and the cotangent bundle, are the only natural vector bundles that we can associate with any smooth projective variety. The canonical line bundle, denoted by \( \omega_X \), is the \( n \)-th exterior power of the cotangent bundle, i.e., \( \omega_X := \bigwedge^n \Omega_X \). Over the complex numbers, \( \omega_X \) is the determinant bundle of holomorphic forms on \( X \). The previous bundles are independent of the chosen projective embedding of \( X \). The canonical divisor \( K_X \) is a divisor on \( X \) for which \( \omega_X \cong \mathcal{O}_X(K_X) \). There are three pure classes of smooth projective varieties, depending on the positivity or negativity of its canonical line bundle:

1. A smooth projective variety \( X \) is said to be Fano if \( \omega_X \) is anti-ample, i.e., \( \omega_X \cdot C < 0 \) for every curve \( C \subset X \).

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(2) A smooth projective variety $X$ is said to be Calabi–Yau if $\omega_X$ is numerically trivial, i.e., $\omega_X \cdot C = 0$ for every curve $C \subset X$.

(3) A smooth projective variety $X$ is said to be canonically polarized if $\omega_X$ is ample, i.e., $\omega_X \cdot C > 0$ for every curve $C \subset X$.

The previous trichotomy generalizes to higher-dimensional varieties the classic trichotomy of Riemann surfaces: the Riemann sphere, complex tori, and curves of genus at least two. A smooth hypersurface $H$ of degree $d$ is Fano (resp. Calabi–Yau and canonically polarized) if $d < n+1$ (resp. $d = n+1$ and $d > n+1$). Among algebraic varieties, we expect that Fano varieties are easier to understand due to multiple reasons. In dimension one there is only one them, so the initial expectation is that there are fewer Fano varieties than Calabi–Yau and canonically polarized varieties in any given dimension. On the other hand, they tend to behave similarly to a projective space. They tend to be simply connected, covered by holomorphic Riemann spheres, and rigid under small deformations. In this note, we focus on the study of Fano varieties, although this often intertwinds with a better understanding of both Calabi–Yau and canonically polarized varieties.

1.2. Fano manifolds. The classification of smooth surfaces with ample anti-canonical divisor was achieved by Pasquale del Pezzo in 1887. These surfaces are currently known as del Pezzo surfaces and they form 9 families depending on the degree of the anti-canonical divisor $-K_X$. The concept of Fano manifolds was studied by Leonard Roth [107] and by Gino Fano [41] in the 30’s and 40’s, respectively. The latter formally introduces the concept of Fano manifolds. Thus, del Pezzo surfaces are smooth Fano surfaces. During the 50’s and the 60’s there were some mentions to Fano manifolds, but it was nothing near a central topic in algebraic geometry. The first systematic study of Fano varieties was led by Iskovskih in [59, 60]. In this paper, Iskovskih combines the recent techniques introduced by Grothendieck to algebraic geometry and Fano’s brilliant intuition to give a complete treatment of smooth Fano 3-folds. In these papers, the main invariant to classify smooth Fano 3-folds is the index. The index of a Fano manifold $X$ is defined to be:

$$i(X) := \max\{r \in \mathbb{Z}_{\geq 1} \mid rH \sim K_X \text{ for } H \text{ ample Cartier on } X\}.$$ 

The main gadget to study these manifolds was to find smooth elements $S \in \mid H\mid$. The existence of these smooth elements was proved by Shokurov [118].

The index of a Fano manifold of dimension 3 is an integer in $\{1, 2, 3, 4\}$. If $i(X) = 4$, then $X \simeq \mathbb{P}^3$. If $i(X) = 3$, then $X$ is isomorphic to a quadric in $\mathbb{P}^4$. If $i(X) = 2$, then $H^3 \in \{1, \ldots, 7\}$. If $H^3 \in \{5, 6, 7\}$, then there is a unique Fano 3-fold $V_d \subset \mathbb{P}^{d+1}$. If $H^3 \in \{1, 2, 3, 4\}$ gives weighted hypersurfaces or complete intersections. Most Fano manifolds have index $i(X) = 1$ and the anti-canonical map, defined by $| - K_X|$, is a morphism $\phi_{-K_X} : X \to \mathbb{P}^N$. In what follows, we set $2g - 2 = -K_X^3$. There are three possible behaviours in the case of index one:

- the anti-canonical map $\phi_{-K_X}$ is an embedding into $\mathbb{P}^{2g-2}$ and the image is an intersection of quadrics;
- hyperelliptic threefolds: the anti-canonical map $\phi_{-K_X}$ maps $X$ into $\mathbb{P}^{g+1}$ with a degree two map onto its image; and
- trigonal threefolds: the image of the anti-canonical map $\phi_{-K_X}$ embeds $X$ in $\mathbb{P}^{g+1}$ and its image is contained in a 4-fold scroll.

Using the previous description, Iskovskih gave a classification of smooth Fano 3-folds of Picard rank one. In [9], Batyrev classified smooth toric Fano 3-folds. In [95], Mori and Mukai classified 3-dimensional Fano manifolds with second Betti number of at least two. Conte started the study of Fano 3-folds with Gorenstein singularities in [34]. Soon later, Maeda considered log Fano 3-folds, i.e., log smooth pairs $(X, \Delta)$ for which
In a few words, the morphism is not an isomorphism), then we say that the previous context, we say that

Thus, nice singularities should correspond to larger values of log discrepancies. Let $\phi: Y \rightarrow X$ be a projective birational morphism and $E \subset Y$ be a prime divisor. The log discrepancy of $(X, \Delta)$ at $E$, is defined to be

$$a_E(X, \Delta) = 1 - \text{coeff}_E(K_Y - \phi^*(K_X + \Delta)).$$

In a few words, $a_E(X, \Delta)$ measures how singular is $(X, \Delta)$ along the tangent directions corresponding to the points of $E$. Thus, nice singularities should correspond to larger values of $a_E(X, \Delta)$ for every possible $E$. In the previous context, we say that $E$ is a prime divisor over $X$. If $E$ lies in the exceptional locus of $\phi$ (where the morphism is not an isomorphism), then we say that $E$ is exceptional over $X$. This leads to the following definitions:

1. We say that $(X, \Delta)$ is terminal if $a_E(X, \Delta) > 1$ for every prime divisor $E$ exceptional over $X$.
2. We say that $(X, \Delta)$ is canonical if $a_E(X, \Delta) \geq 1$ for every prime divisor $E$ over $X$.
3. We say that $(X, \Delta)$ is log terminal (or Kawamata log terminal) if $a_E(X, \Delta) > 0$ for every prime divisor $E$ over $X$. We write klt for short.
4. We say that $(X, \Delta)$ is log canonical if $a_E(X, \Delta) \geq 0$ for every prime divisor $E$ over $X$. We write lc for short.

The class of terminal singularities is the largest class of singularities obtained by the minimal model program for smooth varieties. On the other hand, log canonical singularities are the wider class of singularities in...
which we expect that the MMP can be run. Quotient singularities are klt [66]. Symplectic singularities are canonical [105]. The Reid-Tai criterion allows us to discern whether a quotient singularity is canonical or terminal [106]. The affine cone over a rational normal curve is also klt. On the other hand, a cone over an elliptic curve is lc but not klt (see, e.g., [66]). The boundary $\Delta$ can often be thought of as a correction term for the canonical divisor $K_X$ not being $\mathbb{Q}$-Cartier. We say that a variety $X$ is klt type (resp. lc type) if there exists a boundary $\Delta$ for which $(X, \Delta)$ is klt (resp. lc).

We say that variety $X$ is Fano (also known as $\mathbb{Q}$-Fano) if $X$ has klt singularities and $-K_X$ is ample. We say that a variety $X$ is of Fano type if there exists a boundary $\Delta$ on $X$ for which $(X, \Delta)$ has klt singularities and $-(K_X + \Delta)$ is ample. From Example 1.1, we deduce that every finite quotient of a Fano manifold is of Fano type. Furthermore, every projective toric variety is of Fano type. Even if del Pezzo surfaces are classified, we can find far more Fano varieties of dimension two (see, e.g., [98, 86]).

The books by Prokhorov and Iskovskikh [109, 62] gives a very good account for the study of Fano varieties, especially those of dimension three. Many results regarding rationally connectedness, rigidity, and simply connectedness of Fano manifolds were generalized to the Fano type setting [121, 113, 122, 37]. Fano varieties have also been extensively studied from the perspective of birational rigidity [104, 33]. It is worth mentioning that Fano varieties are tightly related to mirror symmetry [28]. In [2], Alexeev proved that Fano type surfaces with mild singularities form bounded families.

1.4. Calabi–Yau pairs. Analogously, a Calabi–Yau pair is the singular generalization of a Calabi–Yau manifold. Again, we consider pairs instead of varieties.

A log Calabi–Yau pair is a pair $(X, \Delta)$ with log canonical singularities for which $K_X + \Delta = 0$, i.e., the intersection of every curve with the $\mathbb{Q}$-divisor $K_X + \Delta$ is zero. We say that a variety $X$ is of log Calabi–Yau type if there exists a boundary $\Delta$ on $X$ for which the pair $(X, \Delta)$ is log Calabi–Yau. Every Calabi–Yau manifold is already a variety of log Calabi–Yau type. Analogously as in Example 1.1, the finite quotient of a Calabi–Yau manifold is a variety of log Calabi–Yau type.

Every Fano type variety is log Calabi–Yau. Indeed, we can proceed as follows: if $-(K_X + \Delta)$ is an ample divisor, then we can take a large multiple for which $-m(K_X + \Delta)$ is a very ample divisor. So we can consider a general section

$$\Gamma \in |-m(K_X + \Delta)|$$

which intersects the support of $\Delta$ transversally. By choosing $\Gamma$ general enough, we achieve that the pair $(X, \Delta + \Gamma/m)$ is klt. Furthermore, by construction, we have that

$$K_X + \Delta + \Gamma/m \equiv 0.$$ 

The previous argument is in general not very effective, as we have no control over the integer $m$ which we need for this construction to work.

**Example 1.2.** For instance, if we consider the projective plane $\mathbb{P}^2$, there are several ways to produce a boundary $\Gamma$ for which $(\mathbb{P}^2, \Gamma)$ is log Calabi–Yau. We can take $\Gamma$ to be:

1. the sum of three transversal lines,
2. the sum of a line and a conic intersecting transversally, or
3. an elliptic curve.

Of course, if we allow fractional coefficients on the boundary (i.e, consider elements of $|-mK_{\mathbb{P}^2}|$ for $m \geq 2$), then we can construct far more examples.
In general, we try to seek for those boundaries \( \Gamma \) which have more components and smaller denominators in their coefficients. In other words, that belong to \( \lfloor -m(K_X + \Delta) \rfloor \) for smaller values of \( m \).

The singularities introduced in the previous subsection: Kawamata log terminal and log canonical are the local versions of Fano type and log Calabi–Yau type varieties. Indeed, the affine cone over a Fano type variety is of klt type and the affine cone over a log Calabi–Yau type variety is of lc type.

**Remark 1.3.** When we work with singularities of a pair, i.e., a precise closed point \( x \in (X, \Delta) \), we may write \( (X, \Delta; x) \) to mean that the statement that we are writing holds for the pair \( (X, \Delta) \) around the closed point \( x \).

In the other direction, given a klt singularity \( (X, \Delta; x) \) (resp. lc singularity), we can find a projective birational morphism that contracts a unique prime divisor \( E \) mapping onto \( x \) so that \( E \) admits the structure of a Fano type (resp. log Calabi–Yau type) variety. The previous statements are proved in [99, 71, 119]. The understanding of klt singularities has been closely intertwined with the understanding of Fano varieties.

1.5. **Theory of complements.** As pointed out in the previous subsection, every Fano type variety admits a boundary which turns it into a log Calabi–Yau pair. This choice always exists but it is highly non-unique. One of the aim goals of the theory of complements is to make this choice effective in terms of certain invariants of \( X \). The second aim is to make this choice in a somewhat canonical way, or at the very least, find certain distinguished log Calabi–Yau structures the Fano type variety.

A boundary \( B \) with \( \mathbb{Q} \)-coefficients on a normal projective variety \( X \) for which \( p_X, B \) is log Calabi–Yau, is called a \( \mathbb{Q} \)-complement. A \( \mathbb{Q} \)-complement is said to be a \( N \)-complement if the linear equivalence

\[ N(K_X + B) \sim 0 \]

holds. This means that \( NB \in \lfloor -NK_X \rfloor \). Note that in the previous definition we are not requiring the divisor \( K_X \) to be Cartier, not even that \( NK_X \) is Cartier. Hence, \( B \sim -NK_X \) is linear equivalence of Weil divisors, i.e., their difference is the divisor associated to a rational function on \( X \). Note that this notion even works for \( \mathbb{Q} \)-divisors.

The theory of complements was initiated by Prokhorov and Shokurov in the early 2000’s. In [110], Shokurov proved that surfaces of Fano type almost always admit a \( N \)-complement where \( N \in \{1, 2, 3, 6\} \).

More precisely, Shokurov proves the following theorem.

**Theorem 1.4.** Let \( X \) be a Fano type surface. Then, one of the following statements hold:

1. The variety \( X \) admits a \( N \)-complement where \( N \in \{1, 2, 3, 6\} \), or
2. For every \( B \) such that \( (X, B) \) is log Calabi–Yau, the pair \( (X, B) \) has klt singularities.

In other words, either we can find a 6-complement or it is not possible to produce a strictly log canonical singularity on \( (X, B) \). By Alexeev’s boundedness of Fano surfaces [2], the Fano surfaces in the second class of Theorem 1.4 belong to a bounded family, i.e., they can be described using a fixed number of variables and polynomial equations. These surfaces are known as exceptional Fano type surfaces. Although exceptional Fano type surfaces belong to a bounded family, until now we do not have a complete classification of these surfaces. Nevertheless, the fact that they belong to a bounded family implies that there exists a positive integer \( N \) for which all exceptional Fano type surfaces admit a \( N \)-complement. This implies the following theorem, due to Shokurov.

**Theorem 1.5.** There exists a positive integer \( N_2 \) satisfying the following. Let \( X \) be a Fano type surface. Then, \( X \) admits a \( N_2 \)-complement.
In [101], there is a good account of Shokurov’s proof of the boundedness of complements for surfaces and some further results are proved. As of today, it is unknown what is the optimal value for $N_2$. However, it is expected that $N_2 = 66$ suffices and this number is related to Sylvester’s sequence as we explain below.

The previous theorem motivated the problem known as “boundedness of complements”, i.e., find a constant $N_n$ only depending on the dimension $n$, such that every Fano type variety $X$ of dimension $n$ admits a $N_n$-complement. This problem is also known as the “existence of bounded complements”. In some words, the problem asks about an effective way of constructing a log Calabi–Yau variety in a Fano type variety. In [102, 103], Prokhorov and Shokurov started an approach to tackle the existence of bounded complements. The strategy consists of two main steps:

- lifting complements from log canonical places, and
- lifting complements from the base of the fibration by using the canonical bundle formula.

Hence, by running a suitable argument, one could aim to lift the so-called bounded complement from lower-dimensional Fano type varieties. In [12], Birkar settled the existence of bounded complements for $n$-dimensional Fano type varieties.

**Theorem 1.6.** There exists a constant $N_n$, only depending on $n$, satisfying the following. Let $X$ be a Fano type variety of dimension $n$. Then, there exists a $N_n$-complement.

This achievement used crucially some deep theorems in the minimal model program [16, 55]. Moreover, it is a vital step for the boundedness of Fano varieties [15]. A similar statement holds for klt singularities via global-to-local arguments (see subsection 1.7).

**Theorem 1.7.** There exists a constant $N_n$, only depending on $n$, satisfying the following. Let $(X; x)$ be a $n$-dimensional klt singularity. Then, there exists a boundary $B$ on $X$ for which $(X, B; x)$ is log canonical, $x$ is a log canonical center of $(X, B)$, and $N(K_X + B) \sim 0$ holds on a neighborhood of the point $x \in X$.

The boundary $B$ constructed in the previous statement is called a $N$-complement of the klt singularity $(X; x)$. It is worth mentioning, that we expect the existence of bounded complements to hold for log Calabi-Yau varieties as well. More precisely, whenever a variety $X$ admits a log Calabi–Yau structure it also admits an effective log Calabi–Yau structure. In the language of complements: if a $n$-dimensional variety $X$ admits a $\mathbb{Q}$-complement, then it admits a $N_n$-complement. In [45], the authors proved this statement in dimension at most three.

**Theorem 1.8.** There exists a constant $N_3$ satisfying the following. Let $X$ be a normal projective variety of log Calabi–Yau type and dimension at most 3. Then, $X$ admits a $N_3$-complement.

The existence of bounded complements for log Calabi-Yau varieties of dimension at least 4 remains open. The theory of complements helped to make big breakthroughs in the study of Fano varieties: The boundedness of Fano varieties with mild singularities [12, 14] and the algebro-geometric K-stability theory [79]. In the latter, the authors introduced the concept of special complements which behaves well with respect to the normalized volume. It is expected that further understanding of complements will be crucial for the development of Fano type varieties and klt singularities. In what follows, we propose some new special classes of complements related to dual complexes of Calabi-Yau type varieties.

### 1.6 Dual complexes

A dual complex is a combinatorial object that encrypts the data of the intersection of divisors on a normal variety.

Given a normal variety $X$ and a reduced divisor $E = \sum_{i \in I} E_i$ with simple normal crossing support. We define the dual complex $D(E)$ to be the CW complex whose $k$-dimensional cells $v_W$ correspond to irreducible
components $W$ of $\bigcap_{j \in J} E_j$ where $J \subset I$ and $|J| = k + 1$. Given $j_0 \in J$, the irreducible variety $W$ is contained in a unique irreducible component $Z$ of $\bigcap_{\ell \neq j_0} E_{\ell}$. This inclusion induces a gluing map $v_Z \leftrightarrow v_W$. The dimension of the dual complex $\mathcal{D}(E)$ is the largest dimension of its cells.

Now, we turn to define the dual complex of a log Calabi–Yau pair $(X, B)$. The naive idea is to define it as the dual complex of $[B]$. However, there are two issues with this. First, this definition does not behave well under birational modifications as a blow-up of $X$ may introduce new divisors with coefficient one in the boundary of the log Calabi–Yau pair. On the other hand, the divisor $[B]$ may not have simple normal crossing support. The concept of divisorially log terminal models was introduced, in part, to fix these two issues.

Let $(X, B)$ be a log canonical pair. A log canonical place of $(X, B)$ is a prime divisor $E$ over $X$ for which $a_E(X, B) = 0$. We say that $(X, B)$ is strictly log canonical if it is log canonical and admits a log canonical center. A log canonical center of $(X, B)$ is the center on $X$ of a log canonical place of $(X, B)$. A log canonical pair $(X, B)$ is said to be divisorially log terminal (or dlt) if there exists an open subset $U \subseteq X$ satisfying the following conditions:

1. the divisor $[B]|_U$ has simple normal crossing support,
2. every log canonical center of $(X, B)$ intersects $U$, and
3. the log canonical centers of $(X, B)$ are given by strata of $[B]$.

In a few words, the pair $(X, B)$ is dlt if it looks simple normal crossing at the generic point of every log canonical center. If $(X, B)$ is dlt, then blowing up a stratum of $[B]$ induces a barycenter subdivision on one of the simplices of the dual complex $\mathcal{D}([B])$. Hence, the dual complex of $(X, B)$, seen as a topological space, stabilizes when the pair has dlt singularities. The following theorem due to Hacon allows us to birationally transform log canonical pairs into dlt pairs (see, e.g., [67]).

**Theorem 1.9.** Let $(X, B)$ be a log canonical pair. There exists a projective birational morphism $\phi: Y \to X$ satisfying the following conditions:

1. the pair $(Y, B_Y)$ has dlt singularities, where $\phi^*(K_X + B) = K_Y + B_Y$, and
2. the projective birational morphism $\phi$ only contracts divisors with log discrepancy zero with respect to $(X, B)$.

The previous theorem is known as the existence of dlt modifications. The dlt modification allows us to define the dual complex of a log Calabi–Yau pair as follows. This motivates the following definition.

Let $(X, B)$ be a log Calabi–Yau pair. Let $\phi: Y \to X$ be a dlt modification of $(X, B)$. We define the dual complex $\mathcal{D}(X, B)$ of $(X, B)$ to be $\mathcal{D}([B_Y])$. A priori, the dual complex depends on the chosen dlt modification. Indeed, as explained before, we can obtain different triangulations by performing more blow-ups. However, De Fernex, Kollár, and Xu proved that two such dual complexes are simple-homotopy equivalent (see, e.g., [38]).

**Theorem 1.10.** Let $(X, B)$ be a log Calabi–Yau pair. Then its dual complex $\mathcal{D}(X, B)$ is well-defined up to simple-homotopy equivalence.

Is expected that the dual complexes of log Calabi–Yau pairs are PL-homeomorphic to PL-spheres up to a finite cover. More precisely, we have the following conjecture.

**Conjecture 1.11.** Let $(X, B)$ be a n-dimensional log Calabi–Yau pair. There exists a finite cover $\pi: Y \to X$ such that $\mathcal{D}(Y, B_Y) \simeq_{PL} S^{n-1}$.

By [69], this conjecture is known for varieties of dimension at most 4. This conjecture is also known for Mori fiber spaces whenever the base has dimension at most two or when the total space has Picard at most
two, by the work of Mauri [83]. By [90], there are only finitely many possible fundamental groups for the
dual complexes of $n$-dimensional log Calabi–Yau threefolds. Furthermore, by [20], these fundamental groups
are finite. Log Calabi–Yau varieties also appear when compactifying character varieties. In this direction,
the previous conjecture has been studied by Simpson [111] and Goldman and Toledo [50]. Even though the
previous conjecture is known up to dimension 4, very little is known about the possible triangulations of the
sphere that we can obtain in dual complexes of log Calabi–Yau pairs. Any smooth lattice polytope can be
achieved. Indeed, we can consider the projective toric variety corresponding to its dual fan with its reduced
toric boundary.

The following definition will be useful to state some of the results in this article.

Definition 1.12. Let $(X, \Delta)$ and $(X', \Delta')$ be two log pairs such that $X$ and $X'$ are birational. We say that
$(X, \Delta)$ and $(X', \Delta')$ are log crepant equivalent if there is a common resolution $p: Z \to X$ and $q: Z \to X'$ for
which the equality

$$p^*(K_X + \Delta) = q^*(K_{X'} + \Delta')$$

holds. Observe that log crepant pairs share many common properties. For instance, they have the same log
discrepancies, $(X, \Delta)$ is log Calabi–Yau if and only if $(X', \Delta')$ is log Calabi–Yau. In this case, they have the
same coregularity and dual complex.

1.7. Fano varieties and klt singularities. Throughout the preliminaries, we have discussed both Fano
varieties and klt singularities. We culminate the preliminary by presenting some global-to-local and local-
to-global principle that enlighten the tight relation between these global and local objects.

Given a projective variety $X$ and an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $A$ on $X$, we can consider the orbifold cone
$C_X(A)$ of $X$ with respect to the $\mathbb{Q}$-polarization $A$ to be the spectrum of the following ring:

$$\bigoplus_{m \in \mathbb{Z}} H^0(X, \mathcal{O}_X(mA)),$$

where we multiply the sections as elements of $K(X)$. The affine variety $C_X(A)$ is endowed with a $\mathbb{G}_{m^\times}$-action
and has a unique fixed point which we usually call the vertex of the action. We denote the vertex $x_0$ if there
is no room for confusion. The singularities obtained in the previous way are called cone singularities. Let
$\pi: Y \to C_X(A)$ be the blow-up of the maximal ideal at $x_0$. Then $\pi$ extracts a unique prime divisor $E$ over
$x_0$ which is isomorphic to $X$. If we perform adjunction of $K_X + E$ to $E$, under this isomorphism, we obtain

$$(1.2) \quad K_X + E|_E \sim_{\mathbb{Q}} K_X + \sum_{P \in X} \left(1 - \frac{1}{n_P}\right) P$$

where the sum runs over all the prime divisors $P$ of $X$ and $n_P$ is the Weil index of $A$ at $P$. In what follows,
we write $\Delta_A$ for the boundary in (1.2). In particular, if $A$ is a Cartier divisor, then we have

$$K_X + E|_E \sim_{\mathbb{Q}} K_X.$$

The previous adjunction procedure allows comparing the singularities of $X$ with those of $C_X(A)$. The
following result is proved in [66].

Proposition 1.13. Let $X$ be a projective variety and $A$ be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Then, the
following statements hold:

1. the singularity $(C_X(A); x_0)$ is klt if and only if $(X, \Delta_A)$ is log Fano,
2. the singularity $(C_X(A); x_0)$ is canonical if and only if $(X, \Delta_A)$ is log Fano and $-(K_X + \Delta_A) \sim_{\mathbb{Q}} rA$
   for $r \geq 1$, and
(3) the singularity $(C_X(A); x_0)$ is terminal if and only if $(X, \Delta_A)$ is log Fano and $-(K_X + \Delta_A) \sim_{\mathbb{Q}} rA$ for $r > 1$.

The previous procedure allows constructing a singularity $C_X(A)$ out of a projective variety $X$ and a $\mathbb{Q}$-polarization. In general, we may obtain the same singularity $(C_X(A); x_0)$ for different choices of $X$ and the polarization. However, if we consider cone singularities with the endowed $\mathbb{G}_m$-action, then two cone singularities $(C_X(A); x_0)$ and $(C_Y(A_Y); y_0)$ are equivariantly isomorphic if and only if there is an isomorphism $\phi: X \to Y$ for which $\phi^* A_Y = A$. At any rate, the procedure of producing a klt type singularity out of a Fano variety $X$ is not unique as we may choose several different ample divisors on $X$. Anyways, the cone construction is a powerful tool to reduce projective problems to local problems.

In the other direction, given a klt singularity $(X; x)$. One possible way to associate a projective variety to $(X; x)$ is to find a strictly log canonical complement $(X, B; x)$ and take a dlt modification $(Y, B_Y) \to (X, B; x)$. Then, the exceptional divisors that map to $x$ are projective varieties. However, there are some issues with this construction. First, we may extract several divisors over the singularity $x \in X$. Second, these divisors may not be Fano. However, we know that the fibers of the dlt modification (more generally, every Fano type morphism) are rationally connected (see, e.g., [54]). In this situation, the solution is to run a suitable minimal model program on $Y$ over $X$ that will contract all divisors except a single one. This leads to the following proposition proved by Prokhorov and Xu (see, e.g., [101, 119]).

**Proposition 1.14.** Let $(X, B; x)$ be a klt singularity. There exists a projective birational morphism $\pi: Y \to X$ satisfying the following conditions:

- the exceptional locus of $\pi$ consists of a unique prime divisor $E$ mapping onto $x$,
- the divisor $-E$ is ample over $X$, and
- the pair $(Y, B_Y + E)$ has plt singularities,

where $B_Y$ is the strict transform of $B$ on $Y$.

The projective birational morphism $\pi: Y \to X$ as in the previous proposition is called a purely log terminal blow-up or plt blow-up for short. Hence, the previous statement says that every klt singularity admits a plt blow-up. The plt blow-up may not be unique. For instance, for a toric singularity $(T; t)$, every toric projective morphism $Y \to T$ that extracts a unique toric divisor over $t$ is a plt blow-up. Plt blow-ups often help to reduce problems about klt singularities to problems about Fano varieties.

Both constructions; the cone construction and plt blow-ups, are in some sense inverse to each other. On one side, if $X$ is a Fano variety and $A$ is an ample Cartier divisor, then $(C_X(A); x_0)$ is a klt singularity and the blow-up of $x_0$ is a plt blow-up whose exceptional divisor is isomorphic to $X$. On the other hand, if $(X; x)$ is a klt singularity and $Y \to X$ a plt blow-up extracting an exceptional divisor $E$, then the singularity $(X; x)$ degenerates to an orbifold cone over $E$. In general, it is not true that $(X; x)$ is itself an orbifold cone singularity.

2. The Coregularity

In this section, we turn to introduce the main object of this note: the coregularity. The coregularity measures the difference between the dimension of $X$ and the dimension of the largest dual complex among log Calabi–Yau structures of $X$. For simplicity, we first define its counterpart, the regularity.

**Definition 2.1.** Let $X$ be a normal projective variety. The regularity of $X$, denoted by $\text{reg}(X)$, is defined to be

$$\max \{ \dim \mathcal{D}(X, B) \mid (X, B) \text{ is log Calabi–Yau} \}.$$
If the previous set is empty, i.e., if $X$ is not of log Calabi–Yau type, then we say that the regularity is infinite. We set the dimension of the empty set to be $-1$. Note that $\text{reg}(X) = -1$ if and only if $X$ admits a log Calabi–Yau structure and every log Calabi-Yau structure is klt. In other words, $X$ is exceptional. If the regularity is a positive integer $X$ admits a log Calabi–Yau structure $(X, B)$ which admits a log canonical center.

The coregularity of $X$ is defined to be

$$\text{coreg}(X) := \dim X - \text{reg}(X) - 1.$$ 

The coregularity of $X$ is negative infinite if $X$ does not admit a log Calabi–Yau structure. If the coregularity of $X$ is non-negative, then it equals the dimension of the smallest log canonical center among dlt modifications of log Calabi–Yau pairs $(X, B)$. If $(X, B)$ is a log Calabi–Yau pair, then we set $X$ to be a log canonical center of the pair $(X, B)$. Even if this seems unnatural, it does fit the adjunction formula.

Analogously, we may define the local regularity of a normal projective variety $X$ at a closed point $x \in X$. The local regularity, denoted by $\text{reg}(X; x)$, is defined to be

$$\max\{\dim D(X, B; x) \mid (X, B; x) \text{ is log canonical}\}.$$ 

If the previous set is empty, i.e., if $(X; x)$ is not of log canonical type, then we set the local regularity to be infinite. Otherwise, $(X; x)$ admits a log canonical singularity structure. If $(X; x)$ is a klt type singularity, then the regularity is always at least zero. Again, the coregularity is the dimension of the smallest minimal log canonical center among the dlt modifications of lc singularities $(X, B; x)$.

In [110, Section 7], Shokurov defined the regularity of a contractions. He mentions that the regularity characterizes the topological difficulty of the pair. The previous definitions can be given in a more general context of Fano type morphisms $X \to Z$ with a marked closed point $z \in Z$. If $Z$ is a point, then this recovers the global definition. If $X \to Z$ is the identity, then this recovers the local definition. Our definition of regularity tries to maximize the dimension among all the possible dual complexes. On the other hand, our definition of coregularity tries to minimize the difference between $\dim X$ and $\text{reg}(X, B)$.

As for invariants of Fano varieties, the most well-known invariants are the index and the anti-canonical volume $(-K_X)^n$. As explained before, the index measures how divisible the anti-canonical divisor $-K_X$ is in the Picard group of $X$. The volume controls the asymptotic growth of sections of multiples of the anti-canonical divisor. It is expected that most Fano varieties have index one. And, although this statement can not be made precise in a general setting, it is often the case when looking at precise families of examples. However, we do not expect any general framework for Fano varieties of index one. On the other hand, there is no such expectation for volumes. The volume, even in the toric case, can take many different values in a fixed dimension. Usually, bounding the volume from below and/or above is related to boundedness statements in algebraic geometry. As for the coregularity, we expect that most Fano varieties have coregularity zero.

In the following sections, we give some examples and propose structural conjectures about Fano varieties of coregularity zero.

3. Examples and Properties

This section explores examples and properties of the coregularity. In the first part of this section, we focus on examples. In subsection 3.1, we discuss del Pezzo surfaces, while in subsection 3.4, we give examples of singular del Pezzo surfaces. In subsection 3.2, we discuss finite quotient singularities and in subsection 3.3, we study the projective analogue; finite quotients of the projective space. In subsection 3.5, we discuss terminal 3-fold singularities, while in subsection 3.6, we study klt 3-fold singularities in a more general
setting. In this second part of this section, we focus on properties. We study Fano varieties with torus actions in subsection 3.7, the coregularity under morphisms in subsection 3.8, and the coregularity under deformations and degenerations in subsection 3.9. Finally, in subsection 3.10 and subsection 3.11, we give further properties and examples of dual complexes.

3.1. Del Pezzo surfaces. Del Pezzo surfaces are classified by the anti-canonical volume $(-K_X)^2$ which is also called the degree of the del Pezzo surface. The degree of a del Pezzo surface is an integer in \{1, 2, \ldots, 9\}.

There is a unique del Pezzo surface of degree 9, the projective plane $\mathbb{P}^2$. In this case, we can take the $B = L_1 + L_2 + L_3$ so that $\mathcal{D}(\mathbb{P}^2, B)$ is a circle with three marked points corresponding to the lines. Then, the coregularity of $\mathbb{P}^2$ is zero. A del Pezzo surface of degree $d$ in $\{8, 7, 6\}$ is the blow-up of $\mathbb{P}^2$ at $9 - d$ points.

For these surfaces, we may apply an automorphism of $\mathbb{P}^2$ and assume that they are the intersection points of $L_1 \cap L_2$, $L_1 \cap L_3$, and $L_2 \cap L_3$. Hence, the dual complex of the log pull-back of $(\mathbb{P}^2, L_1 + L_2 + L_3)$ to these models are circles with $12 - d$ marked points. Indeed, the log discrepancy of the exceptional divisors is 2 with respect to $\mathbb{P}^2$. Indeed, these are just the blow-ups of smooth points. Moreover, each line decreases this log discrepancy exactly by one. Hence, the del Pezzo surfaces of degree $d \in \{8, 7, 6\}$ have coregularity zero and the number of points in the dual complex is maximal up to isomorphism. All the previous examples are toric.

Up to isomorphism, there is a unique del Pezzo surface of degree 5. This surface can be obtained from $\mathbb{P}^2$ by blowing up 4 points with no three of them collinear. We may assume, up to an automorphism, that the three points are the intersections $L_1 \cap L_2$, $L_1 \cap L_3$, and $L_2 \cap L_3$. The log pull-back of $(\mathbb{P}^2, L_1 + L_2 + L_3)$ to this blow-up is not a log Calabi-Yau pair, so we need to choose a different boundary. We can choose a conic $C$ through the 4 points and a line $L$ that can be chosen to be in general position. The log pull-back of $(\mathbb{P}^2, C + L)$ to this del Pezzo is a log Calabi-Yau pair of the form $(X, C_X + L_X)$, where $C_X$ and $L_X$ are the strict transform of the conic and the line respectively. Since we are not blowing up any intersection point we have that $C_X \cap L_X$ consists of two points. The dual complex $\mathcal{D}(X, C_X + L_X)$ is a circle with two marked points, corresponding to $C_X$ and $L_X$. The self-intersection of $C_X$ is $-3$ and the self-intersection of $L_X$ is 1.

A del Pezzo surface of degree 4 is a Segre surface in $\mathbb{P}^4$ defined by the intersection of two quadrics. They form a 2-dimensional family and they have sixteen $(-1)$-curves. Del Pezzo surfaces of degree 4 can be described as the blow-up of $\mathbb{P}^2$ at 5 points with no three collinear. A del Pezzo surface of degree 3 is a cubic surface of degree three in $\mathbb{P}^3$. They form a 4-dimensional family and they have twenty-seven $(-1)$-curves. A del Pezzo surface of degree 2 is a double cover of $\mathbb{P}^2$ branching over a quartic plane curve. They form a 6-dimensional family and they have fifty-six $(-1)$-curves. Del Pezzo surfaces of degree $d \in \{2, 3, 4\}$ can be described as blow-ups of $\mathbb{P}^2$ at $9 - d$ points with no three collinear and no six on a conic. In this case, we can mimic the construction from the previous paragraph. We choose a conic $C$ through five of the $9 - d$ points and a line $L$ passing through two of them. Then, the log pull-back of $(\mathbb{P}^2, C + L)$ is log Calabi-Yau of coregularity zero. Hence, all del Pezzo’s of degree at least two have coregularity zero.

A del Pezzo surface of degree 1 is a double cover of a quadratic cone in $\mathbb{P}^3$ branched over a smooth genus 4-curve. This surface can be described as the blow-up of $\mathbb{P}^2$ at eight points with no three collinear, no six lying on a conic, and no eight lying on a cubic with a node at one of them. In this case, we can choose a cubic $C_3$ containing the 8 points. Then, the log pull-back of $(\mathbb{P}^2, C_3)$ to the del Pezzo of degree 1 is log Calabi-Yau of coregularity one. Indeed, the dual complex is just a point corresponding to $C_3$. Hence, a del Pezzo of degree one has coregularity at most one.

The following proposition is proved by Prokorov and Shokurov in [101, 110].

**Proposition 3.1.** Let $X$ be a Fano type surface of coregularity zero. Then $X$ either admits a 1-complement or a 2-complement of coregularity zero.
Now, we turn to study the 1-complements and 2-complements of a log del Pezzo of degree one. Let \( X \) be a log del Pezzo of degree one and \( B \) a boundary on \( X \) for which \( (X, B) \) is log Calabi-Yau and \( 2(K_X + B) \sim 0 \). We have a projective birational morphism \( \pi: X \rightarrow \mathbb{P}^2 \) which is the blow-up at 8 points. The push-forward of \( K_X + B \) to \( \mathbb{P}^2 \) induces a boundary \( B_2 \) on \( \mathbb{P}^2 \) for which \( 2(K_{\mathbb{P}^2} + B_2) \sim 0 \) and \( \pi^*(K_{\mathbb{P}^2} + B_2) = K_X + B \). Note that each of the eight points must be contained in the support of \( B_2 \).

We study the possible structures that the divisor \( B_2 \) can have. By counting the degree, we can write

\[
B_2 = \frac{1}{2} \left( \sum_{i=1}^{k} C_i \right)
\]

where \( k \) is at most 6. We analyze the case in which \( k = 2 \). In this case, we have a log Calabi-Yau pair

\[
(\mathbb{P}^2, \frac{1}{2} C_a + \frac{1}{2} C_b),
\]

where \( C_a \) is a curve of degree \( a \), \( C_b \) is a curve of degree \( b \), and \( a + b = 6 \). Note that \( C_1 + C_2 \) must pass with multiplicity at least two at each of the eight points of the blow-up \( X \rightarrow \mathbb{P}^2 \). If the pair \( (\mathbb{P}^2, (C_a + C_b)/2) \) has coregularity one, then \( C_a + C_b \) must pass through one point with multiplicity at least four. If the pair \( (\mathbb{P}^2, (C_a + C_b)/2) \) has coregularity zero, then \( C_a + C_b \) must pass through one point with multiplicity at least six. The possible pairs for the dimension of the spaces of curves of degree \( a \) and \( b \) are \((28,0),(21,3),(15,6)\), and \((10,10)\), up to permutation. Note that the seven double points impose 21 conditions on these spaces while the triple point imposes 6 conditions. In total, we have 27 independent conditions imposed on the space of curves. This implies that no pair of the form (3.1) can have coregularity zero. A similar analysis concludes the following theorem.

**Theorem 3.2.** A del Pezzo surface has coregularity zero if and only if it has degree at least two.

In summary, del Pezzo surfaces of coregularity one form a 8-dimensional family. On the other hand, del Pezzo surfaces of coregularity zero are the union of a 2-dimensional family, a 4-dimensional family, a 6-dimensional family, and 6 other examples, from which four are toric. No del Pezzo surface has coregularity two.

There has been a lot of work on classifying Gorenstein del Pezzo surfaces, i.e., surfaces \( X \) with \( -K_X \) ample and canonical singularities. These surfaces are classified by the fundamental group of its smooth locus (which is finite) in the work of Miyanishi and Zhang (see, e.g., [84, 85]). It would be interesting to extend the previous characterization of del Pezzo surfaces of coregularity zero to Gorenstein del Pezzo.

**Problem 3.3.** Classify Gorenstein del Pezzo surfaces of coregularity zero, one, and two.

Using the work of Prokhorov and Shokurov [101, 110], one can prove the following result which characterizes Fano type surfaces of coregularity zero.

**Theorem 3.4.** Let \( X \) be a Fano type surface of coregularity zero. Then there exists a 2-complement \( (X, B) \) and a cover \( \pi: Y \rightarrow X \) of degree at most two, such that the log pull-back \( (Y, B_Y) \) of \( (X, B) \) to \( Y \) is log crepant to \( (\mathbb{P}^2, L_1 + L_2 + L_3) \).

In the previous theorem, we say that \( (Y, B_Y) \) is log crepant to \( (\mathbb{P}^2, L_1 + L_2 + L_3) \) if there exists a common resolution of singularities \( p: Z \rightarrow Y \) and \( q: Z \rightarrow \mathbb{P}^2 \) for which we have

\[
p^*(K_Y + B_Y) = q^*(K_{\mathbb{P}^2} + L_1 + L_2 + L_3).
\]

Furthermore, in the previous case we can go from \( (\mathbb{P}^2, L_1 + L_2 + L_3) \) to \( (Y, B_Y) \) by blowing up a sequence of points in the total transform of \( L_1 + L_2 + L_3 \) and blowing down a sequence of curves.
The following examples shows that complements computing the coregularity can have arbitrarily small coefficients.

**Example 3.5.** Consider $n$ conics $C_1, \ldots, C_n$ in $\mathbb{P}^2$ passing smoothly through the origin $[0 : 0 : 1]$ with common tangent direction. See Figure 1. Let $L_1, \ldots, L_n$ be general lines passing through the origin with general tangent directions. Then, the pair

$$
\left( \mathbb{P}^2, \frac{1}{n}(L_1 + \cdots + L_n) + \frac{1}{n}(C_1 + \cdots + C_n) \right)
$$

is log Calabi-Yau. We claim that it has coregularity zero. For simplicity we let $L := (L_1 + \cdots + L_n)/n$ and $C := (C_1 + \cdots + C_n)/n$. First, we show that the origin is a log canonical center and more precisely the exceptional extracted by blowing-up its maximal ideal is a log canonical place. Indeed, each line and conic pass through the point with multiplicity one, so the sum $(L_1 + \cdots + L_n + C_1 + \cdots + C_n)/n$ pass through the origin with multiplicity two. Let $p: Y \rightarrow \mathbb{P}^2$ be the blow-up at the origin. Then, we have that

$$p^* (K_{\mathbb{P}^2} + L + C) = K_Y + L_Y + C_Y + E,$$

where $E$ is the exceptional divisor, $L_Y$ (resp. $C_Y$) the strict transform of $L$ (resp. $C$). Analogously, each $L_{Y,i}$ (resp. $C_{Y,i}$) is the strict transform of $L_i$ (resp. $C_i$). Note that the intersection of each component of $L_Y$ and $E$ is transversal. The intersection of each component of $C_Y$ with $E$ is transversal. The intersection of all the components of $C_Y$ is a point $e \in E$. Let $q: Z \rightarrow Y$ be the blow-up of $Y$ at $e$. Then, we have that

$$q^*(K_Y + L_Y + C_Y + E) = K_Z + L_Z + C_Z + E_Z + F_Z$$

where $E_Z$ is the strict transform of $E$ on $Y$ and $F_Z$ is the exceptional of $q$. Analogously, we denote by $L_{Z,i}$ (resp. $C_{Z,i}$) the strict transform of $L_{Y,i}$ (resp. $C_{Y,i}$) on $Z$. Note that $f = E_Z \cap F_Z \neq \emptyset$. This model is actually a log resolution as all components of $L_Z, C_Z, E_Z$ and $F_Z$ intersect transversally. Hence, the pair (3.2) has coregularity zero.

**Figure 1.** Sequence of blow-ups.
3.2. Surface quotient singularities. In dimension two, klt singularities are finite quotient singularities. The most well-known of these are the DuVal singularities described by the following equations:

\[
\begin{align*}
A_n & : x^2 + y^2 + z^{n+1} = 0, \\
D_n & : x^2 + y^2 z + z^{n-1} = 0, \quad n \geq 4, \\
E_6 & : x^2 + y^3 + z^4 = 0, \\
E_7 & : x^2 + y^3 + yz^3 = 0, \\
E_8 & : x^2 + y^3 + z^5 = 0.
\end{align*}
\]

The \(A_n\) singularity is toric, indeed it can also be written as \(xy + z^{n+1} = 0\) and the pair \((A_n, \{x = 0\} + \{y = 0\})\) has coregularity zero. Indeed, every curve in the minimal resolution of the \(A_n\) appear with coefficient one in the log pull-back of the previous pair. Thus, the dual complex in the minimal resolution is a closed segment with \(n + 2\) marked points, from which \(n\) correspond to exceptional curves and 2 correspond to \(\{x = 0\}\) and \(\{y = 0\}\). Hence, the \(A_n\) singularity has coregularity zero.

The \(D_n\) singularity admits a 2-complement given by the pair \((D_n, \{y = z = 0\})\). The strict transform of the curve \(\{y = z = 0\}\) to the minimal resolution intersects a single curve which is the endpoint of one of the branches of the curve that has length different than two. The index one cover of \(K_{D_n} + \{y = z = 0\}\) is a \(A_n\) singularity with its torus invariant boundary. Hence, the \(D_n\) singularity has coregularity zero.

The singularities \(E_6, E_7\) and \(E_8\) are called exceptional singularities. They have coregularity one and the only divisor over them which can compute a log canonical place is the center of the fork. Nevertheless, these singularities admit 6-complements. From now on, we will say that a singularity is exceptional if its coregularity is its dimension minus one. If a klt singularity is exceptional, then we can perform a blow-up which has a unique prime exceptional divisor that is an exceptional Fano variety \cite{87,56,91}. Exceptional Fano varieties of dimension \(n\) are in a bounded family by the work of Birkar \cite{12}. Exceptional singularities of dimension \(n\) can be deformed into orbifold cones over exceptional Fano varieties. This leads to the following theorem due to Han, Liu, and the author \cite{56}:

**Theorem 3.6.** Let \(n\) be a positive integer and \(\epsilon > 0\) be a positive real number. The class of \(n\)-dimensional exceptional singularities \((X; x)\) with log discrepancy at least \(\epsilon\) are bounded up to deformation.

In the previous statement bounded up to deformation means that there exists a bounded set of singularities \(\mathcal{B}\) so that every element as in the statement deforms to an element in \(\mathcal{B}\). Due to the previous theorem, we do not expect interesting behaviours on exceptional singularities. Most invariants take only finitely many possible values on \(n\)-dimensional exceptional singularities with log discrepancy at least \(\epsilon > 0\). The following theorem characterizes klt surface singularities of coregularity zero.

**Theorem 3.7.** Let \((X; x)\) be a 2-dimensional klt singularity of coregularity zero. Then, there exists a reduced 2-complement \((X, B; x)\) for which the index one cover of \(K_X + B\) is toric around the inverse image of \(x\).

**Proof.** By definition, we can find a \(\mathbb{Q}\)-complement \(\Gamma\) so that the singularity \((X, \Gamma; x)\) has coregularity zero. Let \(\pi: Y \to X\) be a dlt modification of \((X, \Gamma; x)\). Note that, since \((X; x)\) is klt, then \(\Gamma\) is non-trivial at \(x\). Write

\[
\pi^* (K_X + \Gamma) = K_Y + S_1 + \cdots + S_r + \Gamma_Y,
\]

where \(\Gamma_Y\) is the strict transform of \(\Gamma\) on \(Y\) and the \(S_i\)'s are the exceptional divisors. By \cite[Corollary 7.12]{110}, the dual complex \(\mathcal{D}(S_1 + \cdots + S_r)\) is a segment of a line. We assume \(S_1\) and \(S_r\) are its endpoints. Let \((S_1, S_2|S_1 + B_{S_1})\) and \((S_r, S_{r-1}|S_r + B_{S_r})\) be the log pairs induced by adjunction of \((K_Y, S_1 + \cdots + S_r)\)
to \(S_1\) and \(S_r\), respectively. Note that the coefficients of \(B_{S_1}\) and \(B_{S_r}\) are standard. Hence, we may assume that one of the following situations hold:

- Each boundary divisor \(B_{S_1}\) and \(B_{S_r}\) have a unique prime component with coefficients \(1 - \frac{1}{m_i}\) and \(1 - \frac{1}{m_r}\), or
- the divisor \(B_{S_1}\) has a unique prime component and \(B_{S_r}\) has two components of coefficient \(\frac{1}{2}\).

In the former case, we can find curves \(B_1\) and \(B_r\) with coefficient one on \(Y\) such that \(B_1\) (resp. \(B_r\)) intersects \(S_1 + \cdots + S_r\) at \(B_{S_1}\) (resp. \(B_{S_r}\)) with multiplicity one. Then, the pair \((Y, S_1 + \cdots + S_r + B_1 + B_r)\) is log Calabi–Yau over \(X\). We set \(B\) to be the push-forward of \(B_1 + B_r\) to \(X\). Then, \((X, B)\) is a toric singularity by Theorem 4.8 below. Indeed, here the two irreducible components of \(B\) play the role of the torus invariant divisors for the action.

In the latter case, we can find a curve \(B_1\) which intersects \(S_1 + \cdots + S_r\) at \(B_{S_1}\) with multiplicity one. Then, the log pair \((Y, S_1 + \cdots + S_r + B_1)\) is log Calabi–Yau. Let \(B\) be the push-forward of \(B_1\) to \(X\). Then, the pair \((X, B; x)\) is log canonical. Indeed, its log pull-back to \(Y\) is simply \((Y, S_1 + \cdots + S_r + B_1)\) which has log canonical singularities. Let \(X' \to X\) be the index one cover of \(K_X + B\). By construction, we have that \(2(K_X + B) \sim 0\). Indeed, we have that

\[
2(K_Y + S_1 + \cdots + S_r + B_1) \sim_X 0
\]

and this is preserved under push-forwards. Hence, the index one cover of \(K_X + B\) is a two-to-one cover \(X' \to X\) that is unramified outside the point \(x \in X\). Indeed, \(B\) is reduced. Let \(\phi : Y' \to Y\) be the normalization of the main component of the fiber product \(X' \times_X Y\). Then, we can write

\[
\phi^*(K_Y + S_1 + \cdots + S_r + B_1) = K_{Y'} + E_1 + \cdots + E_r + \cdots + E_{2r-1} + B'_1 + B'_2.
\]

The involution acts on \(E_1 + \cdots + E_r + \cdots + E_{2r-1}\) by swapping \(E_i\) with \(E_{r-i}\) for \(i \neq r\) and on \(E_r\) the involution sends \(e\) to \(e^{-1}\). The boundary \(B'_1 + B'_2\) is simply the pull-back of \(B_1\) to \(Y'\) and each component intersects \(E_1\) and \(E_{2r-1}\), respectively. Let \(B'\) be the push-forward of \(B'_1 + B'_2\) to \(X'\). Then, the pair \((X', B'; x')\) is log canonical. The same argument as in the first case implies that \((X', B'; x')\) is a toric singularity. \(\square\)

### 3.3. Finite quotients of \(\mathbb{P}^n\)

As discussed in the introduction, quotients of the projective space by finite groups are Fano varieties. Often, these have singularities coming from the fixed points of the action. Analogously, we can consider the local picture, i.e., quotients of the affine space (or a smooth germ) by the action of a finite group fixing the origin. In the same vein, we can consider quotients of del Pezzo surfaces by finite group actions.

The quotient of \(\mathbb{P}^1\) by the action of a finite group \(G\) is exceptional if and only if \(G\) is either \(E_6\), \(E_7\) or \(E_8\) group. Here, we are considering the quotient with the usual logarithmic structure endowed by the Riemann-Hurwitz formula. We recall that by the Chevalley-Shephard-Todd theorem, the quotient of \(\mathbb{C}^n\) by a finite group \(G\) generated by pseudo-reflections is isomorphic to \(\mathbb{C}^n\). Hence, when discussing quotient singularities, we may always assume that \(G\) does not contain pseudo-reflections. In [82], Markushevich and Prokhorov proved that a surface quotient singularity \(\mathbb{C}^2/G\) is exceptional if and only if \(G\) has no semi-invariants of degree at most 2. Similarly, the quotient \(\mathbb{C}^3/G\) is exceptional if and only if \(G\) has no semi-invariants of degree at most 3.

The del Pezzo surface \(X_5\) of degree 5 has an automorphism group isomorphic to \(S_5\). The quotient of \(X_5\) by \(A_5\) and \(S_5\) are exceptional (see, e.g., [29]). Using the Miller–Blichfeldt–Dickson classification of finite subgroups of \(\text{PGL}_3(\mathbb{C})\), the exceptional Fanos obtained as finite quotients \(\mathbb{P}^2/G\) can be classified. Indeed, all these exceptional Fanos are quotients by an element in one of the four imprimitive classes \(F, G, I, J\). This
statement has a local analog for canonical exceptional singularities. In [82], the authors prove that a quotient singularity \( \mathbb{C}^d/G \) is a canonical exceptional singularity if and only if \( G \) is one of the following groups:

1. The Klein’s simple group,
2. the unique central extension of the Klein simple group in \( \text{SL}_3(\mathbb{C}) \),
3. the Hessian group,
4. the normal subgroup of the Hessian group, or
5. a central extension of \( \text{U}_6 \).

In [30], the authors study exceptional quotients \( \mathbb{P}^n/G \) for higher-dimensional projective spaces. In dimension 3, Cheltsov shows that the quotient is exceptional provided that \( |G| \geq 169 \) and \( G \) does not have semi-invariants of degree at most 4. There is further work on the exceptionality of quotients \( \mathbb{P}^n/G \) in dimension at most 9 by Cheltsov and Shramov [31, 32]. We propose a problem in the opposite direction:

**Problem 3.8.** Characterize, in terms of semi-invariants, the quotients \( \mathbb{A}^n/G \) and \( \mathbb{P}^n/G \) that have coregularity zero.

The expected answer is that, whenever \( G \) has enough semi-invariants of small degree, then the quotient should have coregularity zero. For instance, if the group \( G \) fixes the sum of the hyperplanes \( H_1 + \cdots + H_n \), then the quotient \( \mathbb{A}^n/G \) has coregularity zero. However, this is an easy example. It would be interesting to have complete classifications even in low dimensions. In general, we have the following inequality between coregularities of quotients:

**Proposition 3.9.** Let \( (X; x) \) be a klt type singularity and \( G \) be a finite group acting on it. Let \( Y = X/G \) and \( y \) be the image of \( x \). Write \( p: X \to Y \) for the quotient and \( p^*(K_Y + \Delta_Y) = K_X \) for some divisor \( \Delta_Y \) with standard coefficients. Then, we have that

\[
\text{coreg}(X; x) \leq \text{coreg}(Y, \Delta_Y; y).
\]

**Proof.** Consider a complement \( \Gamma_Y \) on \( (Y, \Delta_Y; y) \) which computes the coregularity at \( y \). Then, the pair \( (Y, \Delta_Y + \Gamma_Y; y) \) is log canonical and \( y \in Y \) is a log canonical center. Then, the pull-back

\[
p^*(K_Y + \Delta_Y + \Gamma_Y) = K_X + \Gamma_X,
\]

defines a log canonical pair \( (X, \Gamma_X; x) \) such that \( x \) is a log canonical center. The dual complex \( \mathcal{D}(X, \Gamma; x) \) quotients to the dual complex \( \mathcal{D}(Y, \Delta_Y + \Gamma_Y; y) \) by the induced action of \( G \). In particular, both dual complexes has the same dimension. This implies that

\[
\text{coreg}(Y, \Delta_Y; y) = \text{coreg}(Y, \Delta_Y + \Gamma_Y; y) = \text{coreg}(X, \Delta_X; x) \geq \text{coreg}(X; x).
\]

This finishes the proof. \( \square \)

We note that the previous inequality is not sharp. Indeed, for exceptional quotient singularities of dimension \( n \) the affine space \( \mathbb{C}^n \) has coregularity zero while the quotient has coregularity \( n - 1 \). Thus, the difference between these coregularities can be as large as possible. A more ambitious question in this direction is to try to determine for which finite actions \( G \) on klt singularities \( (X; x) \) the quotient \( (X/G, \bar{x}) \) has the same coregularity as \( (X; x) \).

### 3.4. Singular del Pezzo surfaces

In this subsection, we discuss complements and coregularity of singular del Pezzo surfaces, i.e., surfaces \( X \) with klt singularities for which \( -K_X \) is ample. We will focus on surfaces of Picard rank one. Before proceeding to construct examples, we will review the ADE classification of quotient singularities.
As explained above, Du Val singularities are classified by the ADE classification, i.e., they are either $A_n$, $D_n$, $E_6, E_7$ or $E_8$ singularities. This classification generalizes to klt surface singularities.

**Definition 3.10.** Let $(X; x)$ be a klt surface singularity. We say that $(X; x)$ is of $A$-type if the graph of its minimal resolution is an interval with vertices. We say that $(X; x)$ is of $D$-type if the graph of its minimal resolution is a fork with three branches and two branches have length one. In the $D$-type case, we assume that the third branch is non-empty. We say that $(X; x)$ is of $E$-type if it is not of $A$-type or $D$-type.

It follows from the work of Alexeev that every klt surface singularity belongs to one of the previous classes (see, e.g., [1]). We have the following proposition (see, e.g., [110]).

**Proposition 3.11.** Let $(X; x)$ be a klt surface singularity. Then, one of the following statements hold:

1. $A$-type: there exists a 1-complement $(X; B; x)$ and $(X; B)$ is formally toric around $x$.
2. $D$-type: there exists a reduced 2-complement $(X; B; x)$ and the index one cover of $K_X + B$ is formally toric at the pre-image of $x$, or
3. $E$-type: there exists a 6-complement $(X; B; x)$ and the dual complex $D(X; B; x)$ is a point.

Furthermore, in the case of $D$-type singularities there is no non-trivial 1-complement through $x \in X$.

**Proof.** The statements (1), (2), and (3) follow from the theory of complements for surfaces [110]. We argue that a $D$-type singularity admits no 1-complement. Let $(X; x)$ be a $D$-type singularity. Let $(X; B; x)$ be a 1-complement. Assume that $B$ is non-trivial, i.e., it passes through $x$. Then, every divisor of the minimal resolution of $(X; x)$ has log discrepancy zero with respect to $(X; B; x)$. Let $(Y; B_Y)$ be a dlt modification of $(X; B; x)$. Then, $B_Y$ contains a curve $C$ which intersects transversally three other curves of $|B_Y|$ ($C$ is the fork point in the minimal resolution). Then, the pair $(C, B_C)$ obtained from adjunction of $K_Y + B_Y$ to $C$ satisfies that $B_C$ has three points of coefficient one. This leads to a contradiction. We conclude that $(X; x)$ admits no 1-complement.

The previous proposition states that the only difference between klt surface singularities and Du Val singularities are the self-intersections of the curves in the minimal resolution. Furthermore, if two singularities have the same resolution and self-intersections of the curves in the minimal resolution, then they are analytically isomorphic. The $A$-type and $D$-type singularities have coregularity zero while $E$-type singularities have coregularity one.

We turn to study some examples of complements on Gorenstein del Pezzo surfaces. We recall that Gorenstein del Pezzo surfaces of Picard rank one are classified by the work of Miyanishi and Zhang (see, e.g., [84]). The following example is a Gorenstein del Pezzo surface of coregularity zero that admits a 1-complement and a 2-complement, both of coregularity zero.

**Example 3.12.** Let $X$ be a Gorenstein del Pezzo surface of Picard rank one with two singular points of type $A_3$ and $D_5$. We write $x$ and $y$ for these points, respectively. Let $V \rightarrow X$ be the minimal resolution of $X$. In Figure 2, the solid lines correspond to the $(-2)$-curves extracted on the minimal resolution, while the dotted lines are strict transforms of curves on $X$ which become $(-1)$-curves on $V$. The curves $E_1, \ldots, E_5$ are the exceptional divisors of the resolution of the $D_5$ singularity. The curves $F_1, F_2,$ and $F_3$ are the exceptional divisors of the resolution of the $A_3$ singularity. The curves $C_1, C_2,$ and $C_3$ are strict transforms of curves on $X$. We can contract the $(-1)$-curves on $V$ repeatedly until we obtain a Hirzebruch surface $\Sigma_2$. There are some interesting complements on $X$.

First, we can consider the reduced 2-complement $(X; C_2)$. Locally around the $D_5$-singularity $(X; y)$, this complement is just the standard reduced 2-complement $(X; C_2; x)$. On the other hand, around the $A_3$-singularity $(X; x)$, this complement is a 2-complement which corresponds to the usual 2-complement when
considering this singularity as a $D_3$-singularity. The dual complex $\mathcal{D}(X, C_2)$ is an interval with five vertices, corresponding to $E_3, E_2, E_1, C_2,$ and $F_2.$ Thus, $X$ has coregularity zero.

On the other hand, we can consider the 1-complement $(X, C_3).$ Note that $C_3$ is disjoint from the $D_5$-singularity $y.$ On the other hand, the dual complex $\mathcal{D}(X, C_3)$ is a circle with four vertices corresponding to $F_1, F_2, F_3, C_3.$ We conclude that $X$ is a Gorenstein del Pezzo surface of Picard rank one and coregularity zero. Furthermore, it admits both a 1-complement and a 2-complement that compute the coregularity.

The following is an example of a Gorenstein del Pezzo surface of coregularity zero. It admits a 2-complement of coregularity zero and every 1-complement on it has coregularity one.

**Example 3.13.** Let $X$ be a Gorenstein del Pezzo surface of Picard rank one with two singular points of type $D_4.$ We will write $x$ and $y$ for these points. Let $V \rightarrow X$ be the minimal resolution of $X.$ In Figure 3, the solid lines correspond to the $(-2)$-curves extracted on the minimal resolution, while the dotted lines are strict transforms of curves on $X$ which become $(-1)$-curves on $V.$ The curves $E_1, E_2, E_3, E_4$ are the exceptional divisors over $x$ and the curves $F_1, F_2, F_3, F_4$ are the exceptional divisors over $y.$ We can contract the $(-1)$-curves of $V$ repeatedly until we obtain the Hirzebruch surface $\Sigma_2.$ We can contract the $(-1)$-curves in such a way that $X \rightarrow \Sigma_2$ is an isomorphism around $E_1.$ Moreover, we may assume that the curve $F_1, \Sigma_2 \subset \Sigma_2,$ the
image of $F_1$ on $\Sigma_2$, is a section with self-intersection two. The image of the exceptional divisor of $X \to \Sigma_2$ on $\Sigma_2$ is four points along $F_1, \Sigma_2$. We denote by $E_{i, \Sigma_2}$ the image of $E_i$ on $\Sigma_2$. We denote by $C_{4, \Sigma_2}$ the image of $C_4$ on $\Sigma_2$. We let $\pi: \Sigma_2 \to \mathbb{P}^1$ be the projection of $\Sigma_2$ to $\mathbb{P}^1$.

The pair $(X, C_1)$ is a reduced 2-complement. Locally around both $x$ and $y$, this is the standard reduced 2-complement for the D-type singularity. The dual complex $\mathcal{D}(X, C_1)$ is an interval with five vertices corresponding to the curves $E_1, E_4, C_1, F_4$, and $F_1$.

We study the 1-complements of $X$. Let $(X, C)$ be a 1-complement. By Proposition 3.11, the curve $C$ does not contain $x$ nor $y$ on its support. Let $C_V$ be the strict transform of $C$ on $V$. Then $(V, C_V)$ is a 1-complement and the curve $C_V$ does not intersect any of the curves $E_i$’s nor $F_i$’s. Let $C_{\Sigma_2}$ be the push-forward of $C_V$ on $\Sigma_2$. The following conditions are satisfied:

1. the pair $(\Sigma_2, C_{\Sigma_2})$ is a 1-complement,
2. the curve $C_{\Sigma_2}$ does not contain $E_{1, \Sigma_2}$ nor $F_{1, \Sigma_2}$ on its support,
3. the curve $C_{\Sigma_2}$ contains no vertical components over $\mathbb{P}^1$,
4. every component of $C_{\Sigma_2}$ intersects $F_{1, \Sigma_2}$ in either three or four points.

The third statement follows from the fact that $C_V$ does not intersect any of the $E_i$’s. The last statement holds as every component of $C_{\Sigma_2}$ is horizontal over $\mathbb{P}^1$ and must intersect $F_{1, \Sigma_2}$ along $F_{1, \Sigma_2} \cap E_{i, \Sigma_2}$ for $i \in \{2, 3, 4\}$. Furthermore, every such component may intersect $F_{1, \Sigma_2} \cap C_{4, \Sigma_2}$.

We turn to analyze the class of the components of $C_{\Sigma_2}$ in the Picard group of $\Sigma_2$. Let $s$ be the class of the $(−2)$-section and $f$ be the class of a fiber. The classes $s$ and $f$ generate the cone of effective curves and $−K_{\Sigma_2} ≅ 2s + 4f$. Note that every irreducible curve equivalent to either $s$ or $s + f$ contains $E_{1, \Sigma_2}$. Every irreducible curve equivalent to $s + 2f$ intersects $F_{1, \Sigma_2}$ twice. We conclude that $C_{\Sigma_2}$ must have a single smooth component equivalent to $2s + 4f$ which ramifies at the points $F_{1, \Sigma_2} \cap E_{i, \Sigma_2}$ for $i \in \{2, 3, 4\}$. Hence, the only possible 1-complement $(X, C)$ of $X$ consists of a single elliptic curve $C$ which does not contain $x$ nor $y$ on its support. From the construction, it follows that there is a unique such a curve. We conclude that every 1-complement of $X$ has coregularity one.

To conclude this subsection, we show an example of a singular del Pezzo surface of coregularity zero and Picard rank one which admits a 2-complement of coregularity zero and no 1-complement. Indeed, the following lemma allows us to construct several such examples.

**Lemma 3.14.** Let $X$ be a Fano type surface. Assume that $X$ contains a D-type singularity which is not Gorenstein. Then, $X$ admits no 1-complement.

**Proof.** Assume that $X$ is a Fano type surface and $x \in X$ is a D-type singularity which is not Gorenstein. Let $E$ be the exceptional divisor over $X$ with center $x \in X$ which computes the minimal log discrepancy of $X$ at $x$. Since $(X; x)$ is a D-type singularity which is not Gorenstein, we have that $0 < a_E(X; x) < 1$. Let $(X, B)$ be a 1-complement. Since $K_X + B ≅ 0$, then every log discrepancy of $(X, B)$ is integral. In particular, we must have that $a_E(X, B; x) = 0$. Hence, $(X, B; x)$ is a non-trivial 1-complement of the D-type singularity $(X; x)$. This contradicts Proposition 3.11. We conclude that $X$ admits no 1-complement.

The following is a modification of Example 3.13.

**Example 3.15.** Let $X$ be a Gorenstein del Pezzo surface with two singularities of type $D_4$. Let $V \to X$ be the minimal resolution of $X$. We use the notation of Figure 3. Let $V'$ be the variety obtained by blowing-up the point $C_3 \cap E_2$ in $V$. We call $E$ the exceptional divisor of $V' \to V$. We identify the curves on $V$ with their strict transforms on $V'$. We obtain the diagram in Figure 4. The double curve $E_2$ is a $(-3)$-curve. Let $p: V' \to X'$ be the morphism obtained by contracting the disjoint set of curves $\{E_1, \ldots, E_4\}$ and
\{$F_1, \ldots, F_4, C_3\}$. Note that $C_3$ is a $(-2)$-curve on $V'$. By abuse of notation, we use the same notation for the divisor of $V'$ which are not contracted on $X'$ and their images on $X'$. Then, the image of $F_1 \cup \cdots \cup F_4 \cup C_3$ on $X'$ is a $D_5$-singularity $x \in X'$ and the image of $E_1 \cup \cdots \cup E_4$ on $X'$ is a $D$-type singularity $y \in X'$. Since $E_2$ is a $(-3)$-curve, then $y \in X'$ is a $D$-type singularity which is not Gorenstein. We conclude that $X'$ is a singular del Pezzo surface of Picard rank one with two singularities; a $D$-type singularity which is not Gorenstein and a $D_5$-singularity. By Lemma 3.14, we conclude that $X'$ admits no 1-complement. The pair $(X', E)$ is a reduced 2-complement. Near both $x$ and $y$, the curve $E$ is just a reduced 2-complement of the $D$-type singularities. Indeed, we can compute the pull-back:

\[
p^* (K_{X'} + E) = K_{V'} + E_1 + E_2 + E + C_3 + F_2 + F_1 + \frac{1}{2} (E_3 + E_4 + F_3 + F_4).
\]

We conclude that the dual complex $D(X', E)$ correspond to an interval with six vertices corresponding to the divisors $E_1, E_2, E, C_3, F_2$, and $F_1$. Thus, $X'$ is a singular Del Pezzo surface of Picard rank one and coregularity zero which admits a 2-complement but no 1-complement.

The construction of Example 3.15 can be generalized to obtain countably many singular del Pezzo surfaces of Picard rank one and coregularity zero which admits a 2-complement but no 1-complement. Indeed, we can blow up the intersection $E \cap E_2$ or $E \cap C_3$ and proceed to blowing-up inductively the intersection of the new exceptional divisors with the strict transforms of the previous divisors in the diagram. Afterward, we may contract every single curve in this smooth model, except for the last exceptional curve $E, C_1, C_2, C_4$ and $C_5$. By doing so, we obtain a singular del Pezzo $X$ of Picard rank one with two $D$-type singularities. Furthermore, one of them is non-Gorenstein. By Lemma 3.14, the variety $X$ admits no 1-complement. The pair $(X, E)$ is a reduced 2-complement.
3.5. Terminal 3-fold singularities. In dimension two, terminal singularities are just smooth points. Then, the minimal model program for a smooth surface yields a smooth outcome. However, the ample model (if it exists) of a minimal smooth surface has canonical singularities. In dimension two, the canonical singularities are the Du Val singularities explained above. In dimension three, terminal singularities were classified by Mori (see, e.g., [106, Theorem 6.1]). All these singularities are hyperquotient singularities, i.e., the finite quotient of a hypersurface singularity. We recall the classification of terminal 3-fold singularities. In the following theorem, \( \mu_r \) is the group acting on the hypersurface.

**Theorem 3.16.** Let \((X; x)\) be a terminal 3-fold singularity. Then, \((X; x)\) is a hyperquotient singularity so we can write \((H; 0) / \mu_r \cong (X; x)\), where \(H = \{ f = 0 \}\) in \(\mathbb{K}[x, y, z, w]\). Then, up to a \(\mu_r\)-equivariant change of variables, \(f\) is one of the following:

1. \(cA_r\) singularities: \(f = xy + g(z, t)\) with \(g \in m^2\) or \(f = x^2 + y^2 + g(z, t)\) with \(g \in m^3\).
2. \(cD_4\) singularities: \(f = x^2 + g(y, z, t)\) with \(g \in m^3\) and \(g_3\), the cubic part of \(g\), is an irreducible cubic.
3. \(cD_n\) singularities: \(f = x^2 + yz^2 + g(z, t)\) with \(g \in m^3\).
4. \(cE\) singularities: \(f = x^2 + y^3 + yg(z, t) + h(z, t)\) where \(g \in m^3\) and \(h \in m^4\).

For the weights of the actions in each case, we refer the reader to [106]. On the other hand, canonical singularities are not classified. However, we have the following theorem due to Reid about the general hyperplane sections on a rational Gorenstein singularity.

**Theorem 3.17.** Let \((X; x)\) be a rational Gorenstein singularity. Let \(H \subset X\) be a general hyperplane section containing \(x\). Then, the singularity \((H; x)\) is a rational or elliptic Gorenstein singularity.

Recall that canonical Gorenstein singularities are the same as rational Gorenstein singularities. In particular, we obtain the following theorem for canonical 3-fold singularities.

**Theorem 3.18.** Let \((X; x)\) be a canonical Gorenstein singularity of dimension 3. Let \(H \subset X\) be a general hyperplane section containing \(x\). Then, the singularity \((H; x)\) is either a Du Val singularity or a Gorenstein elliptic singularity.

A singularity \((X; x)\) for which a general hyperplane section \((H; x)\) is a Gorenstein elliptic singularity will be called a compound elliptic singularity, following the similar notation for compound Du Val singularities. The following theorem can be proved using the previous classification, Theorem 3.18, and a similar argument to that used in the proof of Theorem 3.7.

**Theorem 3.19.** Let \((X; x)\) be a 3-fold singularity of coregularity zero. Then, there exists a 2-complement \((X, B; x)\) and a finite cover \(\phi: Y \rightarrow X\) such that \(Y\) is either \(cA_r\) singularity, \(cD_r\) singularity, or a compound elliptic singularity.

As discussed in the previous section, the quotient of a singularity may have lower coregularity. The singularities of type \(cA_r\), \(cD_r\) and compound elliptic singularities have coregularity zero. The fact that the quotient still has coregularity zero imposes a non-trivial condition on the acting group. Especially, if there is a complement that this group is fixing, as this complements behaves as a semi-invariant of low degree. Let us note that in the case of dimension 3 no canonical singularity is exceptional. This is different from what happens to surfaces in which there are already exceptional Du Val singularities.

**Problem 3.20.** Find the exceptional 3-fold singularity with largest log discrepancy.

It is expected that the previous example is computed by a 3-fold quotient singularity.
3.6. Klt 3-fold singularities. In this subsection, we discuss klt 3-fold singularities from a more general perspective via complements and coregularity. The following theorem follows from the theory of complements for surfaces.

**Theorem 3.21.** Let \((X, \Delta; x)\) be a 3-fold klt singularity. Assume that \(\Delta\) has standard coefficients. Then, one of the following statements hold:

1. the singularity \((X, \Delta; x)\) is exceptional,
2. the singularity \((X, \Delta; x)\) has coregularity one and it admits a 1, 2, 3, 4 or 6-complement, or
3. the singularity \((X, \Delta; x)\) has coregularity zero and it either admits a 1-complement or 2-complement.

In the first case, we expect that \((X, \Delta; x)\) admits a \(N\)-complement, where \(N \leq 66\). In the case of exceptional singularities, we know that these can be put together in bounded up to deformation families. In the case of coregularity one, even if the minimal log discrepancy is bounded away from zero, we do not expect this behavior. For instance, one can consider \(cA_n\)-singularities. Instead, for 3-fold singularities of coregularity one we can attempt to construct certain birational models in which the singularity is simplified. In order to do so, we need to introduce the concept of quotient-dlt model:

**Definition 3.22.** Let \((X, \Delta)\) be a log canonical pair. We say that \((X, \Delta)\) has quotient divisorially log terminal singularities (or qdlt for short) if there exists an open subset \(U \subset X\) satisfying the following:

1. every log canonical center of \((X, \Delta)\) intersects \(U\),
2. every log canonical center of \((X, \Delta)\) is a strata of \(|\Delta|\), and
3. for every strata \(Z \subset |\Delta|\) of codimension \(z\) there is an isomorphism
   \[
   (X_Z, |\Delta|_Z) \simeq (\mathbb{A}^z/\mathbb{Z}_k, H_1 + \cdots + H_z),
   \]
   where \(\mathbb{Z}_k\) is acting on \(\mathbb{A}^z\) as multiplication by roots of unity and each \(H_i\) is the image of the corresponding hyperplane on \(\mathbb{A}^z\). We ask the previous isomorphism to hold at the level of formal completions, i.e., the subscript \(Z\) of \(X_Z\) denotes the corresponding formal completion of the localization of \(X\) at \(Z\). In other words, the log canonical centers of \((X, \Delta)\) may not be snc but at the very least it behaves as a finite quotient of a snc pair.

In the case of 3-fold singularities of coregularity one, we can prove the following theorem regarding bounded qdlt models (see, e.g., [89]).

**Theorem 3.23.** Let \((X, \Delta; x)\) be a 3-fold klt singularity of coregularity one. There exists a 6-complement \((X, B; x)\) which is strictly log canonical at \(x\). There exists a projective birational morphism \(\pi: Y \to X\) satisfying the following:

and satisfies the following condition:

- the projective morphism \(\pi\) contracts two irreducible divisors \(E_1\) and \(E_2\),
- we have that \(K_Y + B_Y + E_1 + E_2 = \pi^*(K_X + B)\), where \(B_Y\) is the strict transform of \(B\) on \(Y\), and
- the pair \((Y, B_Y + E_1 + E_2)\) is qdlt.

The previous theorem can be used to compute invariant of singularities of \((X, \Delta)\). The pair \((Y, B_Y + E_1 + E_2)\) is refered as a qdlt modification of \((X, B)\), following the corresponding notation for dlt pairs. In general, it is not known, even in dimension 3 if we can find a “bounded” qdlt model for bounded complements. We propose this as a question that would enhance our understanding of 3-fold klt singularities.

**Problem 3.24.** Find a constant \(N_3\) satisfying the following property: for every klt 3-fold singularity \((X; x)\) there is a strictly log canonical \(N_3\)-complement \((X, B; x)\) which admits a qdlt model \(\pi: Y \to X\) for which \(\rho(Y/X)\) is bounded above by \(N_3\).
3.7. Varieties with torus actions. In this subsection, we discuss the coregularity of Fano $T$-varieties.

A $T$-variety is a normal variety $X$ endowed with the effective action of an algebraic torus $\mathbb{G}^k_m$. The torus complexity of the $T$-variety is the dimension of the variety minus the dimension of the acting torus. In the case that the torus complexity is zero, then we say that $X$ is a toric variety. Varieties with torus action often appear in algebraic geometry as the central fiber of some special degenerations. The torus complexity is also the dimension of the normalized Chow quotient of $X$ by the acting torus.

Affine toric varieties can be described by polyhedral cones in a rational vector space [35]. The combinatorics of this polyhedral cone determine the geometry of the affine toric variety. To go from the theory of affine toric varieties to the general setting of toric varieties, one needs to find a way to glue these polyhedral cones. By Sumihiro’s theorem, we know that every $T$-variety is covered by affine $T$-invariant open sets [112]. Using the previous theorem and the cone description of affine toric varieties leads naturally to the concept of fans of polyhedral cones in rational vector spaces. Again, the combinatorics of the fan encrypt the geometry of the toric variety. A projective toric variety $T$ is the prototype of a variety with coregularity zero. Indeed, if $B_T$ is the reduced sum of the prime torus invariant divisors, then $(T, B_T)$ is log Calabi–Yau and the dual complex $D(T, B_T)$ is a sphere of dimension $\dim T - 1$. Similarly, every toric singularity has coregularity zero.

In [4], the authors started a theory that generalized the language of toric varieties to $T$-varieties in general. In order to describe an affine $T$-variety $X$, the authors define a polyhedral divisor $D$ on the normalized Chow quotient $Y$. This is a divisor whose coefficients are rational polyhedra instead of rational numbers. Then, the authors prove that there is a divisorial sheaf $\mathcal{A}(D)$ associated with this polyhedral divisor. The spectrum of the ring of sections of $\mathcal{A}(D)$ recovers the variety $X$ equivariantly. Furthermore, the authors prove that every affine $T$-variety comes from a polyhedral divisor. In [5], the authors generalize the theory of polyhedral divisors to the concept of divisorial fans to recover all possible $T$-varieties. The philosophy, is that, whenever we can understand the normalized Chow quotient of $X$ and the divisorial fan of $X$, we can reduce problems on $X$ to this lower-dimensional variety. In this direction, we prove that the torus complexity behaves well with respect to the coregularity.

Theorem 3.25. Let $X$ be a Fano variety of torus complexity $c$. Then, we have that $\text{coreg}(X) \leq c$.

Proof. Let $\mathbb{G}^k_m$ be the acting torus and $n$ be the dimension of $X$ so $c = n - k$. For each $r \in \mathbb{Z}_{>2}$, we consider $\mu_r$ the group of roots of unity of $\mathbb{G}^k_m$. Quotients of Fano varieties by finite groups are Fano type (see, e.g., [90]). Hence, we conclude that $Y_r := X/\mu_r$ is a Fano type variety. We write $\pi_r : X \to Y_r$ for the quotient. By Theorem 1.6, there is a $N_n$-complement $(Y_r, B_r)$ which only depends on $n$, i.e., we have that $N_n(K_{Y_r} + B_r) \sim 0$. We write

$$K_X + \Gamma_r = \pi_r^*(K_{Y_r} + B_r).$$

Then, $(X, \Gamma_r)$ is a $\mu_r$-equivariant $N_n$-complement. This means that $N_n(K_X + \Gamma_r) \sim 0$ and $\mu_r \leq \text{Aut}(X, \Gamma_r)$. Since $X$ is Fano, the automorphism group $\text{Aut}(X, \Gamma_r)$ is linear algebraic. Since the set $\{(X, \Gamma_r)\}_{r \in \mathbb{Z}_{>2}}$ belongs to a log bounded family, then there are only finitely many possible isomorphism types for $\{(\text{Aut}(X, \Gamma_r))\}_{r \in \mathbb{Z}_{>2}}$. Hence, we can set $\Gamma_X := \Gamma_r$ for some $r$ large enough and assume that $\mu_r \leq \text{Aut}(X, \Gamma)$ for every $r$. This implies that $\mathbb{G}^k_m \leq \text{Aut}(X, \Gamma)$. Then, we can find a $\mathbb{G}^k_m$-equivariant log resolution $\tilde{X}$ of $(X, \Gamma)$ that dominates the normalized Chow quotient $Y$ of $X$ by $\mathbb{G}^k_m$. We let $(\tilde{X}, \bar{\Gamma})$ be the log pull-back of $(X, \Gamma)$ to $\tilde{X}$. Then, the restriction of $(\tilde{X}, \bar{\Gamma})$ to the general fiber $(F, \Gamma_F)$ of $\tilde{X} \to Y$ is a $(n-c)$-dimensional toric log pair which is log Calabi–Yau. We conclude that all the components of $\Gamma_F$ appear with coefficient one. Hence, we conclude that on $\tilde{X}$ the pair $(X, \Gamma)$ has at least $n - c$ log canonical places whose intersection dominates $Y$. This finishes the proof. \qed
The previous theorem implies that a complexity one Fano variety has coregularity zero or one. In particular, in any case, it admits a 6-complement. Although toric singularities are always klt type, singularities of torus complexity one are not necessarily log terminal, rational, Cohen-Macaulay, nor Du Bois (see, e.g., [78, 73, 74, 75]). The coregularity and torus complexity have a common goal: reduce higher-dimensional geometry problems to low-dimensional geometry and combinatorics.

**Theorem 3.26.** Let $X$ be a Fano type variety of torus complexity one and coregularity zero. Then $X$ admits an equivariant 2-complement $(X, B)$ and a finite cover $Y \to X$ of degree at most two for which the log pull-back $(Y, B_Y)$ of $(X, B)$ degenerates to a projective toric variety.

**Proof.** The normalized Chow quotient of $X$ is $\mathbb{P}^1$. We can find an equivariant projective birational map $\tilde{X} \to X$ for which the torus action admits a good quotient to $\mathbb{P}^1$. Let $\Gamma$ be the sum of all the torus invariant divisors that dominate $\mathbb{P}^1$. Then the log pair $(\tilde{X}, \Gamma)$ has a log canonical center which dominates $\mathbb{P}^1$ and is isomorphic to $\mathbb{P}^1$. We denote one of such log canonical centers $C$. Since $X$ has coregularity zero, the pair defined by adjunction $(K_{\tilde{X}} + \Gamma)|_C = K_C + \Gamma_C$ has coregularity zero as well. Hence, we conclude that $\Gamma_C$ has either two fractional coefficients or three fractional coefficients of the form $1 - \frac{1}{2} \cdot \frac{1}{2}$, and $\frac{1}{2}$. Note that the pair $(\tilde{X}, \Gamma)$ is log Calabi-Yau over $\mathbb{P}^1$. The log pair obtained by the canonical bundle formula is isomorphic to $(C, \Gamma_C)$. We can find a 2-complement for $(C, \Gamma_C)$ which we denote $(C, \Gamma_C + B_C)$. Then, the pair $(\tilde{X}, \Gamma + q^* B_C)$ is reduced 2-complement. We define $B$ to be the push-forward of $\tilde{\Gamma} + q^* B_C$ to $X$. Then, $(X, B)$ is a 2-complement. We take the index one cover of $K_X + B$ and call it $Y$. Then, the log pull-back $(Y, B_Y)$ is a 1-complement and $Y$ is a Fano type variety of torus complexity one. Then, any torus invariant affine variety of $Y$ is described by a polyhedral divisor on $\mathbb{P}^1$ with exactly two fractional coefficients. By [57, Theorem 2.8], each such invariant affine variety can be deformed into an affine toric variety. These deformations are compatible, so they glue together to a deformation of $(Y, B_Y)$ to a projective toric variety. \hfill \Box

### 3.8. Coregularity under morphisms.

In subsection 3.3, we studied the behavior of the coregularity under finite quotients. In this subsection, we study its behavior under birational and contractions. We start with the first proposition, which states that the coregularity can only drop under a birational contraction.

**Proposition 3.27.** Let $X$ be a variety of coregularity $c$. Let $X \dashrightarrow Y$ be a birational contraction. Then, the coregularity of $Y$ is at most $c$.

**Proof.** If the coregularity of $c$ is infinite, i.e., $X$ does not admit a log Calabi–Yau structure, then the proposition holds trivially. Assume that the coregularity of $X$ is $c \in \mathbb{Z}_{>0}$. Let $\Delta$ be a boundary on $X$ which computes the coregularity. We denote by $\Delta_Y$ the push-forward of $\Delta$ on $X$. Note that $(X, \Delta)$ is log Calabi-Yau. Let $p: Z \to X$ and $q: Z \to Y$ be a common log resolution of both $(X, \Delta)$ and $(Y, \Delta_Y)$. Then, we can write

$$p^* (K_X + \Delta) - q^* (K_Y + \Delta_Y) = E - F,$$

where $E$ and $F$ are two $q$-exceptional effective divisors with no common components. Applying the negativity lemma twice, we conclude that $E = F = 0$. Hence, we have that $(X, \Delta)$ and $(Y, \Delta_Y)$ are log crepant equivalent. In particular, $(Y, \Delta_Y)$ is a log Calabi–Yau pair and $\text{coreg}(X, \Delta) = \text{coreg}(Y, \Delta_Y)$. This finishes the proof. \hfill \Box

Observe that the inequality between the coregularities proved in Proposition 3.27 can be strict. Indeed, a del Pezzo of degree one $X_1$ admits a projective birational morphism to $\mathbb{P}^2$. The coregularity of $\mathbb{P}^2$ is zero while the coregularity of $X_1$ is one. The following proposition explains the behavior of the coregularity under
fibrations, i.e., morphisms between normal projective varieties with connected fibers. In the following proof, we will use the language of generalized pairs.

**Proposition 3.28.** Let $\phi: X \to Y$ be a fibration. Assume that $X$ is of Fano type. Then, we have that the inequality

$$\text{coreg}(X) \geq \text{coreg}(Y)$$

holds. In particular, if $X$ has coregularity zero, then $Y$ has coregularity zero.

**Proof.** Let $\Delta$ be a boundary on $X$ which computes the coregularity. Then, $(X, \Delta)$ is a log Calabi–Yau pair. In particular, we have that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Let $(Y, \Delta_Y + M_Y)$ be the pair obtained by the canonical bundle formula. Then, $(Y, \Delta_Y + M_Y)$ is a generalized log canonical pair. By construction, we have that

$$K_X + \Delta \sim_{\mathbb{Q}} \phi^* (K_Y + \Delta_Y + M_Y).$$

In particular, $(Y, \Delta_Y + M_Y)$ is a generalized log Calabi–Yau pair. By [46, Theorem 2.9], we can take a generalized dlt modification $(Y', \Delta_{Y'} + M_{Y'})$ of $(Y, \Delta_Y + M_Y)$. Proceeding as in [90, Lemma 2.36], we obtain a commutative diagram,

$$(X, \Delta) \xrightarrow{\phi} (X', \Delta')$$

$$(Y, \Delta_Y + M_Y) \xleftarrow{\phi} (Y', \Delta_{Y'} + M_{Y'})$$

where $(X', \Delta')$ is a dlt modification of $(X, \Delta)$. By [43], we know that every log canonical centers of $(X', \Delta')$ maps to a generalized log canonical center of $(Y', \Delta_{Y'} + M_{Y'})$. Since $(X, \Delta)$ has coregularity $c$, then $(X', \Delta')$ has a log canonical center of dimension $c$. In particular, we conclude that $(Y', \Delta_{Y'} + M_{Y'})$ has a generalized log canonical center of dimension at most $c$. In particular, the minimal dlt center of $(Y', \Delta_{Y'} + M_{Y'})$ has dimension at most $c$. This implies that the generalized pair $(Y, \Delta_Y + M_Y)$ has coregularity at most $c$.

Now, we turn to prove that $Y$ has coregularity at most $c$. In order to do so, we want to turn the nef part $M_{Y'}$ into an effective divisor. By [12, Lemma 2.12], we have that $Y'$ is of Fano type. We conclude that $Y'$ is also of Fano type as well. In particular, $Y'$ is a Mori dream space. The diminished base locus of $M_{Y'}$ has codimension at least two. We run a $M_{Y'}$-MMP which terminates with a good minimal model $Y' \dasharrow Y''$. All the steps of this minimal model are flips. Furthermore, we have that $M_{Y''}$ is a semiample divisor. Let $\Delta_{Y''}$ be the push-forward of $\Delta_{Y'}$ to $Y''$. Then, $(Y'', \Delta_{Y''} + M_{Y''})$ is a generalized log Calabi–Yau pair, with generalized dlt singularities, and $M_{Y''}$ is semiample. Hence, for a general effective element $0 \leq \Gamma_{Y''} \sim_{\mathbb{Q}} M_{Y''}$, the pair $(Y'', \Delta_{Y''} + \Gamma_{Y''})$ has dlt singularities. Since $[\Delta_{Y''}]$ has at least $c$ components with a common intersection point, so the same holds for $[\Delta_{Y''} + \Gamma_{Y''}]$. We conclude that $(Y'', \Delta_{Y''} + \Gamma_{Y''})$ is a dlt pair of coregularity at most $c$. Let $\Gamma_{Y''}$ be the strict transform of $\Gamma_{Y''}$ on $\Gamma_{Y''}$. Then, the pair $(Y'', \Delta_{Y''} + \Gamma_{Y''})$ is a log canonical pair of coregularity at most $c$. Thus, we have that $\text{coreg}(Y') \leq c$. By Proposition 3.27, we conclude that $\text{coreg}(Y) \leq c$. This finishes the proof. \qed

We note that the previous proof holds without the assumption that $X$ is of Fano type, provided that we have a canonical bundle formula for log canonical pairs (see, e.g., [6]). We also mention that the previous inequality can be strict. Indeed, the del Pezzo surface of degree one $X_1$ admits a fibration to $\mathbb{P}^1$. The elliptic curve $C$ of the 1-complement $(X_1, C)$ dominates $\mathbb{P}^1$. In the following example, we show that in general, the coregularity of the general fiber of a fibration may be higher than the coregularity of the domain.
Example 3.29. Let \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). Consider the projection \( p : X \to \mathbb{P}^1 \) onto the first component. We define the divisors

\[
\Delta_{\text{vert}} = (\{0\} \times \mathbb{P}^1) + \frac{1}{2}(\{x\} \times \mathbb{P}^1), \quad \text{and}
\Delta_{\text{hor}} = \frac{1}{2}(\mathbb{P}^1 \times \{0\}) + \frac{1}{2}(\mathbb{P}^1 \times \{1\}) + \frac{1}{2}(\mathbb{P}^1 \times \{x\}) + \frac{1}{2}D,
\]

where \( D \) is the diagonal. The general fiber is \( \mathbb{P}^1 \) with four points of coefficient \( \frac{1}{2} \), so its coregularity is one. On the other hand the coregularity of \( p \) is zero.

In [42, Example 5.4], the authors show examples in which the coregularity of the general fiber equals the dimension of the general fiber while the pair itself has coregularity zero.

3.9. Coregularity under deformations. In this subsection, we discuss the behaviour of the coregularity under deformations and degenerations. The following example shows that the coregularity can decrease on the special fiber of a flat family of singularities.

Example 3.30. Consider the canonical exceptional Brieskorn singularity

\[ x^3 + y^3 + z^4 + w^5 = 0. \]

Then, consider the following smoothing:

\[ \mathcal{X} := \{x^3 + y^3 + z^4 + w^5 + tw = 0\} \to \mathbb{A}^1_1. \]

The central fiber \( \mathcal{X}_0 \) is exceptional so its coregularity equals two. On the other hand, for every \( t \neq 0 \), we have that \( \mathcal{X}_t \) is smooth at the origin, so its coregularity is zero.

Similar examples show that coregularity zero \( n \)-dimensional singularities can degenerate to exceptional \( n \)-dimensional singularities, i.e., singularities of dimension \( n \) and coregularity \( n - 1 \). The following example shows that the coregularity can increase in the special fiber of a flat family of Fano varieties.

Example 3.31. Let \( X_d \) be a del Pezzo surface of degree \( d \). Fix \( d \leq 4 \). We may find a flat family \( \mathcal{X} \to \mathbb{A}^1 \) for which \( \mathcal{X}_0 \) is a toric variety and \( \mathcal{X}_1 \approx X \). For instance, we can consider the blow-up \( X \to \mathbb{P}^2 \) and deform the blown-up points in such a way that each blow-up occurs at a torus invariant point of \( \mathbb{P}^2 \). Hence, for there is a flat family \( \mathcal{X} \) for which the central fiber \( \mathcal{X}_0 \) is a Fano type variety of coregularity zero and nearby fiber \( \mathcal{X}_t \), with \( t \neq 0 \), are del Pezzo surfaces of degree at most 4, so they have positive coregularity.

3.10. Properties of dual complexes. In this subsection, we discuss some further properties about dual complexes. In order to introduce the first theorem of this subsection, we need to recall the concept of standard \( \mathbb{P}^1 \)-link.

Definition 3.32. Let \( X \to S \) be a projective contraction. Let \( (X, D_1 + D_2 + \Delta) \) be a pair. A morphism \( X \to T \) over \( S \) is said to be a standard \( \mathbb{P}^1 \)-link if the following conditions are satisfied:

1. we have that \( K_X + D_1 + D_2 + \Delta \sim_{\text{Q}, S} 0 \),
2. the morphism \( \pi \) induce isomorphisms \( \pi|_{D_i} : D_i \to T \) for each \( i \),
3. the pair \( (X, D_1 + D_2 + \Delta) \) is plt, and
4. every reduced fiber of \( \pi \) is isomorphic to \( \mathbb{P}^1 \).

Remark 3.33. If \( (X, \Delta) \) is a log Calabi–Yau pair which is birational to standard \( \mathbb{P}^1 \)-link, then the dual complex \( \mathcal{D}(X, \Delta) \) is just two points. In particular, it is not connected.
The following theorem states that the previous is essentially the only case in which the dual complex of a log Calabi–Yau pair can be disconnected. The following theorem is proved by Kollár and Kovacs in the setting of pairs [67]. In the case of generalized pairs it is proved by Filipazzi and Svaldi and independently by Birkar [46, 13].

**Theorem 3.34.** Let \( f : X \to S \) be a projective morphism. Let \((X, \Delta)\) be a log Calabi–Yau pair over \( S \). Fix \( s \in S \). Assume that \( f^{-1}(s) \) is connected. Assume moreover that

\[
\tag{3.3} f^{-1}(s) \cap \text{nklt}(X, \Delta)
\]

is disconnected. Then, the set (3.3) has exactly two components. Moreover, there exists a dlt modification of \((X, \Delta)\) which is birational to a standard \( \mathbb{P}^1 \)-link over \( s \in S \) up to étale base change.

It is easy to show that the statement of Theorem 3.34 does not hold if we drop the log Calabi–Yau assumption as shown in the following example.

**Example 3.35.** Given a point \( p \in \mathbb{P}^2 \), we define the following curves:

1. we let \( C_p := \frac{1}{n}(C_{1,p} + \cdots + C_{n,p}) \) be the average of \( n \) smooth conics through \( p \) with the same tangent at \( p \), and
2. we let \( L_p := \frac{1}{n}(L_{1,p} + \cdots + L_{n,p}) \) be the average of \( n \) lines through \( p \) with different tangent direction at \( p \) and also different from the tangent direction of the conics.

Now, for \( k \) different points \( p_1, \ldots, p_k \) in \( \mathbb{P}^2 \), we can consider the log pair

\[
(\mathbb{P}^2, C_{p_1} + \cdots + C_{p_k} + L_{p_1} + \cdots + L_{p_k}).
\]

As explained in Example 3.5, the previous pair is log canonical and its minimal dlt modification extracts \( 2k \) curves so that the pre-image of each \( p_i \) is the union of two such curves. Then, the dual complex

\[
\mathcal{D}(\mathbb{P}^2, C_{p_1} + \cdots + C_{p_k} + L_{p_1} + \cdots + L_{p_k})
\]

is just \( k \) points. The previous pair is log Calabi–Yau if and only if \( k = 1 \).

Now, we turn to introduce the concept of \( \mathbb{P}^1 \)-linking of log canonical centers.

**Definition 3.36.** Let \((X, \Delta)\) be a dlt pair. Let \( Z_1 \) and \( Z_2 \) be two log canonical centers of \((X, \Delta)\). We say that \( Z_1 \) and \( Z_2 \) are direct \( \mathbb{P}^1 \)-linked if there exists a log canonical center \( W \) of \((X, \Delta)\) containing both \( Z_1 \) and \( Z_2 \) so that the dlt pair \((W, \Delta_W)\) obtained from adjunction of \((X, \Delta)\) to \( W \) is birational to a standard \( \mathbb{P}^1 \)-link. Due to Theorem 3.34, if \( Z_1 \) and \( Z_2 \) are direct \( \mathbb{P}^1 \)-linked, then they have the same dimension \( c \) and \( W \) must have dimension \( c + 1 \). We may say that \( W \) is a linking center. Furthermore, \( Z_1 \) and \( Z_2 \) are the only log canonical centers of \((W, \Delta_W)\).

The concept of direct \( \mathbb{P}^1 \)-linking induces an equivalence relation on the set of log canonical centers of \((X, \Delta)\), where we assume that every log canonical center is direct \( \mathbb{P}^1 \)-linked to itself. We say that two log canonical centers of \((X, \Delta)\) are \( \mathbb{P}^1 \)-linked if they belong to the same class of this equivalence relation. Note that two \( \mathbb{P}^1 \)-linked centers are birational to each other.

In the case of a toric log Calabi-Yau pair the minimal dlt centers are just points and the linking centers are just the torus invariant curves. In a similar vein as Theorem 3.34, it is proved that two minimal dlt centers are birational equivalent. Even further, they are \( \mathbb{P}^1 \)-linked.

**Theorem 3.37.** Let \( f : X \to S \) be a projective contraction. Let \((X, \Delta)\) be a dlt pair which is log Calabi-Yau over \( S \). Let \( s \in S \) be a closed point and assume that \( f^{-1}(s) \) is connected. Let \( Z \subset X \) be a log canonical center of \((X, \Delta)\) which is minimal with respect to the inclusion among centers intersecting \( f^{-1}(s) \). Let \( W \) be
a log canonical center of \((X, \Delta)\) for which \(s \in f(W)\). Then, there exists a log canonical center \(Z_W \subset W\) of 
\((X, \Delta)\) for which \(Z\) and \(Z_W\) are \(\mathbb{P}^1\)-linked and \(f(Z_W)\) contains \(s\). In particular, two minimal log canonical 
centers of \((X, \Delta)\) whose image on \(S\) contain \(s\) are \(\mathbb{P}^1\)-linked.

Note that the concept of \(\mathbb{P}^1\)-linking fits naturally with the definition of pseudo-manifold.

**Definition 3.38.** A topological space \(X\) with a triangulation \(K\) is a \(n\)-dimensional pseudo-manifold if the 
following conditions hold:

1. we have that \(X = |K|\), i.e., \(X\) is the union of all the \(n\)-simplices,
2. every \((n - 1)\)-simplex is the face of either one or two \(n\)-simplices for \(n > 1\), and
3. for every pair \(\sigma\) and \(\sigma'\) of \(n\)-simplices in \(K\), there is a sequence of \(n\)-simplices \(\sigma = \sigma_0, \ldots, \sigma_k = \sigma'\) 
such that the intersection \(\sigma_i \cap \sigma_{i+1}\) is a \((n - 1)\)-simplex for every \(i \in \{0, \ldots, k - 1\}\).

The \(n\) in the definition is called the dimension of the pseudo-manifold.

The previous theorems together with the work of Kollár and Xu gives us the following structural theorem 
for dual complexes of log Calabi–Yau pairs.

**Theorem 3.39.** Let \((X, \Delta)\) be a log Calabi–Yau pair. Then, the dual complex \(D(X, \Delta)\) is a pseudo-manifold
(possibly with boundary). Furthermore, exactly one of the following cases hold:

1. the dual complex \(D(X, \Delta)\) is disconnected and it consists of two points,
2. the dual complex \(D(X, \Delta)\) is connected and it is collapsible to a point, or
3. the dual complex \(D(X, \Delta)\) is connected, non-collapsible, and

\[ H^i(D(X, \Delta), \mathbb{Q}) = 0 \text{ for } 0 < i < \dim D(X, \Delta). \]

We conclude this subsection with the following theorem due to Nakamura that investigates the behavior 
of dual complexes of numerically trivial pairs whose singularities are worse than log canonical [97].

**Theorem 3.40.** Let \((X, \Delta)\) be a pair. Assume that \((X, \Delta)\) has worse than log canonical singularities and 
\(K_X + \Delta \sim_\mathbb{Q} 0\). Then, the dual complex \(D(X, \Delta)\) is collapsible.

In the previous theorem, we need a concept of dual complexes for pairs whose singularities are worst than log canonical. The definition of dual complex in subsection 1.6 extends naturally by taking a dlt modification 
of a possibly non-lc pair, i.e., the dual complex stores the combinatorial information of all log canonical places 
with non-positive log discrepancy.

### 3.11. Examples of dual complexes

In this subsection, we discuss some examples of dual complexes. The first is the example of smooth toric pairs.

**Example 3.41.** Let \(N\) be a free finitely generated abelian group and \(N_\mathbb{Q} := N \otimes \mathbb{Q}\) be the associated 
\(\mathbb{Q}\)-vector space. Let \(P \subset N_\mathbb{Q}\) be a smooth polytope. We can consider the associated fan \(\Sigma_P\). This associated 
fan corresponds to a projective toric variety \(X(\Sigma_P)\). Let \(\Delta(\Sigma_P)\) be the reduced torus invariant divisor, i.e.,
the reduced sum of all the prime torus invariant divisors. Then, we have that

\[ D(X(\Sigma_P), \Delta(\Sigma_P)) \approx \partial P. \]

The pair \((D(X(\Sigma_P), \Delta(\Sigma_P))\) is already smooth, so the previous equality actually holds with the given 
triangulation. Instead, if \(P\) is a simplicial polytope, then \(X(\Sigma_P)\) has \(\mathbb{Q}\)-factorial singularities and the pair 
\((X(\Sigma_P), \Delta(\Sigma_P))\) is qdlt. In this case, the previous equality holds up to possibly performing some cuts on 
the polytope \(P\).
In order to construct more interesting examples, one can start from a projective toric variety $X$ admitting the action of a finite group $G$ that preserves the torus and consider the quotient $X/G$. Since $G$ preserves the torus, then it also preserves the torus invariant boundary. Thus, the quotient $X/G$ admits a complement which makes it into a pair of coregularity zero. In this case, we expect the dual complex of the induced pair structure on the quotient to equal $\delta P/H$ where $H$ is a homomorphic image of $G$. In some cases, we may have $G = H$. In the following, we show one such example.

**Example 3.42.** Let $X_n := (\mathbb{P}^1)^n$ with coordinates $([x_1 : y_1], \ldots, [x_n : y_n])$. Let $\Delta_n$ be the torus invariant boundary. By Example 3.41, know that $D(X_n, \Delta_n) \simeq \delta[0, 1]^n$.

Consider the involution $\tau : X_n \to X_n$ given by

$$\tau([x_1 : y_1], \ldots, [x_n : y_n]) = ([y_1 : x_1], \ldots, [y_n : x_n]).$$

Observe that $\tau$ preserves the torus. Furthermore, the only fixed points of $\tau$ are points of the torus. Let $Y_n := X_n/\tau$ and denote by $p$ the quotient morphism. Let $\Gamma_n$ be the boundary on $Y_n$ for which

$$K_{X_n} + \Delta_n = p^*(K_{Y_n} + \Gamma_n).$$

Then, the pair $(Y_n, \Gamma_n)$ is dlt and

$$D(Y_n, \Gamma_n) \simeq D(X_n, \Delta_n)/\tau \simeq \mathbb{P}^{n-1}_\mathbb{R}.$$

If $n = 3$, then this is an example of a reduced 2-complement without boundary. In dimension two every 2-complement of coregularity zero has an associated dual complex with boundary.

4. Questions

In this section, we present a handful of questions about the coregularity of Fano varieties. First, we start with some structural questions regarding the coregularity, complements, and dual complexes.

4.1. **Structural questions.** Birkar proved that $n$-dimensional Fano type varieties admit $N_n$-complements (see, e.g., [12]). The strategy often reduces to either one of the following cases: exceptional Fano varieties, lifting complements from the canonical bundle formula, or lifting complements from a non-klt center. We expect that in most cases, we can produce a bounded complement which comes from the minimal log canonical center of a $Q$-complement. In other words, if we can produce a log canonical center of dimension $c$, then we should be able to effectively produce a log canonical center of dimension $c$. This leads to the following conjecture about complements for Fano type varieties with bounded coregularity.

**Conjecture 4.1.** Let $c$ be a nonnegative integer. Let $s(c)$ be the $c$-th Sylvester’s number. Let $X$ be a Fano type variety of coregularity at most $c$. Then, the following statements hold:

- If $c = 0$, then $X$ admits a 2-complement.
- If $c \geq 1$, then $X$ admits a $(2s(c) - 3)(s(c) - 1)$-complement.

Observe that in the previous conjecture we do not bound the dimension of $X$. Furthermore, there is a conjectural explicit bound for the largest complement that we can find in coregularity $c$ in terms of Sylvester’s numbers. We mention that the previous conjecture is motivated by two recent results. First, in [42], Fernando Figueroa, Junyao Peng, and the author generalized results about log canonical thresholds which formerly depended on the dimension of the germ to results that only depend on the coregularity. On the other hand, in [40], the authors give an example of a $c$-dimensional log Calabi-Yau klt variety with...
index \((2s(c) - 3)(s(c) - 1)\). This result is related to previous work in which many existential bounds in algebraic geometry are proved to be either double logarithmic or double exponential \([116, 39]\). Observe that in the cases of low coregularity 0, 1 or 2, we expect 2-complements, 6-complements, and 66-complements, respectively. However, Sylvester’s sequence grows doubly-exponentially. Already its 7th term is larger than \(10^{12}\). Nevertheless, in the case of Fano type varieties of coregularity zero we expect a very nice behavior:

**Conjecture 4.2.** Let \(X\) be a \(n\)-dimensional Fano type variety of coregularity zero. Then, there exists a 2-complement \((X, B)\) and a finite cover \(Y \to X\) satisfying the following:

- The log pull-back \((Y, B_Y)\) of \((X, B)\) to \(Y\) is a log Calabi-Yau pair, and
- we have a PL-homeomorphism \(D(Y, B_Y) \simeq S^{n-1}\).

Example 3.15, shows that the 2-complement in the statement is indeed necessary. On one hand, we know that all projective toric varieties have coregularity zero. Further, if we consider their reduced torus invariant boundary, which is indeed a 1-complement, we obtain a log Calabi-Yau pair whose dual complex is a PL-sphere. In \([63]\), the author gives an example of a log Calabi-Yau pair of Fano type whose dual complex is a sphere, but it is not birational to a toric pair. The previous conjecture says that even if coregularity zero Fano type varieties are not necessarily toric, we can still find a 2-complement which up to a cover behaves like a torus boundary of a toric variety. Note that this 2-complement can be described as a 1-complement quotient by an involution. Indeed, a natural way to construct 2-complemented Fano type varieties is to quotient projective toric varieties by involutions (which are not contained in the torus action). We do not see Conjecture 4.2 as an endgame itself, but rather a starting point to generalize the theory of toric varieties to coregularity zero Fano varieties. Indeed, if we enrich the structure of \(D(Y, B_Y)\) with certain normal bundles of the strata, one obtains a fan-like structure \(\Sigma(Y, B_Y)\) which can be used to describe much of the geometry of \(Y\) (and hence \(X\)) itself. In the case of coregularity one, we expect a similar behaviour than Conjecture 4.2. However, in this case we expect a 6-complement and the dual complex to be the quotient of a sphere of dimension \(n - 2\).

The previous discussion about bounded complements and coregularity zero is also expected for higher coregularity. However, in such a case, the dual complex itself will retrieve less information about the Fano variety \(X\). For instance, the dual complex will tell us nothing about the minimal log canonical centers. In order to introduce the next conjecture, we define the concept of minimal dlt center.

**Definition 4.3.** Let \((X, B)\) be a log Calabi–Yau pair. A minimal dlt center of \((X, B)\) is a minimal log canonical center of any dlt modification of \((X, B)\).

By the work of Kollár and Kovacs \([67]\), we know that any two minimal dlt centers of dimension \(c\) are \(\mathbb{P}^1\)-linked, i.e., they can be connected by a sequence of subvarieties of dimension \(c + 1\) that are birational to \(\mathbb{P}^1\)-bundles (see subsection 3.10). This result implies that the birational class of a minimal dlt center of a log Calabi-Yau pair is well-defined. Of course, as usual, we try to study log Calabi-Yau structures with minimized coregularity. The following conjecture is somewhat related to the boundedness of Fano varieties.

**Conjecture 4.4.** Let \(c\) and \(N\) be positive integers. There exists a birationally bounded family \(\mathcal{M}_{c,N}\) of algebraic varieties satisfying the following. Let \(X\) be a Fano type variety of coregularity \(c\). Let \((X, B)\) be a \(N\)-complement of \(X\) computing the coregularity. Then, the minimal dlt centers of \((X, B)\) belong to \(\mathcal{M}_{c,N}\).

Observe that if the minimal dlt center \(M\) of \((X, B)\) is rationally connected, then we get a rationally connected klt log Calabi-Yau pair \((M, B_M)\) of bounded index. By \([17]\), we know that these belong to birationally bounded families. However, in general, we may get minimal dlt centers that are not rationally connected (see, e.g., \([108]\)). We also expect the previous conjecture to hold if the coefficients of \(B\) are
controlled in a set satisfying the DCC set (i.e., the $N$-complement hypothesis can be weakened). In summary, Conjecture 4.2 and Conjecture 4.4 all-together imply that Fano type varieties of bounded coregularity admit bounded complements with birationally bounded minimal dlt centers. It is worth mentioning that in the case of coregularity zero and one the previous conjecture is trivial. However, even in dimension 2, it is not clear we can only obtain a bounded family of K3 surfaces as minimal dlt centers of coregularity two Fanos.

Let $(X, B)$ be a log Calabi-Yau pair of coregularity zero. In [69], Kollár and Xu proved that every strata of $(X, B)$ is a rationally connected variety. In the case of coregularity one, a similar argument shows that every two general points in each strata can be joined by a sequence of rational curves and possibly a single elliptic curve. This motivates the following question relating the coregularity with the covering genus of algebraic varieties.

**Problem 4.5.** Let $c$ be a nonnegative integer. Show the existence of a constant $g(c)$ satisfying the following. Let $(X, B)$ be a log Calabi–Yau pair of coregularity $c$. Then, every log canonical center of $(X, B)$ has covering genus at most $g(c)$.

Even though complements of minimal coregularity seems to be the best choice in most cases, there could be more than one complement computing the coregularity. For instance, in $\mathbb{P}^2$ we can choose any three transversal lines as we can choose any conic and a line. It is somewhat clear that the most components the complement has the better. For instance, in $\mathbb{P}^n$ any two 1-complements with $(n + 1)$ components differ by an automorphism. Similar principles hold for toric varieties. In the next subsection, we introduce an invariant that allows us to make a precise statement about the number of components of a complement.

**4.2. Complexity of Fano varieties.** The complexity of a Fano variety, or more generally a Fano type morphism, measures the difference between the coefficients of the complements and the dimension plus the Picard rank. We give a precise definition in the following. In order to do so, we introduce Fano type morphisms and log Calabi–Yau morphisms.

**Definition 4.6.** Let $\phi: X \to Z$ be a projective contraction. We say that $\phi$ is a Fano type morphism if there exists a boundary $\Delta$ on $X$, big over $Z$, for which $(X, \Delta)$ is klt and $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$. If $Z$ is a point, then a Fano type morphism is simply a Fano type variety. If $X \to Z$ is the identity and $z \in Z$ a closed point, then a Fano type morphism around $z \in Z$ is simply a klt type singularity structure around the point $z$.

We say that $\phi$ is a log Calabi–Yau morphism if there exists a boundary $B$ on $X$ for which $(X, B)$ is log canonical and $K_X + B \sim_{\mathbb{Q}, Z} 0$. If $Z$ is a point, then a log Calabi–Yau morphism is simply a log Calabi–Yau variety. If $X \to Z$ is the identity and $z \in Z$ a closed point, then a log Calabi–Yau morphism around $z \in Z$ is simply a log canonical type singularity structure around the point $z$.

A $N$-complement for the projective contraction $X \to Z$ is a boundary $B$ for which $(X, B)$ is log canonical and $N(K_X + B) \sim_{Z} 0$.

**Definition 4.7.** Let $X \to Z$ be a Fano type morphism. Let $(X, B)$ be a log Calabi–Yau structure over $Z$. Write $B = \sum_{i=1}^{k} a_i B_i$, where each $B_i$ is a prime reduced divisor and each $a_i$ are nonnegative rational numbers. We denote by $|B|$ the sum of the coefficients $a_i$. Then, the relative complexity of $(X, B)$ over $Z$ is defined to be

$$c((X, B)/Z) := \dim X + \rho(X/Z) - |B|.$$ 

The local complexity of $(X, B)$ over a closed point $z \in Z$ is defined to be

$$c((X, B); z) := \dim X + \rho(X_z) - |B|,$$
where \( X_z \) is obtained by a base change to the localization of \( Z \) at \( z \). We define the absolute relative complexity to be:

\[
    c(X/Z) := \min \{ c((X, B); z) \mid (X, B) \text{ is log Calabi–Yau over } Z \}.
\]

Of course, the dimension and the relative Picard are fixed. Hence, the previous definition just intends to maximize the sum of the boundary coefficients among log Calabi–Yau structures on \( X \) over \( Z \). In the case that \( X/Z \) admits no log Calabi–Yau structure, then we just set the complexity to be \( \dim X + \rho(X/Z) \).

Note that for a toric morphism \( X \to Z \) we can always consider the toric boundary \( B \) of \( X \) and \((X, B)\) is log Calabi-Yau. In this case, \( B \) is reduced and the number of its components is exactly \( \rho(X) + \dim X \).

Hence, the absolute relative complexity is zero around the image of every minimal log canonical center of \( (X, B) \). In general, it can be proved that toric morphisms have relative complexity zero everywhere. In [110], Shokurov conjectured that the complexity was nonnegative and if it was zero, then the morphism was indeed formally toric. In [93], the authors settle this conjecture, obtaining the following result:

**Theorem 4.8.** Let \( X \to Z \) be a projective contraction. Then, we have that \( c(X; z) \geq 0 \). If \( c(X; z) = 0 \), then formally around \( z \) the morphism \( X \to Z \) is toric. Furthermore, if \( c((X, B); Z) = 0 \), then \( (X, [B]) \) together with the morphism \( X \to Z \) are formally toric over \( z \).

It is worth mentioning that in the local setting the previous theorem gives a characterization of toric singularities using the language of log canonical singularities. The previous theorem was obtained based on the work of many other mathematicians. Shokurov proved the theorem for arbitrary morphisms of surfaces in [110]. In [120], Yao gives a proof of the projective statement for log smooth pairs. In [65], Keel and McKernan proved the statement for projective surfaces. In [100], Prokhorov proved the statement for certain projective 3-folds using techniques from the minimal model program. In [25], the authors prove the projective version of the previous theorem. The proof of Theorem 4.8 used the language of local Cox rings (see, e.g., [24, 23]).

We expect that the previous theorem can be generalized to an inequality that also considers the coregularity of a Fano type pair. We propose the following problem.

**Problem 4.9.** Let \( X \) be a Fano type variety. Let \((X, B)\) be a log Calabi-Yau pair. Then, we have that

\[
    |B| + \text{coreg}(X) \leq \dim X + \rho(X).
\]

Furthermore, if the equality holds, then we have that \( \text{coreg}(X) = 0 \).

In other words, the sum of the coefficients of \( B \) is not just bounded by \( \dim X + \rho(X) \), but it is also bounded by a sharper bound \( \dim X + \rho(X) - \text{coreg}(X) \). The previous problem, of course, is more interesting when the coregularity is large (or the regularity is small). For instance, it says that for exceptional Fano varieties the sum of the coefficients of any complement is at most \( \rho(X) \).

Another possible generalization of the complexity is towards the language of generalized pair (see, e.g., [14]). A generalized pair \((X, B + M)\) is, briefly speaking, a pair with an extra summand \( M \) that is the push-forward of a nef divisor on a higher birational model. This divisor \( M \) is often called the moduli divisor of the generalized pair, as it naturally appears in the outcome of the canonical bundle formula (see, e.g., [7, 44]). In [93], the language of generalized pair was used, although the complexity invariant does not depend on the moduli part \( M \). We propose the following problem regarding generalized pairs and complexity.

**Problem 4.10.** Let \( X \to Z \) be a Fano type morphism. Let \((X, B + M)\) be a generalized log canonical pair so that

\[
    K_X + B + M \sim_{Z, \Q} 0.
\]
Let $|B|$ be the sum of the coefficients of $B$. Let $M'$ be the model where $M$ descends. Write $M' = \sum_{i=1}^{k} a_i M'_i$ where each $a_i$ is a nonnegative rational number and each $M'_i$ is a nef Cartier divisor which is not $\mathbb{Q}$-linearly trivial. We write $|M|$ for the sum of the $a_i$'s. Then, we have that
\[ c((X, B + M); Z) := \dim X + \rho(X) - |B| - |M| \geq 0. \]

Furthermore, if the equality holds, then $X$ is a toric variety.

Of course, one expects a statement that generalizes both Problem 4.9 and Problem 4.10. We point out that in the statement of Problem 4.10, if $B = 0$, then we expect $X$ to be a special toric variety, similar to a weighted projective space, as in the statement of Kobayashi-Ochiai Theorem.

The coregularity, by definition, is a discrete invariant and can only take integral values. However, the complexity, on the other hand, may take fractional values as shown in the case of $E_8$-singularities. In [25], the authors prove that if the absolute complexity of a projective variety is less than one then it is zero. In this direction, we know very little about the complexity as an invariant itself, i.e., its values larger than one.

**Problem 4.11.** Let $n$ be a positive integer. Describe the set $C_n$ of absolute complexities of $n$-dimensional Fano type varieties. Does $C_n$ have accumulation points for some $n$? Do the accumulation points come from absolute complexities in lower dimensions?

As of today, we do not know examples in which the complexity has accumulation points. So, maybe the last question is vacuous. The following example explains the behaviour of the complexity for klt surface singularities.

**Example 4.12.** Let $(X,x)$ be a klt surface singularity. Then, we have that
\[ c(X;x) \in \left\{ 0, 1, \frac{4}{3}, \frac{5}{4}, \frac{7}{6} \right\}. \]

If the absolute complexity is zero, then the singularity is toric and the absolute complexity is computed by the toric boundary which is a 1-complement. In the absolute complexity one case, the singularity admits a double cover which is toric. The complexity is computed by a reduced 2-complement which is unibranch through the singularity. In the case that $c(X;x) \in \left\{ \frac{4}{3}, \frac{5}{4}, \frac{7}{6} \right\}$, the singularity is of $E_6$, $E_7$ or $E_8$ type, respectively. In these cases, the complexity is computed by a 3, 4 or 6-complement which is unibranch through the singularity, respectively.

Following the philosophy that complements should be lifted from minimal log canonical centers, the following question seems to be natural.

**Problem 4.13.** Show that the set of absolute complexities of klt singularities of coregularity at most $c$ is contained in the set of absolute complexities of klt pairs with standard coefficients of dimension at most $c$.

The situation of complexity is quite similar to such of coregularity. Both invariants are minimized in the toric case. The second invariant characterizes toricness. The first invariant, on the other hand, could be zero and the pair not be even birationally toric. Among complements computing the coregularity, those which minimize the complexity seem to be of special interest. We propose the following question.

**Question 4.14.** Let $X$ be a Fano type variety. Can we find a complement that computes both the coregularity and the complexity? Furthermore, can we find this complement effectively? i.e., to be a $N_c$-complement where $c$ is the coregularity of $X$. 
The problem of finding a complement computing the coregularity often reduces to finding boundaries with large coefficients (or multiplicities). This gives a very naive connection between the coregularity and the complexity.

We recall the following problem introduced in [25]. It is inspired by the classification of DuVal singularities.

**Problem 4.15.** Let $X$ be a variety of absolute complexity strictly less than two. Prove that $X$ admits a two-to-one cover which is rational.

In [25, Section 7], the authors give an example of an irrational example of absolute coregularity zero. This example is a conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ quotient by an involution.

### 4.3. Exceptional Fanos and singularities.

In this subsection, we further discuss exceptional Fano type varieties and exceptional singularities.

We recall that a Fano type variety $X$ is said to be exceptional if for every complement $(X, B)$ the pair $(X, B)$ has klt singularities. In other words, it is not possible to produce a non-trivial log canonical center with a complement on $X$. Note that we always assume that $X$ is a log canonical center of $(X, B)$, but this can be regarded as a trivial log canonical center. On the other hand, we say that a klt singularity $(X; x)$ is exceptional if for every complement $(X, B; x)$ there exists at most one log canonical place. In particular, if $(X; x)$ is an exceptional klt singularity and $(X, B; x)$ is a strictly log canonical complement, then the dlt modification of $(X, B; x)$ is indeed a plt blow-up for $(X; x)$. Observe that in the local setting we can always produce a log canonical center through $x \in X$, hence the exceptional case correspond to such in which we can produce only one log canonical place. Note that the definition of exceptional singularities does not rule out the possibility that two different complements of $(X; x)$ may have different log canonical places. However, the following lemma implies that this is not the case.

**Lemma 4.16.** Let $(X, B; x)$ be a klt singularity. Assume that it is exceptional. Then, there is a unique exceptional divisor $E$ over $X$ with center $c_X(E) = x$ so that every strictly log canonical complement of $(X, B; x)$ has $E$ as its unique log canonical center.

The previous lemma indicates that an exceptional klt singularity admits a unique plt blow-up. Furthermore, the log pair obtained by adjunction to the exceptional of such plt blow-up is an exceptional log Fano pair. The same argument proves the following lemma.

**Lemma 4.17.** Let $(E, B_E)$ be a log Fano pair with standard coefficients. Assume that $(E, B_E)$ is exceptional. Let $(X; x)$ be the klt singularity obtained by taking the cone on $E$ with respect to an ample $\mathbb{Q}$-divisor $A$ satisfying the following conditions:

- we have a $\mathbb{Q}$-linear equivalence $A \sim -r(K_E + B_E)$, and

- for every prime divisor $P$ on $X$, the Weil index of $A$ at $P$ equals the Weil index of $B_E$ at $P$.

Then, the klt singularity $(X; x)$ is exceptional.

Thus, the plt blow-up and cone construction explained in subsection 1.7 preserve the exceptional condition. In general, finding exceptional Fano varieties is harder than producing exceptional log Fano pairs with standard coefficients. For instance, in dimension one, there is no exceptional Fano variety. However, we have three exceptional log Fano pairs with standard coefficients:

$$\left(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{2}{3}\{\infty\}\right), \quad \left(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{3}{4}\{\infty\}\right), \quad \text{and} \quad \left(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{4}{5}\{\infty\}\right)$$

These three correspond to the $E_6, E_7$ and $E_8$ singularities. In general, given a Fano variety $X$ it is often the case that we can find a boundary with standard coefficients $\Delta$ for which $(X, \Delta)$ is an exceptional log
Fano pair. It is fairly simple to produce examples with $X = \mathbb{P}^n$. Then, using Lemma 4.17 we can produce exceptional klt singularities. However, even simple questions about exceptional quotient singularities remain open. In dimension 7 there is no exceptional quotient singularities [31].

**Conjecture 4.18.** For every dimension $n$ there is an exceptional quotient singularity of dimension larger than $n$.

We also expect that it is possible to classify exceptional Brieskorn singularities.

**Problem 4.19.** Classify exceptional Brieskorn singularities.

Some progress towards the previous problem is achieved in [58]. In general, finding examples of exceptional Fano varieties is more challenging. In [65, Example 21.3], the authors give an example of a smooth exceptional Fano surface. We have no standard procedure to create examples of exceptional Fano varieties in every dimension.

**Problem 4.20.** Find examples, in every dimension, of exceptional Fano varieties. Can we produce non-rational examples in every dimension larger than 2?

We conclude this subsection by giving some positive evidence for Problem 4.9.

**Theorem 4.21.** Let $E$ be an exceptional Fano variety for which $\text{Cl}(E) \cong \mathbb{Z}$. Let $(E, B_E)$ be a complement. Then, the sum of the coefficients of $B_E$ is strictly less than one.

**Proof.** Assume by contradiction that the sum of the coefficients of $B_E$ is larger than or equal to one. Let $A$ be an ample generator of $\text{Cl}(E)$. Let $(X; x)$ be the cone over $E$ with respect to the $\mathbb{Q}$-polarization induced by $A$. By Lemma 4.17, the singularity $(X; x)$ is an exceptional singularity. Let $\pi: Y \to X$ be the blow-up of the maximal ideal at $x$. Then, $\pi$ is a plt blow-up and the exceptional divisor is isomorphic to $E$. Let $B$ be the cone over $B_E$. Write $B = \sum_{i=1}^{k} b_i B_i$ where each $b_i$ is a positive rational number and each $B_i$ is a Cartier divisor through $x$. We can write $B_i = \text{div}(f_i)$ locally around $x$ for each $i \in \{1, \ldots, k\}$. Let

$$\Delta := \text{div}(\lambda_1 f_1 + \ldots + \lambda_k f_k)$$

where the $\lambda_i \in \mathbb{K}$ are general enough. Let $Z \to X$ be a log resolution of $(X, B; x)$. Then, for every divisor $E \subset Z$ exceptional over $X$, we have that

$$\text{mult}_E(\lambda_1 f_1 + \ldots + \lambda_k f_k) \leq \text{mult}_E(B).$$

Hence, we conclude that for every choice of $0 < \lambda_i < b_i$, the pair

$$(X, \lambda \Delta + \sum_{i=1}^{k} (b_i - \lambda_i) B_i)$$

is a klt pair. By Proposition 1.14, we know that the pair (4.1) admits a plt blow-up at $x \in X$. Note that this plt blow-up is also a plt blow-up of $(X; x)$. By Lemma 4.16, we know that $(X; x)$ admits a unique plt blow-up. Hence, the plt blow-up of the pair 4.1 must be $\pi$. In particular, we conclude that the pair

$$(Y, \lambda \Delta_Y + \sum_{i=1}^{k} (b_i - \lambda_i) B_i, Y + E)$$

is plt for every choice of $0 < \lambda_i < b_i$. By assumption, we have that $\sum_{i=1}^{k} b_i \geq 1$. Then, the pair $(Y, E + (1 - \epsilon) \Delta_Y)$ is plt and $K_Y + E + (1 - \epsilon) \Delta_Y$ is anti-ample over $X$, for every $\epsilon$ small enough. In particular, the pair $(E, (1 - \epsilon) \Delta_Y |_E)$

is log Fano for every $\epsilon$ small enough. By Theorem 1.6, there is a $N$-complement for $(E, (1 - \epsilon)\Delta_Y|_E)$. Hence, we may find a log Calabi-Yau pair $(E, \Delta_E + \Gamma_E)$ where the coefficients of $\Gamma_E$ belong to $\mathbb{Z}\left\lceil \frac{1}{N} \right\rceil > 0$ and $\Delta_E := \Delta_Y|_E$ is reduced. This contradicts the fact that $E$ is exceptional. Indeed, every prime divisor in the support of $\Delta_E$ is a log canonical center of $(E, \Delta_E + \Gamma_E)$.

4.4. Explicit computations. As we have discussed above, del Pezzo surfaces have coregularity zero if and only if they have degree at least two. There are many more classes of 3-dimensional Fano manifolds. The following question seems natural:

**Question 4.22.** Classify terminal Fano 3-folds of Picard rank one with coregularity zero.

Given a positive answer for Conjecture 4.2, in order to prove that a terminal Fano 3-fold of Picard rank one has coregularity zero, it suffices to study the linear systems $|-K_X|$ and $|-2K_X|$. Hence, this problem simplifies considerably if a basis for $H^0(-K_X)$ and $H^0(-2K_X)$ are found.

Let us mention that the previous question will help to get a complete classification of 3-folds of coregularity zero. Indeed, if $X$ has coregularity zero, up to a two-to-one cover, we may find a 1-complement $(X, B)$. Let $(Y, B_Y)$ be the dlt modification of $(X, B)$. Then, the variety $Y$ has canonical singularities. Replacing $Y$ with a terminalization, we may assume itself is terminal. Then, we may run a minimal model program for $K_Y$. If it terminates with a Mori fiber space to a positive dimensional base, then the base also has coregularity zero. One can study the structure of the base and general fiber to understand the structure of the Mori fiber space itself. In this case, we expect that the Mori fiber space to be similar to a product of the base and the general fiber. On the other case, the MMP terminates with a terminal Fano variety of Picard rank one with a 1-complement of coregularity zero, i.e., one of the varieties as in the previous question.

In dimension two, we know that there are only two possible birational classes of coregularity zero Fano varieties. The ones that admit 2-complements and the ones that admit 1-complements. The latter class is indeed crepant birational to the projective space with three lines. It is natural to ask how many birational classes of coregularity zero Fano varieties are there.

**Question 4.23.** Classify, up to crepant birational equivalence, all 3-folds of coregularity zero.

The previous question will help us understand which triangulations can we find in the sphere $S^2$ among dual complexes of log Calabi–Yau 3-folds of coregularity zero. Indeed, every log Calabi–Yau 3-fold of coregularity zero is crepant birational to a Fano type variety. In the case of dimension 4 we do not have a classification of Fano varieties of Picard rank one and terminal singularities. Hence, this approach may only be successful in dimension 3. However, a better understanding in dimension three may allow us to conjecture some behavior in higher dimensions.

We recall that in the toric case, smooth Fano varieties are in correspondence with the so-called smooth Fano polytopes. There are exactly 5 smooth Fano toric surfaces, 18 smooth Fano toric 3-folds, and 123 smooth Fano toric 4-folds (see, e.g., [10]). It is not known how the number of smooth Fano polytopes grows with the dimension.

Classically, the invariant that has been used to classify Fano varieties is the index. If a Fano variety of dimension $n$ has index larger than $n$, then it is isomorphic to the projective space. Fano varieties of index $n - 1$ have been classified by Fujita [49] and Fano varieties of index $n - 2$ have been classified by Mukai [96]. We expect Fano varieties of large index to have smaller coregularity. Indeed, whenever the index is large, we can already find some interesting elements in $|-K_X|$ (see, e.g., [3]). This raises the following question.

**Question 4.24.** Is there an inequality involving the coregularity and the index of a Fano variety?
Iskovskikh and Manin proved that a smooth quartic threefold is birationally superrigid [61]. Many Fano complete intersections of index one are proved to be birationally superrigid [36]. Pukhlikov conjectured that any Fano variety of Picard rank one and index one is birationally superrigid. In [27], Castravet gives a counter-example for this conjecture. We recall that the example given by Kaloghiros in [63] has coregularity zero and it is indeed a birationally rigid example. It would be interesting to construct more examples of coregularity zero varieties which are birationally rigid.

**Problem 4.25.** For each dimension \( n \), construct a birationally rigid Fano variety of dimension \( n \) and coregularity zero.

Among Fano varieties, Grassmannians are one of the most well-known examples. It would be interesting to compute the coregularity explicitly for these examples.

**Problem 4.26.** For \( 1 \leq k < n \), compute the coregularity of the Grassmannian \( G(k,n) \).

An answer to the previous question may also help to compute the coregularity of 3-folds. Indeed, in [52], Gushel noted that some Fano 3-folds of Picard rank one are contained in \( G(2,m) \). Using this description, Gushel tried to classify Fano 3-folds. Finally, the following question is natural but probably is quite challenging.

**Question 4.27.** Let \( n \) be a positive integer and \( d \) be a positive integer less than \( n+1 \). Find the coregularity of a general hypersurface \( H_d \) of degree \( d \) in \( \mathbb{P}^n \).

4.5. **Thresholds.** In singularity theory, thresholds are important invariants of singularities. In the realm of birational geometry, the log canonical threshold plays a fundamental role in many characterizations and proofs. In a few words, it measures how many times we can add a divisor through the singularities so the pair still has log canonical singularities. Let \( X \) be a normal quasi-projective variety and \( \Delta \) be a boundary for which \((X, \Delta)\) is log canonical. Let \( \Gamma \) be a \( \mathbb{Q} \)-Cartier divisor on \( X \).

\[
\text{lct}((X, \Delta); \Gamma) := \sup \{ t \mid (X, \Delta + t\Gamma) \text{ has log canonical singularities} \}.
\]

In [55], the authors prove that in a fixed dimension, whenever we control the coefficients of both \( \Delta \) and \( \Gamma \) in a DCC set, the possible log canonical thresholds is an ACC set. In what follows, we discuss about local behavior of singularities, so we write \( \text{lct}((X, \Delta; x); \Gamma) \) for the threshold of the pair around \( x \). We always assume that the divisor \( \Gamma \) contains the point \( x \). We define the set of log canonical thresholds with bounded coregularity to be

\[
\text{LCT}_c(I, J) := \{ t \mid \text{lct}((X, \Delta; x); \Gamma), \text{coeff}(\Delta) \in I, \text{coeff}(\Gamma) \in J, \text{ and coreg}(X, \Delta + t\Gamma; x) \leq c \}.
\]

In [42], the authors show that the set \( \text{LCT}_c(I, J) \) behaves like the set of log canonical thresholds with bounded dimension. In other words, we have the following result.

**Theorem 4.28.** Let \( c \) be a positive integer. Let \( I \) and \( J \) be two sets of real numbers satisfying the descending chain condition. Then, the set \( \text{LCT}_c(I, J) \) satisfies the ascending chain condition.

The authors also prove that accumulation points of log canonical thresholds with coregularity \( c \) come from log canonical thresholds with coregularity less than \( c \). In the case of coregularity zero, the only accumulation points are inherit from the sets \( I \) and \( J \). The main idea is to reduce the threshold computation to a minimal log canonical centers, which by definition must have dimension at most \( c \). It is expected, that not only canonical threshold, but other invariants can be reduced to computations on minimal log canonical centers.

A polytopal version of the log canonical threshold, the so-called log canonical threshold polytope, was introduced by Mustață and Libgober in [77]. Let \( X \) be a normal quasi-projective variety with klt type
singularities. Let $D_1,\ldots,D_r$ be a sequence of effective $\mathbb{Q}$-Cartier divisors on $X$. The log canonical threshold polytope is defined to be:

$$P(X; D_1,\ldots,D_r) := \{(t_1,\ldots,t_r) \in \mathbb{R}_{\geq 0}^r \mid (X, t_1D_1 + \cdots + t_rD_r) \text{ is log canonical}\}.$$  

Analogously as in the previous case, we can define log canonical thresholds polytopes of coregularity $c$. This motivates the following question.

**Question 4.29.** Does the ascending chain condition for log canonical threshold polytopes of dimension $r$ and coregularity $c$ holds?

In the previous problem, we are fixing the number of divisors $r$ so all the polytopes have the same dimension and they can be compared in the same affine space. Here, ascending chain condition, means with respect to the inclusion of polytopes. In a similar vein, one can ask whether the ascending chain condition of the volume of such polytopes hold.

An invariant that behaves similar to log canonical thresholds is the pseudo-effective threshold. The pseudo-effective threshold of a variety $X$ with respect to a divisor $D$ is the smallest real number $t$ for which $K_X + tD$ is pseudo-effective. If such a number does not exist, then we just say that the pseudo-effective threshold is infinite. We denote the pseudo-effective threshold by $p(X; D)$. We can define the set of pseudo-effective thresholds with coregularity at most $c$ to be

$$P_c(I, J) := \{p \mid p(X; D), \text{coeff}(D) \in I, \text{ and } \text{coreg}(X, pD) \leq c\}.$$  

In this direction, we propose the following question.

**Question 4.30.** Let $c$ be a positive integer. Let $I$ be a set satisfying the descending chain condition. Does $P_c(I)$ satisfy the ascending chain condition?

We expect a positive answer to the previous question. In [42], the authors prove a similar statement for numerically trivial thresholds. However, we expect the case of pseudo-effective thresholds to require some new ideas.

4.6. **K-stability.** The algebraic K-stability theory uses techniques from algebraic geometry to decide whether a Fano variety admits a Kähler-Einstein metric.

In [48, 76], Fujita and Li introduced a valuative criterion to detect K-semistability of K-stability of a Fano variety. These criteria can be written in terms of blow-ups of the Fano variety $X$.

**Definition 4.31.** Let $X$ be a $n$-dimensional Fano variety. Let $p: Y \to X$ be a projective birational morphism extracting a prime divisor $E$ over $X$. The **expected multiplicity** $S_E(X)$ of $-K_X$ at $E$ is defined to be

$$\frac{1}{(-K_X)^n} \int_0^\infty \text{vol}(p^*(-K_X) - tE)dt.$$  

This invariant is called the expected multiplicity because somehow it measures the average multiplicity at the divisor $E$ among effective divisors which are $\mathbb{Q}$-linearly equivalent to $-K_X$. The expected multiplicity $S_E(X)$ only depends on $E$ and does not depend on the chosen model $Y$.

**Definition 4.32.** Let $X$ be a $n$-dimensional Fano variety. Let $E$ be a prime divisor over $X$. The **beta invariant** of $E$ is defined to be the difference between the log discrepancy at $E$ and the expected multiplicity at $E$, i.e., we define

$$\beta_E(X) := a_E(X) - S_E(X).$$  

The following theorem follows from the work of Fujita, Li, Blum, and Xu [48, 76, 18].
Theorem 4.33. Let $X$ be a Fano variety.

- The variety $X$ is K-semistable if and only if $\beta_E(X) \geq 0$ for every $E$ over $X$.
- The variety $X$ is K-stable if and only if $\beta_E(X) > 0$ for every $E$ over $X$.

It is known that it suffices to check the condition $\beta_E(X) \geq 0$ (resp. $\beta_E(X) > 0$) for divisors $E$ which are log canonical places of complements of $X$. This means that it suffices to check the divisors $E$ for which $a_E(X, B) = 0$ for some $\mathbb{Q}$-complement $B$ of $X$. Furthermore, in the K-semistable case, there exists a divisor $E_0$ over $X$ which minimizes $\beta$. The following question is proposed by Xu:

**Question 4.34.** Let $X$ be a Fano variety. Can we find a complement $(X, B)$ that computes the coregularity of $X$ for which the minimizer of $\beta$ is a log canonical place of $(X, B)$?

Similar questions can be asked for other invariants related to K-stability: the alpha-invariant, the delta-invariant, and the normalized volume. Note that, in the case of Fano varieties of coregularity zero, a positive answer to Conjecture 4.2 and Question 4.34, implies that, in order to detect K-semistability it suffices the linear systems $H^0(-K_X)$ and $H^0(-2K_X)$ and the associated log canonical places. Analogously, in the case of Fano varieties of coregularity one, we expect that it suffices to study $H^0(-NK_X)$ for $N$ at most 6. This would considerably simplify the work to check the K-stability of Fano varieties of small coregularity.

4.7. **Fundamental groups.** In this subsection, we discuss connections of the coregularity with fundamental groups of klt singularities and Fano type varieties. We also discuss connections with log canonical singularities and log Calabi–Yau pairs. We start with the definition of local fundamental groups. For simplicity, throughout this subsection, we work over the field of complex numbers.

**Definition 4.35.** Let $(X, \Delta; x)$ be a singularity of pairs. The standard approximation $\Delta_s$ of $\Delta$ is the largest effective divisor $\Delta_s \leq \Delta$ whose coefficients have the form $1 - \frac{1}{m^r}$ for positive integers $m$. If $\text{coeff}_P(\Delta_s) = 1 - \frac{1}{m}$, then we say that $\Delta$ has orbifold index $m$ at $P$, which will be denoted by $m_P$. For each prime component $P$ of $\Delta$, we denote by $\gamma_P$ a loop around $P$. Given an open subset $U \subset X$ containing $x$, we denote by $\pi_1(U, \Delta|_U)$ to be the quotient of $\pi_1(U \setminus \text{supp}(\Delta|_U))$ by the smallest normal subgroup generated by the elements $\gamma_{P}^{m_P}$. The regional fundamental group of $(X, \Delta; x)$, denoted by $\pi_1^{\text{reg}}(X, \Delta; x)$ is defined to be the inverse limit of the groups $\pi_1(U, \Delta|_U)$, where $U$ runs over all the neighborhoods of $x \in X$.

Analogously, if $(X, \Delta)$ is a quasi-projective pair, now in the global setting, we define $\pi_1^{\text{reg}}(X, \Delta)$ to be the inverse limit of $\pi_1(U, \Delta|_U)$ for all big open subsets $U \subset X$.

In [64], Kollár and Kapovich proved that any finitely presented group can appear as the regional fundamental group of a normal algebraic singularity. However, the situation is much better when we consider singularities of the minimal model program. In [21], Braun proved that the regional fundamental group of a klt type singularity is finite:

**Theorem 4.36.** Let $(X, \Delta; x)$ be a klt type singularity. Then, the group $\pi_1^{\text{reg}}(X, \Delta; x)$ is finite.

This result is based on an analogous projective statement for Fano type varieties.

**Theorem 4.37.** Let $(X, \Delta)$ be a projective pair of Fano type. Then, the group $\pi_1^{\text{reg}}(X, \Delta)$ is finite.

Once the finiteness is settled, it is natural to try to understand which groups we can find in a fixed dimension. For instance, every finite group $G$ appears as the regional fundamental group of a klt type singularity. Indeed, we can find a representation $G \to \text{GL}_n(\mathbb{C})$ for which the induced action of $G$ on $\mathbb{C}^n$ is free in codimension one. Then, the quotient $X_G := \mathbb{C}^n/G$ satisfies that $\pi_1^{\text{reg}}(X_G; 0) \cong G$. Thus, the regional fundamental groups of quotient singularities are all possible finite groups. On the other hand, if we want
to study quotient singularities of a fixed dimension, we need to understand finite groups in a fixed $GL_n(\mathbb{C})$.

The following result is due to Camille Jordan gives a strong control of such groups.

**Theorem 4.38.** There exists a constant $c(n)$, only depending on $n$, satisfying the following. Let $G \leq GL_n(\mathbb{C})$ be a finite subgroup. Then, there exists a normal abelian subgroup $A \leq G$ of index at most $c(n)$.

Collins proved that for $n \geq 71$, one can indeed take $c(n) = n!$. In some words, the previous theorem says that finite subgroups of $GL_n(\mathbb{C})$ are “almost abelian” with respect to the dimension. Note that the previous theorem can also simplify the study of quotient singularities. Indeed, the rank of $A$ is at most $n$ and the quotient $\mathbb{C}^n/A$ is a toric singularity. Thus, any $n$-dimensional quotient singularity admits a cover of degree at most $c(n)$ which makes it a toric singularity. As a consequence, we have the following corollary.

**Corollary 4.39.** There exists a constant $c(n)$, only depending on $n$ satisfying the following. Let $(X; x)$ be a $n$-dimensional quotient singularity. Then, there exists a normal abelian subgroup $A \leq \pi_1^{\text{reg}}(X; x)$ of index at most $c(n)$ and rank at most $n$.

We expect the behavior of klt type singularities to be similar to such of quotient singularities. The following theorem gives a realization of this principle from the perspective of fundamental groups. This theorem is due to the work of Braun, Filipazzi, Svaldi, and the author (see, e.g., [22, 88, 90]).

**Theorem 4.40.** Let $n$ be a positive integer. There exists a constant $c(n)$, only depending on $n$, satisfying the following. Let $(X, \Delta; x)$ be a klt type singularity of dimension $n$ and regularity $r$. Then, there exists a short exact sequence

$$1 \to A \to \pi_1^{\text{reg}}(X, \Delta; x) \to N \to 1,$$

where $A$ is an abelian group of rank at most $r + 1$ and $N$ is a group of order at most $c(n)$.

Thus, the “abelian part” $A$ of the regional fundamental group of a klt type singularity is determined by its regularity while the “non-abelian part” $N$ is determined by the dimension. We expect that the abelian part $A$ of $\pi_1^{\text{reg}}(X, \Delta; x)$ behaves like the maximal torus of a reductive group. However, we do not have statements about its maximality. Even though we have a good structure theorem, we do not know how to control the constant $c(n)$. It is natural to expect that $N$ is some sort of permutation group. We propose the following question that may enlighten:

**Problem 4.41.** For each dimension $n$, construct examples of $n$-dimensional klt type singularities $(X, \Delta; x)$ for which $|\pi_1^{\text{reg}}(X, \Delta; x)/A| > n!$ for every abelian subgroup $A$.

The previous problem is just asking to check that Collins bound is not optimal for klt singularities. In low dimensions, we even expect that the possible groups can be classified:

**Problem 4.42.** Classify the regional fundamental groups of klt 3-fold singularities.

Rationally connected varieties are quite similar to Fano varieties. Indeed, every Fano type variety is rationally connected. On the other hand, every rationally connected variety can be birationally transformed into a tower of Mori fiber spaces whose general fibers are Fano type varieties. Thus, it is natural to study the regional fundamental group of rationally connected varieties. In the case of singular rationally connected varieties, we do not expect the finiteness of the regional fundamental group. For instance, we have the following example of a rational surface with rational log canonical singularities and infinite fundamental group:
Example 4.43. Let \( X = \{ y^2 - x(x^2 - z^2) = 0 \} \subset \mathbb{P}_x^3 \), i.e., the projective cone over an elliptic curve. Observe that \( X \) is invariant with respect to the involution
\[
\tau : \mathbb{P}^3 \to \mathbb{P}^3, \quad \tau([x:y:z:w]) = [-x:y:-z:t].
\]
The fixed points of \( \tau \) are \([0:0:0:1]\) and four other points. Let \( Y = X/\tau \). Then \( Y \) has four \( A_1 \) singularities and a rational singularity \( y \in Y \) which is the quotient by an involution of a simple elliptic singularity. The quotient \( Y \) is rational. Indeed, it contains the total space of an orbifold bundle over \( \mathbb{P}^1 \) with four orbifold points with isotropy \( 1/2 \). The regional fundamental group is infinite since it is an extension of \( \mathbb{Z}/2\mathbb{Z} \) and the regional fundamental group of the cone over an elliptic curve.

However, the following conjecture seems to be optimal in this direction:

Conjecture 4.44. Let \( X \) be rationally connected. Let \( (X, \Delta) \) be a dlt pair. Then, the regional fundamental group \( \pi_1^{\text{reg}}(X) \) is finite.

The following conjecture relates the fundamental group of rationally connected varieties and the coregularity.

Conjecture 4.45. Let \( X \) be a rationally connected variety of dimension \( n \), regularity \( r \), and coregularity \( c \). Assume that \( G := \pi_1^{\text{reg}}(X) \) is finite. Then, \( G \) admits a subnormal subseries:
\[
A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft G,
\]
such that the following conditions are satisfied:
- \( A_0 \) acts on either \( \mathbb{D}^{r-1} \) or \( \mathbb{S}^{r-1} \),
- \( A_1/A_0 \) is an abelian group of rank at most \( r \), and
- \( A_2/A_1 \) has order at most \( N(c) \).

Here, \( N(c) \) only depends on the coregularity \( c \).

The previous conjecture can be regarded as follows. If \( G \) is finite, then the action of \( G \) on the universal cover splits into three different pieces: the action on the dual complex of a log Calabi–Yau structure, the action that fixes point-wise every stratum of the dual complex, and the action in the minimal log canonical center of the CY structure.

We recall the following conjecture about the regional fundamental group of log Calabi–Yau pairs, which is motivated by Campana’s conjecture [26].

Conjecture 4.46. Let \( (X, \Delta) \) be a log Calabi–Yau pair of dimension \( n \). Then, there exists a short exact sequence:
\[
1 \to A \to \pi_1^{\text{reg}}(X, \Delta) \to N \to 1,
\]
where \( A \) is an abelian group generated by at most \( 2n \) elements and \( N \) is a finite group of order at most \( c(n) \).

We expect that in some universal cover the action of \( A \) is either given by the action of an abelian variety or an algebraic torus. In other words, the abelian action on a Calabi–Yau variety should come from a complexification of the circle. In [70], Kollár proved that the fundamental group of a log canonical singularity is not necessarily finite, and we can find the fundamental groups of Riemann surfaces among them. It is expected that in dimension at least 5, among regional fundamental groups of lc singularities, we can find any finitely presented group:

Problem 4.47. Show that any finitely presented group appears as the regional fundamental group of a log canonical singularity of dimension 5.
On the other hand, we expect that this behavior is mostly related to the dual complex. Hence, groups acting on log canonical singularities and fixing the dual complex should be better behaved:

**Conjecture 4.48.** Let \((X, \Delta; x)\) be a log canonical singularity of dimension \(n\). Then, the normal subgroup

\[ N := \ker(\pi_1(X, \Delta; x) \to \pi_1(D(X, \Delta; X))) \]

is virtually nilpotent of rank at most \(2n + 1\).

The classification of log canonical surface singularities is a source of examples that motivates the previous conjecture. Similar examples can be found by taking orbifold cones over projective Calabi–Yau surfaces.

4.8. **Degeneration coregularity.** Degenerations of log Calabi–Yau pairs and Fano varieties are often found in the literature. In general, being able to degenerate a Fano variety or a Calabi–Yau variety to a special kind of of varieties. In many cases, we can use the structure of the central fiber to deduce information about nearby fibers. We define the degeneration coregularity as follows:

\[ \text{degcoreg}(X) := \min \{ \text{coreg}(\mathcal{X}_0) \mid \mathcal{X}' \to \mathcal{X} \text{ is a flat family with slc fibers and } \mathcal{X}_t \simeq X \text{ for some } t \} \]

In the previous definition, we coregularity of a non-normal pair is defined to be the minimum coregularity among the components of its normalization. One can also define a variation of the degeneration coregularity in which we ask the family to be isotrivial outside the central fiber. A similar definition can be given for log Calabi–Yau pairs \((X, \Delta)\). It is clear that we have an inequality

\[ \text{degcoreg}(X, \Delta) \leq \text{coreg}(X, \Delta). \]

Furthermore, this inequality may be strict. For instance, if we consider an elliptic curve \(E\), then we have that

\[ 0 = \text{degcoreg}(E) < \text{coreg}(E) = 1. \]

Indeed, we can degenerate our elliptic curve \(E\) to a cycle of \(k\) rational curves glued along zero and infinity.

In [72], Kulikov studied the central fiber of degenerations of K3 surfaces. They prove that for semistable degeneration of K3 surfaces the central fiber can only have thee possible types:

- it is a smooth K3 surface,
- the dual complex of the central fiber is a segment of a line, the endpoints correspond to rational surfaces and the interior points correspond to ruled surfaces, or
- the dual complex of the central fiber is the triangulation of a sphere, every component of the central fiber is a rational surface of coregularity zero.

The three cases can be distinguished in terms of monodromy around the central fiber. If the monodromy around the special fiber is trivial, then the central fiber is a smooth K3 surface. In particular, a K3 surface may have degeneration coregularity zero, although it has coregularity two.

In this direction, we propose a question in the opposite direction. Given a polyhedral complex \(\mathcal{P}\) whose gluing functions are linear affine maps and its maximal polyhedra are \(n\)-dimensional smooth, we can associate to it a \(n\)-dimensional toric variety \(X(\mathcal{P})\). In the previous case, we say that \(\mathcal{P}\) is a \(n\)-dimensional linear dual complex. In general, the toric variety \(X(\mathcal{P})\) may not be simple normal crossing, but this condition can be obtained by imposing that every vertex in the dual complex is contained in exactly \(n + 1\) maximal polyhedra. In such a case, we say that \(\mathcal{P}\) is smooth. Finally, if every face of dimension \(n - 1\) is contained in exactly two faces of dimension \(n\), then we say that \(\mathcal{P}\) is a Calabi–Yau polyhedral complex. The following proposition follows from the previous definitions.
Proposition 4.49. Let \( P \) be a \( n \)-dimensional linear snc Calabi–Yau polyhedral complex. Then, the variety \( X(P) \) is a log Calabi–Yau simple normal crossing variety so that each component is a projective irreducible toric variety.

For instance, if \( P \) is just a loop with \( k \) vertices, then the associated toric variety \( X(P) \) is a cycle of \( k \) copies of \( \mathbb{P}^1 \) glued along zero and infinity. Many Calabi–Yau varieties degenerate to snc pairs of coregularity zero. The following question aims to understand the opposite direction:

Problem 4.50. Let \( P \) be a \( n \)-dimensional linear snc Calabi–Yau polyhedral complex. Describe the versal deformation space \( \text{Def} X(P) \) in terms of the combinatorics of \( P \). Can \( X(P) \) be deformed into a klt Calabi–Yau variety?

We expect the previous question to be more accessible when the dual complex of \( X(P) \) is a triangulation of a 2-dimensional sphere.

In a similar vein, we can define the iterated degeneration coregularity. We can define this invariant inductively as follows. If the degeneration coregularity of \( X \) equals its dimension \( n \), then the iterated degeneration coregularity is just \( n \). Otherwise, we define it as the minimum among the iterated degeneration coregularity of the minimal dlt centers of the central fiber of any Calabi–Yau semi-log canonical degeneration. For instance, if we can degenerate a log Calabi-Yau pair so that the minimal log canonical centers of the central fiber is an elliptic curve, then its iterated degeneration coregularity is zero. Indeed, we can degenerate the elliptic curve to a semi-log canonical pair of coregularity zero. In most cases known to the author, the iterated degeneration coregularity equals the degeneration coregularity. It would be interesting to find some examples in which these invariants do not agree.

Question 4.51. Is there a klt Calabi-Yau variety (or pair) for which the iterated degeneration coregularity is not equal to the degeneration coregularity?

We expect the previous question to have a positive answer. However, it would be interesting to have some examples and understand how the dual complexes of the different degenerations relate.

In the case that the central fiber of simple normal crossing degeneration is Fano, then the structure is much simpler. Indeed, in this case, the dual complex is a simplex of dimension at most \( n \), where \( n \) is the dimension of the general fiber. In the case that the dual complex of the central fiber is a \( n \)-dimensional simplex, the author together with Loginov proved that each component of the central fiber is a generalized Bott tower [92]. Furthermore, the way that these generalized Bott towers glue is unique. This kind of varieties have coregularity zero. We propose the following problem which is the coregularity one version of the result due to the author and Loginov.

Problem 4.52. Classify log smooth Fano pairs \( (X, \Delta) \) for which \( \Delta \) is reduced and \( \text{coreg}(X, \Delta) = 1 \).

If the Picard rank of \( X \) equals one, then we expect that the varieties in the previous problem are close to being toric. Indeed, we can consider a complement of \( (X, \Delta) \) and observe that its complexity is at most two. However, this may not be the case when the Picard rank of \( X \) is higher.

4.9. Mirror symmetry. Mirror symmetry is an important topic in the geometry of Calabi–Yau varieties and Fano varieties. It is expected that every Calabi–Yau variety \( X \) has a mirror \( X^\vee \) whose complex structure behaves like the symplectic structure of \( X \) and vice-versa. In the case of Fano varieties, the mirror is an affine variety with a potential, the so-called Landau-Ginzburg models. In many cases, this model can be compactified into a log Calabi–Yau variety. In general, we expect the mirror of a log Calabi–Yau variety to be another log Calabi–Yau variety. This has been proved to be the case in dimension two [51].
The Strominger-Yau-Zaslow conjecture (known as SYZ) states that a Calabi-Yau manifold $X$ can be fibered into simpler objects: special Lagrangian tori, such that the dual $X^\vee$ can be obtained by dualizing this family of Lagrangian tori.

Among the many approaches to Mirror symmetry, we find the Gross-Siebert program. The Gross-Siebert program is an algebraic analog of the SYZ conjecture. In this program, the Calabi-Yau variety is degenerated into a toric variety in order to construct its mirror from the central fiber of the degeneration. More precisely, given a Calabi-Yau variety $X$, we try to find a flat degeneration $X \to \mathbb{A}^1$ for which $X_t \simeq X$ for some $t$ and $X_0$ is toric. Then, the mirror of $X_0$ can be constructed using toric geometry. Finally, we can study appropriate deformations of the toric mirror $X_0^\vee$ in order to find the mirror $X^\vee$ of $X$. In our language, the Gross-Siebert program intends to prove Mirror symmetry for Calabi–Yau’s of degeneration coregularity zero. In the toric setting, the dual of a toric variety is always toric. This leads to our first question regarding mirrors of log Calabi–Yau variety.

**Question 4.53.** Let $(X, \Delta)$ be a log Calabi–Yau variety. Let $(X^\vee, \Delta^\vee)$ be its Mirror. Does the equality $\text{coreg}(X, \Delta) = \text{coreg}(X^\vee, \Delta^\vee)$ hold?

We expect that the previous inequality should hold. This expectation is based on the Mirrors of log Calabi-Yau varieties that we can construct in low dimension. Specially, among those that are complete intersections in weighted projective spaces. Let us mention that Mirror symmetry is also expected to preserve certain flat deformations and degenerations of log Calabi–Yau pairs. Hence, it is natural to expect that the previous equality also hold if we replace the coregularity with the degeneration coregularity. Indeed, the Gross-Siebert program can be understood as a very explicit manifestation of this phenomenon. In such a case, there are two other natural questions that arise from a positive answer to the previous question.

**Question 4.54.** Let $(X, \Delta)$ be a log Calabi–Yau pair of dimension $n$ and coregularity zero. Assume Conjecture 1.11. Let $(X^\vee, \Delta^\vee)$ be its mirror. How do the triangulations of the spheres $S^{n-1}$ given by the dual complexes relate?

Finally, it is natural to compare the minimal log canonical centers between the mirrors. However, in order to do so, we need to pass to a dlt modification. The minimal dlt centers of a log Calabi–Yau pair is only well-defined up to birational equivalence. Hence, the best expectation in this direction is the following:

**Question 4.55.** Let $(X, \Delta)$ be a log Calabi–Yau variety. Let $(X^\vee, \Delta^\vee)$ be its mirror. Is a minimal dlt center of $(X, \Delta)$ birational to the mirror of a minimal dlt center of $(X^\vee, \Delta^\vee)$?

There is little to no evidence for a positive answer to the previous question.

4.10. **Positive characteristic.** Recently, there has been a lot of progress in the minimal model program in positive characteristic (see, e.g., [53]). Hence, more tools to tackle problems about complements on Fano varieties become accessible. However, the lack of Kawamata-Viehweg vanishing [11] still imposes a big constraint to mimic proofs from characteristic zero. Hence, new ideas are often required when tackling these problems in positive characteristics. In the case of Fano geometry, we have a couple of examples in positive characteristic that do not lift to characteristic zero [115]. However, it seems that these examples have positive coregularity. We propose the following question:

**Question 4.56.** Find Fano varieties of coregularity zero in positive characteristic that do not lift to characteristic zero.
Many results for complements, coregularity, and dual complexes, are expected to hold in dimension three when the characteristic is larger or equal to 5. However, there could be some interesting examples in low characteristic that do not lift to characteristic zero.

**Question 4.57.** In dimension 3, can we find a log Calabi–Yau pair \((X, \Delta)\) with \(X\) Fano, such that the triangulation of the dual complex \(D(X, \Delta)\) can not be obtained in characteristic zero?

By the Lefschetz principle, every PL-manifold obtained as \(D(X, \Delta)\) for some Calabi–Yau pair in characteristic zero, is already obtained over the complex numbers. The previous question in some sense aims to understand if the theory of complements can give a combinatorial obstruction for the lifting to characteristic zero.

**4.11. Fano rings.** The definition of Fano varieties is a projective one. On the other hand, the definition of klt singularities is a local one, either in the étale topology or analytic topology. One can define affine Fano varieties as follows:

**Definition 4.58.** An affine variety \(U\) is said to be an **affine Fano variety** if there is a projective compactification \(U \hookrightarrow X\) such that if we define \(\Delta := X\setminus U\) with its reduced structure, the pair \((X, \Delta)\) is log Fano with log canonical singularities.

We can give an analogous definition for **affine Calabi–Yau variety**. These varieties are also called open Fanos and open Calabi–Yau’s respectively in the literature.

**Definition 4.59.** We say that a ring \(R\) is a **Fano ring** (resp. **Calabi–Yau ring**) if Spec\((R)\) is an affine Fano variety (resp. affine Calabi–Yau variety).

In dimension one, the only Fano ring is \(\mathbb{K}[x]\). On the other hand, the only Calabi–Yau ring is \(\mathbb{K}[x^{\pm 1}]\). In higher dimensions, due to blow-ups, the classification of Fano rings and Calabi–Yau rings is probably a difficult task. However, we expect that the situation is somehow manageable in dimension 2, so we propose the following problem:

**Question 4.60.** **Classify Fano rings and Calabi–Yau rings of dimension 2.**

It would be interesting to have a purely commutative algebra definition of Fano rings and Calabi–Yau rings. As the previous definitions heavily rely on projective geometry. In the study of coregularity zero Fano varieties, affine Fano varieties of coregularity zero will play the role that complex tori plays in toric geometry. Thus, a better understanding of these rings is desirable.

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