SMALL SUBSET SUMS
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Abstract. Let \( \| \cdot \| \) be a norm in \( \mathbb{R}^d \) whose unit ball is \( B \). Assume that \( V \subset B \) is a finite set of cardinality \( n \), with \( \sum_{v \in V} v = 0 \). We show that for every integer \( k \) with \( 0 \leq k \leq n \), there exists a subset \( U \) of \( V \) consisting of \( k \) elements such that \( \| \sum_{v \in U} v \| \leq \lceil d/2 \rceil \). We also prove that this bound is sharp in general. We improve the estimate to \( O(\sqrt{d}) \) for the Euclidean and the max norms. An application on vector sums in the plane is also given.

1. Definitions, notation, results

We consider the real \( d \)-dimensional vector space \( \mathbb{R}^d \) with a norm \( \| \cdot \| \) whose unit ball is \( B \). For a finite set \( U \subset \mathbb{R}^d \), \( |U| \) stands for the cardinality of \( U \), and \( s(U) \) for the sum of the elements of \( U \), so \( s(U) = \sum_{u \in U} u \), and \( s(\emptyset) = 0 \) of course.

In 1914 Steinitz \cite{St} proved that, in the case of the Euclidean norm, for every finite set \( V \subset B \) with \( s(V) = 0 \), there is an ordering \( v_1, \ldots, v_n \) of the vectors in \( V \) such that all partial sums have norm at most \( 2d \), that is

\[
\max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} v_i \right\| \leq 2d.
\]

It is important here that the bound \( 2d \) does not depend on \( n \), the size of \( V \). Steinitz’s result implies that for every norm and every finite \( V \subset B \) with \( s(V) = 0 \) there is an ordering along which all partial sums are bounded by a constant that depends only on \( B \). Let \( S(B) \) denote the smallest such constant for a given norm with unit ball \( B \), and set \( S(d) = \sup S(B) \) where the supremum is taken over all norms in \( \mathbb{R}^d \). The best known bounds on \( S(d) \) are: \( S(B) \leq d \), proved by Sevastyanov \cite{Se}, and by Grinberg and Sevastyanov \cite{GS}, and \( S(d) \geq \frac{d+1}{2} \), which is shown by an example coming from the \( \ell_1 \) norm \cite{GS}. For specific norms, stronger results may hold. In particular, for \( \ell_2 \) and \( \ell_\infty \), it is conjectured that the right order of magnitude of \( S(B) \) is \( \sqrt{d} \) – although not even \( o(d) \) is known.

Steinitz’s result immediately implies that for every finite set \( V \subset B \) with \( s(V) = 0 \) and every integer \( k \), \( 0 \leq k \leq |V| \), there is a subset \( U \subset V \) such that \( |U| = k \) and \( \| s(U) \| \) is not greater than a constant depending only on \( d, B, k \), for instance \( S(B) \) is such a constant. Let \( T(B, k) \) be the smallest constant with this property, set \( T(B) = \sup_k T(B, k) \), and \( T(d) = \sup T(B) \) where the supremum is taken over all norms in \( \mathbb{R}^d \). It is evident that \( T(B, k) \leq k \).
In this paper we investigate $T(B, k), T(B)$ and $T(d)$. Here come our main results. First, the estimate for general norms.

**Theorem 1.** Let $B$ be the unit ball of an arbitrary norm on $\mathbb{R}^d$. For any finite set $V \subset B$ with $s(V) = 0$, and for any $k \leq |V|$, there exists a subset $U \subset V$ with $k$ elements, so that

$$\|s(U)\| \leq \left\lceil \frac{d^2}{2} \right\rceil.$$  

In other words, $T(d) \leq \left\lceil \frac{d^2}{2} \right\rceil$.

**Theorem 2.** For every $d \geq 1$, there exists a norm in $\mathbb{R}^d$ with unit ball $B$, so that $T(B, k) = \left\lceil \frac{d^2}{2} \right\rceil$ for infinitely many values of $k$. Also, $T(B, k) = k$ for all $k \leq \left\lfloor \frac{d^2}{2} \right\rfloor$.

Theorems 1 and 2 imply that $T(d) = \left\lceil \frac{d^2}{2} \right\rceil$ for all integers $d \geq 1$.

One expects that for specific norms better estimates are valid. We have proved this in some cases. The unit ball of the norm $\ell^p_d$ will be denoted by $B^d_p$. We have the following results in the cases $p = 1, 2, \infty$.

**Theorem 3.** $\frac{d^2}{2} \leq T(B^d_1) \leq \left\lceil \frac{d^2}{2} \right\rceil$.

**Theorem 4.** $\frac{1}{2}\sqrt{d+2} \leq T(B^d_2) \leq 1 + \frac{\sqrt{5}}{2}\sqrt{d}$

**Theorem 5.** $\frac{1}{3}\sqrt{d} \leq T(B^d_\infty) \leq O(\sqrt{d})$

We mention that in Theorems 4 and 5 the order of magnitude is the same as the conjectured value of the Steinitz constant.

**Remark 1.** Note that there is a "complementary" symmetry here. Namely, for every $U \subset V$, $s(U) = -s(V \setminus U)$, hence $\|s(U)\| = \|s(V \setminus U)\|$, and the cases $k$ and $n-k$ are symmetric. Hence, we may assume $k \leq n/2$.

When establishing Helly-type theorems for sums of vectors in a normed plane, Bárány and Jerónimo-Castro proved the following result [3, Lemma 5], which matches our scheme: *Given 6 vectors in the unit ball of a normed plane whose sum is 0, there always exist 3 among them, whose sum has norm at most 1.* In fact, this statement served as the starting point for our current research. An application of Theorem 1 implies an extension of one of the Helly-type results [3, Theorem 3], which we formulate slightly differently and prove in the last section.

**Theorem 6.** Let $k \geq 2$ be a positive integer, and $n = m(k-1) + 1$ for some $m \geq 1$. Assume $B$ is the unit ball of a norm in $\mathbb{R}^2$, $V \subset B$ is of size $n$ and $\|s(V)\| \leq 1$. Then $V$ contains a subset $W$ of size $k$ such that $\|s(W)\| \leq 1$.

2. **Proof of Theorem 1**

We are to consider linear combinations $\sum_{v \in V} \alpha(v)v$ of the vectors in $V$. The coefficients $\alpha(v)$ form a vector $\alpha \in \mathbb{R}^V$. Define the convex polytope

$$P(V, k) = \left\{ \alpha \in \mathbb{R}^V : \sum_{v \in V} \alpha(v)v = 0, \sum_{v \in V} \alpha(v) = k, 0 \leq \alpha(v) \leq 1 \ (\forall v \in V) \right\}.$$
$P(V, k)$ is non-empty as $a(v) \equiv k/n$ lies in it (here $n = |V|$). From now on let $\alpha$ denote a fixed vertex of $P(V, k)$. The basic idea is to choose $U$ to be the set of vectors from $V$ that have the $k$ largest coefficients $\alpha(v)$. This works directly when $d$ is odd, and some extra care is needed for even $d$.

We note first that $P(V, k)$ is determined by $d + 1$ linear equations and $2n$ inequalities for the coefficients $\alpha(v)$, so at a vertex at most $d + 1$ coefficients are strictly between 0 and 1. Define $U_1 = \{v \in V : \alpha(v) = 1\}$ and $Q = \{v \in V : 0 < \alpha(v) < 1\}$. Set $q = \sum_{v \in Q} \alpha(v)$, $q$ is an integer since $q + |U_1| = k$. Split now $Q$ into two parts, $E$ and $F$, so that $|E| = q$ and $E$ contains the vectors with the $q$ largest coefficients in $Q$, and $F$ the rest (ties broken arbitrarily). Then $U = U_1 \cup E$ has exactly $k$ elements and

$$s(U) = \sum_{v \in U_1} v + \sum_{v \in E} v = \sum_{v \in V} \alpha(v)v + \sum_{v \in E} (1 - \alpha(v))v - \sum_{v \in F} \alpha(v)v.$$  

Here $\sum_{v \in V} \alpha(v)v = 0$, so by the triangle inequality

$$||s(U)|| \leq \sum_{v \in E} (1 - \alpha(v)) + \sum_{v \in F} \alpha(v).$$

The average of the coefficients in $Q$ is $a := q/|Q|$. Thus, the average of the coefficients is at least $a$ in $E$, and it is at most $a$ in $F$. Consequently, the last sum is maximal when $\alpha(v) = a$ for all $v \in Q$:

$$||s(U)|| \leq q(1 - a) + (|Q| - q)a = \frac{2}{|Q|} q(|Q| - q) \leq \frac{|Q|}{2}.$$  

This finishes the proof when $d$ is odd as $|Q| \leq d + 1$, and also when $d$ is even and $|Q| \leq d$.

We are left with the case when $d$ is even and $|Q| = d + 1$. The vectors in $Q$ are linearly dependent, so there is a non-zero $\beta \in \mathbb{R}^V$ with $\beta(v) = 0$ when $v \notin Q$ such that $\sum_{v \in Q} \beta(v)v = 0$. We can assume that $\sum_{v \in Q} \beta(v) \neq 0$. Then $\sum_{v \in V} (\alpha(v) + t\beta(v))v = 0$ for every $t \in \mathbb{R}$. Choose $t > 0$ maximal so that $0 \leq \gamma(v) = (\alpha(v) + t\beta(v)) \leq 1$ for every $v \in V$. This means that, for some $v^* \in Q$, $\gamma(v^*) = 0$ or 1.

Assume for the time being that $q \leq (d + 1)/2$.

Suppose first that $\gamma(v^*) = 0$. This time we split $Q^* := Q \setminus v^*$ again into $E$ and $F$ so that $|E| = q$ and $E$ contains the vectors from $Q^*$ with the $q$ largest coefficients. Note that $\sum_{v \in Q^*} \gamma(v) \leq \sum_{v \in Q} \alpha(v) = q$ and that $|Q^*| = d$, so the average $a^*$ of $\gamma(v)$ over $Q^*$ is at most $q/d$. We use again $U = U_1 \cup E$ and we have, the same way as before,

$$||s(U)|| \leq \sum_{v \in E} (1 - \gamma(v)) + \sum_{v \in F} \gamma(v).$$

The right hand side is maximal again if every $\gamma(v)$ equals their average $a^*$, hence

$$||s(U)|| \leq q(1 - a^*) + (d - q)a^* = q + (d - 2q)a^* \leq q + (d - 2q)\frac{q}{d} \leq \frac{d}{2}.$$
because \( d \) is even so \( q \leq (d + 1)/2 \) implies \( 2q \leq d \). Thus, \( \|s(U)\| \leq d/2 \).

The case when \( \gamma(v^*) = 1 \) is similar: this time \( v^* \) is added to \( U_1 \), \( Q^* = Q \setminus v^* \) is split into \( E \) and \( F \) with \( |E| = q - 1 \) so that \( E \) contains the vectors with the largest \( q - 1 \) coefficients. Now \( \sum_{v \in Q^*} \gamma(v) \leq \sum_{v \in Q} \alpha(v) - 1 = q - 1 \), and thus the average \( a^* \) of \( \gamma(v) \) over \( Q^* \) is at most \((q - 1)/d\). As above, we are led to the inequality
\[
\|s(U)\| \leq (q - 1)(1 - a^*) + (d - (q - 1))a^* = (q - 1) + (d - 2(q - 1))a^*.
\]

Using that \( d - 2(q - 1) \geq 0 \) and \( a^* \leq (q - 1)/d \), we conclude that \( \|s(U)\| \leq d/2 - 2/d < d/2 \).

Finally we consider the case \( q > (d + 1)/2 \). By complementary symmetry \( s(U) = -s(V \setminus U) \). For \( q > (d + 1)/2 \), we consider the complementary problem of finding \( U \subset V \) with \( n - k \) elements so that \( \|s(U)\| \leq \lceil d/2 \rceil \).

It is easy to see that \( 1 - \alpha(\cdot) \in \mathbb{R}^V \) is a vertex of \( P(V, n - k) \), for which \( \sum_{v \in Q}(1 - \alpha(v)) < (d + 1)/2 \). □

The same proof yields a stronger statement.

**Theorem 7.** Let \( W \subset B \) finite. Then for every \( k \leq |W| \) and for every vector \( w_0 \in \text{conv} \ W \), there is a subset \( U \subset W \) of cardinality \( k \), so that
\[
\|s(U) - kw_0\| \leq \left\lfloor \frac{d}{2} \right\rfloor.
\]

The proof is the same as above, except that instead of the convex polytope \( P(V, k) \), we consider the coefficient vectors \( \alpha : W \to [0, 1] \) satisfying
\[
\sum_{w \in W} \alpha(w)w = kw_0 \quad \text{and} \quad \sum_{w \in W} \alpha(w) = k.
\]

The condition \( w_0 \in \text{conv} \ W \) ensures that this set is a non-empty convex polytope. The rest of the argument is unchanged.

**Remark 2.** For later reference we record the fact that the linear dependence \( \alpha \) defines the sets \( U_1 \) and \( Q \), if \( |Q| = d + 1 \), then the new linear dependence \( \gamma \) defines \( v^* \in Q \) and \( Q^* \). Note that this works for even and odd \( d \), we only need \( |Q| = d + 1 \). For later use we define
\[(1) \quad A = \{v \in V : \gamma(v) = 1\} \quad \text{and} \quad C = \{v \in V : 0 < \gamma(v) < 1\}.
\]
Assume that $U$ contains $b_i$ copies of $e_i$ for every $i$, so $k = \sum_0^d b_i$. We have to estimate the norm of the vector $v = \sum_0^d b_i e_i$. Assume that

$$v = \sum_0^d a_i e_i$$

for some $a_i \in \mathbb{R}$. Then $\sum_0^d (b_i - a_i) e_i = 0$. Since the only linear dependence of the vectors $e_0, \ldots, e_d$ is $x \sum_0^d e_i = 0$ for some constant $x \in \mathbb{R}$, we obtain that $a_i = b_i - x$ for every $i$. Set

$$f(x) := \sum_0^d |b_i - x|,$$

Then $\|v\| = \min f(x)$ by the fact from the beginning of this section. We are going to estimate $f(x)$. Since $b_i \in \mathbb{Z}$ for every $i$, the function $f(x)$ is piecewise linear on $\mathbb{R}$ (it is affine on all intervals $(q, q+1)$ for $q \in \mathbb{Z}$). Therefore, there exists $c \in \mathbb{Z}$ so that the minimum of $f(x)$ is attained at $c$.

The facts $k = \sum_0^d b_i \equiv [d/2] \mod (d+1)$ and $c \in \mathbb{Z}$ imply that $\sum_0^d (b_i - c) \equiv [d/2] \mod (d+1)$. Thus,

$$\left\lfloor \frac{d}{2} \right\rfloor \leq \sum_0^d (b_i - c) \leq \sum_0^d |b_i - c|,$$

hence, $\|v\| \geq [d/2]$.

We show next that $T(B, k) = k$ when $1 \leq k < [d/2]$. The unit ball $B$ is the same as above and $V = \{e_0, \ldots, e_d\}$. Assume $U \subset V$ with $|U| = k$ and $\|s(U)\| < k$. Add $[d/2] - k$ vectors from $V \setminus U$ to $U$ to obtain a subset $W$ of $[d/2]$ elements. Every addition increases the norm of the sum by at most one (because of the triangle inequality), so we get $\|s(W)\| \leq \|s(U)\| + [d/2] - k < [d/2]$, contrary to what was established above. Thus $T(B, k) \geq k$, while $T(B, k) \leq k$ follows from the triangle inequality. \qed

Further examples showing $T(B, k) = [d/2]$ will be given in the next section.

**Remark 3.** We mention that for large enough $n$, there is no vector set that works simultaneously for all $k$ with $d/2 \leq k \leq n - d/2$. This follows from Steiniz’s theorem: let $v_1, \ldots, v_n$ be the ordering where all partial sums lie in $B$. Then necessarily two partial sums, with at least $d/2$ summands whose cardinalities differ by at least $d/2$, are close to each other: a standard volume estimate shows that their distance is bounded above by $4dn^{-1/d}$. Then their difference, which is a $k$-sum with some $d/2 \leq k \leq n - d/2$, must be small.

### 4. The $\ell_1$ norm, proof of Theorem 3

The upper bound follows from Theorem 1. For the lower bound let $V$ consist of $e_1, \ldots, e_d$ and $d$ copies of $\frac{1}{d} e_0$ (with the same notation as in the previous section). Assume $U \subset V$ has exactly $d$ elements. If $U$ contains $p$ vectors out of $e_1, \ldots, e_d$, then $s(U)$ has $p$ coordinates equal to $\frac{d}{2}$ and $d - p$ coordinates equal to $\frac{d}{2} - 1$. Thus $\|s(U)\|_1 = \frac{d}{2}(p^2 + (d - p)^2)$. The last
expression is minimal when \( p = \lfloor \frac{d}{2} \rfloor \). The minimum equals \( \frac{d}{2} \) when \( d \) is even and \( \frac{d}{2} + \frac{1}{2} \) when \( d \) is odd. This is slightly better (for \( d \) odd) than the stated lower bound.

This example shows that \( T(B^d_1) = T(B^d_1, d) = d/2 \) for even \( d \). A small modification gives further examples implying \( T(B^d_1, k) = d/2 \) for even \( d \) and for all \( k \geq d \). Namely, given \( d \geq 1 \) and \( k \geq d \), let \( V \) consist of the vectors \( e_1, \ldots, e_d \), and \( 2k - d \) copies of \( \frac{1}{2k-d} e_0 \). Then \( V \subset B^d_1 \) and \( s(V) = 0 \). It is not hard to check that this shows \( T(B^d_1, k) = d/2 \) for every \( k \geq d \) (\( d \) is even).

5. The \( \ell_2 \) norm, proof of Theorem \( \[4\] \)

In this section, \( \| \cdot \| \) stands for the Euclidean norm. For the upper bound we will need two lemmas. The first is Lemma 2.2 in Beck’s paper \[4\]. A similar result is given in \[11\] Theorem 4.1. The second is a Steinitz type statement.

**Lemma 1.** Let \( Q \subset B^d_2 \) be finite, and \( \alpha : Q \to [0,1] \). Then there exists \( \varepsilon : Q \to \{0,1\} \) such that \( \| \sum_{v \in Q} (\varepsilon(v) - \alpha(v)) v \| \leq \sqrt{d}/2 \).

**Lemma 2.** Assume that \( V \subset B^d_2 \) is a finite set and \( \| s(V) \| = \sigma \). Then there exists an ordering \( v_1, \ldots, v_n \) of the elements of \( V \), such that, for all \( h \leq n \),

\[
\left\| \sum_{i=1}^{h} v_i \right\| \leq \sqrt{\sigma^2 + h}.
\]

**Proof.** Choose \( v_1 \in V \) arbitrarily. For \( h \geq 2 \), we select \( v_h \) inductively. We set \( S_h = \sum_{i=1}^{h} v_i \). Assume that \( \| S_{h-1} \| \leq \sqrt{\sigma^2 + h - 1} \), and set \( W = V \setminus \{ v_1, \ldots, v_{h-1} \} \). We consider three cases.

**Case 1.** If \( \| S_{h-1} \| \leq \sigma - 1 \), then choose \( v_h \in W \) arbitrary: \( \| S_h \| \leq \sigma \) holds by the triangle inequality.

**Case 2.** If \( \| S_{h-1} \| \geq \sigma \), then by the assumption \( \| S \| = \sigma \), there exists a vector \( v_h \in W \), for which \( \langle S_{h-1}, v_h \rangle \leq 0 \). Therefore,

\[
\| S_h \|^2 = \| S_{h-1} + v_h \|^2 \leq \| S_{h-1} \|^2 + \| v_h \|^2 \leq (\sigma^2 + h - 1) + 1 = \sigma^2 + h.
\]

**Case 3.** If \( \sigma - 1 < \| S_{h-1} \| < \sigma \), define \( \varepsilon = \sigma - \| S_{h-1} \| \), so \( 0 < \varepsilon < 1 \) and \( \varepsilon \leq \sigma \). Then

\[
\sum_{v \in W} \langle v, S_{h-1} \rangle = \langle S_h - S_{h-1}, S_{h-1} \rangle \leq \sigma(\sigma - \varepsilon) - (\sigma - \varepsilon)^2 = \varepsilon(\sigma - \varepsilon).
\]

Thus, there exists \( v_h \in W \), for which \( \langle v_h, S_{h-1} \rangle \leq \varepsilon(\sigma - \varepsilon) \). Then

\[
\| S_h \|^2 = \| S_{h-1} + v_h \|^2 \leq (\sigma - \varepsilon)^2 + 2\varepsilon(\sigma - \varepsilon) + 1
\]

\[
= \sigma^2 + 1 - \varepsilon^2 < \sigma^2 + h.
\]

**Proof of Theorem \[4\]** For the lower bound let \( V \) be the set of vertices of a regular simplex inscribed in \( B^d_2 \). Then \( s(V) = 0 \). Let \( U \subset V \) have \( \left\lfloor \frac{d}{2} \right\rfloor \) elements. A routine computation shows that \( \| s(U) \| \) equals \( \frac{\sqrt{d+2}}{d} \) when \( d \) is even and \( \frac{\sqrt{d+2}}{d} > \frac{\sqrt{d+2}}{d} \) when \( d \) is odd. This implies the lower bound

\[
T(B^d_2) \geq \frac{\sqrt{d+2}}{d}.
\]
For the upper bound we have to prove the existence of \(U \subset V\) with \(|U| = k\) and \(\|s(U)\| \leq \frac{1}{2}\sqrt{\frac{d}{2}}\). From the proof of Theorem 1 recall the definition of \(P(V, k)\) and its vertex \(\alpha \in \mathbb{R}^V\) and \(U_1 = \{v \in V : \alpha(v) = 1\}\) and \(Q = \{v \in V : 0 < \alpha(v) < 1\}\). Here \(|Q| \leq d + 1\).

If \(|Q| = 0\), then \(|U_1| = k\) and \(s(U_1) = 0\), so we can set \(U = U_1\). The case \(|Q| = 1\) is impossible because the sum of all \(\alpha(v)\) is an integer. From now on we assume that \(2 \leq |Q|\) implying \(|U_1| + 1 \leq |U_1| + |Q| - 1\). Using Lemma 1 for \(\alpha\) restricted to \(Q\) we find \(\varepsilon : Q \to \{0, 1\}\) such that 
\[
\|\sum_{v \in Q}(\varepsilon(v) - \alpha(v))v\| \leq \sqrt{d}/2.
\]

Define \(W = U_1 \cup \{v \in Q : \varepsilon(v) = 1\}\), then \(W\) has the properties that 
\[
\|s(W)\| \leq \sqrt{d}/2 \quad \text{and} \quad |W| - k \leq d.
\]
Because of the complementary symmetry, we can assume that \(k \leq |W| \leq k + d\). Set \(h = |W| - k\). Then Lemma 2 applies to \(W\): writing \(\sigma = \|s(W)\|\) we have \(\sigma \leq \sqrt{d}/2\) and so the elements of \(W\) can be ordered as \(w_1, w_2, \ldots\) so that 
\[
\|\sum_{i=1}^{m} w_i\| \leq \sqrt{\sigma^2 + m}
\]
for every \(m\). In particular, with \(m = h \leq d\), 
\[
\|\sum_{i=0}^{h} w_i\| \leq \sqrt{\sigma^2 + h} \leq \sqrt{d/4 + d}.
\]
Then for \(U = W \setminus \{w_1, \ldots, w_h\}\), we have \(|U| = k\) and \(\|s(U)\| \leq \frac{1}{2}\sqrt{\frac{d}{2}}\). \(\square\)

6. The \(\ell_\infty\) norm, proof of Theorem 5

Here, \(\|\cdot\|\) denotes the maximum norm. We need two lemmas again, the first is similar to Lemma 1.

**Lemma 3.** If \(C \subset B^d\) consists of \(d\) linearly independent vectors, then for every point \(z\) of the parallelootope \(P = \sum_{v \in C}[0, v]\), there is a vertex \(u\) of \(P\) with \(\|z - u\|_\infty = O(\sqrt{d})\).

This is a result of Spencer [10] Corollary 8, and also of Gluskin [6] whose work relies on that of Kashin [8]. Spencer’s proof gives the estimate \(\|z-u\| \leq \frac{2}{6}\sqrt{d}\). The linear independence condition is only needed to ensure that \(P\) is a parallelootope, and so its vertices are of the form \(s(D) = \sum_{v \in D} v\) for some subset \(D \subset C\).

The next statement is the (weaker) analogue of Lemma 2 for the \(\ell_\infty\) norm. Note that we require the set \(W\) to contain only a few vectors. The proof is longer and it uses Chobanyan’s transference theorem (for the \(\ell_\infty\) norm) so we postpone it to Section 7.

**Lemma 4.** Assume \(W \subset B^d, |W| = m \leq 5d, \) and \(\|s(W)\|_\infty = O(\sqrt{d})\). Then there is an ordering \(w_1, \ldots, w_m\) of the vectors in \(W\) such that 
\[
\max_{h=1,\ldots,m} \|\sum_{i=1}^{h} w_i\|_\infty = O(\sqrt{d}).
\]

**Proof of Theorem 5** The lower bound uses Hadamard matrices and is given in [1].

For the upper bound we assume, rather for convenience than necessity, that the set \(V \subset \mathbb{R}^d\) is in general position, for instance, no \(d\) vectors from \(V\) are linearly dependent. The general case follows from this by a limit argument. We assume further that \(|V| = n > 5d\) since for \(n \leq 5d\) the result is a consequence of Lemma 4.

Set \(m = \lfloor n/(2d) \rfloor\).
We are going to define linear dependencies $\gamma_i$, for $i = 1, 2, \ldots, m - 1$ so that the sets

$$A_i = \{v \in V : \gamma_i(v) = 1\}, \quad C_i = \{v \in V : 0 < \gamma_i(v) < 1\}$$

satisfy the conditions

$$A_i \subset A_{i+1}, \quad (2i - 1)d \leq |A_i| \leq h_i := \sum_{v \in V} \gamma_i(v) \leq 2di, \quad |C_i| = d.$$

The construction is recursive and is similar to how $\alpha$ and $\gamma \in \mathbb{R}^V$ were constructed. For $i = 1$ we take an arbitrary vertex $\alpha$ of the convex polytope $P(V, 2d)$, then $|Q| = d + 1$ (because of the general position assumption) and $d \leq |U_i| < 2d$ follows. We construct $\gamma$ as specified in Remark 2 and (11). Then define $\gamma_1 = \gamma$, set $A_1 = \{v \in V : \gamma_1(v) = 1\}$, $C_1 = \{v \in V : 0 < \gamma_1(v) < 1\}$. General position implies that $|C_1| = d$ and then $d \leq |A_1| < h_1 = \sum_{v \in V} \gamma_1(v) \leq 2d$.

Assume next that $\gamma_1, \ldots, \gamma_i$ have been constructed ($1 < i < m - 1$), and the sets $A_j, C_j$ for $j \leq i$ satisfy the required conditions. Define the convex polytope

$$P_{i+1} = \{\alpha \in P(V, 2d(i+1)) : \alpha(v) = 1 \ \forall v \in A_i\}$$

We check that $P_{i+1}$ is non-empty. As $|A_i| < h_i \leq 2di$, the linear dependence $\alpha = \gamma_i + t(1 - \gamma_i)$ lies in $P_{i+1}$ for a suitable $t$, we only have to check that $0 < t < 1$ as this implies $0 \leq \alpha(v) = \gamma_i(v) + t(1 - \gamma_i(v)) \leq 1$. To fulfill the condition $\sum_{v \in V} \alpha(v) = 2d(i + 1)$, we must set

$$t = \frac{2d(i + 1) - h_i}{n - h_i} = 1 - \frac{n - 2d(i + 1)}{n - h_i}.$$

Thus $0 < t < 1$ indeed as $h_i \leq 2di$.

Next, let $\alpha_{i+1}$ be a fixed vertex of $P_{i+1}$. The method recorded in Remark 2 gives another linear dependence $\gamma_{i+1}$ with $|C_{i+1}| = d$. $A_i \subset A_{i+1}$ by the construction. All $v \in V$ with $\alpha_{i+1}(v) = 1$ are in $A_{i+1}$, and there are at least $2d(i + 1) - d$ of them. Thus $(2i + 1)d \leq |A_{i+1}|$. Further $|A_{i+1}| < h_{i+1}$ follows since $\gamma_{i+1}(v) = 1$ for every $v \in A_{i+1}$ and $h_{i+1} \leq 2d(i + 1)$ because $h_{i+1} = \sum_{v \in V} \gamma_{i+1}(v) \leq \sum_{v \in V} \alpha_{i+1}(v) = 2d(i + 1)$.

The construction is almost finished, as a last step we define $A_0 = C_0 = \emptyset$.

We use Lemma 3 next. The parallelotope $P := \sum_{v \in C_0} [0, v]$ contains the point $-s(A_i)$, since $0 = s(A_i) + \sum_{v \in C_0} \gamma_i(v)v$. A vertex of $P$ is of the form $s(D) = \sum_{v \in D} v$, where $D$ is a subset of $C_0$. By Lemma 3 there is a $D \subset C_0$ such that the vertex $s(D_0)$ is at distance $O(\sqrt{d})$ from $-s(A_i)$. Thus the vector $z_i = s(A_i \cup D_i)$ is short, namely, $\|z_i\| = O(\sqrt{d})$. Note that by setting $D_0 = \emptyset$, we have $z_0 = 0$ which is again of norm $O(\sqrt{d})$.

For the next step of the proof we first check that the size of the symmetric difference $(A_{i+1} \cup D_{i+1}) \triangle (A_i \cup D_i)$ is at most $5d$. This holds for $i = 0$. For larger $i$, $D_{i+1}$ and $A_{i+1}$ are disjoint, and $A_{i+1}$ contains $A_i$, so the symmetric difference is the same a $X \triangle D_i$, where $X = (A_{i+1} \setminus A_i) \cup D_{i+1}$. Here $|A_{i+1} \setminus A_i| < 3d$, and both $D_i$ and $D_{i+1}$ have at most $d$ elements, which gives the upper bound $5d$. 


Now $z_i - s(D_i) + s(X) = z_{i+1}$. Thus, adding at most $5d$ vectors from $B^d_\infty$ to $z_i$ one arrives at $z_{i+1}$, and both $z_i, z_{i+1}$ are short. Define

$$W = \{-u : u \in D_i \setminus X\} \cup (X \setminus D_i).$$

Then $W$ is a subset of $B^d_\infty$, of at most $5d$ elements, such that $s(W) = \sum_{w \in W} w = z_{i+1} - z_i$. Thus $\|s(W)\| = O(\sqrt{d})$. By applying Lemma 3 to $W$ we get an ordering $w_1, \ldots, w_m$ such that every partial sum along this ordering is $O(\sqrt{d})$. Then for every $h = 1, \ldots, m$.

$$\|z_i + \sum_{j=1}^{h} w_j\| \leq \|z_i\| + \|\sum_{j=1}^{h} w_j\| = O(\sqrt{d}).$$

In the original problem we have to show that for every $k \leq n$ there is a set $U \subset V$ of size $k$ with $\|s(U)\| = O(\sqrt{d})$. This is clear when $k$ equals the size of some $A_i \cup D_i$, but what is to be done for the other $k$? Well, such a $k$ lies between $|A_i \cup D_i|$ and $|A_{i+1} \cup D_{i+1}|$ for some $i$. Note that $z_i = s(A_i \cup D_i)$. Moreover, each sum $z_i + w_1 + \ldots + w_h$ is the sum of vectors in a subset of $V$. This can be seen by induction on $h$. The case $h = 0$ is clear. The induction step $h \rightarrow h$ is clear again when $w_h$ does not come from $D_i$, simply one more term appears in the sum. If however $w_h$ comes from $D_i$, then it cancels the previous $-w_h$ that is a unique term in $s(A_i \cup D_i)$. So each partial sum is a subset-sum. The number of elements in these subsets increases or decreases by one when the next $w_h$ is added. Then for every $k$ between $|A_i \cup D_i|$ and $|A_{i+1} \cup D_{i+1}|$ there is a partial sum containing exactly $k$ terms.

\[\square\]

**Remark 4.** The above proof yields a slightly stronger statement: we construct a chain of subsets of $V$, each with sum of order of magnitude $O(\sqrt{d})$, so that the cardinality of two consecutive subsets differ by one, and the chain traverses from the empty set to $V$. We have hoped to give a better value for the Steinitz constant $S(B^d_2)$ or $S(B^d_\infty)$ by a suitable modification of the argument (we would need an increasing chain of subsets with the previous properties), but our efforts have failed so far.

**Remark 5.** A simpler proof may be given if one only aims for the existence a $k$-element subset with small sum. We may assume that $k \leq n - d$. Starting from a vertex of $P(v, k - d)$ and using Lemma 3 similarly to the proof of Theorem 3 we can construct a set $W$ so that $\|s(W)\| \leq 6\sqrt{d}$, and $k - 2d \leq |W| \leq k$. Let $\alpha$ be the characteristic function of $W$, i.e. $\alpha(v) = 1$ if $v \in W$, and 0 otherwise. Let $l = |W|$, and set $t$ so that $l + t(n - l) = k + d$. Then $t \leq 1$.

Next, consider the set $P$ of the linear dependencies $\beta : V \rightarrow [0, 1]$ with

$$\sum_{v \in V} \beta(v) = (1 - t)s(W), \sum_{v \in V} \beta(v) = k + d, \beta(v) = 1(\forall v \in W).$$

Then $P$ is a non-empty convex polytope, since $\alpha + t(1 - \alpha)$ satisfies all the above conditions. Take an arbitrary a vertex of $P$. As before, invoking Lemma 3 we find a set $Y$ so that $\|s(Y) - (1 - t)s(W)\|_\infty = O(\sqrt{d})$, and
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\[ |Y| - (k + d)| \leq d \]. Furthermore, the construction implies that \( W \subset Y \). Hence,

\[ k - 2d \leq |W| \leq k \leq |Y| \leq k + 2d, \]

and \( \|s(W)\| = O(\sqrt{d}) \) as well as \( \|s(Y)\| = O(\sqrt{d}) \). We finish the proof by applying Lemma 4 to the set \( Y \setminus W \).

**Remark 6.** The above proofs translate for arbitrary norms as long as the analogues of Lemmas 1 and 2 (or Lemmas 3 and 4) may be established.

### 7. Proof of Lemma 4

For this lemma it is natural to use Chobanyan’s transference theorem [5] (see also [1]), which connects Steinitz’s theorem with sign assignments to vectors in a sequence.

Assume \( v_1, \ldots, v_n \) is a sequence of vectors from the unit ball \( B \) of an arbitrary norm on \( \mathbb{R}^d \). It is proved in [2] that there are signs \( \varepsilon_1, \ldots, \varepsilon_n = \pm 1 \) such that

\[ \max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} \varepsilon_i v_i \right\| \leq 2d - 1. \tag{2} \]

This is a general bound that does not depend on \( n \) and the norm. But better estimates are valid for specific norms and some (small) values of \( n \). For fixed \( B \) and \( n \) let \( F(B, n) \), the sign sequence constant of \( B \), be defined as the smallest number that one can write on the right hand side of (2), and let \( F(B) = \sup_n F(B, n) \). It is quite easy to see for instance that \( F(B_2^d, n) \leq \sqrt{n} \) for all \( n \) (but we don’t need this). What we need is a result of Spencer [11, Theorem 1.4]:

**Fact 1.** \( F(B_2^d, d) \leq K \sqrt{d} \) where \( K \) is a universal constant.

Chobanyan’s transference theorem [5] says that, for every norm with unit ball \( B \), \( S(B) \leq F(B) \), that is, the Steinitz constant is at most as large as the sign sequence constant. We need a slightly stronger variant, so we define \( S(B, n) \) as the smallest number \( R \) such that for every set \( V \subset B \) with \( s(V) = 0 \) and \( |V| = n \) there is an ordering \( v_1, \ldots, v_n \) of the elements in \( V \) such that

\[ \max_{k=1,\ldots,n} \left\| \sum_{i=1}^{k} v_i \right\| \leq R. \]

Of course, \( S(B) = \sup_n S(B, n) \). Here comes the stronger version of Chobanyan’s theorem, and comes without proof as the proof is identical with the original one.

**Theorem 8.** For every norm in \( \mathbb{R}^d \) with unit ball \( B \), \( S(B, n) \leq F(B, n) \).

Theorem 8 and Fact 1 imply the following.

**Fact 2.** Given \( V \subset B_\infty^d \) with \( |V| = m \) where \( m \leq 5d \) and \( s(V) = 0 \), there is an ordering \( v_1, \ldots, v_m \) of \( V \) such that \( \max_{h=1,\ldots,m} \left\| \sum_{i=1}^{h} v_i \right\|_\infty \leq K_1 \sqrt{d} \), where \( K_1 \) is a universal constant.
**Proof.** We note first that for $m \leq d$ this follows directly from Fact 1 and Theorem 3 with $K_1 = K$. For $m \geq d$, take the natural embedding of $\mathbb{R}^d$ into $\mathbb{R}^m$, the set $V$ lies in the $\ell_\infty$ unit ball of $\mathbb{R}^m$. Apply Fact 1 and Theorem 3 there, and you get an ordering of $V$ in $\mathbb{R}^d$ along which all partial sums have norm at most $K\sqrt{m} \leq K\sqrt{5d}$. Thus Fact 2 holds with $K_1 = \sqrt{5}K$.

**Proof of Lemma 4.** We need a concrete bound on $\|s(W)\|_\infty$, so suppose that $\|s(W)\|_\infty \leq K_2\sqrt{d}$. For $w \in W$ define $w^* = w - \frac{1}{m}s(W)$. Then $\|w^*\|_\infty \leq \|w\|_\infty + \frac{1}{m}\|s(W)\|_\infty \leq 2$ as $s(W)$, being the sum of $m$ vectors from $B_\infty^d$, has norm at most $m$. Further, $\sum_{w \in W} w^* = 0$ and $W \subset 2B_\infty^d$. By Fact 2 there is an ordering $w_1, \ldots, w_m$ of the vectors in $W$ such that for every $h$

$$\left\| \sum_{i=1}^h w_i^* \right\|_\infty \leq 2K_1\sqrt{d}.$$ 

We check that $\sum_{i=1}^h w_i = \sum_{i=1}^h w_i^* + \frac{h}{m}s(W)$ and so for every $h$

$$\left\| \sum_{i=1}^h w_i \right\|_\infty \leq \left\| \sum_{i=1}^h w_i^* \right\|_\infty + \left\| s(W) \right\|_\infty \leq 2K_1\sqrt{d} + K_2\sqrt{d} = O(\sqrt{d}).$$

**8. An application: proof of Theorem 5.**

We proceed by induction on $m$. For $m = 1$, the assertion is clearly true. For the induction step $(m - 1) \rightarrow m$ let $V \subset B$ with $|V| = (k - 1)m + 1$ and $\|s(V)\|_\infty \leq 1$. Set $v_0 = -s(V)$ so $\|v_0\| \leq 1$. Define $V_0 = V \cup \{v_0\}$. Then $V_0 \subset B$ and $s(V_0) = 0$. So by Theorem 4 there exists a subset $U$ of size $k$, with $\|s(U)\| \leq 1$. We are done if $v_0 \notin U$. So suppose that $v_0 \in U$. Then $W = V \setminus U$, and $\|s(W)\| \leq 1$ because

$$s(U) = -s(W).$$

Here $W$ is of size $(m - 1)(k - 1) + 1$, so the induction hypothesis implies that $W$ contains a subset $U$ of size $k$ with $\|s(U)\| \leq 1$.

We mention finally that Theorem 5 is equivalent to the following Helly type statement. If $V \subset B$ and $|V| = (k - 1)m + 1$, and $\|s(U)\| > 1$ for every set $U \subset V$ of size $k$, then $\|s(V)\| > 1$.

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