Special Deformed Exponential Functions Leading to More Consistent Klauder’s Coherent States

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Abstract

We give a general approach for the construction of deformed oscillators. These ones could be seen as describing deformed bosons. Basing on new definitions of certain quantum series, we demonstrate that they are nothing but the ordinary exponential functions in the limit when the deformation parameters goes to one. We also prove that these series converge to a complex function, in a given convergence radius that we calculate. Klauder’s Coherent States are explicitly found through these functions that we design by deformed exponential functions.

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1 Introduction

The notion of coherent states (C.S.) saw its origins in the early times of quantum mechanics. In 1926 Schrodinger introduced a set of wave functions to describe some particular wave packets for the harmonic oscillators (H.O.). After that, Van Neuman used these functions to investigate the coordinate and momentum measurement process in quantum theory. These ideas did not attract the attention of the eminent scientists for a long period. In the sixties, Glauber, who is considered as one of the fathers of the theory of C.S., designed these states by coherent states, he also proved that they are adequate to describe a coherent laser beam in the framework of quantum theories. Many works after that presented these states in a more modern form. In the authors defines the properties that are conserved in all these generalisations. This constitutes the minimum set of properties that a state should satisfy to be coherent. These properties are:

- Normalisability.
- Continuity.
- and Resolution of unity.

This last property is certainly the most important and the most restrictive one as we shall see in this paper. In fact our aim here is to construct C.S. associated to deformed bosons, and the main difficulty preventing us from presenting a general method of doing so, is exactly this property.

In the present work we discuss the problem of constructing C.S. associated to deformed bosons. We present a general approach of achieving this. In fact we deal with an algebra unifying all the known deformations of the H.O’s algebra. We carry out the calculations for a particular case (of the unifying algebra) which is nevertheless, itself a generalization of many deformations appeared in the literature.

In the first part of this work are given some preleminaries, on the notions of C.S. and deformed bosons, and some of their properties that will be useful in the second part, where we discuss in more detail the construction of coherent states for deformed bosons using a well defined scheme. The difficulty is located indeed in finding the analogue of the exponential function ensuring the obtention of the C.S. starting from the vacuum state in the Fock space.

We proceed by giving what we call in this section quantum series. These ones can converge to a function that is viewed as a deformed exponential function in our context. We calculate in detail the convergence radius, for which these quantum series could have a sense. This radius is compared with the one already used in . The last part of this paper is a sort of a recapitulation where we give all the main ideas discussed here.
2 Preliminaries

2.1 Coherent States

According to Klauder \[3, 6\], the minimum set of requirements to be imposed on a state \(|z >\), for to be a coherent state C.S. is:

a. \textit{Continuity in the label} i.e.

\[ |z > - |z' >|^2 \rightarrow 0 \quad \text{when} \quad |z - z'|^2 \rightarrow 0 \] (1)

b. \textit{Resolution of unity:}

The states \(|z >\) must provide a decomposition (not necessarily unique) of the identity operator:

\[ \int \int d\mu(z)|z > < z| = I \] (2)

where \(d\mu(z)\) is a measure in the label space.

c. \textit{Normalisability}

\[ < z|z >= 1 \] (3)

this last condition can be imposed, almost always, without affecting the first two requirements. However when this is not the case this condition should be dropped \[3\]

In general, the first and third conditions are easily satisfied. This is not the case for the second. In fact, this condition restricts considerably the choice of the states to be considered.

The most familiar case is, of course, the case for which the C.S. were first constructed i.e. the "bosonic" Harmonic Oscillator (H.O.) and the so-called canonical C.S. associated to it:

We recall that the bosonic H.O.'s annihilation and creation operators satisfy the commutation relation \([a, a^+] = 1\), and the canonical C.S. are given by:

\[ |z > = \exp(-\frac{|z|^2}{2}) \exp(za^+)|0 > \]

\[ = \exp(-\frac{|z|^2}{2}) \sum_{n \geq 0} \frac{z^n}{\sqrt{n!}} |n > \] (4)

where \(|n >; n = 1, 2...n\) are the usual number states generated from the vacuum state \(|0 > (a|0 >= 0)\) using:

\[ |n >= \frac{(a^+)^n}{\sqrt{n!}} |0 > \] (5)
These states do satisfy all the requirements mentioned above, in particular they provide a resolution of unity (2) with

\[ d\mu(z) = d^2 zW(|z|^2) \]  

(6)

where \( d^2 z = d\text{Re}(z) d\text{Im}(z) \) and the weight function is \( W(|z|^2) = \frac{1}{\pi} \).

### 2.2 Deformed bosons

One of the most trivial ways of generalising the concept of C.S. is to construct such states for deformed bosons \[7\]. In fact, during the last decades many deformations of the usual H.O. appeared in the literature \[8, 9\]. In a previous work \[10\], we have discussed the possibility of unifying all these deformations. We have also presented an algebra \( \mathcal{A}_Q \) that seems to englobe most of these cases.

The algebra \( \mathcal{A}_Q \) is generated by the triplet \( \{a, a^+, I\} \), and is defined through the following ”Q-mutation” relations:

\[
\begin{align*}
[a, a^+]_Q &= aa^+ - Qa^+a = \Delta_Q' \\
[a, \Delta_Q]_Q &= a\Delta_Q - Q\Delta_Q a = \Delta_Q' a \\
[a^+, \Delta_Q]_Q &= a^+\Delta_Q - Q\Delta_Q a^+ = -a^+\Delta_Q'
\end{align*}
\]  

(7)

where \( Q \) is a complex parameter (of deformation), \( \Delta_Q = a^+a \), and \( \Delta_Q' \) is to be interpreted as a ”Q-derivative” of \( \Delta_Q \).

The Fock space basis associated is defined in the usual way: Given a vacuum state \(|0\rangle\); \((a|0\rangle = 0)\) the different number states are generated through the action of the creation operator on this state:

\[
(a^+)^n|0\rangle = \sqrt{[n]_Q!}n >
\]  

(8)

where the factoriel function is defined as:

\[
[n]_Q! = [n]_Q[n-1]_Q[...][1]_Q
\]

\([0]_Q! = 1\)

and the function \([n]_Q\) appears in:

\[
\begin{align*}
|n\rangle & = [n]_Q|n-1\rangle[...]|1\rangle_0 \\
[a|n\rangle & = \sqrt{[n]_Q} |n-1\rangle \\
[a^+|n\rangle & = \sqrt{[n+1]_Q} |n+1\rangle \\
\Delta_Q|n\rangle & = [n]_Q|n\rangle \\
\Delta_Q'|n\rangle & = ([n+1]_Q - Q[n]_Q)|n\rangle
\end{align*}
\]  

(9)
This algebra could be seen as generalising all the particular deformed boson’s algebras. These can indeed be obtained from it by choosing the adequate function \( |n|_Q \). In particular in \([10]\) we have discussed the case which is obtained from the algebra \( \mathcal{A}_Q \) by choosing:

\[
|n|_Q = \frac{Q^n - Q^{-n}}{Q - Q^{-1}}
\]

which leads to the following "Q-mutation" relation:

\[
a a^+ - Q a^+ a = Q^{-N}
\]

where \( N \) is the number operator defined by:

\[
N|n > = n|n >
\]

3 C.S. for Deformed Bosons

Let’s now turn to the problem of constructing C.S. associated to these deformed H.O. This problem was treated for most of the deformations existing in the litterature. However, presenting a general method for constructing C.S. associated to the algebra \( \mathcal{A}_Q \) in its general form is not possible as was demonstrated in \([10]\).

The main difficulty in doing so resides; in a large proportion; in the fulfillment of condition (b) i.e. in finding the adequate resolution of unity. Concerning this point we distinguish two main approaches in the litterature:

- Deforming the concept of integration and differentiation in such a manner that these satisfy similar properties of those obeyed by ordinary ones. This was done in \([7, 11]\) for the particular case where \( |n| = \frac{q^n - 1}{q - 1} \), this corresponds to the conventional quons \( a a^+ - q a^+ a = 1 \). Achieving this, for the algebra \( \mathcal{A}_Q \) in its general form is still missing.

- Leaving intact the concept of integration and differentiation, and try to fulfill (b) in an analogous manner as in (6) i.e. find the appropriate weight function, such that (b) holds \([12, 13, 14]\).

Using this last method we have succeeded in \([10]\) to construct C.S. for (10). However the construction was valid only when the parameter of the deformation is a root of unity. We now investigate on the possibilities of extending this construction.

As in \([10]\) we begin by constructing C.S. candidates for the algebra \( \mathcal{A}_Q \) in its general form, then specify the particular algebra we are interested in. Such states are given by:

\[
|Q, z > = \sum_{n \geq 0} \frac{z^n}{\sqrt{|n|_Q!}}|n >
\]
These states are, of course, eigenstates of the annihilation operator

\[ a|Q, z > = z |Q, z > \]

We introduce the deformed exponential function associated to the algebra \( A_Q \) by noticing that (12) can be rewritten as:

\[
||Q, z > = \sum_{n \geq 0} \frac{z^n}{|n|_Q!} (a^+)^n |0 >
\]

\[ \equiv \exp^{(1)}_Q (za^+) |0 > \]  

(13)

The use of the superscript (1) will become clear in what follows.

Let’s now see in which circumstances these vectors are C.S. in the sense of Klauder.

First of all these states are clearly normalisable:

\[
<Q, z ||Q, z > = \sum_{n \geq 0} \frac{|z|^{2n}}{|n|_Q! |[n]_Q|} = \sum_{n \geq 0} \frac{|z|^{2n}}{|[n]_Q|!} \equiv \exp^{(2)}_Q (|z|^2)
\]  

(14)

where the bar means complex conjugation, and we have introduced another deformed exponential function as follows:

\[ \exp^{(2)}_Q (x) = \sum_{n \geq 0} \frac{x^n}{|[n]_Q|!} \]  

(15)

We point out at this step, that in order to continue, the ”quantum series” defining the two exponential functions introduced, must converge, and have a non null radius of convergence. For the moment we impose this condition, we shall return to this point later.

From here and after we shall deal only with the normalized states:

\[
|Q, z > = \mathcal{N}(|z|^2) ||Q, z >
\]

\[
= \left[ \exp^{(2)}_Q (|z|^2) \right]^{-\frac{1}{2}} ||Q, z >
\]

\[ = \left[ \exp^{(2)}_Q (|z|^2) \right]^{-\frac{1}{2}} \exp^{(1)}_Q (za^+) |0 >\]  

(16)

The overlap term of two of these states is given by:

\[ \text{In [10] we have imposed on } [n]_Q \text{ to satisfy } [\bar{n}]_Q = [n]_Q, \text{ which means then } \exp^{(1)}_Q = \exp^{(2)}_Q \]
\[ < Q, z | Q, z' > = \mathcal{N}(|z|^2)\mathcal{N}(|z'|^2) \exp(2) (\bar{z}z') \] (17)

This equation, together with:

\[ |Q, z > - |Q, z' > |^2 = 2(1 - \text{Re} < Q, z | Q, z' >) \] (18)

implies the continuity of the states (16) in their label \( z \).

So far, we have seen that conditions (a) and (c) haven’t, almost, imposed any restrictions on the chosen states. In what follows we will see that this is not the case with condition (b).

The question is, in which cases the states (16) do provide a solution of unity (2)? We shall try to respond to this question in the case (6).

Using (16), (12) and (6) in (2) we obtain:

\[
\int \int_{|z|^2 < R_Q} d^2 z \sum_{n,m \geq 0} \frac{z^n}{\sqrt{|n|Q|!}} \frac{\bar{z}^m}{\sqrt{|m|!}} \mathcal{N}^2(|z|^2)|n > < m|W(|z|^2) = I
\] (19)

where \( d^2 z = d\alpha d\beta = r dr d\theta \) when \( z = \alpha + i\beta = re^{i\theta} \) and \( R_Q \) is the convergence radius of the series in (15).

from which we obtain:

\[
\sum_{n \geq 0} \frac{\pi}{|n|Q|!} \left\{ \int_0^{R_Q} dx x^n \mathcal{N}^2(x)W(x) \right\} |n > < n| = I
\] (20)

where \( x = r^2 \).

The completeness of the states \( |n > \) implies:

\[
\int_0^{R_Q} dx x^n \mathcal{N}^2(x)W(x) = \frac{|n|Q|!|}{\pi}
\] (21)

putting

\[ \tilde{W}(x) = \mathcal{N}^2(x)W(x) \]

we get

\[
\int dx x^n \tilde{W}(x) = \frac{|n|Q|!|}{\pi}
\] (22)

which is the well known power moment problem when \( R_Q = \infty \), or the Stieljes moment problem when \( R_Q < \infty \) \[15, 16\].

It is clear that a general solution for this equation (i.e. for a generic r.h.s) can not be given \[15, 16\].

We propose to solve this equation using the Fourier transforms method: multiplying equation(22) by \( \left( \frac{y^n}{n!} \right) \) and summing over \( n \) yields:
\[
\int_0^{R_Q} dx \ e^{i\gamma x} \tilde{W}(x) = \sum_{n \geq 0} \frac{[n]_Q!(iy)^n}{\pi n!} \\
= \tilde{W}(y)
\]  

To proceed, the series in the r.h.s should converge. This imposes a severe restriction on the \([n]_Q\)'s to be chosen, which means on the corresponding particular algebra \(A_Q\).

In [10] we have been interested in the case (10), we present here the following proposition concerning this choice:

\textbf{Proposition 1:}

The series in (23) with \([n]_Q = \frac{Q^n - Q^{-n}}{Q - Q^{-1}}\) converges only when \(|Q| = 1\). It diverges otherwise.

Too restricted as a choice one could say!

To overcome this situation we introduce another parameter of deformation \(p\). This is equivalent to see the parameter \(Q\) as composed of two parameters \(q\) and \(p\).

We define the function \([n]_{q,p}\) as follows:

\[
[n]_{q,p} = \frac{q^n - p^{-n}}{q - p^{-1}}
\]  

This is clearly a generalisation of the first case (10), but it stills a particular case of the algebra \(A_Q\)

The operators \(a\) and \(a^+\) in this case satisfy:

\[
aa^+ - qa^+a = p^{-N}
\]  

To construct the C.S. associated to this, we have to change the label \(Q\) appearing in all previous steps by \(q,p\)!. For convenience we rewrite the main functions here.

The two deformed exponential functions:

\[
\exp_{q,p}^{(1)}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{q,p}!} \\
\exp_{q,p}^{(2)}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{q,p}!}
\]  

and the series \(\tilde{W}(y)\) in (26) will become

\[
\tilde{W}(y) = \sum_{n \geq 0} \frac{[n]_{q,p}!(iy)^n}{\pi n!}
\]
and we have the following result

**Proposition 2:**
The series in (26), (27) and (28) converges simultaneously in two cases:
(i) $|q| \leq 1$ and $|p| = 1$
(ii) $|q| = 1$ and $|p| \geq 1$
otherwise, at least one of these series diverges.

This proposition makes it clear: to continue we shall take our parameters as in (i) or (ii). It is also clear that when this is the case, the two deformed exponential functions (26) and (27) converges for $|x| < |q - p^{-1}|^{-1} = R_{q,p}$.

Now we return back to equation (23), for the case we are considering, the series $\bar{W}(y)$ converges. Thus the inverse Fourier transform of $\bar{W}(y)$ exists and is given by:

$$\hat{W}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} \bar{W}(y) dy$$

(29)

where $x < R_{q,p}$ is overtone.
and we get the weight function:

$$W(x) = \frac{N^{-2}(x)}{2\pi} \int_{-\infty}^{\infty} e^{-iyx} \bar{W}(y) dy$$

(30)

A resolution of unity, thus exists for the case we considered (24) (with the conditions on $q, p$ stated on Proposition 2), which in turn means that indeed the vectors (16) are coherent states for this system.

## 4 A Recapitulatory

In this paper we have discussed the construction of C.S. associated to deformed bosons oscillators. Since all these deformations can be extracted from the algebra $A_Q$ (7),(9), it will be more interesting to achieve such a construction for this algebra. We have tried to do this, using one of the most trivial ways for constructing C.S. i.e. (12) and (6). We failed in reaching our goal due to two main problems. The first is due to the restrictions that imposes the resolution of unity (2) and the equations that derives from it (22) and (23). The other problem is related to the deformed exponential functions introduced. In fact the two series defining this functions (13) and (15) must converge and have a non null convergence radius.

The series in (23) does not converge for all the $|n|Q$’s that one takes, so is the case for (13) and (15), it is thus impossible to construct C.S. for $A_Q$ in its general form using the approach we’ve taken in this paper. However using this same approach we succeeded to do so for a particular case, namely when $4$ for $q = p$ we obtain the case considered in [10], i.e. (11), and in that case $R_{q} = \infty$.  

\[9\]
\[ [n]_Q = [n]_{q,p} = \frac{q^n - p^n}{q - p}. \] It’s true that this still a particular cases of \( A_Q \), but this deformation generalises at least 3 particular cases:

- when \( p \to q \) we obtain (10) and (11).
- when \( p \to 1 \) we obtain the conventional quons \[8\].
- when \( p \to 1 \) and \( q \to 1 \) we obtain the ordinary bosons.

It will be interesting to investigate on the possibility of achieving our goal (i.e. C.S. for \( A_Q \)) using other approaches, for instance changing our starting point (12), or use the other method mentioned above (begining of section 3) concerning the fulfillment of (2).

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