NON-COMMUTATIVE COURANT ALGEBROIDS
AND QUIVER ALGEBRAS

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Abstract. In this paper, we develop a differential-graded symplectic (Batalin–Vilkovisky)
version of the framework of Crawley-Boevey, Etingof and Ginzburg on noncommutative
differential geometry based on double derivations to construct non-commutative analogues
of the Courant algebroids introduced by Liu, Weinstein and Xu. Adapting geometric
constructions of Ševera and Roytenberg for (commutative) graded symplectic superman-
ifolds, we express the BRST charge, given in our framework by a ‘homological double
derivation’, in terms of Van den Bergh’s double Poisson algebras for graded bi-symplectic
non-commutative 2-forms of weight 1, and in terms of our non-commutative Courant alge-
broids for graded bi-symplectic non-commutative 2-forms of weight 2 (here, the grading,
or ghost degree, is called weight). We then apply our formalism to obtain examples of
exact non-commutative Courant algebroids, using appropriate graded quivers equipped
with bi-symplectic forms of weight 2, with a possible twist by a closed Karoubi–de Rham
non-commutative differential 3-form.

1. Introduction

In this paper, we propose a notion of non-commutative Courant algebroid that satisfies
the Kontsevich–Rosenberg principle, whereby a structure on an associative algebra
has geometric meaning if it induces standard geometric structures on its representation
spaces [22]. Replacing vector fields on manifolds by Crawley-Boevey’s double derivations
on associative algebras [9], this principle has been successfully applied by Crawley-Boevey,
Etingof and Ginzburg [10] to symplectic structures and by Van den Bergh to Poisson
structures [33, 34].

Courant algebroids, introduced in differential geometry by Liu, Weinstein and Xu [24],
generalize the notion of the Drinfeld double to Lie bialgebroids. They axiomatize the
properties of the Courant–Dorfman bracket, introduced by Courant and Weinstein [7,8],
and Dorfman [14], to provide a geometric setting for Dirac’s theory of constrained
mechanical systems [13].

Our approach is based on a well-known correspondence (in commutative geometry)
between Courant algebroids and a suitable class of differential graded symplectic mani-
folds. More precisely, symplectic NQ-manifolds are non-negatively graded manifolds (the
grading is called weight), endowed with a graded symplectic structure and a symplectic
homological vector field \(Q\) of weight 1. They encode higher Lie algebroid structures in
the Batalin–Vilkovisky formalism in physics, where the weight keeps track of the ghost
number. Following ideas and results of Ševera [31], Roytenberg [27] proved that symplec-
tic NQ-manifolds of weights 1 and 2 are in 1-1 correspondence with Poisson manifolds.
and Courant algebroids, respectively. Our method to construct non-commutative Courant algebroids is to adapt this result to a graded version of the formalism of Crawley-Boevey, Etingof and Ginzburg.

After setting out our notation (Section 2), and reviewing several constructions involving graded quivers, non-commutative differential forms and double derivations (Section 3), in Section 4 we start generalizing to graded associative algebras the theories of bi-symplectic forms and double Poisson brackets of Crawley-Boevey–Etingof–Ginzburg and Van den Bergh, respectively. In this framework, we obtain suitable Darboux theorems for graded bi-symplectic forms, and prove a 1-1 correspondence between appropriate bi-symplectic NQ-algebras of weight 1 and Van den Bergh’s double Poisson algebras (Section 5). We then use suitable non-commutative Lie and Atiyah algebroids to describe bi-symplectic N-graded algebras of weight 2 whose underlying graded algebras are graded-quiver path algebras, in terms Van den Bergh’s pairings on projective bimodules (Section 6). To complete the data that determines Courant algebroids, in Section 7, we define bi-symplectic NQ-algebras and use non-commutative derived brackets to calculate the algebraic structure that corresponds to symplectic NQ-algebras of this type. By the analogy with Roytenberg’s correspondence for commutative algebras [28], this structure can be regarded as a double Courant–Dorfman algebra, although we call them simply double Courant algebroids. Finally, we consider examples of non-commutative Courant algebroids obtained by deforming standard non-commutative Courant algebroids associated to graded quivers (Section 8.1).

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2. Notation and Conventions

Throughout the paper, unless otherwise stated, all associative algebras will be unital and finitely generated over a fixed base field \( k \) of characteristic 0. The unadorned symbols \( \otimes = \otimes_k, \text{Hom} = \text{Hom}_k \) will denote the tensor product and the space of linear homomorphisms over the base field. The opposite algebra and the enveloping algebra of an associative algebra \( B \) will be denoted \( B^{\text{op}} \) and \( B^e := B \otimes B^{\text{op}} \), respectively. Given an associative algebra \( R \), an \( R \)-algebra will mean an associative algebra \( B \) together with a unit preserving algebra morphism \( R \rightarrow B \) (note that the image of \( R \) may not be in the centre of \( B \)), and by a morphism \( B_1 \rightarrow B_2 \) of \( R \)-algebras we mean an algebra morphism such that \( R \rightarrow B_2 \) is the composite of the given morphisms \( R \rightarrow B_1 \) and \( B_1 \rightarrow B_2 \).

A graded algebra, and a graded \( A \)-module, will mean an \( N \)-graded associative algebra \( A \), and a \( Z \)-graded left \( A \)-module \( V \), with degree decompositions

\[ A = \bigoplus_{d \in N} A_d, \quad V = \bigoplus_{d \in Z} V_d, \]

where \( N \subset \mathbb{Z} \) are the set of non-negative integers and the set of integers, respectively. An element \( v \in V_d \) is called homogeneous of degree \( |v| = d \). Depending on the context, the degree will be called weight, when it plays the role of the ‘ghost degree’ in the BRST or Batalin–Vilkovisky quantization in physics (see, e.g., §3.1.2). For any \( N \in \mathbb{Z} \), the \( N \)-degree of a homogeneous \( v \in V \) is \( |v|_N := |v| + N \) (this notation will be used in §4.2). The graded \( A \)-module \( V[n] \), with degree shifted by \( n \), is \( V[n] = \bigoplus_{d \in \mathbb{Z}} V[n]_d \) with \( V[n]_d = V_{d+n} \). Given another graded module \( V' \), a graded linear map and a graded \( A \)-module homomorphism
$V \to V'$ are respectively a linear map and an $A$-module homomorphism, carrying $V_d$ to $V'_d$, for all $d \in \mathbb{Z}$. Ungraded modules are viewed as graded modules concentrated in degree 0.

We will use the Koszul sign rule to generalize standard constructions for algebras and modules to graded algebras and graded modules, such as tensor products, or the opposite $A^{op}$ and the enveloping algebra $A^e$ of a graded algebra $A$, and to identify graded left $A^e$-modules with graded $A$-bimodules. For instance, the tensor product of two graded algebras $A$ and $B$ is the graded algebra with underlying vector space $A \otimes B$, and multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}a_1a_2 \otimes b_1b_2,$$

for homogeneous $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Given a bigraded module

$$V = \bigoplus_{d,p} V^p_d,$$

the elements $v$ of $V^p_d$ are called homogeneous of bigrading $([v], [v]) := (d, p)$. In this case, the bigraded Koszul sign rule applied to two homogeneous elements $u, v \in V$ yields a sign $(-1)^{(u, |u|)(v, |v|)}$, where

$$(|u|, |v|) := |v||v| + |u||v|. \quad (2.1)$$

Let $R$ be an associative algebra, and $A$ a graded $R$-algebra, that is, a graded algebra equipped with an algebra homomorphism $R \to A$ with image in $A_0$. The $A$-bimodule $A \otimes A$ has two graded $A$-bimodule structures (see [4]), called the outer graded bimodule structure $(A \otimes A)_{out}$ and the inner graded bimodule structure $(A \otimes A)_{inn}$, corresponding to the left graded $A^e$-module structure $A^eA^e$ and right graded $A^e$-module structure $(A^e)^{op} A^e = (A^e)_A^e$, respectively. In other words, for all homogeneous $a, b, u, v \in A$,

$$a(u \otimes v)b = au \otimes vb \quad \text{in} \ (A \otimes A)_{out},$$

$$a \ast (u \otimes v) \ast b = (-1)^{|a||u|+|b||v|}ub \otimes av \quad \text{in} \ (A \otimes A)_{inn},$$

The dual of a graded $A$-bimodule $V$ is the graded $A$-bimodule

$$V^\vee := \bigoplus_{d \in \mathbb{Z}} V^\vee_d, \quad \text{with} \ V^\vee_d := \text{Hom}_{A^e}(V_d, (A \otimes A)_{out}), \quad (2.2)$$

where the graded $A$-bimodule structure on $V^\vee_d$ is induced by the one on $(A \otimes A)_{inn}$.

The above inner and outer bimodule structures are special cases of the following general construction, that for simplicity we will describe only for ungraded algebras and modules. Let $B$ and $V$ be an (ungraded) associative algebra and an (ungraded) $B$-bimodule, respectively. Then the $n$-th tensor power $V^\otimes n$ has many $B$-bimodule structures (cf. [33, pp. 5718, 5732]). Following [12], the $d$-th left and right $B$-module structures of $V^\otimes n$ are

$$(v_1 \otimes \cdots \otimes v_n) \ast_i b = v_1 \otimes \cdots \otimes v_{i-1} \otimes b \otimes v_{i+1} \otimes \cdots \otimes v_n, \quad (2.3)$$

for all $i = 0, \ldots, n - 1$, and $b \in B, v_1, \ldots, v_n \in V$, where the index denotes the number of ‘jumps’. Then the outer bimodule structure and the inner bimodule structure are given by $a_1 *_0 (a \otimes b) *_1 b_1$ and $a_1 *_1 (a \otimes b) *_1 b_1$, respectively. We use a similar notation for the tensor product of an element $u$ of $V$ and an element $v_1 \otimes \cdots \otimes v_n$ of $V^\otimes n$, namely,

$$u \otimes_i (v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_i \otimes u \otimes v_{i+1} \otimes \cdots \otimes v_n \in V^\otimes n+1, \quad (2.4)$$

$$(v_1 \otimes \cdots \otimes v_n) \otimes_i u = v_1 \otimes \cdots \otimes v_{n-1} \otimes u \otimes \cdots \otimes v_n \in V^\otimes n+1.$$
3. Basics on Graded Non-Commutative Algebraic Geometry and Quivers

3.1. Background on graded quivers.

3.1.1. Quivers. To fix notation, here we recall a few relevant definitions from the theory of quivers (see, e.g., [2, 4] for introductions to this topic). A quiver $Q$ consists of a set $Q_0$ of vertices, a set $Q_1$ of arrows, and two maps $t, h: Q_1 \rightarrow Q_0$ assigning to each arrow $a \in Q_1$, its tail and its head. We write $a: i \rightarrow j$ to indicate that an arrow $a \in Q_1$ has tail $i = t(a)$ and head $j = h(a)$. Given an integer $\ell \geq 1$, a non-trivial path of length $\ell$ in $Q$ is an ordered sequence of arrows $p = a_\ell \cdots a_1$, such that $h(a_j) = t(a_{j+1})$ for $1 \leq j < \ell$. This path $p$ has tail $t(p) = t(a_1)$, head $h(p) = h(a_\ell)$, and is represented pictorially as follows.

$$p: \bullet \leftarrow a_\ell \bullet \leftarrow \cdots \leftarrow a_1 \bullet$$

(3.1)

For each vertex $i \in Q_0$, $e_i$ is the trivial path in $Q$, with tail and head $i$, and length 0. A path in $Q$ is either a trivial path or a non-trivial path in $Q$. The path algebra $\mathbb{k}Q$ is the associative algebra with underlying vector space

$$\mathbb{k}Q = \bigoplus_{\text{paths } p} \mathbb{k}p,$$

that is, $\mathbb{k}Q$ has a basis consisting of all the paths in $Q$, with the product $pq$ of two non-trivial paths $p$ and $q$ given by the obvious path concatenation if $t(p) = h(q)$, $pq = 0$ otherwise, $pe_{t(p)} = e_{h(q)}p = p$, $pe_i = e_jp = 0$, for non-trivial paths $p$ and $i, j \in Q_0$ such that $i \neq t(p), j \neq h(p)$, and $e_ie_i = e_i, e_je_j = 0$ for all $i, j \in Q_0$ if $i \neq j$. We will always assume that a quiver $Q$ is finite, i.e. its vertex and arrow sets are finite, so $\mathbb{k}Q$ has a unit

$$1 = \sum_{i \in Q_0} e_i.$$  

(3.2)

Define vector spaces

$$R_Q = \bigoplus_{i \in Q_0} \mathbb{k}e_i, \quad V_Q = \bigoplus_{a \in Q_1} \mathbb{k}a.$$  

Then $R_Q \subset \mathbb{k}Q$ is a semisimple commutative (associative) algebra, because it is the subalgebra spanned by the trivial paths, which are a complete set of orthogonal idempotents of $\mathbb{k}Q$. Furthermore, as $V_Q$ is a vector space with basis consisting of the arrows, it is an $R_Q$-bimodule with multiplication $e_iae_i = a$ if $a: i \rightarrow j$ and $e_iae_j = 0$ otherwise, and the path algebra is the tensor algebra of the bimodule $V_Q$ over $R := R_Q$, that is

$$\mathbb{k}Q = T_R V_Q,$$  

(3.3)

where a path $p = a_\ell \cdots a_1 \in \mathbb{k}Q$ is identified with a tensor product $a_\ell \otimes \cdots \otimes a_1 \in T_R V_Q$.

Given a quiver $Q$, the double quiver of $Q$ is the quiver $\overline{Q}$ obtained from $Q$ by adjoining a reverse arrow $a^*: j \rightarrow i$ for each arrow $a: i \rightarrow j$ in $Q$. Following [10, §8.1], for convenience we introduce the function

$$\varepsilon: \overline{Q} \rightarrow \{\pm 1\}: \quad a \mapsto \varepsilon(a) = \begin{cases} 1 & \text{if } a \in Q_1, \\ -1 & \text{if } a \in Q_1^*: = \overline{Q_1} \setminus Q_1. \end{cases}$$  

(3.4)
3.1.2. Graded quivers and graded path algebras. The following construction is partially inspired by similar ones in physics [23] and for Calabi–Yau algebras (cf. [18], [35, §10.3]).

Definition 3.1. A graded quiver is a quiver $P$ together with a map

$$|\cdot| : P \rightarrow \mathbb{N} : a \mapsto |a|$$

that to each arrow $a$ assigns its weight $|a|$. The weight of the graded quiver is

$$|P| := \max_{a \in P_1} |a|.$$ 

Given a graded quiver $P$ and an integer $N \geq |P|$, the weight $N$ double graded quiver of $P$ is the graded quiver $\overline{P}$ obtained from $P$ by adjoining a reverse arrow $a^* : j \rightarrow i$ for each arrow $a : i \rightarrow j$ in $P$, with weight $|a^*| = N - |a|$ (note that $|\overline{P}| = N$ if and only if $P$ has at least one arrow of weight 0).

The weight function $|\cdot| : P_1 \rightarrow \mathbb{N}$ induces a structure of graded associative algebra, called the graded path algebra of $P$, on the path algebra $\mathbb{k}P$ of the underlying quiver of $P$, where a trivial path $e_i$ has weight $|e_i| = 0$, and a non-trivial path $p = a_1 \cdots a_\ell$ has weight

$$|p| = |a_1| + \cdots + |a_\ell|.$$ 

Let $R := R_{\overline{P}}$ the algebra with basis the trivial paths in a graded quiver $P$, and

$$V_P = \bigoplus_{a \in P_1} \mathbb{k}a$$

(3.5)

the graded $R$-bimodule with basis consisting of the arrows in $P_1$, where $a \in P_1$ has weight $|a|$, and multiplications $e_j a e_i = a$ if $i = t(a), j = h(a)$, and $e_i a e_j = 0$ otherwise, for all $a \in P_1$. As in (3.3), the graded path algebra $\mathbb{k}P$ is the graded tensor algebra

$$\mathbb{k}P = T_R V_P,$$

(3.6)

where a path $p = a_\ell \cdots a_1 \in \mathbb{k}P$ is identified with a tensor product $a_\ell \otimes \cdots \otimes a_1 \in T_R V_P$.

The graded path algebra $\mathbb{k}P$ can be expressed as a graded tensor algebra in another way, using the following two subquivers of $P$. The weight 0 subquiver of $P$ is the (ungraded) quiver $Q$ with vertex set $Q_0 = P_0$, arrow set $Q_1 = \{a \in P_1 \mid |a| = 0\}$, and tail and head maps $t, h : Q_1 \rightarrow Q_0$ obtained restricting the tail and head maps of $P$. The higher-weight subquiver of $P$ is the graded quiver $P^+$ with vertex set $P^+_0 = P_0$, arrow set $P^+_1 = \{a \in P_1 \mid |a| > 0\}$, tail and head maps $t, h : P^+_1 \rightarrow P^+_0$ obtained restricting the tail and head maps of $P$, and weight function $P^+_1 \rightarrow \mathbb{N}$ obtained restricting the weight function of $P$. Later it will also be useful to consider the graded subquivers $P^+_w \subset P$ with vertex set $P^+_w$ and arrow set $P^+_w$ consisting of all the arrows $a \in P^+_1$ with weight $w$, for $0 \leq w \leq |P|$.

In Lemma 3.2 $BaB \subset A$ denotes the $B$-sub-bimodule of $B A B$ generated by $a \in A$.

Lemma 3.2. Let $B = \mathbb{k}Q$ be the path algebra of $Q$. Define the graded $B$-bimodule

$$M_P := \bigoplus_{a \in P^+_1} BaB.$$ 

(3.7)

Then $M_P$ is a finitely generated projective $B$-bimodule and the graded path algebra $A = \mathbb{k}P$ of $P$ is canonically isomorphic to the graded tensor algebra of $M_P$ over $B$, that is,

$$A = T_B M_P.$$ 

(3.8)
Proof. As \( M_P \) has a basis consisting of paths in \( P \) of weight 1, its tensor algebra \( T_B M_P \) has a basis consisting of paths in \( P \) of arbitrary weight, that is, \( A = T_B M_P \) as graded vector spaces, and hence as graded algebras, with concatenation of paths on \( A \) identified with multiplication of paths in \( T_B M_P \) (see \[16\] Lemma 3.3.7 for further details). \( \square \)

It will be useful to decompose
\[
M_P = \bigoplus_{w=1}^{|P|} M_{P(w)}, \quad \text{with} \quad M_{P(w)} = \bigoplus_{a \in P(w),} BaB,
\]
where \( M_{P(w)} \) is a finitely generated projective \( B \)-bimodule of weight \( w \), because so are the \( B \)-bimodules \( BaB \), and hence \( E_w := M_{P(w)}[w] \) is a finitely generated projective \( B \)-bimodule (concentrated in weight 0).

3.2. Graded non-commutative differential forms and double derivations. Let \( R \) be an associative algebra and \( A \) a graded \( R \)-algebra. Given a graded \( A \)-bimodule \( M \), a derivation of weight \( d \) of \( A \) into \( M \) is an additive map \( \theta: A \to M \), such that \( \theta(A) \subset M_{i+d} \), satisfying the graded Leibniz rule \( \theta(ab) = (\theta a) b + (-1)^{d[a]} a (\theta b) \) for all \( a, b \in A \). It is called an \( R \)-linear graded derivation if, furthermore, \( \theta(R) = 0 \), i.e., it is a graded \( R \)-bimodule morphism of weight \( d \). The graded vector space of graded \( R \)-derivations is
\[
\text{Der}_R(A, M) = \bigoplus_{d \in \mathbb{Z}} \text{Der}_R^d(A, M),
\]
where \( \text{Der}_R^d(A, M) \) is the vector space of \( R \)-linear graded derivations of weight \( d \).

3.2.1. Graded non-commutative differential forms.

Lemma 3.3 (cf. \[26\] §2]). There exists a unique pair \( (\Omega^1_R A, d) \) (up to isomorphism), where \( \Omega^1_R A \) is a graded \( A \)-bimodule and \( d: A \to \Omega^1_R A \) is an \( R \)-linear graded derivation, satisfying the following universal property: for all pairs \( (M, \theta) \) consisting of a graded \( A \)-bimodule \( M \) and an \( R \)-linear graded derivation \( \theta: A \to M \), there exists a unique graded \( A \)-bimodule morphism \( i_\theta: \Omega^1_R A \to M \) such that \( \theta = i_\theta \circ d \).

The elements of \( \Omega^1_R A \) are called relative noncommutative differential 1-forms of \( A \) over \( R \). Concretely, we can construct \( \Omega^1_R A \) as the kernel of the multiplication \( A \otimes A \to A \), and
\[
d: A \to \Omega^1_R A: \quad a \mapsto da = 1 \otimes a - a \otimes 1. \tag{3.9}
\]

Lemma 3.4 (cf. \[11\] Proposition 2.6]). Let \( A = T_B M \) be the graded tensor algebra of a graded bimodule \( M \) over an associative \( R \)-algebra \( B \). Then there is a canonical isomorphism
\[
A \otimes_B M \otimes_B A \xrightarrow{\cong} \Omega^1_R A: \quad a_1 \otimes m \otimes a_2 \mapsto a_1(dm)a_2,
\]
where \( d: A \to \Omega^1_R A \) is the universal graded derivation.

Example 3.5. If \( A \) is the graded path algebra of a graded quiver \( P \), then
\[
\Omega^1_R A = \bigoplus_{a \in P_1} (Ae_{h(a)}) da(e_{t(a)} A).
\]
The relative graded non-commutative differential forms of $A$ over $R$ are the elements of
\[ \Omega^\bullet_R A := T^1_R^1 A, \tag{3.10} \]
i.e., the tensor algebra of the graded $A$-bimodule $\Omega^1_R A$. This is a bigraded algebra, with bigrading denoted $([-], \| - \|)$, where the weight $[-]$ is induced by the grading of $A$ (also called weight), and the degree $\| - \|$ is the “form degree”, i.e., the elements of the $n$-th tensor power $(\Omega^1_R A)^{\otimes n}$ have degree $n$.

As for ungraded associative algebras, $(\Omega^\bullet_R A, d)$ has trivial cohomology (cf., e.g., [17 §11.4]). To obtain interesting cohomology spaces, one defines the non-commutative Karoubi–de Rham complex of $A$ (relative over $R$) as the bigraded vector space
\[ DR^\bullet A = \Omega^\bullet_R A/[\Omega^\bullet_R A, \Omega^\bullet_R A], \tag{3.11} \]
where the bigraded commutator $[-, -]$ is given by the bigraded Koszul sign rule, i.e.,
\[ [\alpha, \beta] := \alpha \beta - (-1)^{(\| \alpha \|)(\| \beta \|)} \beta \alpha, \]
with the sign given by (2.11). Then the differential $d: \Omega^\bullet_R A \to \Omega^{\bullet+1}_R A$ descends to another differential $d: DR^\bullet A \to DR^{\bullet+1} A$, and so $DR^\bullet A$ is a differential bigraded vector space.

3.2.2. Graded double derivations. By Lemma [3,3] there is a canonical isomorphism
\[ \Der_R(A, M) \xrightarrow{\cong} \Hom_A(\Omega^1_R A, M): \theta \mapsto i_\theta \tag{3.12} \]
of graded $A$-bimodules, such that $\theta = i_\theta \circ d$. In particular, when $M = (A \otimes A)_{\text{out}}$,
\[ \Der_R A \xrightarrow{\cong} (\Omega^1_R A)^\vee, \quad \Theta \mapsto i_\Theta, \tag{3.13} \]
where the graded $A$-bimodule of $R$-linear graded double derivations on $A$ is
\[ \Der_R A := \Der_R(A, (A \otimes A)_{\text{out}}), \tag{3.14} \]
and the graded $A$-bimodule structure on both $\Der_R A$ and $(\Omega^1_R A)^\vee$ come from the inner graded $A$-bimodule structure $(A \otimes A)^{\text{inn}}$ (see (2.2)). If we want to consider the outer $A$-bimodule structure instead, we can compose with the graded flip isomorphism
\[ \sigma_{(12)}: (A \otimes A)_{\text{out}} \longrightarrow (A \otimes A)_{\text{inn}}: a' \otimes a'' \mapsto (-1)^{|a_1||a_2|} a'' \otimes a', \tag{3.15} \]
"obtaining another canonical isomorphism (cf. [6 §5.3])
\[ \Der_R A \xrightarrow{\cong} \Hom_A(\Omega^1_R A, (A \otimes A)_{\text{inn}}): \Theta \mapsto \Theta^\vee = \sigma_{(12)} \circ i_\Theta, \tag{3.16} \]
where
\[ \Theta^\vee: \Omega^1_R A \longrightarrow (A \otimes A)_{\text{inn}}: \alpha \mapsto (-1)^{|i_\alpha^\vee(\alpha)||i_\alpha^\vee(\alpha)|} i_\Theta^\vee \alpha \otimes i_\Theta^\vee \alpha. \]
Following [10], in this paper we will systematically use symbolic Sweedler’s notation, writing an element $x$ of $A \otimes A$ as $x' \otimes x''$, omitting the summation symbols. In particular, we write $\Theta: A \to A \otimes A$: $a \mapsto \Theta'(a) \otimes \Theta''(a)$ and $i_\Theta: \Lambda^\vee_R A \to A \otimes A: \alpha \mapsto i_\Theta \alpha = i_\Theta^\vee \alpha \otimes i_\Theta^\vee \alpha$.

Example 3.6. Consider the graded path algebra $\mathbb{k}P$ of a graded quiver $P$. This is a graded $R$-algebra, where $R = R_P$ (see §3.1.2). As in the ungraded case (see [33 §6]), the $\mathbb{k}P$-bimodule of $R$-linear double derivations $\Der_R(\mathbb{k}P)$ is generated by the set of double derivations $\{ \partial/\partial a \}_{a \in P_1}$, which on each arrow $b \in P_1$ act by the formula
\[ \frac{\partial b}{\partial a} = \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17} \]
Note that by convention, we compose arrows from right to left (see (3.1)), whereas Van den Bergh composes arrows from left to right (see, e.g., [33, Proposition 6.2.2]).

3.2.3. **Smooth graded algebras.** Let $A$ be a graded $R$-algebra. A finitely generated graded $A$-bimodule $M$ is projective if it is a projective object of the abelian category $\text{mod}(A)$ of finitely generated graded $A$-bimodules, i.e. the functor $\text{Hom}^0_A(M, -)$ is exact on $\text{mod}(A)$.

**Definition 3.7.** The graded $R$-algebra $A$ is called smooth over $R$ if it is finitely generated over $R$ and the graded $A$-module $\Omega^1_RA$ is projective.

Note that in the above definition, if the algebra $A$ is finitely generated over $R$, then $\Omega^1_RA$ is a finitely generated $A$-module.

**Lemma 3.8** (cf. [11, Proposition 5.3(3)]). If an associative (ungraded) algebra $B$ is smooth over $R$ and $M$ is a finitely generated and projective graded $B$-bimodule, then the graded tensor algebra $A = T_R M$ of $M$ over $B$ is also smooth over $R$.

By the above lemma, graded path algebras of graded quivers are prototypical examples of smooth graded algebras.

3.2.4. The morphism bidual. Given a graded $R$-algebra $A$, the evaluation map gives a canonical $A$-bimodule morphism from any graded $A$-bimodule $M$ into its double dual,

$$ \text{bidual}_M : M \rightarrow M^{\vee \vee}, \quad \text{where} \quad M^{\vee \vee} := (M^\vee)^\vee $$

(see (2.2)). This is an isomorphism when $M$ is a finitely generated projective graded $A$-module (cf. [11, §5.3]). In the special case $M = \Omega^1_RA$, $\text{Der}_R A = (\Omega^1_RA)^\vee$ (see (3.13)), so if $A$ is smooth over $R$, then both $\Omega^1_RA$ and $\text{Der}_R A$ are finitely generated and projective, and the above morphism becomes a graded $A$-bimodule isomorphism

$$ \text{bidual}_{\Omega^1_RA} : \Omega^1_RA \xrightarrow{\cong} (\text{Der}_R A)^\vee = (\Omega^1_RA)^{\vee \vee} : \alpha \mapsto i(\alpha) = \alpha^\vee, \quad (3.18) $$

where

$$ i(\alpha) : \text{Der}_R A \rightarrow (A \otimes A)_{\text{out}} : \Theta \mapsto i_\Theta \alpha. $$

3.3. **Non-commutative differential calculus.** Using symbolic Sweedler’s notation as in (3.2.2) any $\Theta \in \text{Der}_R A$ determines a contraction operator

$$ i_\Theta : \Omega^1_RA \rightarrow A \otimes A : \alpha \mapsto i_\Theta \alpha = i'_\Theta \alpha \otimes i''_\Theta \alpha, \quad (3.19) $$

that is determined by its values on generators, namely

$$ i_\Theta(a) = 0, \quad i_\Theta(db) = \Theta(b) = \Theta'(b) \otimes \Theta''(b), $$

for all $a, b \in A$ (so $db \in \Omega^1_RA$). Since $\Omega^\bullet_RA = T_A \Omega^1_RA$ is the free algebra of the graded $A$-bimodule $\Omega^1_RA$, the graded $A$-bimodule morphism $i_\Theta$ admits a unique extension to a graded double derivation of bidegree $(|\Theta|, -1)$ on $\Omega^\bullet_RA$,

$$ i_\Theta : \Omega^\bullet_RA \rightarrow \bigoplus (\Omega^i_RA \otimes \Omega^j_RA) \subset \Omega^\bullet_RA \otimes \Omega^\bullet_RA, \quad (3.20) $$

where the direct sum is over pairs $(i, j)$ with $i + j = \bullet - 1$, and $\Omega^\bullet_RA \otimes \Omega^\bullet_RA$ is regarded a graded $\Omega^\bullet_RA$-bimodule with respect to the outer graded bimodule structure. Explicitly,

$$ i_\Theta(\alpha_0 \alpha_2 \cdots \alpha_n) = \sum_{k=0}^n (-1)^k (\alpha_1 \cdots \alpha_{k-1}(i'_{\Theta} \alpha_k)) \otimes ((i''_{\Theta} \alpha_k) \alpha_{k+1} \cdots \alpha_n). \quad (3.21) $$
for all $\alpha_0, \ldots, \alpha_n \in \Omega_R^1 A$ (see [10, (2.6.2)]). Sometimes we will also need to view the contraction operator $i_\Theta$ as a map $\Omega_R^1 A \to (T_R(\Omega_R^1 A))^{\otimes 2}$, and then extend it further to a graded double derivation of the tensor algebra $T_R(\Omega_R^1 A)$.

The most interesting properties of the contraction operator are the following.

Lemma 3.9 ([10, Lemma 2.6.3]). Let $\Theta, \Delta \in \mathbb{D}er_R A$, and $\alpha, \beta \in \Omega_R^1 A$. Then we have

(i) $i_\Theta(\alpha \beta) = i_\Theta(\alpha)\beta + (-1)^{|\alpha||\beta|} i_\Theta(|\alpha|\Theta|\beta|)\alpha i_\Theta(\beta)$;

(ii) $i_\Theta \circ i_\Delta + i_\Delta \circ i_\Theta = 0$.

Given $\Theta \in \mathbb{D}er_R A$, the corresponding Lie derivative is the graded double derivation

$$L_\Theta : \Omega_R^1 A \longrightarrow \bigoplus (\Omega_R^1 A \otimes \Omega_R^1 A) \subset \Omega_R^2 A \otimes \Omega_R^1 A,$$

of bidegree $(|\Theta|,0)$, where the sum is over pairs $(i,j)$ with $i + j = \bullet$, and $\Omega_R^1 A \otimes \Omega_R^1 A$ is regarded a graded $\Omega_R^1 A$-bimodule with respect to the outer graded bimodule structure, that is is determined by its values on generators by the following formulae for all $a, b \in A$:

$$L_\Theta(a) = \Theta(a), \quad L_\Theta(db) = d\Theta(b).$$

By a simple calculation on generators, one obtains a Cartan formula (see [10, (2.7.2)]):

$$L_\Theta = d \circ i_\Theta + i_\Theta \circ d.$$  \hspace{1cm} (3.22)

Given a graded $R$-algebra $C$, we define a linear map

$$C \otimes C \longrightarrow C : c = c_1 \otimes c_2 \longmapsto c := (-1)^{|c_1||c_2|} c_1 \otimes c_2.$$

Similarly, given a linear map $\phi : C \longrightarrow C^{\otimes 2}$, we define $\circ \phi : C \longrightarrow C : c \longmapsto \circ \phi(c)$.

Applying this construction to $C = \Omega_R^1 A$, we define for all $\Theta \in \mathbb{D}er_R A$, the corresponding reduced contraction operator and reduced Lie derivative,

$$i_\Theta : \Omega_R^1 A \longrightarrow \Omega_R^1 A : \alpha \longmapsto \circ i_\Theta(\alpha) = (-1)^{|i_\Theta|\Theta|\alpha|} i_\Theta(|\alpha|\Theta) i_\Theta(\alpha),$$

$$L_\Theta : \Omega_R^1 A \longrightarrow \Omega_R^1 A : \alpha \longmapsto \circ (L_\Theta \alpha),$$

respectively. Explicitly, for all $\alpha_0, \alpha_1, \ldots, \alpha_n \in \Omega_R^1 A$,

$$i_\Theta(\alpha_0 \cdots \alpha_n) = \sum_{k=0}^{n} (-1)^{k(n-k)} (i_\Theta^\circ \alpha_k) \alpha_{k+1} \cdots \alpha_n \alpha_0 \cdots \alpha_{k-1} (i_\Theta \alpha_k).$$  \hspace{1cm} (3.25)

Applying $\circ$ to (3.22), we obtain the reduced Cartan identity (cf. [10, Lemma 2.8.8])

$$L_\Theta = d \circ i_\Theta + i_\Theta \circ d.$$  \hspace{1cm} (3.26)
(iv) The weight of an associative \( \mathbb{N} \)-algebra \( A \) is \( |A| := \min \{ a \} a \in G \), where the elements of \( G \) are the finite sets of homogeneous generators of \( A \).

4.2. Double Poisson brackets on graded algebras.

4.2.1. Double Poisson brackets on graded algebras. Commutative Poisson algebras appear in several geometric and algebraic contexts. As a direct non-commutative generalization, one might consider associative algebras that are at the same time Lie algebras under a ‘Poisson bracket’ \( \{-, -\} \) satisfying the Leibniz rules \( \{ab, c\} = a\{b, c\} + \{a, c\}b \), \( \{ab, c\} = b\{a, c\} + \{a, b\}c \). However such Poisson brackets are the commutator brackets up to a scalar multiple, provided the algebra is prime and not commutative [15 Theorem 1.2]. A way to resolve this apparent lack of noncommutative Poisson algebras is provided by Van den Bergh in [33 §2.2]. In this subsection, we will extend his definitions to graded associative algebras (cf. [6]).

Let \( A \) be an associative \( \mathbb{N} \)-algebra over \( R \), and \( N \in \mathbb{Z} \). A double bracket of weight \( N \) on \( A \) is an \( R \)-bilinear map

\[
\{-, -\} : A^p \otimes A^q \longrightarrow \bigoplus_{i+j=p+q+N} A^i \otimes A^j \subset A \otimes A,
\]

for any integers \( p \) and \( q \), which is a double \( R \)-derivation of weight \( N \) (for the outer graded \( A \)-bimodule structure on \( A \otimes A \)) in its second argument, that is,

\[
\{a, bc\} = \{a, b\} c + (-1)^{|a||N|b} b \{a, c\},
\]

(4.1)

for all homogeneous \( a, b, c \in A \), and is \( N \)-graded skew-symmetric, that is,

\[
\{a, b\} = -(-1)^{|a||N|b} \{b, a\},
\]

(4.2)

where \( (u \otimes v)^o = (-1)^{|u||v|} v \otimes u \). Let \( S_n \) be the group of permutations of \( n \) elements. For \( a, b_1, ..., b_n \) in \( A \), and a permutation \( s \in S_n \), we define

\[
\{a, b\}_L := \{a, b_1\} \otimes b_2 \otimes \cdots \otimes b_n,
\]

\[
\sigma_s(b) := (-1)^{s^{-1}(i) > s^{-1}(j)} b_{s^{-1}(1)} \otimes \cdots \otimes b_{s^{-1}(n)},
\]

(4.3)

where \( b = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n} \), and

\[
t = \sum_{s^{-1}(i) > s^{-1}(j)} |b_{s^{-1}(i)}||b_{s^{-1}(j)}|.
\]

A double Poisson bracket of weight \( N \) on \( A \) is a double bracket \( \{-, -\} \) of weight \( N \) on \( A \) that satisfies the graded double Jacobi identity: for all homogeneous \( a, b, c \in A \),

\[
0 = \{a, \{b, c\}\}_L + (-1)^{|a||N|(|b|+|c|)} \sigma_{(123)} \{b, \{c, a\}\}_L
\]

\[
+ (-1)^{|c||N|(|a|+|b|)} \sigma_{(132)} \{c, \{a, b\}\}_L.
\]

(4.4)

A double Poisson algebra of weight \( N \) is a pair \( (A, \{-, -\}) \) consisting of a graded algebra and a double Poisson bracket of weight \( N \). Following [33 §2.7], double Poisson algebras of weight \( -1 \) will be called double Gerstenhaber algebras. Next, given a double bracket \( \{-, -\} \), the bracket associated to \( \{-, -\} \) is

\[
\{-, -\} : A \otimes A \longrightarrow A: (a, b) \mapsto \{a, b\} := m \circ \{a, b\} = \{a, b\}' \{a, b\}'',
\]

(4.5)
where \( m \) is the multiplication map. It is clear that \( \{-, -\} \) is a derivation in its second argument. Furthermore, it follows from (1.2) that
\[
\{a, b\} = -(1)^{|a||b|} [a, b, a] \mod [A, A],
\]

A left Loday algebra is a vector space \( V \) equipped with a bilinear operation \([-, -]\) such that the following Jacobi identity is satisfied: \([a, [b, c]] = [[[a, b]], c] + [b, [a, c]]\), for all \( a, b, c \in V \).

**Lemma 4.2.**
(i) Let \( A \) be a double Poisson algebra. Then
\[
\{\{a, \{-\}, \{b, c\}\} - \{\{a, b\}, c\} - \{\{b, \{-\}, \{a, c\}\}\} = 0,
\]

(ii) Let \( A \) be an associative \( N \)-algebra endowed with a double Poisson bracket of weight \( N \). Then the following identity holds:
\[
(-1)^{|a||c|} \{\{a, b\}, c\} + (-1)^{|b||a|} \{a, \{b, c\}\} + (-1)^{|c||b|} \{a, \{b, c\}\} = 0,
\]
where, in (4.7), \( \{-\} \) acts on tensors by \( \{a, u \otimes v\} = \{a, u\} \otimes v + u \otimes \{a, v\} \), for all \( a, u, v \in A \). In fact, \((A, \{-, -\})\) is a left Loday graded algebra.

**Proof.** Immediate from \([33, \text{Proposition 2.4.2}]\). In fact, (4.8) is \([33, \text{Corollary 2.4.4}]\). □

Consider the bigraded algebra \( T_A \text{Der}_R A \) of \( R \)-linear poly-vector fields on a finitely generated graded algebra \( A \) \([33, \text{§3}]\), with degree \( d \) component \((T_A \text{Der}_R A)_d = (\text{Der}_R A)^{\otimes d}\). Then Van den Bergh \([33, \text{Proposition 4.1.1}]\) constructs a map
\[
\mu: P \mapsto \{\{-, -\}\}_{P},
\]
from \((T_A \text{Der}_R A)_2\) into the space of \( R \)-bilinear double brackets on \( A \), given by
\[
\{\{a, b\}\}_{P} = (\Theta'(a) \ast \Delta \ast \Theta''(a))(b) - (\Delta'(a) \ast \Theta \ast \Delta''(a))(b),
\]
for all \( P = \Theta \Delta \), with \( \Theta, \Delta \in \text{Der}_R A \), and \( a, b \in A \). Furthermore, the map \( \mu \) is isomorphism, provided \( A \) is smooth over \( R \) \([33, \text{Proposition 4.1.2}]\).

**Definition 4.3** \([33, \text{Definition 4.4.1}]\). We say that \( A \) is a differential double Poisson algebra (a DDP for short) over \( R \) if it is equipped with an element \( P \in (T_A \text{Der}_R A)_2 \) (a differential double Poisson bracket) such that
\[
\{P, P\} = 0 \mod [T_A \text{Der}_R A, T_A \text{Der}_R A].
\]

Note that if \( A \) is smooth over \( R \), then the notions of differential double Poisson algebra and double Poisson algebra coincide, because \( \mu \) in (4.9) is an isomorphism in this case.

**Example 4.4** \([33, \text{Theorem 6.3.1}]\). Let \( A = k\overline{P} \) be the graded path algebra of a double graded quiver \( \overline{P} \). Then \( A \) has the following differential double Poisson bracket:
\[
P = \sum_{a \in \overline{P}} \frac{\partial}{\partial a} \frac{\partial}{\partial a^*}.
\]
4.2.2. The double Schouten–Nijenhuis bracket. Suppose \( A \) is a finitely generated graded \( R \)-algebra. Given homogeneous \( \Theta, \Delta \in \text{Der}_R A \), a graded version of [33, Proposition 3.2.1] provides two graded derivations \( A \to A^{\otimes 3} \), for the graded outer structure on \( A^{\otimes 3} \), given by

\[
\{\Theta, \Delta\}_l = (\Theta \otimes 1)\Delta - (1^{|\Theta|} \Delta)(1 \otimes \Delta)\Theta,
\]
\[
\{\Theta, \Delta\}_r = (1 \otimes \Theta)\Delta - (1^{|\Theta|} \Delta)(\Delta \otimes 1)\Theta = -\{\Delta, \Theta\}_r.
\]

(4.13)

Now, the graded \( A \)-bimodule isomorphisms

\[
\tau_{(12)}: A \otimes (A \otimes A)_{\text{out}} \xrightarrow{\cong} A^{\otimes 3}: a_1 \otimes (a_2 \otimes a_3) \mapsto (-1)^{|a_1||a_2|}a_2 \otimes a_1 \otimes a_3,
\]
\[
\tau_{(23)}: (A \otimes A)_{\text{out}} \otimes A \xrightarrow{\cong} A^{\otimes 3}: (a_1 \otimes a_2) \otimes a_3 \mapsto (-1)^{|a_2||a_3|}a_1 \otimes a_3 \otimes a_2,
\]

induce isomorphisms

\[
\tau_{(12)}: \text{Der}_R (A, A^{\otimes 3}) \cong \text{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \xrightarrow{\cong} \text{Hom}_{A^e}(\Omega^1_R A, A \otimes (A \otimes A)_{\text{out}}) \cong A \otimes \text{Der}_R A
\]

(4.14)

\[
\tau_{(23)}: \text{Der}_R (A, A^{\otimes 3}) \cong \text{Hom}_{A^e}(\Omega^1_R A, A^{\otimes 3}) \xrightarrow{\cong} \text{Hom}_{A^e}(\Omega^1_R A, (A \otimes A)_{\text{out}} \otimes A) \cong \text{Der}_R A \otimes A,
\]

(4.15)

because \( \Omega^1_R A \) is finitely generated. Hence we can convert the above triple derivations into

\[
\{\Theta, \Delta\}_l := \tau_{(23)} \circ \{\Theta, \Delta\}_l^\sim \in \text{Der}_R A \otimes A,
\]
\[
\{\Theta, \Delta\}_r := \tau_{(12)} \circ \{\Theta, \Delta\}_r^\sim \in A \otimes \text{Der}_R A,
\]

(4.16)

(4.17)

and hence make decompositions

\[
\{\Theta, \Delta\} = \{\Theta, \Delta\}_l \otimes \{\Theta, \Delta\}_r^\prime,
\]
\[
\{\Theta, \Delta\}_r = \{\Theta, \Delta\}_r \otimes \{\Theta, \Delta\}_l^\prime
\]

with \( \{\Theta, \Delta\}_l, \{\Theta, \Delta\}_r^\prime, \{\Theta, \Delta\}_l^\prime, \{\Theta, \Delta\}_r \in \text{Der}_R A \). Using these constructions, given homogeneous \( a, b \in A \) and \( \Theta, \Delta \in \text{Der}_R A \), we define

\[
\{a, b\} = 0,
\]
\[
\{\Theta, a\} = \Theta(a),
\]
\[
\{\Theta, \Delta\} = \{\Theta, \Delta\}_l + \{\Theta, \Delta\}_r
\]

(4.18)

with the right-hand sides in [4,15] viewed as elements of \( (T_A \text{Der}_R A)^{\otimes 2} \). Now, the graded double Schouten–Nijenhuis bracket is the unique extension

\[
\{-, -\} : (T_A \text{Der}_R A)^{\otimes 2} \to (T_A \text{Der}_R A)^{\otimes 2}
\]

of [4,18] of weight -1 to the tensor algebra \( T_A \text{Der}_R A \) satisfying the graded Leibniz rule

\[
\{\Delta, \Theta \Phi\} = (-1)^{|\Delta|\Theta}(\Theta \{\Delta, \Phi\} + \{\Delta, \Theta\} \Phi),
\]

for homogeneous \( \Delta, \Theta, \Phi \in T_A \text{Der}_R A \).
Example 4.5 ([33, Proposition 6.2.1]). The double Schouten–Nijenhuis bracket has very simple formulae for quiver path algebras. Let \( A = \mathbb{k}Q \) and \( a, b \in Q_1 \). Then
\[
\{ a, b \} = 0,
\]
\[
\left\{ \frac{\partial}{\partial a}, b \right\} = \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}
\]
\[
\left\{ \frac{\partial}{\partial a}, \frac{\partial}{\partial b} \right\} = 0.
\]

One of the most remarkable results in [33] is the following.

Proposition 4.6 ([33, Theorem 3.2.2]). The tensor algebra \( T_A \mathbb{D}er_R(A) \) together with the double graded Schouten-Nijenhuis bracket \( \{-, -\} : (T_A \mathbb{D}er_R(A))^{\otimes 2} \to (T_A \mathbb{D}er_R(A))^{\otimes 2} \) is a double Gerstenhaber algebra.

4.3. Bi-symplectic tensor \( \mathbb{N} \)-algebras.

4.3.1. Bi-symplectic tensor \( \mathbb{N} \)-algebras. Let \( B \) be an associative algebra and \( A \) a tensor \( \mathbb{N} \)-algebra over \( B \).

Definition 4.7. An element \( \omega \in \mathbb{D}er^2_R(A) \) of weight \( N \) which is closed for the universal derivation \( d \) is a bi-symplectic form of weight \( N \) if the following map of graded \( A \)-bimodules is an isomorphism:
\[
\iota(\omega) : \mathbb{D}er_R A \xrightarrow{\cong} \Omega^1_R(A)[-N] : \Theta \mapsto \iota_\Theta \omega.
\]

A tensor \( \mathbb{N} \)-algebra \( (A, \omega) \) equipped with a bi-symplectic form of weight \( N \) is called a bi-symplectic tensor \( \mathbb{N} \)-algebra of weight \( N \) over \( B \) if the underlying tensor \( \mathbb{N} \)-algebra can be written as \( A = T_B M \), for an \( \mathbb{N} \)-graded \( B \)-bimodule \( M = \bigoplus_{i \in \mathbb{N}} M^i \), such that \( M^i = 0 \) for \( i > N \), and the underlying ungraded \( B \)-bimodule of \( M^i \) is finitely generated and projective, for all \( 0 \leq i \leq N \).

4.3.2. Double Hamiltonian derivations. Following [10, 35], if \( (A, \omega) \) is a bi-symplectic tensor \( \mathbb{N} \)-algebra, we define the Hamiltonian double derivation \( H_a \in \mathbb{D}er_R A \) corresponding to \( a \in A \) via
\[
i_{H_a} \omega = da,
\]
and write
\[
\{ a, b \}_\omega = H_a(b) \in A \otimes A;
\]
since \( H_a(b) = i_{H_a}(db) \), we may write this expression as
\[
\{ a, b \}_\omega = i_{H_a} i_{H_b} \omega,
\]

Lemma 4.8. If \( (A, \omega) \) is a bi-symplectic associative \( \mathbb{N} \)-algebra of weight \( N \) over \( R \), then \( \{-, -\}_\omega \) is a double Poisson bracket of weight \( -N \) on \( A \).

Proof. This is a graded version of [33, Proposition A.3.3]. To determine the weight of \( \{-, -\}_\omega \), observe that by (4.19), \( |H_a| + |\omega| = |a| \) and by (4.20), \( |\{ a, b \}_\omega| = |a| + |b| - |\omega| \), so \( |\{-, -\}_\omega| = -N \).
As in [10, §2.7], the grading of $A$ determines the Euler derivation $\text{Eu}: A \rightarrow A$, defined by $\text{Eu}|_{A_j} = j \cdot \text{Id}$ for $j \in \mathbb{N}$. The action of the corresponding Lie derivative operator

$$L_{\text{Eu}} : \text{DR}^+_R (A) \rightarrow \text{DR}^+_R (A)$$

has nonnegative integral eigenvalues. As usual all canonical objects (differential forms, double derivations, etc.) acquire weights by means of this operator, which will be denoted by $|-|$ and called weight (e.g. a double derivation $\Theta \in \text{Der}_R A$ has weight $|\Theta|$). Furthermore, if $\omega$ is a bi-symplectic form of weight $k$ on a graded $R$-algebra $A$, we say that a homogeneous double derivation $\Theta \in \text{Der}_R A$ is bi-symplectic if $\mathcal{L}_\Theta \omega = 0$ where $\mathcal{L}_\Theta$ is the reduced Lie derivative (3.24).

As in the commutative case, bi-symplectic forms of weight $k$ impose strong constraints on the associative $\mathbb{N}$-algebra $A$.

**Lemma 4.9.** Let $\omega$ be a bi-symplectic form of weight $j \neq 0$ on an associative $\mathbb{N}$-algebra $A$ over $R$. Then

(i) $\omega$ is exact.

(ii) if $\Theta$ is a bi-symplectic double derivation of weight $l$, and $j + l \neq 0$, then $\Theta$ is a Hamiltonian double derivation.

**Proof.** For (i), note that $L_{\text{Eu}} \omega = j \omega$, as $\omega$ has weight $j$, so the Cartan identity implies $j \omega = L_{\text{Eu}} \omega = d_{\text{Eu}} \omega$, because $\omega$ is closed, where $i_{\text{Eu}} : \text{DR}_R^+(A) \rightarrow \text{DR}_R^+(A)$. For (ii), we apply (3.26) to a bi-symplectic double derivation $\Theta$, obtaining

$$0 = L_{\Theta} \omega = d(i_{\text{Eu}} \Theta \omega),$$

so defining $H := i_{\text{Eu}} \Theta \omega$, we conclude that

$$dH = d(i_{\text{Eu}} \Theta \omega) = L_{\text{Eu}}(\Theta \omega) = |\Theta \omega| \Theta \omega = (l + j) \Theta \omega,$$

where the second identity follows from (4.23). $\square$

The following result describes how the Hamiltonian double derivations $H_a$ exchange double Poisson brackets and double Schouten–Nijenhuis brackets.

**Lemma 4.10 ([33, Proposition 3.5.1]).** The following are equivalent:

(i) $\{\{\cdot,\cdot\}\}$ is a double Poisson bracket on $A$.

(ii) $\{\{H_a, H_b\}\} = H_{\{a,b\}}$, for all $a, b \in A$. Here, $H_x := H_{x'} \otimes x'' + x' \otimes H_{x''}$ for all $x = x' \otimes x'' \in A \otimes A$.

4.4. The canonical bi-symplectic form for a doubled graded quiver.

4.4.1. Casimir elements. To avoid cumbersome signs, in this subsection we shall deal with a finitely generated projective graded $(A^e)^{\text{op}}$-module $F$. Its Casimir element $\text{cas}_F$ is defined as the pre-image of the identity under the canonical isomorphism

$$F \otimes_{(A^e)^{\text{op}}} F^\vee \longrightarrow \text{End}_{(A^e)^{\text{op}}} F.$$

Note that $F^\vee = \text{Hom}_{A^e}(F, A^e_{A^e})$ is equipped with the graded $A$-bimodule structure induced by the outer $A$-bimodule structure on $A \otimes A$. In the following result, we determine the Casimir element for a graded quiver:
Then \( \text{cas}_{V_P} = \sum_{a \in P_1} \widetilde{a} \otimes a \) is the element Casimir for the \((R^e)^{\text{op}}\)-module \( V_P \).

**Proof.** Observe that \( V_P \) is an \((R^e)^{\text{op}}\)-module. Since, by convention, we compose arrows from right to left, we can check that \( \text{eval} \left( \sum_{a \in P_1} a \otimes \widetilde{a} \right) (b) = b \) for all homogeneous \( b \in P_1 \):

\[
\text{eval} \left( \sum_{a \in P_1} a \otimes \widetilde{a} \right) (b) = \sum_{a \in P_1} a \ast \widetilde{a}(b) = e_{h(b)} b e_{t(b)} = b. \quad \Box
\]

Let \( P \) be a graded quiver, with graded path algebra \( \mathbb{k} P = T_R V_P \). For a homogeneous \( b \in P_1 \), we define \( \widetilde{a} \in V_P^\vee \) by

\[
\widetilde{a}(b) = \begin{cases} e_{h(a)} \otimes e_{t(a)} & \text{if } a = b, \\ 0 & \text{otherwise}. \end{cases} \quad (4.24)
\]

4.4.2. **Duals and biduals.** Let \( \overline{P} \) be the weight \( N \) double graded quiver of a graded quiver \( P \), \( R = R_{\overline{P}} \), and \( A = \mathbb{k} \overline{P} \) its graded path algebra. Since \( R \) is a finite-dimensional semisimple algebra over \( \mathbb{k} \) (see Lemma 4.11), it is well known that there are four sensitive ways of defining the dual of an \( R \)-bimodule, but that all of them can be identified by fixing a trace on \( R \), that is, a \( \mathbb{k} \)-linear map \( \text{Tr}: R \to \mathbb{k} \) such that the bilinear form \( R \otimes R \to \mathbb{k}; (a, b) \mapsto \text{Tr}(ab) \) is symmetric and non-degenerate. More precisely, let \( V \) be an \( R \)-bimodule, \( V^\ast := \text{Hom}(V, \mathbb{k}) \) and \( V^\vee := \text{Hom}(V, R \otimes R) \). Then we can use \( \text{Tr}: R \to \mathbb{k} \) to construct an isomorphism \( B: V^\ast \to V^\vee \), by the following formula, for all \( \psi \in V^\ast, v \in V \):

\[
\psi(v) = \text{Tr}((B(\psi'))(v)) \text{Tr}((B(\psi''))(v)), \quad (4.26)
\]

Consider the graded \( R \)-bimodule \( V_{\overline{P}} \) as a vector space, and the space of linear forms \( V_{\overline{P}}^\ast := \text{Hom}(V_{\overline{P}}, \mathbb{k}) \). Then \( V_{\overline{P}} \) has a basis \( \{ \widetilde{a} \}_{a \in \overline{P}_1} \) consisting of all the arrows of \( \overline{P} \). Let \( \{ \overline{a} \}_{a \in \overline{P}_1} \subset V_{\overline{P}}^\ast \) be its dual basis. Given \( b \in A \), we have

\[
\overline{a}(b) = \delta_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases} = \text{Tr}(\delta_{ab} e_{h(a)}) \text{Tr}(e_{t(a)}) = \text{Tr}(\overline{(a)'(b)}) \text{Tr}(\overline{(a)'')(b)), \quad (4.27)
\]

with \( \{ \overline{a} \}_{a \in \overline{P}_1} \) as in Lemma 4.11. Furthermore, \( (1.26) \) applied to \( \{ \overline{a} \}_{a \in \overline{P}_1} \) gives

\[
\overline{a}(b) = \text{Tr} (B(\overline{a})'(b)) \text{Tr} (B(\overline{a})''(b)). \quad (4.28)
\]

Comparing (4.27) and (4.28), it follows that \( B(\overline{a}) = \overline{a} \), that is,

\[
B^{-1}(\overline{a}) = \overline{a}. \quad (4.29)
\]
Using \((A.5)\), we define
\[
\langle -, - \rangle : V^*_P \times V^*_P \longrightarrow k : (a, b) \longmapsto \langle a, b \rangle = \begin{cases} 
\varepsilon(a) & \text{if } a = b^* \in P_1, \\
1 & \text{if } a = b^* \in P_1^*, \\
-1 & \text{if } a = b^* \in P_1^*, \\
0 & \text{otherwise.}
\end{cases}
\]

It is not difficult to see that \((V^*_P, \langle - , - \rangle)\) is a graded symplectic vector space of weight \(N\). Moreover, the graded symplectic form \(\langle - , - \rangle\) determines an isomorphism of graded \(R\)-bimodules, for the canonical graded \(R\)-bimodule structure on \(V^*_P\) and the induced one on its dual. As in [10], we define an isomorphism, where \(V^*_P\) is regarded as a vector space:
\[
\#: V^*_P \cong V^*_P[-N] : \hat{a} \mapsto \varepsilon(a)a^* \tag{4.30}
\]

4.5. The canonical bi-symplectic form for a double graded quiver.

**Proposition 4.12.** Let \(R = \mathbb{K} \overline{P}\), \(A = \mathbb{K} \overline{P}\), and
\[
\omega := \sum_{a \in P_1} da da^* \in DR^2_R A. \tag{4.31}
\]

Then \(\omega\) is bi-symplectic of weight \(N\).

**Proof.** This result is a routine graded generalization of the ungraded statement [10, Proposition 8.1.1(ii)]. We omit the proof. \(\Box\)

5. Restriction theorems of graded bi-symplectic forms

In this section, we prove two technical results of graded bi-symplectic forms, roughly speaking corresponding to graded non-commutative versions, in weights 1 and 2, of the Darboux Theorem in symplectic geometry. Furthermore, we will describe in \(\S 5.1\) a non-commutative analogue of the cotangent exact sequence relating relative and absolute differential forms, focusing on the case of bi-symplectic tensor \(N\)-algebras.

5.1. The cotangent exact sequence. Let \(R\) be a smooth semisimple associative \(k\)-algebra, \(B\) a smooth graded \(R\)-algebra, \(A = T_B M\) the tensor algebra of a graded \(B\)-bimodule \(M\), and \(\omega \in DR^2_R(A)_N\) a bi-symplectic form of weight \(N\) on \(A\), where \(N \in \mathbb{N}\).

The cotangent exact sequence for an arbitrary graded associative \(B\)-algebra is as follows.

**Lemma 5.1.**

(i) There is a canonical exact sequence of graded \(A\)-bimodules
\[
0 \to \text{Tor}^B_1(A, A) \to A \otimes_B \Omega^1_R B \otimes_B A \to \Omega^1_R A \to \Omega^1_B A \to 0.
\]

(ii) Suppose \(A = T_B M\), where \(M\) is a graded \(B\)-bimodule which is flat as either left or right graded \(B\)-module. Then there is an exact sequence of graded \(A\)-bimodules
\[
0 \to A \otimes_B \Omega^1_R B \otimes_B A \to \Omega^1_R A \to A \otimes_B M \otimes_B A \to 0.
\]

**Proof.** This is a consequence of [11] Proposition 2.6 and Corollary 2.10], because the maps involved preserve weights. \(\Box\)
We will now use an explicit description of the space of noncommutative relative differential forms on $A$ over $R$, following [10] §5.2. Define the graded $A$-bimodule

$$\widetilde{\Omega} := (A \otimes_B \Omega^1_R B \otimes_B A) \bigoplus (A \otimes_R M \otimes_R A).$$

(5.1)

Abusing the notation, for any $a', a'' \in A$, $m \in M$, $\beta \in \Omega^1_R B$, we write

$$a' \cdot \widetilde{m} \cdot a'' := 0 \oplus (a' \otimes m \otimes a'') \in A \otimes_R M \otimes_R A \subset \widetilde{\Omega},$$

$$a' \cdot \widetilde{\beta} \cdot a'' := (a' \otimes \beta \otimes a'') \oplus 0 \in A \otimes_R \Omega^1_R B \otimes_B A \subset \widetilde{\Omega}.$$

Let $Q \subset \widetilde{\Omega}$ be the graded $A$-subbimodule generated by the Leibniz rule in $\widetilde{\Omega}$, that is,

$$Q = \langle \langle b'mb'' - db' \cdot (mb'') - b' \cdot \widetilde{m} \cdot b'' - (b'm) \cdot \widetilde{d}b'' \rangle \rangle_{\nu, \nu' \in B, m \in M},$$

(5.2)

where $\langle \langle - \rangle \rangle$ denotes the graded $A$-subbimodule generated by the set $(-).

The graded algebra structure of $A = T_B M$ induces a graded $A$-bimodule structure on $\widetilde{\Omega}$. Then $Q \subset \widetilde{\Omega}$ is a graded $A$-subbimodule, because it is generated by homogeneous elements, so the quotient $\widetilde{\Omega}/Q$ is a graded $A$-bimodule. The following result follows from [10] Lemma 5.2.3, simply because weights are preserved.

**Proposition 5.2.** Let $B$ be a smooth graded $R$-algebra, $M$ a finitely generated projective graded $B$-bimodule, and $A = T_B M$. Then

(i) There exists a graded $A$-bimodule isomorphism

$$f : \Omega^1_R A \xrightarrow{\cong} \widetilde{\Omega}/Q.$$  

(ii) The embedding of the first direct summand in $\widetilde{\Omega}$ (respectively, the projection onto the second direct summand in $\widetilde{\Omega}$), induces, via the isomorphism in (i), a canonical extension of graded $A$-bimodules

$$0 \rightarrow A \otimes_B \Omega^1_R B \otimes_B A \xrightarrow{\varepsilon} \Omega^1_R A \xrightarrow{\nu} A \otimes_B M \otimes_B A \rightarrow 0$$

(5.3)

(iii) The assignment $B \oplus M = T^0_B M \oplus T^1_B M \rightarrow \widetilde{\Omega}$, $b \oplus m \mapsto \widetilde{d}b + \widetilde{m}$ extends uniquely to a graded derivation $\widetilde{d} : A = T_B M \rightarrow \widetilde{\Omega}/Q$; this graded derivation corresponds, via the isomorphism in (i), to the canonical universal graded derivation $d : A \rightarrow \Omega^1_R A$.

In other words, we have (see [5.2])

$$f(\widetilde{m}) = dm, \quad f(\widetilde{d}b) = db,$$

(5.4)

for homogeneous $m \in M$ and $b \in B$, and the commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{d} & \Omega^1_R A \\
\xrightarrow{\widetilde{d}} & & \\
\widetilde{\Omega}/Q & \xrightarrow{f} & \Omega^1_R A
\end{array}$$

Applying the functor $\text{Hom}_{A^e}(-, A^e)$ to (5.3), we obtain the “tangent exact sequence”.

**Lemma 5.3.** (i) Let $B$ be a smooth graded $R$-algebra, $M$ a finitely generated projective graded $B$-bimodule and $A = T_B M$. Then there is a short exact sequence

$$0 \rightarrow A \otimes_B M^\vee \otimes_B A \xrightarrow{\nu^c} \text{Der}_R A \xrightarrow{\varepsilon^c} A \otimes_B \text{Der}_R B \otimes_B A \rightarrow 0.$$

(5.5)
Theorem 5.6. In the Framework 5.5, the isomorphism $B$ stricts, in weight 1, to a $B$ with inverse or its graded generalization, namely, Theorem 4.12: 

$$
\begin{array}{c}
0 \longrightarrow A \otimes_B M^\vee \otimes_B A \overset{\nu}{\longrightarrow} \text{Der}_R A \overset{\epsilon}{\longrightarrow} A \otimes_B \text{Der}_R B \otimes_B A \longrightarrow 0 \\
0 \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \overset{\epsilon}{\longrightarrow} \Omega^1_R A \overset{\nu}{\longrightarrow} A \otimes_B M \otimes_B A \longrightarrow 0
\end{array}
$$

(5.6)

Proof. This result is a graded version of [10, Lemma 5.4.2]. \qed

5.2. Restriction Theorem in weight 0.

Theorem 5.4. Let $R$ be a semisimple finite-dimensional $k$-algebra, $B$ a smooth associative $R$-algebra, and $E_1, \ldots, E_N$ finitely generated projective $B$-bimodules, where $N > 0$. Define the tensor $N$-algebra $A := T_B M$ as the tensor $B$-algebra of the graded $B$-bimodule $M := M_1 \oplus \cdots \oplus M_N$, where $M_i := E_i[-i]$, for $i = 1, \ldots, N$. Let $\omega \in \text{DR}^2_R (A)$ be a bi-symplectic form of weight $N$ over $A$. Then, the isomorphism $\iota(\omega) : \text{Der}_R A \cong \Omega^1_R A[-N]$ induces another isomorphism $\overline{\iota(\omega)} : A \otimes_B \text{Der}_R B \otimes_B A \cong A \otimes_B M \otimes_B A$, which, in weight zero, restricts to the following isomorphism:

$$
\overline{\iota(\omega)}_{(0)} : \text{Der}_R B \cong E_N.
$$

The technical proof of this result is given in Appendix A.

5.3. Restriction Theorem in weight 1 for doubled graded quivers. For convenience, we fix the following.

Framework 5.5. Let $P$ be a doubled graded quiver of weight 2, with graded path algebra $A := kP$. Let $R = R_P$ be the semisimple finite dimensional algebra with basis the trivial paths in $P$, and let $B$ be the smooth path algebra of the weight 0 subquiver of $P$. Let $\omega$ be the canonical bi-symplectic form $\omega \in \text{DR}^2_R (A)$ of weight 2 on $A$. Then $A = T_B M$, where $M$ is the graded $B$-bimodule $M := E_1[-1] \oplus E_2[-2]$, for finitely generated projective $B$-bimodules $E_1$ and $E_2$.

Theorem 5.6. In the Framework 5.5, the isomorphism $\iota(\omega) : \text{Der}_R A \cong \Omega^1_R A[-2]$ restricts, in weight 1, to a $B$-bimodule isomorphism

$$(\iota(\omega))_1 : E_1^\vee \cong E_1 : \quad \tilde{a} \mapsto \epsilon(a) a^*,$$

(5.7)

with inverse

$$b : E_1 \cong E_1^\vee : \quad a \mapsto \epsilon(a) a^*.$$

(5.8)

Proof. By Lemma 3.2 and (5.4), we know that $A$ can be identified both with $T_R V_P$ and $T_B M_P$. We will use the following isomorphisms from the proof of [10, Proposition 8.1.1], or its graded generalization, namely, Theorem 4.1.2

$$G : A \otimes_R V_P \otimes_R A \cong \Omega^1_R A : \quad \sum_{a \in P_1} f_a \otimes a \otimes g_a \mapsto \sum_{a \in P_1} f_a da g_a,$$

(5.9)
\[ H : \text{Der}_R A \xrightarrow{\cong} A \otimes_R V^\vee \otimes_R A : \Theta \mapsto \sum_{a \in P_1} (-1)^{\square}(a) \otimes \tilde{a} \otimes \Theta'(a), \] (5.10)

where \((-1)^{\square}\) is given by the Koszul sign rule. To shorten notation,

\[ V_1 := (V^\vee_1)_1, \quad V_1^\vee := \text{Hom}_{R^e}(V_1, R^e), \quad M_w := (M^\vee_{w1}), \quad \text{for } w > 0, \]

where the subindexes mean weights. Using (4.30), (5.9) and (5.10), we consider the following commutative diagram (some arrows will be constructed below).

\[ A \otimes_B M_1^\vee \otimes_B A^e \xrightarrow{J} \text{Der}_R A \xrightarrow{t(\omega)} \Omega^1_R A \xrightarrow{P} A \otimes_B M_1 \otimes_B A \]

(5.11)

Claim 5.7. If \(a \in P_1\), we have \(A \otimes_B M_1 \otimes_B A \xrightarrow{\cong} A \otimes_R V_1 \otimes_R A\), and consequently

\[ T : A \otimes_B M_1^\vee \otimes_B A \xrightarrow{\cong} A \otimes_R V_1^\vee \otimes_R A. \]

Proof of Claim 5.7. This follows simply because

\[ A \otimes_B M_1 \otimes_B A = \bigoplus_{|a|=1} A \otimes_B Ba B \otimes_B A \cong \bigoplus_{|a|=1} A a A \]

\[ \cong \bigoplus_{|a|=1} A \otimes_R \text{ka} \otimes_R A = A \otimes_R V_1 \otimes_R A. \] (5.12)

Using Claim 5.7, we can construct the left-hand triangle in (5.11). Let \(a \in P_1\) with \(|a|=1\). Then consider \(\tilde{a} \in M_1^\vee\) (as defined in (1.24)) and \(1 \otimes \tilde{a} \otimes 1 \in A \otimes_R M_1^\vee \otimes_B A\). By Claim 5.7 and the natural injection, this element can be viewed in \(A \otimes_R V_1^\vee \otimes_R A \cong V_1^\vee \otimes_R A^e\). Now, to define \(J\), we use the sequence of isomorphisms

\[ V_1^\vee \otimes_R A^e \cong \text{Hom}_{R^e}(V_1^\vee, \text{Hom}_{A^e}(A^e, A^e)) \]

\[ \cong \text{Hom}_{R^e}(A^e \otimes_R V_1^\vee, A^e) \]

\[ = \text{Hom}_{A^e}(\Omega^1_{RA}, A^e) = \text{Der}_R A. \] (5.13)

Note that \(h = H \circ J\) is given by

\[ h : A \otimes_B M_1^\vee \otimes_B A \xrightarrow{J} \text{Der}_R A \xrightarrow{H} A \otimes_R V_1^\vee \otimes_R A \]

(5.14)

\[ 1 \otimes \tilde{a} \otimes 1 \xrightarrow{\frac{\partial}{\partial a}} e_{t(a)} \otimes \tilde{a} \otimes e_{h(a)}. \]

Next, we will focus on the right-hand square of (5.11), where proj is the canonical projection, and construct \(g\) so that it is commutative. Let \(qap\) be a generator of \(A \otimes_R V_1^\vee \otimes_R A\), i.e., \(a \in P_1\) is such that \(|a|=1\), and \(p, q\) are paths in \(P\), such that \(h(p) = t(a)\), \(h(a) = t(q)\). Then \(\text{proj}|_{A \otimes_R V_1 \otimes_R A} = \text{Id}\), and \(P = \text{pr} \circ \nu \circ f\) by Lemma 5.2 where \(\text{pr} : M = \bigoplus_{w>0} M_w \rightarrow M_1\) is the natural projection. Hence

\[ (\text{pr} \circ \nu \circ f \circ G)(q \otimes a \otimes p) = (\text{pr} \circ \nu \circ f)(q da p) \]

\[ = (\text{pr} \circ \nu)((0 \oplus (q \otimes a \otimes p)) \mod Q) \]

\[ = \text{pr}(q \otimes a \otimes p) = q \otimes a \otimes p, \] (5.15)
so the isomorphism \( g \) restricts to the isomorphism \( G \), because
\[
g: A \otimes_R V_1 \otimes_R A \xrightarrow{\sim} A \otimes_B M_1 \otimes_B A: \quad q \otimes a \otimes p \mapsto q \otimes a \otimes p.
\] (5.16)

Therefore, by (5.14), (4.30) and (5.16), we have
\[
(g \circ p) \circ (\text{Id} \otimes \sharp \otimes \text{Id}) \circ h: A \otimes_B M_1^\vee \otimes_B A \rightarrow A \otimes_B M_1 \otimes_B A
\]
\[
1 \otimes \tilde{a} \otimes 1 \mapsto \epsilon_{\text{Id}}(a) \otimes \epsilon(a) a^* \otimes e_{\text{Id}}(a).
\] (5.17)

Let \((-)_w\) mean the component of weight \( w \in \mathbb{Z} \). Then \((A \otimes_B M_1 \otimes_B A)_0 \cong B \otimes_B M_1 \otimes_B B \cong M_1\). Furthermore, since \( \omega \) is a bi-symplectic form of weight 2, (5.17) has weight -2, so \((A \otimes_B M_1^\vee \otimes_B A)_-1 = B \otimes_B M_1^\vee \otimes_B A \cong M_1^\vee\). Therefore, we obtain the following isomorphism of \( B \)-bimodules:
\[
\mathcal{R}: E_1^\vee \xrightarrow{\cong} E_1:\quad \tilde{a} \mapsto \epsilon(a) a^*,
\] (5.18)

with inverse
\[
b: E_1 \xrightarrow{\cong} E_1^\vee:\quad a \mapsto \epsilon(a) \tilde{a}^*.
\] (5.19)

\[\square\]

6. Bi-symplectic \( \mathbb{N} \)-algebras of weight 2

6.1. The graded algebra \( A \). For convenience, we introduce the following.

**Framework 6.1.** Let \( R \) be a semisimple associative algebra, \( B \) a smooth associative \( R \)-algebra, and \( E_1 \) and \( E_2 \) projective finitely generated \( B \)-bimodules. Let
\[
A := T_B M
\]
be the graded tensor \( \mathbb{N} \)-algebra of the graded \( B \)-bimodule
\[
M := E_1[-1] \oplus E_2[-2].
\]

Let \( \omega \in DR^2_R(A) \) be a bi-symplectic form of weight 2. Thus the pair \((A, \omega)\) is a bi-symplectic tensor \( \mathbb{N} \)-algebra of weight 2 (see Definition 4.7).

In this framework, we have
\[
A = \bigoplus_{n \in \mathbb{N}} A^n,
\]
where
\[
A^0 = B, \quad A^1 = E_1, \quad A^2 = E_1 \otimes_B E_1 \oplus E_2.
\] (6.1)

By Lemma 4.8, the bi-symplectic form \( \omega \) on \( A \) determines a double Poisson bracket \( \{\cdot, \cdot\}_{\omega} \) of weight -2. This bracket satisfies the following relations:
\[
\begin{align*}
\{A^0, A^0\}_{\omega} = \{A^0, A^1\}_{\omega} = 0, \\
\{A^1, A^1\}_{\omega} & \subset (A \otimes A)_{(0)} = B \otimes B, \\
\{A^2, A^0\}_{\omega} & \subset (A \otimes A)_{(0)} = B \otimes B, \\
\{A^2, A^1\}_{\omega} & \subset (A \otimes A)_{(1)} = (E_1 \otimes B) \oplus (B \otimes E_1), \\
\{A^2, A^2\}_{\omega} & \subset (A \otimes A)_{(2)}.
\end{align*}
\] (6.2)

6.2. The pairing.
6.2.1. A family of double derivations. By \((6.1), (6.2)\), \(\{A^2, B\}_\omega \subset B \otimes B\), so we can define
\[
X_a := \{a, -\}_\omega |_B: B \to B \otimes B.
\]
for all \(a \in A^2\). Since \(\{-, -\}_\omega\) is a double Poisson bracket, in particular, it satisfies the graded Leibniz rule in its second argument (with respect to the outer structure), and so
\[
X_a(b_1b_2) = \{a, b_1b_2\}_\omega = b_1\{a, b_2\}_\omega + \{a, b_1\}_\omega b_2 = b_1X_a(b_2) + X_a(b_1)b_2,
\]
for all \(a \in A^2\), and \(b_1, b_2 \in B\). Therefore \(X_a \in \text{Der}_R B\), and we construct a ‘family of double derivations’ parametrized by \(A^2\), namely,
\[
X: A^2 \to \text{Der}_R B: \quad a \mapsto X_a := \{a, -\}_\omega |_B. \quad (6.3)
\]

6.2.2. A family of double differential operators. By \((6.2), \{A^2, A^1\}_\omega \subset (A \otimes A)_{(1)} = E_1 \otimes B \otimes B \otimes E_1\), so for all \(a \in A^2\), we can define a map
\[
D: A^2 \to \text{Hom}_{R^e}(E_1, E_1 \otimes B \otimes B \otimes E): \quad a \mapsto D_a := \{a, -\}_\omega |_{E_1}. \quad (6.4)
\]
Then, given \(b \in B, e \in E_1, a \in A^2\), the graded Leibniz rule applied to \(\{-, -\}_\omega\) yields
\[
\begin{align*}
D_a(be) &= bD_a(e) + X_a(b)e, \quad (6.5a) \\
D_a(eb) &= D_a(e)b + eX_a(b), \quad (6.5b)
\end{align*}
\]
with \(b\) acting via the outer bimodule structure on \(D_a\) and \(X_a\). Therefore, \(D\) can be regarded as a family of ‘covariant’ double differential operators parametrized by \(A^2\), associated to the family of double derivations \(X\).

6.2.3. The pairing. Given a graded algebra \(C\), Van den Bergh [20, Appendix A] defines a pairing between two graded \(C\)-bimodules \(P\) and \(Q\) as a homogeneous map of weight \(n\)
\[
\langle -,- \rangle: P \times Q \to C \otimes C,
\]
such that \(\langle p, - \rangle\) is linear for the outer graded bimodule structure on \(C \otimes C\), and \(\langle - , q \rangle\) is linear for the inner graded bimodule structure on \(C \otimes C\), for all \(p \in P, q \in Q\). We say that the pairing is symmetric if \(\langle p,q \rangle = \sigma(12)\langle q, p \rangle\) (with \(\sigma(12)\) as in \((3.15)\)), and non-degenerate if \(P\) and \(Q\) are finitely generated graded \(C\)-bimodules and the pairing induces an isomorphism
\[
Q \xrightarrow{\cong} P^\vee[-n]: \quad q \mapsto \langle - , q \rangle,
\]
with \(P^\vee = \text{Hom}_{C^e}(P, C^e C^e)\).

Consider now the Framework \((5.3)\) associated to a doubled graded quiver \(\overrightarrow{P}\) of weight 2. Using the isomorphism \(\nu\) in \((5.19)\), we define
\[
\langle -,- \rangle: E_1 \otimes E_1 \to B \otimes B: \quad (a,b) \mapsto \nu(a)(b) = \varepsilon(a)a^\nu(b). \quad (6.6)
\]
Consider also the double Poisson bracket \(\{-, -\}_\omega\) of weight -2 associated to the graded bi-symplectic form \(\omega\) of weight 2 in Proposition \((4.12)\). By the third inclusion in \((6.2)\),
\[
\{ - , - \}_\omega |_{(E_1 \otimes E_1)}: E_1 \otimes E_1 \to B \otimes B.
\]

**Lemma 6.2.**
1. \(\{a, b\}_\omega = \langle a, b \rangle\), for all arrows \(a, b \in P_1\) of weight 1.
2. The map \(\langle -,- \rangle\) in \((6.6)\) is a non-degenerate symmetric pairing.
Proof. To prove (i), we use the formula
\[ \{a, b\}_\omega = i_{a^*} \omega \cdot a, \]
with \( \frac{\partial}{\partial a}, \frac{\partial}{\partial a} \in \text{Der}_R A \) (see (3.10)) and the formula (4.31) for the bi-symplectic form \( \omega \). Then
\[ \{a, b\}_\omega = i_{a^*} \omega \cdot a = i_a \left( \sum_{a \in T_1} \varepsilon(b^*) e_{h(b)}(db) e_{l(b)} \right) \]
\[ = \sum_{a \in T_1} i_a \left( \varepsilon(b^*) e_{h(b)}(db) e_{l(b)} \right) = \varepsilon(a) a^*(b) = \langle a, b \rangle. \]

The fact that \( \langle - , - \rangle \) is a pairing follows from (i) and properties of double brackets. This pairing is symmetric, because \( \sigma_{(12)}(a, b) = \varepsilon(b) e_{l(b)} \otimes e_{h(b)} = \langle b, a \rangle \), and non-degenerate, because \( b \) is an isomorphism (see Theorem 5.6).

6.2.4. Preservation of the pairing. Extend the pairing (6.6) to two maps
\[ \langle - , - \rangle_L: E_1 \times (A \otimes A) \rightarrow B^\otimes 3, \]
\[ \langle - , - \rangle_R: E_1 \times (A \otimes A) \rightarrow B^\otimes 3, \]
given, for all \( e_1, e_2 \in E_1, b \in B \), by
\[ \langle e_1, e_2 \otimes b \rangle_L = \langle e_1, e_2 \rangle \otimes b, \quad \langle e_1, b \otimes e_2 \rangle_L = 0, \quad (6.7) \]
\[ \langle e_1, e_2 \otimes b \rangle_R = 0, \quad \langle e_1, b \otimes e_2 \rangle_R = b \otimes \langle e_1, e_2 \rangle. \quad (6.8) \]

We extend similarly the pairing in the first argument with the inner \( \otimes \)-product in (2.4), so
\[ \langle - , - \rangle_L: (A \otimes A) \rightarrow B^\otimes 3 \]
is given by
\[ \langle e_1 \otimes b, e_2 \rangle_L = \langle e_1, e_2 \rangle \otimes b, \quad \langle b \otimes e_1, e_2 \rangle_L = 0. \quad (6.9) \]

We will also extend double derivations \( \Theta: B \rightarrow B^\otimes 2 \) to maps
\[ \Theta: B^\otimes 2 \rightarrow B^\otimes 3: b_1 \otimes b_2 \mapsto \Theta(b_1) \otimes b_2. \quad (6.10) \]

Then the family \( X \) of double derivations in (6.3) and the family \( \mathcal{D} \) of covariant double operators in (6.4), preserve the pairing \( \langle - , - \rangle \), that is,
\[ X_a(\langle e_1, e_2 \rangle) = \tau_{(132)} \langle e_2, \mathcal{D}_a(e_1) \rangle - \tau_{(123)} \langle e_1, \mathcal{D}_a(e_2) \rangle, \quad (6.11) \]
for all \( a \in A^2, e_1, e_2 \in E_1 \). This follows because
\[ X_a(\langle e_1, e_2 \rangle) = \{ \{ a, \{ e_1, e_2 \} \} \omega \}_L = \tau_{(123)} \{ e_1, \{ e_2, a \} \} \omega + \tau_{(132)} \{ e_2, \{ a, e_1 \} \} \omega \]
\[ = \tau_{(132)} \{ e_2, \mathcal{D}_a(e_1) \} - \tau_{(123)} \langle e_1, \mathcal{D}_a(e_2) \rangle, \]
where the first identity follows from the definition (6.10), the second identity is the graded double Jacobi identity, and the third identity follows by graded skew-symmetry of \( \{ - , - \} \).

6.3. Twisted double Lie–Rinehart algebras. A key ingredient in Ševera–Roytenberg’s characterization of symplectic \( \mathbb{N} \)-manifolds of weight 2 can be interpreted algebraically as saying that the Atiyah algebroids and the commutative analogue of \( A^2 \) have structures of Lie–Rinehart algebras (27, Theorem 3.3). In this subsection, we define a slight generalization of Van den Bergh’s double Lie algebroid [34, Definition 3.2.1], that fits both the underlying algebraic structure of \( A^2 \) and a suitable non-commutative version of the Atiyah algebroid that will be introduced in [6.3].
6.3.1. Definition of double Lie–Rinehart algebras. Let $N$ be a $B$-bimodule. Following \[33\] §2.3, given $n, n_1, n_2 \in N$ and $b, b_1 b_2 \in B$, we define

$$
\begin{align*}
\{n_1, b \otimes n_2\}_L &= \{n_1, b\} \otimes n_2, & \{n_1, n_2 \otimes b\}_L &= \{n_1, n_2\} \otimes b, \\
\{n, b_1 \otimes b_2\}_L &= \{n, b_1\} \otimes b_2, & \{b_1, n \otimes b_2\}_L &= \{b_1, n\} \otimes b_2.
\end{align*}
$$

(6.12)

The rest of combinations will be zero by definition. Given a permutation $s \in S_n$, we will use the notation $\tau_s$ for the map in (6.3), to emphasize that permutations act on mixed tensor products of algebras and bimodules. For instance, $\tau_{(12)} : N \otimes B \rightarrow B \otimes N : n \otimes b \mapsto b \otimes n$.

Definition 6.3. A twisted double Lie–Rinehart algebra over $B$ is a 4-tuple

$$(N, \overline{N}, \rho, \{ -, -, \}_N),$$

where $N$ is a $B$-bimodule, $\overline{N} \subset N$ is a $B$-subbimodule called the twisting subbimodule, $\rho : N \rightarrow \mathcal{D}er_R B$ is a $B$-bimodule map, called the anchor, and

$$\{ -, -, \}_N : N \times N \rightarrow N \otimes B \oplus B \otimes N \oplus \overline{N}' \otimes \overline{N} \oplus \overline{N} \oplus \overline{N}'$$

is a bilinear map, called the double bracket, satisfying the following conditions:

(a) $\{n_1, n_2\}_N = -\tau_{(12)} \{n_2, n_1\}_N$,

(b) $\{n_1, bn_2\}_N = b \{n_1, n_2\}_N + \rho(n_1)(b)n_2$,

(c) $\{n_1, n_2 b\}_N = \{n_1, n_2\}_N b + n_2 \rho(n_1)(b)$,

(d) $0 = \{n_1, \{n_2, n_3\}_N\}_N + \{n_2, \{n_3, n_1\}_N\}_N$,

(e) $\rho(\{n_1, n_2\}_N) = \{\rho(n_1), \rho(n_2)\}_{SN}$,

for all $n_1, n_2, n_3 \in N, b \in B$. If the twisting subbimodule is zero (i.e. $\overline{N} = 0$), then we say that the triple $(N, \rho, \{ -, -, \}_N)$ is a double Lie–Rinehart algebra.

In Definition 6.3 all products involved use the outer bimodule structure. Also, in (e), $\{ -, -, \}_{SN}$ denotes the double Schouten–Nijenhuis bracket (see \[11\]), and by convention, $\rho$ acts by the Leibniz rule on tensor products.

Example 6.4. By Proposition \[4\] \[1\] \[\mathcal{D}er_R B\] is a double Lie–Rinehart algebra when it is equipped with the double Schouten–Nijenhuis bracket restricted to $(\mathcal{D}er_R B) \otimes^2$ and the identity as anchor.

6.3.2. $A^2$ as a twisted double Lie–Rinehart algebra. As we showed in (6.2), $\{A^2, A^2\}_\omega \subset (A \otimes A)_2 = E_2 \otimes B \oplus B \otimes E_2 \oplus E_1 \otimes E_1$, so we can define

$$\{ -, -, \}_{A^2} := \{ -, -, \}_N|_{A^2 \otimes A^2} : A^2 \otimes A^2 \rightarrow (A \otimes A)_2.$$

(6.13)

In Framework \[5\] by Lemma \[6\] we know that $E_1$ is endowed with a non-degenerate symmetric pairing. Then, in particular, $E_1 \cong E_1'$.

Proposition 6.5. In the setting of Framework \[5\] $A^2$ is a twisted double Lie–Rinehart algebra, with the bracket $\{ -, -, \}_{A^2}$, the anchor $\rho : A^2 \rightarrow \mathcal{D}er_R B : a \mapsto X_a$ (see (6.3)), and the twisting subbimodule $E_1$.

Proof. Conditions (a) and (d) in Definition 6.3 are automatic, because $\{ -, -, \}_{A^2}$ is the restriction of a double Poisson bracket, and (e) follows by applying Lemma 4.10. Finally,
(b) and (c) are consequences of the graded Leibniz rule applied to $\{-, -\}_\omega$. For instance, (b) follows because, given $a_1, a_2 \in A^2$, $b \in B$,  
\[
\{a_1, ba_2\}_{A^2} = b \{a_1, a_2\}_{A^2} + \{a_1, b\}_{A^2} a_2
\]
\[
= b \{a_1, a_2\}_{A^2} + \mathcal{X}_{a_1}(b)a_2 = b \{a_1, a_2\}_{A^2} + \rho(a_1)(b)a_2
\]

\[\square\]

6.3.3. Morphisms of twisted double Lie–Rinehart algebras. Let $(N, \overline{N}, \{-, -\}_N, \rho_N)$ be a twisted double Lie–Rinehart algebra. Suppose that $\overline{N}$ is endowed with a non-degenerate pairing. Then $\overline{N} \cong \overline{N}'$, and we can perform the following decomposition of the double bracket $\{-, -\}_N$:

\[
\{n_1, n_2\}_N = \{n_1, n_2\}_1^1 + \{n_1, n_2\}_1^2 + \{n_1, n_2\}_2^m
\]

\[
= \{n_1, n_2\}_1^1 \otimes \{n_1, n_2\}_1^2 + \{n_1, n_2\}_1^2 \otimes \{n_1, n_2\}_1^1 + \{n_1, n_2\}_2^m \otimes \{n_1, n_2\}_2^m,
\]

with $\{n_1, n_2\}_1^1 \in N \otimes B, \{n_1, n_2\}_1^2 \in B \otimes N, \{n_1, n_2\}_2^m \in \overline{N} \otimes \overline{N}$, and $\{n_1, n_2\}_1^2, \{n_1, n_2\}_1^1 \in \overline{N}, \{n_1, n_2\}_2^m, \{n_1, n_2\}_2^m \in \overline{N}$.

**Definition 6.6.** Let $(N, \overline{N}, \{-, -\}_N, \rho_N)$ and $(N', \overline{N}', \{-, -\}_{N'}, \rho_{N'})$ be two twisted double Lie–Rinehart algebras over $B$, and $(\langle-\rangle)$ a non-degenerate pairing on $\overline{N}$, so the double bracket $\{\langle -\rangle, -\}_N$ admits the decomposition (6.14). Then a morphism of twisted double Lie–Rinehart algebras between them is a pair $\phi = (\phi_1, \phi_2)$, where $\phi_1 : N \rightarrow N'$ and $\phi_2 : \overline{N} \rightarrow (\overline{N}')^\vee$ are $B$-bimodule morphisms, such that for all $n_1, n_2 \in N$,

(i) $\{n_1, n_2\}_N \\overline{N}', \{n_1, n_2\}_N \overline{N}' \in \overline{N}'$;

(ii) $\phi(\{n_1, n_2\}_N) = \{\phi(n_1), \phi(n_2)\}_{N'}$, where in the left-hand side, by convention,

\[
\phi(\{n_1, n_2\}_N) = \phi_1(\{n_1, n_2\}_N^1 \otimes \{n_1, n_2\}_N^2 + \{n_1, n_2\}_N^2 \otimes \phi_1(\{n_1, n_2\}_N^1))
\]

\[
+ \phi_2(\{n_1, n_2\}_N^m) \otimes \{n_1, n_2\}_N^m + \{n_1, n_2\}_N^m \otimes \phi_2(\{n_1, n_2\}_N^m);
\]

(iii) the following diagram commutes.

\[\begin{array}{ccc}
N & \xrightarrow{\rho_N} & B \\
\downarrow{\phi_1} & & \uparrow{\rho_{N'}} \\
N' & \xrightarrow{\phi_2} & \text{End}_R B \\
\end{array}\]

6.4. The double Atiyah algebra. We will now define a non-commutative analogue of Atiyah algebroids, and use the square-zero construction to show that they are twisted double Lie–Rinehart algebras.

6.4.1. The definition of double Atiyah algebra. Let $R$ be an associative algebra, $B$ be an associative $R$-algebra, and $E$ a finitely generated projective $B$-bimodule equipped with a symmetric non-degenerate pairing $\langle-\rangle$ (see (6.2)). Define

\[
\text{End}_R(E) := \text{Hom}_{R^e}(E, E \otimes B \oplus B \otimes E),
\]

(6.15)
with the outer $R$-bimodule structure on $E \otimes B \oplus B \otimes E$. The surviving inner $B$-bimodule structure on $E \otimes B \oplus B \otimes E$ makes $\text{End}_R(E)$ into a $B$-bimodule. Its elements will be called $R$-linear double endomorphisms. Given $e \in E, D \in \text{End}_R(E)$, we will use the decomposition

$$D(e) = (D^l + D^r)(e) = \left(D^l \otimes D^r + D^r \otimes D^r\right)(e), \quad (6.16)$$

omitting the summation symbols, where $D^l(e) \in E_1 \otimes B$, $D^r(e) \in B \otimes E_1$, and $D^l(e), D^r(e) \in B$.

The following conditions (i) and (ii) should be compared with (6.5).

**Definition 6.7.** The (R-linear) double Atiyah algebra $\mathfrak{At}_B(E)$ is the set of pairs $(X, D)$ with $X \in \text{Der}_R B$ and $D \in \text{End}_R(E)$, satisfying the following conditions for all $b, e \in E$:

(i) $D(be) = bD(e) + X(b)e$,

(ii) $D(eb) = D(e)b + eX(b)$.

Here, all the products are taken with respect to the outer structure.

It is easy to see that $\mathfrak{At}_B(E_1)$ is a $B$-subbimodule of the direct sum of the $B$-bimodules $\text{Der}_R B$ and $\text{End}_R(E)$. Using now the symmetric non-degenerate pairing of $E$, we can impose preservation of the pairing, in the sense of (6.11), on elements of $\mathfrak{At}_B(E)$.

**Definition 6.8.** Let $E$ be a finitely generated projective $B$-bimodule equipped with a symmetric non-degenerate pairing $(\cdot, \cdot)$. The (R-linear) metric double Atiyah algebra of $E$ is the subspace $\mathfrak{At}_B(E, (-,-)) \subset \mathfrak{At}_B(E)$ of pairs $(X, D)$ that preserve the pairing, i.e. $X((e_2, e_1)) = \tau_{(123)}(e_1, D(e_2))L - \tau_{(132)}(e_2, D(e_1)^\circ)L$, for all $e_1, e_2 \in E, X \in \text{Der}_R B$ and $D \in \text{End}_R(E)$.

6.4.2. The bracket. Using the square-zero construction (see, e.g., [17, §3.2]), we define a graded associative $R$-algebra

$$C := B[1],$$

with underlying graded $R$-bimodule $B \oplus (E[-1])$, and multiplication

$$(b, e) \cdot (b', e') = (bb', be + b'e),$$

for all $b, b' \in B, e, e' \in E$. Observe that $E$ is a nilpotent ideal, i.e. $e \cdot e' = 0$, and $C$ has unit $1_C = (1_B, 0)$. Given a graded $C$-bimodule $F$, we obviously have

$$\text{Der}_R(C, F) = \left\{ D : C \rightarrow F \left| \begin{array}{l}
D(bb') = D(b)b' + bD(b'), \\
D(be) = D(b)e + bD(e), \text{ for all } b, b' \in B, e \in E \\
D(0) = 0
\end{array} \right. \right\}, \quad (6.17)$$

so when $F = (C \otimes C)_{\text{out}}$, the subspace of derivations of weight 0 is

$$\left(\text{Der}_R(C')\right)_{(0)} \cong \mathfrak{At}_B(E),$$

where the isomorphism maps a derivation $\Theta : C \rightarrow C \otimes C$ of weight 0 into the pair $(X, D)$ consisting of its restrictions to $B$ and $E[-1]$ (with the appropriate weight shift). The inverse will be denoted

$$\Xi : \mathfrak{At}_B(E_1) \xrightarrow{\cong} \left(\text{Der}_R(C')\right)_{(0)}. \quad (6.18)$$

It also follows from (6.17) that the subspace of double derivations of weight -1 is

$$\left(\text{Der}_R C\right)_{(-1)} \cong E^\vee.$$
Now, we extend the symmetric non-degenerate pairing $\langle \cdot, \cdot \rangle$ from $E$ to $C$ by the formulae $\langle b, b' \rangle = \langle e, b \rangle = \langle b, e \rangle = 0$ (this should be compared with Lemma (6.21)), in the Framework \(\Xi\). Then (6.18) restricts to another isomorphism

$$\Xi : \text{At}_B(E, \langle \cdot, \cdot \rangle) \xrightarrow{\cong} (\text{Der}_R(C, \langle \cdot, \cdot \rangle))_{(0)},$$

(6.19)

where $(\text{Der}_R(C, \langle \cdot, \cdot \rangle))_{(0)}$ is the $C$-bimodule of double derivations of weight 0 that preserve the pairing (extended to $C$).

Observe now that the double Schouten–Nijenhuis bracket on $T_C \text{Der}_R C$ preserves weights, so it restricts to another bracket on the tensor subalgebra $T_C((\text{Der}_R C)_{(-1)})$, that also satisfies skew-symmetry and the Leibniz and Jacobi rules. In the rest of this subsection, we will use this restricted double Schouten–Nijenhuis bracket to construct another double bracket on $T_B \text{At}_B(E)$ via the isomorphisms (6.18) and (6.19).

Let $T_1, T_2 \in (\text{Der}_R C)_{(0)}$. Then formulae (4.13) define weight 0 $R$-linear triple derivations

$$\langle T_1, T_2 \rangle \sim C \rightarrow C^{\otimes 3} \in (\text{Der}_R(C, C^{\otimes 3}))_{(0)},$$

for the outer bimodule structure on $C^{\otimes 3}$, and formulae (4.14), (4.15) define isomorphisms

$$\tau_{(12)} : \text{Der}_R(C, C^{\otimes 3}) \xrightarrow{\cong} C \otimes \text{Der}_R C, \quad \tau_{(23)} : \text{Der}_R(C, C^{\otimes 3}) \xrightarrow{\cong} \text{Der}_R C \otimes C.$$

Since these isomorphisms preserve weights, they restrict to isomorphisms in weight 0,

$$\Xi_{(12)} : \text{Der}(C, C^{\otimes 3})_{(0)} \xrightarrow{\cong} (C \otimes \text{Der}_R C)_{(0)} \xrightarrow{\cong} C_{(0)} \otimes (\text{Der}_R C)_{(0)} \oplus C_{(1)} \otimes (\text{Der}_R C)_{(-1)}$$

$$\xrightarrow{\cong} C_{(0)} \otimes (\text{Der}_R C)_{(0)} \oplus C_{(1)} \otimes (\text{Der}_R C)_{(-1)} \oplus C_{(0)} \oplus (\text{Der}_R C)_{(-1)} \oplus C_{(1)}$$

(6.20)

$$\Xi_{(23)} : \text{Der}(C, C^{\otimes 3})_{(0)} \xrightarrow{\cong} (\text{Der}_R C \otimes C)_{(0)} \xrightarrow{\cong} \text{Der}_R C_{(0)} \otimes C_{(0)} \oplus (\text{Der}_R C)_{(-1)} \otimes C_{(1)}$$

$$\xrightarrow{\cong} \text{Der}_R C_{(0)} \otimes C_{(0)} \oplus (\text{Der}_R C)_{(-1)} \otimes C_{(1)}$$

(6.21)

Hence the double brackets $\langle T_1, T_2 \rangle_{l}$ and $\langle T_1, T_2 \rangle_{r}$, defined as in (6.16), (6.17), restrict to

$$\langle T_1, T_2 \rangle_{l,0} := \Xi_{(23)} \circ \langle T_1, T_2 \rangle_{l} \sim \text{At}_B(E) \otimes B \oplus E' \otimes E,$$

$$\langle T_1, T_2 \rangle_{r,0} := \Xi_{(12)} \circ \langle T_1, T_2 \rangle_{r} \sim B \otimes \text{At}_B(E) \oplus E \otimes E',$$

whereas formulae (4.18), for all $c, c_1, c_2 \in C, T, T_1, T_2 \in (\text{Der}_R C)_{(0)}$, restrict to

$$\langle c_1, c_2 \rangle_0 = 0,$$

$$\langle T, c \rangle_0 = T(c),$$

$$\langle c, T \rangle_0 = -\sigma_{(12)}(T(c)),$$

$$\langle T_1, T_2 \rangle_0 = \langle T_1, T_2 \rangle_{l,0} + \langle T_1, T_2 \rangle_{r,0}.$$ (6.22)

Now, it follows from (6.17) that

$$(\text{Der}_R(C, C^{\otimes 3})_{(0)} \cong \left\{ \begin{array}{ll} X : B \rightarrow B^{\otimes 3} & \text{if } X(bb') = X(b)b' + bX(b') \\ D : E_1 \rightarrow E_1 \otimes B \otimes B + c.p. & \text{if } D(be) = X(b)e + bD(e) \\
\end{array} \right\},$$

$$\Xi(bb') = X(b)b' + bX(b')$$

$$D(be) = X(e)b + eX(b)$$
where “c.p.” denotes cyclic permutations of the triple tensor product. Therefore the restricted double Schouten–Nijenhuis bracket \((6.22)\) corresponds, via the isomorphisms \((6.18)\) and \((6.19)\), to a double bracket on \(T_B \mathcal{A}_B(E)\) given on generators by

\[
\begin{align*}
[b_1, b_2]_{\mathcal{A}_B} &= 0, \\
[(X, \mathcal{D}), b]_{\mathcal{A}_B} &= X(b), \\
[b, (X, \mathcal{D})]_{\mathcal{A}_B} &= -\sigma(b)(X(b)), \\
[(X_1, \mathcal{D}_1), (X_2, \mathcal{D}_2)]_{\mathcal{A}_B} &= \Xi((X_1, \mathcal{D}_1)), \Xi((X_2, \mathcal{D}_2)) \Gamma_{1,0} + \Xi((X_1, \mathcal{D}_1)), \Xi((X_2, \mathcal{D}_2)) \Gamma_{r,0}.
\end{align*}
\]

6.4.3. The double Atiyah algebra as a twisted double Lie–Rinehart algebra.

**Proposition 6.9.** The 4-tuple \((\mathcal{A}_B(E), E, [-,-]_{\mathcal{A}_B}, \rho)\) is a twisted double Lie–Rinehart algebra, where the bracket is defined by \((6.23)\), and the anchor is

\[
\rho: \mathcal{A}_B(E) \longrightarrow \text{Der}_R B: (X, \mathcal{D}) \mapsto X.
\]

**Proof.** The 4-tuple \((\mathcal{A}_B(E), E, [-,-]_{\mathcal{A}_B}, \rho)\) satisfies the properties in Definition \(6.3\) due to the isomorphism \((6.18)\) and the fact that we can restrict the canonical double Schouten–Nijenhuis bracket on \(T_C \text{Der}_R C\) (which is a double Gerstenhaber algebra) to \((\text{Der}_R C)_{(0)}\) preserving the required properties. \(\blacksquare\)

6.5. The map \(\Psi\). Consider the setting of Framework\(6.1\). Here, we will construct a map \(\Psi\) of twisted double Lie–Rinehart algebras between \(A^2\) and \(\mathcal{A}_B(E_1)\), using the isomorphism \((6.18)\). Let \(a \in A^2, (b, e) \in C = B^\#(E_1[-1])\) (see \(6.3.2\)). Define

\[
\begin{align*}
\{a, (b, e)\}_\omega &= (\{a, b\}_\omega, \{a, e\}_\omega).
\end{align*}
\]

Then \(\{a, \cdot\}_\omega \in (\text{Der}_R C)_{(0)}\), because \(\{-,-\}_\omega\) is a double Poisson bracket of weight -2 and \(\{a, (b, e)\}_\omega = (X_a(b), D_a(e))\) (see \(6.3\) and \(6.4\)), so we can define a map

\[
\Psi_1: A^2 \longrightarrow (\text{Der}_R C)_{(0)}: a \mapsto T_a = \{a, \cdot\}_\omega.
\]

Given \(c \in C\), we write \(T_a(c) = T_a(c) \otimes T_a(c) \in C \otimes C\). Similarly, we define

\[
\Psi_2: E_1 \longrightarrow E_1: e \mapsto T_e := \{e, \cdot\}_\omega.
\]

**Proposition 6.10.** The pair \(\Psi = (\Psi_1, \Psi_2)\) is a morphism of twisted double Lie–Rinehart algebras.

**Proof.** We will partially adapt Lemma \(6.10\). Since \((A^2, E_1, \{-,-\}_{A^2}, X)\) is a twisted double Lie–Rinehart algebra, we can perform the following decomposition

\[
\begin{align*}
\{a_1, a_2\}_{A^2} &= \{a_1, a_2\}^l + \{a_1, a_2\}^r + \{a_1, a_2\}^m \\
&= \{a_1, a_2\}^l \otimes \{a_1, a_2\}^r + \{a_1, a_2\}^r \otimes \{a_1, a_2\}^r + \{a_1, a_2\}^m \otimes \{a_1, a_2\}^m,
\end{align*}
\]

with \(\{a_1, a_2\}^l \in A^2 \otimes B, \{a_1, a_2\}^r \in B \otimes A^2, \{a_1, a_2\}^m \in E_1 \otimes E_1\), and hence

\[
\begin{align*}
\{a_1, a_2\}^l, \{a_1, a_2\}^r, \{a_1, a_2\}^m \in A^2, \{a_1, a_2\}^r, \{a_1, a_2\}^m \in B, \{a_1, a_2\}^m, \{a_1, a_2\}^m \in E_1.
\end{align*}
\]

In view of Definition \(6.6\) we need to prove

\[
\Psi(\{a_1, a_2\}_{A^2}) = \{\Psi_1(a_1), \Psi_1(a_2)\}_{(0)} + \Psi_2(\{a_1, a_2\}^m) \otimes \{a_1, a_2\}^m + \{a_1, a_2\}^m \otimes \Psi_2(\{a_1, a_2\}^m).
\]
Claim 6.11. The bracket $\{ -, - \}_{0,0}$ is skew-symmetric and satisfies the double Jacobi identity.

Proof. Straightforward, because $\{ -, - \}_{0,0}$ is defined in terms of the double Poisson bracket $\{ -, - \}_\omega$ on $A$, that already satisfies the required properties. \qed

Let $c = (b, e) \in C$, with $b \in B, e \in E_1$. Let $\Sigma_{(123)}$ and $\Sigma_{(132)}$ the permutations in $S_3$ that acts on $(\mathcal{D}_{\mathcal{R}} C)_0 \otimes C \otimes C + c.p$. Then by Claim 6.11, we have the identity

$$0 = \left\{ \left\{ a_1, \left\{ a_2, c, a_1 \right\}_{\omega,0} \right\}_{\omega,0,L} + \Sigma_{(123)} \left\{ a_2, \left\{ c, a_1 \right\}_{\omega,0} \right\}_{\omega,0,L} + \Sigma_{(132)} \left\{ e, \left\{ a_1, a_2 \right\}_{A^2} \right\}_{\omega,0,L} \right\} . \tag{6.28}$$

The first summand in (6.28) can be written as

$$\left\{ \left\{ a_1, \left\{ a_2, c \right\}_{\omega,0} \right\}_{\omega,0,L} = \left\{ a_1, T_{a_2} (c) \right\}_{\omega,0,L} = \left\{ a_1, T_{a_2} (c) \right\}_{\omega,0} \otimes T_{a_2} (c) = (T_{a_1} \otimes \text{Id}_{C}) T_{a_1} (c). \tag{6.29}$$

Using the skew-symmetry of $\{ -, - \}_{\omega,0}$, we transform the second summand:

$$\left\{ a_2, \left\{ c, a_1 \right\}_{\omega,0} \right\}_{\omega,0,L} = - \left\{ a_2, \left( T_{a_1} (c) \right)^{\circ} \right\}_{\omega,0,L} = - \left\{ a_2, T_{a_1}^\nu (c) \right\}_{\omega,0} \otimes T_{a_1} \nu (c) = -(T_{a_2} \otimes \text{Id}_{C}) (T_{a_1} (c))^\nu. \tag{6.30}$$

Consequently,

$$\Sigma_{(123)} \left\{ a_2, \left\{ c, a_1 \right\}_{\omega,0} \right\}_{\omega,0,L} = - \Sigma_{(123)} \left( (T_{a_2} \otimes \text{Id}_{C}) (T_{a_1} (c))^\circ \right) = - \Sigma_{(123)} \Sigma_{(132)} \left( (\text{Id}_{C} \otimes T_{a_2}) T_{a_1} \right) = -(\text{Id}_{C} \otimes T_{a_2}) T_{a_1}. \tag{6.31}$$

To calculate the third summand, note first that $\{ -, - \}_{\omega,0} |_{A^2 \otimes A^2} = \{ -, - \}_\omega |_{A^2 \otimes A^2}$. Also, $\{ a_1, a_2 \}^\nu \in B$, and so $\left\{ e, \left\{ a_1, a_2 \right\}_{A^2}^{\nu} \right\}_{\omega} = 0$, as $\{ -, - \}_\omega$ is a double Poisson bracket of weight -2. Similarly, by (6.23), $\left\{ \left\{ a_1, a_2 \right\}_{A^2}^{\nu}, c \right\}_{\omega,0} = \left\{ \left\{ a_1, a_2 \right\}_{A^2}^{\nu}, c \right\}_{\omega}$. Hence

$$\left\{ e, \left\{ a_1, a_2 \right\}_{A^2} \right\}_{\omega,0,L}$$

$$= -(\left\{ \left\{ a_1, a_2 \right\}_{A^2}^{\nu}, c \right\}_{\omega,0})^\circ \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} + \left\{ \left\{ a_1, a_2 \right\}_{A^2}^{\nu}, c \right\}_{\omega,0} \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} \right\} = -(T_{a_1} \otimes \text{Id}_{C}) (T_{a_1} (c))^\nu. \tag{6.32}$$

Next,

$$\Sigma_{(132)} \left\{ e, \left\{ a_1, a_2 \right\}_{A^2} \right\}_{\omega,0L}$$

$$= - \Sigma_{(132)} \Sigma_{(12)} \left( T_{a_1,a_2} (c) \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} + T_{a_1,a_2} (e) \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} \right) = - \Sigma_{(123)} \left( T_{a_1,a_2} (c) \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} + T_{a_1,a_2} (e) \otimes \left\{ a_1, a_2 \right\}_{A^2}^{\nu} \right). \tag{6.33}$$
Therefore, paths in $B$ where the rows are the short exact sequences of $\mu$.

Summing up, from (6.28), applying (6.29), (6.30), (6.31), we obtain

$$0 = \left\{ a_1, \{ a_2, c \} \right\}_{\omega,0,L}(c) + \Xi_{(123)} \left\{ a_2, \{ c, a_1 \} \right\}_{\omega,0} + \Xi_{(132)} \left\{ c, \{ a_1, a_2 \} \right\}_{\omega,0,L}$$

$$= \left\{ T_{a_1}, T_{a_2} \right\}_{1,0}(c) - \left( T_{\{ a_1, a_2 \}^{\mu}}(c) \right) \otimes \left\{ a_1, a_2 \right\}^{\mu'}(c) \otimes \left\{ a_1, a_2 \right\}^{\mu''}(c) \right\}_{\omega,0,L}$$

Therefore

$$\left\{ T_{a_1}, T_{a_2} \right\}_{r,0} = - \left\{ T_{a_2}, T_{a_1} \right\}_{1,0}$$

$$= - \left( T_{\{ a_2, a_1 \}^{\mu}}(c) \otimes \left\{ a_2, a_1 \right\}^{\mu'}(c) \right) \otimes \left\{ a_2, a_1 \right\}^{\mu''}(c) \right\}_{\omega,0,L}$$

Finally, (6.27) is the sum of (6.32) and (6.33), as required.

6.6. The isomorphism between $A^2$ and $A\oplus (E_1)$. Consider the setting of Framework 6.1. Here, we will show that the map $\Psi$ constructed in §6.5 is an isomorphism of twisted double Lie–Rinehart algebras. As a consequence, this will imply the following non-commutative version of [27, Theorem 3.3].

**Theorem 6.12.** Let $(A, \omega)$ be the pair consisting of the graded path algebra of a double quiver $P$ of weight 2, and the bi-symplectic form $\omega \in \text{DR}_2^0(A)$ of weight 2 defined in §4.4. Let $B$ be the path algebra of the weight 0 subquiver of $P$. Then $(A, \omega)$ is completely determined by the pair $(E_1, (\cdot, \cdot))$ consisting of the $B$-bimodule $E_1$ with basis given by paths in $P$ of weight 1, and the symmetric non-degenerate pairing

$$(\cdot, \cdot) := \{ \cdot, \cdot \}_{\omega,0} |_{E_1 \otimes E_1} : E_1 \otimes E_1 \to B \otimes B.$$

To prove that $\Psi$ is an isomorphism, we will construct the following commutative diagram, where the rows are the short exact sequences of $B$-bimodules given by the definitions of
the following tasks:

(i) Construction of an isomorphism \( \End_{B^p}(E_1) \cong E_1 \otimes_B E_1 \oplus E_1 \otimes_B E_1 \) (Lemma 6.13).
(ii) Description of a basis of \( \mathfrak{ad}_B(E_1) \) (Lemma 6.13).
(iii) Description of \( \Psi|_{E_1 \otimes_B E_1} \) in the basis of (ii).

6.7. Explicit description of \( \End_{B}(E_1) \).

**Lemma 6.13.** There is a canonical isomorphism

\[
\End_{B}(E_1) \cong E_1 \otimes_B E_1 \oplus E_1 \otimes_B E_1.
\]

**Proof.** We will need the canonical isomorphisms

\[
\Hom_B(Be_i, M) \cong e_i M: \quad f \mapsto f(e_i),
\]

\[
\Hom_{B^p}(e_i B, N) \cong Ne_i: \quad g \mapsto g(e_i),
\]

with inverse isomorphisms

\[
e_i M \cong \Hom_B(Be_i, M): \quad e_i m \mapsto f_m,
\]

\[
Ne_i \cong \Hom_{B^p}(e_i B, N): \quad ne_i \mapsto g_n,
\]

where, for all \( b \in B \),

\[
f_m: Be_i \longrightarrow M: \quad be_i \longmapsto f_m(be_i) = be_im,
\]

\[
g_n: e_i B \longrightarrow N: \quad ne_i \longmapsto g_n(e_i b) = ne_ib.
\]

Since \( E_1 = \bigoplus_{|c|=1} BcB = \bigoplus_{|c|=1} Be_{h(c)} \otimes e_{l(c)} B \), we have

\[
E_1 \otimes_B E_1 \cong \bigoplus_{|c|=|d|=1} Be_{h(c)} \otimes e_{l(c)} Be_{h(d)} \otimes e_{l(d)} B \cong \bigoplus_{|c|=|d|=1} BcBdB.
\]
where we sum over arrows $c,d$ of weight 1. This explicit description of $E_1 \otimes_B E_1$ in the setting of quivers enables the following explicit description of $\text{Hom}_{B^e}(E_1, E_1 \otimes B)$:

$$\text{Hom}_{B^e}(E_1, E_1 \otimes B) \cong \bigoplus_{|c|=|d|=1} \text{Hom}_{B^e} \left( B e_{h(c)} \otimes e_{t(c)} B, B e_{h(d)} \otimes e_{t(d)} B \otimes B \right);$$

$$\cong \bigoplus_{|c|=|d|=1} \text{Hom}_B \left( B e_{h(c)}, B e_{h(d)} \otimes e_{t(d)} B \right) \otimes \text{Hom}_{B^e} \left( e_{t(c)} B, B \right);$$

$$\cong \bigoplus_{|c|=|d|=1} \left( e_{h(c)} B e_{h(d)} \otimes e_{t(d)} B \right) \otimes B e_{t(c)};$$

$$\cong \bigoplus_{|c|=|d|=1} B e_{t(c)} \otimes e_{h(c)} B e_{h(d)} \otimes e_{t(d)} B;$$

$$\cong \bigoplus_{|c|=|d|=1} B c^* B d B \cong \bigoplus_{|c|=|d|=1} B c B d B,$$

where we used (6.36). Also, the last isomorphism is due to the fact that $\mathcal{T}$ is a doubled graded quiver of weight 2; hence there exists an isomorphism between the set of arrows $\{a\}$ such that $|a| = 1$ and the set of reverse arrows $\{a^*\}$. In conclusion,

$$\text{Hom}_{B^e}(E_1, E_1 \otimes B) \cong \bigoplus_{|c|=|d|=1} B c B d B \cong E_1 \otimes_B E_1.$$ 

Similarly,

$$\text{Hom}_{B^e}(E_1, B \otimes E_1) \cong \bigoplus_{|c|=|d|=1} \text{Hom}_{B^e} \left( B e_{h(c)} \otimes e_{t(c)} B, B \otimes B e_{h(d)} \otimes e_{t(d)} B \right);$$

$$= \bigoplus_{|c|=|d|=1} \text{Hom}_B \left( B e_{h(c)}, B \right) \otimes \text{Hom}_{B^e} \left( e_{t(c)} B, B e_{h(d)} \otimes e_{t(d)} B \right);$$

$$\cong \bigoplus_{|c|=|d|=1} e_{h(c)} B \otimes B e_{h(d)} \otimes e_{t(d)} B e_{t(c)};$$

$$\cong \bigoplus_{|c|=|d|=1} B e_{h(d)} \otimes e_{t(d)} B e_{t(c)} \otimes e_{h(c)} B;$$

$$\cong \bigoplus_{|c|=|d|=1} B d B c^* B \cong \bigoplus_{|c|=|d|=1} B d B c B,$$

and we obtain that $\text{Hom}_{B^e}(E_1, B \otimes E_1) \cong \bigoplus_{|c|=|d|=1} B d B c B \cong E_1 \otimes_B E_1$. 

Let $a, b$ be arrows of weight 1, and $r, q, p$ paths in $Q$ that compose, that is,

$$h(p) = t(b), \quad h(b) = t(q), \quad h(q) = h(a), \quad t(a) = h(r).$$

Then we consider the path $ra^*qbp \in \bigoplus_{|c|=|d|=1} B c^* B d B \cong E_1 \otimes_B E_1$. By Lemma 6.13 we need to determine an explicit basis of the $B$-bimodule $\text{End}_{B^e}(E_1)$; the image of the path $ra^*qbp$ under the isomorphism (6.40) (resp. (6.39)) will be denoted $[ra^*qbp]_2 \in$
Lemma 6.14. A basis of \( \text{Hom}_{B^e}(E_1, B \otimes E_1) \) (resp. \([ra^* qbp]_1 \in \text{Hom}_{B^e}(E_1, E_1 \otimes B)\)). Focusing on (6.39),

\[
\bigoplus_{|c|=|d|=1} B e_{t(c)} \otimes e_{h(c)} B e_{h(d)} \otimes e_{t(d)} B 
\]

\[
\cong
\bigoplus_{|c|=|d|=1} e_{h(c)} B e_{h(d)} \otimes e_{t(d)} B \otimes B e_{t(c)}
\]

\[
\bigoplus_{|c|=|d|=1} \text{Hom}_B(B e_{h(c)}, B e_{h(d)} \otimes e_{t(d)} B) \otimes \text{Hom}_{B^e}(e_{t(c)} B, B)
\]

where, by (6.37), we have for \( s, s' \in B \),

\[
f_{e_{h(a)} q e_{h(b)} p} : B e_{h(c)} \rightarrow B e_{h(d)} \otimes e_{t(d)} B : \quad s e_{h(a)} \mapsto s e_{h(a)} q e_{h(b)} \otimes e_{t(b)} p;
\]

\[
f_{r e_{t(a)}}' : e_{t(c)} B \rightarrow B : \quad e_{t(a)} s' \mapsto r e_{t(a)} s'.
\]

Hence, the first isomorphism in (6.39) enables us to write \([ra^* qbp]_1 \in \text{Hom}_{B^e}(E_1, E_1 \otimes B)\):

\[
[ra^* qbp]_1 : E_1 \longrightarrow B E_1 \otimes B B : \quad s e_{t(a)}' \mapsto \delta_{a c}(s e_{h(a)} q e_{h(b)} \otimes e_{t(b)} p) \otimes (r e_{t(a)} s'). \quad (6.42)
\]

Finally, a generic element of a basis of \( \text{End}_{B^e}(E_1) \), in view of (6.42) and (??), shall be written as

\[
f = \sum_{|a|=|b|=1} \alpha_{ra^* qbp}[ra^* qbp]_1 + \alpha_{ra^* qbp}'[ra^* qbp]_2,
\]

where \( \alpha_{ra^* qbp}, \alpha_{ra^* qbp}' \in k \).

6.8. Description of a basis of \( \text{ad}_{B^e}(E_1) \). In this subsection, we will describe a basis of \( \text{ad}_{B^e}(E_1) \). Note that \( e \in \text{ad}_{B^e}(E_1) \) if and only if \( e \in \text{End}_{B^e}(E_1) \) (see (6.7)) satisfies the following additional condition for all \( a, b \) arrows of weight 1:

\[
\langle a, f(b) \rangle_L = -\sigma_{(123)} \langle b, f(a)^\circ \rangle_L,
\]

where \( \sigma_{(123)} : B \otimes B \otimes B \rightarrow B \otimes B \otimes B : b_1 \otimes b_2 \otimes b_3 \mapsto b_3 \otimes b_2 \otimes b_1 \).

Lemma 6.14. A basis of \( \text{ad}_{B^e}(E_1) \) consists of the elements

\[
\varepsilon(b)[ra^* qbp]_2 - \varepsilon(a)[ra^* qbp]_1,
\]

where \( p, q, r \) are paths in \( Q \) and \( a, b \) arrows of weight 1, which satisfy the following compatibility conditions:

\[
h(p) = t(b), \quad h(b) = t(q), \quad h(q) = h(a), \quad t(a) = t(r).
\]

Proof. To prove (6.44), we will write explicitly \( f(b) \) and \( f(a)^\circ \) for \( a, b \) some arrows of weight 1. By (6.7),

\[
f(b) = \sum_{|c|=|d|=1} \alpha_{rc^* qdp}[rc^* qdp]_1(b) + \alpha_{rc^* qdp}'[rc^* qdp]_2(b)
\]

\[
= \sum_{|d|=1} (\alpha_{rb^* qdp} e_{h(b)} q e_{h(d)} \otimes e_{t(d)} p \otimes r e_{t(b)} + \alpha_{rb^* qdp}' e_{h(b)} p \otimes r e_{t(a)} \otimes e_{h(a)} q e_{t(b)}).
\]
Next, we can compute the left hand side of (6.44) (using (6.6) and (6.7)):

\[
\langle a, f(b) \rangle_L = \langle a, \sum_{|d| = 1} \alpha_{rb^*qdp} e_{h(b)} q e_{h(d)} \otimes e_{t(d)} p \otimes r e_{t(b)} \rangle
= \varepsilon(a) \alpha_{rb^*qdp} e_{h(b)} q e_{t(a)} \otimes e_{h(a)} p \otimes r e_{t(b)}.
\]

Let \( h \in E_1 \) and \( s, s' \in B \). Then using the maps

\[
[r c^* qdp]_1^0: E_1 \to B_B \otimes_B E_1: shs' \mapsto \delta_{dh} (r e_{t(c)} s') \otimes (s e_{h(c)} q e_{h(d)} \otimes e_{t(d)} p)
\]

and

\[
[r dq c^* p]_2^0: E_1 \to (E_1)_B \otimes_B B: shs' \mapsto \delta_{dh} (r e_{t(c)} \otimes e_{h(c)} q e_{t(d)} s') \otimes (s e_{h(d)} p),
\]

and the fact that \((-)^\circ\) is linear, we can calculate

\[
f(a) = \left( \sum_{|c| = |d| = 1} (\alpha_{rc^*qdp} [rc^*qdp]_1(a) + \alpha'_{rc^*qdp} [rc^*qdp]_2(a)) \right)^\circ
= \sum_{|c| = |d| = 1} (\alpha_{rc^*qdp} [rc^*qdp]_1(a) + \alpha'_{rc^*qdp} [rc^*qdp]_2(a))
= \sum_{|c| = 1} \alpha_{rc^*qdp} r e_{t(c)} \otimes e_{h(c)} q e_{h(a)} \otimes e_{t(a)} p + \alpha'_{rc^*qdp} r e_{t(c)} \otimes e_{h(c)} q e_{t(a)} \otimes e_{h(a)} p.
\]

Then we compute \( \sigma_{(132)} \langle b, f(a) \rangle_L \):

\[
\sigma_{(132)} \langle b, f(a) \rangle_L = \sigma_{(132)} \sum_{|c| = 1} \langle b, \alpha'_{rc^*qdp} r c^* q e_{t(a)} \rangle \otimes e_{h(a)} p
= \sigma_{(132)} \alpha'_{rb^*qdp} \varepsilon(b) r e_{t(b)} \otimes e_{h(b)} q e_{t(a)} \otimes e_{h(a)} p
= e_{h(b)} q e_{t(a)} \otimes e_{h(a)} p \otimes \alpha'_{rb^*qdp} \varepsilon(b) r e_{t(b)}.
\]

Therefore, by (6.44), we obtain the condition

\[
\alpha_{rb^*qdp} = -\frac{\varepsilon(b)}{\varepsilon(a)} \alpha'_{rb^*qdp},
\]

from which we conclude that (6.35) is a basis of \( \mathfrak{ad}_{B^*}(E_1) \).

Observe that the above descriptions of \( \mathfrak{ad}_{B^*}(E_1) \) and \( E_1 \otimes_B E_1 \) given in Lemma 6.14 and (6.38) provide an isomorphism between these \( B \)-bimodules:

\[
E_1 \otimes_B E_1 \to \mathfrak{ad}_{B^*}(E_1)
ra^*qbp \mapsto \varepsilon(b)[ra^*qbp]_1 - \varepsilon(a)[ra^*qbp]_2.
\]

6.9. The isomorphism \( \Psi|_{E_1 \otimes_B E_1} \). We can now compute \( \Psi|_{E_1 \otimes_B E_1} \) at the basis elements.

Lemma 6.15. \( \Psi \) restricts to an isomorphism \( \Psi: E_1 \otimes_B E_1 \to \mathfrak{ad}_{B^*}(E_1) \), given by

\[
\Psi(ra^*qbp) = \varepsilon(b)[ra^*qbp]_2 - \varepsilon(a)[ra^*qbp]_1,
\]

where \( p, q, r \) are paths in \( Q \) and \( a, b \) arrows of weight 1, that compose, that is,

\[
h(p) = t(b), \quad h(b) = t(q), \quad h(q) = h(a), \quad t(a) = t(r).
\]
Proof. First, we note that \( \{ -, - \}_\omega \) is a double Poisson bracket of weight -2, so \( \mathbb{X}_{ra^*qbp} (b') = 0 \) for all \( b' \in B \), because by a simple application of the Leibniz rule and (6.3),
\[
\mathbb{X}_{ra^*qbp} (b') = -\sigma_{(12)} (ra^* \{ b', qbp \}_\omega + \{ b', ra^* \}_\omega qbp) = 0.
\]
We can apply similarly the (graded) Leibniz rule when \( c \) is an arrow of weight 1:
\[
\mathcal{D}_{ra^*qbp} (c) = \{ ra^*qbp, c \}_\omega = -\sigma_{(12)} (c, ra^*qbp)_\omega = -\sigma_{(12)} (ra^*q^{P,b}p + r^{P,b}a^*)_\omega qbp.
\]
To compute \( \{ c, b \}_\omega \) and \( \{ c, a^* \}_\omega \) we will need an explicit description of the differential double Poisson bracket \( P \in (T_A \text{Der}_B A)^2 \) (see (7.12)). Recall that in our convention, arrows compose from to left. Applying now Propositions 4.12 and 4.5, we compute
\[
\{ c, b \}_\omega = \{ c, \{ P, b \} \}_L = - \left\{\left\{ c, \varepsilon (b) \frac{\partial}{\partial b} \right\} \right\}_\omega = -\varepsilon (b) e_h(b) \otimes e_t(b),
\]
where \( \{ -, - \} \) is the associated bracket to \( \{ -, -, \}_\omega \). Replacing \( b \) by \( a^* \) in (6.48), we obtain
\[
\{ c, a^* \}_\omega = \varepsilon (a) e_t(a) \otimes e_h(a).
\]
Using now (6.48) and (6.49) in (6.47), we conclude
\[
\mathcal{D}_{ra^*qbp} (c) = -\sigma_{(12)} (\varepsilon (b) ra^* q e_h(b) \otimes e_t(b) p - \varepsilon (a) re_t(a) \otimes e_h(a) qbp) = \varepsilon (b) e_t(b) p \otimes re_t(a) \otimes e_h(a) q e_h(b) - \varepsilon (a) e_h(a) q e_h(b) \otimes e_t(b) p \otimes re_t(a)
\]
Therefore, we obtain the formula in the statement of Lemma 6.15.

\section{Non-commutative Courant algebroids}

In this section, we determine the non-commutative geometric structures associated to a bi-symplectic \( NQ \)-algebra, namely, a graded bi-symplectic tensor \( N \)-algebra \((A, \omega)\) equipped with a homological double derivation \( Q \). We will focus on bi-symplectic \( NQ \)-algebras of weight 2 attached to a double graded quiver \( \overline{P} \), obtaining in this case a non-commutative analogue of [27, Theorem 4.5], expressible in terms of the so-called double Courant algebroids over the path algebra of the weight 0 subquiver of \( \overline{P} \), as introduced in (7.1).

\subsection{Definition of double Courant algebroids}

Recall that, in the the setting of Framework 6.1, the data of the bi-symplectic tensor \( N \)-algebra \((A, \omega)\) of weight 2 is equivalent to a pair \((E, \langle -, - \rangle)\), where \( E := E_1 \) is a projective finitely generated \( B \)-bimodule and \( \langle -, - \rangle \) is the symmetric non-degenerate pairing defined in (6.6) (see Theorem 6.12).

\begin{definition}
Let \( R \) be a finite-dimensional semisimple associative algebra and \( B \) a smooth \( R \)-algebra. A \textit{double pre-Courant algebroid over} \( B \) is a 4-tuple \((E, \langle -, - \rangle, \rho, [\ [-, - ] ]\) consisting of a projective finitely generated \( B \)-bimodule \( E \) endowed with a symmetric non-degenerate pairing, called the \textit{inner product},
\[
\langle -, - \rangle : E \otimes E \to B \otimes B,
\]
a \( B \)-bimodule morphism
\[
\rho : E \to \text{Der}_R B,
\]
\end{definition}
called the anchor, and an operation
\[ [-, -] : E \otimes E \to E \otimes B \oplus B \otimes E, \]  
(7.2)
called the double Dorfman bracket, that is \( R \)-linear for the outer (resp. inner) bimodule structure on \( B \otimes B \) in the second (resp. first) argument. This data must satisfy the following conditions:
\[
\begin{align*}
[e_1, be_2] &= \rho(e_1)(b)e_2 + b[e_1, e_2], \\
[e_1, e_2b] &= e_2\rho(e_1)(b) + [e_1, e_2]b, \\
\rho^\vee d(\langle e_2, e_2 \rangle) &= 2([e_2, e_2] + [e_2, e_2]^\sigma), \\
\rho(e_1)(\langle e_2, e_2 \rangle) &= \langle [e_1, e_2], e_2 \rangle_L + \langle e_2, [e_1, e_2] \rangle_R
\end{align*}
\]
(7.3a,b,c,d)
for all \( b \in B \) and \( e_1, e_2 \in E \). If the bracket \([-,-]\) satisfies the double Jacobi identity,
\[
[e_1, [e_2, e_3]]_L = [e_2, [e_1, e_3]]_R + [[e_1, e_2], e_3]_L,
\]
(7.4)
for all \( e_1, e_2, e_3 \in E \), then \( (E, \langle -,-\rangle, \rho, [-,-]) \) is called a double Courant algebroid.

As usual, the products in (7.3a) and (7.3b) are taken with respect to the outer bimodule structure. In (7.3c), the universal derivation \( d : B \to \Omega^1_R B \) acts on tensor products by the Leibniz rule. Furthermore, \( \rho^\vee : \Omega^1_R B \to E \) is composite
\[
\rho^\vee : \Omega^1_R B \xrightarrow{\text{bidual}} (\text{Der}_R B)^{\text{Hom}_{\mathcal{B}^e}(\rho, B^e)} \xrightarrow{E^\vee} \cong E,
\]
where the first map was defined in (3.18), and the isomorphism between \( E \) and \( E^\vee \) is induced by the inner product \( \langle -,-\rangle \). As for commutative Courant algebroids [24], given \( b \in B, e \in E \), we use of the identification
\[
\langle e, \rho^\vee db \rangle = \rho(e)(b).
\]
Finally, we use the notation \( (\cdot)^\sigma := \sigma_{(12)}(\cdot) \) (see (4.3)).

Given \( e_1, e_2 \in E \), we perform the following decomposition of the double Dorfman bracket:
\[
\begin{align*}
\llbracket e_1, e_2 \rrbracket &= [e_1, e_2]^\prime + [e_1, e_2]^\sigma \\
&= [e_1, e_2]^\prime \otimes [e_1, e_2]^\prime + [e_1, e_2]^\prime \otimes [e_1, e_2]^\prime \otimes [e_1, e_2]^\prime \otimes [e_1, e_2]^\prime \otimes [e_1, e_2]^\prime \in E \otimes B \oplus B \otimes E,
\end{align*}
\]
with \([e_1, e_2]^\prime, [e_1, e_2]^\prime \in E \) and \([e_1, e_2]^\prime, [e_1, e_2]^\prime \in B \). Then in (7.3c),
\[
\begin{align*}
[e_2, e_2]^\prime &= \sigma_{(12)}([e_2, e_2]) = [e_2, e_2]^\prime \otimes [e_2, e_2]^\prime + [e_2, e_2]^\prime \otimes [e_2, e_2]^\prime \in B \otimes E \oplus E \otimes B.
\end{align*}
\]
In (7.3d), if \( \langle e_2, e_2 \rangle = \langle e_2, e_2 \rangle^\prime \otimes \langle e_2, e_2 \rangle^\prime \in B \otimes B \), \( \rho(e_1)(\langle e_2, e_2 \rangle) = \rho(\langle e_2, e_2 \rangle^\prime) \otimes \langle e_2, e_2 \rangle^\prime \).

The notation \( \langle -,- \rangle_L \) and \( \langle -,- \rangle_R \) was defined in (6.8) and (6.9), respectively.

Finally, in the double Jacobi identity (7.4), we used the following extensions of the double Dorfman bracket
\[
\begin{align*}
[e_1, e_2 \otimes b]_L &= [e_1, e_2] \otimes b, \\
[e_1, b \otimes e_2]_L &= [e_1, b] \otimes e_2, \\
[e_1, e_2 \otimes b]_R &= e_2 \otimes [e_1, b], \\
[e_1, b \otimes e_2]_R &= b \otimes [e_1, e_2], \\
[e_1 \otimes b, e_2]_L &= [e_1, e_2] \otimes b, \\
[b \otimes e_1, e_2]_L &= [b, e_2] \otimes_1 e_2,
\end{align*}
\]
where in the last two identities, we used the inner \( \otimes \)-product in (2.4).
7.2. Bi-symplectic NQ-algebras. Let \( R \) be an associative algebra, and \( B \) an \( R \)-algebra. We will add now more structure to the data of Definition 4.1.

**Definition 7.2.** (i) An associative NQ-algebra \( (A,Q) \) over \( B \) is an associative \( \mathbb{N} \)-algebra \( A \) of weight \( N \) over \( B \) endowed with a double derivation \( Q : A \to A \otimes A \) of weight +1 which is homological, that is,

\[
\{ Q, Q \} = 0,
\]

where \( \{-,-\} \) is the double Schouten–Nijenhuis bracket.

(ii) A tensor NQ-algebra \( (A,Q) \) over \( B \) is an associative NQ-algebra over \( B \) whose underlying associative \( \mathbb{N} \)-algebra is a tensor \( \mathbb{N} \)-algebra over \( B \).

(iii) A bi-symplectic NQ-algebra \( (A,\omega,Q) \) of weight \( N \) is a tensor NQ-algebra \( (A,Q) \) over \( B \), endowed with a graded bi-symplectic form \( \omega \in \text{DR}^2_R(A) \) of weight \( N \), such that

(a) the underlying tensor \( \mathbb{N} \)-algebra \( A \) over \( B \), equipped with the graded bi-symplectic form \( \omega \), is a bi-symplectic tensor \( \mathbb{N} \)-algebra of weight \( N \) over \( B \);

(b) the homological double derivation \( Q \) is bi-symplectic, that is,

\[
\mathcal{L}_Q \omega = 0,
\]

where \( \mathcal{L}_Q \) is the reduced Lie derivative.

7.3. Bi-symplectic NQ-algebras and double Courant algebroids. We will now explore the interplay between bi-symplectic NQ-algebras of weight 2 and double Courant algebroids. As for commutative manifolds, by Lemma 4.9 a bi-symplectic double derivation is a Hamiltonian double derivation, so \( Q = \{ S, \omega \} \), where \( \{-,-\} \) is the double Poisson bracket of weight -2 induced by \( \omega \) and \( S \in A^3 \) is a ‘cubic’ non-commutative polynomial. Then \( S \) encodes the structure of a double pre-Courant algebroid, recoverable via a non-commutative version of derived brackets and, if in addition, \( \{ S,S \}\omega = 0 \) (here \( \{-,-\}\omega \) denotes the associated bracket to \( \{-,-\} \)), then we will have a double Courant algebroid.

Let \( (A,\omega,Q) \) be a bi-symplectic NQ-algebra of weight 2 over \( B \). In particular, \( Q \) is a homological double derivation, that is,

\[
|Q| = 1, \quad \mathcal{L}_Q \omega = 0, \quad \{ Q, Q \} = 0,
\]

where \( \{-,-\} \) is the graded double Schouten–Nijenhuis bracket on \( T_A \text{Der}_R A \).

**Lemma 7.3.** The identity \( \{ Q, Q \} = 0 \) is equivalent to \( \{ S,S \}\omega = 0 \).

**Proof.** By Lemma 4.10(ii), \( \iota_Q \omega = dS \) for some \( S \in A^1 \). It is easy to see that \( S \in A^3 \) because \( \{\{ -,-\}\omega \} = -2 \), whereas \( |Q| = +1 \). For all \( a \in A \), we see that \( \{ S,a \}\omega = Q(a) \) implies \( |S| = |Q| - \{\{ -,-\}\omega \} = 3 \), that is, \( S \in A^3 \). Now, the identity \( H_{H_{a,b},\omega} = \{ H_a, H_b \} \) of Proposition 4.10 applied to \( a = b = S \) gives

\[
H_{\{ S,S \},\omega} = \{ Q, Q \}.
\]

Therefore the identity \( \{ Q, Q \} = 0 \) is equivalent to \( \{ S,S \}\omega \in B \otimes B \) because, by 4.19, \( H_{\{ S,S \},\omega} = 0 \) implies \( 0 = d(\{ S,S \}\omega) = (d\{ S,S \}'\omega) \otimes \{ S,S \}''\omega + \{ S,S \}''\omega \otimes (d\{ S,S \}'\omega) \), which implies \( \{ S,S \}'\omega, \{ S,S \}''\omega \in R \). Finally, \( |\{ S,S \}\omega | = 0 \), as \( B \) is an associative \( R \)-algebra. However, \( \{ S,S \}\omega \) has weight 4 because \( |S| = 3 \), so \( \{ S,S \}\omega = 0 \). 

By the following result, double pre-Courant algebroids can be recovered using non-commutative derived brackets.
Proposition 7.4. Every weight 3 function \( S \in A^3 \) induces a double pre-Courant algebroid structure on \((E, (-, -))\), given by

\[
\rho(e_1)(b) := \{\{S, e_1\}_\omega, b\}_\omega, \quad \text{(7.5a)}
\]
\[
\begin{bmatrix} e_1, e_2 \end{bmatrix} := \{\{S, e_1\}_\omega, e_2\}_\omega, \quad \text{(7.5b)}
\]

for all \( b \in B \) and \( e_1, e_2 \in E \). Here, \( \{\cdot, \cdot\}_\omega = m \circ \{\cdot, \cdot\} \) is the (graded) associated bracket in \( A \) (see (1.13)).

Remark 7.5. In this section, we will need the graded associated bracket \( \{\cdot, \cdot\} := \{\cdot, \cdot\}_\omega \), and particularly, the graded version of (4.8) (cf. [3]), namely,

\[
\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\}. \tag{7.6}
\]

Recall also that \( \{a, -\} \) acts on tensors by \( \{a, u \otimes v\} := \{a, u\} \otimes v + u \otimes \{a, v\} \).

Proof. To simplify the notation, the double Poisson bracket of weight -2 induced by the bi-symplectic form \( \omega \) will be denoted \( \{\cdot, \cdot\} := \{\cdot, \cdot\}_\omega \), and its associated bracket \( \{\cdot, \cdot\} := \{\cdot, \cdot\}_\omega \). To prove (7.3a) in Definition 7.1, we will use the fact that \( \{\cdot, \cdot\} \) is a double derivation in its second argument with respect to the outer structure, so

\[
\begin{aligned}
\{e_1, be_2\} &= \{\{S, e_1\}, be_2\} \\
&= b\{\{S, e_1\}, e_2\} + \{\{S, e_1\}, b\}e_2 \\
&= b[e_1, e_2] + \rho(e_1)(b)e_2.
\end{aligned}
\]

Similarly, we can prove that (7.3a) holds. To prove (7.3c), first note that \( \{\{e_1, \{e_2, e_2\}_\omega\}'', \{e_2, e_2\}_\omega\}' \) holds. Then, by (1.7),

\[
0 = \{\{S, \{e_1, \{e_2, e_2\}_\omega\}'', \{e_2, e_2\}_\omega\}'\}'
\]

\[
= \{\{\{S, e_1\}', \{e_2, e_2\}_\omega\}'\} \otimes \{\{S, e_2\}_\omega\}'' + \{\{S, \{e_1, \{e_2, e_2\}_\omega\}'\}', \{\{S, e_2\}_\omega\}''\}
\]

\[
= (\{\{S, e_1\}', \{e_2, e_2\}_\omega\}'\} - \{\{S, \{e_2, e_2\}_\omega\}'\}) \otimes \{\{S, e_2\}_\omega\}''
\]

\[
= \{\{\{S, e_1\}', \{e_2, e_2\}_\omega\}'\}L - \{\{e_1, \{\{S, e_2\}_\omega\}''\}, \{\{S, e_2\}_\omega\}''\}
\]

\[
= \{\{S, e_1\}', \{e_2, e_2\}_\omega\}'\}L - \{e_1, \{\{S, e_2\}_\omega\}''\}L
\]

\[
= \{\{S, e_1\}, \{e_2, e_2\}_\omega\}_R - \{e_1, \{\{S, e_2\}_\omega\}''\}_R
\]

\[
= \{\{S, e_1\}, \{e_2, e_2\}_\omega\}_R - \{e_1, \{\{S, e_2\}_\omega\}''\}_R,
\]

where \( \{\cdot, \cdot\}'' = \{\cdot, \cdot\}'' \otimes \{\cdot, \cdot\}'' \). By definition, if \( e, e' \in E \), \( \{e, e'\}' = \langle e, e' \rangle \), so

\[
\rho(e_1)(\{e_2, e_2\}'') \otimes \{e_2, e_2\}'' = \langle e_1, [e_2, e_2] + [e_2, e_2]'' \rangle.
\]

Next, since \( \langle \cdot, \cdot \rangle \) is non-degenerate, the identity

\[
\langle e_1, \rho''(d\{e_2, e_2\}') \otimes \{e_2, e_2\}''' \rangle_L = \langle e_1, [e_2, e_2] + [e_2, e_2]'' \rangle_L
\]

implies that

\[
\rho''(d\{e_2, e_2\}') \otimes \{e_2, e_2\}''' = [e_2, e_2] + [e_2, e_2]''.
\]

Similarly, the identity \( \{e_2, e_2\}''' \otimes \{\{e_1, \{e_2, e_2\}'''\}''' \} = 0 \) implies that

\[
0 = \{\{S, e_1\}, \{e_2, e_2\}'''\}''' \otimes \{\{S, e_2\}_\omega\}'' \otimes \{\{S, e_2\}_\omega\}'' \otimes \{\{S, e_2\}_\omega\}''\}
\]

which is equivalent to

\[
\langle e_2, e_2 \rangle' \otimes \rho''(d\{e_2, e_2\}'') = [e_2, e_2] + [e_2, e_2]''.
\]

The sum of (7.7) and (7.8) gives (7.3c).
Finally, the key fact needed to prove (7.3d) is the double Jacobi identity for \(\{ -, - \}\):
\[
\rho(e_1)(\langle e_2, e_2 \rangle') \otimes \langle e_2, e_2 \rangle'' = \{ \{ S, e_1 \}, \{ e_2, e_2 \} \} \otimes \{ e_2, e_2 \}
\]
\[
= \{ \{ S, e_1 \}, \{ e_2, e_2 \} \}_L
\]
\[
= \{ \{ \{ S, e_1 \}, e_2 \}, e_2 \}_L + \{ e_2, \{ S, e_1 \}, e_2 \}_R
\]
\[
= \langle [e_1, e_2], e_2 \rangle_L + \langle e_2, [e_1, e_2] \rangle_R.
\]

In Lemma 7.3 we showed that the fact that the identity \(\{ Q, Q \} = 0\) is equivalent to \(\{ S, S \} = 0\). In the next proposition, we prove that the weaker condition \(\{ S, S \} = 0\) implies the double Jacobi identity (7.3). As before, \(\{ -, - \}\) is the associated bracket.

**Proposition 7.6.** If \(\{ S, S \} = 0\) then the double Jacobi identity (7.3) holds.

**Proof.** To simplify the notation, we set \(\{ -, - \} := \{ -, - \}\). By (7.3d),
\[
[e_1, [e_2, e_3]]_L = \{ \{ S, e_1 \}, \{ \Theta, e_2 \}, e_3 \}_L,
\]
where the double Jacobi identity implies
\[
\{ \{ S, e_1 \}, \{ \Theta, e_2 \}, e_3 \}_L = \{ \{ S, e_1 \}, \{ S, e_2 \}, e_3 \}_L + \{ \{ S, e_2 \}, \{ S, e_1 \}, e_3 \}_R
\]

Regarding the term \(\{ \{ S, e_1 \}, \{ S, e_2 \}, e_3 \}_L\), it follows from (1.7) that
\[
\{ \{ S, e_1 \}, \{ S, e_2 \} \} = \{ S, \{ S, e_1 \}, e_2 \} - \{ \{ S, S \}, e_1 \}, e_2 \}.
\]

Since \((A, \{-, -\})\) is a graded Loday algebra, by (7.6),
\[
\{ S, \{ S, e_1 \} \} = \frac{1}{2} \{ S, S \}, e_1 \}
\]

Thus, since \(\{ \Theta, \Theta \} = 0\) by hypothesis, we obtain the identity
\[
\{ \{ S, e_1 \}, \{ S, e_2 \} \}, e_3 \}_L = \{ S, \{ S, e_1 \}, e_2 \}, e_3 \}_L
\]
\[
= [[e_1, e_2], e_3]_L.
\]

Putting all together
\[
[e_1, [e_2, e_3]]_L = \{ \{ S, e_1 \}, \{ S, e_2 \} \}, e_3 \}_L + \{ \{ S, e_2 \}, \{ S, e_1 \}, e_3 \}_R
\]
\[
= [[e_1, e_2], e_3]_L + \{ \{ S, e_2 \}, [e_1, e_3] \}_R
\]
\[
= [[e_1, e_2], e_3]_L + \{ e_2, [e_1, e_3] \}_R.
\]

so (7.3) follows. □

In conclusion, we have proved the following.

**Theorem 7.7.** Let \((A, \omega, Q)\) be a bi-symplectic NQ-algebra of weight 2, where \(A\) is the graded path algebra of a double quiver \(\overline{P}\) of weight 2 endowed with a bi-symplectic form \(\omega \in DR^2_P (A)\) of weight 2 defined in (4.4) and a homological double derivation \(Q\). Let \(B\) be the path algebra of the weight 0 subquiver of \(\overline{P}\), and \((E, \{-, -\})\) the pair consisting of the \(B\)-bimodule \(E\) with basis weight 1 paths in \(\overline{P}\) and the symmetric non-degenerate pairing
\[
\langle -, - \rangle := \{ -, - \}_\omega|_{E \otimes E}: E \otimes E \to B \otimes B.
\]

Then the bi-symplectic NQ-algebra \((A, \omega, Q)\) of weight 2 induces a double Courant algebroid \((E, \{-, -\}, \rho, [ -, - ])\) over \(B\), where
\[
\rho(e_1)(b) := \{ \{ \Theta, e_1 \}_\omega, b \}_\omega, \quad [ e_1, e_2 ] := \{ \{ \Theta, e_1 \}_\omega, e_2 \}_\omega,
\]
for all \( b \in B \) and \( e_1, e_2 \in E \). Here \( \Theta \in A^3 \) is determined by the triple \((A, \omega, Q)\), and \( \{ -, - \}_\omega = m \circ \{ -, - \}_\omega \) is the associated bracket in \( A \).

### 8. Exact non-commutative Courant algebroids

#### 8.1. The standard non-commutative Courant algebroid

Let \( B \) be the path algebra of a quiver \( Q \), i.e., \( B = kQ = T_RV_Q \), \( R = R_Q \) the corresponding semisimple finite-dimensional \( k \)-algebra, and \( d_B : B \to \Omega^1_R B \) the universal derivation. By Lemma 3.4,

\[
E_1 := \Omega^1_R B = B \otimes_R V_Q \otimes_R B = \bigoplus_{a \in Q_1} B B a B
\]

To simplify the notation, we define \( \hat{a} := d_B a \), so \( E_1 = \bigoplus_{a \in Q_1} B \hat{a} B \), and

\[
S := \Omega^1_R[1] B := T_B(E_1[-1]).
\]

Then we have an identification \( S = kP \) with the graded path algebra of a graded quiver \( P \) with vertex set \( P_0 = Q_0 \) and arrow set \( P_1 = Q_1 \sqcup \hat{Q}_1 \) obtained from \( Q \) by adjoining an arrow \( \hat{a} \in \hat{Q}_1 \) for each arrow \( a \in Q_1 \), with the same tail and head, i.e., \( t(\hat{a}) = t(a) \), \( h(\hat{a}) = h(a) \) for all \( a \in Q_1 \), and weight function given by \( |a| = 0 \) and \( |\hat{a}| = 1 \), for all \( a \in Q_1 \). Then the graded path algebra of the double quiver \( \mathcal{P} \) of \( P \) of weight 2 can be written

\[
A = k\mathcal{P} = T_S(\text{Der}_R S[-2]).
\]

Note that the arrows of \( \mathcal{P} \) have weights given by

\[
|a| = 0, \quad |a^*| = 2, \quad |\hat{a}| = 1, \quad |\hat{a}^*| = 1
\]

for all \( a \in Q_1 \). In particular, \( |P| = 2 \) if \( Q_1 \) is non-empty (as we will assume). Furthermore, by Proposition 1.2 there is a canonical bi-symplectic form of weight 2 on \( A \), given by

\[
\omega_0 = \sum_{a \in Q_1} (da \, da^* + d\hat{a} \, d\hat{a}^*) \in DR^2_R(A).
\]

where \( d \) is the universal derivation on \( A \).

**Lemma 8.1.** The double derivation of weight +1

\[
Q_0 = \sum_{a \in Q_1} \left( \frac{\partial}{\partial a} \hat{a} - a^* \frac{\partial}{\partial \hat{a}^*} \right)
\]

is a homological, bi-symplectic Hamiltonian double derivation, with Hamiltonian

\[
S_0 = \sum_{a \in Q_1} a^* \hat{a}.
\]

**Proof.** Since \( |\partial/\partial a| = -|a| \), it follows that \( |Q_0| = +1 \). Note also that

\[
\begin{align*}
& i_{Q_0}(da) = Q_0(a) = (e_{h(a)} \otimes e_{t(a)}) \ast \hat{a} = e_{h(a)} \otimes \hat{a}; \\
& i_{Q_0}(d\hat{a}) = Q_0(\hat{a}) = -e_{t(a)} \otimes a^*; \\
& i_{Q_0}(da^*) = Q_0(a^*) = 0; \\
& i_{Q_0}(d\hat{a}) = Q_0(\hat{a}) = 0.
\end{align*}
\]

(8.4)
To prove that $Q_0$ is Hamiltonian (i.e. $\iota_{Q_0}\omega = dS_0$ for some $S_0 \in A^3$), we apply a graded version of (8.3) and (8.4):

\[
\iota_{Q_0}\omega = \iota_{Q_0} \left( \sum_{a \in Q_1} (da\,da^* + d\hat{a}\,d\hat{a}^*) \right) = \sum_{a \in Q_1} \left( \iota_{Q_0}''(da)\,da^*\iota_{Q_0}(da) + \iota_{Q_0}''(d\hat{a}^*)\,d\hat{a}\iota_{Q_0}'(d\hat{a}^*) \right) = \sum_{a \in Q_1} (da^*\hat{a} + a^* \,d\hat{a}) = d \left( \sum_{a \in Q_1} a^*\hat{a} \right).
\]

Therefore, (8.2) implies that $Q_0$ is bi-symplectic, because it is Hamiltonian.

It remains to show that $Q_0$ is homological. Using (8.4),

\[
\{Q_0, Q_0\}_t(a) = \tau_{(23)} \left( \left( (Q_0 \otimes \text{Id})Q_0(a) - (\text{Id} \otimes Q_0)Q_0(a) \right) \right) = \tau_{(23)} \left( Q_0(\hat{a}) \otimes e_{t(a)} - \hat{a} \otimes Q_0(e_{t(a)}) \right) = 0,
\]

so $\{Q_0, Q_0\}_t(a) = 0$ too, by skew-symmetry, and hence $\{Q_0, Q_0\}(a) = 0$. One can show similarly that $\{Q_0, Q_0\}(\hat{a}^*) = 0$. Finally, $\{Q_0, Q_0\}(a^*) = \{Q_0, Q_0\}(\hat{a}) = 0$, by (8.2).

We define the non-commutative derived brackets (see Proposition 7.4)

\[
\rho_0(e) = \{\{S_0, e\}, b\}_\omega; \quad [e_1, e_2]_0 = \{\{S_0, e\}, b\}_\omega;
\]

for all $b \in B$, $e, e_1, e_2 \in E$, with $S_0$ given by (8.3). Using Theorems 5.4 and 7.1 and the above results, we can conclude the following.

**Proposition 8.2.** Let $Q$ be a quiver with path algebra $B$. Let $A = T_B M$, with $M = \Omega^1_R B[-1] \oplus \text{Der}_R B[-2]$. We endow the graded algebra $A$ with the graded bi-symplectic form $\omega_0$ defined in (8.1) and the double derivation $Q_0$ defined in (8.2). Then

(i) The triple $(A, \omega_0, Q_0)$ is a bi-symplectic $\mathbb{N}Q$-algebra of weight 2.

(ii) The 4-tuple $(\Omega^1_R B, \langle -,- \rangle, \rho_0, [-,-]_0)$ (where $\rho_0, [-,-]_0$ were defined in (8.5) and $\langle -,- \rangle$ in (6.6)) is a double Courant algebroid.

The double Courant algebroid obtained in Proposition 8.2 will be called the standard double Courant algebroid.

**8.2. Deformations of the standard non-commutative Courant algebroid.** Ševera and Weinstein [32] showed that the standard Courant bracket on $TM \oplus T^*M$, over any smooth manifold $M$, can be ‘twisted’ in the following way. Given a 3-form $H$, define a bracket $[-,-]_H$ on $TM \oplus T^*M$ by $[X + \xi, Y + \eta]_H := [X + \xi, Y + \eta] + iYiX H$, for vector fields $X, Y$ and 1-forms $\xi, \eta$ on $M$. Then $[-,-]_H$ defines a Courant algebroid structure on $TM \oplus T^*M$ (using the standard inner product and anchor) if and only if the 3-form $H$ is closed. Roytenberg [27, §5] developed an approach to the deformation theory of Courant algebroids using the language of derived brackets in the context of differential graded symplectic manifolds. We will explain now how Roytenberg’s approach can be adapted to our noncommutative formalism.
Consider the Framework 5.5, for a quiver $Q$ and the double graded quiver $\overline{P}$ defined in §8.1. Define an injection
\[
\lambda: \Omega^n_{R}B \hookrightarrow A^n.
\] (8.6) by the formulae
\[
\lambda(a) = a, \quad \lambda(da) = \hat{a},
\]
and the rule $\lambda(adb) = \lambda(a)\lambda(db)$, for all arrows $a, b \in Q_1$ (so $db \in \Omega^1_{R}B$).

Let $\text{Alg}_k$ (respectively $\text{CommAlg}_k$) be the category of associative algebras (resp. the category of commutative algebras). It is well-known that the inclusion functor $\text{CommAlg}_k \hookrightarrow \text{Alg}_k$ has a left adjoint functor, given by the abelianization $(-)_{ab}: \text{Alg}_k \rightarrow \text{CommAlg}_k: A \mapsto A_{ab} := A/(A[A,A]A)$.

Using the Karoubi-de Rham complex (3.11), we get the following by inspection.

Lemma 8.3. We have the following commutative diagram.
\[
\begin{array}{ccc}
\Omega^3_{R}B & \xrightarrow{\lambda} & A^3 \\
\downarrow & & \downarrow \\
\text{DR}^3 A & \xrightarrow{\lambda_{ab}} & A^3_{ab}
\end{array}
\]

Given any $\alpha \in \Omega^*_{R}A$, the corresponding element in $\text{DR}^*_{R}A$ will be denoted $[\alpha]$. Note that $A_{ab} = \text{DR}^0_{R}A$. Let $a, b$ and $c$ be three arrows of weight 1, and $p, q, r, s$ be three paths in $Q$ that compose, that is,
\[
h(p) = t(a), \quad h(a) = t(q), \quad h(q) = t(b), \quad h(b) = t(r), \quad h(r) = t(c), \quad h(c) = t(s).
\]

Define a noncommutative differential 3-form on $B$,
\[
\phi := s(d c)r(d b)q(d a)p \in \Omega^3_{R}B,
\]
with corresponding cubic polynomial on $A$,
\[
\hat{\phi} := \lambda(\phi) = \hat{s}\hat{c}\hat{b}\hat{q}\hat{a}\hat{p} \in A^3.
\]

As usual, $m: A \otimes A \rightarrow A$ denotes multiplication.

Lemma 8.4. (i) $\lambda(\text{d}\phi) = m(Q_0(\lambda(\phi)))$;
(ii) $\{\{\hat{a}, \hat{b}\}\}_{\omega} = 0$.

Proof. To prove (i), by the definition 8.6 of $\lambda$, it suffices to show the identity when $\phi = a \in \Omega^1_{R}B$, with $a \in Q_1$. In this case, it is clear that the left-hand side is $\lambda(\text{d}a) = \hat{a}$, while the right-hand side is
\[
m(Q_0(\lambda(a))) = m(Q_0(a)) = m(e_{h(a)}\hat{a} \otimes e_{t(a)}) = \hat{a},
\]
as required. For (ii), by §8.1, we have the differential double Poisson bracket
\[
P = \sum_{a \in Q_1} \left( \frac{\partial}{\partial \hat{a}} \frac{\partial}{\partial \hat{a}^*} + \frac{\partial}{\partial \hat{a}^*} \frac{\partial}{\partial \hat{a}} \right).
\]
Then (ii) follows by inspection, applying (4.10). \qed
We now define the ‘deformed Hamiltonian’

$$S_{\tilde{\phi}} := S_0 + \tilde{\phi}. \tag{L.1}$$

**Lemma 8.5.** Let $\{-,-\}_\omega$ be the associated bracket to $\{-,-\}_\omega$. Then

$$\{\{S_{\tilde{\phi}}, S_{\tilde{\phi}}\}_\omega = 0 \iff d_{DR}[\tilde{\phi}] = 0 \text{ in } DR^4_R A.$$ 

**Proof.** By the properties of the associated bracket, we have

$$\{\{S_{\tilde{\phi}}, S_{\tilde{\phi}}\}_\omega = \{\{S_0, S_0\}_\omega + \{S_0, \tilde{\phi}\}_\omega + \{\tilde{\phi}, S_0\}_\omega + \{\tilde{\phi}, \tilde{\phi}\}_\omega$$

Let $a$ be an arrow of weight 1, and $p$ a path in $Q$. Lemma 8.4(ii) and the fact that $\{\{a, p\}_\omega = 0$ (since $\|\{a, -\}_\omega\| = -2$) imply that $\{\{\tilde{\phi}, \tilde{\phi}\}_\omega = 0$ and, consequently, $\{\tilde{\phi}, \tilde{\phi}\}_\omega = 0$. Also, Lemmas 7.3 and 8.3 enable us to conclude that $\{S_0, S_0\} = 0$. So,

$$\{\{S_{\tilde{\phi}}, S_{\tilde{\phi}}\}_\omega = 2\{\{S_0, \tilde{\phi}\}_\omega, where we applied (L.6) in the last identity. Then by Lemmas (8.4(i) and Lemma 8.3)

$$\{\{S_0, \tilde{\phi}\}_\omega = m(S_0(\lambda(\phi)))$$

$$= [\lambda(d \phi)]$$

$$= \lambda_{ab}(d_{DR}[\tilde{\phi}]).$$

Since $\lambda_{ab}$ is injective, $\{\{S_{\tilde{\phi}}, S_{\tilde{\phi}}\}_\omega = 0$ if and only if $d_{DR}[\tilde{\phi}] = 0$ in $DR^4_R A$, as required. □

In conclusion, any noncommutative differential 3-form $\lambda \in \Omega^3_R B$ that is closed in the Karoubi–de Rham complex determines a cubic noncommutative polynomial $\tilde{\phi} \in A^3$ and hence a deformation of the standard double Courant algebroid.

**APPENDIX A. PROOF OF THEOREM 5.4**

Consider the diagram (5.6) constructed in (5.1). Let $a', a'' \in A$ and $\sigma \in M_N^\tau$. Then $|\sigma| = -N$ and $a' \otimes \sigma \otimes a'' \in A \otimes B M_N^\tau \otimes_B A$ has weight $|a'| + |a''| - N$, when viewed as an element of the space $A \otimes B M^\gamma \otimes_B A$ under the injection $A \otimes B M_N^\tau \otimes_B A \hookrightarrow A \otimes B M^\gamma \otimes_B A$.

We will write explicitly the morphism $\nu^\gamma$ of Proposition 5.3 using the isomorphisms $\kappa$ and $F$ which will be defined below. Also, these maps will make the square in the following diagram commute.

$$\begin{array}{ccc}
(A \otimes_B M \otimes_B A)^\gamma & \xrightarrow{\nu^\gamma} & (\Omega/Q)^\gamma \\
\kappa \downarrow & & \downarrow F \\
0 & \longrightarrow & A \otimes_B M^\gamma \otimes_B A \longrightarrow \text{Der}_R A \longrightarrow A \otimes_B \text{Der}_R B \otimes_B A \longrightarrow 0
\end{array} \tag{A.1}
$$

Firstly, define

$$\kappa: A \otimes_B M^\gamma \otimes_B A \xrightarrow{\tilde{\kappa}} (A \otimes_B M \otimes_B A)^\gamma: \quad a' \otimes \sigma \otimes a'' \longmapsto \kappa(a' \otimes \sigma \otimes a''),$$

by

$$\kappa(a' \otimes \sigma \otimes a''): A \otimes_B M \otimes_B A \longrightarrow A \otimes A
\quad a_1^{(1)} \otimes m_1 \otimes a_1^{(2)} \longmapsto \pm(a'(a''(m_1)a_1^{(2)}) \otimes (a_1^{(1)}\sigma_1(m_1))a''),$$

where $m_1$ and $a_1^{(1)} \otimes a_1^{(2)}$ are elements of $A \otimes_B M \otimes_B A$ and $A \otimes_{\mathbb{R}} M \otimes_{\mathbb{R}} A$, respectively.
where $a_1^{(1)}, a_1^{(1)} \in A$ and $m_1 \in M$. As usual, we use Sweedler’s convention. In addition, we will use the sign ± to indicate the signs involved since we will construct a map which will turn out to be zero hence signs can be ignored for this purpose. By Proposition 5.2, the morphism $\nu$ is the natural projection:

$$
\nu: \tilde{\Omega}/Q \longrightarrow A \otimes_B M \otimes_B A: \left( \frac{a_2^{(1)} \otimes \beta_2 \otimes a_2^{(2)}}{a_2^{(1)} \otimes m_2 \otimes a_2^{(2)}} \right) \mod Q \longmapsto a_2^{(1)} \otimes m_2 \otimes a_2^{(2)},
$$

where $a_2^{(1)}, a_2^{(2)}, \beta_2, \alpha_2^{(2)} \in A$, $m_2 \in M$ and $\beta_2 \in \Omega_R^1 A$. Define the map

$$
\varphi := \kappa(a' \otimes \sigma \otimes a'') \left( \nu \left( \frac{a_2^{(1)} \otimes \beta_2 \otimes a_2^{(2)}}{a_2^{(1)} \otimes m_2 \otimes a_2^{(2)}} \right) \mod Q \right) \in (\tilde{\Omega}/Q)^\vee
$$

To define $F$ in (A.1), we will need the action of $\tilde{d}$ on homogeneous generators $b \in B = T_B^0 M$ and $m_i \in M = T_B^i M$, for all $i = 1, ..., r$:

$$
\tilde{d} a = \begin{cases} 
(1_A \otimes \tilde{d} b \otimes 1_A) \mod Q & \text{if } a = b \\
0 \otimes \sum_{i=1}^{r} m_1 \cdot \ldots \cdot m_{i-1} \otimes m_i \otimes m_{i+1} \cdot \ldots \cdot m_r \mod Q & \text{if } a = m_1 \otimes \ldots \otimes m_r
\end{cases}
$$

Define now

$$
F: (\tilde{\Omega}/Q)^\vee \longrightarrow \text{Der}_R A
$$

by

$$
F(\varphi)(a) = \begin{cases} 
0 & \text{if } a = b, \\
(\varphi(\tilde{d} a)^o) & \text{otherwise}
\end{cases}
$$

where $\varphi \in (\tilde{\Omega}/Q)^\vee$ and $a \in A$. In particular, if $a = m_1 \otimes \ldots \otimes m_r$ with $r > 0$:

$$(\varphi(\tilde{d} a)^o) = \sum_{i=1}^{r} \sigma_{i(2)} ((a' \sigma''(m_i) m_{i+1} \cdot \ldots \cdot m_r) \otimes (m_1 \cdot \ldots \cdot m_{i-1} \sigma'(m_i)a''))$$

$$
= \sum_{i=1}^{r} (m_1 \cdot \ldots \cdot m_{i-1} \sigma'(m_i)a'') \otimes (a' \sigma''(m_i)m_{i+1} \cdot \ldots \cdot m_r)
$$

Claim A.1. $F(\varphi) \in \text{Der}_R A$.

Proof. Straightforward application of the graded Leibniz rule. \qed

Next, we will focus on the vertical arrow, $\iota(\omega): \text{Der}_R A \xrightarrow{\sim} \Omega_R^1 A$ given by the bi-symplectic form $\omega$ of weight $N$ on $A$. We use the canonical isomorphism $f^{-1}: \Omega_R^1 A \cong \tilde{\Omega}/Q$ (see (5.3)), which induces another one $\Omega_R^2 A \xrightarrow{\sim} (\tilde{\Omega}/Q)^{\otimes N^2}$: $\beta \longmapsto \tilde{\beta}$. In particular, the bi-symplectic form $\omega$ determines an element $\tilde{\omega}$ that can be decomposed, using $\tilde{\Omega} = (A \otimes_B \Omega_R^1 B \otimes_B A) \oplus (A \otimes M \otimes A)$, as follows:

$$
\tilde{\omega} = (\tilde{\omega}_{MM} + \tilde{\omega}_{BB} + \tilde{\omega}_{MB} + \tilde{\omega}_{BM}) \mod Q.
$$
We have to distinguish two cases:

isomorphism $f$ but we do know how it acts on elements of $\Omega_1$

Proof. Claim A.2

Omitting summation signs, we write

\[ (i) \quad \text{Firstly,} \quad \iota F(\varphi)(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = \iota F(\varphi)(d_A m_i) a_i^{(2)} \]

\[ (ii) \quad \text{This case is similar. By definition of} \quad \tilde{\omega}_{MB} = \tilde{\omega}_{BM} \mod Q, \]

\[ \tilde{\omega}_{BB} = (\tilde{\beta}_1 \otimes \tilde{\beta}_2) \mod Q, \]

\[ \tilde{\omega}_{BB} = (\tilde{\beta}_1 \otimes \tilde{\beta}_2) \mod Q, \]

\[ \tilde{\omega}_{BB} = (\tilde{\beta}_3 \otimes \tilde{\beta}_3) \mod Q, \]

\[ \tilde{\omega}_{BB} = (\tilde{\beta}_3 \otimes \tilde{\beta}_3) \mod Q, \]

where $\tilde{m}_i := a_i^{(1)} \otimes m_i \otimes a_i^{(2)} \in A \otimes_R M \otimes_R A$ and $\tilde{\beta}_i := \varpi_i^{(1)} \otimes \beta_i \otimes \varpi_i^{(2)} \in A \otimes_R \Omega^1_R B \otimes_B A$

for $i = 1, 2, 3, 4$, with $a_i^{(1)}, a_i^{(2)}, \varpi_i^{(1)}, \varpi_i^{(2)} \in A, m_i \in M$ and $\beta_i \in \Omega^1_R B$ for $i = 1, 2, 3, 4$. Using this decomposition and the previous isomorphism, we can now calculate

\[ \iota F(\varphi)(\tilde{\omega}) = \iota(\tilde{\omega})(F(\varphi)) = \iota ((\tilde{\omega}_{MM} + \tilde{\omega}_{BB} + \tilde{\omega}_{MB} + \tilde{\omega}_{BM}) \mod Q)(F(\varphi)) \quad (A.2) \]

Claim A.2.

(i) $\iota F(\varphi)(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = \begin{cases} 0 & \text{if } |m_i| < N \\ a_i^{(1)} \circ (d' \sigma''(m_i) \otimes \sigma'(m_i) a''_i) a_i^{(2)} & \text{if } |m_i| = N \end{cases}$

(ii) $\iota F(\varphi)(\tilde{\omega}_i^{(1)} \otimes \beta \otimes \tilde{\omega}_i^{(2)}) = 0$

Proof. Observe that we do not know how the operator $\iota F(\varphi)$ acts on elements of $\tilde{\Omega}/Q$, but we do know how it acts on elements of $\Omega^1_R A$. Hence we have to use the canonical isomorphism $f$ between these objects and then apply the operator $\iota F(\varphi)$.

(i) Firstly,

\[ \iota F(\varphi)(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = \iota F(\varphi)(d_A m_i) a_i^{(2)} \]

\[ = a_i^{(1)} \iota F(\varphi)(d_A m_i) a_i^{(2)} \]

\[ = a_i^{(1)} \circ (F(\varphi)(m_i)) a_i^{(2)} \]

We have to distinguish two cases:

(a) **Case** $|m_i| < N$: Since $\sigma \in M_N^\circ$, $\sigma(m_i) = 0$ since $(A \otimes A)_{(j)} = \{0\}$ with $j < 0$.

Thus, $\iota F(\varphi)(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = 0$.

(b) **Case** $|m_i| = N$:

\[ \iota F(\varphi)(a_i^{(1)} \otimes m_i \otimes a_i^{(2)}) = a_i^{(1)} \circ (F(\varphi)(m_i)) a_i^{(2)} \]

\[ = a_i^{(1)} \circ (a' \sigma''(m_i) \otimes \sigma'(m_i) a''_i) a_i^{(2)} \in A \]

(ii) This case is similar. By definition of $F(\varphi)$,

\[ \iota F(\varphi)(\tilde{\omega}_i^{(1)} \otimes \beta_i \otimes \tilde{\omega}_i^{(2)}) = \iota F(\varphi)(d_A b_i^{(2)}) a_i^{(2)} \]

\[ = \tilde{a}_i^{(1)} b_i^{(1)} \circ (F(\varphi)(b_i^{(2)})) a_i^{(2)} = 0 \]

\[ \square \]

Now, we will use the Claim [A.2] to calculate each summand in [A.2]:

- **Case** $\tilde{\omega}_{BB}$:

As $|\tilde{\omega}_{BB}| = N$ and $|\beta_1| = |\beta_2| = 0$, $|\varpi_i^{(1)}| + |\varpi_i^{(2)}| = N$, with $\varpi_i^{(1)}, \varpi_i^{(2)} \in A$ for $i = 1, 2$. 


In conclusion, without loss of generality, in this case, we can assume that $|\bar{a}_1^{(1)}| = N$. Then

$$\iota(\bar{\omega}_{BB})(F(\varphi)) = \iota(F(\varphi))(\bar{\omega}_{BB})$$
$$= \iota(F(\varphi))(\bar{\beta}_1 \otimes \bar{\beta}_2)$$
$$= (\iota(F(\varphi)) (\bar{\beta}_1)) \bar{\beta}_2 + \bar{\beta}_1 \left( \iota(F(\varphi))(\bar{\beta}_2) \right)$$
$$= \left( \iota(F(\varphi))(\bar{a}_1^{(1)} \otimes \bar{a}_1^{(2)}) \right) \bar{\beta}_2 + \bar{\beta}_1 \left( \iota(F(\varphi))(\bar{a}_2^{(1)} \otimes \bar{a}_2^{(2)}) \right)$$
$$= 0$$

- **Case $\bar{\omega}_{MM}$:**

As $|\bar{\omega}_{MM}| = N$ and $|m_i| \geq 1$, then $|m_i| < N$ for $i = 1, 2$. Then, using Claim [A.2]

$$\iota(\bar{\omega}_{MM})(F(\varphi)) = \iota(F(\varphi))(\bar{\omega}_{MM})$$
$$= \iota(F(\varphi))(\bar{m}_1 \otimes \bar{m}_2)$$
$$= \left( \iota(F(\varphi))(\bar{m}_1) \right) \bar{m}_2 + \bar{m}_1 \left( \iota(F(\varphi))(\bar{m}_2) \right)$$
$$= \left( \iota(F(\varphi))(\bar{a}_1^{(1)} \otimes m_1 \otimes a_1^{(2)}) \right) \bar{m}_2 + \bar{m}_1 \left( \iota(F(\varphi))(\bar{a}_2^{(1)} \otimes m_2 \otimes a_2^{(2)}) \right)$$
$$= 0.$$  

- **Case $\bar{\omega}_{MB}$:**

In this case, $|\beta_3| = 0$, so $N \geq |m_3| \geq 1$. Again, by the Leibniz rule and Claim [A.2]

$$\iota(\bar{\omega}_{MB})(F(\varphi)) = \iota(F(\varphi))(\bar{\omega}_{MB})$$
$$= \iota(F(\varphi))(\bar{m}_3 \otimes \bar{\beta}_3)$$
$$= \left( \iota(F(\varphi))(\bar{m}_3) \right) \bar{\beta}_3 + \bar{m}_3 \left( \iota(F(\varphi))(\bar{\beta}_3) \right)$$
$$= \left( \iota(F(\varphi))(\bar{a}_3^{(1)} \otimes m_3 \otimes a_3^{(2)}) \right) \bar{\beta}_3 + \bar{m}_3 \left( \iota(F(\varphi))(\bar{a}_3^{(1)} \otimes \beta_3 \otimes a_3^{(2)}) \right)$$
$$= \left( \bar{a}_3^{(1)} \circ (F(\varphi))(m_3) \right) a_3^{(2)} \bar{\beta}_3$$

Now we have to distinguish two cases depending on the weight of $m_3$:

(a) **Case $|m_3| < N$:** by Claim [A.2]

$$\iota(\bar{\omega}_{MB})(F(\varphi)) = 0.$$  

(b) **Case $|m_3| = N$:** by the same Claim,

$$\iota(\bar{\omega}_{MB})(F(\varphi)) = \left( \bar{a}_3^{(1)} \circ (a' \sigma'(m_3) \otimes \sigma'(m_3) a'' \bar{a}_3^{(2)}) \right) \bar{\beta}_3$$

$$\in \left( \left( A \otimes_B \Omega_R^1 B \otimes_B A \oplus 0 \right) \mod Q \subset \bar{\Omega}/Q \right)$$

- **Case $\bar{\omega}_{MB}$**

It is similar to the previous case.

In conclusion, $\iota(\bar{\omega})(F(\varphi)) \in (A \otimes_B \Omega_R^1 B \otimes_B A \oplus 0) \mod Q$. For the last step, we define the map $g$ making the following diagram commutative:

$$0 \rightarrow A \otimes_B \Omega_R^1 B \otimes_B A \xrightarrow{\epsilon} \Omega_R^1 A \xrightarrow{\nu} A \otimes_B M \otimes_B A \rightarrow 0$$

$$\xrightarrow{\cong}$$

$$\bar{\Omega}/Q \xrightarrow{g}$$
By Proposition 5.2, we know that \( \nu \) is the projection onto the second direct summand of \( \tilde{\Omega}/Q \) so \( g(\iota(\omega_{MB})(F(\varphi))) \) is zero in \( A \otimes_B M \otimes_B A \). The universal property of the kernel allows us to conclude the existence of the dashed maps

\[
\begin{array}{c}
A \otimes_B M_N^\nu \otimes_B A \\
\downarrow \\
0 \longrightarrow A \otimes_B M_N^\nu \otimes_B A \; \xrightarrow{\nu^\nu} \; \text{Der}_R A \; \xrightarrow{\epsilon^\nu} \; A \otimes_B \text{Der}_R B \otimes_B A \; \longrightarrow \; 0 \\
\downarrow \\
0 \longrightarrow A \otimes_B M_N^\nu \otimes_B A \; \xrightarrow{\epsilon} \; A \otimes_B M \otimes_B A \; \longrightarrow \; 0
\end{array}
\]

making this diagram commutes. Finally, it follows that we constructed the following map:

\[
A \otimes_B M_N^\nu \otimes_B A \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \quad (A.3)
\]

Next, we will consider the ‘inverse’ diagram:

\[
\begin{array}{c}
0 \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \; \xrightarrow{\epsilon} \; \Omega^1_R A \; \xrightarrow{\nu} \; A \otimes_B M \otimes_B A \; \longrightarrow \; 0 \\
\downarrow \iota(\omega)^{-1} \\
0 \longrightarrow A \otimes_B M^\nu \otimes_B A \; \xrightarrow{\nu^\nu} \; \text{Der}_R A \; \xrightarrow{\epsilon^\nu} \; A \otimes_B \text{Der}_R B \otimes_B A \; \longrightarrow \; 0
\end{array}
\]

(A.4)

In a first stage, our aim is to construct the following dashed arrow:

\[
A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow A \otimes_B M^\nu \otimes_B A
\]

which makes the previous diagram commutative.

We begin by recalling that since \( \tilde{\Omega} = (A \otimes_B \Omega^1_R B \otimes_B A) \oplus (A \otimes_R M \otimes_R A) \), \( h \) is the imbedding of the first direct summand in \( \tilde{\Omega} \) (see Proposition 5.2), \( \text{proj} \) is the natural projection and the isomorphism \( f \) was defined in (5.4).

\[
\begin{array}{cc}
\text{proj} & \tilde{\Omega} \\
\downarrow h & \downarrow f \\
\tilde{\Omega}/Q & 0 \\
0 \longrightarrow A \otimes_B \Omega^1_R B \otimes_B A \; \xrightarrow{\epsilon} \; \Omega^1_R A \; \xrightarrow{\nu} \; A \otimes_B M \otimes_B A \; \longrightarrow \; 0
\end{array}
\]

Let \( a', a'' \in A \), \( b \in B \) and \( d_B b \in \Omega^1_R B \). Then \( a' \otimes d_B b \otimes a'' \in A \otimes_B \Omega^1_R B \otimes_B A \). It is a simple calculation that

\[
\epsilon: A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A : \quad a' \otimes d_B b \otimes a'' \mapsto a'(d_A b)a'' 
\]

(A.5)

Now, we focus on the vertical arrow of the diagram (A.4). Observe that since \( \omega \) is a bi-symplectic form of weight \( N \), \( \iota(\omega)^{-1} \) has weight \( -N \). In fact, using (4.20), we can write this double Poisson bracket in terms of the Hamiltonian double derivation. Nevertheless, since \( \{\{\cdot, \cdot\}\}_\omega \) is \( A \)-bilinear with respect to the outer bimodule structure on \( A \otimes A \) in the
second argument and $A$-bilinear with respect to the inner bimodule structure on $A \otimes A$ in the first argument, it is enough to consider $a' = a'' = 1_A$. Then
\[
(i(\omega)^{-1} \circ \varepsilon) (1_A \otimes d_B b \otimes 1_A) = \langle b, -\rangle_\omega = H_b \quad (A.6)
\]
Observe that $H_b \in \mathbb{D}er_R A$ has weight $-N$. Finally, since $\text{inj}$ is the imbedding of $A \otimes_B \Omega^1 B \otimes_B A$ in the first direct summand of $\Omega$, we will determine $\Psi$ to the square in the following diagram commutes:
\[
\begin{array}{cccccc}
0 & \longrightarrow & A \otimes_B M' \otimes_B A & \overset{\nu}{\longrightarrow} & \mathbb{D}er_R A & \overset{\nu^\vee}{\longrightarrow} & A \otimes_B \mathbb{D}er_R B \otimes_B A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \Psi & & \downarrow \cong & & \downarrow \\
(\Omega^1_R A)^\vee & & (\Omega^1_R A)^\vee & & (\Omega^1_R A)^\vee & & (\Omega^1_R A)^\vee & & (A \otimes_B \Omega^1_R A \otimes_B A)^\vee \\
\downarrow & & \downarrow f^\vee & & \downarrow & & \downarrow \lproj \circ (\text{inj})^\vee & & \downarrow \\
(\Omega^1_R A)^\vee & & (\Omega^1_R A)^\vee & & (\Omega^1_R A)^\vee & & (A \otimes_B \Omega^1_R A \otimes_B A)^\vee & & (A \otimes_B \Omega^1_R A \otimes_B A)^\vee \\
\end{array}
\]
In this diagram, we define
\[
\Psi: \mathbb{D}er_R A \longrightarrow (\Omega^1_R A)^\vee: \quad \Theta \longmapsto \Psi(\Theta)
\]
given by
\[
\Psi(\Theta): \Omega^1_R A \longrightarrow A \otimes A: \quad \alpha \longmapsto \Psi(\Theta)(\alpha) = (i_{\Delta} \alpha)^o = \pm i^o_{\Theta}(\alpha) \otimes i^o_{\Theta}(\alpha),
\]
where $\pm := (-1)^{(\|\iota^o_{\Theta}(\alpha)\| + \|\iota^o_{\Theta}(\alpha)\|)}$ (see (A.2)). When we apply $\Psi$ to the element in (A.6):
\[
\Psi: \mathbb{D}er_R A \longrightarrow (\Omega_R^1 A)^\vee: \quad H_b \longmapsto i_{H_b}, \quad (A.7)
\]
such that
\[
i_{H_b}: (\Omega_R^1 A) \longrightarrow A \otimes A:
\]
\[
\bar{e}_1 d_A \bar{e}_2 \longmapsto (\bar{e}_1 H_b(\bar{e}_2))^o \quad (A.8)
\]
Next, applying $f^\vee$, we obtain the following:
\[
(f^\vee \circ \Psi)(H_b): \Omega/Q \longrightarrow A \otimes A:
\]
\[
\left( \begin{array}{c}
\bar{\omega}_2^{(1)} \otimes b_1^{(1)} b_1^{(2)} \otimes \bar{\omega}_2^{(2)} \\
\bar{\omega}_2^{(1)} \otimes m \otimes \bar{\omega}_2^{(2)}
\end{array} \right) \mod Q \rightarrow \left( \begin{array}{c}
im_{b_1^{(1)}} H_b(b_1^{(2)}) \bar{\omega}_2^{(2)} \\
im_{b_1^{(1)}} H_b(b_2^{(2)}) \bar{\omega}_2^{(2)}
\end{array} \right)^o \quad (A.9)
\]
To shorten the notation, we make the following definition:
\[
L := (\lproj \circ f^\vee \circ \Psi)(H_b).
\]
Finally, since
\[
\text{inj}^\vee \circ L: A \otimes_B \Omega_R^1 A \otimes_B A \longrightarrow A \otimes A:
\]
\[
\bar{\omega}_2^{(1)} \otimes b_1^{(1)} b_1^{(2)} \otimes \bar{\omega}_2^{(2)} \longmapsto \left( \begin{array}{c}
im_{b_1^{(1)}} H_b(b_1^{(2)}) \bar{\omega}_2^{(2)} \\
im_{b_1^{(1)}} H_b(b_2^{(2)}) \bar{\omega}_2^{(2)}
\end{array} \right)^o
\]
The key point is to observe that since $b, b_1^{(2)} \in B$, $|b| = |b_1^{(2)}| = 0$. Thus, $|H_{b'}(b_1^{(2)})| = -N < 0$. Thus, $H_b(b_1^{(2)}) = 0$ because $A$ is a bi-symplectic tensor $\mathbb{N}$-algebra so, in particular, it is
non-negatively graded. By the universal property of the kernel, we conclude the existence of the dashed arrows which makes the following diagram commutes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_B \Omega^1_R B \otimes_B A & \xrightarrow{\varepsilon} & \Omega^1_R A & \xrightarrow{\nu} & A \otimes_B M \otimes_B A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \otimes_B M^\vee \otimes_B A & \xrightarrow{\nu^\vee} & \text{Der}_R A & \xrightarrow{\varepsilon^\vee} & A \otimes_B \text{Der}_R B \otimes_B A & \longrightarrow & 0
\end{array}
\]

In special, we are interested in the map

\[A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow A \otimes_B M^\vee \otimes_B A \tag{A.10}\]

Finally, in (A.9), we point out that \( (a_2^{(1)} H_b(m) a_2^{(2)})^0 = 0 \) unless \( m \in M_N \) since \( |H_b| = |m| - N \). Hence, as a consequence of this discussion and using (A.10), we obtain:

\[A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow A \otimes_B M_N \otimes_B A \tag{A.11}\]

By construction, (A.3) and (A.11) are inverse to each other. So, we proved the existence of the following isomorphism:

\[A \otimes_B \Omega^1_R B \otimes_B A \cong A \otimes_B M_N \otimes_B A\]

Or, equivalently using the fact that, by hypothesis, \( B \) is a smooth associative \( R \)-algebra,

\[A \otimes_B \text{Der}_R B \otimes_B A \cong A \otimes_B M_N \otimes_B A \tag{A.12}\]

For reasons that we will clarify below, we make precise this isomorphism:

**Claim A.3.** Let \((A, \omega)\) be a bi-symplectic tensor \( N \)-algebra of weight \( N \). Then \( \iota(\omega)^{-1} \) restricts to a \( B \)-bimodule isomorphism

\[E_N \cong \text{Der}_R B. \tag{A.13}\]

**Proof.** Observe that in the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & A \otimes_B \Omega^1_R B \otimes_B A & \xrightarrow{\varepsilon} & \Omega^1_R A & \xrightarrow{\nu} & A \otimes_B M \otimes_B A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \otimes_A M^\vee \otimes_B A & \xrightarrow{\nu^\vee} & \text{Der}_R A & \xrightarrow{\varepsilon^\vee} & A \otimes_B \text{Der}_R B \otimes_B A & \longrightarrow & 0
\end{array}
\]

the dashed arrow is \( \iota(\omega)^{-1} \circ \varepsilon \). We will see that the weight of this map is \(-N\). This is equivalent to prove that \(|\varepsilon| = 0\) since \( \omega \) is a bi-symplectic form of weight \( N \). As we discussed in (A.5),

\[\varepsilon : A \otimes_B \Omega^1_R B \otimes_B A \longrightarrow \Omega^1_R A : a' \otimes B b'' \otimes a'' \mapsto a'(b' d_A b'') a''.\]

Thus, it is immediate that \(|\varepsilon| = 0\).

Finally, observe that \( A \) is non-negatively graded and \( M_N \) has weight \( N \) while \( \text{Der}_R B \) has weight 0. Note that the part of weight 0 of \( A \otimes_B \text{Der}_R B \otimes_B A \) is \( B \otimes_B \text{Der}_R B \otimes_B B \) which is isomorphic to \( \text{Der}_R B \). Similarly, \( (A \otimes_B M_N \otimes_B A)_N = B \otimes_B M_N \otimes_B B \), where \((-)_N\) denotes the part of weight \( N \). Thus, we obtain the following isomorphism of \( B \)-bimodules

\[E_N \cong \text{Der}_R B\]

**Claim A.3** concludes the proof of Theorem 5.4. \( \square \)
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