On a connection between formulas about $q$–gamma functions

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In this short communication, we want to pay attention to a few wrong formulas which are unfortunately cited and used in a dozen papers afterwards. We prove that the provided relations and asymptotic expansion about the $q$-gamma function are not correct. This is illustrated by numerous concrete counterexamples. The error came from the wrong assumption about the existence of a parameter which does not depend on anything. Here, we apply a similar procedure and derive a correct formula for the $q$-gamma function.

**Keywords:** $q$-Gamma function; asymptotic expansion; boundary functions.

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1. Introduction

Since J. Thomae (1869) and F. H. Jackson (1904) defined the $q$-gamma function, it plays an important role in the theory of the basic hypergeometric series [5] and its applications [8]. Its properties and different representations were discussed in numerous papers, such as in [4], [12] and [11]. A few successful algorithms for its numerical evaluation are introduced in [7] and [6] and [1]. An asymptotic expansion of the $q$-gamma function was provided in [3].

Here, we will make observations on the asymptotic expansions given in [9, 10].

Let $q \in [0, 1)$. A $q$-number $[a]_q$ is

$$[a]_q := \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}.$$

The factorial of a positive integer number $[n]_q$ is given by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n - 1]_q \cdots [1]_q, \quad (n \in \mathbb{N}).$$
An important role in $q$–calculus plays the $q$-Pochhammer symbol defined by
\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \quad (n \in \mathbb{N} \cup \{+\infty\}),
\]
and
\[
(a; q)_\lambda = \frac{(a; q)_\infty}{(aq^\lambda; q)_\infty} \quad (|q| < 1, \lambda \in \mathbb{C}).
\]
The $q$-gamma function
\[
\Gamma_q(z) = (q; q)_{z-1} (1 - q)^{1-z} = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad (0 < q < 1, z \notin \mathbb{Z}^-)
\]
has the following properties:
\[
\Gamma_q(z + 1) = [z]_q \Gamma_q(z) \quad (z \in \mathbb{C}), \quad \Gamma_q(n + 1) = [n]_q! \quad (n \in \mathbb{N}_0).
\]
In particular,
\[
\lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z).
\]
The exact $q$–Gauss multiplication formula can be found in [5] or [4]:
\[
\Gamma_q(nx) \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = [n]_q^{nx-1} \prod_{k=0}^{n-1} \Gamma_q\left(x + \frac{k}{n}\right) \quad (x > 0; n \in \mathbb{N}).
\]
Equivalently, substituting $z = nx$, it can be written in the form
\[
\Gamma_q\left(z\right) \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = [n]_q^{z-1} \prod_{k=0}^{n-1} \Gamma_q\left(\frac{z+k}{n}\right) \quad (z > 0; n \in \mathbb{N}).
\]

2. Our corrections to the paper [9]
Starting from the definition
\[
\Gamma_q(x) = (q; q)_\infty (1 - q)^{1-x} (q^x; q)_\infty^{-1},
\]
we can write
\[
\Gamma_q(x) = (q; q)_\infty (1 - q)^{1/2} (1 - q)^{1/2-x} e^{-\log(q^x)\lambda}.
\]
Hence the function $\Gamma_q(x)$ can be written in the form
\[
\Gamma_q(x) = a(q) \cdot (1 - q)^{1/2-x} e^{\mu(x)} \quad (a(q) \in \mathbb{R}),
\]
where
\[
0 < a(q) = (q; q)_\infty (1 - q)^{1/2} < 1, \quad \mu(x, q) = -\log(q^x; q)_\infty.
\]
Let
\[
\psi(x, q) = \frac{q^x}{(1-q)(1-q^x)}.
\]
Using estimate (2.3), we get

\[ 0 < \mu(x, q) < \psi(x, q) \quad (0 < q < 1, \ x > 0), \]

it exists \( \theta(x, q) \in (0, 1) \) such that

\[ \mu(x, q) = \theta(x, q) \cdot \psi(x, q). \]

Therefore, relation (2.1) becomes

\[ \Gamma_q(x) = a(q) \cdot (1 - q)^{1/2 - x} e^{\theta(x, q) \cdot \psi(x, q)}. \quad (2.3) \]

On the other hand, formula (1.2) can be written in the form

\[ a_p(q)\Gamma_q(x) = \left[ \sum_{k=0}^{p-1} \Gamma_{q^p} \left( \frac{x+k}{p} \right) \right] \quad (x > 0; \ p \in \mathbb{N}), \quad (2.4) \]

where

\[ a_p(q) = \left[ \sum_{k=0}^{p-1} \Gamma_{q^p} \left( \frac{k}{p} \right) \Gamma(p \cdot \frac{2x+k}{p}) \cdots \Gamma(p \cdot \frac{p}{p}) \right]. \]

Substituting \( q \to q^p \) and \( x \to k/p \) into the definition (1.1) of the \( q \)-gamma function, we have

\[ \Gamma_{q^p} \left( \frac{k}{p} \right) = \frac{(q^p; q^p)^n}{(q^p; q^p)^n} (1 - q^p)^{1-k/p} = (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^p; q^p)_n}. \]

Moreover, using

\[ \prod_{k=1}^{p} (1 - q^p)^{1-k/p} = (1 - q^p)^{p-1}, \]

the following holds:

\[ a_p(q) = \left[ \sum_{k=0}^{p-1} \Gamma_{q^p} \left( \frac{k}{p} \right) \right] = \left[ \sum_{k=0}^{p-1} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^p; q^p)_n} \right] = [p]_q \prod_{k=1}^{p} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^p; q^p)_n} = [p]_q (1 - q^p)^{p-1} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^p; q^p)_n}. \]

The following identity is valid

\[ \prod_{k=1}^{p} (q^k; q^p)_n = (q; q)_n. \]

Using estimate (2.3), we get

\[ \Gamma_{q^p}(n + 1) = a(q^p) \cdot (1 - q^p)^{-n-1/2} e^{\theta(n+1,q^p) \cdot \psi(n+1,q^p)} \]

Since

\[ \frac{(q^p; q^p)_n}{(1 - q^p)^{np}} = \Gamma_{q^p}(n + 1) = a(q^p) \cdot (1 - q^p)^{p(-1/2 - n)} e^{p \cdot \theta(n+1,q^p) \cdot \psi(n+1,q^p)}, \]

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and
\[ \prod_{k=1}^{p}(q^k; q^p)_n \frac{(q^k q)^{np}}{(1-q)^{np}} = \Gamma_q(np+1) = a(q) \cdot (1-q)^{-1/2-np} \cdot e^{\theta(np+1,q)} \psi(np+1,q), \]
we have
\[ a_p(q) = \frac{a^p(q^p)}{a(q)} [p]^{1/2} \lim_{n \to \infty} \frac{e^{\theta(n+1,q^p) \psi(n+1,q^p)}}{e^{\theta(np+1,q) \psi(np+1,q)}}. \]
From
\[ \lim_{n \to \infty} \psi(n+1,q^p) = \lim_{n \to \infty} \psi(np+1,q) = 0 \quad (0 < q < 1; \ p \in \mathbb{N}), \]
we find
\[ a_p(q) = [p]^{1/2} \frac{a^p(q^p)}{a(q)}. \]
In that manner, the parameter \(a_p(q)\) from formula (2.4) is expressed via the parameter \(a(q)\) from formula (2.3).

3. Faults in paper [9]

In the very beginning of this section, we wish to express our opinion that in [9] an excellent approach was exposed, but a few mistakes were made in its realization. So, we have decided to refer to them.

In [9], the author has supposed that \(\Gamma_q(x)\) for \(0 < q < 1; \ x > 0\), can be written in the form
\[ \Gamma_q(x) = a \cdot (1-q)^{1/2-x} e^{\mu(x)} \quad (a \in \mathbb{R}), \]
where
\[ \mu(x) = -\log(q^x; q) \]  
\(> 0\).

His efforts in looking for \(\mu(x)\) we shortened a lot by starting from the definition of \(\Gamma_q(x)\). From the fact that
\[ 0 < \mu(x) < \frac{q^x}{(1-q)(1-q^x)}, \]
and
\[ (1-q)(1-q^x) = 1 - q - q^x + q^{x+1} > 1 - q - q^x, \]
the author in [9] concluded wrongly that
\[ 0 < \mu(x) < \frac{q^x}{(1-q)-q^x}. \]
But, expression \(1 - q - q^x\) is not positive for all \(q \in (0, 1)\) and \(x > 0\). Indeed,
\[ 1 - q - q^x \leq 0 \iff 1 - q \leq q^x \iff x \cdot \log q \geq \log(1-q) \iff x \leq \frac{\log(1-q)}{\log q}. \]

Example 3.1. We examined the sign changes of the function \(h_q(x) \equiv 1 - q - q^x\) for different \(q\) and \(x\). Notice that \(x \to +\infty\) if \(q \to 1^−\).
Table 1. Unique real zero of the function \( h_q(x) \) and the sign changes for random values of \( q \) and \( x \)

| \( q \) | \( x : 1 - q - q^x = 0 \) | \( x \) | \( q \) | \( 1 - q - q^x \) |
|---|---|---|---|---|
| 0.1 | 0.045758 | 1.10500 | 0.592727 | -0.15378 |
| 0.3 | 0.296248 | 2.27287 | 0.752038 | -0.275286 |
| 0.5 | 1.0000 | 6.47584 | 0.816692 | -0.0861563 |
| 0.7 | 3.37555 | 43.2362 | 0.946066 | -0.0370453 |
| 0.9 | 21.8543 | 60.1635 | 0.954814 | -0.0167368 |

This estimate should be written in the form

\[
0 < \mu(x) < \frac{q^x}{(1-q) - q^x}, \quad \left( 0 < q < 1; \ x > \frac{\log(1-q)}{\log q} \right).
\]

Furthermore, from the estimate

\[
0 < \mu(x) < \frac{q^x}{(1-q) - q^x},
\]

the author in [9] concluded wrongly that

\[
\mu(x) = \frac{\theta q^x}{(1-q) - q^x},
\]

where \( \theta \) is a number independent of \( x \) between 0 and 1.

**Example 3.2.** We find counterexamples which show that \( \theta \) depends on \( x \) and \( q \). In the first table, we fixed \( q = 0.9 \) and take a few random values for \( x \). In another we changed the rule of variables.

Table 2. The dependence of parameter \( \theta \) from \( x \) and \( q \)

| \( x \) | \( q \) | \( \theta \) | \( x \) | \( q \) | \( \theta \) |
|---|---|---|---|---|---|
| 3.78377 | 0.9 | -7.27980 | 10.5 | 0.063920 | 1.00000 |
| 13.2544 | 0.9 | -1.58344 | 10.5 | 0.234682 | 1.00000 |
| 20.6473 | 0.9 | -0.139893 | 10.5 | 0.494904 | 0.99898 |
| 25.7471 | 0.9 | 0.342512 | 10.5 | 0.618621 | 0.98504 |
| 32.2948 | 0.9 | 0.673069 | 10.5 | 0.806515 | 0.473541 |
| 43.8850 | 0.9 | 0.904181 | 10.5 | 0.915828 | -0.419682 |

In continuation, the author in [9] got the wrong formulas (2.21)-(2.27). He concluded that

\[
a_p = \sqrt{\frac{2}{[2]q}}\Gamma_{q^2}(1/2),
\]

and

\[
\Gamma_q(x) = \sqrt{\frac{2}{[2]q}}\Gamma_{q^2}(1/2)(1-q)^{1/2-x} e^\theta \frac{q^x}{x!} \quad (0 < \theta < 1).
\]

The following wrong version of the \( q \)-Gauss multiplication formula was provided

\[
[n]^{1/2-x} [2]^{(n-1)/2} [q^{n-1}/q^2} (1/2) \Gamma_q(x) = \prod_{k=0}^{n-1} \Gamma_q \left( \frac{x+k}{n} \right) \quad (x > 0; \ n \in \mathbb{N}).
\]

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In a special case, for $n = 2$, it agrees with the exact $q$–Legendre relation. Also, when $q \to 1$, it reduces to well-known formulas for the gamma function.

4. Bounds of the $q$-gamma function

Let

$$g(x) = \ln \Gamma_q(x)$$

Since

$$g(x + 1) = \ln \Gamma_q(x + 1) = \ln ([x]_q \Gamma_q(x)) = \ln [x]_q + g(x),$$

(4.1)

by induction, we get

$$g(x + n) = \sum_{k=0}^{n-1} \ln [x + k]_q + g(x) \quad (n \in \mathbb{N}).$$

In the paper [2] it was proven that $g(x)$ is a convex function.

**Lemma 4.1.** If $x \in (0, 1)$ and $n \in \mathbb{N}$, then

$$g(n) + x \ln [x + n - 1]_q \leq g(x + n) \leq (1 - x)g(n) + xg(n + 1)$$

**Proof.** Since

$$x + n = (1 - x)n + x(n + 1),$$

we can write

$$g(x + n) = g((1 - x)n + x(n + 1)) \leq (1 - x)g(n) + xg(n + 1).$$

Let us find a lower bound for $\Gamma_q(x)$. Since

$$n = (1 - x)(x + n) + x(x + n - 1),$$

and because of the convexity of the function $g(x)$, we have

$$g(n) \leq (1 - x)g(x + n) + xg(x + n - 1).$$

Applying (4.1), for $x \to x + n - 1$, we can write

$$g(x + n) = \ln [x + n - 1]_q + g(x + n - 1),$$

wherefrom

$$g(n) \leq (1 - x)g(x + n) + x(g(x + n) - \ln [x + n - 1]_q) = g(x + n) - x \ln [x + n - 1]_q,$$

i.e.,

$$g(n) + x \ln [x + n - 1]_q \leq g(x + n). \square$$

**Theorem 4.1.** The following bounds are valid:

$$[n - 1]_q! [n - 1 + x]_q^* \leq \Gamma_q(n + x) \leq [n - 1]_q! [n]_q^* \quad (n \in \mathbb{N}; 0 \leq x < 1).$$
Proof. According to the upper bound for $g(x)$, we get e.g.
\[ \ln \Gamma_q(x + n) \leq (1 - x) \ln \Gamma_q(n) + x \ln \Gamma_q(n + 1). \]
Since the real logarithm is an increasing and continuous function, we have
\[ \Gamma_q(x + n) \leq (\lceil n - 1 \rceil_q !)^{1 - x} \lceil n \rceil_q ! x, \]
wherefrom
\[ \Gamma_q(x + n) \leq (\lceil n - 1 \rceil_q ! \lceil n \rceil_q) ! x. \]

According to the lower bound for $g(x)$, we get
\[ \ln \Gamma_q(n) + x \ln [n + n - 1]_q \leq \ln \Gamma_q(x + n), \]
i.e.,
\[ \Gamma_q(n) [n + n - 1]_q^x \leq \Gamma_q(n + x). \]

**Theorem 4.2.**
\[ [n - (1 - x)]_q \leq \left( \frac{\Gamma_q(n + x)}{\lceil n \rceil_q !} \right)^{1/x} \leq [n]_q \quad (n \in \mathbb{N}_0; 0 \leq x < 1). \]

Fig. 1. $\Gamma_q(x)$ and its boundary functions (green and blue) for $q = 0.5$.  

**Theorem 4.3.** For any $n \in \mathbb{N}$ and $x \in (0, 1)$ there exists $\theta = \theta(n, x, q) \in (0, 1)$ such that
\[ \Gamma_q(n + x) = \lceil n - 1 \rceil_q ! \lceil n - \theta (1 - x) \rceil_q ! x. \]
Introducing $y = n + x \; (n \in \mathbb{N}_0; 0 \leq x < 1)$ and denoting $n = \lfloor y \rfloor$, we can write
\[ \lceil \lfloor y \rfloor - 1 \rceil_q ! \lfloor y - 1 \rfloor_q^{- \lfloor y \rfloor} \leq \Gamma_q(y) \leq \lceil \lfloor y \rfloor - 1 \rceil_q ! \lceil \lfloor y \rfloor \rceil_q^{- \lfloor y \rfloor} \quad (y > 1). \]

**Theorem 4.4.** For any $y \in (1, +\infty) \setminus \mathbb{N}$, there exists $\theta = \theta(y, q) \in (0, 1)$ such that
\[ \Gamma_q(y) = \lceil \lfloor y \rfloor - 1 \rceil_q ! \lfloor y \rfloor - \theta (1 - (y - \lfloor y \rfloor)) \rceil_q^{- \lfloor y \rfloor}. \]
Example 4.1. For $y = 15.5$ and $q = 0.1(0.1)0.9$, we have got the following values for $\theta$:

$$\hat{\theta} = \{0.9851, 0.4021, 0.4259, 0.4432, 0.4569, 0.47762, 0.4855, 0.4917\}.$$

Also, for $q = 0.5$ and $y = 2.31(2)22.31$, we have got $\theta \in (0.4468, 0.4623)$.

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