1 Introduction

Convex relaxations play an important role in designing efficient learning and recovery algorithms, as well as in statistical learning and online optimization. It is thus desirable to understand the convex hull of hypothesis sets, to obtain tractable relaxation to these convex hulls, and to understand the tightness of such relaxations.

In this paper we consider convex relaxations of two important problems, namely clustering and hamming embedding, and study the convex hulls of the corresponding hypothesis classes: of cluster incidence matrices and of similarity measures with a short hamming embedding. In section 2 we introduce these classes formally, and understand the relationship between them, showing how hamming embedding can be seen as a generalization of clustering. In section 3 we discuss their convex hull and its relationship to notion of Locality Sensitive Hashing (LSH) as studied by [6]. There has been several studies on different aspects of LSH (e.g. [16,10,7]).

More specifically, we focus on the asymmetric versions of these classes, which correspond to co-clustering (e.g. [11,3]) and asymmetric hamming embedding as recently introduced by [15]. We define the corresponding notion of an Asymmetric LSH, and show how it could be much more powerful then standard (symmetric) LSH (section 4).

Our main conclusion is that the convex hull of asymmetric clustering and hamming embedding is tightly captured by a shift-invariant modification of the max-norm—a tractable SDP-representable relaxation (Theorem 2 in section 5). We contrast this with the symmetric case, in which the corresponding SDP relaxation is not tight, highlighting an important distinction between symmetric and asymmetric clustering, embedding and LSH.
2 Clustering and Hamming Embedding

In this section we introduce the problems of clustering and hamming embedding, providing a unified view of both problems, with hamming embedding being viewed as a direct generalization of clustering. Our starting point, in any case, is a given similarity function \( \text{sim} : S \times S \rightarrow [-1, +1] \) over a (possibly infinite) set of objects \( S \). “Clustering”, as we think of it here, is the problem of partitioning the elements of \( S \) into disjoint clusters so that items in the same cluster are similar while items in different clusters are not similar. “Hamming Embedding” is the problem of embedding \( S \) into some hamming space such that the similarity between objects is captured by the hamming distance between their mappings.

2.1 Clustering

We represent a clustering of \( S \) as a mapping \( h : S \rightarrow \Gamma \), where \( \Gamma \) is a discrete alphabet. We can think of \( h \) as a function that assigns a cluster identity to each element, where the meaning of the different identities is arbitrary. The alphabet \( \Gamma \) might have a fixed finite cardinality \( |\Gamma| = k \), if we would like to have a clustering with a specific number of clusters. E.g., a binary alphabet corresponds to standard graph partitioning into two clusters. If \( |\Gamma| = k \), we can assume that \( \Gamma = [k] \). The alphabet \( \Gamma \) might be infinitely countable (e.g. \( \Gamma = \mathbb{N} \)), in which case we are not constraining the number of clusters.

The cluster incidence function \( \kappa_h : S \times S \rightarrow \{\pm 1\} \) associated with a clustering \( h \) is defined as \( \kappa_h(x, y) = 1 \) if \( h(x) = h(y) \) and \( \kappa_h(x, y) = -1 \) otherwise. For a finite space \( S \) of cardinality \( n = |S| \) we can think of \( \kappa_h \in \{\pm 1\}^{n \times n} \) as a permuted block-diagonal matrix. We denote the set of all valid cluster incidence functions over \( S \) with an alphabet of size \( k \) (i.e. with at most \( k \) clusters) as \( M_{S,k} = \{\kappa_h | h : S \rightarrow [k]\} \), where \( k = \infty \) is allowed.

With this notion in hand, we can think of clustering as a problem of finding a cluster incidence function \( \kappa_h \) that approximates a given similarity \( \text{sim} \), as quantified by objectives \( \min_{x,y} \mathbb{E}_{x,y}[|\kappa_h(x, y) - \text{sim}(x, y)|] \) or \( \max_{x,y} \mathbb{E}_{x,y}[|\text{sim}(x, y)\kappa_h(x, y)|] \) (this is essentially the correlation clustering objective). Since objectives themselves are convex in \( \kappa \), but the constraint that \( \kappa \) is a valid cluster incidence function is not a convex constraint, a possible approach is to relax the constraint that \( \kappa \) is a valid cluster incidence function, or in the finite case, a cluster incidence matrix. This is the approach taken by, e.g. [12,13], who relax the constraint to a trace-norm and max-norm constraint respectively. One of the questions we will be exploring here is whether this is the tightest relaxation possible, or whether there is a significantly tighter relaxation.

2.2 Hamming Embedding and Binary Matrix Factorization

In the problem of binary hamming embedding (also known as binary hashing), we want to find a mapping from each object \( x \in S \) to binary string \( b(x) \in \{\pm 1\}^d \).
such that similarity between strings is approximated by the hamming distance between their images:

$$\text{sim}(x, y) \approx 1 - \frac{2\delta_{\text{Ham}}(b(x), b(y))}{d}$$  \hspace{1cm} (1)

Calculating the hamming distance of two binary hashes is an extremely fast operation, and so such a hash is useful for very fast computation of similarities between massive collections of objects. Furthermore, hash tables can be used to further speed up retrieval of similar objects.

Binary hamming embedding can be seen as a generalization of clustering as follows: For each position $i = 1, \ldots, d$ in the hash, we can think of $b_i(x)$ as a clustering into two clusters (i.e. with $\Gamma = \{\pm1\}$). The hamming distance is then an average of the $d$ cluster incidence functions:

$$1 - \frac{2\delta_{\text{Ham}}(b(x), b(y))}{d} = \frac{1}{d} \sum_{i=1}^{d} \kappa_{b_i}(x, y).$$

Our goal then is to approximate a similarity function by an average of $d$ binary clusterings. For $d = 1$ this is exactly a binary clustering. For $d > 1$, we are averaging multiple binary clusterings.

Since we have $\langle b(x), b(y) \rangle = d - 2\delta_{\text{Ham}}(b(x), b(y))$, we can formulate the binary hashing problem as a binary matrix factorization where the goal is to approximate the similarity matrix by a matrix of the form $RR^\top$, where $R$ is a $d$-dimensional binary matrix:

$$\min_R \sum_{ij} \text{err}(\text{sim}(i, j), X(i, j)) \hspace{1cm} (2)$$

s.t. $X = RR^\top$

$$R \in \{\pm1\}^{n \times d}$$

where $\text{err}(x, y)$ is some error function such as $\text{err}(x, y) = |x - y|$.

Going beyond binary clustering and binary embedding, we can consider hamming embeddings over larger alphabets. That is, we can consider mappings $b : S \to \Gamma^d$, where we aim to approximate the similarity as in (1), recalling that the hamming distance always counts the number of positions in which the strings disagree. Again, we have that the length $d$ hamming embeddings over a (finite or infinitely countable) alphabet $\Gamma$ correspond to averages of $d$ cluster incidence matrices over the same alphabet $\Gamma$.

3 Locality Sensitive Hashing Schemes

Moving on from a finite average of clusterings, with a fixed number of components, as in hamming embedding, to an infinite average, we arrive at the notion of LSH as studied by [6].
Given a collection $S$ of objects, an alphabet $\Gamma$ and a similarity function $\text{sim} : S \times S \to [-1, 1]$ such that for any $x \in S$ we have $\text{sim}(x, x) = 1$, a locality sensitive hashing scheme (LSH) is a probability distribution on the family of clustering functions (hash functions) $\mathcal{H} = \{ h : S \to \Gamma \}$ such that:

$$E_{h \in \mathcal{H}}[\kappa_h(x, y)] = \text{sim}(x, y). \quad (3)$$

discuss similarity functions $\text{sim} : S \times S \to [0, 1]$ as so require

$$\mathbb{P}_{h \in \mathcal{H}}[h(x) = h(y)] = \text{sim}(x, y).$$

The definition (3) is equivalent, except it applies to the transformed similarity function $2\text{sim}(x, y) - 1$.

The set of all locality sensitive hashing schemes with an alphabet of size $k$ is nothing but the convex hull of the set $\mathcal{M}_{S,k}$ of cluster incidence matrices.

The importance of an LSH, as an object in its own right as studied by [6], is that a hamming embedding can be obtained from an LSH by randomly generating a finite number of hash functions from the distribution over the family $\mathcal{H}$. In particular, if we draw $h_1, \ldots, h_d$ i.i.d. from an LSH, then the length-$d$ hamming embedding $b(x) = [h_1(x), \ldots, h_d(x)]$ has expected square error

$$E[(\text{sim}(x, y) - \frac{1}{d} \sum \kappa_{h_d}(x, y) - \theta)^2] \leq \frac{1}{d}, \quad (4)$$

where the expectation is w.r.t. the sampling, and this holds for all $x, y$, and so also for any average over them.

### 3.1 $\alpha$-LSH

If the goal is to obtain an low-error embedding, the requirement (3) might be too harsh. If we are willing to tolerate a fixed offset between our embedding and the target similarity, we can instead require that

$$\alpha E_{h \in \mathcal{H}}[\kappa_h(x, y)] - \theta = \text{sim}(x, y). \quad (5)$$

where $\alpha, \theta \in \mathbb{R}$, $\alpha > 0$. A distribution over $h$ that obeys (5) is called an $\alpha$-LSH. We can now verify that, for $h_1, \ldots, h_d$ drawn i.i.d. from an $\alpha$-LSH, and any $x, y \in S$:

$$E \left[ (\text{sim}(x, y) - \frac{\alpha}{d} \sum \kappa_{h_d}(x, y) - \theta)^2 \right] \leq \frac{\alpha^2}{d}. \quad (6)$$

The length of the LSH required to achive accurate approximation of a similarity function thus scales quadartically with $\alpha$, and it is therefor desireable to obtain an $\alpha$-LSH with as low an $\alpha$ as possible (note that $\text{sim}(x, x) = 1$, implies $\theta = \alpha - 1$, and so we must allow a shift if we want to allow $\alpha \neq 1$).

Unfortunately, even the requirement (5) of an $\alpha$-LSH is quite limiting and difficult to obey, as captured by the following theorem, which is based on lemmas 2 and 3 of [6].
Claim 1. For any finite or countable alphabet $\Gamma$, $k = |\Gamma| \geq 2$, a similarity function $\text{sim}$ has an $\alpha$-LSH over $\Gamma$ for some $\alpha$ if and only if $D(x, y) = \frac{1 - \text{sim}(x, y)}{2}$ is embeddable to hamming space with no distortion.

Proof. Given metric spaces $(X, d)$ and $(X, d')$ any map $f : X \to X'$ is called a metric embedding. The distortion of such an embedding is defined as:

$$\beta = \max_{x, y \in X} \frac{d(x, y)}{d'(f(x), f(y))}, \max_{x, y \in X} \frac{d'(f(x), f(y))}{d(x, y)}$$

We first show that if there exist an $\alpha$-LSH for function $\text{sim}(x, y)$ then $\frac{1 - \text{sim}(x, y)}{2}$ is embeddable to hamming space with no distortion. An $\alpha$-LSH for function $\text{sim}(x, y)$ corresponds to an $\text{LSH}$ for function $1 - \frac{1 - \text{sim}(x, y)}{\alpha}$. Using lemma 3 in [6], we can say that $\frac{1 - \text{sim}(x, y)}{\alpha}$ can be isometrically embedded in the Hamming cube which means $1 - \text{sim}(x, y)$ can be embedded in Hamming cube with no distortion.

As a result of Claim 1, it can be shown that given any large enough set of low dimensional unit vectors, there is no $\alpha$-LSH for the Euclidian inner product.

Claim 2. Let $\{x^{(1)}, \ldots, x^{(n)}\}$ be an arbitrary set of unit vectors in the unit sphere. Let $Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle$ for $1 \leq i, j \leq n$. If $d < \log_2 n$, then there is no $\alpha$-LSH for $Z$.

Proof. According to [8] (see also [4]), if $d < \log_2 n$ then in any set of $n$ points in $d$-dimensional Euclidian space, there exist at least three points that form an obtuse triangle. Equivalently, there exist three vectors $x, y$ and $z$ in any set of $n$ different $d$-dimensional unit vectors such that:

$$\langle z - x, z - y \rangle < 0$$

We rewrite the above inequality as:

$$(1 - \langle z, x \rangle) + (1 - \langle z, y \rangle) < (1 - \langle x, y \rangle)$$
The above inequality implies that the distance measure \( \Delta_{ij} = (1 - Z_{ij})/2 \) is not a metric. Consequently, according to Claim 4 since \( \Delta_{ij} = (1 - Z_{ij})/2 \) is not a metric, there is no \( \alpha \)-LSH for the matrix \( Z \).

As noted by [6] (and stated in claim 2), we can therefore unfortunately conclude that there is no \( \alpha \)-LSH for several important similarity measures such as the Euclidean inner product, Overlap coefficient and Dice’s coefficient. Note that based on Claim 4 even a finite positive semidefinite similarity matrix is not necessarily \( \alpha \)-LSHable.

3.2 Generalized \( \alpha \)-LSH

In the following section, we will see how to break the barrier imposed by Claim 4 by allowing asymmetry, highlighting the extra power asymmetry affords us. But before doing so, let us consider a different attempt at relaxing the definition of an \( \alpha \)-LSH, motivated by the work of [5] and [1]: in order to uncouple the shift \( \theta \) from the scaling \( \alpha \), we will allow for a different, arbitrary, shift on the self-similarities \( \text{sim}(x, x) \) (i.e. on the diagonal of \( \text{sim} \)).

We say that a probability distribution over \( \mathcal{H} = \{ h : S \to \Gamma \} \) is a Generalized \( \alpha \)-LSH, for \( \alpha > 0 \) if there exist \( \theta, \gamma \in \mathbb{R} \) such that for all \( x, y \):

\[
\alpha E_{h \in \mathcal{H}}[\kappa_h(x, y)] = \text{sim}(x, y) + \theta + \gamma 1_{x=y}
\]

With this definition, then any symmetric similarity function, at least over a finite domain, admits a Generalized \( \alpha \)-LSH, with a sufficiently large \( \alpha \):

Claim 3. For a finite set \( S, |S| = n \), for any symmetric \( \text{sim} : S \times S \to [-1, 1] \) with \( \text{sim}(x, x) = 1 \), there exists a Generalized \( \alpha \)-LSH over a binary alphabet \( \Gamma \) (\( |\Gamma| = 2 \)) where \( \alpha = O((1 - \lambda_{\min}) \log n) \)-LSH, and \( \lambda_{\min} \) is the smallest eigenvalue of the matrix \( \text{sim} \).

Proof. We observe that \( \text{sim} - \lambda_{\min} I \) is a positive semidefinite matrix. According to [5], if a matrix \( Z \) with unit diagonal is positive semidefinite, then there is a probability distribution over a family \( \mathcal{H} \) of hash functions such that for any \( x \neq y \):

\[
E_{h \in \mathcal{H}}[h(x)h(y)] = \frac{Z(x, y)}{C \log n}
\]

We let \( Z(x, y) = (\text{sim}(x, y) - \lambda_{\min} 1_{x=y})/(1 - \lambda_{\min}) \). Matrix \( Z \) is positive semidefinite and has unit diagonal. Hence, there is a probability distribution over a family \( \mathcal{H} \) of hash functions such that

\[
E_{h \in \mathcal{H}}[h(x)h(y)] = \frac{\text{sim}(x, y) - \lambda_{\min} 1_{x=y}}{C(1 - \lambda_{\min}) \log n},
\]

equivalently

\[
(C(1 - \lambda_{\min}) \log n) \cdot E_{h \in \mathcal{H}}[\kappa_h(x, y)] = \text{sim}(x, y) - \lambda_{\min} 1_{x=y}.
\]
It is important to note that $\lambda_{\min}$ could be negative, and as low as $\lambda_{\min} = -\Omega(n)$. The required $\alpha$ might therefore be as large as $\Omega(n)$, yielding a terrible LSH.

4 Asymmetry

In order to allow for greater power, we now turn to Asymmetric variants of clustering, hamming embedding, and LSH.

Given two collections of objects $S, T$, which might or might not be identical, and an alphabet $\Gamma$, an asymmetric clustering (or co-clustering [11]) is specified by pair of mappings $f : S \to \Gamma$ and $g : T \to \Gamma$ and is captured by the asymmetric cluster incidence matrix $\kappa_{f,g}(x, y)$ where $\kappa_{f,g}(x, y) = 1$ if $f(x) = g(y)$ and $\kappa_{f,g}(x, y) = -1$ otherwise. We denote the set of all valid asymmetric cluster incidence functions over $S, T$ with an alphabet of size $k$ as $M_{(S,T),k} = \{\kappa_{f,g} | f : S \to [k], g : T \to [k]\}$, where we again also allow $k = \infty$ to correspond to a countable alphabet $\Gamma = \mathbb{N}$.

Likewise, an asymmetric binary embedding of $S, T$ with alphabet $\Gamma$ consists of a pair of functions $f : S \to \Gamma^d, g : T \to \Gamma^d$, where we approximate a similarity as:

$$\text{sim}(x, y) \approx 1 - \frac{2\delta_{\text{Ham}}(f(x), g(y))}{d} = \frac{1}{d} \sum_{i=1}^{d} \kappa_{f_i,g_i}(x, y).$$  (7)

That is, in asymmetric hamming embedding, we approximate a similarity as an average of $d$ asymmetric cluster incidence matrices from $M_{(S,T),k}$.

In a recent work, [15] showed that even when $S = T$ and the similarity function sim is a well-behaved symmetric similarity function, asymmetric binary embedding could be much more powerful in approximating the similarity, using shorter lengths $d$, both theoretically and empirically on data sets of interest. That is, these concepts are relevant and useful not only in an a-priory asymmetric case where $S \neq T$ or sim is not symmetric, but also when the target similarity is symmetric, but we allow an asymmetric embedding. We will soon see such gaps also when considering the convex hulls of $M_{S,k}$ and $M_{(S,T),k}$, i.e. when considering LSHs. Let us first formally define an asymmetric $\alpha$-LSH.

Given two collections of objects $S$ and $T$, an alphabet $\Gamma$, a similarity function $\text{sim} : S \times T \to [-1,1]$, and $\alpha > 0$, we say that an $\alpha$-ALSH is a distribution over pairs of functions $f : S \to \Gamma$, $g : T \to \Gamma$, or equivalently over $M_{(S,T),|\Gamma|}$, such that for some $\theta \in \mathbb{R}$ and all $x \in S, y \in T$:

$$\alpha\mathbb{E}_{(f,g) \in \mathcal{F} \times \mathcal{G}}[\kappa_{f,g}(x, y))] - \theta = \text{sim}(x, y).$$  (8)

To understand the power of asymmetric LSH, recall that many symmetric similarity functions do not have an $\alpha$-LSH for any $\alpha$. On the other hand, any similarity function over finite domains necessarily has an $\alpha$-ALSH:

Claim 4. For any similarity function $\text{sim} : S \times T \to [-1,1]$ over finite $S, T$, there exists an $\alpha$-ALSH with $\alpha \leq \min\{|S|, |T|\}$
This is corollary of Theorem 2 that will be proved later in section 5. The proof follows from Theorem 2 the following upper bound on the max-norm:

\[ \|Z\|_{\text{max}} \leq \text{rank}(Z) \cdot \|Z\|_{\infty}^2 \]

where \( \|Z\|_{\infty}^2 = \max_{x,y} |Z(x, y)| \).

In section 3, we saw that similarity functions that do not admit an \( \alpha \)-LSH, still admit Generalized \( \alpha \)-LSH. However, the gap between the \( \alpha \) required for a Generalized \( \alpha \)-LSH and that required for an \( \alpha \)-ALSH might be as large as \( \Omega(|S|) \):

**Theorem 1.** For any even \( n \), there exists a set \( S \) of \( n \) objects and a similarity \( Z: S \times S \to \mathbb{R} \) such that

- there is a binary \( 3K_R \)-ALSH for \( Z \), where \( K_R \approx 1.79 \) is Krivine’s constant;
- there is no Generalized \( \alpha \)-LSH for any \( \alpha < n - 1 \).

**Proof.** Let \( S = [n] \) and \( Z \) be the following similarity matrix:

\[
Z = 2I_{n \times n} + \begin{bmatrix}
-1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -1
\end{bmatrix}
\]

Now we use Theorem 2, which we will prove later (our proof of Theorem 2 does not rely on the proof of this theorem). Using triangle inequality property of the norm, we have \( \|Z\|_{\text{max}} \leq \|Z - 2I_{n \times n}\|_{\text{max}} + \|2I_{n \times n}\|_{\text{max}} = 3 \); and by Theorem 2 there is a \( 3K_R \)-ALSH for \( Z \). Looking at the decomposition of \( Z \), it is not difficult to see that the smallest eigenvalue of \( Z \) is \( 2 - n \). So in order to have a positive semidefinite similarity matrix, we need \( \gamma \) to be at least \( n - 2 \) and \( \theta \) to be at least \( -1 \) (otherwise the sum of elements of \( Z + \theta + (n - 2)I \) will be less than zero and so \( Z + \theta + (n - 2)I \) will not be positive semidefinite). So \( \alpha = \theta + \gamma \) is at least \( n - 1 \).

### 5 Convex Relaxations, \( \alpha \)-LSH and Max-norm

We now turn to two questions which are really the same: can we get a tight convex relaxation of the set \( M_{(S, T), k} \) of (asymmetric) clustering incidence functions, and can we characterize the values of \( \alpha \) for which we can get an \( \alpha \)-ALSH for a particular similarity measure.

For notational simplicity, we will now fix \( S \) and \( T \) and use \( M_k \) to denote \( M_{(S, T), k} \).

#### 5.1 The Ratio Function

The tightest possible convex relaxation of \( M_k \) is simply its convex hull \( \text{conv} M_k \). Assuming \( P \neq NP \), \( \text{conv} M_k \) is not polynomially tractable. What we ask here is
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whether he have a tractable tight relaxation of conv $M_k$. To measure tightness of some convex $B \supseteq M_k$, for each $Z \in B$, we will bound its cluster ratio:

$$\rho_k(Z) = \min\{r | Z \in r \text{conv } M_k\} = \min\{r | Z/r \in \text{conv } M_k\}.$$

That is, by how much do we have to inflate $M_k$ so that includes $Z \in B$. The supremum $\rho_k(B) = \sup_{Z \in B} \rho_k(Z)$ is then the maximal inflation ratio between $\text{conv } M_k$ and $B$, i.e. such that $\text{conv } M_k \subseteq B \subseteq \rho_k \text{conv } M_k$. Similarly, we define the centralized cluster ratio as:

$$\hat{\rho}_k(Z) = \min_{\theta \in \mathbb{R}} \min\{r | Z - \theta \in r \text{conv } M_k\}.$$

This is nothing but the lowest $\alpha$ for which we have an $\alpha$-ALSH:

**Claim 5.** For any similarity function $\text{sim}(x, y)$, $\hat{\rho}_k(\text{sim})$ is equal to the smallest $\alpha$ s.t. there exists an $\alpha$-ALSH for $\text{sim}$ over alphabet of cardinality $k$.

**Proof.** We write the problem of minimizing $\alpha$ in $\alpha$-ALSH as:

$$\min_{\theta \in \mathbb{R}, \alpha \in \mathbb{R}^+} \alpha \quad \text{s.t. } \text{sim}(x, y) = \alpha E_{(f, g) \in \mathcal{F} \times \mathcal{G}}[\kappa_{f, g}(x, y)] - \theta\quad (9)$$

We know that:

$$E_{(f, g) \in \mathcal{F} \times \mathcal{G}}[\kappa_{f, g}(x, y)] = \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} \kappa_{f, g}(x, y)p(f, g)$$

where $p(f, g)$ is the joint probability of hash functions $f$ and $g$. Define $\mu(f, g) = \alpha p(f, g)$ and write:

$$\alpha = \alpha \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} p(f, g) = \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} \alpha p(f, g) = \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} \mu(f, g)$$

We have:

$$\alpha \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} \kappa_{f, g}(x, y)p(f, g) - \theta = \sum_{f \in M_{S, k}} \sum_{g \in M_{T, k}} \kappa_{f, g}(x, y)\mu(f, g) - \theta$$

Substituting the last two equalities into formulation 9 gives us the formulation for centralized cluster ratio.

Our main goal in this section is to obtain tight bounds on $\rho_k(Z)$ and $\hat{\rho}_k(Z)$.

*The Ratio Function and Cluster Norm* The convex hull conv $M_k$ is related to the cut-norm, and its generalization the cluster-norm, and although the two are not identical, its worth understanding the relationship.
For \( k = 2 \), the ratio function is a norm, and is in fact the dual of a modified cut-norm:

\[
\rho^*_2(W) = \|W\|_{C,2} = \max_{u:S \to \{\pm 1\}, v:S \to \{\pm 1\}} \sum_{x \in S, y \in T} W(x, y)u(x)v(y) \tag{10}
\]

The norm \( \|W\|_{C,2} \) is a variant of the cut-norm, and is always within a factor of four from the cut-norm as defined in, e.g. [1]. The set \( \text{conv}M_2 \) in this case is the unit ball of the modified cut-norm.

For \( k > 2 \), the ratio function is not a norm, since \( M_k \), for \( k > 2 \), is not symmetric about the origin: we might have \( Z \in M_k \) but \( -Z \not\in M_k \) and so \( \rho_k(Z) \neq \rho_k(-Z) \). A ratio function defined with respect the symmetric convex hull of \( \text{conv}(M_k \cup -M_k) \), is a norm, and is dual to the following cluster norm, which is a generalization of the modified cut-norm:

\[
\|W\|_{C,k} = \max_{u:S \to \Gamma, v:S \to \Gamma} \sum_{x \in S, y \in T} W(x, y)\kappa_{u,v}(x, y) \tag{11}
\]

5.2 A Tight Convex Relaxation using the Max-Norm

Recall that the max-norm (also known as the \( \gamma_2 : \ell_1 \to \ell_\infty \) norm) of a matrix is defined as [18]:

\[
\|Z\|_{\max} = \min_{U,V} \max(U_{2,\infty}, V_{2,\infty})
\]

where \( \|U\|_{2,\infty} \) is the maximum \( \ell_2 \) norm of rows of the matrix \( U \). The max-norm is SDP representable and thus tractable [17]. Even when \( S \) and \( T \) are not finite, and thus \( \text{sim} \) is not a finite matrix, the max-norm can be defined as above, where now \( U \) and \( V \) can be thought of as mappings from \( S \) and \( T \) respectively into a Hilbert space, with \( \text{sim}(x, y) = \langle UV^\top(x, y) = \langle U(x), V(y) \rangle \) and \( \|U\|_{2,\infty} = \sup_x \|U(x)\| \).

We also define the centralized max-norm, which, even though it is not a norm, we denote as:

\[
\|Z\|_{\max}^{\theta} = \min_{\theta} \|Z - \theta\|_{\max}
\]

The centralized max-norm is also SDP-representable.

Our main result is that the max-norm provides a tight bound on the ratio function:

**Theorem 2.** For any similarity function \( \text{sim} : S \times T \to \mathbb{R} \) we have that:

\[
\frac{1}{2} \|\text{sim}\|_{\max} \leq \rho_2(\text{sim}) \leq \rho(\text{sim}) \leq \hat{\rho}_k(\text{sim}) \leq \rho_2(\text{sim}) \leq K \|\text{sim}\|_{\max}
\]

and also

\[
\frac{1}{3} \|\text{sim}\|_{\max} \leq \rho(\text{sim}) \leq \rho_k(\text{sim}) \leq \rho_2(\text{sim}) \leq K \|\text{sim}\|_{\max}
\]

where all inequalities are tight and we have \( 1.67 \leq K_G \leq K \leq K_R \leq 1.79 \) (\( K_G \) is Grothendieck’s constant and \( K_R \) is Krivine’s constant).
Clustering, Hamming Embedding, Generalized LSH and the Max Norm

Considering the dual view of \( \rho(\text{sim}) \), the theorem can also be viewed in two ways: First, we see that the centralized max-norm provides a tight characterization (up to a small constant factor) of the smallest \( \alpha \) for which we can obtain an \( \alpha \)-LSH. In particular, since for domains (i.e. finite matrices) the max-norm is always finite, this establishes that we always have an \( \alpha \)-LSH, as claimed in Claim 4. We also used it in Theorem 1 to establish the existence of an \( \alpha \)-LSH for a specific, small, \( \alpha \).

Second, bounding the ratio function establishes that the max-norm ball is a tight tractable relaxation of \( \text{conv} \mathcal{M}_k \):

\[
\{ Z \mid \|Z\|_{\text{max}} \leq 1/K \} \subseteq \text{conv} \mathcal{M}_k \subseteq \{ Z \mid \|Z\|_{\text{max}} \leq 3 \} \tag{12}
\]

Third, we see the effect of the alphabet size \( k \) (number of clusters) on the convex hull is very limited.

The Symmetric Case. It is not difficult to show that the lower bounds for \( \alpha \)-LSH are the same as for \( \alpha \)-ALSH and the inequalities are tight. However, there are no upper bounds for \( \alpha \)-LSH similar to those for \( \alpha \)-ALSH. Specifically, let \( \hat{\alpha} \) and \( \hat{\alpha}_g \) be the smallest values of \( \alpha \) such that there is an \( \alpha \)-LSH for \text{sim} and there is a generalized \( \alpha \)-LSH for \text{sim}, respectively. Note that for some similarity functions \text{sim} there is no \( \alpha \)-LSH at all; that is, \( \hat{\alpha} = \infty \) and \( \|\text{sim}\|_{\text{max}} < \infty \). Also, as Theorem 1 shows, there is a similarity function \text{sim} such that

\[
\|\text{sim}\|_{\text{max}} = O(1) \quad \text{but} \quad \hat{\alpha}_g \geq n - 1.
\]

Moreover, it follows from the result of [2] that there is no efficiently computable upper bound \( \beta \) for \( \hat{\alpha}_g \) such that

\[
\frac{\beta}{\log n} \leq \hat{\alpha}_g \leq \beta
\]

(under a standard complexity assumption that \( NP \not\subseteq \text{DTIME}(n^{\log^3 n}) \)). That is, neither the max-norm nor any other efficiently computable norm of \text{sim} gives a constant factor approximation for \( \hat{\alpha}_g \).

In the remainder of this section we prove a series of lemmas corresponding to the inequalities in Theorem 2.

5.3 Proofs

Lemma 1. For any two sets \( S \) and \( T \) of objects and any function \( \text{sim} : S \times T \rightarrow R \), we have that \( \rho_2(\text{sim}) \leq 2\rho(\text{sim}) \) and the inequality is tight.

Proof. Using Claim 5 all we need to do is to prove that given the function \text{sim}, if there exist an \( \alpha \)-ALSH with arbitrary cardinality, then we can find a binary \( 2\alpha - \text{ALSH} \). In order to do so, we assume that there exists an \( \alpha \)-ALSH for family \( \mathcal{F} \) and \( \mathcal{G} \) of hash functions such that:

\[
\alpha\mathbb{E}_{(f,g) \in \mathcal{F} \times \mathcal{G}}[\kappa_{f,g}(x,y)] = \text{sim}(x,y) + \theta
\]


where \( f : S \to \Gamma \) and \( g : T \to \Gamma \) are hash functions. Now let \( \mathcal{H} \) be a family of pairwise independent hash functions of the form \( \Gamma \to \{\pm 1\} \) such that each element \( \gamma \in \Gamma \), has the equal chance of being mapped into -1 or 1. Now, we have that:

\[
2\alpha \mathbb{E}_{h \in \mathcal{H}, (f,g) \in F \times G} [\kappa_{ho,f,hog}(x,y)] = 2\alpha \mathbb{E}_{h \in \mathcal{H}, (f,g) \in F \times G} [h(f(x))h(g(y))] = 2\alpha (2P_{h \in \mathcal{H}, (f,g) \in F \times G} [h(f(x)) = h(g(y))] - 1)
\]

The tightness can be demonstrated by the example \( \text{sim}(x,y) = 2 \cdot x - y - 1 \) when \( S \) is not finite.

**Lemma 2.** For any two sets \( S \) and \( T \) of objects and any function \( \text{sim} : S \times T \to \mathbb{R} \), we have that \( \|\text{sim}\|_{\max} \leq \rho_2(\text{sim}) \) and the inequality is tight.

**Proof.** Without loss of generality, we assume that \( \Gamma = \{\pm 1\} \). We want to solve the following optimization problem:

\[
\rho_2(\text{sim}) = \min_{\mu : M_{S,2} \times M_{T,2} \to \mathbb{R}^+} \sum_{f \in M_{S,2}} \sum_{g \in M_{T,2}} \mu(f,g) \\
\text{s.t. } \text{sim}(x,y) = \sum_{f \in M_{S,2}} \sum_{g \in M_{T,2}} \kappa_{f,g}(x,y) \mu(f,g)
\]

For any \( x \in S \) and \( y \in T \), we define two new function variables \( \ell_x : M_{S,2} \times M_{T,2} \to \mathbb{R} \) and \( r_y : M_{S,2} \times M_{T,2} \to \mathbb{R} \):

\[
\ell_x(f,g) = \sqrt{\mu(f,g)f(x)} \\
r_y(f,g) = \sqrt{\mu(f,g)g(y)}
\]

Since cluster incidence matrix can be written as \( \kappa_{f,g}(x,y) = f(x)g(y) \), we have \( \text{sim}(x,y) = (\ell_x, r_y) \) and \( \|\ell_x\|^2 = \sum_{f \in M_{S,2}} \sum_{g \in M_{T,2}} \mu(f,g) \). Therefore, we rewrite the optimization problem as:

\[
\rho_2(\text{sim}) = \min_{t, \ell_x, r_y, \mu : M_{S,2} \times M_{T,2} \to \mathbb{R}^+} \ t \\
\text{s.t. } (\ell_x, r_y) = \text{sim}(x,y) \\
\|\ell_x\|^2 \leq t \\
\|r_y\|^2 \leq t \\
\ell_x(f,g) = \sqrt{\mu(f,g)f(x)} \\
r_y(f,g) = \sqrt{\mu(f,g)g(y)}
\]
Finally, we relax the above problem by removing the last two constraints:

$$\|\text{sim}\|_{\text{max}} = \min_{t, t', x} t$$

s.t. \( \langle \ell_x, r_y \rangle = \text{sim}(x, y) \)

$$\|\ell_x\|_2^2 \leq t$$

$$\|r_x\|_2^2 \leq t$$

The above problem is a max-norm problem and the solution is \(\|\text{sim}\|_{\text{max}}\). Therefore, \(\|\text{sim}\|_{\text{max}} \leq \rho_2(\text{sim})\). Taking the function \(\text{sim}(x, y)\) to be a binary cluster incidence function will indicate the tightness of the inequality.

\[\text{Lemma 3. (Krivine’s lemma)}\]

For any two sets of unit vectors \(\{u_i\}\) and \(\{v_j\}\) in a Hilbert space \(H\), there are two sets of unit vectors \(\{u_i'\}\) and \(\{v_j'\}\) in a Hilbert space \(H'\) such that for any \(u_i\) and \(v_j\), \(\sin(c(u_i, v_j)) = \langle u_i', v_j' \rangle\) where \(c = \sinh^{-1}(1)\).

\[\text{Lemma 4.}\]

For any two sets \(S\) and \(T\) of objects and any function \(\text{sim} : S \times T \rightarrow R\), we have that \(\rho_2(\text{sim}) \leq K\|\text{sim}\|_{\text{max}}\) where \(1.67 \leq K_G \leq K \leq K_R \leq 1.79\) (\(K_G\) is Grothendieck’s constant and \(K_R\) is Krivine’s constant).

\[\text{Proof.}\]

A part of the proof is similar to \[1\]. Let \(\ell_x\) and \(r_y\) be the solution to the max-norm formulation \[13\]. If we use Lemma \[3\] on the normalized \(\ell_x/\|\ell_x\|_2\) and \(r_y/\|r_y\|_2\) in Hilbert space \(H\) and we call the new vectors \(\ell_x'\) and \(r_y'\) in Hilbert space \(H'\), we have that:

$$\sin \left( \frac{c.S(x, y)}{\|\ell_x\|_2\|r_x\|_2} \right) = \langle \ell_x', r_y' \rangle.$$

If \(z\) is a random vector chosen uniformly from \(H'\), by Lemma \[3\] we have:

$$\mathbb{E}(\text{sign}(\langle \ell_x', z \rangle), \text{sign}(\langle r_y', z \rangle)) = \frac{2}{\pi} \arcsin(\langle \ell_x', r_y' \rangle) = \frac{2c}{\pi\|\ell_x\|_2\|r_y\|_2} \sin(x, y)$$

Now if we set the hashing function \(f(x) = s(x) \\text{sign}(\langle \ell_x', z \rangle)\) where \(s(x) = 1\) with probability \(\frac{1}{2} + \frac{\|\ell_x\|_2}{2\sqrt{t}}\) and \(s(x) = -1\) with probability \(\frac{1}{2} - \frac{\|\ell_x\|_2}{2\sqrt{t}}\), we have that:

$$\mathbb{E}[f(x) \\text{sign}(\langle r_y', z \rangle)] = \left(\frac{1}{2} + \frac{\|\ell_x\|_2}{2\sqrt{t}}\right) \frac{2c}{\pi\|\ell_x\|_2\|r_y\|_2} \sin(x, y)$$

$$- \left(\frac{1}{2} - \frac{\|\ell_x\|_2}{2\sqrt{t}}\right) \frac{2c}{\pi\|\ell_x\|_2\|r_y\|_2} \sin(x, y)$$

$$= \frac{2c}{\pi\sqrt{t}\|r_y\|_2} \sin(x, y)$$

If we do the same procedure on \(g(y) = s'(x) \\text{sign}(\langle r_y', z \rangle)\), we will have:

$$\mathbb{E}[f(x), g(y)] = \frac{2c}{t\pi} \sin(x, y)$$
By setting $\mu(f, g) = \pi \|sim\|_{\max} p(f, g)$ where $p(f, g)$ is the probability distribution over the defined $f$ and $g$, we can see that such $\mu(f, g)$ is a feasible solution for the formulation of cluster ratio and we have:

$$
\rho_2(sim) \leq \sum_{f \in M_{S, 2}} \sum_{g \in M_{T, 2}} \mu(f, g) = \frac{\pi}{2c} \|sim\|_{\max} = K_R \|sim\|_{\max}
$$

The inequality $K_G \leq K$ is known due to [1].

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A Random Matrices

In this section we investigate the locality sensitive hashing schemes on random p.s.d matrices. We generate a random \( n \times n \) positive semidefinite matrix \( Z \) of rank at most \( d \) by choosing \( n \) \( d \)-dimensional unit vectors \( x^{(i)} \) uniformly at random from the unit ball and set \( Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle \). Since we are generating the data randomly and \( E[Z_{ij}] = 0 \), we don’t expect to observe major changes by thresholding the matrix. So our analysis is limited to the LSH without thresholding, i.e. \( \theta = 0 \).

Since based on Theorem 2, we already know given any set of unit vectors \( x^{(1)}, \ldots, x^{(n)} \) and \( Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle \), there is a \( k_{R} \)-ALSH for the matrix \( Z \), we are just interested in investigating the symmetric LSH for these random vectors.

A.1 LSH

For the symmetric LSH, we only have two possibilities: either having LSH with \( \alpha = 1 \) or not having any LSH. We also know from Claim 2 that there is no \( \alpha \)-LSH if \( d < \log_2 n \) because in that case \( D(x^{(i)}, x^{(j)}) = 1 - Z_{ij} \) is not metric. So we want to know the conditions under which the distance will be a metric and also the conditions for having \( \alpha \)-LSH with high probability.

Lemma 5. If \( x \) is a \( d \)-dimensional unit vector and \( \tilde{x} \) is its projection onto another unit vector that is sampled uniformly at random from the unit sphere, then for any \( t > 1 \), we have \( E[\|\tilde{x}\|_2^2] = \frac{1}{t} \) and moreover, \( P(\|\tilde{x}\|_2^2 \geq \frac{1}{t}) \leq e^{\frac{1}{1 - t log_2 t}} \).

Lemma 6. Let \( \{x^{(1)}, \ldots, x^{(n)}\} \) be a set of unit vectors sampled uniformly at random from the unit sphere and for any \( 1 \leq i, j \leq n \) let \( Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle \). If \( d \geq 72 \log_2 n + \log_2 \frac{1}{\delta} \), then the distance measure \( \Delta_{ij} = 1 - Z_{ij} \) is metric with probability at least \( 1 - \delta \).

Proof. The distance measure \( \Delta_{ij} = 1 - \langle x^{(i)}, x^{(j)} \rangle \) is not a metric if and only if there exist \( i, j \) and \( k \) such that

\[
(1 - \langle x^{(i)}, x^{(j)} \rangle) + (1 - \langle x^{(i)}, x^{(k)} \rangle) < (1 - \langle x^{(j)}, x^{(k)} \rangle)
\]

A simple reordering of the above inequality gives us:

\[
C_{ijk} = \langle x^{(i)}, x^{(j)} \rangle + \langle x^{(i)}, x^{(k)} \rangle - \langle x^{(j)}, x^{(k)} \rangle > 1
\]

For this inequality to hold, the absolute value of at least one of the inner products \( \langle x^{(i)}, x^{(j)} \rangle, \langle x^{(i)}, x^{(k)} \rangle, \langle x^{(j)}, x^{(k)} \rangle \) must be at least \( \frac{1}{4} \). Now we have:

\[
P(\Delta \text{ is not a metric}) = P(\exists i,j,k: C_{ijk} > 1)
\leq P(\exists i,j: |\langle x^{(i)}, x^{(j)} \rangle| > 1/3)
\leq \frac{n^2}{2} P(|\langle x^{(1)}, x^{(2)} \rangle| > 1/3)
\]
Since both $x^{(1)}$ and $x^{(2)}$ are random vectors, the probability $\mathbb{P}(|\langle x^{(1)}, x^{(2)} \rangle| > 1/3)$ is equal to the probability that the projection of a random $d$-dimensional vector onto a 1-dimensional subspace is at least $1/3$ in absolute value. By Lemma 5, we have:

$$P(\Delta \text{ is not a metric}) \leq \frac{n^2}{2} P(|\langle x^{(1)}, x^{(2)} \rangle| > 1/3) \leq \frac{n^2}{2} P(|\langle x^{(1)}, x^{(2)} \rangle|^2 > 1/9) \leq \frac{n^2}{2} e^{1+\log(d/9)-(d/9)} \leq n^2 e^{-d/36} \leq \delta$$

Lemma 7. ([19], Theorem 5.39) Let \{x^{(1)}, \ldots, x^{(n)}\} be a set of unit vectors sampled uniformly at random from the unit sphere and $t \in (0,1)$. Let $Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle$ for $1 \leq i, j \leq n$. If $d \geq C_1 n/t^2$ then with probability at least $1 - e^{-C_2 n/t^2}$, we have $|\lambda_i - 1| \leq t$ for all eigenvalues $\lambda_i$ of $Z$. Here, $C_1 > 0$ and $C_2 > 0$ are some absolute constants.

Theorem 3. Let \{x^{(1)}, \ldots, x^{(n)}\} be a set of unit vectors sampled uniformly at random from the unit sphere. Let $Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle$ for $1 \leq i, j \leq n$. If $d \geq C n \log^2 n$ then with probability at least $1 - e^{-C n/\log^2 n}$, there is an LSH for $Z$. Here, $C > 0$ and $C' > 0$ are some absolute constants.

Proof. Apply Lemma 7 with $t = \frac{1}{C_0 \log n}$ (where $C_0$ is a sufficiently large constant). We get that if $d \geq (C_0^2 C_1) n \log^2 n$ then with probability at least $1 - e^{-(C_2/C^2) n/\log^2 n}$ the smallest eigenvalue is greater than or equal to $1 - \frac{1}{C \log n}$. Therefore, matrix $Y = C \log n \left( Z - (1 - \frac{1}{C \log n}) I \right)$ is a positive semidefinite matrix with unit diagonal. Now according to [5], there exists a distribution over a family $\mathcal{H}$ of hash functions such that for any $i \neq j$, $E_{h \in \mathcal{H}}[h_i h_j] = \frac{Y_{ij}}{C \log n}$. We have,

$$E_{h \in \mathcal{H}}[h_i h_j] = \frac{Y_{ij}}{C \log n} = Z_{ij} - \left(1 - \frac{1}{C \log n}\right) I_{ij} = Z_{ij}$$

Moreover, for every $i$, we have $E_{h \in \mathcal{H}}[h_i h_i] = 1 = Z_{ii}$.

A.2 Generalized LSH

In this section, we try to investigate the conditions to have Generalized LSH with high probability.

Lemma 8. Let \{x^{(1)}, \ldots, x^{(n)}\} be a set of unit vectors. Let $Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle$ for $1 \leq i, j \leq n$. There is a generalized $\alpha$-LSH for matrix $Z$ with $\alpha = O(\log n)$. 
Proof. Matrix $Z$ is positive semi-definite and thus its smallest eigenvalue $\lambda_{\min}$ is non-negative. Applying Claim 3, we get the statement of the lemma.

**Theorem 4.** Let $\{x^{(1)}, \ldots, x^{(n)}\}$ be a set of unit vectors sampled uniformly at random from the unit sphere, let $0 < \alpha < O(\log n)$. Let $Z_{ij} = \langle x^{(i)}, x^{(j)} \rangle$ for $1 \leq i, j \leq n$. If $d \geq Cn \log^2 n / \alpha^2$ then with probability at least $1 - e^{Cn\alpha^2 / \log^2 n}$, there is a generalized $\alpha$-LSH for $Z$. Here $C > 0$ and $C' > 0$ are some absolute constants.

Proof. The proof is a straightforward generalization of Theorem 3.