Decoherence Control in Open Quantum System via Classical Feedback

Narayan Ganesan\* and Tzyh-Jong Tarn\†
Electrical and Systems Engineering,
Washington University in St. Louis

In this work we propose a novel strategy using techniques from systems theory to completely eliminate decoherence and also provide conditions under which it can be done so. A novel construction employing an auxiliary system, the bait, which is instrumental to decoupling the system from the environment is presented. Our approach to decoherence control in contrast to other approaches in the literature involves the bilinear input affine model of quantum control system which lends itself to various techniques from classical control theory, but with non-trivial modifications to the quantum regime. The elegance of this approach yields interesting results on open loop decouplability and Decoherence Free Subspaces (DFS). Additionally, the feedback control of decoherence may be related to disturbance decoupling for classical input affine systems, which entails careful application of the methods by avoiding all the quantum mechanical pitfalls. In the process of calculating a suitable feedback the system has to be restructured due to its tensorial nature of interaction with the environment, which is unique to quantum systems. The results obtained are qualitatively different and superior to the ones obtained via master equations. Finally, a methodology to synthesize feedback parameters itself is given, that technology permitting, could be implemented for practical 2-qubit systems to perform decoherence free Quantum Computing.

PACS numbers:

I. INTRODUCTION

Various authors have studied control of decoherence of an open quantum system. Decoherence Free Subspaces (DFS) help preserve quantum information in an open quantum system. However, the presence of symmetry breaking perturbations or control hamiltonians acting on an open quantum system which is essential to performing arbitrary transforms in the system hilbert space \( \mathcal{H}_s \), could also lead to loss of information by inevitable transfer of states out of DFS, due to the nature of the control hamiltonians. Hence this renders the quantum system at best a noiseless memory, much less a dynamic quantum computer, whose state needs to be transformed in order to perform computations. Recently Lidar and Wu \[20, 27\], Kielpinski et. al. \[32\], Brown et. al \[33\] have proposed a combination of open loop bang-bang pulses, universal control in order to perform computation within the DFS via control pulses. In this work we propose a novel strategy, exploiting the geometry of the bilinear control system on the analytic manifold to completely eliminate decoherence in the presence of symmetry breaking control hamiltonians and still preserve complete controllability of the system in order to perform arbitrary transforms. We also explore the possibilities and provide conditions under which it can be done so. This unified approach to control of decoherence lets us analyze the open loop decoupling problem which directly leads us to the existence of DFS and secondly closed loop decoupling via a classical feedback to the control system which leads us to robust decoherence control. This work is a continuation of the previous results\[14\] wherein some of the theoretical groundwork was laid to study the problem of open loop decoupling, which are now extended to closed loop control and feedback design here. The approach used here is fundamentally different from approaches adopted by other authors in that \( (i) \) the bilinear form of control system is used which is amenable to classical systems theoretical results instead of the stochastic master equation for the state evolution, \( (ii) \) the approach does not aim at mitigating or slowing down the decoherence rate rather aims at completely eliminating via a suitable non-linear feedback. The experimental feasibility is discussed for a finite state environment acting on a two qubit system which is a rather reasonable approximation. A procedure to compute the feedback using the invariant subspace for a system is provided. A detailed step by step algorithm to determine the invariant subspace itself on the tangent space \( T_\xi (\mathcal{M}) \) is also provided. In order to compute the feedback parameters a good estimate of state of the system is essential. A reliable information extraction scheme utilizing indirect continuous measurement via a quantum probe in the context of a decohering quantum system was studied in \[15\].

II. PREVIOUS WORK

Consider an open quantum system interacting with the environment described by,

\[
\frac{\partial \xi(t,x)}{\partial t} = [H_0 \otimes \mathcal{I}_c(t,x) + \mathcal{I}_s \otimes H_{c}(t,x) + H_{SE}(t,x) + \sum_{i=1}^{r} u_i(t) H_i \otimes \mathcal{I}_c(t,x)]\xi(t,x)
\]
Here the argument $x$ denotes the spatial dependence of the combined system-environment state $\xi(t, x)$ as well as control hamiltonians $H_i$, and where $u_i$ are the strength of the control respectively. $H_0, H_E, H_{SE}$ are the system, environment and interaction hamiltonian acting on $\mathcal{H}_s, \mathcal{H}_e$ and $\mathcal{H}_s \otimes \mathcal{H}_e$ (system, environment and the combined) Hilbert spaces respectively. For ease of notation we will suppress the spatial dependance. Define an output equation which could either be a non-demolition measurement or a general bilinear form given by,

$$y(t) = \langle \xi(t)|C(t)|\xi(t)\rangle$$

(1)

where again $C(t, x)$ is assumed to be time-varying operator acting on the system Hilbert space. For instance for a finite system the non-hermitian operator $C = |m\rangle\langle n|$ when plugged in eq. (1) would yield the coherence between the respective states of the system or for an electro-optic system the operator $C = a \exp(i\omega t) + a^\dagger \exp(-i\omega t)$ would yield the output of a real non-demolition observation performed on the system. In order to study the invariance properties with respect to the system dynamics of the above time dependent quantum system, we define $f(t, x, u_1, \ldots, u_r, H_{SB}) = y(t, \xi)$ for $t \in [t_0, t_f]$ to be a complex scalar map as a function of the control functions and the interaction Hamiltonian $H_{SB}$ over a prescribed time interval. The function $f$ is said to be invariant or the signal $y(t, \xi)$ is said to be decoupled from the interaction Hamiltonian $H_{SB}$ if,

$$f(t, x, u_1, \ldots, u_r, H_{SB}) = f(t, x, u_1, \ldots, u_r, 0)$$

(2)

for all admissible control functions $u_1, \ldots, u_r$ and a given interaction Hamiltonian $H_{SB}$. Then the condition for such an output signal to be decoupled from the interaction hamiltonian in the open loop case is given by the following theorem[14], which follows an iterative construction in terms of system operators.

The vector fields $K_0 = \left( \begin{array}{c} 1 \\ (H_0 + H_e)\xi(x,t) \end{array} \right), K_i = \left( \begin{array}{c} 0 \\ H_i\xi(x,t) \end{array} \right), K_p = \left( \begin{array}{c} 0 \\ H_{SP}\xi(x,t) \end{array} \right)$ and $K_I = \left( \begin{array}{c} 0 \\ H_{SB}\xi(x,t) \end{array} \right)$ corresponding to drift, control and interaction can be identified to contribute to the dynamical evolution. It was already noted that the the system was said to be decoupled if it satisfied equations[3] namely,

$$L_{K_i} y(t, \xi) = 0$$

$$L_{K_i} L_{K_{i0}} \cdots L_{K_{in}} y(t, \xi) = 0$$

(3)

Recalling,

**Theorem II.1.** Let

$$C_0 = C(t)$$

$$\vdots$$

$$\tilde{C}_n = \text{span}\{ad_{H_i}C_{n-1}(t)|j = 0, 1, \ldots; i = 1, \ldots, r\}$$

$$\vdots$$

Define a distribution of quantum operators, $\tilde{C}(t) = \Delta(C_1(t), C_2(t), \ldots, C_n(t), \ldots \}$. The output equation (2) of the quantum system is decoupled from the environmental interactions if and only if,

Case (I): Open Loop,

$$[\tilde{C}(t), H_{SB}(t)] = 0$$

(4)

Case (II): Whereas the necessary conditions for Closed Loop control is,

$$[C, H_{SB}] = 0$$

$$[\tilde{C}(t), H_{SB}(t)] \subset \tilde{C}(t)$$

In this work we will be primarily concerned with designing feedback for quantum systems of the form $u = \alpha(\xi) + \beta(\xi)v$ where $\alpha$ and $\beta$ are real vector and a full rank real matrix of the state (or its estimate thereof) of dimension $1 \times r$ and $r \times r$ respectively. We examine a few systems of interest with control hamiltonians, that might be decoupled via feedback of the above form.

**Definition II.1.** The vector field $K_\tau$ satisfying equations [3] is said to be in the orthogonal subspace of the observation space spanned by the one-forms

$$dy(t, \xi), dL_{K_{i0}} y(t, \xi), \ldots, dL_{K_{in}} \cdots L_{K_{in}} y(t, \xi), \cdots$$

(5)

$\forall 0 \leq i_0, \ldots, i_n \leq r$ and $n \geq 0$

Denoted by $K_\tau \in \mathcal{O}^1$

**Lemma II.2.** The distribution $\mathcal{O}^1$ is invariant with respect to the vector fields $K_0, \ldots, K_\tau$ under the Lie bracket operation. (i.e) if $K_\tau \in \mathcal{O}^1$, then $[K_\tau, K_i] \in \mathcal{O}^1$ for $i = 0, \ldots, r$

**III. A SINGLE QUBIT SYSTEM**

Consider a single qubit spin-$1/2$ system coupled to a bath of infinite harmonic oscillators through an interaction hamiltonian $H_{SB}$. The hamiltonian of the system+bath can be written as,

$$H = \frac{\omega_0}{2}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k)$$
where the system is acted upon by the free hamiltonian \( H_0 \) and the decoherence hamiltonian \( H_{SB} \). As is well known there is a rapid destruction of coherence between \( |0\rangle \) and \( |1\rangle \) according to the decoherence function given by [11]. In order to cast the above problem in the present framework we consider a bilinear form of an operator \( C \) that monitors coherence between the basis states. Considering \( C \) to be the non-hermitian operator \( |0\rangle \langle 1 | \) we have a function \( y(t) \) given by \( y(t) = \langle \xi(t)|C|\xi(t)\rangle \) that monitors coherence between the states \( |0\rangle \) and \( |1\rangle \). The problem now reduces to analyzing the applicability of the theorem [11] to the given system. It can be seen right away that the condition \( [\tilde{C}, H_{SB}] \neq 0 \) for the distribution \( \tilde{C} \) defined previously, as calculated in the previously [11]. This implies that the coherence is not preserved under free dynamics or in presence of open loop control. In order to eliminate this decoherence by feedback we now assume the system to be acted upon by suitable control hamiltonians \( \{ H_1, \cdots, H_r \} \) and corresponding control functions \( \{ u_1, \cdots, u_r \} \). As we pointed out earlier the necessary condition is relaxed to \( [\tilde{C}, H_{SB}] \subset \tilde{C} \), with the operators \( C \) and \( H_{SB} \) still required to commute with each other \( [C, H_{SB}] = 0 \). For the single qubit example the second condition fails to hold, again as outlined [11], thus leaving the system unable to be completely decoupled and hence vulnerable to decoherence even in the presence of closed loop and feedback control.

\[ \frac{\partial|\xi(t)\rangle}{\partial t} = \left( \sum_{j=1}^{2} \frac{\sin{\omega_j}}{2} \sigma_z^{(j)} + \sum_k \omega_k b_k^\dagger b_k \right)|\xi(t)\rangle \]

\[ + \sum_k \left( \sum_j \sigma_z^{(j)} \right) (g^{(j)} b_k^\dagger + g^{(j)} b_k)|\xi(t)\rangle \]

\[ + (u_1(t)\sigma_x^{(1)} + u_2(t)\sigma_x^{(2)} + u_3(t)\sigma_y^{(2)})|\xi(t)\rangle \]

which satisfies the basic necessary condition \( [C, H_{SB}] = 0 \) but not the stronger condition provided in Case(ii) of the theorem. Hence the system would eventually leave the DFS and is susceptible to decoherence in the presence of arbitrary control, in other words, not entirely decoupled from \( H_{SB} \). In order to analyze the system and the conditions in the presence of a classical state feedback \( u = \alpha \langle \xi(t)| + \beta \langle \xi(t)| \) the corresponding conditions (ii) of the theorem are to be examined. Since the operator \( H_{SB} \in B(H_s \otimes H_e) \), the set of skew hermitian linear operators acting non-trivially on both system and environment hilbert space, whereas the operators in the distribution \( \tilde{C} \) for the above control system is confined to \( B(H_s) \) that act trivially on the environment hilbert space. Hence the necessary condition specified in Theorem [11] would not be satisfied non-trivially unless the distribution \( \tilde{C} \) acted non-trivially on both \( H_s \) and \( H_e \). In other words the distribution includes operators of the form \( \sum A_\alpha \otimes B_\beta \) for a countable index set \( \{ \alpha \} \) and operators \( A_\alpha \) and \( B_\beta \) operating on system and environment respectively. The above forms cannot be achieved by control hamiltonians acting only on the system. However the situation can be salvaged if one considered a ”bait” qubit whose rate of decoherence or the environmental interaction can be modulated externally at will and the bait qubit is now allowed to interact with our qubits of interest through an Ising type coupling. With the help of the following construction we will be able to generate vector fields of the form \( K_I \) artificially, which will be seen to provide great advantage. With the coherence functional \( y(t) = \langle \xi(t)|01\rangle \langle 10|\xi(t)\rangle \) where \( |\xi(t)\rangle \), the state vector is now the total wave function of system+bait+
environment. Both the qubit systems are assumed to interact with the same environment with the additional requirement that the bait qubit’s decoherence rate be controllable. Physically this amounts to a coherent qubit with controllable environmental interaction. The

\[ i\hbar \frac{\partial |\xi(t)\rangle}{\partial t} = \left( \sum_{j=1}^{2} \frac{\omega_0}{2} \sigma_z^{(j)} + \sum_k \omega_k b_k^\dagger b_k \right) |\xi(t)\rangle + \sum_k \left( \sum_j \sigma_z^{(j)} (g_k b_k^\dagger + g_k^* b_k) |\xi(t)\rangle + \left( u_1(t) \sigma_x^{(1)} + u_2(t) \sigma_y^{(1)} \right) \right) \tag{7} \]

+ u_3(t) \sigma_z^{(2)} + u_4(t) \sigma_y^{(2)} + \frac{\omega_0}{2} \sigma_z^{(b)} + u_6 \sigma_y^{(b)} + u_7 J_1 \sigma_z^{(1)} \sigma_z^{(b)} + u_8 J_2 \sigma_z^{(2)} \sigma_z^{(b)} + u_9 \sum_k \sigma_z^{(j)} (w_k b_k^\dagger + w_k^* b_k) |\xi(t)\rangle \]

where \( \sigma_x, \sigma_y, \sigma_z \) are regular hermitian operators and \( u_1(t) \) to \( u_9(t) \) are time-dependent piecewise constant control functions. The terms of controls \( u_1 \) and \( u_2 \) are generated by the Ising type coupling between qubits 1, 2 and the bait with the corresponding coupling constants \( J_1 \) and \( J_2 \) respectively. The last term in the above control system is due to the interaction of the bait qubit with the environment whose interaction enters the system in a controllable way, hence can be treated as a separate control hamiltonian. Keeping in mind the following commutation relations between different pairs of operators,

\[
\begin{align*}
[\sigma_x, \sigma_y] &= 2i\sigma_z & [\sigma_y, \sigma_z] &= 2i\sigma_x & [\sigma_z, \sigma_x] &= 2i\sigma_y \\
[b_k, b_{k'}^\dagger] &= \delta_{kk'} & [b_k, b_{k'}^\dagger b_k] &= b_k & [b_k^\dagger, b_{k'} b_k] &= -b_k^\dagger \\
\sigma_z &= |1\rangle \langle 1| - |0\rangle \langle 0|, & \sigma_x &= |0\rangle \langle 1| + |1\rangle \langle 0| & \sigma_y &= i|0\rangle \langle 1| - i|1\rangle \langle 0| 
\end{align*}
\]

and \( C = |01\rangle \langle 01| = (\sigma_x^{(1)} - i\sigma_y^{(1)}) \otimes (\sigma_x^{(2)} + i\sigma_y^{(2)}) / 4 \), we have \( [C, H_{SB}] = 0 \) and \( \tilde{C}, H_{SB} \) for instance contains terms of the form \( \sigma_x \otimes I^{(2)} \otimes \sum (g_k b_k^\dagger + g_k^* b_k) \) which are not zero. Fortunately with the above construction these terms can be seen to be present in the distribution \( \tilde{C} \), which can be obtained under the sequence of operations \( [C, H_1] = c_1 \sigma_x \otimes \sigma_y, [C, H_3], H_5] = c_2 \sigma_x \otimes \sigma_y \otimes \sum (g_k b_k^\dagger + g_k^* b_k), [[C, H_1], H_5], H_2] = c_3 \sigma_x \otimes I^{(2)} \otimes \sum (g_k b_k^\dagger + g_k^* b_k) \) and the corresponding \( \sigma_y \) term is obtained via the sequence, \([[[C, H_2], H_5], H_1] \). Since both terms are present in \( \tilde{C} \), so is their linear combination. Hence both the necessary conditions as outlined by the theorem for closed loop decouplability are satisfied for the above system. Hence we are one step closer to decoupling the coherence between the qubits from \( H_{SB} \). In fact it can be seen that the operator \( H_{SB} \) itself can be generated by the control hamiltonians through the lie bracket operation \( H_{SB} = [H_5, H_2], H_1] \) or \([H_5, H_1], H_2] \). Hence any term in \([\tilde{C}, H_{SB}] \) is trivially contained in \( \tilde{C} \). Hence, it might seem at first that the effects of \( H_{SB} \) on the system could be nullified by generating an equivalent \(-H_{SB}\) through control hamiltonians alone. But in order to generate such a vector field one has to know beforehand and as time progresses the exact values of the environmental coupling coefficients \( g_k \) which at best could only be described by a stochastic process. Hence in the light of the aforementioned difficulty, just rendering the coherence independent of \( H_{SB} \) seems like a much better alternative.

\section{V. Scalability}

It can also be seen that the above approach works for finite number of qubits coupled to only one bait qubit through the same \( \sigma_x^{(j)} \), \( \sigma_y^{(j)} \) interactions. Such an interaction can be implemented using the same technology necessary for multi-qubit quantum computers wherein a finite number of qubits are entangled to a single qubit that is capable of readout and storage of an oracle’s query results. With the underlying theory of disturbance decoupling in place all that remains now is synthesis of the feedback control itself. Since the conditions \( [C, H_{SB}] \subset \tilde{C} \) and \([C, H_{SB}] = 0 \) turn out to be necessary conditions, with the proof of sufficiency requiring further insight into design and construction of appropriate control fields we will for the next few sections follow an alternative formalism called an Invariant Subspace which is a part of the tangent space \( T_\xi(M) \) of the analytic manifold. It will be seen later that the two seemingly different approaches viz. (i) the conditions in terms of operators of the system and (ii) The tangent space formalism, complement one another in terms of obtaining a complete solution to the problem of disturbance decoupling.
VI. INVARIANT SUBSPACE FORMALISM

Consider the necessary and sufficient conditions for decouplability

\[ L_{K_1} y(t) = 0 \]
\[ L_{K_1} L_{K_2} y(t) = 0 \]

Hence \( L_{K_1} L_{K_2} y(t) = 0 \). The above equations after subtraction imply \( L_{[K_0, K_1]} y(t) = 0 \). The other necessary conditions viz. \( L_{[K_0, K_1]} L_{K_1} y(t) = 0 \) and \( L_{K_1} L_{[K_0, K_1]} y(t) = 0 \) imply that \( L_{[K_0, K_1]} y(t) = 0 \). In fact the above pattern of equations could be extended to any number of finite vector fields to conclude that

\[ L_{[[\cdots[K_0, K_{i+1}], K_{i+2}] \cdots K_{i+k}]} y(t) = 0 \]

which leads us to define a set of vector fields or distribution \( \Delta \) that share the same property,

\[ \Delta \ni \forall K_\nu \ni L_{K_\nu} y(t) = 0 \]

It is observed immediately that \( K_1 \ni \Delta \). Such a distribution \( \Delta \) is said to belong to null space of the function \( y(\xi, t) \). And from the necessary conditions listed above the distribution is observed to be invariant under the control and drift vector fields \( K_0, \cdots, K_m \), (i.e) \( \forall K_\nu \ni \Delta \).

\[ [K_\nu, K_i] \ni \Delta, \forall i \ni 0, \cdots, m \]

Simply stated,

\[ [\Delta, K_i] \ni \Delta, \forall i \ni 0, \cdots, m \]

We will henceforth refer to the distribution as the invariant distribution. It is also to be noted that the above calculations are reversible and the original necessary and sufficient conditions can be derived starting from the invariant distribution. Hence the necessary and sufficient conditions for open loop decouplability can now be restated in terms of the invariant distribution.

**Theorem VI.1.** The output \( y(t) \) is unaffected by the interaction vector field \( K_i \) if and only if there exists a distribution \( \Delta \) with the following properties,

(i) \( \Delta \) is invariant under the vector fields \( K_0, K_1, \cdots, K_m \)

(ii) \( K_i \ni \Delta \ni \ker(dy(t)) \)

Hence existence of the invariant subspace is essential to decouplability of the system in question. It is now more important to determine the invariant subspace (if any) for the given system and output equation. In order to compute the invariant distribution it properties discussed above comes in handy and provides a means to go about computing the distribution as well.

The procedure starts out by assigning the entire null space \( \ker(dy(t)) \) to invariant distribution \( \Delta \) and successively removing parts of the distribution that don’t satisfy the other properties (i.e), invariance with respect the vector fields \( K_0, \cdots, K_m \). In other words, remove parts of \( \Delta \) whose lie brackets with \( K_0, \cdots, K_m \) do not lie within \( \Delta \). Of course, the above mentioned procedure involves computing inverse image of Lie brackets as described below.

**A. Invariant Distribution Algorithm**

Algorithm 1:

*Step 1:* Let \( \Delta_0 = \ker(dy(t, \xi)) \).

*Step 2:* \( \Delta_{i+1} = \Delta_i - \{ \delta, K_j \ni \Delta_i, 0 \leq j \leq r \} \)

*Step 3:* Maximal invariant distribution is such that \( \Delta^* = \Delta_i \) when \( \Delta_i = \Delta_{i+1} \).

The above is an iterative procedure that computes distributions \( \Delta_i \) in order to arrive at the final invariant distribution \( \Delta^* = \Delta \). Where the ‘−’ is the set removal operation. Let us redefine the set to be removed as,

\[ \mathcal{S}_i = \{ \delta \ni [\delta, K_j] \ni \Delta_i, \forall 0 \leq j \leq r \} \]

Hence the set \( \mathcal{S}_i \) can also be written as,

\[ \mathcal{S}_i = \text{inv}(\Delta_i, K_j) \ni 0 \leq j \leq r \]

Figure 5 outlines the schematic of the algorithm.

One of the foremost issues to be addressed is the convergence of the algorithm. However at this point we are not fully equipped to study the converge as the proof below will introduce additional ideas to discuss convergence. It is to be noted here that \( \Delta_i \) is always a distribution (a vector space) for all \( i \). Hence the set \( \mathcal{S}_i \) is such that, the removal of \( \mathcal{S}_i \) from \( \Delta_i \) results in a distribution of lower dimension \( \Delta_{i+1} \). Hence removal of the set \( \mathcal{S}_i \) removes a subspace \( \Delta_i \) contained within the distribution \( \Delta_i \). Hence we have \( \Delta_{i+1} \ni \Delta_i \) and

\[ \Delta_{i+1} + \Delta_i = \Delta_i \]
in order to locate the subspace $\tilde{\Delta}_i$ we have to determine the complementary subspace (look for vectors that are orthogonal) to eq. (13) and within $\Delta_i$. From the identities of Lie derivatives,

$$L_{K_j} (\omega, \delta) = \langle L_{K_j} \omega, \delta \rangle + \langle \omega, [\delta, K_j] \rangle$$

(15)

Hence for $\omega \in \Omega_i$ and $\delta \in \Delta_{i+1} \subset \Delta_i$, we have $\langle \omega, \delta \rangle = 0$ and $\langle \omega, [\delta, K_j] \rangle = 0$ (eq. (14)). Hence $\langle L_{K_j} \omega, \delta \rangle = 0$ from the previous identity (10). In other words $L_{K_j} \Omega_i$ is orthogonal to $\Delta_{i+1}$. Since $\Delta_{i+1}$ is orthogonal to $(\tilde{\Delta}_i)^*$ and $\Omega_i$, we have,

$$L_{K_j} \Omega_i \subseteq \Omega_i + (\tilde{\Delta}_i)^*$$

(16)

Now consider the same equation (15), for all $\delta \in \tilde{\Delta}_i$ and all $\omega \in \Omega_i$ we have $\langle \omega, \delta \rangle = 0$. But since $\delta \in \tilde{\Delta}_i$, we have $\langle \omega, [\delta, K_j] \rangle \neq 0 \text{ for some } \omega \in \Omega_i$, hence $\langle L_{K_j} \omega, \delta \rangle \neq 0$ as well. Hence for any $\delta \in \tilde{\Delta}_i$ there exists an $\omega \in \Omega_i$ such that $\langle L_{K_j} \omega, \delta \rangle \neq 0$. (i.e)

$$(\tilde{\Delta}_i)^* \subseteq L_{K_j} \Omega_i$$

(17)

Hence from eq. (16) and (17) we conclude that

$$(\tilde{\Delta}_i)^* + \Omega_i = \Omega_i + L_{K_j} \Omega_i, \forall 0 \leq j \leq r.$$  

(18)

although it is possible to prove the stronger condition, $\Omega_i + \tilde{\Delta}_i = L_{K_j} \Omega_i$. We now state the algorithm without proof:

Step 1: Set $\Omega_0 = \text{span}(dy(t, \xi))$.

Step 2: $\Omega_{i+1} = \Omega_i + L_{K_0} (\Omega_i) + \sum_{j=1}^{r} L_{K_j} (\Omega_i)$.

Step 3: The Algorithm converges to $\Omega^* = \Omega_i$ when $\Omega_{r+1} = \Omega_i, \forall i$.

Maximal invariant distribution $\Delta$ is such that $\Delta^* = \Omega^*$. As seen in the proof each step of Algorithm 1, removes a set from a vector which amounts to removing a finite dimension limited by dimension of the tangent space. Hence the convergence of the algorithm is dependent on the finite dimensionality of the tangent space at point $\xi(t)$ which can be guaranteed by the finiteness of control Lie Algebra, which will be studied in the following sections.

B. Observation space and Tangent Space

In definition (11) the observation space spanned by $dy(t, \xi), dL_{K_{0}} y(t, \xi), \cdots, dL_{K_{n}} y(t, \xi)$, $\forall 0 \leq i_0, \cdots, i_n \leq r$ and $n \geq 0$ was defined and it can be easily seen that the necessary and sufficient condition for open loop decouplability (3) is equivalent to being orthogonal to the observation space according to def. (11). The orthogonality relation also follows from the simple Lie derivative identity,

$$L_{K_{r}, y(t, \xi)} = [K_{r}, dy(t, \xi)]$$

(19)
From [14], it can be seen that the one forms \(dy(t, \xi), dL_{K_{\omega}}y(t, \xi)\) etc can be expressed in terms of the commutators of operators and hamiltonians, \(C, H_0, H_t\). Infact the operations performed in the observation space provide an alternative formulation to the theory developed in terms of the tangent space and invariant distributions. As can be seen the structure of the output equation \(y(t) = \langle \xi(t)|C(t)|\xi(t)\rangle\) made possible the simplifications of Lie derivatives of scalar functions to commutators of operators and enjoys ease of calculations when compared to computing Lie derivatives of vector and co-vector fields, if one were to compute the invariant subspace. Hence it is to be noted that the necessary and sufficient conditions for open loop decouplability can just be stated in terms of the observation space without ever having to calculate the invariant distribution which is precisely what Theorem VII.1 sets out to do. And it is also to be noted that the Theorem is a consequence of the orthogonality relation in the observation space (Definition II.1).

However when it comes to feedback decouplability the two different formalisms play equally important roles in constructing a quantum system that might be decoupled using feedback. The observation space formalism provides important necessary conditions (in terms of the commutators of operators) while designing a quantum control system while the tangent space formalism is indispensable to calculating the feedback parameters \(\alpha(\xi(t)), \beta(\xi(t))\) once the system of interest is known to be decouplable using feedback.

VII. SYNTHESIS OF FEEDBACK PARAMETERS \(\alpha(\xi), \beta(\xi)\)

In this section we study the explicit formulation of the feedback control that ensures complete decoupling of the coherence functional from \(H_{SB}\). It is to be seen that this formulation can be applied to outputs other than the coherence functional we wish to monitor, like that of a non-demolition observable.

**Definition VII.1.** A distribution is said to be controlled invariant on the analytic manifold \(D_\omega\) if there exists a feedback pair \((\alpha, \beta)\), \(\alpha\), vector valued and \(\beta\), matrix valued functions such that

\[
[\tilde{K}_0, \Delta](\xi) \subset \Delta(\xi) \\
[\tilde{K}_1, \Delta](\xi) \subset \Delta(\xi)
\]

where,

\[
\tilde{K}_0 = K_0 + \sum_{j=1}^r \alpha_jK_j
\]

and

\[
\tilde{K}_1 = \sum_{j=1}^r \beta_{ij}K_j
\]

It is to be noted that \(\tilde{K}_0\) and \(\tilde{K}_1\) are the new drift and control vector fields of the control system after application of feedback \((\alpha, \beta)\). The problem of decoupling via feedback can now be cast in the original framework of open loop decouplability by requiring that the feedback vector fields now satisfy the open loop decouplability conditions viz.

\[
[\tilde{K}_0, \Delta](\xi) \subset \Delta(\xi) \\
[\tilde{K}_1, \Delta](\xi) \subset \Delta(\xi)
\]

and that \(\Delta\) be contained entirely within the null space of the output function (i.e),

\[
\Delta \subset \ker(dy)
\]

With the above characterization of feedback decouplability the task now reduces to finding a distribution that might satisfy the above invariance conditions with respect to the feedback vector fields, \((\tilde{K}_0, \tilde{K}_1, \cdots, \tilde{K}_r)\), which in turn requires the knowledge of the feedback functions \(\alpha\) and \(\beta\). What seems to be a deadlock situation can now be resolved by further simplifying the invariance condition stated above.

**Lemma VII.1.** An involutive distribution \(\Delta\) defined on the analytic manifold \(D_\omega\) is invariant with respect to the closed loop vector fields \((\tilde{K}_0, \tilde{K}_1, \cdots, \tilde{K}_r)\) for some suitable feedback parameters \(\alpha(\xi)\) and \(\beta(\xi)\) if and only if,

\[
[K_0, \Delta] \subset \Delta + G \\
[K_1, \Delta] \subset \Delta + G
\]

Where \(G\) is the distribution created by the control vector fields.

\[
G = \text{span}\{K_1, \cdots, K_r\}
\]

At this point it is possible to express the necessary and sufficient conditions for the feedback control system \((\tilde{K}_0, \tilde{K}_1, \cdots, \tilde{K}_r)\) to be decoupled from the interaction vector field \(K_I\) just as we were able to provide conditions for open loop decouplability. Moreover the conditions can be expressed entirely in terms of the open loop vector fields and the controlled invariant distribution without ever having to involve the feedback parameters \(\alpha(\xi)\) and \(\beta(\xi)\). The following theorem provides the conditions,

**Theorem VII.2.** The output \(y(t, \xi) = \langle \xi|C(t)|\xi\rangle\) can be decoupled from interaction vector field \(K_I\) via suitable feedback \((\alpha, \beta)\) if and only if there exists an involutive distribution \(\Delta\) such that,

\[
[K_0, \Delta] \subset \Delta + G \\
[K_1, \Delta] \subset \Delta + G
\]

and \(\Delta \subset \ker(dy)\)
Proof. (\(\iff\)) The following proof covers the lemma as well as the theorem above. Assuming that \(\Delta\) is locally controlled invariant or in other words invariant with respect to the closed loop vector fields \((\hat{K}_0, \hat{K}_1, \ldots, \hat{K}_r)\) for some feedback parameters \(\alpha(\xi)\) and \(\beta(\xi)\) within an open set in \(D_\omega\). If \(\tau \in \Delta\), then it can be seen that,

\[
[\hat{K}_i, \tau] = [\beta_{ij} K_j, \tau] = \sum_{j=1}^{r} \beta_{ij} [K_j, \tau] - \sum_{j=1}^{r} (L_\tau \beta_{ij}) K_j
\]

as we know the left hand side is still contained within \(\Delta\) and the last term on the right side is a linear combination of vectors that generate \(G\). Hence

\[
\sum_{j=1}^{r} \beta_{ij} [K_j, \tau] \in \Delta + G
\]

and since \(\beta\) is assumed to be nonsingular it is possible to solve for individual \([K_j, \tau]\) by mere inversion of the matrix \(\beta_{ij}\) and can be found to be linear combination of vectors in \(\Delta + G\) and hence,

\[
[K_i, \tau] \in \Delta + G
\]

Now consider,

\[
[\hat{K}_0, \tau] = [K_0 + \sum \alpha_j K_j, \tau]
\]

\[
= [K_0, \tau] + \sum \alpha_j [K_j, \tau] - \sum_{j=1}^{r} (L_\tau \alpha_j) K_j
\]

Since the left hand side belongs to \(\Delta\) and since \([K_j, \tau] \in \Delta + G, 1 \leq j \leq r\) it can be immediately seen that \([K_0, \tau] \in \Delta + G\) as well.

\((\iff)\) For the proof of sufficiency the following geometric visualization is helpful. Let the dimension of the distribution \(\Delta\) be \(d\). Since \(\Delta\) is involutive there exist \(d\) vectors fields, locally non-vanishing in a neighborhood \(U \subset S_H \cap D_\omega\) of \(\xi\), \(\{v_1, \ldots, v_d\}\) \(\in T_\xi(M)\) that are linearly independent and,

\[
\Delta = \text{span}\{v_1, \ldots, v_d\}\]

s.t \([v_i, v_j] \in \Delta, \forall 1 \leq i, j \leq d\). Now let the dimension of \(\Delta + G\) at \(\xi\) be \(d + q\). It is now possible to find another \(q\) linearly independent vector fields labeled \(\{v_{d+1}, \ldots, v_{d+q}\}\), such that \([v_i, v_j] \in \Delta, \forall 1 \leq i \leq d, d + 1 \leq j \leq d + q\). As a special case one could think of a local co-ordinate basis that are mutually commuting and linearly independent. Let the dimension of the tangent space at the point \(\xi\) be \(N\). Finally it is possible to find \(N - d - q\) additional linearly independent vectors that complete the vector space \(T_\xi(M)\), by Gram-Schmidt procedure or otherwise (i.e),

\[
T_\xi(M) = \text{span}\{v_1, \ldots, v_d, v_{d+1}, \ldots, v_{d+q}, v_{d+q+1}, \ldots, v_N\}\]

It will be seen that the above requirement will be easily satisfied for the extension to control algebra to be discussed following this proof. It is also to be noted that we haven’t imposed any non-singularity restrictions on the distributions above. Now the control vector fields \(K_i \in G\) could be written as a linear combination of the vector fields \([v_1], \ldots, [v_N]\) at each point \(\xi\).

\[
K_i = \sum_{j=1}^{d} c_{ij} [v_j] + \sum_{j=d+1}^{N} c_{ij} [v_j], \forall 1 \leq i \leq r
\]

\[
K_i = K_i^d + K_i^q\]

where \(K_i^d \in \Delta\) and \(K_i^q \notin \Delta\).

The vector fields are devoid of components in \(\Delta\). And since dimension of \(\Delta + G\) is \(d + q\) it can be seen that the \(r\) vectors \(K_i^d, \ldots, K_r^d, \ldots, K_r^q\) span a \(q\) dimensional subspace. Hence it is always possible to generate \(q\) linearly independent vectors and \(r - q\) zero vectors via suitable linear combinations of \(K_1^d, \ldots, K_r^d\). Let the linear combinations be such that,

\[
\sum_{j=1}^{r} \beta_{ij} K_j^d = [v_{d+1}] + \sum_{j=d+q+1}^{N} \tilde{c}_{ij} [v_j]
\]

\[
\sum_{j=1}^{r} \beta_{ij} K_j^q = [v_{d+2}] + \sum_{j=d+q+1}^{N} \tilde{c}_{ij} [v_j]
\]

\[
\vdots
\]

\[
\sum_{j=1}^{r} \beta_{ij} K_j^q = [v_{d+q}] + \sum_{j=d+q+1}^{N} \tilde{c}_{ij} [v_j]
\]

and

\[
\sum_{j=1}^{r} c_{ij} K_i^d = 0
\]

\[
\vdots
\]

\[
\sum_{j=1}^{r} c_{ij} K_i^q = 0
\]

The \(\beta_{ij}\) matrix so formed is precisely the feedback parameter that is used to generate the closed loop vector fields \(\hat{K}_i, 1 \leq i \leq r\).

\[
\hat{K}_i = \beta_{ij} K_j , \text{ denoted by } \beta.K
\]

In order to prove this we note the action of the above linear combination on the open loop vector fields
FIG. 4: The dimension of controlled invariant distribution is $d$ and the control distribution $G$ is partitioned into $\{K'_i\}$ and $\{K''\}$. The basis vectors $|v_1\rangle, \ldots, |v_d\rangle$ span $\Delta$.

$K_1, \ldots, K_r, \text{(i.e.,)}$

\[
\sum_{j=1}^{r} \beta_{1j}K_j = \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + |v_{d+1}\rangle + \sum_{j=d+q+1}^{N} \tilde{c}_{1j}|v_j\rangle
\]

\[
\sum_{j=1}^{r} \beta_{2j}K_j = \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + |v_{d+2}\rangle + \sum_{j=d+q+1}^{N} \tilde{c}_{2j}|v_j\rangle
\]

\[\vdots\]

\[
\sum_{j=1}^{r} \beta_{qj}K_j = \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + |v_{d+q}\rangle + \sum_{j=d+q+1}^{N} \tilde{c}_{qj}|v_j\rangle \quad (27)
\]

and

\[
\sum_{j=1}^{r} \beta_{q+1,j}K_i = \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + 0
\]

\[\vdots\]

\[
\sum_{j=1}^{r} \beta_{r,j}K_j = \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + 0
\]

where the first terms on the right hand side of the above equations can be seen to be from $\beta K^d$ and the later terms from $\beta K''$. We will suppress the summation for ease of notation and all the following terms below are assumed to be summations from $1, \ldots, r$ in the recurring index variable. Now from the necessary conditions we have,

$$[\tau, K_j] \in \Delta + G, \forall \tau \in \Delta \text{ and } 1 \leq j \leq r. \quad (28)$$

and hence,

$$[\tau, \beta_{ij}K_j] = \beta_{ij}[\tau, K_j] + L_\tau(\beta_{ij})K_j \in \Delta + G. \quad (29)$$

and for $1 \leq i \leq q$,

$$[\tau, \beta_{ij}K_j] = [\tau, \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle + |v_{d+i}\rangle + \sum_{j=d+q+1}^{N} \tilde{c}_{1j}|v_j\rangle]$$

$$= [\tau, \sum_{j=1}^{d} \tilde{c}_{1j}|v_j\rangle] + [\tau, |v_{d+i}\rangle] + [\tau, \sum_{j=d+q+1}^{N} \tilde{c}_{1j}|v_j\rangle]$$

By noting that $\tau \in \Delta$, $\Delta$ is involutive and $|v_1\rangle, \ldots, |v_d\rangle$ commute with $|v_{d+1}\rangle, \ldots, |v_N\rangle$, $|v_i\rangle, |v_j\rangle \in \Delta, \forall 1 \leq i \leq d, d+1 \leq j \leq d+q$ the above equation can be seen to simplify to,

$$[\tau, \sum_{j=1}^{r} \beta_{ij}K_j] \in \Delta + \sum_{j=d+q+1}^{N} L_\tau(\tilde{c}_{1j})|v_j\rangle$$

but since we already have $[\tau, \beta_{ij}K_j] \in \Delta + G$ from (28), the above relation is possible only if $L_\tau(\tilde{c}_{1j}) = 0$. Hence we have,

$$[\tau, \sum_{j=1}^{r} \beta_{ij}K_j] = [\tau, \tilde{K}_i] \in \Delta.$$

The argument is trivial for $q + 1 \leq i \leq r$ and it can be easily seen that $[\tau, \sum_{j=1}^{r} \beta_{ij}K_j] \in \Delta$ for all $1 \leq i \leq r$ and $\tau \in \Delta$. Now in order to construct the feedback parameter $\alpha$, by an argument analogous to (29), we can show that,

$$[\tau, \tilde{K}_0] = [\tau, K_0 + \sum_{i=1}^{r} \alpha_iK_i] \in \Delta + G$$

because both $[\tau, K_0]$ and $[\tau, K_i]$ belong to $\Delta + G$. Let,

$$K_0 = \sum_{j=1}^{d} c_j|v_j\rangle + \sum_{j=d+1}^{d+q} c_j|v_j\rangle + \sum_{j=d+q+1}^{N} c_j|v_j\rangle \quad (30)$$

It is now possible to find a suitable linear combination of right hand side of equation set (27) and the above (30) in order to form $\tilde{K}_0$,

$$\tilde{K}_0 = K_0 + \tilde{\alpha}_t\tilde{K}_t = \sum_{j=1}^{d} \tilde{c}_j|v_j\rangle + \sum_{j=d+1}^{d+q} k_j|v_j\rangle + \sum_{j=d+q+1}^{N} \tilde{c}_j|v_j\rangle$$

where $k_j$'s are constants w.r.t $\xi$ and $t$ where as $\tilde{c}_j$'s are some functions of $\xi(t)$. In particular by a suitable linear combination, $k_j$'s can all be made zero. It can again be seen that for all $\tau \in \Delta$,

$$[\tau, \tilde{K}_0] \in \Delta + \sum_{j=d+q+1}^{N} L_\tau(\tilde{c}_j)|v_j\rangle$$

and hence $L_\tau(\tilde{c}_j)$ are equal to zero in order to satisfy the necessary conditions and hence,

$$[\tau, \tilde{K}_0] \in \Delta$$

The closed loop drift vector field was formed by setting $\tilde{K}_0$ equal to $\tilde{K}_0 + \tilde{\alpha}_t\tilde{K}$ for a suitable row vector $\tilde{\alpha}$. Hence the feedback parameter $\alpha = \tilde{\alpha}_t\beta$. \Box
In addition to proving the necessary and sufficient conditions we have also outlined a procedure to compute the feedback parameters α(ξ) and β(ξ) from the maximal controllability invariant distribution Δ, which elicits the application of Tangent space formalism in output decoupling. Hence it is imperative that we compute the maximal invariant distribution Δ for the synthesis of feedback. From the necessary and sufficient conditions we see that the distribution Δ has to satisfy conditions \( \sum_{i} \mathcal{V}_{i} \), or equivalently \( \sum_{i} \mathcal{V}_{i} \), and that Δ ⊂ ker(dy) for complete decouplability. Obviously \( \sum_{i} \mathcal{V}_{i} \) has the advantage that we do not need the knowledge of feedback parameters. Now, similar to the open loop case we can formulate an algorithm in order to arrive at the much sought after invariant distribution, the general idea being: Start out by assigning the whole of null space of y(t) to Δ and iteratively remove the part of the distribution that does not satisfy conditions \( \sum_{i} \mathcal{V}_{i} \).

Step 1: Let \( \Delta_{0} = \ker(dy(t, \xi)) \).
Step 2: \( \Delta_{i+1} = \Delta_{i} - \{ \delta \in \Delta_{i} : [\delta, K_{j}] \notin \Delta_{i} + G, \forall 0 \leq j \leq r \} \).
Step 3: Maximal invariant distribution is such that \( \Delta^{\ast} = \Delta_{i+1} = \Delta_{i} \).

Employing the same logic as before in determining the open loop invariant distribution we can perform the computation in the dual space \( T^{\ast}_{t}(M) \) and arrive at the following algorithm which is easier to compute.

Step 1: Let \( \Omega_{0} = \text{span}(dy(t, \xi)) \).
Step 2: \( \Omega_{i+1} = \Omega_{i} + L_{K_{i}}(\Omega_{i} \cap G^{\perp}) + \sum_{j=1}^{r} L_{K_{j}}(\Omega_{i} \cap G^{\perp}) \).
Step 3: The Algorithm converges to \( \Omega^{\ast} = \Omega_{i+1} = \Omega_{i} \).

VIII. EXTENSION TO CONTROL ALGEBRA

In the previous sections we provided a state feedback given by the vector α(ξ) and matrix β(ξ) which were assumed to be analytical functions of the state ξ. In particular, the analyticity is required for the proof of necessity as well as sufficient conditions. However, the class of analytic functions is too restrictive in terms of feedback that can actually be implemented on the system. For example, by rapid pulses which are arbitrarily strong and fast one can generate Lie bracket of the vector control vector fields which can act as a new control to the system available for feedback. In the light of non-analytic feedback it might be necessary to modify the conditions that guarantee decouplability of the system. Another approach which is sufficiently general would be to use the theory already developed for analytic feedback to systems whose control vector fields belong to the control algebra of the original system. (i.e) we propose to use the system, where \( \hat{K}_{i} \in \{ K_{1}, \cdots, K_{r} \}_{LA} = \mathcal{G} \). The theory of analytic feedback can now be extended to controls from the control algebra instead of just the original set of controls. Hence we can restate the conditions for decouplability in terms of the control algebra, which follows directly from the previous theorem as,

\[
\begin{align*}
\Delta, \mathcal{G} & \subset \Delta \oplus \mathcal{G} \quad (31) \\
[\Delta, \mathcal{C}] & \subset \Delta \oplus \mathcal{G} \quad (32)
\end{align*}
\]

where \( \mathcal{C} = \{ ad_{K_{i}}^{j}, K_{0}, i = 1, \cdots, r; j = 0, 1 \cdots \} \) and \( \mathcal{G} = \{ K_{1}, \cdots, K_{r} \}_{LA} \).

The above lemma just states a condition and does not provide an explicit formulation of the application of feedback. In order to provide the analytic feedback we consider a modified system with additional control vector fields generated from the original system. Consider the following modified system with finite dimensional control algebra \( \mathcal{G} \),

\[
\frac{\partial \xi(t)}{\partial t} = K_{0}(\xi(t)) + \sum_{i=1}^{m} u_{i} K_{i}(\xi(t)) + K_{i}(\xi(t)) \quad (33)
\]

where the vector fields \( \hat{K}_{i} \in \mathcal{G} \) which are generated by the vector fields of the original system are such that \( \mathcal{G} = \text{span}\{ K_{1}, \cdots, K_{m} \} \). (i.e) the set of vector fields \( \hat{K}_{i} \), not necessary a linearly independent set form a vector space basis for \( \mathcal{G} \). This is a required condition as the analytic feedback functions which can only generate utmost linear combinations of the existing control vector fields, (i.e) \( \text{span}\{ K_{1}, \cdots, K_{r} \} \) is inadequate to leverage the set of all possible controls. Hence it is necessary to modify the original system in order to utilize the repertoire of all possible controls for efficient feedback control. It is also to be noted that in so doing we do not alter the set of reachable or controllable set of the original system, but altering the output decouplability instead which is an observability property of the system.

IX. EXAMPLES

As an example of the above formalism consider a single qubit and a two qubit system coupled to the environment,

\[
\frac{\partial \xi(t)}{\partial t} = \frac{\omega}{2} z^{\ast} \xi(t) + \sum_{k} \omega_{b_{k}^{+}} b_{k} \xi(t) + u_{1} \sigma_{z} \xi(t) + u_{2} \sigma_{y} \xi(t) + \sum_{k} \sigma_{z}(g_{k} b_{k}^{+} + g_{k}^{\ast} b_{k}) \xi(t)
\]

with the output,

\[
y(t) = \langle \xi(t) | C | \xi(t) \rangle
\]

When we check against the necessary condition, \( \sum_{k} \sigma_{z}(g_{k} b_{k}^{+} + g_{k}^{\ast} b_{k}) \xi(t) \in \ker(dy(t)) \) which we notice the
single qubit system fails to satisfy, the conclusion that a single qubit system is not decouplable coincides with results obtained earlier by operator algebra. Now, consider the following two-qubit system eq. (34)

\[
\frac{\partial |\xi(t)\rangle}{\partial t} = \left( \sum_{j=1}^{2} \frac{\omega_0}{2} \sigma_z^{(j)} + \sum_k \omega_k b_k^\dagger b_k \right) |\xi(t)\rangle + \sum_k \left( \sum_j \sigma_z^{(j)} \right) (g_k b_k^\dagger + g_k^* b_k) |\xi(t)\rangle + (u_1(t)\sigma_x^{(1)} + u_2(t)\sigma_y^{(1)} + u_3(t)\sigma_z^{(2)}) |\xi(t)\rangle + u_4(t)\sigma_y^{(2)} + \frac{\omega_0}{2} \sigma_z^{(b)} + u_5 \sigma_x^{(b)} + u_6 \sigma_y^{(b)} + u_7 J_1 \sigma_z^{(1)} \sigma_x^{(b)} + u_8 J_2 \sigma_z^{(2)} \sigma_y^{(b)} \right) |\xi(t)\rangle + u_9 \sum_k \sigma_z^{(b)} (w_k b_k^\dagger + w_k^* b_k) |\xi(t)\rangle
\]

with \(\sigma_{x,y,z}\) now skew hermitian and the same output equation as before. It is seen that \(K_I \in \ker(dy(t))\) and

\[
[K_I, K_I] = [\sigma_{x,y,z}^{(1)} \xi, \sum_j \sigma_z^{(j)} (g_k b_k^\dagger + g_k^* b_k)] \xi = c \sum_k \sigma_{y,z}^{(1)} (g_k^* b_k + g_k b_k^\dagger) |\xi\rangle
\]

now belongs to the control algebra generated by the additional vector fields introduced by the bait system. Hence the system which was designed in order to meet the necessary condition, \([\mathcal{C}, H_{SE}] \subset \mathcal{C}\), given by the observation space formalism is also seen to meet the conditions given by tangent space or controllability invariant distribution formalism. A rather interesting scenario arises when the drift vector field \(K_0\) is a part of the ideal of \(\mathcal{G}\) and the interaction vector field \(K_I\) which is a part of the invariant subspace \(\Delta \subset \ker(dy(t))\), is already contained within the control algebra, \(i.e. K_I \subset \mathcal{G}\). The necessary and sufficient conditions for decouplability using feedback are trivially satisfied as \([K_I, K_I] \subset \mathcal{G} \cap \mathcal{K}_I \subset \mathcal{G}\) and

It can be clearly seen that the interaction vector field in deed belongs to \(K_I = \sum_{j,k} \sigma_z^{(j)} (g_k b_k^\dagger + g_k^* b_k) \xi(t) \in \ker(dy(t))\), where \(j = 0, 1, k = 0, 1, \cdots\), but

\[
[K_I, K_I] = [\sigma_{x,y}^{(1)} \xi, \sum_j \sigma_z^{(j)} (g_k b_k^\dagger + g_k^* b_k)] \xi = c \sum_k \sigma_{y,z}^{(1)} (g_k^* b_k + g_k b_k^\dagger) |\xi\rangle
\]

up to a constant \(c\), neither belongs to the span of the control vector fields, control algebra generated by the above vector fields or the controllability invariant distribution \(\Delta\). The last condition can be seen by the fact that \([K_I, K_I]\) does not belong to \(\ker(dy(t))\) and hence does not belong to \(\Delta \subset \ker(dy(t))\) either. Now consider the two qubit system with bait, which was discussed in the earlier section. The control system governing the mechanics following the Schrödinger eq. (7) is given by,

\[
[K_I, K_0] \in \mathcal{G}. \quad \text{Hence,}
\]

\[
[\Delta, K_I] \subset \Delta \oplus \mathcal{G}
\]

and the invariant subspace \(\Delta\) can now be guaranteed to exist and at least one dimensional equal to span\{\(K_0\}\}. Hence existence of feedback and decouplability is guaranteed for the above system.

\section{The Control System}

In the previous section we had only discussed a brief outline of the implementation of disturbance decoupling for quantum systems. In this section we present the construction of actual control system and the control vector fields. The bait qubit as discussed before was primarily used to get a handle on the environment so we may generate vector fields that could help decouple the system from the vector field \(K_I\). Let the following denote the
various hamiltonians acting on the system,
\[
H_0 = \sum_{j=1}^{2} \frac{\omega_j}{2} \sigma_j^{(j)} + \sum_k \omega_k b_k^\dagger b_k,
\]
\[
H_{SB} = \sum_k \left( \sum_j \sigma_j^{(j)} \right) (g_k b_k^\dagger + g_k^* b_k)
\]
\[
H_1 = \sigma_x^{(1)}, \quad H_2 = \sigma_y^{(1)}, \quad H_3 = \sigma_x^{(2)}, \quad H_4 = \sigma_y^{(2)}
\]
\[
H_5 = \sigma_x^{(b)}, \quad H_6 = \sigma_y^{(b)}, \quad H_7 = J_1 \sigma_x^{(1)} \sigma_x^{(b)}, \quad H_8 = J_2 \sigma_x^{(2)} \sigma_x^{(b)}
\]
\[
H_9 = \sum_k \sigma_x^{(b)} (w_k b_k^\dagger + w_k^* b_k)
\]
and let us denote by \(K_i\), the vector fields generated by the hamiltonian \(H_i\), (i.e), \(K_i = H_i(\psi)\). Now consider the particular back and forth maneuver via controls \(u_6\) and \(u_9\),
\[
u_6(\tau) = 1, \quad u_9(\tau) = 0, \quad \text{for } \tau \in [0,t],
\]
\[
u_6(\tau) = 0, \quad u_9(\tau) = 1, \quad \text{for } \tau \in [t,2t],
\]
\[
u_6(\tau) = -1, \quad u_9(\tau) = 0, \quad \text{for } \tau \in [2t,3t],
\]
\[
u_6(\tau) = 0, \quad u_9(\tau) = -1, \quad \text{for } \tau \in [3t,4t]
\]
The corresponding unitary time evolution operator at the end of time instant \(4t\) is given by,
\[
U(4t) = e^{(-iH_0 t)} e^{(-iH_2 t)} e^{(iH_1 t)} e^{(iH_3 t)}
\]
\[
= \exp(-i[H_6, H_9]) t^2 + O(t^3)
\]
the series expansion by Campbell-Baker-Hausdorff formula. In the limit that \(t = dt \to 0\). The effective direction of evolution is given by the commutator of the corresponding hamiltonians, but to the second order in time. Hence we could devise a control vector field in the direction given by the commutators of the corresponding hamiltonians \(H_6\) and \(H_9\), where,
\[
[H_6, H_9] = c.\sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
where \(c\) is a real constant for a skew hermitian \(H_6\) and \(H_9\). In fact it is possible to generate any direction of evolution with arbitrary strength corresponding to repeated commutators of the hamiltonians \(H_1 \cdots H_9\) of the physical system [34]. In order to compute commutators of tensor product operators we use the following identity,
\[
[A \otimes B, C \otimes D] = CA \otimes [B, D] + [A, C] \otimes BD
\]
With another control field \(H_8\) entering the picture we could generate the following direction in conjunction with the previous maneuver \([H_8, H_5], [H_6, H_9]\),
\[
= c_1 [J_2 \sigma_x^{(2)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)]
\]
\[
= c.\sigma_x^{(2)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
Consider the similar maneuver between controls \(u_4, u_6\) and \(u_8\), which generates the direction of evolution corresponding to the following repeated commutator,
\[
[H_4, H_8] = [\sigma_y^{(2)}, J_2 \sigma_x^{(2)} \sigma_x^{(b)}] = c.\sigma_x^{(2)} \sigma_x^{(b)}
\]
where \(c\) is a real constant for a skew hermitian \(H_4, H_8\). Again, from operating on equations (38) and (39) we get,
\[
c_1 [\sigma_x^{(2)} \sigma_x^{(b)}, \sigma_x^{(2)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)]
\]
\[
= c_1 [\sigma_x^{(2)} \sigma_x^{(b)}, (\sigma_x^{(b)})^2 \sum_k (w_k b_k^\dagger + w_k^* b_k)]
\]
\[
= c.\sigma_x^{(2)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
Hence we have generated an effective coupling between qubit 2 and the environment with the help of the bait qubit and its interaction with the environment and qubit 2. It is important to note that the hamiltonian so obtained by the above control maneuver now acts trivially on the hilbert space of the bait qubit, a property which will be found to be extremely useful later. It is also possible to generate the \(\sigma_x^{(2)}\) counterpart of the above coupling by a similar maneuver, given by,
\[
\sigma_x^{(2)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
Again by a symmetric and totally similar argument we could generate a coupling between the environment and qubit 1, which would be given by,
\[
\sigma_x^{(1)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
\[
\sigma_x^{(1)} \sigma_x^{(b)} \sum_k (w_k b_k^\dagger + w_k^* b_k)
\]
Now noting that the constants \(c\) in the above equations could be controlled independently and arbitrarily, we can write the preliminary form of the actual control system which achieves disturbance decoupling. Gathering terms [40], we construct the following control system for
By restructuring the control vector fields as above we are hoping to capture the entire control algebra by a simple linear span of the control vector fields which is essential to analytical feedback theory. Let us again, investigate the decouplability of the above control system from its necessary conditions, that (i)$K_I \in \Delta \subset \ker(dy)$, (ii)$[K_I,K_J] \in \Delta + G$, where $G = \text{span}(K_1 \cdots , K_n)$, is the distribution generated by the control vector fields above. By considering $[K_I,K_J] = \sigma^{(1)}_x \sum (w_k b^*_k + w^*_k b_k)\xi(t)$, which is already contained within $G$. The conditions are also satisfied for the vector fields $K_2, K_3$ and $K_4$. However with the vector field $K_5$, we note that $[K_I,K_5]$,

$$
\tau = \left[ \sum_k \left( \sum_j \sigma^{(j)}_z \right) (g_k b^*_k + g^*_k b_k) \xi(t), \\
\sigma^{(1)}_x \sum_k (w_k b^*_k + w^*_k b_k) \xi(t) \right] \\
= c.\sigma^{(1)}_y \left( \sum_k (w_k b^*_k + w^*_k b_k) \right)^2 \xi(t)
$$

where w.l.o.g $w_k = c_1g_k$ for an arbitrary constant $c_1 \in \mathbb{C}$. For an infinite dimensional environment the vector fields that contain higher powers of $\sum_k (w_k b^*_k + w^*_k b_k)$, cannot be expressed as a linear combination of its lower powers as can be seen from its action on a particular number state $|n\rangle$,

\[
(w_k b^*_k + w^*_k b_k)|n\rangle = w_k \sqrt{n}|n-1\rangle + w^*_k \sqrt{n+1}|n+1\rangle \\
(w_k b^*_k + w^*_k b_k)^2|n\rangle = 2w_k^2|2n\rangle + w_k^2 \sqrt{n(n-1)(n+1)}|n-2\rangle + w^*_k^2 \sqrt{n(n+1)(n+2)}|n+2\rangle
\]

for some $n$. In other words the above term is neither contained in $G$ nor in $\Delta \subset \ker(dy)$, because $L_{\tau}\xi(t) \neq 0$. The only way to correct the above situation is to include the vector $\tau$ as a control vector field in the control system above. This can be achieved by similar maneuvers between the vector fields above, (i.e), $\tau = c.\left[ K_I, [K_1, H_2\xi] \right]$. Now again, since $\tau$ is a new control vector field, it must satisfy condition (ii) above. But $[K_I,\tau] = c.\sigma^{(1)}_y \left( \sum_k (w_k b^*_k + w^*_k b_k) \right)^3 \xi(t)$, now generates the next higher power of the same environmental term, which necessitates us to find a way to include that in our control vector fields as well. In fact, it is possible to generate any power of the environmental term by repeated commutators, which is linearly independent of all the previous terms and hence generates a new direction of flow within the analytic manifold. And it is impossible to include all the successive powers in our control vector fields. Hence the best we could hope to achieve under the present circumstance is to obtain an approximate solution to disturbance decoupling. It is to be noted that the above problem arises only in an infinite dimensional environment and restricting the dimension of environment is a reasonably good approximation. Hence we present a experimentally realizable scheme to demonstrate the theory of disturbance decoupling to practical quantum systems. The following system captures the essence of the problem as well as the solution itself. Before we present the example we summarize the results obtained thus far in a concise form. The following table is helpful in noting the above decouplability results,

|                        | Open Loop | Closed Loop | Closed Loop Restructured |
|------------------------|-----------|-------------|--------------------------|
| Single Qubit           | NO        | NO          | NO                       |
| Two Qubit              | NO        | NO          | NO                       |
| Two Qubit or higher    | NO        | NO          | YES*                     |
| with bait qubit        | NO        | NO          | YES*                     |

*The system can be completely decoupled under the additional assumption of a finite dimensional environment.

We note that the conditions for decouplability from Open loop to Closed loop to Closed Loop Restructured are progressively relaxed. Hence a system that is not Closed Loop Restructured decouplable cannot be Closed Loop or Open Loop decoupled.

**Finite State Environment** Environment always appears to be in a stationary state(also called the Gibbs State). An essential element of the stationary state which is most stable and extremely resilient is the coherent state of an electromagnetic system. Coherent states is generated by the action of the displacement operator $D(\alpha) = e^{(\alpha a^* - \alpha^* a)}$ on the vacuum state $|0\rangle$. An electromagnetic system when perturbed from one coherent state simply settles in another coherent state. It is labeled by a complex number $\alpha$, that denotes the strength.
of the state. The state is given by,

\[ |\alpha\rangle = e^{-1/2|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]  \hspace{1cm} (45)

where \(|n\rangle\) is the number state. It can be the seen that the coefficients of higher \(n\) decrease rapidly and since squared sum of the coefficients is convergent, with major contribution from lower states it is a reasonable approximation to neglect higher energy states of the electromagnetic system. In fact, this is the basis for the experimental realization of “dual rail optical photon quantum gates”, where in only the \(|0\rangle\) and \(|1\rangle\) photon states are used to represent the system under the premise that contributions from higher energy photons are negligible. Hence we consider the following model for a finite state harmonic oscillator with \(N\) energy states which will be later dubbed as the environment. The creation and annihilation operators act on the system as follows,

\[ a|n\rangle = \sqrt{n+1}|n+1\rangle \text{ for } n < N \]
\[ a|n\rangle = 0 \text{ for } n \geq N \]
\[ a^\dagger |n\rangle = \sqrt{n}|n-1\rangle \text{ for } n \leq N \text{ and } n > 0 \]
\[ a^\dagger |n\rangle = 0 \text{ for } n > N \text{ and } n = 0 \]  \hspace{1cm} (46)

It was recently shown by Fu et al.\[18\] in their model of truncated harmonic oscillator that such a system up to energy state \(N\) was feasible. Hence the schematic presented here can be readily implemented if one were able to create and sustain a controllable interaction between the electromagnetic and spin system( the bath). Now consider a single spin-1/2 system with hamiltonians \(\sigma_z, \sigma_x,\) and \(\sigma_y\). The state of the system is represented as \(\psi = |c_0, c_1\rangle\) in the vector form where the coefficients correspond to the two states and \(c_0, c_1 \in \mathbb{C}\text{ s.t. }|c_0|^2 + |c_1|^2 = 1\). The tangent vector to the system is given by the action of the skew hermitian operators on the state \(\psi\), (i.e., \(\dot{\psi} = \sigma_x |\psi\rangle \text{ and } C |\psi\rangle\text{ for any }C \in \mathbb{C}\text{ s.t. }|C|^2 = 1\)). In other words, in order to express any vector in the tangent space as a real linear combination of other vectors we require at least 4 linearly independent vectors given by \(\sigma_z |\psi\rangle, \sigma_x |\psi\rangle, \sigma_y |\psi\rangle, \overline{\psi}\). For the case of a 2 spin-1/2 system the number of linearly independent vectors required is 8, given by a subset of \(\sigma_i \otimes \sigma_j |\psi\rangle\) for \(i, j \in \{x, y, z, 0\}\).

For the case of 2 spin-1/2 system coupled to a 3 state environment, the tangent space is \(4 \times 3\) dimensional and the number of linearly independent vectors required to span the entire tangent space is \(4 \times 3 \times 2 = 24\). In other words we require 24 linearly independent control vector fields to make absolutely sure that the conditions for decouplability are met. Let the environment be governed by a single 3 level harmonic oscillator. The different energy levels are given by \(|0\rangle, |1\rangle, |2\rangle\), and a general state in this basis is given by \(|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle\). We can now examine the linearly independent vectors generated by the powers of the bath/environment operator for the three level system by taking into account the defining relations \([18]\). As it can be seen that the following 6 vectors,

\[ \text{I}\psi = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \]
\[ (wb^\dagger + w^*b)|\psi\rangle = wc_0|0\rangle + w\sqrt{2}c_2|1\rangle + w^*c_0|1\rangle + w^*\sqrt{2}c_2|2\rangle \]
\[ (wb^\dagger + w^*b)^2|\psi\rangle = [w^2(b^\dagger)^2 + w^2w^*(b^\dagger)b + w^2w^*b^\dagger b + w^2w^*b^\dagger b] |\psi\rangle \]
\[ (wb^\dagger + w^*b)^3|\psi\rangle = [w^3w^*(b^\dagger)^2b + w^3w^*b^\dagger b + w^2w^*b^\dagger b + w^2w^*b^\dagger b] |\psi\rangle \]
\[ (wb^\dagger + w^*b)^4|\psi\rangle = [w^4w^*(b^\dagger)^2b + w^4w^*b^\dagger b + w^3w^*b^\dagger b + w^3w^*b^\dagger b] |\psi\rangle \]
\[ (wb^\dagger + w^*b)^5|\psi\rangle = [w^5w^*(b^\dagger)^2b + w^5w^*b^\dagger b + w^4w^*b^\dagger b + w^4w^*b^\dagger b] |\psi\rangle \]
\[ (wb^\dagger + w^*b)^6|\psi\rangle = [w^6w^*(b^\dagger)^2b + w^6w^*b^\dagger b + w^5w^*b^\dagger b + w^5w^*b^\dagger b] |\psi\rangle \]

Expressed in terms of the creation and annihilation operators of the bath, \(b\) and \(b^\dagger\) and do not contain powers higher than 3 in their respective expansions and generate as many linearly independent vectors as possible on \(T_3(M)\), while operating on the state \(\xi\). With the above linearly independent vectors we could construct the new control system given by,

\[ \frac{\partial |\xi(t)\rangle}{\partial t} = \left( \sum_{j=1}^{2} \frac{\omega_j}{2} \sigma_z^{(j)} + \sum_k \omega_k b^\dagger_k b_k \right) |\xi(t)\rangle + \sum_k \sigma_z^{(j)} (gb^\dagger + g^*b) |\xi(t)\rangle + \sum_{i=0}^{5} u_{i1} \sigma_x^{(1)} (wb^\dagger + w^*b)^i |\xi(t)\rangle \]
\[ + \sum_{i=0}^{5} u_{i2} \sigma_y^{(1)} (wb^\dagger + w^*b)^i |\xi(t)\rangle + \sum_{i=0}^{5} u_{i3} \sigma_y^{(2)} (wb^\dagger + w^*b)^i |\xi(t)\rangle + \sum_{i=0}^{5} u_{i4} \sigma_y^{(3)} (wb^\dagger + w^*b)^i |\xi(t)\rangle \]  \hspace{1cm} (47)

For the control system described above where the control vector fields \(\{K_{ji}\}, 0 \leq i \leq 5 \text{ and } 1 \leq j \leq 4\), span the entire control algebra and hence,

\[ [\Delta, K_{ji}] \subset \Delta + \mathcal{G}, 0 \leq i \leq 5 \text{ and } 1 \leq j \leq 4 \]  \hspace{1cm} (48)
where \( \mathcal{G} = \{ K_1, \ldots, K_{24} \} \), \( \mathcal{L}_A = \text{span}\{ K_1, \ldots, K_{24} \} \). It now remains to know if there exists a controlled invariant distribution \( \Delta \), that satisfies the condition stated above. It can be seen that since \( \Delta \) is a subspace of the tangent space \( T_{\xi}(\mathcal{M}) \) at \( \xi \), the equation above is trivially satisfied. The only additional constraint that \( \Delta \) needs to satisfy is that \( \delta \) be a part of the \( \text{ker}(dy) \), the nullspace of \( y \) at the point \( \xi \), which is a subspace of the tangent space \( T_{\xi}(\mathcal{M}) \) itself. \( \text{ker}(dy) \) is comprised of vectors of the form \( H[\xi] \) where \( H \) is a real linear combination(with coefficients possibly a function of the state \( \xi \)) of skew hermitian operators, with the additional constraint that, \( L(H[\xi])y = 0 \), which translates to the commutator, \( [C,H] = 0 \). Since the covector \( dy \) is one dimensional for a scalar function \( y \), the corresponding nullspace \( \text{ker}(dy) \), would be \( n-1 \) dimensional where \( n \) is the dimension of the tangent space. Some of the vectors in \( \text{ker}(dy) \) are,

\[
(I^{(1)} \otimes I^{(2)})(wb^i + w^i b)\xi(t), 1 \leq i \leq 5 \\
(\sigma_2^{(3)} + \sigma_2^{(2)})(wb^i + w^i b)\xi(t), 1 \leq i \leq 5 \\
i(\sigma_2^{(1)} + \sigma_2^{(2)})(wb^i - w^i b)\xi(t), 1 \leq i \leq 5 \text{ etc}
\]

where the operators \( \sigma_2, I \), above are to understood as skew hermitian operators as before. It is to be noted that the algorithm presented in the previous section would terminate after the first iteration as the condition is already satisfied and would yield \( \text{ker}(dy) \) as \( \Delta^* \), the maximal controlled invariant distribution. The least value that \( \Delta \) could take according to the necessary conditions of theorem V.1, \( K_i \in \Delta \subset \text{ker}(dy) \) is, the one dimensional vector space \( \text{span}\{ K_i \} \), itself. The algorithm presented in the previous section is designed to yield the maximal invariant subspace, which guarantees decouplability. But in order to compute the feedback we could work with any \( \Delta \) that is a subspace of maximal \( \Delta^* \) and contains the minimal \( \text{span}\{ K_i \} \), as long as the condition (48) is satisfied.

Feedback Synthesis In order to determine the feedback let us work with the minimal \( \Delta = \text{span}\{ K_i \} \). It is possible to construct \( n - 1 \) vectors where, \( n = 2 \times \dim(T_{\xi}(\mathcal{M})) \) vectors \( v_2, \ldots, v_n \in T_{\xi}(\mathcal{M}) \) that commutes with \( v_1 = K_i \) (i.e) \( [v_1, v_j] = 0 \). Reindexing the control vector fields as \( K_1, \ldots, K_r \), where \( r = n = 24 \) in this case, and since \( K_i \) span the tangent space we can write,

\[
v_j = \sum_{i=1}^{r} d_{ij} K_i \tag{49}
\]

where \( d \) is a non-singular real matrix. Hence we could rewrite,

\[
\begin{pmatrix}
K_1 \\
K_2 \\
\vdots \\
K_r
\end{pmatrix} =
\begin{pmatrix}
d_{11} & \cdots & d_{1r} \\
\vdots & \ddots & \vdots \\
d_{r1} & \cdots & d_{rr}
\end{pmatrix}^{-1}
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_r
\end{pmatrix}
\]

Following the proof of Theorem [V.11], we can form the vectors, \( K_{\alpha}^* = S \nu \), where, \( S = d^{-1} \) but with first column replaced by zeros (i.e),

\[
S = \begin{pmatrix}
0 & s_{12} & \cdots & s_{1r} \\
\vdots & \ddots & \vdots \\
0 & s_{r2} & \cdots & s_{rr}
\end{pmatrix}
\]

Now, the feedback parameter \( \beta \) is such that

\[
\beta \times
\begin{pmatrix}
K_{\alpha}^* \\
K_2^* \\
\vdots \\
K_r^*
\end{pmatrix} =
\begin{pmatrix}
v_2 \\
v_3 \\
\vdots \\
v_r
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_r
\end{pmatrix}
\]

Since the above equations holds for all \( \nu \) and \( K \) vectors we could write \( \beta(\xi)S = J \), but since the above equation remains unaltered when \( S \) is replaced with \( d^{-1} \), we can calculate the feedback parameter as, \( \beta = Jd \). The closed loop vector fields are given by \( \tilde{K} = \beta(\xi)K \). Similarly the parameter \( \alpha \) can be calculated by incorporating \( K_0 \) in the equation. For any \( K_0 = \sum_{i=1}^{r} c_i v_j \), we can find an \( \tilde{\alpha} \) such that,

\[
K_0 + \tilde{\alpha} J = c_1 v_1 \tag{50}
\]

for some \( c_1 \) as a function of the state \( \xi \). The parameter \( \alpha \) is given by \( \alpha = \tilde{\alpha} \hat{\beta} \) and the closed loop drift vector field is given by, \( \tilde{K}_0 = \sum_{i=1}^{r} \alpha_i K_i \). It can be seen that the above closed loop vector fields as in the proof does satisfy invariance w.r.t \( \Delta \), (i.e) \( [\Delta, \tilde{K}_0] \in \Delta, 0 \leq i \leq r \). Hence the system is completely decoupled even in the presence of symmetry breaking control Hamiltonians via classical state feedback.

**We cannot find a suitable basis transformation using real matrices to a known set of commuting vectors such as \( c_1[000], c_1[102], \cdots \in T_{\xi}(\mathcal{M}) \) etc where \( c_1, c_2 \in \mathbb{C} \), as performed in [4], where vectors were transformed to coordinate basis in \( \mathbb{R}^n \) in order to determine the feedback. Hence the task of finding commuting vectors were simplified by such a transformation in the classical case. The difficulty is due to fact that (i) coefficients of the states complex and effectively carry twice the dimension, (ii) tangent vectors at point \( \xi \) is different from that of another point \( \xi_1 \), hence a fixed coordinate transformation does not work for every \( \xi \). Whereas in the case of \( \mathbb{R}^n \) tangent space at every point \( x \) is the same.**
decoupling problem thus rendering the decoherence acting on the system unobservable on the states of interest. However in order to accomplish the goals we had to introduce additional couplings and a bait subsystem that were not a part of the system initially.

XI. INTERNAL MODEL PRINCIPLE

In order to decouple the output from the environment one needs to determine the feedback coefficients $\alpha(\xi)$ and $\beta(\xi)$ where both depend on the combined state of the system and environment. Hence one needs to have a good estimate of the system as well as the environment itself for successful implementation of feedback decoupling. In other words the state observer must include a model of the environment which would enable us estimate its state. At this point, the important differences between classical and quantum decoupling problems can be understood at the outset. The necessary condition in terms of the operator algebra $[C, H_{SB}] \subset C$ was instrumental in design of the bait subsystem. However the structure of the system needed to be altered in order to,

(i) Artificially induce coupling between qubits 1, 2 and the environment with the help of the bait.

(ii) Generate vector fields in higher power of the environment operator to as to generate linearly independent vectors.

Hence it was necessary to modify the core system in more ways than one in order to perform decoupling. Hence, even though environment is an undesirable interaction the higher powers of the same helped us generate linearly independent vectors in the tangent space, which was absolutely necessary for decoupling. Hence the environmental coupling here befits the description of necessary evil. In classical dynamic feedback the design of controller depends on the exosystem. In contrast the state observer/estimator needs to know the model of environment in order to estimate the combined state $\xi$ and calculate the feedback. Hence the model discussed above could be thought of as the Internal Model Principle analog of quantum control systems. In addition classical output regulation problem concerns with following a reference signal in the presence of environmental disturbance that depends on a prescribed exosystem. On the other hand the disturbance decoupling problem focuses on eliminating the effects of the environment.

XII. BILINEAR INPUT AFFINE REPRESENTATION OF QUANTUM SYSTEMS

In this section we will attempt to highlight a few more important differences between the decoherence control in quantum systems and disturbance decoupling of classical input affine systems in $\mathbb{R}^n$.

(i) Classical noise is additive, $\dot{x} = f(x) + u_i g_i(x) + w p(x)$ and operate on the same vector space. Whereas quantum noise is tensorial. The noise parameter $g_k$ and $g_k^*$ dictate the coupling between the environment and the system, (i.e), $K_i = (\sigma_z^{(1)} + \sigma_z^{(2)}) \otimes (g_k b_k + g_k^* b_k^*)|\xi\rangle$ corresponds to the classical noise vector $p(x)$, and it can be easily seen that there is no noise operating on the system in the classical sense. Hence decoherence is not classical noise.

(ii) Vector spaces in quantum control systems are over complex fields. This increases the dimensionality by 2 fold in many instances where linearly combination has to be taken. Hence in order to generate every vector in a vector space of $n$ independent states, we require $2n$ linearly independent vectors.

(iii) The necessary and sufficient conditions impose restrictions on the form of control hamiltonian that could help decouple the system. From the conditions derived above, it is impossible to decouple one part of the system from the other unless our control hamiltonians operate on the both the hilbert spaces non-trivially (i.e $H_i \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the set of linear operators in the joint hilbert space of both the systems. It was in light of this condition that the bait system was originally introduced.

(iv) Distributions need not necessarily be singular. For instance the tangent space of an $SU(2)$ system is spanned by $\sigma_z|\xi\rangle, \sigma_x|\xi\rangle, \sigma_y|\xi\rangle, I|\xi\rangle$, where $|\xi\rangle = c_0|0\rangle + c_1|1\rangle$ and the operators are again assumed to be skew hermitian counterparts of hermitian $\sigma_z, \sigma_x, \sigma_y$. Even though the four vectors are linearly independent for almost all non-zero values of $c_0$ and $c_1$ the distribution is non-singular. Consider $|\xi\rangle = |0\rangle$ and the corresponding tangent vectors are $-i|0\rangle, i|1\rangle, |0\rangle, i|0\rangle$, whose real linear combination is rank deficient. Hence it can be seen that the vector $|0\rangle$ does not belong to tangent space $T_{|0\rangle}$ at the point $|\xi\rangle = |0\rangle$. In general the tangent vectors at point $\xi$ is different from that of another point $\xi_1$. One of the most serious implications is that we cannot find a linear map that transforms the distribution $\Delta$ to a constant $d$ dimensional distribution,

$T.\Delta = \begin{bmatrix} I_{d \times d} & 0 \\ 0 & 0 \end{bmatrix}$

at every point $\xi$, an approach that was used in Isidori[4] to greatly simplify finding commuting vectors $|v_1\rangle, \cdots |v_n\rangle$ in an $n$ dimensional tangent space. The commuting vectors were just taken to be the co-ordinate basis at every point $x$.

XIII. CONCLUSION

In this work we provided the conditions and a step by step procedure to calculate a classical deterministic feedback under which the 2-qubit system could be successfully decoupled from decoherence. As mentioned before the analysis carried out in the bilinear form only helped us learn about the control hamiltonians helpful in decoupling the system but also provided a solution under which the system would be completely decoupled as
tum computation thus enabling us to exploit the com-
immensely helpful in performing decoherence free quan-
tual computation thus enabling us to exploit the com-
putational speed up provided by quantum parallelism. However in order to determine the feedback one needs to have a good estimate of the state of the system.

[1] H.-P. Breuer and F. Petruccione, The Theory of open quantum systems, Oxford University Press, 2002.
[2] William H. Louisell, Quantum Statistical Properties of Radiation, John Wiley & Sons, Inc, 1973.
[3] G. Mahler, V.A Weberruß, Quantum Networks, Springer-Verlag, 1998.
[4] Alberto Isidori, Nonlinear Control Systems, Springer-Verlag, 1995.
[5] Michael B. Mensky, Quantum Measurements and Decoherence, Models and Phenomenology, Kluwer Academic Publishers, 2000.
[6] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, H. D. Zeh, Decoherence and the Appearance of a Classical World in Quantum Theory, Springer, 1996.
[7] Jie Huang, Nonlinear Output Regulation, Theory and Applications, SIAM, 2004.
[8] R. B-Kohout and W. H. Zurek, “A Simple Example of Quantum Darwinism”, European Physical Journal, No. 27, pp 2, 2002.
[9] W. H. Zurek, Rev. of Modern Physics, 75(3), 715-775, 2003.
[10] F. Xue et. al, Phys. Rev. A, 73, 013403, 2006.
[11] J. M. Raimond, M. Brune and S. Haroche, “Reversible Decoherence in an Optical Quantum Computing Algorithm”, Phys. Rev. Lett., 91(18), 187903, 2003.
[12] M. Mohseni, K. Jacobs, G. J. Milburn, and S. Lloyd, “Experimental application of Decoherence-Free Subspaces in an Optical Quantum Computing Algorithm”, Phys. Rev. Lett., 91(18), 187903, 2003.
[13] J. E. Ollivier, D. A. Lidar and L. E. Kay, “Quantum Resonance Realization of Decoherence-Free Computation”, 91(21), 217904, 2003.
[14] A. C. Doherty, K. Jacobs and G. Jungman, “Information, disturbance and Hamiltonian feedback control”, Phys. Rev. A, 63, 062306, 2001.
[15] V. Bruss, A. Dréau, C. Gómez, R. Derka, G. Adam, and H. Widemann, “Quantum State Reconstruction From Incomplete Data”, Phys. Rev. A, 73, 013403, 2006.
[16] J. L. Ollivier and S. Drummond, “Quantum Resonance Realization of Decoherence-Free Computation”, 91(21), 217904, 2003.
[17] D. Kielpinski, C. Monroe and D. J. Wineland, “Architecture for a large-scale ion-trap quantum computer”, Nature, 417, pp 709-711, 2002.
[18] S. Wallentowitz, “Quantum theory of feedback of bosonic gases”, Phys. Rev. A, 66, 032114, 2002.
[19] N. Ganesan, T. J. Tarn, “Control of decoherence in open quantum systems using feedback”, Proc. of the 44th IEEE CDC-ECC, pp.427-433, Dec 2005.
[20] N. Ganesan, T. J. Tarn, “Feedback Control of Decoherence by continuous measurements”, 91(18), 187903, 2003.
[21] N. Ganesan, T. J. Tarn, “Control of decoherence in open quantum systems using feedback”, Proc. of the 44th IEEE CDC-ECC, pp.427-433, Dec 2005.
[22] V. Bužek, G. Drobný, R. Derka, G. Adam, and H. Widemann, “Quantum State Reconstruction From Incomplete Data”, arXiv:quant-ph/9905020.
[23] G. M. Huang, T. J. Tarn, J. W. Clark, “Decoherence and the transition from quantum to classical - Revisited”, Los Alamos Science, No. 27, pp 2, 2002.
[24] J. M. Geremia, J. Stockton, H. Mabuchi, arXiv:quant-ph/0401107v4.
[25] H. M. Wiseman, Phys. Rev. A, 49(3), pp 2133-2150, 1994.