On the localization of four-dimensional brane-world black holes

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Abstract

In the context of brane-world models, we pursue the question of the existence of five-dimensional solutions describing regular black holes localized close to the brane. Employing a perturbed Vaidya-type line-element embedded in a warped fifth dimension, we attempt to localize the extended black-string singularity, and to restore the regularity of the AdS spacetime at a finite distance from the brane by introducing an appropriate bulk energy–momentum tensor. As a source for this bulk matter, we are considering a variety of non-ordinary field-theory models of scalar fields either minimally coupled to gravity, but including non-canonical kinetic terms, mixing terms, derivative interactions and ghosts, or non-minimally coupled to gravity through a general coupling to the Ricci scalar. In all models considered, even in those characterized by a high degree of flexibility, a negative result was reached. Our analysis demonstrates how difficult the analytic construction of a localized brane-world black hole may be in the context of a well-defined field-theory model. Finally, with regard to the question of the existence or not of a static classical black-hole solution on the brane, our analysis suggests that such solutions could in principle exist; however, the associated field configuration itself has to be dynamic.

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1. Introduction

The idea that our world could be a four-dimensional hypersurface, a brane, embedded in a higher dimensional spacetime, the bulk, dates back already to the 1980s [1, 2]. More recently, however, it received a widespread renewed interest when novel theories [3, 4] incorporated gravity into the brane-world scenario in an attempt to solve the hierarchy problem. These proposals have prompted an intensive research activity investigating their implications on gravity, particle physics and cosmology. Gravity, in particular, has seen one of the most important pillars of the general theory of relativity, the concept of four-dimensional spacetime, being modified in order to accommodate the potential existence of extra spacelike dimensions.
This inevitably led to the reviewing of all known solutions and predictions of four-dimensional gravity, the most studied ones being the black-hole solutions. In the context of the large extra dimension scenario [3], where the extra dimensions were assumed to be flat, the study of black holes was straightforward since higher dimensional versions of the Schwarzschild [5] and Kerr solutions [6] were known for decades. However, in the context of the warped extra dimension scenario [4], the task to derive a black hole on a brane embedded in a curved five-dimensional background has proven to be unexpectedly difficult (for reviews, see [7]).

The first attempt to derive a brane-world black-hole solution in a higher dimensional spacetime with a warped extra dimension appeared in [8] where the four-dimensional Minkowski line-element in the Randall–Sundrum (RS) metric was substituted by the Schwarzschild solution, i.e.

\[ ds^2 = e^{2A(y)} \left[ -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \]  

(1)

The above line-element satisfies the five-dimensional Einstein’s field equations of the RS model, since the Schwarzschild solution, just like the Minkowski one, is a vacuum solution. However, it was demonstrated that the above ansatz does not describe a regular black hole localized on the brane, since the solution is characterized by a string-like singularity extended along the fifth dimension. This becomes manifest in the expression of the five-dimensional curvature invariant quantity

\[ R_{MNR}^{\text{MNR}} = \frac{48e^{-4A(y)}M^2}{r^6} + \cdots . \]  

(2)

For \( A(y) = -k|y| \), where \( k \) is the AdS curvature radius, as in the RS model, or for any other warp function decreasing away from the brane, the above quantity blows up at \( y \)-infinity; more importantly, it reveals the existence of a singularity at \( r = 0 \) for every slice \( y = \) const of the five-dimensional AdS spacetime. The above solution was therefore a black string, rather than a black hole, and was soon proven to be plagued by the Gregory–Laflamme instability [9, 10].

In the years that followed, other attempts to derive a regular black-hole solution in a warped five-dimensional background proved how tricky the nature of the problem was: no analytical solution that would satisfy the five-dimensional field equations and describe a four-dimensional black hole on the brane was found, despite numerous different approaches that were used (for some of them, see [11–20])\(^1\). One of these approaches [14] was to assume that the black-hole mass has a non-trivial \( y \)-profile along the extra dimension: if \( M \) in equation (1) is not a constant quantity but a function of \( y \), then, upon a convenient choice, the expression on the rhs of equation (2) could die out at a finite distance from the brane. However, the line-element inside the square brackets in (1) with \( M = M(y) \) is no longer a vacuum solution. A bulk matter distribution must be introduced for the five-dimensional line-element to satisfy the field equations. The corresponding energy–momentum tensor was found [13, 14] to describe a shell-like distribution of matter engulfing the brane with a stiff-fluid equation of state that satisfied all energy conditions on the brane and vanished, as expected, away from the brane. Unfortunately, no field configuration, in the context of scalar or gauge field models, was found that could support such an energy–momentum tensor.

During the same period, numerical solutions were found [23–25] in the context of five- and six-dimensional warped models that exhibited the existence of black-hole solutions with horizon radius smaller than or at most of the order of the AdS length \( \ell = 1/k \). No larger black-hole solutions were found, and that led to arguments [26–31] and counter-arguments

\(^1\) In contrast, analytical solutions describing black holes localized on a 2-brane embedded in a (3+1)-dimensional bulk were constructed in [21, 22] by using a C-metric in the AdS\(_4\).
[32–35] for the non-existence of large, classical, static black-hole solutions on the brane. Even in the case of small black holes, no closed-form analytic solutions, that would allow us to study their topological and physical properties in a comprehensive way, were ever found—in addition, the very existence of the numerical solutions describing small vacuum black holes was put into question in recent works [36, 37]. Recently, new numerical solutions employing novel numerical techniques have been presented [38, 39] that describe both small and large black holes in the context of the RS model: the solutions have been constructed starting from an AdS$_5$/CFT$_4$ solution with an exact Schwarzschild metric at the AdS infinite boundary; the boundary background is then rewritten in a more general way and expanded along the bulk to derive an RS brane at a finite proper distance whose induced metric is a perturbed Schwarzschild metric.

It is an intriguing fact that, contrary to the findings of the numerical works [23–25], all analytical attempts to derive a five-dimensional regular black hole localized on the brane have been forced to introduce some form of matter in the theory, either in the bulk [13, 14, 16, 35] or on the brane [18–20, 34, 40], or even geometrical terms [12, 41]. In one of the most recent numerical works that have presented brane-localized black-hole solutions [37], the existence of a distribution of matter also plays an important role: the solutions exist only upon the introduction of an external electromagnetic field on the brane. In [38, 39], as well as in the lower dimensional constructions [21, 22], no additional matter is introduced; however, the induced geometry on the brane is not a vacuum solution of the 4D equations—rather, it is sourced by the energy–momentum tensor of the conformal field theory (CFT) residing on it (for an introduction to the AdS/CFT correspondence in the brane-world context, see [42]). In our opinion, it is clear that the localization of the black-hole topology—as we know it—close to the brane demands support from some additional form of matter and cannot be realized by itself. For this reason, in this work, we will turn again to the approach of [13, 14] in order to investigate potential field-theory models that could yield the well-behaved energy–momentum tensor that supported a regular, localized black hole. The mass of the black hole will be assumed again to have a non-trivial profile along the extra dimension: this is motivated primarily by the need to eliminate the singular term of equation (2) and turn the singular black-string spacetime to a regular AdS one at a finite distance from the brane; in addition, as the question of whether a purely Schwarzschild line-element should be recovered on the brane still remains open, the $\gamma$-dependence of the mass function will keep the model general enough to accommodate solutions that either resemble the Schwarzschild line-element on the brane or deviate from it. In addition, a time dependence will be introduced in the line-element in an attempt to investigate whether the outcome of the gravitational collapse can indeed be static or not.

The outline of our paper is as follows: in section 2, we present the theoretical framework of our work—we also make a link with previous analyses [13, 14] and justify the changes in our assumptions. In the following two sections, we proceed to the investigation of the scalar field-theory models that we have considered in this work: in section 3, we discuss the first class of models based on one or more scalar fields minimally coupled to gravity; in section 4, we turn to the case of a non-minimally coupled scalar field with a general coupling to the Ricci scalar. In both classes of models, we investigate the existence of viable black-hole solutions in the context of a generalized RS model and determine the obstacles that appear while following this approach. We discuss our results and present our conclusions in section 5.

2. The theoretical framework

As mentioned in the introduction, the factorized metric ansatz (1) leads to a black-string, rather than a black-hole, solution. Therefore, throughout this work, we will consider a
non-factorized metric with \( y \)-dependence in the four-dimensional part of the line-element and more specifically in the mass parameter \( M \). The obvious choice, to substitute the constant \( M \) into equation (1) by a function of the fifth coordinate, however, leads to the appearance of an additional singularity in the five-dimensional spacetime at the location of the horizon [13]. In [14], it was demonstrated that this is due to the non-analyticity of the four-dimensional line-element: employing an analytic ansatz, i.e. a four-dimensional line-element without a horizon, leads to a five-dimensional spacetime without additional singularities.

Therefore, in what follows, we will consider the following analytic Vaidya-type line-element:

\[
\left[ -\left( 1 - \frac{2m(v, y)}{r} \right) dv^2 + 2\epsilon dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + dy^2, \quad \text{(3)}
\]

For a constant value of \( v \), the line-element inside the square brackets is a non-static Vaidya metric that can be used to describe the dynamical process of a collapsing (\( \epsilon = +1 \)) or an expanding (\( \epsilon = -1 \)) shell of matter. If we also ignore the \( v \)-dependence, the four-dimensional static Vaidya metric is related to the Schwarzschild one by a mere coordinate transformation. Although we will be interested in final states that describe a static black hole (thus, we set \( \epsilon = +1 \)), during this work we will keep the \( v \)-dependence, as we would like to investigate whether static configurations can exist at all or whether some type of dynamical evolution is necessarily present in the model even after the formation of the black hole—as a matter of fact, it was Vaidya-type metrics that were used in some of the original works addressing this question [26, 27].

The modified Vaidya-type line-element (3) was also shown to exhibit some attractive characteristics in the quest of localized black holes [14]. Not only is the necessary bulk energy–momentum tensor fairly simple, but also the structure of the five-dimensional spacetime closely resembles that of the factorized spacetime of the black-string solution—indeed, the five-dimensional curvature invariant quantities for the ansatz (3) have the form

\[
R = -20A'^2 - 8A'', \quad R_{MN}R^{MN} = 4(20A'^4 + 16A'^2A'' + 5A'^2), \quad \text{(4)}
\]

\[
R_{MNRS}R^{MNRS} = 8 \left( 5A'^4 + 4A'^2A'' + 2A'^2 + \frac{6 e^{-3A} m^2 (v, y)}{r^6} \right), \quad \text{(5)}
\]

and are formally identical to those for the metric (1) with no extra terms appearing due to the assumed \( y \)-dependence, a behaviour not observed for any other choice of non-factorized line-elements. On the other hand, the assumed scaling of the mass function with \( y \) can in principle eliminate the last singular term of equation (5) and restore the finiteness of the five-dimensional spacetime at a moderate distance from the brane—indeed, any function decreasing faster than the square of the warp factor could achieve the localization of the black-hole singularity.

The components of the Einstein tensor \( G^M_N \) for the line-element (3) are found to be

\[
G^v_v = G^r_r = G^\theta_\theta = G^\phi_\phi = 6A'^2 + 3A'', \quad \text{(6)}
\]

\[
G^v_v = \frac{2}{r^2} e^{-2A} \partial_v m - \frac{1}{r} (\partial_r m + 4A' \partial_v m), \quad \text{(7)}
\]

\[
G^v_v = e^{2A} G^v_v = \frac{1}{r^2} \partial_v m, \quad \text{(8)}
\]

\[
G^v_v = G^r_r = G^v_v = 0, \quad \text{(9)}
\]

\[
G^v_v = 6A'^2. \quad \text{(10)}
\]

Einstein’s field equations in the bulk will follow by equating the above components of \( G^M_N \) with the corresponding ones of the energy–momentum tensor \( T^M_N \). The latter will be determined
once the bulk Lagrangian is defined, in the next section. However, the form of the above Einstein tensor components allows us to make some basic observations. The assumed \( y \)-dependence of the mass function introduces off-diagonal, non-isotropic pressure components. The dependence on \( v \) does not by itself introduce a new pressure component but contributes to one of the non-isotropic ones. In [14], the assumption was made that the warp factor has the form of the RS model, \( A(y) = -k|y| \), which is supported by the bulk cosmological constant. In that case, the Einstein equations corresponding to the diagonal components (6) and (10) are trivially satisfied, and no energy density or diagonal pressure components are necessary in the bulk. Here, however, we will assume that the warp factor has a general form \( A(y) \) in order to allow for less restricted field configurations that, in general, generate both diagonal and off-diagonal components. Since \( G^0_0 = G^\theta_\theta = G^\phi_\phi \), the bulk energy–momentum tensor will satisfy, by construction, a stiff equation of state.

In the following sections, we will study a variety of field theory models in an attempt to find the one that could support the aforementioned line-element (3). It is already known [13] that the desired Vaidya-type metric cannot be supported by conventional forms of matter (realized by either scalar or gauge fields). Motivated by previous considerations of non-ordinary scalar field theories, which aimed to produce additional pressure components necessary for the stabilization of brane-world models [43–45], we will focus our attention on scalar fields and consider a variety of models. These will include one or more scalar fields minimally coupled to gravity but with a general Lagrangian, admitting non-canonical kinetic terms, derivative interactions, mixing terms or the presence of ghosts, as well as a scalar field non-minimally coupled to gravity with a general coupling to the Ricci scalar.

Once a consistent solution in the bulk is found, a single brane will then be introduced in the model that in general contains a localized energy–momentum tensor \( S_{\mu\nu} \). The spacetime will be assumed to be invariant under the mirror transformation \( y \rightarrow -y \). The bulk equations will then be supplemented by the junction conditions [46]

\[
[K_{\mu\nu} - h_{\mu\nu} K] = -\kappa_5^2 S_{\mu\nu},
\]

relating the extrinsic curvature \( K_{\mu\nu} \), the induced metric tensor \( h_{\mu\nu} \), and the energy–momentum tensor \( S_{\mu\nu} \) on the brane—the brackets denote the discontinuity across the brane. The discontinuity of the lhs of the above equation will be a function of the warp factor \( A(y) \), the mass function \( m(v, y) \) and their derivatives with respect to \( y \). With the help of the bulk solution, if existent, the above equation will give us the necessary matter content of the brane for its consistent embedding in the five-dimensional warped spacetime.

3. A field theory with minimally coupled scalars

In this section, we focus on the case of models with minimally coupled scalar fields with a general form of Lagrangian. The action functional of the gravitational theory therefore reads

\[
S = \int d^4x dy \sqrt{-g} \left( \frac{R}{2\kappa_5^2} - \mathcal{L}_{\text{sc}} - \mathcal{L}_m \right),
\]

where \( g_{\mu\nu} \) and \( R \) are the metric tensor and Ricci scalar, respectively, of the five-dimensional spacetime described by (3), and \( \kappa_5^2 = 8\pi G_5 \) is the five-dimensional gravitational constant. The action contains in addition the general Lagrangian \( \mathcal{L}_{\text{sc}} \), associated with one or more scalar fields, and \( \mathcal{L}_m \) stands for any other form of matter or energy in the theory—throughout this work, we will assume that this term describes the distribution of a uniform, negative energy density and thus \( \mathcal{L}_m = -\Lambda_5 \), where \( \Lambda_5 \) the bulk cosmological constant. The field equations resulting from the aforementioned action have the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_5^2 (T_{\mu\nu} - g_{\mu\nu} \Lambda_5),
\]
with $T_{MN}$ being the energy–momentum tensor associated with the scalar fields
\[ T_{MN} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_{sc})}{\delta g^{MN}}. \] (14)
In the following subsections, we consider particular choices for $L_{sc}$, and we examine the existence of a viable solution of the field equations in the bulk.

### 3.1. A single scalar field with a non-canonical kinetic term

As a first step, we consider the following theory of a single scalar field with a non-canonical kinetic term:
\[ L_{sc} = \sum_{n=1} f_n(\phi) (\partial^M \phi \partial_M \phi)^n + V(\phi), \] (15)
where $f_n(\phi)$ are arbitrary, smooth functions of the scalar field $\phi$. The components of the corresponding energy–momentum tensor follow from the expression
\[ T^A_B = 2 \sum_{n=1} n f_n(\phi) (\partial^M \phi \partial_M \phi)^n - \delta^A_B L_{sc}. \] (16)
The off-diagonal components $T^v_r$, $T^y_r$, and $T^v_y$ of the energy–momentum tensor must trivially vanish since the corresponding components of the Einstein tensor, equation (9), do the same. These conditions however impose strict constraints on the form of the scalar field: the vanishing of the $T^v_r$ component, for instance,
\[ T^v_r = 2 \sum_{n=1} n f_n(\phi) (\partial^M \phi \partial_M \phi)^n - 1 (\partial_r \phi)^2 e^{-2A} \] (17)
demands that the scalar field be not a function of the radial coordinate, $\partial_r \phi = 0$. But then it is not possible to satisfy the remaining Einstein’s equations: assuming\(^2\) that $\phi = \phi(v, y)$, the expression of the non-vanishing off-diagonal component $T^v_r$, when combined with the corresponding component of the Einstein tensor (8), leads to the equation
\[ \frac{\partial_m}{\kappa^2} = 2 \sum_{n=1} n f_n(\phi) (\partial_v \phi)^{2n-1} \partial_v \phi. \] (18)
An incompatibility problem arises immediately: the field $\phi$ and, therefore, the right-hand side of the above equation is independent of $r$, but the left-hand side has an explicit dependence on that coordinate. As a result, the case of a single, minimally coupled scalar field, even with a general non-canonical kinetic term, does not lead to a solution.

### 3.2. Two interacting scalar fields

We are thus forced to consider a multi-field model. We will study first the case of two scalar fields $\phi$ and $\chi$ whose dynamics and interactions are described by the Lagrangian
\[ L_{sc} = f^{(1)}(\phi, \chi) \partial^M \phi \partial_M \phi + f^{(2)}(\phi, \chi) \partial^M \chi \partial_M \chi + V(\phi, \chi), \] (19)
where $f^{(1,2)}$ are arbitrary smooth functions of the two fields. The energy–momentum tensor now reads
\[ T^A_B = 2 f^{(1)}(\phi, \chi) \partial^A \phi \partial_B \phi + 2f^{(2)}(\phi, \chi) \partial^A \chi \partial_B \chi - \delta^A_B L_{sc}. \] (20)
\(^2\) Throughout this work, and in order to preserve the spherical symmetry of any potential solution, we assume that the scalar fields do not depend on the angular coordinates $\theta$ and $\phi$.\)
The vanishing of the off-diagonal components $G^r_v, G^r_r,$ and $G^r_v,$ implies again the vanishing of the corresponding components of the energy–momentum tensor, which now results in the following two constraints on the fields:

\[ f^{(1)}(\phi, \chi)(\partial_v \phi)^2 + f^{(2)}(\phi, \chi)(\partial_r \chi)^2 = 0, \quad (21) \]

\[ f^{(1)}(\phi, \chi)\partial_v \phi \partial_r \phi + f^{(2)}(\phi, \chi)\partial_r \chi \partial_v \chi = 0. \quad (22) \]

From the constraint (21), it is clear that if one of the fields were not dependent on $r,$ neither would the other one. Although in this case both constraints would be trivially satisfied, the $(\nu \nu)$-component of the Einstein field equations along the brane. By using expression (20) and the field configurations is imposed, namely

\[ G^r_r = - \kappa^2 m^2 \left[ f^{(1)}(\phi, \chi)(\partial_v \phi)^2 + f^{(2)}(\phi, \chi)(\partial_r \chi)^2 \right], \]

would again present an inconsistency, the rhs being necessarily $r$-independent and the lhs a function of $r.$ Similarly, the constraint (22) implies that if one of the fields were not dependent on $\chi,$ neither would the other one. However, we note from equation (23), that, in order for a solution to exist, the fields

\[ T^v_v = T^r_r = 2e^{-2A}(f^{(1)}(\phi, \chi)\partial_v \phi \partial_r \phi + f^{(2)}(\phi, \chi)\partial_r \chi \partial_v \chi) - L_{\text{sc}}, \quad (24) \]

\[ T^v_\theta = T^\theta_v = -L_{\text{sc}}. \quad (25) \]

The components of the Einstein tensor along the brane, given in equation (6), satisfy the relation $G^v_v = G^r_r = G^\theta_\theta$, therefore the corresponding components of $T^M_N$ should also be equal. Comparing (24) and (25), it is obvious that this holds if an additional constraint on the field configurations is imposed, namely

\[ f^{(1)}(\phi, \chi)\partial_v \phi \partial_r \phi + f^{(2)}(\phi, \chi)\partial_r \chi \partial_v \chi = 0. \quad (26) \]

From the above constraint, we may again conclude that if one of the fields were not dependent on $v,$ neither would the other one. However, we note from equation (23), that, in order for a solution with a non-trivial profile of the mass distribution $m = m(\chi)$ to exist, the fields must necessarily depend on $v.$ In other words, if such a solution exists, the matter distribution around a black hole must be dynamical and not static, even if the mass of the black hole itself is not time-evolving and thus independent of $v.$

Coming back to the existence of the solution and assuming that $\phi = \phi(v, r, \chi)\chi = \chi(v, r, \chi),$ we proceed as follows: we solve the new constraint (26) for the coupling function $f^{(2)}(\phi, \chi),$ and then substitute it into the $(\nu v)$-component (23) to obtain the following alternative form for that equation:

\[ \partial_v m = 2\kappa^2 f^{(1)}(\phi, \chi)\partial_v \phi \partial_r \chi - \partial_r \phi \partial_v \chi. \quad (27) \]

However, a similar rearrangement of equation (21) and substitution into the constraint (22) leads to

\[ \partial_v \phi \partial_r \chi - \partial_r \phi \partial_v \chi = 0, \quad (28) \]

which unfortunately causes the rhs of equation (27) to be zero and thus the mass function loses the desired $\nu$-dependence. We note that the absence of the solution holds independently of the signs of the coupling functions $f^{(1,2)}(\phi, \chi),$ i.e. of whether the two scalars are positive-norm fields or whether they are ghosts—or of the form of the potential $V(\phi, \chi)$ that determines the interaction between the two fields.

\[ T^\nu_v = g^{\nu v} T^v_v, \]

and as a result there are only two independent constraints.
Two interacting scalar fields with general kinetic terms

We now combine the two previous models considered to construct a Lagrangian of two scalar fields interacting through an arbitrary potential \( V(\phi, \chi) \) and admitting general non-canonical kinetic terms. The Lagrangian of the scalar fields then reads

\[
\mathcal{L}_\text{sc} = \sum_{n=1}^{\infty} f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^n + \sum_{n=1}^{\infty} f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^n + V(\phi, \chi),
\]

while the energy momentum tensor assumes the form

\[
T^A_B = 2 \sum_{n=1}^{\infty} f_n^{(1)}(\phi, \chi) n (\partial^M \phi \partial M \phi)^{n-1} \partial^A \phi \partial B \phi
+ 2 \sum_{n=1}^{\infty} f_n^{(2)}(\phi, \chi) n (\partial^M \chi \partial M \chi)^{n-1} \partial^A \chi \partial B \chi - \delta^A_B \mathcal{L}_\text{sc}.
\]

Working, as in the previous subsection, from the vanishing of the off-diagonal components \( G^v_r, G^r_v \) and \( G^r_r \), we derive the following two constraints on the fields:

\[
\sum_{n=1}^{\infty} n [f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^{n-1} (\partial \phi)^2 + f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^{n-1} (\partial \chi)^2] = 0,
\]

\[
\sum_{n=1}^{\infty} n [f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^{n-1} \partial \phi \partial_\phi \phi + f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^{n-1} \partial \chi \partial_\chi \chi] = 0.
\]

Also, the equality of the diagonal components of the Einstein tensor along the brane results into the additional constraint

\[
\sum_{n=1}^{\infty} n [f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^{n-1} \partial_\phi \partial \phi \phi + f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^{n-1} \partial_\chi \partial \chi \chi] = 0,
\]

while the \((v v)\)-component of the Einstein field equations now has the form

\[
\frac{\partial m}{r^2} = 2\kappa^2 \sum_{n=1}^{\infty} n [f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^{n-1} \partial_\phi \partial \phi \phi + f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^{n-1} \partial_\chi \partial \chi \chi].
\]

The following observation makes the attempt to find a viable solution in the context of this model obsolete: if we define the following functions:

\[
\tilde{f}^{(1)}(\phi, \chi) = \sum_{n=1}^{\infty} n f_n^{(1)}(\phi, \chi) (\partial^M \phi \partial M \phi)^{n-1},
\]

\[
\tilde{f}^{(2)}(\phi, \chi) = \sum_{n=1}^{\infty} n f_n^{(2)}(\phi, \chi) (\partial^M \chi \partial M \chi)^{n-1},
\]

then equations (31), (32), (33) and (34) reduce to equations (21), (22), (26) and (23), respectively, with the \( f^{(1,2)}(\phi, \chi) \) coupling functions being replaced by \( \tilde{f}^{(1,2)}(\phi, \chi) \). As a result, upon a similar rearrangement of the three constraints, the rhs of the \((v v)\)-component vanishes, a result that eliminates again the \( y \)-dependence of the mass function.

Two interacting scalar fields with mixed kinetic terms

We now increase the complexity of the model by allowing the scalar fields to have mixed kinetic terms and thus consider the following generalized form of the scalar Lagrangian:

\[
\mathcal{L}_\text{sc} = f^{(1)}(\phi, \chi) \partial^M \phi \partial M \phi + f^{(2)}(\phi, \chi) \partial^M \chi \partial M \chi + f^{(3)}(\phi, \chi) \partial^M \phi \partial M \chi + V(\phi, \chi).
\]
Then, the energy–momentum tensor reads
\[
T^A_B = 2f^{(1)}(\phi, \chi) \partial^A \phi \partial_B \phi + 2f^{(2)}(\phi, \chi) \partial^A \chi \partial_B \chi \\
+ f^{(3)}(\phi, \chi)[\partial^A \phi \partial_B \chi + \partial^A \chi \partial_B \phi] - \delta^A_B \mathcal{L}_{\text{ec}}.
\]  
(38)

The vanishing of the off-diagonal components \(G^r_r\), \(G^\gamma_\gamma\) and \(G^r_\gamma\) imposes again the vanishing of the corresponding components of the energy–momentum tensor, which, in this case, results in the following two constraints:
\[
f^{(1)}(\phi, \chi) (\partial_r \phi)^2 + 2f^{(2)}(\phi, \chi) (\partial_r \chi)^2 + f^{(3)}(\phi, \chi) \partial_r \phi \partial_r \chi = 0,
\]
(39)
\[
2f^{(1)}(\phi, \chi) \partial_r \phi \partial_r \phi + 2f^{(2)}(\phi, \chi) \partial_r \chi \partial_r \chi + f^{(3)}(\phi, \chi)[\partial_r \phi \partial_r \chi + \partial_r \phi \partial_r \chi] = 0.
\]
(40)

From the first of the above two equations, it is clear that either both fields must simultaneously depend on the radial coordinate \(r\) or they both must be \(r\)-independent. If they both are independent of \(r\), then the two constraints are satisfied, but the non-vanishing off-diagonal \((\gamma \gamma)\)-component, which now takes the form
\[
\frac{\partial_m}{\partial^2} = \kappa^2 [2f^{(1)}(\partial_r \phi \partial_r \phi + 2f^{(2)}(\partial_r \chi \partial_r \chi + f^{(3)})(\partial_r \phi \partial_r \chi + \partial_r \phi \partial_r \chi)],
\]
(41)
becomes inconsistent due to the explicit \(r\)-dependence on its lhs. Equation (41) seems to allow for certain combinations of the partial derivatives \(\partial_r \phi\), \(\partial_r X\), \(\partial_r \phi\), \(\partial_r X\) to vanish. However, in what follows, we will assume the most general case, i.e. that \(\phi = \phi(r, v, y)\) and \(X = X(r, v, y)\), and we will comment on particular cases at the end of this subsection.

We now turn to the diagonal components of the Einstein field equations. The diagonal components of the Einstein tensor along the brane are equal, and thus the same must hold for the components of the energy–momentum tensor that now have the form
\[
T^r_r = T^\gamma_\gamma = e^{-2A}[2f^{(1)}(\partial_r \phi \partial_r \phi + 2f^{(2)}(\partial_r \chi \partial_r \chi + f^{(3)})(\partial_r \phi \partial_r \chi + \partial_r \phi \partial_r \chi)] - \mathcal{L}_{\text{ec}}.
\]
(42)

Demanding the equality of the above expressions, the following additional constraint is obtained:
\[
2f^{(1)}(\phi, \chi) \partial_r \phi \partial_r \phi + 2f^{(2)}(\phi, \chi) \partial_r \chi \partial_r \chi + f^{(3)}(\phi, \chi)[\partial_r \phi \partial_r \chi + \partial_r \phi \partial_r \chi] = 0.
\]
(44)

Let us now consider the system of constraints (39), (40) and (44): it is a homogeneous system of linear equations for \(f^{(1)}(\phi, \chi)\), \(f^{(2)}(\phi, \chi)\) and \(f^{(3)}(\phi, \chi)\)—the necessary condition for this system to possess a solution other than the trivial one is the vanishing of the determinant of the matrix of coefficients:
\[
\begin{vmatrix}
(\partial_r \phi)^2 & (\partial_r \chi)^2 & \partial_r \phi \partial_r \chi \\
2\partial_r \phi \partial_r \phi & 2\partial_r \chi \partial_r \chi & \partial_r \phi \partial_r \chi \\
2\partial_r \phi \partial_r \phi & 2\partial_r \chi \partial_r \chi & \partial_r \phi \partial_r \chi
\end{vmatrix} = 0.
\]
(45)

One may easily check that the above condition indeed holds and therefore the system may be solved to yield the values of two coupling functions in terms of the third one. In this way, we find
\[
f^{(1)} = f^{(2)} \frac{(\partial_r \chi)^2}{(\partial_r \phi)^2}, \quad f^{(3)} = -2f^{(2)} \frac{\partial_r \chi}{\partial_r \phi}.
\]
(46)

If we then use the above relations in the expression of the \((\gamma \gamma)\)-component (41), we obtain the alternative form
\[
\frac{\partial_m}{\partial^2} = \frac{2\kappa^2 f^{(2)}}{(\partial_r \phi)^2} (\partial_r \phi \partial_r \chi - \partial_r \phi \partial_r \chi)(\partial_r \phi \partial_r \chi - \partial_r \phi \partial_r \chi).
\]
(47)
We observe that, contrary to what happens in the previous two models considered, the rearrangement of the three constraints (39), (40) and (44) in this model does not by itself cause the vanishing of the rhs of the above equation. Clearly, as the Lagrangian of the model becomes more complex, the system of field equations becomes more flexible.

The remaining independent off-diagonal component that we have not considered yet follows by combining the $G^r_v$ component (7) of the Einstein tensor with the corresponding component of the energy–momentum tensor. Then, we obtain the field equation

$$\frac{2\partial_v m}{r^2} - \frac{e^{2\Lambda}}{r}(\partial^2 r + 4A' \partial_r m) = 2\kappa^2 [ f^{(1)}(\partial_r \phi)^2 + f^{(2)}(\partial_r \chi)^2 + f^{(3)} \partial_v \phi \partial_v \chi] .$$

(48)

Similarly, if we use relations (46) in the above equation, this may be rewritten as

$$\frac{2\partial_v m}{r^2} - \frac{e^{2\Lambda}}{r}(\partial^2 r + 4A' \partial_r m) = 2\kappa^2 \frac{f^{(2)}}{(\partial \phi)^2} (\partial_v \phi \partial_v \chi - \partial_v \phi \partial_v \chi)^2 .$$

(49)

Finally, the last diagonal component, the $(yy)$-component, assumes the form

$$6A^2 = \kappa^2 [-\Lambda_B + 2 f^{(1)}(\partial_r \phi)^2 + 2 f^{(2)}(\partial_r \chi)^2 + 2 f^{(3)} \partial_v \phi \partial_v \chi - \mathcal{L}_{sc}] .$$

(50)

At this point, we will need the explicit expression of the Lagrangian $\mathcal{L}_{sc}$. By making use of the constraints (39) and (44), this turns out to be

$$\mathcal{L}_{sc} = f^{(1)}(\partial_r \phi)^2 + f^{(2)}(\partial_r \chi)^2 + f^{(3)} \partial_v \phi \partial_v \chi + V(\phi, \chi) .$$

(51)

If we use the above expression, then equation (50) and the diagonal components of the field equations along the brane reduce to the following two independent differential equations:

$$6A^2 = \kappa^2 [-\Lambda_B + f^{(1)}(\partial_r \phi)^2 + f^{(2)}(\partial_r \chi)^2 + f^{(3)} \partial_v \phi \partial_v \chi - V(\phi, \chi)] .$$

(52)

$$6A^2 + 3A'' = \kappa^2 [-\Lambda_B - f^{(1)}(\partial_r \phi)^2 - f^{(2)}(\partial_r \chi)^2 - f^{(3)} \partial_v \phi \partial_v \chi - V(\phi, \chi)] ,$$

(53)

respectively. Subtracting the first of the above equations from the second, the latter takes the simpler form

$$3A'' = -2\kappa^2 [ f^{(1)}(\partial_r \phi)^2 + f^{(2)}(\partial_r \chi)^2 + f^{(3)} \partial_v \phi \partial_v \chi] - \frac{2\kappa^2 f^{(2)}}{(\partial \phi)^2} (\partial_v \phi \partial_v \chi - \partial_v \phi \partial_v \chi)^2 ,$$

(54)

where, in the last part, we have again used relations (46). If we now take the square of equation (47) and combine it with equations (49) and (54), we obtain a differential equation for the mass function with no dependence on the fields and their coupling functions, namely

$$\frac{(\partial m)^2}{r^2} = 3A'' \left[ - \frac{2\partial_v m}{r} + e^{2\Lambda}(\partial^2 r + 4A' \partial_r m) \right] .$$

(55)

Unfortunately, this equation is again inconsistent as it involves an explicit dependence on the radial coordinate on which the mass function is assumed not to be dependent.

In addition to the above, this model is plagued by another problem following from the restrictions that the field equations impose on the field configurations: as the warp factor is solely a function of the $y$-coordinate, then through equations (54) and (52), the potential $V(\phi, \chi)$ should also be a function of $y$. Assuming that the potential depends on both fields, and that these are general functions of the $(r, v, y)$ coordinates, $V(\phi, \chi)$ ought to have a particular form, so that its dependence on the $(r, v)$ coordinates vanishes. These forms could be

- $V(\phi, \chi) = F(\chi^n + \phi^m)$, where $n$ is an arbitrary integer and $F$ is an arbitrary function of the combination $\chi^n + \phi^m$. For the latter to be a function of $y$, we should also have
\( \chi^\alpha = \chi_1(y) + \chi_2(r, v) \) and \( \phi^a = \phi_1(y) + \phi_2(r, v) \), with \( \phi_2(r, v) = -\chi_2(r, v) \). But then, one may easily show that
\[
\partial_\alpha \partial_\beta \phi - \partial_\alpha \partial_\beta \chi = \frac{(\chi f)^{\alpha}}{n^2} - (\partial_\alpha \chi \partial_\beta \chi - \partial_\beta \chi \partial_\alpha \chi) = 0, \tag{56}
\]
in which the rhs of the \((yv)\)-component (47) and the assumed dependence of the mass function on \(y\) vanish.

* \( V(\phi, \chi) = G(\chi^n \phi^{n_1}) \), where \( G \) is an arbitrary function of the combination \( \chi^n \phi^{n_1} \) and \((n_1, n_2)\) are arbitrary integers. In this case, we should have \( \chi = \chi_1(y) \chi_2(r, v) \) and \( \phi = \phi_1(y) \phi_2(r, v) \), with \( \phi_2(r, v) = c \chi_2(r, v)^{-n_1/n_2} \) and \( c \) a constant. Once again, the combination \((\partial_\alpha \chi \partial_\beta \phi - \partial_\alpha \phi \partial_\beta \chi)\) is easily found to be zero.

Let us finally investigate whether more specific assumptions on the form of the fields or the potential are allowed. Clearly, the case where the potential \( V \) depends only on one of the two fields, i.e. \( V = V(\chi) \), is excluded: \( \chi \) must necessarily depend on \( r \), as discussed below equations (39)–(40), and the presence of \( \phi \) in the expression of the potential is imperative in order for this \( r \)-dependence to cancel. The same argument excludes the case where only one of the two fields depends on the time-coordinate \( v \), as in that case \( V(\chi, \phi) \) would carry this \( v \)-dependence. The case where none of the two fields depends on \( v \) is also rejected, since then the rhs of equation (47) would be zero—the same holds if we assume that both fields are not functions of the extra coordinate. The assumption that only one of the two fields may depend on \( y \) is the only one allowed with equations (47), (52) and (54) assuming then simpler, yet non-trivial forms—nevertheless, this assumption does not alter the arguments presented above regarding the form of the potential and thus fails to lead to a viable solution.

The analysis presented in this subsection may be easily generalized to allow for more general kinetic terms along the lines of subsection 3.3. Then, the Lagrangian would read
\[
\mathcal{L}_{\infty} = \sum_{n=1} \left[ f_n^{(1)}(\phi, \chi)(\partial^M \phi \partial_M \phi)^n + f_n^{(2)}(\phi, \chi)(\partial^M \chi \partial_M \chi)^n \right.
\]
\[
+ \left. f_n^{(3)}(\phi, \chi)(\partial^M \partial_M \chi)^n \right] + V(\phi, \chi). \tag{57}
\]

Although the expressions of all constraints and non-vanishing field equations would become complicated, one may again show that these, upon conveniently redefining the coupling functions, reduce to the ones presented in this subsection. As the same arguments regarding the restrictions on the potential and form of fields would still hold, no viable solution would emerge in the context of this model either.

4. A non-minimally coupled scalar field theory

Let us now turn to the case of a scalar–tensor theory of gravity with a non-minimally coupled scalar field present in the bulk. We consider the following general form of the action:
\[
S = \int d^4x \, dy \sqrt{-g} \left[ \frac{f(\Phi)}{2\kappa^2} R - \frac{1}{2} (\nabla \Phi)^2 - V(\Phi) - \Lambda_B \right], \tag{58}
\]
where \( f(\Phi) \) is an arbitrary, smooth, positive-definite function of the scalar field \( \Phi \), and \( g_{MN} \) is the five-dimensional metric given again by equation (3). The equations of motion resulting from the aforementioned action have the form
\[
f(\Phi) \left( R_{MN} - \frac{1}{2} g_{MN} R \right) = \kappa^2 \left( -g_{MN} \Lambda_B + \tau^{(\Phi)}_{MN} \right), \tag{59}
\]
with \( \tau^{(\Phi)}_{MN} \) being the generalized energy–momentum tensor of the scalar field defined as
\[
\tau^{(\Phi)}_{MN} = \nabla_M \Phi \nabla_N \Phi - g_{MN} \left[ \frac{1}{2} (\nabla \Phi)^2 + V(\Phi) \right] + \frac{1}{\kappa^2} \left[ \nabla_M \nabla_N f(\Phi) - g_{MN} \nabla^2 f(\Phi) \right]. \tag{60}
\]
In order to derive the explicit form of the above field equations, we need to combine the non-vanishing components of the energy–momentum tensor with those of the Einstein tensor $G_{MN}$ presented in equations (6)–(10). First, the off-diagonal components $T^r_v$, $T^v_r$, $T^v_v$ lead, respectively, to the following four equations:

\begin{align}
(1 + f'')\partial_r\Phi\partial_r\Phi + f'\partial_\theta\partial_\Phi - A' f' \partial_\Phi &= 0, \\
(1 + f'') & (\partial_r \Phi)^2 + f' \partial_\Phi^2 = 0, \\
(1 + f'') \partial_\Phi^2 \partial_\Phi + f' \partial_\theta\partial_\Phi - A' f' \partial_\Phi - \frac{\partial_\Phi m}{r} f' \partial_\Phi &= f \frac{\partial_\Phi m}{r}, \\
(1 + f'') & (\partial_\Phi^2 + f' \partial_\Phi^2 - \frac{m}{r^2} f'_r \partial_\Phi = f \frac{\partial_\Phi m}{r} + \frac{e^{2A}}{r} \frac{\partial_\Phi m}{r} \left( 1 - \frac{2m}{r} \right) \left[ (1 + f'') \partial_\Phi \partial_r \Phi + f' \partial_\theta \partial_\Phi \right] = f \frac{2}{r^2} \partial_\Phi m - \frac{e^{2A}}{r} \left( \partial_\Phi^2 + 4A' \partial_\Phi \right). 
\end{align}

In the above, $f'$ and $f''$, respectively, denote the first and second derivatives of the coupling function $f$ with respect to $\Phi$, and, for simplicity, $\kappa_5^2$ has been set to unity. Also, note that the off-diagonal components of the energy–momentum tensor $T^r_v$ and $T^v_r$ are not independent and their corresponding equations reduce again to equations (61) and (63).

Furthermore, the diagonal components provide us with three additional equations:

\begin{align}
e^{-2A} \left[(1 + f'') \partial_\Phi \partial_\Phi + f' \partial_\theta \partial_\Phi + \frac{m}{r^2} f'_r \partial_\Phi \right] + A' f' \partial_\Phi &= (\mathcal{L}_\Phi + \Box f + \Lambda_B) = 3 f (2A^2 + A'), \\
e^{-2A} \frac{r}{r'} f' \left[ \partial_\Phi + \left( 1 - \frac{2m}{r} \right) \partial_\Phi \right] + A' f' \partial_\Phi &= (\mathcal{L}_\Phi + \Box f + \Lambda_B) = 3 f (2A^2 + A''), \\
(1 + f'') & (\partial_\Phi)^2 + f' \partial_\Phi^2 - (\mathcal{L}_\Phi + \Box f + \Lambda_B) = 6 f A^2. 
\end{align}

The above equations contain the complicated expressions of $\mathcal{L}_\Phi$ and $\Box f$, which are given by

\begin{align}
\mathcal{L}_\Phi &= \frac{1}{2} (\nabla \Phi)^2 + V (\Phi) = \frac{e^{-2A}}{2} \left[ 2 \partial_\Phi \partial_\Phi \Phi + \left( 1 - \frac{2m}{r} \right) (\partial_\Phi \Phi)^2 \right] + \frac{1}{2} (\partial_\Phi \Phi)^2 + V (\Phi) 
\end{align}

and

\begin{align}
\Box f &= e^{-2A} \partial_\Phi f + \frac{e^{-2A}}{r^2 \partial_\Phi} \left[ r^2 \partial_\Phi f + r^2 \left( 1 - \frac{2m}{r} \right) \partial_\Phi f \right] + e^{-4A} \partial_\Phi (e^{4A} \partial_\Phi f), 
\end{align}

respectively, and are thus cumbersome to use. However, the combination of equations (65) and (66) results in a simpler and more useful condition, namely

\begin{align}
(1 + f'') \partial_\Phi \partial_\Phi \partial_\Phi + f' \partial_\theta \partial_\Phi \partial_\Phi = \frac{f'}{r} \left[ \partial_\Phi \Phi + \left( 1 - \frac{3m}{r} \right) \partial_\Phi \Phi \right]. 
\end{align}

In the above analysis, we have once again assumed that the scalar field $\Phi$ and consequently the coupling function $f$ do not depend on the angular coordinates $\theta$ and $\phi$ in order to preserve the spherical symmetry of the solutions on the brane. We have nevertheless retained their dependence on all remaining coordinates $(r, v, y)$. It is easy to see that any simpler ansatz fails to pass the field equations: if we assume that the scalar field $\Phi$ depends only on the bulk coordinate $y$, then equation (63) leads to the result $\partial_\Phi m = 0$—the same equation is inconsistent due to its explicit $r$-dependence in the case where $\Phi$ is assumed to be only a function of the time-coordinate $v$; finally, if the field depends only on the radial coordinate $r$,
then equation (61) demands that \( f' = 0 \)—but this takes us back to the minimal-coupling case that has already been excluded [14].

The above arguments clearly indicate that the scalar field \( \Phi \) must depend at least on a pair of coordinates. In fact, even the assumption that it depends on only two coordinates is inconsistent with the field equations, since

- if \( \Phi = \Phi(v, y) \) and thus \( \partial_y \Phi = 0 \), equation (70) leads to either \( \partial_v \Phi = 0 \) (excluded above) or \( f' = 0 \)—but the latter option again reduces equation (63) to an inconsistent equation;
- if \( \Phi = \Phi(v, r) \) and thus \( \partial_r \Phi = 0 \), equation (61) demands, for \( \partial_r \Phi \neq 0, f' = 0 \)—then, equation (70) leads to \( \partial_r \Phi = 0 \) which is in contradiction with our assumption.
- if \( \Phi = \Phi(r, y) \) and thus \( \partial_y \Phi = 0 \), equation (70) demands, for \( \partial_r \Phi \neq 0, f' = 0 \)—then, equation (61) leads to \( \partial_y \Phi = 0 \) which is again in contradiction with our assumption.

Therefore, we conclude that any attempted simplification in the form of the scalar field does not conform with the field equations, and this, interestingly enough, holds regardless of the form of the coupling function \( f(\Phi) \). We are thus led to consider whether the only remaining possibility \( \Phi(r, v, y) \), in conjunction with an appropriate choice of \( f(\Phi) \), could support the existence of a solution with a mass function \( m = m(v, y) \) that would perhaps localize a black hole together with its singularity close to the brane. Therefore, in what follows we consider a number of natural choices for the coupling function \( f(\Phi) \) and investigate whether these can lead to any viable solutions.

4.1. The \( f(\Phi) = a \Phi \) case

Postulating that \( f(\Phi) = a \Phi \), with \( a \) being a constant, gives \( f'(\Phi) = a \) and \( f''(\Phi) = 0 \), which significantly simplifies the field equations. Looking for a solution for \( \Phi(v, r, y) \), we immediately see that a purely factorized form, e.g., \( \Phi(v, r, y) = U(v)R(r)Y(y) \), or any other form in which at least one of the coordinates is factorized out, are excluded as they fail to satisfy the field equations.

As a matter of fact, for the particular choice of the coupling function \( f \), equation (62) can be analytically integrated to determine the form of \( \Phi \). For \( f(\Phi) = a \Phi \), it takes the form

\[
\frac{\partial^2 \Phi}{(\partial \Phi)^2} = -\frac{1}{a},
\]

(71)

and, upon integrating twice, it yields the general solution

\[
\Phi(v, r, y) = a \ln[r + aB(v, y)] + C(v, y),
\]

(72)

where \( B(v, y) \) and \( C(v, y) \) are arbitrary functions. However, the above solution again fails to satisfy condition (70): this takes the form \( a \partial_v B + B \partial_y C + 1 - 3m/r = 0 \) that cannot be satisfied due to the explicit dependence on \( r \). This result therefore excludes the particular choice for the coupling function.

4.2. The \( f(\Phi) = a \Phi^2 \) case

Also in this case, upon substituting \( f'(\Phi) = 2a \Phi \) and \( f''(\Phi) = 2a \), where \( a \) is again a constant, equation (62) takes the form

\[
-\frac{(1 + 2a)}{2a} \partial_y \Phi = \frac{\partial^2 \Phi}{\partial r \Phi}.
\]

(73)

This can be analytically integrated twice to yield the general solution for \( \Phi \), namely

\[
\Phi(v, r, y) = [B(v, y)r + C(v, y)]^{2a/(1+4a)},
\]

(74)
where again \(B(v, y)\) and \(C(v, y)\) are arbitrary functions. Interestingly enough, the above form of the scalar field together with the assumption \(f(\Phi) = a \Phi^2\) manages to satisfy all off-diagonal equations (61)–(64), with the latter providing constraints that determine the unknown functions \(B(v, y)\) and \(C(v, y)\) in terms of the warp factor \(A(y)\) and the mass function \(m(v, y)\). However, the diagonal equations (65)–(67) are more difficult to satisfy with the constraint (70) proving the particular configuration of \(f\) and \(\Phi\) once again inconsistent by taking the form \(\partial_v C + B(1 - 3m/r) = 0\) and thus demanding the trivial result \(B(v, y) = 0\).

4.3. The \(f(\Phi) = a \Phi^n\) case

In this case, we have \(f'(\Phi) = an \Phi^{n-1}\) and \(f'(\Phi) = an(n - 1) \Phi^{n-2}\), and equation (62) takes the form

\[
-\frac{1}{an} \left[ \Phi^{1-n} + \frac{an(n - 1)}{\Phi} \right] \partial_\tau \Phi = \frac{\partial^2 \Phi}{\partial y^2}.
\]

Integrating the above, we obtain

\[
\partial_\tau \Phi(v, r, y) = b(v, y)\Phi^{1-n} \exp \left[ \frac{\Phi^{2-n}}{an(n - 2)} \right],
\]

where \(b(v, y)\) is an arbitrary function. Unfortunately, the solution of the above first-order differential equation for \(n \geq 3\) cannot be written in a closed form. However, the following integral form

\[
\int d\Phi \Phi^{n-1} \exp \left[ -\frac{\Phi^{2-n}}{an(n - 2)} \right] = b(v, y)r + c(v, y),
\]

where \(c(v, y)\) is another arbitrary function, will prove to be more than adequate for our purpose. Although an explicit form for the scalar field \(\Phi\) cannot be found, differentiating both sides of the above equation with respect to \(v\) yields

\[
\partial_\tau \Phi(v, r, y) = \Phi^{1-n} \exp \left[ \frac{\Phi^{2-n}}{an(n - 2)} \right] \left[ \partial_v b(v, y)r + \partial_v c(v, y) \right].
\]

Differentiating also equation (76) with respect to \(v\) yields \(\partial_\tau \partial_v \Phi\) and upon substitution of the relevant quantities in equation (70), we obtain once again the, condemning for our ansatz, constraint \(\partial_v c + b(1 - 3m/r) = 0\).

It is worth noting that the case where the coupling function \(f(\Phi)\) is a linear combination of different powers of \(\Phi\), i.e. \(f(\Phi) = \sum_{k=0}^n a_k \Phi^k\), was also considered\(^4\). For \(n = 1\) and \(n = 2\), the analyses followed closely the ones for the cases with \(f(\Phi) = a\Phi\) and \(f(\Phi) = a\Phi^2\), respectively, leaving no space for a viable solution. For \(n = 3\), equation (62) could again be integrated once to yield the result

\[
\partial_\tau \Phi(v, r, y) = \frac{b(v, y)}{a_1 + 2a_2 \Phi + 3a_3 \Phi^2} \exp \left[ -\frac{1}{\lambda} \arctan \left( \frac{a_2 + 3a_3 \Phi}{\lambda} \right) \right],
\]

where \(\lambda = \sqrt{3a_1a_3 - a_2^2}\). Integrating once more, we again obtain an integral equation. Following a similar analysis as above, we arrive again, from equation (70), at the constraint \(\partial_v c + b(1 - 3m/r) = 0\) and the trivial result \(b(v, y) = 0\). For \(n \geq 4\), our set of equations do not give a closed form even for \(\partial_\tau \Phi\).

\(^4\) This particular choice for the coupling of a bulk scalar field to the Ricci scalar was considered in [47] in the context of a brane-world cosmological solution that could produce accelerated expansion on the brane at late times.
4.4. The \( f(\Phi) = e^{k\Phi} \) case

We finally consider the case of an exponential coupling function for which \( f'(\Phi) = k e^{k\Phi} \) and \( f''(\Phi) = k^2 e^{k\Phi} \), where \( k \) is a constant—note than an arbitrary constant multiplying the exponential function can be absorbed into the value of \( \Phi \) and thus is set to unity. Then, equation (62) takes the form

\[
-\frac{1}{k} (e^{-k\Phi} + k^2) \partial_r \Phi = \frac{\partial^2 \Phi}{\partial \Phi^2},
\]

with the solution

\[
\partial_r \Phi(v, r, y) = b(v, y) e^{-k\Phi} \exp \left[ \frac{-e^{-k\Phi}}{k^2} \right].
\]

Integrating once more, we obtain

\[
\int d\Phi e^{k\Phi} \exp \left[ -\frac{e^{-k\Phi}}{k^2} \right] = b(v, y) r + c(v, y).
\]

Deriving, from equations (81) and (82), the expressions for \( \partial_v \partial_r \Phi \) and \( \partial_v \Phi \), respectively, and substituting them together with \( \partial_r \Phi \) into equation (70), we again obtain the constraint

\[
\partial_v c + b(1 - 3m/r) = 0
\]

that clearly excludes the exponential ansatz as well.

4.5. A general no-go argument

The failure of finding a viable solution, after a variety of forms for the coupling function \( f(\Phi) \) have been considered, seems to hint that perhaps a theory of a non-minimally coupled scalar field is altogether inconsistent with the realization of the additional bulk matter necessary to support a spacetime described by the line element (3). In that case, one should be able to develop a general argument that would exclude the emergence of a solution independent of the form of the coupling function \( f(\Phi) \).

To this end, we bring equation (62) to the form

\[
1 + f''(\Phi) = -f'(\Phi) \frac{\partial^2 \Phi}{(\partial \Phi)^2},
\]

which we can replace into equation (61) to obtain

\[
A' = \partial_r \left( \frac{\partial_r \Phi}{\partial \Phi} \right).
\]

The above differential equation can be integrated with respect to \( r \) to give

\[
\partial_r \Phi = \partial_r [A'(y) r + F(v, y)].
\]

Similarly, equation (70) can be brought to the following form:

\[
\left( \partial_r - \frac{1}{r} \right) \partial_r \Phi = \frac{1}{r} \left( 1 - \frac{3m}{r} \right),
\]

which upon integration with respect to \( r \) yields

\[
\partial_r \Phi = \partial_r \left[ -1 + \frac{3m}{2r} + D(v, y) r \right].
\]

The functions \( F(v, y) \) and \( D(v, y) \) appearing in equations (85) and (87) are, at the moment, completely arbitrary. It can, however, be easily checked that there exists a relation between them. To establish this relation, we proceed as follows. First, we differentiate equation (85) with respect to \( v \) and equation (87) with respect to \( y \) to obtain

\[
\partial_v \partial_y \Phi = \partial_v F(v, y) \partial_y \Phi + (A' r + F) \partial_v \partial_y \Phi,
\]
\[ \partial_y \partial_v \Phi = r \frac{\partial_r D(v, y)}{\Phi} \partial_r + \left( -1 + \frac{3m}{2r} + D(v, y)r \right) \partial_y \partial_v \Phi. \] (89)

Equating the right-hand sides of the above two equations, we arrive at the relation
\[ \partial_v F(v, y) \partial_r \frac{\partial_r D(v, y)}{\Phi} + \left( A' + \frac{\partial_r m}{2m} + \frac{3m}{2r} + D(v, y)r \right) \partial_y \partial_r \frac{\partial_r D(v, y)}{\Phi}. \] (90)

Taking finally the derivatives of equations (85) and (87) with respect to \( r \), these yield the expressions of the double derivatives \( \partial_y \partial_v \Phi \) and \( \partial_y \partial_r \Phi \) that appear above. Substituting and simplifying leads to the final constraint
\[ -3m \frac{\partial_r D(v, y)}{2r^2} F(v, y) - \frac{3m}{r} \left( A' + \frac{\partial_r m}{2m} \right) + \partial_v F(v, y) + A' + F(v, y) D(v, y) - r \partial_r D(v, y) = 0. \] (91)

However, the above is catastrophic for the existence of the desired solution. The only way the above relation can hold is if the coefficients of all powers of \( r \) identically vanish. This leads to the result that \( F(v, y) = 0 \), which subsequently demands that \( A'(y) = 0 \) which is clearly in contradiction with our assumption as it eliminates the warp factor from the model. In addition, the desired dependence of the mass term on the extra coordinate \( y \) is also forced to vanish, once we assume that \( A'(y) = 0 \), which destroys the localization of the black-hole singularity.

Although of secondary importance, let us finally note that even if the function \( A(y) \) were not forced to be trivial, the constraint following from the second term of equation (91) would lead to the result \( m(v, y) \sim e^{-2A(v)} \)—thus, for a decreasing warp factor, the mass term would have to increase away from the brane, thus invalidating the idea of the localization of black hole. Therefore, a viable field-theory model should not only support a non-trivial profile of the mass function of the black hole but also a profile that could localize the black hole close to the brane.

5. Discussion and conclusions

Despite an intensive research activity over a period of almost 15 years, a closed-form analytical solution that would describe a five-dimensional regular black hole localized on a brane is still missing. Although numerical solutions that reassure us of their existence have appeared in the literature, the way to proceed in order to derive a complete analytical solution remains unclear. As almost all of those numerical solutions rely on the presence of some type of matter, either on the brane or in the bulk, in this work, we turned to a previous idea, introduced by one of the authors and collaborators, that a type of bulk matter can help to localize the extended black-string singularity close to the brane and thus restore the finiteness of the five-dimensional AdS spacetime at a small distance from the brane.

However, the metric ansatz that would describe a five-dimensional spacetime of this form had to be carefully constructed. The black-string spacetime was associated with a factorized metric ansatz; therefore, the localization of the extended singularity would be realized only through a non-factorized ansatz, in which the four-dimensional part would exhibit dependence on the fifth coordinate. Previous attempts [13, 14] had shown that such line-elements characterized by the presence of a horizon in their four-dimensional part led to spacetimes with additional singularities apart from the extended black-string one. A modified Vaidya-type four-dimensional line-element was finally chosen and embedded in a five-dimensional warped spacetime. Being analytic in four dimensions, this metric ansatz was free from any additional singularities. Moreover, its mass being a function of both the fifth and the time-coordinate, provided a reasonable ansatz for a perturbed Schwarzschild background.
on the brane, ideal for investigating both the localization of the black-hole singularity and the existence of a static solution.

Once the gravitational part of our model was decided, we turned to the determination of the field theory model that would support such a spacetime. Previous attempts to find such a model based on ordinary theories of scalar or gauge fields had led to a negative result [13]. Therefore, in this work, we decided to study instead a variety of field theories that could be characterized as non-ordinary—for simplicity, we focused on the case of scalar field theories. In section 3, we examined the case of a field theory with one or more scalar fields minimally coupled to gravity but otherwise described by a general Lagrangian. The cases studied included a single scalar field with a non-canonical kinetic term and two interacting scalar fields with either canonical, non-canonical or mixed kinetic terms. Our analysis allowed for general forms of potentials as well as the case where one or both of the scalar fields were ghosts. In section 4, we turned to the field theory of a single scalar field non-minimally coupled to gravity and studied the cases where its coupling function was a power law of the field, a polynomial, an exponential function or of a completely arbitrary form.

In order to avoid any unreasonable restrictions on the field configurations, we allowed the warp factor to assume a $y$-dependent, but otherwise arbitrary form. We also imposed no fine-tunings between bulk and brane parameters. A viable bulk solution, if emerged, would be subsequently used, to determine, through the junction conditions, the brane content. Nevertheless, our analysis never reached that point: all the field theory models studied, no matter how general, were shown not to be able to support the assumed gravitational background. Considering only the set of gravitational equations in the bulk, we were able to demonstrate that in each and every case, the scalar field-theory model chosen was not compatible with the basic assumptions for the metric ansatz necessary for the localization of the black-string singularity.

Our analysis has, nevertheless, confirmed that such a localization demands the synergetic action of both the bulk and the brane part of spacetime. The chosen metric ansatz introduces in the bulk, apart from an energy density and an isotropic diagonal pressure that satisfy a stiff equation of state, additional off-diagonal, non-isotropic pressure components. The dependence of the mass function on both the fifth- and the time-coordinate contributes to these. It becomes therefore clear that gravitational degrees of freedom tend to leak from the brane—similar to the black-hole singularity—and, although the models considered in this work have failed to localize them, a mechanism must exist that will achieve this. Another important point that has emerged from our analysis is the necessity of the time dependence of the field configurations in all the models we studied—even when the mass parameter is assumed to be time-independent; according to our findings, a static black-hole configuration may indeed exist; however, the associated field configuration itself has to be dynamic.

In the previous related work [13], configurations involving also gauge fields were studied; the arguments however that excluded the existence of a viable solution were identical to the ones used for the case of scalar models. Although, in the present work, we have restricted our study in scalar field-theory models, we anticipate that similar results would follow even in the case of non-ordinary gauge field-theory models—we have postponed this study for a future work. Finally, one should note that all of the above observations are independent of the sign of the parameter $\epsilon$ that appears in our metric ansatz and, thus, hold not only for the creation of a brane-world black hole but also for any expanding distribution of matter in a brane-world set-up.

Our analysis is by no means exhaustive. Nevertheless, in our attempt to generate the bulk energy–momentum tensor necessary for the localization of the black-hole topology close to the brane, we have considered a general selection of non-ordinary scalar field-theory models with a high degree of flexibility and reached a negative result in each case. We have also considered
a particular non-factorized metric ansatz—no matter how well motivated this choice was, we cannot exclude the possibility that the five-dimensional line-element assumes a different form that may perhaps be related to the Schwarzschild black-hole metric on the brane in a more subtle way (see, for example, the construction of brane-localized black holes in the lower dimensional case [21, 22] based on the use of a C-metric—there, a Schwarzschild-like metric for the geometry on the brane was derived; however, it was not a vacuum solution). Our results demonstrate how difficult, if at all possible, the construction of a localized five-dimensional black hole may be in the context of a well-defined field-theory model.

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