A Reverse Hardy–Hilbert’s Inequality Containing Multiple Parameters and One Partial Sum

Bicheng Yang, Shanhe Wu and Xingshou Huang

Abstract: In this work, by introducing multiple parameters and utilizing the Euler–Maclaurin summation formula and Abel’s partial summation formula, we first establish a reverse Hardy–Hilbert’s inequality containing one partial sum as the terms of double series. Then, based on the newly proposed inequality, we characterize the equivalent conditions of the best possible constant factor associated with several parameters. At the end of the paper, we illustrate that more new inequalities can be generated from the special cases of the reverse Hardy–Hilbert’s inequality.

Keywords: reverse Hardy–Hilbert’s inequality; partial sum; multiple parameters; best possible constant factor

MSC: 26D15; 26D20

1. Introduction

Let \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, \ a_m \geq 0, b_n \geq 0, \ 0 < \sum_{m=1}^{\infty} a_m^p < \infty \) and \( 0 < \sum_{n=1}^{\infty} b_n^q < \infty. \)

Then,

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left( \sum_{n=1}^{\infty} b_n^q \right)^{1/q},
\]

(1)

where \( \frac{\pi}{\sin(\pi/p)} \) is the best possible constant factor. Inequality (1) is known in the literature as Hardy–Hilbert’s inequality (see [1]).

By introducing parameters \( \lambda_i \in (0,2) \) \( (i=1,2), \lambda_1 + \lambda_2 = \lambda \in (0,4] \), Krnić and Pečarić [2] provided a generalization of Hardy–Hilbert’s inequality (1) as follows:
where the constant \( B(\lambda, \lambda) \) given by the beta function is the best possible one.

By introducing more parameters, Yang, Wu and Chen [3] established a further generalization of Hardy–Hilbert’s inequality (1) as follows:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_m}{(m^\alpha + n^\beta)^{1/p}} < \frac{1}{\lambda} \left( k_{\lambda_1}(\lambda_2) \right)^{1/\lambda} \left( k_{\lambda_2}(\lambda_1) \right)^{1/\lambda} \left( \sum_{m=1}^{\infty} m^{p(1-\lambda_2)-1} a_m^p \right)^{1/\lambda} \left( \sum_{n=1}^{\infty} n^{q(1-\lambda_1)-1} b_n^q \right)^{1/\lambda},
\]

(3)

where \( \frac{1}{p} + \frac{1}{q} = 1, p > 1, \alpha, \beta \in (0,1], \lambda \in (0, \lambda), \lambda_1 \in (0, \frac{1}{\lambda}], \lambda_2 \in (0, \frac{1}{\lambda}) \cap (0, \lambda), \lambda_1 + \lambda_2 = \lambda \). Inequality (4) is the other kind of (2) involving two partial sums inside the two terms of series.

Recently, Liao, Wu and Huang [5] considered a variation of inequality (3); one partial sum \( B_n = \sum_{k=1}^{n} b_k \) was embedded inside the terms of series, i.e.,

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_m}{(m^\alpha + n^\beta)^{1/p}} < \lambda \left( \frac{1}{\lambda} k_{\lambda_1}(\lambda_2 + 1) \right)^{1/\lambda} \left( \frac{1}{\lambda} k_{\lambda_2}(\lambda_1) \right)^{1/\lambda} \left( \sum_{m=1}^{\infty} m^{p(1-\lambda_2)-1} A_m^p \right)^{1/\lambda} \left( \sum_{n=1}^{\infty} n^{q(1-\lambda_1)-1} B_n^q \right)^{1/\lambda},
\]

(4)

where \( \frac{1}{p} + \frac{1}{q} = 1, p > 1, \alpha, \beta \in (0,1], \lambda \in (0, \lambda), \lambda_1 \in (0, \frac{1}{\lambda}], \lambda_2 \in (0, \frac{1}{\lambda}) \cap (0, \lambda + 1), \lambda_1 + \lambda_2 = \lambda \). Inequality (4) is the other kind of (2) involving two partial sums inside the two terms of series.

Yang, Wu and Huang [6] established a reverse Hardy–Hilbert’s inequality with one partial sum \( B_n = \sum_{k=1}^{n} b_k \) as the term of the double series, as follows:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n b_m}{(m^\alpha + n^\beta)^{1/p}} > \frac{1}{\lambda} \left( \frac{1}{\lambda} k_{\lambda_1}(\lambda_2) \right)^{1/\lambda} \left( \frac{1}{\lambda} k_{\lambda_2}(\lambda_1) \right)^{1/\lambda} \left( \sum_{m=1}^{\infty} m^{p(1-\lambda_2)-1} A_m^p \right)^{1/\lambda} \left( \sum_{n=1}^{\infty} n^{q(1-\lambda_1)-1} B_n^q \right)^{1/\lambda},
\]

(5)

As a further study of the development methods of Hardy–Hilbert-type inequalities, some unconventional methods are adopted. For example, a half-discrete Hilbert-type inequality with the multiple upper limit function and the partial sums was provided by [7]. A reverse Hardy–Hilbert-type integral inequality involving one derivative function was published by [8]. Inequalities (4)–(6) and the work of [7,8] are meaningful extensions of (2) based on the Euler–Maclaurin summation formula, Abel’s partial summation formula.
and the techniques of real analysis. Some applications of Hardy–Hilbert-type inequalities in the real analysis and operator theory can be found in the monograph [9]. In [10], Hong gave an equivalent condition between the best possible constant factor and the parameters in the extension of (4). Some other similar results are provided by [11–13].

Inspired by the work of [4–10], in this paper, we construct a reverse Hardy–Hilbert’s inequality which contains one partial sum and some extra parameters inside the weight coefficients, the reverse Hardy–Hilbert’s inequality has different structural forms by comparing with existing results mentioned above. Our method is mainly based on some skillful applications of the Euler–Maclaurin summation formula and Abel’s partial summation formula. By means of the newly proposed inequality, we then discuss the equivalent conditions of the best possible constant factor associated with several parameters. As applications, we deal with some equivalent forms of the obtained inequality and illustrate how to derive more reverse inequalities of Hardy–Hilbert type from the current results.

2. Preliminaries

For convenience, let us first state the following conditions (C1) that would be used repeatedly in subsequent section:

\[(C1)\] \[ p < 0, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 3], \eta_i \in [0, \frac{1}{2}], \lambda_i = (0, \frac{1}{2}] \cap (0, \lambda) (i = 1, 2),
\]
\[\eta_1 + \eta_2 = \eta, \lambda_1 := \frac{\lambda - \lambda_i}{p} + \frac{\lambda_i}{q}, \lambda_2 := \frac{\lambda - \lambda_i}{q} + \frac{\lambda_i}{p}, \quad a_m, b_n \geq 0 (m, n \in \mathbb{N} = \{1, 2, \ldots\}),\]
\[0 < \sum_{m=1}^{\infty} (m - \eta_i)^{p(1 - \lambda_i - 1)} a_m^p < \infty, \quad 0 < \sum_{n=1}^{\infty} (n - \eta_i)^{q(1 - \lambda_i - 1)} b_n^q < \infty, \quad A_m = \sum_{j=1}^{m} a_j \quad \text{with} \quad A_m = o(e^{(m - \eta_i)}) \quad (t > 0).\]

Lemma 1. (cf. [9], (2.2.3)) (i) If \((-1)^i \frac{d^i}{dt^i} g(t) > 0, \quad t \in [m, \infty) \quad (m \in \mathbb{N}) \quad \text{with} \quad g^{(i)}(\infty) = 0 \quad (i = 0, 1, 2, 3), \quad P_i(t), B_i \quad (i \in \mathbb{N}) \quad \text{are Bernoulli functions and Bernoulli numbers of i-order, then}
\[\int_{m}^{\infty} P_2q^{-1}(t)g(t)dt = -\varepsilon_q \frac{B_q}{2q} g(m) \quad (0 < \varepsilon_q < 1; q = 1, 2, \ldots).\]

In particular, for \( q = 1, B_1 = \frac{1}{2} \), we have:
\[-\frac{1}{12} g(m) < \int_{m}^{\infty} P_1(t) g(t) dt < 0; \quad (7)\]
for \( q = 2, B_2 = -\frac{1}{12} \), it follows that:
\[0 < \int_{m}^{\infty} P_3(t) g(t) dt < \frac{1}{120} g(m). \quad (8)\]

(ii) (cf. [9], (2.3.2)) If \( f(t)(> 0) \in C^1[m, \infty), \lim_{t \to \infty} f^{(i)}(t) = f^{(i)}(\infty) = 0 \quad (i = 0, 1, 2, 3), \)
then we have the following Euler–Maclaurin summation formula:
\[\sum_{k=m}^{\infty} f(k) = \int_{m}^{\infty} f(t)dt + \frac{1}{2} f(m) + \int_{m}^{\infty} P_1(t) f'(t)dt, \quad (9)\]
\[\int_{m}^{\infty} P_1(t) f''(t)dt = -\frac{1}{12} f''(m) + \frac{1}{6} \int_{m}^{\infty} P_3(t) f'''(t)dt. \quad (10)\]

Lemma 2. Suppose that \( s \in (0, 3], \ s_2 \in (0, \frac{1}{2}] \cap (0, s), \ k_1(s_2) := B(s_2, s - s_2), \) we define the following weight coefficient:
\[ \varpi_s(s_2, m) := (m - \eta_1)^{s_2 - s} \sum_{n=1}^{\infty} \frac{(s - \eta_2)^{s_2}}{(m + n - \eta)^s} \quad (m \in \mathbb{N}). \]  

(11)

Then, we have the following inequalities:

\[ 0 < k_s(s_2)(1 - O_s(\frac{1}{(m - \eta)^2})) < \varpi_s(s_2, m) < k_s(s_2) \quad (m \in \mathbb{N}) \]  

(12)

where \( O_s(\frac{1}{(m - \eta)^2}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1 - \eta_2}{m - \eta_1}} u^2 \, du \), which satisfies

\[ 0 < O_s(\frac{1}{(m - \eta)^2}) \leq \frac{1}{2k_s(s_2)} \left( \frac{1 - \eta_2}{m - \eta_1} \right)^2. \]

**Proof.** For fixed \( m \in \mathbb{N} \), we set the following real function:

\[ g(m, t) := \frac{(t - \eta_2)^{s_2}}{(m - \eta)^2} \quad (t > \eta_2). \]  

In the following, we divide two cases of \( s_2 \in (0,1) \cap (0, s) \) and \( s_2 \in [\frac{1}{2}, \frac{3}{2}] \cap (0, s) \) to prove (12).

(i) For \( s_2 \in (0,1) \cap (0, s) \), since \((-1)^i g^{(i)}(m, t) > 0 \quad (t > \eta_2; i = 0,1,2)\), by using Hermite–Hadamard’s inequality (cf. [10]) and setting \( u = \frac{t - \eta_2}{m - \eta_1} \), we find:

\[ \varpi_s(s_2, m) = (m - \eta_1)^{s_2 - s} \sum_{n=1}^{\infty} g(m, n) < (m - \eta_1)^{s_2 - s} \int_{s_2}^{\infty} g(m, t) \, dt \]

\[ = (m - \eta_1)^{s_2 - s} \left[ \int_{s_2}^{\infty} \frac{u^{s_2 - 1}}{(m - \eta_1 + t - \eta)^s} \, dt \right] \]

\[ = \int_{s_2}^{\infty} \frac{u^{s_2 - 1}}{(m - \eta_1 + u)^s} \, du = B(s_2, s - s_2) = k_s(s_2). \]

On the other hand, in view of the decreasing property of the series, setting \( u = \frac{t - \eta_2}{m - \eta_1} \), we obtain:

\[ \varpi_s(s_2, m) = (m - \eta_1)^{s_2 - s} \sum_{n=1}^{\infty} g(m, n) > (m - \eta_1)^{s_2 - s} \int_{s_2}^{\infty} g(m, t) \, dt \]

\[ = \left[ \int_{s_2}^{\infty} \frac{u^{s_2 - 1}}{(m - \eta_1 + u)^s} \, du \right] = B(s_2, s - s_2) = \frac{1 - \eta_2}{m - \eta_1} \int_{s_2}^{\infty} \frac{u^{s_2 - 1}}{(1 + u)^s} \, du \]

\[ = k_s(s_2)(1 - O_s(\frac{1}{(m - \eta)^2})) > 0, \]

where \( O_s(\frac{1}{(m - \eta)^2}) = \frac{1}{k_s(s_2)} \int_{s_2}^{\infty} \frac{u^{s_2 - 1}}{(1 + u)^s} \, du > 0 \), which satisfies the following inequality:

\[ 0 < \int_0^{\infty} \frac{u^{s_2 - 1}}{(1 + u)^s} \, du < \int_0^{\infty} u^{s_2 - 1} \, du = \frac{1}{s_2} \left( \frac{1 - \eta_2}{m - \eta_1} \right)^{s_2} \quad (m \in \mathbb{N}). \]

Hence, we obtain (12).

(ii) For \( s_2 \in [\frac{1}{2}, \frac{3}{2}] \cap (0, s) \), by (9), we have:

\[ \sum_{n=1}^{\infty} g(m, n) = \int_{s_2}^{\infty} g(m, t) \, dt + \frac{1}{2} g(m, 1) + \int_0^{\eta_2} P(t) g(m, t) \, dt \]

\[ = \int_{s_2}^{\infty} g(m, t) \, dt - h(m), \]

where \( h(m) \) is indicated as
\[ h(m) := \int_{\eta_2}^{t} g(m,t) dt - \frac{1}{2} g(m,1) - \int_{\eta_2}^{\eta_2} P(t) g'(m,t) dt. \]

It is easy to observe that \(-\frac{1}{2} g(m,1) = \frac{\left(\frac{t}{m+q}\right)^{\eta_2-2}}{2(m+q)^{\eta_2-1}}.\) Furthermore, integrating by parts, it follows that \[
\int_{\eta_2}^{t} g(m,t) dt = \int_{\eta_2}^{\left(\frac{t}{m+q}\right)^{\eta_2-1}} d\left(\frac{t}{m+q}\right)^{\eta_2} = \frac{1}{\eta_2} \int_{\eta_2}^{(\frac{t}{m+q})^{\eta_2}} d(\frac{t}{m+q})^{\eta_2} + \frac{s}{\eta_2 (s+1)} \int_{\eta_2}^{(\frac{t}{m+q})^{\eta_2}} \left( t - \eta_2 \right)^{(s+1)} dt
\]

We find that:

\[-g'(m,t) = \frac{(s+1)(t - \eta_2)^{(s-2)}}{(m+q)^t} + \frac{s(t - \eta_2)^{(s-2)}}{(m+q)^t} = \frac{(s+1)(t - \eta_2)^{(s-2)}}{(m+q)^t} + \frac{s(t - \eta_2)^{(s-2)}}{(m+q)^t} = \frac{(s+1)(t - \eta_2)^{(s-2)}}{(m+q)^t} - \frac{s(t - \eta_2)^{(s-2)}}{(m+q)^t},\]

and for \(s_2 \in [1, \frac{1}{2}] \cap (0, s),\) we deduce that

\[ (-1)^i \frac{d^i}{dt^i} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} > \frac{s_2+1}{m+q} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} > 0 \quad (t > \eta_2; i = 0, 1, 2, 3).\]

By utilizing (8)–(10), for \(a := 1 - \eta_2 \in [\frac{3}{4}, 1],\) we obtain:

\[ (s+1-s_2) \int_{\eta_2}^{t} P(t) \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} dt > -\frac{(s+1)\eta_2}{12(m+q)^t} \alpha^{t-2}, \]

\[ -(m-\eta_2) s_2 \int_{\eta_2}^{t} P(t) \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} dt > \frac{(m-\eta_2) s_2}{12(m+q)^t} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} \alpha^{t-2}.\]

and then we have:

\[ h(m) > \frac{\alpha^{t-2}}{(m+q)^t} h_1 + \frac{\alpha^{t-2}}{(m+q)^t} h_2 + \frac{\alpha^{t-2}}{(m+q)^t} h_3, \]

where \(h_i (i = 1, 2, 3)\) are formulated as

\[ h_1 := \frac{s_2}{\eta_2} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} - \frac{(2s_2 - (6a + 1)s_2 + 12a^2)}{720}, \quad h_2 := \frac{s_2}{\eta_2 (s+1)} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} - \frac{2(s+1)(2s_2 - (6a + 1)s_2 + 12a^2)}{720}, \quad h_3 := \frac{s_2}{\eta_2 (s+2)(s+3)} \frac{(t - \eta_2)^{(s_2-2)}}{(m+q)^t} - \frac{2(s+1)(3s_2 - (6a + 1)s_2 + 12a^2)}{720}.\]

Moreover, for \(s \in (0, 3], s_2 \in [1, \frac{1}{2}] \cap (0, s), a \in [\frac{3}{4}, 1],\) we find

\[ h_1 > \frac{s_2}{12s_2} [s_2^2 - (6a + 1)s_2 + 12a^2] - \frac{1}{120}, \]

In view of \(\frac{s_2}{20} [s_2^2 - (6a + 1)s_2 + 12a^2] = 6(4a - s_2) \geq 6(4 \cdot \frac{1}{2} - \frac{1}{2}) > 0,\) and

\[ \frac{\alpha^{t-2}}{(m+q)^t} [s_2^2 - (6a + 1)s_2 + 12a^2] = 2s_2 - (6a + 1) \leq 2 \cdot \frac{1}{2} - (6 \cdot \frac{1}{4} + 1) < 0, \] we obtain:
\[ h_1 \geq \frac{(3/4)^2}{2(3/4)^2} \left( \frac{3}{4} \right)^2 - \left( 6 \cdot \frac{3}{4} + 1 \right) \frac{1}{12} + 12 \left( \frac{3}{4} \right)^2 - \frac{1}{120} = \frac{1}{120} > 0, \]

\[ h_2 > a^2 \left( \frac{4a^2}{15} - \frac{1}{120} \right) - \frac{1}{90} > \left( \frac{4}{5} \right)^2 \left[ \frac{4}{5} \right] - 1 = \frac{3}{80} - \frac{1}{90} > 0, \]

\[ h_3 > \frac{8a^4}{105} - \frac{5}{720} \geq \frac{8}{105} \left( \frac{3}{4} \right)^2 - \frac{1}{90} = \frac{27}{1210} - \frac{1}{144} > 0, \]

and hence we have \( h(m) > 0. \)

On the other hand, we have:

\[
\sum_{n=1}^{\infty} g(m,n) = \int_0^\infty g(m,t)dt + \frac{1}{2} g(m,1) + \int_0^\infty P_1(t)g'(m,t)dt
\]

\[
= \int_0^\infty g(m,t)dt + H(m),
\]

where \( H(m) \) is indicated as \( H(m):= \frac{1}{2} g(m,1) + \int_0^\infty P_1(t)g'(m,t)dt. \)

We have already obtained that \( \frac{1}{2} g(m,1) = \frac{a^{n+2}}{2(a-n+1)^3} \) and

\[ g'(m,t) = -\int_0^s \int_0^{(m-\eta)t} \int_0^{(m-\eta)t} \left( \frac{s}{s+n} \right)^{\frac{1}{2}} \left( \frac{m-\eta}{m-\eta+n} \right)^{\frac{1}{2}} dt > 0, \]

\[ (m-\eta) \int_0^\infty P_1(t) \left( \frac{(m-\eta)t}{(m-\eta+n)t} \right)^{\frac{1}{2}} dt > \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} = \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} \]

\[ \geq \left( \frac{1}{2} \right) \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} = \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} > 0. \]

Then, we have:

\[ H(m) > \frac{a^{n+2}}{2(a-n+1)^3} - \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} \]

\[ \geq \left( \frac{1}{2} \right) \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} = \left( \frac{1}{2} \right) \frac{a^{\frac{n+2}{2}}}{12(a-n+1)^3} > 0. \]

Therefore, we derive the inequalities:

\[ \int_0^\infty g(m,t)dt < \sum_{n=1}^{\infty} g(m,n) < \int_0^\infty g(m,t)dt. \]

By virtue of the results of the case (i), we obtain (12). The proof of Lemma 2 is complete. □

**Lemma 3.** Under the assumption (C1), we have the following reverse Hardy–Hilbert’s inequality:

\[ I_0 := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_{mn}}{(m+n-n)} > (k_{\lambda}(\lambda_2))^{\frac{1}{2}} (k_{\lambda}(\lambda_3))^{\frac{1}{2}} \]

\[ \times \left[ \sum_{m=1}^{\infty} (m-\eta) \left( \frac{1}{m-n} \right)^{\frac{1}{2}} \right]^{\frac{1}{4}} \left[ \sum_{n=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{(n-n)^2} \right) \right) (n-\eta_2) \left( \frac{1}{n^2} \right)^{\frac{1}{4}} \right]^{\frac{1}{4}}. \]

**Proof.** By symmetry, for \( s_1, \in (0, \frac{1}{2}) \cap (0, s) \), \( k_\lambda(s_1) = B(s_1, s-s_1) \), we can obtain the following inequalities for the next weight coefficient:
$$0 < k_i(s_i)(1 - O_i\left(\frac{1}{(a_{-\eta_2})^m}\right))$$

(14)

$$< \omega_i(s_i, n) := (n - \eta_2)^{s_i-\frac{1}{2}} \sum_{m=1}^{\infty} \frac{(m-\eta_2)^{\frac{1}{2}-1}}{(m+n-\eta)^{\frac{1}{2}}} < k_i(s_i) \ (n \in \mathbb{N}),$$

where

$$O_i\left(\frac{1}{(a_{-\eta_2})^m}\right) := \frac{1}{\omega_i(s_i, n)} \int_{0}^{1} \frac{u^{n+1}}{(1+u)} \, du \ (> 0).$$

By applying the reverse Hölder’s inequality (cf. [14]), we obtain:

$$I_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\frac{1}{2}}} \left[ (m-\eta_2)^{1-\frac{1}{2}} a_m \right] \left[ (n-\eta_2)^{1-\frac{1}{2}} b_n \right]$$

$$\geq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{\frac{1}{2}}} \left[ (m-\eta_2)^{1-\frac{1}{2}} a_m \right] \left[ (n-\eta_2)^{1-\frac{1}{2}} b_n \right]$$

$$= \sum_{m=1}^{\infty} \sigma_\lambda(\lambda_2, m)(m-\eta_2)^{\rho(1-\lambda)-1} a_m \left[ \sum_{n=1}^{\infty} \omega_i(\lambda_i, n)(n-\eta_2)^{\rho(1-\lambda)-1} b_n \right].$$

Now, by using (12) and (14) for $s = \lambda, s_i = \lambda_i \ (i = 1, 2)$, we obtain (13). Lemma 3 is proved. □

**Lemma 4.** If $t > 0,$ then we have the following inequality:

$$\sum_{m=1}^{\infty} e^{-t(m-\eta_2)} a_m \leq t \sum_{m=1}^{\infty} e^{-t(m-\eta_2)} A_m.$$  

(15)

**Proof.** In view of $A_m e^{-t(m-\eta_2)} = o(1) \ (m \to \infty),$ using Abel’s summation by parts formula, we find:

$$\sum_{m=1}^{\infty} e^{-t(m-\eta_2)} a_m = \lim_{m \to \infty} A_m e^{-t(m-\eta_2)} + \sum_{m=1}^{\infty} A_m [e^{-t(m-\eta_2)} - e^{-t(m-\eta_2+1)}]$$

$$= \sum_{m=1}^{\infty} A_m [e^{-t(m-\eta_2)} - e^{-t(m-\eta_2+1)}] = (1 - e^{-t}) \sum_{m=1}^{\infty} e^{-t(m-\eta_2)} A_m.$$  

Since $1 - e^{-t} < t \ (t > 0),$ we acquire inequality (15). This completes the proof of Lemma 4. □

3. Main Results

**Theorem 1.** Under the assumption (C1), we have the following reverse Hardy–Hilbert’s inequality:

$$I := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\omega_i(s_i)}{(m+n)^{\frac{1}{2}}} > \frac{1}{2} \left( k_\lambda(\lambda_2) \right)^2 \left( k_\lambda(\lambda_1) \right)^2$$

$$\times \left[ \sum_{m=1}^{\infty} (m-\eta_2)^{\rho(1-\lambda)-1} a_m \right] \left[ \sum_{n=1}^{\infty} (1 - O_i\left(\frac{1}{(a_{-\eta_2})^m}\right))(n-\eta_2)^{\rho(1-\lambda)-1} b_n \right].$$

(166)
where \( O_{\lambda}(\frac{1}{(x-\eta)^{\lambda}}) := \frac{1}{\Gamma(\lambda)} \int_{0}^{\infty} \frac{u^{\lambda-1}}{(1+u)^{\lambda}} \, du. \) In particular, for \( \lambda_1 + \lambda_2 = \lambda \), we have

\[
0 < \sum_{m=1}^{\infty} (m-\eta)^{\rho(1-\lambda)} a_m^p < \infty, \quad 0 < \sum_{n=1}^{\infty} (1-O_{\lambda}(\frac{1}{(x-\eta)^{\lambda}}))(n-\eta)^{\rho(1-\lambda)} b_n^q < \infty,
\]

and the following reverse inequality:

\[
I = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda+1}} > \frac{1}{B(\lambda_1, \lambda_2)}
\]

\[
\times \left[ \sum_{m=1}^{\infty} (m-\eta)^{\rho(1-\lambda)} a_m^p \right] \left[ \sum_{n=1}^{\infty} (1-O_{\lambda}(\frac{1}{(x-\eta)^{\lambda}}))(n-\eta)^{\rho(1-\lambda)} b_n^q \right]^\frac{1}{p}.
\]  

(17)

**Proof.** In view of the formula \( \frac{1}{(m+n-\eta)^{\lambda+1}} = \frac{1}{\Gamma(\lambda+1)} \int_{0}^{\infty} t^{(\lambda+1)-1} e^{-(m+n-\eta)t} \, dt \), by using (15), it follows that:

\[
I = \frac{1}{\Gamma(\lambda+1)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \int_{0}^{\infty} t^{(\lambda+1)-1} e^{-(m+n-\eta)t} \, dt
\]

\[
= \frac{1}{\Gamma(\lambda+1)} \int_{0}^{\infty} t^{\lambda-1} \left[ \sum_{m=1}^{\infty} e^{-(m-\eta)t} a_m \right] \sum_{n=1}^{\infty} e^{-(n-\eta)t} b_n \, dt
\]

\[
\geq \frac{1}{\Gamma(\lambda+1)} \int_{0}^{\infty} t^{\lambda-1} \sum_{m=1}^{\infty} e^{-(m-\eta)t} a_m \sum_{n=1}^{\infty} e^{-(n-\eta)t} b_n \, dt
\]

\[
= \frac{1}{\Gamma(\lambda+1)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \int_{0}^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} \, dt
\]

\[
= \frac{1}{\Gamma(\lambda+1)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda+1}} = \frac{I_0}{\lambda},
\]

Furthermore, by means of (13), we obtain (16). The proof of Theorem 1 is complete. \( \square \)

**Remark 1.** For \( s = \lambda + 1 \in (1,3), \ S_2 = \tilde{\lambda}_2 \in (0, \frac{1}{s}] \cap (0, \lambda + 1) \) from (11) and (12), we have \( \lambda \in (0, 2] \), and the following inequality:

\[
\sigma_{\lambda+1}(\tilde{\lambda}_2, m) = (m-\eta)^{\lambda+1} \sum_{n=1}^{\infty} \frac{(m-n)^{\lambda+1}}{(m+n-\eta)^{\lambda+1}} < k_{\lambda+1}(\tilde{\lambda}_2) \quad (m \in N).
\]  

(18)

**Theorem 2.** If \( \lambda_1 + \lambda_2 = \lambda \in (0, 2] \), \( \lambda_1 \in (0,1] \cap (0, \lambda) \), \( \lambda_2 \in (0, \frac{1}{2}] \cap (0, \lambda) \), then the constant factor \( \frac{1}{B(\lambda_1, \lambda_2)} \) in (16) is the best possible.

**Proof.** (i) For the case of \( \lambda_1 \in (0,1) \cap (0, \lambda) \), we prove that the constant factor \( \frac{1}{B(\lambda_1, \lambda_2)} \) in (17) is the best possible.

For any \( 0 < e < \min \{ p \mid (1 - \lambda_1), q \lambda_2 \} \), we set

\[
\tilde{a}_m := m^{\lambda_1-\frac{1}{p}} b_n := n^{\lambda_2-\frac{1}{p}} \quad (m, n \in N).
\]

Since \( 0 < e < \frac{1}{p} \), we have

\[
0 < \lambda_1 - \frac{1}{p} < 1 \quad (p < 0), \text{ and } f(t) := t^{\lambda_1-\frac{1}{p}} \text{ is strictly decreasing with respect to } t > 0.
\]

Thus, by the decreasing property of the series, we have
\[
\widetilde{A}_m := \sum_{i=1}^{m} \tilde{a}_i = \sum_{i=1}^{m} i^{\lambda-i-1} < \int_0^m t^{\lambda-i-1} dt = \frac{1}{\lambda-i} m^{\lambda-i}.
\]

If there exists a constant \( M \geq \frac{1}{\lambda} B(\lambda_1, \lambda_2) \) such that (17) is valid when we replace \( \frac{1}{\lambda} B(\lambda_1, \lambda_2) \) by \( M \) then, in particular, for \( \eta_i = \eta = 0 \) \((i = 1, 2)\), using a substitution of \( a_n = \tilde{a}_n, b_n = \tilde{b}_n \) and \( A_m = \widetilde{A}_m \) in (17), we have:

\[
\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{a}_n \tilde{b}_m}{(m+n)^{\gamma-1}} > M \left[ \sum_{m=1}^{\infty} m^{\rho(1-\lambda_1)-1} \tilde{a}_m \right] \frac{1}{\gamma} \left[ \sum_{n=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{n^\gamma} \right) \right) n^{q(1-\lambda_2)-1} \tilde{b}_n \right]^{\frac{1}{\gamma}}. \tag{19}
\]

By (19) and the decreasing property of the series, we obtain:

\[
\widetilde{I} > M (1 + \sum_{m=1}^{\infty} m^{1-\varepsilon}) \frac{1}{\gamma} \left[ \sum_{n=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{n^\gamma} \right) \right) n^{q(1-\lambda_2)-1} \tilde{b}_n \right]^{\frac{1}{\gamma}}
\]

\[
> M (1 + \int_1^{\infty} x^{-\varepsilon} dx) \frac{1}{\gamma} \left( 1 - O(1) \right) \frac{1}{\gamma}
\]

\[
> \frac{M}{\varepsilon} (\varepsilon + 1) \frac{1}{\gamma} (1 - \varepsilon O(1)) \frac{1}{\gamma}.
\]

By (18), for \( \eta_i = \eta = 0 \) \((i = 1, 2)\), \( \lambda_2 = \lambda_2 - \varepsilon \in (0, \frac{1}{\lambda}) \cap (0, \lambda) \), we have:

\[
\widetilde{I} < \frac{1}{\lambda-i} \sum_{m=1}^{\infty} \left( m^{-(\lambda_2-\varepsilon)+1} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{\gamma-1}} m^{-\varepsilon}\right)
\]

\[
= \frac{1}{\lambda-i} \sum_{m=1}^{\infty} a_{j+1} (\lambda_2) n m^{-\varepsilon-1} \left( 1 + \sum_{m=2}^{\infty} n^{-\varepsilon} \right)
\]

\[
< \frac{1}{\lambda-i} k_{j+1} (\lambda_2) (1 + \int_1^{\infty} x^{-\varepsilon} dx) = \frac{1}{\lambda-i} k_{j+1} (\lambda_2) (\varepsilon + 1).
\]

This yields:

\[
\frac{1}{\lambda-i} k_{j+1} (\lambda_2) (\varepsilon + 1) > \varepsilon \tilde{\lambda} > M (\varepsilon + 1) \frac{1}{\gamma} (1 - \varepsilon O(1)) \frac{1}{\gamma}.
\]

Putting \( \varepsilon \to 0^+ \) into the above inequality, by virtue of the continuity of the beta function, we obtain \( \frac{1}{\lambda} B(\lambda_1, \lambda_2) = \frac{1}{\lambda-i} B(\lambda_1 + 1, \lambda_2) = \frac{1}{\lambda-i} k_{j+1} (\lambda_2) \geq M \).

Hence, \( M = \frac{1}{\lambda} B(\lambda_1, \lambda_2) \) is the best possible constant factor in (17).

(ii) For the case of \( \lambda_1 = 1 \) \((1 < \lambda_2 \leq 2)\), for any \( 0 < \varepsilon < 1 \), replacing \( \lambda \) by \( \lambda - \varepsilon \) in (17), setting \( \lambda_1 = 1 - \varepsilon, \lambda_2 = \lambda - 1 \), by case (i), we have the following inequality with the best possible constant factor \( \frac{1}{\lambda-i} B(1 - \varepsilon, \lambda - 1) \):
\[(m - \eta_i)^{p_{i-1}} \leq (m - \eta_i)^{-\gamma_i-1},\]
\[\sum_{m=1}^{\infty} (m - \eta_i)^{p_{i-1}} a_m^{p_{i}} \leq \sum_{m=1}^{\infty} (m - \eta_i)^{p_{i-1}-1} a_m^{p_{i}} < \infty,\]

it follows that \(\lim_{\varepsilon \to 0^+} \sum_{m=1}^{\infty} (m - \eta_i)^{p_{i-1}} a_m^{p_{i}} = \sum_{m=1}^{\infty} (m - \eta_i)^{p_{i-1}-1} a_m^{p_{i}}\), and in the same way, we conclude that \(\lim_{\varepsilon \to 0^+} K_{q,\varepsilon} = K_{q,0}\) is valid.

If there exists a constant factor \(M \geq \frac{1}{\lambda}B(1, \lambda - 1) = \frac{1}{\lambda(\lambda-1)} (1 < \lambda \leq 2)\), such that (17) (for \(\lambda_1 = 1, \lambda_2 = \lambda - 1\)) is valid when we replace \(\frac{1}{\lambda(\lambda-1)}\) by \(M\), namely
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{m,n}}{(m+n-q)^{p_{i-1}}/\varepsilon} > M \left[ \sum_{m=1}^{\infty} (m - \eta_i)^{-1} a_m^{p_{i}} \right]^2 K_{q,0},
\]

Then, by using Fatou lemma (cf. [15]) and (20), it follows that
\[
\lim_{\varepsilon \to 0^+} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{m,n}}{(m+n-q)^{p_{i-1}}/\varepsilon} / \left[ \sum_{m=1}^{\infty} (m - \eta_i)^{-1} a_m^{p_{i}} \right]^2 K_{q,\varepsilon} \right\} \geq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{m,n}}{(m+n-q)^{p_{i-1}}/\varepsilon} / \left[ \sum_{m=1}^{\infty} (m - \eta_i)^{-1} a_m^{p_{i}} \right]^2 K_{q,0} > M.
\]

By the property of limitation, there exists a constant \(\delta_0 \in (0,1)\), such that for any \(\delta \in (0, \delta_0)\),
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{m,n}}{(m+n-q)^{p_{i-1}}/\varepsilon} / \left[ \sum_{m=1}^{\infty} (m - \eta_i)^{-1} a_m^{p_{i}} \right]^2 K_{q,0} > M,
\]

namely,
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_{m,n}}{(m+n-q)^{p_{i-1}}/\varepsilon} > M \left[ \sum_{m=1}^{\infty} (m - \eta_i)^{-1} a_m^{p_{i}} \right]^2 K_{q,0}.
\]

Since the constant factor \(\frac{1}{\lambda(\lambda-1)} B(1, \lambda - 1)\) in (20) (for \(\varepsilon = \delta\)) is the best possible, we have \(\frac{1}{\lambda(\lambda-1)} B(1, \lambda - 1) \geq M\). Letting \(\delta \to 0^+\), we have \(\frac{1}{\lambda(\lambda-1)} = \frac{1}{B(1, \lambda - 1)} \geq M\), which implies that \(M = \frac{1}{\lambda(\lambda-1)}\) is the best possible factor of (17) (for \(\lambda_1 = 1, \lambda_2 = \lambda - 1\)). This completes the proof of Theorem 2. \(\square\)

**Theorem 3.** Under the assumption (C1), if the constant factor \(\frac{1}{\lambda} (k_1(\lambda_2))^{\frac{1}{2}} (k_1(\lambda_1))^{\frac{1}{2}}\) in (16) is the best possible, then for
\[
\lambda - \lambda_1 - \lambda_2 \in [(\lambda - \lambda_2 - \frac{1}{2}) q, (\lambda - \lambda_2) q] \cap (-\lambda_2 q, (\frac{1}{2} - \lambda_2) q) (\supset \{0\}),
\]
we have \(\lambda_1 + \lambda_2 = \lambda\).

**Proof.** Note that for \(\hat{\lambda}_1 = \frac{\lambda - \lambda_1}{p} + \frac{\lambda_1}{q}, \hat{\lambda}_2 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_2}{q}\), we find
\[
\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{\lambda - \lambda_1}{p} + \frac{\lambda_1}{q} + \frac{\lambda - \lambda_2}{q} + \frac{\lambda_2}{p} = \lambda.
\]
If \(\lambda - \lambda_1 - \lambda_2 \in [(\lambda - \lambda_2 - \frac{1}{2}) q, (\lambda - \lambda_2) q) (\supset \{0\})\), then we have
\[
0 < \hat{\lambda}_i = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} \leq \frac{1}{2}; \text{ if } \lambda - \hat{\lambda}_1 - \hat{\lambda}_2 \in (-\lambda_2 q ((\frac{1}{2} - \lambda_2)q)(\supset \{0\}),
\]
then we have \(0 < \hat{\lambda}_2 \leq \frac{1}{2}\). By using (22), we obtain \(0 < \hat{\lambda}_i < \lambda \) (\(i = 1,2\)), and then we deduce that \(\hat{\lambda}_i \in (0, \frac{1}{2}] \cap (0, \lambda) \) (\(i = 1,2\)).

By applying (17), we have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_{m,n}}{(m+n-q)^{s+1}}
\]

\[
> \frac{1}{\pi} B(\hat{\lambda}_1, \hat{\lambda}_2) \left[ \sum_{m=1}^{\infty} (m-\eta_i)^{p(1-\hat{\lambda}_1)-1}a_m \right]^{\frac{1}{2}} \left[ \sum_{n=1}^{\infty} (1-O_\lambda(\frac{1}{n-\eta_2}))^p (n-\eta_2)^{q(1-\hat{\lambda}_2)-1}b_n \right]^{\frac{1}{2}}.
\]

If the constant factor \(\frac{1}{\pi} (k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}}\) in (16) is the best possible, then by using (23), we have the following inequality:

\[
\frac{1}{\pi} (k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}} \geq \frac{1}{\pi} B(\hat{\lambda}_1, \hat{\lambda}_2) = \frac{1}{\pi} k_\lambda(\hat{\lambda}_1) \in \mathbb{R}_+ = (0, \infty),
\]

\(k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}} \geq k_\lambda(\hat{\lambda}_1).

By employing the reverse Hölder’s inequality (cf. [14]), we obtain:

\[
k_\lambda(\hat{\lambda}_1) = \frac{\frac{1}{2} - \lambda_2}{p} + \frac{\lambda_1}{q}
\]

\[
= \int_0^\infty \frac{1}{(1+u)^{\frac{1}{2} - \lambda_1 - 1}} du = \int_0^\infty \frac{1}{(1+u)^{\frac{1}{2} - \lambda_1 - 1}}(u^{\frac{1}{2} - \lambda_1}) du
\]

\[
\geq \left[ \int_0^\infty \frac{1}{(1+u)^{\frac{1}{2} - \lambda_1 - 1}} du \right]^{\frac{1}{2}} \left[ \int_0^\infty \frac{1}{(1+u)^{\frac{1}{2} - \lambda_1 - 1}} du \right]^{\frac{1}{2}}
\]

\[
= \left( \int_0^\infty \frac{1}{(1+u)^{\frac{1}{2} - \lambda_1 - 1}} du \right) \left( \int_0^\infty \frac{1}{(1+u)^{\lambda_1 - 1}} du \right)
\]

\[
= (k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}},
\]

which implies that \(k_\lambda(\hat{\lambda}_1) = (k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}}\), namely, (24) keeps the form of equality.

Note that (24) keeps the form of equality if and only if there exist constants \(A\) and \(B\) such that they are not both zero satisfying (cf. [15]) \(Au^{\frac{1}{2} - \lambda_1} = Bu^{\lambda_1 - 1}\) a.e. in \(\mathbb{R}_+\).

Assuming that \(A \neq 0\), we have \(u^{\frac{1}{2} - \lambda_1 - \lambda_1} = \frac{A}{B}\) a.e. in \(\mathbb{R}_+\), and \(\lambda - \lambda_2 - \lambda_1 = 0\). Hence, we have \(\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda\). Theorem 3 is proved. \(\square\)

4. Equivalent Forms and Some Particular Inequalities

**Theorem 4.** Under the assumption (C1), we have the following reverse inequality equivalent to (16):

\[
J := \left\{ \sum_{m=1}^{\infty} \frac{(m-\eta_1)^{p(1-\hat{\lambda}_1)-1}}{(1-O_\lambda(\frac{1}{m-\eta_1}))^p (m+n-q)^{s+1}} \left[ \sum_{n=1}^{\infty} \frac{\lambda_{m,n}}{(m+n-q)^{s+1}} \right]^{\frac{1}{p}} \right\}^{\frac{1}{p}}
\]

\[
> \frac{1}{\pi} (k_\lambda(\hat{\lambda}_2))^{\frac{1}{2}} (k_\lambda(\hat{\lambda}_1))^{\frac{1}{2}} \left[ \sum_{m=1}^{\infty} (m-\eta_1)^{p(1-\hat{\lambda}_1)-1}a_m \right]^{\frac{1}{2}}.
\]
In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have $0 < \sum_{m=1}^{\infty} (m - \eta_1) \rho^{(1-\lambda_1)-1} a_m^p < \infty$, and the following reverse inequality equivalent to (19):

$$
\left\{ \sum_{n=1}^{\infty} \frac{(n-\eta_2)^{\frac{1}{2}-1}}{\log \left( \frac{1}{n-\eta_2} \right)^{p_1+1}} \left[ \frac{\sum_{m=1}^{n} A_m}{\sum_{m=1}^{n} (m+n-q)^{p_2+1}} \right]^{p_1} \right\}^{\frac{1}{p_2}} > \frac{1}{\lambda} B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} (m - \eta_1) \rho^{(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}.
$$

(26)

**Proof.** Suppose that (25) is valid. By using the reverse Hölder’s inequality (cf. [14]), we have

$$
I = \sum_{n=1}^{\infty} \left[ \frac{(n-\eta_2)^{\frac{1}{2}-1}}{\log \left( \frac{1}{n-\eta_2} \right)^{p_1+1}} \sum_{m=1}^{n} \frac{A_m}{(m+n-q)^{p_2+1}} \right]^{p_1} (1 - O_2 \left( \frac{1}{(n-\eta_2)^{\frac{1}{2}}} \right))^{\frac{q}{q-1}} b_n^{\frac{q}{q-1}}.
$$

(277)

Then, from (25) and (27), we obtain (16).

On the other hand, assuming that (16) is valid, we set

$$
b_n := \frac{(n-\eta_2)^{\frac{1}{2}-1}}{\log \left( \frac{1}{n-\eta_2} \right)^{p_1+1}} \sum_{m=1}^{n} \frac{A_m}{(m+n-q)^{p_2+1}}, \quad n \in \mathbb{N}.
$$

Then, it follows that

$$
J = \left[ \sum_{n=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{(n-\eta_2)^{\frac{1}{2}}} \right) \right) (n - \eta_2) \right]^{q-1} \left[ \sum_{n=1}^{\infty} b_n \right]^{\frac{q}{q-1}}.
$$

If $J = \infty$, then (25) is naturally valid; if $J = 0$, then it is impossible that it makes (25) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By virtue of (16), we have

$$
\sum_{n=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{(n-\eta_2)^{\frac{1}{2}}} \right) \right) (n - \eta_2) \left[ \sum_{m=1}^{\infty} (m - \eta_1) \rho^{(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} = J
$$

$$
> \frac{1}{\lambda} (k_1(\lambda_2))^{\frac{1}{p}} (k_1(\lambda_1))^{\frac{1}{q}} \left[ \sum_{m=1}^{\infty} (m - \eta_1) \rho^{(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} J^{p-1},
$$

(25)

$$
J > \frac{1}{\lambda} (k_1(\lambda_2))^{\frac{1}{p}} (k_1(\lambda_1))^{\frac{1}{q}} \left[ \sum_{m=1}^{\infty} (m - \eta_1) \rho^{(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}.
$$

Thus, we obtain (25), which implies that (25) is equivalent to (16). The Theorem 4 is proved. □

**Remark 2.** By the same way as above, in view of assumption (C1), if $0 < p < 1, q < 0, \frac{1}{p} + \frac{1}{q} = 1$, then we can obtain the following reverse equivalent inequalities containing one partial sums:

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m}{(m+n-q)^{p_2+1}}.
$$

$$
J > \frac{1}{\lambda} (k_1(\lambda_2))^{\frac{1}{p}} (k_1(\lambda_1))^{\frac{1}{q}} \left[ \sum_{m=1}^{\infty} \left( 1 - O_2 \left( \frac{1}{(n-\eta_2)^{\frac{1}{2}}} \right) \right) (n - \eta_2) \right]^{q-1} \left[ \sum_{n=1}^{\infty} b_n \right]^{\frac{q}{q-1}},
$$

(25)
\[
\left\{ \sum_{n=1}^{\infty} (n-\eta_2)^{p\lambda_2-1} \left[ \sum_{m=1}^{\infty} \frac{\lambda_2}{(m+n-\eta_2)^{\frac{p}{\lambda_2}}} \right]^p \right\}^{\frac{1}{p}} > \frac{1}{\lambda} (k_\lambda(\lambda_2))^\frac{1}{\lambda} (k_\lambda(\lambda_1))^\frac{1}{\lambda} \left[ \sum_{m=1}^{\infty} (1-O_2(\frac{1}{\eta_2^p})) (m-\eta_1)^{p(1-\lambda_1)-1} a_m^p \right]^\frac{1}{\lambda}.
\]

**Theorem 5.** If \( \lambda_1 + \lambda_2 = \lambda (\in (0, 2]) \) satisfying \( \lambda_1 \in (0, 1] \cap (0, \lambda) \) and \( \lambda_2 \in (0, \frac{1}{2}] \cap (0, \lambda) \), then the constant factor \( \frac{1}{\lambda} (k_\lambda(\lambda_2))^\frac{1}{\lambda} (k_\lambda(\lambda_1))^\frac{1}{\lambda} \) in (25) is the best possible. On the other hand, by virtue of the assumption (C1), if the constant factor \( \frac{1}{\lambda} (k_\lambda(\lambda_2))^\frac{1}{\lambda} (k_\lambda(\lambda_1))^\frac{1}{\lambda} \) in (25) is the best possible, then for

\[
\lambda - \lambda_1 - \lambda_2 \in (\lambda - \lambda_2 - \frac{1}{2}) q, (\lambda - \lambda_2) q) \cap (-\lambda_2 q, (\frac{1}{2} - \lambda_2) q) \cup \{0\},
\]

we have \( \lambda_1 + \lambda_2 = \lambda \).

**Proof.** If \( \lambda_1 + \lambda_2 = \lambda (\in (0, 2]) \) satisfying \( \lambda_1 \in (0, 1] \cap (0, \lambda) \) and \( \lambda_2 \in (0, \frac{1}{2}] \cap (0, \lambda) \), then by using Theorem 2, we conclude that the constant factor \( \frac{1}{\lambda} (k_\lambda(\lambda_2))^\frac{1}{\lambda} (k_\lambda(\lambda_1))^\frac{1}{\lambda} \) in (16) is the best possible. By employing (27), we can prove that the constant factor in (25) is still the best possible.

On the other hand, if the same constant factor in (25) is the best possible, then by the equivalency of (25) and (16), in view of \( J^\pi = I \) (in the proof of Theorem 4), it follows that the same constant factor in (16) is still the best possible. By applying Theorem 2, in view of the assumption, we have \( \lambda_1 + \lambda_2 = \lambda \). The proof of Theorem 5 is complete.

**Remark 3.** (i) Taking \( \eta = \eta_1 = \eta_2 = 0 \) in (17) and (26), we obtain the following reverse equivalent inequalities:

\[
\sum_{m=1}^{\infty} \frac{a_m^p}{(m+n)^{\frac{p}{\lambda}}} > \frac{1}{\lambda} \left( \sum_{m=1}^{\infty} \frac{a_m^p}{(m+n)^{\frac{p}{\lambda}}} \right)^{\frac{1}{\lambda}} B(\lambda_1, \lambda_2) \left( \sum_{m=1}^{\infty} \frac{a_m^p}{(m+n)^{\frac{p}{\lambda}}} \right)^{\frac{1}{\lambda}}.
\]

Hence, (17) (resp. (16)) is an extension of inequality (28).

In particular, for \( \lambda = 2, \lambda_1 = \lambda_2 = 1 \), we have

\[
\sum_{m=1}^{\infty} \frac{a_m^p}{(m+n)^{\frac{p}{\lambda}}} > \frac{1}{\lambda} \left( \sum_{m=1}^{\infty} \frac{a_m^p}{(m+n)^{\frac{p}{\lambda}}} \right)^{\frac{1}{\lambda}} \left( \sum_{m=1}^{\infty} (1-O_2(\frac{1}{\eta_2^p})) (m-\eta_1)^{p(1-\lambda_1)-1} b_m^p \right)^{\frac{1}{\lambda}}.
\]

(ii) Putting \( \lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2} \) in (17) and (26), we obtain the following reverse inequalities with the best possible constant factor \( \pi \):
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_n b_m}{(m+n)^2} > \pi \left( \sum_{m=1}^{\infty} \frac{(m-\eta_t)^{1-p}}{m^p} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{1}{n^{1-q}} (n-\eta)^{1-q} b_n \right)^{\frac{1}{q}}, \quad (29)
\]

Choosing \( \eta_1 = \eta_2 = \eta = 0 \) in (29) and (30), we have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_n b_m}{(m+n)^2} > \pi \left( \sum_{m=1}^{\infty} \frac{a_m^{1-p}}{m^p} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{1}{n^{1-q}} (n-\eta)^{1-q} b_n \right)^{\frac{1}{q}},
\]

Choosing \( \eta_1 = \eta_2 = \frac{1}{4}, \eta = \frac{1}{2} \) in (29) and (30), we obtain:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_n b_m}{(m+n)^2} > \pi \left( \sum_{m=1}^{\infty} \frac{(m-\frac{1}{4})^{1-p}}{m^p} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{1}{n^{1-q}} (n-\frac{1}{4})^{1-q} b_n \right)^{\frac{1}{q}},
\]

5. Conclusions

In this paper, inspired by the work of [4–10], we construct a reverse Hardy–Hilbert’s inequality which contains one partial sum and some extra parameters inside the weight coefficients in Theorem 1. Our method is mainly based on some skillful applications of the Euler–Maclaurin summation formula and Abel’s partial summation formula. By means of the newly proposed inequality, we then discuss the equivalent conditions of the best possible constant factor associated with several parameters in Theorems 2 and 3. As applications, we deal with some equivalent forms of the obtained inequality and illustrate how to derive more reverse inequalities of the Hardy–Hilbert type from the current results in Theorems 4 and 5. The lemmas and theorems reveal rich connotations and significance of this type of inequality.

Author Contributions: B.Y. carried out the mathematical studies and drafted the manuscript. S.W. and X.H. participated in the design of the study and performed the numerical analysis. All authors contributed equally in the preparation of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation (Nos.11961021, 11561019), Hechi University Research Foundation for Advanced Talents under Grant (2021GCC024), and the Characteristic Innovation Project of Guangdong Provincial Colleges and Universities (No.2020KTSCX088).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.
References
1. Hardy, G.H.; Littlewood, J.E.; Polya, G. Inequalities; Cambridge University Press: Cambridge, UK, 1934.
2. Krnić, M.; Pečarić, J. Extension of Hilbert’s inequality. J. Math. Anal. Appl. 2006, 324, 150–160.
3. Yang, B.; Wu, S.; Chen, Q. A new extension of Hardy–Hilbert’s inequality containing kernel of double power functions. Mathematics 2020, 8, 894.
4. Adiyasuren, V.; Batbold, T.; Azar, L.E. A new discrete Hilbert-type inequality involving partial sums. J. Inequal. Appl. 2019, 2019, 127.
5. Liao, J.; Wu, S.; Yang, B. A multiparameter Hardy–Hilbert-type inequality containing partial sums as the terms of series. J. Math. 2021, 2021, 5264623.
6. Yang, B.; Wu, S.; Huang, X. A reverse Hardy–Hilbert’s inequality involving one partial sum as the terms of double series. J. Inequal. Appl. 2020, 2020, 259.
7. Qi, L. Some new Hardy–Hilbert-type inequalities with multiparameters. J. Inequal. Appl. 2020, 2020, 299.
8. Yang, B.C. The Norm of Operator and Hilbert-Type Inequalities; Science Press: Beijing, China, 2009.
9. Kuang, J.C. Real and Functional Analysis; Higher Education Press: Beijing, China, 2015.