River Flow Dynamics with Two-dimensional Shallow-water Equations

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Abstract. This study proposes a numerical model to simulate unsteady flows in rivers and associated riverine processes. The mathematical model is based on Saint-Venant equations. The governing equations are discretized by Godunov-type finite volume method. Use of total variation diminishing (TVD) scheme and MUSCL-Hancock technique satisfies second-order temporal and spatial accuracy requirements, respectively. Hydrodynamic flux components associated are computed by Harten-Lax-van Leer (HLL) method. A modified method is introduced for approximating flow features along the shear wave. The proposed scheme is used to model an idealized dam-break case, and a case of dam-break in a channel with a 90° bend which results in bore formation and complex two-dimensional flow; the predictions show excellent agreement with available experimental results and analytical solutions.

1. Introduction
River flows are extensively modelled by non-linear shallow-water equations. The similarity of the numerical predictions to the real flows depends upon the structure on depth averaged equations considered to model the problem. Two-dimensional shallow-water equations are widely applied in modeling hydraulic, coastal, and environmental engineering flows due to lower computational cost compared to three-dimensional models [1,2].

To ensure steady solution, the well-balanced condition has to be satisfied; to ensure well-balancing of proposed numerical scheme, the bed slope terms are discretized by revised surface gradient method (RGSM) so that when added with momentum terms, the system remains conserved [3]. A cell centered finite volume method is used for discretization of governing equations. MUSCL-Hancock technique is employed for reconstruction of flow variables on unstructured computational domain. The proposed method performs very well for various benchmark test cases.

2. Numerical Method
For flows in natural rivers, open channels, and floodplain systems over a horizontal plane, the depth averaged shallow-water equation are given as:

\[
\frac{\partial h}{\partial t} + \frac{\partial (hu_j)}{\partial x_j} = 0
\]

\[
\frac{\partial (hu_j)}{\partial t} + \frac{\partial}{\partial x} \left( hu_j^2 + \frac{1}{2}gh^2 \right) + \frac{\partial}{\partial x_j} \left( u_i u_j \right) = -gh \left( \frac{\partial z}{\partial x_j} - \tau_i \right)
\]

(1)
where \( i=j=1,2; \) \( h \) is water surface elevation, \( u \) and \( v \) represent velocity vectors in \( x \) and \( y \) direction, respectively, \( z \) is bed elevation, and \( \tau \) is the bed friction. The equation is discretized by Godunov type finite volume method with Runge-Kutta scheme for time discretization. Discretization of Equation (1) by finite volume method is expressed as:

\[
\frac{U_{i}^{n+1} - U_{i}^{n}}{\Delta t} = - \left( \frac{\partial F_{x}}{\partial x} + \frac{\partial F_{y}}{\partial y} \right) + S_{i}^{n}
\]

where \( U \) represents conserved variables in Equation (1), superscript \( n \) refers to time level , \( S \) is a vector of bed inclination and friction entities and \( F \) represent flux vector.

\[
S = S_{h} + S_{f} = \begin{bmatrix}
ghS_{ax} \\
ghS_{ay} \\
-ghS_{fx} \\
-ghS_{fy}
\end{bmatrix}
\]

where \( S_{ax}, \ S_{ay} \) and \( S_{fx}, \ S_{fy} \) are bathymetry slope and frictional terms in the \( x \)- and \( y \)-axis directions, respectively. Manning’s relation is used to compute friction terms.

\[
S_{fx} = \frac{n^{2}u\sqrt{u^{2} + v^{2}}}{h^{4/3}}, \quad S_{fy} = \frac{n^{2}v\sqrt{u^{2} + v^{2}}}{h^{4/3}}
\]

In Equation (4) ‘\( n \)’ is the Manning’s roughness coefficient. Equation (2) is then discretized over the control volume (in this study triangle) to approximate the solution. Fluxes in Equation (2) are approximated HLL scheme (Toro, 2001).

\[
F_{HLL} = \begin{cases}
F_{L}, & \text{if } 0 \leq S_{L} \\
S_{R}F_{L} - S_{L}F_{R} + S_{R}S_{L}(U_{R} - U_{L}) & \text{if } S_{R} \leq 0 \leq S_{L} \\
F_{R}, & \text{if } 0 \geq S_{R}
\end{cases}
\]

In Equation (5) \( S_{R} \) and \( S_{L} \) are the right and left propagating waves, respectively. Depth of water and wave speeds needed to compute numerical fluxes by HLL scheme [4]. Small water depth, near wet–dry interfaces cause numerical instabilities, which result in negative water depths or unphysical high velocities; therefore, a threshold water depth of \( \varepsilon = 10^{-5} \) is set to avoid instabilities. Since the proposed scheme is time-explicit, CFL condition is used for limiting time-step.

\[
\Delta t \leq \min \left( \frac{R_{i}}{2 \max_{j}(U + c)_{j}} \right)
\]

where \( R_{i} \) is the shortest distance between cell center and vertex, \( U \) is velocity vector, and \( c \) is wave celerity.

3. Results

3.1. Dam-Break in L-Shape Channel

L-shaped channel is considered to test the ability of numerical scheme of modeling complex dam-break flows. This test case has been designed by CADAM team [5]. The initial water depth in the reservoir is 0.58 m while the channel is dry. Manning’s roughness coefficient equal to 0.019 is fixed throughout the domain. A thin wall (dam) at \( x = 2.39 \) m separates the reservoir from channel.
Figure 1. Wave propagation after \( t = 5 \) sec of dam-break.

Figure 2. Wave propagation after \( t = 12 \) sec of dam-break.

After the removal of dam the wave propagation is depicted in the channel at \( t = 5 \) and \( 12 \) sec in figure 1 and 2, respectively. The observed pattern of wave propagation and bore reflection from the channel wall at the end agrees very well with the experimental and numerical studies [6] confirming the proposed scheme’s ability to model dam-break flows in two dimensions.

3.2. Tidal flow

The flow over large step is simulated to test model’s ability for simulating tidal flows. A channel of length \( L \) of 1500 m is considered. The bathymetry of the problem is given as

\[
z(x) = \begin{cases} 8.0 \text{m} & \text{if } |x - 750| \leq 187.5 \text{m} \\ 0.0 & \text{otherwise} \end{cases}
\]  

(7)

For initial flow conditions, water depth \( H(x, 0) \) is set equal to 16 m in the whole domain. For boundary conditions the velocity \( u(L, t) \) is set as zero, and water depth is set equal to

\[
h(0, t) = H(x, 0) + 4 - 4\sin\left(\frac{4t}{86400} + 0.5\right) \]

(8)

The analytical solution for this problem was proposed by [7], and is given as

\[
h(x, t) = H(x) + 4 - 4\sin\left(\frac{4t}{86400} + 0.5\right)
\]

\[
u(x, t) = \frac{(x - L)\pi}{5400h(x, t)} \cos\left(\frac{4t}{86400} + 0.5\right)
\]

(9)

Simulations are run for a total of 35,000 s; the computed water surface elevation and velocity (at 10,800 s, 21,600 s, 32,400 s) are compared with analytical solution, as shown in Figure 3 and 4, respectively. Excellent agreement is observed between computed and analytical solution.
4. Conclusions

A two-dimensional depth averaged formulation is proposed in this study to model open channel flows. The model is based on non-linear shallow-water equations with a treatment for wet-dry fronts. The spurious oscillations associated with bathymetry are avoided by employing high resolution reconstruction technique. Proposed scheme is used to model complicated dam-break flows involving shocks making it suitable for wide applications in modeling open-channel flows without sediment. Numerical results agree well with experimental studies and analytical solutions. However, to model the effect of eddies associated with flows undergoing rapid changes in direction and varying bed conditions, the proposed scheme will be coupled with suitable turbulence model which will enable modelling complicated river and estuarine flows.

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