Stability and quasinormal modes of the massive scalar field around Kerr black holes

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We find quasinormal spectrum of the massive scalar field in the background of the Kerr black holes. We show that all found modes are damped under the quasinormal modes boundary conditions when \( \mu M \) is not large, thereby implying stability of the massive scalar field. This complements the region of stability determined by the Beyer inequality for large masses of the field. We show that, similar to the case of a non-rotating black holes, the massive term of the scalar field does not contribute in the regime of high damping. Thereby, the high damping asymptotic should be the same as for the massless scalar field.

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I. INTRODUCTION

Scalar field in a black hole backgound is a subject of active investigation \(^1\), yet the massive scalar field has been investigated only in very few studies as to its quasinormal spectrum \(^2,3\), late-time behavior \(^4\) and scattering properties \(^5\). Behavior of the massive scalar field in a black hole background is quite different from that of the massless one in many aspects: first, it demonstrates the so-called superradiant instability \(^6\), which is absent for massless fields. This super-radiant regime happens provided the real oscillation frequency satisfies the inequality \(|\Re \omega| < ma/(2Mr_+)\), where \(m\) is the azimuthal number of the mode, and \(M\) is the mass of the black hole. Another research, indicating the instability of massive scalars was that by Detweiler \(^7\) where slowly growing instability was suggested for values \(\mu M << 1\), i.e. in the regime, when particle Compton wavelength is much larger then the size of the black hole (\(\mu\) is the mass of the scalar field). Second, it may cause existence of infinitely long-living modes called quasi-resonances \(^8\). Finally, at asymptotically late times the massive fields show universal behavior independent on spin of the field \(^9\). Note, that scalar field with the mass term can be also interpreted as a self-interacting scalar field within regime of small perturbations \(^10\). One more recent application of dynamic of the massive scalar field in a black hole background comes from different brane-world theories where massless fields can be considered as effectively massive fields on the brane.

Yet, the strongest motivation to make an extensive search for quasinormal modes of massive scalar field is that until now the stability for massive scalar field perturbations in the Kerr background is not studied under non-super-radiant (quasinormal) boundary conditions. Usually, stability of spherically symmetric black holes, like Schwarzschild or Reissner-Nordstrom ones, can be relatively easily proved by using positivity of the conserved energy. On the contrary to the Schwarzschild case, Kerr black holes have negative energy density inside the ergosphere. Therefore, fields can grow in parts of the space-time, leading to instability. The only result on stability of massive scalar field in Kerr space-time, we are aware of, is that of Beyer \(^11\), where it was proved that the modes of field with the mass obeying the inequality

\[
\mu_0 \geq \frac{|m|a}{2Mr_+} \sqrt{1 + \frac{2M}{r_+} + \frac{a^2}{r_+^2}},
\]

are stable. Note, that the inequality found by Beyer is not compatible with conditions for super-radiance, so it does not contradict super-radiant instability of massive fields. In addition, the above inequality does not mean that fields of lower mass are unstable, and Beyer suggested the ways to lower the inequality \(^11\). If such an instability exists one could find an accurate threshold value of field mass \(\mu\) when the instability begins. This could be done by an extensive computing of the quasinormal modes and by finding of the growing ones. Some low laying WKB quasinormal modes were found by Simone and Will \(^12\), yet neither fundamental modes \((\ell = n = 0, \ell\) is multi-pole number), nor modes with \(n \geq \ell\) can be found with trustable accuracy by WKB approach \(^12,13,14\), even despite very good accuracy of WKB method for low overtones \(^13\). Therefore, being limited by WKB accuracy, one cannot judge about stability and high damping asymptotic in this case. Recently the quasinormal damping and late time regimes in time domain were investigated by L. Burko and G. Khanna in \(^4\), and no instability was observed in the considered range of values of field mass \(\mu\).

Because of the above reasons, in this paper we used the accurate Frobenius method to make an extensive search for quasinormal modes of the massive scalar field around Kerr black holes. The paper organized as follows: Sec. II introduces the main formulas for the wave equations and the Frobenius procedure. Sec. III gives numerical data for found quasinormal modes, and Sec. IV discusses the obtained results and the possibility for (un)stability.
II. BASIC EQUATIONS

The background metric in the Boyer-Lindquist coordinates \( t, r, \theta, \phi \) has the form:
\[
 ds^2 = \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 + \frac{4aMr \sin^2 \theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2
\]
\[
 -\Sigma d\theta^2 - \left( r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2, \tag{2}
\]
where, \( M \) is the mass of the black hole,
\[
 \Delta = r^2 + a^2 - 2Mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta.
\]
and
\[
 S = r^2 + a^2 \cos^2 \theta.
\]
The massive scalar field obeys the equation:
\[
 \Box \Phi + \mu^2 \Phi = 0. \tag{3}
\]
The radial wave equation \( (\Phi = e^{-i\omega t} e^{i\omega \phi} R(r) S(\phi, \theta)) \) is well-known:
\[
 \frac{d}{dr} \left( \Delta \frac{dR(r)}{dr} \right) + \left( \frac{K^2}{\Delta} - \lambda - \mu^2 r^2 \right) R(r) = 0, \tag{4}
\]
where
\[
 K = \omega (r^2 + a^2) - am.
\]
The separation constant \( \lambda \) satisfies the equation
\[
 \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS(\theta)}{d\theta} \right) + \left( - \frac{m^2}{\sin^2 \theta} \right) - a^2 \omega^2 \sin^2 \theta - a^2 \mu^2 \cos^2 \theta + 2am \omega + \lambda \right) S(\theta) = 0, \tag{6}
\]
and can be found numerically for any given \( \omega \). Since \( \lambda(\omega) \) can be obtained numerically, we are able to solve the equation for the radial part \( 4 \), using the continued fraction method \( 17 \). In order to satisfy the quasinormal mode boundary conditions, which corresponds to purely in-going waves at the event horizon and purely outgoing waves at spatial infinity, we require that
\[
 R(r \to \infty) \propto \exp(i\chi r) r^{i\sigma - 1}, \quad \sigma = \frac{(\omega^2 + \chi^2)(r_+^2 + a^2)}{2r_+^2}, \]
\[
 R(r \to r_+) \propto (r - r_+)^{-i\alpha}, \quad \alpha = \frac{\omega r_+ (r_+^2 + a^2) - mar_+}{r_+ - a^2}, \tag{7}
\]
where \( r_+ \) marks the event horizon:
\[
 r_+ = M + \sqrt{M^2 - a^2},
\]
and
\[
 \chi = \pm \sqrt{\omega^2 - \mu^2}.
\]

The sign for \( \chi \) should be chosen in order to remain in the same complex quadrant as \( \omega \).

The equation \( 14 \) has three singularity points: the spatial infinity \( (r = \infty) \), the event horizon \( (r = r_+) \) and the internal horizon \( (r = r_- = a^2/r_+ < r_+) \). Therefore the appropriate Frobenius series has the form:
\[
 R(r) = \exp(i\chi r) \left( \frac{r}{r_+} \right)^{-i\sigma} \left( \frac{r_+ - r_+^2}{r_+ - a^2} \right)^{-i\alpha} \times \sum_{i=0}^{\infty} a_i \left( \frac{r_+ - r_+^2}{r_+ - a^2} \right)^i. \tag{7}
\]

And the coefficients satisfy the three-term recurrence relation:
\[
 \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0, \quad n \geq 0, \quad \gamma_0 = 0, \tag{8}
\]
where \( \alpha_n, \beta_n, \gamma_n \) can be found in analytical form. We do not present them here because they have rather cumbersome form.

By comparing the ratio of the series coefficients
\[
 \frac{a_{n+1}}{a_n} = \frac{\gamma_n}{\alpha_n} - \frac{\alpha_{n-1}}{\alpha_n}, \quad \frac{a_{n+1}}{a_n} = \frac{\alpha_n \gamma_{n+1}}{\beta_n + 1} - \frac{\gamma_n + 1}{\beta_n}, \tag{9}
\]
we obtain an equation with a convergent infinite continued fraction on its right side:
\[
 \beta_n - \frac{\alpha_{n-1} \gamma_n}{\beta_n - \frac{\alpha_{n-2} \gamma_{n-1}}{\beta_n - \frac{\alpha_{n-3} \gamma_{n-2}}{\beta_n - \cdots}}} = \frac{\alpha_n \gamma_{n+1}}{\beta_n + 1} - \frac{\gamma_n + 1}{\beta_n}, \tag{10}
\]
that can be solved numerically by minimizing the absolute value of the difference between its left and right sides. The equation \( 10 \) has infinite number of roots, but the most stable root depends on \( n \). Generally the larger number \( n \) corresponds to the larger imaginary part of the root \( \omega \).

Note, that the case under consideration allows to use the Nollert procedure \( 18 \), in order to improve convergence of the infinite continued fraction, that is useful for searching roots with a very large imaginary part.

III. NUMERICAL RESULTS

Since we are limited by Kaluza-Klein equation, we do not take into consideration the effect of the mass of the scalar field onto the black hole. Thereby, we can consider only small values of mass of the field \( \mu \), in comparison with the mass of the black hole \( M \), i.e. \( \mu M \ll 1 \). Yet, as large values of \( \mu \) are possible in brane-world theories, we shall touch this case as well.

The quasinormal modes of the Kerr black hole depend on several parameters: the multi-pole number \( \ell \), the over-tone number \( n \), the azimuthal number \( m \), and on the black
hole and scalar field parameters: \( M, a, \mu \). Note that \( m \) can take values \(-\ell, -\ell + 1, \ldots, \ell - 1, \ell\). From here and on we measured \( \mu \) and \( a \) in units of black hole mass \( M \).

Let us look at tables I-III: the damping rate determined by the imaginary part of the \( \omega \) into the ringing signal.

To check the correctness of the applied procedure we considered particular cases of massless scalar field in the Kerr background and of the massive field in the Schwarzschild background. For these two cases the quasinormal modes are well-known for example from the papers [19], [2], [3], [8]. One can see, for instance in the table III that QNMs in case of \( \mu = 0 \), coincide with corresponding data in of [19], [2], [3]. Also data in tables I-IV for \( a = 0 \) coincide with modes found in [5].

As the number of the overtone is increased, the QN frequency of the massive field becomes closer to that of the massless field. For the case of Schwarzschild black hole, we proved in [5], both analytically and numerically, that this tendency of QN modes to diminish difference stipulated by mass \( \mu \), ends with a universal high damping asymptotic, which is independent on the mass of the field. Thus, the massive term \( \mu \) does not contribute to high overtone behavior of Schwarzschild black holes. We shall show numerically that the same is true for Kerr black holes. Indeed, from Table II, one can see that already for relatively small overtone number, the difference between quasinormal modes with different values of mass \( \mu \) becomes negligible. This is important for us, because now, we are in position to say that it is enough to study low overtone behavior to judge about stability of Kerr case within our numerical approach.

From Tables I-V and figures (1-2), we see that all found modes for different values of parameters \( \ell \) and \( m \) are damped in the regime of small \( \mu \). Larger \( \mu \) (see Fig. 3) leads to considerable decreasing of the damping rate and finally to the same infinitely long living modes (i.e. modes with infinitesimally small imaginary part) found in [5] for Schwarzschild black holes and called quasi-

resonances. Therefore, when increasing \( \mu \), the strict line in Fig. 3 does not have continuation into negative half-plane. We see here that, the larger mass of the field \( \mu \) is, the more low laying modes will change into quasi-resonances. Meanwhile, the rest of the spectra consists of ordinary modes (i.e. modes with finite time of damping). Moreover, at sufficiently high overtone number, the difference between massive and massless case will be neglected (see Table II). This behavior mimics the QN behavior of non-rotating black holes. Yet, potential for massive scalar field around non-rotating black holes is

| \( \ell = m = 0 \) | \( \mu = 0.1 \) | \( \mu = 0.2 \) | \( \mu = 0.3 \) |
|-----------------|-----------------|-----------------|-----------------|
| a | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | -Im(\( \omega_0 \)) | -Im(\( \omega_0 \)) | -Im(\( \omega_0 \)) |
| 0 | 0.112361 | 0.096823 | 0.116207 | 0.075358 | 0.122060 | 0.043605 |
| 0.1 | 0.112447 | 0.096741 | 0.116311 | 0.075312 | 0.122442 | 0.043495 |
| 0.2 | 0.112707 | 0.096490 | 0.116625 | 0.075167 | 0.123554 | 0.043800 |
| 0.3 | 0.113140 | 0.096050 | 0.117153 | 0.074910 | 0.123993 | 0.044905 |
| 0.4 | 0.113747 | 0.095387 | 0.117903 | 0.074511 | 0.123805 | 0.044021 |
| 0.5 | 0.114521 | 0.094442 | 0.118883 | 0.073989 | 0.126101 | 0.045025 |
| 0.6 | 0.115442 | 0.093120 | 0.120093 | 0.073056 | 0.128069 | 0.044110 |
| 0.7 | 0.116442 | 0.091261 | 0.121514 | 0.071782 | 0.130044 | 0.044193 |
| 0.8 | 0.117301 | 0.088592 | 0.123033 | 0.069849 | 0.132843 | 0.043152 |
| 0.9 | 0.117181 | 0.084742 | 0.124136 | 0.066839 | 0.135784 | 0.041884 |
| 0.95 | 0.115848 | 0.082697 | 0.123910 | 0.064981 | 0.136616 | 0.042938 |
| 0.99 | 0.114478 | 0.082285 | 0.123197 | 0.064010 | 0.134994 | 0.040595 |
| 0.995 | 0.114388 | 0.082240 | 0.123144 | 0.063928 | 0.136560 | 0.040180 |

| \( n \) | \( \mu = 0.3 \) | \( \mu = 0.1 \) |
| 1 | 0.330086 - 0.267718 i | 0.325510 - 0.288297 i |
| 2 | 0.292386 - 0.489967 i | 0.296315 - 0.501634 i |
| 3 | 0.266009 - 0.720273 i | 0.270472 - 0.726246 i |
| 4 | 0.245565 - 0.952427 i | 0.249182 - 0.955699 i |
| 5 | 0.228436 - 1.184966 i | 0.231248 - 1.186916 i |
| 10 | 0.164599 - 2.343964 i | 0.165873 - 2.344194 i |
| 15 | 0.117999 - 3.499102 i | 0.118599 - 3.499165 i |

Table I: Values of the quasinormal frequencies for the fundamental mode \( n = 0, \ell = m = 0 \) for different values of mass \( \mu \), and rotation \( a \).

Table II: QNMs for \( \mu = 0.3 \) and \( \mu = 0.1, a = 0.5, \ell = m = 1 \).
definitely positive everywhere outside black hole, while for Kerr solution, the potential depends upon complex frequency and such a straightforward analysis is impossible.

Now we are in position to state that the high damping asymptotic for quasinormal modes of massive scalar field is the same as for massless one, i.e. the real part increases with \( m \) according to the numerical law described in [20].

To exclude possible instability we should search for QNMs for different values of \( \ell \) and \( m \). As one can see from representative plots Fig. 1, at large \( \ell \) the imaginary part of the QN frequencies remains bounded and, thereby, does not show any tendency to instability. The dependence of the real oscillation frequency on the azimuthal number \( m \) can be found on Fig. 3.

After careful investigating of region of small values \( \mu M \), we conclude that there are no unstable modes. As to the case of intermediate \( \mu M \), our extensive search of quasinormal modes implies that there are no unstable modes at least as far as \( \mu M \) is not much larger than 1. We certainly cannot find quasinormal modes for any large value of \( \mu \). This happens because the numerical procedure used here requires slow changing of \( \mu \), when searching for new modes, in order not to jump occasionally into another overtone. Thereby, considerably large values of \( \mu \) can be reached only for very long time of computation.

| \( \ell = 1; m = 0 \) | \( \mu = 0.1 \) | \( \mu = 0.2 \) | \( \mu = 0.3 \) |
|-----------------|-----------------|-----------------|-----------------|
| \( a \) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) | Re(\( \omega_0 \)) |
| 0.0 | 0.297416 | 0.094957 | 0.310957 | 0.086593 | 0.333777 | 0.071658 | 0.333924 | 0.071646 |
| 0.1 | 0.297602 | 0.094884 | 0.311128 | 0.086544 | 0.333924 | 0.071646 | 0.334366 | 0.071608 |
| 0.2 | 0.298164 | 0.094661 | 0.311646 | 0.086392 | 0.335116 | 0.044905 | 0.336190 | 0.071408 |
| 0.3 | 0.299116 | 0.094273 | 0.312523 | 0.086124 | 0.335116 | 0.044905 | 0.336190 | 0.071408 |
| 0.4 | 0.300478 | 0.093692 | 0.313780 | 0.085717 | 0.336190 | 0.071408 | 0.336190 | 0.071408 |
| 0.5 | 0.302285 | 0.092873 | 0.315446 | 0.085132 | 0.337622 | 0.711202 | 0.339444 | 0.070869 |
| 0.6 | 0.304579 | 0.091742 | 0.317564 | 0.084308 | 0.339444 | 0.070869 | 0.339444 | 0.070869 |
| 0.7 | 0.307414 | 0.090182 | 0.320185 | 0.083141 | 0.341707 | 0.070335 | 0.341707 | 0.070335 |
| 0.8 | 0.310836 | 0.087989 | 0.323557 | 0.081461 | 0.344645 | 0.069472 | 0.344645 | 0.069472 |
| 0.9 | 0.314815 | 0.084810 | 0.327070 | 0.078967 | 0.347735 | 0.068059 | 0.347735 | 0.068059 |
| 0.95 | 0.316919 | 0.082696 | 0.329061 | 0.077284 | 0.347735 | 0.068059 | 0.347735 | 0.068059 |
| 0.99 | 0.318576 | 0.080718 | 0.351004 | 0.066054 | 0.351004 | 0.066054 | 0.351004 | 0.066054 |
| 0.995 | 0.318779 | 0.080453 | 0.351189 | 0.065918 | 0.351189 | 0.065918 | 0.351189 | 0.065918 |

IV. CONCLUSION

In the present paper we have considered the massive scalar field perturbations for the Kerr black holes.
different values of quasinormal modes for values of mass $\mu M$ not much larger than 1. In this regime all found low laying modes are damped.

We have shown that similar to the Schwarzschild case, the massive term does not contribute into the high damping part of the spectrum. Therefore we can conclude that for not large values of $\mu M$ there is no instability for massive scalar field under the quasinormal mode boundary conditions. From the inequality (1) it follows that for a given black hole mass $M$ and rotation $a$, for any large $\mu$, there is sufficiently large $m$, so that inequality is not performed. Therefore we could have instability for sufficiently large values of $m$, even for large $\mu$. Yet, we have shown in Fig. 1, that QN spectrum for larger values of $m$ does not show any tendency to instability, because the $Im\omega$ for increasing values of $m$ remains bounded within some region of negative values corresponding to damping. Therefore inequality (1) found in [11] means stability for fields with sufficiently large values of $\mu$.

In addition, we have shown that the quasi-resonances, which exist for the Schwarzschild metric, exist also for Kerr black holes.

### Table IV: Values of the quasinormal frequencies for the fundamental mode $n = 0$, $\ell = 1$, $m = 1$ for different values of mass $\mu$, and rotation $a$.

| $\ell = 1; m = 1$ | $\mu = 0$ | $\mu = 0.1$ | $\mu = 0.2$ |
|-------------------|-------------|-------------|-------------|
| $a$               | $Re(\omega_0)$ | $-Im(\omega_0)$ | $Re(\omega_0)$ | $-Im(\omega_0)$ | $Re(\omega_0)$ | $-Im(\omega_0)$ |
| 0.0               | 0.292936     | 0.097660     | 0.297416     | 0.094957     | 0.310957     | 0.086593     |
| 0.1               | 0.301045     | 0.097547     | 0.305329     | 0.095029     | 0.318274     | 0.087228     |
| 0.2               | 0.310043     | 0.097245     | 0.314119     | 0.094920     | 0.326433     | 0.087609     |
| 0.3               | 0.320126     | 0.096691     | 0.323981     | 0.094569     | 0.335621     | 0.087979     |
| 0.4               | 0.331567     | 0.095792     | 0.335181     | 0.093883     | 0.346905     | 0.087950     |
| 0.5               | 0.344753     | 0.094395     | 0.348105     | 0.092714     | 0.358230     | 0.087478     |
| 0.6               | 0.360285     | 0.092243     | 0.363345     | 0.090805     | 0.372594     | 0.086320     |
| 0.7               | 0.379159     | 0.088848     | 0.381888     | 0.087678     | 0.390141     | 0.084014     |
| 0.8               | 0.403273     | 0.083132     | 0.405606     | 0.082262     | 0.412675     | 0.079526     |
| 0.9               | 0.437234     | 0.071848     | 0.439045     | 0.071342     | 0.444549     | 0.076737     |
| 0.95              | 0.462261     | 0.060091     | 0.463691     | 0.059825     | 0.468050     | 0.058968     |
| 0.99              | 0.493423     | 0.056712     | 0.494284     | 0.056756     | 0.496939     | 0.053679     |
| 0.995             | 0.490424     | 0.073874     | 0.490872     | 0.073846     | 0.492223     | 0.073740     |

### Table V: Values of the quasinormal frequencies for higher overtones for $\mu M = 0.1$, $\ell = 1$, $m = 1$.

| $\ell = 1; m = 1$ | $n = 1$ | $n = 2$ | $n = 3$ |
|-------------------|-------------|-------------|-------------|
| $a$               | $Re(\omega_0)$ | $-Im(\omega_0)$ | $Re(\omega_0)$ | $-Im(\omega_0)$ | $Re(\omega_0)$ | $-Im(\omega_0)$ |
| 0.0               | 0.264689     | 0.302851     | 0.228650     | 0.538599     | 0.202558     | 0.787632     |
| 0.1               | 0.274260     | 0.301711     | 0.239156     | 0.534670     | 0.212898     | 0.780859     |
| 0.2               | 0.284872     | 0.299936     | 0.250838     | 0.529466     | 0.224537     | 0.772046     |
| 0.3               | 0.296738     | 0.297332     | 0.264079     | 0.522633     | 0.237749     | 0.760444     |
| 0.4               | 0.310140     | 0.293608     | 0.279072     | 0.513640     | 0.252894     | 0.745820     |
| 0.5               | 0.325510     | 0.288297     | 0.296315     | 0.501635     | 0.270471     | 0.726246     |
| 0.6               | 0.343441     | 0.280603     | 0.316481     | 0.485169     | 0.291244     | 0.699664     |
| 0.7               | 0.360932     | 0.269019     | 0.340644     | 0.461505     | 0.316511     | 0.661385     |
| 0.8               | 0.391790     | 0.250187     | 0.370780     | 0.424542     | 0.348980     | 0.603557     |
| 0.9               | 0.428112     | 0.213833     | 0.411782     | 0.355839     | 0.382103     | 0.544245     |
| 0.95              | 0.441334     | 0.287111     | 0.431664     | 0.390149     | 0.382103     | 0.544245     |
| 0.99              | 0.473604     | 0.275655     | 0.468124     | 0.395139     | 0.460787     | 0.576199     |
| 0.995             | 0.480613     | 0.250136     | 0.477664     | 0.381687     | 0.471222     | 0.604783     |

Figure 3: $Re\omega$ vs $Im\omega$ for $n = 1$, $a = 0.5$, $\ell = m = 0$ for different values of $\mu$. 
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