The Similarity of Dynamic Behavior between Different Chaotic Systems

Jizhao Liu,1,2 Xiangzi Zhang,2 Jing Lian,3 Yide Ma,4 Pengbin Chang,4 and Fangjun Huang1
1) School of Data Science and Computer Science, Sun Yat-sen University, No.132, East Outer Ring Road, Guangzhou, Guangdong, P. R. China.
2) School of Information Science and Technology, Jinan University, No.601, West Huangpu Avenue, Guangzhou, Guangdong, P. R. China.
3) School of Electronics and Information Engineering, Lanzhou Jiaotong University, 88 West Anning Road, Lanzhou, Gansu 730070, P. R. China.
4) School of Information Science and Engineering, Lanzhou University, Lanzhou, Gansu, China.

Chaos is associated with stochasticity, complex, irregular motion, etc. It has some peculiar properties such as ergodicity, highly initial value sensitivity, non-periodicity and long-term unpredictability. These pseudo-random features lead chaotic systems to enormous applications such as random number generator, image encryption and secure communication. In general, the concept of chaos is never associated with similarity. However, we found the chaotic systems belonging to one chaos family (OCF) have similar dynamic behavior, which is a novel characteristic of chaos.

In this work, three classical chaotic system family are studied, which are Lorenz family, Chua family and hyperbolic sine family. These systems contain different derived chaotic systems (Lorenz system, Chen system and Lü system), different order chaotic systems (Chua family and hyperbolic sine family), and different kinds of chaotic systems (chaos and hyper-chaos). Their PSPs demonstrate that there exist strong correlation in OCF. Moreover, we found that high order/dimensional chaotic systems will inherit all dynamic behavior of lower ones, and the similarity will decrease as the order/dimensional goes higher, which is analogous to genetic process in biology. All of these features are quantitatively evaluated by PPMCC and SSIM.

Keywords: chaos; dynamic behavior; similarity; genetic process

I. INTRODUCTION

Since Lorenz discovered chaos in a simple system of ordinary differential equations in 1959, a new field of science which has grown ever larger with each passing year has been unleashed.1,2 Over decades, chaos has been observed in nature (weather and climate,3 dynamics of satellites in the solar system,4,5 time evolution of the magnetic field of celestial bodies,6 and population growth in ecology7) and laboratory (electrical circuits,8 lasers,9 chemical reactions,10 fluid dynamics,11 mechanical systems,12 and magneto-mechanical devices).13 Chaotic behavior has also found numerous applications in electrical and engineering,12 information and communication technologies,13 biology and medicine.14 By now, many chaotic models have been developed and studied in great detail, but they continue to present surprises and raise questions. The completely features of chaos is not already known.

There is no universally accepted mathematical definition of chaos. Usually, chaos can often be qualitatively identified with some confidence by observing the strange attractor or chaotic sea in a state space plot or Poincaré section, or quantitatively identified by Lyapunov exponent.15 This is mainly due to the features and manifestations of chaos. In detail, the main features of a nonlinear dynamical system, exhibiting deterministic chaos for given values of the parameters, are the following:16

- Sensitive dependence on initial conditions - two nearby initial conditions on the attractor or in the chaotic sea is separated by a distance which grows exponentially in time, and leads to long-term unpredictability.

- Ergodicity - the trajectory winds around forever, never repeating, and a sufficient long trajectory will almost fill the square for most of the initial conditions.

- Strange attractors - a strange attractor contains an infinite number of points bounded within a definite region of the state space (an attractor with fractal dimension). Moreover, many dynamical systems have multiple coexisting attractors, which can be observed by changing initial conditions17.

These stochastic-like features lead chaotic systems to enormous cryptographic applications. In general, the concept of chaos is never associated with similarity, and there is no correlation among different chaotic systems (DCS). However, we found that the dynamic behaviors of DCS belonging to one chaos family (OCF) are similar, and thereby suggest there exist correlation among DCS of OCF. To illustrate this phenomenon, we have studied Lorenz family, Chua family, and hyperbolic sine family. First, their phase space plots (PSPs) are numerical calculated by 4th-order Runge-Kutta (RK4) method with fixed step size, then the related PSPs among DCS are selected. At last, Pearson correlation coefficient (PPMCC) and structural similarity (SSIM) are utilized to quantitatively evaluate the similarity of PSPs.

The rest of the paper is organized as follows: In Section 2, the similarity evaluation methods utilized in this paper, i.e., the Pearson correlation coefficient (PPMCC) and structural similarity (SSIM) are introduced. In Section 3, the PSPs of

1Electronic mail: liujizhao@mail.sysu.edu.cn
Lorenz family, Chua family, hyperbolic sine family are calculated, and the similarity among these families are quantitatively evaluated. Conclusions and future work are drawn in Section 5.

II. SIMILARITY CALCULATION METHOD

A. Pearson correlation coefficient

The Pearson correlation coefficient (PPMCC) is one of the most commonly used statistical tools to measure the degree of correlation between two data sets $X$ and $Y$:\footnote{PPMCC}

$$PPMCC = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}}$$  \hspace{1cm} (1)

Here $(x_i, y_i)$ are individual paired samples from the data sets $X$ and $Y$, and $n$ is the total number of pairs; $\bar{x}$ and $\bar{y}$ are the mean values of the samples in data sets $X$ and $Y$.

The degree of PPMCC always lies in the range of $-1$ to $1$. PPMCC $> 0$ implies the two variable have positive correlation, PPMCC $< 0$ means a negative correlation. When PPMCC is higher than 0.5 (or lower than -0.5) indicate a strong relationship. PPMCC = 0 implies no relationship.

B. Structural Similarity

The structural similarity (SSIM) index is used for measuring the similarity between two images. Basically, SSIM consists of three local comparison functions between two signal $x$ and $y$, namely luminance comparison, contrast comparison, and structure comparison, which are computed as follows:

$$I(x, y) = \frac{2\mu_x\mu_y + C_1}{\mu_x^2 + \mu_y^2 + C_1}$$
$$C(x, y) = \frac{2\sigma_x\sigma_y + C_2}{\sigma_x^2 + \sigma_y^2 + C_2}$$
$$S(x, y) = \frac{\sigma_{xy} + C_3}{\sigma_x\sigma_y + C_3}$$ \hspace{1cm} (2)

In Eq. (2), $\mu_x$ and $\mu_y$ are the sample means of $x$ and $y$, $\sigma_x$ and $\sigma_y$ are the sample standard deviations of $x$ and $y$, and $\sigma_{xy}$ is the sample correlation coefficient between $x$ and $y$. The constants $C_1$, $C_2$ and $C_3$ are used to stabilize the algorithm when the denominators approach to zero.

The general form of SSIM index is given by combining the three comparison functions:

$$SSIM = I(x, y)^\alpha \cdot C(x, y)^\beta \cdot S(x, y)^\gamma$$ \hspace{1cm} (3)

where $\alpha$, $\beta$ and $\gamma$ are parameters which define the relative importance of the three components.

Usually, $\alpha = \beta = \gamma = 1$. Therefore, SSIM can be rewritten as:

$$SSIM = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}$$ \hspace{1cm} (4)

III. THE SIMILARITY OF DYNAMIC BEHAVIOR AMONG DIFFERENT CHAOTIC SYSTEMS

In order to demonstrate the similarity of dynamic behavior among different chaotic systems, Chua family, Chen family and hyperbolic sine are taken as example in this paper. These is because these three chaotic families have been well studied which involve chaos and hyperchaos, different derived and different order/dimensional chaotic systems.

First, Chua family is studied, the PSPs indicate the dynamic behavior is similar, and thereby suggest there is strong correlation among this OCF. Then the hyperbolic sine family is studied, the resulting PSPs demonstrate that that high order/dimensional chaotic system inherit all dynamic behavior of lower ones, and the similarity will decrease as the order/dimensional goes higher, which is analogous to genetic process in biology. At last, Lorenz family is studied, the PSPs indicate that this phenomenon is widely exist in different derived chaotic systems (Lorenz system, Chen system and Lü system), different order chaotic systems, and different kinds of chaotic systems (chaos and hyper-chaos). All of these features have been quantitatively evaluated by PPMCC and SSIM.

A. Chua family case

The dimensionless state equations of canonical Chua’s system are as follows:\footnote{Chua family}

$$\dot{x} = \alpha(y - x - f(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = \beta z$$ \hspace{1cm} (5)

where

$$f(x) = m_1 x + \frac{1}{2} (m_0 - m_1) (|x+1| - |x-1|)$$ \hspace{1cm} (6)

and $\alpha = 10$, $\beta = 14.87$, $m_0 = -1.27$ and $m_1 = -0.68$.

Forth-order Chua’s system is as follows:\footnote{Chua family}

$$\dot{x} = \alpha(y - x - f(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = \beta z$$
$$\dot{u} = \gamma (z + ru)$$ \hspace{1cm} (7)

Here, $\alpha = 10$, $\beta = -14.87$, $\gamma = -0.0497$, $r = 27.3333$, $m_0 = -1.27$ and $m_1 = -0.68$. 
Fifth-order Chua’s system is described as follows:

\[
\begin{align*}
\dot{x} &= a(y - x - f(x)) \\
\dot{y} &= b(x - y + z) \\
\dot{z} &= cy \\
u &= d(z + r_1u) \\
\dot{v} &= e(u - r_2v)
\end{align*}
\] (8)

Here, \(a = 10\), \(b = 1\), \(c = -14.87\), \(d = -0.0497\), \(e = 66.7962\), \(r_1 = 27.3333\), \(r_2 = 0.05\), \(m_0 = -1.27\) and \(m_1 = -0.68\).

When the step size is set to be 0.001 and all initial conditions are set to be 0.1, one can calculate their PSPs as shown in Fig. 1. From Fig. 1 the corresponding PSPs are almost the same, which indicate the dynamic behavior are very similar to each other.

In order to quantitatively describe these features, PPMCC and SSIM are utilized to calculate the similarity of PSPs. The resulting scores are shown in Tab. I and Tab. II. From Tab. I and Tab. II some PPMCC and SSIM scores of corresponding PSPs are 1, which indicate there exist extreme strong correlation among DCS of Chua family.

**TABLE I. PPMCC score of N-dimensional Chua family**

| System | 3D system | 4D system | 5D system | 3D system | 4D system | 5D system | 3D system | 4D system | 5D system |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| x - y plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| y - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - y plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| y - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |

**TABLE II. SSIM score of N-dimensional Chua family**

| System | 3D system | 4D system | 5D system | 3D system | 4D system | 5D system | 3D system | 4D system | 5D system |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| x - y plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| y - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - y plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| x - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |
| y - z plane | 0.8776 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 | 0.8495 | 0.9531 | 0.9107 |

**B. Hyperbolic Sine family case**

Ref.23 introduced a class of simple chaotic systems with hyperbolic sine nonlinearity. With general nth-order ordinary differential equations (ODEs), any desirable order of hyperbolic sine chaotic systems could be constructed.

The general form is described by

\[
\begin{align*}
\dot{x}_1 &= x_2 - x_1 \\
\dot{x}_2 &= x_3 - x_2 \\
\vdots \\
\dot{x}_{n-3} &= x_{n-2} - x_{n-3} \\
\dot{x}_{n-2} &= x_{n-1} - x_{n-2} \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -x_n - f(x_{n-1}) - nx_{n-2} - nx_{n-3} - \ldots - \frac{1}{2n}x_1
\end{align*}
\] (9)

where \(n\) is the order of the system. In this system, the nonlinear function is \(f(x_{n-1})\), which is defined by \(f(x_{n-1}) = p\sinh(\psi x)\), where \(\sinh(\psi x) = \frac{1}{2}e^{\psi x} - e^{-\psi x}\), and \(p = 1.2 \times 10^{-6}\). According to this equations, third order hyperbolic sine chaotic system is given by:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -x_3 - p\sinh(\psi x_2) - 3x_1
\end{align*}
\] (10)

Fourth order hyperbolic sine chaotic system is given by:

\[
\begin{align*}
\dot{x}_1 &= x_2 - x_1 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_4 - p\sinh(\psi x_3) - 4x_2 - 0.125x_1
\end{align*}
\] (11)
Fifth order chaotic system is given by:

\[
\begin{aligned}
  x_1 &= x_2 - x_1 \\
  x_2 &= x_3 - x_2 \\
  x_3 &= x_4 \\
  x_4 &= x_5 \\
  x_5 &= -x_5 - \rho sinh(\psi x_4) - 5x_3 - 5x_2 - 0.1x_1
\end{aligned}
\]

And tenth order chaotic system is given by:

\[
\begin{aligned}
  x_1 &= x_2 - x_1 \\
  x_2 &= x_3 - x_2 \\
  x_3 &= x_4 - x_3 \\
  x_4 &= x_5 - x_4 \\
  x_5 &= x_6 - x_5 \\
  x_6 &= x_7 - x_6 \\
  x_7 &= x_8 - x_7 \\
  x_8 &= x_9 \\
  x_9 &= x_{10} \\
  x_{10} &= -x_{10} - \rho sinh(\psi x_9) - 10x_8 - 10x_7 - 10x_6 - 10x_5 - 10x_4 - 10x_3 - 10x_2 - 0.05x_1
\end{aligned}
\]

When the step size is set to be 0.001 and all initial conditions are set to be 0.1, one can calculate their PSPs as shown Fig. 2. From the Fig. 2 the corresponding PSPs are similar, and one can note that the PSPs of lower chaotic system (third order chaotic system) appear in high order chaotic system, which suggest that high order chaotic system inherit all dynamic behavior of lowers ones. However, this similarity is weaken as the orders goes higher.

**TABLE III.** PPMCC score of 3D~10D hyperbolic sine systems

| System                  | 3D system 3-4 plane | 3D system 3-5 plane | 3D system 3-3 plane | 3D system 5-4 plane | 3D system 5-5 plane |
|-------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 4D system 2-4 plane     | 0.9443              | 0.9484              | 0.2206              |                     |                     |
| 4D system 3-4 plane     | 0.2606              | 0.3299              | 0.2805              |                     |                     |
| 5D system 3-4 plane     | 0.3542              | 0.3786              | 0.4083              |                     |                     |
| 10D system 3-4 plane    | 0.5789              | 0.5263              | 0.4154              |                     |                     |
| 10D system 3-5 plane    | 0.4833              | 0.3865              |                     |                     |                     |

**TABLE IV.** SSIM score of 3D~10D hyperbolic sine systems

| System                  | 3D system 3-4 plane | 3D system 3-5 plane | 3D system 3-3 plane | 3D system 5-4 plane | 3D system 5-5 plane |
|-------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 4D system 2-4 plane     | 0.4562              | 0.7929              | 0.3971              |                     |                     |
| 4D system 3-4 plane     | 0.5183              | 0.5196              | 0.5864              |                     |                     |
| 5D system 3-4 plane     | 0.5163              | 0.5183              | 0.5864              |                     |                     |
| 10D system 3-4 plane    | 0.6224              | 0.7243              | 0.5796              |                     |                     |
| 10D system 3-5 plane    | 0.5361              | 0.5390              | 0.5883              |                     |                     |
| 10D system 5-4 plane    | 0.9407              | 0.4302              | 0.3947              |                     |                     |
| 10D system 5-5 plane    | 0.3062              | 0.5146              | 0.4500              |                     |                     |
| 10D system 5-5 plane    | 0.5142              | 0.5105              | 0.6351              |                     |                     |

To quantitatively describe this feature, PPMCC and SSIM are utilized to evaluate the similarity scores of PSPs. From Tab. III and IV the corresponding PSPs got higher scores than non-corresponding PSPs. However, these scores decrease as order goes higher, which suggested that, high order chaotic system inherit all dynamic behavior of lowers ones, and the correlation is weaken as the orders goes higher.

**FIG. 2.** (a) is the phase space plot of $x_1 - x_2$ plane of 3D hyperbolic sine system, (b) is the phase space plot of $x_1 - x_3$ plane of 3D hyperbolic sine system, (c) is the phase space plot of $x_2 - x_3$ plane of 3D hyperbolic sine system, (d) is the phase space plot of $x_2 - x_3$ plane of 4D hyperbolic sine system, (e) is the phase space plot of $x_2 - x_4$ plane of 4D hyperbolic sine system, (f) is the phase space plot of $x_3 - x_4$ plane of 4D hyperbolic sine system, (g) is the phase space plot of $x_3 - x_4$ plane of 5D hyperbolic sine system, (h) is the phase space plot of $x_3 - x_5$ plane of 5D hyperbolic sine system, (i) is the phase space plot of $x_3 - x_4$ plane of 5D hyperbolic sine system, (j) is the phase space plot of $x_3 - x_4$ plane of 10D hyperbolic sine system, (k) is the phase space plot of $x_3 - x_4$ plane of 10D hyperbolic sine system, (l) is the phase space plot of $x_3 - x_4$ plane of 10D hyperbolic sine system

**C. Lorenz family case**

1. **Lorenz canonical family**

The equation of canonical Lorenz system is given by:

\[
\begin{aligned}
  \dot{x} &= \sigma (y - x) \\
  \dot{y} &= -xz + rx - y \\
  \dot{z} &= xy - bz
\end{aligned}
\]

where $\sigma = 10$, $r = 28$, and $b = 8/3$.

In 1999, Chen found a similar but nonequivalent chaotic attractor, which is now known to be the dual of Lorenz system.
The standard form is as follows:\cite{35}:

\[
\begin{align*}
\dot{x} &= a(y-x) \\
\dot{y} &= (c-a)x - xz + cy \\
\dot{z} &= xy - bz
\end{align*}
\]  \hfill (15)

where \((a, b, c) = (35, 3, 28)\).

In 2002, Lü and Chen reported a new chaotic system which bridging the gap between Lorenz and Chen systems. The equations are given by:\cite{20}:

\[
\begin{align*}
\dot{x} &= a(y-x) \\
\dot{y} &= -xz + cy \\
\dot{z} &= xy - bz
\end{align*}
\]  \hfill (16)

Here, \(a = 36, b = 3\) and \(c = 20\).

\begin{table}[h]
\caption{PPMCC score of three dimensional Lorenz canonical family}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\text{Lorenz system} & \text{Lorenz system} & \text{Lorenz system} & \text{Chen system} & \text{Chen system} & \text{Lü system} & \text{Lü system} \\
\hline
x - y plane & x - y plane & x - y plane & x - z plane & x - z plane & x - plane & x - plane \\
\hline
1 & 0.2951 & 0.0521 & 0.6851 & 0.2002 & 0.1232 & 0.4965 \\
0.2951 & 1 & 0.3085 & 0.3046 & 0.4796 & 0.3957 & 0.2833 & 0.4295 \\
0.0521 & 0.3085 & 1 & 0.1545 & 0.6204 & 0.8109 & 0.0379 & 0.5877 & 0.7832 \\
0.6851 & 0.3046 & 0.1545 & 1 & 0.3098 & 0.2966 & 0.7108 & 0.2872 & 0.2126 \\
0.2002 & 0.4796 & 0.6204 & 0.3098 & 1 & 0.6555 & 0.2021 & 0.6126 & 0.6444 \\
0.1232 & 0.0379 & 0.1402 & 0.8109 & 0.2021 & 1 & 0.1402 & 0.5733 & 0.7003 \\
0.4965 & 0.5877 & 0.2872 & 0.0379 & 0.6555 & 0.1402 & 1 & 0.1669 & 0.0906 \\
0.193 & 0.6126 & 0.6444 & 0.7108 & 0.2872 & 0.1402 & 0.1669 & 1 & 0.7115 \\
0.0625 & 0.6444 & 0.5733 & 0.2872 & 0.0379 & 0.1402 & 0.1669 & 0.7115 & 1 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\caption{SSIM score of three dimensional Lorenz canonical family}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\text{Lorenz system} & \text{Lorenz system} & \text{Lorenz system} & \text{Chen system} & \text{Chen system} & \text{Lü system} & \text{Lü system} \\
\hline
x - y plane & x - y plane & x - y plane & x - z plane & x - z plane & x - plane & x - plane \\
\hline
1 & 0.6369 & 0.5902 & 0.7658 & 0.6074 & 0.5322 & 0.7482 & 0.5507 & 0.3001 \\
0.6369 & 1 & 0.6583 & 0.7664 & 0.6124 & 0.6515 & 0.6739 & 0.5723 & 0.5898 \\
0.5902 & 0.6583 & 1 & 0.6597 & 0.6212 & 0.5802 & 0.5345 & 0.6074 & 0.6187 \\
0.7658 & 0.7664 & 0.6597 & 1 & 0.6597 & 0.6199 & 0.6597 & 0.5345 & 0.6074 \\
0.6074 & 0.6199 & 0.6597 & 0.6597 & 1 & 0.6735 & 0.5776 & 0.6789 & 0.6444 \\
0.5522 & 0.6735 & 0.5776 & 0.6789 & 0.6597 & 1 & 0.1402 & 0.5917 & 0.7709 \\
0.7056 & 0.6515 & 0.5343 & 0.7635 & 0.5776 & 0.1402 & 1 & 0.5278 & 0.4761 \\
0.5507 & 0.6735 & 0.6597 & 0.7635 & 0.5776 & 0.5278 & 0.5278 & 1 & 0.5938 \\
0.5001 & 0.5723 & 0.6687 & 0.6597 & 0.5317 & 0.6747 & 0.4761 & 0.5938 & 1 \\
\hline
\end{tabular}
\end{table}

\section{High dimensional Lorenz family}

Stenflo derived a fourth order chaotic ordinary differential equations which is known as Lorenz-Stenflo System\cite{28}. The equations is as follows:

\[
\begin{align*}
\dot{x} &= \sigma(y-x) + sv \\
\dot{y} &= rx - xz - y \\
\dot{z} &= xy - bz \\
\dot{v} &= -x - sv
\end{align*}
\]  \hfill (17)

Here, \(\sigma = 10, s = 28, r = 28\) and \(b = 8/3\).

The 5D hyperchaotic Lorenz system is given by:\cite{28}:

\[
\begin{align*}
\dot{x} &= \sigma(y-x) + su \\
\dot{y} &= rx - xz - y - v \\
\dot{z} &= xy - bz \\
\dot{u} &= k_1 u - xz \\
\dot{v} &= k_2 y
\end{align*}
\]  \hfill (18)

where \(\sigma = 10, s = 1, r = 28, b = 8/3, k_1 = 1.36\) and \(k_2 = 64\).

When the step size is set to be 0.001 and all initial conditions are set to be 0.1, one can calculate their PSPs as shown Fig. 4. From Fig. 4, the corresponding PSPs are similar, which indicate there exist correlation among the corresponding dynamic behavior.

Tab. VII and Tab. VIII are the testing results of PPMCC and SSIM. One can note that, the similarity is also exits in...
FIG. 4. (a) is the phase space plot of $x - y$ plane of 3D Lorenz system, (b) is the phase space plot of $x - z$ plane of 3D Lorenz system, (c) is the phase space plot of $y - z$ plane of 3D Lorenz system, (d) is the phase space plot of $x - y$ plane of 4D Lorenz system, (e) is the phase space plot of $x - z$ plane of 4D Lorenz system, (f) is the phase space plot of $y - z$ plane of 4D Lorenz system, (g) is the phase space plot of $x - y$ plane of 5D Lorenz system, (h) is the phase space plot of $x - z$ plane of 5D Lorenz system, (i) is the phase space plot of $y - z$ plane of 5D Lorenz system.

### TABLE VII. PPMCC score of N-dimensional Lorenz family

| Dimension | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Plane     | x - y     | x - z     | y - z     | x - y     | x - z     | y - z     | x - y     | x - z     | y - z     |
| 3D system | 1         | 0.2990    | 0.0709    | 0.0131    | 0.1722    | 0.0126    | 0.7759    | 0.5444    | 0.1281    |
| 4D system | 0.2990    | 1         | 0.4505    | 0.2034    | 0.7873    | 0.7001    | 0.3678    | 0.8414    | 0.4616    |
| 5D system | 0.0709    | 0.4505    | 1         | 0.0054    | 0.4856    | 0.7878    | 0.1922    | 0.4120    | 0.6872    |

### TABLE VIII. SSIM score of N-dimensional Lorenz family

| Dimension | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system | 3D system |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Plane     | x - y     | x - z     | y - z     | x - y     | x - z     | y - z     | x - y     | x - z     | y - z     |
| 3D system | 1         | 0.6244    | 0.6244    | 0.7095    | 0.6244    | 0.7095    | 0.6244    | 0.7095    | 0.6244    |
| 4D system | 0.6244    | 1         | 0.8072    | 0.7982    | 0.8027    | 0.8181    | 0.7716    | 0.8257    | 0.7778    |
| 5D system | 0.6244    | 0.8072    | 1         | 0.8027    | 0.8181    | 0.7716    | 0.8257    | 0.7778    | 0.8181    |

### IV. CONCLUSION AND FUTURE WORK

Chaos is famous for its stochastic-like properties such as ergodicity, highly initial value sensitivity, non-periodicity and long-term unpredictability. These pseudo random features lead chaotic systems to enormous applications such as random number generator, image encryption and secure communication.

In this paper, we found the dynamic behavior among DCS belonging to OCF is analogous to genetic process in biology, that is: 1. The dynamic behavior is similar in OCF. 2. High order/dimensional chaotic systems will inherit all dynamic behavior of lowers ones, and the similarity will decrease as the orders/dimensional goes higher. 3. The dynamic behavior of different derived chaotic systems and different types of chaotic system are less similar than the dynamic behavior among different orders of chaotic system. This phenomenon is widely exist in derived chaotic systems (Chen system, Lü system and Lorenz system), different order of chaotic systems (Chua system and hyperbolic sine chaotic system), and different dimensional chaotic systems/hyper-chaotic systems (N-dimensional Lorenz family). These novel features has been quantitatively evaluated by PPMCC and SSIM.

There is some work which is worth to study in future.

1. The results need to verify by rigorous mathematical proof.
2. The method to evaluate the similarity among different chaotic systems should be improved.

### ACKNOWLEDGMENT

Conflict of Interest: The authors declare that they have no conflict of interest.

1. J. C. Sprott, *Elegant chaos: algebraically simple chaotic flows* (World Scientific, 2010).
2. H. Yuan, Z. Toth, M. Peña, and E. Kalnay, "Overview of weather and climate systems," Handbook of Hydrometeorological Ensemble Forecasting, 1–31 (2018).
3. C. Gilpin and M. W. Feldman, "Phase transition induces chaos in a predator-prey ecosystem with a dynamic fitness landscape," Physical Review D 97, 064029 (2018).
4. W. Gilpin and M. W. Feldman, "A phase transition induces chaos in a predator-prey ecosystem with a dynamic fitness landscape," Physical Review D 97, 064029 (2018).
5. G. Qi, "Energy cycle of brushless dc motor chaotic system," Applied Mathematical Modelling 51, 686–697 (2017).
6. Q. Zhao, H. Yin, and H. Zhu, “Simultaneous recognition of two channels of optical packet headers utilizing reservoir computing subject to mutual-coupling optoelectronic feedback,” Optik-International Journal for Light and Electron Optics 157, 951–956 (2018).
7. S. Vaidyanathan, “Adaptive integral sliding mode control of a chemical chaotic reactor system,” in *Applications of Sliding Mode Control in Science and Engineering* (Springer, 2017) pp. 371–392.
9. Z. Zhang and G. Chen, “Liquid mixing enhancement by chaotic perturbations in stirred tanks,” Chaos, Solitons & Fractals 36, 144–149 (2008).

10. Z. Song, K. Sun, and S. Ling, “Stabilization and synchronization for a mechanical system via adaptive sliding mode control,” ISA transactions 68, 353–366 (2017).

11. P. Malaji and S. F. Ali, “Analysis and experiment of magnetoe-mechanically coupled harvesters,” Mechanical Systems and Signal Processing 108, 304–316 (2018).

12. J. Liu, J. C. Sprott, S. Wang, and Y. Ma, “Simplest chaotic system with a hyperbolic sine and its applications in dcsk scheme,” IET Communications 12, 809–815 (2018).

13. Y. Wang, Z. Zhu, H. Yu, and B. Yang, “Risk analysis on multi-granular flow network for software integration testing,” IEEE Transactions on Circuits and Systems II: Express Briefs 65, 1059–1063 (2018).

14. D. S. Coffey, “Self-organization, complexity and chaos: the new biology for medicine,” Nature medicine 4, 882–885 (1998).

15. J. C. Sprott, Strange attractors: Creating patterns in chaos (M&T Books New York, 1993).

16. L. Kocarev and S. Lian, Chaos-based cryptography: theory, algorithms and applications, Vol. 354 (Springer Science & Business Media, 2011).

17. C. Li, F. Min, and C. Li, “Multiple coexisting attractors of the serial-parallel memristor-based chaotic system and its adaptable generalized synchronization,” Nonlinear Dynamics, 1–22 (2018).

18. H. Xu and Y. Deng, “Dependent evidence combination based on shearmann coefficient and pearson coefficient,” IEEE Access 6, 11634–11640 (2018).

19. Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, “Image quality assessment: from error visibility to structural similarity,” IEEE transactions on image processing 13, 600–612 (2004).

20. X.-G. Zhang, H.-T. Sun, J.-L. Zhao, J.-Z. Liu, Y.-D. Ma, T.-W. Han, et al., “Equivalent circuit in function and topology to chua’s circuit and the design methods of these circuits,” (2014).

21. X. Liu, J. Wang, and L. Huang, “Attractors of fourth-order chua’s circuit and chaos control,” International Journal of Bifurcation and Chaos 17, 2705–2722 (2007).

22. Y. Simin and L. Jinhu, “High order chua’s circuit and its fpga realization,” in Control Conference, 2007. CCC 2007. Chinese (IEEE, 2007) pp. 409–413.

23. J. Liu, J. Ma, J. Lian, P. Chang, and Y. Ma, “An approach for the generation of an nth-order chaotic system with hyperbolic sine,” Entropy 20, 230 (2018).

24. Z. Xingqiu, M. Yide, and S. Li, Nonlinear Circuit-Based Analysis and Design (Beijing:Higher Education Press, 2011).

25. G. Chen and T. Ueta, “Yet another chaotic attractor,” International Journal of Bifurcation and chaos 9, 1465–1466 (1999).

26. J. Lü and G. Chen, “A new chaotic attractor coined,” International Journal of Bifurcation and chaos 12, 659–661 (2002).

27. C. Zhou, C. Lai, and M. Yu, “Chaos, bifurcations and periodic orbits of the lorenz-stenflo system,” Physica Scripta 55, 394 (1997).

28. Q. Yang and C. Chen, “A 5d hyperchaotic system with three positive lyapunov exponents coined,” International Journal of Bifurcation and Chaos 23, 1350109 (2013).

29. J. Liu, G. Chen, D. Cheng, and S. Celikovsky, “Bridge the gap between the lorenz system and the chen system,” International Journal of Bifurcation and Chaos 12, 2917–2926 (2002).