Entanglement spectrum: Identification of the transition from vortex-liquid to vortex-lattice state in a weakly interacting rotating Bose-Einstein condensate

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We use entanglement to investigate the transition from vortex liquid phase to vortex lattice phase in weakly interacting rotating Bose-Einstein condensate (BEC). For the torus geometry, the ground state entanglement spectrum is analyzed to distinguish these two different phases. The low-lying part of ground state entanglement spectrum, as well as the behavior of its lowest level change clearly when the transition occurs. For the sphere geometry, the entanglement gap in the conformal limit (CL) is also studied. We also show that the decrease of entanglement between particles can be regarded as a signal of the transition.

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I. INTRODUCTION

Recently trapped rotating ultracold atoms have attracted considerable interest \cite{1,2}. When the rotating frequency is moderate, the formation of vortex lattice is observed in experiments. However, with the increase of rotating frequency, it is predicted that vortex lattice will melt and the system will enter into a regime of strongly correlated many-body physics, namely vortex liquid phase \cite{3,4}. The ground states in this liquid phase are the bosonic version of quantum Hall states, leading to the invalidity of mean field theory \cite{5,6}. The parameter controlling this quantum phase transition from vortex lattice phase to vortex liquid phase is the average filling factor \( \nu \equiv N/N_V \), where \( N \) is the number of bosons and \( N_V \) is the number of vortices. Exact diagonalization on the torus geometry provides an evidence that this transition occurs at the critical filling factor \( \nu_c \sim 6 \). For \( \nu > \nu_c \), the ground state is a vortex lattice while for \( \nu < \nu_c \), the ground state is a strongly correlated liquid \cite{4}.

The description of condensed matter phases using quantities such as entanglement entropy and fidelity borrowed from quantum information theory is another promising research field attracting great attention \cite{10}. It has been found that when probing topologically ordered states, topological entanglement entropy, which is the sub-leading term of the ground state entanglement entropy, can provide some information that cannot be obtained by conventional condensed matter methods \cite{11,12}. Quantum Hall states of two-dimensional electrons in magnetic field and two-dimensional bosons in rotating traps are very important topologically ordered states, whose topological entanglement entropies are calculated recently \cite{13,14}. Comparing with generally used entanglement entropy which is only a single number, the entanglement spectrum (ES) contains more information and is found very recently to be useful in probing whether a quantum state has topological order \cite{15}. In these two years, a series of papers focus on the ES of fermionic fractional quantum Hall states on the sphere geometry and torus geometry \cite{15,17}, the ES of topological insulators and superconductors \cite{18}, the ES of spin model such as Heisenberg model and Kitaev model \cite{19,21} and the complete definition of entanglement gap in ES \cite{22}. However, the ES of rotating BEC still lacks enough study especially for large filling factors beyond quantum Hall regime and we will study it in this work.

The ES is not only useful for systems of condensed matter physics. It is also important in quantum information theory. By majorization scheme of the ES, we can find deterministically whether a bipartite quantum state can be transformed to another by local quantum operations and classical communication while the entanglement entropy is generally not sufficient \cite{23}.

In this work, we analyze the ground state ES of rotating BEC to investigate the transition from vortex lattice phase to vortex liquid phase from an entanglement perspective. Exact diagonalization method is used to obtain the ground state on the torus and sphere geometry. Then the ES is extracted from the ground state. Comparing with Ref. \cite{4}, where the excitation energy is used to identify the phase transition, we just utilize ground state properties, namely ES to indicate it. For the torus geometry, when bosons interact through contact interaction, we find that with the increase of filling factor, the low-lying part of ground state ES as well as the behavior of its lowest level undergo a qualitative change. The critical filling factor where this change happens is just \( \nu_c \sim 6 \). This phenomenon strongly indicates that a transition of ground state indeed occurs at \( \nu_c \sim 6 \). This is consistent with the conclusion of Ref. \cite{4}. When we change the interaction to Coulomb potential, we find that the transition occurs at a little larger \( \nu_c \). For the sphere geometry and contact interaction, the evolution of the entanglement gap in CL with filling factors is studied. Finally, the entanglement between two bosons and other \( N - 2 \) bosons is also calculated. When \( \nu > \nu_c \), this entanglement decreases monotonically with \( \nu \), im-
Lanczos algorithm is used to diagonalize the interaction and find that there is a conserved quantity (mod \( \mathbb{Z} \)).

The lowest Landau level (LLL) wavefunction is approximated in statistical physics. A density operator can be related to a mean field regime.

Before we study the transition from vortex liquid to vortex solid, we want to show some ES data of typical bosonic quantum Hall states. We use two-body contact interaction \( \sum_{i<j} \delta(r_i - r_j) \) and three-body contact interaction \( \sum_{i<j<k} \delta(r_i - r_j) \delta(r_j - r_k) \) to obtain the exact Laughlin state at \( \nu = 1/2 \) and Pfaffian state at \( \nu = 1 \) respectively.

In this paper, we first study a rotating \( N \)-boson system on the torus with periods \( a \) and \( b \) in the \( x \) and \( y \) directions. Each boson has a mass \( m \). A consistent imposition of periodic boundary conditions requires that \( ab/\ell^2 = 2\pi N_V \), where \( \ell = \sqrt{h/(2\pi m \omega)} \) with \( \omega \) the rotating frequency and \( N_V \) the number of vortex. The interaction is weak enough, the lowest Landau level approximation is valid. The normalized single-particle lowest Landau level (LLL) wavefunction is

\[
\psi_j = \left( \frac{1}{4\pi a^{1/2}} \right)^{1/2} \sum_{n=-\infty}^{\infty} \exp \left[ i \left( \frac{2\pi j}{a} + \frac{nb}{\ell^2} \right) x \right] - \frac{1}{2\ell^2} \left( y + nb + \frac{2\pi j}{a} \right)^2 ,
\]

where \( j = 0, 1, 2, ..., N_V - 1 \). Because \( \psi_j \) is centered along the line \( y = -\frac{2\pi j}{\ell^2} \), the whole system can be divided into \( N_V \) localized orbits spatially. The interaction Hamiltonian \( V(r_1, r_2) \) after standard second quantization procedure becomes \( V = \sum_{k_1, k_2, k_3, k_4} V_{k_1, k_2, k_3, k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \).

On the torus, the matrix element \( V_{k_1, k_2, k_3, k_4} \) takes the form as

\[
V_{k_1, k_2, k_3, k_4} = \delta_{k_1, k_2, k_3, k_4} \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \delta_{k_1-t, k_2-s} \tilde{V} \left( \frac{2\pi s}{a}, \frac{2\pi t}{b} \right) \exp \left\{ -\frac{\ell^2}{2} \left( \frac{(2\pi s/a)^2}{2} + \left( \frac{2\pi t/b} \right)^2 \right) - 2\pi it \frac{k_1 - k_3}{N_V} \right\} ,
\]

where \( \delta' \) is the periodic Kronecker delta function with period \( N_V \) and \( \tilde{V}(q_x, q_y) \) is the Fourier transformation of interaction \( V(r) \) \([24, 25]\).

For fermion Laughlin state, this V-shape can be explained by two chiral edge theory for a properly small \( a \) depending on the system size \([17]\).
the Pfaffian state with $N = N_V = 4$ in $K = 0$ sector, the thin torus states are \((|0202\rangle \pm |2020\rangle)/\sqrt{2}\), so the reduced density operator of part $A$ is $\rho_A = \langle 02|02\rangle + |20\rangle\langle 20\rangle/2$, leading to two equal ES levels in $N_A = 2$ sector with $K_A = 0$ and 2 respectively. So the ES has a flat bottom with three degenerate levels in $\Delta K_A = 0, \pm 2$. When we enlarge $a$, more levels come down but the flat bottom with three degenerate levels is still there [Fig.1(b)]. However, this flat bottom can disappear if we choose another system size. For the Pfaffian state with $N = N_V = 6$ in $K = 0$ sector, the thin torus states are \((|020202\rangle \pm |202020\rangle)/\sqrt{2}\), so the reduced density operator of part $A$ is $\rho_A = \langle 0202|0202\rangle + |2020\rangle\langle 2020\rangle/2$, leading to two equal ES levels again. But they are in different $N_A$ sectors ($N_A = 2$ and $N_A = 4$ respectively). So, when plotting the ES in one $N_A$ sector, the flat bottom disappears and V-shape revives [Fig.1(d)]. At last, for the Pfaffian state with $N = N_V = 4$ in $K = 2$ sector, the thin torus state is $|1111\rangle$. So it is easy to see that only one ES level in the bottom in $N_A = 2$ sector, meaning a V-shape will appear when $a$ increases [Fig.1(e)].

In Ref. [4], a trial wavefunction for vortex liquid phase at $\nu = k/2$ with $k$ an integer is proposed:

$$
\Psi^k = \mathcal{S} \prod_{l<k}^{N/k} |z_i - z_j|^2 \prod_{l<m}^{N/k} (z_i - z_m)^2,
$$

where we express it in disk geometry and the Gaussian exponential factor is omitted. In every set $A_i$, there are $N/k$ bosons and $\mathcal{S}$ means the symmetrization over all possible partition of the $N$ bosons into $k$ sets. When $k = 1(2)$, this state is Laughlin (Pfaffian) state. We expect that the ES of incompressible quantum Hall states at other filling factors on the torus have similar shapes with Laughlin and Pfaffian states. Therefore if we see a V-shape or a flat bottom with three degenerate levels in the ES of one state, we can conjecture that this state has a close relationship with quantum Hall state.

Now we consider a system on the torus with the aspect ratio $a/b$ commensurate with a triangular lattice, which can help us to stabilize the vortex lattice. We suppose the interactions are contact interaction $V(\mathbf{r}) = \delta(\mathbf{r})$ and Coulomb interaction $V(\mathbf{r}) = 1/|\mathbf{r}|$, for which $\tilde{V}(q_x, q_y) \propto 1$ and $\tilde{V}(q_x, q_y) \propto (q_x^2 + q_y^2)^{-1/2}$ respectively. Because we will consider the ground state at large filling factors, the number of vortex cannot be too large. First we study the cases at integer filling factors. When $\nu$ is a small integer (e.g., $\nu \lesssim 10$ for $N_V = 4$), the ground state is unique and for our choice of aspect ratio it is in $K = 0$ sector. When $\nu$ is a large integer (deeply in vortex lattice phase, e.g., $\nu \gtrsim 10$ for $N_V = 4$), the ground state energies in different $K$ sectors become nearly the same. After checking we find all the ES of these nearly-degenerate ground states have qualitatively the same structure, so we still choose the ground state in $K = 0$ sector to extract the ES.

In Figs.2 and 3 we plot the ground state ES for $N_V = 4$ and $N_V = 6$ at integer filling factors $\nu = 1, 2, \ldots, 11, 12$ respectively. For $N_V = 6$, let us first consider contact interaction. One can see that the qualitative behavior of the low-lying part of ES clearly changes after $\nu = 6$, distinguishing the vortex lattice phase and vortex liquid phase. For $\nu \leq 6$, the ground state ES has a structure of V-shape, while for $\nu \geq 7$, this V-shape structure vanishes and the ground levels of ES in every $\Delta K_A$ sector have a tendency to become degenerate, implying a transition from quantum Hall region. If we change the interaction to Coulomb interaction, the structure of the ground state ES changes after a little larger filling factor $\nu = 9$. Similar phenomena occur for $N_V = 4$. The qualitative behavior of the low-lying part of the ES clearly changes after $\nu = 7$ and $\nu = 9$ for contact interaction and Coulomb interaction respectively. Before the transition, the shape of the ES is either a V-shape (at even filling factors) or a flat bottom with three degenerate levels (at odd filling factors). After the transition, the ground levels of ES in every $\Delta K_A$ sector have a tendency to become degenerate.

When the filling factor is a fraction $p/q$ where $q \neq 1$ and $p$ and $q$ are coprime, the ground state is at least $q$-fold degenerate. For our choice of aspect ratio, there is always one of these degenerate ground states in $K = 0$ sector. After calculating the ES of all degenerate ground states (e.g., for $q = 2$), we find that just like in the case of integer filling factors, with the increase of $\nu$, the ground levels of the ES in every $\Delta K_A$ sector have a tendency to become degenerate.

In Ref. [4], a ground energy gap of the Hamiltonian is defined which can indicate the transition from vortex liquid to vortex lattice (Fig.2 and Fig.4). One can see for $N_V = 6$ the ground energy gap vanishes after $\nu = 6$ and $\nu = 8$ for contact and Coulomb interaction respectively, which is consistent with the conclusion obtained by ground state ES.
FIG. 2: The ground state ES for \( N_V = 4 \) at integer filling factor \( \nu = 1, 2, ..., 11, 12 \) from left to right. The subsystem \( A \) consists of orbits 0 and 1 and the subsystem \( B \) consists of orbits 2 and 3. The \( N_A \) sector is chosen as the one where the lowest ES level (corresponding to the largest eigenvalue of \( \rho_A \)) locates. The aspect ratio is chosen as \( \sqrt{3}/2 \). The interaction can be either contact interaction (the first row) or Coulomb interaction (the second row).

FIG. 3: The ground state ES for \( N_V = 6 \) at integer filling factor \( \nu = 1, 2, ..., 11, 12 \) from left to right. The subsystem \( A \) consists of orbits 0, 1 and 2 and the subsystem \( B \) consists of orbits 3, 4 and 5. The \( N_A \) sector is chosen as the one where the lowest ES level (corresponding to the largest eigenvalue of \( \rho_A \)) locates. The aspect ratio is chosen as \( 1/\sqrt{3} \). The interaction can be either contact interaction (the first row) or Coulomb interaction (the second row).

Considering the critical point of a quantum phase transition can be indicated by the singular behavior of the ground energy (or its derivative) of the physical Hamiltonian and the ES corresponds to the spectrum of a pseudo-Hamiltonian \( \tilde{H} \), we hope to see the ground energy of ES can indicate the transition from vortex liquid to vortex lattice. We investigate the evolution of the lowest level \( E_{\text{en}} \) of ground state ES with filling factor \( \nu \). Here we take a system containing six vortices as an example. For contact interaction, from Fig. 3(a) we can see when \( \nu \leq 5 \), \( E_{\text{en}} \) increases slowly and oscillates obviously with \( \nu \). When \( \nu \geq 7 \), \( E_{\text{en}} \) also increases very slowly with \( \nu \).
unnormalization, the ground state is expressed as \( \tilde{\Psi} = \sum_{n_0, n_1, \ldots, n_{2S}} \langle \psi_{n_0, n_1, \ldots, n_{2S}} | n_0, n_1, \ldots, n_{2S} \rangle \), where \( | n_0, n_1, \ldots, n_{2S} \rangle \) is the new basis state that does not contain \( N_j \). The ES in CL is extracted from \( \tilde{\Psi} \). For an incompressible quantum Hall state, the ES in CL can be divided into two parts, one of which is the low-lying conformal field part and the other is the generic part caused by realistic interaction. There is an entanglement gap between two parts [18, 22]. If a state undergoes a transition from a quantum Hall state to a compressible state, this entanglement gap closes [16, 22].

Here we study the evolution of ground state ES in CL with the increase of filling factor on the sphere geometry (Fig. 4). When the filling factor is small (\( \nu = 1 \)), there is a clear entanglement gap separating the low-lying conformal field theory part from the generic part. However, when the filling factor becomes large (\( \nu = 3/2, 5/2, 3 \)), the gap is unclear and hard to identify. Therefore there are not many hints for us to say whether the ground state is an incompressible quantum Hall state or not for \( \nu \geq 3/2 \) in thermodynamic limit. Our conclusion obtained from the entanglement gap is consistent with the one obtained from the energy gap of the Hamiltonian [24], where the authors found for \( \nu \geq 3/2 \) the energy gap is very small showing no clear signs of convergence towards thermodynamic limit and the two-particle correlation function shows very strong oscillations and no hint of incompressibility. The possible reason is that the quantum Hall states at larger filling factors have very large correlation lengths which exceed our largest finite-size system. Therefore, the ES, like the spectrum of the Hamiltonian, can detect the quantum Hall states at small filling factors on the sphere. However, probably it is helpful to observe a possible transition from an incompressible quantum Hall state to a compressible state only in much larger systems.

**IV. PARTICLE ENTANGLEMENT**

We know that in vortex lattice phase, the Gross-Pitaevskii mean field theory is essentially correct. The many-body ground wavefunction in mean field theory is just a product of wavefunctions of every particle, meaning that in mean field theory there is no ground state entanglement between particles. But in vortex liquid phase, the ground state is strongly correlated. So it’s natural to ask whether particle entanglement can be an indicator of the transition from vortex liquid to vortex lattice. Here we consider the ground state particle entanglement for bosons and other \( N - 2 \) bosons, defined as \( S_2 = -\text{Tr}(\rho_2 \ln \rho_2) \). \( \rho_2 \) is the reduced density operator of two bosons, whose matrix element is \( (\rho_2)_{j,k} = \langle \phi_k| \hat{a}_j^\dagger \hat{a}_k | \psi_{\text{GS}} \rangle \), where \( | \psi_{\text{GS}} \rangle \) is ground state average. Considering the case \( N_V = 6 \) on the torus, one can see that \( S_2 \) decreases obviously after \( \nu = 6 \) for contact interaction and after \( \nu = 8 \) for Coulomb interaction.
FIG. 6: (color online) The ground state ES in CL on the sphere geometry for contact interaction with filling factors $\nu = 1, 3/2, 2, 5/2$ and $3$ from left to right. Here $N = (N_V + 2)\nu$. Subsystem $A$ consists of orbits from $0$ to $N_V/2 - 1$ and subsystem $B$ consists of orbits from $N_V/2$ to $N_V$. Unlike what we do on the torus geometry, we choose the sector $N_A = N/2$ for even $N$ [$N_A = (N - 1)/2$ for odd $N$] to plot ES when considering sphere geometry. (a) $\nu = 1$, $N = 14$, $N_V = 12$. (b) $\nu = 3/2$, $N = 18$, $N_V = 10$. (c) $\nu = 2$, $N = 20$, $N_V = 8$. (d) $\nu = 5/2$, $N = 25$, $N_V = 8$. (e) $\nu = 3$, $N = 30$, $N_V = 8$. The entanglement gap is clear at $\nu = 1$ (indicated by the red lines), but hard to identify for larger $\nu$.

FIG. 7: (color online) The ground state particle entanglement $S_2$ on the torus for both contact interaction (black cubic) and Coulomb interaction (red circle). Here $N_V = 6$ and the aspect ratio is $1/\sqrt{3}$.

In summary, we investigate the quantum phase transition from vortex liquid phase to vortex lattice phase in rotating BEC from an entanglement perspective. The ground state entanglement spectrum is a useful tool in detecting this transition. On the torus geometry its low-lying part has different structures in vortex liquid phase and vortex lattice phase. Its lowest level also behaves differently in these two phases. Therefore ground state ES distinguishes these two phases very well. Our results are consistent with the conclusion that for bosons interacting through contact potential, a quantum phase transition happens at $\nu_c \sim 6$. When considering Coulomb interaction, $\nu_c$ will become a little larger. On the sphere geometry, at $\nu \geq 3/2$, the entanglement gap of ES in CL is unclear, showing a need to investigate much larger systems if one wants to observe the melt of vortex lattice. Finally, we show that particle entanglement can also be a signal to indicate the transition.

In this work, we choose the aspect ratio as the one commensurate with triangular lattice. If we change the aspect ratio and the interaction, we can study the ES of other possible ground states, such as smectic state and vortex lattice state with other symmetry. Moreover, it’s also interesting to investigate the ES of the simplest non-Abelian bosonic Moore-Read state at $\nu = 1$. In a very recent paper [27], a new kind of ES is defined where spatial bipartition of the system is replaced by particle bipartition. We hope this new ES can also indicate the transition from vortex liquid to vortex lattice.

Current experiments are deep in the regime of vortex lattice, so it is still a big challenge to create strongly-correlated states in vortex liquid phase. Formation of these new states requires faster rotation and smaller particle number than most current experiments. However, some method to solve this problem has been proposed [28]. We hope the transition from vortex lattice to vortex liquid may be detected experimentally in the future.

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V. SUMMARY

In summary, we investigate the quantum phase transition from vortex liquid phase to vortex lattice phase in rotating BEC from an entanglement perspective. The ground state entanglement spectrum is a useful tool in detecting this transition. On the torus geometry its low-lying part has different structures in vortex liquid phase and vortex lattice phase. Its lowest level also behaves differently in these two phases. Therefore ground state ES distinguishes these two phases very well. Our results are consistent with the conclusion that for bosons interacting through contact potential, a quantum phase transition happens at $\nu_c \sim 6$. When considering Coulomb interaction, $\nu_c$ will become a little larger. On the sphere geometry, at $\nu \geq 3/2$, the entanglement gap of ES in CL is unclear, showing a need to investigate much larger systems if one wants to observe the melt of vortex lattice. Finally, we show that particle entanglement can also be a signal to indicate the transition.

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