Exotic Differentiable Structures
and
General Relativity

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Abstract

We review recent developments in differential topology with special concern for their possible significance to physical theories, especially general relativity. In particular we are concerned here with the discovery of the existence of non-standard ("fake" or "exotic") differentiable structures on topologically simple manifolds such as \(S^7\), \(R^4\) and \(S^3 \times R^1\). Because of the technical difficulties involved in the smooth case, we begin with an easily understood toy example looking at the role which the choice of complex structures plays in the formulation of two-dimensional vacuum electrostatics. We then briefly review the mathematical formalisms involved with differentiable structures on topological manifolds, diffeomorphisms and their significance for physics. We summarize the important work of Milnor, Freedman, Donaldson, and others in developing exotic differentiable structures on well known topological manifolds. Finally, we discuss some of the geometric implications of these results and propose some conjectures on possible physical implications of these new manifolds which have never before been considered as physical models.
1 Introduction

Recently there have been some significant breakthroughs in differential topology and global analysis of
manifolds which may very well have considerable influence for space-time models in physics. We refer
to investigations of the various possible differentiable structures that can be put on a given topological
manifold. A very natural, basic, but sometimes very difficult, question is the following:

**Question 1:** If one smooth manifold is homeomorphic to another, need it be diffeomorphic?

Thus, if two manifolds have the same topology, must they necessarily have the same (up to diffeomor-
phisms) differentiable structure? While this question is clearly a fundamental one for mathematics, it
must surely also have physical content, since the basic model of space-time is that of a smooth manifold,
and diffeomorphisms are generally regarded physically as re-coordinatizations, i.e., changing of (local)
reference frames. As it stands, Question 1 obviously is too general for much progress to be made, so
it is natural to restrict it to certain simple classes of manifolds, specifically spheres, $S^n$, and Euclidean
spaces, $\mathbb{R}^n$.

When restricted to spheres and restated as a conjecture,

**Conjecture 1:** Any manifold homeomorphic to $S^n$ is necessarily diffeomorphic to this
manifold with its standard differentiable structure.

For dimension $n = 1$ the question is easy to resolve, with the answer in the affirmative. For higher di-
dimensions, however, the problem becomes more difficult. A breakthrough was made in 1956 by Milnor[1]
who was able to provide a counter example by explicitly constructing manifolds, $\Sigma_7$, which are topo-
logical seven-dimensional spheres but which are not diffeomorphic to the standard one. This work led
to further classification results for such exotic differentiable structures on higher dimensional spheres,
reducing the problem to one of homotopy[2]. However, as of today, Conjecture 1 for four-spheres is still
an open question.

For Euclidean spaces, $\mathbb{R}^n$, the version would be

**Conjecture 2:** Any manifold homeomorphic to $\mathbb{R}^n$ is necessarily diffeomorphic to this
manifold with its standard differentiable structure.

At first glance, it would seem that the topological triviality of $\mathbb{R}^n$ would make this an easy matter
to settle. In fact, for $n = 1, 2, 3$, the question can be answered in the affirmative by straightforward
computational techniques, trivial for $n = 1$, but more arduous for $n = 2, 3$. Indeed, for $n \leq 3$, any
topological n- manifold can be given a differentiable structure, compatible with its topology, unique up
to diffeomorphism. On the other hand, for $n \geq 5$, Smale's h-cobordism theorem provides a basic tool
for settling the question in the affirmative, for $\mathbb{R}^n$. However, these results failed to give any insight for
the $n = 4$ special case. To summarize, the best that could be said until 1982 was

**Theorem 1:** Conjecture 2 is true for any $n \neq 4$.

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1 There are equally intriguing questions in the homotopy category, but for simplicity, we focus here only on the topological one.
2 We emphasize that we are not referring to merely different structures, but ones which are not diffeomorphic. These
points will be discussed in more detail in Section 2 and Section 3.
It is probably reasonable to say that most workers expected that a similar affirmative result would ultimately be obtained for \( n = 4 \). After all, the topology is trivial, so what reason would there be to suppose that any non-trivial structure could be imposed on this space, when the same cannot be done for any other dimension?

Thus, it came as quite a surprise when the work of Freedman and Donaldson building on earlier results of Casson, established the existence of an “exotic” (“fake” or “non-standard”) differentiable structure on the topological manifold \( \mathbb{R}^4 \). Shortly thereafter Gompf was able to show the existence of several \( \mathbb{R}^4 \), and then even an infinity of distinct (i.e. not diffeomorphic) structures on \( \mathbb{R}^4 \). Freedman and Taylor were even able to demonstrate the existence of a universal exotic \( \mathbb{R}^4 \) in which all others could be smoothly embedded. Some reviews of the subject are available.

The existence of such exotic structures is a strikingly counter-intuitive result. It means that although each of these manifolds is topologically equivalent to \( \mathbb{R}^4 \), there is no local coordinate patch structure in which the global topological coordinates, ordered sets of four numbers, are everywhere smooth. Other strange effects occurring in these manifolds will be discussed in Section 4 below.

The path to the discovery of such manifolds, which we denote generically by \( \mathbb{R}^4_x \), is much more circuitous and mathematically involved than for the \( \Sigma^4_x \). The bad news then is that following the argument in detail requires a great deal of mastery of many branches of mathematics. The good news, from our viewpoint, is that this wandering journey involves mathematical excursions touching on such strongly physics-based topics as Dirac spinors, moduli spaces of Yang-Mills instantons and even an intersection form, \( E_8 \), identical to the Cartan form for the exceptional group recently studied in superstring theory. In view of the exceptional role played by the physically significant dimension four, and of all of the input from physics-based tools to the mathematical developments, it is somewhat surprising that the impact of these mathematical discoveries on physics has not been more widely explored in the general physics literature. To date, mention of exotic structures in physics papers has been mainly confined to problems related to quantization of gravity or supergravity. For example, Witten discusses them in the context of supergravity and superstring theory in ten dimensions, although he argues that their effects cancel out for certain theories. Rohm continues the discussion of these structures as “topological defects” in the context of quantum gravity. Rohm also briefly mentions possible ramifications of exotic structures on the classical Einstein dynamics. Bugajaska pointed out that the existence of non-diffeomorphic copies of \( \mathbb{R}^4 \) affects the homotopy classification (kinks) of Lorentzian metrics on the manifold.

On the classical level, can these models really give rise to “new” physics, since they are merely new manifolds? It would certainly seem so, since the manifold idea underlies all physics. We might equally ask whether “new” topologies might lead to new physics. They certainly do, as evidenced by the impact of “wormhole” models, etc. Of course, it might be argued that there can really be no new physics here, since standard general relativity can be expressed in terms of arbitrary smooth manifolds, which necessarily includes these exotic structures. However, these exotic manifolds are new, and have never been explored before. They exist as physically distinct manifolds having the same topology and this interplay between smoothness and topology has not been explored in the context of physics before.

Historically, the progress of theoretical physics has been marked by increasingly more general relativity principles, involving increasingly weaker pre-assumptions about reality. Classical physics was based on Galilean relativity with its assumption of absolute time, Maxwell’s electromagnetism cum ether involved absolute rest. Einstein generalized these assumptions with his special and then general theories of relativity. The latter began with the questioning of the necessity of restricting physics to inertial reference frames (principle of general relativity), followed by the questioning of the need for flatness, i.e., trivial geometry. The consequence was the idea of “geometry as physics” with all of the attendant theoretical structure of general relativity. Later in the development of the theory further generalizations suggested themselves. For example non-trivial topologies, with closed surfaces not boundaries of volumes, internal symmetries, bundle and gauge theories, etc.

A common thread in this development is the decreasing set of unquestioned mathematical assumptions in the model. “Flat” is the easiest geometry, but does that mean that nature must use it? Must nature use topologically Euclidean space? Must fields be cross sections of trivial, product bundles? Corresponding to this decreasing set of assumptions is an increasing set of mathematical structures.
available to serve as physical fields, geometric, topological, and gauge.

But now, the surprising discovery of \( \mathbb{R}^4_\Theta \)'s implies that there exists an infinity of non-diffeomorphic, thus physically distinct manifolds, each with the simple topology of \( \mathbb{R}^4 \), but not one of which has yet been investigated as a physical model! An interesting exercise is to imagine what would have happened if Einstein had used one of these \( \mathbb{R}^4_\Theta \)'s in his early investigation of general, or even special, relativity. This conjecture is not realistic, of course, since even today, such manifolds have not been explicitly constructed in the sense of a coordinate patch presentation. Nevertheless, many of their properties are known, enough hopefully to begin an investigation of their possible impact on physical theories.

However, we do have an explicit construction of exotic structures provided by the Milnor spheres. These might well provide manageable models to investigate explicitly the influence of exotic smoothness structures on physical theories. For example, let \( \Sigma^7_\Theta \) denote any of the exotic differentiable versions of the seven-sphere, constructed by Milnor\[1\]. Standard \( S^7 \) is of course the Hopf bundle of Yang-Mills fields over compactified \( \mathbb{R}^4 \). The exotic versions are no longer principal \( SU(2) \) bundles, but rather associated bundles with group \( SO(4) \). This fact may well have significant physical implications. Also, \( \Sigma^7_\Theta \), may provide some understanding of the differential geometric restrictions inherent in exotic structures. For example, it is clear that no constant curvature complete metric can be put on the Milnor spheres. The obstruction to continuation of the differential equations expressing constant curvature can be explicitly analyzed in these cases, hopefully providing some insight into what may happen in attempts to continue Einstein metrics in \( \mathbb{R}^4_\Theta \).

Unfortunately, the differential topology involved in studying these questions is far from trivial, and mathematical complexities can often hide physical simplicities. Thus, we begin by studying an easily understood toy model involving complex structures and representations of the plane vacuum electrostatic equations. Certainly, we make no claim that in itself there is any new physics in this model. We only present it to serve as a readily accessible analog to what might occur in the case of differentiable structures.

## 2 Complex Structures as a Toy Model

Consider the two-dimensional physics defined by vacuum, plane electrostatic fields, fully defined by vector fields \( \mathbf{E}(x, y) \) described by component functions \( E_x(x, y) \) and \( E_y(x, y) \). The Maxwell vacuum electrostatic equations are

\[
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \tag{1}
\]

\[
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{2}
\]

These are just the Cauchy conditions for the real and (negative) imaginary parts of an analytic function of the complex variable \( z \equiv x + iy \). The most general solution to (1) and (2) can be obtained from the complex equation,

\[
E_x - i E_y = f(z), \tag{3}
\]

where \( f(z) \) is an arbitrary analytic function.

These facts are well-known and discussed in most introductory texts on electromagnetism. Apart from a few illustrative boundary value problems, however, they do not seem to lead to any significant physical consequences or further insights, probably because the introduction of a complex structure on the space model is possible only for two-dimensional problems. Certainly, it is clear that the physics is contained in (1) and (2), rather than in (3).

However, the problem of complex structures on \( \mathbb{R}^2 \) is relatively easy, compared to that of the differentiable structures on \( \mathbb{R}^4 \), so we can explicitly explore the relationship between the mathematical structure and its physical implications. Hence, for illustrative purposes only, let us assume that the true “physical” electrostatic vacuum equations are reduced to (3). We will then explore what, if any, physical consequences the choice of “structure”, complex in this case, has.
Recall that a complex structure, $CS$, on a two-dimensional manifold, $M$, is defined by covering $M$ with an atlas of charts, $U_i$, together with maps, $f_i$ taking $U_i$ (smoothly and invertibly) onto open balls in $\mathbb{R}^2$ identified with the complex plane $\mathbb{C}$ in the “standard” way, i.e.,

$$(x, y) \in \mathbb{R} \leftrightarrow z \equiv x + iy, \ z \in \mathbb{C}.$$  \hfill (4)

Furthermore, where defined, $f_i \circ f_j^{-1}$ must be analytic in $\mathbb{C}$ in the usual complex sense. The charts, $U_i$, are sometimes called “coordinate patches” and, for $p \in U_i \subset M$, the value $z_i \equiv f_i(p) \in \mathbb{C}$ is the “coordinate of $p$ relative to the patch $U_i$.” A complex valued function, $F : V \to \mathbb{C}$, for some neighborhood $V \subset M$, is “analytic”, or “holomorphic”, if it is complex analytic when expressed in the local coordinates, $z_i$, over the $U_i$ covering $V$, that is, $F \circ f_i^{-1}$ is analytic (where defined) in the usual sense on $\mathbb{C}$.

Two such structures on a given $M$, say $CS'$ given by $\{U'_i, f'_i\}$ and $CS$ given by $\{U_i, f_i\}$ are equivalent (biholomorphic) if and only if there exists a homeomorphism, $F$, of $M$ onto itself such that $f_i \circ F \circ f'_i^{-1}$ and $f'_i \circ F^{-1} \circ f_j^{-1}$ are both holomorphic where defined. Another way to express this is to say that the $CS$ expression of $F$ is analytic in terms of $CS'$ and vice versa. Note that it is not necessary that the $CS$ coordinates themselves be analytic in terms of $CS'$, but only when combined with a homeomorphism. Thus, let $U_1 = U'_1 = \mathbb{R}^2$, with $f_1(x, y) = x + iy$ and $f'_1 = x - iy$. Then clearly the primed coordinate is not analytic in terms of the unprimed one. However, these are equivalent complex structures since the homeomorphism, $F(x, y) = (x, -y)$ satisfies the above condition for equivalence. That is,

$$f_1 \circ F \circ f'_1^{-1} : z = x + iy \to (x, -y) \to (x, y) \to x + iy = z.$$  \hfill (5)

Thus, the physical content of $CS$ and $CS'$ is identical.

If $M$ is $\mathbb{R}^2$, the “standard” complex structure, $CS_0$, is defined by the minimal atlas consisting of only the single $U_1 = M$, and $f_1(x, y) = x + iy$. We can recast the physical theory of two-dimensional vacuum electrostatics by saying that the $x$ and $y$ components of the electric field must be the real and (negative) imaginary parts of an analytic function on $M$, as defined by the standard complex structure, $CS_0$. One consequence of this is that no non-constant vacuum electrostatic field can be bounded.

However, the standard complex structure is not unique! There is precisely one other inequivalent one. One presentation of this second structure, $CS_1$, can be defined by using some smooth diffeomorphism from $[0, \infty)$ onto $[1, 0)$, say $p$, with the property that $xp(x^2)$ is bounded. A simple example is provided by $p(x) = e^{-x}$. $CS_1$ is then defined by

$$(x, y) \to z_1 = p(x^2 + y^2)(x + iy) \in \mathbb{C}.$$  \hfill (6)

It is now easy to show that $CS_0$ is not equivalent to $CS_1$. In fact, if it were then there would exist a function, $F(x, y) = (F_x(x, y), F_y(x, y))$, of the plane onto itself such that $p(F_x(x, y)^2 + F_y(x, y)^2)(F_x(x, y) + i F_y(x, y))$ would be a global analytic function of $x + iy$ in the usual sense. Clearly however this cannot be since this function is non-constant, but bounded on the entire plane, violating a well known property of global analytic functions.

Now we can state the physical implications of the choice of structure, complex in this case: If the physical theory is expressed by the statement that the $x$ and $y$ components of the electrostatic vacuum two-dimensional field are real and (negative) imaginary parts of a function analytic relative to the chosen complex structure, then $CS_0$ and $CS_1$ lead to different fields, with physically measurable differences. In other words,

**Physical content of complex structure:** Experiment could distinguish $CS_0$ from $CS_1$.

However, experiment cannot distinguish $CS_0$ from $CS'$ described earlier, since these are biholomorphic.

We repeat that this discussion was intended to be illustrative rather than of likely physical significance itself. The description of electrostatic field theory in terms of analyticity requirements is certainly not the basis of a general physical theory. In fact, it could be argued that changing the complex structure
results in a changed metric and that the correct theory should include this metric. However, we believe that this model provides some motivation for investigating the possible physical significance of the choice of differentiable structures, whose role in all field theories is indisputable.

3 Differentiable structures and Manifolds

For convenience, we here review some of the basic definitions and facts about differential topology using the following notation and terminology.

- **Standard topological** $\mathbb{R}^n$ is defined as the set of points, $p$, each of which can be identified with an $n$-tuple of real numbers, $\{p^\alpha\}$. The topology is induced by the usual product topology of the real line. For $n = 4$, the range of $\alpha$ is $0, 1, 2, 3$.

- **Topological Manifold**: This is a topological space which is locally homeomorphic to $\mathbb{R}^n$. We assume all manifolds are Hausdorff. For the most part we will be concerned with the $n = 4$ case.

- **Differentiability**: In this paper, assumed to be $C^\infty$, i.e., continuous together with all derivatives. Smooth is a synonym for differentiable in this sense.

- **A smooth atlas of charts** on a topological space, $M$, is a covering of $M$ with open sets, $U_a$, (coordinate patches), together with maps, $x_a$ taking $U_a$ homeomorphically onto an open ball in $\mathbb{R}^n$. For each $a$, $x_a$ is in fact a $n$-tuple of real numbers, $\{x_a^\alpha\}$. Such a pair, $(U_a, x_a)$ is called a chart and the $\{x_a^\alpha(p)\}$ are the local coordinates of $p$ relative to the chart $U_a$. Moreover, where defined, $x_a \circ x_b^{-1}$ must be smooth in terms of the usual $\mathbb{R}^n$ sense.

- **A differentiable structure**, $\mathcal{D}(M)$, is a maximal smooth atlas, i.e., the set of all charts compatible with those of a given smooth atlas. In the following, we will often define a $\mathcal{D}$ by giving one atlas, without referring to the maximalization process which should be carried out to induce $\mathcal{D}$ from a single member.

- **A map between two manifolds** is smooth if it is smooth in the usual real variable sense when expressed in terms of local charts. A smoothly invertible smooth map onto is a diffeomorphism. Two manifolds are diffeomorphic if there is a diffeomorphism between them. Clearly a diffeomorphism is a homeomorphism, so diffeomorphic manifolds are topologically identical. It is easy to see, as in the example below, that different $\mathcal{D}$'s can be placed on a given manifold. The fundamental question of differential topology which concerns us in this paper is whether these different $\mathcal{D}$'s are actually diffeomorphic.

- $\Theta$ used as a subscript indicates an exotic, fake, or non-standard differentiable construction.

- $\mathbb{R}^4_\Theta$ is some exotic $\mathbb{R}^4$. Points of $\mathbb{R}^4_\Theta$ will again be labelled by $p$, but now $p_\alpha$ are globally defined continuous functions, but not globally smooth. $\mathbb{R}^4_\Theta$ has a $\mathcal{D}_\Theta$ with coordinate patches $(U_a, x_a)$. The functions $p_\alpha$ restricted to $U_a$ are continuous functions of $x_a^\alpha$, but cannot be smooth for all $a$, and this is true for any diffeomorphic copy of $\mathcal{D}_\Theta$.

- Finally, we point out that the mathematical notion of diffeomorphism is normally identified with the physical notion of equivalence under re-coordinatization. Thus, if two manifolds are diffeomorphic, they provide fully equivalent physical models, simply presented in different coordinates.

The concept of differentiable structure is key to the matter of this paper. However, it is a subject which can easily be misunderstood. In particular, a very natural confusion can arise over the distinction between the situation in which a given topological manifold has different, but diffeomorphic, differentiable structures and that in which the different structures are not diffeomorphic. This is of course fundamentally important to physical applications since the notion diffeomorphism is generally taken to signify physical equivalence associated with “re-coordinatization.”
A counter example can help clarify the somewhat elusive concept of equivalence and inequivalence of differentiable structures. Consider the real line, $\mathbb{R}^1$, replacing the matrix, $p$, with its single element, $p$, and $x$ with $x$. The standard structure, $\mathcal{D}_0$, is generated from the single global chart with $x = p$. Relative to $\mathcal{D}_0$, $f(p)$ is thus differentiable if and only if $f(x)$ is in the usual real variable sense. Suppose now we define a “new” differentiability structure, $\mathcal{D}_1$, by using another global chart with $u$ as the global coordinate, where $u = p^3$. Clearly this is acceptable since $p \to u$ is a homeomorphism of the manifold onto $\mathbb{R}^1$, and there is only one chart. It is easy to see that these are indeed different structures. If not, their union would also be an atlas. But this would require that the transition, $x \circ u^{-1}$ be smooth. However, this map takes $y \to y^{1/3}$, which is not smooth at the origin. Thus, $\mathcal{D}_0 \neq \mathcal{D}_1$.

However, these two structures are actually equivalent, since the homeomorphism $f : p \to p^3$ of $\mathbb{R}^1$ onto itself is a diffeomorphism of the first structure onto the second. To see this note that $f$ expressed in local charts becomes $u \circ f \circ x^{-1} : y \to (y^{1/3})^3 = y$, the trivial identity map on the coordinate space $\mathbb{R}^1$. In fact, it can be shown rather easily that any differentiable structure on $\mathbb{R}^1$ is equivalent to the standard one. Thus, there can be no new physical content to any other $\mathcal{D}$ on $\mathbb{R}^1$, nor for any $\mathbb{R}^n$ except for the exceptional case of $n = 4$.

## 4 Exotic Differentiable structures

The first breakthrough in the exploration of exotic differentiable structures came in 1956 when Milnor [1] was able to use an extension of the Hopf fibering of spheres [13] to construct an exotic seven-sphere, $\Sigma^7_{\mathcal{D}}$. Consider the $\mathbb{S}^3$ bundles over $\mathbb{S}^4$,

$$\mathbb{S}^3 \to M^7 \xrightarrow{p} \mathbb{S}^4$$

with the rotation group, $SO(4)$, acting on $\mathbb{S}^3$, as bundle group. A classification of such bundles is provided by $\pi_3(SO(4)) \approx \mathbb{Z} + \mathbb{Z}$, as discussed in §18 of [13]. This construction can be described in terms of the normal form for $M^7$ in which the base $\mathbb{S}^4$ is covered by two coordinate patches, say upper and lower hemispheres. The overlap is then $\mathbb{R}^3 \times \mathbb{S}^3$ which has $\mathbb{S}^3$ as a retract. Thus, the bundle transition functions are defined by their value on this subset, defining a map from $\mathbb{S}^3$ into $SO(4)$ and thus generating an element of $\pi_3(SO(4))$. The group action of $SO(4)$ on the fiber $\mathbb{S}^3$ can be conveniently described in the well-known quaternion form,

$$u \to u' = vu\overline{w},$$

where $u, v, w$ are all unit quaternions and $\overline{w}$ is quaternion conjugate of $w$. Thus $u \in \mathbb{S}^3$ and $(v, w) \in SU(2) \times SU(2) \approx Spin(4)$. Standard $\mathbb{S}^7$ is obtained from the element of $\pi_3(SO(4))$ generated by $(v, 1)$, so that the group action reduces to one $SU(2) \approx \mathbb{S}^3$ and the bundle is in fact an $SU(2)$ principal one. In fact, this is precisely the principle $SU(2)$ Yang-Mills bundle over compactified space-time, $\mathbb{S}^4$. For more details on classifying sphere bundles, see [13], §20 and, from the physics viewpoint, [14].

Milnor’s breakthrough in 1956 involved his proving that $M^7$ for the transition function map, an element of $\pi_3(SO(4))$, given by

$$u \to (u^h, \overline{u^j}) \in Spin(4),$$

with $h + j = 1$ and $h - j = k$, and $k^2 \equiv 1$ mod 7, is in fact exotic, i.e., homeomorphic to $\mathbb{S}^7$, but not diffeomorphic to it. Clearly, the constructive part is fairly easy, but the proof of the exotic nature of the resulting sphere is more involved, drawing from several important results in differential topology including the Thom bordism result, cohomology theory, Pontrjagin classes, etc. Later, Kervaire and Milnor [2] and others [14] expanded on these results, leading to a good understanding of the class of exotic spheres in dimensions seven and greater. Kervaire and Milnor classified the set of h-cobordism classes of smooth homotopy n-spheres, which can also be described as the set of diffeomorphism classes of differentiable structures on $\mathbb{S}^n$. Moreover, this latter set can also be identified with the n-th homotopy group of PL/O by smoothing theory for manifolds.
For lower dimensions, early work by Cerf established that if a smooth structure on $S^4$ is obtained by gluing two copies of the standard disk along their $S^3$ boundary by some orientation-preserving diffeomorphism, then this smooth structure is diffeomorphic to the standard one. Also, there is no known example of a smooth compact four-dimensional manifold whose underlying topological manifold admits only a finite number of distinct differentiable structures. On the other hand, bundle theory ensures that any compact topological n-manifold in all dimensions $n \geq 5$ can have at most a finite number of distinct differentiable structures.

Unfortunately, the path to $R^4$ is much less easy to describe. First, we recall that the intersection form of a compact oriented manifold without boundary, obtained by the Poincare duality pairing of homology classes in $H_{n-k}$ and $H_k$ can be simply represented in dimension $n = 4 = 2 + 2$ by a symmetric square matrix of determinant $\pm 1$. This form basically reflects the way in which pairs of oriented two-dimensional closed surfaces fill out the full (oriented) four-space at their intersection points. Physicists are perhaps more familiar with deRham cohomology involving exterior forms for which this intersection pairing is the volume integral of the exterior product of a pair of closed two-forms representing the individual cohomology classes, which again makes sense only in dimension four. Unfortunately, deRham cohomology necessarily involves real coefficients and is thus too coarse for our applications, which need integral homology theory. At any rate, this integral intersection form, $\omega$, plays a central role in classifying compact four manifolds. Whitehead used it to prove that one-connected closed $4$-manifolds are determined up to homotopy type by the isomorphism class of $\omega$. Later, Freedman proved that $\omega$ together with the Kirby-Siebenmann invariant classifies simply-connected closed $4$-manifolds up to homeomorphism. For our purposes, the important result was that there exists a topological four manifold $\big|E_8\big|$, having intersection form $\omega = E_8$, the Cartan matrix for the exceptional lie algebra of the same name. As it stands, Freedman’s work is in the topological category, and does not address smoothness questions. The theorem of Rohlin states that the signature of a closed connected oriented smooth $4$-manifold must be divisible by $16$, so that $\big|E_8\big|$ cannot exist as a smooth manifold since its signature is $8$. Next, Donaldson’s theorem provides the crucial (for our purposes) generalization of this result to establish that $\big|E_8 \oplus E_8\big|$ is not smoothable, even though its signature is $16$. The work of Donaldson is based on the moduli space of solutions to the $SU(2)$ Yang-Mills equations on a four-manifold, which first occur in physics literature.

Having established some algebraic machinery, the next step involves an algebraic variety, the Kummer surface, $K$, a real four-dimensional smooth manifold in $CP^3$. It is known that

$$ K = | -E_8 \oplus -E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |. \quad (10) $$

The last part of this intersection form is easily seen to be realizable by $3(S^2 \times S^2)$, which is smooth. Thus, Donaldson’s theorem implies that it is impossible to do smooth surgery on $K$ in just such a way as to excise the smooth $3(S^2 \times S^2)$, leaving a smooth (reversing orientation) $\big|E_8 \oplus E_8\big|$. In the following, we refer to these two parts as $V_1$ (smoothable) and $V_2$ (not smoothable) respectively, so smooth $K = V_1 \cup V_2$. In investigating the failure of this smooth surgery Freedman found the first fake $R^4$. Using a topological $S^3$ to separate $V_1$ from $V_2$, Donaldson’s result showed that this $S^3$ cannot be smoothly embedded, since otherwise $V_2$ would have a smooth structure. However, by further surgery, it is found that this dividing $S^3$ is also topologically embedded in a topological $R^4$ and actually includes a compact set in its interior. Thus we have

**Existence of exotic $R^4$:** This topological $R^4$ contains a compact set which cannot be contained in any smoothly embedded $S^3$. This surprising result then implies that this manifold is indeed an $R^4$ since in any diffeomorphic image of $R^4$ every compact set is included in the interior of a smooth sphere.

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3. We use the standard notation in which $|\omega|$ is a topological manifold having $\omega$ as intersection form.
Since then, there have been many developments, some of which are summarized in the book by Kirby[8]. Unfortunately, none of the uncountable infinity of $R^4_\Theta$'s has been presented in explicit atlas of charts form, so most of the properties can only be described indirectly, through existence or non-existence type of theorems.

For example, some information about differential geometry on such a manifold can be obtained, such as

**Theorem 1:** There can be no geodesically complete metric (of any signature) with non-positive sectional curvature on $R^4_\Theta$.

Proof: If there were such a metric, the Hadamard-Cartan theorem could be used to show that the exponential map would provide a diffeomorphism of the tangent space at a point onto $R^4_\Theta$. In particular, there can be no flat geodesically complete metric. For more discussion on exotic geometry, see [18]. Natural questions then arise concerning the nature of the obstructions to continuing the solutions to the differential equations expressing flatness in the natural exponential coordinates. In physics, obstructions to continuation of solutions are often of considerable significance, e.g., wormhole sources. However, up to now, such obstructions generally have been a result of either topology, (incompleteness caused by excision), or some sort of curvature singularity. Neither of these is present here. This problem is particularly interesting for those $R^4_\Theta$’s which cannot be smoothly embedded in standard $R^4$, which thus cannot be geodesically completed with a flat metric.

Another useful result is

**Theorem 2:** There exists a smooth copy of each $R^4_\Theta$ for which the global $C^0$ coordinates are smooth in some neighborhood. That is, there exists a smooth copy, $R^4_\Theta = \{(p^\alpha)\}$, for which $p^\alpha \in C^\infty$ for $|p| < \epsilon$.

Proof: This may be obvious, but we seem to need a rather involved argument using the Annulus Theorem[19].

What this gives is a local smooth coordinate patch, on which standard differential geometry can be done, but which cannot be extended indefinitely. The obstruction should be physically interesting. Also, this theorem leads naturally to the following construction.

By puncturing $R^4_\Theta$, we get a “semi-exotic” cylinder, i.e., $R^4_\Theta - \{0\} \simeq R^4 \times_\Theta S^3$, where $\times_\Theta$ means topological but not smooth product. By “semi-exotic” we mean that the product is actually smooth for a semi-infinite extent of the first coordinate. This might be a very interesting cosmological model for physics, which after the big bang is $R^4 \times S^3$. Here we would run into an obstruction to continuing the smooth product structure at some finite time (first coordinate) for some unknown, but potentially very interesting, reason.

An even more interesting possibility to consider would involve localizing the “fakeness” in some sense. One version that comes to mind would happen if we could smoothly glue two such semi-exotic cylinders at their exotic ends. Of course a second gluing at their smooth ends would then give an exotic smoothness on the topological product, $S^1 \times S^3$. The existence of such an $S^1 \times_\Theta S^3$ is not known, as far as we know. We proceed to summarize a set of conjectures.

## 5 Conjectures

What are the possible physical implications of the existence of the exotic spaces? First, consider the $\Sigma^7_\Theta$, which can be explicitly constructed. Perhaps they could be considered as possible models for exotic Yang-Mills theory. Some $\Sigma^7_\Theta$ are $SU(2)$ bundles, but not principle ones, since their groups must be $SO(4)$. This would contrast with standard Yang-Mills structure[20] in which the total space is $S^7$ regarded as a principle $SU(2)$ bundle. Next, $\Sigma^7_\Theta$ can be used as toy space-time models, serving as the base manifolds for various geometric and other field theories. Various questions of physical interest can then be asked on these models and the answers compared to those obtained from standard $S^7$. For example, the non-existence of a constant curvature metric on $\Sigma^7_\Theta$ has already been thoroughly explored[13]. The analysis of such differential geometric problems on $\Sigma^7_\Theta$ as compared to $S^7$ should give some indication of the type of results that could come from physics on $R^4_\Theta$ as compared to that on standard $R^4$. 


There are also questions concerning the physical implications of doing general relativity on $\mathbb{R}^4_\Theta$. First, several questions of physical significance but of a more mathematical nature come to mind:

**Question:** Does there exist an $\mathbb{R}^4_\Theta$ which is standard outside a compact set?

Thus, in the above notation, there would be a copy for which the global continuous coordinates are smooth outside a sphere, i.e., $p^\alpha \in C^\infty$ for $|p| > k$. Clearly this would be an inversion of the result above. Theorem 2. Of course, this is likely to be a very difficult question since if such an $\mathbb{R}^4_\Theta$ were found, exotic structures for many 4-manifolds could be obtained from given smooth structures by using the $\mathbb{R}^4_\Theta$ as a chart in a new atlas. Perhaps an easier question, but one of even greater physical significance would be the following:

**Question:** Does there exist an $\mathbb{R}^4_\Theta$ for which the global continuous coordinates are smooth outside of a cylinder, i.e., $p^\alpha \in C^\infty$ for $p^0 > 0$ and $(p^1)^2 + (p^2)^2 + (p^3)^2 > k$?

Physically, such a structure could provide an interesting model for the world line of a particle. At spatial infinity, everything is standard, geometry can be flat, but this flat geometry cannot be continued into the world line at the origin. This is basically the way particle sources occur in general relativity. A variation on this, still of physical interest is

**Question:** Does there exist an $\mathbb{R}^4_\Theta$ for which the $p^\alpha \in C^\infty$ for $p^0 < 0$?

This could be of physical interest if $p^0$ is time, so that the model is standard for semi-infinite time, but cannot be continued this way indefinitely.

**Question:** What is the nature of Cauchy development in light of the existence of $\mathbb{R}^4_\Theta$? Specifically, does there exist a closed smooth $\mathbb{R}^3$ in $\mathbb{R}^4_\Theta$?

Actually, the entire problem of developing a manifold from a coordinate patch piece on which an Einstein metric is known, still has many unanswered aspects. Recall for example the evolution of our understanding of the appropriate manifold to support the (vacuum) Schwarzschild metric. Originally, the solution was expressed using $(t, r, \theta, \phi)$ coordinates as differentiable outside of the usual “coordinate singularities” well known for spherical coordinates. However, the Schwarzschild metric form itself in these coordinates exhibits another singularity on $r = 2m$, sometimes referred to as the “Schwarzschild singularity.” Later work, culminating in the Kruskal representation, showed that the Schwarzschild singularity could be regarded as merely another coordinate one in the same sense as is the $z$-axis for $(r, \theta, \phi)$. This example helps to illustrate that in general relativity our understanding of the physical significance of a particular metric often undergoes an evolution as various coordinate representations are chosen. In this process, the topology and differentiable structure of the underlying manifold may well change. In other words, as a practical matter, the study of the completion of a locally given metric often involves the construction of the global manifold structure in the process. Could any conceivable local Einstein metric lead to an $\mathbb{R}^4_\Theta$ by such a process?

Of course, local coordinate patch behavior is of great importance to physics, so another set of physically interesting questions would relate to the coordinate patch study of $\mathbb{R}^4_\Theta$. This may be too difficult of a task for present mathematical technology, but some questions may be reasonable. For example, can some $\mathbb{R}^4_\Theta$’s be covered by only a finite number of coordinate patches? If so, what is the minimum number? What are the intersection properties of the coordinate patch set which makes it non-standard?

A directly physical set of questions to be considered would stem from attempts to embed known solutions to the Einstein equations in $\mathbb{R}^4_\Theta$, then asking what sort of obstruction intervenes to prevent their indefinite, complete, continuation, in this space. A particular example would be the embedding of a standard homogeneous and isotropic cosmological metric in $\mathbb{R}^1 \times_\Theta S^3$ discussed above. Clearly the isotropy cannot be continued indefinitely. Why? What is the physical significance of this obstruction?

6 Conclusion

From the time of Einstein, the importance of separating physically invariant statements from those that depend on the choice of the observer has been generally recognized in principle, if not in practice. Mathematically, this means first taking care to identify those transformation groups in the mathematical model of a theory with the physical operation of performing an (idealized) coordinate transformation.
Only statements whose validity is invariant under these transformation groups can then be regarded as having absolute physical significance and testability. Of course, in practice, it is often convenient to restrict a particular argument to a subclass of these transformations, but, in principle, this restriction should be kept in mind. In most applications, these transformations must be smooth, so investigation into smoothness properties of given topological manifolds surely has physical significance.

Finally, it is indeed true that the existence of $\mathbb{R}^4$’s does not in any way change the local physics of general relativity or any other field theory. However, it has long been known that global questions can have profound effects on a physical theory. Until recently, physicists have thought of global matters almost exclusively as being of purely topological significance, whereas we now know that at least in the physically important case of $\mathbb{R}^4$, there are very exciting global questions related to differentiability structures, the way in which local physics is patched together smoothly to make it global. Certainly, the $\mathbb{R}^4$’s are essentially just “other” manifolds. However, there are an infinity of them which have never been remotely considered in the physical context of classical space-time physics on Einstein’s original model, $\mathbb{R}^4$. It would be surprising indeed if none of these had any conceivable physical significance.
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