Confluent $A$-hypergeometric functions and rapid decay homology cycles

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Abstract

We study confluent $A$-hypergeometric functions introduced by Adolphson [1]. In particular, we give their integral representations by using rapid decay homology cycles of Hien [12] and [13]. The method of toric compactifications introduced in [24] and [28] will be used to prove our main theorem. Moreover we apply it to obtain a formula for the asymptotic expansions at infinity of confluent $A$-hypergeometric functions.

1 Introduction

The theory of $A$-hypergeometric systems introduced by Gelfand-Kapranov-Zelevinsky [9] is a vast generalization of that of classical hypergeometric differential equations. As in the case of hypergeometric equations, the holomorphic solutions to their $A$-hypergeometric systems (i.e. the $A$-hypergeometric functions) admit $\Gamma$-series expansions ([9]) and integral representations ([10]). Moreover this theory has deep connections with many other fields of mathematics, such as toric varieties, projective duality, period integrals, mirror symmetry, enumerative algebraic geometry and combinatorics. Also from the viewpoint of the $D$-module theory (see [17] and [18] etc.), $A$-hypergeometric $D$-modules are very elegantly constructed in [10]. For the recent development of this subject see [37] and [38] etc. In [9], [10], [15] and [41] etc. the monodromies of their $A$-hypergeometric functions were studied. In [1] Adolphson generalized the hypergeometric systems of Gelfand-Kapranov-Zelevinsky [9] to the confluent (i.e. irregular) case and proved many important results. However the construction of the confluent $A$-hypergeometric $D$-modules is not functorial as in [9] and [10]. This leads us to some difficulties in obtaining the integral representations of their holomorphic solutions (i.e. the confluent $A$-hypergeometric functions). Indeed, in the confluent case almost nothing is known about their global properties. In this paper, we first construct Adolphson’s confluent $A$-hypergeometric $D$-modules functorially.
as in [10]. Note that recently the same problem was solved more completely in Saito [36] and Schulze-Walther [39, 40] by using commutative algebras. Our approach is based on sheaf-theoretical methods and totally different from theirs. Moreover we also construct an integral representation of the confluent $A$-hypergeometric functions by using the theory of rapid decay homology groups introduced recently in Hien [12] and [13]. Recall that $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$ is a finite subset of a lattice $\mathbb{Z}^n$ and Adolphson’s confluent $A$-hypergeometric system is defined on $\mathbb{C}^A = \mathbb{C}^\#A = \mathbb{C}^N$. Then our integral representation of its holomorphic solutions

\[ u(z) = \int_{\gamma_z} \exp\left(\sum_{j=1}^{N} z_j x^{a(j)}_j\right) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n \tag{1.1} \]

coincides with the one in Adolphson [1, Equation (2.6)], where $\gamma = \{\gamma^z\}$ is a family of real $n$-dimensional topological cycles $\gamma^z$ in the algebraic torus $T = (\mathbb{C}^*)^n$ on which the function $\exp(\sum_{j=1}^{N} z_j x^{a(j)}_j) x_1^{c_1-1} \cdots x_n^{c_n-1}$ decays rapidly at infinity. More precisely $\gamma^z$ is an element of Hien’s rapid decay homology group. See Sections 3 and 4 for the details. Adolphson used the formula (1.1) without giving any geometric condition on the cycles $\gamma^z$ nor proving the convergence of the integrals. In our Theorem 4.5 we could give a rigorous justification to Adolphson’s formula [1, Equation (2.6)] by using rapid decay homology cycles. This integral representation can be considered as a natural generalization of those for the classical Bessel and Airy functions etc. Note that in the case of hypergeometric functions associated to hyperplane arrangements the same problem was precisely studied by Kimura-Haraoka-Takano [22] etc. We hope that our geometric construction would be useful in the explicit study of Adolphson’s confluent $A$-hypergeometric functions. In the proof of Theorem 4.5 we shall use the method of toric compactifications introduced in [24] and [28] for the study of geometric monodromies of polynomial maps. Moreover we introduce Proposition 3.4 which enables us to calculate Hien’s rapid decay homologies by usual relative twisted homologies. By Proposition 3.4 and Lemmas 3.5 and 3.6 we can calculate the rapid decay homologies very explicitly in many cases. Let $\Delta \subset \mathbb{R}^n$ be the convex hull of $A \cup \{0\}$ in $\mathbb{R}^n$ and $h_z : T = (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ the Laurent polynomial on $T$ defined by $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$. Then in Section 5, assuming the condition $0 \in \text{Int}(\Delta)$ and using the twisted Morse theory we construct also a natural basis of the rapid decay homology group indexed by the critical points of $h_z$. Furthermore we apply it to obtain a precise formula for the asymptotic expansions at infinity of Adolphson’s confluent $A$-hypergeometric functions. The formula that we obtain in Theorem 5.6 will be very similar to that of the classical Bessel functions. Finally in Sections 6 and 7, removing the condition $0 \in \text{Int}(\Delta)$ we construct another natural basis of the rapid decay homology group. We thus partially solve the famous problem in Gelfand-Kapranov-Zelevinsky [10] of constructing a basis of the twisted homology group in their integral representation of non-confluent $A$-hypergeometric functions, in the more general case of confluent ones.

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2 Adolphson’s results

First of all, we recall the definition of the confluent $A$-hypergeometric systems introduced by Adolphson [1] and their important properties. In this paper, we essentially follow the terminology of [17] and [18] etc. Let $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$ be a finite subset of the lattice $\mathbb{Z}^n$. Assume that $A$ generates $\mathbb{Z}^n$ as in [9] and [10]. Following [1] we denote by $\Delta$ the convex hull of $A \cup \{0\}$ in $\mathbb{R}^n$. By definition $\Delta$ is an $n$-dimensional polytope. Let $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ be a parameter vector. Moreover consider the differential operators

$$A := (t^1 a(1), t^1 a(2), \ldots, t^1 a(N)) = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq N} \in M(n, N, \mathbb{Z}) \quad (2.1)$$

whose $j$-th column is $t^1 a(j)$. Then Adolphson’s confluent $A$-hypergeometric system on $X = \mathbb{C}^A = \mathbb{C}^N_z$ associated with the parameter vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ is

$$\left\{ \prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} \right\} u(z) = 0 \quad (\mu \in \text{Ker} A \cap \mathbb{Z}^N). \quad (2.3)$$

**Remark 2.1.** The above $A$-hypergeometric system was introduced first by Gelfand-Kapranov-Zelevinsky [9] under the homogenous condition on $A$ i.e. when there exists a linear functional $l : \mathbb{R}^n \to \mathbb{R}$ such that $l(\mathbb{Z}^n) = \mathbb{Z}$ and $A \subset l^{-1}(1)$. In this case, Hotta [10] proved that it is regular holonomic i.e. non-confluent.

**Remark 2.2.** In [1] Adolphson does not assume that $A$ generates $\mathbb{Z}^n$. However we need this condition to obtain a geometric construction of his confluent $A$-hypergeometric systems. Even when $A$ does not generate $\mathbb{Z}^n$, by a suitable linear coordinate change of $\mathbb{R}^n$ we can get an equivalent system for $A' \subset \mathbb{Z}^n$ and $c' \in \mathbb{C}^n$ such that $A'$ generates $\mathbb{Z}^n$. Namely our condition is not restrictive at all.

Let $D(X)$ be the Weyl algebra over $X$ and consider the differential operators

$$Z_{i,c} := \sum_{j=1}^N a_{i,j} z_j \frac{\partial}{\partial z_j} + c_i \quad (1 \leq i \leq n), \quad (2.4)$$

$$\Box_\mu := \prod_{\mu_j > 0} \left( \frac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left( \frac{\partial}{\partial z_j} \right)^{-\mu_j} \quad (\mu \in \text{Ker} A \cap \mathbb{Z}^N) \quad (2.5)$$

in it. Then the above system is naturally identified with the left $D(X)$-module

$$M_{A,c} = D(X) / \left( \sum_{1 \leq i \leq n} D(X) Z_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^N} D(X) \Box_\mu \right). \quad (2.6)$$

Let $\mathcal{D}_X$ be the sheaf of differential operators over the “algebraic variety” $X$ and define a coherent $\mathcal{D}_X$-module by

$$\mathcal{M}_{A,c} = \mathcal{D}_X / \left( \sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X \Box_\mu \right). \quad (2.7)$$
Then $\mathcal{M}_{A,c}$ is the localization of the left $D(X)$-module $M_{A,c}$ (see [17] Proposition 1.4.4 (ii) etc.). Adolphson [1] proved that $M_{A,c}$ is holonomic. In fact, he proved the following more precise result.

**Definition 2.3.** (Adolphson [1, page 274], see also [32] etc.) For $z \in X = \mathbb{C}^A$ we say that the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_{j}x^{a(j)}$ is non-degenerate if for any face $\Gamma$ of $\Delta$ not containing the origin $0 \in \mathbb{R}^n$ we have

$$\left\{ x \in T = (\mathbb{C}^*)^n \mid \frac{\partial h^\Gamma_z}{\partial x_1}(x) = \cdots = \frac{\partial h^\Gamma_z}{\partial x_n}(x) = 0 \right\} = \emptyset,$$

where we set $h^\Gamma_z(x) = \sum_{j:a(j) \in \Gamma} z_{j}x^{a(j)}$.

**Remark 2.4.** Since in the definition above we consider only faces $\Gamma \prec \Delta$ such that $\dim \Gamma \leq n - 1$ and $h_z$ are quasi-homogeneous, our condition (2.8) is equivalent to the weaker one in [1, page 274]. See Kouchnirenko [23, Definition 1.19] etc.

Let $\Omega \subset X$ be the Zariski open subset of $X$ consisting of $z \in X = \mathbb{C}^A$ such that the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_{j}x^{a(j)}$ is non-degenerate. Then Adolphson’s result in [1] Lemma 3.3 asserts that the holonomic $D_X$-module $M_{A,c}$ is an integrable connection on $\Omega$ (i.e. the characteristic variety of $M_{A,c}$ is contained in the zero section of the cotangent bundle $T^{\ast}\Omega$). Now let $X^{\mathbb{R}}$ (resp. $\Omega^{\mathbb{R}}$) be the underlying complex analytic manifold of $X$ (resp. $\Omega$) and consider the holomorphic solution complex $\text{Sol}_X(M_{A,c}) \in D^b(X^{\mathbb{R}})$ of $M_{A,c}$ defined by

$$\text{Sol}_X(M_{A,c}) = R\text{Hom}_{D_X^{\mathbb{R}}}(\langle (M_{A,c})^{\mathbb{R}} \rangle, \mathcal{O}_X^{\mathbb{R}})$$

(see [17] etc. for the details). Then by the above Adolphson’s result, $\text{Sol}_X(M_{A,c})$ is a local system on $\Omega^{\mathbb{R}}$. Moreover he proved the following remarkable result. Let $\text{Vol}_{\mathbb{Z}}(\Delta) \in \mathbb{Z}$ be the normalized $n$-dimensional volume of $\Delta$ i.e. the $n!$ times of the usual one $\text{Vol}(\Delta) \in \mathbb{Q}$ with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

**Theorem 2.5.** (Adolphson [1, Corollary 5.20]) Assume that the parameter vector $c \in \mathbb{C}^n$ is semi-nonresonant (for the definition see [1, page 284]). Then the rank of the local system $H^0\text{Sol}_X(M_{A,c})|_{\Omega^{\mathbb{R}}}$ on $\Omega^{\mathbb{R}}$ is equal to $\text{Vol}_{\mathbb{Z}}(\Delta) \in \mathbb{Z}$.

This is a generalization of the famous result of Gelfand-Kapranov-Zelevinsky in [50] to the confluent case. The sections of the local system $H^0\text{Sol}_X(M_{A,c})|_{\Omega^{\mathbb{R}}}$ are called $A$-hypergeometric functions (associated to the parameter $c \in \mathbb{C}^n$).

### 3 Hien’s rapid decay homologies

In this section, we review Hien’s theory of rapid decay homologies invented in [12] and [13]. For the theory of twisted homology groups we refer to Aomoto-Kita [2] and Pajitnov [33] etc. Let $\mathcal{E}$ be a smooth quasi-projective variety of dimension $n$ and $(\mathcal{E}, \nabla) (\nabla : \mathcal{E} \to \Omega_U^1 \otimes_{\mathcal{O}_U} \mathcal{E})$ an integrable connection on it. We consider $(\mathcal{E}, \nabla)$ as a left $D_U$-module and set

$$\text{DR}_U(\mathcal{E}) = \Omega^{\mathbb{R}}_U \otimes_{D^{\mathbb{R}}_U} \mathcal{E}^{\mathbb{R}} \simeq \Omega^{\mathbb{R}}_U \otimes_{D^{\mathbb{R}}_U} \mathcal{E}^{\mathbb{R}}[n].$$

Assume that $i : U \hookrightarrow Z$ is a smooth projective compactification of $U$ such that $D = Z \setminus U$ is a normal crossing divisor and the extension $i_* \mathcal{E}$ of $\mathcal{E}$ to $Z$ admits a good lattice in the
sense of Sabbah \cite{35} and Mochizuki \cite{30}. Such a good compactification of $U$ for $(E, \nabla)$ always exists by the fundamental theorem recently established by Mochizuki \cite{30}. Now let $\pi : \tilde{Z} \to Z^an$ be the real oriented blow-up of $Z^an$ in \cite{12, 13} and set $\tilde{D} = \pi^{-1}(D^an)$. Recall that $\pi$ induces an isomorphism $\tilde{Z} \setminus \tilde{D} \sim Z^an \setminus D^an$. More precisely, for each point $q \in D^an$ by taking a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of $q$ such that $q = (0, \ldots, 0)$ and $D^an = \{x_1 \cdots x_k = 0\}$ the morphism $\pi$ is explicitly given by

$$
(\{0, \varepsilon\} \times S^1)^k \times B(0; \varepsilon)^{n-k} \to B(0; \varepsilon)^k \times B(0; \varepsilon)^{n-k}$$

where we set $B(0; \varepsilon) = \{x \in \mathbb{C} \mid |x| < \varepsilon\}$ for $\varepsilon > 0$. For $p \geq 0$ and a subset $B \subset \tilde{Z}$ denote by $S_p(B)$ the $\mathbb{C}$-vector space generated by the piecewise smooth maps $c : \Delta^p \to B$ from the $p$-dimensional simplex $\Delta^p$. We denote by $C_{\tilde{Z}, \tilde{D}}^{-p}(\cdot, \mathcal{L})$ the sheaf on $\tilde{Z}$ associated to the presheaf

$$
V \mapsto S_p(\tilde{Z}, (\tilde{Z} \setminus V) \cup \tilde{D}) = S_p(\tilde{Z})/S_p((\tilde{Z} \setminus V) \cup \tilde{D}).
$$

Namely $C_{\tilde{Z}, \tilde{D}}^{-p}(\cdot, \mathcal{L})$ is the sheaf of the relative $p$-chains on the pair $(\tilde{Z}, \tilde{D})$. Now let $\mathcal{L} := H^{-p}DR_U(E) = \text{Ker}\{\nabla^an : E^an \to \Omega^1_{Z, an} \otimes \mathcal{O}_{Z^an} E^an\}$ be the sheaf of horizontal sections of the analytic connection $(E^an, \nabla^an)$ and $\iota : U^an \to \tilde{Z}$ the inclusion. Then $\iota_*\mathcal{L}$ is a local system on $\tilde{Z}$. We define the sheaf $C_{\tilde{Z}, \tilde{D}}^{-p}(\iota_*\mathcal{L})$ of the relative twisted $p$-chains on the pair $(\tilde{Z}, \tilde{D})$ with coefficients in $\iota_*\mathcal{L}$ by $C_{\tilde{Z}, \tilde{D}}^{-p}(\iota_*\mathcal{L}) = C_{\tilde{Z}, \tilde{D}}^{-p} \otimes_{\mathbb{C}} \iota_*\mathcal{L}$.

**Definition 3.1.** (Hien \cite{12} and \cite{13}) A section $\gamma = c \otimes s \in \Gamma(V; C_{\tilde{Z}, \tilde{D}}^{-p}(\iota_*\mathcal{L}))$ is called a rapid decay chain if for any point $q \in \text{c}(\Delta^p) \cap \tilde{D} \cap V$ the following condition holds:

In a local coordinate $(x_1, \ldots, x_n)$ on a neighborhood of $q$ in $Z$ such that $q = (0, \ldots, 0)$ and $D^an = \{x_1 \cdots x_k = 0\}$ by taking a local trivialization $(i_*\mathcal{E})^an \simeq \bigoplus_{i=1}^r \mathcal{O}_{Z^an}(\star D^an)e_i$ with respect to a basis $e_1, \ldots, e_r$ and setting $s = \sum_{i=1}^r f_i \cdot \iota_*i^{-1}e_i$ ($f_i \in \iota_*\mathcal{O}_{Z^an}$), for any $1 \leq i \leq r$ and $N = (N_1, \ldots, N_k) \in \mathbb{N}^k$ there exists $C_N > 0$ such that

$$
|f_i(x)| \leq C_N|x_1|^{N_1} \cdots |x_k|^{N_k}
$$

for any $x \in (c(\Delta^p) \setminus \tilde{D}) \cap V$ with small $|x_1|, \ldots, |x_k|$.

In particular, if $c(\Delta^p) \cap \tilde{D} \cap V = \emptyset$ we do not impose any condition on $s \in \iota_*\mathcal{L}$.

Note that this definition does not depend on the local coordinate $(x_1, \ldots, x_n)$ nor the local trivialization $(i_*\mathcal{E})^an \simeq \bigoplus_{i=1}^r \mathcal{O}_{Z^an}(\star D^an)e_i$. We denote by $C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -p}(\iota_*\mathcal{L})$ the subsheaf of $C_{\tilde{Z}, \tilde{D}}^{-p}(\iota_*\mathcal{L})$ consisting of rapid decay chains. According to Hien \cite{12} and \cite{13}, $C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -p}(\iota_*\mathcal{L})$ is a fine sheaf. Then we obtain a complex of fine sheaves on $\tilde{Z}$:

$$
C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -(p+1)}(\iota_*\mathcal{L}) \to C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -p}(\iota_*\mathcal{L}) \to C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -(p-1)}(\iota_*\mathcal{L}) \to \cdots
$$

**Definition 3.2.** (Hien \cite{12} and \cite{13}) For $p \in \mathbb{Z}$ we set

$$
H^p_{\text{rd}}(U; \mathcal{E}) := H^{-p}\Gamma(\tilde{Z}; C_{\tilde{Z}, \tilde{D}}^{\text{rd}, -(p+1)}(\iota_*\mathcal{L}))
$$

and call it the $p$-th rapid decay homology group associated to the integrable connection $\mathcal{E}$.
In [13] Hien proved that $H_p^{rd}(U; E)$ is isomorphic to the dual of the $p$-th algebraic
de Rham cohomology of the dual connection $E^* \otimes \nabla$ of $E$. For a different approach to this
problem see Kashiwara-Schapira [19]. In this paper, we use only some special integrable
connections $(E, \nabla)$ as the following example.

**Example 3.3.** Let $U \simeq \mathbb{C}^*$ and $E = \mathcal{O}_U \exp(-h(x))x^{-c}$, where $h(x) = \sum_{i \in \mathbb{Z}} a_i x^i$ ($a_i \in \mathbb{C}$)
is a Laurent polynomial and $c \in \mathbb{C}$. As usual we endow $E = \mathcal{O}_U \exp(-h(x))x^{-c}$ with the
connection $\nabla : E \rightarrow \Omega^1_U \otimes \mathcal{O}_U E$ defined by

$$\nabla \{ f \exp(-h(x))x^{-c} \} = df \otimes \exp(-h(x))x^{-c} - \left( dh + \frac{c}{x} dx \right) \otimes f \exp(-h(x))x^{-c} \quad (3.8)$$

for $f \in \mathcal{O}_U$. Then we have $L = H^{-1} \mathcal{D}R_U(E) \simeq \mathbb{C}_{U_{an}}\exp(h(x))x^c \subset \mathcal{O}_{U_{an}}$. In this case,
to define the rapid decay homology groups $H_p^{rd}(U; E)$ we consider (relative) twisted frames on
which the function $\exp(h(x))x^c$ decays rapidly at infinity.

If $E \simeq \mathcal{O}_U(\frac{1}{Z})$ and we have an isomorphism $L = H^{-n} \mathcal{D}R_U(E) \simeq \mathbb{C}_{U_{an}}g \subset \mathcal{O}_{U_{an}}$ for
a possibly multi-valued holomorphic function $g : U_{an} \rightarrow \mathbb{C}$ as the example above, we call $H_p^{rd}(U; E)$ the $p$-th rapid decay homology group associated to the function $g$. In the
special case where $g(x) = \exp(h(x))g_0(x)$ for a meromorphic function $h$ on $Z_{an}$ with poles in
$D_{an}$ and a (possibly multi-valued) function $g_0$ on $U_{an}$ such that at each point of $Z_{an}$ there
exists a local coordinate $x = (x_1, \ldots, x_n)$ satisfying $g_0(x) = x_1^{c_1} \cdots x_n^{c_n}$ ($c_i \in \mathbb{C}$) and
$D = \{ x_1 \cdots x_k = 0 \}$, we shall give a purely topological interpretation of $H_p^{rd}(U; E)$. Since
$Z$ is a good compactification of $U$ for $E$, the meromorphic function $h$ has no point of
indeterminacy on the whole $Z_{an}$ (see [13] Section 2.1 etc.). By $\iota : U_{an} \hookrightarrow \tilde{Z}$ we consider
$U_{an}$ as an open subset of $\tilde{Z}$ and set

$$P = \tilde{D} \cap \{ x \in U_{an} \mid \text{Re}h(x) \geq 0 \}. \quad (3.9)$$

Let $D = D_1 \cup \cdots \cup D_d$ be the irreducible decomposition of $D$. For $1 \leq i \leq d$ let $b_i \in \mathbb{Z}$ be
the order of the meromorphic function $h$ along $D_i$. If $b_i \geq 0$ we say that the irreducible
component $D_i$ is irrelevant. Namely along a relevant component $D_i$ the function $h$ has a
pole of order $-b_i > 0$. Denote by $D'$ the union of the irrelevant components of $D$. Then
we set $Q = \tilde{D} \setminus \{ P \cup \pi^{-1}(D')_{an} \}$. Note that $Q$ is an open subset of $\tilde{D}$ (i.e. the set of the
rapid decay directions of the function $g$ in $\tilde{D}$).

**Proposition 3.4.** In the situation as above, we have an isomorphism

$$H_p^{rd}(U; E) \simeq H_p(U_{an} \cup Q, Q; \iota_*(\mathbb{C}_{U_{an}}g_0)) \quad (3.10)$$

for any $p \in \mathbb{Z}$, where the right hand side is the $p$-th relative twisted homology group of the
pair $(U_{an} \cup Q, Q)$ with coefficients in the rank-one local system $\iota_*(\mathbb{C}_{U_{an}}g_0)$ on $\tilde{Z}$.

**Proof.** Since the function $\exp(h(x))$ is single-valued, we have an isomorphism $L \simeq \mathbb{C}_{U_{an}}g_0$.
First let us consider the case $n = 1$. Locally we may assume that $U = \mathbb{C}^*, Z = \mathbb{C}_x = \mathbb{C}^* \cup \{ 0 \}, D = \{ x = 0 \} \subset \mathbb{Z}$ and $h(x) = x^{-m}$ ($m > 0$). Let $\pi : \tilde{Z} \rightarrow Z_{an}$ be the
real oriented blow-up of $Z_{an}$ along $D_{an}$. In this case we have $\tilde{D} = \pi^{-1}(D_{an}) \simeq S^1$ and
$\tilde{Z} = \{ (r, e^{\sqrt{-1}\theta}) \mid r \geq 0 \simeq [0, \infty) \times S^1 \}$. For $1 \leq i \leq m$ and sufficiently small $\varepsilon > 0$ we set

$$Q_i = \{ e^{\sqrt{-1}\theta} \in \tilde{D} \simeq S^1 \mid \frac{(2i - \frac{3}{2})\pi}{m} - \varepsilon < \theta < \frac{(2i - \frac{1}{2})\pi}{m} + \varepsilon \}, \quad (3.11)$$
and \( Q^\varepsilon = \bigcup_{i=1}^m Q^\varepsilon_i \subset \tilde{D} \). Note that \( Q^\varepsilon \subset \tilde{D} \) contains all the rapid decay directions of \( g(x) = \exp(h(x))g_0(x) \) in \( \tilde{D} \). Now let us consider the two topological subspaces \( U^\text{an} \cup Q^\varepsilon \) and \( \tilde{D} \) of \( \tilde{Z} \). We patch them on their intersection \( Q^\varepsilon \) and construct a new topological space as follows. By identifying the points of \( Q^\varepsilon \subset U^\text{an} \cup Q^\varepsilon \) and those of \( Q^\varepsilon \subset \tilde{D} \) naturally, we obtain a quotient space \( \tilde{Z}^\varepsilon \) of the disjoint union \((U^\text{an} \cup Q^\varepsilon) \sqcup \tilde{D})\). Recall that \( \tilde{Z}^\varepsilon \) is endowed with the strongest topology for which the quotient map \((U^\text{an} \cup Q^\varepsilon) \sqcup \tilde{D} \to \tilde{Z}^\varepsilon \) is continuous. Note also that \( \tilde{D} \) is naturally identified with a close subspace of \( \tilde{Z}^\varepsilon \). We denote the local system on \( \tilde{Z}^\varepsilon \) naturally constructed from \( t_*\mathcal{L} \) by the same letter \( t_*\mathcal{L} \). For \( p \in \mathbb{Z} \) let \( S_p(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L}) \) be the \( \mathbb{C} \)-vector space of the twisted (piecewise smooth) relative \( p \)-chains on the pair \((\tilde{Z}^\varepsilon, \tilde{D})\) with coefficients in \( t_*\mathcal{L} \). Then by the definition of rapid decay chains, for any \( p \in \mathbb{Z} \) we obtain a natural morphism

\[
\Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \to S_p(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L}).
\] (3.12)

We can easily show that the chain map

\[
\Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \to S_{\ldots}(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L})
\] (3.13)

obtained in this way is a homotopy equivalence. Indeed, its homotopy inverse \( S_{\ldots}(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L}) \to \Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \) can be constructed by smooth deformations of chains in \( S_{\ldots}(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L}) \) in the angular direction \( \theta = \text{arg} x \). We can construct them by a smooth vector field on \( \tilde{Z} \). Hence we obtain an isomorphism

\[
H^{-p}\Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \simto H_p(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L})
\] (3.14)

for any \( p \in \mathbb{Z} \). Moreover by excision and homotopy, we have an isomorphism

\[
H_p(\tilde{Z}^\varepsilon, \tilde{D}; t_*\mathcal{L}) \simto H_p(U^\text{an} \cup Q^\varepsilon, Q^\varepsilon; t_*\mathcal{L})
\] (3.15)

for any \( p \in \mathbb{Z} \). Combining (3.14) with (3.15), we obtain an isomorphism

\[
H^{-p}\Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \simto H_p(U^\text{an} \cup Q^\varepsilon, Q^\varepsilon; t_*\mathcal{L})
\] (3.16)

for any \( p \in \mathbb{Z} \). Finally let us consider the case \( n \geq 2 \). First we assume that for some \( 1 \leq k \leq n \) we have \( Z = \mathbb{C}^n_x, D = \{x_1, \ldots, x_k = 0\} \subset \mathbb{Z}, U = \mathbb{Z}\setminus D \) and \( h(x) = x_1^{m_1} \cdots x_k^{m_k} \) (\( m_i > 0 \)). Let \( \pi : \tilde{Z} \to \mathbb{Z}^\text{an} \) be the real oriented blow-up of \( \mathbb{Z}^\text{an} \) along \( D^\text{an} \). In this case we have \( \tilde{Z} = \{(x_i, e^{-\sqrt{-1} \theta_i})_{i=1}^k, x_{k+1}, \ldots, x_n) \mid r_i \geq 0 \simeq (0, \infty) \times S^1)^k \times \mathbb{C}^{n-k} \). For sufficiently small \( \varepsilon > 0 \) we define an open subset \( Q^\varepsilon \subset \tilde{D} \) by

\[
\{(r, e^{-\sqrt{-1} \theta})_{i=1}^k, x_{k+1}, \ldots, x_n) \in Q^\varepsilon \quad \iff \quad \text{Re} e^{\sqrt{-1}(m_1 \theta_1 + \cdots + m_k \theta_k)} < \varepsilon |\text{Im} e^{\sqrt{-1}(m_1 \theta_1 + \cdots + m_k \theta_k)}|
\] (3.17)

for \( \{e^{-\sqrt{-1} \theta_i})_{i=1}^k, x_{k+1}, \ldots, x_n) \in \tilde{D} \). Then \( Q^\varepsilon \) contains all the rapid decay directions of \( g(x) = \exp(h(x))g_0(x) \) in \( \tilde{D} \). As in the case \( n = 1 \), by smooth deformations of chains and excision etc. we obtain an isomorphism

\[
H^{-p}\Gamma(\tilde{Z}; C^\text{rd,}^-_{\tilde{Z}, \tilde{D}}(t_*\mathcal{L})) \simto H_p(U^\text{an} \cup Q^\varepsilon, Q^\varepsilon; t_*\mathcal{L})
\] (3.19)

for any \( p \in \mathbb{Z} \). The general case can be proved similarly by patching local smooth deformations of chains (smooth vector fields) as above by a partition of unity. This completes the proof. \( \square \)
Lemma 3.5. In the situation as above, for \( q \in D^n \) let \( k \geq 0 \) be the number of the relevant irreducible components of \( D^n \) passing through \( q \). Assume that \( k \geq 2 \). Then for a small open neighborhood \( V \) of \( q \) in \( Z^n \) we have

\[
\sum_{p \in \mathbb{Z}} (-1)^p \dim H_p((V \cap U^n) \cup (\pi^{-1}(V) \cap Q), (\pi^{-1}(V) \cap Q); \iota_*(\mathbb{C}_{U^n} g_0)) = 0. \tag{3.20}
\]

Proof. By Mayer-Vietoris exact sequences for relative twisted homology groups, we can reduce the problem to the well-known vanishing of the Euler characteristic of the circle \( S^1 \).

In the sequel, we consider the more special case where \( U = \mathbb{C}_x^* \) and \( (E, \nabla) \) is an integrable connection on \( U \) such that \( \mathcal{L} = H^{-1}DR_U(E) \simeq \mathbb{C}_{U^n} \exp(h(x))x^c \) for a Laurent polynomial \( h(x) = \sum_{i \in \mathbb{Z}} a_i x^i \) (\( a_i \in \mathbb{C} \) and \( c \in \mathbb{C} \)). Then we can take the projective line \( \mathbb{P} \) to be the good compactification \( \bar{Z} \) of \( U = \mathbb{C}_x^* \) for \( (E, \nabla) \). In this case, we have \( D = Z \setminus U = D_1 \cup D_2 \), where we set \( D_1 = \{0\} \) and \( D_2 = \{\infty\} \). For the real oriented blow-up \( \pi : \bar{Z} \rightarrow Z^n \) of \( Z^n \) the subset \( \bar{D} = \pi^{-1}(D^n) \) of \( \bar{Z} \) is a union of two circles \( \bar{D}_i := \pi^{-1}(D^n) \simeq S^1 \) \((i = 1, 2)\). Moreover the open subset \( Q \subset \bar{D} \) is a union of open intervals in \( \bar{D}_1 \cup \bar{D}_2 \simeq S^1 \cup S^1 \). Let \( NP(h) \subset \mathbb{R} \) be the Newton polytope of \( h \) i.e. the convex hull of the set \( \{i \in \mathbb{Z} \mid a_i \neq 0\} \subset \mathbb{R} \). Finally denote by \( \Delta \subset \mathbb{R} \) the convex hull of \( NP(h) \cup \{0\} \) in \( \mathbb{R} \). Then by Proposition [3.4] and Mayer-Vietoris exact sequences for relative twisted homology groups we can easily prove the following result.

Lemma 3.6. In the situation as above, we have

(i) The dimension of the rapid decay homology group \( H_p^{rd}(U; E) \) is \( \text{Vol}_2(\Delta) \) if \( p = 1 \) and zero otherwise.

(ii) Assume that \( \Delta = [-m, 0] \) (resp. \( \Delta = [0, m] \)) for some \( m > 0 \). Then \( Q \subset \bar{D} \) is a union of \( m \) open intervals \( Q_1, Q_2, \ldots, Q_m \) in \( \bar{D}_1 \simeq S^1 \) (resp. in \( \bar{D}_2 \simeq S^1 \)) and the first rapid decay homology group \( H_1^{rd}(U; E) \) has a basis formed by the \( m \) elements

\[
[\gamma_i] \in H_1^{rd}(U; E) \quad (i = 1, 2, \ldots, m), \tag{3.21}
\]

where \( \gamma_i \) is a 1-dimensional twisted chain with values in \( \iota_*\mathcal{L} \) starting from a point in \( Q_i \) and going directly to that in \( Q_{i+1} \) (here we set \( Q_{m+1} = Q_1 \)).

(iii) Assume that \( \Delta = [-m_1, m_2] \) for some \( m_1, m_2 > 0 \). Then \( Q \subset \bar{D} \) is a union of open intervals \( Q_1, Q_2, \ldots, Q_m \) in \( \bar{D}_1 \simeq S^1 \) and the ones \( Q'_1, Q'_2, \ldots, Q'_m \) in \( \bar{D}_2 \simeq S^1 \). If moreover the function \( g_0 \) has a non-trivial monodromy around the origin, then the first rapid decay homology group \( H_1^{rd}(U; E) \) has a basis formed by the \( m_1 + m_2 \) elements

\[
[\gamma_i] \in H_1^{rd}(U; E) \quad (i = 1, 2, \ldots, m_1) \tag{3.22}
\]

and

\[
[\gamma'_i] \in H_1^{rd}(U; E) \quad (i = 1, 2, \ldots, m_2), \tag{3.23}
\]

where \( \gamma_i \) (resp. \( \gamma'_i \)) is a 1-dimensional twisted chain with values in \( \iota_*\mathcal{L} \) starting from a point in \( Q_i \) (resp. \( Q'_i \)) and going directly to that in \( Q_{i+1} \) (resp. \( Q'_{i+1} \)).
4 A geometric construction of integral representations

In this section we give a geometric construction of Adolphson’s confluent $A$-hypergeometric $D$-module $M_{A,c}$ and apply it to obtain the integral representations of $A$-hypergeometric functions. Let $Y = (\mathbb{C}^A)^* = \mathbb{C}^N_\zeta$ be the dual vector space of $X = \mathbb{C}^A = \mathbb{C}^N_\zeta$, where $\zeta$ is the dual coordinate of $z$. As in [10], to $A \subset \mathbb{Z}^n$ we associate a morphism

$$j : T = (\mathbb{C}^*)^n_x \rightarrow Y = (\mathbb{C}^A)^* = \mathbb{C}^N_\zeta$$

(4.1)
defined by $x \mapsto (x^{a(1)}, x^{a(2)}, \ldots, x^{a(N)})$. Since we assume here that $A$ generates $\mathbb{Z}^n$, $j$ is an embedding. Let $I \subset \mathbb{C}[\xi_1, \ldots, \xi_N]$ be the defining ideal of the closure $\overline{\{j(T) \mid j(T) \subset Y\}}$ in $Y$. Moreover denote by $D(Y)$ the Weyl algebra over $Y$. Then we have a ring isomorphism

$$\wedge : D(X) \xrightarrow{\sim} D(Y)$$

(4.2)
defined by

$$(\partial / \partial z_j)^\wedge = \zeta_j, \quad (z_j)^\wedge = -\partial / \partial \zeta_j \quad (j = 1, 2, \ldots, N).$$

(4.3)

We call $\wedge$ the Fourier transform (see Malgrange [25] etc. for the details). Via this $\wedge$, the Adolphson’s system $M_{A,c}$ is transformed to the one

$$(Z_{i,c})^\wedge v(\zeta) = 0 \quad (1 \leq i \leq n), \quad f(\zeta)v(\zeta) = 0 \quad (f \in I)$$

(4.4)
on $Y = (\mathbb{C}^A)^*$. Note that this system has no holomorphic solution in general. Let

$$N_{A,c} = M_{A,c}^\wedge = D(Y)/\left(\sum_{1 \leq i \leq n} D(Y)(Z_{i,c})^\wedge + \sum_{f \in I} D(Y)f\right)$$

(4.5)
be the corresponding left $D(Y)$-module and $N_{A,c}$ the coherent $D_Y$-module associated to it. By a theorem of Hotta [10], $N_{A,c}$ is regular holonomic. Now on the algebraic torus $T = (\mathbb{C}^*)^n_x$ we define a holonomic $D_T$-module $R_c$ by

$$R_c = D_T / \sum_{1 \leq i \leq n} D_T \left\{ x_i \partial / \partial x_i + (1 - c_i) \right\} \simeq \mathcal{O}_T x_1^{c_1-1} \cdots x_n^{c_n-1}. \quad (4.6)$$

This is an integrable connection on $T$ and we have

$$\text{DR}_T(R_c) \simeq (\mathbb{C}^{T+n} x_1^{-c_1+1} \cdots x_n^{-c_n+1})[n]. \quad (4.7)$$

Let $v = [1] \in N_{A,c}$ and $w_0 = [1] \in \mathcal{R}_c$ be the canonical generators. Recall that the transfer bimodule $D_{T \rightarrow Y}$ has the canonical section $1_{T \rightarrow Y}$. We define a section $1_{Y \leftarrow T}$ of $D_{Y \leftarrow T} = \Omega_T \otimes_{\mathcal{O}_T} D_{T \rightarrow Y} \otimes_{j^{-1}\mathcal{O}_Y} j^{-1}\Omega_Y^{-1}$ by

$$1_{Y \leftarrow T} = (dx_1 \wedge \cdots \wedge dx_n) \otimes 1_{T \rightarrow Y} \otimes j^{-1}(d\zeta_1 \wedge \cdots \wedge d\zeta_N)^{-1}. \quad (4.8)$$

Note that this definition of $1_{Y \leftarrow T}$ depends on the coordinates of $Y$ and $T$. Then we obtain a section $w$ of the regular holonomic $D_Y$-module

$$S_{A,c} := \int_j R_c = j_*(D_{Y \leftarrow T} \otimes_{D_T} R_c) \quad (4.9)$$
defined by \( w = j_* (1_{Y_T} \otimes w_0) \). We can easily check that this section \( w \in \mathcal{S}_{A,c} \) satisfies the system (4.4). Hence as in [10, page 268-269], we obtain a morphism

\[
\Psi : \mathcal{N}_{A,c} \rightarrow \mathcal{S}_{A,c} = \int_j \mathcal{R}_c
\]

of left \( \mathcal{D}_Y \)-modules which sends the canonical generator \( v = [1] \in \mathcal{N}_{A,c} \) to \( w \in \mathcal{S}_{A,c} \).

**Definition 4.1.** (Gelfand-Kapranov-Zelevinsky [11, page 262]) For a face \( \Gamma \) of \( \Delta \) containing the origin \( 0 \in \mathbb{R}^n \) we denote by \( \text{Lin}(\Gamma) \subset \mathbb{C}^n \) the \( \mathbb{C} \)-linear span of \( \Gamma \). We say that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant (with respect to \( A \)) if for any face \( \Gamma \) of \( \Delta \) of codimension 1 such that \( 0 \in \Gamma \) we have \( c \notin \{ \mathbb{Z}^n + \text{Lin}(\Gamma) \} \).

Recall that if \( c \in \mathbb{C}^n \) is nonresonant then it is semi-nonresonant in the sense of [11, page 284]. The following result was proved by Saito [36] and Schulze-Walther [39], [40] by using commutative algebras. Here we give a geometric proof to it.

**Lemma 4.2.** Assume that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant. Then the regular holonomic \( \mathcal{D}_Y \)-module \( \mathcal{S}_{A,c} \) is irreducible.

**Proof.** Note that \( \text{DR}_T(\mathcal{R}_c) \simeq (\mathbb{C}_{T^\text{an}}x_1^{c_1} \cdots x_n^{c_n})[n] \) is an irreducible perverse sheaf on \( T^\text{an} \). Then also its minimal extension by the locally closed embedding \( j \) is irreducible (see [17, Corollary 8.2.10] etc.). As in [10, Theorem 3.5 and Propositions 3.2 and 4.4] it suffices to show that the canonical morphism

\[
j_!(\mathbb{C}_{T^\text{an}}x_1^{c_1} \cdots x_n^{c_n}) \rightarrow Rj_!(\mathbb{C}_{T^\text{an}}x_1^{c_1} \cdots x_n^{c_n})
\]

is a quasi-isomorphism. For this, we have only to prove the vanishing \( Rj_* (\mathbb{C}_{T^\text{an}}x_1^{c_1} \cdots x_n^{c_n}) q = 0 \) for any \( q \in j(T) \setminus j(T) \). Note that by the nonresonance of \( c \in \mathbb{C}^n \) for any \( p \in \mathbb{Z} \) and the local system \( \mathcal{L} := \mathbb{C}_{T^\text{an}}x_1^{c_1} \cdots x_n^{c_n} \) on \( T^\text{an} \) we have \( H^p(T^\text{an}; \mathcal{L}) = 0 \). Let \( S(A) \subset \mathbb{Z}^n \) (resp. \( K(A) \subset \mathbb{R}^n \)) be the semigroup (resp. the convex cone) generated by \( A \). Then by (the proof of) [11, Chapter 5, Theorem 2.3] we have \( j(T) \simeq \text{Spec}(\mathbb{C}[S(A)]) \). Let us define an action of \( T \) on \( Y = \mathbb{C}_x^N \) by

\[
(\zeta_1, \ldots, \zeta_N) \mapsto (x^{a(1)}\zeta_1, \ldots, x^{a(N)}\zeta_N)
\]

for \( x \in T \). Then by [11, Chapter 5, Theorem 2.5] there exists a natural bijection between the faces of \( K(A) \) and the \( T \)-orbits in \( j(T) \). In particular, if \( K(A) = \mathbb{R}^n \) we have \( j(T) = j(T) \) and there is nothing to prove. First consider the case where \( 0 \in \mathbb{R}^n \) is an apex of \( K(A) \) and \( q = 0 \in Y = \mathbb{C}_x^N \). If \( 0 \in A \) i.e. \( 0 = a(j) \) for some \( 1 \leq j \leq N \) we have \( j(T) \subset \{ \zeta_j = 1 \} \simeq \mathbb{C}^{N-1} \). Hence we may assume that \( 0 \notin A \) from the first. In this case, \( \{0\} \subset j(T) \) is the unique 0-dimensional \( T \)-orbit in \( j(T) \) which corresponds to \( \{0\} \subset K(A) \). From now on, we will prove that \( Rj_!(\mathcal{L})_0 \simeq 0 \). By our assumption there exists a linear function \( l : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( l(\mathbb{Z}^n) \subset \mathbb{Z} \) and \( K(A) \setminus \{0\} \subset \{ l > 0 \} \). We define a real-valued function \( \varphi : Y = \mathbb{C}_x^N \rightarrow \mathbb{R} \) by

\[
\varphi(\zeta) = |\zeta_1|^{l(\zeta_1)} + \cdots + |\zeta_N|^{l(\zeta_N)},
\]

(4.13)
where we take $C \in \mathbb{Z}_{>0}$ large enough so that $\varphi$ and its level sets $\varphi^{-1}(b)$ ($b > 0$) are smooth. Let $(l_1, l_2, \ldots, l_n) \in \mathbb{Z}^n$ be the coefficients of the linear function $l$. Define an action of the multiplicative group $\mathbb{R}_{>0}$ on $T$ by

$$r \cdot (x_1, \ldots, x_n) = (r^{l_1}x_1, \ldots, r^{l_n}x_n)$$ (4.14)

for $r \in \mathbb{R}_{>0}$. Then we have

$$j(r \cdot x) = (r^{l(a(1))}x_1^{a(1)}, \ldots, r^{l(a(N))}x_N^{a(N)})$$ (4.15)

and hence

$$\varphi(j(r \cdot x)) = r^\varphi\varphi(j(x)).$$ (4.16)

Therefore by the action of $\mathbb{R}_{>0}$ on $Y = C^N_\zeta$ defined by

$$r \cdot (\zeta_1, \ldots, \zeta_N) = (r^{l(a(1))}\zeta_1, \ldots, r^{l(a(N))}\zeta_N),$$ (4.17)

a level set $\varphi^{-1}(t)$ ($t > 0$) of $\varphi$ is sent to the one $\varphi^{-1}(r^C\cdot t)$. Moreover this action preserves the $T$-orbits in $j(T)$. Let $O \subset j(T)$ be such a $T$-orbit. Then all the level sets $\varphi^{-1}(t)$ ($t > 0$) of $\varphi$ are transversal to $O$, or all are not. But the latter case cannot occur by the Sard theorem. Then we obtain an isomorphism

$$H^pR_{j_*}(L)_0 \simeq H^p(C^n; R_{j_*}(L)) \simeq H^p(T^n; L) \simeq 0$$ (4.18)

for any $p \in \mathbb{Z}$. Next consider the remaining case where $q \in O$ for a $T$-orbit $O$ in $j(T)$ such that $\dim O \geq 1$. Then in a neighborhood of $q$, the variety $j(T)$ is a product $W \times O$ for an affine toric variety $W \subset C^N_\zeta$ and $j(T) = (T_1 \cup \cdots \cup T_k) \times O$ for some tori $T_i \simeq (\mathbb{C}^*)^{n-\dim O}$. See [11, Chapter 5, Theorem 3.1] and the proof of [26, Theorem 4.9] etc. for the details. Moreover for the semigroup $S(A_O) \subset \mathbb{Z}^{n-\dim O}$ generated by a finite subset $A_O \subset \mathbb{Z}^{n-\dim O}$ we have $\overline{T_i} \simeq \text{Spec}(\mathbb{C}[S(A_O)]) \subset W$ ($i = 1, 2, \ldots, k$). These varieties $\overline{T_i}$ are the irreducible components of $W$. For the explicit construction of $\overline{T_i}$ see the proof of [26, Theorem 4.9]. By this construction $0 \in \mathbb{R}^{n-\dim O}$ is an apex of the convex cone $K(A_O) \subset \mathbb{R}^{n-\dim O}$ generated by $A_O$. Let $p_2 : W \times O \rightarrow O$ and $q_2 : T_i \times O \rightarrow O$ be the second projections. Then it follows from the nonresonance of $c \in C^n$ the restriction of $L$ to $q_2^{-1}p_2(q) \simeq T_i$ is a non-constant local system. So we can apply our previous arguments and prove $R_{j_*}(L)_q \simeq 0$ in this case, too. This completes the proof.

By Lemma [4.2], if $c \in C^n$ is nonresonant the non-trivial morphism $\Psi$ should be surjective. According to Schulze-Walther [39, Corollary 3.8] the morphism $\Psi$ is also an isomorphism in this case. Let $\vee : D(Y) \rightarrow D(X)$ be the inverse of the Fourier transform $\wedge$. Then we have an isomorphism $N_{A,c}^\vee \simeq M_{A,c}$ of left $D(X)$-modules. The corresponding coherent $D_X$-module $N_{A,c}^\vee \simeq M_{A,c}$ can be more geometrically constructed as follows. Let $\sigma = \langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$ be the canonical pairing defined by $\langle z, \zeta \rangle = \sum_{j=1}^N z_j\zeta_j$ and $p_1 : X \times Y \rightarrow X$ (resp. $p_2 : X \times Y \rightarrow Y$) the first (resp. second) projection. Then we have the following theorem due to Katz-Laumon [20].

**Theorem 4.3.** (Katz-Laumon [20]) In the situation as above, we have an isomorphism

$$N_{A,c}^\vee \simeq \int_{p_1} \left\{ (p_2^*N_{A,c}) \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y} \sigma \right\},$$ (4.19)

where $\mathcal{O}_{X \times Y} \sigma$ is the integrable connection associated to $\sigma : X \times Y \rightarrow \mathbb{C}$ (see [25] etc.).
In the same way, we have

$$S_{A,c}^\vee = \int_{p_1} \left\{ (p_2^* S_{A,c}) \otimes \mathcal{O}_{X \times Y} \mathcal{O}_{X \times Y} e^\sigma \right\}. \quad (4.20)$$

From now on, we assume that $c \in \mathbb{C}^n$ is nonresonant. Then by Lemma 4.2 we obtain surjective morphisms $N_{A,c} \rightarrow S_{A,c}(Y)$ and $\mathcal{M}_{A,c}(X) \simeq N_{A,c}^\vee \rightarrow S_{A,c}^\vee(X)$. Hence we obtain a surjective morphism

$$\mathcal{M}_{A,c} \simeq N_{A,c}^\vee \rightarrow S_{A,c}^\vee \quad (4.21)$$

of left $\mathcal{D}_X$-modules. Let $e^\tau: X \times T \rightarrow \mathbb{C}$ be the function defined by

$$e^{\tau}(z,x) = \exp(\sum_{j=1}^N z_j x_{a(j)}^1 \cdots x_{a(n)}^1). \quad (4.24)$$

Then by the results of Hien-Roucairol [14] the holomorphic solution complex

$$\text{Sol}_X(S_{A,c}^\vee) = \text{RHom}_{\mathcal{D}_X^\an}((S_{A,c}^\vee)^\text{an}, \mathcal{O}_X^\text{an}) \quad (4.25)$$

of $S_{A,c}^\vee$ is expressed by the rapid decay homology groups associated the function $g$. Indeed, for $z \in \Omega$ let $K_z$ (resp. $g_z: T \rightarrow \mathbb{C}$) be the restriction of the connection $K$ (resp. the function $g$) to $U_z := q_1^{-1}(z) \simeq T \subset \Omega \times T$. Namely we set

$$g_z(x) = \exp(\sum_{j=1}^N z_j x_{a(j)}^1 \cdots x_{a(n)}^1). \quad (4.26)$$

Moreover for any $p \in \mathbb{Z}$, by Proposition 3.4 (see also the proof of Theorem 4.5 below) the rapid decay homology groups

$$H^p_{\text{rd}}(U_z; K_z) \quad (z \in \Omega^\text{an}) \quad (4.28)$$

associated to the integrable connections $K_z$ (or to the functions $g_z: T \rightarrow \mathbb{C}$) are isomorphic to each other and define a local system $\mathcal{H}^p_{\text{rd}}$ on $\Omega^\text{an}$. See [14] for the details. The following result is essentially due to Hien-Roucairol [14].
Theorem 4.4. (Hien-Roucairol [14]) In the situation as above, for any $p \in \mathbb{Z}$ we have an isomorphism
\[
\mathcal{H}^{\text{rd}}_{n+p} \simeq H^p \text{Sol}_X \left( \int_{q_1} \mathcal{K} \right) \simeq H^p \text{Sol}_X (\mathcal{S}'_{A,c})
\]  
(4.29)
of local systems on $\Omega^\text{an}$.

In [1, Section 3] Adolphson proved that $\mathcal{M}_{A,c}$ is an integrable connection on $\Omega$. Then by the surjective morphism $\mathcal{M}_{A,c} \simeq \mathcal{N}'_{A,c} \rightarrow \mathcal{S}'_{A,c}$ we find that $\mathcal{S}'_{A,c}$ is also an integrable connection on $\Omega$. This in particular implies that for any $p \neq 0$ we have $H^p \text{Sol}_X (\mathcal{S}'_{A,c}) \simeq 0$. Hence we get
\[
H^p (\mathcal{S}'_{A,c}) \simeq 0 \quad (p \neq 0, \ z \in \Omega^\text{an}).
\]  
(4.30)

It follows also from the surjection $\mathcal{M}_{A,c} \rightarrow \mathcal{S}'_{A,c}$ that we have an injection
\[
\Phi : \mathcal{H}^{\text{rd}}_n \simeq \text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}}) \hookrightarrow \text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}}).
\]  
(4.31)

By using the generator
\[
u = [1] \in \mathcal{M}_{A,c} = \mathcal{D}_X / \left( \sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \text{Ker} A \cap \mathcal{Z} N} \mathcal{D}_X \Box \mu \right)
\]  
(4.32)
of $\mathcal{M}_{A,c}$ we regard $\text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}})$ as a subsheaf of $\mathcal{O}_{X^\text{an}}$. Then we have the following result.

Theorem 4.5. Assume that the parameter vector $c \in \mathbb{C}^n$ is nonresonant. Then the morphism $\Phi$ induces an isomorphism
\[
\mathcal{H}^{\text{rd}}_n \simeq \text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}})
\]  
(4.33)
of local systems on $\Omega^\text{an}$. Moreover this isomorphism is given by the integral
\[
\gamma \mapsto \left\{ \Omega^\text{an} \ni z \mapsto \int_{\gamma^z} \exp \left( \sum_{j=1}^N z_j x^{a(j)} \right) x_1^{c_1} \cdots x_n^{c_n} dx_1 \wedge \cdots \wedge dx_n \right\},
\]  
(4.34)
where for a continuous family $\gamma$ of rapid decay $n$-cycles in $\Omega^\text{an} \times T^\text{an}$ and $z \in \Omega^\text{an}$ we denote by $\gamma^z$ its restriction $\gamma \cap U_z$ to $U_z = q_1^{-1}(z) \simeq T$.

Note that this integral representation of the confluent $A$-hypergeometric functions $\text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}})$ coincides with the one in Adolphson [1, Equation (2.6)].

\textbf{Proof.} Recall that the sheaf $\text{Hom}_{\mathcal{D}_{X^\text{an}}}(\mathcal{M}_{A,c}^\text{an}, \mathcal{O}_{X^\text{an}})$ is a local system on $\Omega^\text{an}$. Moreover by [1, Corollary 5.20] its rank is $\text{Vol}_Z(\Delta)$. So it suffices to show that for any $z \in \Omega^\text{an}$ the dimension of the $n$-th rapid decay homology group $H^{\text{rd}}_n (U_z; \mathcal{K}_z^*)$ is also $\text{Vol}_Z(\Delta)$. Let
\[
\text{Eu}^{\text{rd}} (U_z; \mathcal{K}_z^*) := \sum_{p \in \mathbb{Z}} (-1)^p \dim H^{\text{rd}}_p (U_z; \mathcal{K}_z^*)
\]  
(4.35)
be the rapid decay Euler characteristic. Then by [4.30] we have only to prove the equality
\[
\text{Eu}^{\text{rd}} (U_z; \mathcal{K}_z^*) = (-1)^n \text{Vol}_Z(\Delta).
\]  
(4.36)
Let $\Sigma_0$ be the dual fan of $\Delta$ in $\mathbb{R}^n$ and $\Sigma$ its smooth subdivision. Denote by $Z\Sigma$ the smooth toric variety associated to the fan $\Sigma$. Then $Z\Sigma$ is a smooth compactification of $U_z \simeq T$ such that $Z\Sigma \setminus U_z$ is a normal crossing divisor. By using the non-degeneracy of the Laurent polynomial $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$, as in [28, Section 3] we can construct a complex blow-up $Z := \tilde{Z}\Sigma$ of $Z\Sigma$ such that the meromorphic extension of $h_z$ to it has no point of indeterminacy. For the reader’s convenience, we briefly recall the construction of $Z$. Recall that $T$ acts on $Z\Sigma$ and the $T$-orbits are parametrized by the cones in $\Sigma$. For a cone $\sigma \in \Sigma$ we denote by $T_\sigma \simeq (\mathbb{C}^\ast)^{n-\dim \sigma}$ the corresponding $T$-orbit. Let $\rho_1, \ldots, \rho_m \in \Sigma$ be the rays i.e. the one-dimensional cones in $\Sigma$. By using the primitive vectors $\kappa_i \in \rho_i \cap (\mathbb{Z}^n \setminus \{0\})$ on $\rho_i$ we set

$$m_i = -\min_{\alpha \in \Delta} \langle \kappa_i, \alpha \rangle \geq 0.$$  \hspace{1cm} (4.37)

We renumber $\rho_1, \ldots, \rho_m$ so that $m_i > 0$ if and only if $1 \leq i \leq l$ for some $1 \leq l \leq m$. Then for any $1 \leq i \leq l$ the meromorphic extension of $h_z$ to $Z\Sigma$ has a pole of order $m_i > 0$ along the toric divisor $D_i = T_{\rho_i} \subset Z\Sigma$. By the non-degeneracy of $h_z$ the hypersurface $\bar{h}_z^{-1}(0) \subset Z\Sigma$ intersects $D_I = \cap_{i \in I} D_i$ transversally for any subset $I \subset \{1, 2, \ldots, m\}$ such that $I \cap \{1, 2, \ldots, l\} \neq \emptyset$ (see Definition 2.3). The meromorphic extension of $h_z$ has points of indeterminacy in $\bigcup_{i=1}^{l} (\bar{h}_z^{-1}(0) \cap D_i)$.

![Figure 1](image)

First we construct a tower of $m_i$ codimension-two blow-ups over $\bar{h}_z^{-1}(0) \cap D_1$ (see [28, Section 3] and [29, Section 3 and Lemma 4.9] for the details). Then the indeterminacy of $h_z$ over $D_1 \setminus (\bigcup_{j \neq 1} D_j)$ is eliminated. By repeating this construction also over (the proper transforms of) $D_2, D_3, \ldots, D_l$ we finally obtain the desired proper morphism $Z = \tilde{Z}\Sigma \longrightarrow Z\Sigma$ of $Z\Sigma$ as the figure below.
Then we have an isomorphism the rapid decay Euler characteristic Eu

d homology groups to the geometric situation in Figure 2 above, we can easily calculate by applying Lemmas 3.5 and 3.6 and Mayer-Vietoris exact sequences for relative twisted relative algebraic de Rham complex of $K$ we prove the remaining assertion. Denote the distinguished section ($\rho_i$), and prove the equality (4.36). This completes the proof of the isomorphism (4.33). Let $x_n$), $\Omega_{X \times T/X} \otimes \mathcal{O}_{X \times T} \mathcal{K}$ be the relative algebraic de Rham complex of $\mathcal{K}$ associated to the morphism $q_i : X \times T \to X$. Then we have an isomorphism

$$S_{\lambda, c} \simeq \int_{q_i} \mathcal{K} \simeq H^n \{ (q_i)_* (\Omega_{X \times T/X} \otimes \mathcal{O}_{X \times T} \mathcal{K}) \}.$$

(4.40)
For a relative $n$-form $\omega \in (q_1)_*\Omega^n_{X \times T/X}$ denote by $\text{cl}(\omega \otimes t)$ the section of $S^\vee_{A,c}$ which corresponds to the cohomology class $[(q_1)_*(\omega \otimes t)] \in H^n((q_1)_*(\Omega^1_{X \times T/X} \otimes_{O_{X \times T}} K))$ by the above isomorphism. According to the result of [14], by the isomorphism
\begin{equation}
\mathcal{H}^n_{\text{rd}} \simeq \text{Hom}_{D^{\text{an}}}(S^\vee_{A,c}, O_{X^{\text{an}}})
\end{equation}
of local systems on $\Omega^{an}$, a family of rapid decay cycles $\gamma \in \mathcal{H}^n_{\text{rd}}$ is sent to the section
\begin{equation}
((S^\vee_{A,c})^{an} \ni f \otimes \text{cl}(\omega \otimes t) \longmapsto \left\{ \Omega^{an} \ni z \longmapsto f(z) \int_{\gamma} \exp(\sum_{j=1}^N z_j x^{a(j)}_1 \cdots x^{a(n)}_n \omega) \right\})
\end{equation}
\begin{equation}
(f \in O_{X^{an}}) \in \text{Hom}_{D^{\text{an}}}(S^\vee_{A,c}, O_{X^{an}})
\end{equation}. Then the remaining assertion follows from the lemma below. This completes the proof. □

**Remark 4.6.** When $0 \in \text{Int}(\Delta)$ the irrelevant components of $D$ in the proof above are the last $\mathbb{P}^1$-bundles on $\bar{h}_z^{-1}(0) \cap D_i$ ($1 \leq i \leq l = m$). By the construction of the morphism $Z = \bar{Z}_\Sigma \longrightarrow Z$ we can easily see that for any $t \in \mathbb{C}$ the hypersurface $\bar{h}_z^{-1}(t) \subset Z$ intersects them transversally.

**Lemma 4.7.** By the morphism
\begin{equation}
\mathcal{M}_{A,c} \longrightarrow S^\vee_{A,c} \simeq H^n \left\{ (q_1)_*(\Omega^1_{X \times T/X} \otimes_{O_{X \times T}} K) \right\}
\end{equation}
the canonical section $u = [1] \in \mathcal{M}_{A,c}$ is sent to the cohomology class $\text{cl}(\omega \otimes t)$.

**Proof.** First note that the morphism $\Psi^{\vee}(X): \mathcal{M}_{A,c}(X) \simeq M_{A,c} \simeq N^\vee_{A,c} \longrightarrow S^\vee_{A,c}(X) \simeq S_{A,c}(Y)$ sends the canonical generator $u = [1] \in \mathcal{M}_{A,c}(X)$ to $w = j_*(1_{Y \hookrightarrow T} \otimes w_0) \in S_{A,c}(Y)$. On the other hand, by (4.20) we have an isomorphism $S^\vee_{A,c} \simeq H^N \left\{ (p_1)_* \left\{ \Omega^1_{X \times Y/X} \otimes_{O_{X \times Y}} \left( p_2^* S_{A,c} \otimes_{O_{X \times Y}} O_{X \times Y} e^\sigma \right) \right\} \right\}.$ Then by Malgrange's simple proof [25, page 135] of Theorem 4.3, via this isomorphism the section $w \in S^\vee_{A,c}(X) \simeq S_{A,c}(Y)$ corresponds to the cohomology class
\begin{equation}
[(p_1)_* \left\{ (d\xi_1 \wedge \cdots \wedge d\xi_N) \otimes (p_2^* w) \otimes e^\sigma \right\}].
\end{equation}
Let $\tilde{j}: X \times T \hookrightarrow X \times Y$ be the embedding induced by $j$. By the isomorphism
\begin{equation}
S^\vee_{A,c} \simeq H^N \left\{ (p_1)_* \left\{ \Omega^1_{X \times Y/X} \otimes_{O_{X \times Y}} \tilde{j}_*(\mathcal{D}_{X \times Y \hookrightarrow X \times T} \otimes_{O_{X \times T}} K) \right\} \right\}
\end{equation}
the above cohomology class corresponds to the one
\begin{equation}
\rho := [(p_1)_* \left\{ (d\xi_1 \wedge \cdots \wedge d\xi_N) \otimes \tilde{j}_*(1_{X \times Y \hookrightarrow X \times T} \otimes t) \right\}],
\end{equation}
where the section $1_{X \times Y \hookrightarrow X \times T} \in \mathcal{D}_{X \times Y \hookrightarrow X \times T}$ is defined similarly to $1_{Y \hookrightarrow T} \in \mathcal{D}_{Y \hookrightarrow T}$. Then it suffices to show that via the isomorphism
\begin{equation}
S^\vee_{A,c} \simeq \int_{p_1} \int_{\tilde{j}} K \simeq \int_{q_1} K
\end{equation}
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the cohomology class $\rho$ is sent to the one $\text{cl}((dx_1 \wedge \cdots \wedge dx_n) \otimes t) = [(q_1)_*\{(dx_1 \wedge \cdots \wedge dx_n) \otimes t\}]$ in

$$\int_{q_1} \mathcal{K} \simeq H^n \{ (q_1)_*(\Omega_{X \times T/X} \otimes \mathcal{O}_{X \times T}) \mathcal{K} \}. \quad (4.49)$$

Since $X$ and $X \times Y$ are affine, we have only to prove that via the isomorphism

$$H^N \Gamma(X \times Y; \Omega_{X \times Y/X} \otimes \mathcal{O}_{X \times Y} \int \mathcal{K}) \simeq H^n \Gamma(X \times T; \Omega_{X \times T/X} \otimes \mathcal{O}_{X \times T} \mathcal{K}) \quad (4.50)$$

the cohomology class

$$[(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_*(1_{X \times Y \leftarrow X \times T} \otimes t)] \quad (4.51)$$

is sent to the one $[(dx_1 \wedge \cdots \wedge dx_n) \otimes t]$. Indeed, we have isomorphisms

$$H^0 \Gamma(X \times Y; \Omega_{X \times Y/X}^{N+} \otimes \mathcal{O}_{X \times Y} \int \mathcal{K}) \quad (4.52)$$

by which the element $[(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes \tilde{j}_*(1_{X \times Y \leftarrow X \times T} \otimes t)]$ is sent to the one $[q_2^{-1} \{ j^{-1}(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes 1_{Y \leftarrow T} \} \otimes t \}$. Let $\mathcal{P} \rightarrow \mathcal{K}$ be a free resolution of the left $\mathcal{D}_{X \times T}$-module $\mathcal{K}$. Since $X \times T$ is affine, we obtain a surjective homomorphism

$$\Gamma(X \times T; \mathcal{P}^0) \rightarrow \Gamma(X \times T; \mathcal{K}) \quad (4.56)$$

and can take a lift $\tilde{t} \in \Gamma(X \times T; \mathcal{P}^0)$ of $t \in \Gamma(X \times T; \mathcal{K})$. Moreover by the flatness of the right $\mathcal{D}_T$-module $\mathcal{D}_{Y \leftarrow T}$ and the well-known formula

$$j^{-1} \Omega_Y^{N+} \otimes j^{-1} \mathcal{O}_Y \mathcal{D}_{Y \leftarrow T} \simeq j^{-1} \Omega_Y^N \otimes j^{-1} \mathcal{O}_Y \mathcal{D}_{Y \leftarrow T} \simeq \Omega_T^n \quad (4.57)$$

there exists an isomorphism

$$H^0 \Gamma(X \times T; q_2^{-1} \{ j^{-1} \Omega_Y^{N+} \otimes j^{-1} \mathcal{O}_Y \mathcal{D}_{Y \leftarrow T} \otimes q_2^{-1} \mathcal{D}_T \} \mathcal{K}) \quad (4.58)$$

by which $[q_2^{-1} \{ j^{-1}(d\zeta_1 \wedge \cdots \wedge d\zeta_N) \otimes 1_{Y \leftarrow T} \} \otimes \tilde{t}]$ is sent to $[q_2^{-1}(dx_1 \wedge \cdots \wedge dx_n) \otimes \tilde{t}]$. Similarly, by the isomorphism

$$H^0 \Gamma(X \times T; q_2^{-1} \mathcal{P} \otimes q_2^{-1} \mathcal{D}_T \mathcal{P}) \simeq H^0 \Gamma(X \times T; \Omega_{X \times T/X}^{n+} \otimes \mathcal{O}_{X \times T} \mathcal{K}) \quad (4.61)$$

the element $[q_2^{-1}(dx_1 \wedge \cdots \wedge dx_n) \otimes \tilde{t}]$ is sent to $[(dx_1 \wedge \cdots \wedge dx_n) \otimes t]$. This completes the proof.

As a corollary of Theorem 4.5, we recover the following Saito and Schulze-Walther’s geometric (functorial) construction of Adolphson’s confluent $A$-hypergeometric $\mathcal{D}$-module $\mathcal{M}_{A,c}$ on $\Omega \subset X = \mathbb{C}^A$. 

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Corollary 4.8. \( (\text{Saito }[36]\text{ and Schulze-Walther }[39, 40]) \) Assume that the parameter vector \( c \in \mathbb{C}^n \) is nonresonant. Then we have an isomorphism \( \mathcal{M}_{A,c} \xrightarrow{\sim} S^\vee_{A,c} \) of integrable connections on \( \Omega \). In particular, \( \mathcal{M}_{A,c} \) is an irreducible connection there.

This result was first obtained in Saito [36] and Schulze-Walther [39, 40] by using totally different methods. In fact, they proved moreover that we have an isomorphism \( \mathcal{M}_{A,c} \xrightarrow{\sim} S^\vee_{A,c} \) on the whole \( X \).

Remark 4.9. Since \( \mathcal{N}_{A,c} \) is regular holonomic by a theorem of Hotta [16], it is also regular at infinity in the sense of Daia [4]. Then by using the Fourier-Sato transforms (see [18] and [25] etc.), we can apply the main theorem of Daia [4] to get a topological construction of the sheaf of the confluent \( A \)-hypergeometric functions \( \text{Hom}_{\mathcal{D}_{\mathbb{C}^n}}((\mathcal{M}_{A,c})^\text{an}, \mathcal{O}_{\mathbb{C}^n}) \). This construction is valid even when the parameter \( c \in \mathbb{C}^n \) is not nonresonant.

Example 4.10. Assume that \( n = 1 \) and \( T = \mathbb{C}_x^* \).

(i) If \( A = \{1, -1\} \subset \mathbb{Z} \) our integral representation of the \( A \)-hypergeometric functions \( u(z_1, z_2) \) on \( \mathbb{C}_x^2 \) is
\[
u(t) = \frac{1}{2\pi i} \int_{\gamma(z, -\frac{1}{2})} \exp \left(\frac{tx}{2} - \frac{t}{2x}\right)x^{-\nu-1}dx \quad (4.63)
\]
for the parameter \( \nu = -c \).

(ii) If \( A = \{3, 1\} \subset \mathbb{Z} \) our integral representation of the \( A \)-hypergeometric functions \( u(z_1, z_2) \) on \( \mathbb{C}_x^2 \) is
\[
u(t) = \frac{1}{2\pi i} \int_{\gamma(z, -\frac{1}{3})} \exp \left(\frac{x^3}{3} - tx\right)dx \quad (4.65)
\]
for \( c = 1 \).

5 Asymptotic expansions at infinity of confluent \( A \)-hypergeometric functions

In this section, assuming the condition \( 0 \in \text{Int}(\Delta) \) we construct natural bases of the rapid decay homology groups \( (\mathcal{H}_n^\text{rd})_z \simeq H^\text{rd}_n(U_z; \mathbb{C}_X^\text{an}) \) and apply them to prove a formula for the asymptotic expansions at infinity of Adolphson’s confluent \( A \)-hypergeometric functions. For the construction of them, we first prove that the following subset \( \Omega_0 \) of \( \Omega \) is open dense in \( X = \mathbb{C}_x^N \).
**Definition 5.1.** We define a subset \( \Omega_0 \) of \( \Omega \subset X = \mathbb{C}_z^N \) by: \( z \in \Omega_0 \iff z \in \Omega \) and the Laurent polynomial \( h_\alpha(x) = \sum_{j=1}^{N} z_j x^{a(j)} \) has only non-degenerate (Morse) critical points in \( T = (\mathbb{C}^*)^n \).

It is clear that \( \Omega_0 \subset X = \mathbb{C}_z^N \) is stable by the multiplication of \( \mathbb{C}^* \) (i.e. homothecy) on \( X = \mathbb{C}_z^N \). Let \( b(1), b(2), \ldots, b(n) \in A \) be elements of \( A \) such that \( \{b(1), b(2), \ldots, b(n)\} \) is a basis of the vector space \( \mathbb{R}^n \). By our assumption that \( A \) generates \( \mathbb{Z}^n \), we can take such elements of \( A \).

**Proposition 5.2.** Let \( h(x) = \sum_{j=1}^{N} z_j x^{a(j)} \) be a Laurent polynomial with support in \( A \subset \mathbb{Z}^n \) on \( T = (\mathbb{C}^*)^n \). Assume that \( h \) is non-degenerate i.e. \( z = (z_1, z_2, \ldots, z_N) \in \Omega \). Then for generic \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \) the perturbation

\[
\tilde{h}(x) = h(x) - \sum_{i=1}^{n} \alpha_i x^{b(i)}
\]  

(5.1)

of \( h \) is non-degenerate and has only non-degenerate (Morse) critical points in \( T \).

**Proof.** It is clear that \( \tilde{h} \) is non-degenerate for generic \( \alpha \in \mathbb{C}^n \) (see for example [29, Lemma 5.2] etc.). Let \( l_1, l_2, \ldots, l_n \in (\mathbb{R}^n)^* \) be the dual basis of \( b(1), b(2), \ldots, b(n) \) and set

\[
g_i(x) = \sum_{j=1}^{N} l_i(a(j)) z_j x^{a(j)-b(i)} \quad (i = 1, 2, \ldots, n).
\]

(5.2)

Note that for \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) we have

\[
a_j = \sum_{i=1}^{n} l_i(a)b(i) \quad (j = 1, 2, \ldots, n).
\]

(5.3)

Then we can easily prove the equality

\[
(x_1 \frac{\partial \tilde{h}}{\partial x_1}, \ldots, x_n \frac{\partial \tilde{h}}{\partial x_n}) = (x^{b(1)}(g_1 - \alpha_1), \ldots, x^{b(n)}(g_n - \alpha_n)) \cdot B,
\]

(5.4)

where \( B \in GL_n(\mathbb{C}) \) is an invertible matrix defined by \( B = (b(j))_{i,j=1}^{n} = (b(i))_{i,j=1}^{n} \). Hence we obtain

\[
\{x \in T \mid \frac{\partial \tilde{h}}{\partial x_1}(x) = \cdots = \frac{\partial \tilde{h}}{\partial x_n}(x) = 0\} = \{x \in T \mid g_i(x) = \alpha_i \ (1 \leq i \leq n)\}.
\]

(5.5)

Moreover degenerate critical points of \( \tilde{h} \) in \( T \) correspond to critical points \( x \in T \) of the map \( (g_1, g_2, \ldots, g_n) : T \to \mathbb{C}^n \) such that \( g_i(x) = \alpha_i \ (1 \leq i \leq n) \). By the Bertini-Sard theorem, generic \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n \) are not such critical values. This implies that for generic \( \alpha \in \mathbb{C}^n \) the Laurent polynomial \( \tilde{h} \) has no degenerate critical point. This completes the proof.

**Corollary 5.3.** The subset \( \Omega_0 \) of \( \Omega \) is open dense in \( X = \mathbb{C}_z^N \) and stable by the multiplication of \( \mathbb{C}^* \) (i.e. homothecy) on \( X = \mathbb{C}_z^N \).
Proposition 5.4. Assume that $0 \in \text{Int}(\Delta)$. Then for any $z \in \Omega_0$ the Laurent polynomial $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$ has exactly $\text{Vol}_z(\Delta)$ non-degenerate (Morse) critical points in $T$.

Proof. Let us fix $z \in \Omega_0$ and set $h(x) = h_z(x)$. By an invertible matrix $C \in GL_n(\mathbb{C})$ we define new Laurent polynomials $h_1, h_2, \ldots, h_n$ on $T$ by

$$(h_1, \ldots, h_n) = (x_1 \frac{\partial h}{\partial x_1}, \ldots, x_n \frac{\partial h}{\partial x_n}) \cdot C. \quad (5.6)$$

By our assumption $0 \in \text{Int}(\Delta)$, taking sufficiently generic $C$ we may assume that all the Newton polytopes of $h_1, h_2, \ldots, h_n$ are equal to $\Delta$. Then for any face $\Gamma < \Delta$ of $\Delta$ the set

$$\{x \in T \mid h_1^\Gamma (x) = \cdots = h_n^\Gamma (x) = 0\} \quad (5.7)$$

coincides with that of the critical points of $h^\Gamma$ in $T$. In this correspondence for the special case $\Gamma = \Delta$, multiple roots of the equation $h_1(x) = \cdots = h_n(x) = 0$ in $T$ correspond to degenerate critical points of $h : T \to \mathbb{C}$. But by our assumption $z \in \Omega_0$ there is no such point in $T$. Moreover by the non-degeneracy of $h$ ($\iff z \in \Omega$), for any face $\Gamma < \Delta$ of $\Delta$ such that $0 \notin \Gamma$ (i.e. $\Gamma \neq \Delta$ when $0 \in \text{Int}(\Delta)$) we have

$$\{x \in T \mid h_1^\Gamma (x) = \cdots = h_n^\Gamma (x) = 0\} = \emptyset. \quad (5.8)$$

This means that the (0-dimensional) subvariety $\{x \in T \mid h_1(x) = \cdots = h_n(x) = 0\}$ of $T$ is a non-degenerate complete intersection (for the definition, see [21, Definition 2.7] and [32] etc.). Then by Bernstein’s theorem its cardinality is equal to $\text{Vol}_z(\Delta)$. \qed

From now on, assuming the condition $0 \in \text{Int}(\Delta)$, for any $z \in \Omega_0$ we construct a natural basis of the rapid decay homology group $(\mathcal{H}^{rd}_z)_{\alpha} \simeq H^{rd}_z(U_z; \mathbb{K}^*_z)$ by using the (relative) twisted Morse theory for the function $\text{Re}(h_z) : T^\text{an} \to \mathbb{R}$. For the twisted Morse theory and its applications to period integrals, we refer to Aomoto-Kita [2], Pajitnov [33] and Pham [34] etc. Our construction of the basis is similar to the ones of Dubrovin [5] and Tanabe-Ueda [32] etc. in the untwisted case. Note that by our assumption $0 \in \text{Int}(\Delta)$ any parameter vector $c \in \mathbb{C}^n$ is nonresonant. For $z \in \Omega_0$ let $\alpha(i) \in T$ ($1 \leq i \leq \text{Vol}_z(\Delta)$) be the non-degenerate (Morse) critical points of the Laurent polynomial $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$ in Proposition 5.4. By the Cauchy-Riemann equation, they are also non-degenerate (Morse) critical points of the real-valued function $\text{Re}(h_z) : T^\text{an} \to \mathbb{R}$. We can observe this fact more explicitly by taking a holomorphic Morse coordinate around each $\alpha(i) \in T$ as follows. For a fixed $1 \leq i \leq \text{Vol}_z(\Delta)$ let $y = (y_1, \ldots, y_n)$, $y_j = \xi_j + \sqrt{-1} \eta_j$ ($1 \leq j \leq n$) be a holomorphic Morse coordinate for $h_z$ around its critical point $\alpha(i) \in T$ such that $h_z(x) = h_z(\alpha(i)) + y_1^2 + y_2^2 + \cdots + y_n^2$ in a neighborhood of $\alpha(i) \in T$. Since we have

$$\text{Re}(h_z)(x) = \text{Re}(h_z)(\alpha(i)) + (\xi^2_1 + \cdots + \xi^2_n) - (\eta^2_1 + \cdots + \eta^2_n), \quad (5.9)$$

we regard the smooth submanifold $\{\xi_1 = \cdots = \xi_n = 0\}$ in it as the stable manifold of the gradient flow of the Morse function $\text{Re}(h_z) : T^\text{an} \to \mathbb{R}$ in a neighborhood of $\alpha(i) \in T^\text{an}$ and denote it by $S_i$. By shrinking $S_i$, if necessary, we may assume that $S_i$ is homeomorphic to the $n$-dimensional disk. For $1 \leq i \leq \text{Vol}_z(\Delta)$ let $R_i \subset \mathbb{C}$ be the ray in $\mathbb{C}$ defined by

$$R_i = \{\lambda \in \mathbb{C} \mid \text{Re} \lambda \leq \text{Re}(h_z)(\alpha(i)), \text{ Im} \lambda = \text{Im}(h_z)(\alpha(i))\}. \quad (5.10)$$
Namely $R_i$ emanates from the critical value $h_z(\alpha(i)) \in \mathbb{C}$ of $h_z$ and goes to the left in the complex plane $\mathbb{C}$ so that we have $\text{Re}\lambda \to -\infty$ along it. By shrinking the stable manifold $S_i$ if necessary, we may assume also that the image of $\overline{S}_i \subset T^a$ by the map $h_z : T^a \to \mathbb{C}$ is the closed interval

$$R^*_i = \{ \lambda \in R_i \mid \text{Re}(h_z)(\alpha(i)) - \varepsilon \leq \text{Re}\lambda \leq \text{Re}(h_z)(\alpha(i)) \}$$

(5.11)
in $R_i$ for some $\varepsilon > 0$ and $h_z(\partial S_i)$ is just the one point $\{h_z(\alpha(i)) - \varepsilon\}$ in $R_i$. We drag $\partial S_i \simeq S^{n-1}$ over the complement of $R^*_i$ in $R_i$ to construct a tube $M_i \simeq (-\infty, 0] \times S^{n-1}$ in $T^a$. Then $\gamma_i := \overline{S}_i \cup M_i$ is a singular $n$-chain in $T^a$ whose boundary in the real oriented blow-up $\hat{Z}$ is contained in the subset $Q \subset \hat{D}$. We thus constructed a rapid decay $n$-cycle $\gamma_i$ in $T^a$ for the function

$$g_z(x) = \exp(h_z(x))x_1^{(\alpha_i)} \cdots x_n^{(\alpha_i)}$$

(5.12)
such that

(i) : $S_i \subset \gamma_i$,

(5.13)

(ii) : $\gamma_i \setminus \overline{S}_i \subset \{ x \in T^a \mid \text{Re}(h_z)(x) < \text{Re}(h_z)(\alpha(i)) - \varepsilon \}$ for some $\varepsilon > 0$. (5.14)

**Theorem 5.5.** In the situation as above (i.e. $0 \in \text{Int}(\Delta)$ and $z \in \Omega_0$ etc.), the elements $[\gamma_1, [\gamma_2], \ldots, [\gamma_{\text{Vol}(\Delta)}] \in (H^r_n)^* \simeq H^r_n(U_z; K_z^*)$ form a basis of the rapid decay homology group $H^r_n(U_z; K_z^*)$.

**Proof.** We use the notations in Proposition 3.3 and the proof of Theorem 4.5 by setting

$$\mathcal{L} = \mathbb{C}_{T^a}x_1^{(\alpha_i)} \cdots x_n^{(\alpha_i)}.$$ 

(5.15)

Then by Proposition 3.4 for $U_z = T$ we have an isomorphism

$$H^r_n(U_z; K_z^*) \simeq H_n(T^a \cup Q, Q; \iota_*\mathcal{L}).$$

(5.16)

Note that by (4.30) we have

$$\dim H_n(T^a \cup Q, Q; \iota_*\mathcal{L}) = \text{Vol}_Z(\Delta) = \sharp \{ \alpha(i) \}.$$ 

(5.17)

For $t \in \mathbb{R}$ we define an open subset $T^a_t \subset T^a$ of $T^a$ by

$$T^a_t = \{ x \in T^a \mid \text{Re}(h_z)(x) < t \}.$$ 

(5.18)

Then by Remark 4.6 for any $t \in \mathbb{R}$ the closure of $\partial T^a_t \subset T^a$ in $Z$ intersects each irrelevant divisor $D_i \subset Z$ transversally. This implies that for any $p \in Z$ and $t << 0$ we have

$$H_p(T^a_t \cup Q, Q; \iota_*\mathcal{L}) \simeq 0.$$ 

(5.19)

Moreover for any $p \in Z$ and $t >> 0$ we have an isomorphism

$$H_p(T^a_t \cup Q, Q; \iota_*\mathcal{L}) \simeq H_p(T^a_t \cup Q, Q; \iota_*\mathcal{L}).$$

(5.20)

Now let $-\infty < t_1 < t_2 < \cdots < t_r < +\infty$ be the critical values of $\text{Re}(h_z) : T^a \to \mathbb{R}$. Then by Remark 4.6 for any $p \in Z$ and $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, $[s_1, s_2] \cap \{ t_1, t_2, \ldots, t_r \} = \emptyset$ we have a natural isomorphism

$$H_p(T^a_{s_1} \cup Q, Q; \iota_*\mathcal{L}) \simeq H_p(T^a_{s_2} \cup Q, Q; \iota_*\mathcal{L}).$$

(5.21)
For $1 \leq j \leq r$ let $\alpha(i_1), \alpha(i_2), \ldots, \alpha(i_n) \in T^{an}$ be the critical points of $\text{Re}(h_z) : T^{an} \to \mathbb{R}$ such that $\text{Re}(h_z)(\alpha(i_j)) = t_j$. Then, for sufficiently small $0 < \varepsilon << 1$ we obtain a short exact sequence

$$0 \to H_n(T^a_{t_j - \varepsilon} \cup Q, Q; \iota_s \mathcal{L}) \to H_n(T^a_{t_j - \varepsilon} \cup (\cup_{q=1}^{r_j} S_{i_q}) \cup Q, Q; \iota_s \mathcal{L}) \to \bigoplus_{q=1}^{r_j} H_n(S_{i_q}, \partial S_{i_q}; \iota_s \mathcal{L}) \to 0$$

(5.22)

in view of (5.17), (5.19), (5.20) and the fact $\dim R_{\text{group}}(\iota_s \mathcal{L}) = 1$. Moreover, the canonical generators $[S_{i_q}] \in H_n(S_{i_q}, \partial S_{i_q}; \iota_s \mathcal{L}) = \mathbb{C}$ can be lifted to the elements $[\gamma_{i_q}]$ of $H_n(T^a_{t_j - \varepsilon} \cup (\cup_{q=1}^{r_j} S_{i_q}) \cup Q, Q; \iota_s \mathcal{L}) \subset H_n(T^{an} \cup Q, Q; \iota_s \mathcal{L})$. This implies that $[\gamma_1], [\gamma_2], \ldots, [\gamma_{\text{Volz}(\Delta)}] \in H_n(T^{an} \cup Q, Q; \iota_s \mathcal{L})$ form a basis of $H_n(T^{an} \cup Q, Q; \iota_s \mathcal{L})$. □

Note that for a connected open neighborhood $V$ of the point $z$ in $(\Omega_0)^{an}$ the basis $[\gamma_1], \ldots, [\gamma_{\text{Volz}(\Delta)}] \in (\mathcal{H}_n^{\text{rd}})_z$ constructed in Theorem 5.5 can be naturally extended to a family of the bases $[\gamma_{i_1}^{\text{rd}}], \ldots, [\gamma_{i_r}^{\text{rd}}] \in (\mathcal{H}_n^{\text{rd}})_w (w \in V)$ i.e. a basis of the local system $\mathcal{H}_n^{\text{rd}}$ on $V$. We can extend it so that $V \subset (\Omega_0)^{an}$ is stable by the multiplication of the group $\mathbb{R}_{>0}$ on $X = \mathbb{C}^N$ and the rapid decay $n$-cycles $\gamma_{i_1}^{\text{rd}}, \ldots, \gamma_{i_r}^{\text{rd}} \in \mathbb{C}$ satisfy the conditions

$$(i) : \quad S_{i_r}^{\text{rd}} \subset \gamma_{i_r}^{\text{rd}},$$

$$(ii) : \quad \gamma_{i_r}^{\text{rd}} \setminus S_{i_r}^{\text{rd}} \subset \{ x \in T^{an} \mid \text{Re}(h_w)(x) < \text{Re}(h_w)(\alpha(i_r)^{\text{rd}}) - \varepsilon \} \quad \text{for some} \, \varepsilon > 0.$$

(5.24)

(5.25)

where $S_{i_r}^{\text{rd}} \subset T^{an}$ is the stable manifold of the gradient flow of $\text{Re}(h_w)$ passing through its $i$-th non-degenerate critical point $\alpha(i_r)^{\text{rd}} \in T^{an}$. For $1 \leq i \leq \text{Volz}(\Delta)$ we define a confluent $A$-hypergeometric function $u_i$ on $V \subset (\Omega_0)^{an}$ by

$$u_i(w) = \int_{\gamma_{i_r}^{\text{rd}}} \exp\left(\sum_{j=1}^{N} w_j x^{(j)}_1 x_n^{(i_1)} \cdots x_n^{(i_r)} dx_1 \wedge \cdots \wedge dx_n \right) $$

(5.26)

for $w \in V$. Then by applying the higher-dimensional saddle point (steepest descent) method to holomorphic Morse coordinates around the critical points of $\text{Re}(h_w) : T^{an} \to \mathbb{R}$ ($w \in V$) in $T^{an}$ we obtain the following result. For $\delta > 0$ let $\Lambda_\delta \subset \mathbb{C}$ be the open sector in $\mathbb{C}$ defined by $\Lambda_\delta = \{ \lambda \in \mathbb{C} \mid -\delta < \arg \lambda < \delta \}$. By taking a sufficiently small $\delta > 0$ such that $\lambda \cdot z \in V$ for any $\lambda \in \Lambda_\delta$ we set $\Lambda := \Lambda_\delta$.

**Theorem 5.6.** In the situation as above (i.e. $0 \in \text{Int}(\Delta)$), if $\delta > 0$ is sufficiently small, for any $1 \leq i \leq \text{Volz}(\Delta)$ and $\lambda \in \Lambda$ we have an asymptotic expansion:

$$u_i(\lambda \cdot z) = \int_{\gamma_{i_r}^{\text{rd}}} \exp\left(\sum_{j=1}^{N} z_j x^{(j)}_1 x_n^{(i_1)} \cdots x_n^{(i_r)} dx_1 \wedge \cdots \wedge dx_n \right) \sim \left(\sqrt{-1}\right)^n \alpha(\iota_1)^{\text{rd}} \cdots \alpha(\iota_n)^{\text{rd}} \times \exp(\lambda \cdot h_z(\alpha(i)))$$

(5.27)

$$\times \left\{ \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{H_i(z)}} \cdot \frac{1}{\lambda^{\frac{n}{2}}} + \frac{\beta_1(z)}{\lambda^{\frac{n}{2}+1}} + \frac{\beta_2(z)}{\lambda^{\frac{n}{2}+2}} + \cdots \right\}$$

(5.28)

(5.29)

as $|\lambda| \to +\infty$ in the sector $\Lambda$, where $\beta_i(z) \in \mathbb{C}$ are functions of $z$ and

$$H_i(z) = \det \left( \frac{\partial^2 h_z}{\partial x_j \partial x_k} \right)_{x=\alpha(i)}$$

(5.30)

is the Hessian of $h_z$ at $x = \alpha(i) \in T^{an}$. 

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Proof. First, it is clear that for any \( \lambda \in \Lambda \) the critical points of the function \( \text{Re}(h_{\lambda,z}) = \text{Re}(\lambda \cdot h_z) : T^\text{an} \to \mathbb{R} \) are \( \alpha(i) \) \( (1 \leq i \leq \text{Vol}_z(\Delta)) \). Fix \( 1 \leq i \leq \text{Vol}_z(\Delta) \). Let \( \lambda = (y_1, \ldots, y_n) \), \( y_j = \xi_j + \sqrt{-1}\eta_j \) \( (1 \leq j \leq n) \) be the holomorphic Morse coordinate for the function \( h_z \) around its \( i \)-th critical point \( \alpha(i) \in T^\text{an} \) such that \( h_z(x) = h_z(\alpha(i)) + y_1^2 + \cdots + y_n^2 \). For sufficiently small \( \varepsilon > 0 \) we define an open neighborhood \( W_\varepsilon \) of \( \alpha(i) \in T^\text{an} \) by \( W_\varepsilon = \{ y = (y_1, \ldots, y_n) \mid |y_j| < \varepsilon \ (1 \leq j \leq n) \} \simeq B(0;\varepsilon) \times \cdots \times B(0;\varepsilon) \subset T^\text{an} \) and set

\[
S_i^\varepsilon = \{ \xi_1 = \cdots = \xi_n = 0 \} = \{ \eta \in \mathbb{R}^n \mid |\eta_j| < \varepsilon \ (1 \leq j \leq n) \} \subset W_\varepsilon
\]

in it. Then \( S_i^\varepsilon \) is the stable manifold of the gradient flow of the function \( \text{Re}(h_z) \) passing through its non-degenerate critical point \( \alpha(i) \in T^\text{an} \). For \( \lambda = |\lambda| e^{\sqrt{-1}\theta} \in \Lambda \subset \mathbb{C} \) \( (-\delta < \theta = \arg \lambda < \delta) \) we set \( \{ y_1', \ldots, y_n' \} = (e^{\sqrt{-1}\theta} y_1, \ldots, e^{\sqrt{-1}\theta} y_n \). Then the Laurent polynomial \( h_{\lambda,z} = \lambda \cdot h_z \) can be rewritten as

\[
\lambda \cdot h_z(x) = \lambda \cdot h_z(\alpha(i)) + |\lambda|(y_1')^2 + \cdots + |\lambda|(y_n')^2.
\]

By setting \( y_j' = \xi_j' + \sqrt{-1}\eta_j' \) \( (1 \leq j \leq n) \) we see also that the subset

\[
S^\lambda_i = \{ \xi_1' = \cdots = \xi_n' = 0 \} = \{ \eta' \in \mathbb{R}^n \mid |\eta_j'| < \varepsilon \ (1 \leq j \leq n) \} \subset W_\varepsilon
\]

of \( W_\varepsilon \) is the stable manifold of the gradient flow of \( \text{Re}(h_{\lambda,z}) \) through \( \alpha(i) \in T^\text{an} \). By our construction of the rapid decay n-cycle \( \gamma_i^\lambda \) we may assume that \( S_i^\lambda \subset \gamma_i^\lambda \) and

\[
\text{Re}(h_{\lambda,z})(x) - \text{Re}(h_{\lambda,z})(\alpha(i)) = -\varepsilon^2|\lambda|, \quad \text{Im}(h_{\lambda,z})(x) = \text{Im}(h_{\lambda,z})(\alpha(i))
\]

for any \( x \in \gamma_i^\lambda \setminus S^\lambda_i \). We may assume also that for any \( \lambda, \lambda' \in \Lambda \) such that \( \arg \lambda = \arg \lambda' \) we have \( \gamma_i^\lambda \cup \gamma_i^\lambda' = \gamma_i^\lambda \). This implies that (if \( \delta > 0 \) is sufficiently small) there exists a positive real numbers \( C > 0 \) such that

\[
\int_{\gamma_i^\lambda \setminus S^\lambda_i} \left| \exp(h_z(x) - h_z(\alpha(i))) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n \right| < C
\]

for any \( \lambda \in \Lambda \). Then we have

\[
\int_{\gamma_i^\lambda \setminus S^\lambda_i} \exp(\lambda \cdot h_z(x)) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n = \left| \exp(\lambda \cdot h_z(\alpha(i))) \right|
\]

\[
	imes \int_{\gamma_i^\lambda \setminus S^\lambda_i} \exp(\lambda \cdot \{ h_z(x) - h_z(\alpha(i)) \}) x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n
\]

\[
\leq C |\exp(\lambda \cdot h_z(\alpha(i)))| \times \sup_{x \in \gamma_i^\lambda \setminus S^\lambda_i} \left| \exp\left( \frac{\lambda - 1}{\lambda} (h_{\lambda,z}(x) - h_{\lambda,z}(\alpha(i))) \right) \right|
\]

\[
\leq C |\exp(\lambda \cdot h_z(\alpha(i)))| \times \exp\left( -\frac{\varepsilon^2}{2} |\lambda| \right)
\]

for any \( \lambda \in \Lambda \) satisfying \( |\lambda| > 0 \). Hence, to prove the theorem, it suffices to calculate the asymptotic expansion of the integral

\[
\tilde{u}_i(\lambda \cdot z) = \int_{S_i^\lambda} \exp(\lambda \sum_{j=1}^N z_j x^{a(j)} x_1^{c_1-1} \cdots x_n^{c_n-1} dx_1 \wedge \cdots \wedge dx_n)
\]

for any \( \lambda \in \Lambda \) satisfying \( |\lambda| > 0 \).
as $|\lambda| \longrightarrow +\infty$ in the sector $\Lambda$. For the Morse coordinate $y = (y_1, \ldots, y_n)$ of $h_z$ we can easily show
\[
\det \left( \frac{\partial y_j}{\partial x_k} \right)_{x=a(i)} = \sqrt{\frac{H_i(z)}{2^n}}.
\] (5.41)

Also by using the coordinate $y = (y_1, \ldots, y_n)$ set
\[
f(y_1, \ldots, y_n) := x_1^{e_1-1} \cdots x_n^{e_n-1} \times \det \left( \frac{\partial x_j}{\partial y_k} \right)
\] (5.42)
and let
\[
f(y_1, \ldots, y_n) = \sum_{a \in \mathbb{Z}_n^+} f_a y^a \quad (f_a \in \mathbb{C})
\] (5.43)
be its Taylor expansion at $y = 0$ i.e. $x = \alpha(i)$. Then by (5.41) we obtain
\[
f_0 = f(0, \ldots, 0) = \alpha(i)^{e_1-1} \cdots \alpha(i)^{e_n-1} \times \sqrt{\frac{2^n}{H_i(z)}}.
\] (5.44)

Now the restriction of the $n$-form
\[
\exp(\lambda \cdot h_z(x)) x_1^{e_1-1} \cdots x_n^{e_n-1} dx_1 \wedge \cdots \wedge dx_n
\] (5.45)
to the stable manifold $S^{nz}_i = \{ \eta' \in \mathbb{R}^n \mid |\eta'_j| < \epsilon \ (1 \leq j \leq n) \} \subset \mathbb{R}^n$ has the following form:
\[
(\sqrt{-1})^n e^{-\frac{\sqrt{-1} a_0}{2}} \exp \left\{ \lambda \cdot h_z(\alpha(i)) - |\lambda|(|\eta'_1|^2 - \cdots - |\lambda|(|\eta'_n|^2)\right\} \times \\
\sum_{a \in \mathbb{Z}_n^+} f_a \cdot e^{-\frac{\sqrt{-1} a_0}{2}} \cdot \{ \sqrt{-1} \eta' \}^a \ \mathrm{d}\eta'_1 \wedge \cdots \wedge \mathrm{d}\eta'_n.
\] (5.46)

For any $a = (a_1, \ldots, a_n) \in \mathbb{Z}_n^+$ we can show that the integral of the $n$-form
\[
\omega_a := (\sqrt{-1})^n e^{-\frac{\sqrt{-1} a_0}{2}} \exp \left\{ \lambda \cdot h_z(\alpha(i)) - |\lambda|(|\eta'_1|^2 - \cdots - |\lambda|(|\eta'_n|^2)\right\} \times \\
f_a \cdot e^{-\frac{\sqrt{-1} a_0}{2}} \cdot \{ \sqrt{-1} \eta' \}^a \ \mathrm{d}\eta'_1 \wedge \cdots \wedge \mathrm{d}\eta'_n
\] (5.47)
over the whole $\mathbb{R}^n_{\eta'}$ is equal to
\[
(\sqrt{-1})^n \lambda^{-\frac{a_0}{2}} \exp(\lambda \cdot h_z(\alpha(i))) \times \int_{\mathbb{R}^n} \exp(-t_1^2 - \cdots - t_n^2) \{ \sqrt{-1} t \}^a \ \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n
\] (5.48)
by setting $(t_1, \ldots, t_n) = (\sqrt{|\lambda|} \eta'_1, \ldots, \sqrt{|\lambda|} \eta'_n)$. Note that the integral
\[
\int_{\mathbb{R}^n} \exp(-t_1^2 - \cdots - t_n^2) \{ \sqrt{-1} t \}^a \ \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n
\] (5.49)
is zero if $a_i \in \mathbb{Z}_+$ is odd for some $1 \leq i \leq n$. As the previous part of this proof, we can show also that there exists $M > 0$ such that
\[
\left| \int_{\mathbb{R}^n \setminus S^{nz}_i} \omega_a \right| \leq M \exp(\lambda \cdot h_z(\alpha(i))) \times \exp(\frac{\epsilon^2}{2} |\lambda|)
\] (5.50)
for any $\lambda \in \Lambda$ satisfying $|\lambda| >> 0$. Then the result follows immediately from (5.44). This completes the proof. \qed
Remark 5.7. If $z \in \Omega_0$ and the critical point $\alpha(i)$ of $\text{Re}(h_z) : T^{an} \to \mathbb{R}$ in $T^{an}$ is given, by using the holomorphic Morse coordinate in the proof above we can calculate also the coefficients $\beta_1(z), \beta_2(z), \ldots \in \mathbb{C}$ explicitly.

For the point $z \in \Omega_0$ let

$$L_z = \{ \lambda \cdot z \in X = \mathbb{C}^N \mid \lambda \in \mathbb{C} \} \simeq \mathbb{C}_\lambda$$

be the complex line in $X = \mathbb{C}^N$ passing through $z \in \Omega_0$. Then by Theorems 5.5 and 5.6 we can observe Stokes phenomena for the restrictions $u_i|_{L_z}$ of the functions $u_i$ $(1 \leq i \leq \text{Vol}_Z(D))$ to the line $L_z \simeq \mathbb{C}_\lambda$. Indeed, by Theorem 5.6 the dominance ordering of the functions $u_i|_{L_z}$ at infinity (i.e. where $|\lambda| >> 0$) changes as $\text{arg}(\lambda)$ increases. More precisely, the asymptotic expansions at infinity of the restrictions of the $A$-hypergeometric functions to $L_z \simeq \mathbb{C}_\lambda$ may jump at the Stokes lines:

$$\{ \lambda \in \mathbb{C} \mid \text{Re} [\lambda \cdot \{h_z(\alpha(i)) - h_z(\alpha(j))\}] = 0 \} \quad (i \neq j).$$

It would be an interesting problem to determine the Stokes multipliers in this case.

6 The two-dimensional case

In this section, we shall construct a natural basis of the rapid decay homology group $H_n(T^{an} \sqcup Q; Q; \iota_*L)$ in the two-dimensional case i.e. $n = 2$. First, we prepare some elementary results on relative twisted homology groups. Set $Z = \mathbb{C}^2_1, z_2$ and let $h_0$ be the meromorphic function on $Z^{an}$ defined by

$$h_0(x_1, x_2) = \frac{1}{x_1^{m_1}x_2^{m_2}} \quad (m_1, m_2 \in \mathbb{Z}_{>0}).$$

Let $\pi_0 : \tilde{Z}_0 \to Z^{an}$ be the real oriented blow-up of $Z^{an}$ along the normal crossing divisor $D^{an}_0 = \{x_1 = 0\} \cup \{x_2 = 0\}$ and set $\tilde{D}_0 = \pi_0^{-1}(D^{an}_0) \subset \tilde{Z}_0$ and $U^{an}_0 = Z^{an} \setminus D^{an}_0 \simeq (\mathbb{C}^*)^2$. By the inclusion map $\iota_0 : U^{an}_0 \to \tilde{Z}_0$ we consider $U^{an}_0$ as an open subset of $\tilde{Z}_0$ and set

$$P_0 = \tilde{D}_0 \cap \{ x \in U^{an}_0 \mid \text{Re}h_0(x) \geq 0 \}$$

and $Q_0 = \tilde{D}_0 \setminus P_0$. Finally let $L_0$ be the local system of rank one on $U^{an}_0$ defined by

$$L_0 = \mathbb{C}_{U^{an}_0}x_1^{\beta_1}x_2^{\beta_2} \quad (\beta = (\beta_1, \beta_2) \in \mathbb{C}^2).$$

Then by homotopy and Lemma 3.6 we obtain the following lemma.

Lemma 6.1. (i) For $0 < \varepsilon << 1$ set

$$U^{an}_0(\varepsilon) = \{ x = (x_1, x_2) \in U^{an}_0 \mid \varepsilon < |x_1| < \frac{1}{\varepsilon} \} \subset U^{an}_0.$$

Then for any $p \in \mathbb{Z}$ the natural morphism

$$H_p(U^{an}_0(\varepsilon) \cup Q_0, Q_0; (\iota_0)_*L_0) \to H_p(U^{an}_0 \cup Q_0, Q_0; (\iota_0)_*L_0)$$

is an isomorphism.

(ii) Assume that $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$ satisfies the condition $m_2\beta_1 - m_1\beta_2 \notin \mathbb{Z}$. Then we have

$$H_p(U^{an}_0 \cup Q_0, Q_0; (\iota_0)_*L_0) \simeq 0$$

for any $p \in \mathbb{Z}$.
Proof. (i) can be easily shown by homotopy. We will prove (ii). Let $S^1$ be the unit circle $\{x_1 \in \mathbb{C} \mid |x_1| = 1\}$ in $\mathbb{C}_{x_1}$. Then by (i) and homotopy we have an isomorphism

$$H_p((S^1 \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0) \xrightarrow{\sim} H_p(U_0^{an} \cup Q_0, Q_0; (t_0)_*L_0)$$

(6.7)

for any $p \in \mathbb{Z}$. Let us take the base point $e := 1 \in S^1$ of $S^1$. Then by Lemma 3.6 we have

$$H_p(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0) \simeq 0$$

(6.8)

for $p \neq 1$ and there exists a natural basis $[\gamma_1], \ldots, [\gamma_{m_2}]$ of $H_1(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0) \simeq \mathbb{C}^{m_2}$. Let

$$\Psi_0 : H_1(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0) \xrightarrow{\sim} H_1(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0)$$

(6.9)

be the linear automorphism, i.e. the monodromy of $H_1(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0)$ induced by the (clockwise) rotation along the circle $S^1$. By the matrix representation of $\Psi_0$ with respect to the basis $[\gamma_1], \ldots, [\gamma_{m_2}]$ we see that the eigenvalues of $\Psi_0$ are contained in the set

$$\{ t \in \mathbb{C} \mid t^{m_2} = \exp[2\pi \sqrt{-1}(m_2 \beta_1 - m_1 \beta_2)] \}.$$  

(6.10)

In particular, our assumption $m_2 \beta_1 - m_1 \beta_2 \notin \mathbb{Z}$ implies that $\text{id} - \Psi_0$ is an automorphism of $H_1(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0)$. Now for $0 < \varepsilon << 1$ we define two arcs $G_{\pm} \subset S^1$ in $S^1$ by

$$G_{\pm} = \{ x_1 \in S^1 \mid \pm \text{Re}x_1 > -\varepsilon|\text{Im}x_1| \} \subset S^1.$$  

(6.11)

Then $S^1 = G_+ \cup G_-$. By the Mayer-Vietoris exact sequence for relative twisted homology groups associated to the open covering $S^1 \times \mathbb{C}^* = (G_+ \times \mathbb{C}^*) \cup (G_- \times \mathbb{C}^*)$ of $S^1 \times \mathbb{C}^*$ we can calculate $H_p((S^1 \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0)$ ($p \in \mathbb{Z}$) from $H_p((G_{\pm} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0) \simeq H_p(\{e\} \times \mathbb{C}^*) \cup Q_0, Q_0; (t_0)_*L_0)$ ($p \in \mathbb{Z}$). Then the assertion (ii) follows from the invertibility of $\text{id} - \Psi_0$. \hfill \square

Next consider the meromorphic function $h_1$ on $Z^{an} = \mathbb{C}^2$ defined by

$$h_1(x_1, x_2) = \frac{1}{(x_1 - \lambda_1)^{n_1} \cdots (x_1 - \lambda_k)^{n_k} x_1^{m_1} x_2^{m_2}} \quad (n_j, m_1, m_2 \in \mathbb{Z}_{>0}),$$

(6.12)

where $\lambda_1, \ldots, \lambda_k$ are distinct non-zero complex numbers. Let $\tau_1 : \widetilde{Z}_1 \rightarrow Z^{an}$ be the real oriented blow-up of $Z^{an}$ along $D_1^{an} = \bigcup_{j=1}^{k} \{x_1 = \lambda_j\} \cup \{x_1 = 0\} \cup \{x_2 = 0\}$ and define $D_1 \subset \widetilde{Z}_1$, $U_1^{an} = Z^{an} \setminus D_1^{an} \hookrightarrow \widetilde{Z}_1$, $P_1 \subset D_1$ and $Q_1 = D_1 \setminus P_1$ as above. Moreover let $L_1$ be the local system of rank one on $U_1^{an}$ defined by

$$L_1 = \mathbb{C}_{U_1^{an}}x_1^{\beta_1} x_2^{\beta_2} \prod_{j=1}^{k} (x_1 - \lambda_j)^{\beta_j'} \quad (\beta = (\beta_1, \beta_2) \in \mathbb{C}^2, \beta' = (\beta_1', \ldots, \beta_k') \in \mathbb{C}^k).$$

(6.13)

Then by the proof of Lemma 6.1 (ii) and Mayer-Vietoris exact sequences for relative twisted homology groups we obtain the following proposition.
Proposition 6.2. (i) For $0 < \varepsilon \ll 1$ set

$$U_1^{an}(\varepsilon) = \{(x_1, x_2) \in U_1^{an} \mid \varepsilon < |x_1| < \frac{1}{\varepsilon}, \ |x_1 - \lambda_j| > \varepsilon \ (1 \leq j \leq k)\} \subset U_1^{an}. \quad (6.14)$$

Then for any $p \in \mathbb{Z}$ the natural morphism

$$H_p(U_1^{an}(\varepsilon) \cup Q_1; (t_1)_*\mathcal{L}_1) \to H_p(U_1^{an} \cup Q_1; (t_1)_*\mathcal{L}_1) \quad (6.15)$$

is an isomorphism.

(ii) Assume that $k \geq 1$ and $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$, $\beta' = (\beta'_1, \ldots, \beta'_p) \in \mathbb{C}^k$ satisfy the conditions $m_2\beta_1 - m_1\beta_2 \notin \mathbb{Z}$ and $m_2\beta'_j - n_2\beta_2 \notin \mathbb{Z}$ for any $1 \leq j \leq k$. Then we have

$$\dim H_p(U_1^{an} \cup Q_1; (t_1)_*\mathcal{L}_1) = \begin{cases} k \times m_2 & (p = 2), \\ 0 & (p \neq 2) \end{cases} \quad (6.16)$$

and can explicitly construct a basis of the vector space $H_2(U_1^{an} \cup Q_1; (t_1)_*\mathcal{L}_1)$ over $\mathbb{C}$.

In the special but important case where $k \geq 2$ and $m_1 = m_2 = \cdots = m_k > 0$, we can construct the basis of $H_2(U_1^{an} \cup Q_1; (t_1)_*\mathcal{L}_1)$ in Proposition 6.2 (ii) very elegantly as follows. By homotopy we may assume that $\lambda_j = \exp(\frac{2\pi j}{k} \sqrt{-1}) \ (1 \leq j \leq k)$ from the first. For $1 \leq j \leq k$ let $G_j \subset S^1 = \{x_1 \in \mathbb{C} \mid |x_1| = 1\}$ be the arc in the unit circle $S^1$ between the two points $\lambda_j, \lambda_{j+1} \in S^1$, where we set $\lambda_{k+1} = \lambda_1$. For sufficiently small $\varepsilon > 0$ let $F_j$ be the boundary of the set

$$B(\lambda_j; \varepsilon) \cup F_j \subset B(\lambda_{j+1}; \varepsilon) \subset \mathbb{C}^\times \quad (6.17)$$

and denote the central point of the arc $G_j$ by $e_j \in G_j$. We regard $e_j \in F_j$ as the base point of the one-dimensional complex $F_j$. Then by Lemma 3.6 we have

$$H_p((e_j) \times \mathbb{C}^\times) \cup Q_1; (t_1)_*\mathcal{L}_1) \simeq 0 \quad (6.18)$$

for $p \neq 1$ and there exists a natural basis $[\gamma_{j1}], \ldots, [\gamma_{jm_2}]$ of $H_1((e_j) \times \mathbb{C}^\times) \cup Q_1; (t_1)_*\mathcal{L}_1) \simeq \mathbb{C}^{m_2}$. Moreover by the proof of Lemma 6.1 (ii) and Mayer-Vietoris exact sequences we obtain

$$H_p(F_j \times \mathbb{C}^\times \cup Q_1; (t_1)_*\mathcal{L}_1) \simeq 0 \quad (6.19)$$

for $p \neq 2$. Now we start from the base point $e_j \in F_j$, draw the figure 8 along $F_j$ and come back to the same place $e_j \in F_j$. By our assumption $n_j = n_{j+1}$ we can drag the twisted chains $\gamma_{j1}, \ldots, \gamma_{jm_2}$ over the figure 8 (keeping their end points in the rapid decay direction $Q_1$ of $\exp(h_1)$) to get a basis $[\delta_{j1}], \ldots, [\delta_{jm_2}]$ of $H_2(F_j \times \mathbb{C}^\times \cup Q_1; (t_1)_*\mathcal{L}_1) \simeq \mathbb{C}^{m_2}$. On the other hand, by Proposition 6.2 (i) and homotopy there exists an isomorphism

$$H_p(\{(\cup_{j=1}^k F_j) \times \mathbb{C}^\times\} \cup Q_1; (t_1)_*\mathcal{L}_1) \cong H_p(U_1^{an} \cup Q_1; (t_1)_*\mathcal{L}_1) \quad (6.20)$$

for any $p \in \mathbb{Z}$. Moreover it follows from our assumption $m_2\beta'_j - n_j\beta_2 \notin \mathbb{Z}$ that we have

$$H_p((F_j \cap F_{j-1}) \times \mathbb{C}^\times \cup Q_1; (t_1)_*\mathcal{L}_1) \simeq 0 \quad (6.21)$$

for any $1 \leq j \leq k$ and $p \in \mathbb{Z}$. Hence by the Mayer-Vietoris exact sequences associated to the covering $(\cup_{j=1}^k F_j) \times \mathbb{C}^\times = \cup_{j=1}^k (F_j \times \mathbb{C}^\times)$ of $(\cup_{j=1}^k F_j) \times \mathbb{C}^\times$ we obtain the following result.
Proposition 6.3. Assume that $k \geq 2$, $n_1 = n_2 = \cdots = n_k > 0$ and $\beta = (\beta_1, \beta_2) \in \mathbb{C}^2$, $\beta' = (\beta'_1, \ldots, \beta'_k) \in \mathbb{C}^k$ satisfy the condition $m_2 \beta'_j - n_1 \beta_2 \notin \mathbb{Z}$ for $1 \leq j \leq k$. Then the elements $[\delta_1], \ldots, [\delta_{m_2}] \in H_2(U_1^\text{an} \cup Q_1; (t_1)_*\mathcal{L}_1)$ $(1 \leq j \leq k)$ constructed above are linearly independent over $\mathbb{C}$ and form a basis of $H_2(U_1^\text{an} \cup Q_1; (t_1)_*\mathcal{L}_1)$.

Now let us return to our original situation in Sections 4 and 5 in the two-dimensional case. For $z \in \Omega$ we define $Q \subset D \subset \tilde{Z}$ in the real oriented blow-up $\pi: \tilde{Z} \rightarrow Z^\text{an}$ of $Z^\text{an} = (\tilde{Z}_\Sigma)^\text{an}$ as in the proof of Theorem 4.5. For the local system $\mathcal{L} = \mathbb{C}_{(T^\text{an}, x_1^{c_1-1}x_2^{c_2-1})}$ on $T^\text{an}$ we shall construct a basis of the rapid decay homology group $H^\text{rd}_2(T^\text{an}) := H_2(T^\text{an} \cup Q, Q; \iota_*\mathcal{L})$. By abuse of notations, for an open subset $W$ of $T^\text{an}$ and $p \in Z$ we set

$$H^\text{rd}_p(W) := H_p(W \cup Q, Q; \iota_*\mathcal{L})$$

for short. Recall that $\Sigma$ is a smooth subdivision of the dual fan of $\Delta = \text{conv}(\{0\} \cup A) \subset \mathbb{R}^2$ and $\rho_1, \ldots, \rho_l \in \Sigma$ are the rays i.e. the one-dimensional cones in $\Sigma$ which correspond to the relevant divisors $D_1, \ldots, D_l$ in $Z = \tilde{Z}_\Sigma$. We renumber $\rho_1, \ldots, \rho_l$ in the clockwise order so that we have $D_i \cap D_{i+1} \neq \emptyset$ for any $1 \leq i \leq l - 1$. By the primitive vector $\kappa_i \in \rho_i \cap (\mathbb{Z}^2 \setminus \{0\})$ on $\rho_i$ the order $m_i > 0$ of the pole of $h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)}$ along $D_i$ is explicitly given by

$$m_i = -\min_{a \in \Delta} (\kappa_i, a).$$

For $1 \leq i \leq l$ we set

$$\beta_i = (\kappa_i, (c_1 - 1, c_2 - 1)) \in \mathbb{C}.$$  

Then at each point of $D_i \setminus (\cup_{j \neq i} D_j)$ there exists a local coordinate system $(y_1, y_2)$ of $Z_\Sigma$ such that $D_i = \{y_1 = 0\}$ and the function $x_1^{c_1-1}x_2^{c_2-1}$ has the form $y_2^{\beta_i}$. Namely the function $x_1^{c_1-1}x_2^{c_2-1}$ has the order $\beta_i \in \mathbb{C}$ along $D_i$. By the non-degeneracy of $h_z$ the complex curve $h_z^{-1}(0) \subset Z_\Sigma$ intersects each relevant divisor $D_i$ transversally. Set $v_i = \#(D_i \cap h_z^{-1}(0)) \geq 0$ and $\{q_{i1}, \ldots, q_{iv_i}\} = D_i \cap h_z^{-1}(0)$. By our construction of the complex blow-up $Z = \tilde{Z}_\Sigma \rightarrow Z_\Sigma$ of $Z_\Sigma$, the fiber of the point $q_{ij}$ is a union $E_{ij} = E_{ij1} \cup \cdots \cup E_{ijm_i}$ of exceptional divisors $E_{ijk} \simeq \mathbb{P}^1$ along which $h_z$ has a pole of order $m_i - k$. See Figure 3 below. Moreover the order of the function $x_1^{c_1-1}x_2^{c_2-1}$ along $E_{ijk}$ is $\beta_i$ for any $1 \leq k \leq m_i$. Let $T_i \simeq \mathbb{C}^* \subset D_i$ be the one-dimensional $T$-orbit in $Z_\Sigma$ which corresponds to $\rho_i$ and denote by the same letter $T_i$ its strict transform in the blow-up $Z = \tilde{Z}_\Sigma$. Assume that $\beta_i \notin \mathbb{Z}$ $(1 \leq i \leq l)$ and $m_{i+1} \beta_i - m_i \beta_{i+1} \notin \mathbb{Z}$ $(1 \leq i \leq l - 1)$. Then by Propositions 6.2 (ii) and 6.3 there exists a sufficiently small neighborhood $W_i$ of $T_i^\text{an}$ in $Z^\text{an}$ such that for its open subset $W_i^\circ = W_i \cap T_i^\text{an} \subset T^\text{an}$ we have

$$\dim H^\text{rd}_p(W_i^\circ) = \begin{cases} v_i \times m_i & (p = 2), \\ 0 & (p \neq 2). \end{cases}$$

Furthermore we can explicitly construct a basis $\delta_{ijk} \in H^\text{rd}_2(W_i^\circ)$ $(1 \leq j \leq v_i$, $1 \leq k \leq m_i)$ of the vector space $H^\text{rd}_2(W_i^\circ) = H_2(W_i^\circ \cup Q, Q; \iota_*\mathcal{L})$ over $\mathbb{C}$.  

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Theorem 6.4. Assume that \( c = (c_1, c_2) \in \mathbb{C}^2 \) is nonresonant, \( \beta_i \notin \mathbb{Z} \) \((1 \leq i \leq l)\) and \( m_{i+1}\beta_i - m_i\beta_{i+1} \notin \mathbb{Z} \) \((1 \leq i \leq l-1)\). Then the natural morphisms

\[
\Theta_i : H^d_2(W^\circ_i) \longrightarrow H^d_2(T^{\text{an}}) \quad (1 \leq i \leq l)
\]

are injective and induce an isomorphism

\[
\Theta : \bigoplus_{i=1}^l H^d_2(W^\circ_i) \sim \longrightarrow H^d_2(T^{\text{an}}).
\]

In particular, the cycles \( \gamma_{ijk} := \Theta_i(\delta_{ijk}) \in H^d_2(T^{\text{an}}) \) \((1 \leq i \leq l, 1 \leq j \leq v_i, 1 \leq k \leq m_i)\) form a basis of the vector space \( H^d_2(T^{\text{an}}) = H_2(T^{\text{an}} \cup Q, Q; \iota_\ast L) \) over \( \mathbb{C} \).

Proof. By the repeated use of Lemma 6.1 (i) and homotopy, we can find a small neighborhood \( \tilde{W}_i \) of \( T^{\text{an}} \) in \( Z^{\text{an}} \) containing \( W_i \) such that for its open subset \( \tilde{W}_i = \tilde{W}_i \cap T^{\text{an}} \subset T^{\text{an}} \) the natural morphism

\[
H^d_p(W^\circ_i) \longrightarrow H^d_p(\tilde{W}_i^\circ)
\]

is an isomorphism for any \( p \in \mathbb{Z} \). By our assumption \( (c_1, c_2) \notin \mathbb{Z}^2 \) we have the vanishing of the usual twisted homology group \( H^d_p(T^{\text{an}}, \mathcal{L}) \) for any \( p \in \mathbb{Z} \). Similarly we obtain the vanishings of \( H^d_p(\tilde{W}_i^\circ; \mathcal{L}) \) etc. Then the assertion can be proved by patching these results with the help of Lemma 6.1 (i) and Mayer-Vietoris exact sequences of relative twisted homology groups. \( \square \)

Let \( \Gamma_i \prec \Delta \) be the supporting face of \( \rho_i \) in \( \Delta \). Then the lattice length of \( \Gamma_i \) is equal to \( v_i \geq 0 \) and we have the equality \( \sum_{i=1}^l (v_i \times m_i) = \text{Vol}_Z(\Delta) \) as expected from the result of Theorem 6.4.

7 Higher-dimensional cases

In this section, we shall extend the construction of the basis of the rapid decay homology group \( H_n(T^{\text{an}} \cup Q, Q; \iota_\ast L) \) for \( n = 2 \) in Section 6 to higher-dimensional cases. Let \( T_0 = (\mathbb{C}^*)_x^k \) be a \( k \)-dimensional algebraic torus and \( h_0(x) \) a Laurent polynomial on it whose Newton polytope \( \Delta_0 = NP(h_0) \subset \mathbb{R}^k \) is \( k \)-dimensional. We assume that \( h_0 \) is non-degenerate in the sense of Kouchnirenko [23]. Namely we impose the condition in Definition 2.3 for any face \( \Gamma \prec \Delta_0 \) of \( \Delta_0 \).
Proposition 7.1. In the situation as above, for generic \( a = (a_1, \ldots, a_k) \in \mathbb{C}^k \) the (possibly multi-valued) function \( g(a, x) = h_0(x)x^{-a} \) on \( T_0 \) has exactly \( \text{Vol}_2(\Delta_0) \) critical points in \( T_0 \) and all of them are non-degenerate (i.e. of Morse type) and contained in \( T_0 \setminus \{h_0 = 0\} = \{x \in T_0 \mid g(a, x) \neq 0\} \).

Proof. For \( 1 \leq i \leq k \) set \( \partial_i = \partial_{x_i} \). Then for \( x \in T_0 \) we have

\[
\partial_i g(a, x) = 0 \iff x_i \partial_i h_0(x) - a_i h_0(x) = 0. \tag{7.1}
\]

Since the hypersurface \( \{h_0 = 0\} \subset T_0 \) is smooth, by (7.1) all the critical points of the function \( g(a, \ast) \) are contained in \( T_0 \setminus \{h_0 = 0\} \). Set \( f_i(a, x) = x_i \partial_i h_0(x) - a_i h_0(x) \). Then by Bernstein’s theorem we have

\[
\# \{x \in T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k) \} = \text{Vol}_2(\Delta_0) \tag{7.2}
\]

if the Newton polytopes \( \text{NP}(f_i(a, \ast)) \) of the Laurent polynomials \( f_i(a, \ast) \) \( (1 \leq i \leq k) \) are equal to \( \Delta_0 \) and the subvariety \( K = \{x \in T_0 \mid f_i(a, x) = 0 \quad (1 \leq i \leq k)\} \) of \( T_0 \) is a non-degenerate complete intersection (see 32 etc.). From now on, we will show that these two conditions are satisfied for generic \( a \in \mathbb{C}^k \). First of all, it is clear that \( \text{NP}(f_i(a, \ast)) = \Delta_0 \) \( (1 \leq i \leq k) \) for generic \( a \in \mathbb{C}^k \). Note that \( x \in T_0 \) is in \( K \) if and only if \( x \in T_0 \setminus \{h_0 = 0\} \) and

\[
\frac{x_i \partial_i h_0(x)}{h_0(x)} = a_i \quad (1 \leq i \leq k), \tag{7.3}
\]

that is, \( x \in T_0 \setminus \{h_0 = 0\} \) is sent to the point \( a \in \mathbb{C}^k \) by the map \( T_0 \setminus \{h_0 = 0\} \mapsto \mathbb{C}^k \) defined by

\[
x \mapsto \left( \frac{x_1 \partial_1 h_0(x)}{h_0(x)}, \ldots, \frac{x_k \partial_k h_0(x)}{h_0(x)} \right). \tag{7.4}
\]

By the Bertini-Sard theorem generic \( a \in \mathbb{C}^k \) are regular values of this map. If \( a \in \mathbb{C}^k \) is such a point, we can easily show that \( \det \{\partial_{\ast} f_i(a, x)\}^{k}_{j=1} \neq 0 \) for any \( x \in K \subset T_0 \setminus \{h_0 = 0\} \). Now let \( \Gamma \nsubseteq \Delta_0 \) be a proper face of \( \Delta_0 \). Then for generic \( a \in \mathbb{C}^k \) we have

\[
\{x \in T_0 \mid f_i(a, x)^\Gamma = 0 \quad (1 \leq i \leq k)\} = \emptyset. \tag{7.5}
\]

Indeed, let us assume the converse. Then the first projection from the variety

\[
\{(a, x) \in \mathbb{C}^k \times T_0 \mid f_i(a, x)^\Gamma = 0 \quad (1 \leq i \leq k)\} \subset \mathbb{C}^k \times T_0 \tag{7.6}
\]

to \( \mathbb{C}^k \) is dominant. Moreover by \( \Gamma \neq \Delta_0 \) this variety is quasi-homogeneous with respect to the second variables \( x = (x_1, \ldots, x_k) \). In particular, its dimension is greater than \( k \). Then by considering the second projection from it to \( T_0 \), we find that there exist \( x \in T_0 \) and \( a \neq a' \in \mathbb{C}^k \) such that \( f_i(a, x)^\Gamma = f_i(a', x)^\Gamma = 0 \) for any \( 1 \leq i \leq k \). Namely we have

\[
x_i \partial_i h_0^\Gamma(x) - a_i h_0^\Gamma(x) = 0 \quad (1 \leq i \leq k), \tag{7.7}
\]

and

\[
x_i \partial_i h_0^\Gamma(x) - a_i' h_0^\Gamma(x) = 0 \quad (1 \leq i \leq k). \tag{7.8}
\]

Comparing (7.7) with (7.8) for \( 1 \leq i \leq k \) such that \( a_i \neq a_i' \), we obtain \( h_0^\Gamma(x) = \partial_i h_0^\Gamma(x) = \cdots = \partial_k h_0^\Gamma(x) = 0 \) for the point \( x \in T_0 \). This contradicts our assumption that the
Laurent polynomial $h_0$ is non-degenerate. We thus proved that the subvariety $K = \{ x \in T_0 \mid f_i(a, x) = 0 \ (1 \leq i \leq k) \}$ of $T_0$ is a non-degenerate complete intersection and its cardinality is $\text{Vol}_Z(\Delta_0)$ for generic $a \in \mathbb{C}^k$. Let us fix a point $a \in \mathbb{C}^k$. Recall that $K = \{ x \in T_0 \mid f_i(a, x) = 0 \ (1 \leq i \leq k) \}$ is the set of the critical points of the function $g(a, \ast)$ in $T_0 \setminus \{ h_0 = 0 \}$. At such a critical point $x \in T_0 \setminus \{ h_0 = 0 \}$ we have

$$\frac{\partial^2 g}{\partial x_i \partial x_j}(a, x) = \partial_j \left\{ f_i(a, x)\frac{x^{-a}}{x_i} \right\} = \partial_j f_i(a, x) \cdot \frac{x^{-a}}{x_i}. \quad (7.9)$$

Since $\det \{ \partial_j f_i(a, x) \}_{j,i=1}^k \neq 0$, we obtain

$$\det \left\{ \frac{\partial^2 g}{\partial x_i \partial x_j}(a, x) \right\}_{j,i=1}^k \neq 0. \quad (7.10)$$

Namely all the critical points of the function $g(a, \ast)$ are non-degenerate.

Let $h_0$ be as above and $\mathcal{L}_0$ a local system of rank one on $T_0^\text{an}$ defined by

$$\mathcal{L}_0 = \mathbb{C}_{T_0^\text{an}}x_1^{\beta_1} \cdots x_k^{\beta_k} \quad (\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k). \quad (7.11)$$

From now on, we will calculate the twisted homology groups $H_p(T_0^\text{an} \setminus \{ h_0 = 0 \}^\text{an}; \mathcal{L}_0)$ ($p \in \mathbb{Z}$) by using our twisted Morse theory. Taking a sufficiently generic $a \in \text{Int}(\Delta_0) \subset \mathbb{R}^k$ we set $h_1(x) = h_0(x)x^{-a}$. Then by Proposition 7.1 the real-valued function $f := |h_1|^{-2} : T_0^\text{an} \setminus \{ h_0 = 0 \}^\text{an} \longrightarrow \mathbb{R}$ has only $\text{Vol}_Z(\Delta_0)$ non-degenerate (Morse) critical points in $M := T_0^\text{an} \setminus \{ h_0 = 0 \}^\text{an}$. Moreover we can easily verify that the index of such a critical point is always $k$. For $t \in \mathbb{R}_{>0}$ we define an open subset $M_t \subset M$ of $M$ by

$$M_t = \{ x \in M \mid f(x) = |h_1|^{-2}(x) < t \}. \quad (7.12)$$

Then we have the following result.

**Proposition 7.2.** For generic $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$ and $0 < \varepsilon << 1$ we have

$$H_p(M_{\varepsilon}; \mathcal{L}_0) \simeq 0 \quad (7.13)$$

for any $p \in \mathbb{Z}$.

**Proof.** Let $\Sigma'$ be a smooth subdivision of the dual of $\Delta_0$ and $Z_{\Sigma'}$ the (smooth) toric variety associated to it. Then the divisor at infinity $D' = Z_{\Sigma'} \setminus T_0$ is normal crossing. By the non-degeneracy of $h_0$ the divisor $\overline{h_0^{-1}(0)} \cup D'$ is also normal crossing in a neighborhood of $D'$. Moreover by the condition $a \in \text{Int}(\Delta_0)$ the neighborhood $M_{\varepsilon} \cup (D' \setminus h_0^{-1}(0))$ of $D' \setminus h_0^{-1}(0)$ retracts to $D' \setminus h_0^{-1}(0)$ as $\varepsilon \longrightarrow +0$. Since for generic $\beta \in \mathbb{C}^k$ the local system $\mathcal{L}_0$ has a non-trivial monodromy around each irreducible component of $D'$, the assertion follows.

By this proposition we can apply the argument in the proof of Theorem 5.5 to the Morse function $f = |h_1|^{-2} : M = T_0^\text{an} \setminus \{ h_0 = 0 \}^\text{an} \longrightarrow \mathbb{R}$ and obtain the following theorem.
Theorem 7.3. For generic $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$ we have

$$\dim H_p(T^\text{an}_0 \setminus \{ h_0 = 0 \}^\text{an}; \mathcal{L}_0) = \begin{cases} \text{Vol}_\mathbb{C}(\Delta_0) & (p = k), \\ 0 & (p \neq k) \end{cases}$$  \hspace{1cm} (7.14)$$

and there exists a basis of $H_k(T^\text{an}_0 \setminus \{ h_0 = 0 \}^\text{an}; \mathcal{L}_0)$ indexed by the Vol$_\mathbb{C}(\Delta_0)$ non-degenerate (Morse) critical points of the (possibly multi-valued) function $h_1(x) = h_0(x) x^{-a}$ in $T^\text{an}_0 \setminus \{ h_0 = 0 \}$.

Note that Theorem 7.3 partially solves the famous problem in the paper Gelfand-Kapranov-Zelevinsky [10] of constructing a basis of the twisted homology group in their integral representation of $A$-hypergeometric functions. Indeed Theorem 7.3 holds even if we replace the local system $\mathcal{L}_0$ with the one

$$\mathcal{L}_1 = \mathbb{C}_{T^\text{an}_0 \setminus \{ h_0 = 0 \}^\text{an}} \cdot h_0(x)^{\alpha_1} x_1^{\beta_1} \ldots x_k^{\beta_k} \quad (\alpha \in \mathbb{C}, \ \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^k)$$  \hspace{1cm} (7.15)$$
on $T^\text{an}_0 \setminus \{ h_0 = 0 \}^\text{an}$.

Now we return to our original situation in Sections 4 and 5. We fix a point $z \in \Omega$ and define $Q \subset \bar{D} \subset \bar{Z}$ etc. in the real oriented blow-up $\pi : \bar{Z} \to Z^\text{an}$ of $Z^\text{an} = (\bar{Z}_0)^\text{an}$ as in the proof of Theorem 4.5. For the local system $\mathcal{L} = \mathbb{C}_{T^\text{an}} x_1^{a_1} \ldots x_n^{a_n}$ on $T^\text{an}$ we shall construct a basis of the rapid decay homology group $H^\text{rd}_n(T^\text{an}) := H_n(T^\text{an} \cup Q, Q; t_*\mathcal{L})$. As in Section 6 for an open subset $W$ of $T^\text{an}$ and $p \in \mathbb{Z}$ we set

$$H^\text{rd}_p(W) := H_p(W \cup Q, Q; t_*\mathcal{L})$$  \hspace{1cm} (7.16)$$

for short. Recall that $\rho_1, \ldots, \rho_l$ are the rays in the smooth fan $\Sigma$ which correspond to the relevant divisors $D_1, \ldots, D_l$ in $Z = \bar{Z}_\Sigma$. By the primitive vector $\kappa_i \in \rho_i \cap (\mathbb{Z}^n \setminus \{0\})$ on $\rho_i$ the order $m_i > 0$ of the pole of $h_z(x) = \sum_{j=1}^N z_j x^{a(j)}$ along $D_i$ is explicitly described by

$$m_i = -\min_{a \in \Delta}(\kappa_i, a).$$  \hspace{1cm} (7.17)$$

By the non-degeneracy of $h_z$ the complex hypersurface $\bar{h}_z^{-1}(0) \subset Z_\Sigma$ intersects each relevant divisor $D_i$ transversally. Let $T_i \simeq (\mathbb{C}^*)^{n-1} \subset D_i$ be the $T$-orbit in $Z_\Sigma$ which corresponds to $\rho_i$ and denote by the same letter $T_i$ its strict transform in the blow-up $Z = \bar{Z}_\Sigma$. Recall also that for $1 \leq i \leq l$ the Euler characteristic of the hypersurface $\{ \bar{h}_z = 0 \} = T_i \cap \bar{h}_z^{-1}(0)$ in $T_i$ is equal to $(-1)^n v_i = (-1)^n \text{Vol}_\mathbb{C}(T_i)$. Let $y = (y_1, \ldots, y_n)$ be the coordinates on an affine chart $U_i \simeq \mathbb{C}^n \subset Z_\Sigma$ of $Z_\Sigma$ containing $T_i$ such that $T_i = \{ y_n = 0 \}$ and $\mathcal{L} \simeq \mathbb{C}_{T^\text{an}} y_1^{\beta_1} \ldots y_{n-1}^{\beta_{n-1}}$. Define a local system $\mathcal{L}_i$ on $T^\text{an}_i$ by $\mathcal{L}_i = \mathbb{C}_{T^\text{an}} y_1^{\beta_1} \ldots y_{n-1}^{\beta_{n-1}}$.

Proposition 7.4. If $(\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1}$ is generic, we have

$$\dim H_p(T^\text{an}_i \setminus \{ h_z = 0 \}^{\text{an}}; \mathcal{L}_i) = \begin{cases} v_i & (p = n-1), \\ 0 & (p \neq n-1) \end{cases}$$  \hspace{1cm} (7.18)$$

and can construct a basis of $H_{n-1}(T^\text{an}_i \setminus \{ h_z = 0 \}^{\text{an}}; \mathcal{L}_i)$ by the twisted Morse theory.
Proof. If dim\(\Gamma_i = n - 1 \quad (\iff v_i > 0)\), the assertion follows immediately from Theorem 7.3. If dim\(\Gamma_i < n - 1 \quad (\iff v_i = 0)\), we have \(T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an} \simeq \mathbb{C}^* \times W\) for an open subset \(W\) of \((\mathbb{C}^*)^{n-2}\). Hence for generic \((\beta_1, \ldots, \beta_{n-1}) \in \mathbb{C}^{n-1}\) there exists an isomorphism \(H_p(T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}; \mathcal{L}_i) \simeq 0\) for any \(p \in \mathbb{Z}\).

Similarly we can prove also the following proposition.

**Proposition 7.5.** For each generic parameter vector \(c \in \mathbb{C}^n\) and \(1 \leq i \leq l\) there exists a sufficiently small neighborhood \(W_i\) of \(T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\) such that for its open subset \(W_i^o = W_i \cap T^a_n \subset T^a_n\) we have

\[
\text{dim} H^d_p(W_i^o) = \begin{cases} v_i \times m_i & (p = n), \\ 0 & (p \neq n). \end{cases} \tag{7.19}
\]

and can construct a basis \(\delta_{ijk} \in \text{H}^d_n(W_i^o)\) (\(1 \leq j \leq v_i, 1 \leq k \leq m_i\)) of \(\text{H}^d_n(W_i^o) = H_n(W_i^o \cup Q; Q; L)\) by the twisted Morse theory.

**Proof.** Let \(f_i : T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an} \rightarrow \mathbb{R}\) be the function on \(T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\) defined by \(f_i(x) = |h_{\Gamma_i}^x(x) - x - \alpha|^{-2}\) for a sufficiently generic \(a \in \text{Int}(\Gamma_i) \subset \mathbb{R}^{n-1}\). Then by Proposition 7.3 the function \(f_i\) has only \(v_i\) non-degenerate (Morse) critical points in \(T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\).

By the product decomposition \(\text{H}^d_1 \simeq \mathbb{C}^n \simeq \mathbb{C}^{n-1} \times \mathbb{C}_{\Gamma_i}\) we consider \(f_i\) also as a function on the open subset

\[
U_i^o = (T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}) \times \mathbb{C}_{\Gamma_i} \subset U_i^{an} \tag{7.20}
\]

of \(U_i^{an}\). For \(t \in \mathbb{R} > 0\) we set

\[
W_{i,t}^o = \{y \in U_i^o \cap U_i^o \mid f_i(y) < t\}. \tag{7.21}
\]

Then it follows from Lemma 6.1(i) that by shrinking \(W_i\) and taking large enough \(t_0 >> 1\) we obtain isomorphisms

\[
H^d_p(W_{i,t_0}^o) \xrightarrow{\sim} H^d_p(W_i^o) \quad (p \in \mathbb{Z}). \tag{7.22}
\]

In the same way as the proof of Proposition 7.2 for generic \(c \in \mathbb{C}^n\) and sufficiently small \(0 < \varepsilon << 1\) we can show that \(H^d_p(W_{i,t_0}^o) \simeq 0\) (\(p \in \mathbb{Z}\)). Let \(0 < t_1 < t_2 < \cdots < t_r < +\infty\) be the critical values of \(f_i : T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an} \rightarrow \mathbb{R}\). We may assume that \(t_r < t_0\). For \(1 \leq j \leq r\) let \(\alpha(1), \alpha(2), \ldots, \alpha(n_j) \in T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\) be the critical points of \(f_i\) such that \(f_i(\alpha(q)) = t_j\). Then for sufficiently small \(0 < \varepsilon << 1\) there exists a long exact sequence

\[
\cdots \cdots \cdots \rightarrow H^d_n(W_{i,t_{j-\varepsilon}}^o) \rightarrow H^d_n(W_{i,t_j}^o) \rightarrow \cdots \cdots, \tag{7.23}
\]

\[
\oplus_{q=1}^{n_j} H_n((\mathcal{S}_q \times B_{c_q}^*) \cup Q_i \cup \partial \mathcal{S}_q \times B_{c_q}^* \cup Q_i \cup \mathcal{L}) \rightarrow \cdots \cdots, \tag{7.24}
\]

where \(S_q \subset T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\) is the stable manifold of the gradient flow of \(f_i\) passing through its critical point \(\alpha(q) \in T^a_n \setminus \{h_{\Gamma_i}^x = 0\}^{an}\) and \(B_{c_q}^* \subset \mathbb{C}_{\Gamma_i}\) is the punctured disk \(\{y_n \in \mathbb{C} \mid 0 < |y_n| < \varepsilon\}\). Moreover we assume here that \(S_q\) are homeomorphic to the \((n - 1)\)-dimensional disk. Then by Lemma 3.6 the dimension of the vector space \(H_n((\mathcal{S}_q \times B_{c_q}^*) \cup Q_i \cup \partial \mathcal{S}_q \times B_{c_q}^* \cup Q_i \cup \mathcal{L})\) is \(m_i\). As a result we obtain \(\sum_{i=1}^l \text{dim} H^d_n(W_i^o) \leq \sum_{i=1}^l (v_i \times m_i) = \text{Vol}_2(\Delta) = \text{dim} H^d_n(T^a_n)\) (see (4.30)). Since for generic \(c \in \mathbb{C}^n\) we
have the vanishings of the usual twisted homology groups $H_p(T^\alpha; \mathcal{L})$ and $H_p(W^\circ; \mathcal{L})$ etc., by Mayer-Vietoris exact sequences for relative twisted homology groups we obtain also isomorphisms

$$\bigoplus_{i=1}^l H^\text{rd}_p(W^\circ_i) \xrightarrow{\sim} H^\text{rd}_p(T^\alpha) \quad (p \in \mathbb{Z}).$$

(7.25)

Therefore by (4.30) we get

$$\dim H^\text{rd}_p(W^\circ_i) = \begin{cases} v_i \times m_i & (p = n), \\ 0 & (p \neq n). \end{cases}$$

(7.26)

This implies that for any $1 \leq j \leq r$ and generic $c \in \mathbb{C}^n$ there exists a short exact sequence

$$0 \to H^\text{rd}_n(W^\circ_{i,t_j-\epsilon}) \to H^\text{rd}_n(W^\circ_{i,t_j+\epsilon}) \to \bigoplus_{q=1}^n H_n((\overline{S}_q \times B^*_v) \cup Q, (\partial S_q \times B^*_v) \cup Q; i_* \mathcal{L}) \to 0.$$  

(7.27)

Moreover by Lemma 3.6 we can construct a natural basis of the vector space $H_n((\overline{S}_q \times B^*_v) \cup Q, (\partial S_q \times B^*_v) \cup Q; i_* \mathcal{L}) \simeq \mathbb{C}^m$. Lifting these bases to $H^\text{rd}_n(W^\circ_{i,t_0}) \simeq H^\text{rd}_n(W^\circ_i)$ with the help of the above short exact sequences we obtain the one $\delta_{ijk} \in H^\text{rd}_n(W^\circ_i) (1 \leq j \leq v_i, 1 \leq k \leq m_i)$ of $H^\text{rd}_n(W^\circ_i)$. 

In the course of the proof of Proposition 7.5 we proved also the following theorem.

**Theorem 7.6.** For generic nonresonant parameter vectors $c \in \mathbb{C}^n$ the natural morphisms

$$\Theta_i : H^\text{rd}_n(W^\circ_i) \to H^\text{rd}_n(T^\alpha) \quad (1 \leq i \leq l)$$

are injective and induce an isomorphism

$$\Theta : \bigoplus_{i=1}^l H^\text{rd}_n(W^\circ_i) \xrightarrow{\sim} H^\text{rd}_n(T^\alpha).$$

(7.29)

(7.30)

In particular, the cycles $\gamma_{ijk} := \Theta_i(\delta_{ijk}) \in H^\text{rd}_n(T^\alpha) (1 \leq i \leq l, 1 \leq j \leq v_i, 1 \leq k \leq m_i)$ form a basis of the vector space $H^\text{rd}_n(T^\alpha) = H_n(T^\alpha \cup Q; i_* \mathcal{L})$ over $\mathbb{C}$.

**Remark 7.7.** Let $G \subset \mathbb{C}^n$ be the set of nonresonant parameters and $G_0 \subset G$ its subset consisting of parameters for which Theorem 7.6 holds. It is clear that $G_0$ is open dense in $G$. For a point $c \in G \setminus G_0$ assume that there exists a point $c_0 \in G_0$ close to $c$ such that the Vol$_\mathbb{Z}(\Delta)$ linearly independent rapid decay $n$-cycles constructed at $c_0$ can be continuously extended to $c$. Then we can show that the resulting ones at $c$ are also linearly independent as follows. Let $\mathcal{F}$ be the local system of rank Vol$_\mathbb{Z}(\Delta)$ on $G$ defined by the $n$-th rapid decay homology groups. Then the above families of rapid decay $n$-cycles define sections $s_i, s_2, \ldots, s_{\text{Vol}_n(\Delta)}$ of $\mathcal{F}$ on a connected open neighborhood $V$ of $c_0$ such that $c \in V$. We thus obtain a homomorphism of sheaves $\mathbb{C}^n_{\text{Vol}_\mathbb{Z}(\Delta)} \to \mathcal{F}|_V$ on $V$ whose stalk at $c_0 \in V$ is an isomorphism. By sheaf theory, it is an isomorphism over the whole $V$. This implies that the Vol$_\mathbb{Z}(\Delta)$ rapid decay $n$-cycles at $c$ are also linearly independent.
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