Bosonic binary mixtures with Josephson-type interactions

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Motivated by experiments in bosonic mixtures composed of a single element in two different hyperfine states, we study bosonic binary mixtures in the presence of Josephson interactions between species. We focus on a particular model with $O(2)$ isospin symmetry, lifted by an imbalanced population parametrized by a Rabi frequency, $\Omega_R$, and a detuning, $\nu$, which couples the phases of both species. We have studied the model at mean-field approximation plus Gaussian fluctuations. We have found that both species simultaneously condensate below a critical temperature $T_c$ and the relative phases are locked by the detuning phase, $\alpha_\nu$. Moreover, the condensate fractions are strongly dependent on the ratio $\Omega_R/|\nu|$.

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I. INTRODUCTION

Multicomponent quantum gases are fascinating systems [1]. Basic research in this area has enormously grown in the last few years [2]. Due to the ability of optically trapping and cooling gases to extremely low temperatures, it is possible to study different phenomena in bosonic [3, 4] as well as fermionic mixtures [5]. Important quantum effects like Bose-Einstein condensation (BEC) and superconductivity can now be studied in a very controlled way in multicomponent atomic systems.

Interesting experiments with mixed bosonic quantum fluids have been done by simultaneously trapping $^{87}$Rb atoms in two different hyperfine states [6, 7]. The relative population is reached by applying a coupling field characterized by a Rabi frequency $\Omega_R$ and a detuning $\nu$ with respect to the spacing between the energy levels of the two hyperfine states. In this way, it is possible to transfer atoms from one hyperfine state to the other, producing a Josephson-type interaction between species [10–12].

In general, the name “Josephson interaction” refers to the interaction of a large number of bosonic degrees of freedom allowed to occupy two different quantum states. Although it was originally proposed in superconductor systems [13], where the bosons are Cooper pairs, there are many other systems where this effect shows up. A review covering different physical systems can be found in Ref. [14]. We can distinguish two types of Josephson effects [13]: the so-called “external”, where the two states are spatially separated, like, for instance, in BEC trapped in a double-well potential [16–19], or the “internal”, where the two bosonic states are interpenetrated, without geometrical distinction, like, for instance, the experiments in Refs. [7, 8]. In this paper, we are mainly interested in the latter case of internal Josephson-type interactions.

Static and dynamical properties of binary bosonic mixtures in different trap geometries have been studied theoretically by essentially using Gross-Pitaevskii equations [20–23]. Moreover, to study properties of uniform condensates, especially those issues related with fluctuations, such as symmetry restoration, reentrances, etc., quantum field theory at finite density and temperature [24–27] is a useful technique. Related models, such as $O(N)$ models, have also been extensively studied by using large-$N$ approximation and renormalization-group techniques [28, 29]. These papers are mostly concentrated in multicomponent systems which conserve the particle number of each species independently.

Motivated by these results, we decided to address the effect of Josephson-like interactions in uniform bosonic mixtures. For simplicity, we have considered an $O(2)$ model, perturbed with an explicit symmetry-breaking term parametrized by the Rabi frequency $\Omega_R$ and the detuning term $\nu$. This model is analyzed in mean-field approximation plus Gaussian fluctuations. We show that, in a definite temperature regime, the two atomic species condensate at the same critical temperature $T_c$ and their relative phase is locked by the phase of the detuning parameter $\nu$. On the other hand, the relative population of each condensate strongly depends on the ratio $\Omega_R/|\nu|$. We have found that, due to the original $O(2)$ symmetry, the effective Rabi frequency, given by $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$, is strongly renormalized by thermal fluctuations. However, the ratio $\Omega_R/|\nu|$ remains unaffected under quantum as well as thermal fluctuations. Interestingly, we have found a reentrance of the $O(2)$ symmetry at a very small temperature in which $\Omega_{\text{eff}}(T) = 0$.

The paper is organized as follows. In section II we describe a general model for a binary mixture using quantum field theory language. In section III we concentrate on the $O(2)$ model perturbed with Josephson interactions. In section IV we present the mean-field solution, while in section V we analyze the effect of fluctuations. Numerical results are presented in section VI and, finally, we discuss our results in section VII.

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II. A QUANTUM FIELD THEORY FOR BINARY BOSONIC MIXTURES

We will consider two bosonic species described by two complex fields, \( \phi(\vec{x},t) \) and \( \psi(\vec{x},t) \). The model is defined by the action

\[
S = \int d^3x dt \left\{ \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_I \right\},
\]

where \( \mathcal{L}_\psi \) and \( \mathcal{L}_\phi \) are the non-relativistic quadratic Lagrangian densities

\[
\mathcal{L}_\psi = \psi^* \left( i\partial_t - \frac{\nabla^2}{2m} + \mu_\psi \right) \psi,
\]

\[
\mathcal{L}_\phi = \phi^* \left( i\partial_t - \frac{\nabla^2}{2m} + \mu_\phi \right) \phi.
\]

\( \mu_\phi = \mu + \Omega_R \) and \( \mu_\psi = \mu - \Omega_R \) are the chemical potentials. Throughout the paper, we have used a unit system in which \( \hbar = 1 \). The parameter \( \mu \) controls the overall particle density at the time that the Rabi frequency \( \Omega_R \) controls the population imbalance. We choose the same mass \( m \) for both species, since we are interested in mixtures composed by a single element in two different hyperfine states.

The interaction Lagrangian density \( \mathcal{L}_I \) can be split into two terms,

\[
\mathcal{L}_I = \mathcal{L}_c + \mathcal{L}_J.
\]

The first term, \( \mathcal{L}_c \), contains two-body interactions that preserves the particle number of each species individually. For diluted gases, it can be approximated as a local quartic polynomial of the form

\[
\mathcal{L}_c = -\frac{g_\psi}{2} (\psi^* \psi)^2 - \frac{g_\phi}{2} (\phi^* \phi)^2 - g_{\phi\psi} \psi^* \psi \phi^* \phi.
\]

where the coupling constants \( g_\psi = 4\pi a_\psi/m, g_\phi = 4\pi a_\phi/m \) and \( g_{\phi\psi} = 8\pi a_{\phi\psi}/m \) are written in terms of the intraspecies s-wave scattering lengths \( a_\psi, a_\phi \) and the interspecies s-wave scattering length \( a_{\phi\psi} \). Note that this interaction term is invariant under \( U(1) \otimes U(1) \) transformations.

The second term of Eq. (4) does not conserve the particle number of each species individually. It conserves, however, the total particle number. This term explicitly breaks the symmetry of Eq. (4) as \( U(1)_\phi \otimes U(1)_\psi \rightarrow U(1)_{\phi+\psi} \). We generally call these terms as Josephson interactions, since they couple the phases of each bosonic component. The simplest terms can be written as

\[
\mathcal{L}_J = \nu \psi^* \phi + \nu^* \phi^* \psi - \frac{g_J}{2} (\psi^* \psi \phi + \phi^* \phi^* \psi). \tag{6}
\]

The quadratic term, proportional to \( \nu \), and the quartic two-body interaction term have in general very different origins. The one-particle term is proportional to the detuning \( \nu \). For generality, we have considered a complex parameter in such a way to control the relative phases of the condensates. Considering the two species as components of an isospin doublet, this term arises like an effective spin-orbit interaction. Of course, we could also consider one-body terms of this type with derivative couplings \( (\partial_t \phi + \partial_t \psi) \). However, to keep matters as simple as possible, we will consider only this term. The second term in Eq. (6) represents scattering processes in which the internal hyperfine state of the atoms is not conserved. In the absence of \( \nu \), these processes are unlikely to occur, since both hyperfine states are energetically well separated. However, in the presence of a laser with small detuning between the frequency differences, a very small coupling constant \( g_J \) could produce qualitatively different results.

Some aspects of the phase diagram of the model of Eqs. (4), (5) and (6), without Josephson couplings \( (\mathcal{L}_J = 0) \), was previously studied in Ref. [24]. The zero-temperature mean-field analysis provides constraints on the coupling constants in order to have Bose-Einstein condensates. In particular, it is simple to show that

\[
g_\phi g_\psi - g_{\phi\psi}^2 > 0 \tag{7}
\]

in order to have two coexisting condensates. If the condition (7) is not satisfied, both condensates cannot coexist and they tend to spatially separate, as observed in Ref. [31]. Interestingly, quantum as well as thermal fluctuations do not change this general behavior [24]. However, the presence of Josephson couplings qualitatively changes the phase diagram of this model.

III. O(2) MODEL WITH JOSEPHSON ANISOTROPY

The model described in the preceding section has a very rich phase diagram depending on the relative values of the coupling constants and of the temperature. However, there is a special point of maximum symmetry where the analysis gets simpler. Thus, let us analyze model (4) in its maximum symmetry point given by \( g_{\phi,\psi} = g_\phi = g_\psi = g, \Omega_R = 0, \nu = 0 \) and \( g_J = 0 \). At this point, the interaction term, Eq. (5), takes the simpler form

\[
\mathcal{L}_c = -\frac{g}{2} (\psi^* \psi + \phi^* \phi)^2. \tag{8}
\]

Thus, in addition to the \( U(1) \otimes U(1) \) phase symmetry, there is an emergent \( O(2) \) symmetry, corresponding with rotations in the isospin space \( (\phi, \psi) \). On the one hand, the particle number of each species is independently conserved. On the other hand, the two species are indistinguishable, since any isospin rotation mixing the two species has exactly the same action.

Next, we minimally break the \( O(2) \) symmetry by considering a term proportional to \( \Omega_R \) and a Josephson term of the form

\[
\mathcal{L}_\nu = \nu \psi^* \phi + \nu^* \phi^* \psi. \tag{9}
\]
For simplicity, we ignore two-body Josephson interactions (given by the term proportional to $g_{ij}$ in Eq. (5)), since, in principle, it is of higher order than the one-body interaction term we are considering.

The structure of this model is clearly visualized by defining new fields $(\varphi_1, \varphi_2)$ obtained by an isospin rotation of the original fields $(\phi, \psi)$,

$$
(\varphi_1 
\varphi_2
) = M 
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix},
$$

where the rotation matrix is

$$
M = \frac{1}{D}
\begin{pmatrix}
\Omega_{\text{eff}} - \Omega_R & -\nu \\
\nu^* & \Omega_{\text{eff}} - \Omega_R
\end{pmatrix},
$$

with

$$
D = \sqrt{\Omega_{\text{eff}}^2 - \Omega_R^2 + |\nu|^2}.
$$

and $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$. Of course, one can immediately check that $\text{det}(M) = 1$. With this transformation the Lagrangian density takes the form

$$
\mathcal{L} = \varphi_1^* \left( i \partial_t - \frac{\nabla^2}{2m} + \mu_+ \right) \varphi_1 + \\
+ \varphi_2^* \left( i \partial_t - \frac{\nabla^2}{2m} + \mu_- \right) \varphi_2 - \\
- \frac{g}{2} (\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2)^2,
$$

where $\mu_{\pm} = \mu \pm \Omega_{\text{eff}}$. We see that, while the terms proportional to $\nu$ and $\Omega_R$ in Eq. (8) break the $O(2)$ and $U(1)_\phi \otimes U(1)_\psi$ symmetries, the system still has an $U(1)_{\varphi_1} \otimes U(1)_{\varphi_2}$ symmetry in the new variables. Thus, there is a direction in isospin space in which the particle number of both species is conserved independently. This simple behavior is a consequence of the $O(2)$ symmetry of the two-body interaction term, Eq. (8). It is not difficult to realize that, in the case of imbalanced two-body interactions, $g_{\phi \psi} \neq g$, a term proportional to $\varphi_1 \varphi_1 \varphi_2^* \varphi_2^*$ would be generated upon an isospin rotation, in this way breaking $U(1)_{\varphi_1} \otimes U(1)_{\varphi_2} \to U(1)$. In this sense, the model of Eq. (13) implements a minimal perturbation of the complex $O(2)$ model.

Interestingly, Eq. (13) does not depend on $\Omega_R$ and $\nu$ independently, but only depends on the effective Rabi frequency $\Omega_{\text{eff}} = \sqrt{\Omega_R^2 + |\nu|^2}$. On the other hand, the rotation matrix of Eq. (11) depends only on the ratio $\Omega_R/|\nu|$. It is instructive to see the form of the rotation matrix in two different limits.

Let us consider, for instance, $|\nu| \ll \Omega_R$. In this case,

$$
M = \begin{pmatrix}
\frac{|\nu|}{2\Omega_R} & -e^{i\alpha \nu} \\
e^{-i\alpha \nu} & \frac{|\nu|}{2\Omega_R}
\end{pmatrix},
$$

where we have defined $\nu = |\nu| \exp(i\alpha \nu)$. In the extreme limit of $\nu \to 0$, the two species are decoupled, as expected, and the mixture is proportional to $|\nu|/2\Omega_R$.

In the opposite limit, $|\nu| \gg \Omega_R$,

$$
M = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 - \frac{\Omega_R}{2|\nu|} & -e^{i\alpha \nu} \left( 1 + \frac{\Omega_R}{2|\nu|} \right) \\
e^{-i\alpha \nu} \left( 1 + \frac{\Omega_R}{2|\nu|} \right) & 1 - \frac{\Omega_R}{2|\nu|}
\end{pmatrix}.
$$

In the extreme limit $\Omega_R \to 0$, the fields are symetrically superposed, depending just on the phase of the detuning parameter,

$$
\varphi_1 = \frac{1}{\sqrt{2}} \left( \phi - e^{i\alpha \nu} \psi \right),
$$

$$
\varphi_2 = \frac{1}{\sqrt{2}} \left( e^{-i\alpha \nu} \phi + \psi \right).
$$

Small values of $\Omega_R$ produce corrections of order $\Omega_R/|\nu|$.

### IV. MEAN-FIELD APPROXIMATION

Let us analyze the model of Eq. (13) in the mean-field approximation. This leads to the Gross-Pitaevskii equations

$$
\left( i \partial_t - \frac{\nabla^2}{2m} + \mu_+ - g (\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2) \right) \varphi_1 = 0,
$$

$$
\left( i \partial_t - \frac{\nabla^2}{2m} + \mu_- - g (\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2) \right) \varphi_2 = 0.
$$

Looking for uniform and static solutions $\varphi_{1,2}(x, t) \equiv \varphi_{1,2}^0$ we have

$$
(\mu_+ - g |\varphi_1^0|^2 + |\varphi_2^0|^2) \varphi_1^0 = 0,
$$

$$
(\mu_- - g |\varphi_1^0|^2 + |\varphi_2^0|^2) \varphi_2^0 = 0.
$$

Supposing that $\varphi_{1,2}^0 \neq 0$, we can subtract eq. (20) from eq. (21), obtaining $\Delta \mu = \mu_+ - \mu_- = 0$. Therefore, the two fields $\varphi_{1,2}$ cannot condensate simultaneously, since a solution $\varphi_{1,2}^0 \neq 0$ does not exist, except in the case $\Delta \mu = 2\Omega_{\text{eff}} = 0$. Instead, we have two possible solutions,

$$
\varphi_1^0 = 0 \quad , \quad |\varphi_2^0|^2 = \mu_-/g
$$

or

$$
|\varphi_1^0|^2 = \mu_+/g \quad , \quad \varphi_2^0 = 0
$$

Let us consider the solution $\varphi_2^0 = 0$, Eq. (23). Using the matrix $M^{-1}$, given by the inverse of Eq. (11), it is simple to turn back to the original fields, obtaining

$$
\phi_0 = -\frac{\Omega_{\text{eff}} - \Omega_R}{\sqrt{(\Omega_{\text{eff}} - \Omega_R)^2 + |\nu|^2}} \varphi_1^0,
$$

$$
\psi_0 = -\frac{\nu^*}{\sqrt{(\Omega_{\text{eff}} - \Omega_R)^2 + |\nu|^2}} \varphi_0^0.
$$
where $\phi_0$ and $\psi_0$ are the condensate amplitudes of the fields $\phi(x)$ and $\psi(x)$, respectively. The first observation is that the two original species $\phi$ and $\psi$ condense simultaneously and the relative phase between these condensates, $\Delta \alpha$, is fixed by the phase of the parameter $\nu$,

$$\Delta \alpha = \alpha_\nu + \pi.$$  

Moreover, the condensate fractions of the two species depend on the ratio $\Omega_R/|\nu|$. It is instructive to parametrize $\Omega_R$ and $|\nu|$ in the following way,

$$\Omega_R = \frac{\Delta \mu}{2} \sin \theta,$$

$$|\nu| = \frac{\Delta \mu}{2} \cos \theta,$$

with $0 \leq \theta \leq \pi/2$. In terms of this parametrization, the ratio between the condensate densities takes the form

$$\frac{|\phi_0|^2}{|\psi_0|^2} = \sec^2 \theta \left(1 - \sin \theta \right)^2,$$

which does not depend on $\Delta \mu$ but only on $\tan \theta = \Omega_R/|\nu|$. We depict this function in Fig. 11. For $\theta \to 0$, or $\Omega_R \to 0$ with $|\nu| \neq 0$, the two condensates have essentially the same fraction. On the other hand, for $\theta \to \pi/2$, or $|\nu| \to 0$ with $\Omega_R \neq 0$, only one of the fields condenses.

An important question is how thermal fluctuations affect the general structure of the mean-field zero-temperature result. To see this, let us consider the effective potential

$$U_{\text{eff}}(\varphi_1, \varphi_2) = U_0(\varphi_1, \varphi_2) + \Delta U(\varphi_1, \varphi_2, T),$$

where $\varphi_{1,2}$ represent the order parameters, $U_0$ is the bare potential and $\Delta U$ codifies the effect of fluctuations. Two important symmetries constrain the general form of this potential. On one hand, the $U(1) \otimes U(1)$ symmetry implies that the effective potential is a function of $|\varphi_1|^2$ and $|\varphi_2|^2$. On the other hand, since $O(2)$ symmetry is broken by the parameter $\Delta \mu = \mu_+ - \mu_-$, the potential should be a function of $\chi_s = |\varphi_1|^2 + |\varphi_2|^2$ and $\chi_a = |\varphi_1|^2 - |\varphi_2|^2$ independently. The subindices $s$ and $a$ stand for symmetric and antisymmetric, respectively. Then,

$$U_{\text{eff}} = U_0(\chi_s, \chi_a) + \Delta U(\chi_s, \chi_a, T).$$

Minimizing the effective potential and assuming that $\varphi_{1,2} \neq 0$, we immediately find

$$\mu_+ + \frac{\partial \Delta U}{\partial \chi_s} - g \left(|\varphi_1|^2 + |\varphi_2|^2\right) + \frac{\partial \Delta U}{\partial \chi_a} = 0$$

$$\mu_- + \frac{\partial \Delta U}{\partial \chi_s} - g \left(|\varphi_1|^2 + |\varphi_2|^2\right) - \frac{\partial \Delta U}{\partial \chi_a} = 0.$$

These equations imply that

$$\Delta U(T) = -\frac{1}{2} \Delta \mu(T) \chi_a + F(\chi_s, T),$$

where $F$ is an arbitrary function. Since there is no reason to expect a linear function $\Delta U \sim \chi_a$, we conclude that, in this model, both fields $\varphi_1$ and $\varphi_2$ cannot condensate simultaneously, provided $\Delta \mu(T) \neq 0$. By carefully studying thermal fluctuations, we will see in the next section that, depending on density ratio, it could be a reentrance of the $O(2)$ symmetry, $\Delta \mu(T_r) = 0$, at extremely low temperatures $T_r \ll T_c$, changing the mean-field structure solution.

V. EFFECT OF FLUCTUATIONS

To study thermal as well as quantum fluctuations, we start by considering the following Euclidean finite temperature field theory:

$$S_{\text{E}}(\beta) = \int_0^\beta d\tau \int d^3x \left[ \varphi_1^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu_+ \right) \varphi_1 
+ \varphi_2^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu_- \right) \varphi_2 
- \frac{g}{2} \left( \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2 \right)^2 \right],$$

with $\beta = 1/T$. The partition function reads

$$Z(\beta, \vec{J}) = \int D\varphi_1 D\varphi_2^* D\varphi_2 D\varphi_2^* e^{-S_{\text{E}} + \int d^3x d\tau \vec{J} \cdot \vec{\varphi}},$$

where we have introduced a source $\vec{J}$ in order to compute field correlation functions. The functional integration measure implicitly contains the cyclic bosonic boundary condition in Euclidean time, $\varphi_{1,2}(0, x) = \varphi_{1,2}(\beta, x)$. $W[\beta, \vec{J}] = -\frac{1}{ \beta V } \text{ln} Z$ is the Helmholtz free energy density.

The main purpose of this section is to compute $W[\beta, \vec{J}]$ in mean-field approximation plus Gaussian fluctuations. As discussed in the last section, we expect that, at least in a certain temperature range to be determined later,
fluctuations will not change the general mean-field structure. With this in mind, in order to compute $W[\beta, J]$ we replace in Eq. (36) the following decomposition

$$
\varphi_1(x, \tau) = \varphi_1^0 + \tilde{\varphi}_1(x, \tau) \\
\varphi_2(x, \tau) = \tilde{\varphi}_2(x, \tau)
$$

in which $\int d^3x \tilde{\varphi}_{1,2} = 0$ and $\varphi_1^0(J)$ is a solution of

$$
\frac{\delta S_E}{\delta \varphi_1} = J, \\
\frac{\delta S_E}{\delta \varphi_2} = 0.
$$

where we have chosen $J = \text{const.}$ in the $\varphi_1$ direction.

Retaining up to second-order terms in the fluctuations we obtain

$$
Z(\beta) = e^{-\beta V_{(0)}(\varphi_1^0)} \int [D\tilde{\varphi}] e^{-\int d^3x \sum_{ij} \tilde{\varphi}_i^* S^{(2)}_{ij} \tilde{\varphi}_j},
$$

where

$$
U_0 = -\mu + \varphi_1^0|^2 + \frac{g}{2} \varphi_1^0|^4.
$$

The integration measure is

$$
[D\tilde{\varphi}] = D\tilde{\varphi}_1 D\varphi_1^* D\tilde{\varphi}_2 D\varphi_2^*
$$

and the quadratic kernel

$$
S^{(2)}_{ij} = \frac{\delta^2 S_E}{\delta \varphi_i \delta \varphi_j} \bigg|_{\varphi_1 = \varphi_1^0, \varphi_2 = 0},
$$

with $i, j = 1, 2$.

Integrating out quadratic fluctuations, we find an expression for the free energy density,

$$
W[J, \beta] = U_0 + \Delta W
$$

with

$$
\Delta W[J, \beta] = \frac{1}{2} \ln \det \hat{S}^{(2)} = \frac{1}{2} \text{Tr} \ln \hat{S}^{(2)}.
$$

The matrix $\hat{S}^{(2)}$ in the (Re($\tilde{\varphi}_1$), Im($\tilde{\varphi}_1$), Re($\tilde{\varphi}_2$), Im($\tilde{\varphi}_2$)) basis decouples into two independent $2 \times 2$ blocks,

$$
\hat{S}^{(2)} = \begin{pmatrix}
\hat{S}_a^{(2)} & 0 \\
0 & \hat{S}_b^{(2)}
\end{pmatrix},
$$

with

$$
\hat{S}_a^{(2)} = \begin{pmatrix}
-\frac{\nu_1^2}{2m} - \mu_+ + 3g|\varphi_1^0|^2 & i\partial_\tau \\
-i\partial_\tau & -\frac{\nu_1^2}{2m} - \mu_+ + g|\varphi_1^0|^2
\end{pmatrix}
$$

and

$$
\hat{S}_b^{(2)} = \begin{pmatrix}
-\frac{\nu_1^2}{2m} - \mu - g|\varphi_1^0|^2 & -i\partial_\tau \\
-i\partial_\tau & -\frac{\nu_1^2}{2m} - \mu - g|\varphi_1^0|^2
\end{pmatrix}.
$$

It is not difficult to compute the trace in Fourier space, obtaining

$$
\Delta W = \frac{1}{2\beta} \sum_{n=0}^{\infty} \int \frac{d^3q}{(2\pi)^3} \ln \left\{ (\omega_n^2 + E_1^2) (\omega_n^2 + E_2^2) \right\},
$$

where $\omega_n = 2\pi n \beta$ are the Matsubara frequencies,

$$
E_1 = \sqrt{\frac{q^2}{2m} - \mu_+ + 3g|\varphi_1^0|^2} \left( \frac{q^2}{2m} - \mu_+ + g|\varphi_1^0|^2 \right),
$$

and

$$
E_2 = \frac{q^2}{2m} - \mu - g|\varphi_1^0|^2.
$$

Summing up the Matsubara frequencies, using

$$
\frac{1}{\beta} \sum_n \ln (\omega_n^2 + E_1^2) = E_1 + \frac{2}{\beta} \ln (1 - e^{-\beta E_1}),
$$

we obtain

$$
\Delta W = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \sum_i \left\{ E_i + \frac{2}{\beta} \ln (1 - e^{-\beta E_i}) \right\}.
$$

It is interesting to note that, if we substitute the mean-field value for $\varphi_1^0$, given by Eq. (23), into Eqs. (51) and (52), we immediately obtain

$$
E_1 = \sqrt{\left( \frac{q^2}{2m} \right) \left( \frac{q^2}{2m} + 2g|\varphi_1^0|^2 \right)},
$$

and

$$
E_2 = \frac{q^2}{2m} + \Omega_R^2 + |\mu|^2.
$$

Eqs. (55) and (56) are the usual energy excitations computed in the Bogoliubov approximation. Note that $\lim_{q \to 0} E_1 = 0$, corresponding with the Goldstone mode associated with the spontaneous breakdown of the $U(1)$ symmetry, while Eq. (56) is a gapped mode corresponding to non-condensate fluctuations.

It is useful to express the free energy $W(\beta, J)$ in terms of the order parameter:

$$
\varphi = \delta W/\delta J = \varphi_1^0 + \frac{1}{2} \text{Tr} \begin{pmatrix}
\frac{1}{2} & \delta \hat{S}^{(2)} \\
\delta \varphi_1^0 & \delta J
\end{pmatrix}.
$$

At mean-field level, the order parameter is exactly the mean-field solution $\varphi_1^0$. However, when fluctuations are taken into account, the result given by Eq. (57) is more involved.
We define the Gibbs free energy as a functional of the order parameter \( \bar{\varphi} \) by making a Legendre transformation
\[
\Gamma[\beta, \bar{\varphi}] = \bar{\varphi}J - W ,
\]
where \( \delta \Gamma/\delta \bar{\varphi} = J \). In Eq. (58), \( J \) is a function of the order parameter \( \bar{\varphi} \) obtained by inverting Eq. (57). To leading order in the fluctuations the result is
\[
\Gamma[\beta, \bar{\varphi}] = \mu_+ |\bar{\varphi}|^2 - \frac{\beta}{2} |\bar{\varphi}|^4 - \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \sum_n \left\{ E_i + \frac{2}{\beta} \ln (1 - e^{-\beta E_i}) \right\} .
\]
(59)
This is the Gibbs free energy computed at mean field plus Gaussian fluctuations or, in the language of quantum field theory, the finite temperature one-loop effective action.

The actual condensate amplitude \( \bar{\varphi}_m \) is computed by minimizing the free energy,
\[
\frac{\partial \Gamma[\beta, \bar{\varphi}]}{\partial \bar{\varphi}} \bigg|_{\bar{\varphi} = \bar{\varphi}_m} = 0. 
\]
(60)
Using eq. (59) and defining an effective chemical potential \( \bar{\mu}_+ = g |\bar{\varphi}_m|^2 \), we find an expression for \( \bar{\mu}_+ \) in terms of the original \( \mu_+ \) and of the temperature:
\[
\bar{\mu}_+ = \mu_+ - \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} x \left\{ \left( \frac{\bar{\mu}_+}{2} + \mu_+ \right)(1 + 2n(E_1)) \right\} / \sqrt{\frac{\pi}{m}} \left( \frac{\bar{\mu}_+}{2} + 2\mu_+ \right) + 1 + 2n(E_2) \}
\]
(61)
where \( n(E_i) \) is the usual Bose distribution
\[
n(E_i) = \frac{1}{e^{E_i/T} - 1} 
\]
(62)
with \( i = 1, 2 \).

The total particle density of each species is given by
\[
\rho_{\varphi_1} = \frac{\partial \Gamma}{\partial \mu_+} \bigg|_{\bar{\varphi}_m} = \frac{\mu_+}{g} - \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{q^2/2m}{E_1} \left( 1 + 2n(E_1) \right) - \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left( 1 + 2n(E_2) \right), 
\]
(63)
\[
\rho_{\varphi_2} = \frac{\partial \Gamma}{\partial \mu_-} \bigg|_{\bar{\varphi}_m} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left( 1 + 2n(E_2) \right). 
\]
(64)
Using the relation between \( \mu_+ \) and \( \bar{\mu}_+ \) given by Eq. (61), we finally get
\[
\rho_{\varphi_1} = \frac{\bar{\mu}_+}{g} + \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\bar{\mu}_+}{E_+} \coth(\beta E_+/2), 
\]
(65)
\[
\rho_{\varphi_2} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \coth(\beta E_-/2), 
\]
(66)
with
\[
E_+ = \sqrt{\frac{g^2}{2m} \left( \frac{g^2}{2m} + 2\bar{\mu}_+ \right)}, 
\]
(67)
\[
E_- = \frac{g^2}{2m} + \bar{\mu}_+ - \bar{\mu}_-. 
\]
(68)
From Eq. (67), it is evident that the Goldstone mode is preserved at finite interaction and temperature, as it should be. Since \( \rho_{\varphi_1} \) and \( \rho_{\varphi_2} \) are constants, Eqs. (65), (66), (67) and (68) are coupled equations for the variables \( \bar{\mu}_+ \) and \( \bar{\mu}_- \) as functions of the temperature \( T \).

Expressions (65) and (66) have the usual ultraviolet divergences of a field theory at \( T = 0 \). As is well known, temperature fluctuations are always convergent. The usual way to deal with this divergence is to regularize the integral and then renormalize the bare constants, \( \mu_+ \) and \( g \), in order to obtain finite results. A convenient procedure, in the non-relativistic scalar case, is the cut-off technique. If we simply limit the momentum integrals using an ultraviolet cut-off, \( 0 \leq |q| \leq \Lambda \), the results are obviously \( \Lambda \)-dependent. However, if we begin the calculations with renormalized constants, \( \mu_+^R = \mu_+ + \delta \mu_+ (\Lambda) \), we can adjust \( \delta \mu_+ (\Lambda) \) to make the result independent of \( \Lambda \) . At the end, we can safely take the limit \( \Lambda \to \infty \). After this procedure, the renormalized expressions read
\[
\rho_{\varphi_1} = \frac{\bar{\mu}_+}{g} + \frac{(m\bar{\mu}_+)^{3/2}}{3\pi^2} + \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{\bar{\mu}_+}{e^{\beta E_+} - 1}, 
\]
(69)
\[
\rho_{\varphi_2} = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{e^{\beta E_-} - 1}. 
\]
(70)
Eq. (69) implicitly defines the condensate density \( \bar{\varphi}_m(T) \) or, equivalently, the effective chemical potential \( \bar{\mu}_+(T) \). This equation coincides with that derived from a one-loop effective potential of a single self-interacting field \[24\]. Moreover, eq. (70) determines the effective Rabi frequency, \( \Omega_{\text{eff}}(T) = (\bar{\mu}_+ - \bar{\mu}_+)/2 = \Delta \mu/2 \). The critical temperature, \( T_c \), is easily computed by fixing \( \bar{\mu}_+ = 0 \) in Eq. (69), obtaining the usual expression for an ideal gas,
\[
T_c = \frac{2\pi}{m\zeta(3/2)^2} \bar{\mu}_{\varphi_1}^{3/2}, 
\]
(71)
with \( \zeta(3/2) \sim 2.612 \). We expect corrections of \( T_c \) only at a two-loop approximation\[32, 33\]. Since \( E_- \) represents gapped excitations, the integral in eq. (70) can be safely done in the classical limit, obtaining
\[
\rho_{\varphi_2} \sim \frac{1}{4\sqrt{2\pi}} \left( m\tau^3 \right)^{3/2} g \frac{\Delta \mu}{\bar{\mu}_+}. 
\]
(72)
In terms of the constant densities \( \rho_{\varphi_1}, \rho_{\varphi_2} \), and the critical temperature \( T_c \), given by Eq. (71), we obtain from Eq. (72)
\[
\Delta \mu = 2\Omega_{\text{eff}}(T) = T \ln \left( \frac{\rho_{\varphi_1}}{\rho_{\varphi_2}} \left( \frac{T}{T_c} \right)^{3/2} \right). 
\]
(73)
Since $\Omega_{\text{eff}}(T) > 0$, this calculation is valid only for
\[
\frac{T}{T_c} > \left( \frac{\rho_c}{\rho_{c2}} \right)^{2/3} \left( \frac{\rho_{c2}}{\rho_{c1}} \right)^{2/3} \left( \frac{\rho_{c2}}{\rho_{c3}} \right)^{2/3}.
\] (74)

Surprisingly, there is a minimum temperature $T_\rho = (\rho_{c2}/\rho_{c3})^{2/3}T_c$ where $\Delta \mu(T_\rho) = 0$, the $O(2)$ symmetry is restored and then our mean-field analysis is no longer valid for temperatures $T < T_\rho$. Therefore, in order to have the condensate structure given by eqs. (24) and (25), we need to fix $\rho_{c2}/\rho_{c3} < 1$ and $T_\rho < T < T_c$. This interesting reentrance is not an artifact of the classical approximation used to compute the integral in Eq. (70).

In fact, by solving the integral exactly using numerical methods, we find the same general behavior. In the next section, we numerically compute the condensate fractions as functions of temperature for different values of the parameters.

VI. NUMERICAL RESULTS

To compute the condensate density profile we rewrite Eq. (69) in dimensionless form. For this, we define the condensate fraction $\rho_c = \bar{\rho}_c / (g\rho_{c1})$. The dimensionless temperature is defined as $\bar{T} = T/T_c$, and we introduce the diluteness parameter $n_{c1} = \rho_{c1} a^3$, where $a$ is the s-wave scattering length. Using these definitions, we can write Eq. (69) in the following form:

\[
1 = \rho_c + \frac{8}{3\pi^{1/2} n_{c1}^{1/2} \rho_c^{3/2}} + \frac{4}{\pi^{1/2} (3/2)^{3/2}} \int_0^\infty dy y^2 e^{-y^2/2} \frac{2\zeta(3/2)^{2/3} n_{c1}^{1/3} \rho_c \bar{T}^{-1}}{\bar{T}^{2/3} n_{c1}^{1/3} \rho_c \bar{T}^{-1}} \left( e^{-y^2/2} + \frac{4\zeta(3/2)^{2/3} n_{c1}^{1/3} \rho_c \bar{T}^{-1}}{\bar{T}^{2/3} n_{c1}^{1/3} \rho_c \bar{T}^{-1}} - 1 \right)^{-1}.
\] (75)

It is simple to check that the limit $n_{c1} \to 0$ leads to the ideal gas result $\rho_c = 1 - \bar{T}^{3/2}$. The second term of the r.h.s. of Eq. (75) gives the quantum depletion of the condensate, while the third term represents the temperature dependence. Numerically solving Eq. (75), we can obtain the condensate fraction $\rho_c(\bar{T})$ for different values of the diluteness parameter $n_{c1}$. From this result, it is simple to compute the condensate fractions for the fields $\phi$ and $\psi$ using Eqs. (24) and (25).

We define the condensate fractions for the fields $\phi$ and $\psi$ as $\rho_{\phi} = |\phi|^2 / (\rho_{c2} + \rho_{c3})$ and $\rho_{\psi} = |\psi|^2 / (\rho_{c1} + \rho_{c3})$, where we chose the total particle density $\rho_{\phi} + \rho_{\psi} = \rho_{c1} + \rho_{c2}$ to normalize the fractions. Then, we use Eqs. (24) and (25) to relate $\rho_{c2}$ and $\rho_{c3}$ with $\rho_{c}$, given by Eq. (69).

There are two interesting regimes to focus on. For $|\nu|/\Omega_R \ll 1$, the condensate fractions become

\[
\rho_{\phi} \sim \left( 1 - \frac{\rho_{c2}}{\rho_{c1}} \right) \frac{|\nu|}{2\Omega_R} \rho_c,
\] (76)

\[
\rho_{\psi} \sim \left( 1 - \frac{\rho_{c2}}{\rho_{c1}} \right) \rho_c.
\] (77)

The first factor compensates the normalizations of $\rho_{\phi,\psi}$ and $\rho_c$. To obtain it, we have considered $\rho_{c2}/\rho_{c3} < 1$ and we have dropped terms proportional to $(\rho_{c2}/\rho_{c3})^2$. The condensate fraction is determined by the factor $|\nu|/2\Omega_R$ and the next corrections to eqs. (76) and (77) are terms proportional to $|\nu|/2\Omega_R^2$.

In the opposite regime $\Omega_R/|\nu| \ll 1$, the condensate densities of both species are essentially equal, with small corrections, given by

\[
\rho_{\phi0} \sim \frac{1}{2} \left( 1 - \frac{\rho_{c2}}{\rho_{c1}} \right) \left( 1 - \frac{\Omega_R}{|\nu|} \right) \rho_c,
\] (78)

\[
\rho_{\psi0} \sim \frac{1}{2} \left( 1 - \frac{\rho_{c2}}{\rho_{c1}} \right) \left( 1 + \frac{\Omega_R}{|\nu|} \right) \rho_c,
\] (79)

where we have discarded corrections of order $(\Omega_R/|\nu|)^2$.

We show these curves in Fig. (2) for $n_{c1} = 10^{-5}$, $\rho_{c2}/\rho_{c3} = 10^{-2}$, and $\Omega_R/|\nu| = 0.24$.

VII. DISCUSSION

We have addressed the problem of equilibrium properties of a uniform mixture of two bosonic fields in the presence of Josephson-type interactions. We have considered a quantum field theory built by two non-relativistic complex bosonic fields with general two-body local interactions. We have focused on a particular symmetry point, in which, in addition to the $U(1) \otimes U(1)$ phase
number of each bosonic species denoted in this framework. In this basis, the particle $\phi$ the we have shown that there is a special direction for which

\[
\text{symmetry}, \text{there is an emergent } O(2) \text{ symmetry, related with rotations in the isospin space } (\phi, \psi). \text{ In the absence of Josephson interactions, this model generally presents phase separation. However, interactions that explicitly breaks the } U(1) \otimes U(1) \otimes O(2) \text{ symmetry changes this scenario.}
\]

We have minimally perturbed this model by considering the effect of Josephson couplings that unbalance the species population by transferring charge from one species to the other. These interactions are parametrized by the Rabi frequency $\Omega_R$ and the detuning $\nu$. By making a rotation in the isospin space, $(\phi, \psi) \rightarrow (\varphi_1, \varphi_2)$, we have shown that there is a special direction for which the $U(1) \otimes U(1) \otimes O(2)$ phase symmetry is recovered and only one of the bosonic species (say $\varphi_1$) could eventually condensate in this framework. In this basis, the particle number of each bosonic species $\rho_{\varphi_1}$ and $\rho_{\varphi_2}$ is conserved independently. Of course, the $O(2)$ symmetry is still broken, provided the difference between chemical potentials $\Delta \mu = \mu_+ - \mu_- = 2\Omega_{\text{eff}} \neq 0$.

In the $(\varphi_1, \varphi_2)$ basis, it is simpler to compute fluctuations. Specifically, we have computed finite temperature one-loop effective action (the Gibbs free energy) as a function of the order parameter and of the temperature. In this way, by minimizing the free energy, we have obtained the condensate fraction. Interestingly, we have found a reentrance of the $O(2)$ symmetry at a specific temperature $T_r$, for which $\Omega_{\text{eff}}(T_r) = 0$. This temperature essentially depends on the particle number imbalance between the two species, $\rho_{\varphi_1}/\rho_{\varphi_2}$. Below this temperature, the mean-field approach is unstable under thermal fluctuations. Thus, our approach is valid for $T_r < T < T_c$.

To obtain the condensate profiles of the original fields, we rotated back to the original basis $(\phi, \psi)$. This rotation only depends on the ratio $\Omega_R/|\nu|$. It is interesting to note that, due to the $O(2)$ symmetry of the two-body interaction, fluctuations only renormalize the effective Rabi frequency $\Omega_{\text{eff}} = \sqrt{\Omega_R + |\nu|^2}$, while the ratio $\Omega_R/|\nu|$ remains unaffected. Thus, the isospin rotation coefficients are independent of the temperature.

In Figures (2) and (3) we show the condensate profiles of the $\psi$ and $\phi$ species as functions of the temperature for different values of the parameter $\Omega_R/|\nu|$. We have shown that, for a temperature interval $T_r < T < T_c$, both bosonic species condensate and the relatives phases are locked by the detuning phase $\alpha_\nu$. We also have shown that the ratio between the condensates essentially depends on the temperature-independent parameter $\Omega_R/|\nu|$. We clearly see that, for $|\nu|/\Omega_R \rightarrow 0$, only one condensate survives, while in the opposite limit $\Omega_R/|\nu| \rightarrow 0$, both condensates are essentially equal, with small corrections of order $\Omega_R/|\nu|$.

The results presented in this paper are valid, provided the two-body interaction is invariant under isospin rotations. Consider, for instance, a small deviation from the $O(2)$ model, $g_\phi = g_{\phi \phi} = g$, but $g_{\psi \phi} = g + \Delta g$. Upon rotation to the $(\varphi_1, \varphi_2)$ basis, a term proportional to $\Delta g \varphi_1^* \varphi_1 \varphi_2^* \varphi_2$ will be generated. Thus, even though we have ignored this type of terms in the original model, they will be generated in a more general two-body interaction case. Thus, for $\Delta g \neq 0$, there is no isospin direction in which the $U(1) \otimes U(1)$ symmetry is recovered. This fact makes the study of quantum and thermal fluctuations more involved. We hope to report on this issue shortly.

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