A REMARK ON ANALYTIC FREDHOLM ALTERNATIVE

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Abstract. We apply a recent result of Borichev–Golinskii–Kupin on the Blaschke-type conditions for zeros of analytic functions on the complex plane with a cut along the positive semi-axis to the problem of the eigenvalues distribution of the Fredholm-type analytic operator-valued functions.

Introduction and main results

The goal of this note is to refine partially (for a certain range of parameters) a recent result of R. Frank [5, Theorem 3.1] on some quantitative aspects of the analytic Fredholm alternative. Precisely, the problem concerns the distribution of eigenvalues of finite type of an operator-valued function \( W(\cdot) = I + T(\cdot) \), analytic on a domain \( \Omega \) of the complex plane. We always assume that \( T \in \mathcal{S}_\infty \), the set of compact operators on the Hilbert space. A number \( \lambda_0 \in \Omega \) is called an eigenvalue of finite type of \( W \) if \( \ker W(\lambda_0) \neq \{0\} \), (i.e., \(-1\) is an eigenvalue of \( T(\lambda_0) \)), if \( W(\lambda_0) \) is Fredholm (that is, both \( \dim \ker W(\lambda_0) \) and \( \codim \text{ran} W(\lambda_0) \) are finite), and if \( W \) is invertible in some punctured neighborhood of \( \lambda_0 \). The function \( W \) admits the following expansion at any eigenvalue of finite type, see [6, Theorem XI.8.1],

\[
W(\lambda) = E(\lambda)(P_0 + (\lambda - \lambda_0)^{k_1} P_1 + \ldots + (\lambda - \lambda_0)^{k_l} P_l)G(\lambda),
\]

where \( P_1, \ldots, P_l \) are mutually disjoint projections of rank one, \( P_0 = I - P_1 - \ldots - P_l \), \( k_1 \leq \ldots \leq k_l \) are positive integers, and \( E, G \) are analytic operator-valued functions, defined and invertible in some neighborhood of \( \lambda_0 \). The number

\[
\nu(\lambda_0, W) := k_1 + \ldots + k_l
\]

is usually referred to as an algebraic multiplicity of the eigenvalue \( \lambda_0 \).

The following result, Theorem 3.1, is a cornerstone of the paper [5]. By \( \{\lambda_j\} \) we always denote the eigenvalues of \( W = I + T \) of finite type, repeated accordingly to their algebraic multiplicity.

**Theorem A.** Let \( T(\cdot) \) be an analytic operator-valued function on the domain \( \Omega = \mathbb{C}\setminus \mathbb{R}_+ \), so that \( T \in \mathcal{S}_p \), \( p \geq 1 \), the set of the Schatten–von Neumann operators of order \( p \). Assume that for all \( \lambda \in \mathbb{C}\setminus \mathbb{R}_+ \)

\[
||T(\lambda)||_p \leq \frac{M}{d^\rho(\lambda, \mathbb{R}_+) |\lambda|^\sigma}, \quad \rho > 0, \quad \sigma \in \mathbb{R}, \quad \rho + \sigma > 0,
\]

\( d(\lambda, \mathbb{R}_+) \) is the Euclidean distance from \( \lambda \) to the positive semi-axis. Then for all \( \varepsilon, \varepsilon' > 0 \) and \( \nu \geq 1 \)

\[
\sum_{|\lambda_j| \leq M^{\rho(\rho+\sigma)}} d^{\rho \nu + 1 + \varepsilon}(\lambda_j, \mathbb{R}_+) |\lambda_j|^{\frac{\rho + 1 - \varepsilon}{\rho}} \leq CM^{\frac{\rho (\rho + \sigma) + 1}{\rho}},
\]
where \( q := (p\rho + 2p\sigma - 1 + \varepsilon)_+ \), and
\[
\sum_{|\lambda_j| \geq 2^{M^1/(p+\sigma)} \mu} d^{p\rho + 1 + \varepsilon}(\lambda_j, \mathbb{R}_+) |\lambda_j|^{p\rho - p\rho - 1 - \varepsilon - 1} \leq \frac{C}{\nu^2} e^{2\pi M^{2\rho + \varepsilon}/(p+\sigma)}.
\]

Here \( C \) is a generic positive constant which depends on \( p, \rho, \sigma, \varepsilon, \varepsilon' \).

The similar results for \( \rho = 0 \) are also available.

The proof of this result is based on the identification of the eigenvalues of finite type of \( \mathbb{W} \) with the zeros of certain scalar analytic functions, known as the regularized determinants
\[
f(\lambda) := \det_p (I + T(\lambda)) = \det p(I + T(\lambda)),
\]
see [7, 9] for their definition and basic properties. The point is that the set of eigenvalues of finite type of \( \mathbb{W} \) agrees with the zero set of \( f \), and moreover, \( \nu(\lambda_0, \mathbb{W}) = \mu_f(\lambda_0) \), the multiplicity of zero of \( f \) at \( \lambda_0 \) (see [5] Lemma 3.2) for the rigorous proof.

Thereby, the problem is reduced to the study of the zero distributions of certain analytic functions, the latter being a classical topic of complex analysis going back to Jensen [8] and Blaschke [1].

A key ingredient of the proof in [5] is a result of [2, Theorem 0.2] on the Blaschke-type conditions for zeros of analytic functions in the unit disk which can grow at the direction of certain (finite) subsets of the unit circle. In a recent manuscript [3] some new such conditions on zeros of analytic functions in the unit disk and on some other domains, including the complex plane with a cut along the positive semi-axis, are suggested. Here is a particular case of [3, Theorem 4.5] which seems relevant. We use a convenient shortening
\[
\{u\}_{c,\varepsilon} := (u_- - 1 + \varepsilon)_+ - \min(c, u_+), \quad c \geq 0, \quad \varepsilon > 0, \quad u = u_+ - u_- \in \mathbb{R}.
\]

**Theorem B.** Let \( h \) be an analytic function on \( \Omega = \mathbb{C} \setminus \mathbb{R}_+ \), \( |h(-1)| = 1 \), subject to the growth condition
\[
\log |h(\lambda)| \leq \frac{K}{|\lambda|^r} \frac{(1 + |\lambda|)^b}{d^a(\lambda, \mathbb{R}_+)} \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \quad a, b \geq 0, \quad r \in \mathbb{R}.
\]

Let \( Z(h) \) be its zero set, counting multiplicities (the divisor of \( h \)). Denote
\[
s := 3a - 2b + 2r.
\]

Then for each \( \varepsilon > 0 \) there is a positive number \( C \) which depends on all parameters involved such that the following inequality holds
\[
\sum_{z \in Z(h)} d^{a + 1 + \varepsilon}(z, \mathbb{R}_+) \frac{|z|^s}{(1 + |z|)^r} \leq C \cdot K,
\]
where the parameters \( s_1, s_2 \) are defined by the relations
\[
s_1 := \frac{-2r - a}{a_0} - a - 1 - \varepsilon, \quad s_2 := a + 1 + \varepsilon + \frac{-2r - a}{a_0} + \{s\}_{a,\varepsilon}.
\]

We are aimed at proving the results, which refine Theorem A for a certain range of parameters, by using Theorem B.

**Theorem 0.1.** Let \( T(\cdot) \) be an analytic operator-valued function on the domain \( \Omega = \mathbb{C} \setminus \mathbb{R}_+ \), which satisfies the hypothesis of Theorem A. Assume that
\[
0 < \rho + \sigma \leq \frac{p}{2}.
\]

Then for all \( 0 < \varepsilon < 1 \)
\[
\sum_{|\lambda_j| \leq 2^{M^1/(p+\sigma)}} d^{p\rho + 1 + \varepsilon}(\lambda_j, \mathbb{R}_+) |\lambda_j|^{p\rho - p\rho - 1 - \varepsilon} \leq CM^{p^{1 + \varepsilon}/(2p+\sigma)}.
\]
Theorem 0.2. Theorem 0.2 is not treated in [5].

We follow the line of reasoning from [5]. The scaling

Proof of Theorem 0.1. \[ ρ > \]

the values of \( ρ \) Theorem B gives the same results, (0.2) and (0.3), as in Theorem A, for the rest of the values of \( ρ \) and \( σ \), and the eigenvalues tending to infinity.

The case

(0.7) \[ ρ > 0, \quad ρ + σ < 0, \]

is not treated in [5].

Theorem 0.2. Under conditions (0.7) assume that for all \( λ ∈ \mathbb{C}\backslash \mathbb{R}_+ \)

(0.8) \[ \| T(λ) \|_p \leq \frac{M}{d^p(λ, \mathbb{R}_+) |λ|^\sigma}. \]

Then for \( -ρ/2 < ρ + σ < 0 \) and all \( ε > 0 \)

(0.9) \[ \sum_{|λ_j| \geq M^{1/(ρ+σ)}} d^{pp+1+ε}(λ_j, \mathbb{R}_+) |λ_j|^{θρ - \frac{1}{2} (1+ε)} \leq C Mρ - \frac{1}{2} pρ, \]

and for \( ρ + σ < -ρ/2 \) and all \( ε > 0 \)

(0.10) \[ \sum_{|λ_j| \geq M^{1/(ρ+σ)}} d^{pp+1+ε}(λ_j, \mathbb{R}_+) |λ_j|^{θρ - \frac{1}{2} (1+ε)} \leq C M^{- \frac{1}{2} pρ}, \]

where \( l := (3pp - 2ρσ - 1 + ε)_+ \). Moreover, under conditions (0.7), for all \( ε, ε’ > 0 \)

and \( 0 < μ \leq 1 \)

(0.11) \[ \sum_{|λ_j| \leq M^{1/(μ+σ)}} d^{pp+1+ε}(λ_j, \mathbb{R}_+) |λ_j|^{θρ - pp - p - ε + ε’} \leq C μ^{ε’} M^{\frac{pp+1+ε}{μ ρ}}. \]

1. Proof of main results

Proof of Theorem 0.7

We follow the line of reasoning from [5]. The scaling \( T_1(λ) := T(M^{1/(ρ+σ)} λ) \) looks reasonable, so

(1.1) \[ \| T_1(λ) \|_p \leq \frac{1}{d^p(λ, \mathbb{R}_+) |λ|^\sigma}, \]

and, by [9] Theorem 9.2, (b), we have for the determinant \( f_1 = \det_p(I + T_1) \)

(1.2) \[ |f_1(λ) - 1| \leq φ(\| T_1(λ) \|_p), \quad φ(t) := t \exp(Γ_p(t + 1)^ρ), \quad t ≥ 0, \]

holds with a suitable constant \( Γ_p \) which depends only on \( p \), and provides a lower bound for \( f_1 \) whenever the right side is small enough. We have for \( t ≥ 1 \) and \( λ = -t ∈ \mathbb{R}_- \)

(1.3) \[ |f_1(-t) - 1| \leq \frac{C_1}{t^ρ}, \]

Note that under assumption (0.5)

(0.7) \[ pσ - \frac{1 + ε}{2} \leq -pp + 1 + ε < 0, \]

so for \( |ξ| \leq 1 \)

\[ |ξ|^pσ - \frac{1 + ε}{2} ≥ |ξ|^{pp + 1 + ε} ≥ |ξ|^{pσ - \frac{1 + ε}{2}}, \]

that is, (0.6) is stronger than (0.2) with regard to eigenvalues tending to zero.
(in the sequel $C_k$ stand for generic positive constants depending on the parameters involved). If $t \geq (2C_1)^{1/(p+\sigma)} = C_2$, then $|f_1(-t)| \geq 1/2$, and so

$$\log |f_1(-t)| \geq -2(1 - |f_1(-t)|) \geq \frac{2C_1}{t^{p+\sigma}}.$$  

Next, put

$$h(\lambda) := \frac{f_1(tl)}{f_1(-1)}, \quad h(-1) = 1.$$  

It follows from (1.1) and (1.3) that for $t \geq C_2$

$$\log |h(\lambda)| = \log |f_1(tl)| - \log |f_1(-t)| \leq \frac{1}{t^{p+\sigma}} \left( \frac{1}{d^{p\rho}(\lambda, \mathbb{R}_+)} |\lambda|^{-\rho} + \frac{2C_1}{t^{p+\sigma}} \right) \leq \frac{C_3}{t^{p+\sigma}} \left( \frac{1}{d^{p\rho}(\lambda, \mathbb{R}_+)} |\lambda|^{-\rho} + 1 \right) \leq \frac{C_3}{t^{p+\sigma}} \left( \frac{1}{d^{p\rho}(\lambda, \mathbb{R}_+)} |\lambda|^{-\rho} \right).$$  

Theorem B applies now with

$$a = pp, \quad r = p\sigma, \quad b = p(p + \sigma), \quad K = \frac{C_3}{t^{p+\sigma}},$$

and $s = a$, $\{s\}_{a, \varepsilon} = -a$. In view of (1.5) one has $2r + a = p(p + 2\sigma) \leq 0$, so

$$\{-2r - a\}_{a, \varepsilon} = -\min(a, -2r - a) = 2r + a = pp + p\sigma,$$

(recall that, by the assumption, $a > -2r - a$). Hence

$$s_1 = \frac{2p\sigma - 1 - \varepsilon}{2}, \quad s_2 = pp + p\sigma + 1 + \varepsilon,$$

and (1.4) implies

$$\sum_{z \in Z(h)} d^{pp+1+\varepsilon}(z, \mathbb{R}_+) \frac{|z|^{2p\sigma-1-\varepsilon}}{(1 + |z|)^{pp+p\rho+1+\varepsilon}} \leq \frac{C_4}{t^{p+\sigma}},$$

or 

$$\sum_{\zeta \in Z(f_1)} d^{pp+1+\varepsilon}(\zeta, \mathbb{R}_+) \frac{|\zeta|^{2p\sigma-1-\varepsilon}}{(t + |\zeta|)^{pp+p\rho+1+\varepsilon}} \leq \frac{C_4}{t^{p+\sigma}}.$$  

For $|\zeta| \leq 1$ we fix $t$, say, $t = C_2$, and since $t + |\zeta| \leq C_2 + 1$, we come to

$$\sum_{\zeta \in Z(f_1)\cap \mathbb{D}} d^{pp+1+\varepsilon}(\zeta, \mathbb{R}_+) |\zeta|^{p\sigma - \frac{1}{t^{p+\sigma}}} \leq C_5,$$

which, after scaling, is (1.6). The proof is complete.

Proof of Theorem B

The idea is much the same with the only technical differences. In the above notation relation (1.1) still holds, and the function $T_1$ tends to zero as $t \to 0$—whenever $\rho + \sigma < 0$. So

$$\log |f_1(-t)| \geq -2(1 - |f_1(-t)|) \geq -\frac{2C_1}{t^{p+\sigma}} = -2C_1 t^{p+\sigma}, \quad 0 < t \leq C_2.$$  

For the function $h$ (1.4) we now have

$$\log |h(\lambda)| \leq C_3 t^{p+\sigma} \left( \frac{1}{d^{p\rho}(\lambda, \mathbb{R}_+)} |\lambda|^{-\rho} + 1 \right),$$

and as

$$\frac{1}{d^{pp}(\lambda, \mathbb{R}_+)} |\lambda|^{-\rho} + 1 \leq \frac{|\lambda|^{p\rho} + |\lambda|^{p\rho}}{d^{pp}(\lambda, \mathbb{R}_+)} \leq \frac{|\lambda|^{p\rho} (1 + |\lambda|)^{p\rho + \sigma}}{d^{pp}(\lambda, \mathbb{R}_+)},$$

therefore
we come to the bound
\begin{equation}
\log |h(\lambda)| \leq C_3 |\rho + \sigma| |\lambda|^{p+\sigma} \frac{(1 + |\lambda|)^{p+\sigma}}{d^{p+\sigma}(\lambda, \mathbb{R}_+)}.
\end{equation}

Theorem B applies with
\[ a = pp, \quad r = -a = -pp, \quad b = -p(\rho + \sigma), \quad K = \frac{C_3}{t^{p+\sigma}}, \]
and \(-2r - a = a > 0, so\]
\[ \{ -2r - a \}_a, \varepsilon = -a = -pp, \quad s_1 = -pp - \frac{1 + \varepsilon}{2}. \]

The sign of \( s = 3a - 2b + 2r = p(3\rho + 2\sigma) \) (which can be either positive or negative) affects the computation of \( \{ s \}_a, \varepsilon \), so we will differ two situations. In the case \(-\rho/2 \leq \rho + \sigma < 0\) we have
\begin{equation}
\{ s \}_a, \varepsilon = - \min(a, s_+) = -s,
\end{equation}
since, by \( (0.7) \), \( s_+ = s = p(3\rho + 2\sigma) < pp = a \). So \( s_2 = -p(\rho + \sigma) + 1 + \varepsilon \), and \( (0.4) \) leads to
\begin{equation}
\rho|\rho + \sigma + \frac{1}{2} |^d \sum_{\zeta \in \mathbf{Z}(f_t)} \frac{|\zeta|^{-p\rho - \frac{1}{2}}}{(t + |\zeta|)^{p\rho + \sigma + 1 + \varepsilon}} \leq C_4 \frac{1}{t^{p+\sigma}}, \quad 0 < t \leq C_2.
\end{equation}

A simple bound \( (C_2 + |\zeta|)^{-1} \geq C_5 |\zeta|^{-1} \) for \( |\zeta| \geq 1 \) and fixed \( t = C_2 \) gives
\[ \sum_{\zeta \in \mathbf{Z}(f_t) \cap \mathbb{D}} d^{p+1+\varepsilon}(\zeta, \mathbb{R}_+) |\zeta|^{\rho\sigma - \frac{3}{2} (1+\varepsilon)} \leq C_6, \quad \mathbb{D} := \{ |\zeta| \geq 1 \}, \]
which, after scaling, is \( (0.9) \).

If \( |\zeta| \leq \mu \leq 1 \), we multiply \( (1.8) \) through by \( t^{\rho + \sigma - 1 + \varepsilon} \) and integrate it termwise with respect to \( t \) from 0 to \( t^{C_2} \) (the idea comes from [3])
\[ \int_0^{t^{C_2}} \frac{(p-1)|\rho + \sigma + \frac{1}{2} |^{d-1+\varepsilon}}{(t + |\zeta|)^{p\rho + \sigma + 1 + \varepsilon}} dt = |\zeta|^{\rho\sigma - \frac{3}{2} (1+\varepsilon)} \int_0^{t^{C_2}/|\zeta|} \frac{t^{p\rho + \sigma + 1 + \varepsilon}}{(1 + t)^{p\rho + \sigma + 1 + \varepsilon}} dx \]
\[ \geq C_7 |\zeta|^{\rho\sigma - \frac{3}{2} (1+\varepsilon)}, \]
to obtain
\[ \sum_{\zeta \in \mathbf{Z}(f_t) \cap \mathbb{D}_\mu} d^{p+1+\varepsilon}(\zeta, \mathbb{R}_+) |\zeta|^{\rho\sigma - \rho - 1 + \varepsilon} \leq C_8 (t^{C_2})^{\varepsilon}, \quad \mathbb{D}_\mu := \{ |\zeta| \leq \mu \}, \]
which, after scaling, gives \( (0.11) \).

In the case \( \rho + \sigma < -\rho/2 \) the proof is the same with \( s \leq 0 \) and
\[ \{ s \}_a, \varepsilon = (-3pp - 2\rho \sigma - 1 + \varepsilon)_+ = l, \quad s_2 = \frac{pp + l}{2} + 1 + \varepsilon. \]

\[ \square \]

**Remark 1.1.** The case \( (0.7) \) can be reduced to the one considered in Theorem A by means of the transformation (the change of variables) \( \lambda \to 1/\lambda \) and the general formula
\[ d(1/\lambda, \mathbb{R}_+) = \frac{d(\lambda, \mathbb{R}_+)}{|\lambda|^2}, \quad \lambda \in \mathbb{C} \backslash \mathbb{R}_+. \]

**Remark 1.2.** The general form of \( [3] \) Theorem 4.5] allows a finite number of singularities on \( \mathbb{R}_+ \). So we can obtain the similar results on eigenvalues of finite type for analytic operator-valued functions \( W = I + T \) on \( \mathbb{C} \backslash \mathbb{R}_+ \) subject to the bound
\[ \| T(\lambda) \|_p \leq M \frac{(1 + |\lambda|)^{\tau}}{d^{\rho}(\lambda, \mathbb{R}_+)/|\lambda|^{\rho}} \prod_{j=1}^{n} |\lambda - t_j|^{c_j}, \quad \rho, \tau, c_j, \varepsilon_j \geq 0, \quad \sigma \in \mathbb{R}, \]
where \( \{t_j\} \) and \( \{t'_k\} \) are two disjoint finite sets of distinct positive numbers.

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