Characteristic polynomials of random matrices at edge singularities

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Abstract

We have discussed earlier the correlation functions of the random variables $\det(\lambda - X)$ in which $X$ is a random matrix. In particular the moments of the distribution of these random variables are universal functions, when measured in the appropriate units of the level spacing. When the $\lambda$'s, instead of belonging to the bulk of the spectrum, approach the edge, a cross-over takes place to an Airy or to a Bessel problem, and we consider here these modified classes of universality.

Furthermore, when an external matrix source is added to the probability distribution of $X$, various new phenomenons may occur and one can tune the spectrum of this source matrix to new critical points. Again there are remarkably simple formulae for arbitrary source matrices, which allow us to compute the moments of the characteristic polynomials in these cases as well.

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1 Introduction

In the theory of random matrices, the n-point correlation functions of the eigenvalues are known to be expressible as the determinant of a two-point kernel \[^1, 2\]. The expressions for those kernels depend on the various classes of universality: it is a simple sine-kernel within the bulk of unitary invariant ensembles, an Airy kernel at the edge of the spectrum or a Bessel kernel for other invariance properties of the measure. The level spacing probability \( p(s) \), has also been computed recently for those different kernels \[^2, 3\].

Another interesting object is given by the average moments of the characteristic polynomial of the random matrix. These characteristic polynomials have been first investigated in \[^4, 5\] for a uniform probability measure on unitary matrices, in connection with the moments of the Riemann zeta-function. These results have been generalized to random hermitian \( N \times N \) matrices \( X \) with a unitary invariant probability measure

\[
P(X) = \frac{1}{Z} \exp \left( -N \text{Tr} V(X) \right). \tag{1}
\]

Explicit formulae for the \( 2K \)-point functions

\[
F_{2K}(\lambda_1, \cdots, \lambda_{2K}) = \left< \prod_{1}^{2K} \det(\lambda_l - X) \right> \tag{2}
\]

have been derived, which show that these functions are universal in the Dyson limit, in which the size \( N \) of the matrices goes to infinity, the distances between the \( \lambda \)'s go to zero, and the products \( N(\lambda_i - \lambda_j) \) remain finite. In particular the moments

\[
F_{2K}(\lambda, \cdots, \lambda) = \left< \left[ \det(\lambda - M) \right]^{2K} \right> \tag{3}
\]

of the distribution of the characteristic polynomials were given in the large \( N \) limit by \[^3, 4\]

\[
\exp - (NK V(\lambda)) F_{2K}(\lambda, \cdots, \lambda) = (2\pi N \rho(\lambda))^{K^2} e^{-NK \gamma_K}, \tag{4}
\]

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with
\[ \gamma_K = \prod_{0}^{K-1} \frac{l!}{(K+l)!}, \tag{5} \]
provided \( \lambda \) belongs to the bulk of the support of the distribution of the eigenvalues, i.e. provided \( \rho(\lambda) \) does not vanish. Then one sees explicitly that the only dependence upon the probability measure is through the average density of eigenvalues \( \rho(\lambda) \), and even the coefficient \( \gamma_K \) is a universal number.

However the result does take different forms for different universality classes. Our previous investigations for the three classical Lie groups, \( \text{U}(N), \text{Sp}(N) \) and \( \text{O}(N) \), are extended here to the Bessel kernel and Airy kernel, for which the density of states \( \rho(\lambda) \) presents a singularity at the edge of the spectrum. Furthermore we have considered a Gaussian case in which an external matrix source is present\[8\] in the probability distribution of the matrix
\[ P(X) = \frac{1}{Z} \exp (-N\text{Tr} \frac{1}{2} X^2 + N\text{Tr} AX). \tag{6} \]
Explicit and simple formulae will be derived here again for the correlation functions and the moments of the characteristic polynomials of the matrix \( X \), which depend on the spectrum of the matrix \( A \). By tuning the spectrum of \( A \) appropriately, one can generate a number of different situations. For instance, we have investigated in the past the case in which the average spectrum of \( X \) presents a gap in the presence of \( A \), and by tuning \( A \) one can study the critical point at which this gap vanishes. This creates again a new class of universality, and a new kernel \[9, 10\]. Other cases, such as 2D gravity in the double scaling limit, or the Penner model, would certainly be of interest as well.

2 Sine-kernel

For completeness, and for later use, we begin with the bulk unitary case, governed by the sine-kernel, but with a derivation which differs from our previous one \[6\]. An interesting geometric interpretation of this problem will also be provided. The
kernel, from which all the correlation functions may be obtained, is given in terms of orthogonal polynomials for finite \(N\), but reduces in the Dyson large \(N\)-limit to the sine-kernel

\[
K(x, y) = \frac{\sin(x - y)}{x - y}
\]

in which \(x\) and \(y\) are the eigenvalues measured in the scale of the average spacing \((2\pi\rho(\lambda)N)^{-1}\). Then, one obtains the normalized moments

\[
I_K = e^{-NKV(\lambda)} \frac{\langle [\det(\lambda - X)]^{2K} \rangle}{(2\pi N\rho(\lambda))^{K^2}} = \lim_{\lambda_i \to 0} \frac{\det K(\lambda_i, \lambda_j)}{\Delta^2(\Lambda)}
\]

where \(\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)\) and \(i, j = 1, \ldots, K\). The r.h.s. may be expressed as a contour integral, following eq.(52) of [3],

\[
I_K = \oint \oint K \prod u_i dv_i (2\pi i)^2 \prod_{i=1}^{K} u_i^K \prod_{i=1}^{K} v_i^K \prod_{i=1}^{K} \sin(u_i - v_i)
\]

This may be further reduced to

\[
I_K = \det(a_{nm})
\]

\[
a_{nm} = \frac{1}{n!m!} \frac{\partial^n \partial^m \sin(u - v)}{u - v} \bigg|_{u=v=0}
\]

where \(n, m = 0, 1, \ldots, K - 1\). The explicit evaluation of the determinant of \(a_{n,m}\) gives

\[
\det(a_{nm}) = 2^{K^2-K} \prod_{l=0}^{K-1} \frac{l!}{(K+l)!}
\]

We do recover in this way the factor \(\gamma_K\) [3] (up to a factor \(2^{K^2-K}\) due to a different normalization of the kernel).

It is quite remarkable that this universal normalizing factor \(\gamma_K\) has a geometric interpretation as a Fredholm determinant of the Dirac Laplacian on the two dimensional sphere \(S^2\). The determinant of the Laplacian has been discussed in the connection to string theory [11, 12], and the relation of \(\gamma_K\) to this Fredholm determinant of the Laplacian has been noticed in [3]. Indeed let us show that

\[
\gamma_K = \frac{e^{K^2(1+\gamma)}}{\Delta^+(-K)}
\]
where $\gamma$ is Euler’s constant and $\Delta^+(z)$ the determinant of a Dirac operator, defined below. The derivation goes as follows: let us introduce a function $G(z)$ which satisfies the functional relation

$$G(z + 1) = \Gamma(z)G(z). \quad (14)$$

It is then straightforward to verify that

$$\gamma_K = \prod_{l=0}^{K-1} \frac{l!}{(K+l)!} = 2^{K-2K^2} \frac{\pi^{K+\frac{1}{2}}}{\Gamma(K+\frac{1}{2})} \left[ \frac{G(\frac{1}{2})}{G(K+\frac{1}{2})} \right]^2. \quad (15)$$

A function $G$, satisfying the functional relation (14), is known in the literature as a Barnes function (or as the inverse of a di-gamma function). It is defined by

$$G(z + 1) = \frac{1}{\Gamma_2(z+1)} = (2\pi)^{z/2} e^{-\frac{1}{2}[z+(1+\gamma)z^2]} \prod_{l=1}^{\infty} \left[ (1 + \frac{z}{n}) e^{-z^2/(2n)} \right]. \quad (16)$$

It has been noticed earlier ([13]) that this Barnes function is related to the Fredholm determinant of the Laplacian on $S^2$. Indeed this Fredholm determinant is the (regularized) product

$$\Delta(z) = \prod_l (1 - \frac{z}{\lambda_l})^{g_l} \quad (17)$$

where the $\lambda_l$ are the eigenvalues of the Laplacian, and $g_l$ their degeneracy, i.e. $\lambda_l = l(l+1)$ with multiplicity $g_l = 2l+1$, $l = 0, 1, 2, \ldots$. It is convenient to shift $z$ by 1/4, since this yields the spectrum of the Dirac operator

$$\sqrt{\lambda_l + \frac{1}{4}} = l + \frac{1}{2} \quad (18)$$

Then the regularized (shifted) Fredholm determinant

$$\Delta(z) = \prod_{l=0}^{\infty} \left[ (1 - \frac{z}{l+\frac{1}{2}}) e^{\frac{z^2}{4(l+1/2)^2}} \right]^{2l+1}, \quad (19)$$

factorizes as

$$\Delta(-y^2) = \Delta^+(iy)\Delta^+(-iy) \quad (20)$$

with the determinant of the Dirac operator $\Delta^+(z)$ given by [13]

$$\Delta^+(z) = \prod_{l=0}^{\infty} \left[ (1 - \frac{z}{l+\frac{1}{2}}) e^{\frac{z^2}{4(l+1/2)^2}} \right]^{2l+1}. \quad (21)$$
Then this Dirac determinant $\Delta^+$ is related to the Barnes function by

$$\Delta^+(z) = \pi^{-\frac{1}{4}} (2\pi)^z e^{(1+\gamma+2\log 2)z^2} \frac{\Gamma\left(\frac{1}{2} - z\right) G\left(\frac{1}{2} - z\right)^2}{G\left(\frac{1}{2}\right)^2}. \quad (22)$$

We thereby recover the expression relating the moment $\gamma_K$ to the determinant (13).

This relation between the moments of the distribution and the determinant of the Dirac operator on $S^2$ is in fact general. For instance in the simplest case of a single Gaussian random variable, the moments are

$$\int_{-\infty}^{\infty} x^{2K} e^{-x^2} dx = \Gamma(K + \frac{1}{2}); \quad (23)$$

$\Gamma(K + 1/2)$ is thus the equivalent of $\gamma_K$ for this trivial problem. If we consider the ”Laplacian”, i.e. the harmonic oscillator whose eigenvalues are $\lambda_n = n$, then the Fredholm determinant $\Delta(\lambda)$ is

$$\Delta(\lambda) = -\lambda \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n}\right) e^{\frac{\lambda}{n}}$$

$$= \frac{e^{\gamma \lambda}}{\Gamma(-\lambda)} \quad (24)$$

Hence, we have

$$<x^{2K}> = \frac{e^{\gamma \lambda}}{\Delta(\lambda)} \bigg|_{\lambda = -(K + \frac{1}{2})} \quad (25)$$

The expression (13) is a multi-variable version of this Gaussian integral.

An additional point of interest is that the Fredholm determinant of this Laplacian on $S^2$ may be factorized further into a product of two factors; it turns out that each factor enters into the corresponding expression for the symplectic and orthogonal cases respectively. This will be seen below when we examine the moments related to the Bessel kernel.

### 3 Bessel kernel

We have discussed in our previous work the ensembles invariant under the unitary symplectic and unitary orthogonal Lie groups [6]. The kernels for those ensembles...
are \[14, 13, 16\]

\[K(x, y) = \frac{1}{2\pi} \left( \frac{\sin(x - y)}{x - y} \pm \frac{\sin(x + y)}{x + y} \right)\]  

(26)

where the minus sign corresponds to the Sp and the plus sign to the O ensemble. It is convenient to introduce the Bessel kernel defined by

\[K_\alpha(x, y) = \frac{J_\alpha(x)J'_\alpha(y) - J'_\alpha(x)J_\alpha(y)}{x - y}\]  

(27)

Since \(J_{1/2}(x) = \sqrt{2/\pi x} \sin x\), \(J_{-1/2}(x) = \sqrt{2/\pi x} \cos x\), the kernels for the Sp and O ensembles are both related to this Bessel kernel

\[K_\pm(x, y) = \sqrt{xy}K_{\pm 1/2}(x^2, y^2)\]  

(28)

denamely, \(\alpha = 1/2\) and \(\alpha = -1/2\) represent respectively the Sp and the O ensemble.

We consider from now on an arbitrary \(\alpha\). The 2K-th moment at the origin \((\lambda = 0)\) is expressed as

\[I_K = \oint \oint dudv (2\pi)^2 \Delta(u_2^2)\Delta(v_2^2) \prod_{i=1}^{K} \frac{\Gamma(\alpha + 1)}{(u_i v_i)^\alpha} K_\alpha(u_i, v_i)\]  

(29)

We define now the two functions \(\phi(z)\) and \(\psi(z)\) by

\[J_\alpha(\sqrt{z}) = \left(\frac{\sqrt{z}}{2}\right)^\alpha \frac{1}{\Gamma(\alpha + 1)} \phi(z)\]  

(30)

and

\[\sqrt{z}J'_\alpha(\sqrt{z}) = \frac{z^{\alpha/2}}{2^\alpha \Gamma(\alpha)} \psi(z)\]  

(31)

Their expansions in powers of \(x\) are given by

\[\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n n! \prod_{l=1}^{n} (\alpha + l)}\]  

(32)

\[\psi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n (\alpha + 2n)}{4^n n! \prod_{l=0}^{n} (\alpha + l)}\]  

(33)

Keeping aside trivial factors we are then led to the kernel \(\tilde{K}_\alpha(x, y)\) defined as

\[\tilde{K}_\alpha(x, y) = \frac{1}{2(x - y)}[\phi(x)\psi(y) - \phi(x)\psi(y)]\]  

(34)
As before, we have
\[ I_K = \det(a_{nm}) \] (35)
with
\[ a_{nm} = \frac{1}{n!m!} \partial^n \partial^m K_\alpha(u,v)|_{u=v=0} \] (36)
This determinant may be computed explicitly, and it is given by
\[ I_K = 4^{-K^2-\alpha K} \prod_{l=0}^{2K-1} \frac{1}{(\alpha + l)!} \] (37)
(We have \( I_1 = \frac{1}{4}, \frac{1}{8\pi}, \frac{1}{2\pi} \) for \( \alpha = 0, \frac{1}{2}, \) and \( -\frac{1}{2}, \) respectively.)

It is interesting to relate the three determinants that we have introduced hereabove for the sine-kernel and for the \( Sp \) and \( O \) cases. The determinant for the sine-kernel (11) is
\[ I_U = \det \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 & \ldots \\ 0 & \frac{1}{3} & 0 & -\frac{1}{20} & \ldots \\ -\frac{1}{6} & 0 & \frac{1}{20} & 0 & \ldots \\ 0 & -\frac{1}{30} & 0 & 20 \pi & \ldots \end{pmatrix}. \] (38)
In the symplectic case, \( \alpha = \frac{1}{2} \), we have
\[ I_{Sp} = \det \begin{pmatrix} \frac{1}{3} & -\frac{1}{30} & \ldots \\ -\frac{1}{30} & \frac{20}{3} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}. \] (39)
In the orthogonal case, the determinant becomes for \( \alpha = -\frac{1}{2}, \)
\[ I_O = \det \begin{pmatrix} 1 & -\frac{1}{6} & \ldots \\ -\frac{1}{6} & \frac{1}{20} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}. \] (40)
Thus, we find the factorization of (38) as the product of (39) and (40), up to a trivial numerical factor due to the normalizations,
\[ I_U = I_{Sp} \times I_O \] (41)
The factors \( \gamma_K \) for the unitary, symplectic and orthogonal case are related as \( 2^{K^2-1} \gamma_K^{(U)} = \gamma_K^{(Sp)} \gamma_K^{(O)} \), and \( \gamma_K^{(U)} = (\prod_{l=1}^{K-1} l!)^2/(\prod_{l=1}^{2K-1} l!), \gamma_K^{(Sp)} = 2^{K(K+1)/2} \prod_{l=1}^{K} l/\prod_{l=1}^{K} (2l)! \) and \( \gamma_K^{(O)} 2^{K(K+1)/2-1} \prod_{l=1}^{K-1} l/\prod_{l=1}^{K-1} (2l)! ). \) It is again remarkable that, for arbitrary \( \alpha, \) \( \gamma_K \) may still be expressed as the Fredholm determinant of the Laplacian, in which the eigenvalues are shifted by the amount \( \alpha \).
4 Airy kernel

When the eigenvalues lie near an edge \( \lambda_c \) of the support of the asymptotic density of states (an edge of Wigner’s semi-circle in the Gaussian case), in a neighbourhood of size \( N^{-2/3} \) of that edge, there is a cross-over from the sine-kernel to the Airy kernel. In terms of the Airy function \( A_i(x) \), defined by

\[
A_i(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{\frac{i}{3}z^3 + izx},
\]

which satisfies the differential equation

\[
A_i''(x) = xA_i(x),
\]

one has

\[
K(x, y) = \frac{A_i(x)A'_i(y) - A'_i(x)A_i(y)}{x - y}.
\]

In (44) we have used the scaling variables \( x \) and \( y \) proportional to \( N^{2/3}(\lambda - \lambda_c) \).

There are two ways to obtain the moments under consideration. The first one is to write as before

\[
I_K = <[\det(\lambda_c - X)]^{2K}> = \oint \frac{du}{2\pi i} \Delta(u) \Delta(v) \prod_{i=1}^{K} u_i^{K} v_i^{K} \prod_{i=3D1} K(u_i, v_i)
\]

but in this case, there are three periodic structure due to three valleys of Airy functions, and the result is more complicated. It does not seem to be expressible as simple products of gamma-functions. However we can use a direct method starting with the expression

\[
I_K = <[\det(\lambda_c - X)]^{2K}> = \frac{1}{(2\pi)^{2K}} \int_{-\infty}^{\infty} dz \Delta^2(z) e^{\sum_{i=1}^{2K} z_i^3}
\]

This representation is the edge limit \( \lambda_l \to 0 \) of

\[
F_{2K} = \int_{-\infty}^{\infty} dz_l \oint \frac{du_i}{2\pi i} \prod_{i} \frac{\Delta(z)\Delta(u)}{\prod_{i}(u_i - \lambda_c + \lambda_l)} e^{N \sum_{i=1}^{2K} \frac{i}{3}z_i^3 + iz_l u_i}
\]
The sums and products over $l$ run from $l = 1$ to $l = 2K$. The dependence of $F_{2K}$ on $N$ is of order $N^{\frac{3}{2}K^2 - K}$.

We may then use the standard orthogonal polynomial method. To the complex measure

$$d\mu = dz e^{\frac{1}{3}z^3}$$

we associate the orthogonal polynomials $p_n$ defined as

$$p_n(x) = x^n + \text{lower degree},$$

and

$$\int d\mu p_n(x)p_m(x) = h_n \delta_{n,m}$$

The integral of (42) is then simply

$$I = K!h_0 h_1 \cdots h_{K-1}$$

Note that this looks similar to the partition function of a matrix model, but here it is the partition function of a $K \times K$ matrix model, instead of $N \times N$ ($K$ is finite, since it is the order of the moment that we are considering, whereas $N$ goes to infinity). Therefore this is for any $K$ a completely explicit expression of the moments at the edge. Those coefficients $h_n$ are expressible in terms of ratios of determinants constructed with the moments of the measure:

$$h_n = \frac{d_n}{d_{n-1}}$$

with

$$d_n = \det \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}$$

in which the $m_n$ are the moments of the measure. Those determinants are constants along anti-diagonal lines (Hankel determinants). Then the product $h_0 h_1 \cdots h_{K-1}$ is
reduced to a single determinant. For example, we have for $K = 4$

\[ h_0 h_1 h_2 h_3 = \det \begin{pmatrix} C_1 & -iC_2 & 0 & iC_1 \\ -iC_2 & 0 & iC_1 & 2C_2 \\ 0 & iC_1 & 2C_2 & 0 \\ iC_1 & 2C_2 & 0 & -4C_1 \end{pmatrix}. \tag{54} \]

with $C_1 = A_i(0) = 3^{-2/3}/\Gamma(2/3)$, $C_2 = A'_i(0) = -3^{-1/3}/\Gamma(1/3)$, since all the moments up to $m_6$ are easily expressible in terms of $m_0$ and $m_1$ alone.

More generally we have

\[ m_n = \int z^n d\mu = (-i)^n(n-2)(n-5)(n-8)\cdots \tilde{A}_n \tag{55} \]

where $\tilde{A}_n = C_1$ for $n=0$ (modulo 3), and $\tilde{A}_n = C_2$ for $n=1$ (modulo 3) and $\tilde{A}_n = 0$ for $n=2$ (modulo 3). The last parenthesis of the product in the r.h.s. of (55) is the rest of the division of $n-2$ by 3. Then, $d_n$ is the determinant of a Hankel matrix, whose matrix elements in the first row are $[< z^0 >, < z >, < z^2 >, \cdots] = [C_1, -iC_2, 0, iC_1, 2C_2, 0, -4C_1, 10iC_2, 0, -28iC_1, -80C_2, 0, \cdots]$, and all the others are given by the Hankel rule. In this way we obtain successively,

\[ \begin{align*}
    h_0 &= C_1 = 0.355028053 \\
    h_0 h_1 &= C_2^2 = 0.066987483 \\
    \prod_{l=0}^{2} h_l &= 2C_2^3 + C_1^3 = 0.010074161 \\
    \prod_{l=0}^{3} h_l &= -8C_1 C_2^3 - 3C_1^4 = 0.001580882 \\
    \prod_{l=0}^{4} h_l &= 72C_2^5 + 28C_1^3 C_2^2 = 0.00031309517 \\
    \prod_{l=0}^{5} h_l &= -2160C_2^6 - 1952C_1^2 C_2^3 - 432C_1^6 = 0.000090756324 \tag{56} 
\end{align*} \]

Therefore for the edge problem we have found moments given by $\gamma_K$’s which are more complicated since $\gamma_K = \prod_{l=0}^{2K-1} h_l$. The result is explicit for any finite $K$, but we have not succeeded to continue it to non-integer $K$. The numerical values indicate a smooth curve in a logarithmic plot.
5 Finite N results

We have derived in our previous paper [3] the correlation functions of the characteristic polynomials under the form of a determinant.

\[ F_K(\lambda_1, \cdots, \lambda_K) = \langle \prod_1^K \det(\lambda_l - X) \rangle \]

\[ = \frac{1}{\Delta(\lambda_1, \cdots, \lambda_K)} \det \begin{vmatrix} p_M(\lambda_1) & p_{M+1}(\lambda_1) & \cdots & p_{M+K-1}(\lambda_1) \\ p_M(\lambda_2) & p_{M+1}(\lambda_2) & \cdots & p_{M+K-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_M(\lambda_K) & p_{M+1}(\lambda_K) & \cdots & p_{M+K-1}(\lambda_K) \end{vmatrix}, \quad (57) \]

in which \( X \) is an \( M \times M \) random matrix.

The polynomial \( p_n(x) \) are the (monic) orthogonal polynomials, whose coefficients of highest degree are equal to unity

\[ p_n(x) = x^n + \text{lowerdegree}. \quad (58) \]

If we are concerned simply with the moments of the distribution of a single characteristic polynomial, we obtain from (57)

\[ \mu_K(\lambda) = F_K(\lambda, \cdots, \lambda) = \langle [\det(\lambda - X)]^K \rangle \]

\[ = \frac{(-1)^{K(K-1)/2} \prod_{l=0}^{K-1}(l!)}{\Delta(\lambda_1, \cdots, \lambda_K)} \det \begin{vmatrix} p_M(\lambda) & p_{M+1}(\lambda) & \cdots & p_{M+K-1}(\lambda) \\ p'_M(\lambda) & p'_{M+1}(\lambda) & \cdots & p'_{M+K-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ p_M^{(K-1)}(\lambda) & p_{M+1}^{(K-1)}(\lambda) & \cdots & p_{M+K-1}^{(K-1)}(\lambda) \end{vmatrix}. \quad (59) \]

For the Gaussian distribution,

\[ P(X) = \frac{1}{Z_M} \exp -\frac{N}{2} \text{Tr} X^2, \quad (60) \]

with

\[ M = N - K, \quad (61) \]

the polynomial \( p_n(x) \) are the Hermite polynomials \( H_n(x) \), defined with our normalization as

\[ H_n(x) = \frac{(-1)^n}{N^n} e^{N x^2/2} \left( \frac{d}{dx} \right)^n e^{-N x^2/2} = x^n + \text{l.d.}. \quad (62) \]
The integral representation

\[ H_n(x) = \frac{(-1)^n n!}{N^n} \oint \frac{dz}{2\pi i} e^{-N(z^2/2+xz)} z^{(n+1)} \]  

(63)

over a contour which circles around the origin in the z-plane, turns out to be well adapted.

Note that all these expressions are all valid for finite N. We may thereby recover readily several results that we have discussed in the previous sections. For instance, let us assume that \( M \) is an even number, and consider the center value \( \lambda = 0 \) (since the dependence in \( \lambda \) is known to be contained entirely in the overall factor \([\rho(\lambda)]^{K^2}\), as far as the coefficient \( \gamma_K \) is concerned, it is sufficient to put simply \( \lambda = 0 \)).

The Hermite polynomials \( H_n(x) \) vanish for odd \( n \) at \( x = 0 \). Similarly the odd derivatives of \( H_n(x) \) for even \( n \), also vanish at \( x = 0 \). Hence, the elements of the determinant (59) are alternatively non-zero then zero. Thus the determinant is decomposed into a product of two determinants: this is the exact phenomenon for \( N \) finite of the factorization of the symplectic and orthogonal determinants that we have seen earlier for large \( N \). Since the matrix elements of (59) at \( \lambda = 0 \) are all expressed as derivatives of Hermite polynomials at the origin, it is possible to compute this determinant exactly for finite and arbitrary \( M \) and \( K \). For the even \( M \) case,

\[ F_{2K}(0) = \frac{(-1)^{K(2K-1)\prod_{l=3}^{2K-1}D_0(l)}}{\prod_{l=3D_0(l)}^{2K-1}} \det \begin{vmatrix} H_M(0) & H_{M+2}(0) & \cdots \\ H_M''(0) & H_{M+2}''(0) & \cdots \\ \vdots & \vdots & \ddots \\ H_{M+2}^{(2K-2)}(0) & H_{M+2}^{(2K-2)}(0) & \cdots \end{vmatrix} \]

\times \det \begin{vmatrix} H_{M+1}'(0) & H_{M+3}'(0) & \cdots \\ H_{M+1}''(0) & H_{M+3}''(0) & \cdots \\ \vdots & \vdots & \ddots \\ H_{M+3}^{(2K-1)}(0) & H_{M+3}^{(2K-1)}(0) & \cdots \end{vmatrix} \quad (64)

We denote each determinant as \( I^{(1)}/N^{KM/2} \) and \( I^{(2)}/N^{KM/2} \) respectively, and

\[ F_{2K}(0) = I^{(1)}I^{(2)} \frac{1}{N^{KM}} \frac{1}{\prod_{l=0}^{2K-1} l!}. \quad (65) \]
The above two determinants are easily computed through the explicit expressions for the \( H_n(x) \)'s,

\[
H_{2n}(x) = \frac{1}{n} (-1)^n (2n - 1)!! \sum_{m=0}^{\infty} \frac{(-n)(-n+1) \cdots (-n+m-1)}{(\frac{1}{2})(\frac{3}{2}+1) \cdots (\frac{1}{2}+m-1)} \frac{1}{m!} \left( \frac{Nx^2}{2} \right)^m, \tag{66}
\]

\[
H_{2n+1}(x) = \frac{1}{Nn} (-1)^n (2n+1)!! x \sum_{m=0}^{\infty} \frac{(-n)(-n+1) \cdots (-n+m-1)}{(\frac{1}{2}+\frac{1}{2}) \cdots (\frac{1}{2}+m-1)} \frac{1}{m!} \left( \frac{Nx^2}{2} \right)^m. \tag{67}
\]

The two determinants contain overall products of factors of the form \((2n-1)!!\); once they are extracted one finds

\[
I^{(1)} = C(M + 2K - 3)!!(M + 2K - 5)!! \cdots (M - 1)!!
\]

\[
= C \frac{1}{2^{K(M+K-3)}} \prod_{l=1}^{K} \left\{ \frac{\Gamma(M+2l-2)}{\Gamma(M+2l-1)} \right\}, \tag{68}
\]

\[
I^{(2)} = C(M + 2K - 1)!!(M + 2K - 3)!! \cdots (M + 1)!!
\]

\[
= C \frac{1}{2^{K(M+K-1)}} \prod_{l=1}^{K} \left\{ \frac{\Gamma(M+2l)}{\Gamma(M+2l+1)} \right\}, \tag{69}
\]

with

\[
C = 2^{\frac{K(K-1)}{2}} \prod_{l=0}^{K-1} l! \tag{70}
\]

which is independent of \( M \). In the large \( M \) limit, from the Stirling formula, we have

\[
I^{(1)} \simeq CM^{\frac{MK+K(K-1)}{2}} e^{-\frac{MK^2}{2}} \tag{71}
\]

\[
I^{(2)} \simeq CM^{\frac{MK+K(K+1)}{2}} e^{-\frac{MK^2}{2}} \tag{72}
\]

It is remarkable that, even for finite \( M \) (\( M \) is the size of the random matrix), \( F_{2K}(0) \) for this Gaussian distribution, already exhibits the factor \( \gamma_K = \prod_{l=0}^{K-1} l! / ((K + l)! = \left[ \prod_{l=0}^{K-1} l! \right] / \prod_{l=0}^{2K-1} l! \), which is known to be universal in the large \( M \)-limit. It is indeed obtained from the product of the factor \( C \) and \( 1/(\prod_{l=0}^{2K-1} l!) \) in (65). This means that at each order of the \( 1/N \) expansion, we keep this universal factor for \( F_{2K}(0) \). In the large \( N \) limit (\( M = N - K \)), \( F_{2K}(0) \) becomes

\[
F_{2K}(0) \simeq (2N)^{K^2} e^{-NK} \prod_{l=0}^{K-1} \frac{l!}{(K+l)!} \tag{73}
\]
In the previous paper, we have derived $F_{2K}(\lambda)$, in the large $N$ limit, as

$$F_{2K}(\lambda, \cdots, \lambda) \simeq (2\pi \rho(\lambda)N)^{K^2} e^{-NK} \prod_{l=0}^{K-1} \frac{l!}{(K+l)!}$$  \hspace{1cm} (74)$$

At the band center, $\lambda = 0$, the density of state is $\rho(0) = \frac{1}{\pi}$ for the Gaussian distribution. Therefore, (73) is indeed consistent with (74).

It may be interesting to note that one of the factors of (74), namely $\prod_{l=0}^{2K-1} l!$, appears in $F_{2K}(\lambda)$ in (59). This factor, a product of gamma-functions, remains for any set of orthogonal polynomials, since it stands in the front of the determinant of (59).

The factor $e^{-NK}$ is cancelled by the normalization [6]. For $\lambda \neq 0$, we have evaluated $F_{2K}(\lambda)$. We have here considered the finite $N$ case to see the universal factor $\gamma_K$.

One can recover again the Airy limit by the use of eq.(59). We use once more the properties of the Hermite polynomials such as

$$H_n'(x) = nH_n(x)$$  \hspace{1cm} (75)$$

and their explicit integral representation

$$H_n(x) = \frac{1}{\sqrt{2\pi}} N^{\frac{n}{2}} e^{\frac{N}{2}x^2} \int_{-\infty}^{\infty} ds s^n e^{-\frac{N}{2}s^2 - isN}.$$  \hspace{1cm} (76)$$

We set $n = \delta + N$, and after exponentiation, we have

$$H_n(x) = \frac{1}{\sqrt{2\pi}} N^{\frac{n}{2}} e^{\frac{N}{2}x^2} \int_{-\infty}^{\infty} ds s^\delta e^{-Nf(s)}$$  \hspace{1cm} (77)$$

where $f(s) = \frac{1}{2}s^2 + isx - \log s$. The saddle points are degenerate at the edge $x = 2$. The vicinity of this point is blown out through a change of variables, with a scaling ansatz,

$$x = 2 + N^{-\alpha}y$$  \hspace{1cm} (78)$$

and

$$s = -i + N^{-\beta}z$$  \hspace{1cm} (79)$$
If one expands $f(s)$ up to order $z^3$, one sees that in the proper scaling choice $\alpha = 2/3$ and $\beta = 1/3$, one recovers the Airy limit which governs the properties of the system in a neighbourhood of size $N^{-2/3}$ of the edge of Wigner's semi-circle. Then, the integral becomes

$$I = (-i)^d N^{-\frac{1}{3}} \int_{-\infty}^{\infty} dy e^{iy^{3/2} + izy}$$ (80)

This is indeed the Airy function $A_i(z)$ of (42).

$$H_{N+\delta}(x) = \sqrt{2\pi N} e^{2N(-i)^d A_i((x-2)N^{\frac{2}{3}})}$$ (81)

We now consider all the $\lambda_i = 2$, and the determinant (59) becomes in the large $N$ limit a determinant of Airy functions. If we replace $H_{M+2K-1}$ at the right-up corner of the determinant by the Airy function $A_i(0)$, the other matrix elements become derivatives of the Airy function, since there is a the recursion relation (75). For example, in the $K = 1$ case, we have

$$\text{det} \begin{vmatrix} H_M(2) & H_{M+1}(2) \\ H'_M(2) & H'_{M+1}(2) \end{vmatrix} \sim \text{det} \begin{vmatrix} \frac{N^{\frac{2}{3}}}{M+1} A'_i(0) & A_i(0) \\ \frac{N^{\frac{2}{3}}}{M+1} A''_i(0) & N^{\frac{2}{3}} A'_i(0) \end{vmatrix}$$ (82)

Then, we find in the large $N$ limit, with $N = M - K$,

$$F_{2K}(2) = \frac{N^{\frac{2}{3}K(K+1)}}{\Pi_{l=0}^{2K-1} l!} \text{det} \begin{vmatrix} \cdots & A'_i(0) & A_i(0) \\ \cdots & A''_i(0) & A'_i(0) \\ \cdots & \cdots & \cdots \end{vmatrix}$$ (83)

The above determinant was discussed earlier. Note the factor $1/\Pi_{l=0}^{2K-1} l!$ in front.

## 6 Derivative moments

The same techniques may also be used if one is interested in the moments of the D-th derivatives ($D = 1, 2, \ldots$) of the characteristic polynomials. Let us consider for instance

$$F_{2K}^{(D)}(\lambda_1, \ldots, \lambda_{2K}) = \langle \prod_{l=3}^{2K} \frac{\partial^D}{\partial \lambda_i^D} \text{det}(\lambda_i - X) \rangle$$ (84)
From (85), one sees immediately that it has also the form of a determinant:

\[
F_{2K}^{(D)}(\lambda_1, \ldots, \lambda_{2K}) = \frac{1}{\Delta(\lambda_1, \ldots, \lambda_{2K})} \det \begin{vmatrix}
  p_M^{(D)}(\lambda_1) & p_{M+1}^{(D)}(\lambda_1) & \cdots & p_{M+2K-1}^{(D)}(\lambda_1) \\
  p_M^{(D)}(\lambda_2) & p_{M+1}^{(D)}(\lambda_2) & \cdots & p_{M+2K-1}^{(D)}(\lambda_2) \\
  \vdots & \vdots & \ddots & \vdots \\
  p_M^{(D)}(\lambda_{2K}) & p_{M+1}^{(D)}(\lambda_{2K}) & \cdots & p_{M+2K-1}^{(D)}(\lambda_{2K})
\end{vmatrix}.
\]

(85)

When all the \(\lambda_i\)'s are equal, we have

\[
F_{2K}^{(D)}(\lambda, \ldots, \lambda) = \left( \frac{d^D}{d\lambda^D} \det(\lambda - X) \right)^{2K} \]

\[
= \frac{(-1)^{K(2K-1)}}{\prod_{l=0}^{2K-1}(l!)} \det \begin{vmatrix}
  p_M^{(D)}(\lambda) & p_{M+1}^{(D)}(\lambda) & \cdots & p_{M+2K-1}^{(D)}(\lambda) \\
  p_M^{(D+1)}(\lambda) & p_{M+1}^{(D+1)}(\lambda) & \cdots & p_{M+2K-1}^{(D+1)}(\lambda) \\
  \vdots & \vdots & \ddots & \vdots \\
  p_M^{(D+2K-1)}(\lambda) & p_{M+1}^{(D+2K-1)}(\lambda) & \cdots & p_{M+2K-1}^{(D+2K-1)}(\lambda)
\end{vmatrix}.
\]

(86)

If we set \(\lambda = 0\) it may be again decomposed into a product of two determinants.

Let us assume for definiteness that both \(M\) and \(D\) are even. Then, we have

\[
I^{(1)} = \det \begin{vmatrix}
  H_M^{(D)}(0) & H_{M+2}^{(D)}(0) & \cdots \\
  H_M^{(D+2)}(0) & H_{M+2}^{(D+2)}(0) & \cdots \\
  \vdots & \vdots & \ddots \\
  H_M^{(D+2K-2)}(0) & H_{M+2}^{(D+2K-2)}(0) & \cdots
\end{vmatrix}
\]

(87)

\[
I^{(2)} = \det \begin{vmatrix}
  H_{M+1}^{(D+1)}(0) & H_{M+3}^{(D+1)}(0) & \cdots \\
  H_{M+1}^{(D+3)}(0) & H_{M+3}^{(D+3)}(0) & \cdots \\
  \vdots & \vdots & \ddots \\
  H_{M+1}^{(D+2K-1)}(0) & H_{M+3}^{(D+2K-1)}(0) & \cdots
\end{vmatrix}
\]

(88)

Using the explicit expressions for the Hermite polynomials, we can compute these determinants. We find for arbitrary \(M, D\) and \(K\),

\[
F_{2K}^{(D)}(0) = \frac{1}{N^{K(M-D)}} I^{(1)} I^{(2)} \frac{1}{\prod_{l=0}^{2K-1} l!}
\]

(89)

\[
I^{(1)} = (M + 2K - 3)!!(M + 2K - 5)!! \cdots (M - 1)!!
\]
\begin{align*}
&\times \prod_{l=0}^{K-1} [(M/2 + l)(M/2 + l - 1) \cdots (M/2 - D/2 + l + 1)] \\
&\times 2^{2D+K(K-1)} \prod_{l=0}^{K-1} l!
\end{align*}

$$I^{(2)} = (M+2K-1)!!(M+2K-3)!! \cdots (M+1)!!$$

\begin{align*}
&\times \prod_{l=0}^{K-1} [(M/2 + l + 1)(M/2 + l) \cdots (M/2 - D/2 + l + 2)] \\
&\times 2^{2D+K(K-1)} \prod_{l=0}^{K-1} l!
\end{align*}

One may easily check these results for $D = M$, since the matrix elements below the diagonal vanish, i.e. the determinants are then simply given by the product of the diagonal elements, $\prod_{l=0}^{K} (M + 2l)!$ which agrees with (90). When $D = 0$, it reduces to the previous expression (68). $I^{(2)}$ is obtained from $I^{(1)}$ by the shift $M \rightarrow M + 2$.

In the large $N$ limit, we have

$$F_{2K}^{(D)}(0) \sim (2N)^{K^2+2KD} e^{-KN} \frac{1}{2^{2KD}} \prod_{l=0}^{K-1} \frac{l!}{(K+l)!}$$

Hence, for this derivative moments at finite $M$, again the universal factor $\gamma_K$ is present, and it persists of course in the large $N$ limit.

These results lead to the conjecture that the average of the moment of derivatives of the Riemann zeta-function along the critical line

$$I = \frac{1}{T} \int_0^T dt \frac{d^D}{dt^D} \zeta((1/2 + it)^{2K},$$

also have this universal factor $\gamma_K$.

\section{External source}

We now consider the case in which the external source matrix $A$ is coupled to the random matrix $X$. The measure of the random matrix $X$ is

$$d\mu(X) = \frac{1}{Z} e^{-\frac{N}{2} Tr X^2 + N Tr XA} d^N X$$

\begin{align*}
\times \prod_{l=0}^{K-1} [(\frac{M}{2} + l)(\frac{M}{2} + l - 1) \cdots (\frac{M}{2} - \frac{D}{2} + l + 1)] \\
\times 2^{2D+K(K-1)} \prod_{l=0}^{K-1} l!
\end{align*}

\[I^{(2)} = (M+2K-1)!!(M+2K-3)!! \cdots (M+1)!!

\begin{align*}
&\times \prod_{l=0}^{K-1} [(\frac{M}{2} + l + 1)(\frac{M}{2} + l) \cdots (\frac{M}{2} - \frac{D}{2} + l + 2)] \\
&\times 2^{2D+K(K-1)} \prod_{l=0}^{K-1} l!
\end{align*}

\[\begin{align*}
F_{2K}^{(D)}(0) &\sim (2N)^{K^2+2KD} e^{-KN} \frac{1}{2^{2KD}} \prod_{l=0}^{K-1} \frac{l!}{(K+l)!} \end{align*}

\text{Hence, for this derivative moments at finite } M, \text{ again the universal factor } \gamma_K \text{ is present, and it persists of course in the large } N \text{ limit.}

\text{These results lead to the conjecture that the average of the moment of derivatives of the Riemann zeta-function along the critical line}

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The eigenvalues of the matrix $A$ are denoted by $a_i$, $i = 1, \ldots, N$. In such cases, the standard orthogonal polynomial method cannot be used. However, the n-point correlation functions $\rho(\lambda_1, \ldots, \lambda_n)$ have been found to be described again by the determinant of a kernel; from there the level spacing probability $p(s)$ has been also investigated \cite{9}. If we specialize to a source which has two opposite eigenvalues, namely $a_i = +a$ for $i = 1, \ldots, N/2$ and $a_i = -a$ for $i = N/2 + 1, \ldots, N$, one finds a support for the eigenvalues made of two disconnected segments for $a > 1$. If one tunes the external source so that $a = 1$, i.e. $a_i = \pm 1$, the gap between the two segments closes and the spectrum consists of a single segment for $a < 1$. We want to investigate here the critical point $a = 1$ which gives rise to yet another class of universality.

The moments $F_{2K}(\lambda, \cdots, \lambda)$ at $\lambda = 0$ at this closing gap point may turn out to have interesting applications.

Since $X$ and $A$ are Hermitian matrices, we write

\[
\text{Tr}XA = \text{Tr}U^{-1}X_0UA_0
\]  

where $X_0 = \text{diag}(x_1, \ldots, x_M)$, $A_0 = \text{diag}(a_1, \cdots, a_M)$, and $U$ belongs to the unitary group. The integration over this unitary group $U$ is well known from the work of Harish-Chandra, Itzykson-Zuber \cite{18, 19}, and this is the starting point of the formulae found in \cite{8}. For instance the n-point correlation functions are given by the determinant of the $n \times n$ matrices made with the kernel $K(\lambda_i, \lambda_j)$ with

\[
K(\lambda, \mu) = \int_{-\infty}^{\infty} dt \frac{du}{2\pi} \prod_{i=1}^{N} a_i - it u - a_i u - it e^{-\frac{N}{2} t^2 + N t u + N u^2 + Nu + Nu^2} 
\]  

where the contour encloses all the $a_i$’s.

However we may proceed without that here and compute the correlation functions of the characteristic polynomials directly. Indeed

\[
F_K(\lambda_1, \cdots, \lambda_K) = \left< \prod_{\alpha=1}^{K} \det(\lambda_\alpha - X) \right> 
= \frac{1}{Z} \int dX \prod_{\alpha=1}^{K} \det(\lambda_\alpha - X) e^{-\frac{N}{2} \text{Tr}X^2 + N \text{Tr}XA} 
\]
In the above equation, the random matrix $X$ is assumed to be an $M \times M$ matrix, with $M = N - K$, as before. When $K = 1$, this gives a polynomial, which was investigated before [17].

The explicit integration over the unitary group [18, 19], leads to

$$
F_K(\lambda_1, \cdots, \lambda_K) = \int \prod_{i=1}^M dx_i \frac{\Delta(x_1, \cdots, x_M; \lambda_1, \cdots, \lambda_K)}{\Delta(a) \Delta(\lambda)} e^{-\frac{N}{2} \sum_{i=1}^M x_i^2 + N \sum_{i=1}^M x_i a_i} \tag{98}
$$

where $\Delta(x_1, \cdots, x_M; \lambda_1, \cdots, \lambda_K)$ is the Van der Monde determinant $(M + K) \times (M + K)$ made with the $x$’s and the $\lambda$’s. This determinant may be replaced by a determinant of (monic) polynomials, and we choose the Hermite polynomials defined in (62). It is then straightforward to verify that

$$
\int_{-\infty}^\infty H_n(x) e^{-\frac{N}{2} x^2 + N a x} dx = a^n e^{\frac{N}{2} a^2} \sqrt{\frac{2\pi}{N}} \tag{99}
$$

Therefore we can explicitely integrate over the $M$ variables $x_i$’s in (98) and one obtains

$$
F_K(\lambda_1, \cdots, \lambda_K) = \frac{1}{\Delta(a) \Delta(\lambda)} \times \det \begin{vmatrix}
1 & \cdots & 1 & H_0(\lambda_1) & \cdots & H_0(\lambda_K) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_1^{M+K-1} & \cdots & a_M^{M+K-1} & H_{M+K-1}(\lambda_1) & \cdots & H_{M+K-1}(\lambda_K)
\end{vmatrix} \tag{100}
$$

Let us first check that in the limit of a vanishing source in which all the $a_i \to 0$, we do recover the previous formula (57). The column which depends upon $a_i$ is expanded in Taylor series around $a_1$, and subtracting the successive columns, we obtain, after factoring the Van der Monde determinant $\Delta(a)$ which cancels the denominator, a vanishing upper triangle (up to the $M$-th column), ones on the diagonal and powers of the $a_i$’s below the diagonal. We can now let the $a_i$’s go to zero and we are left with the $K \times K$ of (57). (In [6] we gave a different derivation of this same formula).

If we return to an arbitrary non vanishing external source, we may proceed by returning to (100) and define $G_K(b_1, \cdots, b_K)$,

$$
G_K(b_1, \cdots, b_K) = \int F_K(\lambda_1, \cdots, \lambda_K) \Delta(\lambda) e^{-\frac{N}{2} \sum \lambda_i^2 + N \sum \lambda_i b_i} \prod d\lambda_i = \frac{\Delta(a; b)}{\Delta(a)} e^{\frac{N}{2} \sum b_i^2} \tag{101}
$$
We may now recover $F_K$ by taking the Fourier transform of $G_K(i b_1, \ldots, i b_K)$,

$$\int G_K(i b_1, \ldots, i b_K) e^{-i N \sum \lambda_i b_i} \prod_{i=1}^{K} \frac{db_i}{2\pi} = \left(\frac{1}{N}\right)^K \Delta(\lambda) F_K(\lambda_1, \ldots, \lambda_K) e^{-\frac{N}{2} \sum \lambda_i^2} \quad (102)$$

Therefore, we obtain the following explicit formula,

$$F_K(\lambda_1, \ldots, \lambda_K) = \frac{N^K}{\Delta(\lambda)} e^{\frac{N}{2} \sum \lambda_i^2} \frac{1}{K!} \times \int \prod_{i=1}^{K} \frac{db_i}{2\pi} \prod_{j=1}^{M} (ib_l - a_j) \prod_{l<l'}^{K} (ib_l - ib_{l'}) e^{-\frac{N}{2} \sum \lambda_i^2} \det(e^{-i N \lambda_l b_{l'}}) \quad (103)$$

Note that we could replace in the integrand of (103) $\det(e^{-i N \lambda_l b_{l'}})$ by the diagonal term $e^{-i N \sum \lambda_l b_l}$ and cancel the $K!$ in the denominator. Again we can examine the limit of this formula when the external source goes to zero, and putting all $\lambda_i = \lambda$, we obtain

$$F_{2K}(\lambda) = \frac{N^{K(2K+1)}}{\prod_{l=0}^{2K-1} l!} \left(\frac{1}{2K}\right)! e^{KN\lambda^2} \int \prod_{l=1}^{2K} b_l^M \Delta^2(b) e^{-\frac{N}{2} \sum \lambda_l^2} \sum b_l \prod_{l=1}^{2K} \frac{db_l}{2\pi}, \quad (104)$$

(we have considered $F_{2K}$ instead of $F_K$ in order to compare with our previous results).

In the large $N$ limit, we exponentiate $b_l^M$, ($M = N - K$), and look for the saddle points which are the roots of the equation $b^2 + i \lambda b - 1 = 0$; let us call the two roots $b^+$ and $b^-$. The difference $|b^+ - b^-| = 2\pi \rho(\lambda)$. The leading saddle-point for the $b_l$'s, $(l = 1, \ldots, 2K)$, is obtained by choosing half of them equal to $b^+$, and $b^-$ for the another half. The following Gaussian integral with a Van der Monde determinant,

$$\frac{1}{K!} \int \prod_{i=1}^{K} db_i e^{-\frac{N}{2} f''(b)} \prod_{i<j}^{K} (b_i - b_j)^2 = \left(\frac{2\pi}{N f''(\lambda)}\right)^K \prod_{l=0}^{K-1} \frac{l!}{(N f''(\lambda))^{K(2K-1)}}, \quad (105)$$

where $f''$ is the second derivative of $f$ at the saddle-point, allows us to complete the calculation. Integrating then around the saddle-points $b^+$ and $b^-$, and keeping in mind the combinatorial factor $\frac{(2K)!}{K!K!}$, which is the number of choices of $K b^+$ and $K b^-$ among the $2K b_l$'s, we recover precisely our previous result,

$$e^{-NKV(\lambda)} F_{2K}(\lambda) = (2\pi N \rho(\lambda))^{K^2} e^{-NK \gamma_K} \quad (106)$$
where $\gamma_K = (\prod_{l=0}^{K-1} l!)^2 / (\prod_{l=0}^{2K-1} l!) = (\prod_{l=0}^{K-1} l!) / \prod_{l=0}^{K-1} (K + l)!$, and $V(\lambda) = \lambda^2$.

From the expression (103), it is also easy to obtain the moments at the critical point corresponding to the closure of the gap:

$$F_K(0) = \frac{N^K}{K!} e^{\frac{NM}{2}} \int \prod_{l=1}^{K} \frac{db_l}{2\pi} (1 + b_l^2)^{\frac{M}{2}} \Delta^2(b) e^{-\frac{N}{2} \sum_{l=1}^{K} b_l^2}$$

Note that this expression is exact for finite $N$. In the large $N$ limit, we exponentiate the logarithmic term and expand the exponent about $b_l$ up to order $b_l^4$ term. The critical point, is precisely the point at which the coefficient of the quadratic term $b_l^2$ vanishes. We then have

$$F_K(0) = \frac{N^K}{K!} e^{\frac{NM}{2}} \int \prod_{l=1}^{K} \frac{db_l}{2\pi} e^{-\frac{N}{2} \sum_{l=1}^{K} b_l^2} \Delta^2(b)$$

As in all the cases which appeared in the previous sections, this integral is expressed by a Hankel determinant, in which the matrix elements are $\Gamma\left(\frac{2n-1}{4}\right)$. The determinant is

$$I = \det \begin{bmatrix} \Gamma\left(\frac{1}{4}\right) & 0 & \Gamma\left(\frac{3}{4}\right) & 0 & \ldots \\ 0 & \Gamma\left(\frac{3}{4}\right) & 0 & \Gamma\left(\frac{5}{4}\right) & \ldots \\ \Gamma\left(\frac{1}{4}\right) & 0 & \Gamma\left(\frac{3}{4}\right) & 0 & \ldots \\ \ldots & \ldots & \Gamma\left(\frac{3}{4}\right) & 0 & \ldots \\ & & & & \end{bmatrix}.$$  

Note that we have considered the case $a_l = \pm 1$ case, but the formulae are explicit for any spectrum of the source and they could be easily used to study for instance multi-critical situations which were discussed in [9].

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