Fully Bayesian Analysis of the
Relevance Vector Machine Classification
for Imbalanced Data

Wenyang Wang∗
School of Maritime Economics and Management
Dalian Maritime University
e-mail: ww424@mail.missouri.edu

Dongchu Sun∗
School of Statistics
East China Normal University
e-mail: dsun90@unl.edu

Zhuoqiong He
Department of Statistics
University of Missouri
e-mail: hezh@missouri.edu

Abstract: Relevance Vector Machine (RVM) is a supervised learning algo-
rithm extended from Support Vector Machine (SVM) based on the Bayesian
sparsity model. Compared with the regression problem, RVM classification
is difficult to be conducted because there is no closed-form solution
for the weight parameter posterior. Original RVM classification algorithm
used Newton’s method in optimization to obtain the mode of weight param-
eter posterior then approximated it by a Gaussian distribution in Laplace’s
method. It would work but just applied the frequency methods in a Bayesian
framework. This paper proposes a Generic Bayesian approach for the RVM
classification. We conjecture that our algorithm achieves convergent esti-

∗Key Laboratory of Advanced Theory and Application in Statistics and Data Science (East
China Normal University), Ministry of Education.
mates of the quantities of interest compared with the nonconvergent estimates of the original RVM classification algorithm. Furthermore, a Fully Bayesian approach with the hierarchical hyperprior structure for RVM classification is proposed, which improves the classification performance, especially in the imbalanced data problem. By the numeric studies, our proposed algorithms obtain high classification accuracy rates. The Fully Bayesian hierarchical hyperprior method outperforms the Generic one for the imbalanced data problem.

**Keywords and phrases:** Classification, RVM, Bayesian Analysis, Imbalanced data problem.

### Contents

1. Introduction .................................................. 1
2. RVM Classification ........................................... 4
3. Generic Bayesian RVM Classification ....................... 10
4. Fully Hierarchical Bayesian RVM Classification .......... 12
5. Numeric Studies ............................................... 15
   5.1 Simulation Data Studies ................................. 15
   5.2 Real Data Studies ........................................ 17
6. Conclusion and Discussion .................................... 18
7. Appendix ...................................................... 19
8. Acknowledgments ............................................. 22
References ...................................................... 22

### 1. Introduction

The Kernel method has become popular with the proposal of the Support Vector Machine (SVM) [6] and motivated the rapid growth of the classification algorithms [17, 20]. SVM is designed based on the Structural Risk Minimization (SRM) and it projects the data to the high dimensional linear separable space. It was indicated that SVM constructs an optimal separating hyperplane as the
classification borderline, which can obtain the maximum distance between two
classes for a binary dataset [3]. A binary classification training dataset can be
described as follows: Let \( S^{\text{train}} = \{ (x_1^{\text{train}}, y_1^{\text{train}}), (x_2^{\text{train}}, y_2^{\text{train}}), ..., (x_n^{\text{train}}, y_n^{\text{train}}) \} \)
be the training data, where \( x_i^{\text{train}} \in X \subseteq \mathbb{R}^l \), \( X \) is a subset of \( l \)-dimensional vec-
tor space. The response \( y_i^{\text{train}} \in \{-1, 1\} \) indicates two classes, and \( i = 1, ..., n \).
The output of SVM for an arbitrary data point \( x_0 \) can be expressed as a weighted
sum of the form [26, 19]

\[
f(x_0; w) = \sum_{i=1}^{n} w_i K(x_0, x_i^{\text{train}}) + w_0, \tag{1.1}
\]

where \( K(\cdot, \cdot) \) is the Kernel function, \( w_0 \) and \( w_i \) are weight parameters, \( i = 1, 2, ..., n \).
Note that \( x_0 \) can be a training data point or a test data point. The
Radial Basis Function (RBF) Gaussian kernel, namely

\[
K(x_i, x_j) = \exp \left( -\frac{\|x_i - x_j\|^2}{2\gamma^2} \right),
\]

is used throughout this paper. RBF Gaussian Kernel is the most popular Kernel
in Statistics and Machine Learning fields. The SVM output for the training data
\( S^{\text{train}} \) can be expressed as the matrix form

\[
f(x^{\text{train}}; w) = \Phi^{\text{train}} w, \tag{1.2}
\]

where \( w = (w_0, w_1, ..., w_n)^T \) is the weight parameter and

\[
\Phi^{\text{train}} = \begin{pmatrix}
1 & K(x_1^{\text{train}}, x_1^{\text{train}}) & \cdots & K(x_1^{\text{train}}, x_n^{\text{train}}) \\
1 & K(x_2^{\text{train}}, x_1^{\text{train}}) & \cdots & K(x_2^{\text{train}}, x_n^{\text{train}}) \\
\vdots & \vdots & \ddots & \vdots \\
1 & K(x_n^{\text{train}}, x_1^{\text{train}}) & \cdots & K(x_n^{\text{train}}, x_n^{\text{train}})
\end{pmatrix}. \tag{1.3}
\]

Note that given the training data \( S^{\text{train}} \), the kernel matrix \( \Phi^{\text{train}} \) is fixed. The
classification goal is to obtain \( \text{sign}(f(x^{\text{train}}; \hat{w})) = y^{\text{train}} \), where \( \hat{w} \) is the proper
estimate of \( w \), \( \text{sign}(z) = 1 \) if \( z \geq 0 \), and \( \text{sign}(z) = -1 \) if \( z < 0 \).
Although SVM has been the leading method in pattern recognition and machine learning, it still has a few limitations [24]: (1) The numbers of kernels and parameters increase linearly with the growth of the training data size. SVM would be inefficient in a large data case; (2) The classification result is not a probability and does not provide an accuracy rate. The Relevance Vector Machine (RVM) [24, 23] was originally proposed based on SVM to solve the above two limitations by conducting a sparsity prior to the weight parameter \( w \). Afterward, a fast sequence RVM algorithm was proposed [25] and speeded up the training process significantly. Later, the RVM algorithm was extended to multiple-output regression and multiple-output classification [22]. RVM has obtained more applications in text image recognition [21], image classification [7], time series analysis [15], mechanical fault diagnosis [8], and electric demand forecasting [28]. But most consequential and influential work related to RVM is in the regression field. RVM classification is hard to be conducted in a strict Bayesian framework because there is no closed-form solution for the posterior of weight parameter \( w \). The original RVM classification applies Newton’s method to seek the mode of posterior of \( w \), then approximates it with a normal distribution in the Laplace’s method instead of directly sampling \( w \) from its posterior. The original RVM classification was built in a Bayesian framework but conducted with frequency methods.

RVM is studied in this paper for the classification problem. Class imbalance is one of the most serious and challenging problems in the classification field. The class imbalance situation commonly appears in many fields such as facial detection [11], financial fraud detection [5], and oil exploration [13]. For the imbalanced problem, define \( S_{+}^{train} = \{(x_i^{train}, y_i^{train}) \in S^{train} : y_i = 1, i = 1, ..., n\} \) is the positive or minority training class and \( S_{-}^{train} = \{(x_i^{train}, y_i^{train}) \in S^{train} : y_i^{train} = -1, i = 1, ..., n\} \) is the negative or majority training class. The class type, \{minority, majority\} and \{positive, negative\} are used to describe \( \{S_{+}^{train}, S_{-}^{train}\} \).
Definition 1. Let $|A|$ denote the number of the elements in a set $A$. Define the number of samples in the positive class and the negative class as $N_{p}^{\text{train}} = |S_{+}^{\text{train}}|$ and $N_{n}^{\text{train}} = |S_{-}^{\text{train}}|$, respectively. The imbalance degree of data is defined as $b = N_{n}^{\text{train}}/N_{p}^{\text{train}}$.

The condition of $N_{n}^{\text{train}} > N_{p}^{\text{train}}$ is called the imbalanced data problem. Note that $b > 1$ in imbalanced data. For the imbalanced data classification problem, most of algorithms still can obtain the advantageous global accuracy rates, but they could have a low classification accuracy rate of the positive class. When the positive class is our interest, some particular algorithms are desired to obtain the high positive class accuracy rates without sacrificing the global accuracy rates. Only the binary imbalanced class problems are studied in this paper. Multiple-class issues can be solved based on the one-versus-all scheme [18]. This paper proposes a Generic Bayesian analysis for RVM classification which has a better parameter convergence property than the original one. For the imbalanced data problem, we propose a hierarchical prior structure for the RVM classification and obtain a higher accuracy rate than the Generic algorithm. The detailed introduction of the original RVM classification algorithm is stated in Section 2. Section 3 proposes our Generic Bayesian RVM classification algorithm. The Fully Bayesian RVM classification algorithm with the hierarchical hyperprior is illustrated in Section 4. The numeric studies are posted in Section 5 including the simulated and real data studies. Section 6 concludes this paper with some future research discussions.

2. RVM Classification

Relevance Vector Machine (RVM) is a Bayesian treatment for the output of the Support Vector Machine (SVM). This paper only focuses on the RVM classification. It applies the Bernoulli distribution to the output of SVM in (2) and constructs the probability density function $p(y|x)$ for the classification prob-
lems. The logistic sigmoid link function \( \sigma(z) = 1/(1 + \exp(-z)) \) is used to map \( f(x; \mathbf{w}) \) into \([0, 1]\). The likelihood of the training data set is

\[
p(y^{\text{train}} | \mathbf{w}) = \prod_{i=1}^{n} \sigma\left(f(x_i^{\text{train}}; \mathbf{w})\right)^{y_i^{\text{train}}} \left[1 - \sigma\left(f(x_i^{\text{train}}; \mathbf{w})\right)\right]^{1-y_i^{\text{train}}},
\]

(2.1)

where \( \phi_i^{\text{train}} \) and \( \mathbf{w} \) are defined in (2) and (3). RVM classification introduced a zero-mean Gaussian prior distribution over \( \mathbf{w} \), namely

\[
p(\mathbf{w} | \alpha) = \prod_{s=0}^{n} N(w_s | 0, \alpha_s^{-1}) = N(\mathbf{w} | \mathbf{0}, \mathbf{A}^{-1}),
\]

(2.2)

where \( \alpha = (\alpha_0, \alpha_1, ..., \alpha_n)^T \), \( \mathbf{A} = \text{diag}(\alpha_0, \alpha_1, ..., \alpha_n) \), \( \alpha_s \) is the hyperparameter associated with weight \( w_s \), and \( s = 0, 1, 2, ..., n \). This prior helps to obtain the sparsity constraint. Compared with SVM, RVM classification has fewer relevant vectors because of the sparsity prior. The Bayesian model provides a posterior distribution for \( \mathbf{w} \) as

\[
p(\mathbf{w} | y^{\text{train}}, \alpha) = \frac{p(y^{\text{train}} | \mathbf{w})p(\mathbf{w} | \alpha)}{\int p(y^{\text{train}} | \mathbf{w})p(\mathbf{w} | \alpha)d\mathbf{w}} = \frac{g(\mathbf{w})}{p(y^{\text{train}} | \alpha)},
\]

(2.3)

where \( g(\mathbf{w}) = p(y^{\text{train}} | \mathbf{w})p(\mathbf{w} | \alpha) \), which implies that

\[
p(\mathbf{w} | y^{\text{train}}, \alpha) \propto g(\mathbf{w}).
\]

(2.4)

The limitations of SVM are solved by RVM in the Bayesian framework. The original RVM classification obtained \( \hat{\mathbf{w}} \), which is the estimation of \( \mathbf{w} \), by maximizing \( g(\mathbf{w}) \). The classification function for the training data \( S^{\text{train}} \) is

\[
y_s^{\text{train}} = \text{sign}\left(\frac{1}{1 + \exp(-\Phi^{\text{train}} \hat{\mathbf{w}})} - \frac{1}{2}\right),
\]

(2.5)

where \( \Phi^{\text{train}} \) is defined in (3). A test data can be defined as \( S^{\text{test}} = \{(x_1^{\text{test}}, y_1^{\text{test}}), (x_2^{\text{test}}, y_2^{\text{test}}), ..., (x_m^{\text{test}}, y_m^{\text{test}})\} \), where \( x_j^{\text{test}} \in X \subseteq \mathbb{R}^l \). The response \( y_j^{\text{test}} \in \{-1, 1\} \) indicates two classes,
In the imbalanced data problem, we define $S^\text{test}_+ = \{(x^\text{test}_j, y^\text{test}_j) \in S^\text{test} : y^\text{test}_j = 1, j = 1, ..., m\}$ and $S^\text{test}_- = \{(x^\text{test}_j, y^\text{test}_j) \in S^\text{test} : y^\text{test}_j = -1, j = 1, ..., m\}$. The classification function for a test data $S^\text{test}$ is

$$y^* = \text{sign}\left(\frac{1}{1 + \exp(-\Phi^\text{test} w)} - \frac{1}{2}\right),$$

(2.6)

where

$$\Phi^\text{test} = (\phi^\text{test}_1, \phi^\text{test}_2, \cdots, \phi^\text{test}_m)^T = \begin{pmatrix}
1 & K(x^\text{test}_1, x^\text{train}_1) & \cdots & K(x^\text{test}_1, x^\text{train}_n) \\
1 & K(x^\text{test}_2, x^\text{train}_1) & \cdots & K(x^\text{test}_2, x^\text{train}_n) \\
\vdots & \vdots & \ddots & \vdots \\
1 & K(x^\text{test}_m, x^\text{train}_1) & \cdots & K(x^\text{test}_m, x^\text{train}_n)
\end{pmatrix}. \tag{2.7}
$$

Eight criteria listed in Table 1 are used to evaluate the performance of algorithms in this paper.

| Table 1 |
|---|

| Training Data Global Accuracy Rate | $r^\text{train}_g = \frac{|y^\text{train} = y^*|}{n}$ |
| Training Data Positive Class Accuracy Rate | $r^\text{train}_p = \frac{|y^\text{train} = y^* \land y^\text{train} = 1|}{n}$ |
| Same Size Test Data Global Accuracy Rate | $r^\text{test}_g = \frac{|y^\text{test} = y^*|}{n}$ |
| Same Size Test Data Positive Class Accuracy Rate | $r^\text{test}_p = \frac{|y^\text{test} = y^* \land y^\text{test} = 1|}{n}$ |
| Smaller Size Test Data Global Accuracy Rate | $r^\text{stest}_g = \frac{|y^\text{stest} = y^*|}{n_p}$ |
| Smaller Size Test Data Positive Class Accuracy Rate | $r^\text{stest}_p = \frac{|y^\text{stest} = y^* \land y^\text{stest} = 1|}{n_p}$ |
| Larger Size Test Data Global Accuracy Rate | $r^\text{ltest}_g = \frac{|y^\text{ltest} = y^*|}{n_l}$ |
| Larger Size Test Data Positive Class Accuracy Rate | $r^\text{ltest}_p = \frac{|y^\text{ltest} = y^* \land y^\text{ltest} = 1|}{n_l}$ |

The calculations of $r^\text{test}_g$ and $r^\text{test}_p$ use the same-sized test data as the training data, $n^\text{train} = n^\text{test} = n, r^\text{train}_p = n_p$. Smaller-sized and larger-sized
test data are used for the calculations of \((r_{g,\text{test}}, r_{p,\text{test}}^*)\) and \((r_{g,\text{test}}, r_{p,\text{test}}^*)\), which means \(n^s < n < n^t, n^p < n_p < n^t\). The simulation data studies use all these eight criteria. The real data studies only apply \(r_{\text{train}}, r_{\text{train}}, r_{\text{test}}^t,\) and \(r_{\text{test}}^t\) because it is hard to obtain different-sized real test data. All the test data sets in this paper keep the same imbalance index \(b\) as the training data, namely
\[
\frac{|S^*_\text{test}|}{|S^*_\text{train}|} = \frac{|S_{\text{train}}|}{|S_{\text{train}}|} = b.
\]

The original RVM classification algorithm is stated as follows:

**Algorithm 1. The Original RVM Classification Algorithm**

**Input.** The training data: \((x_1, y_1), (x_2, y_2), ..., (x_n, y_n), x_i \in X \subseteq \mathbb{R}^l\) and \(y_i \in \{-1, 1\}, i = 1, ..., n\).

0. Let \(t = 1\) and initialize \(w\) and \(\alpha\) to obtain the started values \(w^1\) and \(\alpha^1\), calculate

\[
\begin{align*}
    h &= \nabla_w \log g(w) = \Phi^\top (y - [\sigma(\phi_1 w), \ldots, \sigma(\phi_n w)]^\top) - A w, \\
    H &= -\nabla^2 w \log g(w) = \Phi^\top B \Phi + A, \quad (2.8)
\end{align*}
\]

where \(B\) is a \((n + 1) \times (n + 1)\) diagonal matrix with diagonal elements \(b_{ii} = \sigma(\phi_i w) [1 - \sigma(\phi_i w)]\);

1. Fix \(\alpha\) and update \(w\) with

\[
    w^{t+1} = w^t + (H)^{-1} h|_{w=w^t}; \quad (2.9)
\]

2. Fix \(w\) and update \(\alpha\) with

\[
    \alpha^{t+1}_s = \frac{\gamma^t_s}{w^2_s}, \quad (2.10)
\]

where \(\gamma^t_s = 1 - \alpha^t_s H_{ss}, s = 0, 1, 2, ..., n\);

3. Repeat setps 1 and 2 until suitable convergence and obtain \(w_0\), the mode of \(w\);
**Output.** The final estimation of $w$ is $w_{MP} = H^{-1} \Phi^T B y_{|w=w_0}$.

Note that $w_{MP}$, the maximum posterior of $w$, is obtained by the Laplace’s Method [23], which approximates a normal distribution with the mean value $w_0$ to the posterior of $w$. It was concluded that the RVM is better than SVM in the fields of classification and regression [27, 16, 4]. They also showed that the conduction speed of RVM is faster than SVM. But the original RVM classification algorithm still has several shortcomings:

1. Step 1 in Algorithm 1 is to maximize the numerator $g(w)$ in (6); Step 2 is obtained by maximizing the denominator $p(y_{\text{train}}|\alpha)$ in (6). This iteration process cannot ensure the maximum of posterior, $p(w|y_{\text{train}}, \alpha)$;

2. Laplace’s Method is used to approximate $p(w|y_{\text{train}}, \alpha)$ as a normal distribution with the mean value $w_0$. Although it was indicated that the posterior of $w$ is approximately normally distributed [2], the Laplace’s method is still not stable under a strong normal distribution assumption;

3. The suitable convergence criterion is cryptic. An original RVM classification convergence study is stated as follows:

The simulated training dataset is

$$X_{ij} = \begin{pmatrix} X_{ij1} \\ X_{ij2} \end{pmatrix} \overset{\text{indep}}{\sim} N_2(\mu_i, \Sigma_i),$$

(2.11)

where $i = -1, 1$ and $j = 1, \ldots, n_i$. We set

$$\mu_{-1} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}, \mu_1 = \begin{pmatrix} 13 \\ 15 \end{pmatrix}, \Sigma_{-1} = \begin{pmatrix} 10 & 3 \\ 3 & 8 \end{pmatrix}, \text{and } \Sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$  

(2.12)

All the variables in $X_{-1,j}$ and $X_{1,j}$ have the class labels $-1$ (Majority) and $1$ (Minority), respectively. Choose $n_p = n_n = 3$, Figure 1 indicates this simulated training data:
Fig 1: Simulated Data for Original RVM Classification ($n_p = n_n = 3$).

Run the Algorithm 1 50000 iterations and check the parameter convergence in Figure 2. The plots take out 5000 burn-in and the red lines indicate 0. Although the classification accuracy rate is 100% in this case, we cannot determine that the parameter obtains the suitable convergence.

Fig 2: Convergence Plot of $w$ in Algorithm 1.
3. Generic Bayesian RVM Classification

In this section, we propose a Generic Bayesian RVM classification method with the likelihood and prior for $w$ in (4) and (5). Our Generic RVM classification algorithm samples the parameter directly from the posterior instead of Newton’s method in Algorithm 1. A Gamma hyperprior is called for each $\alpha_s$ and it yields a Student-$t$ marginal prior for $w$ when $\alpha$ is integrated out. The Gamma hyperprior is

$$(\alpha_s|a,b) \sim \text{Gamma}(\alpha_s|a,b),$$

the marginal prior for $w_s$ is

$$p(w_s) = \int p(w_s|\alpha_s) p(\alpha_s) d\alpha_s = \frac{b^a \Gamma(a+\frac{1}{2})}{(2\pi)^{\frac{d}{2}} \Gamma(a)} (b + w_s^2)^{-(a+\frac{3}{2})},$$

where $s = 0, 1, 2, ..., n$. The marginal prior for the vector $w$ is a product of independent Student-$t$ distributions in (17). This density induces more sparsity pressure than the LASSO prior [9]. Recall (7), the posterior of $w$ is

$$p(w|y^{train}, \alpha) \propto g(w)$$

$$\propto \exp(-\frac{1}{2}w^T A w) \prod_{i=1}^{n} \left( \frac{1}{1 + \exp(-\phi_{i}^{train} w)} \right)^{\frac{1+y_{i}^{train}}{2}} \times$$

$$\left( \frac{\exp(-\phi_{i}^{train} w)}{1 + \exp(-\phi_{i}^{train} w)} \right)^{\frac{1-y_{i}^{train}}{2}},$$

where $s = 0, 1, 2, ..., n$. Although this posterior has no closed-form, it has the desirable log-concave property from a computational perspective.

**Theorem 1.** The conditional posterior of $w_s$, $p(w_s|y^{train}, \alpha_s)$ is log-concave.

**Proof.** See Appendix A.

With this log-concavity, drawing samples from the posterior of $w_s$ becomes faster and easier with the adaptive rejection sampling algorithm [10]. The pos-
The posterior of $\alpha_s$ is

$$
p(\alpha_s|w_s, a, b) \propto \alpha_s^{\frac{1}{2}} \exp\left(-\frac{1}{2} \alpha_s w_s^2\right) \cdot \alpha_s^{a-1} \exp(-b \alpha_s)
= \alpha_s^{a + \frac{1}{2} - 1} \exp\left[-(b + \frac{1}{2} w_s^2)\alpha_s\right]
\propto \text{Gamma}(a + \frac{1}{2}, b + \frac{1}{2} w_s^2). \tag{3.4}
$$

The following pseudo-code is implemented to perform the Generic Bayesian RVM classification.

---

**Algorithm 2. The Generic Bayesian RVM Classification Algorithm**

**Input.** The training data: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, $x_i \in \mathcal{X} \subseteq \mathbb{R}^l$ and $y_i \in \{-1, 1\}$, $i = 1, \ldots, n$.

0. Let $t = 1$ and initialize $w$ and $\alpha$ to obtain the started values $w^t$ and $\alpha^t$.

Choose $(a, b)$, the number of burn-in $B$, and the number of iterations $T$;

1. Fix $\alpha^t$, draw a new $w^{t+1}$ according to (3.3);

2. Fix $w^t$, draw a new $\alpha^{t+1}$ according to (3.4);

3. Repeat steps 1 and 2 until suitable convergence is obtained by $T$ iterations;

**Output.** The final estimation of $w$ is $\hat{w} = (T - B)^{-1} \sum_{t=B+1}^{T} w^t$.

---

Algorithm 2 is a strict Bayesian method and conducted by the Gibbs sampling method, which could obtain more stable parameter estimates than the original RVM classification regarding parameter convergence. Although Algorithm 2 builds a Bayesian framework for the RVM classification and ends up yielding a sparsity representation, the complete freedom of $\alpha$ given to the parameters makes it difficult to find the unique solution because the number of parameters grows with the sample size. This is a typical case of the Neyman-Scott problem [14]. In RVM’s context, the Neyman-Scott problem means that the prior of (16) makes the estimate of $w$ not consistent [9]. A dimension reduction via the coefficient structure can solve this problem.
4. Fully Hierarchical Bayesian RVM Classification

This section follows the hierarchical prior structure \cite{9} but applied to RVM classification instead of the regression problem. One of the main contributions of the hierarchical prior structure is to add another layer random-coefficient structure for prior of $\alpha$ which reduces the parameter dimensions. The dimensions reduction can give consistent estimates of $w$. This Fully Bayesian method could relate $\alpha$,s with the coefficient parameter and enhance the inner connection of parameters. This paper makes some improvements. In previous work \cite{9}, only $n$ dimensions of the parameters were considered, the error term of the parameters, $w_0$ and $\alpha_0$, were ignored. This paper considers all $n + 1$ dimensions in the parameters. In the numeric study part, the previous work \cite{9} specified all the hyperparameters and only sampled $w$ and $\alpha$ in the Gibbs sampling process. The studies in Section 5 of this paper run the full Gibbs sampling iterations including all the parameters. Recall (5), the prior for $w_s$ is

$$p(w_s | \alpha_s ) = N(w_s | 0, \alpha_s^{-1}). \tag{4.1}$$

Reparametrize $\alpha$ as $\eta = (\eta_0, \eta_1, ..., \eta_n)$, where $\eta_s = log(\alpha_s)$, and $s = 0, 1, 2, ..., n$. It defined the hyperprior for $\eta$ is \cite{9}

$$\eta \sim N_{n+1}(\mu \mathbf{1}_{n+1}, \tau^2 \Sigma_{n+1}), \tag{4.2}$$

where $\Sigma_{n+1} = (1 - \rho)\mathbf{I}_{n+1} + \rho \mathbf{1}_{n+1} \mathbf{1}^\top_{n+1}$, $\mathbf{I}_{n+1}$ is an identity matrix, and $\mathbf{1}_{n+1}$ is a vector with all values of 1. Note that $\rho$ should remain in the interval of $(0, 1)$. The interpretation of $\rho$ is to maintain the trade-off between absolute freedom of $\alpha_s$’s when $\rho$ is close to 0 and the total tightness of $\alpha_s$’s when $\rho$ is close to 1. $\tau^2$ should be relatively large because sparsity is still an important goal in RVM classification. The value of $\rho$ indicates the relative contribution of the joint effects between all the $\alpha_s$’s, the value of $\tau^2$ controls the magnitude of information in $\alpha$. Based on their expected effect, we propose the constant priors
for $\rho$ and $\mu$, a conjugate prior for $\tau^2$, namely

$$p(\rho) = \text{Uniform}(0, 1), \ p(\mu) = \text{Uniform}(0, 1), \text{ and } p(\tau^{-2}) = \text{Gamma}(c, d).$$  (4.3)

Since we only add a new layer to the prior, the Fully conditional posterior for $w$ remains unchanged as (3.3). For the joint posterior of $\alpha$, we can reach it through its reparametrized version $\eta$, $\eta = \log(\alpha)$,

$$p(\eta | \text{others}) \propto p(w | \alpha(\eta)) p(\eta | \mu, \rho, \tau^2)$$

$$\propto \left( \prod_{s=1}^{n+1} \frac{1}{\sqrt{2\pi}} e^{\eta_s^2/2} \right) \exp \left( -\frac{1}{2} \sum_{s=1}^{n+1} \eta_s w_s^2 \right) \exp \left\{ -\frac{1}{2\tau^2(1-\rho)} \times \right.$$

$$\left. \sum_{s=1}^{n+1} (\eta_s - \mu)^2 + \frac{\rho}{2\tau^2(1-\rho)(1+n\rho)} \left[ \sum_{s=1}^{n+1} (\eta_s - \mu) \right]^2 \right\}. \quad (4.4)$$

It seems hard to draw samples for this posterior but it also has the desired log-concave property. The adaptive rejection sampling algorithm can be applied again.

**Theorem 2.** The conditional posterior of $\eta_s, p(\eta_s | \text{others})$ is log-concave.

**Proof.** See Appendix B. \hfill \Box

The prior for $\rho$ allows us to write

$$p(\rho | \text{others}) \propto p(\rho)p(\eta | \mu, \rho, \tau^2)$$

$$\propto \frac{1}{(1-\rho)^{2}(1+n\rho)^{2}} \exp \left\{ -\frac{1}{2\tau^2(1-\rho)} \sum_{s=1}^{n+1} (\eta_s - \mu)^2 + \right.$$

$$\left. \frac{\rho}{2\tau^2(1-\rho)(1+n\rho)} \left[ \sum_{s=1}^{n+1} (\eta_s - \mu) \right]^2 \right\}. \quad (4.5)$$

The method of ratio of uniforms is used to sample from this conditional poste-
rior. The posterior of $\mu$ is

$$p(\mu | \text{others}) \propto p(\mu)p(\alpha|\mu, \rho, \tau^2)$$

$$\propto \exp \left\{ -\frac{1}{2\tau^2(1-\rho)} \sum_{s=1}^{n+1} (\eta_s - \mu)^2 + \frac{\rho}{2\tau^2(1-\rho)}(1+n\rho) \left[ \sum_{s=1}^{n+1} (\eta_s - \mu) \right]^2 \right\}$$

$$\propto \exp \left\{ -\frac{n+1}{2\tau^2(1+n\rho)} \left( \mu - \frac{\sum_{s=1}^{n+1} \eta_s}{n+1} \right)^2 \right\}$$

$$\propto N \left( \frac{\sum_{s=1}^{n+1} \eta_s}{n+1}, \frac{\tau^2(1+n\rho)}{n+1} \right). \quad (4.6)$$

For $\tau^2$, we have

$$p(\tau^2 | \text{others}) \propto p(\tau^2)p(\eta|\mu, \rho, \tau^2)$$

$$\propto (\tau^{-2})^{c-1} \exp(-d\tau^{-2})(\tau^{-2})^{\frac{n+2}{2}} \exp \left\{ -\frac{1}{2\tau^2}(\eta - \mu_{1,n+1})^T \Sigma^{-1}(\eta - \mu_{1,n+1}) \right\}$$

$$\propto \text{Gam} \left( c + \frac{n+1}{2}, d + \frac{1}{2} \left\{ 1 - \rho \right\} \sum_{s=1}^{n+1} (\eta_s - \mu)^2 - \frac{\rho}{(1-\rho)(1+n\rho)} \left[ \sum_{s=1}^{n+1} (\eta_s - \mu) \right]^2 \right\}. \quad (4.7)$$

The samples of $\mu$ and $\tau^2$ are easy to obtain from their special closed-forms. Based on the above derivation of full conditional posteriors, we have an alternative algorithm:

**Algorithm 3. The Fully Hierarchical Bayesian RVM Classification**

**Input.** The training data: $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$, $x_i \in \mathbf{X} \subseteq \mathbb{R}^d$ and $y_i \in \{-1, 1\}$, $i = 1, ..., n$.

0. Let $t = 1$ and initialize $w$, $\alpha$, $\mu$, $\rho$ and $\tau^2$ to obtain the started values $w_t$, $\alpha_t$, $\mu_t$, $\rho_t$ and $\tau_t^2$. Choose $(c, d)$, the number of burn-in $B$, and the number of iterations $T$;

1. Fix other parameters and draw a new $w_{t+1}$ according to (3.3);
2. Fix other parameters and draw a new $\alpha_{t+1}$ according to (4.4);
3. Fix other parameters and draw a new $\rho_{t+1}$ according to (4.5);
4. Fix other parameters and draw a new $\mu_{t+1}$ according to (4.6);
5. Fix other parameters and draw a new $\tau^2_{t+1}$ according to (4.7);
6. Repeat steps 1 – 5 until suitable convergence is obtained by $T$ iterations;
Output. The final estimation of $w$ is $\hat{w} = (T - B)^{-1} \sum_{t=B+1}^{T} w_t$.

5. Numeric Studies

5.1. Simulation Data Studies

The simulated training dataset is the same as (2.11), (2.12). We chose five kinds of sizes, $(n_p, n_n) = (30, 30)$, $(15, 30)$, $(12, 30)$, $(6, 30)$, $(3, 30)$, to illustrate the performance of different algorithms in different-sized data. $b = 1, 2, 2.5, 5, 10$ for these five cases and a larger $b$ indicates a more imbalanced dataset. Following Figure 3 shows the training data sets.

Fig 3: Simulated Gaussian Data
We run the Algorithm 2 and 3 with $T = 5000$, $B = 500$, $(a, b) = (1, 1/999)$, and $(c, d) = (1, 1/999)$. The evaluation criteria come from Table 1. For both Algorithm 2 and 3, we repeat the experiments 100 times for every case in Figure 3 to reduce the randomness impact of the data simulation. Table 2–6 display the mean values and standard deviation values (shown in the bracket) of 100 repeated results, the larger accuracy rate is indicated by boldface. The simulation studies show that for the balanced data and mildly imbalanced data as $b = 1, 2, 2.5$, Algorithms 2 and 3 perform similarly. But for the seriously imbalanced data as $b = 5, 10$, Algorithm 3 outperforms 2 significantly. Especially for $b = 10$, the accuracy rates for the positive class are all zero under Algorithm 2, but Algorithm 3 improves the classification performance in this case.

### Table 2

The Classification Result for $b = 1$.

| Algorithm | $r^{train}_g$ | $r^{test}_g$ | $r^{train}_p$ | $r^{test}_p$ | $r^{train}_c$ | $r^{test}_c$ | $r^{train}_d$ | $r^{test}_d$ |
|-----------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| Algorithm 2 | 0.9823 (0.0148) | 0.9710 (0.0254) | 0.9732 (0.0209) | 0.9993 (0.0047) | 0.9980 (0.0080) | 0.9980 (0.0141) | 0.9996 (0.0022) |
| Algorithm 3 | 0.9770 (0.0170) | 0.9678 (0.0328) | 0.9674 (0.0390) | 0.9993 (0.0067) | 1.0000 (0.0000) | 0.9990 (0.0100) | 0.9998 (0.0016) |

$a$ ($n^{train}_n = 30, n^{train}_p = 30$), $b$ ($n^{test}_n = 30, n^{test}_p = 30$), $c$ ($n^{test}_n = 90, n^{test}_p = 90$), $d$ ($n^{test}_n = 30, n^{test}_p = 15$)

### Table 3

The Classification Result for $b = 2$.

| Algorithm | $r^{train}_g$ | $r^{test}_g$ | $r^{train}_p$ | $r^{test}_p$ | $r^{train}_c$ | $r^{test}_c$ | $r^{train}_d$ | $r^{test}_d$ |
|-----------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| Algorithm 2 | 0.9796 (0.0257) | 0.9773 (0.0418) | 0.9680 (0.0047) | 0.9633 (0.0069) | 0.9760 (0.0542) | 0.9698 (0.0870) | 0.9698 (0.0378) |
| Algorithm 3 | 0.9760 (0.0214) | 0.9822 (0.0443) | 0.9707 (0.0138) | 0.9707 (0.00616) | 0.9767 (0.0477) | 0.9700 (0.0823) | 0.9798 (0.0322) |

$a$ ($n^{train}_n = 30, n^{train}_p = 15$), $b$ ($n^{test}_n = 30, n^{test}_p = 15$), $c$ ($n^{test}_n = 10, n^{test}_p = 5$), $d$ ($n^{test}_n = 90, n^{test}_p = 45$)
Table 4: The Classification Result for $b = 2.5$.

|           | $r_{\text{train}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ | $r_{\text{test}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ |
|-----------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| Algorithm 2 | 0.9693 (0.0296) | 0.9745 (0.0294) | 0.9732 (0.0186) | 0.9403 (0.0920) | 0.9550 (0.1088) | 0.9433 (0.0580) |
| Algorithm 3 | 0.9679 (0.0376) | 0.9710 (0.0286) | 0.9731 (0.0171) | 0.9731 (0.0171) | 0.9375 (0.1212) | 0.9403 (0.0920) |

a $(n_{\text{train}} = 30, n_{\text{test}} = 30)$, b $(n_{\text{train}} = 12, n_{\text{test}} = 12)$, c $(n_{\text{train}} = 10, n_{\text{test}} = 4)$, d $(n_{\text{train}} = 90, n_{\text{test}} = 36)$

Table 5: The Classification Result for $b = 5$.

|           | $r_{\text{train}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ | $r_{\text{test}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ |
|-----------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| Algorithm 2 | 0.8956 (0.0642) | 0.8694 (0.0465) | 0.8733 (0.0421) | 0.8703 (0.0421) | 0.8703 (0.0421) | 0.8703 (0.0421) | 0.8250 (0.2807) | 0.2306 (0.2600) |
| Algorithm 3 | 0.9419 (0.0667) | 0.8964 (0.0674) | 0.8883 (0.0711) | 0.8893 (0.0711) | 0.8893 (0.0711) | 0.8893 (0.0711) | 0.5300 (0.3846) | 0.5489 (0.3689) |

a $(n_{\text{train}} = 30, n_{\text{test}} = 30)$, b $(n_{\text{test}} = 6)$, c $(n_{\text{test}} = 12)$, d $(n_{\text{test}} = 90, n_{\text{test}} = 18)$

Table 6: The Classification Result for $b = 10$.

|           | $r_{\text{train}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ | $r_{\text{test}}^a$ | $r_{\text{test}}^b$ | $r_{\text{test}}^c$ | $r_{\text{test}}^d$ |
|-----------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| Algorithm 2 | 0.9991 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) | 0.9091 (0.0000) |
| Algorithm 3 | 0.9503 (0.0357) | 0.9118 (0.0417) | 0.9118 (0.0375) | 0.9187 (0.0332) | 0.9187 (0.0332) | 0.9187 (0.0332) | 0.7200 (0.2485) | 0.1922 (0.2118) |

a $(n_{\text{train}} = 30, n_{\text{test}} = 3)$, b $(n_{\text{test}} = 3)$, c $(n_{\text{test}} = 10, n_{\text{test}} = 1)$, d $(n_{\text{test}} = 90, n_{\text{test}} = 9)$

5.2. Real Data Studies

Six binary real data sets are studied in this paper and the imbalance index $b$ changes from 1.82 to 11.59. The datasets are obtained from the Knowledge Extraction based on Evolutionary Learning (KEEL) Dataset Repository [1]. KEEL is an open source Java software tool used for data discovery tasks. It includes plenty of datasets which can be used for imbalanced data problem studies. The detailed descriptions of every dataset can be found on the website...
of KEEL [12]. For every dataset, we randomly split the positive and negative classes, where half is the training part, the other half is the test part. **Algorithm 2** and **3** are applied to all the datasets. Table 7 lists some basic information of the datasets and the classification results of the four criteria. The real data studies show that **Algorithm 3** indeed outperforms **2**, especially for seriously imbalanced datasets when we are interested in the positive class.

**Table 7**

*The Classification Result for Real Datasets.*

| Dataset   | b       | Dimension | Total Data Size | $r_{\text{train}}^{\text{g}}$ | $r_{\text{test}}^{\text{g}}$ | $r_{\text{train}}^{\text{p}}$ | $r_{\text{test}}^{\text{p}}$ |
|-----------|---------|-----------|-----------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| glass1    | 1.82    | 9         | 214             | 0.6449                        | 0.6449                        | 0.0000                        | 0.0000                        |
|           |         |           |                 | **0.6542**                    | **0.0263**                    |                               |                               |
| iris0     | 2.00    | 4         | 150             | 1.0000                        | 1.0000                        | 1.0000                        | 1.0000                        |
|           |         |           |                 | **1.0000**                    | **1.0000**                    |                               |                               |
| newthyroid1 | 5.14   | 5         | 215             | 0.8318                        | 0.8333                        | 0.0000                        | 0.0000                        |
|           |         |           |                 | **0.8999**                    | **0.0588**                    |                               |                               |
| glass6    | 6.38    | 9         | 214             | 0.8679                        | 0.8611                        | 0.0000                        | 0.0000                        |
|           |         |           |                 | **0.8679**                    | **0.0000**                    |                               |                               |
| ecoli0345 | 9.00    | 7         | 200             | 0.8600                        | 0.9000                        | 0.0000                        | 0.0000                        |
|           |         |           |                 | **0.9100**                    | **0.1000**                    |                               |                               |
| glass2    | 11.59   | 9         | 214             | 0.9245                        | 0.9167                        | 0.0000                        | 0.0000                        |
|           |         |           |                 | **0.9379**                    | **0.0133**                    |                               |                               |

*Results in the table are listed by Algorithm 2 on the top, Algorithm 3 on the bottom.*

**6. Conclusion and Discussion**

Two Bayesian RVM classification algorithms are proposed in this paper and they make two-fold contributions. The first Generic Bayesian RVM algorithm conducts a strict Bayesian RVM classification algorithm compared to the original RVM classification method and makes it possible to sample the weight parameter directly from its posterior. On the other hand, the Fully Hierarchical Bayesian algorithm follows the hyperprior structure in previous RVM regression work [9] but is applied to the classification problem to improve the classification performance compared to the Generic one, especially in the imbalanced data
problem. We have provided the theoretical justification of the log-concavity for the conditional posterior of some parameters, which helps to set up a fast and stable sampling process. The simulated data studies use the data from the same distribution but with different imbalance indexes. The experiment results show the favorable performance of our two proposed algorithms and indicate that the Fully Hierarchical Bayesian RVM algorithm can classify the seriously imbalanced data more strongly than the Generic one. The real data studies explore six datasets and check the performance of our two proposed algorithms in practice. They both perform well and the Fully Hierarchical one indeed outperforms the Generic one in imbalanced data when we are more interested in the positive class.

7. Appendix

Appendix A. Proof of Theorem 1.

First, the logarithm of the posterior for \( w \) is

\[
l^w = \log p(w|\text{others})
\]

\[
= \sum_{i=1}^{n} \left[ \frac{1 + y_i^{\text{train}}}{2} \log \left( \frac{1}{1 + \exp(-\phi_i^{\text{train}}w)} \right) + \frac{1 - y_i^{\text{train}}}{2} \log \left( \frac{\exp(-\phi_i^{\text{train}}w)}{1 + \exp(-\phi_i^{\text{train}}w)} \right) \right] - \frac{1}{2} w^T A w + C
\]

\[
= \sum_{i=1}^{n} \left[ \frac{1 + y_i^{\text{train}}}{2} \log \left( \frac{1}{1 + \exp(-\sum_{s=0}^{n} \phi_{i,s}^{\text{train}}w_s)} \right) + \frac{1 - y_i^{\text{train}}}{2} \log \left( \frac{\exp(-\sum_{s=0}^{n} \phi_{i,s}^{\text{train}}w_s)}{1 + \exp(-\sum_{s=0}^{n} \phi_{i,s}^{\text{train}}w_s)} \right) \right] - \frac{1}{2} \sum_{s=0}^{n} \alpha_s w_s^2 + C,
\]
where $C$ is some constant. For any $w_k$ in $w$, we have

$$ I_k^w = \log p(w_k | \text{others}) $$

$$ = \sum_{i=1}^{n} \left[ \frac{1 + y_i^{\text{train}}}{2} \log \left( \frac{1}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) + \right.$$

$$ \frac{1 - y_i^{\text{train}}}{2} \log \left( \frac{\exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) \left. \right] - \frac{1}{2} \sum_{s=0,s \neq k}^{n} \alpha_s w_s^2 - \frac{1}{2} \alpha_k w_k^2 + C,$$

where $k = 0, 1, 2, \ldots, n$. Then we calculate the first and second divergency,

$$ \frac{\partial}{\partial w_k} I_k^w = \sum_{i=1}^{n} \left[ \frac{1 + y_i^{\text{train}}}{2} \left( \frac{\phi_{i,k}^{\text{train}} \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) + \right.$$

$$ \frac{1 - y_i^{\text{train}}}{2} \left( \frac{-\phi_{i,k}^{\text{train}}}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) \right] - \alpha_k w_k,$$

$$ \frac{\partial^2}{\partial w_k^2} I_k^w = \sum_{i=1}^{n} \left[ \frac{1 + y_i^{\text{train}}}{2} \left( \frac{-\phi_{i,k}^{\text{train}}^2}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) + \right.$$

$$ \frac{1 - y_i^{\text{train}}}{2} \left( \frac{-\phi_{i,k}^{\text{train}}^2 \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)}{1 + \exp\left( - \sum_{s=0,s \neq k}^{n} \phi_{i,s}^{\text{train}} w_s - \phi_{i,k}^{\text{train}} w_k \right)} \right) \right] - \alpha_k$$

$$ < 0,$$

as desired.
Appendix B. Proof of Theorem 2.

The conditional posterior of any $\eta_k$ in $\eta$ is

$$p(\eta_k \mid \text{others}) \propto \exp \left[ \frac{\eta_k}{2} - \frac{1}{2} \exp(\eta_k)w_k^2 - \frac{(\eta_k - \mu)^2}{2\tau^2(1-\rho)} + \frac{\rho(\eta_k - \mu)^2}{2\tau^2(1-\rho)(1+n\rho)} + \frac{\rho(\eta_k - \mu)}{\tau^2(1-\rho)(1+n\rho)} \sum_{s=0,s\neq k}^{n} (\eta_s - \mu) \right],$$

where $k = 0, 1, 2, \ldots, n$. For a constant $C$, the log-posterior of $\eta_k$ is

$$l^n_k = \log p(\eta_k \mid \text{others}) = C + \frac{\eta_k}{2} - \frac{1}{2} \exp(\eta_k)w_k^2 - \frac{1 + (n-1)\rho}{2\tau^2(1-\rho)(1+n\rho)}(\eta_k - \mu)^2 + \frac{\rho(\eta_k - \mu)}{\tau^2(1-\rho)(1+n\rho)} \sum_{s=0,s\neq k}^{n} (\eta_s - \mu).$$

The second divergence is

$$\frac{\partial^2}{\partial \eta_k^2} l^n_k = -\frac{1}{2} \exp(\eta_k)w_k^2 - \frac{1 + (n-1)\rho}{\tau^2(1-\rho)(1+n\rho)}.$$

Because $1 + (n-1)\rho > 0$ for any $\rho \in (-1, 1)$, $\frac{\partial^2}{\partial \eta_k^2} l^n_k < 0$ always holds.
8. Acknowledgments

This work was supported by the Open Research Fund of Key Laboratory of Advanced Theory and Application in Statistics and Data Science (East China Normal University), Ministry of Education.

References

[1] Jesús Alcalá-Fdez, Alberto Fernández, Julián Luengo, Joaquín Derrac, Salvador García, Luciano Sánchez, and Francisco Herrera. Keel data-mining software tool: data set repository, integration of algorithms and experimental analysis framework. *Journal of Multiple-Valued Logic & Soft Computing*, 17, 2011.

[2] Christopher M Bishop and Michael E Tipping. Variational relevance vector machines. In *Proceedings of the Sixteenth conference on Uncertainty in artificial intelligence*, pages 46–53. Morgan Kaufmann Publishers Inc., 2000.

[3] Bernhard E Boser, Isabelle M Guyon, and Vladimir N Vapnik. A training algorithm for optimal margin classifiers. In *Proceedings of the fifth annual workshop on Computational learning theory*, pages 144–152. ACM, 1992.

[4] Andreas Ch Braun, Uwe Weidner, and Stefan Hinz. Classification in high-dimensional feature spaces: assessment using svm, ivm and rvm with focus on simulated enmap data. *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing*, 5(2):436–443, 2012.

[5] Philip K Chan and Salvatore J Stolfo. Toward scalable learning with non-uniform class and cost distributions: A case study in credit card fraud detection. In *KDD*, volume 1998, pages 164–168, 1998.

[6] Corinna Cortes and Vladimir Vapnik. Support-vector networks. *Machine learning*, 20(3):273–297, 1995.

[7] Begüm Demir and Sarp Erturk. Hyperspectral image classification using
relevance vector machines. IEEE Geoscience and Remote Sensing Letters, 4(4):586–590, 2007.

[8] Wu Dinghai, Zhang Peilin, Zhang Yingtang, et al. Study on diesel engine faults diagnosis based on time frequency singular value spectrum and rvm. Journal of Mechanical Strength, 33(3):317–323, 2011.

[9] Ernest Fokoué, Dongchu Sun, and Prem Goel. Fully bayesian analysis of the relevance vector machine with an extended hierarchical prior structure. Statistical Methodology, 8(1):83–96, 2011.

[10] Walter R Gilks and Pascal Wild. Adaptive rejection sampling for gibbs sampling. Journal of the Royal Statistical Society: Series C (Applied Statistics), 41(2):337–348, 1992.

[11] Daniella Hubl, S Böste, S Feineis-Matthews, H Lanfermann, Andrea Federspiel, W Strik, F Poustka, and Thomas Dierks. Functional imbalance of visual pathways indicates alternative face processing strategies in autism. Neurology, 61(9):1232–1237, 2003.

[12] KEEL. Keel dataset respository. http://sci2s.ugr.es/keel/imbalanced.php.

[13] Miroslav Kubat, Robert C Holte, and Stan Matwin. Machine learning for the detection of oil spills in satellite radar images. Machine Learning, 30(2-3):195–215, Feb 1998.

[14] Jerzy Neyman, Elizabeth L Scott, et al. Consistent estimates based on partially consistent observations. Econometrica, 16(1):1–32, 1948.

[15] N Nikolaev and P Tino. Sequential relevance vector machine learning from time series. In Proceedings. 2005 IEEE International Joint Conference on Neural Networks, 2005., volume 2, pages 1308–1313. IEEE, 2005.

[16] Mahesh Pal and Giles M Foody. Evaluation of svm, rvm and smlr for accurate image classification with limited ground data. IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing, 5(5):1344–1355, 2012.

[17] Stephan R Sain. The nature of statistical learning theory, 1996.
[18] Robert E Schapire and Yoram Singer. Improved boosting algorithms using confidence-rated predictions. *Machine learning*, 37(3):297–336, 1999.

[19] Bernhard Schölkopf, Christopher JC Burges, Alexander J Smola, et al. *Advances in kernel methods: support vector learning*. MIT press, 1999.

[20] John Shawe-Taylor, Nello Cristianini, et al. *Kernel methods for pattern analysis*. Cambridge university press, 2004.

[21] Catarina Silva and Bernardete Ribeiro. Scaling text classification with relevance vector machines. In *2006 IEEE International Conference on Systems, Man and Cybernetics*, volume 5, pages 4186–4191. IEEE, 2006.

[22] Arasanathan Thayananthan. *Template-based pose estimation and tracking of 3D hand motion*. PhD thesis, University of Cambridge, 2006.

[23] Michael E Tipping. The relevance vector machine. In *Advances in neural information processing systems*, pages 652–658, 2000.

[24] Michael E Tipping. Sparse bayesian learning and the relevance vector machine. *Journal of machine learning research*, 1(Jun):211–244, 2001.

[25] Michael E Tipping, Anita C Faul, et al. Fast marginal likelihood maximisation for sparse bayesian models. In *AISTATS*, 2003.

[26] Vladimir Vapnik and Vlamimir Vapnik. *Statistical learning theory*, 1998.

[27] Xu Xiang-min, Mao Yun-feng, Xiong Jia-ni, and Zhou Feng-le. Classification performance comparison between rvm and svm. In *2007 International Workshop on Anti-Counterfeiting, Security and Identification (ASID)*, pages 208–211. IEEE, 2007.

[28] Liu Zunxiong, Zhang Deyun, Sun Qindong, and Xu Zheng. Mid-term electric load prediction based on the relevant vector machine. *Journal of Xian Jiaotong University*, 38(10):1005–1008, 2004.