The Spin Content of the Nucleon*

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The fraction of the nucleon spin that is carried by the quarks, ∆Σ, is computed in lattice QCD with dynamical staggered fermions. We obtain the value ∆Σ = 0.18 ± 0.02.

1. INTRODUCTION

The EMC measurement [1] of the spin-dependent structure function of the proton, g1(x, Q2), has provoked many speculations [2] about the internal spin structure of the nucleon. The first moment of g1(x, Q2), which through the operator product expansion is given by the proton matrix element of the axial vector current weighted by the square of the quark charges (modulo radiative corrections), was found to be

\[ \int_0^1 dx g_1(x, Q^2) = \frac{1}{2} (\Delta u + \frac{1}{3} \Delta d + \frac{1}{3} \Delta s) = 0.126 \pm 0.010 \pm 0.015 \]  

with

\[ \Delta q s_\mu = \langle \vec{p}, s | \bar{q} \gamma_\mu \gamma_5 q | \vec{p}, s \rangle, \quad q = u, d, s \]  

where s_μ is the covariant spin vector of the proton. The average value of Q^2 of the EMC data is ≈ 11 GeV^2. If we combine the result (1) with information from hyperon decays, neutron β-decay and the assumption of flavor SU(3), we obtain [3]

\[ \Delta \Sigma = \Delta u + \Delta d + \Delta s = 0.04 \pm 0.16. \]  

The quantity ∆Σ is the axial baryonic charge of the nucleon,

\[ \Delta \Sigma s_\mu = \langle \vec{p}, s | \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s | \vec{p}, s \rangle \]

which in a naive wave function picture can be interpreted as the fraction of the nucleon spin that is carried by the quarks. For example, in a SU(6)-type model of the nucleon we would obtain \( \Delta \Sigma = 1 \). The vanishingly small experimental value of \( \Delta \Sigma \) is referred to as the spin crisis of the nucleon.

The nucleon matrix element of the axial baryonic current can be written [4]

\[ \langle \vec{p}, s | \bar{u} \gamma_\mu \gamma_5 u | \vec{p}', s' \rangle = \bar{u}(\vec{p}, s) [G_1(k^2) \gamma_\mu \gamma_5 - G_2(k^2) k_\mu \gamma_5] u(\vec{p}', s'), \]

where k is the momentum transfer and

\[ G_1(0) = \Delta \Sigma. \]

Unlike the octet current, \( j^{05}_\mu \) is not conserved due to the anomaly. In the chiral limit

\[ \partial_\mu j^{05}_\mu = N_f \frac{1}{8\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}, \]

where \( N_f \) is the number of flavors. As a result, the form factor \( G_2(k^2) \) does not have a (Goldstone
boson) pole at \( k^2 = 0 \). Writing

\[
\langle \bar{p}, s | N_f \frac{1}{8 \pi^2} \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu} | \bar{p}', s' \rangle = 2 m_N A(k^2) \bar{u}(\bar{p}, s) \gamma_5 u(\bar{p}', s'),
\]

where \( m_N \) is the nucleon mass, one then finds

\[
A(0) = G_1(0) = \Delta \Sigma.
\]

Thus, the quark spin fraction of the nucleon spin is given by the matrix element of the anomalous divergence of the axial baryonic current.

2. LATTICE CALCULATION

A meaningful lattice calculation of \( \Delta \Sigma \) requires (i) the inclusion of dynamical fermions and (ii) a proper definition of \( \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu} \) in order to account for the topological origin of the anomalous divergence. None of these requirements were fulfilled in earlier calculations \( [5] \).

The calculations in this paper are done on a 16\(^3 \cdot 24\) lattice at \( \beta = 5.35 \) and \( ma = 0.01 \) \( [6] \) with four flavors of dynamical staggered fermions using the hybrid Monte Carlo algorithm. Our data sample consists of 85 configurations separated by five trajectories. These configurations were used in ref. 8 to compute the hadron mass spectrum. The lattice parameters correspond to a renormalization group invariant quark mass of \( m_{\text{RGI}} = 35 \) MeV in the \( MS \) scheme at a scale of 1 GeV.

For \( \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu} \) we use Lüscher’s definition 7.

\[
\text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu}(x) = \frac{2}{3} \sum_{\mu \nu \rho \sigma} \epsilon_{\mu \nu \rho \sigma} \\
\times \left( 3 \int_{p(x+\hat{\mu}+\hat{\nu}, \mu, \nu)} d^2 z \\
\times \text{Tr} \left[ P^x_{x+\hat{\nu}, \mu} \partial_{\hat{\nu}} P^x_{x+\hat{\mu}, \mu} \right] \\
\times R^x_{x+\hat{\mu}, \nu} \partial_{\hat{\nu}} R^x_{x+\hat{\mu}, \nu} \right) \\
- \int f(x+\hat{\mu}) d^3 z \left( S^x_{x+\hat{\mu}, \nu} \partial_{\hat{\mu}} S^x_{x+\hat{\nu}, \mu} \right) \\
\times S^x_{x+\hat{\mu}, \nu} \partial_{\hat{\mu}} S^x_{x+\hat{\nu}, \mu} \\
+ \int f(x) d^3 z \left( S^x_{x, \nu} \partial_{\hat{\mu}} S^x_{x, \mu} \right) \\
\times S^x_{x, \nu} \partial_{\hat{\mu}} S^x_{x, \mu} - 1ight),
\]

\(^1\)

where \( P, R \) and \( S \) are certain parallel transporters extrapolated to the interior of the plaquettes, \( z \in p \), and faces, \( z \in f \). This expression proceeds from the principle bundle which is reconstructed from the lattice gauge field. The resulting topological charge,

\[
Q = - \frac{1}{16 \pi^2} \sum_x \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu},
\]

assumes integer values as in the continuum. The major drawback of eq. (10) is that it involves a three-dimensional integral over the faces of the hypercubes. The more elegant algorithm of Phillips and Stone \( [8] \) does not lead naturally to a charge density. It should be noted that in the presence of light dynamical fermions the topological susceptibility, \( \chi_t = \langle Q^2 \rangle / V \) (\( V \): volume), is not affected by dislocations \( [10] \). In the large volume limit the minimal action for a \( |Q| = 1 \) configuration becomes \( [11] \)

\[
S_{\text{min}} \propto \frac{V}{\beta} \ln(ma)^{-1},
\]

which grows beyond any bound for fixed bare mass \( m \) (in physical units). According to eqs. (8) and (9), \( \Delta \Sigma \) is given by

\[
\Delta \Sigma = \lim_{\vec{p} \to 0} \frac{i |\vec{s}|}{|\vec{p}|^2} \times \langle \bar{p}, s | \frac{1}{2 \pi^2} \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu} | 0, s \rangle.
\]

On a finite lattice we cannot reach \( \vec{p} = 0 \). We therefore shall evaluate \( \Delta \Sigma \) at the smallest (non-zero) momentum transfer, i.e. \( |\vec{p}| = 2 \pi / 16 \). This is of the order of the pion mass. Choosing \( \vec{s}' / |\vec{s}| = \pm \vec{p} / |\vec{p}| \), we thus have to compute

\[
C(t) = \pm \frac{i}{|\vec{p}|} \times \\
< B_p(t) P_{\pm} \frac{1}{2 \pi^2} \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu}(x_4) \tilde{B}(0) >,
\]

where \( \tilde{B}, B \) are the baryon creation and annihilation operators, \( P_{\pm} \) is the spin projection operator and

\[
\text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu}(x_4) = \sum_{\vec{x}} \langle \bar{p}^2 \rangle \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu}(x).
\]
Equation (14) contains 12 independent correlation functions. Averaging over all of them gives

\[ C(t) = \Delta \Sigma A_+ e^{-m_N x_4} E_N(t-x_4) + \Delta \Sigma A_- (-1)^i e^{-m_{N,i} x_4} E_{N,i}(t-x_4) + \ldots, \]

where \( \Delta \Sigma_{\Lambda} \) is the quark spin fraction (mass) of the \( \Lambda \), the opposite parity partner of the nucleon \( \Xi \), and \( E_{N,\Lambda} = \sqrt{p^2 + m_N^2} \). The amplitudes \( A_+ \) and \( A_- \) are those of the ordinary correlation function \( < B(t) \tilde{B}(0) > \). The dots in eq. (16) stand for contributions from higher excitations and from baryons backtracking around the boundary.

A non-trivial problem is the computation of \( \text{Tr} F_{\mu\nu} F_{\mu\nu} \) for a given gauge field configuration. For gauge group \( SU(2) \) we could do one integral in \( \text{Tr} F_{\mu\nu} F_{\mu\nu} \) analytically \([12]\), which made its computation just feasible. In the present case of gauge group \( SU(3) \) we shall make use of the fact that the calculation can be reduced to the case of \( SU(2) \) by means of the so-called reduction of the structure group \([13,14]\). The argument goes as follows.

Since \( \text{Tr} F_{\mu\nu} F_{\mu\nu} = \partial_\mu K_\mu \), we can write \( s_0 G_1(0) \propto \langle \tilde{p}, s | K_0 | \tilde{p}, s \rangle \). It turns out \([13]\) that \( \sum_\vec{x} K_0 = \sum_\vec{x} \partial_\mu K_{0i} \in \mathbb{Z} \) is a topological invariant. As in the case of the topological charge we then may continuously deform the \( SU(3) \) gauge field to a \( SU(2) \) gauge field without changing \( \sum_\vec{x} K_0 \). The reason is that \( \pi_3(SU(3)/SU(2)) = 0 \). The reduction of the \( SU(3) \) link matrices to \( SU(2) \) link matrices is done by a coset decomposition \([14]\) \( \tilde{U}(x, \mu) = \omega(x, \mu) U(x, \mu) \), where \( U(x, \mu) \in SU(3), \tilde{U}(x, \mu) \in SU(2) \). In order to be able to use a geometric definition of \( \text{Tr} F_{\mu\nu} F_{\mu\nu} \) on the \( SU(2) \) matrices, we must make the latter as smooth as possible. This is done by fixing to a maximal \( SU(2) \) gauge, which amounts to minimizing \( \sum_{x,\mu} (3 - \text{Re} \text{Tr} \omega(x, \mu)) \) \([14]\). For the minimizing procedure we apply a combination of Metropolis and overrelaxation steps. We have checked for most of our configurations that this results in a gauge invariant \( \text{Tr} F_{\mu\nu} F_{\mu\nu} \).

For \( \tilde{B} \) we take the wall source, while for \( B \) we take the ordinary local baryon operator (i.e. \#1 in ref. \([1]\)). The current is placed at \( x_4 = 4 \). The result for the correlation function (14), averaged over all possible momentum and spin combinations, is shown in fig. 1. The data is fitted by the function (16) with \( A_+, A_-, m_N \) and \( m_\Lambda \) taken from a fit \([8]\) of \( < B_0(t) \tilde{B}_0(0) > \). The result is indicated by the solid line. It leads to

\[ \Delta \Sigma = 0.18 \pm 0.02, \]

while the quark spin fraction of the \( \Lambda \) comes out to be \( \Delta \Sigma_{\Lambda} = 0.22 \pm 0.04 \).

In order to check our results for systematic errors, we have repeated the calculation for \( x_4 = 3 \). We found the same result within errors. Thus we can assume that \( x_4 = 4 \) is far enough away from the source, so that the excited states have died out. Since our calculation involves slowly moving (infrared) modes, it is important to test the efficiency of the algorithm. We have found an autocorrelation time of \( \approx 25 \) trajectories \([1]\). This translates into an autocorrelation time of five configurations.
3. DISCUSSION

Our result (17) lies within the errors of the experimental value (3). One might object that our calculations are not entirely realistic, as we work with four light quarks instead of three and the nucleon matrix element is computed in the chiral limit. But we have reasons to believe that none of these approximations will have a significant effect. As far as the flavor dependence is concerned, one expects $\Delta \Sigma \propto \sqrt{N_f}$, which leads to a small correction only. The quark mass and momentum dependence, on the other hand, is controlled by the $\eta'$ mass which is large compared to the pion mass.

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