Notes on conservation laws in chiral hydrodynamics

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We consider chiral fluids within the standard framework of a chiral-invariant underlying field theory, anomalous in presence of electromagnetic fields. Apart from the Noether axial current of the underlying theory, in the limit of ideal fluid there exist extra conserved currents, corresponding to classical helical motions. The extra conservation laws are known to break down once viscosity is non-vanishing. Which looks puzzling, as if introduction of viscosity were inconsistent with chiral invariance. As a resolution of the puzzle, we argue that locally one can introduce an inertial frame where an extra conservation law still holds. In other words, the extra currents are covariantly conserved. The emergent gravitational field is determined by dynamics of the viscous fluid. We turn then to instabilities of chiral plasma against decays into helical magnetic or vortical configurations.

We emphasise similarity between the two cases in the far infrared region, responsible for the decays. This similarity is not apparent within the standard counting of orders in derivative expansion. The material was originally prepared for a review talk by the author.

INTRODUCTION.

Chiral media attracted a lot of attention recently, see, in particular, the collection of review articles \cite{1} and references therein. There are a few remarkable effects to be observed in such media. First of all, the chiral anomaly \cite{2} is manifested macroscopically, already in the hydrodynamic approximation \cite{3,4}. In more detail, one starts with a fundamental theory possessing an anomalous U(1) chiral symmetry. The fundamental conservation laws read as:

\[
\partial_{\mu}\Theta^{\mu\nu} = eF^{\nu\rho}j_{\rho},
\]

\[
\partial_{\sigma}j_{\sigma}^{\mu} = 0, \quad \partial_{\sigma}j_{\sigma}^{\mu} = e^{2}CF^{\mu\nu}F^{\rho\nu},
\]

where $\Theta^{\mu\nu}, j_{\sigma}^{\mu}, j_{\sigma}^{\mu}$ are the energy-momentum tensor, electromagnetic current (with $e$ being the corresponding coupling) and axial-vector current, respectively, and $F^{\mu\nu}$ is the electromagnetic field-strength tensor, $F^{\mu\nu} = 1/2\rho\sigma\rho\sigma F_{\sigma\rho}$ and $C$ is the coefficient in front of the chiral anomaly, $C = 1/(12\pi^2)$ for one flavour doublet.

The hydrodynamic equations of motion follow from (1) provided that $\Theta^{\mu\nu}, j_{\sigma}^{\mu}, j_{\sigma}^{\mu}$ are put into constituent form. Furthermore, Eqs (1) are supplemented by an equation of state. In most cases, non-vanishing chemical potentials, $\mu, \mu_5$ and finite temperature $T$ are introduced at this point. For our purposes, we do not need to specify the equation of state. Moreover, to simplify presentation, we will mostly omit temperature-dependent terms.

Remarkably enough, using only such a general framework one can fix \cite{4,5} the so called chiral magnetic and chiral vortical effects. By the chiral magnetic effect \cite{4,5,8} one understands a flow of electric current along an external magnetic field. In case of an electrically neutral but chirally asymmetric plasma one gets:

\[
j_{el}^{\mu} = Ce\mu_5 B^{\mu},
\]

where $\mu$ is the chiral chemical potential, $B^{\mu} = (1/2)e^{\alpha\beta\gamma\delta}u_{\alpha}F_{\beta\gamma\delta}$, $u^{\mu}$ is the 4-velocity of the liquid, and the constant $C$ is anomaly-related, as defined above. The chiral vortical effect \cite{4,5,8}, in turn, refers to the axial-vector current proportional to the vorticity $\omega^{\alpha}$ of the liquid, $\omega^{\alpha} = 1/2e^{\alpha\beta\gamma\delta}u_{\beta}\partial_{\gamma}u_{\delta}$. If both $\mu$ and $\mu_5$ are nonvanishing the axial current is given by:

\[
j_{el}^{\mu} = n_5u^{\mu} + C(\mu^2 + \mu_5^2)\omega^{\alpha} + O(e),
\]

where $n_5 \equiv 1/2(n_R - n_L)$ is the the density of the chiral charge, and $O(e)$ are terms vanishing in the limit of the electromagnetic coupling tending to zero.

Solutions \cite{2,3} exhibit a few interesting features. In particular, one can switch off electromagnetic fields so that the axial current is non-anomalous. Nevertheless, the chiral vortical effect (see \cite{3}) survives and is proportional to the coefficient $C$ which apparently encodes the anomaly. Moreover, since the naïve axial current is conserved in this limit, Eq. \cite{3} suggests that the chiral vortical current, or the second term in the r.h.s. of Eq. \cite{3} is separately conserved as well.

And, indeed, it is known since long, see, e.g. \cite{10}, that in the non-dissipative limit fluids possess extra conservation laws. In other words, hydrodynamics of ideal fluids possesses higher symmetry than the fundamental field theory behind it. In view of this, one could argue that the ideal-fluid approximation might be a wrong starting point to study the general case of dissipative fluids. However, the ideal fluid approximation is singled out since it is the leading term in the hydrodynamic expansion (in derivatives). Also, for ideal liquids one can develop a fully field-theoretic description in the sense that infrared degrees of freedom can be identified in all the generality and reduce to a set of scalar fields, see, e.g., \cite{12,13} and references therein. To include dissipation, one needs to introduce in field-
theoretic language violation of unitarity and relation of hydrodynamics to fundamental field theory is much less transparent.

This extra symmetry is broken once viscosity is taken into account \([10]\). Indeed, upon inclusion of effects of viscosity \(\eta \neq 0\), the motion would end up in most cases with a trivial equilibrium configuration of the whole fluid being at rest (for exceptional cases see \([14]\)). Thus, it seems rather obvious that the fluid helicity is not conserved for viscous fluids. It is not straightforward at all, however, to appreciate this observation in field-theoretic terms. For example, a recipe to construct an anomalous axial current in non-equilibrium is suggested in Ref. \([15]\). The current is formulated in terms of the Keldysh-Schwinger contour and is highly non-local.

In these notes we try to bridge the cases with \(\eta \equiv 0\) and \(\eta \neq 0\), but small. The basic idea is that in case of chiral fluids inclusion of effects of viscosity intrinsically, or dynamically does not trigger violation of chirality conservation. There is, however, a kinematical effect which can be described in terms of locally inertial frames. In more detail, Eqs. \((2)\), \((3)\) assume a universal, Lorentz-covariant description of the whole fluid in terms of the 4-vector \(u_\mu(x)\). Inclusion of viscosity introduces acceleration \(a^\mu = du^\mu/ds\) which depends on position \(x^\mu\) and is determined by dynamics of the fluid (say, through the Navier-Stokes equations). Under these circumstances, the conservation laws would preserve their form in a locally inertial frame while in other frames the conservation laws would look as vanishing of the covariant derivative of the corresponding current.

In particular, introduce the fluid-helicity current \([10]\) defined as:

\[
\mathcal{J}_{\text{fluid hel}}^\alpha \equiv \epsilon^{\alpha\beta\gamma\delta}(\mu \cdot u_\beta) \partial_\gamma(\mu \cdot u_\delta), \quad (4)
\]

where \(\mu\) is the chemical potential (and \(\mu_5 = 0\), for simplicity). In case of ideal fluid the current \([14]\) is conserved, for a recent discussion and further references see, e.g., \([16]\):

\[
\partial_\alpha j_{\text{fluid hel}}^\alpha = 0 \quad \text{(ideal fluid)} . \quad (5)
\]

We suggest that (at least for a small viscosity), the current \(j^\alpha_{\text{fluid hel}}\) is covariantly conserved for viscous fluid:

\[
D_\alpha j_{\text{fluid hel}}^\alpha = 0 \quad \text{(viscous fluid)}, \quad (6)
\]

where the “effective” covariant derivative \(D_\alpha\) is defined algebraically the same as in general relativity but the standard external gravitational field entering the covariant derivative is replaced by an effective gravitational field \(h_{\mu\nu}^{\text{eff}}\) determined by the fluid dynamics and, for small viscosity \(\eta\) the effective field \(h_{\mu\nu}^{\text{eff}} \sim \eta\).

The physical meaning of \(\eta\) is transparent. Namely, in a chiral invariant theory viscosity does not introduce any dynamical violation of conservation of the axial current. In the viscous case, however, it is not possible any longer to introduce a global inertial frame where Eq. \((3)\) would hold.

We do check \((6)\) in a trivial case when the physical effect is linear in \(h_{\mu\nu}^{\text{eff}}\) (and \(\eta\)). As for the higher order terms, validity of \((6)\) remains a hypothesis \([30]\).

The organisation of the paper is as follows. In Sect. 2 we discuss conservation of various types of helicity in case of ideal fluid. In Sect. 3 we present in more detail motivation to introduce emergent gravitational field. In Sect. 4 we present a simple example of duality between description of motion of an ideal fluid in external gravitational field and of dynamics of fluid with small viscosity in flat space. In Sect. 5 we argue that the chiral anomaly results in unique predictions for \(D_\alpha j_{\text{fluid hel}}^\alpha\), where \(j_{\text{fluid hel}}^\alpha\) is the total axial current, see, e.g., Eq. \((3)\), and \(D_\alpha\) is the covariant derivative in external gravitational field. Sect. 6 is devoted to a simplified treatment of fluid instabilities.

**CONSERVATION LAWS IN NON-DISSIPATIVE LIMIT**

In this section we introduce various axial currents, or helical motions and review conservation laws of chiral hydrodynamics in the non-dissipative limit.

Let us start with a well-known remark that the chiral anomaly can be cast into a form of conservation of a generalised chiral charge which is non-vanishing for helical configurations of magnetic field:

\[
Q_{5}^{\text{conserved}} = Q_{5}^{\text{naive}} + \frac{e^2}{4\pi^2} \mathcal{H}_{\text{magn hel}} , \quad (7)
\]

where \(\mathcal{A}\) is the vector potential of electromagnetic field and \(\mathcal{B}\) is the magnetic field and \(\mathcal{H}_{\text{magn hel}}\) is the so called magnetic helicity.

At this point, there is no obvious place yet for the chiral vortical current, see Eq \((3)\). But there do exist various ways to appreciate the emergence of a unified chiral current, like \([3]\). In particular, generalisation of \((7)\) to the hydrodynamic case can be obtained \([15]\) through the substitution:

\[
e^A_\mu \rightarrow e^A_\mu + \mu u_\mu , \quad (8)
\]

where \(\mu\) is the chemical potential associated with the charge \(e\). Eq. \((8)\), in turn, is substantiated on the grounds of similarity between the chemical-potential term in the Hamiltonian, \(\delta H = -\mu \cdot Q\) and the electromagnetic interaction with external field \(\delta H = -e j_{\mu}^\alpha A_\mu\). If we treat both terms perturbatively then Eq. \((8)\) seems quite obvious.
Upon using (8), one arrives at the expression of a quantum-mechanically conserved axial charge suited for applications to hydrodynamics:

$$Q_{5}^{\text{hydro}} = Q_{\text{naive}}^{5} + \frac{e^{2}}{4\pi^{2}} H_{\text{magn hcl}} + \frac{1}{4\pi^{2}} H_{\text{fluid hcl}} + \frac{1}{2\pi^{2}} H_{\text{fluid-magn hcl}},$$ (9)

where the fluid helicity $H_{\text{fluid hcl}}$ and fluid-magnetic helicity $H_{\text{fluid-magn hcl}}$ are defined as

$$H_{\text{fluid hcl}} = \int d^{3}x \epsilon^{0ijk}(\mu u_{i})\partial_{j}(\mu u_{k}) ,$$ (10)

and

$$H_{\text{fluid-magn hcl}} = \int d^{3}x \epsilon^{0ijk}(\mu u_{i})\partial_{j}A_{k} = \int d^{3}x \epsilon^{0ijk}(A_{i})\partial_{j}(\mu u_{k}).$$ (11)

Note that all the kinds of helicities entering (9) were introduced first long time ago in the context of magnetohydrodynamics, see, e.g., [10] and references therein.

One might question the validity of (8) since it is well known that introduction of the chemical potential does not result in any extension of the chiral anomaly. The famous triangle graph, with external electromagnetic legs is known to be the only source of non-conservation of the original axial charge $Q_{\text{naive}}$. And Eq. (9) is, at first sight, in contradiction with this assertion.

To address this question, turn to the specific case of ideal fluid. Then, the central point is that various terms in the r.h.s. of Eq. (9) can be separately conserved on equations of motion for ideal fluid. This is simplest to demonstrate in case of the magnetic helicity entering Eq. (7). Indeed, kinematically we get:

$$\frac{d}{dt} H_{\text{magn hcl}} = \int d^{3}x (E \cdot \vec{B}) ,$$ (12)

where $\vec{E}$ is the electric field.

However, as is commonly known, electric field is completely screened inside an ideal conductor, $\vec{E} = 0$. Indeed, in general case of finite conductivity $\sigma_{E}$ electric current is given by $j_{el} = \sigma_{E} \vec{E}$. Since the current is to stay finite also in the limit of the conductivity tending to infinity, $\sigma_{E} \to \infty$, one concludes that

$$\lim_{\sigma_{E} \to \infty} \vec{E} \to 0$$ (13)

in this limit. Thus, we come to a paradoxical conclusion that the chiral anomaly resulting in $\partial_{\mu}j_{5}^{\mu} \sim e^{2}(\vec{E} \cdot \vec{B})$ does not signal any non-conservation of axial charge if one evaluates matrix element of it over a state of ideal fluid in equilibrium.

Clearly enough, the conservation of the magnetic helicity in case of ideal liquid, just discussed, is of a different nature than, say, conservation of the original axial charge $Q_{\text{naive}}$ in field theory. The magnetic helicity does not correspond, at least within the derivation given above, to any Noether charge. Its conservation is not a manifestation of any symmetry of the underlying Lagrangian but is rather of dynamical nature.

The condition of vanishing of the electric field inside an ideal conductor is modified if one allows for variations of chemical potential $\mu$ and/or temperature $T$. In this, more general case one has

$$\vec{j}_{el} = e\sigma_{E} \left( \vec{E} - T\vec{\nabla} (\mu/T) \right).$$

The condition (13) for the fluid being ideal is changed respectively: it is the combination of the two terms in the right-hand side which is to vanish. This change is immediately reflected in the helicities conservation laws. Namely, for the divergence of the fluid-helicity current one gets:

$$\partial_{\alpha}j_{\alpha}^{\text{fluid hcl}} = \frac{T^{2}\mu s}{(p + \epsilon)} \epsilon^{\alpha\beta\gamma\delta} \partial_{\alpha} \left( \frac{\mu}{T} \right) ,$$ (14)

where $p$ and $\epsilon$ are pressure and energy density, respectively, $s$ is entropy density. It is clear now that in case $\mu \neq 0$ (so that the fluid is charged) and absence of external fields ($\vec{E} = 0$) the fluid helicity is conserved, $\partial_{\alpha}j_{\alpha}^{\text{fluid hcl}} = 0$, in the dissipation-free limit. This is, probably, the simplest case when imposing the dissipation-free limit brings in a new conservation law. And in the next section we will mostly concentrate on this case. If both external electromagnetic fields and gradient of the $\mu/T$ ratio are present, it is the combination of the three helicities entering the r.h.s. of Eq. (2) which is conserved for ideal fluid [37].

Within the approach outlined above, the fundamental field theory is invoked only on the level of derivation of the chiral anomaly. There exists a more ambitious field theoretic approach to hydrodynamics which introduces universal infrared degrees of freedom relevant to any fluid, see, e.g., recent papers [12] and references therein. Essentially, these infrared degrees of freedom are deviations of the coordinates of an element of fluid from their equilibrium values. Typically, one introduces scalar fields $\varphi^{I}$ with non-trivial expectation values in the equilibrium:

$$< \varphi^{I} > = x^{I} ,$$ (15)

where $x^{I}$ are the equilibrium positions of coordinates of an element of fluid. There are certain symmetries imposed on the interaction of the fields $\varphi^{I}$ and invariants can be ordered according to the number of derivatives from the fields. The lowest-order invariant is

$$B \sim \left( \epsilon^{\alpha\beta\gamma\delta} \partial_{\alpha} \varphi^{I} \partial_{\beta} \varphi^{J} \partial_{\gamma} \varphi^{K} \epsilon_{IJK} \epsilon_{\delta} \right) ^{2}.$$ Respectively, the field theoretic action density in this approximation is
given in terms of a function \( f(B) \) of the invariant \( B \):

\[
S_{\text{hydro}} = \int d^4 x f(B) .
\]  

(16)

Starting from this action one can develop the Hamiltonian formalism and so on.

Yet another approach utilises variational approach and Hamiltonian formalism, for a recent presentation and further references see [13]. Again, all the effective infrared degrees of freedom of an ideal liquid are included into consideration. One demonstrates [13] that the “naive” axial charge and the fluid helicity are separately conserved, modulo the chiral anomaly.

To summarise, in case of ideal fluid there are extra conservation laws protecting various helicities. Account for the anomaly in presence of external electromagnetic fields unifies all kinds of classically conserved axial charges into a single charge (9) conserved on the quantum level. This beautiful picture is challenged, however, upon inclusion of non-vanishing viscosity.

**MOTIVATION TO INTRODUCE EFFECTIVE GRAVITATIONAL FIELD**

It is known since long, see, e.g., [10], that, say, fluid helicity (10) is no longer conserved once non-vanishing viscosity is accounted for. Indeed, in most cases dissipation brings fluid into the equilibrium state which is nothing else but the whole of the fluid at rest in a particular frame. In this, trivial state the fluid helicity disappears independent of its initial value [38]. The conclusion that conservation of the hydrodynamic axial charge (9) is inconsistent with \( \eta \neq 0 \) sounds puzzling, since it implies, at first sight, that viscosity cannot be triggered by a chiral invariant interaction [16]. In these notes, we are looking primarily for a resolution of this paradox.

To get a better insight into the problem, let us start with approximation of ideal fluid and then “switch on” a small shear viscosity \( \eta \). Furthermore, assume that there is a region where the fluid is rotating with angular velocity \( \vec{\Omega} \). Then the contribution of this region into the fluid helicity is given by:

\[
H_{\text{fluid hel}} \sim \mu^2 \int d^3 x \vec{v}(x) \cdot \vec{\Omega} ,
\]  

(17)

and we assume that it is non-vanishing. In equilibrium and in absence of the viscosity we have \( \vec{v}(x) \) constant as a function of time. Thus, the charge (17) is conserved as a consequence of the equation of motion of ideal liquid which, under the simplifications specified, reads as

\[
\frac{d\vec{v}}{dt} = 0 ,
\]

where \( \vec{v} \) is the velocity of the fluid parallel to the vector of angular velocity \( \vec{\Omega} \). Note that this equation of motion is not specific for chiral fluids.

Include now a (small) shear viscosity. Rotation is consistent with non-vanishing viscosity and, therefore, for variation with time of the fluid helicity we get:

\[
\frac{d}{dt} H_{\text{fluid hel}} \sim \mu^2 \int d^3 x \vec{a}_{\text{fluid}} \cdot \vec{\Omega} ,
\]  

(18)

where \( \vec{a}_{\text{fluid}} \) is the acceleration of the element of fluid considered. In general, the fluid helicity is no longer conserved. Indeed, one can have the product \( \vec{v} \cdot \vec{\Omega} \) non-zero but dependent on the distance \( \rho \) from the axis of the rotation. Then the viscosity \( \eta \neq 0 \) induces non-conservation of \( H_{\text{fluid hel}} \), see [18].

Our central point is that the estimate [18] can be viewed as a hint that the non-conservation of the axial current (in presence of shear viscosity) can be imitated by equations of motion of ideal fluid in external gravitational field. Indeed, according to the equivalence principle, the effect of going into an accelerated frame can be imitated by introduction of an external gravitational field. Therefore, in presence of gravitational field (and in the non-relativistic limit) we expect that the equation of motion of the ideal liquid would look as

\[
\frac{d\vec{v}}{dt} = \vec{a}_{\text{grav}}
\]

where \( \vec{a}_{\text{grav}} \) is the acceleration imposed by the external gravitational field. Choosing the gravitational field such that

\[
\vec{a}_{\text{grav}} = \vec{a}_{\text{fluid}}
\]

would reproduce [18] and we could trade viscosity for a certain external gravitational field.

Note that in our case, the acceleration \( \vec{a}_{\text{fluid}} \) entering Eq. (18) is determined by dynamics of (viscous) fluid. Thus, we introduce an “effective” gravitational field, to reproduce the same acceleration \( \vec{a}_{\text{fluid}} \). Mathematically, our hypothesis can be formulated as Eq. (19) and we will discuss it in length in the next section.

Let us mention that it was argued first long time ago by Luttinger [19] that introduction of a gravitational potential \( \varphi_{gr} \) can be useful for hydrodynamic studies. Namely, to imitate the effect of the temperature gradient one can introduce an effective gravitational potential \( \Psi \) coupled to energy density \( \mathcal{E} \) through the Hamiltonian:

\[
H_L = \int d^3 x \Psi \mathcal{E} ,
\]  

(19)

where the potential \( \Psi \) is adjusted to balance the thermal force, \( \nabla \Psi = \nabla T/T \). For recent applications of this idea to dynamics of chiral fluids see [20, 21]. Reduction of the problem with a non-vanishing gradient of temperature to
the problem of motion in an external gravitational field allows to use macroscopic language to describe thermal effects. Similarly, we are proposing to mimic the effect of (small) shear viscosity by a gravitational field. This analogy allows us to use well known equations from the general relativity.

Remarkably enough, it was argued very recently, see [22] and references therein, that consideration of dissipative fluids within field-theoretic approach to the general theory of fluids brings in the notion of “emergent spacetime”. It seems plausible that the metric tensor introduced in [22] to describe the emergent “emergent spacetime”. It seems plausible that the metric coincides with the “effective” gravitational field introduced above. However, such an identification is not crucial for our purposes here and detailed discussion of the issue is beyond the scope of the present notes.

CLASSICAL GRAVITATIONAL “ANOMALIES”

Thus, we turn to considering motion in an external gravitational field which induces acceleration on matter. Moreover, for our purposes now there is no need to distinguish between “genuine” and “effective” gravitational fields and we will drop the index “effective” from the metric tensor for the moment.

Imagine that in the flat-space limit there exists a conserved current, \( \partial_\mu j^\mu = 0 \). Then, in presence of external gravitational field the ordinary derivatives \( \partial_\mu \) are replaced by covariant ones, \( D_\mu \):

\[
\partial_\mu j^\mu = 0 \rightarrow D_\mu j^\mu = 0 , \tag{20}
\]

where \( j^\mu \) transforms as a vector under general coordinate transformations. Note that in the limit of vanishing gravitational field the current \( j^\mu \) coincides with the current \( j^\mu \) of special relativity. However, \( j^\mu \) might contain terms of first order in gravitational potentials (which are uniquely fixed by the requirement that \( j^\mu \) is transformed like a vector under general coordinate transformations).

For an arbitrary vector \( j^\mu \) and in standard notations, see [23] one has:

\[
\mathcal{J}^\mu = \frac{\partial j^\mu}{\partial x^\mu} + \Gamma^\mu_{\nu\rho} j^\nu , \tag{21}
\]

where \( \mathcal{J}^\mu \equiv D_\mu j^\mu \) and \( \Gamma^\mu_{\nu\rho} \) are Christoffel symbols. Moreover,

\[
(\mathcal{J}^\mu)_\nu = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \mathcal{J}^\mu)}{\partial x^\nu} \tag{22}
\]

where \( g \) is the determinant constructed on the metric tensor \( g_{\mu\nu} \).

The physical meaning of \( (22) \) is of course absolutely transparent. Indeed, in presence of a gravitational field an element of physical volume is written as \( dV^{(4)}_{\text{phys}} = \sqrt{-g} d^4 x \) and Eq \( (22) \) corresponds to conservation of the number of particles in a unit physical volume, \( d\eta = n(x) dV^{(3)}_{\text{phys}} \), where \( n(x) \) is the density of charged particles, the same as in, say, Eq. \( (4) \). However, if one treats the gravitational field as a usual field in the flat space-time \( x^\mu \), then Eq. \( (22) \) looks so as if the current \( j^\mu \) is no longer conserved:

\[
\partial_\mu j^\mu + (1/2) \mathcal{J}^\rho \partial_\rho (\ln(-g)) = 0 . \tag{23}
\]

Note that Eq. \( (23) \) is exact classically.

If there is a gravitational chiral anomaly then one has [24]:

\[
D_\mu j^\mu = C_{\text{gr}} R_{\alpha\beta\gamma\delta} \tilde{R}^{\alpha\beta\gamma\delta} , \tag{24}
\]

where \( R_{\alpha\beta\gamma\delta} \) is the Riemann tensor and \( C_{\text{gr}} \) is a constant. The r.h.s. of \( (24) \) of second order in gravitational field while the effect of acceleration, discussed above, can be imitated by an external field treated as a first-order correction. Thus, for our purposes we can neglect the r.h.s. of \( (24) \).

Note that the chiral anomaly is quite commonly quoted as containing an ordinary derivative in the l.h.s., \( \partial_\mu j^\mu = C_{\text{gr}} R \tilde{R} \). Such a form assumes perturbative evaluation of the matrix element of the axial current with emission/absorption of gravitons in flat space-time. Eq. \( (24) \), on the other hand, refers to a general, \( x \)-dependent gravitational field, or non-vanishing condensate of the graviton field,

\[
<h_{\mu\nu} > \neq 0 \tag{25}
\]

where \( h_{\mu\nu} \equiv g_{\mu\nu}(x) - \eta_{\mu\nu} \). We will come back to continue this discussion later.

Let us now apply \( (24) \) to the case of weak gravitational field, \( |<h_{\mu\nu}(x)>| \ll 1 \). Then we get:

\[
\partial_\mu j^\mu \approx -(1/2) j^\rho \partial_\rho h , \tag{26}
\]

where \( h \equiv h_{00} - \Sigma_i h_{ii} \). Moreover, the \( h_{00} \) component is fixed by the equivalence principle:

\[
g_{00} - 1 \approx 2 \phi_{gr} \approx -2\ddot{a} \cdot \ddot{x} . \tag{27}
\]

where \( \phi_{gr} \) is the Newtonian potential and \( \ddot{a} \) is the acceleration.

As is well known, the components \( h_{ii} \) are not fixed by the equivalence principle and gauge dependent. In other words, the result of calculation is locally sensitive to the gauge fixing, if the problem is not entirely governed by non-relativistic physics. The best known example of this type is deflection of light by the sun. Keeping only \( h_{00} \neq 0 \) underestimates the effect by a factor of 2 in this case. Using the gauge

\[
h_{11} = h_{22} = h_{33} = h_{00} \tag{28}
\]

restores the lacking factor of 2 locally. (A small deflection angle is evaluated by integrating the effect of gravity over
the unperturbed light trajectory in flat space. Only the integral over the whole trajectory is gauge independent. “Locally” means that the correct numerical factor is reproduced already prior to the integration). Thus, we will explore also the gauge (25).

With Eq. (25) in hand, we can come back to the example of motion described by Eqs. (17), (15) and check whether it actually fits Eq. (25). For this purpose, we compare results of two calculations. First, we evaluate divergence of the fluid-helicity current for the fluid configuration (17) and then compare the result with Eq. (26).

Let us fix first notations. We consider the following (non-relativistic) fluid configuration:

\[ u^{(i)} = (1/2)\Omega \cdot x^{(i)} , \quad u^{(2)} = -(1/2)\Omega \cdot x^{(1)} , \quad u^{(3)} = u^{(3)}(t) , \]

where \( \Omega \) is the angular velocity considered to be a constant, \( u^{(i)}(i = 1, 2, 3) \) are components of the 4-velocity, \( u^{(0)} = 1 + 1/2(u^{(1)})^2 + 1/2(u^{(2)})^2 + 1/2(u^{(3)})^2 \). Furthermore, we are evaluating the current \( j_{\text{fluid hel}} \) introduced in eq. (1). The notations are such that \( \epsilon^{0123} = 1 \) and the minkowskian metric tensor is chosen to be \( g_{00} = 1, g_{ii} = -1 \). Also, for the purpose of the present exercise the chemical potential is kept constant.

For the components of the current \( j^{(i)} \equiv \mu^{-2}j_{\text{fluid hel}}^{(i)} \) we find:

\[ j^{(1)} = -(1/2)\Omega x^{(1)}(3) , \quad j^{(2)} = -(1/2)\Omega x^{(2)}(3) , \quad j^{(3)} = -\Omega , \]

where \( a^{(3)} = du^{(3)}/dt \). And, finally, for the divergence of the current we get:

\[ \mu^{-2} \partial_{\mu} j_{\text{fluid hel}}^{(\mu)} = -2a^{(3)}\Omega , \]

where the acceleration \( a^{(3)} \) is determined by dynamics of the viscous fluid. We expect that this relation can be reproduced by considering ideal fluid in the effective gravitational field.

The curved-space counterpart of the fluid-helicity current is given by:

\[ \mu^{-2}(j_{\text{fluid hel}}^{\mu})_{\text{cov}} = (-g)^{-1/2} \epsilon^{\mu\nu\alpha\beta} u_{\nu} D_{\alpha} u_{\beta} . \]

The condition of vanishing of the covariant derivative (22) becomes

\[ \partial_{\mu}(\epsilon^{\mu\nu\alpha\beta} u_{\nu} D_{\alpha} u_{\beta}) = 0 . \]

Using Eqs (27), (28) with \( \vec{a} = (0, 0, a^{(3)}) \) we arrive to the same Eq. (31), as expected.

Two comments concerning this result are now in order. First, Eq. (22) is sensitive only to the component \( h_{00} \) of the weak gravitational field. This is not obvious apriori since the divergence of the fluid-helicity current is equally contributed by the zeroth and spatial components of the current, while non-relativistically we could expect dominance of the zeroth component, see discussion in Sect. 2. Our second comment is that effect of an external gravitational potential on the chiral currents was evaluated in Ref. [21]. In this paper, the authors consider the Fermi-sphere model for the chiral fluid and evaluate the axial current starting from first principles. In our approach, the chiral vortical current (3) arises quantum mechanically while the last step, that is inclusion of the gravitational potential, is an application of a classical equation (22). Algebraically, results appear to coincide while their interpretation is different (25).

It is worth emphasising that presence of linear in the acceleration \( \vec{a} \) terms is specific for chiral effects. In case of the most standard hydrodynamic current, \( j^{\mu} = n u^{\mu} \), the 4-vectors of the velocity and acceleration are orthogonal to each other, \( u^{\mu} \partial_{\mu} u_{\mu} = 0 \) because of the normalisation condition, \( u^{\mu} u_{\mu} = -1 \). As a result the r.h.s. of Eq. (26) vanishes in this case (at least, for a constant \( n \)).

To summarise this section, for a particular example considered we can reinterpret the non-conservation of the fluid helicity due to dissipation as a change of geometry of space due to presence of an external gravitational field. The gravitational field itself is an implicit function of initial distribution of velocities and masses and of viscosity of the fluid. The basic limitation of the example considered is that it is linear in the effective gravitational field triggered by viscosity. Such an approximation can be justified by smallness of viscosity. On the other hand, on physical grounds one can expect that the relation between dissipation and geometry, encoded in Eq. (4) remains true also in higher orders in the gravitational field.

**REMARKS ON HIGHER ORDERS IN GRAVITATIONAL FIELD**

In this section, we will clarify a few points concerning covariant conservation of the axial current. Our remarks are algebraic in nature and apply both in case of “genuine” and “effective” gravitational fields, and we do not distinguish between these two cases in this section.

Let us go back to evaluation of divergence of axial current in gravitational background. For simplicity, we will not introduce external electromagnetic fields. In this approximation and in flat space the axial current is given by (4). Next, we introduce a non-trivial position-dependent background \( h_{\mu\nu} \neq 0 \) where \( h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} \) and evaluate divergence of axial current. Our results of the preceding section can be summarized in the
Generally speaking, the divergence of the axial current conservation law to the case of gravitational background formulation of generalization of a (non-anomalous) coefficient in fact a single term proportional to the anomaly-related coefficient. However, we found out that, in our simplified case there is perturbative Lagrangian on the substitution (8). To justify it one starts with the graph.

Statement that the non-conservation of the total axial Eq. (3) is consistent both with substitution (8) and the field theory, see discussion in Sect. 2. That is why analysis of ideal-fluid approximation is essential since only then would exhibit non-conservation of chirality. Moreover, sense that there are no further matrix elements which does not affect the fundamental triangle anomaly, in the is a manifestation of the well-known theorem that of ideal fluid (see Sect. 2 for discussion). And this helicity is conserved on the equations of motion, from the fundamental chiral anomaly since the generalisation of where the second term in the r.h.s. is the hydrodynamic derivative of a current, $D_\alpha j_5^\alpha = 0$ is a textbook formulation of generalization of a (non-anomalous) conservation law to the case of gravitational background. Generally speaking, the divergence of the axial current $\partial_\alpha j_5^\alpha$ could receive at this step various contributions. However, we found out that, in our simplified case there is in fact a single term proportional to the anomaly-related coefficient $C$.

To elucidate the origin of this factor of $C$ concentrate on the substitution on the perturbative Lagrangian $\delta L = cA_\mu j_5^\mu + \mu u_\mu j_5^\mu$, where the second term in the r.h.s. is the hydrodynamic generalisation of $\delta L = -\delta H = -\mu \cdot Q$. Using the language of Feynman graphs we come to a hydrodynamic analogy of the chiral anomaly where $cA_\mu$ is replaced by $\mu u_\mu$. This is the origin of the fluid-helicity current $j_{\mathrm{fluid\, hel}}^\alpha$ in Eq. (4) and this is how the coefficient $C$ enters the expression $\delta L$.

The fluid-helicity current is not kinematically conserved, $\partial_\alpha j_{\mathrm{fluid\, hel}}^\alpha \neq 0$. This property is inherited from the fundamental chiral anomaly since the $j_{\mathrm{fluid\, hel}}$ term in (4) is generated by the triangle graph, with two $\mu u_\alpha j_5^\alpha$ vertices. However, the fluid helicity is conserved on the equations of motion of ideal fluid (see Sect. 2 for discussion). And this is a manifestation of the well-known theorem that introduction of chemical potentials (and temperature) does not affect the fundamental triangle anomaly, in the sense that there are no further matrix elements which would exhibit non-conservation of chirality. Moreover, the ideal-fluid approximation is essential since only then theory of fluids can consistently be reduced to a unitary field theory, see discussion in Sect. 2. That is why analysis of the ideal fluid in the language of Feynman graphs is granted to obey general theorems of field theory. Thus, Eq. (4) is consistent both with substitution (8) and the statement that the non-conservation of the total axial charge is entirely determined by the original triangle graph.

Next, switch on gravitational field (still within the ideal-fluid approximation). Then for the axial current of the fundamental theory and to first order in gravitational interaction we have $\partial_\alpha j_5^\alpha = 0$ and can use here the ordinary derivative instead of the covariant derivative. This is a well-known manifestation of chirality conservation by gravitational interaction of (fundamental) massless fermions. However, if we generalise conservation of the fluid-helicity current to the presence of external gravitational fields we have to use the general relation (4) and keep the covariant derivative, not replacing it by ordinary derivative. This is because the chiral vortical current in Eq. (3) represents a hydrodynamic analogy of the fundamental triangle graph. In this way we can trace the origin of the anomaly-related coefficient $C$ in the r.h.s. of Eq. (34).

In Sect. 2 we mentioned similarity between calculations in external gravitational field and coordinate-dependent condensate of a scalar (Higgs) field. In conclusion of this section we elaborate further on this remark. Let us remind the reader that currents induced by coordinate-dependent condensates were introduced by Goldstone and Wűlczek [26]. As is observed in [27] the electromagnetic current

$$j_{el}^\mu = (\text{Const}) e^{\mu\nu\sigma} \partial_\mu F_{\nu\sigma}$$  (35)

is in fact not conserved if there is a vortex (such that the vacuum expectation value of $\phi$ is not zero in the vacuum, $<\phi> \neq 0$, but vanishes along a line $<\phi(x) >_{x=0} = 0$). In this case

$$\partial_\mu j_{el}^\mu \sim \delta^{(2)} (x) = 0$$  (36)

Note that the current in (36) is a non-anomalous one-loop effect.

The singular nature of the divergence of the current in (36) implies that the current in (35) cannot be treated classically and there is production of particles. The particles correspond to zero modes of massless fermions in the background of the vortex field, and production of particles by electric fields corresponds to the chiral anomaly in $(1+1)$ dimensions.

A natural question arises whether there is a similar effect in the gravitational case, when $<h_{\mu\nu}> \neq 0$. Eq. (22) might indicate that the classical treatment fails if the determinant $g \rightarrow 0$. The determinant $g$ vanishes near the horizon for a Rindler observer. Indeed, the metric induced is given by $ds^2 = r^2 dt^2 - (dx^\perp)^2$, with $r_e \rightarrow 0$ at the horizon. The physics in Rindler space, in turn, is similar to physics of a large black holes near the horizon. Thus, basing on Eq. (22) one could speculate that production of particles and quantum anomaly are relevant in case $g \rightarrow 0$ [39]. And there is, indeed, a deep connection between the Hawking radiation from the horizon and chiral anomaly in $(1+1)$ dimensions, as is discovered by authors of papers in Ref. [28].
PHASE TRANSITIONS

Instabilities of ideal fluids

Phase transitions reveal structure of conservation laws and of effective degrees of freedom in a dramatic way. In this section we present an oversimplified version of “theory” of instabilities, or phase transitions of chiral fluids. One of such instabilities is a spontaneous production of helical configurations of magnetic field from chiral asymmetric plasma, $\mu_5 \neq 0$. In other words, chirality of microscopic degrees of freedom, elementary fermions, is transformed into helicity of macroscopic magnetic fields. Theory of this instability has been elaborated in quite great detail, see, in particular, [29, 30] and we do not have anything to add to this theory.

We are rather interested in theory of another type of phase transitions which is the transfer of chirality of elementary constituents to macroscopic, helical motion of the fluid. Discussion of this phase transition and of its possible astrophysical applications is very recent [16, 31]. First-principles theory of this transition is much less developed.

The reason is that instabilities with respect to generation of magnetic fields can be studied within (quantum) electrodynamics while field theory for ideal fluids is developed only in case of small fluctuations (quantum) electrodynamics while field theory for ideal fluids is developed only in case of small fluctuations. A very interesting numerical result which might be relevant to our discussion is obtained in Ref. [32]. One starts with elementary constituents to macroscopic, helical motion of the fluid. Discussion of this phase transition and of its possible astrophysical applications is very recent [16, 31]. First-principles theory of this transition is much less developed.

Instabilities of ideal fluids

A crucial point for a phase transition to take place is energy balance. Conservation of the total axial charge provides an extra constraint on the dynamics of the instabilities.

We have the following problem to consider. There are, say, two classically conserved charges, $Q_1, Q_2$ which are mixed up into a single conserved charge $Q_{\text{total}} = Q_1 + Q_2$ because of a generic “anomaly”. The chemical potential associated with $Q_{\text{total}}$ is assumed to be non-vanishing and only transitions consistent with conservation of $Q_{\text{total}}$ are allowed. We start with a state where the whole of the charge is equal to $Q_1$, so that

\begin{equation}
(Q_{\text{total}})_{\text{initial}} = Q_1
\end{equation}

\begin{equation}
E_{\text{initial}} = \mu \cdot Q_1
\end{equation}

Imagine that degrees of freedom associated with $Q_2$ get excited, $Q_2 = |\delta Q_1|$, where the change of $Q_1$ is small $|\delta Q_1| \ll Q_1$. Then it is a reasonable guess that the change in energy is given by:

\begin{equation}
\delta E = -\mu \cdot |\delta Q_1| + (\text{const})(\delta Q_1)^2,
\end{equation}

where the term proportional to $(\delta Q_1)^2$ corresponds to kinetic energy of the excited degrees of freedom. If Eq. (39) indeed holds, then phase transition is favoured at least for small $|\delta Q_1|$. Moreover, it is quite obvious that the final state would have charges $Q_1, Q_2$ of the same order, $Q_1 \sim Q_2$, unless there is a hidden large parameter inherent to the problem [16].

Our central point is that possibility is realised in case of decay of chiral fluid both into helical magnetic field and vortices.

General mechanism of instability

Let us now approach the problem of chiral fluid instability, within framework just outlined, in case when axial charge can be approximated by a sum of two terms, see eq. (7). Thus, we have

\begin{equation}
Q_1 \equiv Q_{\text{naive}}, \quad Q_2 \equiv e^2/(4\pi^2)\mathcal{H}_{\text{magn hel}}.
\end{equation}

We start with an initial state such that $Q_{\text{naive}} \neq 0$ and there is no magnetic field. The phase transition in point is a spontaneous generation of magnetic fields from chiral plasma, see, in particular, [29, 30] and references therein.

Imagine that a helical magnetic field is generated with charge $\delta Q_2$. From definition of the magnetic helicity we have in the momentum space:

\begin{equation}
\delta Q_2 \sim \alpha_{\text{al}} p_B A^2,
\end{equation}

Effective gravitational field introduced in Sect. 3.
where \( \alpha = e^2/4\pi \), \( p_B \) is a characteristic momentum of the magnetic field (simplest helical magnetic field is a combination of three standing waves), and \( \vec{A} \) is the vector potential.

Generation of the magnetic field would cost energy density of order

\[
\vec{B}^2 \sim p_B^2 \vec{A}^2
\]

(40)

The phase transition is energetically favoured provided that \( p_B^2 \leq \mu \alpha q \), or

\[
p_B \sim \alpha q \mu .
\]

(41)

Thus, we see that Eq. (38) does hold, as far as orders of magnitude are concerned.

We come also to a new point. Namely, Eq. (41) exhibits a generic feature of all the hydrodynamic instabilities of chiral fluids. Instabilities arise from a far infrared region, with a large correlation length. In case (41) the correlation length is parametrically enhanced as \( \alpha^{-1} \).

There is another important point worth mentioning. The estimate (40) does not tell us which quantity (if any) is “of order unit”, magnetic field, \( \vec{B}^2 \) or vector potential, \( \vec{A}^2 \). The correct answer seems to be that it is \( \vec{B}^2 \sim \mu^2 \) which is of order unit. The estimate for \( \vec{A}^2 \) then reads as:

\[
\vec{A} \sim r_{typical} \times \vec{B} , \text{ or } [\vec{A}] \sim \alpha^{-1} \mu ,
\]

(42)

where \( \mu \) is the chemical potential. It is amusing that we count \( [\vec{A}] \) as being “large” and extract observable consequences from that. Despite of the fact that \( [\vec{A}] \) is apparently non-gauge invariant. Nevertheless, the estimates do make sense since the charge \( \mathcal{H}_{magn \ hel} \) is gauge invariant.

Combining (41) and (42) we find out that after the phase transition

\[
(\vec{B})^2 \sim \mu^2 , \quad q_2 \sim \alpha q [\vec{A}][\vec{B}] \sim \mu^3,
\]

(43)

where \( q_2 \) is the density of the magnetic helicity. Note that the small parameter \( \alpha q \) is canceled from the expressions for the energy and charge density after the decay of the original chiral plasma. Qualitatively, this picture was advocated in the preceding subsection.

Decay of chiral fluid into vortices

Proceed now to the case:

\[
Q_1 \equiv Q_{naive}, \quad Q_2 \equiv \mathcal{H}_{fluid \ hel}, \quad Q_{initial} = Q_1 .
\]

(44)

The instability to be discussed is the decay of the chiral fluid into vortices \( \text{[14] [31]} \).

Our central point is that estimates of energies and charges in this case are very similar to the case of decay of the plasma into helical magnetic fields. This similarity is not obvious but comes up naturally in the far-infrared region.

At first sight, the case \( \text{[14]} \) is very different from the preceding one. Indeed, the density of the fluid helicity is proportional to

\[
\mathcal{H}_{fluid \ hel} \sim \vec{v} \cdot \text{curl} \vec{v}
\]

and is quadratic in velocity \( \vec{v} \) in the non-relativistic limit. Since the energy is also quadratic in \( \vec{v} \) the condition \( \text{[38]} \) is apparently not satisfied. Moreover, the fluid helicity is apparently suppressed in hydrodynamic approximation by an extra power in the gradient expansion (since we have \( \text{curl} \vec{v} \)).

However, all these objections to the possibility of the phase transition are invalidated by infrared divergences inherent to hydrodynamics. As a result, the actually relevant estimate of the fluid helicity is provided by our toy example, see Eq. (13), with vorticity \( \Omega \) being of zeroth order of smallness in the hydrodynamic approximation. Then, \( \delta Q_1 \) is of first order in the non-relativistic velocity \( \vec{v} \) while the energy is quadratic in \( \vec{v} \). Thus, we have generically the same Eq. (38), and the phase transition is favoured energetically.

The infrared divergences appear both within the framework of perturbation theory (for fluctuations near equilibrium) and in terms of classical solutions for vortices. As textbooks emphasise, vorticity classically costs energy tending to zero in the far infrared, or in the limit of large size of the vortex. This follows from elementary estimates. One can readily appreciate this point by calling on the analogy, suggested by \( \text{[5]} \), between magnetic and vortical cases:

\[
(\vec{A} \sim \vec{B} \times \vec{r}) \rightarrow (\vec{v}_\perp \sim \Omega \times \vec{r}) .
\]

(45)

We see that the component \( v_\perp \) of the velocity perpendicular to \( \Omega \) is “infrared divergent” because of an explicit coordinate dependence. In other words, the vortex is described by a solution of classical hydrodynamic equations. Unfortunately, numerical estimates of the effect of this infrared divergence are difficult to perform. The reason is that vortical classical solutions are difficult to enumerate. In other words, the phase space associated with the vortical solutions is poorly known.

In perturbation theory, to the contrary, the infrared divergences are readily identifiable within the field-theoretic approach to ideal fluids \( \text{[12]} \) mentioned in the Introduction, see Eq. (16). On the other hand, perturbation theory does not tell us, what is the ultimate configuration which the infrared instabilities drive the fluid to.

In more detail, one expands in deviations \( \pi'(x) \) of positions of elements of fluid from their equilibrium values:

\[
\phi' = x' + \pi' ,
\]

(46)
where \( I = 1, 2, 3, \phi^I \) are scalar fields, \( x^I \) are equilibrium positions, or \( x \)-dependent vacuum expectation values in the language of the scalar fields, \( \langle \partial_\mu \phi^I \rangle = \delta^I_\mu \).

Moreover, using action (19) one can quantise the theory and evaluate various correlator functions perturbatively. In particular, one finds (12) for the Fourier transforms of the propagators of \( \pi^I \) in the limit of the frequency \( \omega \to 0 \):

\[
\lim_{\omega \to 0} \langle \partial_\mu \pi^I, \partial_\nu \pi^L \rangle \sim \frac{P^I_L p^5}{\omega_0 \omega} + \frac{P^I_L p^3}{\omega_0 \omega^3 \omega},
\]

where \( \omega_0 \) is a constant, \( P_T \) and \( P_L \) are the transverse and longitudinal projectors corresponding to the decomposition \( \pi^I \equiv (\partial^I \pi_L) / \sqrt{-\partial^2 + \pi^I} \).

Eq. (47) demonstrates clearly that at \( \omega \sim p^3 \) the correlation between variables \( \pi^I_I \) becomes strong and actually cannot be treated perturbatively. The origin of this infrared divergence is the perturbative pole at \( \omega = 0 \). It is worth emphasising that the transverse fluctuations \( \pi^I_T \) correspond, in the language of perturbation theory, to vortices. And the pole at \( \omega = 0 \), see Eq. (47), is a manifestation of the same phenomenon of absence of barrier for creation of vortices of large size, as discussed above. Infrared divergences of perturbation theory indicate emergence of a classical solution, or a new vacuum state in the infrared region. Note that this “classicalization” seems to be a general field-theoretic phenomenon, as argued recently in [34] (in connection with UV divergences).

Finally, appearance of the pole at \( \omega = 0 \) (see Eq. (47)) can be traced back to the fact that quantisation is performed, as usual, in the quadratic approximation. If one keeps non-linearities then one gets an estimate (12):

\[
\frac{\partial \Omega}{\partial t} \sim \eta p^3 \Omega,
\]

where \( p_{typ} \) is a typical momentum and \( \eta \) is the viscosity. Observation (15) can again be traced back to the fact that creation of a vortex does not cost energy in the leading approximation. Eq. (15) can also be used to estimate corrections to our toy model for leading fluid configurations in the infrared region, see Eq. (17).

To summarise, the cases of spontaneous production of macroscopic configurations of magnetic field and of vortices have much in common in the far-infrared region. In both cases the instabilities are associated with far infrared. Attempting to quantise excitations near the “naive” equilibrium state (with \( Q^I_{init} = Q^I_{total} \)) brings to light strong interactions between excitations in some regions of the phase space (see, e.g., [12, 30]). This inconsistency is apparently resolved by formation of coordinate-dependent “condensate” which is nothing else but classical solutions of the corresponding differential equations. In case of the magnetic helicity one deals with solutions of the Beltrami equation, while in case of helical motions one considers solutions of the Navier-Stokes equations. Processes in presence of coordinate-dependent backgrounds are described by a kind of generalisation of the Callan-Harvey currents.

Emergent gravity?

In the preceding subsection we argued that, dynamically, decay of a chiral fluid into helical configurations of magnetic field and into vortices have much in common. However, in case of magnetic fields we were able to derive also an energy balance. A crucial point is that for a small variation \( \partial Q_{naive} \) of the charge the gain in energy is linear in this small variation while the loss of energy is quadratic. The loss of energy is associated with the energy of the generated magnetic field. Note that as far as we consider external magnetic fields, say, relation (7), the energy of magnetic field plays no dynamical role. But once we allow for instabilities, or generation of magnetic fields the energy density \( \epsilon \sim B^2 \) becomes crucial. Moreover, we observed that conservation of axial charge for viscous fluids can be viewed as a modification of the naive charge conservation due to presence of an effective gravitational field \( g^{\mu \nu}_{eff} \).

Then the non-conservation of the naive charge takes the form:

\[
\partial_\alpha (j^\alpha_{naive}) \sim \partial_0 (q^{naive}_5) \sim (\partial_0 g^{eff}_{00}) \Omega,
\]

where we kept the leading-order contribution, linear in the gravitational field, and \( q^5_{naive} \) is the density of the naively conserved axial charge.

Eq. (49) looks as an analogy of, say, fluid-magnetic helicity (14). Namely, we have a contribution to the axial charge expressed as an integral over spatial coordinates. The integrand is a product of a potential, or gauge-noninvariant term and of a “gauge invariant” term. The potential-type term is represented now by a Christoffel symbol \( \Gamma^0_0 \), as it should be in case of gravitational field. The vorticity \( \Omega \) is to be considered as a gauge-invariant term, as is explained in the preceding subsection.

Thus, the expression for axial charge— with account of the emergent gravitational field— looks similar to the electromagnetic case. However, the dynamics of phase transitions is governed also by the energy balance. In the examples we considered the correction to axial charge was linear in a small parameter while loss of energy is quadratic in the same parameter. Thus, for the analogy to be held we are invited to speculate that the effective gravitational energy contains also quadratic terms:

\[
\epsilon_{gravity} \sim \Gamma + ...
\]

The standard expression for the energy of (fundamental) gravitational fields does have such terms. In our case,
these quadratic terms would correspond to the energy of ultra-violet degrees of freedom of the fluid. The same is true in case of the electromagnetic decay of the chiral plasma. Namely, it is the dynamics of the infrared degrees of freedom which drives the decay while the dynamics at the ultra-violet scale ensures stability at short times.

Thus, we hypothesise that the emergent gravitational field is to be treated as a dynamical one once physics of phase transitions is included into the consideration. In other words, we are led to introduce emergent gravity. Much more work is of course needed to make this hypothesis convincing.

Note that the phase transition we are discussing now (that is, decay of chiral plasma into helical states of the effective gravitational field) would also signify emergence of viscosity, even if one starts from an ideal fluid. Phenomenologically, it would be of great interest to check whether the phase transition observed on the lattice [32] results in emergence of a non-vanishing viscosity.

CONCLUSIONS

We have reviewed conservation laws inherent to ideal fluids emphasising the point that there are extra conserved currents (apart from the Noether current of the underlying field theory).

We used this observation to suggest that introduction of viscosity can be imitated by an emergent external gravitational field. In more detail, we have considered the following construction. We start with a state of ideal fluid in equilibrium and non-trivial fluid helicity. Then we switch on shear viscosity, \( \eta \neq 0 \). The corresponding dissipative force induces acceleration which is a function of the initial distribution of velocities (and densities). If we treat the problem in flat space, then the hydrodynamic charge [9] is no longer conserved. At least superficially, this non-conservation is in variance with expectations based on field theory. To elucidate the physical meaning of this non-conservation we introduce gravitational potential which reproduces the field of acceleration induced by viscosity. Then we demonstrate that the non-conservation of the charge under discussion does correspond to the classical equation [24] so that the would be non-conservation of charge corresponds in fact to a change in the physical volume induced by the effective gravitational field.

From a more general perspective, we find out that field-theoretic formulation of dissipative media assumes introduction of curved space. The metric is a function of the viscosity and of distribution of velocities. So far, we could indicate the algorithm of evaluating the effective metrics only to first order in viscosity. Note that our conclusion on emergence of the curved space-time in description of dissipative media is in accord with recent developments, see in particular [22]. The justification given above is, however, is independent.

The approach discussed now is somewhat similar to the now-famous gauge-string duality. In the latter case, to describe dissipation one introduces an extra (curved) dimension. Propagation to the extra dimension corresponds to a kind of “disappearance” from the physical space and describes dissipation. Within the approach considered here, curved space is also introduced “everywhere”. The metric tensor is determined by the fluid dynamics. No further symmetry, like supersymmetry, is required from the underlying fundamental theory at this stage. In projection to the problem of axial current, or helicity conservation, the crucial effect is the change of the physical 4-volume as a function of the emergent gravitational field. Note that in the field theoretic approach to theory of ideal fluid the freedom of the volume reparametrization is a crucial element of the whole construction, see Eq. [10] and discussion around it.

According to the views presented here the physical volume is no longer an invariant of the motion if shear viscosity is taken into account. Rather, the volume becomes a function of the emergent metric. This change of the volume corresponds to dissipation in the real world. The equivalence between dissipation and introduction of external gravitational field seems apparent only at the first step, once we “switch on" viscosity. The gravitational field is a function of the initial distribution of masses an their velocities. On the next step, this adjustment should be reiterated. We have no proof that it is in fact possible. Validity of our equation [6] in higher orders in the effective gravitational field is a guess made on physical grounds. Note also that generically dissipation results in appearance of imaginary parts of various correlators. Eq. [6] does not involve any imaginary parts. Physiciswise, the reason is that conservation of charge is not affected by dissipation. Loss of unitarity is manifested through evolution of a real quantity, that is physical volume.

In the next section, devoted to instabilities of chiral fluids we tried to make two points. First, we emphasised similarity of the dynamics in far infrared which might drive the chiral fluid to decay into helical magnetic and vortical configurations. Since the magnetic-field instability seems to be established theoretically, this similarity supports the idea on possible decay into vortical configurations as well. Finally, we argued that the same similarity suggests that the emergent gravitational field becomes a dynamical degree of freedom once the vortical instability is considered.

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[1] D. E. Kharzeev, K. Landsteiner, A. Schmitt, and H.-U. Yee, “Strongly Interacting Matter in Spinor Field, Lect. Notes Phys. 871, 1 (2013).

[2] S. Adler, “Axial-Vector Vertex in Spinor Electrodynamics” Phys. Rev. 177 (1969) 2426; J.S. Bell and R. Jackiw, “A PCAC puzzle: π^0 → γγ in the σ-model”. Nuovo Cim. A 60 (1969) 47.

[3] A. Yu. Alekseev, V. V. Cheianov, and J. Frohlich, “Universality of transport properties in equilibrium, Goldstone theorem and chiral anomaly”, Phys. Rev. Lett. 81 (1998) 3503, cond-mat/9805346.

[4] D.T. Son and P. Surowka, “Hydrodynamics with Triangle Anomalies”, Phys. Rev. Lett. 103 (2009) 191601, arXiv:0906.5041 [hep-th].

[5] Ya. Neiman and Ya. Oz, “Relativistic Hydrodynamics with General Anomalous Charge”, JHEP 1103 (2011) 023, arXiv:1011.5107 [hep-th].

[6] J. Erdmenger, M. Kaminski, and A. Yarom, “Fluid dynamics of R-charged black holes, JHEP 0901 (2009) 055, arXiv:0809.2488 [hep-th]; N. Banerjee, J. Bhattacharyya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, “Hydrodynamics from charged black branes, JHEP 1101 (2011) 094, arXiv:0809.2596 [hep-th].

[7] A. Vilenkin, “Equilibrium Parity Violating Current In A Magnetic Field, Phys. Rev. D22 (1980) 3080.

[8] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, “The Effects of topological charge change in heavy ion collisions: ‘Event by event P and CP violation’”, Nucl. Phys. A 803 (2008) 227, arXiv:0711.0950 [hep-ph]; K. Fukushima, D. E. Kharzeev, and H. J. Warringa, “The Chiral Magnetic Effect”, Phys. Rev. D 78 (2008) 074033, arXiv:0808.3382 [hep-ph]; D. Kharzeev, “Parity violation in hot QCD: Why it can happen, and how to look for it”, Phys. Lett. B 633 (2006) 260, hep-ph/0406125.

[9] A. Vilenkin, “Quantum Field Theory At Finite Temperature In A Rotating System”, Phys. Rev. D21 (1980) 2260.

[10] H.K. Moffatt, “The degree of knottedness of tangled vortex lines”, J. Fluid Mech. 35. (1969) 117; J. D. Bekenstein, “Helicity conservation laws for fluids and plasmas” Astrophys. Journ. 319 (1987) 207.

[11] S. Golkar and D.T. Son, “(Non-)renormalization of the Quantum Field Theory At Finite temperature”, JHEP 1502 (2015) 169, arXiv:1207.5806 [hep-th].

[12] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, “Null energy condition and supertunnel propagation”, JHEP 0603 (2006) 025, hep-th/0512260; S. Endlich, A. Nicolis, R. Rattazzi, and J. Wang, “The quantum mechanics of perfect fluids”, JHEP, 1104, 102 (2011), arXiv:1011.4996 [hep-th]; S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, “Effective field theory for hydrodynamics: thermodynamics, and the derivative expansion”, Phys. Rev. D85 (2012) 085029, arXiv:1107.0731 [hep-th].

[13] G. M. Monteiro, A. G. Abanov, V.P. Nair, “Hydrodynamics with gauge anomaly: Variational principle and Hamiltonian formulation” Phys. Rev. D91 (2015), 125033, arXiv:1410.4833.

[14] S. Bhattacharyya, J. R. David, and S. Thakur, “Second order transport from anomalies”, JHEP 1401 (2014) 010, arXiv:1305.0340 [hep-th].

[15] F. M. Haehl, R. Loganayagam, and M. Rangamani, “Effective actions for anomalous hydrodynamics”, JHEP 1403 (2014) 034, arXiv:1312.0610 [hep-th].

[16] A. Avdoshkin, V.P. Kirilin, A.V. Sadofyev, and V.I. Zakharov, “On consistency of hydrodynamic approximation for chiral media”, Phys. Lett. B755 (2016) 1, arXiv:1402.3587 [hep-th].

[17] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, “The Holographic fluid dual to vacuum Einstein gravity”, JHEP 1107 (2011) 050, arXiv:1103.3022 [hep-th].

[18] A.V. Sadofyev, V.I. Shevchenko, and V.I. Zakharov, “Notes on chiral hydrodynamics within effective theory approach”, Phys. Rev. D83 (2011) 105025, arXiv:1012.1938 [hep-th].

[19] J. M. Luttinger, “Theory of Thermal Transport Coefficients”, Phys. Rev. 135A (1964) 1505.

[20] M. Stone, “Gravitational Anomalies and Thermal Hall effect in Topological Insulators”, Phys. Rev. B85 (2012) 184503, arXiv:1201.4095 [cond-mat.me-sci].

[21] I. Zahed, “Anomalous Chiral Fermi Surface”, Phys. Rev. Lett. 109 (2012) 091603, arXiv:1204.1955; G. Basar, D. E. Kharzeev, and I. Zahed, “Chiral and Gravitational Anomalies on Fermi Surfaces”, Phys. Rev. Lett. 111 (2013) 161601, arXiv:1307.2234.

[22] M. Crossley, P. Glorioso, and H. Liu, “Effective field theory of dissipative fluids”, arXiv:1511.03646 [hep-th].

[23] L. D. Landau and E. M. Lifshitz, “Classical theory of Fields”, Pergamon.

[24] L. Alvarez-Gaume and E. Witten, “Gravitational Anomalies”, Nucl. Phys. B234 (1984) 269.

[25] P. G. Minkin and V.I. Zakharov, in preparation.

[26] J. Goldstone and F. Wilczek, “Fractional Quantum Numbers on Solitons”, Phys. Rev. Lett. 47 (1981) 986.

[27] C. G. Callan, Jr. and J. A. Harvey, “Anomalies and Fermion Zero Modes on Strings and Domain Walls”, Nucl. Phys. B250 (1985) 427.

[28] S. P. Robinson and F. Wilczek, “A Relationship between Hawking radiation and gravitational anomalies”, Phys. Rev. Lett. 95 (2005) 011303.

[29] A.N. Redlich, “Gauge Noninvariance and Parity Violation of Three-Dimensional Fermions”, Phys. Rev. Lett. 52 (1984) 18; Yu. Akamatsu and N. Yamamoto, “Chiral Plasma Instabilities”, Phys. Rev. Lett. 111 (2013) 052002, arXiv:1302.2125 [nucl-th].

[30] Z.V. Khaidukov, V.P. Kirilin, A.V. Sadofyev, and V.I. Zakharov, “On Magnetostatics of Chiral Media”, arXiv:1307.0138 [hep-th].

[31] N. Yamamoto, “Chiral transport of neutrinos in supernovae: Neutrino-induced fluid helicity and helical plasma instability”, Phys. Rev. D93 (2016) 065017, arXiv:1511.00933 [astro-ph.HE].

[32] T. Burch and G. Torrieri, “Indications of a non-trivial vacuum in the effective theory of perfect fluids”, Phys. Rev. D92 (2015) 1, 016009, arXiv:1502.05121 [hep-lat].
[33] V.P. Kirilin, A.V. Sadofyev and V.I. Zakharov, in preparation.

[34] G. Dvali, G. F. Giudice, C. Gomez, and A. Kehagias, “UV-Completion by Classicalization”, JHEP 1108 (2011) 108, arXiv:1010.1415 [hep-ph].

[35] S.W. Hawking and D. N. Page, “Thermodynamics of Black Holes in anti-De Sitter Space”, Commun. Math. Phys. 87 (1983) 577.

[36] Note, though, that, in other contexts, a somewhat similar problem of finding a mapping of solutions of the Navier-Stokes equations into solutions of Einstein equations has been discussed in great detail and the mapping explicitly constructed, see, in particular [17] and references therein.

[37] If one includes temperature-dependent terms as well, there is one more current, proportional to $T^2$ which is conserved in the dissipation-free limit, see, e.g., [11, 16].

[38] It is actually less trivial that there exist motions corresponding to non-vanishing chiral effects and consistent with viscosity $\eta \neq 0$, see [14].

[39] Note that while the determinant $g$ is gauge dependent its vanishing is gauge independent.

[40] In more detail, one considers ideal fluid in absence of vortices. The two descriptions are equivalent or dual to each other. In the expression of free energy one replaces $Ts$ by $\mu \cdot n$, or vice versa [12].