Exact decoherence to pointer states in free open quantum systems is universal

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In this letter it is shown that exact decoherence to minimal uncertainty Gaussian pointer states is generic for free quantum particles coupled to a heat bath. More specifically, the paper is concerned with damped free particles linearly coupled under product initial conditions to a heat bath at arbitrary temperature, with arbitrary coupling strength and spectral densities covering the Ohmic, subohmic, and supraohmic regime. Then it is true that there exists a time $\tau_c$ such that for times $t > \tau_c$ the state can always be exactly represented as a mixture (convex combination) of particular minimal uncertainty Gaussian states, regardless of and independent from the initial state. This exact ‘localisation’ is hence not a feature specific to high temperatures and weak damping limit, but is rather a generic property of damped free particles.

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There is long tradition of approaching the questions of how and to what extent classical properties of quantum systems emerge dynamically due to the unavoidable coupling to their environment. Essentially any quantum system interacts to some extent with other external degrees of freedom, which in turn may be said to monitor certain properties of the quantum systems $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. This yields decoherence, which results in a loss of purity of initially pure states of a distinguished quantum system coupled to an environment. Not all initial quantum states in such a dilation are yet equally ‘fragile’ to this interaction: there is a small set of initial states that is often relatively robust with respect to this interaction. The term pointer states has been coined for such states, owing the name to models for quantum measurement where the pointer basis is essentially determined by the interaction of the apparatus with the external degrees of freedom $1$.

For harmonic and free quantum systems linearly coupled to a heat bath consisting of harmonic systems this general mechanism is very well-understood indeed. For example, if one prepares a single mode in a pure state in order to let it very weakly interact with an environment in the Gibbs state corresponding to a very high temperature, which one is the state that produces the least entropy over one cycle of the oscillator? In retrospect it hardly comes as a surprise that this is a coherent state, a minimal uncertainty Gaussian state $3$. Most attention has probably been devoted to thoroughly understanding the dynamics of harmonic and free quantum systems in this limit of weak coupling and high temperatures $1, 2, 3, 4$. In this limit in particular, decoherence time scales have been identified $1$. But also exact quantum master equations, generators of dynamical maps, have been derived and scrutinized in great detail $3, 11, 12, 13, 14$. After all, the dynamics of open harmonic or free quantum systems cannot be described other than being well-understood. What else is there to ask for?

A question that seems to have been overlooked so far, yet, is the following: To what extent is exact decoherence in free quantum systems to pointer states actually generic? This question is most appealing in case of the free damped quantum particle $12, 13$, where there is no equilibrium Gibbs state. More specifically: is it true that starting from an arbitrary quantum state, after a fixed finite time $\tau_c$ (independent of the initial state), the state of the system is exactly indistinguishable from a mixture, a convex combination, of minimal uncertainty Gaussian states for all times $t > \tau_c$? In this sense the free quantum system may be said to be in a situation that can operationally not be distinguished locally from the following situation: the particle is somewhere, in a minimal uncertainty Gaussian state, one simply does not know where in phase space. That this is the case seems fairly plausible for the case of high temperatures and weak damping. A significant first step in this direction has indeed been achieved very recently by Diosi and Kiefer in Ref. $7$, showing that this intuition is indeed correct for the approximate generator for the dynamical map in the limit of negligible friction and at high temperatures. But is this a generic feature of free quantum systems that are linearly coupled to an environment in a dilation, and true not only for specific regimes, but for any coupling strength, any non-zero temperature, and Ohmic, sub-ohmic, as well as supraohmic damping? This is the question that will be addressed (and answered) in this paper.

A free quantum system linearly coupled to a heat bath of oscillators will be investigated, where the distinguished system is initially in an arbitrary (and potentially very ‘non-classical’ state), whereas the environment is prepared in the Gibbs state, which corresponds to an initial product state, such that the time evolution of the state of the free quantum system amounts to a completely positive dilation $15$. No assumptions will be made concerning the temperature of the environment and the strength of the coupling; for the class of non-vanishing spectral densities any $C^1$-function $I: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ could be allowed for with

$$\ln I(t!)=t^p = > 0$$

for some $p \geq 0$. This will be referred to as Ohmic damping when $p = 1$, otherwise as subohmic (for $p < 1$) or supraohmic (for $p > 1$). This is an already solved problem in the sense that quantum master equations are known, and hence, the argument draws heavily from known results on generators of dynamical maps $11, 12, 13$, and from earlier results on the long-time behavior in quantum Brownian motion $14$. The starting point is the equation of motion of the
reduced density operators as derived in Ref. [11], in the integrated form as presented in the recent paper Ref. [17]. Later, ideas will be used very similar to the ones in Ref. [7].

The equation of motion of the free particle is for the subsequent purposes most conveniently be expressed in phase space in terms of the Wigner function \( W : \mathbb{R}^2 \rightarrow \mathbb{R} \) [13], which is for each \( t \in \mathbb{R}^+ \) the Fourier transform of the characteristic function, dependent on \( = (\tau; z) \in \mathbb{R}^2 \), where \( \tau \) and \( z \) correspond to position and momentum coordinates in phase space, respectively. As a partial differential equation the Hu-Paz-Zhang equation [11] reads [12]

\[
\partial_t W (\tau; z) = \partial_{\tau} W (\tau; z) + 2\partial_z W (\tau; z) + \partial_z W (\tau; z) W (\tau; z);
\]

where the \( \partial_{\tau} W : \mathbb{R}^+ ! \mathbb{R} \) are time-dependent coefficients for which explicit expressions are known. The formal solution of this partial differential equation can be found for all system parameters [17, 20]. The solution of the differential equation with time-dependent coefficients as presented in Ref. [17] is given by

\[
W (\tau; z) = \int_0^\infty \! d\tau' \, W (\tau'; z) + \int_0^\infty \! d\tau' \, W (\tau'; z) \exp \left\{ -i \left[ (\tau - \tau') M (\tau') \right] \right\};
\]

where dots represent time derivatives. Here, \( G : \mathbb{R} \rightarrow \mathbb{R} \) is the Green’s function, which is \( G (0) = 0 \) for \( t < 0 \) and is for \( t > 0 \) the solution of the integral equation

\[
G (t) = \int_0^\infty \! ds \, G (s) e^{-i \omega t}; \quad G (0) = 0;
\]

with initial conditions \( G (0) = 0, G (0) = 1 \), in terms of the so-called damping kernel. The 2 × 2-matrix

\[
M (t) = \begin{pmatrix} A (t) & C (t) \\ C (t) & B (t) \end{pmatrix};
\]

has coefficients that have in Ref. [17] been expressed in terms of correlation functions. On using the function \( K : \mathbb{R}^+ ! \mathbb{R} \),

\[
K (t) = \begin{cases} 1 & \text{if} \quad \text{Re} [1 + i \omega t] > 0 \\
0 & \text{otherwise}
\end{cases};
\]

with \( \omega = \int_0^\infty \! dt \, e^{it\omega} \), the coefficients \( A (t), B (t), \) and \( C (t) \) can be expressed as

\[
A (t) = \int_0^\infty \! ds \, G (t, s) K (s, s^0); \\
B (t) = \int_0^\infty \! ds \, G (t, s) K (s^0, s); \\
C (t) = \int_0^\infty \! ds \, G (t, s) K (s, s^0);
\]

as \( G (0) = 0 \). Eq. [2], together with the subsequent specifications forms the starting point of our analysis.

Eq. [2] can, using the transformation rule for multiple integrals, be written in form of a product of a time dependent determinant and a convolution with a Gaussian as

\[
W (\tau; z) = \int_0^\infty \! W (\tau; z) \, V (\delta; z) \, \varphi (\delta; z) = \int_0^\infty \! W (\tau; z) \, V (\delta; z) \, \varphi (\delta; z)
\]

where the 2 × 2-matrix \( V \) is given by

\[
V (t) = \begin{pmatrix} G (t) & G (t) \\ G (t) & G (t) \end{pmatrix}
\]

The Green’s function \( G \) can not be evaluated in general in a closed form, the case of Ohmic damping being an exception, where the spectral density is for small frequencies linear in the frequencies. The Laplace transform \( \hat{G} \) of \( G \) is related to the Laplace transform of \( G \) as \( \hat{G} (z) = (z^2 + z + (z^2) \). In order to specify the long time behavior of the Green’s functions, it is sufficient to know the power law for the spectral density for small frequencies only. Using Eq. [1], one arrives for \( p > 0 \) at \( \lim_{t \to 1} G (t) = \varphi (t) = 1 \) (see also Ref. [14]), where \( \varphi (t) = \sin (\pi (p-1) t) \). From the asymptotic behaviour of \( \varphi \) as \( t \to 1 \) it can be seen after a few steps that

\[
\lim_{t \to 1} A (t) = A^0 (t) = 1,
\]

This quantity in turn happens to be a quantity investigated in Ref. [14], where it has been shown that

\[
\lim_{t \to 1} A^0 (t) = A^0 (t) = 1;
\]

which yields \( \lim_{t \to 1} A (t) = A^0 (t) = 1 \), with

\[
A^0 (t) = \frac{2 \sin (\pi (p-1) t)}{(p+1)^2};
\]

In order to find the long time behaviour of the function \( C \), we may use the fact that \( C (t) = 2A (t) \) for all \( t \in \mathbb{R} \), which holds since \( G (0) = 0 \), and apply l’Hospital’s rule to arrive at \( \lim_{t \to 1} C (t) = C^0 (t) = 1 \), with

\[
C^0 (t) = \frac{2 p \sin (\pi (p-1) t)}{(p+1)^2};
\]

To get the long term behaviour of \( B \), we can again start with that

\[
\lim_{t \to 1} B (t) = B^0 (t) = 1,
\]

where

\[
B^0 (t) = \frac{\sin (\pi (p-1) t)}{(p+1)^2};
\]

This, in turn, is nothing but the momentum uncertainty in the stationary setting, which is well-defined even in this free case (compare also Ref. [13, 22]).
with \( z = 1 = (z^2 + iz - z) \). \( B_1 \) is a (time-independent) positive real number. So we have determined the long-time behaviour of the entries of the symmetric 2 \( \times 2 \) matrix \( M(t) \).

Subsequently, a pointer state is taken to be a minimal uncertainty Gaussian state with particular second moments that reflect a small uncertainty in the position canonical coordinate. The statements will be formulated in a language common in quantum optics and continuous-variable quantum information theory. The first moments are \( \langle \xi_1 ; \xi_2 \rangle = 0 \) and \( \langle \eta_1 \rangle = 0 \), the second moments are collected in the covariance matrix

\[
\begin{pmatrix}
2\Omega_1^2 & \hbar \Omega_1 O_2 + O_2 O_1 \hbar \\
\hbar O_1 O_2 + O_2 O_1 & 2\Omega_1^2
\end{pmatrix}
\]

where \( O_1 = X \) and \( O_2 = P \). The second moments for the pointer states are taken to be

\[
\begin{pmatrix}
B_1 & 0 \\
0 & B_1
\end{pmatrix}
\]  

This is a covariance matrix of a minimal uncertainty state, as the first moments are \( 0 \) and the second moments are taken to be \( B_1 \). So we have determined the long-time behaviour of the entries of the symmetric 2 \( \times 2 \) matrix \( M(t) \).

This is a generic result for arbitrary non-zero temperatures, arbitrary coupling strengths and all the spectral densities as in Eq. (1). For specific choices for the spectral density, bounds on the state can be represented as a mixture of pointer states with second moments as in Eq. (6).

We then simply obtain

\[
Z = e^{\frac{1}{2} \hbar \Omega_1 \langle \xi_1^2 \rangle} e^{\hbar \Omega_1 \langle \xi_1 \xi_2 \rangle} e^{\hbar \Omega_1 \langle \eta_1 \eta_2 \rangle} e^{-\frac{1}{2} \hbar \Omega_1 \langle \xi_2^2 \rangle}
\]

for all \( 2 \in \mathbb{R}^2 \). If \( M(t) \) is a legitimate Wigner function, as can be read off the definition of the Wigner function. Then, \( M(t) \) and Eq. (8) imply that

\[
\begin{align*}
W_1(\xi) &= \frac{1}{2} e^{\frac{1}{2} \hbar \Omega_1 \langle \xi_1^2 \rangle} e^{\hbar \Omega_1 \langle \xi_1 \xi_2 \rangle} e^{\hbar \Omega_1 \langle \eta_1 \eta_2 \rangle} e^{-\frac{1}{2} \hbar \Omega_1 \langle \xi_2^2 \rangle} \\
W (\xi, t) &= e^{\frac{1}{2} \hbar \Omega_1 \langle \xi_1^2 \rangle} e^{\hbar \Omega_1 \langle \xi_1 \xi_2 \rangle} e^{\hbar \Omega_1 \langle \eta_1 \eta_2 \rangle} e^{-\frac{1}{2} \hbar \Omega_1 \langle \xi_2^2 \rangle}
\end{align*}
\]

In turn, given the time dependence of the coefficients of \( M(t) \), demonstrated in Eqs. (3), (4), and (5), there exists a finite time \( t_c > 0 \) such that \( Z \) is valid for all \( t > 0 \). This time \( t_c \), in turn, is the time from which on \( W_1 \) is strictly positive, and the state can certainly exactly be represented as a mixture of pointer states with second moments as in Eq. (6).

This is a generic result for arbitrary non-zero temperatures, arbitrary coupling strengths and all the spectral densities as in Eq. (1). For specific choices for the spectral density, bounds on the time \( t_c \) can be found from which on the state can be represented as a mixture of pointer states. For Ohmic damping in particular, the Green’s function is given by \( G(t) = \langle 1, e^z \rangle \), i.e., \( \langle z \rangle = 0 \). The behaviour becomes particularly transparent in the high temperature case. We then simply obtain

\[
\begin{align*}
\log T_c &= \log \frac{1}{\hbar \Omega_1} \\
W_0(\xi, t) &= W_0(\xi, t) e^{\frac{1}{2} \hbar \Omega_1 \langle \xi_1^2 \rangle} e^{\hbar \Omega_1 \langle \xi_1 \xi_2 \rangle} e^{\hbar \Omega_1 \langle \eta_1 \eta_2 \rangle} e^{-\frac{1}{2} \hbar \Omega_1 \langle \xi_2^2 \rangle}
\end{align*}
\]

for all \( 2 \in \mathbb{R}^2 \). This depicts \( T_c = \lim_{t \to \infty} t_c \) where \( t_c \) is the smallest time for which \(\langle z \rangle = 0 \) is satisfied for strictly Ohmic damping.

To conclude, it has been shown that if one couples a free particle linearly to a heat bath prepared in the Gibbs state of some temperature, then, under very general conditions and without approximations, the state of the system becomes after some finite time exactly indistinguishable from an exact mixture of particular minimal uncertainty Gaussian pointer states. In this sense it can be said that exact decoherence to these localized pointer states is generic, and not only a feature of a
limit that can be regarded as being classical. Locally, hence, we arrive at the situation as if we had merely classical ignorance about the position of the particle. Needless to say, care is required in the interpretation of the result, and one should not be tempted by a realistic interpretation in terms of classical alternatives. In turn, the total state of both the system and its environment is very different in structure and is typically a highly correlated and often, but not necessarily, entangled state. It is the hope that this paper can contribute to the debate on the dynamical appearance of classical properties in quantum theory. This debate is potentially becoming more timely than ever with the availability of novel experiments on decoherence, let it be with microwave cavities, ion traps, or nano-electromechanical systems.

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[1] W.H. Zurek, Rev. Mod. Phys. 75, 715 (2003); J.-P. Paz and W.H. Zurek, quant-ph/0010011; D. Giulini, E. Joos, C. Kiefer, I.-O. Stamatescu, and H.D. Zeh, Decoherence and the Appearance of a Classical World in Quantum Theory (Springer, Heidelberg, 1996); D. Giulini, C. Kiefer, and H.D. Zeh, Phys. Lett. A 199, 291 (1995); J.R. Anglin, J.P. Paz, and W.H. Zurek, Phys. Rev. A 55, 4041 (1997); H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Cambridge University Press, Cambridge, 2002).

[2] A.O. Caldeira and A.J. Leggett, Physica A 121, 587 (1983).

[3] N. Gisin and I.C. Percival, J. Phys. A 26, 2233 (1993); J.J. Halliwell and A. Zoupas, Phys. Rev. D 52, 7294 (1995).

[4] D. Braun, F. Haake, and W.T. Strunz, Phys. Rev. Lett. 86, 2913 (2001).

[5] J. Eisert and M.B. Plenio, Phys. Rev. Lett. 89, 137902 (2002); H. McAneney, J. Lee, and M.S. Kim, Phys. Rev. A 68, 063814 (2003).

[6] W.H. Zurek, S. Habib, and J.P. Paz, Phys. Rev. Lett. 70, 1187 (1993).

[7] L. Diosi and C. Kiefer, J. Phys. A 35, 2675 (2002).

[8] S. Bose, K. Jacobs, and P. L. Knight, Phys. Rev. A 59, 3204 (1999); A.V. Rau, J.A. Dunningham, and K. Burnett, Science 301, 1081 (2003).

[9] F. Haake and R. Reibold, Phys. Rev. A 32, 2462 (1985); W.G. Unruh and W.H. Zurek, Phys. Rev. D 40, 1071 (1989); R. Karl Heinrich and F. Haake, Phys. Rev. E 55, 153 (1997).

[10] S. Haroche, Physica Scripta T76, 159 (1998); Q.A. Turchette et al., Phys. Rev. A 62, 053807 (2000); D.A. Kokorowski, A.D. Greilich, T.D. Roberts, and D.E. Pritchard, Phys. Rev. Lett. 86, 2191 (2001); K. Hornberger, S. Uttenhauser, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, ibid. 90, 160401 (2003); X.M.H. Huang, C.A. Zorman, M. Mehregany, and M.L. Roukes, Nature 421, 496 (2003).

[11] B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992).

[12] V. Hakimi and V. Ambegaokar, Phys. Rev. A 32, 423 (1995).

[13] W. Weidlich and F. Haake, Z. Phys. 185, 30 (1965); P. Ullersma, Physica 32, 27 (1966); H. Grabert, U. Weiss, and P. Hänggi, Phys. Rev. Lett. 52, 2193 (1984); P.S. Riseborough, P. Hänggi, and U. Weiss, Phys. Rev. A 31, 471 (1985); P. Talkner, Ann. Phys. 167, 390 (1986).

[14] H. Grabert, P. Schramm, and G.L. Ingold, Phys. Rep. 168, 155 (1988); H. Grabert, P. Schramm, and G.L. Ingold, Phys. Rev. Lett. 58, 1285 (1987).

[15] Note that it is an interesting problem in its own right to investigate the influence of correlated initial conditions, such that time evolution of the state of the distinguished system no longer corresponds to a completely positive map. This is relevant, e.g., when considering a charged quantum particle coupled to the radiation field, where the assumption of product initial conditions is typically not invariant under gauge transformations.

[16] H.-P. Breuer and F. Petruccione, Phys. Rev. A 63, 032102 (2001).

[17] G.W. Ford and R.F. O’Connell, Phys. Rev. D 64, 105020 (2001).

[18] The convention for the Wigner function is

\[ W(\gamma) = \frac{1}{\pi} e^{2i\mathcal{V}} \] with being the symplectic matrix, such that \( \mathcal{V} \) is the standard symplectic scalar product between elements of \( \mathbb{R}^2 \).

[19] For simplicity of the argument, the mass of the particle will set the limit that can be regarded as being classical. Locally, hence, we arrive at the situation as if we had merely classical ignorance about the position of the particle. Needless to say, care is required in the interpretation of the result, and one should not be tempted by a realistic interpretation in terms of classical alternatives. In turn, the total state of both the system and its environment is very different in structure and is typically a highly correlated and often, but not necessarily, entangled state. It is the hope that this paper can contribute to the debate on the dynamical appearance of classical properties in quantum theory. This debate is potentially becoming more timely than ever with the availability of novel experiments on decoherence, let it be with microwave cavities, ion traps, or nano-electromechanical systems.

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[10] S. Haroche, Physica Scripta T76, 159 (1998); Q.A. Turchette et al., Phys. Rev. A 62, 053807 (2000); D.A. Kokorowski, A.D. Greilich, T.D. Roberts, and D.E. Pritchard, Phys. Rev. Lett. 86, 2191 (2001); K. Hornberger, S. Uttenhauser, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, ibid. 90, 160401 (2003); X.M.H. Huang, C.A. Zorman, M. Mehregany, and M.L. Roukes, Nature 421, 496 (2003).

[11] B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992).

[12] V. Hakimi and V. Ambegaokar, Phys. Rev. A 32, 423 (1995).

[13] W. Weidlich and F. Haake, Z. Phys. 185, 30 (1965); P. Ullersma, Physica 32, 27 (1966); H. Grabert, U. Weiss, and P. Hänggi, Phys. Rev. Lett. 52, 2193 (1984); P.S. Riseborough, P. Hänggi, and U. Weiss, Phys. Rev. A 31, 471 (1985); P. Talkner, Ann. Phys. 167, 390 (1986).