On the intersection of dynamical covering sets with fractals

Zhang-nan Hu¹ · Bing Li¹ · Yimin Xiao²

Received: 19 May 2021 / Accepted: 22 October 2021 / Published online: 18 January 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
Let \((X, \mathcal{B}, \mu, T, d)\) be a measure-preserving dynamical system with exponentially mixing property, and let \(\mu\) be an Ahlfors \(s\)-regular probability measure. The dynamical covering problem concerns the set \(E(x)\) of points which are covered by the orbits of \(x \in X\) infinitely many times. We prove that the Hausdorff dimension of the intersection of \(E(x)\) and any regular fractal \(G\) with \(\dim H G > s - \alpha\) equals \(\dim H E(x) + \alpha - s\), where \(\alpha = \dim H E(x) \mu\)-a.e. Moreover, we obtain the packing dimension of \(E(x) \cap G\) and an estimate for \(\dim H (E(x) \cap G)\) for any analytic set \(G\).

Keywords Dynamical covering sets · Exponentially mixing · Hausdorff dimension

Mathematics Subject Classification Primary 37A50; Secondary 28A80 · 60A10

1 Introduction

Let \((X, d)\) be a compact metric space and let \((X, \mathcal{B}, T, \mu, d)\) be a metric measure preserving system (m.m.p.s. for short). The distribution of the orbit of a point in \(X\) is an important topic in ergodic theory and has been studied by many authors. See, for example [1,3,5,10,11,13,18]. The well-known Poincaré recurrence theorem shows that \(\mu\)-a.e. \(x \in X\) is recurrent, that is

\[
\lim \inf_{n \to \infty} d(T^n x, x) = 0.
\]
Boshernitzan [3] proved that if there is some $\tau > 0$ such that the $\tau$-dimensional Hausdorff measure $H_\tau$ of $X$ is $\sigma$-finite, then for $\mu$-a.e. $x \in X$,

$$\liminf_{n \to \infty} n^{\frac{1}{\tau}} d(T^n x, x) < \infty.$$ 

If $\mu$ is ergodic, then for every fixed point $y \in X$, we have, for $\mu$-a.e. $x \in X$,

$$\liminf_{n \to \infty} d(T^n x, y) = 0.$$ 

For an exponentially mixing metric measure preserving system, Fan, Langlet and Li [10] proved that if $t < 1/\alpha_{\text{max}}$, then for $\mu$-a.e. $x \in X$, we have $\liminf_{n \to \infty} n^{t} d(T^n x, y) = 0$ for uniformly for all $y \in X$, where $\alpha_{\text{max}}$ is the maximal local dimension of $\mu$.

Hill and Velani [15] introduced the shrinking targets theory, which concerns the following set of points whose orbits are close to a given point, that is for any given $y \in X$,

$$S(y) = \{ x \in X : T^n x \in B(y, \ell_n) \text{ i.o.} \}, \quad (1.1)$$

where $\{\ell_n\}_{n \geq 1}$ is a sequence of positive real numbers tending to 0 and i.o. stands for infinitely often. Li, Wang, Wu and Xu [28] studied the shrinking target problem in the case when $T$ is the Gauss map and determined the Hausdorff dimension of the set $S(y)$ in (1.1) for certain choices of $\{\ell_n\}$. Bugeaud and Wang [5] studied the problem for the case when $T$ is the $\beta$-transformation. Aspenberg and Persson [1] extended their results to piecewise expanding maps.

Motivated by the Diophantine approximation, Fan, Schmeling and Troubetzkoy [11] proposed the dynamical covering set defined by

$$E(x) = \{ y \in X : T^n x \in B(y, \ell_n) \text{ i.o.} \}, \quad (1.2)$$

which is the set of points $y$ that are well approximated by the orbit of $x$. Among other interesting results, they considered the case when $X$ is the unit interval and $T : x \mapsto 2x \pmod{1}$ and computed $\dim_H E(x)$, where $\dim_H$ denotes the Hausdorff dimension. In [29], Liao and Seuret determined $\dim_H E(x)$ when $T$ is an expanding Markov map. Later, Persson and Rams [34] considered more general piecewise expanding maps than the Markov maps.

In 2017, Wang, Wu and Xu [40] considered the case when $X$ is the middle-third Cantor set, $T x = 3x \pmod{1}$, and $\mu$ is the standard Cantor measure. They gave a complete characterization of the size $E(x)$ for $\mu$-almost all $x$. In [17], Hu and Li investigated the dynamical covering sets in $(X, \mathcal{B}, T, \mu, d)$ with exponentially mixing property, where $\mu$ is an Ahlfors $s$-regular Borel probability measure. They showed that the measure $\mu(E(x))$ is 0 or 1 for $\mu$-a.e. $x$ according to the convergence or divergence of the series $\sum_{n=1}^{\infty} \ell_n^s$, and for $\mu$-a.e. $x$,

$$\dim_H E(x) = \alpha,$$

where $\alpha$ is the upper Besicovitch–Taylor index of $\{\ell_n\}_{n \geq 1}$ defined by

$$\alpha := \inf \left\{ t \leq s : \sum_{n=1}^{\infty} \ell_n^t < \infty \right\} = \sup \left\{ t \leq s : \sum_{n=1}^{\infty} \ell_n^t = \infty \right\}. \quad (1.3)$$

In particular, these results hold when $X$ is the middle-third Cantor set and $\mu$ is the standard Cantor measure.

Motivated by the aforementioned research, we are interested in following natural questions for the dynamical covering set.
Question 1.1 For a given set $G \subset X$, are there points in $G$ which can be well approximated by the orbit of $x \in X$? Equivalently, when is $E(x) \cap G \neq \emptyset$ for $x \in X$?

Question 1.2 If $E(x) \cap G \neq \emptyset$, then how large is the intersection?

Question 1.1 is also closely related to Mahler’s question [31] which is concerned with approximating the numbers in the middle-third Cantor set $C_{1/3}$ by rational numbers and bears some analogy with the dynamical covering set. Several authors have investigated Mahler’s question by measuring the size of the intersection set $W_A(\psi) \cap C_{1/3}$, where $A$ is an infinite subset of $\mathbb{N}$ and

$$W_A(\psi) = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| \leq \psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z} \times A \right\},$$

and $\psi$ is the approximation speed. For example, when $A := \{3^n : n = 0, 1, 2, \ldots\}$, Levesley, Slap and Velani [25] studied the $f$-Hausdorff measure $\mathcal{H}^f$ of the set $W_A(\psi) \cap C_{1/3}$, where $f$ is a measure function, and provided a criterion for $\mathcal{H}^f(W_A(\psi) \cap C_{1/3})$ to be 0 or $\mathcal{H}^f(C_{1/3})$. As for the case when $A = \mathbb{N}$, the problem is still open. Levesley, Salp and Velani [25] conjectured that if $\psi(q) = q^{-r}$ with $r \geq 2$ then

$$\dim_H(W_3(q^{-r}) \cap C_{1/3}) = \frac{2}{r} \dim_H C_{1/3}. \quad (1.4)$$

However, Bugeaud and Durand [4] disagreed with (1.4) and proposed another conjecture:

$$\dim_H(W_3(q^{-r}) \cap C_{1/3}) = \max \left\{ \dim_H(W_3(q^{-r})), \dim_H C_{1/3} - 1, \frac{1}{r} \dim_H C_{1/3} \right\}. \quad (1.5)$$

Bugeaud and Durand [4] provided some results to support their conjecture. In particular, they showed that (1.5) holds for a natural probabilistic model which is a random covering set on the unit circle $\mathbb{T}$ generated by intervals whose centers are uniformly distributed independent random variables in $\mathbb{T}$ (see [4, Section 2]). Recently, Yu [41] proved that the conjecture (1.5) holds for the middle-$p$th Cantor set when $p > 10^7$ is odd and $r \in (1, 1+c)$ for some number $c > 0$. Here, for each odd integer $p > 2$, the middle-$p$th Cantor set is the set of numbers whose base $p$ expansions do not have digit $(p - 1)/2$.

Before stating the main results of this paper, we recall some definitions that will be used throughout this paper.

**Definition 1** A Borel measure $\mu$ on $(X, \mathcal{B})$ is called Ahlfors $s$-regular ($0 < s < \infty$) if there exists a constant $1 \leq c_1 < \infty$ such that

$$c_1^{-1}r^s \leq \mu(B(x, r)) \leq c_1 r^s \quad (1.6)$$

for all $x \in X$ and $0 < r \leq \text{diam } X$, where $B(x, r)$ is the closed ball in metric $d$ whose center is $x$ with radius $r$ and diam $X$ is the diameter of $X$.

**Definition 2** A m.m.p.s. $(X, \mathcal{B}, \mu, T, d)$ is exponentially mixing if there exist two constants $C > 0$ and $0 < \rho < 1$ such that

$$|\mu(E|T^{-n}F) - \mu(E)| \leq C\rho^n$$

for all $n \geq 1$, balls $E \subseteq X$, and measurable sets $F \in \mathcal{B}$ with $\mu(F) > 0$. Here $\mu(A|B)$ denotes the conditional measure $\frac{\mu(A \cap B)}{\mu(B)}$. Sometimes we say $\mu$ is exponentially mixing.
Throughout we denote packing dimension and upper box dimension in the metric space \((X, d)\) by \(\dim_P\) and \(\dim_B\), respectively. We adopt the convention that the Hausdorff dimension and packing dimension of empty sets are equal to \(-\infty\) as in [4] to distinguish the empty set from a non-empty set with dimension 0.

Theorem 1 provides a criterion for Question 1.1. The condition (C) in Theorem 1 is stated in the Sect. 2.

**Theorem 1** Let \((X, \mathcal{B}, \mu, T, d)\) be an exponentially mixing m.m.p.s. and the measure \(\mu\) be Ahlfors \(s\)-regular with \(0 < s < \infty\). Let \(\{\ell_n\}_{n \geq 1}\) be a sequence of positive numbers tending to 0 with the upper Besicovitch–Taylor index \(\alpha < s\) and, for any \(x \in X\), let \(E(x)\) be the dynamical covering set defined in (1.2). Then for any analytic set \(G \subset X\) we have, for \(\mu\)-almost every \(x \in X\)

\[
E(x) \cap G \begin{cases} = \emptyset & \text{if } \dim_P(G) < s - \alpha, \\ \neq \emptyset & \text{if } \dim_H(G) > s - \alpha. \end{cases}
\]

If, in addition, the condition (C) holds, then

\[
E(x) \cap G \neq \emptyset \text{ if } \dim_P(G) > s - \alpha.
\]

**Remark 1** Theorem 1 does not provide complete answers in the critical case of \(\dim_P(G) = s - \alpha\). Even if the condition (C) holds, it is not clear for the case of \(\dim_P(G) = s - \alpha\).

Our Theorem 2 is concerned with Question 1.2 and measures the size of \(E(x) \cap G\).

**Theorem 2** Let \(E(x)\) be the dynamical covering set as in Theorem 1. For any analytic set \(G \subset X\) we have, for \(\mu\)-almost every \(x \in X\)

\[
\dim_H(E(x) \cap G) \begin{cases} \leq \dim_P(G) + \alpha - s & \text{if } \dim_P(G) \geq s - \alpha, \\ = -\infty & \text{if } \dim_P(G) < s - \alpha, \\ \geq \dim_H(G) + \alpha - s & \text{if } \dim_H(G) > s - \alpha. \end{cases}
\]

Moreover, if \(\dim_P(G) > s - \alpha\) and the condition (C) holds, then

\[
\dim_P(E(x) \cap G) = \dim_P(G), \text{ a.e.}
\]

**Remark 2** When \(\dim_P(G) > s - \alpha \geq \dim_H(G)\), the Hausdorff dimension \(E(x) \cap G\) is not explicit. When \(\dim_P(G) = s - \alpha\), \(E(x) \cap G\) is an empty set or a set with Hausdorff dimension 0.

As an immediate consequence of Theorem 2, we have

**Corollary 1** For any regular analytic set \(G \subset X\) in the sense that \(\dim_H(G) = \dim_P(G)\), we have for \(\mu\)-almost every \(x \in X\)

\[
\dim_H(E(x) \cap G) = \begin{cases} \dim_P(G) + \alpha - s & \text{if } \dim_P(G) > s - \alpha, \\ -\infty & \text{if } \dim_P(G) < s - \alpha. \end{cases}
\]

The rest of this article is organized as follows. In Sect. 2, we describe a more general probabilistic setting and prove results on the hitting probabilities of the random covering set. Then Theorems 1–2 follow from Theorems 3–5. This probabilistic setting is closely related to the classical Dvoretzky covering problem concerning the set \(\limsup_{n \to \infty} B(x_n, \ell_n)\), where the centers \(\{x_n\}\) are independent and uniformly distributed. Järvenpää et al. [20] studied the
On the intersection of dynamical covering sets with fractals

hitting probability of the Dvoretzky covering set in a general metric space. We extend their results from the independence setting to a stationary process which is exponentially mixing, and also the uniform distribution is generalized to Ahlfors regular distribution. These results will give the lower and upper bounds for $\dim_H(E(x) \cap G)$ in Theorem 4. In Sect. 4, we derive Theorem 5 by extending the general method of Khoshnevisan, Peres, and Xiao [24] to limsup random fractals in metric spaces. Moreover, we obtain the packing dimension of $E(x) \cap G$ in Theorem 5. The proofs of our main results in Sect. 2 are given in Sects. 3 and 5, respectively. Finally, in Sect. 6, we provide some examples of dynamical systems that satisfy our assumptions so that the main theorems in the paper are applicable to them.

2 General results for stationary processes

In this section, we investigate Questions 1.1 and 1.2 in a general probabilistic setting, which extends the results in [4] and [20].

Let $\{\xi_n\}_{n \geq 1}$ be a stationary process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a compact metric space $(X, d)$. Let $\mu$ be the distribution of $\xi_1$ which is the probability measure defined by

$$\mu(A) = \mathbb{P}(\xi_1 \in A) \quad (2.1)$$

for all Borel sets $A \subset X$.

Definition 3 We say that $\{\xi_n\}_{n \geq 1}$ is exponentially mixing if there exist two constants $C > 0$ and $0 < \rho < 1$ such that

$$|\mathbb{P}(\xi_1 \in A | D) - \mathbb{P}(\xi_1 \in A)| \leq C \rho^n$$

for all $n \geq 1$, balls $A \subset X$ and $D \in \mathcal{F}^{n+1}$, where $\mathcal{F}^{n+1}$ is the sub-$\sigma$-field generated by $\{\xi_{n+i}\}_{i \geq 1}$.

Remark 3 If $(X, \mathcal{B}, \mu, T, d)$ is an exponentially mixing m.m.p.s., define $\xi_n := T^{n-1}x$ for every $x \in X$, then the process $\{\xi_n\}_{n \geq 1}$ is an exponentially mixing process on probability space $(X, \mathcal{B}, \mu)$. Hence the probabilistic model described above covers the dynamical case.

Let $\{\ell_n\}_{n \geq 1}$ be a sequence of positive numbers decreasing to zero. For every $n \geq 1$, denote $I_n := B(\xi_n, \ell_n)$. Define

$$E := \limsup_{n \to \infty} I_n = \{y \in X : y \in I_n \text{ i.o.}\}.$$ 

The set $E$ is a random covering set and consists of the points which are covered by $\{I_n\}_{n \geq 1}$ infinitely often.

The following theorem is concerned with the hitting probabilities of the random covering set $E$.

Theorem 3 Let $\{\xi_n\}_{n \geq 1}$ be an exponentially mixing stationary process taking values in $X$ with probability distribution $\mu$. We assume that $\mu$ is Ahlfors $s$-regular with $0 < s < \infty$. Let $\alpha$ be the upper Besicovitch–Taylor index of $\{\ell_n\}_{n \geq 1}$ with $\alpha < s$. Then for every analytic set $G \subset X$, we have

$$\mathbb{P}(E \cap G \neq \emptyset) = \begin{cases} 0 & \text{if } \dimP(G) < s - \alpha, \\ 1 & \text{if } \dimH(G) > s - \alpha. \end{cases}$$
The theorem below provides an estimate on the Hausdorff dimension of the intersection $E \cap G$.

**Theorem 4** Under the setting of Theorem 3, for every analytic set $G \subset X$, with probability one we have

$$\dim_H(E \cap G) \begin{cases} \leq \dim_P(G) + \alpha - s & \text{if } \dim_P(G) \geq s - \alpha, \\ = -\infty & \text{if } \dim_P(G) < s - \alpha, \\ \geq \dim_H(G) + \alpha - s & \text{if } \dim_H(G) > s - \alpha. \end{cases}$$

The following corollary is an immediate consequence of Theorem 4.

**Corollary 2** Under the setting of Theorem 3, for any analytic set $G \subset X$ with $\dim_H(G) = \gamma$ we have, with probability one

$$\dim_H(E \cap G) = \begin{cases} \gamma + \alpha - s & \text{if } \gamma > s - \alpha, \\ -\infty & \text{if } \gamma < s - \alpha. \end{cases}$$

In general, in Theorem 3, $\mathbb{P}(E \cap G \neq \emptyset) = 1$ may not hold if $\dim_H G > s - \alpha$ is replaced by the weaker condition $\dim_P G > s - \alpha$. A counter-example was given by Li and Suomala [27] when $\{\xi_n\}$ is a sequence of independent and uniformly distributed random variables on the circle. Therefore in the result below, we will make use of the following condition on the sequence $\{\ell_n\}$:

(C) Let $b \in (0, \frac{1}{2})$ be a constant. There exists an increasing sequence of positive integers $\{k_i\}$ such that $k_i \to \infty$ as $i \to \infty$,

$$\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1$$

and

$$\lim_{i \to \infty} \frac{\log_{b^{-1}} n_{k_i}}{k_i} = \alpha,$$

where

$$n_k = \#\{n \geq 1: \ell_n \in [b^{k-1}, b^{k-2})\}.$$

**Remark 4** By Li, Shieh and Xiao [26], the upper Besicovitch–Taylor index $\alpha$ of $\{\ell_n\}_{n \geq 1}$ can be expressed as

$$\alpha = \limsup_{k \to \infty} \frac{\log_{b^{-1}} n_k}{k}.$$  (2.4)

Under Condition (C), we are able to improve Theorem 3 by showing that the hitting probability of the random covering set $E$ with an arbitrary analytic set $G \subset X$ is determined by the packing dimension of $G$.

**Theorem 5** Under the setting of Theorem 3, if the condition (C) holds, then for every analytic set $G \subset X$ with $\dim_P(G) > s - \alpha$, we have

$$\mathbb{P}(E \cap G \neq \emptyset) = 1.$$  

Moreover, if $\dim_P(G) > s - \alpha$, then $\dim_P(E \cap G) = \dim_P(G)$ a.s.
We end this section with some remarks about studies of random covering sets. The random covering problem goes back to 1897 when Borel investigated questions related to random placement of circular arcs in the unit circle [2]. Let $\xi = \{\xi_n\}$ be a sequence of independent and uniformly distributed random variables on the circle $T$ and $\{\ell_n\}$ be a sequence of positive numbers decreasing to 0, define the random covering set

$$E(\xi) = \limsup_{n \to \infty} B(\xi_n, \ell_n).$$

In 1956, Dvoretzky [8] called the attention on the study of $E(\xi)$. He asked the question when $E(\xi) = T$ a.s. or not. In 1971, Shepp [38] gave a sufficient and necessary condition: $E(\xi) = T$ a.s. if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty$. For the high dimensional case, the Dvoretzky covering problem for balls is still open (more details see [23]).

The Hausdorff dimensions of random covering sets were firstly investigated by Fan and Wu [9]. Järvenpää et al. [19] introduced the self-affine covering sets and obtained the dimension formula in terms of the singular value function, which was generalized to any Lebesgue measurable set covering by Feng et al. [12].

The random covering problem is related to many other fields such as number theory. For example, in 2017, Haynes and Koivusalo [14] used a covering argument in Dvoretzky [8] to prove the randomized version of the Littlewood Conjecture.

### 3 Proofs of Theorem 3 and Theorem 4

#### 3.1 A nesting family in a metric space

We start this section by recalling Theorem 2.1 of Käenmäki, Rajala and Suomala [22], which provides a nesting family of “cubes” in a metric space with the finite doubling property [i.e., every ball $B(x, 2r) \subset X$ may be covered by finitely many balls of radius $r$]. This family shares most of the good properties of dyadic cubes of Euclidean spaces.

**Theorem 6** [22] Let $(X, d)$ be a metric space with the finite doubling property and let $0 < b < \frac{1}{2}$ be a constant. Then there exists a collection $\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k \subset \mathbb{N}\}$ of Borel sets that have the following properties:

1. $X = \bigcup_{k \in \mathbb{N}_k} Q_{k,i}$ for every $k \in \mathbb{Z}$.
2. $Q_{k,i} \cap Q_{m,j} = \emptyset$ or $Q_{k,i} \subset Q_{m,j}$, where $k, m \in \mathbb{Z}$, $k \geq m$, $i \in \mathbb{N}_k$ and $j \in \mathbb{N}_m$.
3. For every $k \in \mathbb{Z}$ and $i \in \mathbb{N}_k$, there exists a point $x_{k,i} \in X$ such that

   $$U(x_{k,i}, c_2 b^k) \subset Q_{k,i} \subset B(x_{k,i}, c_2' b^k).$$

   where $c_2 = \frac{1}{2} - \frac{b}{1-b}$, $c_2' = \frac{1}{1-b}$ and $U(x_{k,i}, c_2 b^k)$ is the open ball with center $x_{k,i}$ and radius $c_2 b^k$.

4. There exists a point $x_0 \in X$ so that for every $k \in \mathbb{Z}$, there is an index $i \in \mathbb{N}_k$ with $U(x_0, c_2 b^k) \subset Q_{k,i}$.

5. $\{x_{k,i} : i \in \mathbb{N}_k\} \subset \{x_{k+1,i} : i \in \mathbb{N}_{k+1}\}$ for all $k \in \mathbb{Z}$.

**Remark** 5 From the construction of $\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k \subset \mathbb{N}\}$ in [22] we see that for any $k \in \mathbb{Z}$, $Q_{k,i} \cap Q_{k,j} = \emptyset$ for $i \neq j \in \mathbb{N}_k$. When $\mu$ is Ahlfors $s$-regular, then by (1) and (3) in Theorem 6 and the equalities (1.6), we have

$$c_1^{-1} c_2'^{-s} b^{-ks} \leq \# \mathbb{N}_k = \# \{Q_{k,i} : i \in \mathbb{N}_k\} \leq c_1 c_2^{-s} b^{-ks}.$$  

(3.2)

We will use this fact in our proofs of Theorems 3, 5 and 7 below.
If \((X, d)\) is a compact metric space endowed with an Ahlfors \(s\)-regular measure \(\mu\), then it can be verified that \(X\) has the finite doubling property. Let \(b \in (0, \frac{1}{s})\) be the constant in the condition (C) and let \(\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k\}\) be the nesting family as in Theorem 6 which we call “generalized dyadic cubes” of \((X, d)\). For convenience, we write \(Q_0 = \{X\}\) and \(Q_k = \{Q_{k,i} : i \in \mathbb{N}_k\}\) for \(k \geq 1\), and \(B_k = \{B(x_{k,i}, c_2^lib^k) : i \in \mathbb{N}_k\}\), where \(x_{k,i} (i \in \mathbb{N}_k)\) are the points in Part (3) of Theorem 6.

**Lemma 1** Let \((X, d)\) be a compact metric space endowed with an Ahlfors \(s\)-regular measure \(\mu\), where \(0 < s < \infty\) is a constant. Then, for any constants \(a_0 > 0\) and \(k \geq 1\), a ball \(B\) of radius \(a_0b^k\) may intersect at most \(\frac{c_1^2(2c'_2 + a_0)^r}{c_2^r}\) elements in \(Q_k\), where \(c_1, c_2\) and \(c'_2\) are the constants given in (1.6) and Theorem 6, respectively.

**Proof** Write \(B = B(x_B, a_0b^k)\) and
\[ A = \left\{ Q_{k,i} \subset Q_k : Q_{k,i} \cap B \neq \emptyset, i \in \mathbb{N}_k \right\}. \]

For \(Q_{k,i} \in A\), from Theorem 6 (3), there exists one point \(x_{k,i} \in Q_{k,i}\) so that (3.1) holds. We denote the collection of such points by \(\Upsilon\), and so \(#A = \#\Upsilon\). Notice that
\[ B \subset \bigcup_{Q_{k,i} \in A} Q_{k,i} \subset B(x_B, (2c'_2 + a_0)b^k), \]
and
\[ \bigcup_{x_{k,i} \in \Upsilon} U(x_{k,i}, c_2b^k) \subset \bigcup_{Q_{k,i} \in A} Q_{k,i}. \]

Therefore
\[ \sum_{x_{k,i} \in \Upsilon} \mu(U(x_{k,i}, c_2b^k)) \leq \mu\left( \bigcup_{Q_{k,i} \in A} Q_{k,i} \right) \leq \mu\left( B(x_B, (2c'_2 + a_0)b^k) \right). \]

Since \(\mu\) is Ahlfors \(s\)-regular, we have
\[ \sum_{x_{k,i} \in \Upsilon} \mu(U(x_{k,i}, c_2b^k)) \geq (#A)(c_2b^k)^s c_1^{-1}, \]
and
\[ \mu(B(x_B, (2c'_2 + a_0)b^k)) \leq c_1(2c'_2 + a_0)^s b^{ks}. \]

Hence \(#A \leq \frac{c_1^2(2c'_2 + a_0)^r}{c_2^r}\). This proves Lemma 1. \(\square\)

### 3.2 Proofs of Theorem 3 and Theorem 4

For \(k \geq 1\), denote by \(\mathcal{J}_k = \{j \geq 1 : \ell_j \in [b^{k-1}, b^{k-2})\}\) and \(n_k = \#\mathcal{J}_k\). Given a constant \(c > \frac{s-a_0}{\log b}\), where \(\rho\) is the constant in Definition 3, let \(\mathcal{J}'_k\) be a maximal collection of \(\mathcal{J}_k\) having mutual distances at least \(ck\). One important property of \(\mathcal{J}'_k\) is that any pair of integers \(n, m \in \mathcal{J}'_k\) are at least of distance \(ck\) from each other. We will use this fact to prove Theorems 3 and 5. See Remark 6 below for more details.

Write \(m_k = \#\mathcal{J}'_k\). Since the sequence \(\{\ell_n\}_{n \geq 1}\) is decreasing, the elements in \(\mathcal{J}_k\) are consecutive to each other. Then \(m_k = \lceil(ck)^{-1}n_k\rceil\), where \(\lceil\cdot\rceil\) stands for the ceiling function.
Lemma 2 Let \((X, d)\) be a compact metric space, and \(G \subset X\) be an analytic set.

(1) If \(\dim_H G > t\), there is a nonempty compact subset \(G^* \subset G\) such that
\[
\dim_H (G^* \cap V) > t
\]
for all open sets \(V \subset X\) with \(G^* \cap V \neq \emptyset\).

(2) If \(\dim_P G > t\), there is a nonempty compact subset \(G_* \subset G\) such that
\[
\dim_P (G_* \cap V) > t
\]
for all open sets \(V \subset X\) with \(G_* \cap V \neq \emptyset\).

The conclusions in Lemma 2 are known. We include a proof for completeness.

Proof Since \(G\) is analytic, from [16], there exists a compact set \(K \subset G\) with \(0 < \mathcal{H}^t(K) < \infty\) for some \(t' > t\). Let \(\{V_i\}_{i \geq 1}\) be a countable basis of \(X\). Denote
\[
V = \{i : \dim_H(K \cap V_i) \leq t\}. \tag{3.3}
\]
Then \(\bigcup_{i \in V} V_i\) is relatively open in \(K\), and hence \(G^* = K \setminus \bigcup_{i \in V} V_i\) is compact. Note that
\[
\mathcal{H}^{t'}(K) = \mathcal{H}^{t'} \left(K \cap \bigcup_{i \in V} V_i\right) + \mathcal{H}^{t'} \left(K \setminus \bigcup_{i \in V} V_i\right) \\
\leq \mathcal{H}^{t'} \left(K \cap \bigcup_{i \in V} V_i\right) + \mathcal{H}^{t'}(G^*) \\
\leq \sum_{i \in V} \mathcal{H}^{t'}(K \cap V_i) + \mathcal{H}^{t'}(G^*).
\]
From (3.3) and \(t' > t\), we get \(\mathcal{H}^{t'}(K \cap V_i) = 0\). Therefore
\[
\mathcal{H}^{t'}(G^*) = \mathcal{H}^{t'}(K) > 0.
\]
In particular, \(G^* \neq \emptyset\). Since \(\{V_i\}\) is a basis of \(X\), for any open set \(V \subset X\) with \(V \cap G^* \neq \emptyset\), we have \(\dim_H(G^* \cap V) > t\).

The statement (2) can be obtained similarly with (1) by applying Corollary 1 of Joyce and Preiss [21]. We omit the details. \(\square\)

The following lemma is directly derived from the definitions of Hausdorff dimension and Hausdorff measure.

Lemma 3 If \(G \subset X\) with \(\dim_H(G) > t\), then there is \(k_0 \geq 1\) such that for \(k \geq k_0\), there are at least \(b^{-kt}\) elements in \(Q_k\) intersecting \(G\), that is
\[
\#\{Q \in Q_k : Q \cap G \neq \emptyset\} \geq b^{-kt}.
\]

For \(k \geq 1\), and \(Q \in Q_k\), denote by \(B_Q(\subset B_k)\) the closed ball containing \(Q\) given in Sect. 3.1. Write \(X_Q\) for the indicator function of the event \(\{\xi_n \in B_Q\text{ for some }n \in I'_k\}\). Then
\[
\mathbb{E}(X_Q) = \mathbb{P}\left(\bigcup_{n \in I'_k} \{\xi_n \in B_Q\}\right).
\]
Remark 6 In the lemma below, we estimate $\text{Cov}(X_Q, X_{Q'})$ for some $Q', Q \in Q_k$. Here we use $J'_k$ instead of $J_k$. Hence for any pair $n, m \in J'_k$, we have $\text{dist}(n, m) \geq ck$ so that we can apply the exponential mixing property of $\{\xi_n\}$ to derive that

$$\sum_{n \in J'_k} \sum_{m \in J'_k, n > m} \rho^{n-m} p(\xi_n \in B_Q) \left( \sum_{n \in J'_k} p(\xi_n \in B_Q) \right)^2 \to 0,$$

as $k \to \infty$, which is crucial in our proofs.

Lemma 4 Suppose $\alpha < s$ and $\epsilon > 0$. There exists a constant $k_0 \geq 1$ such that for $k \geq k_0$, the following statements hold.

(1) If $B_Q \cap B_{Q'} = \emptyset$, where $Q, Q' \in Q_k$, then

$$\text{Cov}(X_Q, X_{Q'}) < \epsilon E(X_Q) E(X_{Q'}).$$

(2) There is a constant $0 < M_0 < \infty$ such that

$$\max_{Q \in Q_k} \# \{ Q' \in Q_k : B_Q \cap B_{Q'} \neq \emptyset \} \leq M_0.$$

Proof Let $Q, Q' \in Q_k$ such that $B_Q \cap B_{Q'} = \emptyset$. Then

$$\text{Cov}(X_Q, X_{Q'}) = E(X_Q X_{Q'}) - E(X_Q) E(X_{Q'}) = \mathbb{P} \left( \bigcup_{n \in J'_k} \bigcup_{m \in J'_k} \{ \xi_n \in B_Q, \xi_m \in B_{Q'} \} \right) - \mathbb{P} \left( \bigcup_{n \in J'_k} \{ \xi_n \in B_Q \} \right) \mathbb{P} \left( \bigcup_{m \in J'_k} \{ \xi_m \in B_{Q'} \} \right).$$

Observe that

$$\mathbb{P} \left( \bigcup_{n \in J'_k} \bigcup_{m \in J'_k} \{ \xi_n \in B_Q, \xi_m \in B_{Q'} \} \right) = \mathbb{P} \left( \bigcup_{n \in J'_k} \{ \xi_n \in B_Q \cap B_{Q'} \} \cup \bigcup_{n \in J'_k} \bigcup_{m \in J'_k, m \neq n} \{ \xi_n \in B_Q, \xi_m \in B_{Q'} \} \right) = \mathbb{P} \left( \bigcup_{n \in J'_k} \bigcup_{m \in J'_k, m \neq n} \{ \xi_n \in B_Q, \xi_m \in B_{Q'} \} \right),$$

where the last equality follows from the fact that $B_Q \cap B_{Q'} = \emptyset$.

Since $\{\xi_j\}_{j \geq 1}$ is an exponentially mixing stationary process, if $n > m$, we obtain

$$\mathbb{P}(\xi_n \in B_Q, \xi_m \in B_{Q'}) \leq \mathbb{P}(\xi_n \in B_Q) \mathbb{P}(\xi_m \in B_{Q'}) + C \rho^{n-m} \mathbb{P}(\xi_n \in B_Q).$$

Whence we get an upper bound for (3.5):

$$\mathbb{P} \left( \bigcup_{n \in J'_k} \bigcup_{m \in J'_k, m \neq n} \{ \xi_n \in B_Q, \xi_m \in B_{Q'} \} \right) \leq \left( \sum_{n \in J'_k} \mathbb{P}(\xi_n \in B_Q) \right) \left( \sum_{m \in J'_k} \mathbb{P}(\xi_m \in B_{Q'}) \right) + \frac{C \rho^{ck}}{1 - \rho^{ck}} \sum_{n \in J'_k} \mathbb{P}(\xi_n \in B_Q),$$

\(\text{Springer}\)
and a lower bound for
\[
\mathbb{P}\left( \bigcup_{n \in \mathcal{I}_k} (\xi_n \in B_Q) \right) \geq \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) - \sum_{n \in \mathcal{I}_k} \sum_{n' \in \mathcal{I}_k, n' \neq n} \mathbb{P}(\xi_n \in B_Q, \xi_{n'} \in B_Q) \\
\geq \left( \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \right) \left( 1 - \sum_{n' \in \mathcal{I}_k} \mathbb{P}(\xi_{n'} \in B_Q) - \frac{C \rho^{ck}}{1 - \rho^{ck}} \right). 
\]  
(3.7)

Since \( \alpha < s \), we can pick \( q \in (\alpha, s) \). It follows from equality (2.4) that there exists \( k_1 \geq 1 \) such that \( n_k \leq b^{-kq} \) for all \( k \geq k_1 \). Hence for \( k \geq k_1 \),
\[
\sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \leq c_1 c_2^s b^{ks} m_k \leq c_1 c_2^s \left( b^{ks} + (ck)^{-1} b^{k(s-q)} \right),
\]
which yields \( \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \to 0 \) as \( k \to \infty \). From (3.7), there is a \( k_2 \geq 1 \) such that for \( k \geq k_2 \),
\[
\mathbb{E}(X_Q) \geq M_1 \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q), \]
(3.8)

where \( 0 < M_1 < 1 \) is a constant. By combining (3.4)–(3.8), we see that
\[
\text{Cov}(X_Q, X_{Q'}) \leq \left( \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \right) \left( \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \right) \left( \sum_{n \in \mathcal{I}_k, n \neq m} \mathbb{P}(\xi_n \in B_Q) \right) \\
+ \sum_{m \in \mathcal{I}_k, m \neq n} \mathbb{P}(\xi_m \in B_{Q'}) \frac{4C \rho^{ck}}{1 - \rho^{ck}} + \frac{2C \rho^{ck}}{1 - \rho^{ck}} \left( \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \right) \\
\leq \left( \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \right) \left( \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \right) \left\{ \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) \right\} \\
+ \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \frac{2C \rho^{ck}}{1 - \rho^{ck}} \left( 2 + \left( \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \right)^{-1} \right) \right\}. 
\]

The inequality above together with (3.8) implies that for \( k \geq k_2 \), we have
\[
\frac{\text{Cov}(X_Q, X_{Q'})}{\mathbb{E}(X_Q) \mathbb{E}(X_{Q'})} \leq \frac{1}{M_1^2} \left\{ \sum_{n \in \mathcal{I}_k} \mathbb{P}(\xi_n \in B_Q) + \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \right\} \\
+ \frac{2C \rho^{ck}}{1 - \rho^{ck}} \left( 2 + \left( \sum_{m \in \mathcal{I}_k} \mathbb{P}(\xi_m \in B_{Q'}) \right)^{-1} \right) \right\}. 
\]  
(3.9)

Since \( c > \frac{s - \alpha}{\log b \rho} \), we derive that the right-hand side of (3.9) tends to 0 as \( k \to \infty \).

Thereby for any \( \epsilon > 0 \), there is a \( k_0 \geq 1 \) satisfying for all \( k \geq k_0 \), if \( Q, Q' \in Q_k \) with \( B_Q \cap B_{Q'} = \emptyset \), we always have
\[
\text{Cov}(X_Q, X_{Q'}) < \epsilon \mathbb{E}(X_Q) \mathbb{E}(X_{Q'}). 
\]
Notice that if the distance $\text{dist}(Q, Q') \geq 2(c_2' - c_2)b^k$ for $Q, Q' \in Q_k$, we have $B_Q \cap B_{Q'} = \emptyset$. Then for $Q \in Q_k$,
\[
\#\{Q' : B_Q \cap B_{Q'} \neq \emptyset\} \leq \#\{Q' : \text{dist}(Q, Q') \leq 2(c_2' - c_2)b^k\}.
\]
From Lemma 1, there exists a constant $0 < M_0 < \infty$ independent of $k$ such that
\[
\max_{Q \in Q_k} \#\{Q' : B_Q \cap B_{Q'} \neq \emptyset\} \leq M_0
\]
holds for $k \geq k_0$.

Now we are ready to prove Theorems 3 and 4.

**Proof of Theorem 3** Firstly we show that $\dim_p(G) < s - \alpha$ implies $\mathbb{P}(E \cap G \neq \emptyset) = 0$. By Tricot [39], it suffices to show that whenever $\dim(G) < s - \alpha$, then $E \cap G = \emptyset$ a.s.

We denote by $C_{\ell_n} = C_{\ell_n}(G)$ a collection of the smallest number of the closed balls with radius $\ell_n$ that cover the set $G$. Let $N_{\ell_n}(G) = \#C_{\ell_n}$. Fixing an arbitrary $\eta > 0$ such that $\eta \in (\dim(G), s - \alpha)$, we have
\[
\limsup_{n \to \infty} \frac{\log N_{\ell_n}(G)}{-\log(\ell_n)} \leq \overline{\dim}(G) < \eta,
\]
so there exists an integer $n_0 \in \mathbb{N}$ such that
\[
N_{\ell_n}(G) < \ell_n^{-\eta} \quad (3.10)
\]
for all $n \geq n_0$. For any ball $B$ in $X$ with radius $\ell_n$, since $\{\xi_n\}_{n \geq 1}$ is a stationary process, we have
\[
\mathbb{P}\{I_n \cap B \neq \emptyset\} = \mathbb{P}(\xi_n \in B(x_B, 2\ell_n)) = \mu(B(x_B, 2\ell_n)) \leq c_1 2^s \ell_n^s,
\]
where $I_n = B(\xi_n, \ell_n)$ and $x_B$ is the center of $B$. Note that the event
\[
\{I_n \cap G \neq \emptyset\} \subset \bigcup_{B \in C_{\ell_n}} \{I_n \cap B \neq \emptyset\}.
\]
We derive from this and (3.10) that
\[
\mathbb{P}\{I_n \cap G \neq \emptyset\} \leq \sum_{B \in C_{\ell_n}} \mathbb{P}\{I_n \cap B \neq \emptyset\} \leq N_{\ell_n}(G)c_1 2^s \ell_n^s < c_1 2^s \ell_n^{-\eta}
\]
for all $n \geq n_0$. Hence the series $\sum_{n=1}^{\infty} \mathbb{P}\{I_n \cap G \neq \emptyset\}$ converges by the definition of $\alpha$ and $\eta < s - \alpha$. By the Borel–Cantelli Lemma, we have
\[
\mathbb{P}\{I_n \cap G \neq \emptyset \text{ i.o.}\} = 0.
\]
That is, $E \cap G = \emptyset$ a.s.

Now we prove that if $\dim_H(G) > s - \alpha$, then $\mathbb{P}(E \cap G \neq \emptyset) = 1$. By Lemma 2, we may assume that $G$ is compact and satisfies $\dim_H(G \cap V) > s - \alpha$ whenever $V \subset X$ is an open set with $G \cap V \neq \emptyset$. We choose constants $\beta$ and $t$ such that $\dim_H(G \cap V) > t > s - \beta > s - \alpha$.

Denote by $U(\xi_n, \ell_n)$ the open ball with center $\xi_n$ and radius $\ell_n$. It suffices to show that
\[
\mathbb{P}\{U(\xi_n, \ell_n) \cap G \neq \emptyset \text{ i.o.}\} = 1. \quad (3.11)
\]
Letting $V$ run over a countable basis of $X$, we have $G \cap \bigcup_{n=1}^{\infty} B(\xi_n, \ell_n)$ is dense a.s. and relatively open in $G$ for any $k \geq 1$. Then from Baire’s category theorem we derive $G \cap \limsup_{n \to \infty} U(\xi_n, \ell_n) \neq \emptyset$ a.s., which implies that $E \cap G \neq \emptyset$ a.s.
Now we prove that the equality (3.11) holds. By (2.4) we have
\[ \alpha = \limsup_{k \to \infty} \frac{\log b^{b^{-1} n_k}}{k} > \beta, \]
then there are infinitely many \( k \) such that \( n_k \geq b^{-k \beta} \). This implies that the set defined as
\[ \mathcal{N} = \{ k \geq 1 : n_k \geq b^{-k \beta} \} \]
satisfies \( \# \mathcal{N} = \infty \).

Fix an open set \( V \) with \( G \cap V \neq \emptyset \). Define
\[ Z_k = \{ Q \in Q_k : Q \cap G \cap V \neq \emptyset \}. \]
From Lemma 3, we have \( N_k = \# Z_k \geq b^{-tk} \) for all \( k \) large enough. For \( Q \in Q_k \), there exists \( B(x_Q, c'_2 b^k) \in B_k \), denoted by \( B_Q \). For \( k \in \mathcal{N} \), define
\[ S_k = \# \{ Q \in Z_k : \xi_n \in B_Q \text{ for some } n \in I_k \}, \]
that is \( S_k = \sum_{Q \in Z_k} X_Q \). For \( \epsilon > 0 \) and \( Q \in Z_k \), write
\[ D_k(Q) = \{ Q' \in Z_k : \text{Cov}(X_Q, X_{Q'}) \geq \epsilon \mathbb{E}(X_Q) \mathbb{E}(X_{Q'}) \}, \]
than from Lemma 4, there is a constant \( 0 < M'_0 < \infty \) such that \( \max_{Q \in Z_k} \# D_k(Q) \leq M'_0 \) for \( k \) large enough. Since
\[ \text{Var}(S_k) = \sum_{Q \in Z_k} \sum_{Q' \in Z_k} \text{Cov}(X_Q, X_{Q'}), \]
and
\[ \text{Cov}(X_Q, X_{Q'}) = \mathbb{E}(X_Q X_{Q'}) - \mathbb{E}(X_Q) \mathbb{E}(X_{Q'}) \leq \mathbb{E}(X_Q), \]
it follows that
\[ \text{Var}(S_k) \leq \sum_{Q \in Z_k} \left( \sum_{Q' \in Z_k \setminus D_k(Q)} \epsilon \mathbb{E}(X_Q) \mathbb{E}(X_{Q'}) + M'_0 \mathbb{E}(X_Q) \right) \]
\[ \leq \epsilon \left( \sum_{Q \in Z_k} \mathbb{E}(X_Q) \right)^2 + M'_0 \sum_{Q \in Z_k} \mathbb{E}(X_Q). \]

From (3.7), we have
\[ \mathbb{E}(S_k) = \sum_{Q \in Z_k} \mathbb{E}(X_Q) \geq \sum_{Q \in Z_k} \left( \sum_{n \in I_k} \mathbb{P}(\xi_n \in B_Q) \right) \left( 1 - \sum_{n \in I_k} \mathbb{P}(\xi_n \in B_Q) - \frac{2C \rho^{ck}}{1 - \rho^{ck}} \right). \]

It yields that there is a constant \( 0 < M_2 < \infty \) such that for all \( k \in \mathcal{N} \) large enough, we have
\[ \mathbb{E}(S_k) \geq M_2 N_k m_k b^{ks} \geq M_2 (ck)^{-1} b^{k(s-t-\beta)}. \]
Recall \( t + \beta > s \), if \( k \in \mathcal{N} \) and \( k \to \infty \), we get \( \mathbb{E}(S_k) \to \infty \).

Combining these and Chebyshev’s inequality, we obtain
\[ \mathbb{P}(S_k = 0) \leq \frac{\text{Var}(S_k)}{\mathbb{E}^2(S_k)} \leq \epsilon + \frac{M'_0}{\mathbb{E}(S_k)}. \]
Therefore
\[
\limsup_{k \to \infty} \mathbb{P}(S_k = 0) = 0.
\]
We observe that
\[
\{ U(\xi_n, \ell_n) \cap G \cap V \neq \emptyset \ \text{i.o.} \} \supset \{ S_k > 0 \ \text{i.o.} \}.
\]
Finally this together with Fatou’s lemma implies that
\[
\mathbb{P}(U(\xi_n, \ell_n) \cap G \cap V \neq \emptyset \ \text{i.o.}) \geq \limsup_{k \to \infty} \mathbb{P}(S_k > 0) = 1.
\]
This finishes the proof of Theorem 3. \( \square \)

For proving Theorem 4, we will make use of the following lemma, which is an analogue of Lemma 3.4 in [24], where \( X = [0, 1]^N \) was considered. Here \( (X, d) \) is an Ahlfors regular metric space.

**Lemma 5** Equip \( X \) with the Borel \( \sigma \)-algebra. Suppose that \( A = A(\omega) \) is a random set in \( X \) (i.e., the indicator function \( \chi_{A(\omega)}(x) \) is jointly measurable) such that for any analytic set \( G \subset X \) with \( \dim_H G > \gamma \), we have
\[
\mathbb{P}(A \cap G \neq \emptyset) = 1.
\]
Then
\[
\dim_H(A \cap G) \geq \dim_H G - \gamma, \quad \text{a.s.}
\]
if \( \dim_H G > \gamma \).

We postpone the proof of Lemma 5 to the end of this section. Let us first prove Theorem 4.

**Proof of Theorem 4** Firstly we prove that \( \dim_H(E \cap G) \leq \dim_p G + \alpha - s \) a.s. when \( \dim_p G + \alpha - s \geq 0. \) It suffices to show that
\[
\dim_H(E \cap G) \leq \dim_B G + \alpha - s \quad \text{a.s.}
\]
Let \( C_{\ell_n} \) be a collection of the closed balls with radius \( \ell_n \) whose union covers \( G \) such that \( \mathcal{N}_{\ell_n}(G) = \# C_{\ell_n} \) is the smallest. Let \( \xi := \dim_B G + \alpha - s \), then for any \( \epsilon > 0 \), we have
\[
\xi + s - \alpha + \epsilon > \dim_B G \geq \limsup_{n \to \infty} \frac{\log \mathcal{N}_{\ell_n}(G)}{-\log(\ell_n)},
\]
which implies that
\[
\mathcal{N}_{\ell_n}(G) < \ell_n^{\alpha - s - \xi - \epsilon}
\]
for all \( n \) large enough. For any \( \delta > 0 \), we choose \( N \geq 1 \) such that for all \( n \geq N \), we have \( \ell_n < \delta \) and inequality (3.12) holds.

Let \( A_n = \{ B \in C_{\ell_n} : B \cap I_n \neq \emptyset \} \).

Notice that
\[
\mathbb{E}(\#A_n) \leq \sum_{B \in A_n} \mathbb{P}(I_n \cap B \neq \emptyset) \leq c_1 2^\xi \ell_n^{\alpha - s - \xi - \epsilon},
\]
and

\[ E \cap G \subset \bigcup_{n=N}^{\infty} I_n \cap G \subset \bigcup_{n=N}^{\infty} B. \]

Hence for any \( \theta > \xi \), by choosing \( \epsilon > 0 \) with \( 2\epsilon < \theta - \xi \), we have

\[ \mathbb{E}(\mathcal{H}_\theta^\theta(E \cap G)) \leq \sum_{n=n_1}^{\infty} \mathbb{E}(\#A_n)(2\ell_n)^\theta \leq \sum_{n=n_1}^{\infty} c_1 2^{\theta+s} \ell_n^{\alpha-s}. \]

By the definition of \( \alpha \), we obtain that

\[ \mathbb{E}(\mathcal{H}_\theta^\theta(E \cap G)) < \infty. \]

Therefore \( \mathcal{H}_\theta^\theta(E \cap G) < \infty \) a.s. which implies \( \dim_H(E \cap G) \leq \theta \) a.s. Hence \( \dim_H(G \cap E) \leq \dim_p G + \alpha - s \) a.s.

When \( \dim_p G + \alpha - s < 0 \), by Theorem 3, \( \mathbb{P}(E \cap G = \emptyset) = 1 \). Hence \( \dim_H(E \cap G) = -\infty \) a.s.

The lower bound of \( \dim_H(E \cap G) \) in Theorem 4 follows from Lemma 5. This finishes the proof of Theorem 4.

It remains to prove Lemma 5. The fractal percolation \( \Gamma_i \) used in Lemma 5 can be constructed as follows.

Let \( 0 < p < 1 \). For each \( Q \in \mathcal{Q} \), let \( Z(Q) \) be a random variable taking value 1 with probability \( p \) and value 0 with probability \( 1 - p \). We assume that these random variables are independent for different \( Q \in \mathcal{Q} \). We define the random fractal percolation set as

\[ \Gamma(p) = \bigcap_{n \in \mathbb{N}} \bigcup_{Q \in \mathcal{Q}_n, Z(Q) = 1} \overline{Q}. \]

where \( \overline{Q} \) is the closure of the set \( Q \). When \( p = 2^{-t} < 1 \), for convenience, we denote \( \Gamma_t = \Gamma(2^{-t}) \).

Notice that there is a nature tree structure behind the definition of \( \mathcal{Q} \), which we describe now.

Label each \( Q \in \mathcal{Q} \) with a vertex \( v_Q \) and let \( T \) be a graph with vertex set \( \{v_Q\}_{Q \in \mathcal{Q}} \). There exists an edge between vertices \( v_{Q_{n,i}} \) and \( v_{Q_{m,j}} \) if and only if \( |n - m| = 1 \) and \( Q_{n,i} \cap Q_{m,j} \neq \emptyset \). Then \( T \) is a tree with \( v_X \) as its root. The boundary of the tree \( \partial T \) consists of all infinite paths \( (v_{0,i_0}v_{1,i_1}v_{2,i_2} \ldots) \), where \( v_{m,i_m} = v_{Q_{m,i_m}} \) for \( Q_{m,i_m} \in \mathcal{Q}_m \) and \( Q_{m,i_m} \subset Q_{n,i_n} \), if \( m \geq n \). We call these infinite paths rays. Then we can define a projection \( \Pi: \partial T \to X \) as

\[ \Pi: (v_{0,i_0}v_{1,i_1}v_{2,i_2} \ldots) \mapsto \bigcap_{n=0}^{\infty} \overline{Q_{n,i_n}}. \quad (3.13) \]

Note that \( \Pi(\partial T) = X \). For \( v = (v_{0,i_0}v_{1,i_1}v_{2,i_2} \ldots) \), \( u = (u_{0,i_0}u_{1,i_1}u_{2,i_2} \ldots) \in \partial T \), we define

\[ \kappa(v, u) = \begin{cases} 0 & \text{if } v = u, \\ \min\{j: v_{j,i_j} \neq u_{j,i_j}\} & \text{if } u \neq v. \end{cases} \]

Then \( (\partial T, \kappa) \) is a metric space. We claim that for every \( G \subset X \), we have

\[ \dim_H G = \dim_H^\kappa(\Pi^{-1} G), \quad (3.14) \]
where \( \text{dim}_H^t \) is the Hausdorff dimension in the metric space \((\partial T, \kappa)\).

Now we prove this claim. For any \( x, y \in X \), let \( n \) be the maximal integer with \( x, y \in Q_n \in Q_n \). By Theorem 6, we have \( \text{diam} (Q_n) \leq 2c_2^t b^n \). Then we get

\[
d(x, y) \leq \text{diam} (Q_n) \leq 2c_2^t b^n = 2c_2^t b^{-1} \kappa (\Pi^{-1} x, \Pi^{-1} y),
\]

which derives \( \text{dim}_H G \leq \text{dim}_H^t (\Pi^{-1} G) \).

Further, let \( t > \text{dim}_H G \). For \( \epsilon > 0 \), there exists a \( \delta \)-covering \( \{U_i\}_{i \geq 1} \) of \( G \) which satisfies \( \sum_{i \geq 1} (\text{diam } U_i)^t < \epsilon \). For \( i \geq 1 \), let \( n(i) \) be the integer with

\[
2c_2^t b^{n(i)} \leq \text{diam} (U_i) < 2c_2^t b^{n(i)-1}.
\]

Write

\[
\mathcal{A}(U_i) = \{ Q \in Q_n(i) : Q \cap U_i \neq \emptyset \}.
\]

Then using (3.15), there is a constant \( 0 < M_3 < \infty \) such that for all \( i \geq 1 \), \#\( \mathcal{A}(U_i) \leq M_3 \).

Note that

\[
\Pi^{-1} G \subset \Pi^{-1} \left( \bigcup_{i \geq 1} U_i \right) \subset \Pi^{-1} \left( \bigcup_{i \geq 1} Q \in \mathcal{A}(U_i) \right).
\]

Hence

\[
\mathcal{H}_\delta^t (\Pi^{-1} G) \leq \sum_{i \geq 1} \sum_{Q \in \mathcal{A}(U_i)} \kappa \left( (\Pi^{-1} Q)^t \right) \leq \frac{M_3 b^t}{2c_2^t} \sum_{i \geq 1} (\text{diam } U_i)^t < \epsilon.
\]

The second inequality in (3.16) holds due to (3.15). Then we have \( \mathcal{H}_\delta^t (\Pi^{-1} G) = 0 \) which derives that \( t \geq \text{dim}_H^t (\Pi^{-1} G) \). Therefore \( \text{dim}_H G \geq \text{dim}_H^t (\Pi^{-1} G) \). This proves (3.14).

For \( 0 < p < 1 \), let \( T \) be the tree defined above. Percolation at level \( p \) on \( T \) is obtained by removing each edge of \( T \) with probability \( 1 - p \) and retaining it with probability \( p \), with mutual independence among edges. The random graph connecting the root which is left will be denoted by \( \tilde{T}(p) \). Then the law of \( \Gamma(p) \) given by \( \mathbb{P} \) is the same as that of \( \Pi(\tilde{T}(p)) \). By combining this with (3.14), we obtain the following analogue of the result of [30, p. 957] in our metric space setting. See also Lemma 5.1 in [33] for the case of \( X = [0, 1]^N \).

**Lemma 6** [30] Let \( p = 2^{-t} < 1 \). For any analytic set \( G \subset X \), the following statements hold.

1. If \( \text{dim}_H(G) < t \), then \( \Gamma_t \cap G = \emptyset \) almost surely.
2. If \( \text{dim}_H(G) > t \), then \( \Gamma_t \cap G \neq \emptyset \) with positive probability.
3. If \( \text{dim}_H(G) > t \), then \( \| \text{dim}_H(G \cap \Gamma_t) \|_\infty = \text{dim}_H G - t \), where the \( L^\infty \) norm is the essential supremum in the underlying probability space.

We end this section with a proof of Lemma 5, by extending the method in [24] to the Ahlfors regular metric spaces.

**Proof of Lemma 5** For \( t < \text{dim}_H(G) - \gamma \), let \( \Gamma_t \) be a fractal percolation at level \( 2^{-t} \) in \( X \) [30,33], which is independent of \( A \), and \( \text{dim}_H(\Gamma_t) = s - t \) a.s. By Lemma 6, we have \( \mathbb{P}(\Gamma_t \cap G \neq \emptyset) > 0 \) whenever \( \text{dim}_H(G) > t \), whereas \( \mathbb{P}(\Gamma_t \cap G \neq \emptyset) = 0 \) with \( \text{dim}_H(G) < t \), and

\[
\| \text{dim}_H(G \cap \Gamma_t) \|_\infty = \text{dim}_H(G) - t.
\]
Let \( \hat{\Gamma}_t \) be a union of countably many independent and identically distributed copies of \( \Gamma_t \). The Borel-Cantelli lemma implies that

\[
P(\hat{\Gamma}_t \cap G \neq \emptyset) = \begin{cases} 
0 & \text{if } \dim_H(G) < t, \\
1 & \text{if } \dim_H(G) > t.
\end{cases}
\] (3.17)

Also we have

\[
\dim_H(\hat{\Gamma}_t \cap G) = \dim_H(G) - t > \gamma, \quad \text{a.s.}
\]

By the condition given in the lemma, we have \( A \cap G \cap \hat{\Gamma}_t \neq \emptyset \) a.s. in the product space. Using (3.17), we get \( \dim_H(A \cap G) \geq t \) a.s. Letting \( t \) tend to \( \dim_H G - \gamma \) along rational numbers, we complete our proof. \( \square \)

4 Limsup random fractals in metric space

In order to prove Theorem 5, we first extend the results on hitting probabilities of limsup random fractals in [24] to metric spaces.

Let \( Q = \{ Q_n \}_{n \geq 0} \) be the collection of generalized dyadic cubes in \((X, d)\) given in Sect. 3.1. To make sure the boundaries of sets in \( Q_n \) are covered, we will use \( 2Q \), \( Q \in Q_n \) instead of generalized dyadic cubes in this section. However, we still denote them by \( Q \) and \( Q_n \) respectively for simplicity of notation. For each \( n \geq 1 \), let \( \{ Z_n(Q), Q \in Q_n \} \) be a collection of random variables, each taking values in \( \{0, 1\} \).

Let

\[
A(n) = \bigcup_{Q \in Q_n, Z_n(Q) = 1} Q^o,
\]

where \( Q^o \) is the interior of \( Q \). The random set

\[
A = \limsup_{n \to \infty} A(n)
\]

is called a limsup random fractal associated to \( \{ Z_n(Q), n \geq 1, Q \in Q_n \} \).

We assume the following conditions (H1)–(H2), where (H1) is more general than Condition 4 in [24]. We allow the probability \( P_n(Q) \) to depend not only on the level \( n \), but also on the cubes \( Q \in Q_n \).

(H1) Suppose that for every \( n \geq 1 \), and \( Q \in Q_n \) the probability \( P_n(Q) := P(Z_n(Q) = 1) \) satisfies

\[
\lim_{n \to \infty} \frac{\min_{Q \in Q_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_1,
\]

and

\[
\lim_{n \to \infty} \frac{\max_{Q \in Q_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_2,
\]

where \( \gamma_1, \gamma_2 > 0 \) are constants.

Remark 7 From the proofs of Theorems 7, 8 and Corollary 3, we see that they still hold if the condition (H1) is replaced by the weaker condition (H1′):
(H1') For some constants $\gamma_1, \gamma_2 > 0$,

$$\limsup_{n \to \infty} \min_{Q \in Q_n} \frac{\log_b^{-1} P_n(Q)}{n} = -\gamma_1,$$

$$\limsup_{n \to \infty} \max_{Q \in Q_n} \frac{\log_b^{-1} P_n(Q)}{n} = -\gamma_2,$$

and there exists an increasing sequence of positive integers $\{n_i\}$ with $\lim_{i \to \infty} \frac{n_{i+1}}{n_i} = 1$ such that

$$\lim_{i \to \infty} \min_{Q \in Q_{n_i}} \frac{\log_b^{-1} P_{n_i}(Q)}{n_i} = -\gamma_1,$$

and

$$\lim_{i \to \infty} \max_{Q \in Q_{n_i}} \frac{\log_b^{-1} P_{n_i}(Q)}{n_i} = -\gamma_2.$$

The next condition is concerned with the strength of dependence among the random variables $\{Z_n(Q), n \geq 1, Q \in Q_n\}$. It is a slight modification of Condition 5 in [24].

(H2) A bound on correlation length: for any $\epsilon > 0$, define

$$f(n, \epsilon) = \max_{Q \in Q_n} \#\{Q' \in Q_n: \text{Cov}(Z_n(Q), Z_n(Q')) \geq \epsilon P_n(Q)P_n(Q')\}.$$

Suppose that there is a constant $\delta \geq 0$ such that for all $\epsilon > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log_b^{-1} f(n, \epsilon) \leq \delta. \quad (*)$$

The following theorem characterizes the hitting probability of the limsup random set $A$. It extends Theorem 3.1 in [24].

**Theorem 7** Assume that $A = \limsup_{n \to \infty} A(n)$ is a limsup random set that satisfies the conditions (H1) and (H2). Then for any analytic set $G \subset X$,

$$\mathbb{P}(A \cap G \neq \emptyset) = \begin{cases} 0 & \text{if } \dim_G(G) < \gamma_2, \\ 1 & \text{if } \dim_G(G) > \gamma_1 + \delta. \end{cases}$$

For proving Theorem 7, we will use the following lemma on upper box dimension. From [32], for any $r > 0$ and any bounded set $G \subset X$, let $N_r(G)$ be the smallest number of closed balls with radius $r$ covering $G$. Then

$$\overline{\text{dim}}_B(G) = \limsup_{r \to 0} \frac{\log N_r(G)}{-\log r}.$$

From it, we have the following lemma whose proof is standard and thus omitted.

**Lemma 7** Let $\{k_i\}_{i \geq 1}$ be an increasing sequence of positive integers satisfying (2.2). Then for any bounded set $G \subset X$,

$$\overline{\text{dim}}_B(G) = \limsup_{i \to \infty} \frac{\log N_{a_1 r^{k_i}}(G)}{-\log(a_1 r^{k_i})}, \quad (4.1)$$

where $a_1 > 0$ is a constant.
Remark 8 For any bounded set $G \subset X$, consider $A' = \{ Q \in \mathbb{Q}_n : Q \cap G \neq \emptyset \}$, then $\#A' \geq N_{2c' b^n}(G)$ which derives from

$$G \subset \bigcup_{Q \in A'} Q \subset \bigcup_{x_Q \in Q \in A'} B(x_Q, 2c' b^n),$$

where $x_Q \in Q$.

Proof of Theorem 7 The proof is a modification of that of Theorem 3.1 in [24]. We include it for the sake of completeness. Firstly, we show that $\dim P(G) < \gamma_2$ implies that $A \cap G = \emptyset$, a.s. It suffices to show that whenever $\overline{\dim B} G < \gamma_2$, then $A \cap G = \emptyset$ a.s. Fix an arbitrary but small $\eta > 0$ so that $\overline{\dim B}(G) < \gamma_2 - \eta$.

We denote by $C_{b^n} = C_{b^n}(G)$ a collection of the smallest number of the closed balls with radius $b^n$ that cover the set $G$. Let $N_{b^n}(G) = \#C_{b^n}$. For any $\theta \in (\overline{\dim B}(G), \gamma_2 - \eta)$, by Lemma 7, we have

$$\limsup_{n \to \infty} \frac{\log N_{b^n}(G)}{-\log(b^n)} = \overline{\dim B}(G) < \theta,$$

then there exists an integer $n(\theta) \geq 1$ such that

$$N_{b^n}(G) < b^{-n\theta} \quad (4.2)$$

for all $n \geq n(\theta)$. For any $W \in C_{b^n}$, denote

$$A(W) = \{ Q \in \mathbb{Q}_n : Q \cap W \neq \emptyset \}.$$

Then we have

$$G \subset \bigcup_{W \in C_{b^n}} \bigcup_{Q \in A(W)} Q.$$

In the meanwhile, by Lemma 1, there is a constant $0 < M < \infty$, which does not depend on $n$, such that for all $W \in C_{b^n}$, we have $\#A(W) \leq M$.

On the other hand, by condition (H1), for any $\eta > 0$, there exists $n(\eta)$ such that for all $n \geq n(\eta)$,

$$\max_{Q \in \mathbb{Q}_n} P_n(Q) \leq b^{n(\gamma_2 - \eta - \theta)} \quad (4.3)$$

It follows from (4.2) and (4.3) that for any $n \geq \max\{n(\theta), n(\eta)\}$,

$$\mathbb{P}(G \cap A(n) \neq \emptyset) \leq \mathbb{P}\left( \bigcup_{W \in C_{b^n}} \bigcup_{Q \in A(W)} Q \cap A(n) \neq \emptyset \right) \leq Mb^{-n\theta} \max_{Q \in \mathbb{Q}_n} \mathbb{P}(Q \cap A(n) \neq \emptyset) \leq Mb^{-n\theta} \max_{Q \in \mathbb{Q}_n} P_n(Q) \leq Mb^{n(\gamma_2 - \eta - \theta)}.$$
Fix an open set $V$ such that $V \cap G_\ast \neq \emptyset$. Denote

$$\tilde{A}_n = \{ Q \in \mathcal{Q}_n : Q^c \cap V \cap G_\ast \neq \emptyset \}.$$ 

Let $\mathcal{N}_n$ be the total number of $\tilde{A}_n$. Then

$$G_\ast \cap V \subseteq \bigcup_{Q \in \tilde{A}_n} Q \subseteq \bigcup_{Q \in \tilde{A}_n} B(x_Q, 2c_2' b^n),$$

where $x_Q \in Q$. Using Lemma 7, we have

$$\limsup_{n \to \infty} \frac{\log N_{2c_2' b^n} (V \cap G_\ast)}{- \log (2c_2' b^n)} = \overline{\dim}(V \cap G_\ast) > \gamma_1 + \delta.$$

For any $\eta \in (\gamma_1 + \delta, \overline{\dim}(V \cap G_\ast))$, we derive that

$$N_{2c_2' b^n} (V \cap G_\ast) \geq (2c_2')^{-\eta} b^{-\eta n}$$

holds for infinitely many $n$. Hence $\mathcal{N}_n \geq (2c_2')^{-\eta} b^{-\eta n}$ for infinitely many $n$. This implies the set

$$\mathcal{M} := \{ i \geq 1 : \mathcal{N}_{ni} \geq (2c_2')^{-\eta} b^{-\eta ni} \}$$

satisfies $\#\mathcal{M} = \infty$. We define

$$S_i := \sum_{Q \in \tilde{A}_{ni}} Z_{ni} (Q),$$

where $S_i$ is the total number of sets $Q \in \mathcal{Q}_{ni}$ with $Q \cap V \cap G_\ast \cap A(ni) \neq \emptyset$. Observe that

$$\{ A(n) \cap G_\ast \cap V \neq \emptyset \ i.o. \} \supseteq \{ S_i > 0 \ i.o. \}. $$

We need only show that $\mathbb{P}(S_i > 0 \ i.o. ) = 1$. Firstly, we estimate

$$\text{Var}(S_i) = \sum_{Q \in \tilde{A}_{ni}} \sum_{Q' \in \tilde{A}_{ni}} \text{Cov}(Z_{ni} (Q), Z_{ni} (Q')).$$

Fix $\epsilon > 0$ and for each $Q \in \mathcal{Q}_{ni}$, let $\mathcal{G}_{ni} (Q)$ denote the collection of all $Q' \in \mathcal{Q}_{ni}$ such that

1. $Q' \cap V \cap G_\ast \neq \emptyset$;
2. $\text{Cov}(Z_{ni} (Q), Z_{ni} (Q')) \leq \epsilon P_{ni} (Q) P_{ni} (Q')$.

That is

$$\mathcal{G}_{ni} (Q) = \{ Q' \in \tilde{A}_{ni} : \text{Cov}(Z_{ni} (Q), Z_{ni} (Q')) \leq \epsilon P_{ni} (Q) P_{ni} (Q') \}.$$ 

If $Q' \in \mathcal{Q}_{ni}$ satisfies (1) but not (2), then we say $\mathcal{B}_{ni} (Q)$,

$$\mathcal{B}_{ni} (Q) = \{ Q' \in \tilde{A}_{ni} : \text{Cov}(Z_{ni} (Q), Z_{ni} (Q')) > \epsilon P_{ni} (Q) P_{ni} (Q') \}.$$ 

Then

$$\text{Var}(S_i) \leq \epsilon \left( \sum_{Q \in \tilde{A}_{ni}} P_{ni} (Q) \right)^2 + \left( \max_{Q \in \mathcal{Q}_{ni}} \# \mathcal{B}_{ni} (Q) \right) \left( \sum_{Q \in \tilde{A}_{ni}} P_{ni} (Q) \right).$$  

(4.5)

The last term follows from the fact that $\text{Cov}(Z_n (Q), Z_n (Q')) \leq \mathbb{E}(Z_n (Q)) = P_n (Q)$. Recalling the notation of (H2), we have

$$\text{Var}(S_i) \leq \epsilon \left( \sum_{Q \in \tilde{A}_{ni}} P_{ni} (Q) \right)^2 + f(n_i, \epsilon) \left( \sum_{Q \in \tilde{A}_{ni}} P_{ni} (Q) \right).$$

$\square$ Springer
Combining this with the Paley–Zygmund inequality, we obtain
\[ P(S_i \geq 0) \geq \frac{1}{1 + \frac{\text{Var}(S_i)}{E(S_i)^2}} \]
(4.6)
since \( E(S_i) = \sum_{Q \in \tilde{A}_n} P_{n_i}(Q) \). By the conditions (H1) and (H2), for any \( \theta > 0 \) with \( 2\theta < \eta - \delta - \gamma_1 \), there exists \( N \) such that for all \( n \geq N \), we have
\[ f(n, \epsilon) \leq b^{- (\delta + \theta)n} \text{ and } \min_{Q \in \tilde{Q}_n} P_n(Q) \geq b^{(\gamma_1 + \theta)n}. \]
Thus from (4.4) and arbitrariness of \( \theta \), we have
\[ \limsup_{i \in \mathbb{N}} f(n_i, \epsilon) \leq \limsup_{i \in \mathbb{N}} (2c'_2)^{\epsilon b^{-n_i(2\theta + \delta + \gamma_1 - \eta)}} = 0. \]
(4.7)
By inequalities (4.6), (4.7), and Fatou’s lemma, we derive that
\[ P(S_i > 0 \text{ i.o.}) \geq \lim sup_{i \in \mathbb{N}} P(S_i > 0) = 1. \]
Define the open set \( B(n) := \bigcup_{k=n}^\infty A(k) \) for \( n \geq 1 \). It follows that
\[ P(B(n) \cap G \neq \emptyset, \forall n \geq 1) = 1 \]
for every open set \( V \) with \( G \cap V \neq \emptyset \). Since compact metric spaces are separated, then there exists a countable basis for open sets of \((X, d)\). Letting \( V \) run over the countable basis, we obtain that for all \( n \geq 1 \), the set \( B(n) \cap G \) is a.s. dense in \( G \). Since \( G \) is a complete metric space, by Baire’s category theorem, we have \( \bigcap_{n=1}^\infty B(n) \cap G \) is a.s. dense in \( G \). In particular, \( A \cap G \neq \emptyset \) with probability one.

For \( n \geq 1 \), let \( B_n = \{B(x_{n,i}, 2c'_2 b^n) : i \in \mathbb{N}_n\} \) where \( x_{n,i}, c'_2 \) are given in Sect. 3.1, and \( B = \{B_n\}_{n \geq 0} \). For each \( n \geq 1 \), let \( \{Z_n(B), B \in B_n\} \) be a collection of random variables, each taking values in \([0, 1]\). Let
\[ F(n) = \bigcup_{B \in B_n, Z_n(B) = 1} B^o. \]
Define the random set
\[ F = \limsup_{n \to \infty} F(n). \]

**Theorem 8** Assume that \( F \) satisfies the corresponding conditions (H1) and (H2). Then for any analytic set \( G \subset X \), if \( \dim_p(G) > \gamma_1 + \delta \), we have
\[ P(F \cap G \neq \emptyset) = 1. \]

**Proof** By replacing the generalized dyadic cubes \( Q \) by balls in \( B \), we obtain the theorem directly from the proof of Theorem 7. □
Corollary 3 Suppose A is a discrete limsup random fractal satisfying conditions (H1) and (H2) with δ = 0. Then for any analytic set G ⊂ X, with probability one,
\[
\dim_H(A \cap G) \begin{cases} 
\leq \dim_p(G) - \gamma_2 & \text{if } \dim_p(G) \geq \gamma_2, \\
= -\infty & \text{if } \dim_p(G) < \gamma_2, \\
\geq \dim_H(G) - \gamma_1 & \text{if } \dim_p(G) > \gamma_1.
\end{cases}
\]
In particular, if \(\gamma_1 = \gamma_2 < s\), then \(\dim_H(A) = s - \gamma_1\), a.s.

Proof It suffices to prove \(\dim_H(A \cap G) \leq \overline{\dim}_B G - \gamma_2\) a.s., if \(\dim_p(G) \geq \gamma_2\). Let \(C_{p^n}\) and \(A(W)\), \(W \in C_{p^n}\) be the same as described in the proof of Theorem 7. Define
\[
S_n = \sum_{W \in C_{p^n}} \sum_{Q \in A(W)} Z_n(Q).
\]
Let \(\xi = \overline{\dim}_B G - \gamma_2\), then for \(n\) large enough, we have
\[
\mathbb{E}(S_n) = \sum_{W \in C_{p^n}} \sum_{Q \in A(W)} P_n(Q) \leq Mb^{-n(\xi + 2\epsilon)},
\]
where \(0 < M < \infty\) is a constant independent of \(n\). Hence from the arbitrariness of \(\epsilon\), it follows that for any \(\theta > \xi\), \(\mathbb{E}(\sum_n S_n b^{n\theta}) < \infty\).

Hence
\[
\mathcal{H}^\theta(A \cap G) \leq \sum_{n=0}^{\infty} S_n b^{n\theta} < \infty \text{ a.s.}
\]
which derives that \(\dim_H(A \cap G) \leq \dim_p G - \gamma_2\) a.s.

Finally, the lower bound of \(\dim_H(A \cap G)\) follows from Lemma 5. The proof is complete.

\[\square\]

5 Proof of Theorem 5

Proof of Theorem 5 The proof is divided into three parts. In order to show that \(\dim_p(G) > s - \alpha\) implies \(\mathbb{P}(E \cap G \neq \emptyset) = 1\), we will use the hitting probability of limsup random fractals in Sect. 4. This is done in Parts (i) and (ii). Part (iii) determines the packing dimension of \(E \cap G\).

(i) Construction of a limsup random fractal \(E_\bullet \subset E\).

Firstly, we recall some notations from Sect. 4. For any \(k \geq 2\), \(B_k = \{B(x_k, i, 2\epsilon_2 b^k) : i \in \mathbb{N}_k \subset \mathbb{N}\}, \mathcal{J}_k = \{n \geq 1 : \ell_n \in [b^{k-1}, b^{k-2})\}, n_k = \#\mathcal{J}_k\) and \(\mathcal{J}'_k\) is a maximal collection of points in \(\mathcal{J}_k\) having mutual distances at least \(ck\), where \(c\) is a given constant with \(c > \frac{a - s}{\log_{b-1} \rho}\), and \(p\) appears in Definition 3. In this way, any pair of integers \(n, m \in \mathcal{J}'_k\) are at least of distance \(ck\) from each other. Also, recall that \(m_k = \#\mathcal{J}'_k = \lceil (ck)^{-1} n_k \rceil\).

For every \(J \in B_k\), define
\[
Z_k(J) = \begin{cases} 
1 & \text{if } \exists n \in \mathcal{J}'_k \text{ such that } J \subset I_n = B(\xi_n, \ell_n), \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(A(k)\) be the union of the interiors of sets in \(B_k\) that are contained in some \(I_n\) in \(\mathcal{J}'_k\) with length \(\ell_n \in [b^{k-1}, b^{k-2})\), that is
\[
A(k) = \bigcup_{J \in B_k, Z_k(J) = 1} J^\circ.
\]

\[\square\] Springer
We observe that
\[ A(k) \subset \bigcup_{n \in \mathcal{J}_k} I_n. \]

Define \( E_* := \limsup_{k \to \infty} A(k) \). From the above, we have \( E_* \subset E \).

(ii) Hitting probability of \( E_* \).
Now let \( G \subset X \) be an analytic set such that \( \dim \mathcal{P}(G) > s - \alpha \). We show \( \mathbb{P}(E_* \cap G \neq \emptyset) = 1 \).

For every \( J \in B_k \), the probability
\[ \mathbb{P}(Z_k(J) = 1) = \mathbb{P}\{ \exists n \in \mathcal{J}_k' \text{ such that } J \subset I_n \}. \]

Denote the above probability by \( P_k(J) \).
Write \( J = B(x_J, 2c_2 b^k) \). By using the stationarity on \( \{\xi_n\} \), we derive that,
\[
P_k(J) \leq \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) = \sum_{n \in \mathcal{J}_k'} \mathbb{P}(\xi_n \in B(x_J, \ell_n - 2c_2 b^k)) \]
\[
\leq c_1 \sum_{n \in \mathcal{J}_k'} (\ell_n - 2c_2 b^k)^s \leq c_3 m_k b^{ks},
\]

where \( c_3 = c_1\left(\frac{1}{1 - 2c_2^s}\right) \) is a constant.

On the other hand,
\[
P_k(J) \geq \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) - \sum_{n \in \mathcal{J}_k'} \sum_{m \in \mathcal{J}_k', \ m \neq n} \mathbb{P}(J \subset I_n, J \subset I_m). \tag{5.2}
\]

Since \( \{\xi_n\}_{n \geq 1} \) is stationary and exponentially mixing, if \( n > m \), we have
\[
\mathbb{P}(J \subset I_n, J \subset I_m) \leq \mathbb{P}(J \subset I_m)\mathbb{P}(J \subset I_n) + C \rho^{n-m}\mathbb{P}(J \subset I_n),
\]
where \( C, \rho \) are constants. We notice that \( n, m \in \mathcal{J}_k' \) derives \( n - m \geq ck \), then
\[
\sum_{n \in \mathcal{J}_k'} \sum_{m \in \mathcal{J}_k', \ m \neq n} \mathbb{P}(J \subset I_n, J \subset I_m)
\]
\[
\leq \left( \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) \right) \left( \sum_{m \in \mathcal{J}_k', \ m \neq n} \mathbb{P}(J \subset I_m) \right) + 2C \sum_{n \in \mathcal{J}_k'} \sum_{m \in \mathcal{J}_k', \ m < n} \rho^{n-m}\mathbb{P}(J \subset I_n) \tag{5.3}
\]
\[
\leq \left( \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) \right) \left( \sum_{m \in \mathcal{J}_k', \ m \neq n} \mathbb{P}(J \subset I_m) \right) + 2C \rho^{ck} \frac{1}{1 - \rho^{ck}} \left( \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) \right).
\]

It follows from (5.2) and (5.3) that
\[
P_k(J) \geq \left( \sum_{n \in \mathcal{J}_k'} \mathbb{P}(J \subset I_n) \right) \left( 1 - \sum_{m \in \mathcal{J}_k', \ m \neq n} \mathbb{P}(J \subset I_m) - \frac{2C \rho^{ck}}{1 - \rho^{ck}} \right) \tag{5.4}
\]
\[
\geq c_4 m_k b^{ks} \left( 1 - c_3 m_k b^{ks} - \frac{2C \rho^{ck}}{1 - \rho^{ck}} \right).
\]
where $c_4 = c_1^{-1} \left( \frac{1}{b} - 2c_2 r \right)$ is a constant. Combining (5.1) and (5.4), together with the condition (C), we derive that

$$\limsup_{k \to \infty} \frac{\max_{j \in Q_k} \log_b P_k(j)}{k} = \limsup_{k \to \infty} \frac{\min_{j \in Q_k} \log_b P_k(j)}{k} = \alpha - s,$$

and there is an increasing sequence of integers $\{k_i\}$ that satisfies (2.2) such that

$$\lim_{i \to \infty} \frac{\max_{j \in Q_{k_i}} \log_b P_{k_i}(j)}{k_i} = \lim_{i \to \infty} \frac{\min_{j \in Q_{k_i}} \log_b P_{k_i}(j)}{k_i} = \alpha - s.$$  (5.6)

Next we verify that there is a bound on correlation length.

First we estimate $\text{Cov}(Z_k(J_1)Z_k(J_2))$, where $J_1, J_2 \in B_k$ with $d(J_1, J_2) \geq b^{k-3}$. From (5.3), we have

$$\mathbb{E}(Z_k(J_1)Z_k(J_2)) = \mathbb{P}(Z_k(J_1) = 1, Z_k(J_2) = 1)
= \mathbb{P}\{\exists \ m, n \in \mathcal{J}_k \text{ such that } J_1 \subset I_n \text{ and } J_2 \subset I_m\}
\leq \sum_{n \in \mathcal{J}_k} \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n, J_2 \subset I_m)
\leq \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m) + \frac{2C\rho^{ck}}{1 - \rho^{ck}} \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n).$$  (5.7)

Since

$$\text{Cov}(Z_k(J_1), Z_k(J_2)) = \mathbb{E}(Z_k(J_1)Z_k(J_2)) - \mathbb{E}(Z_k(J_1))\mathbb{E}(Z_k(J_2))
= \mathbb{E}(Z_k(J_1)Z_k(J_2)) - P_k(J_1)P_k(J_2),$$  (5.8)

from the first inequality in (5.4) and (5.7), we get

$$\text{Cov}(Z_k(J_1), Z_k(J_2)) \leq \left( \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) \right) \left( \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m) \right) \left( \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) + \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m) + \frac{4C\rho^{ck}}{1 - \rho^{ck}} \right)
+ \frac{2C\rho^{ck}}{1 - \rho^{ck}} \left( \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) \right)
\leq \left( \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) \right) \left( \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m) \right) \left( \sum_{n \in \mathcal{J}_k} \mathbb{P}(J_1 \subset I_n) + \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m) + \frac{4C\rho^{ck}}{1 - \rho^{ck}} \right)
+ \frac{2C\rho^{ck}}{1 - \rho^{ck}} \sum_{m \in \mathcal{J}_k} \mathbb{P}(J_2 \subset I_m).$$  (5.9)

We notice that from (5.4) there exists a constant $0 < M_4 < \infty$ such that

$$\mathbb{E}(Z_k(J)) = P_k(J) \geq M_4 \sum_{n \in \mathcal{J}_k} \mathbb{P}(J \subset I_n)$$
holds for all $J \in \mathcal{B}_k$ and all large enough $k$. Then the inequality (5.9) reads as follows

$$\text{Cov}(Z_k(J_1), Z_k(J_2)) \leq \frac{1}{M_4} \mathbb{E}(Z_k(J_1)) \mathbb{E}(Z_k(J_2)) \left( 2c_3 m_k b^{k,s} + \frac{4C \rho^{c_k}}{1 - \rho^{c_k}} + \frac{2C \rho^{c_k}}{c_4(1 - \rho^{c_k}) m_k b^{k,s}} \right).$$

Here choosing $k_i$ in (5.6), then $\lim_{i \to \infty} m_i b^{k_i,s} = 0$, and

$$\lim_{i \to \infty} \frac{\rho^{c_k}}{m_i b^{k_i,s}} \leq \lim_{i \to \infty} \frac{c_k \rho^{c_k}}{n_k b^{s-a} k_i} = \lim_{i \to \infty} \frac{c_k \rho^{c_k}}{b^{(s-a)k_i}} = 0,$$

due to $\rho^c < b^{s-a}$. We derive from the equalities above that for any $\epsilon > 0$,

$$\text{Cov}(Z_{k_i}(J_1), Z_{k_i}(J_2)) < \epsilon \mathbb{E}(Z_{k_i}(J_1)) \mathbb{E}(Z_{k_i}(J_2))$$

holds for $i$ large enough. From Lemma 1, this implies that there is a constant $0 < M_5 < \infty$ independent of $k_i$ such that $f(k_i, \epsilon) \leq M_5$, and recall

$$f(k, \epsilon) = \max_{J_1 \in \mathcal{B}_k} \# \left\{ J_1 \in \mathcal{B}_k : \text{Cov}(Z_k(J_1), Z_k(J_2)) \geq \epsilon \mathbb{E}(Z_k(J_1)) \mathbb{E}(Z_k(J_2)) \right\}.$$ 

In particular,

$$\lim_{i \to \infty} \log_b^{b-1} \frac{f(k_i, \epsilon)}{k_i} = 0.$$

Thus we have shown that the condition $(H2)$ in Sect. 4 is satisfied with $\delta = 0$.

Now that we have verified that $E_*$ satisfies the conditions $(H1')$ and $(H2)$ in Sect. 4, we apply Theorem 8 to conclude $E_* \cap G \neq \emptyset$ a.s., which yields $\mathbb{P}(E \cap G \neq \emptyset) = 1$.

(iii) The packing dimension of $E \cap G$.

On the same probability space, let $E'$ be a random covering set that is independent of $E$ and is associated with $\{\xi'_n\}$ and $\{\ell'_n\}$, where $\{\xi'_n\}$ is an exponentially mixing stationary process, and $\{\ell'_n\}$ satisfies the condition (C) with the Besicovitch–Taylor index $\alpha' < s$.

Let the compact set $G_*$ and the open sets $A(k)$ be as described in the proofs of Theorems 7 and 5, respectively. Let $\{A'(k)\}$ be the sequence of open sets corresponding to $E'$. If $\dim_p G > s - \min\{\alpha, \alpha'\}$, by the first part of Theorem 5, we have

$$\mathbb{P}\left( \bigcup_{k=n}^{\infty} A(k) \cap V \cap G_* \neq \emptyset, \forall n \geq 1 \right) = \mathbb{P}\left( \bigcup_{k=n}^{\infty} A'(k) \cap V \cap G_* \neq \emptyset, \forall n \geq 1 \right) = 1$$

for all open set $V$ satisfying $V \cap G_* \neq \emptyset$. By independence, we have

$$\mathbb{P}\left( \bigcup_{k=n}^{\infty} A(k) \cap V \cap G_* \neq \emptyset, \bigcup_{k=n}^{\infty} A'(k) \cap V \cap G_* \neq \emptyset, \forall n \geq 1 \right) = 1.$$

Let $V$ run over a countable basis of $X$, then $\left\{ \bigcup_{k=n}^{\infty} A(k) \cap G_* \right\}_{n \geq 1} \cup \left\{ \bigcup_{k=n}^{\infty} A'(k) \cap G_* \right\}_{n \geq 1}$ is a countable collection of open, dense subsets of the complete metric space $G_*$. Baire’s category theorem implies that

$$\mathbb{P}(E \cap E' \cap G_* \text{ is dense in } G_*) = 1.$$ 

In particular, $E \cap E' \cap G_* \neq \emptyset$ a.s. Hence $\mathbb{P}(E \cap E' \cap G \neq \emptyset) = 1$.

Now we consider the random covering set $E'$, and regard $E \cap G$ as the target set. By Theorem 3, we must have $\dim_p(E \cap G) \geq s - \alpha'$ a.s. Hence we have proved that $\dim_p(G) > \cdots$
$s - \min\{\alpha, \alpha'\}$ implies $\dim_P(E \cap G) \geq s - \alpha'$ a.s. Consequently, if $\dim_P(G) > s - \alpha$, then for $\alpha' \in (s - \dim_P G, \alpha)$ we have $\dim_P(E \cap G) \geq s - \alpha'$ a.s. Letting $\alpha'$ tend to $s - \dim_P G$ along rational numbers, we get

$$\dim_P(G \cap E) \geq \dim_P G,$$ a.s.

Hence $\dim_P(G \cap E) = \dim_P G$ a.s. This finishes the proof of Theorem 5. $\square$

6 Applications

In this section we present some dynamical systems which satisfy all conditions of our results.

6.1 Continued fraction dynamical system

Let $T_G$ be the Gauss map on $(0, 1]$. The Gauss measure $\mu$ on $(0, 1]$ are given by

$$d\mu = \frac{1}{\log 2} \frac{dx}{(1 + x)}.$$

From [37], $T_G$ preserves the Gauss measure $\mu$ which is equivalent to the Lebesgue measure $\mathcal{L}$. By Philipp [35], the system $((0, 1], T_G)$ is exponentially mixing with respect to the Gauss measure $\mu$. Hence our results are applicable to the continued fraction dynamical system. For a given point $x \in (0, 1]$, let

$$E_G(x) = \{y \in (0, 1] : y \in B(T^n_G x, \ell_n) \text{ i.o.}\}.$$

**Theorem 9** Let $\mu$ be the Gauss measure. For any analytic set $G \subset (0, 1]$ we have, for $\mu$-a.e. $x \in (0, 1]$

$$E_G(x) \cap G = \emptyset \text{ if } \dim_P(G) < 1 - \alpha,$$

$$E_G(x) \cap G \neq \emptyset \text{ if } \dim_H(G) > 1 - \alpha.$$

Furthermore under the condition (C), if $\dim_P(G) > 1 - \alpha$,

$$E_G(x) \cap G \neq \emptyset \text{ a.e.}$$

**Theorem 10** For any analytic set $G \subset (0, 1]$ we have a.e.

$$\dim_H(E_G(x) \cap G) \begin{cases} \leq \dim_P(G) + \alpha - 1 & \text{if } \dim_P(G) \geq 1 - \alpha, \\ = -\infty & \text{if } \dim_P(G) < 1 - \alpha, \\ \geq \dim_H(G) + \alpha - 1 & \text{if } \dim_H(G) > 1 - \alpha. \end{cases}$$

Moreover, if $\dim_P(G) > 1 - \alpha$ and the condition (C) is satisfied, then $\dim_P(E_G(x) \cap G) = \dim_P(G)$ a.e.

6.2 The $\beta$-dynamical system

For a real number $\beta > 1$, define the transformation $T_\beta : [0, 1] \to [0, 1]$ by

$$T_\beta : x \mapsto \beta x \mod 1.$$
Let $\mu$ be the Parry measure with the density

$$h(x) = \left(\int_0^1 \frac{1}{\beta^n} \, dx \right)^{-1} \sum_{n:T^n x \leq x} \frac{1}{\beta^n}.$$

It was shown by [36] that the Parry measure $\mu$ is invariant under $T_\beta$ and equivalent to the Lebesgue measure $L$. Hence $\mu$ is Ahlfors regular. From Philipp [35], $T_\beta$ is exponentially mixing. Combining these, we obtain that the $\beta$-dynamical system $([0, 1], T_\beta)$ satisfies all the conditions stated in our results. For a given $x \in [0, 1]$, define the dynamical covering set

$$E_\beta(x) = \{y \in [0, 1]: y \in B(T_\beta^n x, \ell_n) \text{ i.o.}\}.$$

**Theorem 11** Let $([0, 1], T_\beta)$ be the $\beta$-dynamical system endowed with the Parry measure $\mu$. Then for any analytic set $G \subset [0, 1]$ we have

$$E_\beta(x) \cap G = \emptyset \quad \text{if } \dim_P(G) < 1 - \alpha,$$

$$E_\beta(x) \cap G \neq \emptyset \quad \text{if } \dim_H(G) > 1 - \alpha$$

for $\mu$-a.e. $x \in [0, 1]$. Under the corresponding condition (C), if $\dim_P(G) > 1 - \alpha$,

$$E_\beta(x) \cap G \neq \emptyset \quad \text{a.e.}$$

**Theorem 12** Let $([0, 1], T_\beta)$ be the $\beta$-dynamical system endowed with the Parry measure $\mu$. For any analytic set $G \subset [0, 1]$ we have a.e.

$$\dim_H(E_\beta(x) \cap G) \begin{cases}
\leq \dim_P(G) + \alpha - 1 & \text{if } \dim_P(G) \geq 1 - \alpha, \\
= -\infty & \text{if } \dim_P(G) < 1 - \alpha, \\
\geq \dim_H(G) + \alpha - 1 & \text{if } \dim_H(G) > 1 - \alpha.
\end{cases}$$

Moreover, if $\dim_P(G) > 1 - \alpha$ and the condition (C) is satisfied, then $\dim_P(E_\beta(x) \cap G) = \dim_P(G)$ a.e.

### 6.3 The middle-third Cantor set

Our results are applicable to the middle-third Cantor set $C_{1/3}$. In fact, the results also hold for a range of homogeneous self-similar sets satisfying the open set condition.

Let $T_3 x = 3x \mod 1$ be the natural map on $C_{1/3}$, and $\mu$ be the standard Cantor measure. Let $\gamma = \log_3 2$ be the Hausdorff dimension of $C_{1/3}$. From Lemma 3.2 in [40], $\mu$ is exponentially mixing. Also $\mu$ is Ahlfors $\gamma$-regular. Then all the conditions are fulfilled for Theorem 1.1 and 1.2. Define the dynamical covering set

$$E_3(x) = \{y \in C_{1/3}: y \in B(T_3^n x, \ell_n) \text{ i.o.}\},$$

where $x \in C_{1/3}$ is a given point.

**Theorem 13** Let $\mu$ be the standard Cantor measure. Then for any analytic set $G \subset C_{1/3}$, we have

$$E_3(x) \cap G = \emptyset \text{ if } \dim_P(G) < \gamma - \alpha,$$

$$E_3(x) \cap G \neq \emptyset \text{ if } \dim_H(G) > \gamma - \alpha$$

for $\mu$-a.e. $x \in C_{1/3}$. Under the corresponding condition (C), if $\dim_P(G) > \gamma - \alpha$, $E_3(x) \cap G \neq \emptyset$ holds for $\mu$-a.e. $x \in C_{1/3}$. 

Springer
Theorem 14 For any analytic set $G \subset C_{1/3}$ we have a.e.

$$\dim_H(E_3(x) \cap G) = \begin{cases} 
\leq \dim_P(G) + \alpha - \gamma & \text{if } \dim_P(G) \geq \gamma - \alpha, \\
= -\infty & \text{if } \dim_P(G) < \gamma - \alpha, \\
\geq \dim_H(G) + \alpha - \gamma & \text{if } \dim_H(G) > \gamma - \alpha.
\end{cases}$$

Moreover, if $\dim_P(G) > \gamma - \alpha$ and the condition (C) is satisfied, then $\dim_P(E_3(x) \cap G) = \dim_P(G)$ a.e.

References

1. Aspenberg, M., Persson, T.: Shrinking targets in parametrised families. Math. Proc. Camb. Philos. Soc. 166(2), 265–295 (2019)
2. Borel, E.: Sur les séries de Taylor. Acta Math. 21(1), 243–247 (1897)
3. Boshernitzan, M.D.: Quantitative recurrence results. Invent. Math. 113(3), 617–631 (1993)
4. Bugeaud, Y., Durand, A.: Metric Diophantine approximation on the middle-third Cantor set. J. Eur. Math. Soc. (JEMS) 18(6), 1233–1272 (2016)
5. Bugeaud, Y., Wang, B.-W.: Distribution of full cylinders and the diophantine properties of the orbits in $\beta$-expansions. J. Fractal Geom. 1(2), 221–241 (2014)
6. Durand, A.: Sets with large intersection and ubiquity. Math. Proc. Camb. Philos. Soc. 144(1), 119–144 (2008)
7. Durand, A.: On randomly placed arcs on the circle. In: Recent Developments in Fractals and Related Fields, 343–351, Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston (2010)
8. Dvoretzky, A.: On covering a circle by randomly placed arcs. Proc. Natl. Acad. Sci. USA 42, 199–203 (1956)
9. Fan, A.-H., Wu, J.: On the covering by small random intervals. Ann. Inst. H. Poincaré Probab. Stat. 40(1), 125–131 (2004)
10. Fan, A.-H., Langlet, T., Li, B.: Quantitative uniform hitting in exponentially mixing systems, In: Recent Developments in Fractals and Related Fields. 251–266, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston (2010)
11. Fan, A.-H., Schmeling, J., Troubetzkoy, S.: A multifractal mass transference principle for Gibbs measures with applications to dynamical Diophantine approximation. Proc. Lond. Math. Soc. (3) 107(5), 1173–1219 (2013)
12. Feng, D.-J., Järvenpää, E., Järvenpää, M., Suomala, V.: Dimensions of random covering sets in Riemann manifolds. Ann. Probab. 46(3), 1542–1596 (2018)
13. Furstenberg, H.: Recurrence in Ergodic Theory and Combinatorial Number Theory. M. B. Porter Lectures, Princeton University Press, Princeton (1981)
14. Haynes, A., Koivusalo, H.: A randomized version of the Littlewood conjecture. J. Number Theory 178, 201–207 (2017)
15. Hill, R., Velani, S.: The ergodic theory of shrinking targets. Invent. Math. 119(1), 175–198 (1995)
16. Howroyd, J., D.: On dimension and on the existence of sets of finite positive Hausdorff measure. Proc. Lond. Math. Soc. (3) 70(3), 581–604 (1995)
17. Hu, Z.-N., Li, B.: Random covering sets in metric space with exponentially mixing property. Stat. Probab. Lett. 168, 108922 (2021)
18. Hussain, M., Li, B., Simmons, D., Wang, B.-W.: Dynamical Borel–Cantelli lemma for recurrence theory. Ergod. Theory Dyn. Syst. (2021). https://doi.org/10.1017/etds.2021.23
19. Järvenpää, E., Järvenpää, M., Koivusalo, H., Li, B., Suomala, V.: Hausdorff dimension of affine random covering sets in torus. Ann. Inst. Henri Poincaré Probab. Stat. 50(4), 1371–1384 (2014)
20. Järvenpää, E., Järvenpää, M., Koivusalo, H., Li, B., Suomala, V., Xiao, Y.-M.: Hitting probabilities of random covering sets in tori and metric spaces. Electron. J. Probab. 22(1), 18 (2017)
21. Joyce, H., Preiss, D.: On the existence of subsets of finite positive packing measure. Mathematika 42(1), 15–24 (1995)
22. Kaenmäki, A., Rajala, T., Suomala, V.: Existence of doubling measures via generalised nested cubes. Proc. Am. Math. Soc. 140(9), 3275–3281 (2012)
23. Kahane, J.-P.: Random coverings and multipliﬁcative processes. Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 125–146, Progr. Probab., vol. 46, Birkhäuser, Basel (2000)
24. Khoshnevisan, D., Peres, Y., Xiao, Y.-M.: Limsup random fractals. Electron. J. Probab. 5(5), 24 (2000)
On the intersection of dynamical covering sets with fractals

25. Levesley, J., Salp, C., Velani, S.: On a problem of K. Mahler: Diophantine approximation and Cantor sets. Math. Ann. 338(1), 97–118 (2007)

26. Li, B., Shieh, N.-R., Xiao, Y.-M.: Hitting probabilities of the random covering sets. In: Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics. II. Fractals in Applied mathematics, 307–323. Contemp. Math., vol. 601. Amer. Math. Soc., Providence (2013)

27. Li, B., Suomala, V.: A note on the hitting probabilities of random covering sets. Ann. Acad. Sci. Fenn. Math. 39(2), 625–633 (2014)

28. Li, B., Suomala, V.: A note on the hitting probabilities of random covering sets. Ann. Acad. Sci. Fenn. Math. 39(2), 625–633 (2014)

29. Liao, L., Seuret, S.: Diophantine approximation by orbits of expanding Markov maps. Ergod. Theory Dyn. Syst. 33(2), 585–608 (2013)

30. Lyons, R.: Random walks and percolation on trees. Ann. Probab. 18(3), 931–958 (1990)

31. Mahler, K.: Some suggestions for further research. Bull. Austral. Math. Soc. 29(1), 101–108 (1984)

32. Myjak, J., Rudnicki, R.: Box and packing dimensions of typical compacts sets. Monatsh. Math. 131(3), 223–226 (2000)

33. Peres, Y.: Intersection-equivalence of Brownian paths and certain branching processes. Commun. Math. Phys. 177(2), 417–434 (1996)

34. Persson, T., Rams, M.: On shrinking targets for piecewise expanding interval maps. Ergod. Theory Dyn. Syst. 37(2), 646–663 (2017)

35. Philipp, W.: Some metrical theorems in number theory. Pac. J. Math. 20, 109–127 (1967)

36. Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hung. 8, 477–493 (1957)

37. Ryll-Nardzewski, C.: On the ergodic theorems. II. Ergodic theory of continued fractions. Stud. Math. 12, 74–79 (1951)

38. Shepp, L.: Covering the circle with random arcs. Isr. J. Math. 11, 328–345 (1972)

39. Tricot, C.: Two definitions of fractional dimension. Math. Proc. Camb. Philos. Soc. 91(1), 57–74 (1982)

40. Wang, B.-W., Wu, J., Xu, J.: Dynamical covering problems on the triadic Cantor set. C. R. Math. Acad. Sci. Paris 355(7), 738–743 (2017)

41. Yu, H.: Rational points near self-similar sets (2021). arXiv:2101.05910

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.