RATIONAL SEMISTANDARD TABLEAUX AND CHARACTER FORMULA FOR THE LIE SUPERALGEBRA $\hat{\mathfrak{gl}}_{\infty}\mid_{\infty}$

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Abstract. A new combinatorial interpretation of the Howe dual pair $(\hat{\mathfrak{gl}}_{\infty}\mid_{\infty}, \mathfrak{gl}_n)$ acting on an infinite dimensional Fock space $F^n$ of level $n$ is presented. The character of a quasi-finite irreducible highest weight representation of $\hat{\mathfrak{gl}}_{\infty}\mid_{\infty}$ occurring in $F^n$ is realized in terms of certain bitableaux of skew shapes. We study a general combinatorics of these bitableaux, including Robinson-Schensted-Knuth correspondence and Littlewood-Richardson rule, and then its dual relation with the rational semistandard tableaux for $\mathfrak{gl}_n$. This result also explains other Howe dual pairs including $\mathfrak{gl}_n$.

1. Introduction

The Lie superalgebras and their representations appear naturally as the fundamental algebraic structures in various areas of mathematics and mathematical physics, and they have been studied by many people since the fundamental work of Kac [13].

Recently, from the viewpoint of Howe duality [11] [12], a close relation between the representations of Lie algebras and Lie superalgebras has been observed (see [5] [6] [21] [23] and other references therein), by which various character formulas for certain Lie superalgebras have been obtained [3] [4] [7].

The purpose of this paper is to give a unified combinatorial interpretation of the Howe dualities of the pairs $(\mathfrak{g}, \mathfrak{gl}_n)$ including a general linear Lie algebra. Especially, our work will be devoted to the cases when $\mathfrak{g} = \hat{\mathfrak{gl}}_{\infty}\mid_{\infty}$ or $\hat{\mathfrak{gl}}_{\infty}$ acting on an infinite dimensional Fock space which have been studied in [3] [5] [6] [15] [16]. Let $\mathfrak{F}^n (n \geq 1)$ be the infinite dimensional Fock space generated by $n$ pairs of free fermions and $n$ pairs of free bosons (see [3] [6]). On $\mathfrak{F}^n$, there exists a natural commuting actions of the infinite dimensional Lie superalgebra $\hat{\mathfrak{gl}}_{\infty}\mid_{\infty}$ and the finite dimensional Lie algebra $\mathfrak{gl}_n$. Using Howe duality, Cheng and Wang derived a multiplicity-free decomposition

$$\mathfrak{F}^n \cong \bigoplus_{\lambda \in \mathbb{Z}^n_+} L_\lambda \otimes L_n(\lambda),$$

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as a $\hat{\mathfrak{gl}}_{\infty|\infty} \oplus \mathfrak{gl}_n$-module, where the sum ranges over all generalized partitions $\lambda$ of length $n$, $L_n(\lambda)$ is the rational representation of $\mathfrak{gl}_n$ corresponding to $\lambda$, and $L_\lambda$ is the associated quasi-finite highest weight representation of $\hat{\mathfrak{gl}}_{\infty|\infty}$. From the above decomposition and the classical Cauchy identities for hook Schur polynomials (cf. [1, 24]), Cheng and Lam derived a character formula of $L_\lambda$, in terms of hook Schur polynomials [3]. They also described the tensor product decomposition of $L_\lambda \otimes L_\mu$ for $\lambda \in \mathbb{Z}^m_+$ and $\mu \in \mathbb{Z}^n_+$. In fact, this is a natural super-analogue of the Kac and Radul’s works on the construction of quasi-finite irreducible highest weight representation $L_0^\lambda$ of the infinite dimensional Lie algebra $\hat{\mathfrak{gl}}_\infty$, which is also parameterized by a generalized partition [16].

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are linearly ordered $\mathbb{Z}_2$-graded sets. Motivated by the character formula of $L_\lambda$ or $L_0^\lambda$ in [3, 16], we introduce the notion of $\mathcal{A}/\mathcal{B}$-semistandard tableaux of shape $\lambda$, which is our main object in this paper. Roughly speaking, an $\mathcal{A}/\mathcal{B}$-semistandard tableau of shape $\lambda$ is a pair of tableaux $(T^+, T^-)$ such that $T^+$ (resp. $T^-$) is a semistandard tableau with letters in $\mathcal{A}$ (resp. $\mathcal{B}$), where the shapes of $T^+$ and $T^-$ are not necessarily fixed ones but satisfy certain conditions determined by $\lambda$. We develop the insertion scheme for $\mathcal{A}/\mathcal{B}$-semistandard tableaux, and derive analogues of Robinson-Schensted-Knuth (or simply RSK) correspondence and Littlewood-Richardson (or simply LR) rule.

As in the case of ordinary semistandard tableaux, we define skew $\mathcal{A}/\mathcal{B}$-semistandard tableaux of shape $\lambda/\mu$ for arbitrary two generalized partitions $\lambda$ and $\mu$ of the same length, and describe the corresponding skew LR rule. Then it turns out that the combinatorics of $\mathcal{A}/\mathcal{B}$-semistandard tableaux is dual to that of rational semistandard tableaux for general linear Lie algebra introduced by Stembridge [25] in the sense that the skew LR rule (resp. LR rule) of $\mathcal{A}/\mathcal{B}$-semistandard tableaux are completely determined by the LR rule (resp. skew LR rule) of rational semistandard tableaux.

Next, we show that the character of $\text{SST}_{\mathcal{A}/\mathcal{B}}(\lambda)$, the set of all $\mathcal{A}/\mathcal{B}$-semistandard tableaux of shape $\lambda$, reduce to the character of $L_\lambda$ or $L_0^\lambda$ under suitable choices of $\mathcal{A}$ and $\mathcal{B}$. This is done by observing that we have another expression of the character of $\text{SST}_{\mathcal{A}/\mathcal{B}}(\lambda)$, which is a kind of branching formula very similar to the Cheng and Lam’s formula (or the Kac and Radul’s formula). As a corollary, we immediately obtain new combinatorial interpretations of the decomposition of the Fock space representation $\hat{\mathfrak{g}}^n$ and the tensor product $L_\lambda \otimes L_\mu$ from RSK correspondence and LR rule, respectively. We also obtain a Jacobi-Trudi type character formula for $L_\lambda$ or $L_0^\lambda$, which has not been observed yet as far as we know. The dual relationship between $\mathcal{A}/\mathcal{B}$-semistandard tableaux and rational semistandard tableaux now explains the duality between the tensor product decomposition and the branching rule of the pairs $(\hat{\mathfrak{g}}_{\infty|\infty}, \mathfrak{g}_n)$ and $(\hat{\mathfrak{g}}_\infty, \mathfrak{g}_n)$, which is a general feature in any Howe dual pair. We
expect a combinatorial construction of $L_\lambda$ as a vector space spanned by $SST_{A/B}(\lambda)$, and also a $q$-analogue of $L_\lambda$ as a representation of the associated quantum group $U_q(\widehat{\mathfrak{gl}_\infty})$ with a crystal graph $SST_{A/B}(\lambda)$.

Finally, we would like to remark that the notion of $A/B$-semistandard tableaux can be applied to other classes of representations of Lie (super)algebras. For example, when $A$ is finite and $B$ is empty, our results explain the classical Howe duality of the $(\mathfrak{gl}_p|q, \mathfrak{gl}_n)$ pair acting on $S(\mathbb{C}^p|q \otimes \mathbb{C}^n)$, the supersymmetric algebra generated by $\mathbb{C}^p|q \otimes \mathbb{C}^n$ (cf. [5, 19]). Moreover, when $A$ and $B$ are both non-empty finite sets, we can recover the combinatorial picture of the Howe dual pair $(\mathfrak{gl}_p|q, \mathfrak{gl}_n)$ on a supersymmetric algebra [4] (see also [18]), and $SST_{A/B}(\lambda) (\lambda \in \mathbb{Z}^n_{\geq 0})$ realizes the character of an infinite-dimensional unitarizable representation of $\mathfrak{gl}_p|q$. In general, we expect that to arbitrary $A$ and $B$, there corresponds a contragredient Lie superalgebra, where the character of $SST_{A/B}(\lambda)$ gives the character of an irreducible representation parameterized by $\lambda$ (cf. [19]).

This paper is organized as follows. In Section 2, we briefly recall a necessary background on semistandard tableaux for Lie superalgebras and their insertion scheme including LR rule. In Section 3, we review the notion of rational semistandard tableaux and define $A/B$-semistandard tableaux. Some properties of the characters are also discussed. In Section 4, we introduce an insertion algorithm for $A/B$-semistandard tableaux, and then derive analogues of RSK correspondence, and (skew) LR rule. Finally, in Section 5, we show that the character of $SST_{A/B}(\lambda)$ reduces to the character of $L_\lambda$ or $L_\lambda^0$ under a particular choice of $A$ and $B$, and then explain the relationship between the combinatorial results established in the previous sections and the representations of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ or $\widehat{\mathfrak{gl}}_{\infty}$.

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2. Preliminaries

Throughout the paper, we assume that $A$ and $B$ are linearly ordered sets which are at most countable, and also $\mathbb{Z}_2$-graded (that is, $A = A_0 \sqcup A_1$ and $B = B_0 \sqcup B_1$). By convention, we let $\mathbb{N} = \{1 < 2 < \cdots\}$, and $[n] = \{1 < \cdots < n\}$ for $n \geq 1$, where all the elements are of degree 0.

In this section, we introduce the notion of $A$-semistandard tableaux, and describe the associated Littlewood-Richardson rule. In fact, the combinatorics of $A$-semistandard tableaux is essentially the same as that of semistandard tableaux with entries in $\mathbb{N}$. Since the only difference is that we assume the column strict condition
on entries of degree 0 and the row strict condition on entries of degree 1, most of the results in this section can be verified directly by modifying the arguments of the corresponding results in case of ordinary semistandard tableaux (cf. [8, 17, 24]). So, we leave the detailed verifications to the readers.

2.1. **Semistandard tableaux.** Let us recall some basic terminologies (cf. [22]). A partition of a non-negative integer $n$ is a non-increasing sequence of non-negative integers $\lambda = (\lambda_k)_{k \geq 1}$ such that $\sum_{k \geq 1} \lambda_k = n$. We also write $|\lambda| = n$. Each $\lambda_k$ is called a part of $\lambda$, and the number of non-zero parts is called the length of $\lambda$ denoted as $\ell(\lambda)$. We also write $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$, where $m_i$ is the number of occurrences of $i$ in $\lambda$. We denote by $\mathcal{P}$ the set of all partitions, and denote by $\mathcal{P}_\ell$ ($\ell \geq 1$) the set of all partitions with length no more than $\ell$. Recall that a partition $\lambda = (\lambda_k)_{k \geq 1}$ is identified with a Young diagram which is a collection of nodes (or boxes) in left-justified rows with $\lambda_k$ nodes in the $k$th row. For $\lambda \in \mathcal{P}$, $\lambda'$ is the conjugate of $\lambda$. For $\lambda, \mu \in \mathcal{P}$ with $\lambda \supset \mu$ (that is, $\lambda_i \geq \mu_i$ for all $i$), $\lambda/\mu$ is the skew Young diagram obtained from $\lambda$ by removing $\mu$, and $|\lambda/\mu|$ is defined to be the number of nodes in $\lambda/\mu$.

**Definition 2.1.** For a skew Young diagram $\lambda/\mu$, a tableau $T$ obtained by filling $\lambda/\mu$ with entries in $\mathcal{A}$ is called $\mathcal{A}$-semistandard if

1. the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom),
2. the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row).

We say that $\lambda/\mu$ is the shape of $T$, and write $\text{sh}(T) = \lambda/\mu$. We denote by $SST_\mathcal{A}(\lambda/\mu)$ the set of all $\mathcal{A}$-semistandard tableaux of shape $\lambda/\mu$. We assume that $SST_\mathcal{A}(\lambda) = \{\emptyset\}$ when $\lambda \in \mathcal{P}$ is a partition of 0, where $\emptyset$ denotes the empty tableau. We set $\mathcal{P}_\mathcal{A} = \{\lambda \in \mathcal{P} | SST_\mathcal{A}(\lambda) \neq \emptyset\}$. Note that $\mathcal{P}_{[n]} = \mathcal{P}_n$, and $\mathcal{P}_\mathcal{A} = \mathcal{P}$ when $\mathcal{A}$ is an infinite set.

For $T \in SST_\mathcal{A}(\lambda/\mu)$, we let

$$w_{\text{col}}(T) = a_1a_2\ldots a_n$$

be the word with letters in $\mathcal{A}$ obtained by reading the entries of $T$ column by column from right to left, and from top to bottom in each column. Also, we let

$$w_{\text{row}}(T) = b_1b_2\ldots b_n$$

be the word obtained by reading the entries of $T$ row by row from bottom to top, and from left to right in each row.
2.2. Operations on tableaux. Let us define several operations on semistandard tableaux. Let $\lambda/\mu$ be a skew Young diagram.

1. **transpose**: We define $A'$ to be the linearly ordered $\mathbb{Z}_2$-graded set such that $A'_0 = A_1$, $A'_1 = A_0$. For $T \in \text{SST}_A(\lambda/\mu)$, we denote by $T^t$ the transpose of $T$. Then $T^t \in \text{SST}_{A'}(\lambda'/\mu')$.

2. **rotation**: We define $A^\pi$ to be the linearly ordered $\mathbb{Z}_2$-graded set such that $A^\pi_0 = A_1$, $A^\pi_1 = A_0$. For $T \in \text{SST}_A(\lambda/\mu)$, we define $T^\pi$ to be the tableau obtained by applying $180^\circ$-rotation to $T$. Then $T^\pi \in \text{SST}_{A^\pi}(\lambda/\mu^\pi)$, where $(\lambda/\mu)^\pi$ is the skew diagram obtained from $\lambda/\mu$ after $180^\circ$-rotation.

3. **reverse transpose**: We set $A^\# = (A')^\pi$. For $T \in \text{SST}_{A'}(\lambda)$, we define $T^\# = (T^t)^\pi$. Then $T^\# \in \text{SST}_{A^\#}(\lambda)$.

We define $A \ast B$ to be the $\mathbb{Z}_2$-graded set $A \sqcup B$ with the extended linearly ordering given by $a < b$ for all $a \in A$ and $b \in B$. For $S \in \text{SST}_A(\mu)$ and $T \in \text{SST}_B(\lambda)$, we define $S \ast T$ to be the $A \ast B$-semistandard tableau of shape $\lambda$ obtained by gluing $S$ and $T$ so that the right-most node in each row of $S$ is placed next to the left-most node in the same row of $T$.

2.3. Insertions. Let us describe the Schensted's column and row bumping algorithms for $A$-semistandard tableaux (cf. [3, 24]): Suppose that $a \in A$ and $T \in \text{SST}_A(\lambda)$ ($\lambda \in \mathcal{P}_A$) are given.

First, we define $(T \leftarrow a)$ to be the tableau obtained from $T$ by applying the following procedure (called the column bumping algorithm):

1. If $a \in A_0$, let $a'$ be the smallest entry in the first (or the left-most) column, which is greater than or equal to $a$. If $a \in A_1$, let $a'$ be the smallest entry in the first column, which is greater than $a$. If there are more than one $a'$, choose the one in the highest position.
2. Replace $a'$ by $a$. If there is no such $a'$, put $a$ at the bottom of the first column and stop the procedure.
3. Repeat (i) and (ii) on the next column with $a'$.

Next, we define $(a \rightarrow T)$ to be the tableau obtained from $T$ by applying the following procedure (called the row bumping algorithm):

1. If $a \in A_0$, let $a'$ be the smallest entry in the first (or the top) row, which is greater than $a$. If $a \in A_1$, let $a'$ be the smallest entry in the first row, which
is greater than or equal to \( a \). If there are more than one \( a' \), choose the one in the left-most position.

(ii) Replace \( a' \) by \( a \). If there is no such \( a' \), put \( a \) at the right-most end of the top row and stop the procedure.

(iii) Repeat (i) and (ii) on the next row with \( a' \).

Suppose that \( \mu, \nu \in \mathcal{P}_A \) are given. For \( T \in SST_A(\mu) \) and \( T' \in SST_A(\nu) \), let \( w_{\text{col}}(T') = c_1c_2 \ldots c_n \). We define

\[
(T \leftarrow T') = (((T \leftarrow c_1) \leftarrow c_2) \cdots) \leftarrow c_n).
\]

Similarly, let \( w_{\text{row}}(T') = r_1r_2 \ldots r_m \). We define

\[
(T' \rightarrow T) = (r_m \rightarrow (\cdots (r_2 \rightarrow (r_1 \rightarrow T)))).
\]

We define \( (T \leftarrow T')_R \) to be the semistandard tableau in \( SST_N(\lambda/\mu) \) (\( \lambda = \text{sh}(T \leftarrow T') \)) such that if \( c_i \) is in the \( k \)th row of \( T' \) and inserted into \(((\cdots (T \leftarrow c_1) \leftarrow c_2) \cdots) \leftarrow c_{i-1}) \) to create a node in \( \lambda/\mu \), then we fill the node with \( k \). We call \( (T \leftarrow T')_R \) the \textit{recording tableau} of \( (T \leftarrow T') \).

Similarly, we define \( (T' \rightarrow T)_R \) to be the semistandard tableau in \( SST_N'(\eta/\mu) \) (\( \eta = \text{sh}(T' \rightarrow T) \)) such that if \( r_i \) is in the \( k \)th column of \( T' \) and inserted into \((r_{i-1} \rightarrow (\cdots (r_2 \rightarrow (r_1 \rightarrow T))) \) to create a node in \( \eta/\mu \), then we fill the node with \( k \). We also call \( (T' \rightarrow T)_R \) the \textit{recording tableau} of \( (T' \rightarrow T) \).

Given \( \lambda, \mu \) and \( \nu \) in \( \mathcal{P}_A \) such that \( \mu \subset \lambda \) and \( |\lambda| = |\mu| + |\nu| \), a tableau \( T \in SST_N(\lambda/\mu) \) is called a \textit{Littlewood-Richardson tableau} of shape \( \lambda/\mu \) with content \( \nu \) if

1. the number of occurrences of \( k \) in \( T \) is equal to \( \mu_k \) for \( k \geq 1 \),
2. \( w_{\text{col}}(T) \) is a lattice permutation (see [22] for its definition).

We denote by \( LR_{\mu \nu}^{\lambda} \) the set of all Littlewood-Richardson tableaux of shape \( \lambda/\mu \) with content \( \nu \), and put \( |LR_{\mu \nu}^{\lambda}| = N_{\mu \nu}^{\lambda} \), which is called a \textit{Littlewood-Richardson coefficient}. By standard arguments as in the case of \( N \)-semistandard tableaux, we can check the following.

\begin{lemma} \text{(cf. [17, 24, 27])} \end{lemma}

For \( T \in SST_A(\mu) \), \( T' \in SST_A(\nu) \) (\( \mu, \nu \in \mathcal{P}_A \)),

1. \( (T \leftarrow T')_R \in LR_{\mu \nu}^{\lambda} \), where \( \lambda = \text{sh}(T \leftarrow T') \),
2. \( [(T' \rightarrow T)_R]_T \in LR_{\mu' \nu'}^{\eta} \), where \( \eta = \text{sh}(T' \rightarrow T) \).

By Lemma 2.2, we obtain the \textit{Littlewood-Richardson rule} (or simply LR rule) for \( A \)-semistandard tableaux.

\begin{theorem} \text{(cf. [17, 24, 27])} \end{theorem}

Suppose that \( \mu, \nu \in \mathcal{P}_A \) are given.
(1) The map $\rho_{\text{col}} : (T, T') \mapsto ((T \leftarrow T'), (T \leftarrow T')_R)$ gives a bijection

$$
\rho_{\text{col}} : \text{SST}_A(\mu) \times \text{SST}_A(\nu) \rightarrow \bigsqcup_{\lambda \in P_A} \text{SST}_A(\lambda) \times LR_{\mu, \nu}^A.
$$

(2) The map $\rho_{\text{row}} : (T, T') \mapsto ((T' \rightarrow T), [(T' \rightarrow T)_R]^T)$ gives a bijection

$$
\rho_{\text{row}} : \text{SST}_A(\mu) \times \text{SST}_A(\nu) \rightarrow \bigsqcup_{\eta \in P_A} \text{SST}_A(\eta) \times LR_{\mu', \nu'}^T.
$$

For an $r$-tuple of non-negative integers $\nu = (\nu_1, \ldots, \nu_r)$ such that $\nu_i \in P_A (1 \leq i \leq r)$, consider

$$(T_1, \ldots, T_r) \in \text{SST}_A(\nu_1) \times \cdots \times \text{SST}_A(\nu_r).$$

For $1 \leq i \leq r$, put

$$S_i = (((T_1 \leftarrow T_2) \leftarrow T_3) \cdots) \leftarrow T_i,$$

and $S = S_r$. If we put $\mu^{(i)} = \text{sh}(S_i) \in P$, then we have $\nu_1 = \mu^{(1)} \subset \cdots \subset \mu^{(r)} = \mu$, and $\mu^{(i)}/\mu^{(i-1)} (1 \leq i \leq r)$ is a horizontal strip of length $\nu_i$ (we assume that $\mu^{(0)}$ is the empty partition). Filling $\mu^{(i)}/\mu^{(i-1)}$ with $i$, we obtain an $[r]$-semistandard tableau $S_R \in \text{SST}_{[r]}(\mu)$, with content $\nu$ (that is, each entry $i$ occurs as many times as $\nu_i$ for $1 \leq i \leq r$). The correspondence $(T_1, \ldots, T_r) \mapsto (S, S_R)$ is reversible by Theorem 2.3.

Similarly, put

$$S'_i = (T_i \rightarrow \cdots (T_3 \rightarrow (T_2 \rightarrow T_1))),$$

for $1 \leq i \leq r$ and $S'_r = S'_r$. If we put $\mu^{(i)} = \text{sh}(S'_i) \in P$, then we have $\nu_1 = \mu^{(1)} \subset \cdots \subset \mu^{(r)} = \mu$, and $\mu^{(i)}/\mu^{(i-1)} (1 \leq i \leq r)$ is also a horizontal strip of length $\nu_i$.

So, as in the case of $S_R$, we may define an $[r]$-semistandard tableau of shape $\mu$ with content $\nu$, say $S'_R$. The correspondence $(T_1, \ldots, T_r) \mapsto (S', S'_R)$ is also reversible.

Summarizing the arguments, we have

**Proposition 2.4.** Under the above hypothesis, we have two bijections

$$
\varrho_{\text{col}}, \varrho_{\text{row}} : \text{SST}_A(\nu_1) \times \cdots \times \text{SST}_A(\nu_r) \rightarrow \bigsqcup_{\mu \in P_A} \text{SST}_A(\mu) \times \text{SST}_{[r]}(\mu)\nu,
$$

where $\varrho_{\text{col}}(T_1, \ldots, T_r) = (S, S_R)$, $\varrho_{\text{row}}(T_1, \ldots, T_r) = (S', S'_R)$, and $\text{SST}_{[r]}(\mu)\nu$ is the set of all $[r]$-semistandard tableaux of shape $\mu$ with content $\nu$. \qed
2.4. **Switching algorithm.** Let us describe the skew LR rule for \(A\)-semistandard tableaux. To do this, we will use the *switching algorithm* introduced by Benkart, Sottile, and Stroomer [2].

Let \(\lambda/\mu\) be a skew Young diagram. Let \(U\) be a tableau of shape \(\lambda/\mu\) with entries in \(A \sqcup B\), satisfying the following conditions:

(S1) if \(u, u' \in A\) (resp. \(B\)) are entries of \(U\) and \(u\) is northwest of \(u'\), then \(u \leq u'\),
(S2) in each column of \(U\), entries in \(A_0\) or \(B_0\) increase strictly,
(S3) in each row of \(U\), entries in \(A_1\) or \(B_1\) increase strictly,

where we say that \(u\) is northwest of \(u'\) provided the row and column indices of \(u\) are no more than those of \(u'\).

Suppose that \(a \in A\) and \(b \in B\) are two adjacent entries in \(U\) such that \(a\) is placed above or to the left of \(b\). Interchanging \(a\) and \(b\) is called a *switching* if the resulting tableau still satisfies the conditions (S1), (S2) and (S3).

**Theorem 2.5** (Theorem 2.2 and 3.1 in [2]). Let \(\lambda/\mu\) be a skew Young diagram. For \(S \in SST_A(\mu)\) and \(T \in SST_B(\lambda/\mu)\), let \(U\) be a tableau obtained from \(S \ast T\) by applying switching procedures as far as possible. Then

1. \(U = T' \ast S'\), where \(T' \in SST_B(\nu)\) and \(S' \in SST_A(\lambda/\nu)\) for some \(\nu\).
2. \(U\) is uniquely determined by \(S\) and \(T\).
3. When \(A = \mathbb{N}\), \(S' \in LR_{\nu}^{\lambda/\mu}\) if and only if \(S = H^\mu\), where \(H^\mu\) is the unique \(\mathbb{N}\)-semistandard tableau of shape \(\mu\) with content \(\mu\).

□

**Example 2.6.** Suppose that \(A = \mathbb{N}\) and \(B = \mathbb{N}' = \{1' < 2' < 3' < \ldots\}\) (see 2.2). Consider

\[
S \ast T = \begin{pmatrix}
1 & 1 & 2 \\
2 & 3 & 3' \\
1' & 2' & 3'
\end{pmatrix} = S'T' \ast S'', \quad \text{where} \quad S \in SST_N(3, 2), \quad T \in SST_N((3^3)/(3, 2)).
\]

Then

\[
S \ast T = \begin{pmatrix}
1 & 1 & 2 & 1' & 1 & 2 & 1' & 1 & 2 \\
2 & 3 & 3' & 1' & 3 & 3' & 1' & 3 & 3' \\
1' & 2' & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\end{pmatrix}
\]

\[
2' \leftrightarrow 3' 
\]

\[
1' & 1 & 2 & 1' & 2' & 2 & 1' & 2' & 3' \\
2 & 3 & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\]

\[
3' \leftrightarrow 3 
\]

\[
1 & 1 & 2 & 1' & 2' & 3' & 1' & 2' & 3' \\
2 & 3' & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]

\[
S \ast T = \begin{pmatrix}
1 & 1 & 2 & 1' & 1 & 2 & 1' & 1 & 2 \\
2 & 3 & 3' & 1' & 3 & 3' & 1' & 3 & 3' \\
1' & 2' & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\end{pmatrix}
\]

\[
2' \leftrightarrow 3' 
\]

\[
1 & 1 & 2 & 1' & 3' & 3' & 1' & 2' & 3' \\
2 & 3 & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]

\[
3' \leftrightarrow 3 
\]

\[
1 & 1 & 2 & 1' & 2' & 3' & 1' & 2' & 3' \\
2 & 3' & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]

\[
S \ast T = \begin{pmatrix}
1 & 1 & 2 & 1' & 1 & 2 & 1' & 1 & 2 \\
2 & 3 & 3' & 1' & 3 & 3' & 1' & 3 & 3' \\
1' & 2' & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\end{pmatrix}
\]

\[
2' \leftrightarrow 3' 
\]

\[
1' & 1 & 2 & 1' & 2' & 2 & 1' & 2' & 3' \\
2 & 3 & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\]

\[
3' \leftrightarrow 3 
\]

\[
1 & 1 & 2 & 1' & 3' & 3' & 1' & 2' & 3' \\
2 & 3' & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]

\[
S \ast T = \begin{pmatrix}
1 & 1 & 2 & 1' & 1 & 2 & 1' & 1 & 2 \\
2 & 3 & 3' & 1' & 3 & 3' & 1' & 3 & 3' \\
1' & 2' & 3' & 2 & 2' & 3' & 2 & 2' & 3'
\end{pmatrix}
\]

\[
2' \leftrightarrow 3' 
\]

\[
1 & 1 & 2 & 1' & 3' & 3' & 1' & 2' & 3' \\
2 & 3' & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]

\[
3' \leftrightarrow 3 
\]

\[
1 & 1 & 2 & 1' & 2' & 3' & 1' & 2' & 3' \\
2 & 3' & 3' & 2 & 3' & 3' & 2 & 3' & 3'
\]
Remark 2.7. (1) Theorem 2.2 in [2] is shown when $A = B = \mathbb{N}$ having only elements of degree 0. But we may naturally extend this result to arbitrary $A$ and $B$. We leave the detailed verifications to the readers (see also [24]).

(2) The resulting tableau $T'$ in Theorem 2.5 is independent of the choice of $S$, and the algorithm of producing $T'$ from $T$ is known as the Schützenberger’s jeu de taquin slides.

Suppose that $T \in \text{SST}_A(\lambda/\mu)$ is given for a skew Young diagram $\lambda/\mu$. Consider

$$H^\mu \ast T \in \text{SST}_{\mathbb{N} \ast A}(\lambda).$$

Using the switching procedures in Theorem 2.5 we obtain a unique $\mathbb{N} \ast \mathbb{N}$-semistandard tableau given by

$$T' \ast U \in \text{SST}_{\mathbb{N} \ast \mathbb{N}}(\lambda),$$

where $T' \in \text{SST}_A(\nu)$ for some $\nu$, and $U \in \text{LR}_A^{\lambda \nu \mu}$ (see also Example 3.3 in [2]). Let us put

$$j(T) = T', \quad j(T)_R = U.$$

Corollary 2.8 ([2]). Under the above hypothesis,

1. the map $J : T \mapsto (j(T), j(T)_R)$ gives a bijection

$$J : \text{SST}_A(\lambda/\mu) \rightarrow \bigsqcup_{\nu \in \mathcal{P}} \text{SST}_A(\nu) \times \text{LR}_A^{\lambda \nu \mu},$$

2. if $A = \mathbb{N}$, then the map $Q \mapsto j(Q)_R$ restricts to a bijection from $\text{LR}_{\mathbb{N}}^{\lambda \nu \mu}$ to $\text{LR}_{\mathbb{N}}^{\lambda \nu \mu}$. In particular, we have $N_{\lambda \nu \mu} = N_{\lambda \nu \mu}.$

□

Let us remark another symmetry of Littlewood-Richardson tableaux, a bijective proof of which has been given in [10]. For self-containedness, we give another proof using switching algorithm (see also [2]).

Corollary 2.9 (cf.2 [10]). There exists a bijection $\tau : \text{LR}_{\mu \nu}^{\lambda} \rightarrow \text{LR}_{\mu' \nu'}^{\lambda'}$ for $\lambda, \mu, \nu \in \mathcal{P}$.

Proof. Given $T \in \text{LR}_{\mu \nu}^{\lambda}$, consider $S \ast T^t$ where $S = H^{\nu'} \in \text{SST}_{\mathbb{N}}(\mu')$. Note that $S \ast T^t$ is an $\mathbb{N} \ast \mathbb{N}'$-semistandard tableau of shape $\lambda'$. Applying the switching procedures to the pair $(S, T^t)$ as far as possible, we obtain by Theorem 2.5 (3) an $\mathbb{N}' \ast \mathbb{N}$-semistandard tableau of shape $\lambda'$

$$(H^\nu)^t \ast S',$$

where $S' \in \text{LR}_{\nu' \mu'}^{\lambda'}$. Now, if we define $\tau(T) = j(S')_R$ (see Corollary 2.8 (2)), then $\tau$ is a bijection between $\text{LR}_{\mu \nu}^{\lambda}$ and $\text{LR}_{\mu' \nu'}^{\lambda'}$. □
One may define semistandard tableaux with entries from a given $\mathbb{Z}_2$-graded set $A$ with different linear orderings $<$ and $'$. Then the switching algorithm enables us to construct easily a bijection between these two kinds of semistandard tableaux of a given skew shape (cf. [2, 9, 24]).

First, consider $A_0 \ast A_1$, where the linear orderings on $A_0$ and $A_1$ remain the same. We may view $A_0 \ast A_1$ as a shuffling of $A$.

**Proposition 2.10.** For a skew Young diagram $\lambda/\mu$, there exists a bijection between $\text{SST}_A(\lambda/\mu)$ and $\text{SST}_{A_0 \ast A_1}(\lambda/\mu)$.

**Proof.** Let $<$ denotes the linear ordering on $A$. Given $T \in \text{SST}_A(\lambda/\mu)$, suppose that $T$ is not $A_0 \ast A_1$-semistandard. Then there exists an entry $a$ of degree 1 in $T$ such that $a < a'$ for some entry $a'$ of degree 0 in $T$. Let $a_{\text{max}}$ be the largest such entry of degree 1 in $T$. Also, we let $a_{\text{min}}$ be the smallest entry of degree 1 in $T$ such that there exists no entry of degree 0 in $T$ greater than $a_{\text{min}}$. Note that $a_{\text{max}} < a_{\text{min}}$. By definition, we can check that there is no entry $a$ of degree 1 in $T$ such that $a_{\text{max}} < a < a_{\text{min}}$.

Consider the tableau $S$ obtained by removing all the entries of $T$ smaller than $a_{\text{max}}$ or no less than $a_{\text{min}}$ (if there is no such $a_{\text{min}}$, then we assume $a_{\text{min}}$ to be a formal symbol greater than any element in $A$). Then

$$S = S' \ast S'',$$

where $S'$ is a tableaux of a skew shape with the entry $a_{\text{max}}$, and $S'$ is an $A_0$-semistandard tableau with entries in $\{ a \in A_0 | a_{\text{max}} < a < a_{\text{min}} \}$. Now if we apply the switching algorithm in Theorem 2.5 to the pair $(S', S'')$, then we get a new tableau $U$ of the same shape as $S$ such that

$$U = U' \ast U'' ,$$

where $U'$ is an $A_0$-semistandard tableau with entries in $\{ a \in A_0 | a_{\text{max}} < a < a_{\text{min}} \}$, and $U''$ is a tableaux of a skew shape with the entry $a_{\text{max}}$.

Repeating the above argument, we obtain a unique tableau $T^* \in \text{SST}_{A_0 \ast A_1}(\lambda/\mu)$. By construction, this gives the required bijective correspondence. $\square$

By similar arguments, we also have

**Proposition 2.11.** For a skew Young diagram $\lambda/\mu$, there exists a bijection between $\text{SST}_A(\lambda/\mu)$ and $\text{SST}_{A^*=}(\lambda/\mu)$.

3. $A/B$-SEMISTANDARD TABLEAUX
3.1. **Rational semistandard tableaux.** First, let us recall the notion of rational \([n]\)-semistandard tableaux \((n \geq 1)\) which was introduced by Stembridge \([25]\).

For \(n \geq 1\), let

\[
Z^n_+ = \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}
\]

be the set of all *generalized partitions of level* \(n\) (we will call \(n\) as the level of \(\lambda\) rather than its length to avoid a confusion with the length of a partition). We have a natural embedding \(\mathcal{P}_n \subset Z^n_+\). For \(\lambda \in Z^n_+\), we write \(\langle \lambda \rangle = \sum_{1 \leq i \leq n} \lambda_i\), and \(|\lambda| = \sum_{1 \leq i \leq n} |\lambda_i|\). We define

\[
\lambda^* = (-\lambda_n, \ldots, -\lambda_1).
\]

Clearly, \(\lambda^* \in Z^n_+\), and \(\lambda^{**} = \lambda\). We put \(0_n = (0, \ldots, 0) \in Z^n_+\).

Set \([-n] = \{-n < \cdots < -1\}\) with \([-n]_0 = [-n]\). Given \(\lambda = (\lambda_1, \cdots, \lambda_n) \in Z^n_+\), we may identify \(\lambda\) with a *generalized Young diagram* in the following way. First, we fix a vertical line. Then for each \(\lambda_k\), we place \(|\lambda_k|\) nodes in the \(k\)th row in a left-justified (resp. right-justified) way with respect to the vertical line if \(\lambda_k \geq 0\) (resp. \(\lambda_k \leq 0\)). For example,

\[
\lambda = (3, 2, 0, -1, -2) \leftrightarrow \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \\
-2 & -1 & 1 & 2 & 3
\end{array}
\]

where – denotes the empty row. We enumerate the columns of a diagram as in the above figure. We put

\[
\lambda^+ = (\max\{\lambda_1, 0\}, \cdots, \max\{\lambda_n, 0\}),
\lambda^- = (\max\{-\lambda_n, 0\}, \cdots, \max\{-\lambda_1, 0\}),
\]

where \(\lambda^\pm\) are understood as partitions (or Young diagrams).

**Definition 3.1** \([25]\). Let \(T\) be a tableau obtained by filling a generalized Young diagram \(\lambda\) of level \(n\) with entries in \([n] \cup [-n]\). We call \(T\) a *rational* \([n]\)-semistandard tableau of shape \(\lambda\) if

1. the entries in the columns indexed by positive (resp. negative) numbers belong to \([n]\) (resp. \([-n]\)),
2. the entries in each row (resp. column) are weakly (resp. strictly) increasing from left to right (resp. from top to bottom),
if $b_1 < \cdots < b_s$ (resp. $-b'_1 < \cdots < -b'_s$) are the entries in the 1st (resp. $1$-st) column ($s + t \leq n$), then

$$b''_i \leq b_i,$$

for $1 \leq i \leq s$, where $\{b''_1 < \cdots < b''_n\} = [n] \setminus \{b'_1, \cdots, b'_t}\}$.

We denote by $\text{SST}_n(\lambda)$ the set of all rational $[n]$-semistandard tableaux of shape $\lambda \in \mathbb{Z}^n_+$. 

**Example 3.2.** For $\lambda = (3, 2, 0, -1, -2)$, we have

$$\begin{array}{cccc}
2 & 3 & 5 \\
4 & 4 \\
- & - & - & - \\
-3 & - & 5 \\
\end{array} \in \text{SST}_5(\lambda).$$

Let us explain the relation between the rational $[n]$-semistandard tableaux and ordinary $[n]$-semistandard tableaux. Let $T$ be a rational $[n]$-semistandard tableau of shape $(0^{n-t}, 1^t)$ ($0 \leq t \leq n$) with the entries $-b_1 < \cdots < -b_t$. We define $\sigma(T)$ to be the $[n]$-semistandard tableau of shape $(1^{n-t}, 0^t)$ with the entries $b'_1 < \cdots < b'_{n-t}$, where $\{b'_1 < \cdots < b'_{n-t}\} = [n] \setminus \{b_1 < \cdots < b_t\}$. If $t = n$, then we define $\sigma(T)$ to be the empty tableau.

Generally, for $\lambda \in \mathbb{Z}^n_+$ and $T \in \text{SST}_n(\lambda)$, we define $\sigma(T)$ to be the tableau obtained by applying $\sigma$ to the $1$-st column of $T$. For example, when $n = 5$, we have

$$\begin{array}{cccc}
2 & 3 & 5 \\
4 & 4 \\
- & - & - & - \\
-5 & - & 5 \\
-4 & - & 2 \\
\end{array} = \begin{array}{cccc}
1 & 2 & 3 & 5 \\
3 & 4 & 4 \\
- & - & - & - \\
4 & - & - & - \\
\end{array}.$$

By Definition 3.1 it is straightforward to check that $\sigma(T) \in \text{SST}_n(\lambda + (1^n))$, where we define $\mu + \nu = (\mu_k + \nu_k)_{k \geq 1}$ for $\mu = (\mu_k)_{k \geq 1}$ and $\nu = (\nu_k)_{k \geq 1}$ in $\mathbb{Z}^n_+$.

**Lemma 3.3.** For $\lambda \in \mathbb{Z}^n_+$, the map $\sigma : \text{SST}_n(\lambda) \to \text{SST}_n(\lambda + (1^n))$ is a bijection. 

Next, let us introduce the notion of a rectangular complement of an $[n]$-semistandard tableau. Fix $n \geq 1$. For $\lambda \in \mathcal{P}_n$ and a tableau $T \in \text{SST}_n(\lambda)$, we define

$$\delta^n_k(T) = (\sigma^{-k}(T))^\pi,$$

for $k \geq \lambda_1$. We put

$$\delta^n_k(\lambda) = (k - \lambda_n, \ldots, k - \lambda_1).$$
Then $\delta_n^k(\lambda)$ is the shape of $\delta_n^k(T)$. By definition, we have $\delta_n^k(T) \in \text{SST}_{[-n]^\pi}(\delta_n^k(\lambda))$.

Identifying $[-n]^\pi$ with $[n]$ ($-i$ with $i$ for each $i$), we may view $\delta_n^k(T)$ as an element in $\text{SST}_{[n]}(\delta_n^k(\lambda))$.

**Example 3.4.** If $\lambda = (4, 3, 1) \in \mathcal{P}_4$, and $T = \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & 4 \\ 4 & & \end{array}$, then

$$\sigma^{-5}(T) = \begin{array}{ccc} -4 & & \\ -4 & -3 & \\ -3 & -2 & -2 \\ -2 & -1 & -1 & -1 & -1 \end{array}$$

$$\delta_5^4(T) = [\sigma^{-5}(T)]^\pi = \begin{array}{ccc} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & & \\ 4 & & & & \end{array}.$$
By Theorem 2.3 and the linear independence of Schur polynomials, it follows that for \( \mu, \nu \in \mathbb{Z}^n_+ \),

\[
(3.7) \quad s_{\mu}(x_{[n]})s_{\nu}(x_{[n]}) = \sum_{\lambda \in \mathbb{Z}^+_n} c^\lambda_{\mu\nu}s_{\lambda}(x_{[n]}),
\]

where \( c^\lambda_{\mu\nu} = N_{\lambda+((p+q)n)}^{\mu+(p^n)\nu+(q^n)} \) for all sufficiently large \( p, q > 0 \). By definition, we also have \( c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu} \). For later use, let us characterize \( c^\lambda_{\mu\nu} \) more explicitly in terms of Littlewood-Richardson tableaux.

**Lemma 3.7.** Suppose that \( \lambda, \mu \) and \( \nu \in \mathcal{P}_n \) are given. For \( p, q \geq 0 \), there exists a bijection

\[
\pi^n_{p,q} : LR^\lambda_{\mu\nu} \rightarrow LR^\lambda_{(p+q)n}\mu+(p^n)\nu+(q^n),
\]

**Proof.** Suppose that \( LR^\lambda_{\mu\nu} \) is non-empty and \( Q \in LR^\lambda_{\mu\nu} \) is given. Clearly, we may view \( Q \) as an element in \( LR^\lambda_{\mu+(k^n)\nu} \) for \( k \geq 0 \). This implies the bijection

\[
\pi^n_{k,0} : LR^\lambda_{\mu\nu} \rightarrow LR^\lambda_{\mu+(k^n)\nu}.
\]

Then \( \pi^n_{p,q} \) is given by \( \pi^n_{p,q} = \theta \circ \pi^n_{q,0} \circ \theta \circ \pi^n_{p,0} \), where \( \theta \) denotes the map given in Corollary 2.8 (2). \( \square \)

Suppose that \( \lambda, \mu \) and \( \nu \in \mathbb{Z}^n_+ \) are given. Let \( p \) and \( q \) be the smallest non-negative integers such that \( \mu + (p^n), \nu + (q^n), \lambda + ((p + q)n) \in \mathcal{P}_n \). Then we define

\[
(3.8) \quad LR_{\nu/\mu}^\lambda = \{ [Q] \mid Q \in LR^\lambda_{\mu+(p^n)\nu+(q^n)} \},
\]

where \( [Q] = \{ \pi^n_{s,t}(Q) \mid s, t \geq 0 \} \).

**Lemma 3.8.** Under the above hypothesis, \( |LR_{\nu/\mu}^\lambda| = c^\lambda_{\mu\nu} \)

**Proof.** It follows directly from (3.7). \( \square \)

Next, if we put \( x_{[m+n]} = x_{[m]} \sqcup y_{[n]} \) for \( m, n > 0 \), where \( y_{[n]} = \{ y_i = x_{m+i} \mid i \in [n] \} \), then by Corollary 2.8 (1), we have for \( \lambda \in \mathbb{Z}^{m+n}_+ \)

\[
(3.9) \quad s_\lambda(x_{[m+n]}) = s_\lambda(x_{[m]}, y_{[n]}) = \sum_{\mu, \nu} \hat{c}^\lambda_{\mu\nu}s_\mu(x_{[m]})s_\nu(y_{[n]}),
\]

where \( \hat{c}^\lambda_{\mu\nu} = N_{\mu+(p^n)\nu+(p^n)}^{\lambda+((p+q)n)} \) for all sufficiently large \( p > 0 \). We may also characterize \( \hat{c}^\lambda_{\mu\nu} \) in terms of Littlewood-Richardson tableaux.

**Lemma 3.9.** For \( \lambda \in \mathcal{P}_{m+n}, \mu \in \mathcal{P}_m \) and \( \nu \in \mathcal{P}_n \), there exists a bijection

\[
\pi^{m,n}_\ell : LR^{\lambda+((m+n)n)'}_{(\mu+(m)n)'}(\nu+(n)') \rightarrow LR^\lambda_{\mu'\nu'},
\]

where \( \ell \geq 0 \).
Proof. Suppose that $LR_{(\mu+(\ell m^{\nu}))}^{(\lambda+(\ell^m+n))'}$ is non-empty and $Q \in LR_{(\mu+(\ell m^{\nu}))}^{(\lambda+(\ell m^{\nu}))'}$ is given. Note that the $i$th row of $Q$ is filled with $i$ for $1 \leq i \leq \ell$ (in fact, the first $\ell$ rows of $Q$ is of the form $H^{(n^i)}$), and the other entries in $Q$ are greater than $\ell$ since the content of $Q$ is $(\nu+(\ell m^{\nu}))'$. Now, we define $\pi_{\ell}^{m,n}(Q)$ to be the tableau obtained from $Q$ by

1. removing the first $\ell$ rows of $Q$,
2. replacing each entry $i$ in the remaining part of $Q$ by $i - \ell$.

Then it is not difficult to see that $\pi_{\ell}^{m,n}(Q) \in LR_{\mu',\nu'}^{\lambda',\nu}$, Since the first $\ell$ rows in any $Q \in LR_{(\mu+(\ell m^{\nu}))'}^{(\lambda+(\ell m^{\nu}))'}$ is of the form $H^{(n^i)}$, any $Q' \in LR_{\mu',\nu'}^{\lambda',\nu}$ is given by $\pi_{\ell}^{m,n}(Q)$ for a unique $Q \in LR_{(\mu+(\ell m^{\nu}))'}^{(\lambda+(\ell m^{\nu}))'}$. Hence $\pi_{\ell}^{m,n}$ is a bijection.

Suppose that $\mu \in \mathbb{Z}_+^m$, $\nu \in \mathbb{Z}_+^n$, and $\lambda \in \mathbb{Z}_+^{m+n}$ are given. Let $d$ be the smallest non-negative integer such that $\lambda + (d^m+n), \mu + (d^m), \nu + (d^n) \in \mathcal{P}$. Then we define

\begin{equation}
LR_{\mu,\nu}^{\lambda} = \{ [Q] \mid Q \in LR_{(\mu+(d^m+n))'}^{(\lambda+(d^m+n))'} \},
\end{equation}

where $[Q]$ is the set of all the bijective images $(\pi_{\ell}^{m,n})^{-1}(Q)$ ($\ell \geq 0$). From (3.9), it follows that

Lemma 3.10. Under the above hypothesis, $|LR_{\mu,\nu}^{\lambda}| = e_{\mu,\nu}^{\lambda}$.

Proof. It follows from (3.9) and Corollary 2.9.

3.2. $\mathcal{A}/\mathcal{B}$-semistandard tableaux. Now, let us introduce a certain class of bitableaux, which is our main object in this paper.

Definition 3.11. Suppose that $\lambda \in \mathbb{Z}_+^n$ is given. An $\mathcal{A}/\mathcal{B}$-semistandard tableau of shape $\lambda$ is a pair of tableaux $(T^+, T^-)$ such that

\begin{equation}
T^+ \in \text{SST}_A((\lambda + (d^m))/\mu), \quad T^- \in \text{SST}_B((d^n)/\mu),
\end{equation}

for some integer $d \geq 0$ and $\mu \in \mathcal{P}_n$ satisfying

1. $\lambda + (d^m) \in \mathcal{P}_n$,
2. $\mu \subset (d^n)$, and $\mu \subset \lambda + (d^n)$.

We denote by $\text{SST}_{\mathcal{A}/\mathcal{B}}(\lambda)$ the set of all $\mathcal{A}/\mathcal{B}$-semistandard tableaux of shape $\lambda$. For $(T^+, T^-) \in \text{SST}_{\mathcal{A}/\mathcal{B}}(\lambda)$, we say that $(T^+, T^-)$ is of level $n$. We set

\begin{equation}
\mathcal{P}_{\mathcal{A}/\mathcal{B}} = \bigsqcup_{n \geq 1} \mathcal{P}_{\mathcal{A}/\mathcal{B},n}
\end{equation}

where $\mathcal{P}_{\mathcal{A}/\mathcal{B},n} = \{ \lambda \in \mathbb{Z}_+^n \mid \text{SST}_{\mathcal{A}/\mathcal{B}}(\lambda) \neq \emptyset \}$
Example 3.12. Suppose that $\mathcal{A} = \mathcal{B} = \mathbb{N}$. Consider

$$(T^+, T^-) = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 4 & 1 \\ 2 & 3 & 3 & 2 & 4 \\ 4 & 3 & 3 & 5 & \end{pmatrix} \in SST_{\mathcal{A}/\mathcal{B}}((3, 2, 0, -2)),$$

where the vertical lines in $T^+$ and $T^-$ correspond to the one in the generalized partition $(3, 2, 0, -2)$. Note that

$$sh(T^+) = ((3, 2, 0, -2) + (3^4)) / (2, 1, 0, 0),$$
$$sh(T^-) = (3^4) / (2, 1, 0, 0).$$

Example 3.13. Let $\lambda = 0_n \in \mathbb{Z}_+^n$. Then for $(T^+, T^-) \in SST_{\mathcal{A}/\mathcal{B}}(0_n)$,

$$sh(T^+) = sh(T^-) = (d^n) / \mu$$

for some $d \geq 0$ and $\mu \in \mathcal{P}_n$. If we identify $(T^+, T^-)$ with $((T^+)^\pi, (T^-)^\pi)$, then we have

$$(3.12) \quad SST_{\mathcal{A}/\mathcal{B}}(0_n) = \bigsqcup_{\lambda \in \mathcal{P}_n} SST_{\mathcal{A}}(\lambda) \times SST_{\mathcal{B}}(\lambda),$$

by Lemma 2.11.

The decomposition (3.12) can be viewed as a branching rule, and it can be generalized as follows.

Proposition 3.14. For $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B}, n}$, there exists a bijection between $SST_{\mathcal{A}/\mathcal{B}}(\lambda)$ and

$$\bigsqcup_{\mu, \nu \in \mathcal{P}_n} LR_{d^\pi}^{\lambda/\mu} \times SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{B}}(\nu).$$

Proof. By Definition 3.11 we have

$$SST_{\mathcal{A}/\mathcal{B}}(\lambda)$$

\[\overset{1\sim 1}{\longleftrightarrow}\] $\bigsqcup_{d \geq 0}^{\eta \in \mathcal{P}_n} SST_{\mathcal{A}}((\lambda + (d^n))/\eta) \times SST_{\mathcal{B}}((d^n)/\eta)$

\[\overset{1\sim 1}{\longleftrightarrow}\] $\bigsqcup_{d \geq 0}^{\eta, \mu \in \mathcal{P}_n} LR_{\mu/\eta}^{\lambda + (d^n)} \times SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{B}}((d^n)/\eta)$ by Corollary 2.8

\[\overset{1\sim 1}{\longleftrightarrow}\] $\bigsqcup_{d \geq 0}^{\eta, \mu \in \mathcal{P}_n} LR_{\mu/\eta}^{\lambda/\mu} \times SST_{\mathcal{A}}(\mu) \times SST_{\mathcal{B}}((d^n)/\eta)$ by (3.8).
where the union is taken over \( d \geq 0 \) and \( \eta \in \mathcal{P}_n \), which satisfy the conditions (1) and (2) in Definition 3.11. For each \( d \geq 0 \) and \( \eta \in \mathcal{P}_n \) above, there exists a unique \( \nu \in \mathcal{P}_n \) such that \((d^n)/\eta = \nu^\circ\), and \( \eta - (d^n) = \nu^* \). Hence, we have

\[
SST_B((d^n)/\eta) \xrightarrow{1-1} SST_B^* (\nu) \xrightarrow{1-1} SST_B (\nu),
\]

where the first correspondence is given by \( \pi \), and the second one is given by Proposition 2.11. This completes the proof. \(\square\)

3.3. Characters. From now on, let \( x_{\mathcal{A}} = \{ x_a \mid a \in \mathcal{A} \} \) be the set of variables indexed by \( \mathcal{A} \). Let \( P_{\mathcal{A}} = \bigoplus_{a \in \mathcal{A}} \mathbb{Z} \epsilon_a \) be the free abelian group with the basis \( \{ \epsilon_a \mid a \in \mathcal{A} \} \). For \( \lambda \in \mathcal{P}_\mathcal{A} \) and \( T \in SST_\mathcal{A}(\lambda) \), we define the \( \mathcal{A} \)-weight of \( T \) by \( wt_\mathcal{A}(T) = \sum_{a \in \mathcal{A}} m_a \epsilon_a \in P_\mathcal{A} \), where \( m_a \) is the number of occurrences of \( a \) in \( T \). Put \( x_\mathcal{A}^T = \prod_{a \in \mathcal{A}} x_a^{m_a} \). Now, we define the character of \( SST_\mathcal{A}(\lambda) \) to be

\[
(3.13) \quad S_\lambda(x_\mathcal{A}) = \sum_{T \in SST_\mathcal{A}(\lambda)} x_\mathcal{A}^T \in \mathbb{Z}[[x_\mathcal{A}]].
\]

For simplicity, let us often write \( S_\lambda^A = S_\lambda(x_\mathcal{A}) \). We assume that \( S_\lambda^A = 0 \) unless \( \lambda \in \mathcal{P}_\mathcal{A} \). The character of \( SST_\mathcal{A}(\lambda/\mu) \) of a skew Young diagram is defined similarly.

**Example 3.15.** (1) If \( \mathcal{A} = [n] \) and \( \lambda \in \mathcal{P}_n \), then \( S_\lambda(x_{[n]}) = s_\lambda(x_{[n]}) \) is the Schur polynomial corresponding to \( \lambda \) (cf. [22]).

(2) Suppose that \( \mathcal{A} = [m] \ast [n]' \). To distinguish \([m] \) and \([n]'\) as sets, let us write \([n]' = \{ 1' < 2' < \cdots < n' \} \). Then \( S_\lambda(x_\mathcal{A}) \) is the \((m,n)\)-hook Schur polynomial corresponding to \( \lambda \), which is the character of an irreducible representation of the general linear Lie superalgebra \( \mathfrak{gl}_{m|n} \) (see [1]). Note that \( S_\lambda(x_\mathcal{A}) \neq 0 \) if and only if \( \lambda_{m+1} \leq n \), that is, \( \lambda \) is an \((m,n)\)-hook partition.

**Lemma 3.16.** For \( \lambda \in \mathcal{P}_\mathcal{A} \), \( S_\lambda^A = S_\lambda^{A_0 \ast A_1} = S_\lambda^{A''} \).

**Proof.** It follows directly from Proposition 2.10 and 2.11. \(\square\)

**Lemma 3.17.** The set \( \{ S_\lambda^A \mid \lambda \in \mathcal{P}_\mathcal{A} \} \) is linearly independent over \( \mathbb{Z} \).

**Proof.** Suppose that

\[
\sum_{1 \leq i \leq m} a_{\lambda(i)} S_{\lambda(i)}^A = 0,
\]

where \( \lambda^{(i)} \in \mathcal{P}_\mathcal{A} \) and \( a_{\lambda(i)} \in \mathbb{Z} \) for \( 1 \leq i \leq m \).

Choose a finite subset \( \hat{\mathcal{A}} \subset \mathcal{A} \) such that \( S_{\lambda(i)}^{\hat{\mathcal{A}}} \neq 0 \) for all \( 1 \leq i \leq m \). If we put \( x_a = 0 \) for all \( a \notin \hat{\mathcal{A}} \) in \( S_{\lambda(i)}^{\hat{\mathcal{A}}} \), then we get

\[
\sum_{1 \leq i \leq n} a_{\lambda(i)} S_{\lambda(i)}^{\hat{\mathcal{A}}} = 0.
\]
By Lemma 3.16 $S^\lambda_{\mu}$ are hook Schur polynomials (see Example 3.15). Then the linear independence follows from that of ordinary hook Schur polynomials (see [1]).

Since the bijections in Theorem 2.3 and Corollary 2.8 (1) preserve the weights of $A$-semistandard tableaux, it follows that

**Corollary 3.18.**

1. For $\mu, \nu \in \mathcal{P}_A$, we have $S^A_\mu S^A_\nu = \sum_{\lambda \in \mathcal{P}_A} N^\lambda_{\mu \nu} S^A_\lambda$.
2. For a skew Young diagram $\lambda/\mu$, we have $S^A_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}_A} N^\lambda_{\mu \nu} S^A_\nu$.

In particular, for $\lambda \in \mathcal{P}_{A/B}$, we have

$$S^A_{\lambda/B} = \sum_{\mu \in \mathcal{P}_B} S^A_\mu S^B_{\lambda/\mu} = \sum_{\mu \in \mathcal{P}_A, \nu \in \mathcal{P}_B} N^\lambda_{\mu \nu} S^A_\mu S^B_\nu.$$ 

Now, for $\lambda \in \mathbb{Z}_+^n$, we define the *character* of $SST_{A/B}(\lambda)$ to be

$$S_{\lambda}(x_A, x_B) = \sum_{(T^+, T^-) \in SST_{A/B}(\lambda)} x_A^{T^+} (x_B^{-1})^{T^-} \in \mathbb{Z}[[x_A, x_B^{-1}]],$$

where $x_B^{-1} = \{ x_b^{-1} \mid b \in B \}$. For simplicity, let us write $S^A_{\lambda/B} = S_{\lambda}(x_A, x_B)$. We assume that $S^A_{\lambda/B} = 0$ unless $\lambda \in \mathcal{P}_{A/B}$. It is easy to check that $S^A_{\lambda/B}$ is a well-defined element in $\mathbb{Z}[[x_A, x_B^{-1}]]$.

Note that each non-zero monomial in $S^A_{\lambda/B}$ is of homogeneous degree $\langle \lambda \rangle = \sum_{i=1}^n \lambda_i$, and the degree of each monomial in $x_A$ (resp. $x_B^{-1}$) is at least $|\lambda^+|$ (resp. $|\lambda^-|$). For $(T^+, T^-) \in SST_{A/B}(\lambda)$, we also define the *$A/B$-weight of $T$* by $\text{wt}_{A/B}(T) = \text{wt}_A(T^+) - \text{wt}_B(T^-) \in P_A \oplus P_B$.

**Remark 3.19.** If we identify $\mathbb{Z}[[x_A, x_B^{-1}]]$ with $\mathbb{Z}[[x_A]] \otimes \mathbb{Z}[[x_B^{-1}]]$, then the set

$$S_{A/B} = \{ S_{\lambda}(x_A) S_\mu(x_B^{-1}) \mid \lambda \in \mathcal{P}_A, \mu \in \mathcal{P}_B \}$$

is linearly independent over $\mathbb{Z}$. Consider

$$\sum_{\lambda, \mu} c_{\lambda \mu} S_{\lambda}(x_A) S_\mu(x_B^{-1}),$$

for $c_{\lambda \mu} \in \mathbb{Z}$, which is not necessarily a finite sum. Then we can check that it is a well-defined element in $\mathbb{Z}[[x_A, x_B^{-1}]]$ since each monomial in $x_A$ and $x_B^{-1}$ occurs only in finitely many $\lambda$ and $\mu$’s. From the linear independence of $S_{A/B}$, $c_{\lambda \mu}$ is uniquely determined for all $\lambda, \mu$.

Now, we can express $S_{\lambda}(x_A, x_B)$ as a (possibly infinite) linear combination of $S_{\lambda}(x_A) S_\mu(x_B^{-1})$’s as follows.
For \( \lambda \in \mathcal{P}_{A/B,n} \), we have
\[
S_\lambda(x_A, x_B) = \sum_{\mu, \nu \in \mathcal{P}_n} c_{\mu \nu}^\lambda S_\mu(x_A)S_\nu(x_B^{-1}).
\]

**Proof.** It follows from Proposition 3.14. \( \square \)

**Example 3.21.** When \( \lambda = 0_n \), we have
\[
S_{0_n}(x_A, x_B) = \sum_{\lambda \in \mathcal{P}_n} S_\lambda(x_A)S_\lambda(x_B^{-1}).
\]

**Proposition 3.22.** The set \( \{ S_\lambda^{A/B} | \lambda \in \mathcal{P}_{A/B} \} \) is linearly independent over \( \mathbb{Z} \).

**Proof.** Suppose that
\[
\sum_{1 \leq i \leq m} a_{\lambda(i)} S_{\lambda(i)}^{A/B} = 0,
\]
where \( \lambda(i) \in \mathcal{P}_{A/B} \) and \( a_{\lambda(i)} \in \mathbb{Z} \) for \( 1 \leq i \leq m \). Suppose that the level of \( \lambda(i) \) is \( n_i \) for \( 1 \leq i \leq m \). We will use induction on \( m \) to show that \( a_{\lambda(i)} = 0 \) for \( 1 \leq i \leq m \). Clearly, it is true when \( m = 1 \).

Let \( n = \max\{n_1, \ldots, n_m\} \), and let \( I = \{ i \mid n_i = n \} \). Choose a positive integer \( d \) such that \( \lambda(i) + (d^n) \in \mathcal{P} \) for all \( i \in I \). Then if we write \( S_{\lambda(i)}^{A/B} (i \in I) \) as a linear combination of \( S_\lambda(x_A)S_\mu(x_B^{-1}) \)'s, then the coefficient of \( S_{(d^n)}(x_B^{-1}) \) in \( S_{\lambda(i)}^{A/B} \) is \( S_{\lambda(i)}^{(d^n)}(x_A) \) for \( i \in I \) (cf. Proposition 3.14 and 3.20). Since \( S_{(d^n)}(x_B^{-1}) \) occurs only in the expansion of \( S_{\lambda(i)}^{A/B} \) for \( i \in I \), we have
\[
\sum_{i \in I} a_{\lambda(i)} S_{\lambda(i)}^{(d^n)}(x_A) = 0,
\]
from the linear independence of \( S_{A,B} \) (see also Remark 3.19). Since \( \lambda(i) + (d^n) \) are mutually different for \( i \in I \), we have \( a_{\lambda(i)} = 0 \) for \( i \in I \), and hence by induction hypothesis, \( a_{\lambda(i)} = 0 \) for all \( 1 \leq i \leq m \). \( \square \)

**Corollary 3.23.** For \( n \geq 1 \), consider \( \sum_{\lambda \in \mathcal{P}_{A/B,n}} a_\lambda S_\lambda^{A/B} \), where \( a_\lambda \in \mathbb{Z} \), which is not necessarily a finite sum. Then it is a well-defined element in \( \mathbb{Z}[[x_A, x_B^{-1}]] \), and the coefficient \( a_\lambda \) is uniquely determined for \( \lambda \in \mathcal{P}_{A/B,n} \).

**Proof.** Note that for \( \mu, \nu \in \mathcal{P}_n \), \( S_\mu(x_A)S_\nu(x_B^{-1}) \) occurs in the expansion of \( S_\lambda^{A/B} \) only if
\[
|\mu| \geq |\lambda^+| \quad \text{and} \quad |\nu| \geq |\lambda^-| \quad \text{(or} |\mu| + |\nu| \geq |\lambda| \text{)}.
\]
Since the level of \( \lambda \) is fixed, there are only finitely many \( \lambda \)'s satisfying (3.15), and hence the coefficient of \( S_\mu(x_A)S_\nu(x_B^{-1}) \) in \( \sum_{\lambda \in \mathcal{P}_{A/B,n}} a_\lambda S_\lambda^{A/B} \) is a well-defined integer. This implies that \( \sum_{\lambda \in \mathcal{P}_{A/B,n}} a_\lambda S_\lambda^{A/B} \) is a well-defined element in \( \mathbb{Z}[[x_A, x_B^{-1}]] \) (see Remark 3.19).
Next, suppose that \( \sum_{\lambda \in \mathcal{P}_{A/B}, n} a_{\lambda} S_{\lambda}^{A/B} = 0 \). For \( d > 0 \), the coefficient of \( S_{(d^n)}(x_B^{-1}) \) in \( \sum_{\lambda \in \mathcal{P}_{A/B}, n} a_{\lambda} S_{\lambda}^{A/B} \) is given by

\[
\sum_{\lambda + (d^n) \in \mathcal{P}_n} a_{\lambda} S_{\lambda+(d^n)}(x_A).
\]

Then we have \( a_{\lambda} = 0 \) for all \( \lambda + (d^n) \in \mathcal{P}_n \) from the linear independence of \( \{ S_{\lambda}^A \mid \lambda \in \mathcal{P}_A \} \) (even if it is not a finite sum). Since \( d \) is an arbitrary positive integer, it follows that \( a_{\lambda} = 0 \) for all \( \lambda \in \mathcal{P}_{A/B}, n \).

\[\square\]

4. Insertion scheme

In this section, we will describe the combinatorial behavior of \( A/B \)-semistandard tableaux, which is closely related to that of rational \([n]\)-semistandard tableaux. We will introduce an algorithm of inserting an \( A/B \)-semistandard tableau into another, and derive analogues of Robinson-Schensted-Knuth correspondence and Littlewood-Richardson rule. We also obtain a Jacobi-Trudi type character formula for \( SST_{A/B}(\lambda) \).

4.1. Robinson-Schensted-Knuth correspondence. Let

\[(4.1) \quad \mathcal{F}_{A/B} = \bigsqcup_{c \in \mathbb{Z}} SST_{A/B}(c)\]

be the set of all \( A/B \)-semistandard tableaux of level 1.

Note that for \( (w^+,w^-) \in \mathcal{F}_{A/B} \), \( w^+ \) (resp. \( w^- \)) is a semistandard tableau of a single row, and \( \text{sh}(w^+) \in \mathbb{Z}_{\geq 0} \), and \( (w^+,w^-) \in SST_{A/B}(c) \) if and only if \( \text{sh}(w^+)-\text{sh}(w^-) = c \).

**Theorem 4.1.** For \( n \geq 1 \), there exists a bijection

\[
\kappa_{A/B} : \mathcal{F}_{A/B}^n \rightarrow \bigsqcup_{\lambda \in \mathcal{P}_{A/B,n}} SST_{A/B}(\lambda) \times SST_{[n]}(\lambda),
\]

where \( \mathcal{F}_{A/B}^n \) is the set of all \( n \)-tuples of \( A/B \)-semistandard tableaux of level 1.

**Proof.** Fix \( n \geq 1 \). To each ordered \( n \)-tuple \( w = ((w_i^+),w_i^-))_{1 \leq i \leq n} \in \mathcal{F}_{A/B}^n \), we will associate a pair \( \kappa_{A/B}(w) = (P_w,Q_w) \in SST_{A/B}(\lambda) \times SST_{[n]}(\lambda) \) for some \( \lambda \in \mathcal{P}_{A/B,n} \), as follows.

**Step 1.** First, we let

\[
T^- = (((w_1^-)^{\pi} \leftarrow (w_2^-)^{\pi}) \cdots) \leftarrow (w_n^-)^{\pi})^{\pi}.
\]
Let \( Q \) be the recording tableau for \(((w_1^-)^n \leftarrow (w_2^-)^n) \cdots \leftarrow (w_n^-)^n\) (see Proposition 2.4). That is, \( g_{\text{col}}((w_1^-)^n, \ldots, (w_n^-)^n) = ((T^-)^n, Q) \). We assume that \( \text{sh}(T^-) = (d^n)/\mu \) for some \( d \geq 0 \) and \( \mu \in \mathcal{P}_n \). Then \( \text{sh}(Q) = \delta_d^\mu(\mu) \).

**Step 2.** Next, we will define \( T^+ \) using \( w_i^+ \) (\( 1 \leq i \leq n \)). Since the shape of \( T^+ \) must be of the form \((\lambda + (d^n))/\mu\) for some \( \lambda \in \mathbb{Z}_+^n \), we will consider the row insertions of \( S_i * w_i^+ \)'s instead of \( w_i^+ \)'s, where \( S_i \) is an \( \mathbb{N} \)-semistandard tableau of a single row such that the recording tableau of the row insertions of \( S_i \)'s is a rectangular complement of \( Q \). Let us explain it more precisely.

Set

\[
Q' = \delta_d^\mu(Q).
\]

Note that \( \text{sh}(Q') = \mu \).

Suppose that \( \text{wt}_{[n]}(Q') = \sum_{i=1}^n \nu_i \epsilon_i \) (or the content of \( Q' \) is \( \nu = (\nu_1, \ldots, \nu_n) \)). Applying Proposition 2.4 when \( A = [n] \), there exists a unique \((S_1, \ldots, S_n)\) such that \( S_i \in SST_{[n]}(\nu_i) \) for \( 1 \leq i \leq n \), and

\[
\theta_{\text{row}}(S_1, \ldots, S_n) = (H^\mu, Q^\prime) \in SST_{[n]}(\mu) \times SST_{[n]}(\mu)_{\nu}.
\]

We assume that \( S_i \) is empty if \( \nu_i = 0 \).

For \( 1 \leq i \leq n \), we put

\[
U_i = S_i * w_i^+,
\]

which is an \([n] * A\)-semistandard tableau of a single row with length \( \nu_i + \text{sh}(w_i^+) \).

Applying Proposition 2.4 once again to \( U_i \)'s, we have

\[
\theta_{\text{row}}(U_1, \ldots, U_n) = (U, U_R),
\]

where \( U \in SST_{[n]*A}(\lambda + (d^n)) \) and \( U_R \in SST_{[n]}(\lambda + (d^n)) \) for some \( \lambda \in \mathbb{Z}_+^n \). Since \( i < a \) for all \( i \in [n] \) and \( a \in A \) with respect to the linear ordering on \([n] * A\), we have

\[
U = H^\mu * T^+,
\]

where \( T^+ \in SST_A((\lambda + (d^n))/\mu) \).

Now, we define

\[
4.2 \quad P_w = (T^+, T^-), \quad Q_w = \sigma^{-d}(U_R).
\]

Then, we have \( P_w \in SST_A(\lambda) \) and \( Q_w \in SST_{[n]}(\lambda) \). Since the correspondence \( w \mapsto (P_w, Q_w) \) is reversible by construction, \( \kappa_{A/B} \) is a bijection.

**Example 4.2.** Suppose that \( A = \{a_1 < a_2 < a_3 < \cdots\} \) and \( B = \{b_1 < b_2 < b_3 < \cdots\} \), where all the elements are of degree 0. Let \( w = ((w_i^+, w_i^-))_{i=1,2} \) be given by

\[
(w_1^+, w_1^-) = (a_1 a_1 a_2 a_4 a_5, b_3 b_3 b_4 b_6) \in SST_{A/B}(1),
\]

\[
(w_2^+, w_2^-) = (a_1 a_3 a_6, b_2 b_3 b_6) \in SST_{A/B}(0).
\]
Then,

\[ T^- = ((w^-_1)^\pi \leftarrow (w^-_2)^\pi)^\pi = (b_0b_1b_3b_3 \leftarrow b_0b_3b_2)^\pi = b_3 b_4 b_6 b_6. \]

Since the recording tableau for \((b_6b_3b_3 \leftarrow b_0b_3b_2)\) is \(Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 \end{pmatrix}\), we have \(Q' = \delta_5^2(Q) = 122\), where we put \(d = 5\). If we put \(S_1 = 1\) and \(S_2 = 11\), then \(\varrho_{row}(S_1, S_2) = (H^{(3,0)}, Q')\). Put

\[
U_1 = 1 * a_1a_1a_2a_4a_5, \quad U_2 = 11 * a_1a_3a_6.
\]

Then

\[
(U_2 \rightarrow U_1) = H^{(3,0)} * T^+ = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ a_1 & a_2 \end{pmatrix} a_3, a_5, a_6,
\]

\[
(U_2 \rightarrow U_1)_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.
\]

Therefore, we have

\[
P_w = \begin{pmatrix} a_1 & a_1 & a_3 & a_4 & a_5 & a_6 & b_3 & b_4 & b_5 \end{pmatrix},
\]

\[
Q_w = \begin{pmatrix} 1 & 2 \\ -2 \end{pmatrix},
\]

where \((P_w, Q_w) \in \text{SST}_{A/B}(\langle 2, -1 \rangle) \times \text{SST}_{[2]}(\langle 2, -1 \rangle)\).

Let us consider the character identity associated to RSK correspondence in Theorem 4.1. Given \(w = ((w^+_i, w^-_i))_{1 \leq i \leq n} \in \mathcal{F}_{A/B}^n\), we set

\[
\text{wt}_{A/B}(w) = \sum_{1 \leq i \leq n} \text{wt}_{A/B}(w^+_i, w^-_i) \in P_A \oplus P_B,
\]

\[
\text{wt}_{[n]}(w) = \sum_{1 \leq i \leq n} m_i \epsilon_i \in P_{[n]},
\]

where \(m_i = \text{sh}(w^+_i) - \text{sh}(w^-_i)\), the shape of \((w^+_i, w^-_i)\) for \(1 \leq i \leq n\). Then the character of \(\mathcal{F}_{A/B}\) is given by

\[
\prod_{i \in [n]} \frac{\prod_{a \in A_1} (1 + x_a x_i) \prod_{b \in B_1} (1 + x_b^{-1} x_i^{-1})}{\prod_{a \in A_0} (1 - x_a x_i) \prod_{b \in B_0} (1 - x_b^{-1} x_i^{-1})}.
\]

**Corollary 4.3.** The map \(\kappa_{A/B} : w \mapsto (P_w, Q_w)\) in Theorem 4.1 is a bijection preserving weights, that is,

\[
\text{wt}_{A/B}(w) = \text{wt}_{A/B}(P_w), \quad \text{wt}_{[n]}(w) = \text{wt}_{[n]}(Q_w).
\]
Hence, we obtain the following identity:

$$\prod_{i \in [n]} \frac{\prod_{a \in A_i} (1 + x_a x_i)}{\prod_{a \in A_0} (1 - x_a x_i)} \prod_{b \in B_i} (1 + x_b^{-1} x_i) \prod_{b \in B_0} (1 - x_b^{-1} x_i) = \sum_{\lambda \in \mathcal{P}_{A/B,n}} S^A/B_{\lambda} s_{\lambda}(x_{[n]}).$$

**Proof.** It follows directly from Theorem 4.1 and Corollary 4.3 that $\text{wt}_{A/B}(w) = \text{wt}_{A/B}(P_w)$. So, it suffices to show that

$$\text{wt}_{[n]}(w) = \text{wt}_{[n]}(Q_w).$$

Following the notations in Theorem 4.1, we have

$$\text{wt}_{[n]}(Q) = \sum_{1 \leq i \leq n} \text{sh}(w_i^-) \epsilon_i, \quad \text{wt}_{[n]}(Q^\vee) = -\text{wt}_{[n]}(Q) + d(\epsilon_1 + \cdots + \epsilon_n),$$

$$\text{wt}_{[n]}(U_R) = \text{wt}_{[n]}(Q^\vee) + \sum_{1 \leq i \leq n} \text{sh}(w_i^+) \epsilon_i,$$

$$\text{wt}_{[n]}(Q_w) = \text{wt}_{[n]}(U_R) - d(\epsilon_1 + \cdots + \epsilon_n),$$

(recall that $\text{wt}_{[n]}(\sigma(T)) = \text{wt}_{[n]}(T) + (\epsilon_1 + \cdots + \epsilon_n)$ for $T \in \text{SST}_{[n]}(\lambda)$). Since $\text{wt}_{[n]}(w) = \sum_{1 \leq i \leq n} (\text{sh}(w_i^+) - \text{sh}(w_i^-)) \epsilon_i$, we have $\text{wt}_{[n]}(w) = \text{wt}_{[n]}(Q_w)$. \hfill $\square$

**Corollary 4.4.** For $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ such that $\nu_i \in \mathcal{P}_{A/B}$, the map $\kappa_{A/B}$ also gives the following bijection

$$\kappa_{A/B} : \text{SST}_{A/B}(\nu_1) \times \cdots \times \text{SST}_{A/B}(\nu_n) \longrightarrow \bigsqcup_{\mu \in \mathcal{P}_{A/B,n}} \text{SST}_{A/B}(\mu) \times \text{SST}_{[n]}(\mu),$$

where $\text{SST}_{[n]}(\mu)_{\nu}$ is the set of all rational $[n]$-semistandard tableaux of shape $\mu$ with content $\nu$, or weight $\sum_{i=1}^n \nu_i \epsilon_i$.

**Proof.** It follows directly from Theorem 4.1 and Corollary 4.3. \hfill $\square$

Now, we have a Jacobi-Trudi type character formula for $S^A/B_{\lambda}$.

**Theorem 4.5.** Suppose that $A$ and $B$ are infinite sets. For $\lambda \in \mathbb{Z}_+^n$, we have

$$S^A/B_{\lambda} = \det \left( S^A/B_{\lambda_{i-1} + j} \right)_{1 \leq i, j \leq n}.$$  

**Proof.** Since $A$ and $B$ are infinite sets, we have $\mathcal{P}_{A/B,k} = \mathbb{Z}_+^k$ for all $k \geq 1$. Fix $n \geq 1$. For $\lambda, \mu \in \mathbb{Z}_+^n$, we define $\lambda > \mu$ if and only if there exists an $i$ such that $\lambda_k = \mu_k$ for $1 \leq k < i$ and $\lambda_i > \mu_i$. Then $\lambda > \mu$ is a linear ordering on $\mathbb{Z}_+^n$, called the reverse lexicographic ordering.

Given $\mu \in \mathbb{Z}_+^n$, put $H^A/B_{\mu} = \prod_{1 \leq i \leq n} S^A/B_{\mu_i} \in \mathbb{Z}[x_A, x_B^{-1}]$. By Corollary 4.3, we have $H^A/B_{\mu} = \sum_{\lambda \in \mathbb{Z}_+^n} K_{\lambda \mu} S^A/B_{\lambda}$, where $K_{\lambda \mu} = |\text{SST}_{[n]}(\lambda)_{\mu}|$. By Lemma 3.3 we
have for all $d > 0$,
\begin{equation}
K_{\lambda \mu} = K_{\lambda + (d^n) \mu + (d^n)},
\end{equation}
which is equal to the ordinary Kostka number of shape $\lambda + (d^n)$ with content $\mu + (d^n)$ whenever $\lambda + (d^n)$ and $\mu + (d^n)$ are ordinary partitions. This implies that $K_{\lambda \mu}$ is zero unless $\lambda \geq \mu$ (cf. \[22\]). Hence, we may write
\begin{equation}
H^{A/B}_\mu = \sum_{\lambda \geq \mu} K_{\lambda \mu} S^{A/B}_\lambda + \sum_{\lambda \geq \mu} K_{\lambda \mu} S^{A/B}_\lambda.
\end{equation}

For $\lambda, \mu \in \mathcal{P}_n$, let $h_\mu(x_{[n]})$ (resp. $s_\lambda(x_{[n]})$) be the complete symmetric polynomial (resp. Schur polynomial) in $n$ variables corresponding to $\mu$ (resp. $\lambda$). Recall that
\begin{equation}
h_\mu(x_{[n]}) = \sum_{\lambda \geq \mu} K_{\lambda \mu} s_\lambda(x_{[n]}),
\end{equation}

\begin{equation}
s_\lambda(x_{[n]}) = \det (h_{\lambda_i - i + j}(x_{[n]}))_{1 \leq i, j \leq n},
\end{equation}
where we assume that $h_k(x_{[n]}) = 0$ for $k < 0$ (see \[22\]).

**Case 1.** Suppose that $\lambda \in \mathcal{P}_n$ is given, where $\lambda_i - i + j > 0$ for all $1 \leq i, j \leq n$. Comparing (4.4) and (4.5), we have
\begin{equation}
S^{A/B}_\lambda - \det \left( H^{A/B}_{\lambda_i - i + j} \right)_{1 \leq i, j \leq n} = \sum_{\nu \geq \lambda} a_{\nu} S^{A/B}_{\nu}
\end{equation}
for some $a_{\nu} \in \mathbb{Z}$. If we apply the same argument to $\lambda + (d^n)$ for $d \geq 0$, we have
\begin{equation}
S^{A/B}_{\lambda + (d^n)} - \det \left( H^{A/B}_{\lambda_i - i + j + d} \right)_{1 \leq i, j \leq n} = \sum_{\nu \geq \lambda} b_{\nu} S^{A/B}_{\nu}
\end{equation}
for some $b_{\nu} \in \mathbb{Z}$. On the other hand, by (4.3), the equation (4.4) still holds when we replace $\lambda$ and $\mu$ by $\lambda + (d^n)$ and $\mu + (d^n)$ $(d \geq 0)$, respectively. Hence, from (4.6), we also obtain
\begin{equation}
S^{A/B}_{\lambda + (d^n)} - \det \left( H^{A/B}_{\lambda_i - i + j + d} \right)_{1 \leq i, j \leq n} = \sum_{\nu \geq \lambda} a_{\nu} S^{A/B}_{\nu + (d^n)}.
\end{equation}

Comparing (4.7) and (4.8), it follows from Corollary 3.28 that
\begin{itemize}
    \item[(1)] $a_{\nu} = b_{\nu + (d^n)}$,
    \item[(2)] $a_{\nu} = 0$ whenever $\nu + (d^n) \in \mathcal{P}_n$,
\end{itemize}
where we assume that $b_{\nu} = 0$ for $\nu \in \mathcal{P}_n$. Since $d > 0$ is arbitrary, we conclude that $a_{\nu} = 0$ for all $\nu$ such that $\nu \geq \lambda$ and $\nu \in \mathbb{Z}_+^n \setminus \mathcal{P}_n$.
Case 2. Suppose that \( \lambda \in \mathbb{Z}_+^n \) is given and

\[
S^{A/B}_\lambda - \det \left( H^{A/B}_{\lambda,-i+j} \right)_{1 \leq i,j \leq n} = \sum_{\nu \in \mathbb{Z}_+^n} c_\nu S^{A/B}_{\nu},
\]

for some \( c_\nu \in \mathbb{Z} \). By (4.3), we also have

\[
S^{A/B}_{\lambda+(dn)} - \det \left( H^{A/B}_{\lambda,-i+j+d} \right)_{1 \leq i,j \leq n} = \sum_{\nu \in \mathbb{Z}_+^n} c_\nu S^{A/B}_{\nu+(dn)},
\]

for all \( d \geq 0 \). By Case 1, the above equation is zero if \( d \) is sufficiently large, and hence \( c_\nu = 0 \) for all \( \nu \in \mathbb{Z}_+^n \). This completes the proof.

Example 4.6. Suppose that \( A \) and \( B \) are infinite sets. By Corollary 4.3, we have

\[
H^{A/B}_{(1,1)} = S^{A/B}_{(1,1)} + S^{A/B}_{(2,0)} + S^{A/B}_{(3,-1)} + S^{A/B}_{(4,-2)} + \cdots,
\]

\[
H^{A/B}_{(2,0)} = +S^{A/B}_{(2,0)} + S^{A/B}_{(3,-1)} + S^{A/B}_{(4,-2)} + \cdots.
\]

Hence,

\[
S^{A/B}_{(1,1)} = H^{A/B}_{(1,1)} - H^{A/B}_{(2,0)} = S^{A/B}_1 S^{A/B}_0 - S^{A/B}_2 S^{A/B}_0 = \det \begin{pmatrix} S^{A/B}_1 & S^{A/B}_2 \\ S^{A/B}_0 & S^{A/B}_1 \end{pmatrix}.
\]

4.2. Littlewood-Richardson rule. Now, we are in a position to describe the LR rule for \( A/B \)-semistandard tableaux. For \( \mu \in \mathcal{P}_{A/B,m} \) and \( \nu \in \mathcal{P}_{A/B,n} \), we will introduce an algorithm of inserting a bitableau \( T_1 = (T_1^+, T_1^-) \in SST_{A/B}(\mu) \) into another bitableau \( T_2 = (T_2^+, T_2^-) \in SST_{A/B}(\nu) \) to create a new bitableau \( T = (T^+, T^-) \in SST_{A/B}(\lambda) \) for some \( \lambda \in \mathcal{P}_{A/B,m+n} \) together with a recording tableau in \( LR_{\lambda}^{\mu,\nu} \).

Let us give a brief sketch of our algorithm, which is very similar to RSK correspondence in Theorem 4.1. First, we define a \( B \)-semistandard tableau \( T^- \) by applying the column insertion algorithm to \( (T_i^-)^\pi \) \((i = 1, 2)\). Next, we consider \( \mathbb{N} \)-semistandard tableaux \( U_i \) \((i = 1, 2)\) whose recording tableau with respect to row insertion forms a rectangular complement to that of \( (T_i^-)^\pi \) \((i = 1, 2)\) with respect to column insertion. Finally, we apply the row insertion algorithm to \( U_i * T_i^+ \) instead of \( T_i^+ \) \((i = 1, 2)\) to obtain an \( A \)-semistandard tableau \( T^+ \) of a skew shape. Then the pair \( T = (T^+, T^-) \) becomes an \( A/B \)-semistandard tableau of shape \( \lambda \in \mathbb{Z}_+^{m+n} \), and the recording tableau is given as an element in \( LR_{\lambda}^{\mu,\nu} \) corresponding to the row insertion of \( U_2 * T_2^+ \) into \( U_1 * T_1^+ \).
Theorem 4.7. For $\mu \in \mathcal{P}_{A/B,m}$ and $\nu \in \mathcal{P}_{A/B,n}$, there exists a bijection
\[
\rho_{A/B} : SST_{A/B}(\mu) \times SST_{A/B}(\nu) \longrightarrow \bigcup_{\lambda \in \mathcal{P}_{A/B,m+n}} SST_{A/B}(\lambda) \times LR_{\mu,\nu}^\lambda.
\]
In terms of characters, we have
\[
S_{\mu}^{A/B}S_{\nu}^{A/B} = \sum_{\lambda \in \mathcal{P}_{A/B,m+n}} \hat{c}_{\mu,\nu}^{\lambda}S_{\lambda}^{A/B}.
\]
\[\square\]

Proof. Let $T_1 = (T_1^+, T_1^-) \in SST_{A/B}(\mu)$ and $T_2 = (T_2^+, T_2^-) \in SST_{A/B}(\nu)$. We will define a pair
\[
\rho_{A/B}(T_1, T_2) = (T, T_R),
\]
where $T = (T^+, T^-) \in SST_{A/B}(\lambda)$ and $T_R \in LR_{\mu,\nu}^\lambda$ for some $\lambda \in \mathcal{P}_{A/B,m+n}$.

Step 1. First, let us define $T^-$. Suppose that
\[
sh(T_1^+) = (\mu + (d^m))/\zeta^{(1)}, \quad sh(T_2^+) = (\nu + (d^m))/\zeta^{(2)},
\]
for some $d > 0$ and $\zeta^{(1)} \in \mathcal{P}_m$, $\zeta^{(2)} \in \mathcal{P}_n$. We may assume that $d$ is sufficiently large. Then
\[
sh(T_1^-) = (d^m)/\zeta^{(1)}, \quad sh(T_2^-) = (d^m)/\zeta^{(2)}.
\]
We define
\[
T^- = (T_1^-)^\pi \leftarrow (T_2^-)^\pi.
\]
Then we have $sh(T^-) = (d^{m+n})/\zeta$ for some $\zeta \in \mathcal{P}_{m+n}$.

Step 2. Next, let us define $T^+$. Consider $(T_i^-)^\pi$ for $i = 1, 2$. Set $\eta^{(i)} = sh(T_i^-)^\pi$ for $i = 1, 2$. Note that
\[
\eta^{(1)} = \delta_d^{\mu}(\zeta^{(1)})', \quad \eta^{(2)} = \delta_d^{\nu}(\zeta^{(2)})'.
\]
By Theorem 2.3 there exists a unique $(S_1, S_2) \in SST_{|d|}(\eta^{(1)}) \times SST_{|d|}(\eta^{(2)})$ such that
\[
(S_2 \rightarrow S_1) = H^n, \quad (S_2 \rightarrow S_1)_R = ((T_2^-)^\pi \rightarrow (T_1^-)^\pi)_R,
\]
where $\eta = sh((T_2^-)^\pi \rightarrow (T_1^-)^\pi)$. Since
\[
((T_2^-)^\pi \rightarrow (T_1^-)^\pi) = ((T_1^-)^\pi \leftarrow (T_2^-)^\pi)^\pi = [(T^-)^\pi]^\pi = (T^-)^\pi,
\]
we have
\[
\eta = \delta_d^{m+n}(\zeta)'.
\]
Set $U_i = \delta_d^{m}(S_i)^\pi$ ($i = 1, 2$). Then
\[
sh(U_i) = sh(\delta_d^{m}(S_i))' = \delta_d^{m}(\delta_d^{m}(\zeta^{(i)}))' = \delta_d^{m}(\delta_d^{m}(\zeta^{(i)}))' = \zeta^{(i)},
\]
for some $\delta_d^{m}$.
and hence $U_i \in SST[d'_i](\zeta^{(i)})$ ($i = 1, 2$). Then
\[
(U_2 \to U_1) = (\delta^d(S_2) \to \delta^d(S_1)^t) = [\delta^d(S_1) \leftarrow \delta^d(S_1)]^t
\]
\[
= [\delta^d_{m+n}(S_2 \to S_1)]^t \quad \text{by Theorem 3.6}
\]
\[
= [\delta^d_{m+n}(H^n)]^t,
\]
and
\[
\text{sh}(U_2 \to U_1) = \delta^d_{m+n}(\eta)' = \delta^d_{m+n}(\delta^{m+n}_d(\zeta)')'
\]
\[
= \delta^d_{m+n}(\delta^{m+n}_d(\zeta')') = \zeta.
\]
Hence, $(U_2 \to U_1) \in SST[d'_i](\zeta)$.

Now, we set
\[
\hat{U}_1 = U_1 \ast T_1^+ \in SST[d'_1 \ast A](\mu + (d^m)),
\]
\[
\hat{U}_2 = U_2 \ast T_2^+ \in SST[d'_2 \ast A](\nu + (d^n)).
\]
Then, we have
\[
(\hat{U}_2 \to \hat{U}_1) = (U_2 \to U_1) \ast T^+,
\]
where $T^+ \in SST_A((\lambda + (d^{m+n})/\zeta)$ for some $\lambda \in \mathbb{Z}_m^{m+n}$. We define
\[
(4.9) \quad T = (T^+, T^-).
\]
We can check that $T$ does not depend on the choice of $d$, and $T \in SST_A/B(\lambda)$.

**Step 3.** By Lemma 2.2, we have $((\hat{U}_2 \to \hat{U}_1)_R)^t \in LR_{(\mu + (d^m))}^{(\lambda + (d^{m+n}))'}$. Now, we define
\[
(4.10) \quad T_R = [Q] \in LR_{\mu \nu}^{\lambda},
\]
where $[Q]$ is the element in $LR_{\mu \nu}^{\lambda}$ including $((\hat{U}_2 \to \hat{U}_1)_R)^t$ (see (3.10)).

Since the construction of $(T, T_R)$ is reversible, $\rho_{A/B}$ is a bijection. This completes the proof.

**Example 4.8.** Suppose that $A = \{a_1 < a_2 < a_3 < \cdots\}$ and $B = \{b_1 < b_2 < b_3 < \cdots\}$. For convenience, we assume that all the elements are of degree 0. Suppose that
\[
T_1 = (T_1^+, T_1^-) = \left( \begin{array}{c|cc}
  a_1 & a_2 & b_2 \\
  & b_3 & b_4 \\
  a_3 & b_1 & b_2 \\
 \end{array} \right) \in SST_A/B((2, -1)),
\]
\[
T_2 = (T_2^+, T_2^-) = \left( \begin{array}{c|cc}
  a_1 & a_4 & b_1 \\
  & b_2 & b_3 \\
  a_3 & b_1 & b_2 \\
 \end{array} \right) \in SST_A/B((1, -1)).
\]
Then we have

\[
T^- = \begin{pmatrix}
 b_1 \\
 b_1 \\
 b_2 \\
 b_2 \\
 b_3 \\
 b_2
\end{pmatrix}.
\]

Note that

\[
(T_1^-)^{\sharp} = b_3 \ b_2 , \quad (T_2^-)^{\sharp} = b_2 \ b_1,
\]

and

\[
(T_2^-)^{\sharp} \rightarrow (T_1^-)^{\sharp} = b_3 \ b_2 \ b_1 \quad \text{and} \quad ((T_2^-)^{\sharp} \rightarrow (T_1^-)^{\sharp})_R = \begin{bmatrix}
 \bullet & \bullet & 1 \\
 1 & 2
\end{bmatrix}.
\]

Hence, if we put \( S_1 = \begin{pmatrix} 1 & 2 \end{pmatrix} \) and \( S_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \), then \( S_2 \rightarrow S_1 = H^{(3,2)} \) and \( (S_2 \rightarrow S_1)_R = ((T_2^-)^{\sharp} \rightarrow (T_1^-)^{\sharp})_R \). Now, we put

\[
U_1 = \delta_2^4(S_1)^t = \begin{pmatrix} 1 & 3 & 4 \\
 2 & 3 & 4
\end{pmatrix}, \quad U_2 = \delta_2^4(S_2)^t = \begin{pmatrix} 2 & 3 & 4 \\
 3 & 4
\end{pmatrix}.
\]

Then we have

\[
\hat{U}_1 = U_1 \ast T_1^+ = \begin{pmatrix} 1 & 3 & 4 & a_1 & a_2 & a_2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & a_1 & a_2 & a_3 & a_4 \\
 2 & 3 & 4 & a_2 \\
 3 & 4 \\
 \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 4 & 5 \\
 \end{pmatrix}
\]

\[
\hat{U}_2 = U_2 \ast T_2^+ = \begin{pmatrix} 2 & 3 & 4 & a_3 & a_4 \end{pmatrix}.
\]

Note that \(((\hat{U}_2 \rightarrow \hat{U}_1)_R)^t \in LR_{(6,3)}^{(8,4,3,2)} \). Therefore,

\[
T = (T^+, T^-) = \begin{pmatrix}
 a_1 & a_2 & a_3 & a_4 \\
 a_2 & b_1 \\
 b_1 & b_2 & b_2 & b_3
\end{pmatrix}.
\]
where $T \in \text{SST}_{A/B}(4,0,-1,-2)$, and

$T_R = \begin{bmatrix}
  \bullet & \bullet & 1 \\
  \bullet & 1 \\
  \bullet \\
  2 \\
  3
\end{bmatrix} \in \text{LR}^{(4,0,-1,-2)}_{(2,-1)(1,-1)},$

(see Lemma 3.9).

4.3. Skew Littlewood-Richardson rule. We have seen that the products of characters of $\text{SST}_{A/B}(\lambda)$’s is determined by the branching coefficients of rational Schur polynomials. This might be understood as a dual relation between $A/B$-semistandard tableaux and rational $[n]$-semistandard tableaux. To complete this dual relationship, we will define $A/B$-semistandard tableaux of skew shapes, and then obtain their skew LR rule, which is completely determined by the LR rule of rational $[n]$-semistandard tableaux.

**Definition 4.9.** Suppose that $\lambda$ and $\mu \in \mathbb{Z}_n^+$ are given. An $A/B$-semistandard tableau of skew shape $\lambda/\mu$ is a pair of tableaux $(T^+, T^-)$ such that

$T^+ \in \text{SST}_A((\lambda + (d \mu))/\nu), \quad T^- \in \text{SST}_B((\mu + (d \mu))/\nu),$

for some integer $d \geq 0$ and $\nu \in \mathcal{P}_n$ satisfying

1. $\lambda + (d \mu), \mu + (d \mu) \in \mathcal{P}_n$,
2. $\nu \subset (d \mu), \lambda + (d \mu)$, and $\mu + (d \mu)$.

We denote by $\text{SST}_{A/B}(\lambda/\mu)$ the set of all $A/B$-semistandard tableaux of skew shape $\lambda/\mu$. Note that $\text{SST}_{A/B}(\lambda/0_n) = \text{SST}_{A/B}(\lambda)$. The character of $\text{SST}_{A/B}(\lambda/\mu)$ is defined similarly, and denoted by $S_{\lambda/\mu}$.

To discuss the skew LR rule, we need to consider a rectangular complement of a Littlewood-Richardson tableau, which is given in the following lemma.

**Lemma 4.10.** For $\lambda, \mu,$ and $\nu \in \mathcal{P}_n$, there exists a bijection

$\delta_{p,q}^n : LR^\lambda_{\mu,\nu} \rightarrow LR_{\delta_{p,q}^n(\lambda)}^{\delta_{p,q}^n(\mu,\nu)},$

for $p \geq \mu_1$ and $q \geq \nu_1$.

**Proof.** Suppose that $LR^\lambda_{\mu,\nu}$ is non-empty, and $Q \in LR^\lambda_{\mu,\nu}$ is given. Then $\tau(Q) \in LR_{\mu',\nu'}^\lambda$, where $\tau$ is the map given in Corollary 2.9. By Theorem 2.9, there exists
a unique pair \((T_1, T_2) \in \text{SST}_{\nu}(\mu) \times \text{SST}_{\nu}(\nu)\) such that \((T_2 \to T_1) = H^\lambda\) and \(((T_2 \to T_1)_R)^t = \tau(Q)\). Now, we define

\[
\delta^p_{p,q}(Q) = (\delta^p_T(T_1) \leftarrow \delta^q_T(T_2))_R,
\]

where \(p \geq \mu_1\) and \(q \geq \nu_1\). By Theorem 16, \((\delta^p_T(T_1) \leftarrow \delta^q_T(T_2)) = \delta^p_{p+q}(T_2 \to T_1) = \delta^p_{p+q}(H^\lambda)\), and hence \(\delta^p_{p,q}(Q) \in LR_{\delta_{\mu}^p(\lambda)}(\delta_{\nu}^q(\nu))\).

Suppose that \(\delta^p_{p,q}(Q) = \delta^p_{p,q}(Q')\) for some \(Q' \in LR_{\mu}^\lambda\). Let \((T_1', T_2') \in \text{SST}_{\nu}(\mu) \times \text{SST}_{\nu}(\nu)\) be the associated pair such that \((T_2' \to T_1') = H^\lambda\) and \(((T_2' \to T_1')_R)^t = \tau(Q')\).

Note that

\[
(\delta^p_T(T_1) \leftarrow \delta^q_T(T_2)) = \delta^p_{p+q}(T_2 \to T_1) = \delta^p_{p+q}(H^\lambda) = (\delta^p_T(T_1) \leftarrow \delta^q_T(T_2)),
\]

\[
(\delta^p_T(T_1) \leftarrow \delta^q_T(T_2))_R = \delta^p_{p,q}(Q') = \delta^p_{p,q}(Q) = (\delta^p_T(T_1) \leftarrow \delta^q_T(T_2))_R.
\]

Since \(\rho_{\text{col}}\) is a bijection, we have \(\delta^p_{p}(T_1) = \delta^p_{p}(T_1)\) and \(\delta^q_{q}(T_2) = \delta^q_{q}(T_2)\). Since \(\delta^p_{p}\) is also bijective for \(k = p, q\), we have \((T_1, T_2) = (T_1', T_2')\), and hence \(Q = Q'\), which implies that \(\delta^p_{p,q}\) is one-to-one.

Since \(\delta^p_{p,q}\) also gives a one-to-one map from \(LR_{\delta_{\mu}^p(\mu)}(\delta_{\nu}^q(\nu))\) to \(LR_{\mu}^\lambda\), \(\delta^p_{p,q}\) is a bijection.

\[\square\]

**Theorem 4.11.** For \(\lambda, \mu \in \mathbb{Z}_+^n\), there exists a bijection

\[J_{A/B} : \text{SST}_{A/B}(\lambda/\mu) \longrightarrow \bigsqcup_{\nu \in \mathcal{P}_{A/B}} \text{SST}_{A/B}(\nu) \times LR_{\nu}^{\lambda/\mu}.
\]

In terms of characters, we have

\[S_{\lambda/\mu}^{A/B} = \sum_{\nu \in \mathcal{P}_{A/B}} c_{\lambda, \mu, \nu} S_{\nu}^{A/B}.
\]

**Proof.** To each \(T = (T^+, T^-) \in \text{SST}_{A/B}(\lambda/\mu)\), we will associate a pair \(J_{A/B}(T) = (j(T), j(T)_R) \in \text{SST}_{A/B}(\nu) \times LR_{\nu}^{\lambda/\mu}\) for some \(\nu \in \mathcal{P}_{A/B}\).

First, consider \((T^-)^\pi\). Suppose that \(\text{sh}((T^-)^\pi) = \alpha/\beta\) for some \(\alpha, \beta \in \mathcal{P}_{n}\). In fact, one may assume that \(\beta = \delta^p_{\nu}(\mu + (p^n))\), where \(p = \max\{-\mu_n, 0\}\) and \(r = \mu_1 + p\).

Applying Corollary 16 to \((T^-)^\pi\), we have

\[J((T^-)^\pi) = (j((T^-)^\pi), j((T^-)^\pi)_R) \in \text{SST}_{B^\pi}(\gamma) \times LR_{\beta}^{\gamma},\]

for some \(\gamma \in \mathcal{P}_{n}\). We define

\[\widehat{T}^- = j((T^-)^\pi)^\pi\]

Note that \(\text{sh}(\widehat{T}^-) = \gamma^\pi\).

Next, consider \(Q = j((T^-)^\pi)_R\). If we put \(q = \gamma_1\), then by Lemma 4.10 we have

\[\delta_{p,q}^n(Q) = Q^\nu \in LR_{\delta_{\nu}(\beta)}^{\lambda/\mu}(\delta_{\nu}(\gamma)).\]
By definition of $\alpha$ and $\beta$, we can check the following facts:

1. $\text{sh}(T^+) = \lfloor \lambda + ((p + q)^n) \rfloor / \delta^n_{p+q}(\alpha)$,
2. $\delta^p_{\nu}(\beta) = \mu + (p^n)$,
3. $Q^* T^+ \in \text{SST}_{\mathbb{N}+A}(\lfloor \lambda + ((p + q)^n) \rfloor / \mu + (p^n))$.

Since $j(Q^*) = H^{\mu}_{\nu}(\gamma)$ by Theorem 2.5 (3), we obtain

$$J(Q^* T^+) = (H^{\mu}_{\nu}(\gamma) * \hat{T}^+, \hat{Q}),$$

where

1. $\hat{T}^+ \in \text{SST}_A((\nu + (q^n)) / \delta^p_{\nu}(\gamma))$ for some $\nu \in \mathbb{Z}_+^n$,
2. $\hat{Q} \in \text{LR}^{\lambda+((p+q)^n)}_{\mu+(p^n) \nu+(q^n)}$.

Now, we define

$$j(T) = (\hat{T}^+, \hat{T}^-).$$

Then $j(T) \in \text{SST}_{A/B}(\nu)$ since $\text{sh}(\hat{T}^-) = \gamma^\pi = (q^n) / \delta^p_{\nu}(\gamma)$. And we define $j(T)_R$ to be the element in $\text{LR}^{\lambda+\mu}_{\nu+(p^n)}$ containing $\hat{Q}$ (see (3.8)).

Since our construction is reversible, the correspondence $T \mapsto (j(T), j(T)_R)$ is bijective. Moreover, we obtain the corresponding identity from the characters of the both sides since $\text{wt}_{A/B}(T) = \text{wt}_{A/B}(j(T))$ for all $T \in \text{SST}_{A/B}(\lambda/\mu)$. This completes the proof. \[\square\]

**Corollary 4.12.** For $\lambda \in \mathbb{Z}_+^n$, we have

$$S^A/B_{0_n/\lambda} = S^A/B_{\lambda^*}.$$

**Proof.** By Theorem 4.11, it suffices to show that

$$c^0_{\lambda \mu} = \begin{cases} 1, & \text{if } \mu = \lambda^*, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $c^0_{\lambda \mu} = N^{(p+q)^n}_{\lambda+(p^n)+\mu+(q^n)}$ for sufficiently large $p, q > 0$ whenever $\lambda + (p^n), \mu + (q^n) \in \mathcal{P}_n$ and $\lambda + (p^n) \subset ((p+q)^n)$. Fix such $p$ and $q$. Then it is not difficult to see that there exists a unique Littlewood-Richardson tableau $Q$ of shape $(p+q)^n / (\lambda + (p^n))$, whose content should be $\delta^p_{p+q}(\lambda + (p^n))$. Hence, we have

$$\mu + (q^n) = \delta^p_{p+q}(\lambda + (p^n)) = \lambda^* + (q^n),$$

which implies that $\mu = \lambda^*$. \[\square\]
In this section, we discuss a relation between \( A/B \)-semistandard tableaux and a certain class of representations of infinite dimensional Lie superalgebra \( \hat{gl}_{\infty|\infty} \) (or \( \hat{gl}_{\infty} \)). More precisely, we will show that the characters of certain quasi-finite irreducible representations of \( \hat{gl}_{\infty|\infty} \) (or \( \hat{gl}_{\infty} \)) parameterized by generalized partitions (see [3, 16]) are realized as those of \( A/B \)-semistandard tableaux of the corresponding shapes with suitable choices of \( A \) and \( B \). Using the combinatorial results established in the previous sections, we will characterize the Grothendieck rings for certain categories of semi-simple representations of \( \hat{gl}_{\infty|\infty} \) (or \( \hat{gl}_{\infty} \)), whose irreducible factors are parameterized by generalized partitions.

5.1. **Fock space representations.** Let \( \frac{1}{2}\mathbb{Z} = \{ \frac{n}{2} \mid n \in \mathbb{Z} \} \) be a \( \mathbb{Z}_2 \)-graded set with \( (\frac{1}{2}\mathbb{Z})_0 = \mathbb{Z} \) and \( (\frac{1}{2}\mathbb{Z})_1 = \frac{1}{2} + \mathbb{Z} \). A linear ordering on \( \frac{1}{2}\mathbb{Z} \) is given as the ordinary one. Let \( C_{\infty|\infty} \) be the associated superspace with a basis \( \{ \varepsilon_k \mid k \in \frac{1}{2}\mathbb{Z} \} \). Let

\[
\hat{gl}_{\infty|\infty} = \{ (a_{ij})_{i,j \in \frac{1}{2}\mathbb{Z}} \mid a_{ij} \in \mathbb{C}, a_{ij} = 0 \text{ for } |i - j| \gg 0 \}.
\]

Since an element in \( gl_{\infty|\infty} \) is a linear transformation of \( C_{\infty|\infty} \), \( gl_{\infty|\infty} \) is naturally endowed with a \( \mathbb{Z}_2 \)-grading, and becomes a Lie superalgebra with respect to supercommutator. For \( i, j \in \frac{1}{2}\mathbb{Z} \), we denote by \( e_{ij} \) the elementary matrix with 1 at the \( i \)th row and the \( j \)th column and 0 elsewhere.

Let \( \hat{gl}_{\infty|\infty} = gl_{\infty|\infty} \oplus \mathbb{C}K \) be a central extension of \( gl_{\infty|\infty} \) with respect to the following two-cocycle

\[
\alpha(A, B) = \text{str}([J, A]B) \quad (A, B \in gl_{\infty|\infty}),
\]

where \( J = \sum_{r \leq 0} e_{rr} \), and \( \text{str} \) is the supertrace defined by \( \text{str}((a_{ij})) = \sum_{i \in \frac{1}{2}\mathbb{Z}} (-1)^{2i} a_{ii} \).

Then we have a triangular decomposition

\[
\hat{gl}_{\infty|\infty} = n_+ \oplus \mathfrak{h} \oplus n_-,
\]

where \( \mathfrak{h} \) is the subalgebra spanned by diagonal matrices and \( K \), and \( n^+ \) (resp. \( n^- \)) is the subalgebra of strictly upper (resp. lower) triangular matrices. With this, one can define a Verma module \( M(\Lambda) \) of \( \hat{gl}_{\infty|\infty} \) with highest weight \( \Lambda \in \mathfrak{h}^* \). Then we denote by \( L(\Lambda) \) the unique irreducible quotient of \( M(\Lambda) \) with highest weight \( \Lambda \). If we define \( \text{deg} e_{ij} = j - i \) for \( i, j \in \frac{1}{2}\mathbb{Z} \), then \( gl_{\infty|\infty} \) becomes a \( \frac{1}{2}\mathbb{Z} \)-graded Lie superalgebra. And if we define the degree of the highest weight vector in \( L(\Lambda) \) to be 0, then \( L(\Lambda) \) is also naturally \( \frac{1}{2}\mathbb{Z} \)-graded \( L(\Lambda) = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} L(\Lambda)_k \). We say that \( L(\Lambda) \) is quasi-finite if \( \dim L(\Lambda)_k \) is finite for all \( k \in \frac{1}{2}\mathbb{Z} \) (cf. [3, 15]).

For \( n \geq 1 \), let \( \mathfrak{F}^n \) be the infinite dimensional Fock space generated by \( n \) pairs of free fermions and \( n \) pairs of free bosons (see [3, 6, 14] for a detailed description). Then we have a natural commuting action of \( \hat{gl}_{\infty|\infty} \) and \( gl_n \) on \( \mathfrak{F}^n \). Using Howe
duality, Cheng and Wang proved the following multiplicity-free decomposition of $\mathfrak{g}_n$.

**Theorem 5.1** ([6]). As a $(\widehat{\mathfrak{g}}_{\infty|\infty}, \mathfrak{g}_n)$-module,

$$\mathfrak{g}_n \simeq \bigoplus_{\lambda \in \mathbb{Z}_n^+} L(\Lambda(\lambda)) \otimes L_n(\lambda),$$

where $L_n(\lambda)$ is the irreducible rational representation of $\mathfrak{g}_n$ corresponding to $\lambda$, and $\Lambda(\lambda) \in \mathfrak{h}^*$ is the highest weight determined by

$$\Lambda(\lambda)(e_{kk}) = \begin{cases} \max\{\lambda'_k - k, 0]\}, & \text{if } k \in \mathbb{Z}_{>0}, \\ -\max\{\lambda'_{k-1} + k, 0\}, & \text{if } k \in \mathbb{Z}_{\leq 0}, \\ \max\{\lambda_k + \frac{1}{2} - k + \frac{1}{2}, 0\}, & \text{if } k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}, \\ -\max\{-\lambda_n + k + \frac{1}{2} + k - \frac{1}{2}, 0\}, & \text{if } k \in -\frac{1}{2} - \mathbb{Z}_{\geq 0}, \end{cases}$$

( $\lambda'_i$ is the number of nodes in the $i$th column of $\lambda$ for $i \in \mathbb{Z} \setminus \{0\}$).

For $k \in \frac{1}{2}\mathbb{Z}$, let $\omega_k$ be the fundamental weight given by $\omega_k(e_{ll}) = \delta_{kl}$ and $\omega_k(K) = 0$, and let $x_k = e^{\omega_k}$ be the formal variable. Then we can define the character of $L(\Lambda(\lambda))$ ($\lambda \in \mathbb{Z}_n^+$) with respect to the action of the abelian subalgebra $\bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathbb{C} e_{kk}$.

In [3], using the classical Cauchy identities of (hook) Schur functions (cf. [22, 24]), Cheng and Lam showed that $L(\Lambda(\lambda))$ is given as a linear combination of product of two hook Schur functions.

Put $A = (\frac{1}{2}\mathbb{Z}_{>0})'$ and $B = (\frac{1}{2}\mathbb{Z}_{\leq 0})'$. Then we may view $x_A = \{ e^{\omega_k} | k \in \frac{1}{2}\mathbb{Z}_{>0} \}$ and $x_B = \{ e^{\omega_k} | k \in \frac{1}{2}\mathbb{Z}_{\leq 0} \}$. Now, we obtain a new combinatorial realization of $\text{ch} L(\Lambda(\lambda))$.

**Theorem 5.2.** For $\lambda \in \mathbb{Z}_n^+$, we have

$$\text{ch} L(\Lambda(\lambda)) = S_{\lambda}^{A/B},$$

where $A = (\frac{1}{2}\mathbb{Z}_{>0})'$ and $B = (\frac{1}{2}\mathbb{Z}_{\leq 0})'$.

**Proof.** From the Cheng and Lam’s formula (Theorem 3.2 in [3]), we have

$$\text{ch} L(\Lambda(\lambda)) = \sum_{\mu, \nu \in \mathfrak{P}_n} c_{\mu \nu}^\lambda \cdot S_\mu(x_{A_0 \ast A_1}) S_\nu(x_{B_0 \ast B_1}^{-1}).$$

Note that $S_\mu(x_{A_0 \ast A_1})$ and $S_\nu(x_{B_0 \ast B_1}^{-1})$ are hook Schur functions with countably many even and odd variables. By Lemma 3.16, we have $S_{A_0 \ast A_1} = S_\mu^A$ and $S_{B_0 \ast B_1}^{-1} = S_\nu^B$. Therefore, $\text{ch} L(\Lambda(\lambda)) = S_{\lambda}^{A/B}$ by Proposition 3.20. \qed
Remark 5.3. It is not difficult to see that the left-hand sides in the character identities of RSK correspondence in Theorem 4.1 (or Corollary 4.3) and the Fock space decomposition in Theorem 5.1 are equal. Comparing these two identities, we can also prove that \( \text{ch} L(\Lambda(\lambda)) = S_{\lambda}^{A/B} \) from the linear independence of rational Schur polynomials.

Now, we have a Jacobi-Trudi type character formula for \( L(\Lambda(\lambda)) \).

Corollary 5.4. Under the above hypothesis, we have

\[
\text{ch} L(\Lambda(\lambda)) = \det (\text{ch} L(\Lambda(\lambda_i - i + j)))_{1 \leq i,j \leq n}.
\]

Proof. It follows directly from Theorem 4.5. \( \square \)

Remark 5.5. (1) For \( \lambda \in \mathbb{Z}^n_+ \), the tableau \( T^\lambda \) in \( SST_{A/B}(\lambda) \) corresponding to the highest weight vector can be found easily. For example, if \( \lambda = (4,3,2,-2,-3) \), then \( T^\lambda \) is given by filling the generalized Young diagram \( \lambda \) in the following pattern

\[
T^\lambda = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{3}{2} & \frac{3}{2} & - \frac{1}{2} \\
1 & 2 & 1 & 0 \\
-1 & 0 & 1 & 2 \\
-\frac{1}{2} & -\frac{1}{2} & 0 & 1
\end{pmatrix}.
\]

In this case, we have

\[
\Lambda(\lambda) = \omega_2 + 2\omega_3 + 2\omega_1 + 4\omega_{\frac{1}{2}} - 2\omega_0 - 2\omega_{-\frac{1}{2}} - \omega_{-1} + 5\Lambda_0,
\]

where \( \Lambda_0 \in \mathfrak{h}^* \) is defined by \( \Lambda_0(e_{kk}) = 0 \) for all \( k \in \frac{1}{2}\mathbb{Z} \), and \( \Lambda_0(K) = 1 \).

(2) The LR rule for \( \mathcal{A}/B \)-semistandard tableaux (Theorem 4.7) gives a combinatorial interpretation of decomposition of the tensor product \( L(\Lambda(\lambda)) \otimes L(\Lambda(\mu)) \) (cf. Theorem 6.1 in [3]). In particular, the RSK correspondence (Theorem 4.1) corresponds to the decomposition of \( \mathfrak{g}^n \) as a \( (\widetilde{\mathfrak{g}}_\infty, \mathfrak{g}_n) \)-module (Theorem 5.1).

(3) It would be interesting to construct an explicit basis of \( L(\Lambda(\lambda)) \) whose elements are parameterized by \( \mathcal{A}/B \)-semistandard tableaux of shape \( \lambda \).

Next, let \( \mathfrak{g}l_\infty \) be the subalgebra of \( \mathfrak{g}l_{\infty|\infty} \) consisting of matrices \( (a_{ij})_{i,j \in \frac{1}{2}\mathbb{Z}} \) such that \( a_{ij} = 0 \) unless \( i, j \in \mathbb{Z} \). Put \( \widetilde{\mathfrak{g}}l_\infty = \mathfrak{g}l_\infty \oplus CK \), which is an infinite dimensional Lie algebra. The triangular decomposition is naturally induced from \( \widetilde{\mathfrak{g}}l_{\infty|\infty} \), say, \( \mathfrak{g}l_\infty = n_+^i \oplus \mathfrak{h}^0 \oplus \mathfrak{n}_0^i \). As in the case of \( \mathfrak{g}l_{\infty|\infty} \), we can define the Verma module \( M^0(\Lambda) \), and the associated irreducible highest weight module \( L^0(\Lambda) \) for \( \Lambda \in (\mathfrak{h}^0)^* \). The fundamental weights \( \omega_k \) (\( k \in \mathbb{Z} \)) and \( \Lambda_0 \) are still available.
For \( n \geq 1 \), let \( \mathfrak{F}_n^0 \) be the infinite dimensional Fock space generated by \( n \) pairs of free bosons (see [16] for a detailed description). In [16], using a natural commuting action of \( \hat{\mathfrak{gl}}_{\infty|\infty} \) and \( \mathfrak{gl}_n \) on \( \mathfrak{F}_n^0 \), Kac and Radul derived a multiplicity-free decomposition as follows:

**Theorem 5.6 ([16])**. As a \((\hat{\mathfrak{g}}\mathfrak{l}_{\infty}, \mathfrak{g}\mathfrak{l}_n)\)-module,

\[
\mathfrak{F}_n^0 \cong \bigoplus_{\lambda \in \mathbb{Z}_+^n} L^0(\Lambda(\lambda)) \otimes L_n(\lambda),
\]

where \( \Lambda(\lambda) \in (\mathfrak{h}^0)^* \) is the highest weight determined by

\[
\Lambda(\lambda)(e_{kk}) = \begin{cases} 
\lambda_k, & \text{if } k \in \mathbb{Z}_{>0} \text{ and } \lambda_k > 0, \\
\lambda_{n+k}, & \text{if } k \in \mathbb{Z}_{\leq0} \text{ and } \lambda_{n+k} < 0, \\
0, & \text{otherwise},
\end{cases}
\]

\( \Lambda(\lambda)(K) = -n. \) □

Let \( \mathcal{A} = \mathbb{Z}_{>0} \) and \( \mathcal{B} = \mathbb{Z}_{\leq0} \) be the sets with the usual linear ordering such that all the elements are of degree 0, that is, \( \mathcal{A}_0 = \mathcal{A} \) and \( \mathcal{B}_0 = \mathcal{B} \). Comparing the Kac and Radul’s formula for \( L^0(\Lambda(\lambda)) \) with Proposition 3.20, we obtain the following.

**Theorem 5.7.** For \( \lambda \in \mathbb{Z}_+^n \), we have

\[
\text{ch} L^0(\Lambda(\lambda)) = S^{\mathcal{A}/\mathcal{B}}_\lambda,
\]

where \( \mathcal{A} = \mathbb{Z}_{>0} \) and \( \mathcal{B} = \mathbb{Z}_{\leq0} \). □

**Corollary 5.8.** Under the above hypothesis, we have

\[
\text{ch} L^0(\Lambda(\lambda)) = \det \left( \text{ch} L^0(\Lambda(\lambda_i - i + j)) \right)_{1 \leq i, j \leq n}.
\]

□

5.2. Grothendieck rings. Let us summarize the previous results in terms of categories of representations.

For \( n \geq 1 \), let \( \mathcal{R}_n \) be the Grothendieck group for the category of rational representations of \( \mathfrak{g}\mathfrak{l}_n \). We may view \( \mathcal{R}_n \) \((n \geq 1)\) as the free \( \mathbb{Z} \)-module spanned by the rational Schur polynomials \( s_\lambda = s_\lambda(x_{[n]}) \) for \( \lambda \in \mathbb{Z}_+^n \). Let \( \mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n \), where \( \mathcal{R}_0 = \mathbb{Z} \). If we identify the elements in \( \mathcal{R} \otimes \mathcal{R} \) with the functions in two sets of variables \( x_N \) and \( y_N \), then there exists a natural comultiplication \( \Delta : \mathcal{R} \to \mathcal{R} \otimes \mathcal{R} \) defined by \( \Delta(1) = 1 \otimes 1 \) and \( \Delta(s_\lambda(x_{[n]})) = \sum_{p+q=n} s_{\lambda}(x_{[p]}, y_{[q]}) \) for \( \lambda \in \mathbb{Z}_+^n \); where \( 1 \) denotes the unity in \( \mathcal{R}_0 \) and \( y_{[q]} = \{ x_{p+1}, \ldots, x_n \} \). By (3.9), we have

\[
\Delta(s_\lambda(x_{[n]})) = s_\lambda(x_{[n]}) \otimes 1 + 1 \otimes s_\lambda(x_{[n]}) + \sum_{\mu, \nu} \hat{c}^\lambda_{\mu, \nu} s_\mu(x_{[p]}) \otimes s_\nu(x_{[q]}),
\]
where the sum is taken over all $\mu \in \mathbb{Z}_+^p$ and $\nu \in \mathbb{Z}_+^q$ such that $p + q = n$. Also, the counit $\varepsilon : R \to \mathbb{Z}$ is the $\mathbb{Z}$-linear map which vanishes on $R_n$ for $n \geq 1$ with $\varepsilon(1) = 1$. Note that $\Delta$ preserves the grading, that is, $\Delta(R_n) \subset (R \otimes R)_n = \bigoplus_{p+q=n} R_p \otimes R_q$. Hence $R$ is a graded cocommutative coalgebra over $\mathbb{Z}$ (cf. [28]).

Let $R^* = \bigoplus_{n \geq 0} R_n^*$ be the graded dual of $R$, where $R_n^* = \text{Hom}_\mathbb{Z}(R_n, \mathbb{Z})$. Let $\sigma_\lambda$ be the element in $R^*$ dual to $s_\lambda$ for $\lambda \in \mathbb{Z}_+^n$ (that is, $\sigma_\lambda(s_\mu) = \delta_{n,n'}\delta_{\lambda,\mu}$ for $\mu \in \mathbb{Z}_+^n$). Then for any $\varphi \in R^*$, we may write $\varphi = \varphi(1)\varepsilon + \sum_{\lambda} \varphi(\lambda)\sigma_\lambda$.

Since $R$ is a cocommutative coalgebra over $\mathbb{Z}$, $R^*$ naturally becomes a graded commutative $\mathbb{Z}$-algebra with the multiplication $\Delta^*$ given by

$$
\Delta^*(\sum_{\mu} a_{\mu} \sigma_\mu \otimes \sum_{\nu} b_{\nu} \sigma_{\nu}) = \sum_{\lambda} \left( \sum_{\mu,\nu} a_{\mu} b_{\nu} \hat{c}_{\mu,\nu}^\lambda \right) \sigma_\lambda,
$$

$$
\Delta^*(\sum_{\mu} a_{\mu} \sigma_\mu \otimes \varepsilon) = \Delta^*(\varepsilon \otimes \sum_{\mu} a_{\mu} \sigma_\mu) = \sum_{\mu} a_{\mu} \sigma_\mu,
$$

$$
\Delta^*(\varepsilon \otimes \varepsilon) = \varepsilon,
$$

where we assume that $\hat{c}_{\mu,\nu}^\lambda = 0$ unless $\lambda \in \mathbb{Z}_+^{p+q}$, $\mu \in \mathbb{Z}_+^p$ and $\nu \in \mathbb{Z}_+^q$.

From now on, we assume that $g = \hat{g}|_{\mathbb{Z}}$ or $\hat{g}_\infty$. For each $\lambda \in \mathbb{Z}_+^n$, we denote by $L_\lambda$ the associated irreducible representation of $g$ (that is, $L_\lambda = L(\Lambda(\lambda))$ or $L^0(\Lambda(\lambda)))$. For $n \geq 1$, let $\mathcal{O}_n$ be the category of $g$-modules $V$ which are isomorphic to

$$
\bigoplus_{\lambda \in \mathbb{Z}_+^n} L_\lambda^\otimes m_\lambda
$$

for some $m_\lambda \in \mathbb{Z}_{\geq 0}$. For convenience, we denote by $\mathcal{O}_0$ the category of finite dimensional trivial representations of $g$.

**Lemma 5.9.** For $V \in \mathcal{O}_m$ and $W \in \mathcal{O}_n$, we have $V \otimes W \in \mathcal{O}_{m+n}$.

**Proof.** We may assume that $m, n > 0$. Suppose that

$$
V = \bigoplus_{\mu \in \mathbb{Z}_+^m} L_\mu^\otimes a_\mu, \quad W = \bigoplus_{\nu \in \mathbb{Z}_+^n} L_\nu^\otimes b_\nu,
$$

for some $a_\mu$ and $b_\nu \in \mathbb{Z}_{\geq 0}$. From (3.3), we see that for each $\lambda \in \mathbb{Z}_+^{m+n}$, there exist only finitely many $\mu$ and $\nu$'s such that $c_{\mu,\nu}^\lambda \neq 0$. By Theorem 4.7, the multiplicity of $L_\lambda$ ($\lambda \in \mathbb{Z}_+^{m+n}$) in $V \otimes W$ is given by

$$
\sum_{\mu,\nu} a_\mu b_\nu c_{\mu,\nu}^\lambda,
$$

which is a non-negative integer, and hence $V \otimes W \in \mathcal{O}_{m+n}$. \hfill \Box
Let
\[(5.3) \quad K(g) = \bigoplus_{n \geq 0} K(O_n)\]
be the direct sum of the Grothendieck groups of \(O_n\). By Lemma 5.9, \(K(g)\) naturally becomes a commutative \(\mathbb{Z}\)-algebra with the multiplication given as a tensor product.

We denote by \([V] \in K(O_n)\) the isomorphism class of \(V \in O_n\).

For \(n \geq 1\), we define a \(\mathbb{Z}\)-linear map \(\chi_n : R^* \rightarrow K(O_n)\) by
\[(5.4) \quad \chi_n(\varphi) = \sum_{\lambda \in \mathbb{Z}^{m+n}} \varphi(s_{\lambda}) [L_{\lambda}],\]
for \(\varphi \in R^*\). Also, we define \(\chi_0(\varepsilon) = [C]\), the isomorphism class of the one-dimensional trivial representation. So we have a \(\mathbb{Z}\)-linear map
\[(5.5) \quad \chi = \bigoplus_{n \geq 0} \chi_n : R^* \rightarrow K(g).\]

**Theorem 5.10.** \(\chi\) is an isomorphism of commutative \(\mathbb{Z}\)-algebras.

**Proof.** It follows from Theorem 5.2, 5.7, and 4.7. \(\square\)

Let us end this section with a remark on an involution on \(R^*\). We define a \(\mathbb{Z}\)-linear map \(\omega : R^* \rightarrow R^*\) by
\[(5.6) \quad \omega(\sum_{\lambda} a_{\lambda} \sigma_{\lambda}) = \sum_{\lambda} a_{\lambda} \sigma_{\lambda^*},\]
for \(\sum_{\lambda} a_{\lambda} \sigma_{\lambda} \in R^*,\) and \(\omega(\varepsilon) = \varepsilon\). It is clear that \(\omega\) is an isomorphism of \(\mathbb{Z}\)-modules.

**Lemma 5.11.** There exists a bijection from \(LR^\lambda_{\mu \nu}\) to \(LR^{\lambda^*}_{\mu^* \nu^*}\) for \(\lambda \in \mathbb{Z}_+^{m+n}, \mu \in \mathbb{Z}_+^m,\) and \(\nu \in \mathbb{Z}_+^n\).

**Proof.** Given \(\lambda \in \mathbb{Z}_+^n\), let \(p, q\) be sufficiently large positive integers such that \(\lambda + (p^n), \lambda^* + (q^n) \in \mathcal{P}_n\). Then we can check that
\[
\delta_{p+q}^n([\lambda + (p^n)]') = [\lambda^* + (q^n)]',
\]
as Young diagrams.

Now, suppose that \(\lambda \in \mathbb{Z}_+^{m+n}, \mu \in \mathbb{Z}_+^m, \nu \in \mathbb{Z}_+^n\) are given, and \(p, q\) are sufficiently large positive integers. Then, the required bijection comes from the following one-to-one correspondences
\[
LR^\lambda_{\mu \nu} \overset{1-1}{\longleftrightarrow} LR^{[\lambda+(p^n)]'}_{[\mu+(p^n)]' [\nu+(p^n)]'}
\]
\[
\overset{1-1}{\longleftrightarrow} LR^{[\lambda^*+(q^n)]'}_{[\mu^*+(q^n)]' [\nu^*+(q^n)]'} \quad \text{by Lemma 4.10}
\]
\[
= LR\overset{1-1}{\longleftrightarrow} LR^\lambda_{\mu^* \nu^*}.
\]
Remark 5.12. The map $\vartheta : R \to R$ sending $s_\lambda(x_{[n]})$ to $s_\lambda(x_{[n]}^{-1})$ for $\lambda \in \mathbb{Z}_+^n$ with $\vartheta(1) = 1$, is an automorphism of a coalgebra over $\mathbb{Z}$. In fact, $\vartheta(s_\lambda(x_{[n]})) = s_\lambda(x_{[n]}^{-1})$, and $\omega$ is the map induced from $\vartheta$. From [8.39] and the linear independence of rational Schur polynomials, it follows directly that $\hat{c}_{\mu\nu}^\lambda = \hat{c}_{\mu\nu}^\lambda$ for $\lambda \in \mathbb{Z}^{m+n}, \mu \in \mathbb{Z}_{+}^m, \nu \in \mathbb{Z}_+^n$, while Lemma 5.11 gives a bijective proof of this.

Therefore, it follows that

**Proposition 5.13.** $\omega$ is an involution of $R^*$ as a $\mathbb{Z}$-algebra. 

\[ \Box \]

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