CONSTRUCTIVE EQUIVARIANT OBSERVER DESIGN FOR INERTIAL VELOCITY-AIDED ATTITUDE

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ABSTRACT

Inertial Velocity-Aided Attitude (VAA) is an important problem in the control of Remotely Piloted Aerial Systems (RPAS), and involves estimating the velocity and attitude of a vehicle using gyroscope, accelerometer, and inertial-frame velocity (e.g. GPS velocity) measurements. Existing solutions tend to be complex and provide limited stability guarantees, relying on either high gain designs or assuming constant acceleration of the vehicle. This paper proposes a novel observer for inertial VAA that exploits Lie group symmetries of the system dynamics, and shows that the observer is synchronous with the system trajectories. This is achieved by adding a virtual state of only three dimensions, in contrast to the larger virtual states typically used in the literature. The error dynamics of the observer are shown to be almost globally asymptotically and locally exponentially stable. Finally, the observer is verified in simulation, where it is shown that the estimation error converges to zero even with an extremely poor initial condition.

1 Introduction

Attitude estimation is a fundamental problem in the control of Remotely Piloted Aerial Systems RPAS. Many of the most popular approaches from the last 15 years rely on inertial measurement unit (IMU) signals, including gyroscope, accelerometer, and sometimes magnetometer measurements. A common assumption used in observer design, such as in [Mahony et al., 2008, Martin and Salaün, 2010, Pilimlin et al., 2010, Hua et al., 2014], is that the accelerometer signal is dominated by gravity vector. While this assumption has proven useful in many practical situations, it is unreliable when a RPAS experiences large accelerations such as when sharply changing direction or making turns at high speeds. A number of authors have incorporated either body-fixed frame or inertial-frame measurements of the RPAS’ velocity to overcome this problem. Estimating the attitude of a RPAS using velocity measurements is referred to as the velocity-aided attitude (VAA) problem. Velocity aided attitude estimation algorithms typically include an estimator for the vehicle velocity, as well as the primary estimate of the vehicle attitude, that acts as auxiliary state and provides the connection between the velocity measurement and the attitude estimate.

There are two important variations of the VAA problem, depending on whether the velocity of the vehicle is measured in the body-fixed frame (such as provided by an air-data system or a doppler radar), or in the inertial frame (such as provided by a...
GPS). One of the first approaches to body-fixed VAA was proposed in [Bonnabel et al. 2008], which applied a general theory of symmetry-preserving observers to derive an estimator with a local convergence guarantee. Similarly, Martin and Salaün [2016] used the same approach to derive an observer by linearisation of the symmetry-based error dynamics. They showed the resulting algorithm to be computationally simpler than an EKF, although the convergence could still only be guaranteed locally. By using a body-fixed frame measurement of velocity provided by an underwater doppler system, Troni and Whitcomb [2013] developed an explicit complementary filter and showed that including the velocity measurement improves performance in practical experiments. Allibert et al. [2014] proposed a novel constructive observer design for estimating the velocity and attitude of a quadrotor from and IMU using a body-fixed frame velocity measurement, and showed that using a body-fixed measurement meant that the yaw of the RPAS is unobservable. Allibert et al. [2016] then simplified and extended this observer, and showed that the attitude estimate (including yaw) can be made almost-globally asymptotically stable when a magnetometer measurement is used in addition to the velocity measurement. Similarly, Hua et al. [2016] exploited a magnetometer and a body-fixed frame velocity measurement in the framework of [Bonnabel et al. 2008], and proposed innovation terms that lead to almost-global asymptotic stability of the observer error dynamics. Renallegue et al. [2020] provided a constructive observer design using a body-fixed measurements of velocity and a magnetometer, such that the direction of gravity in the body-fixed frame can be estimated independently from the magnetometer measurements. Finally, Wang and Tayebi [2021] considered the VAA problem with intermittent measurements of velocity and inertial directions in the body-fixed frame, and showed almost-global stability of their proposed hybrid observer.

The solutions to inertial VAA problem solutions are more complex and provide fewer stability guarantees, in general, than the solutions to body-fixed VAA. In some of the first work on the topic, Hua [2010] provided two observers for the inertial VAA problem. The first of these observers features semi-global stability by using a high gain design, and the second features almost-global convergence when the vehicle’s acceleration is constant by using a virtual $3 \times 3$ matrix in the design. Roberts and Tayebi [2011] additionally used a magnetometer measurement to provide two observers with semi-global convergence guarantees by using high gain designs. Gopi et al. [2012] developed an observer for the position, velocity, and attitude of a RPAS using GPS and IMU measurements, and showed the observer error dynamics to be semi-globally exponentially stable by embedding $SO(3) \rightarrow \mathbb{R}^3$ to overcome the topological constraints of the rotation group Bhat and Bernstein [2000]. However, the gains must be tuned carefully to ensure stability, and the attitude estimated provided by the observer cannot be guaranteed to be continuous. Dukan and Swensen [2013] followed a similar approach to create an observer for attitude, position and velocity that takes accelerometer and gyroscope biases into account. While their experimental results showed that the observer can perform well in practice, the authors did not provide any proof of stability. A simplified model of quadrotor dynamics was used by Martin and Sarras [2016] to develop a semi-globally asymptotically stable observer for the vehicle’s pitch, roll, and horizontal velocity, where a gyroscope, an accelerometer, and knowledge of the thrust and rotor drag provide measurements of the velocity and gravity direction directly. Recently, Hua et al. [2017] used a mixed measurement of the vertical component of the inertial frame velocity and the horizontal components of the body-fixed frame velocity to propose a Riccati observer for the inertial VAA problem. They showed this to be locally exponentially stable, and their simulation results indicated a large domain of attraction.

In this paper, we consider the inertial VAA problem without magnetometer measurements. The design procedure developed by van Goor and Mahony [2021] is applied to propose an observer architecture with synchronous error dynamics. It is shown that the resulting observer has almost-globally asymptotically stable attitude error dynamics and globally exponentially stable velocity error dynamics. To the authors’ knowledge, this observer is the first that guarantees almost-global convergence independently of the chosen gains, and without assuming constant acceleration. Moreover, the auxiliary state used in the proposed observer design is naturally connected to the Lie group structure of the system dynamics viewed through the framework of van Goor and Mahony [2021]. Simulation results are provided to demonstrate the observer’s performance.

2 Preliminaries

The special orthogonal group is the Lie group of 3D rotations, defined

$$SO(3) := \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I_3, \det(R) = 1 \}.$$  

For any vector $\Omega \in \mathbb{R}^3$, define

$$\Omega^\times = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}.$$  

Then $\Omega^\times v = \Omega \times v$ for any $v \in \mathbb{R}^3$ where $\times$ is the usual vector (cross) product. The Lie algebra of $SO(3)$ is defined

$$so(3) := \{ \Omega^\times \in \mathbb{R}^{3 \times 3} \mid \Omega \in \mathbb{R}^3 \}.$$  

The projector from $\mathbb{P}_{so(3)} : \mathbb{R}^{3 \times 3} \rightarrow so(3)$ is defined

$$\mathbb{P}_{so(3)}(M) := \frac{1}{2}(M - M^T).$$  

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For any two vectors $a, b \in \mathbb{R}^3$, one has the following identities:
\begin{align*}
    a \times b &= -b \times a, \\
    a \times b^\top &= ba^\top - a^\top bI_3, \\
    (a \times b)^\top &= ba^\top - ab^\top.
\end{align*}
(1)

The special Euclidean group and its Lie algebra are defined
\begin{align*}
    \text{SE}(3) &:= \left\{ \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \middle| R \in \text{SO}(3), \ v \in \mathbb{R}^3 \right\}, \\
    \text{se}(3) &:= \left\{ \begin{pmatrix} \Omega^\times & a \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4} \middle| \Omega, a \in \mathbb{R}^3 \right\}.
\end{align*}

An element of $\text{SE}(3)$ may be denoted $X = (R, v)$, where $R \in \text{SO}(3)$ and $v \in \mathbb{R}^3$ for convenience. Likewise, an element of $\text{se}(3)$ may be denoted $\Delta = (\Omega_\Delta, u_\Delta)$, where $\Omega_\Delta, u_\Delta \in \mathbb{R}^3$.

3 Problem Description

Consider a vehicle equipped with an inertial measurement unit (IMU). Let $R \in \text{SO}(3)$ denote the vehicle’s attitude and $v \in \mathbb{R}^3$ denote the vehicle’s velocity, both with respect to some given inertial frame $\{0\}$. The angular velocity $\Omega \in \mathbb{R}^3$ and the linear acceleration $a \in \mathbb{R}^3$ are measured by the IMU. The dynamical model of $R$ and $v$ considered is
\begin{align*}
    \dot{R} &= R\Omega^\times, \\
    \dot{v} &= Ra + g,
\end{align*}
(2)
where $g \in \mathbb{R}^3$ is the gravity vector in the inertial frame. The problem is to design an observer for $R$ and $v$ using a measurement of the vehicle’s velocity in the inertial frame; that is,
\begin{equation}
    h(R, v) = v.
\end{equation}
(3)

4 Observer Design

4.1 Equivariant Observer Architecture

Identify the homogeneous matrix
\begin{equation}
    X = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \in \text{SE}(3)
\end{equation}
with the state $(R, v)$, noting that although this is not a rigid-body transformation, the symmetry properties of $\text{SE}(3)$ can still be exploited for the VAA state. We will write $X = (R, v) \in \text{SE}(3)$ to save space in the sequel. Similarly we write $U = (\Omega, a)$ and $G = (0, g)$ for
\begin{equation}
    U = \begin{pmatrix} \Omega^\times & a \\ 0 & 0 \end{pmatrix} \in \text{se}(3), \quad \text{and} \quad G = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix} \in \text{se}(3).
\end{equation}

The lifted dynamics (2) on the Lie group $\text{SE}(3)$ may be written as
\begin{equation}
    \dot{X} = XU + GX.
\end{equation}
(4)
In other words, the Lie group dynamics have both left- and right-invariant components corresponding to body- and spatial-velocities.

Following the observer design procedure described in [van Goor and Mahony 2021], let $\hat{X}, \hat{\Omega} \in \text{SE}(3)$ and define
\begin{equation}
    \dot{\hat{X}} = \hat{X}U + G\hat{X} + \Delta\hat{X}, \quad \dot{\hat{\Omega}} = G\hat{\Omega} + \hat{\Gamma}\hat{\Omega},
\end{equation}
(5)
where $\Delta, \Gamma \in \text{se}(3)$ are correction terms that remain to be chosen.

Lemma 4.1. Define an error $\hat{E} := \hat{\Omega}^{-1}X\hat{\Omega}^{-1}Z$. The system dynamics (4) and the observer internal model (5) are $E$-synchronous [van Goor and Mahony 2021].

Proof. Direct computation provides
\begin{align*}
    \dot{\hat{E}} &= -\hat{Z}^{-1}\hat{\Omega}\hat{X}^{-1}\hat{\Omega}\hat{Z}^{-1}\hat{X}^{-1}\hat{Z} + \hat{Z}^{-1}\hat{\Omega}\hat{X}^{-1}\hat{Z} \\
    &\quad - \hat{Z}^{-1}\hat{\Omega}\hat{X}^{-1}\hat{\Omega}\hat{Z}^{-1}\hat{X}^{-1}\hat{Z} + \hat{Z}^{-1}\hat{X}^{-1}\hat{X}^{-1}\hat{Z}, \\
    &= -\hat{Z}^{-1}\hat{\Omega}G\hat{X}\hat{X}^{-1}\hat{Z} - \hat{\Gamma}\hat{E} + \hat{Z}^{-1}(XU + GX)\hat{X}^{-1}\hat{Z} \\
    &\quad - \hat{Z}^{-1}\hat{X}\hat{X}^{-1}(XU + G\hat{X} + \Delta\hat{X})\hat{X}^{-1}\hat{Z} \\
    &\quad + \hat{Z}^{-1}\hat{X}\hat{X}^{-1}G\hat{Z} + \hat{E}\hat{\Gamma}, \\
    &= -\hat{\Gamma}\hat{E} - \hat{Z}^{-1}\hat{X}\hat{X}^{-1}(\Delta\hat{X})\hat{X}^{-1}\hat{Z} + \hat{E}\hat{\Gamma}, \\
    &= -\hat{\Gamma}\hat{E} + \hat{E}\hat{\Gamma} - \hat{E}(\hat{Z}^{-1}\Delta\hat{Z}).
\end{align*}
This shows that, indeed, the $\hat{E} = 0$ whenever the correction terms $\Delta$, $\Gamma$ are chosen to be zero, proving the result.

From $[5]$, let $\dot{Z} = (R_Z, z) \in \text{SE}(3)$ and $\Gamma = (\Omega_\Gamma, u_\Gamma) \in \mathfrak{se}(3)$. Then,

$$\dot{R}_Z = R_Z \Omega_\Gamma^x,$$

By choosing $R_Z(0) = I_3$ and $\Omega_\Gamma \equiv 0$, it follows that $R_Z \equiv I_3$ and

$$\dot{z} = g + R_Z u_\Gamma.$$ 

(6)

It is only the $z$ term that is important in the observer design and we will only consider $\Gamma = (0, u_\Gamma) \in \mathfrak{se}(3)$ in the sequel. Similarly, we will write $[6]$ instead of the full $\dot{Z}$ dynamics $[5]$ to emphasise the simplicity of the proposed observer. Moreover, in this case $\hat{E} := \dot{Z}^{-1}X^{-1}\dot{Z} = (R_E, v_E)$ expands to

$$R_E = R\hat{R}^\top,$$

and $v_E = v - z - RR^\top(\hat{v} - z).$

4.2 Observer Design

**Theorem 4.2.** Consider the system dynamics $[2]$. Let $\hat{R} \in \text{SO}(3)$, $\hat{v}, z \in \mathbb{R}^3$, and define the observer dynamics $[5]$ written in coordinate form

$$\begin{align*}
\dot{\hat{R}} &= \hat{R} \Omega^x + \Omega_\Delta^x \hat{R}, \\
\dot{\hat{v}} &= \hat{R}a + g + \Omega_\Delta^x \hat{v} + u_\Delta, \\
\dot{z} &= g + u_\Gamma,
\end{align*}$$

(8a, 8b, 8c)

Choose the innovation terms $(\Omega_\Delta, u_\Delta, u_\Gamma)$ as follows:

$$\begin{align*}
\Omega_\Delta &= c(\hat{v} - z) \times (v - z), \\
u_\Delta &= k(v - \hat{v}) - c((\hat{v} - z) \times (v - z)) \times z, \\
u_\Gamma &= k(v - z),
\end{align*}$$

(9a, 9b, 9c)

with $k, c > 0$ positive gains. Suppose that the measured velocity $[3]$ is bounded and $Ra$ is a uniformly continuous and persistently exciting signal; that is, there exist constants $\mu, \delta > 0$ such that, for all $b \in S^2$ and time $t \geq 0$, there is a $\tau \in [t, t + \delta]$ satisfying

$$|b^x (R(\tau)a(\tau))| \geq \mu.$$  

(10)

Let $E = (R_E, v_E)$ as in $[7]$. Then

1. The solution $z$ is uniformly continuous $\forall t \geq 0$, and $\dot{E}$ converges to $E_s \cup E_u$ such that $E_s = \{(I, 0)\}$ and $E_u = \{(Q, 0) \in \text{SE}(3) \mid \text{tr}(Q) = -1\}$.

2. The set $E_u$ is the set of unstable equilibria. That is, for any point in $E_u$ and any neighbourhood $\forall t$ of that point, there exists an initial condition in $\forall t$ such that the error dynamics with that initial condition converge to $E_s$.

3. $E_s = \{(I, 0)\}$ is almost-globally asymptotically and locally exponentially stable. Moreover, if $\dot{E} \in E_s$ then $\dot{R} = R$ and $\dot{v} = v$.

**Proof.** Proof of item 1): Recalling $[8c]$ along with $[9]$ and the fact that $v$ is bounded by assumption, it is straightforward to verify that $z$ is bounded and uniformly continuous.

By direct computation, the dynamics of $R_E$ are

$$\begin{align*}
\dot{R}_E &= (R\Omega^x)\hat{R}^\top + R(R\Omega^x + \Delta^x \hat{R}), \\
&= R\Omega^x \hat{R}^\top - R\Omega^x \hat{R}^\top - RR^\top \Delta^x, \\
&= -R_E \Omega_\Delta^x.
\end{align*}$$

Likewise, the dynamics of $v_E$ are

$$\begin{align*}
\dot{v}_E &= (Ra + g) - (g + u_\Gamma) - (-R_E \Omega_\Delta^x)\hat{v} \\
&= -R_E(\hat{R}a + g + \Omega_\Delta^x \hat{v} + u_\Delta) \\
&+ (-R_E \Omega_\Delta^x)z + R_E(g + u_\Gamma), \\
&= Ra - u_\Gamma + R_E \Omega_\Delta^x \hat{v} \\
&- Ra - R_E g - R_E \Omega_\Delta^x \hat{v} - R_E u_\Delta \\
&- R_E \Omega_\Delta^x z + R_E g + R_E u_\Gamma, \\
&= -u_\Gamma - R_E u_\Delta - R_E \Omega_\Delta^x z + R_E u_\Gamma.
\end{align*}$$
By substituting in the correction terms \([9]\),
\[
\begin{align*}
\dot{v}_E &= -u_T - R_E u_\Delta - R_E \Omega_\Delta z + R_E u_T, \\
&= -k(v - z) - R_E (k(v - \hat{v}) - c((\hat{\nu} - z) \times (v - z)) \times z) \\
&\quad - R_E (c(\hat{\nu} - z) \times (v - z))^T z + R_E k(v - z), \\
&= -k(v - z) - kR_E (v - \hat{v}) + kR_E (v - z), \\
&= -k(v - z) + kR_E (\hat{v} - z), \\
&= -k(v - z) - R \hat{R}^T \hat{\nu} + R \hat{R}^T z), \\
&= -kv_E, 
\end{align*}
\]
which implies that \(v_E\) exponentially converges to zero.

Similarly, by substituting for \(\Omega_\Delta\) and using some of the identities \([1]\),
\[
\begin{align*}
\dot{R}_E &= -R_E \Omega_\Delta^T, \\
&= -cR_E ((\hat{\nu} - z) \times (v - z))^T, \\
&= -cR_E ((v - z)(\hat{\nu} - z)^T - (\hat{\nu} - z)(v - z)^T), \\
&= -cR_E ((v - z)(R_E^T(v - z - v_E))^T \\
&\quad - (R_E^T(v - z - v_E))(v - z)^T), \\
&= -cR_E(v - z)(v - z - v_E)^T R_E \\
&\quad + c(v - z - v_E)(v - z)^T.
\end{align*}
\]

Consider the following candidate Lyapunov function,
\[
L := \frac{1}{2} \text{tr} \left( [(\dot{E} - I)A(\dot{E} - I)^T] \right), \\
A := \begin{pmatrix} I_3 & 0 \\ 0 & \alpha \end{pmatrix}, 
\]
with \(\alpha > \frac{\alpha}{2|v_E|} > 0\). One has,
\[
\begin{align*}
L &= \frac{1}{2} \text{tr} \left( \left( R_E^2 - I_3 \right) \left( \dot{R}_E \right) + \alpha \langle v_E, \dot{v}_E \rangle \right) \\
&= \frac{1}{2} \text{tr} \left( \left( R_E^2 - I_3 \right) \left( \dot{R}_E \right) + \alpha \langle v_E, \dot{v}_E \rangle \right), \\
&= \frac{1}{2} |R_E - I_3|^2 + \frac{\alpha}{2} |v_E|^2.
\end{align*}
\]

The dynamics of \(L\) are
\[
\dot{L} = \left( R_E - I_3, \dot{R}_E \right) + \alpha \langle v_E, \dot{v}_E \rangle, \\
= \langle R_E - I_3, -cR_E(v - z)(v - z - v_E)^T R_E \rangle \\
+ \langle R_E - I_3, c(v - z - v_E)(v - z)^T \rangle \\
- \alpha \langle v_E, kv_E \rangle, \\
= c \langle R_E^T R_E - R_E^T, (v - z)(v - z - v_E)^T \rangle \\
+ c \langle R_E^T - I_3, (v - z)(v - z - v_E)^T \rangle - k\alpha |v_E|^2, \\
= c \langle R_E^T R_E - I_3, (v - z)(v - z + v_E)^T \rangle - k\alpha |v_E|^2, \\
= c(v - z - v_E)^T (R_E^T R_E - I_3)(v - z) - k\alpha |v_E|^2, \\
= c(v - z)^T (R_E^T R_E - I_3)(v - z) \\
+ c(v - z)^T (R_E^T R_E - I_3)(v - z) - k\alpha |v_E|^2, \\
= -c \langle (R_E^2 - I)(v - z)^2 - k\alpha |v_E|^2 \\
+ c(v - z)^2 \rangle (R_E^2 - I)(v - z) + |v_E|^2, \\
= \frac{c}{2} |(R_E^2 - I)(v - z)| + |v_E|^2,
\]
Then, by using the definition of \(\alpha\) and the fact that the Frobenius norm is submultiplicative,
\[
\dot{L} \leq \frac{c}{2} |(R_E^2 - I)(v - z)|^2 - \frac{c}{2} |v_E|^2 \\
+ c |v_E||((R_E^2 - I)(v - z))|, \\
= \frac{c}{2} |(R_E^2 - I)(v - z)| + |v_E|^2,
\]
\[
= \frac{c}{2} |(R_E^2 - I)(v - z)| + |v_E|^2,
\]
which is clearly negative semi-definite.

It is easy to see that $\dot{\mathcal{L}}$ is uniformly continuous as it is the product, sum and composition of uniformly continuous functions. It follows from Barbabal’s lemma [Slotine and Li [1991], Lemma 4.2/4.3] that $\dot{\mathcal{L}} \to 0$ and $\mathcal{L} \to \mathcal{L}_{\text{lim}}$, ($\mathcal{L}_{\text{lim}} \leq \mathcal{L}(0)$) a positive constant value. Combining this with the fact that the equilibrium of the sub-state $v_E = 0$ is uniformly exponentially stable, one ensures that:

$$\mathcal{L} \to \text{tr}(I - R_E) \to \mathcal{L}_{\text{lim}}, \quad \text{and } \frac{d}{dt} \text{tr}(I - R_E) \to 0.$$ 

From there, one has

$$\frac{d}{dt} \text{tr}(R_E) = -\text{tr}(R_E\Omega_{\Delta}^2) = -\text{tr}(P_{\text{so}(3)}(R_E)\Omega_{\Delta}^2) \to 0,$$

which implies that (i) $P_{\text{so}(3)}(R_E) \to 0$ or (ii) $\Omega_{\Delta} \to 0$. The first case implies that $R_E \to R_E^T$ and hence $R_E^2 \to R_E^T R_E = I_3$.

The second case directly implies that $\frac{d}{dt} R_E^2 \to 0$. Therefore, in either case one concludes that $\frac{d}{dt} R_E^2 \to 0$.

Now, using the fact $\frac{d}{dt}(v - z) = Ra - k(v - z)$ along with the assumption that $Ra$ is persistently exciting, direct application of Lemma 6.1 shows that $(v - z)$ is also persistently exciting.

Since $\dot{\mathcal{L}} \to 0$ in (15) implies that $R_E^2(v - z) \to v - z$, it follows that $R_E^2 \to I_3$ by direct application of Lemma 6.3. It follows that $R_E \to R_E^T$, and thus $R_E$ converges to a symmetric matrix. From this one concludes that $R_E \to I_3$, or $R_E \to UDU^T$ with $D = \text{diag}(1, -1, -1)$ and $U \in \text{SO}(3)$. For the latter case, note that for any $Q \in \text{SO}(3)$, $\text{tr}(Q) = \det Q$ if and only if $Q = UDU^T$ for some $U \in \text{SO}(3)$. Therefore, $E$ converges asymptotically to $E_s$ (for which $\mathcal{L}_{\text{lim}} = 0$) or to $E_u$ (for which $\mathcal{L}_{\text{lim}} = 0$).

Proof of item 2): To see that $\dot{E} = (R_E, 0) = (UDU^T, 0)$ is an unstable equilibrium for $D = \text{diag}(1, -1, -1)$ and any $U \in \text{SO}(3)$, recall the definition of $\mathcal{L}$ in (12). Let $\omega = U e_1 \in \mathbb{R}^3$ so that

$$R_E \omega = UDU^T U e_1 = U e_1 = \omega,$$

and define $Q(s) = R_E^2 \exp(s\omega \times)$. Define also $L_u(s) = \mathcal{L}(Q(s), 0)$. Then,

$$L_u(s) = \frac{1}{2} |Q(s) - I_3|^2 + \frac{a}{2} |\omega|^2,
= \frac{1}{2} \text{tr} \left( (R_E e^{s\omega \times} - I_3)(R_E e^{s\omega \times} - I_3)^T \right),
= \text{tr} \left( I_3 - R_E e^{s\omega \times} \right).$$

By taking a 2nd order Taylor expansion,

$$L_u(s) \approx \text{tr} \left( I_3 - R_E (I_3 + s\omega \times) \right),
= \text{tr} (I_3 - R_E) + s \text{tr}(R_E \omega \times) - \frac{s^2}{2} \text{tr}(R_E \omega \times \omega \times).$$

Noting that $L_u(0) = 4, |\omega| = 1$, and $\text{tr}(R_E \omega \times) = 0$ as $R_E$ is symmetric, this simplifies to

$$L_u(s) \approx L_u(0) - \frac{s^2}{2} \text{tr}(R_E (\omega \omega^T - 2\omega^T \omega I_3)),
= 4 - \frac{s^2}{2} \text{tr}(2\omega^T - R_E),
= 4 - \frac{s^2}{2}(1 - \text{tr}(R_E)),
= 4 - s^2.$$ 

Hence, in any neighbourhood of the equilibrium $(R_E, 0) = (UDU^T, 0)$ of $\mathcal{L}$, there exists a perturbation $Q(s)$ such that $\mathcal{L}(Q(s), 0) < \mathcal{L}(R_E, 0)$. Therefore, the equilibrium $\dot{E} = (UDU^T, 0)$ is unstable. Moreover, in any neighbourhood $\mathcal{U}$ of $(UDU^T, 0)$, there exist initial conditions $(R_E(0), v_E(0)) = (Q(s), 0)$ that guarantee $(R_E, v_E) \to (I_3, 0)$. Since the original choice of $\dot{E} \in E_u$ was arbitrary, this result holds for all elements of the equilibrium set $E_u$.

Proof of item 3): The almost-global asymptotic stability and local exponential stability of $\dot{E}_s = (I_3, 0)$ follow from direct application of Trumpf et al. [2012] Theorem 4.3 along with the persistence of excitation of $(v - z)$ from Lemma 6.2. Finally, supposing that $E = (R_E, v_E) = (I_3, 0)$, the definitions (17) provide

$$\dot{R} = RR^T R = R_E^2 R = R,$$
Constructive Equivariant Observer Design for Inertial Velocity-Aided Attitude

Figure 1: A comparison of the true (solid blue) and estimated (dashed red) velocity and attitude of the example system over time. The pitch and roll components of the attitude converge more quickly than the yaw.

and,

\[ \dot{v} = (RR^T)(-v_E + v - z + RR^T z), \]

\[ = R^T_E(v - v_E) + (I - R^T_E)z = v. \]

This completes the proof.

5 Simulations

The proposed observer is verified in the following simulation. Define \( R(0) = I_3, v(0) = 0 \) and let the input signals be

\[ \Omega(t) = (0 \ 0 \ 1)^T, \]
\[ a(t) = (5\sin(5t) \ 0 \ -9.81)^T, \]
\[ g = (0 \ 0 \ 9.81)^T. \]

Consider the observer defined in Theorem 4.2 and let the initial state be

\[ \hat{R}(0) = \exp(((2 -1 \ 1.5)^T)\times), \]
\[ \hat{v}(0) = (3 \ -2 \ 2)^T, \]
\[ z(0) = (0 \ 0 \ 0)^T. \]

Finally, choose the observer gains \( k = 5, c = 1 \).

The true system and the observer equations were integrated over a period of 15 s using Euler integration with a time-step of 0.1 s. Figure 1 shows the estimated and true system trajectories over time. It is clear to see that both the velocity and attitude converge to the true values. Interestingly, the pitch and roll of the attitude converge more quickly than the yaw, as these are the directions associated with estimating the gravity vector in the body-fixed frame. Figure 2 shows the attitude error, velocity error, and Lyapunov value of the observer over time. From these, it can be seen that there is a fast initial convergence followed by a slower second phase of convergence. Regardless, the Lyapunov value is clearly decreasing, and the observer is able to estimate the true attitude and velocity despite a large initial attitude error of more than 150°.

6 Conclusion

This paper presents a novel constructive observer design for inertial VAA that exploits a modelling of the system dynamics motivated by recent advances in equivariant observer theory [van Goor and Mahony, 2021]. This theory provides an observer architecture that is synchronous with the system trajectories, and this paper proposes correction terms that are shown to lead to almost globally asymptotically and locally exponentially stable error dynamics. To the authors’ knowledge, this design is the first to feature such stability properties independently of the chosen gains. Finally, the provided simulations verify the observer is indeed able to converge from even a large initial error.
Figure 2: The evolution of the error metrics of the observer state over time. The attitude error, velocity error, and Lyapunov value all show a sharp initial decrease, followed by a slower second phase of convergence.

Appendix

Lemma 6.1. Let $a(t) \in \mathbb{R}^3$ be uniformly continuous and bounded. Suppose $a(t)$ is persistently exciting; that is, there exist $\mu, T > 0$ such that, for all $t \geq 0$ and $b \in S^2$, $|b^\times a(\tau)| \geq \mu$ for some $\tau \in [t, t + T)$. If $x(t)$ satisfies $\dot{x} = a - kx$ for some fixed $k > 0$, then $x(t)$ is also uniformly continuous, bounded and persistently exciting.

Proof. The uniform continuity and boundedness of $x$ follow immediately from its dynamics. Fix $\mu, T$ as in the definition of the persistence of excitation of $a$. Since $a$ is uniformly continuous, there exists $\delta \in (0, T)$ such that, if $|t_2 - t_1| < \delta$, then $|a(t_2) - a(t_1)| < \mu/2$.

Let $T' := T + \delta$, and let $b \in S^2$ and $t \geq 0$ be arbitrary. There must exist some $\tau \in [t, t + T)$ such that $|b^\times a(\tau)| > \mu$. Moreover, for any $s \in [0, 1)$, one has that

$$|b^\times a(\tau + s\delta)| = |b^\times a(\tau) + b^\times (a(\tau + s\delta) - a(\tau))|,$$

$$\geq |b^\times a(\tau)| - |b^\times (a(\tau + s\delta) - a(\tau))|,$$

$$\geq |b^\times a(\tau)| - |b||a(\tau + s\delta) - a(\tau)|,$$

$$> \mu - \mu/2,$$

$$= \mu/2.$$

From there, one ensures that $|b^\times a(\tau + s\delta)| > 0$ for all $s \in [0, 1)$.
Let $\mu' := \frac{\mu}{2(1+k^2)}$ and suppose $|b^x)(\tau + s\delta)| \leq \mu'$ for all $s \in [0, 1)$. Then,

$$|b^x)(\tau + \delta) - b^x)(\tau)| = \left| \int_0^1 b^x a(\tau + s\delta) \, ds \right|,$$

$$\geq \left| \int_0^1 b a(\tau + s\delta) \, ds \right| - \left| \int_0^1 k b^x(x(\tau + s\delta)ds \right|,$$

$$\geq \int_0^1 |b a(\tau + s\delta)| \, ds - k \int_0^1 b^x(x(\tau + s\delta)ds,$$

$$\geq \int_0^1 \mu/ ds - k \int_0^1 \mu' ds,$$

$$= \mu/2 - k\mu',$$

$$= \mu/2 - k\mu/(2(k + 2)),$$

$$= (\mu/2 - k\mu)/(2(k + 2)),$$

$$= \mu/2(k + 2),$$

$$= \mu''/2(k + 2).$$

This contradicts the assumption, and therefore there must exist $s \in [0, 1)$ such that $|b^x)(\tau + s\delta)| > \mu'$. Put differently, there exists $\tau' = \tau + s\delta \in [0, T']$ such that $|b^x)(\tau')| > \mu'$. Hence $x$ is persistently exciting, as required.

\[ \square \]

**Lemma 6.2.** Assume that $x$ is a uniformly continuous, bounded and persistently exciting signal. Then there exist $\mu', T' > 0$ such that, for all $t \geq 0$:

$$\lambda_2 \left( \int_t^{t+T'} xx^\top d\tau \right) \geq \mu' ,$$

with $\lambda_2(S)$ denotes the second largest eigenvalue of a symmetric matrix $S \in \mathbb{R}^{3 \times 3}$.

**Proof.** Since $x(t)$ is a persistently exciting signal, then for all $t \geq 0$ and $b \in \mathbb{S}^2$ there exists $\tau \in [t, t+T)$ such that $|b^x)(\tau)| > \mu$. This implies that:

$$|b^x)(\tau)|^2 = b^\top (|x|^2I - xx^\top) b > \mu^2 .$$

Now, since $x$ is uniformly continuous and bounded, it follows that there exists $\mu''$, such that:

$$\int_t^{t+T'} |b^x)(\tau)|^2 d\tau = b^\top \left( \int_t^{t+T'} (|x|^2I - xx^\top) d\tau \right) b > \mu'' .$$

Equivalently, one has that

$$\lambda_{\min} \left( \int_t^{t+T'} (|x|^2I - xx^\top) d\tau \right) > \mu'' .$$

(16)

(17)

Taking the trace of both sides of (16),

$$\text{tr} \left( \int_t^{t+T'} (|x|^2I - xx^\top) d\tau \right) = 2 \text{tr} \left( \int_t^{t+T'} xx^\top d\tau \right) > 3\mu'' .$$

Let $(\lambda_1, \lambda_2, \lambda_3)$ denote the eigenvalues of $\int_t^{t+T'} xx^\top d\tau$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then $\lambda_1 + \lambda_2 + \lambda_3 > 3\mu''$. Additionally, from (10), $\lambda_{\min} = \lambda_2 + \lambda_3 > \mu''$. Since $\lambda_2 \geq \lambda_3$, this ensures that $\lambda_2 \geq \frac{1}{2} \mu''$. 

\[ \square \]
Lemma 6.3. Let $Q \in \text{SO}(3)$ such that $\dot{Q}$ converges to zero. Consider a bounded and uniformly continuous persistently exciting signal $x \in \mathbb{R}^3$. That is, there exist constants $\mu, \delta > 0$ such that

$$\lambda_2 \left( \int_t^{t+T'} xx^\top d\tau \right) \geq \mu',$$

(18)

according to lemma 6.2. If $(I - Q)x \to 0$, then $Q$ converges to $I$.

Proof. Define $\epsilon, \delta > 0$ so that, for all $t \geq 0$, one has

$$M(t) := \lambda_2 \left( \int_t^{t+\delta} xx^\top d\tau \right) > \epsilon.$$

Integrating by parts yields

$$M(t) = \int_t^{t+\delta} xx^\top d\tau,$$

$$= \int_t^{t+\delta} Qxx^\top d\tau,$$

$$= \left[ Q(t + \tau) \int_t^{t+\tau} x(s)x(s)^\top ds \right]_0^\delta$$

$$- \int_t^{t+\delta} \left( \frac{d}{d\tau} Q(\tau) \right) \left( \int_t^{t+\tau} x(s)x(s)^\top ds \right) d\tau,$$

$$\to \left[ Q(t + \tau) \int_t^{t+\tau} x(s)x(s)^\top ds \right]_0^\delta,$$

$$= Q(t + \delta) \int_t^{t+\delta} x(\tau)x(\tau)^\top d\tau,$$

$$= Q(t + \delta) M(t).$$

Then it follows that $QM \to M$. Let $(m_1, m_2, m_3)$ be the three orthonormal eigenvectors of $M$ associated with the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 > \lambda_2 \geq \epsilon$. It is straightforward to verify that $(m_1, m_2)$ are also eigenvectors of $Q$ associated to the eigenvalue 1. Combining this with the fact that $\det(Q) = 1$, one concludes that $Q \to I$. 

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