Research Article

V-Probximal Trustworthy Banach Spaces

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In a recent work (2016), the first author proved the fuzzy sum rule for the V-proximal subdifferential under some natural assumptions on an equivalent norm of the Banach spaces. In the present paper, we are going to prove that the class of Banach spaces satisfying the fuzzy sum rule is very large and contains all \( L^p \) spaces satisfying the fuzzy sum rule as well as the sequence spaces \( W^{p,\infty}(1 < p < \infty) \), the Sobolev spaces \( W^{p,\infty}(1 < p < \infty) \), and the Schatten trace ideals \( C^p(1 < p < \infty) \).

1. Preliminaries

Let \( X \) be a Banach space with dual \( X^* \), \( f : X \rightarrow \mathbb{R} \cup \{ \infty \} \) a function, \( \bar{x} \in \text{dom } f : \{ x \in X : f(x) < \infty \} \). Throughout the paper, \( B \) denotes the closed unit ball in \( X \) and \( \langle \cdot, \cdot \rangle \) is the dual pairing between \( X \) and its dual \( X^* \). We define (see [1, 2]) the analytic (resp., the geometric) V-proximal subdifferential of \( f \) at \( \bar{x} \) as follows:

\[
\partial^v f(\bar{x}) := \{ x^* \in X^* : \exists \delta, \sigma > 0, (x^*, x - \bar{x}) \leq f(x) - f(\bar{x}) + \sigma \| x - \bar{x} \|^2, \forall x \in \bar{x} + \delta B \} \tag{1}
\]

(resp. \( \partial^g f(\bar{x}) := \{ x^* \in X^* : (x^*, -1) \in N^\sigma(\text{epi} f ; (\bar{x}, f(\bar{x})) \} \), where \( N^\sigma(S ; u) = \partial^g \pi_S(u) \) is the V-proximal normal cone associated with \( S \) at \( u \in S \). Here, \( J : X \rightarrow X^* \) is the normalized duality mapping and \( V : X^* \times X \rightarrow \mathbb{R} \) is a functional defined by

\[
V(x^*, x) = \| x^* \|^2 - 2\langle x^*, x \rangle + \| x \|^2, \text{ for any } x^* \in X^*, x \in X. \tag{2}
\]

We recall, respectively, the well-known concepts of proximal subdifferential and Fréchet subdifferential (see, for instance, [3]):

(i) \( x^* \in \partial^pf(x) \) if and only if there exist \( \delta, \sigma > 0 \) such that

\[
\langle x^*, x' - x \rangle \leq f(x') - f(x) + \sigma \| x' - x \|^2, \text{ for any } x' \in x + \delta B \notag
\]

(ii) \( x^* \in \partial^gf(x) \) if and only if for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\langle x^*, x' - x \rangle \leq f(x') - f(x) + \epsilon \| x' - x \|, \text{ for any } x' \in x + \delta B \tag{3}
\]

The Fréchet and proximal normal cones are defined as

\[
N^F(S ; \bar{x}) = \partial^f \pi_S(\bar{x}) \quad \text{and} \quad N^p(S ; \bar{x}) = \partial^p \pi_S(\bar{x}). \notag
\]

Notice that (see [3–5]) Fréchet and proximal subdifferential can be defined geometrically by the formulas

\[
\partial^F f(\bar{x}) = \{ x^* \in X^* : (x^*, -1) \in N^F(S ; \bar{x}) \}, \tag{4}
\]

\[
\partial^p f(\bar{x}) = \{ x^* \in X^* : (x^*, -1) \in N^p(S ; \bar{x}) \}. \tag{5}
\]

Using the same terminology used in Ioffe [6], we will say that \( X \) is a V-proximal trustworthy space provided that for any \( \epsilon > 0 \), any two functions \( f_1, f_2 : X \rightarrow \mathbb{R} \cup \{ \infty \} \) and any \( u \in X \) such that \( f_1 \) is lower semicontinuous and \( f_2 \) is Lipschitz around \( u \), the following fuzzy sum rule holds:
\[ \partial^2_u(f_1 + f_2)(u) \subset \{ \partial^2_u f_i(u_1) + \partial^2_u f_i(u_2) : u_i \in U_i(u, \varepsilon), \ i = 1, 2 \} + \varepsilon B. \]

(6)

Here, \( U_i(u, \varepsilon) = \{ x \in u + \varepsilon B \text{ such that } |f_i(x) - f_i(u)| < \varepsilon \} \) and \( B \) denotes the closed unit ball in \( X^* \). It has been proved in Theorem 2.3 in [1] that if \( X \) is a Banach space with an equivalent norm \( \| \cdot \| \) such that \( \| \cdot \|^s \) (for some \( s \geq 2 \)) is \( C^2 \)-differentiable on \( X \) \( \setminus \{ 0 \} \) and \( V \) be the functional associated to that norm \( \| \cdot \| \), then \( X \) is \( V \)-proximal trust-worthy space. Indeed, we have proved the following result.

**Theorem 1.** Let \( X \) be a Banach space with an equivalent norm \( \| \cdot \| \) such that \( \| \cdot \|^s \) (for some \( s \geq 2 \)) is \( C^2 \)-differentiable on \( X \) \( \setminus \{ 0 \} \), and let \( V \) be the functional associated to that norm \( \| \cdot \| \). For any \( \varepsilon > 0 \), any two functions \( f_1, f_2 : X \to \mathbb{R} \cup \{ \infty \} \) and any \( u \in X \) such that \( f_1 \) is lower semicontinuous and \( f_2 \) is Lipschitz around \( u \), the following fuzzy sum rule holds:

\[ \partial^2_u (f_1 + f_2)(u) \subset \{ \partial^2_u f_i(u_1) + \partial^2_u f_i(u_2) : u_i \in U_i(u, \varepsilon), \ i = 1, 2 \} + \varepsilon B. \]

(7)

For a positive measure space \( (\Omega, \Sigma, \mu) \), we denote by \( L^p \), \( p \in [1, \infty) \), the Banach space \( L^p(\Omega, \Sigma, \mu) \) with its canonical norm \( \| x \| = (\int_\Omega |x(w)|^p d\mu(w))^{1/p} \).

We recall the following result from Theorem 1.1 in Section 5.1 in [7] (see also [8]).

**Theorem 2.**

(i) If \( p \) is an even integer, then \( \| \cdot \|^p \) is \( C^m \)-differentiable on \( L^p \setminus \{ 0 \} \)

(ii) If \( p \) is an odd integer, then \( \| \cdot \|^p \) is \( C^{m-1} \)-smooth on \( L^p \setminus \{ 0 \} \)

(iii) If \( p \) is not integer, then \( \| \cdot \|^p \) is \( C^{[p]} \)-smooth on \( L^p \setminus \{ 0 \} \), where \( [p] \) is the integer part of \( p \)

The following corollary follows directly from Theorem 2.

**Corollary 3.** For the canonical norm of \( L^p \), we have \( \| \cdot \|^p \) which is \( C^2 \)-differentiable on \( L^p \setminus \{ 0 \} \), for any \( p \geq 2 \).

Unfortunately, for the case of \( p \in (1, 2) \), the function \( \| \cdot \|^p \) is not \( C^2 \)-differentiable on \( L^p \setminus \{ 0 \} \) and so the fuzzy sum rule cannot be covered by Theorem 1. This our objective in the next few lines, and so we obtain that \( L^p \) is \( V \)-proximal trust-worthy, for any \( p \in (1, \infty) \). In the sequel of this section, we assume that \( X = L^p \) with \( p \in (1, 2) \). It is well-known that \( X \) is \( 2 \)-uniformly convex (see, for instance, [7]), that is, there is a constant \( c > 0 \) such that

\[ \delta_X(\varepsilon) \geq c\varepsilon^2, \quad \text{for all } 0 < \varepsilon \leq 2. \]

(8)

The following lemma is taken from [9].

**Lemma 4.** If \( E \) is a uniformly convex Banach space, then the inequality

\[ V(J(x), y) \geq 8C^2\delta_E \left( \frac{\| x - y \|}{4C} \right) \]

holds for all \( x \) and \( y \) in \( E \), where \( C = \sqrt{(||x||^2 + ||y||^2)/2} \).

In page 51 in [9], the following parallelogram inequality is presented:

\[ 2||x||^2 + 2||y||^2 - ||x - y||^2 \geq (p - 1)||x + y||^2 \quad (1 < p \leq 2), \quad \text{for any } x, y \in X. \]

(10)

It follows that

\[ \frac{||x - y||}{4C} \leq \frac{1}{2}, \quad \text{for any } x, y \in X, \]

(11)

and so the previous lemma with the inequality (8) yields the following important result.

**Proposition 5.** For some positive constant \( c > 0 \), we have for any \( x \) and \( y \) in \( X = L^p \) \( (1 < p \leq 2) \),

\[ V(J(x), y) \geq \frac{c}{2}||x - y||^2. \]

(12)

Using this proposition, we get obviously, in our setting \( X = L^p \) \( (1 < p \leq 2) \), the inclusions

\[ \partial^p f(\bar{x}) \subset \partial^p_u f(\bar{x}), \]

\[ N^p(S; \bar{x}) \subset N^p(S; \bar{x}). \]

(13)

(14)

However, the inclusions \( \partial^p x(\bar{x}) \subset \partial^p u(\bar{x}) \subset \partial^p f(\bar{x}) \) and \( N^p(S; \bar{x}) \subset N^p(S; \bar{x}) \) have been proved, in [1], for Banach spaces with the assumption: For some \( \sigma, \delta > 0 \) and \( q \in (1, 2] \),

\[ V(J(\bar{x}), x) \leq \delta \| x - \bar{x} \|^q, \quad \text{for all } x \in \bar{x} + \delta B. \]

(15)

This assumption is valid whenever the space is \( q \)-uniformly convex Banach spaces. For reason of completeness, we present their proofs.

**Proposition 6.** Assume that there exist constants \( \bar{\sigma}, \bar{\delta} > 0 \) and \( q \in (1, 2] \) such that 1.8 holds. Then,

\[ N^p(S; \bar{x}) \subset N^p(S; \bar{x}), \]

\[ \partial^p u(\bar{x}) \subset \partial^p f(\bar{x}). \]

(16)

Proof. We prove only the first relation; the second one can be conducted similarly. Let \( \bar{x} \in N^p(S; \bar{x}) \). Then, by Proposition 3.4 in [2], there exists \( \sigma > 0 \) such that

\[ \langle x^*, \bar{x} - \bar{x} \rangle \leq \sigma V(J(\bar{x}), x), \quad \text{for all } x \in S. \]

(17)
Now, let $\varepsilon > 0$ be given and take $\mu = (1/2) \min \{ \delta, \varepsilon/\sigma \}$.

Then, by (15),

$$
\| x, x - \tilde{x} \| \leq \sigma V(J(\tilde{x}), x) \leq \sigma \tilde{\sigma} \| x, x - \tilde{x} \|^q
= \sigma \tilde{\sigma} \| x, x - \tilde{x} \|^{q - 1} \| x - \tilde{x} \| < \varepsilon \| x, x - \tilde{x} \|
$$

(18)

Hence, for any $\varepsilon > 0$, there exists $\mu > 0$ such that $\| x, x - \tilde{x} \| \leq \varepsilon \| x - \tilde{x} \|$, for any $x \in S \cap (\tilde{x} + \mu B)$, that is $x^* \in N^F(S; \tilde{x})$ and so $N^\mu(S; \tilde{x}) \subset N^F(S; \tilde{x})$.

Using Remark 7.7 in [9] and the fact that $X = L^p(\rho \in (1, 2))$ is $p$-uniformly smooth, we obtain

$$
V(J(\tilde{x}), y) \leq \beta \rho_{X_{1}} \left( \frac{4 \| x - y \|}{R^p} \right) \leq \beta \frac{4 \| x - y \|}{R^p}, \ \forall x, y \in \mathbb{R} B
$$

(19)

and hence, for any $\tilde{x} \in X$ and any $\delta > 0$, we can write

$$
V(J(\tilde{x}), y) \leq \beta \| x - \tilde{x} \|^p, \ \forall x \in \tilde{x} + \delta B
$$

(20)

where $\tilde{\sigma}$ depends on $\tilde{x}, \delta$, and $\beta$. Consequently, as a direct corollary of Proposition 6, we have, in our setting of $L^p$ spaces ($\rho \in (1, 2)$), both inclusions

$$
N^\mu(S; \tilde{x}) \subset N^F(S; \tilde{x})
$$

(21)

$$
\partial_{\tilde{\sigma}}^\mu f(x) \subset \partial_G^\mu f(x) \subset \partial F f(\tilde{x})
$$

(22)

that hold for any lower semicontinuous function $f$ and any nonempty closed set $S$.

Now, we recall from lojko [10] and Fabian [11] the following two important results.

**Proposition 7.** Suppose that $X$ is uniformly convex and that $S$ is a closed subset of $X$. Let $x^* \in S$ and $x^* \in N^F(S; \tilde{x})$. Then, for any $\varepsilon > 0$, there exist $x_1 \in S$ and $x_2 \in N^F(S; x_1)$ such that $\| x - x_1 \| < \varepsilon$ and $\| x^* - x_1^* \| < \varepsilon$.

**Theorem 8.** Let $X$ be an Asplund space. For any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{ \infty \}$ and any $u \in X$ such that $f_1$ is lower semicontinuous and $f_2$ is Lipschitz around $u$, the following fuzzy sum rule holds:

$$
\partial^F (f_1 + f_2)(u) \subset \left\{ \partial^F f_2(u_2), u_i \in U_{\partial_f}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon B
$$

(23)

Now we are ready to prove the main result of this section.

**Theorem 9.** Let $X = L^p(\rho \in (1, 2))$. For any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{ \infty \}$ and any $u \in X$ such that $f_1$ is lower semicontinuous and $f_2$ is Lipschitz around $u$, the following fuzzy sum rule holds:

$$
\partial^F (f_1 + f_2)(u) \subset \left\{ \partial^F f_2(u_2), u_i \in U_{\partial_f}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon B
$$

(24)

**Proof.** Fix any $\varepsilon > 0$. Let $x^* \in \partial_{\tilde{\sigma}}^\mu (f_1 + f_2)(u)$. Then, by (21) we have $x^* \in \partial_{\tilde{\sigma}}^F (f_1 + f_2)(u)$. Then, by Theorem 8, there exist $v_1 \in U_{\partial_f}(u, \varepsilon/2)$ and $x_1^* \in \partial_{\tilde{\sigma}}^\mu (f_1(u)), i = 1, 2$ such that $\| x^* - (x_1^* + x_2^*) \| < \varepsilon/2$. Thus, $v_2 \in U_{\partial_f}(v_1, \varepsilon/2)$ and $(x_1^*, x_2^*) \in \partial_{\tilde{\sigma}}^\mu (f_1(u) + f_2(u))$ such that

$$
\| (x_1^*, x_2^*) - (v_1, v_2) \| < \frac{\varepsilon}{2},
$$

(25)

$$
\| (x_1^*, x_2^*) - (x_1^* - 1) \| < \frac{\varepsilon}{2}.
$$

(26)

Clearly, $x_1^*, x_2^* \in \partial_{\tilde{\sigma}}^\mu f_i(x_i^*)$ and $u_i = x_i^*$ for $i = 1, 2$. Thus, by the inclusions (5) and (13), we obtain

$$
\frac{x_1^*}{\alpha_{i, \delta}} \in \partial_{\tilde{\sigma}}^F (\partial_{\tilde{\sigma}}^F f_i(x_i^*))
$$

(27)

$$
\| (x_1^*, x_2^*) - (v_1, v_2) \| < \frac{\varepsilon}{2},
$$

(26)

$$
\| (x_1^*, x_2^*) - (x_1^* - 1) \| < \frac{\varepsilon}{2}.
$$

(27)

Also, we have

$$
\| y_1^* - x_1^* \| \leq \| y_1^* - \alpha_{i, \delta} x_1^* \| + \| \alpha_{i, \delta} x_1^* - x_1^* \|
$$

(28)

$$
\leq \alpha_{i, \delta} \| x_1^* - x_1^* \| + \| x_1^* \|
$$

$$
\leq \frac{\delta}{\alpha_{i, \delta}} \| x_1^* \| \leq \frac{\varepsilon}{2} (1 + \| x_1^* \|)
$$

Thus,

$$
\| x^* - (y_1^* + y_2^*) \| \leq \| x^* - (x_1^* + x_2^*) \| + \sum_{i=1}^2 \| y_i^* - x_i^* \|
$$

(29)

$$
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$


Therefore, for any $\varepsilon > 0$, there exist $u_1 \in U_{\eta_1}(u, \varepsilon), u_2 \in U_{\eta_2}(u, \varepsilon)$, $y_1^* \in \partial_q f_1(u_1)$, and $y_2^* \in \partial_q f_2(u_2)$ such that

$$x^* \in y_1^* + y_2^* + \varepsilon B_\ast.$$

(30)

This completes the proof.

Remark 10.

(i) An inspection of the previous proof shows that Theorem 9 holds for any 2-uniformly convex and p-uniformly smooth. Indeed, all the assumptions of Propositions 6–7 and Theorem 8 are satisfied whenever the space is 2-uniformly convex and p-uniformly smooth. Consequently, all the spaces $F, C^p, W^{p,m}, L^p$ for $p \in (1, 2]$ are V-proximal trustworthy. For the proof of the uniform convexity and uniform smoothness of the spaces $F, L^p, W^{p,m}$, we refer, for instance, to Remark 1.6.9 in [9] and for the Schatten trace ideals $C^p$, we refer to [12]

(ii) For the case $p \geq 2$, we have the following:

(i) All Sobolev spaces $W^{p,m}$ and $L^p$ are V-proximal trustworthy for any $p \geq 2$ by combining Theorem 2 and Theorem 1.

(ii) The spaces $C^p$ for $p \geq 2$ are V-proximal trustworthy by combining Theorem 1. and the following result proved in Theorem 2 in [13].

Theorem 11.

(i) If $p$ is an even integer, then the norm in $C^p$ is $C^{\infty}$-differentiable away from zero

(ii) If $p$ is an odd integer, then the norm in $C^p$ is $C^{p-1}$-differentiable away from zero and is not $C^p$-differentiable

(iii) If $p$ is not integer, then the norm in $C^p$ is $C^{|p|}$-differentiable away from zero and is not $C^{|p|+1}$-differentiable

2. Approximate Mean Value Theorem

This section is concerned with the approximate mean value theorem for both analytic and geometric V-proximal subdifferentials in V-proximal trustworthy spaces. It is well-known that approximate mean value theorem (AMVT) and its variants are very important in nonsmooth analysis and optimization. It has been proved for all the existing subdifferentials (see, for instance, [14]). Since the analytic V-proximal subdifferential $\partial^\pi Q$ is the smallest one in some important cases (for instance, $L^p, C^p, W^{p,m}$ with $p \geq 2$), we cannot deduce that the AMVT form the ones proved for the other existing subdifferentials ([14]). For this reason, it is needed to prove the AMVT for the analytic V-proximal subdifferential $\partial^\pi Q$ in V-proximal trustworthy spaces. For the geometric V-proximal subdifferential $\partial^\pi Q$, the AMVT follows directly from the inclusion $\partial^\pi F \subset \partial^\pi Q$.

Theorem 12 (Approximate mean value theorem). Let $X$ be a smooth Banach space which is V-proximal trustworthy and let $f : X \rightarrow R \cup \{\infty\}$, be l.s.c. function finite at two distinct points $a \neq b$. Let $r \in R$ with $r < f(b) - f(a)$. Then, there exist $c \in [a, b)$ and a sequence $x_n \rightarrow c$ with $f(x_n) \rightarrow f(c)$ and $x_n^* \in \partial^\pi f(x_n)$ such that

$$\liminf_{n} \left( x_n^* - x_n - c \right) \geq 0,$$

$$\liminf_{n} \left( x_n^* - b - a \right) \geq r$$

(31)

$$f(c) - f(a) \geq |r|. $$

Proof. Since $X$ is smooth, we can take $x^* \in X^*$ so that $r = \langle x^*, a - b \rangle$. Set $g(x) := f(x) + \langle x^*, x \rangle + \psi_{[a, b]}(x)$. Clearly, $g(b) \geq g(a)$ and so $g$ attains its minimum at some point $c \in [a, b)$ and so $0 \in \partial_{\pi}^Q f(c)$ and hence, we can take $y_n^* \in \partial_{\pi}^Q f(x_n^*)$ such that $\|y_n^* + x_n^* + x^*\| < 1/n$. Then, $\{y_n\} \subset [a, b]$ for $n$ sufficiently large and hence,

$$\langle y_n^*, z - y_n \rangle \leq 0, \quad \forall z \in [a, b], \forall n. $$

(32)

Hence,

$$\langle y_n^*, c - x_n \rangle = \langle y_n^* + x_n^* + y_n^* + c - x_n \rangle + \langle -y_n^*, c - y_n \rangle + \langle -y_n^*, y_n - x_n \rangle + \langle -x_n^*, c - x_n \rangle + \langle y_n^*, y_n - x_n \rangle + \langle -x_n^*, c - x_n \rangle $$

(by (32)).

Taking the liminf on both sides of the previous inequality yields

$$\liminf_{n} \langle x_n^*, c - x_n \rangle \geq \liminf_{n} \left[ \langle x_n^* + x_n^* + y_n^* + c - x_n \rangle + \langle -y_n^*, y_n - x_n \rangle + \langle -x_n^*, c - x_n \rangle \right] \geq 0,$$

(33)

and hence, (i) is proved. Let us show (ii). Since $y_n \in [a, b)$, we can write $y_n = t_n b + (1 - t_n) a$ for a sequence $\{t_n\} \subset [0, 1)$ converging to some $t_0 \in [0, 1)$. Since $c = t_0 b + (1 - t_0) a$. Hence, $b - y_n = (1 - t_0) (b - a)$ and so $b - a = (1/(1 - t_0)) (b - y_n)$. Hence,

$$\liminf_{n} \langle x_n^*, b - a \rangle = \langle x^*, a - b \rangle + \liminf_{n} \langle x_n^* + x^*, b - a \rangle$$

$$= r + \liminf_{n} \langle y_n^*, b - a \rangle$$

$$= r + \frac{1}{1 - t_0} \liminf_{n} \langle y_n^*, y_n - b \rangle \geq (29).$$

(34)

To prove (iii), we use the fact that $c$ is the minimum of $g$ on $[a, b]$ and hence, $g(c) \leq g(a)$; that is, $f(c - f(a) \leq \langle x^*, a - c \rangle \leq 1/(1 - t_0) \langle x^*, a - b \rangle = (1 - t_0)|r|$. 


We close this section with the following result which is an application of the AMVT. Many other applications of the previous version of AMVT can be proved.

**Theorem 13** (Lipschitz behavior). Let \( X \) be a smooth Banach space which is \( V \)-proximal trustworthy and let \( U \) be an open convex subset of \( X \). Let \( f: X \to \mathbb{R} \cup \{+\infty\} \) be a l.s.c. function with \( U \cap \text{dom} \, f = \emptyset \). The following two assertions are equivalent:

1. For some positive constant \( L > 0 \), we have \( \{\|x^*\| : x^* \in \partial \alpha f(x)\} \leq L \) for all \( x \in U \).
2. \( f \) is Lipschitz on \( U \) with rank \( L \).

**Proof.** The implication (2) \( \Rightarrow \) (1) follows directly from the fact that \( \partial \alpha f(x) \) is always included in the Clarke subdifferential. So, we have to prove the implication (1) \( \Rightarrow \) (2). Let \( x, y \in U \) and let \( r = f(y) - f(x) \). Applying Theorem 12 to get a point \( c \in [x, y] \subset U \) and a sequence \( x_n \to c \) with \( f(x_n) \to f(c) \) and \( x^*_n \in \partial \alpha f(x_n) \) such that

\[
\lambda = \lim \inf_{n \to \infty} \langle x^*_n, y - x \rangle.
\]

Let \( \varepsilon > 0 \). By the definition of the liminf, there exist \( z \in U \) and \( z^* \in \partial \alpha f(z) \) such that

\[
f(y) - f(x) \leq \langle z^*, y - x \rangle + \varepsilon \leq \|z^*\| \|y - x\| + \varepsilon \leq L \|y - x\| + \varepsilon.
\]

By taking \( \varepsilon \to 0 \), we obtain

\[
f(y) - f(x) \leq L \|y - x\|,
\]

and by exchanging the roles of \( x \) and \( y \) we arrive to

\[
|f(y) - f(x)| \leq L \|y - x\|, \quad \text{for all } x, y \in U.
\]

This completes the proof.

**Data Availability**

Not applicable.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally to this work.

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