Addendum to
“Spherical structures on torus knots and links” *

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Abstract

The present paper considers an infinite family of cone-manifolds endowed with spherical metric. The singular strata is the torus knot or link \( t(p, q) \) depending on \( \gcd(p, q) = 1 \) or \( \gcd(p, q) > 1 \). In the latter case one obtains a link with \( \gcd(p, q) \) components. Cone angles along all components of the singular strata are supposed to be equal. Domain of existence for a spherical metric is found and a volume formula is presented.

1 Introduction

Three-dimensional cone-manifold is a metric space obtained from a collection of disjoint simplices in the space of constant sectional curvature \( k \) by isometric identification of their faces in such a combinatorial fashion that the resulting topological space is a manifold (also called the underlying space for a given cone-manifold).

Such the metric space inherits the metric of sectional curvature \( k \) on the union of its 2- and 3-dimensional cells. In case \( k = +1 \) the corresponding cone-manifold is called spherical (or admits a spherical structure). By analogy, one defines euclidean \( (k = 0) \) and hyperbolic \( (k = -1) \) cone-manifolds.

The metric structure around each 1-cell is determined by a cone angle that is the sum of dihedral angles of corresponding simplices sharing the 1-cell under identification. The singular locus of a cone-manifold is the closure of all its 1-cells with cone angle different from \( 2\pi \). For the further account

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we suppose that every component of the singular locus is an embedded circle with constant cone angle along it. For the further account, see [1].

The present paper is an addendum to [2] and comprises more general case of torus knot and link cone-manifolds. The cone angles are supposed to be equal for all components of the singular strata. Denote a cone-manifold of torus knot type singularity and cone angle(s) $\alpha$ by $\mathbb{T}_{p,q}(\alpha)$. This cone-manifold is rigid: it is Seifert fibred due to [3] and the base is a turnover of cone angles $\alpha$, $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$. Domains of existence for a spherical metric are given in terms of cone angle(s) and volume formula is presented.

2 Torus $(p, q)$ knots

Denote by $t(p, q)$ a torus knot or link depending on the case of $\gcd(p, q) = 1$ or $\gcd(p, q) > 1$. In the latter case one obtains a link with $\gcd(p, q)$ components, see [4].

As far as $t(p, q)$ and $t(q, p)$ torus links are isotopic one may assume that $p \leq q$ without loss of generality. Denote $\mathbb{T}_{p,q}(\alpha)$ a cone–manifold with singular set the torus knot $t(p, q)$ and cone angle $\alpha$ along its component(s).

The following theorem holds for $\mathbb{T}_{p,q}(\alpha)$ cone-manifolds:

**Theorem 1** The cone-manifold $\mathbb{T}_{p,q}(\alpha)$, $1 \leq p \leq q$ admits a spherical structure if

$$2\pi \left(1 - \frac{1}{p} - \frac{1}{q}\right) < \alpha < 2\pi \left(1 - \frac{1}{p} + \frac{1}{q}\right).$$

The volume of $\mathbb{T}_{p,q}(\alpha)$ equals

$$\text{Vol} \; \mathbb{T}_{p,q}(\alpha) = \frac{p \cdot q}{2} \left(\frac{\alpha}{2} - \pi \left(1 - \frac{1}{p} - \frac{1}{q}\right)\right)^2.$$

**Proof.** We prove this theorem using the result on two–bridge torus links obtained earlier in [2] and the covering theory. Every two–bridge torus link is a $(2, 2p)$ torus link as shown at the Fig. [1]

Divide the proof into three subsequent steps:

1st step. Using the Reidemeister moves rearrange the diagram of $\mathbb{T}_{2,2p}$, given at the Fig. [1] link in order to place one of its components around the other. The diagram obtained is depicted at the Fig. [2] Denote by $\mathbb{T}_{2,2p}(\alpha, \beta)$ a cone-manifold with cone angles $\alpha$ and $\beta$ along the components of $\mathbb{T}_{2,2p}$.
2nd step. The diagram of $T_{p,q}$ is depicted at the Fig. 3. It is clearly seen that $T_{p,q}(\alpha)$ forms a $q$-folded cyclic covering of $T_{2,2p}(\alpha, \frac{2\pi}{q})$ branched along its central component, as depicted at the Fig. 2.

3rd step. The formula of [2, Theorem 2] provides that

$$\text{Vol}_{T_{2,2p}}(\alpha, \beta) = \frac{1}{2p} \left( \frac{\alpha + \beta}{2} \cdot p - \pi(p-1) \right)^2.$$ 

As far as $T_{p,q}(\alpha)$ is a $q$-folded cyclic branched covering of $T_{2,2p}(\alpha, \frac{2\pi}{q})$ one has that

$$\text{Vol}_{T_{p,q}}(\alpha) = q \cdot \text{Vol}_{T_{2,2p}} \left( \alpha, \frac{2\pi}{q} \right) = \frac{p \cdot q}{2} \left( \frac{\alpha}{2} - \pi \left( 1 - \frac{1}{p} - \frac{1}{q} \right) \right)^2.$$ 

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Recall, that the number of components equals $\gcd(p, q)$. By the Schl"afli formula (see [1]), the singular strata component’s length for $T_{p,q}(\alpha)$ equals

$$\ell_\alpha = \frac{1}{\gcd(p, q)} \cdot \frac{d}{d\alpha} \Vol T_{p,q}(\alpha) = \frac{\alpha}{2} - \pi \left( 1 - \frac{1}{p} - \frac{1}{q} \right).$$

The proof is completed. $\square$

References

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