Bottom-up reconstruction of non-singular bounce in F(R) gravity from observational indices

S. D. Odintsov, V. K. Oikonomou, Tanmoy Paul

1) ICREA, Passeig Lluís Companys, 23, 08010 Barcelona, Spain
2) Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain
3) Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
4) International Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia
5) Tomsk State Pedagogical University, 634061 Tomsk, Russia
6) Department of Physics, Chandernagore College, Hooghly - 712 136.
7) Department of Theoretical Physics, Indian Association for the Cultivation of Science, 2A & 2B Raja S.C. Mullick Road, Kolkata - 700 032, India

We apply the bottom-up reconstruction technique in the context of bouncing cosmology in F(R) gravity, where the starting point is a suitable ansatz of observable quantity (like spectral index or tensor to scalar ratio) rather than a priori form of Hubble parameter. In inflationary scenario, the slow roll conditions are assumed to hold true, and thus the observational indices have general expressions in terms of the slow-roll parameters, as for example the tensor to scalar ratio in F(R) inflation can be expressed as $r = 48\epsilon_F^2$ with $\epsilon_F = -H_F^2 \frac{dH_F}{dt}$ and $H_F, t_F$ are the Hubble parameter, cosmic time respectively. However, in the bouncing cosmology (say in F(R) gravity theory), the slow-roll conditions are not satisfied, in general, and thus the observable quantities do not have any general expressions that will hold true irrespective of the form of F(R). Thus, in order to apply the bottom-up reconstruction procedure in F(R) bouncing model, we use the conformal correspondence between F(R) and scalar-tensor model where the conformal factor in the present context is chosen in a way such that it leads to an inflationary scenario in the scalar-tensor frame. Due to the reason that the scalar and tensor perturbations remain invariant under conformal transformation, the observable viability of the scalar-tensor inflationary model confirms the viability of the conformally connected F(R) bouncing model. Motivated by these arguments, here we construct a viable non-singular bounce in F(R) gravity directly from the observable indices of the corresponding scalar-tensor inflationary model.

I. INTRODUCTION

The current observations indicate, with no doubt, that the present Universe is expanding in an accelerating way. Its expansion rate is quantified by the evolution of the Hubble parameter $H = \dot{a}/a$, where $a(t)$ is the scale factor of the Universe. So, when we go back in time, there are two possibilities, firstly that the scale factor reaches the value zero and therefore the Kretschmann scalar diverges at the time when the scale factor becomes zero. This indicates a spacetime curvature singularity known as Big-Bang singularity. It is a common thought that the yet to be found quantum theory of gravity may have a significant role in removing the Big-Bang singularity, just as happens in quantum electrodynamics where the quantum corrections remove the classical divergence of the Coulomb potential. The second possibility however to describe the early-time era, that overrides the quantum gravity era assumption, is the bouncing cosmology description \cite{1, 15}, where the scale factor never becomes zero and thus the spacetime singularity is absent. In the case of bouncing scenario, the Universe starts from a contracting era, then it bounces off when it reaches a minimum size of the scale factor, and starts to expand again. Thereby, the bounce occurs at the time when $H = 0$ and $\dot{H} > 0$. Moreover, bounce cosmology is also appealing since it can be obtained as a cosmological solution of the theory of Loop Quantum Cosmology \cite{59, 64}.

Among the non-singular bouncing models proposed so far, the matter bounce scenario \cite{6, 14, 15, 54, 64, 79} gained a lot of attention because of the fact that the Universe evolves in a way similar to a matter dominated epoch even at late times in this scenario. However the matter bounce scenario in a scalar-tensor theory has some problematic implications, like the fact that the scalar power spectrum is scale invariant, so the scalar spectral index is exactly...
equal to one, and the corresponding running of the index becomes zero, which is not compatible with Planck 2018 observations, and also the amplitude of the tensor and scalar perturbations are of the same order, which in turn makes the tensor-to-scalar ratio to be of the order of unity, so it is incompatible too with the Planck constraints. Moreover, the scalar and tensor perturbations are not stable. Here it may be mentioned that such problems in scalar-tensor theory can not be even resolved in a standard $F(R)$ model, because a scalar-tensor model can be thought as an equivalent dual theory of a $F(R)$ model, connected by a conformal transformation of the metric (it may be mentioned that the duality between scalar-tensor and $F(R)$ model can be used to solve the $F(R)$ gravitational equation of motion i.e one can solve the scalar-tensor equation of motion which are relatively easier to solve and then transform back the solutions into the corresponding $F(R)$ model by inverse conformal transformation, see $[83]$). However these problems can be resolved in a Lagrange multiplier $F(R)$ gravity model, which clearly indicates the importance of Lagrange multiplier term in making the observable indices of a matter bounce scenario compatible with Planck constraints $[66]$. But the energy conditions are violated near the bouncing era (like in most of the bouncing models) in a Lagrange multiplier $F(R)$ matter bounce model. It is the holonomy improved Lagrange multiplier $F(R)$ gravity model which rescues the energy condition and also makes the observable quantities compatible with Planck results $[67]$. Actually in the holonomy corrected model, the Hubble squared parameter is proportional to quadratic and to linear powers of the effective energy density ($\rho_{\text{eff}}$), unlike to the usual Friedmann case where $H^2$ is proportional only to the linear power of $\rho_{\text{eff}}$. This difference in the field equations becomes significant near the bouncing point era and helps to rescue the energy conditions.

In the earlier literature of bouncing cosmology, the scale factor or equivalently the Hubble parameter was assumed to have an a priori specific form (maybe it is matter bounce or quasi-matter bounce or some other models) and then the observational quantities were determined in a specific background theory. However, in the present paper, we use a different approach to study the bouncing cosmology. In particular, we use a bottom-up reconstruction technique for non-singular bounce in an $F(R)$ gravity model, in which the observable indices are assumed to have a specifically chosen form. Such a bottom-up approach has been also used earlier, however in the context of $F(R)$ inflationary cosmology. In the case of inflation, the slow-roll conditions hold true and thus the observable quantities can be, in general, expressed in terms of the slow-roll parameters, as for example the tensor to scalar ratio in $F(R)$ inflation has a general expression like
\[ r = 48\epsilon_F^2, \]
where $\epsilon_F = -\frac{\ddot{H}}{H^2}$ (during the inflationary epoch $\epsilon_F$ remains less than unity and moreover $\epsilon_F = 1$ indicates the exit of inflation) and $H_F, \ t_F$ are the Hubble parameter, cosmic time respectively. The authors of $[84]$ used this slow-roll expression of $r$ to construct a viable $F(R)$ inflationary model from bottom-up reconstruction procedure. However in the bouncing cosmology in a specific theory, say in $F(R)$ gravity, the scenario is different, in particular the slow-roll conditions do not in general hold true and hence the observable indices do not have general expressions that will hold for any form of $F(R)$. Thus in order to incorporate the bottom-up reconstruction technique in the $F(R)$ bouncing model, one may use the conformal correspondence between $F(R)$ and scalar-tensor model, where the conformal factor should be chosen in such a way that it leads to an inflationary scenario in the scalar-tensor frame. This type of conformal equivalence between bounce and inflation has been demonstrated in $[21]$ $[91]$. Moreover as shown in $[91]$, the scalar and tensor perturbations remain invariant under conformal transformation and thus the observable viability of the scalar tensor inflationary scenario confirms the viability of the conformally connected $F(R)$ bouncing scenario. Motivated by these arguments, in the present paper, we construct a viable non-singular bounce in $F(R)$ gravity theory directly from the observable indices of the corresponding scalar-tensor inflationary frame. The ansatz of the tensor to scalar ratio we will consider for the scalar-tensor frame provides an inflationary era which also has an exit at a finite time.

The paper is organized as follows: after discussing some essential features of $F(R)$ gravity in Sec. II, we will describe the bouncing cosmological perturbation in terms of generation era of the perturbation modes in Sec. III. Then we will reveal the bottom-up reconstruction method in $F(R)$ bouncing cosmology in Sec. IV. The conclusions follow in the end of the paper.

II. ESSENTIAL FEATURES OF $F(R)$ GRAVITY

Let us briefly recall some basic features of $F(R)$ gravity, which are necessary for our presentation, for reviews on this topic see $[83]$ $[87]$. The gravitational action of $F(R)$ gravity in vacuum is equal to,
\[ S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}F(R) \]
(1)
where $\kappa^2$ stands for $\kappa^2 = 8\pi G = 1/M_p^2$ and also $M_p$ is the reduced Planck mass. By using the metric formalism, we vary the action with respect to the metric tensor $g_{\mu\nu}$, and the gravitational equations read,
\[ F'(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F'(R) + g_{\mu\nu} \Box F'(R) = 0 \]
(2)
where $R_{\mu\nu}$ is the Ricci tensor constructed from $g_{\mu\nu}$. Since the present article is devoted to cosmological context, in particular, to non-singular bouncing cosmology, the background metric of the Universe will be assumed to be a flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt_F^2 + a_F^2(t_F) [dx^2 + dy^2 + dz^2]$$

with $t_F$ is the cosmic time and $a_F(t_F)$ being the scale factor of the Universe. For this metric, the temporal and spatial components of Eq.(2) become,

$$0 = -\frac{F(R)}{2} + 3 \left( H_F^2 + \frac{dH_F}{dt_F} \right) F'(R) - 18 \left( 4 H_F^2 \frac{dH_F}{dt_F} + H_F \frac{d^2H_F}{dt_F^2} \right) F''(R)$$

$$0 = \frac{F(R)}{2} - \left( 3 H_F^2 + \frac{dH_F}{dt_F} \right) F'(R) + 6 \left( 8 H_F^2 \frac{dH_F}{dt_F} + 4 \frac{d^2H_F}{dt_F^2} \right)^2 + 6 H_F \frac{d^2H_F}{dt_F^2} + \frac{d^3H_F}{dt_F^4} \right) F''(R)$$

with $F(R)$ the cosmic time and $a_F(t_F)$ being the scale factor of the Universe. For this metric, the temporal and spatial components of Eq.(2) become,

$$0 = -\frac{F(R)}{2} + 3 \left( H_F^2 + \frac{dH_F}{dt_F} \right) F'(R) - 18 \left( 4 H_F^2 \frac{dH_F}{dt_F} + H_F \frac{d^2H_F}{dt_F^2} \right) F''(R)$$

$$0 = \frac{F(R)}{2} - \left( 3 H_F^2 + \frac{dH_F}{dt_F} \right) F'(R) + 6 \left( 8 H_F^2 \frac{dH_F}{dt_F} + 4 \frac{d^2H_F}{dt_F^2} \right)^2 + 6 H_F \frac{d^2H_F}{dt_F^2} + \frac{d^3H_F}{dt_F^4} \right) F''(R)$$

respectively, where $H_F = \frac{1}{a_F} \frac{da_F}{dt_F}$ is the Hubble parameter of the Universe. Comparing the above equations with usual Friedmann equations, it is easy to understand that $F(R)$ gravity provides a contribution in the energy-momentum tensor, with its effective energy density ($\rho_{eff}$) and pressure ($p_{eff}$) given by,

$$\rho_{eff} = \frac{1}{k^2} \left[ -\frac{f(R)}{2} + 3 \left( H_F^2 + \frac{dH_F}{dt_F} \right) f'(R) - 18 \left( 4 H_F^2 \frac{dH_F}{dt_F} + H_F \frac{d^2H_F}{dt_F^2} \right) f''(R) \right]$$

$$p_{eff} = \frac{1}{k^2} \left[ f(R) + 3 H_F^2 + \frac{dH_F}{dt_F} \right] f'(R) + 6 \left( 8 H_F^2 \frac{dH_F}{dt_F} + 4 \frac{d^2H_F}{dt_F^2} \right)^2 + 6 H_F \frac{d^2H_F}{dt_F^2} + \frac{d^3H_F}{dt_F^4} \right) f''(R)$$

respectively, where $f(R)$ is the deviation of $F(R)$ gravity from the Einstein gravity, that is $F(R) = R + f(R)$. Thus, the effective energy-momentum tensor (EMT) depends on the form of $F(R)$, as expected.

### III. COSMOLOGICAL PERTURBATION: AN ATTEMPT FOR A GENERAL EXPRESSION OF TENSOR-TO-SCALAR RATIO IN F(R) BOUNCING SCENARIO

The Universe’s evolution in a general bouncing cosmology, consists of two eras, an era of contraction and an era of expansion. Some of the well known scale factor which correspond to a non-singular bounce, have of the form, $a_F(t_F) = e^{\alpha t_F}$, $a_F(t_F) = \cosh t_F$, $a_F(t_F) = a_0(t_F^2 + 1)^n$, $a_F(t_F) = e^{\frac{1}{\alpha} (t_F - t_0)^n}$ and so on. At the bouncing point, the Hubble parameter becomes zero and thus the comoving Hubble radius, defined by $r_h = \frac{1}{a_F H_F}$, diverges at the bouncing point, in all of the aforementioned models. However the asymptotic behavior of the comoving Hubble radius makes a difference in the above bouncing models, specifically for some bouncing scale factors like $a_F(t_F) = e^{\alpha t_F}$, $a_F(t_F) = \cosh t_F$, $a_F(t_F) = (a_0 t_F^2 + 1)^n$ for $n > 1/2$, the Hubble radius is zero at both sides of the bounce and finally shrinks to zero size asymptotically (see the left plot of Fig. 4), which corresponds to an accelerating late time Universe. Therefore in such cases, the Hubble horizon goes to zero at large values of the cosmic time, and only for cosmic times near the bouncing point the Hubble horizon has an infinite size. So the primordial perturbation modes relevant for present time era are generated for cosmic times near the bouncing point, because only at that time all the primordial modes are contained in the horizon. As the horizon shrinks, the modes exit the horizon and become relevant for present time observations. On other hand, some bouncing models scale factor, like for example $a_F(t_F) = \ln (t_F^2 + t_0^2)$ (with $t_0$ being a constant arbitrary time), $a_F(t_F) = (a_0 t_F^2 + 1)^n$ for $n < 1/2$ lead to a divergent Hubble radius asymptotically (see the right plot of Fig. 4), which corresponds to a decelerating Universe (due to the fact that the Hubble radius increases) at late time. In such cases, the perturbation modes are generated at very large negative cosmic times, corresponding to the low curvature regime of the contracting era, unlike to the previous situations, where the perturbation modes are generated near the bouncing era. More explicitly, in the latter case, the comoving wave number $k$ begins its propagation through spacetime at large negative cosmic times, in the contracting phase on sub-Hubble scales, and exits the Hubble radius during this phase , and re-enters the Hubble radius during the low-curvature regime in expanding phase at the time $t_h(k)$ (the exit and entry time are symmetric
FIG. 1: Left plot : The Hubble radius $r_h = \frac{H(t_F)}{a(t_F)}$ as a function of the cosmic time $t_F$ for $a_F(t_F) = (t_F^2 + 1)^{4/5}$, where the Hubble radius decreases monotonically at both sides of the bounce and shrinks to zero asymptotically. Right plot : The Hubble radius $r_h = \frac{H(t_F)}{a(t_F)}$ as a function of the cosmic time $t_F$ for $a_F(t_F) = (t_F^2 + 1)^{1/3}$, where the Hubble radius diverges asymptotically.

about the bouncing point as the scale factor is itself symmetric) thus being relevant for present time observations. Therefore, the physical picture in the two cases is very different with regard to when the perturbation modes are generated. However, in both cases, the comoving curvature perturbation has to be evolved from the contracting phase to the expanding one, followed by the bouncing phase, in order to get the power spectrum at later times. In the large scale limit (i.e in the super-Hubble scale $k \ll a_F H_F$) of the contracting phase, the comoving curvature perturbation ($\Re(k, \eta)$) satisfies the cosmological perturbation equation

$$v''(k, \eta) - \frac{z''(\eta)}{z(\eta)} v(k, \eta) = 0$$

where $\eta$ is the conformal time defined as $dt_F = a(t_F) d\eta$ and prime denotes the differentiation with respect to $\eta$ throughout the paper. The above equation is written in terms of the canonical variable : $v(k, \eta) = z(\eta) \Re(k, \eta)$, and the variable $z(\eta)$ depends on the specific model. Since in the present context, we are interested in F(R) model, the variable $z(\eta)$ has the form :

$$z(\eta(t_F)) = \frac{a_F(t_F)}{\kappa \left( H_F(t_F) + \frac{F''(R)}{2F(R)} \frac{dR}{dt_F} \right)} \sqrt{\frac{3(F''(R))^2}{2F(R)}} \left( \frac{dR}{dt} \right)^2$$

However in terms of a general $z(\eta)$, the solution of Eq.(6) is given by,

$$v_c(k, \eta) = z(\eta) \left[D_c(k) + S_c(k) \int^\eta d\eta \frac{1}{z^2} \right]$$

where the suffix 'c' denotes the contracting phase and $D_c(k), S_c(k)$ are independent of time and carry the information about the spectra of the two modes. The above solution of $v(k, \tau)$ immediately leads to the curvature perturbation in the super-Hubble scale of the contracting phase as,

$$\Re_c(k, \eta) = \frac{v(k, \eta)}{z(\eta)} = D_c(k) + S_c(k) \int^\eta d\eta \frac{1}{z^2}$$

As is evident from the above expression, the $D$ mode is a constant mode and generally the $S$ mode behaves as an increasing mode. Similarly in the large scale limit of the expanding phase, the curvature perturbation has the following solution,

$$\Re_e(k, \eta) = D_e(k) + S_e(k) \int^\eta d\eta \frac{1}{z^2}$$

The $D_e$ mode of the curvature perturbation is constant in time, as is the $D_c$ mode in the contracting phase. However generally the role of the $S$ mode becomes very different. In the expanding phase $S_e$ is the sub-dominant decreasing
mode, whereas in the contracting phase, the opposite situation occurs, that is, the $S_c$ mode is the decreasing mode. Therefore, the dominant mode of the curvature perturbations in the period of expansion is $D_c$. This leads to the following power spectrum of the curvature perturbations at late times (which is useful for the present observation),

$$P_c(k) = \frac{k^3}{2\pi^2}|D_c(k)|^2$$  \hspace{1cm} (10)

Similarly the power spectrum for the tensor perturbation is given by $P_T(k) = \frac{k^3}{2\pi^2}|D_T(k)|^2$. Moreover the curvature perturbation should be continuous and thus $R_c$, $R_e$ have to be matched through the bouncing point as explicitly performed in [63, 70]. During this matching procedure, the $D_c$ mode may inherit a contribution from both $D_e$ and $S_e$ modes [69, 70]. At this stage it may be mentioned that a model will be a viable one if the observable quantities like the spectral index ($n_s$) and the tensor-to-scalar ratio ($r$) defined by $n_s - 1 = \left. \frac{\partial \ln P}{\partial \ln k} \right|_{h.c}$ and $r = \frac{P_T(k)}{P_T^{(T)}(k)}$, are compatible with the latest observations. Note that the subscript “h.c” indicates that these must be evaluated at the horizon crossing time instance, corresponding to a cosmic time when the perturbation mode $k$ crosses the horizon i.e. when $k = a_p H_F$. Up to this point, we do not consider any certain form of $F(R)$. At this stage it deserves mentioning that the extraction of various observable quantities from the above mentioned power spectrums need an explicit solution of the Mukhanov-Sasaki variable governed by Eq. (6) which in turn demands a particular form of $F(R)$, in a bouncing scenario where the slow roll conditions are not valid. This spoils the generality for the tensor to scalar ratio expression that is supposed to be true for any form of $F(R)$. Thus in the bouncing context, one may not be able to extract an expression for the tensor to scalar ratio which is supposed to valid for a general $F(R)$. On other hand, one may think about the power law parametrization where the scalar and tensor power spectrums can be parametrized as $P_c(k) \propto \left(\frac{k}{a_p H_F}\right)^{n_s-1}$ and $P_T^{(T)}(k) \propto \left(\frac{k}{a_p H_F}\right)^{n_T}$ respectively or equivalently $P_c(k) = A_s \left(\frac{k}{a_p H_F}\right)^{n_s-1}$ and $P_T^{(T)}(k) = A_T \left(\frac{k}{a_p H_F}\right)^{n_T}$ with $A_s$ being the scalar power spectrum at the horizon crossing and thus known as scalar perturbation amplitude, for similar reason $A_T$ stands for the tensor perturbation amplitude. However in the case of power law parametrization, the spectral index becomes independent of the wave number $k$ and thus the running of the spectral index vanishes i.e $\frac{d \ln n_s}{d \ln k} = 0$. The vanishing $\alpha$ is not compatible with the Planck 2018 observations, which indicates that the power law parametrization is not a viable consideration.

IV. BOTTOM-UP RECONSTRUCTION IN F(R) BOUNCING COSMOLOGY

The bottom-up reconstruction technique is actually motivated from the work [84] where the bottom-up approach is considered to check the viability of $F(R)$ gravity in the context of slow roll inflationary scenario. The authors of [84] considered some specific forms, in particular, the exponential and logarithmic forms of the tensor to scalar ratio ($r$) as a function of the e-folding number and then compared such ansatzs of the tensor to scalar ratio with its general slow roll expression in a $F(R)$ model i.e with $r = 48 \epsilon_F^2$ (where $\epsilon_F = -\frac{\dot{a}}{a_H} \frac{\dot{H} F}{H^2 F}$) to determine the corresponding Hubble parameter in terms of e-folding number or at a same time in terms of the cosmic time. The determination of the Hubble parameter in turn helps to obtain the form of $F(R)$ realizing such evolution of the Hubble parameter from the gravitational equation of motion, which further reveals the other observable quantities like the spectral index, the running of index etc. and consequently the model can be directly confronted with the Planck constraints. Thus the viability of slow roll inflationary $F(R)$ model can be judged directly from a viable ansatz of the tensor to scalar ratio rather than starting from a Hubble parameter expression. Because the starting point is a specific form of $r = r(N)$ in such bottom-up technique, the slow roll conditions play an important role to determine the evolution of the Hubble parameter from the general ansatz of the tensor to scalar ratio.

If we want to apply the same procedure in the present context i.e in the context of $F(R)$ bouncing scenario, then we will be hinged at some intermediate stage and the demonstration goes as follows : suppose we start from some specific form of tensor to scalar ratio which, in fact, lies within the Planck constraints for some viable parametric regimes. According to [84], the next step is to determine the Hubble parameter by comparing the ansatz $r = r(N)$ with a general slow roll expression of the tensor to scalar ratio, if any, valid in a $F(R)$ bouncing scenario. However this step is problematic, because in the case of bounce, the slow roll conditions do not hold true in general, and thus there is no such general expression of the tensor to scalar ratio in a $F(R)$ bouncing model, unlike to the $F(R)$ inflationary case where the slow roll conditions are indeed true and consequently the tensor to scalar ratio has a general expression like $r = 48 \epsilon_F^2$ irrespective of the form of $F(R)$ [88, 90].
explicitly, the observable quantities like the spectral index, tensor to scalar ratio in F(R) inflationary scenario can be expressed in terms of the slow roll parameters irrespective of the form of F(R), however this is not the case, in general, in a bouncing scenario. Because the slow roll conditions are not valid in a bounce model, the bottom-up reconstruction technique considered in [21] is problematic to apply in a F(R) bouncing scenario in the present context.

The above arguments clearly reveal that the validity/invalidity of the slow roll conditions is the sole reason that one can apply the bottom-up method in an inflationary scenario but seems problematic in a bouncing model. Thus as a next attempt for applying the bottom-up reconstruction procedure in a bouncing model, we may think the conformal correspondence of a bounce model with an inflationary one where the slow roll conditions are indeed true. It is well known that a F(R) theory can be equivalently mapped to a scalar-tensor theory by a conformal transformation of the spacetime metric where the scalar field potential depends on the form of F(R). Due to such conformal relation, the scale factor and the proper time of one frame also get connected with that of the other frame. To demonstrate it briefly, let us start with a F(R) action,

$$S = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} \right]$$

The above action can be mapped to a scalar-tensor action by applying the following transformation of the metric

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{-\sqrt{\kappa} \phi} g_{\mu\nu}$$

where $f(\phi)$ (an arbitrary function of $\phi$) is the conformal factor which is further related to the higher curvature degrees of freedom as $F'(R) = e^{-\sqrt{\kappa} \phi}$. If $\tilde{R}$ and $R$ are the Ricci scalars formed by the metric $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ respectively, then they are connected by,

$$R = e^{-\sqrt{\kappa} \phi} \left[ \tilde{R} - \kappa^2 f'(\phi) \tilde{g}^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - \sqrt{6\kappa} \Box f(\phi) \right]$$

where $\Box$ is the d’Alembertian operator for $\tilde{g}_{\mu\nu}$. Using the above expression along with the aforementioned relation between $f(\phi)$ and $F'(R)$, the following scalar-tensor action is achieved:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \omega(\phi) \tilde{g}^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - \frac{(AF'(A) - F(A))}{F'(A)^2} \right]$$

with $\omega(\phi) = f'(\phi)^2$ and $A(x)$ being given by $F'(A) = e^{-\sqrt{\kappa} \phi}$. Eq. [12] clearly indicates that the field $\phi(x)$ acts as a scalar field with the potential $\frac{(AF'(A) - F(A))}{F'(A)^2} = V(A(\phi))$. Thus the higher curvature degree of freedom manifests itself as a scalar degree of freedom with a potential $V(\phi)$ which actually depends on the form of F(R). If the F(R) model spacetime is characterized by a FRW metric with $\eta$ be the conformal time and $a_F(\eta)$ is the scale factor i.e

$$ds^2 = a_F^2(\eta) \left[ -d\eta^2 + \delta_{ij} dx^i dx^j \right]$$

then the metric in the corresponding scalar-tensor model becomes

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \delta_{ij} dx^i dx^j \right]$$

with $a(\eta) = e^{-\sqrt{\kappa} f(\phi)} a_F(\eta)$ is the scale factor in the scalar tensor model. It may be observed that the conformal time remains unchanged in both the frames, however the cosmic time transforms by the way $dt = e^{-\sqrt{\kappa} f(\phi)} dt_F$ with $t$ being the cosmic time in the scalar tensor space. Before moving further, we want to clarify the notations that we will use throughout the paper: $(t_F, a_F(t_F))$ and $(t, a(t))$ are the cosmic time, scale factor in the F(R) and scalar-tensor frame respectively. Regarding the Hubble parameter, $H_F$ is reserved for the F(R) frame while $H$ is for the scalar-tensor one. $R$ and $\tilde{R}$ are the Ricci scalar in F(R) and scalar-tensor frame respectively. Moreover $\frac{d}{d\eta}$ is represented by an “overdot” (as for example $\dot{H} = \frac{dH}{d\eta}$), $\frac{dx}{d\eta}$ is represented by itself and the other derivatives are shown by the respective arguments. Coming back to Eq. [12], if $a(\eta)$ provides an inflationary scenario in the scalar tensor frame, then by properly choosing the conformal factor $f(\phi)$, we may get a bouncing universe in respect to the F(R) frame scale factor $a_F(\eta)$. This type of conformal equivalence between bounce and inflationary models has been demonstrated in [21] [31].
Moreover as shown in Ref. [91], the scalar and tensor perturbations remain invariant under conformal transformation and thus the viability of the scalar tensor inflationary scenario confirms the viability of the conformally connected F(R) bouncing model can be investigated by looking into the corresponding scalar-tensor inflationary frame where, due to the slow roll conditions, the bottom-up reconstruction technique can be easily applied. For the purpose of applying the bottom-up reconstruction procedure in the scalar tensor model, we start with a certain ansatz of the tensor-to-scalar ratio \( r \) which leads to an inflationary scenario. However before moving to the explicit ansatz of \( r \), we present the gravitational and scalar field equation of motion for the action (13) in FRW spacetime as,

\[
3H^2 = \kappa^2 \left[ \frac{1}{2} \omega \dot{\phi}^2 + V(\phi) \right]
\]

and

\[
\omega \ddot{\phi} + \frac{1}{2} \omega'(\phi) \dot{\phi}^2 + 3H \omega \dot{\phi} + V'(\phi) = 0
\]

respectively, where the “dot” represents \( \frac{d}{dt} = \frac{1}{a(t)} \frac{da}{dt} \). The spatial component of the Einstein equation i.e \( 2 \dot{H} = -\kappa^2 \omega \dot{\phi}^2 \) can be derived from the above two equations and hence is not an independent one. Moreover the off-diagonal Einstein equations are trivial as the off-diagonal components of the Einstein tensor vanishes for the FRW metric. As mentioned earlier, we deal with an inflationary scenario in the scalar-tensor (ST) model and thus the slow roll conditions hold true in the ST frame. The slow roll conditions are put by introducing some slow roll parameters which is regarded to be less than unity during the inflationary period. In the case of action (13), the slow roll parameters are defined as Ref. [84-88],

\[
\epsilon_1 = -\frac{\dot{H}}{H^2} \quad , \quad \epsilon_2 = \frac{\dot{\phi}}{H \phi} \quad , \quad \epsilon_4 = \frac{\dot{\omega}}{2H \omega}
\]

In a more general action like \( S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} G(\tilde{R}, \phi) - \frac{1}{2} \omega(\phi) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \) (where \( G(\tilde{R}, \phi) \) is any analytic function of \( \tilde{R} \) and \( \phi \)), there is another slow roll parameter defined as \( \epsilon_3 = -\frac{\dot{G}_{RR}}{2H \dot{\phi} R} \) (with \( G_R = \frac{\partial G}{\partial \phi} \)), however in the present case i.e for action (13), \( G(\tilde{R}, \phi) = \tilde{R} \) and thus the slow roll parameter \( \epsilon_3 \) vanishes. With the conditions \( \epsilon_i \ll 1 \), the spectral index for curvature perturbation and the tensor to scalar ratio of the ST model (13) are given by Ref. [84-88],

\[
\begin{align*}
n_s &= 1 - 4\epsilon_1 - 2\epsilon_2 - 2\epsilon_4 \\
r &= 8\kappa^2 \left( \frac{\omega \dot{\phi}^2}{H^2} \right)
\end{align*}
\]

respectively. Incorporating the gravitational equation \( 2 \dot{H} = -\kappa^2 \omega \dot{\phi}^2 \) into the above expression yields a simplified form of the tensor to scalar ratio as follows,

\[
r = -16 \frac{\dot{H}}{H^2} = 16\epsilon_1
\]

Furthermore, the equations of motion, due to the slow roll conditions, can be approximated as follows,

\[
3H^2 = \kappa^2 V(\phi)
\]

and

\[
\frac{1}{2} \omega'(\phi) \dot{\phi}^2 + 3H \omega \dot{\phi} + V'(\phi) = 0
\]

Having set the stage, let us consider an ansatz of tensor-to-scalar ratio in terms of the e-folding number as,

\[
r(N) = 16\epsilon^\beta (N(t) - N_i)
\]

where \( \beta \) is a dimensionless model parameter. Here it may be mentioned that the e-folding number can be defined as either \( N(t) = \int_{t_i}^t H dt \) or \( N(t) = \int_{t_{end}}^t H dt \) where \( t_i \) and \( t_{end} \) are the onset and the end point of inflation respectively. Thereby in the former case, \( N > 0 \) i.e. the e-foldings number monotonically increases with the cosmic time, while
for the latter case, the e-folding number decreases with $t$. However in the present paper, we follow the convention for which $\dot{N} > 0$ i.e $N(t) = \int_{t_0}^{t} H dt$. In principle, we can start with any form of $r(N)$ i.e. any combination of functions are allowed in the expression of the tensor to scalar ratio to start with. We choose the particular form \(r(T)\) of $r$ in order to proceed our calculations analytically. There may exist some other forms of $r$ (other than \(r(T)\)) for which analytic calculations can be performed, some of them are discussed later. The most important part is to check whether the choice of $r$ leads to the observable compatibility with the Planck constraints. As we now demonstrate the above choice of $r = r(N)$ leads to an inflationary cosmology in the scalar tensor frame. Comparing Eqs. (23) and \(N\), we get a first order differential equation for the Hubble parameter as \(N\)

\[
\frac{1}{H(N)} \frac{dH}{dN} = -e^{\beta(N-N_f)}
\]

where we use the $\frac{d}{dt} = H(N) \frac{d}{dN}$. Solving Eq. (24), we obtain

\[
H(N) = H_0 \exp \left[ -\frac{1}{\beta} e^{\beta(N-N_f)} \right]
\]

with $H_0$ is an integration constant having mass dimension $[+1]$. Such evolution of the Hubble parameter immediately leads to the acceleration factor of the universe as $\frac{1}{H} = H^2(N) \left(1 + \frac{1}{H(N) \frac{dH}{dN}}\right) = H^2(N) \left(1 - \beta e^{\beta(N-N_f)}\right)$. Thereby the inflationary era of the universe continues as long as the condition $1 - e^{\beta(N-N_f)} > 0$ holds, which becomes,

\[
N < N_f (\text{say})
\]

Therefore, the evolution of the Hubble parameter in Eq. (25) leads to an inflationary era of the universe and moreover the inflation has an exit at $N(t_{\text{end}}) = N_f$ with $t_{\text{end}}$ is the cosmic time when the inflation ends. Thus the total e-folding of the inflationary epoch is given by $N_T = N(t_{\text{end}}) - N(t_h) = \int_{t_h}^{t_{\text{end}}} H dt$ (the subscript “T” denotes the total e-folding) which is considered to be around $N_T \approx 60$ for the CMB scale perturbation mode having horizon crossing time is $t_h$. We will use this constraint on $N_T$ later.

The de-Sitter evolution of the Hubble parameter becomes more prominent if we determine the Hubble parameter in terms of the cosmic time. Using the relation $\dot{N} = H(N)$ along with Eq. (25), one can find $H = H(t)$ in the leading order of $(t - t_h)$ i.e near the onset of inflation as,

\[
H(t) = H_0 \exp \left[ -\frac{1}{\beta} e^{-\beta N_f} \right] \left\{ 1 - H_0(t-t_h) \exp \left[ -\frac{1}{\beta} e^{-\beta N_f} - \beta N_f \right] \right\}
\]

We fix the integration constant during solving $\dot{N} = H(N)$ in a way such that $N(t_h) = 0$ which is also true from the definition of $N(t) = \int_{t_0}^{t} H dt$ we considered. Hence the resulting evolution of $H(t)$ near the beginning of inflation (i.e $t \to t_h$) is a quasi de-Sitter evolution. Thus as a whole, the ansatz of $r(N)$ in Eq. (23) allows an inflationary scenario of the universe having an exit at $N = N_f$ (or $t = t_{\text{end}}$) and moreover the Hubble parameter evolution near the beginning of the inflation follows a quasi de-Sitter evolution. The Hubble parameter can also be expressed in terms of the conformal time $\eta$ by using the following relation,

\[
\eta = \int \frac{dt}{a(t)} = \int \frac{e^{-N}}{N} dN = \frac{1}{H_0} \int e^{-N} \exp \left[ -\frac{1}{\beta} e^{\beta(N-N_f)} \right] dN
\]

The integral in the right hand side is troublesome to perform, however can be done in the limit $N \to 0$ i.e near the beginning of the inflation and as a result, one gets

\[
\eta(N) = -\frac{\exp \left[ \frac{1}{\beta} e^{-\beta N_f} \right]}{H_0 \left( 1 - e^{-\beta N_f} \right)} e^{-\left(1-e^{-\beta N_f}\right)}
\]

We will use this expression later. Having confirmed the inflationary scenario in the ST frame, the next task is to determine the conformal factor $f(\phi)$ (see Eq. (12)) in such a way that the conformally transformed F(R) frame scale factor leads to a non-singular bounce. We choose

\[
f(\phi(N)) = \frac{\sqrt{6}}{\kappa} \ln \left[ e^{-N} \cosh \left( \gamma \eta(N) \right) \right]
\]
where $\gamma$ is an arbitrary parameter for the moment and $\eta = \eta(N)$ is given in Eq. (25). Using the aforementioned relation between $a(\eta)$ and $a_F(\eta)$ (see Eq. (15)), it is easy to see that due to the above form of $f(\phi)$, the conformally connected F(R) frame scale factor behaves as

$$a_F(\eta) = \cosh (\gamma \eta)$$

(31)

which indeed leads to a non-singular bounce at $\eta = 0$. Moreover near $\eta = 0$, the F(R) scale factor can be approximated as $a_F(\eta) = 1 + \frac{\gamma^2}{2} \eta^2$ and consequently the conformal time is related to the F(R) cosmic time by $t_F = \int a_F(\eta) d\eta = \eta + \frac{\gamma^2 \eta^2}{6} \approx \eta$. Thus the scale factor in terms of the cosmic time turns out to be $a_F(t_F) = 1 + \frac{\gamma^2}{2} t_F^2$ from which the bouncing behaviour (at $t_F = 0$) in the F(R) frame becomes more prominent with respect to its cosmic time. Thereby the $f(\phi)$ in Eq. (30) connects an inflationary ST frame where the Hubble parameter follows Eq. (25) with a F(R) bouncing frame having the scale factor given in Eq. (31). With the F(R) bouncing scale factor $a_F$, one can reconstruct the form of F(R) by using the corresponding Jordan frame gravitational equation of motion. For the slow roll field equations of the ST model (i.e Eqs. (21) and (22)), the slow roll parameter $\omega H$ has been determined directly from the observational index, in particular from the tensor-to-scalar ratio ansatz.

As mentioned earlier, the observable viability of the inflationary ST model confirms the viability of the F(R) bouncing model. Thus, in the following, we investigate the observational viability of the inflationary ST frame where, recall, the Hubble parameter and the Ricci scalar as

$$H_F(t_F) = \frac{\gamma^2 t_F}{1 + \frac{\gamma^2}{2} t_F^2} \approx \gamma^2 t_F$$

$$R(t_F) = 12 H_F^2 + 6 \frac{dH_F}{dt_F} = \frac{6 \gamma^2 (1 + \frac{\gamma^2}{2} t_F^2)}{1 + \frac{\gamma^2}{2} t_F^2} \approx 6 \gamma^2 + 3 \gamma^2 t_F^2$$

(32)

respectively, with the $H_F(t_F)$ and $R(t_F)$ being considered up to $O(t_F^2)$, similar to the case of the scale factor. However Eq. (22) clearly indicates that the Hubble parameter varies linearly with $t_F$ and goes to zero at the bouncing point, while the Ricci scalar, on the other hand, becomes $R(0) = 6 \gamma^2$. At a later part, we will give an estimation of the Ricci scalar at the bouncing point. In regard to the primordial perturbation, we will determine the form of F(R) near the bouncing regime. The near-bounce scale factor $a_F(t_F) = 1 + \frac{\gamma^2}{2} t_F^2$ that we have obtained immediately leads to the Hubble parameter and the Ricci scalar as,

$$12 \gamma^2 (R - 6 \gamma^2) F''(R) + (R - 12 \gamma^2) F'(R) + F(R) = 0$$

(33)

Solving the above equation for $F(R)$, we get,

$$F(R) = \frac{6 \gamma^2 D}{\sqrt{\kappa}} R - D \sqrt{3 \gamma^2 \pi} e^{-\frac{\pi}{\sqrt{\kappa}} (R - 6 \gamma^2)^{3/2}} Erfi \left[ \frac{\sqrt{R - 6 \gamma^2}}{2 \sqrt[3]{3 \gamma^2}} \right]$$

(34)

where $Erfi[z]$ is the imaginary error function defined as $Erfi[z] = -iErf[iz]$ with $Erf[z]$ being the error function and 'i' is the imaginary unit. Moreover $D$ is an integration constant having mass dimension $[-2]$. Recall, as mentioned in Sec. 11 that the $F(R)$ gravity contributes an effective energy-momentum tensor where the effective energy density ($\rho_{eff}$) and the pressure ($p_{eff}$) are given in Eq. (35). Using these expressions of $\rho_{eff}$ and $p_{eff}$, it is easy to show that at the bounce $\rho_{eff} = 12 [F(R) - R(1)]$ and $p_{eff} = \rho_{eff} - 24 [\frac{d^2F}{dt_F^2} - 2F']^2$. The form of F(R) as determined in Eq. (30) leads to

$$\rho_{eff} = 0 \quad , \quad \rho_{eff} + p_{eff} = \frac{-2 \gamma^2 D}{\kappa^2} \left[ 1 + \frac{6 \gamma^2 D}{\sqrt{\kappa}} \right]$$

at $t_F = 0$. These indicate a violation of energy condition which in turn ensures a bouncing phenomena at $t_F = 0$. As mentioned earlier, the observable viability of the inflationary ST model confirms the viability of the F(R) bouncing model. Thus, in the following, we investigate the observational viability of the inflationary ST frame where, recall, the Hubble parameter has been determined directly from the observational index, in particular from the tensor-to-scalar ratio ansatz.

Using the slow roll field equations of the ST model (i.e Eqs. (21) and (22)), the slow roll parameter $\epsilon_2$ turns out to be

$$\epsilon_2 = \frac{\ddot{\phi}}{H \dot{\phi}} = \frac{3 \dot{H} + 3 H \dot{\omega} + \frac{1}{2} \dot{\phi}}{H (3 H \dot{\omega} + \frac{1}{2} \dot{\phi})}$$

where $\omega(\phi)$ is the self kinetic coupling of the scalar field, which is further related to the conformal factor as $\omega(\phi) = f'(\phi)^2$. Plugging the above expression of $\epsilon_2$ into Eq. (23) yields the spectral index as follows,

$$n_s = 1 + \frac{4 \dot{H}}{H^2} + \frac{2 (3 \dot{H} + 3 H \dot{\omega} + \frac{1}{2} \dot{\phi})}{H (3 H \dot{\omega} + \frac{1}{2} \dot{\phi})} - \frac{\dot{\omega}}{\dot{H}}$$

(35)
with, recall, the “dot” symbolizes $\frac{d}{dt}$ (i.e. the derivative with respect to the ST frame cosmic time). We will determine the scalar spectral index in terms of e-folding number and for this purpose what we need is the following identities:

$$\frac{d}{dt} = H(N) \frac{d}{dN}, \quad \frac{d^2}{dt^2} = H^2(N) \frac{d^2}{dN^2} + H \frac{dH}{dN} \frac{d}{dN}$$  \hspace{1cm} (36)

Using the above identities along with the aforementioned relation between $\omega(\phi)$ and $f(\phi)$, we determine various terms present in the right hand side of Eq. (35), as follows:

$$\left[ \frac{3H\dot{\omega} + 3H^2 \dot{\omega} + \frac{1}{2} \ddot{\omega}}{3H\omega + \frac{3}{2} \dot{\omega}} \right] = \frac{H}{f''(N) + (3 - e^{\beta(N-N_f)}) f'(N)} \times \left[ -3 f'(N)e^{\beta(N-N_f)} \right.$$  \hspace{1cm} (37)

$$+ \left( f''(N) - f'(N)e^{\beta(N-N_f)} \right) \left( 6 - 3e^{\beta(N-N_f)} + \frac{f''(N)}{f'(N)} \right) + \left( f''(N) - f'(N)e^{\beta(N-N_f)} - \beta f'(N)e^{\beta(N-N_f)} \right) \right]$$

and

$$\frac{\dot{\omega}}{\omega H} = \frac{2 \left( f''(N) - f'(N)e^{\beta(N-N_f)} \right)}{f'(N)}$$  \hspace{1cm} (38)

where $f'(N) = \frac{df}{dN}$ (also the higher derivatives have the respective meaning) and in determining the above expressions, we neglect the acceleration term of the scalar field due to the slow roll conditions. Recall, the conformal factor $f(N)$ is chosen in such a way in Eq. (30) that it leads to a non-singular bounce in the F(R) frame. However in order to determine the explicit form of $f(N)$ we need the functional behaviour of $\eta = \eta(N)$ which in turn demands to perform the integral of Eq. (28). As demonstrated earlier in Eq. (29), this integral can be performed in the limit $N \rightarrow 0$ i.e near the horizon crossing time, which is indeed sufficient in the present context as the observable quantities like the spectral index, tensor-to-scalar ratio are eventually determined at the horizon crossing instance. As a result, the conformal factor in terms of the e-folding number takes the following form:

$$f(N) = \frac{\sqrt{6}}{\kappa} \left\{ -N + \ln \left[ \cosh \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) \right] \right\}$$  \hspace{1cm} (39)

with $P(N) = \exp \left[ -\left( 1 - e^{-\beta N_f} \right) N + \frac{1}{\beta} N \right]$. Consequently, $f'(N)$, $f''(N)$ and $f'''(N)$ are obtained as,

$$f'(N) = \frac{\sqrt{6}}{\kappa} \left\{ -1 - \frac{\gamma}{H_0} P(N) \tanh \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) \right\}$$  \hspace{1cm} (40)

$$f''(N) = \frac{\sqrt{6}}{\kappa} \left\{ \frac{\gamma^2}{H_0^2} P^2(N) \cosh^2 \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) + \frac{\gamma \left( 1 - e^{-\beta N_f} \right)}{H_0} P(N) \tanh \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) \right\}$$  \hspace{1cm} (41)

and

$$f'''(N) = \frac{\sqrt{6}}{\kappa} \left\{ -3 \gamma^2 \left( 1 - e^{-\beta N_f} \right) P^2(N) \cosh^2 \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) - \frac{\gamma \left( 1 - e^{-\beta N_f} \right)}{H_0} P(N) \tanh \left( \frac{\gamma P(N)}{H_0(1 - e^{-\beta N_f})} \right) \right\}$$  \hspace{1cm} (42)

respectively. Plugging the expressions of Eqs. (37) and (38) into Eq. (35), one gets the final form of the spectral index in terms of the e-folding number as,

$$n_s = 1 - 2e^{\beta(N-N_f)} - \frac{2 f''(N)}{f'(N)} + \frac{2}{f''(N) + (3 - e^{\beta(N-N_f)}) f'(N)} \times \left[ -3 f'(N)e^{\beta(N-N_f)} \right.$$  \hspace{1cm} (43)

$$+ \left( f''(N) - f'(N)e^{\beta(N-N_f)} \right) \left( 6 - 3e^{\beta(N-N_f)} + \frac{f''(N)}{f'(N)} \right) + \left( f''(N) - f'(N)e^{\beta(N-N_f)} - \beta f'(N)e^{\beta(N-N_f)} \right) \right]$$
where \( f^{(i)}(N) \) (with \( i = 0, 1, 2, 3 \)) are given above and recall that \( t_h \) is the horizon crossing instance. Thus Eqs. (23) and (24) provide the final forms of the scalar spectral index and the tensor-to-scalar ratio (as a function of e-folding number) in the ST frame respectively. With these expressions of \( n_s \) and \( r \), now we can confront the model with the Planck 2018 constraints (22), which constrain the observational indices as follows,

\[
n_s = 0.9649 \pm 0.0042, \quad r < 0.064.
\]  

Eq. (23) clearly indicates that \( r \) depends on \( N_f - N(t_h) = N_T \) i.e the total e-folding of the inflationary era) and \( \beta \), while from Eq. (24), it is easy to observe that the spectral index depends on \( N_T \) and the dimensionless parameters \( \beta, \frac{H_0}{f} \). The dependence of \( n_s \) on the parameter \( \gamma/H_0 \) actually arises from the conformal factor which has \( \gamma \) dependent term. Considering \( N_T = 60 \), the tensor to scalar ratio lies within the Planck constraints for \( \beta > 0.092 \). Thus taking \( \beta = 0.1 \) and \( N_T = 60 \), the spectral index is compatible with the Planck results if the parameter \( \frac{H_0}{f} \) lies within the range given by \( 10^{-3} \lesssim \frac{H_0}{f} \lesssim 0.1 \); this is depicted in Fig. 2. The parameter \( H_0 \) is approximately the de-Sitter Hubble parameter during inflationary epoch (see Eq. (24)), which is generally considered as \( 10^{-6} \) for the former case i.e for \( \beta < 1 \), the Hubble parameter comes as \( H(N) = H_0 e^{-N/\beta} \). However the Hubble parameter \( H(N) = H_0 (\beta - N)^\alpha \) vanishes at a finite e-folding \( N = \beta \) which is not physical at all. On other hand, \( H(N) = H_0 e^{-N/\beta} \) leads to an ever accelerating universe i.e the inflationary scenario of the scalar-tensor frame gets no exit. Thus it is clear that although the ansatz of Eq. (25) provide analytic results, such forms of \( r = r(N) \) suffer with some severe problems, unlike the form \( r(N) = 16 e^{\beta(N(t) - N_f)} \) that we have considered in the present paper in Eq. (26) which seems to free from such problems.

Thus the scalar-tensor inflationary observable quantities are simultaneously compatible with the Planck constraints for the parametric ranges given by \( N_T = 60, \beta = 0.1 \) and \( 10^{-3} \lesssim \frac{H_0}{f} \lesssim 0.1 \) respectively. Being the scalar and tensor perturbations remain invariant under conformal transformation, the observable viability of the scalar-tensor inflationary model in turn confirms the viability of the conformally connected F(R) bouncing model where the scale factor behaves as \( a_F(\eta) = \cosh (\gamma \eta) \). Thus a viable F(R) bouncing model can be constructed directly from the observable indices of the corresponding scalar-tensor inflationary frame.

Before concluding we would like to mention that apart from the ansatz (28) of the tensor to scalar ratio, some other forms of \( r = r(N) \) also lead to analytic results. Some of them are given by,

\[
r(N) = \frac{16\alpha}{\beta - N}, \quad r(N) = \frac{1}{\beta^2}
\]  

eq 45)

etc. For the former case i.e for \( r(N) = \frac{16\alpha}{\beta - N} \), the Hubble parameter comes as \( H(N) = H_0 (\beta - N)^0 \), while comparing \( r = -\frac{16H(N)}{H_0(\beta - N)^0} \) with the latter one yields \( H(N) = H_0 e^{-N/\beta} \). However the Hubble parameter \( H(N) = H_0 (\beta - N)^\alpha \) vanishes at a finite e-folding \( N = \beta \) which is not physical at all. On other hand, \( H(N) = H_0 e^{-N/\beta} \) leads to an ever accelerating universe i.e the inflationary scenario of the scalar-tensor frame gets no exit. Thus it is clear that although the ansatz of Eq. (25) provide analytic results, such forms of \( r = r(N) \) suffer with some severe problems, unlike the form \( r(N) = 16 e^{\beta(N(t) - N_f)} \) that we have considered in the present paper in Eq. (26) which seems to free from such problems.
In the present work, we have applied the bottom-up reconstruction technique to construct a viable non-singular bounce in F(R) gravity, where the starting point is to consider a suitable ansatz of observational quantities, like the scalar spectral index or tensor to scalar ratio as function of e-foldings number, rather than a priori form of Hubble parameter. The bottom-up procedure can be directly applied in inflationary context where, due to the slow roll conditions, the observable quantities can be expressed in terms of the slow roll parameters in general. However in bouncing case (say in F(R) gravity), the scenario is different, in particular the slow roll conditions in a bouncing model are not true and hence the observable indices do not have any general expressions that will be valid for any form of F(R). Thus in order to apply the bottom-up reconstruction technique in F(R) bouncing model, we have used the conformal equivalence between F(R) and scalar-tensor model, where the conformal factor is chosen in such a way so that it leads to an inflationary era in the scalar-tensor frame. Thereby the conformal factor bridges a F(R) non-singular bounce model to a scalar-tensor inflationary model. Moreover the observational viability of the scalar-tensor inflationary frame, where the bottom-up reconstruction can be applied, confirms the viability of the conformally connected F(R) bouncing model. Keeping these arguments in mind, we try to construct a viable F(R) bouncing scenario directly from the observable indices of the corresponding scalar-tensor (ST) model, in particular we start with a suitable ansatz of the tensor-to-scalar ratio of the ST frame in terms of e-foldings number. The ansatz of \( r = r(N) \) corresponds to an inflationary era in the scalar-tensor frame, which also has an exit at a finite time. On other hand, due to the suitably considered conformal factor, the F(R) frame scale factor behaves as \( a_F(\eta) = \cosh(\gamma \eta) \) which indicates a non-singular bounce at \( \eta = 0 \). With the ansatz \( r = r(N) \) along with the conformal factor, we investigate the viability of the ST inflationary model in respect to the Planck constraints and as a result, the observable quantities like the spectral index, tensor-to-scalar ratio are found to lie within the constraints for a certain parametric ranges. This in turn confirms the observable viability of the F(R) bouncing model. Thus a viable F(R) bouncing model is constructed directly from the tensor-to-scalar ratio ansatz of the corresponding scalar-tensor inflationary model.

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