From Nonperturbative SUSY Gauge Theories to Integrable Systems

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Abstract

I review the appearance of integrable structures in the formulation of exact nonperturbative solutions to 4d supersymmetric quantum gauge theories. Various examples of $\mathcal{N} \geq 2$ SUSY Yang-Mills nonperturbative solutions are adequately described in terms of the (deformations of the) finite-gap solutions to integrable models: through the generating differential and the $\tau$-function. One of the basic definitions of generating differential is discussed and its role in the theory of integrable systems is demonstrated.

1. INTRODUCTION. Recent investigations showed that not only the simplest (topological) string model’s generating functions but also the low-energy effective actions and the BPS massive spectra for 4d (at least) $\mathcal{N} \geq 2$ SUSY Yang-Mills theories possess a nice description in terms of the integrable systems. The relation between the Seiberg-Witten solutions (SW) and integrable theories is already established in detail for two families of models: the $\mathcal{N} = 2$ SYM theory with one ($N_a = 1$) “matter” $\mathcal{N} = 2$ hypermultiplet in the adjoint representation – related to the Calogero-Moser integrable models and to the $\mathcal{N} = 2$ SYM QCD – theory with ($N_f$) fundamental matter hypermultiplets – connected with the family of integrable spin chains.

The SW solution can be formally defined as a map

$$G, \tau, h_i \rightarrow T_{ij}, a_i, a_i^D$$

(1)

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3It is known also that the $N_c = 3, N_f = 2$ curve can be associated with the Goryachev-Chaplygin top.
and the solution to this problem has an elegant description in the following terms [1]: one associates with the data $G$ (gauge group), $\tau$ (the UV coupling constant), $h_k = \frac{1}{k} \langle \mathrm{Tr} \Phi^k \rangle$ – the v.e.v.’s of the Higgs fields) a family of complex curves $\Sigma$ with $h_i$ parameterizing (some) moduli of their complex structures, and a meromorphic 1-form $dS$ on every $\Sigma$. The periods $a_i = \oint A_i dS$, $a_i^D = \oint B_i dS$ determine the BPS massive spectrum, $\frac{\partial F}{\partial a_i} = a_i^D$ the low-energy effective action $F$ and the set of low-energy coupling constants $T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j} = \frac{\partial a_i^D}{\partial a_j}$.

In terms of integrable systems the curves $\Sigma$ are interpreted [2] as spectral curves of certain integrable systems, and $a_i$, $a_i^D$ are related to the action integrals ($\oint pdq$) of the system. To describe the solution one should present the explicit construction

$$G, \tau, h_k \rightarrow (\Sigma, dS) \{h_i\},$$

and this turns to be equivalent to a selection of particular integrable model.

2. CURVES AND INTEGRABLE SYSTEMS. First let us discuss the fact of appearance of (finite-dimensional) integrable systems in the framework of the quantum 4d field theories. Indeed, since the solutions are formulated in terms of periods of some differential on a complex curve – it means that an integrable system (moreover an integrable system of KP/Toda type) arises more or less by definition. The arrow from curves to the finite-gap solutions is provided by the Krichever construction, while the arrow in the opposite direction by the Novikov hypothesis proven by Shiota. Since till the moment no real progress has been achieved in deducing the integrable equations directly from the basic definitions of the quantum field (string) theory – this relation can be considered just as an observation. Below in this section, reviewing the main statements of refs. [3, 4], I stress that in fact this is a useful observation leading to the possibility of applying rather simple technique of $\leq 2$-dimensional integrable systems (the Lax pairs, spectral curves, symplectic forms and Hamiltonian flows, $\tau$-functions, etc) to 4d quantum field theories.

To be more concrete, let us start with the observation [4] that the nonperturbative solution to the pure $\mathcal{N} = 2$ SYM theory with $SU(N_c)$ gauge groups is described in terms of the periodic Toda-chain spectral curves. The periodic problem in this model can be formulated in two different elegant ways, which can be naturally deformed in two different directions. The useful observation is that exactly these two deformations are related to the two physically interesting deformations of the 4d theory by coupling it to the adjoint and fundamental matter hypermultiplets.
The Toda-chain system is defined by the equations of motion with the pairwise exponential interaction
\[ \frac{\partial q_i}{\partial t} = p_i, \quad \frac{\partial p_i}{\partial t} = e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}} \]  
(3)
and the \( N_c \)-periodic problem, corresponding to the situation when one has exactly \( N_c \) particles living on a circle (that just means that \( N_c \)-th particle interacts as well with the first one) can be equivalently formulated in terms of two different Lax representations. The spectral curve itself arises as a condition of the common spectrum of the Lax operator \( \mathcal{L} \) and the shift operator \( T_N \) responsible for the periodic boundary conditions. In the first version the Lax operator of the auxiliary linear problem
\[ \lambda \psi^\pm_n = \sum_k \mathcal{L}_{nk} \psi^\pm_{n+k} = e^{\frac{1}{2}(q_{n+1} - q_n)} \psi^\pm_{n+1} + p_n \psi^\pm_n + e^{\frac{1}{2}(q_n - q_{n-1})} \psi^\pm_{n-1} \quad (= \pm \frac{\partial}{\partial t} \psi^\pm_n) \]  
(4)
becomes the \( N_c \times N_c \) matrix-valued 1-form [7] – that corresponds to the fact that it is defined on the eigenvectors of the shift operator \( T_N \), and the eigenvalues of the Lax operator are defined from the spectral equation
\[ \det_{N_c \times N_c} \mathcal{L}^T \mathcal{C}(w) - \lambda = 0 \quad \Rightarrow \quad w + \frac{1}{w} = 2P_{N_c}(\lambda) \]  
(5)
where \( w \) is the eigenvalue of the \( T_N \)-operator and \( P_{N_c}(\lambda) \) – a polynomial of degree \( N_c \) with the coefficients being the symmetric functions of the Toda chain Hamiltonians – identified with the v.e.v.’s of the Higgs field \( h_k \).

An alternative description of the same system arises when one solves explicitly the auxiliary linear problem [8] which is just a second-order difference equation and rewrites the solution \( \tilde{\psi}_{i+1} = L_i^{TC}(\lambda) \tilde{\psi}_i \) (after a simple ”gauge” transformation) with the help of a chain of \( 2 \times 2 \) Lax matrices [8]
\[ L_i^{TC}(\lambda) = \begin{pmatrix} p_i + \lambda e^{q_i} & e^{q_i} \\ e^{-q_i} & 0 \end{pmatrix}, \quad i = 1, \ldots, N_c \]  
(6)
manifestly depending on the Lax eigenvalue – in contrast to the previous picture now we work with the Lax eigenfunctions. The shift operator becomes
\[ T_N(\lambda) = \prod_{N_c \leq i \leq 1} L_i(\lambda) \]  
(7)
and the (same!) curve and integrals of motion of the Toda chain are generated by another form of spectral equation (with \( 2 \times 2 \) instead of \( N_c \times N_c \) matrices)
\[ \det_{2 \times 2} \left( T_N^{TC}(\lambda) - w \right) = w^2 - w \text{Tr}T_N^{TC}(\lambda) + 1 = 0 \quad \Rightarrow \quad w + \frac{1}{w} = \text{Tr}T_N^{TC}(\lambda) \equiv 2P_N(\lambda) \]  
(8)
The generating 1-form (to be discussed in detail in the next section)

\[ ds \cong \lambda d \log w = \text{Tr} L d \log T_{N_c} \]  

(9)
is defined by the eigenvalues of two (commuting on a spectral curve) operators.

The \( N_c \times N_c \) matrix-valued Lax 1-form comes naturally from the \( GL(N_c) \) Calogero system \( \mathcal{L}(\xi) d\xi \) which is defined on elliptic curve \( E(\tau) \). The Calogero coupling constant in 4d interpretation plays the role of the mass of the adjoint matter \( \mathcal{N} = 2 \) hypermultiplet breaking \( \mathcal{N} = 4 \) SUSY down to \( \mathcal{N} = 2 \) \[3\]. The spectral curve \( \Sigma^{\text{Cal}} \) for the Calogero system:

\[ \det_{N_c \times N_c} (\mathcal{L}^{\text{Cal}}(\xi) - \lambda) = 0 \]  

(10)
and the periods \( a_i \) and \( a_i^D \) of the generating 1-differential

\[ dS^{\text{Cal}} \cong \lambda d\xi \]  

(11)
in the double-scaling limit \( (x_i^{\text{Cal}} - x_j^{\text{Cal}}) \sim [(i - j) \log \mathcal{g}^{\text{Cal}} + (q_i - q_j)] \to \infty \) \[10\], recover the Toda-chain data \( \mathcal{g} \) and \( \mathcal{g} \). In this limit, the elliptic curve \( E(\tau) \) degenerates into the (two-punctured) Riemann sphere with coordinate \( w = e^{\xi} e^{i\tau} \) so that

\[ dS^{\text{Cal}} \to dS^{\text{TC}} \cong \lambda \frac{dw}{w} \]  

(12)
In contrast to the Toda case, \( \Sigma^{\text{Cal}} \) \[10\] can not be rewritten in the form \( \mathcal{g} \) and the specific \( w \)-dependence of the spectral equation \( \mathcal{g} \) is not preserved by embedding of Toda into Calogero-Moser system. However, the form \( \mathcal{g} \) is preserved by the alternative deformation of the Toda-chain system when considering it as (a particular case of) a spin-chain model.

The full spectral curve for the periodic spin chain is given by:

\[ \det_{2 \times 2} (T_{N_c}(\lambda) - w) = 0, \]  

(13)
with the inhomogeneous \( T \)-matrix

\[ T_{N_c}(\lambda) = \prod_{i=N_c}^1 L_i(\lambda - \lambda_i) \]  

(14)
(in general \( \det T_{N_c}(\lambda) \equiv Q(\lambda) \neq 1! \)) and generating differential is now

\[ dS = \lambda \frac{d\tilde{w}}{\tilde{w}} \quad \tilde{w} = w \cdot (\det T_{N_c})^{-1/2} \]  

(15)
or

\[ w + \frac{Q(\lambda)}{w} = 2P_{N_c}(\lambda) \quad \tilde{w} + \frac{1}{\tilde{w}} = \frac{2P_{N_c}(\lambda)}{\sqrt{Q(\lambda)}} \]  

(16)
which is a proposed form of a curve for $\mathcal{N} = 2$ SUSY QCD.

3. SYMPLECTIC FORM. Now let us turn to the discussion of a more subtle point – why the generating 1-form (11) indeed describes an integrable system. To do this I will discuss the symplectic structure on the space of the finite-gap solutions. This symplectic structure was introduced in [11] and recently proposed in [13] as coming directly from the symplectic form on the space of all the solutions to the hierarchy. Below, a very simple and straightforward proof of this result is presented and the relation with the analogous object in low-dimensional non-perturbative string theory [14] is discussed.

To prove that (11) is a generating one-form of the whole hierarchy one starts with the variation of the generating function

$$S(\Sigma, \gamma) = \sum_i \int_{\gamma_i} Edp$$

(17)

(where $dE$ and $dp$ ($= d\lambda$ and $= \frac{dw}{\psi}$ in the particular case above) are two meromorphic differentials on a spectral curve $\Sigma$ and $\gamma$ is the divisor of the solution (poles of the BA function))

$$\delta S = \sum_i (Edp)(\gamma_i) + \sum_i \int_{\gamma_i} \delta Edp$$

$$\delta^2 S = \delta \left( \sum_i (Edp)(\gamma_i) \right) + \sum_i (\delta Edp)(\gamma_i)$$

(18)

From $\delta^2 S = 0$ it follows that

$$\varpi = \delta E \wedge \delta p = \delta \left( \sum_i (Edp)(\gamma_i) \right) = - \sum_i (\delta Edp)(\gamma_i)$$

(19)

Now, the variation $\delta E$ (for constant $p$) follows from the Lax equation (auxiliary linear problem)

$$\frac{\partial}{\partial t} \psi = \mathcal{L} \psi (= E \psi)$$

(20)

so that

$$\delta E = \frac{\langle \psi^\dagger \delta \mathcal{L} \psi \rangle}{\langle \psi^\dagger \psi \rangle}$$

(21)

and one concludes that

$$\varpi = - \langle \delta \mathcal{L} \sum_i \left( dp \frac{\psi^\dagger \psi}{\langle \psi^\dagger \psi \rangle} \right)(\gamma_i) \rangle$$

(22)

Let us turn to several important examples.

**KP/KdV.** In the KP-case the equation (20) looks as

$$\frac{\partial}{\partial t} \psi = \left( \frac{\partial^2}{\partial x^2} + u \right) \psi (= E \psi)$$

(23)

therefore the equation (22) implied by $\langle \psi^\dagger \psi \rangle = \int dx \psi^\dagger(x, P) \psi(x, P)$ and $\delta \mathcal{L} = \delta u(x)$ gives

$$\varpi = - \int dx \delta u(x) \sum_i \left( \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger(x)(\gamma_i) \right)$$

(24)
The differential $d\Omega = \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger(x) \psi(x)$ is holomorphic on $\Sigma$ except for the "infinity" point $P_0$ where it has zero residue \(^\text{[12]}\). Its variation \(^\text{[1]}\)

$$\tilde{\delta} \left( \text{res}_{P_0} d\Omega + \sum_i \text{res}_{\gamma_i} d\Omega \right) = 0 \quad (25)$$

can be rewritten as

$$\delta v(x) + \sum_i d\Omega(\gamma_i) = 0 \quad (26)$$

where $v(x)$ is a "residue" of the BA function at the point $P_0$ obeying $v'(x) = u(x)$. Substituting \(^\text{[23]}\) into \(^\text{[22]}\) one gets

$$\varpi = \int dx \delta u(x) \int_{dx'} \delta u(x') \quad (27)$$

or the first symplectic structure of the KdV equation.

**Toda chain/lattice.** (The case directly related to the pure SYM theory). One has $\langle \psi^\dagger \psi \rangle = \sum_n \psi^+_n(P) \psi^-_n(P)$, and the Lax equation acquires the form \(^\text{[4]}\) where $t = t_+ + t_-$ and $t_1 = t_+ - t_-$ is the first time of the Toda chain, so that

$$\delta \lambda = \frac{\sum_k \psi^+_k \delta p_k \psi^-_k}{\langle \psi^+ \psi^- \rangle} \quad (28)$$

and \(^\text{[22]}\) becomes

$$\varpi = -\sum_k \delta p_k \sum_i \left( \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi^+_k \psi^-_k \right) (\gamma_i) \quad (29)$$

and to get

$$\varpi = \sum_k \delta p_k \wedge \delta q_k \quad (30)$$

one has to prove

$$\sum_i \left( \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi^+_k \psi^-_k \right) (\gamma_i) = \delta q_k \quad (31)$$

To do this one considers again

$$\tilde{\delta} \left( \text{res}_{P_+} + \text{res}_{P_-} + \sum_i \text{res}_{\gamma_i} \right) d\Omega_n = 0 \quad \text{d}\Omega_n = \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi^+_n \psi^-_n \quad (32)$$

where the first two terms for $\psi^\pm_n \sim e^{\pm q_n \lambda^\pm_0 (1 + O(\lambda^{-1}))}$ satisfying two "shifted" equations \(^\text{[4]}\) (with $\tilde{q}_n$ and $q_n$ correspondingly) give $\delta q_n = \tilde{q}_n - q_n$ while the rest – the l.h.s. of \(^\text{[31]}\).

**Calogero-Moser system.** Introducing the "standard" $dE$ and $dp$ one the curve $\Sigma$ \(^\text{[11]}\) with the 1-form \(^\text{[11]}\) where $dp = d\xi$ is holomorphic on torus $\int_A dp = \omega$, $\int_B dp = \omega'$ and $E = \lambda$

\(^4\text{It should be pointed out that the variation } \tilde{\delta} \text{ corresponds to a rather specific situation when one shifts only } \psi \text{ keeping } \psi^\dagger \text{ fixed.}\)
has \( n - 1 \) poles with \( \text{residue} = 1 \) and 1 pole with \( \text{residue} = -(n - 1) \), the BA function is defined by

\[
\mathcal{L}^{Cal}(\xi) a = \lambda a
\]

with the essential singularities

\[
a_i \sim e^{x_i \xi(\xi)} (1 + \mathcal{O}(\xi)) \quad a_i \sim e^{x_i \xi(\xi)} \left(-\frac{1}{n-1} + \mathcal{O}(\xi)\right)
\]

and (independent of dynamical variables) poles \( \gamma \). Hence, similarly to the above case for the eq. (22) one has

\[
\langle \psi^\dagger \psi \rangle = \sum_i a_i^\dagger(P) a_i(P), \quad \delta \mathcal{L}^{Cal} = \sum_i a_i^\dagger(P) \delta p_i a_i(P)
\]

so that

\[
\omega = -\sum_k \delta p_k \sum_i \left( \frac{d\xi}{\langle a_i^\dagger a_i \rangle} - a_k^\dagger a_k \right) (\gamma_i)
\]

and the residue formula

\[
\delta \left( \sum_{P, p=0} \text{res}_{P_j} + \sum_i \text{res}_{\gamma_i} \right) d\Omega_k = 0 \quad d\Omega_k = \frac{d\xi}{\langle a_i^\dagger a_i \rangle} a_k^\dagger a_k
\]

where the first sum is over all ”infinities” \( p = 0 \) at each sheet of the cover (10). After variation and using (34) it gives again

\[
\omega = \sum_k \delta p_k \wedge \delta x_k
\]

The general proof of the more cumbersome analog of the above derivation can be found in [13].

To show how the above formulas work explicitly, let us, finally, demonstrate the existence of (25), (32) and (36) for the 1-gap solution. Let

\[
\psi = e^{x \zeta(z)} \frac{\sigma(x - z + \kappa)}{\sigma(x + \kappa) \sigma(z - \kappa)} \quad \psi^\dagger = e^{-x \zeta(z)} \frac{\sigma(x + z + \kappa)}{\sigma(x + \kappa) \sigma(z + \kappa)}
\]

be solutions to

\[
(\partial^2 + u) \psi = (\partial^2 - 2\varphi(x + \kappa)) \psi = \varphi(z) \psi
\]

Then

\[
\langle \psi^\dagger \psi \rangle = \frac{\sigma(x - z + \kappa) \sigma(x + z + \kappa)}{\sigma^2(x + \kappa) \sigma(z - \kappa) \sigma(z + \kappa)} = \frac{\sigma^2(z)}{\sigma(z + \kappa) \sigma(z - \kappa)} (\varphi(z) - \varphi(x + \kappa))
\]

and let us take the average over a period \( 2\bar{\omega} \) to be \( \langle \varphi(x + \kappa) \rangle = 2\bar{\eta} \). Also

\[
dp = d (\zeta(z) + \log \sigma(2\bar{\omega} - z + \kappa) - \log \sigma(\kappa - z)) = -dz (\varphi(z) + \zeta(2\bar{\omega} - z + \kappa) - \zeta(\kappa - z)) = -dz (\varphi(z) - 2\bar{\eta})
\]
and
\[ \frac{dp}{\langle \psi^\dagger \psi \rangle} = dz \frac{\sigma(z + \kappa)\sigma(z - \kappa)}{\sigma^2(z)} \]
\[ d\Omega = \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger \psi = dz \frac{\sigma(x + z + \kappa)\sigma(x + z - \kappa)}{\sigma^2(z)\sigma^2(x + \kappa)} = dz (\wp(z) - \wp(x + \kappa)) \]

(42)

Now, the variation \( \dot{d} \) explicitely looks as
\[ \dot{d}\Omega \equiv \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger \psi + \delta \kappa \]
\[ = dz \frac{\sigma(x + \kappa + z)\sigma(x + \kappa - z)}{\sigma^2(z)\sigma^2(x + \kappa)} \left[ 1 + \delta \kappa (\zeta(x + \kappa) + \zeta(z - \kappa) - \zeta(x + \kappa)) + O\left((\delta \kappa)^2\right) \right] \]
\[ = dz (\wp(z) - \wp(x + \kappa)) \left[ 1 + \delta \kappa (\zeta(x + \kappa) + \zeta(z - \kappa) - \zeta(x + \kappa)) + O\left((\delta \kappa)^2\right) \right] \]

(43)

It is easy to see that (43) has non-zero residues at \( z = 0 \) and \( z = \kappa \) (the residue at \( z = x + \kappa \) is suppressed by \( \wp(z) - \wp(x + \kappa) \)). They give
\[ \text{res}_{z=0}\delta d\Omega \sim \delta \kappa \int_{z \to 0} dz \wp(z) (\zeta(x - z + \kappa) + \zeta(z - \kappa)) \sim \delta \kappa \int_{z \to 0} \zeta(z) d(\zeta(x - z + \kappa) + \zeta(z - \kappa)) \sim \delta \kappa (\wp(x + \kappa) + \wp(\kappa)) \sim \delta (\zeta(x + \kappa) + \zeta(\kappa)) \equiv \delta \nu(x) \]

(44)

and
\[ \text{res}_{z=\kappa}\delta d\Omega = \delta \kappa (\wp(\kappa) - \wp(x + \kappa)) = d\Omega(\kappa) \]

(45)

which follows from the comparison to (42).

4. CONCLUSION. The quantization of the symplectic form (13) is known to correspond to the complete description of the effective theory (not only its low-energy part) at least in the simplest case when \( E = W(\mu) \) and \( p = Q(\mu) \) were two functions (polynomials) on a complex sphere. The corresponding generating function (9) was essential in the definition of the duality transformation between two dual points with completely different behavioir (see [14] for details). The exact answer for the partition function \( \log T = \log T_0 + \log T_\theta \equiv F + \log T_\theta \) should also include the deformation of the oscillating part, corresponding to the massive excitations.

An advantage of the language of the integrable systems is that it allows one at least in principle to use a strict (in many interesting cases explicit) formulation of the hypothetical properties of quantum field and string theories where the basic one is given by the already mentioned duality.

In all interesting integrable models the Liouville torus of an integrable system is a real section of a complex torus being for KP/Toda-theories a Jacobian of a (spectral) Riemann curve \( \Sigma \). It is clear that there exists several possibilities to choose a real section for the same
Jacobian – these different choices correspond to a priori different integrable systems which however may be related in a simple way having the same or related spectral curves. Such sort of duality imposed by the "exchange" of different cycles on the same Jacobian (or the spectral curve itself) is defined globally for the integrable system. Exchange between different sections of Jacobian (different trajectories) corresponds to the exchange between the particles and "collective excitations" (like monopoles) in field theory (see [15] for the stringy explanation of this effect).

Another sort of duality works only locally in moduli space for a completely integrable system defined by a set of Hamiltonians \( \{ h_k \} \) commuting with respect the Poisson bracket determined by \( \{ h_k, h_l \} = 0 \). In many physical cases the phase space has a structure of a cotangent bundle to a configuration space, then in addition to hamiltonians one can find another distinguished set of Poisson-commuting variables, for example, the co-ordinates on configuration space \( \{ q_k \} : \{ q_k, q_l \} = 0 \). Again, the dual transformation preserves the symplectic form. However, now the "integrals of motion" are no longer constants on trajectories and one can study the variation of the partition function with respect to moduli \( h_k \) or \( a_k \) variables. This gives rise to the Whitham deformations of the finite-gap solutions, producing the exact answer for the whole theory.

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\(^5\)For example the Jacobi map might not be able to distinguish the curves \( \Sigma \) and \( \tilde{\Sigma} \) if \( \Sigma \) is a cover of \( \tilde{\Sigma} \).
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