Multiple Randomization Designs *

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Abstract

In this study we introduce a new class of experimental designs. In a classical randomized controlled trial (RCT), or A/B test, a randomly selected subset of a population of units (e.g., individuals, plots of land, or experiences) is assigned to a treatment (treatment A), and the remainder of the population is assigned to the control treatment (treatment B). The difference in average outcome by treatment group is an estimate of the average effect of the treatment. However, motivating our study, the setting for modern experiments is often different, with the outcomes and treatment assignments indexed by multiple populations. For example, outcomes may be indexed by buyers and sellers, by content creators and subscribers, by drivers and riders, or by travelers and airlines and travel agents, with treatments potentially varying across these indices. Spillovers or interference can arise from interactions between units across populations. For example, sellers’ behavior may depend on buyers’ treatment assignment, or vice versa. This can invalidate the simple comparison of means as an estimator for the average effect of the treatment in classical RCTs. We propose new experiment designs for settings in which multiple populations interact. We show how these designs allow us to study questions about interference that cannot be answered by classical randomized experiments. Finally, we develop new statistical methods for analyzing these Multiple Randomization Designs.

1 Introduction

In a randomized experiment designed to evaluate the causal effect of a binary treatment, the starting point is typically a single population of units. Members of such a population can be individuals, products, firms, plots of land, time periods, etc. A random subset of this population is assigned

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to the treatment group, which will be exposed to the treatment, or treatment $A$. The complement of this subset is assigned to the control group, and exposed to treatment $B$. Average outcomes for units in the treatment group are then compared to average outcomes in the control group to estimate the average causal effect of the treatment. See Fisher [1937], Neyman [1923/1990], Holland [1986], Wu and Hamada [2011], Imbens and Rubin [2015] for general discussions, and Kohavi et al. [2009], Gupta et al. [2019] for modern applications.

Modern experiments, however, often involve multiple populations with outcomes (and treatment assignments) indexed by members of each population. For example, outcomes may be indexed by buyers and sellers, renters and properties, content creators and subscribers, or services and customers. These settings are challenging for traditional experimental designs because the members of both populations often act strategically in response to the treatment assignment in ways that create interference or spillovers between two different buyer-seller pairs. In this paper we introduce a general class of experiments, which we call *Multiple Randomization Designs* (MRDs), wherein we allow the treatment to vary systematically within units in each of the populations in order to assess the presence and magnitude of such interference. We also develop new statistical methods for analyzing such experiments, including estimators for average effects, variances, and measures of precision.

Consider a specific example of a two-sided marketplace where multiple sellers make offers, and where multiple buyers can make bids. Suppose buyers search for — or are presented with — a particular seller’s offers that they can then choose to bid on, or not. The experimenter may be interested in evaluating the effect of a new information policy on the likelihood of a buyer buying the item offered by a given seller. When the buyer is presented with information on the item, the different policies may consist of either posting details on the item and seller (treatment $A$), or summary information about the item but no information about the seller (treatment $B$). In a traditional experimental design (A/B test), the experimenter might choose to randomize buyers to treatment $A$ or treatment $B$ in what we will call a *buyer experiment*. Alternatively, the experimenter might choose to randomize sellers to treatment $A$ or treatment $B$ in a *seller experiment*. Both of these *Single Randomization Designs* fit into standard RCT approach. Here, we propose *Multiple Randomization Designs*, where the assignment to treatment $A$ or treatment $B$ is determined at the level of the buyer/seller pair. While traditional (single-population) randomized experiments are a special case of MRDs, the latter allow for substantially richer designs.

The two key features that differentiate the settings we consider from traditional RCTs are (a) outcomes are observed for pairs (or more generally, tuples) of units, where each element of the tuple belongs to a different population (e.g., sellers and buyers); and (b) the assignment can happen at the level of the tuple. Although in the current paper we use the case of buyers and sellers as the running example, there are many other settings in which our proposed MRDs are applicable, such as: (a) ride-sharing where drivers choose to accept ride requests from riders at different times (e.g., Uber, Lyft), (b) travel websites where air travelers search for flights offered by airlines or travel agencies (e.g., Expedia, Kayak), (c) short-term rental marketplaces where owners offer properties to renters (e.g., AirBnB, VRBO), (d) housing sales where properties are offered for viewing by real estate agents to potential buyers (e.g., Zillow, Redfin), (e) social media where content creators and subscribers interact (e.g., Instagram, TikTok, YouTube), and (f) employment centers where job applicants search for or are matched with job vacancies.
To show the information content of the proposed experiments, consider an example with eight sellers and five buyers. This is the simplest setting for the MRDs we consider in this paper, and in fact this particular design has also been independently proposed in Johari et al. [2020] in their study of biases in traditional RCTs. We assign the pair corresponding to buyer \(i\) and seller \(j\) to a binary treatment, \(W_{ij} \in \{C, T\}\). An example of an MRD treatment assignment matrix in this setting is shown in Equation (1.1).

\[
W = \begin{pmatrix}
C & C & C & C & C & C & C & C & 1 \\
C & C & C & C & C & C & C & C & 2 \\
C & C & C & T & T & T & T & T & 3 \\
C & C & C & T & T & T & T & T & 4 \\
C & C & C & T & T & T & T & T & 5
\end{pmatrix}
\]

(1.1)

In comparison, a conventional buyer randomized experiment would have an assignment matrix \(W\) with all columns identical and rows consisting of either all \(T\) or all \(C\), and a seller randomized experiment would correspond to an assignment matrix with all rows identical and columns consisting of either all \(T\) or all \(C\).

There are two key insights to setting up the experimental design as a choice for the distribution of a matrix \(W\) jointly randomizing across buyers and sellers, rather than for the distribution of a vector, randomizing only at the level of buyers or at the level of sellers:

First, by allowing for a richer set of potential assignment matrices \(W\), MRDs can be more efficient than conventional designs in answering standard questions. Specifically, under some conditions, designs where the fraction of treated buyer-seller pairs is the same in all columns and all rows are more efficient than either a buyer or a seller experiment. This insight is related to the motivation underlying stratification, Latin squares and factorial designs in agricultural experiments.

Second and our main focus, MRDs can generate information about spillovers and interference that cannot be learned from single randomization designs. Specifically, MRDs allow for tests for the presence of spillovers and estimation of their magnitude. They can estimate average effects that take some forms of such interference into account. Consider the example treatment assignment matrix in Equation (1.1) in which three sets of buyer-seller pairs receive the control treatment, but differ systematically in terms of their general exposure. The blue \(C\) buyer-seller pairs (buyers 1–2, and sellers 5–8) correspond to pairs where the sellers are in the treatment group for other buyers. The green \(C\) buyer-seller pairs (buyers 3–5, and sellers 1–4) correspond to pairs where the buyers are in the treatment group for other sellers. Finally, the red \(C\) buyer-seller pairs (buyers 1–2, and sellers 1–4) consist of pairs where the buyers and corresponding sellers are always in the control group. Comparing outcomes for these three sets of controls pairs is informative about the extent and nature of spillovers, and allows us to correct for spillovers under some assumptions on the nature of the interference.

The framework we develop allows for much richer experiments than the simple one in Equation (1.1). In Section 8 we discuss a number of experiments that allow us to estimate the direct effect of exposing sellers to the treatment for a buyer, versus the indirect effect of the buyer having
been exposed to the treatment for other sellers under various assumptions on interference. Such
direct and indirect effects are at the heart of equilibrium effects that arise from interacting pop-
ulations that are difficult to infer in standard experimental designs. As an illustration, suppose
we are concerned that there are direct effects of the treatment on buyer-seller pair \((i, j)\) from the
treatment \(W_{ij}\), but also indirect effects from treatments \(W_{km}\) for other pairs \((k, m)\) with either
\(k = i\) or \(m = j\). To get at these direct and indirect effects we first randomize sellers into two
groups \(\{X_S^j \in \{B, S\}\}\), and then conduct a buyer experiment for the first set of sellers, those with
\(X_S^j = B\) (with assignment for buyer \(i\) equal to \(W^B_i \in \{C, T\}\)), and conduct a seller experiment for
the second set of sellers, those with \(X_S^j = S\), (with assignment for seller \(j\) equal to \(W^S_j \in \{C, T\}\)).

An example of the assignment matrix under such a design is given in Equation (1.2).

\[
\begin{array}{cccccccc}
\text{(Sellers)} & j & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
X^S_j & \rightarrow & B & B & B & S & S & S & S & S \\
W^S_j & \rightarrow & T & C & T & C & C & C & C & C \\
\end{array}
\]

\[W = \begin{pmatrix}
C & C & C & T & C & T & C & C \\
C & C & C & T & C & T & C & C \\
T & T & T & C & T & C & C & C \\
C & C & C & T & C & T & C & C \\
T & T & T & C & T & C & C & C
\end{pmatrix}\]

Interpreting and analyzing such experiments requires careful consideration of the types and
mechanisms for interference and spillovers. We discuss some of these more general settings in detail
in Section 8.

Although in this paper we primarily focus on settings with two dimensions (e.g., buyers and
sellers), the framework we develop naturally extends to settings with three or more dimensions, in
which the assignment is a tensor instead of a matrix. For example, we could define the interaction
taking place between buyers, sellers, and real-estate agents in a housing example and consider
changing the structure of the exchange at the level of the buyer-seller-agent. Similarly, we could
define the interaction between buyers, sellers, agents, and time periods. We consider extensions to
such tensors of higher order in Section 8.5.

The multi-dimensional nature of the population then gives the researcher the opportunity to
create multiple groups that have three characteristics: (a) all groups are exposed to the same
treatment, (b) the groups are ex ante comparable to each other because of the randomization, and
(c) the groups differ systematically in their exposure patterns, so that their outcome comparisons
are informative about the nature of spillovers.

2 Related Literature

Special cases of MRDs, or related designs, have been discussed previously in the literature:

One prominent design with a long history is the crossover design or switchback design, see
Cochran [1939], Cochran and Cox [1948], Brown Jr [1980], Cook and DeMets [2007], Bojinov et al. [2020]. In such designs, each unit is assigned to both treatment and control, but at different times. This is a special case of the MRD set up where the first population is the population of units, and the second population is the set of time periods. Crossover designs are intended to improve precision over designs with fixed treatments over time. In practice, the use of such experimental designs requires a careful consideration of the length of the time periods used. It may take individuals some time to respond to a change in their experimental status, and if the time period over which the assignment changes is short relative to the adjustment period the designs may be biased. A related design corresponds to the case where units can only switch from the control group to the treatment group, but not backwards. This design is considered in Xiong et al. [2019]. Such settings often involve dynamic treatment effects and allow for richer questions than the standard cross-over designs.

A second design that has a long tradition, especially in agricultural and industrial settings, is the *split-plot design* [Yates, 1935, Brandt, 1938, Jones and Nachtsheim, 2009]. In such designs there are multiple primary units (i.e., plots of land) that each consist of multiple units (subplots) that can separately be assigned to a treatment and for whom we can measure the outcome. The concern is that even in the absence of any treatment, the outcomes for the subplots that are part of the same plot are correlated. Taking such correlations into account can lead to more effective experimental designs. Note that in the split-plot designs the concern is typically not about estimating spillovers and interference for such effects, which are our primary concerns.

In recent years there have been a number of innovative new experimental designs that, like our MRDs, are motivated by the settings in modern companies with some type of interference. Here we mention three strands of this emerging literature.

One example of such designs is the equilibrium design proposed by Wager and Xu [2021], Munro et al. [2021]. In these studies the treatment is continuous and assigned uniformly, e.g., price. It can change over time but at an point in time all units are exposed to the same treatment.

In a study of two-sided marketplaces, say with buyers and sellers, Johari et al. [2020], Li et al. [2021] investigate the biases arising from (single randomization design) buyer experiments or seller experiments, and show when each of them is likely to be severely biased. They also independently propose the special case of MRD in Equation (1.1) as a strategy to address some of the biases that arise from buyer or seller experiments. Johari et al. [2020] also consider dynamic settings where the fraction of treated and control units changes over time.

A different approach to dealing with spillovers or interference is given by *clustered experiments* ([Donner and Klar, 2000, Roberts and Roberts, 2005]), or, in network settings, by *network bucket testing* [Backstrom and Kleinberg, 2011, Katzir et al., 2012, Ugander et al., 2013, Aronow and Samii, 2017]. In network bucket testing there is a population of units that are assigned to the treatment or control group. Interference between the units is measured through their graph adjacency, and the population of units is partitioned into subpopulations so that units close in the network are likely to belong to the same subpopulation. The subpopulations are then assigned to treatment or control as in a clustered randomized experiment. When the graph is a bi-partite graph based on two populations, network bucket testing has some similarity with the setting we consider in the present paper, although in the network bucket testing literature the focus is traditionally on assignments where the primary units are either in the treatment group or the control group, in
contrast to the approach we consider.

3 Set-Up and Notation

In what follows, we use the two-population buyer-seller case as our generic example throughout, but we emphasize that the ideas presented here apply to other settings, e.g., drivers/riders, properties/rentals, content creators/subscribers, applicants/vacancies, and others including settings involving more than two populations, e.g., passengers, airlines, and travel agents. There are $I$ buyers, indexed by $i \in \mathbb{I} := \{1, \ldots, I\}$. There are $J$ sellers, indexed by $j \in \mathbb{J} := \{1, \ldots, J\}$. Both $I$ and $J$ may be large, although we present some finite sample results that are valid irrespective of the absolute and relative magnitude of $I$ and $J$. For house rental websites, ‘buyers’ would correspond to guests, and ‘sellers’ to hosts. For ride sharing services, ‘buyers’ are riders, and ‘sellers’ are drivers. For social media, ‘buyers’ are subscribers, and ‘sellers’ are content creators. For ease of exposition, we refer to one population as the buyers and the other population as the sellers. Over a fixed period of time, say a week or a month, there is for each buyer-seller pair a measure of engagement, for example the amount of money spent by a buyer with a particular seller, or an indicator that buyer and seller have interacted, which we use as the outcome, denoted by $Y_{ij}$.

We consider a binary intervention that can be assigned at the level of pair $(i,j)$ of a buyer $i$ and a seller $j$, denoted by $W_{ij} \in \{C, T\}$, with $W$ the $I \times J$ matrix with element $W_{ij}$. There may also be buyer and seller characteristics that are associated with the outcomes, but we ignore these for the moment. An example assignment matrix is shown in Equation (1.2), where the columns correspond to the $J$ sellers, and the rows correspond to the $I$ buyers. The focus of this paper is on choices for distributions for $W$.

$$W = \begin{pmatrix}
1 & C & C & C & C & C & C & C \\
2 & C & T & T & T & C & T & T \\
3 & C & T & C & T & C & C & T \\
4 & T & C & C & T & C & C & T \\
5 & T & T & C & T & C & C & T
\end{pmatrix}, \quad (3.1)$$

We refer to a distribution $p : \mathbb{W} \mapsto [0,1]$ as an experimental design, where $\mathbb{W}$ is the set of values that the matrix $W$ can take on. Many conventional RCTs impose substantial restrictions on these assignment matrices. More specifically, conventional designs restrict all elements within rows (or within columns) of the matrix to be identical, with variation only between rows (or columns). As discussed in Section 1, experimental designs that allow for variation in the treatment both within rows and columns can be more efficient in answering questions that can already be answered using conventional designs; and, more importantly, they can inform about spillover effects, thus answering questions that conventional designs cannot address. (Whenever the distinction matters and is not clear from context, we refer to a draw of an assignment matrix from $\mathbb{W}$ as $w$, and to $W$ as a matrix-valued random variable.)

The buyer-seller pair $(i,j)$ is the unit of observation as well as the unit of analysis. In principle
each outcome \(Y_{ij}\) can depend on the full matrix \(W\), so that the potential outcomes are \(Y_{ij}(W)\), with \(Y(W)\) denoting the full \(I \times J\) matrix of potential outcomes for a given \(I \times J\) matrix of assignments \(W\). Special values for the assignment matrix are \(W = T\), with typical element \(T_{ij} = T\) for all \(i \in I\) and \(j \in J\), corresponding to all pairs/interactions being exposed to the new treatment, and \(W = C\), with typical element \(C_{ij} = C\) for all \(i \in I\) and \(j \in J\), corresponding to all pairs being exposed to the control treatment. The realized outcomes correspond to the potential outcomes evaluated at the actual assignment: \(Y = Y(W)\).

Define the sum of the realized outcomes for each buyer and each seller:

\[
Y^B_i = \sum_{j=1}^J Y_{ij}, \quad \text{and} \quad Y^S_j = \sum_{i=1}^I Y_{ij}.
\]

In conventional RCTs the unit that we randomize over is the buyer (or the seller), and the basic outcome is the buyer total outcome \(Y^B_i\) (or seller total outcome \(Y^S_j\)). Such buyer (seller) experiments are a special case of the set-up we introduce here next.

One of the most interesting estimands is the effect of universal policies, that is, exposing all interactions to the treatment versus exposing none of the interactions. It is common, and it will be particularly useful to scale the average causal effect by the average outcome without the intervention, so we take as the primary object of interest the \textit{lift}:

\[
\theta = \frac{\frac{1}{IJ} \sum_{i=1}^I \sum_{j=1}^J (Y_{ij}(T) - Y_{ij}(C))}{\frac{1}{IJ} \sum_{i=1}^I \sum_{j=1}^J Y_{ij}(C)}, \quad (3.2)
\]

rather than the average causal effect

\[
\tau = \frac{1}{IJ} \sum_{i=1}^I \sum_{j=1}^J (Y_{ij}(T) - Y_{ij}(C)) = \theta \times \frac{1}{IJ} \sum_{i=1}^I \sum_{j=1}^J Y_{ij}(C). \quad (3.3)
\]

One reason it is helpful to focus on the lift rather than on the average treatment effect is that the lift \(\theta\) at the interaction level is identical to the lift at the buyer level (and also to the lift at the seller level). Define the buyer-level lift \(\theta^B\) and the buyer-level average treatment effect \(\tau^B\)

\[
\theta^B = \frac{\frac{1}{I} \sum_{i=1}^I (Y^B_i(T) - Y^B_i(C))}{\frac{1}{I} \sum_{i=1}^I Y^B_i(C)}, \quad \tau^B = \frac{1}{I} \sum_{i=1}^I (Y^B_i(T) - Y^B_i(C)),
\]

and similarly for the sellers,

\[
\theta^S = \frac{\frac{1}{J} \sum_{j=1}^J (Y^S_j(T) - Y^S_j(C))}{\frac{1}{J} \sum_{j=1}^J Y^S_j(C)}, \quad \tau^S = \frac{1}{J} \sum_{j=1}^J (Y^S_j(T) - Y^S_j(C)).
\]

Then the lifts are equal for all three outcomes, but the average treatment effects differ for the three
outcomes by a scaling factor that depends on the number of buyer and sellers:

\[ \theta = \theta^B = \theta^S \quad \text{and} \quad \tau = \frac{1}{J} \tau^B = \frac{1}{J} \tau^S. \]

For some of the formal results, however, we focus on average effects because we can obtain exact finite sample results, and we rely on large sample approximations to obtain the results for the corresponding lifts.

4 Buyer Experiments, Seller Experiments, and Multiple Randomization Designs

In this section we characterize classes of experiments: First, we define a general class of experiments, and show how conventional buyer- and seller- randomized experiments fit into this class. Then, we discuss experimental designs in this class that go beyond buyer- and seller- experiments. Towards these goals, let \( W \) denote the set of \( 2^{IJ} \) possible values assumed by the \( I \times J \) matrix \( W \), where each element \( w_{ij} \) of the matrix \( W \) belongs to \( \{C, T\} \).

Definition 1 (A Marketplace Experimental Design). A marketplace experimental design is a distribution over \( W \):

\[ p : \mathbb{W} \mapsto [0, 1), \quad \text{with} \quad \sum_{W \in \mathbb{W}} p(W) = 1, \]

that is row and column exchangeable, and such that there is a value \( \overline{w} \in (0, 1) \) that satisfies

\[ p(W) > 0 \Rightarrow \frac{1}{IJ} \sum_{i=1}^{N} \sum_{j=1}^{J} 1_{w_{ij}=T} = \overline{w}. \]

Because we take the potential outcomes \( Y_{ij}(W) \) as fixed and the assignment function is row and column exchangeable, the assignments are by definition independent of the potential outcomes. The condition that all assignments with positive probability have the same fraction of treated buyer/seller pairs is not strictly necessary, but it allows us later to present exact finite sample results. It therefore clarifies what can be learned without relying on large sample approximations. Similarly the condition that rules out that there is a single assignment \( W \) that has probability one is not strictly necessary but it is helpful in that it rules out degenerate experiments.

Given an experimental design \( p(\cdot) \) define the set of assignments with positive probability:

\[ \mathbb{W}_p^+ = \left\{ W \in \mathbb{W} \mid p(W) > 0 \right\}. \]

Given an assignment \( W \), define for each buyer the fraction of treated sellers (and vice versa):

\[ \overline{w}_i^B = \frac{1}{J} \sum_{j=1}^{J} 1_{w_{ij}=T}, \quad \overline{w}_j^S = \frac{1}{I} \sum_{i=1}^{I} 1_{w_{ij}=T}. \]
By the definition of a marketplace experimental design, it follows that $w = \sum_i w_i^B / I = \sum_j w_j^S / J$.

A key feature of a marketplace experimental design is the presence of variation in the treatment for buyer across sellers, and similarly, the presence of variation in the treatment for sellers across buyer. We refer to absence of variation in the assignment as consistency of the buyer (or seller) experience.

Definition 2 (Consistent Experiences). An assignment $W$ induces a consistent experience for buyer $i$ if $w_i^B \in \{0, 1\}$, and induces a consistent experience for seller $j$ if $w_j^S \in \{0, 1\}$.

For buyer $i$ an assignment $W$ corresponds to an inconsistent experience if $w_i^B \in (0, 1)$, and similarly for a seller. It is possible that all buyers have a consistent experience, all buyers have an inconsistent experience, or some buyers have a consistent and some have an inconsistent experience. The same holds for sellers. However, it is not possible for all buyers and all sellers to have a consistent experience, unless there is no variation in the treatment (and so $W \in \{C, T\}$), which is ruled out by our definition of a marketplace experiment.

Inconsistent experiences are at the heart of spillover concerns in the current set up. To illustrate how inconsistent experiences may result in spillovers, suppose the two populations are travelers and airlines, with the outcome the number of trips a traveler takes with an airline. The treatment is an offering of free drinks and meals on flights over four hours. Airlines can choose to offer that to all their travelers, or to only some of their travelers. One question is what would happen if all airlines offered free drinks and meals to all travelers. One might expect that such a practice would not change the amount of travel done on each airline. However, if one did an airline experiment where only some airlines offered free meals and others did not, travelers may switch flights from airlines with no free meals to airlines with free meals. On the other hand, suppose one did a traveler experiment, with a large fraction of treated travelers. If the treatment changes the flights chosen by the travelers, with an increased preference for flights over four hours, the airlines may change the flights they offer, leading to a different experience for travelers in the control group.

To classify the marketplace experiments it is useful to consider the possible extent of inconsistent experiences. Each assignment $W$ leads to a set of values for the share of treated interactions for a given buyer $i$, $\bar{w}_i^B$. Denote the set of all values $\bar{w}_i^B$ across buyers $i \in I$ by $\forall^B(W)$. By taking the union over all assignments $W$ such that $p(W) > 0$, we obtain the set $\forall^B = \bigcup_{W \in W_p^+} \forall^B(W)$ of all possible fractions of treated viewers. We can define the same sets based on the fraction of treated buyers for a given seller. Formally:

Definition 3 (Consistency Sets).

$$\forall^B(W) = \{\bar{w}_1^B, \bar{w}_2^B, \ldots, \bar{w}_I^B\}, \quad \forall^B = \bigcup_{W \in W_p^+} \forall^B(W),$$

$$\forall^S(W) = \{\bar{w}_1^S, \bar{w}_2^S, \ldots, \bar{w}_J^S\}, \quad \forall^S = \bigcup_{W \in W_p^+} \forall^S(W).$$

Next, we define classes of marketplace experiments in terms of restrictions on the assignment mechanism. Specifically we consider classes that restrict $W_p^+$ to sets $\forall^B$ and $\forall^S$ with a certain structure. We focus in particular on three classes of experiments: First, buyer (and seller) experi-
ments, where all buyers (sellers) have a consistent experience. These are the traditional A/B tests or RCTs. Second, completely randomized experiments, where we randomize over all interactions. Third, Simple Multiple Randomization Designs (SMRDs), an instance of MRDs in which we randomize both buyers and sellers separately, and then determine the treatment as a function of the outcomes of the two randomizations. This third class of designs is the main focus of the current paper.

Let us now formally define these three classes of experimental designs. Along the way, we illustrate assignment matrices for the different classes of experiments in a very simple setting with eight sellers and four buyers where always 25% of the buyer/seller pairs are treated, so that \( \bar{w} = \frac{1}{4} \).

**Definition 4 (Seller Experiment).** A Single Randomization Design (SRD) is a marketplace experiment where either each seller or each buyer has a consistent experience with probability one, and so either

\[ \forall^S = \{0, 1\}, \quad \text{and} \quad \forall^B = \{\bar{w}\}, \]

or

\[ \forall^B = \{0, 1\}, \quad \text{and} \quad \forall^S = \{\bar{w}\}. \]

A seller experiment is a SRD marketplace experiment where all columns either consist of all C’s or all T’s, and all rows of \( W \) are identical. In that case all sellers (and no buyers) have a consistent experience, with either all their interactions being treated or all their interactions being control. Therefore, for all assignments with positive probability, it holds \( \bar{w}^S_j = \bar{w} \). An example of an assignment matrix for a seller experiment is

\[
W = \begin{pmatrix}
C & T & C & C & C & C & T \\
C & T & C & C & C & C & T \\
C & T & C & C & C & C & T \\
C & T & C & C & C & C & T \\
\end{pmatrix}
\]

Seller Experiment.

Notice that in the example above, \( \forall^B = \{1/4\} \) and \( \forall^S = \{0, 1\} \), which reflects that every buyer has an inconsistent experience, and every seller has a consistent experience.

An example of an assignment matrix that could arise in a buyer experiment is

\[
W = \begin{pmatrix}
C & C & C & C & C & C & C \\
T & T & T & T & T & T & T \\
C & C & C & C & C & C & C \\
C & C & C & C & C & C & C \\
\end{pmatrix}
\]

Buyer Experiment.

Here the second buyer is in the treatment group, and the first, third and fourth buyer are in the control group. In this case \( \forall^B = \{0, 1\} \) and \( \forall^S = \{1/4\} \). Every buyer has a consistent experience, and every seller has an inconsistent experience. An assignment mechanism for this case would be

\[
p(W) = \begin{cases} 
\frac{1}{4} & \text{if } \bar{w}^S_j = \frac{1}{4}, \forall j \in J \text{ and } \bar{w}^B_j \in \{0, 1\} \forall i \in I, \\
0 & \text{otherwise.}
\end{cases}
\]
If an experiment is both a seller experiment and a buyer experiment, it must be the case that either \( \mathbb{V}^S = \mathbb{V}^B = \{0\} \), or \( \mathbb{V}^S = \mathbb{V}^B = \{1\} \), and neither case is of much interest, and is ruled out by our definition of a marketplace experiment.

In this paper we are most interested in experiments that are neither buyer nor seller experiments, in the sense that neither group has a consistent experience.

**Definition 5** (General Marketplace Design (GMD)). A General Marketplace Design is a marketplace experiment such that both \( \mathbb{V}^B \) and \( \mathbb{V}^S \) contain elements different from 0 and 1.

By definition, GMDs do not offer consistent experiences to all buyers and do not offer a consistent experience to all sellers. Consider, for example, a setting in which we are confident that there is no interference between sellers for a given buyer, but for a given seller there is unrestricted interference among outcomes for interactions corresponding to different buyers. In this case, GMDs would be less appealing than a classic seller experiment, as they would fail to provide us with unbiased estimates of the average causal effect of some policy for sellers. However, as we discuss next, if we are willing to make some assumptions on the extent to which interference across buyers and sellers is present, then MRDs have the ability to deliver much richer information regarding the presence of interference both within seller and buyer interactions.

The class of GMDs covers many special cases of interests. The first, simplest, design, is one in which we randomize over all interactions \((i,j)\).

**Definition 6** (Completely Randomized Marketplace Design (CRMD)). A Completely Randomized Marketplace Design is a GMD such that

\[
\mathbb{W}^+_p = \bigg\{ \mathbf{W} \in \mathbb{W} \biggm| \forall i \in I, \forall j \in J, \quad w^B_i = w^S_j = \frac{1}{4} \bigg\},
\]

and \( \mathbb{V}^B = \mathbb{V}^S = \{\frac{1}{4}\} \).

In a CRMD the set of buyer/seller pairs that are treated are selected at random from the full set of pairs, subject to the restriction that each buyer and each seller have the same fraction of treated pairs. In the case with four buyer and eight sellers, an example of such an assignment is:

\[
\mathbf{W} = \begin{pmatrix}
C & T & C & C & T & C & C & C \\
C & C & T & T & C & C & C & C \\
T & C & C & C & T & C & C & C \\
C & C & C & C & C & C & T & T
\end{pmatrix}.
\]

In this case \( \mathbb{V}^S = \mathbb{V}^B = \{1/4\} \). An example of an assignment mechanism for this case is

\[
p(W) = c, \quad \forall W \text{ s.t. } \bar{w}^S_j = \frac{1}{4}, \quad \forall j \in J, \quad \text{and} \quad \bar{w}^B_i = \frac{1}{4}, \quad \forall i \in I,
\]

and 0 otherwise. This design has some similarities to Latin Squares and factorial designs [Fisher, 1937, Hinkelman et al., 1996, Keedwell and Dénes, 2015]. However, in that literature the focus is on efficiency of the design, not on the presence or absence of interference. Latin Squares and factorial designs are attractive if the researcher is confident that there are no spillovers of any type.
In that case randomizing over the interactions rather than the buyers or sellers generally improves precision. However, if part of the goal is to assess the extent of spillovers, these designs are not attractive.

Here both buyers and sellers are randomized separately. First each buyer $i \in I$ is randomly assigned an integer $W^B_i$ in a set $W^B = \{0,1,\ldots,N^B\}$, possibly just with two values, $W^B = \{0,1\}$. Second, independently of the assignments to the buyer, each seller is randomly assigned an integer $W^S_j$ in a set $W^S = \{0,1,\ldots,N^S\}$. We denote by $I^n_B$ the number of buyer assigned $W^B_i = n$, and similarly by $J^n_S$ the number of sellers assigned $W^S_j = n$. Then the assignment for the pair $(i,j)$ is a function of the buyer and seller assignment.

**Definition 7** (Multiple Randomization Designs (MRD)). A Multiple Randomization Design is a GMD in which, for some $f : \mathbb{W}^B \times \mathbb{W}^S \to \{C,T\}$,

$$W_{ij} = f(W^B_i, W^S_j),$$

with $\mathbb{W}^B$ and $\mathbb{W}^S$ (with typical elements $W^B_i$ and $W^S_j$) independent random vectors.

A special case of an MRD with only two types of buyer and sellers is what we refer to as a Simple Multiple Randomization Design (SMRD). Such designs do not have the richness of the full class of MRDs, but they contain many of the insights into the general case. This special case of MRDs has also been independently proposed in Johari et al. [2020], where the focus is on the bias of the difference in means estimator for average treatment effect.

**Definition 8** (Simple Multiple Randomization Design (SMRD)). A Simple Multiple Randomization Design is a MRD where $W^B_i \in \{0,1\}$, $W^S_j \in \{0,1\}$, and either (conjunctive)

$$W_{ij} = \begin{cases} C & \text{if } W^B_i = 0 \text{ or } W^S_j = 0 \\ T & \text{if } W^B_i = 1 \text{ and } W^S_j = 1, \end{cases}$$

or (disjunctive)

$$W_{ij} = \begin{cases} C & \text{if } W^B_i = 0 \text{ and } W^S_j = 0 \\ T & \text{if } W^B_i = 1 \text{ or } W^S_j = 1. \end{cases}$$

In a conjunctive SMRD an interaction $(i,j)$ is treated if and only if both the buyer $i$ and the seller $j$ are assigned the value 1, while in a disjunctive SMRD an $(i,j)$ is treated if either $i$ or $j$ take the value 1. In a conjunctive SMRD we have $\mathbb{V}^B = \{0,p^B\}$ and $\mathbb{V}^S = \{0,p^S\}$, for some $p^B$ and $p^S$, and the assignments may look like

$$W = \begin{pmatrix} C & C & C & C & C & C & C \\ C & C & C & C & C & C & C \\ C & C & C & C & T & T & T \\ C & C & C & C & T & T & T \end{pmatrix}.$$  

In the second case of a disjunctive SMRD $\mathbb{V}^B = \{p^B,1\}$ and $\mathbb{V}^S = \{p^S,1\}$, for some $p^B$ and $p^S$, and
the assignments may look like
\[
W = \begin{pmatrix}
C & C & C & T & T & T & T \\
C & C & C & T & T & T & T \\
T & T & T & T & T & T & T \\
T & T & T & T & T & T & T \\
\end{pmatrix}.
\]

A buyer experiment can be viewed as a special case of a disjunctive SMRD where \( W^B_j = 1 \) for all sellers, and similarly a seller experiment corresponds to the special case of a disjunctive SMRD where \( W^B_i = 1 \) for all buyers.

Note that a conjunctive SMRD may be converted into a disjunctive SMRD by exchanging 0 and 1 for \( W^M_i \) and \( W^V_j \), and T and C. Consequently, without loss of generality we focus on conjunctive SMRDs. We refer to a buyer with \( W^B_i = 1 \) as a selected buyer, and the buyer with \( W^B_i = 0 \) as a non-selected buyer, and similarly for the sellers. We fix the number of selected buyers at \( I_T \) and the number of non-selected buyers at \( I_C = I - I_T \). Similarly, we fix the number of selected sellers at \( J_T \) and the number of non-selected buyers at \( J_C = J - J_T \).

Define \( \mathcal{W}(I_T, J_T) \) to be the subset of the set of all possible assignment matrices \( \mathcal{W} \) with assignment matrices that correspond to a SMRD of this type:
\[
\mathcal{W}(I_T, J_T) = \left\{ W \in \mathcal{W} \middle| w^S_j \in \left\{ 0, \frac{I_T}{T} \right\}, w^S_j \in \left\{ 0, \frac{J_T}{J} \right\} \right\}.
\]

Let \( p \) be the marketplace experimental design that uniformly weights assignments \( W \in \mathcal{W}(I_T, J_T) \):
\[
p(W) = \begin{cases} 
(i_T)(J_T) & \text{if } W \in \mathcal{W}(I_T, J_T) \\
0 & \text{else.}
\end{cases}
\]

In a SMRD, the pair of binary values \((W^B_i, W^S_j)\) defines four assignment types of buyer/seller pairs/interactions:
\[
T_{ij} = \begin{cases}
\text{c} & \text{if } W^B_i = 0, W^S_j = 0 \text{ (and } W_{ij} = 0) \\
\text{ib} & \text{if } W^B_i = 1, W^S_j = 0 \text{ (and } W_{ij} = 0) \\
\text{is} & \text{if } W^B_i = 0, W^S_j = 0 \text{ (and } W_{ij} = 0) \\
\text{t} & \text{if } W^B_i = 1, W^S_j = 1 \text{ (and } W_{ij} = 1). 
\end{cases}
\]

Here, \( \text{c} \) stays for “consistent control”, \( \text{ib} \) for “inconsistent buyer control”, \( \text{is} \) for “inconsistent seller control”, and \( \text{t} \) for “treated”. Note that \( W_{ij} = T \) if \( T_{ij} = t \) and \( W_{ij} = C \) otherwise. Under a SMRD we have assignment type matrices of the form
\[
W = \begin{pmatrix}
C & C & C & C & C & C & C \\
C & C & C & C & C & C & C \\
C & C & C & C & T & T & T \\
C & C & C & C & T & T & T \\
\end{pmatrix},
\]
and

\[ T = \begin{pmatrix}
  c & c & c & c & is & is & is & is \\
  c & c & c & c & is & is & is & is \\
  ib & ib & ib & ib & t & t & t & t \\
  ib & ib & ib & ib & t & t & t & t \\
\end{pmatrix}. \]

These assignment types play an important role in our designs. \textit{Ex ante} the four assignment types are comparable because of the randomization, implying that their potential outcome distributions are the same. Of the four types of buyer/seller pairs there is one (type t) that receives the treatment. The other three \((c, ib, is)\) are all exposed to the control treatment. Under a buyer or seller experiment there would only be two types, with only one exposed to the control treatment. Having multiple sets of \textit{ex ante} comparable groups, three in this SMRD case, all exposed to the control treatment, is what gives the SMRD design the ability to detect the presence of different types of interference.

Though multiple control groups have been considered previously by Rosenbaum [1987], this was in the context of observational studies, where the goal was to provide a means of assessing the extent of unobserved differences between those receiving (or selecting) treatment versus control. In contrast, here the goal is both to relax the assumption of no interference and to allow the measurement of spill-over effects.

5 Interference

In this section, to make our discussion concrete, we introduce different assumptions on the extent of the spillovers between buyer/seller pairs. These assumptions are formulated, for a given pair \((i, j)\) and an assignment matrix \(W\), in terms of restrictions on other assignment matrices \(W'\) that guarantee that the potential outcomes for pair \((i, j)\) and assignments \(W\) and \(W'\) are identical. That is, restrictions on \(W\) and \(W'\) such that

\[ Y_{ij}(W) = Y_{ij}(W'). \]

Consider buyer \(i\) and seller \(j\): interference corresponds to differences between \(Y_{ij}(W)\) and \(Y_{ij}(W')\) for assignments \(W\) and \(W'\) where the treatment for the pair \((i, j)\) is identical, \(w_{ij} = w'_{ij}\), but some other elements of the assignment matrices \(W\) and \(W'\) differ. Without any further assumption, there are for each buyer/seller pair as many potential outcomes as there are elements in \(W\), making inference intractable. In order to be able to do inference, we need to restrict the extent of the interference. The first assumption we consider — the strongest — rules out \textit{any} type of interference.

\textbf{Assumption 5.1 (Strong No-Interference).} The potential outcomes satisfy the strong no-interference assumption if, for any pair \((i, j)\), and any \(W, W'\) such that \(w_{ij} = w'_{ij}\)

\[ Y_{ij}(W) = Y_{ij}(W'). \]

Under Assumption 5.1, a natural approach would be to do a CRMD where we randomize all pairs. This would generally be more efficient than using a buyer or seller experimental design.

A weaker assumption allows for interference between the pairs involving the same seller but different buyers or the other way around. Let \(W, W'\) be assignments such that the treatment for

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the pair \((i, j)\) coincides, \(w_{ij} = w'_{ij}\), but there is a buyer, \(i'\), for which the treatments to which seller \(j\) is exposed differ: \(w'_{i'j} \neq w'_{ij}\). Then, under this type of interference, it may be the case that \(Y_{ij}(W) \neq Y_{ij}(W')\). However, for any assignment \(W''\) with \(w''_{i'j} = w_{i'j}\) for all \(i' \in \mathbb{I}\), then it follows \(Y_{ij}(W) = Y_{ij}(W'')\) — irrespective of the values \(w''_{ij}'\), for any \(j' \neq j\). We formalize this form of interference below.

**Assumption 5.2 (No-Interference for Sellers).** The potential outcomes satisfy the no-interference for sellers assumption if for any pair \((i, j)\), and any \(W, W'\) such that \(w_{i'j} = w'_{i'j}\) for all \(i'\),

\[
Y_{ij}(W) = Y_{ij}(W').
\]

If Assumption 5.2 holds, then changing one or more of the treatments for another seller \(j'\) does not change the outcomes for seller \(j\). But, changing one or more of the treatments for buyer \(i'\) may change the outcomes for buyer \(i\). If the No-Interference-for-Sellers assumption holds, doing a seller experiment is a natural strategy.

**Assumption 5.3 (No-Interference for Buyers).** The potential outcomes satisfy the no-interference for buyer assumption if, for any pair \((i, j)\), and any \(W, W'\), such that \(w_{ij} = w'_{ij}\) for all \(j'\),

\[
Y_{ij}(W) = Y_{ij}(W').
\]

If Assumption 5.3 holds, then changing one or more of the treatments for another buyer \(i'\) does not change the outcomes for buyer \(i\). But, changing one or more of the treatments for seller \(j'\) may change the outcomes for seller \(j\). If this assumption holds, using a buyer experimental design is a natural strategy.

Further weakening the assumptions we allow for some interference between both sellers and buyers, but restrict it to be local. This is a novel assumption that does not appear to have been considered in the literature.

**Assumption 5.4 (Local Interference).** Potential outcomes satisfy the local interference assumption if, for any pair \((i, j)\), and any \(W, W'\), such that (a) the assignments for the pair \((i, j)\) coincide, i.e. \(w_{ij} = w'_{ij}\), (b) the fraction of treated sellers for the same buyer coincide, i.e. \(w^{B}_i = w'^{B}_i\), and (c) the fraction of treated buyers for the same seller coincide, i.e. \(w^{S}_j = w'^{S}_j\),

\[
Y_{ij}(W) = Y_{ij}(W').
\]

To illustrate the content of this assumption, consider the following assignment matrices in a six seller, four buyer example. Under the assumption of local interference the outcome for buyer/seller pair \((3, 4)\) is identical for the assignment matrices, because (a) the \((3,4)\) elements are identical, and (b) the fourth column of the assignment matrices (given in red) have the same fraction of treated pairs, and (c) the third row of the assignment matrices (also given in red) have the same fraction
of treated pairs:

\[
W = \begin{pmatrix}
C & C & T & C & C & C \\
C & T & C & T & C & T \\
C & T & C & T & T & T \\
T & C & C & C & C & C
\end{pmatrix}, \quad W' = \begin{pmatrix}
T & C & T & T & C & C \\
T & C & T & C & C & C \\
T & C & C & T & T & T \\
T & C & C & C & T & T
\end{pmatrix}.
\]

Thus, under local interference, \( Y_{34}(W) = Y_{34}(W') \). In Section 7, we discuss and present simulation results for a specific parametric choice that fits Assumption 5.4, where we assume that the causal effects can be decomposed into three additive components, \((i)\) the direct effect, \((ii)\) the effect of buyer spillovers, and \((iii)\) the effect of seller spillovers.

6 Estimation and Inference for SMRDs under Local Interference

In this section, we focus on SMRDs introduced in Definition 8. We assume throughout this section the “local interference” scenario described in Assumption 5.4, in which the potential outcomes for interaction \((i,j)\) are indexed by one of four types \(\omega \in \{c, ib, is, t\}\). We study estimation of causal effects under this form of interference, providing finite population results.

Define the population averages by assignment type for \(\omega \in \{c, ib, is, t\}\):

\[
\bar{Y}(\omega) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} Y_{ij}(\omega),
\]

and the average realized outcome over buyer/seller pairs:

\[
\bar{Y}_\omega = \frac{1}{N_\omega} \sum_{i=1}^{I} \sum_{j=1}^{J} Y_{ij}(\omega) \mathbf{1}_{T_{ij}=\omega},
\]

with \(N_\omega = \sum_{i,j} \mathbf{1}_{T_{ij}=\omega}\). Note that \(N_\omega\) is fixed under the experimental designs we consider, and so \(\bar{Y}_\omega\) is stochastic only because the assignment types \(T_{ij}\) are.

**Lemma 1.** Suppose we have a SMRD and Assumption 5.4 holds. Then, for all \(\omega \in \{c, ib, is, t\}\):

\[
E[\bar{Y}_\omega] = \bar{Y}(\omega).
\]

We next define the four spillover effects that we consider in this paper. These are all obtained as linear combinations of the outcomes \(\bar{Y}(\omega)\). First:

\[
\tau_{direct} = \bar{Y}(t) - \bar{Y}(ib) - \bar{Y}(is) + \bar{Y}(c).
\]

This estimator has a difference-in-differences form (e.g., Angrist and Krueger [2000]): we take the difference in average outcomes for buyer and seller pairs who were both assigned to treatment, and subtract the average outcome for buyer who were assigned to the treatment and sellers who were
not. From that difference we subtract the difference between the average for buyer assigned to the
control group and sellers assigned to the treatment group and the average for buyer assigned to the
control group and sellers assigned to the control group.

Second, we define the seller spillovers $\tau^S_{\text{spillover}}$, and buyer spillovers $\tau^B_{\text{spillover}}$:

$$
\tau^S_{\text{spillover}} = \overline{Y}(\text{is}) - \overline{Y}(\text{c}) \quad \text{ and } \quad \tau^B_{\text{spillover}} = \overline{Y}(\text{ib}) - \overline{Y}(\text{c}).
$$

Last, we define the average effect for the treated pairs $\tau$:

$$
\tau(p^B, p^S) = \overline{Y}(\text{t}) - \overline{Y}(\text{c}).
$$

For this last estimator, we write $\tau(p^B, p^S)$ to indicate that this is the average effect on a buyer-
seller pairs $(i, j)$ of both $i$ and $j$ being assigned to treatment versus both being assigned to control
under an SMRD design in which a fraction $p^B \in (0, 1)$ of all the buyers are treated, and a fraction
$p^S \in (0, 1)$ of all the sellers are treated. If $p^B$ and $p^S$ are close to one, this would estimate something
close to the average effect of exposing all pairs to the treatment versus exposing no pair to the
treatment. This is because we are considering a conjunctive SMRD. For a disjunctive SMRD, the
average effect of exposing all pairs versus exposing no pairs would be approximated by taking $p^B$
and $p^S$ are close to zero. In the context of the SMRD, where $p^B, p^S < 1$, we cannot directly estimate
$\tau(1, 1)$ without additional assumptions.

Define the plug-in sample estimates of the spillover effects introduced above:

$$
\hat{\tau}_{\text{direct}} := \overline{Y}_t - \overline{Y}_\text{ib} + \overline{Y}_\text{is} + \overline{Y}_c,
\hat{\tau}(p^B, p^S) := \overline{Y}_t - \overline{Y}_c,
\hat{\tau}^B_{\text{spillover}} := \overline{Y}_\text{ib} - \overline{Y}_c,
\hat{\tau}^S_{\text{spillover}} := \overline{Y}_\text{is} - \overline{Y}_c.
$$

Since all the spillover effects we consider are linear combinations of the estimators $\overline{Y}_\omega$, $\omega \in \{\text{c, ib, is, t}\}$, Lemma 1 together with the linearity of the expectation give us that the plug-in estimates of all the spillover effects we consider, are unbiased. We present this result in Theorem 1.

**Theorem 1.** Suppose we have a SMRD as in Definition 8, and Assumption 5.4 holds. The sample plug-in estimate $\hat{\tau}(p^B, p^S)$ is unbiased for $\tau(p^B, p^S) = \overline{Y}(\text{t}) - \overline{Y}(\text{c})$:

$$
\mathbb{E} [\hat{\tau}(p^B, p^S)] = \mathbb{E} [\overline{Y}_t - \overline{Y}_c] = \tau(p^B, p^S).
$$

Corresponding results hold true for the sample plug-in estimates of the other spillover effects.
Namely, $\hat{\tau}_{\text{direct}}$ is unbiased for $\tau_{\text{direct}}$, $\hat{\tau}^B_{\text{spillover}}$ is unbiased for $\tau^B_{\text{spillover}}$, and $\hat{\tau}^S_{\text{spillover}}$ is unbiased
for $\tau^S_{\text{spillover}}$.

A remark worth making, is that $\hat{\tau}(p^B, p^S)$ is not unbiased for $\tau = \tau(1, 1)$, the average effect of exposing all buyer-seller pairs to the treatment versus no-one. Although we can estimate the average outcome when no pair is exposed to the treatment, with the data from a SMRD we cannot estimate the average outcome when every pair is exposed. Doing so would require two additional assumptions. One, we would need a richer experiment where we vary the share of other buyers or other sellers exposed, and second, we would need a functional form that allows us to extrapolate to full exposure. We discuss such richer designs in Section 8.1.
We now characterize the variances of the estimators of the spillover effects. Towards this goal, we introduce some additional notation. Define the average outcome for each buyer and each seller, for a given type $\omega$

$$\bar{Y}_i^B(\omega) := \frac{Y_i^B(\omega)}{J} = \frac{1}{J} \sum_{j=1}^{J} Y_{ij}(\omega), \quad \text{and} \quad \bar{Y}_j^S(\omega) := \frac{Y_j^S(\omega)}{I} = \frac{1}{I} \sum_{i=1}^{I} Y_{ij}(\omega).$$

Next, define the deviations from population averages for the four assignment types $\omega \in \{c, ib, is, t\}$ for buyers, sellers, and interactions:

$$\hat{Y}_i^B(\omega) := \bar{Y}_i^B(\omega) - \bar{Y}(\omega), \quad \hat{Y}_j^S(\omega) := \bar{Y}_j^S(\omega) - \bar{Y}(\omega), \quad \hat{Y}_{ij}(\omega) := Y_{ij}(\omega) - \bar{Y}_i^B(\omega) - \bar{Y}_j^S(\omega) + \bar{Y}(\omega).$$

Let the corresponding population variances be defined as follows:

$$\Sigma_{\omega}^{2,B} := \frac{1}{I-1} \sum_{i=1}^{I} \left( \hat{Y}_i^B(\omega) \right)^2, \quad \Sigma_{\omega}^{2,S} := \frac{1}{J-1} \sum_{j=1}^{J} \left( \hat{Y}_j^S(\omega) \right)^2,$$

and a similar quantity for $\hat{Y}_{ij}(\omega)$ as

$$\Sigma_{\omega}^{2,BS} := \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \hat{Y}_{ij}(\omega) \right)^2.$$ 

Finally, define the variances of the differences, for all $\omega, \omega' \in \{c, ib, is, t\}, \omega \neq \omega'$, for the buyers

$$\Sigma_{\omega,\omega'}^{2,B} := \frac{1}{I-1} \sum_{i=1}^{I} \left( \hat{Y}_i^B(\omega) - \hat{Y}_i^B(\omega') \right)^2,$$

for the sellers,

$$\Sigma_{\omega,\omega'}^{2,S} := \frac{1}{J-1} \sum_{j=1}^{J} \left( \hat{Y}_j^S(\omega) - \hat{Y}_j^S(\omega') \right)^2,$$

and for the interactions

$$\Sigma_{\omega,\omega'}^{2,BS} := \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \hat{Y}_{ij}(\omega) - \hat{Y}_{ij}(\omega') \right)^2.$$

With this notation in place, we can characterize the variance $\text{V}(\bar{Y}_\omega)$ and covariances $\text{C}(\bar{Y}_\omega, \bar{Y}_{\omega'})$ of the sample averages $\bar{Y}_\omega$, for $\omega, \omega' \in \{c, ib, is, t\}$. As a consequence of the linear representation of the estimators of the spillover effects, we can obtain their variance using the quantities above. We present this exact finite sample result, similar to corresponding results for standard experiments (e.g., Neyman [1923/1990], Imbens and Rubin [2015]), in Theorem 2.

**Theorem 2.** Suppose we have a SMRD as in Definition 8, with $I \times J$ total units, $I > I_T \geq 1, J > \ldots$
\[ J_T \geq 1, \text{ and for which Assumption 5.4 holds. Then:} \]

\[
\mathbb{V}(\hat{\tau}(p^B, p^S)) = \frac{1}{I_T} \Sigma_t^{2, B} + \frac{1}{J_T} \Sigma_t^{2, S} + \frac{I_C J_C + I_T J_T}{I_T I_T J_T} \Sigma_t^{2, BS} \\
+ \frac{1}{I_C} \Sigma_c^{2, B} + \frac{1}{J_C} \Sigma_c^{2, S} + \frac{I_C J_C}{I_C I_C J_C} \Sigma_c^{2, BS} \\
- \frac{1}{I_T} \Sigma_{t,c}^2 - \frac{1}{J_T} \Sigma_{t,c}^2 - \frac{1}{I_J} \Sigma_{t,c}^2 .
\]

Similar characterizations hold true for the variances of the other spillover effects. Details and expressions are deferred to the Appendix.

For the lift we cannot get exact variances, but we can approximate the variance using an expansion:

**Lemma 2.**

\[
\mathbb{V}\left(\frac{\bar{Y}_t - \bar{Y}_c}{\bar{Y}_c}\right) \approx \frac{\bar{Y}(c)^2 \mathbb{V}(\bar{Y}_t) + \bar{Y}(t)^2 \mathbb{V}(\bar{Y}_c) - 2\bar{Y}(c)(\bar{Y}(t)C(\bar{Y}_c, \bar{Y}_t))}{\bar{Y}(c)^4}.
\]

The full expressions for the components of this variance are presented in the Appendix.

### 6.1 Estimates for the variance of the spillover effects

We next present results in which we show how it is (a) possible to obtain unbiased estimates for the variance of the (sample) average potential outcomes \( \bar{Y}_\omega \) for each of the four types \( \omega \in \{c, ib, is, t\} \), and (b) lower and upper bounds on the variance of the spillover effects \( \hat{\tau} \), as well as \( \hat{\tau}^{\text{direct}}, \hat{\tau}^{\text{spillover}, B}, \hat{\tau}^{\text{spillover}, S} \). Our results are the direct counterpart of “classic” characterizations that can be obtained in the context of simple randomized experiments — in which the goal is to infer causal effects for a single population.

**Lemma 3.** Consider a SMRD as in Definition 8, with \( I \times J \) total units, \( I > I_T \geq 2, J > J_T \geq 2 \), and for which Assumption 5.4 holds. For all \( \omega \in \{c, ib, is, t\} \), there exists a linear estimator \( \hat{\Sigma}_\omega \) such that

\[
\mathbb{E}\left[\hat{\Sigma}_\omega\right] = \mathbb{V}(\bar{Y}_\omega).
\]

The expression of the estimator requires additional notation: details are deferred to the Appendix.

Next, we show that, analogous to the case of simple single-population randomization [Neyman, 1923/1990], direct application of the Cauchy-Schwarz inequality provides us with lower and upper bounds on the variance of the causal estimates of interest.

**Lemma 4.**

\[
\hat{\mathbb{V}}^{lo}(\hat{\tau}^{\text{spillover}}) = \hat{\Sigma}_{ib} + \hat{\Sigma}_c - \sqrt{\hat{\Sigma}_{ib} \hat{\Sigma}_c} \leq \mathbb{V}(\hat{\tau}^{\text{spillover}}),
\]
and

\[ \hat{\psi}^{hi}(\hat{\tau}^\text{B}_{\text{spillover}}) = \hat{\Sigma}_{ib} + \hat{\Sigma}_{c} + \sqrt{\hat{\Sigma}_{ib}\hat{\Sigma}_{c}} \geq \sqrt{V(\hat{\tau}^\text{B}_{\text{spillover}})}, \]

where \( \hat{\Sigma}_\omega \) was defined in Lemma 3. The expressions for the estimates of the variances of other spillover effects \( \hat{\tau}, \hat{\tau}_\text{direct}, \hat{\tau}^\text{S}_{\text{spillover}} \) can be derived analogously and are deferred to the Appendix.

### 7 An example of local interference: Additive Local Interference

For the sake of concreteness, we make a parametric assumption on the form of the local interference (if any).

**Assumption 7.1** (Additive Local Interference). There are functions \( h_{ij} : \{C,T\} \rightarrow \mathbb{R}, h^B_{ij} : [0,1] \rightarrow \mathbb{R} \) and \( h^S_{ij} : [0,1] \rightarrow \mathbb{R} \) such that for all \( i \in I, j \in J, \) and \( W \in \mathbb{W}, \)

\[
Y_{ij}(W) = h_{ij}(w_{ij}) + h^B_{ij}(w^B_i) + h^S_{ij}(w^S_j). \tag{7.1}
\]

Without loss of generality, in what follows we normalize so that \( h^B_{ij}(0) = h^S_{ij}(0) = 0. \)

Notice that, under Assumption 7.1, in a SMRD the potential outcome \( Y_{ij} \) is a function of the type \( \omega \in \{c,ib,is,t\} \). Indeed, the outcome for interaction \((i,j)\) is determined by the values of the triple \((w_{ij}, w^B_i, w^S_j)\), which are in a one-to-one correspondence with the types \( \omega \in \{c,ib,is,t\} \).

Hence, we write

\[
Y_{ij}(\omega) = \begin{cases} 
 h_{ij}(C) & \text{if } \omega = c, \\
 h_{ij}(C) + h^B_{ij}(p^B) & \text{if } \omega = is, \\
 h_{ij}(C) + h^S_{ij}(p^S) & \text{if } \omega = ib, \\
 h_{ij}(T) + h^B_{ij}(p^B) + h^S_{ij}(p^S) & \text{if } \omega = t. 
\end{cases}
\]

Under this assumption, the average population outcomes for the different types are given by:

\[
\bar{\bar{Y}}(c) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}(C), \\
\bar{\bar{Y}}(is) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}(C) + \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h^B_{ij}(p^B), \\
\bar{\bar{Y}}(ib) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}(C) + \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h^S_{ij}(p^S), \\
\bar{\bar{Y}}(t) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}(T) + \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h^B_{ij}(p^B) + \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h^S_{ij}(p^S). 
\]
We can characterize the lift as
\[ \theta = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij}(T) - Y_{ij}(C)) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( h_{ij}(T) - h_{ij}(C) + h_{ij}^B(1) + h_{ij}^S(1) \right). \]
Moreover, we can characterize the interference effects as follows
\[ \tau_{\text{direct}} = \overline{Y}(t) - \overline{Y}(ib) - \overline{Y}(is) + \overline{Y}(c) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \{ h_{ij}(T) - h_{ij}(C) \}, \]
\[ \tau_{\text{spillover}}^{B} = \overline{Y}(ib) - \overline{Y}(c) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}^B(p^B), \]
\[ \tau_{\text{spillover}}^{S} = \overline{Y}(is) - \overline{Y}(c) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} h_{ij}^S(p^S), \]
\[ \tau(p^B, p^S) = \overline{Y}(t) - \overline{Y}(c) = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} (h_{ij}(T) - h_{ij}(C)) + (h_{ij}^S(p^S) + h_{ij}^B(p^B)). \]

### 7.1 Simulations

We here show simulation results for the SMRD under additive local interference introduced in Assumption 7.1. Incidentally, this allows us to verify the correctness of the formulae provided in Section 6 on synthetic data.

For fixed \( I_T \leq I, J_T \leq J \), let \( P_T \) denote the distribution over the matrix of types \( T \) induced by sampling \( W \) from a SMRD \( \mathbb{W}(I_T, J_T) \) as per Equation (4.1). We generate a dataset as follows:

\[ T \sim P_T(\cdot) \]
\[ Y_{ij} \mid T \sim F_{T_{ij}}(\cdot), \quad \text{independently across } i, j, \]

where
\[ F_{T_{ij}}(\cdot) = F_{1(T_{ij}=t)}(\cdot) + F_B(\cdot)1(T_{ij} \in \{ib, t\}) + F_S(\cdot)1(T_{ij} \in \{is, t\}). \]

I.e.,
\[ Y_{ij} \mid T_{ij} \sim \left\{ \begin{array}{ll}
F_0(\cdot) & \text{if } \omega = c, \\
F_0(\cdot) + F_B(\cdot) & \text{if } \omega = ib, \\
F_0(\cdot) + F_S(\cdot) & \text{if } \omega = is, \\
F_1(\cdot) + F_B(\cdot) + F_S(\cdot) & \text{if } \omega = t,
\end{array} \right. \]

where \( F_\ell \) are distributions, \( \ell \in \{0, B, S, 1\} \). In our simulations, we assume \( F_0, F_1, F_B, F_S \) are the laws of Gaussian random variables, in which, for \( p^0 = 1, p^1 = 1, \) and \( p^B \) and \( p^S \) the proportions of
buyers and sellers in the MRD under consideration,

\[ F_{\ell}(\cdot) = \mathcal{N}(p^T \mu_{\ell}, \sigma_{\ell}^2). \]  

(7.4)

Under the data generating assumption of Equation (7.2) then, for every parametric choice made on \( F_0, F_1, F_B, F_S \), and for every interaction \( (i, j) \in \{1, \ldots, I\} \times \{1, \ldots, J\} \), there are four potential outcomes, one for each type \( \omega \in \{c, ib, is, t\} \). We conduct an experiment — i.e. conditionally on a single assignment matrix \( \mathbf{W} \) — we however only get to observe one of the four potential outcomes for every interaction \( Y_{i,j} \).

**Simple draws from Equation (7.3) and SMRDs**

First, we show in Figure 1 how the parametrization of Equation (7.3) is sufficiently flexible to capture different distributions of the data for appropriate configurations of parameters governing the mean and standard deviation of each type.

**Checking unbiasedness and variance formulae**

Next, we verify empirically the correctness of Lemma 1, namely that \( \overline{Y}_{\omega} \) is an unbiased estimate of \( \overline{Y}(\omega) \). To do so, we appeal to the parametric formulation in Equation (7.3): for every type \( \omega \in \{c, ib, is, t\} \) we draw a matrix \( Y(\omega) = [Y_{ij}(\omega)] \) of \( I \times J \) conditionally i.i.d. draws, all from the same distribution (associated to the corresponding type \( s \) as per Equation (7.3)). We then generate a large number \( N_{MC} \) of draws from the SMRD \( \mathcal{W}(I_T, J_T) \), and create the empirical (sampling) distribution for the estimates \( \overline{Y}_{\omega} \). We compare the average of \( \overline{Y}_{\omega} \) across the \( N_{MC} \) replicas to the true population value \( \overline{Y}(\omega) \). We also compare the empirical \( \alpha = 2.5\% \) and \( \alpha = 97.5\% \) quantiles of \( \overline{Y}_{\omega} \) to the corresponding true value, obtained by noting that \( \overline{Y}_{\omega} \) is Gaussian distributed under Equation (7.3). We show in Figure 2 results for the Gaussian additive local case, where we set \( I = 100, J = 110, p^B = p^S = 0.5 \) and the parameters \( \mu_c = 0, \sigma_c = 2, \mu_{ib} = 1, \sigma_{ib} = .5, \mu_{is} = 3, \sigma_{is} = .2, \mu_t = 4, \sigma_t = 8 \).

We also check that we can obtain unbiased estimates also for the spillover effects – that are simply linear combinations of such average effects (see Figure 3).

Next, we move to the analysis of the sample variance. In Figure 4 we show that, consistently with what is stated in Lemma 3, there exists an unbiased unbiased estimator for the variances of such average effects. Having unbiased estimates for the variance of the average effects is crucial, as it allows to obtain a confidence statement about the observed effect.

Moreover, having unbiased estimates for the variances of the average potential outcomes allows us to derive upper and lower bounds for the spillover effects detected in the sample. This is a simple application of Cauchy-Schwarz, obtained as described in Lemma 4.
Figure 1: Distribution of potential outcomes under Equation (7.3). We fix $I = 200$, $J = 100$, and $I_T = 150$, $J_T = 50$, so that $p^B = 3/4$, and $p^S = 1/2$. Along the columns, we vary the specification of the parameters $\mu_\omega, \sigma_\omega$. Left column: means $\mu_t = 1, p^S \mu_{is} = -1, p^B \mu_{ib} = 2, \mu_t = 6$, and standard deviations $\sigma_t = 1, \sigma_{is} = 1, \sigma_{ib} = 0.5, \sigma_t = 0.5$. Right column: same parameters as the left column, except for $\mu_{ib} = 2$. Each subplot in the first row displays the population distribution of the four types $\{c, ib, is, t\}$ (i.e., for each type, all the $I \times J$ interactions in the population). In the second row, each subplot displays the observed interactions from a single draw from a SMRD $\mathbb{W}(I_T, J_T)$.
Figure 2: Unbiasedness of average effects (Lemma 1) for data generated using Equation (7.3). For \( \omega \in \{c, ib, is, t\} \), we compare the empirical distribution of \( \bar{Y}_\omega \) over \( N_{MC} = 5000 \) draws from \( \mathcal{W}(I_T, J_T) \) to the true distribution of \( \bar{Y}_\omega \). Vertical solid lines are plotted in correspondence of the empirical average of \( \bar{Y}_\omega \), as well as the empirical 2.5%, and 97.5% quantiles. Orange dotted vertical lines are plotted in correspondence of the true population mean \( \bar{Y}(\omega) \), as well as the \( \alpha = 2.5\% \) and \( \alpha = 97.5\% \) quantiles of \( \bar{Y}_\omega \) over the \( N_{MC} \) draws, where we denote by \( z_\alpha(\bar{Y}_\omega) = t_\alpha \times \sqrt{\operatorname{Var}(\bar{Y}_\omega)} \) to be the \( \alpha\% \) quantile of the estimate \( \bar{Y}_\omega \) of the average effects. Here, \( \alpha = 97.5\% \) and the closed-form distribution of \( \bar{Y}_\omega \) follows from the Gaussian assumption of Equation (7.4).

Figure 3: Unbiasedness of spillover effects (Theorem 1). Synthetic data generated using the Gaussian ALI assumption of Equation (7.4). The same quantities as in Figure 2 are plotted, now for the spillover effects (left to right, \( \hat{\tau}_{\text{direct}}, \hat{\tau}_{\text{spillover}}, \hat{\tau}_{\text{spillover}}, \hat{\tau} \)).
Figure 4: Unbiased estimate for the variance of the average effects (Lemma 3). Synthetic data generated using the Gaussian ALI assumption of Equation (7.3).

Figure 5: Lower and upper bounds on the variance of the estimated spillover effects (Lemma 4). Synthetic data generated using the Gaussian ALI assumption of Equation (7.3). In this case, for $\tau_{\text{direct}}, \tau^B_{\text{spillover}}, \tau^S_{\text{spillover}}$, the lower bounds on the corresponding variance terms are vacuous (negative), and omitted.
Scaling with the sample size

As already discussed, in the same way in which in a classic randomized experiment it is only possible to get upper and lower bounds on the variance of the average treatment effect, also in the double randomized setting unbiased estimates of the spillover effects introduced in Section 6 are not available. However, just like for the single population setting, the lower and upper bounds on the variance estimates of the spillover effects provided in Lemma 4 tend to 0 when the sample size diverges. We here investigate the relationship between the bounds provided and the sample size of the experiment.

For fixed Gaussian kernels with location and scale parameters $\mu_c = 3, \sigma_c = 7, \mu_{ib} = -6, \sigma_{ib} = 1, \mu_{is} = 0.4, \sigma_{is} = 2, \mu_t = 10, \sigma_t = 4$, and sample sizes,

$$(I, J) \in \{(60, 80), (105, 135), (155, 190), (200, 245), (250, 300)\},$$

and randomization proportions $I_T / I = 0.2, J_T / J = 0.4$, we show in Figures 6 and 7 how the estimates of the variances of the average effects, as well as the upper and lower bound on the spillover effects (y-axis, in log scale) scale as the sample size increases (x-axis).
8 Generalizations and Extensions

In this section, we discuss four additional designs that highlight the richness of the MRDs. This is not an exhaustive set, but underlines that one one allows for assignment at the level of the buyer/seller pair, there are many complex designs that can answer questions about particular types of spillovers.

8.1 General Multiple Randomization Designs

In the simple multiple randomized experiments we allowed for two types of buyers and two types of sellers. This can be generalized. For example, we could have an experiment that randomly assigns buyers to $WB = \{0, 1\}$, and sellers to $WS = \{0, 1, 2\}$, with the assignment for the interaction $(i, j)$ determined as:

$$W_{ij} = \begin{cases} T & \text{if } W^S_i + W^S_j \geq 2, \\ C & \text{otherwise.} \end{cases}$$

In a three buyers, four sellers case, with $W^B_i = 0$ for $i = 1$, $W^B_i = 1$ for $i = 2$, and $W^B_i = 2$ for $i = 3$, and $W^S_j = 0$ for $j = 1, 2$, $W^S_j = 1$ for $j = 3$ and $W^S_j = 2$ for $j = 4$, this would lead to the assignment matrix

$$W = \begin{pmatrix} C & C & C & T \\ C & C & T & T \\ C & T & T & T \end{pmatrix}.$$ 

Such experiments generate a number of types equal to the product of the cardinality of the sets $WB$ and $WS$. This type of design would allow us to estimate values of $\tau(p^B, p^S)$ for more pairs of values $(p^B, p^S)$ which would make the extrapolation to $\tau(1, 1)$ more credible.

8.2 Equilibrium Designs

We next discuss an additional class of MRDs that can shed light on direct and indirect (equilibrium) effects of treatments. As an illustration, suppose that we are worried that both direct effects of the treatment on buyer/seller pair $(i, j)$ from the treatment $W_{ij}$, as well as indirect effects from treatments $W_{km}$ for other pairs $(k, m)$ with either $k = i$ or $m = j$ are potentially present. Concretely, let us continue to think of the treatment as an informational one, that can be applied at the buyer/seller level. In addition, let us think of each seller maker making a separate promotion decision that is applied to every buyer for that seller. This decision takes into account the demand for that seller given the treatment assignment for all buyer/seller pairs. As such, it is an equilibrium decision. As a result, the decision on the promotion for each seller may depend on the fraction of buyers in the treatment group. Suppose the treatment has a direct positive effect on the demand for a particular seller. Then, the equilibrium promotion may be different if the fraction of treated buyers for a particular seller is high. Suppose buyers make their decisions to buy from a seller influenced both by the promotion and the treatment status. Let us denote the buying decision by buyer $i$ for seller $j$, given promotion $p$ and given the treatment $w \in \{C, T\}$ as $Y_{ij}(w; p)$. Underlying this structure is the assumption that the treatments for other buyer/seller pairs only affect the buying decision for buyer $i$ from seller $j$ through the promotion decision for seller $j$. 

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The promotion decision by the decision maker for buyer \( i \) is \( P_i(W_i^B) \), where \( W_i^B = \sum_j W_{ij}/J \) is the fraction of treated sellers for buyer \( i \), with the promotion depending on the fraction of sellers exposed to the treatment for that buyer. Of particular interest are \( P^T_i = P_i(1) \), the promotion decision if (almost) all the sellers for buyer \( i \) are exposed to the treatment, and \( P^C_i = P_i(0) \), the promotion decision if no or few sellers for buyer \( i \) are exposed to the treatment.

Consider the following experimental design that will allow us to assess direct seller and buyer responses to the treatment, and to disentangle these direct effect of the sellers from the indirect (equilibrium) effects through the effect of the treatment on promotions under the local interference assumption. First we split the sellers randomly into two groups, \( B \) and \( S \), with fractions \( p^B_S \) and \( p^S_S = 1 - p^B_S \). Let \( X_j^S \in \{B, S\} \) denote the binary assignment for sellers to these two groups. For the first group of sellers (sellers with assignment \( X_j^S = B \)), we run a buyer experiment where we randomly assign the buyers to the treatment or control group, with the sellers in this group all having the same inconsistent experience. Let \( W_{ij}^B \in \{C, T\} \) be the assignment for the buyers in this experiment. For the second group of sellers (sellers with assignment \( X_j^S = S \)) we run a seller experiment, with assignment \( W_{ij}^S \in \{C, T\} \). Note that the value of \( W_{ij}^S \) does not matter for sellers with \( X_j^S = B \). An example of the assignment matrix under such a design is given in 1.2.

The assignment matrix in 1.2 is an example with 8 sellers (the columns) and 5 buyers (the rows). The fraction of sellers for whom we run a buyer experiment is \( p^B_S = \text{pr}(X_j^S = B) = 3/8 \).

There are a number of interesting comparisons between sets of seller/buyer pairs we can perform in 1.2 that are informative about the effects of the treatment through different channels. For example, there are three different types of triples \((W_{ij}^B, W_{ij}^S, X_j^S)\) such that the pair \((i,j)\) is in the control group, \( W_{ij} = C \), as highlighted by the different color coding. Comparing outcomes for these types provides a measure of the indirect effects of the treatment.

First, consider the average outcome for pairs of buyers and sellers with \( W_{ij}^B = C, W_{ij}^S = C \) and \( X_j^S = S \) (in the assignment matrix these are denoted by the red \( C \)). Let \( \mathcal{I}_{CC,S} = \{(i,j) : W_{ij}^B = C, W_{ij}^S = C, X_j^S = S\} \), and \( N_{CC,S} = |\mathcal{I}_{CC,S}| \), then

\[
\bar{Y}_{CC,S} = \frac{1}{N_{CC,S}} \sum_{(i,j) \in \mathcal{I}_{CC,S}} Y_{ij}.
\]

This average involves buyers 1, 2, 4. For these buyers the fraction of control sellers is high (6 out of 8), so let we approximate the promotion as \( P^C_i \). Hence, we approximate the average above as

\[
\bar{Y}_{CC,S} = \frac{1}{N_{CC,S}} \sum_{(i,j) \in \mathcal{I}_{CC,S}} Y_{ij}(C; P^C_i).
\]

Similarly, we compute the average outcome for pairs with \( W_{ij}^B = T, W_{ij}^S = C \), and \( X_j^S = S \) (these are denoted in the assignment matrix by the blue \( C \)). Let \( \mathcal{I}_{TC,S} = \{(i,j) : W_{ij}^B = C, W_{ij}^S = C, X_j^S = S\} \), and

\[
\bar{Y}_{TC,S} = \frac{1}{N_{TC,S}} \sum_{(i,j) \in \mathcal{I}_{TC,S}} Y_{ij}.
\]
For the buyers in these pairs, where the fraction of treated sellers is 5 out of 8, we approximate
again the promotion with $P_i^T$, and the average outcome is
\[
Y_{T,C;S} = \frac{1}{N_{T,C;S}} \sum_{(i,j) \in I_{T,C,S}} Y_{ij}(C; P_i^T).
\]

The comparison between the average outcomes for these two sets of pairs of sellers/buyers is in-
formative about the indirect effect of the treatment on purchases through buyer responses on
promotions, that is, it is estimating an average of $Y_{ij}(C; P_i^T) - Y_{ij}(C; P_i^C)$. In contrast, if we
compare $Y_{CC,S}$ with treated units in the seller experiment with the same assignment in the buyer
experiment, letting $I_{C,T;S} = \{(i, j) : W_i^B = C, W_j^S = T, X_j^S = S\}$
\[
Y_{C,T;S} = \frac{1}{N_{C,T;S}} \sum_{(i,j) \in I_{C,T,S}} Y_{ij}(T; P_i^C),
\]
we get an estimate of the direct effect of the treatment on the sellers in an environment with control
promotion decisions, that is, an average of $Y_{ij}(T; P_i^C) - Y_{ij}(C; P_i^C)$.

There are a number of other comparisons between different sets of pairs of buyers and sellers
we could perform here. For example, comparing control units in the seller experiment who would
have been controls in the buyer experiment (in the assignment matrix these are denoted by the red
C). with control units in the buyer experiment (in the assignment matrix these are denoted by the
green C). Letting $I_{C,\bullet;B} = \{(i, j) : W_i^S = C, X_j^S = B\}$
\[
\overline{Y}_{C,\bullet;B} = \frac{1}{N_{C,\bullet;B}} \sum_{(i,j) \in I_{C,\bullet;B}} Y_{ij}
\]
is informative about spillover effects for sellers on purchases for in a control environment from
having been exposed to the treatment for other buyers/items.

8.3 Detection of Synergistic Spill-Over Under Local Interference

In this section, we show that there are types of spillover effects that are detectable only using
double randomization. This is the case when there is a synergistic interaction, where effects are
manifested only for interactions where both the buyer and seller have inconsistent experiences.

Consider a double randomization whereby buyers and sellers are randomized to three levels so
that $W_i^B, W_j^S \in \{-1, 0, 1\}$. Interactions are treated if and only if $W_i^B W_j^S = 1$. Thus interactions
are treated if $W_i^B = W_j^S = -1$ or if $W_i^B = W_j^S = 1$. Further suppose that:
\[
\text{pr}(W_i^B = w) = \begin{cases} 
\pi/2 & \text{if } w = \pm 1, \\
1 - \pi & \text{if } w = 0,
\end{cases} \quad \text{and} \quad \text{pr}(W_j^S = w) = \begin{cases} 
q/2 & \text{if } w = \pm 1, \\
1 - q & \text{if } w = 0.
\end{cases}
\]
We now have five types of interaction:

\[
T_{ij} = \begin{cases} 
    c & \text{if } W^B_i = 0, W^S_j = 0, \text{ and } W_{ij} = 0 \\
    ib & \text{if } W^B_i = \pm 1, W^S_j = 0, \text{ and } W_{ij} = 0 \\
    is & \text{if } W^B_i = 0, W^S_j = \pm 1, \text{ and } W_{ij} = 0 \\
    ibs & \text{if } W^B_i W^S_j = -1, \text{ and } W_{ij} = 0 \\
    t & \text{if } W^B_i W^S_j = 1 \text{ and } W_{ij} = 1,
\end{cases}
\]

where “ibs” denotes “inconsistent buyer and seller control”. The five types \{c, ib, is, ibs, t\} arise in the following proportions:

\[
\Pr(T_{ij} = \omega) = \begin{cases} 
    (1 - \pi)(1 - q) & \text{if } \omega = c \\
    \pi(1 - q) & \text{if } \omega = ib \\
    (1 - \pi)q & \text{if } \omega = is \\
    \pi q/2 & \text{if } \omega = ibs \\
    \pi q/2 & \text{if } \omega = t.
\end{cases}
\]

Notice that under this design, whether \(T_{ij}\) is is or ibs, the same proportion of interactions with other \(W_{ij'}\), \(j' \neq j\), are treated; similarly whether \(T_{ij}\) is ib or ibs, the same proportion of interactions \(W_{i'j}\), \(i' \neq i\), are treated. From the perspective of sellers they are either in control or treatment, if the former then no buyers are treated, if the latter then \((\pi/2) \times 100\%\) of buyers that they see are treated. Similarly, from the perspective of buyers they are either control with no sellers treated, or they are in treatment with \((q/2) \times 100\%\) of sellers treated.

**Definition 9** (Synergistic Spillover). *Synergistic spillover is present if there are interactions that respond in the same way if assigned to c, is, or ib, but differently if assigned to ibs.*

Note that the presence of synergistic spillover is compatible with the assumption that there is only local interference. Since it relates to interactions that only respond to the presence of spillovers both from buyers and sellers it is clearly only detectable in a doubly-randomized experiment.

We now show that it is possible to detect the presence of synergistic spillover without making any assumption beyond local interference. The argument here relates to Bell’s inequality in quantum mechanics; see Robins et al. [2015] for a similar argument.

**Lemma 5.** Suppose that the response \(Y\) is binary, taking values 0 and 1. If

\[
\alpha(Y(ibs)) - \alpha(Y(c)) - \alpha(Y(is)) - \alpha(Y(ib)) > 0
\]

then synergistic spillover is present.

Proof: Suppose there were no synergistic spillover, then for all units \(Y(ibs) - Y(c) - Y(is) - Y(ib) \leq 0\). This implies \(\alpha(Y(ibs)) - \alpha(Y(c)) - \alpha(Y(is)) - \alpha(Y(ib)) \leq 0\).
8.4 Multiple Randomization Clustered Designs

Double randomized approaches are also relevant in settings where we are concerned with spillovers between nearby buyers. They can shed light on the presence and magnitude of such spillovers within a single experiment. Consider a setting where buyers are clustered. In the examples that follow, there are 8 buyer and 10 sellers, with buyers partitioned into 4 clusters of 2 buyers each.

First, consider a simple buyer experiment as illustrated in (8.1), where buyers 1, 2, 4, 6 and 8 are assigned to the treatment group. All buyers have consistent experiences in this experiment, but not all clusters of buyers. Clusters 1 and 2 have all buyers in the same treatment group, but clusters 3 and 4 have both treated and control buyers.

\[
\begin{array}{cccccccccc}
\text{Buyers} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{Clusters} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array}
\]

\[
W = \begin{pmatrix}
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & T & T & T & T & T \\
\end{pmatrix}
\]

There may be concerns that outcomes for buyer 5 (which is in the control group) are affected by the fact that buyer 6, in the same cluster, is in the treatment group. In order to address such spillover concerns, one can do a modified buyer experiment where the randomization is over clusters of buyers, rather than over the buyers themselves. This is illustrated in (8.2). Here all buyers in clusters 2 and 4 are treated, and all buyers in clusters 1 and 3 are in the control group. As in 8.1, every buyer is faced with the same set of treated buyers, just as in a seller experiment, but the set of possible assignments is constrained by the clustering. In this example the first and third cluster are assigned to the treatment group for this assignment.

\[
\begin{array}{cccccccccc}
\text{Buyers} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{Clusters} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\end{array}
\]

\[
W = \begin{pmatrix}
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
\end{pmatrix}
\]
These first two designs fit into the traditional designs typically used. Specifically, they would fit in with the buyer experiments discussed earlier. However, we can enrich this design by also randomizing over the sellers. This will allow us to learn more from a single experiment. In particular it allows for estimating within the same experiment both treatment effects that allow for spillovers and the magnitude of the spillovers. In the first extension, in 8.3 we assign the clusters to treatment and control groups separately for each seller, in what is essentially a clustered version of a completely randomized two-sided experiment. For each seller, buyers in two randomly chosen clusters out of the four are assigned to the treatment group, and buyers in the remaining two clusters are assigned to the control group. Such a design could improve precision relative to the design in 8.2. It would not, however, allow us to learn new effects.

$$W_{j}^{S} = \begin{pmatrix}
C & C & C & C & C & T & T & T & T & T \\
C & C & C & C & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
\end{pmatrix}$$

$$W = \begin{pmatrix}
C & C & C & C & C & T & T & T & T & T \\
C & C & C & C & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
\end{pmatrix}$$

We can further enrich the designs by randomly splitting sellers into two groups, denoted by $c$ for cluster-randomized or $u$ for unit-randomized. For sellers in the first group, we assign treatments to buyers based on a clustered design as in 8.3. For sellers in the second group, we assign the buyers to treatment and control randomly without regard to the clusters. These randomization type groups are shown as $R_{j}^{S}$ in 8.4.

$$W_{j}^{S} = \begin{pmatrix}
C & C & C & C & C & T & T & T & T & T \\
C & C & C & C & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
\end{pmatrix}$$

$$W = \begin{pmatrix}
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
T & T & T & T & T & T & T & T & T & T \\
C & C & C & C & C & C & C & C & C & C \\
C & C & C & C & C & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
T & T & T & T & T & C & C & C & C & C \\
\end{pmatrix}$$
Ex ante control items for the cluster-randomized sellers are comparable to control items for the unit-randomized sellers. Systematic differences in their outcomes are the result of spillovers. The most natural interpretation is the systematic presence of spillovers between items within clusters.

Last, we discuss mixed Buyer/Seller experiments. Here we split the sellers into two groups, \( B \) and \( S \), with fractions \( p_B \) and \( p_S = 1 - p_B \). Let \( T^B_j \in \{B, S\} \) denote the assignment for sellers to these groups. For the first group of sellers we do a clustered buyer experiment. Let \( W^B_i \in \{C, T\} \) be the assignment for the buyers in this experiment. For the second group of sellers we do a seller experiment, with assignment \( W^S_j \in \{C, T\} \). This design is informative about seller responses.

\[
W = \begin{pmatrix}
C & C & C & C & T & C & T & T & C & C

C & C & C & C & T & C & T & T & C & C

T & T & T & T & T & T & C & T & T & C

T & T & T & T & T & T & C & T & T & C

C & C & C & C & T & C & T & T & C & C

C & C & C & C & T & C & T & T & C & C

T & T & T & T & T & T & C & T & T & C

T & T & T & T & T & T & C & T & T & C
\end{pmatrix}
\]

8.5 Triple Randomization Designs

In the previous sections, we have focused on the case in which the outcomes of interest are indexed by two populations (e.g., buyers and sellers), and the randomization can be performed across both populations. However, extensions to the case in which experimenters can (i) observe outcomes resulting from the interaction of more than two populations, and (ii) intervene by means of randomization across these multiple populations, are possible.

Concretely, suppose that in addition to buyers and sellers, we can also assign, as in crossover designs, the treatment to different levels depending on the time at which the interaction happens, or depending on which unit of a third population of agents takes part in the interaction (e.g., real-estate agents in the home-buying example). In that case, denoting by \( k = 1, \ldots, K \) the time period (or agent), we would observe an outcome for each triple \((i, j, k)\), denoted by \( Y_{ijk} \). The generality of the framework provided in the earlier sections naturally lends itself to this setting, in which more than two populations are interacting. For example, in the case of a MRD involving three populations, an experiment would correspond to a distribution over an order-three tensor \( W \) with dimensions \( I \times J \times K \), in which the typical element \( W_{ijk} \) would encode the treatment assigned to seller \( i \), buyer \( j \), at time \( k \).

Importantly, in a complex experiment involving multiple populations, we must be careful when formulating the assumptions describing how interference might be at play. While enriching the ex-
periment with additional dimensions could allow us to study and reveal interesting spillovers across
the population being analyzed, it will also increase the complexity of the estimation task, as the
number of estimands and assignment “types” will in general increase (potentially, at an exponential
rate) with the experiment’s dimensionality. A rigorous analysis of the assumptions under which
spillover effects across multiple populations can be learned, as well as related computational and
statistical tradeoffs, is an exciting avenue for future work.

9 Conclusions

We have propose a new class of experimental designs that is intended to allow the researcher to learn
about spillovers in a setting. The key feature of the settings we consider is that we have multiple
populations and can assign treatments to pairs (or tuples) with each tuple element corresponding
to the identity of a member of each population. This allows for much richer designs than the
conventional designs. We demonstrated how such designs can lead to more precise inferences about
standard estimands such as the overall average effect of the intervention and that they can generate
information about spillovers that conventional designs cannot reveal. We also propose methods for
estimation and inference.

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APPENDIX

In what follows, we present proofs of the results stated in Section 6. In our proofs, we always assume that an experiment is a conjunctive SMRD (as per Definition 8), with a total of \(I \times J\) units, in which each buyer \(i\) is endowed with a random variable \(W^B_i \in \{0, 1\}\), and each seller \(j\) is endowed with a random variable \(W^S_j \in \{0, 1\}\), such that \(I > I_T > 1\) and \(J > J_T > 1\) \((I_T = 1, J_T = 1\) would suffice to prove the results related to the first moments). Moreover, we assume the local interference assumption introduced in Assumption 5.4. Corresponding results for disjunctive SMRDs are straightforward to obtain.

Useful definitions for the analysis of potential outcomes in MRDs. We start with some preliminaries. Recall the definitions of the average outcomes for each buyer and each seller given in Section 3:

\[
\bar{Y}_i^B(\omega) = \frac{Y_i^B}{J}, \quad \bar{Y}_j^S(\omega) = \frac{Y_j^S}{I}, \quad \bar{Y}_i^B(\omega) = \frac{1}{I} \sum_{j=1}^I Y_{ij}(\omega), \quad \bar{Y}_j^S(\omega) = \frac{1}{J} \sum_{i=1}^J Y_{ij}(\omega),
\]

For each of the four assignment types \(\omega \in \{c, ib, is, t\}\), each buyer \(i\) and seller \(j\), define the following deviations:

\[
\hat{Y}_i^B(\omega) := \bar{Y}_i^B(\omega) - \bar{Y}(\omega), \quad \text{and} \quad \hat{Y}_i^S(\omega) := \bar{Y}_i^S(\omega) - \bar{Y}(\omega),
\]

as well as

\[
\hat{Y}_{ij}(\omega) = Y_{ij}(\omega) - \bar{Y}_i^B(\omega) - \bar{Y}_j^S(\omega) + \bar{Y}(\omega).
\]

By definition, the sum of these deviations is equal to zero:

\[
\sum_{i=1}^I \hat{Y}_i^B(\omega) = 0, \quad \sum_{i=1}^I \hat{Y}_{ij}(\omega) = 0, \quad \sum_{j=1}^J \hat{Y}_j^S(\omega) = 0, \quad \sum_{j=1}^J \hat{Y}_{ij}(\omega) = 0.
\]

Notice that, with this notation in place, we can decompose the potential outcomes \(Y_{ij}(\omega)\) as

\[
Y_{ij}(\omega) = \bar{Y}(\omega) + \hat{Y}_i^B(\omega) + \hat{Y}_j^S(\omega) + \hat{Y}_{ij}(\omega).
\]

Linear representation of the plug-in estimates \(\bar{Y}_\omega\). Recall from Definition 8 that \(W^B_i, W^S_j\) are random variables which determine whether buyer \(i\) and seller \(j\) are eligible to be exposed to the treatment.

The following lemma leverages these random variable to decomposes the averages of observed outcomes into a linear combination of four terms.

**Lemma A.1.** The average outcomes \(\bar{Y}_\omega\) by assignment type \(\omega \in \{c, ib, is, t\}\) can be decomposed as

\[
\bar{Y}_c = \bar{Y}(t) + \frac{1}{I_T} \sum_{i=1}^I W^B_i \hat{Y}_i^B(t) + \frac{1}{J} \sum_{j=1}^J W^S_j \hat{Y}_j^S(t) + \frac{1}{I_T J_T} \sum_{i=1}^I \sum_{j=1}^J W^B_i W^S_j \hat{Y}_{ij}(t),
\]

\[
\bar{Y}_{ib} = \bar{Y}(ib) + \frac{1}{I_T} \sum_{i=1}^I W^B_i \hat{Y}_i^B(ib) + \frac{1}{J} \sum_{j=1}^J (1 - W^S_j) \hat{Y}_j^S(ib) + \frac{1}{I_T J_C} \sum_{i=1}^I \sum_{j=1}^J W^B_i (1 - W^S_j) \hat{Y}_{ij}(ib),
\]

\[
\bar{Y}_{is} = \bar{Y}(is) + \frac{1}{I_C} \sum_{i=1}^I (1 - W^B_i) \hat{Y}_i^B(is) + \frac{1}{J} \sum_{j=1}^J W^S_j \hat{Y}_j^S(is) + \frac{1}{I_C J_T} \sum_{i=1}^I \sum_{j=1}^J W^B_i (1 - W^S_j) \hat{Y}_{ij}(is),
\]

37
Consider inspecting the form of the sample averages themselves reveals some insights. For the first two moments of these sample averages, as well as their cross-moments, under the SMRD design, coefficients non-stochastic (equal to the potential outcomes). This will allow us to calculate the exact values.

This completes the proof of the first result. The others are similar and are omitted.

Proof of Lemma A.1. Consider the first equality:

\[ \bar{\Psi}_c = \bar{\Psi}(c) + \frac{1}{J_C} \sum_{i=1}^{I_C} (1 - W_i^B \bar{Y}_i^B(c)) + \frac{1}{J_C} \sum_{j=1}^{J} (1 - W_j^S) \bar{Y}_j^S(c) + \frac{1}{I_C J_C} \sum_{i=1}^{I_C} \sum_{j=1}^{J} (1 - W_i^B)(1 - W_j^S) \bar{Y}_{ij}(c). \]

\( \bar{\Psi}_c \) is non-stochastic, and equal to \( \sum_{i,j} Y_{ij}(C)/(IJ) \). The second term, \( \sum_{i=1}^{I_C} (1 - W_i^B \bar{Y}_i^B(c))/I_C \), is an average over buyers. For its variance to be small, one needs to have many buyers. The third term, \( \sum_{j=1}^{J} (1 - W_j^S) \bar{Y}_j^S(c)/J_C \), is an average over sellers. For the variance of this term to become small, we need many sellers. The fourth term, \( \sum_{i=1}^{I_C} \sum_{j=1}^{J} (1 - W_i^B)(1 - W_j^S) \bar{Y}_{ij}(c)/(I_C J_C) \), is an average over both buyers and sellers. Its variance will go to zero as long as there are many terms in this average, and typically this average will be small relative to the second and third components. Note that the terms in this average sum to zero over buyers and sum to zero over sellers. In practice, it is likely that the variance is dominated by either the second or third term.

Incidentally, it is interesting to see what happens if we have a buyer experiment with \( W_j^S = 1 \) for all \( j \). In

\[ \bar{\Psi}_c = \bar{\Psi}(c) + \frac{1}{J_C} \sum_{i=1}^{I_C} (1 - W_i^B \bar{Y}_i^B(c)) + \frac{1}{J_C} \sum_{j=1}^{J} (1 - W_j^S) \bar{Y}_j^S(c) + \frac{1}{I_C J_C} \sum_{i=1}^{I_C} \sum_{j=1}^{J} (1 - W_i^B)(1 - W_j^S) \bar{Y}_{ij}(c). \]
that case, the third and fourth terms vanish, and
\[
\bar{Y}_c = \bar{Y}(c) + \frac{1}{I_c} \sum_{i=1}^{I_c} (1 - W_{B_i}^B) \dot{Y}^B_i(c).
\]
Similarly if we have a seller experiment with \(W_{S_j}^S = 1\) the second and fourth term vanish, and we have
\[
\bar{Y}_c = \bar{Y}(c) + \frac{1}{J_c} \sum_{j=1}^{J_c} (1 - W_{S_j}^S) \dot{Y}^S_j(c).
\]
The next step towards calculating the moments of the estimates \(\bar{Y}_\omega, \omega \in \{c, ib, is, t\}\), is to define the demeaned treatment indicators
\[
D_{B_i}^B = W_{B_i}^B - \frac{I_T}{I}, \quad \text{and} \quad D_{S_j}^S = W_{S_j}^S - \frac{J_T}{J}.
\]
To calculate the moments of the sample averages \(\bar{Y}_\omega\) for \(\omega \in \{c, ib, is, t\}\) we characterize the first two moments of \(D_{B_i}^B\) and \(D_{S_j}^S\).

**Lemma A.2.** Let \(W_{B_i}^B, W_{S_j}^S\) be the binary random variables denoting if buyer \(i\) and seller \(j\) can be exposed to the treatment. Then, for all \(i, i' \in I, i \neq i'\),
\[
E[D_{B_i}^M] = 0, \quad \forall (D_{B_i}^M) = \frac{I_C I_T}{I^2}, \quad C(D_{B_i}^M, D_{B_i'}^M) = -\frac{I_C I_T}{I^2 (I - 1)}.
\]
Similarly, for all \(j, j' \in J, j \neq j'\),
\[
E[D_{S_j}^M] = 0, \quad \forall (D_{S_j}^M) = \frac{J_C J_T}{J^2}, \quad C(D_{S_j}^M, D_{S_j'}^M) = -\frac{J_C J_T}{J^2 (J - 1)}.
\]
Finally, because \(D_{B_i}^M\) and \(D_{S_j}^M\) are independent, we have
\[
C(D_{B_i}^M, D_{S_j}^M) = 0, \quad \forall i, j.
\]

**Proof of Lemma A.2.** To prove that \(E[D_{B_i}^M] = 0\), notice that \(W_{B_i}^B\) is a Bernoulli random variable with bias given by \(p^B = \frac{I_T}{I}\). Moreover,
\[
\forall (D_{B_i}^M) = \forall (W_{B_i}^B) = \frac{I_T}{I} \left( 1 - \frac{I_T}{I} \right) = \frac{I_C I_T}{I^2}.
\]
For the covariance,
\[
C(D_{B_i}^M, D_{B_i'}^M) = C(W_{B_i}^B, W_{B_i'}^B) = E[W_{B_i}^B W_{B_i'}^B] - E[W_{B_i}^B] E[W_{B_i'}^B]
= \frac{I_T}{I} \frac{I_T - 1}{I - 1} - \frac{I_T^2}{I^2 (I - 1)} = -\frac{I_C I_T}{I^2 (I - 1)}.
\]
Corresponding proofs for \(D_{S_j}^M\) are analogous and omitted.

Note that the covariance between \(D_{B_i}^M\) and \(D_{B_i'}^M\) for \(i \neq i'\) differs from zero because we fix the number of selected buyers at \(I_T\), rather than tossing a coin for each buyer. Fixing the number of selected buyers is important for getting exact finite sample results for the variances.
Now define the average residuals by assignment type, for $\omega \in \{c, ib, is, t\}$:

$$\varepsilon^B_\omega = \frac{1}{I_T} \sum_{i=1}^J D^B_i \hat{Y}_i^B(\omega), \quad \varepsilon^S_\omega = \frac{1}{J_T} \sum_{j=1}^J D^S_j \hat{Y}_j^S(\omega),$$

and

$$\varepsilon^{BS}_\omega = \frac{1}{I_T J_T} \sum_{i=1}^J \sum_{j=1}^J D^B_i D^S_j \hat{Y}_{ij}(\omega).$$

These representations allow us to split the averages of observed values $\bar{Y}_\omega$ into deterministic and stochastic components.

**Lemma A.3.** We divide this lemma in three parts.

(a) The sample estimates $\bar{Y}_\omega$, $\omega \in \{c, ib, is, t\}$ can be written as the sums of four terms:

$$\bar{Y}_t = \bar{Y}(t) + \varepsilon^B_t + \varepsilon^S_t + \varepsilon^{BS}_t,$$

$$\bar{Y}_{ib} = \bar{Y}(ib) + \varepsilon^B_{ib} - \varepsilon^S_{ib} - \varepsilon^{BS}_{ib},$$

$$\bar{Y}_{is} = \bar{Y}(is) - \varepsilon^B_{is} + \varepsilon^S_{is} - \varepsilon^{BS}_{is},$$

$$\bar{Y}_c = \bar{Y}(c) - \varepsilon^B_c - \varepsilon^S_c + \varepsilon^{BS}_c.$$

(b) For each $\omega \in \{c, ib, is, t\}$, the $\varepsilon$ in the decomposition above are mean-zero error terms:

$$\mathbb{E} \left[ \varepsilon^B_\omega \right] = \mathbb{E} \left[ \varepsilon^S_\omega \right] = \mathbb{E} \left[ \varepsilon^{BS}_\omega \right] = 0.$$

(c) For all $\omega, \omega' \in \{c, ib, is, t\}$, the error terms above are uncorrelated:

$$\mathbb{C} \left( \varepsilon^B_\omega, \varepsilon^S_{\omega'} \right) = \mathbb{C} \left( \varepsilon^B_\omega, \varepsilon^{BS}_{\omega'} \right) = \mathbb{C} \left( \varepsilon^S_\omega, \varepsilon^{BS}_{\omega'} \right) = 0.$$

Before proving this lemma, let us just provide an intuition about the decomposition of the four averages $\bar{Y}_t$, $\bar{Y}_{ib}$, $\bar{Y}_{is}$, and $\bar{Y}_c$ described above, as this is a key step to obtaining the variance of the estimator for the average treatment effect. In particular, looking at (i), the first term $\bar{Y}(\omega)$ is deterministic (the unweighted average of potential outcomes over all pairs $(i, j)$, not depending on the assignment). The other three terms, $\varepsilon^B_\omega$, $\varepsilon^S_\omega$, and $\varepsilon^{BS}_\omega$, are mutually uncorrelated stochastic terms with expectation equal to zero. The variances of the four averages will depend on the variances of the three stochastic terms, and the covariances will depend on the covariances of the corresponding stochastic terms, e.g., the covariance of $\varepsilon^B_t$ and $\varepsilon^B_{ib}$, or the covariance of $\varepsilon^{BS}_c$ and $\varepsilon^{BS}_{is}$.

**Proof of Lemma A.3.** For part (a) consider $\bar{Y}_t$. Now consider for the treated type the average of the observed outcomes, decomposed as in Lemma A.1:

$$\bar{Y}_t = \bar{Y}(t) \pm \frac{1}{I_T} \sum_{i=1}^J W^B_i \hat{Y}_i^B(t) \pm \frac{1}{J_T} \sum_{j=1}^J W^S_j \hat{Y}_j^S(t) \pm \frac{1}{I_T J_T} \sum_{i=1}^J \sum_{j=1}^J W^B_i W^S_j \hat{Y}_{ij}(t).$$
Using now Lemma A.2, and substituting $D^M_i + I_T/I$ for $W^B_i$ and $D^Y_j + J_T/J$ for $W^S_j$, we can write

$$\overline{Y}_t = \overline{Y}(t) + \frac{1}{I_T} \sum_{i=1}^I \left( D^B_i + \frac{I_T}{T} \right) \dot{Y}^B_i(t) + \frac{1}{J_T} \sum_{j=1}^J \left( D^S_j + \frac{J_T}{J} \right) \dot{Y}^S_j(t)$$

$$+ \frac{1}{I_T J_T} \sum_{i=1}^I \sum_{j=1}^J \left( D^B_i + \frac{I_T}{T} \right) \left( D^S_j + \frac{J_T}{J} \right) \dot{Y}_{ij}(t).$$

Because by definition the $\dot{Y}_{ij}(t)$, $\dot{Y}^B_i(t)$ and $\dot{Y}^S_j(t)$ sum exactly to zero, the equation above simplifies to

$$\overline{Y}_t = \overline{Y}(t) + \frac{1}{I_T} \sum_{i=1}^I D^B_i \dot{Y}^B_i(t) + \frac{1}{J_T} \sum_{j=1}^J D^S_j \dot{Y}^S_j(t) + \frac{1}{I_T J_T} \sum_{i=1}^I \sum_{j=1}^J D^B_i D^S_j \dot{Y}_{ij}(t)$$

$$= \overline{Y}(t) + \varepsilon^B_t + \varepsilon^S_t + \varepsilon^{BS}_t.$$

This concludes the proof of the first part of (a). The proofs of the other parts of (a) follow the same argument and are omitted. Given part (a), (b) follows immediately because the $D^M_i$ and $D^Y_j$ have expectation equal to zero. The same holds for the covariances in (c). \qed

With this result in place, we can show Lemma 1 and Theorem 1.

**Proof of Lemma 1 and Theorem 1.** Apply Lemma A.3, and linearity of the expectation operator. \qed

We now move to the variance characterization. For $\omega \in \{c,ib,is,t\}$, recall the definitions of the population variances of $\dot{Y}^B_i(\omega)$ and $\dot{Y}^S_j(\omega)$ given in Section 6:

$$\Sigma^2_{\omega} := \frac{1}{I-1} \sum_{i=1}^I \left( \dot{Y}^B_i(\omega) \right)^2, \quad \Sigma^2_{\omega} := \frac{1}{J-1} \sum_{j=1}^J \left( \dot{Y}^S_j(\omega) \right)^2,$$

and a similar quantity for $\dot{Y}_{ij}(\omega)$ as

$$\Sigma^2_{\omega} := \frac{1}{(I-1)(J-1)} \sum_{i=1}^I \sum_{j=1}^J \left( \dot{Y}_{ij}(\omega) \right)^2.$$
Proof of Lemma A.4. We prove the first part of the lemma by showing the three equalities

\[ \mathbb{V}_t^B := \mathbb{V}(\pi_t^B) = \frac{I_C}{I_T I} \Sigma_t^{2,B}, \]  
(A.2)

\[ \mathbb{V}_t^S := \mathbb{V}(\pi_t^S) = \frac{J_C}{J_T J} \Sigma_t^{2,S}, \]  
(A.3)

and

\[ \mathbb{V}_t^{BS} := \mathbb{V}(\pi_t^{BS}) = \frac{I_C}{I_T I} \frac{J_C}{J_T J} \Sigma_t^{2,BS}. \]  
(A.4)

Because Lemma A.3 implies that

\[ \mathbb{V}_t = \mathbb{V}(\bar{\pi}_t) = \mathbb{V}(\pi_t^B) + \mathbb{V}(\pi_t^S) + \mathbb{V}(\pi_t^{BS}), \]

the three equalities Equations (A.2) to (A.4) imply the first result. First, we show Equation (A.2).

\[ \mathbb{V}_t^B := \mathbb{V}(\pi_t^B) = E \left( \left( \frac{1}{I_T} \sum_{i=1}^{I} D_t^B \dot{Y}_i^B(t) \right)^2 \right) \]

\[ = \frac{1}{I_T^2} E \left[ \sum_{i=1}^{I} \sum_{i' = 1}^{I} D_t^B D_t^B \dot{Y}_i^B(t) \dot{Y}_{i'}^B(t) \right] \]

\[ = \frac{1}{I_T^2} \sum_{i=1}^{I} \sum_{i' = 1}^{I} E \left[ D_t^B D_t^B \dot{Y}_i^B(t) \dot{Y}_{i'}^B(t) \right] - \frac{1}{I_T^2} \sum_{i=1}^{I} \sum_{i' = 1}^{I} \frac{I_T I_C}{I_T (I - 1)} \dot{Y}_i^B(t) \dot{Y}_{i'}^B(t) \]

\[ + \frac{1}{I_T^2} \sum_{i=1}^{I} \frac{I_T I_C}{I_T^2 (I - 1)} \dot{Y}_i^B(t) \dot{Y}_i^B(t) + \frac{1}{I_T^2} \sum_{i=1}^{I} E \left( (D_t^B)^2 \right) \left( \dot{Y}_i^B(t) \right)^2 \]

Because \( \sum_i \dot{Y}_i^B(t) = 0 \), the first of the three terms involving the double sum is equal to zero, and this expression simplifies to two single sums:

\[ \mathbb{V}_t^B = \frac{1}{I_T} \sum_{i=1}^{I} \frac{I_T I_C}{I_T^2 (I - 1)} \dot{Y}_i^B(t) \dot{Y}_i^B(t) + \frac{1}{I_T^2} \sum_{i=1}^{I} \frac{I_T I_C}{I_T^2} \left( \dot{Y}_i^B(t) \right)^2 \]

\[ = \frac{I_T I_C}{I_T^2 (I - 1)} \sum_{i=1}^{I} \left( \dot{Y}_i^B(t) \right)^2 \]

\[ = \frac{I_C}{I_T I} \Sigma_t^{2,B}. \]

The proof for Equation (A.3) follows the same exact argument, and is omitted. Next, consider Equation (A.4). The term \( \pi_t^{BS} \) is a double sum, and we write it out in full.

\[ \mathbb{V}_t^{BS} := \mathbb{V}(\pi_t^{BS}) = \mathbb{V} \left( \frac{1}{I_T J_T} \sum_{i=1}^{I} \sum_{j=1}^{J} D_t^B D_t^S \dot{Y}_{ij}(t) \right) \]

\[ = E \left[ \frac{1}{I_T J_T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i' = 1}^{I} \sum_{j' = 1}^{J} D_t^B D_t^B D_t^S D_t^S \dot{Y}_{ij}(t) \dot{Y}_{i'j'}(t) \right] . \]
By independence of $D^B_i$ and $D^S_j$, this is equal to

$$
\forall_{t}^{BS} = \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i'=1}^{I} \sum_{j'=1}^{J} \mathbb{E} \left[D^B_i D^B_{i'} \right] \sum_{j=1}^{J} \sum_{j'=1}^{J} \mathbb{E} \left[D^S_j D^S_{j'} \right] \hat{Y}_{ij}(t) \hat{Y}_{ij'}(t).
$$

Using the covariances and variances for $D^B_i$ and $D^S_j$ and for $D^B_j$ and $D^S_i$, this is equal to

$$
\forall_{t}^{BS} = \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i'=1}^{I} \sum_{j'=1}^{J} \frac{I C I_T}{T^2(I-1)} \frac{J C J_T}{J^2(J-1)} \hat{Y}_{ij}(t) \hat{Y}_{ij'}(t)
- \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i'=1}^{I} \sum_{j'=1}^{J} \frac{I C I_T}{T^2(I-1)} \frac{J C J_T}{J^2(J-1)} \hat{Y}_{ij}(t) \hat{Y}_{ij'}(t)
- \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{i'=1}^{I} \sum_{j'=1}^{J} \frac{I C I_T}{I(I-1)} \frac{J C J_T}{J^2(J-1)} \hat{Y}_{ij}(t) \hat{Y}_{ij'}(t)
+ \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{I C I_T}{I(I-1)} \frac{J C J_T}{J(J-1)} \left( \hat{Y}_{ij}(t) \right)^2.
$$

Because $\sum_i \sum_j \hat{Y}_{ij}(t) = 0$, the first three terms are equal to zero, and so this sum is equal to

$$
\forall_{t}^{BS} = \frac{1}{I^2 J^2 T} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{I C I_T}{I(I-1)} \frac{J C J_T}{J(J-1)} \left( \hat{Y}_{ij}(t) \right)^2
= \frac{I C J_C}{I_T J_T} \forall_{t}^{BS}.
$$

The proofs of the other parts of the lemma are similar and are omitted.

In order to characterize the variance of the spillover effects, we need to characterize the covariance between the estimators $\overline{Y}_\omega, \overline{Y}_{\omega'}$, for $\omega, \omega' \in \{c, ib, is, t\}$. Towards this goal, recall the definitions provided in Section 6: for all $\omega, \omega' \in \{c, ib, is, t\}$, $\omega \neq \omega'$, for the buyers

$$
\Sigma^2_{\omega, \omega'}^B := \frac{1}{I-1} \sum_{i=1}^{I} \left( \hat{Y}_{i}^B(\omega) - \bar{Y}_{i}^B(\omega') \right)^2,
$$

for the sellers,

$$
\Sigma^2_{\omega, \omega'}^S := \frac{1}{J-1} \sum_{j=1}^{J} \left( \hat{Y}_{j}^S(\omega) - \bar{Y}_{j}^B(\omega') \right)^2,
$$

and for the interactions

$$
\Sigma^2_{\omega, \omega'}^{BS} := \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \hat{Y}_{ij}(\omega) - \bar{Y}_{ij}(\omega') \right)^2.
$$

With these definitions in place, we characterize the covariance of the sample averages $\overline{Y}_\omega, \overline{Y}_{\omega'}$ in Lemma A.5.
Lemma A.5.

\[ C_{t,ib} := C \left( \overline{\gamma}_t, \overline{\gamma}_{ib} \right) = \frac{I_C}{2I_T I} \left( \Sigma_{2,B_t} + \Sigma_{2,B_{ib}} - \Sigma_{2,B_{t,ib}} \right) - \frac{1}{2J} \left( \Sigma_{2,S_t} + \Sigma_{2,S_{ib}} - \Sigma_{2,S_{t,ib}} \right) - \frac{I_C}{2I_T J} \left( \Sigma_{2,BS_t} + \Sigma_{2,BS_{ib}} - \Sigma_{2,BS_{t,ib}} \right) \]

\[ C_{t,is} := C \left( \overline{\gamma}_t, \overline{\gamma}_{is} \right) = -\frac{1}{2I} \left( \Sigma_{2,B_t} + \Sigma_{2,B_S} - \Sigma_{2,B_{t,ib}} \right) + \frac{J_C}{2J T_J} \left( \Sigma_{2,S_t} + \Sigma_{2,S_{is}} - \Sigma_{2,S_{t,ib}} \right) - \frac{J_C}{2I T_J} \left( \Sigma_{2,BS_t} + \Sigma_{2,BS_{is}} - \Sigma_{2,BS_{t,ib}} \right) \]

\[ C_{t,c} := C \left( \overline{\gamma}_t, \overline{\gamma}_c \right) = -\frac{1}{2I} \left( \Sigma_{2,B_t} + \Sigma_{2,B_c} - \Sigma_{2,B_{t,ic}} \right) - \frac{1}{2J} \left( \Sigma_{2,S_t} + \Sigma_{2,S_{c}} - \Sigma_{2,S_{t,ic}} \right) + \frac{1}{2I T_J} \left( \Sigma_{2,BS_t} + \Sigma_{2,BS_{c}} - \Sigma_{2,BS_{t,ic}} \right) \]

\[ C_{ib,ic} := C \left( \overline{\gamma}_{ib}, \overline{\gamma}_{ic} \right) = -\frac{1}{2I} \left( \Sigma_{2,B_{ib}} + \Sigma_{2,BS_{ib}} - \Sigma_{2,BS_{t,ib,ic}} \right) - \frac{1}{2J} \left( \Sigma_{2,S_{ib}} + \Sigma_{2,SB_{ib}} - \Sigma_{2,SB_{t,ib,ic}} \right) + \frac{1}{2I T_J} \left( \Sigma_{2,BS_{ib}} + \Sigma_{2,BS_{ib,ic}} - \Sigma_{2,BS_{t,ib,ic}} \right) \]

\[ C_{ib,c} := C \left( \overline{\gamma}_{ib}, \overline{\gamma}_c \right) = \frac{I_C}{2I T_I} \left( \Sigma_{2,B_{ib}} + \Sigma_{2,B_S} - \Sigma_{2,B_{t,ib,c}} \right) - \frac{1}{2J} \left( \Sigma_{2,S_{ib}} + \Sigma_{2,S_{c}} - \Sigma_{2,S_{t,ib,c}} \right) - \frac{I_C}{2I T_I J} \left( \Sigma_{2,BS_{ib}} + \Sigma_{2,BS_{is}} - \Sigma_{2,BS_{t,ib,c}} \right) \]

and last

\[ C_{is,c} := C \left( \overline{\gamma}_{is}, \overline{\gamma}_c \right) = \frac{I_C}{2I T_I} \left( \Sigma_{2,B_{is}} + \Sigma_{2,B_S} - \Sigma_{2,B_{t,ic,c}} \right) - \frac{1}{2J} \left( \Sigma_{2,S_{is}} + \Sigma_{2,S_{c}} - \Sigma_{2,S_{t,ic,c}} \right) - \frac{I_C}{2I T_I J} \left( \Sigma_{2,BS_{is}} + \Sigma_{2,BS_{is}} - \Sigma_{2,BS_{t,ic,c}} \right) \]

Proof of Lemma A.5. We prove the first result for \( C_{t,ib} \) by showing the following three equalities

\[ C^B_{t,ib} := C \left( \sigma^B_t, \sigma^B_{ib} \right) = \frac{I_C}{2I_T I} \left( \Sigma^{2,B}_{2,B_t} + \Sigma^{2,B}_{2,B_{ib}} - \Sigma^{2,B}_{2,B_{t,ib}} \right), \quad (A.5) \]

\[ C^S_{t,ib} := C \left( \sigma^S_t, \sigma^S_{ib} \right) = \frac{1}{2J} \left( \Sigma^{2,S}_{2,S_t} + \Sigma^{2,S}_{2,S_{ib}} - \Sigma^{2,S}_{2,S_{t,ib}} \right), \quad (A.6) \]

and

\[ C^{BS}_{t,ib} := C \left( \sigma^{BS}_t, \sigma^{BS}_{ib} \right) = \frac{I_C}{2I_T J} \left( \left( \Sigma^{2,BS}_{2,BS_t} + \Sigma^{2,BS}_{2,BS_{ib}} - \Sigma^{2,BS}_{2,BS_{t,ib}} \right) \right), \quad (A.7) \]
In combination with the fact that
\[ C(\overline{Y}_t, \overline{Y}_{ib}) = C(\overline{\epsilon}_t, \overline{\epsilon}_{ib}) - C(\overline{\epsilon}_S, \overline{\epsilon}_{ib}) - C(\overline{\epsilon}_{BS}, \overline{\epsilon}_{ib}), \]
this proves the first result.

First (A.5):
\[
C_{t,ib}^B = \mathbb{E} \left[ \left( \frac{1}{I_T} \sum_{i=1}^{I} D_i^B \dot{Y}_i^B(t) \right) \left( \frac{1}{I_T} \sum_{i=1}^{I} D_i^B \dot{Y}_i^B(ib) \right) \right]
= \mathbb{E} \left[ \frac{1}{I_T} \sum_{i=1}^{I} \sum_{i' = 1}^{I} D_i^B D_{i'}^B \dot{Y}_i^B(t) \dot{Y}_{i'}^B(ib) \right]
= \frac{1}{I_T^2} \sum_{i=1}^{I} \sum_{i' = 1}^{I} \mathbb{E} \left[ D_i^B D_{i'}^B \dot{Y}_i^B(t) \dot{Y}_{i'}^B(ib) \right]
= -\frac{1}{I_T^2} \sum_{i=1}^{I} \sum_{i' = 1}^{I} \left( \frac{I_C I_T}{I^2(I-1)} \right) \dot{Y}_i^B(t) \dot{Y}_{i'}^B(ib) + \frac{1}{I_T^2} \sum_{i=1}^{I} \left( \frac{I_C I_T}{I^2(I-1)} + \frac{I_C I_T}{I^2} \right) \dot{Y}_i^B(t) \dot{Y}_i^B(ib).
\]
Because \( \sum_i \dot{Y}_i^B(t) = 0 \) the first term is equal to zero. Thus,
\[
C_{t,ib}^B = \frac{I_C}{I_T I} \left( \frac{1}{I-1} \sum_{i=1}^{I} \dot{Y}_i^B(t) \dot{Y}_i^B(ib) \right).
\]
Because
\[
\Sigma_{t,ib}^{2,B} = \frac{1}{I-1} \sum_{i=1}^{I} \left( \dot{Y}_i^B(t) - \dot{Y}_i^B(ib) \right)^2
= \frac{1}{I-1} \sum_{i=1}^{I} \dot{Y}_i^B(t)^2 + \frac{1}{I-1} \sum_{i=1}^{I} \dot{Y}_i^B(ib)^2 - \frac{2}{I-1} \sum_{i=1}^{I} \dot{Y}_i^B(t) \dot{Y}_i^B(ib)^2
= \Sigma_t^{2,B} + \Sigma_{ib}^{2,B} - \frac{2I_T I}{I_C} C_{t,ib}^B,
\]
we have
\[
C_{t,ib}^B = \frac{I_C}{2I_T I} \left( \Sigma_t^{2,B} + \Sigma_{ib}^{2,B} - \Sigma_{t,ib}^{2,B} \right).
\]
This completes the proof of (A.5). Similarly, to prove (A.6), we have

\[ C_{t,ib}^{S} = \mathbb{E} \left[ \left( \frac{1}{J} \sum_{j=1}^{J} D_{j}^{S} \dot{Y}_{j}^{S} (t) \right) \left( \frac{1}{J} \sum_{j=1}^{J} D_{j}^{S} \dot{Y}_{j}^{S} (ib) \right) \right] \]

\[ = \mathbb{E} \left[ \frac{1}{J} \sum_{j=1}^{J} \sum_{j'=1}^{J} D_{j}^{S} D_{j'}^{S} \dot{Y}_{j}^{S} (t) \dot{Y}_{j'}^{S} (ib) \right] \]

\[ = \frac{1}{J} \sum_{j=1}^{J} \sum_{j'=1}^{J} \mathbb{E} \left[ D_{j}^{S} D_{j'}^{S} \dot{Y}_{j}^{S} (t) \dot{Y}_{j'}^{S} (ib) \right] \]

\[ = -\frac{1}{J} \sum_{j=1}^{J} \sum_{j'=1}^{J} \frac{J_{C} J_{T}}{J^{2} (J-1)} \dot{Y}_{j}^{S} (t) \dot{Y}_{j'}^{S} (ib) + \frac{1}{J} \sum_{j=1}^{J} \left( \frac{J_{C} J_{T}}{J^{2} (J-1)} + \frac{J_{C} J_{T}}{J^{2}} \right) \dot{Y}_{j}^{S} (t) \dot{Y}_{j}^{S} (ib) \]

\[ = \frac{1}{J} \sum_{j=1}^{J} \left( \frac{J_{C} J_{T}}{J^{2} (J-1)} + \frac{J_{C} J_{T}}{J^{2}} \right) \dot{Y}_{j}^{S} (t) \dot{Y}_{j}^{S} (ib) \]

\[ = \frac{1}{J} \left( \frac{1}{J-1} \sum_{j=1}^{J} \dot{Y}_{j}^{S} (t) \dot{Y}_{j}^{S} (ib) \right) . \]

Because

\[ \Sigma_{t,ib}^{2,S} = \frac{1}{J-1} \sum_{j=1}^{J} \left( \dot{Y}_{j}^{S} (t) - \dot{Y}_{j}^{S} (ib) \right)^{2} \]

\[ = \frac{1}{J-1} \sum_{j=1}^{J} \dot{Y}_{j}^{S} (t)^{2} + \frac{1}{J-1} \sum_{j=1}^{J} \dot{Y}_{j}^{S} (ib)^{2} - \frac{2}{J-1} \sum_{j=1}^{J} \dot{Y}_{j}^{S} (t) \dot{Y}_{j}^{S} (ib)^{2} \]

\[ = \Sigma_{t}^{2,S} + \Sigma_{t,ib}^{2,S} + 2JC_{t,ib}^{S}, \]

it follows that

\[ C_{t,ib}^{S} = \frac{1}{2J} \left( \Sigma_{t}^{2,S} + \Sigma_{t,ib}^{2,S} - \Sigma_{t,ib}^{2,S} \right) . \]

This finishes the proof of (A.6). Third, consider (A.7):

\[ C_{t,ib}^{BS} = \mathbb{E} \left[ \frac{1}{I_{T}^{2} J_{T} J_{C}} \sum_{i,i'=1}^{I_{T}} \sum_{j,j'=1}^{J} D_{i}^{B} D_{j}^{S} D_{i}^{B} D_{j}^{B} \dot{Y}_{ij}^{B} (t) \dot{Y}_{i'j'}^{B} (ib) \right] . \]

By independence of \( D_{i}^{B} \) and \( D_{j}^{S} \), this is equal to

\[ C_{t,ib}^{BS} = \frac{1}{I_{T}^{2} J_{T} J_{C}} \sum_{i,i'=1}^{I_{T}} \sum_{j,j'=1}^{J} \mathbb{E} \left[ D_{i}^{B} D_{i}^{B} \right] \mathbb{E} \left[ D_{j}^{S} D_{j}^{S} \right] \dot{Y}_{ij}^{B} (t) \dot{Y}_{i'j'}^{B} (ib) \].

Using the covariances and variances for \( D_{i}^{B} \) and \( D_{j}^{B} \) and for \( D_{j}^{S} \) and \( D_{j}^{S} \), this is equal to
Because \( \sum_i \sum_j \dot{Y}_{ij}(t) = 0 \), the first three terms are equal to zero, and so this sum is equal to

\[
C_{\text{lib}}^{\text{BS}} = \frac{1}{I_T I_C J_I J_T} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{I_C I_T}{I(I-1) J(J-1)} \dot{Y}_{ij}(t) \dot{Y}_{ij}(ib)
\]

\[
- \frac{1}{I_T I_C J_I J_T} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{I_C I_T}{I(I-1) J(J-1)} \dot{Y}_{ij}(t) \dot{Y}_{ij}(ib)
\]

\[
+ \frac{1}{I_T I_C J_I J_T} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{I_C I_T}{I(I-1) J(J-1)} \dot{Y}_{ij}(t) \dot{Y}_{ij}(ib).
\]

Because

\[
\Sigma_{t,ib}^{2,\text{BS}} = \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \dot{Y}_{ij}(t) - \dot{Y}_{ij}(ib) \right)^2
\]

\[
= \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \dot{Y}_{ij}(t) \right)^2 + \frac{1}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \left( \dot{Y}_{ij}(ib) \right)^2
\]

\[
- \frac{2}{(I-1)(J-1)} \sum_{i=1}^{I} \sum_{j=1}^{J} \dot{Y}_{ij}(t) \dot{Y}_{ij}(ib)
\]

\[
= \Sigma_{t,ib}^{2,\text{BS}} + \Sigma_{2,ib}^{2,\text{BS}} - \Sigma_{t,ib}^{2,\text{BS}}
\]

\[
= \frac{2I_T I_J}{I_C} C_{t,ib}^{\text{BS}},
\]

it follows that

\[
C_{t,ib}^{\text{BS}} = \frac{I_C}{2I_T I_J} \left( \Sigma_{t,ib}^{2,\text{BS}} + \Sigma_{2,ib}^{2,\text{BS}} - \Sigma_{t,ib}^{2,\text{BS}} \right).
\]

This finishes the proof of (A.7). The proofs for the other covariance matrices follow the same pattern and are omitted.

With these characterizations in place, we can now prove Theorem 2.

**Proof of Theorem 2.** Using the fact that spillovers are linear combinations of the sample averages \( \overline{\omega} \), \( \omega \in \{c, ib, is, t\} \), Theorem 2 follows by noting that for scalars \( \alpha, \beta \), \( \mathbb{V}(\alpha X + \beta Y) = \alpha^2 \mathbb{V}(X) + \beta^2 \mathbb{V}(Y) + 2\alpha\beta \mathbb{C}(X, Y) \).

\( \square \)
9.1 Proofs in Section 6.1

9.1.1 Review: variance estimates for a single population

To present our results on estimates of the variance, we start by reviewing a classic result to characterize the estimate of variances of a simple two-arms experiment, when a single population is present.

**Lemma A.6.** Let \( Y_i, i = 1, \ldots, I \) be a population of \( I \) units with potential outcomes \( Y_i(\omega) \) (if unit \( i \) is in the control group) and \( Y_i(t) \) (if unit \( i \) is in the treatment group). Let the treatment group be identified by and index set \( I_t = \{i_1, \ldots, i_{I_t}\} \subset \{1, \ldots, I\} \), of size \( |I_t| = I_t \), with \( 2 \leq I_t \leq I - 2 \). Let \( I_c = \{1, \ldots, I\} \setminus I_t \) be the index set of the \( I_c := I - I_t \) units assigned to the control group. For \( \omega \in \{c, t\} \), let

\[
\bar{Y}(\omega) = \frac{1}{I} \sum_{i=1}^{I} Y_i(\omega) \quad \text{and} \quad \Sigma_{\omega}^2 = \frac{1}{I} \sum_{i=1}^{I} (Y_i(\omega) - \bar{Y}(\omega))^2.
\]

be the mean and variance of the potential outcomes in the population. Define the corresponding plug-in estimates for these to be

\[
\hat{\bar{Y}}_\omega = \frac{1}{I_\omega} \sum_{i \in I_\omega} Y_i(\omega), \quad \text{and} \quad \hat{\Sigma}_\omega^2 = \frac{1}{I_\omega} \sum_{i \in I_\omega} (Y_i(\omega) - \hat{\bar{Y}}(\omega))^2.
\]

Then it holds

\[
E\left[\hat{\bar{Y}}_\omega\right] = \bar{Y}(\omega), \quad (A.8)
\]

and

\[
\mathbb{V}\left(\hat{\bar{Y}}_\omega\right) = \frac{I - I_\omega}{I_\omega} \frac{1}{I - 1} \Sigma_{\omega}^2,
\]

and

\[
E\left(\hat{\Sigma}_\omega^2\right) = \frac{I_\omega - 1}{I_\omega} \frac{I}{I - 1} \Sigma_{\omega}^2.
\]

I.e., \( \hat{\bar{Y}}_\omega \) is an unbiased estimate of the population mean \( \bar{Y}(\omega) \). We can obtain an unbiased estimate of the variance of this estimator by reweighing \( \hat{\Sigma}_\omega^2 \):

\[
\hat{\mathbb{V}}\left(\hat{\bar{Y}}_\omega\right) := \frac{I - I_\omega}{I_\omega - 1} \frac{1}{I - 1} \hat{\Sigma}_\omega^2
\]

satisfies

\[
E\left[\hat{\mathbb{V}}\left(\hat{\bar{Y}}_\omega\right)\right] = \mathbb{V}\left(\hat{\bar{Y}}_\omega\right).
\]

**Proof of Lemma A.6.** The proof of this classic result can be found, e.g., in Cochran [1977, Theorems 2.1, 2.2, 2.4].

**Remark** For the single population case, simpler expressions can be obtained by defining the variance to be \( \Phi_{\omega}^2 = \frac{I}{I - 1} \Sigma_{\omega}^2 \) and its estimate \( \hat{\Phi}_{\omega}^2 = \frac{I_\omega}{I_\omega - 1} \hat{\Sigma}_{\omega}^2 \). In the double randomized setting, however, the former parametrization is more convenient and we will henceforth adopt this one.
9.1.2 Estimates for the double randomized experiments

We now tackle the problem of proving lower and upper bound on the variance of causal effects in SMRDs, and prove Lemmas 3 and 4. Given $I$ buyers and $J$ sellers, consider an assignment matrix $W$ with positive probability under a SMRD $\mathcal{W}(I_T, J_T)$. By construction, entries $(i,j)$ in $W$ are of one of four types $s \in \{c, ib, is, t\}$. For a given $W$, denote by $I_\omega \subseteq \{1, \ldots, I\}$ the subset of buyers’ indices for which there exists at least one seller $j$ such that unit $(i,j)$ has type $\omega$: $I_\omega := \{i \in \{1, \ldots, I\} : T_{ij} = s$ for some $j\}$. Symmetrically, let $J_\omega \subseteq \{1, \ldots, J\}$ the subset of sellers’ indices for which there exists at least one buyer $i$ such that unit $(i,j)$ has type $\omega$. Let $I_\omega = |I_\omega|$ and $J_\omega = |J_\omega|$ denote the sizes of these index sets. From the properties of SMRDs, it follows that $I_\omega = I_{ib} = I_C$, and $I_{ib} = I_t = I_T$, and $J_\omega = J_{ib} = J_C$, and $J_{is} = I_t = J_T$. Then, for each type $\omega$, exactly $I_\omega J_\omega$ units are assigned type $\omega$. In what follows, we will always assume that $I_\omega, J_\omega \geq 2$ for all $\omega$.

Recall the definition of the (nonrandom) row and column means of the matrix of potential outcomes,

$$\bar{Y}_i^B(\omega) = \frac{1}{J} \sum_{j=1}^J Y_{i,j}(\omega) \quad \text{and} \quad \bar{Y}_j^S(\omega) = \frac{1}{I} \sum_{i=1}^I Y_{i,j}(\omega).$$

Define then, the (nonrandom) row and column partial mean of the matrix of potential outcomes. That is, for a given row $i$, the average over a fixed index set of columns $J_\omega \subseteq J$ — symmetrically, for a given column $j$, the average over a fixed set of rows $I_\omega \subseteq I$:

$$\bar{Y}_{i,J_\omega}^B(\omega) = \frac{1}{J_\omega} \sum_{j \in J_\omega} Y_{i,j}(\omega) \quad \text{and} \quad \bar{Y}_{I_\omega,j}^S(\omega) = \frac{1}{I_\omega} \sum_{i \in I_\omega} Y_{i,j}(\omega).$$

For a given SMRD $\mathcal{W}(I_T, J_T)$, with (random) assignment matrix $W$ and characterized by (random) index sets $I_\omega, J_\omega$ for each $s \in \{c, ib, is, t\}$, $i \in I_\omega, j \in J_\omega$, define the random average over the columns selected by the set $J_\omega$ (or the rows selected by $I_\omega$):

$$\hat{Y}_{i,J_\omega}^B(\omega) := \frac{1}{J_\omega} \sum_{j \in J_\omega} Y_{i,j}(\omega), \quad \text{and} \quad \hat{Y}_{I_\omega,j}^S(\omega) := \frac{1}{I_\omega} \sum_{i \in I_\omega} Y_{i,j}(\omega).$$

**Remark** The quantities $\hat{Y}_{i,J_\omega}^B(\omega)$ and $\hat{Y}_{I_\omega,j}^S(\omega)$ are both averages over $J_\omega$ elements of the $i$-th row of the matrix of potential outcomes $Y(\omega)$. However, the former ($\hat{Y}_{i,J_\omega}^B(\omega)$) is a random quantity — an estimator resulting from the random selection of $J_\omega$ distinct columns, whereas the latter ($\hat{Y}_{I_\omega,j}^S(\omega)$) is a fixed population value, obtained by averaging over the fixed $J_\omega$ distinct indices $\{j_1, \ldots, j_\omega\} = J_\omega$.

Next, define the row and column (nonrandom) variance terms $\Delta_{\omega}^{2,B}, \Delta_{\omega}^{2,S}, \Delta_{\omega}^{2,BS}$, which are just a reweighing of the quantities introduced in Section 6. This reweighing simplifies notation in the proofs:

$$\Delta_{\omega}^{2,B} := \frac{1}{I} \sum_{i=1}^I \left\{ \bar{Y}_{i}^B(\omega) - \bar{Y}(\omega) \right\}^2,$$

$$\Delta_{\omega}^{2,S} := \frac{1}{J} \sum_{j=1}^J \left\{ \bar{Y}_{j}^S(\omega) - \bar{Y}(\omega) \right\}^2,$$

and

$$\Delta_{\omega}^{2,BS} := \frac{1}{IJ} \sum_{i=1}^I \sum_{j=1}^J \left\{ Y_{i,j}(\omega) - \bar{Y}_{i}^B - \bar{Y}_{j}^S + \bar{Y}(\omega) \right\}^2.$$
Define the sample “plug-in” counterparts of the population quantities $\Delta_2^{B}, \Delta_2^{S}, \Delta_2^{BS}$:

\[
\hat{\Delta}_2^{B} = \frac{1}{I_\omega} \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \overline{Y}_\omega \right)^2,
\]

\[
\hat{\Delta}_2^{S} = \frac{1}{J_\omega} \sum_{j \in J_\omega} \left( \hat{Y}_j^S(\omega) - \overline{Y}_\omega \right)^2,
\]

and

\[
\hat{\Delta}_2^{BS} := \frac{1}{I_\omega J_\omega} \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j}(\omega) - \hat{Y}_i^B(\omega) - \hat{Y}_j^S(\omega) + \overline{Y}_\omega \right)^2.
\]

Notice that $\hat{\Delta}_2^{B}, \hat{\Delta}_2^{S}, \hat{\Delta}_2^{BS}$ are stochastic, as their value depends on the (random) assignment $W$ through the induced index sets $I_\omega, J_\omega$. Last, define

\[
\eta_\omega^B := \frac{1}{I} \left( \frac{J}{J_\omega} \right)^{-1} \sum_{J_\omega} \sum_{i} \{ \bar{Y}_{i,J_\omega}(\omega) - \bar{Y}_i^B(\omega) \}^2,
\]

and symmetrically

\[
\eta_\omega^S := \frac{1}{J} \left( \frac{I}{I_\omega} \right)^{-1} \sum_{I_\omega} \sum_{j} \{ \bar{Y}_{I_\omega,j}(\omega) - \bar{Y}_j^S(\omega) \}^2.
\]

In Lemmas A.8, A.9 and A.11, we analyze the expectation of each term $\hat{\Delta}_2^{B}, \hat{\Delta}_2^{S}, \hat{\Delta}_2^{BS}$ separately. First, we state a useful result in Lemma A.7.

**Lemma A.7.** Let

\[
\chi_\omega^{2,B} := \mathbb{E} \left[ \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right)^2 \right].
\]

It holds

\[
\chi_\omega^{2,B} = I_\omega (\hat{\Delta}_2^{2,B} + \eta_\omega^B),
\]

where $\eta_\omega^B$ was defined in Equation (A.10).

**Proof of Lemma A.7.** Consider $\chi_\omega^{2,B}$ as defined in Equation (A.12): notice that this expectation is taken with respect to the random assignment matrices $W$. Under the sampling scheme of (simple) double randomization, every assignment matrix $W$ supported by $\mathcal{W}(I_T, J_T)$ is equivalently characterized by the index sets $I_\omega, J_\omega$, for $\omega \in \{c, ib, is, t\}$. That is, to each $W$, there is one and only one collection of index sets $I_\omega, J_\omega$ for $\omega \in \{c, ib, is, t\}$, and viceversa. Notice that there are exactly $\binom{I}{I_\omega} \binom{J}{J_\omega}$ such assignments. Moreover, each assignment can be determined by first forming an index set $I_\omega$ by selecting at random $I_\omega$ rows, and then by forming an index set $J_\omega$ by selecting at random $J_\omega$ columns. Notice that, as far as the rows go, every row $i \in \{1, \ldots, I\}$ appears in exactly $\binom{I}{I_\omega} \binom{I}{I_{\omega-1}}$ index sets $I_\omega$. Hence,

\[
\chi_\omega^{2,B} = \frac{(I_{\omega-1})}{(I)} \left( \frac{J}{J_\omega} \right)^{-1} \sum_{i=1}^{I_\omega} \sum_{J_\omega} \left\{ \left( \bar{Y}_{i,J_\omega}(\omega) - \bar{Y}(\omega) \right)^2 \right\}
\]

\[
= \frac{I_\omega}{T} \left( \frac{J}{J_\omega} \right)^{-1} \sum_{i=1}^{I_\omega} \sum_{J_\omega} \left\{ \left( \bar{Y}_{i,J_\omega}(\omega) - \bar{Y}(\omega) \right)^2 \right\},
\]

\[(A.13)\]
where the second sum is over all \( \binom{J}{J_ω} \) subsets \( J_ω \) of \( J_ω \) distinct indices in \( \{1, \ldots, J\} \).

We can further decompose the term \( χ^2_B \) defined above. Fix a row \( i \) and a selection of \( J_ω \) disjoint indices \( J_ω = \{j_1, \ldots, j_{J_ω}\} \subseteq \{1, \ldots, J\} \). It holds

\[
\{ \bar{Y}_{i,j_ω} - \bar{Y}(\omega) \}^2 = \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) + \bar{Y}_i(\omega) - \bar{Y}(\omega) \right\}^2 \\
= \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2 + \left\{ \bar{Y}_i(\omega) - \bar{Y}(\omega) \right\}^2 \\
+ 2 \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\} \left\{ \bar{Y}_i(\omega) - \bar{Y}(\omega) \right\}.
\]

Summing over all choices \( J_ω \) of \( J_ω \) disjoint indices in the set \( \{1, \ldots, J\} \),

\[
\sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2 = \sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2 + \left( \frac{J}{J_ω} \right) \left\{ \bar{Y}_i(\omega) - \bar{Y}(\omega) \right\}^2,
\]

where we exploited the fact that

\[
\sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\} = 0,
\]

since \( \bar{Y}_i(\omega) = \frac{\sum_{J_ω} \bar{Y}_{i,j_ω}}{\binom{J}{J_ω}} \).

If we now sum over all buyers \( i \),

\[
\sum_{i=1}^{I} \sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}(\omega) \right\}^2 = \sum_{i=1}^{I} \sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2 + \left( \frac{J}{J_ω} \right) \sum_{i=1}^{I} \left\{ \bar{Y}_i(\omega) - \bar{Y}(\omega) \right\}^2 \\
= I \left( \frac{J}{J_ω} \right) (\eta^B + \Delta^2_B),
\]

where \( \eta^B := \frac{1}{J} \sum_{J_ω} \sum_{i=1}^{I} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2 \). Hence, plugging this in Equation (A.13),

\[
χ^2_B = \frac{I}{J} \left( \frac{J}{J_ω} \right)^{-1} \sum_{i=1}^{I} \sum_{J_ω} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}(\omega) \right\}^2 \\
= \frac{I}{J} \left( \frac{J}{J_ω} \right)^{-1} \left[ I \left( \frac{J}{J_ω} \right) (\eta^B + \Delta^2_B) \right] \\
= I \omega \left( \Delta^2_B + \eta^B \right). \tag{A.14}
\]

**Lemma A.8.** It holds

\[
\mathbb{E} \left[ \Delta^2_B \right] = \Delta^2_B - \mathbb{V} \left( \bar{Y}(\omega) \right) + \eta^B, \tag{A.15}
\]

where

\[
\eta^B := \frac{1}{I} \left( \frac{J}{J_ω} \right)^{-1} \sum_{J_ω} \sum_{i=1}^{I} \left\{ \bar{Y}_{i,j_ω} - \bar{Y}_i(\omega) \right\}^2.
\]
Proof of Lemma A.8.

\[
\mathbb{E} \left[ \hat{\Delta}^2_B \right] = \mathbb{E} \left[ \frac{1}{I_\omega} \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \overline{Y}_\omega \right)^2 \right] \\
= \frac{1}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left( \left[ \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right] - \left[ \overline{Y}_\omega - \overline{Y}(\omega) \right] \right)^2 \right] \\
= \frac{1}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left[ \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right]^2 \right] + \frac{1}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left[ \overline{Y}_\omega - \overline{Y}(\omega) \right]^2 \right] \\
- \frac{2}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left( \overline{Y}_\omega - \overline{Y}(\omega) \right) \left( \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right) \right] \\
= \frac{1}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left[ \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right]^2 \right] + \frac{1}{I_\omega} \mathbb{E} \left[ \left[ \overline{Y}_\omega - \overline{Y}(\omega) \right]^2 \right] \\
- \frac{2}{I_\omega} \mathbb{E} \left[ \left\{ \overline{Y}_\omega - \overline{Y}(\omega) \right\} \left\{ \overline{Y}_\omega - \overline{Y}(\omega) \right\} \right] \\
= \frac{1}{I_\omega} \mathbb{E} \left[ \sum_{i \in I_\omega} \left\{ \hat{Y}_i^B(\omega) - \overline{Y}(\omega) \right\} \right] - \mathbb{E} \left[ \left\{ \overline{Y}_\omega - \overline{Y}(\omega) \right\}^2 \right].
\]

Hence, we write

\[
\mathbb{E} \left[ \hat{\Delta}^2_B \right] = \frac{1}{I_\omega} \chi^2_B - \mathbb{V}(\overline{Y}_\omega), \tag{A.16}
\]

where we have used the definition of \( \chi^2_B \) given in Equation (A.12), Lemma A.7. Now going back plugging in Equation (A.14) in Equation (A.16) it follows

\[
\mathbb{E} \left[ \hat{\Delta}^2_B \right] = \frac{1}{I_\omega} \chi^2_B - \mathbb{V}(\overline{Y}_\omega) = \Delta^2_B - \mathbb{V}(\overline{Y}_\omega) + \eta^B.
\]

\[\square\]

Lemma A.9. It holds

\[
\mathbb{E} \left[ \hat{\Delta}^2_S \right] = \Delta^2_S - \mathbb{V}(\overline{Y}_\omega) + \eta^S, \tag{A.17}
\]

where \( \eta^S \) was defined in Equation (A.11).

Proof of Lemma A.9. The proof is identical to Lemma A.8, where we let \( \chi^2_S \) be the column counterpart to Equation (A.14),

\[
\chi^2_S := \mathbb{E} \left[ \sum_{j \in J_\omega} \left( \hat{Y}_j^S(\omega) - \overline{Y}(\omega) \right)^2 \right],
\]

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where, by the same argument of Lemma A.7, it holds
\[
\chi^2 S = J_\omega \Delta^2 S + \frac{J_\omega}{I} \left( \frac{1}{I} \sum_{j=1}^{I/\omega} (\hat{Y}^S_{I/\omega,j} - \hat{Y}^S_j)^2 \right)
\]
\[
= J_\omega \left( \Delta^2 S + \eta^S \right),
\] (A.18)
in which we sum over all \((I/\omega)\) index sets \(I_\omega\) of \(I_\omega\) disjoint indices in \(\{1, \ldots, I\}\).

It remains for us to characterize the cross-term \(\Delta^{2,\text{BS}}_S\). Let us start by stating a useful (and generic) decomposition for matrices.

**Lemma A.10.** Let \(x \in \mathbb{R}^{B \times S}\) be a matrix. Let
\[
\bar{x} := (BS)^{-1} \sum_{b,s} x_{b,s}
\]
be the grand mean of the matrix, where averaging is uniform across entries. Let
\[
\bar{x}^B_b := S^{-1} \sum_s x_{b,s} \quad \text{and} \quad \bar{x}^S_s := B^{-1} \sum_b x_{b,s}
\]
be the average of the \(b\)-th row and of the \(s\)-th column respectively. It holds
\[
\sum_{b,s} (x_{b,s} - \bar{x})^2 = S \sum_b (\bar{x}^B_b - \bar{x})^2 + B \sum_s (\bar{x}^S_s - \bar{x})^2
\]
\[
+ \sum_{b,s} \left( x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x} \right)^2.
\]

**Proof of Lemma A.10.**
\[
\sum_{b,s} (x_{b,s} - \bar{x})^2 = \sum_{b,s} \left( x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x} \right)^2
\]
\[
= \sum_{b,s} \left( (\bar{x}^B_b - \bar{x} + \bar{x}^S_s - \bar{x})^2
\]
\[
= \sum_{b,s} \left( (\bar{x}^B_b - \bar{x})^2 + (\bar{x}^S_s - \bar{x})^2 + (x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x})^2 \right)
\]
\[
= \sum_{b,s} (\bar{x}^B_b - \bar{x})^2 + \sum_{b,s} (\bar{x}^S_s - \bar{x})^2 + \sum_{b,s} (x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x})^2,
\]
where we have noted that all the cross terms in the square cancel since
\[
\sum_{b,s} (\bar{x}^B_b - \bar{x}) = 0, \quad \sum_{b,s} (\bar{x}^S_s - \bar{x}) = 0, \quad \sum_{b,s} (x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x}) = 0.
\]
Hence,
\[
\sum_{b,s} (x_{b,s} - \bar{x})^2 = S \sum_b (\bar{x}^B_b - \bar{x})^2 + B \sum_s (\bar{x}^S_s - \bar{x})^2 + \sum_{b,s} (x_{b,s} - \bar{x}^B_b - \bar{x}^S_s + \bar{x})^2.
\]
For our matrix of potential outcomes $Y(\omega)$, direct application of Lemma A.10 gives us

$$\frac{1}{IJ} \sum_{i,j} (Y_{i,j}(\omega) - \bar{Y}(\omega))^2 = \frac{1}{I} \sum_i \left( \hat{Y}_i^B(\omega) - \bar{Y}(\omega) \right)^2 + \frac{1}{J} \sum_j \left( \hat{Y}_j^S(\omega) - \bar{Y}(\omega) \right)^2$$

$$+ \frac{1}{IJ} \sum_{i,j} \left( Y_{i,j}(\omega) - \hat{Y}_i^B(\omega) - \hat{Y}_j^S(\omega) + \bar{Y}(\omega) \right)^2$$

$$= \Delta_{2,B}^2 + \Delta_{2,S}^2 + \Delta_{2,BS}^2.$$

(A.19)

We now analyze the expectation of the crossed term $\hat{\Delta}_{2,BS}^2$.

**Lemma A.11.** It holds

$$\mathbb{E} \left[ \hat{\Delta}_{2,BS}^2 \right] = \Delta_{2,BS}^2 + \mathbb{V}(\bar{Y}(\omega)) - \eta_{BS}^B \frac{I}{I_\omega} - \eta_{BS}^S \frac{J}{J_\omega}.$$

**Proof of Lemma A.11.**

$$\hat{\Delta}_{2,BS}^2 = \frac{1}{I_\omega J_\omega} \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j}(\omega) - \hat{Y}_i^B(\omega) - \hat{Y}_j^S(\omega) + \bar{Y}(\omega) \right)^2$$

$$= \frac{1}{I_\omega J_\omega} \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j}(\omega) - \bar{Y}(\omega) \right)^2 + \frac{1}{I_\omega} \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \bar{Y}(\omega) \right)^2$$

$$+ \frac{1}{J_\omega} \sum_{j \in J_\omega} \left( \hat{Y}_j^S(\omega) - \bar{Y}(\omega) \right)^2 + \frac{1}{I_\omega J_\omega} \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j}(\omega) - \hat{Y}_i^B(\omega) - \hat{Y}_j^S(\omega) + \bar{Y}(\omega) \right)^2$$

We can then analyze the expectation of $\hat{\Delta}_{2,BS}^2$. The proof of the lemma involves expanding terms and using properties of expectation and variance. The detailed steps involve breaking down the expression into components that can be analyzed using linearity of expectation and properties of the matrices of potential outcomes. The final result is given by the equation above, which expresses the expectation of the crossed term as a function of the variances and expected values of the potential outcomes and their differences.
Under the expectation operator,
\[
E \left[ \hat{\Delta}_\omega^{2,BS} \right] = \frac{1}{I_\omega J_\omega} E \left[ \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j}(\omega) - \bar{Y}(\omega) \right)^2 \right] - \frac{1}{I_\omega} E \left[ \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \bar{Y}(\omega) \right)^2 \right]
- \frac{1}{J_\omega} E \left[ \sum_{j \in J_\omega} \left( \hat{Y}_j^S(\omega) - \bar{Y}(\omega) \right)^2 \right] + \mathcal{V} \left( \bar{Y}_\omega \right)
- \frac{1}{I_\omega} E \left[ \sum_{i \in I_\omega} \left( \hat{Y}_i^B(\omega) - \bar{Y}(\omega) \right)^2 \right] - \frac{1}{J_\omega} E \left[ \sum_{j \in J_\omega} \left( \hat{Y}_j^S(\omega) - \bar{Y}(\omega) \right)^2 \right] + \mathcal{V} \left( \bar{Y}_\omega \right).
\]

Now, leveraging Equation (A.19) for the first summation, Equation (A.14) for the second summation, and Equation (A.18) for the third summation,
\[
E \left[ \hat{\Delta}_\omega^{2,BS} \right] = \Delta_\omega^{2,B} + \Delta_\omega^{2,S} + \hat{\Delta}_\omega^{2,BS} - \left[ \Delta_\omega^{2,B} + \eta_\omega^B \right] - \left[ \Delta_\omega^{2,S} + \eta_\omega^S \right] + \mathcal{V}(\hat{Y}_\omega)
= \Delta_\omega^{2,BS} + \mathcal{V}(\bar{Y}_\omega) - \eta_\omega^B - \eta_\omega^S.
\tag{A.20}
\]

We now use the characterizations Equations (A.15), (A.17) and (A.20), to define an unbiased estimator for \( \mathcal{V}(\bar{Y}_\omega) \), as stated in Lemma 3.

**Lemma A.12** (A longer version of Lemma 3). Let
\[
\alpha_\omega^B = \sqrt{\frac{1}{I_\omega} \frac{T - I_\omega}{I_\omega} \frac{1}{J_\omega} \frac{T - J_\omega}{J_\omega} \frac{I_\omega}{J_\omega}} \quad \text{and} \quad \alpha_\omega^S = \sqrt{\frac{1}{J_\omega} \frac{T - J_\omega}{J_\omega} \frac{1}{I_\omega} \frac{T - I_\omega}{I_\omega}}.
\]

Define
\[
\hat{\Sigma}_\omega := \frac{I_\omega^{-1} (\alpha_\omega^B)^2 \hat{\Delta}_\omega^{2,B} + J_\omega^{-1} (\alpha_\omega^S)^2 \hat{\Delta}_\omega^{2,S} + (I_\omega J_\omega)^{-1} (\alpha_\omega^B \alpha_\omega^S)^2 \hat{\Delta}_\omega^{2,BS}}{1 - (\alpha_\omega^B)^2 - (\alpha_\omega^S)^2 + (\alpha_\omega^B \alpha_\omega^S)^2} 
- \frac{(\alpha_\omega^B)^2}{1 - (\alpha_\omega^B)^2} \frac{1}{J_\omega - 1} \sum_{i \in I_\omega} \sum_{j \in J_\omega} \left( Y_{i,j} - \hat{Y}_i^B(\omega) \right)^2
- \frac{(\alpha_\omega^S)^2}{1 - (\alpha_\omega^S)^2} \frac{1}{I_\omega - 1} \sum_{j \in J_\omega} \sum_{i \in I_\omega} \left( Y_{i,j} - \hat{Y}_j^S(\omega) \right)^2.
\]

It holds
\[
E \left[ \hat{\Sigma}_\omega \right] = \mathcal{V} \left( \bar{Y}_\omega \right).
\]

**Proof of Lemma 3 and Lemma A.12.** Let us start by observing that, given the definition of \( \alpha_\omega^B, \alpha_\omega^S \), a simple re-writing of the results presented in Lemma A.4 allows us to write
\[
\mathcal{V} \left( \bar{Y}_\omega \right) = (\alpha_\omega^B)^2 \Delta_\omega^{2,B} + (\alpha_\omega^S)^2 \Delta_\omega^{2,S} + (\alpha_\omega^B \alpha_\omega^S)^2 \Delta_\omega^{2,BS}.
\]

Define
\[
\hat{\Gamma}_\omega = (\alpha_\omega^B)^2 \hat{\Delta}_\omega^{2,B} + (\alpha_\omega^S)^2 \hat{\Delta}_\omega^{2,S} + (\alpha_\omega^B \alpha_\omega^S)^2 \hat{\Delta}_\omega^{2,BS},
\]
and apply the expectation operator, leveraging the results in Lemmas A.8, A.9 and A.11,
\[
\mathbb{E} \left[ \hat{\Gamma}_\omega \right] = (\alpha^B_\omega)^2 \mathbb{E} \left[ \hat{\Delta}^{2,B} \right] + (\alpha^S_\omega)^2 \mathbb{E} \left[ \hat{\Delta}^{2,S} \right] + (\alpha^B_\omega \alpha^S_\omega)^2 \mathbb{E} \left[ \hat{\Delta}^{2,BS} \right] \\
= (\alpha^B_\omega)^2 \left( \Delta^{2,B} + \mathbb{V} \left( \bar{Y}_\omega \right) + \eta^B_\omega \right) + (\alpha^S_\omega)^2 \left( \Delta^{2,S} - \mathbb{V} \left( \bar{Y}_\omega \right) + \eta^S_\omega \right) \\
+ (\alpha^B_\omega \alpha^S_\omega)^2 \left( \Delta^{2,BS} + \mathbb{V} \left( \bar{Y}_\omega \right) - \eta^B_\omega - \eta^S_\omega \right).
\]
Rearranging,
\[
\mathbb{E} \left[ \hat{\Gamma}_\omega \right] = \mathbb{V} \left( \bar{Y}_\omega \right) \left\{ (1 - (\alpha^B_\omega)^2 - (\alpha^S_\omega)^2 + (\alpha^B_\omega \alpha^S_\omega)^2) + (\alpha^B_\omega)^2 \left( 1 - (\alpha^S_\omega)^2 \right) \eta^B_\omega + (\alpha^S_\omega)^2 \left( 1 - (\alpha^B_\omega)^2 \right) \eta^S_\omega \right\}.
\]
Hence, observing that
\[
\frac{x(1 - y)}{1 - x - y + xy} = \frac{x(1 - y)}{(1 - x)(1 - y)} = \frac{x}{1 - x},
\]
and rescaling the quantity above,
\[
\frac{\mathbb{E} \left[ \hat{\Gamma}_\omega \right]}{1 - (\alpha^B_\omega)^2 - (\alpha^S_\omega)^2 + (\alpha^B_\omega \alpha^S_\omega)^2} = \mathbb{V} \left( \bar{Y}_\omega \right) + \frac{(\alpha^B_\omega)^2}{1 - (\alpha^S_\omega)^2} \eta^B_\omega + \frac{(\alpha^S_\omega)^2}{1 - (\alpha^B_\omega)^2} \eta^S_\omega.
\]
We now leverage classic results on randomized experiments to obtain unbiased estimates for $\eta^B_\omega, \eta^S_\omega$.

First, the variance of the row-mean estimate follows from Lemma A.6:
\[
\mathbb{V} \left( \bar{Y}_i^B (\omega) \right) = \mathbb{E} \left[ \left( \bar{Y}_i^B (\omega) - \bar{Y}_i^B (\omega) \right)^2 \right] = \frac{J - J_\omega}{J_\omega} \frac{1}{J - 1} \sum_{j=1}^{J} \left( Y_{i,j} (\omega) - \bar{Y}_i^B (\omega) \right)^2,
\]
where the expression Equation (A.21) is implied by Lemma A.6 since in a SMRD, we can see each row $i$ as its own population with mean $\bar{Y}_i^B$ and corresponding estimate $\hat{Y}_i^B$. Then, for those rows which feature at least two columns of type $\omega$, we can provide an unbiased estimate of the variance term in Equation (A.21). Define the sample estimate
\[
\hat{\mathbb{V}} \left( \bar{Y}_i^B (\omega) \right) := \frac{J - J_\omega}{J_\omega} \frac{1}{J - 1} \sum_{j=1}^{J} \left( Y_{i,j} (\omega) - \bar{Y}_i^B (\omega) \right)^2.
\]
From Equation (A.8),
\[
\mathbb{E} \left[ \left\{ \frac{1}{J_\omega - 1} \sum_{j \in J_\omega} \left( Y_{i,j} (\omega) - \bar{Y}_i^B (\omega) \right)^2 \right\} \right] = \frac{1}{J - 1} \sum_{j=1}^{J} \left( Y_{i,j} (\omega) - \bar{Y}_i^B (\omega) \right),
\]
which directly implies that
\[
\mathbb{E} \left[ \hat{\mathbb{V}} \left( \bar{Y}_i^B (\omega) \right) \right] = \mathbb{V} \left( \bar{Y}_i^B (\omega) \right).
\]
Averaging these estimates over the rows,
\[
\hat{\eta}^B_\omega = \frac{1}{J_\omega} \sum_{i \in I_\omega} \hat{\mathbb{V}} \left( \bar{Y}_i^B (\omega) \right),
\]
satisfying
\[
\mathbb{E}\left[\hat{\eta}_B^2\right] = \mathbb{E}\left[\frac{1}{I_\omega} \sum_{i \in I_\omega} \hat{\psi}\left(\hat{Y}_i^B(\omega)\right)\right] = \frac{1}{I}\binom{J_\omega}{I}^{-1} \sum_{i=1}^I (\hat{Y}_i^B - \bar{Y}_i^B)^2 =: \eta_B^2.
\]

Symmetrically for the sellers,
\[
\mathbb{V}\left(\hat{Y}_j^S(\omega)\right) = \mathbb{E}\left[\left(\hat{Y}_j^S(\omega) - \bar{Y}_j^S(\omega)\right)^2\right] = \frac{I - I_\omega}{I\omega} - \frac{1}{I - I_\omega} \sum_{i \in I_\omega} (Y_{i,j}(\omega) - \bar{Y}_j^S(\omega))^2.
\]
then
\[
\hat{\psi}\left(\hat{Y}_j^S(\omega)\right) := \frac{I - I_\omega}{I\omega} - \frac{1}{I - I_\omega} \sum_{i \in I_\omega} (Y_{i,j}(\omega) - \bar{Y}_j^S(\omega))^2.
\]
\[
\mathbb{E}\left[\hat{\psi}\left(\hat{Y}_j^S(\omega)\right)\right] = \mathbb{V}\left(\hat{Y}_j^S(\omega)\right).
\]

Average these estimates over the columns,
\[
\hat{\eta}_S^2 = \frac{1}{J_\omega} \sum_{j \in J_\omega} \hat{\psi}\left(\hat{Y}_j^S(\omega)\right), \text{ satisfying } \mathbb{E}\left[\hat{\eta}_S^2\right] = \eta_S^2.
\]

Therefore,
\[
\hat{\Sigma}_\omega = \hat{\Gamma}_\omega - \frac{(\alpha_B^2)}{1 - (\alpha_B^2)^2} \hat{\eta}_B^2 - \frac{(\alpha_S^2)}{1 - (\alpha_S^2)^2} \hat{\eta}_S^2 \text{ satisfies } \mathbb{E}\left[\hat{\Sigma}_\omega\right] = \mathbb{V}\left(\bar{\Sigma}_\omega\right).
\]

Proof of Lemma 4. The proof is a direct application of the Cauchy-Schwarz inequality: for any two random quantities \(X, Y\), it holds \(-\sqrt{\mathbb{V}(X)\mathbb{V}(Y)} \leq \mathbb{C}(X, Y) \leq \sqrt{\mathbb{V}(X)\mathbb{V}(Y)}\). Then, we can lower and upper bound any unknown covariance term \(\mathbb{C}(\bar{\Sigma}_\omega, \bar{\Sigma}_{\omega'})\) with the square root of the product of the corresponding variance estimates, and the result follows.