COMPACT QUANTUM METRIC SPACES FROM FREE GRAPH ALGEBRAS

KONRAD AGUILAR, MICHAEL HARTGLASS, AND DAVID PENNEYS

Abstract. Starting with a vertex-weighted pointed graph $(\Gamma, \mu, v_0)$, we form the free loop algebra $S_0$ defined in Hartglass-Penneys’ article on canonical C$^*$-algebras associated to a planar algebra. Under mild conditions, $S_0$ is a non-nuclear simple C$^*$-algebra with unique tracial state. There is a canonical polynomial subalgebra $A \subset S_0$ together with a Dirac number operator $N$ such that $(A, L^2 A, N)$ is a spectral triple. We prove the Haagerup-type bound of Ozawa-Rieffel to verify $(S_0, A, N)$ yields a compact quantum metric space in the sense of Rieffel.

We give a weighted analog of Benjamini-Schramm convergence for vertex-weighted pointed graphs. As our C$^*$-algebras are non-nuclear, we adjust the Lip-norm coming from $N$ to utilize the finite dimensional filtration of $A$. We then prove that convergence of vertex-weighted pointed graphs leads to quantum Gromov-Hausdorff convergence of the associated adjusted compact quantum metric spaces.

As an application, we apply our construction to the Guionnet-Jones-Shyakhtenko (GJS) C$^*$-algebra associated to a planar algebra. We conclude that the compact quantum metric spaces coming from the GJS C$^*$-algebras of many infinite families of planar algebras converge in quantum Gromov-Hausdorff distance.

1. Introduction

In Connes’ noncommutative geometry [Con89, Con94], the notion of a spectral triple is an analog of a space of smooth functions on a non-commutative manifold. In [Rie98, Rie99], Rieffel initiated the study of noncommutative metric geometry via the notion of a compact quantum metric space. He then introduced quantum Gromov-Hausdorff distance as a noncommutative analogue of Gromov-Hausdorff distance to provide a framework for establishing convergence of certain spaces arising in the operator algebra and high-energy physics literature [Rie04a, Rie04b].

To the best of our knowledge, all results proving convergence in quantum Gromov-Hausdorff distance do so for sequences of nuclear C$^*$-algebras, where finite-dimensional approximations are crucial in demonstrating convergence [Rie04a, KL09, Agu19, Lat17, JRZ18, KK21]. In this article, we prove a result about quantum Gromov-Hausdorff convergence for compact quantum metric spaces associated to non-nuclear free graph algebras produced from vertex-weighted pointed graphs.

Given an unoriented connected graph $\Gamma = (V, E)$ with an arbitrary vertex weighting $\mu : V \to (0, \infty)$, we replace each edge $\epsilon \in E$ between two distinct vertices by two oriented edges $\epsilon, \epsilon^{op} \in \vec{E}$ in opposite directions, and we replace each loop by a single oriented loop to obtain a strongly connected directed graph $\vec{\Gamma} = (V, \vec{E})$ which inherits the same weighting $\mu$. One forms the Toeplitz-Cuntz-Krieger graph algebra $T(\vec{\Gamma})$ [FR99] with generators $\ell(\epsilon)$ for $\epsilon \in \vec{E}$, and the free graph algebra [HP17] is given by

\[ S(\Gamma, \mu) = C^*(C_0(V) \cup \{ a_\epsilon \ell(\epsilon) + a_\epsilon^{-1} \ell(\epsilon^{op}) \mid \epsilon \in \vec{E} \}) \subset T(\vec{\Gamma}). \]

Here, each $a_\epsilon \in (0, \infty)$ depends on the weighting of the source and target of $\epsilon$, which is chosen so that $S(\Gamma, \mu)$ has a semifinite trace $\text{Tr}$. By [Har17], $S(\Gamma, \mu)$ is simple exactly when

\[ \mu(v) < \sum_{\epsilon \in \vec{E}, s(\epsilon) = v} \mu(t(\epsilon)); \quad (1) \]

we assume this condition in the sequel.

Date: September 16, 2021.
Now there are canonical projections \( p_v \in \mathcal{S}(\Gamma, \mu) \) for the vertices \( v \in V \), and by simplicity [HP17, HP14, Har17], each compression \( p_v \mathcal{S}(\Gamma, \mu) p_v \) is Morita equivalent to \( \mathcal{S}(\Gamma, \mu) \). We thus consider pointed weighted graphs, which are equipped with a basepoint \( v_0 \in V \) such that \( \mu(v_0) = 1 \). We consider the free loop algebra \( \mathcal{S}_0(\Gamma, \mu) := p_{v_0} \mathcal{S}(\Gamma, \mu) p_{v_0} \), which can be described as generated by loops on \( \Gamma \) based at \( v_0 \). Under condition (1), \( \mathcal{S}_0(\Gamma, \mu) \) also has unique trace [Har17].

The loop algebra \( \mathcal{S}_0 \) has a dense \( * \)-subalgebra \( A \) of finite linear combinations of loops, which acts by bounded operators on \( L^2(A, \tau_n) \cong L^2(\mathcal{S}_0, \tau_n) \). Moreover, \( A \) is filtered by finite dimensional \( * \)-closed subspaces \( A_n \) of linear combinations of loops of length at most \( n \), which satisfy \( A_m \cdot A_n \subset A_{m+n} \) and \( A_0 = \mathbb{C} 1_A \). In this situation, by [OR05, Lemma 1.1], the formula

\[
N = \sum_{n \geq 0} n \text{Proj}_{A_n \oplus A_{n-1}},
\]

defines a Dirac number operator which has bounded commutator with elements of \( A \). Thus \( (A, L^2 A, N) \) is a spectral triple in the sense of Connes [Con94]. We prove the Haagerup-type inequality of [OR05, Theorem 1.2], which gives the following theorem.

**Theorem A.** The Dirac number operator \( N \) induces a Lip-norm \( L \) on \( A \), making \( (\mathcal{S}_0, A, L) \) a compact quantum metric space in the sense of Rieffel [Rie04a].

Thus given a connected, vertex-weighted pointed graph \( (\Gamma, \mu, v_0) \), we get a canonical compact quantum metric space. Given a sequence of connected vertex-weighted graphs \( (\Gamma_n, \mu_n) \), we say it converges locally uniformly to a limit \( (\Gamma, \nu) \) if essentially on every ball of radius \( R \) about \( v \), the graphs \( \Gamma_n \) eventually coincide with \( \Gamma \), and the weights converge pointwise. This is a weighted analog of Benjamini-Schramm convergence [BS01]. (See Definition 3.9 for the precise definition.) With this definition in hand, we can ask whether the associated compact quantum metric spaces \( (\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), L_n) \) converge in quantum Gromov-Hausdorff distance to \( (\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), L) \).

Unfortunately, we were unable to solve this question due to two main problems. First, projecting an element in \( A_n \) onto \( A_{n-1} \) can increase the operator norm, similar to how truncating a Fourier series can increase the sup norm. Second, these algebras are non-nuclear, so we are missing the finite dimensional approximations which were essential to the results [Rie04a, KL09, Agu19, Lat17, JRZ18, KK21].

In analogous situations [Rie99, Agu16], one replaces the Lip norm \( L \) with another Lip norm \( \mathcal{L} \) produced by a Minkowski functional. In our setup, we choose \( \mathcal{L} \) so that it agrees with \( L \) on the spaces \( A_n \oplus A_{n-1} \) of homogeneous loops, i.e., spans of loops of the same length \( n \). While this produces a less canonical compact quantum metric space, these adjusted quantum metrics take advantage of the intrinsic finite-dimensional spaces \( A_n \oplus A_{n-1} \) of homogeneous loops, which replace the finite dimensional approximations in the nuclear setting. In §3.3 below, we are able to prove that these compact quantum metric spaces converge in quantum Gromov-Hausdorff distance to the desired limit.

**Theorem B.** If the sequence of vertex-weighted pointed graphs \( (\Gamma_n, \mu_n, v_0^n) \) converges locally uniformly to \( (\Gamma, \mu, v_0) \), then the induced compact quantum metric spaces \( (\mathcal{S}_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n) \) converge in quantum Gromov-Hausdorff distance to \( (\mathcal{S}_0(\Gamma, \mu), A(\Gamma, \mu), \mathcal{L}) \).

**Application to subfactor theory**
The original motivation in our two articles [HP17, HP14] was to develop a connection between subfactor theory and \( C^* \)-algebras with a view toward connections to Connes’ non-commutative geometry [Con94]. The standard invariant of a finite index subfactor forms a shaded subfactor planar algebra [Jon99]. Here, we work with unshaded unitary factor planar algebras, which correspond to symmetrically self-dual bifinite bimodules over some factor [BHP12, Pen20]. The more categorically minded reader may choose to work directly with a unitary tensor category as in [HHP20].

A special application of the setup in this article is when:

- \( \Gamma \) is the principal graph of an unshaded unitary factor planar algebra \( \mathcal{P}_\bullet \),
- \( \mu \) is a quantum dimension vertex-weighting which satisfies the Frobenius-Perron condition, and
- \( v_0 = * \), the distinguished vertex corresponding to the empty diagram/monoidal unit object.

2
In this case, the cutdown $S_0$ of $S(\Gamma, \mu)$ at $v_0 = \ast$ is isomorphic to the Guionnet-Jones-Shlyakhtenko (GJS) C*-algebra [HP17, HP14]. This algebra is the C*-completion of the graded algebra $Gr_0$ arising from their diagrammatic reproof [GJS10] of Popa’s celebrated subfactor reconstruction theorem [Pop95].

When $P_\ast = NC_\ast$, the factor planar algebra of non-commuting polynomials on self-adjoint variables $X_1, \ldots, X_n$, $Gr_0$ is exactly the algebra of non-commutative polynomials, and $S_0$ is Voiculescu’s reduced C*-algebra generated by $n$ free semi-circular elements. Thus we may view $Gr_0 \subset S_0$ as a smooth subalgebra of polynomials inside the algebra of non-commutative algebraic functions. We are thus in a position to study Connes’ non-commutative geometry via Dirac operators and spectral triples [Con94].

In subfactor theory, there are many examples of local uniform graph convergence. For instance, we have examples coming from quantum groups at roots of unity [Jon83, Wen88, GdlHJ89, Wen90, Xu98], continuous families of subfactors [BNP07], and from composites at a fixed index [BH96, Liu15]. A particular application to subfactor theory is the following corollary of Theorem B, which holds in much more generality than stated.

**Corollary C.** For a fixed $n$, the GJS C*-algebra $S_0(A_n, \exp(\pi i/(n + 1)))$ of the Temperley-Lieb-Jones (TLJ) (sub)factor planar algebra gives a compact quantum metric space when equipped with the Lip norm $L_n$ from the Dirac number operator. Adjusting our Lip norm to $L_n$ as in Theorem B, the associated compact quantum metric spaces converge in quantum Gromov-Hausdorff distance to the adjusted compact quantum metric space of the GJS C*-algebra of TLJ at $q = 1$.

**Acknowledgements.** The authors would like to thank Farzad Fathizadeh, Matilde Marcolli, Marc Rieffel, and Robin Tucker-Drob for helpful conversations. David Penneys was supported by NSF DMS grants 1500387/1655912 and 1654159.

2. **Background**

2.1. **Compact quantum metric spaces.** We rapidly recall the notions of Gromov-Hausdorff distance, order unit space, compact quantum metric space, and quantum Gromov-Hausdorff distance from [Rie04a].

**Definition 2.1.** Suppose we have two compact subsets $X, Y$ of a metric space $(Z, \rho)$. The Hausdorff distance between $X$ and $Y$ is given by

$$\text{dist}_H(X, Y) := \inf \{r > 0 | X \subset N_r(Y) \text{ and } Y \subset N_r(X)\},$$

where for $A \subset Z$, $N_r(A)$ is the $r$-neighborhood of $A$:

$$N_r(A) := \{z \in Z | \text{there is an } a \in A \text{ with } \rho(z, a) < r\}.$$ 

**Definition 2.2.** Now suppose $(X, \rho_X)$ and $(Y, \rho_Y)$ are independent compact metric spaces. Let $X \sqcup Y$ be the disjoint union of $X$ and $Y$, and let $\mathcal{M}(\rho_X, \rho_Y)$ be the set of all metrics $\rho$ on $X \sqcup Y$ such that

- $\rho$ induces the disjoint union topology on $X \sqcup Y$,
- $\rho|_X = \rho_X$ and $\rho|_Y = \rho_Y$.

The Gromov-Hausdorff distance between $(X, \rho_X)$ and $(Y, \rho_Y)$ is given by

$$\text{dist}_{GH}(X, Y) = \inf \{\text{dist}_H(X, Y) | X, Y \subset (X \sqcup Y, \rho) \text{ and } \rho \in \mathcal{M}(\rho_X, \rho_Y)\}.$$ 

**Definition 2.3.** An order unit space $(V, e)$ is a real vector space $V$ together with a partial order $\leq$ with an element $e$ called the order unit which satisfies

- (order unit) For every $v \in V$, there is an $r > 0$ such that $v \leq re$.
- (Archimedean property) If $v \leq re$ for all $r > 0$, then $v \leq 0$.

An order unit space has a norm, which is given by $\|v\| = \inf \{r > 0 | -re \leq v \leq re\}$.

**Example 2.4.** Suppose $A$ is a unital C*-algebra. Then the self-adjoint elements $A_{s.a.}$ of $A$ form an order unit space with order unit $1_A$.

**Definition 2.5.** Suppose $(V, e)$ is an order unit space.
• A state of \((V, e)\) is a continuous linear functional \(\mu \in V^*\) such that \(\mu(e) = 1 = \|\mu\|\). The space of states on \((V, e)\) is denoted \(S(V)\). Given a seminorm \(L\) on \((V, e)\), it induces a \([0, \infty]\)-valued metric on \(S(V)\) by
\[
\rho_L(\mu, \nu) = \sup \{|\mu(v) - \nu(v)|L(v)| \leq 1\}.
\]
• A Lip-norm on \((V, e)\) is a seminorm \(L\) on \(V\) such that
  1. \(L(v) = 0\) if and only if \(v \in \mathbb{R}e\).
  2. The topology on \(S(V)\) induced by \(\rho_L\) is the weak-* topology.

**Definition 2.6.** A compact quantum metric space is a triple \((V, e, L)\) where \((V, e)\) is an order unit space and \(L\) is a Lip-norm on \((V, e)\).

The following criterion will be useful in determining whether \(L\) is a Lip-norm on a unital C*-algebra \(A\). A *unital pre C*-algebra* is a pair \((A, \phi)\) where \(A\) is a unital complex *-algebra and \(\phi: A \to \mathbb{C}\) is a positive linear functional \((\phi(a^*a) \geq 0)\) with \(\phi(1_A) = 1\mathbb{C}\) such that the left action of \(A\) on \(L^2(A, \phi)\) is by bounded operators.

**Proposition 2.7 ([OR05, Prop. 1.3]).** Let \((A, \phi)\) be a unital pre C*-algebra, and let \(L\) be a seminorm on \(A\). Then \(L\) is a Lip-norm if and only if
\[
\{a \in A | L(a) \leq 1 \text{ and } \phi(a) = 0\}
\]
is a norm totally bounded subset of \(A\).

**Definition 2.8 ([Rie04a, Sections 3 and 4]).** Suppose we have compact quantum metric spaces \((V, e, V_L)\) and \((W, e_W, W_L)\). Then \((V \oplus W, (e, e_W))\) is an order unit space. Let \(\mathcal{M}(V, W)\) be the set of all Lip-norms \(L\) on \(V \oplus W\) which induce \(L_V\) on \((V, e_V)\) and \(L_W\) on \((W, e_W)\). The quantum Gromov-Hausdorff distance between \((V, e_V, V_L)\) and \((W, e_W, W_L)\) is
\[
\text{dist}_q(V, W) := \inf \{\text{dist}^p_{\mathcal{H}}(S(V), S(W)) | L \in \mathcal{M}(L_V, L_W)\}.
\]

The following lemma will help in providing important estimates later.

**Lemma 2.9.** Let \((V, e, L)\) be a compact quantum metric space, and let \(W \subseteq V\) be a unital subspace \((e \in W)\) such that \((W, e, L|_W)\) is a compact quantum metric space. If \(\phi \in S(V)\), then
\[
\text{dist}_q(V, W) \leq 2 \text{dist}^H_{\mathcal{H}}(\{a \in V | L(a) \leq 1 \text{ and } \phi(a) = 0\}, \{a \in W | L|_W(a) \leq 1 \text{ and } \phi(a) = 0\}).
\]

**Proof.** By [Lat16, Thm. 6.3], we have that
\[
\text{dist}_q(V, W) \leq 2 \text{dist}^H_{\mathcal{H}}(\{a \in V | L(a) \leq 1\}, \{a \in W | L|_W(a) \leq 1\}).
\]
However, by the discussion preceding [Lat16, Def. 3.14], we have that
\[
\text{dist}^H_{\mathcal{H}}(\{a \in V | L(a) \leq 1\}, \{a \in W | L|_W(a) \leq 1\})
\]
\[
\leq \text{dist}^H_{\mathcal{H}}(\{a \in V | L(a) \leq 1 \text{ and } \phi(a) = 0\}, \{a \in W | L|_W(a) \leq 1 \text{ and } \phi(a) = 0\}),
\]
which completes the proof. \(\square\)

### 2.2. The Ozawa-Rieffel criterion.
In [OR05], Ozawa and Rieffel give criteria to determine when a filtered *-algebra with a tracial state gives a compact quantum metric space. We now recall their setup and theorem.

**Assumptions 2.10.** For this section, \(A\) is a unital complex *-algebra equipped with a trace \(\text{tr}: A \to \mathbb{C}\) such that \((A, \text{tr})\) is a pre C*-algebra. We further assume:
• \(A\) is filtered by *-closed finite dimensional subspaces. That is, there are finite dimensional *-closed subspaces \(A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots\) whose union is \(A\) which satisfy \(A_m \cdot A_n \subseteq A_{m+n}\).
• The ground algebra \(A_0\) is trivial, i.e., \(A_0 = \mathbb{C}1_A\).
The left (and right) action(s) of \( A \) on \((A, \text{tr})\) is bounded in \( \| \cdot \|_2 \), and thus extends to an action on \( \mathcal{H} = L^2(A, \text{tr}) \) by bounded operators.

Under these assumptions, we set \( W_n = A_n \oplus A_{n-1} \) which is finite dimensional, and we let \( P_n \) be the orthogonal projection from \( \mathcal{H} \) onto \( W_n \).

**Definition 2.11.** The Dirac number operator is defined by \( N := \sum_{n \geq 0} nP_n \), which is closable with dense domain.

One has the following lemma due to [OR05].

**Lemma 2.12 ([OR05, Lemma 1.1]).** For every \( a \in A \), \([N, a]\) is densely defined and extends to a bounded operator on \( \mathcal{H} \).

Using this lemma, we define a seminorm \( L \) on \( A \) by \( L(a) = \|[N, a]\| \). Observe that \( L \) vanishes exactly on \( A_0 = \mathbb{C}1_A \). We set \( \mathcal{A} = \overline{\mathcal{A}}^{\| \cdot \|} \), and we denote by \( S(\mathcal{A}) \) the state space of \( \mathcal{A} \). In the theory of compact quantum metric spaces, one induces a metric \( \rho \) on \( S(\mathcal{A}) \) with values in \([0, \infty)\) by the formula

\[
\rho(\phi, \psi) = \sup \{ |\phi(x) - \psi(x)| \mid x \in A \text{ with } L(x) \leq 1 \}.
\]

If the topology induced by \( \rho \) induces the weak-* topology on \( S(\mathcal{A}) \), we say that \((\mathcal{A}, A, L)\) is a compact quantum metric space in the sense of Rieffel [Rie04a]. A main result of [OR05] is the following theorem:

**Theorem 2.13 ([OR05, Theorem 1.2]).** If there exists a \( C > 0 \) such that for all \( m, n, k \in \mathbb{N} \) and \( a_k \in W_k \)

\[
\|P_m a_k P_n\| \leq C \|a_k\|_2,
\]

then \((\mathcal{A}, A, L)\) is a compact quantum metric space.

2.3. Free graph algebras. Let \( \Gamma = (V, E) \) be a countable, connected, locally finite, undirected graph, and let \( \mu : V \to \mathbb{R}_{>0} \) be a weighting on the vertices. (The examples in the later part of this article will be principal graphs of (sub)factor planar algebras with a quantum dimension weighting which satisfies the Frobenius-Perron condition.)

We now follow the construction from [HP17], summarized in [Har17, Section 2.1].

**Definition 2.14.** From our undirected graph \( \Gamma \), we form a directed graph \( \vec{\Gamma} = (V, \vec{E}, s, t) \) as follows.

(1) For each \( e \in E \) which has endpoints \( \alpha \neq \beta \) in \( V \), we get two directed edges \( e \) and \( e^\text{op} \) in \( \vec{E} \) such that \( s(e) = t(e^\text{op}) = \alpha \) and \( s(e^\text{op}) = t(e) = \beta \).

(2) For each \( e \in E \) which is a loop at the vertex \( \gamma \), we get one directed edge \( e \in \vec{E} \) with \( s(e) = t(e) = \gamma \). Note that \( \vec{\Gamma} \) inherits the weighting \( \mu \) from \( \Gamma \).

The algebra \( C_0(V) \) is the C*-algebra generated by the indicator functions \( p_v \) for \( v \in V \) acting on \( \ell^2(V) \). The \( C_0(V) \) Hilbert bimodule \( \mathcal{X} \) is the completion of the space of formal finite \( \mathbb{C} \)-linear combinations of edges of \( \vec{E} \), under the \( C_0(V) \)-valued inner product given by \( \langle \epsilon, \epsilon' \rangle = \delta_{\epsilon, \epsilon'} p_{t(\epsilon)} \). The action of \( C_0(V) \) on \( \mathcal{X} \) is given by \( p_v \epsilon = \delta_{v, s(\epsilon)} \epsilon \) and \( p_v \epsilon = \delta_{v, t(\epsilon)} \epsilon \).

We then form the Pimsner-Fock space

\[
\mathcal{F}(\vec{\Gamma}) = C_0(V) \oplus \bigoplus_{n \geq 1} \bigotimes_{\mathbb{C}_0(V)}^n \mathcal{X}.
\]

The spaces \( \bigotimes_{\mathbb{C}_0(V)}^n \mathcal{X} \) are spanned by elements of the form \( \epsilon_1 \otimes \cdots \otimes \epsilon_n \) such that \( \epsilon_1 \cdots \epsilon_n \) is a path in \( \vec{\Gamma} \).

For each edge \( e \in \vec{E} \), we get bounded creation and annihilation operators on \( \mathcal{F}(\vec{\Gamma}) \) given by

\[
\ell(e)(\epsilon_1 \otimes \cdots \otimes \epsilon_n) := \epsilon \otimes \epsilon_1 \otimes \cdots \otimes \epsilon_n,
\]

\[
\ell(e)^* (\epsilon_1 \otimes \cdots \otimes \epsilon_n) := \langle e\epsilon_1, \cdots, e\epsilon_n \rangle_{C_0(V)} \epsilon_2 \otimes \cdots \otimes \epsilon_n.
\]

The Pimsner-Toeplitz algebra \( \mathcal{T}(\vec{\Gamma}) \) is the C*-algebra generated by the \( \ell(e), \ell(e)^* \).
**Definition 2.15.** The free graph algebra $S = S(\Gamma, \mu)$ is generated by the edge elements

$$X_\epsilon = a_\epsilon \ell(\epsilon) + a_\epsilon^{-1} \ell(\epsilon^\text{op}) \quad \text{where} \quad a_\epsilon = \sqrt{\frac{\mu(s(\epsilon))}{\mu(t(\epsilon))}}.$$  

Note that $a_\epsilon^{-1} = a_{\epsilon^\text{op}}$. We set $X_\epsilon := X_\epsilon + X_{\epsilon^\text{op}}$.

We now give the structure of the free graph algebra. There is a conditional expectation $E : S(\Gamma, \mu) \to C_0(V)$ given by $E(\chi) = \sum_{v \in V(\Gamma)} \langle v | x v \rangle C_0(v)$. We have the following lemma.

**Lemma 2.16 ([GJS11, HP17]).** The algebras $S_{e, \mu} = C^*(C_0(V), X_\epsilon)$ are free with amalgamation over $C_0(V)$ with respect to the conditional expectation $E$, i.e.

$$S(\Gamma, \mu) = \,* (S_{e, \mu}, E).$$

Furthermore, $\mu \circ E$ defines a (semifinite) a trace $\text{Tr}$ on $S$.

One can check that $\text{Tr}(p_v) = \mu(v)$ for all $v \in V$.

3. **Free loop algebras and compact quantum metric spaces**

3.1. **Free loop algebras give compact quantum metric spaces.** Let $H_{\Gamma, \mu} = F(\bar{\Gamma}) \otimes C_0(V) \ell^2(V(\Gamma), \mu)$.

Here, $\ell^2(V(\Gamma), \mu)$ is the Hilbert space spanned by $V$, whose inner product is given by $\langle v, w \rangle = \delta_{v=w} \mu(v)^2$.

Note that paths in $\bar{\Gamma}$ give an orthogonal basis for $H_{\Gamma, \mu}$, and $\|\epsilon_1 \otimes \cdots \otimes \epsilon_n\|_{H_{\Gamma, \mu}} = \sqrt{\mu(t(\epsilon_n))}$.

We introduce the following notation.

**Notation 3.1.** Let $\Pi$ denote the set of all paths in $\bar{\Gamma}$, and $\sigma = \epsilon_1 \cdots \epsilon_n \in \Pi$. Let $|\sigma| = n$ denote the length of $\sigma$. We set:

- $\sigma^\text{op} := \epsilon_n^\text{op} \cdots \epsilon_1^\text{op}$
- $\ell(\sigma) := \ell(\epsilon_1) \cdots \ell(\epsilon_n)$.
- $a_\sigma := \sqrt{\frac{\mu(s(\sigma))}{\mu(t(\sigma))}} = \sqrt{\frac{\mu(s(\epsilon_1))}{\mu(t(\epsilon_n))}}$
- $X_\sigma := X_{\epsilon_1} \cdots X_{\epsilon_n}$.
- $Y_\sigma := \sum_{\sigma = \rho \tau} a_\rho a_\tau^{-1} \ell(\rho) \ell(\tau^\text{op})^\ast$.

Once it is shown in Proposition 3.2 below that $Y_\sigma \in S(\Gamma, \mu)$, it will follow from faithfulness of the trace that $Y_\sigma$ is the unique element in $S(\Gamma, \mu)$ whose right support is under $\rho(t(\sigma))$ and satisfies $Y_\sigma \cdot \rho(t(\sigma)) = a_\sigma \rho$. The element $Y_\sigma$ is known as the Wick word of $a_\sigma \rho$. Observe that $Y_{\sigma^*} = Y_{\sigma^\text{op}}$.

We now perform a change of basis from the $X$’s to $Y$’s. These $Y$’s will be useful later on as they are eigenvectors of the number operator.

**Proposition 3.2 (Change of basis).** Suppose $\sigma$ is a path in $\bar{\Gamma}$ of length $n$.

1. $Y_\sigma \in S(\Gamma, \mu)$
2. $X_\sigma = X_\sigma + Q$ where $Q$ is a linear combination of the $X_{\sigma'}$ with $|\sigma'| < n$.
3. $X_\epsilon = X_\epsilon + P$ where $P$ is a linear combination of the $Y_{\sigma'}$ with $|\sigma'| < n$.

**Proof.** We will prove this by induction on $|\sigma|$, the length of $\sigma$. If $|\sigma| = 1$, then $\sigma = \epsilon$ for some $\epsilon \in \bar{\Gamma}$ and it is apparent that $Y_\epsilon = X_\epsilon$.

Given, $\sigma$ with $|\sigma| > 1$, write $\sigma = \epsilon \tau$ for $\epsilon \in \bar{\Gamma}$, and write $\tau = \epsilon' \tau'$ for $\epsilon' \in \bar{\Gamma}$. We see that

$$X_\epsilon Y_\tau = (a_\epsilon \ell(\epsilon) + a_{\epsilon^\text{op}} \ell(\epsilon^\text{op})^\ast) \sum_{\tau = \tau_1 \tau_2} a_{\tau_1} a_{\tau_2}^{-1} \ell(\tau_1) \ell(\tau_2^\text{op})^\ast$$

$$= \sum_{\tau = \tau_1 \tau_2} a_{\epsilon \tau_1} a_{\tau_2} \ell(\epsilon \tau_1) \ell(\tau_2^\text{op})^\ast + \delta_{\epsilon \tau_1 \tau_2} a_{\tau_1} a_{\tau_2}^{-1} \ell(\tau_1^\text{op}) \ell(\tau_2^\text{op})^\ast + a_{\tau_1} a_{\tau_2}^{-1} \ell(\epsilon^\text{op})^\ast \ell(\tau^\text{op})^\ast$$

$$= Y_\sigma + \delta_{\epsilon \tau_1 \tau_2} Y_{\tau_1 \tau_2}.$$
Assumption 3.3. Our vertex-weighted graph \((\Gamma, \mu)\) comes with a pointing \(v_0 \in V\) with \(\mu(v_0) = 1\) which is minimal amongst vertex weights, i.e., \(\mu(v_0) \leq \mu(v)\) for all \(v \in V\).

Definition 3.4. From the free graph algebra \(S = S(\Gamma, \mu)\), we define the free loop algebra at \(v_0\), denoted \(S_0\), as the cutdown at \(p_{v_0}\), i.e., \(S_0 = p_{v_0}SP_{v_0}\).

Remark 3.5. In [HP17, Har17] the K-theory of \(S(\Gamma, \mu)\) was shown to be given by \(K_0(S(\Gamma, \mu)) = \mathbb{Z}\{[p_v]|v \in V\}\), the free abelian group generated by the equivalence classes of the projections \(p_v\), and \(K_1(S(\Gamma, \mu)) \cong \{0\}\). Under the mild assumption (1), \(S(\Gamma, \mu)\) is simple, and thus \(S_0(\Gamma, \mu)\) is Morita equivalent to \(S(\Gamma, \mu)\). Moreover, under condition (1), \(S_0(\Gamma, \mu)\) has unique trace. It follows that if \((\Gamma, \mu, v_0)\) and \((\Gamma', \mu', v_0)\) are two pointed weighted graphs, and the additive groups generated by \(\{\mu(v)|v \in V\}\) and \(\{\mu(v')|v' \in V'\}\) do not agree, then \(S_0(\Gamma, \mu) \not\cong S_0(\Gamma', \mu')\).

Let \(H_0\) be the GNS Hilbert space of \(S_0\) under the finite trace \(tr_0 = Tr|S_0\). Let \(\Pi_0\) denote the set of loops based at \(v_0\). Note that \(\Pi_0\) is an orthonormal basis for \(H_0 = L^2(S_0, tr_0)\). Furthermore, if \(\sigma \in \Pi_0\), then \(Y_\sigma\) is the unique element in \(S_0\) satisfying \(Y_\sigma v_0 = \sigma\) (note that \(a_\sigma\) is necessarily 1 if \(\sigma\) is a loop). This means that we have the following important fact:

Fact 3.6. The set \(\{Y_\sigma|\sigma \in \Pi_0\}\) is an orthonormal basis for \(L^2(S_0, tr_0)\).

We define an unbounded operator \(N) in \(H_0\) by the closure of the operator satisfying \(N(\sigma) = n\sigma\) whenever \(\sigma\) is a loop of length \(n\). Let \(A\) be the unital \(*\)-algebra generated by the elements \(Y_\sigma\), which by Proposition 3.2, is also generated by the elements \(X_\sigma\). Notice that under the identification of \(H_0\) with \(L^2(S_0, tr_0)\), we may realize \(N : A \to A\) as an unbounded operator satisfying \(N(Y_\sigma) = nY_\sigma\) whenever \(\sigma\) is a loop of length \(n\). Observe that due to cutting down \(S\) by \(p_{v_0}\), it follows that the null space of \(N\) is precisely scalar multiples of \(p_{v_0}\), the identity in \(S_0\). Set \(A_n := \text{span}\{Y_\sigma|\sigma \leq n\}\), which is \(*\)-closed and finite dimensional. Furthermore, it is straightforward to see that \(A_n \cdot A_m \subseteq A_{n+m}\), giving a \(*\)-filtration of \(A\) by finite dimensional subspaces. We are now in position to use the Ozawa-Rieffel framework as in §2.2.

As above, we set \(W_n := \text{span}\{Y_\sigma|\sigma = n\}\) = \(A_n \cap A_{n-1}\), and we define \(P_n\) to be the orthogonal projection from \(H_0\) onto \(W_n\).

Lemma 3.7. If \(x \in W_k\), then \(\|P_n x P_n\| \leq \|x\|\).

Proof. Write \(x = \sum_{|\sigma|=k} b_\sigma Y_\sigma\). Note that \(\|x\|^2 = \sum_{|\sigma|=k} |b_\sigma|^2\). We need to show that if \(\xi \in P_n H_0\) then we have

\[
\|P_n x \xi\|_{H_0} \leq \|x\|_{H_0} \|\xi\|_{H_0}
\]

Write \(\xi = \sum_{|\tau|=n} c_\tau \tau\). The term \(P_n x P_n\) is zero unless \(|m-n| \leq k \leq m+n\). Choose \(j\) such that \((k-j) + (n-j) = m\), and write

\[
x = \sum_{|\rho|=|k-j|} b_{\rho_1} \rho_1 Y_{\rho_1} \quad \text{and} \quad \xi = \sum_{|\tau|=n-j} c_{\rho_2} \rho_2 \tau_2.
\]

This means that

\[
P_n x \xi = \left( \sum_{|\rho_1|=k-j|} b_{\rho_1} \rho_1 \ell(\rho_1) \ell(\tau_1)^* \right) \cdot \sum_{|\rho_2|=j} c_{\rho_2} \rho_2 \tau_2 = \sum_{|\rho|=|k-j|} a_{\rho\sigma} b_{\rho\sigma} c_{\sigma} \rho \tau.
\]
From this, we see that
\[
\|P_m x \xi\|_2^2 = \sum_{|\rho|=k-j, |\tau|=n-j} \left| \sum_{|\sigma|=j} a_{\sigma}^2 b_{\rho\sigma} c_{\sigma\tau} \right|^2 \\
\leq \sum_{|\rho|=k-j, |\tau|=n-j} \left( \sum_{|\sigma|=j} a_{\sigma}^2 b_{\rho\sigma} c_{\sigma\tau} \right)^2 \\
\leq \sum_{|\rho|=k-j, |\tau|=n-j} \left( \sum_{|\sigma|=j} b_{\rho\sigma} c_{\sigma\tau} \right)^2 \\
\leq \left( \sum_{|\sigma|=k} |b_{\sigma'}|^2 \right) \cdot \left( \sum_{|\sigma'|=n} |c_{\sigma'}|^2 \right) \\
= \|x\|_2^2 \cdot \|\xi\|_2^2
\]
as desired. \(\square\)

Lemma 3.7 immediately implies Theorem A.

**Proof of Theorem A.** Lemma 3.7 allows us to use the Ozawa-Rieffel criterion in Theorem 2.13 (for \(C=1\)) to conclude that \((S_0, A, N)\) is a compact quantum metric space. \(\square\)

3.2. Convergence for weighted pointed graphs. We now discuss a type of convergence for vertex-weighted pointed graphs, which is a weighted analog of Benjamini-Schramm convergence [BS01]. As in the previous sections, the graphs we consider are countable, connected, locally finite, undirected, vertex-weighted, and pointed, where the base-point has minimal weight 1. The following notation will be handy.

**Notation 3.8.** Suppose \(\Gamma = (V, E, v)\) is such a graph and \(R \in \mathbb{N}\). We denote by \(\Gamma(R)\) the truncation of \(\Gamma\) to the ball of radius \(R\) of \(\Gamma\) based at \(v\).

**Definition 3.9.** Suppose we have a sequence of such graphs \((\Gamma_n = (V_n, E_n, v_n), \mu_n)\), and another graph \((\Gamma = (V, E, v), \mu)\). We say that \(\Gamma_n\) converges locally to \(\Gamma\) if for all \(R \in \mathbb{N}\), there is an \(N_R > 0\) such that for every \(n > N_R\),

- there is a pointed graph isomorphism \(\varphi_n^R : \Gamma_n(R) \rightarrow \Gamma(R)\), and
- these graph isomorphisms satisfy for all \(n > \max\{N_R, N_{R+1}\}\), \(\varphi_n^{R+1}\mid_{\Gamma_n(R)} = \varphi_n^R\).

Moreover, we say \(\Gamma_n \rightarrow \Gamma\) locally uniformly if \(\Gamma_n \rightarrow \Gamma\) locally and the isomorphisms \(\varphi_n^R\) satisfy
- for every vertex \(w \in \Gamma\) with \(\text{dist}(v, w) \leq R\), \(\lim_{n \rightarrow \infty} \mu_n((\varphi_n^R)^{-1}(w)) = \mu(w)\).

**Examples 3.10.** We give several examples of local uniform graph convergence.

1. (Subgraphs converging to a limit graph) Consider the Coxeter-Dynkin diagrams \(A_n\) with their unique normalized Frobenius-Perron weighting, where the base-point is at the left:

```
[1] [2] [3] \cdots [n-1] [n]
```

where \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\) and \(q = \exp\left(\frac{2\pi i}{2n}\right)\).
It is easy to see that the $A_n$ converge to the Coxeter-Dynkin diagram $A_\infty$ with its Frobenius-Perron weighting

$$
\begin{array}{cccccc}
1 & 2 & 3 & \cdots
\end{array}
$$

Just observe that as $\theta \to 0$, we have $q = e^{i\theta} \to 1$, so $[n] \to n$.

(2) (Weightings converging on the same graph) We fix the graph $A_\infty$, but we consider the continuous family of Frobenius-Perron weightings given by

$[1]^{\star} [2] [3] \cdots$ where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $q \geq 1$.

It is easily verified that any convergent sequence $q_n \to q_0$ gives a convergent sequence of graphs.

(3) (Existence of only local isomorphisms) Consider the affine Coxeter-Dynkin diagrams $D_n^{(1)}$ with their unique normalized Frobenius-Perron weighting:

$$
\begin{array}{cccc}
1 & \cdots & 2 & 1
\end{array}
$$

It is easily verified that these graphs converge to the affine Coxeter-Dynkin diagram $D_\infty$ with its Frobenius-Perron weighting

$$
\begin{array}{cccc}
1 & \cdots & 2 & 1
\end{array}
$$

3.3. Adjusting the Lip-norm. Nuclearity is often used to establish quantum Gromov-Hausdorff convergence of infinite-dimensional quantum metric spaces (see [Rie04a, KL09, Lat17, Agu19, JRZ18, KK21] where nuclearity is either implicitly or explicitly used to provide finite-dimensional approximations). Since $S_0$ is exact but non-nuclear, we do not have contractive completely positive maps for finite dimensional approximations. Instead, we pass to a new Lip-norm $L$ on $S_0$ defined from the finite dimensional spaces $A_n \oplus A_{n-1}$ from our filtration $(A_n)$ of $A \subset S_0$. The spaces $A_n \oplus A_{n-1}$ of homogeneous loops provide an appropriate finite dimensional approximation.

**Definition 3.11.** Let $W_n := \text{span} \{ Y_\sigma \mid |\sigma| = n \}$ i.e., the span of the Wick words of length $n$ in $S_0$, and observe $A_n = \text{span} \bigcup_{k=0}^{n} W_k$. Set $B_n := \{ x \in W_n | L(x) \leq 1 \}$, and define

$$
C' := \text{conv} \bigcup_{k=0}^{\infty} B_k, \quad C := \text{conv} \bigcup_{k=1}^{\infty} B_k, \quad \text{and} \quad C_n := \text{conv} \bigcup_{k=1}^{n} B_k = \text{conv} \bigcup_{k=1}^{n} B_k.
$$

We then define $L$ on $S_0$ to be the Minkowski functional associated to $C'$, i.e.

$$
L(x) := \inf \{ r > 0 \mid r^{-1} x \in C' \}.
$$

Observe that $L(x) = \infty$ whenever $r^{-1} x \notin C'$ for all $r > 0$. Clearly $L$ is finite on $A$.

**Remarks 3.12.**

(1) By construction, $L \leq L$.

(2) By lower semi-continuity of $L$, we have that

$$
\{ x \in S_0 \mid L(x) \leq 1 \} \subset \{ x \in S_0 \mid L(x) \leq 1 \}.
$$

Thus, as $L$ is a Lip-norm, it follows from Proposition 2.7 that $L$ is a Lip-norm.

(3) Observe that if $x \in W_n$, then $L(x) = L(x)$. Indeed, by lower-semicontinuity of $L$, we see that $L \leq 1$ on $C$. So if $L(x) = 1$, then $\alpha x \notin C$ for all $\alpha > 1$, and thus $L(x) = 1$. 

9
Lemma 3.13. For every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that $x \in A \ominus A_K$ and $L(x) \leq 1$ implies $\|x\| < 2\varepsilon / 3$. Moreover, for all $n > m > K$, $\text{dist}_H(C_m, C_n) < 4\varepsilon / 3$. This result holds independent of the graph $\Gamma$ and the vertex-weighting $\mu$ on $S_0$.

Proof. By [OR05, §3], given $a \in A$ with $L(a) \leq 1$ and $\varepsilon > 0$, there are $K > M > 0$ depending only on the constant $C$ in Theorem 2.13 ([OR05, Theorem 1.2]) so that

- if $a^{(M)} = \sum_{m-n \geq M} P_m a P_n$ and $a^M = a - a^{(M)}$, then $\|a^{(M)}\| < \varepsilon / 3$
- if $\tilde{a}_K = \sum_{k \leq K} a_K$ and $\tilde{a}_K = a - \tilde{a}_K$ (which are both in $A(t)$), $\|\tilde{a}_K\|^M < \varepsilon / 3$.

Thus for $x \in A \ominus A_K$, we have

$$\|x\| = \|(\tilde{x}_K)^M + (\tilde{x}_K)^M + x^M\| = \|(\tilde{x}_K)^M + x^M\| < 2\varepsilon / 3.$$  

It was shown in Lemma 3.7 that $C = 1$ regardless of the graph $\Gamma$ and the vertex-weighting $\mu$ (as long as the base point $v_0$ has minimal weighting). Therefore, it follows that if $x \in A \ominus A_K$ and $L(x) \leq 1$, then $L(x) \leq 1$ and hence $\|x\| < 2\varepsilon / 3$.

Now by definition, $C_m \subset C_n$. Observe that when $V$ is a vector space and $S \subset T \subset V$, then every element in $\text{conv}(T)$ is a convex combination of an element in $\text{conv}(S)$ and an element in $\text{conv}(T \setminus S)$. Hence if $x \in C_n$, we have $x = \lambda y + (1 - \lambda)z$ where $y \in C_m$ and $z \in \text{conv} \bigcup_{m=1}^n B_k$. Since $z \in A \ominus A_K$ and $L(z) \leq 1$, $\|z\| < 2\varepsilon / 3$ by the preceding paragraph. We conclude that $\text{dist}_H(C_m, C_n) < 4\varepsilon / 3$ as desired. □

Corollary 3.14. For every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that $\text{dist}_q((S_0, A, L), (A_k, L_{A_k})) < \varepsilon$ for all $k \geq K$. Again, this result holds independent of the graph $\Gamma$ and the vertex-weighting $\mu$ on $S_0$.

Proof. Immediate from Lemmas 2.9 and 3.13. □

Lemma 3.15. Let $(M, \rho)$ be a normed linear space. Suppose we have $k > 0$ sequences $(X^n_i)$, $i = 1, \ldots, k$, of compact convex subsets of $Z$ and compact convex subsets $X^i \subset Z$ such that $X^n_i \to X^i$ in Hausdorff distance for each $i = 1, \ldots, k$. Then

$$\text{conv} \left( \bigcup_{i=1}^k X^n_i \right) \to \text{conv} \left( \bigcup_{i=1}^k X^i \right)$$

in Hausdorff distance.

Proof. Let $\varepsilon > 0$. Choose $N$ sufficiently large such that $\text{dist}_H(X^n_i, X^i) < \varepsilon$ for all $n > N$ and $i = 1, \ldots, k$. Then for $x^i \in X^i$, $i = 1, \ldots, k$, we can choose $x^n_i \in X^n_i$ for $n > N$ with $\text{dist}_H(x^n_i, x^i) < \varepsilon$. Hence

$$\left| \sum_{i=1}^k \lambda_i x^n_i - \sum_{i=1}^k \lambda_i x^i \right| = \sum_{i=1}^k \lambda_i \|x^n_i - x^i\| < \varepsilon.$$

This means every point in $\text{conv} \left( \bigcup_{i=1}^k X^i \right)$ can be $\varepsilon$-approximated by a point in $\text{conv} \left( \bigcup_{i=1}^k X^n_i \right)$ where $n > N$. But the above argument clearly works swapping the roles of $\text{conv} \left( \bigcup_{i=1}^k X^n_i \right)$ and $\text{conv} \left( \bigcup_{i=1}^k X^i \right)$ for $n > N$. The result follows. □

Theorem 3.16. Let $(\Gamma_n, \mu_n)$ be a sequence of vertex-weighted pointed graphs converging locally uniformly to $(\Gamma, \mu)$, each of which have base point $v_0$ (here, we suppress the isomorphism data $\varphi_n^R$). Let $L_n$ be the Lip-norm constructed on $S_0(\Gamma_n, \mu_n)$ as in (2) above. Then $(S_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), L_n)$ converges in quantum Gromov-Hausdorff distance to $(S_0(\Gamma, \mu), A(\Gamma, \mu), L)$.

Proof. For any fixed $K \in \mathbb{N}$, $A_K(\Gamma_n, \mu_n)$ is finite-dimensional. For sufficiently large $n$, all of the $\Gamma_n(K)$ coincide (again, we suppress the isomorphism data $\varphi_n^R$). It follows that for these values of $n$, the vector spaces $A_K(\Gamma_n, \mu_n)$ are canonically isomorphic to the complex linear span of all loops in $\Gamma$ of length at most $K$ based at $v_0$. Setting

$$V_K := \text{span}_{\mathbb{R}} \{ a\sigma + a\sigma^{op} | a \in \mathbb{C} \text{ and } \sigma \in \Pi_0 \text{ such that } |\sigma| \leq K \},$$


$V_K$ is canonically isomorphic to $A_K(\Gamma, \mu)^{sa}$ and $A_K(\Gamma_n, \mu_n)^{sa}$ for sufficiently large $n$.

To prove quantum Gromov-Hausdorff convergence, we use [Rie04a, Theorem 11.2] on a continuous field of order unit spaces whose underlying vector space is $V_K$ obtained by transporting the norms from the compact quantum metric spaces $(S_0(\Gamma_n, \mu_n), A(\Gamma_n, \mu_n), \mathcal{L}_n)$. Fix a loop $\sigma$ of $\Gamma$ based at $v_0$, and let $Y_\sigma(n)$ be the Wick-word for $\sigma$ in $S_0(\Gamma_n, \mu_n)$:

$$Y_\sigma(n) = \sum_{\sigma = \rho \tau} a_\rho(n) a_\tau^{-1}(n) \ell(\rho) \ell(\tau)^{op}.$$  

since $\mu_n \to \mu$ as $n \to \infty$, it follows that $a_\rho(n) \to a_\rho$ for any path $\rho$ in $\Gamma$. For $\xi \in V_K$, we write $Y_\xi$ for the corresponding linear combination of Wick words in $S_0(\Gamma, \mu)$, and we write $Y_\xi(n)$ for the corresponding linear combination of Wick words in $S_0(\Gamma_n, \mu_n)$ for sufficiently large $n$. Setting $\|\xi\| := \|Y_\xi(n)\|_{S_0(\Gamma_n, \mu_n)}$ and $\|\xi\| := \|Y_\xi\|_{S_0(\Gamma, \mu)}$, we have $\|\xi\|_n \to \|\xi\|$ as $n \to \infty$. Moreover, for each $k$ between 1 and $K$, note that for any $\xi \in V_K$ which is a linear combination of loops of length exactly $k$, $\mathcal{L}_n(Y_\xi(n))$ converges to $\mathcal{L}(Y_\xi)$. This is due to the fact that on the space $W_K$, $L$ and $\mathcal{L}$ coincide.

Now by [Rie04a, Theorem 11.2] and Lemma 3.15, we have that

$$\lim_{n \to \infty} \text{dist}_q((A_K(\Gamma_n, \mu_n)^{sa}, \mathcal{L}|_{A_K(\Gamma_n, \mu_n)^{sa}}), (A_K(\Gamma, \mu)^{sa}, \mathcal{L}|_{A_K(\Gamma, \mu)^{sa}})) = 0.$$  

Finally, using Corollary 3.14 and a standard $\varepsilon/3$ argument, the result follows. \hfill $\Box$

4. Application to subfactor theory

We refer the reader to [Jon12] for the definition of a subfactor planar algebra and its principal graphs and to [BHP12] for the definition of a factor planar algebra and its principal graph.

4.1. The Guionnet-Jones-Shlyakhtenko C*-algebras. Let $\mathcal{P}_\bullet$ be a (sub)factor planar algebra. We now give the construction from [HP17, HP14] of the Guionnet-Jones-Shlyakhtenko (GJS) C*-algebras based on the constructions [GJS10, JSW10, GJS11]. A similar construction starting from a unitary tensor category and chosen symmetrically self-dual generator was given in [HHP20].

First, we form the graded algebra $Gr_0 = \bigoplus_{n \geq 0} \mathcal{P}_n$ with the Bacher-Walker product

$$x \star y = \sum_{j=0}^{\min\{m,n\}} \binom{m-j}{j} \binom{n-j}{m-j} x^j y^{n-j} \quad \text{for} \quad x \in \mathcal{P}_m, y \in \mathcal{P}_n,$$

and trace given by

$$\text{tr}(x) = \delta_{m=0} x \quad \text{for} \quad x \in \mathcal{P}_m.$$  

(Since $\mathcal{P}_0 = \mathbb{C}$, the above expression gives us a number.) We note that under the GNS inner product $\langle x, y \rangle = \text{tr}_0(y^* \star x)$, the spaces $\mathcal{P}_n$ are orthogonal for distinct $n$.

Observe that since each $\mathcal{P}_n$ is *-closed and finite dimensional, the subspaces $A_n = \bigoplus_{k=0}^n \mathcal{P}_n$ give $Gr_0$ the structure of a *-filtration by finite dimensional subspaces. Moreover, since $\mathcal{P}_\bullet$ is connected, $\mathcal{P}_0 = \mathbb{C}1_{Gr_0}$. Finally, by [GJS10, JSW10], the action of $Gr_0$ on $(Gr_0, \text{tr}_0)$ is bounded in $\| \cdot \|_2$. Hence Assumptions 2.10 hold, and we are in the position to use the Ozawa-Rieffel criterion from Theorem 2.13 ([OR05, Theorem 1.2]).

**Definition 4.1.** The C*-algebra $A_0 = \overline{Gr_0}^{\| \cdot \|}$ acting on $L^2(Gr_0, \text{tr}_0)$ is called the GJS C*-algebra of $\mathcal{P}_\bullet$.

Let $(\Gamma, \mu)$ be the principal graph of $\mathcal{P}_\bullet$ with its quantum dimension weighting, which satisfies the Frobenius-Perron condition. Note that the distinguished vertex $*$ has minimal weight 1, so Assumption 3.3 holds. We have the following lemma from [HP14] which connects the GJS C*-algebra to the free loop algebras discussed in Section 3.1.

**Lemma 4.2** ([HP14, Cor. 3.4]). The C*-algebra $A_0$ is isomorphic to the free loop algebra $S_0(\Gamma, \mu)$. 11
Remark 4.3. The examples of local uniform graph congerence in Examples 3.10 are all examples of principal graphs of subfactors with weightings given by quantum dimension functions which satify the Frobenius-Perron condition. We may thus interpret Theorem B as giving quantum Gromov-Hausdorff convergence of the compact quantum metric spaces associated to GJS C*-algebras.

4.2. The number operator. As in §2.2, we have the number operator \( N = \sum_{n \geq 0} n P_n \) on \( \text{Gr}_0 \), where \( P_n \) is the projection with range \( B_n = A_n \ominus A_{n-1} = P_n \). We end our article with some further observations about the properties of the number operator in our setup.

To begin, we give a supplementary diagrammatic proof that the number operator has bounded commutator with \( \text{Gr}_0 \), although it follows directly from Lemma 2.12 ([OR05, Lemma 1.1]).

Lemma 4.4. The number operator \( N \) has bounded commutator with every \( x \in \text{Gr}_0 \).

Proof. To show \( \|[N, x]\|_\infty \) is bounded for an arbitrary \( x \in \text{Gr}_0 \), it suffices to consider a fixed \( x \in \mathcal{P}_m \).

Suppose \( y \in \mathcal{P}_n \), and we write \( y \) for it image in \( L^2(\text{Gr}_0, \text{tr}_0) \). We need only treat the case \( m < n \), since we may ignore the behavior of \( [N, x] \) on a finite dimensional subspace. We have

\[
[N, x] y = N(x \ast y) - x \ast (Ny).
\]

We now see we can write this sum at the end as

\[
\left( \sum_{j=0}^{m} (m + n - 2j) m - j \begin{pmatrix} x \ y \ n-j \end{pmatrix} - \sum_{j=0}^{m} n - j \begin{pmatrix} x \ y \ n-j \end{pmatrix} \right) y
\]

where the sum in parentheses is a finite sum of bounded operators in the \( \mathcal{P}_* \)-Toeplitz algebra \( T_0(\mathcal{P}_*) \) [HP17], which is independent of \( y \). We are finished. \( \square \)

Proposition 4.5. The number operator \( N \) has compact resolvent and is \( \theta \)-summable.

Proof. We must show that \( e^{-tN^2} \) is trace class for all \( t > 0 \). Since \( \Gamma \) is the principal graph of \( \mathcal{P}_* \), \( \dim(\mathcal{P}_n) \) is the number of loops of length \( 2n \) on \( \Gamma \) based at \(*\). Letting \( A_{\Gamma} \) be the adjacency matrix of \( \Gamma \), we have that \( A_{\Gamma} \) acts on \( \ell^2(V) \), and \( \|A_{\Gamma}\| \leq \delta \) by [Pop94, §1.3.5]. Define \( e_s \in \ell^2(V) \) by \( e_s(v) = \delta_{v=\ast} \), and note that the number of loops based at \(*\) of length \( 2n \) is \( \langle A^2_{\Gamma} e_s, e_s \rangle \). Hence, we see

\[
\dim(\mathcal{P}_n) = \langle A^2_{\Gamma} e_s, e_s \rangle = |\langle A_{\Gamma} e_s, e_s \rangle| \leq \|A_{\Gamma}\|^{2n} \leq \delta^{2n}.
\]

Thus, on \( \mathcal{P}_n \), \( e^{-tN^2} \) has trace bounded above by

\[
\dim(\mathcal{P}_n)e^{-tn^2} \leq \delta^{2n}e^{-tn^2}.
\]

It is now clear by the root test that

\[
\sum_{n\geq0} \delta^{2n}e^{-tn^2} < \infty.
\]

We now use techniques from [CJS14] to show that the number operator arises as \( \partial \ast \partial \) where \( \partial \) is a derivation from \( \text{Gr}_0 \) into a Hilbert space.
Definition 4.6. We define \( K_0 = \bigoplus_{i,j \geq 0} P_{i,j,1} \), where \( P_{i,j,1} = P_{i+j+1} \), but we think of elements as having \( i \) strings out the top, \( j \) strings out the bottom, and one string to the right:

\[
P_{i,j,1} \ni x \longleftrightarrow \begin{array}{c}
\uparrow \\
\downarrow \\
\hline \\
\uparrow \\
\downarrow \\
\hline
x
\end{array}
\]

We define \( K \) to be the completion of \( K_0 \) using the inner product

\[
\langle x_{i,j}, y_{k,l} \rangle = \delta_{i=k} \delta_{j=l} x_{i,j}^* y_{k,l}^* .
\]

We have an action of \( \text{Gr}_0 \otimes \text{Gr}_0^{op} \) on \( K \) by bounded operators. Given \( x \in P_m \) and \( y \in P_n \), we think of \( x \otimes y^{op} \) as the following diagram:

\[
x \otimes y^{op} = \begin{array}{c}
m \\
\hline \\
x
\downarrow \\
\hline \\
m
\end{array}
\]

which acts on \( K \) by left multiplication in a variation of the Bacher-Walker product. If \( z \in P_{k,l,1} \), we have

\[
(x \otimes y^{op}) \cdot z = \sum_{i=0}^{\min\{m,k\}} \sum_{j=0}^{\min\{n,l\}} x_{m-i,j} y_{n-j,l} z_{i,j,l} .
\]

It is easy to see that this action is bounded using the Fock space argument of [HP17]. This means that \( K \) has the natural structure of a \( \text{Gr}_0 - \text{Gr}_0 \) bimodule. We use the notation \( x \star \xi \star y = (x \otimes y^{op}) \cdot \xi \).

Definition 4.7. We define a map \( \partial : \text{Gr}_0 \to K \) by the linear extension of

\[
P_m \ni \begin{array}{c}
m \\
\hline \\
x
\downarrow \\
\hline \\
m
\end{array} \mapsto \sum_{j=0}^{m-1} \begin{array}{c}
m-1 \\
\hline \\
x
\downarrow \\
\hline \\
m-1
\end{array} = \sum_{j=0}^{m-1} \begin{array}{c}
m-j-1 \\
\hline \\
x
\downarrow \\
\hline \\
m-j-1
\end{array} .
\]

Lemma 4.8. The map \( \partial \) is a closable derivation in the Bacher-Walker product.

Proof. First, we show that \( \partial \) is a derivation. We need to show that \( \partial(x \star y) = x \star \partial(y) + \partial(x) \star y \). It is straightforward to compute

\[
\partial \left( \sum_{i=0}^{\min\{m,n\}} \begin{array}{c}
m-i \\
\hline \\
x
\downarrow \\
\hline \\
n-i
\end{array} \right) = \sum_{i=0}^{\min\{m,n\}} \sum_{j=0}^{n-i} \begin{array}{c}
m-i \\
\hline \\
x
\downarrow \\
\hline \\
n-j-i-1
\end{array} + \sum_{i=0}^{\min\{m,n\}} \sum_{j=0}^{m-j} \begin{array}{c}
m-j-1 \\
\hline \\
x
\downarrow \\
\hline \\
m-j-1
\end{array}
\]

The right hand side is easily seen to be equal to \( x \star \partial(y) + \partial(x) \star y \) after switching the order of summation. To show \( \partial \) is closable, it is easy to calculate that in the Bacher-Walker product,

\[
\partial^* \left( \begin{array}{c}
m-1 \\
\hline \\
x
\downarrow \\
\hline \\
m-j-1
\end{array} \right) = \begin{array}{c}
m \\
\hline \\
x
\downarrow \\
\hline \\
m
\end{array} .
\]

Hence \( \partial^* \) defined, and we are finished.
The following corollary is now immediate.

**Corollary 4.9.** The number operator $N = \partial^*\partial$.

**References**

[Agu16] K. Aguilar. AF algebras in the quantum Gromov-Hausdorff propinquity space, 2016. [arXiv:1612.02404]

[Agu19] Konrad Aguilar. Fell topologies for AF-algebras and the quantum propinquity. *J. Operator Theory*, 82(2):469–514, 2019. [MR4015960 DOI:10.7900/jot]

[BH96] Dietmar Bisch and Uffe Haagerup. Composition of subfactors: new examples of infinite depth subfactors. *Ann. Sci. École Norm. Sup. (4)*, 29(3):293–383, 1996. [MR1386923]

[BHP12] Arnaud Brothier, Michael Hartglass, and David Penneys. Rigid C*-tensor categories of bimodules over interpolated free group factors. *J. Math. Phys.*, 53(12):123525, 43, 2012. [MR3405915 DOI:10.1063/1.4769178 arXiv:1208.5505]

[BNP07] Dietmar Bisch, Remus Nicoara, and Sorin Popa. Continuous families of hyperfinite subfactors with the same standard invariant. *Internat. J. Math.*, 18(3):255–267, 2007. [MR2314611 arXiv:math.OA/0604460 DOI:10.1142/S0129167X07004011]

[BS01] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13, 2001. [MR1873300 DOI:10.1214/EJP.v6-96 arXiv:math/0011019]

[CJS14] Stephen Curran, Vaughan F. R. Jones, and Dimitri Shlyakhtenko. On the symmetric enveloping algebra of planar algebra subfactors. *Trans. Amer. Math. Soc.*, 366(1):113–133, 2014. [MR3118393 DOI:10.1090/S0002-9947-2013-05910-7 arXiv:1105.1721]

[Con89] A. Connes. Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems*, [488] 8(2):469–514, 1989. [MR1007407 DOI:10.1017/S0143385700004934]

[FR99] Neal J. Fowler and Iain Raeburn. The Toeplitz algebra of a Hilbert bimodule. *Indiana Univ. Math. J.*

[GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. Coxeter graphs and towers of algebras. *Mathematical Sciences Research Institute Publications*, 14. Springer-Verlag, New York, 1989. x+288 pp. ISBN: 0-387-96979-9, MR999799.

[GJS10] Alice Guionnet, Vaughan F. R. Jones, and Dimitri Shlyakhtenko. Random matrices, free probability, planar algebras and subfactors. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 201–239. Amer. Math. Soc., Providence, RI, 2010. [MR2732052, arXiv:0712.2904v2]

[GJS11] Alice Guionnet, Vaughan F. R. Jones, and Dimitri Shlyakhtenko. A semi-finite algebra associated to a subfactor planar algebra. *J. Funct. Anal.*, 261(5):1345–1360, 2011. [arXiv:0911.4728, MR2807103, DOI:10.1016/j.jfa.2011.05.004]

[Har17] Michael Hartglass. Free product C*-algebras associated with graphs, free differentials, and laws of loops. *Can. J. Math.*, 69(3):548–578, 2017. [MR3676867 DOI:10.4153/CJM-2016-022-6 arXiv:1509.02553]

[HHP20] Michael Hartglass and Roberto Hernández Palomares. Realizations of rigid C*-tensor categories as bimodules over GJS C*-algebras. *J. Math. Phys.*, 61(8):081703, 32, 2020. [MR4139893 DOI:10.1063/5.0015294 arXiv:2005.09821]

[HP14] Michael Hartglass and David Penneys. C*-algebras from planar algebras II: The Guionnet–Jones–Shlyakhtenko C*-algebras. *J. Funct. Anal.*, 267(10):3859–3893, 2014. [MR3266249, DOI:10.1016/j.jfa.2014.08.024, arXiv:1401.2486]

[HP17] Michael Hartglass and David Penneys. C*-algebras from planar algebras I: Canonical C*-algebras associated to a planar algebra. *Trans. Amer. Math. Soc.*, 369(6):3977–4019, 2017. [MR3624399 DOI:10.1090/tran/6781 arXiv:1401.2485]

[Jon83] Vaughan F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. [MR696688, DOI:10.1007/BF01389127]

[Jon99] Vaughan F. R. Jones. Planar algebras I, 1999. [arXiv:math.QA/9909027]

[Jon12] Vaughan F. R. Jones. Quadratic tangles in planar algebras. *Duke Math. J.*, 161(12):2257–2295, 2012. [MR2972458, arXiv:1007.1158, DOI:10.1215/00127094-1726308]

[JRZ18] Marius Junge, Sepideh Rezvani, and Qiang Zeng. Harmonic analysis approach to Gromov-Hausdorff convergence for noncommutative tori. *Comm. Math. Phys.*, 358(3):919–994, 2018. [MR3778347 DOI:10.1007/s00220-017-3017-4 arXiv:1612.02735]

[JSW10] Vaughan Jones, Dimitri Shlyakhtenko, and Kevin Walker. An orthogonal approach to the subfactor of a planar algebra. *Pacific J. Math.*, 246(1):187–197, 2010. [MR2665882, DOI:10.2140/pjm.2010.246.187 arXiv:0807.4146]

[KK21] Jens Kaad and David Kyed. Dynamics of compact quantum metric spaces. *Ergodic Theory Dynam. Systems*, 41(7):2069–2109, 2021. [MR4266364 DOI:10.1017/etds.2020.34 arXiv:1904.13278]

[KL09] David Kerr and Hanfeng Li. On Gromov-Hausdorff convergence for operator metric spaces. *J. Operator Theory*, 62(1):83–109, 2009. [MR2520541 arXiv:math.OA/0411157]

[Lat16] Frédéric Latrémolière. The quantum Gromov-Hausdorff propinquity. *Trans. Amer. Math. Soc.*, 368(1):365–411, 2016.
[Lat17] F. Latrémolière. A compactness theorem for the dual Gromov-Hausdorff propinquity. Indiana Univ. Math. J., 66(5):1707–1753, 2017. MR3718439 DOI:10.1512/iumj.2017.66.6151 arXiv:1501.06121.

[Liu15] Zhengwei Liu. Composed inclusions of $A_3$ and $A_4$ subfactors. Adv. Math., 279:307–371, 2015. MR3345186 DOI:10.1016/j.aim.2015.03.017 arXiv:1308.5691.

[OR05] Narutaka Ozawa and Marc A. Rieffel. Hyperbolic group $C^*$-algebras and free-product $C^*$-algebras as compact quantum metric spaces. Canad. J. Math., 57(5):1056–1079, 2005. MR2164594 DOI:10.4153/CJM-2005-040-0 arXiv:math/0302310.

[Pen20] David Penneys. Unitary dual functors for unitary multienriched categories. High. Struct., 4(2):22–56, 2020. MR413163 arXiv:1808.00323.

[Pop94] Sorin Popa. Classification of amenable subfactors of type II. Acta Math., 172(2):163–255, 1994. MR1278111 DOI:10.1007/BF02392646.

[Pop95] Sorin Popa. An axiomatization of the lattice of higher relative commutants of a subfactor. Invent. Math., 120(3):427–445, 1995. MR1334479 DOI:10.1007/BF01241137.

[Rie98] Marc A. Rieffel. Metrics on states from actions of compact groups. Doc. Math., 3:215–229, 1998. MR1647515 arXiv:math/9807084.

[Rie99] Marc A. Rieffel. Metrics on state spaces. Doc. Math., 4:559–600, 1999. MR1727499, arXiv:math/9906151.

[Rie04a] Marc A. Rieffel. Gromov-Hausdorff distance for quantum metric spaces. Mem. Amer. Math. Soc., 168(796):1–65, 2004. Appendix 1 by Hahnfeng Li, Gromov-Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. MR2055927 DOI:10.1090/memo/0796 arXiv:math/0011063.

[Rie04b] Marc A. Rieffel. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. volume 168, pages 67–91. 2004. Gromov-Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. MR2055928 DOI:10.1090/memo/0796 arXiv:math/0108005.

[Wen88] Hans Wenzl. Hecke algebras of type $A_n$ and subfactors. Invent. Math., 92(2):349–383, 1988. MR936866 DOI:10.1007/BF01404457.

[Wen90] Hans Wenzl. Quantum groups and subfactors of type $B$, $C$, and $D$. Comm. Math. Phys., 133(2):383–432, 1990. MR1090432.

[Xu98] Feng Xu. Standard $\lambda$-lattices from quantum groups. Invent. Math., 134(3):455–487, 1998. MR1660937 DOI:10.1007/s002220050271.