Schensted type correspondence for type $G_2$ and computation of the canonical basis of a finite dimensional $U_q(G_2)$-module

Cédric Lecouvey
lecouvey@math.unicaen.fr

Abstract

We use Kang-Misra’s combinatorial description of the crystal graphs for $U_q(G_2)$ to introduce the plactic monoid for type $G_2$. Then we describe the corresponding insertion algorithm which yields a Schensted type correspondence. Next we give a simple algorithm for computing the canonical basis of any finite dimensional $U_q(G_2)$-module.

1 Introduction

The quantum algebra $U_q(g)$ associated to a semisimple Lie algebra $g$ is the $q$-analogue introduced by Drinfeld and Jimbo of its universal enveloping algebra $U(g)$. According to Kashiwara [8] each finite dimensional irreducible $U_q(g)$-module $V$ has a unique crystal basis which can be regarded as a basis at $q = 0$. This crystal basis can be extended to obtain a true basis of $V$ called the global crystal basis of $V$. The global crystal basis coincides with the canonical basis of $V$ introduced by Lusztig [20]. Moreover the module structure on $V$ induces a combinatorial structure on its crystal basis $V$ called crystal graph. Crystal graphs permit to reduce many problems in the representation theory of $U_q(g)$ to combinatorics.

In this article we restrict ourselves to $U_q(G_2)$. Write $\Lambda_1$ and $\Lambda_2$ for the two fundamentals weights of $U_q(G_2)$ and let $P_+$ be the set of its dominants weights. For $\lambda \in P_+$ denote by $V(\lambda)$ the $U_q(G_2)$-module of weight $\lambda$ and by $B(\lambda)$ the crystal graph of $V(\lambda)$. The purpose of this article is two-fold.

In a first part we use the explicit description of the crystal graphs for type $G_2$ [3] (based on a notion of tableaux for type $G_2$) to introduce a monoid structure $\equiv$ on the vertices of $\Gamma = \bigoplus_{l \geq 0} B(\Lambda_1)^{\otimes l}$ analogous to the plactic monoid of Lascoux and Schützenberger. Given $w_1$ and $w_2$ two vertices of $\Gamma$, this monoid structure is such that $w_1 \equiv w_2$ if and only if the vertices $w_1$ and $w_2$ occur at the same place in two isomorphic connected components of $\Gamma$. It will be called the plactic monoid of type $G_2$ and denoted $Pl(G_2)$. By using this monoid, we describe the corresponding column insertion algorithm which yields a Schensted type correspondence in $\Gamma$. Note that such a correspondence also exists for types $C$, $D$, $B$ and $D$

The second part of this paper is devoted to the computation of the global basis of $V(\lambda)$. We introduced a $q$-analogue $W(\Lambda_2)$ of the $2$-th wedge product of $V(\Lambda_1)$. The representation $W(\Lambda_2)$ is not irreducible but contains an irreducible component isomorphic to $V(\Lambda_2)$. To make the notation homogeneous set $W(\Lambda_1) = V(\Lambda_1)$. For $p = 1, 2$, $W(\Lambda_p)$ has a simple crystal basis naturally indexed in terms of column shaped Young tableaux of height $p$. Then we give explicit formulas for the expansion of the global basis of $V(\Lambda_p)$, $p = 1, 2$ on these bases. In the general case, we embed $V(\lambda)$ with $\lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2$ in

$$W(\lambda) = W(\Lambda_1)^{\otimes \lambda_1} \otimes W(\Lambda_2)^{\otimes \lambda_2}.$$ 

The tensor product of the crystal bases of the modules $W(\Lambda_p)$, $p = 1, 2$ occurring in $W(\lambda)$ is a natural basis $\{v_\tau\}$ of $W(\lambda)$ indexed by combinatorial objects $\tau$ called tabloids. Then we describe an algorithm similar to those given in [4], [17] and [18] which provides the expansion of the global basis of $V(\lambda)$ on the basis $\{v_\tau\}$. Note that the coefficients of this expansion are integral that is belong to $\mathbb{Z}[q, q^{-1}]$. 

1
2 Background on $U_q(G_2)$

In this section we briefly review the basic facts that we shall need concerning the representation theory of $U_q(G_2)$ and the notions of crystal basis and canonical basis of their representations. The reader is referred to [2], [4], [9] and [10] for more details.

2.1 The quantum enveloping algebras $U_q(G_2)$

The Dynkin diagram of the Lie algebra of type $G_2$ is

$$
\begin{array}{c}
\text{type } G_2
\end{array}
$$

Given a fixed indeterminate $q$ set

$$
q_i = \begin{cases} q & \text{if } i = 1, \\ q^2 & \text{if } i = 2 \end{cases},
$$

and $[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$ and $[m]_i! = [m]_i [m - 1]_i \cdots [1]_i$.

The quantized enveloping algebras $U_q(G_2)$ is the associative algebra over $\mathbb{Q}(q)$ generated by $e_i, f_i, t_i, t_i^{-1}$, $i = 1, 2$, subject to relations determined by the Cartan matrix

$$
\begin{pmatrix}
2 & -3 \\
-3 & 2
\end{pmatrix}
$$

of type $G_2$. Given $i \in \{1, 2\}$ and $m \in \mathbb{N}$ we set $e_i^{(m)} = e_i^m / [m]_i!$ and $f_i^{(m)} = f_i^m / [m]_i!$ For any $i = 1, 2$ the subalgebra of $U_q(G_2)$ generated by $e_i, f_i$ and $t_i$ is isomorphic to $U_q(\mathfrak{sl}_2)$ the quantum enveloping algebra associated to $\mathfrak{sl}_2$.

The representation theory of $U_q(G_2)$ is closely parallel to that of $G_2$. The weight lattice $P$ of $U_q(G_2)$ is the $\mathbb{Z}$-lattice generated by the fundamentals weights $\Lambda_1, \Lambda_2$. Write $P_+$ for the set of dominant weights of $U_q(G_2)$. We denote by $V(\lambda)$ the irreducible finite dimensional $U_q(G_2)$-module with highest weight $\lambda \in P^+$.

Given two $U_q(G_2)$-modules $M$ and $N$, we can define a structure of $U_q(G_2)$-module on $M \otimes N$ by putting:

$$
\begin{align*}
t_i(u \otimes v) &= t_i u \otimes t_i v, \\
e_i(u \otimes v) &= e_i u \otimes t_i^{-1} v + u \otimes e_i v, \\
f_i(u \otimes v) &= f_i u \otimes v + t_i u \otimes f_i v.
\end{align*}
$$

In the sequel we need the following general lemma (see [4] p.32). Let $V(l)$ be the irreducible $U_q(\mathfrak{sl}_2)$-module of dimension $l + 1$.

**Lemma 2.1.1** Consider $v_r \in V(r)$ and $v_s \in V(s)$. Set $t(v_r) = q^a v_r$ with $a \in \mathbb{Z}$. Then for any integer $r$ one has:

$$
(f^{(m)})(v_r \otimes v_s) = \sum_{k=0}^{m} q^{(m-k)(a-k)} f^{(k)}(v_r) \otimes f^{(m-k)}(v_s).
$$

2.2 Crystal basis and crystal graph of $U_q(G_2)$

The reader is referred to [4] and [10] for basic definitions on crystal bases and crystal graphs. Given $(L, B)$ and $(L', B')$ two crystal bases of the finite-dimensional $U_q(G_2)$-modules $M$ and $M'$, $(L \otimes L', B \otimes B')$ with $B \otimes B' = \{b \otimes b'; b \in B, b' \in B'\}$ is a crystal basis of $M \otimes M'$. The action of $\bar{e}_i$ and $\bar{f}_i$ on $B \otimes B'$ is given by:

$$
\begin{align*}
\bar{f}_i(u \otimes v) &= \begin{cases} 
\bar{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v) \\
\bar{u} \otimes \bar{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v)
\end{cases} \\
&\text{and} \\
\bar{e}_i(u \otimes v) &= \begin{cases} 
u \otimes \bar{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v) \\
\bar{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v)
\end{cases}
\end{align*}
$$
where \( \varepsilon_i(u) = \max \{ k; \varepsilon_i^k(u) \neq 0 \} \) and \( \varphi_i(u) = \max \{ k; \varphi_i^k(u) \neq 0 \} \). We say that a vertex \( b \) is a highest weight vertex if \( \varepsilon_i(b) = 0 \) for \( i = 1, 2 \). Then \( u \otimes v \) is a highest weight vertex if and only if

\[
\text{u is a highest weight vertex and } \varepsilon_i(v) \leq \varphi_i(u) \text{ for } i = 1, 2.
\]

### 2.3 Combinatorics of crystal graphs

In this paragraph we recall Kang-Misra’s combinatorial description of \( B(\lambda) \) \cite{KangMisra}. It is based on the notion of tableau of type \( G_2 \) analogous to Young tableau for type \( A \).

The crystal graphs of the representation \( B(\Lambda_1) \) is

\[
\begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \\
\rightarrow 0 \rightarrow \overline{2} \rightarrow \overline{3} \rightarrow \overline{1}.
\end{array}
\]

Set \( \Gamma = \bigoplus_{l \geq 0} B(\Lambda_1)^{\otimes l} \). Each vertex \( x_1 \otimes x_2 \cdots \otimes x_l \) of \( \Gamma \) may be identified with the word \( x_1 \cdots x_l \) on the totally ordered alphabet \( \mathcal{G} = \{ 1 \prec 2 \prec 3 \prec \overline{3} \prec \overline{2} \prec \overline{1} \} \).

For any word \( w \) we denote by \( B(w) \) the connected component of \( \Gamma \) containing \( w \) and we write \( l(w) \) for the length of \( w \). For \( i = 1, 2, 3 \) let \( d_i \) be the number of letters \( i \) in \( w \) minus the number of letters \( i \).

Then the weight of the vertex \( w \) is given by

\[
\text{wt}(w) = (d_1-d_2+2d_3)\Lambda_1 + (d_2-d_3)\Lambda_2.
\]

A column \( C \) on the alphabet \( \mathcal{G} \) is a Young diagram of shape column and height 1 or 2 filled by letters of \( \mathcal{G} \) and such that

\[
C = \begin{array}{c}
a \end{array} \text{ with } a \in \mathcal{G} \text{ or } C = \begin{array}{c}
a \\
b
\end{array} \text{ with } a \prec b \in \mathcal{G} \text{ or } C = \begin{array}{c}
0
\end{array}
\]

Write \( h(C) \) for the height of \( C \). Let \( \mathcal{C}(p) \), \( p = 1, 2 \) be the set of columns of height \( p \). To each column \( C \), we associate its reading \( w(C) \) that is the word obtained by reading the letters of \( C \) from top to bottom.

Denote by \( \text{dist}(a, b) \) the number of arrows between the vertices \( a \) and \( b \) in the crystal \( (\overline{3}) \) of \( B(\Lambda_1) \). A column \( C \) is said admissible if \( h(C) = 1 \) or \( C = \begin{array}{c}
0
\end{array} \) with

\[
\left\{ \begin{array}{l}
\text{dist}(a, b) \leq 2 \text{ if } a = 1 \text{ or } 0 \\
\text{dist}(a, b) \leq 3 \text{ otherwise}
\end{array} \right.
\]

Given two admissible columns \( C_1 \) and \( C_2 \) we write \( C_1 \preceq C_2 \) when the juxtaposition \( C_1 C_2 \) of these columns verifies one of the following assertions

\[
\left\{ \begin{array}{l}
(i) \quad C_1 C_2 = \begin{array}{c}
a \\
b
\end{array} \text{ with } a \preceq b \text{ and } (a, b) \neq (0, 0) \\
(ii) \quad C_1 C_2 = \begin{array}{c}
a \\
b \\
c
\end{array} \text{ with } a \preceq c \text{ and } (a, c) \neq (0, 0) \\
(iii) \quad C_1 C_2 = \begin{array}{c}
a \\
b \\
c \\
d
\end{array} \text{ with } a \preceq c \text{ and } (a, c) \neq (0, 0) \quad \text{and} \quad \text{dist}(a, d) \geq 3 \text{ if } a = 2, 3, 0 \quad \text{and} \quad \text{dist}(a, d) \geq 2 \text{ if } a = 3
\end{array} \right.
\]

Consider \( \lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2 \) a dominant weight. Then the vertex

\[
b_\lambda = (1 \otimes 2)^{\otimes \lambda_2} \otimes (1)^{\otimes \lambda_1}
\]

is of highest weight \( \lambda \) in \( \Gamma \). In the sequel we will identify \( B(\lambda) \) with \( B(b_\lambda) \). We associate to \( \lambda \) the Young diagram \( Y(\lambda) \) having \( \lambda_i \) \( i = 1, 2 \) columns of height \( i \). By definition, a tableau \( T \) of type \( G_2 \) and shape \( \lambda \) is a filling of \( Y(\lambda) \) by letters of \( \mathcal{G} \) such that the columns \( C_i \) of \( T = C_1 \cdots C_s \) are admissible and \( C_i \preceq C_{i+1} \) for \( i = 1, \ldots, s-1 \). We write \( T_G(\lambda) \) for the set of tableaux of type \( G_2 \) and shape \( \lambda \). The reading of the tableau \( T = C_1 \cdots C_s \) is the word \( w(T) = w(C_s) \cdots w(C_1) \) obtained by reading the columns of \( T \) from right to left.
Theorem 2.3.1 (Kang-Misra)

(i): The vertices of \( B(\Lambda_p) \) \( p = 1, 2 \) are the readings of the admissible columns of height \( p \).

(ii): The vertices of \( B(\lambda) \) are the readings of the tableaux of type \( G_2 \) and shape \( \lambda \).

\[
\begin{array}{cccccccccccc}
11 & \downarrow & 21 & \rightarrow & 31 & \downarrow & 01 & \rightarrow & 31 & \rightarrow & 21 & \rightarrow & 11 \\
12 & \downarrow & 22 & \rightarrow & 32 & \rightarrow & 02 & \rightarrow & 32 & \rightarrow & 22 & \rightarrow & 12 \\
13 & \downarrow & 23 & \rightarrow & 33 & \rightarrow & 03 & \rightarrow & 33 & \rightarrow & 23 & \rightarrow & 13 \\
10 & \downarrow & 20 & \rightarrow & 30 & \rightarrow & 00 & \rightarrow & 30 & \rightarrow & 20 & \rightarrow & 10 \\
13 & \downarrow & 23 & \rightarrow & 33 & \rightarrow & 03 & \rightarrow & 33 & \rightarrow & 23 & \rightarrow & 13 \\
12 & \downarrow & 22 & \rightarrow & 32 & \rightarrow & 02 & \rightarrow & 32 & \rightarrow & 22 & \rightarrow & 12 \\
10 & \downarrow & 21 & \rightarrow & 31 & \rightarrow & 01 & \rightarrow & 31 & \rightarrow & 21 & \rightarrow & 10 \\
\end{array}
\]

the crystal \( B(\Lambda_1)^{\otimes 2} \simeq B(2\Lambda_1) \oplus B(\Lambda_2) \oplus B(\Lambda_1) \oplus B(0) \)

2.4 Canonical basis of \( V(\lambda) \)

Denote by \( F \mapsto \overline{\mathbf{f}} \) the involution of \( U_q(G_2) \) defined as the ring automorphism satisfying

\[
\overline{q} = q^{-1}, \quad t_i = t_i^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i \quad \text{for } i = 1, 2.
\]

By writing each vector \( v \) of \( V(\lambda) \) in the form \( v = Fv_\lambda \) where \( F \in U_q(G_2) \), we obtain an involution of \( V(\lambda) \) defined by

\[
\overline{v} = \overline{Fv_\lambda}.
\]

Let \( U_Q^- \) be the subalgebra of \( U_q(G_2) \) generated over \( \mathbb{Q}[q, q^{-1}] \) by the \( f_i^{(k)} \) and set \( V_Q(\lambda) = U_Q^- v_\lambda \). We can now state:

Theorem 2.4.1 (Kashiwara) There exists a unique \( \mathbb{Q}[q, q^{-1}] \)-basis \( \{ G(T); T \in T_G(\lambda) \} \) of \( V_Q(\lambda) \) such that:

\[
G(T) \equiv w(T) \mod qL(\lambda), \quad (10)
\]

\[
\overline{G(T)} = G(T). \quad (11)
\]

Note that \( G(T) \in V_Q(\lambda) \cap L(\lambda) \). The basis \( \{ G(T); T \in T_G(\lambda) \} \) is called the lower global (or canonical) basis of \( V(\lambda) \).

3 Plactic monoid and Schensted’s type correspondence

3.1 The plactic monoid

Definition 3.1.1 Let \( w_1 \) and \( w_2 \) be two words on \( \mathcal{G} \). We write \( w_1 \sim w_2 \) when these two words occur at the same place in two isomorphic connected components of the crystal \( \Gamma \).

From Theorem 2.3.1 it follows that for any word \( w \in \mathcal{G}^* \) there exists a unique tableau of type \( G_2 \) \( P(w) \) such that \( w \sim w(P(w)) \). So the set \( \mathcal{G}^*/\sim \) can be identified with the set of tableaux of type \( G_2 \). Our aim is now to show that \( \sim \) are in fact a congruences \( \equiv \) so that \( \mathcal{G}^*/\sim \) is in a natural way endowed with a multiplication.
Set
\[ S = \{21, 31, 01, 32, 21, 22, 11, 12, 23, 13, 10, 1\} \].

To defined the plactic relations for type \( G_2 \) we need the bijection \( \Theta \) from \( S \) to \( B(12) \) defined by

\[
\begin{array}{c|ccccccccccc}
  w & 21 & 31 & 01 & 32 & 21 & 22 & \bar{11} & \bar{12} & \bar{23} & \bar{13} & \bar{10} & \bar{1} & 3 & \bar{1} & 2 \\
  \Theta(w) & 12 & 13 & 23 & 20 & 30 & 33 & 00 & 03 & 32 & 02 & 32 & 31 & 21 \\
\end{array}
\]  

(12)

**Definition 3.1.2** The monoid \( Pl(G_2) \) is the quotient of the free monoid \( G^* \) by the relations:

\[ 10 \equiv 1, \quad 13 \equiv 2, \quad 12 \equiv 3, \quad 22 \equiv 0, \quad 01 \equiv \bar{1}, \quad 31 \equiv \bar{3}, \quad 21 \equiv \bar{3} \].  

\( (R_1) \)

\[ 1\bar{T} \equiv 0. \]  

\( (R_2) \)

\[ abc \equiv \begin{cases} 
  a\Theta(bc) & \text{if } bc \in S \\
  \Theta^{-1}(ab)c & \text{otherwise}
\end{cases} \text{ with } ab \in B(12) \text{ and } bc \in B(11). \]  

\( (R_3) \)

\[ xyz \equiv \Theta^{-1}(xy)z \text{ with } xy \in B(\Lambda_2) \text{ and } yz \in B(\Lambda_2). \]  

\( (R_4) \)

For any word \( w \) occurring in the left hand side of a relation \( R_i, \ i = 1, ..., 4 \) we write \( \xi_i(w) \) the word occurring in the right hand side of this relation.

**Proposition 3.1.3** The maps \( \xi_1, \xi_2, \xi_3 \) and \( \xi_4 \) are respectively the crystal graph isomorphisms

(i) : \( B(10) \xrightarrow{\sim} B(1) \), (ii) : \( B(1\bar{T}) \xrightarrow{\sim} B(\emptyset) \), (iii) : \( B(121) \xrightarrow{\sim} B(112) \) and (iv) : \( B(123) \xrightarrow{\sim} B(110) \).

**Proof.** The simplest way to prove this proposition consists in the computation of the crystals above. The crystals \( B(10) \) and \( B(1\bar{T}) \) occur in [1] and we see that \( \xi_1 \) and \( \xi_2 \) are the crystal graph isomorphisms (i) and (ii). We obtain that \( \xi_4 \) is the crystal graph isomorphism (iv) from the crystals (13) and (14) below. Similarly we can prove that \( \xi_3 \) is the crystal graph isomorphism between \( B(121) \) and \( B(112) \) by drawing these two crystals which contain 64 vertices. 

**Remark** Write \( \xi'_1 \) for the crystal graph isomorphism \( B(110) \xrightarrow{\sim} B(11) \), then \( \xi'_4 = \xi'_1\xi_4 \) is the crystal isomorphism \( B(123) \xrightarrow{\sim} B(11) \). By replacing Relation \( R_4 \) in the definition of \( Pl(G_2) \) above by a relation \( R'_4 \) describing the isomorphism \( \xi'_4 \) we obtain Littelmann’s presentation of \( Pl(G_2) \) [19]. The two presentations coincide excepted for the description of \( \xi_3 \) which is not totally exact in [19].
Theorem 3.1.4  Given two words $w_1$ and $w_2$

$$w_1 \sim w_2 \iff w_1 \equiv w_2$$

To prove this, we need the following lemma.

Lemma 3.1.5  Let $w$ be a highest weight vertex of $\Gamma$. Then $w(P(w)) \equiv w$.

Proof. We proceed by induction on $l(w)$. If $l(w) = 1$ then $w = 1$ and $P(w) = \begin{array}{c}
1
\end{array}$. Suppose the proposition true for the highest weight vertices of length $\leq l$ and consider a highest weight vertex $w$ of length $l + 1$. Write $w = vx$ where $x \in G$ and $v$ is a word of length $l$. Then if follows from (6) that $v$ is a highest weight vertex and $\varepsilon_i(x) \leq \varphi_i(v)$ for $i = 1, 2$. So by induction $v \equiv w(P(v))$. Hence $w \equiv w(P(v))x$. Let $P'$ be the tableau obtained by erasing the first column of $P(v)$. The condition $\varepsilon_i(x) \leq \varphi_i(v)$ implies that only the following situations can occur:

1. $x = 1$ and $P(w)$ is obtained by adding a column of height 1 in $P(v)$. We have $w \equiv w(P(v))1 \equiv w(P(w))$. Indeed, by relation $R_3$, 1 commutes in $Pl(G_2)$ with all the column words 12 occurring in $w(P(v))$.

2. $x = 2$ and $P(w)$ is obtained from $P(v)$ by erasing a column of height 1 and adding a column of height 2. Similarly we have $w \equiv w(P(v))$ because in $Pl(G_2)$, 2 commute with all the columns words 12 occurring in $w(P(v))$.

3. $x = 3$, $P(v)$ has at least a column of height 2 and $P(w)$ is obtained from $P(v)$ by erasing a column of height 2 and adding two columns of height 1. We can write $w(P(v)) = w(P')12$. Then $w \equiv w(P')123 \equiv w(P')11$ by applying relations $R_4$ and $R_1$. We obtain $w \equiv w(P(w))$ by using the same argument than in case 1.
4. \( x = 0 \), \( P(v) \) has at least a column of height 1 and \( P(w) = P(v) \). If \( w(P(v)) = w(P')1 \) then \( w \equiv w(P')10 \equiv w(P')1 \) by relation \( R_1 \). If \( w(P(v)) = w(P')12 \) then \( w \equiv w(P')120 \equiv w(P')21 \) by relations \( R_1 \) and \( R_4 \). In the both cases \( w \equiv w(P(w)) \).

5. \( x = 3 \), \( P(v) \) has at least two columns of height 1 and \( P(w) \) is obtained from \( P(v) \) by erasing two columns of height 1 and adding a column of height 2. If \( w(P(v)) = w(P')1 \) then \( w \equiv w(P')13 \equiv w(P')2 \) by applying relation \( R_1 \). If \( w(P(v)) = w(P')12 \) then \( w \equiv w(P')123 \equiv w(P')22 \) by applying relations \( R_4 \) and \( R_1 \). In the both cases \( w \equiv w(P(w)) \).

6. \( x = 2 \), \( P(v) \) has at least a column of height 2 and \( P(w) \) is obtained from \( P(v) \) by erasing a column of height 2 and adding a column of height 1. We can write \( w(P(v)) = w(P')12 \). So \( w \equiv w(P')122 \equiv w(P')1 \equiv w(P(w)) \) by applying relation \( R_1 \) two times.

7. \( x = 4 \), \( P(v) \) has at least a column of height 1 and \( P(w) \) is obtained from \( P(v) \) by erasing a column of height 1. If \( w(P(v)) = w(P')1 \) then \( w \equiv w(P')11 \equiv w(P') \) by applying relation \( R_2 \). If \( w(P(v)) = w(P')12 \) then \( w \equiv w(P')121 \equiv w(P')2 \) by applying relation \( R_1 \) two times. In the both cases \( w \equiv w(P(w)) \).

\begin{proof} (of Theorem 3.1.4) By Proposition 3.1.3 and \( R \), \( R \), the plactic relations of Definition 3.1.2 are compatible with Kashiwara’s operators, that is, for any words \( w_1 \) and \( w_2 \) such that \( w_1 \equiv w_2 \) one has:

\[
\begin{align*}
\tilde{e}_i(w_1) &\equiv \tilde{e}_i(w_2) \quad \text{and} \quad \varepsilon_i(w_1) = \varepsilon_i(w_2) \\
\tilde{f}_i(w_1) &\equiv \tilde{f}_i(w_2) \quad \text{and} \quad \varphi_i(w_1) = \varphi_i(w_2).
\end{align*}
\] (16)

Hence:

\[ w_1 \equiv w_2 \implies w_1 \sim w_2. \]

From Lemma 3.1.3, we obtain that two highest weight vertices \( w_1^0 \) and \( w_2^0 \) with the same weight \( \lambda \) verify \( w_1^0 \equiv w_2^0 \). Indeed there is only one tableau of type \( G_2 \) whose reading is a highest vertex of weight \( \lambda \). Now suppose that \( w_1 \sim w_2 \) and denote by \( w_1^0 \) and \( w_2^0 \) the highest weight vertices of \( B(w_1) \) and \( B(w_2) \). We have \( w_1^0 \equiv w_2^0 \). Set \( w_1 = \tilde{F} w_1^0 \) where \( \tilde{F} \) is a product of Kashiwara’s operators \( \tilde{f}_i \), \( i = 1, 2 \). Then \( w_2 = \tilde{F} w_2^0 \) because \( w_1 \sim w_2 \). So by \( R \) we obtain \( w_1^0 \equiv w_2^0 \implies \tilde{F} w_1^0 \equiv \tilde{F} w_2^0 \implies w_1 \equiv w_2 \).

\end{proof}

### 3.2 Bumping algorithm

Now we are going to see how the orthogonal tableau \( P(w) \) may be computed for each vertex \( w \) by using an insertion scheme analogous to bumping algorithm for type \( A \). As a first step, we describe \( P(w) \) when \( w = w(C)x \), where \( x \) and \( C \) are respectively a letter and an admissible column. This will be called “the insertion of the letter \( x \) in the admissible column \( C \)” and denoted by \( x \rightarrow C \). Then we will be able to obtain \( P(w) \) when \( w = w(T)x \) with \( x \) a letter and \( T \) an orthogonal tableau. This will be called “the insertion of the letter \( x \) in the orthogonal tableau \( T \)” and denoted by \( x \rightarrow T \). Our construction of \( P \) will be recursive, in the sense that if \( P(u) = T \) and \( x \) is a letter, then \( P(ux) = x \rightarrow T \).
3.2.1 Insertion of a letter in an admissible column

When $h(C) = 1$ and $C = \begin{array}{|c|} \hline a \\hline \end{array}$ we have

$$\begin{equation}
  x \rightarrow C = \begin{cases}
    (i) : \begin{array}{|c|} \hline a \hline \end{array} x & \text{if } ax \in B(11) \\
    (ii) : \begin{array}{|c|} \hline a \hline \end{array} x & \text{if } ax \in B(12) \\
    (iii) : \begin{array}{|c|} \hline a' \hline \end{array} & \text{with } a' = \xi_1(ax) \text{ if } ax \in B(10) \\
    (iv) : \emptyset \text{ if } ax = 1T 
  \end{cases}
\end{equation}$$

Indeed in each cases (i) to (iv) $x \rightarrow C$ is a tableau of type $G_2$ such that $w(x \rightarrow C) \equiv w(C)x$.

When $h(C) = 2$ and $C = \begin{array}{|c|c|} \hline a & b \\hline \end{array}$ we have

$$\begin{equation}
  x \rightarrow C = \begin{cases}
    (v) : \begin{array}{|c|} \hline a' \hline \end{array} x' & \text{with } x' a' = \xi_3(abx) \text{ if } bx \text{ is not a column word} \\
    (vi) : \begin{array}{|c|} \hline x' \hline \end{array} y' & \text{with } a' x' = \xi_4(abx) \text{ if } bx \text{ is an admissible column word} \\
    (vii) : \begin{array}{|c|} \hline x' \hline \end{array} & \text{with } x' = \xi_1(a\xi_1(bx)) \text{ if } bx \text{ is a non admissible column word}
  \end{cases}
\end{equation}$$

Indeed in cases (v) and (vi) $x \rightarrow C$ is a tableau of type $G_2$ such that $w(x \rightarrow C) \equiv w(C)x$ by Proposition 3.1.3. In case (vii), we obtain by (5) that the highest weight vertex of $B(abx)$ may be written $12u$ with $u$ a letter such that $\varepsilon_1(u) = 0$ and $\varepsilon_2(u) \leq 1$. So $u \in \{1, 3, \overline{2}\}$. We have $u = 3$ otherwise $B(abx) = B(121)$ and $x \preceq b$, or $B(abx) = B(123)$ and $bx$ is an admissible column word. Hence $B(abx) = B(122)$. We have

$$B(122) : 12 \overline{2} \rightarrow 12 \overline{1} \rightarrow 13 \overline{1} \rightarrow 23 \overline{1} \rightarrow 20 \overline{1} \rightarrow 30 \rightarrow 00$$

and it is easy to verify that $\xi_1(a\xi_1(bx))$ is the image of $abx$ by the crystal isomorphism $B(122) \xrightarrow{\sim} B(1)$. In cases (iii), (iv), (vi) and (vii) we have $l(x \rightarrow C) < l(w(C)x)$. We will say that the insertion procedure causes a contraction. Note that if the words $w(C_1)x_1$ and $w(C_2)x_2$ (where $C_1$, $C_2$ are admissible columns and $x_1$, $x_2$ are letters) belongs to the same connected component, the insertions $x_1 \rightarrow C_1$ and $x_2 \rightarrow C_2$ are of the same type (i) to (vii).

3.2.2 Insertion of a letter in a tableau of type $G_2$

Consider a tableau $T = C_1C_2 \cdots C_r$ of type $G_2$. The insertion $x \rightarrow T$ is characterized by the following proposition.

Proposition 3.2.1 Set $T' = C_2 \cdots C_s$.

1. If the insertion $x \rightarrow C_1$ is of type (i), (ii) or (iv) in 3.2.1 then $x \rightarrow T = (x \rightarrow C_1)T'$.

2. If the insertion $x \rightarrow C_1$ is of type (v) in 3.2.1 with $C_1 = \begin{array}{|c|} \hline a \hline \end{array} \setminus \begin{array}{|c|} \hline b \hline \end{array}$ and $C_1' = \begin{array}{|c|} \hline a' \hline \end{array} \setminus \begin{array}{|c|} \hline b \hline \end{array}$ then $x \rightarrow T = C_1'(x' \rightarrow T')$ and is obtained by computing successively insertions of type (v).

3. If the insertion $x \rightarrow C_1$ is of type (iii) in 3.2.1 and $x \rightarrow C_1 = \begin{array}{|c|} \hline a' \hline \end{array}$ then $x \rightarrow T = a' \rightarrow T'$.

4. If the insertion $x \rightarrow C_1$ is of type (vi) in 3.2.1 then $x \rightarrow T = x' \rightarrow (y' \rightarrow T')$. Moreover the insertion of the letters $x'$ and $y'$ in $T'$ does not cause a new contraction.

5. If the insertion $x \rightarrow C_1$ is of type (vii) in 3.2.1 then $x \rightarrow T = x' \rightarrow T'$. Moreover the insertion of the letter $x'$ in $T'$ does not cause a new contraction.
The insertion procedure terminates because in cases 2, 3, 4 and 5 we are reduced to the insertion of a letter in a tableau whose number of boxes is strictly less than that of \( T \).

**Proof.** It follows from the proof of lemma 3.1.3 that the difference between the number of boxes in the shapes of \( T \) and \( x \to T \) belongs to \( \{-1,0,1\} \). So only one contraction can occur during the insertion \( x \to T \).

In case 1, \( T \) contains only columns of height 1. Hence the proposition follows immediately from the definition of the insertions of type (i), (ii), and (iv).

In case 2, we can write by (17) \( w^0 = v^0 x^0 \) where \( v^0 \) is the highest weight vertex of \( w(T) \) and \( x^0 \) is a letter. The word \( v^0 \) is the reading of \( T^0 \) a tableau of type \( G_2 \). Write \( T^0 = C^0_0 \cdots C^0_i \). The insertion \( x^0 \to C^0_i \) is of type (v) since it is true for the insertion \( x \to C \). Hence \( x^0 \in \{1,2\} \) since \( w(C^0_i) = 12 \). Let \( T^0 \) be the tableau obtained from \( T^0 \) by adding a box containing the letter \( x^0 \). Denote by \( C^0_k \) the column of \( T^0 \) where this new box appears. There exists a unique sequence \( w_0 = w(T)x, \ldots, w_{k-1} \) such that \( w(T) \) is the highest weight vertex of \( B(w_{k-1}) \) and such that \( w_{j-1} = w(C_r) \cdots w(C_j) x_j^{-1} w(C_{j+1}) \cdots w(C^0_i) \) is transformed into the congruent word \( w_j = w(C_r) \cdots w(C_{j+1}) x_j^{-1} w(C^0_i) \) where \( x_j \) and \( C^0_j \) are determined by the insertion of type (v) \( x_j^{-1} \to C_j = C^0_j x_j^{-1} \). The word \( w_{k-1} \) is the reading of a tableau of type \( G_2 \) and is obtained by computing \( k-1 \) insertions of type (v) so is equal to \( w(x \to T) \).

The rest of the proof is obtained by induction on \( l(w(T)) \). If \( l(w(T)) = 1 \), the proposition follows from (17). Now suppose the proposition true for any tableau \( T \) such that \( l(w(T)) \leq m \) with \( m \in \mathbb{N} \). Then consider a tableau \( T \) with \( l(w(T)) = m + 1 \) and a letter \( x \).

In case 3, \( x \to T = a' \to T' \) since, by induction, \( a' \to T' \) is a tableau and \( w(a' \to T') \equiv w(T)x \).

In cases 4 and 5 the result is obtained by using the induction hypothesis as in case 2. \( \square \)

**Example 3.2.2** Consider the tableau of type \( G_2 \)

\[
T = \begin{array}{ccc}
2 & 0 & 3 \\
0 & 2 & 1 \\
\end{array}
\]

Then the insertion \( \overline{3} \to T \) may be computed as follows:

\[
\begin{align*}
\overline{3} & \to \begin{array}{ccc}
2 & 0 & 3 \\
0 & 2 & 1 \\
\end{array} \quad = \quad 3 \to \begin{array}{ccc}
0 & 3 \\
2 & 1 \\
\end{array} \quad = \quad 3 \to \begin{array}{ccc}
3 & 3 & 1 \\
3 & 1 \\
\end{array} \quad = \quad \begin{array}{ccc}
3 & 3 & 1 \\
3 & 1 \\
\end{array} \quad = \quad 2 \to \begin{array}{ccc}
3 & 1 \\
3 & 1 \\
\end{array} \quad = \quad \begin{array}{ccc}
2 & 3 & 1 \\
3 & 2 \\
\end{array} \\
\end{align*}
\]

Finally for any vertex \( w \in \Gamma \), we will have:

\[
P(w) = \begin{array}{c} \hat{w} \end{array} \text{ if } w \text{ is a letter,} \\
P(w) = x \to P(u) \text{ if } w = ux \text{ with } u \text{ a word and } x \text{ a letter.}
\]

### 3.3 Robinson Schensted correspondence

In this section a bijection is established between words \( w \) of length \( l \) on \( G \) and pairs \( (P(w),Q(w)) \) where \( P(w) \) is the tableau of type \( G_2 \) defined above and \( Q(w) \) is an oscillating tableau of type \( G_2 \).

**Definition 3.3.1** An oscillating tableau \( Q \) of type \( G_2 \) and length \( l \) is a sequence \( (Q_1,\ldots,Q_l) \) of Young diagrams whose columns have height 1 or 2 satisfying for \( k = 1, \ldots, l \) one of the following assertions:

1. \( Q_{k+1} \) is obtained by adding one box to \( Q_k \).
2. \( Q_{k+1} \) is obtained by deleting one box in \( Q_k \).
3. \( Q_{k+1} = Q_k \).
4. \( Q_{k+1} \) is obtained from \( Q_k \) by moving one box from height 2 to height 1.
5. \( Q_{k+1} \) is obtained from \( Q_k \) by moving one box from height 1 to height 2
Let \( w = x_1 \cdots x_l \) be a word. The construction of \( P(w) \) involves the construction of the \( l \) tableaux of type \( G_2 \) defined by \( P_i = P(x_1 \cdots x_i), \ i = 1, \ldots, l. \) For \( w \in \mathcal{G}^* \) we denote by \( Q(w) \) the sequence of shapes of the tableaux \( (P_1, \ldots, P_l) \).

**Proposition 3.3.2** \( Q(w) \) is an oscillating tableau of type \( G_2 \).

**Proof.** It follows immediately from the proof of Lemma 3.1.5. \( \blacksquare \)

**Theorem 3.3.3** Two vertices \( w_1 \) and \( w_2 \) of \( \Gamma \) belong to the same connected component if and only if \( Q(w_1) = Q(w_2) \).

**Proof.** The proof is the same than in Theorem 3.4.3 of \( \text{[10]} \). \( \blacksquare \)

**Corollary 3.3.4** Let \( \mathcal{G}_1^* \) and \( \mathcal{O}_l \) be the set of words of length \( l \) on \( \mathcal{G} \) and the set of pairs \( (P, Q) \) where \( P \) is a tableau of type \( G_2 \) and \( Q \) an oscillating tableau of type \( G_2 \) and length \( l \) such that \( P \) has shape \( Q_l \) (\( Q_l \) is the last shape of \( Q \)). Then the map:

\[
\Psi : \mathcal{G}_1^* \rightarrow \mathcal{O}_l \quad w \mapsto (P(w), Q(w))
\]

is a bijection.

**Proof.** The proof is analogous to that of Theorem 5.2.2 of \( \text{[16]} \). \( \blacksquare \)

## 4 Computing the global basis of \( V(\Lambda) \)

### 4.1 The representation \( V(\Lambda_1) \)

The vector representation \( V(\Lambda_1) \) of \( U_q(G_2) \) is the vector space of basis \( \{v_x, x \in \mathcal{G}\} \) where

\[
t_i(v_x) = q_i^{<\text{wt}(x), \alpha_i>} \quad \text{for } i = 1, 2.
\]

and

\[
\begin{align*}
&f_i(v_x) = v_y \quad \text{if } f_i(x) = y, \ \text{otherwise } f_i(v_x) = 0 \\
&e_i(v_x) = \delta_{i,1}(q + q^{-1})v_x \\
&\text{for } x \neq 0, \ e_i(v_x) = v_y \quad \text{if } e_i(x) = y, \ \text{otherwise } e_i(v_x) = 0 \\
&\text{otherwise}
\end{align*}
\]

\( \text{(18)} \)

Note that, with our definition of the action of \( f_1 \) on \( v_n \) and \( v_0 \), we have \( f_1^{(2)}(v_n) = v_\varpi \).

**Remark 4.1.1** Set \( L_1 = \bigoplus_{x \in \mathcal{G}} Av_x. \) If we identify the image of \( v_x \) by the canonical projection \( L_1 \rightarrow L_1/qL_1 \) with \( x \) then \( (L_1, B_1 = B(\Lambda_1)) \) is the crystal basis of \( V(\Lambda_1) \).

The basis \( \{v_x, x \in \mathcal{G}\} \) satisfies conditions \( (11) \) and \( (11) \) of Theorem 2.4.1 thus is the canonical basis of \( V(\Lambda_1) \). We have \( G(\mathcal{L}(x)) = v_x \) for any \( x \in B(\Lambda_1) \).

### 4.2 Canonical basis of \( V(\Lambda_2) \)

Similarly to \( \text{[13]} \) we introduce \( W(\Lambda_2) \) the \( q \)-analogue to the 2-th wedge product of \( V(\Lambda_1) \) defined by

\[
W(\Lambda_2) = V(\Lambda_1)^{\otimes 2}/N
\]

where \( N \) is the sub-module isomorphic to \( V(2\Lambda_1) \) appearing in the decomposition

\[
V(\Lambda_1)^{\otimes 2} \simeq V(2\Lambda_1) \bigoplus V(\Lambda_2) \bigoplus V(\Lambda_1) \bigoplus V(0)
\]

to \( V(\Lambda_1)^{\otimes 2} \) into its irreducible components. Let \( \Psi \) be the canonical projection \( V(\Lambda_1)^{\otimes 2} \rightarrow W(\Lambda_2) \). Set

\[
\Psi(v_{x_1} \otimes v_{x_2}) = v_{x_1} \wedge v_{x_2}. \quad (19)
\]
Proposition 4.2.1 In \( W(\Lambda_2) \) we have the relations:

1. for \( x \neq 0 \), \( v_x \wedge v_x = 0 \),
2. for \( xy \in B(\Lambda_2) \) and \( x \neq \overline{y} \), \( v_y \wedge v_x = \begin{cases} -q^2 v_y \wedge v_y & \text{if } x = 0 \text{ or } y = 0 \\ -q^4 v_x \wedge v_y & \text{if } (x, y) = (2, 3) \text{ or } (x, y) = (\overline{3}, \overline{2}) \end{cases} \),
3. \( v_0 \wedge v_1 = -q^2 v_1 \wedge v_0 + (q^5 - q) v_2 \wedge v_3 \), \( v_5 \wedge v_1 = -q^3 v_1 \wedge v_5 + (q^3 - q) v_2 \wedge v_0 \), \( v_5 \wedge v_1 = -q^3 v_1 \wedge v_5 + (q^3 - q) v_3 \wedge v_0 \) and \( v_7 \wedge v_3 = -q^2 v_0 \wedge v_7 + (q^5 - q)v_5 \wedge v_7 \), \( v_7 \wedge v_3 = -q^3 v_3 \wedge v_7 + (q^3 - q)v_0 \wedge v_7 \),
4. \( v_7 \wedge v_3 = -q^4 v_3 \wedge v_7 - qv_0 \wedge v_0 \),
5. \( v_7 \wedge v_2 = -q^4 v_2 \wedge v_7 + (q^7 - q)v_3 \wedge v_7 + q^4 v_0 \wedge v_0 \),
6. \( v_7 \wedge v_1 = -q^4 v_1 \wedge v_7 + (q^5 - q^3)v_2 \wedge v_7 + (-q^8 + q^6 + q^4 + q^2)v_3 \wedge v_7 + (-q^5 + q^3 - q)v_0 \wedge v_0 \).

Proof. The proof is analogous to that of Proposition 3.1.1 of [18]. Starting from the highest weight vector \( v_1 \otimes v_1 \) of \( N \) we apply Chevalley's operators \( f_i \) \( i = 1, 2 \) to compute a basis of vectors of \( N \). These vectors are annihilated by \( \Psi \) which give equalities in \( W(\Lambda_2) \). The relations of the Proposition are then obtained by linear combinations of these equalities. ■

For any column \( C \) of reading \( w = c_1 c_2 \) where the \( c_i \)'s are letters, we set \( v_C = v_{c_1} \wedge v_{c_2} \). Then each vector \( \Psi(v_1 \otimes v_2) = v_{x_1} \wedge v_{x_2} \) can be decomposed into a linear combination of vectors \( v_C \) by applying from left to right a relation given in the above proposition.

Lemma 4.2.2 The vectors of \( \{ v_C, C \in C(2) \} \) form a basis of \( W(\Lambda_2) \).

Proof. Each vector of \( W(\Lambda_2) \) can be decomposed into a linear combination of vectors \( v_C \). So it suffices to proves that \( \dim(W(\Lambda_2)) = \text{card}(C(2)) \). It follows from the definition of \( W(\Lambda_2) \) that \( \dim(W(\Lambda_2)) = \dim(V(\Lambda_1))^2 - \dim(V(2\Lambda_1)) = 49 - 27 = 22 \) which is exactly the number of columns of height 2 on \( G \). ■

The coordinates of a vector \( v_{x_1} \wedge v_{x_2} \) on the basis \( \{ v_C, C \in C(2) \} \) are all in \( \mathbb{Z}[q] \) since it is true for the coefficients appearing in the relations of Proposition 3.1.1. Consider the \( A \)-lattice \( L_2 = \bigoplus_{C \in C(2)} v_C \) of \( W(\Lambda_2) \) and denote by \( \pi_2 \) the projection \( L_2 \rightarrow L_2/qL_2 \). We identify \( \pi_2(v_C) \) with the word \( w(C) \).

Lemma 4.2.3 \( \langle L_2, B_2 = \{ w(C), C \in C(2) \} \rangle \) is a crystal basis of \( W(\Lambda_2) \).

Proof. The proof is the same than in Lemma 3.1.3 of [18]. ■

The vector \( v_{\Lambda_2} = v_1 \wedge v_2 \) is of highest weight \( \Lambda_2 \) in \( W(\Lambda_2) \). We identify \( V(\Lambda_2) \) with the sub-module of \( W(\Lambda_2) \) isomorphic to \( V(\Lambda_2) \). In the sequel we need the explicit description of the action of the Chevalley operators \( f_1 \) and \( f_2 \) on the basis \( \{ v_C, C \in C(2) \} \). They are given by the following tables.
(where we have written for short $C$ in place of $v_C$) obtained from (18), (19) and Proposition 4.2.1:

$$f_1(v_C) = \begin{cases} 
\begin{array}{c|c}
C & f_1(v_C) \\
\hline
\frac{1}{3} & \frac{2}{3} + q \frac{1}{0} \\
\frac{1}{0} & \frac{2}{0} + (q^2 + 1) \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} \\
\frac{1}{2} & \frac{2}{2} + q \frac{1}{1} \\
\frac{1}{1} & \frac{2}{1} \\
\frac{2}{3} & q^{-1} \frac{2}{0} \\
\frac{0}{0} & (1 + q^{-2}) \frac{2}{3} \\
\frac{2}{2} & q^{-1} \frac{2}{1} \\
\end{array} 
\end{cases}$$

and $f_1(v_C) = 0$ otherwise.
and \(f_2(v_C) = 0\) otherwise.

Now we are going to give the explicit decomposition of the canonical basis of \(V(\Lambda_2)\) on \(\{v_C, C \in \mathbf{C}(2)\}\). Consider an admissible column \(C\) of height 2. If \(w(C) \neq \emptyset\) there is a unique path in \(B(\Lambda_2)\) joining 12 to \(w(C)\). Otherwise we choose the path \(\emptyset = \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 (12)\). Then (with our choice of the path joining 12 to \(\emptyset\)) we can write \(w(C) = \tilde{f}_{i_1} \tilde{f}_{i_2} \tilde{f}_{i_3} (12)\) with \(i_k \neq i_{k+1}\) for \(k = 1, \ldots, r - 1\).

**Theorem 4.2.4** For any admissible column \(C\) of height 2, \(G(C) = f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} (v_{\Lambda_2})\).

**Proof.** The vectors \(f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} (v_{\Lambda_2})\) belong to \(V_0(\Lambda_2)\) and are fixed by the involution \(\sigma\). So it suffices to prove that the coordinates of the decomposition of each vector \(f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} (v_{\Lambda_2})\) on the basis \(\{v_C, C \in \mathbf{C}(2)\}\) are all in \(\mathbb{Z}[q]\) and such that

\[
f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} (v_{\Lambda_2}) = w(C) \mod (q).
\]

This is shown by an explicit computation from the action of the operators \(f_1\) and \(f_2\) given above. The results are given in the table below.
4.3 Algorithm for the global basis of $V(\lambda)$.

4.3.1 The representation $W(\lambda)$

To make our notation homogeneous write $W(\Lambda_1) = V(\Lambda_1)$.

Consider $\lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2 \in P_+$ and set

$$W(\lambda) = W(\Lambda_1)^{\otimes \lambda_1} \otimes W(\Lambda_2)^{\otimes \lambda_2}. \tag{20}$$

The natural basis of $W(\lambda)$ consists of the tensor products $v_{C_1} \otimes \cdots \otimes v_{C_r}$ of basis vectors $v_{C_i}$ of the previous section appearing in (20). The juxtaposition of the columns $C_1, \ldots, C_r$ is called a tabloid of shape $Y(\lambda)$. We can regard it as a filling $\tau$ of the Young diagram of shape $\lambda$ the $i$-th column of which is equal to $C_i$. We shall write $v_\tau = v_{C_r} \otimes \cdots \otimes v_{C_1}$. Note that the columns of $\tau$ are not necessarily admissible and there is no condition on the rows. The reading of the tabloid $\tau = C_1 \cdots C_r$ is $w(\tau) = w(C_r) \cdots w(C_1)$. We denote by $T(\lambda)$ the set of tabloids of shape $\lambda$.

Let $L_\lambda$ be the $A$-submodule of $W(\lambda)$ generated by the vectors $v_\tau$, $\tau \in T(\lambda)$. We identify the image of the vector $v_\lambda$ by the projection $\pi_\lambda : L_\lambda \rightarrow L_\lambda/qL_\lambda$ with the word $w(\tau)$. The pair $(L_\lambda, B_\lambda = \{w(\tau), \tau \in T(\lambda)\})$ is then a crystal basis of $W(\lambda)$. Indeed by Remark 4.1.1 and Lemma 4.2.3, it is the tensor product of the crystal bases of the representations $W(\Lambda_p)$ $p = 1, 2$ occurring in $W(\lambda)$.

Set

$$v_\lambda = v_{\Lambda_1}^{\otimes \lambda_1} \otimes v_{\Lambda_2}^{\otimes \lambda_2}. \tag{21}$$

We identify $V(\lambda)$ with the submodule of $W(\lambda)$ generated by $v_\lambda$. Then, with the above notations, $v_\lambda = v_T$ where $T_\lambda$ is the orthogonal tableau of shape $\lambda$ whose $k$-th row is filled by letters $k$ for
$k = 1, 2$. By Theorem 4.2 of [10], we know that

$$B(\lambda) = \{ \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_r}^{a_r} w(T_\lambda); \ i_1, \ldots, i_r = 1, 2; \ a_1, \ldots, a_r > 0 \} - \{ 0 \}.$$  

The actions of $\tilde{e}_i$ and $\tilde{f}_i$, $i = 1, 2$, on each $w(\tau) \in B_2$ are identical to those obtained by considering $w(\tau)$ as a vertex of $\Gamma$ since it is true on $B_1$ and $B_2$. Hence, by Theorem 2.3.1 $B(\lambda) = \{ w(T); T \in T_G(\lambda) \}$. For each vector $G(T)$ of the canonical basis of $V(\lambda)$ we will have:

$$G(T) \equiv v_T \mod qL_\lambda.$$  

The aim of this section is to describe an algorithm computing the decomposition of the canonical basis $\{ G(T); T \in T_G(\lambda) \}$ onto the basis $\{ v_T; \tau \in T(\lambda) \}$ of $W(\lambda)$.

In the sequel we will need a total order on the readings of the tabloids. Let $w_1 = x_1 \cdots x_l$ and $w_2 = y_1 \cdots y_l$ be two distinct vertices of $\Gamma$ with the same length and $k$ the lowest integer such that $x_k \neq y_k$. We write $w_1 \preceq w_2$ if $x_k \leq y_k$ in $G$ and $w_1 \succ w_2$ otherwise that is, $\preceq$ is the lexicographic order on $G^*$. We endow the set $T(\lambda)$ with the total order:

$$\tau_1 \preceq \tau_2 \iff w(\tau_1) \preceq w(\tau_2).$$

In fact $\preceq$ is a lexicographic order defined on the readings of the tabloids of $T(\lambda)$ such that $x \preceq \tilde{f}_i(x)$ for any letter $x$ with $\varphi_i(x) \neq 0$.

We are going to compute the canonical basis $\{ G(T); T \in T_G(\lambda) \}$ in two steps. First we obtain an intermediate basis $\{ A(T); T \in T_G(\lambda) \}$ which is fixed by the involution (condition [11]). When $T = C$ is an admissible column, $A(C) = G(C)$. In the general case, we have to correct $A(T)$ in order to verify condition [10]. This second step is easy because we can prove that the transition matrix from $\{ A(T); T \in T_G(\lambda) \}$ to $\{ G(T); T \in T_G(\lambda) \}$ is unitriangular once the orthogonal tableaux and the tabloids are ordered by $\preceq$.

We will need the following lemma whose proof is the same than in Lemma 4.1.1 of [18].

**Lemma 4.3.1** Let $v \in V(\lambda)$ be a vector of the type

$$v = \tilde{f}_{i_1}^{(r_1)} \cdots \tilde{f}_{i_s}^{(r_s)} v_\lambda$$

where $(i_1, \ldots, i_s)$ and $(r_1, \ldots, r_s)$ are two sequences of integers. Then the coordinates of $v$ on the basis $\{ v_\tau; \tau \in T(\lambda) \}$ belong to $\mathbb{Z}[q, q^{-1}]$.

**4.3.2 The basis $A(T)$**

The basis $\{ A(T) \}$ will be a monomial basis, that is, a basis of the form

$$A(T) = \tilde{f}_{i_1}^{(r_1)} \cdots \tilde{f}_{i_m}^{(r_m)} v_\lambda.$$  

By Lemma 4.3.1 the coordinates of $A(T)$ on the basis $\{ v_\tau \}$ of $W(\lambda)$ belongs to $\mathbb{Z}[q, q^{-1}]$. To find the two sequences of integers $(i_1, \ldots, i_m)$ and $(r_1, \ldots, r_m)$ associated to $T$, we proceed as follows.

Write $T = C_1 \cdots C_s \neq T_\lambda \in T_G(\lambda)$. Let $C_k$ be the rightmost column of $T$ such that $w(C_k)$ is not a highest weight vertex (i.e. $w(C_k) \neq 1, 12$). When $w(C_k) \neq 0 \tilde{f}_i$ let $i_1 \in \{ 1, 2 \}$ be the unique integer such that $\tilde{e}_{i_1}(w(C_k)) \neq 0$. When $w(C_k) \neq 0 \tilde{f}_i$ we choose $i_1 = 1$. If $k = 1$, set $l = 1$. If $k > 1$ and $\tilde{e}_{i_1} v_{C_k} \neq 0$ or $\tilde{e}_{i_1}(w(C_{k-1})) = 0$ set $l = k$. Otherwise let $l$ be the lowest integer $l < k$ satisfying the two conditions

$$\begin{cases} (i) : f_{i_1} v_{C_j} = 0 \text{ for } j = l + 1, \ldots, k \\ (ii) : \tilde{e}_{i_1}(w(C_j)) \neq 0 \text{ for } j = l, \ldots, k \end{cases}$$  

Set $\varepsilon_{l,j} = \varepsilon_{i_1}((w(C_j)))$ for $j = l, \ldots, k$ and $r_1 = \sum_{j=l}^k \varepsilon_{l,j}$. Write $T_1$ for the tableau obtained by changing in $T$ each column $C_j$, $j = l, \ldots, k$ into the column of reading $\tilde{e}_{i_1}^{\varepsilon_{l,j}}(w(C_j))$.

**Lemma 4.3.2** $T_1 \in T_G(\lambda)$. 

\[ \text{15} \]
Proof. Set $T_1 = D_1 \cdots D_s$. Then by construction of $T_1$, we have $D_i = c_i$ for $i \notin \{l, ..., k\}$. The columns $C_i$ with $i = k + 1, ..., s$ are of highest weight. One has $w(D_k \cdots D_s) = \bar{e}^{\bar{e}_{i_1}}(w(C_k \cdots C_s))$ by (3) because $e_{i_1,k} = e_{i_1}w(C_k \cdots C_s)$ and $\varphi_{i_1}w(C_j) = 0$ for $j = k + 1, ..., s$ (otherwise we would have $\bar{e}_{i_1}(w(C_k)) = 0$ since the letters $1, ..., i_1$ would occur in $C_k$). So $D_k \cdots D_s$ is a tableau of type $G_2$.

We have $w(D_l \cdots D_k) = \bar{e}_i \bar{e}_j(\bar{e}^{\bar{e}_{i_1}}(w(C_k \cdots C_s)))$ because $\varphi_{i_1}w(C_j) = 0$ for $j = l + 1, ..., k$ (from (i) of (23)) so is the reading of a tableau of type $G_2$. Hence $D_l \cdots D_k$ is a tableau of type $G_2$.

The proof will be complete if we show that $D_l \cdots D_k = C_l \cdots C_{l-1}D_l$ is a tableau of type $G_2$, that is if we prove that $C_{l-1} \leq D_l$. We can suppose $l > 1$. Then we have either $e_{i_1}w(C_l) \neq 0$ and $\bar{e}_{i_1}(w(C_l)) \neq 0$ or $\bar{e}_{i_1}(w(C_{l-1})) = 0$. Suppose $\bar{e}_{i_1}(w(C_{l-1})) = 0$. Then we have $\bar{e}_{i_1}(w(C_l)w(C_{l-1})) = w(D_l)w(C_{l-1})$ by (3). So $C_{l-1} \leq D_l$.

Now suppose $e_{i_1}w(C_l) \neq 0$ and $\bar{e}_{i_1}(w(C_l)) \neq 0$. If $i_1 = 2$, $C_l$ is necessarily the column of reading $\overline{33}$. We have $C_{l-1}C_l = \begin{array}{c} 2 \\ 3 \end{array}$ and $C_{l-1}D_l = \begin{array}{c} 1 \\ 2 \end{array}$ with $x \geq 3$ and $\text{dist}(x, \overline{3}) \geq 3$. Hence $x \geq 2$ and $C_{l-1} \leq D_l$ by using that $C_{l-1} \leq C_l$. If $i_1 = 1$ and $h(C_l) = 2$, only the following configurations can appear

\[
\begin{align*}
(i) : C_{l-1}C_l &= \begin{array}{c} 2 \\ 3 \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 1 \\ 3 \end{array} \\
(ii) : C_{l-1}C_l &= \begin{array}{c} 2 \\ 0 \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 1 \\ 3 \end{array} \\
(iii) : C_{l-1}C_l &= \begin{array}{c} 0 \\ 2 \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 1 \\ 2 \end{array} \\
(iv) : C_{l-1}C_l &= \begin{array}{c} 0 \\ 2 \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 1 \\ 2 \end{array} \\
(v) : C_{l-1}C_l &= \begin{array}{c} 3 \\ 2 \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 3 \\ 2 \end{array}
\end{align*}
\]

In case (i), (ii) we have $x = 1$ and $y \geq 3$ by (6). In cases (iii), (iv) and (v) we have similarly $x \geq 3$. By using that $C_{l-1} \leq C_l$, we obtain $C_{l-1} \leq D_l$. If $i_1 = 1$ and $h(C_l) = 1$ only the following configurations may appear:

\[
\begin{align*}
(vi) : C_{l-1}C_l &= \begin{array}{c} 0 \\ y \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 3 \\ y \end{array} \\
(vii) : C_{l-1}C_l &= \begin{array}{c} 0 \\ y \end{array} \quad \text{and} \quad C_{l-1}D_l = \begin{array}{c} 0 \\ y \end{array}
\end{align*}
\]

with $x \geq 3$ by (8). So $C_{l-1} \leq D_l$ and the proposition holds. □

Once the tableau $T_1$ defined, we do the same with $T_1$ getting a new tableau $T_2$ of type $G_2$ and a new integer $i_2$. And so on until the tableau $T_s$ obtained is equal to $T_\lambda$. Notice that we can not write $w(T_1) = \bar{e}_{i_1}(w(T))$ in general, that is, our algorithm does not provide a path in the crystal graph $B(\lambda)$ joining the vertex $w(T)$ to the vertex of highest weight $w(T_\lambda)$.

Example 4.3.3 Consider the tableau of type $G_2$

\[
T = \begin{array}{c} 3 \\ 3 \\ 3 \\ 1 \\ 2 \\ 1 \end{array}
\]

We obtain successively:

\[
T_1 = \begin{array}{cc} 3 & 3 \\ 2 & 2 \\ 2 & 2 \end{array}, \quad T_2 = \begin{array}{cc} 2 & 2 \\ 2 & 3 \\ 3 & 3 \end{array}, \quad T_3 = \begin{array}{cc} 1 & 1 \\ 3 & 3 \\ 3 & 3 \end{array}, \quad T_4 = \begin{array}{cc} 1 & 1 \\ 2 & 3 \\ 2 & 2 \end{array}
\]

and $A(T) = f_j^{(4)} f_{l_1}^{(5)} f_{l_2}^{(6)} v_{T_\lambda}$. 

16
Proposition 4.3.4 The expansion of $A(T)$ on the basis $\{v_\tau; \tau \in T(\lambda)\}$ of $W(\lambda)$ is of the form

$$A(T) = \sum_\tau \alpha_{\tau,T}(q)v_\tau$$

where the coefficients $\alpha_{\tau,T}(q)$ satisfy:

(i): $\alpha_{\tau,T}(q) \neq 0$ only if $\tau$ and $T$ have the same weight,

(ii): $\alpha_{\tau,T}(q) \in \mathbb{Z}[q, q^{-1}]$ and $\alpha_{\tau,T}(q) = 1$,

(iii): $\alpha_{\tau,T}(q) \neq 0$ only if $\tau \leq T$.

Proof. The proof is the same than in Proposition 4.3.4 of [18].

It follows from (iii) that the vectors $A(T)$ are linearly independent in $V(\lambda)$. This implies that $\{A(T); T \in T_G(\lambda)\}$ is a $\mathbb{Q}[q]$-basis of $V(\lambda)$. Indeed by Theorem 2.3.1, $\dim V(\lambda) = \text{card}(T_G(\lambda))$. As a consequence of (22), we obtain $A(T) = A(T)$.

4.3.3 From $A(T)$ to $G(T)$

To obtain $G(T)$ from $A(T)$ we proceed as in [14], [17] and [18]. The reader is referred to [14] for the proofs. Set

$$G(T) = \sum_{\tau \in T_G(\lambda)} d_{\tau,T}(q)v_\tau$$

Our aim is to describe a simple algorithm for computing the rectangular matrix of coefficients

$$D = [d_{\tau,T}(q)], \quad \tau \in T(n, \lambda), \quad T \in T_G(\lambda).$$

Lemma 4.3.5 The coefficients $d_{\tau,T}(q)$ belong to $\mathbb{Q}[q]$. Moreover $d_{\tau,T}(0) = 0$ if $\tau \neq T$ and $d_{T,T} = 1$.

Now write

$$G(T) = \sum_{S \in T_G(\lambda)} \beta_{S,T}(q)A(S) \quad (24)$$

the expansion of the basis $\{G(T)\}$ on the basis $\{A(T)\}$.

Lemma 4.3.6 The coefficients $\beta_{S,T}(q)$ of (24) satisfy:

(i): $\beta_{S,T}(q) = \beta_{S,T}(q^{-1})$,

(ii): $\beta_{S,T}(q) = 0$ unless $S \leq T$,

(iii): $\beta_{T,T}(q) = 1$.

Let $T_\lambda = T^{(1)} \leq T^{(2)} \leq \cdots \leq T^{(i)}$ be the sequence of tableaux of $T_G(\lambda)$ ordered in increasing order. We have $G(T_\lambda) = A(T_\lambda)$, i.e. $G(T^{(1)}) = A(T^{(1)})$. By the previous lemma, the transition matrix $M$ from $\{A(T)\}$ to $\{G(T)\}$ is upper unitriangular once the two bases are ordered with $\leq$. Since $\{G(T)\}$ is a $\mathbb{Q}[q, q^{-1}]$ basis of $V_\lambda$ and $A(T) \in V_\lambda(\lambda)$, the entries of $M$ are in $\mathbb{Q}[q, q^{-1}]$. Suppose by induction that we have computed the expansion on the basis $\{v_\tau; \tau \in T(\lambda)\}$ of the vectors

$$G(T^{(1)}), ..., G(T^{(i)})$$

and that this expansion verifies $d_{\tau,T^{(p)}}(q) = 0$ if $\tau \triangleright T^{(p)}$ for $p = 1, ..., i$. The inverse matrix $M^{-1}$ is also upper unitriangular with entries in $\mathbb{Q}[q, q^{-1}]$. So we can write:

$$G(T^{(i+1)}) = A(T^{(i+1)}) - \gamma_1(q)G(T^{(i)}) - \cdots - \gamma_1(q)G(T^{(1)}). \quad (25)$$

It follows from condition (11) and Proposition 4.3.4 that $\gamma_{m}(q) = \gamma_{m}(q^{-1})$ for $m = 1, ..., i$. By Lemma 4.3.3, the coordinate $d_{T^{(i)}, T^{(i+1)}}(q)$ of $G(T^{(i+1)})$ on the vector $v_{T^{(i)}}$ belongs to $\mathbb{Q}[q]$, $d_{T^{(i)}, T^{(i+1)}}(0) = 0$ and the coordinate $d_{T^{(i)}, T^{(i)}}(q)$ of $G(T^{(i)})$ on the vector $v_{T^{(i)}}$ is equal to 1. Moreover $v_{T^{(i)}}$ can only occur in $A(T^{(i+1)}) - \gamma_{1}(q)G(T^{(i)})$. If

$$\alpha_{T^{(i)}, T^{(i+1)}}(q) = \sum_{j=-r}^{s} a_j q^j \in \mathbb{Z}[q, q^{-1}]$$
then we will have
\[
\gamma_i(q) = \sum_{j=-r}^{0} a_j q^j + \sum_{j=1}^{r} a_{-j} q^j \in \mathbb{Z}[q, q^{-1}].
\]
Next if the coefficient of \(v_{T(i-1)}\) in \(A(T^{(i+1)}) - \gamma_i(q)G(T^{(i)})\) is equal to
\[
\sum_{j=-k}^{k} b_j q^j
\]
using similar arguments we obtain
\[
\gamma_{i-1}(q) = \sum_{j=-l}^{0} b_j q^j + \sum_{j=1}^{l} b_{-j} q^j,
\]
and so on. So we have computed the expansion of \(G(T^{(i+1)})\) on the basis \(\{v_{\tau}\}\) and this expansion verifies \(d_{\tau, T^{(i+1)}}(q) = 0\) if \(\tau \nleftrightsquigarrow T^{(i+1)}\). Finally notice that \(\gamma_i(q) \in \mathbb{Z}[q, q^{-1}]\) by Proposition 4.3.4.

**Theorem 4.3.7** Let \(T \in T_G(\lambda)\). Then \(G(T) = \sum d_{\tau, T}(q)v_{\tau}\) where the coefficients \(d_{\tau, T}(q)\) verify:

(i): \(d_{\tau, T}(q) \in \mathbb{Z}[q]\),
(ii): \(d_{T, T}(q) = 1\) and \(d_{\tau, T}(0) = 0\) for \(\tau \neq T\),
(iii): \(d_{\tau, T}(q) \neq 0\) only if \(\tau\) and \(T\) have the same weight, and \(\tau \leq T\).

**References**

[1] T. H. Baker, *An insertion scheme for \(C_n\) crystals*, in M. Kashiwara and T. Miwa, eds., Physical Combinatorics, Birkhäuser, Boston, 191 (2000), 1-48.

[2] V. Chari, A. Presley, *A guide to quantum groups*, Cambridge University Press 1994.

[3] J. Hong, S. J. Kang, *Introduction to quantum groups and crystals bases*, A.M.S 2002, GSM/12.

[4] J. C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Math. 6, A.M.S 1995

[5] N. Jing, K. C. Misra, M. Okado, *q-Wedge modules for quantized enveloping algebras of classical type*, Journal of Algebra, 230 (2000), 518-539.

[6] S. J. Kang, K. C. Misra, *Crystal bases and tensor product decompositions of \(U_q(G_2)\)-modules*, Journal of Algebra, 163 (1994), 675-691.

[7] M. Kashiwara, *Crystallizing the \(q\)-analogue of universal enveloping algebra*, Commun. Math. Phys, 133 (1990), 249-260.

[8] M. Kashiwara, *On crystal bases of the \(q\)-analogue of universal enveloping algebras*, Duke Math. J, 63 (1991), 465-516.

[9] M. Kashiwara, *Crystallization of quantized universal enveloping algebras*, Sugaku Expositiones, 7 (1994), 99-115

[10] M. Kashiwara, *On crystal bases*, Canadian Mathematical Society, Conference Proceedings, 16 (1995), 155-197.

[11] M. Kashiwara, T. Miwa, J-U. H. Petersen, C. M. Yung, *Perfect crystals and \(q\)-deformed Fock spaces*, Selecta Mathematica, 2 (1996), 415-499.

[12] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the \(q\)-analogue of classical Lie algebras*, Journal of Algebra, 165 (1994), 295-345.
[13] A. Lascoux, M. P Schützenberger, *Le monoïde plaxique*, in non commutative structures in algebra and geometric combinatorics A. de Luca Ed., Quaderni della Ricerca Scientifica del C.N.R., Roma, 1981.

[14] B. Leclerc, P. Toffin, *A simple algorithm for computing the global crystal basis of an irreducible* $U_q(sl_n)$-*module*, Int. J. Algebra Computation, 10 (2000), 191-208.

[15] C. Lecouvey, *Schensted-type correspondence, plactic monoid and Jeu de Taquin for type* $C_n$: J. Algebra, 247 (2001),

[16] C. Lecouvey, *Schensted-type correspondences and plactic monoids for types* $B_n$ and $D_n$, Journal of Combinatoric Algebra (to appear).

[17] C. Lecouvey, *An algorithm for computing the global basis of an* $U_q(sp_{2n})$-*module*, Advances in Applied Math. (to appear).

[18] C. Lecouvey, *An algorithm for computing the global basis of a finite dimensional irreducible* $U_q(so_{2n+1})$ or $U_q(so_{2n})$-*module* (submitted).

[19] P. Littelmann, *A plactic algebra for semisimple Lie algebras*, Adv. in Math, 124 (1996), 312-331.

[20] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Am. Math. Soc, 4 (1991), 365-421.

[21] R. Marsh, *Algorithms to obtain the canonical basis in some fundamental modules of quantum groups*, Journal of Algebra 196, 831-860 (1996).