Exact Solutions of a (2+1)-Dimensional Nonlinear Klein-Gordon Equation

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Abstract
The purpose of this paper is to present a class of particular solutions of a C(2,1) conformally invariant nonlinear Klein-Gordon equation by symmetry reduction. Using the subgroups of similitude group reduced ordinary differential equations of second order and their solutions by a singularity analysis are classified. In particular, it has been shown that whenever they have the Painlevé property, they can be transformed to standard forms by Moebius transformations of dependent variable and arbitrary smooth transformations of independent variable whose solutions, depending on the values of parameters, are expressible in terms of either elementary functions or Jacobi elliptic functions.

1 Introduction
In a recent paper [1], we constructed second order differential equations invariant under the Poincaré, similitude and conformal groups in (2+1)-dimensional space-time. For instance, the planar nonlinear Klein-Gordon (NLKG) equation

\[ \Box_3 u = H(u), \]

where \( H \) is an arbitrary sufficiently smooth function of its argument and \( \Box_3 = \partial_t^2 - \Delta_2 \) is the wave operator in the (2+1)-dimensional Minkowski space, is invariant under Poincaré group \( P(2,1) \) and more specifically, NLKG equation with a power nonlinearity

\[ \Box_3 u = au^k \quad (1.1) \]
is invariant under the Poincaré group extended by dilations, also called similitude group. Among equations of the form (1.1), the special case \( k = 5 \) plays a privileged role in classical and quantum field theory. In this context Eq. (1.1) with \( k = 5 \), namely

\[
\Box_3 u = a u^5
\]

(1.2)

where \( a \) is an arbitrary constant, arises as the equation of motion (Euler-Lagrange equation) obtained minimizing the action corresponding to a Lagrangian density

\[
\mathcal{L} = \frac{1}{2} (\nabla u)^2 - 6 a u^5 = \frac{1}{2} (u_x^2 + u_y^2) - 6 a u^5.
\]

This equation is also called classical \( \phi^6 \)-field equation. For further physical motivation of this equation, the reader is referred to Ref. [2]. Another remarkable property of Eq. (1.2) is that, in addition to being similitude invariant, it is also invariant under the conformal group \( \text{C}(2,1) \) of space-time.

It will be of interest to find exact solutions of the NKLG equation (1.2), not only from mathematical but also from physical point of view. While there exists an extensive literature on exact solutions of the NLKG equation they are mainly devoted to translation invariant solutions in 1+1 dimension. The study of exact solutions in higher dimensions appears in a few papers only. For example, in [2] exact solutions are studied in (3+1)-dimensional case. Another article dedicated to A.O.Barut [3] investigated translationally invariant solutions which actually live in a (1+1)-dimensional case and static spherically symmetric and similarity solutions in 3+1 dimensions. To our knowledge, there exists no systematic study of exact solutions of the NKLG equation in 2+1 dimensions and the present paper aims at obtaining exact solutions of the NLKG equation.

The method to be used for solving (1.2) is the symmetry method. This method is described in various books [4, 5] and lectures [6, 7]. Applications of the method to find exact solutions of numerous equations of nonlinear mathematical physics are given in [8]. The first step is to reduce the considered PDE to an ODE expressed in terms of symmetry variables. Next, we integrate, whenever possible, the obtained ODEs. For the NLKG equation (1.2), except for degenerate cases simplifying either to an algebraic or to a first order equation that can be integrated directly, most of the reduced ODEs will be second order and nonlinear.

Often, two approaches are adopted for solving these ODEs. The first is to find the symmetry group of the reduced equation, if one exists, and then using it to lower the order of the equation. The second one consists of performing a singularity analysis in order to establish whether the equation is of the Painlevé type meaning that the general solution of the corresponding equation has no movable singularities (branch points, essential singularities) other than
the poles. Note that the second method gives more satisfactory results than the first one. While the equation has no nontrivial symmetries, it may well belong to the class of the Painlevé type equations. Equations of the form

$$w''(z) = f(z, w, w')$$

where $f$ is analytic in $z$, rational in $w'$, and algebraic in $w$, possessing the Painlevé property was classified by Painlevé and Gambier. They showed that such an ODE can be reduced to one of the 50 equations listed in [9, 10] whose solutions can be expressed in terms of either elementary or Jacobi elliptic functions or of solutions of linear equations. Six of them are irreducible and known to be Painlevé transcendent which often occur in a host of physical problems. Equations that do not possess the Painlevé property will in general contain moving (logarithmic) singularities and we have no systematic method for integrating them.

In order to obtain in a systematic manner symmetry reductions of Eq. (1.2) we need a classification of subgroups of the symmetry group into conjugacy classes under the action of the symmetry group. The subgroups of similitude groups are known only for small dimensions, while those of Poincaré groups having generic orbits of codimension one are known for arbitrary dimension. The subgroups of $S(2,1)$ are classified in [14]. Making use of this result we obtain all possible reduced ODEs, mostly of second order nonlinear. In order to facilitate the tedious computations the MATHEMATICA package has been used.

2 The Symmetry Group and its Lie Algebra

The symmetry group of Eq. (1.2) called similitude group or extended Poincaré group $S(2,1)$ is the group of transformations leaving the Lorentz metric form invariant. Its structure is

$$S(2,1) = D \triangleright (SL(2,\mathbb{R}) \triangleright T_3)$$

(2.1)

where $T_3$ are space-time translations, $SL(2,\mathbb{R})$ is the special linear group, $D$ dilations and $\triangleright$ denotes a semi-direct product. A convenient basis for the Lie algebra of the extended Poincaré group is given by translations $\{P_0, P_1, P_2\}$, Lorentz boosts $\{K_1, K_2\}$, rotation $L_3$, and dilation $D$. For Eq. (1.2) the basis is realized as

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad P_2 = \partial_y, \quad K_1 = t\partial_x + x\partial_t, \quad K_2 = t\partial_y + y\partial_t, \quad L_3 = y\partial_x - x\partial_y, \quad D = x\partial_x + y\partial_y + t\partial_t - u/2\partial_u.$$  

(2.2a)
In addition to generators (2.2a), Eq. (1.2) has also conformal symmetries generated by

\[ C_0 = 2xt \partial_x + 2yt \partial_y + (x^2 + y^2 + t^2) \partial_t - tu \partial_u \]
\[ C_1 = (t^2 + x^2 - y^2) \partial_x + 2xy \partial_y + 2xt \partial_t - ux \partial_u \]
\[ C_2 = 2xy \partial_x + (t^2 - x^2 + y^2) \partial_y + 2yt \partial_t - uy \partial_u. \]  

(2.2b)

3 Symmetry Reductions of NLKG Equation

In this section we give a classification of symmetry reductions of (1.2) with respect to invariance under the similitude group. Applying the method of symmetry reduction we will derive all the solutions invariant under subgroups with generic orbits of codimension 1 in the space of independent variables. However for the sake of completeness, first, we obtain all reductions of (1.2) to lower dimensional equations with two independent variables.

3.1 Symmetry Reductions to PDEs in Two Variables

We restrict ourselves to subgroups with generic orbits of codimension two in the space-time \((x, y, t)\) and of codimension three in the space \((x, y, t, u)\). The corresponding three invariants \(I_i(x, y, t, u), i = 1, 2, 3\) of the group action on \(X \otimes U\) must provide an invertible transformation from the space of dependent variables to that of the invariants. Hence the invariants of the subgroup \(H\) of the symmetry group can be written in the form

\[ I_1 = \xi(x, y, t), \quad I_2 = \eta(x, y, t), \]
\[ I_3 = f(x, y, t, u) = uv\phi(x, y, t). \]

This permits us to reduce (1.2) to a partial differential equation for \(f(\xi, \eta)\) which is a function of the symmetry variables \(\xi\) and \(\eta\) and write the solution of (1.2) as

\[ u(x, y, t) = f(\xi, \eta)\phi(x, y, t) \]  

(3.1)

where \(\phi, \xi\) and \(\eta\) are known functions whose precise forms are to be determined by the choice of subgroup. Substituting the reduction formula (3.1) into (1.2) we obtain a partial differential equation for the function \(f\).

The classification of all subgroups of the similitude group is well known \[11\]. Using these classification results we classify symmetry reductions. In the following we give reduction formulas and reduced equations for all possible subgroups:
Subgroup \( \{ K_2 + L_3 \} \) :

\[
    u = f(\xi, \eta), \quad \xi = x + t, \quad \eta = y^2 - 2t(x + t),
\]

\[
    4(\xi^2 - \eta)f_{\eta\eta} - 4\xi f_{\xi\eta} - 6f_\eta + af_5 = 0. \tag{3.2}
\]

Subgroup \( \{ K_1 \} \) :

\[
    u = f(\xi, \eta), \quad \xi = t^2 - x^2,
\]

\[
    4\xi^2 f_{\xi\xi} - f_{\eta\eta} + 4f_\xi - af_5 = 0. \tag{3.3}
\]

Subgroup \( \{ D \} \) :

\[
    u = t^{-1/2}f(\xi, \eta), \quad \xi = \frac{y}{t}, \quad \eta = \frac{x}{t},
\]

\[
    4(\xi^2 - 1)f_{\xi\xi} + 4(\eta^2 - 1)f_{\eta\eta} + 8\xi\eta f_{\xi\eta}
\]

\[
    + 12\xi f_\xi + 12\eta f_\eta + 3f - 4af_5 = 0. \tag{3.4}
\]

Subgroup \( \{ P_2 \} \) :

\[
    u = f(x, t),
\]

\[
    f_{tt} - f_{xx} - af_5 = 0, \quad \text{y-independent equation.} \tag{3.5}
\]

Subgroup \( \{ L_3 \} \) :

\[
    u = f(\xi, t), \quad \xi = (x^2 + y^2)^{1/2},
\]

\[
    f_{tt} - f_{\xi\xi} - 1/\xi f_\xi - af_5 = 0. \tag{3.6}
\]

Subgroup \( \{ P_0 \} \) :

\[
    u = f(y, t),
\]

\[
    f_{tt} - f_{yy} - af_5 = 0, \quad \text{x-independent equation.} \tag{3.7}
\]

Subgroup \( \{ P_0 - P_1 \} \) :

\[
    u = f(\xi, y), \quad \xi = x + t,
\]

\[
    f_{yy} + bf_5 = 0. \tag{3.8}
\]
Subgroup \( \{K_1 + P_2\} \):

\[
\begin{align*}
    u &= f(\xi, \eta), \quad \xi = t^2 - x^2, \quad \eta = (x + t)e^{-y}, \\
    \eta^2 f_{\eta\eta} - 4\eta f_{\xi\eta} - 4\xi^2 f_{\xi\xi} + \eta f_{\eta} - 4\xi f_{\xi} + af^5 &= 0.
\end{align*}
\]  
(3.9)

Subgroup \( \{L_3 + P_0\} \):

\[
\begin{align*}
    u &= f(\xi, \eta), \quad \xi = x^2 + y^2, \quad \eta = \arctan \frac{y}{x} - t, \\
    4\xi^2 f_{\xi\xi} + (1 - \frac{1}{\xi^2}) f_{\eta\eta} - 4f_{\xi} - af^5 &= 0.
\end{align*}
\]  
(3.10)

Subgroup \( \{D + bK_1\} \quad 0 < b \leq 1 \):

\[
\begin{align*}
    u &= y^{-1/2} f(\xi, \eta), \quad \xi = (x + t)y^{-b-1}, \quad \eta = (x + t)y^{-(b+1)}, \\
    (b - 1)^2 \xi^2 f_{\xi\xi} - 2[(1 - b^2)\xi \eta - 2]f_{\xi\eta} + (b + 1)^2 \eta^2 f_{\eta\eta} \\
    + (b - 1)(b - 3)\xi f_{\xi} + (b + 1)(b + 3)\eta f_{\eta} + 3/4f + af^5 &= 0.
\end{align*}
\]  
(3.11)

Subgroup \( \{D + bL_3\}, \quad b > 0 \):

\[
\begin{align*}
    u &= t^{-1/2} f(\xi, \eta), \quad \xi = (x^2 + y^2)^{-b/2}e^{\arctan y/x}, \quad \eta = (x^2 + y^2)t^{-2}, \\
    (1 + b^2)\xi^2 f_{\xi\xi} - 4b\eta f_{\xi\eta} - 4\eta^2(\eta - 1) f_{\eta\eta} \\
    + (1 + b^2)\xi^2 f_{\xi} - 4\eta(2\eta - 1) f_{\eta} - (3/4)\eta f + af f^5 &= 0.
\end{align*}
\]  
(3.12)

Subgroup \( \{D + K_1 + P_0 + P_1\} \):

\[
\begin{align*}
    u &= y^{-1/2} f(\xi, \eta), \quad \xi = x - t, \quad \eta = (1 + x + t)y^{-2}, \\
    16\eta^2 f_{\eta\eta} + 16f_{\xi\xi} + 32\eta f_{\eta} + 3f + 4af^5 &= 0.
\end{align*}
\]  
(3.13)

3.2 Symmetry Reductions to Ordinary Differential Equations

Subgroups of the symmetry group that have generic orbits of codimension one in the space of independent variables \((x, y, t)\) and of codimension two in \((x, y, t, u)\)
space will provide reductions to ordinary differential equation. The invariants of subgroup $H$ have the form

$$I_1 = \xi(x, y, t) \quad \text{and} \quad I_2 = f(x, y, t, u) = \tilde{\phi}(x, y, t)u.$$  

In this case the reduction formula will be

$$u(x, y, t) = \phi(x, y, t)f(\xi)$$

where $\phi$ and $\xi$ are again known functions. If the action of the subgroup $H$, restricted to the time-space variables, is transitive the ordinary differential equation is of first order, otherwise of second order. We mention that subgroups with codimension zero in the time-space variables will reduce the original equation to an algebraic equation that may or may not admit nontrivial solutions.

We run through the individual subgroups and obtain the following reductions:

1. Subgroup $\{P_1, D + bK_2\}$:
   
   $$u = (t^2 - y^2)^{-1/4} f(\xi), \quad \xi = (t - y)^{b+1}(t + y)^{b-1},$$
   
   $$16(b^2 - 1)\xi^2 f'' + 8(2b^2 - b - 2)\xi f' + f - 4af^5 = 0. \quad (3.14)$$

2. Subgroup $\{D + bL_3, P_0\}$:
   
   $$u = (x^2 + y^2)^{-1/4} f(\xi), \quad \xi = \arctan y/x - b/2 \log(x^2 + y^2),$$
   
   $$4(1 + b^2)f'' + 4b f' + 4af^5 + f = 0. \quad (3.15)$$

3. Subgroup $\{D + bK_1, P_0 - P_1\}$, $-1 < b \leq 1, \quad b \neq 0$:
   
   $$u = y^{-1/2} f(\xi), \quad \xi = (x + t)y^{-(b+1)},$$
   
   $$b^2\xi^2 f'' + b(b + 2)\xi f' + 3/4 f + af^5 = 0. \quad (3.16)$$

4. Subgroup $\{D + K_1 + P_0 + P_1, P_0 - P_1\}$:
   
   $$u = y^{-1/2} f(\xi), \quad \xi = (x + t + 1)y^{-2},$$
   
   $$16\xi^2 f'' + 32\xi f' + 3f + 4af^5 = 0. \quad (3.17)$$

5. Subgroup $\{D + K_2 + P_0 + P_1, P_1\}$:
   
   $$u = (t + y + 1)^{-1/4} f(\xi), \quad \xi = t - y,$$
   
   $$f' + af^5 = 0. \quad (3.18)$$
(6) Subgroup \( \{D + bK_1, K_2 + L_3\}, \; b > 0 \):
\[
u = (t^2 - x^2 - y^2)^{-1/4} f(\xi), \quad \xi = (t^2 - x^2 - y^2)^{b+1}(t + x)^{-2},
\]
\[
4(b^2 - 1) \xi^2 f'' + 2(2b^2 - 1)\xi f' - 1/4 f - af^5 = 0.
\] (3.19)

(7) Subgroup \( \{D - K_1 + P_0 - P_1, K_2 + L_3\} \)
\[
u = (x^2 + y^2 - t^2 - x - t)^{-1/4} f(\xi), \quad \xi = x + t,
\]
\[
4\xi f' + f - 4af^5 = 0.
\] (3.20)

(8) Subgroup \( \{D, P_0\} \):
\[
u = t^{-1/2} f(\xi), \quad \xi = y/t,
\]
\[
4(1 - \xi^2) f'' - 12\xi f' - 3f + 4af^5 = 0.
\] (3.21)

(9) Subgroup \( \{P_0 - P_1, D\} \):
\[
u = y^{-1/2} f(\xi), \quad \xi = (x + t)/y,
\]
\[
4\xi^2 f'' + 12\xi f' + 3f + 4af^5 = 0.
\] (3.22)

(10) Subgroup \( \{D, P_2\} \):
\[
u = t^{-1/2} f(\xi), \quad \xi = x/t,
\]
\[
4(1 - \xi^2) f'' - 12\xi f' - 3f + 4af^5 = 0.
\] (3.23)

(11) Subgroup \( \{K_1, K_2 + L_3\} \):
\[
u = f(\xi), \quad \xi = x^2 + y^2 - t^2,
\]
\[
4\xi f'' + 6f' + af^5 = 0.
\] (3.24)

(12) Subgroup \( \{P_0 - P_1, K_1 + P_2\} \)
\[
u = f(\xi), \quad \xi = e^{-y}(x + t),
\]
\[
\xi^2 f'' + \xi f' + af^5 = 0.
\] (3.25)
Subgroup \( \{ P_0 - P_1, D + K_2 + L_3 \} \):
\[
u = (x + t)^{-1/2} f(\xi), \quad \xi = \frac{y}{x + t} - \log(x + t),
\]
\[f'' + af^5 = 0. \quad (3.26)\]

Subgroup \( \{ K_1, K_2, L_3 \} \):
\[
u = f(\xi), \quad \xi = t^2 - x^2 - y^2,
\]
\[4\xi f'' + 6 f' - af^5 = 0. \quad (3.27)\]

Subgroup \( \{ L_3, P_1, P_2 \} \):
\[
u = f(t), \quad f'' - af^5 = 0. \quad (3.28)\]

Subgroup \( \{ K_1, P_0, P_1 \} \):
\[
u = f(y), \quad f'' + af^5 = 0. \quad (3.29)\]

4 Discussion of the Reduced Ordinary Differential Equations

Once the reduced ODEs were obtained the remaining task will be to transform them, whenever they have the Painlevé property, into one of the standard forms that can be integrated once with the exception of the Painlevé transcendents. By a rescaling of independent and dependent variables all second order ODEs obtained through the symmetry reduction can be written in a unified manner as

\[A(\xi)f''(\xi) + B(\xi)f'(\xi) + C(\xi)f(\xi) + D(\xi)f^5(\xi) = 0. \quad (4.1)\]

We now pick out those having the Painlevé property. To achieve this task we subject the reduced equations to the Painlevé test which provides the necessary conditions for having the Painlevé property. Eq. (4.1) itself does not directly have the Painlevé property. However, a leading order analysis indicates that if we make the substitution
\[f(\xi) = \sqrt{h(\xi)}, \quad h(\xi) > 0\]
then the equation for \( h \) may have the Painlevé property.

In the following we run through all the ODEs separately:
Equation (3.14):

\[ B\xi^2 f'' + A\xi f' + f^5 = 0 \quad B = 16(b^2 - 1), \quad A = 8(2b^2 - b - 2). \quad (4.2) \]

By a change of independent variable \( \eta = \ln \xi \) it reduces to

\[ B\ddot{f} + (A - B)\dot{f} + f - 4af^5 = 0 \quad (4.3) \]

where dot denotes derivative with respect to \( \eta \). For \( b = \pm 1 \) after scaling variables we have

\[ \dot{f} = f(f^4 - 1). \quad (4.4) \]

Its solution with the original variable is

\[ f = (1 - \xi_0 \xi)^{-1/4} \]

where \( \xi_0 \) is an integration constant. For \( b = 0 \), it has the form

\[ f'' + f + f^5 = 0. \quad (4.5) \]

This equation passes the Painlevé test. For \( b \neq \pm 1 \), putting

\[ z = f \quad w(z) = \dot{f} \]

transforms equation (4.3) to

\[ 16(b^2 - 1)ww_z - 8bw + z - 4az^5 = 0 \]

which is an Abel equation of the second kind. Some remarks on this equation are noteworthy. This equation is not tractable by standard methods, meaning that there is no systematic method for solving it in closed form, neither a substitution transforming it into a linear equation. A list of solvable examples can be found in the collection of [12]. In a very recent paper [13], F. Schwarz studied symmetry analysis of Abel equation and showed that, when the coefficients of the rational normal form of the equation satisfy some constraint, Abel equation admits a one-parameter structure-preserving symmetry group reducing it to a quadrature. Existence of a two-parameter symmetry group implies that equation is actually equivalent to a Bernoulli equation.

Equation (3.15):

\[ f'' + Af' + f + f^5 = 0, \quad A = 2b(1 + b^2)^{-1/2}. \quad (4.6) \]

Eq. (4.4) passes the Painlevé test only for \( b = 0(A = 0) \). For \( b \neq 0 \), a transformation from \((\xi, f(\xi)) \rightarrow (z, w(z))\) by setting \( z = f \), \( w(z) = f' \) brings (4.6) to an Abel equation of the second kind

\[ w w_z + Aw + z + z^5 = 0. \]
Equations (3.16), (3.17), (3.19), (3.22):
They are all treated as similar to equation (3.14).

Equation (3.18):
Eq. (3.18) is immediately integrated to give the singular solution
\[ f = \left\{4a(\xi_0 - \xi)\right\}^{-1/4} \]
where \(\xi_0\) is an integration constant.

Equation (3.20):
\[ 4\xi f' + f - 4af^5 = 0. \] (4.7)
Let us transform the independent variable from \(\xi\) to \(\zeta = \ln \xi\). Scaling variables lead to
\[ f_\zeta = f(f^4 - 1) \]
which is equation (4.4) again.

Equation (3.21):
\[ (1 - \xi^2)f'' - 3\xi f' - \frac{3}{4}f + f^5 = 0. \] (4.8)
This equation has the Painlevé property. Eq. (3.23) is treated similarly.

Equation (3.24):
\[ \xi f'' + \frac{3}{2}f' + f^5 = 0. \] (4.9)
This equation has the Painlevé property. Eq. (3.27) is treated similarly.

Equations (3.25), (3.26), (3.28), (3.29):
All of these equations can be transformed to
\[ f'' + f^5 = 0 \] (4.10)
which has the first integral
\[ f'^2 + 1/3f^6 = C. \]
where $C$ is an integration constant. Setting $\phi = f^2$ this equation is further transformed to the elliptic function equation

$$\phi'^2 = 4C\phi - 4/3\phi^4$$

The form of solutions will depend on the values of $C$. The solutions of a more general equation of this type will be discussed below.

In summary, among the considered equations the only ones that pass the Painlevé test are

$$f'' + f + f^5 = 0,$$  \hspace{1cm} (4.11a)

$$(1 - \xi^2)f'' - 3\xi f' - \frac{3}{4}f + f^5 = 0,$$  \hspace{1cm} (4.11b)

$$\xi f'' + \frac{3}{2}f' + f^5 = 0.$$  \hspace{1cm} (4.11c)

The substitution $f = \sqrt{h}$ transforms these equations to

$$h'' = \frac{h'^2}{2h} - 2(h^3 + h),$$  \hspace{1cm} (4.12a)

$$h'' = \frac{h'^2}{2h} - \frac{1}{2(1 - \xi^2)}(4h^3 - 3h - 6\xi h'),$$  \hspace{1cm} (4.12b)

$$h'' = \frac{h'^2}{2h} - \frac{1}{\xi}(2h^3 + \frac{3}{2}),$$  \hspace{1cm} (4.12c)

respectively. By a rescaling of variables (4.12a) can be brought to a standard form [3]

$$h'' = \frac{h'^2}{2h} + \frac{3}{2}h^3 - \frac{h}{2}$$  \hspace{1cm} (4.13)

which has the first integral

$$h'^2 = h^4 - h^2 + Ch = h(h^3 - h + C),$$  \hspace{1cm} (4.14)

where $C$ is an integration constant. Solutions of this equation can be expressed in terms of Jacobi elliptic functions depending on the value of $C$. By a linear transformation

$$h(\xi) = \frac{3}{2}(1 - \xi^2)^{-1/2}W(\eta), \quad \eta = \ln(\xi + \sqrt{\xi^2 + 1})$$
the standard form of (4.12b) becomes (4.13). Finally, transforming (4.12c) by
\[ h(\xi) = \sqrt{-\frac{16}{3}} \xi^{-1} W(\eta), \quad \eta = -\frac{16}{3} \xi^{-3/2} \]
we have again (4.13). We rewrite Eq. (4.14) as
\[ h' = P(h) = h(h - h_1)(h - h_2)(h - h_3) \quad (4.15) \]
with
\[ h_1 + h_2 + h_3 = 0 \]
\[ h_1h_2 + h_1h_3 + h_2h_3 = -1 \]
\[ h_1h_2h_3 = -C. \]
The above equation can be simplified by a Moebius transformation of the dependent variable
\[ h(\xi) = \frac{\rho Z(\xi) + \sigma}{\mu Z(\xi) + \nu}, \]
where \( \rho, \sigma, \mu, \nu \) are constants. If all four roots of \( P(h) \) are distinct we choose \( \rho, \sigma, \mu, \nu \) so as to transform the zeros at 0, \( h_1, h_2 \) and \( h_3 \) into zeros at \( \pm 1 \) and \( \pm M \), where \( M \) is some constant. In other words, we transform (4.13) into the standard form
\[ Z'^2 = K(1 - Z^2)(M^2 - Z^2). \quad (4.16) \]
The form of solution of this equation depends on the multiplicity of the roots of the polynomial \( P(h) \). Solutions can be real or complex, finite or singular, periodic or localized. If the four roots are distinct, we obtain solutions in terms of Jacobi elliptic functions. The general solution of (4.16) is given by
\[
\begin{align*}
Z &= \text{sn}(\sqrt{K}M(\xi - \xi_0), M^{-1}), & \text{for } M^2 > 1, \ M^2 \in \mathbb{R} \\
Z &= \text{cn}(\sqrt{-K(1-M^2)}(\xi - \xi_0), (1 - M^2)^{-1/2}), & \text{for } M^2 < 0 \\
Z &= \text{dn}(\sqrt{-K}(\xi - \xi_0), (1 - M^2)^{1/2}), & \text{for } 0 < M^2 < 1.
\end{align*}
\]
They are hence always periodic rather than localized, and they may be finite or have periodically spaced singularities (poles) on the real axis. A detailed treatment of these functions can be found in any book on elliptic functions (for example, see Ref. [14]).

If any of the roots have multiplicity higher than one, we obtain elementary solutions. These can be localized, namely solitary waves or kinks. They can
also be periodic and hence delocalized. They can have singularities on the real axis. Some elementary solutions of (4.15) are listed below:

a.) \( h_1 = h_2 = h_3 = 0 \)
\[ h = (\xi - \xi_0)^{-1} \]

b.) \( h_2 = h_3 = 0, \ h_1 \neq 0 \)
\[ h = h_1 \{1 - h_2^2(\xi - \xi_0)^2/4\}^{-1} \]

c.) \( h_1 = h_2 = h_3 \neq 0 \)
\[ h = h_1(\xi - \xi_0)^2\{((\xi - \xi_0)^2 - 4h_1^{-2}\}^{-1} \]

d.) \( h_3 = 0, \ h_1 = h_2 \neq 0 \)
\[ h = h_1[1 - e^{(\xi-\xi_0)}]^{-1} \]

e.) \( h_3 = 0, \ h_1 \neq h_2 \neq 0 \)
\[ h = h_1h_2\{(h - h_1) \cosh \sqrt{h_1h_2}(\xi - \xi_0) + h_1 + h_2\}^{-1} \]

f.) \( h_2 = h_3 \neq 0, \ h_1 \neq h_3 \neq 0 \)
\[ h = h_1h_2 \tanh^2 X\{h_1 - h_2 + h_2 \tanh^2 X\}^{-1}, \quad X = \sqrt{h_2(h_2 - 1)/2(\xi - \xi_0)}. \]

In addition to similitude invariant exact solutions, conformal transformations generated by (2.2b) can be used to obtain conformally invariant solutions. For example, solutions invariant under a combination of conformal generators \( C_0, C_1, C_2 \), namely \( K = \alpha_0 C_0 + \alpha_1 C_1 + \alpha_2 C_2 \) are provided by the following reduction formula

\[ u = r^{-1}f(\omega), \quad \omega = r^{-2}(\beta_0 t + \beta_1 x + \beta_2 y), \quad r^2 = t^2 - x^2 - y^2 \quad (4.17) \]

with constants satisfying
\[ \beta_2\alpha_2 + \beta_1\alpha_1 - \beta_0\alpha_0 = 0. \]

Substitution of (4.17) into (1.2) gives rise to the second order ODE

\[ \beta^2 f'' + af^5 = 0, \quad \beta^2 = \beta_2^2 + \beta_1^2 - \beta_0^2. \quad (4.18) \]

Again, by a change of dependent variable this equation can be transformed to the elliptic function equation. On the other hand conformal symmetry allows for new solutions to be produced from Poincaré and similitude invariant solutions. More precisely, whenever \( f(\bar{r}) \) is a solution to (1.2), so is

\[ u(r) = \sigma^{-1/2}f(\bar{r}), \quad \sigma = 1 - \theta_3 r + \theta_2 r^2, \quad \bar{r} = \sigma^{-1}(r - \theta), \quad r = (t, x, y) \]

where \( \theta = (\theta_0, \theta_1, \theta_1) \) are group parameters.
5 Conclusions

In this paper we combine group theory with singularity analysis to obtain exact solutions to a conformally invariant nonlinear Klein-Gordon equation arising in classical and quantum field theory in (2+1)-dimensions. We first classify symmetry reductions of the equation based on a subgroup classification, rather than to choose intuitively obvious subgroups. In subsection 3.1 using the subgroups of the similitude group (i.e. the Poincaré group extended by dilations) we list all the reduced equations in two variables. In subsection 3.2 all possible symmetry reductions to first and second order ordinary differential equations are given. In the final section solutions of the reduced ordinary differential equations are discussed. In particular, it has been shown that, whenever they pass the Painlevé test, they can be transformed to the equations for elliptic functions. Hence, in general, we integrate them in terms of Jacobi elliptic functions. As the limiting cases of these functions we obtain interesting elementary solutions which may be periodic or localized. In cases when the reduced second order equations do not have the Painlevé property we reduce them to Abel type equations of the second kind. A well-known characteristic of them is that it is exceptionally that they are integrable with quadratures. That is why it is no surprise that the ones appearing in the present paper are far from being integrated by standard methods and they often lack a symmetry which will enable to find suitable coordinates reducing them to quadratures. Note that they may have movable branch points other than movable poles. This means that an Abel equation does not have the Painlevé property.

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