UNIFORMLY RECURRENT SUBGROUPS AND THE IDEAL STRUCTURE OF REDUCED CROSSED PRODUCTS

TAKUYA KAWABE

Abstract. We study the ideal structure of reduced crossed product of topological dynamical systems of a countable discrete group. More concretely, for a compact Hausdorff space $X$ with an action of a countable discrete group $\Gamma$, we consider the absence of a non-zero ideals in the reduced crossed product $C(X) \rtimes_r \Gamma$ which has a zero intersection with $C(X)$. We characterize this condition by a property for amenable subgroups of the stabilizer subgroups of $X$ in terms of the Chabauty space of $\Gamma$. This generalizes Kennedy’s algebraic characterization of the simplicity for a reduced group $C^*$-algebra of a countable discrete group.

1. Introduction

Throughout this paper, $\Gamma$ denotes a countable discrete group. We say $X$ is a compact $\Gamma$-space if $X$ is a compact Hausdorff space with a continuous $\Gamma$-action $\Gamma \times X \to X$, $(t,x) \mapsto tx$. We study the ideal structure of the reduced crossed product $C(X) \rtimes_r \Gamma$. The simplest situation is the following.

Definition 1.1. Let $X$ be a compact $\Gamma$-space. We say $C(X)$ separates the ideals in $C(X) \rtimes_r \Gamma$ if for every ideal $I$ in $C(X) \rtimes_r \Gamma$, we have $I = (I \cap C(X)) \rtimes_r \Gamma$. The simplest situation is the following.

In other words, there is one-to-one correspondence between the ideals in $C(X) \rtimes_r \Gamma$ and the $\Gamma$-invariant ideals in $C(X)$ (see [15, Proposition 1.1]).

Definition 1.2. We say that a compact $\Gamma$-space $X$ satisfies the intersection property if every non-zero ideal in $C(X) \rtimes_r \Gamma$ has a non-zero intersection with $C(X)$.

Then we have the following result.

Theorem 1.3 (Sierakowski, [15, Theorem 1.10]). Let $X$ be a compact $\Gamma$-space. Then $C(X)$ separates the ideals in $C(X) \rtimes_r \Gamma$ if and only if $X$ satisfies the following properties.

(i) The action of $\Gamma$ on $X$ is exact.
(ii) Every $\Gamma$-invariant closed set in $X$ has the intersection property.

The purpose of this paper is to characterize the intersection property of $\Gamma$-spaces in terms of dynamical systems. For an amenable group $\Gamma$, Kawamura and Tomiyama showed that the intersection properties of compact $\Gamma$-spaces is equivalent to topological freeness.

Theorem 1.4 (Kawamura–Tomiyama, [8, Theorem 4.1]). If $\Gamma$ is amenable, the following are equivalent.

(i) The space $X$ has the intersection property.
(ii) For every $t \in \Gamma \setminus \{e\}$, we have $\text{Fix}(t)^\circ = \emptyset$. 

We say that $\Gamma$ is $C^*$-simple if its reduced group $C^*$-algebra $C^*_r\Gamma$ is simple. In recent work [7], Kalantar and Kennedy established a dynamical characterization of $C^*$-simplicity, and Breuillard, Kalantar, Kennedy and Ozawa proved that many groups are $C^*$-simple. In more recent work [9], Kennedy showed an algebraic characterization of $C^*$-simplicity, as follows.

**Theorem 1.5** (Kennedy, [9, Theorem 6.3]). A countable discrete group is $C^*$-simple if and only if it satisfies the following condition: For every amenable subgroup $\Lambda \leq \Gamma$, there exists a sequence $(g_n)$ such that for every subsequence $(g_{n_k})$ of $(g_n)$, we have

$$\bigcap_k g_{n_k} \Lambda g_{n_k}^{-1} = \{e\}.$$  

Equivalently, the sequence $(g_n \Lambda g_n^{-1})$ converges to $\{e\}$ in the Chabauty topology.

The set $\text{Sub}(\Gamma)$ of all subgroups of $\Gamma$ admits a natural topology, called Chabauty topology. We treat $\text{Sub}(\Gamma)$ as a compact $\Gamma$-space with this topology and the $\Gamma$-action by conjugation (see Definition 5.1).

The first main result of this paper is the characterization of the intersection property by a property for stabilizer subgroups, which is motivated by the above results Theorem 1.4 and 1.5.

**Theorem 1.6.** Let $X$ be a compact $\Gamma$-space. The following are equivalent.

(i) Every $\Gamma$-invariant closed set in $X$ has the intersection property.

(ii) For every point $x$ in $X$ and every amenable subgroup $\Lambda$ in $\Gamma_x$, there is a net $(g_i)$ in $\Gamma$ such that $(g_i x)$ converges to $x$ and $(g_i \Lambda g_i^{-1})$ converges to $\{e\}$ in the Chabauty topology.

If $X$ is minimal, the simplicity of $C(X) \rtimes_r \Gamma$ is characterized by purely algebraic conditions for the stabilizer subgroups of $X$, as follows.

**Theorem 1.7.** Let $X$ be a minimal compact $\Gamma$-space. The following are equivalent.

(i) The reduced crossed product $C(X) \rtimes_r \Gamma$ is simple.

(ii) For every point $x$ in $X$ and every amenable subgroup $\Lambda$ in $\Gamma_x$, there is a sequence $(g_i)$ in $\Gamma$ such that $(g_i \Lambda g_i^{-1})$ converges to $\{e\}$ in the Chabauty topology.

(iii) There is a point $x$ in $X$ such that for every amenable subgroup $\Lambda$ in $\Gamma_x$, there is a sequence $(g_i)$ in $\Gamma$ such that $(g_i \Lambda g_i^{-1})$ converges to $\{e\}$ in the Chabauty topology.

To prove these results, the equivariant injective envelope $C(\tilde{X})$ of $C(X)$ plays a central role. The $\Gamma$-space $\tilde{X}$ has some properties analogous to those of the Hamana boundary (or universal Furstenberg boundary, [11, §3]).

The simplicity of reduced crossed products is also characterized in terms of uniformly recurrent subgroups (URS in short) as with the $C^*$-simplicity of countable discrete groups [9]. The notion of URS’s is introduced by Glasner–Weiss [3] as a topological dynamical analogue of the notion of invariant random subgroups, which is an ergodic theoritic concept. A URS of $\Gamma$ is defined as a minimal component of the $\Gamma$-space $\text{Sub}(\Gamma)$. The set of all URS’s of $\Gamma$ has a natural partial order (denoted by $\preceq$), introduced by Le Boudec–Matte Bon [11, §2.4].

The second main result of this paper is a property for amenable URS’s from the aspect of its order structure.
Theorem 1.8. Let $X$ be a compact $\Gamma$-space. Suppose that $S_X$ is a URS ($X$ is not necessarily minimal). Then $S_X$ contains a unique URS $A_X$. Moreover, $A_X$ is the largest amenable URS dominated by $S_X$. Namely, for every amenable URS $U$ such that $U \leq S_X$, we have $U \leq A_X$.

The notation $S_X$ denotes the closed $\Gamma$-invariant subspace of $\text{Sub}(\Gamma)$ arising from stabilizer subgroups of $X$, called the stability system of $X$ (see [3, §1] or Definition 6.3). If $X$ is minimal, the space $S_X$ is a URS. On the other hand, every URS is a stability system of a transitive $\Gamma$-space, but it is not known whether every URS is a stability system of a minimal $\Gamma$-space. Using the above result, we prove that it is true for amenable URS's.

In this paper, we also study the ideals in the group $C^*$-algebra of $\Gamma$. In particular, we see the relationship between amenable URS's of $\Gamma$ and the ideals of $C^*_r \Gamma$. For an amenable subgroup $\Lambda$ of $\Gamma$, we have the continuous $*$-representation $\pi_\Lambda$ of $C^*_r \Gamma$ on the Hilbert space $\ell^2(\Gamma/\Lambda)$ extending the canonical action of $\Gamma$ on the coset space $\Gamma/\Lambda$. We show that for stabilizer subgroup $\Lambda$ of the Hamana boundary, the ideal $\ker(\pi_\Lambda)$ is maximal.

In Section 2 we recall the notion of stabilizer subgroups and study its relationship to the intersection property. In Section 3 we recall the $\Gamma$-injective envelope and show some properties from the viewpoint of operator algebras which are analogous to those of the Hamana boundary. In Section 4 we prove a technical result to prove the main result Theorem 1.6 and we prove it in Section 5. In Section 6 we establish the characterization of simplicity of reduced products. In Section 7 we show a property for the $\Gamma$-injective envelope from the viewpoint of topological dynamical system to prove the main result Theorem 1.8. Finally, in Section 8 and 9 we study the ideals arising from amenable URS's.

Acknowledgements. The author would like to thank his supervisor, Professor Narutaka Ozawa for his support and many valuable comments.

2. Preliminaries

Definition 2.1. For a compact $\Gamma$-space $X$ and a point $x$ in $X$, we denote by $\Gamma_x$ the stabilizer subgroup, i.e. $\Gamma_x = \{t \in \Gamma : tx = x\}$. Let $\Gamma^0_x$ denote the subgroup consisting the elements in $\Gamma$ which act as identity on a neighborhood of $x$. We say that a compact $\Gamma$-space is topologically free if $\Gamma^0_x = \{e\}$ for every $x \in X$. Note that a $\Gamma$-space $X$ is topologically free if and only if $\text{Fix}(t)^0 = \emptyset$ for every $t \in \Gamma \setminus \{e\}$, where $\text{Fix}(t)$ denotes the fixed point set in $X$ of the homeomorphism $t$.

Let $X$ be a compact $\Gamma$-space. There is a canonical conditional expectation $E_X$ from $C(X) \rtimes_\gamma \Gamma$ to $C(X)$ defined by

$$E_X(f\lambda_t) = \begin{cases} f & t = e \\ 0 & t \neq e \end{cases}$$

and extended by linearity. Note that $E_X$ is faithful (see [2, Chapter 4.1]). For every $x$ in $X$, we define a conditional expectation $E_x$ from $C(X) \rtimes_\gamma \Gamma$ to $C^*_r(\Gamma_x)$ by

$$E_x(f\lambda_t) = f(x)E_{\Gamma_x}(\lambda_t)$$

where $E_{\Gamma_x}$ is the canonical conditional expectation from $C^*_r \Gamma$ to $C^*_r(\Gamma_x)$ (given by $E_{\Gamma_x}(\lambda_t) = \lambda_t$ if $t \in \Gamma_x$ and $E_{\Gamma_x}(\lambda_t) = 0$ if $t \in \Gamma \setminus \Gamma_x$, see [2, Corollary 2.5.12]).
In this paper, we often use the following fact about unital completely positive maps. See [2] Proposition 1.5.7 for proof.

**Definition 2.2.** Let $A$ and $B$ be unital $C^*$-algebras and $\phi$ be a unital completely positive map. The **multiplicative domain** of $\phi$ is the subspace $\text{mult}(\phi)$ of $A$ defined by

$$\text{mult}(\phi) = \{a \in A : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}.$$ 

**Proposition 2.3.** Let $A$ and $B$ be unital $C^*$-algebras and $\phi$ be a unital completely positive map. Then, for every $a \in \text{mult}(\phi)$ and $b \in A$, one has $\phi(ab) = \phi(a)\phi(b)$ and $\phi(ba) = \phi(b)\phi(a)$. In particular, $\text{mult}(\phi)$ is the largest $C^*$-subalgebra of $A$ to which the restriction of $\phi$ is multiplicative.

The following is a generalization of the result Theorem 1.4 and [13, Theorem 14].

**Lemma 2.4.** Let $X$ be a compact $\Gamma$-space. Then we have the following.

(i) If the set $\{x \in X : \Gamma_x \text{ is } C^*\text{-simple}\}$ is dense in $X$, then $X$ has the intersection property. In particular, if $X$ is topologically free, then $X$ has the intersection property.

(ii) If $X$ has the intersection property and $\Gamma^0_x$ is amenable for every point $x$ in $X$, then $X$ is topologically free.

**Proof.** We prove (i) by contradiction. Suppose that there is no non-zero closed ideal $I$ in $C(X)\rtimes_r \Gamma$ such that $I \cap C(X) = 0$. Then $E_X(I)$ is a non-zero since $E_X$ is faithful. Therefore $ev_x \circ E_X(I) \neq 0$ for some $x$ in $X$ such that $\Gamma_x$ is $C^*$-simple (otherwise, we have $ev_x(E_X(I)) = 0$ densely, this implies that $E_X(I) = 0$). It follows that $E_x(I) \neq 0$ since $ev_x \circ E_X = \tau_x \circ E_x$, where $\tau_x$ is the canonical tracial state on $C^*_x(\Gamma_x)$ defined by $\tau_x(a) = \langle a\delta_x, \delta_x \rangle$ for any $a \in C^*_x \Gamma$. We observe that $E_x(I) \subset C^*_x(\Gamma_x)$ is an ideal of $C(X)$ since $C^*_x(\Gamma_x)$ is contained in the multiplicative domain of $E_x$. We show that $E_x(I)$ is not dense in $C^*_x(\Gamma_x)$, which yields the desired contradiction with $C^*$-simplicity of $C^*_x(\Gamma_x)$. The $*$-homomorphism

$$C(X) + I \to (C(X) + I)/I \cong C(X)/(C(X) \cap I) = C(X) \xrightarrow{ev_x} \mathbb{C}$$

extends a state $\phi_x$ on $C(X)\rtimes_r \Gamma$. We show that $\phi_x \circ E_x = \phi_x$. This implies that $\ker \phi_x \supset E_x(I)$ since $\phi_x(I) = 0$ (hence $E_x(I)$ is not dense). Let $t$ be an element of $\Gamma \setminus \Gamma_x$. There is a function $f \in C(X)$ such that $f(x) = 1$ and $f(tx) = 0$. Since $C(X)$ is contained in the multiplicative domain of $\phi_x$, we have

$$\phi_x(\lambda_t) = f(x)\phi_x(\lambda_t) = \phi_x(f)\phi_x(\lambda_t) = \phi_x(f\lambda_t) = \phi_x(\lambda_t (t^{-1}f)) = \phi_x(\lambda_t)\phi_x(t^{-1}f) = \phi_x(\lambda_t)f(tx) = 0,$$

therefore, $\phi_x = \phi_x \circ E_x$ on $C^*_x \Gamma$. This implies that for every $f \in C(X)$ and $t \in \lambda$, we obtain

$$\phi_x \circ E_x(f\lambda_t) = \phi_x(f)\phi_x(\lambda_t)) = f(x)\phi_x(E\Gamma_x(\lambda_t)) = \phi_x(f)\phi_x(\lambda_t) = \phi_x(f\lambda_t),$$

thus we have $\phi_x \circ E_x = \phi_x$.

Next, we show (ii). Since $\Gamma^0_x$ is amenable for any $x$, we define the representation $\pi_x$ of $C(X)\rtimes_r \Gamma$ on $l_2(\Gamma/\Gamma^0_x)$, which given by $\pi_x(f\lambda_t)\delta_p = f(tpx)\delta_p$ for $p \in \Gamma/\Gamma^0_x$. Note that for $t$ and $s$ in $\Gamma$ such that $s^{-1}t \in \Gamma^0_x$, we have $tx = sx$, thus the notation $px$ is well-defined. Set $\pi = \bigoplus_{x \in X} \pi_x$. The representation $\pi$ is faithful by the intersection property since $\ker(\pi) \cap C(X) = 0$. This implies that $X$ is topologically
free. Otherwise, there is an element \( t \) in \( \Gamma \setminus \{ e \} \) and a non-zero function \( f \) in \( C(X) \) such that \( \text{supp}(f) \) is contained in \( \text{Fix}(t) \), which implies that \( \pi(f(1 - \lambda)) = 0 \) in contradiction with faithfulness.

3. Equivariant injective envelopes

**Definition 3.1.** We say that an operator system (resp. unital C*-algebra) \( V \) is a \( \Gamma \)-operator system (resp. unital \( \Gamma \)-C*-algebra) if it comes together with a complete order isomorphic (resp. unital *-isomorphic) \( \Gamma \)-action on \( V \). A \( \Gamma \)-equivariant unital complete positive map between \( \Gamma \)-operator systems is called a \( \Gamma \)-morphism.

**Definition 3.2.** We say that \( \Gamma \)-operator system \( V \) is \( \Gamma \)-injective if \( V \) is an injective object in the category of all \( \Gamma \)-operator systems with \( \Gamma \)-morphisms. Namely, for any \( \Gamma \)-operator systems \( W_0 \subset W \) and any \( \Gamma \)-morphism \( \phi \) from \( W_0 \) to \( V \), there is a \( \Gamma \)-morphism \( \tilde{\phi} \) from \( W \) to \( V \) such that \( \tilde{\phi}|_{W_0} = \phi \).

For every compact \( \Gamma \)-space \( X \), we denote by \( \tilde{X} \) the Gelfand spectrum of the \( \Gamma \)-injective envelope of \( C(X) \), i.e. \( C(\tilde{X}) \) satisfies the following properties (see [6]).

- The \( \Gamma \)-C*-algebra \( C(\tilde{X}) \) is a \( \Gamma \)-injective operator system.
- The \( \Gamma \)-C*-algebra \( C(X) \) is contained in \( C(\tilde{X}) \) as a unital \( \Gamma \)-C*-subalgebra and \( C(X) \subset C(\tilde{X}) \) is rigid, i.e. the identity map is the only \( \Gamma \)-morphisms on \( C(\tilde{X}) \) which is the identity map on \( C(X) \).

If \( X \) is the one-point \( \Gamma \)-space, \( \tilde{X} \) is called the Hamana boundary, denoted by \( \partial_H \Gamma \).

We prove some facts for \( \tilde{X} \), a generalization of the properties for the Hamana boundary ([13] Proposition 8 and Lemma 9). Recall that a subgroup \( \Lambda \subseteq \Gamma \) is relatively amenable if there is a \( \Lambda \)-invariant state on \( \ell_{\infty, \Lambda} \). Since there is a \( \Lambda \)-morphism from \( \ell_{\infty, \Lambda} \) to \( \ell_{\infty, \Gamma} \), the notions of amenability and relative amenability coincide for discrete groups. We denote by \( q \) the \( \Gamma \)-equivariant continuous surjection \( \tilde{X} \to X \).

**Proposition 3.3.** Let \( X \) be a compact \( \Gamma \)-space. Then, one has the following.

(i) The space \( \tilde{X} \) is a Stonean space.

(ii) For any closed \( \Gamma \)-invariant set \( Z \) in \( \tilde{X} \), we have \( Z = \tilde{X} \) if \( q(Z) = X \).

(iii) The group \( \Gamma_y \) is amenable for every point \( y \) in \( \tilde{X} \).

In particular, for any \( t \in \Gamma \), the set \( \text{Fix}(t) \) is clopen, hence \( \Gamma_y = \Gamma_y^t \) for any \( y \in \tilde{X} \).

**Proof.** There is an including \( \Gamma \)-equivariant unital *-homomorphism from \( C(X) \) to the \( \Gamma \)-injective C*-algebra \( \ell_{\infty}(\Gamma, \ell_{\infty}X) \), which is defined by \( f \mapsto (tf)_{t \in \Gamma} \). It follows that there are \( \Gamma \)-morphisms \( \phi \colon \ell_{\infty}(\Gamma, \ell_{\infty}X) \to C(\tilde{X}) \) and \( \psi : C(\tilde{X}) \to \ell_{\infty}(\Gamma, \ell_{\infty}X) \), which extend the identity map on \( C(X) \). Since \( \ell_{\infty}(\Gamma, \ell_{\infty}X) \) is also an injective operator system, \( C(\tilde{X}) \) is an injective operator system, thus \( \tilde{X} \) is Stonean. Then \( \text{Fix}(t) \) is clopen by Frolik’s theorem.

Next we show the condition (ii). Suppose that there is a closed \( \Gamma \)-invariant set \( Z \subsetneq \tilde{X} \) such that \( q(Z) = X \), then the corresponding \( \Gamma \)-equivariant quotient map \( \pi \) from \( C(\tilde{X}) \) to \( C(Z) \) is not faithful. Since \( q(Z) = X \), there is a \( \Gamma \)-morphism \( \phi \) from \( C(Z) \) to \( C(\tilde{X}) \) such that \( \phi \circ \pi|_{C(Z)} = \text{id}_{C(Z)} \) by \( \Gamma \)-injectivity of \( C(\tilde{X}) \). This implies that \( \phi \circ \pi = \text{id}_{C(\tilde{X})} \) by rigidity, hence \( \pi \) is faithful, a contradiction.

Next, we prove amenability of \( \Gamma_y \). There is a inclusion \( \iota \) from \( \ell_{\infty, \Gamma} \) to \( \ell_{\infty}(\Gamma, \ell_{\infty}X) \) as a unital \( \Gamma \)-C*-subalgebra. Since the map \( \text{ev}_x \circ \phi \circ \iota \) is a \( \Gamma_y \)-invariant state on \( \ell_{\infty, \Gamma} \), we obtain (relative) amenability of \( \Gamma_y \).
We obtain the following result the case $X$ being trivial (see [7, Theorem 6.2]).

**Theorem 3.4.** Let $X$ be a compact $\Gamma$-space. Then the following are equivalent.

(i) The space $X$ has the intersection property.

(ii) The space $\hat{X}$ has the intersection property.

(iii) The space $\hat{X}$ is (topologically) free.

**Proof.** First, we prove that (i) implies (ii). Suppose $X$ has the intersection property. We show that every quotient map $\pi$ from $C(\hat{X}) \rtimes \Gamma$ to a C*-algebra $A$ is faithful if $\ker(\pi) \cap C(X) = 0$. Since $\ker(\pi) \cap C(X) = 0$, the quotient map $\pi$ is faithful on $C(X) \rtimes \Gamma$ by the intersection property for $X$. By $\Gamma$-injectivity of $C(X)$, there is a $\Gamma$-morphism $\phi$ from $A$ to $C(\hat{X})$ such that $\phi \circ \pi|_{C(X) \rtimes \Gamma} = E_X$. This implies that $\phi \circ \pi|_{C(\hat{X})} = \text{id}_{C(\hat{X})}$ by rigidity of $C(X) \subset C(\hat{X})$. Therefore, we obtain $C(\hat{X}) \subset \text{mult}(\phi \circ \pi)$. It follows that $\phi \circ \pi = E_X$, hence $\pi$ is faithful.

Next, we prove that (ii) implies (i). Suppose $\hat{X}$ has the intersection property. Let $\pi$ be a representation of $C(X) \rtimes \Gamma$ on a Hilbert space $\mathcal{H}$ such that $\ker(\pi) \cap C(X) = 0$. We prove that $\pi$ is injective. By Arveson’s extension theorem, we extend $\pi$ to a unital completely positive map $\tilde{\pi}$ from $C(\hat{X}) \rtimes \Gamma$ to $\mathcal{B}(\mathcal{H})$. We consider a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ defined by

$$D = C^*(\tilde{\pi}(C(\hat{X}) \rtimes \Gamma)) = \text{closure}(\tilde{\pi}(C(\hat{X})) \cdot \pi(C_\star \Gamma)).$$

We define $\Gamma$-action on $D$ as Ad $\pi(\cdot)$, then $\tilde{\pi}$ is $\Gamma$-equivariant. Since $\pi$ is faithful on $C(X)$, there is a $\Gamma$-morphism $\phi$ from $C^*(\tilde{\pi}(C(\hat{X})))$ to $C(\hat{X})$ such that $\phi \circ \pi = \text{id}_{C(X)}$ by $\Gamma$-injectivity of $C(X)$, which implies that $\phi \circ \tilde{\pi}|_{C(\hat{X})} = \text{id}_{C(\hat{X})}$ by rigidity. It follows that $C^*(\tilde{\pi}(C(\hat{X}))) \subset \text{mult}(\phi)$, hence $\phi$ is a $*$-homomorphism. Now, consider a subset of $D$ given by

$$L = \text{closure}(\ker(\phi) \cdot \pi(C_\star \Gamma)).$$

Since $\phi$ is $\Gamma$-equivariant, the set $\ker(\phi)$ is $\Gamma$-invariant, therefore we have

$$\ker(\phi) \cdot \pi(C(\Gamma)) = \pi(C(\Gamma)) \cdot \ker(\phi).$$

This implies that $L$ is an ideal of $D$. Since $L \cap C^*(\tilde{\pi}(C(\hat{X}))) = \ker(\phi)$, the map $\phi$ extends the quotient map $\tilde{\phi}$ from $D$ to $D/L$. It follows that $\tilde{\phi} \circ \tilde{\pi}$ is a $*$-homomorphism which is faithful on $C(X)$. Thus, we have $\ker(\pi) \subset \ker(\tilde{\phi} \circ \tilde{\pi}) = 0$ since $\hat{X}$ has the intersection property.

The equivalence of (ii) and (iii) follows from Lemma 3.3 and Proposition 3.3.

4. $\Gamma$-MORPHISMS TO INJECTIVE ENVELOPES

In this section, we prove equivalence of the intersection property and the “unique trace property” for crossed products. First, we show a lemma to prove the theorem.

**Lemma 4.1.** Let $Y$ be compact $\Gamma$-space. If $Y$ is topologically free, then the only conditional expectation from $C(Y) \rtimes \Gamma$ to $C(Y)$ is the canonical conditional expectation $E_Y$. Moreover, if $Y$ is Stonean and $\Gamma_y$ is amenable for every $y$ in $Y$, then the converse is also true.

**Proof.** Suppose that $Y$ is topologically free and let $\Phi$ be a conditional expectation from $C(Y) \rtimes \Gamma$ to $C(Y)$. The space $Y$ is topologically free if and only if $\{y \in Y : \Gamma_y = \{e\}\}$ (denoted by $Y_0$) is dense in $Y$ since $\bigcup_{t \in \Gamma} \partial \text{Fix}(t)$ has no interior
by Baire category theorem. Fix an element $t$ in $\Gamma \setminus \{e\}$. For every $y$ in $Y_0$, we have $ty \neq y$. Then there is a non-zero function in $C(X)$ such that $f(y) = 1$ and $f\lambda ty = 0$. It follows that $\Phi(\lambda y) = \Phi(f\lambda ty) = 0$, hence $\Phi(t) = 0$. This implies that $\Phi = E_Y$.

Next, we show the converse. Suppose that $Y$ is Stonean space and $\Gamma_y$ is amenable for every $y$ in $Y$. Then there is a conditional expectation $\Phi$ from $C(Y) \rtimes_r \Gamma$ to $C(Y)$, defined by $\Phi(f\lambda y) = f \cdot \chi_{\text{Fix}(t)}$. Continuity of $\Phi$ follows from the equality $\Phi(t)(y) = \tau_0 \circ E_y$, where $\tau_0$ is the unit character of $\Gamma_y$. Note that $\tau_0$ is continuous on $C^*_\text{r}(\Gamma_y)$ since $\Gamma_y$ is amenable. It follows that there is a non-canonical conditional expectation if $Y$ is not (topologically) free. 

\begin{theorem}
Let $X$ be a compact $\Gamma$-space. Then the following are equivalent.
\begin{enumerate}[label=(\roman*)]
\item The space $X$ has the intersection property.
\item The only $\Gamma$-morphism from $C(X) \rtimes_r \Gamma$ to $C(\tilde{X})$ which is the identity map on $C(X)$ is the canonical conditional expectation $E_X$.
\end{enumerate}
\end{theorem}

\begin{proof}
Let $\phi$ be a $\Gamma$-morphism from $C(X) \rtimes_r \Gamma$ to $C(\tilde{X})$ such that $\phi|_{C(X)} = \text{id}_{C(X)}$. There is a $\Gamma$-morphism $\Phi$ from $C(\tilde{X}) \rtimes_r \Gamma$ onto $C(\tilde{X})$ extending $\phi$. Then $\Phi$ is a conditional expectation since $C(X) \subset C(\tilde{X})$ is rigid. Hence (ii) is equivalent to the uniqueness of conditional expectations from $C(X) \rtimes_r \Gamma$ onto $C(\tilde{X})$, that is equivalent to the (topological) freeness of $\tilde{X}$ by Proposition 33 and Lemma 44. It follows that (i) and (ii) are equivalent by Theorem 33.
\end{proof}

5. Stabilizer subgroups and the intersection property

In this section, we establish a characterization of the intersection property in terms of stabilizer subgroups.

\begin{definition}
The Chabauty space of $\Gamma$ is the set $\text{Sub}(\Gamma)$ of all subgroups in $\Gamma$ with the relative topology of the product topology on $\{0, 1\}^\Gamma$.
\end{definition}

Note that a sequence $(\Lambda_i)_i$ of subgroup in $\Gamma$ converges to a subgroup $\Lambda$ in the Chabauty topology if and only if it satisfies the following conditions.

\begin{enumerate}[label=(\roman*)]
\item For every $t \in \Lambda$, one has $t \in \Lambda_i$ eventually.
\item For every subsequence $(\Lambda_{i_k})_k$ of $(\Lambda_i)_i$, one has $\bigcap_k \Lambda_{i_k} \subset \Lambda$.
\end{enumerate}

Let $X$ be a compact $\Gamma$-space. We set the compact $\Gamma$-space
\[ S(X, \Gamma) = \{(x, \Lambda) \in X \times \text{Sub}(\Gamma) : \Lambda \leq \Gamma_x \} \]
with the relative topology of the product topology on $X \times \text{Sub}(\Gamma)$. We consider the closed $\Gamma$-invariant subspace of $S(X, \Gamma)$, defined by
\[ S_a(X, \Gamma) = \{(x, \Lambda) \in X \times \text{Sub}(\Gamma) : \Lambda \leq \Gamma_x, \Lambda \text{ is amenable}\}. \]
We denote by $p_X$ the $\Gamma$-equivariant continuous surjection from $S_a(X, \Gamma)$ to $X$ defined by
\[ p_X(x, \Lambda) = x \]
for $(x, \Lambda) \in S_a(X, \Gamma)$, hence $C(X) \subset C(S_a(X, \Gamma))$ as a unital $\Gamma$-$C^*$-subalgebra.

\begin{theorem}
Let $X$ be a compact $\Gamma$-space. Then the following are equivalent.
\begin{enumerate}[label=(\roman*)]
\item The space $X$ has the intersection property.
\item For every closed $\Gamma$-invariant set $Y$ in $S_a(X, \Gamma)$ such that $p_X(Y) = X$, the space $Y$ contains $X \times \{e\}$.
\end{enumerate}
\end{theorem}
Proof. Suppose that $X$ does not have the intersection property. We denote by $q$ the $\Gamma$-equivariant continuous surjection from $\tilde{X}$ to $X$. We define a $\Gamma$-equivariant continuous map $\Phi$ from $\tilde{X} \to S_\theta(X, \Gamma)$ by $\Phi(y) = (q(y), \Gamma_y)$ for $y \in \tilde{X}$. We claim that $\Phi(\tilde{X}) \not\supset X \times \{e\}$, which means that (ii) is not true. Otherwise, the closed $\Gamma$-invariant set $Z := \{y \in \tilde{X} : \Gamma_y = \{e\}\}$ satisfies that $q(Z) = \tilde{X}$, therefore we have $Z = X$ by Proposition 3.3. Since $X$ does not have the intersection property, the space $\tilde{X}$ is not free by Theorem 5.2, a contradiction.

On the other hand, let $Y$ be a closed $\Gamma$-invariant set in $S_\theta(X, \Gamma)$ such that $p_X(Y) = X$ and $Y \not\supset X \times \{e\}$. There is a $\Gamma$-morphism $\theta$ from $C(X) \rtimes_r \Gamma$ to $C(Y)$, defined by

$$\theta(f, \lambda)(x, \Lambda) = \begin{cases} f(x) & t \in \Lambda, \\ 0 & t \notin \Lambda \end{cases}.$$ 

There is a $\Gamma$-morphism $\mu$ from $C(Y)$ to $C(\tilde{X})$ which is the identity map on $C(X)$ since $C(\tilde{X})$ is $\Gamma$-injective. We show that $\mu \circ \theta \neq E_X$. Suppose that $\mu \circ \theta = E_X$. We claim that for every $x \in X$, one has $p_X^{-1}(x) \cap Y = \{x\} \times Y_x$ for a $Y_x \subset \text{Sub}(\Gamma)$. Since $Y \not\supset X \times \{e\}$, there is a point $x$ in $X$ such that $Y_x \not\supset \{e\}$. Let $\tilde{x}$ be a point in $\tilde{X}$ such that $q(\tilde{x}) = x$. We observe that the support of $ev_\tilde{x} \circ \mu$ is contained in $\{x\} \times Y_x$. Indeed, let $f$ be a function in $C(Y)$ such that $\int_\{x\} Y_x = 0$. For every open neighborhood $U$ of $x$, we take a continuous function $h_U$ on $X$ such that $0 \leq h_U \leq 1$, the support of $h_U$ contained in $U$ and $h_U(x) = 1$. We denote by $\tilde{h}_U$ the function $h_U \circ \mu$. For every $\varepsilon > 0$, there is a neighborhood $U$ of $x$ such that $|\int_{\{x\}} Y_x| < \varepsilon$ for every $y \in p_X^{-1}(U) \cap Y$, hence we obtain

$$|ev_{\tilde{x}} \circ \mu(f)| = |ev_{\tilde{x}} \circ \mu(f) \cdot \tilde{h}_U(\tilde{x})| = |ev_{\tilde{x}} \circ \mu(\tilde{h}_U)| \leq \|\tilde{h}_U\| \leq \varepsilon.$$ 

Therefore $ev_{\tilde{x}} \circ \mu(f) = 0$, which implies that $\mathbf{supp}(ev_{\tilde{x}} \circ \mu) \subset \{x\} \times Y_x$. We denote by $\mu_X$ the Radon probability measure on $Y_x$, which is the restriction of $ev_{\tilde{x}} \circ \mu$ on $Y_x$. Then, for any $t \in \Gamma$, we obtain the following equation.

$$\mu_X\{\Lambda \in Y_x : t \in \Lambda\} = \mu_X(\theta(\lambda_t)|_{Y_x})$$

$$= ev_{\tilde{x}} \circ \mu \circ \theta(\lambda_t)$$

$$= \begin{cases} 1 & (t = e) \\ 0 & (t \neq e) \end{cases}.$$ 

It contradicts that $Y_x \not\supset \{e\}$. \hfill \Box

**Theorem 5.3.** Let $X$ be a compact $\Gamma$-space. The following are equivalent.

(i) Every $\Gamma$-invariant closed set in $X$ has the intersection property.

(ii) For every point $x$ in $X$ and every amenable subgroup $\Lambda$ in $\Gamma_x$, there is a net $(g_i)$ in $\Gamma$ such that $(g_i x)$ converges to $x$ and $(g_i \Lambda g_i^{-1})$ converges to $\{e\}$ in the Chabauty topology.

**Proof.** Suppose that (ii) is true. Let $Z$ be a $\Gamma$-invariant closed subset of $X$. Then, for any $(z, \Lambda) \in S(Z, \Gamma)$, we have $\Gamma(z, \Lambda) \ni (z, \{e\})$. This implies that for every $\Gamma$-invariant closed subset $Y$ in $S_\theta(Z, \Gamma)$, one has $Z \times \{e\} \subset Y$, therefore $Z$ has the intersection property by Theorem 5.2.

Conversely, suppose that (i) is true. Let $x$ be a point in $X$ and $\Lambda$ be an amenable subgroup in $\Gamma_x$. Then, we have $\Gamma(x, \Lambda) \ni p_X \left(\Gamma(x, \Lambda)\right) \times \{e\}$ by Theorem 5.2.
In particular, there is a net \((g_i)\) in such that \((g_ix)\) converges to \(x\) and \((g_i\Lambda g_i^{-1})\) converges to \(\{e\}\).

\[\square\]

6. Minimal Case

We consider the case where the compact \(\Gamma\)-space \(X\) is minimal, i.e. there are no non-trivial closed \(\Gamma\)-invariant subspaces in \(X\). Equivalently, there are no non-trivial \(\Gamma\)-invariant closed ideals in \(C(X)\). For a minimal compact \(\Gamma\)-space \(X\), the space \(\hat{X}\) is also minimal by Proposition 5.3 (ii).

We claim that for a minimal compact \(\Gamma\)-space \(X\), the reduced crossed product \(C(X)\rtimes\Gamma\) is simple if and only if \(X\) has the intersection property. Since for every ideal \(I\) in \(C(X)\rtimes\Gamma\), \(C(X)\cap I\) is \(\Gamma\)-invariant ideal.

**Theorem 6.1.** Let \(X\) be a minimal compact \(\Gamma\)-space. The following are equivalent.

1. The reduced crossed product \(C(X)\rtimes\Gamma\) is simple.
2. For every point \(x\) in \(X\) and every amenable subgroup \(\Lambda\) in \(\Gamma_x\), there is a sequence \((g_i)\) in \(\Gamma\) such that \((g_i\Lambda g_i^{-1})\) converges to \(\{e\}\) in the Chabauty topology.
3. There is a point \(x\) in \(X\) such that for every amenable subgroup \(\Lambda\) in \(\Gamma_x\), there is a sequence \((g_i)\) in \(\Gamma\) such that \((g_i\Lambda g_i^{-1})\) converges to \(\{e\}\) in the Chabauty topology.

**Proof.** We show that (i) and (ii) are equivalent to existence of a minimal \(\Gamma\)-invariant subspace \(Y\) in \(\mathcal{S}_n(X,\Gamma)\) such that \(Y \neq X \times \{e\}\). This implies the desired equivalence by Theorem 5.2. Note that for every \(\Gamma\)-invariant subspace \(Z\) of \(X\), we have \(p_X(Z) = X\) since \(X\) is minimal.

Suppose that there is a minimal \(\Gamma\)-invariant subspace \(Y\) in \(\mathcal{S}_n(X,\Gamma)\) such that \(X \times \{e\} \neq Y\). Let \((x,\Lambda)\) be an element in \(Y\). We claim that \(\text{Ad}(\Gamma)\Lambda \neq \{e\}\), which means that (i) and (ii) are not true. Otherwise, there is a net \((g_i)\) in \(\Gamma\) such that \(g_i\Lambda g_i^{-1} \to \{e\}\). We may assume that \(g_i x \to y\) for a point \(y\) in \(X\). Then we have \(g_i(x,\Lambda) \to (y,\{e\})\), this implies that \(Y \supset X \times \{e\}\). By minimality of \(Y\), we obtain \(Y = X \times \{e\}\), a contradiction.

Next, we show the converse. Suppose that there is an element \((x,\Lambda)\) in \(\mathcal{S}_n(X,\Gamma)\) such that \(\text{Ad}(\Gamma)\Lambda \neq \{e\}\). Then we have \(\Gamma(x,\Lambda) \cap X \times \{e\} = \emptyset\), hence there is a minimal \(\Gamma\)-invariant subspace \(Y\) in \(\mathcal{S}_n(X,\Gamma)\) such that \(Y \neq X \times \{e\}\) (take a minimal component of \(\Gamma(x,\Lambda)\)).

We also characterize simplicity for reduced crossed products in terms uniformly recurrent subgroups (Glasner–Weiss [3]).

**Definition 6.2.** A subset \(U\) of \(\text{Sub}(\Gamma)\) is called a uniformly recurrent subgroup (URS) of \(\Gamma\) if \(U\) is a minimal closed subset of the Chabauty space \(\text{Sub}(\Gamma)\). A URS \(U\) is amenable if any subgroup contained in \(U\) is amenable.

**Definition 6.3.** For a compact \(\Gamma\)-space \(X\), we define the subspace \(\mathcal{S}_X\) of \(\text{Sub}(\Gamma)\) as the closure of the set \(\{\Gamma_x : x \in X\}, \Gamma_x = \Gamma_x^*\}.\) We call \(\mathcal{S}_X\) by stability system of \(X\). If \(X\) is minimal, the set \(\mathcal{S}_X\) is a URS ([5 Proposition 1.4]).

For a normal subgroup \(N\) in \(\Gamma\), the singleton \(\{N\}\) in \(\text{Sub}(\Gamma)\) is a URS. By [9 Theorem 4.1], \(\text{C}^*\)-simplicity of \(\Gamma\) is equivalent to absence of non-trivial amenable URS’s. There is a non-\(\text{C}^*\)-simple countable group which has no non-trivial normal
Corollary 6.4. Let $X$ be a minimal compact $\Gamma$-space. The following are equivalent.

(i) The reduced crossed product $C(X)\rtimes_\gamma \Gamma$ is simple.

(ii) For any non-trivial amenable URS $\mathcal{U}$, one has $\mathcal{U} \not\leq \mathcal{S}_X$.

Proof. If $C(X)\rtimes_\gamma \Gamma$ is not simple, $\tilde{X}$ is not topologically free by Theorem 6.3. Hence $\mathcal{S}_{\tilde{X}}$ is a non-trivial amenable URS and $\mathcal{S}_{\tilde{X}} \not\leq \mathcal{S}_X$.

We show the converse. Suppose that $C(X)\rtimes_\gamma \Gamma$ is simple. Let $\mathcal{U}$ be an amenable URS such that $\mathcal{U} \not\leq \mathcal{S}_X$. For $\Lambda \in \mathcal{U}$, there is a point $x \in X$ such that $\Lambda \leq \Gamma_x$, this implies that $\text{Ad}(\Gamma)\Lambda \ni \{\epsilon\}$ by Theorem 6.1. Therefore we have $\mathcal{U} = \{\epsilon\}$ by minimality of $\mathcal{U}$. □

7. STRONGLY PROXIMALITY AND AMENABLE URS’S

Definition 7.1. Let $X$ be a compact $\Gamma$-space. $X$ is strongly proximal if for every Radon probability measure $\mu$ on $X$, the weak$^*$-closure of $\Gamma \mu$ contains a point mass. $X$ is called a $\Gamma$-boundary if $X$ is minimal and strongly proximal.

It is known that the Hamana boundary $\partial_H \Gamma$ is a $\Gamma$-boundary (Kalantar–Kennedy [7]). In this section, we proof an analogous property for every compact $\Gamma$-space $X$.

We denote by $\tilde{X}$ the inverse image of a point $z \in X$ under the $\Gamma$-equivariant continuous surjection from $\tilde{X}$ to $X$. For any compact Hausdorff space $Y$, we denote by $\mathcal{M}(Y)$ the set of all Radon probability measures on $Y$.

Theorem 7.2. Let $X$ be a compact $\Gamma$-space and $Z$ be a subset in $X$ such that $\Gamma Z$ is dense in $X$. Then for every family $\{\mu_z\}_z \in Z$ such that $\mu_z \in \mathcal{M}(\tilde{X})$, the space $\tilde{X}$ is contained in the weak$^*$-closure of $\{t\mu_z : t \in \Gamma, z \in Z\}$ in $\mathcal{M}(\tilde{X})$.

Proof. We define $\Gamma$-morphism $\phi$ from $C(\tilde{X})$ to $\ell_\infty(\Gamma \times Z)$ by

$$\phi(f)(t, z) = (f, t\mu_z), \quad t \in \Gamma, z \in Z.$$ 

Observe that for $f \in C(X)$ one has $\phi(f) = (f(tz))_{t, z}$. Since $\Gamma Z$ is dense in $X$, $\phi|_{C(X)}$ is the inclusion map from $C(X)$ to $\ell_\infty(\Gamma \times Z)$ as a unital $\Gamma$-$C^*$-subalgebra. Hence by $\Gamma$-injectivity of $C(\tilde{X})$, there is a $\Gamma$-morphism $\psi$ from $\ell_\infty(\Gamma \times Z)$ to $C(\tilde{X})$ which satisfies that $(\psi \circ \phi)|_{C(X)} = \text{id}_{C(X)}$ (then $\psi \circ \phi = \text{id}_{C(\tilde{X})}$ by rigidity). It implies that for any $y \in \tilde{X}$, there is a state $\omega$ on $\ell_\infty(\Gamma \times Z)$ such that $\omega \circ \phi = \text{ev}_y$. Since there is a net $(\xi_t)$ in $\ell_1(\Gamma \times Z)$ such that $\xi_t \to \omega$ in weak$^*$-topology, it implies that

$$\xi_t \circ \phi(f) = \sum_{t, z} \xi_t(t, z)\langle f, t\mu_z \rangle \to f(y)$$

for any $f \in C(\tilde{X})$. It means that $\text{ev}_y \in \text{conv}(\{t\mu_z\})$, therefore we have $\text{ev}_y \in \{t\mu_z\}$ by Milman’s converse since $\text{ev}_y$ is an extreme point of $\mathcal{M}(\tilde{X})$. □

Next, we consider some applications to properties for amenable URS’s, inspired from Le Boudec–Matte Bon [11 §2.4].
Proof. Since $\Gamma_x$ acts on $\bar{X}_x$ and $U$ is amenable, there is a $H_x$-invariant measure $\mu_x$ on $\bar{X}_x$ since there is a $\Gamma_x$-morphism from $C(\bar{X}_x)$ to $\ell_\infty \Gamma_x$ by $\Gamma_x$-injectivity of $\ell_\infty \Gamma_x$ and there is a $H_x$-invariant state on $\ell_\infty \Gamma$ by amenability of $H_x$. Hence, for any $y \in \bar{X}$, there is a net $(t_i, x_i)$ in $\Gamma \times X$ such that $t_i \mu_{x_i} \to \text{ev}_y$ by Theorem 7.4. We may assume that the net $(t_i H_{x_i} t_i^{-1})_i$ converges to $K_y \in U$. Then $K_y$ fixes $y$ since $t_i \mu_{x_i}$ is $(t_i H_{x_i} t_i^{-1})$-invariant. \hfill $\blacksquare$

**Theorem 7.4** (see also Theorem [18]). Let $X$ be a compact $\Gamma$-space. Suppose that $S_X$ is a URS ($X$ is not necessarily minimal). Then $S_X$ contains a unique URS $A_X$. Moreover, for every amenable URS $U$ such that $U \equiv S_X$, we have $U \equiv A_X$.

**Proof.** Let $U$ be a URS such that $U \equiv S_X$. Since $S_X$ is a URS, for any $x \in X$ there is a subgroup $H \in U$ such that $H \leq \Gamma_x$. Hence for every $y \in \bar{X}$, there is a subgroup $K_y \in U$ such that $K_y \leq \Gamma_y$ by Lemma 7.3. It implies that $U \equiv V$ for every URS $V$ in $S_X$. In particular, for every URS $V$ in $S_X$, one has $V \equiv S_X$ since $\Gamma_y = \Gamma_y^0$ for every $y \in \bar{X}$ by Proposition [19]. This implies that $V_1 \equiv V_2$ for URS's $V_1$ and $V_2 \subset S_X$, hence $V_1 = V_2$. Therefore, there is a unique URS contained in $S_X$. \hfill $\blacksquare$

For every URS $U$, there is a compact $\Gamma$-space $X$ such that $S_X = U$ ([3 Proposition 6.1]). Hence we get the following.

**Corollary 7.5.** For every URS $U$, there is a unique amenable URS $A_U \equiv U$ which satisfies that $V \equiv A_U$ for every amenable URS $V \equiv U$.

It is not known whether for every URS $U$, there exists a minimal $\Gamma$-space $X$ such that $S_X = U$. Here, we prove that it is true for amenable URS's.

**Corollary 7.6.** For every amenable URS $U$, there is a minimal compact $\Gamma$-space $X$ such that $S_X = U$.

**Proof.** There is a compact $\Gamma$-space $X$ such that $S_X = U$. We take a minimal $\Gamma$-subspace $Y$ in $\bar{X}$, then $S_Y \subset S_X$ since $\Gamma_y = \Gamma_y^0$ for every $y \in \bar{X}$. Hence we have $U \equiv S_Y = A_X \equiv U$ by Theorem 7.4. \hfill $\blacksquare$

**8. The maximal ideal arising from stabilizer subgroups**

Let $X$ be a minimal compact $\Gamma$-space (recall that $\bar{X}$ is also minimal in this situation). For $x \in \bar{X}$, we define a representation $\pi_x$ of $C(\bar{X}) \rtimes_r \Gamma$ on $\ell_2(\Gamma_x)$ as follows.

\[
\begin{aligned}
\pi_x(f)\delta_y &= f(y)\delta_y & f \in C(\bar{X}) \\
\pi_x(\lambda_t)\delta_y &= \delta_{ty} & t \in \Gamma,
\end{aligned}
\]

where $y \in \Gamma_x$. In other words, $\pi_x$ is the GNS representation with respect to $1_x := \gamma_0 \circ E_x$, where $\gamma_0$ is the unit character of $\Gamma_x$. Since $\gamma_0$ is continuous since $\Gamma_x$ is amenable by Proposition [3] the state $1_x$ is continuous.

Note that $\ker(\pi_x) = \ker(\pi_y)$ for every $x$, $y \in \bar{X}$ since $1_{tx} = 1_x \circ \Ad(t^{-1})$ for every $t \in \Gamma$ and the map $\bar{X} \to S(C(\bar{X}) \rtimes_r \Gamma)$ given by $x \mapsto 1_x$ is continuous, where $S(C(\bar{X}) \rtimes_r \Gamma)$ is the state space of $C(\bar{X}) \rtimes_r \Gamma$.
Theorem 8.1. For every minimal compact $\Gamma$-space $X$ and every $x \in \tilde{X}$, the C*-algebra $\pi_x(C(X) \rtimes \Gamma)$ is simple.

First, we prove a lemma. We define the unital completely positive map $\tilde{E}$ on $C(\tilde{X}) \rtimes \Gamma$ by $\tilde{E}(f \lambda_t) = f_{\text{Fix}(t)} \lambda_t$ for $f \in C(\tilde{X})$ and $t \in \Gamma$. We see that $\tilde{E}$ is continuous. Let $B \subset C(\tilde{X}) \rtimes \Gamma$ the closed linear span of $\{ f \lambda_t : \text{supp}(f) \subset \text{Fix}(t) \}$. Then, $B$ is a C*-subalgebra of $C(\tilde{X}) \rtimes \Gamma$, which is contained in the multiplicative domain of $E_x$ for every $x \in X$. Since $E_x \circ \tilde{E} = E_x$ and $\{ E_x \}_x$ is a faithful family of *-homomorphisms on $B$ (because $\tau_\lambda \circ E_x = \text{ev}_x \circ E_\tilde{X}$), the map $\tilde{E}$ is continuous on $C(\tilde{X}) \rtimes \Gamma$. Note that $\pi_x \circ \tilde{E}(\lambda_t) = \pi_x(\chi_{\text{Fix}(t)})$ for every $t \in \Gamma$.

Lemma 8.2. For every conditional expectation $\Phi: C(\tilde{X}) \rtimes \Gamma \to C(\tilde{X})$, one has $\Phi \circ \tilde{E} = \Phi$.

Proof. It suffices to show that $\Phi(\lambda_t)(x) = 0$ for every $x \in \tilde{X} \setminus \text{Fix}(t)$. Since $x \notin \text{Fix}(t)$, there is a $f \in C(\tilde{X})$ such that $f(x) = 1$ and $f(tx) = 0$. Then $\Phi(\lambda_t)(x) = f(x)\Phi(\lambda_t)(x) = (\Phi|_{\text{Fix}(t)})(x) = \Phi(\lambda_t)(x)f(tx) = 0$. \qed

Proof of Theorem 8.1. Suppose that there is a quotient map $\rho$ from $C(\tilde{X}) \rtimes \Gamma$ to a unital C*-algebra $A$. It suffices to show that $\rho$ is faithful. Since $X$ is minimal, the map $\rho \circ \pi_x$ is injective on $C(X)$, hence there is $\Gamma$-morphism $\phi: A \to C(\tilde{X})$ such that $\phi \circ \rho \circ \pi_x|_{C(X)} = \text{id}_{C(X)}$ by $\Gamma$-injectivity. We extend $\phi \circ \rho$ to a $\Gamma$-morphism $\Phi: \pi_x(C(\tilde{X}) \rtimes \Gamma) \to C(\tilde{X})$ such that $\Phi \circ \pi_x|_{C(X)} = \text{id}_{C(X)}$ by $\Gamma$-injectivity of $C(\tilde{X})$. Then $\Phi \circ \pi_x$ is a conditional expectation by rigidity. By Lemma 8.2 and the fact that $\pi_x \circ \tilde{E}(\lambda_t) = \pi_x(\chi_{\text{Fix}(t)})$, for $t \in \Gamma$, we obtain the following equality.

\[
\begin{align*}
\text{ev}_x \circ \phi \circ \rho \circ \pi_x(\lambda_t) &= \text{ev}_x \circ \Phi \circ \pi_x(\lambda_t) \\
&= \text{ev}_x \circ \Phi \circ \pi_x \circ \tilde{E}(\lambda_t) \\
&= \text{ev}_x \circ \Phi \circ \pi_x(\chi_{\text{Fix}(t)}) \\
&= \chi_{\text{Fix}(t)}(x) \\
&= \begin{cases} 1 & t \in \Gamma \\ 0 & t \notin \Gamma \end{cases} \\
&= 1_x(\lambda_t).
\end{align*}
\]

Since $C(X)$ is contained in the multiplicative domains of $1_x$ and $\phi \circ \rho$, for any $f \in C(X)$ and $t \in \Gamma$, we have

\[
1_x(f \lambda_t) = f(x)1_x(\lambda_t) = f(x)\text{ev}_x \circ \phi \circ \rho \circ \pi_x(\lambda_t) = \text{ev}_x(f \phi \circ \rho(\pi_x(\lambda_t))) = \text{ev}_x \circ \phi \circ \rho(\pi_x(\lambda_t)) = \text{ev}_x \circ \phi \circ \rho \circ \pi_x(f \lambda_t).
\]

This implies that $1_x = (\pi_x(\cdot)\delta_x, \delta_x) = \text{ev}_x \circ \phi \circ \rho \circ \pi_x$ on $C(X) \rtimes \Gamma$. Since $\delta_x$ is a cyclic vector, $\rho$ is faithful. \qed

In particular, for any $x \in \partial_{\tilde{X}} \Gamma$ (recall that $C(\partial_{\tilde{X}} \Gamma)$ is the $\Gamma$-injective envelope of $\mathbb{C}$), the C*-algebra $\pi_x(C(\tilde{X}) \rtimes \Gamma)$ is simple. Moreover, we have a stronger conclusion in this situation, a generalization of the Powers’ averaging property ([14]), which is equivalent to the C*-simplicity (see [5, 9]). First, we show the following lemma.
Lemma 8.3. Let $x$ be a point in $\partial H\Gamma$. Then for every finite family $\{\phi_k\}_{k=0}^N$ of states on $\pi_x(C(\partial H\Gamma)\rtimes_r\Gamma)$, there is a net $(\alpha_i)$ in $\text{conv}\{\text{Ad}(t) : t \in \Gamma\}$ such that $\phi_k \circ \pi_x \circ \alpha_i \to 1_x$ for every index $k$.

Proof. First we show that for every state $\phi$ on $\pi_x(C(\partial H\Gamma)\rtimes_r\Gamma)$, we have

$$1_x \in \text{conv}\{\phi \circ \pi_x \circ \text{Ad}(t) : t \in \Gamma\}.$$ 

By [4, Theorem 2.3], there is a $\Gamma$-boundary $X \subset \text{conv}\{\phi \circ \pi_x \circ \text{Ad}(t) : t \in \Gamma\}$. Hence there is a $\Gamma$-equivariant continuous surjection $p : \partial H\Gamma \to X$ by [4, Theorem 3.11]. Since there is a natural $\Gamma$-morphism from $\pi_x(C(\partial H\Gamma)\rtimes_r\Gamma)$ to $C(X)$, we have a $\Gamma$-morphism $\Phi$ from $\pi_x(C(\partial H\Gamma)\rtimes_r\Gamma)$ to $C(\partial H\Gamma)$ such that $\Phi(\pi_x(a))(x) = p(x)(a)$ for any $a \in C(\partial H\Gamma)\rtimes_r\Gamma$. Then $\Phi \circ \pi_x$ is conditional expectation from $C(\partial H\Gamma)\rtimes_r\Gamma$ to $C(\partial H\Gamma)$. Hence by Lemma 8.2 we have

$$p(x)(\pi_x(\lambda_t)) = \Phi \circ \pi_x(\lambda_t)(x) = \Phi \circ \pi_x \circ \tilde{E}(\lambda_t)(x) = \chi_{\text{Fix}(t)}(x) = 1_x(\lambda_t),$$

hence $1_x \in X$.

Next, we show the theorem. Take a net $(\alpha_i)$ in $\text{conv}\{\text{Ad}(t) : t \in \Gamma\}$ such that

$$\frac{1}{N} \left( \sum_{k=0}^N \phi_k \right) \circ \pi_x \circ \alpha_i \to 1_x.$$ 

We may assume that $\phi_k \circ \alpha_i \to \psi_k$, where $\psi_k \in S(\pi_x(C(\partial H\Gamma)\rtimes_r\Gamma))$, then we have

$$\frac{1}{N} \left( \sum_{k=0}^N \psi_k \right) \circ \pi_x = 1_x.$$ 

This implies that $\psi_k \circ \pi|_{C^*_r(\Gamma_x)} = 1_x|_{C^*_r(\Gamma_x)} = \tau_0$ because $\tau_0$ is a character, hence it is extremal in $S(C^*_r(\Gamma_x))$. Similarly, we obtain $\psi_k \circ \pi|_{C(\partial H\Gamma)} = 1_x|_{C(\partial H\Gamma)} = ev_x$ because $ev_x$ is an extreme point of $M(\hat{X})$. We claim that for any $\theta \in S(C(\partial H\Gamma)\rtimes_r\Gamma)$, $\theta|_{C^*_r(\Gamma_x)} = \tau_0$ and $\theta|_{C(\partial H\Gamma)} = ev_x$ imply that $\theta = 1_x$. Since $C(\partial H\Gamma) \subset \text{mult}(\theta)$, it suffices to show that $\theta(t) = 0$ for every $t \in \Gamma \setminus \Gamma_x$. Take a function $f \in C(\hat{X})$ such that $f(x) = 1$ and $f(tx) = 0$, we have $1_x(\lambda_t) = f(x)1_x(\lambda_t) = 1_x(f\lambda_t) = 1_x(\lambda_t(t^{-1}f)) = 1_x(\lambda_t)f(tx) = 0$. Hence we have $\psi_k \circ \pi_x = 1_x$. \hfill \Box

Theorem 8.4. Let $x$ be a point in $\partial H\Gamma$. Then for every $a \in C^*_r(\Gamma)$, the element $1_x(a)$ is contained in the norm closed convex hull of $\{\pi_x(\lambda_a\lambda^*_\alpha) : \alpha \in \Gamma\}$.

Proof. We show it by contradiction. Suppose that there is an element $a \in \pi_x(C^*_r(\Gamma))$ such that $1_x(a) \not\in \text{conv}_{\text{norm}}\{\pi_x(\lambda_a\lambda^*_\alpha) : \alpha \in \Gamma\}$. Then there is a bounded linear functional $\phi$ on $\pi_x(C^*_r(\Gamma))$ such that $\text{Re}(\phi \circ \pi_x(b - \phi(1)1_x(a))) > \varepsilon > 0$ for every $b \in \text{conv}_{\text{norm}}\{\pi_x(\lambda_a\lambda^*_\alpha) : \alpha \in \Gamma\}$ by Hahn–Banach separation theorem. By Hahn–Jordan decomposition, we can write $\phi = \sum_{k=0}^3 i^kc_k\phi_k$, where $\phi_k$ is a state on $\pi_x(C^*_r(\Gamma))$ and $c_k$ is non-negative scalar. Then by Lemma 8.3, there is a net $(\alpha_i)$ in $\text{conv}\{\text{Ad}(t) : t \in \Gamma\}$ such that $\phi_k \circ \pi_x \circ \alpha_i \to 1_x|_{C^*_r(\Gamma)}$ for $k = 1, 2, 3, 4$. But we have

$$\text{Re}(\phi \circ \pi_x \circ \alpha_i(a) - \phi(1)1_x(a)) = \text{Re} \left( \sum_{k=0}^3 i^kc_k(\phi_k \circ \pi_x \circ \alpha_i(a) - 1_x(a)) \right) \geq \varepsilon,$$

a contradiction. \hfill \Box

Corollary 8.5. For every point $x$ in $\partial H\Gamma$, the $C^*$-algebra $\pi_x(C^*_r(\Gamma))$ is simple.

Proof. For every non-zero positive element $\pi_x(a) \in \pi_x(C^*_r(\Gamma))$, we have $1_x(a) \neq 0$ since $1_x$ is faithful on $\pi_x(C^*_r(\Gamma))$. This implies that $1_x(a) \in \text{Ideal}(\pi_x(a))$ by theorem 8.4, where $\text{Ideal}(\pi_x(a))$ is the ideal in $\pi_x(C^*_r(\Gamma))$ generated by $\{\pi_x(a)\}$. \hfill \Box
9. AMENABLE URS’S AND IDEALS IN THE GROUP C*-ALGEBRA

In this section, we see the relationship between amenable URS’s of $\Gamma$ and ideals in $C^*_r \Gamma$. For an amenable subgroup $\Lambda$ in $\Gamma$, we define a representation $\pi_{\Lambda}$ of $C^*_r \Gamma$ on $\ell_2(\Gamma/\Lambda)$ by

$$\pi_{\Lambda}(\lambda_t)\delta_x = \delta_{tx}, \ x \in \Gamma/\Lambda.$$ 

Since $\langle \pi_{\Lambda}(\cdot)\delta_\Lambda, \delta_\Lambda \rangle = \tau_0 \circ E_\Lambda$ (where $\tau_0$ is the unit character), the representation $\pi_{\Lambda}$ is unitarily equivalent to the GNS representation with respect to $1_\Lambda := \tau_0 \circ E_\Lambda$. Note that for $x \in \partial_\Gamma \Gamma$, one has $\pi_x|_{C^*_r \Gamma} = \pi_{\Gamma_x}$. Since $1_\Lambda \circ \text{Ad}(t^{-1}) = 1_{t\Lambda^{-1}}$, we have the following equality.

$$\ker(\pi_{\Lambda}) = \{ a \in C^*_r \Gamma : \langle \pi_{\Lambda}(a) \xi, \eta \rangle = 0, \text{ for every } \xi, \eta \in \ell_2(\Gamma/\Lambda) \}$$

$$= \{ a \in C^*_r \Gamma : \langle \pi_{\Lambda}(\cdot)\delta_\Lambda, \delta_\Lambda \rangle = 1_\Lambda(\lambda^*_a a \lambda_t) = 0, \text{ for every } s, t \in \Gamma \}$$

$$= \bigcap_{s, t \in \Gamma} \{ a \in C^*_r \Gamma : 1_\Lambda(\lambda^*_a a \lambda_t) = 0 \}$$

$$= \bigcap_{\Delta \in \text{Ad}(\Gamma) \Lambda} \bigcap_{\tau \in \Gamma} \{ a \in C^*_r \Gamma : 1_\Delta(at) = 0 \}$$

$$= \bigcap_{\Delta \in \text{Ad}(\Gamma) \Lambda} \bigcap_{\tau \in \Gamma} \{ a \in C^*_r \Gamma : 1_\Delta(at) = 0 \}.$$ 

In particular, for every amenable URS $U$ and every elements $H_1$ and $H_2$ in $U$, we have $\ker(\pi_{H_1}) = \ker(\pi_{H_2})$, hence we set $I_U := \ker(\pi_H)$ for $H \in U$. Note that $I_{S_{\partial_\Gamma} \Gamma}$ is maximal by Corollary 8.5. From the above equality, we obtain the following easily.

**Proposition 9.1.** Let $\Lambda$ be an amenable subgroup of $\Gamma$. Then for every amenable URS $U$ contained in $\text{Ad}(\Gamma)\Lambda$, we have $\ker(\pi_{\Lambda}) \subset I_U$.

In particular, if $\{ e \} \in \text{Ad}(\Gamma)\Lambda$, the representation $\pi_{\Lambda}$ is faithful, but the converse need not be true in general, i.e. there is a group which has a non-trivial amenable URS $U$ such that $I_U = 0$. The following example was communicated to us by Koichi Shimada.

**Example 9.2.** Let $A_4$ denote the alternating group on 4 letters $\{1, 2, 3, 4\}$. Then, the group algebra $\mathbb{C}(A_4)$ is isomorphic to $\mathbb{C}^3 \oplus M_3(\mathbb{C})$. Indeed, the derived subgroup of $A_4$ is $K := \{ e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) \}$ and $A_4/K \cong \mathbb{Z}/3\mathbb{Z}$, which accounts for the abelian quotient $\mathbb{C}(A_4/K) \cong \mathbb{C}^3$. Since the standard action of $A_4$ on the $4$ letters is doubly transitive, it gives rise to an irreducible representation on the $3$-dimensional space $\{ \xi \in \mathbb{C}(\{1, 2, 3, 4\}) : \sum_k \delta_\xi(k) = 0 \}$, which accounts for the factor $M_3(\mathbb{C})$. Since $\dim \mathbb{C}(A_4) = |A_4| = 12 = \dim \mathbb{C}^3 \oplus M_3(\mathbb{C})$, we have $\mathbb{C}(A_4) \cong \mathbb{C}^3 \oplus M_3(\mathbb{C})$. Now we consider the subgroup $\Lambda := \{ e, (1, 2)(3, 4) \}$ of order $2$. From the above description, it is not difficult to see that the representation $\pi_{\Lambda}$ of $\mathbb{C}(A_4)$ on $\mathbb{C}(A_4/\Lambda)$ is faithful.

**Proposition 9.3.** Let $U$ and $V$ be amenable URS’s. If $U \subset V$, then $I_U \subset I_V$.

**Proof.** It suffices to show that for amenable subgroups $\Lambda_1$ and $\Lambda_2$ in $\Gamma$ such that $\Lambda_1 \lhd \Lambda_2$, we have $\ker(\pi_{\Lambda_1}) \subset \ker(\pi_{\Lambda_2})$. By amenability of $\Lambda_2$, there is an approximately invariant vector $\langle \xi_t \rangle$ in $\ell_2(\Lambda_2/\Lambda_1)$, i.e. $\langle \xi_t \rangle$ is a net in $\ell_2(\Lambda_2/\Lambda_1) \subset \ell_2(\Gamma/\Lambda_1)$ such that $\| \pi_{\Lambda_1}(\lambda_t)\xi_t - \xi_t \| \to 0$ for any $t \in \Lambda_2$. Take a Følner net $(F_i)$ of $\Lambda_2$, then
the net of vectors
\[ \xi_i := |F_i|^{-1} \sum_{t \in F_i} \delta_{tA} \]
is approximately invariant in \( \ell_1(A_2/\Lambda_1) \), hence the net \( (\xi_i^{1/2}) \) is an approximately invariant in \( \ell_2(A_2/\Lambda_1) \). Then we have the net \( ((\pi_{\Lambda_1}(\cdot)\xi_i, \xi_i)) \) of state on \( C_1^* \Gamma \) converges to \( 1_{A_2} \). This implies that there is a state \( \phi \) on \( \pi_{\Lambda_1}(C_1^* \Gamma) \) such that \( 1_{A_2} = \phi \circ \pi_{\Lambda_1} \), hence we have \( \ker(\pi_{\Lambda_1}) \subset \ker(\pi_{\Lambda_2}) \).

We show a relaxed form of the converse of Proposition 9.3 (Note that the converse of Proposition 9.3 is not true. Example 9.2 is a counter example.) For a subset \( S \) in \( \Gamma \), we set \( T(S) := \{ t \in \Gamma : t^n \in S \text{ for a non-zero integer } n \} \).

**Theorem 9.4.** Let \( \Lambda \) and \( \Lambda' \) be amenable subgroups of \( \Gamma \) such that \( \ker(\pi_{\Lambda}) \subset \ker(\pi_{\Lambda'}) \). Then, there is an amenable subgroup \( \Delta \in \overline{\text{Ad}(\Gamma)\Lambda} \) such that \( \Delta \subset T(\Lambda') \).

**Proof.** It suffices to show that for every finite set \( F \subset \Gamma \setminus T(\Lambda') \), there is an element \( t_F \in \Gamma \) such that \( F \subset \Gamma \setminus t_F \Lambda'^{-1} \). Indeed, let \( (F_n) \) be an increasing sequence of finite subset in \( \Gamma \setminus T(\Lambda') \) such that \( \bigcup F_n = \Gamma \setminus T(\Lambda') \). Take a cluster point \( \Delta \) of \( \{t_{F_n} \Lambda'^{-1} \} \) where \( t_{F_n} \) satisfies that \( F_n \subset \Gamma \setminus t_{F_n} \Lambda'^{-1} \), then we have \( \Delta \subset T(\Lambda') \).

We show it by contradiction. Suppose that there is a finite set \( F \subset \Gamma \setminus T(\Lambda') \) such that \( t_\Lambda^{-1} \cap F \neq \emptyset \) for every \( t \in \Gamma \). It is easy to see that \( t_\Lambda^{-1} \cap F \neq \emptyset \) for every \( t \in \Gamma \) if and only if for all \( x \in \Gamma/\Lambda \), there exists an element \( g \in F \) such that \( gx = x \). We define \( p_g \) as the orthogonal projection onto the closed linear span of \( \{a_x : x \in \Gamma/\Lambda, gx = x\} \). Then we obtain the following conditions.

- \( \pi_{\Lambda'}(\lambda_g) p_g = p_g = p_g \pi_{\Lambda'}(\lambda_g) \) for every \( g \in F \).
- \( \sum_{g \in F} p_g \geq 1 \).

Since \( \ker(\pi_{\Lambda}) \subset \ker(\pi_{\Lambda'}) \), the map \( \pi_{\Lambda}(C_1^* \Gamma) \ni \pi_{\Lambda}(a) \to \pi_{\Lambda'}(a) \in \pi_{\Lambda'}(C_1^* \Gamma) \) is a \(*\)-homomorphism. We extend it to a unital completely positive map \( \Theta \) from \( B(\ell_2(\Gamma/\Lambda)) \) to \( B(\ell_2(\Gamma/\Lambda')) \) by Arveson's extension theorem. Since \( \pi_{\Lambda}(C_1^* \Gamma) \subset \mult(\Theta) \), the element \( a_g := \Theta(p_g) \) satisfies the following conditions.

- \( 0 \leq a_g \leq 1 \) for every \( g \in F \).
- \( \pi_{\Lambda'}(\lambda_g) a_g = a_g = a_g \pi_{\Lambda'}(\lambda_g) \) for every \( g \in F \).
- \( \sum_{g \in F} a_g \geq 1 \).

The sequence \( n^{-1} \sum_{k=1}^n \pi_{\Lambda'}(\lambda_g)^k \) converges in the strong operator topology to the orthogonal projection onto the \( \pi_{\Lambda'}(\lambda_g) \)-invariant vectors, which will be denoted by \( q_g \).

The second condition implies that \( q_g a_g = a_g q_g \), therefore we have \( \text{supp}(a_g) \leq q_g \).

Since \( g^n \notin \Lambda' \) for every non-zero integer \( n \), we have \( \langle \pi_{\Lambda'}(\lambda_g)^n \delta_{\Lambda'}, \delta_{\Lambda'} \rangle = 0 \). This implies that
\[ \langle \text{supp}(a_g) \delta_{\Lambda'}, \delta_{\Lambda'} \rangle \leq \langle q_g \delta_{\Lambda'}, \delta_{\Lambda'} \rangle = 0. \]

Hence we have \( \langle a_g \delta_{\Lambda'}, \delta_{\Lambda'} \rangle = 0 \), it contradicts that
\[ \langle \sum_{g \in F} a_g \delta_{\Lambda'}, \delta_{\Lambda'} \rangle \geq \langle \delta_{\Lambda'}, \delta_{\Lambda'} \rangle = 1. \]

□

**Corollary 9.5.** Let \( \Lambda \) be an amenable subgroup of \( \Gamma \) such that the representation \( \pi_{\Lambda} \) is faithful. Then, there is a torsion group \( \Delta \) contained in \( \overline{\text{Ad}(\Gamma)\Lambda} \). In particular, for any amenable URS \( \mathcal{U} \), the condition \( I_{\mathcal{U}} = 0 \) implies that \( \mathcal{U} \) consists of torsion groups.
Proof. It is easy to see the first part of the theorem by Theorem [9.4] Let $U$ be an amenable $URS$ such that $I_U = 0$. Then, there is a torsion group $\Delta \in U$. Since $\Ad(\Gamma)\Delta = U$, every $H \in U$ is a torsion group. $\square$

Note that the converse of the above corollary need not be true in general. Let $N$ be a non-trivial finite normal subgroup of $\Gamma$. Then, it is clear that $\pi_N$ is not faithful, but $N$ is a torsion group since $|N|$ is finite.

References

[1] E. Breuillard, M. Kalantar M. Kennedy and N. Ozawa; $C^*$-simplicity and the unique trace property for discrete groups. Preprint (2014). (arXiv:1410.2518)
[2] N. Brown and N. Ozawa; $C^*$-algebras and Finite-Dimensional Approximations. Graduate studies in Mathematics, 88. American Mathematical Society, Providence, RI. 2008.
[3] E. Glasner and B. Weiss; Uniformly recurrent subgroups. Recent Trends in Ergodic Theory and Dynamical Systems, Contemp. Math. 631, RI. 2015. pp. 63–75.
[4] S. Glasner; PloxAimal flows. Lecture Notes in Mathematics, Vol. 517. Springer-Verlag, Berlin, Heidelberg, New York, 1976.
[5] U. Haagerup; A new look at $C^*$-simplicity and the unique trace property of a group. Operator Algebras and Applications. Abel Symposia, vol 12. Springer, Cham.
[6] M. Hamana; Injective envelopes of $C^*$-dynamical systems. Tohoku Math. Journ. 37 (1985), 463–487.
[7] M. Kalantar and M. Kennedy; Boundaries of reduces $C^*$-algebras of discrete groups. Journal für die reine und angewandte Mathematik. To appear.
[8] S. Kawamura and J. Tomiyama; Properties of topological dynamical systems and corresponding $C^*$-algebras. Tokyo. J. Math. 13 (1990), 251–257.
[9] M. Kennedy; Characterizations of $C^*$-simplicity. Preprint (2015). (arXiv:1509.0187)
[10] A. Le Boudec; $C^*$-simplicity and the amenable radical. Invent. Math. To appear.
[11] A. Le Boudec and N. Matte Bon; Subgroup dynamics and $C^*$-simplicity of groups of homeomorphisms. Preprint (2016). (arXiv:1605.01651)
[12] J. Renault; Cartan subalgebras in $C^*$-algebras. Irish Math. Soc. Bulletin 61 (2008), 29–63.
[13] N. Ozawa; Lecture on the Furstenberg boundary and $C^*$-simplicity. http://www.kurims.kyoto-u.ac.jp/~narutaka/notes/yokou2014.pdf (2014).
[14] R. Powers; Simplicity of the $C^*$-algebra associated with the free group on two generators. Duke Math. Journ. 42 (1975), 151–156.
[15] A. Sierakowski; The ideal structure of reduced crossed products. Münster Journal of Mathematics 3 (2010), 237–262.