MONOTONE ODEs WITH DISCONTINUOUS VECTOR FIELDS IN SEQUENCE SPACES

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Abstract. We consider a system of ODE in a Fréchet space with unconditional Schauder basis. The right side of the ODE is a discontinuous function. Under certain monotonicity conditions we prove an existence theorem for the corresponding initial value problem. We employ an idea of the partial order which seems to be new in this field.

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1. Introduction. The analysis of ordinary differential equations (ODEs) with non Lipschitz right hand side has a long history. Without any claims on completeness of exposition we just note some principle points of this history. A detailed discussion of further developments in any of these points requires a separate survey.

The first result belongs to G. Peano (1890). G. Peano considered an initial value problem

\[ \dot{x} = f(t, x), \quad x(t_0) = x_0 \]  

(1.1)

where \( f \) is a continuous mapping of some domain 

\[ D \subset \mathbb{R}^{m+1} = \{(t, x)\}, \quad x = (x_1, \ldots, x_m) \]

with values in \( \mathbb{R}^m \).

G. Peano stated that this problem has a solution that is defined locally for small \( |t - t_0| \). This solution may not be unique.

C. Carathéodory relaxed the conditions of this theorem up to measurability of the function \( f \) in \( t \).

All these results are essentially based on the fact that a closed ball in \( \mathbb{R}^m \) is compact. In an infinite dimensional Banach space they are in general invalid.

The corresponding example was first constructed by J. Dieudonné [1]. His example is as follows:

\[ \dot{x}_k = \sqrt{|x_k|} + \frac{1}{k}, \quad x_k(0) = 0, \quad k \in \mathbb{N}, \quad t \geq 0. \]

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By direct integration one can easily check that this IVP does not have solutions
\[ x(t) = \{ x_k(t) \} \in c_0. \]

In the infinite dimensional case besides the continuity one must impose some
extra compactness conditions on \( f \) [8], [9] in order to recover the existence.

Observe that all the existence results mentioned above follow in one way or
another from the Schauder-Tychonoff fixed point theorem.

Note also that in case of non-Lipschitz equations a solution is in general not
unique and one has to make separate efforts to study the problem of uniqueness.

When the function \( f \) is discontinuous in \( x \) then even examples that are pretty
innocuous from the first glance can provide nonexistence. Indeed, consider a scalar
IVP
\[ \dot{x} = h(x), \quad x(0) = 1, \] (1.2)
where
\[ h(x) = \begin{cases} 1, & \text{if } x \leq 1, \\ -1, & \text{if } x > 1. \end{cases} \]

This problem does not have a continuous solution \( x(t) \) in the sense of integral
equation:
\[ x(t) = 1 + \int_0^t h(x(\xi))d\xi, \quad t > 0. \]

To show this assume the converse: this solution exists for some \( t > 0 \). It is clear
that it can not be equal to 1 identically. Thus for some \( t' > 0 \) we have \( x(t') > 1 \).
(The case \( x(t') < 1 \) is accomplished similarly.) Then there exists an interval \((t_1, t')\)
such that
\[ t \in (t_1, t') \implies x(t) > 1 \]
and \( x(t_1) = 1 \).

For \( t \in (t_1, t') \) we can write
\[ x(t) = x(t_1) + \int_{t_1}^t h(x(\xi))d\xi = 1 - (t - t_1) < 1. \]

This is a contradiction.

Such examples prompt an idea to change the concept of a solution itself. Note
in addition that if the right side of equation (1.1) is just a measurable function then
even for continuous \( x(t) \) a mapping \( t \mapsto f(t, x(t)) \) is not obliged to be measurable
[5].

The corresponding transformation of the notion of a solution was proposed
by A. Filippov [4]. According to him an absolutely continuous function \( x(t) \) is a
solution to (1.1) with \( f \) measurable if the following inclusion holds
\[ \dot{x}(t) \in \bigcap_{r > 0} \bigcap_N \conv f(t, B_r(x(t)) \setminus N) \text{ for almost all } t. \]

Here \( B_r(x) \subset \mathbb{R}^m \) is an open ball of the radius \( r \) and the center at \( x \). The
intersection \( \bigcap_N \) is taken over all measure-null sets \( N \); and conv stands for the closed
convex hull of a set.
Once we have denied the classical concept of a solution then a lot of reasonable generalizations arise. Filippov’s concept is good for control and for dry friction mechanics [10], [12]. A very different approach by DiPerna and Lions is good for PDE and fluid mechanics [2].

In this article we try to save the classical concept of a solution for some class of discontinuous ODE.

In the proofs we essentially use the Lebesgue integration theory for functions that take values in Fréchet spaces.

The Lebesgue integration theory of functions with values in Banach spaces is developed in [6]. The construction of [6] can easily be generalized to the case of Fréchet space valued functions. Since we have not seen such a generalization elsewhere and for completeness sake we collect a necessary theory in Section 4. There by means of minor modifications we adapt the theory from [6] to the case of Fréchet spaces.

2. The main theorems. Let $E$ stand for a Fréchet space. Its topology is defined by a collection of seminorms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. Recall that a Fréchet space is Hausdorff: for any $x \neq 0$ there exists $i$ such that $\|x\|_i \neq 0$. Moreover a topology of a Fréchet space is completely metrizable by the following metrics

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x - y\|_k\}.$$ 

Assume that the space $E$ possesses an unconditional Schauder basis $\{e_k\}_{k \in \mathbb{N}}$. Recall several definitions.

**Definition 1.** A sequence $\{e_k\}_{k \in \mathbb{N}} \subset E$ is called a Schauder basis if for every $x \in E$ there is a unique sequence of scalars $\{x_k\}_{k \in \mathbb{N}}$ such that

$$x = \sum_{k=1}^{\infty} x_k e_k. \quad (2.1)$$

This series is convergent in the topology of $E$.

We shall say that $\{e_k\}_{k \in \mathbb{N}}$ is an unconditional basis if for any $x \in E$ and for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ the sum

$$\sum_{k=1}^{\infty} x_{\pi(k)} e_{\pi(k)}$$

is convergent to the same element.

Introduce linear functions $e^j : E \to \mathbb{R}$ by the formula $e^j(x) = x_j$. These functions are continuous [3].

We use $I_T$ to denote $[0, T], \quad T > 0$.

Equip the space $E$ with a partial order $\ll$ as follows

$$x = \sum_{k=1}^{\infty} x_k e_k \ll y = \sum_{k=1}^{\infty} y_k e_k \iff x_i \leq y_i, \quad i \in \mathbb{N}.$$
**Definition 2.** We shall say that a function $g : E \to \mathbb{R}$ is left continuous if for all $x' \in E$ and for all sequences

$$x_k \to x', \quad x_k \ll x_{k+1}, \quad k \in \mathbb{N}$$

one has

$$\lim_{k \to \infty} g(x_k) = g(x').$$

The main object of our study is the following initial value problem

$$\dot{x}(t) = f(t, x(t)) = \sum_{k=1}^{\infty} f_k(t, x(t)) e_k, \quad x(0) = \hat{x} \in E. \quad (2.2)$$

Here the function $f : I_T \times E \to E$ is such that all the functions $f_k : I_T \times E \to \mathbb{R}$ are left continuous in the second argument when the first one is fixed.

For any fixed $x$ the function $t \mapsto f(t, x)$ is integrable on $I_T$.

Assume that there exists an element $C = \sum_{k=1}^{\infty} C_k e_k \in E$ such that for any $(t, x) \in I_T \times E$ the following inequality holds

$$f(t, x) \ll C. \quad (2.3)$$

Assume that there exists an element $x_* \in E$ that satisfies the inequality

$$x_* \ll \hat{x} + \int_0^t f(\xi, x_*) d\xi, \quad t \in I_T. \quad (2.4)$$

**Definition 3.** We shall say that a function $x \in C(I_T, E)$ is a solution to IVP (2.2) if a function $t \mapsto f(t, x(t))$ is integrable in $I_T$ and the following equation

$$x(t) = \hat{x} + \int_0^t f(\xi, x(\xi)) d\xi, \quad t \in I_T$$

is satisfied.

**Theorem 1.** In addition to the hypotheses above assume that $f$ is monotone:

$$x \ll y \implies f(t, x) \ll f(t, y), \quad \forall x, y \in E, \quad \forall t \in I_T. \quad (2.5)$$

Then problem (2.2) has a solution $x(t)$.

Theorem 1 is proved in Section 3.2.

Condition (2.5) is essential and can not be dropped: see example (1.2).

To show that condition (2.3) also matters we modify Dieudonné’s example.

The IVP

$$\dot{x}_k = q(x_k) + \frac{1}{k}, \quad x_k(0) = 0, \quad k \in \mathbb{N}, \quad t \geq 0$$

does not have solutions in $c_0$; here the function

$$q(\xi) = \begin{cases} \sqrt{\xi}, & \text{if } \xi \geq 0, \\ 0, & \text{if } \xi < 0 \end{cases}$$
is monotone.

Let $V \subseteq E$ be a nonempty set. We formally put

$$
\sup V := \sum_{k=1}^{\infty} v_k^* e_k, \quad v_k^* = \sup \{e^k(v) \mid v \in V\}.
$$

Let $Q \subset C(I_T, E)$ stand for a set of solutions to problem (2.2). From Theorem 1 we know that $Q \neq \emptyset$.

For each $t \in I_T$ introduce a set $Q(t) = \{v(t) \mid v \in Q\} \subset E$.

**Theorem 2.** Assume that the conditions of Theorem 1 are fulfilled.

Then for each $t \in I_T$ an element

$$
\overline{x}(t) = \sup Q(t) \in E
$$

is well defined and $\overline{x}$ is a solution to problem (2.2).

This theorem is proved in Section 3.3.

**Theorem 3.** Assume in addition that $E$ is a reflexive space. Then for almost all $t$ the solution $x(t)$ from Theorem 1 is weakly differentiable: there exists $\dot{x}(t) \in E$ such that for almost all $t$ one has

$$
\left(\psi, \frac{x(t + h) - x(t)}{h}\right) \to (\psi, \dot{x}(t)), \quad h \to 0, \quad h \neq 0, \quad \forall \psi \in E'
$$

and

$$
\dot{x}(t) = f(t, x(t)).
$$

Theorem 3 is proved in Section 3.4.

**Remark 1.** Theorems 1, 2, 3 remain valid if the functions $f_k$ are right continuous in the second argument.

2.1. An example. Introduce a function

$$
H(\eta) = \begin{cases} 
-1, & \text{if } \eta \leq 0, \\
1, & \text{if } \eta > 0.
\end{cases}
$$

Let functions

$$
n : \mathbb{Z}_+ \to \mathbb{Z}_+, \quad \rho_k \in L^1(0, \infty), \quad k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}
$$

be given.

Let $B = \{z \in \mathbb{C} \mid |z| < 1\}$ stand for an open ball. Let $E$ be a space of analytic functions

$$
u : B \to \mathbb{C}, \quad u(z) = \sum_{k=0}^{\infty} u_k z^k, \quad u_k \in \mathbb{R}
$$
with a collection of seminorms
\[ \|u\|_n = \max\{|u(z)| \mid |z| \leq 1 - 1/n\}, \quad n \in \mathbb{N}. \]

A set \( \{z^k\}, \quad k \in \mathbb{Z}_+ \) is an unconditional Schauder basis in \( E \).

Consider the following initial value problem
\[ u_t = \frac{\partial^p}{\partial z^p} \left( \sum_{k=0}^{\infty} H(u_{n(k)} + \rho_k(t)) z^k \right), \quad u|_{t=0} = \hat{u}(z), \quad (2.6) \]

where the constant \( p \) belongs to \( \mathbb{N} \).

It is not hard to show that for any \( \hat{u} \in E \) and \( T > 0 \) problem (2.6) satisfies all the conditions of Theorem 1 and has a solution
\[ u(t, z) = \sum_{k=0}^{\infty} u_k(t) z^k \]
from \( C(I_T, E) \). Particularly to satisfy condition (2.3) one should take
\[ C = \frac{\partial^p}{\partial z^p} \sum_{k=0}^{\infty} z^k \in E. \]

3. Proofs of the theorems. We denote all inessential positive constants by the same letter \( c \).

3.1. Auxiliary facts. The following theorem is essentially based on the assumption that the Schauder basis is unconditional. This theorem generalizes the corresponding result for Banach spaces [7].

**Theorem 4.** ([11]) Fix a sequence \( \lambda = \{\lambda_j\}_{j \in \mathbb{N}} \in \ell_\infty \). Then
\[ \mathcal{M}_\lambda x = \sum_{k=1}^{\infty} \lambda_k x_k e_k, \quad x = \sum_{k=1}^{\infty} x_k e_k \in E \]
is a bounded linear operator of \( E \) to \( E \) and for any number \( i' \) there exists a number \( i \) and a positive constant \( c \) both independent on \( \lambda \) such that
\[ \|\mathcal{M}_\lambda x\|_{i'} \leq c\|\lambda\|_{\ell_\infty} \cdot \|x\|_i, \quad \forall x \in E. \]

Particularly this theorem implies
\[ y = \sum_{k=1}^{\infty} y_k e_k \in E \implies |y| := \sum_{k=1}^{\infty} |y_k| e_k \in E. \]

Other consequence of this theorem is as follows.
Lemma 1. Assume that constant vectors
\[ a = \sum_{k=1}^{\infty} a_k e_k, \quad b = \sum_{k=1}^{\infty} b_k e_k \in E \]
are such that \( a \ll b \). Then for any sequence of reals \( y_k \), \( a_k \leq y_k \leq b_k \) an element
\[ y = \sum_{k=1}^{\infty} y_k e_k \in E \]
is well defined.

Indeed,
\[ y = \mathcal{M}_a a + \mathcal{M}_b b, \]
where
\[ \alpha = \{a_k\}, \quad \beta = \{\beta_k\} \in \ell_\infty, \quad \alpha_k + \beta_k = 1, \quad \alpha_k, \beta_k \geq 0. \]

Lemma 2. Assume that constant vectors
\[ a = \sum_{k=1}^{\infty} a_k e_k, \quad b = \sum_{k=1}^{\infty} b_k e_k \in E \]
are such that \( a \ll b \). Then an interval
\[ [a, b] := \{x \in E \mid a \ll x \ll b\} \]
is a compact set. Moreover for any \( i' \in \mathbb{N} \) there exists \( i \in \mathbb{N} \) and a positive constant \( c > 0 \) such that
\[ x \in [a, b] \implies \|x\|_{i'} \leq c(\|a + b\|_{i'} + \|a - b\|_i). \quad (3.1) \]
The constant \( c \) and the index \( i \) do not depend on \( a, b \).

Remark 2. In the sequel we do not use compactness of an interval.

Proof of Lemma 2. Let us shift the set \([a, b]\) and consider a set
\[ J = [a - s, b - s], \quad s = \frac{a + b}{2}. \]
The set \([a, b]\) is compact iff the set \( J \) is compact.

Consider a projection \( P_n y = \sum_{k=1}^{n} y_k e_k \). Each set
\[ K_n = P_n(J) \subset J \]
is compact since it is a closed and bounded subset of \( \mathbb{R}^n \).

Show that the sets \( \{K_n\} \) form \( \varepsilon \)-nets in \( J \).

Indeed, take any element \( y \in J \) and present it as follows
\[ y = P_n y + q_n, \quad q_n = \sum_{k=n+1}^{\infty} y_k e_k, \quad |y_k| \leq r_k = \frac{b_k - a_k}{2}. \]
A series $R_n = \sum_{k=n+1}^{\infty} r_k e_k \in E$ is a tail of the expansion of the element $(b - a)/2$ and thus for all $i$ it follows that $\|R_n\|_i \to 0$. Observe that

$$q_n = M_\lambda R_n,$$

where $\lambda = \{\lambda_j\}$, $\lambda_j = y_j/r_j$ provided $r_j \neq 0$ and $\lambda_j = 0$ otherwise.

Theorem 4 implies that for any $i'$ there exists $i$ such that

$$\|q_n\|_{i'} = \|M_\lambda R_n\|_{i'} \leq c \|R_n\|_i \to 0. \quad (3.2)$$

The limit in the last part of formula (3.2) is uniform in $y \in J$. This proves the lemma in the part of compactness.

From the formulas above it also follows that $y = M_\lambda R_0$, $y \in J$,

$$\|y\|_{i'} \leq c \|R_0\|_i = c \left\|{a - b \over 2}\right\|_i.$$

After the back shift we readily obtain estimate (3.1).

Lemma 2 is proved. \hfill $\square$

**Lemma 3.** Let $W \subset E$ be a nonempty set with an upper bound $\overline{w} \in E$:

$$w \in W \implies w \ll \overline{w}.$$

Then the element $\sup W$ is well defined and $\sup W \ll \overline{w}$.

Indeed, the assertion follows from Lemma 1: take any $x \in W$; then $\sup W \in [x, \overline{w}]$.

**Lemma 4.** Let a function $F : I_T \times E \to \mathbb{R}$ be left continuous in the second argument and a function $t \mapsto F(t, x)$ be integrable on $I_T$ for each $x \in E$.

Suppose also that $F$ is monotone:

$$\forall x, y \in E, \quad \forall t \in I_T \quad \text{one has} \quad x \ll y \implies F(t, x) \leq F(t, y).$$

Assume that a function $u : I_T \to E$,

$$u(t) = \sum_{k=1}^{\infty} u_k(t)e_k$$

is such that all the functions $u_k : I_T \to \mathbb{R}$ are Lebesgue measurable and for some $a, b \in E$, $a \ll b$ and for all $t \in I_T$ one has

$$u(t) \in [a, b].$$

Then a mapping $t \mapsto F(t, u(t))$ is integrable on $I_T$. 
Proof of Lemma 4. From [5] we know that for each \(k\) there exists a sequence of simple functions \(\varphi_{k,j}(t)\) such that

\[
a_k \leq \varphi_{k,j}(t) \leq \varphi_{k,j+1}(t) \leq b_k, \quad a = \sum_{r=1}^{\infty} a_r e_r, \quad b = \sum_{r=1}^{\infty} b_r e_r
\]

and \(\varphi_{k,j} \to u_k\) pointwise for each \(t \in I_T\) as \(j \to \infty\).

Introduce the following functions

\[
[a,b] \ni U_j(t) = \sum_{k=1}^{j} \varphi_{k,j}(t) e_k + \sum_{r=j+1}^{\infty} a_r e_r, \quad F_j(t) = F(t, U_j(t)).
\]

Each function \(U_j\) has a finite set of values in \(E\) and

\[
U_j(t) \ll U_{j+1}(t), \quad t \in I_T.
\]

Thus \(F_j : I_T \to \mathbb{R}\) is integrable.

Let us show that \(U_j \to u\) pointwise in \(E\). Indeed, consider an estimate:

\[
\|U_j(t) - u(t)\|_i \leq \left\| \sum_{k=1}^{N} (\varphi_{k,j}(t) - u_k(t)) e_k \right\|_i + \left\| \sum_{k=N+1}^{j} (\varphi_{k,j}(t) - u_k(t)) e_k + \sum_{r=j+1}^{\infty} (a_k - u_k(t)) e_k \right\|_i, \quad j > N.
\]

The first summand in the right hand side of this formula vanishes as \(j \to \infty\). By Lemma 2 the second summand is estimated from above in terms of

\[
\left\| \sum_{k=N+1}^{j} a_k e_k \right\|_{i_1}, \quad \left\| \sum_{k=N+1}^{j} a_k e_k \right\|_{i_2}, \quad \left\| \sum_{k=N+1}^{j} b_k e_k \right\|_{i_1}, \quad \left\| \sum_{k=N+1}^{j} b_k e_k \right\|_{i_2}
\]

and

\[
\left\| \sum_{k=j+1}^{\infty} a_k e_k \right\|_{i_1}, \quad \left\| \sum_{k=j+1}^{\infty} a_k e_k \right\|_{i_2}, \quad \left\| \sum_{k=j+1}^{\infty} b_k e_k \right\|_{i_1}, \quad \left\| \sum_{k=j+1}^{\infty} b_k e_k \right\|_{i_2}.
\]

These terms vanish as \(N \to \infty\).

So that \(F_j(t) \to F(t, u(t))\) pointwise. On the other hand

\[
F(t,a) \leq F_j(t) \leq F(t,b).
\]

Therefore the assertion of the lemma follows from the Dominated convergence theorem.

Lemma 4 is proved. \(\square\)
Lemma 5. Take a function \( u : I_T \to [a, b] \subset E \),
\[
    u(t) = \sum_{k=1}^{\infty} u_k(t)e_k
\]
with \( u_k \) Lebesgue measurable. Then a function \( t \mapsto f(t, u(t)) \) is integrable in \( I_T \).

Proof of Lemma 5. Indeed, from Lemma 4 we know that the functions \( f_k(\cdot, u(\cdot)) : I_T \to \mathbb{R} \) are integrable and \( f_k(t, a) \leq f_k(t, u(t)) \leq f_k(t, b) \).

Introduce functions
\[
    \phi_n(\cdot) = \sum_{k=1}^{n} f_k(\cdot, u(\cdot))e_k
\]
and observe that \( \phi_n(\cdot) \to f(\cdot, u(\cdot)) \) pointwise in \( E \).

The functions \( \phi_n \) are integrable. Moreover,
\[
    \mathcal{M}_{\lambda_n} f(t, a) \ll \phi_n(t) \ll \mathcal{M}_{\lambda_n} f(t, b), \quad \lambda_n = (1, \ldots, 1, 0, 0, \ldots) \text{ \( n \) times}
\]

and from Lemma 2 and Theorem 4 it follows that for any \( i \in \mathbb{N} \) there are \( i', i'' \in \mathbb{N} \) and a constant \( c > 0 \) such that for all \( t \) one has
\[
    \| \phi_n(t) \|_i \leq c(\| f(t, a) \|_{i'} + \| f(t, b) \|_{i'} + \| f(t, a) \|_{i''} + \| f(t, b) \|_{i''}).
\]
By the statement of the problem the function in the right side of this inequality is integrable (for details see Section 4).

The Dominated convergence theorem (Theorem 10 below) concludes the proof. Lemma 5 is proved. \( \square \)

3.2. Proof of Theorem 1. Introduce a set
\[
    S = \{ u : I_T \to E \mid u(t) = \sum_{k=1}^{\infty} u_k(t)e_k \in E \}
\]

where \( \Phi(u)(t) = \hat{x} + \int_0^t f(\xi, u(\xi))d\xi \). The set \( S \) is not empty: \( x_* \in S \).

By formula (2.3) if \( u \in S \) then one has
\[
    \Phi(u)(t) \in [x_*, \hat{x} + T|C|], \quad u(t) \in [x_*, \hat{x} + T|C|], \quad t \in I_T. \tag{3.3}
\]
Observe also that \( \Phi(S) \subset S \). Indeed, this follows by Lemma 5 from the first inclusion of (3.3) and monotonicity of the mapping \( f \).
The set $S$ is partially ordered by the following binary relation. For any $u, v \in S$ by definition put

$$u \prec v \iff u(t) \ll v(t) \quad \forall t \in I_T.$$  

Let $S(t)$ denote the following set $\{ \eta(t) \mid \eta \in S \} \subset [x_*, \hat{x} + T|C|]$. By Lemma 3 a function $\bar{x}(t) = \sup S(t)$ is correctly defined. We then obtain

$$\bar{x}(t) = \sum_{k=1}^{\infty} \bar{x}_k(t)e_k \in [x_*, \hat{x} + T|C|].$$

From [3] we know that all the functions $\bar{x}_k$ are lower semicontinuous.

Furthermore for any $w \in S$ and for any $t \in I_T$ it follows that

$$w(t) \ll \bar{x}(t) \implies f(t, w(t)) \ll f(t, \bar{x}(t)) \implies \int_0^t f(\xi, w(\xi))d\xi \ll \int_0^t f(\xi, \bar{x}(\xi))d\xi.$$  

And thus

$$w(t) \ll \hat{x} + \int_0^t f(\xi, w(\xi))d\xi \ll \hat{x} + \int_0^t f(\xi, \bar{x}(\xi))d\xi.$$  

The last estimate holds for all $w \in S$. This implies

$$\bar{x}(t) \ll \hat{x} + \int_0^t f(\xi, \bar{x}(\xi))d\xi$$

and $\bar{x} \in S$.

By definition of $\bar{x}$ one obtains

$$\bar{x}(\cdot) \prec \Phi(\bar{x}(\cdot)) \in S \implies \bar{x}(\cdot) = \Phi(\bar{x}(\cdot))$$

and $\bar{x}$ is the desired solution to problem (2.2).

The theorem is proved. $\square$

### 3.3. Proof of Theorem 2.

Let $x \in Q$ be a solution. Then for all $t \in I_T$ we have

$$x(t) = \hat{x} + \int_0^t f(\xi, x(\xi))d\xi \ll \hat{x} + T|C|$$

and thus $\bar{x}(t) \ll \hat{x} + T|C|$. Since $\bar{x} \in Q$ we also obtain $x_* \ll \bar{x}(t) \ll \bar{x}(t)$.

Recall that all the functions $\bar{x}_k(t)$ from the expansion

$$\bar{x}(t) = \sum_{k=1}^{\infty} \bar{x}_k(t)e_k$$

are lower semicontinuous [3].
Therefore by Lemma 5 the function $\xi \mapsto f(\xi, x(\xi))$ is integrable and the following formula is correct:
\[
x(t) = \dot{x} + \int_0^t f(\xi, x(\xi))d\xi \ll \dot{x} + \int_0^t f(\xi, \bar{x}(\xi))d\xi
\]
and thus
\[
\dot{x}(t) \ll \dot{x} + \int_0^t f(\xi, \bar{x}(\xi))d\xi.
\]
So that $\bar{x} \in S$ and then $\dot{x} = \bar{x}$.

The theorem is proved. \[ \square \]

3.4. Proof of Theorem 3. Recall that a function $w : I_T \to \mathbb{R}$ is absolutely continuous iff it can be presented in the form
\[
w(t) = \int_0^t p(\xi)d\xi, \quad p \in L^1(I_T).
\]
In this case the classical derivative $\dot{w}$ exists for almost all $t$ and
\[
\dot{w}(t) = p(t) \quad \text{almost everywhere.}
\]
From Lemma 5 we know that $f(\cdot, x(\cdot)) \in L^1(I_T, E)$. Thus
\[
(\psi, x(t)) = (\psi, \dot{x}) + \left(\psi, \int_0^t f(\xi, x(\xi))d\xi\right) = (\psi, \dot{x}) + \int_0^t (\psi, f(\xi, x(\xi)))d\xi
\]
is an absolutely continuous function and
\[
\frac{(\psi, x(t + h)) - (\psi, x(t))}{h} = \left(\psi, \frac{x(t + h) - x(t)}{h}\right), \quad h \neq 0. \tag{3.3}
\]
For almost all $t$ one has
\[
\frac{d}{dt}(\psi, x(t)) = (\psi, \dot{x}(t)). \tag{3.4}
\]
The space $E$ is separable so that $E''$ is also separable and thus $E'$ is separable.

Let $\Psi \subset E'$ be a countable dense set.

Since $x_* \ll x(t)$ we obtain $f(t, x(t)) \in [f(t, x_*), C]$ and for any $i$ and for almost all $t$ we have
\[
\left\| \frac{1}{h} \int_t^{t+h} f(\xi, x(\xi))d\xi \right\|_i \leq \frac{1}{|h|} \left| \int_t^{t+h} \left\| f(\xi, x(\xi)) \right\|_i d\xi \right|
\]
\[
\leq \frac{1}{|h|} \left| \int_t^{t+h} \tilde{c}_i (\| f(\xi, x_*) + C \|_i + \| f(\xi, x_*) - C \|_i ) d\xi \right| \leq c_i(t). \tag{3.5}
\]
Here $\tilde{c}_i$ is a positive constant.
Let $\Theta_\psi$ be a set of values $t \in I_T$ for which formula (3.4) does not hold. The measure of $\Theta_\psi$ is equal to zero so that the measure of a set

$$\Theta = \bigcup_{\psi \in \Psi} \Theta_\psi$$

equals zero as well.

By the same reason a set

$$\tilde{\Theta} = \Theta \cup \{t \text{ for which formula (3.5) does not hold}\}$$

is of measure zero too.

Introduce a set $I_T' = I_T \setminus \tilde{\Theta}$. Consider an element

$$X_{t,h} = \frac{x(t + h) - x(t)}{h}$$

as a linear function on $E'$.

For each $t \in I'$ one therefore obtains $\|X_{t,h}\|_i \leq c_i(t)$.

Fix $t \in I_T'$. From formula (3.3) for any $\psi \in \Psi$ one gets

$$(\psi, X_{t,h}) \to \frac{d}{dt}(\psi, x(t)), \quad h \to 0.$$ 

From the Banach-Steinhaus theorem [3] it follows that the limit

$$(q(t), \psi) = \lim_{h \to 0} (\psi, X_{t,h})$$

exists for all $\psi \in E'$. For each $t \in I_T'$ the element $q(t) \in E''$. By reflexivity of $E$ we can regard it as $q(t) \in E$ and put $\dot{x} = q$.

The theorem is proved. $\square$

4. Appendix: A sketch of the Lebesgue integration theory for functions with values in Fréchet spaces. Let $(M, \Sigma, \mu)$ be a measure space with $\sigma-$algebra $\Sigma$ and a measure $\mu : \Sigma \to [0, \infty]$; and let $(F; \{\| \cdot \|_i\}_{i \in \mathbb{N}})$ be a Fréchet space.

**Definition 4.** We shall say that $f : M \to F$ is a step function if

1) $f$ has a finite set of values $\{a_1, \ldots, a_\nu\} \subset F$;

2) for each $k = 1, \ldots, \nu$ a set $A_k = f^{-1}(a_k)$ is measurable;

3) $\mu\{x \in M \mid f(x) \neq 0\} < \infty$.

The integral of a step function is defined as follows

$$\int f = \sum_{a_k \neq 0} a_k \mu(A_k).$$
The integral possesses the standard elementary properties.

It is clear that the step functions form a vector space which we denote by \( \text{St}(M, F) \). Being equipped with a collection of seminorms

\[
|f|_i = \int \|f\|_i, \quad \|f\|_i \in \text{St}(M, \mathbb{R})
\]

the space \( \text{St}(M, F) \) becomes a locally convex space.

**Definition 5.** We shall say that \( \{f_n\} \subset \text{St}(M, F) \) is an approximating sequence for a function \( g : M \to F \) if

1) \( \{f_n\} \) is an \( L^1 \)–Cauchy sequence: for each \( \varepsilon > 0 \) and for each \( s \in \mathbb{N} \) there exists a number \( l \) such that

\[
i, m > l \implies |f_i - f_m|_s < \varepsilon,
\]

and

2) \( f_n \to g \) almost everywhere.

We use \( \mathcal{L} \) to denote the space of functions \( g \) that have an approximating sequence. The functions from \( \mathcal{L} \) we call integrable functions.

**Theorem 5.** Let \( \{f_n\} \subset \text{St}(M, F) \) be an \( L^1 \)–Cauchy sequence. Then it contains a subsequence \( \{f_{j_p}\} \) that converges almost everywhere to a function \( g : M \to F \) and for any \( \varepsilon > 0 \) there exists a measurable set \( P, \mu(P) < \varepsilon \) such that \( f_{j_p} \) converges to \( g \) uniformly in \( M \setminus P \) i.e. for any \( s \in \mathbb{N} \) one has

\[
\lim_{j, i \to \infty} \sup_{x \in M \setminus P} \|f_{j_p}(x) - g(x)\|_s = 0.
\]

**Proof of Theorem 5.** The idea of Lemma 3.1 [6] is not destroyed if we substitute a seminorm \( \| \cdot \|_s \) instead of the norm in its proof. Consequently from the proof of this lemma we conclude that the sequence \( \{f_n\} \) contains a subsequence \( f_{n}^1 \) such that for almost all \( x \in M \) one has

\[
\|f_{n}^1(x) - f_{m}^1(x)\|_1 \to 0 \quad \text{as} \quad n, m \to \infty
\]

and for some measurable \( P_1, \mu(P_1) < \varepsilon/2 \) it follows that

\[
\lim_{j, i \to \infty} \sup_{x \in M \setminus P_1} \|f_{j}^1(x) - f_{i}^1(x)\|_1 = 0.
\]

Repeating the argument we extract a subsequence \( \{f_{n}^2\} \subset \{f_{n}^1\} \) such that for almost all \( x \in M \) one has

\[
\|f_{n}^2(x) - f_{m}^2(x)\|_2 \to 0 \quad \text{as} \quad n, m \to \infty
\]

and for some measurable \( P_2, \mu(P_2) < \varepsilon/2^2 \) it follows that

\[
\lim_{j, i \to \infty} \sup_{x \in M \setminus P_2} \|f_{j}^2(x) - f_{i}^2(x)\|_2 = 0
\]
and so forth.

Thus for almost all \( x \in M \) the diagonal sequence \( f_{j_p}(x) := f_p^p(x) \) is a Cauchy sequence in \( F \) and therefore it is convergent to some function \( g : M \to F \) almost everywhere. It is also clear that

\[
P = \bigcup_{i=1}^{\infty} P_i, \quad \mu(P) < \varepsilon.
\]

The theorem is proved.

\[\square\]

**Theorem 6.** Let \( \{f_n\}, \{h_n\} \subset \text{St}(M, F) \) be \( L^1 \)-Cauchy sequences that converge almost everywhere to the same function \( f \).

Then one has

\[
\lim_{n \to \infty} \int f_n = \lim_{n \to \infty} \int h_n.
\]

The existence of the limits above is obvious: for any \( s \in \mathbb{N} \) we have

\[
\left\| \int f_n - \int f_m \right\|_s \leq \int \|f_n - f_m\|_s \to 0 \quad \text{as} \quad m, n \to \infty.
\]

Thus \( \int f_n \) is a Cauchy sequence in \( F \).

**Proof of Theorem 6.** The argument of Lemma 3.2 [6] is adapted in the same way: we carry out all the steps of the proof replacing the norm with an arbitrary seminorm from the family \( \{\| \cdot \|_s\} \). This gives

\[
|f_n - h_n|_s \to 0 \quad \text{as} \quad n \to \infty, \quad \forall s \in \mathbb{N}.
\]

The theorem is proved.

\[\square\]

The last theorem allows to define the integral of a function \( f \in \mathcal{L} \). Indeed, let \( f_n \) be an approximating sequence for \( f \). Then by definition put

\[
\int f = \lim_{n \to \infty} \int f_n.
\]

Observe also that for each \( s \) we have

\[
\|f_n\|_s - \|f\|_s \leq \|f_n - f\|_s \to 0
\]

almost everywhere and

\[
\int \|f_n\|_s - \|f_m\|_s \leq \int \|f_n - f_m\|_s \to 0
\]

as \( m, n \to \infty \). So that \( \|f\|_s \) is an integrable function in the standard Lebesgue sense (\( \|f_n\|_s \) is an approximating sequence for \( \|f\|_s \)).

**Lemma 6.** If \( \{f_n\} \) is an approximating sequence for \( f \) then it converges to \( f \) relative the seminorms \( |\cdot|_s \) i.e. for any \( s \in \mathbb{N} \) it follows that \( |f - f_n|_s \to 0 \).
Indeed, fix \( n \); a sequence \( h_{in}^s = \| f_i - f_n \|_s \) is an approximating sequence for \( \| f - f_n \|_s \):

\[
\int |h_{in}^s - h_{jn}^s| \leq \int \| f_i - f_j \|_s;
\]

\( h_{in}^s \to \| f - f_n \|_s \) almost everywhere as \( i \to \infty \)

and

\[
\int \| f - f_n \|_s = \lim_{i \to \infty} \int h_{in}^s.
\]

By the definition of an approximating sequence, the integral \( \int h_{in}^s \) vanishes as \( i, n \to \infty \).

**Theorem 7.** The space \( \mathcal{L} \) is complete relative the seminorms \( | \cdot |_s \).

**Proof of Theorem 7.** Let \( \{ g_n \} \subset \mathcal{L} \) be an \( L^1 \)–Cauchy sequence. There exists a sequence \( \{ f_n \} \subset \text{St}(M, F) \) such that

\[
\max_{1 \leq i \leq n} |f_n - g_n|_i < \frac{1}{n}.
\]

The sequence \( f_n \) is an \( L^1 \)–Cauchy sequence: for any \( s \) it follows that

\[
|f_j - f_i|_s \leq |f_j - g_j|_s + |f_i - g_i|_s + |g_j - g_i|_s.
\]

From Theorem 5 we see that there is a subsequence \( \{ f_{n_l} \} \subset \{ f_n \} \) such that \( f_{n_l} \to g \) almost everywhere.

The sequence \( \{ f_{n_l} \} \) is an approximating sequence for \( g \). By Lemma 6 it follows that \( |f_{n_l} - g|_s \to 0 \).

To conclude the proof observe that

\[
|g_n - g|_s \leq |g_n - f_n|_s + |f_{n_l} - f_n|_s + |f_{n_l} - g|_s.
\]

**Theorem 8.** Suppose that a sequence \( \{ f_n \} \subset \mathcal{L} \) converges to an element \( f \in \mathcal{L} \) relative the seminorms \( \{ | \cdot |_s \} \). Then it contains a subsequence \( \{ f_{j_p} \} \) that converges almost everywhere to \( f \) and for any \( \varepsilon > 0 \) there exists a measurable set \( P, \mu(P) < \varepsilon \) such that \( f_{j_p} \) converges to \( f \) uniformly in \( M \setminus P \).

The proof of this theorem repeats the proof of Theorem 5.

Theorem 8 implies in particular that for \( f \in \mathcal{L} \) the following two facts are equivalent:

\[
(f = 0 \text{ almost everywhere}) \iff (|f|_s = 0 \ \forall s \in \mathbb{N}).
\]

Combining Theorems 8 and 7 we obtain

**Theorem 9.** Let a sequence \( \{ f_n \} \subset \mathcal{L} \) be an \( L^1 \)–Cauchy sequence that converges almost everywhere to a function \( f \). Then \( f \in \mathcal{L} \) and \( |f_n - f|_s \to 0 \) for any \( s \).
Theorem 10. (Dominated convergence theorem) Assume that a sequence \( \{f_n\} \subset \mathcal{L} \) converges almost everywhere to a function \( f \). Assume also that there exists a sequence of Lebesgue integrable functions \( g_s : M \to \mathbb{R} \) such that the inequalities
\[
\|f_n\|_s \leq g_s
\]
hold for all \( n, s \in \mathbb{N} \). Then \( f \in \mathcal{L} \) and \( \{f_n\} \) converges to \( f \) relative the seminorms \( | \cdot |_s \).

Proof of Theorem 10. Indeed, introduce functions
\[
\psi^s_n(x) = \sup_{i,j \geq n} \|f_i(x) - f_j(x)\|_s.
\]
For almost all \( x \) we have
\[
\psi^s_n(x) \to 0
\]
as \( n \to \infty \) and \( 0 \leq \psi^s_n(x) \leq 2g_s(x) \). By the standard finite dimensional Lebesgue integration theory these functions are measurable. From the finite dimensional Dominated convergence theorem we obtain \( \int \psi^s_n \to 0 \) for each \( s \in \mathbb{N} \). Therefore \( \{f_n\} \) is an \( L^1 \)-Cauchy sequence and Theorem 9 concludes the proof.

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