Factorization theorems for the representations of the fundamental groups of quasiprojective varieties and some applications

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Abstract

In this paper, using Gromov-Jost-Korevaar-Schoen technique of harmonic maps to nonpositively curved targets, we study the representations of the fundamental groups of quasiprojective varieties. As an application of the above considerations we give a proof of a weak version of the Shafarevich Conjecture.

1 Introduction

It is known [11] that to every representation of the fundamental group of a smooth projective variety $X$ to a simple complex Lie group $G$ we can assign an equivariant harmonic map from the universal cover of $X$ to the corresponding symmetric space. If $G$ is a simple Lie group over an arbitrary local field $K$, then we can try to work with equivariant harmonic maps from the universal cover of $X$ to the corresponding Euclidean building $B$ instead.

Developing the theory of the equivariant harmonic maps from the universal cover of a Riemannian manifold $X$ to $B$, a Euclidean building, Gromov and Schoen in [15] and Korevaar and Schoen in [25] proved the following result:

**Theorem 1.1** Consider a Zariski dense representation of the fundamental group of $X$, $\pi_1(X)$, to some simple Lie group $G$ over $K$. Then if $\pi_1(X)$ acts on the Euclidean building $B$ corresponding to $G$ without fixed points, there exists an equivariant nonconstant harmonic map $U$ from the universal cover of the Riemannian manifold $X$ to $B$.

The main goal of this paper is to study the factorizations of Zariski dense representations of the algebraic fundamental group of $X$ to some simple Lie group $G$ using the theory of harmonic maps. The pioneer ideas in this direction belong to Eells and Sampson, Carlson and Toledo and Yau and Jost (see [23] for more detailed account on that). Later these ideas were developed by Corlette and Simpson [42], [41] and Mok [16], who uses the existence of a Kähler form on a symmetric space. Since there is no Kähler form on the building $B$ we cannot use Mok’s
technique [14] to get a holomorphic foliation on $X$. The Higgs bundles technique from [25] and [38] also does not work. Instead we use a generalization of the Clemens-Lefschetz-Simpson theorem from [38] and the Gromov-Schoen theory to prove the main theorem:

**Theorem 1.2** Let $\rho : \pi_1(X) \to G$ be a Zariski dense representation of the fundamental group of $X$ to some simple Lie group $G$ over $K$. Then

A. either the image of $\rho$ is in a maximal compact subgroup of $G$ or,

B. there exist:

1) a finite etale cover $X'$ of a blow up of $X$;

2) a smooth projective variety $Y$ of positive dimension less then or equal to the rank $r$ of $G$ over $K$;

3) a holomorphic map $h : X' \to Y$ such that $\rho : \pi_1(X') \to G$ factors through a representation of $\pi_1(Y)$.

We will always assume that the representation $\rho : \pi_1(X) \to G$ is centerless, namely the intersection of $\rho(\pi_1(X))$ and the center of $G$, $Z(G)$ is trivial.

We also partially extend this result to the case when $X \setminus D$ is a quasiprojective variety and $\rho : \pi_1(X) \to G$ is a representation with unipotent monodromy at infinity.

Recently Simpson and Corlette have proven the following theorem:

**Theorem 1.3** Let $S$ be a smooth projective variety and $\rho$ be a Zariski dense representation of $\pi_1(S)$ in $SL(2, \mathbb{C})$. Then one of the following is true:

1) $\rho$ is a pullback from an orbicurve;

2) $\rho$ is a pullback from a Hilbert modular variety.

This result was partially generalized in [25] to higher rank complex groups.

Let $\rho : \pi_1(X) \to G$ be a Zariski dense nonrigid representation of the fundamental group of $X$ to some complex simple Lie group $G$. Then there exists a curve $I$ in the moduli space of representations passing through $\rho : \pi_1(X) \to G$ in direction of which $\rho$ deforms nontrivially. Since the moduli space of representations is an affine variety the curve $I$ is affine as well. Let $\bar{I}$ be the compactification of $I$. Consider now the representation $\bar{\rho}$ to $\bar{G}$, where $\bar{G}$ is defined over the field of fractions of $I$. Let $O_p$ be the localization at some point $p \in \bar{I} \setminus I$. Let $Z[T]$ be an extension of the ring of integers of $O_p$ which contains all coefficients of $\bar{\rho}$ localized at the point $p$. Consider now $I$ as a curve over $\text{Spec}Z[T]$. The fact that $\rho : \pi_1(X) \to G$ is a Zariski dense nonrigid representation
implies that $\bar{\rho}$ is a Zariski dense nonrigid representation. Observe that there is still going to be a point $q$ in the curve $\bar{I}$ which is not in Spec$Z[T]$. Let $\chi(\bar{\rho})$ be a character of $\bar{\rho}$. Moving over Spec$Z[T]$ in $\bar{I}$ we see that $\chi(\bar{\rho})$ is unbounded at $q$. Therefore we conclude that the representation $\bar{\rho}$ to $\bar{G}$ is not contained in any bounded subgroup in $\bar{G}$.

Using Theorem 1.2 we were able to prove the following:

**Corollary 4.2** Let $\varrho : \pi_1(X) \longrightarrow G$ be a Zariski dense nonrigid representation of the fundamental group of $X$ to some complex simple Lie group $G$. Then there exist:

1) a finite etale cover $X'$ of a blow up of $X$;

2) a smooth projective variety $Y$ of positive dimension $l$ less or equal to the rank $r$ of $G$ over $\mathbb{C}$;

3) and a holomorphic map $h : X' \longrightarrow Y$ such that $\varrho : \pi_1(X') \longrightarrow G$ factors through a representation of $\pi_1(Y)$.

In the complimentary situation, namely when we consider a Zariski dense rigid representation of the fundamental group of $X$ to a complex simple Lie group $G$, following Simpson [38] we can assign to every Zariski dense rigid representation a new Zariski dense rigid representation $\bar{\varrho} : \pi_1(X) \longrightarrow G^1$ into a group $G^1$ defined over a local field. The procedure goes as follows.

Observe that the moduli space of representations is defined over $\mathbb{Q}$, and since we are working with a rigid representation we can find an isomorphic representation defined over $\mathbb{Q}$. Therefore we can assume that $K = \mathbb{Q}$. Let $E$ be a finite extension of $\mathbb{Q}$ defined to be the extension which contains all coefficients of our representation, and let $O$ denote the ring of integers in $E$. Let $E_p$ denote the field of fractions of the completion of $O$ at $p$, for some prime $p$. Let $G^1$ be the group of $E_p$ valued points of $G$ and we again use $\varrho$ for the representation $\varrho : \pi_1(X) \longrightarrow G^1$. Therefore we get:

**Corollary 7.1** Let $\varrho : \pi_1(X) \longrightarrow G$ be a Zariski dense rigid representation. Then one of the following holds:

A. For every prime $p$ the image of $\varrho : \pi_1(X) \longrightarrow G^1$ is contained in a maximal compact subgroup in $G^1$.

B. For some prime $p$ the image of $\varrho : \pi_1(X) \longrightarrow G^1$ is not contained in a maximal compact subgroup in $G^1$. Then there exist:

1) a finite etale cover $X'$ of a blow up of $X$;

2) a smooth projective variety $Y$ of positive dimension $l$ less or equal to the rank $r$ of $G$ over $\mathbb{C}$;
3) and a holomorphic map \( h : X' \to Y \) such that \( \varphi : \pi_1(X') \to G \) factors through a representation of \( \pi_1(Y) \).

The paper is organized as follows:

The first part of the paper contains 4 sections. After the introduction in section 2 we prove the Clemens-Lefschetz-Simpson theorem which generalizes a previous result by Simpson and is a main tool in our proofs. We give a brief explanation of the Gromov-Korevaar-Schoen-Jost theory in section 3. The proof of our main theorem is given in section 4.

We consider some applications of the above results in the second part of the paper. Section 5 contains the proof of the main theorem in the quasiprojective case.

Following the general philosophy of studying the fundamental groups of projective varieties by studying their representations, we consider in section 6 the representations of the fundamental groups of projective varieties onto the fundamental groups of some negatively curved polyhedra defined by Benakli in [4]. Using the Clemens-Lefschetz-Simpson theorem we give a partial answer to a question posed by Gromov in [17], proving the following theorem for the Benakli’s complexes:

**Theorem 6.1** Every representation of the fundamental group of a projective variety onto the fundamental group of some negatively curved 2-dimensional polyhedra, comes from the representation of the fundamental group of an orbicurve.

In section 7 using a recent work of N. Katz [24] we prove some generalizations of the result of Corlette and Simpson. We say that a representation is integral if it is conjugate to a representation whose matrix coefficients are algebraic integers and that a representation is of dimension one if the corresponding harmonic map to the building \( B \) is of rank one.

We were not able to get further then the following:

**Theorem 7.1** Let \( \varphi : \pi_1(X) \to G \) be a rigid Zariski dense representation of dimension one. Then it is integral, in other words it is conjugate to a representation whose matrix coefficients are algebraic integers.

The idea is that since the algebraic curves do not have many rigid representations something which factors through them does not have either.

In the last section we make a connection with the theory of Shafarevich maps

\[ Sh : X \to Sh(X) \]

developed by János Kollár [27], [28]. We consider also the relative version of this map.
Sh^H : X \rightarrow Sh^H(X),

where $H$ is a normal subgroup in $\pi_1(X)$.

Using Theorems 1.2 and 1.3 we were able to estimate the dimension of the image of the Shafarevich map:

**Theorem 8.2** Let $X$ be a smooth projective variety which has a type B (from Theorem 1.2) Zariski dense representation $\rho$ of its fundamental group to a Lie group $G$ defined over a local field $K$. Define $H = \ker(\rho)$. Then:

$$\dim_{\mathbb{C}} Sh^H(X) = \dim_{\mathbb{C}}(Y) \leq \text{rank}_{\mathbb{C}}(G),$$

where $Y$ is the variety defined in Theorem 1.2. Moreover there exists a finite nonramified covering of $Y$ which is birationally isomorphic to a finite nonramified covering of $Sh^H(X)$.

Using the above theorem we are able to show that in some cases the Shafarevich morphism (in the sense of J. Kollár) exists. Namely, we prove the following theorem:

**Theorem 8.3** Let $X$ be a smooth projective variety which has a type B (from Theorem 1.2) Zariski dense representation $\rho$ of its fundamental group to a Lie group $G$ defined over a local field $K$. Define $H = \ker(\rho)$. Then: $Y$ is isomorphic to $Sh^H(X')$ and the map

$$Sh^H : X' \rightarrow Sh^H(X'),$$

is a morphism. Here $X'$ and $Y$ are the same as in Theorem 1.2.

The Shafarevich conjecture says that for every smooth projective variety $X$ there exists a Stein manifold $Sh(\tilde{X})$ and a proper map with connected fibers $Sh : \tilde{X} \rightarrow Sh(\tilde{X})$. It was shown in [27] and [28] that the following are equivalent:

1) There exists a space $Sh(\tilde{X})$ with no compact complex subspaces and a proper map with connected fibers $Sh : \tilde{X} \rightarrow Sh(\tilde{X})$. This differs from the Shafarevich conjecture since we have weaken the requirement about $Sh(\tilde{X})$ being Stein.

2) There exists a holomorphic map $Sh : X \rightarrow Sh(X)$, which contracts all subvarieties $Z$ in $X$ having the property that $\text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$ is finite.

(We use $Sh : X \rightarrow Sh(X)$ for the Shafarevich morphism, instead of $Sh : X \rightarrow Sh(X)$, which we use for the rational map defined by János Kollár.) The existence of the morphism

$$Sh : X \rightarrow Sh(X)$$
for $X$ smooth projective was conjectured by János Kollár in [27] and [28]. This conjecture we call a weak version of the Shafarevich conjecture or Shafarevich-Kollár conjecture.

Therefore using theorem 8.3 we obtain the following:

**Corollary 8.1** If in the condition of the previous theorem we have that $H = \ker(\rho)$ is finite then Shafarevich-Kollár conjecture is true.

The Shafarevich-Kollár conjecture in general is connected with another question of Gromov:

**Question.** Can we find a faithful discrete cocompact action of every word hyperbolic Kähler group on a space with $K < 0$?

The theorem of Rips [17] gives an answer to this question in some cases. In these cases, using the technique developed in this paper one can try to show that the Shafarevich-Kollár conjecture follows for every Kähler group for which the map:

$$U : \tilde{X} \rightarrow T$$

is pluriharmonic.

We were able to realize the above idea for fundamental groups, which have faithful discrete cocompact action on trees and 2-dimensional negatively curved 2-complexes discussed in section 6.

As a consequence we obtain the following nonvanishing theorem:

**Theorem 8.4** Let $X$ be a smooth fourfold of general type which has a type B. (Theorem 1.2) Zariski dense representation of its algebraic fundamental group, $\rho : \pi_1(X) \rightarrow G$, where $G$ is a simple Lie group of rank two or three over a local field $K$. Then either:

1) $\rho : \pi_1(X) \rightarrow G$ factors through a representation of the fundamental group of an orbicurve, or

2) $P_n(X) := H^0(X, nK_X)$ is not zero for $n \geq 4$.

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**Remark 1.1** Recently we were informed by J. Jost and K. Zuo that they have obtained results similar to our results from section 4.
2 Lefschetz theorem

In this section we generalize a result of Simpson [38]. The Clemens-Lefschetz-Simpson theorem we prove is of independent interest but it is also one of the key arguments in the next sections.

Let $X$ be a smooth projective variety and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be linearly independent holomorphic one-forms in $H^0(X, \Omega_X^1)$. Denote by $\tilde{X}$ the universal covering of $X$. Then by integrating the pullbacks of the forms $\alpha_1, \alpha_2, \ldots, \alpha_n$ over $\tilde{X}$ we can define a map

$$g : \tilde{X} \rightarrow \mathbb{C}^n.$$ 

Let us first consider the case $n = 2$.

**Theorem 2.1** Let $\alpha_1, \alpha_2$ be two linearly independent holomorphic one forms. Then one of the following cases holds:

a) The map $g$ has connected fibers.

b) There is a holomorphic map with connected fibers from $X$ to a normal projective variety $Y$ such that $1 \leq \dim Y \leq 2$ and the forms $\alpha_1, \alpha_2$ are pullbacks from $Y$.

c) The third case is a combination of the previous two. Namely, up to a linear change of the coordinates in $\mathbb{C}^2$ the first form $\alpha_1$ is a pullback by a map $X \rightarrow Y$, where $Y$ is an algebraic curve. The second form $\alpha_2$ gives us a map $\tilde{X} \rightarrow \mathbb{C}$ with connected fibers.

**Proof.** We borrow heavily from Simpson’s arguments from [38], where he proves the case $n = 1$.

Let $Alb(X)$ be the Albanese variety of $X$. The forms $\alpha_1, \alpha_2$ are pullbacks of forms on $Alb(X)$, which we will also call $\alpha_1, \alpha_2$. Define $B$ as the biggest abelian subvariety over which both forms $\alpha_1$ and $\alpha_2$ considered as a sections of $\Omega^1_{Alb(X)}$ are zero. Let

$$A = Alb(X)/B.$$ 

We have the natural map

$$a : X \rightarrow A.$$ 

Let $l : \tilde{X} \rightarrow X$ be the universal covering map.

Suppose first that the dimension of $A$ is zero. But this means that the forms $\alpha_1, \alpha_2$ are the zero forms. Therefore $\dim_{\mathbb{C}} A \geq 1$.

Consider the map $g$. Let $Q$ be the subset of $\mathbb{C}^2$ over which the map $g$ has multiple fibers. From the construction of $g$ it follows that $Q$ is a constructible set in $\mathbb{C}^2$. We show now that if $\dim_{\mathbb{C}} Q > 0$ then either case b) or c) of the theorem holds.
Assume that $\dim \mathbb{C} Q > 0$. Then we have that $Q$ is an analytic subvariety of $\mathbb{C}^2$ and $\dim \mathbb{C} Q = 1$.

We consider now the composition of the maps

$$g : \tilde{X} \to \mathbb{C}^2 \to \mathbb{C}^1,$$

where the map

$$p : \mathbb{C}^2 \to \mathbb{C}^1 = \mathbb{C}^2/l$$

is just a quotient by some line $l$ in $\mathbb{C}^2$.

Let us choose coordinates $t_1$ and $t_2$ in $\mathbb{C}^2$ so that $\alpha_1 = g^*(dt_1)$ and $\alpha_2 = g^*(dt_2)$. There are two main cases to be dealt with.

Case $\alpha$) There is a line

$$l : c_1 t_1 + c_2 t_2 = 0,$$

contained in $Q$. Here $c_1$ and $c_2$ are some complex numbers.

In this case the map

$$g : \tilde{X} \to \mathbb{C}^2 \to \mathbb{C}^1$$

is given by integration of a holomorphic one form. Applying Simpson’s version from [38] we obtain a map $X \to Y$, where $Y$ is an algebraic curve. This places us in cases b) or c) of the theorem. We will be in case b) if there exists another line in $Q$, which is transversal to $l$. Otherwise we get case c) of the theorem.

Case $\beta$) $Q$ does not contain a line.

Let $x$ be a smooth point in $Q$. Then we need to consider two possibilities.

1. The differential $g_*\gamma$ is the zero map on $T\tilde{X}_y$ for every $y$ from $g^{-1}(x)$. We show that in this situation we have a map $X \to Y$, where $Y$ is an algebraic surface. Consider the map

$$a : X \to A.$$

Let the map $f : X \to Y$ be the Stein factorization of $a$. Pulling back the map $f : X \to Y$ to the universal coverings of $X$ and $Y$, $\tilde{X}$ and $\tilde{Y}$, we get that the map

$$g : \tilde{X} \to \mathbb{C}^2$$

factors through the map

$$\tilde{f} : \tilde{X} \to \tilde{Y}.$$
Suppose now that \( \dim_{\mathbb{C}} Y > 2 \). Obviously a connected component of the fiber of \( \tilde{f} \) maps to a component of \( v \), where \( v \) is the map

\[
v : \tilde{Y} \rightarrow \mathbb{C}^2,
\]

with the property that \( g = \tilde{f} \circ v \). Note that under the assumption \( \dim_{\mathbb{C}} Y > 2 \) we have that dimension of the fiber of \( v \) is at least one. But the fact that \( T\tilde{X}_y \) for every \( y \) from \( g^{-1}(x) \) goes to a point under \( g_* \) means that the restrictions and projections of the forms \( \alpha_1 \) and \( \alpha_2 \) on a connected component of \( g^{-1}(x) \) are equal to zero. This implies that \( g^{-1}(x) \) goes to a discrete set of points under \( \tilde{f} \), which contradicts the fact that the dimension of the fiber of \( v \) is at least one. Therefore there exists a map \( X \rightarrow Y \), where \( Y \) is an algebraic surface, so we are in case b) of the theorem.

2. For every \( y \) from \( g^{-1}(x) \), \( g_* \) sends \( T\tilde{X}_y \) to the same line in \( T \times \mathbb{C}^2 \). Let \( N \) be the irreducible component of \( Q \) passing through \( x \). Since \( Q \) does not contain a line we can assume that the two form

\[
dt_1 \wedge dt_2
\]

is zero when restricted to \( N \) and \( dt_1, dt_2 \) are linearly independent at every point \( x \) in \( N \). In this case the Castelnuovo-De Francis theorem gives us a holomorphic map \( s : l(g^{-1}(N)) \rightarrow Y \), where \( Y \) is an algebraic curve. We can see that the map \( g : g^{-1}(N) \rightarrow \tilde{Y} \) factors through the map \( \tilde{s} : g^{-1}(N) \rightarrow \tilde{Y} \). Consider now the map

\[
a : X \rightarrow A.
\]

Let \( J = a(X) \). We prove next that \( \dim_{\mathbb{C}} J \leq 2 \). Suppose not. From the construction of \( a \) we see that the map

\[
g : \tilde{X} \rightarrow \mathbb{C}^2
\]

factors through the maps

\[
\tilde{a} : \tilde{X} \rightarrow \tilde{J},
\]

\[
v : \tilde{J} \rightarrow \mathbb{C}^2.
\]

To get a contradiction we need to show that there exists a fiber \( V \) of the map \( v \) such that \( \dim_{\mathbb{C}} V = 0 \). We know that \( \tilde{Y} \) is contained in \( \tilde{J} \). But by the universality of the Albanese map for \( l(g^{-1}(N)) \) we can see that the map

\[
s : l(g^{-1}(N)) \rightarrow Y
\]
is nothing else but the map
\[ a : X \longrightarrow A, \]
restricted on \( l(g^{-1}(N)) \). Recall that the map
\[ g : \tilde{X} \longrightarrow \mathbb{C}^2 \]
was the pullback of \( a : X \longrightarrow A \) to the universal covers of \( X \) and \( A \). Consider now a point \( x \) in \( N \). It follows from the construction that \( g^{-1}(x) \) is contained \( g^{-1}(N) \). We conclude that \( l(g^{-1}(x)) \) has discrete image under \( s \) and therefore it has a discrete image under the map
\[ \tilde{a} : \tilde{X} \longrightarrow \tilde{J}. \]
In this way we obtain a fiber \( V = \tilde{a}(g^{-1}(x)) \) of the map \( v \) such that \( \dim_{\mathbb{C}} V = 0 \). Therefore \( \dim_{\mathbb{C}} J \leq 2 \) and we are in case b) of the theorem.

So far we have shown that the map
\[ g : \tilde{X} \longrightarrow \mathbb{C}^2 \]
has no multiple fibers over an open set whose complement is of codimension greater than two. Now we show that in this case part a) of the theorem holds. To finish the argument we use the discussion of Gromov and Schoen \[ 15 \] paragraph 9 about the Stein factorization for nonproper varieties. The only thing we need to check is if the corresponding leave space is Hausdorff. This follows easily from Lemma 9.3 in \[ 15 \]. After that the version of the Stein factorization theorem from (see Theorem 3 \[ 7 \]) implies that the map \( g \) exists, namely we have \( g \) factoring as
\[ g' : \tilde{X} \longrightarrow \mathbb{C}^2 \]
and a finite covering
\[ m : \mathbb{C}^2 \longrightarrow \mathbb{C}^2. \]
To show that the map \( g \) has connected fibers we need to show that \( m \) is an isomorphism. But since the branch locus of \( m \) produces a multiple fibers of \( g \) we conclude that \( m \) is etale outside some subset in \( \mathbb{C}^2 \) of codimension two. Then due to the purity theorem (SGA) we conclude that \( m \) is etale. But \( \pi_1(\mathbb{C}^2) = 0 \) and hence \( m \) is biholomorphic.

Another way to see that \( m \) is biholomorphic is to observe that since \( Q \) has real codimension at least 4 then \( \mathbb{C}^2 \setminus Q \) is simply connected and therefore it has one, namely the trivial nonramified covering. Therefore \( g \) has connected fibers in codimension two. We finish the proof by observing that semicontinuity ensures all fibers of \( g \) are connected. \( \square \)
Now we state the Clemens-Lefschetz-Simpson theorem for arbitrary \( n \). We sketch a proof emphasizing the details where it differs from the case \( n = 1 \).

Recall that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are \( n \) are holomorphic one forms on \( X \) and the map

\[
g : \tilde{X} \to \mathbb{C}^n
\]

was defined on \( \tilde{X} \) by integrating them on \( \tilde{X} \).

**Theorem 2.2 (Clemens-Lefschetz-Simpson)** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be linearly independent holomorphic one forms on \( X \). Then one of the following cases holds:

a) The map \( g \) has connected fibers.

b) There is a holomorphic map with connected fibers from \( X \) to the projective normal variety \( Y \) such that \( 1 \leq \dim Y \leq n \) and the forms \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are pullbacks from \( Y \).

c) The third case is a combination of the previous two. Namely after a linear change of the coordinates in \( \mathbb{C}^n \) some of the forms \( \alpha_1, \ldots, \alpha_k \) come as a pullbacks of a map \( X \to Y \), where \( Y \) is an algebraic variety \( 1 \leq \dim Y \leq k < n \). The rest of the forms give us a map \( g' : \tilde{X} \to \mathbb{C}^{n-k} \), with connected fibers.

**Proof.** The proof splits again into two cases.

1. \( Q \) contains a hyperplane - then we reduce to the Simpsons’s version of the theorem.

2. \( Q \) does not contain a hyperplane. We argue in the same way as in the case of \( n = 2 \). Instead of applying the Castelnuovo De Francis theorem we apply a generalization of it due to Z. Ran.

**Theorem 2.3 (Z. Ran)** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be linearly independent holomorphic one forms on \( X \) and such that

\[
\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n = 0.
\]

Then there exists a complex torus \( A \) and an analytic map

\[
f : X \to A,
\]

such that \( f(X) \) is a proper analytic subvariety of \( A \), \( \dim f(X) \leq n \), and the forms \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are pullbacks from \( A \).

\[\square\]

In section 5 we give a version of the Clemens-Lefschetz-Simpson theorem for quasiprojective varieties.
3 Harmonic maps to buildings—basic facts

3.1

Following Gromov-Schoen and Korevaar-Schoen-Jost we briefly describe some facts of the theory of the harmonic maps to buildings, which are essential for our discussion. For a more detailed treatment one can look at their original papers [15], [29]. Also in this section we define a new object: a spectral covering, which we assign to every nonconstant harmonic map to a building.

Suppose from now on that $X$ is a smooth projective variety and $\tilde{X}$ is the universal cover of $X$. Let $G$ be a simple Lie group of rank $r$ over $\mathbb{Q}_p$ and let $B$ be the corresponding Euclidean building. (For definitions and more detailed account on buildings see [3].) $G$ acts on $B$ by automorphisms and if there exists a representation of $\pi_1(X)$ in $G$, so does $G$.

Following [15] one can define an energy functional for every equivariant continuous nonconstant map $U: \tilde{X} \to B$ and we can ask if there exists a minimum for this functional - the harmonic map. The following remarkable result gives the answer to this question.

**Theorem 3.1 (Gromov-Schoen)** If the action of $\pi_1(X)$ on the building has no fixed point, then there exists an equivariant harmonic map $U: \tilde{X} \to B$.

**Proof.** See [15].

Another important result from [15] which we will need is the following:

**Theorem 3.2 (Gromov-Schoen)** The singular set of the equivariant harmonic map $U: \tilde{X} \to B$ is of codimension at least two.

All these results hold in the case when the building is locally compact. For some of our considerations we need the same results in the nonlocally compact situation. In this case we use the following:

**Theorem 3.3 (Korevaar-Schoen)** The previous two theorems hold in the case of length spaces of non-positive curvature, including nonlocally compact buildings.

**Proof.** The theorem follows from proposition 2.6.5 in [29]. The actual proof was given by R. Schoen in his course at IAS Princeton 1992-1993 and will appear elsewhere. R. Schoen informed me that the same result was proven J. Jost.

Using this map $U$ we construct a new object - a spectral covering. As we will see later this construction is not unique.

In every apartment $Ap$ of the building we have locally a canonical choice of coordinates modulo similarity maps and actions of the affine Weyl group $\tilde{W}$. This means that locally we can choose
one coordinate $z_1$ and consider also $e_1(z_1), \ldots, e_w(z_1)$, where $e_1 \ldots e_w$ are the elements of the usual Weyl group, and $w=$ number of the elements in $W$. Let $U^*T^*Ap$ be the pullback of the cotangent bundle of $Ap$ and let $d$ denote the corresponding exterior derivative in $T^*Ap$. Then we can define, modulo the action of $W$, the differential forms $de_1(z_1), \ldots, de_w(z_1)$ globally on the whole building. Observe that after taking differentials we can make everything invariant under the translations of the affine Weyl group. Consider the complexified differentials $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. Due to the harmonicity of $U$ the $(1,0)$ part of each of the forms $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ is holomorphic. (This follows essentially from the fact that due to the Corlette vanishing theorem (see [15]), $U$ is actually pluriharmonic outside the singular set). We will denote these new holomorphic one forms again by $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. Since $U$ is an equivariant map these holomorphic one forms descend to forms on $X$, defined modulo a $W$-action. Let $h_1 \ldots h_r$ be a basis for all invariant polynomials of $G$. We apply them to $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. According to [15] the map $U$ is Lipschitz and therefore we get that the holomorphic differentials $h_1, \ldots, h_r$ are bounded. Using Theorem 3.2 we conclude that $h_1, \ldots, h_r$ extend to a holomorphic differentials on the whole $X$, namely $h_1, \ldots, h_r$ extend to elements of $H^0(X, \text{Symm}^{d_1} \Omega^1_X), \ldots, H^0(X, \text{Symm}^{d_r} \Omega^1_X)$ respectively.

**Remark 3.1** If the group $G$ is not simply connected one needs include in $W$ the fundamental group of $G$ as well.

### 3.2

Here we give one way of defining a spectral covering:

We define a spectral covering corresponding to the elements $h_1, \ldots, h_r$ from $H^0(X, \text{Symm}^{d_1} \Omega^1_X), \ldots, H^0(X, \text{Symm}^{d_r} \Omega^1_X)$ respectively to be the zero scheme $\mathcal{S}$ of the section $\det (\rho(\theta) - \lambda \cdot \text{id}_V)$, where $\text{id}_V$ is the identity in a vector space $V$ such that $\dim V = w$. Here by $\det (\rho(\theta) - \lambda \cdot \text{id}_V)$ we mean the following:

1. For every element of $G$ the coefficients of the characteristic polynomial $\det (\rho(\theta) - \lambda \cdot \text{id}_V)$ (here $\lambda$ is a number ) are combinations of the invariant polynomials $h_1, \ldots, h_r$.

2. Now if we interpret $h_1, \ldots, h_r$ as elements from $H^0(X, \text{Symm}^{d_1} \Omega^1_X), \ldots, H^0(X, \text{Symm}^{d_r} \Omega^1_X)$ and $\lambda$ as the tautological section of $T^*X$ we obtain $\det (\rho(\theta) - \lambda \cdot \text{id}_V)$ as an element from $H^0(T^*X, \text{Symm}^w \pi^* \Omega^1_X)$.

For more detailed account on spectral coverings see [15].

Some of these coverings may be nonreduced and reducible. In such cases we work with $S' = \mathcal{S}_{\text{red}}$. Finally we take $S$ - the equivariant desingularization of $S'$. (Note that the existence of this desingularisation follows from the equivariant version of the famous Hironaka’s theorem.)
**Definition 3.1** S is called the factorizing spectral covering corresponding to U.

For the proof of the main theorem we need only the following three properties of S:

1) The forms $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ are well defined holomorphic one forms on S;

2) There is a $W$ action on at least an open set of S;

3) S is smooth.

Therefore we can define S as a finite covering of a blow-up of X with the above three properties. An alternative way of constructing S is given in [13] and [45]. We describe this construction now.

First choose r generically linearly independent holomorphic one forms out of $U^*de_1(z_1), \ldots, U^*de_w(z_1)$. The multivalued form $U^*de_1(z_1), \ldots, U^*de_r(z_1)$ defines an $r$-fold covering $S'$ in $T^*X$. Now we define S to be the Galois closure of the function field extension $\mathbb{C}(S')/\mathbb{C}(X)$.

It is clear that from the above definitions that the spectral covering is not unique. In the proof of the main theorem we show that the factorizing properties of a given representation do not depend on the spectral covering we have chosen.

The following fact is going to play an essential role in the proof of the Shafarevich-Kollár Conjecture.

**Lemma 3.1** The singular set $Q$ of the map $U' : S \rightarrow B$ is contained in the union of the zeros of $U^*de_1(z_1), \ldots, U^*de_w(z_1)$.

**Proof.** The proof follows from the fact that according to Gromov and Schoen $\text{codim}_S Q \leq 2$ and that $U'(Q)$ is contained in the faces of the chambers of B. The existence of the intrinsic derivative, proved in [13] and [29], implies existence of a kernel $T^*S$ and therefore $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ are equal to zero on $Q$. \hfill $\square$

**Remark 3.2** Observe that over some open set in $S$, namely outside $Q$ we can use the Castelnuovo-de Francis theorem to get a factorization (see [25]).

**3.3 Example**

Let us consider the case of the group $SL(3, \mathbb{Q}_p)$. The Weyl group in this case is going to be $S_3$. Then the coordinates $e_1(z_1) \ldots e_6(z_1)$ can be thought as six roots of unity on the face of $B$, which in this case is nothing else but an equilateral triangle. The complexified differentials $U^*de_1(z_1), \ldots, U^*de_6(z_1)$ are $U^*de_1(z_1), \ldots, U^*de_3(z_1), -U^*de_1(z_1), \ldots, -U^*de_3(z_1)$.

In this case we have
$$h_1 = -(U^*de_1(z_1)^2 + U^*de_2(z_2)^2 + U^*de_3(z_3)^2)$$
$$h_2 = U^*de_1(z_1)^2. U^*de_2(z_2)^2. U^*de_3(z_3)^2.$$ 

The spectral covering $\tilde{S}$ in this case is given by the following section in $H^0(X, \text{Symm}^6\Omega^1_X)$:
$$\lambda^6 - h_1\lambda^4 + h_2^2/2\lambda^2 - h_2^2,$$
where $\lambda$ is the tautological one form.

If, for example, $U^*de_1(z_1) = U^*de_2(z_2) \neq 0$, then the factorizing spectral covering $S$ is the desingularization of the zeros of the section $\lambda^2 - U^*de_1(z_1)^2$, which belongs to $H^0(X, \text{Symm}^2\Omega^1_X)$.

4 The main theorem and its proof

4.1

The main idea is to use the abundance of holomorphic one forms over the factorizing spectral covering and to apply to it the Clemens-Lefschetz-Simpson theorem we have proven above. This gives the factorization over an open set of $X$. To finish the proof we use a new technique developed by János Kollár in [27] and [28].

Recall some notation from section 2. Let $X$ be a smooth projective variety and let $\tilde{X}$ be the universal cover of $X$. Let $G$ be a simple Lie group of rank $r$ over an arbitrary local field of characteristic zero, and let $B$ be the corresponding Euclidean building ($B$ could be non locally finite). $G$ acts on $B$ by isometries and if we have a representation $\varrho : \pi_1(X) \rightarrow G$, so does $\pi_1(X)$. If $\pi_1(X)$ acts on $B$ without a fixed point then it follows from the theory of Gromov-Korevaar-Schoen-Jost that there exists a nonconstant equivariant harmonic map $U : \tilde{X} \rightarrow B$.

**Theorem 4.1** Let $U : \tilde{X} \rightarrow B$ be an equivariant harmonic map which corresponds to the Zariski dense representation $\varrho : \pi_1(X) \rightarrow G$. Then there exists a holomorphic map $x : S \rightarrow Y$ where $Y$ is a normal projective variety such that $1 \leq \dim(Y) \leq r$. Moreover there exist $S^0$ and $Y^0$ - Zariski open sets in $S$ and $Y$ respectively such that $\varrho : \pi_1(S^0) \rightarrow G$ factors through a representation $\varrho' : \pi_1(Y^0)) \rightarrow G$.

The fact that $\pi_1(X)$ acts on $B$ without fixed points allows us to define spectral covering $\tilde{S}$ the corresponding to $\varrho : \pi_1(X) \rightarrow G$. Take the corresponding $S$. Since the map $U : \tilde{X} \rightarrow B$ is a nonconstant map we have at least one holomorphic one form $\alpha_1$ on $S$.

If we have the forms $\alpha_1, \ldots, \alpha_q$, $1 \leq q \leq r$, then we are in position to apply the Clemens-Lefschetz-Simpson theorem.

Suppose that part a) or a) of this theorem holds in our situation.

Let $\tilde{S}$ be the universal cover of $S$. Consider the map $g : \tilde{S} \rightarrow \mathbb{C}^q$ defined as in section 2 by integration of $\alpha_1, \ldots, \alpha_q$ over $\tilde{S}$. Let $\text{re}(g)$ be the real part of this map. Obviously $U : \tilde{X} \rightarrow B$
provides us with the $\pi_1(S)$-equivariant harmonic map $U' : \tilde{S} \to B$.

**Lemma 4.1** The fiber of $re(g) : \tilde{S} \to \mathbb{R}^q$ maps to a point under $U'$.

**Proof.** We use an argument of [15], namely the local version of the Stein factorization theorem, which says that in a small neighborhood $\Omega$ in $\tilde{S}$ the map $g$ decomposes into a composition of a holomorphic map $\Omega \to D$, where $D$ is an open ball in $\mathbb{C}^q$, followed by a harmonic map $u : D \to B$. So we get that each intersection of the fiber of $re(g)$ with a small neighborhood $\Omega$ goes to a point in $B$. But now using the fact that the fiber is connected and all the maps are continuous we see that the lemma holds. Observe that this argument also works in the neighborhoods around the critical points of this map. What one does is first approximate the singular fibers by nonsingular ones, and then use the fact that $g$ is a continuous map. $\blacksquare$

Now using the above Lemma we construct an isometry $\mathbb{R}^q \to B$. Before we do this we show how we are going to use this isometry.

Recall that we are in case a) or c) of the Clemens-Lefschetz-Simpson theorem with $l = q$. The image of $U$ is all of $\mathbb{R}^q$. But this means that the action of $\pi_1(X)$ on the building $B$ preserves $\mathbb{R}^q$, so the action on the building at infinity preserves a flat. Hence it is contained in some parabolic subgroup in $G$. This contradicts the Zariski density of $\rho$. In fact one can prove this without going to the building at infinity. Fixing $\mathbb{R}^q$ is equivalent to fixing $\mathbb{R}^q$ pointwise up to the action of some finite group and this is again a contradiction.

This rules out the cases a) and c) of the Clemens-Lefschetz-Simpson theorem.

Now we prove the existence of the isometry $\mathbb{R}^q \to B$.

**Lemma 4.2** In the cases when part a) or c) of the Clemens-Lefschetz-Simpson theorem holds we always have an isometry $\mathbb{R}^q \to B$.

**Proof.** The case $q = 1$ was proven by Simpson [39]. Following our proof of the Clemens-Lefschetz-Simpson theorem we are going to work for simplicity with the case $q = 2$.

Start with the map $f : \tilde{S} \to \mathbb{C}$ defined by integration of $\alpha_1$ over $\tilde{S}$. Since the singular sets of the map $f$ are compact sets in $\tilde{S}$ and the fibers $F$ of $f$ are connected (recall we are in case a) or c) of the Clemens-Lefschetz-Simpson theorem), we can find a real line $\mathbb{R}$ in $\mathbb{C}$ over which the map $f$ has only smooth points. Now through every point of this real line we can define another real line in the fiber of the map $t : F \to \mathbb{C}$. Here we again use the fact that the singular sets of the map $t$ are compact sets in $F$ and the fibers $F'$ of $t$ are connected. This means that we can not only find such a line but also require that it passes through the point of our initial line $\mathbb{R}$ in $\mathbb{C}$. We can do this in a continuous way using the fact that the singular sets of the maps above are in codimension at least two. Since every $\mathbb{R}$-bundle over $\mathbb{R}$ is trivial we have a smooth $\mathbb{R}^2$ in $\tilde{S}$. Now using Lemma
4.1 we find smooth map $\mathbb{R}^2 \rightarrow B$. To show that this is an isometry we apply Simpson’s argument, which uses that the forms $\alpha_1$ and $\alpha_2$ are defined by $U$. Thus the differentials of $U$ and $\text{re}(g)$ are the same and therefore the differential of $U$ is equal to the identity. □

From now on we are going to work only with case B) of the Clemens-Lefschetz-Simpson theorem. We need first to construct the factorization map. Part b) of the Clemens-Lefschetz-Simpson theorem gives us a map $S \rightarrow Y$. So we have a $\pi_1(S)$-equivariant harmonic map $U : S \rightarrow B$. Let $\tilde{Y}$ be the universal cover of $Y$.

**Lemma 4.3** The map $U \iota : \tilde{S} \rightarrow B$ factors through a map $u_0 : \tilde{Y} \rightarrow B$, namely $U \iota = u_0.a$, where $a$ is the map $a : \tilde{S} \rightarrow \tilde{Y}$.

**Proof.** For simplicity we again consider the case $\dim_{\mathbb{R}} B = 2$ and two holomorphic one-forms $\alpha_1$, $\alpha_2$ such that $\alpha_1 \wedge \alpha_2 \neq 0$ generically on $\tilde{S}$. The forms $\alpha_1$, $\alpha_2$ come from holomorphic one-forms $\beta_1$, $\beta_2$ on $\tilde{Y}$. This way we get that the map $r_1 : \tilde{S} \rightarrow \mathbb{R}^2$, given by integration of $\alpha_1$, $\alpha_2$ over $\tilde{S}$, factors through the map $r_2 : \tilde{Y} \rightarrow \mathbb{R}^2$, given by integration of $\beta_1$, $\beta_2$ on $\tilde{Y}$. Namely we have $r_1 = r_2.a$. But then the fiber of the map $a : \tilde{S} \rightarrow \tilde{Y}$ is contained in the fiber $J$ of the map $r_1 : \tilde{S} \rightarrow \mathbb{R}^2$. By Lemma 4.1 $J$ is mapped to a point under $U$ and so is the fiber of the map $a$. This proves the factorization.

Define $X^0$ to be the Zariski dense set in $X$ obtained by throwing away the branch locus of the map $S \rightarrow X$ and the images in $X$ of the exceptional sets in $S$. Let $S^0$ be the preimage of $X^0$ in $S$. The fundamental group $\pi_1(S^0)$ maps to a group with finite index in $\pi_1(X^0)$. The representation $g(\pi_1(X))$ is Zariski dense in $G$ as is $g(\pi_1(X^0))$ since $\pi_1(X^0)$ surjects to $\pi_1(X)$. But since we know that $\pi_1(S^0)$ maps to a group with finite index in $\pi_1(X^0)$, we can conclude that $g(\pi_1(S^0))$ is Zariski dense in $G$.

From the previous lemma we have that the harmonic map $\tilde{U} : \tilde{S}^0 \rightarrow B$ factors through the map $\tilde{u}_0 : \tilde{Y^0} \rightarrow B$, namely $\tilde{U} = \tilde{u}_0.a$. (Here $\tilde{S}^0$ and $\tilde{Y^0}$, are the universal covers of $S^0$ and $Y^0$ respectively.) Using the properties of the pullback map for the inclusion $i : \tilde{S}^0 \rightarrow \tilde{S}$ and the fact that $\pi_1(S^0)$ surjects onto $\pi_1(S)$ we conclude that the map $\tilde{U} : \tilde{S}^0 \rightarrow B$ is equivariant with respect to $\pi_1(S^0)$.

We show that the action of $g : \pi_1(S^0)$ factors through an action of $g : (\pi_1(Y^0))$. Take $\gamma$ an element of $\pi_1(S^0)$ such that $a_\ast(\gamma) = 1$. Then for any $x$ in $\tilde{S}^0$ we have $a_\ast(\gamma.x) = a_\ast(x)$. We use the same notation: $a$ for both maps $a : S \rightarrow Y$ and $a : \tilde{S} \rightarrow \tilde{Y}$. Then using the equivariance of $U$ we have

$$g(\gamma)\tilde{U}(x) = \tilde{U}(\gamma.x) = \tilde{u}_0.a(\gamma.x) = \tilde{u}_0.a(x) = \tilde{U}(x).$$
In the same way one can see that $\bar{u}_0$ is equivariant for the action of $\varrho(\pi_1(Y^0))$ on $B$.

$$\bar{u}_0(a_*(\gamma)a(x)) = \bar{u}_0a(\gamma x) = \bar{U}(\gamma x) = \varrho(\gamma)\bar{U}(x) = \varrho(a_*(\gamma))\bar{U}(ax).$$

We need to show that $\varrho(\gamma) = 1$ if $\gamma$ is in $\text{Ker}(a_*)$. Recall that $a$ has connected fibers, so the map $a_* : \pi_1(S^0) \rightarrow \pi_1(Y^0)$ is a surjective map. But $\text{Ker}(a_*)$ is a normal subgroup in $\pi_1(S^0)$ thus $\varrho(\text{Ker}(a_*)))$ is normal in $G$. Since $G$ is a simple Lie group over $K$ there are two possibilities:

1) $\varrho(\text{Ker}(a_*)) = G$;

2) $\varrho(\text{Ker}(a_*))$ is contained in the center $Z(G)$.

Suppose that we are in case 1). But from the computation above we know that $\varrho(\text{Ker}(a_*))$ fixes $\bar{U}(X^0)$, an open set in $U(X)$, which is a contradiction since $G$ does not fix any point in $B$.

In case 2) we obtain that $\varrho(\gamma) = 1$ since we are working only with centerless representations.

Let $\varrho : \pi_1(X) \rightarrow G$ be a Zariski dense nonrigid representation of the fundamental group of $X$ to some complex simple Lie group $G$. Then there exists a curve $I$ in the moduli space of representations passing through $\varrho : \pi_1(X) \rightarrow G$ in direction of which $\varrho$ deforms nontrivially. Since the moduli space of representations is an affine variety the curve $I$ is affine as well and let $\bar{I}$ be the compactification of $I$. Consider now the representation $\bar{\varrho}$ to $\bar{G}$, where $\bar{G}$ is defined over the field of fractions of $I$. Let $O_p$ be the localization at some point $p \in \bar{I}$. Let $Z[T]$ be an extension of the ring of integers of $O_p$ which contains all coefficients of $\bar{\varrho}$ localized at the point $p$. Consider now $I$ as a curve over $\text{Spec}Z[T]$. The fact that $\varrho : \pi_1(X) \rightarrow G$ is a Zariski dense nonrigid representation implies that $\bar{\varrho}$ is a Zariski dense nonrigid representation. Observe that there are still going to be a point $q$ in the curve $\bar{I}$ which is not in $\text{Spec}Z[T]$. Let $\chi(\bar{\varrho})$ is a character of $\bar{\varrho}$. Moving over $\bar{I} \subset \text{Spec}Z[T]$ we see that $\chi(\bar{\varrho})$ is unbounded at $q$. Therefore we conclude that the representation $\bar{\varrho}$ to $\bar{G}$ is not contained in any bounded subgroup in $\bar{G}$.

The following corollary is an easy consequence of the previous result:

**Corollary 4.1** Let $U$ be an equivariant harmonic map $U : \bar{X} \rightarrow B$ and $\varrho : \pi_1(X) \rightarrow G$ be a Zariski dense nonrigid representation, and let $X$ be a projective variety. Then one of the following possibilities holds:

A) The $\text{Im}(\varrho(\pi_1(X)))$ is contained in a maximal compact subgroup in $G$ or

B) There exists a holomorphic map $S \rightarrow Y$ and a map $u_0 : S^0 \rightarrow Y^0$ with $S^0$ and $Y^0$ Zariski open sets in $S$ and $Y$ respectively, such that $\varrho(\pi_1(S^0))$ factors through a representation $\varrho' : \pi_1(Y^0) \rightarrow G$, where $Y$ is a normal projective variety such that $1 \leq \text{dim}(Y) \leq r$.

Using this corollary we give a proof of Theorem 1.2. It follows from remark 3.1 and [25],

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paragraph 4 that using the properties of the spectral covering we can mod out $S^0$ by $W$ and get a map

$$h : X^0 \to Y^0,$$

which comes from the morphism $h : X \to Y/W$, where $X^0$ is an open set in $X$ and $Y^0$ is an open set in $Y/W$. From now on we denote all modifications of $Y$, namely blow ups and finite nonramified covers, by $Y$ if not stated otherwise.

Denote the generic fiber of $h : X^0 \to Y^0$ by $Z^1$.

Let us first resolve the singularities of $Y$. This might change the fundamental group of $Y$ but we need only that the fundamental group of $X$ does not change, which follows from the fact that $X$ is a smooth projective variety.

We finish the proof of the main theorem by generalizing 4.8.1. in [27]. According to corollary 4.1 we have the following sequence:

$$\pi_1(X^0) \to \pi_1(Y^0) \to G.$$

What we need to show is that $\pi_1(Z^1)$ belongs to the kernel of the map

$$\pi_1(X^0) \to \pi_1(Y^0).$$

We cannot do that directly on $X$ but by generalizing 4.8.1. in [27] we show that this is possible on some finite nonramified covering of $X$, $X(T)$.

Observe that

$$im[\pi_1(X^0) \to \pi_1(Y^0)] \subset Ker \rho.$$

**Definition 4.1** Let $P$ be a subgroup of $\pi_1(X)$ and let $X(P)$ be a covering of $X$ with fundamental group $P$. Consider now the Stein factorization of the map $X(P) \to Y$ and define $X(P) \to Y(P)$ to be the map with connected fibers in this factorization.

Note that $Y(P)$ will be an analytic variety even when the covering $Y(P) \to Y$ is infinite. This follows from the most general version of the Stein factorization theorem (see Theorem 3 [7]).

**Definition 4.2** Define $\Omega$ to be the intersection of all subgroups $P$ in $\pi_1(X)$ such that $H \subseteq P$, where $H = im[\pi_1(Z^1) \to \pi_1(X)]$ and the covering $Y(P) \to Y$ has finite ramification indexes.

Such an $\Omega$ is well defined due to the fact that there exists at least one such a $P - \pi_1(X)$.

Define $K = Ker(\rho)$. 

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Lemma 4.4 The covering $Y(K) \to Y$ has finite ramification indexes.

Proof. After intersecting $X$ with sufficiently many generic hyperplanes we get a finite ramified covering $X \cap H \to Y$. This covering obviously has finite ramification index. But since the covering $X(K) \to X$ is nonramified we know the indeces of the covering $Y(K) \to Y$ are also finite.

Lemma 4.5 There exists a finite index subgroup $T$ in $\pi_1(X)$ such that:

$$H \subseteq R \subseteq \Omega \subseteq \pi_1(X),$$

where $R$ is the kernel of the map $\pi_1(X(T)) \to \pi_1(Y(T))$.

Proof. The proof is the same as in 4.8.1 [27].

To finish the proof of 1.2 we need only to observe that

$$R \subseteq \Omega \subseteq K$$

and this gives us the complete factorization

$$X(T) \to Y(T).$$

As an almost immediate corollary we have:

Corollary 4.2 Let $\rho : \pi_1(X) \to G$ be a Zariski dense nonrigid representation of the fundamental group of $X$ to some complex simple Lie group $G$. Then there exist:

1) a finite etale cover $X'$ of a blow up of $X$;

2) a smooth projective variety $Y$ of positive dimension $l$ less then or equal to the rank $r$ of $G$ over $\mathbb{C}$;

3) and a holomorphic map $h : X' \to Y$

such that $\rho : \pi_1(X') \to G$ factors through a representation of $\pi_1(Y)$.

Proof. The fact that $\rho : \pi_1(X) \to G$ is a Zariski dense nonrigid representation of the fundamental group of $X$ to some complex simple Lie group $G$ implies that there exists a curve in the moduli space of representations passing through $\rho : \pi_1(X) \to G$, which intersects infinity in the moduli space of representations in the point $p$. Let $O_p$ be the local ring of this point as a point of the curve described above. This way we obtain a representation $\tilde{\rho}$ to $\tilde{G}$, where $\tilde{G}$ is defined over
the field of fractions of the completion of $O_p$, which is also Zariski dense and is not contained in any bounded subgroup in $\overline{G}$. We finish the proof by applying Theorem 1.2.

\[\square\]

**Remark 4.1** In the last section we give a different proof of the last part of Theorem 1.2. There we use the theory of the Shafarevich maps.

**Remark 4.2** Observe that if we work with group of finite index in $\pi_1(X)$ we get again the conclusions of the main theorem.

## 5 The quasiprojective case.

In this section we are going to work with $X = X_1 \setminus D$ - a quasiprojective variety, where $D$ is the divisor at infinity and $X_1$ is the compactification of $X$. Our goal is to prove theorems similar to that of the previous section. Of course to start the whole procedure we need to show that there exists an equivariant continuous map of finite energy $U : \tilde{X} \to B$. To be able to show this we require that our divisor at infinity is a divisor with normal crossing with unipotent monodromy around it. We can cover $X$ by finitely many open sets since it is compactly embedded. Using the fact that we have a unipotent monodromy at infinity we see that the fundamental groups of the open sets which cover the divisor at infinity are contained in maximal compact subgroups. A simple computation shows that by simultaneous conjugation by some elements we can make all of the finitely many generators of these groups have integer coefficients and hence they are contained in the same maximal compact subgroup. Using the fact that the maximal compact subgroups fix a point in $B$ we get an equivariant continuous map of finite energy from every open set which covers the divisor at infinity to $B$, namely the map to the fixed point. Using the standard center of mass construction we obtain the equivariant continuous map of finite energy $U : \tilde{X} \to B$.

We are going to make sense of spectral covering in the quasiprojective case. First we formulate and give a sketch of the proof of Corlette vanishing theorem in the quasiprojective case.

Let $X$ be a quasiprojective variety with universal covering $\tilde{X}$. Consider now a representation of the fundamental group of $X$ to some Lie group defined over an arbitrary nonarchimedean field. Let $B$ be the corresponding Euclidean building. Let us assume also that the divisor at infinity in $X$ is a divisor with normal crossings and the monodromy around it is unipotent. We have the following theorem, the proof of which is similar to the proof of the original theorem of Gromov and Schoen. It is clear that in the situation above, provided that $\pi_1(X)$ acts on $B$ without fixed points, to every Zariski dense representation we assign a harmonic map $U : \tilde{X} \to B$ for which the whole theory of Gromov, Korevaar, Schoen and Jost works.
Now take a regular point $x_0$ of $\tilde{X}$ (for the definition of regular point see [15]). Then the image of the ball $B - \sigma(x_0)$ is contained in at least one flat $F$ in $B$. Let $\nabla$ denote the pullback connection and let $d\nabla$ denote the corresponding exterior derivative operator on $p$-forms with values in $U^*TF$. Let $\delta\nabla$ denote its formal adjoint. The differential $dU$ then defines a 1-form with values in $U^*TF$, and we have the following:

**Theorem 5.1** (Corlette) Let $X$ be a quasiprojective variety, let $\omega$ be a parallel $p$-form on $\tilde{X}$, and let $U$ be a harmonic map $U : \tilde{X} \rightarrow B$. Then in the neighborhood of $x_0$, a regular point for $U$, the form $w \wedge dU$ satisfies $\delta\nabla (w \wedge dU) \equiv 0$.

**Proof.** The statement of this theorem is local so the proof is almost the same as in [15]. First we exhaust $X$ by compact sets $X_i$. After that one needs to choose the right cutfunction and apply to each $X_i$ theorem 7.2 from [15]. But since the statement is local we just choose the same functions as in [15].

\[ \square \]

**Corollary 5.1** In the situation above $U$ is pluriharmonic.

**Proof.** Proof is as in [15]. \[ \square \]

First we define the spectral covering. As in the compact case we have the one forms $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ defined on $X = X_1 \setminus D$ up to action of the Weyl group. (Here $z_1$ is an arbitrary choice of a coordinate on a given chamber of $B$, which we extend using the action of the group.) Again we apply to them the basis of the $G$-invariant for $G$ polynomials and obtain the forms $h_1, \ldots, h_r$. Due to the fact that $U$ is a finite energy map it follows from [15] that the forms $h_1, \ldots, h_r$ are $L_2$ bounded and therefore have at most log poles. The unipotency of the loops around infinity gives us that the residues of $h_1, \ldots, h_r$ are rational. Therefore over some finite covering of $X_1$ they are holomorphic. Using the procedure of section 3 we build now the spectral covering $S$ over $X_1$.

The theorem above gives us the forms $U^*de_1(z_1), \ldots, U^*de_w(z_1)$ defined on $X = X_1 \setminus D$ up to an action of the Weyl group. Therefore over the spectral covering $S$ we obtain the forms $\alpha_1, \alpha_2, \ldots, \alpha_r$, where $r$ is the rank of the group $G$. Observe that the forms $\alpha_1, \alpha_2, \ldots, \alpha_r$ are $L_2$ bounded and therefore have at most log poles. This can be seen in the following way: Using [15](Theorem 6.3) we see that a loop around $D$ fixes a point $s$ in $B$. Therefore the map $U : D \rightarrow B$ gives us a weakly subharmonic map $d^2(U(x), s)$ and then the same arguments as in [15](Theorem 6.3) imply the above statement.

Now we state the quasiprojective version of the Clemens -Lefschetz -Simpson theorem.

Observe that if $l : S \rightarrow X$ is the spectral coveing then $S = S^1 \setminus l^*(D)$. 

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Following Itaka [21] we introduce the Albanese map \( alb : S \rightarrow Alb(S) \) for quasiprojective varieties. As in the case of projective varieties it is defined by integrals of holomorphic one forms on \( S \). Here \( Alb(S) \) is a semiabelian variety- a group extension of \( Alb(S^1) \) by \((\mathbb{C}^*)^{\times l}\).

Define also
\[
A = Alb(S)/B.
\]

We have the natural map
\[
a : S \rightarrow A.
\]

Here \( B \) is again the maximal abelian subvariety over which the forms \( \alpha_1, \alpha_2, ..., \alpha_n \) are zero as forms on \( Alb(S^1) \).

Let us also define the map
\[
g : \tilde{S} \rightarrow \mathbb{C}^n
\]
as a pullback of \( a \). In this situation we have the following:

**Theorem 5.2** (Clemens - Lefschetz - Simpson) Let \( S \) be a smooth quasiprojective variety. Let \( \alpha_1, \alpha_2, ..., \alpha_r \) be holomorphic one forms on \( X \) with at most log poles at \( l^* (D) \). Then one of the following cases holds:

A) The map \( g \) has connected fibers.

B) There is a holomorphic map with connected fibers from \( S \) to the projective normal variety \( Y \) such that \( 1 \leq \dim Y \leq n \) and the forms \( \alpha_1, \alpha_2, ..., \alpha_n \) are pullbacks from \( Y \).

C) The third case is a combination of the previous two. Namely after a linear change of the coordinates in \( \mathbb{C}^n \) some of the forms \( \alpha_1, ..., \alpha_k \) come as a pullbacks of a map \( X \rightarrow Y \), where \( Y \) is an algebraic variety \( 1 \leq \dim Y \leq k < n \). The rest of the forms give us a map \( g' : \tilde{X} \rightarrow \mathbb{C}^l \), where \( l \leq n-k \), with connected fibers.

The proof is similar to the proof in the projective case.

Now we formulate the version of our main theorem for a quasiprojective variety \( X \).

**Theorem 5.3** Let \( X = X^1 \setminus D \) be a smooth quasiprojective variety and \( \varrho : \pi_1 (X) \rightarrow G \) be a Zariski dense representation of the fundamental group of \( X \) with unipotent monodromy around \( D \), where \( D \) is a divisor with a normal crossing. Let \( G \) be a simple Lie group over \( K \). Then

A. either the image of \( \varrho \) is in a maximal compact subgroup of \( G \) or,

B. there exist:

1) a blow up \( X' \) of a finite etale cover of \( X^1 \);
2) a smooth projective variety $Y$ of positive dimension $l$ less than or equal to the rank $r$ of $G$ over $K$;

3) A holomorphic map $h : X' \rightarrow Y$ such that $\varphi : \pi_1(X') \rightarrow G$ factors through a representation of $\pi_1(Y)$ and such that the pullback of $D$ in $X'$ is a pullback from a divisor on $Y$.

We give an application of the above theorem, which was suggested by J. Kollár.

According to N. Mok (see [34]) every real Zariski dense discrete representation in $SL(n, C)$ of noncompact type of the fundamental group of any compact Kähler manifold after some blow up and finite nonramified covering factors though the representation of the fundamental group of projective algebraic variety of general type.

Let $X = X^1 \setminus D$ be a quasiprojective variety such that $X^1$ has Kodaira dimension zero and let $\varphi : \pi_1(X) \rightarrow SL(n, C)$ be a real Zariski dense discrete representation of the fundamental group of $X$. The hypothesis of the Mok’s theorem send us in case B) of the above theorem. Since $X'$ is a finite etale cover of blow up of $X^1$ it has also Kodaira dimension equal to zero. Mok’s theorem also tells us that $Y$ from part B) of the previous theorem is an algebraic variety of general type. Therefore we obtain a holomorphic map $h : X' \rightarrow Y$ from a variety with Kodaira dimension zero to a variety of general type and this impossible, due to a theorem of Kawamata [1]. We have obtained the following:

**Corollary 5.2** Let $X = X^1 \setminus D$ be a quasiprojective variety such that $X^1$ has Kodaira characteristic zero. Then:

$$\varphi : \pi_1(X) \rightarrow SL(n, R)$$

is a finite group.

6 **Factorization theorems for complexes of groups.**

In [15] Gromov and Schoen proved that if the fundamental group of a quasiprojective variety $X$ admits a decomposition as an amalgamated product of groups, then $X$ admits a surjective holomorphic map to an algebraic curve. In this section we extend this result to higher dimensional complexes of groups. This gives a partial answer to a question Gromov stated in [17] (section 7).

In what follows we are working with the negatively curved complexes of groups defined by Benakli in [4].

The key idea in [15] is that the Baas-Serre theory assigns to an amalgamated product of groups a tree to which one applies the theory of harmonic maps. According to [1] and [15] (Theorem 6.4) we can assign to every negatively curved two dimensional complex of groups a connected negatively
curved simplicial cell complex of dimension 2 with a finite set of isometry types of cells \(T\). Let \(G(T)\) be the universal covering of \(T\). The fundamental group \(\pi_1(T)\) (see \([18]\)) acts on \(G(T)\) by simplicial isometries. Let \(X\) be a projective variety such that there exists a surjective homomorphism \(\theta : \pi_1(X) \rightarrow \pi_1(T)\). Therefore \(\pi_1(X)\) acts on \(G(T)\) without fixed points. Using the fact that \(G(T)\) is contractible (\([6]\) and \([18]\)) we apply the Korevaar-Schoen theorem to get a harmonic map \(U : \tilde{X} \rightarrow G(T)\). In this section we work only with \(G(T)\) of the type defined by Benakli in \([4]\) paragraph 7. Namely they are constructed by using the baricentric subdivision of a regular hyperbolic polyhedra.

The following theorem is a consequence of the previous sections:

**Theorem 6.1** Every representation of the fundamental group of projective varieties onto the fundamental group of some negatively curved 2-dimensional polyhedra, comes from the representation of the fundamental group of an orbicurve.

**Proof.** The only thing we need to show is how to construct the spectral covering. To do this we first choose local coordinates on \(G(T)\). Using the symmetries, defined on \(G(T)\) (see \([6]\) and \([18]\)) we can extend these coordinates to the all of \(G(T)\). To obtain holomorphic forms, and consequently a spectral covering, we need vanishing theorems. But since in the case of \(G(T)\) we do not have a statement about the codimension of the singularities of the corresponding harmonic map, we need to do some extra work, which is the essence of the theorem. First we use the following theorem (\([4]\)):

**Theorem 6.2** (Benakli) \(G(T)\) can be embedded isometrically in a hyperbolic space \(\mathbb{H}^3\) in such a way that \(G(T)\) is a "deformation retract" of \(\mathbb{H}^3_1\).

Here we denote by \(\mathbb{H}^3_1\) the 3-hyperbolic space \(\mathbb{H}^3\) with the vertexes of every ideal polytop thrown away. Observe that \(\mathbb{H}^3_1\) is contractible.

We prove a version of the vanishing theorem for \(G(T)\) closely following \([15]\). We omit the details of the argument in \([15]\), emphasizing only the differences.

Let us first mention that Benakli’s construction can be made \(\pi_1(T)\) - equivariant. For our argument we need a special kind of retraction. Namely to show that the harmonic maps \(U_t : \tilde{X} \rightarrow \mathbb{H}^3_t\) exist for every \(t\), we need to make sure that the curvature of \(\mathbb{H}^3_t\) is nonpositive for every \(t\). We do that by retracting \(\mathbb{H}^3_t\) to \(G(T)\) equidistantly. Observe that the \(\mathbb{H}^3_t\) are going to be singular for every \(t\). The crucial fact, which helps us avoid these difficulties is that \(\mathbb{H}^3_1\) has dimension greater than \(G(T)\).

**Lemma 6.1** \(\mathbb{H}^3_t\) is negatively curved for every \(t\).
The proof of this statement is a standard application of the Gauss formula for the curvature of a hypersurface.

**Theorem 6.3** The map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ is pluriharmonic for every $t > 0$.

**Proof.** Observe that the singular set of $G(T)$ consists of the centers of the ideal hyperbolic polyhedra and the edges, which come out of them. Therefore we need to worry only about ball $b$ around a regular points $x_0$ in $\tilde{X}$ which maps to the ideal polyhedra on the boundary of $\mathbb{H}_t^3$ which can be retracted to $G(T)$. We consider 3 different cases:

A) The map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ maps a ball $b$ around a regular point $x_0$ in $\tilde{X}$ to a face in the boundary of $\mathbb{H}_t^3$ but far from the edges. In this case the proof of the theorem follows from Theorem (7.3) \[15\].

B) The image of a ball $b$ around a regular point $x_0$ in $\tilde{X}$ under the map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ contains an edge coming out from the center of an ideal hyperbolic polyhedra. In this situation $\mathbb{H}_t^3$ looks locally like a product of a tree and $\mathbb{R}^2$. Therefore Theorem (7.2) \[15\] applies and we can find for every $t$ a sequence of functions $\psi_{i,t}$ which:

1) vanish in a neighborhood of the set $S_t$ of $U_t$;
2) tend to 1 on $X \setminus S_t$ and;
3) such that:

$$\lim_{t \to \infty} \lim_{i \to 0} \int_X \| \nabla^2 U_t \| \psi_{i,t} d\mu = 0.$$ 

Following Theorem (7.2) \[15\] we get:

**Theorem 6.4** Let $X$ be a smooth projective variety, $\omega$ a parallel p-form on $\tilde{X}$ and let the image of a ball $b$ around a regular point $x_0$ in $\tilde{X}$ under the map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ contain an edge of an ideal hyperbolic polyhedra but be far from a vertex. Then in some small neighborhood around $x_0$ the form $\omega \wedge dU$ satisfies

$$\delta_{\nabla} (\omega \wedge dU) \equiv 0.$$ 

Here the notations are the same as in theorem 5.2.

Now in the same way as in Theorem (7.3) \[15\] we obtain:

**Theorem 6.5** The map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ is pluriharmonic.

C) The image of a ball $b$ around a regular point $x_0$ in $\tilde{X}$ under the map $U_t : \tilde{X} \rightarrow \mathbb{H}_t^3$ contains a translated center of an ideal hyperbolic polyhedra. This case cannot be done as the previous two.
$\mathbb{H}^3_t$ looks locally like a product $P$ of an open polyhedra (namely we have taken the bottom face away) and an open interval. We can approximate $P$ by products $C_t$ of intervals and cones. Following the last example in the Introduction of [13], we can also approximate the singular metric $g_t$ on $C_t$ by regular metrics $g_{ts}$ with $K \leq 0$. This construction is similar to the polyhedral immersion of a 2-disk into $\mathbb{H}^4$ described in [44]. To show that the maps $U_{ts}: \tilde{X} \to b$ are pluriharmonic for every regular metric $g_{ts}$ on $C_t$, we use the same argument as in B). Namely we use Theorem (7.2) [15] to find for every $t$ a sequence of functions $\psi_{i,t}$.

According to [20] the limit of pluriharmonic maps is a pluriharmonic map so we obtain a pluriharmonic map to $C_t$ with singular metric $g_t$ on it. Now we approximate $P$ by the $C_t$, and using again that limit of pluriharmonic maps is a pluriharmonic map, we get that the map $U_t: \tilde{X} \to \mathbb{H}^3_t$ is pluriharmonic.

$\square$

**Remark 6.1** The above argument works in the same way for a truncated two dimensional polyhedra ([4]). Part C) of the argument does not generalize in higher dimensions.

An easy corollary of the above theorem is the following:

**Corollary 6.1** The map $U_t: \tilde{X} \to G(T)$ is pluriharmonic.

We are now ready to finish the proof of Theorem 6.1. After we get the pluriharmonicity of the map $U: \tilde{X} \to G(T)$ we can construct the spectral covering. As before the pluriharmonicity implies that if we take the $(0,1)$ part of the complexified differentials $du_1, du_2$ we obtain holomorphic differentials over some spectral covering.

The same argument as in the proof of Theorem 1.2 gives a complete factorization $h: X \to Y$. Let say that we are in case A) of the Clemens-Lefschetz-Simpson theorem. Then we have a local isometry $l: \mathbb{R}^2 \to G(T)$ as in Lemma 4.2. We obtain $\mathbb{R}^2$ as the real part of the map defined by the integration of the holomorphic forms over the spectral covering $S$. The fact that $G(T)$ is strictly negatively curved make the existence of such a local isometry impossible. Following Lemma 4.2 we obtain in the same way a contradiction in the case when $\text{dim}_c Y = 2$.

According to the Clemens-Lefschetz-Simpson theorem there is a possibility of an isometry $l: \mathbb{R}^1 \to G(T)$. But then if the image is isometric to $\mathbb{R}^1$ we will have subcomplexes fixed under the action of $\pi_1(X)$ and this contradicts the surjectivity of $\theta: \pi_1(X) \to \pi_1(T)$.

Therefore the Clemens-Lefschetz-Simpson theorem that there is a factorization through the fundamental group of an orbicurve.

$\square$
A conjecture stated by J. Carlson and D. Toledo \[28\] says that if $\Gamma$ is a Kähler group then $H^2(\Gamma)$ is nontrivial. As a simple corollary of the above theorem 6.1 we have:

**Corollary 6.2** If there exists a surjective homomorphism $\theta : \pi_1(X) \rightarrow \pi_1(T)$ then the above conjecture is true for $\pi_1(X)$.

Inspired by \[17\] we describe a construction which we use in the next section. According to the previous theorem, if we work with negatively curved length spaces we get a nice description of the representations, namely they are coming from the representations of the fundamental group of an orbicurve. It is natural to try to hyperbolize the objects we are working with. This can be done in the case of a building $B$. (Following \[8\] we can replace every chamber in every apartment of the building by a negatively curved complex with boundary the boundary of the chamber. )

According to the result of R. Charney and M. Davis \[8\] we obtain the hyperbolized building $HB$, which is strictly negatively curved. The functoriality of the construction of Charney and Davis provides us with an action of $G$, the corresponding group of the initial building $B$, over $HB$. Unfortunately there is one disadvantage of the whole construction - $HB$ has a nontrivial topology. So we can use the technique developed above only on representations $\varrho : \pi_1(X) \rightarrow G$, for which we know there exists a $G$-equivariant continuous map $U : \tilde{X} \rightarrow HB$. Along the lines of the previous corollary we have:

**Corollary 6.3** Every Zariski dense representation $\varrho : \pi_1(X) \rightarrow G$ for which there exists an $G$-equivariant continuous map $U : \tilde{X} \rightarrow HB$, such that $G$ acts on $HB$ without a fixed point, factors through the representation of the fundamental group of an orbicurve.

Let us give some sufficient conditions for the existence of a $G$-equivariant continuous map $U : \tilde{X} \rightarrow HB$.

**Theorem 6.6** There exists a $G$-equivariant continuous map $U : \tilde{X} \rightarrow HB$ if the group homomorphism $\pi_1(X) \rightarrow \text{Out}(PI)$ induced by $\varrho : \pi_1(X) \rightarrow G$ can be lifted to a group homomorphism $\pi_1(X) \rightarrow \text{Aut}(PI)$. Here $PI$ is the fundamental group of $HB$ and $\text{Out}(PI)$ and $\text{Aut}(PI)$ denote the groups of outer automorphisms and automorphisms of $PI$ respectively.

**Proof.**

It is easy to see the necessity of this condition directly. It has a natural interpretation in terms of gerbes (see \[12\] and \[14\]). Let us explain the construction of the gerbes we use in this situation. For this we need to choose a universal covering $\tilde{HB} \rightarrow HB$ of $HB = K(PI, 1)$.

Then we have an exact sequence of groups:
where $\text{Homeo}(\tilde{HB}, HB)$ is the group of homeomorphisms of $\tilde{HB}$, which cover some homeomorphism of $HB$.

Since we have a principal bundle over $HB$ with group $\text{Homeo}(HB)$, there is a gerbe whose objects are local liftings of the structure group to $\text{Homeo}(\tilde{HB})$, and whose arrows are isomorphisms of $\text{Homeo}(\tilde{HB})$-bundles, which induce the identity on the $\text{Homeo}(HB)$-bundle. The band of this gerbe is locally isomorphic to the band associated to the constant sheaf of $PI$, but there is an outer twisting represented by a class in $H^1(X, \text{Out}(PI))$, namely the class of the homomorphism $\varrho : \pi_1(X) \to G$. The obstruction to realizing the given band as arising from some sheaf of groups over $X$ is exactly the obstruction to lifting the above class to a class in $H^1(X, \text{Aut}(PI))$.

Let us see how all this applies to the situation, where we let $HB$ be the hyperbolized $SL(3, (Q)_p)$ building. It easy to see that in this case $PI = F_{SL(3,(Q)_p)}$, where $F_{SL(3,(Q)_p)}$ is a free group with $SL(3,(Q)_p)$ many generators. $\text{Out}(F_{SL(3,(Q)_p)})$ contains $SL(3,(Q)_p))$. What we need now, to claim the existence of a $G$- equivariant continuous map $U : \tilde{X} \to HB$, is the splitting of the following exact sequence:

$$1 \to F_{SL(3,(Q)_p)} \to \text{Aut}(F_{SL(3,(Q)_p)}) \to SL(3,(Q)_p) \to 1,$$

over $T$, where $T$ is the set theoretic image of $SL(3,(Q)_p)$ in $Out(F_{SL(3,(Q)_p)}))$.

To have this splitting we need to make $SL(3,(Q)_p)$ act on the graph we have assigned to $F_{SL(3,(Q)_p)}$ (the graph in this case looks like rose with one vertex and $SL(3,(Q)_p)$ edges) while fixing the vertex. For this we need to make a canonical choice of an initial point for $PI$ in $HB$. A sufficient condition for this is, for example, if the image of $\tilde{X}$ under $U$ is homotopy equivalent to a tree which is fixed under the action of the image of $\varrho : \pi_1(X) \to G$. If this condition is satisfied, we change in a canonical way the generators of $F_{SL(3,(Q)_p)}$ when we are change the initial point for $F_{SL(3,(Q)_p)}$ in $HB$. One case when the above condition is satisfied is when the whole image of $U$ is a tree which is fixed under the action of the image of $\varrho : \pi_1(X) \to G$. This is an equivariant harmonic map $U$ having dimension 1. Unfortunately in general this condition is hard to check.

Using the technique of F. Paulin [35] we prove:

**Theorem 6.7** Let $X$ be a smooth projective variety, and let $\Gamma$ be a word hyperbolic group acting on the corresponding $\mathbb{R}$-tree without fixing a vertex. Let us assume that $\text{Out}(\Gamma)$ is an infinite group and let $\rho : \pi_1(X) \to \Gamma$ be a surjective homomorphism. Then $\rho : \pi_1(X) \to \Gamma$ factors through a representation of the fundamental group of an orbicurve.
Proof. According to Paulin [35] $\Gamma$ has an fixed point free action on $T$, an $\mathbb{R}$ tree. Using a theorem of Reeps [35] we conclude that this action is simplicial. We obtain $T$ by deforming the Cayley graph of $\Gamma$, using that $Out(\Gamma)$ is an infinite group. From the Schoen-Korevaar-Jost theorem we know that there exists a harmonic $\pi_1(X)$-equivariant map to $T$. In the same way as in the proof of theorem 6.1 we show that $\rho: \pi_1(X) \rightarrow \Gamma$ factors through a representation of the fundamental group of an orbicurve. We rule out the possibility of part A) of the Clemens-Lefschetz-Simpson theorem by using the fact that in the situation above we do not have parabolic action of $\Gamma$ on infinity of $T$, namely $\Gamma$ does not have fixed point of the infinity of $T$.

Another way to show that $\rho(\pi_1(X)) : \pi_1(X) \rightarrow \Gamma$ factors through a representation of the fundamental group of an orbicurve, is to follow the proof of theorem 8.1 [20].

Remark 6.2 We think that one can use the harmonic map technique to prove the Morgan-Shalen conjecture for Kähler groups. The conjecture states [35], that every group with free action on an $\mathbb{R}$ tree is a free product of an abelian and a surface group.

7 Some results about the integrality of representations

In this section we discuss a partial verification of the following two conjectures of Simpson [41]. We are going to consider representations $\varrho: \pi_1(X) \rightarrow G$, where $G$ is simple Lie group defined over $k$, an algebraically closed field of characteristic zero. In this section we work only with $G = SL(n,k)$.

Conjecture 7.1 Let $\varrho: \pi_1(X) \rightarrow G$ be a rigid semisimple representation. Then $\varrho \otimes k$ is a direct summand over $k$ in the monodromy representation of a motive (i.e. comes from geometry) over $X$.

Conjecture 7.2 Let $\varrho: \pi_1(X) \rightarrow G$ be a rigid semisimple representation. Then it is integral, in other words it is conjugate to a representation whose matrix coefficients are algebraic integers.

(Note that the first Conjecture would imply the second.)

Here rigid representation means that every representation which is nearby in the affine variety of representations is conjugate to it. In the case of an irreducible representation (the case we are working with), rigid is equivalent to the fact that the corresponding point in the moduli space of representations is isolated.

The goal of this section is to verify the second conjecture in some cases.

We proceed following Simpson [38]. We assign to every Zariski dense rigid representation a new Zariski dense rigid representation $\varrho: \pi_1(X) \rightarrow G^1$ over a local nonarchimedean field. The
procedure goes as follows. Observe that the moduli space of representations is defined over $\mathbb{Q}$, and since we are working with a rigid representation we can find an isomorphic representation defined over $\overline{\mathbb{Q}}$. Let $E$ be a finite extension of $\mathbb{Q}$ defined to be the extension which contains all coefficients of our representation, and let $O$ denote the ring of integers in $E$. Let $E_p$ denote the field of fractions of the completion of $O$ in $p$, for some prime $p$. Let $G^1$ be the new group over $E_p$ and use $\rho$ again for the representation $\rho : \pi_1(X) \to G^1$. Since $E_p$ is a local field then we are in the situation of the Theorem 4.2. Namely, using the Bruhat-Tits theory we can attach to $G^1$ a building $B$. Following the Gromov and Schoen theory we can attach a harmonic map to $\rho$. Therefore we get:

**Corollary 7.1** Let $\rho : \pi_1(X) \to G$ be a Zariski dense rigid representation. Then one of the following holds:

A) For every prime $p$ the image of $\rho : \pi_1(X) \to G^1$ is contained in some maximal compact subgroup in $G^1$.

B) For some prime $p$ the image of $\rho : \pi_1(X) \to G^1$ is not contained in some maximal compact subgroup in $G^1$. Then there exist:

1) a finite etale cover $X'$ of a blow up of $X$;
2) a smooth projective variety $Y$ of positive dimension less or equal to the rank $r$ of $G$ over $\mathbb{C}$;
3) and a holomorphic map $h : X' \to Y$

such that $\rho : \pi_1(X') \to G$ factors through a representation of $\pi_1(Y)$.

In case A) we apply theorem os Simpson’s [43] to obtain that $\rho$ is an integer representation, namely a representation $\rho : \pi_1(X) \to GL(r, \mathbb{Z})$, $r \geq n$.

Now we define special type of representations for which case B) of the above theorem does not occur, namely conjecture 7.2 is satisfied. The idea is that since curves do not have many rigid representations then if something factors through a curve should not have either.

**Definition 7.1** Take $\rho : \pi_1(X) \to G^1$ to be a Zariski dense representation in $G^1$ which is assigned to $\rho : \pi_1(X) \to G$. Define the corresponding forms $de_1(z_1), \ldots, de_1(z_1)$, and the holomorphic one forms $\alpha_1, \alpha_2, \ldots, \alpha_n \neq 0$ on $\tilde{S}$. If all of these forms span a one dimensional subbundle of the cotangent bundle of $\tilde{S}$ over a Zariski open set we say that $\rho : \pi_1(X) \to G$ is of dimension one.

**Theorem 7.1** Let $\rho : \pi_1(X) \to G$ be a rigid Zariski dense representation of dimension one. Then it is integral, in other words it is conjugate to a representation whose matrix coefficients are algebraic integers.

**Proof.**
The case of $G = SL(2, \mathbb{C})$ - representations was proven by Simpson [38] for any rigid representation. Note that every Zariski dense representation to $SL(2, \mathbb{C})$ is of rank one. For groups of higher ranks we use the fact that our representation is of dimension one. In this situation we can use Theorem 1.1, namely, using the Bruhat-Tits theory we can attach to $G^1$ a building $B$ and then attach, following Gromov-Schoen theory, a harmonic map to $B$. Since our representation is of dimension one, Theorem 1.2 gives us that there exists a finite nonramified covering $X^1 \to X$ such that the fundamental group of $X^1$ factors though the fundamental group of some smooth algebraic curve $Y^1$. Now following [38] we can actually factor the original representation $\rho : \pi_1(X) \to G^1$ working with the fundamental group of an orbicurve, $\pi_1(Y, O)$, instead of with the fundamental group of $\pi_1(Y^1)$.

We proceed by applying a theorem of N. Katz.

**Theorem 7.2** *(Katz)* If $\rho : \pi_1(Y, O) \to G$ is cohomologically rigid then it is motivic.

To be able to apply the above theorem we need to show that rigidity in terms of moduli spaces and the cohomological rigidity define the same objects. This is the subject of the next two lemmas.

Let us first give the precise definitions.

**Definition 7.2** Let $Y \cong \mathbb{P}^1$. We say that $\rho : \pi_1(Y, O) \to G$ is a physically rigid representation if and only if the conjugacy class $\rho : \pi_1(Y, O) \to G$ is uniquely determined by its local monodromies.

Let us introduce a new notion-the notion of cohomological rigidity.

**Definition 7.3** Let $Y \cong \mathbb{P}^1$ and let $\rho : \pi_1(Y, O) \to G$ be a representation. Let $F$ be the bundle over $Y \cong \mathbb{P}^1$ corresponding to $\rho : \pi_1(Y, O) \to G$. We say that $\rho : \pi_1(Y, O) \to G$ is cohomologically rigid if and only if

$$\chi(X, j_*(\text{End}F)) = 2.$$

For more detailed treatment of the notions of physical and cohomological rigidity see [24].

Now we show that the notions of rigidity in terms of moduli spaces and the cohomological rigidity coincide.

**Lemma 7.1** Let $\rho : \pi_1((Y, O)) \to G$ be a Zariski dense representation which is rigid in the moduli sense. Then $Y \cong \mathbb{P}^1$ with orbistructure in finitely many points.
Proof. Let \( a_1, \ldots, a_g, b_1, \ldots, b_g, r_1, \ldots, r_t \) be the generators of \( \pi_1((Y, O)) \). (Here \( g \) is the genus of \( Y \), and \( t \) is the number of the points with an orbistructure). Consider the map:

\[
s : G_1 \times \ldots \times G_{2g} \times C_1 \times \ldots \times C_t \to G.
\]

where \( G_1, \ldots, G_{2g} \) are \( 2g \) copies of \( G \) and \( C_1, \ldots, C_t \) are the conjugacy classes \( SL(n, \mathbb{C})/Z(r_1), \ldots, SL(n, \mathbb{C})/Z(r_t) \). Here the \( Z(r_i) \) are the centralizers of the \( \rho(r_i) \) in \( SL(n, \mathbb{C}) \). This map is defined to be just the multiplication of the corresponding elements. The dimension of the maximal component of the preimage of 1 under this map gives an estimate for the dimension of the tangent space to the moduli space of representations \( \varrho : \pi_1(Y, O) \to G \). This dimension is equal to:

\[
2g \dim G + \sum \dim SL(n, \mathbb{C})/Z(r_i) - \dim G.
\]

Since \( \varrho : \pi_1(Y, O) \to G \) is a rigid representation, then this dimension should be equal to zero. But the number we have computed above is always positive (\( n \geq 3 \)) unless \( Y \cong \mathbb{P}^1 \).

\[ \square \]

The following was shown in [24]:

**Theorem 7.3 (Katz)** Let \( Y \cong \mathbb{P}^1 \). Then the notions of physical rigidity and cohomological rigidity over \( Y \) are the same.

The second lemma we need is the following:

**Lemma 7.2** Let \( Y \cong \mathbb{P}^1 \). Then the notions of physical rigidity and rigidity in terms of the moduli space of representations for an orbigroup over \( Y \) are the same.

**Proof.** Let us first show that physical rigidity implies moduli space rigidity. Observe that \( r_1, \ldots, r_t \) are semisimple elements in \( G \). We need to show that if we have two representations of \( \pi_1(Y, O) \) that we can deform one to another, then these two representations are conjugate, provided we know they are physically rigid. But since \( r_1, \ldots, r_t \) are semisimple, if we deform them they need to stay in the same conjugacy classes. Therefore they have the same local monodromies, and the physical rigidity implies that they are actually conjugate.

Now we show that moduli space rigidity implies physical rigidity. Let us start with a rigid, in terms of moduli spaces, representation \( \varrho : \pi_1(Y, O) \to G \). We have already estimated the dimension of the component of the moduli space of representations which contains \( \varrho \). Namely this dimension is less than or equal to:

\[
\sum \dim SL(n, \mathbb{C})/Z(r_i) - \dim G.
\]
Observe that:

\[
\sum_{i=1}^{t} \dim Z(r_i) = \left( \sum_{i=1}^{t} \sum_{j=1}^{k_i} n_{ij}^2 \right) - 1.
\]

(Here \(k_i\) is the number of the different eigenvalues of \(\rho(r_i)\) and \(n_{ij}\) are the sizes of the corresponding Jordan blocks.)

We also have:

\[
\dim SL(n, \mathbb{C})/Z(r_i) = (n^2 - 1) - \left( \sum_{j=1}^{n} n_{ij}^2 \right) - 1.
\]

Since \(\varrho : \pi_1(Y,O) \to G\) is rigid in terms of the moduli spaces we have that:

\[
\dim SL(n, \mathbb{C}) \geq n^2 - 1 = \sum_{i=1}^{t} \left( (n^2 - 1) - \left( \sum_{j=1}^{n} n_{ij}^2 \right) - 1 \right) - (n^2 - 1).
\]

This is equivalent to:

\[
2(n^2 - 1) \geq \sum_{i=1}^{t} (n^2 - \sum_{j=1}^{n} n_{ij}^2).
\]  

But from the Grothendieck-Ogg-Shafarevich formula (see [24]) we know that:

\[
\chi(X, j_*(EndF)) = (2 - t)(n^2) + \sum_{i=1,j=1}^{t,n} n_{ij}^2.
\]  

(Here \(\chi(X, j_*(EndF))\) is the Euler characteristic of \(j_*(EndF)\) and \(j\) is the embedding \(j : \mathbb{P}^1 \to \mathbb{P}^1\).)

Combining (1) and (2) we obtain:

\[
\chi(X, j_*(EndF)) \geq 2.
\]

On the other hand we have started with an irreducible representation \(\varrho : \pi_1(Y,O) \to G\) we have:

\[
\dim H^0(X, j_*(EndF)) = \dim H^2(X, j_*(EndF)) = 1
\]

Then \(\dim H^1(X, j_*(EndF)) = 0\), which means that the representation \(\varrho : \pi_1(Y,O) \to G\) is cohomologically rigid. Now we are in a position to apply theorem 7.2 and we conclude that \(\varrho : \pi_1(Y,O) \to G\) is motivic.

But if \(\varrho : \pi_1(Y,O) \to G\) is motivic then the image of \(\varrho : \pi_1(Y,O) \to G^1\) is in a bounded subgroup in \(G^1\). This can be seen as follows:
The fact \( \varrho : \pi_1(Y,O) \to G \) means that as representation \( \varrho \) extends from a discrete representation of the fundamental group of a complex variety extends to a representation of the profinite completion of the fundamental group i.e. it extends to a representation of \( X \) as an algebraic variety. But under continuous representation the compact profinite completion of the fundamental group goes to a compact image. So we get that the image of \( \varrho : \pi_1(Y,O) \to G^1 \) is in a bounded subgroup in \( G^1 \).

Therefore the corresponding harmonic map is the constant map so the forms \( \alpha_1, \alpha_2, \ldots, \alpha_n \neq 0 \) do not exist. This contradicts our assumption that we have a factorization through an orbicurve. So the second case does not occur.

\[ \square \]

In the process of the proof of this theorem we have obtained the following corollary:

**Corollary 7.2** The preimage of 1 under the map defined above
\[ s : C_1 \times \ldots \times C_t \to G, \]
is connected.

**Remark 7.1** The argument above depends a lot on the fact that we are working with \( SL(n,\mathbb{C}) \) (the dimensions of \( SL(n,\mathbb{C})/Z(r_i) \), a duality argument which is hidden in the computation of \( \dim H^2(X,j_*(\text{End}F)) = 1 \)), but we believe that it is true for all simple complex Lie groups.

Now we are going to extend the field of application of the above theorem to the quasiprojective case.

**Theorem 7.4** Let \( X \) be a quasiprojective variety and let \( \varrho : \pi_1(X) \to G \) be a rigid Zariski dense representation of dimension one and type B) with unipotent monodromy at infinity.

Then \( \varrho : \pi_1(X) \to G \) is integral.

**Proof.** From section 5 we know that we can define a spectral covering for quasiprojective variety. The proof of theorem 7.4 just repeats the arguments in the proof of theorem 7.2.

\[ \square \]

Now we are going to formulate 3 different sufficient conditions for which we get a factorization through an orbicurve:

1) It follows from the work of Arapura, Bressler and Ramachandran that if \( H^1EL_2(\pi_1(X)) \neq 0 \) then every Zariski dense representation \( \varrho : \pi_1(X) \to G \) factors through a representation of the fundamental group of an orbicurve.
2) It follows from [16] that if $H^1EL_2(S) \neq 0$, where $S$ is the factorizing spectral covering, then every Zariski dense representation $\rho : \pi_1(X) \to G$ factors through a representation of the fundamental group of an orbicurve.

3) It follows from [25] that if $Prym_\sigma(S,X)$ is nontrivial then every Zariski dense representation $\rho : \pi_1(X) \to G$ factors through a representation of the fundamental group of an orbicurve.

Therefore we arrive at:

**Corollary 7.3** If $\rho : \pi_1(X) \to G$ is a rigid Zariski dense representation and one of the above conditions is satisfied, then this representation is integral.

There is one more case for which we were able to prove the fact that every rigid representation is integral.

**Corollary 7.4** Let $X$ be a smooth projective variety such that $\pi_1(X)$ is a word hyperbolic group, acting on the corresponding $\mathbb{R}$-tree. Assume that $Out(\Gamma)$ is an infinite group. Let $\rho : \pi_1(X) \to SL(n,\mathbb{C})$ be a rigid Zariski dense representation such that $Ker(\rho) \supseteq Ker(R)$. Then $\rho : \pi_1(X) \to SL(n,\mathbb{C})$ is integral.

Here $Ker(R)$ is the kernel of the map

$$R : \pi_1(X) \to Isom(T),$$

where $Isom(T)$ is the group of isometries of the tree $T$, defined in the proof of theorem 6.7.

**Proof.** It follows from theorems 6.1, 7.1 and 6.7.

In some cases it is possible to prove that every rigid representation is motivic without assuming that the harmonic map to the building $B$ is of dimension one. It follows from Corollary 6.2 that:

**Corollary 7.5** Every rigid Zariski dense representation $\rho : \pi_1(X) \to G$ for which there exists an $G$-equivariant nonconstant continuous map $U : \tilde{X} \to HB$, is integral.

**Proof.** If the action of $G^1$ fixes a point in $HB$ then the image of $\rho : \pi_1(X) \to G$ is contained in a maximal compact subgroup of $G$. Using again the theorem of Baas we get that our representation is integral. Now if we have an action of $G$ without a fixed point we are in a position to apply Corollary 6.2. In this case it is very easy to prove (see [29]) the pluriharmonicity of $U$ since we keep almost the same metric on the boundary of every chamber. Then we get a spectral covering as in Theorem 7.1. Also the spectral covering, which we obtain has an involution on it. Following [38] and [39] we obtain an involution which acts on the curve $Y$ as well. We mod out the spectral
covering $S$ and the curve $Y$ by $\mathbb{Z}_2$ and we have a holomorphic map between the quotients. The rest of the proof is the same as in the main theorem. We finish it by applying theorem 7.3.

The above corollary actually implies that if there exists an $\rho$-equivariant nonconstant continuous map $U: \tilde{X} \to HB$ and $\rho$ is a rigid representation then the action of $\rho$ on $B$ should have a fixed point.

We think of the above corollary as a generalization of the tree case, since the tree is negatively curved. Unfortunately as we saw in Theorem 6.3 it is in general impossible to make $G$ act on simply connected negatively curved spaces. So one should try another approach for the general Simpson conjecture. It seems one way of doing that would be to generalize the result of Katz to higher dimensions. While at the moment the Conjecture 7.1 and Conjecture 7.2 look out of reach in their complete generality the following conjecture looks more doable at this moment.

**Conjecture 7.3** Every rigid Zariski dense representation $\varphi: \pi_1(X) \to G$ of a group $\pi_1(X)$ with the property $T$ is integrable.

### 8 The Shafarevich conjecture

Inspired by the results in [27] we make in this section a connection between the Shafarevich map and the results of this paper. We recall the definition of the Shafarevich map [27], and describe some of its properties.

**Definition 8.1** Let $X$ be a smooth projective variety. Then we call a rational map

$$Sh: X \dashrightarrow Sh(X),$$

the Shafarevich map of $X$ iff:

1) $Sh(X)$ is a normal projective variety;

2) The rational map $Sh: X \to Sh(X)$ has connected fibers and;

3) there are countable many closed subvarieties $D_i$ in $X$ ($D_i \neq X$) such that for every irreducible subvariety $Z$ in $X$, not in the union of the $D_i$ we have:

$$Sh(Z) = \text{point if } \text{im}[\pi_1(Z') \to \pi_1(X)] \text{ is finite.}$$

It is easy to see that if the rational map $Sh: X \to Sh(X)$ exists, it is unique up to birational equivalence.

One can give a relative version of this definition with respect to a normal subgroup $H$ of $\pi_1(X)$. To do that one needs to require in part 3) of the definition above that $\text{im}[\pi_1(Z') \to \pi_1(X)] \cap H$
has a finite index in $\text{im}[\pi_1(Z') \rightarrow \pi_1(X)]$. In this case we write $Sh^H : X \rightarrow Sh^H(X)$ for the corresponding rational map.

The following theorem is proven in [27].

**Theorem 8.1** (János Kollár) The rational maps $Sh : X \rightarrow Sh(X)$ and $Sh^H : X \rightarrow Sh^H(X)$ exist.

The properties of the Shafarevich map we have described above provide us with enough information to prove the following theorem:

**Theorem 8.2** Let $X$ be a smooth projective variety which has type B (from Theorem 1.1) Zariski dense representation of its fundamental group to a Lie group $G$ defined over a field with discrete valuation. Then:

$$\dim_{\mathbb{C}} Sh^H(X) = \dim_{\mathbb{C}}(Y) \leq \text{rank}_{\mathbb{C}}(G),$$

where $Y$ is the variety defined in Theorem 1.1 and $H = \text{Ker}(\varrho)$. Moreover we have that some finite nonramified coverings of $Y$ and $Sh(X(\Gamma))$ (defined below) are birationally isomorphic.

**Proof.** From the construction of the holomorphic map $X \rightarrow Y$ in the proof of Theorem 1.2 we know that if $Z$ is the general fiber of this map we have that the intersection of the subgroups $R = \text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$ and $H = \text{Ker}(\varrho)$ is a subgroup of finite index in $R$. This implies that $Sh^H : X \rightarrow Sh^H(X)$ factors through $X \rightarrow Y$ and we get:

$$\text{rank}_{\mathbb{C}}(G) \geq \dim_{\mathbb{C}} Y \geq \dim_{\mathbb{C}} Sh^H(X).$$

To get the inverse inequality we argue as follows:

Let us first do two reductions.

Denote by $L$ the intersection of all subgroups of finite index in $\pi_1(X)$. According to remark 4.4 the conclusions of the main theorem are still true if we work with $\pi_1(X)/L$. Therefore we can assume that

$$H = \bar{H}.$$ 

Define now $R = \text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$.

The second reduction is to show that we can always assume that $R$ is contained in $H$. If $R$ is not contained in $H$ then $R = \text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$ intersects $H = \text{Ker}(\varrho)$ in a subgroup of finite index in $\text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$. Then let the image of this finite subgroup be the finite subgroup $\Pi$ in $\pi_1(X)/H$. Now we do the following modification:
Observe that \( \pi_1(X)/H \) is a residually finite group due to the fact that \( G \) is an affine algebraic group. Therefore since we have mod out \( \pi_1(X) \) by \( L \) and \( H = \overline{H} \) we can find a subgroup of finite index \( \Theta \) in \( H = \overline{H}/H \) which does not intersect \( \Pi \). We conclude that for the covering, which corresponds to the group \( \Theta \), \( R \) is contained in \( H \).

So from now on we assume that \( R \) is contained in \( H \).

Applying 4.8 \cite{27}, we see that there exists some subgroup \( \Gamma \) of finite index in \( \pi_1(X) \) such that for the finite nonramified covering \( X(\Gamma) \) we have a complete factorization through \( Sh^H(X(\Gamma)) \).

The construction goes as follows. We have already constructed \( X(\Gamma) \) in section 4. According to 4.8 there exists a map:

\[
 f_* : \pi_1(X(\Gamma)) \to \pi_1(Sh^H(X(\Gamma)))
\]

such that

\[
 Ker f_* \subseteq \overline{R} \subseteq \overline{H} = H.
\]

But \( Y \) was defined as the variety of minimal dimension, through which we have a factorization. Therefore:

\[
 dim_C Sh^H(X) \geq dim_C Y.
\]

Combining this with the previous inequality we get:

\[
 dim_C Sh^H(X) = dim_C(Y) \leq rank_C(G).
\]

\[\square\]

We prove now a weak version of the Shafarevich conjecture. The Shafarevich conjecture says that for every smooth projective variety \( X \) there exists a there exists a Stein manifold \( Sh(\overline{X}) \) and a proper map with connected fibers \( Sh : \overline{X} \to Sh(\overline{X}) \).

An easy consequence of previous theorem is the following theorem:

**Theorem 8.3** Let \( X \) be a smooth projective variety which has a type B (from Theorem 1.2) Zariski dense representation \( \rho \) of its fundamental group to a Lie group \( G \) defined over a local field \( K \). Define \( H = \text{Ker}(\rho) \). Then \( Y \) is isomorphic to \( Sh^H(X') \) and the map

\[
 Sh^H : X' \to Sh^H(X'),
\]

is a morphism. Here \( X' \) and \( Y \) are the same as in Theorem 1.2.
Proof. Without lost of generality we can work with the case when $H = \text{Ker}(\rho)$ is finite.

It is clear that if $Z$ is a subvariety in $X$ and the fundamental group of $Z$ is finite then $Z$ goes to a point in $B$, where $B$ is the corresponding building. This follows from a very general theorem of M. Bridson saying that every finite group, which acts on a CAT space has a fixed point. Therefore the corresponding harmonic maps are just constants. We need to show that if $Z$ is a subvariety of $X$ with a finite fundamental group then then it goes to a point in $Y$.

According to Gromov and Schoen the map $U : \bar{X} \rightarrow B$ is essentially regular, namely the intrinsic derivative exists everywhere (see [15]). We claim that $U^*\alpha_i$ is zero on $Z$. On the intersection of the smooth locus of $U$ and $Z$ this is obviously true. For the singular points we use the same argument as in the proof of Lemma 3.1, namely the existence of the intrinsic derivative. This existence implies existence of a kernel of $U^*$ in $T^*X'$ and we conclude $U^*\alpha_i$ is zero on $Z$. This implies that $Z$ will go to a point in $Y$ even if $Z$ it is contained in the singular set of $U$.

Another way to show that $Z$, a subvariety of $X$ with a finite fundamental group goes to a point in $Y$ is the following. Using theorem 6.3 [15] we obtain the following estimate, which follows from the existence of the intrinsic derivative

$$\lim_{x \rightarrow S} \|dU(x)\| = 0,$$

where $S$ is the singular set of $U$.

According to [15] $U(S)$ is contained in the closed faces of the simplices of the highest dimension. Now using the above estimate and approximating $U(S)$ in the normal directions of the faces containing it we get that $U^*\alpha_i$ is zero on $Z$. □

Therefore we obtain the following :

**Corollary 8.1** If the conditions of the above theorem we have $H = \text{ker}(\rho)$ is a finite group then the Shafarevich-Kollár conjecture is true.

**Proof.** According to the previous theorem the map

$$\text{Sh} : X' \rightarrow \text{Sh}(X'),$$

is a morphism. Consider the morphism

$$h : X \rightarrow Y$$

from the proof of the main theorem in section 4.

Assume that there is a subvariety $Z$ in $X$ with the property that $\text{im}[\pi_1(Z) \rightarrow \pi_1(X)]$ is finite and it does not go to a point in $Y$. But then pulling it back to $X'$ we get a subvariety $Z'$ in $X'$ such
that $\pi_1(Z')$ is a finite group due to a theorem of Campana. Therefore $im[\pi_1(Z') \to \pi_1(X')]$ is a finite group and $Z'$ goes to a point in $\text{Sh}(X')$. Now using the map $\text{Sh}(X') \to Y$ we obtain a contradiction since the image of $Z'$ in $Y$ is the same as the image of $Z$ in $Y$. Therefore $Y = \text{Sh}(X)$.

The Shafarevich-Kollár conjecture in general is connected with another question of Gromov:

Can we find a faithful discrete cocompact action of every Kähler word hyperbolic group to a space with $K \leq 0$?

We believe that the Shafarevich-Kollár conjecture follows for every Kähler group for which this could be done.

Now we show two examples for which the above condition is satisfied and we can actually prove the Shafarevich conjecture.

**Corollary 8.2** Let $X$ be a compact Kähler manifold and $\pi_1(X)$ is an amalgamated product of two groups and $\pi_1(X)$ is word hyperbolic. Then the Shafarevich-Kollár conjecture is true for $X$.

**Proof.** According to [15] we have a faithful discrete cocompact action of $\pi_1(X)$ on a tree. The rest is just repeating the argument from the proof of Theorem 8.3. \qed

In the same way Theorem 6.1 implies:

**Corollary 8.3** Let $X$ be a compact Kähler manifold and $\pi_1(X)$ has an a faithful discrete cocompact action on a two dimensional complexes defined in section 6. Then the Shafarevich-Kollár conjecture is true for $X$.

Now we discuss some possible applications of the Shafarevich-Kollár conjecture.

One possible application of the above statement is to try to answer the following question of M. Ramachandran [37].

**Conjecture 8.1** (M. Ramachandran) Let $X$ be a smooth projective variety which has a type B (from Theorem 1.2) Zariski dense representation $\rho$ of its fundamental group to some Lie group $G$ defined over a local field $K$. Define $H = \text{Ker}(\rho)$ and assume $H$ is a finite group. Then one of the following holds:

1) The universal cover of $X$ satisfies the Bochner - Hartogs property, namely for every $\alpha \in A_{c}^{0,1} (\tilde{X})$ satisfying

$$\bar{\partial} \alpha = 0,$$

there exists $u \in C_{c}^{\infty} (\tilde{X})$ such that
\[ \tilde{\partial} u = \alpha. \]

2) \( \pi_1(X) \) is comesurable to the fundamental group of a compact Riemann surface.

Here \( A^0_{c}(\tilde{X}) \) is the space of compactly supported smooth (0,1) forms on \( \tilde{X} \) and \( C^\infty_c(\tilde{X}) \) is the space of all compactly supported smooth functions on \( \tilde{X} \) with complex values.

**Remark 8.1** M. Ramachandran actually stated the conjecture in much bigger generality namely for every smooth projective variety.

We hope to answer this question by using the maps

\[ U : \tilde{X} \rightarrow B \]

and

\[ Sh^H : X' \rightarrow Sh^H(X'). \]

to produce enough plurisubharmonic functions on \( \tilde{X} \).

We study now some questions connected with the topological nature of the map

\[ Sh^H : X' \rightarrow Sh^H(X'). \]

We give a partial answer to some questions posed by János Kollár at the end of [1]. These questions are about how much the Shafarevich variety \( Sh(X) \) depends on the original variety \( X \) e.g. does a deformation of \( X \) give a deformation of \( Sh(X) \) an so on.

**Corollary 8.4** Let \( X \) be a smooth projective variety which has nonrigid finite kernel Zariski dense representation in \( SL(2, \mathbb{C}) \). Let \( X_t, t \in \text{to the unit disc } \Delta \), be a holomorphic deformation of \( X \). Then \( Sh(X_t) \) is a holomorphic deformation of \( Sh(X) \).

**Proof.** The moduli space of \( \lambda \) - connections (see [14]) deforms together with \( X \). The moduli space of \( \lambda \)-connections itself is a deformation of the moduli space of \( SL(2, \mathbb{C}) \) representations to the moduli space of \( SL(2, \mathbb{C}) \) Higgs bundles. Since we have fixed the representation we get a constant holomorphic section over \( \Delta \times \mathbb{P}_1 \), which specializes at 0 of \( \mathbb{P}_1 \) to an Higgs bundle. As we said the moduli space of \( SL(2, \mathbb{C}) \) Higgs bundles is a deformation of the moduli space \( \lambda \) - connections so we get a holomorphic section to the family of moduli spaces of \( SL(2, \mathbb{C}) \) Higgs bundles. over
\[ \Delta. \] Also we have a holomorphic map from the moduli space of \( SL(2, \mathbb{C}) \) Higgs bundles to the \( Hilb(T^*(X)) \), which corresponds to every Higgs bundle a spectral covering (see [BS]). We get this way a holomorphic deformation of the corresponding spectral coverings and as a consequence a family of curves \( C_t \) through which according to our main theorem the representation factors.

But since this is an finite kernel Zariski dense representation we have \( Sh(X_t(\Gamma)) = C_t(\Gamma) \), where \( C_t(\Gamma) \) is the spectral covering of \( X_t(\Gamma) \). Observe that the map \( Sh(X_t(\Gamma)) \to Sh(X_t) \) has the same Galois group as \( X_t(\Gamma) \to X_t \). Therefore we can mod out by this group and get that \( Sh(X_t) \) deforms itself.

\[ \square \]

Even more is true:

**Corollary 8.5** Let \( X \) be a smooth projective variety which has nonrigid finite kernel Zariski dense representation \( \rho : \pi_1(X) \to SL(2, \mathbb{C}) \). Let \( Y \) be another smooth projective varieties which is homeomorphic to \( X \). Then \( Sh(X) \) is homeomorphic to \( Sh(Y) \).

**Proof.** Due to the fact that we have a finite kernel Zariski dense representation we conclude that the \( \pi_1(X) \) is infinite and therefore \( Sh(X(\Gamma)) \) is a curve. Therefore \( \pi_1(X) = \pi_1(Y) \) and let say that \( C_X(\Gamma) = Sh(X(\Gamma)) \) and \( C_Y(\Gamma) = Sh(Y(\Gamma)) \) are the curves through which our representation factors. The only thing we need to show is that \( g(C_X) = g(C_Y) \), where \( g(C_X) \) is the genus of \( C_X \). According to a theorem of Siu (see for example [40]) if we have a map from \( \pi_1(X) \) to \( \pi_1(C_Y) \) we have a holomorphic map \( h : X(\Gamma) \to C'_Y \) to some other curve \( C'_Y \). Due to the fact that \( C_X = Sh(X(\Gamma)) \) we get that \( g(C'_Y) \geq g(C_X) \). Now to finish the proof we change the places of \( X \) and \( Y \) in the above argument.

\[ \square \]

**Remark 8.2** The above corollary works for \( X_t \) homotopy equivalent to \( X \) as well as for \( Sh_K(X) \) instead of \( Sh(X) \), where \( K \) is the kernel of \( \rho \) and \( K \) is an infinite group.

The following two corollaries are straightforward consequences from section 7.

**Corollary 8.6** The corollary above is true if instead of a Zariski dense representation in \( SL(2, \mathbb{C}) \) we require the existence of a finite kernel surjective homomorphism of the fundamental group of \( X \) to the fundamental group of a 2 dimensional negatively curved complex of groups (see paragraph 6).

**Corollary 8.7** The corollary above is true if instead of a Zariski dense representation in \( SL(2, \mathbb{C}) \) we require the existence of a nonrigid Zariski dense representation of dimension 1 (see paragraph 7) to any simple complex Lie group.
In the same way we treat the case when \( Y \) is a surface.

**Theorem 8.4** Let \( X \) be a smooth projective variety which has a finite kernel Zariski dense representation \( \rho : \pi_1(X) \to G \) of type B in \( G \), where \( G \) is a simple Lie group such that \( \text{rank}_\mathbb{C} G = 2 \). Let \( X_t \) be a holomorphic deformation of \( X \). Then \( \text{Sh}(X_t(\Gamma)) \) is a holomorphic deformation of \( \text{Sh}(X(\Gamma)) \).

**Proof.** The assumptions of the theorem imply that we have a factorization through the representation of the fundamental group of \( Y \), where \( Y \) is either an algebraic curve or an algebraic surface. We have considered the first case in the previous corollary.

Let us consider now the case \( \dim \text{Sh}(X(\Gamma)) = 2 \). Observe that \( \text{Sh}(X_t(\Gamma)) \) is defined only up to birational isomorphism. Therefore when we say that \( \text{Sh}(X_t(\Gamma)) \) is a holomorphic deformation of \( \text{Sh}(X(\Gamma)) \), we mean that the minimal model of \( \text{Sh}(X_t(\Gamma)) \) is a holomorphic deformation of the minimal model of \( \text{Sh}(X(\Gamma)) \). Since \( X_t \) is a holomorphic deformation of \( X \) and we work with the same representation in the same way as in the previous corollary, we can show that the corresponding spectral coverings of \( X_t \) and \( X \) are holomorphic deformations of each other. What we need to show is that if \( Y \) is a holomorphic deformation of \( Y_t \) then the minimal model of \( Y \) is a holomorphic deformation of \( Y_t \). But in case of surfaces with \( \kappa(Y) \geq 0 \) (this is exactly our case since \( \text{Sh}(X(\Gamma)) \) has a large fundamental group) this follows from a theorem of Iitaka. The fact that \( Y \) and \( \text{Sh}(X(\Gamma)) \) are isomorphic proves the theorem. \( \square \)

The following nonvanishing theorem is a consequence of (3).

**Theorem 8.5** Let \( X \) be a smooth fourfold of general type with a type B (from Theorem 1.1) Zariski dense representation of its fundamental group \( \rho : \pi_1(X) \to G \). Here \( G \) is a rank 2 or 3 simple Lie group over field with a discrete valuation. Then either:

1) \( \rho : \pi_1(X) \to G \) factors through a representation of the fundamental group of an orbicurve or;

2) \( P_n=H^0(X,nK_X) \) for \( n \geq 4 \) is not zero.

**Proof.** According to Theorem 1.2 our representation factors through a representation of the fundamental group of some variety \( Y \) of dimension less than or equal to 3. If this \( \dim Y = 1 \) then we are in part 1) of the theorem.

Consider the case where the dimension of \( Y \) is at least 2. The previous theorem tells us that \( Y \) is birational to \( \text{Sh}^H(X) \). Following János Kollár [27] (4.5 and 5.8) we can work with a morphism \( X^1 \to S \), where \( X^1 \) is a nonramified finite covering of \( X \) and \( S \), a smooth variety with generically large fundamental group. It follows from [27] that in this case nonvanishing theorems for \( X^1 \) imply nonvanishing theorems for \( X \). We need to consider the following cases:
1) $\dim S = 3$ and $S$ is of general type. By \cite{27} (10.1) we have that $P_m(S) \geq 1$ for $m \geq 2$. Observe also that the fibers $X_s$ of the map $X^1 \to S$ are curves of general type and therefore we have $P_1(X_s) \geq 1$. Using \cite{27} (10.4) we obtain:

$$P_n(X) \geq P_{n-2}(S) \geq 1, \text{ for } n \geq 4.$$  

2) $\dim S = 3$ and $S$ is abelian variety. Then we can apply \cite{27} (8.10), which gives a strong nonvanishing. Strong nonvanishing is equivalent to:

$$h^0(X^1, K_{X^1} \otimes D) \neq 0 \text{ for } D \text{ big divisor on } X^1 \text{ and } X^1 \text{big birational to } X.$$ 

In our situation we can make $3K_X = D$ since $X$ is of general type and therefore:

$$P_n(X) \geq 1 \text{ for } n \geq 4.$$ 

This argument actually applies in the previous case too. We need only that the fundamental group of $S$ is generically large.

3) $\dim S = 2$ and $S$ is abelian. We know that the fiber $X_s$ of $X^1 \to S$ is a surface of general type. If we have a complete factorization $X^1 \to S$ of the representation $\rho$ we get a contradiction, since $\rho$ is a Zariski dense representation and we cannot have a Zariski dense representation of a free abelian group in a simple Lie group $G$. Therefore we need to work with a finite ramified cover of $S$ - a surface of general type with generically large fundamental group, which we will also denote by the letter $S$. We are in a position to apply \cite{27} (10.4).

We have:

$$P_n(X_s) \geq 1 \text{ for } n \geq 2,$$

and also

$$P_n(S) \geq 1 \text{ for } n \geq 2.$$ 

Therefore we obtain from \cite{27} (10.4):

$$P_n(X) \geq P_{n-2}(S) \geq 1, \text{ for } n \geq 4.$$  

4) $\dim S = 2$ and $S$ is of general type. Then the fiber $X_s$ of $X^1 \to S$ is also a surface of general type. In this case we argue as before, again using \cite{27} (10.4).

**Remark 8.3** Obviously the parts 1) and 2) are not mutually exclusive.
In the last application we are going to be concerned with the nonexistence of some representations for a special class of algebraic varieties. If we think of $\text{Hom}(\pi_1(X), G)$ as the first nonabelian cohomology group, then the nonexistence of certain representations is a sort of vanishing theorems.

**Definition 8.2** We say that an algebraic variety $X$ has a generically large fundamental group iff for every subvariety $Z$ of $X$ outside some union of divisors $\bigcup D_i$ we have that:

$$\text{im}[\pi_1(Z') \to \pi_1(X)]$$

is an infinite group.

**Corollary 8.8** Let $X$ be a variety with a generically large fundamental group. Then:

1) A finite nonramified covering of $X$ is birationally isomorphic to $\text{Sh}(X)$ and

2) $X$ does not have Zariski dense, finite kernel representations in a complex Lie group $G$ if $\dim_{\mathbb{C}}(X) > \text{rank}_{\mathbb{C}}(G)$.

**Proof.** Part 1) of this corollary follows immediately from the definition of the Shafarevich variety. Part 2) of this corollary follows from theorem 8.3 since:

$$\dim_{\mathbb{C}}X = \dim_{\mathbb{C}}\text{Sh}(X) = \dim_{\mathbb{C}}(Y) \leq \text{rank}_{\mathbb{C}}(G),$$

and this contradicts our assumption that:

$$\dim_{\mathbb{C}}(X) > \text{rank}_{\mathbb{C}}(G).$$

\[\square\]

9 Final remarks

By definition all of our spectral coverings are zero schemes of sections in the cotangent bundle of $X$. Then obviously they can be deformed to the zero section. The zero section taken with some multiplicity (in the case of buildings this multiplicity was equal to $w$) corresponds to a harmonic map to a point in the Euclidean building $B$ corresponding to $G$. Therefore we have an action of $\pi_1(X)$ on $B$ which fixes a point. In all of our arguments before we have excluded the case where $\pi_1(X)$ fixes a point of $B$. But if the fixed point $P$ is not a point at infinity of $B$ we assign to the action of $\pi_1(X)$ the constant equivariant harmonic map $\bar{X} \to P$. Obviously all holomorphic differentials in this case are equal to zero and the spectral covering $S$ is just $X$, the zero section.
of the cotangent bundle of $X$. Therefore every nonrigid subgroup $G(B)$ (the group of isometries of $B$) which cannot be deformed to another subgroup of $G(B)$ fixing a point of $B$, cannot be the fundamental group of a smooth projective variety. Of course one needs to find (if possible) the proper notion of deformation. It is clear that such a statement might then be true in much more general situations, namely for nonrigid subgroups of the group of isometries of quite general nonpositively curved length spaces.

Another direction one can try to apply the techniques developed in this paper is to study the p-adic uniformization defined in [33], [30], [20].

Using the techniques described above one can try to study the discrete subgroups of the infinite dimensional Lie groups $SDiff(D)$, where $D$ is a Riemannian domain. It follows from [1], [31] and [32] that these groups are nonpositively curved in some sense if $D = T^2$, $D = T^n$ or $D = S^2$. Then using the harmonic map technique one obtains some finiteness results. These remarks are going to be an object of future considerations.

References

[1] V.I. Arnold Mathematical methods in classical mechanics.. Springer -Verlag , Berlin, 1990.

[2] D. Arapura, P. Bressler, M.Ramachandran On the fundamental group of a compact Kähler manifold . Duke Mathematical Journal, v.68, No.3, 1992, pp. 477-488.

[3] H. Bass Groups of integral representation type. . Pacific Jurnal of Mathematics, 86, 1980 , 15-51.

[4] N. Benakli Thesis . Universite de Paris VII Sud, Orsay, France, 1992 .

[5] K. Brown Buildings. Springer, 1989.

[6] M. Bridson Geodesics and curvature in Metric Simplicial Complexes . Preprint ,Cornel University, 1991.

[7] H. Cartan Collected works . V.2, p. 806 ,France, 1979 .

[8] R. Charney, M. Davis Strict hyperbolization . Preprint IAS.,1993 .

[9] H. Clemens Degeneration of Kählerian manifolds. Duke Mathematical Journal., 1977, v. 44, pp. 215-290.

[10] K. Corlette Nonabelian Hodge theory. Preprint, University of Chicago, 1990.
[11] K. Corlette Archimedean superrigidity and hyperbolic geometry. Annals of Mathematics, 135 (1992) ,165-182.

[12] P. Deligne Hodge theory. Springer LNM, V. 900, pp. 806 - 900 . 1982 .

[13] R. Donagi Spectral covers. to appear in Journ. de Algebr . Geometrie ,Orsay, 1993.

[14] H. Girout Cohomologie nonabelian e. Springer - Verlag, 1982 .

[15] M. Gromov , R. Schoen Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one . IHES - Publications Mathematiques, 76, 1992, pp. 165-246.

[16] M. Gromov Kähler hyperbolicity and $L^2$ Hodge theory. Journal of Differential Geometry, 33 ,1991, pp. 263-292.

[17] M. Gromov Assymptotic invariants of hyperbolic groups . Geometric group theory, vol. 2, London Mathematical Society, LNS, 182 , 1994.

[18] A. Haefliger Complexes of group and orbihedra . Preprint, 1991.

[19] R. Hamilton Harmonic maps of manifolds with boundary. Lecture notes in Mathematics, 471, Springer -Verlag , Berlin, 1975.

[20] B. Hunt A Siegel modular 3 - fold that is a Picard modular 3- fold . Compositio Mathematica , v. 76 , 1990 , pp. 203-242.

[21] S. Iitaka Logarithmic forms of algebraic varieties. Journal. Fac. Univ. Tokyo, 1A, 23, 1976, pp. 525-544.

[22] J. Jost Equilibrium maps between metric spaces. Preprint, Ruhr- Universitat Bochum, 1993 .

[23] J. Jost, S. T. Yau Harmonic maps and supperrigidity. Proc. Symp. Pure. Math., 54, pp. 245-280, 1993 .

[24] N. Katz Rigid local systems. Preprint , Princeton University, 1993 .

[25] L. Katzarkov , T. Pantev Representations of the fundamental groups , which are pullbacks. to appear in Journal of Differential Geometry.

[26] Y. Kawamata Characterization of abelian varieties. Compositio Mathematicae, Vol.43, Fasc. 2, 1981, pp. 253-276.
[27] János Kollár *Shafarevich maps and plurigenera of algebraic varieties*. Inventiones Mathematicae, 113, Fasc. 1, 1993, pp. 165-215.

[28] János Kollár *Shafarevich maps and automorphic forms*. Lecture notes, SGI, Park City, Utah, 1993.

[29] N. Korevaar, R. Schoen *Sobolev spaces and harmonic maps for metric target spaces*. Preprint, IAS-Princeton, 1993.

[30] A. Kurihara *Construction of p-adic unit balls and the Hirzebruch proportionality*. American Journal of Mathematics, v. 102, 1980, pp. 565-648.

[31] A. I. Lukatskii *Curvature of the diffeomorphisms preserving the measure*. Functional analysis and its applications, Vol. 13, 1979, pp. 174-177.

[32] A. I. Lukatskii *Curvature of the diffeomorphisms preserving the measure of the n-torus*. Uspechi Mat. Nauk, Vol. 36, N. 12, 1981, pp. 187-189.

[33] D. Mumford *An algebraic surface with $K$ ample, $K^2 = 9, p_g = q = 0* . American Journal of Mathematics, v. 101, 1979, pp. 233-244.

[34] N. Mok *Factorization of semisimple discrete representations of the Kähler groups*. Inventiones Mathematicae, v. 110, fasc.3, 1992, pp. 557-614.

[35] F. Paulin *Outer Automorphisms of Hyperbolic Groups and Small Actions on $\mathbb{R}$-Trees*. MSRI - publications, 19, Arboreal Group Theory, 1992, pp. 331-345.

[36] Z. Ran *On subvarieties of abelian varieties*. Inventiones Mathematicae, 62, 1981, pp. 459-479.

[37] M. Ramachandran *A Bochner-Hartogs type of theorem for coverings of compact Kähler manifolds*. Preprint, SUNY at Buffalo, 1993.

[38] C.T. Simpson *A Lefschetz theorem for $\pi_0$ of the integral leaves of holomorphic one-form*. Compositio Mathematicae, 87, 1993, pp. 99-113.

[39] C.T. Simpson *Integrality of rigid local systems of rank two on a smooth projective variety*. Preprint, Princeton University, 1991.

[40] C.T. Simpson *The ubiquity of the variations of Hodge structure*. Proceedings of the 1989 AMS Summer conference in Sundance, Utah.

[41] C.T. Simpson *Nonabelian Hodge theory*. Proceedings of the International Congress of Mathematicians - Kyoto 1990. Springer, Tokyo, 1991.
[42] C.T. Simpson *Some families of local systems over smooth projective varieties*. Annals of Mathematics, 138 (1993), pp. 337-425.

[43] C.T. Simpson *Higgs bundles and local systems*. IHES - Publications Mathematiques, 75, 1992, pp. 5-95.

[44] W. Thurston, H. B. Lawson, M. Gromov *Hyperbolic 4-manifolds and coformally flat 3-manifolds*. IHES - Publications Mathematiques, 68, 1988, pp. 27-45.

[45] K.-Zuo *Factorization of nonrigid Zariski dense representations of π₁ of projective manifolds*. Preprint, University of Kaiserslautern, 1992.