POWERS OF IDEALS ASSOCIATED TO \((C_4, 2K_2)\)-FREE GRAPHS

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Abstract. Let \(G\) be a \((C_4, 2K_2)\)-free graph with edge ideal \(I(G) \subset k[x_1, \ldots, x_n]\). We show that \(I(G)^s\) has linear resolution for every \(s \geq 2\). Also, we show that every power of the vertex cover ideal of \(G\) has linear quotients. As a result, we describe the Castelnuovo-Mumford regularity of powers of \(I(G)^s\) in terms of the maximum degree of \(G\).

1. Introduction

Let \(G\) be a simple graph with the vertex set \(V = \{x_1, \ldots, x_n\}\) and let \(S = k[x_1, \ldots, x_n]\) be the polynomial ring over a field \(k\). The edge ideal \(I(G)\) of \(G\) is the ideal generated by the monomials \(x_ix_j\) where \(\{x_i, x_j\}\) is an edge of \(G\). A set of vertices \(C\) is called a vertex cover if every edge of \(G\) contains a vertex of \(C\). The vertex cover ideal of \(G\), denoted by \(I(G)^\triangledown\), is generated by the squarefree monomials \(x_{i_1} \cdots x_{i_k}\) where \(\{x_{i_1}, \ldots, x_{i_k}\}\) is a vertex cover of \(G\). The vertex cover ideal of \(G\) is also known as Alexander dual of \(I(G)\).

Minimal free resolutions of edge ideals and vertex cover ideals are extensively studied and recently some attention has been given to regularity of powers of these ideals. It is well-known \([5, 20]\) that for any graded ideal \(I\) there exist non-negative integers \(c, d, s_0\) such that \(\text{reg}(I^s) = ds + c\) for all \(s \geq s_0\). Although the constant \(d\) can be determined from the generators of the ideal, no explicit formulas are known for \(c\) and \(s_0\). When \(I\) is an edge or cover ideal, the integers \(c\) and \(s_0\) were computed for some families of graphs \([1, 2, 3, 8, 12, 17, 18, 19, 21]\).

By Fröberg’s Theorem \([10]\) the edge ideal \(I(G)\) has a linear resolution if and only if \(G\) is co-chordal, i.e., the complement graph \(G^c\) is chordal. Herzog, Hibi and Zheng \([17]\) showed that if \(G\) is a co-chordal graph, then all powers of \(I(G)\) have linear resolutions. Recall that a graph is chordal if it has no induced cycle of length 4 or more. Francisco, Hà and Van Tuyl showed that if some power of \(I(G)\) has linear resolution, then the complement graph \(G^c\) has no induced cycle of length 4, i.e., \(G\) is gap-free. In this direction Peeva and Nevo \([25]\) asked the following question regarding this wider family of graphs: If \(G\) is a gap-free graph, then does \(I(G)^s\) have a linear resolution for every \(s \gg 0\)?

In Section \([2]\) we answer the question above in the affirmative under the additional assumption that \(G^c\) is gap-free. If both \(G\) and \(G^c\) are gap-free, then \(G\) is also known as \((C_4, 2K_2)\)-free. Our proof is based on a structural characterization of such graphs given in \([4]\) and a method given in \([2]\) to bound the regularity of powers of edge ideals.

A graded ideal \(I\) is called componentwise linear if for each \(d\), the ideal generated by all degree \(d\) elements of \(I\) has a linear resolution. The notion of componentwise linear ideal generalizes that of ideal with linear resolution. For a graded ideal, a stronger

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property than being componentwise linear is having linear quotients:

\[
\text{linear quotients } \implies \text{componentwise linear}
\]

equigenerated ideal with linear quotients \(\implies\) linear resolution

A graph \(G\) is called (sequentially) Cohen-Macaulay if the quotient ring \(S/I(G)\) is (sequentially) Cohen-Macaulay over every field \(k\). Due to a result of Eagon and Reiner [7] it is known that a graph is Cohen-Macaulay if and only if its vertex cover ideal has a linear resolution. More generally, it was proved in [13] that a graph is sequentially Cohen-Macaulay if and only if its vertex cover ideal is componentwise linear.

In general, powers of an ideal with linear resolution may not have linear resolutions. Sturmfels [26] gave an example of an ideal which has linear quotients but the second power of the ideal has no linear resolution. On the other hand, one may ask the following question: Given a (sequentially) Cohen-Macaulay graph, what can be said about the powers of its vertex cover ideal? Francisco and Van Tuyl [9] proved that the vertex cover ideal of a Cohen-Macaulay chordal graph have linear resolutions. Furthermore, they conjectured that all powers of the vertex cover ideal of a chordal graph are componentwise linear. It is also known that all powers of vertex cover ideals of Cohen-Macaulay families of bipartite [15, 24] and cactus [22] graphs have linear resolutions.

In Section 3 we show that all powers of the vertex cover ideal of a \((C_4, 2K_2)\)-free graph are componentwise linear. Our proof is based on showing that such ideals have linear quotients. Since the regularity of a componentwise linear ideal can be determined from its generators, we obtain a formula for the regularity of powers of the vertex cover ideal in terms of the maximum degree of the graph.

2. Powers of Edge Ideals of \((C_4, 2K_2)\)-Free Graphs

2.1. Background on Graph Theory. Throughout this paper \(G\) will denote a finite simple graph. We write \(G = (V, E)\) where \(V\) and \(E\) are respectively the sets of vertices and edges of the graph. We say a vertex \(v\) is a neighbor of \(u\) if \(\{u, v\} \in E\). In this case, we simply write \(uv \in G\) and say \(u\) and \(v\) are adjacent. The neighbor set of \(v\), denoted by \(N(v)\), consists of all vertices of \(G\) that are adjacent to \(v\). A vertex is isolated if it has no neighbors. The complement graph \(G^c\) has the same vertices as \(G\) and \(uv \in G^c\) if \(uv \notin G\). We say \(G\) is connected if there is a path between every pair of vertices.

A graph \(H\) is called a subgraph of \(G\) if the vertex and edge sets of \(H\) are contained respectively in those of \(G\). A subgraph \(H\) of \(G\) is called an induced subgraph if \(uv \in G\) implies \(uv \in H\) for all vertices \(u\) and \(v\) of \(H\). We say \(G\) is \(H\)-free if \(G\) has no induced subgraph isomorphic to \(H\). If \(A\) is a set of vertices of \(G\), then \(G - A\) denotes the induced subgraph which is obtained from \(G\) by removing the vertices in \(A\). We say \(A\) is an independent set if no two vertices of \(A\) are adjacent in \(G\). A complete graph (or clique) is a graph such that every pair of vertices are adjacent. A complete graph on \(n\) vertices is denoted by \(K_n\). A cycle graph with vertices \(v_1, \ldots, v_n\) and edges \(v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\) is denoted by \(C_n = (v_1v_2 \ldots v_n)\). A graph is chordal if it has no induced cycle of length 4 or more. The complement graph of \(C_4\) is denoted by
Remark 2.2. Observe that any cycle of length at least \( H \) in the induced subgraph of \( (C_4, 2K_2) \)-free graphs is a split graph. Therefore, a \((C_4, 2K_2)\)-free graph has no induced \( C_n \) for \( n \geq 6 \). Thus a \((C_4, 2K_2)\)-free graph has no induced \( C_n \) for \( n \geq 6 \).

2.2. Bounding the Regularity. The (Castelnuovo-Mumford) regularity of a monomial ideal \( I \subset S = k[x_1, \ldots, x_n] \) is given by

\[
\text{reg}(I) = \max\{j - i : b_{i,j}(I) \neq 0\}
\]

where \( b_{i,j}(I) \) are the graded Betti numbers of \( I \). An ideal \( I \) generated in degree \( d \) is said to have a linear resolution if for all \( i \geq 0 \) \( b_{i,j}(I) = 0 \) for all \( j \neq i + d \).

If \( G \) is a graph on the vertices \( x_1, \ldots, x_n \), then the edge ideal of \( G \) is defined as

\[
I(G) = (xy : xy \text{ is an edge of } G).
\]

The method of polarization reduces the study of minimal free resolutions of monomial ideals to that of square-free monomial ideals. Therefore quadratic monomial ideals can be studied via edge ideals. The following classical result of Fröberg [10] gives a combinatorial characterization of edge ideals which have linear resolutions.

Theorem 2.3. [10] Theorem 1] The minimal free resolution of \( I(G) \) is linear (i.e., \( \text{reg}(I(G)) = 2 \)) if and only if the complement graph \( G^c \) is chordal.

We recall the following well-known results.

Theorem 2.4. [14] Corollary 1.6.3] Let \( I \subset S \) be a monomial ideal and let \( I^{\text{pol}} \) be its polarization. Then \( \text{reg}(I) = \text{reg}(I^{\text{pol}}) \).

Lemma 2.5. If \( I \subset S \) is a monomial ideal, then \( \text{reg}(I, x) \leq \text{reg}(I) \) for any variable \( x \).

Lemma 2.6. Let \( I \subset S \) be a monomial ideal, and let \( m \) be a monomial of degree \( d \). Then

\[
\text{reg}(I) \leq \max\{\text{reg}(I : m) + d, \text{reg}(I, m)\}.
\]

Lemma 2.7. [6] Lemma 3.1] Let \( x \) be a vertex of \( G \) with neighbours \( y_1, \ldots, y_m \). Then

\[
\text{reg}(I(G)) \leq \max\{\text{reg}(I(G - \{N(x) \cup \{x\}\})) + 1, \text{reg}(I(G - \{x\}))\}.
\]

Moreover, \( \text{reg}(I(G)) \) is equal to one of these terms.

Given an edge ideal, the following order was defined on its powers:
Definition 2.8. [2] Discussion 4.1] Let $I$ be an edge ideal which is minimally generated by the monomials $L_1, \ldots, L_k$. Consider the order $L_1 > L_2 > \cdots > L_k$ and let $s \geq 1$. Given two minimal monomial generators $M, N$ of $I^s$ we set $M > N$ if there exists an expression $L_n^a L_2^b \cdots L_k^b = M$ such that $(a_1, \ldots, a_k) > \text{lex}(b_1, \ldots, b_k)$ for every expression $L_1^b \cdots L_k^b = N$.

The next result follows from Theorem 4.12 in [2]. The last part of the theorem was stated in a different form in [2] Corollary 5.3]. We state the theorem in a more general form and provide the proof for the last part.

Theorem 2.9. [2] Let $I$ be an edge ideal which is minimally generated by the monomials $L_1, \ldots, L_{r_1}$. For each $s \geq 1$, let $L_1^{(s)} > L_2^{(s)} > \cdots > L_{r_1}^{(s)}$ be the order on the minimal monomial generators of $I^s$ induced by the order $L_1 > \cdots > L_{r_1}$ as defined in Definition 2.8. Then for all $s \geq 1$ and $1 \leq \ell \leq r_s - 1$,

$((I^{s+1}, L_1^{(s)}, \ldots, L_{\ell}^{(s)}): L_{\ell+1}^{(s)}) = ((I^{s+1}, L_{\ell+1}^{(s)}), \text{some variables}).$

In particular, if $\text{reg}(I) \leq 4$ and $\text{reg}(((I^{s+1}, L_1^{(s)}, \ldots, L_{\ell}^{(s)}): L_{\ell+1}^{(s)})) \leq 2$ for every $0 \leq \ell \leq r_s - 1$ and $s \geq 1$, then $I^t$ has linear resolution for every $t \geq 2$.

Proof. The first part of the theorem follows from [2] Theorem 4.12]. To prove the last part suppose that $\text{reg}(I) \leq 4$ and $\text{reg}(((I^{s+1}, L_1^{(s)}, \ldots, L_{\ell}^{(s)}): L_{\ell+1}^{(s)})) \leq 2$ for every $0 \leq \ell \leq r_s - 1$ and $s \geq 1$. Let $t \geq 1$. Using Lemma 2.6 we get

$\text{reg}(I^{t+1}) \leq \max\{\text{reg}(I^{t+1}, L_1^{(t)}) + 2t, \text{reg}(I^{t+1}, L_1^{(t)})\}$

$\leq \max\{2 + 2t, \text{reg}((I^{t+1}, L_1^{(t)}): L_2^{(t)}) + 2t, \text{reg}((I^{t+1}, L_1^{(t)}, L_2^{(t)}))\}$

$= \max\{2 + 2t, \text{reg}((I^{t+1}, L_1^{(t)}, L_2^{(t)}))\}$

$\cdots$

$\leq \max\{2 + 2t, \text{reg}(I^{t+1}, L_1^{(t)}, \ldots, L_{r_1}^{(t)})\}$

$= \max\{2 + 2t, \text{reg}(I^t)\}.$

Since $\text{reg}(I) \leq 4$, the result follows inductively. □

Theorem 2.10. [2] Lemmas 6.14 and 6.15] Let $G$ be a gap-free graph with edge ideal $I = I(G)$ and let $e_1, \ldots, e_s$ be some edges of $G$ where $s \geq 1$. Then $(I^{s+1}: e_1 \cdots e_s)_{\text{pol}}$ is the edge ideal of some gap-free graph $G'$. Also, if $C_n = (u_1, \ldots, u_n)$ is a cycle on $n \geq 5$ vertices such that $C_n^c$ is an induced subgraph of $G'$, then $C_n^c$ is an induced subgraph of $G$ as well.

2.3. Regularity of Powers of Edge Ideals of $(C_4, 2K_2)$-free Graphs. We first bound the regularity of the edge ideal of a $(C_4, 2K_2)$-free graph. We use Lemma 2.1 and the description of $(C_4, 2K_2)$-free graphs given in Theorem 2.1.

Proposition 2.11. Let $G$ be a $(C_4, 2K_2)$-free graph. Then $\text{reg}(I(G)) \leq 3$.

Proof. We may assume that $G$ has no isolated vertices as the removal of isolated vertices does not change the edge ideal. We proceed by induction on the number of vertices of $G$. If $G$ is $K_2$, then the result is clear. Let $V = V_1 \cup V_2 \cup V_3$ be a partition of the vertices of $G$ as in Theorem 2.1. If $V_3 = \emptyset$, then $G^c$ is chordal and the result follows from Theorem 2.3. Therefore let us assume that $V_3 \neq \emptyset$. Let $x$ be a vertex that belongs to $V_3$. Then both $G - \{x\}$ and $(G - \{x\})^c$ are also gap-free.
By induction we have \( \text{reg}(I(G - \{x\})) \leq 3 \). Also \( G - \{N(x) \cup \{x\}\} \) consists of an edge and possibly some isolated vertices. Thus \( \text{reg}(I(G - \{N(x) \cup \{x\}\})) = 2 \) and the proof follows from Lemma 2.7. 

We are now ready to prove our main result in this section.

**Theorem 2.12.** If \( G \) is a \((C_4,2K_2)\)-free graph, then \( I(G)^s \) has a linear resolution for all \( s \geq 2 \).

**Proof.** Let \( V = V_1 \cup V_2 \cup V_3 \) be a partition of the vertices of \( G \) as in Theorem 2.11. If \( V_3 = \emptyset \), then \( G \) and \( G^c \) are chordal and the result follows from [17, Theorem 3.2] and Theorem 2.12. Let us assume that \( V_3 \neq \emptyset \).

Let \( A \) be the set of edges of \( G \) that contain a vertex of \( V_3 \). Let \( B \) be the set of remaining edges of \( G \). Fix a total order on the edges of \( G \) such that \( a > b \) for any \( a \in A \) and \( b \in B \). Let \( s \geq 1 \) be arbitrarily fixed and consider the total order \( M_1 > \cdots > M_r \) on the minimal monomial generators of \( I(G)^s \) induced by the order on the edges of \( G \) as described in Definition 2.8. By Theorem 2.9 and Proposition 2.11 it suffices to show that

\[
\text{reg}(\langle I(G)^{s+1}, M_1, \ldots, M_{t+1} \rangle : M_{t+1}) \leq 2 \quad \text{for every } 0 \leq t \leq r - 1.
\]

First, we claim that if \( M \) is a minimal monomial generator of \( I(G)^s \) which is divisible by a vertex of \( V_3 \), then \( \text{reg}(\langle I(G)^{s+1} : M \rangle^{\text{pol}}) = 2 \). To this end, let \( G' \) be the gap-free graph with edge ideal \( \langle I(G)^{s+1} : M \rangle^{\text{pol}} \) as in Theorem 2.10. From Theorem 2.3, it suffices to show that \( G' \) has no induced \( C_n \) for \( n \geq 5 \). Assume for a contradiction \( C_n \) is an induced subgraph of \( G' \) for some \( n \geq 5 \). Then \( C_n \) is an induced subgraph of \( G \) as well by Theorem 2.10. By Remark 2.2 we must have \( n = 5 \). Notice that the complement of a cycle of length 5 is again a cycle of length 5. Observe that the induced subgraph of \( G \) on \( V_3 \) is the only induced cycle of \( G \) of length 5. Therefore \( V(C^*_{5}) = V_3 \). Then by [8, Lemma 3.6] no vertex of \( V_3 \) divides \( M \), which is a contradiction.

Let \( t \) be the largest index such that \( M_t \) is divisible by a vertex in \( V_3 \). Then each of \( M_1, \ldots, M_t \) is divisible by a vertex in \( V_3 \) and we have \( \text{reg}(\langle I(G)^{s+1} : M_i \rangle^{\text{pol}}) = 2 \) for all \( 1 \leq i \leq t \) by the previously proved claim. Combining Lemma 2.5, Theorem 2.4 and Theorem 2.3 we get

\[
\text{reg}(\langle I(G)^{s+1}, M_1, \ldots, M_{t+1} \rangle : M_{t+1}) \leq 2 \quad \text{for every } 0 \leq t \leq r - 1.
\]

Let \( 0 \leq j \leq r - t - 1 \) be fixed. We will show that

\[
\text{reg}(\langle I(G)^{s+1}, M_1, \ldots, M_{t+j} \rangle : M_{t+j+1}) \leq 2.
\]

We claim that

\[
\{ z : z \in V_3 \} \subseteq \langle I(G)^{s+1}, M_1, \ldots, M_{t+j} \rangle : M_{t+j+1} \rangle.
\]

Indeed, let \( M_{t+j+1} = (vw)N \) where \( v \in V_3 \) and \( N = 1 \) for \( s = 1 \) and \( N \) is a minimal monomial generator of \( I(G)^{s+1} \) for \( s \geq 2 \). Then for every \( u \in V_3 \) the monomial \( (uv)N \) is a minimal generator of \( I(G)^s \) and \( (uv)N > M_{t+j+1} \). Thus \( (uv)N : (M_{t+j+1}) = (u) \).

Now, using Theorem 2.4 we get

\[
\langle I(G)^{s+1}, M_1, \ldots, M_{t+j} \rangle : M_{t+j+1} \rangle = \langle I^{s+1} : M_{t+j+1} \rangle, \text{ some variables}
\]

where \( J \) is the edge ideal of the induced subgraph of \( G \) on \( V_1 \cup V_2 \). By Theorem 2.3, the ideal \( J \) has a linear resolution. From the proof of [2, Theorem 6.16] and Theorem 2.4 it is known that \( \text{reg}(\langle I^{s+1} : M_{t+j+1} \rangle) = 2 \). Thus the proof follows from Lemma 2.5. □
Remark 2.13. In Theorem [2.12] the order on the generators of \( I(G) \) is crucial. For example, let \( V_1 = \{a\}, V_2 = \{b\} \) and \( V_3 = \{c, d, e, f, g\} \) with
\[
I(G) = (ab, bc, bd, be, bf, bg, cd, de, ef, fg, gc).
\]
Using Macaulay2 [11] we get \( \text{reg}(I(G)^2 : (ab)) = 3 \). Therefore an order on the edges which starts with \( ab \) does not allow one to apply Theorem [2.9].

3. Powers of Vertex Cover Ideals of \((C_4, 2K_2)\)-free graphs

Let \( G \) be a graph on the vertices \( x_1, \ldots, x_n \). A set \( C \) of vertices of \( G \) is called a vertex cover if every edge of \( G \) contains a vertex from \( C \). The vertex cover \( C \) is called minimal if no proper subset of \( C \) is a vertex cover of \( G \). The vertex cover ideal of \( G \), denoted by \( I(G)^\vee \), is defined as
\[
I(G)^\vee = (x_1 \ldots x_k : \{x_1, \ldots, x_k\} \text{ is a minimal vertex cover of } G).
\]

In the next lemma, we describe the minimal vertex covers of \((C_4, 2K_2)\)-free graphs.

Lemma 3.1. Let \( G \) be a \((C_4, 2K_2)\)-free graph with the vertex set \( V = V_1 \cup V_2 \cup V_3 \) partitioned as in Theorem [2.2] and let \( V_3 \neq \emptyset \). Then any \( A \subseteq V \) is a minimal vertex cover of \( G \) if and only if \( A \) has one of the following forms:

(i) \( A = V_2 \cup \{a, b, c\} \) where \( a, b, c \in V_3 \), \( ab \in G \), \( ac \notin G \) and \( bc \notin G \),

(ii) \( A = N(a) \) for some \( a \in V_2 \).

Proof. One can show that if \( A \) satisfies one of the conditions above, then it is a minimal vertex cover. Conversely, suppose that \( A \) is a minimal vertex cover. Let \( C_5 \) be the induced subgraph on \( V_3 \). We consider cases.

Case 1: Suppose that there exists a vertex \( u \in V_3 \) such that \( u \notin A \). Since \( u \) is adjacent to every vertex in \( V_2 \) we must have \( V_2 \subseteq A \). This implies \( A \cap V_1 = \emptyset \) and \( A \setminus V_2 \) is a minimal vertex cover of \( C_5 \). Then \( A \) must be of the form given in (i) since any minimal vertex cover of \( C_5 \) consists of three vertices \( a, b, c \) such that \( ab \in C_5 \), \( ac \notin C_5 \) and \( bc \notin C_5 \).

Case 2: Suppose that \( V_3 \subseteq A \). Since any minimal vertex cover of \( C_5 \) contains 3 vertices, there exists \( a \in V_2 \) such that \( a \notin A \). Observe that any minimal vertex cover of \( K_n \) has \( n-1 \) vertices. Therefore \( V_2 \setminus \{a\} \subseteq A \). Since \( A \) covers \( G \), we have \( N(a) \cap V_1 \subseteq A \). By minimality of \( A \) no vertex of \( V_1 \) except the neighbors of \( A \) can belong to \( A \). Thus \( A = N(a) \) as in (ii). \( \square \)

Definition 3.2 (Linear quotients). A monomial ideal \( I \subset \mathbb{k}[x_1, \ldots, x_n] \) is said to have linear quotients if there is an ordering \( m_1, \ldots, m_q \) on the minimal monomial generators of \( I \) such that for every \( i = 2, \ldots, q \) the ideal \( (m_1, \ldots, m_{i-1}) : m_i \) is generated by a subset of \( \{x_1, \ldots, x_n\} \).

A connected graph is called a cactus if each edge belongs to at most one cycle. A generalized star graph based on \( G_{m,n_1,\ldots,n_k} \) is a special type of chordal graph, see [23, Definition 1.3]. Mohammadi [22, 23] proved the following result regarding the powers of vertex cover ideals of these graphs.

Theorem 3.3. [22, Theorem 3.3] [23, Theorem 1.5] If \( G \) is either a Cohen-Macaulay cactus graph or a generalized star graph based on \( G_{m,n_1,\ldots,n_k} \), then all powers of the vertex cover ideal of \( G \) are weakly polymatroidal. In particular, they have linear quotients.
Lemma 3.4. Let \( f_1, \ldots, f_5 \) be the minimal vertex covers of \( C_5 \) and let \( s \geq 1 \). Then every minimal monomial generator \( M \) of \((I(C_5))^s\) has a unique expression of the form \( M = f_1^{\alpha_1} \cdots f_5^{\alpha_5} \).

Proof. Let \( I(C_5) = (u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_1) \) and let \( f_1 = u_1u_2u_4, f_2 = u_4u_5u_2, f_3 = u_2u_3u_5, f_4 = u_3u_4u_1, f_5 = u_5u_1u_3 \). Suppose \( M = f_1^{\beta_1} \cdots f_5^{\beta_5} \) is another expression. Since each \( f_i \) has the same degree, \( \sum_{i=1}^{5} \alpha_i = \sum_{i=1}^{5} \beta_i \). The exponent of \( u_2 \) in \( M \) is \( \sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \beta_i \). Therefore \( \alpha_4 + \alpha_5 = \beta_4 + \beta_5 \). The exponent of \( u_1 \) in \( M \) is \( \alpha_1 + \alpha_4 + \alpha_5 = \beta_1 + \beta_4 + \beta_5 \). Thus \( \alpha_1 = \beta_1 \). From the symmetry of the graph it follows that \( \alpha_i = \beta_i \) for each \( i = 1, \ldots, 5 \). \( \square \)

Lemma 3.5. Let \( G \) be a \((C_4, 2K_2)\)-free graph with the vertex set \( V = V_1 \cup V_2 \cup V_3 \) partitioned as in Theorem 2.7 and let \( V_3 \neq \emptyset \). Let \( M_1, \ldots, M_q \) be the minimal monomial generators of \((I(G))^s\). Then for every \( s \geq 1 \) and non-negative integers \( m_1, \ldots, m_q \) with \( \sum q_{i=1}^{q} m_i = s \), the monomial \( M_1^{m_1} \cdots M_q^{m_q} \) is a minimal generator of \((I(G))^s\).

Proof. If \( V_2 = \emptyset \), then the result is clear since \((I(G))^s\) is generated in degree \( 3s \). So, let us assume that \( V_2 = \{z_1, \ldots, z_k\} \) for some \( k \geq 1 \). Let \( I(C_5)^s = (f_1, \ldots, f_5) \). Using Lemma 3.4 let

\[
G = \prod_{i=1}^{5} (V_2 f_i)^{\alpha_i} \prod_{j=1}^{k} N(z_j)^{\beta_j}
\]

be a generator of \((I(G))^s\) for some \( \alpha_i, \beta_j \geq 0 \) with \( \sum \alpha_i = \sum \beta_j = s \). We claim that \( G \) is minimal. Let

\[
H = \prod_{i=1}^{5} (V_2 f_i)^{\gamma_i} \prod_{j=1}^{k} N(z_j)^{\kappa_j}
\]

be a minimal generator of \((I(G))^s\) such that \( H \) divides \( G \). Observe that for every \( j = 1, \ldots, k \) the exponents of \( z_j \) in \( G \) and \( H \) are respectively \( s - \beta_j \) and \( s - \kappa_j \). Since \( H \) divides \( G \) we get \( \kappa_j \geq \beta_j \) for all \( j = 1, \ldots, k \). Assume for a contradiction that \( \kappa_0 > \beta_0 \) for some \( j_0 \). Then we get \( \sum \alpha_i > \sum \beta_j \). Let \( u = \gcd(H, V_3^s) \) and \( v = \gcd(G, V_3^s) \) so that \( u \) divides \( v \). Observe that \( u \) has degree

\[
3 \sum_{i=1}^{5} \gamma_i + 5 \sum_{j=1}^{k} \kappa_j = 3 \sum_{i=1}^{5} \gamma_i + 5(s - \sum_{i=1}^{5} \gamma_i) = 5s - 2 \sum_{i=1}^{5} \gamma_i.
\]

Similarly, \( v \) has degree \( 5s - 2 \sum_{i=1}^{5} \alpha_i \). Thus degree of \( u \) is greater than degree of \( v \), which is a contradiction. Now, we have \( \kappa_j \geq \beta_j \) for all \( j = 1, \ldots, k \). This implies \( \sum \alpha_i = \sum \gamma_i \). Then \( \prod_{i=1}^{5} f_i^{\alpha_i} \) and \( \prod_{i=1}^{5} f_i^{\gamma_i} \) are minimal generators of \((I(C_5))^A\) where \( A = \sum \alpha_i \). Since \( H \) divides \( G \) we get \( \prod_{i=1}^{5} f_i^{\alpha_i} = \prod_{i=1}^{5} f_i^{\gamma_i} \). Thus \( G = H \) and the minimality of \( G \) is established. \( \square \)

Lemma 3.6. If \( G \) is a connected split graph with at least one edge, then \( G \) is a generalized star graph based on \( G_{m, n_1} \) for some \( m, n_1 \).

Proof. Suppose that \( G \) is a split graph with the vertex set \( V = V_1 \cup V_2 \) where \( V_1 \) is independent and \( V_2 \) is a clique. We may assume that \( V_2 \) is a maximal clique of \( G \). If \( V_1 = \emptyset \), then \( G = G_{1, |V_2| - 1} \). Let \( u_1 \in V_1 \) and let \( N(u_1) = \{y_1, \ldots, y_m\} \). Since \( V_2 \) is a maximal clique we have \( V_2 \setminus N(u_1) \neq \emptyset \). Let \( V_2 \setminus N(u_1) = \{x_{1,1}, \ldots, x_{1,n_1}\} \) and \( V_1 \setminus \{u_1\} = \{u_{1,1}, \ldots, u_{1,m_1}\} \). Now we have
which shows that \( G \) is a generalized star graph based on \( G_{m,n} \). 

We now prove our main result in this section.

**Theorem 3.7.** Let \( G \) be a \((C_4, 2K_2)\)-free graph with at least one edge. Then \( (I(G)^\vee)^s \) has linear quotients for every \( s \geq 1 \).

**Proof.** We may assume that \( G \) has no isolated vertices. Suppose that the vertex set \( V = V_1 \cup V_2 \cup V_3 \) of \( G \) is partitioned as in Theorem 2.1. If \( V_2 = \emptyset \), then \( I(G) = (C_5) \). Since \( C_5 \) is a connected cactus graph and it is Cohen-Macaulay by [9, Proposition 4.1] it follows from Theorem 3.3 that every power of \( I(C_5)^\vee \) has linear quotients. If \( V_3 = \emptyset \), then the proof follows from Lemma 3.6 and Theorem 3.3. Therefore let us assume that both \( V_2 \) and \( V_3 \) are non-empty. Let \( V_2 = \{z_1, \ldots, z_k\} \) and let \( I(C_5)^\vee = (f_1, \ldots, f_5) \). We claim that every minimal generator \( M \) of \( (I(C_5)^\vee)^s \) has a unique expression of the form \( M = \prod_{i=1}^5 (V_2 f_i)^{\gamma_i} \prod_{j=1}^k N(z_j)^{\beta_j} \). Note that \( M \) has such expression by Lemma 3.3. Suppose that \( M = \prod_{i=1}^5 (V_2 f_i)^{\alpha_i} \prod_{j=1}^k N(z_j)^{\gamma_j} \). For each \( j \in \{1, \ldots, k\} \) the exponent of \( z_j \) in \( M \) is \( s - \beta_j = s - \kappa_j \) and thus \( \beta_j = \kappa_j \). Then \( \sum_{i=1}^5 \alpha_i = \sum_{i=1}^5 \gamma_i \) and from Lemma 3.4 it follows that \( \alpha_i = \gamma_i \) for each \( i \in \{1, \ldots, 5\} \).

Let \( L_1^s < L_2^s < \cdots < L_r^s \) be a linear quotients order on the minimal monomial generators of \((I(C_5)^\vee)^p\) for each \( p \). Consider a total order \( M_1, \ldots, M_q \) on the minimal monomial generators of \((I(G)^\vee)^s\) such that for any

\begin{equation}
(3.1) \quad M_t = \prod_{i=1}^5 (V_2 f_i)^{\alpha_i} \prod_{j=1}^k N(z_j)^{\beta_j} \quad \text{and} \quad M_t = \prod_{i=1}^5 (V_2 f_i)^{\gamma_i} \prod_{j=1}^k N(z_j)^{\kappa_j}
\end{equation}

\( M_t \) precedes \( M_\ell \) in the order (i.e., \( t < \ell \)) if one of the following holds:

(i) \( \sum_{i=1}^5 \alpha_i > \sum_{i=1}^5 \gamma_i \)

(ii) \( (\alpha_1, \ldots, \alpha_5) = (\gamma_1, \ldots, \gamma_5) \) and \( (\beta_1, \ldots, \beta_k) \succ \text{lex} (\kappa_1, \ldots, \kappa_k) \)

(iii) \( A := \sum_{i=1}^5 \alpha_i = \sum_{i=1}^5 \gamma_i \) and \( \prod_{i=1}^5 f_i^{\alpha_i} < \prod_{i=1}^5 f_i^{\gamma_i} \) in the linear quotients order of \((I(C_5)^\vee)^4\).

Let \( 2 \leq \ell \leq q \) be as in Eq. (3.1). We will show that \((M_1, \ldots, M_{\ell-1}) : M_\ell \) is generated by variables. If \( \kappa_j = 0 \) for all \( j = 1, \ldots, k \), then for all \( t < \ell \) with \( M_t \) as in Eq. (3.1) the condition (iii) is satisfied. In this case, the colon ideal

\[ (M_1, \ldots, M_{\ell-1}) : M_\ell = (L_1^s, \ldots, L_{\ell-1}^s) : L_\ell^s \]

is generated by some variables. So, let us assume that \( \kappa_j \neq 0 \) for at least one \( j \). We claim that \( z_j \) is a generator of \((M_1, \ldots, M_{\ell-1}) : M_\ell \) for every \( \kappa_j \neq 0 \). Indeed, if \( \kappa_j \neq 0 \), then by Lemma 3.5 \( M_r = (M_r V_2 f_5) / N(z_j) \) is a minimal monomial generator for some \( r < \ell \). Observe that \( M_r / \gcd(M_r, M_\ell) = z_j \) which shows that \( z_j \) is a generator of the colon ideal.

Let \( t < \ell \) and let \( M_t \) be as in Eq. (3.1). Observe that if \( \beta_j < \kappa_j \) for some \( j \), then \( z_j \) divides \( M_t / \gcd(M_t, M_\ell) \). If \( \beta_j \geq \kappa_j \) for all \( j \), then we have \( (\beta_1, \ldots, \beta_k) = (\kappa_1, \ldots, \kappa_k) \)
and the condition (iii) holds. Thus using Lemma 3.5 we get

\((M_1, \ldots, M_{\ell-1}) : M_\ell = (z_j : \kappa_j \neq 0) + (\text{possibly some vertices of } V_3)\)

as desired. \(\Box\)

If \(v\) is a vertex of a graph \(G\), then the degree of \(v\) is the number of neighbors of \(v\). The maximum degree of \(G\), denoted by \(\Delta(G)\), is the maximum degree of its vertices.

**Corollary 3.8.** If \(G\) is a \((C_4, 2K_2)\)-free graph, then for each \(s \geq 1\)

\[\text{reg}((I(G)^v)^s) = \begin{cases} 3s & \text{if } G \text{ is a } C_5 \text{ with isolated vertices} \\ s\Delta(G) & \text{otherwise.} \end{cases}\]

**Proof.** It is known [14, Theorem 8.2.15] that if an ideal has linear quotients, then it is componentwise linear. Therefore by [14, Corollary 8.2.14] for each \(s \geq 1\), \(\text{reg}((I(G)^v)^s)\) is equal to the highest degree of a generator in a minimal set of generators of \((I(G)^v)^s\). Let \(V = V_1 \cup V_2 \cup V_3\) be partitioned as in Theorem 2.1. Let us assume that \(V_3 \neq \emptyset\) since otherwise the result is already known.

**Case 1:** Suppose that \(V_3 \neq \emptyset\). Then by Lemma 3.1 for any minimal vertex cover \(A\) of \(G\) and for all \(v_i \in V_1\) we have

\[|N(v_1)| < |N(v_3)| < |A| \leq |N(v_2)|\]

which shows that maximum cardinality of a minimal vertex cover of \(G\) is \(\Delta(G)\).

**Case 2:** Suppose that \(V_3 = \emptyset\). First note that \(N(a)\) is a minimal vertex cover of \(G\) for every \(a \in V_2\). Also, notice that \(V_2\) is a minimal vertex cover of \(G\) if and only if \(N(a) \cap V_1 \neq \emptyset\) for every \(a \in V_2\). If \(A\) is a minimal vertex cover of \(G\), then either \(A = V_2\) or \(A = N(a)\) for some \(a \in V_2\). Therefore the maximum cardinality of a minimal vertex cover of \(G\) is \(\Delta(G)\), which completes the proof. \(\Box\)

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