COMPLETENESS OF THE ISOMORPHISM PROBLEM
FOR SEPARABLE C*-ALGEBRAS

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Abstract. We prove that the isomorphism problem for separable nuclear C*-algebras is complete in the class of orbit equivalence relations. In fact, already the isomorphism of simple, separable AI C*-algebras is a complete orbit equivalence relation. This means that any isomorphism problem arising from a continuous action of a separable completely metrizable group can be reduced to the isomorphism of simple, separable AI C*-algebras. As a consequence, we get that the isomorphism problems for separable nuclear C*-algebras and for separable C*-algebras have the same complexity. This answers questions posed by Elliott, Farah, Paulsen, Rosendal, Toms and Törnquist.

1. Introduction

Broadly speaking, a problem $P$ in a class $\Gamma$ is called complete in $\Gamma$ if any other problem in $\Gamma$ can be reduced to $P$. Complete problems typically appear in logic and computer science, perhaps with the most prominent examples of NP-complete problems.

In the continuous setting, a descriptive complexity theory for problems arising as Borel and analytic equivalence relations on standard Borel spaces, has been developed by Kechris, Louveau, Hjorth and others [35, 45, 1, 37, 38] over the last 30 years. The classification problems arising in this setting are of the following form: given an analytic or Borel equivalence relation $E$ on a standard Borel space $X$, decide whether two points in $X$ are $E$-equivalent. Of great interest here are the equivalence relations given by Borel actions of separable completely metrizable (i.e. Polish) groups on standard Borel spaces or, equivalently, continuous actions of Polish groups on Polish spaces (see [8]). Typically, isomorphism problems arising in various areas of mathematics are easily translated into this language. The relative complexity is measured in terms of Borel reducibility: an equivalence relation $E$ on $X$ is Borel reducible to an equivalence relation $F$ on $Y$ if there is a Borel map $f : X \to Y$ such that $x_1 E x_2$ if and only if $f(x_1) F f(x_2)$ for every $x_1, x_2 \in X$. The meaning of this notion is that the function $f$, being computable (Borel), gives a way of reducing the problem of $E$-equivalence
of points in $X$ to that of $F$-equivalence of points in $Y$. We say that an equivalence relation $E$ is complete in a class $\Gamma$ of equivalence relations if it belongs to $\Gamma$ and every relation $F$ in $\Gamma$ is Borel-reducible to $E$. For examples of complete analytic equivalence relations see [53, 36]. Two relations $E$ and $F$ are bi-reducible if $E$ is Borel reducible to $F$ and $F$ is Borel reducible to $E$. For group actions, also a stronger notion (in the measure-theoretic context) is used: two group actions on standard Borel measure spaces $X$ and $Y$ are called orbit equivalent if there is a Borel isomorphism of $X$ and $Y$ which maps (a.e.) orbits to orbits (see [29]). We say that an equivalence relation is an orbit equivalence relation if it is bi-reducible with an equivalence relation induced by a Borel action of a Polish group. Descriptive complexity theory has applications to various classification problems arising in many areas and it has enjoyed spectacular successes, for instance the striking result of Thomas [65] on the relative complexity of isomorphism problems for torsion-free abelian groups or the results of Foreman, Rudolph and Weiss [28] on the conjugacy problem in ergodic theory.

The isomorphism problem for separable $C^*$-algebras has been studied since the work of Glimm in the 1960’s and evolved into the Elliott program that classifies $C^*$-algebras via their $K$-theoretic invariants. Glimm’s result [32], restated in modern language, implies that the isomorphism relation for UHF algebras is smooth (see [30, Chapter 5.4]). In the 1970’s the classification has been pushed forward to AF algebras via the $K_0$ group [17]. The Elliott invariant, which consists of the groups $K_0$ and $K_1$ together with the tracial simplex and the pairing map, was conjectured (see [19, 24]) to completely classify all infinite-dimensional, separable, simple nuclear $C^*$-algebras. The conjecture has been verified for various classes of $C^*$-algebras, e.g. certain classes of real rank zero algebras, AH algebras of slow dimension growth or separable, simple, purely infinite, nuclear algebras (modulo the universal coefficients theorem) [18, 21, 22, 16, 23, 63, 17] and there have been dramatic breakthroughs in the program, including the counterexamples to the general classification conjecture constructed by Rørdam [64] and Toms [67].

Classification program of $C^*$-algebras can be studied from the point of view of descriptive complexity theory (cf [20]) and the framework has been set up by Kechris [44] and Farah, Toms and Törnquist [25]. Next, generalizing the results of Thomsen [66] and using Elliott’s classification for AI algebras [16], Farah Toms and Törnquist showed [25, Corollary 5.2] that the isomorphism of separable simple AI algebras is an orbit equivalence relation and is not classifiable by countable structures (see [30, Chapter 10]). Recently, the former has been generalized by Elliott, Farah, Paulsen, Rosendal, Toms and Törnquist [15], who showed that in fact, the isomorphism of separable $C^*$-algebras is an orbit equivalence relation. Farah, Toms and Törnquist [25] and Elliott, Farah, Paulsen, Rosendal, Toms and Törnquist [15] asked what is the complexity of the isomorphism problem of separable and separable nuclear $C^*$-algebras. Understanding the complexity of these
problems can shed new light to the whole classification problem. In this paper we prove the following.

**Theorem 1.1.** The isomorphism relation of separable C*-algebras is complete in the class of orbit equivalence relations. In fact, already the isomorphism of separable nuclear (and even simple, separable, AI) C*-algebras is a complete orbit equivalence relation.

A perhaps less sophisticated way of stating this result is to say that any possible classification of separable nuclear C*-algebras must essentially use C*-algebras as the invariants. On the other hand, one could say that, in a way, Elliott’s conjecture turns out to be true: any separable C*-algebra is classified by the Elliott invariant, though of a perhaps different algebra. This solves the following problems [25, Problem 9.3], [25, Problem 9.7], [15, Question 4.1], [15, Question 4.2].

The proof of Theorem 1.1 must be geometric. In [25, Corollary 5.2] Farah, Toms and Törnquist showed that the relation of affine homeomorphism of Choquet simplices is Borel reducible to the isomorphism relation of simple, separable, AI algebras. Thus, Theorem 1.1 follows from the following.

**Theorem 1.2.** The relation of affine homeomorphism of Choquet simplices is complete in the class of orbit equivalence relations.

Gao and Kechris and, independently, Clemens [12] showed that the isometry relation of separable complete metric spaces is a complete orbit equivalence relation. Afterwards, using the theory of Lipschitz free spaces [5] of Weaver [70] and the results of Mayer-Wolf [56], Melleray [57] showed that the isometry of separable Banach spaces is a complete orbit equivalence relation. Our proof of Theorem 1.2 reduces the relation of isometry of separable complete metric spaces to the affine homeomorphism of Choquet simplices.

Haydon [36] (see also [10, Theorem 29.9] and [13, Page 143]) showed that any Polish space is homeomorphic to the extreme boundary of a simplex. One should note, however, that this construction is by no means unique and indeed, there are simplices with homeomorphic extreme boundaries which are not affinely homeomorphic (see [2, Page 119]). The general problem of determining a simplex from the structure of its extreme boundary is called the Dirichlet extension problem and it has a highly nontrivial solution [3].

In this paper, we take a different approach and, given a complete metric space, we construct a simplex whose extreme boundary only contains the metric space as a subset. The advantage over Haydon’s construction is that this construction is invariant under the isometry. Our simplices, which we call S-extensions of metric spaces, seem to be different from most of the typical constructions of simplices appearing in the literature (which are either formed as inverse limits or as quotients).

Our proof reveals also an interesting connection between three categories of objects: separable metric spaces, metrizable Choquet simplices and separable Banach spaces. It is somewhat parallel to the connection between
metric spaces and Banach spaces given by the Arens–Eells extensions. Recall that given a separable metric space, its Arens–Eells extension is a separable Banach space and the assignment of the Arens–Eells extensions is isometry invariant. On the other hand, given a simplex, one can look at the space of affine continuous functions on it. Now, the $S$-extensions give an invariant assignment of separable Banach spaces to separable metric spaces that factors through simplices as follows:

$$\text{metric space} \xrightarrow{S\text{-extension}} \text{Choquet simplex} \xrightarrow{\text{affine space}} \text{Banach space}$$

The three classes of separable metric spaces, metrizable Choquet simplices and separable Banach spaces contain universal objects that have been studied independently. The Urysohn space $U_{[68]}$ is the universal ultrahomogeneous separable metric space and the Urysohn sphere $U_1$ is its counterpart of diameter 1. Both these spaces are constructed in the same way and have finite isometry extension properties. For more on the Urysohn space and sphere, see [60, Chapter 5] or [58, 59, 51]. The Gurari˘ı space [34] is a similar object in the category of separable Banach spaces: a separable Banach space with an almost isometric extension property. This space turns out to be unique [54] (see also [48] for a recent proof of this result). The Poulsen simplex [62] is the unique (see [51]) metrizable Choquet simplex with dense extreme boundary. All these spaces can be formed via Fraïssé constructions from finite (or finite-dimensional) objects. It is known that the Gurari˘ı space is (isomorphic to) the dual of the space of affine functions on the Poulsen simplex [51] (see also [55]) but the relations between the Urysohn space and the Gurari˘ı space or the Poulsen simplex remain unclear. Fonf and Wojtaszczyk [27] showed that the Gurari˘ı space is not isomorphic to the Arens–Eells extension of the Urysohn space. It seems plausible, however, that a modification of the techniques of this paper can produce an invariant assignment of simplices of metric spaces so that the Poulsen simplex is associated to the Urysohn space. Then, the dual affine space would be the Gurari˘ı space.

This paper is organized as follows. In Section 2 we recall some basic facts from convex analysis and Choquet’s theory. In Section 3 we recall and slightly strengthen the results of Clemens, Gao and Kechris on the isometry of separable metric spaces. Section 4 contains some elementary back-and-forth constructions for building affine homeomorphism. The $S$-extension construction appears in Section 5. Sections 6 and 7 are a preparation for a coding construction of metrics that appears in Section 8. Section 9 contains some concluding remarks and questions.

1.1. Acknowledgement. Part of this work was done during the author’s stay at the Fields Institute during the thematic program in the fall 2012. The author is grateful for the hospitality of the Fields Institute and would like
to thank Antonio Avilés, George Elliott, Ilijas Farah and Stevo Todorcević for many inspiring discussions.

2. Convex analysis and Choquet’s theory

For basic concepts of convex analysis and Choquet’s theory we refer the reader to [41, Chapter 15] and to [61, 4, 6, 10]. In this paper we use the terms simplex and Choquet simplex interchangeably and all convex compact sets that we consider are metrizable.

We consider the Hilbert cube $[0, 1]^\mathbb{N}$ as a compact convex subset of a locally convex topological vector space (e.g. $\ell_\infty$ with the weak* topology) and with the standard metric on $[0, 1]^\mathbb{N}$ given by $d_{[0,1]^\mathbb{N}}(x, y) = \sum_n 2^{-n} |x(n) - y(n)|$. Given a set $A$ in a locally convex vector topological space, we write $\text{conv}(A)$ for the closed convex hull of $A$. For a compact convex set $C$, we write $\text{ext}(C)$ for the set of extreme points of $C$. Given two convex compact sets $C$ and $D$, we write $C \simeq D$ to denote that $C$ and $D$ are affinely homeomorphic. We write $\Delta^n$ for the $n$-dimensional simplex.

Whenever we consider a metric on a topological space, we assume it is compatible with the topology. Although convex compact sets are typically considered only with the affine and topological structure, we will sometimes use metrics on convex compact sets in locally convex vector topological spaces. We will always assume that if $C$ is a compact convex set with a metric $d_C$ on it, then there is a bigger convex compact set $D$ with $C \subseteq D$ and a norm $|| \cdot ||$ on $D$ such that $d_C(x, y) = ||x - y||$ for every $x, y \in C$. For convex compact subsets of the Hilbert cube, we can use the metric $d_{[0,1]^\mathbb{N}}$ and for subsets of $\mathbb{R}$ the standard distance on $\mathbb{R}$. Note that the open balls in such metrics are convex.

Given an inverse system $(S_n, \pi_n)$ of simplices (with $\pi_n : S_{n+1} \to S_n$ an affine continuous surjection) and $x \in \lim S_n$, we write $x \upharpoonright S_i$ (or even $x \upharpoonright i$ if it does not cause confusion) for the image of $x$ under the canonical projection map from $\lim S_n$ to $S_i$. Sometimes, we use the notation $x \upharpoonright S_i$ for $x \in S_j$ with $j > i$. It is worth noting here that, in general, it is not true that any continuous affine surjection is open and the exact characterization of when this happens has been given by Vesterstrøm [69]. The inverse limit of a system of simplices is again a simplex [40] and Lazar and Lindenstrauss [49] proved that any (metrizable) simplex is an inverse limit of a sequence of finite-dimensional simplices. Recently, López-Abad and Todorcević [52] showed that a generic inverse limit of simplices is affinely homeomorphic to the Poulsen simplex. For more on the Poulsen simplex we refer the reader to [41, Chapter 15] and [6] Chapter 3 Section 7.

The analysis on convex compact sets is dual to the theory of order unit Banach spaces via the spaces of affine functions (see [6] Chapter 2 and [14]).

The standard Borel structure on the space of simplices was introduced in [25] and is based on the parametrization of simplices proved by Lazar and Lindenstrauss [49] and the duality to the order unit spaces. Alternately, one
can also use the induced Borel structure from the space of compact subsets of the Hilbert cube (see [25, Section 4.1.4]) or of the Poulsen simplex. Note that for a convex compact set $C$, the space of compact convex subsets of $C$ is closed in $K(C)$. The space of simplices contained in $C$ is not closed but it is Borel in $K(C)$ [25, Lemma 4.7].

We will need a simple observation about affine continuous functions on convex compact sets. Note that if $(C,d_C)$ is a compact convex set in a locally convex vector topological space and $f,g : \Delta^n \rightarrow C$ are two affine functions such that $d_C(f(e), g(e)) \leq \varepsilon$ for some $\varepsilon > 0$ and every $e \in \text{ext}(\Delta^n)$, then $\|f - g\|_\infty \leq \varepsilon$. This follows by a simple induction on the dimension of the simplex (case $n = 1$ is obvious as the metric is given by a norm, and the induction step follows from the fact that every point in $\Delta^n$ belongs to a one-dimensional simplex whose one vertex belongs to $\Delta^{n-1}$ and the other is an extreme point of $\Delta^n$). Note that since any finite-dimensional convex compact set is an affine image of $\Delta^n$ for some $n \in \mathbb{N}$, the above observation is true for any finite-dimensional convex compact set in place of $\Delta^n$. Now, since the convex span of extreme points is dense in any convex compact set, the following is true as well.

**Proposition 2.1.** Let $C, D$ be convex compact sets in locally convex vector topological spaces and let $d_C$ be a metric on $C$. If $f,g : D \rightarrow C$ are two affine continuous functions such that $d_C(f(e), g(e)) \leq \varepsilon$ for some $\varepsilon > 0$ and every $e \in \text{ext}(D)$, then $\|f - g\|_\infty \leq \varepsilon$.

3. **Isometry of perfect Polish spaces of bounded diameter**

The standard Borel space of separable complete metric spaces is identified with the space of closed subsets of the Urysohn space $U$ with its Effros Borel structure. Gao and Kechris [31], and independently Clemens [11, 12], proved that the isometry relation on the space of separable complete metric spaces is complete in the class of orbit equivalence relations.

Similarly, we consider the standard Borel space of separable complete metric spaces of diameter bounded by 1 as the space of closed subsets of the Urysohn sphere $U_1$.

For a Polish space $X$, write $F_p(X)$ for the space of perfect (nonempty) subsets of $X$. Note if $X$ is compact, then $F_p(X)$ is $G_\delta$ in $K(X)$, which implies that for every Polish space $X$ the set $F_p(X)$ is Borel in $F(X)$. Given this, the standard Borel space of perfect separable metric spaces of diameter bounded by 1 is identified with $F_p(U_1)$.

The following result is essentially due to Gao, Kechris [31] and Clemens [11, 12] and we only indicate the necessary changes in their proof to obtain the stronger statement.

**Proposition 3.1.** The isometry relation of perfect subspaces of $U_1$ is complete in the class of orbit equivalence relations.
Then clearly \( X \) is a perfect Polish space.

Now, the map \( s \) replacing \( x \) (with diameter 1), for the union of \( C \in \mathbb{N} \) define the metric \( d \) on \( X \). It is routine \([31, \text{Section 2G}]\) to check that the map \( s \mapsto (X, d_s) \) is Borel. Now, the map \( s \mapsto d_s \) is a reduction of \( E(X_1) \) to the isometry of perfect separable complete metric spaces. If \( s, t \in F(X)^{\mathbb{N}} \) are \( E(X_1) \)-equivalent, then clearly \( X_s \) and \( X_t \) are isometric. On the other hand, suppose \( \varphi : X_s \rightarrow X_t \) is an isometry. Then it must be the case that \( \phi''|X_1 = X_1 \) since the points in \( X_1 \) are the only points \( y \in X_s \) such that the set \( \{ d_s(x, y) : x \in X_0, d_s(x, y) \in \mathbb{N} \} \) consists of all positive natural numbers, and the same is true for \( y \in X_t \).

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The above proposition can be also proved by adapting the proof of Clemens \([11, 12]\). Gao and Kechris \([31, \text{Section 2D}]\) show also that the isometry relation of subspaces of \( U \) is bireducible with the equivalence relation induced by the action of \( \text{Iso}(U) \) on \( F(U) \). Gao and Kechris deduce it from the fact that for every separable complete metric space \( X \) there is an isometric copy \( Z(X) \) of \( X \) in \( U \) such that \( F(U) \ni X \mapsto Z(X) \in F(U) \) is Borel and for every \( X, Y \in F(U) \) if \( X \) and \( Y \) are isometric, then there is an isometry \( \varphi \in \text{Iso}(U) \) such that \( \varphi'' Z(X) = Z(Y) \). This follows from the fact (see Gromov \([33, \text{Lemma 2.5}]\)).
Page 79] and [31, Lemma 2.2]) that for every separable metric space there is a metric extension $X^*$ of $X$ (obtained via the Katětov construction) that is canonically isometric to the Urysohn space $\mathbb{U}$ and such that any isometry $\varphi : X \to Y$ extends to an isometry $\varphi^* : X^* \to Y^*$. For more details on this construction see [30, Page 325]. Exactly the same arguments apply to metric spaces of diameter bounded by 1 and the Urysohn sphere $\mathbb{U}_1$, which gives the following.

**Proposition 3.2.** There is a Borel map $Z : \mathcal{F}_p(\mathbb{U}_1) \to \mathcal{F}_p(\mathbb{U}_1)$ such that $X$ is isometric to $Z(X)$ and if $X, Y \in \mathcal{F}_p(\mathbb{U}_1)$ are isometric, then there is an isometry $\varphi \in \text{Iso}(\mathbb{U}_1)$ with $\varphi''Z(X) = Z(Y)$.

4. Twisted homeomorphisms and approximate intertwinings

Say that an inverse system $(S_n, \pi_n : n \in \mathbb{N})$ of simplices is increasing if $S_n \subseteq S_{n+1}$ is a face of $S_{n+1}$ for each $n \in \mathbb{N}$ and $\pi_{n+1}(s) = s$ for each $s \in S_{n+1}$. Note that if $(S_n : n \in \mathbb{N})$ is an increasing system of simplices, then also $S_i \subseteq \lim S_n$ is a face of $\lim S_n$, for each $i \in \mathbb{N}$.

An approximate intertwining between two increasing inverse systems of simplices $(S_i : i \in \mathbb{N})$ and $(T_i : i \in \mathbb{N})$ is a sequence of affine continuous injective maps $\varphi_i : S_{n_i} \to T_{m_i}$ and $\psi_i : T_{m_i} \to S_{n_{i+1}}$ (for some increasing sequences $n_i$ and $m_i$ of natural numbers) such that $\varphi_n \subseteq \psi_{n-1}$, $\psi_n \subseteq \varphi_{n-1}$ (recall that the systems are increasing) and for each $\varepsilon > 0$ there exists $i$ such that for each $j > i$ and for each $x \in S_{m_j}$ and $y \in T_{m_j}$ we have

- $|d_{T_{m_j}}(\varphi_i(x \upharpoonright S_{n_i}), \varphi_j(x \upharpoonright T_{m_j}))| < \varepsilon$,
- $|d_{S_{n_{i+1}}}(\psi_i(y \upharpoonright T_{m_i}), \psi_j(y \upharpoonright S_{n_{i+1}}))| < \varepsilon$.

**Proposition 4.1.** Suppose $(S_i : i \in \mathbb{N})$ and $(T_i : i \in \mathbb{N})$ are two increasing inverse systems of simplices and there is an approximate intertwining between these systems. Then $\lim S_i$ and $\lim T_i$ are affinely homeomorphic via a map which extends all the maps in the approximate intertwining.

**Proof.** Write $S = \lim S_i$ and $T = \lim T_i$. Let $(\varphi_n : n < \omega)$ and $(\psi_n : n < \omega)$ form an approximate intertwining and for simplicity assume that $n_i = m_i = i$. Note that for each $x \in S$ and for each $n \in \mathbb{N}$ the sequence of points $(\varphi_k(x \upharpoonright S_k) \upharpoonright T_n : k \in \mathbb{N})$ in $T_n$ is Cauchy, by the property of approximate intertwining. For $x \in S$ write $x_n = \lim_k \varphi_k(x \upharpoonright S_k) \upharpoonright T_n$. Note that given $n < m \in \mathbb{N}$ we have $x_m \upharpoonright T_n = x_n$ since $(\lim_k \varphi_k(x \upharpoonright S_k) \upharpoonright T_m \upharpoonright T_n = \lim_k (\varphi_k(x \upharpoonright S_k) \upharpoonright T_m) \upharpoonright T_n).$ Thus, there exists a point $y \in T$ such that for each $y \upharpoonright T_n = x_n$, for each $n \in \mathbb{N}$. Write $\varphi(x) = y$. Note that the map $\varphi$ is affine since $(\upharpoonright T_n) \circ \varphi_n$ is affine for each $n \in \mathbb{N}$. Note also that for each $n \in \mathbb{N}$ we have $\varphi \upharpoonright S_n = \varphi_n$ since if $x \in S_n$, then $\varphi_k(x \upharpoonright S_k) = \varphi_n(x)$ for each $k > n$.

**Claim 4.2.** The map $\varphi$ is continuous.

**Proof.** Fix $\varepsilon > 0$. We need $\delta > 0$ such that if $x_1, x_1 \in S$ are such that $d_S(x_1, x_2) < \delta$, then $d_T(\varphi(x_1), \varphi(x_2)) < \varepsilon$. Fix $n \in \mathbb{N}$ big enough so that
$T_n$ is $\varepsilon/2$-dense in $T$ and $\varphi(x)$ is $\varepsilon/2$-close to $\varphi_n(x | S_n)$. Let $\delta > 0$ be such that if $d_{S_n}(x_1, x_2) < \delta$, then $d_{T_n}(\varphi_n(x_1), \varphi_n(x_2)) < \varepsilon/2$. Then, clearly, $\delta$ is as needed. \HRule

Analogously define the map $\psi : T \to S$ and argue that it is affine and continuous. Now, the facts that $\bigcup_n S_n$ is dense in $S$ and $\bigcup_n T_n$ is dense in $T$, $\varphi$ extends $\varphi_n$ and $\psi$ extends $\psi_n$ for each $n \in \mathbb{N}$, imply that $\varphi^{-1} = \psi$, as $\varphi_n \subseteq \psi_n^{-1}$ and $\psi_n \subseteq \varphi_{n+1}^{-1}$. \HRule

Given a sequence $C_n$ of compact convex sets of a compact convex set $C$ in a locally convex topological vector space, say that $C_n$ is convergent if it convergent in $K(C)$ (i.e. in the Hausdorff metric) and write $\lim_n C_n$ for the limit. Note that any increasing sequence of compact convex sets is convergent and then $\lim_n C_n$ is equal to the closure of its union. Suppose that $C_n \subseteq X$ and $D_n \subseteq X$ are two increasing sequences of compact convex sets. Say that a sequence of affine homeomorphic embeddings $\varphi_n : C_n \to C$ is a twisted approximation if for each $\varepsilon > 0$ there exists $n_0$ such that for every $k, l > n_0$ we have

\begin{enumerate}[(i)]
  \item $d_C(\varphi(x), \varphi(x)) < \varepsilon$ for every $x \in C_k$,
  \item $d_C(rng \varphi_k, \lim_n D_n) < \varepsilon$,
  \item $\varphi_k$ is an $\varepsilon$-isometry.
\end{enumerate}

In (ii) above $d_C$ stands for the Hausdorff metric extending $d_C$.

**Proposition 4.3.** Suppose $(C_n : n \in \mathbb{N})$ and $(D_n : i \in \mathbb{N})$ are two increasing sequences of convex compact subets of a convex compact set $C$ in a locally convex topological vector space. If there is a twisted approximation from $(C_n : n \in \mathbb{N})$ to $(D_n : n \in \mathbb{N})$, then $\lim_n C_n$ and $\lim_n D_n$ are affinely homeomorphic.

**Proof.** The proof is analogous to the proof of Proposition 4.1. Note that by (i), for every $x \in \bigcup_n C_n$ the sequences $\varphi_n(x)$ is convergent. Thus, we can define $\varphi' : \bigcup_n C_n \to C$ as $\varphi(x) = \lim_n \varphi_n(x)$. Note that (iii) implies that $\varphi'$ is continuous (even an isometry). Moreover, (iii) implies that given any $x \in \lim_n C_n$ and any sequence $x_k \in \bigcup_k C_k$ convergent to $x$, the sequence $\varphi'(x_k)$ is convergent and the limit does not depend on the choice of the sequence $x_k$. Thus, $\varphi'$ extends uniquely to $\varphi : \lim_n C_n \to C$, which again is continuous. The condition (ii) implies that $\text{rng}(\varphi) = \lim_n D_n$. By (iii) we also get that $\varphi$ is one-to-one and hence a homeomorphism. Finally, $\varphi'$ is affine as a limit of affine maps and hence so is $\varphi$. \HRule

Typically, convex compact sets are considered up to affine homeomorphism and that is why we stated Proposition 4.3 in the above form. However, we will apply twisted approximations to get actual equality of convex compact sets. Say that a sequence of affine homeomorphic embeddings $\varphi_n : C_n \to C$ is a strong twisted approximation if it is a twisted approximation and for each $\varepsilon > 0$ there exists $n_0$ such that for every $k > n_0$ we have
(i') \( d_C(\varphi_k(x), x) < \varepsilon \) for every \( x \in C_k \)

Constructing a strong twisted approximation thus boils down to showing that the Hausdorff distance between \( \lim_n C_n \) and \( \lim_n D_n \) is zero.

**Proposition 4.4.** Suppose \((C_n : n \in \mathbb{N})\) and \((D_n : i \in \mathbb{N})\) are two convergent sequences of convex compact subsets of a convex compact set \( C \) in a locally convex vector space. If there is a strong twisted approximation from \((C_n : n \in \mathbb{N})\) to \((D_n : n \in \mathbb{N})\), then \( \lim_n C_n \) and \( \lim_n D_n \) are equal.

**Proof.** Note that the affine homeomorphism constructed in Proposition 4.3 is actually the identity on \( \bigcup_n C_n \) and hence on all of \( \lim_n C_n \). \( \square \)

5. \( S \)-extensions of metric spaces

In this section we define the \( S \)-extensions of metric spaces. The definitions below stem from a simple observation that any finite metric space \((X, d_X)\) is isometric to the extreme boundary of a convex compact subset of \( \ell_\infty^n \) for some \( n \in \mathbb{N} \). Indeed, if \( X = \{x_1, \ldots, x_n\} \), then the map \( x_i \mapsto a_i = (d_X(x_i, x_1), \ldots, d_X(x_i, x_n)) \) maps \( X \) isometrically into the extreme boundary of the convex hull of \( \{a_i : i \leq n\} \) taken with the \( \ell_\infty \) metric.

Given a metric space \((X, d_X)\), together with its (enumerated) countable dense set \( D \subseteq X \) and a countable (enumerated) family \( F \subseteq C(X, [0, 1]) \) of continuous functions with values in the unit interval, we will form a convex compact set \( S(X, d_X, D, F) \). Let \( D = (d_n : n \in \mathbb{N}) \) and \( F = (f_n : n \in \mathbb{N}) \). Consider the vectors \( a_n = (f_1(d_n), f_2(d_n), \ldots) \in [0, 1]^\mathbb{N} \) and let \( S_n(X, d_X, D, F) \) be the convex hull of the set \( \{a_i : i \leq n\} \) in \([0, 1]^\mathbb{N}\). Note that the sequence \( S_n(X, d_X, D, F) \) is increasing, hence convergent in \( K([0, 1]^\mathbb{N}) \).

**Definition 5.1.** Given a metric space \((X, d_X)\) define \( S(X, d_X, D, F) \) as \( \lim_{n \to \infty} S_n(X, d_X, D, F) \).

We will always consider \( S(X, d_X, D, F) \) with the metric \( d_{[0,1]^\mathbb{N}} \) induced from the Hilbert cube. Note that since all \( S_n(X, d_X, D, F) \)'s are convex, so is \( S(X, d_X, D, F) \). In general, this is all we know about this set. We will show, however, that if the functions \( F \) are carefully chosen, then \( S(X, d_X, D, F) \) is a simplex and its set of extreme points contains a dense homeomorphic copy of \( X \).

For each \( n \) write \( S_n^a(X, d_X, D, F) \) for the projection of \( S_n(X, d_X, D, F) \) into the first \( n \)-many coordinates. Treating \([0, 1]^n\) as a subset of the Hilbert cube, we see that \( S(X, d_X, D, F) \) is also the limit of the sets \( S_n^a(X, d_X, D, F) \).

**Definition 5.2.** We say that a countable family \( F \) of Lipschitz 1 functions on a metric space \((X, d_X)\) is **saturated** if

- for every \( x \in X \) the distance function \( z \mapsto d_X(z, x) \) belongs to the uniform closure of \( F \)
- for every \( x_1, \ldots, x_n \in X \) there are \( f_1, \ldots, f_n \in F \) such that the vectors \( (f_1(x_1), \ldots, f_n(x_1)), \ldots, (f_1(x_n), \ldots, f_n(x_n)) \in \mathbb{R}^n \) are linearly independent.
Note that for any metric space the set of distance functions to a dense countable subset of the space satisfies the first condition above. However, in general, the set of distance functions does not satisfy the second condition above (with a minimal example being a space having four elements).

We first show that the definition of \(S(X, d_X, D, F)\) does not depend on the choice of the dense set \(D\), in particular it does not depend on the enumeration of \(D\). From the point of view of further applications, we would need the fact that if \(D\) and \(E\) are two countable dense sets in \(X\), then \(S(X, d_X, D, F)\) and \(S(X, d_X, E, F)\) are affinely homeomorphic. However, they are actually the same set.

**Lemma 5.3.** If \(F\) is saturated, and \(D, E\) are two countable dense subsets of \(X\), then \(S(X, d_X, D, F)\) and \(S(X, d_X, E, F)\) are equal.

**Proof.** Let \(D = \{d_n : n \in \mathbb{N}\}\) and \(E = \{e_n : n \in \mathbb{N}\}\). Write \(b_n = (f_1(d_n), f_2(d_n), \ldots) \in [0, 1]^\mathbb{N}\) and \(c_n = (f_1(e_n), f_2(e_n), \ldots) \in [0, 1]^\mathbb{N}\). Write \(B_n\) for \(S_n(X, d_X, D, F)\) and \(C_n\) for \(S_n(X, d_X, E, F)\). Note that \(\{b_n : n \in \mathbb{N}\}\) as well as \(\{c_n : n \in \mathbb{N}\}\) are affinely independent. Now \(S(X, d_X, D, F) = \lim_n B_n\) and \(S(X, d_X, E, F) = \lim_n C_n\). We will construct a strong twisted approximation from \((B_n : n \in \mathbb{N})\) to \((C_n : n \in \mathbb{N})\).

For each \(n\) choose a subsequence \((k^n_m : m \in \mathbb{N})\) such that

\[
d_X(d_{k^n_m}, e_{k^n_m}) < 1/m
\]

for each \(m \in \mathbb{N}\). For \(m \in \mathbb{N}\), let \(\varphi_m : B_m \to [0, 1]^\mathbb{N}\) be the affine map which maps \(b_i\) to \(c_{k^n_m}\) for \(i \leq n\). We claim that \((\varphi_k : k \in \mathbb{N})\) is a strong twisted approximation.

Note that \(d_{[0,1]^\mathbb{N}}(a_i, \varphi_m(a_i)) < 1/m\) for each \(m \in \mathbb{N}\) since the functions in \(F\) are Lipschitz 1. By Proposition 2.1, this implies (i') and (iii). To see (ii), note that \(\text{rng}\varphi_n \subseteq C\) for each \(n\), so it is enough to show that for each \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(\text{rng}\varphi_n\) is \(\varepsilon\)-dense in \(C\) for every \(n > n_0\). Pick \(\varepsilon > 0\) and let \(n_1 \in \mathbb{N}\) be big enough so that \([0, 1]^{n_1}\) is \(\varepsilon/4\)-dense in \([0, 1]^\mathbb{N}\) (treat \([0, 1]^\mathbb{N}\) as a subset of \([0, 1]^{n_1}\) via the embedding \((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, 0, \ldots)\)). Write \(C^{n_1}\) for the projection of \(C\) to \([0, 1]^{n_1}\) and note that there is \(n_2 > n_1\) such that \(C_{n_2}^{n_1}\) is \(\varepsilon/4\)-dense in \(C^{n_1}\) for every \(n > n_2\). Thus, \(C_{n_2}\) is \(\varepsilon/2\)-dense in \(\lim_n C_n\). For each \(i \leq n_2\) pick \(l_i\) so that \(d_X(e_i, d_{l_i}) < \varepsilon/4\). Pick \(n_0 > n_2\) so that \(n_0 > l_i\) for each \(i \leq n_2\) and \(1/n_0 < \varepsilon/4\) and hence

\[
d_X(e_i, e_{k^n_{n_0}}) < \varepsilon/2 \quad \text{for each } i \leq n_2.
\]

Then

\[
d_{[0,1]^\mathbb{N}}(e_i, \varphi_{n_0}(b_i)) < \varepsilon/2 \quad \text{for each } i \leq n_2,
\]

which implies that \(\text{rng}\varphi_{n_0}\) is \(\varepsilon\)-dense in \(B\) as well as is \(\text{rng}\varphi_n\) for every \(n > n_0\) since the ranges increase. This ends the proof. □

In the sequel, we write \(S(X, d_X, F)\) rather than \(S(X, d_X, D, F)\). Another way of stating the previous lemma is then to say that \(S(X, d_X, F)\) is the
closed convex hull of the set \( \{(f_1(x), f_2(x), \ldots) : x \in X\} \). In principle, this can be taken as the definition of \( S(X, d_X, F) \) but in further arguments we will use the approximation of \( S(X, d_X, F) \) by \( S_n(X, d_X, D, F) \).

Note that \( D \) can be seen as a subset of \( S(X, d_X, D, F) \) via the map \( d \mapsto (f_1(d), f_2(d), \ldots) \). Note also that since all distance functions to the points of \( D \) are in the uniform closure of \( F \), the above map is an embedding. Denote this map by \( i_D^F \).

**Lemma 5.4.** If \( F \) is saturated, then there is a canonical homeomorphic embedding \( i_F \) of \( X \) into \( S(X, d_X, F) \) which extends \( i_D^F \) for all countable dense \( D \subseteq X \) and is Lipschitz 1.

**Proof.** Write \( i_F(x) = (f_1(x), f_2(x), \ldots) \) and note that \( i_F(x) \in S(X, d_X, F) \) for every \( x \in X \) by Lemma 5.3. It is clear that, when viewed as a map to \( S(X, d_X, D, F) \), \( i_F \) extends \( i_D^F \). The fact that \( i_F \) is an embedding of \( X \) into \( S(X, d_X, D, F) \) follows from the fact that all distance functions to the points in \( D \) are in the uniform closure of \( F \).

To see that \( i_F \) is a homeomorphism, suppose that \( x_n \in X \) and \( x \in X \) are such that \( i_F(x_n) \rightarrow i_F(x) \). This means that \( f(x_n) \rightarrow f(x) \) for every \( f \in F \). Again, since the distance functions from all points in \( X \) are in the uniform closure of \( F \) we have \( d(x_n, y) \rightarrow d(x, y) \) for every \( y \in X \) and hence \( x_n \rightarrow x \), which shows that \( i_F \) is a homeomorphism.

Finally, the fact that \( i_F \) is Lipschitz 1 follows immediately from the fact that all functions in \( F \) are Lipschitz 1. \( \square \)

In the sequel we will abuse notation and treat \( X \) as a subset of \( S(X, d_X, F) \), unless this can cause confusion.

Now, we need to locate the extreme points of the sets \( S(X, d_X, F) \). Notice that if \( F_1 \subseteq F_2 \) are two families of Lipschitz 1 functions, then there is a natural affine continuous projection from \( S(X, d_X, F_2) \) to \( S(X, d_X, F_1) \) (which forgets the coordinates from \( F_2 \setminus F_1 \)).

Note also that if \( K \) and \( L \) are convex compact sets and \( \varphi : K \rightarrow L \) is an affine continuous surjection, then \( \text{ext}(L) \subseteq \varphi''\text{ext}(K) \). This follows from the fact that if \( x \in L \) is an extreme point of \( L \), then \( \varphi^{-1}(\{x\}) \) is a compact convex set, so it contains a relative extreme point, say \( y \). Now \( y \) must be extreme in \( K \) since otherwise \( y \) is a nontrivial affine combination of \( y_1, y_2 \in K \). Since \( y \) was extreme in \( \varphi^{-1}(\{x\}) \), these points cannot belong to \( \varphi^{-1}(\{x\}) \) and hence \( \varphi(y_1) \neq \varphi(y_2) \), which contradicts the fact that \( x \) was extreme in \( L \).

The above implies that the larger the family \( F \) we take, the more extreme points we get in \( S(X, d_X, F) \). The lemma below says that if \( F \) is saturated (in fact here we only need the first item from the definition), then we get quite enough of them.

**Lemma 5.5.** If \( F \) is saturated, then the points of \( X \) are extreme points in \( S(X, d_X, F) \).
Given a compact convex set $C$ in a locally convex vector topological space, with a metric $d_C$ on $C$, and a subset $A \subseteq C$ we say that $A$ is an $\varepsilon$-face provided that for every $x,y \in C$ if $\frac{1}{2}(x+y) \in A$, then both $d_C(x,A)$ and $d_C(y,A)$ are smaller than $\varepsilon$. Note that if $A \subseteq C$ is written as $A = \bigcap_n A_n$ so that each $A_n$ is closed convex and for each $\varepsilon > 0$ there is $n_0$ such that $A_n$ is an $\varepsilon$-face for each $n > n_0$, then $A$ is a face.

Proof of Lemma 5.6. We will again look at $S(X,d_X,D,F)$ for a fixed countable dense set $D \subseteq X$. By Proposition 5.3 it is enough to show that the points of $D$ are extreme in $S(X,d_X,D,F)$. Moreover, it is enough to show that given the enumeration of $D$ as $(d_1,d_2,\ldots)$, the point $d_1$ is extreme in $S(X,d_X,D,F)$.

Claim 5.6. For any $A \subseteq X$ we have $\text{diam}_{S(X,d_X,F)} (\text{conv}(i_FA)) \leq \text{diam}_X (A)$.

Proof. This follows directly from the fact that $i_F$ is Lipschitz 1 and from local convexity of the Hilbert cube. □

Write $A_n$ for the closed convex hull of the ball in $X$ around $d_1$ of diameter $1/n$. Claim 5.6 implies that $\bigcap_n A_n$ contains only one point, and thus is the singleton $\{d_1\}$. We will show that for each $\varepsilon > 0$ there is $n_0$ such that $A_n$ is an $\varepsilon$-face for every $n > n_0$.

Claim 5.7. Suppose $\delta \geq 0$ and $a_1,\ldots,a_k \in [0,1]$, $\alpha_1,\ldots,\alpha_k \in [0,1]$ are such that $\sum_i \alpha_i = 1$. If $b \in [0,1]$ and $b_1,\ldots,b_k \in [0,1]$ are such that $|b-b_i| \leq a_i + \delta$ for each $i \leq k$, then

$$|b - \sum_{i=1}^k \alpha_i b_i| \leq \sum_{i=1}^k \alpha_i a_i + \delta.$$

Proof. This is just a straightforward computation. Note that we have $|b - \sum_i \alpha_i b_i| = |\sum_i \alpha_i (b-b_i)| \leq \sum_i |\alpha_i b_i| \leq \sum_i \alpha_i (a_i + \delta) = \sum_i \alpha_i a_i + \delta$. □

Fix $\varepsilon > 0$ and find a function $f \in F$ which is $\varepsilon/8$-uniformly close to the distance function $z \mapsto d(z,d_1)$. Without loss of generality assume that $f = f_1$.

Suppose now that $n > 8/\varepsilon$ and $y,z \in S(X,d_X,F)$ are such that $\frac{1}{2}(y+z) \in A_n$. Approximate $y$ with $y'$ and $z$ with $z'$ such that $d_{S(X,d_X,F)}(y,y') < \varepsilon/8$, $d_{S(X,d_X,F)}(z,z') < \varepsilon/8$ and $y',z' \in S_k(X,d_X,D,F)$ for some $k \in \mathbb{N}$. Note that then $\frac{1}{2}(y'+z')$ is $\varepsilon/8$-close to $A_n$. Since the first coordinate of every point in $A_n$ is smaller than $1/n + \varepsilon/8$, the first coordinate of $\frac{1}{2}(y'+z')$ is smaller than $1/n + \varepsilon/4$. Hence, the first coordinates of both $y'$ and $z'$ are smaller than $2(1/n + \varepsilon/4)$, which is smaller than $3\varepsilon/4$. Since the first coordinate of points in $A_n$ is given by the function that is $\varepsilon/8$-close to the the distance function from $d_1$ and all other coordinates are given by functions that extend Lipschitz 1 functions, Claim 5.7 implies that

$$d_{S(X,d_X,F)}(y',d_1) < 3\varepsilon/4 + \varepsilon/8$$

as well as

$$d_{S(X,d_X,F)}(z',d_1) < 3\varepsilon/4 + \varepsilon/8.$$
This implies that \( d_{S(X,d,X,F)}(y,d_1) < 3\varepsilon/4 + \varepsilon/4 = \varepsilon \) and \( d_{S(X,d,X,F)}(z,d_1) < 3\varepsilon/4 + \varepsilon/4 = \varepsilon \), which shows that \( A_n \) is an \( \varepsilon \)-face. \( \square \)

The fact that \( X \) is dense in the set of extreme points of \( S(X,d_X,F) \) will follow from the following general lemma.

**Lemma 5.8.** Suppose \( C_n \) is an increasing sequence of compact convex subsets of a metrizable convex compact set \( C \) in a locally convex topological vector space. Then \( \bigcup_n \text{ext}(C_n) \) is dense in \( \text{ext}(\lim_n C_n) \).

*Proof.* Write \( K \) for the closure of \( \bigcup_n \text{ext}(C_n) \) and suppose \( x \in \lim_n C_n \) is such that \( x \notin K \). We need to show that \( x \) is not an extreme point of \( \lim_n C_n \). Note that [41, Chapter 15, Proposition 2.3] \( x \) is a barycenter of a probability measure \( \mu \) concentrated on \( K \). By a theorem of Bauer [7] (see also [61, Proposition 1.4]) if \( x \) is an extreme point and a barycenter of a probability measure \( \mu \), then \( \mu = \delta_x \). But \( \mu \) is concentrated on \( K \), so cannot be equal to \( \delta_x \). \( \square \)

This immediately gives the following.

**Corollary 5.9.** If \( F \) is saturated, then \( X \) is dense in the set of extreme points of \( S(X,d_X,F) \).

Let us now see some examples of convex compact sets that can arise as \( S(X,d_X,F) \). The next proposition will not be used later in the proof but it shows some ideas behind the constructions that follow.

In the following proposition, for a compact metric space \( X \), write \( P(X) \) for the Bauer simplex of all Borel probability measures on \( X \). The extreme boundary of \( P(X) \) is canonically homeomorphic to \( X \) and any simplex whose extreme boundary is homeomorphic to \( X \) is canonically affinely homeomorphic to \( P(X) \) (this follows for example from the positive solution to the Dirichlet extension problem for Bauer simplices [3]).

As a comment to the assumption of the following proposition, note that if \( (X,d_X) \) is compact, then Lipschitz functions are dense in \( C(X) \), by the Stone–Weierstrass theorem. Thus, if \( (X,d_X) \) is compact, then the set of all Lipschitz 1 functions is linearly dense in \( C(X) \).

**Proposition 5.10.** Suppose \( (X,d_X) \) is compact, \( F \) is saturated and linearly dense in \( C(X) \). Then \( S(X,d_X,F) \) is affinely homeomorphic to the Bauer simplex \( P(X) \).

*Proof.* Note that the set of extreme points of \( S(X,d_X,F) \) is equal to \( X \) by Lemma 5.5 and Corollary 5.9. We need to prove that \( S(X,d_X,F) \) is a simplex. For that, suppose \( \mu, \nu \) are two distinct probability measures on \( X \). By linear density of \( F \) in \( C(X) \), there is \( f \in F \) such that \( \int f d\mu \neq \int f d\nu \). Without loss of generality, assume that \( f = f_1 \). Pick a countable dense set \( D \subseteq X \) and choose two sequences of atomic measures \( \mu_n \) and \( \nu_n \) concentrated on \( D \) such that \( \mu_n \to \mu \) and \( \nu_n \to \nu \). Note that each \( \mu_n \) and \( \nu_n \) has a
barycenter in one of the sets $S_k(X, d_X, D, F)$, for if
\[
\mu_n = \sum_{i=1}^{k_n} \alpha_i^n \delta_{d_i^n} \quad \text{with} \quad d_i^n \in D, \alpha_i^n \geq 0, \sum_{i=1}^{k_n} \alpha_i^n = 1,
\]
\[
\nu_n = \sum_{i=1}^{l_n} \beta_i^n \delta_{e_i^n} \quad \text{with} \quad e_i^n \in D, \beta_i^n \geq 0, \sum_{i=1}^{l_n} \beta_i^n = 1,
\]
then the barycenter of $\mu_n$ is $\sum_{i=1}^{k_n} \alpha_i^n d_i^n$ and the barycenter of $\nu_n$ is $\sum_{i=1}^{l_n} \beta_i^n e_i^n$.

Note that then $\sum_{i=1}^{k_n} \alpha_i^n f_1(d_i^n)$ and $\sum_{i=1}^{l_n} \beta_i^n f_1(e_i^n)$ are the first coordinates in $[0, 1]^\mathbb{N}$ of the barycenters of $\mu_n$ and $\nu_n$. Thus, the first coordinate of the barycenter of $\mu$ is the limit of the first coordinates of the barycenters of $\mu_n$ [11, Chapter 15, Proposition 2.2] and is equal to
\[
\lim_n \sum_{i=1}^{k_n} \alpha_i^n f_1(d_i^n) = \lim_n \int f_1 d\mu_n = \int f_1 d\mu
\]
and the first coordinate of the barycenter of $\nu$ is the limit of the first coordinates of the barycenters of $\nu_n$ and is equal to
\[
\lim_n \sum_{i=1}^{l_n} \beta_i^n f_1(e_i^n) = \lim_n \int f_1 d\nu_n = \int f_1 d\nu.
\]
Thus, $\mu$ and $\nu$ have different barycenters, which shows that $S(X, d_X, F)$ is a simplex. \hfill \Box

Now we need to see how $S(X, d_X, F)$ depends on the choice of $F$. First note that it does not depend on the enumeration of $F$.

**Lemma 5.11.** Suppose $F$ is saturated and $G$ enumerates the same set of functions as $F$. Then $S(X, d_X, F)$ and $S(X, d_X, G)$ are affinely homeomorphic.

*Proof.* Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation such that $G = (f_{\pi(1)}, f_{\pi(2)}, \ldots)$. Let $h : [0, 1]^\mathbb{N} \to [0, 1]^\mathbb{N}$ be defined as $h(x_1, x_2, \ldots) = (x_{\pi(1)}, x_{\pi(2)}, \ldots)$ and note that $h$ is an affine homeomorphism of the Hilbert cube which maps $S(X, d_X, F)$ to $S(X, d_X, G)$. \hfill \Box

Suppose now that $(X, d_X)$ is a metric space, $D \subseteq X$ is a countable dense set and that $F$ consists of distance functions from the points in $D$. In such a case, we denote $F$ by $d_X D$. Note that $d_X D$ always consists of Lipschitz 1 functions and its uniform closure contains all distance functions from the points in $X$. In general, still, $d_X D$ need not be saturated. We will see, however, that if $X$ has certain additional property (e.g. is the Urysohn space or the Urysohn sphere), then this is the case.

**Definition 5.12.** We say that a separable metric space $(X, d_X)$ is *saturated* if $d_X D$ is saturated for every countable dense set $D \subseteq X$. 
Proposition 5.13. Suppose \((X, d_X)\) is separable, complete and saturated and \(D, E \subseteq X\) are two countable dense sets. Then there is an affine homeomorphism \(\tau_D^E : S(X, d_X, d_X D) \to S(X, d_X, d_X E)\) such that the following diagram commutes.

\[
\begin{array}{ccc}
S(X, d_X, d_X D) & \xrightarrow{\tau_D^E} & S(X, d_X, d_X E) \\
i_{d_X D} & & i_{d_X E} \\
X & & \\
\end{array}
\]

The map \(\tau_D^E\) should be treated as the transition map between the coordinate systems of \(D\) and \(E\).

Proof. First note that isolated points of \(X\) must belong to both \(D\) and \(E\) and hence if \(X_1\) is the set of isolated points, then by Lemma 5.11 we can assume that \(X_1\) is enumerated in the same way in \(D\) and \(E\). Thus, without loss of generality we can assume that \(X\) has no isolated points. Moreover, we can assume that \(D\) and \(E\) are disjoint since we can always use a third countable dense set which is disjoint from both \(D\) and \(E\). Write \(D = (d_0, d_1, \ldots)\) and let \(D' = (d_1, d_2, \ldots)\).

Claim 5.14. \(S(X, d_X, d_X D)\) and \(S(X, d_X, d_X D')\) are affinely homeomorphic via a map \(\tau_{D'}^D : S(X, d_X, d_X D) \to S(X, d_X, d_X D')\) such that \(\tau_{D'}^D \circ i_{d_X D} = i_{d_X D'}\).

Proof. Write \(C_n = S_n(X, d_X, D, d_X D)\) and \(B_n = S_n(X, d_X, D, d_X D')\) for each \(n \in \mathbb{N}\). Write also \([0, 1]^{[0,1]}\) as \([0, 1]^{[2,3,\ldots]} \times [0, 1]\) interpreting the second coordinate of the product as the first coordinate in the Hilbert cube. Pick a subsequence \(d_{k_n} \to d_0\) in \(D\) and note that the distance function to \(d_0\) is the uniform limit of the distance functions to \(d_{k_n}\)'s. By Proposition 2.1 this implies that on \(\bigcup_n C_n\) the first coordinate is the uniform limit of the coordinates numbered with \(k_n\)'s. Thus, \(\bigcup_n C_n\) treated as a subset of \([0, 1]^{[2,3,\ldots]} \times [0, 1]\), is a graph of an affine function, say \(a\), defined on \(\bigcup_n B_n\). Moreover, since \(a\) is a uniform limit of the functions given by coordinates in \(\{k_1, k_2, \ldots\}\), and these functions clearly extend to the closure of \(\bigcup_n B_n\), the function \(a\) also uniquely extends to an affine function defined on the closure of \(\bigcup_n B_n\), which is equal to \(S(X, d_X, d_X D')\). Note that the graph of the unique extension is equal to the closure of \(\bigcup_n C_n\), i.e. \(S(X, d_X, d_X D)\). Given that, the projection function from \([0, 1]^{[2,3,\ldots]} \times [0, 1]\) to \([0, 1]^{[2,3,\ldots]}\) is an affine isomorphism from \(S(X, d_X, d_X D)\) to \(S(X, d_X, d_X D')\). Write \(\tau_{D'}^D\) for this isomorphism and note that since it just erases the first coordinate, we have \(\tau_{D'}^D \circ i_{d_X D} = i_{d_X D'}\). \(\square\)

Write now \(D_n\) for the sequence \((e_1, e_2, \ldots, e_n, d_{n+1}, d_{n+1}, \ldots)\).
Claim 5.15. For each \( n \in \mathbb{N} \), there is an affine homeomorphism \( \tau_n : S(X, d_X, d_X D) \to S(X, d_X, d_X D_n) \) such that \( \tau_n \circ i_{d_X D} = i_{d_X D_n} \).

Proof. It is enough to show that there is an affine homeomorphism \( \tau'_n : S(X, d_X, d_X D_n) \to S(X, d_X, d_X D_{n+1}) \) with \( \tau'_n \circ i_{D_n} = i_{D_{n+1}} \) and without loss of generality assume that \( n = 0 \). But this follows from Claim 5.14 since both \( S(X, d_X, d_X D) \) and \( S(X, d_X, d_X D_1) \) are affinely homeomorphic to \( S(X, d_X, d_X D') \) via maps which make the appropriate diagrams commute.

Now note that since \( \tau_{n+1}^{-1} \tau_n^{-1} \) changes only the \( n \)-th coordinate, the sequence \( \tau_n \) is uniformly Cauchy with respect to the Hilbert cube metric, and thus converges to an affine map \( \tau_D^E : S(X, d_X, d_X D) \to [0,1]^N \). Moreover, since the distance of \( \text{rng} \tau_n \) to \( B_n \) is smaller than \( 2^{-n} \), we have that \( \tau_D^E \) is an affine map from \( S(X, d_X, d_X D) \) to \( S(X, d_X, d_X E) \). In the same way construct maps \( \tau^n : S(X, d_X, d_X E) \to S(X, d_X, d_X E_n) \) with \( E_n = (d_1, d_2, \ldots, d_n, e_{n+1}, e_{n+2}, \ldots) \) and note that their limit is an affine function \( \tau_D^E : S(X, d_X, d_X E) \to S(X, d_X, d_X D) \). The maps are clearly inverse to each other, so \( \tau_D^E \) is an affine homeomorphism. Finally, \( \tau_D^E \circ i_{d_X D} = i_{d_X E} \) follows from the fact that \( \tau_n \circ i_{d_X D} = i_{d_X D_n} \) holds for each \( n \) and \( i_{d_X D_n} \to i_{d_X E} \). This ends the proof.

The following proposition is stated for the Urysohn sphere \( U_1 \) but it also holds true (with the same proof) for the Urysohn space \( U \). We say that a function \( f : X \to [0,\infty] \) defined on a metric space \( (X, d_X) \) is a Katětov function if for every \( x, y \in X \) we have \( |f(x) - f(y)| \leq d_X(x, y) \leq f(x) + f(y) \) (cf. [60] Lemma 5.1.22 and [30] Definition 1.2.1).

Proposition 5.16. The Urysohn sphere \( U_1 \) is saturated.

Proof. Write \( d \) for the metric \( d_{U_1} \) on \( U_1 \). Pick a countable dense set \( D \subseteq U_1 \) and let \( x_1, \ldots, x_n \in U_1 \). We show that there are \( d_1, \ldots, d_n \in D \) such that the matrix

\[
\begin{pmatrix}
  d(x_1, d_1) & \ldots & d(x_1, d_n) \\
  \vdots & \ddots & \vdots \\
  d(x_n, d_1) & \ldots & d(x_n, d_n)
\end{pmatrix}
\]

is invertible. The proof is by induction. For \( n = 1 \), any \( d_1 \neq x_1 \) will do. Suppose \( n > 1 \) and \( x_1, \ldots, x_{n-1} \in U \) are given. Pick any \( d_1, \ldots, d_{n-1} \) that witness the inductive assumption for \( x_1, \ldots, x_{n-1} \) and let \( d \in D \) be such that \( d(x_n, d) < d(x_i, d) \) for all \( i \neq n \). Consider the function

\[
\varepsilon \mapsto \det\left(\begin{pmatrix}
  d(x_1, d_1) & \ldots & d(x_1, d_{n-1}) & d(x_1, d) \\
  \vdots & \ddots & \vdots & \vdots \\
  d(x_n, d_1) & \ldots & d(x_n, d_{n-1}) & d(x_n, d) + \varepsilon
\end{pmatrix}\right)
\]

and note that it is a nonzero linear function, so there are arbitrarily small \( \varepsilon > 0 \) at which it does not vanish. Pick such \( \varepsilon_0 > 0 \) which is smaller than \( \min\{d(x_i, d) - d(x_n, d) : i < n\} \). Note now that the function \( f : U_1 \to \mathbb{R} \)
\{x_1, \ldots, x_n\} \to [0,1] given by \( f(x_i) = d(x_i, d) \) if \( i < n \) and \( f(x_n) = d(x_n, d) + \varepsilon_0 \) is a Katětov function with values in \([0,1]\). Thus, there is \( y \in U_1 \) which realizes \( f \). This means that
\[
\det \begin{pmatrix}
    d(x_1, d_1) & \ldots & d(x_1, d_{n-1}) & d(x_1, y) \\
    \vdots & \ddots & \vdots & \vdots \\
    d(x_n, d_1) & \ldots & d(x_n, d_{n-1}) & d(x_n, y)
\end{pmatrix} > 0
\]
Since the set of such \( y \)'s is clearly open, we can find one, say \( d_n \), in \( D \). \( \square \)

The Urysohn space has the extension property saying that every Katětov function defined on its finite subset is realized as a distance function to some point in the space. Huhunaïšvili \([39]\) (cf also \([33, 42, 9, 58, 59]\)) showed that the same is true for Katětov functions defined on compact subsets of the Urysohn space. This is probably the strongest result that can guarantee that certain Katětov functions can be realized in the Urysohn space (or the Urysohn sphere). The following construction is motivated by the need of realizing Katětov functions defined on non-compact subspaces of the Urysohn sphere.

Let \((X, d_X)\) be a separable metric space of diameter bounded by 1 and let \(D \subseteq X\) be its dense countable subset. Let \(R(D)\) be the ring of functions generated by the distance functions \( d \mapsto d_X(x, d) \) for \( d \in D \) and all rational constant functions. Note that \(R(D)\) is countable and all functions in \(R(D)\) are Lipschitz. Let \(R_1(D)\) the the family of functions in \(R(D)\) which are Lipschitz 1 and have the range contained in \([\tfrac{1}{2}, 1]\). Note that dividing a bounded Lipschitz function by an appropriately large constant, we get a Lipschitz 1 function with the range contained in \([-\frac{1}{4}, \frac{1}{4}\]). Next, adding \(\frac{3}{4}\) we get a Lipschitz 1 function with \(\text{rng}(f) \subseteq \left[\tfrac{1}{2}, 1\right]\). This shows that \(R_1(D)\) is linearly dense in \(R(D)\). On the other hand, since \(d_X\) is bounded by 1, every function in \(R_1(D)\) is a Katětov function. Recall that the elements of \(X\) are identified with Katětov functions on \(X\) by the Kuratowski construction \([39]\), Chapter 1.2], i.e. \(x \in X\) is identified with the function \( y \mapsto d_X(x, y) \). Write \(F(X)\) for the family of all finitely supported (see \([30, \text{Definition 1.2.2}]\)) Katětov functions on \(X\) with values in \([0,1]\). Let \(E(X, d_X, D)\) be the completion of the space \(F(X) \cup R_1(D)\) with the sup metric. Note that \(E(X, d_X, D)\) is an extension of \(X\) (as \(F(X)\) contains all functions \( y \mapsto d_X(x, y) \) for \(x \in X\), is separable and realizes all finitely supported Katětov functions on \(X\). However, it is slightly bigger than the usual one-step Katětov extension since we also have realized the functions in \(R_1(D)\). A standard argument shows that if \(\varphi : X \to X\) is an isometry, then \(\varphi\) extends to an isometry \(\varphi'\) of \(E(X, d_X, D)\) and the definition of \(E(X, d_X, D)\) does not depend on the choice of the dense set \(D\). Thus, slightly abusing notation, we write \(E(X, d_X)\) for \(E(X, d_X, D)\). Note, however, that given \(D\), we have a canonical countable dense set \(D'\) in \(E(X, d_X, D)\) consisting of \(R_1(D)\) and all Katětov functions finitely supported on a subset of \(D\) and assuming rational values on their support.
Now, similarly as in the Katětov construction of the Urysohn space, we iterate the above extension construction infinitely many times.

**Definition 5.17.** Given a separable metric space \((X,d_X)\) and its countable dense subset \(D \subseteq X\) define inductively \(E^n(X,d_X,D)\) and \(D^n(X,d_X,D)\) as follows. \(E^0(X,d_X,D) = (X,d_X)\) and \(D_0(X,d_X,D) = D\). Given \(E^n(X,d_X,D)\) and \(D^n(X,d_X,D)\), which is a countable dense subset of \(E^n(X,d_X,D)\) let \(E^{n+1}(X,d_X,D) = E(E^n(X,d_X,D), D^n(X,d_X,D))\) and let \(D^{n+1}(X,d_X,D)\) be the canonical extension of \(D^n(X,d_X,D)\) to a countable dense subset of \(E^{n+1}(X,d_X,D)\). Write \(E^\infty(X,d_X,D)\) for the completion of the space \(\bigcup E^n(X,d_X,D)\) and \(D^\infty(X,d_X,D)\) for \(\bigcup_n D^n(X,d_X,D)\).

Again, the construction of \(E^\infty(X,d_X,D)\) does not depend on the initial choice of the countable dense set \(D\) and we will abuse notation writing \(E^\infty(X,d_X)\) for \(E^\infty(X,d_X,D)\), unless this can cause confusion. Note that the space \(E^\infty(X,d_X)\) is actually isometric to the Urysohn sphere since it realizes all finitely supported Katětov functions with values in \([0,1]\). Note also that if \(\varphi : X \rightarrow X\) is an isometry, then \(\varphi\) extends canonically to an isometry \(\varphi^\infty : E^\infty(X,d_X) \rightarrow E^\infty(X,d_X)\). This follows in the same way as the analogous extension property is proved for the Katětov extensions [60], Page 115 from the (above mentioned) fact that \(\varphi\) extends to \(E(X,d_X)\).

Given a metric space \((X,d_X)\) and its countable dense subset \(D\), write \(d_{E^\infty(X,d_X)}D^\infty(X,d_X)\) for the family of functions on \(X\) of the form \(x \mapsto d_{E^\infty(X,d_X)}(x,d)\) for \(d \in D^\infty(X,d_X,D)\). Write also \(R^\infty(X,d_X,D)\) for the ring of functions on \(X\) generated by \(d_{E^\infty(X,d_X)}D^\infty(X,d_X)\). The next proposition follows rather immediately from the construction.

**Proposition 5.18.** Let \((X,d_X)\) be a separable metric space of diameter bounded by \(1\) and let \(D \subseteq X\) be a countable dense set. Then \(R^\infty(X,d_X,D)\) is contained in the linear span of \(d_{E^\infty(X,d_X)}D^\infty(X,d_X)\).

**Proof.** Write \(D^\infty\) for \(D^\infty(X,d_X,d)\) and \(dD^\infty\) for \(d_{E^\infty(X,d_X)}D^\infty(X,d_X)\). Every function \(f\) in \(R^\infty(X,d_X,D)\) is a polynomial in finitely many functions in \(dD^\infty\), which in turn are the distance functions to finitely many points in \(D^\infty\). Thus, there is \(n \in \mathbb{N}\) such that \(f\) belongs to the ring generated by the distance functions to \(D^n(X,d_X,D)\). Then, \(f\) belongs to the linear span of distance functions to the points in \(D^{n+1}(X,d_X,D)\). \(\square\)

Now, fix a countable dense set \(D \subseteq \mathbb{U}_1\) and write \(D^\infty\) for \(D^\infty(\mathbb{U}_1,d_{\mathbb{U}_1})\) and \(d_E\) for the metric on \(E^\infty(\mathbb{U}_1,d_{\mathbb{U}_1})\). Consider the convex compact set \(S(E^\infty(\mathbb{U}_1,d_{\mathbb{U}_1}),d_E D^\infty(\mathbb{U}_1,d_{\mathbb{U}_1},D))\) and write \(S(\mathbb{U}_1,d_{\mathbb{U}_1})\) for the closed convex hull of \(\mathbb{U}_1\) in \(S(E^\infty(\mathbb{U}_1,d_{\mathbb{U}_1}),d_E D^\infty(\mathbb{U}_1,d_{\mathbb{U}_1},D))\). Note that \(S(\mathbb{U}_1,d_{\mathbb{U}_1})\) is equal to \(S(\mathbb{U}_1,d_{\mathbb{U}_1},d_E D^\infty)\). Note also that by Proposition 4.3 and the fact that \(E^\infty(\mathbb{U}_1,d_{\mathbb{U}_1})\) is isometric to the Urysohn sphere, the family \(d_E D^\infty\) is saturated.

Recall (Proposition 4.2) that given a subspace \((X,d_X)\) of \(\mathbb{U}_1\) we write \(Z(X,d_X)\) for an isometric copy of \((X,d_X)\) appropriately embedded into \(\mathbb{U}_1\).
Proof. Write which are supported on ext(S).

Definition 5.19. For a subspace (X, dX) of the Urysohn sphere (U1, dU1), write $S_Z(X, d_X)$ for $S(Z(X, d_X), d_E D^\infty)$.

Note that by Proposition 5.13 and the remarks proceeding Proposition 5.18, up to affine homeomorphism, the above definition does not depend on the choice of the dense countable set $D \subseteq U_1$.

Similarly as with Proposition 5.16, the following result is stated for the Urysohn sphere but holds true for the Urysohn space as well.

Proposition 5.20. The convex compact set $S(U_1, d_{U_1})$ is a simplex.

Proof. Write $S$ for $S(U_1, d_{U_1})$ and $d_S$ for the metric on $S$ (induced from the Hilbert cube). Recall that we have fixed a countable dense set $D \subseteq U_1$ and write $d_E$ for the metric on $E^\infty(U_1, d_{U_1}, D)$.

Suppose $\mu$ and $\nu$ are two distinct Borel probability measures in $P(S)$ which are supported on $\text{ext}(S)$. Since as $U_1 \subseteq \text{ext}(S)$ is dense by Corollary 5.9, we can pick two sequences of Borel probability measures $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$ such that $\mu_n$ and $\nu_n$ are finitely supported on $U_1$.

Since $\mu$ and $\nu$ are distinct as elements of $P(S)$, and the coordinate functions (from [0, 1]] separate points in $S$, by the Stone–Weierstrass theorem, there is a function $f \in C(S)$ which belongs to the ring generated by the coordinate functions and is such that $\int f d\mu \neq \int f d\nu$. Note that the restrictions of the coordinate functions to $U_1$ are equal to the $d_E$-distance functions to the points in $D^\infty$. By Proposition 5.18 and the fact that $U_1$ is dense in $\text{ext}(S)$, we can assume that the restriction of $f$ to $U_1$ is actually equal to the $d_E$-distance function to, say, $z \in D^\infty$. Thus, we assume that the function $f$ on $\text{ext}(S)$ is just one of the coordinate functions. Say it is the $k$-th coordinate, i.e. $z$ is the $k$-th element of $D^\infty$.

Write $x$ for the barycenter of $\mu$ and $y$ for the barycenter of $\nu$ and let $x_n$ be the barycenter of $\mu_n$ and $y_n$ of $\nu_n$. Note that $x_n \rightarrow x$ and $y_n \rightarrow y$ [11, Chapter 15, Proposition 2.2]. Now, since $\mu_n$ and $\nu_n$ are supported on $U_1$, similarly as in Proposition 5.10, we get that $x_n(k) = \int f d\mu_n$ and $y_n(k) = \int f d\nu_n$. But $\int f d\mu_n \rightarrow \int f d\mu$ and $\int f d\nu_n \rightarrow \int f d\nu$. Thus, $x(k) = \int f d\mu$ and $y(k) = \int f d\nu$ and hence $x$ and $y$ are distinct, as needed. This ends the proof.

Corollary 5.21. For every closed subspace $(X, d_X)$ of $U_1$ the set $S_Z(X, d_X)$ is a simplex and $Z(X, d_X)$ is a dense subset of $\text{ext}(S_Z(X, d_X))$.

Proof. For simplicity identify $(X, d_X)$ with $Z(X, d_X)$. Note that $S_Z(X, d_X)$ is equal to the closed convex hull of $X$ in $S(U_1, d_{U_1})$ and $X$ is contained in $\text{ext}(S(U_1, d_{U_1}))$. Thus, $S_Z(X, d_X)$ is a face of $S(U_1, d_{U_1})$. By Proposition 5.20, this implies [11, Chapter 15, Corollary 3.3] that $S_Z(X, d_X)$ is a simplex. Clearly, $X$ is contained in $\text{ext}(S_Z(X, d_X))$. The fact that $X$ is dense in $\text{ext}(S_Z(X, d_X))$ follows directly from Corollary 5.9.

Finally, we need to see that $S$-extensions of metric spaces are invariant under the isometry of the metric spaces.
Proposition 5.22. Suppose \((X, d_X)\) and \((Y, d_Y)\) subspaces of \(\mathbb{U}_1\). If \((X, d_X)\) and \((Y, d_Y)\) are isometric, then the simplices \(S_Z(X, d_X)\) and \(S_Z(Y, d_Y)\) are affinely homeomorphic via a map that extends the isometry of \(Z(X, d_X)\) and \(Z(Y, d_Y)\).

Proof. For simplicity again identify \((X, d_X)\) with \(Z(X, d_X)\) and \((Y, d_Y)\) with \(Z(Y, d_Y)\). Let \(\varphi : \mathbb{U}_1 \to \mathbb{U}_1\) be an isometry such that \(\varphi^\prime X = Y\). Let \(D \subseteq \mathbb{U}_1\) be the fixed countable dense set and let \(E = \varphi^\prime D\). Write \(\varphi^\infty : E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}) \to E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})\) for the extension of \(\varphi\) and note that \((\varphi^\infty)^\prime D^\infty = E^\infty\). Write \(d\) for the metric on \(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1})\). Note that since \(\varphi^\infty\) is an isometry, the sets \(S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty)\) and \(S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), E^\infty, dE^\infty)\) are equal and we get the following diagram:

\[
\begin{array}{c}
S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) \xrightarrow{id} S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), E^\infty, dE^\infty) \\
U_1 \xrightarrow{id_{E^\infty}} \mathbb{U}_1
\end{array}
\]

Composing this with the diagram in Proposition 5.13 (where we take the restrictions of the maps \(id_{D^\infty}\) and \(id_{E^\infty}\) to \(\mathbb{U}_1\)), we get

\[
\begin{array}{c}
S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) \xrightarrow{\tau_{E^\infty}^{D^\infty}} S(E^\infty(\mathbb{U}_1, d_{\mathbb{U}_1}), D^\infty, dD^\infty) \\
U_1 \xrightarrow{id_{D^\infty}} \mathbb{U}_1
\end{array}
\]

and since all the maps are affine and continuous, this immediately implies that \(\tau_{E^\infty}^{D^\infty}\) maps \(conv(id_{D^\infty}X) = S_Z(X, d_X)\) to \(conv(id_{D^\infty}Y) = S_Z(Y, d_Y)\).

The simplex \(S(\mathbb{U}_1, d_{\mathbb{U}_1})\) plays now the role of a universal homogeneous Choquet simplex. However, it does not seem to be affinely homeomorphic to the Poulsen simplex. On the other hand, that there is another convex compact set, which seems to be related to the Poulsen simplex. It is the set \(S(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)\). Write \(S'(\mathbb{U}_1, d_{\mathbb{U}_1})\) for \(S(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)\) (for a countable dense set \(D \subseteq \mathbb{U}_1\)). Again, up to affine homeomorphism, it does not depend on the choice of the countable set \(D\). Now, the map from \(S(\mathbb{U}_1, d_{\mathbb{U}_1})\) to \(S'(\mathbb{U}_1, d_{\mathbb{U}_1})\), which forgets about the coordinates corresponding to \(D^\infty \setminus D\) is clearly affine and continuous and hence \(S(\mathbb{U}_1, d_{\mathbb{U}_1})\) can be treated as an unfolded version of \(S'(\mathbb{U}_1, d_{\mathbb{U}_1})\). We do not know if \(S'(\mathbb{U}_1, d_{\mathbb{U}_1})\) is a simplex but it seems to be more closely related to the Poulsen simplex than \(S(\mathbb{U}_1, d_{\mathbb{U}_1})\).

Proposition 5.23. \(S'(\mathbb{U}_1, d_{\mathbb{U}_1})\) has a dense set of extreme points.

The proof follows from the following simple claim.
Claim 5.24. If $g$ and $h$ are Katetov functions and $\alpha \in [0, 1]$, then $\alpha f + (1 - \alpha)g$ is also a Katetov function.

Proof. This is an elementary computation. The Katetov inequalities for $\alpha f + (1 - \alpha)g$ follow directly from the Katetov inequalities for $f$ and $g$. □

Proof of Proposition 5.23. By the fact that $\mathbb{U}_1 \subseteq \text{ext}(\mathcal{S}(\mathbb{U}_1, d_{\mathbb{U}_1}))$, it is enough to check that $\mathbb{U}_1$ is dense in $\mathcal{S}(\mathbb{U}_1, d_{\mathbb{U}_1})$. Let $s \in \mathcal{S}(\mathbb{U}_1, d_{\mathbb{U}_1})$ and $\varepsilon > 0$. Find $n$ such that $2^{-(n+1)} < \varepsilon / 2$ and $s$ is $\varepsilon$-close to $\mathcal{S}_n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$. Write $D = (d_1, d_2, \ldots)$ and $\mathbb{U}_1 \upharpoonright n$ for the image of $\mathbb{U}_1 \subseteq \mathcal{S}(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ under the projection map from $[0, 1]^N$ to $[0, 1]^n$. Note that it suffices to show that $\mathcal{S}_n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ is contained in $\mathbb{U}_1 \upharpoonright n$. Clearly, the vertices of the simplex $\mathcal{S}_n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ belong to $\mathbb{U}_1 \upharpoonright n$. For each $i \leq n$ let $f_i : \{d_1, \ldots, d_n\} \to [0, 1]$ be defined $f_i(z) = d_i(z)$. Now, Claim 5.24 implies that for every $\alpha_1, \ldots, \alpha_n \in [0, 1]$ with $\sum \alpha_i = 1$ the function $\sum \alpha_i f_i$ is a Katetov function with values in $[0, 1]$, thus realized in $\mathbb{U}_1$. This implies that the set of points $(d(x, d_1), \ldots, d(x, d_n)) \in [0, 1]^n$ for $x \in \mathbb{U}_1$ is convex. But the latter set is equal to $\mathbb{U}_1 \upharpoonright n$. Thus, since the vertices of $\mathcal{S}_n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ belong to $\mathbb{U}_1 \upharpoonright n$ and $\mathbb{U}_1 \upharpoonright n$ is convex, we have that $\mathcal{S}_n(\mathbb{U}_1, d_{\mathbb{U}_1}, D, d_{\mathbb{U}_1} D)$ is contained in $\mathbb{U}_1 \upharpoonright n$, as needed. This ends the proof. □

6. Iterated Cone Construction

Given a simplex $S$ and a point $s \in S$ we define the cone of $S$ over $s$ as follows. Consider $Y = S \times [0, 1]$ and let

$$\text{cone}(S, s) = \text{conv}((S \times \{0\}) \cup (s, 1)),$$

where the convex hull is taken in $Y$ (recall our convention on metrics in Section 2 and note that if $S$ is a subset of the Hilbert cube, then the cone is embedded in the Hilbert cube as well).

The point $(s, 1)$ is called the cone point of the cone and we will identify $S$ with $S \times \{0\} \subseteq \text{cone}(S, s)$. We also denote the cone point as $c(s)$. The cone admits a natural affine continuous map $\pi : \text{cone}(S, s) \to S$ which is just the projection map in $Y$. It maps the cone point $c(s)$ to the point $s$.

We state now a couple of basic facts on the structure of cones.

Lemma 6.1. If $S$ is a simplex and $s \in S$, then $\text{cone}(S, s)$ is a simplex, $S$ is a face of $\text{cone}(S, s)$ and $\text{ext}(\text{cone}(S, s)) = \text{ext}(S) \cup \{c(s)\}$.

Proof. Note that if two points $z_1, z_2 \in S \times [0, 1]$ have an affine combination that lies in $S$, then both $z_1$ and $z_2$ must belong to $S$. This implies that $S$ is a face of the convex compact set $\text{cone}(S, s)$ and that $\text{ext}(S) \subseteq \text{ext}(\text{cone}(S, s))$. It is clear that $c(s)$ is an extreme point of $\text{cone}(S, s)$, so indeed $\text{ext}(\text{cone}(S, s)) = \text{ext}(S) \cup \{c(s)\}$. To see that $\text{cone}(S, s)$ is a simplex note that if $\mu$ is a measure concentrated on $\text{ext}(S) \cup \{c(s)\}$ such that $\int f \, d\mu = 0$ for every affine continuous function on $\text{cone}(S, s)$, then $\mu(\{c(s)\}) = 0$ and hence $\mu$ must be concentrated on $\text{ext}(S)$. □
Lemma 6.2. Given a simplex $S$ and two points $s_1, s_2 \in S$ we have
$$\text{cone}(\text{cone}(S, s_1), s_2) \simeq \text{cone}(\text{cone}(S, s_2), s_1).$$

Proof. Both simplices are affinely homeomorphic to the closed convex hull of $S \times \{(0,0,1)\} \cup \{(s_1,0,1)\}$ and $(s_2,1,0)$ in $S \times [0,1]^2$. To see the affine homeomorphism of $\text{cone}(\text{cone}(S, s_1), s_2)$ and $\text{cone}(\text{cone}(S, s_2), s_1)$ directly, write $c(s_1)$ for the cone point of $\text{cone}(S, s_1), c'(s_2)$ for the cone point of $\text{cone}(S, s_1), s_2), c(s_2)$ for the cone point of $\text{cone}(S, s_2)$ and $c'(s_1)$ for the cone point of $\text{cone}(S, s_2), s_1)).$ The affine homeomorphism then maps $(1-\beta)((1-\alpha)x + \alpha c(s_1)) + \beta c'(s_2)$ to $(1-\gamma)((1-\delta)x + \delta c(s_2)) + \gamma c'(s_1),$ where $\gamma = (1-\beta)\alpha$ and $\delta = \beta/(1-\alpha(1-\beta)).$ \hfill \Box

Given the above lemma, we use the notation $\text{cone}(S, s_1, \ldots, s_n)$ to denote an iterated cone of the form $\text{cone}(\ldots \text{cone}(S, s_1), \ldots, s_n)$ (or with any other permutation). We also call a cone of the form $\text{cone}(S, s_1, s_2)$ a double cone. Given a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\},$ the simplex $\text{cone}(S, s_{i_1}, \ldots, s_{i_k})$ is a subset of $\text{cone}(S, s_1, \ldots, s_n)$ in the same way $S$ is a subset of $\text{cone}(S, s)$. The simplex $\text{cone}(S, s_{i_1}, \ldots, s_{i_k})$ is then a face of $\text{cone}(S, s_1, \ldots, s_n).$ We call a face of the form $\text{cone}(S, s_{i_1}, \ldots, s_{i_k})$ in $\text{cone}(S, s_1, \ldots, s_n)$ a subcone.

Lemma 6.3. Given a simplex $S$ and $s_1, \ldots, s_n \in S$, if $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\},$ then
$$\text{cone}(S, s_1, \ldots, s_n) \simeq \text{cone}(\text{cone}(S, s_{i_1}, \ldots, s_{i_k}), s_{j_1}, \ldots, s_{j_{n-k}}).$$

Proof. This follows directly from the definitions and Lemma 6.2 \hfill \Box

Definition 6.4. Given a simplex $S$ and its countable (enumerated) subset $D = \{d_n : n \in \mathbb{N}\},$ define the iterated cone over $D$ as follows. Let $S_0 = S$ and for each $n \in \mathbb{N}$ let $S_n = \text{cone}(S_{n-1}, d_n)$ and let $\pi_n : S_n \to S_{n-1}$ be the projection map. Define
$$\text{cone}(S, D) = \lim_{\leftarrow} (S_n, \pi_n).$$

Note that if $S$ is a subset of the Hilbert cube, then $\text{cone}(S, D)$ can be naturally embedded into the Hilbert cube as well. Note also that the inverse system in the above definition is increasing. Thus, we treat $S_n$ as a face of $\text{cone}(S, D),$ for each $n \in \mathbb{N}$. Given that, all cone points of the simplices $S_n$ belong to $\text{cone}(S, D)$ and we refer to them as to the cone points of the iterated cone. In principle, the iterated cone over $D$ may depend on the ordering of the set $D$. As we will see, this does not happen.

Lemma 6.5. Let $S$ be a simplex, $\varepsilon > 0$ and $\varphi : S \to S$ be an affine homeomorphism. Given two points $s_1, s_2 \in S$ if $d_S(\varphi(s_1), s_2) < \varepsilon,$ then there is an affine homeomorphism $\varphi' : \text{cone}(S, s_1) \to \text{cone}(S, s_2)$ extending $\varphi$ such that
$$d_S(\varphi(\pi_1(s)), \pi_2(\varphi'(x))) < \varepsilon$$
for every $s \in \text{cone}(S, s_1),$ where $\pi_1 : \text{cone}(S, s_1) \to S$ and $\pi_2 : \text{cone}(S, s_2) \to S$ are the projection maps.
Lemma 6.7. Given a simplex \( S \) and a countable subset \( D \) and \( d \in S \setminus D \) we have
\[
\text{cone}(S, \{d\} \cup D) \simeq \text{cone}(\text{cone}(S, D), d).
\]
Suppose Lemma 6.8. such that $x \in X_0$ then cone($\lim_{\rightarrow} X_n, x$) $\simeq \lim_{\rightarrow} \text{cone}(X_n, x)$, where the latter inverse system has the natural projection maps $\pi_n : \text{cone}(X_n, x) \to \text{cone}(X_{n-1}, x)$ that extend $\pi_n$ and map the cone point to the cone point.

Write $D = \{d_1, d_2, \ldots\}$ and $X_n = \text{cone}(S, d_1, \ldots, d_n)$. By the remark above, cone$(\text{cone}(S, D), d) \simeq \lim_{\rightarrow} \text{cone}(X_n, d)$ and by Lemma 6.2 we have cone$(X_n, d) \simeq \text{cone}(S, d, d_1, \ldots, d_n)$. Thus, cone$(\text{cone}(S, D), d)$ is affinely homeomorphic to $\lim_{\rightarrow} \text{cone}(S, d, d_1, \ldots, d_n)$, which is equal to cone$(S, \{d\} \cup D)$. \qed

Now we look at the extreme point of the iterated cones.

**Lemma 6.8.** Suppose $S$ is a simplex and $s_1, s_2 \in S$. If $y \in \text{cone}(S, s_1)$ is such that

$$y = \alpha x + (1 - \alpha)s_1$$

for some $x \in S$ and $0 < \alpha < 1$, then for every $y' \in \text{cone}(\text{cone}(S, s_1), s_2)$ with $\pi(y') = y$ there exists a unique $x' \in \text{cone}(S, s_2)$ such that

$$y' = \alpha x' + (1 - \alpha)s_1,$$

where $\pi : \text{cone}(\text{cone}(S, s_1), s_2) \to \text{cone}(S, s_1)$ is the projection map.

**Proof.** Uniqueness of $x'$ is immediate. We only need to find $x'$ in the cone. Put $y'' = \alpha s_2 + (1 - \alpha)s_1$ and note that $\pi(y'') = y$. Since cone$(\text{cone}(S, s_1), s_2)$ is convexely generated by cone$(S, s_1)$ and $y''$, there is $\beta \in [0, 1]$ such that $y' = \beta y + (1 - \beta)y''$. Put $x' = \beta x + (1 - \beta)s_2$. We claim that $x'$ is as needed, i.e. that $\alpha x' + (1 - \alpha)s_1 = y'$. And indeed,

$$y' = \beta y + (1 - \beta)y'' = \beta (\alpha x + (1 - \alpha)s_1) + (1 - \beta)(\alpha s_2 + (1 - \alpha)s_1)
= \alpha \beta x + \beta (1 - \alpha)s_1 + (1 - \beta)\alpha s_2 + (1 - \beta)(1 - \alpha)s_1
= \alpha \beta x + (1 - \alpha)s_1 + (1 - \beta)\alpha s_2
= \alpha (\beta x + (1 - \beta)s_2) + (1 - \alpha)s_1 = \alpha x' + (1 - \alpha)s_1$$

\qed

**Lemma 6.9.** Given a simplex $S$ and its countable subset $D$ let $\{v_n : n \in \mathbb{N}\}$ be the set of cone points of cone$(S, D)$. Then

$$\text{ext}(\text{cone}(S, D)) = \text{ext}(S) \cup \{v_n : n \in \mathbb{N}\}.$$

**Proof.** By Lemma 6.1, $S$ is a face of cone$(S, D)$, so $\text{ext}(S) \subseteq \text{ext}(\text{cone}(S, D))$. Also, for each $n$ the simplex cone$(S, d_0, \ldots, d_n)$ is a face of cone$(S, D)$, which shows that $v_n \in \text{ext}(\text{cone}(S, D))$. We need to show that there are no more extreme points in cone$(S, D)$. Let $y \in \text{cone}(S, D)$ be such that $y \notin S$ and $y \neq v_n$ for each $n$. Pick $n$ such that $y \upharpoonright n \notin S$. This means that there exists $\lambda$ with $0 < \lambda < 1$ and $s \in S$ such that

$$y \upharpoonright n = \lambda s + (1 - \lambda)v_n.$$
By Lemma [6.8] for each \( m > n \) there exists \( s_m \in \text{cone}(S, d_{n+1}, \ldots, d_m) \) such that
\[
y \upharpoonright m = \lambda s_m + (1 - \lambda)v_n.
\]
Uniqueness of \( s_m \) implies that \( s_{m_2} \upharpoonright m_1 = s_{m_1} \) for each \( m_2 > m_1 > n \). Let \( x \in \text{cone}(S, \{d_{n+1}, d_{n+2}, \ldots\}) \) be such that \( x \upharpoonright k = s_k \) for each \( k > n \). By Lemma [6.7], we have
\[
\text{cone}(S, D) = \text{cone}(\text{cone}(S, \{d_{n+1}, d_{n+2}, \ldots\}), d_0, \ldots, d_n)
\]
and in the latter simplex we have \( y = \lambda x + (1 - \lambda)v_n \), which shows that \( y \) is not an extreme point of \( \text{cone}(S, D) \).

**Lemma 6.10.** Given a simplex \( S \) and its countable subset \( D \), let \( \{v_n : n < \omega\} \) be the set of cone points of \( \text{cone}(S, D) \). Then each \( v_n \) is an isolated point of \( \text{ext}(\text{cone}(S, D)) \).

**Proof.** Write \( D = \{d_1, d_2, \ldots\} \) so that \( v_n \) is the cone point over \( d_n \), for each \( n \in \mathbb{N} \). We need to show that \( v_n \) is isolated in \( \text{ext}(S) \cup \{v_i : i \neq n\} \). Write \( \pi_n^\infty : \text{cone}(S, D) \to \text{cone}(S, d_1, \ldots, d_n) \) for the projection map. Clearly, \( v_n = \pi_n^\infty(v_n) \) has positive distance from \( S \cup \{v_i : i < n\} \) in \( \text{cone}(S, d_1, \ldots, d_n) \), so let \( U \) be an open neighborhood of \( v_n \) in \( \text{cone}(S, d_1, \ldots, d_n) \) that is disjoint from \( S \) and \( \{v_i : i < n\} \). Then \( (\pi_n^\infty)^{-1}(U) \) is an open neighborhood of \( v_n \) that is disjoint from \( \text{ext}(S) \) and does not contain any \( v_i \) with \( i \neq n \). \(\square\)

7. The blow-up construction

Recall that given a simplex \( S \) and two points \( x_1, x_2 \in S \) the double cone \( \text{cone}(S, x_1, x_2) \) is the simplex \( \text{cone}(\text{cone}(S, x_1), x_2) \). Now, given a simplex \( S \) and its subset \( X \subseteq S \) together with a metric \( d_X \) on \( X \) we will define another, bigger, simplex, which will be used to encode the metric in its affine structure.

Given a countable dense subset \( D \) of a metric space \((X, d_X)\), enumerate as \( (p_n = (x_n, y_n, U_n) : n \in \mathbb{N}) \), with infinite repetitions, all triples \((x, y, U)\) with \( x, y \in D \) distinct and \( U \subseteq [0, 1] \) basic open with \( d_X(x, y) \in U \). Call the sequence \( (p_n : n \in \mathbb{N}) \) the metric scheme of \((X, d_X, D)\) and denote it by \( \text{Sch}(X, d_X, D) \). For every \( n \in \mathbb{N} \) and \( p_n = (x_n, y_n, U) \) in the metric scheme, write \( p_n(1) \) for \( x_n \) and \( p_n(2) \) for \( y_n \). Write also \( p_n(D) \) for the \( n \)-th element of \( \text{Sch}(X, d_X, D) \).

Define an increasing inverse system of simplices as follows. Let \( B_0 = S \) and \( B_n = \text{cone}(B_{n-1}, p_n(1), p_n(2)) \). Define the blow-up of \( S \) with respect to \((X, d_X)\) (slightly abusing the notation), denoted by \( B(S, X, d_X) \) as \( \lim B_n \) and write \( c_1(p_n) \) for the cone point over \( p_n(1) \) in \( B(S, X, d_X) \) and \( c_2(p_n) \) for the cone point over \( p_n(2) \) in \( B(S, X, d_X) \).

**Definition 7.1.** Given a closed subspace \((X, d_X)\) of \( U_1 \) define \( B(X, d_X) \) as \( B(S_Z(X, d_X), X, d_X) \).

In principle, the blow-up construction depends only on the enumeration (with repetitions) of pairs of points in \( D \). We keep, however, the sequence of
Proposition 7.2. Suppose $S$ and $T$ are simplices, $X \subseteq S$ and $Y \subseteq T$ are their subsets, $d_X$ and $d_Y$ are metrics on $X$ and $Y$, respectively and $D \subseteq X$ and $E \subseteq Y$ are countable dense sets. Given an affine homeomorphism $\varphi : S \to T$ such that $\varphi''X = Y$ and $\varphi \upharpoonright X$ is an isometry of $(X, d_X)$ and $(Y, d_Y)$, there is an affine homeomorphism $\bar{\varphi} : B(S, X, d_X) \to B(T, Y, d_Y)$ such that $\bar{\varphi}$ extends $\varphi$ and

(a) for every $p \in \text{Sch}(X, d_X, D)$ there exists $q \in \text{Sch}(Y, d_Y, E)$ such that if $p = (x_1, x_2, U)$, then $q = (y_1, y_2, U)$ and $\bar{\varphi}$ maps $c_1(p)$ to $c_1(q)$ and $c_2(p)$ to $c_2(q)$,

(b) for every $q \in \text{Sch}(Y, d_Y, E)$ there exists $p \in \text{Sch}(X, d_X, D)$ such that if $q = (y_1, y_2, V)$, then $p = (x_1, x_2, V)$ and $\bar{\varphi}^{-1}$ maps $c_1(q)$ to $c_1(p)$ and $c_2(q)$ to $c_2(p)$.

Proof. The proof is essentially the same as that of Proposition 6.6 and we only sketch it. Write $B_n$ for the simplices in the inverse system of $B(S, X, d_X)$ and $C_n$ for the simplices in the inverse system of $B(T, Y, d_Y)$. By induction on $i \in \mathbb{N}$, construct an approximate intertwining $\psi_i : B_{n_i} \to C_{m_i}$ and $\psi : C_{m_i} \to B_{n_{i+1}}$ so that $\psi_0 = \varphi$. At the inductive construction, when extending a map from $B_n$ to $B_{n+1}$, make sure that if $p_n(D) = (x_1, x_2, U)$, then for some $m \in \mathbb{N}$ with $p_m(E) = (y_1, y_2, U)$, the point $c_1(p_m(E))$ is mapped to $c_1(p_m(E))$, $c_2(p_m(E))$ is mapped to $c_2(p_m(E))$ so that

$$d_T(\varphi(x_1), y_1), d_T(\varphi(x_2), y_2) < 2^{-m}$$

and $d_Y(y_1, y_2) \in U$. Analogous conditions apply when we extend a map from $C_m$ to $C_{m+1}$. The fact that the above is possible follows at once from the fact that $\psi$ was an isometry and the triplets in the schemes are enumerated with infinite repetitions.

Once the construction is finished, (2) and the symmetric condition involving $d_S$ imply that the sequence forms an approximate intertwining. Then, an application of Proposition 4.3 gives an affine homeomorphism $\bar{\varphi} : \lim B_i \to \lim C_i$. The conditions (a) and (b) follow directly from the construction. $\square$

8. Coding a metric into the affine structure of a simpliex

Given a metric space $(X, d_X)$ of diameter bounded by 1, which we assume is a subspace of the Urysohn sphere $\overline{U}_1$, and a dense countable subset $D$ of $X$, let $(p_n = ((x_n, y_n) : n \in \mathbb{N})$ be the associated metric scheme. Consider the blow-up $B(X, d_X)$ and let $M(X, d_X, D)$ be the subset of $B(X, d_X)$ consisting of the points $qc_1(p_n) + (1-q)c_2(p_n)$ for $q \in U_n \cap \mathbb{Q}$. 

$p_n$'s for reference to the further construction. Note that each pair of distinct points in $D$ appears infinitely often in the sequence. Analogous arguments as in Section 6 show that the blow-up depends neither on the ordering $p_n$ nor on the choice of the dense set.
Use the Kuratowski–Ryll-Nardzewski theorem \cite[Theorem 12.13]{43}, to find a countable dense subset \( D(X) \) for every (nonempty) closed subset \( X \) of \( U_1 \) so that the map \( D : X \mapsto D(X) \) is Borel. Define now \( \Phi(X, d_X) \) as cone\((B(X, d_X), M(X, d_X, D(X)))\). It is easy to check that \( \Phi \) is a Borel map from the space of closed (nonempty) subsets of \( U_1 \) to the space of separable Choquet simplices (cf. Section \( 2 \) for the Borel structure on the space of simplices). In fact, the map \((X, D) \mapsto \text{cone}(B(X, d_X), M(X, d_X, D(X)))\), which maps a pair of a Polish space and its dense countable subset to a simplex, is continuous.

**Proposition 8.1.** Suppose \((X, d_X)\) and \((Y, d_Y)\) are subspaces of \( U_1 \). If \((X, d_X)\) and \((Y, d_Y)\) are isometric, then the simplices \( \Phi(X, d_X) \) and \( \Phi(Y, d_Y) \) are affinely homeomorphic.

**Proof.** For simplicity, identify \( X \) with \( Z(X) \) and \( Y \) with \( Z(Y) \) (see Proposition \( 3.2 \)) and let \( \varphi : U_1 \rightarrow U_1 \) be an isometry such that \( \varphi'' X = Y \). By Propositions \( 5.22 \) and \( 7.2 \), there is an affine homeomorphism \( \bar{\varphi} : B(X, d_X) \rightarrow B(Y, d_Y) \) which extends \( \varphi \) and such that

- for each \( p = (x_1, x_2, U) \) in the scheme of \((X, d_X)\) there is \( q = (y_1, y_2, V) \) in the scheme of \((Y, d_Y)\) such that \( U = V \) and \( \bar{\varphi} \) maps the cone points \( c_1(p) \) to \( c_1(q) \) and \( c_2(p) \) to \( c_2(q) \),
- for each \( q = (y_1, y_2, V) \) in the scheme of \((Y, d_Y)\) there is \( p = (x_1, x_2, U) \) in the scheme of \((X, d_X)\) that \( U = V \) and \( \bar{\varphi}^{-1} \) maps \( c_1(q) \) to \( c_1(p) \) and \( c_2(q) \) to \( c_2(p) \).

The above imply that \( \varphi'' \text{cl}(M(X, d_X, D(X))) = \text{cl}(M(Y, d_Y, D(Y))) \) and hence, by Proposition \( 6.6 \) there is an affine homeomorphism \( \bar{\varphi} : \Phi(X, d_X) \rightarrow \Phi(Y, d_Y) \) that extends \( \bar{\varphi} \). \( \square \)

Proposition \( 8.1 \) shows that \( \Phi \) is a homomorphism from the isometry of separable metric spaces to the affine homeomorphism of simplices. To show that \( \Phi \) is a reduction, we will restrict attention to perfect separable metric spaces.

**Claim 8.2.** If \((X, d_X)\) is perfect, then the set of nonisolated extreme points of \( \Phi(X, d_X) \) is equal to \( \text{ext}(S(X, d_X)) \).

**Proof.** \( X \) is perfect and dense in \( \text{ext}(S_Z(X, d_X)) \) by Corollary \( 5.21 \), so all extreme points of \( S_Z(X, d_X) \) are nonisolated in \( \text{ext}(S_Z(X, d_X)) \). Since the simplex \( S_Z(X, d_X) \) is a face of \( \Phi(X, d_X) \), the extreme points of \( S_Z(X, d_X) \) are still nonisolated extreme points of \( \Phi(X, d_X) \). Now, Lemmas \( 6.9 \) and \( 6.10 \) (and analogous statements for the blow-up) imply that all the other extreme points of \( \Phi(X, d_X) \) are isolated. \( \square \)
Lemma 8.3. Suppose \((X,d_X)\) is a perfect metric subspace of \(\mathbb{U}_1\). If \(x,y \in X\), then \(d_X(x,y)\) is the only \(\alpha \in [0,1]\) such that

\[
\forall U \subseteq [0,1] \text{ basic open neighborhood of } \alpha \\
\exists V_x, V_y \subseteq \Phi(X,d_X) \text{ open neighborhoods of } x \text{ and } y \in \Phi(X,d_X) \\
\forall c_1 \in V_x \forall c_2 \in V_y \text{ isolated extreme points} \\
(3) \quad [(\exists \lambda \in [0,1]) \lambda c_1 + (1 - \lambda)c_2 \\
\text{is a limit of isolated extreme points of } \Phi(X,d_X)] \\
\Rightarrow (\exists \lambda \in U \lambda c_1 + (1 - \lambda)c_2 \\
\text{is a limit of isolated extreme points of } \Phi(X,d_X)]
\]

Proof. Write \(D\) for the countable dense subset \(D(X)\) of \(X\). By Claim 8.2 and Lemma 6.3, the only isolated extreme points of \(\Phi(X,d_X)\) are in the sets \(E_1 = \{c_1(p),c_2(p) : p \in \text{Sch}(X,d_X,D)\}\) and \(E_2 = \{c(z) : z \in M(X,d_X,D)\}\). It follows from the iterated cone construction that if \(z \in M(X,d_X,D)\), then for any isolated extreme point \(e\) of \(\Phi(X,d_X)\), no point in the set \(\{\lambda c(z) + (1 - \lambda)e : \lambda \in [0,1]\}\) is a limit of isolated extreme points of \(\Phi(X,d_X)\).

On the other hand, if \(e_1,e_2 \in E_1\) are such that for some \(\lambda \in [0,1]\) the point \(\lambda e_1 + (1 - \lambda)e_2\) is a limit of isolated extreme points of \(\Phi(X,d_X)\), then there is \(p \in \text{Sch}\) such that \(e_1 = c_1(p), e_2 = c_2(p)\). The latter follows from the fact that the intersection of \(B(X,d_X)\) with the closure of the set \(\{c(z) : z \in M(X,d_X,D)\}\) is exactly the closure of \(M(X,d_X,D)\).

Pick \(x,y \in X\). We claim that \(\alpha = d_X(x,y)\) satisfies the condition (3). Pick any \(U \subseteq [0,1]\) basic open neighborhood of \(\alpha\) and let \(\varepsilon > 0\) be such that \((\alpha - \varepsilon,\alpha + \varepsilon) \subseteq U\). Since \(X\) is a topological subspace of \(S(X,d_X)\), there are \(V_x^1\) and \(V_y^1\) open neighborhoods of \(x\) and \(y\) in \(S(X,d_X)\), respectively, such that \(V_x^1 \cap X \subseteq \text{ball}(X,d_X)(x,\varepsilon/2)\) and \(V_y^1 \cap X \subseteq \text{ball}(X,d_X)(y,\varepsilon/2)\). Let \(V_x\) and \(V_y\) be open neighborhoods of \(x\) and \(y\) in \(\Phi(X,d_X)\) such that if \(c_1 \in V_x\) and \(c_2 \in V_y\), then \(\pi(c_1) \in V_x^1\) and \(\pi(c_2) \in V_y^1\), where \(\pi : \Phi(X,d_X) \to S(X,d_x)\) denotes the projection map. We claim that \(V_x\) and \(V_y\) are as needed. Let \(c_1 \in V_x\) and \(c_2 \in V_y\) be arbitrary extreme isolated points such that for some \(\lambda \in [0,1]\) the point \(\lambda c_1 + (1 - \lambda)c_2\) is a limit of isolated extreme points of \(\Phi(X,d_X)\). By the remarks in the previous paragraph, there is \(p \in \text{Sch}(X,d_X,D)\) such that \(p = (d_1,d_2,V)\) for some \(d_1,d_2 \in D\) with \(\pi(c_1) = d_1\) and \(\pi(c_2) = d_2\) and \(V\) is a basic open neighborhood of \(d_X(d_1,d_2)\). Now, since \(c_1 \in V_x\) and \(c_2 \in V_y\) we have that \(d_X(d_1,d_2) \in (\alpha - \varepsilon,\alpha + \varepsilon) \subseteq U\), so \(V \cap U\) is nonempty. Pick any \(\lambda \in V \cap U\) and note that \(\lambda c_1 + (1 - \lambda)c_2\) is also a limit of isolated extreme points of \(\Phi(X,d_X)\) since \(\lambda \in V\).

We also need to show that \(\alpha\) is the only number in \([0,1]\) satisfying (3) for \(x\) and \(y\). Pick any \(\beta \in [0,1]\) distinct from \(\alpha\). Let \(\varepsilon > 0\) be smaller than \(|\alpha - \beta|\). Pick a basic open neighborhood \(U \subseteq (\beta - \varepsilon/2,\beta + \varepsilon/2)\) of \(\beta\). We claim that \(U\) witnesses that (3) is not satisfied. Let \(V_x\) and \(V_y\) be arbitrary open neighborhoods of \(x\) and \(y\), respectively, in \(\Phi(X,d_X)\). Pick \(d_1,d_2 \in D\) such that \(d_X(d_1,x) < \varepsilon/2, d_X(d_2,y) < \varepsilon/2\) and \(d_1 \in V_x, d_2 \in V_y\). Note that
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d_X(d_1, d_2) \in (\alpha - \varepsilon/2, \alpha + \varepsilon/2) \) and that \( U \) is disjoint from \((\alpha - \varepsilon/2, \alpha + \varepsilon/2)\).

Find a basic open neighborhood \( V \) of \( d_X(d_1, d_2) \) in \([0, 1]\) such that \( \overline{V} \cap U = \emptyset \).

Find \( n \in \mathbb{N} \) big enough so that \( p_n \in \text{Sch}(X, d_X, D) \) is equal to \((d_1, d_2, V)\) and such that letting \( c_1 = c_1(p_n), c_2 = c_2(p_n) \) we have \( c_1 \in V_x \) and \( c_2 \in V_y \).

Now, there is \( \lambda \in [0, 1] \) such that \( \lambda c_1 + (1 - \lambda)c_2 \) is a limit of isolated extreme points of \( \Phi(X, d_X) \) but the set of such \( \lambda \) is equal to \( V \), which is disjoint from \( U \). This shows that \( \beta \) does not satisfy (3) and ends the proof. \( \square \)

Now we are ready to finish the proof that \( \Phi \) is a reduction.

**Proposition 8.4.** If \((X, d_X)\) and \((Y, d_Y)\) are perfect closed subspaces of \( \mathbb{U}_1 \) and \( \Phi(X, d_X) \) is affinely homeomorphic to \( \Phi(Y, d_Y) \), then \((X, d_X)\) is isometric to \((Y, d_Y)\).

**Proof.** Let \( \varphi : \Phi(X, d_X) \to \Phi(Y, d_Y) \) be an affine homeomorphism. Note that \( \varphi \) maps nonisolated extreme points of \( \Phi(X, d_X) \) to nonisolated extreme points of \( \Phi(Y, d_Y) \), so, by Claim 8.2, \( \varphi''\text{ext}(S(X, d_X)) = \text{ext}(S(Y, d_Y)) \).

Now, \( X \subseteq S(X, d_X) \) and \( Y \subseteq S(Y, d_Y) \) are dense \( G_\delta \) sets, so there is a comeager set \( X' \subseteq X \) such that \( Y' = \varphi''X' \) is comeager in \( Y \). Note that since \( \varphi \) preserves the topological and affine structure, \( \varphi'' \) is preserved by \( \varphi \) and so Lemma 8.3 implies that for \( x_1, x_2 \in X' \) we have \( d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \). This means that \((X', d_X)\) and \((Y', d_Y)\) are isometric. Since \( X' \) is comeager in \( X \) and \( Y' \) is comeager in \( Y \), the spaces \( X \) and \( Y \) are isometric as well. \( \square \)

Theorem 1.2 now follows from Propositions 3.1, 8.1 and 8.4.

9. Concluding remarks and open questions

A major problem [12, Problem 5.2] that is still left open is that of the complexity of the homeomorphism relation for compact metric spaces. As shown by Kechris and Solecki (see also [25, Theorem 1.4] for a new proof of this result), it is an orbit equivalence relation that is bi-reducible with an action of the group of homeomorphisms of the Hilbert cube. Homeomorphism of compact metric spaces is Borel reducible to the affine homeomorphism of Bauer simplices and it is not known whether it is also a complete orbit equivalence relation. As a comment to this problem, let us note that from the point of view of Banach space theory, this is the question whether the isometry problems for general separable Banach spaces and for separable Banach spaces of the form \( C(K) \), have the same complexities.

In [25, Theorem 7.3] Farah, Toms and Tönnquist showed that the isomorphism of separable simple nuclear C*-algebras is below an action of the automorphism group of the Cuntz algebra \( \text{Aut}(O_2) \). Since complete orbit equivalence relations are typically induced by actions of universal Polish groups, the following question is natural.

**Question 9.1.** Is the group \( \text{Aut}(O_2) \) a universal Polish group?
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