Conformal symmetry and quantum localization in space-time

Marc-Thierry Jaekel and Serge Reynaud

1Laboratoire de Physique Théorique de l’Ecole Normale Supérieure, CNRS, UPMC, 24 rue Lhomond, F75251 Paris Cedex 05
2Laboratoire Kastler Brossel, Université Pierre et Marie Curie, CNRS, ENS, case 74, Campus Jussieu, F75252 Paris Cedex 05

The classical procedures which define the relativistic notion of space-time can be implemented in the framework of Quantum Field Theory. Only relying on the conformal symmetries of field propagation, time-frequency transfer and localization lead to the definition of time-frequency references and positions in space-time as quantum observables. Quantum positions have a non vanishing commutator identifying with spin, both observables characterizing quantum localization in space-time. Frame transformations to accelerated frames differ from their classical counterparts. Conformal symmetry nevertheless allows to extend the covariance rules underlying the formalism of general relativity under an algebraic form suiting the quantum framework.

I. INTRODUCTION

Different domains of physics actually use different representations of the notions of time and space. The space-time parameter manifold underlying modern physical theories, such as quantum field theories or general relativity, is alien to the physical observables used by modern metrology to build reference systems in space-time. This situation can hardly be sustained, as it is the source of strong difficulties when domains lying at the interface between general relativity, quantum theory and metrology are explored with an ever increasing accuracy in space and time measurements.

The different notions of time and space underlying physics were first clearly stated by Newton in his *Principia* [1]:

"I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration. Relative, apparent and common time, is some sensible and external measure of duration by the means of motion, which is commonly used instead of true time.

II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces, which our senses determine by its position to bodies, and which is commonly taken for immovable space."

The representation of time and space as mathematical parameters, independent of physical observables, was privileged by Newton and still lies at the basis of the differential formalism of modern physical theories. On another hand, when he founded relativity [2], Einstein insisted on the necessity to connect space and time to physical observables in order to reach a consistent theoretical framework. This program is nowadays implemented by specific physical systems, namely clocks and light signals, which deliver and disseminate time observables, and specific procedures, clock synchronization and event localization, to coordinate events both in time and in space. The relativistic conception of space-time and its constructive procedures lie at the heart of modern metrology [3] and coordination systems, such as the Global Positioning System (GPS) [4, 5].

The relativistic conception must however face the quantum nature of physical observables. The incompatibility of the notions of time and space used in the relativistic and quantum formalisms has been early recognized by Schrödinger [6], reviving the fundamental distinction made by Newton. As underlined by Schrödinger, the formalism of quantum mechanics preserves Newton mathematical time while it represents space positions by operators, a difference of status which cannot be accepted from the point of view of relativity. Even if space positions may be given a representation in terms of quantum operators conjugate to momentum [7, 8, 9], it is commonly admitted that time cannot be given a similar description as an operator conjugate to energy [10, 11]. Modern Quantum Field Theory manages to restore compatibility, but at the price of representing both time and space as parameters and of losing their connection with physical observables. This choice is at the origin of strong conflicts between relativistic and quantum requirements when attempting to built a consistent theory at their interface [12, 13].

In fact, as we discuss here, the relativistic approach of space-time may be implemented within the standard framework of Quantum Field Theory, by applying the latter to a suitable line of reasoning [14, 15, 16, 17, 18, 19]. Let us first briefly recall the steps followed by a relativistic construction of space-time. It begins with a local definition of time, as a physical observable delivered by a clock located at a give place in space [2]. In today’s applications, this local time is provided by the most accurate clocks available, *i.e.* by atomic clocks [4]. The local notion of time must then be extended over all space or, equivalently, the different local times associated with remote clocks must be synchronized. For that purpose, two observers share a piece of information which allows them to compare the indications of their respective local clocks. This is accomplished by the exchange of propagating electromagnetic fields.
Radio links are used by today’s most efficient systems for disseminating time references or for synchronizing clocks located in satellites (as in the GPS constellation) or in stations on the Earth surface [4]. A first observer encodes a time reference on an electromagnetic pulse, representing the time delivered by his clock. Comparing the received time reference with the indications of his own clock, the second observer then proceeds to the identification of the two time variables, i.e. to the synchronization of his clock. To ensure a faithful comparison, encoding should be performed using physical quantities which are preserved by field propagation.

Space coordination follows time transfer or synchronization. An event is completely localized both in space and time using several transfers of time, at least the same number as the space-time dimension. Concrete realizations like the GPS [4, 5] use a higher number for raising degeneracies between solutions. An event in space-time is then defined by merging several exchanges of electromagnetic signals. Classically, the positions of an event, both in space and time, is deduced from the different transferred time references. This is illustrated in two-dimensional space-time by simple relations

\[ t - x/c = u^- \quad t + x/c = u^+ \]  

The values of light-cone variables \((u^\pm)\) are the quantities preserved by field propagation and they are sufficient to provide the positions localizing an event in space-time. The implementation of these procedures with quantum fields hits upon fundamental limitations imposed by quantum theory. Heisenberg inequalities limit the possibility to focus field energies at a given value of a light-cone variable. References used in a time transfer should be represented as non commuting operators defined in a quantum algebra. Transfer and localization should then be performed using quantum observables which are preserved by field propagation. Such physical quantities exist, due to the symmetries of field propagation equations.

In the following, we show how the conformal symmetries underlying the propagation of quantum fields allow one to realize a relativistic space-time construction and to obtain time-frequency references and space-time positions as quantum observables. We briefly discuss their resulting properties under relativistic transformations and point at consequences for a potential extension of the formalism found in general relativity.

II. QUANTUM TIME-FREQUENCY TRANSFER

As a first step, we show how observables used in time-frequency transfer or synchronization can be implemented in the framework of Quantum Field Theory. In order to discuss the quantum properties induced by fields on transfer or synchronization observables, it will be sufficient to use a simplified model in terms of a real scalar quantum field in two-dimensional space-time. This simplification amounts to project fields on the direction of propagation and hence to discard all effects related to transverse directions or to photon polarizations [14]. Such effects, which do not affect two-dimensional space-time. This simplification amounts to project fields on the direction of propagation and hence to discard all effects related to transverse directions or to photon polarizations [14]. Such effects, which do not affect

\[ [\phi_\omega(t, x), \phi_{\omega'}(t', x')] = 2\pi \delta(\omega - \omega') \delta(t - t' - x/c) \quad [\phi_\omega(t, x), \phi_{\omega'}^\dagger(t', x')] = 0 \]  

\[ [\phi_\omega^\dagger(t, x), \phi_{\omega'}(t', x')] = 2\pi \delta(\omega - \omega') \delta(t - t' + x/c) \quad [\phi_\omega^\dagger(t, x), \phi_{\omega'}^\dagger(t', x')] = 0 \]  

These commutation rules are sufficient to determine all commutation properties of operators built on fields, in particular the algebra of observables.
Propagation preserves some quantities which are built on the field and can be obtained from the field energy-momentum tensor (\(e_\sigma(u) = (\partial_u \varphi_\sigma(u))^2\)):

\[
P_\sigma = \int_{-\infty}^{\infty} e_\sigma(u) du, \quad J_\sigma = \int_{-\infty}^{\infty} u e_\sigma(u) du, \quad C_\sigma = \int_{-\infty}^{\infty} u^2 e_\sigma(u) du
\]  

(5)

The three conserved quantities \(P_\sigma, J_\sigma, C_\sigma\) respectively describe, for each propagation direction, the momentum, angular momentum and quadripolar momentum of the quantum field. According to Noether theorem, these conserved quantities correspond to symmetries of propagation equations \(^{[24]}\) and identify with the generators of these symmetries. The action on fields of these conserved quantities is deduced from canonical commutation rules \(^{[24]}\) and indeed identifies with the action of symmetry generators

\[
\frac{i}{\hbar}[P_\sigma, \varphi_\sigma(u)] = \partial_u \varphi_\sigma(u), \quad \frac{i}{\hbar}[J_\sigma, \varphi_\sigma(u)] = u \partial_u \varphi_\sigma(u), \quad \frac{i}{\hbar}[C_\sigma, \varphi_\sigma(u)] = u^2 \partial_u \varphi_\sigma(u)
\]

(6)

For each field component, the algebra generated by conserved quantities coincides with a conformal algebra. In two-dimensional space-time, the conformal symmetry algebra contains an infinite number of generators (describing arbitrary changes of the light cone variable \(u\)). But only the three momenta written in (5) will allow for a generalization to four dimensions and will be needed for defining localization observables. These three conserved quantities generate a special conformal algebra

\[
\frac{i}{\hbar}[P_\sigma, J_\sigma] = P_\sigma, \quad \frac{i}{\hbar}[P_\sigma, C_\sigma] = 2J_\sigma, \quad \frac{i}{\hbar}[J_\sigma, C_\sigma] = C_\sigma
\]

(7)

These generators \(^{[3]}\) respectively represent translations, dilations or Lorentz transformations, and transformations to accelerated frames. At this point, it should be noted that the photon number \(N_\sigma\) is preserved by these conformal transformations \(^{[14]}\)

\[
N_\sigma = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} a_\omega^\dagger a_\omega = 0
\]

(8)

The photon number for each component is defined so that the corresponding energy spectral density identifies with Planck constant (see eq. (3)), so that it is a non local expression of the light cone variable describing the quantum field. It is invariant under frame transformations corresponding to translations, Lorentz transformations and accelerations \(^{[5]}\). This means in particular that observers which can be related by such frame transformations will give the same physical interpretation to processes involving photons. Other invariants are also obtained from the Casimir invariants of the conformal algebra \(^{[6]}\) (\(\cdot\) denotes the symmetrized product)

\[
\alpha_\sigma^2 = C_\sigma \cdot P_\sigma - J_\sigma^2
\]

(9)

Using the definition of conserved quantities \(^{[3]}\), one sees that the Casimir invariant \(\alpha_\sigma^2\) \(^{[6]}\) measures the field dispersion transversely to the direction of propagation.

In a classical framework, each field component may be seen as propagating while preserving the value of a light cone variable \(u\), which can then be used as the reference to be shared by remote observers. The energy of classical fields can be focussed at a given value of \(u\) with a precision which is only limited by technology. This allows one to encode a time reference provided by \(u\) and which can be defined, at least in principle, with an arbitrary precision. This situation does not hold any more in the quantum framework, due to Heisenberg inequalities which constrain conjugate degrees of freedom, hence field and energy densities. The quantum framework nevertheless allows one to define quantum observables \(\Omega_\sigma\) and \(U_\sigma\) representing the frequency and the position of a propagating field component and taking the place of the classical frequency \(\omega\) and the classical light cone variable \(u\) \(^{[14]}\). Quantum definitions generalize the classical ones while taking into account the fundamental relations entailed by Planck constant \(^{[2]}\) between energy and frequency, and by relativity \(^{[24]}\) between the energy barycenter (center of inertia) and Lorentz boost

\[
\Omega_\sigma = \frac{P_\sigma}{\hbar N_\sigma}, \quad U_\sigma = J_\sigma \cdot \frac{1}{P_\sigma}
\]

(10)

The relativistic quantum observables defined by \(^{[10]}\) provide the required quantum references to be used in frequency or time transfer. The usual quantum limits encountered in time and frequency transfers by means of quantum fields
may then be described in a universal way by the quantum commutator of the time and frequency observables

\[ [\Omega_\sigma, U_\sigma] = -\frac{i}{\hbar} \]

(11)

The quantum commutator (11) limiting time and frequency transfers only depends on the photon number, so that it is identical for different observers which can be related through a conformal frame transformation (5). As expected, the quantum references defined by (10) undergo relativistic transformations which follow the classical ones in the case of translations and Lorentz transformations

\[
\begin{align*}
\frac{i}{\hbar} [P_\sigma, \Omega_\sigma] &= 0, \\
\frac{i}{\hbar} [P_\sigma, U_\sigma] &= 1, \\
\frac{i}{\hbar} [J_\sigma, \Omega_\sigma] &= -\Omega_\sigma, \\
\frac{i}{\hbar} [J_\sigma, U_\sigma] &= U_\sigma
\end{align*}
\]

(12)

The first equations in (12) represent the conjugation relation between positions and momenta, while the second ones correspond to the Doppler shift and time dilation which characterize Lorentz transformations. Furthermore, the field propagation equations (2) admit a larger group of symmetries than the Weyl group defined by equations (12). These symmetries include in particular transformations between observers in relative acceleration. The corresponding transformations of quantum references are then deduced from their definition (10) and the conformal algebra (7)

\[
\frac{i}{\hbar} [C_\sigma, \Omega_\sigma] = -2 U_\sigma \cdot \Omega_\sigma, \\
\frac{i}{\hbar} [C_\sigma, U_\sigma] = U_\sigma^2 - (\alpha_\sigma^2 + \frac{\hbar^2}{4}) \frac{1}{P_\sigma}
\]

(13)

The relativistic transformation of the frequency observable corresponds to a position dependent shift of frequency, thus providing a quantum generalization of the classical redshift undergone by frequencies when seen by an accelerated observer. The first relation in (13) then realizes a quantum version of Einstein effect on clocks in an accelerated frame [21]. Meanwhile, the relativistic transformation of the position observable no longer identifies with its classical analog but involves an additional term which may be seen as reflecting the fundamental constraints entailed on field dispersions by Heisenberg inequalities. Using equations (10), the Casimir invariants \(\alpha_\sigma^2\) (9) allow one to rewrite the special conformal generators \(C_\sigma\) in terms of positions and momenta

\[ C_\sigma = U_\sigma P_\sigma U_\sigma + (\alpha_\sigma^2 + \frac{\hbar^2}{4}) \frac{1}{P_\sigma} \]

(14)

Last relations in (13) then follow from conjugation (12) and relation (14). It can be shown that the term \(\hbar^2/4\) corresponds to a minimum of field dispersions, the latter being attained for single photon states [14].

The scheme described in the previous section may be applied using quantum fields. Combining time transfers in different directions allows one to define localization observables for positions in space and time in terms of quantum fields. In a two-dimensional space-time, relations (1) may be used and relativistic transformations of localization observables simply follow from those of transfer observables (12) and (13), that is from the whole conformal algebra generated by both propagation directions. In fact, the properties of quantum localization observables quite generally follow from the conformal algebra, as discussed in next section.

III. QUANTUM LOCALIZATION

In order to provide a more realistic description of quantum localization observables in four-dimensional space-time, we now discuss their implementation using electromagnetic fields, thus taking into account polarization properties [22].

Electromagnetic fields satisfy Maxwell equations and the solutions may be decomposed as plane waves, with wave vectors \(k\) lying on the light-cone \((k^2 = 0)\) and corresponding amplitudes \((a_{k\sigma}, a_{k\sigma}^\dagger)\) themselves decomposing over two helicity components \((\sigma = \pm)\) which describe photon polarizations [22]. To circumvent gauge ambiguities, electromagnetic fields will be described by their real antisymmetric tensor

\[ F_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \delta(k^2)\theta(k_0) \Sigma_{\sigma} \{ \epsilon_{\mu\nu}^\sigma(k)a_{k\sigma}e^{-ikx} + \epsilon_{\mu\nu}^\sigma(k)^*a_{k\sigma}^\dagger e^{ikx} \} \]

(15)

\(\epsilon_{\mu\nu}^\sigma(k)\) denotes the expression taken by the polarization tensor of mode \((k, \sigma)\), as determined by Maxwell equations, and * denotes complex conjugation.
The algebra of quantum fields is obtained from the canonical commutation rules satisfied by photon creation \( (a_k^\dagger) \) and annihilation \( (a_k) \) operators

\[
[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta_{k\sigma'}(2k_0) \delta^3(k - k'), \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0
\]

\((2k_0) \delta^3(k - k')\) stands for the invariant Dirac distribution on the light cone.

Due to the conformal invariance of Maxwell equations \([23, 24]\), the electromagnetic field energy-momentum tensor \( T_{\mu\nu} \) provides quantities \( \Delta_a \) which are preserved by propagation and which correspond to generators of conformal symmetries \([25]\) (indices are lowered and raised using Minkowski metric \( \eta_{\mu\nu} \); \( \Delta_a \) is independent of \( t \))

\[
T_{\mu\nu}(x) = F_{\lambda\mu}(x) F^{\lambda\nu}(x) + \frac{1}{4} \eta_{\mu\nu} F_{\lambda\rho}(x) F^{\lambda\rho}(x), \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)
\]

\[
\Delta_a = \int \delta(x^0 - t) T_{\mu\nu}(x) \delta^\mu_a(x) d^4x
\]

(17)

The set of space-time functions \( \delta^\mu_a \) which determine the conserved quantities is fixed by the symmetries of the light cone \( \eta_{\mu\nu}(x^\mu + \delta^\mu_a(x))(x^\nu + \delta^\nu_a(x)) = \eta_{\mu\nu}x^\mu x^\nu = 0 \) and a basis is given by the following 15 expressions \( (\mu = P_\mu, J_{\mu\nu}, D, C_\mu, \text{for } \mu, \nu = 0, 1, 2, 3) \)

\[
\delta^\mu_{P_\mu}(x) = \delta^\mu
\]

\[
\delta^\mu_{J_{\mu\nu}}(x) = \frac{1}{2} x^\mu \delta^\nu - \delta^\mu x^\nu, \quad \delta^\mu_{D}(x) = x^\mu
\]

\[
\delta^\mu_{C_\mu}(x) = 2 x^\mu x^2 - \delta^\mu x^2
\]

(18)

From canonical field commutation relations \([19]\), the conserved quantities \( \Delta_a \) defined by relations \([17]\) generate conformal transformations of the fields. In fact, Maxwell fields defined by \([15]\) correspond to particular representations (with helicity \( \pm 1 \)) of the conformal algebra \([18]\) on the light cone \([26, 27]\)

\[
-i \hbar \left[ \Delta_a, F_{\mu\nu}(x) \right] = \left( \delta_a F_{\mu\nu}(x) \right), \quad \delta_a = \delta^\mu_a(x) \partial_\mu + \sigma_a, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}
\]

\[
-i \hbar \left[ \Delta_a, \Delta_b \right] = \Delta_{[a,b]}, \quad \delta_{[a,b]} = \left( \delta^\mu_a(x) \partial_\mu \delta^\nu_b(x) - \delta^\mu_b(x) \partial_\mu \delta^\nu_a(x) \right) \partial_\mu + [\sigma_a, \sigma_b]
\]

(19)

\( \sigma_a \), which describes the polarization dependent part of the conformal generator, is a matrix acting on field indices which is determined by Maxwell equations and its symmetries \([18]\). Relations \([19]\) allow one to identify the conserved quantities \( \Delta_a \), with the corresponding symmetry generators \( a \).

As in the two-dimensional case, conformal symmetries describe changes of reference frame. In the four-dimensional case, relativistic transformations include translations \( P_\mu \) and Lorentz transformations \( J_{\mu\nu} \), corresponding to momentum and angular momentum conservations, and satisfying a Poincaré algebra

\[
[P_\mu, P_\nu] = 0, \quad \frac{i}{\hbar} [J_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu
\]

\[
\frac{i}{\hbar} [J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\rho\mu} - \eta_{\rho\mu} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\rho\mu}
\]

(20)

But they also include dilations generated by \( D \)

\[
\frac{i}{\hbar} [D, P_\mu] = -P_\mu, \quad [D, J_{\mu\nu}] = 0
\]

(21)

and transformations to uniformly accelerated frames \( C_\mu \)

\[
\frac{i}{\hbar} [C_\mu, P_\nu] = 2 (J_{\mu\nu} - \eta_{\mu\nu} D)
\]

\[
\frac{i}{\hbar} [J_{\mu\nu}, C_\rho] = \eta_{\mu\rho} C_\nu - \eta_{\nu\rho} C_\mu, \quad \frac{i}{\hbar} [D, C_\mu] = C_\mu
\]

(22)

The photon number provides a first quantum observable built on Maxwell fields and remaining invariant under conformal transformations \([19]\). It may be written as a quadratic form of electromagnetic fields \([15]\) and is non
local in the space-time parameter $x$. This invariance ensures that observers which are related through a conformal transformation will have the same interpretation of processes involving photons.

Classically, positions in four-dimensional space-time are defined from the intersection of at least four light cones (for instance emitted by beacons located on satellites of the GPS constellation). Equivalently, classical positions are defined from the intersection of at least four light rays with different propagation directions. This generalizes the two-dimensional case and provides explicit algebraic expressions relating the position components with the total momentum $P_\mu$, angular momentum $J_{\mu\nu}$ and dilation generator $D$ of the field configuration. The same definition may be applied in the quantum case by considering quantum fields merging on the localization event and possessing momenta components corresponding to different propagation directions. This means that the quantum fields used for localization correspond to a total squared mass ($P^2 = M^2$) which does not vanish. Generalizing the relations satisfied by classical positions leads to the following definition for quantum positions in space-time

$$X_\mu = \frac{1}{P^2} \cdot (P^\lambda \cdot J_{\lambda\mu} + P_\mu \cdot D)$$

(23)

In contrast to the two-dimensional case, the position observables defined by equations (23) do not exhaust all the information contained in the localization fields. Indeed, spin components may further be defined either under the form of a vector $S_\mu$ (Pauli-Lubanski vector) or of an antisymmetric tensor $S_{\mu\nu}$ ($\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor with $\epsilon_{0123} = 1$)

$$S_\mu = \frac{-1}{2} \epsilon_{\mu\nu\lambda\rho} P^\nu J^{\lambda\rho}, \quad S_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho} \frac{P^\lambda}{P^2} S^\rho$$

(24)

Position and spin observables characterize space-time localization. In particular, Weyl generators are recovered from these observables and take their usual form

$$J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu}, \quad D = P_\mu \cdot X_\mu$$

(25)

As was the case for time-frequency observables, quantum positions and momenta correspond to conjugate observables

$$\frac{i}{\hbar} [P_\mu, X_\nu] = \eta_{\mu\nu}$$

(26)

Conjugation relations directly follow from the conformal algebra and the definition of quantum positions. In particular, their time component is a direct extension of the time-frequency conjugation relation. These relations then sustain the existence of a time operator conjugate to energy, contrarily to a common opinion. This paradox disappears when considering that the position observables defined by equations correspond to hermitian (or symmetric operators) which are not self-adjoint. This property is to be connected with the important part of the Hilbert space which is excluded by definition, in particular all field states with a vanishing total mass. Indeed, non self-adjoint operators not only provide quite acceptable representations of physical observables, but even appear to be unavoidable when dealing with localization in a non-commutative space-time. As another characteristic property of quantum positions, one notes that the latter possess a non vanishing commutator which is related to the spin observable

$$\frac{i}{\hbar} [X_\mu, X_\nu] = -\frac{S_{\mu\nu}}{P^2}$$

(27)

As can be seen on the simple example of two-photon states built with electromagnetic fields, both the spatial dispersion of quantum fields and the photon helicities contribute to the position commutator. This commutator shows that quantum localization in space-time is affected by size effects, characterized by spin in a mass dependent way. This points at an intimate relation between position and spin observables and at a scale dependence of the resulting space-time non-commutativity.

Relativistic transformations between observers being determined by conformal symmetries, the definitions of quantum positions and spin result in classical transformations under Lorentz transformations and dilatation.

$$\frac{i}{\hbar} [J_{\mu\nu}, X_\rho] = \eta_{\mu\rho} X_\nu - \eta_{\nu\rho} X_\mu, \quad \frac{i}{\hbar} [D, X_\mu] = X_\mu$$

$$\frac{i}{\hbar} [J_{\mu\nu}, S_\rho] = \eta_{\mu\rho} S_\nu - \eta_{\nu\rho} S_\mu, \quad \frac{i}{\hbar} [D, S_\mu] = -S_\mu$$

(28)
However, transformations between observers in relative acceleration exhibit specific behaviors of quantum localization observables. First, quantum positions are related to the transformation of the mass observable $M$ under an acceleration $a^\mu$ (the mass observable $M$ being Lorentz invariant)

$$\Delta = \frac{a^\mu}{2} C_{\mu}$$

$$\frac{i}{\hbar} [\Delta, M] = -\Phi \cdot M, \quad \Phi = a^\nu X_{\mu}$$

(29)

As in the classical case, the accelerated mass (29) undergoes a shift proportional to position similar to the redshift undergone by frequency references [13]. The mass transformation (29) may also be used as an equivalent definition of quantum positions (23).

As a consequence of the conformal algebra (22) and of the definitions of positions and spin observables (23, 24), transformations between frames in relative acceleration mix the different quantum localization observables [18]. Comparing expressions (30) with their classical analogs shows that the transformation of quantum level, of the classical properties relating fields and space-time coordinates. In particular, the non-vanishing commutator of quantum positions precludes the use of classical covariance rules for representing frame transformations.

The intimate relation between positions and spin also manifests itself in their mixed transformations. Similarly to the transformations of time references (13), transformations of positions involve corrections with respect to classical expressions which depend on spin, quadrupole momenta and momentum [18].

At the classical level, relativistic transformations between observers in relative acceleration are usually treated in the framework of differential geometry. The non-commutative character of quantum positions (27) signals a failure, at the quantum level, of the classical properties relating fields and space-time coordinates. In particular, the non-vanishing commutator of quantum positions precludes the use of classical covariance rules for representing frame transformations. This can also be seen in the occurrence of spin and quadrupole dependent terms in the transformation of quantum positions (30). However, equations (30) show that, besides position independent corrections, transformations under acceleration of spin and momentum are given by the same position dependent linear operator, which takes the same form as its classical analog [15].

$$\delta^\mu_{\Delta}(x) = a^\nu (x_{\nu} x^\mu - \frac{\delta^\mu}{2} x^2)$$

$$\frac{i}{\hbar} [\Delta, P_\mu] = -\partial_\mu \delta^\mu_{\Delta}(X) \cdot P_\nu - a^\nu S_{\mu\nu}$$

$$\frac{i}{\hbar} [\Delta, S_\mu] = -\partial_\mu \delta^\mu_{\Delta}(X) \cdot S_\nu - \epsilon_{\mu\nu\rho\sigma} a^\rho P^\sigma Q^\sigma$$

(31)

Similarly, besides position independent corrections, positions transform classically, provided a symmetrized product is used, and the spin-quadrupole independent part of positions shift has for differential the previous linear operator

$$\delta^\mu_{\Delta}(X) = a^\nu \left( X_{\nu} \cdot X^\mu - \frac{\delta^\mu}{2} X^2 \right)$$

$$\frac{i}{\hbar} [\Delta, X^\mu] = \delta^\mu_{\Delta}(X) - \delta^\mu_{\Delta}(S P^\mu P^2) - \partial_\mu \delta^\mu_{\Delta}(Q) \frac{P^\mu P^2}{P^2}$$

(32)

A trace of classical covariance properties remains at the quantum level, suggesting a possible generalization. Indeed, the algebraic framework of quantum theory bears constitutive rules which allow extensions of the covariance rules. As
a consequence of Jacobi identities applied to the conformal algebra and of the invariance of the commutator between momenta and positions, the following identities are satisfied
\[ [P_\mu, [\Delta, X_\nu]] - [X_\nu, [\Delta, P_\mu]] = [\Delta, [P_\mu, X_\nu]] = 0 \tag{33} \]

The first commutator in the first equation in (33) isolates the position dependent part in the position shift and takes its differential while the second commutator isolates the momentum dependent part in the momentum shift. The algebraic relation (33) may then be seen as an algebraic expression of the relation between equations 31 and 32.

Commutators between momenta and position shifts appearing in (33) have a direct physical interpretation, as they correspond to the shifts undergone by clock rates under changes of reference frames. Equations (33) extend at the quantum level the classical relations between clock rates and frequency shifts according to Einstein effect. Symmetrizing the expressions of these commutators in their indices provides a symmetric tensor which, in the particular case of accelerated frames, extends at the quantum level the corresponding conformal factor.

\[ [P_\mu, [\Delta, X_\nu]] + [P_\nu, [\Delta, X_\mu]] = [X_\mu, [\Delta, P_\nu]] + [X_\nu, [\Delta, P_\mu]] = -2\hbar^2\Phi_{\mu\nu} \tag{34} \]

Relation (34) may be seen as a quantum extension of the classical notion of metric.

IV. CONCLUSION

We have shown that the classical procedures which define the relativistic notion of space-time and which are used in practical realizations of reference frames can be implemented in the framework of Quantum Field Theory. When expressed by means of quantum fields, time-frequency transfer, or synchronization, and localization lead to the definition of references and positions in space-time as quantum observables. This success meets the necessity imposed by relativity and quantum theory to represent space-time positioning with physical observables, hence with quantum operators.

The basic and sufficient structure required for defining space-time positioning appears to be provided by symmetries of field propagation and more precisely conformal symmetries. Characteristic properties of relativity, such as Einstein effect, follow from the conformal algebra. As a bonus with respect to the classical situation, symmetry generators identify with quantities preserved by field propagation so that both relativistic transformations and quantum operators are expressed within a single algebra. If the use of electromagnetic fields imposes itself on practical grounds, gravitational fields would lead to equivalent space-time positions, showing that universality of space-time is preserved at the quantum level. The major difference with classical space-time appears in the non-commutativity of quantum positions, characterized by spin. The latter also enter the transformations of positions under frame transformations between relatively accelerated observers. Considering spin as a fundamental element of a quantum extension of space-time opens new perspectives, for instance in view of applying quantum deformations of space-time symmetries or for reconciling quantum fields with a mechanical description of elementary particles.

Although non-commutativity precludes the ordinary use of differential geometry for quantum positions, the frame transformations of quantum observables keep the trace of the covariance rules which play a crucial role in the formalism of general relativity. Conformal invariance moreover allows one to extend, for uniform accelerations, the covariance rules under a purely algebraic form suited to the quantum formalism. This remarkable property points at an alternative quantum extension of the metric fields founding general relativity as a gravitation theory.

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