Statistical Inference for Generalized Additive Partially Linear Model

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Abstract

The Generalized Additive Model (GAM) is a powerful tool and has been well studied. This model class helps to identify additive regression structure. Via available test procedures one may identify the regression structure even sharper if some component functions have parametric form. The Generalized Additive Partially Linear Models (GAPLM) enjoy the simplicity of the GLM and the flexibility of the GAM because they combine both parametric and nonparametric components. We use the hybrid spline-backfitted kernel estimation method, which combines the best features of both spline and kernel methods for making fast, efficient and reliable estimation under $\alpha$-mixing condition. In addition, simultaneous confidence corridors (SCCs) for testing overall trends and empirical likelihood confidence region for parameters are provided under independent condition. The asymptotic properties are obtained and simulation results support the theoretical properties. For the application, we use the GAPLM to improve the accuracy ratio of the default predictions for 19610 German companies. The quantlet for this paper are available on https://github.com.

JEL Classification: C14 G33

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1. INTRODUCTION

The class of generalized additive models (GAMs) provides an effective semiparametric regression tool for high dimensional data, see [6]. For a response $Y$ and a predictor vector $X = (X_1, \ldots, X_d)^\top$, the pdf of $Y_i$ conditional on $X_i$ with respect to a fixed $\sigma$-finite measure from exponential families is

$$f(Y_i|X_i, \phi) = \exp \{\{Y_im(X_i) - b\{m(X_i)\}\} / a(\phi) + h(Y_i, \phi)\}.$$  

The function $b$ is a given function which relates $m(x)$ to the conditional variance function $\sigma^2(x) = \text{var}(Y|X = x)$ via the equation $\sigma^2(x) = a(\phi)b''\{m(x)\}$, in which $a(\phi)$ is a nuisance parameter that quantifies overdispersion. For the theoretical development, it is not necessary to assume that the data $\{Y_i, X_i^\top\}_{i=1}^n$ come from such an exponential family, but only that the conditional variance and conditional mean are linked by the following equation

$$\text{var}(Y|X = x) = a(\phi)b''\left[(b')^{-1}\{E(Y|X = x)\}\right].$$

More specifically, the model is

$$E(Y|X) = b'\left\{c + \sum_{\alpha=1}^d m_\alpha(X_\alpha)\right\},$$  

with $b'$ is the derivative of function $b$. Model (1) can for example be used in scoring methods and analyzing default of companies (Here $Y = 1$ denotes default and $b' = e^y/1 + e^y$ is the link function). Fitting Model (1) to such a default data set leads to $d$ estimated component functions $\hat{m}_\alpha(\cdot)$ was studied in [11, 25]. Plotting these $\hat{m}_\alpha(\cdot)$ with simultaneous confidence corridors (SCCs) as developed by [25], one can check the functional form and therefore obtain simpler parameterizations of $m_\alpha$.

The typical approach is to perform a preliminary (nonparametric) analysis on the influence of the component functions, and one may improve the model by introducing parametric
components. This will lead to simplification, more interpretability and higher precision in statistical calibration. With these thoughts in mind, the GAM model changes to a Generalized Additive Partially Linear Model (GAPLM):

\[ E(Y|T, X) = b'[m(T, X)] , \]  

with \( m(T, X) = \beta^T T + \sum_{\alpha=1}^{d_2} m_{\alpha}(X_{\alpha}) \) and

\[ \beta = (\beta_0, \beta_1, \ldots, \beta_{d_1})^T, T = (T_0, T_1, \ldots, T_{d_1})^T, X = (X_1, \ldots, X_{d_2})^T, \]

where \( T_0 = 1, T_k \in \mathbb{R} \) for \( 1 \leq k \leq d_1 \). In this paper, we have following equation

\[ \text{var}(Y|T = t, X = x) = a(\phi) b'' \left[ (b')^{-1} \{ E(Y|T = t, X = x) \} \right]. \]

We can write (2) in the usual regression form:

\[ Y_i = b'[m(T_i, X_i)] + \sigma(T_i, X_i) \varepsilon_i \]

with white noise \( \varepsilon_i \) that satisfies \( E(\varepsilon_i|T_i, X_i) = 0, E(\varepsilon_i^2|T_i, X_i) = 1 \). For identifiability,

\[ E\{m_{\alpha}(X_{\alpha})\} = 0, 1 \leq \alpha \leq d_2. \]

As in most works on nonparametric smoothing, estimation of the functions \( \{m_{\alpha}(x_{\alpha})\}_{\alpha=1}^{d_2} \) is conducted on compact sets. Without lose of generality, let the compact set be \( \mathbb{K} = [0, 1]^{d_2} \).

Some estimation methods for Model [2] have been proposed, but are either computationally expensive or lacking theoretical justification. The kernel-based backfitting and marginal integration methods e.g., in [3, 9, 24], are computationally expensive. In the meanwhile, more advanced non- and semiparametric models (without link function) have been studied, such as partially linear model and varying-coefficient model, see [10, 12, 16, 21, 22]. [21] pro-
posed a nonconcave penalized quasi-likelihood method, with polynomial spline smoothing for estimation of $m_\alpha, 1 \leq \alpha \leq d_2$, and deriving quasi-likelihood based estimators for the linear parameter $\beta \in \mathbb{R}^{1+d_1}$. To our knowledge, [21] is a pilot paper since it provides asymptotic normality of the estimators for the parametric components in GAPLM with independent observations. However, asymptotic normality for estimations of the nonparametric component functions $m_\alpha, 1 \leq \alpha \leq d_2$ and SCCs are still missing. Recently, [13] studied more complicated Generalized Additive Coefficient Model by using two-step spline method, but independent and identical assumptions are required for the asymptotic properties of the estimation and inference of $m_\alpha$, and the asymptotic normality of parameter estimations is also missing. [5] developed nonparametric analysis of deviance tools, which can be used to test the significance of the nonparametric term in generalized partially linear models with univariate nonparametric component function. [8] provided empirical likelihood based confidence region for parameter $\beta$ and pointwise confidence interval for nonparametric term in generalized partially linear models.

The spline backfitted kernel (SBK) estimation introduced in [20] combines the advantages of both kernel and spline methods and the result is balanced in terms of theory, computation, and interpretation. The basic idea is to pre-smooth the component functions by spline estimation and then use the kernel method to improve the accuracy of the estimation on a specific $m_\alpha$. In this paper we extend the SBK method to calibrate Model (2) with additive nonparametric components, as a result we obtain oracle efficiency and asymptotic normality of the estimators for both the parametric and nonparametric components under $\alpha$-mixing condition, which complicates the proof of the theoretical properties. With stronger i.i.d assumption, we provide empirical likelihood (EL) based confidence region for parameter $\beta$ due to the advantages of EL such as increase of accuracy of coverage, easy implementation, avoiding estimating variances and studentising automatically, see [8]. In addition we provide SCCs for the nonparametric component functions based on maximal deviation distribution in [2] so one can test the hypothesis of the shape for nonparametric terms.
The paper is organized as follows. In Section 2, we discuss the details of (2). In Section 3, the oracle estimator and their asymptotic properties are introduced. In Section 4, the SBK estimator is introduced and the asymptotics for both the parametric and nonparametric component estimations are given. In addition, SCCs for testing overall trends and entire shapes are considered. In Section 5, we apply the methods to simulated and real data examples. All technical proofs are given in the Appendix.

2. MODEL ASSUMPTIONS

The space of $\alpha$-centered square integrable functions on $[0, 1]$ is defined as in [18],

$$\mathcal{H}_\alpha^0 = \{ g : \mathbb{E} \{ g(X_\alpha) \} = 0, \mathbb{E} \{ g^2(X_\alpha) \} < +\infty \}.$$  

Next define the model space $\mathcal{M}$, a collection of functions on $\mathbb{R}^{d_2}$ as

$$\mathcal{M} = \left\{ g(\mathbf{x}) = \sum_{\alpha=1}^{d_2} g_\alpha(\mathbf{x}) : g_\alpha \in \mathcal{H}_\alpha^0 \right\}.$$  

The constraints that $\mathbb{E} \{ g_\alpha(X_\alpha) \} = 0$, $1 \leq \alpha \leq d_2$ ensure the unique additive representation of $m_\alpha$ as expressed in [3]. Denote the empirical expectation by $\mathbb{E}_n$, then $\mathbb{E}_n \varphi = \sum_{i=1}^n \varphi(X_i) / n$. For functions $g_1, g_2 \in \mathcal{M}$, the theoretical and empirical inner products are defined respectively as $\langle g_1, g_2 \rangle = \mathbb{E} \{ g_1(X) g_2(X) \}$, $\langle g_1, g_2 \rangle_n = \mathbb{E}_n \{ g_1(X) g_2(X) \}$. The corresponding induced norms are $\| g_1 \|_2^2 = \mathbb{E} g_1^2(X)$, $\| g_1 \|_{2,n}^2 = \mathbb{E}_n g_1^2(X)$. More generally, we define $\| g \|_r^r = \mathbb{E} | g(X) |^r$.

In the paper, for any compact interval $[a, b]$, we denote the space of $p$-th order smooth functions as $C^{(p)}[a, b] = \{ g : g^{(p)} \in C[a, b] \}$, and the class of Lipschitz continuous functions for constant $C > 0$ as $\text{Lip}([a, b], C) = \{ g : | g(x) - g(x') | \leq C |x - x'|, \forall x, x' \in [a, b] \}$. For any vector $\mathbf{x} = (x_1, x_2, \ldots, x_d)^\top$, we denote the supremum and $p$ norms as $| \mathbf{x} | = \max_{1 \leq \alpha \leq d} |x_\alpha|$ and $\| \mathbf{x} \|_p = \left( \sum_{\alpha=1}^d x_\alpha^p \right)^{1/p}$. In particular, we use $\| \mathbf{x} \|$ to denote the Euclidean norm, i.e., $p = 2$. We need the following assumptions:
The additive component functions $m_\alpha \in C^1([0, 1], 1 \leq \alpha \leq d_2$ with $m_1 \in C^2([0, 1], m'_\alpha \in \text{Lip}([0, 1], C_m), 2 \leq \alpha \leq d_2$ for some constant $C_m > 0$.

The inverse link function $b'$ satisfies: $b' \in C^2(\mathbb{R}), b''(\theta) > 0, \theta \in \mathbb{R}$ and $C_b > \max_{\theta \in \Theta} b''(\theta) \geq \min_{\theta \in \Theta} b''(\theta) > c_b$ for constants $C_b > c_b > 0$.

The conditional variance function $\sigma^2(x)$ is measurable and bounded. The errors $\{\varepsilon_i\}_{i=1}^n$ satisfy $E(\varepsilon_i|F_i) = 0, E(|\varepsilon_i|^{2+\eta}) \leq C_\eta$ for some $\eta \in (1/2, +\infty)$ with the sequence of $\sigma$-fields:

$F_i = \sigma\{X_j, j \leq i; \varepsilon_j, j \leq i - 1\}$ for $i = 1, \ldots, n$.

The density function $f(x)$ of $(X_1, \ldots, X_{d_2})$ is continuous and $0 < c_f \leq \inf_{x \in \chi} f(x) \leq \sup_{x \in \chi} f(x) \leq C_f < \infty$.

The marginal densities $f_\alpha(x_\alpha)$ of $X_\alpha$ have continuous derivatives on $[0, 1]$ as well as the uniform upper bound $C_f$ and lower bound $c_f$.

Constants $K_0, \lambda_0 \in (0, +\infty)$ exist such that $\alpha(n) \leq K_0 e^{-\lambda_0 n}$ holds for all $n$, with the $\alpha$-mixing coefficients for $\{Z_i = (T_i^\top, X_i^\top, \varepsilon_i)^\top\}_{i=1}^n$ defined as

$\alpha(k) = \sup_{B \in \sigma\{Z_s, s \leq t\}, C \in \sigma\{Z_s, s \geq t+k\}} |P(B \cap C) - P(B)P(C)|, k \geq 1$.

$\{Z_i = (T_i^\top, X_i^\top, \varepsilon_i)^\top\}_{i=1}^n$ are independent and identically distributed.

There exist constants $0 < c_\delta < C_\delta < \infty$ and $0 < c_Q < C_Q < \infty$ such that $c_\delta \leq E(|T_k|^{2+\delta} |X = x) \leq C_\delta$ for some $\delta > 0$, and $c_Q I_{d_1 \times d_1} \leq E(TT^\top |X = x) \leq C_Q I_{d_1 \times d_1}$.

Assumptions (A1), (A2) and (A4) are standard in the GAM literature, see [19, 23], while Assumptions (A3) and (A5) are the same for weakly dependent data as in [11, 20] and
Assumption (A6) is the same with (C5) in [21]. When categorical predictors presents, we can create dummy variables in $T_i$ and Assumption (A6) is still satisfied.

3. ORACLE ESTIMATORS

The aim of our analysis is to provide precise estimators for the component functions $m_\alpha(\cdot)$ and parameters $\beta$. Without loss of generality, we may focus on $m_1(\cdot)$. If all the unknown $\beta$ and other $\{m_\alpha(x_\alpha)\}_{\alpha=2}^{d_2}$ were known, we are in a comfortable situation since the multi-dimensional modelling problem has reduced to one dimension. As in [17], define for each $x_1 \in [h, 1-h]$, $a \in A$ a local quasi log-likelihood function

$$\bar{\ell}_{m_1}(a, x_1) = n^{-1} \sum_{i=1}^{n} [Y_i \{a + m(T_i, X_{i,1})\} - b\{a + m(T_i, X_{i,1})\}] K_h(X_{i,1} - x_1)$$

with $m(T_i, X_{i,1}) = \beta^\top T_i + \sum_{\alpha=2}^{d_2} m_\alpha(X_{i,\alpha})$ and $K_h(u) = K(u/h)/h$ a kernel function $K$ with bandwidth $h$ that satisfy

(A7) The kernel function $K(\cdot) \in C^1[-1, 1]$ is a symmetric pdf. The bandwidth $h = h_n$ satisfies $h = o\left(n^{-1/5}\log n^{-1/5}\right)$, $h^{-1} = O\left(n^{1/5}\log n\delta\right)$ for some constant $\delta > 1/5$.

Since all the $\beta$ and $\{m_\alpha(x_\alpha)\}_{\alpha=2}^{d_2}$ are known as obtained from oracle, one can obtain the so-called oracle estimator

$$\bar{m}_{K,1}(x_1) = \arg\max_{a \in A} \bar{\ell}_{m_1}(a, x_1).$$

Denote $\|K\|_2^2 = \int K^2(u) du$, $\mu_2(K) = \int K(u) u^2 du$ and the scale function $D_1(x_1)$ and bias function $\text{bias}_1(x_1)$

$$D_1(x_1) = f_1(x_1) E\{b''\{m(T, X)\} | X_1 = x_1\},$$

$$\text{bias}_1(x_1) = \mu_2(K) - \int K(u) u^2 du.$$
bias_1 (x_1) = \mu_2 (K) [m''_1 (x_1) f_1 (x_1) E [b'' \{m (T, X)\} \mid X_1 = x_1] \\
+ m'_1 (x_1) \frac{\partial}{\partial x_1} \{f_1 (x_1) E [b'' \{m (T, X)\} \mid X_1 = x_1]\} \\
- \{m'_1 (x_1)\}^2 f_1 (x_1) E [b'' \{m (T, X)\} \mid X_1 = x_1] \]. \tag{6}

**Lemma 1** Under Assumptions (A1)-(A7), for any \(x_1 \in [h, 1 - h]\), as \(n \to \infty\), the oracle kernel estimator \(\tilde{m}_{K,1} (x_1)\) given in (4) satisfies

\[
\sup_{x_1 \in [h, 1-h]} |\tilde{m}_{K,1} (x_1) - m_1 (x_1)| = O_{a.s.} \left(\log n / \sqrt{nh}\right),
\]

\[
\sqrt{nh} \{\tilde{m}_{K,1} (x_1) - m_1 (x_1) - bias_1 (x_1) h^2 / D_1 (x_1)\} \xrightarrow{L} N \left(0, D_1 (x_1)^{-1} v^2_1 (x_1) D_1 (x_1)^{-1}\right),
\]

with

\[
v^2_1 (x_1) = f_1 (x_1) E \{\sigma^2 (T, X) \mid X_1 = x_1\} \|K\|_2^2.
\]

Lemma 1 is given in [11]. The above oracle idea applies to the parametric part as well.

Define the log-likelihood function

\[
\ell_\beta (a) = n^{-1} \sum_{i=1}^{n} \left[ Y_i \{a^\top T_i + m (X_i)\} - b \{a^\top T_i + m (X_i)\}\right], \tag{7}
\]

where \(m (X_i) = \sum_{a=1}^{d_2} m_a (X_{ia})\). The infeasible estimator of \(\beta\) is \(\tilde{\beta} = \arg \max_{a \in \mathbb{R}^{1+d}} \ell_\beta (a)\).

Clearly, \(\nabla \ell_\beta (\beta) = 0\). To maximize (7), we have

\[
n^{-1} \sum_{i=1}^{n} \left[ Y_i T_i - b \{a^\top T_i + m (X_i)\} T_i\right] = 0,
\]

then the empirical likelihood ratio is

\[
\hat{R} (a) = \max \{\Pi_{i=1}^{n} n p_i | \Sigma_{i=1}^{n} \sum_{i=1}^{n} Z_i (a) = 0, p_i \geq 0, \Sigma_{i=1}^{n} p_i = 1\}
\]

where \(Z_i (a) = [Y_i - b \{a^\top T_i + m (X_i)\}] T_i\).
Theorem 1  (i) Under Assumptions (A1)-(A6), as $n \to \infty$,
\[
\left| \tilde{\beta} - \beta - \left[ E b'' \{ m(T, X) \} T T^T \right]^{-1} n^{-1} \sum_{i=1}^{n} \sigma(T_i, X_i) \varepsilon_i T_i \right| = O_{a.s.} \left( n^{-1} (\log n)^2 \right),
\]
\[
\sqrt{n} \left( \tilde{\beta} - \beta \right) \xrightarrow{d} N \left( 0, a(\phi) \left[ E b'' \{ m(T, X) \} T T^T \right]^{-1} \right).
\]

(ii) Under Assumptions (A1)-(A4), (A5') and (A6),
\[
-2 \log \tilde{R} (\beta) \xrightarrow{d} \chi^2_{d_1}.
\]

Although the oracle estimators $\tilde{\beta}$ and $\tilde{m}_{K,1} (x_1)$ enjoy the desirable theoretical properties in Theorem 1 and Lemma 1, they are not a feasible statistic as its computation is based on the knowledge of unavailable component functions $\{ m_\alpha (x_\alpha) \}_{\alpha=2}^{d_2}$.

4. SPLINE-BACKFITTED KERNEL ESTIMATORS

In practice, the rest components $\{ m_\alpha (x_\alpha) \}_{\alpha=2}^{d_2}$ are of course unknown and need to be approximated. We obtain the spline-backfitted kernel estimators by using estimations of $\{ m_\alpha (x_\alpha) \}_{\alpha=2}^{d_2}$ and the unknown $\beta$ by splines and we employ them to estimate $m_1 (x_1)$ as in (4). First, we introduce the linear spline basis as in [10]. Let $0 = \xi_0 < \xi_1 < \cdots < \xi_N < \xi_{N+1} = 1$ denote a sequence of equally spaced points, called interior knots, on $[0,1]$. Denote by $H = (N+1)^{-1}$ the width of each subinterval $[\xi_J, \xi_{J+1}]$, $0 \leq J \leq N$ and denote the degenerate knots $\xi_{-1} = 0, \xi_{N+2} = 1$. We need the following assumption:

(A8) The number of interior knots $N \sim n^{1/4} \log n$, i.e., $c_N n^{1/4} \log n \leq N \leq C_N n^{1/4} \log n$ for some constants $c_N, C_N > 0$. 

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Following [11], for $J = 0, \ldots, N + 1$, define the linear B spline basis:

$$b_J(x) = (1 - |x - \xi_J|/H) = \begin{cases} 
(N + 1)x - J + 1, & \xi_{J-1} \leq x \leq \xi_J \\
J + 1 - (N + 1)x, & \xi_J \leq x \leq \xi_{J+1}, \\
0, & \text{otherwise}
\end{cases}$$

the space of $\alpha$-empirically centered linear spline functions on $[0, 1]$:

$$G_{n,\alpha}^0 = \left\{ g_\alpha : g_\alpha(x_\alpha) = \sum_{J=0}^{N+1} \lambda_J b_J(x_\alpha), E_n \{g_\alpha(X_\alpha)\} = 0 \right\}, 1 \leq \alpha \leq d_2,$$

and the space of additive spline functions on $\chi$:

$$G_n^0 = \left\{ g(x) = \sum_{\alpha=1}^{d_2} g_\alpha(x_\alpha) ; g_\alpha \in G_{n,\alpha}^0 \right\}.$$

Define the log-likelihood function

$$\hat{L}(\beta, g) = n^{-1} \sum_{i=1}^n \left[ Y_i \{\beta^T T_i + g(X_i)\} - b \{\beta^T T_i + g(X_i)\} \right], g \in G_n^0, \quad (8)$$

which according to Lemma 14 of [19], has a unique maximizer with probability approaching 1. The multivariate function $m(x)$ is then estimated by the additive spline function $\hat{m}(x)$ with

$$\hat{m}(t, x) = \hat{\beta}^T t + \hat{m}(x) = \arg\max_{g \in G_n^0} \hat{L}(\beta, g).$$

Since $\hat{m}(x) \in G_n^0$, one can write $\hat{m}(x) = \sum_{\alpha=1}^{d_2} \hat{m}_\alpha(x_\alpha)$ for $\hat{m}_\alpha(x_\alpha) \in G_{n,\alpha}^0$. Next define the log-likelihood function

$$\ell_{m_1}(a, x_1) = n^{-1} \sum_{i=1}^n \left[ Y_i \{a + \hat{m}(T_i, X_{i,1})\} - b \{a + \hat{m}(T_i, X_{i,1})\} \right] K_h(X_{i,1} - x_1) \quad (9)$$
where \( \hat{m}(T_i, X_{i,1}) = \hat{\beta}^\top T_i + \sum_{\alpha=2}^{d_2} \hat{m}_\alpha (X_{i\alpha}) \). Define the SBK estimator as:

\[
\hat{m}_{SBK,1}(x_1) = \arg\max_{a \in A} \ell_{m_1}(a, x_1).
\]  

(10)

**Theorem 2** Under Assumptions (A1)-(A8), as \( n \to \infty \), \( \hat{m}_{SBK,1}(x_1) \) is oracally efficient,

\[
\sup_{x_1 \in [0,1]} |\hat{m}_{SBK,1}(x_1) - m_K(x_1)| = O_{a.s.} \left( n^{-1/2} \log n \right).
\]

The following corollary is a consequence of Lemma 1 and Theorem 2.

**Corollary 1** Under Assumptions (A1)-(A8), as \( n \to \infty \), the SBK estimator \( \hat{m}_{SBK,1}(x_1) \) given in (10) satisfies

\[
\sup_{x_1 \in [h, 1-h]} |\hat{m}_{SBK,1}(x_1) - m_1(x_1)| = O_{a.s.} \left( \log n / \sqrt{nh} \right)
\]

and for any \( x_1 \in [h, 1-h] \), with bias \( _1(x_1) \) as in (6) and \( D_1(x_1) \) in (5)

\[
\sqrt{nh} \left\{ \hat{m}_{SBK,1}(x_1) - m_1(x_1) - \text{bias}_1(x_1) h^2 / D_1(x_1) \right\} \Rightarrow N(0, D_1(x_1)^{-1} v_1^2(x_1) D_1^{-1}(x_1)^{-1})
\]

Denote \( a_h = \sqrt{-2 \log h}, C(K) = \|K'\|_2^2 \|K\|_2^{-2} \) and for any \( \alpha \in (0, 1) \), the quantile

\[
Q_h(\alpha) = a_h + a_h^{-1} \left[ \log \left\{ \sqrt{C(K)} / (2\pi) \right\} - \log \left\{ -\log \sqrt{1-\alpha} \right\} \right].
\]

Also with \( D_1(x_1) \) and \( v_1^2(x_1) \) given in (5), define

\[
\sigma_n(x_1) = n^{-1/2} h^{-1/2} v_1(x_1) D_1^{-1}(x_1).
\]

**Theorem 3** Under Assumptions (A1)-(A4), (A5'), (A6)-(A8), as \( n \to \infty \),

\[
\lim_{n \to \infty} \Pr \left\{ \sup_{x_1 \in [h,1-h]} |\hat{m}_{SBK,1}(x_1) - m_1(x_1)| / \sigma_n(x_1) \leq Q_h(\alpha) \right\} = 1 - \alpha.
\]
A $100 \left(1 - \alpha\right)\%$ simultaneous confidence band for $m_1(x_1)$ is

$$\hat{m}_{SBK,1}(x_1) \pm \sigma_n(x_1) Q_h(\alpha).$$

In fact, $\hat{\beta}$ obtained by maximizing (8) is equivalent to $\hat{\beta}_{SBK} = \text{argmax}_{\alpha \in \mathbb{R}^{1+d_1}} \ell_\beta(a)$ with

$$\ell_\beta(a) = n^{-1} \sum_{i=1}^n \left[ Y_i \left\{ a^\top T_i + \hat{m}(X_i) \right\} - b \left\{ a^\top T_i + \hat{m}(X_i) \right\} \right]$$

in which $\hat{m}(X_i) = \sum_{\alpha=1}^{d_2} \hat{m}_\alpha(X_{i\alpha})$. The empirical likelihood ratio is

$$\hat{R}(a) = \max \left\{ \prod_{i=1}^n n p_i \left| \Sigma_{i=1}^n p_i \hat{Z}_i(a) = 0, p_i \geq 0, \Sigma_{i=1}^n p_i = 1 \right\}$$

where $\hat{Z}_i(a) = \left[ Y_i - b' \left\{ a^\top T_i + \hat{m}(X_i) \right\} \right] T_i$. Similar to Theorem 2, the main result shows that the difference between $\hat{\beta}$ and its infeasible counterpart $\tilde{\beta}$ is asymptotically negligible.

**Theorem 4** (i) Under Assumptions (A1)-(A6) and (A8), as $n \to \infty$, $\hat{\beta}$ is oracularly efficient, i.e., $\sqrt{n} \left( \hat{\beta}_k - \tilde{\beta}_k \right) \overset{d}{\to} 0$ for $0 \leq k \leq d_1$ and hence

$$\sqrt{n} \left( \hat{\beta} - \beta \right) \overset{d}{\to} N \left( 0, \phi \left[ E b'' \left\{ m(\mathbf{T}, \mathbf{X}) \right\} \mathbf{T} \mathbf{T}^\top \right]^{-1} \right).$$

(ii) Under Assumptions (A1)-(A4), (A5'), (A6) and (A8), as $n \to \infty$,

$$\sup \left| -2 \log \hat{R}(\beta) + 2 \log \tilde{R}(\beta) \right| = o_p(1),$$

so

$$-2 \log \hat{R}(\beta) \overset{d}{\to} \chi^2_{d_1}.$$ 

As a reviewer pointing out, an obvious advantage of GAPLM over GAM is the capability of including categorical predictors. Since $m_\alpha$ is not a function of $\mathbf{T}$ in GAPLM, so we can simply create dummy variables to represent the categorical effects and use spline estimation.
proposed spline estimation combined with categorical kernel functions to handle the case when function \( m_\alpha \) depends on categorical predictors.

5. EXAMPLES

We have applied the SBK procedure to both simulated (Example 1) and real (Example 2) data and implemented our algorithms with the following rule-of-thumb number of interior knots

\[
N = N_n = \min \left( \left\lfloor \frac{n^{1/4} \log n}{1} \right\rfloor + 1, \left\lfloor \frac{n}{4d - 1} \right\rfloor - 1 \right)
\]

which satisfies (A8), i.e., \( N = N_n \sim n^{1/4} \log n \), and ensures that the number of parameters in the linear least squares problem is less than \( n/4 \), i.e., \( 1 + d(N + 1) \leq n/4 \). The bandwidth of \( h_\alpha \) is computed as [11] in the asymptotically optimal way.

5.1 Example 1

The data are generated from the model

\[
\Pr(Y = 1|T = t, X = x) = b' \left\{ \beta^T T + \sum_{\alpha=1}^{d_2} m_\alpha(X_\alpha) \right\}, b'(x) = \frac{e^x}{1 + e^x}
\]

with \( d_1 = 2, d_2 = 5, \beta = (\beta_0, \beta_1, \beta_2)^T = (1, 1, 1)^T, m_1(x) = m_2(x) = m_3(x) = \sin(2\pi x), m_4(x) = \Phi(6x - 3) - 0.5 \) and \( m_5(x) = x^2 - 1/3 \), where \( \Phi \) is the standard normal cdf. The predictors are generated by transforming the following vector autoregression (VAR) equation for \( 0 \leq r_1, r_2 < 1, 1 \leq i \leq n, Z_0 = 0 \)

\[
Z_i = r_1Z_{i-1} + \varepsilon_i, \varepsilon_i \sim N(0, \Sigma), \Sigma = (1 - r_2)I_{d\times d} + r_2I_d1_d^T, d = d_1 + d_2, 1 \leq i \leq n,
\]

\[
T_i = (1, Z_{i1}, \ldots, Z_{id_1})^T, X_\alpha = \Phi \left( \sqrt{1 - r_1^2}Z_{i\alpha} \right), 1 \leq i \leq n, 1 + d_1 \leq \alpha \leq d_1 + d_2,
\]

with stationary \( Z_i = (Z_{i1}, \ldots, Z_{id})^T \sim N \left\{ 0, (1 - r_1^2)^{-1} \Sigma \right\}, I_d = (1, \ldots, 1)^T \) and \( I_{d \times d} \) is the \( d \times d \) identity matrix. The \( X \) is transformed from \( Z \) to satisfy Assumption (A4). In this study, we selected four scenarios: \( r_1 = 0, r_2 = 0; r_1 = 0.5, r_2 = 0; r_1 = 0, r_2 = 0.5; r_1 = 0.5, r_2 = 0 \).
0.5, $r_2 = 0.5$. The parameter $r_1$ controls the dependence between observations and $r_2$ controls the correlation between variables. In the selected scenarios, $r_1 = 0$ indicates independent observations and $r_1 = 0.5 \alpha$-mixing observations, $r_2 = 0$ indicates independent variables and $r_2 = 0.5$ correlated variables within each observation. Define the empirical relative efficiency of $\hat{\beta}_1$ with respect to $\tilde{\beta}_1$ as $\text{EFF}_r(\hat{\beta}_1) = \left( \frac{\text{MSE}(\hat{\beta}_1)}{\text{MSE}(\tilde{\beta}_1)} \right)^{1/2}$.

Table 1 shows the mean of bias, variances, MSEs and EFFs of $\hat{\beta}_1$ for $R = 1000$ with sample sizes $n = 500, 1000, 2000, 4000$. The results show that the estimator works as the asymptotic theory indicates, see Theorem 4 (i).

Figure 1 shows the kernel densities of $\hat{\beta}_1$'s for $n = 500, 1000, 2000, 4000$ from 1000 replications, again the theoretical properties are supported.

Table 2 shows the simulation results of the empirical likelihood confidence interval for $\beta$ with $n = 500, 1000, 2000, 4000$ and $r_1 = 0, r_2 = 0$ from 1000 replications. The mean and standard deviation of $-2 \log \hat{R}(\beta) + 2 \log \tilde{R}(\beta)$ (DIFF) support the oracle efficiency in Theorem 4 (ii). The performance of empirical likelihood confidence interval are compared with the wald-type one and it is clear that they have similar performance but empirical likelihood confidence interval has better coverage ratio and shorter average length.

Next for $\alpha = 1, \ldots, 5$, let $X_{\alpha, \text{min}}^i, X_{\alpha, \text{max}}^i$ denote the smallest and largest observations of the variable $X_\alpha$ in the $i$-th replication. The component functions $\{m_\alpha\}_{\alpha=1}^5$ are estimated on equally spaced points $\{x_t\}_{t=0}^{100}$ with $0 = x_0 < \ldots < x_{100} = 1$ and the estimator of $m_\alpha$ in the $r$-th sample as $\hat{m}_{\text{SBK}, \alpha, r}$. The (mean) average squared error (ASE and MASE) are:

$$\text{ASE}(\hat{m}_{\text{SBK}, \alpha, r}) = 101^{-1} \sum_{t=0}^{100} \{\hat{m}_{\text{SBK}, \alpha, r}(x_t) - m_\alpha(x_t)\}^2,$$

$$\text{MASE}(\hat{m}_{\text{SBK}, \alpha}) = R^{-1} \sum_{r=1}^R \text{ASE}(\hat{m}_{\text{SBK}, \alpha, r}).$$
In order to examine the efficiency of $\hat{m}_{SBK,\alpha}$ relative to the oracle estimator $\hat{m}_{K,\alpha}(x_\alpha)$, both are computed using the same data-driven bandwidth $\hat{h}_{\alpha,\text{opt}}$, described in Section 5 of [11]. Define the empirical relative efficiency of $\hat{m}_{SBK,\alpha}$ with respect to $\hat{m}_{K,\alpha}$ as

$$\text{EFF}_r(\hat{m}_{SBK,\alpha}) = \left[ \frac{\sum_{t=0}^{100} \{\hat{m}_{K,\alpha}(x_t) - m_\alpha(x_t)\}^2}{\sum_{t=0}^{100} \{\hat{m}_{SBK,\alpha,r}(x_t) - m_\alpha(x_t)\}^2} \right]^{1/2}.$$ 

EFF measures the relative efficiency of the SBK estimator to the oracle estimator. For increasing sample size, it should increase to 1 by Theorem 2. Table 3 shows the MASEs of $\hat{m}_{K,1}$, $\hat{m}_{SBK,1}$ and the average of EFFs from 1000 replications for $n = 500$, 1000, 2000, 4000. It is clear that the MASEs of both SBK estimator and the oracle estimator decrease when sample sizes increase, and the SBK estimator performs as well asymptotically as the oracle estimator, see Theorem 2.

To have an impression of the actual function estimates, for $r_1 = 0$, $r_2 = 0.5$ with sample size $n = 500$, 1000, 2000, 4000, we have plotted the SBK estimators and their 95% asymptotic SCCs (red solid lines), pointwise confidence intervals (red dashed lines), oracle estimators (blue dashed lines) for the true functions $m_1$ (thick black lines) in Figure 2. Here we use $r_1 = 0$ because we want to give the 95% asymptotic SCCs, which need the observations be i.i.d to satisfy Assumption (A5'). As expected by theoretical results, the estimation is closer to the real function and the confidence band is narrower as sample size increasing.

To compare the prediction performance of GAM and GAPLM, we introduce CAP and AR first. For any score function $S$, one defines its alarm rate $F(s) = \Pr(S \leq s)$ and the hit rate $F_D(s) = \Pr(S \leq s \mid D)$ where $D$ represents the conditioning event of “default”. Define the Cumulative Accuracy Profile (CAP) curve as

$$\text{CAP}(u) = F_D\{F^{-1}(u)\}, u \in (0, 1), \quad (11)$$
which is the percentage of default-infected obligators that are found among the first (according to their scores) 100u% of all obligators. A perfect rating method assigns all lowest scores to exactly the defaulters, so its CAP curve linearly increases up and then stays at 1, in other words, \( \text{CAP}_P(u) = \min(u/p, 1), u \in (0, 1) \), where \( p \) denotes the unconditional default probability. In contrast, a noninformative rating method with zero discriminatory power displays a diagonal line \( \text{CAP}_N(u) = u, u \in (0, 1) \). The CAP curve of a given scoring method \( S \) always locates between these two extremes and give information about its performance.

The area between the CAP curve and the noninformative diagonal \( \text{CAP}_N(u) \equiv u \) is \( a_R \), whereas \( a_P \) is the area between the perfect CAP curve \( \text{CAP}_P(u) \) and the noninformative diagonal \( \text{CAP}_N(u) \). Thus the CAP can be measured for example by Accuracy Ratio (AR): the ratio of \( a_R \) and \( a_P \).

\[
AR = \frac{a_R}{a_P} = \frac{2 \int_0^1 \text{CAP}(u) \, du - 1}{1 - p},
\]

where \( \text{CAP}(u) \) is given in (11). The AR takes value in \([0, 1]\), with value 0 corresponding to the noninformative scoring, and 1 the perfect scoring method. A higher AR indicates an overall higher discriminatory power of a method. Table 4 shows the average and standard deviations of the ARs from 1000 replications using \( k \)-fold cross-validation with \( k = 2, 10, 100 \) for \( r_1 = 0, r_2 = 0 \) and \( n = 500, 1000, 2000, 4000 \). In each replication, we randomly divide the set of observations into \( k \) equal size folds and use the rest \( k - 1 \) folds as training data set to make prediction for each fold. After we obtain all the prediction for each observation in the data set, we compute the CAP and AR based on above formula. It is clear that GAPLM has best predication accuracy.

Last, to show the estimation performance when \( T \) has categorical variables, we generate data using the same model above but add one more categorical variable, i.e., \( d_1 = 3, \beta = (\beta_0, \beta_1, \beta_2, \beta_3)^T = (1, 1, 1, 1)^T, T_3 = \{0, 1\} \) with probability 0.5 for \( T_3 = 1 \) and independent with the other variables \( T \) and \( X \). Table 5 shows the bias, variances, MSEs and EFFs of
\( \hat{\beta}_3 \) for \( R = 1000 \) with sample sizes \( n = 500, 1000, 2000, 4000 \). The results show that the estimator works as the asymptotic theory indicates.

[Table 5 about here.]

5.2 Example 2

The credit reform database, provided by the Research Data Center (RDC) of the Humboldt Universität zu Berlin, was studied by using GAM model in [11]. The data set contains \( d = 8 \) financial ratios, which are shown in Table 6, such as Operating\( \text{Income} / \text{Total Assets} \) and \( \log(\text{Total Assets}) \), of 18610 solvent (\( Y = 0 \)) and 1000 insolvent (\( Y = 1 \)) German companies. The time period ranges from 1997 to 2002 and in the case of the insolvent companies the information was gathered 2 years before the insolvency took place. The last annual report of a company before it went bankrupt receives the indicator \( Y = 1 \) and for the rest (solvent) \( Y = 0 \). In the original data set, the variables are labeled as \( Z_\alpha \). In order to satisfy the Assumption (A4) in [11], we need the transformation:

\[
X_{i\alpha} = F_{n\alpha}(Z_{i\alpha}), \quad \alpha = 1, \ldots, 8,
\]

where \( F_{n\alpha} \) is the empirical cdf for the data \( \{X_{i\alpha}\}_{i=1}^n \). See [4, 11] for more details of this data set.

[Table 6 about here.]

Using GAM and SBK method, we clearly see via the SCCs that the shape of \( m_2(x_2) \) is linear. Figure 3(a) shows that a linear line is covered by the SCCs of \( \hat{m}_2 \). We additionally show the SCCs for another component function of \( \log(\text{Total Assets}) \) in Figure 3(b). The SCCs do not cover a linear line. In fact, among all the 8 financial ratio considered, only \( x_2 \) yields a linear influence. To improve the precision in statistical calibration and interpretability, we can use GAPLM with parametric \( m_2(x_2) = \beta_2 x_2 \).

[Figure 3 about here.]

For the RDC data, the in sample AR value obtained from GAPLM is 62.89%, which is very close to the AR value 63.05% obtained from GAM in [11] and higher than the AR value 60.51% obtained from SVM in [4]. To compare the prediction performance, we use
the AR introduced in Example 1. Then we randomly divide the data set into \( k = 2,10 \) folds and obtain the prediction for each observation using the rest \( k - 1 \) folds as training set. Based on the prediction of all the observation, we can compute prediction AR value. Table 7 shows the mean and standard deviation of the prediction AR values from 100 replications. GAPLM has higher prediction AR value than GAM for 99 replications when \( k = 2 \) and 100 times when \( k = 10 \). It is clear that GAPLM has best prediction accuracy due to the better statistical calibration.

[Table 7 about here.]

6. APPENDIX

A.1 Preliminaries

In the proofs that follow, we use “\( \mathcal{U} \)” and “\( u \)” to denote sequences of random variables that are uniformly “\( \mathcal{O} \)” and “\( \sigma \)” of certain order. Denote the theoretical inner product of \( b_j \) and 1 with respect to the \( \alpha \)-th marginal density \( f_\alpha (x_\alpha) \) as 
\[
\langle b_j (X_\alpha), 1 \rangle = \int b_j (x_\alpha) f_\alpha (x_\alpha) dx_\alpha
\]
and define the centered B spline basis \( b_{J,\alpha} (x_\alpha) \) and the standardized B spline basis \( B_{J,\alpha} (x_\alpha) \) as
\[
b_{J,\alpha} (x_\alpha) = b_j (x_\alpha) - \frac{c_{J,\alpha}}{c_{J-1,\alpha}} b_{J-1} (x_\alpha),
B_{J,\alpha} (x_\alpha) = \frac{b_{J,\alpha} (x_\alpha)}{\| b_{J,\alpha} \|_2}, 1 \leq J \leq N + 1,
\]
so that \( EB_{J,\alpha} (X_\alpha) = 0, EB_{J,\alpha}^2 (X_\alpha) = 1 \). Theorem A.2 in [20] shows that under Assumptions (A1)-(A5) and (A7), constants \( c_0 (f), C_0 (f), c_1 (f) \) and \( C_1 (f) \) exist depending on the marginal densities \( f_\alpha (x_\alpha), 1 \leq \alpha \leq d \), such that
\[
c_0 (f) H \leq c_{J,\alpha} \leq C_0 (f) H,
\]
and
\[
c_1 (f) H \leq \| b_{J,\alpha} \|_2 \leq C_1 (f) H. \tag{A.1}
\]

Lemma A.1 ([1], p.149) For any \( m \in C^1 [0,1] \) with \( m' \in \text{Lip} ([0,1], C_{\infty}) \), there exist a constant \( C_\infty > 0 \) and a function \( g \in G_\infty^{(0)} [0,1] \) such that \( \| g - m \|_\infty \leq C_\infty H^2 \).
A.2 Oracle estimators

PROOF OF THEOREM 1 (i) According to the Mean Value Theorem, a vector $\tilde{\beta}$ between $\beta$ and $\tilde{\beta}$ exists such that 
\[
\left( \tilde{\beta} - \beta \right) \nabla^2 \ell_\beta (\tilde{\beta}) = \nabla \ell_\beta (\tilde{\beta}) - \nabla \ell_\beta (\beta) = - \nabla \ell_\beta (\beta)
\]

where
\[
\nabla \ell_\beta (\tilde{\beta}) = 0,
\]

with $c_b > 0$ according to (A2), and then the infeasible estimator is $\tilde{\beta} = \arg\max_{a \in \mathbb{R}^{1+d_1}} \ell_\beta (a)$.

We have $|n^{-1} \sum_{i=1}^{n} \sigma (T_i, X_i) \varepsilon_i T_i| = O_{a.s.} \left( n^{-1/2} \log n \right)$ by Bernstein’s Inequality as Lemma A.2 in [11], so
\[
\left| \tilde{\beta} - \beta \right| = O_{a.s.} \left( n^{-1/2} \log n \right)
\]

according to $\tilde{\beta} - \beta = - \left\{ \nabla^2 \ell_\beta (\tilde{\beta}) \right\}^{-1} \nabla \ell_\beta (\beta)$. Then
\[
\nabla^2 \ell_\beta (\tilde{\beta}) \xrightarrow{a.s.} \nabla^2 \ell_\beta (\beta) = -n^{-1} \sum_{i=1}^{n} b'' \{ \beta^T T_i + m (X_i) \} T_i T_i^T,
\]

which converges to $-Eb'' \{ m (T, X) \} TT^T$ almost surely at the rate of $n^{-1/2} \log n$. So
\[
\left| \tilde{\beta} - \beta - \left[ Eb'' \{ m (T, X) \} TT^T \right]^{-1} n^{-1} \sum_{i=1}^{n} \sigma (T_i, X_i) \varepsilon_i T_i \right| = O_{a.s.} \left( n^{-1} \left( \log n \right)^2 \right).
\]

Since $n^{-1} \sum_{i=1}^{n} \sigma (T_i, X_i) \varepsilon_i T_i \xrightarrow{P} N (0, a (\phi) \left[ Eb'' \{ m (T, X) \} TT^T \right]^{-1})$ by central limit theorem, so Theorem 1 (i) is proved by Slutsky’s theorem.

(ii) The proof is trivial based on the properties of empirical likelihood ratio for generalized linear model, see Theorem 3.2 in [15] and Corollary 1 in [7].
A.3 Spline backfitted kernel estimators

In this section, we present the proofs of Theorems 2, 3 and 4. We write any \( g \in C_n^0 \) as \( g = \lambda^\top B(X_i) \) with vector \( \lambda_0 = (\lambda_{J,\alpha})_{1 \leq J \leq N+1, 1 \leq \alpha \leq d_2} \in \mathbb{R}^{(N+1)d_2} \) is the dimension of the additive spline space \( C_n^0 \), and

\[
B(x) = \{B_{1,1}(x), \ldots, B_{N+1,1}(x), \ldots, B_{1,d_2}(x_{d_2}), \ldots, B_{N+1,d_2}(x_{d_2})\}^\top.
\]

Denote \( B(t, x) = \{1, t_1, \ldots, t_{d_1}, B_{1,1}(x), \ldots, B_{N+1,1}(x), \ldots, B_{1,d_2}(x_{d_2}), \ldots, B_{N+1,d_2}(x_{d_2})\}^\top \), \( \lambda = (\lambda^0, \lambda_k, \lambda_{J,\alpha})^\top \in \mathbb{R}^{N_d} \) with \( N_d = 1 + d_1 + (N+1)d_2 \) and

\[
\hat{L}(\lambda_0, g) = \hat{L}(\lambda) = n^{-1} \sum_{i=1}^n \left[ y_i \{\lambda^\top B(T_i, X_i)\} - b \{\lambda^\top B(T_i, X_i)\} \right],
\]

which yields the gradient and Hessian formulae

\[
\nabla \hat{L}(\lambda) = n^{-1} \sum_{i=1}^n \left[ y_i B(T_i, X_i) - b' \{\lambda^\top B(T_i, X_i)\} B(T_i, X_i) \right],
\]

\[
\nabla^2 \hat{L}(\lambda) = -n^{-1} \sum_{i=1}^n b'' \{\lambda^\top B(T_i, X_i)\} B(T_i, X_i) B(T_i, X_i)^\top.
\]

The multivariate function \( m(t, x) \) is estimated by

\[
\hat{m}(t, x) = \hat{\beta}_0 + \sum_{k=1}^{d_1} \hat{\beta}_k t_k + \sum_{\alpha=1}^{d_2} \hat{m}_\alpha(x_\alpha) = \hat{\lambda}^\top B(t, x),
\]

\[
\hat{\lambda} = \left(\begin{array}{c}
\lambda^\beta \\
\lambda^g
\end{array}\right)^\top = \left(\begin{array}{c}
\hat{\beta}^\top \\
\hat{\lambda}^g
\end{array}\right)^\top = \left(\begin{array}{c}
\hat{\beta}_k \\
\hat{\lambda}_{J,\alpha}
\end{array}\right)_{0 \leq k \leq d_1, 1 \leq \alpha \leq d_2, 1 \leq J \leq N+1} = \text{argmax}_\lambda \hat{L}(\lambda).
\]

Lemma 14 of Stone (1986) ensures that with probability approaching 1, \( \hat{\lambda} \) exists uniquely and that \( \nabla \hat{L}(\hat{\lambda}) = 0 \). In addition, Lemma A.1 and (A1) provide a vector \( \tilde{\lambda} = (\beta^\top, \lambda^g)^\top \) and an additive spline function \( \tilde{m} \) such that

\[
\tilde{m}(x) = \tilde{\lambda}^\top B(x), \quad \|\tilde{m} - m\|_\infty \leq C_\infty H^2.
\]
We first establish technical lemmas before proving Theorems 2 and 4.

**Lemma A.2** Under Assumptions (A1)-(A6) and (A8), as \( n \to \infty \)

\[
\left| \nabla \hat{L}(\tilde{\lambda}) \right| = \mathcal{O}_{a.s.} \left( H^2 + n^{-1/2} \log n \right), \\
\left\| \nabla \hat{L}(\tilde{\lambda}) \right\| = \mathcal{O}_{a.s.} \left( H^{3/2} + H^{-1/2} n^{-1/2} \log n \right).
\]

**Proof.** See supplement. \( \square \)

Define the following matrices:

\[
V = EB(T, X)B(T, X)^\top, S = V^{-1}, \\
V_n = n^{-1} \sum_{i=1}^n B(T_i, X_i)B(T_i, X_i)^\top, S_n = V_n^{-1},
\]

\[
V_b = Eb'' \{ m(T, X) \} B(T, X)B(T, X)^\top = \begin{bmatrix}
v_{b,0,0} & v_{b,0,k} & v_{b,0,J,\alpha} \\
v_{b,0,k'} & v_{b,k,k'} & v_{b,J,\alpha,k'} \\
v_{b,0,J',\alpha} & v_{b,J',\alpha',k} & v_{b,J,\alpha,J',\alpha'}
\end{bmatrix}_{N_d \times N_d},
\]

where \( N_d = (N + 1) d_2 + 1 + d_1 \), and

\[
S_b = V_b^{-1} = \begin{bmatrix}
s_{b,0,0} & s_{b,0,k} & s_{b,0,J,\alpha} \\
s_{b,0,k'} & s_{b,k,k'} & s_{b,J,\alpha,k'} \\
s_{b,0,J',\alpha} & s_{b,J',\alpha',k} & s_{b,J,\alpha,J',\alpha'}
\end{bmatrix}_{N_d \times N_d}, \quad (A.3)
\]

For any vector \( \lambda \in \mathbb{R}^{N_d} \), denote

\[
V_b(\lambda) = Eb'' \{ \lambda^\top B(T, X) \} B(T, X)B(T, X)^\top, S_b(\lambda) = V_b^{-1}(\lambda)
\]

\[
V_{n,b}(\lambda) = -\nabla^2 \hat{L}(\lambda), S_{n,b}(\lambda) = V_{n,b}^{-1}(\lambda). \quad (A.4)
\]
Lemma A.3 Under Assumptions (A2) and (A4),

\[ c_{V,I_N} \leq V \leq C_{V,I_N}, \quad c_{S,I_N} \leq S \leq C_{S,I_N}, \]
\[ c_{V,b,I_N} \leq V_b \leq C_{V,b,I_N}, \quad c_{S,b,I_N} \leq S_b \leq C_{S,b,I_N}. \]

Under Assumption (A2), (A4), (A5) and (A8), as \( n \to \infty \) with probability increasing to 1

\[ c_{V,I_N} \leq V_n(\lambda) \leq C_{V,I_N}, \quad c_{S,I_N} \leq S_n(\lambda) \leq C_{S,I_N} \]
\[ c_{V,b,I_N} \leq V_{n,b}(\lambda) \leq C_{V,b,I_N}, \quad c_{S,b,I_N} \leq S_{n,b}(\lambda) \leq C_{S,b,I_N}. \]

Proof. Using Lemma A.7 in [12] and boundness of function \( b' \). □

Define three vectors \( \Phi_b, \Phi_v, \Phi_r \) as

\[ \Phi_b = (\Phi_{b,J}, \alpha)_{0 \leq k \leq d_1, 1 \leq \alpha \leq d_2, 1 \leq J \leq N + 1}^T \]
\[ = -S_{b n}^{-1} \sum_{i=1}^n [b' \{ m(T_i, X_i) \} - b' \{ \bar{m}(T_i, X_i) \}] B(T_i, X_i), \]

\[ \Phi_v = (\Phi_{v,J}, \alpha)_{0 \leq k \leq d_1, 1 \leq \alpha \leq d_2, 1 \leq J \leq N + 1}^T \]
\[ = -S_{b n}^{-1} \sum_{i=1}^n [\sigma(T_i, X_i) \epsilon_i] B(T_i, X_i), \]
\[ \Phi_r = (\Phi_{r,J}, \alpha)_{0 \leq k \leq d_1, 1 \leq \alpha \leq d_2, 1 \leq J \leq N + 1}^T \]
\[ = \lambda\bar{\lambda} - \Phi_b - \Phi_v. \]

Lemma A.4 Under Assumptions (A1)-(A6) and (A8), as \( n \to \infty \)

\[ \| \lambda - \bar{\lambda} \| = O_{a.s.}(H^{3/2} + H^{-1/2}n^{-1/2} \log n), \] \hspace{1cm} (A.5)
\[ \| \Phi_r \| = O_p(H^{-3/2}n^{-1} \log n), \] \hspace{1cm} (A.6)
\[ \| \Phi_b \| = O_{a.s.}(H^2), \| \Phi_v \| = O_{a.s.}(H^{-1/2}n^{-1/2} \log n). \]
Proof. See supplement.

**Lemma A.5** Under Assumptions (A1)-(A6) and (A8), as $n \to \infty$

\[
\|\hat{m} - \bar{m}\|_{2,n} + \|\hat{m} - \bar{m}\|_2 = O_{a.s.}\left(H^{3/2} + H^{-1/2}n^{-1/2} \log n\right),
\]

\[
\|\hat{m} - m\|_{2,n} + \|\hat{m} - m\|_2 = O_{a.s.}\left(H^{3/2} + H^{-1/2}n^{-1/2} \log n\right).
\]

**Proof.** Lemma A.3 implies

\[
\|\hat{m} - \bar{m}\|_{2,n} + \|\hat{m} - \bar{m}\|_2 \leq 2CV\|\hat{\lambda}_g - \bar{\lambda}_g\|
\]

\[
= O_{a.s.}\left(H^{3/2} + H^{-1/2}n^{-1/2} \log n\right).
\]

The Lemma follows $\|\bar{m} - m\|_{\infty} + \|\bar{m} - m\|_2 + \|\bar{m} - m\|_{2,n} = O(H^2)$ by (A.2). □

**Proof of Theorem 2.** According to (9) and the Mean Value Theorem, a $\bar{m}_{K,1}(x_1)$ between $\hat{m}_{SBK,1}(x_1)$ and $\bar{m}_{K,1}(x_1)$ exists such that

\[
\hat{\ell}_{m_1}\{\hat{m}_{SBK,1}(x_1), x_1\} - \hat{\ell}\{\bar{m}_{K,1}(x_1), x_1\} = \hat{\ell}_{m_1}\{\bar{m}_{K,1}(x_1), x_1\}\{\hat{m}_{SBK,1}(x_1) - \bar{m}_{K,1}(x_1)\},
\]

Then according to $\hat{\ell}_{m_1}\{\hat{m}_{SBK,1}(x_1), x_1\} = 0$, one has

\[
\hat{m}_{SBK,1}(x_1) - \bar{m}_{K,1}(x_1) = -\frac{\hat{\ell}_{m_1}\{\hat{m}_{K,1}(x_1), x_1\}}{\hat{\ell}_{m_1}\{\hat{m}_{K,1}(x_1), x_1\}}.
\]

The theorem then follows Lemmas A.15 and A.16 in [11] with small modification including variable $T$.

**Proof of Theorem 3.** It follows Theorem 2 and the same proof of Theorem 1 in [25]. □

**Proof of Theorem 4.** See supplement. □
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8. SUPPLEMENTARY MATERIALS

Supplement to “Statistical Inference for Generalized Additive Partially Linear Model”: Supplement containing theoretical proof of Lemmas A.2, A.4 and Theorem 4 referenced in the main article.

gaplmsbk.R: R-package containing code to perform SBK estimation for component functions in generalized additive partially linear model available on https://github.com.

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Supplement to “Statistical Inference for Generalized Additive Partially Linear Model”

Proof of Lemma A.2

\[
\nabla \hat{L}(\bar{\lambda}) = n^{-1} \sum_{i=1}^{n} \left[ Y_i B(T_i, X_i) - b' \left\{ \bar{\lambda}^\top B(T_i, X_i) \right\} B(T_i, X_i) \right]
\]

\[
= n^{-1} \sum_{i=1}^{n} \left[ b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\} + \sigma(X_i) \varepsilon_i \right] B(T_i, X_i)
\]

The first \((1 + d_1)\) elements of the above vector is

\[
n^{-1} \sum_{i=1}^{n} \left[ [b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\}] + \sigma(X_i) \varepsilon_i \right] T_{ik}, 0 \leq k \leq d_1,
\]

with \(T_{i0} = 1\). These elements are \(O_{a.s.} (H^2 + n^{-1/2} \log n)\) according to \((A.2)\). The other elements can be written as

\[
n^{-1} \sum_{i=1}^{n} [\xi_{i,J,\alpha,n} + E [b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\}] B_{J,\alpha}(X_{i\alpha}) + \sigma(X_i) \varepsilon_i B_{J,\alpha}(X_{i\alpha})],
\]

where \(\xi_{i,J,\alpha,n}\) is

\[
[b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\}] B_{J,\alpha}(X_{i\alpha}) - E \left[ [b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\}] B_{J,\alpha}(X_{i\alpha}) \right].
\]

According to \((A.1)\) and \((A.2)\), one has

\[
|E [b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\}] B_{J,\alpha}(X_{i\alpha})| \leq E \left| b' \{m(T_i, X_i)\} - b' \{\bar{m}(T_i, X_i)\} \right| \frac{|b_{J,\alpha}(X_{i\alpha})|}{\|b_{J,\alpha}\|_2}
\]

\[
\leq c \|m - \bar{m}\|_\infty \max_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \|b_{J,\alpha}\|_2^{-1} \max_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} E |b_{J,\alpha}(X_{i\alpha})| = O \left( H^2 \times H^{-1/2} \times H \right) = O \left( H^{5/2} \right),
\]
for some constant $c$ and likewise for any $p \geq 2$

\[
E \left| b' \{ m (T, X) \} - b' \{ \bar{m} (T, X) \} \right|^p \leq (cH^{5/2})^{p-2} E \left| b' \{ m (T, X) \} - b' \{ \bar{m} (T, X) \} \right|^2 \frac{b^2 \{ X \} \| b \|^2}{\| b \|^2},
\]

and

\[
E \left| b' \{ m (T, X) \} - b' \{ \bar{m} (X) \} \right|^2 B^2 \{ X \} \leq c \| m - \bar{m} \|_\infty \max_{1 \leq J \leq N+1, 1 \leq \alpha \leq d} \| b \|_2 \max_{1 \leq J \leq N+1, 1 \leq \alpha \leq d} E \left| b^2 \{ X \} \right| = O \left( H^4 \right).
\]

Using these bounds and applying Lemma A.2 in [11], one has $n^{-1} \sum_{i=1}^n \xi_{i,J,a,n} = O_{a.s.} \left( H^2 n^{-1/2} \log n \right)$ and

\[
n^{-1} \left| \sum_{i=1}^n \sigma (X) \varepsilon_i B^{J,a} (X) \right| = O_{a.s.} \left( n^{-1/2} \log n \right).
\]

The lemma is then proved. \hfill \square

**Proof of Lemma A.4** The Mean Value Theorem implies that an $N_d \times N_d$ diagonal matrix $t$ exists whose diagonal elements are in $[0, 1]$, such that for $\hat{\lambda}^* = t\hat{\lambda} + (I_{N_d} - t) \bar{\lambda}$

\[
\nabla \hat{L} \left( \hat{\lambda} \right) - \nabla \hat{L} \left( \bar{\lambda} \right) = \nabla^2 \hat{L} \left( \hat{\lambda}^* \right) \left( \hat{\lambda} - \bar{\lambda} \right).
\]

Since, as noted before, that $\nabla \hat{L} \left( \hat{\lambda} \right) = 0$, the above equation becomes

\[
\hat{\lambda} - \bar{\lambda} = - \left\{ \nabla^2 \hat{L} \left( \hat{\lambda}^* \right) \right\}^{-1} \nabla \hat{L} \left( \bar{\lambda} \right).
\]

According to (A.4),

\[
- \nabla^2 \hat{L} \left( \lambda \right) = n^{-1} \sum_{i=1}^n b'' \left\{ \lambda \top B \left( T_i, X_i \right) \right\} B \left( T_i, X_i \right) B \left( T_i, X_i \right) \top = V_{n,b} \left( \lambda \right),
\]
Lemma A.3 implies that with probability approaching 1

\[ c_{V,b}I_{N_d} \leq -\nabla^2 \hat{L}(\hat{\lambda}^*) \leq C_{V,b}I_{N_d}. \]

Then (A.5) follows Lemma A.2. Furthermore, \( \|\hat{\lambda}^* - \bar{\lambda}\| = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2}\log n \right) \) as well according to \( \hat{\lambda}^* \)'s definition. Note that Taylor expansion ensures that for any vector \( a \in \mathbb{R}^{N_d} \)

\[ a^\top \left\{ \nabla^2 \hat{L}(\hat{\lambda}^*) - \nabla^2 \hat{L}(\bar{\lambda}) \right\} a \leq \|b''\|_{\infty} \max_{1 \leq i \leq n} \left| \hat{\lambda}^* \top B(T_i, X_i) - \bar{\lambda} \top B(T_i, X_i) \right| a^\top V_n a \]

while by Cauchy Schwartz inequality

\[ \max_{1 \leq i \leq n} \left| \hat{\lambda}^* \top B(T_i, X_i) - \bar{\lambda} \top B(T_i, X_i) \right| \leq \|\hat{\lambda}^* - \bar{\lambda}\| \max_{1 \leq i \leq n} \|B(T_i, X_i)\| \]

\[ = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2}\log n \right) \times O_p \left( H^{-1/2} \right) = O_p \left( H + H^{-1/2}n^{-1/2}\log n \right). \]

Consequently, one has the following bound on the difference of two Hessian matrices

\[ \sup_{a \in \mathbb{R}^{N_d}} \left\| \nabla^2 \hat{L}(\hat{\lambda}^*) - \nabla^2 \hat{L}(\bar{\lambda}) \right\| a \|a\|^{-1} = O_p \left( H + H^{-1/2}n^{-1/2}\log n \right). \]

Denote next

\[ \hat{d} = -\left\{ \nabla^2 \hat{L}(\hat{\lambda}^*) \right\}^{-1} \nabla \hat{L}(\bar{\lambda}) = \hat{\lambda} - \bar{\lambda} \]

\[ \bar{d} = -\left\{ \nabla^2 \hat{L}(\bar{\lambda}) \right\}^{-1} \nabla \hat{L}(\bar{\lambda}) \]

then \( \|\hat{d}\| = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2}\log n \right) \) and so is \( \|\bar{d}\| \) by similar arguments. Furthermore,

\[ \nabla^2 \hat{L}(\hat{\lambda}^*) \left( \hat{d} - \bar{d} \right) = \left\{ \nabla^2 \hat{L}(\bar{\lambda}) - \nabla^2 \hat{L}(\hat{\lambda}^*) \right\} \bar{d} \]
entails that

\[
\| \hat{d} - \bar{d} \| = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2} \log n \right) \times O_p \left( H + H^{-1}n^{-1/2} \log n \right) \\
= O_p \left( H^{5/2} + H^{-3/2}n^{-1} \log^2 n \right).
\]

Denote

\[
\hat{d} = \left[ n^{-1} \sum_{i=1}^{n} b'' \{ m (T_i, X_i) \} B (T_i, X_i) B (T_i, X_i)^\top \right]^{-1} \nabla \hat{L} (\lambda) .
\]

Using similar calculations, one can show that

\[
\| \tilde{d} - \bar{d} \| = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2} \log n \right) \times O_{a.s.} \left( H^2 \right) \\
= O_{a.s.} \left( H^{7/2} + H^{3/2}n^{-1/2} \log n \right),
\]

\[
\| \tilde{d} - \Phi_b - \Phi_v \| = O_{a.s.} \left( H^{3/2} + H^{-1/2}n^{-1/2} \log n \right) \times O_{a.s.} \left( H^{-1/2}n^{-1/2} \log n \right) \\
= O_{a.s.} \left( Hn^{-1/2} \log n + H^{-1}n^{-1} \log^2 n \right),
\]

Putting together the above proves \([A.6]\). Lastly, almost surely

\[
\| \Phi_b \| = \| S_b n^{-1} \sum_{i=1}^{n} [ b' \{ m (T_i, X_i) \} - b' \{ \bar{m} (T_i, X_i) \}] B (T_i, X_i) \| \\
\leq C_{S,b} \| n^{-1} \sum_{i=1}^{n} [ b' \{ m (T_i, X_i) \} - b' \{ \bar{m} (T_i, X_i) \}] B (T_i, X_i) \| = O_{a.s.} \left( H^2 \right)
\]

and

\[
\| \Phi_v \| = \| S_b n^{-1} \sum_{i=1}^{n} [ \sigma (T_i, X_i) \varepsilon_i ] B (T_i, X_i) \| \\
\leq C_{S,b} \| n^{-1} \sum_{i=1}^{n} [ \sigma (T_i, X_i) \varepsilon_i ] B (T_i, X_i) \| = O_{a.s.} \left( H^{-1/2}n^{-1/2} \log^2 n \right),
\]

which completes the proof of the lemma. \(\square\)
Proof of Theorem 4 (i) The Mean Value Theorem implies the existence of \( \hat{\beta} \) between \( \hat{\beta} \) and \( \tilde{\beta} \) such that \( (\hat{\beta} - \tilde{\beta}) = -\left\{ \nabla^2 \ell_{\beta} \left( \hat{\beta} \right) \right\}^{-1} \nabla \ell_{\beta} \left( \hat{\beta} \right) \), where

\[
-\nabla^2 \ell_{\beta} \left( \hat{\beta} \right) = n^{-1} \sum_{i=1}^{n} b'' \left\{ \hat{\beta}^T T_i + m(X_i) \right\} T_i T_i^T > c_0 I_{d_i \times d_i}
\]

according to Assumption (A6). We have

\[
\nabla \ell_{\beta} \left( \hat{\beta} \right) = \left\{ \frac{\partial \ell_{\beta} \left( \tilde{\beta} \right)}{\partial \beta_k} \right\}_k = \nabla \ell_{\beta} \left( \hat{\beta} \right) - \nabla \ell_{\beta} \left( \tilde{\beta} \right) = n^{-1} \sum_{i=1}^{n} \left[ b' \left\{ \hat{\beta}^T T_i + m(X_i) \right\} - b' \left\{ \tilde{\beta}^T T_i + \hat{m}(X_i) \right\} \right] T_i.
\]

So for a given \( 0 \leq k \leq d_1 \),

\[
\frac{\partial \ell_{\beta} \left( \tilde{\beta} \right)}{\partial \beta_k} = n^{-1} \sum_{i=1}^{n} \left[ b' \left\{ \hat{\beta}^T T_i + m(X_i) \right\} - b' \left\{ \tilde{\beta}^T T_i + \hat{m}(X_i) \right\} \right] T_{ik}
\]

\[
= n^{-1} \sum_{i=1}^{n} b'' \left\{ \tilde{\beta}^T T_i + m(X_i) \right\} \left\{ m(X_i) - \hat{m}(X_i) \right\} T_{ik}
\]

\[
+ O \left[ n^{-1} \sum_{i=1}^{n} \left\{ m(X_i) - \hat{m}(X_i) \right\}^2 T_{ik} \right]
\]

\[
= I_k + O_{a.s.} \left( H^3 + H^{-2} n^{-1} \log n \right),
\]

by Lemma A.5, where \( I_k = I_{k1} + I_{k2} \),

\[
I_{k1} = n^{-1} \sum_{i=1}^{n} b'' \left\{ \tilde{\beta}^T T_i + m(X_i) \right\} \left\{ m(X_i) - \hat{m}(X_i) \right\} T_{ik},
\]

\[
I_{k2} = n^{-1} \sum_{i=1}^{n} b'' \left\{ \tilde{\beta}^T T_i + m(X_i) \right\} \left\{ \hat{m}(X_i) - \tilde{m}(X_i) \right\} T_{ik}.
\]

According to Lemma A.1 \( I_{k1} = O_{a.s.} \left( H^2 \right) \), while

\[
I_{k2} = n^{-1} \sum_{i=1}^{n} b'' \left\{ \tilde{\beta}^T T_i + m(X_i) \right\} \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \left( \lambda_{j,\alpha} - \tilde{\lambda}_{j,\alpha} \right) B_{j,\alpha}(X_{i\alpha}) \right\} T_{ik}
\]

\[
= I_{k2,b} + I_{k2,v} + I_{k2,r}
\]
where

\[ I_{k_2,b} = n^{-1} \sum_{i=1}^n b'' \{ \hat{\beta}^\top T_i + m(X_i) \} \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \Phi_{b,J}\alpha B_{J}\alpha (X_{i\alpha}) \right\} T_{ik}, \]

\[ I_{k_2,v} = n^{-1} \sum_{i=1}^n b'' \{ \hat{\beta}^\top T_i + m(X_i) \} \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \Phi_{v,J}\alpha B_{J}\alpha (X_{i\alpha}) \right\} T_{ik}, \]

\[ I_{k_2,r} = n^{-1} \sum_{i=1}^n b'' \{ \hat{\beta}^\top T_i + m(X_i) \} \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \Phi_{r,J}\alpha B_{J}\alpha (X_{i\alpha}) \right\} T_{ik}. \]

We have

\[ |I_{k_2,b}| \leq C_n b^{-1} \sum_{i=1}^n \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} |\Phi_{b,J}\alpha| |B_{J}\alpha (X_{i\alpha})| \right\} T_{ik} \]

\[ \leq C_n Q b \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \Phi_{b,J}\alpha^2 \right\}^{1/2} \times \left[ 1 + \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \left( n^{-1} \sum_{i=1}^n |B_{J}\alpha (X_{i\alpha})| \right)^2 \right]^{1/2} \]

\[ = C_n Q \times O_{a.s.} \left( N_{d}^{1/2} H^{5/2} \right) \times \left\{ O_{a.s.} \left( N + 1 \right) \times d_2 \times O_{a.s.}(H) \right\} \]

\[ = O_{a.s.} \left( H^2 \right) = O_{a.s.} \left( n^{-1/2} \right). \]

according to (A.5). Similarly,

\[ |I_{k_2,r}| = O_p \left( N_d H^{7/2} + N_d H^{-1/2} n^{-1} \log n \right) = o_p \left( n^{-1/2} \right). \]

We have \( I_{k_2,v} = \tilde{I}_{k_2,v} + O_{a.s.} \left( n^{-1/2} \right) \times O_{a.s.} \left( N_{d}^{1/2} n^{-1/2} \log n \right) \times O(N), \) where

\[ \tilde{I}_{k_2,v} = n^{-1} \sum_{i=1}^n b'' \{ m(T_i, X_i) \} \left\{ \sum_{1 \leq j \leq N+1, 1 \leq \alpha \leq d_2} \Phi_{v,J}\alpha B_{J}\alpha (X_{i\alpha}) \right\} T_{ik} \]

\[ = -n^{-1} \sum_{i=1}^n b'' \{ m(T_i, X_i) \} n^{-1} \sum_{j=1}^n \sigma_j (T_j, X_j) \tilde{z}_j B^\top (X_j) S_{k,J} B (X_i) T_{ik} \]

where \( B(x) = \{ B_{1,1} (x_1), \ldots, B_{N+1,d_2} (x_{d_2}) \}^\top \) and \( S_{k,J} \) consists of columns \( 2 + d_1 \) to \( N_d \) of \( S_b \) defined in (A.3). \( \tilde{I}_{k_2,v} = o_{a.s.} \left( n^{-1/2} \right) \) by calculation similarly to the proof of Theorem 5 in [11]. Putting the above together, one has

\[ |\hat{\beta} - \tilde{\beta}| = o_p \left( n^{-1/2} \right). \]
(ii) According to Section 11.2 in [15],

\[
\begin{align*}
w_i &= \frac{1}{n} \frac{1}{1 + \lambda(\beta)^\top Z_i(\beta)} \frac{1}{n} \sum_{i=1}^{n} Z_i(\beta) = 0, \\
&= (A.8)
\end{align*}
\]

where \( Z_i(\beta) = [Y_i - b' \{ \beta^\top T_i + m(X_i) \}] T_i = \sigma(T_i, X_i) \varepsilon_i T_i. \)

\[n^{-1} \sum_{i=1}^{n} \sigma(T_i, X_i) \varepsilon_i T_i \xrightarrow{L} N\left(0, a(\phi) \left[ \mathbb{E} b'' \{ m(T, X) \} TT^\top \right]^{-1} \right)\] by central limit theorem and

\[
\max_{1 \leq i \leq n} \left| \lambda(\beta)^\top Z_i(\beta) \right| = o_p(1).
\]

So

\[
-2 \log \hat{R}(\beta) = -2 \sum_{i=1}^{n} \log (nw_i) = 2 \sum_{i=1}^{n} \log \left[ 1 + \lambda(\beta)^\top Z_i(\beta) \right] = 2 \sum_{i=1}^{n} \left\{ \lambda(\beta)^\top Z_i(\beta) \right\} - \sum_{i=1}^{n} \left\{ \lambda(\beta)^\top Z_i(\beta) \right\}^2 + 2 \sum_{i=1}^{n} \eta_i
\]

where \( \eta_i = O_p \left( \left\{ \lambda^\top(\beta) Z_i(\beta) \right\}^3 \right) \) with

\[
|\sum_{i=1}^{n} \eta_i| \leq C \left\| \lambda(\beta)^\top \right\|^3 \sum_{i=1}^{n} \|Z_i(\beta)\|^3 = o_p(1).
\]

Similarly, \(-2 \log \hat{R}(\beta) = 2 \sum_{i=1}^{n} \left\{ \hat{\lambda}^\top(\beta) \hat{Z}_i(\beta) \right\} - \sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta)^\top \hat{Z}_i(\beta) \right\}^2 \) with

\[
\hat{Z}_i(\beta) = [Y_i - b' \{ \beta^\top T_i + \hat{m}(X_i) \}] T_i.
\]

So the difference
\[-2 \log \hat{R}(\beta) + 2 \log \tilde{R}(\beta)\]
\[
= 2 \sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta)^\top \hat{Z}_i(\beta) \right\} - \sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta)^\top \hat{Z}_i(\beta) \right\}^2 +
-2 \sum_{i=1}^{n} \left\{ \lambda^\top (\beta) Z_i(\beta) \right\} + \sum_{i=1}^{n} \left\{ \lambda^\top (\beta) Z_i(\beta) \right\}^2 + o_p(1)
\]
\[
= 2I_1 + I_2 + o_p(1)
\]

with

\[
I_1 = \sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta)^\top \hat{Z}_i(\beta) - \lambda(\beta)^\top Z_i(\beta) \right\},
\]
\[
I_2 = \sum_{i=1}^{n} \left[ \left\{ \lambda(\beta)^\top Z_i(\beta) \right\}^2 - \left\{ \hat{\lambda}(\beta)^\top \hat{Z}_i(\beta) \right\}^2 \right].
\]

Rewrite

\[
I_1 = \sum_{i=1}^{n} \hat{\lambda}(\beta)^\top \left\{ \hat{Z}_i(\beta) - Z_i(\beta) \right\} + \sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta) - \lambda(\beta) \right\}^\top Z_i(\beta),
\]

then

\[
\begin{align*}
\sum_{i=1}^{n} \hat{\lambda}(\beta)^\top \left\{ \hat{Z}_i(\beta) - Z_i(\beta) \right\} \\
= \hat{\lambda}(\beta)^\top \sum_{i=1}^{n} \left\{ \hat{Z}_i(\beta) - Z_i(\beta) \right\} \\
= \hat{\lambda}(\beta)^\top \sum_{i=1}^{n} \left[ b' \{ \beta^\top T_i + m(X_i) \} - b' \{ \beta^\top T_i + \hat{m}(X_i) \} \right] T_i \\
\leq \left\| \hat{\lambda}(\beta)^\top \right\| \left\| \sum_{i=1}^{n} \left[ b' \{ \beta^\top T_i + m(X_i) \} - b' \{ \beta^\top T_i + \hat{m}(X_i) \} \right] T_i \right\| \\
= O_p(n^{-1/2}) \sigma_p(n^{1/2}) = o_p(1)
\end{align*}
\]

following \( \left\| \hat{\lambda}(\beta)^\top \right\| = O_p(n^{-1/2}) \) and

\[
\sum_{i=1}^{n} \left[ b' \{ \beta^\top T_i + m(X_i) \} - b' \{ \beta^\top T_i + \hat{m}(X_i) \} \right] T_i = o_p(n^{1/2})
\]
from the proof for (A.7). Denote

$$
\hat{S} (\beta) = \frac{1}{n} \sum_{i=1}^{n} [Y_i - b' \{\beta^\top T_i + \hat{m}(X_i)\}]^2 T_i T_i^\top,
$$

$$
S (\beta) = \frac{1}{n} \sum_{i=1}^{n} [Y_i - b' \{\beta^\top T_i + m(X_i)\}]^2 T_i T_i^\top.
$$

According to Section 11.2 in [15],

$$
\hat{\lambda} (\beta) = \hat{S}^{-1} (\beta) n^{-1} \sum_{i=1}^{n} \hat{Z}_i (\beta) + o_p \left( n^{-1/2} \right),
$$

$$
\lambda (\beta) = S^{-1} (\beta) n^{-1} \sum_{i=1}^{n} Z_i (\beta) + o_p \left( n^{-1/2} \right).
$$

We have

$$
\hat{S} (\beta) - S (\beta)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left\{ [Y_i - b' \{\beta^\top T_i + \hat{m}(X_i)\}]^2 - [Y_i - b' \{\beta^\top T_i + m(X_i)\}]^2 \right\} T_i T_i^\top + o_p \left( n^{-1/2} \right)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left( 2Y_i - 2b' \{\beta^\top T_i + \hat{m}(X_i)\} - b' \{\beta^\top T_i + m(X_i)\} \right) T_i T_i^\top + o_p \left( n^{-1/2} \right)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left( 2\sigma (T_i, X_i) \varepsilon_i + b' \{\beta^\top T_i + m(X_i)\} - b' \{\beta^\top T_i + \hat{m}(X_i)\} \right) T_i T_i^\top + o_p \left( n^{-1/2} \right)
$$

$$
\frac{1}{n} \sum_{i=1}^{n} 2\sigma (T_i, X_i) \varepsilon_i [b' \{\beta^\top T_i + m(X_i)\} - b' \{\beta^\top T_i + \hat{m}(X_i)\}] T_i T_i^\top
t + \frac{1}{n} \sum_{i=1}^{n} \left( b' \{\beta^\top T_i + m(X_i)\} - b' \{\beta^\top T_i + \hat{m}(X_i)\} \right)^2 T_i T_i^\top + o_p \left( n^{-1/2} \right)
$$

$$
= o_p \left( n^{-1/2} \right) + o_p \left( n^{-1/2} \right) + o_p \left( n^{-1/2} \right) = o_p \left( n^{-1/2} \right).
$$

So
\[
\hat{\lambda}(\beta) - \lambda(\beta) \\
= \hat{S}^{-1}(\beta) n^{-1} \sum_{i=1}^{n} \hat{Z}_i(\beta) - S^{-1}(\beta) n^{-1} \sum_{i=1}^{n} Z_i(\beta) \\
= \hat{S}^{-1}(\beta) n^{-1} \sum_{i=1}^{n} \left\{ \hat{Z}_i(\beta) - Z_i(\beta) \right\} + \left\{ \hat{S}^{-1}(\beta) - S^{-1}(\beta) \right\} n^{-1} \sum_{i=1}^{n} Z_i(\beta) \\
= \sigma_p \left( n^{-1/2} \right).
\]

Then

\[
\sum_{i=1}^{n} \left\{ \hat{\lambda}(\beta) - \lambda(\beta) \right\}^T Z_i(\beta) \\
= \left\{ \hat{\lambda}(\beta) - \lambda(\beta) \right\}^T \sum_{i=1}^{n} Z_i(\beta) \\
= \sigma_p \left( n^{-1/2} \right) O_p \left( n^{1/2} \right) = \sigma_p \left( 1 \right),
\]

and \( I_1 = \sigma_p \left( 1 \right) \).

\[
I_2 = \sum_{i=1}^{n} \left[ \left\{ \lambda(\beta)^T Z_i(\beta) \right\}^2 - \left\{ \hat{\lambda}(\beta)^T \hat{Z}_i(\beta) \right\}^2 \right] \\
= \sum_{i=1}^{n} \left\{ \lambda(\beta)^T Z_i(\beta) + \hat{\lambda}(\beta)^T \hat{Z}_i(\beta) \right\} \left\{ \lambda(\beta)^T Z_i(\beta) - \hat{\lambda}(\beta)^T \hat{Z}_i(\beta) \right\} \\
= \sigma_p \left( 1 \right).
\]

by similar proof. Putting together, we have

\[
-2 \log \hat{R}(\beta) + 2 \log \tilde{R}(\beta) = \sigma_p \left( 1 \right).
\]
Figure 1: Plots of densities for $\hat{\beta}_1$ with $n = 500$ - dotted line, $n = 1000$ - dashed line, $n = 2000$ - thin solid line, $n = 4000$ - thick solid line for (a) $r_1 = 0, r_2 = 0$, (b) $r_1 = 0, r_2 = 0.5$, (c) $r_1 = 0.5, r_2 = 0$, (d) $r_1 = 0.5, r_2 = 0.5$ from 1000 replications.
Figure 2: Plots of $m_1(x_1)$ - thick black line, $\tilde{m}_{K,1}(x_1)$ - blue dashed line, asymptotic 95% pointwise confidence intervals - red dashed line, $\tilde{m}_{SBK,1}(x_1)$ and 95% simultaneous confidence bands - red solid line for $r_1 = 0$, $r_2 = 0.5$ and (a) $n = 500$, (b) $n = 1000$, (c) $n = 2000$, (d) $n = 4000$. 

Confidence Level = 0.95, $n = 500$

Confidence Level = 0.95, $n = 1000$

Confidence Level = 0.95, $n = 2000$

Confidence Level = 0.95, $n = 4000$
Figure 3: Plots of estimations of component functions (a) $\hat{m}_{SBK,2}(x_2)$ and (b) $\hat{m}_{SBK,8}(x_8)$ and asymptotic 95% simultaneous confidence bands.
| $r$  | $n$  | $10 \times \text{BIAS}$ | $100 \times \text{VARIANCE}$ | $100 \times \text{MSE}$ | EFF ($\hat{\beta}_1$) |
|------|------|------------------------|-----------------------------|------------------------|-------------------|
| $r_1 = 0$ | 500  | 1.509                  | 2.018                       | 4.298                  | 0.8436            |
|       | 1000 | 0.727                  | 1.197                       | 1.726                  | 0.8749            |
|       | 2000 | 0.408                  | 0.626                       | 0.793                  | 0.9189            |
|       | 4000 | 0.240                  | 0.282                       | 0.339                  | 0.9534            |
| $r_1 = 0.5$ | 500  | 1.473                  | 3.136                       | 5.306                  | 0.8392            |
|       | 1000 | 0.834                  | 1.287                       | 1.983                  | 0.8873            |
|       | 2000 | 0.476                  | 0.674                       | 0.901                  | 0.9294            |
|       | 4000 | 0.260                  | 0.202                       | 0.270                  | 0.9665            |
| $r_1 = 0$ | 500  | 1.327                  | 3.880                       | 5.642                  | 0.8475            |
|       | 1000 | 0.699                  | 1.851                       | 2.339                  | 0.8856            |
|       | 2000 | 0.665                  | 0.739                       | 1.182                  | 0.9353            |
|       | 4000 | 0.390                  | 0.290                       | 0.442                  | 0.9479            |
| $r_1 = 0.5$ | 500  | 1.635                  | 4.230                       | 6.903                  | 0.8203            |
|       | 1000 | 0.901                  | 1.190                       | 2.002                  | 0.8758            |
|       | 2000 | 0.529                  | 0.806                       | 1.086                  | 0.9304            |
|       | 4000 | 0.209                  | 0.366                       | 0.410                  | 0.9483            |

Table 1: The mean of $10 \times \text{Bias}$, $100 \times \text{Variances}$, $100 \times \text{MSEs}$ and EFFs of $\hat{\beta}_1$ from 1000 replications.
Table 2: Coverage ratios and average length of the empirical likelihood confidence interval (EL) and Wald-type confidence interval for $\beta_1$ for $n = 500, 1000, 2000, 4000$ with $r_1 = 0$ from 1000 replications. DIFF = $-2 \log \hat{R}(\beta) + 2 \log \tilde{R}(\beta)$ is the difference between $-2 \log \hat{R}(\beta)$ and $-2 \log \tilde{R}(\beta)$.

|                  | $n = 500$ | $n = 1000$ | $n = 2000$ | $n = 4000$ |
|------------------|-----------|------------|------------|------------|
| **Coverage Ratio** |           |            |            |            |
| EL               | 0.923     | 0.941      | 0.946      | 0.951      |
| Wald             | 0.918     | 0.934      | 0.944      | 0.948      |
| **Average Length** |          |            |            |            |
| EL               | 1.2675    | 0.9474     | 0.7105     | 0.5339     |
| Wald             | 1.4073    | 1.0447     | 0.7480     | 0.5625     |
| **DIFF**         |           |            |            |            |
| MEAN             | 0.1213    | 0.1023     | 0.0981     | 0.0726     |
| SD               | 0.5199    | 0.4703     | 0.3667     | 0.3242     |
| $r$   | $n$   | $100 \times \text{MASE}(\hat{m}_{K,\alpha})$ | $100 \times \text{MASE}(\hat{m}_{SBK,\alpha})$ | EFF($\hat{m}_{SBK,1}$) |
|-------|-------|------------------------------------------|------------------------------------------|---------------------|
| $r_1 = 0$ | 500   | 4.482                                     | 4.603                                     | 0.9501              |
|       | 1000  | 2.418                                     | 2.503                                     | 0.9809              |
|       | 2000  | 1.582                                     | 1.613                                     | 0.9854              |
|       | 4000  | 1.212                                     | 1.247                                     | 0.9923              |
|       | 500   | 4.060                                     | 4.322                                     | 0.9445              |
|       | 1000  | 2.592                                     | 2.649                                     | 0.9767              |
|       | 2000  | 1.746                                     | 1.714                                     | 0.9832              |
|       | 4000  | 1.194                                     | 1.218                                     | 0.9936              |
| $r_1 = 0.5$ | 500   | 4.845                                     | 6.348                                     | 0.8827              |
|       | 1000  | 2.935                                     | 3.559                                     | 0.8755              |
|       | 2000  | 1.951                                     | 2.177                                     | 0.9494              |
|       | 4000  | 1.515                                     | 1.648                                     | 0.9795              |
| $r_1 = 0.5$ | 500   | 5.656                                     | 7.114                                     | 0.8722              |
|       | 1000  | 2.804                                     | 3.570                                     | 0.8951              |
|       | 2000  | 1.886                                     | 2.089                                     | 0.9478              |
|       | 4000  | 1.525                                     | 1.634                                     | 0.9744              |

Table 3: The $100 \times \text{MASE}$s of $\hat{m}_{K,1}$, $\hat{m}_{SBK,1}$ and EFFs for $n = 500$, 1000, 2000, 4000 from 1000 replications.
| n   | k = 2   | k = 10  | k = 100 |
|-----|---------|---------|---------|
| 500 | GLM     | 0.6287 (0.0436) | 0.6412 (0.0397) | 0.6438 (0.0390) |
|     | GAM     | 0.6222 (0.0732) | 0.6706 (0.0393) | 0.6756 (0.0400) |
|     | GAPLM   | 0.6511 (0.0479) | 0.6828 (0.0377) | 0.6861 (0.0391) |
| 1000| GLM     | 0.6429 (0.0282) | 0.6476 (0.0268) | 0.6488 (0.0268) |
|     | GAM     | 0.6735 (0.0438) | 0.6863 (0.0326) | 0.6929 (0.0261) |
|     | GAPLM   | 0.6861 (0.0298) | 0.6968 (0.0254) | 0.7001 (0.0258) |
| 2000| GLM     | 0.6474 (0.0204) | 0.6513 (0.0195) | 0.6519 (0.0188) |
|     | GAM     | 0.6842 (0.0615) | 0.6984 (0.0286) | 0.7000 (0.0185) |
|     | GAPLM   | 0.6984 (0.0204) | 0.7067 (0.0178) | 0.7057 (0.0178) |
| 4000| GLM     | 0.6507 (0.0134) | 0.6522 (0.0136) | 0.6529 (0.0132) |
|     | GAM     | 0.6889 (0.0243) | 0.6968 (0.0403) | 0.7079 (0.0164) |
|     | GAPLM   | 0.7056 (0.0130) | 0.7110 (0.0124) | 0.7119 (0.0119) |

Table 4: The mean and standard deviation (in parentheses) of Accuracy Ratio (AR) values for GLM, GAM, GAPLM for $r_1 = 0$, $r_2 = 0$ from 1000 replications.
| $r$   | $n$ | 10 × BIAS | 100 × VARIANCE | 100 × MSE | EFF ($\hat{\beta}_3$) |
|-------|-----|-----------|----------------|-----------|-------------------------|
| $r_1 = 0$ | 500 | 1.476     | 10.129         | 12.309    | 0.7634                  |
|        | 1000| 0.770     | 4.437          | 5.031     | 0.8343                  |
|        | 2000| 0.448     | 1.846          | 2.047     | 0.8929                  |
|        | 4000| 0.315     | 0.937          | 1.037     | 0.9572                  |
| $r_2 = 0$ | 500 | 1.336     | 10.329         | 12.115    | 0.7445                  |
|        | 1000| 0.833     | 4.221          | 4.916     | 0.8267                  |
|        | 2000| 0.423     | 1.952          | 2.132     | 0.8832                  |
|        | 4000| 0.302     | 0.944          | 1.036     | 0.9436                  |
| $r_1 = 0.5$ | 500 | 1.441     | 10.154         | 12.231    | 0.7556                  |
|        | 1000| 0.803     | 4.446          | 5.114     | 0.8430                  |
|        | 2000| 0.489     | 2.136          | 2.376     | 0.8785                  |
|        | 4000| 0.328     | 0.924          | 1.032     | 0.9572                  |
| $r_2 = 0.5$ | 500 | 1.475     | 11.014         | 13.190    | 0.7794                  |
|        | 1000| 0.812     | 4.464          | 5.124     | 0.8314                  |
|        | 2000| 0.524     | 1.970          | 2.245     | 0.8852                  |
|        | 4000| 0.302     | 0.966          | 1.058     | 0.9529                  |

Table 5: The mean of 10×Bias, 100×Variances, 100×MSEs and EFFs of $\hat{\beta}_3$ from 1000 replications.
| Ratio No. | Definition                  | Ratio No. | Definition                  |
|----------|-----------------------------|-----------|-----------------------------|
| $Z_1$    | Net Income/Sales            | $Z_5$    | Cash/Total Assets           |
| $Z_2$    | Operating Income/Total Assets | $Z_6$   | Inventories/Sales           |
| $Z_3$    | Ebit/Total Assets           | $Z_7$    | Accounts Payable/Sales      |
| $Z_4$    | Total Liabilities/Total Assets | $Z_8$  | $\log(\text{Total Assets})$ |

Table 6: Definitions of financial ratios.
|       | $k = 2$          | $k = 10$          |
|-------|-----------------|------------------|
| GLM   | 0.5627 (0.0271) | 0.5751 (0.00162) |
| GAM   | 0.5888 (0.0405) | 0.6123 (0.00219) |
| GAPLM | 0.5928 (0.0408) | 0.6164 (0.00196) |

Table 7: The mean and standard deviation (in parentheses) of AR values for GLM, GAM, GAPLM for $k$-fold Cross-validation with $k = 2$ and 10 from 1000 replications.