Oscillation criteria for first-order dynamic equations with nonmonotone delays

Nurten KILIÇ¹, Özkan ÖCALAN*²

¹Kütahya Dumlupınar University, Faculty of Science and Arts, Department of Mathematics, 43100, Kütahya, Turkey
²Akdeniz University, Faculty of Science, Department of Mathematics, 07058 Antalya, Turkey

Abstract
In this paper, we consider the first-order dynamic equation as following:

\[ x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \in [t_0, \infty)_T \]

where \( p_i \in C_{rd}([t_0, \infty)_T, \mathbb{R}_0^+) \), \( \tau_i \in C_{rd}([t_0, \infty)_T, T) \) (\( i = 1, 2, \ldots, m \)) and \( \tau_i(t) \leq t \), \( \lim_{t \to \infty} \tau_i(t) = \infty \). When the delay terms \( \tau_i(t) \) (\( i = 1, 2, \ldots, m \)) are not necessarily monotone, we present new sufficient conditions for the oscillation of first-order delay dynamic equations on time scales.

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1. Introduction
As is well known, after Stefan Hilger [15], [16] introduced the theory of dynamic equations on time scales (or measure chain) in his Ph.D. thesis in 1988, a lot of papers have been devoted to this subject field. Especially, the oscillatory behaviour of solutions of differential/difference and dynamic equations has been studied by many authors. See, for example, [1-34] and the references cited therein. Consider the first-order delay dynamic equation

\[ x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \in [t_0, \infty)_T, \]

(1.1)

where \( T \) is a time scale unbounded above with \( t_0 \in T \), \( p_i \in C_{rd}([t_0, \infty)_T, \mathbb{R}_0^+) \), \( \tau_i \in C_{rd}([t_0, \infty)_T, T) \) (\( i = 1, 2, \ldots, m \)) are not necessarily monotone such that

\[ \tau_i(t) \leq t \quad \text{for all} \quad t \in T, \quad \lim_{t \to \infty} \tau_i(t) = \infty. \]

(1.2)

A function \( p : T \to \mathbb{R} \) is called positively regressive (we write \( p \in \mathbb{R}_0^+ \)) if it is rd-continuous and satisfies \( 1 + \mu(t)p(t) > 0 \) for all \( t \in T \), where \( \mu : T \to \mathbb{R}_0^+ \) is the graininess function defined by \( \mu(t) := \sigma(t) - t \) with the forward jump operator \( \sigma : T \to T \) defined by \( \sigma(t) = \inf \{ s \in T : s > t \} \) for \( t \in T \). A point \( t \in T \) is called right-dense if \( \sigma(t) = t \) (or equivalently

*Corresponding Author.
Email addresses: nurten.kilic@dpu.edu.tr (N. Kılıç), ozkanocalan@akdeniz.edu.tr (Ö. Öcalan)
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\(\mu(t) = 0\) holds, otherwise it is called right-scattered. The readers are referred to Bohner and Peterson [2] for further details concerning the time scales calculus.

A function \(x : \mathbb{T} \rightarrow \mathbb{R}\) is called a solution of the equation (1.1), if \(x(t)\) is delta differentiable for \(t \in \mathbb{T}^\kappa\) and satisfies equation (1.1) for \(t \in \mathbb{T}^\kappa\). We say that a solution \(x\) of equation (1.1) has a generalized zero at \(t\) if \(x(t) = 0\) or if \(\mu(t) > 0\) and \(x(t)x(\sigma(t)) < 0\).

Let \(\sup T = \infty\) and then a nontrivial solution \(x\) of equation (1.1) is called oscillatory on \([t, \infty)\) if it has arbitrarily large generalized zeros in \([t, \infty)\).

For \(m = 1\), equation (1.1) reduces to

\[x^\Delta(t) + p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_T.\]  
(1.3)

Now, we give some well-known tests on oscillatory behaviour of (1.3). In 2002, Zhang and Deng [32], using the cylinder transforms, proved that if \(\tau(t)\) is eventually nondecreasing and

\[\limsup_{t \to \infty} \sup_{\lambda \in E} \{\lambda e_{-\lambda p}(t, \tau(t))\} < 1,\]

where \(E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0\}\) and in 2005, Bohner [4], using exponential functions notation, proved that if \(\tau(t)\) is eventually nondecreasing and

\[\limsup_{t \to \infty} \sup_{-\lambda p \in \mathbb{R}^+} \{\lambda e_{-\lambda p}(t, \tau(t))\} < 1,\]

where

\[e_{-\lambda p}(t, \tau(t)) = \exp\left\{\int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s)) \Delta s\right\},\]

and

\[\xi_h(z) = \begin{cases} \log(1 + h z) / h, & \text{if } h \neq 0 \\ z, & \text{if } h = 0 \end{cases},\]

then all solutions of equation (1.3) are oscillatory.

In 2005, Zhang et al. [33] and in 2006, Şahiner and Stavroulakis [27], using different technique, obtained that if \(\tau(t)\) is eventually nondecreasing and

\[\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1,\]  
(1.4)

then all solutions of equation (1.3) are oscillatory. In 2005, Zhang et al. [33] (See also Agarwal and Bohner [1, Theorem 1]) established the following result. Assume that \(\tau(t)\) is eventually nondecreasing and

\[m := \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > \frac{1}{e},\]  
(1.5)

then all solutions of (1.3) oscillate.

In 2006, Şahiner and Stavroulakis [27] found out that if \(\tau(t)\) is eventually nondecreasing,

\[\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > c \quad \text{and} \quad \limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > 1 - \frac{c^2}{4},\]  
(1.6)

where \(c \in (0, 1)\), then every solution of equation (1.3) oscillates. Furthermore, Agarwal and Bohner [1] improved the condition (1.6) as follows:
If \( \tau(t) \) is eventually nondecreasing,

\[
\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > c \quad \text{and} \quad \limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > 1 - \left(1 - \sqrt{1 - c}\right)^2
\]  

(1.7)

where \( c \in (0, 1) \mathbb{R} \), then every solution of equation (1.3) oscillates.

Also, in 2016, Karpuz and Öcalan [19] enhanced the condition (1.7) by extending the second integral condition to the larger interval \([\tau(t), t]_{\mathbb{T}}\) as the following:

Assume that \( \tau(t) \) is eventually nondecreasing and

\[
\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > c \quad \text{and} \quad \limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1 - \left(1 - \sqrt{1 - c}\right)^2,
\]  

(1.8)

where \( c \in (0, 1) \mathbb{R} \). Then every solution of equation (1.3) oscillates.

Zhang et al. [33] established the following result. Assume that \( \tau(t) \) is eventually non-decreasing and \( m \in [0, \frac{1}{e}] \) (where \( m \) is defined by (1.5)). Moreover, if

\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1 + \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2},
\]  

(1.9)

where \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{m\lambda} \), then all solutions of equation (1.3) are oscillatory. It is obvious that, since

\[
\frac{1 + \ln \lambda_1}{\lambda_1} \leq 1 \quad \text{for} \quad \lambda_1 \in [1, e],
\]

the condition (1.9) implies

\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1 - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2}.
\]  

(1.10)

Clearly, when \( 0 < c \leq \frac{1}{e} \), it is easy to verify that

\[
\frac{1 - c - \sqrt{1 - 2c - c^2}}{2} > \left(1 - \sqrt{1 - c}\right)^2 > \frac{c^2}{4}
\]

and therefore the condition (1.10) is weaker than the conditions (1.6) and (1.8).

Now, we assume that \( \tau(t) \) is not necessarily monotone. Set

\[
h(t) = \sup_{s \leq t} \tau(s), \quad t \in \mathbb{T}, \quad t \geq 0.
\]  

(1.11)

Clearly, \( h(t) \) is nondecreasing and \( \tau(t) \leq h(t) \) for all \( t \geq 0 \).

In 2017, Öcalan, Özkan and Yıldız [24, Theorem 2.2] studied the equation (1.3) when \( \tau(t) \) is not necessarily monotone and obtained the following result.

**Theorem A.** If

\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} p(s) \Delta s > 1,
\]  

(1.12)

where \( h(t) \) is defined by (1.11), then every solution of (1.3) is oscillatory.

Finally, Öcalan [25, Corollary 2.4] established the following result when \( \tau(t) \) is not necessarily monotone.
Theorem B. If
\[
\liminf_{t \to \infty} \int_{h(t)}^{t} p(s) \Delta s = \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > \frac{1}{e},
\]  
where \( h(t) \) is defined by (1.11), then all solutions of (1.3) oscillate.

A slight modification in the proofs of Theorems A and B leads to the following result.

Theorem 1.1. Assume that all the conditions of Theorems A and B hold. Then

(i) the dynamic inequality
\[
x^\Delta(t) + p(t)x(\tau(t)) \leq 0, \quad t \in [t_0, \infty)_\mathbb{T}
\]

has no eventually positive solutions;

(ii) the dynamic inequality
\[
x^\Delta(t) + p(t)x(\tau(t)) \geq 0, \quad t \in [t_0, \infty)_\mathbb{T}
\]

has no eventually negative solutions.

2. Main results

In this section, we present some new sufficient conditions for the oscillation of all solutions of (1.1), under the assumption that the arguments \( \tau_i(t) \) \( (i = 1, 2, \ldots, m) \) are not necessarily monotone. Set
\[
h_i(t) = \sup_{s \leq t} \{ \tau_i(s) \} \quad \text{and} \quad h(t) = \max_{1 \leq i \leq m} \{ h_i(t) \}, \quad t \in \mathbb{T}, \quad t \geq 0.
\]  
(2.1)

Clearly, \( h_i(t) \) \( (i = 1, 2, \ldots, m) \) are nondecreasing and \( \tau_i(t) \leq h_i(t) \leq h(t) \) \( (i = 1, 2, \ldots, m) \) for all \( t \geq 0 \).

The following lemma was given by Şahiner and Stavroulakis [27].

Lemma 2.1. Assume that \( f : \mathbb{T} \to \mathbb{R} \) is rd-continuous, \( g : \mathbb{T} \to \mathbb{R} \) is nonincreasing and \( \tau : \mathbb{T} \to \mathbb{T} \) is nondecreasing. If \( b < u \), then
\[
\int_{b}^{\sigma(u)} f(s) g(\tau(s)) \Delta s \geq g(\tau(u)) \int_{b}^{\sigma(u)} f(s) \Delta s.
\]

The following result is easily obtained by using the similar way in the proof of Lemma 2.3 in [24].

Lemma 2.2. Assume that (2.1) holds and \( \alpha > 0 \). Then, we have
\[
\alpha := \liminf_{t \to \infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \Delta s = \liminf_{t \to \infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) \Delta s,
\]

where \( \tau(t) = \max_{1 \leq i \leq m} \{ \tau_i(t) \}, \quad t \in \mathbb{T}, \quad t \geq 0 \).

Theorem 2.3. Assume that \( - \sum_{i=1}^{m} p_i \in \mathbb{R}^+ \). If \( \tau_i(t) \) \( (i = 1, 2, \ldots, m) \) are not necessarily monotone and
\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s > 1
\]  
(2.2)
or
\[
\liminf_{t \to \infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) \Delta s > \frac{1}{e},
\]  
(2.3)
where \( h(t) \) is defined by (2.1) and \( \tau(t) = \max_{1 \leq i \leq m} \{ \tau_i(t) \} \). Then all solutions of (1.1) oscillate.

**Proof.** Assume, for the sake of contradiction, that there exists a nonoscillatory solution \( x(t) \) of (1.1). Since \(-x(t)\) is also a solution of (1.1), we can confine our discussion only to the case where the solution \( x(t) \) is eventually positive. Then, there exists \( t_1 > t_0 \) such that \( x(t), \ x(\tau_i(t)) > 0 \) \((i = 1, 2, \ldots, m)\), for all \( t \geq t_1 \). Thus, from (1.1) we have

\[
x^\Delta(t) = -\sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,
\]

which means that \( x(t) \) is an eventually nonincreasing function. In view of this and \( \tau_i(t) \leq \tau(t) \) \((i = 1, 2, \ldots, m)\), (1.1) gives

\[
x^\Delta(t) + \left( \sum_{i=1}^{m} p_i(t) \right) x(\tau(t)) \leq 0, \quad t \geq t_1.
\]

Comparing (2.2) and (2.3), we obtain a contradiction to Theorem 1.1. Thus, the proof of the theorem is completed. \( \square \)

Now, we consider the case where \( 0 < \alpha \leq \frac{1}{e} \). Then, we will obtain new oscillatory condition for all solutions of (1.1). We need the following lemma to establish our result. When the case \( \tau_i(t) \) \((i = 1, 2, \ldots, m)\) are not necessarily monotone, the following lemma can be easily obtained by using the similar process in [33, Lemma 2.4]. So, the proof of the following result is omitted here.

**Lemma 2.4.** Assume that \( \tau_i(t) \) \((i = 1, 2, \ldots, m)\) are not necessarily monotone. Let \( 0 \leq \alpha \leq \frac{1}{e} \) and \( x(t) \) be an eventually positive solution of Eq.(1.1). Then, we get

\[
\liminf_{t \to \infty} \frac{x(\sigma(t))}{x(h(t))} \geq \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.4}
\]

where \( h(t) \) is defined by (2.1) and \( \tau(t) = \max_{1 \leq i \leq m} \{ \tau_i(t) \} \).

**Theorem 2.5.** Assume that \(-\sum_{i=1}^{m} p_i \in \mathbb{R}^+ \) and \( 0 \leq \alpha \leq \frac{1}{e} \). If \( \tau_i(t) \) \((i = 1, 2, \ldots, m)\) are not necessarily monotone and

\[
\limsup_{t \to \infty} \int_{h(t)} \sum_{i=1}^{m} p_i(s) \Delta s > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.5}
\]

where \( h(t) \) is defined by (2.1). Then all solutions of (1.1) oscillate.

**Proof.** Assume, for the sake of contradiction, that there exists a nonoscillatory solution \( x(t) \) of (1.1). Since \(-x(t)\) is also a solution of (1.1), we can confine our discussion only to the case where the solution \( x(t) \) is eventually positive. Then, there exists \( t_1 > t_0 \) such that \( x(t), \ x(\tau_i(t)) > 0 \) \((i = 1, 2, \ldots, m)\), for all \( t \geq t_1 \). Thus, from (1.1) we have

\[
x^\Delta(t) = -\sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,
\]

which means that \( x(t) \) is an eventually nonincreasing function. In view of this and \( \tau_i(t) \leq h_i(t) \leq h(t) \) \((i = 1, 2, \ldots, m)\), Eq.(1.1) gives

\[
x^\Delta(t) + \sum_{i=1}^{m} p_i(t)x(h(t)) \leq 0, \quad t \geq t_1. \tag{2.6}
\]
Integrating (2.6) from \( h(t) \) to \( \sigma(t) \) and taking into account the facts that the function \( h(t) \) is nondecreasing and the function \( x(t) \) is nonincreasing, we obtain

\[
x(\sigma(t)) - x(h(t)) + \sigma(t) \sum_{i=1}^{m} p_i(s) x(h(s)) \Delta s \leq 0.
\]

Therefore, by using Lemma 2.1, we get

\[
x(\sigma(t)) - x(h(t)) + x(h(t)) \sigma(t) \sum_{i=1}^{m} p_i(s) \Delta s \leq 0
\]
or

\[
x(\sigma(t)) + x(h(t)) \left[ \sigma(t) \sum_{i=1}^{m} p_i(s) \Delta s - 1 \right] \leq 0.
\]

Consequently,

\[
\sigma(t) \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \frac{x(\sigma(t))}{x(h(t))},
\]

which gives

\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \liminf_{t \to \infty} \frac{x(\sigma(t))}{x(h(t))}
\]

and by (2.4), (2.7) leads to

\[
\limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2},
\]

which contradicts with (2.5). The proof of the theorem is completed.

\[\square\]

**Example 2.6.** Let \( m = 1 \), \( h \in \mathbb{Z} \) and \( T = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\} \), where \( h > 0 \). Then, we have

\[
\sigma(t) = t + h, \quad \mu(t) = h \quad \text{and} \quad x^\Delta(t) = \frac{x(t+h) - x(t)}{h}
\]
for \( t \in T \). Thus, Eq.(1.1) becomes

\[
\frac{x(t+h) - x(t)}{h} + p(t)x(\tau(t)) = 0, \quad t \in \{hk : k \in \mathbb{Z}\}.
\]

Let \( \tau(t) = t - 2 \) and \( h = 2 \). Since \( p(t) \in \{hk : k \in \mathbb{Z}\} \), we assume

\[
p(2t) = 0.18 \quad \text{and} \quad p(2t+2) = 0.27, \quad t = 0, 2, 4, \ldots
\]

When \( T = h\mathbb{Z} \), from (iii) in Theorem 1.79 [2], we have the following.

\[
\int_{a}^{b} f(t) \Delta t = \sum_{k=\frac{a}{h}}^{\frac{b-1}{h}} f(kh)h \quad \text{for} \quad a < b.
\]

So, by using (2.8), we observe that, for \( \tau(t) \), \( p(t) \in \{hk : k \in \mathbb{Z}\} \).

\[
\alpha := \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s = \liminf_{t \to \infty} \sum_{j=\frac{t}{2h}}^{\frac{t-1}{2h}} p(2j)2 = \liminf_{t \to \infty} 2p(t-2) = 0.36 \neq \frac{1}{\epsilon}
\]
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and

\[
\beta := \limsup_{t \to \infty} \int_{h(t)}^{\sigma(t)} p(s) \Delta s = \limsup_{t \to \infty} \sum_{j=-L}^{L} p(2j)2 = \limsup_{t \to \infty} 2[p(t - 2) + p(t)] = 0.9 \neq 1
\]

demonstrates that Theorem 2.3 fails. On the other hand,

\[
\beta = 0.9 \neq 1 - \left(1 - \sqrt{1 - 0.36}\right)^2 = 0.96
\]

demonstrates that the condition (1.8) doesn’t hold. However, since

\[
\beta = 0.9 > 1 - \frac{1 - 0.36 - \sqrt{1 - 2(0.36) - (0.36)^2}}{2} = 0.87391,
\]
every solution oscillates by Theorem 2.5.

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