DEPENDENCE LOGIC IN PREGEOMETRIES AND $\omega$-STABLE THEORIES

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Abstract. We show that dependence and independence concepts in model theory obey the same complete axiom system as dependence and independence concepts in database theory and logic in general.

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1. Introduction

Dependence and independence concepts are ubiquitous in science. For example, the laboratory work of Mengel on heredity of traits among plants is the basis of modern biology. Galileo’s experiments on how the time of descent of a ball depends on the height from which it is dropped and how it is independent from the weight of the ball, are corner stones of physics. Arrow’s Theorem in economics assumes the independence of the social preference from “irrelevant alternatives”. Independence of random variables belongs to the heart of statistics. In game theory dependence is embedded in the concept of a strategy. During the last thirty years dependency theory has emerged as the basis of effective database design. Last but not least, linear...
and algebraic dependency are cornerstone concepts of algebra. Dependence and independence concepts of algebra have their natural analogues in model theory, more exactly in geometric stability theory, arising from pregeometries and non-forking. In this paper we show that all these dependence and independence concepts have a common core that permits a complete axiomatization. Our focus is on pregeometries in algebra and model theory. We refer to [8] for a general introduction to the team semantics approach to dependence and independence, especially from the database (i.e. horizontal) point of view. We refer to [2] for a survey of dependence relations in models of first order theories.

A technical tool introduced in [11] for the study of dependence phenomena is that of a team [20]. Technically speaking a team is a table of data about the values of some given variables. However, the concept of a team appears in various disguises in all fields of science where dependence and independence make sense. Table 1 gives some examples.

In the general case the values appearing in a team can be completely arbitrary. Sometimes the values are binary, sometimes integers or rational numbers, and sometimes values have no numerical content, e.g. if the values are colours. An interesting special case is when all the values come from a field, e.g. the field of rational numbers: In such a case we may meaningfully observe that e.g. in Table 2 we have $z = x + y$. If the values of the table are arbitrary, e.g. colours, such equations are, of course, meaningless. In fact, if $+$ is part of the vocabulary of dependence logic, then the meaning of the logical formula $z = x + y$ in the team of Table 2 as given by the truth definition (see [20]) is exactly the same as the algebraic meaning, row by row.

Actually, we can think of the team of Table 2 in two different ways. We can think of it as a table of three rows of data about $x$, $y$ and $z$. We can also think of Table 2 as a set of three vectors $x$, $y$ and $z$ in a 3-dimensional vector space over the rational field $\mathbb{Q}$. Suppose the basis of the vector space is $\{e_1, e_2, e_3\}$. Then we can

| $x$  | $y$  | $z$  |
|------|------|------|
| 0    | 1    | 1    |
| 0.5  | -2   | -1.5 |
| 0    | -4   | -4   |

Table 2. A team
think that
\[ x = 0.5e_2 \]
\[ y = e_1 - 2e_2 - 4e_3 \]
\[ z = e_1 - 1.5e_2 - 4e_1. \]

In a sense, \( x, y \) and \( z \) are elements to which each row contributes something. As it turns out, linear dependence and independence of \( x, y \) and \( z \) in the (vertical) sense of vector spaces are closely related to dependence and independence concepts that are associated with Table 2 in the (horizontal) sense of database theory. In Table 2 \( x \) and \( y \) functionally determine \( z \), but if we operate in a vector space we can say more, namely that \( x \) and \( y \) linearly determine \( z \), as \( z = x + y \).

In this paper we make the leap of moving from the horizontal thinking to the vertical thinking entirely, and we in fact consider \( x, y \) and \( z \) as elements of a field, or more generally, elements of a model of a first order theory. The team disappears but we can still think of \( x, y \) and \( z \) as having some kind of coordinatization with respect to a basis, this coordinatization being an analogue of what is called the team.

We consider various logics with new atomic formulas. The new atomic formulas correspond to different concepts of dependence and independence. In this paper we consider these logics on the atomic level only. This may seem strange, as if we had logics with not logical operations. The point is that the atomic formulas alone are so powerful that the questions of their axiomatisation is non-trivial. Similar logics with actual logical operations \( \land, \lor, \neg, \exists, \forall \) were introduced in [20] and their axiomatization was investigated in [13]. See also [12].

2. Abstract Systems

We commence with a completely abstract setup. Here we assume no background geometry and the meaning of dependence and independence are purely combinatorial. This is the way these concepts are treated in database theory.

Atomic Dependence Logic (ADL) is defined as follows. The language of this logic is made only of dependence atoms. That is, if \( \vec{x} \) and \( \vec{y} \) are finite sequences of variables, with \( \vec{y} \neq \emptyset \) if \( \vec{x} \neq \emptyset \), then the formula \( =((\vec{x}, \vec{y})) \) is a formula of the language of ADL. The semantics is defined as in [20]. We denote by \( \text{Var} \) the set of first-order variables. Let \( M \) be a first order structure. Let \( X = \{s_i\}_{i \in I} \) with \( s_i : \text{dom}(X) \to M \) and \( \vec{x}'\vec{y} \subseteq \text{dom}(X) \subseteq \text{Var} \). We say that \( M \) satisfies \( =((\vec{x}, \vec{y})) \) under \( X \), in symbols \( M \models_X =((\vec{x}, \vec{y})) \), if
\[
\forall s, s' \in X \ (s(\vec{x}) = s'(\vec{x}) \to s(\vec{y}) = s'(\vec{y})).
\]

Let \( \Sigma \) be a set of atoms and let \( X \) be such that the set of variables occurring in \( \Sigma \) is included in \( \text{dom}(X) \). We say that \( M \) satisfies \( \Sigma \) under \( X \), in symbols \( M \models_X \Sigma \), if \( M \) satisfies every atom in \( \Sigma \) under \( X \). We say that \( =((\vec{x}, \vec{y})) \) is a logical consequence of \( \Sigma \), in symbols \( \Sigma \models =((\vec{x}, \vec{y})) \), if for every \( M \) and \( X \) such that the set of variables occurring in \( \Sigma \cup \{ =((\vec{x}, \vec{y})) \} \) is included in \( \text{dom}(X) \) we have:
\[
\text{If } M \models_X \Sigma \text{ then } M \models_X =((\vec{x}, \vec{y})).
\]

The deductive system of ADL consists of the following rules:
\[
(a_1.) =((\vec{x}, \vec{x}) \text{ [as a degenerate case of this rule we admit } =((\emptyset, \emptyset))];
(b_1.) \text{ If } =((\vec{x}, \vec{y}), \vec{u} \subseteq \vec{y} \text{ and } \vec{x} \subseteq \vec{z}, \text{ then } =((\vec{z}, \vec{u}));
(c_1.) \text{ If } =((\vec{x}, \vec{y}) \text{ and } =((\vec{y}, \vec{z}), \text{ then } =((\vec{x}, \vec{z}));
\]
(d₁.) If \( \equiv (\vec{x}, \vec{y}) \) and \( \equiv (\vec{x}, \vec{z}) \), then \( \equiv (\vec{x}, \vec{y} \vec{z}) \).

A deduction from a set of atoms \( \Sigma \) is a sequence of atoms \( (\phi_0, ..., \phi_{n-1}) \) such that each \( \phi_i \) is either an element of \( \Sigma \), an instance of axiom \( (a_1.) \), or follows from one or more formulas of \( \Sigma \cup \{\phi_0, ..., \phi_{i-1}\} \) by one of the rules presented above. We say that \( \phi \) is provable from \( \Sigma \), in symbols \( \Sigma \vdash \phi \), if there is a deduction \( (\phi_0, ..., \phi_{n-1}) \) from \( \Sigma \) with \( \phi = \phi_{n-1} \).

**Theorem 2.1** ([1]). Let \( \Sigma \) be a set of atoms. Then

\[ \Sigma \models \equiv (\vec{x}, \vec{y}) \text{ if and only if } \Sigma \vdash \equiv (\vec{x}, \vec{y}). \]

**Atomic Absolute Independence Logic** (AAIndL) is defined as follows. The language of this logic is made only of absolute independence atoms. That is, let \( \vec{x} \) be a finite sequence of variables, then \( \bot (\vec{x}) \) is a formula of the language of AAIndL. The intuition behind the atom \( \bot (\vec{x}) \) is that \( \vec{x} \) consists of independent elements. That is, each element of \( \vec{x} \) is independent of all the other elements of \( \vec{x} \). In particular, we ask that each element of \( \vec{x} \) does not depend on any other element, i.e. that it is not constant, for in our terminology a constant would be determined (in a trivial way) by any of the other variables. Let \( \mathcal{M} \) be a first order structure. Let \( X = \{s_i\}_{i \in I} \) with \( s_i : \text{dom}(X) \to M \) and \( \vec{x} \subseteq \text{dom}(X) \subseteq \text{Var} \). If \( x \in \vec{x} \), we denote by \( \vec{x} - x \) any enumeration of the set \( \{x' \in \vec{x} \mid \exists s \in X(s(x) \neq s(x'))\} \). We say that \( \mathcal{M} \) satisfies \( \bot (\vec{x}) \) under \( X \), in symbols \( \mathcal{M} \models_X \bot (\vec{x}) \), if for all \( x \in \vec{x} \)

\[ \forall s, s' \in X \exists s'' \in X (s''(x) = s(x) \land s''(\vec{x} - x) = s'(\vec{x} - x)) \]

and

\[ \exists s, s' \in X (s(x) \neq s'(x)). \]

Let \( \Sigma \) be a set of atoms and let \( X = \{s_i\}_{i \in I} \) be such that the set of variables occurring in \( \Sigma \) is included in \( \text{dom}(X) \). We say that \( \mathcal{M} \) satisfies \( \Sigma \) under \( X \), in symbols \( \mathcal{M} \models_X \Sigma \), if \( \mathcal{M} \) satisfies every atom in \( \Sigma \) under \( X \). We say that \( \bot (\vec{x}) \) is a logical consequence of \( \Sigma \), in symbols \( \Sigma \models \bot (\vec{x}) \), if for every \( \mathcal{M} \) and \( X \) such that the set of variables occurring in \( \Sigma \cup \{\bot (\vec{x})\} \) is included in \( \text{dom}(X) \) we have that

\[ \text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X \bot (\vec{x}). \]

The deductive system of AAIndL consists of the following rules:

(a₂.) \( \bot (\emptyset) \);

(b₂.) If \( \bot (\vec{x} \vec{y}) \), then \( \bot (\vec{x}) \);

(c₂.) If \( \bot (\vec{x}) \), then \( \bot (\pi \vec{x}) \) [where \( \pi \) is a permutation of \( \vec{x} \)].

The notions of deduction and provability are defined in analogy with ADL.

**Theorem 2.2.** Let \( \Sigma \) be a set of atoms of AAIndL. Then

\[ \Sigma \models \bot (\vec{x}) \text{ if and only if } \Sigma \vdash \bot (\vec{x}). \]

**Proof.** The direction \( (\Rightarrow) \) is obvious. For the direction \( (\Leftarrow) \), suppose \( \Sigma \not\models \bot (\vec{x}) \). Notice that, because of rule \( (a_2.) \), if this is the case, then \( \vec{x} \not\models \emptyset \). We can assume that \( \vec{x} \) is injective. This is without loss of generality because clearly \( \mathcal{M} \models_X \bot (\vec{x}) \) if and only if \( \mathcal{M} \models_X \bot (\pi \vec{x}) \), where \( \pi : \text{Var}^{<\omega} \to \text{Var}^{<\omega} \) is the function that eliminates repetitions in finite sequences of variables. Let then \( \vec{x} = (x_{j_0}, ..., x_{j_n}) \neq \emptyset \) be injective. Let \( \mathcal{M} \) be the structure in the empty signature with \( \{0, 1\} \) as its domain. Define \( X = \{s_t \mid t \in 2^\omega\} \) to be the set of assignments which give all the possible
combinations of 0s and 1s to all the variables but $x_{j_0}$ and which at $x_{j_0}$ are such that

$$s_t(x_{j_0}) = 0$$  if $\vec{x} = \{x_{j_0}\}$

$$s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}))$$  if $\vec{x} \neq \{x_{j_0}\}$

for all $t \in 2^\omega$, where $p : M^{< \omega} \to M$ is the function which assigns 1 to the sequences with an odd numbers of 1s and 0 to the sequences with an even numbers of 1s. We claim that $\mathcal{M} \not \models_X \bot(\vec{x})$. There are two cases.

**Case 1.** For all $t \in 2^\omega$, $s_t(x_{j_0}) = 0$. In this case there is no $s, s' \in X$ such that $s(x) \neq s'(x)$.

**Case 2.** For all $t \in 2^\omega$, $s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}}))$. Notice that if this is the case, then $n \geq 2$. Let $t, d \in 2^\omega$ be such that $s_d(x_{j_1}) = 0$, $s_t(x_{j_1}) = 1$ and $s_i(x_{j_1}) = s_d(x_{j_1})$ for every $i \in \{2, ..., n-1\}$. Clearly

$$p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})) \neq p(s_t(x_{j_1}), ..., s_d(x_{j_{n-1}})).$$

Suppose that $\mathcal{M} \models_X \bot(\vec{x})$. Then there exists $f \in 2^\omega$ such that

$$s_f(x_{j_0}) = s_t(x_{j_0}) \land s_d(\vec{x} \setminus x_{j_0}) = s_d(\vec{x} \setminus x_{j_0}).$$

Notice that under this $X$ we have $\vec{x} \setminus x_{j_0} = \{x_{j_1}, ..., x_{j_{n-1}}\}$, thus

$$s_f(\vec{x} \setminus x_{j_0}) = (s_f(x_{j_1}), ..., s_f(x_{j_{n-1}}))$$

and

$$s_d(\vec{x} \setminus x_{j_0}) = (s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})).$$

Hence

$$p(s_d(x_{j_1}), ..., s_d(x_{j_{n-1}})) = p(s_f(x_{j_1}), ..., s_f(x_{j_{n-1}})) = s_f(x_{j_0}) = s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})),$$

which is a contradiction. Let now $\bot(\vec{v}) \in \Sigma$, we want to show that $\mathcal{M} \models_X \bot(\vec{v})$. As before, we assume, without loss of generality, that $\vec{v}$ is injective. Notice that if $\vec{v} = \emptyset$, then $\mathcal{M} \models_X \bot(\vec{v})$. Thus let $\vec{v} = (v_{h_0}, ..., v_{h_{c-1}}) \neq \emptyset$. We make a case distinction on $\vec{v}$.

**Case 1.** $x_{j_0} \notin \vec{v}$. Let $v \in \vec{v}$. Because of the assumption, $v \neq x_{j_0}$ and $x_{j_0} \notin \vec{v} \setminus v$. Thus for every $t, d \in 2^\omega$ clearly there is $f \in 2^\omega$ such that

$$s_f(v) = s_t(v) \land s_d(\vec{v} \setminus v) = s_d(\vec{x} \setminus x_{j_0}) \land s_d(v).$$

**Case 2.** $x_{j_0} \in \vec{v}$.

**Subcase 2.1.** $\vec{x} \setminus \vec{v} \neq \emptyset$. Notice that $\vec{x} \neq \{x_{j_0}\}$ because if not then $\vec{x} \setminus \vec{v} = \{x_{j_0}\}$ and so $x_{j_0} \notin \vec{v}$. Hence for every $t \in 2^\omega$ we have that

$$s_t(x_{j_0}) = p(s_t(x_{j_1}), ..., s_t(x_{j_{n-1}})).$$

Suppose, without loss of generality, that $\vec{v} = (x_{j_0}, v_{h_1}, ..., v_{h_{c-1}})$ and let $\vec{x}' = \vec{x} \cap \vec{v} = (u_{p_0}, ..., u_{p_{m-1}})$ and $z \in \vec{x} \setminus \vec{v}$. Let $v \in \vec{v}$.

**Subcase 2.1.1.** $v \neq x_{j_0}$. Let $k \in \{1, ..., c-1\}$ and $v = v_{h_k}$. Let $t, d \in 2^\omega$ and let $f \in 2^\omega$ be such that:

i) $s_f(v_{h_k}) = s_d(v_{h_k})$;

ii) $s_f(v_{h_i}) = s_d(v_{h_i})$ for every $i \in \{1, ..., k-1, k+1, ..., c-1\}$;

iii) $s_f(u) = 0$ for every $u \in \vec{x} \setminus \vec{x}' z$;
iv) \( s_f(z) = 0 \), if \( p(s_f(u_{p_0}), ..., s_f(u_{p_{m-1}})) = s_d(x_{j_0}) \) and \( s_f(z) = 1 \) otherwise.

Then \( f \) is such that

\[
s_f(v_{h_k}) = s_t(v_{h_k})
\]

and

\[
(s_f(x_{j_0}), s_f(v_{h_1}), ..., s_f(v_{h_{k-1}}), s_f(v_{h_{k+1}}), ..., s_f(v_{h_{n-1}})) =
(s_d(x_{j_0}), s_d(v_{h_1}), ..., s_d(v_{h_{k-1}}), s_d(v_{h_{k+1}}), ..., s_d(v_{h_{n-1}})).
\]

**Subcase 2.1.2.** \( v = x_{j_0} \).

Let \( t, d \in 2^\omega \) and let \( f \in 2^\omega \) be such that:

i) \( s_f(v_{h_i}) = s_d(v_{h_i}) \) for every \( i \in \{1, ..., c-1\} \); 

ii) \( s_f(u) = 0 \) for every \( u \in \vec{x} - \vec{x}' \); 

iii) \( s_f(z) = 0 \), if \( p(s_f(u_{p_0}), ..., s_f(u_{p_{m-1}})) = s_t(x_{j_0}) \) and \( s_f(z) = 1 \) otherwise.

Then \( f \) is such that

\[
s_f(x_{j_0}) = s_t(x_{j_0})
\]

and

\[
(s_f(v_{h_1}), ..., s_f(v_{h_{n-1}})) = (s_d(v_{h_1}), ..., s_d(v_{h_{n-1}})).
\]

**Subcase 2.2.** \( \vec{x} \subseteq \vec{v} \). This case is not possible. Suppose indeed it is, then by rule \((a_2,)\) we can assume that \( \vec{v} = \vec{x} \vec{v}' \) with \( \vec{v}' \subseteq \text{Var} - \vec{x} \).

Thus by rule \((b_2,)\) we have that \( \Sigma \vdash \bot(\vec{x}) \), contrary to our assumption.

**Atomic Independence Logic (AIndL)** is defined as follows. The language of this logic is made only of independence atoms. That is, let \( \vec{x} \) and \( \vec{y} \) be finite sequences of variables, then the formula \( \vec{x} \perp \vec{y} \) is a formula of the language of AIndL. The intuitive meaning of the atom \( \vec{x} \perp \vec{y} \) in team semantics is that the values of the variables in \( \vec{x} \) give no information about the values of the variables in \( \vec{y} \) and vice versa. The semantics is defined as in [7]. Let \( \mathcal{M} \) be a first order structure. Let \( X = \{s_i\}_{i \in I} \) with \( s_i : \text{dom}(X) \rightarrow M \) and \( \vec{x} \vec{y} \subseteq \text{dom}(X) \subseteq \text{Var} \). We say that \( \mathcal{M} \) satisfies \( \vec{x} \perp \vec{y} \) under \( X \), in symbols \( \mathcal{M} \models_X \vec{x} \perp \vec{y} \), if

\[
\forall s, s' \in X \exists s'' \in X (s''(\vec{x}) = s(\vec{x}) \land s''(\vec{y}) = s'(\vec{y})).
\]

Let \( \Sigma \) be a set of atoms and let \( X \) be such that the set of variables occurring in \( \Sigma \) is included in \( \text{dom}(X) \). We say that \( \mathcal{M} \) satisfies \( \Sigma \) under \( X \), in symbols \( \mathcal{M} \models_X \Sigma \), if \( \mathcal{M} \) satisfies every atom in \( \Sigma \) under \( X \). We say that \( \vec{x} \perp \vec{y} \) is a logical consequence of \( \Sigma \), in symbols \( \Sigma \models \vec{x} \perp \vec{y} \), if for every \( \mathcal{M} \) and \( X \) such that the set of variables occurring in \( \Sigma \cup \{\vec{x} \perp \vec{y}\} \) is included in \( \text{dom}(X) \) we have that

\[
\text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X \vec{x} \perp \vec{y}.
\]

The deductive system of AIndL consists of the following rules:

- \((a_3,)\) \( \vec{x} \perp \emptyset \);
- \((b_3,)\) If \( \vec{x} \perp \vec{y} \), then \( \vec{y} \perp \vec{x} \);
- \((c_3,)\) If \( \vec{x} \perp \vec{y} \vec{z} \), then \( \vec{x} \perp \vec{y} \);
- \((d_3,)\) If \( \vec{x} \perp \vec{y} \) and \( \vec{x} \vec{y} \vec{z} \), then \( \vec{x} \perp \vec{y} \vec{z} \);
- \((e_3,)\) If \( x \perp x \), then \( x \perp \vec{y} \) [for arbitrary \( \vec{y} \)];
- \((f_3,)\) If \( \vec{x} \perp \vec{y} \), then \( \pi \vec{x} \perp \sigma \vec{y} \) [where \( \pi \) and \( \sigma \) are permutations of \( \vec{x} \) and \( \vec{y} \) respectively].

The notions of deduction and provability are defined in analogy with ADL.
Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \bar{x} \perp \bar{y}$$

if and only if $\Sigma \vdash \bar{x} \perp \bar{y}$.

Atomic Absolute Conditional Independence Logic (AACIndL) is defined as follows. The language of this logic is made only of absolute conditional independence atoms. That is, let $\bar{x}$ and $\bar{z}$ be finite sequences of variables, then $\perp_{\bar{x}}(\bar{z})$ is a formula of the language of AACIndL. Let $\mathcal{M}$ be a first order structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \text{dom}(X) \to M$ and $\bar{x} \bar{z} \subseteq \text{dom}(X) \subseteq \text{Var}$. Let $x \in \bar{x}$, we denote by $\bar{x} - x$ any enumeration of the set $\{x' \in \bar{x} \mid \mathcal{M} \not\models x' = x\}$. We say that $\mathcal{M}$ satisfies $\perp_{\bar{x}}(\bar{z})$ under $X$, in symbols $\mathcal{M} \models_X \perp_{\bar{x}}(\bar{z})$, if for all $x \in \bar{x}$

$$\forall s, s' \in X (s(\bar{z}) = s'(\bar{z}) \to \exists s'' \in X (s''(\bar{z}) = s(\bar{z}) \land (s''(x) = s(x) \land s''(\bar{x} - x) = s'(\bar{x} - x)))$$

and

$$\exists s, s' \in X (s(\bar{z}) = s'(\bar{z}) \land s(x) \neq s'(x)).$$

Let $\Sigma$ be a set of atoms and let $X = \{s_i\}_{i \in I}$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(X)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $X$, in symbols $\mathcal{M} \models_X \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $X$. We say that $\perp_{\bar{x}}(\bar{z})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \perp_{\bar{x}}(\bar{z})$, if for every $\mathcal{M}$ and $X$ such that the set of variables occurring in $\Sigma \cup \{\perp_{\bar{x}}(\bar{z})\}$ is included in $\text{dom}(X)$ we have that

$$\text{if } \mathcal{M} \models_X \Sigma \text{ then } \mathcal{M} \models_X \perp_{\bar{x}}(\bar{z}).$$

The deductive system of AACIndL consists of the following rules:

$(a_4.)$ $\perp_{\bar{x}}(\emptyset)$;
$(b_4.)$ If $\perp_{\bar{x}}(\bar{y})$, then $\perp_{\bar{x}}(\bar{z})$;
$(c_4.)$ If $\perp_{\bar{x}}(\bar{x})$, then $\perp_{\bar{x}}(\bar{z})$;
$(d_4.)$ If $\perp_{\bar{x}}(\bar{z})$ and $\perp_{\bar{x}}(\bar{u})$, then $\perp_{\bar{x}}(\bar{z})$;
$(e_4.)$ If $\perp_{\bar{x}}(\bar{x})$, then $\perp_{\bar{x}}(\bar{y})$ [for arbitrary $\bar{z}$ and $\bar{y}$];
$(f_4.)$ If $\perp_{\bar{z}}(\bar{x})$, then $\perp_{\bar{z}}(\sigma \bar{x})$ [where $\pi$ and $\sigma$ are permutations of $\bar{z}$ and $\bar{x}$ respectively].

The notions of deduction and provability are defined in analogy with ADL.

**Theorem 2.4.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \vdash \perp_{\bar{x}}(\bar{z}) \Rightarrow \Sigma \models \perp_{\bar{x}}(\bar{z}).$$

It is at present not known whether a completeness theorem holds for this system relative to finite set of axioms, but in the light of what is known about conditional independence in database theory it seems that this is not the case. Although a finite axiomatization is unlikely, it is still possible to find a recursively enumerable axiomatization by reducing the question, by means of new predicate symbols, to first order logic. In first order logic one can use the Goëdel Completeness Theorem to conclude that there must be some effectively given complete axiom system.

Atomic Conditional Independence Logic (ACIndL) is defined as follows. The language of this logic is made only of conditional independence atoms. That is, let $\bar{x}$, $\bar{y}$ and $\bar{z}$ be finite sequences of variables, then the formula $\perp_{\bar{x}} \perp_{\bar{y}} \bar{z}$ is a formula of the language of ACIndL. The semantics is defined as in [7]. Let $\mathcal{M}$ be a first order
structure. Let \( X = \{ s_i \}_{i \in I} \) with \( s_i : \text{dom}(X) \to M \) and \( \vec{x} \vec{y} \vec{z} \subseteq \text{dom}(X) \subseteq \text{Var.} \) We say that \( M \) satisfies \( \vec{x} \perp_{\vec{x}} \vec{y} \) under \( X \), in symbols \( M \models_X \vec{x} \perp_{\vec{x}} \vec{y} \), if
\[
\forall s, s' \in X (s(\vec{z}) = s'(\vec{z}) \Rightarrow \exists s'' \in X (s''(\vec{z}) = s(\vec{x}) \land s''(\vec{y}) = s'(\vec{y}))).
\]

Let \( \Sigma \) be a set of atoms and let \( X \) be such that the set of variables occurring in \( \Sigma \) is included in \( \text{dom}(X) \). We say that \( M \) satisfies \( \Sigma \) under \( X \), in symbols \( M \models_X \Sigma \), if \( M \) satisfies every atom in \( \Sigma \) under \( X \). We say that \( \vec{x} \perp_{\vec{x}} \vec{y} \) is a logical consequence of \( \Sigma \), in symbols \( \Sigma \models \vec{x} \perp_{\vec{x}} \vec{y} \), if for every \( M \) and \( X \) such that the set of variables occurring in \( \Sigma \cup \{ \vec{x} \perp_{\vec{x}} \vec{y} \} \) is included in \( \text{dom}(X) \) we have that
\[
\text{if } M \models_X \Sigma \text{ then } M \models_X \vec{x} \perp_{\vec{x}} \vec{y}.
\]

The deductive system of ACIndL consists of the following rules:

\( (a_5) \) \( \vec{x} \perp_{\vec{x}} \vec{y} \);  
\( (b_5) \) If \( \vec{x} \perp_{\vec{x}} \vec{y} \), then \( \vec{y} \perp_{\vec{y}} \vec{x} \);  
\( (c_5) \) If \( \vec{x} \vec{z} \perp_{\vec{x}} \vec{y} \), then \( \vec{x} \perp_{\vec{x}} \vec{y} \);  
\( (d_5) \) If \( \vec{x} \perp_{\vec{x}} \vec{y} \), then \( \vec{x} \vec{z} \perp_{\vec{x}} \vec{y} \vec{z} \);  
\( (e_5) \) If \( \vec{x} \perp_{\vec{x}} \vec{y} \) and \( \vec{u} \perp_{\vec{u}} \vec{x} \vec{z} \vec{y} \), then \( \vec{u} \perp_{\vec{u}} \vec{y} \vec{z} \vec{u} \);  
\( (f_5) \) If \( \vec{y} \perp_{\vec{y}} \vec{z} \vec{u} \) and \( \vec{z} \vec{x} \perp_{\vec{z}} \vec{y} \vec{u} \), then \( \vec{x} \perp_{\vec{x}} \vec{u} \);  
\( (g_5) \) If \( \vec{x} \perp_{\vec{x}} \vec{y} \) and \( \vec{x} \vec{y} \perp_{\vec{x}} \vec{u} \), then \( \vec{x} \perp_{\vec{x}} \vec{y} \vec{u} \);  
\( (h_5) \) If \( \vec{x} \perp_{\vec{x}} \vec{y} \), then \( \pi \vec{x} \perp_{\pi \vec{x}} \sigma \vec{y} \) [where \( \pi, \tau \) and \( \sigma \) are permutations of \( \vec{x}, \vec{z} \), and \( \vec{y} \) respectively].

The notions of deduction and provability are defined in analogy with ADL.

**Theorem 2.5.** Let \( \Sigma \) be a set of atoms, then
\[
\Sigma \vdash \vec{x} \perp_{\vec{x}} \vec{y} \Rightarrow \Sigma \models \vec{x} \perp_{\vec{x}} \vec{y}.
\]

Parker and Parsaye-Ghomi [18] proved that it is not possible to find a finite complete axiomatization for the conditional independence atoms. Furthermore, in [10] and [9] Hermann proved that the consequence relation between these atoms is undecidable. It is, a priori, obvious (as pointed out earlier) that there is some recursive axiomatization for the conditional independence atoms, because we can reduce the whole question to first order logic with extra predicates and then appeal to the Completeness Theorem of first order logic. In [10] Naumov and Nicholls developed an explicit recursively axiomatization of them.

### 3. Dependence and Independence in Pregeometries

#### 3.1. Closure Operator Atomic Dependence Logic

**Definition 3.1.** Let \( M \) be a set and \( \text{cl} : \mathcal{P}(M) \to \mathcal{P}(M) \) an operator on the power set of \( M \). We say that \( \text{cl} \) is a closure operator and that \( (M, \text{cl}) \) is a closure system if for every \( A, B \subseteq M \) the following conditions are satisfied:

i) \( A \subseteq \text{cl}(A) \);  
ii) If \( A \subseteq B \) then \( \text{cl}(A) \subseteq \text{cl}(B) \);  
iii) \( \text{cl}(A) = \text{cl}(\text{cl}(A)) \).

**Proposition 3.2.** Let \( (M, \text{cl}) \) be a closure system and \( A, B \subseteq M \). If \( A \subseteq \text{cl}(B) \) then \( \text{cl}(A) \subseteq \text{cl}(B) \).
Let $\Sigma$ be a set of atoms, then

**Theorem 3.6.**

Let $\Sigma$ be a set of atoms, then

$\Sigma \models = (\vec{x}, \vec{y})$ if and only if $\Sigma \vdash = (\vec{x}, \vec{y})$.

**Proof.** Suppose that $A \subseteq \text{cl}(B)$, then by property ii) and iii) of the definition $\text{cl}(A) \subseteq \text{cl}(\text{cl}(B)) = \text{cl}(B)$.

**Example 3.3.** Let $(A, F)$ be an algebra of type $\Omega$, and for every $B \subseteq A$ let $(|B|, F)$ be the subalgebra of $(A, F)$ generated by $B$. Then $[\cdot] : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator.

**Example 3.4.** Let $(X, \tau)$ be a topological space, and for every $Y \subseteq X$ let $\overline{Y}$ be the smallest closed subset of $X$ that contains $Y$. Then $\overline{\cdot} : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator.

**Example 3.5.** Let $\mathcal{M}$ be a first-order structure in the signature $\mathcal{L}$ and $A \subseteq \mathcal{M}$. We say that $b$ is algebraic over $A$ if there is an $\mathcal{L}$-formula $\phi(v, \vec{a})$ and $\vec{a} \in A$ such that $\mathcal{M} \models \phi(b, \vec{a})$ and $\phi(M, \vec{a}) = \{m \in M \mid M \models \phi(m, \vec{a})\}$ is finite. Let $\text{acl}_M(A) = \{m \in M \mid m \text{ is algebraic over } A\}$. Then $\text{acl}_M : \mathcal{P}(M) \to \mathcal{P}(M)$ is a closure operator.

Let $\mathcal{L}$ be a signature (possibly higher-order) and $\mathcal{K}$ be a class of $\mathcal{L}$-structures. We denote by $U(K)$ the class of domains of structures in $\mathcal{K}$ and by $V$ the set theoretical universe. A second-order $n$-ary operator op on $\mathcal{K}$ is a class function $\text{op} : U(\mathcal{K}) \to V$ such that for every $\mathcal{M} \in \mathcal{K}$,

$$\text{op}(M) : \mathcal{P}^n(M) \to \mathcal{P}^n(M).$$

Given such an operator, for every $\mathcal{M} \in \mathcal{K}$ we can consider the second-order structure $(\mathcal{M}, \text{op}(\mathcal{M}))$. For ease of notation we denote the structure $(\mathcal{M}, \text{op}(\mathcal{M}))$ simply as $(\mathcal{M}, \text{op}).$

**Closure Operator Atomic Dependence Logic (COADL)** is defined as follows. The syntax and deductive system of this logic are the same as those of ADL. Let $\mathcal{L}$ be a signature (possibly higher-order) and $\mathcal{K}$ a class of $\mathcal{L}$-structures. Let cl be a unary second-order closure operator on $\mathcal{K}$, such that there exists a structure $\mathcal{M} \in \mathcal{K}$ such that the closure system $(\mathcal{M}, \text{cl})$ has the following two properties:

- (CL1) $\text{cl}(\emptyset) \neq \emptyset$;
- (CL2) $\text{cl}(\emptyset) \neq M$.

Let $\mathcal{M} \in \mathcal{K}$ and $s : \text{dom}(s) \to M$ with $\vec{x} \vec{y} \subseteq \text{dom}(s) \subseteq \text{Var}$. We say that $\mathcal{M}$ satisfies $= (\vec{x}, \vec{y})$ under $s$, in symbols $\mathcal{M} \models_s = (\vec{x}, \vec{y})$, if for every $y \in \vec{y}$

$$s(y) \in \text{cl}(\{s(x) \mid x \in \vec{x}\}).$$

Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$. We say that $= (\vec{x}, \vec{y})$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models = (\vec{x}, \vec{y})$, if for every $\mathcal{M} \in \mathcal{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{= (\vec{x}, \vec{y})\}$ is included in $\text{dom}(s)$ we have that

$$\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s = (\vec{x}, \vec{y}).$$

**Theorem 3.6.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models = (\vec{x}, \vec{y}) \text{ if and only if } \Sigma \vdash = (\vec{x}, \vec{y}).$$

**Proof.** For the direction ($\Rightarrow$), Let $\mathcal{M} \in \mathcal{K}$ and $s$ an appropriate assignment. We only prove the soundness of rules $(a_1)$, $(b_1)$ and $(c_1)$.
Let \( y \) by Definition 3.1 ii) we have that \( s = (\vec{x}, \vec{y}) \) by rule (\( b \)). Let now \( s = (\vec{x}, \vec{y}) \) because for \( \vec{x} \) and \( \vec{y} \).

(c1.) Suppose that \( M \models s = (\vec{x}, \vec{y}) \) and \( M \models s = (\vec{y}, \vec{z}) \). Let \( z \in \vec{x} \), then \( s(z) \in cl(\{s(y) \mid y \in \vec{y}\}) \) because \( M \models s = (\vec{y}, \vec{z}) \). Furthermore we have that \( s(y) \in cl(\{s(x) \mid x \in \vec{x}\}) \) because \( M \models s = (\vec{x}, \vec{y}) \). Thus, by Proposition 3.2 we have that \( cl(\{s(y) \mid y \in \vec{y}\}) \subset cl(\{s(x) \mid x \in \vec{x}\}) \) and hence that \( s(y) \in cl(\{s(x) \mid x \in \vec{x}\}) \).

For the direction (\( \Rightarrow \)), suppose \( \Sigma \not\models (\vec{x}, \vec{y}) \). Let \( V = \{z \in \text{Var} \mid \Sigma \models (\vec{x}, z)\} \) and \( W = \text{Var} - V \). First notice that \( \vec{y} \not\models \emptyset \), indeed if not so, by the syntactic constraints that we put on the system, we have that \( \vec{x}, \vec{y} = \emptyset \) and so by the admitted degenerate case of rule (a1.) we have that \( \Sigma \models (\vec{x}, x) \). Furthermore \( \vec{y} \cap W = \emptyset \), indeed if \( \vec{y} \cap W = \emptyset \) then for every \( y \in \vec{y} \) we have that \( \Sigma \models (\vec{x}, y) \) and so by rule (d1.) we have that \( \Sigma \models (\vec{x}, \vec{y}) \).

By assumption there is \( M \in K \) with properties (CL1) and (CL2), so there are \( a, b \in M \) with \( a \in cl(\emptyset) \) and \( b \notin cl(\emptyset) = cl(\{a\}) \). Let \( s \) be the following assignment:

\[
 s(v) = \begin{cases} 
 a & \text{if } v \in V \\
 b & \text{if } v \in W. 
\end{cases}
\]

We claim that \( M \not\models s = (\vec{x}, \vec{y}) \). In accordance to the semantic we then have to show that there is \( y \in \vec{y} \) such that \( s(y) \notin cl(\{s(x) \mid x \in \vec{x}\}) \). Let \( y \in \vec{y} \cap W \), then

\[
 s(y) = b \notin cl(\{a\}) = cl(\{s(x) \mid x \in \vec{x}\})
\]

because for \( x \in \vec{x} \) we have that \( \Sigma \models (\vec{x}, x) \). Indeed by rule (\( a_1 \)) \( \models (\vec{x}, \vec{x}) \) and so by rule (\( b_1 \)) \( \models (\vec{x}, x) \). Notice that in the case \( \vec{x} = \emptyset \), we have that

\[
 s(y) = b \notin cl(\{a\}) \supseteq cl(\emptyset) = cl(\{s(x) \mid x \in \vec{x}\}).
\]

Let now \( (\vec{x}, \vec{y}) \in \Sigma \), we want to show that \( M \models s = (\vec{x}, \vec{y}) \). If \( \vec{y} = \emptyset \) then also \( \vec{x} = \emptyset \) and so trivially \( M \models s = (\vec{x}, \vec{y}) \). Noticed this, for the rest of the proof we assume \( \vec{y} \not\models \emptyset \).

**Case 1.** \( \vec{x} = \emptyset \). Suppose that \( M \not\models s = (\emptyset, \vec{y}) \), then there exists \( y' \in \vec{y} \) such that \( s(y') = b \), so \( \Sigma \models (\vec{x}, y') \). Notice though that \( \Sigma \models (\emptyset, \vec{y}) \), and by rule (\( b_1 \)) \( \Sigma \models (\emptyset, y') \) and hence again by rule (\( b_1 \)) \( \Sigma \models (\vec{x}, y') \).

**Case 2.** \( \vec{x} \neq \emptyset \) and \( \vec{x} \subseteq V \). If this is the case, then

\[
\forall x' \in \vec{x} \Sigma \models (\vec{x}, x') \implies \Sigma \models (\vec{x}, \vec{y}) \ [\text{by rule (d1.)}]
\]

\[
\Sigma \models (\vec{x}, \vec{y}) \ [\text{by rule (c1.)}]
\]

\[
\forall y' \in \vec{y} \Sigma \models (\vec{x}, y') \ [\text{by rule (b1.)}]
\]

\[
\vec{y} \subseteq V.
\]

If \( \vec{x} \subseteq V \) then for every \( x' \in \vec{x} \) we have that \( s(x') = a \) so

\[
cl(\{s(x') \mid x' \in \vec{x}\}) = cl(\{a\}).
\]

Let \( y' \in \vec{y} \), then we have that \( s(y') = a \) and clearly \( a \in cl(\{a\}) \). Hence \( \Sigma \models s = (\vec{x}, \vec{y}) \).

**Case 3.** \( \vec{x} \cap W \neq \emptyset \). If this is the case, then there exists \( w \in \vec{x} \) such that \( \Sigma \models (\vec{x}, w) \), so we have \( w \in \vec{x} \) such that \( s(w) = b \) and hence \( cl(\{s(x') \mid x' \in \vec{x}\}) \supseteq cl(\{b\}) \). Let now \( y' \in \vec{y} \).
Subcase 3.2. $y' \in W$. In this case we have that $s(y') = b$. Clearly
\[ b \in \cl(\{b\}) \subseteq \cl(\{s(x') \mid x' \in \vec{x}'\}). \]
Hence $\mathcal{M} \models_s = (\vec{x}', \vec{y}')$.

Subcase 3.2. $y' \in V$. In this case we have that $s(y') = a$. By choice of $a$
\[ a \in \cl(\{b\}) \subseteq \cl(\{s(x') \mid x' \in \vec{x}'\}). \]
Hence $\mathcal{M} \models_s = (\vec{x}', \vec{y}')$.

3.2. Pregeometries

Noticing various similarities in which the notion of dependence occurs in linear
algebra, field theory and graph theory, in the mid 1930’s, Hassler Whitney [22]
and Bartel Leendert van der Waerden [21] independently identified few conditions
capable to subsume all these cases of dependence. This led to the definition of the
notion of abstract dependence relation, also known as matroid. In the 1970’s, Gi-
anarlo Rota and Henry H. Crapo [4] introduced the term pregeometry. Although
strictu sensu the two terms are synonymous, sometimes mathematicians refer to
matroids as finite pregeometries. Finite matroids can be characterized in several
equivalent ways, but these equivalences fail in the infinite setting. Thus, the
general definition of a pregeometry generalizes only one of the aspects of these finite
objects, for this reason in some cases it is thought that pregeometries and matroids
are different things. In the model-theoretic community the term pregeometry is
preferred, probably because of the focus on infinite structures.

Definition 3.7. Let $M$ be a set and $\cl : \mathcal{P}(M) \to \mathcal{P}(M)$ a closure operator on the
power set of $M$. We say that $(M, \cl)$ is a pregeometry if for every $A, B \subseteq M$ and
$a, b \in M$ the following conditions are satisfied:

i) If $a \in \cl(A \cup \{b\}) - \cl(A)$, then $b \in \cl(A \cup \{a\})$ [Exchange Principle];

ii) If $a \in \cl(A)$, then $a \in \cl(A_0)$ for some $A_0 \subseteq_{\text{Fin}} A$ [Finite Character].

Given a pregeometry $(M, \cl)$, we say that $A \subseteq M$ is closed if $\cl(A) = A$.

Example 3.8. Let $\mathbb{K}$ be a field and $V$ be a vector space over $\mathbb{K}$. For every $A \subseteq V$ let
$\langle A \rangle$ be the smallest subspace of $V$ containing $A$, i.e. the subspace of $V$ spanned by
$A$. Then $(V, \langle \rangle)$ is a pregeometry. Among the conditions defining a pregeometry we
only show that the Exchange Principle is satisfied. Suppose that $a \in \langle A \cup \{b\} \rangle - \langle A \rangle$
then there exists $\tilde{c} \in K$ and $d \neq 0 \in K$ such that $a = \sum_{i=0}^{n-1} c_i a_i + db$, so
$b = \frac{a}{d} - \sum_{i=0}^{n-1} \frac{c_i}{d} a_i$. Hence $b \in \langle A \cup \{a\} \rangle$.

Example 3.9. Let $\mathbb{K}$ be an algebraically closed field and, for $a \in K$ and $A \subseteq K$,
let $a \in \acl(A)$ if $a$ is algebraic over the subfield of $\mathbb{K}$ generated by $A$. Then $(K, \acl)$
is a pregeometry, see for example [15].

Definition 3.10. Let $(M, \cl)$ be a pregeometry.

i) We say that $(M, \cl)$ is a geometry if $\cl(\emptyset) = \emptyset$ and $\cl(\{m\}) = \{m\}$ for all
$m \in M$.

ii) We say that $(M, \cl)$ is trivial if $\cl(A) = \bigcup_{a \in A} \cl(\{a\})$ for any $A \subseteq M$.

Definition 3.11. Let $(M, \cl)$ be a pregeometry and $A \subseteq M$. We say that $A$ is
independent if for all $a \in A$ we have $a \notin \cl(A \setminus \{a\})$. Let $(M, \cl)$ be a pregeometry
and $B \subseteq A \subseteq M$. We say that $B$ is a basis for $A$ if $B$ is independent and $A \subseteq \cl(B)$.
Proposition 3.12. Let \((M, \text{cl})\) be a pregeometry and \(B \subseteq A \subseteq M\). The following are equivalent:

i) \(B\) is a maximally independent subset of \(A\);

ii) \(B\) is a basis for \(A\);

iii) \(B\) is a minimal subset of \(A\) such that \(A \subseteq \text{cl}(B)\).

Proof. See \cite{3} Proposition 1.5. \(\blacksquare\)

Proposition 3.13. Let \((M, \text{cl})\) be a pregeometry, \(A_1 \subseteq M\) and \(A_0 \subseteq A_1\) independent. Then \(A_0\) can be extended to a maximally independent subset of \(A_1\).

Proof. See \cite{3} Proposition 1.7. \(\blacksquare\)

Lemma 3.14. Let \((M, \text{cl})\) be a pregeometry and \(A, B, C \subseteq M\) with \(A \subseteq C\) and \(B \subseteq C\). If \(A\) and \(B\) are bases for \(C\), then \(|A| = |B|\).

Proof. See \cite{14} Lemma 8.1.3. \(\blacksquare\)

Definition 3.15. Let \((M, \text{cl})\) be a pregeometry and \(A \subseteq M\). The dimension of \(A\) is the cardinality of a basis for \(A\). We let \(\dim(A)\) denote the dimension of \(A\).

If \((M, \text{cl})\) is a pregeometry and \(A, C \subseteq M\), we also consider the localization \(\text{cl}_C(A) = \text{cl}(C \cup A)\). It is easy to see that \((M, \text{cl}_C)\) is also a pregeometry.

Definition 3.16. Let \((M, \text{cl})\) be a pregeometry and \(A, C \subseteq M\). We say that \(A\) is independent over \(C\) if \(A\) is independent in \((M, \text{cl}_C)\) and that \(B \subseteq A\) is basis for \(A\) over \(C\) if \(B\) is a basis for \(A\) in \((M, \text{cl}_C)\). We let \(\dim(A/C)\) be the dimension of \(A\) in \((M, \text{cl}_C)\) and call \(\dim(A/C)\) the dimension of \(A\) over \(C\).

Corollary 3.17. Let \((M, \text{cl})\) be a pregeometry and \(C \subseteq M\). For every \(A \subseteq M\), there exists \(B \subseteq A\) such that \(B\) is a basis for \(A\) over \(C\).

Proof. Immediate from Proposition 3.13. \(\blacksquare\)

The following lemma will be relevant in the proof of Theorem 3.28, this is the reason for which we state it here.

Lemma 3.18. Let \((M, \text{cl})\) be a pregeometry and \(A \subseteq M\) be an independent set.

Let \(D_0, D_1 \subseteq A\) and \(D_0 \cap D_1 = \emptyset\), then

i) \(D_0\) is independent over \(D_1\);

ii) \(\dim(D_0/D_1) = \dim(D_0)\).

Proof. i) Suppose that \(D_0\) is not independent over \(D_1\), then there exists \(d \in D_0\) such that \(d \in \text{cl}_{D_1}(D_0 - \{d\}) = \text{cl}(D_1 \cup (D_0 - \{d\}))\). By hypothesis \(D_0 \cap D_1 = \emptyset\), so \(d \in \text{cl}((D_1 \cup D_0) - \{d\})\). Thus \(D_0 \cup D_1\) is dependent, a contradiction.

ii) By i) \(D_0\) is independent over \(D_1\) and thus it is independent in both the pregeometries \((M, \text{cl})\) and \((M, \text{cl}_{D_1})\), hence we have that \(\dim(D_0/D_1) = |D_0| = \dim(D_0)\). \(\blacksquare\)

The notion of dimension that we have been dealing with allow us to define an independence relation with many desirable properties.
**Definition 3.19.** Let \((M, \text{cl})\) be a pregeometry, \(A, B, C \subseteq M\). We say that \(A\) is independent from \(C\) over \(B\) if for every \(\bar{a} \in A\) we have \(\dim(\bar{a}/B \cup C) = \dim(\bar{a}/B)\). In this case we write \(A \overset{\text{cl}}{\vdash}_B C\).

**Lemma 3.20 (Monotonicity).** If \(A \overset{\text{cl}}{\vdash}_C B\) and \(D \subseteq B\), then \(A \overset{\text{cl}}{\vdash}_C D\).

*Proof.* See [3, Proposition 1.17].

**Lemma 3.21 (Transitivity).** \(A \overset{\text{cl}}{\vdash}_C B \cup D\) if and only if \(A \overset{\text{cl}}{\vdash}_C B\) and \(A \overset{\text{cl}}{\vdash}_{C \cup B} D\).

*Proof.* See [3, Proposition 1.17].

**Lemma 3.22 (Finite Basis).** \(A \overset{\text{cl}}{\vdash}_C B\) if and only if \(A \overset{\text{cl}}{\vdash}_C B_0\) for all finite \(B_0 \subseteq B\).

*Proof.* The direction \((\Rightarrow)\) follows from Monotonicity. For the direction \((\Leftarrow)\), suppose that there exists \(\bar{a} \in A\) such that \(\bar{a} \overset{\text{cl}}{\vdash}_C B\). Then there exists \(\bar{a}' \subseteq \bar{a}\) such that \(\bar{a}'\) is independent over \(C\) but not over \(C \cup B\). Thus there exists \(a' \in \text{cl}(C \cup (\bar{a}' - \{a'\}))\). By Property v) of Definition 3.7 there exists \(C_0 \subseteq \text{Fin} C\) and \(B_0 \subseteq \text{Fin} B\) such that \(a' \in \text{cl}(C_0 \cup B_0) \cup (\bar{a}' - \{a'\}))\), thus \(\bar{a}'\) is independent over \(C\) but not over \(C \cup B\). Hence \(\bar{a} \overset{\text{cl}}{\vdash}_C B_0\).

**Lemma 3.23.** For any \(A \subseteq M\), \(A \overset{\text{cl}}{\vdash}_C \text{cl}(C)\).

*Proof.* Let \(\bar{a} \in A\) and \(\bar{a}' \subseteq \bar{a}\) be independent over \(C\). Then for every \(a' \in \bar{a}'\) we have \(a' \notin \text{cl}(C \cup (\bar{a}' - \{a'\}))\). Clearly \(\text{cl}(C \cup (\bar{a}' - \{a'\})) = \text{cl}(C \cup \text{cl}(C) \cup (\bar{a}' - \{a'\}))\), thus \(\bar{a}'\) is also independent over \(C \cup \text{cl}(C)\).

**Lemma 3.24 (Symmetry).** If \(A \overset{\text{cl}}{\vdash}_C B\), then \(B \overset{\text{cl}}{\vdash}_C A\).

*Proof.* See [3, Proposition 1.7].

**Corollary 3.25 (Exchange).** If \(A \overset{\text{cl}}{\vdash}_D B\) and \(A \cup B \overset{\text{cl}}{\vdash}_D C\), then \(A \overset{\text{cl}}{\vdash}_D B \cup C\).

*Proof.*

\[
\begin{align*}
A \overset{\text{cl}}{\vdash}_D B & \quad \text{and} \quad A \cup B \overset{\text{cl}}{\vdash}_D C \\
\downarrow & \quad \downarrow \\
A \overset{\text{cl}}{\vdash}_D B & \quad \text{and} \quad C \overset{\text{cl}}{\vdash}_D A \cup B \quad \text{[by Lemma 3.21]} \\
\downarrow & \quad \downarrow \\
A \overset{\text{cl}}{\vdash}_D B & \quad \text{and} \quad C \overset{\text{cl}}{\vdash}_{D \cup B} A \quad \text{[by Lemma 3.21]} \\
\downarrow & \quad \downarrow \\
A \overset{\text{cl}}{\vdash}_D B & \quad \text{and} \quad A \overset{\text{cl}}{\vdash}_{D \cup B} C \quad \text{[by Lemma 3.21]} \\
\downarrow & \quad \downarrow \\
A \overset{\text{cl}}{\vdash}_D B & \quad \text{and} \quad A \overset{\text{cl}}{\vdash}_D B \cup C \quad \text{[by Lemma 3.21]}.
\end{align*}
\]

**Proposition 3.26.** If \(A \overset{\text{cl}}{\vdash}_B A\), then \(A \overset{\text{cl}}{\vdash}_B D\) for any \(D \subseteq M\).
Proof. Suppose $A \models B A$. Let $\bar{a} \in A$, then $\dim(\bar{a}/B \cup \bar{a}) = \dim(\bar{a}/B)$, and so $\dim(\bar{a}/B) = 0$ because $\dim(\bar{a}/B \cup \bar{a}) = 0$. Thus $\emptyset$ is basis for $\bar{a}$ over $B$ and hence $\bar{a} \subseteq \cl(B \cup \emptyset) = \cl(B)$. Hence $A \subseteq B$. Let now $D \subseteq M$, by Lemma 3.23 we have that $D \models B \cl(B)$ and so by Lemma 3.20 and Lemma 3.24 we can conclude that $\bar{a} \models D B$.

\section{3.3. Pregeometry Atomic Independence Logics}

Pregeometry Atomic Absolute Independence Logic (PAAIndL) is defined as follows. The syntax and deductive system of this logic are the same as those of AAIndL. Let $\mathcal{L}$ be a signature (possibly higher-order) and $\mathbf{K}$ a class of $\mathcal{L}$-structures. Let $\cl$ be a unary second-order pregeometric operator on $\mathbf{K}$, such that there exists a structure $\mathcal{M} \in \mathbf{K}$ such that the pregeometry $(\mathcal{M}, \cl)$ has the following three properties:

- (P1) $\cl(\emptyset) \neq \emptyset$;
- (P2) for every independent $D_0 \subseteq \text{Fin} M$, $\cl(D_0) \neq \bigcup_{D \subseteq D_0} \cl(D)$;
- (P3) $\dim(M) \geq \omega$.

Notice that conditions (P1) and (P2) put some relevant (but reasonable) restrictions on the pregeometry of $(\mathcal{M}, \cl)$. Indeed, condition (P1) prohibits that the pregeometry is a geometry, while condition (P2) can be seen as a stronger form of the pregeometries that we saw in Examples 3.8 and 3.9 respectively.

In Section 3.4 we will see that in the case of vector spaces and algebraically closed fields conditions (P1) and (P2) are always satisfied (with respect to non-triviality.

Let $\mathcal{M} \in \mathbf{K}$ and $s : \text{dom}(s) \rightarrow M$ with $\bar{x} \subseteq \text{dom}(s) \subseteq \text{Var}$. We say that $\mathcal{M}$ satisfies $\bot(\bar{x})$ under $s$, in symbols $\mathcal{M} \models_s \bot(\bar{x})$, if for every $x \in \bar{x}$

$s(x) \notin \cl(\{s(z) \mid z \in \bar{x}\} - \{s(x)\})$.

Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$. We say that $\bot(\bar{x})$ is a logical consequence of $\Sigma$, in symbols $\mathcal{M} \models \bot(\bar{x})$, if for every $\mathcal{M} \in \mathbf{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\bot(\bar{x})\}$ is included in $\text{dom}(s)$ we have that

if $\mathcal{M} \models_s \Sigma$ then $\mathcal{M} \models_s \bot(\bar{x})$.

Theorem 3.27. Let $\Sigma$ be a set of atoms, then

$\Sigma \models \bot(\bar{x})$ if and only if $\Sigma \vdash \bot(\bar{x})$.

Proof. The direction ($\Rightarrow$) is immediate. For the direction ($\Leftarrow$), suppose $\Sigma \nvdash \bot(\bar{x})$. Notice that if this is the case then $\bar{x} \neq \emptyset$. Indeed if $\bar{x} = \emptyset$ then $\Sigma \vdash \bot(\bar{x})$ because by rule $(a_2) \vdash \bot(\emptyset)$. We can assume that $\bar{x}$ is injective. This is without loss of generality because clearly $\mathcal{M} \models_s \bot(\bar{x})$ if and only if $\mathcal{M} \models_s \bot(\pi \bar{x})$, where $\pi : \text{Var}^{<\omega} \rightarrow \text{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables. Let then $\bar{x} = (x_{j_0}, ..., x_{j_{n-1}}) \neq \emptyset$ be injective.

By assumption there is $\mathcal{M} \in \mathbf{K}$ with properties (P1), (P2) and (P3). Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $\mathcal{M}$ and $\{w_i \mid i \in \omega\}$ an injective enumeration of $\text{Var} - \{x_{j_0}\}$. Notice that, because of property (P3), we have that $\kappa \geq \omega$. Let $s$ be the following assignment:

$s(w_i) = a_i$
and
\[ s(x_{j_0}) = e \quad \text{if } \vec{x} = \{x_{j_0}\} \]
\[ s(x_{j_0}) = d \quad \text{if } \vec{x} \neq \{x_{j_0}\}, \]
where \( e \in \text{cl}(\emptyset) \) and \( d \) is such that \( d \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\}) \) but \( d \not\in \text{cl}(D) \) for every \( D \subseteq \{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\} \). Notice that \( e \) and \( d \) do exist because of properties (P1) and (P2).

We claim that \( M \not\models s \perp (\vec{x}) \). This is immediate because either
\[ s(x_{j_0}) = e \quad \text{or} \quad s(x_{j_0}) = d, \]
and
\[ e \in \text{cl}(\emptyset) \subseteq \text{cl}(\{s(x) \mid x \in \vec{x}\} - \{s(x_{j_0})\}) \]
\[ d \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\}) = \text{cl}(\{s(x) \mid x \in \vec{x}\} - \{s(x_{j_0})\}). \]

Let now \( \perp(\vec{v}) \in \Sigma \), we want to show that \( M \models s \perp (\vec{v}) \). As before, we assume, without loss of generality, that \( \vec{v} \) is injective. Notice that if \( \vec{v} = \emptyset \), then \( M \models s \perp (\vec{v}) \). Thus let \( \vec{v} = (v_{h_0}, \ldots, v_{h_{c-1}}) \neq \emptyset \).

**Case 1**. \( x_{j_0} \notin \vec{v} \). Let \( w_{r_i} = v_{h_i} \) for every \( i \in \{0, \ldots, c-1\} \), we then have that
\[ \{s(v_{h_0}) = s(w_{r_0}) = a_{r_0}, \ldots, s(v_{h_{c-1}}) = s(w_{r_{c-1}}) = a_{r_{c-1}}\} \]
is independent.

**Case 2**. \( x_{j_0} \in \vec{v} \).

**Subcase 2.1.** \( \vec{x} - \vec{v} \neq \emptyset \). Notice that \( \vec{x} \neq \{x_{j_0}\} \) because if not then \( \vec{x} - \vec{v} = \{x_{j_0}\} \) and so \( x_{j_0} \notin \vec{v} \). Hence \( s(x_{j_0}) = d \). Let \( (\vec{v} - \{x_{j_0}\}) \cap \vec{x} = \{v_{h_0}', \ldots, v_{h_{b-1}}'\}, \)
\( \vec{v} - \vec{x} = \{v_{h_0}'', \ldots, v_{h_{b-1}''}\}, \)
\( w_{r_i} = v_{h_i}' \) for every \( i \in \{0, \ldots, b-1\} \) and \( w_{r_i''} = v_{h_i}'' \) for every \( i \in \{0, \ldots, t-1\} \). Suppose now that the set \( \{a_{r_0}''', \ldots, a_{r_{t-1}''}, a_{r_0}''', \ldots, a_{r_{t-1}'''}\} \) is dependant. The set \( \{a_{r_0}''', \ldots, a_{r_{t-1}''}, a_{r_0}''', \ldots, a_{r_{t-1}'''}\} \) is independent, so there are three cases.

**Case 1.** \( a_{r_{t-1}''} \in \text{cl}(\{a_{r_0}''', \ldots, a_{r_{b-1}''}, d, a_{r_0}''', \ldots, a_{r_{t-1}''}, a_{r_{t+1}''}, \ldots, a_{r_{t-1}'''}\}) \). If this is the case, then
\[ a_{r_{t-1}''} \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}}), a_{r_0}''', \ldots, a_{r_{t-1}''}, a_{r_{t+1}''}, \ldots, a_{r_{t-1}'''}\}) \]
because \( d \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\}) \). This is absurd though because the set \( \{s(x_{j_1}), \ldots, s(x_{j_{n-1}}), a_{r_0}''', \ldots, a_{r_{t-1}''}, a_{r_{t+1}''}, \ldots, a_{r_{t-1}'''}\} \) is made of distinct elements of the basis \( B \) and so it is independent.

**Case 2.** \( d \in \text{cl}(\{a_{r_0}''', \ldots, a_{r_{b-1}''}, a_{r_0}''', \ldots, a_{r_{t-1}''}\}) \). Notice that
\[ d \notin \text{cl}(\{a_{r_0}'', \ldots, a_{r_{b-1}''}\}) \]
because \( \{a_{r_0}'', \ldots, a_{r_{b-1}''}\} \subset \{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\} \) and \( d \) has been chosen such that \( d \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\}) \) but \( d \notin \text{cl}(D) \) for every \( D \subseteq \{s(x_{j_1}), \ldots, s(x_{j_{n-1}})\} \). Thus there is \( l \leq t - 1 \) such that
\[ d \in \text{cl}(\{a_{r_0}''', \ldots, a_{r_{b-1}''}, a_{r_0}''', \ldots, a_{r_{t-1}''}\} \cup \{a_{r_{l}'''}\}) - \text{cl}(\{a_{r_0}''', \ldots, a_{r_{b-1}''}, a_{r_0}''', \ldots, a_{r_{t-1}''}\}) \]
Let $\Sigma$ be a set of atoms and let $s$ is included in $\text{dom}(\cdot)$.

**Case 2.** which is impossible as we saw in Subcase 2.2.

Thus by the Exchange Principle we have that $M \models s$ satisfies every atom in $\Sigma$ under $\cdot$

Theorem 3.28. Let $\Sigma$ be a set of atoms, then $M \equiv \Sigma \models$, in symbols $\models \Sigma$, if $M$ satisfies every atom in $\Sigma$ under $s$.

Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(s)$. We say that $M$ satisfies $\Sigma$ under $s$, in symbols $M \models s \models \Sigma$, if $M$ satisfies every atom in $\Sigma$ under $s$. We say that $\Sigma \models s \models \Sigma$, if for every $M \in K$ and $s : \text{dom}(s) \to M$ with $\vec{x} \in \text{dom}(s) \subseteq \text{Var}$. We say that $\Sigma$ satisfies $\vec{x} \models s \models \vec{y}$ under $s$, in symbols $M \models s \models \vec{x} \models \vec{y}$, if

$$s(\vec{x}) \models \Sigma \models s(\vec{y}).$$

Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(s)$. We say that $M$ satisfies $\Sigma$ under $s$, in symbols $M \models s \models \Sigma$, if $M$ satisfies every atom in $\Sigma$ under $s$. We say that $\vec{x} \models s \models \vec{y}$ is a logical consequence of $\Sigma$, in symbols $\models \vec{x} \models s \models \vec{y}$, if for every $M \in K$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\vec{x} \models s \models \vec{y}\}$ is included in $\text{dom}(s)$ we have that

$$M \models s \models \vec{x} \models \vec{y}.$$

Theorem 3.28. Let $\Sigma$ be a set of atoms, then

$\Sigma \models s \models \vec{x} \models \vec{y}$ if and only if $\Sigma \models \vec{x} \models \vec{y}$.

**Proof.** For the direction ($\Leftarrow$), let $M \in K$ and $s$ an appropriate assignment.

(a$_3$.) Obvious.

(b$_3$.)

$$\models M \models s \models \vec{x} \models \vec{y} \quad \Rightarrow \quad s(\vec{x}) \models \Sigma \models s(\vec{y})$$

$$\Rightarrow \quad s(\vec{y}) \models \Sigma \models s(\vec{x}) \quad \text{[By Lemma 3.24]}$$

$$\Rightarrow \quad M \models s \models \vec{y} \models \vec{x}.$$
(d₃.) Let then \( b \) have that
\[
\overrightarrow{x} \perp \overrightarrow{y} \quad \text{and} \quad \overrightarrow{x} \perp \overrightarrow{z}.
\]
\[
s(\overrightarrow{x}) \downarrow s(\overrightarrow{y}) \quad \text{and} \quad s(\overrightarrow{x})s(\overrightarrow{y}) \downarrow s(\overrightarrow{z})
\]
\[
s(\overrightarrow{x}) \downarrow s(\overrightarrow{y})s(\overrightarrow{z}) \quad \text{[By Corollary 3.26]}
\]
\[
\mathcal{M} \models_{s} \overrightarrow{x} \perp \overrightarrow{y} \perp \overrightarrow{z}.
\]

(e₃.) Suppose that \( \mathcal{M} \models_{s} x \perp x \), then \( s(x) \downarrow s(\overrightarrow{x}) \) and so by Proposition 3.20 we have that \( s(x) \downarrow s(\overrightarrow{y}) \) for any \( \overrightarrow{y} \in \text{Var} \).

(f₃.) Obvious.

For the direction \( (\Rightarrow) \), suppose \( \Sigma \not\vdash \overrightarrow{x} \perp \overrightarrow{y} \). Notice that if this is the case then \( \overrightarrow{x} \not\subseteq \emptyset \) and \( \overrightarrow{y} \not\subseteq \emptyset \). Indeed if \( \overrightarrow{y} = \emptyset \) then \( \Sigma \vdash \overrightarrow{x} \perp \overrightarrow{y} \) because by rule \((a₃.)\) \( \vdash \overrightarrow{x} \perp \emptyset \). Analogously if \( \overrightarrow{x} = \emptyset \) then \( \Sigma \vdash \overrightarrow{x} \perp \overrightarrow{y} \) because by rule \((a₃.)\) \( \vdash \emptyset \perp \overrightarrow{y} \) and so by rule \((b₃.)\) \( \vdash \emptyset \perp \overrightarrow{y} \). We can assume that \( \overrightarrow{x} \) and \( \overrightarrow{y} \) are injective. This is without loss of generality because clearly \( \mathcal{M} \models_{s} \overrightarrow{x} \perp \overrightarrow{y} \) if and only if \( \mathcal{M} \models_{s} \pi \overrightarrow{x} \perp \pi \overrightarrow{y} \), where \( \pi : \text{Var}^{\omega} \to \text{Var}^{\omega} \) is the function that eliminates repetitions in finite sequences of variables. Furthermore we can assume that \( \overrightarrow{x} \perp \overrightarrow{y} \) is minimal, in the sense that if \( \overrightarrow{x}' \subseteq \overrightarrow{x} \), \( \overrightarrow{y}' \subseteq \overrightarrow{y} \) and \( \overrightarrow{x}' \neq \overrightarrow{x} \) or \( \overrightarrow{y}' \neq \overrightarrow{y} \), then \( \Sigma \vdash \overrightarrow{x}' \perp \overrightarrow{y}' \). This is for two reasons.

i) If \( \overrightarrow{x} \perp \overrightarrow{y} \) is not minimal we can always find a minimal atom \( \overrightarrow{x}'' \perp \overrightarrow{y}'' \) such that \( \Sigma \not\vdash \overrightarrow{x}'' \perp \overrightarrow{y}'' \), \( \overrightarrow{x}'' \subseteq \overrightarrow{x} \) and \( \overrightarrow{y}'' \subseteq \overrightarrow{y} \) — just keep deleting elements of \( \overrightarrow{x} \) and \( \overrightarrow{y} \) until you obtain the desired property or until both \( \overrightarrow{x}'' \) and \( \overrightarrow{y}'' \) are singletons, in which case, due to the trivial independence rule \((a₃.)\), \( \overrightarrow{x}'' \perp \overrightarrow{y}'' \) is a minimal statement.

ii) For any \( \overrightarrow{x} \subseteq \overrightarrow{x} \) and \( \overrightarrow{y} \subseteq \overrightarrow{y} \) we have that if \( \mathcal{M} \models_{s} \overrightarrow{x} \perp \overrightarrow{y} \) then \( \mathcal{M} \models_{s} \overrightarrow{x} \perp \overrightarrow{y} \), for every \( \mathcal{M} \) and \( s \).

Let instead \( \overrightarrow{x} = \overrightarrow{x}'' \overrightarrow{x}''' \) and \( \overrightarrow{y} = \overrightarrow{y}'' \overrightarrow{y}''' \), then
\[
\mathcal{M} \models_{s} \overrightarrow{x} \overrightarrow{x}'' \perp \overrightarrow{y} \overrightarrow{y}'' \quad \Rightarrow \quad s(\overrightarrow{x})s(\overrightarrow{x}'') \downarrow s(\overrightarrow{y})s(\overrightarrow{y}'') \quad \text{[By Lemma 3.20]}
\]
\[
s(\overrightarrow{x})s(\overrightarrow{x}'') \downarrow s(\overrightarrow{y})s(\overrightarrow{y}'') \quad \text{[By Lemma 3.20 and 3.23]}
\]
\[
\mathcal{M} \models_{s} \overrightarrow{x} \perp \overrightarrow{y}.
\]

Let then \( \overrightarrow{x} = (x_{j_{0}}, \ldots, x_{j_{n-1}}) \) and \( \overrightarrow{y} = (y_{k_{0}}, \ldots, y_{k_{m-1}}) \) be injective and such that \( \overrightarrow{x} \perp \overrightarrow{y} \) is minimal.

Let \( V = \{ v \in \text{Var} \mid \Sigma \vdash v \perp v \} \) and \( W = \text{Var} - V \). We claim that \( \overrightarrow{x}, \overrightarrow{y} \not\subseteq V \). We prove it only for \( \overrightarrow{x} \), the other case is symmetrical. Suppose that \( \overrightarrow{x} \subseteq V \), then for every \( x \in \overrightarrow{x} \) we have that \( \Sigma \vdash x \perp x \) so by rule \((e₃.)\), \((b₃.)\) and \((d₃.)\)
\[
\quad \Sigma \vdash \overrightarrow{y} \perp x_{j_{0}} \quad \text{and} \quad \Sigma \vdash \overrightarrow{y}x_{j_{0}} \perp x_{j_{1}} \quad \Rightarrow \quad \Sigma \vdash \overrightarrow{y} \perp x_{j_{0}}x_{j_{1}},
\]
\[
\quad \Sigma \vdash \overrightarrow{y} \perp x_{j_{0}}x_{j_{1}} \quad \text{and} \quad \Sigma \vdash \overrightarrow{y}x_{j_{0}}x_{j_{1}} \perp x_{j_{2}} \quad \Rightarrow \quad \Sigma \vdash \overrightarrow{y} \perp x_{j_{0}}x_{j_{1}}x_{j_{2}},
\]
\[
\vdots
\]
\[
\quad \Sigma \vdash \overrightarrow{y} \perp x_{j_{0}} \ldots \perp x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \overrightarrow{y}x_{j_{0}} \ldots x_{j_{n-2}} \perp x_{j_{n-1}} \quad \Rightarrow \quad \Sigma \vdash \overrightarrow{y} \perp \overrightarrow{x}.
\]

Hence by rule \((b₃.)\) \( \Sigma \vdash \overrightarrow{x} \perp \overrightarrow{y} \). Thus \( \overrightarrow{x} \cap W \neq \emptyset \) and \( \overrightarrow{y} \cap W \neq \emptyset \). Without loss of generality suppose that \( x_{j_{0}} \in W \) and \( y_{k_{0}} \in W \). Let \( \overrightarrow{x} \cap W = \overrightarrow{x}' = (x_{j_{0}}, \ldots, x'_{j_{n-1}}) = (x_{j_{0}}, \ldots, x'_{j_{n-1}}) \neq \emptyset \) and \( \overrightarrow{y} \cap W = \overrightarrow{y}' = (y_{k_{0}}, \ldots, y'_{k_{m-1}}) \). Notice that \( \overrightarrow{x}' \cap \overrightarrow{y}' = \emptyset \).
Indeed let $z \in \vec{p} \cap \vec{q}$, then by rules $(b_3)$ and $(c_3)$ we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

By assumption there is $M \in K$ with properties (P1), (P2) and (P3). Let then $\{a_i \mid i \in \kappa\}$ be an injective enumeration of a basis $B$ for $M$ and $\{w_i \mid i \in \lambda\}$ be an injective enumeration of $W - \{x_{j_0}\}$. Notice that, because of property (P3), we have that $\kappa \geq \lambda$, indeed either $\lambda = \omega$ or $\lambda = n$ for $n \in \omega$. Let $s$ be the following assignment:

i) $s(v) = e$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \lambda$,

iii) $s(x_{j_0}) = d$,

where $e \in \text{cl}(\emptyset)$ and $d$ is such that

$$d \in \text{cl}\left\{s(x_{j_1}), \ldots, s(x_{j_{n-1}' - 1}), s(y_{k_0}), \ldots, s(y_{k_{m-1}' - 1})\right\}$$

but $d \notin \text{cl}(D)$ for every $D \subset \left\{s(x_{j_1}), \ldots, s(x_{j_{n-1}' - 1}), s(y_{k_0}), \ldots, s(y_{k_{m-1}' - 1})\right\}$. Notice that $e$ and $d$ do exist because of properties (P1) and (P2).

We claim that $M \not\models s \vec{x} \perp \vec{q}$, as noticed this implies that $M \not\models s \vec{x} \perp \vec{y}$. First we show that the set $\left\{s(x') \mid x' \in \vec{x}\right\}$ is independent. By construction $s(x_{j_0}) \notin \text{cl}(\left\{s(x') \mid x' \in \vec{x}\right\} - \{s(x_{j_0})\})$. Let then $i \in \{1, \ldots, n' - 1\}$ and suppose that $s(x_{j_i}') \in \text{cl}(\left\{s(x_{j_0}), \ldots, s(x_{j_{i-1}' - 1}), s(x_{j_{i+1}' - 1}), \ldots, s(x_{j_{n-1}' - 1})\right\})$.

The set $\left\{s(x_{j_1}'), \ldots, s(x_{j_{i-1}' - 1}), s(x_{j_{i+1}' - 1}), \ldots, s(x_{j_{n-1}' - 1})\right\}$ is independent, so

$$s(x_{j_i}') \in \text{cl}(\left\{s(x_{j_1}'), \ldots, s(x_{j_{i-1}' - 1}), s(x_{j_{i+1}' - 1}), \ldots, s(x_{j_{n-1}' - 1})\right\} \cup \{s(x_{j_0})\})$$

but

$$s(x_{j_i}') \notin \text{cl}(\left\{s(x_{j_1}'), \ldots, s(x_{j_{i-1}' - 1}), s(x_{j_{i+1}' - 1}), \ldots, s(x_{j_{n-1}' - 1})\right\})$$.

Hence by the Exchange Principle

$$s(x_{j_0}) \in \text{cl}(\left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1})\right\})$$,

a contradiction. Thus $\dim(s(\vec{x})) = |\{s(x') \mid x' \in \vec{x}\}|$.

We now show that $\left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1})\right\}$ is a basis for $\left\{s(x') \mid x' \in \vec{x}\right\}$ over $\left\{s(y') \mid y' \in \vec{y}\right\}$. As we noticed above $\vec{x} \cap \vec{g} = \emptyset$, so by properties of our assignment $s(\vec{x}) \cap s(\vec{y}) = \emptyset$. Thus, by Lemma 3.18 $\left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1})\right\}$ is independent over $\left\{s(y') \mid y' \in \vec{y}\right\}$, also $\left\{s(x_{j_0}), \ldots, s(x_{j_{n-1}' - 1})\right\} \subseteq \text{cl}(s(\vec{y}) \cup \left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1})\right\})$, because

$$s(x_{j_0}) \in \text{cl}(\left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1}), s(y_{k_0}), \ldots, s(y_{k_{m-1}' - 1})\right\})$$.

Hence

$$\dim(s(\vec{x})/s(\vec{y})) = |\left\{s(x_{j_1}'), \ldots, s(x_{j_{n-1}' - 1})\right\}| = \dim(s(\vec{x})) - 1$$.

Let now $\vec{v} \perp \vec{w} \in \Sigma$, we want to show that $M \models s \vec{v} \perp \vec{w}$. As before, we assume, without loss of generality, that $\vec{v}$ and $\vec{w}$ are injective. Notice also that if $\vec{v} = \emptyset$ or $\vec{w} = \emptyset$, then $M \models s \vec{v} \perp \vec{w}$. Thus let $\vec{v}, \vec{w} \neq \emptyset$.

**Case 1.** $\vec{v} \subseteq V$ or $\vec{w} \subseteq V$. Suppose that $\vec{v} \subseteq V$, the other case is symmetrical, then $s(\vec{v}) \subseteq \text{cl}(\emptyset)$. Thus $\dim(s(\vec{v})/s(\vec{w})) = 0 = \dim(s(\vec{v}))$. 

Case 2. \( \vec{v} \not\subseteq V \) and \( \vec{w} \not\subseteq V \). Let \( \vec{v} \cap W = \vec{v}' \neq \emptyset \) and \( \vec{w} \cap W = \vec{w}' \neq \emptyset \). Notice that
\[
s(\vec{v})^c \not\subseteq s(\vec{w}) \text{ if and only if } s(\vec{v})^c \not\subseteq s(\vec{w}').
\]
Left to right holds in general. As for the other direction, suppose that \( s(\vec{v})^c \not\subseteq s(\vec{w}) \). If \( u \in \vec{v} \vec{w} - \vec{v}' \vec{w}' \), then \( s(u) = e \in \operatorname{cl}(\emptyset) \). Thus
\[
s(\vec{v})^c \not\subseteq s(\vec{w}) \text{ and } s(\vec{v})^c \not\subseteq \operatorname{cl}(\emptyset) \quad \text{[By Lemma 3.23]}
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{w}) \cup \operatorname{cl}(\emptyset)
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{w}) \cup \operatorname{cl}(\emptyset) \cap s(\vec{v})
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{v}).
\]
So
\[
s(\vec{v})^c \not\subseteq s(\vec{v}) \text{ and } s(\vec{v})^c \not\subseteq \operatorname{cl}(\emptyset) \quad \text{[By Lemma 3.23]}
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{v}) \cup \operatorname{cl}(\emptyset)
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{v}) \cup \operatorname{cl}(\emptyset) \cap s(\vec{v})
\]
\[
s(\vec{v})^c \not\subseteq s(\vec{v}).
\]
Subcase 2.1. \( x_{j_0} \notin \vec{v}' \vec{w}' \). Notice that \( \vec{v}' \cap \vec{w}' = \emptyset \), so by properties of our assignment \( s(\vec{v}') \cap s(\vec{w}') = \emptyset \). Thus by Lemma 3.18 it follows that \( \dim(s(\vec{v}'))/s(\vec{w}')) = \dim(s(\vec{v}')) \).

Subcase 2.2. \( x_{j_0} \in \vec{v}' \vec{w}' \).

Subcase 2.2.1. \( \vec{v}' \vec{w}' - (\vec{v}' \vec{w}') \neq \emptyset \). Let \( \vec{v}' \vec{w}' - \{x_{j_0}\} \cap \vec{v}' \vec{w}' = \{u_{k_0}', ..., u_{k_{b-1}}'\} \), \( \vec{v}' \vec{w}' - \vec{v}' \vec{w}' = \{u_{k_0}', ..., u_{k_{b-1}}'\} \), \( w_{r_i} = u_{k_i}' \) for every \( i \in \{0, ..., b - 1\} \) and \( w_{r_i} = u_{k_i} \) for every \( i \in \{0, ..., t - 1\} \). Suppose now that the set \( \{a_{r_0}', ..., a_{r_{b-1}}', d, a_{r_0''}, ..., a_{r_{t-1}}''\} \) is dependent. There are three cases.

Case 1. \( a_{r_i'} \in \operatorname{cl}\left(\{a_{r_0}', ..., a_{r_{b-1}}', d, a_{r_0''}, ..., a_{r_{t-1}}''\}\right) \). If this is the case, then
\[
a_{r_{i'}} \in \operatorname{cl}\left(\{s(x'_{j_1}), ..., s(x'_{j_{b-1}}), s(y'_{k_0}), ..., s(y'_{k_{b-1}}), a_{r_0}'', ..., a_{r_{t-1}}'', a_{r_{i+1}}', ..., a_{r_{i-1}}'\}\right)
\]
because \( d \in \operatorname{cl}\left(\{s(x'_{j_1}), ..., s(x'_{j_{b-1}}), s(y'_{k_0}), ..., s(y'_{k_{b-1}})\}\right) \). This is absurd though because the set \( \{s(x'_{j_1}), ..., s(x'_{j_{b-1}}), s(y'_{k_0}), ..., s(y'_{k_{b-1}}), a_{r_0''}, ..., a_{r_{t-1}}''\} \) is made of distinct elements of the basis \( B \) and so it is independent.

Case 2. \( d \in \operatorname{cl}\left(\{a_{r_0}', ..., a_{r_{t-1}}', a_{r_0''}, ..., a_{r_{t-1}}''\}\right) \). Notice that
\[
d \notin \operatorname{cl}\left(\{a_{r_0}', ..., a_{r_{t-1}}'\}\right)
\]
because \( \{a_{r_0'}, \ldots, a_{r_{t-1}}'\} \subset \{s(x_{j_1'}), \ldots, s(x_{j_{m-1}'})\} \) and \( d \) has been chosen such that \( d \in \text{cl}\left(\{s(x_{j_1'}), \ldots, s(x_{j_{m-1}'})\}\right) \) but \( d \notin \text{cl}(D) \) for every \( D \subseteq \{s(x_{j_1'}), \ldots, s(x_{j_{m-1}'})\} \). Thus there is \( l \leq t - 1 \) such that \( d \in \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}}', a_{r_0''}, \ldots, a_{r_{t-1}''}\}\right) \) and then by the Exchange Principle we have that
\[
\forall i \in \{1, \ldots, l\} \quad a_{r_i''} \in \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}}, a_{r_0''}, \ldots, a_{r_{t-1}''}\}\right).
\]
Thus we have that \( a_{r_{t-1}''} \in \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}'}, a_{r_0''}, \ldots, a_{r_{t-1}''}\}\right) \), which is impossible as we saw in Case 1.

**Case 3.** \( a_{r_{t-1}}' \in \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}'}, a_{r_{t-1}'} \} \right) \). Notice that
\[
a_{r_{t-1}}' \notin \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}'}, a_{r_0''}, \ldots, a_{r_{t-1}''}\}\right).
\]
Thus by the Exchange Principle we have that \( d \in \text{cl}\left(\{a_{r_0'}, \ldots, a_{r_{t-1}'}, a_{r_0''}, \ldots, a_{r_{t-1}''}\}\right) \), which is impossible as we saw in Case 2.

We can then conclude that the set \( \{a_{r_0'}, \ldots, a_{r_{t-1}'}, d, a_{r_0''}, \ldots, a_{r_{t-1}''}\} \) is independent. Clearly \( \{s(v') \mid v' \in \vec{v}'\} \cup \{s(w') \mid w' \in \vec{w}'\} = \{a_{r_0'}, \ldots, a_{r_{t-1}'}, d, a_{r_0''}, \ldots, a_{r_{t-1}''}\} \). Furthermore, as we noticed above, \( s(\vec{v}') \cap s(\vec{w}') = \emptyset \). Hence by Lemma 3.18 we have that \( \dim(s(\vec{v}')/s(\vec{w}')) = \dim(s(\vec{v}')) \).

**Subcase 2.2.2.** \( \vec{x}' \vec{y}' \subseteq \vec{v}' \vec{w}' \). This case is not possible. First notice that if \( \Sigma \vdash \vec{x}' \perp \vec{y}' \) then \( \Sigma \vdash \vec{x} \perp \vec{y} \). Let \( \vec{x} - \vec{x}' = (x_{s_0}, \ldots, x_{s_{m-1}}) \) and \( \vec{y} - \vec{y}' = (y_{s_0}, \ldots, y_{s_{m-1}}) \), then by rule \((e_3)\), \((b_3)\) and \((d_3)\) we have that
\[
\Sigma \vdash \vec{x}' \perp \vec{y}' \quad \text{and} \quad \Sigma \vdash \vec{x}' \vec{y}' \perp y_{s_0}
\]
\[
\Sigma \vdash \vec{x}' \vec{y}' \perp y_0
\]
\[
\vdots
\]
\[
\Sigma \vdash \vec{x}' \vec{y}' y_{s_0} \ldots y_{s_{m-2}} \perp y_{s_{m-1}}
\]
\[
\Sigma \vdash \vec{x}' \vec{y}' y_{s_0} \ldots y_{s_{m-2}} \perp y_{s_{m-1}}
\]
and hence by rule \((f_3)\) and \((b_3)\) we have that \( \Sigma \vdash \vec{y} \perp \vec{x}' \). Thus
\[
\Sigma \vdash \vec{y} \perp \vec{x}' \quad \text{and} \quad \Sigma \vdash \vec{y} \vec{x}' \perp x_{s_0}
\]
\[
\Sigma \vdash \vec{y} \perp \vec{x}' x_{s_0}
\]
\[
\vdots
\]
\[
\Sigma \vdash \vec{y} \perp \vec{x}' x_{s_0} \ldots x_{s_{m-2}} \quad \text{and} \quad \Sigma \vdash \vec{y} \vec{x}' x_{s_0} \ldots x_{s_{m-2}} \perp x_{s_{m-1}}
\]
\[
\vdots
\]
\[
\Sigma \vdash \vec{y} \perp \vec{x}' x_{s_0} \ldots x_{s_{m-1}}
\]
and hence by rule \((f_3)\) and \((b_3)\) we have that \(\Sigma \vdash \vec{x} \perp \vec{y}\).

By rule \((f_3)\) we can assume that \(\vec{v} = \vec{v}' \vec{u}\) and \(\vec{w} = \vec{w}' \vec{u}'\) with \(\vec{u} \vec{u}' \subseteq \text{Var} - \vec{v}' \vec{w}'\). Furthermore because \(\vec{x}' \vec{y}' \subseteq \vec{v}' \vec{u}'\) again by rule \((f_3)\) we can assume that \(\vec{v} = \vec{x}'' \vec{y}'' \vec{z}''\) and \(\vec{u}'' = \vec{x}'''' \vec{g}'''' \vec{z}''''\) with \(\vec{x}'' \vec{y}'' = \vec{x}' \vec{y}'\) and \(\vec{z}'' \vec{z}'''' \subseteq \text{Var} - \vec{x}' \vec{y}'\).

Hence \(\vec{v} = \vec{x}'' \vec{y}'' \vec{z}'' \vec{u}''\) and \(\vec{w} = \vec{x}''' \vec{g}''' \vec{z}''' \vec{u}''\). By hypothesis we have that \(\vec{v} \perp \vec{w} \in \Sigma\) so by rules \((c_3)\) and \((b_3)\) we can conclude that \(\Sigma \vdash \vec{x}'' \vec{y}'' \perp \vec{x}''' \vec{g}'''\). If \(\vec{x}''' = \vec{x}'\) and \(\vec{y}''' = \vec{y}'\), then \(\Sigma \vdash \vec{x} \perp \vec{y}'\) because as we noticed \(\vec{v}'' \vec{w}'' = \emptyset\). Thus \(\Sigma \vdash \vec{x} \perp \vec{y}'\), a contradiction. Analogously if \(\vec{x}''' = \vec{x}'\) and \(\vec{g}''' = \vec{g}'\), then \(\Sigma \vdash \vec{y}' \perp \vec{x}'\). Thus by rule \((b_3)\) \(\Sigma \vdash \vec{x} \perp \vec{y}'\) and hence \(\Sigma \vdash \vec{x} \perp \vec{y}'\), a contradiction. There are then four cases:

i) \(\vec{x}'' \neq \vec{x}'\) and \(\vec{x}''' \neq \vec{x}'\);

ii) \(\vec{y}'' \neq \vec{y}'\) and \(\vec{x}''' \neq \vec{x}'\);

iii) \(\vec{y}'' \neq \vec{y}'\) and \(\vec{g}''' \neq \vec{g}'\);

iv) \(\vec{x}'' \neq \vec{x}'\) and \(\vec{g}''' \neq \vec{g}'\).

Suppose that either i) or ii) holds. If this is the case, then \(\Sigma \vdash \vec{x}'' \perp \vec{y}''\) because by hypothesis \(\vec{x} \perp \vec{y}'\) is minimal. So \(\Sigma \vdash \vec{x}'' \perp \vec{y}''\vec{x}''' \vec{g}'''\), because by rule \((d_3)\)

\[\Sigma \vdash \vec{x}'' \perp \vec{y}''\] and \(\Sigma \vdash \vec{x}'' \perp \vec{x}''' \vec{y}''' \vec{g}''' \Rightarrow \Sigma \vdash \vec{x}'' \perp \vec{y}'' \vec{x}''' \vec{g}'''\).

Hence by rule \((e_3)\) \(\Sigma \vdash \vec{x}'' \perp \vec{x}''' \vec{g}'''\) and then by rule \((b_3)\) \(\Sigma \vdash \vec{x}'' \vec{y}'' \perp \vec{x}''''\). So by rule \((e_3)\) \(\Sigma \vdash \vec{x}'' \vec{y}'' \perp \vec{x}''''\). We are under the assumption that \(\vec{x}'''' \neq \vec{x}'\) thus again by minimality of \(\vec{x} \perp \vec{y}'\) we have that \(\Sigma \vdash \vec{x}'''' \perp \vec{y}'\) and so by rule \((b_3)\) we conclude that \(\Sigma \vdash \vec{y}' \perp \vec{x}''''\). Hence \(\Sigma \vdash \vec{y}' \perp \vec{x}''''\) because by rule \((d_3)\)

\[\Sigma \vdash \vec{y}' \perp \vec{x}''''\]

Then finally by rule \((e_3)\) and \((b_3)\) we can conclude that \(\Sigma \vdash \vec{x}' \perp \vec{y}'\) and so \(\Sigma \vdash \vec{x} \perp \vec{y}'\), a contradiction. The case in which either iii) or iv) holds is symmetrical.

**Pregeometry Atomic Absolute Conditional Independence Logic** (PGAACIndL) is defined as follows. The syntax and deductive system of this logic are the same as those of AACIndL. Let \(\mathcal{L}\) be a signature (possibly higher-order) and \(\mathbf{K}\) a class of \(\mathcal{L}\)-structures. Let \(cl\) be a unary second-order pregeometric operator on \(\mathbf{K}\), such that there exists a structure \(\mathcal{M} \in \mathbf{K}\) such that the pregeometry \((\mathcal{M}, cl)\) has properties (P1), (P2) and (P3). Let \(\mathcal{M} \in \mathbf{K}\) and \(s : \text{dom}(s) \rightarrow \mathcal{M}\) with \(\vec{x} \vec{z} \subseteq \text{dom}(s) \subseteq \text{Var}\). We say that \(\mathcal{M}\) satisfies \(\perp_{\mathcal{E}}(\vec{x})\) under \(s\), in symbols \(\mathcal{M} \models_s \perp_{\mathcal{E}}(\vec{x})\), if for every \(x \in \vec{x}\)

\[s(x) \notin \text{cl}(s(\vec{z}) \cup \{s(u) \mid u \in \vec{x}\} - \{s(x)\})\].

Let \(\Sigma\) be a set of atoms and let \(s\) be such that the set of variables occurring in \(\Sigma\) is included in \(\text{dom}(s)\). We say that \(\mathcal{M}\) satisfies \(\Sigma\) under \(s\), in symbols \(\mathcal{M} \models_s \Sigma\), if \(\mathcal{M}\) satisfies every atom in \(\Sigma\) under \(s\). We say that \(\perp_{\mathcal{E}}(\vec{x})\) is a logical consequence of \(\Sigma\), in symbols \(\Sigma \models \perp_{\mathcal{E}}(\vec{x})\), if for every \(\mathcal{M} \in \mathbf{K}\) and \(s\) such that the set of variables occurring in \(\Sigma\) and \(\{\perp_{\mathcal{E}}(\vec{x})\}\) is included in \(\text{dom}(s)\) we have that

if \(\mathcal{M} \models_s \Sigma\) then \(\mathcal{M} \models_s \perp_{\mathcal{E}}(\vec{x})\).

**Theorem 3.29.** Let \(\Sigma\) be a set of atoms, then

\[\Sigma \vdash \perp_{\mathcal{E}}(\vec{x}) \Rightarrow \Sigma \models \perp_{\mathcal{E}}(\vec{x})\].

**Proof.** Let \(\mathcal{M} \in \mathbf{K}\) and \(s\) an appropriate assignment.

(a4.) Obvious.

(b4.) Suppose that \(\mathcal{M} \models_s \perp_{\mathcal{E}}(\vec{x} \vec{y})\), then for every \(v \in \vec{x} \vec{y}\) we have that \(s(v) \notin \text{cl}(s(\vec{z}) \cup (s(\vec{x} \vec{y}) - \{s(v)\}))\). In particular for every \(x \in \vec{x}\) we have that \(s(x) \notin \text{cl}(s(\vec{z}) \cup (s(\vec{x} \vec{y}) - \{s(v)\}))\).

Hence (and (P3). Let $M \in \mathcal{M}$ such that we can conclude that some means of extra predicates) one can argue that there is $\{ \}$, then there exists $x \in \bar{x}$ such that $s(x) \notin cl(s(\bar{z}) \cup (s(\bar{x}) - \{x\}))$ and so $s(x) \notin cl(s(\bar{z}) \cup (s(\bar{x}) - \{x\}))$ because $s(\bar{z}) \cup (s(\bar{x}) - \{x\})$ is a subset of $s(\bar{z}) \cup (s(\bar{x}) - \{x\})$.

(c4.) Suppose that $M \models_s \perp_{\mathcal{L}, \bar{z}}(\bar{x})$, then for every $x \in \bar{x}$ we have that $s(x) \notin cl(s(\bar{z}) \cup (s(\bar{x}) - \{x\}))$ and so $s(x) \notin cl(s(\bar{z}) \cup (s(\bar{x}) - \{x\}))$ because $s(\bar{z}) \cup (s(\bar{x}) - \{x\})$ is a subset of $s(\bar{z}) \cup (s(\bar{x}) - \{x\})$.

(d4.) Suppose that $M \not\models_s \perp_{\mathcal{L}, \bar{z}}(\bar{x})$, then there exists $x \in \bar{x}$ such that $s(x) \in cl(s(\bar{z}) \cup (s(\bar{x}) - \{x\}))$. Suppose now that $M \models_s \perp_{\mathcal{L}}(\bar{x})$ and let $s(\bar{u}) = \{a_0, ..., a_{n-1}\}$, then there exists $j \in \{0, ..., n - 1\}$ such that $s(x) \in cl(s(\bar{z}) \cup \{a_0, ..., a_j\} \cup (s(\bar{x}) - \{s(x)\}))$ but $s(x) \notin cl(s(\bar{z}) \cup \{a_0, ..., a_{j-1}\} \cup (s(\bar{x}) - \{s(x)\}))$. Thus by the Exchange Principle $a_j \in cl(s(\bar{z}) \cup \{a_0, ..., a_{j-1}\} \cup (s(\bar{x}) - \{s(x)\}))$ and so $a_j \in cl(s(\bar{z}) \cup (s(\bar{x}) - \{a_j\}))$. Hence $M \not\models_s \perp_{\mathcal{L}, \bar{z}}(\bar{u})$.

(e4.) Suppose that $M \models \perp_{\mathcal{L}}(\bar{y})$, then for every $x \in \bar{x}$ we have that $s(x) \notin cl(s(\bar{z}) \cup (s(\bar{x}) - \{s(x)\})) = cl(s(\bar{z}))$, a contradiction. Thus everything follows, in particular we can conclude that $M \models \perp_{\mathcal{L}}(\bar{y})$.

(f4.) Obvious.

The above theorem does not give the desired completeness of our axioms for $\perp_{\mathcal{L}}(\bar{x})$. In fact we do not know whether the given axioms are complete, this is an open problem at the moment. By soft arguments (reduction to first order logic by means of extra predicates) one can argue that there is some effective axiomatization, we just do not have an explicit one.

**Pregeometry Atomic Conditional Independence Logic (PGACIndL)** is defined as follows. The syntax and deductive system of this logic are the same as those of ACIndL. Let $\mathcal{L}$ be a signature (possibly higher-order) and $\mathcal{K}$ a class of $\mathcal{L}$-structures. Let $cl$ be a unary second-order pregeometric operator on $\mathcal{K}$, such that there exists a structure $\mathcal{M} \in \mathcal{K}$ such that the pregeometry $(\mathcal{M}, cl)$ has properties (P1), (P2) and (P3). Let $\mathcal{M} \in \mathcal{K}$ and $s : \text{dom}(s) \to M$ with $\bar{x} \bar{y} \bar{z} \subseteq \text{dom}(s) \subseteq \text{Var}$. We say that $\mathcal{M}$ satisfies $\bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$ under $s$, in symbols $\mathcal{M} \models_s \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$, if

$$\frac{s(\bar{x})}{s(\bar{z})} \models_s \frac{s(\bar{y})}{s(\bar{z})}.$$  

Let $\Sigma$ be a set of atoms and let $s$ be such that the set of variables occurring in $\Sigma$ is included in $\text{dom}(s)$. We say that $\mathcal{M}$ satisfies $\Sigma$ under $s$, in symbols $\mathcal{M} \models_s \Sigma$, if $\mathcal{M}$ satisfies every atom in $\Sigma$ under $s$. We say that $\bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$ is a logical consequence of $\Sigma$, in symbols $\Sigma \models \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$, if for every $\mathcal{M} \in \mathcal{K}$ and $s$ such that the set of variables occurring in $\Sigma \cup \{\bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}\}$ is included in $\text{dom}(s)$ we have that

if $\mathcal{M} \models_s \Sigma$ then $\mathcal{M} \models_s \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$.

**Theorem 3.30.** Let $\Sigma$ be a set of atoms, then

$$\Sigma \models \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z} \implies \Sigma \models \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}.$$  

**Proof.** Let $\mathcal{M} \in \mathcal{K}$ and $s$ an appropriate assignment.

(a5.) $\dim(s(\bar{x})/s(\bar{z})) = 0 = \dim(s(\bar{x})/s(\bar{y}) \cup s(\bar{y}))$, thus $\mathcal{M} \models_s \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z}$.

(b5.)

$$\mathcal{M} \models_s \bar{x} \perp_{\mathcal{L}} \bar{y} \bar{z} \implies s(\bar{x}) \leftarrow[s(\bar{z})] s(\bar{y}) \implies s(\bar{y}) \leftarrow[s(\bar{z})] s(\bar{x}) \implies \mathcal{M} \models_s \bar{y} \perp_{\mathcal{L}} \bar{x}.$$  

[By Lemma]
(e.5.) Suppose that $\mathcal{M} \models \overline{\vec{x}} \perp_{\pi} \overline{\vec{y}}$, then $s(\overline{\vec{x}}) \upharpoonright_{s(\overline{\vec{y}})} s(\overline{\vec{y}})$ and so $s(\overline{\vec{x}}) \upharpoonright_{s(\overline{\vec{y}})} s(\overline{\vec{y}})$ because $\dim(s(\overline{\vec{x}})/s(\overline{\vec{y}}) \cup s(\overline{\vec{y}})) = \dim(s(\overline{\vec{x}})/s(\overline{\vec{y}}) \cup (s(\overline{\vec{y}}) \cup s(\overline{\vec{y}})))$. Furthermore $s(\overline{\vec{x}}) \upharpoonright_{s(\overline{\vec{y}}),s(\overline{\vec{x}})} s(\overline{\vec{y}}) \cup s(\overline{\vec{y}})$ because $\dim(s(\overline{\vec{y}})/s(\overline{\vec{y}}) \cup s(\overline{\vec{y}})) = 0 = \dim(s(\overline{\vec{y}})/s(\overline{\vec{y}}) \cup s(\overline{\vec{y}}))$. Hence by Lemma 3.21 and 3.24 we have that $s(\overline{\vec{x}}) \upharpoonright_{s(\overline{\vec{y}})} s(\overline{\vec{y}})$. Thus $\mathcal{M} \models \overline{\vec{x}} \perp_{\pi} \overline{\vec{y}}$.

(f.5.)
\(g_5.\) \(\mathcal{M} \models \overline{\overline{x}} \perp \overline{\overline{y}}\) and \(\mathcal{M} \models \overline{\overline{x}} \overline{\overline{y}} \perp \overline{\overline{u}}\)

\[s(\overline{x}) \cup s(\overline{y}) \cup s(\overline{u})\]

\[s(\overline{x}) \cup s(\overline{y}) \cup s(\overline{u})\]  
[By Corollary 3.25]

\(h_5.\) Obvious.

Just as with \(\perp \overline{\overline{x}}\), the above theorem does not give the desired completeness of our axioms for \(\perp \overline{\overline{x}}\). In fact we again do not know whether the given axioms are complete, this is another open problem at the moment. By soft arguments (reduction to first order logic by means of extra predicates) one can again argue that there is some effective axiomatization, we just do not have an explicit one.

3.4. Pregeometries in Vector Spaces and Algebraically Closed Fields

Let \(K\) be a field and \(\mathcal{L} = \{+, 0\} \cup \{k : k \in K\}\), where + is a binary function symbol, 0 is a constant and \(k\) is a unary function symbol for each \(k \in K\). Let \(\text{VS}_K\) be the theory which consists of the axioms for additive commutative groups plus the following axioms:

i) \(\forall x \forall y r(x + y) = r(x) + r(y)\) for \(r \in K\);

ii) \(\forall x (r + s)(x) = r(x) + s(x)\) for \(r, s \in K\);

iii) \(\forall x r(s(x)) = rs(x)\) for \(r, s \in K\);

iv) \(\forall x 1(x) = x\).

Any vector space \(\mathcal{V}\) over \(K\) can be seen as model \(\mathcal{V}\) of \(\text{VS}_K\) by interpreting \(k(a)\) as \(ka\) and any model \(\mathcal{V}\) of \(\text{VS}_K\) can be seen as a vector space \(\mathcal{V}\) over \(K\) by defining \(ka\) as \(k(a)\).

**Theorem 3.31.** Let \(\mathcal{V} \models \text{VS}_K\), then \((\mathcal{V}, \langle \rangle)\) is a pregeometry with properties (P1) and (P2). Furthermore, there exists \(\mathcal{V} \models \text{VS}_K\), such that \((\mathcal{V}, \langle \rangle)\) has property (P3).

**Proof.** Let \(\mathcal{V} \models \text{VS}_K\). First of all notice that \((\emptyset) = \{0\}. Let now \(D_0 = \{d_0, \ldots, d_{n-1}\} \subseteq \mathcal{V}\) be an independent set. Clearly

\[\sum_{i=0}^{n-1} d_i \in \langle D_0 \rangle.\]

Suppose that there exists \(j \in \{0, \ldots, n - 1\}\) such that

\[\sum_{i=0}^{n-1} d_i \in \langle d_0, \ldots, d_{j-1}, d_{j+1}, \ldots, d_{n-1}\rangle,\]

then there exists \(\tilde{c} \in K\) such that

\[\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} c_id_i.\]
Hence
\[ d_j = \sum_{i=0}^{n-1} c_i d_i - \sum_{i=0}^{n-1} d_i, \]
a contradiction.

Finally, let \( \kappa > |K| + \aleph_0 \). The theory VS\(_K\) has infinite models, so by the Löwenheim-Skolem theorem there is a structure \( V \models VS_K \) such that \( |V| = \kappa \). Thus \( \dim(V) = \kappa \), because for every \( A \subseteq V \) such that \( |A| < \kappa \), we have that \( \langle A \rangle \leq \aleph_0 + |A| < \kappa \).

We know show that in the case of vector spaces with the span pregeometry, if we restrict to independence over the empty set, we can have a more intuitive characterization of the pregeometric independence relation.

**Lemma 3.32.** Let \( V \models VS_K \), then
\[ \vec{a} \cl \uparrow \emptyset \vec{b} \text{ if and only if } \langle \vec{a} \rangle \cap \langle \vec{b} \rangle = \{0\}, \]
where \( \dim(\vec{a}/\vec{b}) \) and \( \dim(\vec{a}) \) are computed in the pregeometry \( (V, \uparrow) \).

This is a consequence of a more general fact that holds in any modular pregeometry.

**Definition 3.33.** Let \( (M, \cl) \) be a pregeometry. We say that \( (M, \cl) \) is modular if for every closed \( A, B \subseteq M \) we have
\[ \dim(A) + \dim(B) = \dim(A \cup B) + \dim(A \cap B) \]

**Proposition 3.34.** Let \( (M, \cl) \) be a pregeometry. The pregeometry \( (M, \cl) \) is modular if and only if for all \( A, B, C \subseteq M \) such that \( C \subseteq A \cap B \) we have
\[ A \cl \uparrow_C B \text{ if and only if } \cl(A) \cap \cl(B) \subseteq \cl(C) \]

**Proof.** See [3].

Let \( L_r \) be the language of rings \( \{+, -, \cdot, 0, 1\} \), where \(+, -\) and \( \cdot \) are binary function symbols and 0 and 1 are constants. We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences
\[ \forall a_0 \cdots \forall a_{n-1} \exists x \ (x^n + \sum_{i=0}^{n-1} a_i x^i = 0) \]
for \( n = 1, 2, \ldots \). We denote by ACF the axioms for algebraically closed fields.

**Proposition 3.35.** Let \( K \models ACF \) and \( A \subseteq K \). Then \( a \in \acl(A) \) if and only if \( a \) is algebraic over the subfield of \( K \) generated by \( A \).

**Proof.** See [14, Proposition 3.2.15].

**Theorem 3.36.** Let \( K \models ACF \), then \( (K, \acl) \) is a pregeometry with properties (P1) and (P2). Furthermore, there exists \( K \models ACF \), such that \( (K, \acl) \) has property (P3).
Proof. Let $K \models ACF$. First of all, notice that by Proposition 3.35, $acl(\emptyset) = \{a \in K \mid a \text{ is algebraic over } P\} \neq \emptyset$, where $P$ denotes the subfield of $K$ generated by the empty set. Let now $D_0 = \{d_0, \ldots, d_{n-1}\} \subseteq K$ be an independent set. Clearly $\prod_{i=0}^{n-1} d_i \in acl(D_0)$, indeed let $F$ be the subfield of $K$ generated by $D_0$, then $\prod_{i=0}^{n-1} d_i \in F$ and so clearly it is algebraic over $F$. We claim that $\prod_{i=0}^{n-1} d_i \notin \bigcup_{D \subseteq D_0} acl(D)$. It suffices to show that $\prod_{i=0}^{n-1} d_i \notin acl(T)$ for $T \subseteq D_0$ and $|T| = n - 1$. Let then $T = \{d_0, \ldots, d_{l-1}, d_{l+1}, \ldots, d_{n-1}\}$ and suppose that $\prod_{i=0}^{n-1} d_i \in acl(T)$. By Proposition 3.35, $acl(T)$ is the set of algebraic elements over the subfield $F'$ of $K$ generated by $T$. This set is actually a field, so for every $i \in \{0, \ldots, l-1, l+1, \ldots, n-1\}$ we have that $d_i^{-1} \in acl(T)$ and thus that $\prod_{i=0}^{n-1} d_i^{-1} \in acl(T)$. Hence also

$$\prod_{i=0}^{n-1} d_i \cdot \prod_{i \neq l}^{n-1} d_i^{-1} = d_l \in acl(T).$$

Thus $D_0$ is not independent, a contradiction.

Finally, let $\kappa > \aleph_0$. The theory ACF has infinite models, so by the Löwenheim-Skolem theorem there is a structure $K \models ACF$ such that $|K| = \kappa$. Thus $dim(K) = \kappa$, because for every $A \subseteq K$ such that $|A| < \kappa$, we have that $acl(A) \leq \aleph_0 + |A| < \kappa$.

We now show that, when instantiated to the theory ACF, the formal notion of dimension coincides with the algebraic notion of transcendence degree. Let $K$ be an algebraically closed field and $E, F$ subfields of $K$. We denote by $trdg(F/E)$ the transcendence degree of $F$ over $E$.

Lemma 3.37. Let $K \models ACF$, $A, B, C \subseteq K$ and $F$ the subfield of $K$ generated by $C$, then

$$trdg(F(A)/F(B)) = dim(A/C \cup B),$$

where $dim(A/C \cup B)$ is computed in $K$.

Proof. Let $A'$ be a basis for $A$ over $C \cup B$, we want to show that $A'$ is a transcendence basis for $F(A)$ over $F(B)$. Let $a' \in A'$, then $a' \notin acl(C \cup B \cup (A' - \{a'\}))$. By Proposition 3.35, $acl(C \cup B \cup (A' - \{a'\}))$ is the set of algebraic elements over the subfield of $K$ generated by $C \cup B \cup (A' - \{a'\})$, that is $F(B \cup (A' - \{a'\}))$, thus $a'$ is transcendental over $F(B \cup (A' - \{a'\}))$. Hence $A'$ is algebraically independent over $F(B)$. Let $a$ be an element of $F(A)$. By Proposition 3.35, $a \in acl(C \cup A)$. By hypothesis $A \subseteq acl(C \cup B \cup A')$ and so $acl(C \cup A) \subseteq acl(C \cup B \cup A')$. Thus $a \in acl(C \cup B \cup A')$ and then, again by Proposition 3.35, we have that $a$ is algebraic over $F(B \cup A')$. Hence $F(A)$ is algebraic over $F(B \cup A')$.

4. Dependence and Independence in $\omega$-Stable Theories

4.1. Forking and Strongly Minimal Sets

Let $M$ be an $L$-structure and $A \subseteq M$, we denote by $S_n^M(A)$ the space of complete $n$-types over $A$. 
Definition 4.1. Let $T$ be a complete first-order theory in a countable language with infinite models and let $\kappa$ be an infinite cardinal. We say that $T$ is $\kappa$-stable if whenever $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \kappa$, then $|S_n^M(A)| = \kappa$.

Let $T$ be an $\omega$-stable theory in the signature $\mathcal{L}$. Let and $\mathcal{M} \models T$, $A \subseteq M$, $\phi$ an $\mathcal{L}_A$ formula and $p \in S_n^M(A)$, we denote by $RM(\phi)$ and $RM(p)$ the Morley Rank of $\phi$ and $p$ respectively. Furthermore, we denote by $2\mathcal{M}$ the monster model of $T$. For details see [14, Chapter 6].

Definition 4.2. Let $\mathcal{M} \models T$, $A \subseteq B \subseteq M$, $p \in S_n(A)$, $q \in S_n(B)$ and $p \subseteq q$. If $RM(q) < RM(p)$, we say that $q$ is a forking extension of $p$ and that $q$ forks over $A$. If $RM(q) = RM(p)$, we say that $q$ is a non-forking extension of $p$.

Proof. See [14, Theorem 6.3.2].

Definition 4.3. We say that $\bar{a}$ is independent from $B$ over $A$ if $tp(\bar{a}/A \cup B)$ is a non-forking extension of $tp(\bar{a}/A)$. We write $\bar{a} \fork A B$.

This notion of independence as many desirable properties.

Lemma 4.4 (Monotonicity). If $\bar{a} \fork A B$ and $C \subseteq B$, then $\bar{a} \fork A C$.

Proof. $A \subseteq A \cup C \subseteq A \cup B$ so $tp(\bar{a}/A) \subseteq tp(\bar{a}/A \cup C) \subseteq tp(\bar{a}/A \cup B)$ and hence $RM(\bar{a}/A) \geq RM(\bar{a}/A \cup C) \geq RM(\bar{a}/A \cup B)$. Thus if $RM(\bar{a}/A) = RM(\bar{a}/A \cup B)$, then $RM(\bar{a}/A) = RM(\bar{a}/A \cup C)$.

Lemma 4.5 (Transitivity). $\bar{a} \fork A \bar{b} \bar{c}$ if and only if $\bar{a} \fork A \bar{b}$ and $\bar{a} \fork A \bar{b} \bar{c}$.

Proof. $RM(\bar{a}/A \cup \bar{b} \bar{c}) \leq RM(\bar{a}/A \cup \bar{b}) \leq RM(\bar{a}/A)$, so $RM(\bar{a}/A) = RM(\bar{a}/A \cup \bar{b} \bar{c})$ if and only if $RM(\bar{a}/A) = RM(\bar{a}/A \cup \bar{b})$ and $RM(\bar{a}/A \cup \bar{b}) = RM(\bar{a}/A \cup \bar{b} \bar{c})$.

Lemma 4.6 (Finite Basis). $\bar{a} \fork A B$ if and only if $\bar{a} \fork A B_0$ for all finite $B_0 \subseteq B$.

Proof. The direction ($\Rightarrow$) follows from Monotonicity. For the direction ($\Leftarrow$), suppose that $\bar{a} \fork A B$. Then, there is $\bar{\phi}(\bar{v}) \in tp(\bar{a}/A \cup B)$ with $RM(\bar{\phi}) < RM(\bar{a}/A)$. Let $B_0$ be a finite subset of $B$ such that $\bar{\phi}$ is an $\mathcal{L}_{A \cup B_0}$-formula. Then $\bar{a} \fork A B_0$.

Lemma 4.7 (Symmetry). If $\bar{a} \fork A \bar{b}$, then $\bar{b} \fork A \bar{a}$.

Proof. See [14, Lemma 6.3.19].

Corollary 4.8 (Exchange). If $\bar{a} \fork A \bar{b}$ and $\bar{a} \bar{b} \fork A \bar{c}$, then $\bar{a} \fork A \bar{b} \bar{c}$.
Proof.

\[
\begin{align*}
\vec{a} \overset{\text{frk}}{\mapsto} A \vec{b} \quad \text{and} \quad \vec{a} \vec{b} \overset{\text{frk}}{\mapsto} A \vec{c} \\
\vec{b} \overset{\text{frk}}{\mapsto} A \vec{a} \quad \text{by Lemma 4.7} \\
\vec{a} \vec{b} \overset{\text{frk}}{\mapsto} A \vec{c} \quad \text{by Lemma 4.5} \\
\vec{a} \vec{c} \overset{\text{frk}}{\mapsto} A \vec{b} \quad \text{by Lemma 4.7} \\
\vec{a} \vec{b} \vec{c} \overset{\text{frk}}{\mapsto} A \quad \text{by Lemma 4.5}.
\end{align*}
\]

Corollary 4.9. For any \( \vec{b}, \vec{b} \overset{\text{frk}}{\mapsto} A \text{acl}(A) \).

Proof. Let \( \vec{c} \in \text{acl}(A) \), then there exists \( \phi(\vec{v}, \vec{a}) \) and \( \vec{a} \in A \) such that \( \mathfrak{M} \models \phi(\vec{c}, \vec{a}) \) and \( |\phi(\mathfrak{M}, \vec{a})| < \infty \). Thus \( \text{RM}(\phi(\vec{v}, \vec{a})) = 0 \). Furthermore \( \phi(\vec{v}, \vec{a}) \in \text{tp}(\vec{c}/A) \subseteq \text{tp}(\vec{c}/A, \vec{b}) \), so \( \text{RM}(\vec{c}/A, \vec{b}) = \text{RM}(\vec{c}/A) \). Thus \( \vec{c} \overset{\text{frk}}{\mapsto} A \vec{b} \) and, by symmetry, \( \vec{b} \overset{\text{frk}}{\mapsto} A \vec{c} \).

Hence by finite basis \( \vec{b} \overset{\text{frk}}{\mapsto} A \text{acl}(A) \).

Proposition 4.10. If \( \vec{a} \overset{\text{frk}}{\mapsto} B \vec{a} \), then \( \vec{a} \overset{\text{frk}}{\mapsto} B \vec{b} \) for any \( \vec{b} \in \mathfrak{M} \).

Proof. If \( \text{RM}(\vec{a}/B \cup \vec{a}) = \text{RM}(\vec{a}/B) \), then \( \text{RM}(\vec{a}/B) = 0 \) because \( \text{RM}(\vec{a}/B \cup \vec{a}) = 0 \).

So there exists \( \phi(\vec{v}) \in \text{tp}(\vec{a}/B) \) such that \( |\phi(\mathfrak{M})| < \infty \) and thus we have that \( \vec{a} \cup \vec{a} \subseteq \text{acl}(B) \). Let now \( \vec{b} \in \mathfrak{M} \), by Corollary 4.9 we have that \( \vec{b} \overset{\text{cl}}{\mapsto} B \text{cl}(B) \) and hence by Lemma 4.4 and Lemma 4.7 we can conclude that \( \vec{a} \overset{\text{cl}}{\mapsto} B \vec{b} \).

We conclude this section stating a fundamental theorem about Morley Rank in strongly minimal sets, which will play a crucial role in the following.

Definition 4.11. Let \( \mathcal{M} \) be an \( L \)-structure and let \( D \subseteq M^n \) be an infinite definable set. We say that \( D \) is minimal in \( \mathcal{M} \) if for every definable \( Y \subseteq D \) either \( Y \) is finite or \( D - Y \) is finite. If \( \phi(\vec{v}, \vec{a}) \) is the formula that defines \( D \), then we also say that \( \phi(\vec{v}, \vec{a}) \) is minimal. We say that \( D \) and \( \phi \) are strongly minimal if \( \phi \) is minimal in any elementary extension \( \mathcal{N} \) of \( \mathcal{M} \). We say that a theory \( T \) is strongly minimal if the formula \( v = v \) is strongly minimal (i.e. if \( \mathcal{M} \models T \), then \( \mathcal{M} \) is strongly minimal).

If \( \mathcal{M} \) is a model of an \( \omega \)-stable theory and \( F = \phi(\mathcal{M}, \vec{c}) \) is a strongly minimal set we can define a pregeometry \((F, \text{cl}_F)\) by defining \( \text{cl}_F(X) = \text{acl}(\vec{c} \cup X) \cap F \). When \( F = \phi(M, \emptyset) \), we denote the pregeometry \((F, \text{cl}_F)\) simply as \((F, \text{cl})\).

Theorem 4.12. Let \( T \) be an \( \omega \)-stable theory, \( F = \phi(\mathcal{M}, \vec{c}) \) a strongly minimal set, \( \vec{a} \in F \) and \( A \subseteq F \). Then \( \text{RM}(\text{tp}(\vec{a}/A \cup \vec{c})) = \text{dim}(\vec{a}/A) \), where \( \text{dim}(\vec{a}/A) \) is computed in the pregeometry \((F, \text{cl}_F)\).

Proof. Adaptation of [14, Theorem 6.2.19]. See also [19, Section 1.5].
4.2. \(\omega\)-Stable Atomic Independence Logics

\(\omega\)-Stable Atomic Independence Logic (\(\omega\)SAIndL) is defined as follows. The syntax and deductive system of this logic are the same as those of \(A\IndL\). Let \(T\) be a first-order \(\omega\)-stable theory such that there exists a strongly minimal set \(F = \phi(M, \emptyset)\) such that \((F, cl)\) has properties (P1), (P2) and (P3). Let \(\mathcal{M} \models T\) and \(s : \text{dom}(s) \to M\) with \(\vec{x} \vec{y} \subseteq \text{dom}(s) \subseteq \text{Var}\). We say that \(\mathcal{M}\) satisfies \(\vec{x} \perp \vec{y}\) under \(s\), in symbols \(\mathcal{M} \models_s \vec{x} \perp \vec{y}\), if

\[
s(\vec{x}) \downarrow_{\emptyset} s(\vec{y}).
\]

Let \(\Sigma\) be a set of atoms and let \(s\) be such that the set of variables occurring in \(\Sigma\) is included in \(\text{dom}(s)\). We say that \(\mathcal{M}\) satisfies \(\Sigma\) under \(s\), in symbols \(\mathcal{M} \models_s \Sigma\), if \(\mathcal{M}\) satisfies every atom in \(\Sigma\) under \(s\). We say that \(\vec{x} \perp \vec{y}\) is a logical consequence of \(\Sigma\), in symbols \(\Sigma \models \vec{x} \perp \vec{y}\), if for every \(\mathcal{M} \models T\) and \(s\) such that the set of variables occurring in \(\Sigma \cup \{\vec{x} \perp \vec{y}\}\) is included in \(\text{dom}(s)\) we have that

if \(\mathcal{M} \models_s \Sigma\) then \(\mathcal{M} \models_s \vec{x} \perp \vec{y}\).

**Theorem 4.13.** Let \(\Sigma\) be a set of atoms, then

\[\Sigma \models \vec{x} \perp \vec{y}\] if and only if \(\Sigma \models \vec{x} \perp \vec{y}\).

**Proof.** For the direction \((\Leftarrow)\), let \(\mathcal{M} \models T\) and \(s\) an appropriate assignment.

\((a_3.)\) Obvious.

\((b_3.)\)

\[
\mathcal{M} \models_s \vec{x} \perp \vec{y} \quad \implies \quad s(\vec{x}) \downarrow_{\emptyset} s(\vec{y})
\]

\[
\implies \quad s(\vec{y}) \downarrow_{\emptyset} s(\vec{x}) \quad \text{[By Lemma 4.7]}
\]

\[
\implies \quad \mathcal{M} \models_s \vec{y} \perp \vec{x}.
\]

\((c_3.)\)

\[
\mathcal{M} \models_s \vec{x} \perp \vec{y} \quad \implies \quad s(\vec{x}) \downarrow_{\emptyset} s(\vec{y}) s(\vec{z})
\]

\[
\implies \quad s(\vec{z}) \downarrow_{\emptyset} s(\vec{y}) \quad \text{[By Lemma 4.3]}
\]

\[
\implies \quad \mathcal{M} \models_s \vec{x} \perp \vec{y}.
\]

\((d_3.)\)

\[
\mathcal{M} \models_s \vec{x} \perp \vec{y} \quad \text{and} \quad \mathcal{M} \models_s \vec{x} \vec{y} \perp \vec{z}
\]

\[
\Downarrow
\]

\[
s(\vec{x}) \downarrow_{\emptyset} s(\vec{y}) \quad \text{and} \quad s(\vec{z}) s(\vec{y}) \downarrow_{\emptyset} s(\vec{z})
\]

\[
\Downarrow
\]

\[
s(\vec{x}) \downarrow_{\emptyset} s(\vec{y}) s(\vec{z}) \quad \text{[By Corollary 4.8]}
\]

\[
\mathcal{M} \models_s \vec{x} \perp \vec{y} \vec{z}.
\]

\((e_3.)\) Suppose that \(\mathcal{M} \models_s x \perp x\), then \(s(x) \downarrow_{\emptyset}^c s(x)\) and so by Proposition 4.10, we have that \(s(x) \downarrow_{\emptyset}^c s(\vec{y})\) for any \(\vec{y} \in \text{Var}\).

\((f_3.)\) Obvious.

For the direction \((\Rightarrow)\), suppose \(\Sigma \not\models \vec{x} \perp \vec{y}\). Notice that if this is the case then \(\vec{x} \neq \emptyset\) and \(\vec{y} \neq \emptyset\). Indeed if \(\vec{y} = \emptyset\) then \(\Sigma \models \vec{x} \perp \vec{y}\) because by rule \((a_3.)\) \(\models \vec{x} \perp \emptyset\). Analogously if \(\vec{x} = \emptyset\) then \(\Sigma \models \vec{x} \perp \vec{y}\) because by rule \((a_3.)\) \(\models \vec{x} \perp \emptyset\) and so by rule \((b_3.)\) \(\models \emptyset \perp \vec{y}\). We can assume that \(\vec{x}\) and \(\vec{y}\) are injective. This is without loss of generality because clearly \(\mathcal{M} \models_s \vec{x} \perp \vec{y}\) if and only if \(\mathcal{M} \models_s \pi \vec{x} \perp \pi \vec{y}\), where
π : Var<ω → Var<ω is the function that eliminates repetitions in finite sequences of variables. Furthermore we can assume that $\vec{x} \perp \vec{y}$ is minimal, in the sense that if $\vec{x}′ \subseteq \vec{x}$, $\vec{y}′ \subseteq \vec{y}$ and $\vec{x}′ \neq \vec{x}$ or $\vec{y}′ \neq \vec{y}$, then $\Sigma \vdash \vec{x}′ \perp \vec{y}′$. This is for two reasons.

i) If $\vec{x} \perp \vec{y}$ is not minimal we can always find a minimal atom $\vec{x}^* \perp \vec{y}^*$ such that $\Sigma \not\vdash \vec{x}^* \perp \vec{y}^*$, $\vec{x}^* \subseteq \vec{x}$ and $\vec{y}^* \subseteq \vec{y}$ — just keep deleting elements of $\vec{x}$ and $\vec{y}$ until you obtain the desired property or until both $\vec{x}^*$ and $\vec{y}^*$ are singletons, in which case, due to the trivial independence rule $(a_3)$, $\vec{x}^* \perp \vec{y}^*$ is a minimal statement.

ii) For any $\vec{x} \subseteq \vec{x}$ and $\vec{y} \subseteq \vec{y}$ we have that if $M \not\models_s \vec{x} \perp \vec{y}$ then $M \not\models_s \vec{x} \perp \vec{y}$, for every $M$ and $s$.

Let indeed $\vec{x} = \vec{x}′$ and $\vec{y} = \vec{y}′$, then $\Sigma \vdash \vec{x} \perp \vec{y}$.

Let then $\vec{x} = (x_{j_0}, \ldots, x_{j_{n-1}})$ and $\vec{y} = (y_{k_0}, \ldots, y_{k_{m-1}})$ be injective and such that $\vec{x} \perp \vec{y}$ is minimal.

Let $V = \{v \in \text{Var} | \Sigma \vdash v \perp v\}$ and $W = \text{Var} - V$. We claim that $\vec{x}, \vec{y} \subseteq V$. We prove it only for $\vec{x}$, the other case is symmetrical. Suppose that $\vec{x} \subseteq V$, then for every $x \in \vec{x}$ we have that $\Sigma \vdash x \perp x$ so by rule $(c_3)$, $(b_3)$ and $(d_3)$

$$\Sigma \vdash \vec{y} \perp x_{j_0} \quad \text{and} \quad \Sigma \vdash \vec{y} x_{j_0} \perp x_{j_1} \Rightarrow \Sigma \vdash \vec{y} \perp x_{j_0} x_{j_1},$$

$$\Sigma \vdash \vec{y} \perp x_{j_0} x_{j_1} \quad \text{and} \quad \Sigma \vdash \vec{y} x_{j_0} x_{j_1} \perp x_{j_2} \Rightarrow \Sigma \vdash \vec{y} \perp x_{j_0} x_{j_1} x_{j_2},$$

$$\Sigma \vdash \vec{y} \perp x_{j_0} \ldots x_{j_{n-2}} \quad \text{and} \quad \Sigma \vdash \vec{y} x_{j_0} \ldots x_{j_{n-2}} \perp x_{j_{n-1}} \Rightarrow \Sigma \vdash \vec{y} \perp \vec{x}.$$ 

Hence by rule $(b_3)$, $\Sigma \vdash \vec{x} \perp \vec{y}$. Thus $\vec{x} \cap W \neq \emptyset$ and $\vec{y} \cap W \neq \emptyset$. Without loss of generality suppose that $x_{j_0} \in W$ and $y_{k_0} \in W$. Let $\vec{x} \cap W = \vec{x} = (x_{j_0}, \ldots, x_{j_{n-1}}) = (x_{j_0}, \ldots, x_{j_{n-1}'}) \neq \emptyset$ and $\vec{y} \cap W = \vec{y} = (y_{k_0}, \ldots, y_{k_{m-1}'}, \ldots)$. Notice that $\vec{x} \cap \vec{y} = \emptyset$. Indeed let $z \in \vec{x} \cap \vec{y}$, then by rules $(b_3)$ and $(c_3)$ we have that $\Sigma \vdash z \perp z$. Thus $z \in V$, a contradiction.

By assumption there exists a strongly minimal set $F = \phi(\mathfrak{M}, \emptyset)$ such that $(F, \text{cl})$ has properties $(P1)$, $(P2)$ and $(P3)$. Let then $\{a_i | i \in \kappa\}$ be an injective enumeration of a basis $B$ for $F$ and $\{w_i | i \in \lambda\}$ be an injective enumeration of $W - \{x_{j_0}\}$. Notice that, because of property $(P3)$, we have that $\kappa \geq \lambda$, indeed either $\lambda = \omega$ or $\lambda = n$ for $n \in \omega$. Let $s$ be the following assignment:

i) $s(v) = e$ for every $v \in V$,

ii) $s(w_i) = a_i$ for every $i \in \lambda$,

iii) $s(x_{j_0}) = d$,

where $e \in \text{cl}(\emptyset)$ and $d$ is such that

$$d \in \text{cl}(\{s(x_{j_1}), \ldots, s(x_{j_{n-1}'}, \ldots, s(y_{k_0}'), \ldots, s(y_{k_{m-1}'}))\})$$

but $d \notin \text{cl}(D)$ for every $D \subseteq \{s(x_{j_1}), \ldots, s(x_{j_{n-1}'}, \ldots, s(y_{k_0}'), \ldots, s(y_{k_{m-1}'}))\}$. Notice that $e$ and $d$ do exist because of properties $(P1)$ and $(P2)$.
Notice that by Theorem 4.12 for every \( \vec{x}, \vec{y} \in \text{Var} \), we have that

\[
(*) \quad s(\vec{x}) \upharpoonright \emptyset \downarrow s(\vec{y}) \text{ if and only if } s(\vec{x}) \downarrow_1 s(\vec{y}).
\]

We claim that \( \mathfrak{M} \not\models_s \vec{x} \perp \vec{y} \), as noticed this implies that \( \mathfrak{M} \models_s \vec{x} \perp \vec{y} \). First we show that the set \( \{ s(x') \mid x' \in \vec{x} \} \) is independent. By construction \( s(x_j) \notin \text{cl}(\{ s(x') \mid x' \in \vec{x} \} - \{ s(x_j) \}) \). Let then \( i \in \{ 1, ..., n' - 1 \} \) and suppose that \( s(x_{j_i}) \in \text{cl}(\{ s(x_{j_1}), ..., s(x_{j_{i-1}}), s(x_{j_{i+1}}), ..., s(x_{j_{n-1}}) \}) \).

We now show that \( s(x_{j_i}) \notin \text{cl}(\{ s(x_{j_1}), ..., s(x_{j_{i-1}}), s(x_{j_{i+1}}), ..., s(x_{j_{n-1}}) \}) \).

Thus, by Lemma 3.18, \( s(x_{j_i}) \in \text{cl}(\{ s(x_{j_1}), ..., s(x_{j_{i-1}}), s(x_{j_{i+1}}), ..., s(x_{j_{n-1}}) \}) \), a contradiction. Thus \( \dim(s(\vec{x}')) = | \{ s(x') \mid x' \in \vec{x} \} | \).

We now show that \( \{ s(x_{j_1}), ..., s(x_{j_{n-1}}) \} \) is a basis for \( \{ s(x') \mid x' \in \vec{x} \} \) over \( \{ s(y') \mid y' \in \vec{y} \} \). As we noticed above \( \vec{x} \cap \vec{y} = \emptyset \), so by properties of our assignment \( s(\vec{x}) \cap s(\vec{y}) = \emptyset \). Thus, by Lemma 3.18 \( \{ s(x_{j_1}), ..., s(x_{j_{n-1}}) \} \) is independent over \( \{ s(y') \mid y' \in \vec{y} \} \), also \( \{ s(x_{j_0}), ..., s(x_{j_{n-1}}) \} \subseteq \text{cl}(s(\vec{y})) \cup \{ s(x_{j_1}), ..., s(x_{j_{n-1}}) \} \).

Because \( s(x_{j_0}) \in \text{cl}(\{ s(x_{j_1}), ..., s(x_{j_{n-1}}), s(y_{j_0}), ..., s(y_{j_{n-1}}) \}) \).

Hence \( \dim(s(\vec{x}'))/s(\vec{y}) = | \{ s(x_{j_1}), ..., s(x_{j_{n-1}}) \} | = \dim(s(\vec{x}')) - 1 \). So \( s(\vec{x}') \upharpoonright_0 s(\vec{y}) \) and then by (*) we have that \( s(\vec{x}') \upharpoonright_0 s(\vec{y}) \), that is \( \mathfrak{M} \models_s \vec{x} \perp \vec{y} \).

Let now \( \vec{v} \perp \vec{w} \in \Sigma \), we want to show that \( \mathfrak{M} \models_s \vec{v} \perp \vec{w} \). As before, we assume, without loss of generality, that \( \vec{v} \) and \( \vec{w} \) are injective. Notice also that if \( \vec{v} = \emptyset \) or \( \vec{w} = \emptyset \), then \( \mathfrak{M} \models_s \vec{v} \perp \vec{w} \). Thus let \( \vec{v}, \vec{w} \neq \emptyset \).

**Case 1.** \( \vec{v} \subseteq V \) or \( \vec{w} \subseteq V \). Suppose that \( \vec{v} \subseteq V \), the other case is symmetrical, then \( s(\vec{v}) \subseteq \text{cl}(\emptyset) \). Thus \( \dim(s(\vec{v})/s(\vec{w})) = 0 = \dim(s(\vec{v})) \). So \( s(\vec{v}) \upharpoonright_0 s(\vec{w}) \) and hence by (*) we have that \( s(\vec{v}) \upharpoonright_0 s(\vec{w}) \), that is \( \mathfrak{M} \models_s \vec{v} \perp \vec{w} \).

**Case 2.** \( \vec{v} \not\subseteq V \) and \( \vec{w} \not\subseteq V \). Let \( \vec{v} \cap W = \vec{v}' \neq \emptyset \) and \( \vec{w} \cap W = \vec{w}' \neq \emptyset \). Notice that

\[
(**) \quad s(\vec{v}) \upharpoonright_0 s(\vec{w}) \text{ if and only if } s(\vec{v}') \upharpoonright_0 s(\vec{w}').
\]

Left to right holds in general. As for the other direction, suppose that \( s(\vec{v}') \upharpoonright_0 s(\vec{w}') \). If \( u \in \vec{v} \vec{w} - \vec{v}' \vec{w}' \), then \( s(u) = e \in \text{cl}(\emptyset) \). Thus
\[ s(\vec{v}) \upharpoonright \delta \rightarrow s(\vec{w}) \]
\[ s(\vec{v}) \upharpoonright \delta \rightarrow s(\vec{u}) \] and \[ s(\vec{v}) \downarrow \rightarrow s(\vec{v}) \downarrow \text{cl}(\emptyset) \] \[ s(\vec{v}) \downarrow \rightarrow s(\vec{w}) \cup \text{cl}(\emptyset) \]
\[ s(\vec{v}) \downarrow \rightarrow s(\vec{u}) \cup (\text{cl}(\emptyset) \cap s(\vec{w})) \]
\[ s(\vec{v}) \downarrow \rightarrow s(\vec{w}) \]

So
\[ s(\vec{v}) \downarrow \rightarrow s(\vec{v}) \downarrow \rightarrow s(\vec{w}) \] \[ \text{By (⋆)} \text{ and Lemma 6.23} \]

**Subcase 2.1.** \( x_{j_0} \notin \vec{v} \vec{u}' \). Notice that \( \vec{v} \cap \vec{u}' = \emptyset \), so by properties of our assignment \( s(\vec{v}) \cap s(\vec{u}') = \emptyset \). Thus, by Lemma 6.18 it follows that \( \dim(s(\vec{v})/s(\vec{u}')) = \dim(s(\vec{v})) \). So \( s(\vec{v}) \downarrow \rightarrow s(\vec{w}) \) and then by (⋆) we have that \( s(\vec{v}) \upharpoonright \delta \rightarrow s(\vec{w}) \) which by (⋆⋆) implies that \( \mathcal{M} \models \vec{v} \perp \vec{w} \), that is \( \mathcal{M} \models x_{j_0} \vec{v} \perp \vec{w} \).

**Subcase 2.2.** \( x_{j_0} \in \vec{v} \vec{u}' \).

**Subcase 2.2.1.** \( \vec{v}' \vec{y}' \cap \vec{v} \vec{u}' \neq \emptyset \). Let \( \vec{v}' \vec{u}' - \{x_{j_0}\} \cap \vec{v}' \vec{y}' = \{u_{k_0}, \ldots, u_{k_{b-1}}\} \), \( \vec{v}' \vec{u}' - \vec{v}' \vec{y}' = \{u_{k_0}, \ldots, u_{k_{b-1}}\} \), \( w_{r_i} = u_{k_i} \) for every \( i \in \{0, \ldots, b - 1\} \) and \( w_{r_i} = u_{k_i} \) for every \( i \in \{0, \ldots, t - 1\} \). Suppose now that the set \( \{a_{r_0}, \ldots, a_{r_{b-1}}, d, a_{r_{t_i}}, \ldots, a_{r_{i+1}}\} \) is dependent. There are three cases.

**Case 1.** \( a_{r_i} \in \text{cl}\{a_{r_0}, \ldots, a_{r_{b-1}}, d, a_{r_{i-1}}, a_{r_{t_i}}, \ldots, a_{r_{i+1}}\} \). If this is the case, then
\[ a_{r_{i'}} \in \text{cl}\{s(x_{j_1}), \ldots, s(x_{j_{b-1}}), s(y_{k_0}), \ldots, s(y_{k_{b-1}}), a_{r_{i-1}}, a_{r_{i+1}}, \ldots, a_{r_{i' - 1}}\} \]
but \( d \in \text{cl}\{s(x_{j_1}), \ldots, s(x_{j_{b-1}}), s(y_{k_0}), \ldots, s(y_{k_{b-1}})\} \). This is absurd though because the set \( \{s(x_{j_1}), \ldots, s(x_{j_{b-1}}), s(y_{k_0}), \ldots, s(y_{k_{b-1}}), a_{r_0}, \ldots, a_{r_{i-1}}\} \) is made of distinct elements of the basis \( B \) and so it is independent.

**Case 2.** \( d \notin \text{cl}\{a_{r_0}, \ldots, a_{r_{b-1}}\} \). Notice that
\[ d \notin \text{cl}\{a_{r_0}, a_{r_{t_i}}, \ldots, a_{r_{i+1}}\} \]
because \( \{ a_{r_0}', ..., a_{r_{m-1}}' \} \subseteq \{ s(x_{j_1}'), ..., s(x_{j_{m'-1}}'), s(y_{k_0}'), ..., s(y_{k_{m'-1}}') \} \) and \( d \) has been chosen such that \( d \in \text{cl}(\{ s(x_{j_1}'), ..., s(x_{j_{m'-1}}'), s(y_{k_0}'), ..., s(y_{k_{m'-1}}') \}) \) but \( d \not\in \text{cl}(D) \) for every \( D \subseteq \{ s(x_{j_1}'), ..., s(x_{j_{m'-1}}'), s(y_{k_0}'), ..., s(y_{k_{m'-1}}') \} \). Thus there is \( l \leq t - 1 \) such that
\[ d \in \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_0}''', ..., a_{r_{l-1}}''' \} \cup \{ a_{r_l}''' \}) - \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_0}''', ..., a_{r_{l-1}}''' \}) \]
and then by the Exchange Principle we have that
\[ a_{r_l}''' \in \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_0}''', ..., a_{r_{l-1}}''' \} \cup \{ d \}). \]

Thus we have that \( a_{r_l}''' \in \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', d, a_{r_0}''', ..., a_{r_{l-1}}''' \}), \) which is impossible as we saw in Case 1.

Case 3. \( a_{r_l}'' \in \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_{l+1}}', ..., a_{r_{l-1}}''', d, a_{r_0}''', ..., a_{r_{l-1}}''' \}). \) Notice that
\[ a_{r_l}''' \notin \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_{l+1}}', ..., a_{r_{l-1}}''', a_{r_0}''', ..., a_{r_{l-1}}''' \}). \]

Thus by the Exchange Principle we have that \( d \in \text{cl}(\{ a_{r_0}', ..., a_{r_{l-1}}', a_{r_0}''', ..., a_{r_{l-1}}''' \}) \), which is impossible as we saw in Case 2.

We can then conclude that the set \( \{ a_{r_0}', ..., a_{r_{l-1}}', d, a_{r_0}''', ..., a_{r_{l-1}}''' \} \) is independent.

Clearly \( \{ s(v') \mid v' \in \vec{v} \} \cup \{ s(w) \mid w \in \vec{w} \} = \{ a_{r_0}', ..., a_{r_{l-1}}', d, a_{r_0}''', ..., a_{r_{l-1}}''' \} \). Furthermore, as we noticed above, \( s(\vec{v}) \cap s(\vec{w}) = \emptyset \). Thus by Lemma 3.18 we have that \( \dim(s(\vec{v})/s(\vec{w})) = 1 \). So \( s(\vec{v}) \not\subseteq s(\vec{w}) \) and then by \((\star)\) we have that \( s(\vec{v}) \not\subseteq s(\vec{w}) \) which by \((\star \star)\) implies that \( s(\vec{v}) \not\subseteq s(\vec{w}) \), that is \( 2^n \models \vec{v} \perp \vec{w} \).

Subcase 2.2.2. \( \vec{x} \vec{y} \subseteq \vec{v} \vec{w} \). As shown in Theorem 3.28, this case is not possible.

\[ \textbf{ω-Stable Atomic Conditional Independence Logic (ωSACIndL)} \] is defined as follows. The syntax and deductive system of this logic are the same as those of ACIndL. Let \( T \) be a first-order \( \omega \)-stable theory such that there exists a strongly minimal set \( F = \phi(2^n, \emptyset) \) such that \( (F, \text{cl}) \) has properties (P1), (P2) and (P3). Let \( M \models T \) and \( s : \text{dom}(s) \rightarrow M \) with \( \vec{x} \vec{y} \subseteq \text{dom}(s) \subseteq \text{Var} \). We say that \( M \) satisfies \( \vec{x} \perp \vec{y} \) under \( s \), in symbols \( M \models_s \vec{x} \perp \vec{y} \), if

\[ s(\vec{x}) \not\subseteq s(\vec{y}). \]

Let \( \Sigma \) be a set of atoms and let \( s \) be such that the set of variables occurring in \( \Sigma \) is included in \( \text{dom}(s) \). We say that \( M \) satisfies \( \Sigma \) under \( s \), in symbols \( M \models_s \Sigma \), if \( M \) satisfies every atom in \( \Sigma \) under \( s \). We say that \( \vec{x} \perp \vec{y} \) is a logical consequence of \( \Sigma \), in symbols \( M \models \vec{x} \perp \vec{y} \), if for every \( M \models T \) and \( s \) such that the set of variables occurring in \( \Sigma \cup \{ \vec{x} \perp \vec{y} \} \) is included in \( \text{dom}(s) \) we have that

\[ \text{if } M \models_s \Sigma \text{ then } M \models_s \vec{x} \perp \vec{y}. \]

**Theorem 4.14.** Let \( \Sigma \) be a set of atoms, then

\[ \Sigma \models \vec{x} \perp \vec{y} \Rightarrow \Sigma \models \vec{x} \perp \vec{y}. \]
Proof. Let \( M \models T \) and \( s \) an appropriate assignment.

\((a_5.)\) \( \text{RM}(s(\vec{x})/s(\vec{x})) = 0 = \text{RM}(s(\vec{x})/s(\vec{x}) \cup s(\vec{y})) \), thus \( M \models s \vec{x} \perp_\vec{z} \vec{y} \).

\((b_5.)\)
\[
\begin{align*}
M \models s \vec{x} \perp_\vec{z} \vec{y} & \implies s(\vec{x}) \frk_{s(\vec{z})} s(\vec{y}) \\
& \implies s(\vec{y}) \frk_{s(\vec{z})} s(\vec{x}) \quad \text{[By Lemma 4.7]} \\
& \implies M \models s \vec{y} \perp_\vec{z} \vec{x}.
\end{align*}
\]

\((c_5.)\)
\[
\begin{align*}
M \models s \vec{x} \vec{y} \perp_\vec{z} \vec{y} & \implies s(\vec{x})s(\vec{y}) \frk_{s(\vec{z})} s(\vec{y}) \\
& \implies s(\vec{y}) \frk_{s(\vec{z})} s(\vec{y}) \quad \text{[By Lemma 4.4]} \\
& \implies s(\vec{x}) \frk_{s(\vec{z})} s(\vec{y}) \quad \text{[By Lemma 4.4 and 4.7]} \\
& \implies M \models s \vec{x} \perp_\vec{z} \vec{y}.
\end{align*}
\]

\((d_5.)\) Suppose that \( M \models s \vec{x} \perp_\vec{z} \vec{y} \), then \( s(\vec{x}) \frk_{s(\vec{z})} s(\vec{y}) \) and so \( s(\vec{x}) \frk_{s(\vec{z})} s(\vec{y})s(\vec{z}) \) because \( \text{RM}(s(\vec{x})/s(\vec{z}) \cup s(\vec{y})) = \text{RM}(s(\vec{x})/s(\vec{z}) \cup (s(\vec{y}) \cup s(\vec{z}))) \). Furthermore \( s(\vec{z}) \frk_{s(\vec{z})} s(\vec{y})s(\vec{z}) \) because \( \text{RM}(s(\vec{z})/s(\vec{z}) \cup s(\vec{x}) \cup s(\vec{y}) \cup s(\vec{z})) = 0 = \text{RM}(s(\vec{z})/s(\vec{z}) \cup s(\vec{x}) \cup s(\vec{y}) \cup s(\vec{z})) \). Hence by Lemma 4.7 and 4.5 we have that \( s(\vec{x})s(\vec{z}) \frk_{s(\vec{z})} s(\vec{y})s(\vec{z}) \).

\((e_5.)\)
\[
\begin{align*}
M \models s \vec{x} \perp_\vec{z} \vec{y} & \implies s(\vec{x}) \frk_{s(\vec{z})} s(\vec{y}) \\
& \quad \quad \downarrow \quad \downarrow \\
& \quad \quad s(\vec{y}) \quad s(\vec{x}) \\
& \quad \quad \frk_{s(\vec{z})} \quad \frk_{s(\vec{z})} \\
& \quad \quad s(\vec{y})s(\vec{u}) \quad s(\vec{x})s(\vec{u}) \quad \text{[By Lemma 4.7 and 4.5]} \\
& \quad \quad \downarrow \quad \quad \downarrow \\
& \quad \quad s(\vec{y}) \quad s(\vec{x})s(\vec{y}) \\
& \quad \quad \frk_{s(\vec{z})} \quad \frk_{s(\vec{z})} \\
& \quad \quad s(\vec{y}) \quad s(\vec{x}) \\
& \quad \quad \frk_{s(\vec{z})} \quad \frk_{s(\vec{z})} \\
& \quad \quad \quad \quad \quad \text{[By Lemma 4.4 and 4.7]} \\
& \quad \quad \quad \quad \quad M \models \vec{u} \perp_\vec{z} \vec{y}.
\end{align*}
\]

\((f_5.)\)
\[
\begin{align*}
M \models \vec{y} \perp_\vec{z} \vec{y} & \implies \vec{y} \perp_\vec{z} \vec{y} \\
& \quad \quad \downarrow \\
& \quad \quad s(\vec{y}) \\
& \quad \quad \frk_{s(\vec{z})} \\
& \quad \quad s(\vec{y})s(\vec{x}) \quad s(\vec{u}) \quad \text{[By Proposition 4.10]} \\
& \quad \quad \downarrow \quad \quad \downarrow \\
& \quad \quad s(\vec{z})s(\vec{x}) \downarrow s(\vec{u}) \\
& \quad \quad \frk_{s(\vec{z})} \quad \frk_{s(\vec{z})} \\
& \quad \quad s(\vec{y}) \downarrow s(\vec{u}) \\
& \quad \quad \frk_{s(\vec{z})} \\
& \quad \quad s(\vec{y}) \frk_{s(\vec{z})} s(\vec{u}) \quad \text{[By what we showed in (e_5.)]} \\
& \quad \quad \downarrow \\
& \quad \quad M \models \vec{x} \perp_\vec{z} \vec{u}.
\end{align*}
\]
(g5.)
\[\mathcal{M} \models_s \vec{x} \perp \vec{z} \vec{y} \text{ and } \mathcal{M} \models_s \vec{x} \vec{y} \perp \vec{z} \vec{u}\]
\[\Downarrow\]
\[s(\vec{x}) \nabla_{s(\vec{z})} s(\vec{y}) \text{ and } s(\vec{x})s(\vec{y}) \nabla_{s(\vec{z})} s(\vec{u})\]
\[\Downarrow\]
\[s(\vec{x}) \nabla_{s(\vec{z})} s(\vec{y})s(\vec{u})\]
\[\Downarrow\]
\[\mathcal{M} \models_s \vec{x} \perp \vec{z} \vec{y} \vec{u}\]

(h5.) Obvious.

Once again the above theorem does not give the desired completeness of our axioms for \(\vec{x} \perp \vec{z} \vec{y}\). In fact we do not know whether the given axioms are complete, this is another open problem at the moment. By reduction to first order logic by means of extra predicates one can argue that there is some effective axiomatization, we just do not have an explicit one. For particular theories \(T\) we can make a further conclusion\(^1\). Suppose \(T\) is the theory of vector spaces over a fixed finite field. Then \(T\) is decidable, so in this case we get the decidability of the relation \(\Sigma \models \vec{x} \perp \vec{z} \vec{y}\) for finite \(\Sigma\).

### 4.3. \(\omega\)-Stable Atomic Dependence Logic

As known [7], in dependence logic the dependence atom is expressible in terms of the conditional independence atom. Indeed given a FO-structure \(\mathcal{M}\) and an appropriate \(X\) we have that

\[\mathcal{M} \models_X = (\vec{x}, \vec{y}) \text{ if and only if } \mathcal{M} \models_X \vec{y} \perp_{\vec{x}} \vec{y}.\]

\(\omega\)-Stable Atomic Dependence Logic (\(\omega\)SADL) is defined as follows. The syntax and deductive system of this system are the same as those of ADL. Let \(T\) a first-order \(\omega\)-stable theory such that there exists \(\mathcal{M} \models T\) such that the closure system \((\mathcal{M}, \text{acl})\) has the following two properties:

(CL1) \(\text{acl}(\emptyset) \neq \emptyset\);

(CL2) \(\text{acl}(\emptyset) \neq \mathcal{M}\).

Let \(\mathcal{M} \models T\) and \(s : \text{dom}(s) \rightarrow M\) with \(\vec{x} \vec{y} \subseteq \text{dom}(s) \subseteq \text{Var}\). We say that \(\mathcal{M}\) satisfies \(= (\vec{x}, \vec{y})\) under \(s\), in symbols \(\mathcal{M} \models_s = (\vec{x}, \vec{y})\), if

\[s(\vec{y}) \nabla_{s(\vec{x})} s(\vec{y}).\]

Let \(\Sigma\) be a set of atoms and let \(s\) be such that the set of variables occurring in \(\Sigma\) is included in \(\text{dom}(s)\). We say that \(\mathcal{M}\) satisfies \(\Sigma\) under \(s\), in symbols \(\mathcal{M} \models_s \Sigma\), if \(\mathcal{M}\) satisfies every atom in \(\Sigma\) under \(s\). We say that \(= (\vec{x}, \vec{y})\) is a logical consequence of \(\Sigma\), in symbols \(\mathcal{M} \models = (\vec{x}, \vec{y})\), if for every \(\mathcal{M} \models T\) and \(s\) such that the set of variables occurring in \(\Sigma \cup \{= (\vec{x}, \vec{y})\}\) is included in \(\text{dom}(s)\) we have that

\[\text{if } \mathcal{M} \models_s \Sigma \text{ then } \mathcal{M} \models_s = (\vec{x}, \vec{y}).\]

\(^1\)We are indebted to Tapani Hyttinen for pointing this out.
Theorem 4.15. Let $\mathcal{M} \models T$ and $\vec{a}, \vec{b} \in M$, then

\[ \vec{b} \frk \parallel \vec{a} \vec{b} \text{ if and only if } \forall b \in \vec{b} \ b \in \acl(\vec{a}). \]

Proof.

\[ \vec{b} \frk \parallel \vec{a} \vec{b} \iff \RM(\text{tp}(\vec{b}/\vec{a} \cup \vec{b})) = \RM(\text{tp}(\vec{b}/\vec{a})) \]

\[ \iff \RM(\text{tp}(\vec{b}/\vec{a})) = 0 \]

\[ \iff \exists \phi(\vec{a}) \in \text{tp}(\vec{b}/\vec{a}) \text{ s.t. } |\phi(\mathcal{M})| < \infty \]

\[ \iff \vec{b} \in \acl(\vec{a}) \]

\[ \iff \forall b \in \vec{b} \ b \in \acl(\vec{a}). \]

\[ \blacksquare \]

Theorem 4.16. Let $\Sigma$ be a set of atoms, then

\[ \Sigma \models = (\vec{x}, \vec{y}) \text{ if and only if } \Sigma \vdash = (\vec{x}, \vec{y}). \]

Proof. Because of Theorem 4.15, this reduces to Theorem 3.6.

4.4. Forking in Vector Spaces and Algebraically Closed Fields

We denote by $\text{VS}^{\text{inf}}_K$ the theory

\[ \text{VS}_K \cup \{ \exists x_0 \ldots \exists x_{n-1} \bigwedge_{i,j=0, i \neq j} x_i \neq x_j \mid n \in \omega \}. \]

Proposition 4.17. i) The theory $\text{VS}^{\text{inf}}_K$ is strongly minimal.

ii) Let $\mathcal{V} \models \text{VS}^{\text{inf}}_K$ and $A \subseteq V$, then $\acl(A) = \langle A \rangle$.

Proof. See [14, Example 8.1.10].

For the rest of the section let $K$ be a countable field.

Proposition 4.18. The theory $\text{VS}^{\text{inf}}_K$ is $\omega$-stable.

Proof. The theory $\text{VS}^{\text{inf}}_K$ is $\kappa$-categorical for $\kappa \geq \aleph_1$. Thus by [14, Theorem 6.1.18] we have that $T$ is $\omega$-stable. Notice that the countability of $K$ is necessary here, and indeed [14, Theorem 6.1.18] applies only for countable signatures.

Proposition 4.19. The theory $\text{VS}^{\text{inf}}_K$ has a strongly minimal set $F = \phi(\mathcal{M}, \emptyset)$ such that $(F, \cl)$ has properties (P1), (P2) and (P3).

Proof. Let $\mathcal{V} \models \text{VS}^{\text{inf}}_K$. By Proposition 4.17(i), the set $V$ is strongly minimal. Thus the pregeometry $(V, \cl)$ defined as $\cl(X) = \acl(X) \cap V$, is just the pregeometry $(V, \acl)$, which in turn, by Proposition 4.17(ii), is just the pregeometry $(V, \langle \rangle)$. Hence the proof reduces to what we showed in Theorem 3.31.

We denote with $\emptyset \models^\text{VS}^{\text{inf}}_K$ and $\acl \models^\text{VS}^{\text{inf}}_K$ the satisfaction relation of the systems $\text{PGACIndL}$, $\text{PGAIndL}$ and $\text{CLADL}$ relative to the theory $\text{VS}^{\text{inf}}_K$ and the pregeometric operators $\langle \rangle$ and $\acl$ respectively. We denote by $\frk \models^\text{VS}^{\text{inf}}_K$ the satisfaction relation of the systems $\omega \text{SACIndL}$, $\omega \text{SAIndL}$ and $\omega \text{SADL}$ relative to the theory
VS_{\text{inf}}^K$. By what we showed in this section, Theorem 4.12 and Theorem 3.13 it follows directly the following theorem.

**Theorem 4.20.** Let $V \models VS_{\text{inf}}^K$ and $s : \text{dom}(s) \to V$ with $\vec{x}\vec{y}\vec{z} \subseteq \text{dom}(s) \subseteq \text{Var}$, then

\[
\begin{align*}
V \models VS_{\text{inf}}^K \vec{x} \perp \vec{z} \vec{y} & \iff V \models s!VS_{\text{inf}}^K \vec{x} \perp \vec{z} \vec{y}, \\
V \models (\vec{x}, \vec{y}) & \iff V \models s!(\vec{x}, \vec{y}).
\end{align*}
\]

We now pass to the study of forking in $\text{ACF}$. For $p = 0$ or a prime number we denote by $\text{ACF}_p$ the theory of algebraically closed fields of characteristic $p$.

**Proposition 4.21.** i) The theory $\text{ACF}_p$ is strongly minimal.

ii) The theory $\text{ACF}_p$ is $\omega$-stable.

iii) The theory $\text{ACF}_p$ has a strongly minimal set $F = \phi(\mathfrak{M}, \emptyset)$ such that $(F, \text{cl})$ has properties (P1), (P2) and (P3).

**Proof.** i). See [14, Corollary 3.2.9].

ii). The theory $\text{ACF}_p$ is $\kappa$-categorical for $\kappa \geq \aleph_1$. Thus by [14, Theorem 6.1.18] we have that $T$ is $\omega$-stable.

iii). Let $K$ be an algebraically closed field such that $\text{trdg}(K/P) = \aleph_0$, where $P$ denotes the prime field of $K$. By i), the set $K$ is strongly minimal. Thus the pregeometry $(K, \text{cl})$ defined as $\text{cl}(X) = \text{acl}(X) \cap K$, is just the pregeometry $(K, \text{acl})$. Hence the proof reduces to what we showed in Theorem 3.36.

We denote with $\text{acl} \models_{\text{ACF}_p}$ the satisfaction relation of the systems PGACIndL, PGAIndL and PGADL relative to the theory $\text{ACF}_p$ and the pregeometric operator acl. We denote by $\text{frk} \models_{\text{ACF}_p}$ the satisfaction relation of the systems $\omega\text{SACIndL}$, $\omega\text{SAIndL}$ and $\omega\text{SADL}$ relative to the theory $\text{ACF}_p$. By what we showed in this section, Theorem 4.12 and Theorem 4.13 it follows directly the following theorem.

**Theorem 4.22.** Let $K \models \text{ACF}_p$ and $s : \text{dom}(s) \to K$ with $\vec{x}\vec{y}\vec{z} \subseteq \text{dom}(s) \subseteq \text{Var}$, then

\[
\begin{align*}
K \models_{\text{ACF}_p} \vec{x} \perp \vec{z} \vec{y} & \iff K \models_{\text{ACF}_p} (\vec{x}, \vec{y}), \\
K \models_{\text{ACF}_p} (\vec{x}, \vec{y}) & \iff K \models_{\text{ACF}_p} (\vec{x}, \vec{y}).
\end{align*}
\]

5. **Conclusion**

We introduced several forms of dependence and independence logics:

- Dependence Logic with atoms $(\vec{x}, \vec{y})$
- Absolute Independence Logic with atoms $\perp(\vec{x})$
- Independence Logic with atoms $\vec{x} \perp \vec{y}$
- Absolute Conditional Independence Logic with atoms $\perp\vec{z}(\vec{x})$
- Conditional Independence Logic with atoms $\vec{x} \perp \vec{z} \vec{y}$

that we studied on the atomic level, where we already have non-trivial questions about axiomatizability and decidability. We gave these concepts meaning in different contexts such as:

- Team semantics (i.e. databases)
• Closure Operators
• Pregeometries
• ω-Stable first order theories.

In several important cases we found that the common axioms of independence, going back to Whitney [22] and van der Waerden [21], are complete on the atomic level. As pointed out in [12], these are the same axioms that govern central concepts of independence in database theory. Furthermore, it can be argued that these axioms govern the concepts of dependence and independence in a whole body of areas of science and humanities. Thus the concepts of dependence and independence enjoy a remarkable degree of robustness, reminiscent of their central role in mathematics and its applications. The situation of this paper, where variables are interpreted as elements of a structure with a pregeometry, is naturally much richer than the more general environment of database theory, or team semantics, where variables are interpreted as vectors with no field structure on the coefficients. However, our results demonstrate that it is possible to consider algebra, and by the same token model theory of stable theories, in the same framework with databases and other more general structures where dependence and independence concepts make sense. It is possible that database theory benefits from such a common framework, but also that algebra and model theory benefit from this connection. At least the concepts of dependence and independence have, by virtue of their axioms, meaning that crosses over the territory from database theory all the way to algebra and model theory.

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