THE CALDERÓN OPERATOR AND THE STIELTJES
TRANSFORM ON VARIABLE LEBESGUE SPACES WITH
WEIGHTS

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Abstract. We characterize the weights for the Stieltjes transform and the
Calderón operator to be bounded on the weighted variable Lebesgue spaces
$L^{p(\cdot)}_w(0, \infty)$, assuming that the exponent function $p(\cdot)$ is log-Hölder continuous
at the origin and at infinity. We obtain a single Muckenhoupt-type condition
by means of a maximal operator defined with respect to the basis of intervals
$(0, b : b > 0)$ on $(0, \infty)$. Our results extend those in [18] for the constant
exponent $L^p$ spaces with weights. We also give two applications: the first is
a weighted version of Hilbert’s inequality on variable Lebesgue spaces, and
the second generalizes the results in [42] for integral operators to the variable
exponent setting.

1. Introduction and results

In this paper we consider two classical operators: the generalized Stieltjes transform $S_{\lambda}$ and the generalized Calderón operator $C_{\lambda}$, where $0 < \lambda \leq 1$, defined for non-negative functions $f$ on $(0, \infty)$ by

$$S_{\lambda} f(x) = \int_0^\infty \frac{f(y)}{(x+y)^\lambda} \, dy$$

and

$$C_{\lambda} f(x) = \frac{1}{x^\lambda} \int_0^x f(y) \, dy + \int_x^\infty \frac{f(y)}{y^\lambda} \, dy.$$ 

The Calderón operator $C = C_1$ plays an important role in the theory of interpolation: see [5]. More generally, we have that for $\lambda > 0$, $C_{\lambda} = H_{\lambda} + H^*_\lambda$, where

$$H_{\lambda} f(x) = \frac{1}{x^\lambda} \int_0^x f(y) \, dy$$

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is a Hardy-type operator and $H_\lambda^*$ its adjoint. The Stieltjes transform $S = S_1$ is, formally, the same as $L^2 = L \circ L$, where $L$ is the Laplace transform. A classical reference for the Stieltjes transform is the monograph by D. Widder [43].

These two operators clearly satisfy $2^{-\lambda} C_\lambda f(x) \leq S_\lambda f(x) \leq C_\lambda f(x)$, so $S_\lambda$ is bounded on a Banach function space if and only if $C_\lambda$ is. Hereafter, given functions $f, g \geq 0$ we will write $f \lesssim g$ if there exists $c > 0$ such that $f \leq cg$. If $f \lesssim g$ and $g \lesssim f$ hold, we will write $f \sim g$. Thus we have that $S_\lambda f \sim C_\lambda f$.

We shall also consider the operator

$$S^\alpha f(x) = \int_0^\infty \frac{|x-t|^\alpha}{(x+t)^{\alpha+1}} f(t) dt, \quad \alpha \geq 0,$$

and $C^\alpha$, which is the sum of the Riemann-Liouville and Weyl averaging operators:

$$C^\alpha f(t) = I^\alpha f(t) + J^\alpha f(t) = \frac{\alpha+1}{\alpha+1} \int_0^t (t-x)^\alpha f(x) dx + (\alpha+1) \int_t^\infty \frac{(x-t)^\alpha}{x^{\alpha+1}} f(x) dx.$$

It is clear that if $\alpha = 0$, then $S^0, C^0, I^0$ and $J^0$ are $S, C, H_1$ and $H_1^*$, respectively. Moreover, $I^0 f \lesssim H_1 f$, $J^0 f \lesssim H_1^* f$, $C^\alpha f \lesssim C_1 f$ and $S^\alpha f \sim C^\alpha f$ for non-negative measurable functions $f$.

To put our results into context, we briefly review the history of weighted norm inequalities for the Calderón operator $C_\lambda$ and the Stieltjes transform $S_\lambda$, which in turn depend on the weighted norm inequalities for the Hardy operator $H_\lambda$. Muckenhoupt [36] established two-weight norm inequalities for the Hardy operator; this implicitly gave bounds for the Calderón operator using this condition and its dual. A different condition for the Stieltjes transform, expressed in terms of the operator $S_\lambda$ applied to the pair of weights, was discovered by Andersen [1]. As a consequence, he proved the following one-weight condition.

**Theorem 1.1.** Given $0 < \lambda \leq 1$ and $1 < p < \frac{1}{1-\lambda}$, define $q \geq p$ by $\frac{1}{q} = \frac{1}{p} - (1-\lambda)$. Then $S_\lambda : L^p(w^p) \to L^q(w^q)$ if and only if the weight $w$ satisfies the $A_{p,q,0}$ condition:

$$\sup_{b > 0} \left( \frac{1}{b} \int_0^b w^q dx \right)^{\frac{1}{q}} \left( \frac{1}{b} \int_0^b w^{-p'} dx \right)^{\frac{1}{p'}} < \infty,$$

where $p'$ stands for the Hölder conjugate exponent of $p$.

The $A_{p,q,0}$ condition is a weaker version of the $A_{p,q}$ condition introduced by Muckenhoupt and Wheeden [35] to characterize the weighted norm inequalities for fractional integrals and fractional maximal operators. (See also [8].) In the one-weight case the restriction on $p$ and $q$ is natural: by homogeneity, if $S_\lambda : L^p(0, \infty) \to L^q(0, \infty)$, then $\frac{1}{q} = \frac{1}{p} - (1-\lambda)$.

For other results on weighted norm inequalities for the Hardy operator, the Calderón operator and the Stieltjes transform, we refer the reader to Sinnammon [21] and Gogatishvili, et al. [20, 21, 22].

A different approach to the one-weight inequalities for $S_\lambda$ and $C_\lambda$ in the case $\lambda = 1$ was developed by Duoandikoetxea, Martín-Reyes and Ombrosi [18]. They
introduced a maximal operator $N$ defined with respect to the basis $\mathcal{B} = \{(0, b) : b > 0\}$: for $f \in L^1_{\text{loc}}(0, \infty)$ and $x \in (0, \infty)$,

$$Nf(x) = \sup_{b > x} \frac{1}{b} \int_0^b |f(y)| \, dy.$$ 

They proved the following weighted norm inequality.

**Theorem 1.2.** Given $1 < p < \infty$, $N : L^p(w) \to L^p(w)$ if and only if the weight $w$ satisfies the $A_{p,0}$ condition:

$$\sup_{b > 0} \left( \frac{1}{b} \int_0^b w \, dx \right) \left( \frac{1}{b} \int_0^b w^{1-p'} \, dx \right)^{p-1} < \infty.$$ 

The $A_{p,0}$ condition is analogous to the Muckenhoupt $A_p$ condition, which characterizes weighted norm inequalities for the Hardy-Littlewood maximal operator \cite{37} (see also \cite{35}). This class is related to the $A_{p,q,0}$ class given above: if $q = p$ and $w \in A_{p,p,0}$, then $w^p \in A_{p,0}$.

For non-negative functions $f$, we have that $Nf \leq Cf$: given $0 < x < b$

$$\frac{1}{b} \int_0^b f(y) \, dy \leq \frac{1}{x} \int_0^x f(y) \, dy + \int_x^b \frac{f(y)}{y} \, dy \leq Sf(x);$$

if we take the supremum over all such $b$ we get the desired inequality. Similarly, we also have that $Hf \leq Nf$. By a straightforward duality argument using the Hardy operators, in \cite{18} they proved the following result.

**Theorem 1.3.** Given $1 < p < \infty$, $C : L^p(w) \to L^p(w)$ if and only if the weight $w$ satisfies the $A_{p,0}$ condition; a similar result holds for $S$.

In this paper, our goal is to generalize these results in two ways. First, we extend the approach in \cite{19} to give a new proof of Theorem 1.1 using the maximal operator $N$. We will do so using a Hedberg type inequality \cite{28}. More importantly, we extend all of these results to the scale of variable Lebesgue spaces. These are a generalization of the classical Lebesgue spaces, with the constant exponent $p$ replaced by an exponent function $p(\cdot)$. They were introduced by Orlicz \cite{39} in 1931; harmonic analysis on these spaces has been studied intensively for the past 25 years. We refer the reader to the monographs \cite{10, 15} for a comprehensive history.

To state our results we first introduce some basic definitions; for more information we refer the reader to the above books and also to \cite{30}. Let $\mathcal{P}(0, \infty)$ denote the collection of bounded measurable functions $p(\cdot) : (0, \infty) \to [1, \infty)$. For a measurable subset $E$ of $(0, \infty)$, let

$$p^-_E = \inf_{x \in E} p(x), \quad p^+_E = \sup_{x \in E} p(x);$$

for brevity we will simply write $p^- = p^-_{(0, \infty)}$ and $p^+ = p^+_{(0, \infty)}$. Thus, we can write

$$\mathcal{P}(0, \infty) = \{p(\cdot) : (0, \infty) \to [1, \infty) \text{ with } p^+ < \infty\}.$$
As in the constant exponent case, define the conjugate exponent \( p'(\cdot) \) pointwise by

\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1
\]

on every \( x \in (0, \infty) \). Notice that, if \( p(x) = 1 \), then \( p'(x) = \infty \) so \( p'(\cdot) \notin \mathcal{P}(0, \infty) \). However, if \( p(\cdot) \in \mathcal{P}(0, \infty) \) with \( p^- > 1 \), then \( p'(\cdot) \in \mathcal{P}(0, \infty) \).

The variable Lebesgue space \( L^{p(\cdot)}(0, \infty) \) is the set of measurable functions \( f \) such that the modular

\[
\varrho_{p(\cdot)}(f) := \int_0^\infty |f(x)|^{p(x)} \, dx < \infty.
\]

This becomes a Banach function space when equipped with the Luxemburg norm defined by

\[
\|f\|_{p(\cdot)} := \inf \{ \mu > 0 : \varrho_{p(\cdot)}(f/\mu) \leq 1 \}.
\]

If \( p(\cdot) = p \) is constant, then \( L^{p(\cdot)}(0, \infty) = L^p(0, \infty) \) with equality of norms.

For our results we need to impose a regularity condition on \( p(\cdot) \) at 0 and at infinity.

**Definition 1.4.** Given \( p(\cdot) \in \mathcal{P}(0, \infty) \), we say that \( p(\cdot) \) is log-Hölder continuous at the origin, and denote this by \( p(\cdot) \in LH_0(0, \infty) \), if there exist constants \( C_0 > 0 \) and \( p_0 \geq 1 \) such that

\[
|p(x) - p_0| \leq \frac{C_0}{-\log(x)}, \quad \text{for all } 0 < x < 1/2.
\]

We say that \( p(\cdot) \) is log-Hölder continuous at infinity, and denote it by \( p(\cdot) \in LH_\infty(0, \infty) \), if there exist constants \( C_\infty > 0 \) and \( p_\infty \geq 1 \) such that

\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + x)}, \quad \text{for all } x \in (0, \infty).
\]

Observe that if \( p(\cdot) \in LH_0(0, \infty) \), then \( p_0 = \lim_{x \to 0^+} p(x) \), which allows us to define \( p(0) = p_0 \). Similarly, if \( p(\cdot) \in LH_\infty(0, \infty) \), then \( p_\infty = \lim_{x \to \infty} p(x) \). Moreover, if \( p^- > 1 \) then it is easy to see that \( p(\cdot) \in LH_0(0, \infty) \) and \( p(\cdot) \in LH_\infty(0, \infty) \) imply \( p'(\cdot) \in LH_0(0, \infty) \) and \( p'(\cdot) \in LH_\infty(0, \infty) \) with \( p'(\cdot) = (p(\cdot))' \), respectively.

For many results in harmonic analysis to be true in the variable Lebesgue spaces, it is necessary to assume a stronger condition than \( LH_0(0, \infty) \). Instead, we assume that the exponent \( p(\cdot) \) is log-Hölder continuous at every point in \( (0, \infty) \):

\[
|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad \text{for all } 0 < |x - y| < 1/2.
\]

However, for the Hardy operator, it was shown that this condition is not necessary, and the weaker condition \( LH_0(0, \infty) \) is sufficient: see [10].

Given a weight \( w \) –i.e., a non-negative, locally integrable function on \( (0, \infty) \) such that \( 0 < w(x) < \infty \) a.e.–we define the weighted variable Lebesgue space \( L^{p(\cdot)}_w(0, \infty) \) as follows: \( f \in L^{p(\cdot)}_w(0, \infty) \) if \( fw \in L^{p(\cdot)}(0, \infty) \). When \( p(\cdot) = p \) is constant, this becomes the weighted Lebesgue space \( L^p(w^p) \). (In other words, in the variable Lebesgue spaces we define weights as multipliers rather than as measures.)
Given a weight $w$ and an operator $T$, we say that $T$ is strong-type $(p(\cdot), q(\cdot))$ with respect to $w$ if
$$
\|(Tf)w\|_{q(\cdot)} \leq K\|fw\|_{p(\cdot)};
$$
equivalently, $T : L^p_w(0, \infty) \to L^q_w(0, \infty)$. We say that $T$ is weak-type $(p(\cdot), q(\cdot))$ with respect to $w$ if for all $\mu > 0$,
$$
\mu\|w\chi_{\{x \in (0,\infty) : Tf(x) > \mu\}}\|_{q(\cdot)} \leq K\|fw\|_{p(\cdot)}.
$$
Note that if $T$ is strong-type $(p(\cdot), q(\cdot))$ with respect to $w$, then it is automatically of weak-type as well.

The weights we consider are a generalization of the $A_{p,q,0}$ weights defined above.

**Definition 1.5.** Given $0 < \lambda \leq 1$ and $p(\cdot) \in \mathcal{P}(0, \infty)$ such that $p_+ < \frac{1}{1-\lambda}$, define $q(\cdot)$ by $1/q(x) = 1/p(x) - (1-\lambda)$. We say that a weight $w \in A_{p(\cdot), q(\cdot), 0}$ if there exists a constant $C > 0$ such that for every $b > 0$,
$$
\|w\chi_{(0,b)}\|_{q(\cdot)}\|w^{-1}\chi_{(0,b)}\|_{p'(\cdot)} \leqCb^\lambda.
$$
If $\lambda = 1$, then $p(\cdot) = q(\cdot)$ and we write $w \in A_{p(\cdot), 0}$.

The $A_{p(\cdot), q(\cdot), 0}$ condition is a weaker version of the class $A_{p(\cdot), q(\cdot)}$ introduced in [7] (see also [13]) to control weighted norm inequalities for the fractional integral operator. Similarly, the $A_{p(\cdot), 0}$ condition is a weaker version of the $A_{p(\cdot)}$ condition [9] [11] which governs weighted norm inequalities for the maximal operator on weighted Lebesgue spaces. When $p(\cdot)$ and $q(\cdot)$ are constant, then the $A_{p(\cdot), q(\cdot), 0}$ condition becomes the $A_{p,q,0}$ condition defined above.

We can now state our main results. The first is for the maximal operator $N$.

**Theorem 1.6.** Given $p(\cdot) \in \mathcal{P}(0, \infty)$, suppose $p(\cdot) \in LH_0(0, \infty) \cap LH_\infty(0, \infty)$ and $1 < p^- \leq p^+ < \infty$. If $w$ is a weight on $(0, \infty)$, then the following are equivalent:

(i) The maximal operator $N$ is of strong-type $(p(\cdot), p(\cdot))$ with respect to $w$.

(ii) The maximal operator $N$ is of weak-type $(p(\cdot), p(\cdot))$ with respect to $w$.

(iii) $w \in A_{p(\cdot), 0}$.

**Remark 1.7.** In the proof of Theorem 1.6 we do not need to assume the log-Hölder continuity conditions in order to prove the necessity of the $A_{p(\cdot), 0}$ condition. This raises the question of whether there are weaker conditions on $p(\cdot)$ so that the $A_{p(\cdot), 0}$ condition is also sufficient. A similar question has been asked for the Hardy-Littlewood maximal operator: see [11] [31].

Theorem 1.6 is the heart of our work. Our proof is adapted from the proof of the boundedness of the Hardy-Littlewood maximal operator on weighted variable Lebesgue spaces in [11]. However, the fact that $N$ is an operator on the half-line introduces a number of technical obstacles that were not present in that proof.

Given Theorem 1.6 we can deduce the following result that characterizes the weights controlling the boundedness of the generalized Calderón operator $C_\lambda$ and the generalized Stieltjes transform $S_\lambda$ using a Hedberg type inequality.
Theorem 1.8. Given $0 < \lambda < 1$ and $p(\cdot) \in \mathcal{P}(0, \infty)$, suppose $p(\cdot) \in \mathcal{L}H_0(0, \infty) \cap \mathcal{L}H_\infty(0, \infty)$ and $1 < p^- \leq p^+ < \frac{\lambda}{1-\lambda}$. Define $q(\cdot) \in \mathcal{P}(0, \infty)$ by $1/q(x) = 1/p(x) - (1-\lambda)$. If $w$ is a weight on $(0, \infty)$, then the following are equivalent:

(i) The operator $\mathcal{C}_\lambda$ is of strong-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(ii) The operator $S_\lambda$ is of strong-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(iii) The operator $\mathcal{C}_\lambda$ is of weak-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(iv) The operator $S_\lambda$ is of weak-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(v) $w \in A_{p(\cdot), q(\cdot), 0}$.

As a consequence of Theorem 1.8, we immediately get weighted norm inequalities for $\mathcal{C}_\alpha$ and $S_\alpha$. Since $\mathcal{C}_\alpha \lesssim \mathcal{C}$, we have that the $A_{p(\cdot), 0}$ weights are sufficient for the boundedness of $\mathcal{C}_\alpha$ for any $\alpha \geq 0$. Surprisingly, $w \in A_{p(\cdot), 0}$ is also necessary, and it does not depend on $\alpha$.

Theorem 1.9. Given $p(\cdot) \in \mathcal{P}(0, \infty)$, suppose $p(\cdot) \in \mathcal{L}H_0(0, \infty) \cap \mathcal{L}H_\infty(0, \infty)$ and $1 < p^- \leq p^+ < \infty$. If $w$ is a weight on $(0, \infty)$, then the following statements are equivalent:

(i) $w \in A_{p(\cdot), 0}$.
(ii) There exists $\alpha \geq 0$ such that $\mathcal{C}_\alpha$ is of strong-type $(p(\cdot), p(\cdot))$ with respect to $w$.
(iii) For every $\alpha \geq 0$, $\mathcal{C}_\alpha$ is of strong-type $(p(\cdot), p(\cdot))$ with respect to $w$.
(iv) There exists $\alpha \geq 0$ such that $\mathcal{C}_\alpha$ is of weak-type $(p(\cdot), p(\cdot))$ with respect to $w$.
(v) For every $\alpha \geq 0$, $\mathcal{C}_\alpha$ is of weak-type $(p(\cdot), p(\cdot))$ with respect to $w$.

Since $S_\alpha f \sim C_\alpha f$, the same equivalence is true with $\mathcal{C}_\alpha$ replaced by $S_\alpha$.

Since we also have that $\mathcal{C}_\lambda = \mathcal{H}_\lambda + \mathcal{H}_\lambda^*$, as an immediate consequence of Theorem 1.8, we get weighted bounds for the Hardy operators.

Theorem 1.10. Given $0 < \lambda \leq 1$ and $p(\cdot) \in \mathcal{P}(0, \infty)$, suppose $p(\cdot) \in \mathcal{L}H_0(0, \infty) \cap \mathcal{L}H_\infty(0, \infty)$ and $1 < p^- \leq p^+ < \frac{\lambda}{1-\lambda}$. Define $q(\cdot) \in \mathcal{P}(0, \infty)$ by $1/q(x) = 1/p(x) - (1-\lambda)$. If $w$ is a weight on $(0, \infty)$, then the following are equivalent:

(i) The operators $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^*$ are of strong-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(ii) The operators $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^*$ are of weak-type $(p(\cdot), q(\cdot))$ with respect to $w$.
(iii) $w \in A_{p(\cdot), q(\cdot), 0}$.

One-weight norm inequalities for the Hardy operators in the variable Lebesgue spaces do not appear to have been considered before now. For two-weight inequalities, see Mamedov, et al. [12, 27, 32, 33, 41, 35]. These results are not immediately comparable to ours, even in the one-weight case, since they assume log-Hölder continuity conditions that depend on the weight. See [12] for a discussion of cases where this condition overlaps with our regularity assumptions.

Remark 1.11. It is tempting to conjecture that either the strong or weak type inequality for only one of the operators $\mathcal{H}_\lambda$ or $\mathcal{H}_\lambda^*$ implies the $A_{p(\cdot), q(\cdot), 0}$ condition.
However, this is not true even in the constant exponent case. For simplicity we will show this when $p = 2$ and $\lambda = 1$, but our example can easily be modified to work for any $p$ and $\lambda$. By [36], a necessary and sufficient condition for $H_1$ to be bounded on $L^2(w)$ is that

$$
\sup_{r>0} \int_r^\infty \frac{w(x)}{x^2} \, dx \int_0^r w(x)^{-1} \, dx < \infty. \tag{1.4}
$$

Let

$$
w(x) = \begin{cases}
1 & 0 < x \leq 1 \\
e^{-x} & x > 1.
\end{cases}
$$

This weight satisfies (1.4). Indeed, if $r \leq 1$, then

$$\int_r^\infty \frac{w(x)}{x^2} \, dx \int_0^r w(x)^{-1} \, dx \leq \int_r^\infty \frac{dx}{x^2} \int_0^r dx = \frac{1}{r} \cdot r = 1.
$$

And if $r > 1$, the left-hand side is dominated by

$$\int_r^\infty \frac{e^{-x}}{x^2} \, dx \int_0^r e^{x} \, dx \leq \frac{e^{-r}}{r} e^r \leq 1.
$$

On the other hand, $w \notin A_{2,0}$, since for every $r > 1$,

$$\frac{1}{r} \int_0^r w(x) \, dx \frac{1}{r} \int_0^r w(x)^{-1} \, dx \geq \frac{1}{r} \int_1^r e^{-x} \, dx \frac{1}{r} \int_1^r e^{x} \, dx = \frac{e^{-1} - e^{-r}}{r^2} (e^r - 1),
$$

and the right-hand side is unbounded as $r \to \infty$.

For a related instance in which the $A_{p,0}$ condition is sufficient but not necessary, see [2].

We now give two applications of Theorem 1.8 More precisely, we will give an application of a generalization of this theorem to higher dimensions. If we replace $(0, \infty)$ by $\mathbb{R}^n$, then we may define the variable Lebesgue space $L^p(\mathbb{R}^n)$ exactly as above. We define log-Hölder continuity as in Definition 1.4, replacing $x$ by $|x|$ on the right-hand side of each inequality. Finally, we say that a weight $w \in A_{p(\cdot),0}$ if for all $b > 0$,

$$
\|w\chi_{B(0,b)}\|_{p(\cdot)} \|w^{-1}\chi_{B(0,b)}\|_{p'(\cdot)} \leq C b^n.
$$

For a measurable function $f$ on $\mathbb{R}^n$, define the radial operators

$$
Nf(x) = \sup_{b > |x|} \frac{1}{b^n} \int_{B(0,b)} |f(y)| \, dy,
$$

and

$$
Sf(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x|^n + |y|^n} \, dy, \quad x \in \mathbb{R}^n.
$$

Then we can modify the proofs of Theorems 1.6 and 1.8 to get the following result.

**Theorem 1.12.** Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ and $1 < p_- \leq p_+ < \infty$. If $w$ is a weight on $\mathbb{R}^n$, then the following are equivalent:

(i) $N$ is strong $(p(\cdot), p(\cdot))$ with respect to $w$;

(ii) $S$ is strong $(p(\cdot), p(\cdot))$ with respect to $w$;
The first application of Theorem 1.12 is a weighted version of Hilbert’s inequality: for $p > 1$ and non-negative functions $f, g$,

$$
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy \leq C_p \|f\|_{L^p(0,\infty)} \|g\|_{L^{p'}(0,\infty)},
$$

which was first proved by G. Hardy and M. Riesz [25] (also see [26, Chapter IX]).

**Theorem 1.13.** Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, suppose $p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n)$ and $1 < p^- \leq p^+ < \infty$. Then there exists $C > 0$ such that for any non-negative functions $f, g$, $f \in L^p(\cdot)w(\mathbb{R}^n)$ and $g \in L^{p'}(\cdot)w^{-1}(\mathbb{R}^n)$, independent of $f$ and $g$,

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^n + |y|^n} \, dx \, dy \leq C \|f w\|_{p(\cdot)} \|g w^{-1}\|_{p'(\cdot)}
$$

if and only if $w \in A_{p(\cdot),0}$.

Theorem 1.13 appears to be new, even in the constant exponent case. When $n = 1$ it is implicit in [18].

**Remark 1.14.** The sharp constant in Hilbert’s inequality is $\frac{\pi}{\sin(\pi/p)}$; this is due to J. Schur [40]. Here, we are not concerned with finding the best constant. However, this is an interesting problem, especially in the constant exponent case where there has been a great deal of work on sharp constants related to the so-called $A_2$ conjecture. See, for instance, [29].

The second application of Theorem 1.12 is to the continuity of certain integral operators on variable Lebesgue spaces. Given an index set $J$, let $\{T_j\}_{j \in J}$ be a family of (singular) integral operators defined by

$$
T_j f(x) = pv \int_{\mathbb{R}^n} K_j(x,y) f(y) \, dy
$$

where each $K_j$ satisfies a decay estimate,

$$
|K_j(x,y)| \leq \frac{C_0}{|x-y|^n}, \quad x \neq y,
$$

with $C_0$ independent of $j \in J$. We are interested in the boundedness of the associated maximal operator

$$
T^* f(x) = \sup_{j \in J} |T_j f(x)|.
$$

These operators were first considered by Soria and Weiss in [42]. They prove that $T^* : L^p(\cdot) \rightarrow L^p(\cdot)$ provided that $w$ is an $A_p$ weight that is essentially constant over dyadic annuli. More precisely, they assume that there exists a constant $C_1 > 0$ such that

$$
\sup_{2^k-2 \leq |x| \leq 2^{k+1}} w(x) \leq C_1 \inf_{2^k-2 \leq |x| \leq 2^{k+1}} w(x), \quad k \in \mathbb{Z}.
$$

We can extend their result to the variable Lebesgue spaces.
Theorem 1.15. Let \( \{T_j\}_{j \in J} \), \( T^* \) be defined as above. Given \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) suppose \( p(\cdot) \in LH_0(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n), \) \( 1 < p^- \leq p^+ < \infty, \) and for every family of balls \( B \) with bounded overlap,

\[
\sum_{B \in \mathcal{B}} \|f \chi_B\|_{p(\cdot)} \|g \chi_B\|_{p'(\cdot)} \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)},
\]

where the constant \( C \) is independent of \( B \) and only depends on \( p(\cdot) \) and the bound on the overlap. If \( T^* \) is of strong type \( (p(\cdot), p(\cdot)) \), and if \( w \in A_{p(\cdot),0} \) and satisfies (1.8), then \( T^* \) is of strong type \( (p(\cdot), p(\cdot)) \) with respect to \( w \).

Theorem 1.15 is new, but this question has also been considered by Bandaliyev [3, 4]. However, his results have different hypotheses on \( p(\cdot) \) and the weights, and his proofs rely on other techniques.

Remark 1.16. The summation condition (1.9) was introduced by Berezhnoi [6] in the study of Banach function spaces. In [15] this condition was shown to be very closely related to the boundedness of the Hardy-Littlewood maximal operators and singular integrals on the variable Lebesgue spaces. Thus it is a very reasonable assumption in the context of Theorem 1.15. As shown in [15, Theorem 7.3.22], this condition holds if \( p(\cdot) \in LH(\mathbb{R}^n) \cap LH_\infty(\mathbb{R}^n) \), where \( LH(\mathbb{R}^n) \) is the local log-Hölder condition defined by (1.3).

Remark 1.17. In the constant exponent case, Theorem 1.15 appears to be a generalization of the original result of Soria and Weiss, since we only assume \( w \in A^p_0 \) whereas they assume the stronger condition \( w \in A_p \). However, given the additional assumption (1.8), these two conditions are the same: Clearly, we always have \( A_p \subset A^p_0 \). Conversely, given \( w \in A^p_0 \) that satisfies (1.8), fix any ball \( B = B(x, r) \). If \( r > |x|/2 \), then \( B \subset \bar{B} = B(0, s) \), \( s = |x| + r \), and \( |B| \sim |\bar{B}| \). Hence,

\[
\frac{1}{|B|} \int_B w \, dx \left( \frac{1}{|B|} \int_B w^{1-p'} \, dx \right)^{p-1} \lesssim \frac{1}{|\bar{B}|} \int_{\bar{B}} w \, dx \left( \frac{1}{|\bar{B}|} \int_{\bar{B}} w^{1-p'} \, dx \right)^{p-1} \leq C.
\]

On the other hand, if \( r \leq |x| < 2 \), and \( k \in \mathbb{Z} \) is such that \( 2^{k-1} \leq |x| < 2^k \), then for any \( y \in B, 2^{k-2} \leq |y| < 2^{k+1} \), and so \( w \) is essentially constant on \( B \), so the \( A_p \) condition holds on \( B \).

The remainder of this paper is organized as follows. In Section 2 we state and prove a number of technical lemmas on the exponents \( p(\cdot) \) and the weights \( A_{p(\cdot),q(\cdot),0} \) that we will use in the proofs of our main results. The proof of Theorem 1.6 is in Section 3 and the proofs of Theorems 1.8 and 1.9 are in Section 4. Finally, Section 5 contains the proof of Theorems 1.10, 1.12, 1.13 and 1.15.

2. Technical results

In this section we establish some properties of log-Hölder continuous exponents and \( A_{p(\cdot),0} \) weights that we will use in our main proofs. We begin with two lemmas that allow us to apply the \( LH_0 \) and \( LH_\infty \) conditions. The first is a version of [13, Lemma 3.2] (see also [10]) to the basis of intervals \( \{(0, b)\}_{b > 0} \).
Lemma 2.1. Given \( p(\cdot) \in \mathcal{P}(0, \infty) \), suppose \( p(\cdot) \in LH_0(0, \infty) \). Then there exists \( C > 0 \) such that for every \( b > 0 \),
\[
b^{p(0,b) - p(0,b)} \leq C.
\]

Proof. Fix \( p(\cdot) \in LH_0(0, \infty) \). Since \( p^-(0,b) - p^+(0,b) \leq 0 \), we can assume that \( 0 < b < 1/2 \). For if \( b \geq 1/2 \), then
\[
b^b p^-(0,b) - p^+(0,b) \leq (1/2) b^p p^-(0,b) \leq (1/2) p^- - p^+ = 2^{p^+ - p^-}.
\]

Fix \( 0 < b < 1/2 \). We will bound the difference \( p^+(0,b) - p^-(0,b) \). From the definition of \( p^-(0,b) \), given any \( \epsilon > 0 \), there exists \( x_\epsilon < b < 1/2 \) such that \( 0 \leq p(x_\epsilon) - p(0,b) < \epsilon \). Consequently,
\[
0 \leq p(0) - p^-(0,b) \leq |p(0) - p(x_\epsilon)| + p(x_\epsilon) - p^-(0,b) \leq \frac{C_0}{-\log(x_\epsilon)} + \epsilon \leq \frac{C_0}{-\log(b)} + \epsilon,
\]
and if we let \( \epsilon \to 0 \), we get
\[
0 \leq p(0) - p^-(0,b) \leq \frac{C_0}{-\log(b)}.
\]
Similarly, we have that
\[
0 \leq p^+(0,b) - p(0) \leq \frac{C_0}{-\log(b)}.
\]
Therefore,
\[
0 \leq p^+(0,b) - p^-(0,b) \leq \frac{2C_0}{-\log(b)}.
\]
Now, since \( 1/b > 2 \),
\[
b^{p^+(0,b) - p^-(0,b)} = (1/b)^{p^+(0,b) - p^-(0,b)} \leq (1/b)^{2C_0} = b^{2C_0} = e^{2C_0}.
\]
If we take \( C = \max\{2^{p^+ - p^-}, e^{2C_0} \} \), we get the desired inequality. 

The next result allows us to estimate the modular \( \omega_{p(\cdot)}(f) \) by means of the modular \( \omega_{p(\cdot)}(f) \) whenever \( p(\cdot) \in LH_\infty(0, \infty) \). This result is from [11] Lemma 2.7, but as they noted there, the proof is identical to the case with Lebesgue measure [10] Lemma 3.26.

Lemma 2.2. Given \( p(\cdot) \in \mathcal{P}(0, \infty) \), suppose \( p(\cdot) \in LH_\infty(0, \infty) \). Fix a set \( G \subset (0, \infty) \) and a non-negative measure \( \mu \). Then, for every \( t > 1/p^- \), there exists a positive constant \( C_t = C(t, C_\infty) \) such that for all functions \( g \) with \( 0 \leq g(x) \leq 1 \),
\[
\int_G g(x)p(x)d\mu(x) \leq C_t \int_G g(x)p_\infty d\mu(x) + \int_G \frac{1}{(e + x)^t} d\mu(x),
\]
and
\[
\int_G g(x)p_\infty d\mu(x) \leq C_t \int_G g(x)p(x)d\mu(x) + \int_G \frac{1}{(e + x)^t} d\mu(x).
\]

In the next series of results, we establish the properties of \( A_{p(\cdot),0} \) weights. These are similar to the properties of the \( A_p(\cdot) \) weights established in [11] Section 3], which in turn are related to the properties of the Muckenhoupt \( A_p \) weights.
Lemma 2.3. Given \( p(\cdot) \in P(0, \infty) \), if \( w \in A_{p(\cdot),0} \), then there exists \( C > 0 \) such that for any \( b > 0 \) and any measurable set \( E \subset (0, b) \),

\[
\frac{|E|}{b} = \frac{|E|}{|(0, b)|} \leq C \frac{\|w \chi_E\|_{p(\cdot)}}{\|w \chi_{(0,b)}\|_{p(\cdot)}}.
\]

Proof. Fix \( b > 0 \) and \( E \subset (0, b) \). Then by Hölder’s inequality and the \( A_{p(\cdot),0} \) condition we have

\[
|E| = \int_0^b w(x) \chi_E(x) w^{-1}(x) \, dx \leq C \|w \chi_E\|_{p(\cdot)} \|w^{-1} \chi_{(0,b)}\|_{p'(.)} \leq \frac{C b \|w \chi_E\|_{p(\cdot)}}{\|w \chi_{(0,b)}\|_{p(\cdot)}}. \tag*{□}
\]

Lemma 2.4. Given \( p(\cdot) \in P(0, \infty) \) suppose \( p(\cdot) \in LH_0(0, \infty) \). If \( w \in A_{p(\cdot),0} \), then there exists \( C_0 > 0 \), depending on \( p(\cdot) \) and \( w \), such that for every \( b > 0 \),

\[
\|w \chi_{(0,b)}\|_{p(\cdot)}^{p(\cdot)-p^+_0} \leq C_0.
\]

Proof. Fix \( b > 0 \). We will consider two cases: \( b < 1 \) and \( b \geq 1 \).

If \( b < 1 \), then we apply the previous lemma with \( E = (0, b) \subset (0, 1) \) to get

\[
\|w \chi_{(0,b)}\|_{p(\cdot)} \geq C b \|w \chi_{(0,1)}\|_{p(\cdot)}.
\]

Then by Lemma 2.3

\[
\|w \chi_{(0,b)}\|_{p(\cdot)}^{p(\cdot)-p^+_0} \leq (C b \|w \chi_{(0,1)}\|_{p(\cdot)})^{p(\cdot)-p^+_0} \leq C b^{p^+_0-p^+_0} (1 + \|w \chi_{(0,1)}\|_{p(\cdot)}^{-1})^{p^+_0-p^+_0} \leq C (1 + \|w \chi_{(0,1)}\|_{p(\cdot)}^{-1})^{p^+_0-p^+} = C_1.
\]

If \( b \geq 1 \), then we repeat the argument but now take \( E = (0, 1) \subset (0, b) \) and use Lemma 2.3 with \( w^{-1} \in A_{p'_(\cdot),0} \). By Hölder’s inequality,

\[
\|w^{-1} \chi_{(0,1)}\|_{p'(.)} \geq \frac{C \|w^{-1} \chi_{(0,1)}\|_{p'(.)}}{b} \geq \frac{C}{\|w \chi_{(0,b)}\|_{p(\cdot)}}.
\]

Thus,

\[
\|w \chi_{(0,b)}\|_{p(\cdot)}^{p(\cdot)-p^+_0} \leq C \|w^{-1} \chi_{(0,1)}\|_{p'(.)}^{p^+_0-p^+_0} \leq C (1 + \|w^{-1} \chi_{(0,1)}\|_{p'(.)}^{p^+_0-p^+}) = C_2.
\]

If we let \( C = \max\{C_1, C_2\} \) we get the desired inequality. \( \tag*{□} \)

Remark 2.5. In Lemma 2.4 the \( LH_0(0, \infty) \) condition on \( p(\cdot) \) is not required (as in [11] Lemma 3.3) since the intervals involved in the \( A_{p(\cdot),0} \) condition are nested.

We now want to define a condition analogous to the \( A_\infty \) condition but associated with the basis of intervals \( \{(0,b)\}_{b>0} \) (as considered in [19]). Hereafter, given an exponent \( p(\cdot) \) and a weight \( w \), we define the weight \( W(x) = w(x)^{p(x)} \) and denote \( W(E) = \int_E W(x) \, dx \) for any measurable set \( E \subset (0, \infty) \). Similarly, for the dual weight \( w^{-1} \) we write \( \sigma(x) = w(x)^{-p'(x)} \) and \( \sigma(E) = \int_E \sigma(x) \, dx \).
Lemma 2.9. Given a weight $w$ such that $0 < w(0,b) < \infty$ for every $b > 0$, we say that $w \in A_{\infty,0}$ if there exist constants $C, \delta > 0$ such that for every $b > 0$ and each measurable set $E \subset (0,b)$,
\[
\frac{|E|}{b} \leq C \left( \frac{w(E)}{w(0,b)} \right)^\delta.
\]

As an immediate consequence of this definition, we have the following lemma.

Lemma 2.7. If $w \in A_{\infty,0}$, for every $0 < \alpha < 1$, there exists $0 < \beta < 1$ (depending on $\alpha$) such that, given $b > 0$ and a measurable set $E \subset (0,b)$, if $|E| \geq \alpha b$, then $w(E) \geq \beta w(0,b)$.

The next lemma requires the deeper properties of the $A_{\infty}$ condition defined with respect to a basis.

Lemma 2.8. If $w \in A_{\infty,0}$, then $w \notin L^1(0,\infty)$.

Proof. It follows from \cite[Theorems 3.1, 4.1]{19} that if $w \in A_{\infty,0}$, then there exist constants $\gamma, \delta > 1$, such that for any $b > 0$, if $E \subset (0,b)$ and $\gamma|E| \leq b$, then $\delta w(E) \leq w(0,b)$. In particular, if we let $b_k = \gamma^k$ for $k \in \mathbb{N}$, and let $E = (0,1)$, then $w(0,b_k) \geq \delta^k w(0,1)$.

Since the right-hand side tends to infinity as $k \to \infty$ (recall that $0 < w(0,1) < \infty$), we get the desired conclusion. \qed

We will apply these lemmas to the weights $W$ and $\sigma$ using the following result.

Lemma 2.9. Given $p(\cdot) \in \mathcal{P}(0,\infty)$, suppose $p(\cdot) \in LH_{0}(0,\infty) \cap LH_{\infty}(0,\infty)$. If $w \in A_{p(\cdot),0}$, then $W \in A_{\infty,0}$.

Proof. Notice first that from the fact that $0 < w(x) < \infty$ a.e. and the $A_{p(\cdot),0}$ condition, $0 < \|w \chi_{(0,b)}\|_{p(\cdot)} < \infty$ for every $b > 0$. Hence, $0 < W(0,b) < \infty$ for every $b > 0$.

Fix $b > 0$ and a measurable set $E \subset (0,b)$. We consider three cases: $\|w \chi_{E}\|_{p(\cdot)} \leq \|w \chi_{(0,b)}\|_{p(\cdot)} \leq 1$, $\|w \chi_{E}\|_{p(\cdot)} \leq \|w \chi_{(0,b)}\|_{p(\cdot)}$ and $1 < \|w \chi_{E}\|_{p(\cdot)} \leq \|w \chi_{(0,b)}\|_{p(\cdot)}$.

In the first case, by \cite[Corollary 2.23]{10}, we have that $W(E) \leq W(0,b) \leq 1$, $\|w \chi_{E}\|_{p(\cdot)} \leq W(E)^{1/p_k^+} \leq W(0,b)^{1/p_0^+}$ and $\|w \chi_{(0,b)}\|_{p(\cdot)} \geq W(0,b)^{1/p_0^+}$. Thus, by Lemmas 2.3 and 2.4 we get
\[
\frac{|E|}{b} \leq C \frac{\|w \chi_{E}\|_{p(\cdot)}}{\|w \chi_{(0,b)}\|_{p(\cdot)}} \leq C \frac{\|w \chi_{E}\|_{p(\cdot)}}{\|w \chi_{(0,b)}\|_{p(\cdot)}} W(0,b)^{1/p_0^+} \leq C W(0,b)^{1/p_0^+} \frac{\|w \chi_{(0,b)}\|_{p(\cdot)}^{p_0^+} - \|w \chi_{(0,b)}\|_{p(\cdot)}^{p_0^+}}{1/p_0^+}.
\]
\[
\leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+_{(0, b)}}
\leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+}.
\]

In the second case, if \( \|w\chi_E\|_{p(\cdot)} < 1 \leq \|w\chi_{(0, b)}\|_{p(\cdot)} \), then we have \( \|w\chi_E\|_{p(\cdot)} \leq W(E)^{1/p^+_{(0, b)}} \) and \( \|w\chi_{(0, b)}\|_{p(\cdot)} \geq W(0, b)^{1/p^+_{(0, b)}} \), which yields
\[
\frac{|E|}{b} \leq C \frac{\|w\chi_{(0, b)}\|_{p(\cdot)}^{-1}}{\|w\chi_{(0, b)}\|_{p(\cdot)}} \leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+_{(0, b)}} \leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+},
\]
where we have used again Lemma 2.3.

Finally, in the third case, if \( 1 < \|w\chi_E\|_{p(\cdot)} \leq \|w\chi_{(0, b)}\|_{p(\cdot)} \), then we will show that
\[
(2.1) \quad \frac{|E|}{b} \leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+} \leq C \left( \frac{W(E)}{W(0, b)} \right)^{1/p^+},
\]
Since \( p(\cdot) \in LH_\infty(0, \infty) \) and \( \|w\chi_{(0, b)}\|_{p(\cdot)}^{-1} \leq 1 \), we can apply Lemma 2.2 with measure \( dp(x) = w(x)^p(x) \, dx \), \( G = (0, b) \) and \( g(x) \equiv \|w\chi_{(0, b)}\|_{p(\cdot)}^{-1} \). Hence, for every \( t > 1/p^- \),
\[
\int_0^b \frac{1}{\|w\chi_{(0, b)}\|_{p(\cdot)}} w(x)^p(x) \, dx \leq C_t \int_0^b \left( \frac{w(x)}{\|w\chi_{(0, b)}\|_{p(\cdot)}} \right)^{p(\cdot)} \, dx + \int_0^b \frac{w(x)^p(x)}{(e + x)^{tp^-}} \, dx.
\]

By the definition of the norm the first term is equal to \( C_t \). We will now show that we can choose \( t > 1/p^- \), depending only on \( p(\cdot) \) and \( w \), such that the second term is smaller than 1. In fact,
\[
\int_0^\infty \frac{w(x)^p(x)}{(e + x)^{tp^-}} \, dx = \int_0^1 \frac{w(x)^p(x)}{(e + x)^{tp^-}} \, dx + \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} \frac{w(x)^p(x)}{(e + x)^{tp^-}} \, dx
\]
\[
\leq \frac{W(0, 1)}{e^{tp^-}} + \sum_{k=0}^\infty \frac{1}{2^{kp^-}} W(2^k, 2^{k+1})
\leq \frac{W(0, 1)}{e^{tp^-}} + \sum_{k=0}^\infty \frac{1}{2^{kp^-}} W(0, 2^{k+1})
\leq \frac{W(0, 1)}{e^{tp^-}} + \sum_{k=0}^\infty \frac{1}{2^{kp^-}} \max \left\{ \|w\chi_{(0, 2^{k+1})}\|_{p(\cdot)}, \|w\chi_{(0, 2^{k+1})}\|_{p(\cdot)}^{-1} \right\},
\]
where in the last inequality we used [10], Corollary 2.23.

To estimate the norm \( \|w\chi_{(0, 2^{k+1})}\|_{p(\cdot)} \) we use Lemma 2.3 with \( E = (0, 1) \subset (0, 2^{k+1}) \):
\[
\|w\chi_{(0, 2^{k+1})}\|_{p(\cdot)} \leq C \frac{|(0, 2^{k+1})|}{|(0, 1)|} \|w\chi_{(0, 1)}\|_{p(\cdot)} \leq C 2^{k+1}.
\]
Thus, \( \max \left\{ \|w\chi_{(0,2^{k+1})}\|_{p_1}^{p_1^-}, \|w\chi_{(0,2^{k+1})}\|_{p_k}^{p_k^+} \right\} \leq C2^{(k+1)p^+} \leq C2^{kp^+} \); consequently,

\[
\int_0^\infty \frac{w(x)p(x)}{(e + x)^{p^-}} \, dx \leq \frac{W(0,1)}{e^{tp^-}} + C \sum_{k=0}^{\infty} \frac{2^{kp^+}}{2^{ktp^-}}.
\]

If we take \( t > p^+/p^- \), the last sum converges; hence, by the dominated convergence theorem,

\[
\lim_{t \to \infty} \int_0^\infty \frac{w(x)p(x)}{(e + x)^{p^-}} \, dx = 0.
\]

Furthermore, \( \lim_{t \to \infty} W(0,1)e^{-tp^-} = 0 \). Therefore, we can take \( t \) sufficiently large that (2.2) is less than 1. Therefore,

\[
W(0, b) \leq (C_t + 1)\|w\chi_{(0,b)}\|_{p_1}^{p^-}
\]
or, equivalently,

(2.3) \[
W(0, b)^{1/p^-} \leq (C_t + 1)^{1/p^-} \|w\chi_{(0,b)}\|_{p_1}^{p^-}.
\]

We now estimate the term \( W(E) \). We again apply Lemma 2.1 exchanging the roles of \( p(\cdot) \) and \( p_{\infty} \). Thus,

\[
1 = \int_E \left( \frac{w(x)}{\|w\chi_E\|_{p_1}} \right)^{p_1} \, dx \leq C_t \int_E \|w\chi_E\|_{p_1}^{-p^-} w(x)^{p(x)} + \int_E w(x)^{p(x)} (e + x)^{p^-} \, dx.
\]

If we repeat the above argument, we can make the last integral smaller than 1/2, which gives us,

(2.4) \[
\|w\chi_E\|_{p_1}^{p^-} \leq 2C_t W(E).
\]

If we combine (2.3) and (2.4), we get (2.1). This completes the proof.

From inequalities (2.3) and (2.4) with \( E = (0, b) \), we get the following corollary.

**Corollary 2.10.** Given \( p(\cdot) \in \mathcal{P}(0, \infty) \) suppose \( p(\cdot) \in LH_0(0, \infty) \cap LH_{\infty}(0, \infty) \). If \( w \in A_{p(\cdot), 0} \) and \( b > 0 \) such that \( \|w\chi_{(0,b)}\|_{p_1} \geq 1 \), then

\[
\|w\chi_{(0,b)}\|_{p_1} \sim W(0, b)^{1/p^-}.
\]

3. Proof of Theorem 1.6

**Proof.** The implication \((i) \Rightarrow (ii)\) is straightforward. We will next prove \((ii) \Rightarrow (iii)\). Suppose that for every \( \mu > 0 \) and every \( f \in L_0^{p(\cdot)}(0, \infty) \),

\[
\mu \|w\chi_{x \in (0,\infty): Nf(x) > \mu}\|_{p_1} \leq K\|f\|_{p_1}.
\]

Fix \( b > 0 \); then by duality there exists a non-negative function \( g \in L^{p(\cdot)}(0, \infty) \) such that \( \|g\|_{p_1} \leq 1 \) and

\[
\|w^{-1}\chi_{(0,b)}\|_{p_1} \sim \int_0^b w^{-1}(y)g(y) \, dy.
\]

Without loss of generality, we may suppose that \( \|w^{-1}\chi_{(0,b)}\|_{p_1} > 0 \). If we let \( f = \chi_{(0,b)}w^{-1}g \) and \( \mu = \frac{1}{t} \int_0^b w^{-1}(y)g(y) \, dy > 0 \), then for every \( x \in (0, b) \), \( Nf(x) \geq \mu \).
Thus, for every $\nu > 1$, $(0, b) \subset \{ x \in (0, \infty) : Nf(x) > \mu/\nu \}$. From the weak-type inequality, if we let $\nu \to 1$,
\[
\left( \frac{1}{b} \int_0^b w^{-1}(y)g(y) \, dy \right) \| w\chi_{(0,b)} \|_{p(\cdot)} \leq CK\| f\|_{p(\cdot)} = C\| g\|_{p(\cdot)} \leq C.
\]
Therefore, we have that
\[
\| w\chi_{(0,b)} \|_{p(\cdot)} \| w^{-1}\chi_{(0,b)} \|_{p'(\cdot)} \sim \| w\chi_{(0,b)} \|_{p(\cdot)} \int_0^b w^{-1}(y)g(y) \, dy
\]
\[
= Cb\| w\chi_{(0,b)} \|_{p(\cdot)} \left( \frac{1}{b} \int_0^b w^{-1}(y)g(y) \, dy \right) \leq Cb.
\]
Since this is true for all $b > 0$, $w \in A_{p(\cdot),a}$.

We now come to the proof of (iii)$\Rightarrow$(i), which is the most difficult part. Fix $w \in A_{p(\cdot),0}$; without loss of generality we may assume that $f \geq 0$ and $\| fw\|_{p(\cdot)} \leq 1$.
We begin by arguing as in the proof of \cite[Lemma 2.2]{[10]}. From the definition we have that $Nf$ is decreasing and continuous. Thus, given $\mu > 0$, if the level set
\[
\{ x \in (0, \infty) : Nf(x) > \mu \} \neq \emptyset,
\]
either equals $\{ 0, b \}$ for some $b > 0$ or it equals $(0, +\infty)$. In the first case, we have that
\[
\lambda b = \int_0^b f(x) \, dx
\]
while in the second case,
\[
\int_0^\infty f(x) \, dx = +\infty.
\]
To avoid the latter case, we shall further assume that $f$ is bounded and has compact support. The full result then follows by a standard density argument (cf. \cite[Section 3.4]{[10]}).

We now split $f = f_1 + f_2$, where $f_1 = f\chi_{\{ f\sigma^{-1} > 1 \}}$ and $f_2 = f\chi_{\{ f\sigma^{-1} \leq 1 \}}$. Then,
\[
Nf \leq Nf_1 + Nf_2
\]
\[
\int_0^\infty f_i(x)p(x)w(x)p(x) \, dx \leq \int_0^\infty f(x)p(x)w(x)p(x) \, dx \leq 1, \quad i = 1, 2.
\]
Hence, it will suffice to show that
\[
I_i := \int_0^\infty Nf_i(x)p(x)w(x)p(x) \, dx \leq C, \quad i = 1, 2.
\]

**Estimate for $I_1$:** By our choice of $f$, we can find a non-increasing sequence of positive real numbers $\{ b_k \}_{k \in \mathbb{Z}}$ such that $\{ x \in (0, \infty) : Nf_1(x) > 2^k \} = (0, b_k)$, $\{ x \in (0, \infty) : 2^k < Nf_1(x) \leq 2^{k+1} \} = [b_{k+1}, b_k)$ and
\[
2^k b_k = \int_0^b f_1(x) \, dx.
\]
Consequently, we have that $b_{k+1} \leq b_k/2$, and so $||[b_{k+1}, b_k]| \geq |(0, b_k)|/2$. For simplicity, from now on we will write $p_{(0,b_k)}^- = p_k^-$ and $p_{(0,b_k)}^+ = p_k^+$.
Thus, for each $k \in \mathbb{Z}$, we estimate (3.2) by adapting the approach in [11]:

\begin{equation}
I_1 = \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} N f_1(x)^{p(x)} w(x)^{p(x)} \, dx
\end{equation}

\begin{align*}
&\leq \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} (2^{k+1})^{p(x)} w(x)^{p(x)} \, dx \\
&\leq 2^{p^+} \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} (2^{k})^{p(x)} w(x)^{p(x)} \, dx \\
&\lesssim \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left( \frac{1}{b_k} \int_0^{b_k} f_1(y) \, dy \right)^{p(x)} w(x)^{p(x)} \, dx \\
&\lesssim \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left( \int_0^{b_k} (f_1(y)\sigma^{-1}(y)) \sigma(y) \, dy \right)^{p(x)} b_k^{-p(x)} w(x)^{p(x)} \, dx \\
&=: J.
\end{align*}

Since $f_1 \sigma^{-1} > 1$ or $f_1 \sigma^{-1} = 0$, by (3.1) we have that

\[
\int_0^{b_k} (f_1(y)\sigma^{-1}(y)) \sigma(y) \, dy \leq \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy = \int_0^{b_k} (f_1(y)w(y))^{p(y)} \, dy \leq 1.
\]

Thus, for each $k \in \mathbb{Z}$ and $x \in (b_{k+1}, b_k)$ we have

\[
\left( \int_0^{b_k} (f_1(y)\sigma^{-1}(y)) \sigma(y) \, dy \right)^{p(x)} \leq \left( \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy \right)^{p(x)} \leq \left( \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy \right)^{p(x)}.
\]

Hence, by Jensen’s inequality,

\[
J \lesssim \sum_{k \in \mathbb{Z}} \left( \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy \right)^{p(x)} \int_{b_{k+1}}^{b_k} b_k^{-p(x)} w(x)^{p(x)} \, dx
\]

\[
= \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma(0,b_k)} \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy \right)^{p(x)} \int_{b_{k+1}}^{b_k} \sigma(0,b_k) b_k^{-p(x)} w(x)^{p(x)} \, dx
\]

\[
\leq \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma(0,b_k)} \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^{p(y)} \sigma(y) \, dy \right)^{p(x)} b_k^{-p(x)} w(x)^{p(x)} \, dx
\]
Then by Lemma 2.4 and [10, Corollary 2.23], we have

$$\times \int_{b_{k+1}}^{b_k} \sigma(0, b_k) p_k b_k^{-p(x)} w(x)^{p(x)} \, dx.$$  

To complete the proof we will estimate the last integral using the $A_{p(\cdot), 0}$ condition. We will show that

$$\int_{b_{k+1}}^{b_k} \sigma(0, b_k) p_k b_k^{-p(x)} w(x)^{p(x)} \, dx \leq C \sigma(0, b_k), \quad \text{for all } k \in \mathbb{Z},$$

or, more generally,

$$\int_0^b \sigma(0, b) p_{(b, \cdot)} b^{-p(x)} w(x)^{p(x)} \, dx \leq C \sigma(0, b), \quad \text{for all } b > 0.$$  

From the $A_{p(\cdot), 0}$ condition we know that

$$\left\| \frac{w \chi(0, b)}{b} \right\|_{p'(-)} \leq C,$$

so by the definition of the norm,

$$\int_0^b \left( \frac{w(x)}{b} \right)^{p(x)} \, dx \leq C.$$  

Hence, it will suffice to show that

$$\sigma(0, b) p_{(b, \cdot)} \leq C \sigma(0, b) \left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{p(x)}$$

for every $x \in (0, b)$: that is,

$$\frac{\sigma(0, b) p_{(b, \cdot)}^{-1}}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{p(x)}} \leq C, \quad x \in (0, b).$$  

(3.5)

The proof of (3.4) when $\left\| w^{-1} \chi(0, b) \right\|_{p'(-)} > 1$ is simple. By [10] Corollary 2.23, we have $\sigma(0, b) \leq \left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{(p'_{(b, \cdot)})}$ + 1. It is easy to see that

$$p_{(b, \cdot)} + (p_{(b, \cdot)} - 1) = p_{(b, \cdot)},$$

and since the exponent $p_{(b, \cdot)} - p(x)$ is negative,

$$\frac{\sigma(0, b) p_{(b, \cdot)}^{-1}}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{p(x)}} \leq \left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{(p_{(b, \cdot)}) - p(x)} \leq 1.$$

Now suppose that $\left\| w^{-1} \chi(0, b) \right\|_{p'(-)} \leq 1$. Then, $\sigma(0, b) \leq \left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{(p'_{(b, \cdot)})}$. Then by Lemma 2.4 and [10] Corollary 2.23, we have

$$\frac{\sigma(0, b)}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{p_{(b, \cdot)}}} \leq \frac{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{(p'_{(b, \cdot)})}}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}} = \frac{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}^{(p'_{(b, \cdot)}) - 1}}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}} \leq C \sigma(0, b) \frac{(p_{(b, \cdot)} + 1)^{-1}}{\left\| w^{-1} \chi(0, b) \right\|_{p'(-)}} = C \sigma(0, b) \frac{1}{p_{(b, \cdot)}}.$$  

Consequently,
\begin{equation}
\sigma(0, b)^{p(0, b) - 1} \leq C \|w^{-1} \chi_{(0, b)}\|^{p(0, b)}_{p'}.
\end{equation}

We now claim that
\begin{equation}
\|w^{-1} \chi_{(0, b)}\|^{p(x) - p(0, b)}_p \leq C, \quad x \in (0, b).
\end{equation}
To prove this, we first estimate the exponent:
\[
p(x) - p(0, b) = \frac{p'(x)}{p'(x) - 1} - \frac{(p')_{(0, b)} - p'(x)}{(p')_{(0, b)} - 1)} = \frac{(p')_{(0, b)} - p'(x)}{(p'(x) - 1)((p')_{(0, b)} - 1)} \leq \frac{(p')_{(0, b)} - (p')_{(0, b)}}{(p'(x) - 1)((p')_{(0, b)} - 1)} = \frac{p(x) - p(0, b)}{(p'(x) - 1)^2}, \quad x \in (0, b).
\]
Thus, for every \( x \in (0, b) \),
\[
\|w^{-1} \chi_{(0, b)}\|^{p(0, b) - p(x)}_{p'} = \left( \|w^{-1} \chi_{(0, b)}\|^{-1}_{p'} \right)^{p(x) - p(0, b)} \leq \|w^{-1} \chi_{(0, b)}\|^{p(0, b)}_{p'} \leq \frac{(p')_{(0, b)} - (p')_{(0, b)}}{(p'(x) - 1)^2},
\]
and by Lemma 2.4 applied to \( w^{-1} \in A^0_{p'} \), the right-hand term is bounded by a constant. This proves (3.7). Together, (3.6) and (3.7) immediately yield (3.5).

Given (3.5), we can now estimate as follows:
\[
J \leq C \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma(0, b_k)} \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^\frac{p(y)}{p-1} \sigma(y) dy \right)^{p-1} \sigma(0, b_k).
\]
Since \( \sigma \in A_{\infty, 0} \) and \( |[b_{k+1}, b_k]| \geq |(0, b_k)|/2 \), by Lemma 2.7 there exists \( 0 < \beta < 1 \) such that
\[
\sigma(b_{k+1}, b_k) \geq \beta \sigma(0, b_k).
\]
Define the weighted maximal operator
\[
N_\sigma g(x) = \sup_{b > x} \frac{1}{\sigma(0, b)} \int_0^b |g(y)| \sigma(y) dy, \quad x > 0.
\]
From the \( A^p_{(1, 0)} \) condition we have that \( 0 < \sigma(0, b) < \infty \) for every \( b > 0 \) (see the proof of Lemma 2.7). This fact together with [13, Lemma 2.2 (2)] implies that \( N_\sigma \) is bounded on \( L^p \) \((0, \infty), \sigma \) since \( p^- > 1 \). Hence,
\[
J \leq C \sum_{k \in \mathbb{Z}} \left( \frac{1}{\sigma(0, b_k)} \int_0^{b_k} (f_1(y)\sigma^{-1}(y))^\frac{p(y)}{p-1} \sigma(y) dy \right)^{p-1} \sigma(0, b_k) = C \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left[ N_\sigma \left( f_1(\sigma^{-1})^{p(1)} \right)(x) \right]^{p-1} \sigma(x) \, dx \leq C \int_0^\infty (f_1(x)\sigma^{-1}(x))^{p(x)} \sigma(x) \, dx.
\]
Then we can repeat the argument used in (3.3) to get
\[
\int_{0}^{\infty} f_1(x) p(x) w(x) p(x) \, dx 
\leq C.
\]

Estimate for \( I_2 \): As we did for \( f_1 \), we can find a non-increasing sequence \( \{b_k\}_{k \in \mathbb{Z}} \) such that \( \{x \in (0, \infty) : N f_2(x) > 2^k\} = (0, b_k), \{x \in (0, \infty) : 2^k < N f_2(x) \leq 2^{k+1}\} = [b_{k+1}, b_k) \) and
\[
2^k b_k = \int_{0}^{b_k} f_2(x) \, dx.
\]
Then we can repeat the argument used in (3.3) to get
\[
I_2 \leq C \sum_{k \in \mathbb{Z}} \int_{b_{k+1}}^{b_k} \left( \frac{1}{b_k} \int_{0}^{b_k} f_2(y) \, dy \right) ^{p(x)} w(x)^{p(x)} \, dx.
\]

We now estimate \( I_2 \). As we did for \( f_1 \), we can find a non-increasing sequence \( \{b_k\}_{k \in \mathbb{Z}} \) such that \( \{x \in (0, \infty) : N f_2(x) > 2^k\} = (0, b_k), \{x \in (0, \infty) : 2^k < N f_2(x) \leq 2^{k+1}\} = [b_{k+1}, b_k) \) and
\[
2^k b_k = \int_{0}^{b_k} f_2(x) \, dx.
\]
\[ \frac{1}{b_k} \leq C \|w \chi_{(0,b_k)}\|_{p'(\cdot)}^{-1} \|w^{-1} \chi_{(0,b_k)}\|_{p(\cdot)}^{-1} \]

\[ \leq C \|w \chi_{I_0}\|_{p(\cdot)}^{-1} \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)}^{-1} \leq C \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)}. \]

Hence, by Hölder’s inequality and our assumptions on \( f \),

\[ \frac{1}{C b_k} \int_{0}^{b_k} f_2(y) \, dy \leq \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)}^{-1} \|f_2w\|_{p(\cdot)} \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)} \leq 1. \]

Since \( p(\cdot) \in L^{H_\infty}(0, \infty) \), we can apply Lemma 2.2 with \( d\mu(x) = w(x)^{p'(x)} \, dx \), to the function \( g = \frac{1}{C b_k} \int_{0}^{b_k} f_2(y) \, dy \leq 1 \) on \( G = [b_{k+1}, b_k) \), to get

\[ K_2 \leq C \sum_{k:b_k > c_0} \int_{b_{k+1}}^{b_k} \left( \frac{1}{C b_k} \int_{0}^{b_k} f_2(y) \, dy \right)^{p(x)} w(x)^{p(x)} \, dx \]

\[ \leq C_t \sum_{k:b_k > c_0} \int_{b_{k+1}}^{b_k} C^{-p(\cdot)} \left( \frac{1}{b_k} \int_{0}^{b_k} f_2(y) \, dy \right)^{p(\cdot)} w(x)^{p(x)} \, dx \]

\[ + \sum_{k:b_k > c_0} \int_{b_{k+1}}^{b_k} \frac{w(x)^{p(x)}}{(e + x)^{tp}} \, dx \]

\[ \leq C_t \sum_{k:b_k > c_0} \left( \frac{1}{b_k} \int_{0}^{b_k} f_2(y) \, dy \right)^{p(\cdot)} W(b_{k+1}, b_k) + \int_{c_0}^{\infty} \frac{w(x)^{p(x)}}{(e + x)^{tp}} \, dx. \]

Arguing as in the proof of Lemma 2.9, we can choose \( t > 1 \) sufficiently large such that the second integral in the last line is at most 1. To estimate the sum in the last line we start by rewriting it as follows:

\[ \sum_{k:b_k > c_0} \left( \frac{1}{b_k} \int_{0}^{b_k} f_2(y) \, dy \right)^{p(\cdot)} W(b_{k+1}, b_k) \]

\[ = \sum_{k:b_k > c_0} \left( \frac{1}{\sigma(0,b_k)} \int_{0}^{b_k} f_2(y) \sigma^{-1}(y) \sigma(y) \, dy \right)^{p(\cdot)} \left( \frac{\sigma(0,b_k)}{b_k} \right)^{p(\cdot)} W(b_{k+1}, b_k) \]

\[ \leq C \sum_{k:b_k > c_0} \left( \frac{1}{\sigma(0,b_k)} \int_{0}^{b_k} f_2(y) \sigma^{-1}(y) \sigma(y) \, dy \right)^{p(\cdot)} \sigma(b_{k+1}, b_k) \]

\[ \times \frac{\sigma(0,b_k)^{p(\cdot)-1}}{b_k^p} \frac{W(0,b_k)}{b_k^p}. \]

where we have used again that \( \sigma \in A_{\infty,0} \). Since \( W(I_0), \|\sigma(I_0)\| \geq 1 \), by Corollary 2.23 we have \( \|w \chi_{I_0}\|_{p(\cdot)}, \|w^{-1} \chi_{I_0}\|_{p'(\cdot)} \geq 1 \), so \( \|w \chi_{(0,b_k)}\|_{p(\cdot)}, \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)} \geq 1 \) for every \( b_k > c_0 \). Hence, we can apply Corollary 2.10 twice and the \( A_{p(\cdot),0} \) condition to get

\[ \sigma(0,b_k)^{p(\cdot)-1} = \sigma(0,b_k)^{p(\cdot)} \frac{b_k^{p(\cdot)}}{b_k^{p(\cdot)}} \leq C \|w^{-1} \chi_{(0,b_k)}\|_{p'(\cdot)} \leq C \frac{b_k^{p(\cdot)}}{W(0,b_k)}. \]

Thus the final term is bounded.
To estimate the sum, recall that since $p_\infty \geq p^- > 1$, $N_\sigma$ is bounded on $L^{p^-}((0, \infty), d\sigma)$. Therefore, if we apply Lemma 2.2 with $d\mu(x) = \sigma(x) \, dx$ and $g = f_2 \sigma^{-1} \leq 1$ on $G = [b_{k+1}, b_k)$, and use the boundedness of $N_\sigma$, we get

$$\sum_{k:b_k > c_0} \left( \frac{1}{b_k} \int_{b_k}^{b_k} f_2(y) \, dy \right)^{p_\infty} W(b_{k+1}, b_k) \leq C \sum_{k:b_k > c_0} \left( \frac{1}{\sigma(0, b_k)} \int_0^{b_k} f_2(y) \sigma^{-1}(y) \sigma(y) \, dy \right)^{p_\infty} \sigma(b_{k+1}, b_k) \leq C \sum_{k:b_k > c_0} \int_{b_k}^{b_{k+1}} N_\sigma(f_2 \sigma^{-1})(x)^{p_\infty} \sigma(x) \, dx \leq C \int_0^\infty N_\sigma(f_2 \sigma^{-1})(x)^{p_\infty} \sigma(x) \, dx \leq C \int_0^\infty (f_2(x) \sigma^{-1}(x))^{p_\infty} \sigma(x) \, dx \leq C_1 C \int_0^\infty (f_2(x) \sigma^{-1}(x))^{p(x)} \sigma(x) \, dx + \int_0^\infty \frac{\sigma(x)}{(e + x)^{p^-}} \, dx \leq C_1 + 1.$$

In the second to last inequality we again used Lemma 2.2 exchanging the roles of $p(\cdot)$ and $p_\infty$ and replacing $w$ by $\sigma$. In the final inequality we used the fact that

$$\int_0^\infty (f_2(x) \sigma^{-1}(x))^{p(x)} \sigma(x) \, dx = \int_0^\infty f_2(x)^{p(x)} w(x)^{p(x)} \, dx \leq 1.$$

To estimate the final integral, we argued as we did in the proof of Lemma 2.4 with $\sigma$ instead of $W$, to show that we could choose $t$ big enough so that this term is smaller than 1. This completes the proof. □

4. Proofs of Theorems 1.8 and 1.9

We will prove Theorem 1.8 in two steps. First, we will prove it when $\lambda = 1$. Then we will give two lemmas that let us prove it for every $0 < \lambda < 1$.

**Proof of Theorem 1.8** for $\lambda = 1$. As we have remarked in the introduction, $Cf \sim Sf$; hence, it will suffice to prove that (i), (iii) and (v) are equivalent. Clearly, (i) implies (iii). Similarly, (iii)$\Rightarrow$(v) is immediate: since $Nf \lesssim Cf$, if $C$ is of weak-type, then $N$ is weak-type, and by Theorem 1.6 we get that $w \in A_{p(\cdot),0}$.

Finally, we will show that (v)$\Rightarrow$(i). If $w \in A_{p(\cdot),0}$, then $w^{-1} \in A_{p'(\cdot),0}$, and so by Theorem 1.6, $N$ is bounded on $L^{p'(\cdot)}(0, \infty)$ and $L^{p'_w(\cdot)}(0, \infty)$. Since $Hf \leq Nf$ for non-negative $f$, $H$ is bounded on $L^{p'_w(\cdot)}(0, \infty)$ and $L^{p'(\cdot)}(0, \infty)$. Then by duality we also have that $H^*$ is bounded on $L^{p'_w(\cdot)}(0, \infty)$. Therefore,

$$\|(Cf)w\|_{p(\cdot)} \leq \|(Hf)w\|_{p(\cdot)} + \|(H^*f)w\|_{p(\cdot)} \leq K \|fw\|_{p(\cdot)}.$$

This completes the proof when $\lambda = 1$. □
In order to prove Theorem 1.8 when \( \lambda \in (0, 1) \), we need two lemmas. The first lets us relate the \( A_{p(q,1)} \) to the \( A_{p,1} \) condition. This result is analogous to the property of the \( A_{p,q} \) weights in [23] and the \( A_{p(q,1)} \) weights proved in [7].

**Lemma 4.1.** Given \( p(\cdot) \in \mathcal{P} \) and \( \lambda > 0 \), define \( q(\cdot) \) as in the statement of Theorem 1.8. Then \( w \in A_{p(q,1)} \) if and only if \( w^{1/\lambda} \in A_{\lambda p(q,1)} \).

**Proof.** The proof is essentially the same as the proof of the corresponding result for the \( A_{p,q} \) and \( A_{p,1} \) classes. More precisely, it is enough to consider intervals of the form \( \{(0, b) : b > 0\} \), \( n = 1 \), and \( \alpha = 1 - \lambda \) in the proof of [7, Lemma 4.1 (i)].

The second lemma is a Hedberg-type inequality (see [23, Eq.(5)]) which lets us control \( S_\lambda \) with \( S = S_1 \).

**Lemma 4.2.** Given \( p(\cdot) \in \mathcal{P}(0, \infty) \) and \( \lambda > 0 \), define \( q(\cdot) \) as in the statement of Theorem 1.8. Let \( w \) be a weight and let \( f \) be a non-negative function in \( L^{p(\cdot)}(0, \infty) \). Then for every \( x \in (0, \infty) \),

\[
S_\lambda \left( \frac{f}{w} \right)(x) \leq \left[ S \left( g^{1/\lambda} \right)(x) \right]^\lambda \left( \int_0^\infty f(y)^{p(y)} \, dy \right)^{1-\lambda},
\]

where \( g(y) = f(y)^{p(y)/q(y)} w^{-1}(y) \).

**Proof.** We adapt the argument given in [23] for the fractional maximal operator with weights (see also [7, 24]). From the definition of \( g \) and the relation between \( p(\cdot) \) and \( q(\cdot) \) we get

\[
f(y)w^{-1}(y) = g(y)f(y)^{1-\lambda p(y)/q(y)} = g(y)f(y)^{(1-\lambda)p(y)}.
\]

Thus, if we apply Hölder’s inequality with \( 1/\lambda > 1 \) and \((1/\lambda)' = 1/(1 - \lambda)\), we get

\[
S_\lambda \left( \frac{f}{w} \right)(x) = \int_0^\infty \frac{g(y)}{x+y} f(y)^{(1-\lambda)p(y)} \, dy \\
\leq \left( \int_0^\infty g(y)^{1/\lambda} \frac{1}{x+y} \, dy \right)^\lambda \left( \int_0^\infty f(y)^{p(y)} \, dy \right)^{1-\lambda} \\
= \left[ S \left( g^{1/\lambda} \right)(x) \right]^\lambda \left( \int_0^\infty f(y)^{p(y)} \, dy \right)^{1-\lambda}.
\]

**Proof of Theorem 1.8 for \( \lambda \in (0, 1) \).** As in the case \( \lambda = 1 \), since \( S_\lambda \sim C_\lambda \) and the strong-type implies the weak-type, it is enough to prove that \((iii) \Rightarrow (v)\) and \((v) \Rightarrow (i)\).

To prove \((iii) \Rightarrow (v)\) we argue as in the proof of necessity in Theorem 1.6. Fix \( b > 0 \); then there exists a non-negative function \( g \in L^{p(\cdot)}(0, \infty) \) such that \( \|g\|_{p(\cdot)} \leq 1 \) and

\[
\|w^{-1}\chi_{(0,b)}\|_{p(\cdot)} \sim \int_0^b w^{-1}(y)g(y) \, dy.
\]

Without loss of generality we may assume that \( \|w^{-1}\chi_{(0,b)}\|_{p(\cdot)} > 0 \). Define \( f = \chi_{(0,b)}w^{-1}g \); then \( f \in L^{p(\cdot)}_w(0, \infty) \) with \( \|fw\|_{p(\cdot)} = \|\chi_{(0,b)}g\|_{p(\cdot)} \leq 1 \). If \( x \in (0, b) \),
By Lemma 4.1, so \( g \) implies (iv) implies (iii).

Hence, \((0, b) \subset \{ x \in (0, \infty) : S(x) > 0 \}\), so by the weak-type inequality we have that
\[
\frac{\|w^{-1} \chi_{(0,b)}\|_{p^{-1}(\lambda)}}{b^{\lambda}} \leq \mu \|w\chi_{(x \in (0,\infty) : S(x) > \mu)}\|_{q(-)} \leq K \|fw\|_{p(-)} \leq K,
\]

or, equivalently,
\[
\|w\chi_{(0,b)}\|_{q(-)} \leq Cb^{\lambda}.
\]

Since \( b > 0 \) is arbitrary, we get that \( w \in A_{p(-),q(-)} \).

To prove \((v) \Rightarrow (i)\), fix \( w \in A_{p(-),q(-)} \). To show that this implies \( \|S(f)w\|_{q(-)} \leq C\|fw\|_{p(-)} \) for every \( f \in L^{p(-)}_{w(\lambda)}(0, \infty) \), we will prove an equivalent inequality: for every \( f \in L^{p(-)}(0, \infty) \),
\[
\|S(f/w)\|_{q(-)} \leq C\|f\|_{p(-)}.
\]

Without loss of generality, we may assume \( \|f\|_{p(-)} = 1 \), so that \( \int_0^\infty f(y)^p dy = 1 \). We will show that
\[
\|S(f/w)\|_{q(-)} \leq C.
\]

By Lemma 4.2 we have
\[
S_{\lambda} \left( \frac{f}{w} \right)(x) \leq \left[ S \left( g^{1/\lambda} \right) \right] \lambda^{-\lambda}(x)
\]
with \( g(y) = f(y)^{p(y)/q(y)}w^{-1}(y) \). Therefore,
\[
\|S(f/w)\|_{q(-)} \leq \|S(g^{1/\lambda})\|_{q(-)} = \|S(g^{1/\lambda})w^{1/\lambda}\|_{q(-)}.
\]

Observe that
\[
\int_0^\infty \left( g(y) \right)^{1/\lambda}w(y)^{1/\lambda} \lambda q(y) \, dy = \int_0^\infty \left( f(y)^{p(y)/q(y)} \right)^{q(y)} \, dy = 1,
\]
so \( g^{1/\lambda} \in L^{\lambda q(-)}_{w_{1/\lambda}}(0, \infty) \) with \( \|g^{1/\lambda}w^{1/\lambda}\|_{\lambda q(-)} = \|gw\|_{q(-)} = 1 \). Further, we have that \( q(-) \in LH_0(0, \infty) \cap LH_{\infty}(0, \infty) \) since \( p(-) \) belongs to both classes and \( p^+ < 1/(1-\lambda) \).

By Lemma 4.3, \( w^{1/\lambda} \in A_{\lambda q(-)} \). So by the case \( \lambda = 1 \) proved above, \( S \) is bounded on \( L^{\lambda q(-)}_{w_{1/\lambda}} \). Therefore, we have that
\[
\|S(f/w)\|_{q(-)} \leq C\|g^{1/\lambda}w^{1/\lambda}\|_{\lambda q(-)} \leq C.
\]

This completes the proof. \( \square \)

Proof of Theorem 1.3. Since \( C^\alpha f \leq Cf \) for non-negative functions \( f \), by Theorem 1.8 and the fact that the strong-type inequality implies the weak-type, it suffices to show that \( (iv) \) implies \( (i) \).
We argue as we did for the proof of necessity above. Fix \( b > 0 \); then by duality there exists a non-negative function \( g \) such that \( \| g \|_{\varphi(\cdot)} \leq 1 \) and
\[
\| \chi_{(0,b)} \|_{\varphi(\cdot)} \sim \int_0^b w^{-1}(x)g(x)\,dx.
\]
Again we may assume \( \| w^{-1}\chi_{(0,b)} \|_{\varphi(\cdot)} > 0 \). Let \( f = \chi_{(0,b)}gw^{-1} \); then for \( t \in (2b,3b) \),
\[
C_\alpha f(t) \geq \frac{1}{(3b)^{\alpha+1}} \int_0^b (t-x)^\alpha w^{-1}(x)g(x)\,dx
\]
\[
\geq \frac{1}{3^{\alpha+1}b} \int_0^b w^{-1}(x)g(x)\,dx \sim \frac{\| w^{-1}\chi_{(0,b)} \|_{\varphi(\cdot)}}{b} := \mu > 0.
\]
Therefore,
\[(2b,3b) \subset \{ t \in (0,\infty) : C_\alpha f(t) > \mu \}.
\]
By the weak-type inequality and the choice of \( g \), we get that
\[
\| \chi_{(2b,3b)}w \|_{\varphi(\cdot)} \frac{\| w^{-1}\chi_{(0,b)} \|_{\varphi(\cdot)}}{b} \lesssim K \| w\chi_{(0,b)}w^{-1}g \|_{\varphi(\cdot)} \leq K.
\]
On the other hand, if we let \( f = \chi_{(2b,3b)} \), then it follows from (4.1) that \( f \in L_{w}^{p(\cdot)}(0,\infty) \). Thus, if we take \( t \in (0,b) \), we have that \( C_\alpha f(t) \geq 3^{\alpha-1} \), and so
\[(0,b) \subset \{ t \in (0,\infty) : C_\alpha f(t) > 3^{\alpha-1} \}.
\]
Therefore, again by the weak-type inequality, we have that
\[3^{\alpha-1} \| w\chi_{(0,b)} \|_{\varphi(\cdot)} \leq K \| w\chi_{(2b,3b)} \|_{\varphi(\cdot)}.
\]
If we combine this inequality with (4.1), we see that \( w \in A_{\varphi(\cdot),0} \). \( \square \)

5. Proofs of Theorems 1.12, 1.13 and 1.15

**Proof of Theorem 1.12** The proof of these results in \( \mathbb{R}^n \), \( n > 1 \), is essentially the same as the proof of the one-dimensional results on \( (0,\infty) \). In the definition of \( A_{\varphi(\cdot),0} \), we replace \( b \) in the denominator by \( b^n \) or by the volume of the ball \( B(0,b) \). The proof of Lemma 2.5 relies on results from 19, but these are for abstract bases over measure spaces and so hold in higher dimensions. In the proofs of the lemmas in Section 2 and in the proofs of Theorem 1.14 and of Theorem 1.18 for \( \lambda = 1 \), we replace \( (0,\infty) \) by \( \mathbb{R}^n \), the intervals \( (0,b) \) by the balls \( B(0,b) \) and intervals of the form \( (a,b) \) by the annuli \( \{ x \in \mathbb{R}^n : a < |x| < b \} \).

The proofs then go through exactly the same as in the one-dimensional case. We used the fact that the weighted maximal operator \( N_\sigma \) is bounded on \( L^p((0,\infty),d\sigma) \) for any \( 1 < p < \infty \), proved in 18. Now, we need to show that the corresponding operator on \( \mathbb{R}^n \), given by
\[
N_\sigma f(x) = \sup_{b>|x|} \frac{1}{\sigma(B(0,b))} \int_{B(0,b)} |f(y)|\sigma(y)\,dy
\]
is bounded on \( L^p(\mathbb{R}^n,d\sigma) \) for \( 1 < p < \infty \). We include the proof below, which was sketched in [17] pp. 559-560. First, notice that the \( A_{\varphi(\cdot),0} \) condition will guarantee
0 < \sigma(B(0,b)) < \infty. Then, we can show that \( N_\sigma \) satisfies a weak \((1,1)\) inequality, as in the one-dimensional case (see [13, Lemma 2.2]). Suppose \( f \) is a bounded function of compact support. Then, we have that given any \( \mu > 0 \), there exists \( b = b(\mu) > 0 \) such that

\[
\{ x \in \mathbb{R}^n : N_\sigma f(x) > \mu \} = B(0,b),
\]

and

\[
\mu = \frac{1}{\sigma(B(0,b))} \int_{B(0,b)} |f(y)|\sigma(y) \, dy.
\]

But then we immediately get the weak \((1,1)\) inequality:

\[
\sigma(\{ x \in \mathbb{R}^n : N_\sigma f(x) > \mu \}) = \sigma(B(0,b)) \leq \frac{1}{\mu} \int_{\mathbb{R}^n} |f(y)|\sigma(y) \, dy.
\]

That \( N_\sigma \) is bounded on \( L^p(\mathbb{R}^n,d\sigma) \) for \( p > 1 \) now follows from Marcinkiewicz interpolation. \( \square \)

**Proof of Theorem 1.13** By Theorem 1.12 \( w \in A_p(\cdot,0) \) is equivalent to

\[
\|(Sf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.
\]

By duality, this inequality can be rewritten as

\[
\sup_{\|gw^{-1}\|_{p(\cdot)} \leq 1} \int_{\mathbb{R}^n} Sf(x)g(x) \, dx \leq C\|fw\|_{p(\cdot)},
\]

which in turn is equivalent to

\[
\sup_{g \in L^{p(\cdot)}_w(\mathbb{R}^n),g \geq 0} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{f(x)}{|x|^n + |y|^n} \, dy \right) \frac{g(x)}{\|gw^{-1}\|_{p(\cdot)}} \, dx \leq C\|fw\|_{p(\cdot)}.
\]

This in turn is equivalent to the desired inequality (1.16). \( \square \)

**Proof of Theorem 1.16** For each \( k \in \mathbb{Z} \), define the annuli \( I_k = \{ x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k \} \) and \( I_k^* = \{ x \in \mathbb{R}^n : 2^{k-2} \leq |x| < 2^{k+1} \} \). Note that the \( I_k^* \) have bounded overlap. Given \( f \in L^{p(\cdot)}_w(\mathbb{R}^n) \), let \( f_{k,0} = f \chi_{I_k^*} \) and \( f_{k,1} = f - f_{k,0} \). Then we have that

\[
T^* f(x) = \sum_{k \in \mathbb{Z}} T^* f(x) \chi_{I_k}(x)
\]

\[
\leq \sum_{k \in \mathbb{Z}} T^* f_{k,0}(x) \chi_{I_k}(x) + \sum_{k \in \mathbb{Z}} T^* f_{k,1}(x) \chi_{I_k}(x) := T^*_0 f(x) + T^*_1 f(x).
\]

For the operator \( T^*_0 \), we will use duality, [13, the boundedness of \( T^* \) and (1.9) to get

\[
\|wT^*_0 f\|_{p(\cdot)} \leq C \sup_{\|g\|_{p(\cdot)} \leq 1} \int_{\mathbb{R}^n} T^*_0 f(x)g(x)w(x) \, dx
\]

\[
\leq C \sup_{\|g\|_{p(\cdot)} \leq 1} \sum_{k \in \mathbb{Z}} \int_{I_k} |T^* f_{k,0}(x)||g(x)||w(x)| \, dx
\]
Since we have the pointwise estimate 

\[
\|f\|_{p(\cdot)} \leq C \sup_{\|g\|_{p(\cdot)} \leq 1} \sum_{k \in \mathbb{Z}} \sup_{I_k} \sup_{x \in I_k} w(x) \int_{I_k} |T^* f_k,0(x)| g(x) \, dx
\]

Then by (1.7) we have the pointwise estimate 

\[
\|f\|_{p(\cdot)} \leq C \sup_{\|g\|_{p(\cdot)} \leq 1} \sum_{k \in \mathbb{Z}} \sup_{I_k} \sup_{x \in I_k} w(x) \|T^* f_k,0\|_{p(\cdot)} \|g\chi_{I_k}\|_{p(\cdot)}
\]

In order to estimate \( T_1^* \), first note that for \( x \in I_k \) and \( y \in (I_k^c)^c, |x-y| \sim |x|+|y| \). Then by (1.7) we have the pointwise estimate 

\[
T_1^* f(x) \leq C_0 \sum_{k \in \mathbb{Z}} \left( \int_{I_k^c} \frac{f(y)}{|x-y|^n} dy \right) \chi_{I_k}(x) \leq C_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x|^n + |y|^n} dy = C_0 S f(x).
\]

Since \( w \in A_{p(\cdot),0} \), the desired bound follows from Theorem 1.12. \( \square \)

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