Two Dimensional String Theory and the Topological Torus

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ABSTRACT

We analyze topological string theory on a two dimensional torus, focusing on symmetries in the matter sector. Even before coupling to gravity, the topological torus has an infinite number of point-like physical observables, which give rise via the BRST descent equations to an infinite symmetry algebra of the model. The point-like observables of ghost number zero form a topological ground ring, whose generators span a spacetime manifold; the symmetry algebra represents all (ground ring valued) diffeomorphisms of the spacetime. At nonzero ghost numbers, the topological ground ring is extended to a superring, the spacetime manifold becomes a supermanifold, and the symmetry algebra preserves a symplectic form on it. In a decompactified limit of cylindrical target topology, we find a nilpotent charge which behaves like a spacetime topological BRST operator. After coupling to topological gravity, this model might represent a topological phase of \( c = 1 \) string theory. We also point out some analogies to two dimensional superstrings with the chiral GSO projection, and to string theory with \( c = -2 \).

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1. Introduction

Since the discovery of the topological symmetry underlying string theory in $c < 1$ dimensions [1–7], it has become a common feeling that the case of $c = 1$ (see [8,9] for a review) might share at least some topological properties with the simplest exactly solvable models [10,11]. By naive counting, gravity in two dimensions has minus one field-theoretical degrees of freedom, which suggests that at $c = 1$ we are on the verge after which topological symmetry ceases to be an explicit symmetry of the model. Still, there is the long-standing conjecture [12] that quantum gravity and string theory may have an unbroken, topological phase, even in dimensions higher than one. However, for this to be the case, topological symmetry should be hidden in the broken phase in an as yet unknown manner, hence the issue of spontaneous breakdown of topological symmetry would be crucial in this respect.

The theory of strings in two dimensions is presumably the simplest string theory in which topological symmetry of any kind is not manifest. The recent discovery [13] and analysis [14–18] of the ground ring structure in two dimensional string theory provides new support to the idea advocated some time ago [11] that discrete states [19] of $c = 1$ string theory may have a topological origin. Discrete states have been analyzed thoroughly [20] in the framework of stringy BRST cohomologies. Their correlation functions have been discussed in [21]. The discrete states are closely related to spacetime $W_\infty$-like symmetries of the model [13,14,18,21,22–26].

In this paper, we will discuss topological sigma model with the two dimensional toroidal target. The motivation for the present work is the hope that the topological torus coupled to topological gravity might represent a topological phase of two dimensional string theory. This model has been touched upon from several viewpoints in [4,27,28]; some other topological aspects of $c = 1$ string theory have been studied in the context of the Penner model in [29], and the Liouville approach in [30].

Possible connections with $c = 1$ string theory do not represent the only motivation for the study of the topological torus. Indeed, the topological torus is an
excellent example of topological field theory leading to interesting phenomena of their own. It has an infinite number of physical states, which lead to an enormous quantum symmetry of the model, and which might be eventually assembled into a field theoretical degree of freedom. Moreover, the topological torus represents one of the simplest topological sigma models with \( \pi_1(M) \neq 0 \), a class of targets not yet fully understood. Another motivation comes from the fact that the model can be easily extended to higher target dimensions, and may thus eventually shed some light on the issue of higher symmetries in string theory [31].

The paper is organized as follows. In §2, we set up the basic framework of the topological torus sigma model on flat worldsheet, and identify BRST invariant observables (§2.1). We analyze their structure at ghost number zero in §2.2, and at nonzero ghost numbers in §2.3. Point-like observables of ghost number zero form a topological ground ring, which is extended by observables of nonzero ghost numbers to a superring. Bosonic generators of the ground ring span a manifold, which we will refer to as the ‘spacetime’ manifold, to distinguish it from the ‘target’ torus. Fermionic generators extend the spacetime manifold to a supermanifold. Via descent equations, the ground ring observables give rise to a symmetry algebra, formed by BRST invariant conserved charges. We study these symmetries in §3. In §3.1 we show that the bosonic part of the symmetry algebra generates the algebra of all spacetime diffeomorphisms (with coefficients in the topological ground ring), while its supersymmetric extension preserves a symplectic form on the spacetime supermanifold. In §3.2 we study symmetries of the decompactified limit in which the target torus becomes an infinite cylinder. In this limit, the symmetry algebra can be extended in a weak sense to the topologically twisted \( N = 2 \) superconformal algebra, hence realizing topological BRST symmetry in spacetime. The BRST charge of this spacetime topological symmetry is conserved, but not BRST invariant under the topological BRST charge on the worldsheet, hence the target topological symmetry is hidden in the model. §4 offers some summarizing discussion.
2. The Topological Torus

We will start with the following free field Lagrangian,

\[ I_0 = \frac{1}{\pi} \int_{\Sigma} d^2z \left( \partial_z \bar{X} \partial_{\bar{z}} X - \chi_z \partial_{\bar{z}} \psi - \bar{\chi}_{\bar{z}} \partial_z \bar{\psi} \right) , \tag{2.1} \]

which describes the topological sigma model on flat worldsheet \( \Sigma \) and with a flat two dimensional torus as the target. This is a conformal field theory (CFT) even before imposing the BRST constraint, hence we will use frequently some parts of CFT terminology. In (2.1), \( X, \bar{X} \) are complex coordinates on the torus, which has its Kähler structure fixed once and for all. Namely, we assume

\[ X \equiv X + 2\pi m R + 2\pi n R \tau_0, \tag{2.2} \]

with \( R \) real, and \( \text{Im} \tau_0 \neq 0 \). \( \psi, \bar{\psi} \) of (2.1) are fermionic ghosts of conformal weight zero, and the antighosts \( \chi_z, \bar{\chi}_{\bar{z}} \) have conformal weights (0,1) and (1,0) respectively. On-shell we shall write

\[ X(z, \bar{z}) = X(z) + X(\bar{z}), \quad \bar{X}(z, \bar{z}) = \bar{X}(z) + \bar{X}(\bar{z}), \]

\[ \psi(z), \quad \bar{\psi}(\bar{z}), \quad \chi_z(z), \quad \bar{\chi}_{\bar{z}}(\bar{z}). \tag{2.3} \]

With this normalization, we have

\[ X(z) \bar{X}(w) \sim -\ln(z - w), \]

\[ \psi(z) \chi_w(w) \sim -\frac{1}{z - w}, \tag{2.4} \]

and similarly for the right-movers. The energy-momentum tensor of the model is the standard one of free fields,

\[ T_{zz}(z) = -\partial_z X \partial_{\bar{z}} \bar{X} + \chi_z \partial_{\bar{z}} \psi, \tag{2.5} \]

while the exotic form of the bosonic kinetic term in (2.1) only affects the zero-mode
part of $X$ and $\bar{X}$:

$$X(z, \bar{z}) = x - i \left( \frac{in_1}{2R} + \frac{n_1 \text{Re} \tau_0}{2R \text{Im} \tau_0} + \frac{n_2}{2R \text{Im} \tau_0} + m_1 R + m_2 R \tau_0 \right) \ln z$$

$$- i \left( \frac{in_1}{2R} + \frac{n_1 \text{Re} \tau_0}{2R \text{Im} \tau_0} + \frac{n_2}{2R \text{Im} \tau_0} \right) \ln \bar{z} + \text{oscillators.}$$

(2.6)

Physical states will have $n_1, n_2 = 0$, hence only winding states will survive the BRST condition.

2.1. Observables

Topological BRST symmetry of (2.1) acts (on-shell) via its BRST charge by

$$[Q, X] = \psi, \quad [Q, \bar{X}] = \bar{\psi},$$

$$\{Q, \psi\} = 0, \quad \{Q, \bar{\psi}\} = 0, \quad (2.7)$$

$$\{Q, \chi_z\} = \partial_z \bar{X}, \quad \{Q, \bar{\chi}_{\bar{z}}\} = \partial_{\bar{z}} X.$$

As $X, \bar{X}$ do not exist as quantum fields, $\psi$ and $\bar{\psi}$ are BRST nontrivial. Hence, they give rise to the set of nontrivial point-like observables that correspond to the homology ring of the target, quite analogously as in any other topological sigma model. We will refer to these as the ‘homology observables,’ and will denote them by

$$O^{(0)}_0 = 1, \quad P^{(0)}_0 = \psi, \quad Q^{(0)}_0 = \bar{\psi}, \quad R^{(0)}_0 = \psi \bar{\psi}. \quad (2.8)$$

On the other hand, as the fundamental group of the target is nontrivial, we obtain a new set of observables from winding sectors.* From the worldsheet point of view, the existence of these new observables is related to the fact that the following composites,

$$O^{(0)}_k(z, \bar{z}) = e^{ik \bar{X}(z) + i\bar{k} X(\bar{z})} \quad (2.9)$$

with $k = mR + nR \tau_0, \quad \bar{k} = -mR - nR \bar{\tau}_0$, carry conformal weight $(0,0)$ with respect to the energy-momentum tensor (2.5) (and its right-moving counterpart).

* This fact has been pointed out in [4].
These observables can be naturally referred to as ‘homotopy observables.’ General point-like physical observables can be constructed as products of the homotopy and homology observables. We will denote the corresponding fields by

\[ P_k^{(0)} = \psi O_k^{(0)}, \quad Q_k^{(0)} = \bar{\psi} O_k^{(0)}, \quad R_k^{(0)} = \psi \bar{\psi} O_k^{(0)}. \]  

In cohomological field theories [32], BRST invariant fields of conformal weight \((0,0)\) and ghost number \((m,n)\) give rise to a new set of observables, with conformal weights \((p,q)\) and ghost number \((m - p, n - q)\). The key to this hierarchy of observables are the (on-shell) descent equations,

\[
\{Q, \mathcal{O}^{(0)}\} = 0, \\
\{Q, \mathcal{O}^{(1)}\} = d\mathcal{O}^{(0)}, \\
\{Q, \mathcal{O}^{(2)}\} = d\mathcal{O}^{(1)},
\]

which can be split in the case at hand to the left-moving and right-moving part. \(\dagger\) (In (2.11), \(\mathcal{O}^{(0)}\) is an arbitrary physical observable of conformal weight \((0,0)\), and \{ , \} denotes either commutator or anticommutator here.)

To summarize, we have encountered a remarkably rich structure of observables in the pure matter theory, even before coupling to gravity. Thus, we have worldsheet scalars

\[
O_k^{(0)}(z, \bar{z}) = e^{ik\bar{X}(z) + ikX(\bar{z})}, \\
P_k^{(0)}(z, \bar{z}) = \psi e^{ik\bar{X}(z) i\bar{k}X(\bar{z})}, \\
Q_k^{(0)}(z, \bar{z}) = \bar{\psi} e^{ik\bar{X}(z) + i\bar{k}X(\bar{z})}, \\
R_k^{(0)}(z, \bar{z}) = \psi \bar{\psi} e^{ik\bar{X}(z) + i\bar{k}X(\bar{z})}
\]

of ghost numbers \((0,0)\), \((0,1)\), \((1,0)\) and \((1,1)\), parametrized by \(k\), the winding number on the target. These fields have partners of conformal weight \((1,0)\) with

\(\dagger\) Actually, an even more refined splitting exists, as both of the exterior derivatives involved (i.e. \(d\) and \(Q\)) can be split into \(d = \partial + \bar{\partial}\), \(Q = Q_L + Q_R\), and analogously for the ghost number. We will mostly avoid using this splitting in the following, however.
ghost numbers \((-1,0), (0,0), (-1,1)\) and \((0,1)\), and partners of conformal weight \((0,1)\) with ghost numbers \((0,-1), (0,0), (1,-1)\) and \((1,0)\), assembled in the following one-forms:

\[ O_k^{(1)}(z, \bar{z}) = i(k\chi + \bar{k}\bar{\chi}) e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ P_k^{(1)}(z, \bar{z}) = i(i\partial X + k\psi\chi + \bar{k}\psi\bar{\chi}) e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ Q_k^{(1)}(z, \bar{z}) = i(i\bar{\partial} \bar{X} + \bar{k}\bar{\psi}\bar{\chi} + k\psi\chi) e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ R_k^{(1)}(z, \bar{z}) = i ((i\partial X + k\psi\chi)\bar{\psi} + \psi(i\bar{\partial} \bar{X} + k\bar{\psi}\bar{\chi})) e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}. \]

(Here and throughout, we use the following notation: \(\partial = dz \partial_z\), \(\bar{\partial} = d\bar{z} \partial_{\bar{z}}\), \(\chi = \chi_z dz\), \(\bar{\chi} = \bar{\chi}_{\bar{z}} d\bar{\bar{z}}\).) Moreover, partners of \((2.12)\) exist with conformal weight \((1,1)\) and ghost numbers \((-1,-1), (-1,0), (0,-1)\) and \((0,0),\)

\[ O_k^{(2)}(z, \bar{z}) = -k\bar{k} \chi \wedge \bar{\chi} e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ P_k^{(2)}(z, \bar{z}) = -(i\partial X + k\psi\chi) \wedge \bar{k}\bar{\chi} e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ Q_k^{(2)}(z, \bar{z}) = -(i\bar{\partial} \bar{X} + \bar{k}\bar{\psi}\bar{\chi}) \wedge k\chi e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}, \]
\[ R_k^{(2)}(z, \bar{z}) = -(i\partial X + k\psi\chi) \wedge (i\bar{\partial} \bar{X} + \bar{k}\bar{\psi}\bar{\chi}) e^{ik\bar{X}(z)+i\bar{k}X(\bar{z})}. \]

Each of these fields will play its specific role below.

2.2. Ghost Number Zero

Now we will show that an important structure exists already at ghost number zero. Indeed, observables \(O_k^{(0)}\) with ghost number zero and conformal weight \((0,0)\) form a ring which, as will be apparent below, is the topological analog of the ground ring of two dimensional string theory discovered in [13].\[ More precisely, it is the matter part of the corresponding analog, as the Liouville dimension has not yet been included. The full-fledged analog should come from the coupling to topological gravity.\]
the ring is free and commutative, with two generators
\begin{align}
a &= e^{ik_a \hat{X}(z) + i\hat{k}_a \hat{X}(\bar{z})}, \\
b &= e^{ik_b \hat{X}(z) + i\hat{k}_b \hat{X}(\bar{z})},
\end{align}
where \( k_a, k_b \) is an arbitrarily chosen basis of the torus’ lattice. Generic elements of the ring are finite sums of
\begin{equation}
a^m b^n,
\end{equation}
with \( m, n \in \mathbb{Z} \). In geometrical terms, the ring is isomorphic to the group ring of the fundamental group of the target manifold. We will refer to it as the ‘topological ground ring’ of the sigma model, and denote it by \( \mathcal{R} \).

\( P_k^{(1)} \) and \( Q_k^{(1)} \) are one-forms on the worldsheet. Each of them is conserved modulo BRST commutators by virtue of the descent equation (2.11). As the one-forms carry ghost number zero, they generate an algebra of symmetries of the topological ground ring. This algebra can be identified from the operator product expansion (OPE) of these currents (with \( dz, d\bar{z} \) omitted):
\begin{equation}
P_k^{(1)}(z, \bar{z}) P_\ell^{(1)}(w, \bar{w}) \sim i(\ell - k) \frac{P_{k+\ell}^{(1)}(w, \bar{w})}{z - w},
\end{equation}
and similarly for \( Q_k^{(1)} \)'s, with \( (\ell - \bar{k}) \) replacing \( (\ell - k) \) on the right hand side.

The corresponding charges
\begin{equation}
L_k \equiv \frac{1}{2\pi} \oint_C P_k^{(1)}(z, \bar{z})
\end{equation}
form an infinite algebra, which is surprisingly close to the Virasoro algebra:
\begin{equation}
[L_k, L_\ell] = (k - \ell)L_{k+\ell}.
\end{equation}
The action of the algebra on the topological ground ring is encoded in the following
OPE:

\[ P_k^{(1)}(z, \bar{z}) \cdot Q_\ell^{(1)}(w, \bar{w}) \sim i\ell \cdot \frac{O^{(1)}_{k+\ell}(w, \bar{w})}{(z - w)}. \]  

(2.20)

As the topological ground ring is the ring of (positive as well as negative) power series in two generators (2.15), it is sufficient to identify the action of the Virasoro-like algebra (2.19) on the generators \( a, b \) of \( \mathcal{R} \). From (2.20) we obtain

\[
L_{mR+nR\tau_0} \cdot a = -R \ a^{m+1} b^n, \\
L_{mR+nR\tau_0} \cdot b = -R\tau_0 \ a^m b^{n+1}.
\]

(2.21)

Hence, the Virasoro-like algebra (2.19) acts on the \((a, b)\) plane via specific infinitesimal diffeomorphisms, with

\[
L_{mR+nR\tau_0} = -R \ a^{m+1} b^n \frac{\partial}{\partial a} - R\tau_0 \ a^m b^{n+1} \frac{\partial}{\partial b},
\]

(2.22)

while the analogous algebra generated by \( \bar{L}_k \),

\[
L_k \equiv \frac{1}{2\pi} \oint_C Q_k^{(1)}(z, \bar{z}),
\]

(2.23)

acts by

\[
L_{mR+nR\tau_0} = -R \ a^{m+1} b^n \frac{\partial}{\partial a} - R\tau_0 \ a^m b^{n+1} \frac{\partial}{\partial b}
\]

(2.24)

and satisfies

\[
[L_k, L_\ell] = -(\bar{k} - \bar{\ell}) L_{k+\ell}.
\]

(2.25)

Commutation relations between \( L_k \) and \( \bar{L}_\ell \),

\[
[L_k, \bar{L}_\ell] = -\bar{k} \ L_{k+\ell} - \ell \ \bar{L}_{k+\ell},
\]

(2.26)

can be determined either from the corresponding OPE,

\[
P_k^{(1)}(z, \bar{z}) \cdot Q_\ell^{(1)}(w, \bar{w}) \sim i\ell \cdot \frac{Q_{k+\ell}(w, \bar{w})}{z - w} - i\bar{k} \cdot \frac{P_{k+\ell}^{(1)}(w, \bar{w})}{z - \bar{w}},
\]

(2.27)

or by commuting directly (2.22) and (2.24) on the \((a, b)\) plane.
Finally, $R_{k}^{(2)}$, which are two-forms of ghost number zero, can be multiplied by their own coupling constants and added to the Lagrangian (2.1),

$$I(\alpha_k) = I_0 + \sum_k \alpha_k \int_{\Sigma} R_{k}^{(2)}(z, \bar{z}),$$

(2.28)

hence deforming the theory while preserving topological symmetry. We must be careful here however, as it is still necessary to check which values of $\alpha_k$ are permitted, i.e. which $R_{k}^{(2)}$'s are genuine moduli of the theory. In particular, it is important to bear in mind that under these deformations, the on-shell structure we have studied so far may be changed.

The possibility of adding terms $R_{k}^{(2)}$ with nonzero $k$ to the Lagrangian is extremely attractive, as they break translational invariance in the target. On the other hand, this possibility is presumably not related to the non-conservation of momenta in Liouville theory, as the topological analog of Liouville dimension is expected to emerge just after coupling to topological gravity. Note that even the deformed Lagrangian (2.28) still carries ghost number (0,0).

2.3. Nonzero Ghost Numbers

At ghost number one, we have two infinite sequences of fermionic point-like observables, namely $P_{k}^{(0)}$ and $Q_{k}^{(0)}$ of (2.10), each of them parametrized by target winding numbers. Together with the bosonic observables $R_{k}^{(0)}$ of ghost number (1,1), they comprise an extension of the topological ground ring $\mathcal{R}$ to a supersymmetric version thereof, which we will denote by $\mathcal{R}'$. The following OPEs between the observables of conformal weight (0,0),

$$O_{k}^{(0)}(z, \bar{z}) P_{\ell}^{(0)}(w, \bar{w}) \sim P_{k+\ell}^{(0)}(w, \bar{w}),$$

$$O_{k}^{(0)}(z, \bar{z}) Q_{\ell}^{(0)}(w, \bar{w}) \sim Q_{k+\ell}^{(0)}(w, \bar{w}),$$

$$O_{k}^{(0)}(z, \bar{z}) R_{\ell}^{(0)}(w, \bar{w}) \sim R_{k+\ell}^{(0)}(w, \bar{w}),$$

$$P_{k}^{(0)}(z, \bar{z}) Q_{\ell}^{(0)}(w, \bar{w}) \sim R_{k+\ell}^{(0)}(w, \bar{w}),$$

(2.29)

and zero otherwise, show that this fermionic extension of $\mathcal{R}$ is a free ring generated
multiplicatively by the generators $a, b$ of $\mathcal{R}$ (with both positive and negative powers permitted), together with two anticommuting generators $\theta, \bar{\theta}$,

$$\theta \equiv \psi(z), \quad \bar{\theta} \equiv \bar{\psi}(\bar{z}). \quad (2.30)$$

Geometrically, this ring is naturally isomorphic to the tensor product of the homology ring of the target and the group ring of its fundamental group. Elements of $\mathcal{R}'$ are finite sums of

$$a^m b^n \theta^p \bar{\theta}^q, \quad m, n \in \mathbb{Z}, \quad p, q \in \{0, 1\}. \quad (2.31)$$

Analogously, the algebra of symmetries of the topological ground ring $\mathcal{R}$ is extended to a superalgebra of symmetries of the extended ring $\mathcal{R}'$. We will define the charges that correspond to nonzero ghost number currents of (2.13) by

$$Q_k = \frac{1}{2\pi} \oint_C O^{(1)}_k(z, \bar{z}),$$
$$G_k = \frac{1}{2\pi} \oint_C R^{(1)}_k(z, \bar{z}), \quad (2.32)$$

and determine their algebra from the OPEs of the currents.\(^*\) First of all, tensorial properties of the new charges with respect to the Virasoro-like algebra (2.19) and (2.26) can be determined from

$$P_k^{(1)}(z, \bar{z}) O^{(1)}_{\ell}(w, \bar{w}) \sim i\ell \frac{O^{(1)}_{k+\ell}(w, \bar{w})}{z-w},$$
$$P_k^{(1)}(z, \bar{z}) R^{(1)}_{\ell}(w, \bar{w}) \sim i(\ell-k) \frac{R^{(1)}_{k+\ell}(w, \bar{w})}{z-w} \quad (2.33)$$

(and analogously for $P_k^{(1)}$ replaced by $Q_k^{(1)}$, with some obvious conjugations), which shows that $Q_m$ behave like Fourier components of a one-form, while $G_m$ comprise

\(^*\) In the remainder of this section, we will work in the decompactification limit of $\text{Im } \tau_0 \to \infty$, which leads to $k = mR, \bar{k} = -mR$. We can set $k_a = R$, and $b$ is effectively zero. The objects will be scaled such that $R = 1$. 

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a two-tensor with respect to (2.19) and (2.26). The anticommutation relation of the fermionic charges is encoded in

\[ O^{(1)}_k(z, \bar{z}) R^{(1)}_\ell(w, \bar{w}) \sim ik \frac{Q^{(1)}_{k+\ell}(w, \bar{w})}{z-w} - i\bar{k} \frac{P^{(1)}_{k+\ell}(w, \bar{w})}{\bar{z}-\bar{w}}. \] (2.34)

We can thus obtain the complete algebra of quantum symmetries of the topological ground ring \( R' \), as generated by BRST invariant currents (2.13), in the decompactification limit studied here. After redefining

\[
\mathcal{L}_m = \frac{1}{2}(L_m + \bar{L}_m), \\
\mathcal{J}_m = \bar{L}_m - L_m, 
\] (2.35)

we arrive at the following algebra of charges:

\[
[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n}, \quad [\mathcal{J}_m, \mathcal{J}_n] = 0, \\
[\mathcal{L}_m, \mathcal{Q}_n] = -n\mathcal{Q}_{m+n}, \quad [\mathcal{J}_m, \mathcal{Q}_n] = 0, \\
[\mathcal{L}_m, \mathcal{G}_n] = (m - n)\mathcal{G}_{m+n}, \quad [\mathcal{J}_m, \mathcal{G}_n] = 0, \\
\{\mathcal{Q}_m, \mathcal{G}_n\} = -m\mathcal{J}_{m+n}, \quad [\mathcal{L}_m, \mathcal{J}_n] = -n\mathcal{J}_{m+n}. 
\] (2.36)

This is an extension of the Virasoro-like algebra (2.19) by fermionic generators. We will analyze its possible physical meaning in the next section.

By definition, the algebra (2.36) acts on the fermionic extension of the ground ring, \( R' \), as a symmetry algebra. As \( R' \) is free and finitely generated, the action of the symmetry algebra is determined by its action on the superplane of \((a, \theta, \bar{\theta})\), where it acts by (infinitesimal) super-diffeomorphisms. From the corresponding OPEs, of which the first few examples are

\[
P^{(1)}_k(z, \bar{z}) P^{(0)}_{\ell}(w, \bar{w}) \sim i(\ell - k) \frac{P^{(0)}_{k+\ell}(z, \bar{z})}{z-w}, \\
P^{(1)}_k(z, \bar{z}) Q^{(0)}_{\ell}(w, \bar{w}) \sim i\ell \frac{Q^{(0)}_{k+\ell}(z, \bar{z})}{z-w} - i\bar{k} \frac{P^{(0)}_{k+\ell}(z, \bar{z})}{\bar{z}-\bar{w}}, \] (2.37)
we infer

\[
\mathcal{L}_m = -a^{m+1}_{m+1} \frac{\partial}{\partial a} + \frac{1}{2} ma^m (\theta + \bar{\theta}) \left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \bar{\theta}} \right),
\]

\[
\mathcal{J}_m = ma^m (\bar{\theta} - \theta) \left( \frac{\partial}{\partial \bar{\theta}} + \frac{\partial}{\partial \theta} \right),
\]

\[
\mathcal{Q}_m = ma^m \left( \frac{\partial}{\partial \bar{\theta}} + \frac{\partial}{\partial \theta} \right),
\]

\[
\mathcal{G}_m = -a^{m+1}_m (\bar{\theta} - \theta) \frac{\partial}{\partial a} - m \theta \bar{\theta} a^m \left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \bar{\theta}} \right).
\]

The formulae obtained here can be easily extended to the full-fledged torus with both of the target dimensions compactified, as we will see below. We will discuss the decompactification limit again in §3.2.

The analysis of the observables with nonzero ghost numbers is completed by mentioning the existence of two-forms with nonzero ghost numbers in (2.14). At ghost number \((-1,-1)\) we have two-forms \(O_k^{(2)}\), which are BRST invariant up to total derivative. Hence their integrals over \(\Sigma\) are physical observables, and can be used to deform the Lagrangian to

\[
I(\alpha_k, \beta_\ell) = I_0 + \sum_k \alpha_k \int_\Sigma R_k^{(2)}(z, \bar{z}) + \sum_\ell \beta_\ell \int_\Sigma O_\ell^{(2)}(z, \bar{z}).
\]

The ghost number is no longer conserved if at least some of the \(\beta_\ell\)'s are nonzero. \(P_k^{(2)}\) and \(Q_k^{(2)}\) are two-forms as well, but cannot be simply added to the Lagrangian, as they are fermionic.
3. Spacetime Symmetries

3.1. Spacetime Diffeomorphisms

In the previous subsection, we have mainly analyzed the theory in the decom-pactified limit of \( \text{Im} \tau_0 \to \infty \). Now we will study the full-fledged model.

The full algebra of symmetries is now

\[
\begin{align*}
[L_k, L_\ell] &= (k - \ell)L_{k+\ell}, & [\bar{L}_k, \bar{L}_\ell] &= -(\bar{k} - \bar{\ell})\bar{L}_{k+\ell}, \\
[L_k, Q_\ell] &= -\ell Q_{k+\ell}, & [\bar{L}_k, Q_\ell] &= \bar{\ell} Q_{k+\ell}, \\
[L_k, G_\ell] &= (k - \ell)G_{k+\ell}, & [\bar{L}_k, G_\ell] &= -(\bar{k} - \bar{\ell})G_{k+\ell}, \\
\{Q_k, G_\ell\} &= -k \bar{L}_{k+\ell} - \bar{k} L_{k+\ell}, & [L_k, \bar{L}_\ell] &= -\bar{k} L_{k+\ell} - \ell \bar{L}_{k+\ell}.
\end{align*}
\]

This algebra acts on the superplane of generators of the topologic ground ring \((a, b, \theta, \bar{\theta})\) by the following vector fields,

\[
\begin{align*}
L_k &= -Ra^{m+1}b^n \frac{\partial}{\partial a} - R\tau_0 a^m b^{n+1} \frac{\partial}{\partial b} + Ra^m b^n \theta \left( (m + n\tau_0) \frac{\partial}{\partial \theta} + (m + n\bar{\tau}_0) \frac{\partial}{\partial \bar{\theta}} \right), \\
\bar{L}_k &= -Ra^{m+1}b^n \frac{\partial}{\partial \bar{a}} - R\bar{\tau}_0 a^m b^{n+1} \frac{\partial}{\partial \bar{b}} + Ra^m b^n \bar{\theta} \left( (m + n\tau_0) \frac{\partial}{\partial \theta} + (m + n\bar{\tau}_0) \frac{\partial}{\partial \bar{\theta}} \right), \\
G_k &= R(\theta - \bar{\theta})a^m b^n \frac{\partial}{\partial a} + R(\bar{\tau}_0 \theta - \tau_0 \bar{\theta})a^m b^{n+1} \frac{\partial}{\partial b} \\
&\quad - Ra^m b^n \theta \left( (m + n\tau_0) \frac{\partial}{\partial \theta} + (m + n\bar{\tau}_0) \frac{\partial}{\partial \bar{\theta}} \right), \\
Q_k &= Ra^m b^n \left( (m + n\tau_0) \frac{\partial}{\partial \theta} + (m + n\bar{\tau}_0) \frac{\partial}{\partial \bar{\theta}} \right),
\end{align*}
\]

where we have set \( k \equiv mR + nR\tau_0, \) \( k_a \equiv R, \) \( k_b \equiv R\tau_0. \)

We can get some better insight into the structure of the symmetry algebra after switching to its basis with real structure constants, first by performing the fermionic coordinate change to

\[
\Theta_a = \frac{\tau_0 \bar{\theta} - \bar{\tau}_0 \theta}{\tau_0 - \bar{\tau}_0}, \quad \Theta_b = \frac{\theta - \bar{\theta}}{\tau_0 - \bar{\tau}_0}, \quad \Theta_{\bar{a}} = \frac{\bar{\tau}_0 \bar{\theta} - \tau_0 \theta}{\tau_0 - \bar{\tau}_0}, \quad \Theta_{\bar{b}} = \frac{\bar{\theta} - \bar{\bar{\theta}}}{\tau_0 - \bar{\tau}_0},
\]

where \( \Theta_a, \Theta_b \) is the natural basis of the first integral cohomology group of the
target, and then by switching from $L_k, \ldots, G_k$ to

$$
\begin{align*}
\hat{\mathcal{L}}_{m,n}^a &= \frac{\tau_0 L_{mR+nR\tau_0} - \bar{\tau}_0 L_{mR+nR\tau_0}}{R(\tau_0 - \bar{\tau}_0)}, \\
\hat{\mathcal{G}}_{m,n} &= \frac{G_{mR+nR\tau_0}}{R(\tau_0 - \bar{\tau}_0)}, \\
\hat{\mathcal{L}}_{m,n}^b &= \frac{L_{mR+nR\tau_0} - \bar{L}_{mR+nR\tau_0}}{R(\tau_0 - \bar{\tau}_0)}, \\
\hat{\mathcal{Q}}_{m,n} &= \frac{Q_{mR+nR\tau_0}}{R}.
\end{align*}
$$

(3.4)

After these changes, the vector field representation (3.2) simplifies to

$$
\begin{align*}
\hat{\mathcal{L}}_{m,n}^a &= -a^{m+1} b^n \frac{\partial}{\partial a} + a^n b^n \Theta_a \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right), \\
\hat{\mathcal{L}}_{m,n}^b &= -a^n b^{n+1} \frac{\partial}{\partial b} + a^n b^n \Theta_b \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right), \\
\hat{\mathcal{G}}_{m,n} &= a^{m+1} b^n \Theta_b \frac{\partial}{\partial a} - a^n b^{n+1} \Theta_a \frac{\partial}{\partial b} + a^n b^n \Theta_a \Theta_b \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right), \\
\hat{\mathcal{Q}}_{m,n} &= a^n b^n \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right).
\end{align*}
$$

(3.5)

In terms of the redefined generators (3.4), the bosonic part of the symmetry algebra (3.1) takes the following form:

$$
\begin{align*}
[\hat{\mathcal{L}}_{m,n}^a, \hat{\mathcal{L}}_{p,q}^a] &= (m-p) \hat{\mathcal{L}}_{m+p,n+q}^a, \\
[\hat{\mathcal{L}}_{m,n}^b, \hat{\mathcal{L}}_{p,q}^b] &= (n-q) \hat{\mathcal{L}}_{m+p,n+q}^b, \\
[\hat{\mathcal{L}}_{m,n}^a, \hat{\mathcal{L}}_{p,q}^b] &= n \hat{\mathcal{L}}_{m+p,n+q}^a - p \hat{\mathcal{L}}_{m+p,n+q}^b.
\end{align*}
$$

(3.6)

When restricted to the action on $(a, b)$, (3.6) is the full algebra of infinitesimal (ground ring valued) diffeomorphisms of the $(a, b)$ plane. However, each generator of the algebra (3.5) has an important fermionic part, proportional to $m \partial / \partial \Theta_a + n \partial / \partial \Theta_b$. In the superplane of $(a, b, \Theta_a, \Theta_b)$, the algebra (3.6) and its supersymmetric extension to

$$
\begin{align*}
[\hat{\mathcal{L}}_{m,n}^a, \hat{\mathcal{G}}_{p,q}] &= (m-p) \hat{\mathcal{G}}_{m+p,n+q}, \\
[\hat{\mathcal{L}}_{m,n}^b, \hat{\mathcal{G}}_{p,q}] &= -(n-q) \hat{\mathcal{G}}_{m+p,n+q}, \\
[\hat{\mathcal{L}}_{m,n}^a, \hat{\mathcal{Q}}_{p,q}] &= -p \hat{\mathcal{Q}}_{m+p,n+q}, \\
[\hat{\mathcal{L}}_{m,n}^b, \hat{\mathcal{Q}}_{p,q}] &= -q \hat{\mathcal{Q}}_{m+p,n+q}, \\
\{\hat{\mathcal{Q}}_{m,n}, \hat{\mathcal{G}}_{p,q}\} &= -n \hat{\mathcal{L}}_{m+p,n+q}^a + m \hat{\mathcal{L}}_{m+p,n+q}^b
\end{align*}
$$

(3.7)

acts by some specific superdiffeomorphisms. In this way, the algebra of all diffeomorphisms of the $(a, b)$ plane gets extended to an algebra of volume preserving
superdiffeomorphisms, where the preserved volume on the superplane is given by

\[ \text{Vol} = \frac{da}{a} \wedge \frac{db}{b} \wedge d\Theta_a \wedge d\Theta_b. \] (3.8)

Moreover, it is easy to observe that (3.5) does not exhaust the whole set of all Vol-preserving superdiffeomorphisms. Actually, our symmetry algebra preserves in addition to (3.8) also the following symplectic two-form

\[ \omega = \frac{da}{a} \wedge d\Theta_a + \frac{db}{b} \wedge d\Theta_b \] (3.9)

on the superplane. This suggests that a relation to grassmannian particle mechanics might exist: If we interpret the superplane \((a, b, \Theta_a, \Theta_b)\) as the phase space of a mechanical system with two degrees of freedom evolving in a purely grassmannian time coordinate, then (3.9) can be considered the canonical symplectic form of the system.

In the conventional phase of string theory in two dimensions, classical limit of string theory corresponds to the one-particle limit of the free fermions in the target [33], and Witten’s ground ring is an algebra of polynomial functions on the extended phase space of the particle. The corresponding symmetry algebra preserves the symplectic structure on the phase space. When compared with the observations just made, we see that in the matter sector of the topological string theory on the torus, we have arrived at another one-particle system: the grassmannian particle in two dimensions. This correspondence bears some remote resemblance to \(c = -2\) matrix models [34], where the starting point is essentially a grassmannian matrix mechanics.

Note that the algebra of vector fields (3.5) enjoys an important symmetry. After denoting

\[ D_a \equiv \frac{\partial}{\partial \Theta_a}, \quad D_b \equiv \frac{\partial}{\partial \Theta_b}, \] (3.10)

we observe that the full algebra can be generated from \(\hat{\mathcal{G}}\) by consecutive
(anti)commutations with $D$’s. Schematically,

\[
\begin{align*}
\{D_a, \hat{G}_{m,n}\} &= \hat{L}_{m,n}^b, & \{D_b, \hat{G}_{m,n}\} &= -\hat{L}_{m,n}^a, \\
[D_b, \hat{L}_{m,n}^b] &= \hat{Q}_{m,n}, & [D_a, \hat{L}_{m,n}^a] &= \hat{Q}_{m,n}.
\end{align*}
\]

(3.11)

This leads to a succinct superspace formulation of the algebra, which we won’t enter into here.

Instead, note that our symmetry algebra (3.5) does not generate the full algebra of (ground ring valued) $\omega$-preserving superdiffeomorphisms. Actually, (3.5) is the algebra of all \textit{exact} $\omega$-preserving diffeomorphisms, i.e. those diffeomorphisms that are generated from Hamiltonians. To obtain the full algebra of all symplectomorphisms, two new generators, $D_a, D_b$ in the notation of (3.10), should be added. These two generators can be thought of as the missing zero modes of $\hat{Q}$, as we have formally

\[
\begin{align*}
D_a &= \lim_{m \to 0} \frac{1}{m} \hat{Q}_{m,0}, & D_b &= \lim_{m \to 0} \frac{1}{m} \hat{Q}_{0,m}.
\end{align*}
\]

(3.12)

In terms of the underlying conformal field theory, these operators actually do exist, and are given by

\[
\begin{align*}
D_a &= \frac{i}{2\pi} \oint_C (\chi(z) - \bar{\chi}(\bar{z})) \quad D_b = \frac{i}{2\pi} \oint_C (\bar{\tau}_0 \chi(z) - \bar{\tau}_0 \bar{\chi}(\bar{z})).
\end{align*}
\]

(3.13)

Clearly, they are not BRST invariant. It is interesting, however, to note that they are BRST invariant on the instanton moduli space of the model, where $\partial X = 0$.

3.2. Spacetime Topological Symmetry

Now we will return to the analysis of the symmetry algebra in the decompactification limit of $\text{Im} \tau_0 \to \infty$. This limit not only simplifies some expressions, but reveals an interesting (albeit speculative) physical structure.
From the spacetime point of view, the symmetry algebra (2.36) represents a fermionic extension of the Virasoro algebra. It is convenient to introduce a new formal variable $Z$, and define

$$
T(Z) = \sum_m L_m Z^{-m-2}, \quad J(Z) = \sum_m J_m Z^{-m-1},
$$

$$
Q(Z) = \sum_m Q_m Z^{-m-1}, \quad G(Z) = \sum_m G_m Z^{-m-2}.
$$

(3.14)

The commutation relations (2.36) can now be formally rewritten as OPEs in this new variable. Conformal weights of $Q(Z)$ and $G(Z)$ with respect to $T(Z)$ are one and two respectively, which is the situation encountered in the topologically twisted $N = 2$ superconformal algebra in two dimensions. We are thus tempted to identify the symmetry algebra of the decompactified model, (2.36), as the algebra of spacetime topological symmetry. There are important differences between (2.36) and the two dimensional topological algebra, however. First of all, the zero mode of the natural candidate for the BRST current, $Q(Z)$, does not exist. As a result of this, $Q_n$ and $G_m$ cannot anticommute to $L_{m+n}$, and a peculiar modification of the topological symmetry algebra in spacetime would result.

To obtain a closer correspondence to the standard topological symmetry algebra, we will have to go beyond the algebra of exact $\omega$-preserving diffeomorphisms, which is generated by the charges of §2, and invoke one of the non-exact symplectomorphisms of (3.10). Indeed, the (rescaled) operator $D_b$, which we will now denote by

$$
Q \equiv \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \bar{\theta}},
$$

(3.15)

completes the algebra of $L_m$ and $G_n$ to the topological symmetry algebra in spacetime,

$$
[L_m, L_n] = (m - n)L_{m+n}, \quad [L_m, G_n] = (m - n)G_{m+n},
$$

$$
[L_m, Q] = 0, \quad \{Q, G_n\} = 2L_n.
$$

(3.16)

and plays the role of the BRST-like charge of this spacetime topological symmetry.
In the underlying worldsheet CFT we have

$$Q = \frac{1}{2\pi} \oint (\chi(z) + \bar{\chi}(\bar{z})). \quad (3.17)$$

The worldsheet current of $Q$ does not enter the descent equations of $Q$ studied in §2, as its $Q$-commutator is not equal to $d$(something). Consequently, its charge, albeit conserved (on shell), is not physical with respect to the worldsheet topological BRST symmetry. (One more refinement, mentioned in §2.1, should be checked here. In topological CFT, one can make use of the chiral splitting of the topological BRST charge, $Q = Q_L + Q_R$, to define another nilpotent charge,

$$\tilde{Q} \equiv Q_L - Q_R. \quad (3.18)$$

In the case of the topological torus, the topological ground ring is the ring of cohomology classes of both $Q$ and $\tilde{Q}$. The new nilpotent charge defines its own descent equations, hence its own class of currents conserved modulo $\tilde{Q}$ and acting as symmetries on $R'$. However, $Q$ does not enter the descent equations of $\tilde{Q}$ either.)

The fact that the spacetime topological BRST charge is not invariant under the worldsheet topological BRST, yet can it be defined as a conserved charge, may suggest that the spacetime topological symmetry is still present in the model, but is perhaps broken spontaneously.

With the interpretation of $Q$ as the BRST-like operator of spacetime topological symmetry, it is natural to ask what do the physical states of this cohomology operator, satisfying

$$Q|\text{phys}\rangle = 0. \quad (3.19)$$

correspond to. Despite the physical answer to this question is not completely clear, the condition of topological BRST invariance in spacetime acquires a natural geometrical meaning: One linear combination of the topological ghosts $\psi$ and $\bar{\psi}$ corresponds to the would-be cohomology class in the decompactified dimension
of the target, and physical states in the sense of (3.19) are precisely those states that are independent of this linear combination of ghosts – indeed a very plausible condition in the decompactified limit.

In this subsection, we have tried to obtain at least some preliminary interpretation of (2.36) in terms of a topological symmetry in spacetime. Alternatively, we might try to interpret (2.36) as a supersymmetry algebra in its more conventional sense. Indeed, (2.36) resembles the symmetry of the non-critical two dimensional superstring, as studied in [9,17,35,36] (see also [37]). Actually, two dimensional superstring theory with the chiral GSO projection has been conjectured [35,38] to be a topological theory. As a result of two dimensional kinematics, the spacetime supersymmetry charge $Q_{\text{susy}}$ is nilpotent, which suggests that it can be used for imposing a BRST-like condition on physical states,

$$Q_{\text{susy}}|_{\text{phys}} = 0,$$  \hspace{1cm} (3.20)

as has been pointed out in [35]. In such a model, the tachyon is essentially projected out by the GSO projection and the BRST condition, hence only the discrete (special) states survive. The analogy between the topological torus and superstring theory in two dimensions deserves further investigation.

4. Discussion

In this paper we have analyzed symmetries of topological string theory on the two dimensional torus. We have found an interesting structure even before the topological torus is coupled to topological gravity.

Point-like observables in the sigma model form a closed ring under operator multiplication, which is conjectured to be the topological analog of the ground ring of two dimensional string theory.

The model contains an infinite number of conserved BRST invariant quantum currents. The corresponding charges form a supersymmetric algebra of symmetries
of the topological ground ring. A bosonic subsector of this large symmetry algebra generates all (ground ring valued) diffeomorphisms of a two dimensional spacetime. Hence, the invariance of the model under the symmetry algebra may signalize that, after coupling to gravity, the model does realize an unbroken phase of general relativity in spacetime. Unlike in generic topological sigma models, where the invariance under target diffeomorphisms is usually proved as an independence of the target metric with the use of the topological BRST symmetry on the worldsheet, here we have obtained a direct realization of spacetime diffeomorphisms in terms of conserved charges corresponding to quantum currents on the worldsheet.

The appearance of spacetime diffeomorphisms in the set of conserved charges of the model is by no means obvious, and cannot be extended to arbitrary topological sigma models. This enormous quantum symmetry is related closely to the nontrivial fundamental group of the target. Indeed, the first homology group of the target, which is simply

$$\pi_1(M)/[\pi_1(M), \pi_1(M)] \otimes \mathbb{R},$$

gives rise to the bosonic, ghost-number-preserving part of the symmetry algebra of the model. Second, $\pi_1(M)$ itself generates its own class of observables, called homotopy observables in §2, which make the dimension of the space of physical states as well as the dimension of the symmetry algebra infinite.

This realization of spacetime diffeomorphism invariance in terms of conserved charges on the worldsheet might be relevant to string theory also on more general grounds. Note that the restriction to two target dimensions made in this paper is not necessary, the model can be easily extended to an arbitrary (even) number of target dimensions. In this version of topological string theory with a toroidal target, we will obtain, after repeating the steps of §2 and §3, the algebra of spacetime diffeomorphisms as the algebra of BRST invariant conserved worldsheet charges of ghost number zero.
At nonzero ghost numbers, the symmetry algebra extends to the algebra of (ground ring valued) diffeomorphisms that preserve a symplectic form on the superextension of the spacetime manifold, and are exact in the sense that they are generated by Hamiltonians. This algebra can be further extended by two additional operators to the algebra of all (ground ring valued) symplectomorphisms. These additional operators are conserved, but BRST invariant only weakly.

By standard topological arguments, the Lagrangian has its own class of possible deformations that maintain topological invariance.

In §2.3 and §3.2 we have studied in some detail the topological torus in the decompactified limit of $\text{Im} \tau_0 \to \infty$, in which just one dimension of the target remains compact, and the torus becomes an infinite cylinder. In this specific limit, spacetime algebra of topological symmetry emerges, in which one of the non-exact symplectomorphisms mentioned above plays the role of the BRST-like charge of topological symmetry in spacetime. As this charge is BRST invariant only weakly, spacetime topological symmetry is hidden in the model. It is interesting to note that precisely this limit of the topological torus has been studied in [27], where some interesting conjectures about possible relations to more familiar aspects of string theory, in particular to high energy scattering, have been made.

In order to decide whether the topological theory is relevant to the conventional phase of string theory in two dimensions, it is necessary to find out which of these aspects persist after the topological torus is coupled to topological gravity. This issue has two sides – first, we must analyze the theory in a fixed gravitational background, which allows us to extend the local results of this paper globally to surfaces of arbitrary genus and solve the topological sigma model completely. Next, we must construct the full-fledged topological string theory on the torus. Topological gravity becomes dynamical and contributes its own degrees of freedom to the topological ground ring, as well as new generators to the symmetry algebra. Naively, one might expect one more spacetime dimension to be generated by the Mumford-Morita cohomology classes $\sigma_m$ coming from the gravitational sector.
The model we have studied in this paper is a topological string theory in two dimensions which, we believe, will be interesting in itself. Besides, we have found several analogies between this topological theory and the conventional phase of string theory in two dimensions. (Some other analogies can be found in the light of the recent progress in two dimensional string theory reported in [39].) To what extent these analogies are purely formal remains to be seen. Clearly, much more must be done before any detailed comparison of the topological theory studied in this paper to the conventional phase of string theory in two dimensions will be possible.

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