SINGLE POINT BLOW-UP AND FINAL PROFILE FOR A PERTURBED NONLINEAR HEAT EQUATION WITH A GRADIENT AND A NON-LOCAL TERM

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Dedicated to the memory of Ezzeddine Zahrouni.

Abstract. We consider in this paper a perturbation of the standard semilinear heat equation by a term involving the space derivative and a non-local term. In some earlier work [1], we constructed a blow-up solution for that equation, and showed that it blows up (at least) at the origin. We also derived the so called “intermediate blow-up profile”. In this paper, we prove the single point blow-up property and determine the final blow-up profile.

1. Introduction. We consider the problem

\[
\begin{cases}
  u_t &= \Delta u + |u|^{p-1}u + \mu |\nabla u| q, \\
  u(0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N),
\end{cases}
\] (1.1)

where \( u = u(x,t) \in \mathbb{R}, x \in \mathbb{R}^N \) and the parameters \( p, q \) and \( \mu \) are such that

\[ p > 3, \quad \frac{N}{2}(p-1) + 1 < q < \frac{N}{2}(p-1) + \frac{p+1}{2}, \quad \mu \in \mathbb{R}. \] (1.2)

When \( \mu = 0 \), we recover the standard semilinear heat equation with power nonlinearity,

\[ u_t = \Delta u + |u|^{p-1}u, \] (1.3)

which has attracted a lot of attention in the last 50 years (see the book [24] by Quittner and Souplet), in particular, as a model for the study of blow-up in PDEs. Although the analysis of (1.3) is far from being trivial, one may feel that such an equation is too idealized, and may not capture a lot of features one may encounter in real life (parabolic) models. For that reason, some authors tried to study perturbations of that equation with different kind of terms, aiming to be closer to more realistic models. In particular, we mention the following perturbation by a nonlinear gradient term

\[ u_t = \Delta u + |u|^{p-1}u + \mu |\nabla u|^q, \] (1.4)

2020 Mathematics Subject Classification. 35B20, 35B44, 35K55.

Key words and phrases. Blow-up, nonlinear heat equation, gradient term, non-local term.
first introduced by Chipot and Weissler [5]. We also mention the perturbations involving non-local (or integral) terms, as we encounter in PDEs modeling Micro Electro-Mechanical Systems (MEMS):

\[ u_t = \Delta u + \frac{\lambda}{(1-u)^2(1 + \gamma \int_{\Omega} \frac{1}{1-u} dx)^2}, \]  

(1.5)

where \( \frac{1}{1-u} \) may blow up in finite time (see Duong and Zaag [6] and the references therein). Specific difficulties arise in the study of blow-up for both equations (1.4) and (1.5), as one may see from the constructions of singular solutions in Tayachi and Zaag [26] (see also the note [27]) and Duong and Zaag [6].

In this paper, following our earlier work [1] on (1.1), we intend to consider more complicated perturbations of equation (1.3). Specifically, since it shows a product

\[ (1.4) \]  

and (1.5), as one may see from the constructions of singular solutions in Tayachi and Zaag [26] (see also the note [27]) and Duong and Zaag [6].

Equation (1.1) is wellposed in the weighted functional space \( W^{1,\infty}_\beta(\mathbb{R}^N) \) defined as follows:

\[ W^{1,\infty}_\beta(\mathbb{R}^N) = \{ g, (1 + |y|^\beta)g \in L^\infty, (1 + |y|^\beta)\nabla g \in L^\infty \}, \]  

(1.6)

where

\[ 0 \leq \beta < \frac{2}{p-1}, \text{ if } \mu = 0 \text{ and } \frac{N}{q-1} < \beta < \frac{2}{p-1}, \text{ if } \mu \neq 0, \]  

(1.7)

as one may see from Appendix C in [1].

In [1], we constructed a solution \( u(x,t) \) for equation (1.1) which blows up in finite time \( T \) at \( a = 0 \), and we proved that the solution behaves as follows: for all \( (x,t) \in \mathbb{R}^N \times [0,T) \),

\[ \left| u(x,t) - (T-t)^{-\frac{\beta}{p-1}} f \left( \frac{x}{\sqrt{(T-t)\log(T-t)}} \right) \right| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^{\frac{1}{2}p}} \right| \log(T-t)\right|^{\frac{1}{1-q}}, \]  

(1.8)

and

\[ \left| \nabla u(x,t) - (T-t)^{-\frac{1}{2} - \frac{\beta}{p-1}} \nabla f \left( \frac{x}{\sqrt{(T-t)\log(T-t)}} \right) \right| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^{\frac{1}{2}p}} \right| \log(T-t)\right|^{\frac{1}{1-q}}, \]  

(1.9)

(note that \( 0 \leq \beta < 1 \) from (1.7) and (1.2)), where the “intermediate” profile \( f \) is given by

\[ f(z) = (p - 1 + b|z|^2)^{-\frac{1}{p-1}}, \text{ for all } z \in \mathbb{R}^N, \text{ with } b = \frac{(p-1)^2}{4p}. \]  

(1.10)

Note that in the literature (for example in Section 2 page 850 of Giga and Kohn [14]), \( a \) is a blow-up point for \( u \) if \( |u(x_n,t_n)| \rightarrow \infty \) as \( n \rightarrow \infty \), for some sequence \( (x_n,t_n) \rightarrow (a,T) \), with a similar definition for blow-up points of \( \nabla u \).

Our argument in [1] is a non trivial adaptation of the pioneering work performed for equation (1.3) by Bricmont and Kupiainen [4] and Merle and Zaag [20] (see also the note [21]). More precisely, the proof is given in the similarity variables setting: we linearize the PDE around the profile candidate \( f \) defined in (1.10) (actually, around a small perturbation of \( f \)), then, we control the nonpositive part of the spectrum thanks to the decaying properties of the linear operator. As for the positive eigenvalues, we control them thanks to a topological argument based on
index theory. Note that already when $\mu = 0$, the profile in (1.8) is sharper than the profile derived in [20], in the sense that we divide here the bound by $1 + \left(\frac{|x|^2}{t-T}\right)^{\frac{\beta}{2}}$.

Despite this sharp estimate, we left two questions unanswered in [1]:

- Is the origin the only blow-up point of the constructed solution?
- Can we derive an equivalent of the “final profile” (namely, $u^*(x) \equiv \lim_{t \to T} u(x,t)$) near the origin?

In this paper, we positively answer these two open questions. In fact, we adapt the technique developed for equation (1.3) by Giga and Kohn [14], in order to obtain the single point blow-up result for equation (1.1) and give the description of the blow-up final profile $u^*$ such that $u(x,t) \to u^*$ in $C^1$ of every compact set of $\mathbb{R}^N \setminus \{0\}$. More precisely, we prove the following Theorem:

**Theorem 1.1.** Let $\mu \in \mathbb{R}$ and $p, q$ be two real numbers such that (1.2) holds.

Assume that equation (1.1) has a solution $u$ which blows up at the origin in some finite time $T$ and satisfies (1.8) and (1.9) for some $\beta$ satisfying (1.7). Then,

1. Both $u$ and $\nabla u$ blow up at the origin and only at the origin.
2. For all $x \neq 0$, $u(x,t) \to u^*(x)$ as $t \to T$ in $C^1$ of every compact of $\mathbb{R}^N \setminus \{0\}$, with

$$u^*(x) \sim \left[\frac{8p \log |x|}{(p-1)^2|x|^2}\right]^\frac{1}{p-1} \text{ as } x \to 0,$$

and for $|x|$ small, we have

$$|
abla u^*(x)| \leq C|x|^{-\frac{p+1}{p-1}} \log |x|^{\frac{p+3}{p-1}}. \tag{1.12}$$

**Remark 1.2.**

1. As for the intermediate profile (i.e. valid for $0 \leq t < T$) given in (1.8)-(1.9), the final profile (i.e. for $t = T$) given in (1.11)-(1.12) shows no change in the behavior, with respect to the standard semilinear heat equation (1.3). However, one should keep in mind that the proof is much more complicated, as one already sees from [1] and the present paper.

2. From (1.8), we know that $u$ blows up at the origin. It happens that the fact that $\nabla u$ blows up at the origin is a simple consequence of the single point blow-up for $u$ together with the mean value theorem (see the proof below in Section 3). In addition, we could only obtain an upper bound in (1.12), which is however sufficient to assert that $\nabla u$ blows up only at the origin. We conjecture that (1.11) holds also after differentiation in space. Unfortunately, our estimates are still not sharp enough to prove that.

3. As we have already pointed out in [1], the single-point blow-up for $u$ is trivial, if $\mu \leq 0$. Indeed, it happens that we are able to ensure that the constructed solution is nonnegative, hence, our solution is a nonnegative subsolution of the standard semilinear heat equation (1.3), and the argument of Giga and Kohn in Section 2 page 850 of [14] allows us to conclude that $u$ blows up only at the origin. However, this leaves open the non blow-up of $\nabla u$ outside the origin when $\mu \leq 0$, and for both $u$ and $\nabla u$, when $\mu > 0$, not mentioning the derivation of the final blow-up profile for general $\mu \in \mathbb{R}$.

The paper is organized as follows. In Section 2, we control the non-local term in equation (1.1). In Section 3, we prove the single-point blow-up property. Finally, in Section 4, we describe the final profile.
2. **Control of the non-local term.** This section is devoted to the control of the non-local term in equation (1.1). We consider \( u(t) \) a solution of (1.1) which blows up in finite time \( T \) at the point \( a = 0 \) and satisfies (1.8) and (1.9). We consider \( \delta > 0 \) which will be fixed small enough such that various estimates hold. For \( K_0 > 0 \) to be fixed large enough later and any \( x_0 \neq 0 \), we define \( t_0(x_0) \in [0, T) \) by

\[
|x_0| = K_0 \sqrt{(T - t_0(x_0)) \log(T - t_0(x_0))}, \quad \text{if} \quad 0 < |x_0| \leq \delta, \quad \text{(2.1)}
\]

\[
t_0(x_0) = t_0(\delta), \quad \text{if} \quad |x_0| > \delta.
\]

Note that \( t_0(x_0) \rightarrow T \) as \( x_0 \rightarrow 0 \) and \( t_0(x_0) \) is uniquely defined if \( \delta \) is sufficiently small.

Let us introduce for each \( x_0 \neq 0 \) a rescaled version of \( u \)

\[
v(x_0, \xi, \tau) = (T - t_0(x_0))^{\frac{1}{p-1}} u(x, t), \quad \text{(2.2)}
\]

where

\[
x = x_0 + \xi \sqrt{T - t_0(x_0)}, \quad t = t_0(x_0) + \tau (T - t_0(x_0)), \quad \xi \in \mathbb{R}^N, \quad \tau \in [-\frac{t_0(x_0)}{T - t_0(x_0)}, 1). \quad \text{(2.3)}
\]

We also introduce

\[
w = \nabla v. \quad \text{(2.4)}
\]

Since \( u \) is a solution of equation (1.1), it follows that \( v \) and \( w \) satisfy

\[
v_\tau = \Delta v + |v|^{p-1} v + \mu (T - t_0(x_0)) \gamma |w| \int_{B_0} |v|^{q-1}, \quad \text{(2.5)}
\]

\[
w_\tau = \Delta w + p|v|^{p-1} w + \mu (T - t_0(x_0)) \gamma \nabla(|w| \int_{B_0} |v|^{q-1}), \quad \text{(2.6)}
\]

where

\[
\gamma = \frac{p - q}{p - 1} + \frac{N - 1}{2} \quad \text{(2.7)}
\]

and \( B_0 \) is the ball of center \(-K_0 \sqrt{\log(T - t_0(x_0))}\frac{x_0}{|x_0|}\) and radius \(|\xi + K_0 \sqrt{\log(T - t_0(x_0))}\frac{x_0}{|x_0|}|\). Note from (1.2) that

\[
\gamma > 0. \quad \text{(2.8)}
\]

Let us give an idea of the method used to prove Theorem 1.1. We proceed in two steps:

- **First,** arguing as Giga and Kohn did in [14] [Theorem 2.1 Page 850], we consider \( r > 0 \) and \( \phi_r \), a smooth function supported in \( B_r = \{ x \in \mathbb{R}^N; |x| < r \} \) such that \( \phi_r = 1 \) on \( B_{r/2} \) and \( 0 \leq \phi_r \leq 1 \). We then consider cut-off versions of the functions, namely \( v \phi_r \) and \( w \phi_r \).

  We use an iteration process to show that \( v \) and \( w \) are bounded, hence that \( \xi = 0 \) is not a blow-up point of \( v \) and \( w \). From the transformation (2.2)-(2.3), this means that neither \( u \) nor \( \nabla u \) blow up at \( x = x_0 \). Since \( u \) blows up at the origin by (1.8), this implies the single point blow-up property for \( u \). Using the mean value theorem, this yields that \( \nabla u \) blows up at the origin, which proves the single point blow-up property for \( \nabla u \) too.

- **Second,** we prove the existence of a blow-up final profile \( u^* \) such that \( u(x, t) \rightarrow u^*(x) \) as \( t \rightarrow T \) in \( C^1 \) of every compact of \( \mathbb{R}^N \setminus \{0\} \). Next, we find an equivalent of \( u^* \) and an upper bound on \( \nabla u^* \) near the blow-up point.
Furthermore, for all $\mu \neq 0$ and $p, q$ be two real numbers such that (1.2) holds.

Assume equation (1.1) has a solution $u$ which blows up at the origin in some finite time $T$ and satisfies (1.8) and (1.9) for some $\beta$ satisfying (1.7). Then, for all $\eta > 0$, there exists a positive constant $C_\eta$ such that for all $(x, t) \in \mathbb{R}^N \times (0, T)$

$$
\int_{B(0, |x|)} |u(x', t)|^{q-1} dx' \leq C_\eta (T-t)^{\gamma - \frac{1}{2} - \eta}.
$$

**Proof.** From inequality (1.8), we have

$$
\left|u(x', t)\right|^{q-1} \leq C(T-t)^{-\frac{3}{p+1}} \left|f\left(\frac{x'}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|^{q-1} + C \frac{(T-t)^{-\frac{3}{p+1}}}{\log(T-t)} \frac{1}{\left(\frac{|x|}{\sqrt{T-t}|\log(T-t)|}\right)^{\frac{q-1}{2}}}.
$$

(2.9)

By definition (1.10) of the profile $f$, we see that

$$
\int_{B(0, |x|)} \left|f\left(\frac{x'}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|^{q-1} \leq C \int_{B(0, |x|)} \frac{1}{(1 + |x'|^2)^{\frac{q-1}{2}}} dx',
$$

$$
\leq C ((T-t)|\log(T-t)|)^{\frac{2}{q-1}} \int_{B(0, \frac{|x|}{\sqrt{T-t}|\log(T-t)|})} \frac{dy}{(1 + |y|^2)^{\frac{q-1}{2}}}.
$$

Since $2 \frac{q-1}{p+1} > N$ from (1.2), we have

$$
\int_{B(0, |x|)} \left|f\left(\frac{x'}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|^{q-1} \leq C(T-t)^{\frac{2}{q-1}} |\log(T-t)|^{\frac{2}{q-1}},
$$

(2.10)

on the one hand. On the other hand, from the choice of $\beta$ in (1.7), we have $(q-1)\beta > N$, and this yields

$$
\int_{B(0, |x|)} \frac{1}{(1 + |x'|^2)^{\frac{q-1}{2}}}^{q-1} dx' \leq C(T-t)^{\frac{2}{q-1}} \int_{B(0, \frac{|x|}{\sqrt{T-t}|\log(T-t)|})} \frac{dy}{(1 + |y|^2)^{q-1}} \leq C(T-t)^{\frac{2}{q-1}}.
$$

(2.11)

From (2.9), (2.10) and (2.11), we get

$$
\int_{B(0, |x|)} |u(x', t)|^{q-1} dx' \leq C(T-t)^{\frac{2}{q-1} - \frac{1}{q-1} |\log(T-t)|^\frac{2}{q-1}}.
$$

Furthermore, for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$
\int_{B(0, |x|)} |u(x', t)|^{q-1} dx' \leq C_\eta (T-t)^{\frac{2}{q-1} - \frac{1}{q-1} - \eta}.
$$

Using the definition (2.7) of $\gamma$, we conclude the proof of Proposition 2.1. □
Proposition 3.1. (No blow-up under some threshold) The definition given on page 2, we see that contradiction, we assume that the origin is not a blow-up point for $\nabla u$ with the following property:

$$v(x_0, \xi, \tau) = 0, \text{ and that } \nabla u \text{ blows up at the origin. More precisely, this is our main tool:}$$

Proposition 3.1 (No blow-up under some threshold). There exist $\bar{\varepsilon}$ and $M > 1$ with the following property:

Assume that for some $\varepsilon_0 \geq \bar{\varepsilon}$ and $x_0 \in \mathbb{R}^N \setminus \{0\}$, we have for all $|\xi| < 1$ and $\tau \in [0,1)$:

$$|v(x_0, \xi, \tau)| + \sqrt{1 - \tau} |\nabla v(x_0, \xi, \tau)| \leq \varepsilon_0 (1 - \tau)^{-\frac{1}{N+1}},$$

(3.1)

where $v(x_0, \xi, \tau)$ is defined in (2.2). Then, for all $|\xi| \leq \frac{1}{M\tau}$ and $\tau \in (0,1)$, we have

$$|v(x_0, \xi, \tau)| + |\nabla v(x_0, \xi, \tau)| \leq M\varepsilon_0.$$

Let us first use this proposition to prove part 1 of Theorem 1.1, then, we will prove it.

Proof of part 1 of Theorem 1.1, assuming that Proposition 3.1 holds. Note first from (1.8) that $u$ blows up at the origin. It remains then to prove that neither $u$ nor $\nabla u$ blow up outside the origin, and also that $\nabla u$ blows up at the origin.

Let us first note that this latter fact is a consequence of the single point blow-up property of $u$. Indeed, assume that $u$ blows up only at the origin. Then, proceeding as in Merle [19], we derive the existence of a blow-up final profile $u^*$ such that $u(x, t) \to u^*(x)$ as $t \to T$ in $C^1$ of every compact of $\mathbb{R}^N \setminus \{0\}$. Now, arguing by contradiction, we assume that the origin is not a blow-up point for $\nabla u$. Thanks to the definition given on page 2, we see that

$$|\nabla u(x, t)| \leq M, \text{ whenever } |x| \leq \delta \text{ and } t \in \{T - \delta, T\},$$

for some $M > 0$ and $\delta > 0$. Using the mean value, we write for all $t \in [0, T)$,

$$|u(0, t) - u(\delta e_1, t)| = \left| \int_0^\delta \frac{\partial u}{\partial x_1}(\xi, t) d\xi \right| \leq M\delta,$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the canonical basis of $\mathbb{R}^N$. Making $t \to T$, we see that

$$\limsup_{t \to T} |u(0, t)| \leq |u^*(\delta e_1)| + M\delta,$$

which is a contradiction, since we know from (1.8) that $u(0, t) \to \infty$ as $t \to T$.

Thus, it remains to prove that nor $u$ neither $\nabla u$ blow up outside the origin.

Consider $\delta > 0$ to be fixed later small enough. Consider then $x_0 \neq 0$ and $v(x_0, \xi, \tau)$ defined in (2.1)-(2.3). By definition, it is enough to show that this function satisfies the hypotheses of Proposition 3.1 in order to conclude.
From (2.1), 2 cases arise:

**Case 1.** $|x_0| \leq \delta$. Take $|\xi| < 1$ and $\tau \in [0,1)$. By definition (2.1)-(2.3), we write from (1.8)

$$|v(x_0, \xi, \tau)| = (T - t_0(x_0))^{-\frac{1}{1+p}} |u(x,t)|$$

$$\leq (T - t_0(x_0))^{-\frac{1}{1+p}} (T - t)^{-\frac{1}{1+p}} \left\{ f \left( \frac{x}{\sqrt{(T-t)|\log(T-t)|}} \right) + \frac{C}{|\log(T-t)|^{1/2}} \right\}$$

$$\leq C(1 - \tau)^{-\frac{1}{1+p}} \left\{ f \left( \frac{x}{\sqrt{(T-t_0(x_0))|\log(T-t_0(x_0))|}} \right) + \frac{C}{|\log(T-t_0(x_0))|^{1/2}} \right\}$$

$$= C(1 - \tau)^{-\frac{1}{1+p}} \left\{ f \left( \frac{K_0 - 1}{\sqrt{|x-x_0|}} \right) + \frac{C}{|\log(T-t_0(x_0))|^{1/2}} \right\}$$

$$\leq C(1 - \tau)^{-\frac{1}{1+p}} \left\{ f \left( \frac{K_0}{2} \right) + \frac{C}{|\log(T-t_0(x_0))|^{1/2}} \right\}$$

$$\leq \varepsilon_0 (1 - \tau)^{-\frac{1}{1+p}},$$

provided that $\delta$ is small enough, and $K_0$ is large enough.

By a similar calculation, we can prove the estimate on $\nabla v$ starting from (1.9).

**Case 2.** $|x_0| > \delta$. Here, we have $t_0(x_0) = t_0(\delta)$. Concerning $v$, the calculation of the former case works, except for the estimate of

$$\frac{|x_0|}{\sqrt{(T-t_0(x_0))|\log(T-t_0(x_0))|}},$$

which needs this small adjustment:

$$\frac{|x_0|}{\sqrt{(T-t_0(x_0))|\log(T-t_0(x_0))|}} \geq \frac{\delta}{\sqrt{(T-t_0(\delta))|\log(T-t_0(\delta))|}} = K_0.$$

Once again, the estimate for $\nabla v$ follows similarly, starting from (2.12).

This finishes the proof of Part 1 of Theorem 1.1, assuming that Proposition 3.1 holds.

It remains then to prove Proposition 3.1. Throughout this section, we write $v(\xi, \tau)$ instead of $v(x_0, \xi, \tau)$, in order to simplify notations.

**Proof of Proposition 3.1.** To prove Proposition 3.1, we use a cut-off technique: Let $r \in (0,1]$, and consider $\phi_r$, a smooth function supported in $B(0,r)$ such that $\phi_r = 1$ on $B(0,\frac{r}{2})$ and $0 \leq \phi_r \leq 1$. We introduce the cut-off of the solution $v$ and its gradient $w = \nabla v$, by $v_r = \phi_r v$ and $w_r = \phi_r w$.

Let us sketch the main steps of the proof of Proposition 3.1.

- In Step 1, we use the Duhamel formulation of the equation satisfied by the cut-off of the gradient of $v$. Then, starting from hypothesis (3.1) of Proposition 3.1 and using Proposition 2.1 (actually its consequence (2.12)), we prove by an iteration process the existence of some $r_1$ such that for all $\tau \in [0,1)$,

$$\|v(\tau)\|_{L^\infty(B_{r_1})} \leq \varepsilon_0 \left(1 - \tau\right)^{-\frac{1}{1+p}}, \quad (3.2)$$

and

$$\|w(\tau)\|_{L^\infty(B_{r_1})} \leq \frac{C\varepsilon_0}{(1 - \tau)^{1/p}}. \quad (3.3)$$

- In Step 2, applying the Duhamel formulation of the equation satisfied by the cut-off of $v$ and estimate (3.3), we improve estimate (3.2) and prove that for all $\tau \in [0,1)$,

$$\|v(\tau)\|_{L^\infty(B_{r_2})} \leq \frac{C\varepsilon_0}{(1 - \tau)^{1/p}}. \quad (3.4)$$
for some \( r_2 \in (0, r_1) \).

- In the final step, from the inequalities (3.3) and (3.4), we prove that for all \( \tau \in [0, 1) \),
  \[
  \|v(\tau)\|_{L^\infty(B_{r_3})} + \|w(\tau)\|_{L^\infty(B_{r_3})} \leq M\varepsilon_0, \tag{3.5}
  \]
  for some \( r_3 \in (0, r_2] \) and a constant \( M > 0 \).

**Step 1.** Our starting point is the hypothesis (3.1) given in Proposition 3.1. Using
the Duhamel formulation satisfied by the cut-off of the gradient of the solution and
applying Proposition 2.1 for some \( \eta > 0 \), we establish the following Lemma:

**Lemma 3.2.** For any \( r > 0 \), there exists \( C_r > 0 \) such that for any \( \alpha > 0 \), there
exists \( \varepsilon > 0 \) such that for any \( \varepsilon_0 \in (0, \varepsilon) \), if we have for all \( \tau \in [0, 1) \),
\[
\|v\|_{L^\infty(B_{r \tau})} \leq \frac{\varepsilon_0}{(1 - \tau)^{3\alpha}}, \quad \text{and} \quad \|w\|_{L^\infty(B_{r \tau})} \leq \frac{C\varepsilon_0}{(1 - \tau)^{\alpha}}, \tag{3.6}
\]
then, for all \( \tau \in [0, 1) \),
\[
\|w\|_{L^\infty(B_{r \tau})} \leq \begin{cases} 
  C_r \varepsilon_0 \frac{1}{(1 - \tau)^{\alpha}}, & \text{if } \alpha \geq \gamma, \\
  C_r \varepsilon_0 \frac{1}{(1 - \tau)^{\alpha}}, & \text{if } \alpha < \gamma,
\end{cases}
\]
where \( \gamma \) is introduced in (2.7).

Before proving this lemma, we need the following integral result from Giga and
Kohn [14] (see Lemma 2.2 page 851 in [14]):

**Lemma 3.3.** For \( 0 < \alpha < 1 \), \( \theta > 0 \) and \( 0 \leq \tau < 1 \), the integral
\[
I(\tau) = \int_0^\tau (\tau - s)^{-\alpha}(1 - s)^{-\theta} ds
\]
satisfies
i) \( I(\tau) \leq ((1 - \alpha)^{-1} + (\alpha + \theta - 1)^{-1})(1 - \tau)^{1 - \alpha - \theta} \), if \( \alpha + \theta > 1 \),
ii) \( I(\tau) \leq (1 - \alpha)^{-1} + |\log(1 - \tau)| \), if \( \alpha + \theta = 1 \),
iii) \( I(\tau) \leq (1 - \alpha - \theta)^{-1} \), if \( \alpha + \theta < 1 \).

With this lemma, we are ready to give the proof of Lemma 3.2.

**Proof of Lemma 3.2.** Consider \( w_r = \phi_r w \), where the cut-off \( \phi_r \) is introduced right
before (3.2). Since \( w \) satisfies (2.6), it follows that \( w_r \) satisfies the following equation:
for all \( \xi \in \mathbb{R} \) and \( \tau \in [0, 1) \),
\[
\partial_\tau w_r = \Delta w_r + w \Delta \phi_r - 2\nabla(w \nabla \phi_r) + p|w|^{p-1} w_r \\
+ \mu(T - t_0(x_0))|\gamma| \nabla(\phi_r|w| \int_{B_0} |v|^{q-1}) - \mu(T - t_0(x_0))|\gamma| \nabla \phi_r|w| \int_{B_0} |v|^{q-1}.
\]

The Duhamel equation satisfied by \( w_r \) is the following: for all \( \tau \in [0, 1) \),
\[
w_r(\tau) = S(\tau) w_r(0) + \int_0^\tau S(\tau - s) w \Delta \phi_r - 2 \int_0^\tau S(\tau - s) \nabla(w \nabla \phi_r)
+ p \int_0^\tau S(\tau - s)|w|^{p-1} w_r \\
+ \mu(T - t_0(x_0))\gamma \int_0^\tau S(\tau - s) \nabla \phi_r|w| \int_{B_0} |v|^{q-1})
- \mu(T - t_0(x_0))\gamma \int_0^\tau S(\tau - s) \nabla \phi_r|w| \int_{B_0} |v|^{q-1}. \tag{3.7}
\]
Then, applying item i) and Case 1.

In order to apply Lemma 3.3, we distinguish 3 cases:

We recall the following well-known smoothing effect of the heat semigroup:

$$
\|S(t)f\|_{L^\infty} \leq \|f\|_{L^\infty}, \quad \|\nabla S(t)f\|_{L^\infty} \leq \frac{C}{\sqrt{t}} \|f\|_{L^\infty}, \quad \forall t > 0, \quad \forall f \in L^\infty(\mathbb{R}^N). \quad (3.8)
$$

Taking the $L^\infty$-norm on the Duhamel equation (3.7), using the smoothness of $\phi_r$ and inequalities (3.8), we get for all $\tau \in [0,1)$:

$$
\|w_r(\tau)\|_{L^\infty} \leq \|w(0)\|_{L^\infty(B_r)} + C \int_0^\tau \|w\|_{L^\infty(B_r)} + C \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|w\|_{L^\infty(B_r)} \\
+ p \int_0^\tau \|v\|_{L^\infty(B_r)} \|w_r\|_{L^\infty} \\
+ C \mu(T - t_0(x_0))^{\gamma} \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|w\|_{L^\infty(B_r)} \int_{B_0} |v|^{q-1} \|f\|_{L^\infty(B_r)} \\
+ C \mu(T - t_0(x_0))^{\gamma} \int_0^\tau \|w\|_{L^\infty(B_r)} \int_{B_0} |v|^{q-1} \|f\|_{L^\infty(B_r)}. \quad (3.9)
$$

Note that the various $C$ constants may depend on $r$. Since $\gamma > 0$ from (2.8), we may fix some

$$
\eta \in (0, \gamma \frac{2}{\alpha}) \text{ with in addition } \eta < \frac{\gamma - \alpha}{2} \text{ if } \alpha < \gamma. \quad (3.10)
$$

From assumption (3.6) and inequality (2.12), we obtain for all $\tau \in [0,1)$,

$$
\|w_r(\tau)\|_{L^\infty} \leq C \varepsilon_0 + C \varepsilon_0 \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1 - s)^{-\alpha} ds \\
+ C \varepsilon_0^{p-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} ds \quad \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1 - s)^{-\frac{1}{2} + \gamma - \eta - \alpha} ds. \quad (3.11)
$$

In order to apply Lemma 3.3, we distinguish 3 cases:

**Case 1.** When $\alpha > \frac{1}{2}$. Clearly, from (2.7) and (1.2), we have

$$
0 < \gamma < \frac{1}{2} \quad (3.12)
$$

and

$$
\alpha + \eta - \gamma > 0. \quad (3.13)
$$

Then, applying item i) of Lemma 3.3, we obtain

$$
\|w_r(\tau)\|_{L^\infty} \leq C \varepsilon_0 (1 - \tau)^{-\alpha + \frac{1}{2}} + C \varepsilon_0 (T - t_0(x_0))^{\gamma - \eta}(1 - \tau)^{\gamma - \eta - \alpha} \\
+ C \varepsilon_0^{p-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} ds.
$$
Since \( \gamma - \alpha - \eta < -\alpha + \frac{1}{2} \) from (3.10) and (3.12), using (3.10) again, we see that
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 (1 - \tau)^{\gamma - \eta - \alpha} + C_0 \varepsilon_0^{-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} \, ds \tag{3.14}
\]
\[
\leq C_0 \varepsilon_0 + C_0 \varepsilon_0^{-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} \, ds
+ C_0 \varepsilon_0 (\alpha + \eta - \gamma) \int_0^\tau (1 - s)^{\gamma - \alpha - \eta - 1} \, ds.
\]

Let us now recall the following Gronwall Lemma from Giga and Kohn [14] (see Lemma 2.3 page 852 there):

**Lemma 3.4** (A Gronwall lemma; Giga and Kohn [14]). If \( y(t), r(t) \) and \( q(t) \) are continuous functions defined on \([t_0, t_1]\) such that
\[
y(t) \leq y_0 + \int_{t_0}^t y(s) r(s) \, ds + \int_{t_0}^t q(s) \, ds, \quad t_0 \leq t \leq t_1,
\]
then
\[
y(t) \leq \exp \left\{ \int_{t_0}^t r(\tau) \, d\tau \right\} [y_0 + \int_{t_0}^t q(\tau) \exp \left\{ - \int_{t_0}^\tau r(\sigma) \, d\sigma \right\} \, ds].
\]

Applying Lemma 3.4 to estimate (3.14), we obtain for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 (1 - \tau)^{-C_0 \varepsilon_0^{-1}} \left[ 1 + (\alpha + \eta - \gamma) \int_0^\tau (1 - s)^{\gamma - \alpha - \eta + C_0 \varepsilon_0^{-1} - 1} \, ds \right].
\]

From (3.13), we see that we can choose \( \varepsilon_0 \) small enough such that \( \gamma - \alpha - \eta + C_0 \varepsilon_0^{-1} < 0 \), and obtain for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 (1 - \tau)^{\gamma - \alpha - \eta}.
\]

Since \( \eta < \frac{1}{2} \) from (3.10), we see that for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 (1 - \tau)^{-\alpha + \frac{1}{2}}.
\]

**Case 2.** When \( \gamma \leq \alpha \leq \frac{1}{2} \). We see from (3.10) that estimate (3.13) still holds. Therefore, using (3.11) and applying items i) and iii) of Lemma 3.3, we obtain for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 \left[ 1 + \left( \log(1 - \tau) \right] + C_0 \varepsilon_0 (T - t_0(x_0))^{\gamma - \eta} (1 - \tau)^{\gamma - \eta - \alpha}
+ C_0 \varepsilon_0^{-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} \, ds
\leq C_0 \varepsilon_0 (1 - \tau)^{\gamma - \alpha - \eta} + C_0 \varepsilon_0^{-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} \, ds,
\]
which is exactly the same inequality (3.14) encountered in Case 1. Thus, by the same argument, we obtain for \( \varepsilon_0 \) small enough and for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 (1 - \tau)^{-\alpha + \frac{1}{2}}.
\]

**Case 3.** When \( \alpha < \gamma < \frac{1}{2} \). From (3.10), we see that
\[
\gamma - \alpha - \eta > 0. \tag{3.15}
\]
Thanks to (3.11) together with item iii) of Lemma 3.3, we obtain for all \( \tau \in [0, 1] \),
\[
\|w_r(\tau)\|_{L^\infty} \leq C_0 \varepsilon_0 + C_0 \varepsilon_0^{-1} \int_0^\tau (1 - s)^{-1} \|w_r(s)\|_{L^\infty} \, ds.
\]
Using again Lemma 3.4, we see that for all $\tau \in [0, 1)$,
\[\|w_{\tau}(\tau)\|_{L^\infty} \leq C\varepsilon_0(1 - \tau)^{-C\varepsilon_0^p - 1}.\]
Since $\gamma < \frac{1}{2}$ from (3.12), it is easy to see that the cases 1, 2 and 3 mentioned above cover all possibilities. Thus, this concludes the proof of Lemma 3.2.

Starting from the hypothesis (3.1) stated in Lemma 3.1 and using a finite iteration, Lemma 3.2 provides us with a positive constant $r_1$ such that estimates (3.2) and (3.3) are satisfied.

**Step 2.** Here, we start from estimates (3.2) and (3.3) we have just proved in Step 1. Then, we give a Duhamel formulation satisfied by the cut-off of the solution $v$. Using various estimates of Step 1 together with Proposition 2.1, we prove the following result:

**Lemma 3.5.** There exists a positive constant $r_2 < r_1$ such that
\[\|v\|_{L^\infty(B_{r_2})} \leq C\varepsilon_0(1 - \tau)^{-C\varepsilon_0^p - 1}. \] (3.16)

**Proof.** Let $v_{r_1} = \phi_{r_1}v$, where the cut-off $\phi_{r_1}$ is defined right before (3.2). Using equation (2.5), we see that $v_{r_1}$ satisfies the following equation for all $\xi \in \mathbb{R}$ and $\tau \in [0, 1)$:
\[
\partial_{\tau}v_{r_1} = \Delta v_{r_1} + v\Delta \phi_{r_1} - 2\nabla(v\nabla \phi_{r_1}) + |v|^{p-1}v_{r_1} + \mu(T - t_0(x_0))^{\gamma} \phi_{r_1}\nabla v \int_{B_0} |v|^q ds.
\]

Using a Duhamel formulation, we see that for all $\tau \in [0, 1)$,
\[
v_{r_1}(\tau) = S(\tau)v_{r_1}(0) + \int_0^\tau S(\tau - s)v\Delta \phi_{r_1} - 2\int_0^\tau S(\tau - s)v\nabla(v\nabla \phi_{r_1})
+ \int_0^\tau S(\tau - s)|v|^{p-1}v_{r_1} + \mu(T - t_0(x_0))^{\gamma} \int_0^\tau S(\tau - s)\phi_{r_1}\nabla v \int_{B_0} |v|^q ds.
\]

Taking the $L^\infty$–norm, using inequalities (3.8) and the smoothness of $\phi_{r_1}$, we get for all $\tau \in [0, 1)$:
\[
\|v_{r_1}(\tau)\|_{L^\infty} \leq \|v(0)\|_{L^\infty(B_{r_1})} + C\int_0^\tau \|v\|_{L^\infty(B_{r_1})} + C\int_0^\tau (\tau - s)^{-\frac{1}{2}} \|v\|_{L^\infty(B_{r_1})} \] (3.17)
\[+ \int_0^\tau \|v\|_{L^\infty(B_{r_1})} + C\mu(T - t_0(x_0))^{\gamma} \int_0^\tau \|\nabla v\|_{L^\infty(B_{r_1})} \int_{B_0} |v|^{q-1} ds \|L^\infty(B_{r_1})|\]
From estimates (2.12), (3.2) and (3.3), we obtain for all $\tau \in [0, 1)$,
\[
\|v_{r_1}(\tau)\|_{L^\infty} \leq C\varepsilon_0 + C\varepsilon_0 \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1 - s)^{-\frac{1}{2}} ds
+ C\varepsilon_0^{p-1} \int_0^\tau (1 - s)^{-1} \|v_{r_1}(s)\|_{L^\infty} ds
+ C\varepsilon_0(T - t_0(x_0))^{\gamma + \gamma - \eta - C\varepsilon_0^p - 1} \int_0^\tau (1 - s)^{-\frac{1}{2} + \gamma - \eta - C\varepsilon_0^{p-1}} ds.
\]

Using (3.10) and (3.12), we write $\frac{1}{2} - \gamma + \eta + C\varepsilon_0^{p-1} \leq \frac{1}{2} - \frac{1}{2} + C\varepsilon_0^{p-1} \leq \frac{1}{2} + C\varepsilon_0^{p-1} < 1$, for $\varepsilon_0$ small enough. Recalling that $p > 3$ from (1.2), we obtain by item iii) of Lemma 3.3 for all $\tau \in [0, 1)$,
\[
\|v_{r_1}(\tau)\|_{L^\infty} \leq C\varepsilon_0 + C\varepsilon_0^{p-1} \int_0^\tau (1 - s)^{-1} \|v_{r_1}(s)\|_{L^\infty} ds.
\]

Thanks to Lemma 3.4, we conclude the proof of Lemma 3.5, taking $r_2 = \frac{T}{2}$. \qed
Step 3. Estimates (3.4) and (3.3) from Steps 1 and 2 are the starting point in this part. More precisely, we consider Theorem 1.1. More precisely, we consider Step 3.1. BOUTHAINA ABDELHEDI AND HATEM ZAAG

\[ \|v(\tau)\|_{L^\infty(B_{r_2})} \leq \frac{C\varepsilon_0}{(1-\tau)^{C_{r_0}}}, \]

\[ \|w(\tau)\|_{L^\infty(B_{r_2})} \leq \frac{C\varepsilon_0}{(1-\tau)^{C_{r_0}}} \]

Plugging these estimates in inequalities (3.9) and (3.17) (which both hold with \( r \) and \( r_1 \) replaced by \( r_2 \)), taking \( \varepsilon_0 \) small enough, we obtain for all \( \tau \in [0,1) \),

\[ \|v_{r_2}(\tau)\|_{L^\infty} \leq C\varepsilon_0 + C\varepsilon_0^{p-1} \int_0^\tau (1-s)^{C\varepsilon_0^{p-1}(p-1)} \|v_{r_2}(s)\|_{L^\infty} ds, \]

\[ \|w_{r_2}(\tau)\|_{L^\infty} \leq C\varepsilon_0 + C\varepsilon_0^{p-1} \int_0^\tau (1-s)^{C\varepsilon_0^{p-1}(p-1)} \|w_{r_2}(s)\|_{L^\infty} ds. \]

Taking \( \varepsilon_0 \) even smaller so that \( C\varepsilon_0^{p-1}(p-1) < 1 \) and using the Gronwall argument of Lemma 3.4, we deduce that for all \( \tau \in [0,1) \),

\[ \|v_{r_2}(\tau)\|_{L^\infty} \leq M\varepsilon_0, \]

\[ \|w_{r_2}(\tau)\|_{L^\infty} \leq M\varepsilon_0, \]

for some \( M > 0 \). This concludes the proof of Proposition 3.1 and part 1 of Theorem 1.1 too.

4. Existence of the final profile. In this section, we give the proof of part 2) of Theorem 1.1.

Proof of part 2) of Theorem 1.1. We consider \( u \) a solution of equation (1.1) which blows up at the origin and only there in finite time \( T > 0 \). Adapting the method used by Merle [19] and Zaag [29], we prove the existence of a blow-up final profile \( u^* \) such that \( u(x,t) \rightarrow u^* \) in \( C^1 \) of every compact of \( \mathbb{R}^N \setminus \{0\} \).

Consider \( x_0 \neq 0 \) to be taken small enough later. The proof is performed in the rescaled variable framework, \( \psi(x_0, \xi, \tau) \) and \( w(x_0, \xi, \tau) \) introduced in (2.2), (2.4) and (2.1). Let us first put together the estimates we already have:

1-Initialization for \( v \) and \( w \) at \( \tau = 0 \). From estimates (1.8), (1.9), the definitions (2.2) and (2.4) of \( v \) and \( w \), we have the following:

\[ \sup_{|\xi| \leq 6 \log(T-t_0(x_0))} |v(x_0, \xi, 0) - f(K_0)| \leq \frac{C}{|\log(T-t_0(x_0))|^{\frac{1}{4}}}, \quad (4.1) \]

\[ \sup_{|\xi| \leq 6 \log(T-t_0(x_0))} |w(x_0, \xi, 0) - \frac{1}{\sqrt{|\log(T-t_0(x_0))|}} \nabla f(K_0)| \leq \frac{C}{|\log(T-t_0(x_0))|^{\frac{1}{4}}}, \quad (4.2) \]

where \( t_0(x_0) \) is defined in (2.1).

2-A rough bound on \( v \) and \( w \) for \( \tau \in [0,1) \). We claim that for all \( \tau \in [0,1) \) and \( |\xi| \leq 6 \log(T-t_0(x_0)) \), we have

\[ \frac{(1-\tau)^{\frac{1}{4}}}{\sqrt{|\log(T-t_0(x_0))|}} |v(x_0, \xi, \tau)| + \sqrt{1-\tau} |w(x_0, \xi, \tau)| \]

\[ \leq f\left( \frac{K_0}{2} \right) + \frac{C}{\sqrt{|\log(T-t_0(x_0))|}} \nabla f(K_0) + \frac{C}{|\log(T-t_0(x_0))|^{\frac{1}{4}}} \equiv \varepsilon_1(K_0, x_0). \]

\[ \frac{(1-\tau)^{\frac{1}{4}}}{\sqrt{|\log(T-t_0(x_0))|}} |v(x_0, \xi, \tau)| + \sqrt{1-\tau} |w(x_0, \xi, \tau)| \]

\[ \leq f\left( \frac{K_0}{2} \right) + \frac{C}{\sqrt{|\log(T-t_0(x_0))|}} \nabla f(K_0) + \frac{C}{|\log(T-t_0(x_0))|^{\frac{1}{4}}} \equiv \varepsilon_1(K_0, x_0). \]
The proof is exactly the same as for the calculation displayed in the proof of part 1 of Theorem 1.1 on page 7, though we consider here $\xi$ is some larger ball.

**3-A uniform bound on $v$ and $w$ for $\tau \in [0, 1)$.** Let us choose $K_0$ large enough and $|x_0|$ small enough such that $\varepsilon_1(K_0, x_0) < \bar{\varepsilon}$, where $\bar{\varepsilon}$ is the constant introduced in Proposition 3.1.

Applying that proposition, we obtain

$$
\sup_{|\xi| \leq 5|\log(T-t_0(x_0))|^{1/2}} |v(x_0, \xi, \tau)| + |w(x_0, \xi, \tau)| \leq M \varepsilon_1(K_0, x_0) \equiv M_1. \quad (4.4)
$$

Now, using the above mentioned information, we proceed in three steps:

- **First**, we prove that for all $\tau \in [0, 1)$, $|\xi| \leq 2|\log(T-t_0(x_0))|^{1/2}$,

  $$
  \|w(x_0, \xi, \tau)\|_{L^\infty} \leq \frac{C}{|\log(T-t_0(x_0))|^{1/2}}. \quad (4.5)
  $$

- **Next**, we prove that for all $\tau \in [0, 1)$, $|\xi| \leq |\log(T-t_0(x_0))|^{1/2}$,

  $$
  \|v(x_0, \xi, \tau) - v_{K_0}(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T-t_0(x_0))|^{1/2}}, \quad (4.6)
  $$

where $v_{K_0}(\tau) = ((p-1)(1-\tau) + bK_0^2)^{-1/2}$ is the solution of the ordinary differential equation $v'_{K_0}(\tau) = v_{K_0}(\tau)$, with initial data $v_{K_0}(0) = f(K_0)$.

- **Finally**, we use classical parabolic regularity and the definition of $t_0(x_0)$, to get the equivalent of $u^*$ as $x_0 \to 0$.

**Step 1. Smallness of $w$**

Arguing as in Proposition 3.1, we introduce $\phi$ a $C^\infty$ cut-off function, with $\text{supp}(\phi) \subset B(0, 1)$, $0 \leq \phi \leq 1$ and $\phi = 1$ on $B(0, \frac{1}{2})$. We also introduce for every $r > 0$,

$$
\phi_r(\xi) = \phi\left(\frac{\xi}{r|\log(T-t_0(x_0))|^{1/2}}\right).
$$

Note that

$$
\|\nabla \phi_r\|_{L^\infty} \leq \frac{C}{r|\log(T-t_0(x_0))|^{1/2}}, \quad (4.7)
$$

and

$$
\|\Delta \phi_r\|_{L^\infty} \leq \frac{C}{r^2|\log(T-t_0(x_0))|^{1/2}}. \quad (4.8)
$$

Introducing for $r = 4$, $w_4 = \phi_4 w$, and arguing as for (3.9), taking the $L^\infty$-norm on the Duhamel equation satisfied by $w_2$, we obtain for all $\tau \in [0, 1)$,

$$
\|w_4(\tau)\|_{L^\infty} \leq \|w_4(0)\|_{L^\infty} + C \int_0^\tau \|\Delta \phi_4\|_{L^\infty} \|w\|_{L^\infty(B)} + C \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|\nabla \phi_4\|_{L^\infty} \|w\|_{L^\infty(B)}
\quad + p \int_0^\tau \|v\|_{L^{p-1}(B)} \|w_4\|_{L^\infty}
\quad + C \mu(T-t_0(x_0))^{\gamma} \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|w\|_{L^\infty(B)} \int_{B_0} |v|^{q-1} \|w\|_{L^\infty(B)}
\quad + C \mu(T-t_0(x_0))^{\gamma} \int_0^\tau \|\nabla \phi_4\|_{L^\infty} \|w\|_{L^\infty(B)} \int_{B_0} |v|^{q-1} \|w\|_{L^\infty(B)}, \quad (4.9)
$$
where \( B_0 \) already introduced right after (2.7) is the ball of center 
\(-K_0 \sqrt{\log(T - t_0(x_0))} \frac{x_0}{|x_0|}\) and radius \(|\xi + K_0 \sqrt{\log(T - t_0(x_0))} \frac{x_0}{|x_0|}|\), and \( B \) is 
the ball of center 0 and radius \( 4\log(T - t_0(x_0))^{\frac{1}{2}} \).

Consider \( \eta \in (0, \gamma) \). Using (4.2), (4.4), (4.7), (4.8) and (2.12) we deduce for all 
\( \tau \in [0, 1) \):

\[
\|w_4(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{2}}} + \frac{CM_1}{|\log(T - t_0(x_0))|^{\frac{1}{2}}} + \frac{CM_1}{|\log(T - t_0(x_0))|^{\frac{1}{2}}}
\]

\[
+ C \eta (T - t_0(x_0))^{\gamma - \eta} M_1 \left( \int_0^\tau ((\tau - s)^{-\frac{1}{2}} + \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{2}}} - s)^{\gamma - \frac{1}{2} - \eta} ds \right)
\]

\[
+ pM_1^{p-1} \int_0^\tau \|w_4\|_{L^\infty} ds.
\]

If we remark that \( M_1 \leq 1 \) (see (4.4) and (4.3)) and

\[
(T - t_0(x_0))^{\gamma - \eta} \leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{2}}},
\]

for \( x_0 \) small enough and \( K_0 \) large enough, then by Lemma 3.3, we obtain for all 
\( \tau \in [0, 1) \),

\[
\|w_4(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{2}}} + p \int_0^\tau \|w_4(s)\|_{L^\infty} ds.
\]

Applying Lemma 3.4, we deduce that for all \( \tau \in [0, 1) \),

\[
\|w_4(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{2}}}.
\]

Thus, (4.5) follows.

**Step 2. Sharp behavior of \( v \)**

Let

\[
v_{K_0}(\tau) = ((p - 1)(1 - \tau) + bK_0^2)^{-\frac{1}{p-1}}
\]

be the solution of the ordinary differential equation

\[
v'_{K_0}(\tau) = v_{K_0}^p(\tau),
\]

with initial data \( v_{K_0}(0) = f(K_0) \).

Introducing \( \psi = v - v_{K_0} \) and \( \psi_2 = \phi_2 \psi \), we see from equation (2.5) that \( \psi_2 \)

satisfies the following equation for all \( \xi \in \mathbb{R} \) and for all \( \tau \in [0, 1) \):

\[
\partial_\tau \psi_2 = \Delta \psi_2 + \psi \Delta \phi_2 - 2\nabla(\psi \nabla \phi_2) + a\psi_2 + \mu(T - t_0(x_0))^{\gamma} \phi_2 |\nabla v|^p \int_{B_0} |v|^{q-1},
\]

where

\[
a(x_0, \xi, \tau) = \begin{cases} 
|v|^{p-1}v - v_{K_0}^p, & \text{if } v \neq v_{K_0}, \\
v - v_{K_0} \quad \text{otherwise}.
\end{cases}
\]

Using (4.4), (4.3) and (4.11), we see that

\[
\sup_{|\xi| \leq 2|\log(T - t_0(x_0))|^{\frac{1}{2}}, \tau \in [0, 1]} |a(x_0, \xi, \tau)| \equiv \eta_2 \to 0, \quad \text{as } |x_0| \to 0 \quad \text{and } K_0 \to +\infty.
\]
Step 3. Conclusion of the proof of Part 2 of Theorem 1.1

Taking the $L^\infty$-norm on the Duhamel equation satisfied by $\psi_2$, and using estimates (4.1), (2.12), (4.4), (4.7) and (4.8), we get for all $\tau \in [0, 1),$
\[
\|\psi_2(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0)|^{\frac{1}{2}}} + \frac{CM_1}{|\log(T - t_0(x_0))|^\frac{3}{2}} + \frac{CM_1}{|\log(T - t_0(x_0))|^\frac{1}{2}} + C_\eta (T - t_0(x_0))^{\gamma - \eta} M_1 \int_0^\tau (1 - s)^{\gamma - \frac{1}{2} - \eta} ds + C_\eta_2 \int_0^\tau \|\psi_2(s)\|_{L^\infty} ds.
\]
Using (4.10), we obtain for all $\tau \in [0, 1),$
\[
\|\psi_1(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0)|^{\frac{1}{4}}} + C_\eta \int_0^\tau \|\psi_1\|_{L^\infty} ds.
\]
By Lemma 3.4, we deduce that for all $\tau \in [0, 1),$
\[
\|\psi_1(\tau)\|_{L^\infty} \leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{4}}}.
\]
This concludes the proof of (4.6).

Step 3. Conclusion of the proof of Part 2 of Theorem 1.1

Using (4.5), (4.6) and classical parabolic regularity, we see that
\[
\forall \tau \in \left[\frac{1}{2}, 1\right), \ |\xi| \leq \frac{1}{2} |\log(T - t_0(x_0))|^{\frac{1}{4}}, \ |\partial_\tau v(x_0, \xi, \tau)| + |\partial_\tau w(x_0, \xi, \tau)| \leq C.
\]
Therefore, we have the existence of limits for $v(x_0, 0, \tau)$ and $w(x_0, 0, \tau)$ as $\tau \to 1$ for $|x_0|$ sufficiently small, on the one hand.

On the other hand, since neither $u$ nor $\nabla u$ blow up outside the origin, using again classical parabolic regularity, we derive the existence of a limiting profile $u^*$ such that $u(x, t) \to u^*(x)$ as $t \to T$, in $C^1$ of every compact of $\mathbb{R}^N \setminus \{0\}$ (see Merle [19] for a similar argument).

Letting $\tau \to 1$ in (4.5) and (4.6), and using the definitions (2.2) and (2.3), we have
\[
u^*(x_0) = \lim_{t \to T} u(x_0, t) = \lim_{\tau \to 1} \frac{v(x_0, 0, \tau)}{(T - t_0(x_0))^{\frac{1}{p - \eta}}} \sim (bK_0^2)^{-\frac{1}{p - \eta}} (T - t_0(x_0))^{-\frac{1}{p - \eta}}, \text{ as } x_0 \to 0,
\]
and
\[
|\nabla u^*(x_0)| = |\lim_{t \to T} \nabla u(x_0, t)| = |\lim_{\tau \to 1} \frac{w(x_0, 0, \tau)}{(T - t_0(x_0))^{\frac{1}{p - \eta} + \frac{1}{2}}}|
\leq \frac{C}{|\log(T - t_0(x_0))|^{\frac{1}{4}} (T - t_0(x_0))^{\frac{1}{p - \eta} + \frac{1}{2}}}.
\]
From the definition (2.1) of $t_0(x_0)$, we have
\[
\log |x_0| \sim \frac{1}{2} \log(T - t_0(x_0)),
\]
and
\[
T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0^2 |\log(x_0)|^2}, \text{ as } x_0 \to 0.
\]
Hence,
\[
u^*(x_0) \sim \left(\frac{b|x_0|^2}{2 |\log(x_0)|} \right)^{-\frac{1}{p - \eta}}, \text{ as } x_0 \to 0,
\]
and
\[
|\nabla u^*(x_0)| \leq |x_0|^{-\frac{p + 2}{p - \eta}} |\log |x_0||^{\frac{p + 2}{p - \eta}}.
\]
This concludes the proof of part 2) of Theorem 1.1.\]
Acknowledgments. The authors would like to thank Philippe Souplet for pointing out that the gradient blow-up is in fact a direct consequence of the single point blow-up property of the solution, which greatly improved our result.

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Received September 2020; 1st revision September 2020; 2nd revision February 2021.

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