A POTENTIAL SPACE ESTIMATE FOR SOLUTIONS OF SYSTEMS OF NONLOCAL EQUATIONS IN PERIDYNAMICS

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Abstract. We show that weak solutions to the strongly-coupled system of nonlocal equations of linearized peridynamics belong to a potential space with higher integrability. Specifically, we show a function that measures local fractional derivatives of weak solutions to a linear system belongs to $L^p$ for some $p > 2$ with no additional assumption other than measurability and ellipticity of coefficients. This is a nonlocal analogue of an inequality of Meyers for weak solutions to an elliptic system of equations. We also show that functions in $L^p$ whose Marcinkiewicz-type integrals are in $L^p$ in fact belong to the Bessel potential space $L^p_s$. Thus the fractional analogue of higher integrability of the solution’s gradient is displayed explicitly. The distinction here is that the Marcinkiewicz-type integral exhibits the coupling from the nonlocal model and does not resemble other classes of potential-type integrals found in the literature.

Key words. Peridynamics, higher integrability, nonlocal coupled system, fractional Korn’s inequality, potential spaces

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1. Introduction. Nonlocal models are becoming commonplace across application areas. Typically these models involve averaged difference quotients instead of derivatives of quantities. As a result model equations are formulated using integral operators and integral equations, in contrast to classical ones that rely on differential operators and differential equations. This characteristic makes nonlocal models amenable to describe singular and discontinuous physical, social and biological phenomena, see [1,2,9,20] for applications and analysis of nonlocal equations. Our interest is on models in peridynamics, a nonlocal reformulation of the basic equations of motion in continuum mechanics, that have shown promising potential in modeling the spontaneous formation of discontinuities in solids. The present work studies qualitative properties of solutions to the equilibrium equation in the linearized bond-based peridynamic model that first appeared in [23], with a generalization later appearing in [24,25]. To describe the model, a material body occupying a region has undergone the deformation that maps a material point $x$ to $x + u(x)$ in a deformed domain. Clearly, the vector field $u$ represents the displacement field. Treating the material body as a complex mass-spring system, in peridynamics, it is postulated that material points $y$ and $x$ interact through a bond vector $y - x$. Under the uniform small strain theory [24], the strain of the bond $y - x$ is given by the nonlocal linearized strain $\frac{u(x) - u(y)}{|y - x|} \cdot \frac{x - y}{|x - y|}$. A portion of this strain contributes to the volume changing component of the deformation and the remaining is the shape changing component. According to the linearized bond-based peridynamic model [24], the balance of forces is given by a system of nonlocal equations

$$c_0 \int_{B_h(x)} \rho(x,y) \frac{|x - y|}{|y - x|} \otimes \frac{|x - y|}{|x - y|} (u(x) - u(y)) \, dy = F(x)$$

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where \( \mathbf{F}(\mathbf{x}) \) is a vector of applied forces, \( \rho \) is a nonnegative measurable function that represents the strength interactions between material points \( \mathbf{x} \) and \( \mathbf{y} \), and finally the positive number \( c_h \) is a normalizing constant chosen in such a way that for smooth deformations when \( h \to 0 \), the nonlocal operator converges to a differential operator.

The kernel \( \rho(\mathbf{x}, \mathbf{y}) \) contains properties of the modeled material and typically decreases when \( |\mathbf{x} - \mathbf{y}| \) gets larger. It may depend on material points \( \mathbf{x}, \mathbf{y} \), their relative position \( \mathbf{y} - \mathbf{x} \) or for isotropic materials only with their relative distance \( |\mathbf{y} - \mathbf{x}| \). For general \( \rho \), the equation may model heterogeneous and anisotropic materials. In the above system the left hand side represents the linearized internal force density function due to the deformation \( \mathbf{x} \mapsto \mathbf{x} + \mathbf{u}(\mathbf{x}) \) and is a weighted average of the nonlocal linearized strain function associated with the displacement \( \mathbf{u} \). See the paper [17,24] for derivation. See also the papers [5,6,8,29] for some mathematical analysis of linearized models. Under small strain regime nonlocal nonlinear peridynamic evolution models have also been studied [13,14].

This work will focus on system of equations that use interaction kernels \( \rho(\mathbf{x}, \mathbf{y}) \) that behave like \( |\mathbf{x} - \mathbf{y}|^{-(d+2s)} \) for \( |\mathbf{x} - \mathbf{y}| \) close to zero and infinity. To be precise, we study the system of nonlocal equations of the type, formally given by,

\[
L_h \mathbf{u}(\mathbf{x}) := c_h \int_{B_h(\mathbf{x})} \frac{A(\mathbf{x}, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \otimes \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})) \, d\mathbf{y} = \mathbf{F}(\mathbf{x})
\]

where \( d \geq 2, 0 < s < 1 \) are fixed, and \( A(\mathbf{x}, \mathbf{y}) \) is a measurable function, which we refer as coefficient, that is elliptic and symmetric in the sense,

\[
\alpha_1 \leq A(\mathbf{x}, \mathbf{y}) \leq \alpha_2, \quad A(\mathbf{x}, \mathbf{y}) = A(\mathbf{y}, \mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.
\]

The goal of this paper is twofold. The first is to establish the higher integrability of a measure of smoothness of weak solutions to the peridynamic system of nonlocal equations given in (1.1). The notion of weak solution will be be defined in the next section. Roughly speaking, we show that there exists an exponent \( p > 2 \) such that for any weak solution \( \mathbf{u} \) to (1.1) corresponding to \( \mathbf{F} \in L^2(\mathbb{R}^d; \mathbb{R}^d) \), both \( \mathbf{u} \) and \( D^s(\mathbf{u}) \) are in \( L^p \) where the function \( D^s(\mathbf{u}) \) is a measure of local smoothness of \( \mathbf{u} \) given by

\[
D^s(\mathbf{u})(\mathbf{x}) = \left( \int_{\mathbb{R}^d} \frac{(|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 (\mathbf{x} - \mathbf{y})^2}{|\mathbf{x} - \mathbf{y}|^{d+2s}} \, d\mathbf{y} \right)^{\frac{1}{2}}.
\]

The higher integrability result holds under no additional assumption on the coefficient \( A(\mathbf{x}, \mathbf{y}) \) other than ellipticity and measurability. The second goal is to characterize the space of vector fields \( \mathbf{u} \) such that both \( \mathbf{u} \) and \( D^s(\mathbf{u}) \) are in \( L^p \). We will establish that for \( p \in \left( \frac{2d}{d + 2s}, \infty \right) \), the space is the standard Bessel potential space \( \mathcal{L}^p \), which will be defined shortly, that is, \( \{ \mathbf{u} \in L^p(\mathbb{R}^d; \mathbb{R}^d) : D^s(\mathbf{u}) \in L^p(\mathbb{R}^d) \} = \mathcal{L}^p(\mathbb{R}^d) \).

We should note that one may think of this system (1.1) as a fractional analogue of the strongly coupled system of partial differential equations

\[
\text{div } \mathcal{C}(\mathbf{x})\nabla \mathbf{u}(\mathbf{x}) = \mathbf{F}(\mathbf{x})
\]

where \( \mathcal{C}(\mathbf{x}) \) is a fourth-order tensor of bounded coefficients. Systems of differential equations of the above type are commonly used in elasticity and are not necessarily
uniformly elliptic but rather satisfy the weaker Legendre-Hadamard ellipticity condition. For a class of coefficients this connection between the nonlocal system \((1.1)\) and the local system \((1.3)\) is rigorously justified in [6,17,18] in the event of vanishing nonlocality \((\mathbf{h} \to 0)\). As has already been shown in [17, Theorem 3] via a simple calculation using Taylor expansion, for \(u \in C^2_b(\mathbb{R}^d;\mathbb{R}^d)\), fixed \(s\), and \(A(x,y) = a(x)\), there is a coefficient \(\mu(x)\) which is a constant multiple of \(a(x)\) such that as \(\mathbf{h} \to 0\)

\[
L_\mathbf{h} u \to \mathbb{L}_0 u := \text{div}(\mu(x)\nabla u) + 2\nabla(\mu(x)\text{div}u).
\]

In light of this connection, the higher integrability result for weak solution to \((1.1)\) is a fractional analogue of Meyers inequality for systems of differential equations which states that weak solutions of the strongly coupled systems \((1.3)\) with measurable coefficients satisfying the Legendre-Hadamard ellipticity condition and corresponding to highly integrable data are in \(W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)\) for some \(p > 2\), see [7,19].

In what follows we assume that the horizon \(\mathbf{h}\) is given and fixed. We will not track the dependence of generic constants on the horizon. In the next section, we introduce notations, give the definition of weak solutions and state the precise statement of the main results. We will also describe how we prove those results. In Section 3 we prove the higher integrability result. We first establish local estimates for vector fields in the fractional Sobolev spaces, as well as solutions of the nonlocal elliptic system \((1.1)\). For a given weak solution \(u\), we will use the local estimates to prove higher integrability of \(D^\mathbf{s}(u)\). In Section 4 we prove the characterization of the potential spaces.

2. Statement of main results. To state the main results of the paper we first introduce the quadratic bilinear form associated with the operator \(\mathbb{L}\) in \((1.1)\):

\[
E_h(u,v) = \frac{c_h}{2} \int_{\mathbb{R}^d} \int_{B_\mathbf{h}(x)} \frac{A(x,y)}{|x-y|^{d+2s}} (u(x) - u(y)) \cdot (v(x) - v(y)) \cdot \left( \frac{x-y}{|x-y|} \right) \text{d}y \text{d}x.
\]

We use the notation \(\mathcal{E}\) to represent the bilinear form when both integrals are on \(\mathbb{R}^d\), that is formally, the horizon is \(\infty\). In this case we take \(c_\infty = 1\). For \(u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\), as we will show shortly that the energy \(E_h(u,u)\) is finite if and only if \(u \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)\). This assertion follows from the lower and upper bounds of the coefficient \(A(x,y)\) and using the equivalence of spaces that is recently proved in [16]. The notion of weak solution to \((1.1)\) is standard and is given in terms of the bilinear form \(\mathcal{E}\).

DEFINITION 2.1. Given \(F \in W^{-s,2}(\mathbb{R}^d;\mathbb{R}^d)\), we say that \(u \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)\) is a weak solution to \((1.1)\) if

\[
E_h(u,\phi) = \langle F, \phi \rangle
\]

for any \(\phi \in W^{s,2}(\mathbb{R}^d;\mathbb{R}^d)\). In the above \(\langle \cdot,\cdot \rangle\) denotes the duality pairing.

Existence and uniqueness of weak solutions to this particular system has been demonstrated recently in [11] with some complementary boundary conditions. However, the focus of this paper will be the issue of regularity of solutions. We seek to address the following question: If the data is regular, how regular is the solution? In particular, we are interested in data \(F\) coming from a class of high integrability. We use the following standard notations for the fractional Sobolev exponents and its Hölder conjugate (denoted by the prime notation)

\[
2^* = \frac{2d}{d - 2s}, \quad \text{and} \quad (2^*)' = 2_{*} := \frac{2d}{d + 2s}.
\]
Our result on Sobolev regularity of solutions to the system (1.1) states that any weak solution $u$ to the nonlocal system with Poisson data $F$ in the described class in fact belongs to the Bessel potential space $L^p_s(\mathbb{R}^d)$ for some $p > 2$. The space $L^p_s(\mathbb{R}^d)$ will be defined shortly but we note that $W^{s,p}(\mathbb{R}^d) \subset L^p_s(\mathbb{R}^d)$. For precision, we state the result in the following theorem.

**Theorem 2.2.** Suppose that $0 < \alpha_1 \leq \alpha_2 < \infty$, $0 < s < 1$, $d \geq 2$, $\delta_0 > 0$ and $0 < h \leq \infty$ are all given. Let $A(x, y)$ be symmetric and measurable with $\alpha_1 \leq A(x, y) \leq \alpha_2$ and $u \in W^{s,2}(\mathbb{R}^d)$ be any weak solution in the sense of (2.1) corresponding to a given $F \in L^{2s, 2s/h_0}(\mathbb{R}^d) \cap W^{-s, 2}(\mathbb{R}^d)$. Then there exists an exponent $p > 2$ such that $u \in L^p_s(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ such that

$$
\|u\|_{L^p_s} \leq C \left( \|F\|_{L^{2s, 2s/h_0}} + \|u\|_{W^{s, 2}} \right).
$$

The constant $C$ depends only on $s$, $d$, $\delta_0$, $h$, and the ellipticity bounds $\alpha_1$ and $\alpha_2$, and $p$ depends only on $s$, $\delta_0$, and $d$.

Some remarks are in order. First, note that for $\delta_0$ small, the exponent $2s + \delta_0$ can be smaller than 2, and the conclusion of the theorem holds true the rough data $F \in L^{2s, 2s/h_0}(\mathbb{R}^d)$. Second, the fact that the solution has higher integrability follows from Sobolev embedding. Rather, the main implication of the regularity result is that, with no additional smoothness condition on the coefficient $A(x, y)$, a weak solution satisfying (2.1) has a higher integrable fractional “s-derivative”. For scalar equations, this type of nonlocal analogue of Meyers inequality was obtained in the recent work [4]. It has been shown in several subsequent works that solutions to scalar fractional equations surprisingly also improve in differentiability, again with no additional smoothness conditions on the coefficients. Some of these approaches use reverse Hölder inequalities and nonlocal Gehring-type lemmas as in [12], or prove the result via a commutator estimate as in [21] or apply a functional analytic approach as in [3]. There, application of appropriate embedding estimates show improved differentiability will lead to improved integrability.

The proof of Theorem 2.2 we present in this paper follows the argument presented in [4] for nonlocal scalar equations. By using Gehring’s lemma, the authors of [4] have shown that the scalar function

$$
\Upsilon^s(u)(x) := \left( \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, dy \right)^{1/2}
$$

is in $L^p(\mathbb{R}^d)$ for some $p > 2$ for any weak solution $u$ to a nonlocal scalar equations with a $L^2$ right-hand side. This was possible after establishing localized reverse Hölder inequalities for $\Upsilon^s(u)$ using Caccioppoli estimates derived from the equation. It turns out that this higher integrability result of $\Upsilon^s(u)$ for the weak solution $u$ is equivalent to $u$ being in the Bessel potential space $L^p_s$. This is a consequence of the classical embedding results in [26]. Unfortunately, for a vector field weak solution $u$ to (1.1), working with the function $\Upsilon^s(u)$ does not seem to be appropriate as it is not directly related to the energy space associated with the coupled nonlocal system of equations. We rather use the smaller scalar function $D^s(u)$ introduced in (1.2).

The function $D^s(u)$ is related to the energy space $W^{s, 2}(\mathbb{R}^d)$ since for a vector field $u \in L^2(\mathbb{R}^d)$, $u \in W^{s, 2}(\mathbb{R}^d)$ if and only if $D^s(u) \in L^2(\mathbb{R}^d)$ via the fractional Korn-type inequality proved in [16]. With this in hand, we prove localized reverse Hölder inequalities for $D^s(u)$ using Caccioppoli estimates derived from the equation, yielding the higher integrability of $D^s(u) \in L^p(\mathbb{R}^d)$ for some $p > 2$.
via Gehring’s lemma. The latter is possible via application of Sobolev embedding estimates in the space

\[ (2.3) \quad \mathcal{X}_q^s(\mathbb{R}^d) = \left\{ v \in L^q(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^q}{|x-y|^{d+sq}} \, dy \, dx < \infty \right\} \]

which is shown to be equivalent with the standard fractional Sobolev space \( W^{s,q}(\mathbb{R}^d; \mathbb{R}^d) \), for any \( s \in (0,1) \) and \( q \in (1,\infty) \) proved in [22, Theorem 1.1].

The higher integrability of \( D^s(u) \) will be shown to be equivalent with the statement that \( u \) is in a higher-integrable potential space. Establishing this equivalence is the second goal of the paper. To this end, we first introduce the space of Bessel potentials, \( L^q_s \). For \( 0 < s < 1, 1 < q < \infty \)

\[ (2.4) \quad L^q_s(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d; \mathbb{R}^d) : \left( 1 + 4\pi^2|\xi|^2 \right)^{s/2} \hat{f} \right\} \subset L^q(\mathbb{R}^d; \mathbb{R}^d), \]

where \( S'(\mathbb{R}^d; \mathbb{R}^d) \) is the space of vector-valued tempered distributions, and the Fourier transform \( \hat{h} \) is defined for smooth vector fields \( \mathbf{h} \) as

\[ \hat{h}(\xi) = \int_{\mathbb{R}^d} e^{-i2\pi \xi \cdot x} \mathbf{h}(x) \, dx. \]

The notation \( \cdot \vee \) denotes the inverse Fourier transform. We will retain the same Fourier transform notations even for distributions. The norm in \( L^q_s(\mathbb{R}^d) \) is given by

\[ (2.5) \quad \|f\|_{L^q_s(\mathbb{R}^d)} := \left\| \left( 1 + 4\pi^2|\xi|^2 \right)^{s/2} \hat{f} \right\|_{L^q}. \]

Our second main result is the characterization of functions in the potential space \( L^q_s \) in terms of \( D^s(f) \).

**Theorem 2.3.** Let \( 2s_r \leq p < \infty \) and \( 0 < s < 1 \). Then \( f \in L^p(\mathbb{R}^d; \mathbb{R}^d) \) if and only if \( f \in L^q_s(\mathbb{R}^d; \mathbb{R}^d) \) and \( D^s(f) \in L^p(\mathbb{R}^d) \). Moreover, there exist positive constants \( C_1 \) and \( C_2 \) such that

\[ C_1\|f\|_{L^q_s(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} + \|D^s(u)\|_{L^p(\mathbb{R}^d)} \leq C_2\|f\|_{L^q_s(\mathbb{R}^d)}. \]

We emphasize that this characterization is in the same spirit as classical characterizations of vector fields in the Bessel potential space. One such characterization is given by Stein in [26, Theorem 1] or [27, Chapter V] in terms of smallness of the difference \( f(x+y) - f(x) \) via the Marcinkiewicz integral \( \Upsilon^s(f) \). The result states that for \( 0 < s < 1 \) and \( 2s_r < q < \infty \), \( f \in L^q_s(\mathbb{R}^d; \mathbb{R}^d) \) if and only if \( f \in L^q(\mathbb{R}^d) \) and \( \Upsilon^s(f) \in L^q(\mathbb{R}^d) \). A finer characterization that uses means over balls is also given in [28] that is valid for the full range of \( q \in (1,\infty) \). Various other characterizations have also been explored throughout the literature. However, the essence of Theorem 2.3 lies in the fact that to determine if \( f \) is a Bessel potential of an \( L^p \) vector field we do not need the smallness of a measure of \( f(x+y) - f(x) \) but rather a measure of the quantity \( \|f(x+y) - f(x)\| \cdot \frac{Y}{|y|} \) via another Marcinkiewicz-type integral of \( f \), \( D^s(f) \).

Since for any \( x \in \mathbb{R}^d \) the pointwise estimate \( D^s(f)(x) \leq \Upsilon^s(f)(x) \) holds, the right-hand side inequality in Theorem 2.3 follows from the characterization in [26, Theorem 1]. However, the left-hand side inequality is a refined version of [26, Theorem 1] since
we are estimating \( \|f\|_{L^q_p(\mathbb{R}^d)} \) in terms of the smaller function \( D^s(f) \) in place of \( T^s(f) \). We will in fact show that the left hand side inequality in the theorem is valid for \( q \in (1, \infty) \).

Our proof of Theorem 2.3 follows the steps presented in the proof of [26, Theorem 1]. We first develop the necessary technical tools that allow us to relate the Marcinkiewicz-type integral \( D^s(f) \) with the potential function of \( f \). It turns out this is possible by introducing a Poisson-type integral of \( f \) and a corresponding Littlewood-Paley \( g \)-function. We will show that, in parallel with classical results, this new \( g \)-function can be used to characterize \( L^p \) norms of vector fields.

3. Meyers-type higher integrability result for coupled nonlocal systems. In this section we prove Theorem 2.2. As we have discussed in the introduction our proof of Theorem 2.2 follows the argument presented in [4] in obtaining a higher integrability result for scalar nonlocal elliptic equations. The first result we prove here is the higher integrability of \( D^s(u) \) using Gehring’s lemma.

**Theorem 3.1.** Suppose that \( 0 < \alpha_1 \leq \alpha_2 < \infty, 0 < s < 1, d \geq 2, 0 < h \leq \infty, and \delta_0 > 0 \) are given. Then there exist positive constants \( p > 2 \) and \( C > 0 \) such that for any \( A(x, y) \) symmetric, measurable with \( \alpha_1 \leq A(x, y) \leq \alpha_2 \) and any weak solution \( u \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d) \) in the sense of (2.1) corresponding to \( F \in L^{2, s+h_0}(\mathbb{R}^d; \mathbb{R}^d) \cap W^{-s,2}(\mathbb{R}^d; \mathbb{R}^d) \), we have

\[
\|u\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} + \|D^s(u)\|_{L^p(\mathbb{R}^d)} \leq C \left( \|F\|_{L^2(2s, s+h_0)} + \|u\|_{W^{s,2}(\mathbb{R}^d; \mathbb{R}^d)} \right). 
\]

The constant \( C \) depends only on \( \delta_0, s, d \) and the ellipticity bounds \( \alpha_1 \) and \( \alpha_2 \) and \( p \) depends only on \( \delta_0, s, d \).

For a given \( \delta_0 > 0 \), by the data \( F \) belongs to \( L^{2, s+h_0}(\mathbb{R}^d; \mathbb{R}^d) \cap W^{-s,2}(\mathbb{R}^d; \mathbb{R}^d) \) we mean the duality pairing is replaced by an integral and satisfies the inequality

\[
\left| \int_{\mathbb{R}^d} F \cdot v dx \right| \leq C \|F\|_{L^2(2s, s+h_0)} \|v\|_{W^{s,2}}, \quad \forall v \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d)
\]

for some \( C > 0 \). This class is certainly more that \( L^2(\mathbb{R}^d, \mathbb{R}^d) \). Indeed, using Sobolev embedding and interpolation, one can show vector fields in \( L^{2, s+h_0}(\mathbb{R}^d; \mathbb{R}^d) \) are in the above class provided \( \delta_0 \in [0, 4s/(d + 2s)] \).

We notice that Theorem 2.2 now follows as a corollary of this theorem. Indeed, applying the characterization theorem (Theorem 2.3) we see that \( u \in L^p_1(\mathbb{R}^d) \). Moreover, by applying the well-known embedding [27, Chapter V] of the spaces \( W^{s,p}(\mathbb{R}^d; \mathbb{R}^d) \subset L^p(\mathbb{R}^d) \) for \( p > 2 \), Theorem 2.2 implies a higher Besov space regularity result for solutions.

3.1. Local estimates. Before we prove Theorem 3.1, we first establish a series of inequalities that will lead us to a reverse Hölder’s inequality for \( D^s(u) \) over balls. The main technical tool for the estimates is a Sobolev embedding in the space \( \mathcal{X}^s_q(\mathbb{R}^d) \) introduced in (2.3). The embedding is a consequence of the equivalence of \( \mathcal{X}^s_q(\mathbb{R}^d) \) with the standard fractional Sobolev space \( W^{s,q}(\mathbb{R}^d; \mathbb{R}^d) \) for any \( s \in (0, 1) \) and \( q \in (1, \infty) \) proved in [22].

**Theorem 3.2 (Fractional Korn’s inequality).** For any \( s \in (0, 1) \) and \( 1 < q < \infty \),

\[
\mathcal{X}^s_q(\mathbb{R}^d) = W^{s,q}(\mathbb{R}^d; \mathbb{R}^d).
\]
Moreover, there exists a universal constant $C = C(d, q, s)$ such that for all $f \in W^{s,p}(\mathbb{R}^d; \mathbb{R}^d)$

$$[f]_{X_q^s} \leq [f]_{W^{s,q}} \leq C[f]_{X_q^s}. \quad (3.1)$$

For a given bounded domain $\Omega$, the function space $X_q^s(\Omega)$ is defined in the same way as $X_q^s(\mathbb{R}^d)$ but the functions are defined in $\Omega$ and the integrations in the semi-norm are on $\Omega$. We call the above theorem “fractional Korn’s inequality” because it has been shown in [15] that

$$\lim_{s \to 1^-} (1 - s) [f]_{X_q^s(\Omega)} = \|\nabla f\|_{L^q(\Omega)}$$

for every $f \in W^{1,q}_{sym}(\Omega; \mathbb{R}^d)$.

This association suggests that $X_q^s(\Omega)$ is the fractional analogue of $W^{1,q}_{sym}(\Omega; \mathbb{R}^d)$, which in turn is known to coincide with $W^{1,q}(\Omega; \mathbb{R}^d)$ via the classical Korn’s inequality. Theorem 3.2 extends this equivalence to fractional spaces.

The following lemma is a localized Sobolev embedding estimate which we obtain as an application of Theorem 3.2.

**Lemma 3.3.** Let $R > 0$ and $x_0 \in \mathbb{R}^d$. Suppose that $u \in X_q^s(B_R(x_0))$ for $1 < q \leq p$. Let $d > sq$ and $q^{**} := \frac{dq}{d - sq}$. Then $u \in L^{q^{**}}(B_{R/2}(x_0); \mathbb{R}^d)$, and there exists a constant $C = C(d, s, q)$ such that

$$\|u\|_{L^{q^{**}}(B_{R/2}(x_0))} \leq C\left([u]_{X_q^s(B_R(x_0))} + R^s\|u\|_{L^q(B_R(x_0))}\right). \quad (3.2)$$

**Proof.** By a scaling argument it suffices to prove the result for the case $R = 1$. Let $\varphi \in C_\infty^0(\mathbb{R}^d)$ be a cutoff function satisfying $\varphi \equiv 1$ on $B_{\frac{3}{4}}(x_0)$ and $\varphi \equiv 0$ on $\mathbb{C}B_{\frac{1}{4}}(x_0)$. After noting that $\varphi u \in X_q^s(\mathbb{R}^d)$, we can apply the fractional Sobolev embedding theorem followed by Theorem 3.2 to obtain that

$$\|\varphi u\|_{L^{q^{**}}(\mathbb{R}^d)} \leq C\|\varphi u\|_{W^{s,q}(\mathbb{R}^d)} \leq C([\varphi u]_{X_q^s(\mathbb{R}^d)})$$

where $q^{**} := \frac{dq}{d - sq}$, if $d > sq$, and any number bigger than $q$, if $d \leq sq$. Splitting the integral in the semi-norm on the right-hand side and adding and subtracting $\varphi(y)u(x)$ we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(\varphi(y)u(y) - \varphi(x)u(x)) \cdot \frac{x - y}{|x - y|}|^q |x - y|^{d + sq}}{|x - y|^{d + sq}} \varphi(y) \, dy \, dx$$

$$\leq C \int_{B_1(x_0)} \int_{B_1(x_0)} \frac{|u(y) - u(x)| \cdot \frac{x - y}{|x - y|}|^q}{|x - y|^{d + sq}} \varphi(y) \, dy \, dx$$

$$+ C \int_{B_1(x_0)} \int_{B_1(x_0)} \frac{|\varphi(x) - \varphi(y)|^q |u(x)|^q}{|x - y|^{d + sq}} \, dy \, dx$$

$$+ C \int_{B_1(x_0)} \int_{\mathbb{C}B_{\frac{1}{4}}(x_0)} \frac{\varphi(y)|u(x)|^q}{|x - y|^{d + sq}} \, dy \, dx$$

$$:= I_1 + I_2 + I_3.$$  

Clearly, $I_1 \leq [u]_{X_q^s(B_1(x_0))}^q$. To estimate $I_2$, we use the mean value theorem to get

$$I_2 \leq C \|\nabla \varphi\|_{L^\infty} \int_{B_1(x_0)} \int_{B_1(x_0)} \frac{|u(x)|^q}{|x - y|^{d + sq(1 - s)}} \, dy \, dx \leq C\|u\|_{L^q(B_1(x_0))}^q.$$
Lastly, since supp $\varphi \subset B_2(x_0)$ we get
\[
I_3 \leq \int_{B_2(x_0)} \int_{\mathbb{R}^d} \frac{|u(x)|^q}{|x - y|^{d + tq}} \, dy \, dx
\leq \int_{B_1(x_0)} |u(x)|^q \int_{|z| \geq \frac{1}{4}} \frac{1}{|z|^{d + sq}} \, dz \, dx \leq C \|u\|_{L^q(B_1(x_0))}^q.
\]

Collecting the estimates we obtain (3.2) for $R = 1$.

As a corollary we have the following.

\textbf{Corollary 3.4.} Let $0 < s < 1$, and $0 < t < s$. Then there exists a constant $C = C(d, s, t)$ such that if $x_0 \in \mathbb{R}^d$ and $R > 0$, then for any $p > \frac{d}{d - t}$
\[
\|u\|_{L^p(B_{\frac{3}{2}}(x_0))} \leq C \left( R^{s - t} \|D^s(u)\|_{L^q(B_R(x_0))} + R^t \|u\|_{L^q(B_R(x_0))} \right),
\]
where $q = \frac{pd}{d + pt}$.

\textbf{Proof.} The case $t = s$ is precisely Lemma 3.3 for the special choice of $q$. We assume from now on that $t < s$. Note that $tq < d$, and $q \in (1, p)$. Using Lemma 3.3,
\[
\|u\|_{L^p(B_{\frac{3}{2}}(x_0))} \leq C \left( |u|_{L^q(B_R(x_0))} + R^t \|u\|_{L^q(B_R(x_0))} \right).
\]

Using H"{o}lder’s inequality with exponents $\frac{p}{q}, \frac{p}{p - q}$, we obtain that for each $x \in \mathbb{R}^d$
\[
\frac{1}{|x - y|^{d + pt}} \leq \left( \int_{B_R(x_0)} \frac{|(u(x) - u(y)) \cdot \frac{x - y}{|x - y|}|^p}{|x - y|^{d + pq}} \, dy \right)^{\frac{p - q}{p}} \leq C \left( R^{s - t} \|D^s(u)(x)\|_q \right)^q.
\]

The proof of the corollary is complete by integrating with respect to $x$, taking the $q$th root, and combining with (3.4).

\textbf{3.2. Higher integrability when the horizon is infinite.} In this subsection we will prove Theorem 3.1 when the horizon is infinite. In this case we recall that the bilinear form we use is $\mathcal{E}$, and $c_\infty = 1$. Let us begin analyzing the system corresponding to infinite horizon. First the system is invariant under translation and scaling. By that we mean if $u \in W^{s, 2}(\mathbb{R}^d ; \mathbb{R}^d)$ is a weak solution to (2.1), then for any $x_0 \in \mathbb{R}^d$ and $R > 0$, the vector field $u_R(x) = \frac{u(Rx - x_0)}{R^{2s}}$ is a solution to (2.1) corresponding to the coefficient $A(Rx - x_0, Ry - x_0)$ and right hand side $F(Rx - x_0)$.
The coefficient \( A(Rx - x_0, Ry - x_0) \) has same lower and upper bounds as \( A(x, y) \). Note in particular that

\[
(3.5) \quad D^s(u_R)(x) = \frac{1}{R^{2s}}D^s(u)(Rx + x_0), \quad \text{and} \quad \mathcal{M}(|u_R|(x)) = \frac{1}{R^{2s}}\mathcal{M}(|u|(Rx + x_0)),
\]

where in the above \( \mathcal{M} \) denotes the Hardy-Littlewood maximal function.

Second, for \( u \in L^2(\mathbb{R}^d; \mathbb{R}^d) \), using the lower and upper bounded of \( A \), we see that the quantity \( E(u, u) \) is equivalent to the seminorm of \( [f]_{X_2^s} \). Using the fraction Korn’s inequality Theorem 3.2, we can conclude that \( E(u, u) < \infty \) if and only if \( u \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d) \).

We now start obtaining estimates for solutions of the nonlocal system that naturally will be derived from the equation. The first estimate is an energy estimate.

**Lemma 3.5.** Given \( 0 < \alpha_1 \leq \alpha_2 < \infty, 0 < s < 1 \) and \( d \geq 2 \), there exists a positive constant \( C = C(\alpha_1, \alpha_2, p, s, d) \) such that for any \( A(x, y) \) symmetric, measurable with bounds \( \alpha_1 \leq A(x, y) \leq \alpha_2 \), \( x_0 \in \mathbb{R}^d, R > 0 \) and any weak solution \( u \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d) \) in the sense of (2.1) corresponding to \( F \in L^{2s, +h(x)}(\mathbb{R}^d; \mathbb{R}^d) \cap W^{-s,2}(\mathbb{R}^d; \mathbb{R}^d) \) we have

\[
(3.6) \quad \int_{B_R(x_0)} |D^s(u)(x)|^2 \, dx \leq C \int_{\mathbb{R}^d} |u(y)|^2 G_R(y) \, dy + \int_{B_R(x_0)} |F(y) \cdot u(y)| \, dy,
\]

where

\[
G_R(y) = \min \left\{ \frac{1}{R^{2s}}, \frac{R^d}{|y - x_0|^{d+2s}} \right\}.
\]

**Proof.** The proof of the lemma can be done essentially in the same way as in the proof of the Caccioppoli inequality for scalar nonlocal equations proved in [4, Theorem 3.1]. We will give a sketch of the proof highlighting the differences. Define a cutoff function \( \varphi : \mathbb{R}^d \to [0, 1] \) such that

\[
\varphi \equiv 1 \text{ on } B_{\frac{R}{2}}(x_0), \quad \varphi \equiv 0 \text{ on } \mathbb{R}^d \setminus B_R(x_0), \quad |\varphi(x) - \varphi(y)| \leq C \frac{|x - y|}{R}.
\]

Define \( v(x) := \varphi^2(x) u(x) \). Then \( v \in W^{s,2}(\mathbb{R}^d; \mathbb{R}^d) \). Using this as a test function in (2.1) we have

\[
I_1 - I_2 := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(u(y) - u(x)) \cdot \frac{y - x}{|x - y|}|^2 \varphi^2(x) \frac{A(x, y)}{|x - y|^{d+2s}} \, dy \, dx \\
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( (u(y) - u(x)) \cdot \frac{y - x}{|x - y|} \right) (\varphi(y) - \varphi(x))(\varphi(y) + \varphi(x)) \frac{u(y) \cdot (y - x)}{|y - x|} \frac{A(x, y)}{|x - y|^{d+2s}} \, dy \, dx \\
= - \int_{\mathbb{R}^d} F \cdot v \, dy.
\]

Now we estimate \( I_1 \) from below and \( I_2 \) from above. For \( I_1 \) we have the bound

\[
I_1 \geq \alpha_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(u(x) - u(y)) \cdot \frac{x - y}{|x - y|}|^2 \varphi^2(x)}{|x - y|^{d+2s}} \, dy \, dx.
\]

Now for \( I_2 \): Let \( \varepsilon > 0 \). Using Young’s inequality \( ab \leq \varepsilon a^2 + C_\varepsilon b^2 \) and symmetry we
have
\[ I_2 \leq 4\Lambda \varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (u(x) - u(y)) \cdot \frac{x - y}{|x - y|} \right|^2 \frac{\varphi^2(x)}{|x - y|^{d+2s}} \, dy \, dx + \alpha_2 C_\varepsilon \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(x) - \varphi(y)|^2 \frac{|u(y)|^2}{|x - y|^{d+2s}} \, dy \, dx. \]

For \( \varepsilon \) small enough we can absorb the first term on the right hand side to obtain
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (u(x) - u(y)) \cdot \frac{x - y}{|x - y|} \right|^2 \frac{\varphi^2(x)}{|x - y|^{d+2s}} \, dy \, dx \leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(y) - \varphi(x)|^2 \frac{|u(y)|^2}{|x - y|^{d+2s}} \, dy \, dx + \int_{B_R} |F(y) \cdot u(y)| \, dy. \] (3.7)

One then estimates the first term on the left hand side in the exact same way as in the proof of [4, Theorem 3.1] to complete the proof.

**Proof of Theorem 3.1 when \( h = \infty \).** We prove the theorem in two steps.

**Step 1.** We use Gehring’s lemma to prove that there exists an exponent \( p \in (2, 2^*) \) and a constant \( C = C(\alpha_1, \alpha_2, d, s, p) \) such that if \( u \) satisfies (2.1) then for any \( x_0 \in \mathbb{R}^d \) and \( R > 0 \) we have
\[ \left( \int_{B_{R/2}(x_0)} |D^s(u)(x)|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_{2R}(x_0)} |D^s(u)(x)|^2 \, dx \right)^{\frac{1}{2}} + C \left( \int_{B_{2R}(x_0)} |M(|u|^2)(x)|^{\frac{p}{2}} \, dx \right)^{\frac{1}{2}} + C \left( \int_{B_{2R}(x_0)} |F(x)|^{\frac{p}{2}} |M(u(x)|^{\frac{p}{2}} \, dx \right)^{\frac{1}{2}}. \] (3.8)

To that end, we begin by assuming \( x_0 = 0 \) and \( R = 1 \). From Lemma 3.5 we have
\[ \int_{B_{\frac{1}{2}}} |D^s(u)(x)|^2 \, dx \leq C \int_{\mathbb{R}^d} |u(x)|^2 G_1(x) \, dx + C \int_{B_1} |F(x) \cdot u(x)| \, dx \]
\[ \leq C \int_{B_1} |u(x)|^2 \, dx + \int_{B_1} |u(x)|^2 G_1(x) \, dx + \int_{B_1} |F(x) \cdot u(x)| \, dx \]
\[ := J_1 + J_2 + J_3. \]

We proceed to bound \( J_1, J_2, \) and \( J_3 \). First, using Corollary 3.4, for any \( q \in (1, 2) \) we have
\[ J_1 = C \int_{B_1} |u(x)|^2 \, dx \leq C \left( \int_{B_{2}} |D^s(u)(x)|^q \, dx \right)^{\frac{2}{q}} + C \left( \int_{B_{2}} |u(x)|^q \, dx \right)^{\frac{2}{q}} \]
\[ \leq C \left( \int_{B_{2}} |D^s(u)(x)|^q \, dx \right)^{\frac{2}{q}} + C \int_{B_{2}} |u(x)|^2 \, dx \]
\[ \leq C \left( \int_{B_{2}} |D^s(u)(x)|^q \, dx \right)^{\frac{2}{q}} + C \int_{B_{2}} M(|u|^2)(x) \, dx \]
where in the second inequality we have applied Hölder’s inequality and used the fact that for almost every \( x \in \mathbb{R}^d, |u(x)|^2 \leq M(|u|^2)(x) \). Second, noting that if \( y \in B_1 \) and \( x \in \mathbb{C}B_1 \), then \( |x - y| < 2|x| \), we have

\[
J_2 = \int_{\mathbb{C}B_R} |u(x)|^2 G(x) \, dx \leq C \int_{\mathbb{C}B_R} |u(x)|^2 \left( \min \left\{ 1, \frac{1}{|x - y|^{d+2s}} \right\} \right) \, dx
\]

\[
\leq C \left( |u|^2 \ast G_1 \right)(y),
\]

where we recall \( G_1(x) = \min \left\{ 1, \frac{1}{|x|^{d+2s}} \right\} \). The quantity \( J_2 \) can be further estimated using [27, Chapter 3, Section 2.2, Theorem 2] to give the bound

\[
J_2 \leq CM(|u|^2)(y)
\]

for any \( y \in B_1 \). Lastly, we have

\[
J_3 \leq C \int_{B_1} |F(x)||u(x)| \, dx \leq C \int_{B_1} |F(x)||M(u)(x)| \, dx.
\]

Combining the bounds for \( J_1, J_2 \) and \( J_3 \), we have that when \( x_0 = 0, R = 1 \) for any \( y \in B_1 \),

\[
\int_{B_{\frac{1}{2}}} |D^s(u)(x)|^2 \, dx \leq C \left( \int_{B_2} |D^s(u)(x)|^q \, dx \right)^{\frac{2}{q}} + C \int_{B_2} M(|u|^2)(x) \, dx
\]

\[
+ C \int_{B_1} |F(x)||M(|u|)(x)| \, dx.
\]

Integrating (3.9) with respect to \( y \) over \( B_2 \) and enlarging the regions of integration, we obtain

\[
\int_{B_{\frac{1}{2}}} |D^s(u)(x)|^2 \, dx \leq C \left( \int_{B_2} |D^s(u)(x)|^q \, dx \right)^{\frac{2}{q}} + C \int_{B_2} M(|u|^2)(x) \, dx
\]

\[
+ C \int_{B_2} |F(x)||M(|u|)(x)| \, dx.
\]

Now, define

\[
g(x) := |D^s(u)(x)|^q
\]

and

\[
f(x) := \left( M(|u|^2)(x) + |F(x)||M(|u|)(x) \right)^{q/2}.
\]

Then, for \( x_0 = 0 \) and \( R = 1 \) we rewrite (3.10) as

\[
\int_{B_{\frac{1}{2}}(x_0)} g^{\frac{2}{q}}(x) \, dx \leq C \left( \int_{B_{2R}(x_0)} g(x) \, dx \right)^{\frac{2}{q}} + C \int_{B_{2R}} f^{\frac{2}{q}}(x) \, dx.
\]

By a scaling and translation argument and recalling the discussion at the beginning of the subsection, we have that (3.11) in fact holds for every \( x_0 \in \mathbb{R}^d \) and for every \( R > 0 \). Note that (3.11) is precisely the reverse Hölder’s inequality we needed to
We will estimate the last two terms of (3.12). The middle term is easy to control.

Choosing \( r \) such that \( qr < 2^* \), setting \( p = qr \), taking the \( q \)th roots and using the basic inequality \( (a + b)^{1/q} \leq a^{1/q} + b^{1/q} \) we arrive at the expression (3.8).

Step 2: In this step we finalize the proof starting from the inequality (3.8). We set \( R = 4\sqrt{d} \) in the above to obtain that for any \( x_0 \), inequality (3.8) reduces to

\[
\|D^s(u)\|_{L^p(B_{R/2}(x_0))} \leq C \|D^s(u)\|_{L^2(B_2R(x_0))} + C \|\mathcal{M}(|u|^2)\|_{L^{2}\left(\frac{B_2R(x_0)}{2}\right)}^{1/2} \\
+ C \|\mathcal{F}\mathcal{M}(|u|)\|_{L^{p/2}(B_{2R}(x_0))}.
\]

We keep this fixed value of \( R = 4\sqrt{d} \) for the rest of the proof and write the whole space \( \mathbb{R}^d \) as a union of balls centered at grip points \( k \in \mathbb{Z}^d \) and radius \( R \) whose enlargement has finite overlap. To that end, following [4, Page 39] we introduce \( \omega_k = B_{R}(k) \), and \( \Omega_k = B_{2R}(k) \). Then there exists \( N = N(d) \in \mathbb{N} \) such that \( \mathbb{R}^d \subset \cup_{k \in \mathbb{Z}^d} \omega_k \) and \( \sum_{k \in \mathbb{Z}^d} \chi_{\Omega_k} \leq N \). Then we get

\[
\int_{\mathbb{R}^d} |D^s(u)|^p \, dx \leq \sum_{k \in \mathbb{Z}^d} \int_{\omega_k} |D^s(u)|^p \, dx \\
\leq C \sum_{k \in \mathbb{Z}^d} \left( \int_{\omega_k} |D^s(u)|^2 \, dx \right)^{\frac{p}{2}} + C \sum_{k \in \mathbb{Z}^d} \int_{\omega_k} |\mathcal{M}(|u|^2)|^{\frac{p}{2}} \, dx \\
+ C \sum_{k \in \mathbb{Z}^d} \int_{\Omega_k} \left( |\mathcal{F}| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx.
\]

Since \( p > 2 \), using the inequality \( \sum_{k} a_k^{p/2} \leq \left( \sum_{k} a_k \right)^{p/2} \), and the finite overlap property of the covering \( \{\Omega_k\} \), we obtain that

\[
(3.12) \quad \int_{\mathbb{R}^d} |D^s(u)|^p \, dx \leq C \left( \int_{\mathbb{R}^d} |D^s(u)|^2 \, dx \right)^{\frac{p}{2}} + C \int_{\mathbb{R}^d} |\mathcal{M}(|u|^2)|^{\frac{p}{2}} \, dx \\
+ C \int_{\mathbb{R}^d} \left( |\mathcal{F}| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx.
\]

We will estimate the last two terms of (3.12). The middle term is easy to control using the \( L^p \) boundedness of the maximal operator and so we have

\[
\int_{\mathbb{R}^d} |\mathcal{M}(|u|^2)|^{p/2} \, dx \leq C \int_{\mathbb{R}^d} (|u|^2)^{\frac{p}{2}} \, dx = C \|u\|_{L^p(\mathbb{R}^d)}^p.
\]
To estimate the last term of (3.12) we recall that the data \( F \in L^{2s, \delta_0}(\mathbb{R}^d; \mathbb{R}) \cap W^{-s,2}(\mathbb{R}^d; \mathbb{R}) \). Now we use the exponents \( \frac{2(2s + \delta_0)}{p} \), \( \nu := \frac{2(2s + \delta_0)}{2(2s + \delta_0) - p} \) to apply Hölder’s inequality and obtain that

\[
\int_{\mathbb{R}^d} (|F| \mathcal{M}(|u|))^{p/2} \, dx \leq \left( \int_{\mathbb{R}^d} |F|^{2s+\delta_0} \, dx \right)^{\frac{p}{2s+\delta_0}} \left( \int_{\mathbb{R}^d} \mathcal{M}(|u|)^{\frac{p}{2}} \, dx \right)^{\frac{\nu}{2}}.
\]

Applying Young’s inequality as well as the strong \((p, p)\) property of the maximal operator we obtain that

\[
(3.13) \quad \int_{\mathbb{R}^d} (|F| \mathcal{M}(|u|))^{p/2} \, dx \leq C \|F\|_{L^{2s, \delta_0}}^p + C \|u\|_{L^{2s+\delta}}^p.
\]

Using these bounds and taking the \( p^{th} \) power on both sides we obtain from (3.12) that

\[
\left( \int_{\mathbb{R}^d} |D^r(u)|^p \, dx \right)^{1/p} \leq C \left[ \left( \int_{\mathbb{R}^d} |D^r(u)|^2 \, dx \right)^{\frac{p}{2}} + \int_{\mathbb{R}^d} |u|^p \, dx + \|F\|_{L^{2s, \delta_0}}^p + \|u\|_{L^{2s+\delta}}^p \right]^{1/p} \leq C \|D^r(u)\|_{L^2} + \|u\|_{L^p} + \|F\|_{L^{2s, \delta_0}} + \|u\|_{L^{2s+\delta}}.
\]

It is clear that \( \|D^r(u)\|_{L^2} \leq \|u\|_{W^{2,r}(\mathbb{R}^d)} \). To complete the proof of the theorem we will need to bound the \( L^p \) and \( L^p \) norms of \( u \). A simple calculation shows that we choose \( p \) sufficiently close to 2, depending on \( \delta_0 \), such that \( \frac{p\delta}{2} \leq 2^* = \frac{2d}{d-2s} \). Now by Sobolev embedding

\[
\|u\|_{L^{2^*}(\mathbb{R}^d)} \leq C \|u\|_{W^{s,2}(\mathbb{R}^d)}.
\]

By interpolation if \( 2 \leq r \leq 2^* \), then there exists \( t \in [0, 1] \) depending only on \( r \) and \( 2^* \) such that \( \|u\|_{L^r} \leq \|u\|_{L^{2^*}}^{1-t} \|u\|_{L^2}^t \). Combined with the inequality \( a^\delta b^{1-t} \leq a + b \) we get

\[
\|u\|_{L^r(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^{2^*}(\mathbb{R}^d)}.
\]

Therefore, for \( r \in [2, 2^*] \), we have

\[
\|u\|_{L^r(\mathbb{R}^d)} \leq C \|u\|_{W^{s,2}(\mathbb{R}^d)}.
\]

In particular by our choice of \( p \), the above inequality holds for \( r = p \) as well as \( r = \frac{p\delta}{2} \). That completes the proof.

### 3.3. Higher integrability when the horizon is finite.

We now address the remaining case of the proof of Theorem 3.1 where the horizon \( 0 < \delta < \infty \), and \( c_\delta \in (0, \infty) \). We begin with the following simple observation relating the bilinear forms \( \mathcal{E}_h \) and \( \mathcal{E} \).

**Proposition 3.6.** There exists a bounded linear operator \( \mathcal{P}_h : L^2 \rightarrow L^2 \) such that for any \( u, v \in W^{s,2}(\mathbb{R}^d; \mathbb{R}) \), we have

\[
\mathcal{E}_h(u, v) = c_h \mathcal{E}(u, v) + \int_{\mathbb{R}^d} \langle \mathcal{P}_h u(x), v(x) \rangle \, dx.
\]

**Proof.** We begin by writing that

\[
\mathcal{E}_h(u, v) = c_h \mathcal{E}(u, v) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x, y) (u(x) - u(y)) \cdot \left( \frac{x - y}{|x - y|} \right) \cdot \left( \frac{x - y}{|x - y|} \right) \, dx \, dy.
\]
where \( k(x, y) = c_b(1 - \chi_{B_b(0)}(|x - y|)) \frac{A(x, y)}{|x - y|^{d + 2s}} \) is symmetric. Notice also that \( k \) is bounded and has no singularity along the diagonal \( x = y \). Moreover, it decays fast enough at infinity that for a fixed \( x \), the function \( k(x, y) \) is integrable in \( y \). We may then apply Fubini’s theorem, iterate the integrals and get

\[
\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, y)(u(x) - u(y)) \cdot \frac{(x - y)}{|x - y|} (v(x) - v(y)) \cdot \frac{(x - y)}{|x - y|} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^d} \left( -\int_{\mathbb{R}^d} k(x, y) \frac{(x - y) \otimes (x - y)}{|x - y|^2} (u(x) - u(y)) \, dy \right) v(x) \, dx.
\]

We now define the operator \( P_h : L^2 \to L^2 \) as

\[
P_h u(x) = -\int_{\mathbb{R}^d} k(x, y) \frac{(x - y) \otimes (x - y)}{|x - y|^2} (u(x) - u(y)) \, dy, \quad x \in \mathbb{R}^d.
\]

It is clear that the operator is linear. To show that it is bounded, we notice that

\[
P_h u(x) = \int_{\mathbb{R}^d} k(x, y) \frac{(x - y) \otimes (x - y)}{|x - y|^2} u(y) \, dy - \mathbb{K}(x) u(x)
\]

where easy estimates show that \( \mathbb{K}(x) = \int_{\mathbb{R}^d} k(x, y) \frac{(x - y) \otimes (x - y)}{|x - y|^2} \, dy \) is a uniformly bounded matrix valued function. The first term, on the other hand, is a convolution type operator and its magnitude is bounded from above by the function

\[
\alpha_2 \gamma * |u(x)|, \quad \text{where} \quad \gamma(\xi) = c_b(1 - \chi_{B_b(0)}(|\xi|)) \frac{1}{|\xi|^{d+2s}} \in L^1(\mathbb{R}^d).
\]

The boundedness of the operator \( P_h \) on \( L^2 \) now follows from Young’s inequality.

**Remark 3.7.** It is now an easy corollary of the above proposition to state that for \( u \in L^2(\mathbb{R}^d; \mathbb{R}^d), \mathbb{E}_h(u, u) < \infty \) if and only if \( u \in W^{s, 2}(\mathbb{R}^d; \mathbb{R}^d) \).

**Proof of Theorem 3.1 when \( 0 < h < \infty \).** We begin with a solution \( u \) to the system of nonlocal equations in the sense of (2.1) corresponding to \( F \in L^{2, \gamma + \theta_0}(\mathbb{R}^d, \mathbb{R}^d) \). Using Proposition 3.6, we can conclude that \( u \) satisfies

\[
\mathbb{E}(u, v) = \frac{1}{c_h} \left( \mathbb{E}_h(u, v) - \int_{\mathbb{R}^d} \langle P_h u(x), v(x) \rangle \, dx \right) = \langle F_h, v \rangle
\]

for any \( v \in W^{s, 2}(\mathbb{R}^d, \mathbb{R}^d) \), where we have defined \( F_h = \frac{1}{c_h} (F + P_h u) \). That is \( u \) solves a nonlocal system corresponding to infinite horizon in the sense of (2.1) with a modified data \( F_h \) that belongs to the intersection of the sum space \( L^{2, \gamma + \theta_0}(\mathbb{R}^d, \mathbb{R}^d) \) and the dual space \( W^{-s, 2}(\mathbb{R}^d, \mathbb{R}^d) \). As a consequence, we can apply almost all estimates we obtain in the previous subsection up to (3.12) to choose an exponent \( p > 2 \) but close to 2 and a constant \( C > 0 \) such that

\[
\int_{\mathbb{R}^d} |D^s(u)|^p \, dx \leq C \left( \int_{\mathbb{R}^d} |D^s(u)|^2 \, dx \right)^{p/2} + C \int_{\mathbb{R}^d} |u|^p \, dx
\]

\[
+ C \int_{\mathbb{R}^d} \left( \mathbb{F}_h |\mathbb{M}(|u|)) \right)^{p/2} \, dx.
\]
The only term that is new is the third term, and we estimate it now.

\[ \int_{\mathbb{R}^d} \left( |\mathcal{P}_h u| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx \leq C \left( \int_{\mathbb{R}^d} \left( |\mathcal{F}_h u| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx + \int_{\mathbb{R}^d} \left( |\mathcal{P}_h u| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx \right) \]

The second term in the above can be estimated in exactly the same was as (3.13). We apply Hölder’s inequality with the exponent \( \frac{4}{p} \) and \( \frac{4}{4-p} \), and Proposition 3.6 to estimate the last term as

\[
\int_{\mathbb{R}^d} \left( |\mathcal{P}_h u| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx \leq \left( \int_{\mathbb{R}^d} |\mathcal{P}_h u|^2 \, dx \right)^{4/p} \left( \int_{\mathbb{R}^d} \mathcal{M}(|u|)^{2p} \, dx \right)^{4/p} \left( \int_{\mathbb{R}^d} \mathcal{M}(|u|)^{2p} \, dx \right)^{4/p} \]

After noticing that \( \frac{2p}{4-p} > 1 \), we can apply the boundedness of the maximal function to obtain

\[
\int_{\mathbb{R}^d} \left( |\mathcal{P}_h u| \mathcal{M}(|u|) \right)^{\frac{p}{2}} \, dx \leq C \left( \|u\|_{L^2}^p + \|u\|_{L^{\frac{2p}{4-p}}}^p + \|\mathcal{F}_h u\|_{L^{\frac{2p}{4-p}}}^p \right)
\]

Combining all the above we obtain that

\[
\int_{\mathbb{R}^d} |D^s(u)|^p \, dx \leq C \|D^s(u)\|_{L^2}^p + \|u\|_{L^p}^p + C\|\mathcal{F}(u)\|_{L^{(2s_0+4)}}^p + C\|u\|_{L^{(2s_0+4)}}^p + \|u\|_{L^p}^p + \|u\|_{L^{\frac{2p}{4-p}}}^p
\]

Notice that since \( p > 2 \), \( \frac{2p}{4-p} \geq p \) and therefore as before we use Sobolev embedding and interpolation to estimate all the terms in the right hand side of the above inequality only in terms of \( \|u\|_{W^{s_0,2}} \) and \( \|\mathcal{F}(u)\|_{L^{(2s_0+4)}} \) for some \( p > 2 \) but sufficiently close to 2. That completes the proof.

4. Characterization of Potential Spaces. The main objective of this section is to prove Theorem 2.3. Our proof of Theorem 2.3 follows the steps presented in the proof of [26, Theorem 1]. We first develop necessary technical tools that allows us to relate the Marcinkiewicz-type integral \( D^s(f) \) with the potential function of \( f \).

4.1. Poisson-type kernel and integral. We recall the standard Poisson kernel \( p_t(y) \) and introduce the modified Poisson-type kernel \( \mathcal{P}_t(y) \) given by their Fourier transforms, respectively,

\[
\hat{p}_t(\xi) = e^{-2\pi|\xi|t}, \quad \hat{\mathcal{P}}_t(\xi) = e^{-2\pi|\xi|t} \left( I_{d+1} + (2\pi|\xi|t) \begin{bmatrix} -\xi \otimes \xi & -1 & \xi \\ \frac{|\xi|^2}{2} & 1 \\ -1 & 1 \end{bmatrix} \right).
\]

Notice that \( \mathcal{P}_t(y) \) is a \((d+1) \times (d+1)\) matrix of functions which is explicitly given by the formula, see [22]

\[
\mathcal{P}_t(x) = \frac{2(d+1)}{\omega_d} \left( \frac{t}{|x|^2 + t^2} \right)^{\frac{d+1}{2}} \begin{bmatrix} x \otimes x & tx \\ tx & t^2 \end{bmatrix}
\]

where \( x \) is considered both a column and row \( d \)-vector. Several properties of the matrix kernel \( \mathcal{P}_t \) are given in [22]. We list now the properties that we need. First, the
matrix kernel $\mathbb{P}_t$ is in fact an approximation to the identity. For any $t > 0$, if $I_{d+1}$ denotes the $(d + 1) \times (d + 1)$ identity matrix, then

\begin{equation}
\int_{\mathbb{R}^d} \mathbb{P}_t(x) \, dx = \int_{\mathbb{R}^d} \mathbb{P}(x) \, dx = I_{d+1}.
\end{equation}

Moreover, for each $j, k$, and $\ell \in \{1, \ldots, d, d+1\}$ and for every $t > 0$ we have that $\partial_j \mathbb{P}_t^{jk}(x) \in L^1(\mathbb{R}^d)$ and $\partial_\ell \mathbb{P}_t^{jk}(x) \in L^1(\mathbb{R}^d)$. We also have the following point wise estimates: there exists a constant $c = c(d) > 0$ such that for any $j, k = 1, 2, \ldots, d + 1$,

$$|\partial_j \mathbb{P}_t^{jk}(x)| \leq c |x|^{-d-1}, \quad |\partial_\ell \mathbb{P}_t^{jk}(x)| \leq c t^{-d-1}, \quad \forall x \in \mathbb{R}^d, \quad t > 0.$$ 

In addition, one can easily establish $|\partial_t \mathbb{P}_t(x)| \leq \frac{C}{t^{d+2}}$ and $|\partial_t \mathbb{P}_t(x)| \leq \frac{C}{|x|^{d+2}}$.

Throughout, functions $f : \mathbb{R}^d \to \mathbb{R}^{d+1}$ are of the form $f = (f_1, f_2, \ldots, f_d, 0)$. We treat $f$ both as being a vector field in $\mathbb{R}^d$ and $\mathbb{R}^{d+1}$, as well as column and row vectors; it will be clear from context.

We recall the Poisson integral of $f$ given by $u^\wedge(x, t) := p_t \ast f(x)$, where the convolution is component wise. A Poisson-type integral of $f$ can now be naturally defined using the Poisson-type kernel $\mathbb{P}_t = (p_t^{ij})$ as

$$U(x, t) := \mathbb{P}_t \ast f(x).$$

The convolution in the above is taken in the sense of matrix multiplication. That is, the $i^{th}$ entry component of $U$ is given by $U_i = \sum_{j=1}^{d+1} p_t^{ij} \ast f_j$, where $\mathbb{P}_t = (p_t^{ij})$. Notice that taking the Fourier transform in $x$ transforms the convolution into the matrix multiplication

\begin{equation}
\hat{U}(\xi, t) = \hat{\mathbb{P}}_t(\xi) \hat{f}(\xi).
\end{equation}

The key connection of $\mathbb{P}_t$ with $D^s(f)$ is obtained through the following important relation that we will be using below. For any $z, x \in \mathbb{R}^d$, we have

$$\mathbb{P}_t(x) \left( \begin{array}{c} z \\ 0 \end{array} \right) = \mathbb{P}(x, t) \left( z \cdot \frac{x}{|x|} \right), \quad z \in \mathbb{R}^d,$n

where the vector function $\mathbb{P}(x, t)$ is given by

\begin{equation}
\mathbb{P}(x, t) := \frac{2(d + 1)}{\omega_d} \frac{t|x|}{(|x|^2 + t^2)^{d+1}} \left( \frac{x}{t} \right).
\end{equation}

In particular, $\mathbb{P}(x, t)$ and all its derivatives in $t$ satisfy the same estimates as $\mathbb{P}_t(x)$ and corresponding derivatives. Using this relation and (4.2), we see that

$$U(x, t) = f(x) + \int_{\mathbb{R}^d} \mathbb{P}_t(y)(f(x+y) - f(x)) \, dy = f(x) + \int_{\mathbb{R}^d} \mathbb{P}(y, t)(f(x+y) - f(x)) \frac{y}{|y|} \, dy.$$ 

4.2. Littlewood-Paley-type $g$-function. We can define the analogue of the classical Littlewood-Paley $g$-function corresponding to the new Poisson-type integral $U(x, t)$. The following definition is natural:

\begin{equation}
g_1(f)(x) := \left( \int_0^\infty t |\partial_t U(x, t)|^2 \, dt \right)^{\frac{1}{2}}.
\end{equation}
In the definition above $\nabla = (\nabla_x, \partial_t)$, and $|\nabla U(x,t)|^2 = |\partial_t U|^2 + \sum_{k=1}^{d+1} |\nabla_x U_k|^2$. We can use these functions to characterize the $L^p$ norm of a vector field. The following is a result similar to [27, Theorem 1, Chapter IV].

**Theorem 4.1.** Suppose that $1 < p < \infty$. Then there are constants $C_1, C_2$ such that for any $f \in L^p(\mathbb{R}^d; \mathbb{R}^d)$

$$C_1 \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \leq C_2 \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)}$$

To prove the theorem we follow the steps and the approach given in [27] for the proof of [27, Theorem 1, Chapter IV]. We first prove the theorem for $p = 2$.

**Lemma 4.2.** Let $f \in L^2(\mathbb{R}^d)$. Then $\hat{g}_1(f) \in L^2(\mathbb{R}^d)$ with

$$\|\hat{g}_1(f)\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{4} \int_{\mathbb{R}^d} \left| \hat{f_d} + \frac{\xi \otimes \xi}{|\xi|^2} \hat{f}(\xi) \right|^2 d\xi.$$  

The proof is tedious but elementary. It is given in the appendix. In the next proposition we prove one of the inequalities in Theorem 4.1.

**Proposition 4.3.** Let $1 < p < \infty$. If $f \in L^p(\mathbb{R}^d)$, then $\hat{g}_1(f) \in L^p(\mathbb{R}^d)$. Moreover, there exists a positive constant depending only on $p$ and $d$ such that for all $f \in L^p(\mathbb{R}^d)$,

$$\|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}.$$  

**Proof.** We use the theory of singular integrals for Hilbert space-valued functions outlined in [27, Chapter II, Section 5]. We use the notation of that section as well. Define the Hilbert space $\mathcal{H}$ to be the $L^2$ space on $(0, \infty)$ with the functions taking values in $\mathbb{R}^d$ with measure $t dt$, i.e.

$$\mathcal{H} := \left\{ h : (0, \infty) \to \mathbb{R}^d \mid \|h\|^2_{\mathcal{H}} := \int_0^\infty t |h(t)|^2 dt < \infty \right\}.$$  

The absolute value $|h|$ is the norm in $\mathbb{R}^d$. Denote the Banach space of bounded linear operators from $\mathbb{R}^d$ to $\mathcal{H}$ by $B(\mathbb{R}^d, \mathcal{H})$. Let $\varepsilon > 0$ be fixed for now. For each $x$ consider the matrix-valued function

$$K_\varepsilon(x, t) := \partial_t P_{t+\varepsilon}(x).$$  

Then for any fixed $x \in \mathbb{R}^d$, we identify the matrix function $K_\varepsilon(x, \cdot)$ by $K_\varepsilon(x)$. Now we show that $K_\varepsilon(x) \in B(\mathbb{R}^d, \mathcal{H})$. This is equivalent to showing that the integral $\int_0^\infty t |\partial_t P_{t+\varepsilon}(x)|^2 dt$ is finite. Indeed, from the formula (4.1) we have that $|\partial_t P_t(x)| \leq C \frac{(|x|^2 + t^2)^{t+1}}{2}$, and therefore for each $x \in \mathbb{R}^d$ we have the estimate after change of variables that

$$\|K_\varepsilon(x)\|^2_{B(\mathbb{R}^d, \mathcal{H})} = \sup_{|y| \leq 1} \|(K_\varepsilon(x), y)\|^2_{\mathcal{H}} \leq \sup_{|y| \leq 1} \int_0^\infty t |\partial_t P_{t+\varepsilon}(x)|^2 dt \leq C \int_0^\infty \frac{t}{(|x|^2 + (t+\varepsilon)^2)^{t+1}} dt \leq C\varepsilon.$$


and
\[ \|K_\varepsilon(x)\|^2_{B(\mathbb{R}^d, \mathcal{H})} \leq C \int_0^\infty \frac{t}{(\|x\|^2 + (t+\varepsilon)^2)^{d+1}} \, dt \leq \frac{1}{|x|^{2d}} \int_0^\infty \frac{t}{(1+t^2)^{d+1}} \, dt = \frac{C}{|x|^{2d}}. \]

From the above two estimates we also conclude that
\[ (4.8) \quad x \mapsto \|K_\varepsilon(x)\|_{B(\mathbb{R}^d, \mathcal{H})} \in L^2(\mathbb{R}^d). \]

Similarly, for \(1 \leq j \leq d\), again referring to (4.1) that
\[ \|\partial_j K_\varepsilon(x)\|^2_{B(\mathbb{R}^d, \mathcal{H})} \leq C \int_0^\infty \frac{t}{(\|x\|^2 + (t+\varepsilon)^2)^{d+2}} \, dt \leq C \int_0^\infty \frac{t}{(1+t^2)^{d+2}} \, dt = \frac{C}{|x|^{2d+2}}. \]

Thus,
\[ (4.9) \quad \|\partial_j K_\varepsilon(x)\|_{B(\mathbb{R}^d, \mathcal{H})} \leq \frac{C}{|x|^{d+1}}, \quad 1 \leq j \leq d. \]

Now define the operator
\[ T_\varepsilon(f)(x) = \int_{\mathbb{R}^d} K_\varepsilon(y)f(x-y) \, dy. \]

Notice that from the definition of \(K_\varepsilon(y)\), \(T_\varepsilon(f)(x)\) in terms of the Poisson-type kernel \(U\) as \(T_\varepsilon(f)(x) = \partial_t U(x, t + \varepsilon)\). It then follows that \(T_\varepsilon\) is a vector field since the integrand is a matrix multiplying a vector. In fact, \(T_\varepsilon(f)(x)\) take their values in \(\mathcal{H}\) for each \(x \in \mathbb{R}^d\). Moreover, we have
\[ \|T_\varepsilon(f)(x)\|^2_{\mathcal{H}} = \int_0^\infty t \|\partial_t U(x, t + \varepsilon)\|^2 \, dt \]
\[ = \int_0^\infty (t + \varepsilon) \|\partial_t U(x, t + \varepsilon)\|^2 \, dt - \int_0^\varepsilon \varepsilon \|\partial_t U(x, t + \varepsilon)\|^2 \, dt \]
\[ = \int_\varepsilon^\infty t \|\partial_t U(x, t)\|^2 \, dt - \varepsilon \int_\varepsilon^\infty |\partial_t U(x, t)|^2 \, dt \]
\[ \leq \int_0^\infty t \|\partial_t U(x, t)\|^2 \, dt = \left[ \hat{g}_1(f)(x) \right]^2. \]

Therefore, by the previous theorem, \(x \mapsto \|T_\varepsilon(f)(x)\|_{\mathcal{H}}\) is square integrable and
\[ \|T_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{2} \left( \int_{\mathbb{R}^d} \left| \mathbb{I}_d + \frac{\xi \otimes \xi}{|\xi|^2} \right| \hat{f}(\xi) \, d\xi \right)^{1/2} \leq \|f\|_{L^2(\mathbb{R}^d)}, \]
and thus, we obtain
\[ (4.10) \quad \|K_\varepsilon(x)\|_{B(\mathbb{R}^d, \mathcal{H})} \leq 1. \]

Now using (4.8), (4.9) and (4.10) we can use the theory of singular integrals [27, Chapter 2, Section 5] and conclude that
\[ \|T_\varepsilon f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty, \]
with \(C\) independent of \(\varepsilon\). Notice from the above calculations that for each \(x\), the positive function \(\|T_\varepsilon(f)(x)\|_{\mathcal{H}}\) increases to \(\hat{g}_1(f)(x)\) as \(\varepsilon \to 0\) and therefore, we have that
\[ \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty. \]
That completes the proof. \(\square\)
Next, we prove the reverse inequality by establishing a comparison of norms of operators with matrix symbols $I_d$ and $I_d + \frac{\xi \otimes \bar{\xi}}{|\xi|^2}$. We recall that for $1 \leq j \leq d$ and $f$ belonging to the class of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ the $j^{th}$ Riesz transform is defined as

$$R_j(f)(x) := \frac{2}{\omega_d} \text{P.V.} \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} f(x - y) \, dy.$$  

For any $f \in \mathcal{S}(\mathbb{R}^d)$ we have $\hat{R_j(f)}(\xi) = -i \frac{\xi_j}{|\xi|^2} \hat{f}(\xi)$, and $\|R_j f\|_{L^p(\mathbb{R}^d)} \leq C(p) \|f\|_{L^p(\mathbb{R}^d)}$ for $1 < p < \infty$.

**Lemma 4.4.** Let $f \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The translation-invariant operator $L(f)$ defined by

$$[Lf(x)]_k := f_k(x) - 3R_k \left[ \sum_{j=1}^{d} R_j f_j \right](x), \quad 1 \leq k \leq d,$$

satisfies

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \|L f\|_{L^p(\mathbb{R}^d)}.$$

**Proof.** Clearly by the $L^p$ boundedness of the Riesz transforms,

$$\|Lf\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)},$$

so the $L^p$ norm of $Lf$ is finite. Applying the $k^{th}$ Riesz transform to $(Lf)_k$ and summing gives

$$\sum_{k=1}^{d} R_k(Lf)_k = R_k f_k - 3R_k R_k \left( \sum_{j=1}^{d} R_j f_j \right) = -2 \sum_{k=1}^{d} R_k f_k.$$

Thus, by the $L^p$ boundedness of the Riesz transforms we have

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \|Lf\|_{L^p(\mathbb{R}^d)} \tag{4.11}$$

Now writing as

$$f_k(x) = f_k(x) - 3R_k \left( \sum_{j=1}^{d} R_j f_j \right)(x) + 3R_k \left( \sum_{j=1}^{d} R_j f_j \right)(x),$$

and so taking the $L^p$ norm on both sides and using the $L^p$ boundedness of the Riesz transforms gives

$$\|f_k\|_{L^p(\mathbb{R}^d)} \leq \left\| f_k - 3R_k \left( \sum_{j=1}^{d} R_j f_j \right) \right\|_{L^p(\mathbb{R}^d)} + \left\| 3R_k \left( \sum_{j=1}^{d} R_j f_j \right) \right\|_{L^p(\mathbb{R}^d)}$$

$$= \| (Lf)_k \|_{L^p(\mathbb{R}^d)} + \left\| 3R_k \left( \sum_{j=1}^{d} R_j f_j \right) \right\|_{L^p(\mathbb{R}^d)}$$

$$\leq C \| (Lf) \|_{L^p(\mathbb{R}^d)} + C \sum_{j=1}^{d} \left\| R_j f_j \right\|_{L^p(\mathbb{R}^d)} \leq C \| L f \|_{L^p(\mathbb{R}^d)} \quad \text{by (4.11)}.$$
Summing over $k$ finishes the proof. \hfill \Box

Remark 4.5. The symbol associated to $L$ is $\mathbb{I}_d + 3 \frac{\xi \otimes \xi}{|\xi|^2}$, which will appear in the proof of the converse inequalities for $\hat{g}_1$.

The next result proves the remaining inequality in Theorem 4.1.

**Proposition 4.6.** Let $1 < p < \infty$. Then there exists a positive constant $C$ such that for any $f \in L^p(\mathbb{R}^d; \mathbb{R}^d)$,

$$
\|f\|_{L^p(\mathbb{R}^d)} \leq C \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)}.
$$

**Proof.** Let $f_1, f_2$ be in $L^2(\mathbb{R}^d; \mathbb{R}^d)$ with respective Poisson-type integrals $U_1, U_2$. Polarization of the identity (4.6) leads to

$$(4.12) \int_0^\infty \int_{\mathbb{R}^d} t \langle \partial_t U_1(x, t), \partial_t U_2(x, t) \rangle dx dt = \frac{1}{4} \int_{\mathbb{R}^d} \left( (\mathbb{I}_d + \frac{\xi \otimes \xi}{|\xi|^2}) \hat{f}_1(\xi), (\mathbb{I}_d + \frac{\xi \otimes \xi}{|\xi|^2}) \hat{f}_2(\xi) \right) d\xi$$

$$= \frac{1}{4} \int_{\mathbb{R}^d} \left( (\mathbb{I}_d + 3 \frac{\xi \otimes \xi}{|\xi|^2}) \hat{f}_1(\xi), \hat{f}_2(\xi) \right) d\xi$$

$$= \frac{1}{4} \int_{\mathbb{R}^d} \langle LF_1(x), F_2(x) \rangle dx,$$

where the last inequality follows by Parseval’s relation. Now suppose in addition that $f_1 \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ and $f_2 \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ with $\|f_2\|_{L^p(\mathbb{R}^d)} \leq 1$. Then using the Cauchy-Schwarz inequality, Hölder’s inequality and Proposition 4.3 we have

$$\left| \int_{\mathbb{R}^d} \langle LF_1(x), f_2(x) \rangle dx \right| \leq 4 \int_{\mathbb{R}^d} \|\hat{g}_1(f_1)(x)\| \|\hat{g}_1(f_2)(x)\| dx$$

$$\leq 4 \|\hat{g}_1(f_1)\|_{L^p(\mathbb{R}^d)} \|\hat{g}_1(f_2)\|_{L^p(\mathbb{R}^d)} \leq C \|\hat{g}_1(f_1)\|_{L^p(\mathbb{R}^d)}.$$

Taking the supremum on both sides over all $f_2 \in L^2(\mathbb{R}^d; \mathbb{R}^d) \cap L^p(\mathbb{R}^d; \mathbb{R}^d)$ with $\|f_2\|_{L^p(\mathbb{R}^d)} \leq 1$ gives

$$(4.13) \|Lf\|_{L^p(\mathbb{R}^d)} \leq C \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)}$$

for every $f \in (L^2 \cap L^p)(\mathbb{R}^d)$. Using Lemma 4.4,

$$(4.14) \|f\|_{L^p(\mathbb{R}^d)} \leq C \|\hat{g}_1(f)\|_{L^p(\mathbb{R}^d)}$$

for every $f \in (L^2 \cap L^p)(\mathbb{R}^d)$. The passage to the general case $f \in L^p$ follows by density. Let $f_m$ be a sequence of functions in $(L^2 \cap L^p)(\mathbb{R}^d)$ which converge in $L^p$ to an arbitrary function $f \in L^p$. Then

$$|\hat{g}_1(f_m)(x) - \hat{g}_1(f)(x)|^2 = \left( \int_0^\infty t |\partial_t U_m(x, t)|^2 dt \right)^\frac{1}{2} - \left( \int_0^\infty t |\partial_t U(x, t)|^2 dt \right)^\frac{1}{2}$$

$$= \int_0^\infty t \left( |\partial_t U_m(x, t)|^2 + |\partial_t U(x, t)|^2 \right) dt$$

$$- 2 \left( \int_0^\infty t |\partial_t U_m(x, t)|^2 dt \right)^\frac{1}{2} \left( \int_0^\infty t |\partial_t U(x, t)|^2 dt \right)^\frac{1}{2}.$$
By Cauchy-Schwartz inequality the last expression cannot exceed
\[
\int_0^\infty t \left( |\partial_t U_m(x, t)|^2 + |\partial_t U(x, t)|^2 - 2 \langle \partial_t U_m(x, t), \partial_t U(x, t) \rangle \right) dt,
\]
and so
\[
|\hat{g}_1(f_m)(x) - \hat{g}_1(f)(x)|^2 \leq \int_0^\infty t \left| \partial_t (U_m - U)(x, t) \right|^2 dt = |\hat{g}_1(f_m - f)(x)|^2.
\]
Therefore by Theorem 4.3, \( \hat{g}_1(f_m) \) converges to \( \hat{g}_1(f) \) in \( L^p(\mathbb{R}^d) \), so we obtain (4.14) for a general \( f \in L^p(\mathbb{R}^d) \) and the proof is complete.

Now we come to the final preliminary inequality that must be established before proving our main result. Recall that the Riesz potential \( I^s \) and the Bessel potential \( J^s \) acting on a function \( f \) are given by
\[
(4.15) \quad I_s(f)(x) := c_{d,s} \int \frac{f(y)}{|x - y|^{d-2s}} dy,
\]
\[
(4.16) \quad J_s(f)(x) := (G_s \ast f)(x) = G_s * f, \quad G_s(x) := ((1 + 4\pi^2|\xi|^2)^{-s/2}) \hat{f}(\xi),
\]
where \( c_{d,s} \) is an appropriate normalizing constant.

**Theorem 4.7.** Let \( f \in L^p(\mathbb{R}^d) \), \( 1 < p < \infty \) and let \( 0 < s < 1 \). Denote \( f_s := I_s(f) \). Let \( U \) and \( U_s \) be the Poisson-type integrals of \( f, f_s \) respectively. Then for every \( x \in \mathbb{R}^d \) we have
\[
\hat{g}_1(f)(x) \leq C D^s(f_s)(x) .
\]

**Proof.** We first establish the equality
\[
(4.17) \quad \partial_t U(x, t) = \frac{-1}{\Gamma(1-s)} \int_0^\infty \partial_t U_s(x, t + r) r^{-s} dr
\]
where \( \Gamma \) denotes the Gamma function. This identity can be established using the Fourier transform and will be done in the appendix. We will use this to estimate \( \hat{g}_1(f) \) point wise. To that end, we write
\[
\hat{g}_1(f)(x) = \int_0^\infty t |\partial_t U(x, t)|^2 dt = \int_0^\infty t \left| \frac{-1}{\Gamma(1-s)} \int_0^t \partial_t U_s(x, r)(r-t)^{-s} dr \right|^2 dt.
\]
Dividing the intervals of integration in the inside integral, we see that
\[
\hat{g}_1(f)(x) \leq C \int_0^\infty t \left| \int_0^{2t} \partial_t U_s(x, r)(r-t)^{-s} dr \right|^2 dt
\]
\[
+ C \int_0^\infty t \left| \int_{2t}^\infty \partial_t U_s(x, r)(r-t)^{-s} dr \right|^2 dt
\]
\[
\leq C \int_0^\infty t \left( \int_0^{2t} |\partial_t U_s(x, r)|^2 (r-t)^{-s} dr \right) \left( \int_0^{2t} \frac{1}{(r-t)^s} dr \right) dt
\]
\[
+ C \int_0^\infty t \left( \int_{2t}^\infty |\partial_t U_s(x, r)|^2 (r-t)^{-s} dr \right) \left( \int_{2t}^\infty \frac{1}{(r-t)^{1+2s}} dr \right) dt,
\]
and so
\[
|\hat{g}_1(f)(x)| \leq C \int_0^\infty t \left| \partial_t U(x, t) \right|^2 dt.
\]
where we have used the Cauchy-Schwartz inequality in the last inequality. Simplification and interchanging of the integrals via Fubini implies that
\[
\hat{g}_t(f)(x) \leq C \int_0^\infty t^{2-s} \int_t^{2t} |\partial_t U_s(x, r)|^2 (r-t)^{-s} \, dr \, dt 
+ C \int_0^\infty t^{1-2s} \int_0^\infty |\partial_t U_s(x, r)|^2 (r-t) \, dr \, dt 
= C \int_0^\infty |\partial_t U_s(x, r)|^2 \left( \int_r^{r/2} t^{2-s} (r-t)^{-s} \, dt + \int_0^{r/2} t^{1-2s} (r-t) \, dt \right) \, dr 
\leq C \int_0^\infty r^{3-2s} |\partial_t U_s(x, r)|^2 \, dr = C \int_0^\infty t^{3-2s} |\partial_t U_s(x, t)|^2 \, dt.
\]

Notice also that using the fact that
\[
\hat{\partial}_t g_{tt} \leq C \int_0^\infty \hat{\partial}_t \hat{P}(x, t) \left( f_s(x+y) - f_s(x) \right) \cdot \frac{y}{|y|} \, dy.
\]

where we use the relation (4.4). By computation of \( \partial_t \hat{P}_t \) it is not difficult to show that
\[
|\partial_t U_s(x, t)| \leq \int_{\mathbb{R}^d} |\partial_t \hat{P}(y, t)| \left( f_s(x+y) - f_s(x) \right) \cdot \frac{y}{|y|} \, dy.
\]

We now divide the integration region in the right hand side and estimate using Hölder’s inequality to obtain
\[
|\partial_t U_s(x, t)|^2 \leq \left( \int_{|y| \leq t} |\partial_t \hat{P}(y, t)| \left( f_s(x+y) - f_s(x) \cdot \frac{y}{|y|} \, dy + \int_{t \leq |y|} \cdots \, dy \right)^2 
\leq \left( \int_{|y| \leq t} |\partial_t \hat{P}(y, t)| \, dy \right) \left( \int_{|y| \leq t} |\partial(f_s)(x, y)|^2 \, dy \right) 
+ \left( \int_{|y| > t} |\partial_t \hat{P}(y, t)| \, dy \right) \left( \int_{|y| > t} |\partial(f_s)(x, y)|^2 \, dy \right) \cdot \frac{|y|}{|y|}.
\]

where we introduced the notation \( \partial(f_s)(x, y) := f_s(x+y) - f_s(x) \cdot \frac{y}{|y|} \). We now use the estimates for \( |\partial_t \hat{P}(y, t)| \) to get
\[
|\partial_t U_s(x, t)|^2 \leq \left( \int_{|y| \leq t} \frac{C}{|y|^{d+2}} \, dy \right) \left( \int_{|y| \leq t} |\partial(f_s)(x, y)|^2 \, dy \right) 
+ \left( \int_{|y| > t} \frac{C}{|y|^{d+2}} \, dy \right) \left( \int_{|y| > t} |\partial(f_s)(x, y)|^2 \, dy \right) \cdot \frac{C}{t^2} 
\leq \frac{C}{t^2} \left( \int_{|y| \leq t} |\partial(f_s)(x, y)|^2 \, dy \right) 
+ \frac{C}{t^2} \left( \int_{|y| > t} |\partial(f_s)(x, y)|^2 \, dy \right) \cdot \frac{C}{t^2} \left( \int_{|y| > t} |\partial(f_s)(x, y)|^2 \, dy \right).
\]
As a consequence, combining the above with (10), and interchanging the integrals we have that
\[
\left[\mathcal{g}_1(f)(x)\right]^2 \leq C \int_0^\infty t^{3-2s} \left(\frac{1}{t^2}\right) \left(\int_{|y| \leq t} |\mathcal{P}(f)(x,y)|^2 \left|\partial_t \mathcal{P}(y,t)\right| \, dy + \int_{|y| > t} \ldots \, dy\right) \, dt
\]
\[
= C \int_{\mathbb{R}^d} |\mathcal{P}(f)(x,y)|^2 \left(\int_0^{|y|} t^{1-2s} \left|\partial_t \mathcal{P}(y,t)\right| \, dt + \int_0^\infty t^{1-2s} \left|\partial_t \mathcal{P}(y,t)\right| \, dt\right) \, dy
\]
\[
\leq C \int_{\mathbb{R}^d} |\mathcal{P}(f)(x,y)|^2 \left(\int_0^{|y|} t^{1-2s} \, dt + \int_0^\infty t^{-d-1-2s} \, dt\right) \, dy
\]
\[
= C \int_{\mathbb{R}^d} |\mathcal{P}(f)(x,y)|^2 \frac{1}{|y|^{d+2s}} \, dy = C \left[D^s(f)(x)\right]^2.
\]

The proof is complete. □

5. Conclusion. In this paper we have established a qualitative property of solutions to strongly coupled system of nonlocal equations that arise from the linearization of the bond-based peridynamic model. We have obtained a higher integrability potential space estimate for solutions corresponding to measurable and elliptic coefficients.
and possibly rough data. Our proof of the result follows classical arguments via reverse Hölder’s inequality and is inspired by the approach implemented for nonlocal equations in [4]. The regularity estimate, which should be considered as a nonlocal analogue of the celebrated inequality of Meyers that applies to system of PDEs with elliptic and measurable coefficients, is applicable to those nonlocal models with kernel that is locally comparable with fractional kernels of type $|x|^{-(d+2s)}$ and with bounded and unbounded support. In particular, it is applicable for peridynamic models with fractional kernels and finite horizon.

It is anticipated that the higher integrability result we obtained in the current work will be used to obtain estimates for solutions of nonlocal equations with coefficients that have large jump discontinuities as well as highly oscillatory feature. For example, there is an interest in such types of estimates for homogenization of coefficients that have large jump discontinuities as well as highly oscillatory feature. The higher integrability of a measure of smoothness of sequence of solutions guarantee compactness of the superperidynamic models with highly oscillatory coefficients. The higher integrability of a measure of smoothness of sequence of solutions guarantee compactness of the solutions in, say in $L^2$, and assist in establishing convergence rate of solutions to the homogenized solution. We hope to make a rigorous study of this in future work.

**Appendix A. Proof of Lemma 4.2.**

*Proof.* Using Fubini’s Theorem and Plancherel’s Theorem,

$$
\|\hat{g}_1(f)\|_{L^2(R^d)}^2 = \int_0^t \int_{R^d} |\hat{\partial}_t U(x, t)|^2 \, dx \, dt = \int_0^t \int_{R^d} |\hat{\partial}_t U(\xi, t)|^2 \, d\xi \, dt.
$$

By a direct computation, for $1 \leq k \leq d$,

$$
|\hat{\partial}_k \hat{U}_k(\xi, t)|^2 = |\hat{\partial}_k \left( e^{-2\pi|\xi|t} \hat{f}_k + (2\pi|\xi|t)e^{-2\pi|\xi|t} \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right) \left( -\frac{\xi_k}{|\xi|} \right) \right) |^2 
$$

$$
= 4\pi^2|\xi|^2e^{-4\pi|\xi|t} |A_k(\xi, t)|^2,
$$

where

$$
A_k(\xi, t) := \hat{f}_k(\xi) + \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right) \left( \frac{\xi_k}{|\xi|} \right) - (2\pi|\xi|t) \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right) \left( \frac{\xi_k}{|\xi|} \right).
$$

(Note that here we are using the modulus $| \cdot |$ for complex numbers.) Next,

$$
|\hat{\partial}_t \hat{U}_{d+1}(\xi, t)|^2 = \left| \hat{\partial}_t \left( (2\pi|\xi|t)e^{-2\pi|\xi|t} \left( -\frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right) \right) \right|^2 
$$

$$
= 4\pi^2|\xi|^2e^{-4\pi|\xi|t} \left| \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right|^2 (1 + 2\pi|\xi|t)^2,
$$

Expanding the $A_k$

$$
\sum_{k=1}^d |A_k(\xi, t)|^2 = |\hat{f}(\xi)|^2 + 3 \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right)^2 - (8\pi|\xi|t) \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right)^2 + (4\pi^2|\xi|^2t^2) \left( \frac{\xi}{|\xi|} \cdot \hat{f}(\xi) \right)^2.
$$
Now we prove (4.6). By the above formulas,
\[
\left| \partial_t \tilde{U}(\xi, t) \right|^2 \\
= 4\pi^2 |\xi|^2 e^{-4\pi|\xi||t|} \left( \sum_{k=1}^{d} |A_k(\xi, t)|^2 + (1 + 2\pi|\xi|t)^2 \left| \frac{\xi}{|\xi|} \cdot \tilde{f}(\xi) \right|^2 \right) \\
= 4\pi^2 |\xi|^2 e^{-4\pi|\xi||t|} \left( \tilde{f}(\xi) + 3 - 8\pi|\xi|t + 4\pi^2|\xi|^2t^2 + (1 + 2\pi|\xi|t)^2 \right) \left| \frac{\xi}{|\xi|} \cdot \tilde{f}(\xi) \right|^2 \\
= 4\pi^2 |\xi|^2 e^{-4\pi|\xi||t|} \left| \tilde{f}(\xi) \right|^2 + 4\pi^2 |\xi|^2 e^{-4\pi|\xi||t|} \left( 4 - 4\pi|\xi|t + 8\pi^2|\xi|^2t^2 \right) \left| \frac{\xi}{|\xi|} \cdot \tilde{f}(\xi) \right|^2 
\]

Thus,
\[
\int_{0}^{\infty} \int_{\mathbb{R}^d} |\partial_t \tilde{U}(\xi, t)|^2 d\xi dt = \int_{\mathbb{R}^d} \frac{1}{4} \left| \tilde{f}(\xi) \right|^2 + \left( \frac{1}{2} - \frac{1}{2} + \frac{3}{4} \right) \left| \frac{\xi}{|\xi|} \cdot \tilde{f}(\xi) \right|^2 d\xi \\
\quad = \frac{1}{4} \int_{\mathbb{R}^d} \left( \mathbb{I}_d + \frac{\xi \otimes \xi}{|\xi|^2} \right) \tilde{f}(\xi) \left| \frac{\xi}{|\xi|} \cdot \tilde{f}(\xi) \right|^2 d\xi.
\]

where we have used the formulas: by a change of variables \( \eta = 4\pi|\xi||t|, \)
\[
\int_{0}^{\infty} 8\pi^2|\xi|^2 t e^{-4\pi|\xi||t|} dt = \frac{1}{2} \int_{0}^{\infty} \eta e^{-\eta} d\eta = \frac{1}{2} \Gamma(1) = \frac{1}{2} , \\
\int_{0}^{\infty} 16\pi^2|\xi|^2 t e^{-4\pi|\xi||t|} dt = \frac{1}{2} \int_{0}^{\infty} \eta e^{-\eta} d\eta = 1 , \\
- \int_{0}^{\infty} 32\pi^3|\xi|^3 t^2 e^{-4\pi|\xi||t|} dt = - \frac{1}{2} \int_{0}^{\infty} \eta^2 e^{-\eta} d\eta = - \frac{\Gamma(2)}{2} = -1 , \\
\int_{0}^{\infty} 64\pi^4|\xi|^4 t^3 e^{-4\pi|\xi||t|} dt = \frac{1}{4} \int_{0}^{\infty} \eta^3 e^{-\eta} d\eta = \frac{\Gamma(3)}{4} = \frac{3}{2} .
\]

**Appendix B. Proof of (4.17)**. This can be done using the Fourier transform.

Denote
\[
A(\xi) := \begin{bmatrix}
\frac{\xi \otimes \xi}{|\xi|^2} & -i \frac{\xi}{|\xi|} \\
-i \frac{\xi}{|\xi|} & 1
\end{bmatrix}.
\]

Then
\[
\frac{-1}{\Gamma(1-s)} \int_{0}^{\infty} \partial_t \tilde{U}_r(\xi, t + r)^{-s} dr \\
\quad = \frac{-1}{\Gamma(1-s)} \int_{0}^{\infty} 4\pi^2|\xi|^2 e^{-2\pi|\xi||t|} \left( I_{d+1} - 2A(\xi) + 2\pi|\xi|tA(\xi) \right) (2\pi|\xi|)^{-s} \tilde{f}(\xi)^{-s} dr \\
\quad = \frac{-1}{\Gamma(1-s)} 2\pi|\xi| e^{-2\pi|\xi||t|} \left( \int_{0}^{\infty} (2\pi|\xi|)^{-s} e^{-2\pi|\xi||r|} (2\pi|\xi|) dr \right) \left( I_{d+1} - 2A(\xi) + 2\pi|\xi|tA(\xi) \right) \\
\quad + \frac{1}{\Gamma(1-s)} 2\pi|\xi| e^{-2\pi|\xi||t|} \left( \int_{0}^{\infty} (2\pi|\xi|)^{-s} e^{-2\pi|\xi||r|} (2\pi|\xi|) dr \right) A(\xi) \\
\quad = \frac{-1}{\Gamma(1-s)} 2\pi|\xi| e^{-2\pi|\xi||t|} \left( I_{d+1} - 2A(\xi) + 2\pi|\xi|tA(\xi) \right) \\
\quad + \frac{-1}{\Gamma(1-s)} 2\pi|\xi| e^{-2\pi|\xi||t|} \left( \frac{\Gamma(2-s)}{\Gamma(1-s)} A(\xi) \right) \\
\quad = ( -2\pi|\xi| e^{-2\pi|\xi||t|} \left( I_{d+1} - 2A(\xi) + (2\pi|\xi|t)A(\xi) + \frac{\Gamma(2-s)}{\Gamma(1-s)} A(\xi) \right) ) = \partial_t \tilde{U}(\xi, t) .
\]
In the last equality we used the identity $\Gamma(x + 1) = x\Gamma(x)$ for every $x > 0$. By a change of variables we have

$$
\partial_t U(x, t) = \frac{-1}{\Gamma(1 - s)} \int_t^\infty \partial_{tt} U_s(x, r)(r - t)^{-s} dr.
$$

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