POSITIVE SOLUTIONS
OF SEMIPOSITONE ELLIPTIC PROBLEMS
WITH CRITICAL TRUDINGER–MOSER NONLINEARITIES

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Abstract. We prove the existence of a positive solution to a semipositone $N$-Laplacian problem with a critical Trudinger–Moser nonlinearity. The proof is based on obtaining uniform $C^{1,\alpha}$ a priori estimates via a compactness argument. Our result is new even in the semilinear case $N = 2$, and our arguments can easily be adapted to obtain positive solutions of more general semipositone problems with critical Trudinger–Moser nonlinearities.

1. Introduction

Elliptic problems with critical Trudinger–Moser nonlinearities have been widely investigated in the literature. We refer the reader to the survey paper of de Figueiredo et al. [2] for an overview of recent results on Trudinger–Moser type inequalities and related critical problems. A model critical problem of this type is

$$
\begin{cases}
-\Delta_N u = \lambda |u|^{N-2}u e^{\beta |u|^{N'}} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

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where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\Delta_N u = \text{div} (|\nabla u|^{N-2} \nabla u)$ is the $N$-Laplacian of $u$, $N' = N/(N-1)$ and $\lambda, \beta > 0$. This problem is a natural analog of the Brézis–Nirenberg problem for the $p$-Laplacian in the borderline case $p = N$, where the critical growth is of exponential type and is governed by the Trudinger–Moser inequality

$$\sup_{u \in W^{1,N}_0(\Omega), \|u\| \leq 1} \int_{\Omega} e^{\alpha_N |u|^{N'}} \, dx < \infty.$$  

Here $W^{1,N}_0(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u|^N \, dx \right)^{1/N},$$

$\alpha_N = N \omega^{1/(N-1)}_N$ and $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$ (see Trudinger [11] and Moser [10]). A result of Adimurthi [1] gives a positive solution of this problem for $\lambda \in (0, \lambda_1)$, where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of $-\Delta_N$ in $\Omega$ (see also do Ó [5]). Theorem 1.4 in de Figueiredo et al. [3], [4] gives a nontrivial solution for $\lambda \geq \lambda_1$ in the semilinear case $N = 2$. More recently, Yang and Perera [13] obtained a nontrivial solution in the general quasilinear case $N \geq 3$ when $\lambda > \lambda_1$ is not an eigenvalue.

In the present paper we study the related semipositone problem

$$-\Delta_N u = \lambda u^{N-1} e^{\beta u^{N'}} - \mu \quad \text{in } \Omega,$$

$$u > 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\mu > 0$. Since $-\mu < 0$, $u = 0$ is not a subsolution of this problem, which makes finding a positive solution rather difficult (see Lions [8]). This compounds the usual difficulties arising from the lack of compactness associated with critical growth problems. Our main result here is that this problem has a weak positive solution for all sufficiently small $\mu$ when $\lambda < \lambda_1$.

**Theorem 1.1.** If $\lambda \in (0, \lambda_1)$, then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem (1.2) has a weak positive solution $u_\mu \in C^{1,\alpha}_0(\overline{\Omega})$ for some $\alpha \in (0,1)$.

This result seems to be new even in the semilinear case $N = 2$. The outline of the proof is as follows. We consider the modified problem

$$-\Delta_N u = \lambda f(u^+) - \mu g(u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $f(u) = u^+$, $g(u) = u^-$.
where \( f(t) = t^{N-1}e^{\beta t N} \) for \( t \geq 0 \), \( u^+(x) = \max\{u(x), 0\} \) and
\[
g(t) = \begin{cases} 
0 & \text{if } t \leq -1, \\
1 + t & \text{if } -1 < t < 0, \\
1 & \text{if } t \geq 0.
\end{cases}
\]

Weak solutions of this problem coincide with critical points of the \( C^1 \)-functional
\[
E_\mu(u) = \int_\Omega \left[ \frac{\nabla u}{N} - \lambda F(u^+) + \mu G(u) \right] dx, \quad u \in W^{1,N}_0(\Omega),
\]
where
\[
F(t) = \int_0^t f(s) \, ds, \quad t \geq 0, \quad G(t) = \int_0^t g(s) \, ds, \quad t \in \mathbb{R}.
\]
The functional \( E_\mu \) satisfies the (PS)_c condition for all \( c \neq 0 \) satisfying
\[
c < \frac{1}{N} \left( \frac{\alpha N}{\beta} \right)^{N-1} - \frac{\mu}{2} |\Omega|,
\]
where \( |\cdot| \) denotes the Lebesgue measure in \( \mathbb{R}^N \), and it follows from the mountain pass theorem that \( E_\mu \) has a uniformly positive critical level below this threshold for compactness for all sufficiently small \( \mu > 0 \) (see Lemmas 3.1 and 3.2). This part of the proof is more or less standard. The novelty of the paper lies in the fact that the solution \( u_\mu \) of the modified problem (1.3) thus obtained is positive, and hence solves our original problem (1.2), if \( \mu \) is further restricted. Note that this does not follow from standard arguments based on the maximum principle since the perturbation term \( -\mu < 0 \). This is precisely the main difficulty in finding positive solutions of semipositone problems as was pointed out in Lions [8].

We will prove that for every sequence \( \mu_j > 0, \mu_j \rightarrow 0 \), a subsequence of \( u_j = u_{\mu_j} \) is positive in \( \Omega \). The idea of the proof is to show that a subsequence of \( u_j \) converges in \( C^{1,\alpha}_0(\Omega) \) to a solution of the limit problem
\[
\begin{cases} 
-\Delta_N u = \lambda f(u^+) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]
This requires a uniform \( C^{1,\alpha}_0(\Omega) \) estimate of \( u_j \) for some \( \alpha \in (0, 1) \). It is well-known that each \( u_j \) belongs to \( C^{1,\alpha}_0(\Omega) \). However, proving that the sequence \( (u_j) \) remains bounded in \( C^{1,\alpha}_0(\Omega) \) is a nontrivial task in the critical case. We will obtain the required estimate by proving the following compactness result, which is of independent interest.

**Theorem 1.2.** If \( \mu_j > 0, \mu_j \rightarrow \mu \geq 0 \), \( (u_j) \subset W^{1,N}_0(\Omega) \), and
\[
E_{\mu_j}(u_j) \rightarrow c, \quad E'_{\mu_j}(u_j) \rightarrow 0
\]
for some \( c \neq 0 \) satisfying
\[
(1.4) \quad c < \frac{1}{N \left( \frac{\alpha N}{\beta} \right)^{N-1}} - \frac{\mu}{2} ||\Omega||,
\]
then a subsequence of \((u_j)\) converges to a critical point of \( E_\mu \) at the level \( c \).

This theorem implies that
\[
\sup_j \int_{\Omega} e^{b|u_j|^{N'}} \, dx < \infty
\]
for all \( b \) (see Lemma 3.4). This together with the Hölder inequality implies that
\( f(u_j^+) \) is bounded in \( L^s(\Omega) \) for all \( s > 1 \), so \((u_j)\) is bounded in \( L^{N}(\Omega) \) by Guedda and Véron [6, Proposition 1.3]. The global regularity result in Lieberman [7] then gives the desired \( C_{1,0}^{\lambda}(\Omega) \) estimate.

Theorem 1.2 is proved in Section 2 and Theorem 1.1 in Section 3. In closing the introduction we remark that we have confined ourselves to the model problem (1.2) only for the sake of simplicity. The arguments given in this paper can easily be adapted to obtain positive solutions of more general semipositone problems with critical Trudinger–Moser nonlinearities.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. First we collect some elementary estimates for easy reference.

**Lemma 2.1.** For all \( t \geq 0 \),
\( F(t) \)
(\( F(t) \)
(a) \( F(t) \leq \frac{N - 1}{\beta N} \frac{tf(t)}{(N/(N - 1))} \),
(b) \( F(t) \leq F(1) + \frac{N - 1}{N(N + \beta - 1)} tf(t) \),
(c) \( F(t) \leq \frac{1}{N} tf(t) \),
(d) \( F(t) \leq \frac{1}{N} t^N + \frac{\beta}{N} t^{N^2/(N - 1)} e^{\beta t^{N'}} \),
(e) \( F(t) \geq \frac{1}{N} t^N + \frac{\beta(N - 1)}{N^2} t^{N^2/(N - 1)} \).

**Proof.** (a) Integrating by parts,
\[
F(t) \leq \frac{N - 1}{\beta N} \frac{t^{N-N/(N-1)} e^{\beta t^{N'}}}{(N/(N - 1))} - \frac{N - 1}{\beta} \int_0^t s^{N-N/(N-1)} e^{\beta s^{N'}} \, ds
\]
\[
\leq \frac{N - 1}{\beta N} \frac{t^N e^{\beta t^{N'}}}{(N/(N - 1))} = \frac{N - 1}{\beta N} \frac{tf(t)}{(N/(N - 1))}.
\]
(b) For \( t \leq 1 \), \( F(t) \leq F(1) \). For \( t > 1 \),
\[
F(t) = F(1) + \int_1^t f(s) \, ds.
\]
Integrating by parts,
\[
\int_1^t f(s) \, ds \leq \frac{1}{N} t^N e^{\beta^N} - \frac{\beta}{N-1} \int_1^t s^{N-1+(N-1)/N} e^{\beta^N} \, ds
\]
and hence
\[
\int_1^t f(s) \, ds \leq \frac{N-1}{N(N+\beta-1)} tf(t).
\]
(c) Integrating by parts,
\[
F(t) = \frac{1}{N} t^N e^{\beta^N} - \frac{\beta}{N-1} \int_0^t s^{N+(N-1)/N-1} e^{\beta^N} \, ds \leq \frac{1}{N} tf(t).
\]
(d) Since \( e^t \leq 1 + te^t \) for all \( t \geq 0 \),
\[
F(t) \leq \int_0^t s^{N-1} \left(1 + \beta s^{N/(N-1)} e^{\beta^N}\right) ds \leq \frac{1}{N} t^N \left(1 + \beta t^{N/(N-1)} e^{\beta^N}\right).
\]
(e) Since \( e^t \geq 1 + t \) for all \( t \geq 0 \),
\[
F(t) \geq \int_0^t s^{N-1} \left(1 + \beta s^{N/(N-1)}\right) ds = \frac{1}{N} t^N + \frac{\beta(N-1)}{N^2} t^{N/(N-1)}.
\]
Next we prove the following lemma.

**Lemma 2.2.** If \( (u_j) \) is a sequence in \( W_0^{1,N}(\Omega) \) converging almost everywhere to \( u \in W_0^{1,N}(\Omega) \) and

\[
(2.1) \quad \sup_j \int_\Omega u_j^+ f(u_j^+) \, dx < \infty,
\]
then
\[
\int_\Omega F(u_j^+) \, dx \to \int_\Omega F(u^+) \, dx.
\]

**Proof.** For \( M > 0 \), write
\[
\int_\Omega F(u_j^+) \, dx = \int_{\{u_j^+ < M\}} F(u_j^+) \, dx + \int_{\{u_j^+ \geq M\}} F(u_j^+) \, dx.
\]
By Lemma 2.1 (a) and (2.1),
\[
\int_{\{u_j^+ \geq M\}} F(u_j^+) \, dx \leq \frac{N-1}{\beta N M^{N/(N-1)}} \int_\Omega u_j^+ f(u_j^+) \, dx = O\left(\frac{1}{M^{N/(N-1)}}\right)
\]
as \( M \to \infty \). Hence
\[
\int_\Omega F(u_j^+) \, dx = \int_{\{u_j^+ < M\}} F(u_j^+) \, dx + O\left(\frac{1}{M^{N/(N-1)}}\right),
\]
and the conclusion follows by first letting \( j \to \infty \) and then letting \( M \to \infty \). □

We will also need the following result of Lions [9] (see Remark I.18 (i)).
LEMMA 2.3. If \((u_j)\) is a sequence in \(W^{1,N}_0(\Omega)\) with \(\|u_j\| = 1\) for all \(j\) and converging almost everywhere to a nonzero function \(u \in W^{1,N}_0(\Omega)\), then

\[
\sup_j \int_\Omega e^{b|u_j|^{N'}} \, dx < \infty \quad \text{for all } b < \frac{\alpha_N}{(1 - \|u\|^{N})^{1/(N-1)}}.
\]

We are now ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. We have

\[
(2.2) \quad E_{\mu_j}(u_j) = \frac{1}{N} \|u_j\|^N - \lambda \int_\Omega F(u_j^+) \, dx + \mu_j \int_\Omega G(u_j) \, dx = c + o(1),
\]

\[
(2.3) \quad E'_{\mu_j}(u_j) u_j = \|u_j\|^N - \lambda \int_\Omega u_j^+ f(u_j^+) \, dx + \mu_j \int_\Omega u_j g(u_j) \, dx = o(\|u_j\|).
\]

Since

\[
\int_\Omega F(u_j^+) \, dx \leq F(1)|\Omega| + \frac{N - 1}{N(N + \beta - 1)} \int_\Omega u_j^+ f(u_j^+) \, dx
\]

by Lemma 2.1 (b), \((\mu_j)\) is bounded, and

\[
(2.4) \quad \left| \int_\Omega u_j g(u_j) \, dx \right| \leq \int_\Omega |u_j| \, dx, \quad \left| \int_\Omega G(u_j) \, dx \right| \leq \int_\Omega |u_j| \, dx,
\]

it follows from (2.2) and (2.3) that \((u_j)\) is bounded in \(W^{1,N}_0(\Omega)\). Hence a renamed subsequence converges to some \(u\) weakly in \(W^{1,N}_0(\Omega)\), strongly in \(L^p(\Omega)\) for all \(p \in [1, \infty)\), and almost everywhere in \(\Omega\). Moreover,

\[
(2.5) \quad \sup_j \int_\Omega u_j^+ f(u_j^+) \, dx < \infty
\]

by (2.3) and (2.4), and hence

\[
(2.6) \quad \int_\Omega F(u_j^+) \, dx \to \int_\Omega F(u^+) \, dx
\]

by Lemma 2.2. Clearly,

\[
(2.7) \quad \mu_j \int_\Omega u_j g(u_j) \, dx \to \mu \int_\Omega u g(u) \, dx, \quad \mu_j \int_\Omega G(u_j) \, dx \to \mu \int_\Omega G(u) \, dx.
\]

We claim that the weak limit \(u\) is nonzero. Suppose \(u = 0\). Then

\[
(2.8) \quad \int_\Omega F(u_j^+) \, dx \to 0, \quad \mu_j \int_\Omega u_j g(u_j) \, dx \to 0, \quad \mu_j \int_\Omega G(u_j) \, dx \to 0
\]

by (2.6) and (2.7), and hence \(c > 0\) and \(\|u_j\| \to (Nc)^{1/N}\) by (2.2).

Let \((Nc)^{1/(N-1)} < \gamma < \frac{\alpha_N}{\beta}\). Then \(\|u_j\| \leq \gamma^{(N-1)/N}\) for all \(j \geq j_0\) for some \(j_0\). Let \(q = \frac{\alpha_N}{\beta} \gamma > 1\). By the Hölder inequality,

\[
\int_\Omega u_j^+ f(u_j^+) \, dx \leq \left( \int_\Omega |u_j|^N \, dx \right)^{1/p} \left( \int_\Omega e^{q\beta|u_j|^{N'}} \, dx \right)^{1/q},
\]
where $1/p + 1/q = 1$. The first integral on the right-hand side converges to zero since $u = 0$, while the second integral is bounded for $j \geq j_0$ since $q\beta |u_j|^{N'} = \alpha_N |\tilde{u}_j|^{N'}$ with $\tilde{u}_j = u_j/\gamma^{(N-1)/N}$ satisfying $\|\tilde{u}_j\| \leq 1$, so
\[
\int_{\Omega} u_j^+ f(u_j^+) \, dx \to 0.
\]
Then $u_j \to 0$ by (2.3) and (2.8), and hence $c = 0$ by (2.2) and (2.8), a contradiction. So $u$ is nonzero.

Since $E'_{\mu_j}(u_j) \to 0$,
\[
\int_{\Omega} |\nabla u_j|^{N-2} \nabla u_j \cdot \nabla v \, dx - \lambda \int_{\Omega} f(u_j^+) v \, dx + \mu_j \int_{\Omega} g(u_j) v \, dx \to 0
\]
for all $v \in W_0^{1,N}(\Omega)$. For $v \in C_0^\infty(\Omega)$, an argument similar to that in the proof of Lemma 2.2, using the estimate
\[
\left| \int_{\{u_j^+ \geq M\}} f(u_j^+) \, v \, dx \right| \leq \frac{\sup |v|}{M} \int_{\Omega} u_j^+ f(u_j^+) \, dx
\]
and (2.5) shows that
\[
\int_{\Omega} f(u_j^+) \, v \, dx \to \int_{\Omega} f(u^+) \, v \, dx \quad \text{and} \quad \mu_j \int_{\Omega} g(u_j) \, v \, dx \to \mu \int_{\Omega} g(u) \, v \, dx
\]
since $g$ is bounded, so
\[
\int_{\Omega} |\nabla u|^{N-2} \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} f(u^+) \, v \, dx - \mu \int_{\Omega} g(u) \, v \, dx.
\]
Then this holds, for all $v \in W_0^{1,N}(\Omega)$, by density and taking $v = u$ gives
(2.9)
\[
\|u\|_N = \lambda \int_{\Omega} u^+ f(u^+) \, dx - \mu \int_{\Omega} u \, g(u) \, dx.
\]
Next we claim that
(2.10)
\[
\int_{\Omega} u_j^+ f(u_j^+) \, dx \to \int_{\Omega} u^+ f(u^+) \, dx.
\]
We have
(2.11)
\[
u_j^+ f(u_j^+) \leq |u_j|^{N} e^{\beta |u_j|^{N'}} = |u_j|^{N} e^{\beta |\tilde{u}_j|^{N'}} = |\tilde{u}_j|^{N'}
\]
where $\tilde{u}_j = u_j/\|u_j\|$. Setting
\[
\kappa = \lambda \int_{\Omega} F(u^+) \, dx - \mu \int_{\Omega} G(u) \, dx,
\]
we have $\|u_j\|_N \to N(c + \kappa)$ by (2.2), (2.6) and (2.7), so $\tilde{u}_j$ converges almost everywhere to $\tilde{u} = u/[N(c + \kappa)]^{1/N}$. Then
(2.12)
\[
\|u_j\|_N (1 - \|\tilde{u}\|_N) \to N(c + \kappa) - \|u\|_N
\]
By Lemma 2.1 (c),
\[
\int_{\Omega} u^+ f(u^+) \, dx \geq N \int_{\Omega} F(u^+) \, dx,
\]
and it is easily seen that \(tg(t) \leq N(G(t) + 1/2)\) for all \(t \in \mathbb{R}\) and hence
\[
\int_{\Omega} u g(u) \, dx \leq N \left( \int_{\Omega} G(u) \, dx + 1/2|\Omega| \right),
\]
so it follows from (2.9) that \(\|u\|^N \geq N(\kappa - (\mu/2)|\Omega|)\). Hence
\[
(2.13) \quad N(c + \kappa) - \|u\|^N \leq N \left( c + \frac{\mu}{2} |\Omega| \right) < \left( \frac{\alpha_N}{\beta} \right)^{N-1}
\]
by (1.4). We are done if \(\|u\|\) and it is easily seen that
\[
\|tg\|_{250}^K. \text{ Perera — I. Sim}
\]
Then \(\|u\|\) by Lemma 2.3. For \(N\)
\[
(2.13)
\]
so it follows from (2.9) that \(\|u\|^N \geq N(\kappa - (\mu/2)|\Omega|)\). Hence
\[
(2.13) \quad N(c + \kappa) - \|u\|^N \leq N \left( c + \frac{\mu}{2} |\Omega| \right) < \left( \frac{\alpha_N}{\beta} \right)^{N-1}
\]
by (1.4). We are done if \(\|u\| = 1\), so suppose \(\|u\| \neq 1\) and let
\[
\left[ N(c + (\mu/2)|\Omega|) \right]^{1/(N-1)} \geq \frac{\alpha_N/\beta}{(1 - \|u\|^N)^{1/(N-1)}}
\]
Then \(\|u_j\|^{N/(N-1)} \leq \tilde{\gamma} - 2\varepsilon\) for all \(j \geq j_0\) for some \(j_0\) by (2.12) and (2.13), and
\[
(2.14) \quad \sup_j \int_{\Omega} e^{\beta \tilde{\gamma} |u_j|^N} \, dx < \infty
\]
by Lemma 2.3. For \(M > 0\) and \(j \geq j_0\), (2.11) then gives
\[
\int_{\{u_j \geq M\}} u_j^+ f(u_j^+) \, dx \leq \int_{\{u_j \geq M\}} u_j^N e^{\tilde{\gamma} (\tilde{\gamma} - 2\varepsilon) \tilde{u}^N} \, dx
\]
\[
= \|u_j\|^N \int_{\{u_j \geq M\}} \tilde{u}^N e^{-\tilde{\gamma} \tilde{u}^N} e^{-\tilde{\gamma} (\|u_j\|N) e^{\tilde{\gamma} \tilde{u}^N} \, dx}
\]
\[
\leq \left( \max_{t > 0} t^N e^{-\tilde{\gamma} t^N} \right) \|u_j\|^N \int_{\Omega} e^{\tilde{\gamma} \tilde{u}^N} \, dx.
\]
The last expression goes to zero as \(M \to \infty\) uniformly in \(j\) since \(\|u_j\|\) is bounded and (2.14) holds, so (2.10) now follows as in the proof of Lemma 2.2.

Now it follows from (2.3), (2.10), (2.7) and (2.9) that
\[
\|u_j\|^N \to \lambda \int_{\Omega} u^+ f(u^+) \, dx - \mu \int_{\Omega} u g(u) \, dx \to \|u\|^N,
\]
and hence \(\|u_j\| \to \|u\|\). So \(u_j \to u\) by the uniform convexity of \(W^1_0(\Omega)\). Clearly, \(E_\mu(u) = c\) and \(E'_\mu(u) = 0\). \(\square\)

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Recall that \(E_\mu\) satisfies the Palais–Smale compactness condition at the level \(c \in \mathbb{R}\), or the (PS)\(_c\) condition for short, if every sequence \(\{u_j\}\) in \(W^1_0(\Omega)\) such that \(E_\mu(u_j) \to c\) and \(E'_\mu(u_j) \to 0\), called a (PS)\(_c\) sequence, has a convergent subsequence. The following lemma is immediate from the general compactness result in Theorem 1.2.

**Lemma 3.1.** \(E_\mu\) satisfies the (PS)\(_c\) condition for all \(c \neq 0\) satisfying
\[
c < 1 \left( \frac{\alpha_N}{\beta} \right)^{N-1} - \frac{\mu}{2} |\Omega|.
\]
First we show that $E_\mu$ has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 3.1 for all sufficiently small $\mu > 0$. We may assume that $0 \in \Omega$ without loss of generality. Take $r > 0$ so small that $B_r(0) \subset \Omega$ and let

$$v_j(x) = \begin{cases} \frac{1}{\omega_{N-1}^{1/N}} (\log j)^{(N-1)/N} & \text{if } |x| \leq r/j, \\ \log r / |x| & \text{if } r/j < |x| < r, \\ 0 & \text{if } |x| \geq r. \end{cases}$$

It is easily seen that $v_j \in W^{1,N}_0(\Omega)$ with $\|v_j\| = 1$ and

$$\int_\Omega v_j^N \, dx = O(1/\log j) \quad \text{as } j \to \infty. \quad (3.1)$$

**Lemma 3.2.** There exist $\mu_0, \rho, c_0 > 0$, $j_0 \geq 2$, $R > \rho$ and $\vartheta < (\alpha N/\beta)^{N-1/N}$ such that the following hold for all $\mu \in (0, \mu_0)$:

(a) if $\|u\| = \rho$ then $E_\mu(u) \geq c_0$,

(b) $E_\mu(Rv_{j_0}) \leq 0$,

(c) denoting by $\Gamma = \{ \gamma \in C([0,1], W^{1,N}_0(\Omega)) : \gamma(0) = 0, \gamma(1) = Rv_{j_0} \}$ the class of paths joining the origin to $Rv_{j_0}$,

$$c_0 \leq c_\mu := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} E_\mu(u) \leq \vartheta + C_\lambda \mu^{N'}, \quad (3.2)$$

where $C_\lambda = (1 - 1/N)|\Omega|/\lambda^{1/(N-1)}$,

(d) $E_\mu$ has a critical point $u_\mu$ at the level $c_\mu$.

**Proof.** Set $\rho = \|u\|$ and $\tilde{u} = u/\rho$. By Lemma 2.1 (d) and since

$$\lambda_1 = \inf_{u \in W^{1,N}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^N \, dx}{\int_\Omega |u|^N \, dx},$$

we have

$$\int_\Omega F(u^+) \, dx \leq \int_\Omega \left[ \frac{1}{N} |u|^N + \frac{\beta}{N} |u|^{N^2/(N-1)} e^{\beta |u|^N'} \right] \, dx \leq \frac{1}{N \lambda_1} \|u\|^N + \frac{\beta \rho N^{N^2/(N-1)} / 2^{N^2/(N-1)}}{N} \left( \int_\Omega e^{2\beta |u|^N'} \, dx \right)^{1/2} = \frac{\rho N}{N \lambda_1} + O\left( \rho N^{N^2/(N-1)} \right) \quad \text{as } \rho \to 0.$$

since \( W_0^{1,N}(\Omega) \hookrightarrow L^{2N^2/(N-1)}(\Omega) \) and \( \int_{\Omega} e^{2\beta \rho^N|u|^N} \, dx \) is bounded by (1.1) when \( 2\beta \rho^N \leq \alpha_N \). Since \( G(t) \geq -1/2 \) for all \( t \in \mathbb{R} \), then

\[
E_\mu(u) \geq \frac{1}{N} \left( 1 - \frac{\lambda}{\lambda_1} \right) \rho^N + O(\rho^{N^2/(N-1)}) - \frac{\mu}{2} |\Omega|.
\]

Since \( \lambda < \lambda_1 \), (a) follows from this for sufficiently small \( \rho, \mu, c_0 > 0 \).

Since \( v_j \geq 0 \),

\[
E_\mu(tv_j) = \int_{\Omega} \left[ \frac{t^N}{N} |\nabla v_j|^N - \lambda F(tv_j) + \mu tv_j \right] \, dx
\]

for \( t \geq 0 \). By the Hölder and Young’s inequalities,

\[
\mu t \int_{\Omega} v_j \, dx \leq \mu |\Omega|^{1-1/N} \left( \int_{\Omega} v_j^N \, dx \right)^{1/N} \leq C_\lambda \mu^N + \frac{\mu N}{N} \int_{\Omega} v_j^N \, dx,
\]

so \( E_\mu(tv_j) \leq H_j(t) + C_\lambda \mu^N \), where

\[
H_j(t) = \frac{t^N}{N} \left( 1 + \lambda \int_{\Omega} v_j^N \, dx \right) - \lambda \int_{\Omega} F(tv_j) \, dx.
\]

By Lemma 2.1 (c),

\[
\int_{\Omega} F(tv_j) \, dx \geq \frac{t^N}{N} \int_{\Omega} v_j^N \, dx + \frac{\beta(N-1)}{N^2} t^{N^2/(N-1)} \int_{\Omega} v_j^{N^2/(N-1)} \, dx,
\]

so

\[
(3.3) \quad H_j(t) \leq \frac{t^N}{N} - \frac{\lambda \beta(N-1)}{N^2} t^{N^2/(N-1)} \int_{\Omega} v_j^{N^2/(N-1)} \, dx \to -\infty \quad \text{as} \quad t \to \infty.
\]

So to prove (b) and (c), it suffices to show that there exists \( j_0 \geq 2 \) such that

\[
\vartheta := \sup_{t \geq 0} H_{j_0}(t) < \frac{1}{N} \left( \frac{\alpha_N}{\beta} \right)^{N-1}.
\]

Suppose \( \sup_{t \geq 0} H_j(t) \geq (\alpha_N/\beta)^{N-1}/N \) for all \( j \). Since \( H_j(t) \to -\infty \) as \( t \to \infty \) by (3.3), there exists \( t_j \geq 0 \) such that

\[
(3.4) \quad H_j(t_j) = \frac{t_j^N}{N} \left( 1 + \varepsilon_j \right) - \lambda \int_{\Omega} F(t_jv_j) \, dx = \sup_{t \geq 0} H_j(t) \geq \frac{1}{N} \left( \frac{\alpha_N}{\beta} \right)^{N-1}
\]

and

\[
(3.5) \quad H'_j(t_j) = t_j^{N-1} \left( 1 + \varepsilon_j - \lambda \int_{\Omega} v_j^N e^{\beta t_j v_j^N} \, dx \right) = 0,
\]

where

\[
\varepsilon_j = \lambda \int_{\Omega} v_j^N \, dx.
\]
Since $F(t) \geq 0$ for all $t \geq 0$, (3.4) gives $\beta t \geq \alpha N/(1 + \varepsilon_j)$, and then (3.5) gives
\[
\frac{1 + \varepsilon_j}{\lambda} = \int_{\Omega} v_j^N e^{\beta N} v_j^N \, dx \\
\geq \int_{B_{r_j}(0)} e^{\alpha N} e^{\beta N} v_j^N \, dx = r_j N j^{N \varepsilon_j/(1 + \varepsilon_j)}.
\]
By (3.6), $\varepsilon_j \to 0$ and $j^{N \varepsilon_j/(1 + \varepsilon_j)} \leq j^{N \varepsilon_j} = e^{N \varepsilon_j \log j} = O(1)$, so (3.6) is impossible for large $j$.

By (a)–(c), $E_{\mu}$ has the mountain pass geometry and the mountain pass level $c_{\mu}$ satisfies
\[
0 < c_{\mu} \leq \vartheta + C_{\lambda} \mu^{N} < \frac{1}{N} \left( \frac{\alpha N}{\beta} \right)^{N-1} - \frac{\mu}{2} |\Omega|
\]
for all sufficiently small $\mu > 0$, so $E_{\mu}$ satisfies the (PS) $c_{\mu}$ condition by Lemma 3.1. So $E_{\mu}$ has a critical point $u_{\mu}$ at this level by the mountain pass theorem.

Now we show that $u_{\mu}$ is positive in $\Omega$, and hence a weak solution of problem (1.2), for all sufficiently small $\mu \in (0, \mu_0)$. It suffices to show that for every sequence $\mu_j > 0$, $\mu_j \to 0$, a subsequence of $u_j = u_{\mu_j}$ is positive in $\Omega$. By (3.2), a renamed subsequence of $u_{\mu_j}$ converges to some $c$ satisfying
\[
0 < c < \frac{1}{N} \left( \frac{\alpha N}{\beta} \right)^{N-1}.
\]
Then a renamed subsequence of $(u_j)$ converges in $W^{1,N}_0(\Omega)$ to a critical point $u$ of $E_0$ at the level $c$ by Theorem 1.2. Since $c > 0$, $u$ is nontrivial.

**Lemma 3.3.** A further subsequence of $(u_j)$ is bounded in $C^{1,N}_0(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

**Proof.** Since
\[
\begin{cases}
-\Delta_N u_j = \lambda f(u_j^{+}) - \mu_j g(u_j) & \text{in } \Omega, \\
u_j = 0 & \text{on } \partial \Omega,
\end{cases}
\]
it suffices to show that $(u_j)$ is bounded in $L^\infty(\Omega)$ by the global regularity result of Lieberman [7], and this will follow from Proposition 1.3 of Guedda and Véron [6] if we show that $f(u_j^{+})$ is bounded in $L^s(\Omega)$ for some $s > 1$.

Let $s > 1$. By the Hölder inequality,
\[
\left( \int_{\Omega} |f(u_j^{+})|^s \, dx \right)^{1/s} \leq \left( \int_{\Omega} |u_j|^p \, dx \right)^{(N-1)/p} \left( \int_{\Omega} e^{\beta |u_j|^N} \, dx \right)^{1/q},
\]
where $(N-1)/p + 1/q = 1/s$. The first integral on the right-hand side is bounded since $W^{1,N}_0(\Omega) \hookrightarrow L^p(\Omega)$, and so is the second integral by Lemma 3.4 below.

**Lemma 3.4.** If $(u_j)$ is a convergent sequence in $W^{1,N}_0(\Omega)$, then
\[
\sup_j \int_{\Omega} e^{b |u_j|^N} \, dx < \infty \quad \text{for all } b.
\]
Proof. The case \( b \leq 0 \) is trivial, so suppose \( b > 0 \) and let \( u \in W_0^{1,N}(\Omega) \) be the limit of \( (u_j) \). We have

\[
|u_j|^N \leq (|u| + |u_j - u|)^N \leq 2^N \left(|u|^N + |u_j - u|^N\right),
\]

so

\[
\int_\Omega e^{b|u_j|^N} \, dx \leq \left( \int_\Omega e^{2^{N+1}b|u|^N} \, dx \right)^{1/2} \left( \int_\Omega e^{2^{N+1}b|u_j - u|^N} \, dx \right)^{1/2}.
\]

The first integral on the right-hand side is finite, and the second integral equals

\[
\int_\Omega e^{2^{N+1}b\|u_j - u\|^N} |v_j|^N \, dx,
\]

where \( v_j = (u_j - u)/\|u_j - u\| \). Since \( \|v_j\| = 1 \) and \( \|u_j - u\| \to 0 \), this integral is bounded by (1.1).

By Lemma 3.3, a renamed subsequence of \( u_j \) converges to \( u \) in \( C_0^1(\bar{\Omega}) \). Since \( u \) is a nontrivial weak solution of the problem

\[
\begin{cases}
-\Delta_N u = \lambda (u^+)^{N-1} e^\beta (u^+)^N & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

\( u > 0 \) in \( \Omega \) and its interior normal derivative \( \partial u/\partial \nu > 0 \) on \( \partial\Omega \) by the strong maximum principle and the Hopf lemma for the \( p \)-Laplacian (see Vázquez [12]). Since \( u_j \to u \) in \( C_0^1(\bar{\Omega}) \), then \( u_j > 0 \) in \( \Omega \) for all sufficiently large \( j \). This concludes the proof of Theorem 1.1.

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