The thermodynamic uncertainty relations (TURs) provide upper bounds on the precision of an arbitrary current in a system in terms of the entropy production (EP) of that system. All TURs derived so far have concerned a single physical system, varying in the assumptions they make about the dynamics of that system. However, many physical scenarios of interest involve multiple interacting systems, e.g., organelles within a biological cell. Here we show how to extend the previously derived TURs to those scenarios. A common feature of these extended versions of the TURs is that they bound the global EP, jointly generated by the set of interacting systems, in terms of a weighted sum of the precisions of the local currents generated within those systems – plus an information-theoretic correction term. We also exploit these extended TURs to obtain bounds that do not involve the global EP, but instead relate the local EPs of the individual systems and the statistical coupling among the currents generated within those systems. We derive such bounds for both scalar-valued and vector-valued currents within each system. We illustrate our results with numerical experiments.

I. INTRODUCTION

Stochastic thermodynamics has resulted in powerful results concerning the thermodynamic behavior of a broad spectrum of far-from-equilibrium systems, ranging from Brownian particles to molecular motors within living systems [1]. In particular, the thermodynamic uncertainty relations (TURs) are a set of results that provide upper bounds on the precision of a current generated by a dynamical system, in terms of the entropy production (EP) of that dynamics [2–12]. There are many such TURs, differing in their assumptions about the dynamics, e.g., whether it is a non-equilibrium steady state (NESS) [2], a time-homogeneous process [4] or whether the driving protocol is time-symmetric [5] or periodically time-dependent [7]. It is also possible to derive bounds concerning the joint precision of multiple currents within a system [8].

While they vary from one another in many respects, all TURs derived so far have concerned a single physical system, coupled to its own set of one or more reservoirs. This strongly limits their applicability to modeling the physical world, since many physical scenarios of interest involve multiple interacting systems. In particular, in multipartite processes (MPPs) [13–15] there are multiple distinct “subsystems” of an overarching system, each with its own set of reservoirs, independent of the reservoirs of the other subsystems.

Fig. 1 displays a set of scenarios that are suitable to be modeled as MPP. On the left column, a gene regulatory network is shown, where the nodes represent protein complexes (subsystems), and the directed edges signify the dependencies between these nodes. Biological machinery of cells comprises multiple classes of interactions between certain complexes and molecules at a time, suggesting a modular architecture [16]. The information-thermodynamic dynamics of such a tripartite — essentially a multipartite — regulatory structure is investigated in [17]. The right column illustrates the cluster formation of self-propelling particles. As
demonstrated in [13, 19] which follow the Kern-Frenkel model for Janus particles, cluster formation of self-propelling particles depend on the interaction range and thermodynamic parameters. These dependencies can be captured in an MPP, where each particle is a subsystem by itself, composing aggregates as allowed by the dynamics that dictate the local and global behaviors.

The goal of this paper is to extend the conventional TURs, applicable to single systems, to settings of multiple, dependent systems. We begin in the next section by reviewing rate matrix “units”, a key attribute of the thermodynamics of MPPs [20, 22]. In particular we summarize the key concepts arising in the trajectory-wise stochastic thermodynamics of MPPs. We finish this first section by summarizing those TURs that were derived earlier for single systems, which we extend to MPPs in this paper [2, 4, 8].

In the following section we present a formal approach to extend any TUR to MPPs, so long as the TUR involves the global EP jointly generated by a set of interacting systems. We then use this approach to modify the previously mentioned TURs. These extended TURs provide bounds on the global EP in terms of the precisions of the local currents generated within the subsystems – plus an information-theoretic correction term. This information-theoretic correction term is a generalization of the multi-information among the states of the overall system.

In the last section we illustrate our findings through numerical experiments involving multiple quantum dots. We conclude by discussing the implications of our results and suggesting possible future work. All proofs not in the main text are in the appendices.

II. BACKGROUND

A. Rate matrix units

We write \( \mathcal{N} \) for a set of \( N \) subsystems with finite state spaces \( \{ X_i : i = 1, \ldots N \} \). \( X \) is the joint space of \( \mathcal{N} \), and \( x \) is a vector in \( X \). For any \( A \subset \mathcal{N} \), we write \( -A := \mathcal{N} \setminus A \) for the complement. The full distribution over \( X \) at time \( t \) is denoted by \( p^X(t) \), with the particular value for state \( x \) written as \( p_x(t) \). We use \( \delta(\cdot, \cdot) \) to indicate the Kronecker delta function.

We write \( \mathcal{X} \) for the set of all possible trajectories of the system in \([0, t_f]\), a time interval of interest where the system evolves. A particular trajectory of states is written as \( x \), and the probability density for such a trajectory \( x \) is written as \( P(x) \). In general, the Shannon entropy of a distribution is written as \( S(\cdot) \), e.g., \( S(p^X(t)) \) is the Shannon entropy of the distribution over joint states at time \( t \).

In an MPP, the global system’s evolution is determined by a continuous-time Markov chain (CTMC) given by a set of time-varying stochastic rate matrices, \( \{ W_x'(i; t) : i = 1, \ldots, N \} \), where for all \( i \), \( W_{x'}(i; t) = 0 \) if \( x_{i} \neq x_{-i} \). [13, 20, 22]:

\[
\frac{dp_x(t)}{dt} = \sum_{x'} W_x'(t)p_{x'}(t) = \sum_{x'} \sum_{i\in\mathcal{N}} W_{x'}(i; t)p_{x'}(t) \quad (1)
\]

We refer to the (pre-fixed) trajectory of the matrices \( W(i; t) \) across the interval \([0, t_f]\) as the driving protocol of the process. For any \( A \subset \mathcal{N} \), we define:

\[
W_{x'}(i; t) := \sum_{i\in A} W_{x'}(i; t) \quad (3)
\]

For each subsystem \( i \), we write \( r(i; t) \) for any set of subsystems at time \( t \) that includes \( i \), such that it is possible to write:

\[
W_{x'}(i; t) = W_{x'_{r(i; t)}}(i; t)\delta(x'_{\sim r(i; t)}, x_{\sim r(i; t)}) \quad (4)
\]

for appropriate functions \( W_{x'_{r(i; t)}}(i; t) \). \( r(i; t) \) will not be unique in general.

![FIG. 2: A representative MPP.](image)
We use the term *unit* to refer to any set of subsystems \( \omega \) such that \( i \in \omega \) implies \( r(i,t) \subseteq \omega \). Note that any intersection of units is a unit, and any union of units is a unit. A set of units that covers \( \mathcal{N} \) and that is closed under intersections is referred to as a *unit structure*, denoted by \( \mathcal{N}^\ast \). In the following we consider only the unit structures whose topology does not change in time, and we require that \( \mathcal{N} \) itself is not a part of the unit structure. Fig. 2 depicts a representative form of such an MPP that is structured by Eqs. [1] to [4]. Note that the most fine-grained unit structure is given by the transitive closure of the \( r(\cdot) \) relation. For example, in Fig. 2, the unit that contains subsystem \( A \) is \( \{A,B,C\} \), since \( B \in r(A) \) and \( C \in r(B) \).

**B. Trajectory-level thermodynamics of MPPs**

Physically, in general, in an MPP each subsystem is coupled to its own set of one or more reservoirs, and no two reservoirs can interact [13,20,22]. In our analysis of the MPP TURs, we allow the full generality of either one reservoir per subsystem or multiple reservoirs per subsystem, except in the special case of an NESS, where the associated subsystems must have multiple reservoirs.

Choosing units so that \( k_B = 1 \), and assuming LDB, the (local) stochastic entropy of any set of subsystems \( \alpha \) is [13,20,22]:

\[
s^\alpha(x_\alpha(t)) := -\ln p_{x_\alpha(t)}(t)
\]

with its change from \( t = 0 \) to \( t = t_f \) written as:

\[
\Delta s^\alpha(x_\alpha) := -\ln p_{x_\alpha(0)}(0) - \ln p_{x_\alpha(t_f)}(t_f)
\]

Let \( \alpha \) be a set of subsystems (not necessarily a unit). We use \( Q^\omega(x) \) to indicate the *local* entropy flow (EF) into the subsystems in \( \alpha \) from their reservoirs during the interval \( [0,t_f] \) if the full system follows trajectory \( x \). (See Appendix A for formal details.) The *local* EP of any unit \( \omega \) is just the difference between the associated local EF and change in stochastic entropy:

\[
\sigma^\omega(x) := \Delta s^\omega(x) - Q^\omega(x)
\]

The *global* EP is the special case where \( \omega = \mathcal{N} \):

\[
\sigma(x) := \Delta s(x) - Q(x)
\]

From now on we leave the specification \( \mathcal{N} \) implicit if it is clear from context.

Given any unit-indexed function \( f^\omega : X \to \mathbb{R} \), the associated *inclusion-exclusion sum* is defined as:

\[
\sum_{\omega \in \mathcal{N}^\ast} f^{\omega'}(x) := \sum_{j=1}^{n} f^{\omega_j}(x) - \sum_{1 \leq j < j' \leq n} f^{\omega_j \cap \omega_{j'}}(x) + \sum_{1 \leq j < j' < j'' \leq n} f^{\omega_j \cap \omega_{j'} \cap \omega_{j''}}(x) - \ldots
\]

Building on this we define the *in-ex information* over units during \([0,t_f]\) as:

\[
I^{\mathcal{N}^\ast}(x(t)) := \left[ \sum_{\omega \in \mathcal{N}^\ast} s^{\omega}(x(t)) \right] - s(x(t))
\]

\[
= -s(x(t)) + \sum_{j=1}^{n} s^{\omega_j}(x(t)) - \sum_{1 \leq j < j' \leq n} s^{\omega_j \cap \omega_{j'}}(x(t)) + \ldots
\]

Note that the in-ex information is equivalent to multi-information in the case of independent units. In particular, if the unit structure consists solely of two units that are independent, then Eq. [10] simply corresponds to the mutual information between these units.

It is shown in [5] that the global EP is given by the in-ex sum of the local EPs minus the change in in-ex information:

\[
\sigma(x) = \sum_{\omega \in \mathcal{N}^\ast} \sigma^{\omega}(x) - \Delta I^{\mathcal{N}^\ast}(x)
\]

Eq. [11] is a keystone of our analysis below.

**C. Thermodynamic uncertainty relations**

A current generated by any trajectory across a space \( Y \) is any function of the form

\[
J(x) = \sum_{x \neq x'} n_{x,x'}(x)d(x,x')
\]

where \( n_{x,x'}(x) \) corresponds to the total number of transitions from \( x \) to \( y \), and \( d(x,x') = -d(x',x) \) is the anti-symmetric increment associated with a given transition.

Currents vary from one trajectory to another, and so the stochasticity of the trajectories induces a stochasticity of currents. This stochasticity can be quantified by the precision of the current, i.e., by its variance to mean-square ratio. TURs provide upper bounds on the precision of any current generated by a system in terms of the associated expected EP. The TURs are independent of the precise choice of
d. In addition, which precise TUR applies to a given process only depends on high-level properties of the driving protocol and of the associated distribution $P$, independent of their details.

Historically, the first TUR [2] took the form:

$$\langle \sigma \rangle \geq \frac{2\langle J \rangle^2}{\text{Var}[J]}$$

It applies whenever the system is in a NESS throughout $[0, t_f]$. Note that this requires that the driving protocol be time-independent, which limits the applicability of this bound. However, one can avoid such a strong restriction on the physical scenario, at the expense of having a slightly weaker bound. Assume that $P$ evolves under a time-symmetric protocol, and that the starting distribution is identical to the ending distribution. In this case, the fluctuation theorem uncertainty relation (FTUR) [5] holds:

$$\langle \sigma \rangle \geq \ln \left( \frac{2\langle J \rangle^2}{\text{Var}[J]} + 1 \right)$$

More recently, a new type of TUR that applies to a combination of instantaneous currents and time-integrated currents was derived [6]. This TUR holds independent of whether the distributions at the start and end of the interval of evolution are identical:

$$\langle \sigma \rangle \geq \frac{2\langle j \rangle^2}{\text{Var}[J]}$$

Note that these TURs consider single currents only, defined by a single function $d(\ldots)$. However, even in a process that is not an MPP, in general there will be multiple currents, defined by different functions $d(\ldots)$. Write $J(x)$ for such a vector of currents, with expectation value $\langle J \rangle$. Also write $\Sigma^{-1}$ for the associated inverse covariance matrix. In [8] the generalized Cramer-Rao inequality is used to derive a multi-dimensional version of the FTUR in terms of these quantities, where $\Sigma$ is an EP-like term:

$$\langle J \rangle^T \Sigma^{-1} \langle J \rangle \leq \frac{1}{2} \Sigma \tag{16}$$

Several papers have also investigated ways to unify many of the TURs and / or extend them beyond CTMCs. For example, [4] considered the Hilbert space of any observables that are invariant under time-reversal. This let them derive a TUR-like bound that holds for all stationary Markov processes. As another example, [10] showed how to introduce the key variables of stochastic thermodynamics (such as the current and entropy production) into quantum field theory, and then expressed the TUR given by [2] in field-theoretic terms.

The discovery of TURs inspired the derivation of many other bounds that constrain the precision of other observables. In particular, [23] showed that the distribution for the time to first hit a large threshold current must satisfy a TUR-like relation.

III. MAIN RESULTS

A. General approach for arbitrary TURs

As mentioned in the introduction, one major limitation of the TURs reviewed above is that they only apply to single systems, evolving in isolation, and so do not apply to MPPs in particular. To extend them to forms that apply to MPPs, first we define currents over units in a way that parallels Eq. (12).

$$J^\omega(x) = J^\omega(x_\omega) = \sum_{x_\omega \neq x_\omega'} n_{x_\omega}^\omega(x_\omega) d_\omega(x_\omega, x_\omega') \tag{17}$$

where $d_\omega(\ldots)$ is an arbitrary anti-symmetric function. Importantly, because each unit evolves autonomously, as a self-contained CTMC, each local
EP $\sigma^w$ bounds the current $J^w$ generated in the associated unit $\omega$ according to the previously derived TURs of single systems. Building on this observation, the central mathematical tool that we use here is given by averaging both sides of Eq. (11) according to $P_{\omega}$:

$$\langle \sigma \rangle = \sum_{\omega \in N^*} \langle \sigma^\omega \rangle - \left( \Delta J^{N^*} \right)_{P_{\omega}}$$  \hspace{1cm} (18)

The first term on the RHS of Eq. (20) is an in-ex sum of the expected local EPs over individual units, each of which evolves according to its own self-contained CTMC. Therefore the (in-ex sum of the) local EPs over units can be bounded by the precision of the (in-ex sum of the) local current precisions generated in the corresponding units. Our procedure for extending the TURs to apply to MPPs strongly depends on this property.

### B. Extensions of previously derived TURs for scalar-valued currents to MPPs

In the following, we use the scalar-valued TURs summarized in Section [IIC] to expand the local EPs in Eq. (18).

#### 1. Steady-state TUR

Suppose that each unit $\omega$ is in a NESS as it evolves, though the individual steady states over each unit need not be identical. (Note that even though each unit is in a NESS, the joint system need not be.) Then for each unit $\omega$,

$$\langle \sigma^\omega \rangle \geq \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]}$$  \hspace{1cm} (19)

Substituting Eq. (19) into Eq. (18), we get:

$$\langle \sigma \rangle \geq \sum_{\omega \in N^*} \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]} - \left( \Delta J^{N^*} \right)$$  \hspace{1cm} (20)

#### 2. FTUR

Next weaken the assumption that each unit is in an NESS, to only only assume that the starting and ending distributions over each unit $\omega$ are identical, and that the driving protocols over each unit (which, once again, need not be identical) are time-symmetric. With these assumptions we can invoke Eq. (14) to write:

$$\langle \sigma^\omega \rangle \geq \ln \left( \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]} + 1 \right)$$  \hspace{1cm} (21)

for each unit $\omega$. Plugging this expression in for the local EPs in Eq. (18) gives:

$$\langle \sigma \rangle \geq \sum_{\omega \in N^*} \ln \left( \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]} + 1 \right) - \left( \Delta J^{N^*} \right)$$  \hspace{1cm} (22)
3. TUR involving instantaneous current

We noted in Section II that there are also TURs that do not restrict either the initial or final distributions. An example is given in [6]. Define the instantaneous current over the joint system at time \( t \) as 
\[
j_t(x) = \sum_{x \neq x'} W_{z,t}^x (d_{x',x}) j_t(x)
\]
and define \( j_t^+(x) \) to be any instantaneous current over \( X_\omega \) at that time. Furthermore, introduce the time-integrated current over such unit as 
\[
J_\omega^t(x) = \int_{t_{\omega - 1}}^t d_t j_t^+(x).
\]
In [6] it is shown that if the rate matrix for unit is time-homogeneous, then even though the overall system does not have to evolve time-homogeneously, it must be that:

\[
\langle \sigma^\omega \rangle \geq \frac{2 \langle \tau^\omega j^\omega \rangle^2}{\text{Var}[J^\omega]} \tag{23}
\]

Substitution of Eq. (23) to Eq. (18) gives:

\[
\langle \sigma \rangle \geq \sum_{\omega \in \mathcal{N}^*} \frac{2 \langle \tau^\omega j^\omega \rangle^2}{\text{Var}[J^\omega]} - \langle \Delta J^{\mathcal{N}^*} \rangle \tag{24}
\]

It is possible to repeat the procedure above to construct multipartite TURs on different levels — n.b., this extension process would apply to any unit \( \omega \in \mathcal{N}^* \) that would define its own unit structure under subsystem LDB. (See Appendix D of [21] for an inclusive discussion on SLDB.) We could then consider any pair of units, \( \omega \) and \( \omega' \), such that

\[
\omega^* := \{ \omega' \in \mathcal{N}^* : \omega' \subseteq \omega \} \tag{25}
\]

Repeating the steps in Section III B 2, Section III B 3, we obtain Eqs. (26) to (28). These expressions bound the local EPs of units in terms of the current precisions associated with their sub-units, with the corresponding change in the in-ex information over these sub-units.

\[
\langle \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \frac{2 \langle J^\omega' \rangle^2}{\text{Var}[J^\omega']} - \langle \Delta J^{\omega^*} \rangle \tag{26}
\]

\[
\langle \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \ln \left( \frac{2 \langle J^\omega' \rangle^2}{\text{Var}[J^\omega']} + 1 \right) - \langle \Delta J^{\omega^*} \rangle \tag{27}
\]

\[
\langle \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \frac{2 \langle \tau^\omega j^\omega' \rangle^2}{\text{Var}[J^\omega']} - \langle \Delta J^{\omega^*} \rangle \tag{28}
\]

We remark that Eq. (26) is a refinement to Eq. (20), where Eq. (27) is to Eq. (22), and Eq. (28) is to Eq. (24).

C. Conditional TURs based on DFTs for multipartite processes

In [21] a conditional DFT is derived, applicable for any unit \( \omega \), and any associated EP value \( \sigma^\omega \) that has non-zero probability:

\[
\langle \sigma | \sigma^\omega \rangle \geq \sigma^\omega \tag{29}
\]

Eq. (29) means that the expected global EP conditioned on the joint EP of \( A \) and \( B \) cannot be smaller than the joint EP of \( A \) and \( B \). It also means that the expected joint EP of \( A \) and \( B \) conditioned on the EP of \( B \) cannot be smaller than the EP of \( B \).

Note that the bound in Eq. (29) is saturated if the value \( \sigma^\omega \) happens to equal the expected EP of unit \( \omega \), \( \langle \sigma^\omega \rangle \) (assuming that can occur with non-zero probability). For this particular case where the observed EP equals \( \langle \sigma^\omega \rangle \), we can plug in the results of the previous sections into the RHS of Eq. (29), to derive the following conditional TURs for MPPs:

\[
\langle \sigma | \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \frac{2 \langle J^\omega' \rangle^2}{\text{Var}[J^\omega']} - \langle \Delta J^{\omega^*} \rangle \tag{30}
\]

\[
\langle \sigma | \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \ln \left( \frac{2 \langle J^\omega' \rangle^2}{\text{Var}[J^\omega']} + 1 \right) - \langle \Delta J^{\omega^*} \rangle \tag{31}
\]

\[
\langle \sigma | \sigma^\omega \rangle \geq \sum_{\omega' \in \omega^*} \frac{2 \langle \tau^\omega j^\omega' \rangle^2}{\text{Var}[J^\omega']} - \langle \Delta J^{\omega^*} \rangle \tag{32}
\]

D. Extensions of previously derived TURs for vector-valued currents to MPPs

In the analysis of the preceding sections, there was only one current specified for each unit, i.e., each unit \( \omega \) had a single associated increment function \( d_\omega(.,.) \). In general though, any subsystem will have many possible associated increment functions, and the associated currents can be statistically coupled. This results in TURs for “vector-valued” current. In this section we present several results related to such current.

To begin, we review the vector-valued fluctuation theorem for MPPs derived in [21], and a joint fluctuation theorem of EP and current over a single system derived in [4]. We then combine these results to derive FTURs for vector-valued currents that apply to MPPs.

Our first results apply for any pairs of sets of units, \( A, B \), which in general may or may not have subsystems in common. Each of those sets has an associated vector of currents, indexed by the units in that
set, which we write as \( \vec{J}^A \) and \( \vec{J}^B \), respectively. Note that when the sets do indeed overlap, there is statistical coupling between \( \vec{J}^A \) and \( \vec{J}^B \).

We derive a bound on the precisions of both \( \vec{J}^A \) and \( \vec{J}^B \). This bound depends on the covariance of those current vectors, and on the EPs within \( A \) and \( B \) — but does not explicitly involve the global EP. As a special case, we present a bound on the local precision of a vector-valued current over the subsystems within a single unit (i.e., a bound on the overall EP generated in that unit).

Vector-valued currents have already been analyzed previously for the special case of a single system in [8]. We next show how to extend these results to MPPs, concentrating on a few specific scenarios.

Special cases of some of our findings in this section concern scenarios that were already addressed by previous results in the literature. Our findings do not necessarily tighten such earlier results. However, our results extend beyond those special cases, to tease apart some of the different bounds constraining how the thermodynamic properties of different subsystems can interact with one another. These bounds provide us several trade-off relations concerning the thermodynamics of MPPs.

1. The vector-valued FTUR for multipartite processes

Let \( A = \{ \alpha \} \) be any set of units. We write \( \vec{\sigma}^A \) for the associated vector whose components are the local EP values \( \sigma^\alpha \), and \( \vec{J}^A \) for the associated vector whose components are the individual currents \( J^\alpha \). We also write \( \sigma^A(x) \) for the joint EP generated by all the subsystems in \( A \), while \( J^A(x) \) is the total current generated by the subsystems in \( A \).

It was shown in [21] that the following vector-valued fluctuation theorem holds:

\[
\ln \left[ \frac{P(\vec{\sigma}^A)}{P(-\vec{\sigma}^A)} \right] = \sigma^{\cup A} \tag{33}
\]

In Appendix B we show that in fact a vector-valued joint fluctuation theorem over both local EPs and local currents holds:

\[
\ln \left[ \frac{P(\vec{\sigma}^A, \vec{J}^A)}{P(-\vec{\sigma}^A, -\vec{J}^A)} \right] = \sigma^{\cup A} \tag{34}
\]

Building on Eq. (34), we investigate the statistical dependencies between the local currents generated by different units. Recall that for any set of subsystems \( A \), \( \vec{J}^A \) indicates the vector whose components are the individual currents \( J^\alpha \) for units \( \alpha \in A \). Similarly, for any set of subsystems \( B \), \( \vec{J}^B \) indicates the vector whose components are the individual currents \( J^\beta \) for units \( \beta \in B \). If \( A \) and \( B \) overlap, then the vector-valued currents \( \vec{J}^A \) and \( \vec{J}^B \) will be statistically coupled, in general. We denote their covariance as \( \text{Covar}(\vec{J}^A, \vec{J}^B) \):

\[
\text{Covar}(\vec{J}^A, \vec{J}^B) = \mathbb{E}[(\vec{J}^A - \mathbb{E}[\vec{J}^A])(\vec{J}^B - \mathbb{E}[\vec{J}^B])]
\]

Note that the diagonal entries of the covariance matrix correspond to the variances of currents. In Appendix D we derive the vector-valued FTUR for MPPs:

\[
\frac{\text{Covar}(\vec{J}^A, \vec{J}^B)}{\langle \vec{J}^A \rangle \langle \vec{J}^B \rangle} \geq 2 \cdot \frac{1 + e^{(\sigma^{\cup A} - \sigma^{\cup B})}}{e^{(\sigma^{\cup A})} + e^{(\sigma^{\cup B})} - (1 + e^{(\sigma^{\cup A} - \sigma^{\cup B})})} \tag{35}
\]

where \( \sigma^{\cup A} \) is the total EP generated by the subsystems in \( A \), and \( \sigma^{\cup B} \) is defined similarly. Eq. (35) holds in any setting where the driving protocol is time-symmetric. Note that when \( A = B \), Eq. (35) reduces to:

\[
\frac{\text{Var}[\vec{J}^A]}{\langle \vec{J}^A \rangle^2} \geq \frac{2}{e^{(\sigma^{\cup A})} - 1} \tag{36}
\]

(In Appendix C we derive Eq. 37 in a more straightforward manner, by invoking Eq. (34) explicitly.) Eq. (37) shows how \( \sigma^{\cup A} \) bounds the precisions of the local currents.

2. MTUR for multipartite processes

In Section III D 1 we used FTs for MPPs to obtain vector-valued TURs for MPPs. In this subsection we instead vector-valued TURs for MPPs by building on the analysis in [8].

We start with a bound derived in [8]:

\[
\langle J \rangle_T \Xi_j^{-1}(J) \leq \frac{1}{2} \Delta S \tag{38}
\]

Consider each component of \( J \) in Eq. (38) as a local current originated in a single unit. Suppose also that the inverse covariance matrix is diagonal, implying that the units are independent. Given these conditions, we note that LHS equals the sum of the precisions of the current within individual units. Substituting Eq. (20) to the RHS of Eq. (38) gives:

\[
\langle J \rangle_T \Xi_j^{-1}(J) \leq \frac{\sum_{\omega \in x} \sigma^\omega(x_\omega) - \Delta P_j^N(x)}{2} \tag{39}
\]

If the units overlap, but the change in in-ex information is positive, then the LHS of Eq. (39) would
be upper-bounded by the in-ex sum of local EPs that are generated in distinct units. Alternatively, if the in-ex sum of local EPs is non-negative, then the LHS of Eq. (39) would be upper-bounded by the drop in the in-ex information.

Following Eq. (39), we also present Eq. (40) in a parallel form to Eq. (10):
\[
\langle J^A \rangle^{T \Xi_j^{-1}} \langle J^A \rangle \leq \frac{\langle \sigma^{(\Pi \Lambda A)} \rangle - 1}{2}
\]
(Eq. 40)

Equivalently:
\[
\langle \sigma^{(\Pi \Lambda A)} \rangle \geq \ln \left( 2 \langle J^A \rangle^{T \Xi_j^{-1}} \langle J^A \rangle + 1 \right)
\]
(Eq. 41)

In addition, for the global system:
\[
\langle \sigma \rangle \geq \ln \left( 2 \langle J \rangle^{T \Xi_j^{-1}} \langle J \rangle + 1 \right)
\]
(Eq. 42)

Note that if we consider a scalar current only, then this expression boils down to:
\[
\langle \sigma \rangle \geq \ln \left( \sum_{\omega \in N} \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]} + 1 \right)
\]
(Eq. 43)

Also, note that we can express Eq. (39) and Eq. 10 in a different form, by using a finding from given in Eq. (44), which states that the expected EP over a set of subsystems is given by the KL-divergence between the (forward and backward) probability distributions of the EP vector over such subsystems, where each component of the EP vector is mapped to a scalar local EP value over one unit that is a part of the overall unit structure.

\[
\langle \sigma^{(\Pi \Lambda A)} \rangle = D \left( \mathcal{P} (\sigma^A) \parallel \mathcal{P} (-\sigma^A) \right)
\]
(Eq. 44)

Substituting Eq. (44) to Eq. (39), and to Eq. (40) would then give, respectively, Eq. (41) and Eq. (42):
\[
D \left( \mathcal{P} (\sigma^A) \parallel \mathcal{P} (-\sigma^A) \right) \geq 2 \langle J^A \rangle^{T \Xi_j^{-1}} \langle J^A \rangle \geq \ln \left( 2 \langle J^A \rangle^{T \Xi_j^{-1}} \langle J^A \rangle + 1 \right)
\]
(Eq. 45)

IV. EXAMPLES

Here we illustrate our findings in a setting of multiple single-level quantum dots coupled to multiple reservoirs. The particular physical scenarios we consider are depicted in Fig. 4.

We follow the analysis in [21], and construct the rate matrices supposing that all the reservoirs are at the same chemical potential while having different temperatures. We explicitly write down the rate matrices for the two scenarios investigated — \( W_x \) for Fig. 4 (a) and \( W' \) for Fig. 4 (b)— as follows:
\[
W_x (\{A, B, C\}; t) = W_{x,A,B,C}^{\prime} (A, t)
\]
\[
+ W_{x,A,B,C}^{\prime} (B, t)
\]
\[
+ W_{x,A,B,C}^{\prime} (C, t)
\]
\[
W_{x}' (\{C, D\}; t) = W_{x,c,D}^{\prime} (C, t) + W_{x,D}^{\prime} (D, t)
\]
\[
W_{x}' (\{C\}; t) = W_{x,C}^{\prime} (C, t)
\]
(Eq. 47)

In the following, we define \( X^{-1} := \frac{\langle J^\omega \rangle^2}{\text{Var}[J^\omega]} \), which we extend to write \( X^{-1} \) to indicate \( X^{-1} \) computed for the specific unit \( \omega \). Similarly, we define \( Y^{-1} = \ln \left( \frac{\langle J^\omega \rangle^2}{\text{Var}[J^\omega]} + 1 \right) \), and \( Z^{-1} = \frac{2 \langle J^\omega \rangle^2}{\text{Var}[J^\omega]} \). The in-ex sum in bounds given by Eq. (40), Eq. (22) and Eq. (24) is written as \( \text{in-ex} \sum \) in the legends.

Fig. 5 illustrates Eq. (20) for a NESS, where there are three non-overlapping units, denoted by \( \omega \), \( \omega' \), and \( \omega'' \). Next, Fig. 6 shows how the reciprocal form of Eq. (20), the multipartite FTUR given by Eq. (22), and the multipartite TUR for instantaneous currents given by Eq. (24). The curves in Fig. 9 imply that in an NESS generated by three non-overlapping units, the bound given by the multipartite TUR for instantaneous currents is closest to global mean dissipation.

Our findings also reflect the trade-offs between various units in bounding the global EP, that is captured by the change in the in-ex information.

Fig. 10 demonstrates this through a scenario where the global current over a joint system is fixed.
FIG. 5: The components that compose Eq. (20) on the \( y \)-axis. \( 2X^{-1}, 2X^{-1}, 2X^{-1} \) denote the contributions from all three units to the in-ex sum on the RHS of Eq. (20).

FIG. 6: Precision and inverse-mean dissipation obtained from Fig. 5 on the \( y \)-axis.

FIG. 7: The components that compose Eq. (24) on the \( y \)-axis. \( 2Z^{-1}, 2Z^{-1}, 2Z^{-1} \) denote the contributions from all three units to the in-ex sum on the RHS. \( \omega'' \) is the unit that is the intersection of other two units.

FIG. 8: Precision and inverse-mean dissipation obtained from Fig. 7 on the \( y \)-axis.

FIG. 9: Mean dissipation and relevant in-ex sums attached to Eq. (20), Eq. (22), and Eq. (24), respectively labeled by \( X, Y, \) and \( Z \).

V. DISCUSSION

In this paper we introduced a way to extend any conventional TUR, derived for the currents and EP of a single system, to a corresponding TUR that instead applies to the case of multiple, dependent systems, by exploiting some of the results in [20–22]. We illustrated our approach by deriving the multipartite extensions of the TURs that were derived in [2, 5, 6, 8].

There are several avenues of future work suggested by our results. First, it might be possible to use the same kind of approach to extend the speed-limit theorems [9], or bounds on first-passage times [23]—a particular one being dissipation-time uncertainty relation [24]—to MPPs. Note also that the analysis in this paper as well as the analyses in [20, 22] only consider the case where the same unit structure applies at all times. However, in many real-world scenarios the rate matrices change in way that makes it natural to also choose a unit structure that changes with
We analyze a setting of two non-overlapping units, \( \omega \) and \( \omega' \), where the global current precision is fixed. Here, the averaged change in the in-ex information changes to compensate the increase (decrease) in the individual averaged EPs over units.

To conclude, we emphasize that studies on thermodynamic constraints and trade-off relations in MPPs is crucial to obtaining an understanding of the physical systems that are governed by the laws of stochastic thermodynamics. Such systems, including active matter particles that demonstrate aggregation or organelles in a biological cell, are governed by the underlying overall dependency graph combined with the joint dynamics. More work has to be done to elucidate how the topology of a given dependency graph would imply the information-theoretic constraints that are intertwined with the thermodynamic ones.

**ACKNOWLEDGMENTS**

This work was supported in part by the Santa Fe Institute and Grant No. FQXi-RFP-IPW-1912 from the FQXi foundation. Gülce Kardeş thanks Gianmaria Falasco for helpful discussion.

[1] U. Seifert, *Stochastic thermodynamics, fluctuation theorems and molecular machines*, Rep. Prog. Phys. **75**, 126001 (2012).
[2] A. C. Barato and U. Seifert, *Thermodynamic uncertainty relation for biomolecular processes*, Phys. Rev. Lett. **114**, 158101 (2015).
[3] T. R. Gingrich and J. M. Horowitz, *Thermodynamic uncertainty relations constrain non-equilibrium fluctuations*, Nat. Phys. **16**, 15–20 (2019).
[4] G. Falasco, M. Esposito, and J.-C. Delvenne, *Unifying Thermodynamic Uncertainty Relations*, New J. Phys. **22**, 053046 (2020).
[5] Y. Hasegawa and T. V. Vu, *Generalized thermodynamic uncertainty relation via fluctuation theorem*, arXiv preprint arXiv:1902.06376 (2019).
[6] K. Liu, Z. Gong, and M. Ueda, *Thermodynamic Uncertainty Relation for Arbitrary Initial States*, Phys. Rev. Lett. **125**, 140602 (2020).
[7] T. Koyuk and U. Seifert, *Thermodynamic uncertainty relation for time-dependent driving*, arXiv preprint arXiv:2005.02312 (2020).
[8] A. Dechant, *Multidimensional thermodynamic uncertainty relations*, arXiv preprint arXiv:1809.10414 (2018).
[9] V. T. Vo, T. V. Vu, and Y. Hasegawa, *Unified Approach to Classical Speed Limit and Thermodynamic Uncertainty Relation*, arXiv preprint arXiv:2007.03495 (2020).
[10] O. Niggemann and U. Seifert, *Field-Theoretic Thermodynamic Uncertainty Relation*, J. Stat. Phys. **178**, 1142–1174 (2020).
[11] G. Landi, J. Goold, A. M. Timpanaro, and G. Guarnieri, *Thermodynamic uncertainty relations from exchange fluctuation theorems*, Phys. Rev. Lett **123**, 090604 (2019).
[12] A. C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, *A unifying picture of generalized thermodynamic uncertainty relations*, J. Stat. Mech. **2019**, 084017 (2019).
[13] J. M. Horowitz, *Multpartite information flow for multiple Maxwell demons*, J. Stat. Mech. **2015**, P03006 (2015).
[14] J. M. Horowitz and M. Esposito, *Thermodynamics with Continuous Information Flow*, Phys. Rev. X **4**, 031015 (2014).
[15] D. Hartich, A. C. Barato, and U. Seifert, *Stochastic thermodynamics of bipartite systems: transfer entropy inequalities and a Maxwell’s demon interpretation*, J. Stat. Mech. **2014**, P02016 (2014).
[16] Z. Bar-Joseph, G. K. Gerbee, T. I. Lee, et al., *Computational discovery of gene modules and regulatory networks*, Nat. Biotechnol. **21**, 1337–1342 (2003).
[17] S. Otsubo and T. Sagawa, *Information-thermodynamic characterization of stochastic Boolean networks*, arXiv preprint arXiv:2001.02205 (2018).
[18] R. Fantoni, A. Giacometti, F. Sciortino, and G. Pastrone, *Cluster theory of Janus particles*, Soft Matter **7**, 2419 (2011).
[19] Z.-W. Li, Z.-Y. Lu, Z.-Y. Sun, and L.-J. Ana, *Model,
self-assembly structures, and phase diagram of soft Janus particles, Soft Matter 8, 6693 (2012).
[20] D. H. Wolpert, Minimum entropy production in multipartite processes due to neighborhood constraints, arXiv preprint arXiv:2001.02205 (2020).
[21] D. H. Wolpert, Fluctuation theorems for multipartite processes, arXiv preprint arXiv:2003.11144 (2020).
[22] D. H. Wolpert, Uncertainty relations and fluctuation theorems for Bayes nets, arXiv preprint arXiv:1911.02700 (2019).
[23] T. R. Gingrich and J. M. Horowitz, Fundamental bounds on first passage time fluctuations for currents, Phys. Rev. Lett. 119, 170601 (2017).
[24] M. Esposito, Stochastic thermodynamics under coarse graining, Phys. Rev. E 85, 041125 (2012).
[25] G. Falasco and M. Esposito, Dissipation-Time Uncertainty Relation, Phys. Rev. Lett. 125, 120604 (2020).
[26] U. Seifert, From Stochastic Thermodynamics to Thermodynamic Inference, Annu. Rev. Condens. Matter Phys. 10, 171–192 (2019).
[27] T. R. Gingrich and J. M. Horowitz, Thermodynamic uncertainty relations constrain non-equilibrium fluctuations, Nat. Phys. 16, 15–20 (2019).
[28] N. Merhav and Y. Kafri, Statistical properties of entropy production derived from fluctuation theorems, J. Stat. Mech. 2010, P12022 (2010).
[29] M. Esposito and C. V. den Broeck, Ensemble and trajectory thermodynamics: A brief introduction, Phys. A 418, 6 (2015).

Appendix A: Defining trajectory-level entropy flows

Trajectory-level EFs must be defined carefully, which requires that we introduce additional notation that was used in [20,22]. Write \( M(x) \) for the total number of state transitions during \([0,t_f]\) made by all subsystems. If \( M(x) \geq 1 \), then we define \( \eta_x : \{1,\ldots,M(x)\} \to \mathcal{N} \) as the function that maps any integer \( j \in \{1,\ldots,M(x)\} \) to the subsystem that changes its state by the corresponding transition. We write \( k(j) \) for the associated function specifying which reservoir is involved in the \( j \)th transition. Similarly, we define \( \tau_x : \{0,\ldots,M(x)\} \to \mathcal{N} \) to be the function that maps any integer \( j \in \{1,\ldots,M(x)\} \) to the time of the \( j \)th transition (where 0 is mapped to time 0.) So \( \eta^{-1}(i) \) is the set of all state transitions at which subsystem \( i \) changes state in the trajectory \( x \). (Note that this means that for any set of subsystems \( \alpha, \eta^{-1}(\alpha) := \cup_{i \in \alpha} \eta^{-1}(i) \) is the set of all state transitions.) These definitions allow us to precisely express the local entropy flow for any set of subsystems \( \alpha \):

\[
Q^\alpha(x) = \sum_{i \in \alpha} \sum_{j \in \eta^{-1}(i)} \ln \left[ \frac{W_{x(i)}(\tau(j))}{W_{x(i)}(\tau(j-1))} \right] \ln \left[ \frac{W_{x(i)}(\tau(j-1))}{W_{x(i)}(\tau(j))} \right].
\]

(A1)

Appendix B: Proof of Eq. (34)

MPP FTs, including Eq. (33), were derived in [21]. The joint FT of current and EP, in the form of Eq. (B1), was derived in [3]:

\[
P(\sigma, J) = e^{\sigma \tilde{P}} (-\sigma, -J)
\]

(B1)

We can combine these results to obtain joint FTs of vector-valued local currents and vector-valued local EPs in MPPs:

\[
P(\sigma^A, J^A) = \int D x_A P(x_A) \prod_{\omega \in A} \delta \left( \sigma^\omega - \ln \left[ \frac{P(x_\omega)}{\tilde{P}(x_\omega)} \right] \right)
\]

\[
= e^{\sigma^A \tilde{P}} \int D x_A P(x_A) \prod_{\omega \in A} \delta \left( \sigma^\omega - \ln \left[ \frac{P(x_\omega)}{\tilde{P}(x_\omega)} \right] \right)
\]

\[
= e^{\sigma^A \tilde{P}} \int D \tilde{x}_A \tilde{P}(\tilde{x}_A) \prod_{\omega \in A} \delta \left( -\sigma^\omega - \ln \left[ \frac{\tilde{P}(\tilde{x}_\omega)}{\tilde{P}(x_\omega)} \right] \right)
\]

\[
= e^{\sigma^A \tilde{P}} \tilde{P} (-\sigma^A, -J^A).
\]

(B2)

Appendix C: Derivation of Eq. (37)

Following [25,28], we first define the distribution \( Q(\sigma^A, J^A) \):

\[
Q(\sigma^A, J^A) = \left( 1 + e^{-\sigma^A A} \right) P(\sigma^A, J^A)
\]

(C1)

This is the joint probability distribution over all components of \( \sigma^A \) and \( J^A \). Using this notation, the first moment of a vector-valued current emerging in \( A \) is

\[
E[J^A] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(\sigma^A, J^A) J^A d\sigma^A dJ^A
\]

(C2)
Eq. (C2) captures the individual contributions of currents through each unit in \( \mathcal{A} \). Expanding it in terms of \( \mathbf{Q} \), we get:

\[
\mathbf{E}[\mathbf{J}^{\mathcal{A}}] = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \mathbf{Q} \left( \mathbf{E}^{\mathcal{A} \setminus \mathcal{B}} \right) \mathbf{J}^{\mathcal{A}} \left( 1 - e^{-\sigma^{\mathcal{A} \setminus \mathcal{B}}} \right) d\sigma^{\mathcal{A} \setminus \mathcal{B}} d\mathbf{J}^{\mathcal{A}}
\]

\[
= \left\langle \mathbf{J}^{\mathcal{A}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(C3)

Accordingly,

\[
\mathbf{E} \left[ \tilde{\sigma}^{\mathcal{A}} \right] = \left\langle \tilde{\sigma}^{\mathcal{A}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(C4)

\[
\mathbf{E} \left[ (\mathbf{J}^{\mathcal{A}})^2 \right] = \left\langle (\mathbf{J}^{\mathcal{A}})^2 \right\rangle_{\mathbf{Q}}
\]

(C5)

An application of the Cauchy-Schwarz inequality to this result gives

\[
(\mathbf{J}^{\mathcal{A}})^2 = \left\langle \mathbf{J}^{\mathcal{A}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}^2 \leq \left\langle \left( \mathbf{J}^{\mathcal{A}} \right)^2 \right\rangle_{\mathbf{Q}} \left\langle \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}^2
\]

(C6)

We can now apply the inequalities derived in [5] to Eq. (C6):

\[
\left\langle \tanh \left( \frac{\sigma^{\mathcal{A}}}{2} \right) \right\rangle_{\mathbf{Q}}^2 \leq \left\langle \tanh \left( \frac{\sigma^{\mathcal{A}}}{2} \tanh \left( \frac{\sigma^{\mathcal{A}}}{2} \right) \right) \right\rangle_{\mathbf{Q}} \leq \tanh \left( \frac{\left\langle \sigma^{\mathcal{A}} \right\rangle_{\mathbf{Q}}}{2} \right)
\]

(C7)

Combining Eqs. (C3)–(C7) we obtain the inequality, \( \frac{\left\langle (\mathbf{J}^{\mathcal{A}})^2 \right\rangle_{\mathbf{Q}}}{\left\langle \mathbf{J}^{\mathcal{A}} \right\rangle_{\mathbf{Q}}^2} \geq \tanh \left( \frac{\left\langle \sigma^{\mathcal{A}} \right\rangle_{\mathbf{Q}}}{2} \right)^{-1} \), which yields Eqn. Eq. [51] of the main text:

\[
\frac{\text{Var}[\mathbf{J}^{\mathcal{A}}]}{\left\langle \mathbf{J}^{\mathcal{A}} \right\rangle^2} \geq \frac{2}{e^{\left\langle \sigma^{\mathcal{A}} \right\rangle_{\mathbf{Q}}}} - 1.
\]

(C8)

Appendix D: Derivation of Eq. [36]

We consider two set of units \( \mathcal{A} \) and \( \mathcal{B} \). \( \mathbf{J}^{\mathcal{A}} \) indicates the vector whose components are the individual currents \( J^{\alpha} \) for units \( \alpha \in \mathcal{A} \). Similarly, \( \mathbf{J}^{\mathcal{B}} \) indicates the vector whose components are the individual currents \( J^{\beta} \) for units \( \beta \in \mathcal{B} \). We denote the covariance of \( \mathbf{J}^{\mathcal{A}} \) and \( \mathbf{J}^{\mathcal{B}} \) as \( \text{Covar}(\mathbf{J}^{\mathcal{A}}, \mathbf{J}^{\mathcal{B}}) \):

\[
\text{Covar}(\mathbf{J}^{\mathcal{A}}, \mathbf{J}^{\mathcal{B}}) = \mathbf{E}[(\mathbf{J}^{\mathcal{A}} - \mathbf{E}[\mathbf{J}^{\mathcal{A}}])(\mathbf{J}^{\mathcal{B}} - \mathbf{E}[\mathbf{J}^{\mathcal{B}}])]
\]

(D1)

As in Eq. (C1), we construct a new distribution from \( \mathbf{P} \):

\[
\mathbf{Q} \left( \mathbf{J}^{\mathcal{A}}, \mathbf{J}^{\mathcal{B}} \right) = \left( 1 + e^{-\sigma^{\mathcal{A}}} \right) \mathbf{P} \left( \mathbf{J}^{\mathcal{A}}, \mathbf{J}^{\mathcal{B}} \right)
\]

(D2)

Next we define:

\[
\mathbf{E}[\mathbf{J}^{\mathcal{A}}] = \left\langle \mathbf{J}^{\mathcal{A}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(D3)

\[
\mathbf{E}[\mathbf{J}^{\mathcal{B}}] = \left\langle \mathbf{J}^{\mathcal{B}} \tanh \left( \frac{\sigma^{\mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(D4)

\[
\mathbf{E} [\tilde{\sigma}^{\mathcal{A}}] = \left\langle \tilde{\sigma}^{\mathcal{A}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(D5)

\[
\mathbf{E}[\mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}}] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{P} (\tilde{\sigma}, \tilde{\mathbf{J}}) \mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}} d\tilde{\sigma} d\tilde{\mathbf{J}}
\]

(D6)

Repeating the steps in Appendix C, we write Eq. (D6) as:

\[
\mathbf{E}[\mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}}] = \left\langle \mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}} \tanh \left( \frac{\sigma^{\mathcal{A} \setminus \mathcal{B}}}{2} \right) \tanh \left( \frac{\sigma^{\mathcal{B}}}{2} \right) \right\rangle_{\mathbf{Q}}
\]

(D7)

We note that:

\[
\frac{\left\langle \mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}} \right\rangle_{\mathbf{Q}}}{\left\langle \mathbf{J}^{\mathcal{A}} \right\rangle_{\mathbf{Q}} \left\langle \mathbf{J}^{\mathcal{B}} \right\rangle_{\mathbf{Q}}} \leq \left( \frac{\left\langle \sigma^{\mathcal{A}} \right\rangle_{\mathbf{Q}}}{2} \right) \left( \frac{\left\langle \sigma^{\mathcal{B}} \right\rangle_{\mathbf{Q}}}{2} \right)^{-1}
\]

(D8)

We then use the product formula for hyperbolic tangents, \( \tanh(a+b) = \frac{\tanh(a) + \tanh(b)}{\tanh(a) \tanh(b) + 1} \). Hence we express Eq. (D8) as follows:

\[
\frac{\left\langle \mathbf{J}^{\mathcal{A}} \mathbf{J}^{\mathcal{B}} \right\rangle_{\mathbf{Q}}}{\left\langle \mathbf{J}^{\mathcal{A}} \right\rangle_{\mathbf{Q}} \left\langle \mathbf{J}^{\mathcal{B}} \right\rangle_{\mathbf{Q}}} \leq \left( \tanh \left( \frac{\left\langle \sigma^{\mathcal{A} \setminus \mathcal{B}} \right\rangle_{\mathbf{Q}}}{2} \right) \right) \left( \tanh \left( \frac{\left\langle \sigma^{\mathcal{B}} \right\rangle_{\mathbf{Q}}}{2} \right) \right)^{-1}
\]

(D9)

We also note that

\[
\frac{\tanh \left( \frac{\left\langle \sigma^{\mathcal{A} \setminus \mathcal{B}} \right\rangle_{\mathbf{Q}}}{2} \right)}{\tanh \left( \frac{\left\langle \sigma^{\mathcal{A} \setminus \mathcal{B}} + \sigma^{\mathcal{B}} \right\rangle_{\mathbf{Q}}}{2} \right)} = 1 - \frac{2 \left( e^{-\sigma^{\mathcal{A} \setminus \mathcal{B}}} + e^{-\sigma^{\mathcal{B}}} \right)}{1 + e^{-\sigma^{\mathcal{A} \setminus \mathcal{B}}} - e^{-\sigma^{\mathcal{B}}} + e^{-\sigma^{\mathcal{A} \setminus \mathcal{B}}} + e^{-\sigma^{\mathcal{B}}}}
\]
Finally, noting that

\[
\frac{\langle \vec{J}^A \vec{J}^B \rangle - \langle \vec{J}^A \rangle \langle \vec{J}^B \rangle}{\langle \vec{J}^A \rangle \langle \vec{J}^B \rangle} = \frac{\langle \vec{J}^A \vec{J}^B \rangle}{\langle \vec{J}^A \rangle \langle \vec{J}^B \rangle} - 1 \quad (D11)
\]

and combining with the results above, we obtain

\[
\frac{\text{Covar}[\vec{J}^A, \vec{J}^B]}{\langle \vec{J}^A \rangle \langle \vec{J}^B \rangle} \geq \frac{2 \cdot \left(1 + e^{\langle \sigma^A \cup \sigma^B \rangle}\right)}{e^{\langle \sigma^B \rangle} + e^{\langle \sigma^A \rangle} - \left(1 + e^{\langle \sigma^A \cup \sigma^B \rangle}\right)} \quad (D12)
\]