A New Aspect of Representations of $U_q(\hat{sl}_2)$
—Generic Case

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Abstract. An identity is derived from the tensor product representation $V_m(x) \otimes V_n(y)$ of $U_q(\hat{sl}_2)$ and a new basis of $V_m(x) \otimes V_n(y)$ is established.

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In this paper we take $q$ to be a complex number instead of a formal variable, unless otherwise pointed out explicitly. For any complex number $q$ and any integers $r$ and $s$ we use the notations

$$[r]_q = \frac{q^r - q^{-r}}{q - q^{-1}},$$

$$[r]_q! = \prod_{s=1}^{r} [s]_q,$$

$$\begin{bmatrix} r \\ s \end{bmatrix}_q = \frac{[r]_q!}{[s]_q![r-s]_q}, r \geq s,$$

and we set $[0]_q! = 1$. We also assume that $q$ is not a root of unity.

1. Some Basic Facts

In this section we list some basic facts about $U_q(\hat{sl}_2)$ and fix notations. For details we refer the readers to Ref. [1].

**Definition 1.1.** The quantum affine algebra $U_q(\hat{sl}_2)$ is the associative algebra over $\mathbb{C}$ with generators $e_i, f_i, K_i$ and $K_i^{-1} (i = 0, 1)$ and the following relations:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_0K_1 = K_1K_0,$$

$$K_i e_i K_i^{-1} = q^2 e_i, K_i f_i K_i^{-1} = q^{-2} f_i,$$

$$K_i e_j K_i^{-1} = q^{-2} e_j, K_i f_j K_i^{-1} = q^2 f_j, i \neq j,$$

$$[e_i, f_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$[e_0, f_1] = [e_1, f_0] = 0,$$

$$e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0,$$

$$f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0. (i \neq j).$$

Moreover, $U_q(\hat{sl}_2)$ is a Hopf algebra over $\mathbb{C}$ with the comultiplication

$$\triangle(e_i) = e_i \otimes K_i + 1 \otimes e_i,$$

$$\triangle(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i,$$

$$\triangle(K_i) = K_i \otimes K_i, \triangle(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

and the antipode

$$S(K_i) = K_i^{-1}, S(K_i^{-1}) = K_i,$$

$$S(e_i) = -e_i K_i^{-1}, S(f_i) = -K_i f_i.$$
Definition 1.2. The quantum algebra $U_q(sl_2)$ is the associative algebra over $C$ with generators $e, f, K$ and $K^{-1}$ and the following relations:

\begin{align*}
KK^{-1} &= K^{-1}K = 1, \\
KeK^{-1} &= q^2e, KfK^{-1} = q^{-2}f, \\
[e, f] &= \frac{K - K^{-1}}{q - q^{-1}}.
\end{align*}

It is a Hopf algebra over $C$ with the following comultiplication $\Delta$ and antipode $S$:

\begin{align*}
\Delta e &= e \otimes K + 1 \otimes e, \\
\Delta f &= f \otimes 1 + K^{-1} \otimes f, \\
\Delta K &= K \otimes K, \Delta K^{-1} = K^{-1} \otimes K^{-1}, \\
S(K) &= K^{-1}, S(K^{-1}) = K, S(e) = -eK^{-1}, S(f) = -Kf.
\end{align*}

There is an associative algebra homomorphism, known as evaluation map, from $U_q(sl_2)$ to $U_q(sl_2)[2]$.

Definition 1.3. For any $x \in C \setminus \{0\}$, the evaluation map $ev_x$ from $U_q(sl_2)$ to $U_q(sl_2)$ is the associative algebra homomorphism such that

\begin{align*}
ev_x(e_0) &= q^{-1}xf, ev_x(e_1) = e, \\
ev_x(f_0) &= qx^{-1}e, ev_x(f_1) = f, \\
ev_x(K_0) &= K^{-1}, ev_x(K_1) = K.
\end{align*}

It is clear that modules of $U_q(sl_2)$ can be obtained by pulling back modules of $U_q(sl_2)$ by the homomorphism $ev_x$. We denote by $V(x)$ the pull back module of $U_q(sl_2)$ of a module $V$ of $U_q(sl_2)$ by the evaluation map $ev_x$.

Definition 1.4. Let $V$ and $W$ be two modules of $U_q(sl_2)$. The $U_q(sl_2)$ module $W(y) \otimes V(x)$ is the vector space $W \otimes V$ with the module structure defined through the following action

\[ g(w \otimes v) \overset{\Delta}{=} (ev_y \otimes ev_x)\Delta g(w \otimes v), \forall g \in U_q(sl_2). \]

We consider the module $V_m(x) \otimes V_n(y)$ of $U_q(sl_2)$, where $V_n$ is the standard $n + 1$ dimensional module of $U_q(sl_2)$. There is a basis $\{v_i|i = 0, 1, \cdots, m\}$ of $V_m$ such that its module structure is defined through the following actions:

\begin{align*}
Kv_i &= q^{m-2i}v_i, \\
fv_i &= [i + 1]q^{i}v_{i+1}, \\
ev_i &= [m + 1 - i]q^{m-i}v_{i-1}, i = 0, 1, \cdots, m.
\end{align*}

Here we have used the convention $v_{-1} = v_{m+1} = 0$. From now on we denote by $\{w_i|i = 0, 1, \cdots, n\}$ the basis of $V_n$ satisfying these equations to distinguish it.
from that of $V_m$. Finally, we recall from [2] that, as a representation of $U_q(sl_2)$,

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{|m-n|}.$$  

2. An Identity

In this section we will derive an identity from the representation $V_m(x) \otimes V_n(y)$.

For any integer $l \leq \min\{m, n\}$, let

$$\Omega_l = \sum_{i=0}^{l} c_{i,l-i} v_i \otimes w_{l-i}, \Phi_l = \sum_{i=0}^{l} d_{i,l} v_{m-l+i} \otimes w_{n-i},$$

where

$$c_{i,l-i} = (-1)^i q^{i(2l-n-i-1)} \prod_{j=0}^{i} \frac{[n-l+j]}{[m-j+1]} q^{rac{m-n-2l+i}{2}},$$

$$d_{i,l} = (-1)^i q^{i(-m-2l-i-1)} \prod_{j=0}^{i} \frac{[m-l+j]}{[n-j+1]} q^{rac{m-n-2l+i}{2}}.$$

It is readily verified that

$$e \Omega_l = 0, f \Phi_l = 0.$$

Besides, $\Omega_l$ are the only highest vectors (up to a scalar multiple) in $V_m(x) \otimes V_n(y)$ with respect to $U_q(sl_2)$ and $\Phi_l$ the only lowest vectors. As a matter of fact, $\Omega_l$ generates the $U_q(sl_2)$ submodule $V_{m+n-2l}$. It then follows that $f^{m+n-2l} \Omega_l$ must be a scalar multiple of $\Phi_l$,

$$f^{m+n-2l} \Omega_l = \alpha_l \Phi_l,$$

where $\alpha_l$ is a constant.

We find that there are two ways, direct and indirect ones, of determining $\alpha_l$.

Thus by equating the two results from different ways we are able to establish an identity.

First, let us take the direct way. For any positive integer $k$ we have

$$\triangle f^k = \sum_{j=0}^{k} q^{-j(k-j)} \begin{bmatrix} k \\ j \end{bmatrix}_q K^{-j} f^{k-j} \otimes f^j.$$  

Using this formula, after some elementary calculation, we get the following coefficient of the term $v_{m-l} \otimes w_n$ in $f^{m+n-2l} \Omega_l$:

$$q^{-l(n-l)} \min\{l,m-l\} \sum_{i=0}^{\min\{l,m-l\}} (-1)^i q^{-i} \frac{[m+n-2l] q! [m-l] q! [n] q!}{[m-l-i] q! [n-l+i] q! [i] q! [l-i] q!} \times \prod_{j=0}^{i} \frac{[n-l+j]}{[m-j+1]} q^{rac{m-n-2l+i}{2}}.$$  

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On the other hand, the coefficient of the same term in $\Phi_l$ is

$$d_{0,l} = \frac{[m-l]_q}{[n+1]_q}.$$  

It follows that

$$\alpha_l = q^{-l(n-l)} \frac{[n+1]_q}{[m-l]_q} \sum_{i=0}^{\min(l,m-l)} (-1)^i q^{-i} \frac{[m+n-2l]_q! \cdot [m-l]_q! \cdot [n]_q!}{[m-l-i]_q \cdot [n-l+i]_q \cdot [l-i]_q} \times \prod_{j=0}^{i} \frac{[n-l+j]_q}{[m-j+1]_q}.$$  

Now let us turn to probe the detour. We need the following results which can be proved by direct calculation.

**Lemma 2.1.** For $l \geq 1$

$$e_0 \Omega_l = [2]_q [n-l]_q q^{-1} (xq^m - yq^{-n+2l-2}) \Omega_{l-1}.$$  

**Lemma 2.2.** For $l \geq 1$

$$e_0 \Phi_l = [m-l]_q q^{-1} (xq^n - yq^{-m+2l-2}) \Phi_{l-1}.$$  

It follows from Lemma 2.1 that

$$e_0 \Omega_l = c_{l-1} f^2 \Omega_{l-1} + c_l f \Omega_l + c_{l+1} \Omega_{l+1},$$  

where

$$c_{l-1} = \frac{[n-l]_q q^{-1} (xq^m - yq^{-n+2l-2})}{[m+n-2l+1]_q [m+n-2l+2]_q},$$  

and $c_l$ and $c_{l+1}$ are two other constants. We then have

$$e_0 f^{m+n-2l} \Omega_l = c_{l-1} f^{m+n-2l} \Omega_{l-1},$$  

namely,

$$\alpha_l e_0 \Phi_l = c_{l-1} \alpha_{l-1} \Phi_{l-1}.$$  

Combining this result with Lemma 2.2 we get

$$\frac{\alpha_l}{\alpha_{l-1}} = \frac{[n-l]_q}{[m-l]_q} \frac{[m+n-2l+1]_q}{[m+n-2l+2]_q} q^{m-n}, l \geq 1.$$  

This immediately leads to the following expression of $\alpha_l$.

$$\alpha_l = q^{(m-n)} [n]_q [n+1]_q [m+n]_q \prod_{i=1}^{l} \frac{[n-i]_q}{[m-i]_q} \prod_{i=1}^{2l} \frac{1}{[m+n-2l+i]_q}.$$  

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Comparing this expression with the previous one, after some simplification we arrive at the following

**Theorem 2.1.** Let $q$ be an indeterminate and $m$ and $l$ be two positive integers satisfying $m \geq l$. Then

\[
\min\{l, m-l\} \sum_{i=0}^{\min\{l, m-l\}} (-1)^i q^{-i} \left[ \frac{m-i}{i} \right]_q \left[ \frac{m-l-i}{l-i} \right]_q = q^{l(m-l)}
\]

is a polynomial identity.

**Corollary.**

\[
\sum_{i=0}^{\min\{l, m-l\}} (-1)^i C_i^l C_{m-i}^l = 1.
\]

**Proof.** Rewrite Theorem 1 as

\[
\min\{l, m-l\} \sum_{i=0}^{\min\{l, m-l\}} (-1)^i q^{-i} \left[ \begin{array}{c} l \\ i \\ q \\ l-i \end{array} \right] = q^{l(m-l)}
\]

and take the classical limit.

**3. A New Basis of $V_m(x) \otimes V_n(y)$**

In this section we will establish a new basis of $V_m(x) \otimes V_n(y)$ under a certain condition. Without losing generality we assume $n \leq m$.

Let $j, l$ be two non-negative integers and $j \leq l \leq n$. We introduce the notation

\[
\phi_{l,j} = e_j^0 f^l \Omega_0
\]

and define the sets

\[
\Delta_l = \{ \phi_{l,j} | j = 0, 1, \ldots, l, l = 0, 1, \ldots, n \}.
\]

Obviously, elements from different $\Delta_l$ are linearly independent as they belong to different weight spaces. We will prove for each $l \in \{0, 1, \cdots, n\}$ $\Delta_l$ is a linearly independent set under some condition. To this end, we will calculate explicitly the determinant of the coefficient matrix of $\Delta_l$ with respect to the linearly independent set $\{v_i \otimes w_j | i + j = l\}$.

Let

\[
\phi_{l,j} = \sum_{i=0}^{l} \gamma_{l,j}^{i,l-i} v_i \otimes w_{l-i}.
\]

We denote by $(\Delta_l)$ the $l + 1$ by $l + 1$ coefficient matrix $(\gamma_{l,j}^{i,l-i})$ whose row is marked by $j$ and column by $(i, l-i)$ and denote by $|\Delta_l|$ the corresponding determinant. From the equations

\[
\phi_{l+1,0} = e_0 \phi_{l,0}, \phi_{l+1,j} = f \phi_{l,j-1}, j = 1, 2, \cdots, l,
\]
we get by direct calculation
\[ \gamma_{l+1,i}^{i+1-i} = yq^{-1}[l-i+1]q^{\gamma_{l+1,i}} + xq^{-n+2l-2i+1}[l]q^{\gamma_{l+1,i}} - i+1 \]
\[ \gamma_{l+1,j}^{i+1} = q^{-m+2}[l-i+1]q^{\gamma_{l+1,j}} + [l]q^{\gamma_{l+1,j}} - i+1, \quad j = 1, 2, \ldots, l+1. \]

Multiply the column \((l+1,0)\) of \((\Delta_{l+1})\) by \((-q^{-m+2l})/[l+1]q\) and add the result to the column \((l,0)\). Then multiply the column \((l,0)\) of the resulted matrix by \((-q^{-m+2l-2})/[l+2])q\) and add the result to the column \((l-1,0)\). Continue this operation of canceling the "unwanted" terms but keeping the value of the determinant until we reach the first column. This process leads to the following inductive formula:
\[ |\Delta_{l+1}| = c[l+1]q^l |\Delta_l|, \]

where
\[ c = (y - xq^{-m-n+2l})q^{-(l+1)} \sum_{i=0}^{l} (-1)^i [l]q^i y^{l-i} q^{i(l-1)} q^{-i(m+n)}. \]

In deriving this formula, we have used the result:
\[ \gamma_{l+1,i}^{l-i} = x^{l-i} y^{[l]q^{-l}} q^{-i(l-i)(i-n)}, \]
which can easily be obtained by straightforward calculation.

To get a neat expression for \(c\) we need the following lemmas.

**Lemma 3.1.** Let \(l\) be a positive integer. For each \(k \in \{l-1, l-3, \ldots, l-1-2(l-1)\}\)
\[ \sum_{i=0}^{l} (-1)^i [l]q^i q^{ik} = 0. \]

Proof. We use induction method. When \(l = 1\) the formula is trivially true. Using the formula
\[ [l]q^i = q^{-i}[l-1]q^i + q^{-i}[l-1]q^i, \]
we have
\[ \sum_{i=0}^{l} (-1)^i [l]q^i q^{ik} = \sum_{i=0}^{l-1} (-1)^i q^{(k-1)i} [l-1]q^i - q^{l+k-1} \sum_{j=0}^{l-1} (-1)^j q^{(k-1)j} [l-1]q^j. \]

This implies the inductive step. Thus the lemma is proved.
Lemma 3.2. Let $m, n$ and $l$ be integers, $l \geq 1$. Then for any complex numbers $x, y$

$$\sum_{i=0}^{l} (-1)^i \binom{l}{i}_q x^i y^{l-i} q^{(l-1)q^{-i(m+n)}} = \prod_{j=0}^{l-1} (y - xq^{-m-n+2j}).$$

Proof. Regard the left hand side as a polynomial of order $l$ in indeterminate $y$. Then for each $j \in \{0, 1, \cdots, l-1\}$ $xq^{-m-n+2j}$ is a root of this polynomial thanks to Lemma 3.1. So the lemma follows.

With this lemma we are able to write down the determinant $|\Delta_{l+1}|$ in a neat form.

Proposition 3.1.

$$|\Delta_{l+1}| = \frac{[n]_q}{[m+1]_q} q^{\frac{l(l+1)(l+2)}{2}} \prod_{j=1}^{l+1} \prod_{j=0}^{j} (y - xq^{-m-n+2j})^{l+1-j}.$$ 

Proof. This is a direct consequence of the inductive formula and Lemma 3.2.

We are now finally prepared to prove the main result of this section. Let

$$\Delta = \bigcup_{l=0}^{n-1} \Delta_l \bigcup_{l=0}^{n-1} f^{m+n-2l} \Delta_l \bigcup_{i=0}^{m-n} f^i \Delta_n.$$ 

Here for a non-negative integer $i$ $f^i \Delta_i$ is defined to be the set $\{f^i \phi_{l,j} | \phi_{l,j} \in \Delta_l\}$.

Theorem 3.1. $\Delta$ is a basis of $V_m(x) \otimes V_n(y)$ if and only if for each $j \in \{0, 1, \cdots, n-1\}$, $y \neq xq^{-m-n+2j}$.

Proof. The "only if" part follows directly from Proposition 1. The same proposition, together with the decomposition rule of $V_m \otimes V_n$ presented at the end of the first section, implies the "if" part.

Before concluding this paper we would like to state a dual form of Theorem 3.1. Let

$$\varphi_{l,j} = f^l_j e^j \Phi_0$$

$$\Lambda_l = \{\varphi_{l,j} | j = 0, 1, \cdots, l\}, l = 0, 1, \cdots, n$$

$$\Lambda = \bigcup_{l=0}^{n-1} \Lambda_l \bigcup_{l=0}^{n-1} e^{m+n-2l} \Lambda_l \bigcup_{i=0}^{m-n} e^i \Lambda_n.$$ 

We have the following

Theorem 3.2. $\Lambda$ is a basis of $V_m(x) \otimes V_n(y)$ if and only if for each $j \in \{0, 1, \cdots, n-1\}$, $y \neq xq^{m+n-2j}$.

The proof of this theorem is parallel to that of the last theorem. We would rather omit the details.
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