Affine polar spaces derived from polar spaces and Grassmann structures defined on them

K. Prażmowski, M. Żynel

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Abstract

We prove that an affine polar space in the meaning of Cohen and Shult can be recovered from one of the three adjacency relations on a Grassmann structure over it. The result directly generalizes the results of our previous work where we use an affine space over a vector space equipped with a nondegenerate reflexive form as a starting point to the Cohen-Shult affine polar spaces.

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Introduction

Affine polar spaces were first introduced by Cohen and Shult in [6]. The idea of that new concept resembles the affine reduct of a projective space. Start with a polar space (cf. [19], [3], [5] as well as [18] and [12]) and simply delete a fixed hyperplane in it. A hyperplane is understood geometrically, i.e. as a proper subspace with the property that it meets every line in at least one point. The authors in [6] do not only show how to construct an affine polar space in a polar space but they also provide an axiomatic characterization of it. In this note we prefer the first framework where we can refer to the ambient polar space for convenience or clarity.

There are many analogies between polar spaces and affine polar spaces. Following Tits (cf. [18]), the former can be characterized as gamma spaces (cf. [5]) which every strong subspace is a projective space, and the latter are gamma spaces, which every strong subspace is an affine space.

It is one of standard questions posed within framework of geometry of (partial) linear spaces and Grassmann spaces associated with them, whether adjacency relation on $k$-subspaces is sufficient to recover the underlying space. The main goal of [15] was to show that adjacency relation (actually there are three different adjacency relations) on strong $k$-subspaces of an affine polar space is sufficient to reconstruct the underlying affine polar space. In that point the paper [15] solves problems analogous to those solved in [14] in the context of geometry of polar spaces. Note that situation investigated in [14] was even simpler a bit, as there are two distinct different adjacency relations on strong $k$-subspaces of a polar space. In [15] we, however, did not follow the idea of Cohen and Shult strictly. Instead we started from an affine space over a vector space with a reflexive form. Consequently, while
a polar space (with a few exceptions) can be thought of as a structure consisting of self-conjugate points and lines of a metric projective space, an affine polar space of 
15 is a structure of isotropic (i.e. all) points and isotropic lines of a metric affine space. Additional result of 
15 is that in terms of aforementioned adjacencies as single primitive notions we can also reinterpret the underlying affine space.

A natural question here is what is the gain or what is the loss in the approach of 
15 versus that of Cohen and Shult. The gain is that cases where the index of the affine polar space is 1, e.g. Minkowskian geometry, are included. The loss is that: the symplectic geometry, affine polar spaces that arise by deleting nontangent hyperplanes, and all affine polar spaces with nondesarguesian planes are excluded. So, many Cohen-Shult affine polar spaces were not covered by the approach of 
15.

The most important question is whether the main result of 
15 can be carried on to Cohen-Shult affine polar spaces and this paper gives a positive answer. That is we show that, for each reasonable dimension, the underlying Cohen-Shult affine polar space can be recovered from the binary adjacency on strong $k$-subspaces of it (cf. Theorem 
15). We also say that the affine polar space (and other respective structures) are definable in terms of corresponding adjacencies. The term definable, interpretable, reinterpretable or recoverable used here may raise some ambiguity: how to define points and relations on points in terms of subspaces and relations on them. A precise discussion of logical foundations of these problems can be found in 
13, 
15 Sect. 6]. Without coming into details let us say, roughly, that such a definability means that points can be identified with couples of subspaces – elements of corresponding structures, and under this identification suitable relations on points can be expressed as relations on corresponding couples of subspaces. Our reasoning here is standard, like in many papers on Chow-like theorems. The starting point is to determine and characterize the cliques of all considered adjacency relations in respective geometry. To this point we use here the readily apparent relationship between affine polar spaces and polar spaces like the one between affine and projective spaces. In consequence the cliques, as well as strong subspaces, in an affine polar space are reducts of respective cliques and strong subspaces in the "surrounding" polar space in which our affine polar space is defined. These cliques are used to reinterpret other various notions like ternary collinearity relation or lines. Then, the critical point is an induction step that brings the dimension $k$ down by one to dimension 0, which corresponds to the underlying structure. In order to do that we simply identify $(k - 1)$-subspaces with cliques of type star and reconstruct our adjacency relations on them. Such a structure of proof is typical and it was also applied in 
14 and 
15. It can be also applied to investigations on adjacencies of subspaces of a projective and of an affine space, in particular proving the Chow theorem (cf. 
4) for corresponding Grassmann structures (see 
10, 
9, see also 
8). It is also worth to mention that as a direct consequence of a reinterpretability of a sort discussed above a suitable theorem in the spirit of Chow follows: a bijection on $k$-subspaces which preserves (in both directions) the (currently investigated) adjacency is induced by an automorphism of the underlying space. So, this theorem remains valid also for affine polar spaces. In some cases a theorem like that may follow from general properties of the adjacency graph (cf. 
11). In our case we have three adjacencies and no general theorem like that of 
11 can be used.
Our main result, Theorem 1.5, is formulated as generally as possible, but it is new only for the case where Cohen-Shult approach misses that of [15]. We could have deal with this specific case exclusively here but the arguments are the same as in general.

1 Notions, problems

Let $\mathfrak{Q} = (Q, \mathcal{L})$ be a polar space of index $m$ (cf. [3]). The class of polar spaces is characterized axiomatically, but we do not have to quote here an adequate axiom system. It suffices to imagine $\mathfrak{Q}$ as a quadric embedded into a projective space together with the (projective) lines which lay on it. Remember, however, that these structures constitute only a part of models. Traditionally, for points $a, b \in Q$ we write $a \perp b$ when they are collinear. Given two subsets $X, Y \subseteq Q$ we write $X \perp Y$ when $x \perp y$ for all $x \in X$, $y \in Y$. We call a subspace $X$ of $\mathfrak{Q}$ strong if $X \perp X$ (it is called singular in [3]). We write $\wp(\mathfrak{Q})$ ($\wp_k(\mathfrak{Q})$) for the family of all (of all $k$-dimensional, respectively) strong subspaces of $\mathfrak{Q}$. Clearly, $k \leq m$. Let $\mathcal{H}$ be a hyperplane of $\mathfrak{Q}$ (cf. [6]). This means that $\mathcal{H}$ is a proper subspace of $\mathfrak{Q}$ such that any line in $\mathcal{L}$ crosses $\mathcal{H}$. Let $L \in \mathcal{L}$. Note that either $|\mathcal{H} \cap L| \geq 2$ and then $L \subseteq \mathcal{H}$ or $|\mathcal{H} \cap L| = 1$. We write $L^\infty = p$ if $\{p\} = L \cap \mathcal{H}$. The point $L^\infty$ is referred to as the improper point of $L$.

The affine polar space $\mathfrak{A}$ is a reduct of $\mathfrak{Q}$: the pointset of $\mathfrak{A}$ is the set $Q \setminus \mathcal{H}$ and the lineset $\mathcal{G}$ of $\mathfrak{A}$ is the set

$$\{L \setminus \mathcal{H} : L \in \mathcal{L} \} \setminus \{\emptyset\} = \{L \setminus \{L^\infty\} : L \in \mathcal{L}, L \not\subseteq \mathcal{H}\}.$$  

Clearly, if $l \in \mathcal{G}$ then there is the unique $L \in \mathcal{L}$ such that $l \subseteq L$. We see that $L = l \cup \{L^\infty\}$ and write $l = L$ as well as $l^\infty = L^\infty$. Two lines $l_1, l_2 \in \mathcal{G}$ are parallel, in symbols $l_1 \parallel l_2$, iff $l_1^\infty = l_2^\infty$. Recall that the parallelism $\parallel$ is definable in terms of the incidence structure $\mathfrak{A}$ (cf. [6]). The same notation $\perp$, $\wp(\mathfrak{A})$, $\wp_k(\mathfrak{A})$ and related terminology is used with respect to $\mathfrak{A}$ as to $\mathfrak{Q}$.

A strong subspace of $\mathfrak{A}$ is obtained by deleting $\mathcal{H}$ from a strong subspace of $\mathfrak{Q}$, so we can write

$$\wp_k(\mathfrak{A}) = \{X \setminus \mathcal{H} : X \in \wp_k(\mathfrak{Q}), X \not\subseteq \mathcal{H}\}.  \tag{1}$$

Note that if $X \in \wp(\mathfrak{Q})$ and $X \not\subseteq \mathcal{H}$ then dim$_{\mathfrak{Q}}(X) = \dim_{\mathfrak{A}}(X \setminus \mathcal{H})$. In particular, the index of $\mathfrak{A}$ (i.e. the maximal dimension of a strong subspace of $\mathfrak{A}$) is $m$ as well.

Two properties are crucial:

(i) Every $X \in \wp_k(\mathfrak{Q})$ carries (as a substructure of $\mathfrak{Q}$) the geometry of a $k$-dimensional projective space.

(ii) Every $A \in \wp_k(\mathfrak{A})$ carries (as a substructure of $\mathfrak{A}$) the geometry of a $k$-dimensional affine space. In particular, it contains a natural parallelism of lines, which is the restriction of the parallelism $\parallel$ defined on $\mathfrak{A}$ to the set of lines of $A$.

The property (i) can be considered as a characteristic axiom of polar spaces; similarly, (ii) can be considered as a characteristic axiom of affine polar spaces. Many of our results concerning classification of cliques can be proved synthetically under assumption that the underlying space is a $\Gamma$-space which satisfies (ii).
For a subspace $A$ of $\mathfrak{U}$ the least subspace of $\mathfrak{Q}$ that contains $A$ is written as $\overline{A}$. Then $A = \overline{A} \setminus \mathcal{H}$. We write $A^\infty := \overline{A} \cap \mathcal{H}$. Note that if $A \in \mathfrak{v}_k(\mathfrak{U})$ then $A^\infty \in \mathfrak{v}_{k-1}(\mathfrak{Q})$. It is seen that $A^\infty = \{l^\infty : l \in \mathfrak{G}, l \subset A\}$. So, we can extend the parallelism of lines to the parallelism of subspaces $A_1, A_2$, which we write as $A_1 \parallel A_2$, by requirement that $A_1^\infty = A_2^\infty$. Note that for $\mathfrak{v}(\mathfrak{U}) \ni A_1, A_2 \subset A \in \mathfrak{v}(\mathfrak{U})$ we have $A_1^\infty = A_2^\infty$ iff $A_1, A_2$ are parallel subspaces of the affine space $A$.

If $X, Y$ are subspaces of $\mathfrak{Q}$ we denote by $X \cup Y$ the join of $X$ and $Y$ in $\mathfrak{Q}$, i.e. the least subspace of $\mathfrak{Q}$ that contains $X \cup Y$; the meet of $X$ and $Y$, is simply $X \cap Y$. For $X, Y \in \mathfrak{v}(\mathfrak{Q})$ we have always $X \cap Y \in \mathfrak{v}(\mathfrak{Q})$ iff $X \perp Y$.

This paragraph remains true if we replace $\mathfrak{Q}$ the cliques in our affine polar space $\mathfrak{U}$ by $\mathfrak{U}$.

Three types of adjacencies on $\mathfrak{v}_k(\mathfrak{U})$ will be investigated in the sequel:

$$A_1 \sim_+ A_2 : \iff A_1 \cap A_2 \in \mathfrak{v}_{k-1}(\mathfrak{U}),$$
$$A_1 \sim_+ A_2 : \iff A_1 \cup A_2 \in \mathfrak{v}_{k+1}(\mathfrak{U}),$$
$$A_1 \sim_+ A_2 : \iff A_1 \sim_+ A_2 \land A_1 \sim_+ A_2.$$

These adjacencies may degenerate on $\mathfrak{v}_k(\mathfrak{U})$: $\sim_+$ is total for $k = 0$ and $\sim_+^+$ is empty for $k = m$; in other cases they do not degenerate. Consequently, $\sim_+$ is total for $k > 0$, and $\sim_+^+$ we assume that $k < m$.

Let us recall after \cite{14} two other adjacencies defined on $\mathfrak{v}_k(\mathfrak{Q})$:

$$X_1 \sim_+ X_2 : \iff X_1 \cap X_2 \in \mathfrak{v}_{k-1}(\mathfrak{Q}),$$
$$X_1 \sim_+ X_2 : \iff X_1 \cup X_2 \in \mathfrak{v}_{k+1}(\mathfrak{Q}).$$

From $X_1 \sim_+ X_2$ it follows $X_1 \sim_+ X_2$, but the converse implication fails. Note the following evident but useful relations $(A_1, A_2 \in \mathfrak{v}_k(\mathfrak{U}))$:

$$A_1 \sim_+ A_2 \implies \overline{A_1} \sim_+ \overline{A_2}, \quad (2)$$
$$A_1 \sim_+ A_2 \iff \overline{A_1} \sim_+ \overline{A_2}, \quad (3)$$
$$\overline{A_1} \sim_+ \overline{A_2} \iff A_1 \sim_+ A_2 \text{ or } A_1 \parallel A_2. \quad (4)$$

The maximal cliques of the two adjacency relations $\sim_+^+$ and $\sim_-$ in the polar space $\mathfrak{Q}$ has been established in \cite{14}. We give the complete list here as it is needed to get the cliques in our affine polar space $\mathfrak{U}$.

**Fact 1.1** (\cite{14} Prop. 3.3, Prop. 3.4). Let $\mathcal{K}$ be a subset of $\mathfrak{v}_k(\mathfrak{Q})$.

- $\mathcal{K}$ is a maximal $\sim_+^+$-clique iff it has one of the following two forms:
  - (a) $\mathcal{K} = T(B) = \{U \in \mathfrak{v}_k(\mathfrak{Q}) : U \subset B\}$, where $B \in \mathfrak{v}_{k+1}(\mathfrak{Q})$, or
  - (b) $\mathcal{K} = [C, M] = \{U \in \mathfrak{v}_k(\mathfrak{Q}) : C \subset U \subset M\}$, where $C \in \mathfrak{v}_{k-1}(\mathfrak{Q})$ and $C \subset M \in \mathfrak{v}_m(\mathfrak{Q})$.

- $\mathcal{K}$ is a maximal $\sim_-$-clique iff it has one of the following two forms:
  - (a) as above, or
  - (c) $\mathcal{K} = S(C) = \{U \in \mathfrak{v}_k(\mathfrak{Q}) : C \subset U\}$, where $C$ is as in (1).
In view of (2) and (3), if $K \subset \wp_k(\Gamma)$ is a maximal $\sim_-$-clique (a maximal $\sim^+$-clique) then the family $$\tilde{K} := \{A : A \in K\}$$ is a $\sim_-$-clique (a $\sim^+$-clique resp.), whose extension (under suitable dimensional assumptions) to a respective maximal clique $\overline{K}$ is unique. In other words we can simply say that the maximal cliques in $\Upsilon$ are reducts of respective cliques in the polar space $\Omega$. With the help of (1) we obtain the following list of possible forms of maximal cliques on $\wp_k(\Gamma)$.

**Proposition 1.2.** Let $K$ be a subset of $\wp_k(\Gamma)$.

- A maximal $\sim^+$-clique $K$ may have one of the following forms:
  1. $K = T(B) = \{A \in \wp_k(\Gamma) : A \subset B\}$, where $B \in \wp_{k+1}(\Gamma)$. Then $\tilde{K} \subset T(B)$.
  2. $K = [C, M] = \{A \in \wp_k(\Gamma) : C \subset A \subset M\}$, where $C \in \wp_{k-1}(\Gamma)$ and $C \subset M \in \wp_m(\Gamma)$. Then $\tilde{K} \subset [C, M]$.
  3. $K = [A_0, M]^* = \{A \in \wp_k(\Gamma) : A_0 \parallel A \subset M\}$, where $A_0 \in \wp_k(\Gamma)$ and $A_0 \subset M \in \wp_m(\Gamma)$. Then $\tilde{K} \subset [A_0^*, M]$.

A set of the form (g) is a $\sim^+$-clique; in case (h) it is not maximal iff $k = 0$ and $m > 1$, while in cases (e) and (f) it is not maximal iff $0 < k = m - 1$.

- A maximal $\sim_-$-clique $K$ may have one of the following forms:
  4. $K \subset T(B)$ with $T(B)$ defined as in (1) is a selector of $B^\infty$, i.e. for every $A \in T(B)$ there is exactly one $A' \in K$ with $A' \parallel A$.
  5. $K = S(C) = \{A \in \wp_k(\Gamma) : C \subset A\}$, where $C \in \wp_{k-1}(\Gamma)$. Then $\tilde{K} = S(\overline{C})$.

A set of the form (g) does not exist when $k = m$, otherwise sets of both types are maximal $\sim_-$-cliques.

- A maximal $\sim_-$-clique may have the form (e) or (f).

**Proof.** Let $K$ be a maximal $\sim^+$-clique in $\Upsilon$. By (3), $\tilde{K}$ is a $\sim^+$-clique. So, in view of (1) and (2) two possibilities arise: $\tilde{K} \subset T(B)$ for some $B \in \wp_{k+1}(\Omega)$, or $\tilde{K} \subset [C, M]$ for some $C \in \wp_{k-1}(\Omega)$ and $M \in \wp_m(\Omega)$ with $C \subset M$. In the first case $B \not\subset \mathcal{H}$ and therefore $K$ is the maximal $\sim^+$-clique of hyperplanes in the affine space $B \setminus \mathcal{H}$. Thus $K = T(B \setminus \mathcal{H})$. In the second case $M \not\subset \mathcal{H}$, so $M \setminus \mathcal{H}$ is an affine space that contains $K$ and $M \cap \mathcal{H}$ is its horizon. If $C \not\subset \mathcal{H}$, then $K$ is of the form (e), otherwise $K$ is of the form (f).

Now, let $K$ be a maximal $\sim_-$-clique in $\Upsilon$. By (3), $\tilde{K}$ is a $\sim_-$-clique. Again by (1) and (2) we have either $\tilde{K} \subset T(B)$ for some $B \in \wp_{k+1}(\Omega)$, or $\tilde{K} \subset S(C)$ for some $C \in \wp_{k-1}(\Omega)$. In the first case $B \not\subset \mathcal{H}$ and $K$ is a $\sim_-$-clique of hyperplanes in the affine space $B \setminus \mathcal{H}$, so no two elements of $K$ are parallel. Since $K$ is maximal there is a hyperplane in $K$ in every hyperplane direction of $B \setminus \mathcal{H}$. Hence $K$ has form (e). In the second case $C \not\subset \mathcal{H}$ as otherwise we would have parallel elements in $K$ which is impossible. Thus $K \subset S(C \setminus \mathcal{H})$. Since $K$ is maximal we get $K = S(C \setminus \mathcal{H})$.

If $K$ is a maximal $\sim_-$-clique in $\Upsilon$, then it is a clique with respect to both $\sim^+$ and $\sim_-$, so it is of the form (e) or (f).
If a subset $\mathcal{K}$ of $\wp_k(\mathfrak{U})$ is of one the forms \( [1] \) - \( [3] \), then it is evidently a clique of a corresponding adjacency. It is maximal iff $\mathcal{K}$ is maximal.

We denote classes of the sets of the form “$K = \ldots$” introduced above as follows:

\[
[1] = T, \quad [2] = S^\wedge, \quad [3] - S^\ast, \quad [4] - T^\ast, \quad [5] - S.
\]

A polar space $\Omega$ determines a partial linear space $P_k(\Omega)$ (cf. \[5\], \[14\]) with the point set $\wp_k(\Omega)$ and with the line set $P_k(\Omega)$ consisting of the $k$-pencils, i.e. with the sets

\[
p(C, B) = \{X \in \wp_k(\Omega) : C \subset X \subset B\},
\]

where $C \in \wp_{k-1}(\Omega)$ and $C \subset B \in \wp_{k+1}(\Omega)$.

Accordingly, we define on the set $\wp_k(\mathfrak{U})$ two structures of a partial linear space:

\[
P_k(\mathfrak{U}) = \langle \wp_k(\mathfrak{U}), P_k(\mathfrak{U}) \rangle \quad \text{and} \quad P^\ast_k(\mathfrak{U}) = \langle \wp_k(\mathfrak{U}), P^\ast_k(\mathfrak{U}) \rangle,
\]

where $P_k(\mathfrak{U})$ consists of the pencils $p(C, B)$ with $C \in \wp_{k-1}(\mathfrak{U})$ and $C \subset B \in \wp_{k+1}(\mathfrak{U})$, defined analogously as pencils over $\Omega$, and $P^\ast_k(\mathfrak{U})$ consists of the nonvoid sets $p|_{\sim} = \{X \in \mathfrak{H} : X \in p, X \not\subset \mathfrak{H}\}$ with $p \in P_k(\Omega)$. Note that for $C \in \wp_{k-1}(\mathfrak{U})$ and $C \subset B \in \wp_{k+1}(\mathfrak{U})$ we have $p(C, B) = p(\mathfrak{C}, \mathfrak{B})|_{\sim}$ and $p(C, B) = p(\mathfrak{C}, \mathfrak{B})$. Consequently, $P_k(\mathfrak{U}) \subset P^\ast_k(\mathfrak{U})$. Let $A_0 \in \wp_k(\mathfrak{U})$ and $A_0 \subset B \in \wp_{k+1}(\mathfrak{U})$; we set $p^\ast(A_0, B) = \{A \in \wp_k(\mathfrak{U}) : A_0 \parallel A \subset B\}$. Let $P^\ast_k(\mathfrak{U})$ be the class of all the sets of the form $p^\ast(A_0, B)$. It is seen that $p^\ast(A_0, B) = p(A_0^\infty, B)|_{\sim}$ and $p^\ast(A_0, B) = p(A_0^\infty, \mathfrak{B}) \setminus \{B^\infty\}$. Finally, we have $P^\ast_k(\mathfrak{U}) = P_k(\mathfrak{U}) \cup P^\ast_k(\mathfrak{U})$. Structures $P_k(\mathfrak{U})$ and $P^\ast_k(\mathfrak{U})$ are usually referred to as spaces of $k$-pencils or Grassmann spaces over $\mathfrak{U}$. In terms of these structures we can say that

\(~^+\) on $\wp_k(\mathfrak{U})$ is the binary collinearity in $P^\ast_k(\mathfrak{U})$, and

\(~\) on $\wp_k(\mathfrak{U})$ is the binary collinearity in $P_k(\mathfrak{U})$.

Notice that the above definitions of pencils require that $k < m$ and we will assume that implicitly when referring to either $P_k(\mathfrak{U})$ or $P^\ast_k(\mathfrak{U})$. For $k = 0$ we have $\mathfrak{U} \cong P_0(\mathfrak{U}) = P^\ast_0(\mathfrak{U})$.

One could note a similarity of the above construction to the, analogous, construction of the space of pencils $P_k(\mathfrak{A})$ associated with an affine space $\mathfrak{A}$ (cf. definitions in \[17\], \[2\], or, in a more modern paper \[7\]).

Let us point out an evident but useful consequence of \( [11] \).

**Fact 1.3.** Let $D \in \wp(\mathfrak{U})$, $\dim(D) > k$. Then the restriction of $P_k(\mathfrak{U})$ to the segment $[\emptyset, D] = \{A \in \wp(\mathfrak{U}) : A \subset D\}$ is $P_k(D)$, and the restriction of $P^\ast_k(\mathfrak{U})$ is $P^\ast_k(D)$.

One can say that ‘locally’ $P_k(\mathfrak{U})$ is an affine Grassmann space.

One more fact, which is a consequence (simple: verify axioms of a polar space) and an easy (formal) strengthening of \[13\] Theorem 3.5] is needed.

**Fact 1.4.** Let $C \in \wp(\mathfrak{Q})$, $\dim(C) = k - 1 < m$. Then the restriction of $P_k(\mathfrak{Q})$ to the segment $[C, Q] = \{X \in \wp(\mathfrak{Q}) : C \subset X\}$ in $P_k(\mathfrak{Q})$ is a polar space.

Our goal is as follows.
**Theorem 1.5.** If $0 < k \leq m$ then for $\sim = \sim_+$, while if $k \leq m - 1$ then for $\sim \in \{\sim^+, \sim\}$ the affine polar space $\mathfrak{U}$ can be recovered from $\langle \mathfrak{g}_k(\mathfrak{U}), \sim \rangle$.

The case where $m = 1$ is excluded in the Cohen-Shult axiom system in [6] but it is covered by both constructive approaches: the one in [15] and the other in [6] consisting in hyperplane removal.

Note that $P_0(\mathfrak{U}) = P_0^1(\mathfrak{U})$ is, up to an isomorphism, the affine polar space $\mathfrak{U}$ itself. Hence the following is immediate by 1.5.

**Corollary 1.6.** $\mathfrak{U}$ is definable in both $P_k(\mathfrak{U})$ and $P^1_k(\mathfrak{U})$, provided that $k < m$.

2 The reasoning

Provided that $k < m$, let us write $L(A_1, A_2, A_3)$ if $A_1, A_2, A_3$ are collinear in $P_k(\mathfrak{U})$, and $L^1(A_1, A_2, A_3)$ if they are collinear in $P^1_k(\mathfrak{U})$.

We also introduce an auxiliary structure

$$G_k(\mathfrak{U}) := \langle \mathfrak{g}_k(\mathfrak{U}), \mathfrak{g}_{k+1}(\mathfrak{U}), \subset \rangle,$$

which is a partial linear space sometimes called a Grassmannian over $\mathfrak{U}$. Note that $G_0(\mathfrak{U}) \cong \mathfrak{U} \cong P_0(\mathfrak{U}) = P_0^1(\mathfrak{U})$. Later, we will use the first isomorphism to show that $\mathfrak{U}$ can be reconstructed in terms of adjacency on $k$-subspaces.

**Fact 2.1.** Let $K \in T^*$. Then $| \cap K | \leq 1$.

**Proof.** Let $K \subset T(B)$ for $B$ as in [12][13]. Clearly, $\cap K$ is a (affine) subspace of $B$. Suppose that $\cap K$ contains a line $L$ of $\mathfrak{U}$. Then $B^\infty$ contains a hyperplane $C$ that misses $L^\infty$. On the other hand there is $A \in K$ with $A^\infty = C$. From assumption $L \subset A$ and a contradiction arises. ■

**Fact 2.2.** Let $K_1 \in T$, $K_2 \in S^\wedge$, $K_3 \in S^*$, $K_4 \in T^*$, and $K_5 \in S$.

(i) Either $|K_1 \cap K_2| \leq 1$ or $K_1 \cap K_2 \in P_k(\mathfrak{U})$.

(ii) Either $|K_1 \cap K_3| \leq 1$ or $K_1 \cap K_3 \in P^1_k(\mathfrak{U})$.

(iii) $|K_2 \cap K_3| \leq 1$.

(iv) $K_1$ is a strong subspace of $P^1_k(\mathfrak{U})$; it carries geometry of a $(k + 1)$-dimensional dual affine space i.e. of a $(k + 1)$-dimensional projective space with one point deleted. It is a subspace of $P_k(\mathfrak{U})$ but not strong.

(v) $K_2$ is a strong subspace both in $P^1_k(\mathfrak{U})$ and in $P_k(\mathfrak{U})$; it carries geometry of a $(m - k)$-dimensional projective space.

(vi) $K_3$ is a strong subspace in $P^1_k(\mathfrak{U})$ and an anti-clique in $P_k(\mathfrak{U})$; it carries geometry of a $(m - k)$-dimensional affine space.

(vii) If $\cap K_4 \neq \emptyset$, then $K_4$ carries geometry of a projective space. In general, however the geometry of $K_4$ is much more complex.

(viii) If $k < m$ then $K_5$ is a subspace in $P^1_k(\mathfrak{U})$ but it is not strong; the restriction of $P_k(\mathfrak{U})$ to $K_5$ carries geometry of a polar space.

Every line of $P_k(\mathfrak{U})$ has form $K_1 \cap K_2$ for some $K_1 \in T$ and $K_2 \in S^\wedge$; every line of $P^1_k(\mathfrak{U})$ has form $K_1 \cap K_2$ for some $K_1 \in T$ and $K_2 \in S^\wedge \cup S^*$. 
PROOF. The reasoning follows by close inspection of 1.2 so, we use its notation.

(i) If \(|K_1 \cap K_2| > 1\), then \(K_1 \cap K_2 = [C, B] = p(C, B)\), where \(C, B\) are as in 1.2 (4), (5) and \(C \subset B\).

(ii) If \(|K_1 \cap K_3| > 1\), then \(K_1 \cap K_3 = [A^\infty, B]^* = p^*(A_0, B)\), where \(A_0, B\) are as in 1.2 (4), (6) and \(A_0 \subset B\).

(iii) Immediately by 1.2 as no two elements of \(K_2\) are parallel.

In cases (iv), (v), (vi), and (vii) the corresponding \(K_4\) consists of subspaces of an affine space: of \(B, M, M,\) and \(B\) in respective cases. Our claim is a consequence of 1.3 and known properties of affine Grassmann spaces. In (viii) we use, additionally, 2.1 as in this case \(K_4 = [a, B] = \{A \in \wp_k(\Omega) : a \in A \subset B\}\) provided that a point \(a\) is in \(\bigcap K_4\).

Let us point out some deviations for extreme values of \(k\) and \(m\):

(i) If \(m = 1\) then \(\sim\) and \(\sim^+\) are sensible only for \(k = 0\) and then \(T(B) \in T\) is a line of \(\Omega\).

(ii) If \(k = m\) then the classes \(T, S^\wedge, S^*, T^*\) are void.

(iii) If \(k = m - 1\) then \([C, M] \in S^\wedge\) is a line of \(P_k(\Omega)\) and \([A_0, M]^* \in S^*\) is a line of \(P^*_k(\Omega)\). In that case a line of \(P^*_k(\Omega)\) has exactly one extension to a maximal \(\sim^*\)-clique.

(iv) The geometries on the elements of \(T, S^\wedge,\) and \(S^*\) are pairwise distinct provided that \(k \neq 0\) or \(k \neq m - 1\). If \(k = 0\) and \(m = 1\) then \(T\) and \(S^*\) both consist of affine lines.

**Lemma 2.3.** Let \(A_1, A_2 \in \wp_k(\Omega)\) with \(A_1 \sim^+ A_2\). Set \(B := A_1 \cup A_2\) and

\[
\mathcal{X} := \bigcap \{K \in T \cup S^\wedge \cup S^* : A_1, A_2 \in K\}.
\]  

If \(A_1 \sim_\wedge A_2\), then \(\mathcal{X} = p(A_1 \cap A_2, B)\) and if \(A_1 \parallel A_2\), then \(\mathcal{X} = p^*(A_1, B)\). In any case \(\mathcal{X} \in P^*_k(\Omega)\). Moreover,

\[
\mathcal{X} = \bigcap \{K \in S^\wedge \cup S^* : A_1, A_2 \in K\}.
\]  

Let \(k < m - 1\). Then

\(A_3 \in \mathcal{X}\) (i.e. \(L^1(A_1, A_2, A_3)\)) \iff (\forall A)[A \sim^+ A_1, A_2 \implies A \sim^+ A_3]\]  

for arbitrary \(A_3 \in \wp_k(\Omega)\).

**Proof.** In view of 1.2 and 2.2(iii) under our assumptions \(A_1, A_2\) are in two of the three possible maximal \(\sim^*\)-cliques: one of type \(T\) and the other of type \(S^\wedge\) or \(S^*\), depending on whether \(A_1 \sim_\wedge A_2\) or \(A_1 \parallel A_2\) respectively. The first statement follows from 2.2(iii). (iii).

To prove (vi) set \(C := \overline{A_1 \cap A_2}\) and observe that

\[
\bigcap \{\hat{K} : A_1, A_2 \in K \in S^\wedge \cup S^*\} = \bigcap \{[C, M] : M \in \wp_m(\Omega), \overline{A_1} \cup \overline{A_2} \subset M\} = [C, \overline{B}].
\]
Deleting $\mathcal{H}$ from $\mathfrak{Q}$ that set becomes either $p(A_1 \cap A_2, B)$, or $p^*(A_1, B)$ depending on whether $C \not\subset \mathcal{H}$ or $C \subset \mathcal{H}$ respectively.

To prove (7) note that the right hand side of it means that $A_3$ is $\sim^+$-adjacent to every element of all the cliques $A_1, A_2$ belong to. In particular, $A_3$ belongs to each of the maximal $\sim^+$-cliques that contains $A_1, A_2$. This suffices to state that equivalently $A_3$ is in the meet of the two appropriate cliques, i.e. $A_3$ is collinear with $A_1, A_2$ in $P_k^1(\mathfrak{M})$.

\textbf{Corollary 2.4.} Let $k < m - 1$. Then $P_k^1(\mathfrak{M})$ is definable in terms of $\sim^+$ on $\mathfrak{Q}_k(\mathfrak{M})$. If $0 < k = m - 1$, then the structure $G_k^0(\mathfrak{M})$ is definable in terms of $\sim^+$.

\textbf{Corollary 2.5.} If $k < m - 1$, then the three types $T$, $S^\wedge$, and $S^*$ of maximal $\sim^+$-cliques are distinguishable in terms of $\sim^+$, as they (as subspaces of $P_k^1(\mathfrak{M})$) carry distinct geometries.

\textbf{Corollary 2.6.} The class of maximal strong subspaces of $P_k^1(\mathfrak{M})$ is equal to $T \cup S^\wedge \cup S^*$ when $k < m - 1$, and it is equal to $T$ when $k = m - 1$. The classes $P_k^1(\mathfrak{M})$ and $P_k^2(\mathfrak{M})$ are distinguishable in terms of geometry of $P_k^1(\mathfrak{M})$, provided $k \neq 0$; consequently, $P_k(\mathfrak{M})$ is definable in $P_k^1(\mathfrak{M})$.

\textbf{Proof.} The first claim follows immediately from $\mathfrak{R}_2$. To prove the second claim it suffices to note that if $p \in P_k^1(\mathfrak{M})$ then

\[ p \in P_k^1(\mathfrak{M}) \text{ iff } p \subset K \text{ for some } K \in S^* \text{ and} \]

\[ p \in P_k(\mathfrak{M}) \text{ iff } p \subset K \text{ for some } K \in S^\wedge. \]

This closes the proof in case $k < m - 1$. Let $k = m - 1$, then $X = T(B) \in T$ is simply a dual affine space, i.e. a projective space with one point deleted; it is known that “affine” lines on $X$ (which are exactly the elements of $P_k^1(\mathfrak{M})$ contained in $X$) can be distinguished from the class of all the lines on $X$ (cf. [16]).

Note that the two types of pencils within $P_k^1(\mathfrak{M})$ can be directly distinguished as follows. Let $p \in P_k^1(\mathfrak{M})$. We have $p \in P_k^1(\mathfrak{M})$ iff there is a triangle $\Delta$ in $P_k^1(\mathfrak{M})$ such that $p$ misses the vertices of $\Delta$ and crosses exactly two of its sides.

\textbf{Lemma 2.7.} Let $K \in T^* \cup S$ and $A \in K$.

(i) Let $K \in S$. If $k = m$ we assume, additionally, that $\mathfrak{Q}$ is not of type D (which is read, in our terminology, as $|S(C)| \geq 3$ with $C \in \mathfrak{Q}_{k-1}(\mathfrak{Q})$). Then an element of $T^* \cup S$ that contains all the elements of $K$ except, possibly, $A$ contains $A$ as well, and thus it coincides with $K$.

(ii) Let $K \in T^*$ (note that then $k < m$). Then there is $A' \in \mathfrak{Q}_k(\mathfrak{M})$ such that $A \neq A'$ and $K \setminus \{A\} \cup \{A'\} \in T^* \cup S$. Consequently, the two types $T^*$ and $S$ of $\sim_-$-cliques are distinguishable in terms of $\sim_-$. The same property distinguishes $T^*$ from $S^\wedge$ and thus these two types of $\sim$-cliques are distinguishable in terms of $\sim_\wedge$. 


Proof. Recall by 1.2 that a clique \( K \) in \( T^* \) is a selector of \( B^{\infty} \), i.e., the set of distinct representatives from all possible directions of \( k \)-subspaces in \( B \). Every such a representative \( A \) can be selected up to parallelism, in other words every \( A \) can be replaced by \( A' \) such that \( A' \neq A, A' \parallel A \) and \( A' \in T(B) \). This is not doable with elements in cliques of type \( S \) or \( S^\wedge \). \( \square \)

**Lemma 2.8.** Let \( A_1, A_2, A_3 \in \wp_k(\Omega) \) be pairwise distinct. If \( k < m \), then we have

\[
L(A_1, A_2, A_3) \iff (\exists K_1 \in T^*)(\exists K_2 \in S^\wedge)[A_1, A_2, A_3 \in K_1, K_2]
\]

and

\[
L(A_1, A_2, A_3) \iff (\exists K_1 \in T^*)(\exists K_2 \in S^\wedge)[A_1, A_2, A_3 \in K_1, K_2].
\]

**Proof.** \( \Rightarrow \): Assume that \( A_1, A_2, A_3 \in p(C, B) \), where \( C \in \wp_{k-1}(\Omega) \) and \( C \subset B \in \wp_{k+1}(\Omega) \). As no two of \( A_1, A_2, A_3 \) are parallel take a selector \( K_1 \in T^* \) of \( B^{\infty} \) with \( A_1, A_2, A_3 \in K_1 \), and take \( K_2 := S(C) \) to get 8 or \( K_2 := [C, M] \) for some \( M \in \wp_m(\Omega) \) with \( B \subset M \) to get 9. We are through here by 1.2.

\( \Leftarrow \): Assume that \( K_1 \subset T(B) \) and \( K_2 = S(C) \) or \( K_2 = [C, M] \), for some \( B, C, M \) like in 1.2. In both cases \( K_1 \cap K_2 \subset p(C, B) \) and thus \( A_1, A_2, A_3 \) are collinear in \( P_k(\Omega) \). \( \square \)

**Corollary 2.9.** If \( k < m \), then the structure \( P_k(\Omega) \) is definable both in terms of \( \sim \) and in terms of \( \sim^+ \).

**Fact 2.10.** Let \( K \subset T^* \) and take the least subspace \( K' \) of \( P_k(\Omega) \) that contains \( K \). If \( \cap K \neq \emptyset \) then \( K' = K \). If \( \cap K = \emptyset \) then \( K' = T(B) \) for some \( B \in \wp_{k+1}(\Omega) \).

**Proof.** Let \( K \subset T(B) \) for \( B \) as in 1.2. From 2.1 there are two cases to consider. Firstly, let \( a \) be a common point of all \( A \in K \). Then \( K = [a, B] \), which is, already, a subspace of \( P_k(\Omega) \). If \( \cap K = \emptyset \), the claim is evident. \( \square \)

Note that in the first case of 2.10 \( K' (= K \) here) carries geometry of some projective space and in the second case the geometry of a dual affine space.

**Corollary 2.11.** A maximal strong subspace of \( P_k(\Omega) \) is either an element of \( S^\wedge \) or it has form \([a, B] \) with \( B \in \wp_{k+1}(\Omega) \) and a point \( a \) of \( \Omega \) on \( B \).

In particular case where \( k = 1, m = 2 \) maximal strong subspaces of \( P_k(\Omega) \) are lines.

**Corollary 2.12.** The class \( T \) is definable in the structure \( P_k(\Omega) \). Consequently, the structure \( P_k(\Omega) \) and the relation \( \sim^+ \) on \( \wp_k(\Omega) \) are definable in \( P_k(\Omega) \).

Further reasoning to prove \( \[1.23] \) is standard and we will give only a brief overview here. Let us begin with \( k < m - 1 \). We will show the induction step, that is, we start with one of the adjacencies \( \sim_2, \sim^+ \) or \( \sim \) on \( \wp_k(\Omega) \) and in terms of such a system interpret adequate adjacency on \( \wp_{k-1}(\Omega) \). So, consider two maps:

\[
\begin{align*}
f: \wp_{k-1}(\Omega) & \ni C \mapsto S(C) \in S, \\
g: \wp_{k-1}(\Omega) & \ni C \mapsto \{[C, Y]: C \subset Y \in \wp_m(\Omega)\} \subset S^\wedge.
\end{align*}
\]
By 1.2, \( S \cup T^* \) consists of the maximal \( \sim_- \)-cliques, by 2.7 \( S \) and \( T^* \) are distinguishable in terms of \( \sim_- \), and thus the image \( S \) of \( f \) is definable in terms of \( \sim_- \) on \( \wp_k(\mathfrak{U}) \). The map \( f \) sets a one-to-one correspondence between elements of \( S \) and \( \wp_{k-1}(\mathfrak{U}) \). Moreover, \( S(C_1) \cap S(C_2) \neq \emptyset \) iff \( C_1 \sqsubseteq C_2 \) which gives \( \sim^+ \) on \( \wp_{k-1}(\mathfrak{U}) \).

Similarly, by 1.2, \( S^* \cup T^* \cup T \) is the class of the maximal \( \sim^+ \)-cliques, and by 2.5, \( S^\uparrow \) is distinguishable in terms of \( \sim^+ \) on \( \wp_k(\mathfrak{U}) \). Finally, by 1.2, the elements of \( S^\uparrow \cup T^* \) are the maximal \( \sim \)-cliques, and by 2.7, \( S^\uparrow \) and \( T^* \) are distinguishable in terms of \( \sim \). So, the class \( S^\uparrow \) is definable on \( \wp_k(\mathfrak{U}) \) both in terms of \( \sim^+ \) and in terms of \( \sim \). Two stars \( K_1, K_2 \in S^\uparrow \) are said to be related iff \( |K_1 \cap K_2| \geq 2 \). If so, we write \( K_1 \approx K_2 \). If \( K_i = [C_i, V_i] \in S^\uparrow \), \( i = 1, 2 \) and \( K_1 \approx K_2 \) then, clearly, \( C_1 = C_2 \). Let \( C \in \wp_{k-1}(\mathfrak{U}) \). Then \( C \in \wp_{k-1}(\mathfrak{U}) \). It is known that \( [C, Q] \) induces a polar space which is connected and therefore the transitive closure of the relation \( \approx \) partitions the family \( S^\uparrow \) into equivalence classes which uniquely correspond to the elements of \( \wp_{k-1}(\mathfrak{U}) \) via the map \( g \). The same trick as in the previous paragraph gives us \( \sim^+ \) on \( \wp_{k-1}(\mathfrak{U}) \).

Note that for \( C \in \wp_{k-1}(\mathfrak{U}) \) and \( A \in \wp_k(\mathfrak{U}) \) we have \( C \subset A \) iff \( A \in f(C) \) as well as iff \( A \in \bigcup g(C) \). Hence, what we have actually defined is \( G_{k-1}(\mathfrak{U}) \). In turn, the relation \( \sim^+ \) on \( \wp_{k-1}(\mathfrak{U}) \) remains definable in \( G_{k-1}(\mathfrak{U}) \) and we can continue our inductive procedure as long as \( k \geq 1 \) (so, \( k - 1 \geq 0 \)). Proceeding inductively, we end up with \( G_0(\mathfrak{U}) \) which is, up to an isomorphism, our affine polar space \( \mathfrak{U} \) and that way Theorem 1.3 is proved. Note that in case \( k < m - 1 \) Corollary 1.6 is an immediate consequence of the above result.

Now, let us pay attention to the cases \( k = m - 1 \) and \( k = m \). Here, we need some other techniques. In view of 1.2 and 2.2 the maximal cliques of the relation \( \sim^+ \) defined on \( \wp_{m-1}(\mathfrak{U}) \) are the elements of \( T \), and the maximal cliques of \( \sim_- \) defined on \( \wp_m(\mathfrak{U}) \) are the elements of \( S \), which yields that

the structure \( G_{m-1}(\mathfrak{U}) \) is definable in both \( \langle \wp_m(\mathfrak{U}), \sim_- \rangle \) and \( \langle \wp_{m-1}(\mathfrak{U}), \sim^+ \rangle \), and

the structures \( \langle \wp_m(\mathfrak{U}), \sim_- \rangle \) and \( \langle \wp_{m-1}(\mathfrak{U}), \sim^+ \rangle \) are mutually definable (10)

(cf. a particular case of (10) in 2.3).

The case where \( \sim \) is defined on \( \wp_{m-1}(\mathfrak{U}) \) requires a different treatment. In this case the function \( g \) makes sense as previously, but now related stars would coincide, as the elements of \( S^* \) are simply the lines of \( P_{m-1}(\mathfrak{U}) \). So, it is impossible to identify the elements of \( \wp_{m-1}(\mathfrak{U}) \) with the classes of mutually related stars.

**Proposition 2.13.** The relation \( \sim^+ \) on \( \wp_{m-1}(\mathfrak{U}) \) can be characterized in terms of \( \sim \) defined on \( \wp_{m-1}(\mathfrak{U}) \). Consequently, \( G_{m-1}(\mathfrak{U}) \) can be recovered within \( P_{m-1}(\mathfrak{U}) \).

**Proof.** In view of 1.2, the maximal cliques of the relation \( \sim \) on \( \wp_{m-1}(\mathfrak{U}) \) are the selectors in \( T^* \) and the stars in \( S^* \). From 2.7, the elements of \( T^* \) and \( S^* \) are distinguishable.

For \( A_1, A_2, A_3 \in \wp_{m-1} \) write \( \Pi(A_1, A_2, A_3) \) when \( A_1, A_2, A_3 \in \mathcal{K} \) for some \( \mathcal{K} \in T^* \) and there is no \( \mathcal{K}' \in S^* \) with \( A_1, A_2, A_3 \in \mathcal{K}' \). Directly in terms of \( P_{m-1}(\mathfrak{U}) \) one can express this definition as follows: \( \Pi(A_1, A_2, A_3) \) iff \( A_1, A_2, A_3 \) are the vertices of a proper triangle. In any case, if \( \Pi(A_1, A_2, A_3) \), then \( A_1, A_2, A_3 \in T(M) \) for a uniquely determined \( M \in \wp_m(\mathfrak{U}) \). To close the proof, we note that the following
holds
\[ B_1 \sim^+ B_2 \iff (\exists \ A_1, A_2, A_3 \in \wp_{m-1}(\Omega)) (\exists K_1, K_2 \in \mathcal{T}^*) \left[ \Pi(A_1, A_2, A_3) \land A_1, A_2, A_3, B_1 \in K_1 \land A_1, A_2, A_3, B_2 \in K_2 \right] \] (11)
for any \( B_1, B_2 \in \wp_{m-1}(\Omega) \).

Let us consider the following relation \( \preceq^s \) defined for any \( A_1, A_2 \in \wp_{m-1}(\Omega) \):
\[ A_1 \preceq^s A_2 \iff (\forall D_1, D_2 \in \wp_m(\Omega)) \left[ A_1 \subseteq D_1 \Rightarrow D_1 = D_2 \lor D_1 \sim D_2 \lor (\exists D \in \wp_m(\Omega)) \left[ D \sim D_1, D_2 \right] \right]. \] (12)

**Lemma 2.14.** Let \( A_1, A_2 \in \wp_{m-1}(\Omega) \). The following conditions are equivalent.
(i) \( A_1 \preceq^s A_2 \).
(ii) \( \overline{A_1} \sim_\mathcal{A} \overline{A_2} \) holds i.e. either \( A_1 \sim_\mathcal{A} A_2 \) or \( A_1^\mathcal{A} = A_2^\mathcal{A} \) (i.e. \( A_1 \parallel A_2 \)).

**Proof.** Let \( \overline{A_1} \sim_\mathcal{A} \overline{A_2} \) hold, so \( \dim(\overline{A_1} \cap \overline{A_2}) \geq m - 2 \). Take \( D_1, D_2 \) as in (12), so \( \dim(D_1 \cap D_2) \geq m - 2 \). If \( \dim(D_1 \cap D_2) = m - 1 \) we are through; assume that \( \dim(D_1 \cap D_2) = m - 2 \). From the properties of the polar space \( \Omega \) there is a required subspace \( D \) and thus \( A_1 \sim^s A_2 \) holds.

Now, let \( A_1 \sim^s A_2 \) and \( \dim(\overline{A_1} \cap \overline{A_2}) < m - 2 \). There are \( D_1', D_2' \in \wp_m(\Omega) \) with \( D_1' \cap D_2' = \overline{A_1} \sim_\mathcal{A} \overline{A_2} \) and \( A_i \subseteq D_i' \) for \( i = 1, 2 \). Note that \( D_1', D_2' \not\subseteq \mathcal{H} \) as otherwise there would be no \( A_1, A_2 \). So, we have \( D_i := D_i' \setminus \mathcal{H}, i = 1, 2 \) that do not satisfy (12) and thus \( A_1 \sim^s A_2 \) is false.

Clearly, the \( \sim^s \)-cliques are restrictions of \( \sim_\mathcal{A} \)-cliques defined on \( \wp_{m-1}(\Omega) \). In view of (11) there are three classes of the maximal cliques of \( \sim^s \): the class \( \mathcal{T} \), the class \( S \), and the class \( S^\circ = \{ U : U \parallel U_0 \} : U_0 \in \wp_{m-1}(\Omega) \} \).

So, in view (2.14) the class \( S \cup S^\circ \) is definable in terms of \( G_{m-1}(\Omega) \).

**Proposition 2.15.** The structure \( P_{m-1}(\Omega) \) can be defined in terms of \( G_{m-1}(\Omega) \).

Thus \( P_{m-1}(\Omega) \) remains definable in \( G_{m-1}(\Omega) \) and, consequently, one can distinguish \( S \) and \( S^\circ \) in terms of the geometry of \( G_{m-1}(\Omega) \).

In particular, \( \sim_\mathcal{A} \) is definable on \( \wp_{m-1}(\Omega) \) in terms of \( \sim^+ \) and in terms of \( \sim \).

**Proof.** To justify the first statement it suffices to note that the lines of \( P_{m-1}(\Omega) \) are the sets of the form \( K_1 \cap K_2 \), where \( K_2 \in \mathcal{T} \) and \( K_1 \in S \cup S^\circ \).

The second claim is a direct consequence of (2.10). Finally, for \( X \in S \cup S^\circ \) we have \( X \in S \) if each two \( A_1, A_2 \in X \), if joinable in \( P_{m-1}(\Omega) \) lie on a line of \( P_{m-1}(\Omega) \). This enables us to distinguish respective types of cliques.

Evidently, if \( A_1, A_2 \in \wp_{m-1}(\Omega) \) then \( A_1 \sim A_2 \) iff \( A_1, A_2 \in K \) for some \( K \in S \). Summing up (10), 2.12, 2.15, 1.5, 2.14, 2.6 and 1.5 we get (1.6) for \( k = m - 1 \).
3 Final remarks

Our main result, Theorem 1.5, reads (formally) more or less the same way as the main statements of [15]: Theorem 4.1 together with Theorem 5.6. The three adjacencies ~+, ~−, ~ on strong subspaces were also introduced in [15]. The key characterization of adjacency cliques in [12] resembles analogous characterizations with similar formulas in [15]: Fact 2.3 and Proposition 2.4. Another set of important properties of cliques, gathered in Fact 2.2, corresponds to considerations on pages 45–46 in [15]. Definability of pencils in terms of ~+-adjacency in Lemma 2.3 resembles Proposition 3.2 in [15]. The same idea to distinguish selectors from star cliques shown in Lemma 2.7 is also used in Corollary 3.8 in [15]. An analogue of Corollary 2.9 where definability of Pk(U) in terms of ~ is stated, has been proved in Proposition 3.12 in [15]. Finally, definition (12) and use of the relation ~∗ are based on similar idea as those used in the proof of Proposition 5.5 in [15].

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Authors’ address:
Krzysztof Prażmowski, Mariusz Żynel
Institute of Mathematics, University of Białystok
ul. Akademicka 2, 15-267 Białystok, Poland
krzypraz@math.uwb.edu.pl, mariusz@math.uwb.edu.pl