COARSE AND PRECISE $L^p$-GREEN POTENTIAL ESTIMATES ON NONCOMPACT RIEMANNIAN MANIFOLDS †

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Abstract. We are concerned about the coarse and precise aspects of a priori estimates for Green’s function of a regular domain for the Laplacian-Betrami operator on any $3 \leq n$-dimensional complete non-compact boundary-free Riemannian manifold through the square Sobolev/Nash/logarithmic-Sobolev inequalities plus the rough and sharp Euclidean isoperimetric inequalities. Consequently, we are led to evaluate the critical limit of an induced monotone Green’s functional using the asymptotic behavior of the Lorentz norm deficit of Green’s function at the infinity, as well as the harmonic radius of a regular domain in the Riemannian manifold with nonnegative Ricci curvature.

1. Introduction

To highlight the key issues around the current paper, let us recall a few of background materials as follows.

Let $M^n$ be a complete Riemannian manifold of dimension $n \geq 3$ and its length element take in local coordinates the following form

$$ds^2 = \sum_{j,k=1}^{n} g_{jk}(x) dx^j dx^k,$$

where $(g_{jk})$ is a symmetric positive definite matrix leading: the inverse matrix $(g_{jk})^{-1} = (g^{jk})$; the determinant $g = \det(g_{jk})$; the Riemannian volume element $dV(x) = \sqrt{\det(g_{jk})} dx$; and the distance between two points $x, y \in M^n$: $d(x,y) = \inf_{\gamma} L(\gamma)$ in which the infimum is taken over all piece-wise $C^1$-curves $\gamma$ in $M^n$ with $L(\gamma)$ being defined as the sum of the lengths

$$L(\gamma_i) = \int_0^1 |\gamma_i'(t)| dt; \quad \gamma_i(0) = x_i \text{ and } \gamma_i(1) = y_i$$

of the $C^1$ pieces $\gamma_i : [0,1] \mapsto M^n$ making $\gamma$. One way of measuring the degree to which the geometry decided by $ds^2$ might be different from $dx^2$ of the Euclidean space $\mathbb{R}^n$ is the Ricci curvature tensor $Ric(u,v) –$ this tensor, acting on two vectors $u$ and $v$ in the tangent space $T_p M^n$ of $M^n$ at the point $p \in M^n$, is defined as the trace of the linear transformation $w \mapsto R(w,v)u$, a self-mapping of $T_p M^n$, where $R(\cdot,\cdot)$ stands for the Riemann curvature tensor. Since $Ric(u,v) = Ric(v,u)$, the Ricci tensor is completely determined by $Ric(u,u)$ for all vectors $u$ of unit length – this function on the family of unit tangent vectors is usually called the Ricci curvature – when $Ric(u,u)$ is nonnegative for all unit tangent vectors $u$ then $(M^n, ds^2)$ is said to have nonnegative Ricci curvature.

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Next, denote by $\nabla$ and $\Delta$ the gradient and Laplacian-Bertrami operators which are regarded as the most fundamental operators on $(M^n, ds^2)$ and determined respectively in local coordinates via:

$$\nabla = \left( \sum_{k=1}^{n} g^{1k} \frac{\partial}{\partial x_k}, \ldots, \sum_{k=1}^{n} g^{nk} \frac{\partial}{\partial x_k} \right)$$

and

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial}{\partial x_k} \right).$$

Associated with these two operators are the forthcoming three curvature-free quantities: capacity, heat kernel and Green’s function. If $C_0^\infty(E)$ means the class of all $C^\infty$ smooth functions compactly supported in a set $E \subseteq M^n$, then the harmonic/Newtonian/Wiener capacity of a pre-compact open set $\Omega \subseteq M^n$ is defined by

$$\text{cap}_2(\Omega) = \inf \left\{ \int_\Omega |\nabla f|^2 dV : f \in C_0^\infty(M^n), f \geq 1 \text{ on } \Omega \right\}.$$ 

Usually, $f$ is called a $(1 \leq p < \infty, \text{homogeneous})$ Sobolev function on $\Omega$ provided $\int_\Omega |\nabla f|^p dV < \infty$. Moreover, the notation $p_t(x, y)$ is used as the heat kernel on $M^n$ – that is – the smallest positive solution $u(t, x, y)$ to the heat equation

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} u(t, x, y) = \Delta u(t, x, y), \quad (t, x, y) \in (0, \infty) \times M^n \times M^n, \\ \lim_{t \to 0} u(t, x, y) = \delta_x(y), \quad (x, y) \in M^n \times M^n, \end{array} \right.$$ 

where $\delta_x(y)$ is the Dirac measure. At the same time, the symbol $G(x, y) = \sup_{\Omega} G_\Omega(x, y)$ is chosen for the minimal Green’s function of $M^n$ where the supremum is taken over all regular (meaning that the Dirichlet problem is uniquely solvable) bounded open sets $\Omega \subseteq M^n$ containing $x$, and $G_\Omega(x, y)$ is the nonnegative Green’s function of $\Omega$ obeying

$$\left\{ \begin{array}{l} \Delta G_\Omega(x, y) = \delta_x(y), \quad y \in \Omega, \\ G_\Omega(x, y) = 0, \quad y \in M^n \setminus \Omega. \end{array} \right.$$ 

Of course, the last equation on $\Delta$ is understood in the sense of distribution, i.e.,

$$\int_\Omega \langle \nabla G_\Omega(x, \cdot), \nabla \phi \rangle dV = \phi(x) \quad \text{for all functions } \phi \in C_0^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle$ stands for the Riemannian metric on $M^n$. The above-considered manifolds can be classified into two types: non-parabolic or parabolic according to that $G(x, y)$ is finite or infinite for all $y \in M^n \setminus \{x\}$ or all $y \in M^n$. It is interesting, natural and important that the classical relation between the heat kernel and the minimal Green’s function for a non-parabolic manifold is determined by

$$G(x, y) = \int_0^\infty p_t(x, y) \, dt, \quad (x, y) \in M^n \times M^n.$$ 

For more closely-related information see also Varopolulos [51], Li-Yau [39], Li-Tam [37], Li-Tam-Wang [38], Li [36], Schoen [48] and Davies-Safarov [17].

More than that, however, we will employ $X \lesssim Y$ (i.e., $Y \gtrsim X$) to represent that there is a constant $\kappa > 0$ such that $X \leq \kappa Y$, but also make $X \approx Y$ stand for both $X \lesssim Y$ and $Y \lesssim X$.

With the help of the previously-described notions and notations, in the subsequent Sections 2-3 we will, coarsely and precisely, treat a series of monotonic
$L^p(0 \leq p < \frac{n}{n-2})$-estimates for Green’s functions associated with Laplacian-Betrami operators on non-compact complete boundary-free Riemannian manifolds with dimension $n \geq 3$ through three mutually-implied inequalities – $L^2$-Sobolev, -Nash and -logarithmic Sobolev inequalities.

In Theorem 1 we will revisit five known characterizations (Faber-Krahn’s eigenvalue inequality, Maz’ya iso-capacitary inequality, On/Off-diagonal upper bound of heat kernel and distributional behavior of the Green function of the manifold) of the $L^2$-Sobolev/Nash/Log-Sobolev inequalities, and in Theorem 2 we give two more new ones – Lorentz-Green local inequality (i.e., distributional behavior of the Green function of any regular bounded open subset of the manifold) and the following Green potential-volume inequality:

$$\sup_{x \in \Omega} \int_{\Omega} G_{\Omega}(x, y) \, dV(y) \lesssim V(\Omega)^{\frac{2}{n}}.$$  

In the above and below, $\Omega$ is a regular bounded open set $\Omega \subseteq M^n$ (containing $x$ whenever needed). Interestingly, this last estimate leads to a discovery of the comparison principle as stated in Theorem 3, especially saying that if the coarse $L^1$-Sobolev inequality (equivalently, the rough isoperimetric inequality of Euclidean type) is valid then

$$\left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{n}{n-p(n-2)}} \lesssim \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{n}{n-q(n-2)}}$$

holds for $0 \leq q < p < \frac{n}{n-2}$, but also $G_{\Omega}(x, \cdot)$ belongs uniformly to the Lorentz space $L^{\frac{n}{n-2}}(\Omega)$; see also Gr"uter-Widman [26] for an earlier result in $\mathbb{R}^n$.

In Theorems 5 & 6, we will give the sharp inequalities corresponding to those in Theorems 1 & 2 under the assumption that $M^n$ allows the optimal isoperimetric inequality of Euclidean type to exist. In particular, we will establish such a classification result that together with the nonnegative Ricci curvature and the volume $\omega_n$ of the unit ball of $\mathbb{R}^n$, either the sharp Maz’ya’s iso-capacitary inequality

$$V(\Omega) \frac{n-2}{n} \leq (n(n-2)\omega_n^2)^{-1} \text{cap}_2(\Omega)$$

or the sharp Green potential-volume inequality (cf. Weinberger [52] for the $\mathbb{R}^n$-case):

$$\sup_{x \in \Omega} \int_{\Omega} G_{\Omega}(x, y) \, dV(y) \leq (2n\omega_n^2)^{-1} V(\Omega)^{\frac{2}{n}}$$

ensures that $M^n$ is either isometric or diffeomorphic to $\mathbb{R}^n$. Even more interestingly, this last inequality will drive us to obtain the optimal monotone principle for the $L^p$-Green potentials – Theorem 7: under $M^n$ being of the sharp isoperimetric inequality of Euclidean type, the functional

$$\left( \frac{(n(n-2)\omega_n^2)^p}{pB(\frac{n}{n-2} - p, p)} \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{n}{n-p(n-2)}}$$

is monotone decreasing as $p$ varies from 0 to $n/(n-2)$ (viewed as the critical index) where $B(\cdot, \cdot)$ is the classical Beta function. Consequently, we can evaluate the limit of the above monotone functional as $p \to n/(n-2)$ – this limit is fortunately found to equal $\lim_{t \to \infty} D_{\Omega}(x, t)^{\frac{2n}{n-2}}$ where

$$D_{\Omega}(x, t) = V(\{y \in \Omega : G_{\Omega}(x, y) \geq t\}) \frac{2n}{n-2} - n(n-2)\omega_n^2 t.$$
is the so-called Lorentz norm deficit of $G_{\Omega}(x, \cdot)$ at $(x, t)$. Even more fortunately, if $(M^n, ds^2)$ has also nonnegative Ricci curvature then

$$\lim_{t \to \infty} D_{\Omega}(x, t)^{\frac{n}{4}} = \omega_n R_{\Omega}(x)^n,$$

i.e., the volume of the geodesic ball with center $x$ and harmonic radius $R_{\Omega}(x)$ of $\Omega$ at $x$ – it seems worthwhile to indicate that $R_{\Omega}(x)^{\frac{n}{2} - n}$ is the so-called Robin mass of $\Omega$ at $x$ – see also Bandle-Flucher [7], Flucher [21] and Bandle-Brillard-Flucher [6] in regard to this concept of Euclidean type and its applications in geometric potential analysis – and yet, it is perhaps appropriate to mention one more fact that the quantity $R_{\Omega}(x)^{\frac{n}{2} - n}$ is the leading term of the regular part of the Green function $G_{\Omega}(x, y)$ and when $\Omega$ approaches $M^n$, the resulting harmonic radius at $x$:

$$\lim_{y \to x} \left( d(x, y)^{\frac{n}{2} - n} - n(n - 2)\omega_n G(x, y) \right)^{\frac{1}{\frac{n}{2} - n}}$$

can be modified for the Green function of the Yamabe operator on a compact Riemannian manifold in order to properly construct a canonical metric on a locally conformally flat Riemannian manifold – see Habermann-Jost [28] (or Habermann [27]) where the construction of the metric deepens Hersch’s [30] and Leutwiler’s [33] and relies on Schoen-Yau’s positive mass theorem in [49] with connection to Schoen’s final solution to Yamabe problem [47]. Needless to say, the endpoint $n/(n-2)$ is crucial – see also Schoen-Yau’s paper [50] for an account on the crucial role of such an exponent/index playing in the integrability of the minimal Green’s function for the conformally invariant Laplace operator.

The proofs of the above-mentioned theorems (mixing Riemannian geometry and integration theory of Green’s functions) will be well detailed except those already-known parts in Theorems 1 and 5 (whose references will surely be located). Our ideas, techniques and methods determining the sharp constants are adjustable for settling the similar problems on the Green function of the Yamabe operator – this will be an object of our future investigation.

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2. Coarse Estimates

It is well-known that geometry of the rough $L^1$-Sobolev inequality is completely characterized in terms of the isoperimetric inequality with a rough constant – that is to say – the classic Sobolev inequality

$$\left( \int_{M^n} |f|^{\frac{n}{n-1}} \, dV \right)^{\frac{n-1}{n}} \lesssim \int_{M^n} |\nabla f| \, dV$$

for all functions $f \in C_0^\infty(M^n)$, amounts to the generic isoperimetric inequality

$$V(\Omega)^{\frac{n-1}{n}} \lesssim S(\partial \Omega)$$

for all smooth bounded domains $\Omega \subseteq M^n$.

Here and henceforth, $S(\partial \Omega)$ is the area of the boundary $\partial \Omega$ of $\Omega \subseteq M^n$.

While working on some analytic and geometric forms of $L^2$-Sobolev inequality, Nash inequality, and logarithmic-Sobolev inequality without optimal constant, we get a collection of the most-known equivalences:
Theorem 1. Let \((M^n, ds^2)\), \(n \geq 3\), be non-compact complete boundary-free Riemannian manifold. Then the following eight statements with coarse constants are equivalent:

(i) \(L^2\) Sobolev inequality

\[
\left( \int_{M^n} |f|^\frac{2n}{n-2} \, dV \right)^{\frac{n-2}{n}} \lesssim \int_{M^n} |\nabla f|^2 \, dV
\]

holds for all \(f \in C_0^\infty(M^n)\).

(ii) Nash’s inequality

\[
\left( \int_{M^n} |f|^2 \, dV \right)^{\frac{1}{1+\frac{2}{n}}} \lesssim \left( \int_{M^n} |f| \, dV \right)^{\frac{1}{n}} \int_{M^n} |\nabla f|^2 \, dV
\]

holds for all \(f \in C_0^\infty(M^n)\).

(iii) \(L^2\) logarithmic Sobolev inequality

\[
\exp\left( \frac{2}{n} \int_{M^n} |f|^2 \log |f|^2 \, dV \right) \lesssim \int_{M^n} |\nabla f|^2 \, dV
\]

holds for all \(f \in C_0^\infty(M^n)\) with \(\int_{M^n} |f|^2 \, dV = 1\).

(iv) Faber-Krahn’s eigenvalue inequality

\[
\lambda_1(\Omega)^{-1} = \sup_{f \in C_0^\infty(\Omega), f \neq 0} \frac{\int_{\Omega} |f|^2 \, dV}{\int_{\Omega} |\nabla f|^2 \, dV} \lesssim V(\Omega)^{\frac{2}{n}}
\]

holds for all regular bounded open sets \(\Omega \subseteq M^n\).

(v) Maz’ya’s iso-capacitary inequality

\[
V(\Omega)^{\frac{n-2}{n}} \lesssim \text{cap}_2(\Omega)
\]

holds for all pre-compact open sets \(\Omega \subseteq M^n\).

(vi) On-diagonal upper bound of heat kernel

\[
\sup_{x \in M^n} p_t(x, x) \lesssim t^{-\frac{n}{2}}
\]

holds for all \(t > 0\).

(vii) Off-diagonal upper bound of heat kernel

\[
\sup_{(x,y) \in M^n \times M^n} p_t(x, y) \lesssim t^{-\frac{n}{2}}
\]

holds for all \(t > 0\).

(viii) Lorentz-Green’s global inequality

\[
\sup_{x \in M^n} V(\{y \in M^n : G(x, y) \geq t\}) \lesssim t^{-\frac{n-2}{n-1}}
\]

holds for all \(t > 0\), with \(G(x, y)\) being finite for \(y \neq x\) (i.e., \(M^n\) being non-parabolic).

Proof. A simple proof of the equivalence (i)\(\Leftrightarrow\) (ii) can be found in [4] by Bakry, Coulhon, Ledoux and Saloff-Coste; (i)\(\Leftrightarrow\) (vi) is due to Varopoulos [51]; (ii)\(\Leftrightarrow\) (vi) is due to Carlen-Kusuoka-Stroock [8]; (iii)\(\Leftrightarrow\) (vi) is due to Davies [16] and Bakry [2]; (iv)\(\Leftrightarrow\) (i)/(vi) is due to Carron [10] and Grigor’yan [24] (cf. [25]); (v)\(\Leftrightarrow\) (i) is due to Maz’ya [43]; (vi)\(\Leftrightarrow\) (i) is due to Nash [44]; (vii)\(\Leftrightarrow\) (i)/(ii)/(iii) are shown in Varopoulos [51], Carlen-Kusuoka-Stroock [8] and Bakry [2]; (viii)\(\Leftrightarrow\) (iv) is due to Carron [10]. □
As a by-product of the foregoing theorem, we find two more new conditions deciding geometry of \((M^n, ds^2)\).

**Theorem 2.** Let \((M^n, ds^2)\), \(n \geq 3\), be non-compact complete boundary-free Riemannian manifold. Then anyone from (i) to (viii) in Theorem 1 is equivalent to each of the following two statements with coarse constants:

(ix) Lorentz-Green’s local inequality

\[
\sup_{x \in \Omega} V\{ \{ y \in \Omega : G_{\Omega}(x, y) \geq t \} \} \lesssim \left( t + V(\Omega) \right)^{\frac{2-n}{2}}
\]

holds for all regular bounded open sets \(\Omega \subseteq M^n\) and every \(t > 0\).

(x) Green’s potential-volume inequality

\[
\sup_{x \in \Omega} \int_{\Omega} G_{\Omega}(x, y) \, dV \lesssim V(\Omega)^{\frac{2}{n}}
\]

holds for all regular bounded open sets \(\Omega \subseteq M^n\).

**Proof.** Since (iv) \(\leftrightarrow\) (viii) above is valid, it suffices to check (viii) \(\Rightarrow\) (ix) \(\Rightarrow\) (x) \(\Rightarrow\) (iv).

Suppose (viii) is true. Note that for any regular bounded open set \(\Omega \subseteq M^n\) with \(V(\Omega) = \infty\) we have

\[
y \in \Omega \quad \& \quad G_{\Omega}(x, y) \geq t \implies y \in \Omega \quad \& \quad G(x, y) \geq t,
\]

whence reaching (ix) via

\[
V\{ \{ y \in \Omega : G_{\Omega}(x, y) \geq t \} \} \lesssim \min\{ V(\Omega), t^{-\frac{2-n}{n}} \} \approx \left( V(\Omega)^{\frac{2-n}{n}} + t \right)^{\frac{2-n}{n}}.
\]

If (ix) is valid, then an integration by parts and a change of variables yield

\[
\int_{\Omega} G_{\Omega}(x, y) \, dV(y) = \int_0^\infty V\{ \{ y \in \Omega : G_{\Omega}(x, y) \geq t \} \} \, dt
\]

\[
\lesssim \int_0^\infty \left( V(\Omega)^{\frac{2-n}{n}} + t \right)^{\frac{2-n}{n}} \, dt
\]

\[
\lesssim \int_0^\infty t \left( V(\Omega)^{\frac{2-n}{n}} + t \right)^{\frac{2(2-n)}{2-n}} \, dt
\]

\[
\approx V(\Omega)^{\frac{2(2-n)}{n}} \int_0^\infty t^{\frac{2(2-n)}{2-n}} \frac{2(2-n)}{2-n} \, dt
\]

\[
\approx V(\Omega)^{\frac{2}{n}} \int_0^\infty s^{\frac{2(2-n)}{2-n}} \, ds,
\]

namely, (x) holds. Now, suppose (x) is valid. Given a regular bounded open set \(\Omega \subseteq M^n\) with \(V(\Omega) < \infty\) (the condition (iv) is trivially valid for \(V(\Omega) = \infty\)). Assume that \(u \neq 0\) solves

\[
\left\{ \begin{array}{c}
\Delta u(y) = \lambda_1(\Omega) u(y), \quad y \in \Omega, \\
u(y) = 0, \quad y \in \partial \Omega.
\end{array} \right.
\]
Then for each $x \in \Omega$ we have
\[
u(x) = \int_{\Omega} G_{\Omega}(x, y) \Delta u(y) \, dV(y) = \lambda_1(\Omega) \int_{\Omega} G_{\Omega}(x, y) u(y) \, dV(y) \leq \lambda_1(\Omega) \sup_{y \in \Omega} u(y) \int_{\Omega} G_{\Omega}(x, y) \, dV(y),
\]
thereby deriving
\[
1 \leq \lambda_1(\Omega) sup_{x \in \Omega} \int_{\Omega} G_{\Omega}(x, y) \, dV(y) \lesssim \lambda_1(\Omega)V(\Omega)^{\frac{2}{n}}.
\]
In other words, (iv) is true. 

Here it is worth observing that if $(M^n, ds^2)$ allows the coarse/generic isoperimetric inequality above to exist then (vi), (vii) and (viii) hold – see also [11], [12], [13], and [23], and hence the remaining conditions in Theorems 1 & 2 hold. Evidently, the previously-described conditions (iv)-(v)-(vi)-(vii)-(viii)-(ix)-(x) may be regarded as seven different but equivalent geometric criteria (involving no curvature hypotheses) for three different but equivalent inequalities (i)-(ii)-(iii) above to hold. Since the generic $L^1$ Sobolev inequality is strictly stronger than the generic $L^2$ Sobolev inequality, the isoperimetric inequality without best constant implies all the seven geometric inequalities but not conversely in general – nevertheless the implication can be reversed when $(M^n, ds^2)$ has a nonnegative Ricci curvature – see also [29, Lemma 8.1 & Theorem 8.4] as well as [10], [15], [51]. Here it is also worth noticing that from Yau [55], Varopoulos [51] and J. Li [34] we see that if $(M^n, ds^2)$ has non-negative Ricci curvature then the generic $L^p$ ($1 \leq p < n$) Sobolev inequality is true on $M^n$ when and only when $V(B_t(x)) \gtrsim t^n$. Furthermore, Theorem 2 has actually motivated us to establish the following rough comparison principle for Green’s function integrals which is of independent interest.

**Theorem 3.** Let $(M^n, ds^2)$, $n \geq 3$, be a non-compact complete boundary-free Riemannian manifold with anyone of Theorem 1 (i) through Theorem 2 (x) holding. If $0 \leq q < p < \frac{n}{n-2}$ then the following comparison inequality with coarse constant
\[
\left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{n-p(n-2)}{n(n-2)}} \lesssim \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{n-q(n-2)}{n(n-2)}}
\]
holds for all regular bounded open sets $\Omega \subseteq M^n$ containing $x$. Moreover, $G_{\Omega}(x, \cdot)$ belongs uniformly to the Lorentz space $L^{\frac{n}{n-2}}_{\frac{n}{n-2}}(\Omega)$, namely,
\[
\sup_{(t, x) \in (0, \infty) \times \Omega} t^{\frac{n-2}{n}} V(\{ y \in \Omega : G_{\Omega}(x, y) \geq t \}) < \infty.
\]

**Proof.** Due to the conditions from (i) to (x) in Theorems 1 and 2 are all equivalent, we may use any of them in what follows.

Suppose $\int_{\Omega} G_{\Omega}(x, y)^q \, dV(y)$ is finite – otherwise there is nothing to argue.
Case 1: \( q = 0 \). This ensures \( V(\Omega) < \infty \). Using Theorem 2 (ix), we obtain

\[
\int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) = \int_{0}^{\infty} V(\{y \in \Omega : G_{\Omega}(x, y) \geq t\}) \, dt^p \\
\lesssim \int_{0}^{\infty} (t + V(\Omega) \frac{n-q}{n}) \frac{n-q}{p} \, dt^p \\
\lesssim V(\Omega)^\frac{n-q(n-2)}{n} \int_{0}^{\infty} s^{p-1} (1 + s)^{\frac{n-q}{n}} \, ds,
\]

as desired.

Case 2: \( q > 0 \). Note that

\[
\int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) = \frac{p}{q} \int_{0}^{\infty} \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{p}{q}} \, dt^q
\]

and for \( \alpha = p, q \) and \( r > 0 \),

\[
\int_{0}^{\infty} V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \, dt^{\alpha} \geq r^{\alpha} V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}).
\]

So

\[
\int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \leq \frac{p}{q} \int_{0}^{\infty} \left( V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \right)^{\frac{p}{q}} \, dr^q.
\]

This yields

\[
\left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{q}{p}} \leq \frac{p}{q} \int_{0}^{\infty} \left( V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \right)^{\frac{q}{p}} \, dr^q.
\]

Furthermore, using Theorem 2 (ix) again we obtain

\[
\int_{0}^{\infty} \left( V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \right)^{\frac{q}{p}} \, dr^q \lesssim \int_{0}^{t} \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{p}{q}} \, dr^q + \int_{t}^{\infty} r^{\frac{n-q}{n}} \, dr^q \\
\approx \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{p}{q}} t^{q(1-\frac{1}{p})} + t^{\frac{a(n-p(n-2))}{p(2-n)}}.
\]

If

\[
t = \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{n-q}{n-2}}
\]

then under \( 0 < q < p < \frac{n}{n-2} \),

\[
\int_{0}^{\infty} \left( V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \right)^{\frac{q}{p}} \, dr^q \lesssim \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{p}{q} - \frac{n}{n-2}}
\]

and hence

\[
\int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \lesssim \left( \int_{\Omega} G_{\Omega}(x, y)^q \, dV(y) \right)^{\frac{p}{q} - \frac{n}{n-2}}.
\]
Clearly, the second fact stated in Theorem 3 follows immediately from Theorem 2 (ix).

Remark 4. Under the same hypothesis as in Theorem 3, we have that if \( \frac{n}{n-2} < p < 0 \) then \( 0 < -p < \frac{n}{n-2} \) and hence the Cauchy-Schwarz inequality and Theorem 3 yield

\[
V(\Omega) \leq \left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{1}{p}} \left( \int_{\Omega} G_{\Omega}(x, y)^{-p} \, dV(y) \right)^{\frac{1}{p}} \\
\lesssim \left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{1}{p}} V(\Omega)^{\frac{n-p(n-2)}{2n}},
\]

namely,

\[
V(\Omega)^{\frac{n-p(n-2)}{8n}} \lesssim \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y).
\]

Clearly, this last estimate can be regarded as a kind of the isoperimetric inequality involving the Green function. Moreover, for \( 0 \leq p < \frac{n}{n-2} \) one has the following area-volume-type isoperimetric estimate:

\[
\left( \int_{\Omega} G_{\Omega}(x, y)^p \, dV(y) \right)^{\frac{n-p(n-2)}{8(n-1)}} \lesssim S(\partial \Omega),
\]

upon the coarse isoperimetric inequality being valid for all regular bounded domains \( \Omega \subseteq M^n \) containing \( x \).

3. Precise Estimates

In order to find out the sharp versions of the estimates established in Theorems 1-2-3, we need to review the celebrated result (due to Federer-Fleming \[20\] and Maz‘ya \[41\] for \( M^n = \mathbb{R}^n \)) that the sharp \( L^1 \)-Sobolev inequality

\[
\left( \int_{M^n} |f(x)|^{\frac{n}{n-1}} \, dV \right)^{\frac{n-1}{n}} \leq (n\omega_n^{\frac{1}{n}})^{-1} \int_{M^n} |\nabla f(x)| \, dV \quad \text{for all functions } f \in C_0^\infty(M^n),
\]

is equivalent to the optimal isoperimetric inequality of Euclidean type

\[
V(\Omega)^{\frac{n-1}{n}} \leq (n\omega_n^{\frac{1}{n}})^{-1} S(\partial \Omega) \quad \text{for all smooth bounded domains } \Omega \subseteq M^n.
\]

Moreover, according to Hebey \[29\] page 244 and Ledoux \[32\] we see that if \( (M^n, ds^2) \) has nonnegative Ricci curvature then the just-mentioned isoperimetric inequality is valid only when \( M^n \) is isometric to \( \mathbb{R}^n \). So, when studying sharp geometric forms of the \( L^2 \)-Sobolev inequality, -Nash inequality, and -logarithmic Sobolev inequality of Euclidean type, we may naturally derive the following result which is partially known.

**Theorem 5.** Let \( (M^n, ds^2) \), \( n \geq 3 \), be a non-compact complete boundary-free Riemannian manifold with the sharp isoperimetric inequality of Euclidean type. Then the following sharp:

(i) \( L^2 \) Sobolev inequality

\[
\left( \int_{M^n} |f(x)|^{\frac{2n}{2n-2}} \, dV \right)^{\frac{n-2}{n}} \leq (n(n-2))^{-1} \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})\Gamma(1+\frac{2}{n})\omega_n} \right)^{\frac{1}{n}} \int_{M^n} |\nabla f(x)|^2 \, dV
\]

holds for all \( f \in C_0^\infty(M^n) \), where \( \Gamma(\cdot) \) is the standard Gamma function.
(ii) Nash’s inequality
\[
\left( \int_{M^n} |f|^2 \, dV \right)^{1+\frac{2}{n}} \leq \frac{(n+2)\frac{n+2}{n}}{n(2\omega_n)^\frac{2}{n}} \lambda_N \left( \int_{M^n} |f| \, dV \right)^\frac{2}{n} \int_{M^n} |\nabla f|^2 \, dV
\]
holds for all \( f \in C_0^\infty (M^n) \), where \( \lambda_N \) is the first non-zero Neumann eigenvalue of \( \Delta \) on radial functions on the unit ball of \( \mathbb{R}^n \).

(iii) \( L^2 \) logarithmic Sobolev inequality
\[
\exp \left( \frac{2}{n} \int_{M^n} |f|^2 \log |f|^2 \, dV \right) \leq \frac{2}{en\pi} \int_{M^n} |\nabla f|^2 \, dV
\]
holds for all \( f \in C_0^\infty (M^n) \) with \( \int_{M^n} |f|^2 \, dV = 1 \).

(iv) Faber-Krahn’s eigenvalue inequality
\[
\inf_{D \subseteq \mathbb{R}^n} \lambda_1 (D) V_c (D)^\frac{n}{2} \leq \lambda_1 (\Omega) V (\Omega)^\frac{n}{2}
\]
holds for all regular bounded open sets \( \Omega \subseteq M^n \), where the infimum ranges over all regular bounded open sets \( D \subseteq \mathbb{R}^n \) and \( \lambda_1 (D) \) and \( V_c (D) \) are respectively the eigenvalue and volume associated with \( D \) under the Euclidean metric.

(v) Maz’ya’s iso-capacitary inequality
\[
V (\Omega)^{\frac{n-2}{n}} \leq \left( n(n-2)\omega_n^\frac{2}{n} \right)^{-1} \text{cap}_2 (\Omega)
\]
holds for all pre-compact open sets \( \Omega \subseteq M^n \).

(vi) On-diagonal bound of heat kernel
\[
\sup_{x \in M^n} p_t (x, x) \leq (4\pi t)^{-\frac{n}{2}}
\]
holds for all \( t > 0 \).

(vii) Off-diagonal bound of heat kernel
\[
\sup_{(x, y) \in M^n \times M^n} p_t (x, y) \leq (4\pi t)^{-\frac{n}{2}}
\]
holds for all \( t > 0 \).

(viii) Lorentz-Green’s global inequality
\[
\sup_{x \in M^n} V \{ y \in M^n : G(x, y) \geq t \} \leq \left( n(n-2)\omega_n^\frac{2}{n} \right)^{-\frac{n}{n-2}} t^{-\frac{n}{n-2}}
\]
holds for all \( t > 0 \), with \( G(x, y) \) being finite for \( y \neq x \) (i.e., \( M^n \) being non-parabolic).

On the other hand, if anyone of (i), (ii), (iii), (v), (vi), (vii), (iv), (viii) is true and \((M^n, ds^2)\) has nonnegative Ricci curvature then \( M^n \) is isometric (diffeomorphic) to \( \mathbb{R}^n \).

Proof. To begin with, the sharpness of the above eight statements is due to the fact that all equalities there can occur when \( M^n = \mathbb{R}^n \).

Next, let us check each statement. Note that the Euclidean counterpart of (v) is Maz’ya’s sharp isocapacitary estimate in [42, page 105, (7)] whose proof depends only on the sharp Euclidean isoperimetric inequality. So, (v) is true under the
The validity of (v) is used to imply
\[
\int_{M^n} |f|^{2n-2} dV = \int_0^\infty V(\{x \in M^n : |f(x)| \geq t\}) \frac{dt}{t^{n-2}} \leq \frac{\int_0^\infty \text{cap}_2(\{x \in M^n : |f(x)| \geq t\})^{n-2} dt}{(n(n-2))^{\frac{1}{2}} \omega_n^{\frac{1}{2}}}.
\]

According to Maz’ya’s [43, Remark 5 & Proposition 1], we have
\[
\left( \int_0^\infty \text{cap}_2(\{x \in M^n : |f(x)| \geq t\})^{n-2} dt \right)^{\frac{1}{n-2}} \leq \frac{\int_{M^n} |\nabla f|^2 dV}{\left(\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+\frac{n}{2})}{\Gamma\left(\frac{n}{2}+1\right)}\right)^{\frac{1}{n}}},
\]
therefore deriving (i).

According to Ni’s argument for Perelman’s proposition (cf. [45, Proposition 4.1]) we find easily that if \((M^n, ds^2)\) allows the optimal isoperimetric inequality of Euclidean type then (iii) holds. This in turn implies (vii) and so (vi) – see Bakry-Concordet-Ledoux [3, Theorem 1.2].

The verification of (ii) is more or less contained in Druet-Hebey-Vaugon’s argument for [19, Theorem 5.1]. To see this, we may just prove that (ii) is valid for any function \(f \in C_0^\infty(M^n)\) which is continuous and has only non-degenerate critical points in its support. As with such a function \(f\), let \(g : \mathbb{R}^n \mapsto \mathbb{R}\) be nonnegative, radial, decreasing with respect to \(|x|\), and be determined by
\[
V_e(\{x \in \mathbb{R}^n : g(x) \geq t\}) = V(\{x \in M^n : f(x) \geq t\}).
\]

Then \(g\) has a compact support in \(\mathbb{R}^n\) and enjoys
\[
\int_{M^n} f^2 dV = \int_{\mathbb{R}^n} g^2 dV_e, \quad j = 1, 2
\]
and
\[
-\int_{f^{-1}[t]} |\nabla f|^{-1} dS = \frac{d}{dt} V(\{x \in M^n : f(x) \geq t\}) = \frac{d}{dt} V_e(\{x \in \mathbb{R}^n : g(x) \geq t\}) = -\int_{g^{-1}[t]} |\nabla_e g|^{-1} dS_e,
\]
where \(f^{-1}[t]\) and \(g^{-1}[t]\) are the pre-images of \(t\) under \(f\) and \(g\) respectively; \(dS\) and \(dS_e\) denote the area elements associated with \(M^n\) and \(\mathbb{R}^n\) respectively; and \(\nabla_e\) stands for the Euclidean gradient.

Since
\[
(n\omega_n)^{-1} = \frac{V_e(U)^{\frac{2}{n-2}}}{S_e(\partial U)}
\]
holds for any Euclidean ball \(U \subseteq \mathbb{R}^n\), the sharp isoperimetric inequality of Euclidean type is applied to yield
\[
S_e(g^{-1}[t]) \leq S(f^{-1}[t]), \quad t > 0.
\]
Note that $|∇_e g|$ equals a positive constant on $g^{-1}[t]$. So the last inequality plus the Cauchy-Schwarz inequality implies

$$
\left( \int_{g^{-1}[t]} |∇_e g| \, dS_e \right) \left( \int_{g^{-1}[t]} \frac{dS_e}{|∇_e g|} \right) = S_e(g^{-1}[t])^2 \leq S(f^{-1}[t])^2
$$

$$
\leq \left( \int_{f^{-1}[t]} |∇f| \, dS \right) \left( \int_{f^{-1}[t]} \frac{dS}{|∇f|} \right),
$$

and consequently,

$$
\int_{g^{-1}[t]} |∇_e g| \, dS_e \leq \int_{f^{-1}[t]} |∇f| \, dS, \quad t > 0.
$$

Now the co-area formula, along with the last inequality, gives

$$
\int_{\mathbb{R}^n} |∇_e g|^2 \, dV_e = \int_0^\infty \left( \int_{g^{-1}[t]} |∇_e g| \, dS_e \right) \, dt
$$

$$
\leq \int_0^\infty \left( \int_{f^{-1}[t]} |∇f| \, dS \right) \, dt
$$

$$
= \int_\Omega |∇f|^2 \, dV.
$$

Now an application of the sharp Nash’s inequality on $\mathbb{R}^n$ (due to Carlen-Loss [9]) produces

$$
\left( \int_{M^n} f^2 \, dV \right)^{1+\frac{2}{n}} = \left( \int_{\mathbb{R}^n} g^2 \, dV_e \right)^{1+\frac{2}{n}}
$$

$$
\leq \frac{(n+2)^{\frac{2}{n}}}{n(2\omega_n)^\frac{2}{n}} \left( \int_{\mathbb{R}^n} g \, dV_e \right)^\frac{2}{n} \int_{\mathbb{R}^n} |∇_e g|^2 \, dV_e
$$

$$
\leq \frac{(n+2)^{\frac{2}{n}}}{n(2\omega_n)^\frac{2}{n}} \left( \int_{M^n} f \, dV \right)^\frac{2}{n} \int_{M^n} |∇f|^2 \, dV,
$$

as desired.

To reach (iv), let $\Omega$ be any regular bounded open subset of $M^n$ and $f \in C_0^\infty(\Omega)$ be nonnegative. In a similar manner to proving the optimal Nash’s inequality above, we may choose a Euclidean ball $B \subseteq \mathbb{R}^n$ such that $V_e(B) = V(\Omega)$. Given $t > 0$, let $\Omega_t = \{ x \in \Omega : f(x) \geq t \}$ and $g$ be a nonnegative radial $C_0^\infty(B)$ function with

$$
B_t = \{ y \in B : g(y) \geq t \}; \quad V_e(B_t) = V(\Omega_t).
$$

Then

$$
-\int_{\partial B_t} |∇_e g|^{-1} \, dS_e = \frac{dV_e(B_t)}{dt} = \frac{dV(\Omega_t)}{dt} = -\int_{\partial \Omega_t} |∇f|^{-1} \, dS.
$$

Additionally,

$$
\int_{\Omega} f^2 \, dV = \int_0^\infty V(\Omega_t) \, dt^2 = \int_0^\infty V_e(B_t) \, dt^2 = \int_B g^2 \, dV_e.
$$

From the sharp isoperimetric inequality of Euclidean type, the fact that $|∇_e g|$ is equal to a non-zero constant on $\partial B_t$, and the Cauchy-Schwarz inequality, we
As a result, we find
\[
\int F \d S_c \left( \int |\nabla g|^{-1} dS_c \right) = S_c(\partial B_t)^2 \leq S(\partial\Omega_t)^2
\]
and so,
\[
\int_{\partial\Omega_t} |\nabla f| dS = \int_{\partial\Omega_t} |\nabla f|^{-1} dS,
\]
thereby getting
\[
\{ a \text{ sequence} \}
\]
conclude
\[
\left( \int_{\partial B_r} |\nabla g| dS_c \right) \left( \int_{\partial B_r} |\nabla g|^{-1} dS_c \right) = S_c(\partial B_t)^2 \leq S(\partial\Omega_t)^2
\]
\[
\leq \left( \int_{\partial\Omega_t} |\nabla f| dS \right) \left( \int_{\partial\Omega_t} |\nabla f|^{-1} dS \right),
\]
and so,
\[
\int_{\partial B_r} |\nabla g| dS_c \leq \int_{\partial\Omega_t} |\nabla f| dS, \quad t > 0.
\]
Furthermore, the co-area formula is used once again to deduce
\[
\int_B |\nabla g|^2 dV_c = \int_0^\infty \left( \int_{\partial B_r} |\nabla g| dS_c \right) dr \leq \int_0^\infty \left( \int_{\partial\Omega_t} |\nabla f| dS \right) dt = \int_\Omega |\nabla f|^2 dV.
\]
As a result, we find
\[
\frac{\int_\Omega f^2 dV}{\int_B |\nabla g|^2 dV_c} \leq \frac{\int_B g^2 dV_c}{\int_B |\nabla g|^2 dV_c},
\]
thereby getting
\[
\lambda(\Omega)^{-1} \leq \frac{V(\Omega)^{\frac{2}{n}}}{\lambda_1,\epsilon(B)V_c(B)} \leq \frac{V(\Omega)^{\frac{2}{n}}}{\inf_{D \subseteq \mathbb{R}^n} \lambda_1,\epsilon(D)V_c(D)\overline{D}}.
\]
The statement (vii) follows from the forthcoming Theorem 6 (ix) since there is a sequence \( \{\Omega_i\}_{i=1}^{\infty} \) of regular bounded open subsets of \( M^n \) such that
\[
x \in \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_n, \quad \cup_{i=1}^{\infty} \Omega_i = M^n \quad \text{and} \quad G_{\Omega_i}(x, \cdot) / G(x, \cdot)
\]
for a given point \( x \in M^n \); see also [32].

Finally, let us handle the rest of Theorem 5.

It is known that if (i)/(ii)/(iii)/(vi)/(vii) is true and \( (M^n, ds^2) \) has nonnegative Ricci curvature then \( M^n \) is isometric to \( \mathbb{R}^n \) – see Varopoulos [51] and Ledoux [32], Druet-Hebey-Vaugon [19], Xia [54] (cf. Xia’s paper [53] regarding Gagliardo-Nirenberg’s inequalities on manifolds of nonnegative Ricci curvature and Lutwark-Yang-Zhang’s work [40] on the optimal affine Gagliardo-Nirenberg inequality of Euclidean type), Bakry-Concordet-Ledoux [3], Ni [45] (cf. Ni [46] and Kotschwar-Ni [31]), extending the sharp \( L^p(1 < p < n) \) Sobolev logarithmic inequality in DelPino-Dolbeault [15], and Li [35]. Since (v) implies (i), if (v) is valid with \( (M^n, ds^2) \) having nonnegative Ricci curvature then \( M^n \) must be isometric to \( \mathbb{R}^n \).

In accordance with Theorem 4 if anyone of (iv) and (vii) is valid then Theorem 1 (vii) is true and hence from Li-Yau [39] it follows that when \( B_r(x) \) represents the geodesic ball \( \{ y \in M^n : d(y, x) < r \} \) with radius \( r > 0 \) and center \( x \in M^n \) we get
\[
1 \leq \liminf_{t \to \infty} V(B_t(x)) p_t(x, y) \quad \text{and} \quad 1 \leq \liminf_{t \to \infty} \frac{V(B_t(x))}{\omega_n t^n} x, y \in M^n.
\]
Since the Ricci curvature of \( (M^n, ds^2) \) is nonnegative, an application of Gromov’s comparison theorem (cf. [29] page 11) produces a constant \( \kappa \) depending only on \( n \) such that
\[
\kappa \leq \frac{V(B_r(x))}{\omega_{n+1}} \leq 1 \quad \text{for all} \ r > 0 \ \text{and} \ x \in M^n.
\]
This last estimate indicates that \( \kappa \leq 1 \) is true. Furthermore, from Cheeger-Colding [14] it follows that \( M^n \) is diffeomorphic to \( \mathbb{R}^n \).

The follow-up seems quite natural.
**Theorem 6.** Let \((M^n, ds^2)\), \(n \geq 3\), be a non-compact complete boundary-free Riemannian manifold with the sharp isoperimetric inequality of Euclidean type. Then the following sharp:

(ix) Lorentz-Green’s local inequality

\[
\sup_{x \in \Omega} V(\{y \in \Omega : G\Omega(x, y) \geq t\}) \leq (n(n - 2)\omega_n^2 t + V(\Omega)^{\frac{n-2}{n}})^{\frac{n}{n-2}}
\]

holds for all regular bounded open sets \(\Omega \subseteq M^n\) and every \(t > 0\).

(x) Green’s potential-volume inequality

\[
\sup_{x \in \Omega} \int_{\Omega} G\Omega(x, y) dV(y) \leq (2n\omega_n^2 - 1)^{-1} V(\Omega)^{\frac{n}{n-1}}
\]

holds for all regular bounded open sets \(\Omega \subseteq M^n\).

On the other hand, if either (ix) or (x) is true under \((M^n, ds^2)\) being of nonnegative Ricci curvature then \(M^n\) is diffeomorphic to \(\mathbb{R}^n\).

**Proof.** The sharpness of (ix) and (x) can be verified by taking \(M^n = \mathbb{R}^n\) and \(\Omega = B_r(x)\).

To prove (ix), for the sake of simplicity let us write \(V_t = V(\{y \in \Omega : G\Omega(x, y) \geq t\})\) for given \(x \in \Omega\) and any \(t \geq 0\).

Note that

\[
\frac{dV_t}{dt} = -\int_{\{y \in \Omega : G\Omega(x, y) = t\}} |\nabla G\Omega(x, y)|^{-1} dS
\]

and

\[
1 = \int_{\{y \in \Omega : G\Omega(x, y) = t\}} |\nabla G\Omega(x, y)| dS.
\]

So, a combined application of the Cauchy-Schwarz inequality and the sharp isoperimetric inequality of Euclidean type gives

\[
-\frac{dV_t}{dt} \geq \left(\int_{\{y \in \Omega : G\Omega(x, y) = t\}} dS\right)^2 \geq \left((n\omega_n^2 V_t^{\frac{n-1}{n}})^{\frac{n}{n-1}}\right)^2.
\]

Integrating both sides of the last differential inequality over \([t_1, t_2]\) where \(0 \leq t_1 < t_2 < \infty\), we obtain

\[
V_{t_2}^{\frac{2-n}{n}} \geq V_{t_1}^{\frac{2-n}{n}} + n(n - 2)\omega_n^2 (t_2 - t_1),
\]

whence deducing (ix).

To check (x), we apply the foregoing notations and the just-demonstrated (ix) to achieve

\[
\int_{\Omega} G\Omega(x, y) dV(y) = \int_0^\infty V_t dt
\]

\[
\leq n^2 \omega_n^2 V(\Omega) \frac{2(n-1)}{n} \int_0^\infty \left(1 + n(n - 2)\omega_n^2 V(\Omega)^{\frac{n-2}{n}} \frac{2(n-1)}{n}\right) \frac{2(n-1)}{n} dt
\]

\[
= (2n\omega_n^2)^{-1} V(\Omega)^{\frac{n}{n-1}}.
\]

According to Theorem 2 if (iv) or (viii) is true then the condition of Theorem 1 (vii) is valid, and consequently the argument for the rigidity part of Theorem 5 can be used to derive that \(M^n\) is diffeomorphic to \(\mathbb{R}^n\).

In spirit of Theorem 6 we fortunately discover an optimal version of Theorem 3.
Theorem 7. Let \((M^n, ds^2)\), \(n \geq 3\), be a non-compact complete boundary-free Riemannian manifold with the sharp isoperimetric inequality of Euclidean type. If \(0 \leq q < p < \frac{n}{n-2}\) then the following comparison inequality

\[
\left( \int_\Omega G_\Omega(x, y)^p \, dV(y) \right)^{\frac{n}{n-p}} \leq \kappa_{p,q} \left( \int_\Omega G_\Omega(x, y)^q \, dV(y) \right)^{\frac{n}{n-q}}
\]

holds for any regular bounded open set \(\Omega \subseteq M^n\) containing \(x\), with equality when \((M^n, \Omega) = (\mathbb{R}^n, B_r(x))\), where

\[
\kappa_{p,q} = \left( \frac{(n(n-2)\omega_n^2)^q}{q B\left( \frac{n}{n-2}, -q \right)} \right)^{\frac{n}{n-q(n-2)}} \left( \frac{(n(n-2)\omega_n^2)^p}{p B\left( \frac{n}{n-2}, -p \right)} \right)^{\frac{n}{n-p(n-2)}}.
\]

Moreover, if

\[
G(p; x, \Omega) = \left( \frac{(n(n-2)\omega_n^2)^p}{p B\left( \frac{n}{n-2}, -p \right)} \int_\Omega G_\Omega(x, y)^p \, dV(y) \right)^{\frac{n}{n-p}}
\]

then

\[
\lim_{p \to \frac{n}{n-2}} G(p; x, \Omega) = \lim_{t \to \infty} D_\Omega(x, t)^{\frac{n}{n-2}}.
\]

where

\[
D_\Omega(x, t) = V\left( \{ y \in \Omega : G_\Omega(x, y) \geq t \} \right)^{\frac{n}{n-2}} - n(n-2)\omega_n^2 t
\]

is referred to as the Lorentz norm deficit of \(G_\Omega(x, \cdot)\) at \((x, t)\). In particular, if \((M^n, ds^2)\) has nonnegative Ricci curvature then

\[
\lim_{t \to \infty} D_\Omega(x, t)^{\frac{n}{n-2}} = \omega_n R_\Omega(x)^n,
\]

where

\[
R_\Omega(x) = \lim_{y \to x} \left( d(x, y)^{2-n} - n(n-2)\omega_n G_\Omega(x, y) \right)^{\frac{1}{n-2}}
\]

is called the harmonic radius of \(\Omega\) at \(x\).

Proof. Let us still use the notations introduced in the proofs of Theorems 5 & 6. Without loss of generality we may assume \(\int_\Omega G_\Omega(x, y)^q \, dV(y) < \infty\) - otherwise there is nothing to argue. Under the sharp isoperimetric inequality of Euclidean type we have the following monotone inequality:

\[
D_\Omega(x, t) \geq D_\Omega(x, r) \quad \text{for} \quad t \geq r \geq 0.
\]

Case 1: \(q = 0\). Clearly, \(0 < p < \frac{n}{n-2}\), the last inequality and the layer-cake formula imply

\[
\int_\Omega G_\Omega(x, y)^p \, dV(y) \leq p \int_0^\infty t^{p-1} \left( V_0 \omega_n^{\frac{2-n}{n}} + n(n-2)\omega_n^2 t \right)^{\frac{n}{n-2}} \, dt
= \frac{p V(\Omega)}{(n(n-2)\omega_n^2)^p} \int_0^\infty r^{p-1} (1 + \frac{p(n-2)}{n} r)^{\frac{n}{n-2}} \, dr,
\]
as desired.

Case 2: $q > 0$. For simplicity, set

$U_q(r) = -\int_r^\infty r^q dV_t$.

Through integrating by parts, changing variables and using the previous monotone inequality we obtain

$$U_q(r) \leq r^q V_r + q \int_r^\infty \left(D_\Omega(x, r) + n(n-2)\omega_n^\frac{2}{n} t^{\frac{2}{n}} r^{q-1} \right) dt$$

$$= n^2 \omega_n^\frac{2}{n} \int_r^\infty \left(D_\Omega(x, r) + n(n-2)\omega_n^\frac{2}{n} t^{\frac{2}{n}} r^{q-1} \right) dt$$

$$= n^2 \omega_n^\frac{2}{n} D_\Omega(x, r) \frac{n-2}{n-2} \int_{n-2}^\infty \frac{1}{\omega_n^\frac{2}{n}} (1 + t)^{\frac{2}{n}} t^q dr,$$

thereby getting

$$\left(\frac{(n-2)\omega_n^\frac{2}{n}}{qB(\frac{n}{n-2} - q, q)}\right)^{\frac{n}{n(q-2)}} \leq D_\Omega(x, r)^\frac{2}{n}.$$

Note that

$$\frac{dU_q(r)}{dr} = r^q \frac{dV_r}{dr} \leq -(n\omega_n^\frac{2}{n})^{2(n-1)}.$$

So, the foregoing two inequalities produce the following differential inequality

$$r^q \left((\alpha U_q(r))^{\frac{2}{n(q-2)}} + n(n-2)\omega_n^\frac{2}{n} r^{\frac{2}{n}}\right)^{\frac{2(n-1)}{2}} \leq -(n\omega_n^\frac{2}{n})^{2} \frac{dU_q(r)}{dr},$$

where

$$\alpha = \frac{(n-2)\omega_n^\frac{2}{n}}{qB(\frac{n}{n-2} - q, q)}.$$

Since $U_q(r) \leq U_q(0)$, the last differential inequality is used to derive

$$r^q \left(\frac{2}{n(q-2)} + U_q(0))^{\frac{n-2}{n(q-2)}} \left(n(n-2)\omega_n^\frac{2}{n} r^{\frac{2}{n}}\right)^{\frac{2(n-1)}{2}} \leq -(n\omega_n^\frac{2}{n})^{2} \frac{dU_q(r)}{dr},$$

Upon integrating this last inequality over $[0, s]$ against $dr$, we get

$$U_q(s)^{\frac{(2-n)(q+1)}{n-2}} \leq \int_0^s \left(\frac{2}{n(q-2)} + U_q(0))^{\frac{n-2}{n(q-2)}} r^q dr\right)^{\frac{2(n-1)}{2}} + U_q(0)^{\frac{(2-n)(q+1)}{n-2}}$$

$$= U_q(0)^{\frac{(2-n)(q+1)}{n-2}} \left(1 + \int_0^s \frac{(2(n-2)n)\omega_n^\frac{2}{n} s^{(n-2)\omega_n^\frac{2}{n}} (1 + r)^{\frac{2(n-1)}{2}} r^q dr}{(n-2)\omega_n^\frac{2}{n} r^{\frac{2}{n}} (n-2)\omega_n^\frac{2}{n}} \right).$$

To shorten our notation, let

$$\beta = (\alpha U_q(0))^{\frac{n-2}{n(q-2)}} n(n-2)\omega_n^\frac{2}{n}.$$
Then the last inequality, together with an integration-by-parts, yields

$$U_p(0) = (p - q) \int_0^{\infty} U_q(s) s^{p-q-1} \, ds$$

$$\leq U_q(0) \int_0^{\infty} \left( 1 + \frac{\int_0^{\beta s} (1 + r)^{\frac{2(n-1)}{2-n} r^q \, dr}}{\frac{\alpha n(q+1)}{n(n-2)\omega_n} (n-q(n-2))} \right) s^{p-q} \, ds$$

$$= -U_q(0) \int_0^{\infty} s^{p-q} \, ds \left( 1 + \frac{\int_0^{\beta s} (1 + r)^{\frac{2(n-1)}{2-n} r^q \, dr}}{\frac{\alpha n(q+1)}{n(n-2)\omega_n} (n-q(n-2))} \right) \frac{2(n-1)}{(2-n)(q+1)}$$

$$\leq \left( \frac{\alpha U_q(0)}{\left( \frac{n-2}{n} \right) (n(n-2)\omega_n)^{\frac{2}{n}}} \right)^{\frac{n-p(n-2)}{n-q(n-2)}} \int_0^{\infty} u^p(1 + u)^{\frac{2(n-1)}{2-n} \, du}$$

$$= \left( \frac{pB\left( \frac{n}{n-2} - p, p \right)}{(n(n-2)\omega_n)^{\frac{2}{n}}} \right)^{\frac{n-p(n-2)}{n-q(n-2)}} \left( \frac{\alpha n(q+1)}{n(n-2)\omega_n} (n-q(n-2)) \right) U_q(0)^{\frac{n-p(n-2)}{n-q(n-2)}}.$$ 

A simplification of the just-obtained equalities and inequalities, along with

$$U_p(0) = \int_{\Omega} G_\Omega(x, y)^p \, dV(y),$$

gives the desired inequality.

Of course, if $M = \mathbb{R}^n$ and $\Omega = B_R(x) = \{ y \in \mathbb{R}^n : |x - y| < R \}$ (given $x \in \mathbb{R}^n$ and $R > 0$), then

$$G_\Omega(x, y) = \left\{ \begin{array}{ll} \left( n(n-2)\omega_n \right)^{-1} \left( |x - y|^{2-n} - \left( \frac{n-2}{n} \right) \frac{2-n}{2-n} |x| \right), & x \neq 0, \\
\left( n(n-2)\omega_n \right)^{-1} \left( |y|^{2-n} - R^{2-n} \right), & x = 0, \end{array} \right.$$ 

and hence a direct computation yields the equality case of Theorem 7 via

$$\left( \int_{\Omega} G_\Omega(x, y)^p \, dV(y) \right)^{\frac{n}{n-p(n-2)}} = \kappa_{n,p,0} V(\Omega).$$

See also Weinberger [52] or Bandle [5], pages 59-61.

Last of all, let us deal with the limit formulas. Fixing $r \in (0, \infty)$, employing

$$U_p(r)^{\frac{n}{n-p(n-2)}} \leq \kappa_{n,p,0} D_\Omega(x, r)^{\frac{2-n}{2-n}},$$
and the following elementary inequality (cf. [1] page 389, (17)):

\[(a + b)^c \leq a^c + c2^{c-1}(b^c + ba^{c-1})\] where \(a, b \geq 0; \ c \geq 1,\)

we obtain that \(U_p(r) \leq \int_0^r V_i\,dt^p\) is valid for a sufficient large \(r,\) whence reaching

\[
G(p, x, \Omega) \leq \left(\kappa_{n,p,0} \left(U_p(r) + \int_0^r V_i\,dt^p\right)\right)^{\frac{n}{n-p(n-2)}}
\]

\[\leq \left(V_r^{\frac{2-n}{n}} - n(n-2)\omega_n^2 r^{\frac{n}{n-2}}\right) + \left(\kappa_{n,p,0}2^{\frac{n}{n-p(n-2)}}\right) \times \left(\int_0^r V_i\,dt^p\right)\]

\[\leq D_{\Omega}(x, r)^{\frac{n}{n-2}} + \left(\kappa_{n,p,0}2^{\frac{n}{n-p(n-2)}}\right) \left(2\int_0^r V_i\,dt^p\right)^{\frac{n}{n-p(n-2)}}.
\]

Taking the asymptotic behavior of the following Beta function into account:

\[B\left(\frac{n}{n-2} - p, p + 1\right) \sim \left(\frac{\sqrt{2\pi}(\frac{n}{n-2} - p)^{\frac{n}{n-2} - p} - (p + 1)^{p+\frac{1}{2}}}{2(n-1)^{\frac{n}{n-2}}}ight)^{\frac{n}{n-2},p(n-2) - n},\]

we gain

\[
\lim_{p \to \frac{n}{n-2}} \left(\kappa_{n,p,0}2^{\frac{n}{n-p(n-2)}}\right) \left(2\int_0^r V_i\,dt^p\right)^{\frac{n}{n-p(n-2)}} = 0,
\]

and thus

\[
\lim_{p \to \frac{n}{n-2}} G(p, x, \Omega) \leq D_{\Omega}(x, r)^{\frac{n}{n-2}}.
\]

Letting \(r \to \infty\) on the right-hand-side of the last inequality, we find

\[
\lim_{p \to \frac{n}{n-2}} G(p, x, \Omega) \leq \lim_{r \to \infty} D_{\Omega}(x, r)^{\frac{n}{n-2}}.
\]

At the same time, since \(D_{\Omega}(x, s)^{\frac{n}{n-2}}\) decreases with \(s,\) one has

\[
U_p(0) = \int_0^\infty \left(D_{\Omega}(x, s) + n(n-2)\omega_n^2 s\right)^{\frac{n}{n-2}} ds^p
\]

\[
\geq \int_0^\infty \left(\lim_{s \to \infty} D_{\Omega}(x, s) + n(n-2)\omega_n^2 s\right)^{\frac{n}{n-2}} ds^p
\]

\[
= \lim_{s \to \infty} D_{\Omega}(x, s)^{\frac{n-p(n-2)}{n-2}} n(n-2)\omega_n^2 \frac{n}{n-2} - pB\left(\frac{n}{n-2} - p, p,\right),
\]

producing

\[
\lim_{p \to \frac{n}{n-2}} G(p, x, \Omega) \geq \lim_{r \to \infty} D_{\Omega}(x, r)^{\frac{n}{n-2}}.
\]

Therefore, the first limit formula follows.

In order to verify the second limit formula, we observe that \(G_{\Omega}(x, y) = r\) implies

\[d(x, y)^{2-n} = n(n-2)\omega_n r + R_{\Omega}(x)^{2-n} + o(1)\] as \(r \to \infty,\)

and hence

\[B_{r_+}(x) \subseteq \{y \in \Omega : G_{\Omega}(x, y) \geq r\} \subseteq B_{r_+}(x),\]

where

\[r_+ = (n(n-2)\omega_n r + R_{\Omega}(x)^{2-n})^{\frac{n}{n-2}} \pm o(1)\] as \(r \to \infty.\]
Since the sharp isoperimetric inequality of Euclidean type is valid for $(M^n, ds^2)$ which has nonnegative Ricci curvature, we conclude (cf. [29 page 244]):

$$V(B_{r_{\pm}}(x)) = \omega_n r_{\pm}^n.$$  

Consequently, 

$$\omega_n r_{\pm}^n \leq V(\{y \in \Omega : G_{\Omega}(x, y) \geq r\}) \leq \omega_n r_{+}^n.$$  

This yields 

$$D_{\Omega}(x, r)^{\frac{2}{n-2}} = \omega_n R_{\Omega}(x)^n \pm o(1) \quad as \quad r \to \infty.$$  

\[\square\]

Remark 8. Under the same hypothesis as in Theorem 7, we can obtain the sharp isoperimetric-type inequality for $0 < p < \frac{n}{n-2}$:

$$\left(\frac{\left(n(n-2)\omega_n^{\frac{2}{n-2}}\right)^p}{pB\left(\frac{1}{n-2} - p, p\right)} \int_{\Omega} G_{\Omega}(x, y)^p dV(y)\right)^{\frac{n-1}{n-2}} \leq (\omega_n^{\frac{n}{n-2}})^{-1} S(\partial \Omega),$$

but also the non-sharp one for $\frac{n}{n-2} < p < 0$:

$$V(\Omega)^{\frac{n-p(n-2)}{n}} \leq \left(n(n-2)\omega_n^{\frac{2}{n-2}}\right)^p (-p)B\left(\frac{n}{n-2} + p, -p\right) \int_{\Omega} G_{\Omega}(x, y)^p dV(y),$$

for all regular bounded domains $\Omega \subseteq M^n$ containing $x$.

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