Hua type beta-integrals and projective systems of measures on flag spaces

Yury A. Neretin

We construct a family of measures on flag spaces (or, equivalently, on the spaces of upper-triangular matrices) compatible with respect to natural projections. We obtain an \( n(n-1)/2 \)-parametric family of beta-integrals over space of upper-triangular matrices of size \( n \).

1 Formula

1.1. Beta-integrals. Let \( K \) be \( \mathbb{R} \), \( \mathbb{C} \), or the quaternions \( \mathbb{H} \), let \( \kappa = \dim K \) (over \( \mathbb{R} \)). By \( dz \) we denote the Lebesgue measure on \( K \).

Denote by \( Z_n(K) \) the space of strictly upper triangular matrices \( Z = \{z_{km}\} \) of size \( n \), i.e. \( z_{km} = 0 \) for \( k > m \), \( z_{kk} = 1 \), other matrix elements are arbitrary. We write \( Z_{\{n\}} \) if we wish to emphasize the order of a matrix. Denote by \( dZ \) or \( \text{d}Z_{\{n\}} = \text{d}Z \) the Lebesgue measure on \( Z_n(K) \),

\[
dZ = dZ^{(n)} := dz_{12} \ldots dz_{1n} \ dz_{23} \ldots dz_{2n} \ldots dz_{(n-1)n}.
\]

The space \( Z_n(K) \) is identified in the usual way with the space of flags in \( K^n \) (up to a subset of measure 0).

For a matrix \( Z \in Z_n(K) \) we denote by \( [Z]_{pq} \) its left upper corner of size \( p \times q \) (it has \( p \) rows and \( q \) columns). We consider such matrices only for \( p < q \). Denote

\[
s_{pq}(Z) := \det([Z]_{pq} [Z]_{pq}^*).
\]

Notice that the matrix \( [Z]_{pq} [Z]_{pq}^* \) is positive definite and therefore \( s_{pq}(Z) \) are positive.

**Theorem 1.1** Let integer parameters \( p, q \) range in the domain \( 1 \leq p < q \leq n \). For \( \lambda_{pq} \in \mathbb{C} \) set

\[
\nu_{pq} := -\frac{1}{2}(q-p-1)\kappa + \sum_{k,m: p \leq k < q, q \leq m \leq n} \lambda_{mk}.
\]

Then

\[
\int_{Z_n(K)} \prod_{1 \leq p < q \leq n} s_{pq}(Z)^{-\lambda_{pq}} \, dZ^{(n)} = \pi^{n(n-1)/4} \prod_{1 \leq p < q \leq n} \frac{\Gamma(\nu_{pq} - \kappa/2)}{\Gamma(\nu_{pq})}, \tag{1.1}
\]

the integral absolutely converges if and only if

\[
\text{Re} \nu_{pq} > \frac{1}{2}.
\]

\[\text{supported by grant FWF, project P25142}\]
Remark. For $K = \mathbb{C}$ and $K = \mathbb{H}$, i.e., $\kappa = 2, 4$, we have cancellations in the right hand side of (1.1), because
\[ \frac{\Gamma(\nu - 1)}{\Gamma(\nu)} = \frac{1}{\nu - 1}, \quad \frac{\Gamma(\nu - 2)}{\Gamma(\nu)} = \frac{1}{(\nu - 1)(\nu - 2)}. \]

Theorem 1.1 is a corollary of the following statement.

**Theorem 1.2** Consider the map $Z_n(K) \to Z_{n-1}(K)$ forgetting the last column of a matrix $Z^{(n)}$. Consider a measure
\[ \prod_{p=1}^{n-1} s_{p\nu}(z)^{-\lambda_p} dZ^{(n)} \] on $Z_n(K)$. Assume
\[ \lambda_p + \lambda_{p+1} + \cdots + \lambda_{n-1} > \frac{1}{2} (n - p) \kappa \]
for all $p$. Then the pushforward of this measure under the forgetting map is
\[ \pi^{\frac{(n-1)\kappa}{2}} \prod_{1 \leq p \leq n-1} \frac{\Gamma(\lambda_p + \cdots + \lambda_n - (n-p)\kappa/2)}{\Gamma(\lambda_p + \cdots + \lambda_n - (n-p+1)\kappa/2)} \times \prod_{p=1}^{n-2} s_{p(n-1)}(Z^{(n-1)})^{-\lambda_p} dZ^{(n-1)}. \] (1.3)

Remark. On a geometric language the 'forgetting map' is projection of a flag to a $(n-1)$-dimensional subspace.

1.2. Comparison with Hua integrals. In his famous book [5], Hua Loo Keng obtained a collection of matrix integrals in the following spirit:
\[ I_n(\alpha) = \int_{T \in \text{Symm}_n(\mathbb{R})} \det(1 + T^2)^{-\alpha} dT = \pi^{\frac{n(n+1)}{2}} \Gamma(\alpha - n/2) \prod_{p=1}^{n-1} \frac{\Gamma(2\alpha - (n+p)/2)}{\Gamma(2\alpha)}, \]
where the integration is taken over the space $\text{Symm}_n(\mathbb{R})$ of real symmetric matrices of size $n$. Recall that $\text{Symm}_n(\mathbb{R})$ is a chart on the real Lagrangian Grassmannian $\mathcal{L}_n$ (see, e.g., [9], Sect.3.1, 3.3). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real symplectic matrix. The corresponding transformation of $\mathcal{L}_n$ has the form
\[ \tilde{T} = (a + Tc)^{-1}(b +Td). \]

The maximal compact subgroup of the real symplectic group $\text{Sp}(2n, \mathbb{R})$ is isomorphic to $U(n)$, it can be realized as the group of all real matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

\[This is a positive measure on $Z_n(K)$ if $\lambda_p \in \mathbb{R}$ and a complex-valued measure if $\lambda_p \in \mathbb{C}$.\]
such that $a + ib \in U(n)$. The integrand in the Hua integral has nice behavior under the action of the group $U(n)$,

$$
\det(1 + \tilde{T}^2)^{-\alpha} d\tilde{T} = \det(1 + T^2)^{-\alpha} |\det(a - T b)|^{-2\alpha + n+1}dT
$$

(1.4)

Later (see [4], [7]) it became known that Hua integrals can be included to multi-parametric families of matrix beta-integrals. There is another nice property of Hua integrals: the measures

$$
I_n(\sigma + n/2)^{-1} \det(1 + T^2)^{-\sigma - n/2} dT
$$

form a projective system under the natural 'cutting' map

$$
\text{Symm}_n(\mathbb{R}) \to \text{Symm}_{n-1}(\mathbb{R}).
$$

This phenomenon was a standpoint for infinite-dimensional harmonic analysis (see [10], [8], [1]).

Let us return to our expressions (1.2). To be definite, set $K = \mathbb{C}$. The group $GL(n, \mathbb{C})$ acts on the flag space, an element $g \in GL(n, \mathbb{C})$ induces a rational transformation of $\mathbb{C}$. For $g \in GL(n, \mathbb{C})$ denote by $(a_{p}\hspace{1mm}b_{p}\hspace{1mm}c_{p}\hspace{1mm}d_{p}) = g$ its representation as a block matrix of size $p+(n-p)$. Denote by $Z = (u_{p}\hspace{1mm}v_{p}\hspace{1mm}0\hspace{1mm}w_{p}) = Z$ the corresponding splitting of $Z \in \mathbb{C}$. Let $g$ be unitary. Then the image of the measure (1.2) under the corresponding transformation is

$$
\prod_{p=1}^{n-1} \det(u_{p}a_{p} + v_{p}c_{p})^{-\lambda_{p}-4} \prod_{p=1}^{n-1} s_{pn}(Z)^{-\lambda p} dZ
$$

This is an analog of (1.4) for flag spaces.

2 Calculations

2.1. The main lemma. Fix $p$ and consider the corner $[z]_{pn}$. Theorem 1.2 is based on the following formula:

Lemma 2.1

$$
\int_{\mathbb{K}} s_{pn}(Z)^{-\lambda} d z_{pn} = \pi^{\lambda/2} s_{(p-1)(n-1)}(Z) \cdot \Gamma(\lambda - \kappa/2) \Gamma(\lambda - \kappa)
$$

This lemma is a corollary of two following lemmas.

Lemma 2.2 Let $a, c > 0$, $b \in \mathbb{K}$, and $ac - b\overline{b} > 0$. Let $\lambda \in \mathbb{C}$, Re $\lambda > \kappa/2$. Then

$$
\int_{\mathbb{K}}(a u \pi + b \overline{\pi} + \overline{\pi}b + c)^{-\lambda} du = \pi^{\kappa/2} a^{\lambda - \kappa} (ac - \overline{b})^{\kappa/2 - \lambda} \Gamma(\lambda - \kappa/2) \Gamma(\lambda).
$$
Proof. By an affine change of variable we reduce the integral to the form
\[ \int_K |w|^2 + 1 - \lambda \, dw. \]
\[ \Box \]

Lemma 2.3
Denote \( u = z_{pn} \). Then \( s_{pn}(Z) \) has the form
\[ s_{pn}(Z) = au\overline{u} + ub + \overline{b}u + c, \]  
moreover
\[ a = s_{(p-1)(n-1)}(Z), \quad ac - \overline{b}b = s_{(p-1)n}(Z) \cdot s_{p(n-1)}(Z). \]

Proof of this lemma occupies Subsections 2.2-2.4.

2.2. Proof of Lemma 2.3 for \( K = \mathbb{R} \) and \( \mathbb{C} \).
Step 0. We write \( [Z]_{pn} \) as a block matrix of size \((p - 1) + 1 \times (p - 1) + (n - p) + 1\),
\[ [Z]_{pn} = \begin{pmatrix} Q & R & t_1 & u \end{pmatrix}. \]
Therefore
\[ s_{pn}(Z) = \begin{pmatrix} QQ^* + RR^* + tt^* & R\rho^* + t\overline{\rho} \end{pmatrix}. \]

Step 1. We write coefficients \( a, b, c \) in (2.1):
\[ a = \det(QQ^* + RR^* + tt^*) + \det(QQ^* + RR^* + tt^* \begin{pmatrix} t & 1 \end{pmatrix}) = \]
\[ = \det(QQ^* + RR^* + tt^* \begin{pmatrix} t & 1 \end{pmatrix}) = \det \left[ \begin{pmatrix} QQ^* + RR^* & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^* & 1 \end{pmatrix} \right] = \]
\[ = \det(QQ^* + RR^*) = s_{(p-1)(q-1)}(Z); \]
\[ b = \det(QQ^* + RR^* + tt^* \begin{pmatrix} u & 0 \end{pmatrix}) = \det \left[ \begin{pmatrix} QQ^* + RR^* & t \rho R^* \end{pmatrix} \begin{pmatrix} 0 & t \rho \overline{R} \end{pmatrix} \begin{pmatrix} 0 & t \rho \rho \overline{R} \end{pmatrix} \right] = \]
\[ = \det(QQ^* + RR^* \begin{pmatrix} t \rho R^* & t \rho \rho \overline{R} \end{pmatrix}); \]
\[ c = \det(QQ^* + RR^* + tt^* \begin{pmatrix} R^* \rho \end{pmatrix}) \]  
(2.2)

Step 2. Recall the Desnanot-Jacobi formula (see, e.g., [6]). Consider a block matrix \( S \) of size \( m + 1 + 1 \),
\[ S = \begin{pmatrix} U & v_1 & v_2 \\ w_1 & x_{11} & x_{12} \\ w_2 & x_{12} & x_{22} \end{pmatrix}. \]
Then
\[
\det(U) \det(S) = \\
= \det \left( \begin{array}{cc}
U & v_1 \\
w_1 & x_{11}
\end{array} \right) \cdot \det \left( \begin{array}{cc}
U & v_2 \\
w_2 & x_{22}
\end{array} \right) - \det \left( \begin{array}{cc}
U & v_1 \\
w_2 & x_{21}
\end{array} \right) \cdot \det \left( \begin{array}{cc}
U & v_2 \\
w_1 & x_{12}
\end{array} \right).
\]

We apply this identity to the matrix
\[
H := \begin{pmatrix}
QQ^* + RR^* + tt^* & R\rho^* & t \\
\rho R^* & \rho \rho^* & 0 \\
t^* & 0 & 1
\end{pmatrix}
\]
and get
\[
ac - b^b = \det(H) \det(QQ^* + RR^* + tt^*). \tag{2.3}
\]

Next, we represent \(H\) as
\[
H = \begin{pmatrix}
QQ^* + RR^* & R\rho^* & t \\
\rho R^* & \rho \rho^* & 0 \\
t^* & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
t^* & 0 & 1
\end{pmatrix}. \tag{2.4}
\]

Therefore
\[
\det(H) = \begin{pmatrix}
QQ^* + RR^* & R\rho^*
\end{pmatrix}
\rho \rho^* = s_{p(n-1)}(Z). \tag{2.5}
\]

On the other hand, \(\det(QQ^* + RR^* + tt^*) = s_{(p-1)n}(Z)\), and (2.3) implies the desired statement.

### 2.3. Several remarks on quaternionic determinants.

1) **Definition of a quaternionic determinant.** Consider an \(m \times m\) matrix \(A\) over quaternions. It determines an automorphism of an \(H\)-module \(H^m\) and therefore a \(\mathbb{R}\)-linear operator \(A_\mathbb{R}\) in \(\mathbb{R}^{4m}\), it can be easily shown that \(\det(A_\mathbb{R}) \geq 0\). We set
\[
\det(A) := \sqrt[4]{\det(A_\mathbb{R})}.
\]

By definition, \(\det(AB) = \det(A) \det(B)\), for strict upper-triangular (and lower triangular) matrices the determinant is 1, for diagonal matrices with entries \(a_{ii}\) it equals \(\prod |a_{ii}|\). These remarks establish a coincidence of \(\det(A)\) with Dieudonne determinant [2] over \(H\).

2) **Formula for determinant of block matrix** (see, e.g., [3], Sect. II.5) remains valid for quaternionic matrices
\[
\det \begin{pmatrix}
u & v \\
w & x
\end{pmatrix} = \det(u) \cdot \det(x - wu^{-1}v), \tag{2.6}
\]

here \(\begin{pmatrix} u & v \\
w & x \end{pmatrix}\) is a block matrix of size \((m + k) \times (m + k)\), \(u\) is assumed to be invertible. The formula follows from the identity
\[
\begin{pmatrix}
1 & 0 \\
-wu^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
u & v \\
w & x
\end{pmatrix}
\begin{pmatrix}
1 & -u^{-1}v \\
0 & 1
\end{pmatrix}
= \begin{pmatrix} u & 0 \\
0 & x - wu^{-1}v \end{pmatrix}. \tag{2.7}
\]
3) **Positive definite matrices.** Let a block matrix \( \begin{pmatrix} u & v \\ w & x \end{pmatrix} \) be Hermitian positive definite. Then \( x - wu^{-1}v \) also is positive definite (in this case the matrix in the left-hand side of (2.7) is positive definite).

4) For positive definite block \((m+1) \times (m+1)\) matrices we can write

\[
\det \begin{pmatrix} u & v \\ w & x \end{pmatrix} = \det(u) \cdot (x - wu^{-1}v). \tag{2.8}
\]

In particular, for \(2 \times 2\) positive definite matrices we have

\[
\begin{pmatrix} u & v \\ \overline{v} & x \end{pmatrix} = ux - \overline{v}. \tag{2.9}
\]

2.4. **Proof of Lemma 2.3 for** \(K = \mathbb{H}\). **Step 0** is the same.

**Step 1.** Since \(s_{p_n}(Z)\) is positive definite, we have

\[
s_{p_n}(Z) = \begin{pmatrix} QQ^* + RR^* + tt^* & R\rho^* + \overline{t}\pi \\ \rho R^* + ut^* & \rho \rho^* + u\overline{\pi} \end{pmatrix} = \det(QQ^* + RR^* + tt^*) \times \\
\times \left[ \rho \rho^* + u\overline{\pi} - (\rho R^* + ut^*)(QQ^* + RR^* + tt^*)^{-1}(R\rho^* + t\overline{\pi}) \right].
\]

Expanding the expression in square brackets in variables \(u, \pi\), we get

\[
a = \det(QQ^* + RR^* + tt^*) \cdot \left[ 1 - t^*(QQ^* + RR^* + tt^*)^{-1}t \right]; \tag{2.10}
\]

\[
b = \det(QQ^* + RR^* + tt^*) \cdot \left[ -t^*(QQ^* + RR^* + tt^*)^{-1}R\rho^* \right]; \tag{2.11}
\]

\[
c = \det(QQ^* + RR^* + tt^*) \cdot \left[ \rho \rho^* - \rho R^*(QQ^* + RR^* + tt^*)^{-1}R\rho^* \right]. \tag{2.12}
\]

Notice, that expression (2.10) is real and therefore we can write \(ua\overline{\pi} = au\overline{\pi}\).

Next, we transform \(a\) as

\[
a = \det \begin{pmatrix} QQ^* + RR^* + tt^* & t \\ t^* & 1 \end{pmatrix} \tag{2.13}
\]

Indeed, this matrix is positive definite because it equals \(XX^*\) with \(X = \begin{pmatrix} Q & P & t \\ 0 & 0 & 1 \end{pmatrix}\).

Therefore we can apply the transformation (2.8) to (2.13) and get the initial expression (2.10) for \(a\).

This argumentation remains to be valid for \(c\) (we do not need a final expression), but generally

\[
b \neq \det \begin{pmatrix} QQ^* + RR^* + tt^* & R\rho^* \\ t^* & 0 \end{pmatrix} \tag{2.11}
\]

(the expression in the right hand side is real positive, the expression (2.11) for \(b\) is not real).
Transforming $a$ as in $\mathbb{R}$-$\mathbb{C}$-cases, we get
\[ a = s_{(p-1)(n-1)}(Z). \]

**Step 2.** The Desnanot-Jacobi identity does not hold for matrices over $\mathbb{H}$. However, we can adapt previous reasonings for quaternionic case. Again, consider the matrix $H$ given by (2.4) and transform its determinant
\[
\begin{aligned}
\det \begin{pmatrix}
QQ^* + RR^* + tt^* & R\rho^* \\
R\rho^* & \rho \rho^*
\end{pmatrix} = & \det(QQ^* + RR^* + tt^*) \times \\
\times & \det \left( \begin{pmatrix}
\rho \rho^* & 0 \\
0 & 1
\end{pmatrix} \right) + \left( \begin{pmatrix}
\rho \rho^* \\
0
\end{pmatrix}
\right) (QQ^* + RR^* + tt^*)^{-1} \left( \begin{pmatrix}
R\rho^* \\
t^*
\end{pmatrix}
\right) = \\
= & \det(QQ^* + RR^* + tt^*) \times \\
\times & \det \left[ \begin{pmatrix}
1 - t^*(QQ^* + RR^* + tt^*)^{-1}t & -\rho \rho^*(QQ^* + RR^* + tt^*)^{-1}t \\
-t^*(QQ^* + RR^* + tt^*)^{-1}R\rho^* & \rho \rho^* - \rho R^*(QQ^* + RR^* + tt^*)^{-1}R\rho^*
\end{pmatrix}
\right].
\end{aligned}
\]

Denote by $\Xi$ the $2 \times 2$ matrix in square brackets. The matrix $H$ is positive definite since
\[
H = \begin{pmatrix}
Q & R \\
0 & \rho
\end{pmatrix} \begin{pmatrix}
Q & R \\
0 & \rho
\end{pmatrix}^*,
\]
therefore $\Xi$ is positive definite. Hence we can evaluate $\det(\Xi)$ by (2.9). Comparing its matrix elements with (2.10)–(2.12), we get
\[
\det(\Xi) = \det(QQ^* + RR^* + tt^*)^{-2}(ac - b\overline{b}).
\]

Therefore
\[
\det(H) = (ac - b\overline{b}) \cdot \det(QQ^* + RR^* + tt^*)^{-1} = (ac - b\overline{b}) \cdot s_{(p-1)n}(Z)^{-1}.
\]
The reasoning (2.4)–(2.5) remains valid and we get $\det H = s_{p(n-1)}(Z)$. This completes the calculation.

**2.5. Proof of Theorem 1.2.** Consider a function $\Psi$ on $\mathbb{Z}_n(\mathbb{K})$ that does not depend on variables $z_{1n}, \ldots, z_{(n-1)n}$. We denote such functions as $\Psi(Z^{(n-1)})$. Let us transform the integral
\[
\begin{aligned}
\int_{\mathbb{Z}_n(\mathbb{K})} \Psi(Z^{(n-1)}) \cdot \prod_{p=1}^{n-1} s_{pn}(Z)^{-\lambda_p} dZ^{(n)} = & \\
= & \int_{\mathbb{Z}_{n-1}(\mathbb{K}) \times \mathbb{K}^{n-1}} \Psi(Z^{(n-1)}) \cdot \prod_{p=1}^{n-1} s_{pn}(Z)^{-\lambda_p} dZ^{(n-1)} \ dz_{1n} dz_{2n} \ldots dz_{(n-1)n}.
\end{aligned}
\]
The variable $z_{(n-1)n}$ is presented only in the factor $s_{pn}(z)^{-\lambda_n-1}(Z)$. Integrating with respect to $z_{(n-1)n}$ by Lemma 2.1 we get

$$
\pi^{n/2} \frac{\Gamma(\lambda_n - 1 - \kappa/2)}{\Gamma(\lambda_n - 1)} \int_{\mathbb{R}^{n-2}(3) \times \mathbb{R}^{n-1}} \left[ \Psi(Z^{(n-1)}) \cdot s_{(p-1)(n-1)}(Z^{(n-1)})^{\lambda_n - 1 - \kappa/2} \right] \times
$$

$$\times \prod_p s_{pn}(Z^{(n)})^{-\lambda_p} \cdot s_{(n-2)n}(Z^{(n)})^{-\lambda_{n-2} - \lambda_{n-1} + \kappa/2} dZ^{(n-1)} dZ_{1n} \ldots dZ_{(n-2)n}.
$$

Now the variable $z_{(n-2)n}$ is presented only in the factor $s_{(p-1)n}(Z^{(n)})^{-\lambda_{n-2} - \lambda_{n-1} + \kappa/2}$.

We again apply Lemma 2.1. Etc. Finally, we get the integration of $\Psi(Z^{(n-1)})$ with respect to the measure (1.3).

References

[1] Borodin, A.; Olshanski, G. Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes. Ann. of Math. (2) 161 (2005), no. 3, 1319-1422

[2] Dieudonné, J. Les déterminants sur un corps non commutatif. (French) Bull. Soc. Math. France 71, (1943). 27-45.

[3] Gantmacher, F. R. The theory of matrices. Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998.

[4] Gindikin, S. G. Analysis in homogeneous domains. (Russian) Uspehi Mat. Nauk 19 1964 no. 4 (118), 3-92; English transl.: Russ. Math. Surv., 1964, 19:4, 1-89

[5] Hua, L. K. Harmonic analysis of functions of several complex variables in the classical domains. Science Press, Peking, 1958 (Chinese); Russian translation: Izdat. Inostr. Lit., Moscow 1959; English translation: Amer. Math. Soc., Providence, R.I. 1963

[6] Krattenthaler, C. Advanced determinant calculus. The Andrews Festschrift (Maratea, 1998). Sém. Lothar. Combin. 42 (1999), Art. B42q, 67 pp. (electronic).

[7] Neretin, Yu. A. Matrix analogues of the B-function, and the Plancherel formula for Berezin kernel representations. Sb. Math. 191 (2000), no. 5-6, 683-715

[8] Neretin, Yu. A. Hua-type integrals over unitary groups and over projective limits of unitary groups. Duke Math. J. 114 (2002), no. 2, 239-266.
[9] Neretin, Yu. A. Lectures on Gaussian integral operators and classical groups. EMS, Zürich, 2011.

[10] Pickrell, D. Measures on infinite-dimensional Grassmann manifolds. J. Funct. Anal. 70 (1987), no. 2, 323-356.

Math.Dept., University of Vienna,
Oskar-Morgenstern-Platz 1, 1090 Wien;
& Institute for Theoretical and Experimental Physics (Moscow)
& Mech.Math.Dept., Moscow State University,
e-mail: neretin(at) mccme.ru
URL: www.mat.univie.ac.at/~neretin