In quantum groups coproducts of Lie-algebras are twisted in terms of generators of the corresponding universal enveloping algebra. If representations are considered, twists also serve as starproducts that accordingly quantize representation spaces. In physics, requirements turn out to be the other way around. Physics comes up with noncommutative spaces in terms of starproducts that miss a suiting quantum symmetry. In general the classical limit is known, i.e. there exists a representation of the Lie-algebra on a corresponding finitely generated commutative space. In this setup quantization can be considered independently from any representation theoretic issue. We construct an algebra of vector fields from a left cross-product algebra of the representation space and its Hopf-algebra of momenta. The latter can always be defined. The suitingly divided cross-product algebra is then lifted to a Hopf-algebra that carries the required genuine structure to accomodate a matrix representation of the universal enveloping algebra as a subalgebra. We twist the Hopf-algebra of vector fields and thereby obtain the desired twisting of the Lie-algebra. Since we twist with vector fields and not with generators of the Lie-algebra, this is the most general twisting that can possibly be obtained. In other words, we push starproducts to twists of the desired symmetry algebra and to this purpose solve the problem of turning vector fields into a Hopf-algebra. We give some genuine example.

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Studies of quantum groups require for a considerable mathematical framework that historically caused the topic to be turned into a mathematical field on its own. As a consequence it then naturally followed its own mathematical interests - apart from actual physical requirements. In quantum groups deformations of a Lie-algebra $\mathfrak{g}$ are considered in terms of its universal enveloping algebra $U(\mathfrak{g})$. Coproducts of $U(\mathfrak{g})$ are deformed by conjugation with quasitriangular structures $R \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ or twists $F \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. The noncocommutative coproduct of the deformed version of the universal enveloping algebra $U(\mathfrak{g})$ dually implies a noncommutative structure on representation space. As an example see [3]. Thus within the standard workflow of quantum groups, symmetry algebras are first deformed and represented afterwards. Physics, however, requires for the opposite procedure. Theories and models come down with noncommutative spaces, as canonical spacetime in [7, 23, 24], that miss the corresponding quantum symmetry. In most cases the classical limit exists, i.e. there exists a representation of $\mathfrak{g}$ on a finitely generated commutative space. The task at hand is to find the corresponding deformation of the symmetry algebra. But quantum groups do not provide the required techniques. It thus takes quite a time until such quantizations are found - if they are found at all. For the case of canonical commutation relations these were constructed in [22, 5, 25, 13]. While twists can be used as starproducts, the opposite only holds for some specific exceptions. This is the standard situation in physics. Quite often it has been observed that quantization requires for some enhancement of the symmetry algebra [26]. For example, the well-known $\kappa$-deformation of the Poincaré algebra [15, 13, 16] cannot be reduced to that of the Lorentz-algebra alone. The algebra of momenta is a vital component of this deformation. The mathematical setup to this example had been provided by [21]. The same holds for the mentioned $\theta$-deformations of the Poincaré algebra for canonical commutation relations. Obviously only those very specific deformations can merely be performed within the symmetry algebra, that are ruled by a quasitriangular structure $R$. But these only provide quantum spaces with quadratic commutation relations. We can thus observe the physical reason why $\kappa$- and $\theta$-deformations required for some algebraic enhancement: The deformation parameter carries a physical dimension. Thus while the mathematical workflow restricted to a single version of quantum spaces, that turned out quite unhandy for physical applications, physics itself came up with deformations beyond this setup. And mathematics, as often, delivered an explanation afterwards. The universal enveloping algebra of a Lie-algebra is obviously not large enough in order to perform most general quantizations of its coproducts. The authors of [18, 17, 14] incorporated this idea and used the Poincaré algebra as a whole in order to obtain more general twistings. They receive quantum spaces with quadratic as well as Lie-
algebra valued commutation relations. Here we want to push this a little further. Within another example of physics, phase space deformations were considered in order to obtain high energy motivated minimal uncertainty models [12, 11, 10, 9]. The author speculates that the deformation of a corresponding Poincaré-algebra might be obtained by the use of the phase space algebra itself. In contrast to this, the authors of [8] formulate starproducts in terms of vector fields. Vector fields are most fundamental objects of differential geometry and Lie-algebras themselves describe nothing else than the currents on curved manifolds. Apart from this, there is a close relation between noncommutative geometry and quantization over curved spaces. In this respect vector fields also played a crucial role for noncommutative gravity [2, 1]. Vector fields might thus provide the actual and most genuine structure underlying any deformation-quantization. But in order to consider such twist-deformations, an algebra of vector fields would have to be enhanced to a Hopf-algebra. The actual question is, how this is possibly done. A very elegant solution to this problem was provided by the authors of [20]. But they already incorporated a physical interpretation into their setup that we want to avoid here. To any representation space we can formally define an action of a Hopf-algebra of momenta. These can be joined to a left cross-product algebra that we devide in such a way, that we can lift it to an actual Hopf-algebra. In fact this construction provides a very clear and genuine structure that we further denote as a Hopf-algebra of vector fields. This Hopf-algebra is large enough to accomodate any matrix representation of the universal enveloping algebra $U(g)$ as a subalgebra. This is the commutative limit that is well-known in physics and has to be fed into this setup. By twisting the Hopf-algebra of vector fields we thus twist its subalgebra as well - but more general than the generators of $U(g)$ could possibly do. In the mean time the twist is nothing else than the starproduct, that comes with the noncommutative associative space. We thus achieve several goals. Starproducts directly can be used as twists in order to obtain a quantization of the desired symmetry and in parallel we open the formalism for most general quantizations and thus stay as close as possible to the actual requirements of physics. The paper is organised as follows. In the first section we formulate the classical limit that we have to feed as input into our procedure. We take the opportunity to recall basic definitions and properties of required notions in order to be self-contained. In the following section we construct the Hopf-algebra of vector fields and the actual twists will be considered in the third section. We close with the basic example of a deformation of the two-dimensional representation of $U(sl_2)$. The exposition of the matter orients itself to the textbooks [5, 19].
2 Representation of $U(\mathfrak{g})$ on $U(\mathfrak{x})$

As outlined in the introduction, the deformation of a universal enveloping algebra $U(\mathfrak{g})$ of a Lie-algebra $\mathfrak{g}$ and its accordingly deformed representation space $\mathfrak{x}$ is actually independent of any representation theoretic issues, presupposing that the non-quantized limit exists and is well defined.

In this section we concretize this specific undeformed setup and in order to be self-contained we take the opportunity to recall basic definitions and properties of Lie-algebras and their representations.

It is our aim to represent $\mathfrak{g}$ on a finite dimensional $K$-linear vector space $\mathfrak{x}$. As fields $K$ we consider complex or real numbers. Let us shortly recall the definition of a Lie-algebra before we continue.

2.1 Definition (Lie-algebra) Let $\mathfrak{g}$ be a $p$-dimensional vector space over the field $K$. The vector space $\mathfrak{g}$ is called a Lie-algebra if there exists a bracket $[\cdot, \cdot]_\mathfrak{g}: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that holds the following properties:

\[
\forall g, h, k \in \mathfrak{g} : 
\begin{align*}
[g, h]_\mathfrak{g} &= -[h, g]_\mathfrak{g} & \text{(Antisymmetry)} \\
[g + h, k]_\mathfrak{g} &= [g, k]_\mathfrak{g} + [h, k]_\mathfrak{g} & \text{(Bilinearity)} \\
[g, [h, k]_\mathfrak{g}]_\mathfrak{g} + [h, [k, g]_\mathfrak{g}]_\mathfrak{g} + [k, [g, h]_\mathfrak{g}]_\mathfrak{g} &= 0 & \text{(Jacobi-Identity)}
\end{align*}
\]

As an element of the Lie-algebra $\mathfrak{g}$, the bracket can be expressed as a linear combination in terms of basis elements $(g_a)_{a \in \{1, \ldots, p\}}$, i.e.

\[
[g_a, g_b]_\mathfrak{g} = i \sum_{c=1}^{p} f_{abc} g_c, \quad f_{abc} \in K.
\]

Formally a representation of $\mathfrak{g}$ on $\mathfrak{x}$ is much more the representation of its universal enveloping algebra $U(\mathfrak{g})$ on $\mathfrak{x}$, that we define as follows.

2.2 Definition (Universal Enveloping Algebra) Let $\mathfrak{g}$ be a Lie-algebra over the field $K$ with $p$-dimensional basis $(g_a)_{a \in \{1, \ldots, p\}}$ and bracket $[\cdot, \cdot]_\mathfrak{g}$. Then the universal enveloping algebra $U(\mathfrak{g})$ is defined to be the quotient of the tensor algebra $T(\mathfrak{g})$ and the two-sided ideal $\mathcal{I}_\mathfrak{g} \subset T(\mathfrak{g})$

\[
U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\mathcal{I}_\mathfrak{g}}.
\]

The two-sided ideal $\mathcal{I}_\mathfrak{g}$ is generated by relations

\[
\forall g_a, g_b \in \mathfrak{g} : g_a \otimes g_b - g_b \otimes g_a - i \sum_{c=1}^{p} f_{abc} g_c = 0 \quad (2.1)
\]
For \( \varphi(g_a), \omega(g_b) \in U(\mathfrak{g}) \) the bracket 
\[ [\varphi(g_a), \omega(g_b)] := \varphi(g_a) \otimes \omega(g_b) - \omega(g_a) \otimes \varphi(g_b) \]
is called the commutator.

Before we continue to discuss our specific case let us also recall the definition of the representation of an algebra on a \( K \)-linear vector space.

### 2.3 Definition (Representation)

Let \((\mathfrak{A}, \mu, \eta, +; K)\) be an algebra over the field \( K \) and let \((\mathfrak{V}, +; K)\) be a vector space. A left representation of \( \mathfrak{A} \) on \( \mathfrak{V} \) is a pair \((\rho, \mathfrak{V})\) consisting of a map 
\[
\rho : \mathfrak{A} \otimes \mathfrak{V} \rightarrow \mathfrak{V}
\]
\[
a \otimes v \mapsto \rho(a \otimes v) = \rho_a(v) = a \triangleright v
\]
such that for all \( a \in \mathfrak{A} \) the maps \( \rho_a \) realize the algebra \( \mathfrak{A} \) within the endomorphism of \( \mathfrak{V} \), i.e. 
\[
\forall a, b, 1 \in \mathfrak{A}, v \in \mathfrak{V} : (a \cdot b) \triangleright v = a \triangleright (b \triangleright v)
\]
\[
1 \triangleright v = v
\]

The representation \( \rho \) is also called a left action \( "\triangleright" \).

With this little preparation we understand that a representation \( \rho \) of \( U(\mathfrak{g}) \) on the finite dimensional vector space \( \mathfrak{X} \) is more specifically defined in terms of a matrix representation, i.e. for basis elements \( g_a \in U(\mathfrak{g}) \) and \( x_i \in \mathfrak{X} \) we obtain 
\[
\rho(g_a \otimes x_i)_j = (g_a \triangleright x_i)_j = \sum_{i=1}^{n}(g_a)_{ji}x_i,
\]

where \( (g_a)_{ji} \in GL(n, K) \subset \text{Mat}(n, K) \). Moreover, the generating relations of \( U(\mathfrak{g}) \) have to be represented on \( \mathfrak{X} \) by 
\[
\forall g_a, g_b \in \mathfrak{g} : (g_a \cdot g_b - g_b \cdot g_a - [g_a, g_b]_{\mathfrak{g}}) \triangleright x_i =
\]
\[
= g_a \triangleright (g_b \triangleright x_i) - g_b \triangleright (g_a \triangleright x_i) - i \sum_{c=1}^{p} f_{abc}(g_c \triangleright x_i) = 0.
\]

Here we replaced the tensor product \( \"\otimes\" \) by conventional multiplication \( \"\cdot\" \). In terms of matrix representations \((2.2)\) these relations then read
\[
\forall g_a, g_b \in \mathfrak{g} : \sum_{i=1}^{n} \sum_{j=1}^{n} (g_a)_{kj}(g_b)_{ji} - \sum_{j=1}^{n} (g_b)_{kj}(g_a)_{ji} - ((g_a, g_b)_{\mathfrak{g}})_{ki}x_i
\]
\[
= \sum_{j=1}^{n} (g_a)_{kj} \sum_{i=1}^{n} (g_b)_{ji}x_i - \sum_{j=1}^{n} (g_b)_{kj} \sum_{i=1}^{n} (g_a)_{ji}x_i
\]
\[
- i \sum_{c=1}^{p} f_{abc} \sum_{i=1}^{n} (g_c)_{ki}x_i = 0
\]

\( (2.3) \)
Up to this point we consider the Lie-algebra $\mathfrak{g}$ and the vector space $\mathfrak{X}$ to be given and moreover that the representation $\rho$ exists and is well behaved. This setup represents the actual input from outside that we require for our considerations. Of course we want more structure than that. For our purpose we have to enhance $\mathfrak{X}$ to an algebra and thus extend $U(\mathfrak{g})$ to a Hopf-algebra.

Enhancing $\mathfrak{X}$ to an algebra is usually performed in several blends of one and the same idea: enhancing to the tensor algebra of $\mathfrak{X}$ and then dividing by a suitable two-sided ideal. In order to get things straight, we first turn $\mathfrak{X}$ into a Lie-algebra and then as well consider it as a universal enveloping algebra.

We thus fix an $n$-dimensional basis for $\mathfrak{X}$ to be $(x_i)_{i=1,2,...,n}$. Enhancing $\mathfrak{X}$ to a Lie-algebra is easily performed by introducing a $K$-bilinear bracket $\{\cdot,\cdot\} : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X}$.

The easiest choice for a bracket $\{\cdot,\cdot\}$, that satisfies the requirements of a Lie-algebra and later as well delivers the required commutative algebra of coordinates, is the vanishing bracket

$$\forall x_i, x_j \in \mathfrak{X} : [x_i, x_j] = 0.$$  

We thus have turned $\mathfrak{X}$ into a Lie-algebra. As we did for the Lie-algebra $\mathfrak{g}$, we can now consider the universal enveloping algebra $U(\mathfrak{X})$ of $\mathfrak{X}$ and thus enhanced the vector space to a commutative and associative algebra that is generated by relations

$$\forall x_i, x_j \in U(\mathfrak{X}) : x_i \otimes x_j - x_j \otimes x_i = 0.$$  

(2.4)

We once more replace the tensor product "$\otimes$" by a multiplication "$\cdot$". In order to transfer the action of $U(\mathfrak{g})$ on the vector space $\mathfrak{X}$ to an action on the algebra $U(\mathfrak{X})$ we have to enhance $U(\mathfrak{g})$ to a Hopf-algebra by introducing a coproduct, counit and antipode by

$$\forall g_a \in U(\mathfrak{g}) : \Delta(g_a) = g_a \otimes 1 + 1 \otimes g_a, \quad \epsilon(g_a) = 0, \quad S(g_a) = -g_a.$$  

It is quickly verified that this definition of the Hopf-algebra $U(\mathfrak{g})$ satisfies all axioms and requirements of a Hopf-algebra. The following definition then tells us how the representation $\rho$ on $\mathfrak{X}$ is enhanced to that of $U(\mathfrak{X})$.

2.4 Definition Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; K)$ be a Hopf-algebra over the field $K$. Let $(\mathfrak{A}, \mu, \eta, +; K)$ be an algebra. The left representation of $\mathcal{H}$ on $\mathfrak{A}$ is a left action that additionally satisfies

$$\forall h \in \mathcal{H}, a, b, \mathbf{1} \in \mathfrak{A} : h \triangleright (a \cdot b) = \sum(h(1) \triangleright a) \cdot (h(2) \triangleright b)$$  

$$h \triangleright \mathbf{1} = \epsilon(h)$$  

(2.5)

with $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. The algebra $\mathfrak{A}$ then becomes a left $\mathcal{H}$-module algebra.
Since the multiplication of $U(\mathfrak{X})$ is defined by the generating relations $\forall \ x_i, x_j \in U(x_i) : [x_i, x_j] = 0$, we have to verify that the action of $U(\mathfrak{g})$ respects this, i.e. for $g_a \in U(\mathfrak{g})$

$$g_a \cdot (x_i \cdot x_j - x_j \cdot x_i) = \Delta(g_a) \cdot (x_i \cdot x_j - x_j \cdot x_i)$$

$$= (g_a \cdot x_i) x_j + x_i (g_a \cdot x_j) - (g_a \cdot x_i) x_j - x_i (g_a \cdot x_j)$$

$$= (g_a \cdot x_i) x_j - (g_a \cdot x_i) x_j + x_i (g_a \cdot x_j) - x_i (g_a \cdot x_j) = 0,$$

since any $g_a \cdot x_i \in U(\mathfrak{X})$ once more commutes with an $x_j \in U(\mathfrak{X})$. Thus the commutation relations of $U(\mathfrak{X})$ have to be compatible with the coalgebra sector of $U(\mathfrak{g})$. We thus have completed our setup that from now on is denoted by the \textit{commutative limit}. Note that we do \textit{not} enhance $U(\mathfrak{X})$ to a Hopf-algebra as well. In the next section we continue with basic constructions that pave the way to deformations of this setup.

3 \textbf{A Hopf-Algebra of Vector Fields $\mathfrak{W}(\Pi, \mathfrak{X})$}

In this section we construct the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ that we require for general deformations of $U(\mathfrak{g})$ and $U(\mathfrak{X})$. To this purpose we first introduce a Hopf-algebra of momenta $U(\Pi)$ that is represented as a left action on $U(\mathfrak{X})$. We continue with the construction of a left cross-product algebra $U(\mathfrak{X}) \triangleright U(\Pi)$ that we further devide in order to lift it to the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$. In the last subsection we further more define the left action of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$.

3.1 \textbf{A Hopf-Algebra $U(\Pi)$ of Momenta}

We begin this section with one more Hopf-algebra $U(\Pi)$ that we loosely denote as the algebra of momenta. As long $U(\mathfrak{X})$ is actually considered to be an algebra of coordinates, $U(\Pi)$ can actually be considered to be nothing than that.

We introduce $U(\Pi)$ as a copy of $U(\mathfrak{X})$, with the exception that in contrast to $U(\mathfrak{X})$ it is enhanced by coalgebra structure and an antipode. We thus understand $U(\Pi)$ to be generated by a $n$-dimensional basis $(\pi_i)_{i=1,2,...,n}$ with commutation relations

$$\pi_i \pi_j - \pi_j \pi_i = [\pi_i, \pi_j] = 0, \quad (3.1)$$

and a primitive coalgebra structure for all $\pi_i \in U(\Pi)$ as well as a standard antipode

$$\Delta(\pi_i) = \pi_i \otimes 1 + 1 \otimes \pi_i, \quad \epsilon(\pi_i) = 0, \quad S(\pi_i) = -\pi_i. \quad (3.2)$$
We define the left action of $U(\Pi)$ on $U(\mathcal{X})$ by
\[
\forall \pi_i, 1 \in U(\Pi) \land x_j, 1 \in U(\mathcal{X}) : \pi_i \triangleright x_j = -i\delta_{ij}, \quad 1 \triangleright x_j = x_j, \quad \pi_i \triangleright 1 = \epsilon(\pi_i) \quad (3.3)
\]
We could also have omitted the imaginary unit here, but since we are interested in physical applications, we stick as close as possible to physical notions. It is evident that (3.3) is a well defined action, since the relations (3.1) are realized on $U(\mathcal{X})$ by
\[
(\pi_i \pi_j - \pi_j \pi_i) \triangleright x_k = \pi_i \triangleright (\pi_j \triangleright x_k) - \pi_j \triangleright (\pi_i \triangleright x_k) = \pi_i \triangleright (-i\delta_{jk} 1) - \pi_j \triangleright (-i\delta_{ik} 1) = 0 \quad (3.4)
\]
and in turn, $U(\Pi)$ respects the algebra relations (2.4) of $U(\mathcal{X})$ by means of the coalgebra structure (3.2) of $U(\Pi)$ by
\[
\pi_i \triangleright (x_k x_l - x_l x_k) = \Delta(\pi_i) \triangleright (x_k x_l - x_l x_k) = (\pi_i \triangleright x_k) x_l + x_k (\pi_i \triangleright x_l) - (\pi_i \triangleright x_l) x_k - x_l (\pi_i \triangleright x_k) = -i\delta_{ik} x_l - ix_k \delta_{il} + ix_l \delta_{ik} = 0.
\]

### 3.2 The Left Cross-Product $U(\mathcal{X}) \bowtie U(\Pi)$

Within the next step towards a Hopf-algebra of vector fields, we join the algebra $U(\mathcal{X})$ and the Hopf-algebra $U(\Pi)$ to a single left cross-product algebra. Before we do so, we shortly recall its definition-proposition, that can be found in the literature.

#### 3.1 Definition-Proposition

Let $\mathcal{H}$ be a Hopf-algebra and let $\mathfrak{A}$ be a left $\mathcal{H}$-module algebra. Then there exists a left cross-product algebra $\mathfrak{A} \bowtie \mathcal{H}$ on $\mathfrak{A} \otimes \mathcal{H}$ with the associative product
\[
\forall a, b \in \mathfrak{A}, \ h, k \in \mathcal{H} : (a \otimes h) \odot (b \otimes k) = \sum a(h_{(1)} \triangleright b) \otimes h_{(2)} k
\]
and unit element $1 \otimes 1$.

Thus for the algebraic relations of $U(\mathcal{X}) \bowtie U(\Pi)$, by the use of (3.2) and (3.3), we obtain for $x_i \otimes \pi_r, x_j \otimes \pi_s \in U(\mathcal{X}) \otimes U(\Pi)$
\[
(x_i \otimes \pi_r) \odot (x_j \otimes \pi_s) = x_i (\pi_r \triangleright x_j) \otimes \pi_s + x_i x_j \otimes \pi_r \pi_s = -i\delta_{ij} x_i \otimes \pi_s + x_i x_j \otimes \pi_r \pi_s
\]
In particular we compute that with $\Delta(1) = 1 \otimes 1$ we obtain
\[
(x_i \otimes 1) \odot (x_j \otimes 1) = x_i x_j \otimes 1
\]
\[
(1 \otimes \pi_r) \odot (1 \otimes \pi_s) = 1 \otimes \pi_r \pi_s
\]
such that \( U(\mathfrak{X}) \cong U(\mathfrak{X}) \otimes 1 \) and \( U(\Pi) \cong 1 \otimes U(\Pi) \) are contained as subalgebras within \( U(\mathfrak{X}) \rtimes U(\Pi) \). We thus also find that
\[
[x_i \otimes \pi_r, x_j \otimes \pi_s]_\circ = (x_i \otimes \pi_r) \circ (x_j \otimes \pi_s) - (x_j \otimes \pi_s) \circ (x_i \otimes \pi_r) = -i\delta_{rj} x_i \otimes \pi_s + i\delta_{si} x_j \otimes \pi_r.
\]
Moreover, we find in particular that
\[
[x_i \otimes \pi_r, 1 \otimes \pi_s]_\circ = -\delta_{rj} x_i \otimes 1 = 0,
\]
As \( U(\mathfrak{X}) \rtimes U(\Pi) \) provides the algebraic structure on \( U(\mathfrak{X}) \otimes U(\Pi) \), that is a vector space, we can thus once more understand \( U(\mathfrak{X}) \rtimes U(\Pi) \) to be the tensor algebra \( \mathcal{U}(\mathfrak{X}) \otimes \mathcal{U}(\Pi) \) that is divided by a suitable two-sided ideal. Making thus the identification
\[
\mathfrak{w}^0_{ir} \equiv x_i \otimes \pi_r, \quad \mathfrak{w}^+_r \equiv 1 \otimes \pi_r, \\
\mathfrak{w}^-_i \equiv x_i \otimes 1, \quad 1 \equiv 1 \otimes 1,
\]
we regard \( \mathfrak{w}^0, \mathfrak{w}^\pm \) as the generators of \( U(\mathfrak{X}) \rtimes U(\Pi) \) that by relations
\[
\begin{align*}
[x^0_{ir}, x^0_{js}]_\circ = -i\delta_{rj} x^0_{is} + i\delta_{si} x^0_{jr}, \quad [x^+_r, x^-_j]_\circ = -i\delta_{rj} 1 \\
[x^+_r, x^-_i]_\circ = -i\delta_{rj} x^-_i, \quad [x^0_{ir}, x^+_s]_\circ = i\delta_{si} x^+_r, \\
[x^+_i, x^+_j]_\circ = 0, \quad [x^-_i, x^-_j]_\circ = 0,
\end{align*}
\]
constitute the required two-sided ideal \( \mathcal{I}_{\mathfrak{X},\Pi} \). We can thus set
\[
U(\mathfrak{X}) \rtimes U(\Pi) = \frac{T(U(\mathfrak{X}) \otimes U(\Pi))}{\mathcal{I}_{\mathfrak{X},\Pi}},
\]
as for any universal enveloping algebra.

### 3.3 The Hopf-algebra \( \mathcal{W}(\Pi, \mathfrak{X}) \) of vector fields

The Relations (3.5) exhibit a nice structure of subalgebras within the cross-product algebra \( U(\mathfrak{X}) \rtimes U(\Pi) \), that already indicates into the desired direction of our purpose. However, since we would like to lift our construction to a Hopf-algebra, such that we can represent it once more on an algebra, we have to perform further modifications. The second relation of (3.5) does not allow for a Hopf-algebra enhancement,
since it would not confirm for the homomorphy property of the coproduct. Moreover we do not really have a use for a coproduct on \( w_i^- \), i.e. a coproduct on a coordinate. The authors of [20] found an elegant way to deal with a similar issue by a specific bicross-product construction. However, they had to introduce a physical interpretation as well that we avoid here by the pursuing another direction.

We reach our goal by further deviding our algebra \( U(\mathfrak{g}) \rtimes U(\Pi) \) by relation

\[
\mathfrak{W}(\Pi, \mathfrak{g}) = T(U(\mathfrak{g}) \otimes U(\Pi)) / \mathcal{I}_{2g}.
\]

The two-sided ideal \( \mathcal{I}_{2g} \) is generated by relations

\[
[w^0_{ir}, w^0_{js}]_\circ \cdot w^0_{kl} = -i\delta_{ij} w^0_{ks} + i\delta_{is} w^0_{jr}, \quad [w^0_{ir}, w^+_s]_\circ = i\delta_{is} w^+_r,
\]

\[
[w^+_r, w^+_s]_\circ = 0, \quad [w^-_r, w^-_s]_\circ = 0.
\]

We already see that this is very similar to the structure that we, for example, expect from a Poincaré-algebra. But it is much more general in its foundations. And we see how this applies to any desired setup based on the commutative limit we discussed above. It is easily checked that these relations induce a closed algebra, i.e. that the Jacobi-Identities

\[
\begin{align*}
[w^0_{ir}, [w^0_{js}, w^0_{kl}]_\circ]_\circ & + [w^0_{js}, [w^0_{kl}, w^0_{ir}]_\circ]_\circ + [w^0_{kl}, [w^0_{ir}, w^0_{js}]_\circ]_\circ = 0 \\
[w^+_r, [w^+_s, w^+_t]_\circ]_\circ & + [w^+_s, [w^+_t, w^+_r]_\circ]_\circ + [w^+_t, [w^+_r, w^+_s]_\circ]_\circ = 0 \\
[w^-_r, [w^-_s, w^-_t]_\circ]_\circ & + [w^-_s, [w^-_t, w^-_r]_\circ]_\circ + [w^-_t, [w^-_r, w^-_s]_\circ]_\circ = 0 \\
[w^+_r, [w^+_s, w^-_t]_\circ]_\circ & + [w^+_s, [w^-_t, w^+_r]_\circ]_\circ + [w^-_t, [w^+_r, w^-_s]_\circ]_\circ = 0
\end{align*}
\]

are satisfied, as it should for an associative algebra of this kind.

We proceed by the following definition-proposition to enhance \( \mathfrak{W}(\Pi, \mathfrak{g}) \) to a Hopf-algebra.

**3.2 Definition-Proposition** Let \( \mathfrak{W}(\Pi, \mathfrak{g}) \) be an algebra with the two-sided ideal \( \mathcal{I}_{2g} \), defined as above. Then \( \mathfrak{W}(\Pi, \mathfrak{g}) \) is a Hopf-algebra with the following coproduct, counit and antipode

\[
\forall i, r \in 1, 2, \ldots n : \quad \Delta(w^0_{ir}) = w^0_{ir} \otimes 1 + 1 \otimes w^0_{ir}, \quad \epsilon(w^0_{ir}) = 0, \\
S(w^0_{ir}) = -w^0_{ir}, \\
\Delta(w^+_r) = w^+_r \otimes 1 + 1 \otimes w^+_r, \quad \epsilon(w^+_r) = 0, \\
S(w^+_r) = -w^+_r.
\]

(3.7)
3.4 Representation of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$

**Proof:** It is evident that the axioms of coassociativity $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ and that of the counit $(\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta$ are satisfied for both, $w^0_\pi$ and $w^+_\pi$. Moreover the antipode axiom $\mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = \mu \circ (\text{id} \otimes S) \circ \Delta$ is fulfilled as well. Here $\mu$ is the multiplication within $\mathfrak{W}(\Pi, \mathfrak{X})$ and $\eta$ is the unit element, being the map $\eta : K \rightarrow \mathfrak{W}(\Pi, \mathfrak{X})$ such that $\eta(x) = x \cdot 1$.

Since it is an important issue here, we explicitly check on the homomorphy property of the coproduct. Thus we check that

$$
\left[ \Delta(w^0_\pi), \Delta(w^0_{js}) \right] = w^0_\pi \otimes 1 + 1 \otimes w^0_\pi = 0
$$

and

$$
\left[ \Delta(w^0_\pi), \Delta(w^+_{js}) \right] = w^0_\pi \otimes 1 + 1 \otimes w^0_{js} = -i \delta_{rj} \Delta(w^0_\pi) + i \delta_{si} \Delta(w^0_{jr})
$$

The same trivially holds for the counit. The antipode obviously is an anti-algebra homomorphism, as it should, since

$$
- \left[ S(w^0_\pi), S(w^0_{js}) \right] = -i \delta_{rj} S(w^0_\pi) + i \delta_{si} S(w^0_{jr})
$$

$$
- \left[ S(w^0_\pi), S(w^+_{js}) \right] = i \delta_{si} S(w^+_\pi)
$$

$\square$

We are now prepared to consider representations of $\mathfrak{W}(\Pi, \mathfrak{X})$ on algebras.

3.4 Representation of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$

It is our aim within this subsection to represent $\mathfrak{W}(\Pi, \mathfrak{X})$ on the algebra of coordinates $U(\mathfrak{X})$. Remember that we do not treat $U(\mathfrak{X})$ as a Hopf-algebra. As vector fields, we introduce the left action of $w^0_\pi, w^+_s \in \mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$ by

$$
w^0_\pi \triangleright x_j = \pi_r \triangleright x_j = -i \delta_{rj} x_i, \quad w^0_\pi \triangleright 1 = \epsilon(w^0_\pi),
$$

$$
w^+_r \triangleright x_j = \pi_r \triangleright x_j = -i \delta_{rj} 1, \quad w^+_r \triangleright 1 = \epsilon(w^+_r). \quad (3.8)
$$

We have thus to verify that the Hopf-algebra of vector fields $\mathfrak{W}(\Pi, \mathfrak{X})$ is realized as vector space endomorphisms on $U(\mathfrak{X})$. In particular this means that the first two
relations of (3.6) have to be realized by means of (3.8), i.e. we obtain

\[
\left( [w^0_{ir}, w^0_j]_\otimes + i \delta_{rj} w^0_{is} - i \delta_{si} w^0_{jr} \right) \triangleright x_k \\
= w^0_{ir} \triangleright (w^0_j \triangleright x_k) - w^0_{j} \triangleright (w^0_{ir} \triangleright x_k) + i \delta_{rj} (w^0_j \triangleright x_k) - i \delta_{si} (w^0_{jr} \triangleright x_k) \\
= w^0_{ir} \triangleright (-i \delta_{sk} x_j) - w^0_{i} \triangleright (-i \delta_{rk} x_i) + i \delta_{rj} (-i \delta_{sk} x_i) - i \delta_{si} (-i \delta_{rk} x_j) \\
= -\delta_{sk} \delta_{rj} x_i + \delta_{rk} \delta_{si} x_j + \delta_{jr} \delta_{sk} x_i - \delta_{si} \delta_{rk} x_j = 0
\]

and

\[
\left( [w^0_{ir}, w^+_s]_\otimes - i \delta_{si} w^+_r \right) \triangleright x_j \\
= w^0_{ir} \triangleright (w^+_s \triangleright x_j) - w^+_s \triangleright (w^0_{ir} \triangleright x_j) - i \delta_{si} (w^+_r \triangleright x_j) \\
= w^0_{ir} \triangleright (-i \delta_{sj} 1) - w^+_s \triangleright (-i \delta_{rj} x_i) - i \delta_{si} (-i \delta_{rj} 1) \\
= \delta_{rj} \delta_{si} - \delta_{si} \delta_{rj} = 0.
\]

The third relation is already represented on \( U(\mathfrak{X}) \) given by (3.4). We further more have to check whether the representation (3.8) respects the algebra relations (2.4) of \( U(\mathfrak{X}) \), i.e. we have

\[
w^0_{ir} \triangleright (x_j x_k - x_k x_j) = \Delta(w^0_{ir}) \triangleright (x_j x_k - x_k x_j) \\
= (w^0_{ir} \triangleright x_j) x_k + x_j (w^0_{ir} \triangleright x_k) - (w^0_{ir} \triangleright x_k) x_j - x_k (w^0_{ir} \triangleright x_j) \\
= (-i \delta_{rj} x_i) x_k + x_j (-i \delta_{rk} x_i) - (-i \delta_{rk} x_i) x_j - x_k (-i \delta_{rj} x_i) = 0
\]

and

\[
w^+_r \triangleright (x_j x_k - x_k x_j) = \Delta(w^+_r) \triangleright (x_j x_k - x_k x_j) \\
= (w^+_r \triangleright x_j) x_k + x_j (w^+_r \triangleright x_k) - (w^+_r \triangleright x_k) x_j - x_k (w^+_r \triangleright x_j) \\
= (-i \delta_{rj} x_k) x_j + x_j (-i \delta_{rk} x_j) - (-i \delta_{rk} x_j) x_k - x_k (-i \delta_{rj}) = 0.
\]

We thus made all necessary preparations to attack the actual interesting step in the next section.

4 **Representation of \( U(\mathfrak{g}) \) in \( \mathfrak{W}(\Pi, \mathfrak{X}) \)**

In this section we map \( U(\mathfrak{g}) \) as a subalgebra within \( \mathfrak{W}(\Pi, \mathfrak{X}) \) by means of its matrix representation (2.2) and the Hopf-algebra homomorphism

\[
\rho : U(\mathfrak{g}) \longrightarrow \mathfrak{W}(\Pi, \mathfrak{X}) \\
g_a \mapsto i(g_a)_r w^0_{ir}.
\]
We verify that the generating relations (2.1) of $U(\mathfrak{g})$ are realized in terms of (3.7). In particular, we obtain for basis elements $g_a, g_b \in U(\mathfrak{g})$

\[
[g_a, g_b] = [(g_a)_{ri}(g_b)_{sj}] = (g_a)_{ri}(g_b)_{sj} [w^0_{ir}, w^0_{js}] \\
= (g_a)_{ri}(g_b)_{sj} \left( -i\delta_{xr}w^0_{is} + i\delta_{si}w^0_{jr} \right) \\
= -(g_b)_{sk}(g_a)_{kj}w^0_{is} + i(g_a)_{rk}(g_b)_{kj}w^0_{jr} \\
= i((g_a)_{sk}(g_b)_{ki} - (g_b)_{sk}(g_a)_{ki})w^0_{is} = if_{abc}(g_c)_{si}w^0_{is} = if_{abc}g_c
\]

Here we use summation convention for any pair of equal indices. The Hopf structure \(\mathfrak{W}(\Pi, \mathfrak{X})\) corresponds to that of $U(\mathfrak{g})$, i.e.

\[
\Delta(g_a) = \Delta(i(g_a)_{ri}w^0_{ir}) = i(g_a)_{ri}\Delta(w^0_{ir}) = i(g_a)_{ri}(w^0_{ir} \otimes 1 + 1 \otimes w^0_{ir}) \\
= g_a \otimes 1 + 1 \otimes g_a \\
\epsilon(g_a) = \epsilon(i(g_a)_{ri}w^0_{ir}) = i(g_a)_{ri}\epsilon(w^0_{ir}) = 0 \\
S(g_a) = S(i(g_a)_{ri}w^0_{ir}) = i(g_a)_{ri}S(w^0_{ir}) = -i(g_a)_{ri}w^0_{ir} = -g_a
\]

We verify that the representation of $U(\mathfrak{g})$ in $\mathfrak{W}(\Pi, \mathfrak{X})$ also accommodates the correct representation on $U(\mathfrak{X})$. The representation of $\mathfrak{W}(\Pi, \mathfrak{X})$ on $U(\mathfrak{X})$ implies that

\[
(g_a \triangleright x_i)_k = (i(g_a)_{si}w^0_{js}) \triangleright x_i)_k = (i(g_a)_{si}w^0_{js} \triangleright x_i)_k \\
= (i(g_a)_{si}(-i\delta_{si}x_j))_k = (g_a)_{si}x_j
\]

This neatly corresponds to the matrix representation (2.2). We obtain double applications of the represented generators of $U(\mathfrak{g})$ according to

\[
(((g_a g_b) \triangleright x_i)_k = (i(g_b)_{sj}w^0_{js} \triangleright (i(g_a)_{ri}w^0_{ir} \triangleright x_i))_k \\
= (-i(g_b)_{sj}(g_a)_{ri}w^0_{js} \triangleright (-i\delta_{ir}x_i))_k \\
= (-(g_b)_{sj}(g_a)_{ri}(-i\delta_{ir}x_i))_k \\
= ((g_b)_{sj}(g_a)_{ri}(-i\delta_{ir}x_i))_k \\
= (((g_b)_{si}(g_a)_{i}(-i\delta_{is}x_j))_k \\
= ((g_a)_{si}(g_b)_{i}x_j)_k = (g_a)_{si}(g_b)_{i}x_j
\]

Note that the formal reversal of the order of generators $w^0$ is only applied to get indices straight. The actual order of application of generators remains unchanged as one can see from the last equation. We once more verify that this actually realizes the generating relations (2.1) of $U(\mathfrak{g})$ on $U(\mathfrak{X})$ via matrix representation according to (2.3), i.e.

\[
(((g_a g_b - g_b g_a) \triangleright x_i)_k = (((g_b)_{si}(g_a)_{i}(-i\delta_{is}x_j))_k \\
= (((g_a)_{i}(g_b)_{i} - (g_b)_{i}(g_a)_{i})_{x_j})_k \\
= ((if_{abc}(g_c)_{i}x_j)_k = (if_{abc}(i(g_c)_{si}w^0_{js} \triangleright x_i)_k \\
= (if_{abc}(g_c \triangleright x_i)_k
\]
Through the coproduct in $\mathfrak{W}(\Pi, \mathfrak{x})$ it is clear that our realization of $U(g)$ in $\mathfrak{W}(\Pi, \mathfrak{x})$ respects the generating relations of $U(\mathfrak{x})$. We thus have received a left action of the Hopf-algebra $U(g)$ on $U(\mathfrak{x})$ via its matrix representation within $\mathfrak{W}(\Pi, \mathfrak{x})$. We can now proceed to twist $\mathfrak{W}(\Pi, \mathfrak{x})$ and thus to most generally twist its subalgebra $U(g)$ as well.

5 Twisting

In order to obtain deformations $\mathfrak{W}(\Pi, \mathfrak{x})$, we introduce twists in this section. To this purpose we recall some basic properties of twists. Since we want to consider the twists of vector fields to be starproducts of associative algebras of coordinates $U(\mathfrak{x})$ at the same time, it is our intend to clarify that the definition of twists incorporates this demand. We then proceed and give some examples of twists for $\mathfrak{W}(\Pi, \mathfrak{x})$ that we apply to a two-dimensional representation of $U(sl_2)$ in the next section. For this section we recommend [6] as a textbook for reference. We begin by recalling the definition of a twist.

5.1 Definition Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; K)$ be a Hopf-algebra over the field $K$. Then an invertible object $F \in \mathcal{H} \otimes \mathcal{H}$ is called a twist, if the following two conditions hold

\[ F_{12} (\Delta \otimes id) (F) = F_{23} (id \otimes \Delta) (F) \]  \hspace{1cm} (5.1)

\[ (\epsilon \otimes id) (F) = 1 = (id \otimes \epsilon) (F). \]  \hspace{1cm} (5.2)

For $F = \sum F^{(1)} \otimes F^{(2)}$ the objects $F_{12}$ and $F_{23}$ are defined by

\[ F_{12} = \sum F^{(1)} \otimes F^{(2)} \otimes 1 \]

\[ F_{23} = \sum 1 \otimes F^{(1)} \otimes F^{(2)}. \]

Using this definition, we can now recall the required proposition stating how a twist is used to deform the corresponding Hopf-algebra.

5.2 Proposition Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S; K)$ be a Hopf-algebra and let furthermore the objects $\eta, \eta^{-1} \in \mathcal{H}$ be given by

\[ \eta = \mu (id \otimes S) (F) \]

\[ \eta^{-1} = \mu (S \otimes id) (F). \]

Then $(\mathcal{H}, \mu, \eta, \Delta_F, \epsilon, S_F; K)$ with

\[ \Delta_F(h) = F \Delta(h) F^{-1} \]

\[ S_F(h) = \eta S(h) \eta^{-1} \]

and $h \in \mathcal{H}$ is the Hopf-algebra $\mathcal{H}_F$ that is called the twist of $\mathcal{H}$. 
Note that the Hopf-algebra $H$ not necessarily has to be cocommutative. We further elucidate some consequences and properties of the defined twist before we come to specific examples for $\mathfrak{M}(\Pi, \mathfrak{X})$. If the Hopf-algebra $H$ is represented on $U(\mathfrak{X})$ by a left action, then the generating relations (2.4) of $U(\mathfrak{X})$ are preserved under the action of $H$, i.e. for $h \in H$ we have

$$x_i x_j - x_j x_i = 0 \Rightarrow h \triangleright (x_i x_j - x_j x_i) = \Delta(h) \triangleright (x_i x_j - x_j x_i) = \sum (h(1) \triangleright x_i)(h(2) \triangleright x_j) - (h(1) \triangleright x_j)(h(2) \triangleright x_i) = 0.$$ 

Within the representation of $H$ on $U(\mathfrak{X})$ we can consider a twist $F \in H \otimes H$ to deform the product $\mu$ of $U(\mathfrak{X})$ to a noncommutative product $\mu_F$ by

$$\mu_F(x_i, x_j) = x_i \ast_F x_j = F^{-1} \triangleright (x_i \cdot x_j) = \mu \left( (F^{-1}(1) \triangleright x_i), (F^{-1}(2) \triangleright x_j) \right).$$

This implies new generating relations for a deformation of $U(\mathfrak{X})$, that we further denote by $U(\mathfrak{X}_F)$, being

$$x_i \ast_F x_j - x_j \ast_F x_i - [x_i \ast_F x_j] = 0,$$ (5.3) 

where the commutator $[x_i \ast_F x_j]$ has to be replaced by a corresponding right hand side. This nonvanishing commutator reflects the noncocommutativity of the twisted coproduct $\Delta_F$ in $H_F$. The defining relations (5.1) and (5.2) of the twist $F$ thereby ensure that the axiom of coassociativity and the counit axiom of the coproduct $\Delta_F$ are satisfied, i.e. that

$$\Delta_F \otimes \text{id} \circ \Delta_F = (\text{id} \otimes \Delta_F) \circ \Delta_F$$

$$\epsilon \otimes \text{id} \circ \Delta_F = (\text{id} \otimes \epsilon) \circ \Delta_F$$

Covariance of the generating relations (5.3) of $U(\mathfrak{X}_F)$ under the action of $H_F$ is then given by

$$h \triangleright (x_i \ast_F x_j - x_j \ast_F x_i - [x_i \ast_F x_j]) = h \triangleright (F^{-1} \triangleright (x_i \cdot x_j)) - h \triangleright (F^{-1} \triangleright (x_j \cdot x_i)) - h \triangleright [x_i \ast_F x_j]$$

$$= F^{-1} \triangleright (F \Delta(h) F^{-1}) \triangleright (x_i \cdot x_j) - F^{-1} \triangleright (F \Delta(h) F^{-1}) \triangleright (x_j \cdot x_i) - h \triangleright [x_i \ast_F x_j]$$

$$= F^{-1} \triangleright (\Delta_F(h) \triangleright (x_i \cdot x_j) - \Delta_F(h) \triangleright (x_j \cdot x_i)) - h \triangleright [x_i \ast_F x_j]$$

Thus transformation and deformation commute, such that noncommutativity of $U(\mathfrak{X}_F)$ is preserved under the left action of $H_F$. Coassociativity of $\Delta_F$ implies the
associativity of the starproduct $\star_F$, i.e. we have

\[ \mathcal{F} \triangleright (h \triangleright (x_i \star_F (x_j \star_F x_i))) = (\text{id} \otimes \Delta_F) \circ \Delta_F(h) \triangleright (x_i \cdot x_j \cdot x_i) \]
\[ = (\Delta_F \otimes \text{id}) \circ \Delta_F(h) \triangleright (x_i \cdot x_j \cdot x_i) \]
\[ = \mathcal{F} \triangleright (h \triangleright ((x_i \star_F x_j) \star_F x_i)) \]

In the following we consider specific twistings of $\mathcal{W}(\Pi, \mathcal{X})$. It is our intend to merely outline the application of the formalism. We thus stick to some simple but nontrivial and genuine examples. We encourage the reader to derive more sophisticated twists for his very own purpose and use the following consideration as an examplary guiding line. Our first example is given by

\[ F_{\theta} := e^{i \theta r_s m^+_r \otimes m^+_s}, \quad \theta_{rs} = -\theta_{sr} \in \mathbb{R} \tag{5.4} \]

Since $\epsilon(m^+_r) = 0$ relation (5.2) is satisfied. Relation (5.1) is satisfied as well since

\[ e^{\frac{i}{2} \theta_{rs} m^+_r \otimes m^+_s} \otimes 1 (\Delta \otimes \text{id}) (e^{\frac{i}{2} \theta_{rs} m^+_r \otimes m^+_s} \otimes 1 \otimes 1) \]
\[ = e^{\frac{i}{2} \theta_{rs} (m^+_r \otimes m^+_s \otimes 1 + m^+_s \otimes m^+_r \otimes 1 + m^+_r \otimes m^+_s \otimes 1)} \]
\[ = e^{i \theta_{rs} (1 \otimes m^+_r \otimes m^+_s)} (\text{id} \otimes \Delta) (e^{\frac{i}{2} \theta_{rs} m^+_r \otimes m^+_s}) \]

due to the vanishing commutator $[m^+_r, m^+_s] = 0$. In general these computations are performed using the Baker-Campbell-Hausdorff formula

\[ e^A e^B = e^{A+B + \frac{1}{2}[A,B] + \frac{1}{12}(A,[A,B] - [B,A,B]) + \frac{1}{72}([A,[B,A]] - [B,[A,B]]) + \ldots} \]

Using the formula

\[ e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, [A, \ldots [A, B]]]] \tag{5.5} \]

we can now compute the twisted coproducts of $m^+_r$ and $m^+_r$ to be

\[ \Delta_F(m^+_r) = m^+_r \otimes 1 + 1 \otimes m^+_r \]
\[ \Delta_F(m^+_r) = m^+_r \otimes 1 + 1 \otimes m^+_r \]
\[ -\frac{1}{2} \theta_{rs} (m^+_s \otimes m^+_r - m^+_r \otimes m^+_s) \]

These of course correspond to the results of [5], but now this twist can be applied to any representation of a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We obtain the generating
relations of $U(\mathfrak{f}_h)$ by
\[
x_i \ast \mathcal{F}_h x_j - x_j \ast \mathcal{F}_h x_i = e^{-\frac{i}{2} \theta_{rs} w^+_r \otimes w^+_s} \triangleright (x_i x_j) - e^{-\frac{i}{2} \theta_{rs} w^+_r \otimes w^+_s} \triangleright (x_j x_i)
\]
\[
= (1 - \frac{i}{2} \theta_{rs} w^+_r \otimes w^+_s) \triangleright (x_i x_j) - (1 - \frac{i}{2} \theta_{rs} w^+_r \otimes w^+_s) \triangleright (x_j x_i)
\]
\[
= x_i x_j - \frac{i}{2} \theta_{rs} (-i \delta_{ri}) (-i \delta_{sj}) - x_j x_i + \frac{i}{2} \theta_{rs} (-i \delta_{ri}) (-i \delta_{sj})
\]
\[
= i \theta_{ij}
\]
We now come to a more genuine example taken from $\mathfrak{f}_h$. We introduce the twist
\[
\mathcal{F}_h := e^{i h w^+_{11} \otimes w^+_{22}}.
\] (5.6)
The generators $w^+_{11}$ and $w^+_{22}$ commute according to $\mathfrak{so}_3$, i.e. $[w^+_{11}, w^+_{22}] = 0$. Relation $\mathfrak{so}_2$ once more is trivially satisfied. We check for $\mathfrak{so}_1$, i.e.
\[
e^{i h w^+_{11} \otimes w^+_{22}} (\Delta \otimes \text{id})(e^{i h w^+_{11} \otimes w^+_{22}})
\]
\[
= e^{i h (w^+_{11} \otimes w^+_{22} \otimes 1 + w^+_{12} \otimes 1 \otimes w^+_{22} + 1 \otimes w^+_{11} \otimes w^+_{22})}
\]
\[
= e^{i h 1 \otimes w^+_{11} \otimes w^+_{22} (\text{id} \otimes \Delta)(e^{i h w^+_{11} \otimes w^+_{22}})}.
\]
We once more derive the twisted coproducts using formula (5.5). The coproducts of $w^+_s$ remain undeformed for $s \neq 1$. For the coproduct of $w^+_1$, we obtain
\[
\Delta_{\mathcal{F}_h}(w^+_1) = w^+_1 \otimes 1 + 1 \otimes w^+_1 + w^+_1 \otimes \left(e^{-h w^+_{22}} - 1\right)
\] (5.7)
The twisted coproduct of $w^+_r$ also remains undeformed apart from four specific cases, that are
\[
r \neq 1 : \Delta_{\mathcal{F}_h}(w^+_{1r}) = w^+_{1r} \otimes 1 + 1 \otimes w^+_{1r} + w^+_{1r} \otimes \left(e^{+h w^+_{22}} - 1\right)
\]
\[
i \neq 1, 2 : \Delta_{\mathcal{F}_h}(w^+_{11}) = w^+_{11} \otimes 1 + 1 \otimes w^+_{11} + w^+_{11} \otimes \left(e^{-h w^+_{22}} - 1\right)
\]
\[
i = 2, r = 1 : \Delta_{\mathcal{F}_h}(w^+_{21}) = w^+_{21} \otimes 1 + 1 \otimes w^+_{21} + h w^+_{11} \otimes w^+_{21}
\]
\[
+ w^+_{21} \otimes \left(e^{-h w^+_{22}} - 1\right)
\]
\[
r \neq 1 : \Delta_{\mathcal{F}_h}(w^+_{2r}) = w^+_{2r} \otimes 1 + 1 \otimes w^+_{2r} + h w^+_{11} \otimes w^+_{2r}.
\]
We thus see in this final example how the introduced formalism of vector fields \( \mathfrak{W}(\Pi, \mathfrak{X}) \) unfolds its impact. The twist \( F_h \) cannot be expressed in terms of generators of \( U(\mathfrak{g}) \) but through the representation of \( U(\mathfrak{g}) \) in \( \mathfrak{W}(\Pi, \mathfrak{X}) \) we now, nevertheless, use it to twist its coproduct and thus obtain the desired deformation of the symmetry algebra. This is sketched in the next section at the example of \( U(sl_2) \).

6 Deformation of a two-dimensional Representation of \( U(sl_2) \)

In this section we shortly consider the two-dimensional representation of \( U(sl_2) \) that we want to twist by means of (5.6). To this purpose we directly consider the corresponding matrix representation of \( U(sl_2) \) given in terms of Pauli-matrices and a canonical basis for the representation space. The Hopf-algebra of \( U(sl_2) \) can thus be considered to be generated by the basis \((\sigma_i)_{i \in 1,2,3}\) with the Hopf-structure

\[
\Delta(\sigma_i) = \sigma_i \otimes 1 + 1 \otimes \sigma_i, \quad \epsilon(\sigma_i) = 0, \quad S(\sigma_i) = -\sigma_i
\]

In the two-dimensional representation we then identify with the well-known Pauli-matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Making the identification

\[
x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

we obtain the explicit left action of the two-dimensional representation of \( U(sl_2) \) by

\[
\sigma_1 \triangleright x_1 = x_2, \quad \sigma_2 \triangleright x_1 = ix_2, \quad \sigma_3 \triangleright x_1 = x_1,
\]

\[
\sigma_1 \triangleright x_2 = x_1, \quad \sigma_2 \triangleright x_2 = -ix_1, \quad \sigma_3 \triangleright x_2 = -x_2
\]

The Hopf-algebra \( U(sl_2) \) thus gets represented in the accordingly dimensioned Hopf-algebra of vector fields \( \mathfrak{W}(\Pi, \mathfrak{X}) \) by

\[
\sigma_1 = i(\mathfrak{w}_{21}^0 + \mathfrak{w}_{12}^0),
\]

\[
\sigma_2 = \mathfrak{w}_{12}^0 - \mathfrak{w}_{21}^0,
\]

\[
\sigma_3 = i(\mathfrak{w}_{11}^0 - \mathfrak{w}_{22}^0)
\]

For the twist-deformation of these coproducts we now merely have to insert these expressions in those for the coproducts of \( \sigma_i \) from above and afterwards insert the twisted expressions for the vector fields from the last section. In particular for the
We obtain in two dimensions the following explicit expressions for the twisted coproducts of $w_1^+$ and $w_2^+$ to be
\[
\Delta_{\mathcal{F}_h}(w_1^+) = w_1^+ \otimes 1 + 1 \otimes w_1^+ + \left(e^{-\hbar w_2^+} - 1\right)
\]
\[
\Delta_{\mathcal{F}_h}(w_2^+) = w_2^+ \otimes 1 + 1 \otimes w_2^+.
\]
We as well obtain the twisted coproducts of $w_{11}^0$, $w_{12}^0$, $w_{21}^0$ and $w_{22}^0$ to be given by
\[
\Delta_{\mathcal{F}_h}(w_{11}^0) = w_{11}^0 \otimes 1 + 1 \otimes w_{11}^0
\]
\[
\Delta_{\mathcal{F}_h}(w_{12}^0) = w_{12}^0 \otimes 1 + 1 \otimes w_{12}^0 + w_{12}^0 \otimes \left(e^{+\hbar w_2} - 1\right)
\]
\[
\Delta_{\mathcal{F}_h}(w_{21}^0) = w_{21}^0 \otimes 1 + 1 \otimes w_{21}^0 + h w_{11}^0 \otimes w_1^+ + w_{21}^0 \otimes \left(e^{-\hbar w_2} - 1\right)
\]
\[
\Delta_{\mathcal{F}_h}(w_{22}^0) = w_{22}^0 \otimes 1 + 1 \otimes w_{22}^0 + h w_{11}^0 \otimes w_2^+.
\]
The generating relations of $U(\mathfrak{X}_{\mathcal{F}_h})$ then read
\[
x_1 \ast_{\mathcal{F}_h} x_2 - x_2 \ast_{\mathcal{F}_h} x_1 = i \hbar x_1.
\]
The twisted coproducts of the generators $\sigma_i$ of $U_{\mathcal{F}_h}(sl_2)$ are then given by
\[
\Delta_{\mathcal{F}_h}(\sigma_1) = i(\Delta_{\mathcal{F}_h}(w_{21}^0) + \Delta_{\mathcal{F}_h}(w_{12}^0))
\]
\[
\Delta_{\mathcal{F}_h}(\sigma_2) = \Delta_{\mathcal{F}_h}(w_{12}^0) - \Delta_{\mathcal{F}_h}(w_{21}^0)
\]
\[
\Delta_{\mathcal{F}_h}(\sigma_3) = i(\Delta_{\mathcal{F}_h}(w_{11}^0) - \Delta_{\mathcal{F}_h}(w_{22}^0)).
\]

7 Closing Remarks

We introduced a general construction that allows for an introduction of a Hopf-algebra of vector fields on a finitely generated representation space of universal enveloping algebra type. Existing representations of $U(\mathfrak{g})$ can be embedded into the vector fields. Since the latter is larger than $U(\mathfrak{g})$, twisting of the vector fields provides a larger varity of deformations for $U(\mathfrak{g})$ that could not be obtained within $U(\mathfrak{g})$ alone. In the mean time the twists of our vector fields are nothing else than starproducts. In the last section we presented some examples that outline applicability of our construction. However, we emphasize that this setup is of course not restricted to commutative vector fields as the examples might suggest.

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