HERMITE–HADAMARD-TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE CONVEX

F. Qi¹, T.-Yu Zhang², and B.-Ya. Xi²

UDC 517.5

We establish some new Hermite–Hadamard-type inequalities for functions whose first derivatives are of convexity and apply these inequalities to construct inequalities for special means.

1. Introduction

Throughout this paper, we use I and I° to denote an interval on the real line ℝ and the interior of I, respectively.

In [4], the following Hermite–Hadamard-type inequalities were proved for continuously differentiable convex functions:

**Theorem 1.1** ([4], Theorem 2.2). Let f : I° ⊆ ℝ → ℝ be a continuously differentiable mapping on I° and let a, b ∈ I° with a < b. If |f′| is convex on [a, b], then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)(|f′(a)| + |f′(b)|)}{8}.
\]

**Theorem 1.2** ([4], Theorem 2.3). Let f : I° ⊆ ℝ → ℝ be a continuously differentiable mapping on I° and let a, b ∈ I° with a < b and p > 1. If the new mapping |f′|^p/(p-1) is convex on [a, b], then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left[ \frac{|f′(a)|^{p/(p-1)} + |f′(b)|^{p/(p-1)}}{2} \right]^{(p-1)/p}.
\]

In [9], the inequalities presented above were generalized as follows:

**Theorem 1.3** ([9], Theorems 1 and 2). Let f : I ⊆ ℝ → ℝ be continuously differentiable on I° and let a, b ∈ I with a < b and q ≥ 1. If |f′|^q is convex on [a, b], then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left[ \frac{|f′(a)|^q + |f′(b)|^q}{2} \right]^{1/q}.
\]

¹ Institute of Mathematics, Henan Polytechnic University and College of Sciences, Tianjin Polytechnic University, Tianjin, China.
² College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, China.

Published in Ukrain’s’kyi Matematychnyi Zhurnal, Vol. 67, No. 4, pp. 555–567, April, 2015. Original article submitted November 27, 2012, revision submitted August 29, 2014.
and
\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( \frac{4}{p + 1} \right)^{1/p} \left( |f'(a)| + |f'(b)| \right).
\]

In [7], these inequalities were additionally generalized as follows:

**Theorem 1.4** ([7], Theorems 2.3 and 2.4). Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be continuously differentiable on \( I \), let \( a, b \in I \) with \( a < b \), and let \( p > 1 \). If \( |f'|^{p/(p-1)} \) is convex on \( [a, b] \), then
\[
\left| \frac{1}{b - a} \int_a^b f(x) \, dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{b - a}{16} \left( \frac{4}{p + 1} \right)^{1/p} \left( |f'(a)| + |f'(b)| \right).
\]

In [5], an inequality similar to the inequalities presented above was obtained in the following form:

**Theorem 1.5** ([5], Theorem 3). Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous mapping on \( [a, b] \) whose derivative belongs to \( L_p([a, b]) \), then
\[
\left| \frac{1}{b - a} \int_a^b f(x) \, dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{b - a}{4} \left( \frac{4}{p + 1} \right)^{1/p} |f'(a)| + |f'(b)|.
\]

In [5], an inequality similar to the inequalities presented above was obtained in the following form:

**Theorem 1.5** ([5], Theorem 3). Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous mapping on \( [a, b] \) whose derivative belongs to \( L_p([a, b]) \). Then
\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q + 1)} \right]^{1/q} (b - a)^{1/q} \| f' \|_p,
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p > 1 \).

Recently, the following inequalities were obtained in [10, 11].

**Theorem 1.6** [10]. Let \( I \subseteq \mathbb{R} \) be an open interval with \( a, b \in I \) and \( a < b \) and let \( f : I \rightarrow \mathbb{R} \) be twice continuously differentiable mapping such that \( f'' \) is integrable. If \( 0 \leq \lambda \leq 1 \) and \( |f''| \) is a convex function on \( [a, b] \), then
\[
\left| (\lambda - 1) f \left( \frac{a + b}{2} \right) - \lambda \frac{f(a) + f(b)}{2} + \int_a^b f(x) \, dx \right|
\]
Hermite–Hadamard-Type Integral Inequalities for Functions Whose First Derivatives Are Convex

\[
\begin{align*}
\left( b-a \right)^2 & \left\{ \frac{\lambda^4 + (1 + \lambda)(1-\lambda)^3 + 5\lambda - 3}{4} \right\} |f''(a)| \\
& \leq \left\{ \frac{\lambda^4 + (2-\lambda)\lambda^3 + \frac{1-3\lambda}{4}}{4} \right\} |f''(b)|, \quad 0 \leq \lambda \leq \frac{1}{2}, \\
& \frac{(b-a)^2}{48} (3\lambda - 1) (|f''(a)| + |f''(b)|), \quad \frac{1}{2} \leq \lambda \leq 1.
\end{align*}
\]

**Theorem 1.7** [11]. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be continuously differentiable on \( I^o \), let \( a, b \in I \) with \( a < b \), and let \( f' \in L((a, b)) \). If \( |f'|^q \) is convex for \( q \geq 1 \) on \([a, b]\), then

\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f \left( \frac{a+b}{2} \right) \right] - \frac{b-a}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{b-a}{12} \left[ \frac{2^{q+1} + 1}{3(q+1)} \right]^{1/q} \left[ \left( 3 |f'(a)|^q + |f'(b)|^q \right) \frac{1}{4} + \left( |f'(a)|^q + 3 |f'(b)|^q \right)^{1/q} \right]
\]

and

\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f \left( \frac{a+b}{2} \right) \right] - \frac{b-a}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{5(b-a)}{72} \left[ \frac{61 |f'(a)|^q + 29 |f'(b)|^q}{90} \right]^{1/q} + \left( \frac{29 |f'(a)|^q + 61 |f'(b)|^q}{90} \right)^{1/q}.
\]

A function \( f : I \subseteq \mathbb{R} \to [0, \infty) \) is called \( s \)-convex if the inequality

\[
f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)
\]

holds for all \( x, y \in I, \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \) and some fixed \( s \in (0, 1) \).

In [1], some inequalities of Hermite–Hadamard-type for \( s \)-convex functions were established as follows:

**Theorem 1.8** ([1], Theorems 2.3 and 2.4). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a continuously differentiable mapping on \( I^o \) such that \( f' \in L((a, b)) \), where \( a, b \in I \) with \( a < b \).

If \( |f'|^p/(p-1) \) is \( s \)-convex on \([a, b]\) for \( p > 1 \) and some fixed \( s \in (0, 1] \), then

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{1/p} \left( \frac{1}{s+1} \right)^{2/q} \times \left( (2^{1-s} + s + 1) |f'(a)|^q + 2^{1-s} |f'(b)|^q \right)^{1/q}
\]
+ \left[2^{1-s}|f'(a)|^q + (2^{1-s} + s + 1)|f'(b)|^q \right]^{1/q},
\]

where \( p \) is the conjugate of \( q \), i.e., \( \frac{1}{p} + \frac{1}{q} = 1. \)

If \( |f'|^q \) is \( s \)-convex on \([a, b]\) for \( q \geq 1 \) and some fixed \( s \in (0, 1] \), then
\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \right| \leq \frac{b - a}{8} \left[ \frac{2}{(s + 1)(s + 2)} \right]^{1/q} \\
\times \left\{ \left[ (2^{1-s} + 1)|f'(a)|^q + 2^{1-s}|f'(b)|^q \right]^{1/q} \\
+ \left[ (2^{1-s} + 1)|f'(b)|^q + 2^{1-s}|f'(a)|^q \right]^{1/q} \right\}.
\]

Some inequalities of Hermite–Hadamard type were also obtained in [2, 3, 6, 8, 12, 14–17, 19]; see also plenty of references therein.

In the present paper, we establish some new Hermite–Hadamard-type integral inequalities for functions whose first derivatives are of convexity and apply them to derive some inequalities of special means.

2. Lemmas

To establish our new integral inequalities of Hermite–Hadamard type, we need the following lemmas.

**Lemma 2.1.** Let \( I \) be an interval and let \( f : I \to \mathbb{R} \) be continuously differentiable on \( I^\circ \) with \( a, b \in I, \ a < b, \) and \( \lambda, \mu \in \mathbb{R}. \) If \( f' \in L([a, b]), \) then
\[
(1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \\
= (b - a) \left[ \int_0^{1/2} (\lambda - t)f'(ta + (1-t)b)dt + \int_{1/2}^1 (\mu - t)f'(ta + (1-t)b)dt \right].
\]

**Proof.** This follows from the standard integration by parts.

**Lemma 2.2** [18]. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be continuously differentiable on \( I^\circ \) and let \( a, b \in I \) with \( a < b. \) If \( f' \in L([a, b]) \) and \( \lambda, \mu \in \mathbb{R}, \) then
\[
\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x)dx \\
= \frac{b - a}{4} \int_0^1 \left[ (1 - \lambda - t)f'\left(ta + (1-t)\frac{a + b}{2}\right) + (\mu - t)f'\left(t\frac{a + b}{2} + (1-t)b\right) \right] dt.
\]
**Proof.** This can be proved by the standard integration by parts.

**Remark 2.1.** Lemmas 2.1 and 2.2 are equivalent.

3. New Integral Inequalities of Hermite–Hadamard Type

We are now in a position to establish some new integral inequalities of Hermite–Hadamard type for functions whose derivatives are of convexity.

**Theorem 3.1.** Let \( f: I \subseteq \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function on \( I^0 \), let \( a, b \in I \) with \( a < b \), and let \( 0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1 \). If \(|f'|\) is convex on \([a, b]\), then

\[
\left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x)dx \right| \\
\leq \frac{b - a}{24}\left[ (10 - 3\lambda + 8\lambda^3 - 15\mu + 8\mu^3)|f'(a)| \\
+ (8 - 9\lambda + 24\lambda^2 - 8\lambda^3 - 21\mu + 24\mu^2 - 8\mu^3)|f'(b)| \right].
\] (3.1)

**Proof.** By Lemma 2.1 and the convexity of \(|f'|\) on \([a, b]\), we get

\[
\left| (1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x)dx \right| \\
\leq (b - a)\left[ \int_0^{1/2} |\lambda - t||f'(ta + (1 - t)b)|dt + \int_{1/2}^1 |\mu - t||f'(ta + (1 - t)b)|dt \right] \\
\leq (b - a)\left[ \int_0^{1/2} (\lambda - t)(|f'(a)| + (1 - t)|f'(b)|)dt + \int_{1/2}^1 (\mu - t)(|f'(a)| + (1 - t)|f'(b)|)dt \right].
\]

Substituting equations

\[
\int_0^{1/2} (\lambda - t)(|f'(a)| + (1 - t)|f'(b)|)dt = \frac{1}{24}\left[ (1 - 3\lambda + 8\lambda^3)|f'(a)| + (2 - 9\lambda + 24\lambda^2 - 8\lambda^3)|f'(b)| \right]
\]
and

\[
\int_{1/2}^{1} |\mu - t| (|t| f'(a) + (1 - t) |f'(b)|) dt
\]

\[
= \frac{1}{24} \left[ (9 - 15\mu + 8\mu^3) |f'(a)| + (6 - 21\mu + 24\mu^2 - 8\mu^3) |f'(b)| \right]
\]

in the above inequality, we find (3.1).

Theorem 3.1 is proved.

**Theorem 3.2.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function on \( I^o \), let \( a, b \in I \) with \( a < b \), and let \( 0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1 \). If \( |f'|^q \) for \( q > 1 \) is convex on \( [a, b] \) and \( q \geq p > 0 \), then

\[
\left| (1 - \mu) f(a) + \lambda f(b) + (\mu - \lambda) f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) dx \right|
\]

\[
\leq (b - a) \left( \frac{q - 1}{2q - p - 1} \right)^{1-1/q} \left[ \frac{1}{(p + 1)(p + 2)} \right]^{1/q}
\]

\[
\times \left\{ \left[ \left( \frac{1}{2} - \lambda \right)^{(2q - p - 1)/(q - 1)} + \lambda (2q - p - 1)/(q - 1) \right]^{1-1/q}
\right.
\]

\[
\times \left[ \left( \frac{1}{2} (p + 1 + 2\lambda) - \lambda \right)^{p+1} + \lambda^{p+2} \right] |f'(a)|^q
\]

\[
+ \left[ \frac{1}{2} (p + 3 - 2\lambda) - \lambda \right)^{p+1} + (p + 2 - \lambda) \lambda^{p+1} \right] |f'(b)|^q
\]

\[
\left. + \left[ \left( \mu - \frac{1}{2} \right)^{(2q - p - 1)/(q - 1)} + (1 - \mu) (2q - p - 1)/(q - 1) \right]^{1-1/q}
\right)
\]

\[
\times \left[ \left( \frac{1}{2} (p + 1 + 2\mu) - \mu \right)^{p+1} + (p + 1 + \mu) (1 - \mu)^{p+1} \right] |f'(a)|^q
\]

\[
+ \left[ \frac{1}{2} (p + 3 - 2\mu) - \mu \right)^{p+1} + (1 - \mu)^{p+2} \right] |f'(b)|^q \right) \right) \right)^{1/q}.
\]
Proof. By Lemma 2.1, the convexity of $|f'|^q$ on $[a, b]$, and Hölder’s integral inequality, we obtain

\[
(1 - \mu)f(a) + \lambda f(b) + (\mu - \lambda)f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx
\]

\[
\leq (b - a) \left[ \int_0^{1/2} |\lambda - t||f'(ta + (1 - t)b)|dt + \int_{1/2}^1 |\mu - t||f'(ta + (1 - t)b)|dt \right]
\]

\[
\leq (b - a) \left[ \left( \int_0^{1/2} |\lambda - t|^{(q-p)/(q-1)} dt \right)^{1-1/q} \left( \int_0^{1/2} |\lambda - t|^p f'(ta + (1 - t)b)^q dt \right)^{1/q} \right]
\]

\[
+ \left( \int_{1/2}^1 |\mu - t|^{(q-p)/(q-1)} dt \right)^{1-1/q} \left( \int_{1/2}^1 |\mu - t|^p f'(ta + (1 - t)b)^q dt \right)^{1/q}
\]

\[
\leq (b - a) \left\{ \left[ \int_0^{1/2} |\lambda - t|^{(q-p)/(q-1)} dt \right]^{1-1/q} \left[ \int_0^{1/2} |\lambda - t|^p (t|f'(a)|^q + (1 - t)|f'(b)|^q) dt \right]^{1/q} \right. \right.
\]

\[
+ \left. \left[ \int_{1/2}^1 |\mu - t|^{(q-p)/(q-1)} dt \right]^{1-1/q} \left[ \int_{1/2}^1 |\mu - t|^p (t|f'(a)|^q + (1 - t)|f'(b)|^q) dt \right]^{1/q} \right\}. \quad (3.2)
\]

Furthermore, a straightforward computation gives

\[
\int_0^{1/2} |\lambda - t|^{(q-p)/(q-1)} dt = \frac{q - 1}{2q - p - 1} \left[ \left( \frac{1}{2} - \lambda \right)^{(2q-p-1)/(q-1)} + \lambda^{(2q-p-1)/(q-1)} \right], \quad (3.3)
\]

\[
\int_{1/2}^1 |\mu - t|^{(q-p)/(q-1)} dt = \frac{q - 1}{2q - p - 1} \left[ \left( \frac{1}{2} - \mu \right)^{(2q-p-1)/(q-1)} + (1 - \mu)^{(2q-p-1)/(q-1)} \right],
\]

\[
\int_0^{1/2} |\lambda - t|^p (t|f'(a)|^q + (1 - t)|f'(b)|^q) dt
\]

\[
= \frac{1}{(p + 1)(p + 2)} \left\{ \left[ \frac{1}{2}(p + 2\lambda) \left( \frac{1}{2} - \lambda \right)^{p+1} + \lambda^{p+2} \right] |f'(a)|^q \right. \right.
\]

\[
\left. \left. \left[ \frac{1}{2}(p - 2\lambda) \left( \frac{1}{2} + \lambda \right)^{1-p} + \lambda^{-1} \right] |f'(b)|^q \right\}. \quad (3.4)
\]
\[
\left[ \frac{1}{2} (p+3-2\lambda) \left( \frac{1}{2} - \lambda \right)^{p+1} + (p+2-\lambda)\lambda^{p+1} \right] |f'(b)|^q
\]

and

\[
\int_{1/2}^{1} |\mu - t|^p (t |f'(a)|^q + (1-t) |f'(b)|^q) \, dt
\]

\[
= \frac{1}{(p+1)(p+2)} \left\{ \left[ \frac{1}{2} (p+1+2\mu) \left( \mu - \frac{1}{2} \right)^{p+1} + (p+1+\mu)(1-\mu)^{p+1} \right] |f'(a)|^q + \left[ \frac{1}{2} (p+3-2\mu) \left( \mu - \frac{1}{2} \right)^{p+1} + (1-\mu)^{p+2} \right] |f'(b)|^q \right\}.
\]

Substituting the equations (3.3) and (3.4) in (3.2), we arrive at inequality (3.1).

Theorem 3.2 is proved.

**Corollary 3.1.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function on \( I^0 \), let \( a, b \in I \) with \( a < b \), and let \( 0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1 \). If \( |f'|^q \) for \( q \geq 1 \) is convex on \([a, b]\), then

\[
\left| (1-\mu)f(a) + \lambda f(b) + (\mu - \lambda)f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right|
\]

\[
\leq \frac{b-a}{2} \left( \frac{1}{3} \right)^{1/q} \left\{ \left[ \left( \frac{1}{2} - \lambda \right)^2 + \lambda^2 \right]^{1-1/q} \left[ \left( 1 + \lambda \right) \left( \frac{1}{2} - \lambda \right)^2 + \lambda^3 \right] |f'(a)|^q + \left[ (2-\lambda) \left( \frac{1}{2} - \lambda \right)^2 + (3-\lambda)\lambda^2 \right] |f'(b)|^q \right\}
\]

\[
+ \left[ \left( \mu - \frac{1}{2} \right)^2 + (1-\mu)^2 \right]^{1-1/q} \left[ (1+\mu) \left( \mu - \frac{1}{2} \right)^2 + (2+\mu)(1-\mu)^2 \right] |f'(a)|^q
\]

\[
+ \left[ (2-\mu) \left( \mu - \frac{1}{2} \right)^2 + (1-\mu)^3 \right] |f'(b)|^q \right\}
\]

and

\[
\left| (1-\mu)f(a) + \lambda f(b) + (\mu - \lambda)f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right|
\]
\[ \leq \frac{b-a}{2} \left[ \frac{2}{(q+1)(q+2)} \right]^{1/q} \left\{ \left( \left[ \frac{1}{2}(q+1+2\lambda) \left( \frac{1}{2} - \lambda \right) q^1 + \lambda q^2 \right] \right) f'(a)^q \right. \\
+ \left[ \frac{1}{2}(q + 3 - 2\lambda) \left( \frac{1}{2} - \lambda \right) q^1 + (q + 2 - \lambda) \lambda q^2 \right] f'(b)^q \right\}^{1/q} \\
+ \left( \left[ \frac{1}{2}(q + 1 + 2\mu) \left( \mu - \frac{1}{2} \right) q^1 + (q + 1 + \mu)(1 - \mu) q^2 \right] \right) f'(a)^q \right\}^{1/q} \\
+ \left( \left[ \frac{1}{2}(q + 3 - 2\mu) \left( \mu - \frac{1}{2} \right) q^1 + (1 - \mu) q^2 \right] \right) f'(b)^q \right\}^{1/q} \}

**Proof.** This follows from Theorem 3.1 and setting \( p = 1 \) and \( p = q \) in Theorem 3.2.

**Corollary 3.2.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a continuously differentiable function on \( I^0 \), let \( a, b \in I \) with \( a < b \), and let \( m > 0 \), \( m \geq 2\ell \geq 0 \). If \( |f'|^q \) for \( q > 1 \) is convex on \([a, b]\) and \( q \geq p > 0 \), then

\[
\left| \frac{\ell}{m} [f(a) + f(b)] + \frac{m - 2\ell f}{m} \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
\leq \frac{b-a}{4m^2} \left( \frac{q - 1}{2q - p - 1} \right)^{1-1/q} \left( \frac{1}{2m(p+1)(p+2)} \right)^{1/q} \\
\times \left[ (2\ell)^{(2q-p-1)/(q-1)} + (m - 2\ell)^{(2q-p-1)/(q-1)} \right]^{1-1/q} \left\{ \left( (2\ell)^p + (mp + m + 2\ell)(m - 2\ell)^{p+1} \right) \right] f'(a)^q \right\}^{1/q} \\
+ \left[ (2m(2\ell + 4m - 2\ell)(2\ell)^{p+1} + (mp + m + 2\ell + 3m - 2\ell)(m - 2\ell)^{p+1} \right] f'(b)^q \right\}^{1/q} \\
+ \left( (2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1} \right) f'(a)^q \right\}^{1/q} \\
+ \left( (2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1} \right) f'(b)^q \right\}^{1/q} \}

**Proof.** This is obtained by setting \( \lambda = 1 - \mu = \frac{\ell}{m} \) in Theorem 3.2.

**Corollary 3.3.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a continuously differentiable function on \( I^0 \), let \( a, b \in I \) with \( a < b \), and let \( m > 0 \), \( m \geq 2\ell \geq 0 \). If \( |f'|^q \) for \( q \geq 1 \) is convex on \([a, b]\), then
\[
\left| \frac{\ell}{m} [f(a) + f(b)] + \frac{m - 2\ell}{m} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\]
\[
\leq \frac{b - a}{8m^2} \left( \frac{1}{3m} \right)^{1/q} \left[ (m - 2\ell)^2 + (2\ell)^2 \right]^{1-1/q}
\times \left\{ \left[ (m + \ell)(m - 2\ell)^2 + 4\ell^3 \right] |f'(a)|^q + \left[ (2m - \ell)(m - 2\ell)^2 + 4(3m - \ell)\ell^2 \right] |f'(b)|^q \right\}^{1/q}
\]
\[
+ \left[ 4(3m - \ell)\ell^2 + (2m - \ell)(m - 2\ell)^2 \right] |f'(a)|^q + \left[ (m + \ell)(m - 2\ell)^2 + 4\ell^3 \right] |f'(b)|^q \right\}^{1/q}
\]

and
\[
\left| \frac{\ell}{m} [f(a) + f(b)] + \frac{m - 2\ell}{m} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\]
\[
\leq \frac{b - a}{4m} \left[ \frac{1}{2m^2(q + 1)(q + 2)} \right]^{1/q} \left\{ \left[ (2\ell)^q + 2 + (mq + m + 2\ell)(m - 2\ell)^{q+1} \right] |f'(a)|^q
\right.
\]
\[
+ \left[ (2mq + 4m - 2\ell)(2\ell)^{q+1} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1} \right] |f'(b)|^q \right\}^{1/q}
\]
\[
+ \left[ (2mq + 4m - 2\ell)(2\ell)^{q+1} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1} \right] |f'(a)|^q
\]
\[
+ \left[ (2\ell)^q + 2 + (mq + m + 2\ell)(m - 2\ell)^{q+1} \right] |f'(b)|^q \right\}^{1/q}.
\]

**Proof.** This is obtained by setting \( \lambda = 1 - \mu = \frac{\ell}{m} \) in Corollary 3.1.

4. **Applications to Special Means**

For two positive numbers \( a > 0 \) and \( b > 0 \), define

\[
A(a, b) = \frac{a + b}{2}, \quad G(a, b) = \sqrt{ab}, \quad H(a, b) = \frac{2ab}{a + b},
\]

\[
I(a, b) = \begin{cases} 
\frac{1}{e} \left( \frac{b}{a} \right)^{1/(b-a)}, & a \neq b, \\
1, & a = b,
\end{cases}
\]

\[
L(a, b) = \begin{cases} 
\frac{b - a}{ln b - ln a}, & a \neq b, \\
1, & a = b,
\end{cases}
\]
and

$$L_s(a, b) = \begin{cases} \left[ \frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right]^{1/s}, & s \neq 0, -1 \text{ and } a \neq b, \\ L(a, b), & s = -1 \text{ and } a \neq b, \\ I(a, b), & s = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

It is well known that $A, G, H, L = L_{-1}, I = L_0$, and $L_s$ are respectively called the arithmetic, geometric, harmonic, logarithmic, exponential, and generalized logarithmic means of two positive numbers $a$ and $b$.

**Theorem 4.1.** Let $b > a > 0$, $q > 1$, $q \geq p > 0$, $m > 0$, $m \geq 2\ell \geq 0$, and $s \in \mathbb{R}$.

1. If either $s > 1$ and $(s - 1)q \geq 1$ or $s < 1$ and $s \neq 0$, then

$$\left| \frac{2\ell A(a^s, b^s) + (m - 2\ell) [A(a, b)]^s}{m} - [L_s(a, b)]^s \right| \leq \frac{b - a}{4m^2} |s| \left( \frac{q - 1}{2q - p - 1} \right)^{1-1/q} \left[ \frac{1}{2m(p + 1)(p + 2)} \right]^{1/q}$$

$$\times \left[ (2\ell)^{(2q-p-1)/(q-1)} + (m - 2\ell)^{(2q-p-1)/(q-1)} \right]^{1-1/q}$$

$$\times \left\{ \left[ (2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1} \right] a^{(s-1)q}$$

$$+ \left[ (2mp + 4m - 2\ell)(2\ell)^{p+1} + (mp + 3m - 2\ell)(m - 2\ell)^{p+1} \right] b^{(s-1)q}$$

$$+ \left[ (2mp + 4m - 2\ell)(2\ell)^{p+1} + (mp + 3m - 2\ell)(m - 2\ell)^{p+1} \right] a^{(s-1)q}$$

$$+ \left[ (2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1} \right] b^{(s-1)q} \right\}.$$

2. If $s = -1$, then

$$\left| \frac{1}{m} \left[ \frac{2\ell}{H(a, b)} + \frac{m - 2\ell}{A(a, b)} \right] - \frac{1}{L(a, b)} \right|$$

$$\leq \frac{b - a}{4m^2} \left( \frac{q - 1}{2q - p - 1} \right)^{1-1/q} \left[ \frac{1}{2m(p + 1)(p + 2)} \right]^{1/q}$$

$$\times \left[ (2\ell)^{(2q-p-1)/(q-1)} + (m - 2\ell)^{(2q-p-1)/(q-1)} \right]^{1-1/q}$$
Hence, when $s > 1$, Theorem 4.1 follows.

Proof. We set $f(x) = x^s$ for $x > 0$ and $s \neq 0, 1$. Then it is easy to obtain that

$$f'(x) = sx^{s-1}, \quad |f'(x)|^q = |s|^q x^{(s-1)q}, \quad (|f'(x)|^q)^\prime = (s-1)q[|s|-1]|s|^q x^{(s-1)q-2}.$$ 

Hence, when $s > 1$ and $(s-1)q \geq 1$, or when $s < 1$ and $s \neq 0$, the function $|f'|^q$ is convex on $[a, b]$. From Corollary 3.2, Theorem 4.1 follows.

By the argument similar to Theorem 4.1, we further obtain the following conclusions.

Theorem 4.2. Let $b > a > 0$, $q \geq 1$, $m > 0$, $m \geq 2\ell \geq 0$, and $s \in \mathbb{R}$.

1. If either $s > 1$ and $(s-1)q \geq 1$ or $s < 1$ and $s \neq 0$, then

$$\left| \frac{2\ell A(a^s, b^s) + (m - 2\ell)[A(a, b)]^s}{m} - [L_s(a, b)]^s \right|$$

$$\leq \frac{b - a}{8m^2} \left( \frac{1}{3m} \right)^{1/q} \left[ 4\ell^2 + (m - 2\ell)^2 \right]^{1-1/q} |s|$$

$$\times \left\{ \left[ (4\ell^3 + (m + \ell)(m - 2\ell)^2) a^{(s-1)q} + (4(3m - \ell)(m - 2\ell)^2 + (2m - \ell)(m - 2\ell)^2) b^{(s-1)q} \right]^{1/q} + \left[ (4(3m - \ell)(m - 2\ell)^2 + (2m - \ell)(m - 2\ell)^2) a^{(s-1)q} + (4\ell^3 + (m + \ell)(m - 2\ell)^2) b^{(s-1)q} \right]^{1/q} \right\}$$

and

$$\left| \frac{2\ell A(a^s, b^s) + (m - 2\ell)[A(a, b)]^s}{m} - [L_s(a, b)]^s \right|$$

$$\leq \frac{b - a}{4m} |s| \left[ \frac{1}{2m^2(q+1)(q+2)} \right]^{1/q}$$

$$\times \left\{ \left[ ((2\ell)^q + (mq + m + 2\ell)(m - 2\ell)^q+1) a^{(s-1)q} \right]^{1/q} \right\}$$
\[ + \left( (2mq + 4m - 2\ell)(2\ell)^{q+1} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1} \right) b^{(s-1)q} \]
\[ + \left[ (2mq + 4m - 2\ell)(2\ell)^{q+2} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1} \right] a^{(s-1)q} \]
\[ + \left( (2\ell)^{q+1} + (mq + m + 2\ell)(m - 2\ell)^{q+1} \right) b^{(s-1)q} \]\n\[ \left\{ \frac{2\ell}{m} H(a, b) + \frac{2\ell}{m} A(a, b) - \frac{1}{L(a, b)} \right\} \]
\[ \leq \frac{b - a}{8m^2} \left( \frac{1}{3m} \right)^{1/q} \left[ 4\ell^2 + (m - 2\ell)^2 \right]^{1-1/q} \]
\[ \times \left\{ \frac{4\ell^2 + (m + \ell)(m - 2\ell)^2}{a^{2q}} + \frac{4(3m - \ell)\ell^2 + (2m - \ell)(m - 2\ell)^2}{b^{2q}} \right\}^{1/q} \]
\[ + \left[ \frac{4(3m - \ell)\ell^2 + (2m - \ell)(m - 2\ell)^2}{a^{2q}} + \frac{4\ell^2 + (m + \ell)(m - 2\ell)^2}{b^{2q}} \right]^{1/q} \}\]

(2) If \( s = -1 \), then
\[ \left\{ \frac{2\ell}{m} H(a, b) + \frac{2\ell}{m} A(a, b) - \frac{1}{L(a, b)} \right\} \]
\[ \leq \frac{b - a}{4m} \left( \frac{1}{2m^2(q + 1)(q + 2)} \right)^{1/q} \left\{ \frac{(2\ell)^{q+2} + (mq + m + 2\ell)(m - 2\ell)^{q+1}}{a^{2q}} \right\] 
\[ + \frac{(mq + 3m - 2\ell)(m - 2\ell)^{q+1} + (2mq + 4m - 2\ell)(2\ell)^{q+1}}{b^{2q}} \right\}^{1/q} \]
\[ + \left[ \frac{2mq + 4m - 2\ell)(2\ell)^{q+1} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1}}{a^{2q}} \right]^{1/q} \]
\[ + \left[ \frac{(2\ell)^{q+2} + (mq + m + 2\ell)(m - 2\ell)^{q+1}}{b^{2q}} \right]^{1/q} \right\} \]
and
\[
\frac{1}{m} \left[ \frac{2\ell}{H(a,b)} + \frac{m - 2\ell}{A(a,b)} \right] - \frac{1}{L(a,b)} \leq \frac{b - a}{4m^2} \frac{4\ell^2}{H(a^2, b^2)}.
\]

**Theorem 4.3.** Let \( b > a > 0, q > 1, q \geq p > 0, m > 0, \) and \( m \geq 2\ell \geq 0. \) Then
\[
\left| \frac{2\ell \ln G(a,b) + (m - 2\ell) \ln A(a,b)}{m} - \ln I(a,b) \right|
\leq \frac{b - a}{4m^2} \left( \frac{q - 1}{2q - p - 1} \right)^{1-1/q} \left[ 1 \left( \frac{1}{2m(p + 1)(p + 2)} \right)^{1/q}
\times \left[ (m - 2\ell)^{(2q-p-1)/(q-1)} + (2\ell)^{(2q-p-1)/(q-1)} \right]^{1-1/q}
\times \left\{ \left( \frac{(2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1}}{a^q} \right) + \frac{(mp + 3m - 2\ell)(m - 2\ell)^{p+1} + (2mp + 4m - 2\ell)(2\ell)^{p+1}}{b^q} \right. \\
+ \left. \frac{(2mp + 4m - 2\ell)(2\ell)^{p+1} + (mp + 3m - 2\ell)(m - 2\ell)^{p+1}}{a^q} \right. \\
+ \left. \frac{(2\ell)^{p+2} + (mp + m + 2\ell)(m - 2\ell)^{p+1}}{b^q} \right\}^{1/q} \right].
\]

**Proof.** This is obtained by taking \( f(x) = \ln x \) for \( x > 0 \) in Corollary 3.2.

As in Theorem 4.3, we arrive at the following inequalities:

**Theorem 4.4.** Let \( b > a > 0, q > 1, m > 0, \) and \( m \geq 2\ell \geq 0. \) Then
\[
\left| \frac{2\ell \ln G(a,b) + (m - 2\ell) \ln A(a,b)}{m} - \ln I(a,b) \right|
\leq \frac{b - a}{4m} \left[ \frac{1}{2m^2(q + 1)(q + 2)} \right]^{1/q} \left\{ \left( \frac{(2\ell)^{q+2} + (mq + m + 2\ell)(m - 2\ell)^{q+1}}{a^q} \right) + \frac{(mq + 3m - 2\ell)(m - 2\ell)^{q+1} + (2mq + 4m - 2\ell)(2\ell)^{q+1}}{b^q} \right. \\
+ \left. \frac{(2mq + 4m - 2\ell)(2\ell)^{q+1} + (mq + 3m - 2\ell)(m - 2\ell)^{q+1}}{a^q} \right. \\
+ \left. \frac{(2\ell)^{q+2} + (mq + m + 2\ell)(m - 2\ell)^{q+1}}{b^q} \right\}^{1/q} \right].
\]
and

\[
\left| \frac{2\ell \ln G(a, b) + (m - 2\ell) \ln A(a, b)}{m} - \ln I(a, b) \right| \\
\leq \frac{b - a}{8m^2} \left( \frac{1}{3m} \right)^{1/q} \left[ (m - 2\ell)^2 + (2\ell)^2 \right]^{1-1/q} \\
\times \left\{ \left[ \frac{4\ell^3 + (m + \ell)(m - 2\ell)^2}{a^q} + \left( \frac{2m - \ell}{b} \right) (m - 2\ell)^2 + 4(3m - \ell)^2 \right]^{1/q} \\
+ \left[ 4(3m - \ell)^2 + (2m - \ell)(m - 2\ell)^2 \left( \frac{a}{b} \right)^q + \frac{4\ell^3 + (m + \ell)(m - 2\ell)^2}{b^q} \right]^{1/q} \right\}.
\]

In particular,

\[
\left| \frac{2\ell \ln G(a, b) + (m - 2\ell) \ln A(a, b)}{m} - \ln I(a, b) \right| \leq \frac{b - a}{4m^2} \frac{4\ell^2 + (m - 2\ell)^2}{H(a, b)}.
\]

Remark 4.1. This paper is a simplified version of the preprint [13].

The present paper was partially supported by the National Natural Science Foundation under Grant No. 11361038 of China and by the Foundation of the Research Program of Science and Technology at the Universities of Inner Mongolia Autonomous Region under Grant No. NJZY14191, China.

REFERENCES

1. M. W. Alomari, M. Darus, and U. S. Kirmaci, “Some inequalities of Hermite–Hadamard type for s-convex functions,” Acta Math. Sci. Ser. B Engl. Ed., 31, No. 4, 1643–1652 (2011).
2. R.-F. Bai, F. Qi, and B.-Y. Xi, “Hermite–Hadamard type inequalities for the \(m\)- and \((\alpha, m)\)-logarithmically convex functions,” Filomat, 27, No. 1, 1–7 (2013).
3. L. Chun and F. Qi, “Integral inequalities of Hermite–Hadamard type for functions whose third derivatives are convex,” J. Inequal. Appl., 2013, No. 451 (2013).
4. S. S. Dragomir and R. P. Agarwal, “Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,” Appl. Math. Lett., 11, No. 5, 91–95 (1998).
5. S. S. Dragomir, R. P. Agarwal, and P. Cerone, “On Simpson’s inequality and applications,” J. Inequal. Appl., 5, No. 6, 533–579 (2000).
6. J. Hua, B.-Y. Xi, and F. Qi, Hermite–Hadamard type inequalities for geometric-arithmetically s-convex functions,” Comm. Korean Math. Soc., 29, No. 1, 51–63 (2014).
7. U. S. Kirmaci, “Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula,” Appl. Math. Comput., 147, No. 1, 137–146 (2004).
8. W.-H. Li and F. Qi, “Some Hermite–Hadamard-type inequalities for functions whose n-th derivatives are \((\alpha, m)\)-convex,” Filomat., 27, No. 8, 1575–1582 (2013).
9. C. E. M. Pearce and J. Pečarić, “Inequalities for differentiable mappings with application to special means and quadrature formulae,” Appl. Math. Lett., 13, No. 2, 51–55 (2000).
10. M. Z. Sarikaya and N. Aktan, On the Generalization of Some Integral Inequalities and Their Applications, available online at http://arxiv.org/abs/1005.2897.
11. M. Z. Sarikaya, E. Set, and M. E. Özdemir, “On new inequalities of Simpson’s type for convex functions,” RGMIA Res. Rep. Coll., 13, No. 2, Art. 2 (2010).
12. F. Qi, M. A. Latif, W.-H. Li, and S. Hussain, “Some integral inequalities of Hermite–Hadamard type for functions whose \( n \)-times derivatives are \((\alpha, m)\)-convex,” *Turk. J. Anal. Number Theory*, 2, No. 4, 140–146 (2014).
13. F. Qi, T.-Y. Zhang, and B.-Y. Xi, *Hermite–Hadamard Type Integral Inequalities for Functions Whose First Derivatives Are of Convexity*, available online at http://arxiv.org/abs/1305.5933.
14. Y. Shuang, Y. Wang, and F. Qi, “Some inequalities of Hermite–Hadamard type for functions whose third derivatives are \((\alpha, m)\)-convex,” *J. Comput. Anal. Appl.*, 17, No. 2, 272–279 (2014).
15. Y. Shuang, H.-P. Yin, and F. Qi, “Hermite–Hadamard-type integral inequalities for geometric-arithmetically \( s \)-convex functions,” *Analysis (Munich)*, 33, No. 2, 197–208 (2013).
16. S.-H. Wang and F. Qi, “Hermite–Hadamard type inequalities for \( n \)-times differentiable and preinvex functions,” *J. Inequal. Appl.*, 2014, No. 49 (2014).
17. B.-Y. Xi and F. Qi, “Some Hermite–Hadamard type inequalities for differentiable convex functions and applications,” *Hacet. J. Math. Stat.*, 42, No. 3, 243–257 (2013).
18. B.-Y. Xi and F. Qi, “Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means,” *J. Funct. Spaces Appl.*, 2012, Article ID 980438 (2012), 14 p.
19. T.-Y. Zhang, A.-P. Ji, and F. Qi, “Some inequalities of Hermite–Hadamard type for GA-convex functions with applications to means,” *Matematiche (Catania)*, 68, No. 1, 229–239 (2013).