Generalizations of some weighted Opial-type inequalities in conformable calculus

S. H. Saker\textsuperscript{a}, G. M. Ashry\textsuperscript{b}, M. R. Kenawy\textsuperscript{b,∗}

\textsuperscript{a}Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt.
\textsuperscript{b}Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt.

Abstract

In this paper, we prove new $α$-fractional inequalities of Opial type using conformable calculus. From our results we obtain classical integral inequalities as special cases.

Keywords: Opial type inequalities, conformable fractional calculus, Hölder inequalities, chain rule.

2020 MSC: 26A33, 26D10.

1. Introduction

In 1960, Opial in \cite{23} proved that

$$\int_{a}^{b} |f(x)| f'(x) \, dx \leq \frac{b-a}{4} \int_{a}^{b} (f'(x))^2 \, dx,$$

(1.1)

where $f \in C^1 [a, b]$ and $f(x) > 0$, $f(a) = f(b) = 0$. The constant $1/4$ is the best constant. Also he proved that

$$\int_{0}^{b} |f(x)| f'(x) \, dx \leq \frac{b}{2} \int_{0}^{b} (f'(x))^2 \, dx,$$

where $f(0) = 0$.

In 1962, Beesack in \cite{5} generalized (1.1) and proved that

$$\int_{a}^{b} |f(x)||f'(x)| \, dx \leq \frac{1}{2} \int_{a}^{b} \frac{1}{h(x)} \, dx \int_{a}^{b} h(x) (f'(x))^2 \, dx,$$

(1.2)

where $f$ is an absolutely continuous function on $[a, b]$, $f(a) = 0$ and $h$ is a continuous and positive function such that $\int_{a}^{b} \left( \frac{1}{h(x)} \right) \, dx < \infty$.

∗Corresponding author

Email addresses: shaaker@mans.edu.eg (S. H. Saker), gma07@fayoum.edu.eg (G. M. Ashry), mrz000@fayoum.edu.eg (M. R. Kenawy)

doi: 10.22436/jmcs.030.01.04

Received: 2022-07-05 Revised: 2022-09-06 Accepted: 2022-09-09
In 1968, Beesack and Das in [6] proved the following inequality
\[
\int_a^b \left| f(x) \right|^\lambda \left| f'(x) \right|^\mu \, dx \leq K(a, \lambda, \mu) \int_a^b \left| f(x) \right|^{\lambda + \mu} \, dx,
\]
where \( f \) is an absolutely continuous function on \([a, b]\) and \( f(0) = 0 \), \( t' \) is of constant sign, \( \lambda, \mu \) are real numbers with \( \lambda \mu > 0 \) and either \( \lambda + \mu < 0 \) or \( \lambda + \mu > 1 \), \( h, v \) are nonnegative measurable functions with
\[
\int_a^b h^{\frac{1}{\lambda+\mu-1}}(t) \, dt < \infty,
\]
and
\[
K(a, \lambda, \mu) = \left( \frac{\mu}{\lambda+\mu} \right)^{\frac{\mu}{\lambda+\mu}} \left( \int_0^a s^{\frac{\lambda}{\lambda+\mu}}(x) \, dx \right)^{\frac{\lambda}{\lambda+\mu}} \left( \int_0^b r^{\frac{\mu}{\lambda+\mu}}(x) \, dx \right)^{\frac{\mu}{\lambda+\mu}}.
\]
In 1983, Yong in [28] proved that
\[
\int_a^b \left| f(x) \right|^\lambda \left| f'(x) \right|^\mu \, dx \leq \frac{\mu}{\lambda+\mu} (b - a)^\lambda \int_a^b \left| f(x) \right|^{\lambda + \mu} \, dx,
\]
for \( \lambda \geq 0, \mu \geq 1 \), \( f \) is absolutely continuous on \([a, b]\) such that \( f(0) = 0 \) and \( r \) is a bounded and positive function. For more details about Opial type inequalities, we refer readers to [3, 8, 13, 22, 23].

Various integral inequalities and their extensions are important in the study of qualitative behavior of differential equations (see, e.g., [7, 9, 10, 18, 19] for more details) and partial differential equations (see, e.g., [17, 20] for more details). Recently, Opial type inequalities and their extensions have become an important tool for all types of differential equations by using it to prove the uniqueness and existence of initial and boundary value problems.

By utilizing the conformable calculus, many authors proved some integral inequalities like Chebyshev’s type [21, 26], Hardy’s type [24], Hermite-Hadamard’s type [2, 15, 16], and Iyengar’s type [25].

The paper is organized as follows. In Section 2, we present some concepts on conformable calculus. In Section 3, we prove some Opial type inequalities for \( \alpha \)-fractional differentiable functions and obtain the classical ones as special cases when \( \alpha = 1 \).
2. Preliminaries and basic lemmas

In this section, we present some basic definitions and lemmas on conformable calculus. The results are adapted from [14], for more details, we refer the reader to [1, 4, 14].

Definition 2.1. The conformable derivative of order α of a function \( w : [0, \infty) \to \mathbb{R} \) is defined by

\[
D_\alpha w(s) = \lim_{\varepsilon \to 0} \frac{w(s + \varepsilon s^{1-\alpha}) - w(s)}{\varepsilon}, \quad \text{for all } s > 0, \ \alpha \in (0, 1).
\]

Definition 2.2. The conformable integral of order α of a function \( w : [0, \infty) \to \mathbb{R} \) is defined by

\[
(\mathcal{I}_\alpha^w)(x) = \int_a^x w(s) \, ds = \int_a^x s^{\alpha-1}w(s) \, ds, \ 0 < \alpha \leq 1.
\]

Theorem 2.3. Let \( w \) and \( v \) be \( \alpha \)-differentiable such that \( x > 0 \). Then for \( \alpha \in (0, 1) \),

1. \( D_\alpha (aw + bv)(x) = aD_\alpha w(x) + bD_\alpha v(x) \);
2. \( D_\alpha (x^\lambda) = \lambda x^{\lambda-\alpha} \) for all \( \lambda \in \mathbb{R} \);
3. \( D_\alpha (\theta) = 0 \), for all constant functions \( w(x) = \theta \);
4. \( D_\alpha (wv)(x) = wD_\alpha v(x) + vD_\alpha w(x) \);
5. \( D_\alpha (\frac{v}{w})(x) = \frac{vD_\alpha w(x) - wD_\alpha v(x)}{w^2} \);
6. if \( w \) is differentiable, then \( D_\alpha w(x) = x^{1-\alpha} \frac{dw(x)}{dx} \).

Lemma 2.4. Let \( v(x) \) be \( \alpha \)-differentiable with respect to \( x \) and \( w \) be \( \alpha \)-differentiable with respect to \( v \). Then the chain rule by using conformable derivative is defined by

\[
D_\alpha w(v(x)) = v^{\alpha-1}(x) (D_\alpha w(v(x))) \, D_\alpha v(x).
\]

Lemma 2.5. Let \( w \) and \( v \) be \( \alpha \)-differentiable with respect to \( x \) on \([a, b]\). Then the integration by parts using conformable calculus is defined as

\[
\int_a^b D_\alpha^w v(x) \, d_\alpha x = w(x) v(x)|_a^b - \int_a^b w(x) (D_\alpha^v v(x)) \, d_\alpha x.
\]

Lemma 2.6. Let \( 0 < \alpha \leq 1 \) and \( w, v : [a, b] \to \mathbb{R} \). Then the H"older inequality by using conformable integral is defined by

\[
\int_a^b |w(x) v(x)| \, d_\alpha x \leq \left( \int_a^b |w(x)|^\beta \, d_\alpha x \right)^\frac{1}{\beta} \left( \int_a^b |v(x)|^\gamma \, d_\alpha x \right)^\frac{1}{\gamma},
\]

for \( \frac{1}{\beta} + \frac{1}{\gamma} = 1 \) and \( \beta > 1 \).

3. Main results

Theorem 3.1. Let \( \lambda, \mu \in \mathbb{R}^+ \) such that \( \lambda + \mu > 1 \), \( a, x \in \mathbb{R} \), \( g, h \) be nonnegative continuous functions on \((a, x)\) with \( \int_a^x g^{\frac{1}{\lambda+\mu-1}}(s) \, d_\alpha s < \infty \), and \( \Phi : [a, x] \to \mathbb{R} \) be \( \alpha^{th} \) differentiable with \( D_\alpha \Phi \) of constant sign in \((a, x)\) and \( \Phi(a) = 0 \). Then

\[
\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu \, d_\alpha t \leq K_1(a, x, \lambda, \mu) \int_a^x g(t) |D_\alpha \Phi(t)|^{\lambda+\mu} \, d_\alpha t,
\]

where

\[
K_1(a, x, \lambda, \mu) = \left( \frac{\mu}{\lambda + \mu} \right)^{\frac{1}{\lambda+\mu}} \left[ \int_a^x h^{\frac{\lambda+\mu}{\mu}}(t) \left( \frac{1}{g(t)} \right)^\frac{\mu}{\lambda} \left( \int_a^t \frac{1}{g(s)} \, d_\alpha s \right)^{\lambda+\mu-1} \, d_\alpha t \right]^{\frac{\lambda}{\lambda+\mu}}.
\]
Proof. Suppose that
\[
|\Phi(t)| = \int_a^t |D_\alpha \Phi(s)| \, d_\alpha s = \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \cdot |D_\alpha \Phi(s)| \, d_\alpha s.
\]
Since \( g \) is nonnegative on \((a, x)\), then by using the Hölder inequality (2.2) such that \( \beta = \lambda + \mu \) and \( \gamma = \frac{\lambda + \mu}{\lambda + \mu - 1} \), we find that
\[
|D_\alpha \Phi(s)| = g^{\frac{1}{\lambda+\mu}}(s) |D_\alpha \Phi(s)|, \text{ and } \nu(s) = \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)},
\]
where that
\[
\int_a^t |D_\alpha \Phi(s)| \, d_\alpha s \leq \left( \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s \right)^{\frac{1}{\lambda+\mu}} \cdot \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda+\mu-1}{\lambda+\mu}}.
\]
This leads to
\[
|\Phi(t)|^\lambda \leq \left( \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s \right)^{\frac{\lambda}{\lambda+\mu}} \cdot \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}.
\]
Let
\[
\Omega(t) := \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s.
\]
Then \( \Omega(a) = 0 \), and
\[
|D_\alpha \Phi(t)|^\lambda = \frac{|D_\alpha \Phi(t)|^\mu}{g(t)} \quad \text{and} \quad |D_\alpha \Phi(t)|^{\lambda+\mu} = \frac{|D_\alpha \Phi(t)|^\mu}{g(t)}.
\]
Since \( h \) is a nonnegative function on \((a, x)\), then by using (3.2) we find that
\[
h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu \leq h(t) |D_\alpha \Phi(t)|^\mu \left( \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s \right)^{\frac{\lambda}{\lambda+\mu}} \cdot \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}
\]
\[
= \Omega^{\frac{\lambda}{\lambda+\mu}}(t) \left( \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s \right)^{\frac{\mu}{\lambda+\mu}} h(t) \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}.
\]
Integrating the above inequality from \( a \) to \( x \), we have
\[
\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu \, d_\alpha t \leq \int_a^x \Omega^{\frac{\lambda}{\lambda+\mu}}(t) \left( \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} \, d_\alpha s \right)^{\frac{\mu}{\lambda+\mu}} h(t) \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} \, d_\alpha t.
\]
By employing the Hölder inequality (2.2) on the right side of integral of (3.3) with \( \beta = (\lambda + \mu)/\mu \) and \( \gamma = (\lambda + \mu)/\lambda \), we have
\[
\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu \, d_\alpha t \leq \left[ \int_a^x \Omega^{\frac{\lambda}{\lambda+\mu}}(t) \, d_\alpha t \right]^\frac{\mu}{\lambda+\mu} \left[ \int_a^x h^{\frac{\lambda+\mu}{\mu}}(t) \left( \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} \, d_\alpha s \right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} \, d_\alpha t \right].
\]
By using the chain rule (2.1), we get that
\[
D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right) = D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (\Omega (t)) \right) D_\alpha (\Omega (t)) \Omega^{\alpha-1} (t) \\
= \frac{\lambda + \mu}{\mu} \Omega^{\frac{\lambda + \mu}{\mu} - \alpha} (t) D_\alpha (\Omega (t)) \Omega^{\alpha-1} (t) = \frac{\lambda + \mu}{\mu} \Omega^{\frac{\lambda}{\mu}} (t) D_\alpha (\Omega (t)).
\]
Then we have
\[
\Omega^{\frac{\lambda}{\mu}} (t) D_\alpha (\Omega (t)) = \frac{\mu}{\lambda + \mu} D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right).
\]
(3.5)
Since \( \Omega (a) = 0 \) and from (3.4) and (3.5), we deduce that
\[
\int_a^x h (t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d\alpha t \\
\leq \left( \frac{\mu}{\lambda + \mu} \right) ^{\frac{\mu}{\lambda + \mu}} \left[ \int_a^x D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right) \, d\alpha t \right] ^{-\frac{\mu}{\lambda + \mu}} \left[ \int_a^x h^{\frac{\lambda + \mu}{\mu}} (t) \left( \frac{1}{g (t)} \right) ^{\frac{\mu}{\lambda + \mu}} \left( \int_t^1 \frac{1}{g^\frac{\lambda + \mu}{\mu} (s)} \, ds \right) ^{\lambda + \mu - 1} \, d\alpha t \right] ^{\frac{\lambda}{\lambda + \mu}},
\]
which is the inequality (3.1). The proof is complete. \( \square \)

**Corollary 3.2.** In Theorem 3.1, if \( \alpha = 1 \) and \( a = 0 \), then we get
\[
\int_0^x h (t) |\Phi (t)|^\lambda |\Phi' (t)|^\mu \, dt \leq K_2 (x, \lambda, \mu) \int_0^x g (t) |\Phi' (t)|^\lambda + \mu \, dt,
\]
where
\[
K_2 (x, \lambda, \mu) = \left( \frac{\mu}{\lambda + \mu} \right) ^{\frac{\mu}{\lambda + \mu}} \left[ \int_0^x h^{\frac{\lambda + \mu}{\mu}} (t) \left( \frac{1}{g (t)} \right) ^{\frac{\mu}{\lambda + \mu}} \left( \int_0^1 \frac{1}{g^\frac{\lambda + \mu}{\mu} (s)} \, ds \right) ^{\lambda + \mu - 1} \, dt \right] ^{\frac{\lambda}{\lambda + \mu}},
\]
which is the inequality of (1.3).

**Theorem 3.3.** Let \( \lambda, \mu \in \mathbb{R}^+ \) such that \( \lambda + \mu > 1 \), \( x, b \in \mathbb{R} \), \( g, h \) be nonnegative continuous functions on \( (x, b) \) with \( \int_x^b g^{\frac{1}{\lambda + \mu} - 1} (s) \, d\alpha s < \infty \), and \( \Phi : [x, b] \to \mathbb{R} \) be \( \alpha^{th} \) differentiable with \( D_\alpha \Phi \) of constant sign in \((x, b)\) and \( \Phi (b) = 0 \). Then
\[
\int_x^b h (t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d\alpha t \leq K_3 (x, b, \lambda, \mu) \int_x^b g (t) |D_\alpha \Phi (t)|^\lambda + \mu \, d\alpha t,
\]
(3.6)
where
\[
K_3 (x, b, \lambda, \mu) = \left( \frac{\mu}{\lambda + \mu} \right) ^{\frac{\mu}{\lambda + \mu}} \left[ \int_x^b h^{\frac{\lambda + \mu}{\mu}} (t) \left( \frac{1}{g (t)} \right) ^{\frac{\mu}{\lambda + \mu}} \left( \int_t^b \frac{1}{g^{\frac{\lambda + \mu}{\mu} - 1} (s)} \, d\alpha s \right) ^{\lambda + \mu - 1} \, d\alpha t \right] ^{\frac{\lambda}{\lambda + \mu}}.
\]

**Proof.** Suppose that
\[
|\Phi (t)| = \int_t^b |D_\alpha \Phi (s)| \, d\alpha s = \int_t^b \frac{1}{g^{\frac{\lambda + \mu}{\mu} - 1} (s)} \, g^{\frac{\lambda + \mu}{\mu} (s)} \, d\alpha s.
\]
Since $h$ is nonnegative on $(x, b)$, then by using the Hölder inequality (2.2) such that $\beta = \lambda + \mu$ and $\gamma = \frac{\lambda + \mu}{\lambda + \mu - 1}$ and

$$w(s) = g^\frac{1}{\lambda + \mu}(s)|D_\lambda \Phi(s)|,$$ and $v(s) = \frac{1}{r^{\frac{1}{\lambda + \mu}}(s)},$

where that

$$\int_t^b |D_\lambda \Phi(s)| d\alpha s \leq \left( \int_t^b g(s)|D_\lambda \Phi(s)|^{\lambda + \mu} d\alpha s \right)^{\frac{1}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}}. $$

This gets us that

$$|\Phi(t)|^\lambda \leq \left( \int_t^b g(s)|D_\lambda \Phi(s)|^{\lambda + \mu} d\alpha s \right)^{\frac{1}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}}. $$

Letting

$$\Omega(t) := \int_t^b g(s)|D_\lambda \Phi(s)|^{\lambda + \mu} d\alpha s,$$

then we see that $\Omega(b) = 0$, and

$$D_\lambda \Omega(t) = -g(t)|D_\lambda \Phi(t)|^{\lambda + \mu} < 0. $$

Hence we have

$$|D_\lambda \Phi(t)|^\mu = \left( \frac{-D_\lambda \Omega(t)}{g(t)} \right)^{\frac{\mu}{\lambda + \mu}} \text{ and } |D_\lambda \Phi(t)|^{\lambda + \mu} = \frac{-D_\lambda \Omega(t)}{g(t)}. \quad (3.7)$$

Since $h$ is a nonnegative function on $(x, b)$, then by using (3.7) we find that

$$h(t)|\Phi(t)|^\lambda |D_\lambda \Phi(t)|^\mu \leq h(t)|D_\lambda \Phi(t)|^\mu \left( \int_t^b g(s)|D_\lambda \Phi(s)|^{\lambda + \mu} d\alpha s \right)^{\frac{1}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}} \Omega^\frac{\mu}{\lambda + \mu}(t) (-D_\lambda \Omega(t)) \frac{\mu}{\lambda + \mu} h(t) \left( \frac{1}{g(t)} \right)^{\frac{\mu}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}}. $$

Integrating the above inequality from $x$ to $b$, we have

$$\int_x^b h(t)|\Phi(t)|^\lambda |D_\lambda \Phi(t)|^\mu d\alpha t \leq \int_x^b \Omega^\frac{\mu}{\lambda + \mu}(t) (-D_\lambda \Omega(t)) \frac{\mu}{\lambda + \mu} h(t) \left( \frac{1}{g(t)} \right)^{\frac{\mu}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}} d\alpha t. \quad (3.8)$$

By using the Hölder inequality (2.2) on the right side of integral of (3.8) such that $\beta = (\lambda + \mu)/\mu$ and $\gamma = (\lambda + \mu)/\lambda$, we have

$$\int_x^b h(t)|\Phi(t)|^\lambda |D_\lambda \Phi(t)|^\mu d\alpha t \leq \left[ \int_x^b \Omega^\frac{\mu}{\lambda + \mu}(t) (-D_\lambda \Omega(t)) d\alpha t \right] \left[ \int_x^b h^\frac{\lambda + \mu}{\lambda}(t) \left( \frac{1}{g(t)} \right)^{\frac{\mu}{\lambda + \mu}} \left( \int_t^b \frac{1}{g}(s) d\alpha s \right)^{-\frac{\lambda + \mu - 1}{\lambda + \mu}} d\alpha t \right]^\lambda \frac{\lambda}{\lambda + \mu}. \quad (3.9)$$
Using the chain rule (2.1), we get that
\[ D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right) = D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right) \Omega^{\alpha - 1} (t) \]
\[ = \frac{\lambda + \mu}{\mu} \Omega^{\frac{\lambda + \mu - \alpha}{\mu}} (t) D_\alpha (\Omega (t)) \Omega^{\alpha - 1} (t) = \frac{\lambda + \mu}{\mu} \Omega^{\frac{\lambda}{\mu}} (t) D_\alpha (\Omega (t)). \]

Then we have
\[ \Omega^{\frac{\lambda}{\mu}} (t) D_\alpha (\Omega (t)) = \frac{\mu}{\lambda + \mu} D_\alpha \left( \Omega^{\frac{\lambda + \mu}{\mu}} (t) \right). \]

Since \( \Omega (b) = 0 \) and from (3.9) and (3.10), we deduce that
\[ \int_x^b h(t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, dt \leq K_3 (x, b, \lambda, \mu) \int_x^b g(t) |\Phi' (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, dt, \]
which is the inequality (3.6). The proof is complete. \( \square \)

**Corollary 3.4.** In Theorem 3.3, if \( \alpha = 1 \), then we have the following inequality
\[ \int_x^b h(t) |\Phi (t)|^\lambda |\Phi' (t)|^\mu \, dt \leq K_4 (x, b, \lambda, \mu) \int_x^b g(t) |\Phi' (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, dt, \]
where
\[ K_4 (x, b, \lambda, \mu) = (\frac{\mu}{\lambda + \mu})^{\frac{\mu}{\lambda + \mu}} \left[ \int_x^b h(t)^{\lambda + \mu} (t) \left( \frac{1}{g(t)} \right)^{\frac{\mu}{\lambda + \mu}} \left( \int_t^b \frac{1}{g(s) \lambda + \mu} \, ds \right)^{\lambda + \mu - 1} \, dt \right]^{\frac{\mu}{\lambda + \mu}}. \]

Assume that there exists \( x \in (a, b) \) which is the unique solution of the equation
\[ K (\lambda, \mu) = K_1 (a, x, \lambda, \mu) = K_3 (x, b, \lambda, \mu) < \infty, \]
where \( K_1 (a, x, \lambda, \mu) \) and \( K_3 (x, b, \lambda, \mu) \) are given in Theorems 3.1 and 3.3, now since
\[ \int_a^b h(t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d_\alpha t = \int_a^b h(t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d_\alpha t + \int_a^b h(t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d_\alpha t, \]
then we have the following theorem.

**Theorem 3.5.** Let \( \lambda, \mu \in \mathbb{R}^+ \) such that \( \lambda \mu > 0 \) and \( \lambda + \mu > 1 \), \( a, b \in \mathbb{R} \), \( h, g \) be nonnegative continuous functions on \( [a, b] \) with \( \int_a^b g(t)^{\frac{1}{\lambda + \mu}} (s) \, ds < \infty \), and \( \Phi : [a, b] \to \mathbb{R} \) be \( \alpha \)-th differentiable thus \( D_\alpha \Phi \) of constant sign in \( (a, b) \), and \( \Phi (a) = 0 = \Phi (b) \). Then
\[ \int_a^b h(t) |\Phi (t)|^\lambda |D_\alpha \Phi (t)|^\mu \, d_\alpha t \leq K (\lambda, \mu) \int_a^b g(t) |D_\alpha \Phi (t)|^{\lambda + \mu} \, d_\alpha t. \]

**Proof.** The proof can be obtained by making a combination of the proof of Theorems 3.1 and 3.3. \( \square \)
References

[1] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math., 279 (2015), 57–66.
[2] T. Abdeljawad, P. O. Mohammed, A. Kashuri, New modified conformable fractional integral inequalities of Hermite-Hadamard type with applications, J. Funct. Spaces, 2020 (2020), 14 pages.
[3] R. P. Agarwal, P. Y. H. Pang, Opial inequalities with applications in differential and difference equations, Kluwer Academic Publishers, Dordrecht, 320 (1995).
[4] D. R. Anderson, Taylor’s formula and integral inequalities for conformable fractional derivatives, Contrib. Math. Eng. Springer. Cham., (2016), 25–43.
[5] P. R. Beesack, On an integral inequality of Z. Opial, Trans. Amer. Math. Soc., 104 (1962), 470–475.
[6] P. R. Beesack, K. M. Das, Extensions of Opial’s inequality, Pacific J. Math., 26 (1968), 215–232.
[7] M. Bohner, T. S. Hassan, T. Li, Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments, Indag. Math. (N.S.), 29 (2018), 548–560.
[8] D. W. Boyd, J. S. W. Wong, An extension of Opial’s inequality, J. Math. Anal. Appl., 19 (1967), 100–102.
[9] K.-S. Chiu, T. Li, Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, Math. Nachr., 292 (2019), 2153–2164.
[10] J. Džurina, S. Grace, I. Jadlovská, T. Li, Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term, Math. Nachr., 293 (2020), 910–922.
[11] S. Frassu, T. Li, G. Viglialoro, Improvements and generalizations of results concerning attraction-repulsion chemotaxis models, Math. Methods Appl. Sci., 45 (2021), 11067–11078.
[12] S. Frassu, G. Viglialoro, Boundedness criteria for a class of indirect (and direct) chemotaxis-consumption models in high dimensions, Appl. Math. Lett., 132 (2022), 7 pages.
[13] L.-G. Hua, On an inequality of Opial, Sci. Sinica, 14 (1965), 789–790.
[14] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math., 264 (2014), 65–70.
[15] M. A. Khan, Y.-M. Chu, A. Kashuri, R. Liko, G. Ali, Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations, J. Funct. Spaces, 2018 (2018), 9 pages.
[16] Y. Khurshid, M. A. Khan, Y.-M. Chu, Conformable fractional integral inequalities for GG-and GA-convex function, AIMS Math., 5 (2020), 5012–5030.
[17] T. Li, N. Pintus, G. Viglialoro, Properties of solutions to porous medium problems with different sources and boundary conditions, Z. Angew. Math. Phys., 70 (2019), 1–18.
[18] T. Li, Y. V. Rogovchenko, Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations, Monatsh. Math., 184 (2017), 489–500.
[19] T. Li, Y. V. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, Appl. Math. Lett., 105 (2020), 1–7.
[20] T. Li, G. Viglialoro, Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime, Differ. Integral Equ., 34 (2021), 315–336.
[21] K. S. Nisar, G. Rahman, K. Mehrez, Chebyshev type inequalities via generalized fractional conformable integrals, J. Inequal. Appl., 2019 (2019), 1–9.
[22] C. Olech, A simple proof of a certain result of Z. Opial, Ann. Polon. Math., 8 (1960), 61–63.
[23] Z. Opial, Sur une inégalité, Ann. Polon. Math., 8 (1960), 29–32.
[24] S. H. Saker, D. O’Regan, M. R. Kenawy, R. P. Agarwal, Fractional Hardy Type Inequalities via Conformable Calculus, Mem. Differ. Equ. Math. Phys., 73 (2018), 131–140.
[25] M. Z. Sarikaya, H. Yıldız, H. Budak, On weighted Iyengar-type inequalities for conformable fractional integrals, Math. Sci., 11 (2017), 327–331.
[26] E. Set, A. O. Akdemir, I. Mumcu, Chebyshev type inequalities for conformable fractional integrals, Miskolc Math. Notes, 20 (2019), 1227–1236.
[27] G.-S. Yang, On a certain result of Z. Opial, Proc. Japan Acad., 42 (1966), 78–83.
[28] G.-S. Yang, A note on some integrodifferential inequalities, Soochow J. Math., 9 (1983), 231–236.