A REACTION-DIFFUSION-ADVECTION SIS EPIDEMIC MODEL IN A SPATIALLY-TEMPORALLY HETEROGENEOUS ENVIRONMENT

DANHUA JIANG, ZHI-CHENG WANG AND LIANG ZHANG

School of Mathematics and Statistics, Lanzhou University, and
Key Laboratory of Applied Mathematics and Complex Systems of Gansu province
Lanzhou, Gansu 730000, China

(Communicated by Yuan Lou)

Abstract. In this paper, we study the effects of diffusion and advection for an SIS epidemic reaction-diffusion-advection model in a spatially and temporally heterogeneous environment. We introduce the basic reproduction number $R_0$ and establish the threshold-type results on the global dynamics in terms of $R_0$. Some general qualitative properties of $R_0$ are presented, then the paper is devoted to studying how the advection and diffusion of the infected individuals affect the reproduction number $R_0$ for the special case that $\gamma(x,t) - \beta(x,t) = V(x,t)$ is monotone with respect to spatial variable $x$. Our results suggest that if $V_x(x,t) \geq 0, \not\equiv 0$ and $V(x,t)$ changes sign about $x$, the advection is beneficial to eliminate the disease, whereas if $V_x(x,t) \leq 0, \not\equiv 0$ and $V(x,t)$ changes sign about $x$, the advection is bad for the elimination of disease.

1. Introduction. The SIS (susceptible-infected-susceptible) models provide essential frames in studying the dynamics of disease transmission in the field of theoretical epidemiology. Recently Allen et al. [3] proposed a frequency-dependent SIS model with a no-flux boundary condition

$$\begin{align*}
S_t &= d_S \Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, \\
I_t &= d_I \Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, \\
\frac{\partial S}{\partial n} &= \frac{\partial I}{\partial n} = 0, \\
S(x,0) &= S_0(x) \geq 0, \\
I(x,0) &= I_0(x) \geq 0, \\
S(x,t) &= S(x,t), \\
I(x,t) &= I(x,t),
\end{align*}$$

(1)

to investigate the impact of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease, where $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected individuals in a given spatial region $\Omega$, which is assumed to be a bounded domain in $\mathbb{R}^m (m \geq 1)$ with smooth boundary $\partial \Omega$; the positive constants $d_S$ and $d_I$ are diffusion coefficients for the susceptible and infected populations; the positive functions $\beta(x)$ and $\gamma(x)$ are Hölder continuous on $\Omega$ and represent the rates of disease transmission and recovery at location $x$, respectively. The homogeneous Neumann boundary conditions mean that there is

2010 Mathematics Subject Classification. 35K57, 37N25, 92B05.
Key words and phrases. SIS epidemic model, reaction-diffusion-advection, spatial heterogeneity, temporal periodic.
* Corresponding author: Zhi-Cheng Wang.
no population flux across the boundary $\partial \Omega$ and both the susceptible and infected individuals live in a self-contained environment.

Regarding (1), the main results of [3] concern the existence, uniqueness and asymptotic behaviors of the endemic equilibrium as the diffusion rate of the susceptible individuals approaches to zero. Allen et al. [2] also investigated a discrete SIS model associated with (1). Peng and Liu [19] discussed the global stability of the endemic equilibrium in some special cases. Further results concerning the asymptotic behavior of the endemic equilibrium of (1) were derived by [18, 20]. Peng and Zhao [21] treated the system (1) with $\beta$ and $\gamma$ being functions of spatiotemporal variables and temporally periodic, and found that the combination of spatial heterogeneity and temporal periodicity can enhance the persistence of the disease. Huang et al. [11] studied the global dynamics of system (1) subject to the Dirichlet boundary conditions. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in [7]. Li et al. [13] performed qualitative analysis on an SIS epidemic reaction-diffusion system with a linear source in spatially heterogeneous environment, which the main feature lies in that its total population number varies compared to model (1). In these works the populations are assumed to adopt random diffusion in the habitats.

In some circumstance, populations may take biased or passive movement in certain direction, e.g., due to external environmental forces such as water flow [14, 15, 16], wind [6] and so on, which usually can be described by adding an advection term to the existing reaction-diffusion models. Recently, Cui et al. [4, 5] considered an SIS epidemic with advection, the results in [4] are in strong contrast with the case of no advection, and the results in [5] suggest that advection can help speed up the elimination of disease. For the SIS epidemic reaction-diffusion-advection model in [4, 5], it was assumed that the rates of disease transmission and recovery depend only on the spatial variable. However, the rates of disease transmission and disease recovery may be spatially and temporally heterogeneous. Typically, they vary periodically in time, for example, due to the seasonal fluctuation and periodic availability of vaccination strategies.

Here, we consider the following SIS epidemic reaction-diffusion-advection model in a spatially heterogeneous and temporally periodic environment:

$$
\begin{align*}
S_t &= dS S_{xx} - qS_x - \frac{\beta(x,t)S}{S+I}I + \gamma(x,t) I, & 0 < x < L, t > 0, \\
I_t &= dI I_{xx} - qI_x + \frac{\beta(x,t)S}{S+I}S - \gamma(x,t) I, & 0 < x < L, t > 0, \\
dS S_x - qS = dI I_x - qI = 0, & x = 0, L, t > 0, \\
S(x,0) = S_0(x) \geq 0, I(x,0) = I_0(x) \geq 0, & 0 < x < L,
\end{align*}
$$

(2)

where the functions $\beta(x,t)$ and $\gamma(x,t)$ represent the rates of disease transmission and recovery at location $x$ and time $t$, respectively; $L$ is the size of the habitat, and we call $x = 0$ the upstream end and $x = L$ the downstream end; $q$ is the effective speed of the current (sometimes we call $q$ the advection speed/rate, and we remark here that $q$ should be non-negative since $x = L$ is defined to be the downstream end). Here we impose no-flux boundary conditions at the upstream and downstream ends, respectively. It means that there is no population net flux across the boundary $x = 0$ and $x = L$.

As the term $SI/(S+I)$ is a Lipschitz continuous function of $S$ and $I$ in the open first quadrant, we can extend its definition to the entire first quadrant by defining it to be zero when either $S = 0$ or $I = 0$. Following [3], we further assume that at the initial time, there is a positive number of infected individuals, that is,
It follows from the maximum principle for parabolic equations that the endemic persistence and extinction of the infectious disease for (2). Of particular interest is diffusion in a spatially heterogeneous and temporally periodic environment on the habitat if the terminology analogous to that in [3, 21]. We say that the rate

Thus the total population size is constant in time, i.e.,

which also shows that both \( ||S(\cdot,t)||_{L^1([0,L])} \) and \( ||I(\cdot,t)||_{L^1([0,L])} \) are uniformly bounded for \( t \in [0,\infty) \). From now on, we always assume that (A1) and (A2) hold and \( N \) is a given positive constant throughout this paper.

We say \((\tilde{S},\tilde{I})\) is an \( \omega \)-periodic solution of (2), if it is a nonnegative classical solution of the associated periodic-parabolic problem:

By a disease-free \( \omega \)-periodic solution of (2), we mean that \((\tilde{S},\tilde{I})\) is a nonnegative solution to (5) in which \( \tilde{I} \equiv 0 \) on \([0,L] \times \mathbb{R} \); and \((\tilde{S},\tilde{I})\) is said to be an endemic \( \omega \)-periodic solution if \( \tilde{I} \geq 0, \neq 0 \) on \([0,L] \times \mathbb{R} \). It is easy to observe from (5) that the unique disease-free \( \omega \)-periodic solution is \((S^*,0) = \left( \frac{qNe_0}{dS(e^{qN}\omega - 1)}, 0 \right) \) (see [4]). It follows from the maximum principle for parabolic equations that the endemic \( \omega \)-periodic solution \((\tilde{S},\tilde{I})\) is positive on \([0,L] \times [0,\infty)\), that is, \( \tilde{S}(x,t) > 0, \tilde{I}(x,t) > 0, \forall (x,t) \in [0,L] \times [0,\infty) \).

To give biological interpretations of our analytical results more clearly, we adopt the terminology analogous to that in [3, 21]. We say that \( x \) is a low-risk site if the local disease transmission rate \( \int_0^L \beta(x,t) dx \) is lower than the local disease recovery rate \( \int_0^\omega \gamma(x,t) dt \). A high-risk site is defined in a reversed manner. We also call that \((0,L)\) is a low-risk habitat if \( \int_0^L \int_0^L \beta(x,t) dx dt < \int_0^\omega \int_0^L \gamma(x,t) dx dt \) and a high-risk habitat if \( \int_0^\omega \int_0^L \beta(x,t) dx dt \geq \int_0^\omega \int_0^L \gamma(x,t) dx dt \).

The main goal of our current work is to investigate the effect of advection and diffusion in a spatially heterogeneous and temporally periodic environment on the persistence and extinction of the infectious disease for (2). Of particular interest is...
the basic reproduction number which will serve as the threshold value for persistence and extinction of the disease.

In this paper, we first introduce the basic reproduction number $R_0$ for (2), which contains the results in [21] for the case $q = 0$ and in [4, 5] for the case $\beta(x, t) \equiv \beta(x), \gamma(x, t) \equiv \gamma(x)$ as special cases. The threshold-type dynamics for the system (2) is established by monotone dynamics, which says that if $R_0 \leq 1$, the disease-free solution is globally stable, while if $R_0 > 1$, (2) admits at least one endemic $\omega$-periodic solution and the disease is uniformly persistent. Compared to the previous references mentioned above [3, 4, 13, 18, 19, 20, 21], where the global asymptotic stability of the disease-free equilibrium was proved under the stronger assumption $R_0 < 1$, our general results that disease-free solution is globally stable provided $R_0 \leq 1$ for nonautonomous system, seems to be new and applicable for other SIS type PDE models.

Let $\gamma(x, t) - \beta(x, t) = V(x, t)$. In particular, we consider the special case: $V(x, t)$ is monotone with respect to spatial variable $x$. Biologically, if $V(x, t)$ is monotone increasing with respect to $x$, i.e., $V_x(x, t) \geq 0, \neq 0$, it means that for any given time, the spatial change rate of disease recovery $\gamma_x(x, t)$ is greater than the spatial change rate of disease transmission $\beta_x(x, t)$, while if $V(x, t)$ is monotone decreasing with respect to $x$, it has the reversed biological meaning. Then we study the effect of the advection rate $q$ and mobility of the infected individuals $d_I$ on $R_0$, and the results obtained are explicit.

The rest of our paper is arranged as follows. In Section 2, we first introduce the basic reproduction number $R_0$ and then establish the threshold dynamics in the terms of $R_0$. Section 3 is concerned with some general qualitative properties of $R_0$. Section 4 is devoted to the effect of the advection rate $q$ and mobility of the infected individuals $d_I$ on $R_0$ when $V(x, t)$ is monotone with respect to spatial variable $x$.

2. Threshold dynamics in terms of $R_0$. In this section, we first introduce the basic reproduction number $R_0$ for the periodic reaction-diffusion-advection system (2) and then we establish its threshold-type dynamics in terms of $R_0$. As a first step, we need to define the next infection operator for (2), which arises from the combination of the idea in [25] for periodic ordinary differential models and that in [26] for autonomous reaction-diffusion systems, see also [21].

Since the environment of our model (2) is periodic in time, we let $X = C([0, L], \mathbb{R})$ denote the Banach space of continuous functions on the interval $[0, L]$ with the supremum norm $\|u\|_{\infty} = \max_{x \in [0, L]} |u(x)|$ and $C_\omega$ be the ordered Banach space consisting of all $\omega$-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{R}$, which is equipped with the maximum norm $\|\cdot\|$ and the positive cone $C_\omega^+ := \{\phi \in C_\omega : \phi(t)(x) \geq 0, \forall t \in \mathbb{R}, x \in [0, L]\}$. For any given $\phi \in C_\omega$, we also use the notation $\phi(x, t) := \phi(t)(x)$. Let $\Gamma(t, s)$ be the evolution operator of the reaction-diffusion-advection equation
\[
\begin{align*}
I_t - d_I I_{xx} + q I_x &= -\gamma(x, t) I, & 0 < x < L, & t > 0, \\
I_{t} L_x - q I &= 0, & x = 0, L, & t > 0.
\end{align*}
\]

It follows from the standard semigroup theory that there exist positive constants $K$ and $c_0$ such that
\[
\|\Gamma(t, s)\| \leq Ke^{-c_0(t-s)}, \quad \forall t \geq s, \quad t, s \in \mathbb{R}.
\]

Let $\phi \in C_\omega$ and suppose that $\phi(x, s)$ is the density distribution of the infected individuals at the spatial location $x \in (0, L)$ and time $s$. Then the term $\beta(x, s)\phi(x, s)$
means the density distribution of the new infections produced by the infected individuals who were introduced at time \( s \). Thus, for given \( t \geq s \), \( \Gamma(t, s)\beta(x, s)\phi(x, s) \) is the density distribution at location \( x \) of those infected individuals who were newly infected at time \( s \) and remain infected at time \( t \). Therefore,

\[
\int_{-\infty}^{t} \Gamma(t, s)\beta(\cdot, s)\phi(\cdot, s)ds = \int_{0}^{\infty} \Gamma(t, t-a)\beta(\cdot, t-a)\phi(\cdot, t-a)da
\]

represents the density distribution of the accumulative new infections at location \( x \) and time \( t \) produced by all those infected individuals \( \phi(x, s) \) introduced at all the previous time to \( t \).

As in \([25]\) and \([21]\), we introduce the linear operator \( L : C_\omega \rightarrow C_\omega \):

\[
L(\phi)(t) := \int_{0}^{\infty} \Gamma(t, t-a)\beta(\cdot, t-a)\phi(\cdot, t-a)da,
\]

which we call as the next infection operator. Under our assumption on \( \beta \) and \( \gamma \), it is easy to see that \( L \) is continuous, compact on \( C_\omega \) and positive (namely, \( L(C_\omega^+) \subset C_\omega^+ \)). We define the spectral radius of \( L \) as the basic reproduction number

\[
R_0 = \rho(L)
\]

for system \((2)\).

In what follows, we first want to obtain an equivalent characterization of the basic reproduction number \( R_0 \). This thus leads us to consider the following linear periodic-parabolic eigenvalue problem

\[
\begin{cases}
\psi_t - d_I\psi_{xx} + q\psi_x + \gamma(x, t)\psi = \frac{\beta(x, t)}{\mu}\psi, & 0 < x < L, \ t > 0, \\
d_I\psi_x - q\psi = 0, & x = 0, L, \ t > 0, \\
\psi(x, 0) = \psi(x, \omega), & 0 < x < L.
\end{cases}
\]

(6)

By \([10, \text{Theorem 16.1}]\), problem (6) has a unique principal eigenvalue \( \mu_0 \), which is positive and corresponds to an eigenvector \( \psi \in C_\omega \) and \( \psi > 0 \) on \([0, L] \times \mathbb{R}\). Furthermore, we have the following observation, and the proof is similar to the argument of Lemma 2.1 in \([21]\).

**Lemma 2.1.** \( R_0 = \mu_0 > 0 \).

We now set

\[
\varphi(x, t) = e^{-\frac{q}{d_I}x}\psi(x, t).
\]

It then easily follows that \((R_0, \varphi)\) satisfies

\[
\begin{cases}
\varphi_t - d_I\varphi_{xx} - q\varphi_x + \gamma(x, t)\varphi = \frac{\beta(x, t)}{R_0}\varphi, & 0 < x < L, \ t > 0, \\
\varphi_x(x, t) = 0, & x = 0, L, \ t > 0, \\
\varphi(x, 0) = \varphi(x, \omega), & 0 < x < L,
\end{cases}
\]

which can be rewritten as

\[
\begin{cases}
e^{-\frac{q}{d_I}x}\varphi_t - d_I(e^{-\frac{q}{d_I}x}\varphi_x)_x + e^{-\frac{q}{d_I}x}\gamma(x, t)\varphi = \frac{e^{-\frac{q}{d_I}x}\beta(x, t)}{R_0}\varphi, & 0 < x < L, \ t > 0, \\
\varphi_x(x, t) = 0, & x = 0, L, \ t > 0, \\
\varphi(x, 0) = \varphi(x, \omega), & 0 < x < L.
\end{cases}
\]

(8)

It is also well known (see \([10, \text{Theorem 7.2}]\)) that \( R_0 \) is the principal eigenvalue of the adjoint problem of (8):

\[
\begin{cases}
-e^{-\frac{q}{d_I}x}\varphi_t - d_I(e^{-\frac{q}{d_I}x}\varphi_x)_x + e^{-\frac{q}{d_I}x}\gamma(x, t)\varphi = \frac{e^{-\frac{q}{d_I}x}\beta(x, t)}{R_0}\varphi, & 0 < x < L, \ t > 0, \\
\varphi_x(x, t) = 0, & x = 0, L, \ t > 0, \\
\varphi(x, 0) = \varphi(x, \omega), & 0 < x < L,
\end{cases}
\]
that is, one can find a function \( \phi \in C_\omega \) with \( \phi > 0 \) on \([0, L] \times \mathbb{R}\) such that \((\mathcal{R}_0, \phi)\) solves
\[
\begin{cases}
-\phi_t - d_1 \phi_{xx} - q \phi_x + \gamma(x,t) \phi = \frac{\beta(x,t)}{R_0} \phi, & 0 < x < L, \ t > 0, \\
\phi_x(x,t) = 0, & x = 0, L, \ t > 0, \\
\phi(x,0) = \phi(x,\omega), & 0 < x < L.
\end{cases}
\]

To obtain a better understanding of \( \mathcal{R}_0 \), sometimes we need to resort to the following periodic-parabolic eigenvalue problem
\[
\begin{cases}
\theta_t - d_1 \theta_{xx} - q \theta_x + \gamma(x,t) \theta = \lambda \theta, & 0 < x < L, \ t > 0, \\
\theta_x(x,t) = 0, & x = 0, L, \ t > 0, \\
\theta(x,0) = \theta(x,\omega), & 0 < x < L.
\end{cases}
\] (9)

Let \( \lambda_0 \) be the unique principal eigenvalue of (9). Then we have the following result.

**Lemma 2.2.** 1 - \( \mathcal{R}_0 \) has the same sign as \( \lambda_0 \).

**Proof.** We rewrite (9) as the following
\[
\begin{cases}
e^{\frac{\beta}{R_0} x} \theta_t - d_1 (e^{\frac{\beta}{R_0} x} \theta_x)_x = e^{\frac{\beta}{R_0} x} (\gamma(x,t) - \beta(x,t))\theta + \lambda_0 e^{\frac{\beta}{R_0} x} \theta, & 0 < x < L, \ t > 0, \\
\theta_x(x,t) = 0, & x = 0, L, \ t > 0, \\
\theta(x,0) = \theta(x,\omega), & 0 < x < L.
\end{cases}
\] (10)

Due to [10, Theorem 7.2], we know that \( \lambda_0 \) is also the principal eigenvalue of the adjoint problem of (10), that is, there exists \( \theta^* \in C_\omega \) with \( \theta^* > 0 \) on \([0, L] \times \mathbb{R}\) such that \((\lambda_0, \theta^*)\) solves
\[
\begin{cases}
-e^{\frac{\beta}{R_0} x} \theta_t^* - d_1 (e^{\frac{\beta}{R_0} x} \theta^*_x)_x \\
= e^{\frac{\beta}{R_0} x} (\beta(x,t) - \gamma(x,t))\theta^* + \lambda_0 e^{\frac{\beta}{R_0} x} \theta^*, & 0 < x < L, \ t > 0, \\
\theta^*_x(x,t) = 0, & x = 0, L, \ t > 0, \\
\theta^*(x,0) = \theta^*(x,\omega), & 0 < x < L.
\end{cases}
\] (11)

We multiply the equation (8) by \( \theta^* \) and (11) by \( \varphi \), respectively, integrate over \((0, L) \times (0, \omega)\) by parts, and then subtract the resulting equation to obtain
\[
\left(1 - \frac{1}{\mathcal{R}_0}\right) \int_0^\omega \int_0^L e^{\frac{\beta}{R_0} x} \beta \varphi_\theta^* dxdt + \lambda_0 \int_0^\omega \int_0^L e^{\frac{\beta}{R_0} x} \varphi \theta^* dxdt = 0.
\]

Since \( \int_0^\omega \int_0^L e^{\frac{\beta}{R_0} x} \beta \varphi_\theta^* dxdt \) and \( \int_0^\omega \int_0^L e^{\frac{\beta}{R_0} x} \varphi \theta^* dxdt \) are both positive, we conclude that \( 1 - \frac{1}{\mathcal{R}_0} \) and \( \lambda_0 \) have the opposite signs, which thereby deduces our result. \( \square \)

Next, we establish the threshold dynamics behavior of (2) in terms of \( \mathcal{R}_0 \). We start with the uniform bound of the solution when the initial data \((S_0, I_0)\) satisfy (A1), which is fundamental in determining the long-time behavior of solutions of (2). In fact, under hypothesis (3)(and so (4) holds), it can be concluded that for any fixed \( q \geq 0 \), \( ||S(\cdot, t)||_{L^\infty([0,L])} \) and \( ||I(\cdot, t)||_{L^\infty([0,L])} \) are also uniformly bounded in \([0, \infty)\), by following the argument in [1](see also Exercise 4 of Section 3.5 in [9]).

**Lemma 2.3.** There exists a positive constant \( C \) independent of the initial data \((S_0, I_0)\) satisfying (A1) such that for the corresponding unique solution \((S, I)\) of (2), we have
\[
||S(\cdot, t)||_{L^\infty([0,L])} + ||I(\cdot, t)||_{L^\infty([0,L])} \leq C, \quad \forall t \in [0, \infty).
\]

Now we are in a position to prove a threshold type result on the global dynamics of (2), which implies that the basic reproduction number \( \mathcal{R}_0 \) can be used to predict the extinction and persistence of the disease.
Theorem 2.4. The following statements are valid.

(i) If \( R_0 \leq 1 \), then \( (S, I) \to (S^*, 0) \) uniformly on \([0, L] \) as \( t \to \infty \).

(ii) If \( R_0 > 1 \), then (2) has at least one endemic \( \omega \)-periodic solution, and for any solution \((S, I)\) of (2) with the initial data \((S_0, I_0)\) satisfying (A1), there exists a constant \( \eta > 0 \) such that

\[
\liminf_{t \to \infty} S(x, t) \geq \eta \quad \text{and} \quad \liminf_{t \to \infty} I(x, t) \geq \eta
\]

uniformly on \([0, L] \).

Proof. (i) By the strong maximum principle [23], both \( S(x, t) \) and \( I(x, t) \) are positive for \( x \in [0, L] \) and \( t > 0 \). From Lemma 2.3, we see that \( \|S(\cdot, t)\|_{L^\infty([0, L])} \) and \( \|I(\cdot, t)\|_{L^\infty([0, L])} \) are uniformly bounded in \([0, \infty) \). Let

\[
M = \sup_{(x, t) \in [0, L] \times [0, \infty)} S(x, t).
\]

By the equation of \( I(x, t) \) in (2), we have

\[
I_t = d_I I_{xx} - q_I x + \frac{\beta(x, t) SI}{S + I} - \gamma(x, t) I
\leq d_I I_{xx} - q_I x + \left[ \frac{\beta(x, t) M}{M + I} - \gamma(x, t) \right] I.
\]

Let \( \bar{I}(x, t) \) satisfies

\[
\begin{aligned}
\bar{I}_t &= d_I \bar{I}_{xx} - q_I x + \left[ \frac{\beta(x, t) M}{M + I} - \gamma(x, t) \right] \bar{I}, \quad 0 < x < L, \ t > 0, \\
d_I \bar{I}_x - q \bar{I} &= 0, \quad x = 0, L, \ t > 0, \\
\bar{I}(x, 0) &= \bar{I}_0(x) \geq 0, \neq 0, \quad 0 < x < L.
\end{aligned}
\]

Let \( X^+ = C([0, L], \mathbb{R}_+) \) be the positive cone in the Banach space \( X = C([0, L], \mathbb{R}) \) with the usual supremum norm, and we assume that \( \bar{I}(x, 0) = \bar{I}(x, 0) \in X^+ \). Then for every initial value function \( \bar{I}_0 \in X^+ \), there exists a unique classical solution \( \bar{I}(x, t; \bar{I}_0) \) of (13) on \((0, \sigma(\bar{I}_0))\) with \( \bar{I}(x, 0; \bar{I}_0) = \bar{I}_0 \) (see, e.g., [24, Chapter 7, Theorem 3.1]).

Since \( R_0 \leq 1 \), it follows from Lemma 2.2 that \( \lambda_0 \geq 0 \), where \( \lambda_0 \) is the principal eigenvalue of (9). One can then verify that

\[
0 < \bar{I}(x, t) < ce^{-\lambda_0 t} e^{\frac{\beta(x, t) M}{M + I}} \theta(x, t), \quad 0 < x < L, \ t > 0,
\]

where \( \theta(x, t) \) is the positive eigenfunction of (9) corresponding to the principal eigenvalue \( \lambda_0 \geq 0 \), and \( c > 0 \) is a constant satisfying \( ce^{\frac{\beta(x, t) M}{M + I}} \theta(x, 0) \geq \bar{I}(x, 0) \) for every \( x \in (0, L) \). Thus, we have proven the global existence of \( \bar{I}(x, t) \), i.e., \( \sigma(\bar{I}_0) = +\infty \), since there is an \( L^\infty \)-bound on \( \bar{I}(x, t; \bar{I}_0) \). By the parabolic comparison principle, we have \( I(x, t) \leq \bar{I}(x, t) \) for every \( x \in (0, L) \) and \( t \geq 0 \).

Claim. If \( R_0 \leq 1 \), then \( \bar{I}(x, t) \to 0 \) uniformly on \([0, L] \) as \( t \to \infty \).

For given \( \bar{I}_0 \in X_+ \), let \( (\Psi(t)\bar{I}_0)(x) = \bar{I}(x, t; \bar{I}_0) \) be the unique solution of (13). Since \( \sigma(\bar{I}_0) = +\infty \) for all \( \bar{I}_0 \in X_+ \), then \( \Psi(t) \) is a continuous-time semiflow on \( X^+ \). We define the Poincaré map \( \tilde{T} : X^+ \to X^+ \) by \( \tilde{T}(\bar{I}_0) = \Psi(\omega)(\bar{I}_0) \). By an argument similar to that of [10, Proposition 21.2], it follows that \( \tilde{T} : X^+ \to X^+ \) is continuous and strongly monotone. Moreover, \( \tilde{T} \) maps any order interval in \( X^+ \) to a precompact set in \( X^+ \). Clearly, \( \tilde{T}(0) = 0 \).
Linearizing (13) at \( \bar{I} = 0 \), we obtain
\[
\begin{cases}
\dot{I}_x = d_I I_x - q I_x + [\beta(x, t) - \gamma(x, t)] I, & 0 < x < L, \ t > 0, \\
\dot{I}_x = -q I_x = 0, & x = 0, L, \ t > 0.
\end{cases}
\] (14)

According to [10, Chapter II], (14) admits an evolution operator \( \mathcal{U}(t, \tau) \), \( 0 \leq \tau \leq t \leq \omega \), and for any \( 0 \leq \tau \leq t \leq \omega \), \( \mathcal{U}(t, \tau) \) is a compact and strongly positive operator on \( X^+ \). By [10, Proposition 23.1], the Poincaré map \( \mathcal{T} \) associated with (13) is defined in a neighborhood of \( \bar{I}_0 = \bar{I}(x, 0) \) and Fréchet differentiable at \( \bar{I}_0 \), with \( D\mathcal{T}(\bar{I}_0) = \mathcal{U}(\omega, 0) \). Let \( r = r(D\mathcal{T}(\bar{I}_0)) \). Then by [10, Proposition 14.4], \( \lambda_0 = -\frac{1}{\omega} \ln r \) is the unique principal eigenvalue of the periodic-parabolic eigenvalue problem (9). Since \( R_0 \leq 1 \), then \( \lambda_0 \geq 0 \). Thus \( r \leq 1 \).

Let \( f(x, t, \bar{I}) = \left( \frac{\beta(x, t) M}{M + I} - \gamma(x, t) \right) \bar{I} \). Then we have \( f(x, t, \bar{I}) < \frac{\beta f(x, t, 0)}{0} \cdot \bar{I} \), \( \forall (x, t) \in [0, L] \times \mathbb{R}, \bar{I} > 0 \), which implies \( \mathcal{T}(\bar{I}) < D\mathcal{T}(0) \bar{I}, \forall \bar{I} \in X^+ \) with \( \bar{I} \gg 0 \). Therefore, by [28, Theorem 2.2.2], we obtain that \( I(x, t) = 0 \) is globally attractive for \( \mathcal{T} \) in \( X^+ \), and the claim follows.

Since \( I(x, t) \leq I(x, t) \), then \( I(x, t) \to 0 \) uniformly on \( [0, L] \) as \( t \to \infty \). Next, we show that \( S(x, t) \to S^* \) as \( t \to \infty \). By the equation of \( S(x, t) \) in (2), we have
\[
S_t - d SS_{xx} + q S_x = \left[ \gamma(x, t) - \frac{\beta(x, t) S}{S + I} \right] I \leq \gamma(x, t) I, \quad 0 < x < L, \ t > 0.
\]

By the continuity of \( \gamma(x, t) \) on \( [0, L] \times [0, \omega] \) and the above argument that \( I(x, t) \to 0 \) uniformly on \( [0, L] \) as \( t \to \infty \), we have
\[
S_t - d SS_{xx} + q S_x \to 0 \quad \text{as} \quad t \to \infty.
\]

Thus for \( \forall \{t_j\}, t_j \to +\infty \), let \( S_j(x, t) = \lim_{j \to +\infty} S(x, t + t_j) \). Then \( S_j(x, t) \) satisfies
\[
\begin{cases}
(S_j)_t - d SS_{jxx} + q S_j_x = 0, & 0 < x < L, \ t > 0, \\
d S_j(x, t + q S_j)_x - q S_j = 0, & x = 0, L, \ t > 0, \\
\int_0^L S_j(x, t)dx = N, & t \geq 0.
\end{cases}
\]

We obtain that \( S_j(x, t) = \frac{q Ne^{\theta x} e^{-q S_j}}{d e^{\theta x} - q S_j - 1} = S^* \). Then we have \( \lim_{t \to +\infty} S_j(x, t) = S^* \) for each \( j \in \mathbb{R} \). Hence, \( S(x, t) \to S^* \) as \( t \to \infty \). This completes the proof of assertion (i).

(ii) We appeal to the theory of uniform persistence and coexistence states developed in [17, 27] for periodic semiflow to prove (ii). We denote
\[
X_0 = \left\{ (u, v) \in X \times X : \int_0^L (u + v)dx = N \right\} \quad \text{and} \quad U = (X^+ \times X^+) \cap X_0.
\]

By the standard regularity theory for parabolic equations and the property (4), for every \( (S_0, I_0) \in U \), (2) admits a unique solution \( \psi(t, (S_0, I_0)) = (S, I) \in U \), which exists for any \( t \geq 0 \).

We now define an \( \omega \)-periodic semiflow \( \Phi(t) : U \to U \) by
\[
\Phi(t)(S_0, I_0) = \psi(t, (S_0, I_0)), \quad (S_0, I_0) \in U, \ t \geq 0.
\]

Let \( U_0 := \{(S_0, I_0) \in U : I_0 \neq 0 \} \) and \( \partial U_0 := \{(S_0, I_0) \in U : I_0 = 0 \} \). Then \( U = U_0 \cup \partial U_0 \), \( U_0 \) and \( \partial U_0 \) are relatively open and closed subsets of \( U \), respectively, and \( U_0 \) is convex. Clearly, \( \Phi(t)U_0 \subset U_0 \) and \( \Phi(t)\partial U_0 \subset \partial U_0 \) for all \( t \geq 0 \). Hence, by means of Lemma 2.3, we can apply a similar argument to the proof of [10, Proposition 21.2] to conclude that \( \Phi(t) : U \to U \) is continuous and compact for any \( t > 0 \). Actually, with the help of Lemma 2.3, for \( t \geq 0 \), \( \Phi(t) \) is point dissipative.
in $X \times X$, i.e., there exists a constant $C_0 > 0$ such that for any $(S_0, I_0) \in U$, there exists $t_0 = t_0(S_0, I_0) > 0$ such that $\Phi(t)(S_0, I_0)$ satisfies $\|\Phi(t)(S_0, I_0)\|_{X \times X} = \|S(\cdot, t)\|_X + \|I(\cdot, t)\|_X \leq C_0$ for $t \geq t_0$.

We first prove the uniform persistence of the Poincaré map $T : U \to U$ defined by $T(S_0, I_0) = \Phi(\omega)(S_0, I_0), (S_0, I_0) \in U$. With the above argument, it is easy to see that $T : U \to U$ is a continuous, point dissipative and compact map with $T(U_0) \subset U_0$ and $T(\partial U_0) \subset \partial U_0$. By [8, Theorem 2.4.7], $T : U \to U$ has a global attractor. Denote by $\omega(S_0, I_0)$ the associated omega-limit set of $(S_0, I_0) \in U$. When $(S_0, I_0) \in \partial U_0$, we know that $I(x, t) \equiv 0$, and so $S(x, t)$ satisfies the following:

\[
\begin{cases}
S_t - dSS_{xx} + qS_x = 0, & 0 < x < L, \ t > 0,
S_x = 0, & x = 0, L, \ t > 0,
S(x, 0) = S_0(x) \geq 0, & 0 < x < L,
\int_0^L S(x, t)dx = N, & t \geq 0.
\end{cases}
\]

As a consequence, we can use the same analysis as in the proof of [4, Lemma 2.4] to conclude that $S(x, t) \to S^*$ in $C([0, L])$ as $t \to \infty$. This therefore implies that $\cup_{(S_0, I_0)\in \partial U_0}\omega(S_0, I_0) = \{(S^*, 0)\}$, which is the disease-free solution. For simplicity, we denote $Q = (S^*, 0)$. Let $A_0$ be the maximal positively invariant set for $T$ in $\partial U_0$. It then follows that $\hat{A}_0 = \cup_{(S_0, I_0)\in A_0}\omega(S_0, I_0) = \{Q\}$, and $\{Q\}$ is a compact and isolated invariant set for $T$ restricted in $\hat{A}_0$. We further have the following claim.

**Claim.** There exists a real number $\delta > 0$ such that

$$\limsup_{n \to \infty} \|T^n(S_0, I_0) - Q\|_\infty \geq \delta, \quad \forall (S_0, I_0) \in U_0.$$  

It suffices to prove that there exist $\delta_0 > 0$ such that for any $(S_0, I_0) \in B(Q, \delta_0) \cap U_0$, where $B(Q, \delta_0)$ is the $\delta_0$-neighborhood of $Q$, there exists $n = n_0 \geq 1$ such that $T^n((S_0, I_0)) \not\in B(Q, \delta_0)$.

Indeed, let $\lambda_0$ be defined as before. Under the assumption $R_0 > 1$, Lemma 2.2 gives $\lambda_0 < 0$. Consequently, by the continuity of the principal eigenvalue $\lambda_0$ with respect to the function $\beta(x, t)$ in (9), we can choose a small constant $\epsilon_0$ such that $\lambda_0(\epsilon_0) < 0$ and $0 < \epsilon_0 < -\lambda_0(\epsilon_0)$, where $\lambda_0(\epsilon_0)$ is the unique principal eigenvalue of the periodic-parabolic problem

\[
\begin{cases}
\theta_t - d^2\theta_{xx} - q\theta_x = \frac{\beta(x, t)(S^* - \epsilon_0)}{S^* + 2\epsilon_0}\theta - \gamma(x, t)\theta + \lambda\theta, & 0 < x < L, \ t > 0,
\theta_x(x, t) = 0, & x = 0, L, \ t > 0,
\theta(x, 0) = \theta(x, \omega), & 0 < x < L.
\end{cases}
\]

We now fix such chosen $\epsilon_0$. According to the continuous dependence of solution on the initial data, after some regularity argument, we observe that

$$\lim_{(S_0, I_0) \to Q} \Phi(t)(S_0, I_0) = \lim_{(S, I) \to Q} (S(\cdot, t), I(\cdot, t)) = Q$$

in $X \times X$ uniformly for $t \in [0, \omega]$. Then we can find a neighborhood of $Q$ in $X \times X$, say $B(Q, \delta_0)$ with the radius $\delta_0$ satisfying $0 < \delta_0 \leq \delta_0 < \epsilon_0$, such that $\|S(\cdot, t) - S^*\|_{C([0, L])} + \|I(\cdot, t)\|_{C([0, L])} < \epsilon_0$, for any $(S_0, I_0) \in B(Q, \delta_0)$.

Assume that, by contradiction, there exists $(\tilde{S}_0, \tilde{I}_0) \in B(Q, \delta_0) \cap U_0$, such that for all $n \geq 1$, $T^n(\tilde{S}_0, \tilde{I}_0) = \psi(n\omega, (\tilde{S}_0, \tilde{I}_0)) \in B(Q, \delta_0)$. For any $t \geq 0$, let $t = n\omega + t'$ with $t' \in [0, \omega)$ and $n = \lfloor t/\omega \rfloor$ being the integer part of $t/\omega$. Note that $(\tilde{S}(\cdot, t), \tilde{I}(\cdot, t)) = \psi(t, (\tilde{S}_0, \tilde{I}_0)) = \psi(t, \psi(n\omega, (\tilde{S}_0, \tilde{I}_0)))$. Thus, we have

$$\|\tilde{S}(\cdot, t) - S^*\|_{C([0, L])} + \|\tilde{I}(\cdot, t)\|_{C([0, L])} < \epsilon_0, \quad \forall t \in [0, \omega).$$  
\[ (16) \]
Let \( \theta_0 \) be a positive eigenvector corresponding to \( \lambda_0(\epsilon_0) \) in (15). Clearly, \( \theta_0 \gg 0 \) on \([0, L] \times \mathbb{R} \). In particular, \( \theta_0(\cdot, 0) \in \text{int} (X^+) \). On the other hand, as \((\hat{S}_0, \hat{I}_0) \in U_0 \), the strong maximum principle for parabolic equations shows \( \hat{S}(\cdot, t) \gg 0 \) and \( \hat{I}(\cdot, t) \gg 0 \) in \( X^+ \) for all \( t > 0 \). Therefore, without loss of generality, we can assume that \((\hat{S}_0, \hat{I}_0) \in \text{int} (X^+) \times \text{int} (X^+) \). Then there exists \( c^* > 0 \) such that \( \hat{I}_0 \geq c^* e^{\frac{\hat{\beta}}{S^+} x} \theta_0 \) on \([0, L] \). By means of (16) and the choice of \( \epsilon_0 \), we find that \( \hat{I}(x, t) \) is a supersolution to the problem

\[
\begin{cases}
  w_t - dw_{xx} + qw_x = \frac{\beta(x,t)(S^+ - \epsilon_0)}{S^+ + 2\epsilon_0} w - \gamma(x,t)w, & 0 < x < L, \ t > 0, \\
  dwx - qw = 0, & x = 0, \ t > 0, \\
  w(x, 0) = c^* \varphi_0(x, 0), & 0 < x < L.
\end{cases}
\]

Furthermore, it is easily checked that \( c^* e^{-\lambda_0(\epsilon_0) t} e^{\frac{\hat{\beta}}{S^+} x} \theta_0(x, t) \) is the unique solution to (17). By the parabolic comparison principle, we deduce

\[
\hat{I}(x, t) \geq c^* e^{-\lambda_0(\epsilon_0) t} e^{\frac{\hat{\beta}}{S^+} x} \theta_0(x, t) \to \infty \quad \text{uniformly for } x \in [0, L], \quad \text{as } t \to \infty,
\]

which is in contradiction with Lemma 2.3. This proves our claim.

The above claim implies that \( Q \) is an isolated invariant set for \( T \) in \( U \), and \( W^*(Q) \cap U_0 = \emptyset \), where \( W^*(Q) \) is the stable set of \( Q \) for \( T \). As a result, [27, Theorem 2.2] (see also [28, Theorem 1.3.1 and Remark 1.3.1]) asserts that \( T \) is uniformly persistent with respect to \((U, \partial U_0)\). Furthermore, [27, Theorem 2.3] (see also [17, Theorem 4.5]) implies that \( T \) has a fixed point \( \phi^* \) in \( U_0 \), and hence, (2) has an \( \omega \)-periodic solution \( \Phi(t)\phi^* \) in \( U_0 \). In view of [17, Theorem 4.5], we see that \( T : U_0 \to U_0 \) has a compact global attractor \( A_0 \). Clearly, \( \phi^* \in A_0 \). Let \( B_0 \equiv \cup_{t \in [0, \omega]} \Phi(t)A_0 \). It then follows that \( B_0 \subset U_0 \) and \( \lim_{t \to \infty} d(\Phi(t)\phi, B_0) = 0 \) for all \( \phi \in U_0 \), where \( d \) is the norm-induced distance in \( X \times X \). Since \( A_0 \subset U_0 \) and \( A_0 = T(A_0) = \Phi(\omega)A_0 \), we yield \( A_0 \subset \text{int}(P_1) \times \text{int}(P_2) \), and hence \( B_0 \subset \text{int}(X^+) \times \text{int}(X^+) \). Obviously, \( \Phi(t)\phi^* \in B_0 \), and so \( \Phi(t)\phi^* \) is an endemic \( \omega \)-periodic solution of (2). By virtue of the compactness and global attractiveness of \( B_0 \) for \( \Phi(t) \) in \( U_0 \), we can conclude that there exists \( \eta > 0 \) such that \( \lim \inf_{t \to \infty} \Phi(t)\phi \geq (\eta, \eta) \) for all \( \phi \in U_0 \), which implies the persistence statement (12). The proof is complete.

3. Qualitative properties of \( \mathcal{R}_0 \). In this section, we will present some quantitative properties for the basic reproduction number \( \mathcal{R}_0 \). First of all, when \( \beta(x, t) - \gamma(x, t) \) or both \( \beta(x, t) \) and \( \gamma(x, t) \) are spatially homogeneous, we have:

**Lemma 3.1.** The following assertions hold.

(i) If \( \beta(x, t) \equiv \beta(t) \) and \( \gamma(x, t) \equiv \gamma(t) \), then \( \mathcal{R}_0 = \frac{\int_0^\infty \beta(t) dt}{\int_0^\infty \gamma(t) dt} \).

(ii) If \( \beta(x, t) - \gamma(x, t) \equiv h(t) \), then \( \mathcal{R}_0 > 1 \) if \( \int_0^\infty h(t) dt > 0 \), \( \mathcal{R}_0 = 1 \) if \( \int_0^\infty h(t) dt = 0 \), \( \mathcal{R}_0 < 1 \) if \( \int_0^\infty h(t) dt < 0 \).

The proof of Lemma 3.1 is similar to that of [21, Lemma 2.3]. Secondly, if \( \beta(x, t) - \gamma(x, t) \) or both \( \beta(x, t) \) and \( \gamma(x, t) \) depend on the spatial variable alone, we have the following results:

**Lemma 3.2.** The following assertions hold.

(i) Assume that \( q \equiv 0 \) and \( \beta(x, t) - \gamma(x, t) \equiv h(x) \).

   (i-1) If \( \int_0^L h(x) dx \geq 0 \) and \( h \not\equiv 0 \) in \((0, L)\), then \( \mathcal{R}_0 > 1 \) for all \( d_I \);

   (i-2) If \( \int_0^L h(x) dx < 0 \) and \( h \leq 0 \) on \([0, L] \), then \( \mathcal{R}_0 < 1 \) for all \( d_I \);
Theorem 1.2. From now on, we always assume that $\beta$ and temporal factors. The subsequent results present some analytical properties of

\[ R_0 = \sup_{\varphi \in H^1(0,L), \varphi \neq 0} \left\{ \frac{\int_0^L \beta(x)\varphi'^2 \, dx}{d_1 \int_0^L \varphi'^2 \, dx + \int_0^L \gamma(x)\varphi'^2 \, dx} \right\} \]

and $R_0$ is a nonincreasing function of $d_1$ with $R_0 \to \max_{x \in [0,L]} \frac{\beta(x)}{\gamma(x)}$ as $d_1 \to 0$, and $R_0 \to \int_0^L \beta(x) \, dx \int_0^L \gamma(x) \, dx$ as $d_1 \to \infty$.

(ii) Assume that $q > 0$ and $\beta(x,t) \equiv \beta(x), \gamma(x,t) \equiv \gamma(x)$, we have

\[ R_0 = \sup_{\varphi \in H^1(0,L), \varphi \neq 0} \left\{ \frac{\int_0^L \beta(x) e^{\frac{q}{\gamma(x)}t} \varphi'^2 \, dx}{d_1 \int_0^L e^{\frac{q}{\gamma(x)}t} \varphi'^2 \, dx + \int_0^L \gamma(x) e^{\frac{q}{\gamma(x)}t} \varphi'^2 \, dx} \right\}, \quad (18) \]

and the following statements about $R_0$ hold:

(iii-1) For any $q > 0$, there are $R_0 \to \frac{\beta(L)}{\gamma(L)}$ as $d_1 \to 0$ and $R_0 \to \int_0^L \beta(x) \, dx \int_0^L \gamma(x) \, dx$ as $d_1 \to \infty$;

(iii-2) For any $d_1 > 0$, there are $R_0 \to \tilde{R}_0$ as $q \to 0$ and $R_0 \to \frac{\beta(L)}{\gamma(L)}$ as $q \to \infty$, where $\tilde{R}_0$ is the basic reproduction number when $q = 0$ in (18);

(iii-3) If $\beta(x) > (\gamma)\gamma(x)$ on $[0,L]$, then $R_0 > (\gamma)1$ for any $d_1 > 0$ and $q > 0$.

The result (i) of Lemma 3.2 appears in [21, Lemma 2.4], and (ii) is given in [4, Theorem 1.2]. From now on, we always assume that $\beta$ and $\gamma$ depend on both spatial and temporal factors. The subsequent results present some analytical properties of $R_0$ for the general case of $\beta(x,t)$ and $\gamma(x,t)$.

Lemma 3.3. The following statements about $R_0$ hold.

(i) $R_0 \geq \frac{\int_0^L \beta(x) \, e^{\frac{q}{\gamma(x)}t} \, dx \, dt}{\int_0^L \gamma(x) \, e^{\frac{q}{\gamma(x)}t} \, dx \, dt}$ for all $d_1 > 0$ and $q > 0$, and the equality holds if and only if $\frac{\int_0^L \beta(x) \, dx \, dt}{\int_0^L \gamma(x) \, dx \, dt}$ is a constant and $\frac{\beta(x)}{\gamma(x)} - \frac{\gamma(x)}{\gamma(x)} \text{ depends only on the variable } t$ (in such a case, $R_0 = \frac{\int_0^L \beta(x) \, dx \, dt}{\int_0^L \gamma(x) \, dx \, dt}$);

(ii) $R_0 < 1$ for all $d_1 > 0$ and $q > 0$ if $\int_0^L \max_{x \in [0,L]} |\beta(x,t) - \gamma(x,t)| \, dt \leq 0$ and $\beta(x,t) - \gamma(x,t)$ nontrivially depends on the spatial variable $x$.

Proof. We use similar arguments to those in [21, Theorem 2.5] to obtain our assertions with some necessary modifications.

(i) Since $(R_0, \varphi)$ satisfies (8), and $\varphi > 0$ on $[0,L] \times [0,\omega]$, we divide the equation (8) by $\varphi$ and integrate the resulting equation over $(0,L) \times (0,\omega)$ to derive

\[-d_1 \int_0^\omega \int_0^L e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt + \int_0^\omega \int_0^L \gamma(x,t) e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt = \frac{1}{R_0} \int_0^\omega \int_0^L \beta(x,t) e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt,\]

which therefore implies

\[ R_0 \geq \frac{\int_0^\omega \int_0^L \beta(x,t) e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt}{\int_0^\omega \int_0^L \gamma(x,t) e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt}. \]

Clearly, the above equality holds if and only if

\[ \int_0^\omega \int_0^L e^{\frac{q}{\gamma(x,t)}t} \varphi'^2 \, dx \, dt = 0, \]
that is, \( \varphi_x \equiv 0 \) and equivalently, \( \varphi(x, t) \equiv \varphi(t) \). Hence, (7) becomes equivalent to

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d\varphi}{dt} + \gamma(x, t)\varphi = \frac{\beta(x, t)}{R_0} \varphi, \quad t > 0, \\
\varphi(0) = \varphi(\omega).
\end{array} \right.
\end{aligned}
\]

Then it is easy to see that

\[
\mathcal{R}_0 = \frac{\int_0^\omega \beta(x, t)dt}{\int_0^\omega \gamma(x, t)dt}, \quad \forall x \in [0, L],
\]

and \( \frac{\beta(x, t)}{R_0} - \gamma(x, t) \) depends only on the variable \( t \). This equivalently means that \( \frac{\int_0^\omega \beta(x, t)dt}{\int_0^\omega \gamma(x, t)dt} \) must be a constant and

\[
\frac{\beta(x, t)}{\int_0^\omega \beta(x, t)dt} - \frac{\gamma(x, t)}{\int_0^\omega \gamma(x, t)dt}
\]

depends only on \( t \).

(ii) Let \( \lambda_0 \) be defined as before. By taking \( m(x, t) = \beta(x, t) - \gamma(x, t) \) and \( \lambda = 1 \) in [10, Lemma 15.6], we have \( \mu(0) = 0 \) and

\[
\lambda_0 = \mu(1) \geq -\frac{1}{\omega} \int_0^\omega \max_{x \in [0, L]} [\beta(x, t) - \gamma(x, t)] dt \geq 0
\]

under our hypothesis, and the inequalities are strict if \( \beta(x, t) - \gamma(x, t) \) depends nontrivially on \( x \in (0, L) \). This, combined with Lemma 2.2, implies the assertion (ii).

From (7), it can be seen that \( \mathcal{R}_0 \) is a smooth function of \( d_I \) and \( q \). If the advection rate \( q = 0 \), we denote the basic reproduction number by \( \overline{\mathcal{R}}_0 \), so \( \overline{\mathcal{R}}_0 \) is a smooth function of \( d_I \) only. The basic reproduction number \( \overline{\mathcal{R}}_0 \) was introduced in [21], where \( \overline{\mathcal{R}}_0 \) is a threshold value for the global dynamics. Next, by using Propositions 1, 2 and 3 in Appendix, we immediately establish the limiting behaviors of \( \mathcal{R}_0 \) as follows.

**Lemma 3.4.** The following statements about \( \mathcal{R}_0 \) hold.

(i) For any given \( q > 0 \), we have

\[
\lim_{d_I \to 0} \mathcal{R}_0 = \frac{\int_0^\omega \beta(L, t)dt}{\int_0^\omega \gamma(L, t)dt}, \quad \text{and} \quad \lim_{d_I \to \infty} \mathcal{R}_0 = \frac{\int_0^\omega \int_0^L \beta(x, t)dxdt}{\int_0^\omega \int_0^L \gamma(x, t)dxdt}.
\]

(ii) For any given \( d_I > 0 \), we have

\[
\lim_{q \to 0} \mathcal{R}_0 = \overline{\mathcal{R}}_0, \quad \text{and} \quad \lim_{q \to \infty} \mathcal{R}_0 = \frac{\int_0^\omega \beta(L, t)dt}{\int_0^\omega \gamma(L, t)dt},
\]

where \( \overline{\mathcal{R}}_0 = \overline{\mu}_0 \) is the principal eigenvalue of (6) corresponding to \( q = 0 \).

Let \( \gamma(x, t) - \beta(x, t) = V(x, t) \). We see that \( V(x, t) \) may change sign with respect to \( x \) or \( t \). In particular, if \( V(x, t) \) stays the same sign, we have the following results.

**Lemma 3.5.** The following statements about \( \mathcal{R}_0 \) hold.

(i) If \( \beta(x, t) < \gamma(x, t) \) on \([0, L] \times [0, \omega]\), then \( \mathcal{R}_0 < 1 \) for any \( d_I > 0 \) and \( q > 0 \);

(ii) If \( \beta(x, t) > \gamma(x, t) \) on \([0, L] \times [0, \omega]\), then \( \mathcal{R}_0 > 1 \) for any \( d_I > 0 \) and \( q > 0 \);

(iii) If \( \beta(x, t) = \gamma(x, t) \) on \([0, L] \times [0, \omega]\), then \( \mathcal{R}_0 = 1 \) for any \( d_I > 0 \) and \( q > 0 \).
Proof. (i) We subtract both sides of (7) by \( \beta(x,t) \), multiply by \( e^{(q/d_t)x} \varphi \), and integrate by parts over \((0, L) \times (0, \omega)\) to obtain
\[
d_t \int_0^L \int_0^\omega e^{\frac{q}{d_t}x} \varphi_x^2 dx dt + \int_0^L \int_0^\omega [\gamma(x,t) - \beta(x,t)] e^{\frac{q}{d_t}x} \varphi^2 dx dt = \left( \frac{1}{R_0} - 1 \right) \int_0^L \int_0^\omega \beta(x,t) e^{\frac{q}{d_t}x} \varphi^2 dx dt.
\]
Since \( \beta(x,t) < \gamma(x,t) \) on \([0, L] \times [0, \omega]\), we have
\[
\left( \frac{1}{R_0} - 1 \right) \int_0^L \int_0^\omega \beta(x,t) e^{\frac{q}{d_t}x} \varphi^2 dx dt \geq \int_0^L \int_0^\omega [\gamma(x,t) - \beta(x,t)] e^{\frac{q}{d_t}x} \varphi^2 dx dt > 0,
\]
which implies that \( R_0 < 1 \).

(ii) We subtract both sides of (8) by \( e^{\frac{q}{d_t}x} \beta(x,t) \), divide by \( \varphi \), and integrate by parts over \((0, L) \times (0, \omega)\) to obtain
\[
d_t \int_0^L \int_0^\omega \beta(x,t) e^{\frac{q}{d_t}x} \varphi_x^2 dx dt + \int_0^L \int_0^\omega [\gamma(x,t) - \beta(x,t)] e^{\frac{q}{d_t}x} \varphi^2 dx dt = \left( \frac{1}{R_0} - 1 \right) \int_0^L \int_0^\omega \beta(x,t) e^{\frac{q}{d_t}x} \varphi^2 dx dt.
\]
Since \( \beta(x,t) > \gamma(x,t) \) on \([0, L] \times [0, \omega]\), then
\[
\left( \frac{1}{R_0} - 1 \right) \int_0^L \int_0^\omega \beta(x,t) e^{\frac{q}{d_t}x} \varphi_x^2 dx dt \leq \int_0^L \int_0^\omega [\gamma(x,t) - \beta(x,t)] e^{\frac{q}{d_t}x} \varphi^2 dx dt < 0,
\]
which implies that \( R_0 > 1 \).

(iii) By virtue of statements (i) and (ii), we can easily see that (iii) holds. \(\square\)

4. Special case: \( \gamma(x,t) - \beta(x,t) \) is monotone with respect to spatial variable \( x \). In general, the principal eigenvalue \( \lambda_0 \) of problem (9) does not enjoy any monotonicity property on either \( d_t \) or \( q \). In this section, we investigate the monotonicity results of \( \lambda_0 \) on \( d_t \) and \( q \) when \( \gamma(\cdot, t), \beta(\cdot, t) \in C^1([0, L]) \) and \( \gamma(x,t) - \beta(x,t) \) is monotone with respect to spatial variable \( x \). Precisely, let \( \gamma(x,t) - \beta(x,t) = V(x,t) \), we consider \( V(x,t) \) satisfying one of the following assumptions:

(C1) \( V(x,t) \) is monotone increasing with respect to \( x \), i.e., \( V_x(x,t) \geq 0, \neq 0 \);

(C2) \( V(x,t) \) is monotone decreasing with respect to \( x \), i.e., \( V_x(x,t) \leq 0, \neq 0 \).

We remark that it is biologically reasonable for the assumptions of \( \gamma(x,t) - \beta(x,t) \). Due to the spatial heterogeneity of environment and movements of individuals, the rate of disease transmission \( \beta(x,t) \) and the rate of disease recovery \( \gamma(x,t) \) will change with spatial locations, and the spatial change rate of them maybe different. Assumption (C1) implies that for any time, the spatial change rate of disease recovery \( \gamma_x(x,t) \) is greater than the spatial change rate of disease transmission \( \beta_x(x,t) \), while assumption (C2) means that the spatial change rate of disease recovery \( \gamma_x(x,t) \) is less than the spatial change rate of disease transmission \( \beta_x(x,t) \).

4.1. Monotonicity of \( \lambda_0 \) in \( d_t \) and \( q \). In this subsection, we explore the monotonicity of the principal eigenvalue \( \lambda_0 \) with respect to \( d_t \) and \( q \). Note that \( \gamma(x,t) - \beta(x,t) = V(x,t) \), then the eigenvalue problem (9) reads as
\[
\begin{cases}
\theta_t - d_t \theta_{xx} - q \theta_x + V(x,t) \theta = \lambda \theta, & 0 < x < L, \ t > 0, \\
\theta_x(x, t) = 0, & x = 0, L, \ t > 0, \\
\theta(x, 0) = \theta(x, \omega), & 0 < x < L,
\end{cases}
\tag{19}
\]
which can be further written as the following
\begin{equation}
\begin{cases}
  e^{\frac{d_f}{d_I} t} \theta_t - d_I \left( e^{\frac{d_f}{d_I} x} \theta_x \right)_x + e^{\frac{d_f}{d_I} x} V(x, t) \theta = \lambda e^{\frac{d_f}{d_I} t} \theta, & 0 < x < L, t > 0, \\
  \theta_x(x, t) = 0, & x = 0, L, t > 0, \\
  \theta(x, 0) = \theta(x, \omega), & 0 < x < L.
\end{cases}
\end{equation}

(20)

According to [10], one knows that the adjoint problem of (20):
\begin{equation}
\begin{cases}
  -e^{\frac{d_f}{d_I} x} \psi_t - d_I \left( e^{\frac{d_f}{d_I} x} \psi_x \right)_x + e^{\frac{d_f}{d_I} x} V(x, t) \psi = \lambda e^{\frac{d_f}{d_I} x} \psi, & 0 < x < L, t > 0, \\
  \psi_x(x, t) = 0, & x = 0, L, t > 0, \\
  \psi(x, 0) = \psi(x, \omega), & 0 < x < L.
\end{cases}
\end{equation}

(21)

namely,
\begin{equation}
\begin{cases}
  -\psi_t - d_I \psi_{xx} - q \psi_x + V(x, t) \psi = \lambda \psi, & 0 < x < L, t > 0, \\
  \psi_x(x, t) = 0, & x = 0, L, t > 0, \\
  \psi(x, 0) = \psi(x, \omega), & 0 < x < L
\end{cases}
\end{equation}

has the same principal eigenvalue \( \lambda_0 \) with a principal eigenfunction \( \psi \in C^{2,1}([0, L] \times [0, \omega]) \).

Lemma 4.1. For any given \( d_I, q > 0 \), the following assertions hold.

(i) If assumption (C1) holds, then the principal eigenfunction \( \theta(x, t) \) of (19) satisfies \( \theta_x(x, t) < 0 \) in \((0, L)\) for all \( t \);
(ii) If assumption (C2) holds, then the principal eigenfunction \( \theta(x, t) \) of (19) satisfies \( \theta_x(x, t) > 0 \) in \((0, L)\) for all \( t \).

Lemma 4.1 plays an important role in establishing the monotonicity of the principal eigenvalue \( \lambda_0 \) in \( d_I \) and \( q \) in this subsection, and the proof is similar to [22, Lemma 4.1], here we omit it.

Lemma 4.2. For any given \( d_I > 0 \), the following assertions hold.

(i) If assumption (C1) holds, then \( \lambda_0 \) is strictly monotone increasing in \( q \);
(ii) If assumption (C2) holds, then \( \lambda_0 \) is strictly monotone decreasing in \( q \).

Proof. It is well known (see, e.g.,[12]) that \( \lambda_0 \) and the associated principal eigenfunction \( \theta \) are \( C^1 \)-function of \( q \). For notational simplicity, we denote \( \frac{\partial \lambda_0}{\partial q} \) by \( \theta' \) and \( \frac{\partial \lambda_0}{\partial \theta} \) by \( \lambda'_0 \). Then, we differentiate (19) with respect to \( q \) to obtain
\begin{equation}
\begin{cases}
  \theta'_t - d_I \theta'_{xx} - q \theta'_x - \theta_x + V(x, t) \theta' = \lambda'_0 \theta + \lambda_0 \theta', & 0 < x < L, t > 0, \\
  \theta'_x(x, t) = 0, & x = 0, L, t > 0, \\
  \theta'(x, 0) = \theta'(x, \omega), & 0 < x < L.
\end{cases}
\end{equation}

(22)

We rewrite (22) as
\begin{equation}
\begin{cases}
  (e^{\frac{d_f}{d_I} x} \theta'_t)_t - d_I (e^{\frac{d_f}{d_I} x} \theta'_x)_x - e^{\frac{d_f}{d_I} x} \theta'_x + e^{\frac{d_f}{d_I} x} V(x, t) \theta' \\
  = e^{\frac{d_f}{d_I} x} \left( \lambda'_0 \theta + \lambda_0 \theta' \right), & 0 < x < L, t > 0, \\
  \theta'_x(x, t) = 0, & x = 0, L, t > 0, \\
  \theta'(x, 0) = \theta'(x, \omega), & 0 < x < L.
\end{cases}
\end{equation}

(23)

Let \( \psi \) be the principal eigenfunction corresponding to \( \lambda_0 \) in (21). We now multiply the equation of (23) by \( \psi \) and integrate the resulting equation over \((0, L) \times (0, \omega)\).
In view of the equation of $\psi$, Clearly, $\lambda$ is monotone increasing in $\lambda$.

From (24) and (25), it immediately follows that

$$\lambda^0 = \frac{\partial \lambda_0}{\partial q} = -\frac{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt}{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt}.$$  \hspace{1cm} (24)

In view of the equation of $\psi$ in (21), we find

$$-\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta' \psi dxdt$$

$$= -d_I \int_0^\omega \int_0^L e^{\frac{V}{d}} \theta_x \psi dxdt + \int_0^\omega \int_0^L e^{\frac{V}{d}} [-V(x,t) + \lambda_0] \theta' \psi dxdt. \hspace{1cm} (25)$$

From (24) and (25), it immediately follows that

$$\lambda^0 = \frac{\partial \lambda_0}{\partial q} = -\frac{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt}{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt}.$$  \hspace{1cm} (24)

Thanks to Lemma 4.1 and the positivity of $\theta$ and $\psi$, if (C1) holds, we have $\lambda^0 > 0$, while if (C2) holds, we have $\lambda^0 < 0$. So our conclusions are established.

**Lemma 4.3.** For any given $q > 0$, if assumption (C2) holds, then $\lambda_0$ is strictly monotone increasing in $d_I$.

**Proof.** Clearly, $\lambda_0$ and the associated principal eigenfunction $\theta$ are $C^1$-function of $d_I$ (see, e.g., [12]). For simplicity, we denote $\frac{\partial \theta}{\partial d_I}$ by $\theta'$ and $\frac{\partial \lambda_0}{\partial q}$ by $\lambda^0$. Then, differentiating (19) with respect to $d_I$, we have

$$\begin{cases}
\theta' - d_I \theta_{xx} - q \theta' - \theta_{xx} + V(x,t) \theta' = \lambda_0 \theta + \lambda_0 \theta', & 0 < x < L, t > 0, \\
\theta'_x(x,t) = 0, & x = 0, L, t > 0, \\
\theta(x,0) = \theta(x,\omega), & 0 < x < L,
\end{cases}$$

which can be further rewritten as

$$\begin{cases}
(e^{\frac{V}{d}} \theta')_x - d_I (e^{\frac{V}{d}} \theta')_x - e^{\frac{V}{d}} \theta_{xx} + e^{\frac{V}{d}} V(x,t) \theta' = e^{\frac{V}{d}} (\lambda_0 \theta + \lambda_0 \theta'), & 0 < x < L, t > 0, \\
\theta'_x(x,t) = 0, & x = 0, L, t > 0, \\
\theta(x,0) = \theta(x,\omega), & 0 < x < L.
\end{cases}$$  \hspace{1cm} (26)

Let $\psi$ be the principal eigenfunction corresponding to $\lambda_0$ which satisfies (21). Multiplying the equation of (26) by $\psi$ and integrating the resulting equation over $0 < x < L$, we obtain

$$-\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta' \psi dxdt$$

$$= \int_0^\omega \int_0^L e^{\frac{V}{d}} \theta_{xx} \psi dxdt + \int_0^\omega \int_0^L e^{\frac{V}{d}} [-V(x,t) + \lambda_0] \theta' \psi dxdt. \hspace{1cm} (27)$$

Substituting $-\psi_t = d_I \psi_{xx} + q \psi_x - V(x,t) \psi + \lambda_0 \psi$ to (27), we get

$$\lambda^0 = \frac{\partial \lambda_0}{\partial d_I} = -\frac{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta_{xx} \psi dxdt}{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt} = \frac{\int_0^\omega \int_0^L e^{\frac{V}{d}} (\theta \psi_x + \frac{V}{d} \theta_x \psi) dxdt}{\int_0^\omega \int_0^L e^{\frac{V}{d}} \theta \psi dxdt}. \hspace{1cm} (28)$$
Lemma 4.4. The following statements on \( \lambda \) whether From (28), we see that if assumption (C1) holds, we are not sure whether \( \lambda \) or (C2), and and

\[
\begin{align*}
\text{(i)} & \quad \text{if assumption (C1)} \\
\text{(ii)} & \quad \text{if assumption (C2)}
\end{align*}
\]

Remark 1. From (28), we see that if assumption (C1) holds, we are not sure whether \( \lambda \) is monotone with respect to \( d_I \) for any given \( q > 0 \).

By Propositions 1, 2, 3 in the Appendix and the similar arguments to Lemma 3.3, we have the following results.

Lemma 4.4. The following statements on \( \lambda \) hold.

(i) For any given \( q > 0 \), we have

\[
\lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega} \int_0^\omega \left[ \gamma(L, t) - \beta(L, t) \right] dt \quad \text{and} \quad \lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega L} \int_0^L \int_0^\omega \left[ \gamma(x, t) - \beta(x, t) \right] dx dt.
\]

(ii) For any given \( d_I > 0 \), we have

\[
\lim_{q \to 0} \lambda_0 = \lambda_0^* \quad \text{and} \quad \lim_{q \to \infty} \lambda_0 = \frac{1}{\omega} \int_0^\omega \left[ \gamma(L, t) - \beta(L, t) \right] dt,
\]

where \( \lambda_0^* \) is the principal eigenvalue of (19) with \( q = 0 \), and

\[
\lambda_0^* \leq \frac{1}{\omega L} \int_0^\omega \int_0^L \left[ \gamma(x, t) - \beta(x, t) \right] dx dt.
\]

4.2. Effects of \( d_I \) and \( q \) on \( \mathcal{R}_0 \). From Lemma 3.5, we see that if \( \gamma(x, t) - \beta(x, t) = V(x, t) > (\leq, =) 0 \) on \([0, L] \times [0, \omega] \), then \( \mathcal{R}_0 < (\geq, =) 1 \) for any \( d_I > 0 \) and \( q > 0 \).

In this subsection, we study the effects of \( d_I \) and \( q \) on \( \mathcal{R}_0 \) when \( V(x, t) \) satisfies assumption (C1) or (C2). Then we investigate the impact of \( d_I \) and \( q \) on \( \mathcal{R}_0 \) in different habitats.

Specifically, we investigate the impact of \( d_I \) and \( q \) on \( \mathcal{R}_0 \) in the following statements on \( \mathcal{R}_0 \) hold.

(i) If assumption (C1) holds, we have

(1-i) if \( \int_0^\omega \int_0^L V(x, t) dx dt < 0 \), then there exists a unique \( q^* > 0 \) such that

\[
\mathcal{R}_0 > 1 \quad \text{for} \quad 0 < q < q^* \quad \text{and} \quad \mathcal{R}_0 \leq 1 \quad \text{for} \quad q \geq q^*;
\]

(ii-2) if \( \int_0^\omega \int_0^L V(x, t) dx dt > 0 \),

(1-ii) in case of \( \mathcal{R}_0 \leq 1 \), then \( \mathcal{R}_0 < 1 \) for all \( q > 0 \);

(1-ii-2) in case of \( \mathcal{R}_0 > 1 \), then there exists a unique \( q^- > 0 \) such that \( \mathcal{R}_0 > 1 \)

for \( 0 < q < q^- \), and \( \mathcal{R}_0 \leq 1 \) for \( q \geq q^- \).

(ii) If assumption (C2) holds, we have

(2-i) if \( \int_0^\omega \int_0^L V(x, t) dx dt < 0 \), then \( \mathcal{R}_0 > 1 \) for all \( q > 0 \);

(2-ii) if \( \int_0^\omega \int_0^L V(x, t) dx dt > 0 \),

(2-ii-1) in case of \( \mathcal{R}_0 > 1 \), then \( \mathcal{R}_0 > 1 \) for all \( q > 0 \);

(2-ii-2) in case of \( \mathcal{R}_0 = 1 \), then \( \mathcal{R}_0 > 1 \) for all \( q > 0 \);
(c) in case of $R_0 < 1$, there exists a unique $q^+ > 0$ such that $R_0 \leq 1$ for $0 < q \leq q^+$, and $R_0 > 1$ for $q > q^+$.

Proof. (i) If $V(x,t)$ solely changes sign about $x$ on $[0,L]$ and satisfies (C1), then $V(x,t)$ must change sign once on $[0,L]$ and $V(L,t) > 0$. By virtue of Lemma 4.4, we have

$$\lim_{q \to -\infty} \lambda_0 = \frac{1}{\omega} \int_0^\omega V(L,t) dt > 0 \quad \text{and} \quad \lim_{q \to 0} \lambda_0 = \overline{\lambda}_0.$$  

Thus, if $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, we have

$$\overline{\lambda}_0 \leq \frac{1}{\omega L} \int_0^\omega \int_0^L V(x,t) dx dt < 0.$$  

Moreover, it follows from Lemma 4.2 that $\lambda_0$ is monotone increasing in $q$ provided (C1) holds. Then there exist a unique $q^+ > 0$ such that $\lambda_0 = 0$ for $q = q^+$, $\lambda_0 < 0$ for $0 < q < q^+$, and $\lambda_0 > 0$ for $q > q^+$.

If $\overline{\lambda}_0 \geq 0$ and $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then $\lambda_0 > 0$ for all $q > 0$ if $\overline{\lambda}_0 < 0$ and $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, then there exists a unique $q^- > 0$ such that $\lambda_0 = 0$ for $q = q^-$, $\lambda_0 < 0$ for $0 < q < q^-$, and $\lambda_0 > 0$ for $q > q^-$. Therefore, applying Lemma 2.2, we see that assertion (i) holds.

(ii) If $V(x,t)$ solely changes sign about $x$ on $[0,L]$ and satisfies (C2), then $V(x,t)$ must change sign once on $[0,L]$ and $V(L,t) < 0$. Thus, according to Lemma 4.4, we have

$$\lim_{q \to -\infty} \lambda_0 = \frac{1}{\omega} \int_0^\omega V(L,t) dt < 0 \quad \text{and} \quad \lim_{q \to 0} \lambda_0 = \overline{\lambda}_0.$$  

Thus, if $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, we have

$$\overline{\lambda}_0 \leq \frac{1}{\omega L} \int_0^\omega \int_0^L V(x,t) dx dt < 0.$$  

It follows from Lemma 4.2 that $\lambda_0$ is monotone decreasing in $q$ provided (C2) holds. Thus, $\lambda_0 < 0$ for all $q > 0$.

If $\overline{\lambda}_0 < 0$ and $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then $\lambda_0 < 0$ for all $q > 0$; if $\overline{\lambda}_0 = 0$ and $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then $\lambda_0 < 0$ for all $q > 0$; if $\int_0^\omega \int_0^L V(x,t) dx dt > 0$ and $\overline{\lambda}_0 > 0$, then there exists a unique $q^+ > 0$ such that $\lambda_0 = 0$ for $q = q^+$, $\lambda_0 > 0$ for $0 < q < q^+$, and $\lambda_0 < 0$ for $q > q^+$. As a consequence, by virtue of Lemma 2.2, we derive assertion (ii). \hfill \Box

From the biological point of view, Theorem 4.5 reveals that for given diffusion rate of the infected individuals, the advection which causes the individuals to concentrate at the downstream end, has various influences on the disease in different environments.

Statement (i) illustrates that if the spatial change rate of disease recovery is greater than the spatial change rate of disease transmission and the downstream end is always a low-risk site for any time, we see that in the high-risk habitat, there exists a unique critical advection speed such that the disease persists if the advection speed is less than the critical value, and the disease will be eliminated if the advection rate is greater than the critical value, while in the low-risk habitat, if the disease can not persist without advection, then it can not persist with any advection, but if the disease persists without advection, there exists a unique critical advection speed such that the disease can be eliminated if and only if the advection...
speed is greater than the critical value. That is, all in all, the advection is beneficial to eliminate the disease in such a condition.

Statement (ii) displays that if the spatial change rate of disease recovery is less than the spatial change rate of disease transmission and the downstream end is always a high-risk site for any time, we see that in the high-risk habitat, the disease persists for any advection speed, while in the low-risk habitat, if the disease persists without advection, then it persists with any advection, but if the disease can be eliminated without advection, then either the disease persists for any advection speed, or there exists a unique critical advection speed such that the disease will be eliminated if and only if the advection speed is less than the critical value. Therefore, in this case, the advection is bad for the elimination of the disease.

**Theorem 4.6.** Assume that $V(x,t)$ solely changes sign about $x$. Then for given $q > 0$, the following statements on $R_0$ hold.

(i) If assumption (C1) holds, we have

(i-1) if $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, then there exist two numbers $0 < d_I < d_f$, such that $R_0 < 1$ for $0 < d_I < d_f$, and $R_0 > 1$ for $d_I > d_f$;

(ii-2) if $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then either $R_0 < 1$ for any $d_I > 0$, or there exist two numbers $0 < d_I < d_f$ such that $R_0 < 1$ for $d_I > 0$.

(ii) If assumption (C2) holds, we have

(ii-1) if $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, then $R_0 > 1$ for all $d_I > 0$;

(ii-2) if $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then there exists a unique critical number $d_I^* > 0$ such that $R_0 > 1$ for $0 < d_I < d_I^*$, and $R_0 < 1$ for $d_I > d_I^*$.

**Proof.** (i) If $V(x,t)$ solely changes sign about $x$ on $[0,L]$ and satisfies (C1), then $V(x,t)$ must change sign once on $[0,L]$ and $V(L,t) > 0$. Thus, according to Lemma 4.4, we have

$$\lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega} \int_0^\omega V(L,t) dt > 0.$$ 

If $\int_0^\omega \int_0^L V(x,t) dx dt < 0$, we have

$$\lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega L} \int_0^\omega \int_0^L V(x,t) dx dt < 0.$$ 

Then there exist $0 < d_I < d_f$, such that $\lambda_0 > 0$ provided $0 < d_I < d_f$, and $\lambda_0 < 0$ provided $d_I > d_f$. If $\int_0^\omega \int_0^L V(x,t) dx dt > 0$, then

$$\lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega L} \int_0^L \int_0^\omega V(x,t) dx dt > 0.$$ 

Then either $\lambda_0 > 0$ for any $d_I > 0$, or there exists two numbers $0 < d_I < d_f^*$, such that $\lambda_0 > 0$ if $d_I \in (0,d_f^*) \cup (d_f^*,+\infty)$. Therefore, it follows from Lemma 2.2 that assertion (i) holds.

(ii) Since $V(x,t)$ solely changes sign about $x$ on $[0,L]$ and satisfies (C2), then $V(x,t)$ must change sign once on $[0,L]$ and $V(L,t) < 0$. Thus, according to Lemma 4.4, we have

$$\lim_{d_I \to 0} \lambda_0 = \frac{1}{\omega} \int_0^\omega V(L,t) dt < 0.$$
If \( \int_0^\omega \int_0^L V(x,t)dxdt < 0 \), we have

\[
\lim_{d_I \to \infty} \lambda_0 = \frac{1}{\omega L} \int_0^L \int_0^\omega V(x,t)dxdt < 0.
\]
Furthermore, it follows from Lemma 4.3 that \( \lambda_0 \) is monotone increasing in \( d_I \) if (C2) holds. Then \( \lambda_0 < 0 \) for all \( d_I > 0 \) for this case. On the other hand, if \( \int_0^\omega \int_0^L V(x,t)dxdt > 0 \), we have

\[
\lim_{d_I \to \infty} \lambda_0 = \frac{1}{\omega L} \int_0^L \int_0^\omega V(x,t)dxdt > 0.
\]
By virtue of the monotonicity of \( \lambda_0 \) in \( d_I \), there exists a unique critical number \( d^*_I > 0 \) such that \( \lambda_0 = 0 \) for \( d_I = d^*_I \), \( \lambda_0 < 0 \) for \( 0 < d_I < d^*_I \), and \( \lambda_0 > 0 \) for \( d_I > d^*_I \). Hence, it follows from Lemma 2.2 that assertion (ii) holds.

Part (i) of Theorem 4.5 shows that for a given advection speed, if the spatial change rate of disease recovery is greater than the spatial change rate of disease transmission and the downstream end is always a low-risk site for any time, we see that in the high-risk habitat, the disease will be eliminated provided the diffusion rate of the infected individuals is small enough, while in the low-risk habitat, the disease will be eliminated provided the diffusion rate of the infected individuals is small or large enough.

Part (ii) shows that for a given advection speed, if the spatial change rate of disease recovery is less than the spatial change rate of disease transmission and the downstream end is always a high-risk site for any time, we see that in the high-risk habitat, the disease persists for any diffusion rate of the infected individuals, but in the low-risk habitat, there exists a unique critical diffusion rate of the infected individuals, such that the disease will be eliminated if and only if the diffusion rate of the infected individuals is greater than the critical value.

5. Discussion. In this paper, we have investigated an SIS epidemic reaction-diffusion-advection model where spatial heterogeneity and temporal periodicity are incorporated, which makes it more reasonable for describing the transmission of infectious disease. We have introduced the basic reproduction number \( R_0 \) and shown that \( R_0 \) predicts the threshold dynamics of (2) by persistence theory. Then we have discussed in detail the analytical properties of \( R_0 \). The effects of advection and diffusion on the persistence of the disease are further studied for the special case that \( \gamma(x,t) - \beta(x,t) = V(x,t) \) is monotone with respect to spatial variable \( x \) and solely changes sign about \( x \). The results show that if \( V_v(x,t) \geq 0, \neq 0 \) and \( V(x,t) \) changes sign about \( x \), the advection is beneficial to eliminate the disease, if \( V_v(x,t) \leq 0, \neq 0 \) and \( V(x,t) \) changes sign about \( x \), the advection is bad for the elimination of disease. In addition, the effects of diffusion on the persistence of the disease depends on the habitat is high-risk or low-risk.

In the future work, we would like to study the effects of advection and diffusion on the persistence of the disease provided that \( V(x,t) \) changes sign about spatial variable \( x \) and temporal variable \( t \) and \( V(x,t) \) has no monotonicity in \( x \) or \( t \). In particular, we assume that

\[
V(x,t) = f(x)g(t),
\]
where \( f(x) \) is a Hölder continuous function on \([0,L]\), and \( g(t) \) is a \( \omega \)-periodic Hölder continuous function on \( \mathbb{R} \). Then we distinguish three different cases:

(i) Spatial changing sign: \( f(x) \) changes sign in \((0,L)\) and \( g(t) > 0 \) in \( \mathbb{R} \):
(ii) Temporal changing sign: \( f(x) > 0 \) in \((0, L)\) and \( g(t) \) changes sign in \( \mathbb{R} \);
(iii) Full changing sign: both \( f(x) \) changes sign in \((0, L)\) and \( g(t) \) changes sign in \( \mathbb{R} \).

By Lemma 4.4, it is clear that when \( d_I \to 0 \) or \( q \to \infty \), the sign of the principal eigenvalue \( \lambda_0 \) depends on \( V(L, t) \), but how \( \lambda_0 \) varies with \( d_I \) and \( q \) is unknown. We would like to discuss \( f(x) \) and \( g(t) \) change sign once firstly, then twice, and so forth. The key problem is how to present the qualitative properties of \( \lambda_0 \) with respect to \( d_I \) and \( q \) in this case to see the effects of advection and diffusion on the persistence of the disease. Therefore, it is a challenging and interesting work for us.

Acknowledgments. This work is supported by NNSF of China (11371179, 11731005, 11701242) and the Fundamental Research Funds for the Central Universities (lzujbky-2017-ot09, lzujbky-2017-27) and NSF of Gansu Province, China (1606RJZA069).

Appendix. We consider the linear periodic-parabolic eigenvalue problem in one space dimension:

\[
\begin{cases}
\varphi_t - Da(x,t)\varphi_{xx} - \alpha h(x,t)\varphi_x + V(x,t)\varphi = \lambda m(x,t)\varphi, & 0 < x < L, 0 < t < \omega, \\
\varphi_x(x, t) = 0, & x = 0, L, 0 < t < \omega, \\
\varphi(x, 0) = \varphi(x, \omega), & \end{cases}
\]

where \( D, \alpha, L \) are constants with \( D, L > 0 \), the functions \( a, h \) and \( V \) are Hölder continuous and periodic in \( t \) with the same period \( \omega \). Moreover, it is assumed that \( a(x,t), h(x,t) > 0, \forall (x,t) \in [0, L] \times [0, \omega] \). The constants \( D \) and \( \alpha \) stand for the diffusion and advection (or drift) coefficients, respectively.

Let us denote by \( \lambda^N_{\alpha}(D) \) the principal eigenvalue of (29). In [22], for the weight function \( m(x, t) \equiv 1 \), the asymptotic behaviors of the principal eigenvalue \( \lambda = \lambda_1 \) as \( D \) goes to zero or infinity and \( \alpha \) goes to infinity were investigated. By modifying the arguments in [22] slightly, we see that the following three results hold true.

**Proposition 1.** For any given \( D, L > 0 \), there holds

\[
\lim_{\alpha \to \infty} \lambda^N_{\alpha}(D) = \frac{\int_0^\omega V(L, t)dt}{\int_0^\omega m(L, t)dt} \quad \text{and} \quad \lim_{\alpha \to -\infty} \lambda^N_{\alpha}(D) = \frac{\int_0^\omega V(0, t)dt}{\int_0^\omega m(0, t)dt}.
\]

**Proposition 2.** The following statements are valid:

(i) For any given \( L > 0 \) and \( \alpha > 0 \), we have

\[
\lim_{D \to 0} \lambda^N_{\alpha}(D) = \frac{\int_0^\omega V(L, t)dt}{\int_0^\omega m(L, t)dt}.
\]

(ii) For any given \( L > 0 \) and \( \alpha < 0 \), we have

\[
\lim_{D \to 0} \lambda^N_{\alpha}(D) = \frac{\int_0^\omega V(0, t)dt}{\int_0^\omega m(0, t)dt}.
\]

(iii) Assume that \( a(x, t) \) is a positive constant. For any given \( L > 0 \) and \( \alpha = 0 \), we have

\[
\lim_{D \to 0} \lambda^N_{\alpha}(D) = \min_{x \in [0, L]} \frac{\int_0^\omega V(x, t)dt}{\int_0^\omega m(x, t)dt}.
\]
Proposition 3. Assume that $a(x,t) \equiv a(t)$. Then for any given $L > 0$ and $\alpha \in \mathbb{R}$, there holds
\[
\lim_{D \to \infty} \lambda_N^1(\alpha, D) = \frac{\int_0^\omega \int_0^L V(x,t) dx dt}{\int_0^\omega \int_0^L m(x,t) dx dt}.
\]

REFERENCES

[1] N. D. Alikakos, An application of the invariance principle to reaction-diffusion equations, *J. Differential Equations*, 33 (1979), 201–225.

[2] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic disease patch model, *SIAM J. Appl. Math.*, 67 (2007), 1283–1309.

[3] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Discrete Contin. Dyn. Syst.*, 21 (2008), 1–20.

[4] R. Cui and Y. Lou, A spatial SIS model in advective heterogeneous environments, *J. Differential Equations*, 261 (2016), 3305–3343.

[5] R. Cui, K.-Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differential Equations*, 263 (2017), 2343–2373.

[6] K. A. Dahmen, D. R. Nelson and N. M. Shnerb, Life and death near a windy oasis, *J. Math. Biol.*, 41 (2000), 1–23.

[7] J. Ge, K. I. Kim, Z.-G. Lin and H.-P. Zhu, An SIS reaction-diffusion-advection model in a low-risk and high-risk domain, *J. Differential Equations*, 259 (2015), 5486–5509.

[8] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, RI, 1988.

[9] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin-New York, 1981.

[10] P. Hess, *Periodic-parabolic Boundary Value Problems and Positivity*, Pitman Res. Notes Math. , vol. 247, Longman Scientific & Technical, Harlow, 1991.

[11] W. Huang, M. Han and K. Liu, Dynamics of an SIS reaction-diffusion epidemic model for disease transmission, *Math. Biosci. Eng.*, 7 (2010), 51–66.

[12] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag New York, Inc., New York, 1966.

[13] H.-C. Li, R. Peng and F.-B. Wang, Varying total population enhances disease persistence: Qualitative analysis on a diffusive SIS epidemic model, *J. Differential Equations*, 262 (2017), 885–913.

[14] F. Lutscher, M. A. Lewis and E. McCauley, Effects of heterogeneity on spread and persistence in rivers, *Bull. Math. Biol.*, 68 (2006), 2129–2160.

[15] F. Lutscher, E. McCauley and M. A. Lewis, Spatial patterns and coexistence mechanisms in systems with unidirectional flow, *Theor. Popul. Biol.*, 71 (2007), 267–277.

[16] F. Lutscher, E. Pachepsky and M. A. Lewis, The effect of dispersal patterns on stream populations, *SIAM Rev.*, 47 (2005), 749–772.

[17] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, 37 (2005), 251–275.

[18] R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model I, *J. Differential Equations* 247 (2009), 1096–1119.

[19] R. Peng and S.-Q. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, *Nonlinear Anal.*, 71 (2009), 239–247.

[20] R. Peng and F.-Q. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: effects of epidemic risk and population movement, *Phys. D*, 259 (2013), 8–25.

[21] R. Peng and X.-Q. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, 25 (2012), 1451–1471.

[22] R. Peng and X.-Q. Zhao, Effects of diffusion and advection on the principal eigenvalue of a periodic parabolic problem with applications, *Calc. Var. Partial Differential Equations*, 54 (2015), 1611–1642.

[23] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.
[24] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys Monogr., vol. 41, American Mathematical Society, Providence, RI, 1995.

[25] W. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *J. Dynam. Differential. Equations*, 20 (2008), 699–717.

[26] W. Wang and X.-Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of Dengue transmission, *SIAM J. Appl. Math.*, 71 (2011), 147–168.

[27] X.-Q. Zhao, Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications, *Canad. Appl. Math. Quart.*, 3 (1995), 473–495.

[28] X.-Q. Zhao, *Dynamical Systems in Population Biology*, Springer-Verlag, New York, 2003.

Received September 2017; revised February 2018.

E-mail address: jiangdh15@lzu.edu.cn
E-mail address: wangzhch@lzu.edu.cn (Z.-C. Wang) (Corresponding author)
E-mail address: lz@lzu.edu.cn