Optical Bistability in a cavity with one moving mirror

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Abstract
We analyze the behaviour of a coherent field driving a single mode optical cavity with one perfectly reflecting moving mirror and a partially reflecting fixed mirror, and show that this system’s output exhibits optical bistability due to radiation pressure acting over the moving mirror.

1 Introduction
In the last decade cavities with moving mirrors have been extensively studied and used to show how to synthesize Schrödinger-cat states [1] [2], prove quantum properties of macroscopic objects [3] [4] and more. The experimental analysis of such a system started two decades ago with Dorsel et al [5] who found a nonlinear behaviour in a plane Fabry-Perot interferometer with a light mirror. Time has passed and each day the theoretical use of this systems adds on, but we found no quantum description for the optical bistability phenomenon described by Dorsel et al. In this paper we present a model intented to give a quantum description for optical bistability observerd in cavities with one moving mirror and relate its solution to dispersive optical bistability in Kerr like medium. [5]

2 Model
We consider a cavity with one fixed partially reflecting mirror and one perfectly reflecting mirror, which can move in harmonic oscillations under the influence of radiation pressure. The cavity is coherently driven. Considering the closed system, and rotating wave aproximation and general assumptions to evade Doppler shifting and Casimir effects [6] [2], we are able to write down the Hamiltonian for the closed system as:

\[ H = \sum_{q=1}^{4} H_q \]  

(1)

Where \( H_1 \) is the term corresponding to the cavity, \( H_2 \) the term linked to the harmonic oscillating mirror, \( H_3 \) is the coupling between the internal cavity field
and the moving mirror, and \( H_4 \) is the coupling between the coherent driving field and the internal cavity field. Each one can be expressed properly as:

\[
\begin{align*}
H_1 &= \hbar \omega_C \hat{n} \\
H_2 &= \hbar \omega_M \hat{N} \\
H_3 &= -\hbar G \hat{n} (b^\dagger + b) \\
H_4 &= i \hbar (\mathcal{E} e^{-i \omega_L t} a^\dagger - \mathcal{E}^* e^{i \omega_L t} \hat{a})
\end{align*}
\]

Where \( G \) is the coupling constant for the field-mirror interaction; \( \omega_C, \omega_M, \omega_L \) are the frequencies associated to the cavity, mirror and field respectively.

To get information about the behaviour of the output field from this cavity, we have to analyze the master equation for the density operator and consider the partially reflecting mirror coupled to a vacuum field. In a reference frame rotating at \( \omega_L \), the frequency of the coherent driving field, such open system’s master equations is:

\[
\dot{\rho} = -\frac{i}{\hbar} [H_b, \rho] + L_\rho
\]

Where the transformed Hamiltonian looks like:

\[
H_b = \sum_{q=1}^{4} H_{bq}
\]

With components:

\[
\begin{align*}
H_{b1} &= \hbar \delta \hat{n} \\
H_{b2} &= \hbar \omega_M \hat{N} \\
H_{b3} &= -\hbar G \hat{n} (b^\dagger + b) \\
H_{b4} &= i \hbar (\mathcal{E} a^\dagger - \mathcal{E}^* \hat{a}) \\
L_\rho &= \gamma \left( 2 \hat{a} \rho \hat{a}^\dagger - \rho \hat{n} - \hat{n} \rho \right)
\end{align*}
\]

Where \( \delta = \omega_C - \omega_L \) is the detuning between the cavity and the coherent driving field. And \( L_\rho \) represents the interaction with a thermal reservoir through the partially reflecting mirror with coupling constant given by \( \gamma \) or damping rate of the cavity to the external reservoir.

With this, to ease the work we can talk about a coordinate transformation over the total density operator:

\[
\tilde{\rho} = D_b^\dagger(\kappa \hat{n}) \rho D_b(\kappa \hat{n})
\]

Where \( D_b(\kappa \hat{n}) = e^{\kappa \hat{n}(b^\dagger - b)} \) is the Glauber’s displacement operator in mirror coordinates, so our master equation becomes:

\[
\dot{\tilde{\rho}} = -\frac{i}{\hbar} \left[ \tilde{H}, \tilde{\rho} \right] + \tilde{L}_\rho
\]
with \( \tilde{H} = D_b^\dagger(\kappa\hat{n})H D_b(\kappa\hat{n}) \) and expressed by:

\[
\begin{align*}
\tilde{H}_1 &= \hbar \delta \hat{n} \\
\tilde{H}_2 &= \hbar \omega_M \left[ \hat{N} + \kappa \hat{n}(\hat{b}^\dagger + \hat{b}) - \kappa^2 \hat{n}^2 \right] \\
\tilde{H}_3 &= -\hbar G \hat{n}(\hat{b}^\dagger + \hat{b} + \kappa \hat{n}) \\
\tilde{H}_4 &= \hbar g (E D_b^\dagger(\kappa)\hat{n}^\dagger - E^* D_b(\kappa)\hat{n}) \\
L_\tilde{\rho} &= \gamma \left( 2D_b(\kappa)\hat{\rho} D_b^\dagger(\kappa)\hat{\rho}^\dagger - \hat{\rho} \hat{n} - \hat{n} \hat{\rho} \right)
\end{align*}
\]

In order to obtain the behaviour for the mean field amplitude, \( \langle a \rangle \), we start from the simplified master equation, transform it back to the original coordinate system and multiply it by \( \hat{a} \) by the left and trace over the result. And do the same with \( \hat{a}^\dagger \). It can be proved, using fundamental trace properties \cite{8}, that this tracing operation over the backward transformation, \( \text{Tr} \left( \hat{a} D_b(\kappa) \tilde{\rho} D_b^\dagger(\kappa) \right) \) and \( \text{Tr} \left( \hat{a}^\dagger D_b(\kappa) \tilde{\rho} D_b^\dagger(\kappa) \right) \), summarize to:

\[
\text{Tr} \left( \hat{a} D_b(\kappa) \tilde{\rho} D_b^\dagger(\kappa) \right) = \langle \hat{a} \rangle \tilde{\rho} = \text{Tr} \left( D_b^\dagger(\kappa)\hat{\rho} D_b(\kappa) \hat{\rho} \right)
\]

With this, if we consider the initial state of the mirror as a thermal state that will remain like that because the radiation pressure from the field is strong enough to induce the oscillation but not strong enough to change the state, we can express both tracing operations as:

\[
\begin{align*}
\langle \hat{a} \rangle \tilde{\rho} &= e^{\kappa^2(\hat{n} - \frac{1}{2})} \langle \hat{a} \rangle \tilde{\rho} \\
\langle \hat{a}^\dagger \rangle \tilde{\rho} &= e^{\kappa^2(\hat{n} - \frac{1}{2})} \langle \hat{a}^\dagger \rangle \tilde{\rho}
\end{align*}
\]

From Eq.\((7)\) we notice that there’s no need to transform back from Eq.\((??)\) in order to know the behaviour of the mean field amplitude coming out from the cavity, it’s enough to analyze the cavity’s creation and annihilation operators behaviour in the transformed master equation obtained beforehand. Assuming the density operator’s form as \( \tilde{\rho} = \tilde{\rho}_F \tilde{\rho}_M \), where \( \tilde{\rho}_F = |\alpha\rangle \langle \alpha | \) and \( \tilde{\rho}_M = |R\rangle \langle R | \), we obtain a set of Langevin equations in the displaced coordinates:

\[
\begin{align*}
\langle \hat{a} \rangle _{\tilde{\rho}} &= gE e^{\kappa^2(\hat{n} + \frac{1}{2})} - \left[ \gamma^2 + i(\delta + 3 \omega_M \kappa^2) \right] \langle \hat{a} \rangle _{\tilde{\rho}} - 6i \omega_M \kappa^2 \langle a \rangle _{\tilde{\rho}} \langle a^\dagger \rangle _{\tilde{\rho}} \\
\langle \hat{a}^\dagger \rangle _{\tilde{\rho}} &= gE^* e^{\kappa^2(\hat{n} + \frac{1}{2})} - \left[ \gamma^2 - i(\delta + 3 \omega_M \kappa^2) \right] \langle a^\dagger \rangle _{\tilde{\rho}} + 6i \omega_M \kappa^2 \langle a \rangle _{\tilde{\rho}} \langle a^\dagger \rangle _{\tilde{\rho}}
\end{align*}
\]

This Langevin set clearly shows bistable behaviour similar to dispersive optical bistability found in Kerr crystals \cite{9}. Next graphic shows the behaviour of the output field in response to the input intensity for this system’s steady state solution.
Figure 1: Scaled output intensity $|\alpha|^2$ vs scaled input intensity $\frac{g^2|E|^2 e^{2\kappa^2(n+\frac{1}{2})}}{\gamma^2}$. With $\frac{3\omega_M\kappa^2}{\gamma^2} = 0.5$ and $\frac{\delta + 3\omega_M\kappa^2}{\gamma^2} = -3.0$ (thick solid), $= -2.0$ (dot), $= -1.0$ (solid)

3 Analysis

Now we write down the equation that describes the semi-classical behaviour of the output intensity of a coherently driven cavity with one moving mirror free to oscillate under radiation pressure from the Langevin set (8) and plotted in Fig. 2:

$$0 = g^2 E e^{2\kappa^2(n+\frac{1}{2})} - \gamma^2 a - \left[\left(\delta + 3\omega_M\kappa^2\right) - 6\omega_M\kappa^2 a\right]^2 a, \quad (9)$$

and try to analyze and comment about its significance.

The quantity $E e^{2\kappa^2(n+\frac{1}{2})}$ represents the scaled input field intensity, the scaling comes first from the coupling between the internal cavity field and the external field constant, $g$, and then by a factor related to the initial energy of the mirror and how it is coupled to the cavity, exponential term.

The term containing $\gamma^2 a$ is related to the transmitivity of the partial reflecting fixed mirror in the cavity.

Finally, $\left[\left(\delta + 3\omega_M\kappa^2\right) - 6\omega_M\kappa^2 a\right]^2 a$ is the term that describes the action of the radiation pressure over the moving mirror. The first term inside the bracket would describe how the oscillation of the free mirror modifies the detuning between the resonant frequency of the cavity and the frequency of the input laser. The second term would describe the direct action of the field over the moving mirror, so the full bracket term shows how the influence of the radiation pressure over the mirror modifies the physical length of the cavity. This is the term that will change the resonant frequency of the cavity and so tune the internal cavity field and the external field in and out of resonance according to the intensity of the incident field and so the transmitted field will increase or decrease, respectively.

Let us take the steady state solution for the Langevin set for a cavity with fixed mirrors containing a Kerr-like medium of non-linear constant $\chi$.
\[ 0 = g^2 E - \gamma^2 a - (\delta - 2\chi a)^2 a \]

Term by term analysis shows a correlation between the changes in optical length of the cavity produced by the Kerr-like medium, \((\delta - 2\chi a)^2 a\), and the change in the physical length of the cavity because of radiation pressure, \([ (\delta + 3\omega_M \kappa^2) - 6\omega_M \kappa^2 a ]^2 a \).

Finally, taking the classical analysis made by Meystre et al. [10] for the cavity with one moving mirror

\[ I_t = I_o \left[ 1 + R \left( \frac{\beta_0 + \beta_2 I_o}{T^2} \right)^2 \right]. \]

and comparing with our result, we can see that this pair of equations are parallel and we can make a similitude between terms. This comparison is showed in Table 1.

| Classical | Quantum |
|-----------|---------|
| \( \beta_0 \) | \( \delta + 3\omega_M \kappa^2 \) |
| \( \beta_2 I_o \) | \( -6\omega_M \kappa^2 a \) |

Table 1: Parallel between classical and quantum analysis for a cavity with one moving mirror

4 Conclusions

We have proposed a model to analyze the experiment performed by Dorsel et al. [5]. Our model proved practical in helping to describe, from a quantum mechanical approach, the nonlinear response of the output field to incident intensity, the analysis related the behaviour of this system to dispersive optical bistability from fixed cavities containing Kerr-like mediums. Finally it can be proved consistent with models for the limiting case of a cavity with fixed mirrors.

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