Hamiltonian Approach to the Torsional Anomalies and Its Dimensional Ladder

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(Dated: December 13, 2019)

Torsion can cause various anomalies in various dimensions, including the $(3+1)$-d Nieh-Yan anomaly, the $(2+1)$-d Hughes-Leigh-Fradkin (HLF) parity anomaly and the $(3+1)$-d, $(1+1)$-d chiral energy-momentum anomaly. We study these anomalies from the Hamiltonian approach. We derive the $(1+1)$-d chiral energy-momentum anomaly from the single-body Hamiltonian. We then show how other torsional anomalies can be related to the $(1+1)$-d chiral energy-momentum anomaly in a straightforward way. Finally, the Nieh-Yan anomaly and the $(3+1)$-d chiral energy-momentum anomaly are obtained from the parity anomaly and the HLF effective action, respectively. Hence, we have constructed the dimensional ladder for the torsional anomalies from the single-body Hamiltonian picture.

I. INTRODUCTION

Many topological phases of matter can be understood from quantum anomalies [1–5]. A classic example is the integer quantum Hall effect, where the Hall current is captured by the $(2+1)$-dim parity anomaly in the bulk and thus the $(1+1)$-dim gauge anomaly at the edges [6]. Both the electromagnetic anomalies and the gravitational anomalies have been extensively used to characterize various topological phases of matter [5, 7–14]. Especially, the dimensional ladder for these anomalies have been known for a long time [15]. Namely, the chiral anomaly, the parity anomaly and the gauge anomaly in different dimensions relate to each other closely, which has been used to classify topological phases of matter in different dimensions [3–16].

Torsion can cause quantum anomalies as well, but they are less studied compared to the electromagnetic anomalies and the gravitational anomalies. However, torsion is ubiquitous in condensed matter. The torsion tensor can be used as the probe for thermal transport [17–19] and also used to describe dislocations [20–23]. In addition, torsion can also emerge from the order parameters of Fermi superfluid or topological superconductors [24–26]. Hence, torsional effects in topological phases of matter are attracting more and more attention and they can lead to various novel phenomena, including the anomalous thermal Hall effect in Weyl semimetals [27], the chiral torsional magnetic effect [28] and other viscoelastic responses [21–23, 29–35]. Due to the close relation between topological phases of matter and quantum anomalies, it is thus natural to ask what torsional anomalies we have and whether they form a dimensional ladder.

Various torsional anomalies have been obtained in the context of high-energy physics [22, 28, 36–41], including the Nieh-Yan chiral anomaly and the chiral energy-momentum anomaly (or the chiral diffeomorphism anomaly) in four dimensions [23, 36–41], the Hughes-Leigh-Fradkin (HLF) parity anomaly [42] in three dimensions [21, 22] and the chiral energy-momentum anomaly in two dimensions [22]. In the same spirit as Ref. [43], these anomalies can be derived from the Atiyah-Patodi-Singer index theorem or the chiral anomaly [23]. However, the torsional anomalies listed here are all divergent and thus depend on the ultra-violet physics. Hence, an alternative approach based on the Hamiltonian is highly desired, which enables us to appreciate the physics behind and the Hamiltonian approach is more familiar to condensed matter physicists.

In this paper, we have derived various torsional anomalies based on the Hamiltonian approach. Firstly, the $(1+1)$-dim chiral energy-momentum anomaly is derived from the single-body Hamiltonian. Then, all other torsional anomalies can be straightforwardly obtained from this anomaly, including the $(2+1)$-dim HLF effective action, the $(3+1)$-dim Nieh-Yan anomaly and the $(3+1)$-dim chiral energy-momentum anomaly. Especially, we show that the quadratic cut-off in the $(1+1)$-dim chiral energy-momentum anomaly is the energy density, so the cut-off measures the depth of the vacuum. We also reveal the close relation between the $(3+1)$-dim torsional anomalies and the $(2+1)$-dim parity-odd effective action. Namely, assuming translational invariance along one direction, the corresponding momentum is a good quantum number, so it acts like a mass term. The response currents from the $(2+1)$-dim effective action are then used to derive the corresponding $(3+1)$-dim anomaly currents at given momentum. The corresponding $(3+1)$-dim anomaly current can thus be obtained by summing over all the momentum. Finally, motivated by the recently predicted thermal Nieh-Yan anomaly [27, 44–45], we conjecture that there are thermal corrections to the torsional anomalies in the dimensional ladder obtained above, where the dimensionful coefficients are replaced by the temperature. Although we have focused on the $4–3–2$ dimensional ladder, our results can be easily gen-

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eralized to \((2n + 2) - (2n + 1) - (2n)\) dimensional ladder, which is discussed in the two appendices.

The rest of this paper is organized as follows. In Sec. II, we give a brief summary of the torsional anomalies. In Sec. III, we construct the first part of the dimensional ladder shown in Fig. 1 i.e., the \((1 + 1)\)-dim chiral energy-momentum anomaly, the \((2 + 1)\)-dim HLF effective action and the \((3 + 1)\)-dim chiral energy-momentum anomaly. In Sec. IV, we construct the second part of the dimensional ladder for the chiral current, i.e., \((1 + 1)\)-dim chiral anomaly, \((2 + 1)\)-dim parity anomaly and \((3 + 1)\)-dim Nieh-Yan anomaly. We summarize the main results of this paper in Sec. V.

II. TORSIONAL ANOMALIES

It is known that torsion can induce various anomalies, including the Nieh-Yan anomaly in four dimensions \(38, 39, 40\), the chiral energy-momentum anomaly in two and four dimensions \(22, 23\) and the Hughes-Leigh-Fradkin (HLF) parity anomaly in three dimensions. Interestingly, all these anomalies contain a divergent coefficient and thus depend on the ultra-violet cut-off. To be more concrete, the Nieh-Yan anomaly is (for the derivation using Fujikawa's method, please refer to appendix A)

\[
\frac{1}{\sqrt{|g|}} \partial_{\mu} \sqrt{|g|} \eta_{\mu\rho\sigma} = \frac{\Lambda^2}{32\pi^2} \frac{e^{\mu\rho\sigma}}{\sqrt{|g|}} \left( T_{\mu\rho} T_{\rho\sigma} \eta_{ab} - 2 e^{\mu}_{a} e^{\rho}_{b} \Omega_{ab\rho\sigma} \right),
\]

where \(\Lambda\) is the cut-off, \(e^{\mu}_{a}\) is the vielbein and \(g_{\mu\nu}\) is the metric with \(g \equiv \det g_{\mu\nu}\). \(a, \rho, \mu = 0, 1, 2, 3\) are the coordinates indices; \(a\) is called the Lorentz indices and it labels the locally flat coordinates with basis \(\{e_{a}\}\); while \(\mu\) is called the Einstein indices used for the basis \(\{\partial_{\mu}\}\). \(j^{5\mu}\) is the axial current. \(T^{5\mu}_{\mu\nu}\) is the torsion tensor, i.e., \(T^{a}_{\mu\nu} = \frac{1}{2} T^{a}_{\mu\nu} dx^\mu \wedge dx^\nu\) and \(T^{\alpha}_{\mu} = de^{\alpha}_{\mu} + \omega^{\alpha}_{\beta} \wedge e^{\mu}_{\beta}\), where \(\omega^{\alpha}_{\beta} = \omega^{\alpha}_{\beta \mu} dx^\mu\) is the spin connection. \(\Omega_{ab\mu\nu}\) is the curvature tensor, i.e., \(\Omega_{a} = \frac{1}{2} \Omega_{ab\mu\nu} dx^\mu \wedge dx^\nu\) and \(\Omega_{ab} = d\omega_{ab} + \omega_{a} \wedge \omega_{b}\).

As for the massless Dirac Fermions in \((1 + 1)\)-dim spacetime, there is (for details, please refer to appendix B)

\[
\nabla_{\mu} \xi^{5B}_{(c)} + T^{\rho}_{\mu\rho} \xi^{5B}_{(c)} - e^{\nu}_{a} T^{d}_{\mu\nu} \xi^{5A}_{(c)}
\]

where \(\cdots\) means that we have only kept the leading divergent terms. \(T_{\mu\nu}\) and \(\xi^{5B}_{(c)}\) are the canonical energy-momentum tensor and the chiral canonical energy-momentum tensor, respectively. \(\eta_{ab} = \text{diag}(1, -1)\) is the metric. Terms in the first line of Eq. (2) can be derived from the chiral covariant translational symmetry. That is, the right-handed Weyl Fermions transform as \(\psi_{R} \rightarrow \xi^{\mu} D_{\mu} \psi_{R}\), and \(\psi_{L} \rightarrow -\xi^{\mu} D_{\mu} \psi_{L}\) for left-handed Weyl Fermions (for details, please refer to appendix B), where \(D_{\mu}\) is the Lorentz covariant derivative and it acts on the Lorentz indices. So these terms appear even at the classical limit. Compared with the U(1) case, fermions coupled to the electromagnetic field has a classical equation of motion for the energy-momentum tensor as \(\nabla_{\mu} T^{\mu\nu} = F^{\mu\nu} j_{\mu}\), where \(F^{\mu\nu}\) is the field strength tensor of the electromagnetism. In both cases, we treat the electromagnetic field or the torsion field as the external background field. The terms in the second line are the anomalies from the quantum fluctuations. In the same spirit of the Callan-Harvey mechanism \(45\), this term is expected to originate from the current inflow in the bulk, which shall be explored in the next section. This chiral energy-momentum anomaly appears in four dimensions as well. Especially, for the massless Dirac Fermions, there is

\[
\nabla_{\mu} \xi^{5B}_{(c)} + T^{\rho}_{\mu\rho} \xi^{5B}_{(c)} - e^{\nu}_{a} T^{d}_{\mu\nu} \xi^{5A}_{(c)} - 2 e^{\mu}_{a} e^{\rho}_{b} \Omega_{ab\rho\sigma}\Omega_{cd\mu\nu} S^{abcd}
\]

where \(\cdots\) means that only the most divergent terms are kept. Detailed calculations are relegated in Appendix C.

Finally, for the \((2 + 1)\)-dim massive Dirac Fermions, the parity-odd effective action induced by torsion is

\[
S_{\text{HLF}} = \frac{m^{2} \text{sign}(m)}{8\pi} \int d^{3}x e^{\mu\nu\rho} e^{\mu}_{a} T^{b}_{\mu\nu} \eta_{ab},
\]

where \(m\) is the mass of Dirac Fermions. Note that by direct perturbative calculations \(21\), this effective action is divergent and it is proportional to the cut-off \(\Lambda\). However, for reasons that will be discussed in the next section, the HLF action here is written in terms of the mass instead of the cut-off.

Clearly, the coefficients of the anomaly terms in Eq. (2), (3) and Eq. (4) are all dimensionful and divergent. One may be curious whether we can understand these anomalies in a consistent way, or they are simply artificial illness. For Dirac Fermions in curved spacetime without torsion, it is known that the chiral anomaly, the parity anomaly and the gauge anomaly consist of a dimensional ladder \(43\). Namely, the \((2n + 1)\)-dim parity anomaly can be obtained from the \((2n + 2)\)-dim chiral anomaly by using the Atiyah-Patodi-Singer index theorem. Due to the Callan-Harvey mechanism, the current inflow captured by the \((2n + 1)\)-dim parity odd effective action (parity anomaly) leads to the \((2n)\)-dim gauge anomaly at the edges. By employing the same idea, the torsional anomalies can be derived as well \(22\). However, it is also tempting to derive these torsional anomalies from the single-body Hamiltonian, which emphasizes the physical mechanism behind.
Figure 1. Dimensional ladder for the torsional anomalies. "Chiral E. M. Anomaly" stands for the chiral energy-momentum anomaly. "HLF effective action" stands for the Hughes-Leigh-Fradkin effective action obtained in Ref. [21]. The boxes in first line are for the anomalies relating to the energy-momentum tensor based on the Hamiltonian approach (the corresponding derivation based on the Hamiltonian approach is given in Sec. [III]), while boxes in the bottom for the anomalies in the chiral U(1) current (the corresponding derivation is presented in Sec. [IV]).

III. ANOMALIES OF THE CHIRAL ENERGY-MOMENTUM TENSOR

In this section, we shall study the anomalies encoded in the the chiral energy-momentum tensor based on the Hamiltonian approach (boxes in the first row in Fig. [1]). First of all, we shall derive the (1 + 1)-dim chiral energy-momentum anomaly from the single-body Hamiltonian. Then, both the (2 + 1)-dim HLF effective action and the (3 + 1)-dim chiral energy-momentum anomaly can be easily obtained from the (1 + 1)-dim chiral energy-momentum anomaly. Finally, we further show how to derive the (3 + 1)-dim chiral energy-momentum anomaly from the (2 + 1)-dim HLF effective action, which completes the first row of the dimensional ladders for the torsional anomalies shown in Fig. [1].

A. (1 + 1)-dim chiral energy-momentum anomaly from single-body Hamiltonian

The Hamiltonian for the (1 + 1)-dim Dirac Fermions is

\[ H = (-i\partial_z) \sigma^3, \]

where \( \sigma^3 \) is the Pauli matrix, i.e., \( \sigma^3 = \text{diag}(1, -1) \) and the chirality of Fermions is labeled by \( s = \pm 1 \). Then, we turn on the vielbein. For simplicity, we assume that \( \epsilon^{\alpha}_{\mu} = \delta_{\mu}^{\alpha} + \delta_{\delta}^{\alpha} \Phi(t) \) with \( \partial_{t}\Phi \ll 1 \) and \( \Phi \ll 1 \). The corresponding torsional electric field is given as \( T^3 = \partial_t e^3 - \partial_\mu e^\mu = \partial_t \Phi \). Because \( e^{3}_{\mu} \) is defined from \( \epsilon^{\alpha}_{\mu} \) by requiring \( e^{\alpha}_{\mu} e^{\beta}_{\nu} = \delta_{\mu}^{\delta} \delta_{\nu}^{\beta} \), there is \( e^{3}_{\mu} = \delta_{\mu}^{\delta} - \delta_{\mu}^{3} \delta_{\delta}^{3} \Phi \). The coupling between the Pauli matrices and the vielbein is \( \sigma^\alpha \rightarrow \sigma^\alpha e^{\alpha}_{\mu} \), so the Hamiltonian becomes

\[ H = [1 - \Phi(t)] (-i\partial_z) \sigma^3. \] (5)

Notice that \( \partial_t \Phi(t) \ll 1 \), so one can obtain the dispersion relation, i.e., \( \mathcal{E}_s = s(1 - \Phi) p_z \). In addition, the effective Fermi velocity is now \( v_F = (1 - \Phi) \), so \( v_F \) can be tuned by changing the vielbein adiabatically. Hence, if we impose cut-off on the energy, i.e., \(-\Lambda < \mathcal{E}_s < \Lambda\), we can squeeze particles and thus momentum out of the system by tuning \( \Phi \), which leads to energy-momentum anomaly for \( s \)-Fermions. To be more concrete, let us calculate the momentum density for \( s \)-Fermions, i.e.,

\[ \tau^{s\text{st}}_{(c)3} = \int \frac{dp_z}{2\pi} n_F(\mathcal{E}_s(p_z)) p_z \]

\[ = \frac{1}{(1 - \Phi)^2} \int_{-\Lambda}^{\Lambda} d\mathcal{E}_s \frac{1}{2\pi} \exp(\beta\mathcal{E}_s) + \mathcal{E}_s \]

\[ T^{\mathcal{E}_s}_{\tau} = \frac{s}{4\pi(1 - \Phi)^2} \Lambda^2, \] (6)

where \( s = \pm 1 \) stands for the chirality and \( \mathcal{E}_s = s(1 - \Phi) p_z \) is the dispersion relation. \( \Lambda \) is the energy-cut-off, i.e., \(-\Lambda < (1 - \Phi) p_z < \Lambda\), and it measures the depth of the vacuum. \( n_F(\mathcal{E}_s) = [\exp(\beta\mathcal{E}_s) + 1]^{-1} \) is the Fermi-Dirac distribution with \( \beta = \frac{1}{T} \) is the inverse of the temperature \( T \). Eq. (6) tells us that the averaged momentum density of the occupied electrons receives corrections from the vielbein. The energy cut-off is fixed at \( \Lambda \), so the changes of \( \Phi \) will change the Fermi velocity and thus the averaged momentum. As a consequence, there is an anomaly in the momentum current for \( s \)-Fermions.

From Eq. (6), the time derivative of the chiral momentum density at the small \( \Phi \) approximation is

\[ \partial_t \tau^{s\text{st}}_{(c)3} = -\frac{\Lambda^2}{\pi} \partial_t \Phi, \]

or in a covariant form,

\[ \nabla_{\mu} \tau^{s\text{st}}_{(c)3} = \frac{\Lambda^2}{2\pi} \eta_{abc} \frac{\epsilon^{\mu}_{\nu}}{\sqrt{|g|}} T_{b \mu \nu} + \ldots, \] (7)

where \( \ldots \) stands for other terms appear at the classical level. This is similar to the result in Eq. (2) up to a multiplier 2, but we can always rescale the cut-off to match them. The physical mechanism of this anomaly can be straightforwardly appreciated as follow. Let us consider the right-handed Fermions (\( s = +1 \)) for concreteness. Since the chemical potential is set to zero, all the negative-energy states are occupied. The only energy scale in the system is the cut-off, so, by dimensional analysis, the averaged momentum must be proportional to the cut-off square, i.e., \( \tau^{s\text{st}}_{(c)3} \propto \Lambda^2 \). In the presence of vielbeins, i.e., \( e^{3}_{\mu} = 1 - \Phi \), the cut-off for momentum becomes \( (1 + \Phi) \Lambda \). Hence, the change of momentum density is \( \Delta \tau^{s\text{st}}_{(c)3} \propto \Lambda^2 - (1 + \Phi)^2 \Lambda^2 = -2\Phi\Lambda^2 \), which implies that \( \frac{\Delta \tau^{s\text{st}}_{(c)3}}{\Delta \Phi} \propto -\Lambda^2 \partial_t \Phi \) and this is leads to the chiral energy-momentum anomaly.
B. Torsional anomalies from the \((1+1)\)-dim chiral energy-momentum anomaly

After obtaining the \((1 + 1)\)-dim chiral energy-momentum anomaly, one can straightforwardly obtain other torsional anomalies, including the \((2 + 1)\)-dim HLF effective action, the \((3 + 1)\)-dim chiral energy-momentum anomaly and the Nieh-Yan anomaly.

As for the HLF effective action, this can be done by employing the Callan-Harvey mechanism \([22, 23, 46]\). Consider the \((2 + 1)\)-dim Chern insulators with chiral zero modes at the edges. Then, the \((1 + 1)\)-dim chiral energy-momentum anomaly derived in last section implies the existence of the energy-momentum anomalies for the chiral edge modes. This energy-momentum anomaly can only come from the current inflow in the bulk, so the corresponding energy-momentum current in the bulk must be

\[
\tau^\mu_{(c)a} = \frac{m^2 \text{sign}(m) \epsilon^{\mu \nu \rho}}{4\pi} \sqrt{|g|} \eta_{a \nu} T^b_{\nu \rho},
\]

where \(m\) is the gap of the \((2 + 1)\)-dim Chern insulators and it acts as the cut-off for the edges states, so \(\Lambda\) in Eq. 2 is replaced by \(m\). In addition, \(\text{sign}(m)\) is from the chirality of the edges modes. Note that we are considering the continuum model in the bulk rather than the tight-binding model, so the prefactor is \(\text{sign}(m)\). Alternatively, if the tight-binding model is considered, then \(\text{sign}(m)\) should be replaced by \(1 - \text{sign}(m)\). This difference comes from different regularization schemes.

From Eq. 3, one can easily write down the corresponding effective action, i.e.,

\[
S_{HLF} = \frac{m^2 \text{sign}(m)}{8\pi} \int d^3 x \epsilon^{\mu \nu \rho} \epsilon^a_{(c) \mu} T^b_{\nu \rho},
\]

where the canonical energy-momentum tensor is defined as \(T^\mu_{(c)a} = \frac{1}{\sqrt{|g|} \omega_{a \mu}} \frac{\delta S}{\delta a_{\mu}}\) with the spin connection \(\omega_{a \mu}\) kept fixed. Hence, we have recovered the \((2 + 1)\)-dim HLF effective action from the \((1 + 1)\)-dim chiral energy-momentum anomaly.

Interestingly, the \((3 + 1)\)-dim Nieh-Yan anomaly can be derived from the \((1 + 1)\)-dim chiral energy-momentum anomaly as well. Consider Weyl Fermions under a specific configuration of vielbeins, i.e., \(\epsilon^*_{(c) a} = \delta^a_\mu + w^a_\mu\) and \(w^a_\mu = \frac{1}{2} \xi^a \bar{T}^3_{B}(0, -y, x, 0), \bar{T}^3_B > 0\), which means that the torsional magnetic fields are applied along the \(z\)-direction. For simplicity, the spin connection is set to zero. Then, the effective Hamiltonian for Weyl Fermions with \(s\)-chirality is

\[
H_s = s[p_z \sigma^3 + \left(\hat{p}_y + \frac{1}{2} \bar{T}^3_B y p_z\right) \sigma^1 + \left(\hat{p}_y - \frac{1}{2} \bar{T}^3_B y p_z\right) \sigma^2],
\]

where \(p_z\) is a good quantum number and \(s = \pm 1\) denotes the chirality. This Hamiltonian looks like Weyl Fermions under magnetic fields \(\bar{T}^3_B\) with charge \(p_z\). The dispersion relation can be straightforwardly derived, i.e.,

\[
\mathcal{E}_s = \begin{cases} 
  s |p_z| & n = 0 \\
  \pm \sqrt{p_z^2 + 2|n\bar{T}^3_B p_z|} & |n| \geq 1 
\end{cases}.
\]

Compared to the magnetic case, the lowest torsional Landau levels are \(s |p_z|\) rather than \(s p_z\), which is because the charge of the torsional magnetic fields is \(p_z\) and it contributes an extra minus sign when \(p_z < 0\). Notice that only the lowest torsional Landau levels can distinguish Weyl Fermions with different chirality. Thus, the \(|n| > 1\) torsional Landau levels contribute equally to the \(s\)-Weyl Fermion current. That is, as far as the chirality anomaly is concerned, only the lowest torsional Landau levels matter. Consequently, the system is effective reduced from \((3 + 1)\)-dim to \((1 + 1)\)-dim and the corresponding effective \((1 + 1)\)-dim Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \left(\bar{\psi} \Gamma^a \partial_\mu \psi - \bar{\psi} \Gamma^a \partial_\mu \Gamma^\mu \bar{\psi}\right),
\]

where \(\Gamma^\mu\) is the \((1 + 1)\)-dim Gamma matrices, \(\psi'(p_z) = \left(\begin{array}{c} \psi_R \\ \psi_L \end{array}\right)\) for \(p_z > 0\) and \(\psi'(p_z) = \left(\begin{array}{c} \psi_L \\ \psi_R \end{array}\right)\) for \(p_z < 0\). The \((3 + 1)\)-dim chiral current can be defined in terms of the \((1 + 1)\)-dim Dirac Fermions, i.e.,

\[
\bar{j}^{5\mu} = \left(\bar{\psi} \Gamma^\mu \Gamma^5 p_z \psi\right) \left(\bar{T}^3_B \frac{2}{2\pi}\right),
\]

where \(\bar{T}^3_B\) is the torsional magnetic fields, \(\frac{1}{2\pi} \left(\bar{T}^3_B\right)\) is from the torsional Landau level degeneracy and the chirality of \(\psi'\) is twisted by sign \((p_z)\). Especially, \(\left(\bar{\psi} \Gamma^a \Gamma^5 p_z \psi\right)\) is the \((1 + 1)\)-dim chiral energy-momentum density \(\tau_{(c)a}^{5(1+1)\mu}\). Hence, there is \(\nabla_\mu j^{5\mu} = \left(\nabla_\mu \tau_{(z)z}^{5(1+1)\mu}\right) \bar{T}^3_B\), where \(\nabla_\mu \tau_{(c)a}^{5(1+1)\mu}\) is the \((1 + 1)\)-dim chiral energy-momentum anomaly. Consequently, the Nieh-Yan anomaly in Eq. 11 can be derived from the \((1 + 1)\)-dim chiral energy-momentum anomaly.

As for the \((3 + 1)\)-dim chiral energy-momentum anomaly in Eq. 3, it involves a term mixing the torsion tensor and the electromagnetic tensor. This anomaly can also be obtained from the \((1 + 1)\)-dim chiral energy-momentum anomaly. Namely, under external magnetic fields, the \((3 + 1)\)-dim Dirac Fermions are effectively reduced to the \((1 + 1)\)-dim ones consisting of lowest Landau levels. The \((3 + 1)\)-dim chiral energy-momentum anomaly can be written as \(\nabla_\mu \tau_{(c)a}^{5(3+1)\mu} \propto \nabla_\mu \tau_{(c)a}^{5(1+1)\mu} B\), which coincides with the anomaly in Eq. 11. So far, we have shown that most known torsional anomalies can be understood from the \((1 + 1)\)-dim chiral energy-momentum anomaly. It is natural to ask the relation between the \((2 + 1)\)-dim HLF effective action and the \((3 + 1)\)-dim torsional anomalies.
C. (3 + 1)-dim chiral energy-momentum anomaly from (2 + 1)-dim HLF effective action

In this section, we shall reveal the relation between the HLF effective action and the (3 + 1)-dim chiral energy-momentum anomaly based on the single-body Hamiltonian.

The Hamiltonian for the (3 + 1)-dim right-handed Weyl Fermions under torsion is

\[ H = H_1(\hat{p}_x, \hat{p}_y, \epsilon_{i1}^a) + p_z \sigma^3, \]  

(14)

where the background curvature is assumed to be zero, so we can set the spin connection to zero. \( H_1(\hat{p}_x, \hat{p}_y, \epsilon_{i1}^a) \) is the Hamiltonian for the (2 + 1)-dim massless Dirac Fermions with vielbeins and \( \epsilon_{i1}^a = \epsilon_{i1}^a(x, y) \) is the vielbein in the space perpendicular to \( z \). In addition, \( p_z \) is a good quantum number. Note that \( p_z \sigma^3 \) looks like a mass term for the (2 + 1)-dim Dirac Fermions. So from the HLF effective action, the momentum density at given momentum \( p_z \) is

\[ \rho^{(+)}_{(c)a}(p_z) = \frac{p_z^2}{4\pi} \operatorname{sign}(p_z) \eta_{ab} T^b, \]  

(15)

where the superscript \((+)^t\) denotes the chirality, the mass term \( m^2 \operatorname{sign}(m) \) is replaced by \( p_z^2 \operatorname{sign}(p_z) \) and \( T^a_B = - \left( \partial_1 \epsilon_{1a}^x - \partial_2 \epsilon_{1a}^y \right) \) is the torsional magnetic fields. Now let us further turn on the electric fields, i.e., \( p_z \to p_z - A_z(t) \), where \( \partial_1 A_z \ll 1 \). Thus, the momentum density at given momentum \( p_z \) becomes

\[ \rho^{(+)}_{(c)a}(p_z - A_z) = \frac{(p_z - A_z)^2}{4\pi} \operatorname{sign}(p_z - A_z) \eta_{ab} T^b, \]  

(16)

which implies that the momentum density of the right-handed Weyl Fermions can be tuned by varying \( A_z \) adiabatically. Hence, the momentum for the right-handed Weyl Fermions can be pumped up from the Dirac sea by changing \( A_z \). Notice that the effective mass for the right-handed and left-handed Weyl Fermions always has opposite sign. So the increasing of the momentum density of the right-handed Weyl Fermions is accompanied by the decreasing of the momentum density of left-handed ones. Consequently, there is an anomaly in the (3 + 1)-dim chiral energy-momentum tensor, but not the energy-momentum tensor.

Now let us calculate the total momentum density by integrating over \( p_z \), and the total momentum density for right-handed Weyl Fermions is given as

\[ \tau^{(+)}_{(c)a} = \int_{-\Lambda}^{+\Lambda} dp_z \rho^{(+)}_{(c)a} \]

\[ = \frac{\Lambda}{4\pi^2} \eta_{ab} A_z T^b. \]  

(17)

This implies that

\[ \partial_1 \tau^{(+)}_{(c)a} = \frac{\Lambda}{2\pi^2} \eta_{ab} E_z T^a + \ldots, \]  

(18)

and its covariant form is the (3 + 1)-dim chiral energy-momentum anomaly given in Eq. (3). Hence, we have completed the first part of the dimensional ladder shown in Fig. I

IV. ANOMALIES OF THE CHIRAL CURRENT

In the last section, we have completed the dimensional ladder for the chiral energy-momentum anomaly. In this section, we shall show that there is an analogue ladder for the anomalies encoded in the chiral current as well, which is shown in the second row of Fig. I. Namely, the Nieh-Yan anomaly can be understood from the (2 + 1)-dim parity anomaly, and the (2 + 1)-dim parity anomaly can be obtained from the (1 + 1)-dim chiral anomaly.

The relation between the (2 + 1)-dim parity anomaly and the (1 + 1)-dim chiral anomaly is well-known in the context of integer quantum Hall effect, which is used to understand the bulk-edge correspondence. For completeness, we shall briefly review it here. The covariant form of the Hall current in (2 + 1)-dim is

\[ j^\mu = \frac{\operatorname{sign}(m)}{8\pi} \epsilon^{\mu\rho\sigma} F_{\rho\sigma}, \]  

(19)

where \( m \) is the mass of the (2 + 1)-dim Dirac Fermions and \( \operatorname{sign}(m) \) is from the chirality of the edge modes. In addition, the spatial components of the current above can be recast in a more inspiring form, i.e., \( j = \frac{\operatorname{sign}(m)}{2\pi} \hat{z} \times E \). If we consider the tight-binding model in the bulk instead, then the Hall current becomes \( j = \frac{1}{2\pi} \left[ \frac{1}{2} \operatorname{sign}(m) \hat{z} \times E + \frac{1}{2} \left[ 1 + \operatorname{sign}(m) \right] \right] \) is the Chern number in the bulk. So for the topologically non-trivial phase, \( \frac{1}{2} \operatorname{sign}(m) \frac{1}{2} \pi = 1 \), we have obtained the quantized Hall conductance, i.e., \( \sigma_H = \frac{1}{2\pi} \). From the Hall current in Eq. (19), one can write down the parity odd effective action (the parity anomaly), i.e.,

\[ S = \frac{\operatorname{sign}(m)}{16\pi} \int d^3 x \epsilon^{\mu\rho\sigma} A_{\mu} F_{\rho\sigma}. \]  

(20)

In addition, Eq. (19) also tells us that there is current flowing toward the edges, which leads to the current non-conservation at the edges, or the (1 + 1)-dim gauge anomaly.

Now let us turn to the Nieh-Yan anomaly and its relation to the parity anomaly. The Hamiltonian for the right-handed Weyl Fermions is

\[ H = H_1(\hat{p}_x, \hat{p}_y; \epsilon_{i1}^a) + p_z \sigma^z, \]  

(21)

where \( p_z \) is a good quantum number and \( H_1 \) is the Hamiltonian for the (2 + 1)-dim Dirac Fermions under external vielbeins, i.e., \( \epsilon_{i1}^a = \delta_{i1}^a - w_{i1}^a(x, y) \). For concreteness, we further assume that \( H_1 \) is given as

\[ H_1(\hat{p}_x, \hat{p}_y; \epsilon_{i1}^a) = (\hat{p}_x - w_{1}^x p_z) \sigma^x + (\hat{p}_y - w_{1}^y p_z) \sigma^y. \]  

(22)

Hence, Eq. (21) looks like (2 + 1)-dim massive Dirac Fermions (the mass term is \( p_z \sigma^3 \)) under external electromagnetic fields \( w_{1}^i \) with charge \( p_z \). From the parity
anomaly, one can straightforwardly read off the Hall cur-
rent density at given momentum \( p_z \), i.e.,
\[
\rho^{(+)}(p_z) = \frac{1}{4\pi} p_z \text{sign}(p_z) T^3_B,
\]
(23)
where \( T^3_B \) is the torsional magnetic fields, i.e., 
\( T^3_B = \partial_x w_y - \partial_y w_x \). Now we further turn on the torsional elec-
tric fields, i.e., \( T^3_E = \partial_x e^3_z - \partial_z e^3_x \). For simplicity, we set \( e^{*3}_0 = 0 \) and \( e^{*3}_x = 1 + \Phi(t) \) with \( \Phi \) a slowly varying field.
Hence, the Hall density becomes
\[
\rho^{(+)}(p_z) = \frac{1}{4\pi} p_z \text{sign}[(1 - \Phi)p_z] T^3_B,
\]
(24)
where \( e^{*3} \) only modifies the mass term, but not the charge
of the effective electromagnetic fields \( w^a_i \). The total cur-
rent density can be derived by summing over the momentum,
\[
j^{(+)} = \int dp_z 2\pi \rho^{(+)}(p_z) = \frac{1}{8\pi^2 (1 - \Phi)^2} \int d\epsilon d\epsilon' \text{sign} \epsilon T^3_B
\]
where \( \epsilon \equiv (1 - \Phi) p_z \) is the energy and we have imposed
cut-off \( \Lambda \) upon the energy. By recast the result above to a
covariant form, one can obtain the Nieh-Yan anomaly in
Eq. (1). This complete the second part of the dimension-
nal ladder shown in Fig. 1.

It is interesting to notice that the \((3 + 1)\)-dim chiral
energy-momentum anomaly involves the mixing term be-
tween the torsion and the electromagnetic tensor, while
the Nieh-Yan anomaly only involves the torsion tensor.
Furthermore, if we regard the \((3 + 1)\)-dim Weyl Fermions
as the boundary chiral modes of some \((4 + 1)\)-dim topo-
logical phases of matter, then from the Callan-Harvey
mechanism, one can obtain the corresponding \((4 + 1)\)-
dim effective action \( \Box \)
\[
S_{(4+1)} = \frac{m^2 \text{sign}(m)}{64\pi^2} \int d^5 \epsilon \epsilon' T^a T^b \eta_{ab},
\]
whose response charge current and energy-momentum
current are the bulk origin of the boundary Nieh-Yan
anomaly and chiral energy-momentum anomaly, respec-
tively.

V. SUMMARY AND DISCUSSION

The dimensional ladder for torsional anomalies is con-
structed based on the single-body Hamiltonian. We
have shown how to obtain the \((2 + 1)\)-dim HLF effective
action, the \((3 + 1)\)-dim Nieh-Yan anomaly and chiral
energy-momentum anomaly from the \((1 + 1)\)-dim chiral
energy-momentum anomaly. In addition, we have also
clarified the relation between the \((2 + 1)\)-dim HLF effective
action and the \((3 + 1)\)-dim chiral energy-momentum
anomaly, the \((2 + 1)\)-dim parity anomaly and the \((3 + 1)\)-
dim Nieh-Yan anomaly. Our work has provided a com-
plete physical picture for various torsional anomaly in
various dimensions.

Recently, it was pointed out in Ref. [27, 44, 45] that,
at finite temperature, there is an extra thermal term in
the Nieh-Yan anomaly as well, where the cut-off is re-
placed by the temperature. Hence, based on the dimen-
sional ladder obtained here, it is natural to conjecture
that all the torsional anomalies in Fig. 1 that are cut-
off dependence receive thermal corrections as well. That
is, the dimensionful parameter is replaced by the tem-
perature. Specifically, for the \((1 + 1)\)-dim chiral energy-
momentum anomaly calculated in Eq. (6), if we use the
finite-temperature Fermi-Dirac distribution function in-
stead of the zero-temperature one, then there is an extra
thermal term proportional to the temperature square.
If this conjecture is true, then the \((2 + 1)\)-dim thermal
Hall effect can be straightforwardly understood from the
\((2 + 1)\)-dim thermal HLF effective action in the bulk,
or \((1 + 1)\)-dim energy-momentum anomaly at the edges.
In addition, the recently observed negative magneto-
thermoelectric resistance reported in Ref. [13] can be
understood based on this conjecture as well. Namely, if
there is an extra thermal term in the \((3 + 1)\)-dim chiral
energy-momentum anomaly equation in Eq. (3), i.e.,
\[
\nabla_\mu T^{3\mu}_a \equiv T^a \frac{3\mu}{\sqrt{g}} \eta_{ab} F_{\mu\nu} T_{\rho\sigma} + \ldots,
\]
then by considering the inter-valley scatterings, in the steady states, the
chiral chemical potential is proportional to \( \epsilon^{\mu\nu\rho\sigma} \eta_{ab} F_{\mu\nu} T_{\rho\sigma} \),
or \( T(B \cdot \nabla T) \) for \( a = 0 \). Combined with the chiral mag-
netic effect, one can thus obtain the negative magneto-
thermoelectric resistance. However, here, we shall re-
strict ourselves to the zero-temperature limit and leave
the conjecture for future study.

VI. ACKNOWLEDGEMENT

The authors wish to thank Barry Bradlyn for insightful
discussions. Z.-M. H. and M. S. were not directly sup-
ported by any funding agencies, but this work would not
be possible without resources provided by the Depart-
ment of Physics at the University of Illinois at Urbana-
Champaign. B. H. was supported by the ERC Starting
Grant No. 678795 TopInSy.

Appendix A: Fujikawa’s approach to the Nieh-Yan
anomaly

In this section, we shall provide the detailed derivation
of the torsion induced chiral anomaly. For convenience,
we shall focus on the Euclidean spacetime. The conven-
tion of the Wick rotation is
where \( \tau \) is the Euclidean time. The subscript \( E \) stands for the Euclidean spacetime, which will be neglected without causing any confusions, because in this section we will focus on the Euclidean spacetime.

The Jacobian \( J[\theta] \) of the chiral transformation \( \psi \rightarrow e^{i\theta \gamma_5} \psi \) is known to be

\[
\ln J[\theta] = -2i\theta \text{Tr}\gamma_5, \tag{A2}
\]

where \( \text{Tr} \) is the trace of both the internal indices and the spacetime coordinates. \( \text{Tr}\gamma_5 \) is obviously divergent, so we introduce the following regulator,

\[
\exp \left( \frac{\Phi^2}{\Lambda^2} \right),
\]

where

\[
\Phi = \epsilon^\mu_\alpha (\partial_\mu + \frac{\omega_{\alpha b} - \frac{1}{2} T^\rho_{\mu \rho}) e^\mu_\alpha \gamma^a
\]

are skew Hermitian. Note that \( \Phi \) is neither Hermitian nor skew Hermitian, so we use \( D \) rather than \( D \) and the extra term \( \frac{1}{2} T^\rho_{\mu \rho} \) in \( \Phi \) and \( \Phi \) is cancelled in the action.

Then, for \( \Phi^2 \), there is

\[
2\text{Tr} \left( \gamma^5 e^{-\Lambda^2 \Phi} \right)
\]

\[
= -2i \sum_n \frac{1}{n!} \int d^d x \int \frac{d^d k}{(2\pi)^d} e^{-\Lambda^2 k^2} \text{tr} \left( \gamma^5 \sigma^{a_1 b_1} \sigma^{a_2 b_2} \ldots \sigma^{a_n b_n} \right) \prod_{i=1}^n (-i\Lambda^2 T^a_{a_i b_i} T^a_{a_i b_i} + i\Lambda^2 F_{a_i b_i}) \bigg|_{n=1}^n
\]

\[
-2i \sum_n \frac{1}{n!} \int d^d x \int \frac{d^d k}{(2\pi)^d} e^{-\Lambda^2 k^2} \left( -2\Lambda^2 T^d_{b d} + 2\Lambda^2 F \right)^n
\]

\[
-2i \sum_{n=0}^{\infty} \sum_{p=0, 2, 4, \ldots} C_p \frac{1}{n!} \int d^d x \int \frac{d^d k}{(2\pi)^d} e^{-\Lambda^2 k^2} \left( -2\Lambda^2 T^d_{b d} + 2\Lambda^2 F \right)^n
\]

\[
-2i \sum_{p=0, 2, 4, \ldots} C_p \frac{1}{n!} \int d^d x \int \frac{d^d k}{(2\pi)^d} e^{-\Lambda^2 k^2} \left( -2\Lambda^2 T^d_{b d} + 2\Lambda^2 F \right)^n
\]

\[
-2i \left[ \frac{(\Lambda^2)^{\frac{1}{2}(d+p)}}{2^{d+p} (4\pi)^{d/2}} \sum_{\sigma} (\delta_{\sigma(b_1)} \sigma(b_2) \delta_{\sigma(b_3)} \sigma(b_4) \ldots) \right] \frac{(-2\Lambda^2 T^d_{b d})^p}{p!} e^{2\Lambda^2 F}.
\]

\[
\rightarrow \text{in the third line is because that we have replaced } -i\sigma^{a_1 b_1} (T^d_{a_1 b_1} F_{b_1}) \text{ with } 2T^d + 2F \text{ and } T^d = \frac{1}{2} T_{\mu \nu} dx^\mu \wedge dx^\nu,
\]
\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \] which is because of the identity
\[
\text{tr} \left( \gamma^5 \sigma^{a_1 b_1} \sigma^{a_2 b_2} \ldots \sigma^{a_n b_n} \right) = (-i)^n \epsilon^{a_1 b_1 \ldots a_n b_n}.
\]
In the fourth line, we have performed expansion \((a + b)^n = \sum_m c_m^a m^n b^{(n-m)}\). In the last line, we have used the following identity
\[
\int \frac{d^4k}{(2\pi)^4} \exp \left( -k_\alpha \Lambda^2 \delta_{ab} k_b \right) k_{a_1} k_{a_2} \ldots k_{a_{2m}}
= \left( \frac{1}{2} \right)^m \frac{\Lambda^2}{(4\pi)^{d/2}} \left( \sum_{(\sigma)} \delta_{\sigma(a_1)\sigma(a_2)} \ldots \right),
\]
where \(\sum_{(\sigma)}\) means sum over all possible permutations.

Notice that \(T^{b_1} \) and \(T^{b_2} \) in \((-2\Lambda^{-2} T^b)^p\) commute, so the anomaly above can be recast as
\[
-2i \int \sum_{p \text{ even}} \left[ \frac{\Lambda^2}{(4\pi)^{d/2}} \right] \frac{1}{2p} \frac{(p-1)!!}{p!} (4\Lambda^4 T^{c_1} T^{c_2} \eta_{c_1 c_2})^2 e^{2\Lambda^{-2} F},
\]
where \(\sum_{\sigma} (\delta_{\sigma(b_1)\sigma(b_2)} \delta_{\sigma(b_3)\sigma(b_4)} \ldots)\) contains \((p-1)!!\) terms and \((-1)^p = 1\) because that \(p\) is an even integer. That is, we have obtained
\[
2\text{Tr} \gamma_5 = -2i \int \frac{\Lambda^2}{(4\pi)^{d/2}} e^{2\Lambda^{-2} T^{c_1} T^{c_2} \eta_{c_1 c_2}} e^{2\Lambda^{-2} F} + \ldots, \quad (A3)
\]
where \(e^{2\Lambda^{-2} F}\) is the Chern character and \(T^{c_1} T^{c_2} \eta_{c_1 c_2}\) reminds us of the Nieh-Yan term, i.e., \(T^a \wedge T^a - \epsilon^a \wedge \epsilon^a \wedge \Omega_{ab}\). Thus, it natural to conjecture that the full anomaly polynomial should be
\[
2\text{Tr} \gamma_5 = -2i \int \frac{\Lambda^2}{(4\pi)^{d/2}} e^{2\Lambda^{-2} (T^a \wedge T^a - \epsilon^a \wedge \epsilon^a \wedge \Omega_{ab})} e^{2\Lambda^{-2} F} \hat{A} (\Omega),
\]
where \(\hat{A} (\Omega)\) is the Dirac genus.

### Appendix B: Chiral energy-momentum anomaly

In this section, we shall provide the derivation of the chiral energy-momentum anomaly in details. We first derive the constraints from the covariant diffeomorphism. After that, the anomaly in the zero-curvature limit is also derived by using Fujikawa’s method.

1. **Covariant diffeomorphism and the corresponding current**

Consider an action \(S (\bar{\psi}, \psi, e^{*a}_\mu, \omega_{ab})\) with both local Lorentz symmetry and local Einstein symmetry. The local Lorentz transformation is given as
\[
\delta e^{*a}_\mu = -\alpha^a_{\beta} e^{\beta}_\mu,
\]
\[
\delta \omega_{ab} = D_\mu \alpha_{ab},
\]
and
\[
\delta \psi = -\frac{1}{2} \alpha_{ab} \sigma^{ab} \psi,
\]
where \(\omega_{ab}\) transforms as the gauge field for the Lorentz group as we expected. The local Einstein transformation is
\[
\delta e^{*a}_\mu = \mathcal{L}_\xi e^{*a}_\mu,
\]
\[
\delta \omega_{ab} = \mathcal{L}_\xi \omega_{ab},
\]
and
\[
\delta \psi = \xi^\mu \partial_\mu \psi,
\]
where \(\mathcal{L}_\xi\) is the Lie derivative of the vector \(\xi = \xi^\mu \partial_\mu\). The local Einstein transformation is usually called diffeomorphism as well. By performing a Local Einstein transformation and then a local Lorentz transformation with \(\alpha_{ab} = (i\omega)_{ab}\), one can obtain the covariant diffeomorphism, i.e.,
\[
\delta e^{*a}_\mu = T^a_{\mu \nu} \xi^\nu + \nabla_\mu \xi^a,
\]
\[
\delta \omega_{ab} = \Omega_{abc} \xi^c,
\]
\[
\delta \psi = \xi^\mu D_\mu \psi
\]
Then, the variation of the action
\[
\delta S = \int d^4x \sqrt{|g|} \left[ -\frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta e^{*a}_\mu} \delta e^{*a}_\mu + \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \omega_{ab}} \delta \omega_{ab} \right]
= \int d^4x \sqrt{|g|} \left[ (\nabla_\mu + T^a_{\mu \nu}) T^a_{\nu \mu} \xi^a - \Omega_{abc} \xi^c - \left( \tau^a_{(c)} T^a_{\mu \nu} \xi^\nu - S^{abc} \Omega_{abc} \xi^c \right) \right],
\]
where $\tau^\mu_{(c)a}$ is the canonical energy-momentum tensor, i.e.,
\[
\tau^\mu_{(c)a} = -\frac{1}{\sqrt{|g|}} \delta S \biggl/ \delta \omega^\mu_{ab}\biggr|_\mu^a
\]
and $S^{\mu\nu}$ is the spin current, i.e.,
\[
S^{\mu\nu} = \frac{1}{\sqrt{|g|}} \delta S \biggl/ \delta \omega^\mu_{ab}\biggr|_\mu^a.
\]
Hence, we have obtained
\[
(\nabla_\mu + T^\rho_\mu \rho_\mu) \tau^\mu_{(c)a} = \left(\tau^\mu_{(c)a} T^\rho_\mu - S^{\mu\rho\Omega_{ab}\rho\mu} \right) e^\nu_a,
\]
(B2)
and similarly, for the chiral energy-momentum tensor $\tau^{5\mu}_{(c)a}$ of the massless Dirac Fermions, there is
\[
(\nabla_\mu + T^\rho_\mu \rho_\mu) \tau^{5\mu}_{(c)a} = \left(\tau^{5\mu}_{(c)a} T^\rho_\mu - S^{\mu\rho\Omega_{ab}\rho\mu} \right) e^\nu_a.
\]
Note that for massive Dirac Fermions, the mass term would break the chiral symmetry and thus leads to an extra mass term in the equation above.

2. Chiral energy-momentum anomalies

After deriving the classical equation for the chiral energy-momentum tensor, we are ready to explore the corresponding quantum anomalies. For the chiral covariant deffomorphism, i.e.,
\[
\delta \psi = \xi^\mu \gamma^5 D_\mu \psi,
\]
one can straightforwardly obtain the Jacobian, i.e.,
\[
D_\psi D \psi \rightarrow J(\xi) D \psi D \psi
\]
and
\[
\ln J(\xi) = 2Tr \biggl( \overline{D} \gamma_5 \biggr) \xi^a.
\]
Similar to the chiral anomaly, the regularized Jacobian is
\[
2Tr \left( \overline{D}_a \gamma_5 e^{-\Lambda^2 - \Psi} \right) \rightarrow -2i \sum_{m=2}^{\infty} \frac{1}{m!} \int \frac{d^4k}{(2\pi)^4} e^{-i\tau^2 k} \left( -2\Lambda^2 T^a \right) k^m,
\]
where $\rightarrow$ means that we have replaced $T^a \left( -i\sigma^m \right)$ by $\frac{1}{2} T^a \left( -i\sigma^m \right)$ and similar for $F$. In addition, we have only kept the most divergent term and set the spin connection to zero. Similar to the chiral anomaly case, one can find that in the zero-curvature limit, the most divergent terms are
\[
2Tr \left( \overline{D}_a \gamma_5 \right) = \frac{2\Lambda^2}{(4\pi)^{d/2}} \delta_{ab} \int \frac{d^4k}{(2\pi)^4} e^{-\Lambda^2 \left( T^a \right)^2} F^a \Lambda^2 F^b A(\Omega) \ldots,
\]
where $A(\Omega)$ is the Dirac genus.

Then, we have found that in two dimensions, there is
\[
\nabla_\mu \tau^\mu_{(c)a} = -\frac{\Lambda^2}{4\pi} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}} \delta_{ab} T^b_{\mu\rho} + \ldots,
\]
and in $(3+1)$ dimensions, there is
\[
\nabla_\mu \tau^\mu_{(c)a} = -\frac{\Lambda^2}{2\pi} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{|g|}} \frac{T^b_{\mu\rho} F^b_{\rho\sigma}}{2\pi} + \ldots,
\]
where $\ldots$ means that we have only kept the most divergent terms caused by the torsion.

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