Choices and Intervals

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AofA, Paris, June 17, 2014

joint work with

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Random structures formed by adding objects one after the other according to some random rule. Examples:

1. balls-and-bins model: \( n \) bins, place balls one after the other into bins, for each ball choose bin uniformly at random (maybe with size-biasing)

2. random graph growth: \( n \) vertices, add (uniformly chosen) edges one after the other.

3. interval fragmentation: unit interval \([0, 1]\), add uniformly chosen points one after the other \(\rightarrow\) fragmentation of the unit interval.

Extensive literature on these models.
Power of choices

Aim: Changing behaviour of model by applying a different rule when adding objects

1. balls-and-bins model: $n$ bins, at each step choose two bins uniformly at random and place ball into bin with fewer/more balls.
   Azar, Broder, Karlin, Upfal ’99; D’Souza, Krapivsky, Moore ’07; Malyshkin, Paquette ’13

2. random graph growth: $n$ vertices, at each step uniformly sample two possible edges to add, choose the one that (say) minimizes the product of the sizes of the components of its endvertices.
   Achlioptas, D’Souza, Spencer ’09; Riordan, Warnke ’11+’12

3. interval fragmentation: unit interval $[0, 1]$, at each step, uniformly sample two possible points to add, choose the one that falls into the larger/smaller fragment determined by the previous points.
   → this talk
Balls-and-bins model

$n$ bins, place $n$ balls one after the other into bins.

- Model A: For each ball, choose bin uniformly at random.
- Model B: For each ball, choose two bins uniformly at random and place ball into bin with more balls.
- Model C: For each ball, choose two bins uniformly at random and place ball into bin with fewer balls.

How many balls in bin with largest number of balls?

- Model A:
- Model B:
- Model C:
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- Model A: $\approx \log n / \log \log n$
- Model B:
- Model C:
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- **Model B**: $\approx \log n / \log \log n$
- **Model C**: $O(\log \log n)$
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\( \Psi \)-process: definition

\( X \): random variable on \([0, 1]\), \( \psi(x) = P(X \leq x) \).
$\Psi$-process: definition

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\( \Psi \)-process: definition

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2. Step \(n\): \(n - 1\) points in interval, splitting it into \(n\) fragments

**Choices and Intervals**
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3. Step $n + 1$:
   - Order intervals/fragments according to length
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$\Psi$-process: examples

$X$: random variable on $[0, 1]$, $\Psi(x) = P(X \leq x)$.

- $\Psi(x) = x$: uniform process
- $\Psi(x) = 1_{x \geq 1}$: Kakutani process
- $\Psi(x) = x^k$, $k \in \mathbb{N}$: max-$k$-process (maximum of $k$ intervals)
- $\Psi(x) = 1 - (1 - x)^k$, $k \in \mathbb{N}$: min-$k$-process (minimum of $k$ intervals)
Main result

\[ l_1^{(n)}, \ldots, l_n^{(n)}: \text{lengths of intervals after step } n. \]

\[ \mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{n.l_k^{(n)}} \]

Main theorem

Assume \( \psi \) is continuous + polynomial decay of \( 1 - \psi(x) \) near \( x = 1 \).

1. \( \mu_n \) (weakly) converges almost surely as \( n \to \infty \) to a deterministic probability measure \( \mu^\psi \) on \( (0, \infty) \).

2. Set \( F^\psi(x) = \int_0^x y \mu^\psi(dy) \). Then \( F^\psi \) is \( C^1 \) and

\[ (F^\psi)'(x) = x \int_x^\infty \frac{1}{z} d\psi(F^\psi(z)). \]
Properties of limiting distribution

Write $\mu^\Psi(dx) = f^\Psi(x) \, dx$.

**max-$k$-process ($\Psi(x) = x^k$)**

\[ f^\Psi(x) \sim C_k \exp(-kx), \quad \text{as } x \to \infty. \]

**min-$k$-process ($\Psi(x) = 1 - (1 - x)^k$)**

\[ f^\Psi(x) \sim \frac{c_k}{x^2 + \frac{1}{k-1}}, \quad \text{as } x \to \infty. \]

**convergence to Kakutani (cf. Pyke ’80)**

If $(\Psi_n)_{n \geq 0}$ s.t. $\Psi_n(x) \to 1_{x \geq 1}$ pointwise, then

\[ f^{\Psi_n}(x) \to \frac{1}{2} 1_{x \in [0,2]}, \quad \text{as } n \to \infty. \]
Properties of limiting distribution (2)

![Graph showing density distribution over length for different distributions: min-5, min-2, uniform, max-2, max-10. The graph plots density on the y-axis and length on the x-axis.]
Proof of main theorem: the stochastic evolution

Embedding in continuous time: points arrive according to Poisson process with rate $e^t$.

$N_t$: number of intervals at time $t$

$I_1^{(t)}, \ldots, I_{N_t}^{(t)}$: lengths of intervals at time $t$.

Observable: size-biased distribution function

$$A_t(x) = \sum_{k=1}^{N_t} I_k^{(t)} 1_{I_k^{(t)} \leq xe^{-t}}$$

Then $A = (A_t)_{t \geq 0}$ satisfies the following stochastic evolution equation:

$$A_t(x) = A_0(e^{-t}x) + \int_0^t (e^{s-t}x)^2 \left[ \int_{e^{s-t}x}^\infty \frac{1}{Z} d\psi(A_s(z)) \right] ds + M_t(x),$$

for some centered noise $M_t$.

Claim: $A_t$ converges almost surely to a deterministic limit as $t \to \infty$. 

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Deterministic evolution

Let \( F = (F_t)_{t \geq 0} \) be solution of

\[
F_t(x) = F_0(e^{-t}x) + \int_0^t (e^{s-t}x)^2 \left[ \int_{e^{s-t}x}^{\infty} \frac{1}{z} d\psi(Fs(z)) \right] ds
\]

\[=: S^{\psi}(F)_t.\]

Define the following norm:

\[
\| f \|_{x^{-2}} = \int_0^\infty x^{-2} |f(x)| \, dx.
\]

Lemma

Let \( F \) and \( G \) be solutions of the above equation. For every \( t \geq 0 \),

\[
\| F_t - G_t \|_{x^{-2}} \leq e^{-t} \| F_0 - G_0 \|_{x^{-2}}.
\]

In particular: \( \exists! F^{\psi} : F_t \rightarrow F^{\psi} \) as \( t \rightarrow \infty \).
Problem

Cannot control noise $M_t$ using the norm $\| \cdot \|_x^{-2}$!

$\implies$ no quantitative estimates to prove convergence.
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Cannot control noise $M_t$ using the norm $\| \cdot \|_x^{-2}$!\[=> no quantitative estimates to prove convergence.

Still possible to prove convergence by Kushner–Clark method for stochastic approximation algorithms.

1. Shifted evolutions $A^{(n)} = (A_{t-n}^{(n)})_{t \in \mathbb{R}}$. Show: almost surely, the family $(A^{(n)})_{n \in \mathbb{N}}$ is precompact in a suitable functional space.

2. Show $\mathcal{S}^\psi$ is continuous in this functional space.

3. Show $A^{(n)} - \mathcal{S}^\psi(A^{(n)}) \to 0$ almost surely as $n \to \infty$.

This entails that every subsequential limit $A^{(\infty)}$ of $(A^{(n)})_{n \in \mathbb{N}}$ is a fixed point of $\mathcal{S}^\psi$. By previous lemma: $A^{(\infty)} \equiv F^\psi$.

Note: precompactness shown by entropy bounds, already used by Lootgieter ’77; Slud ’78.
Open problem: empirical distribution of points
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Thank you for your attention!