Prime Points in Orbits: Some Instances of the Bourgain-Gamburd-Sarnak Conjecture

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Abstract

We use Vaughan’s variation on Vinogradov’s three-primes theorem to prove Zariski-density of prime points in several infinite families of hypersurfaces, including level sets of some quadratic forms, the Permanent polynomial, and the defining polynomials of some pre-homogeneous vector spaces. Three of these families are instances of a conjecture by Bourgain, Gamburd and Sarnak regarding prime points in orbits of simple algebraic groups. Our approach is based on the formulation of a general condition on the defining polynomial of a hypersurface, which suffices to guarantee that Zariski-density of prime points is equivalent to the existence of an odd point.

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1 Introduction and statement of the main results

Dirichlet’s Theorem on Arithmetic Progressions states that if two integers $\alpha$ and $\beta$ are co-prime, then there are infinitely many primes in the arithmetic progression $\alpha, \alpha + \beta, \alpha + 2\beta, \ldots$. In other words, there are infinitely many primes that are congruent to $\alpha$ modulo $\beta$. In his survey [18], Sarnak has suggested a new perspective on Dirichlet’s theorem: the additive linear algebraic group $\mathbb{G}_a$ over $\mathbb{R}$ acts on the space $\mathbb{A}^1 = \mathbb{R}^1$ by translations, namely $g \cdot x = x + g$ for $x \in \mathbb{A}^1$ and $g \in \mathbb{G}_a$. Write $\mathbb{G}_a(\mathbb{Z})$ for the lattice subgroup of integer points in $\mathbb{G}_a$, and $\mathbb{A}^1(\mathbb{Z})$ for the set of integer points in $\mathbb{A}^1$. As sets, both $\mathbb{G}_a(\mathbb{Z})$ and $\mathbb{A}^1(\mathbb{Z})$ can be identified with $\mathbb{Z}$, and both $\mathbb{G}_a$ and $\mathbb{A}^1$ can be identified with $\mathbb{R}$. $\mathbb{G}_a(\mathbb{Z})$ also acts on $\mathbb{A}^1$ by translations, and so does any cyclic subgroup $\langle \beta \rangle = \beta \mathbb{Z}$ of $\mathbb{G}_a(\mathbb{Z})$. For $\alpha \in \mathbb{A}^1(\mathbb{Z})$ and $\beta \in \mathbb{G}_a(\mathbb{Z})$, the orbit $\langle \beta \rangle \cdot \alpha$ is simply the arithmetic progression $\alpha + \beta \mathbb{Z}$. Since in $\mathbb{A}^1$ a set is Zariski-dense if and only if it is infinite, we see that Dirichlet’s theorem can be stated as follows: the set of prime points in an orbit $\langle \beta \rangle \cdot \alpha$ is Zariski-dense in $\mathbb{A}^1$ if and only if $\gcd(\alpha, \beta) = 1$.

In recent years, several questions pertaining to Zariski-density of prime points in orbits of the group of integral points of an algebraic group defined over $\mathbb{Q}$ have been formulated. These questions continue the longstanding tradition of questions regarding prime values of algebraic curves. An example for the latter is the Euler conjecture, which concerns prime values of the quadratic function $x^2 + 1$, or the twin prime conjecture, which concerns values of the quadratic function $x(x + 2)$ with exactly two prime factors. Concerning the case where the algebraic varieties are orbits of algebraic groups, Bourgain, Gamburd and Sarnak have formulated some conjectures, one of which is the following [3, Conj. 1.4] (see also [18]):

**Conjecture** (Bourgain, Gamburd, Sarnak). Let $G \subset GL_n(\mathbb{R})$ be the group of real points of an algebraically connected, algebraically simply connected, absolutely almost simple linear algebraic group defined over $\mathbb{Q}$. Let $\Lambda = O$ be a subgroup of $G(\mathbb{Z}) = G \cap GL_n(\mathbb{Z})$ which is Zariski-dense in $G$, and $f$ be a non-zero polynomial in the coordinate ring $\mathbb{Q}[G]$. Assume that $f$ is not a unit, that it assumes integral values on $O$, and that it factors into $k$ irreducibles in the coordinate ring $\mathbb{Q}[G]$. Denote by $r_0(O, f)$ the least $r$ such that the set of $x \in O$ for which $f(x)$ has at most $r$ prime factors, is Zariski-dense in $G$, the Zariski-closure of $O$. Assume that for every integer $q \geq 2$, there exists $x \in O$ such that $\gcd(f(x), q) = 1$ (such a pair $(O, f)$ is called primitive). Then $r_0(O, f) = k$.

**Remark.** Note that $\mathbb{Q}[G]$ is indeed a unique factorization domain so that the number of irreducible factors of $f$ is well-defined, see the discussion and references preceding [3, Conj. 1.4]. Also, note that if $(O, f)$ is primitive then in particular $\gcd \{f(x) \mid x \in O\} = 1$.

The Bourgain-Gamburd-Sarnak conjecture has been established in some cases ([11], [3], [8], [1], [7], [17]), one of which is the following [16]. Let $G = SL_n(\mathbb{R})$ and let $\text{Mat}_n(\mathbb{R})$ be the space of $n \times n$ matrices over $\mathbb{R}$, which we identify with the affine space
Consider the action of $G$ on $\text{Mat}_n$ by left matrix multiplication $g \cdot x = gx$ for $g \in G$ and $x \in \text{Mat}_n$. Disregarding the variety of singular matrices, each $\text{SL}_n$-orbit is of the form

$$D_m = \{ x \in \text{Mat}_n(\mathbb{R}) : \det(x) = m \}$$

for $0 \neq m \in \mathbb{R}$. Let $\Lambda = \text{SL}_n(\mathbb{Z})$, and call an integral matrix prime if all of its entries are prime numbers ($\neq 1$) in $\mathbb{Z}$.

**Theorem 1.1** ([16]). For $n \geq 3$ and $m \neq 0$, prime matrices are Zariski-dense in $D_m$ if and only if $m \equiv 0 \pmod{2^{n-1}}$.

If prime matrices are Zariski-dense in $D_m$, then, since $D_m(\mathbb{Z})$ is a union of finitely many $\text{SL}_n(\mathbb{Z})$-orbits, there exists at least one $\text{SL}_n(\mathbb{Z})$-orbit $O \subset D_m$ such that $O$ is Zariski-dense in $D_m$ and prime matrices are Zariski-dense in $O$. This means that when there are no congruence obstructions, $r_0 (O, f) = n^2$ for $f (x_{1,1}, \ldots, x_{n,n}) = \prod_{i,j=1}^{n} x_{i,j}$.

It is a natural problem to find further infinite families of examples where the conjecture holds, and this is the goal of the present paper. We extend the approach of [16] to varieties given as the level set of polynomials with a certain structure, that generalizes the determinant polynomial. More specifically, we consider hyper-surfaces of the form $X_m = \{ x : \Delta (x) = m \}$ where $m \in \mathbb{Z}$, and formulate a sufficient condition on $\Delta$ such that prime points are Zariski-dense in $X_m$ if and only if $X_m$ contains an odd point (Theorem E). While the condition on $X_m$ is clearly necessary, it is interesting that under certain conditions on the structure of $\Delta$ it is also sufficient.

The determinant variety $D_m$ is one instance in which our method holds; we proceed to discuss some further examples.

In the first example, we consider the quadratic form on $\mathbb{R}^{2n+k}$:

$$Q_{n,k}(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_k) = \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{k} z_i^2$$

and the algebraic variety defined as the level set of this form,

$$Q_m = \left\{ (x, y, z) \in \mathbb{R}^{2n+k} : Q_{n,k}(x, y, z) = m \right\},$$

where $m \neq 0$ is an integer. The variety $Q_m$ is an orbit of the orthogonal group of the form $Q_{n,k}$ in its action on $\mathbb{R}^{2n+k}$ by matrix multiplication. Since this group is conjugate to the orthogonal group $\text{SO}_{2n,k}$, for $n \geq 3$ it is a simple (but not simply connected) algebraic group. The following is a necessary and sufficient condition for Zariski density of prime vectors in $Q_m$, namely vectors all of whose entries are prime numbers ($\neq 1$) in $\mathbb{Z}$:

**Theorem A.** Let $n \geq 3$, $k \geq 0$ and $m \neq 0$ be integers. Prime vectors are Zariski-dense in $Q_m$ if and only if $n + k \equiv m \pmod{2}$. 

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In the second example, we consider the variety of $2n \times 2n$ anti-symmetric matrices of fixed Pfaffian $m$:

$$\mathcal{F}_m = \left\{ x \in \text{Mat}_{2n}(\mathbb{R}) \mid x^t = -x, \text{Pf}(x) = m \right\}.$$  

For $m \neq 0$, this variety is an SL$_{2n}$-orbit under the action: $g \cdot x = gxg^t$, $g \in \text{SL}_{2n}$. An anti-symmetric matrix all of whose non-diagonal entries are primes numbers ($\neq 1$) in $\mathbb{Z}$ will be called a prime matrix in $\mathcal{F}_m$. We prove the following:

**Theorem B.** For $n \geq 2$, prime matrices are Zariski-dense in $\mathcal{F}_m$ if and only if $m$ is an odd integer.

While the Bourgain-Gamburd-Sarnak conjecture has been formulated for simple groups $G$, it is of course natural to consider the case where $G$ is semi-simple as well. Indeed, our third infinite family of examples consists of orbits of $G = \text{SP}_\ell(\mathbb{R}) \times \text{SL}_{2n}(\mathbb{R})$ acting on the space of $2\ell \times 2n$ matrices by $(g_1, g_2) \cdot x = g_1xg_2^t$ for $g_1 \in \text{SP}_\ell$, $g_2 \in \text{SL}_{2n}$, and $x \in \text{Mat}_{2\ell \times 2n}(\mathbb{R})$ with $\ell \geq n \geq 1$. Let

$$\Omega_\ell = \begin{bmatrix} 0_\ell & I_\ell \\ -I_\ell & 0_\ell \end{bmatrix}. \quad (1.2)$$

Let $m \in \mathbb{Z}$ and define the variety

$$\mathcal{R}_m = \left\{ x \in \text{Mat}_{2\ell \times 2n}(\mathbb{R}) \mid \text{Pf} \left( x^t \Omega_\ell x \right) = m \right\},$$

for which we prove the following:

**Theorem C.** For $n \geq 1$, $\ell \geq 2$, $\ell \geq n$, and $0 \neq m \in \mathbb{Z}$, prime matrices are Zariski-dense in $\mathcal{R}_m \subset \text{Mat}_{2\ell \times 2n}(\mathbb{R})$ if and only if $m \equiv 0 \pmod{2^{2n-1}}$.

The motivation for this family of varieties comes from pre-homogeneous vector spaces, to which the families $\{D_m\}_{m \neq 0}$, $\{Q_m\}_{m \neq 0}$ and $\{F_m\}_{m \neq 0}$ belong as well, as explained in Section 7.

In section 8, using the same technique, we prove Zariski-density of prime points in some non-homogeneous varieties, which do not support a group action at all. The main example is the variety of $n \times n$ matrices of fixed permanent $m$:

$$\mathcal{P}_m = \left\{ x \in \text{Mat}_n(\mathbb{R}) \mid \text{perm}(x) = m \right\},$$

for which we prove the following:

**Theorem D.** Let $n \geq 3$ and $0 \neq m \in \mathbb{Z}$. Write $2^s - 1 \leq n < 2^{s+1} - 1$ for a unique integer $s \geq 2$ (i.e. $s = \lfloor \log_2(n + 1) \rfloor$). Then prime matrices are Zariski-dense in $\mathcal{P}_m$ if and only if

$$m \equiv \begin{cases} 2^{n-s} \pmod{2^{n-s+1}} & \text{if } n = 2^s - 1 \\ 0 \pmod{2^{n-s}} & \text{if } 2^s - 1 < n < 2^{s+1} - 1. \end{cases}$$
Note that in this case, the set of \( m \in \mathbb{Z} \) where \( \mathcal{P}_m \) is Zariski dense, depends on whether \( n + 1 \) is a power of 2 or not!

**Remark 1.2.** A key ingredient of our method is that, as was noted in [10], establishing Zariski-density of prime points in such varieties \( \mathcal{X}_m \) can be reduced to solving a non-homogeneous linear Diophantine equation

\[
\alpha_1 \xi_1 + \ldots + \alpha_n \xi_n + \alpha_{n+1} = 0
\]

in primes. This is possible by a theorem of Vaughan [23] (based on Vinogradov’s Three-Prime Theorem), when the integers \( \alpha_1, \ldots, \alpha_{n+1} \) satisfy certain congruence conditions and \( n \geq 3 \). It will be discussed extensively in Section [2].

**Connection to the Bourgain-Gamburd-Sarnak conjecture.** We remark that Theorems [A] [B] and [C] are indeed new instances of the conjecture stated above. The common setting of these examples (generalizing that of Theorem [11]) is that of a semi-simple linear algebraic group \( G \) acting on a real finite dimensional linear space \( V(\mathbb{R}) \), realized as a space of matrices \( \text{Mat}_{n_1 \times n_2}(\mathbb{R}) \). There exists a polynomial \( \Delta \) on \( V \cong \mathbb{R}^{\dim V} \) which is invariant under the action of \( G \) (namely \( \Delta(g \cdot x) = \Delta(x) \)) such that the level sets \( \mathcal{X}_m = \{ x \in V : \Delta(x) = m \} \) for \( m \neq 0 \) are orbits of \( G \). (The level set for \( m = 0 \) consists of several \( G \)-orbits and will not be part of our discussion.) When restricting to integral points, \( G(\mathbb{Z}) \) acts on \( V(\mathbb{Z}) \), and \( V(\mathbb{Z}) = \bigcup_{m \in \mathbb{Z}} \mathcal{X}_m(\mathbb{Z}) \), since \( \Delta \) has integer coefficients. Moreover, every \( \mathcal{X}_m(\mathbb{Z}) \) for \( 0 \neq m \in \mathbb{Z} \) is a finite union of \( G(\mathbb{Z}) \)-orbits; hence, since \( \mathcal{X}_m(\mathbb{Z}) \) is Zariski-dense in \( \mathcal{X}_m \), there exists at least one such \( G(\mathbb{Z}) \)-orbit \( \mathcal{O} \) which is also Zariski-dense in \( \mathcal{X}_m \).

Theorems [11] [A] [B] and [C] assert that when there are no congruence obstructions, prime points are Zariski dense in \( \mathcal{X}_m \) (the Zariski closure of \( \mathcal{O} \)); this means that for the polynomial \( f(x_1, \ldots, x_{\dim V}) = \prod_{i=1}^{\dim V} x_i \), \( r_0(\mathcal{O}, f) = \dim V \).

The Bourgain-Gamburd-Sarnak conjecture is formulated specifically for a simply connected simple algebraic group \( \tilde{G} \subset GL_n \) defined over \( \mathbb{Q} \), with the orbit being a Zariski-dense subgroup \( \Lambda \) of \( \tilde{G}(\mathbb{Z}) \). To put this in perspective, note that in [3] §2.2, the example of the double-cover adjoint epimorphism \( \phi: \tilde{G} = SL_2(\mathbb{R}) \rightarrow SO(F) = G \) is considered, where \( F \) is the three variables form \( F(x, y, z) = xz - y^2 \). It is shown there that the conjecture fails for the adjoint group \( SO(F) \) and the primitive polynomial \( f(x, y, z) = x_{1,1} - 1 \in \mathbb{Q}[G] \).

When \( G \) is not simply connected, the covering map \( \tilde{G} \rightarrow G, \tilde{g} \rightarrow g \) is not injective, and to compare the conjecture to our set-up we follow the discussion in [3] §2.2. Given the \( G \)-orbit \( G \cdot v_0 = \mathcal{X} \subset \mathbb{R}^d \cong V \cong G/H \) with \( v_0 \in \mathbb{Z}^d \) and \( H = \text{St}_G(v_0) \), consider the map \( \phi: \tilde{G} \rightarrow \mathcal{X} \) given by \( \phi(\tilde{g}) = g \cdot v_0 \), so that \( \mathcal{O} = \phi(\tilde{G}(\mathbb{Z})) = G \cdot v_0 \subset \mathcal{X}(\mathbb{Z}) \).

There is an injective ring homomorphism between the coordinate rings \( \phi^*: \mathbb{Q}[G/H] = \mathbb{Q}[\mathcal{X}] \rightarrow \mathbb{Q}[\tilde{G}], \) given by \( \phi^*(f) = f \circ \phi \). Given a polynomial \( f \in \mathbb{Q}[\mathcal{X}] = \mathbb{Q}[G/H] \) which is primitive on \( \mathcal{O} \), clearly \( \phi^*(f) \in \mathbb{Q}[\tilde{G}] \) is primitive on \( \tilde{G}(\mathbb{Z}) = \Lambda \), and then the conjecture
asserts that $r_0(\Lambda, \phi^*(f))$ is equal to the number of irreducible factors in the factorization of $\phi^*(f)$ to irreducibles in the unique factorization domain $\mathbb{Q}[\tilde{G}]$.

Let us begin by noting that $r_0(\Lambda, \phi^*(f))$ is equal to $r_0(\mathcal{O}, f)$, arguing as follows. First, the set of points $x \in \mathcal{O} \subset X(\mathbb{Z}) \subset V \cong G/H$ where $f(x)$ has strictly less than $r_0(\mathcal{O}, f)$ prime factors has a non-trivial polynomial vanishing on it, and so the same is true of its inverse image in $\tilde{G}(\mathbb{Z}) = \Lambda$, namely the set of $\tilde{x} \in \Lambda$ where $\phi^*(f)(\tilde{x})$ has less than $r_0(\mathcal{O}, f)$ prime factors. Hence $r_0(\Lambda, \phi^*(f)) \geq r_0(\mathcal{O}, f)$. Second, to see that the set $Z^0$ of $\tilde{x} \in \Lambda$ where $\phi^*(f)(\tilde{x})$ has at most $r_0(\mathcal{O}, f)$ prime factors is Zariski dense in $\tilde{G}$, assume for contradiction that it is not, and let $Z \subset \tilde{G}$ denote its Zariski closure. Clearly, if $\tilde{H}(\mathbb{Z}) = \tilde{H} \cap \tilde{G}(\mathbb{Z})$, then $Z^0 \tilde{h} = Z^0$ for every $\tilde{h} \in \tilde{H}(\mathbb{Z})$, and hence $Z \tilde{h} = Z$ for every $\tilde{h} \in \tilde{H}(\mathbb{Z})$. It follows that $Z \tilde{h} = Z$ for every $\tilde{h} \in \tilde{H}$, since $\tilde{H}$ is the Zariski closure of $\tilde{H}(\mathbb{Z})$. This is a consequence of the Borel density theorem, since $\tilde{H}$ is a (semi)simple algebraic group defined over $\mathbb{Q}$ in the examples under consideration, and $\tilde{H}(\mathbb{Z}) \subset \tilde{H}$ is a lattice subgroup. As a result, $\phi(Z) \subset X \cong G/H$ is a proper Zariski closed subset containing the set of all $x \in \mathcal{O}$ having the property that $f(x)$ is the product at most $r_0(\mathcal{O}, f)$ prime factors. Since the latter set is Zariski dense by definition of $r_0(\mathcal{O}, f)$, we have arrived at a contradiction, and as a result $r_0(\mathcal{O}, f) = r_0(\Lambda, \phi^*(f))$.

Theorems [A] [B] and [C] establish that $r_0(\mathcal{O}, f)$ is equal to the number of irreducible factors of $f$ in $\mathbb{Q}[G/H] = \mathbb{Q}[X]$, namely $d = \dim V$. Thus the verification of the conjecture for the pair $(\Lambda, \phi^*(f))$ will be complete upon showing that the number of irreducible factors of $\phi^*(f)$ in $\mathbb{Q}[\tilde{G}]$ is equal to $r_0(\mathcal{O}, f)$, and no more. Let $\phi^*(f) = h_1 \cdots h_s$ be the decomposition into non-trivial irreducibles in the unique factorization domain $\mathbb{Q}[\tilde{G}]$, and assume for contradiction $s > d = r_0(\mathcal{O}, f)$. The group of real points $\tilde{G}(\mathbb{R})$ is contained in $M_n(\mathbb{R})$ for some $n$, and coincides with the set of common zeros of an ideal $\mathcal{J}$ in the polynomial ring $\mathbb{Q}[\{t_{i,j}\}_{i,j=1}^n]$. We can represent each element $h_i$ in the ring $\mathbb{Q}[\tilde{G}]$ in the form $\frac{a_i}{b_i} h'_i + j_i$, where $h'_i \in \mathbb{Q}[\{t_{i,j}\}_{i,j=1}^n]$ is a polynomial with integral coefficients whose greatest common divisor is 1 (for definiteness), $a_i, b_i \in \mathbb{Z} \setminus \{0\}$, and $j_i \in \mathcal{J}$. Then $\phi^*(f) = \frac{a}{b} h'_1 \cdots h'_s + j$ and for $\tilde{x} \in \Lambda$, and $\phi(\tilde{x}) = x = (x_1, \ldots, x_d) \in \mathcal{O} \subset X(\mathbb{Z})$, we have the identity (since $j(\tilde{x}) = 0$)

$$\phi^*(f)(\tilde{x}) = \frac{a}{b} h'_1(\tilde{x}) \cdots h'_s(\tilde{x}) = f(\phi(\tilde{x})) = f(x) = x_1 \cdots x_d$$

For $\tilde{x} \in \tilde{G}(\mathbb{Z})$, each $h'_i(\tilde{x})$ is an integer, and so the previous identity represents the integer $f(x)$ as a product of $d$ integers, and as a product of $s > d$ integers and the rational number $\frac{a}{b}$, where we assume $(a, b) = 1$.

Consider the set $\tilde{x} \in \tilde{G}(\mathbb{Z}) = \Lambda$ where the integer $f(\phi(\tilde{x}))$ is a product of exactly $d = r_0(\mathcal{O}, f)$ prime factors. This set is clearly Zariski dense by definition of $r_0(\Lambda, \phi^*(f))$ and the fact that it is equal to $r_0(\mathcal{O}, f)$. However, $f(\phi(\tilde{x}))$ is also a product of $s > d$ integer factors $h'_i(\tilde{x})$ and $\frac{a}{b}$. For this to happen $b$ must cancel against some of the factors dividing $h'_i(\tilde{x})$, and in addition possibly some of the factors $h'_i(\tilde{x})$ are equal to $\pm 1$. It follows that the set in question is contained in the union of the zero sets of the
polynomials \( h_i' \pm 1 \) and \( h_i' \pm c \), where \( c \) ranges over all the factors of \( b \), and \( 1 \leq i \leq s \).

The latter condition is a consequence of the fact that \( s > d \). This last set is not Zariski dense in \( \tilde{G} (\mathbb{R}) \), because if it were, one of the polynomials \( h_i' \pm c \) would vanish identically on \( \tilde{G} (\mathbb{R}) \), but we have assumed that \( h_i = \frac{a}{b} h_i' + j_i \) is a non-trivial irreducible element in the ring \( \mathbb{Q}[\tilde{G}] \), namely not a constant and not a unit. Therefore we have arrived at a contradiction, and we can conclude that the conjecture is verified for the pair \( (\tilde{G}(\mathbb{Z}), \phi^* (f)) \).

In fact, the arguments above verify the conjecture for the polynomial \( \phi^* (f) \) and any Zariski dense subgroup \( \Lambda \subset \tilde{G}(\mathbb{Z}) \) which is transitive on \( O \), and satisfies also that \( \Lambda \cap \tilde{H}(\mathbb{Z}) \) is Zariski dense in \( \tilde{H} \).

2 The method of proof: prime solutions to linear equations

The varieties that we consider in the present paper are of the form

\[
X_m = \{ x : \Delta (x) = m \}
\]

with \( 0 \neq m \in \mathbb{Z} \),

\[
x = (\xi, y, z) = ((\xi_1, \ldots, \xi_n), (y_1, \ldots, y_N), (z_1, \ldots, z_k)),
\]

and

\[
\Delta (x) = F_1 (y) \xi_1 + \ldots + F_n (y) \xi_n + G (z),
\]

where \( G \) and \( F_i \) for all \( i = 1, \ldots, n \) are polynomials with integer coefficients, and each \( F_i \), \( i = 1, \ldots, n \) is not the zero polynomial.

For example, the determinant of a matrix \( x \in \text{Mat}_n \) can be expanded along the \( i \)-th row, and is therefore “a linear combination” of the variables \( (\xi_1, \ldots, \xi_n) = (x_{i,1}, \ldots, x_{i,n}) \), where the “coefficients” are polynomials in the remaining entries of \( x \). As for the remaining examples, we shall verify later on that these varieties indeed share this structure.

When the variables \( y \) and \( z \) assume fixed integer values, the equation \( \Delta (x) = m \) becomes a non-homogeneous linear Diophantine equation in the variables \( \{\xi_i\}_{i=1}^n \). Under certain necessary congruence conditions on the coefficients, such equations can be solved in primes:

**Theorem 2.1** (Vaughan, [23]). Let \( \alpha_1, \ldots, \alpha_n, m \in \mathbb{Z} \setminus \{0\} \) where \( n \geq 3 \), and consider the equation:

\[
\alpha_1 \xi_1 + \ldots + \alpha_n \xi_n = m.
\]

Let \( T \) be a large positive integer, and let:

\[
\mathcal{H}_{\text{prime} \leq T} = \left\{ (p_1, \ldots, p_n) \mid \begin{array}{l}
|p_i| \leq T \\
\alpha_1 p_1 + \ldots + \alpha_n p_n = m
\end{array} \right\}.
\]
Then for every fixed large $C > 0$

$$\left| H_{\text{prime} \leq T} \right| \geq \mathcal{G} \cdot \frac{T^{n-1}}{(\log T)^{n}} + O_{C} \left( \frac{T^{n-1}}{(\log T)^{C}} \right) \quad (2.3)$$

where $\mathcal{G} > 0$ if and only if for all $i = 1, \ldots, n$

$$\gcd (\alpha_{1}, \ldots, \alpha_{n}) = \gcd (\{\alpha_{1}, \ldots, \alpha_{n}, m\} \setminus \{\alpha_{i}\}) \quad (2.4)$$

and

$$\alpha_{1} + \ldots + \alpha_{n} - m \equiv 0 \pmod{2 \cdot \gcd (\alpha_{1}, \ldots, \alpha_{n}, m)}. \quad (2.5)$$

Theorem 2.1 is actually a variation on Vinogradov’s three-prime theorem (\cite{24}; see also \cite{23} and \cite{15}) and is stated as an exercise in \cite{23}; for the details of the proof, see \cite{9}. The following is a consequence of Theorem 2.1.

**Corollary 2.2.** Let $H$ be the affine space of solutions to the non-homogeneous linear equation:

$$\alpha_{1}\xi_{1} + \ldots + \alpha_{n}\xi_{n} = m$$

where $\alpha_{1}, \ldots, \alpha_{n}, m \in \mathbb{Z} \setminus \{0\}$ and $n \geq 3$. Let $H_{\text{prime}} \subset H$ be the set of prime points in $H$, namely the set of prime vector solutions to the given equation. Assume the integers $\alpha_{1}, \ldots, \alpha_{n}, m$ satisfy conditions (2.4) and (2.5) stated in Theorem 2.1. Then $H_{\text{prime}}$ is Zariski-dense in $H$.

**Notation:** $Z (h)$. For a polynomial $h (\xi_{1}, \ldots, \xi_{n})$, we let $Z (h)$ denote the zero-set of $h$, namely the set $Z (h) = \{(\xi_{1}, \ldots, \xi_{n}) \mid h (\xi_{1}, \ldots, \xi_{n}) = 0\}$.

**Proof.** $H$ is a translation of a linear space of dimension $n - 1$ and therefore $H \cong \mathbb{A}^{n-1}$. In particular, $H$ is irreducible. Assume $H_{\text{prime}}$ is not Zariski-dense in $H$. Since $H$ is irreducible, this means that there exists a polynomial $h (\xi_{1}, \ldots, \xi_{n})$ such that $h (\xi_{1}, \ldots, \xi_{n}) = 0$ for all $(\xi_{1}, \ldots, \xi_{n}) \in H_{\text{prime}}$, and $h \not\equiv 0$ on $H$. We may assume $h$ to be irreducible. $H$ is an irreducible affine variety of dimension $n - 1$, hence any proper closed hyper-surface inside it is of dimension $n - 2$. In particular, the zero set $Z (h)$ is of dimension $n - 2$. It follows (\cite{13} Lemma 1) that the number of integer points inside

$$H \cap \{(\xi_{1}, \ldots, \xi_{n}) : h (\xi_{1}, \ldots, \xi_{n}) = 0\} \cap \{(\xi_{1}, \ldots, \xi_{n}) : |\xi_{i}| \leq T \text{ for all } i\} \quad (2.6)$$

is bounded by a constant times $T^{n-2}$, namely it is $O (T^{n-2})$. As we assume $H_{\text{prime} \leq T}$ is contained in the set (2.6) this contradicts (2.3) with $\mathcal{G} > 0$. \qed

Theorem 2.1 and specifically Corollary 2.2 are the key ingredients of the proof of Theorem 1.1, as well as of the examples that we consider in this paper. We shall use the technique presented in \cite{16} as follows.

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Theorem 2.3. Let \( m \in \mathbb{Z} \), \( m \neq 0 \) and consider a variety \( \mathcal{X}_m \) of the form 2.1, with \( F_1, \ldots, F_n \) and \( G \) as in 2.2. Assume that \( G(z) \) is not identically equal to \( m \), and that there exists a Zariski-dense subset \( \mathcal{G} \subset \mathbb{R}^{N+k} \) whose elements are prime vectors \((y, z) \in \mathbb{Z}^{N+k}\) that satisfy

\[
F_1(y) + \ldots + F_n(y) + G(z) - m \equiv 0 \pmod{2 \cdot \gcd (F_1(y), \ldots, F_n(y), G(z) - m)} , \\
gcd \left( \{ F_1(y), \ldots, F_n(y), G(z) - m \} \setminus \{ F_j(y) \} \right) = \gcd (F_1(y), \ldots, F_n(y))
\]

for all \( j = 1, \ldots, n \). Then prime points are Zariski-dense in \( \mathcal{X}_m \).

Remark 2.4. Except for the variety \( \mathcal{Q}_m \) in Theorem A, all the examples considered in this paper have \( G(z) \) which identically equals zero.

Theorem 2.3 is a consequence of Corollary 2.2 along with the following two observations — the first is that the varieties \( \mathcal{X}_m \) are irreducible.

Lemma 2.5. The polynomial

\[
\Delta (\xi, y, z) = \sum_{i=1}^{n} F_i(y) \xi_i + G(z) ,
\]

where \( G(z) \) is not the zero polynomial, is irreducible.

Proof. Suppose \( \Delta = f \cdot h \). Assume without the loss of generality that \( \deg_{\xi_i} f \geq 1 \) (namely \( \deg_{\xi_i} f = 1 \)), and therefore \( \deg_{\xi_i} h = 0 \). Since monomials of the form \( \xi_i \xi_j \) do not appear in \( \Delta \), it follows that for every \( i = i, \ldots, n \) \( \deg_{\xi_i} f = 1 \) and \( \deg_{\xi_i} h = 0 \). Thus,

\[
f = \sum_{i=1}^{n} f_i(y, z) \xi_i + g(y, z).
\]

\[
h = h(y, z) .
\]

It follows that

\[
G(z) = g(y, z) \cdot h(y, z) ,
\]

which implies

\[
g(y, z) = g(z) \\
h(y, z) = h(z) .
\]

Hence

\[
\Delta = \left( \sum_{i=1}^{n} f_i(y, z) \xi_i + g(z) \right) \cdot (h(z)) .
\]

Then

\[
F_i(y) = f_i(y, z) \cdot h(z) ,
\]

namely \( f_i(y, z) = F_i(y) \) and \( h(z) \) is a scalar. \( \square \)
We note that the homogeneous varieties we consider each constitutes an orbit of a connected algebraic group, and hence are clearly irreducible. But for the non-homogeneous varieties we consider the previous argument is necessary.

Let us now formulate the context in which Vaughan's criterion will be applied. In what follows, we use the term "algebraic variety" as an abbreviation for the term "the set of real points of an algebraic variety defined over \( \mathbb{R} \)" which describes all the varieties we will consider in the present paper.

**Lemma 2.6.** Let \( \Xi \) and \( Y \) be algebraic varieties, and let \( X \subset \Xi \times Y \) be an irreducible subvariety. Let \( A \) be a Zariski-dense subset of \( Y \) such that for every \( y \in A \) the fiber \( X_y := \{ \xi : (\xi, y) \in X \} \) is non-empty. Assume that for every \( y \in A \) there exists a Zariski-dense subset \( B_y \) in \( X_y \). Then, the set

\[
T = \{ (\xi, y) : y \in A, \xi \in B_y \}
\]

is Zariski-dense in \( X \).

**Proof.** We let \( U \subset X \) be a non-empty open set, and show that \( U \) contains a point from \( T \). Let \( \phi : X \to Y \) be the natural projection, which is defined over \( \mathbb{R} \) and has a Zariski-dense image, by assumption.

Since \( X(\mathbb{C}) \) is irreducible and \( U(\mathbb{C}) \) is open and non-empty, \( U(\mathbb{C}) \) is irreducible and dense in \( X(\mathbb{C}) \). Therefore, \( \phi(U(\mathbb{C})) \) contains a non-empty open set \( W(\mathbb{C}) \) (see e.g. [22, Theorem 1.9.5]) of \( \phi(U(\mathbb{C})) = Y(\mathbb{C}) \). Then \( W = W(\mathbb{C}) \cap Y \) is a non-empty open set of \( Y \), since the set of real points \( Y \) of the variety \( Y(\mathbb{C}) \) which is defined over \( \mathbb{R} \) are Zariski dense in \( Y(\mathbb{C}) \) and thus intersect every non-empty open set (see e.g. [2, Chap. AG, Cor. 13.3]). Since \( A \) is dense in \( Y \) it intersects \( W \); let

\[
y \in A \cap W \subseteq A \cap \phi(U).
\]

Then \( \phi^{-1}(y) \cap U \) is non-empty, and clearly open in \( \phi^{-1}(y) \). By projecting to \( \Xi \), we may identify \( \phi^{-1}(y) \) with \( X_y \), and \( \phi^{-1}(y) \cap U \) with an open subset of \( X_y \); this open subset intersects \( B_y \), which is assumed to be Zariski-dense in \( X_y \). Let \( \xi \) be a point in this intersection; then \( (\xi, y) \) is contained in \( U \cap T \).

The following special case of Lemma 2.6, where all the fibers \( X_y \) coincide, will be used later on.

**Example 2.7.** Let \( A \subset \mathbb{A}^m \) be a Zariski-dense subset such that that for every \( a \in A \) there exists a subset \( B_a \subset \mathbb{A}^n \) which is Zariski-dense in \( \mathbb{A}^n \). Then the set

\[
T = \{ (a, b_a) \mid a \in A, b_a \in B_a \}
\]

is Zariski-dense in \( \mathbb{A}^{m+n} \).
We conclude this section with a proof of Theorem 2.3.

Proof. For every \((y, z)\), let \(\mathcal{H}(y, z)\) denote the space of solutions to the non-homogeneous linear equation:

\[
F_1(y)\xi_1 + \ldots + F_n(y)\xi_n + G(z) = m,
\]

and let \(\mathcal{H}_{\text{prime}}(y, z) \subset \mathcal{H}(y, z)\) denote the subset of prime solutions to this equation. Define

\[
\tilde{G} = \{(y, z) \in G \mid G(z) \neq m \text{ and } \forall j : F_j(y) \neq 0\} \subset G,
\]

and note that \(\tilde{G}\) is Zariski-dense in \(\mathbb{A}^{n+k}\), since it is obtained from \(G\) by removing its intersection with the two Zariski-closed subsets defined by \(\{G(z) = m\}\) and \(\bigcup_{j=1}^n \{F_j(y) = 0\}\), which are proper subsets since we assume each \(F_j(y)\) is not the zero polynomial and \(G(z)\) is not the constant \(m\). Moreover, by Corollary 2.2, for every \((y, z) \in \tilde{G}\), the fiber \(\mathcal{H}(y, z)\) is non-empty, and \(\mathcal{H}_{\text{prime}}(y, z)\) is Zariski-dense in \(\mathcal{H}(y, z)\).

In the notations of Lemma 2.6, take \(\Xi = \mathbb{A}^n\), \(Y = \mathbb{A}^{N+k}\) and \(X = \mathcal{X}_m\); note that \(\mathcal{X}_m \subset \mathbb{A}^n \times \mathbb{A}^{N+k}\) is irreducible by Lemma 2.5. The proof is concluded by Lemma 2.6, when taking \(A = \tilde{G}\), \(X_y = \mathcal{H}(y, z)\) and \(B_y = \mathcal{H}_{\text{prime}}(y, z)\). \(\square\)

In order to apply Theorem 2.3 for establishing Zariski-density of prime points in varieties of the form 2.1, one must establish the existence of a set \(G\) of prime points \((y, z)\) on which the polynomials \(F_i(y)\) satisfy the congruence conditions defined in Theorem 2.3 with respect to \(m\) and \(G(z)\). This is the topic of Section 3.

### 3 Variety defined by a quadratic form

As mentioned above, we shall consider varieties of the form 2.1, where the polynomials \(F_i(y)\) that appear as coefficients in the form 2.1 satisfy some general conditions (to be formulated in Theorem 3). However, in the case of the variety \(Q_m\) considered in Theorem A, the polynomials \(F_i(y)\) are quite simple, and so we begin by analyzing this example explicitly. As we shall see later on, this example already demonstrates the main ideas of the general case.

Recall from Section 1 the following quadratic form on \(\mathbb{R}^{2n+k}\):

\[
Q_{n,k}(\xi_1, \ldots, \xi_n, y_1, \ldots, y_n, z_1, \ldots, z_k) = \sum_{i=1}^n \xi_i y_i + \sum_{i=1}^k z_i^2,
\]

and the variety:

\[
Q_m = \{(\xi, y, z) \in \mathbb{R}^{2n+k} \mid Q_{n,k}(\xi, y, z) = m\}.
\]

**Theorem A** Let \(n \geq 3\), \(k \geq 0\) and \(m \neq 0\) be integers. Prime matrices are Zariski-dense in \(Q_m\) if and only if \(n + k \equiv m \pmod{2}\).
Proof of Theorem A. The condition \( n + k \equiv m \pmod 2 \) is necessary: if prime points are Zariski-dense in \( \mathbb{Q}_m \), then there exists an odd point in \( \mathbb{Q}_m \), namely there exist odd integers \( \xi_i, y_i, z_i \) which satisfy the equation \( \sum_{i=1}^{n} \xi_i y_i + \sum_{i=1}^{k} z_i^2 = m \). Take this equation modulo 2 to obtain \( n + k \equiv m \pmod 2 \).

For sufficiency, we apply Theorem 2.3. The variety \( \mathbb{Q}_m \) is of the form 2.1, with \( N = n \),

\[
F_1(y_1, \ldots, y_N) = y_1 \\
\vdots \\
F_n(y_1, \ldots, y_N) = y_n
\]

and

\[
G(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i^2.
\]

Take \( G_{\text{quad}} \subset \mathbb{R}^{n+k} \) to be the set of integer points \((y_1, \ldots, y_n, z_1, \ldots, z_k)\) such that \( y_3, \ldots, y_n, z_1, \ldots, z_k \) are any odd primes satisfying \( m \neq \sum_{i=1}^{k} z_i^2 \), and \( y_1, y_2 \) are distinct odd primes that are co-prime to \( m - \sum_{i=1}^{k} z_i^2 \). By Lemma 2.6 (the case of Example 2.7), \( G_{\text{quad}} \) is Zariski-dense in \( \mathbb{R}^{n+k} \). For every \((y_1, \ldots, y_n, z_1, \ldots, z_k) \in G_{\text{quad}} \) it holds that

\[
gcd \left( y_1, \ldots, y_n, m - \sum_{i=1}^{k} z_i^2 \right) = 1, \\
gcd (y_1, \ldots, y_n) = 1, \\
gcd \left( \left\{ y_1, \ldots, y_n, m - \sum_{i=1}^{k} z_i^2 \right\} \setminus \{y_j\} \right) = 1
\]

for all \( j = 1, \ldots, n \) and

\[
y_1 + \ldots + y_n + m - \sum_{i=1}^{k} z_i^2 = 0 \pmod 2
\]

because of the condition on \( m, n, k \). By Theorem 2.3, we conclude that prime points are Zariski-dense in \( \mathbb{Q}_m \).

Remark 3.1. It would have been simpler to take \( G_{\text{quad}} \subset \mathbb{R}^{n+k} \) to be the set of integer points \((y_1, \ldots, y_n, z_1, \ldots, z_k)\) such that \( y_4, \ldots, y_n, z_1, \ldots, z_k \) are any odd primes, and \( y_1, y_2, y_3 \) are different odd primes. The choice of \( G_{\text{quad}} \) as in the above proof, however, demonstrates the idea of the proofs to come.
4 Intertwined polynomials

Our goal in the present and the following sections is to formulate sufficient conditions on the coefficients $F_i(y)$ in a variety of the form \ref{2.1} so that they will satisfy the conditions of Theorem \ref{2.3} implying that prime points Zariski-dense in this variety. In particular, we wish to be able to control the gcd’s of every $n$-sized subset of \{\(G(z) - m, F_1(y), \ldots, F_n(y)\)\}, for a Zariski-dense subset of prime vectors \((y, z)\). As a result, we are interested in the common prime factors of \{\(G(z) - m, F_1(y), \ldots, F_n(y)\)\}. For reasons that will be discussed in the next section, the case of the prime 2 should be handled separately.

In this section, we formulate conditions on a pair $F, \tilde{F}$ of polynomials, such that there exists a Zariski-dense subset $G$ of every $n$-sized subset of \{\(G(z) - m, F_1(y), \ldots, F_n(y)\)\} has an "iterated linear structure", as follows.

**Definition 4.1** (A pair of intertwined polynomials). Let $\mathcal{Y}$ be a set of commutative variables. Two polynomials $F, \tilde{F}$ in the variables $\mathcal{Y}$ with integer coefficients are called **intertwined of depth $d = 1$**, if the set of variables $\mathcal{Y}$ has a decomposition to three mutually disjoint sets

\[
\mathcal{Y} = U \cup \tilde{U} \cup W = \{(u_i)_i, (\tilde{u}_i)_i, (w_i)_i\}
\]

with $U$ and $\tilde{U}$ non-empty and of the same size (denoted $k_0$), and $F, \tilde{F}$ of the form

\[
F(\mathcal{Y}) = \sum_{i=1}^{k_0} \alpha_i u_i + \beta(W), \quad \tilde{F}(\mathcal{Y}) = \sum_{i=1}^{k_0} \alpha_i \tilde{u}_i + \beta(W) \tag{4.1}
\]

where $\{\alpha_1, \ldots, \alpha_{k_0}\}$ are integers whose gcd is a power of 2 and $\beta(w)$ is an arbitrary polynomial with integer coefficients.

In particular, note that keeping the variables in $W$ fixed, $F$ and $\tilde{F}$ are linear forms in the two disjoint sets of variables $U$ and $\tilde{U}$, inhomogeneous if $W \neq \emptyset$ and $\beta \neq 0$.

Continuing inductively, two polynomials $F, \tilde{F}$ in a set of commuting variables $\mathcal{Y}$ with integer coefficients are called **intertwined of depth $d \geq 2$**, if $\mathcal{Y}$ has a decomposition to four disjoint sets

\[
\mathcal{Y} = U \cup \tilde{U} \cup \mathcal{V} \cup \mathcal{W} = \{(u_i)_i, (\tilde{u}_i)_i, (v_i)_i, (w_i)_i\}
\]

with $U$ and $\tilde{U}$ non-empty and of the same size (denoted $k_d$), and $F, \tilde{F}$ of the form

\[
F(\mathcal{Y}) = \sum_{i=1}^{k_d} \alpha_i (\mathcal{V}) u_i + \beta(\mathcal{V}, \mathcal{W}), \quad \tilde{F}(\mathcal{Y}) = \sum_{i=1}^{k_d} \alpha_i (\mathcal{V}) \tilde{u}_i + \beta(\mathcal{V}, \mathcal{W})
\]
where for some pair $i_1, i_2$ the polynomials $\alpha_{i_1}(V), \alpha_{i_2}(V)$ are intertwined of depth $d - 1$ (namely the set of variables $V$ itself has a decomposition into disjoint sets of variables with $\alpha_{i_1}(V), \alpha_{i_2}(V)$ - playing the role of $F$ and $\tilde{F}$ - satisfying the foregoing conditions).

Note that fixing the variables in $V$ and $W$ again $F$ and $\tilde{F}$ are linear forms in the two disjoint sets of variables $U$ and $\tilde{U}$, possibly inhomogeneous.

We will also say that $F, \tilde{F}$ are intertwined through the set of polynomials $\{\alpha_i\}_{i=1}^{k_d}$.

Before proceeding to give an example of intertwined polynomials, let us introduce the following

**Notation.** $M_{i_1, \ldots, i_k}^{j_1, \ldots, j_l}(a)$: For a matrix $a$, we let $M_{i_1, \ldots, i_k}^{j_1, \ldots, j_l}(a)$ denote the matrix obtained from $a$ by deleting the rows indexed $i_1, \ldots, i_k$ and the columns indexed $j_1, \ldots, j_l$.

**Example 4.2.** If $a$ and $b$ are two $k \times k$ matrices of variables that are identical except for their $j$-th row (resp. column), and in the $j$-th rows (resp. columns) the sets of variables that appear in $a$ and $b$ are disjoint. Then the polynomials $\det(a)$ and $\det(b)$ are intertwined of depth $k$. Indeed, if $k = 1$, then they are clearly intertwined of depth 1; for $k > 1$ we let $Y$ denote the union of variables appearing in $a$ and $b$, and write

$$F(Y) = \det(a) = \sum_{i=1}^{k} (-1)^{i+j} \det \left( M_{i}^{j}(a) \right) \cdot a_{i,j}$$

and

$$\tilde{F}(Y) = \det(b) = \sum_{i=1}^{k} (-1)^{i+j} \det \left( M_{i}^{j}(b) \right) \cdot b_{i,j}$$

(when the matrices $a$ and $b$ differ by the $j$-th column). Then, setting $V$ to be the set of variables appearing in the matrix $M_j(a)$, $\alpha_i(V) = (-1)^{i+j} \det \left( M_{i}^{j}(a) \right)$, $\beta \equiv 0$, $U = (u_i)_{i=1}^{k} = (a_{i,j})_{i=1}^{k}$ and $\tilde{U} = (\tilde{u}_i)_{i=1}^{k} = (b_{i,j})_{i=1}^{k}$, we have:

$$F(Y) = \det(a) = \sum_{i=1}^{k} \alpha_i(V) u_i, \quad \tilde{F}(Y) = \det(b) = \sum_{i=1}^{k} \alpha_i(V) \tilde{u}_i$$

where by the induction hypothesis, every pair among $\{\alpha_i(V)\}$ is intertwined of depth $k - 1$, since they are determinants of two $(k - 1) \times (k - 1)$ matrices that differ only by one column (resp. row).

Since intertwined polynomials have integral coefficients, they assume integer values on integral substitutions. We are interested in the situation when the integral values obtained by a intertwined couple have no common prime factors other than 2.
Definition 4.3. Integers $\alpha_1, \ldots, \alpha_n$ are called 2-coprime if they have no common prime factors other than 2, namely if $\gcd(\alpha_1, \ldots, \alpha_n)$ is a non-negative power of 2. In particular, if $\{\alpha_i\}_{i=1}^n$ are coprime than they are 2-coprime.

Lemma 4.4. Let $\alpha_1, \ldots, \alpha_n, \beta, \gamma$ be non-zero integers, and let $s_1, \ldots, s_n$ and $q_1 \ldots, q_n$ be non-negative integers such that $\{q_i\}_{i=1}^n$ are odd. Consider the following inhomogeneous linear form in the variables $y_1, \ldots, y_n$:

$$f(y) = \alpha_1 y_1 + \ldots + \alpha_n y_n + \beta.$$ 

If $\gamma$ is 2-coprime to $\gcd(\alpha_1, \ldots, \alpha_n)$, then the set

$$\left\{ y = (y_1, \ldots, y_n) \mid y_i \text{ is an odd prime for all } i, \quad y_i \equiv q_i \ (\text{mod } 2^{s_i}) \text{ for all } i, \quad f(y) \text{ is } 2\text{-coprime to } \gamma \right\} \subset \mathbb{Z}^n$$

is Zariski-dense in $\mathbb{R}^n$.

Remark 4.5. Note that in the special case where $\alpha_1, \ldots, \alpha_n$ are 2-coprime, there are no restrictions on $\beta$.

Remark 4.6. The integer $\gamma$ can be replaced by any finite number of non-zero integers $\gamma_1, \ldots, \gamma_r$, by taking $\gamma = \text{lcm}(\gamma_1, \ldots, \gamma_r)$ (least common multiple); namely, if $\gamma_1, \ldots, \gamma_r$ are non-zero integers such that every $\gamma_j$ is 2-coprime to $\gcd(\alpha_1, \ldots, \alpha_n)$, then the set

$$\left\{ (y_1, \ldots, y_n) \mid y_i \text{ is an odd prime for all } i, \quad y_i \equiv q_i \ (\text{mod } 2^{s_i}) \text{ for all } i, \quad f(y) \text{ is } 2\text{-coprime to } \gamma_j \text{ for all } j \right\} \subset \mathbb{Z}^n$$

is Zariski-dense in $\mathbb{R}^n$.

The proof of Lemma 4.4 is postponed to the Appendix. The concluding result of this section is the following.

Theorem 4.7. Let $m \neq 0$ be an integer, and let $F, \tilde{F}$ be a pair of intertwined polynomials of depth $d \geq 1$ in the set of variables $Y$. There exists a Zariski-dense subset $\mathcal{G}' \subset \mathbb{R}[Y]$ of odd prime points $y$ such that for every $y \in \mathcal{G}'$, any two integers in $\{F(y), \tilde{F}(y), m\}$ are 2-coprime.

Moreover, if $(q_i, s_i)$ are non-negative integers such that $q_i$ is odd, then the elements $y = (y_i)$ of $\mathcal{G}'$ can be chosen such that $y_i \equiv q_i \ (\text{mod } 2^{s_i})$ for every $i$.

Proof. By induction on the depth $d$. If $d = 1$, then $F, \tilde{F}$ are of the form 4.1

$$F(y) = \sum_{i=1}^{k_0} \alpha_i u_i + \beta(w), \quad \tilde{F}(y) = \sum_{i=1}^{k_0} \alpha_i \bar{u}_i + \beta(w),$$
where \( \gcd \{ \alpha_i \}_{i=1}^{k_0} \) is a power of 2, and in particular 2-coprime to \( m \). Hence, by Lemma 4.4, for any integral \( w \) there exists a subset

\[
A_w = \left\{ u = (u_1, \ldots, u_{k_0}) \mid \begin{array}{l}
u_i \text{ are odd primes and } u_i \equiv q_i \mod{2^s_i} \text{ for every } i \\
F(y) = F((u, w)) \text{ is } 2\text{-coprime to } m
\end{array} \right\} \subset \mathbb{Z}^{k_0}
\]

which is Zariski-dense in \( \mathbb{R}^{k_0} \). Also by this Lemma, for every integral \( w \) and \( u \in A_w \) there exists a subset

\[
A_{u,w} = \left\{ \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_{k_0}) \mid \begin{array}{l}
\{\tilde{u}_i\} \text{ are odd primes and } \tilde{u}_i \equiv q_i \mod{2^s_i} \text{ for every } i \\
\tilde{F}(y) = F(\tilde{u}, w) \text{ is } 2\text{-coprime to both } m \text{ and } F(y) = F((u, w))
\end{array} \right\} \subset \mathbb{Z}^{k_0}
\]

which is also Zariski-dense in \( \mathbb{R}^{k_0} \). According to Lemma 2.6 (the case of Example 2.7), the set

\[
G' = \left\{ y = (u, \tilde{u}, w) \mid \begin{array}{l}
\{w_i\} \text{ are odd primes } w_i \equiv q_i \mod{2^s_i} \text{ for every } i \\
u \in A_w \text{ and } \tilde{u} \in A_{u,w}
\end{array} \right\} \subset \mathbb{Z}^{|\mathcal{Y}|}
\]

is Zariski-dense in \( \mathbb{R}^{|\mathcal{Y}|} \).

Let \( d \geq 2 \). Since the pair \( \alpha_1(\mathcal{V}), \alpha_2(\mathcal{V}) \) is intertwined of depth \( d - 1 \), there exists a Zariski-dense set \( G'_{\text{ind}} \subset \mathbb{R}^{|\mathcal{Y}|} \) of odd prime \( v \) (satisfying any desired odd congruence conditions modulo powers of 2) such that for every \( v \in G'_{\text{ind}} \), the integers \( m, \alpha_1(v), \alpha_2(v) \) are pairwise 2-coprime. In particular, \( \gcd \{ \alpha_i(v) \}_{i=1}^{k_0} \) is a power of 2, hence 2-coprime to any given integer, for every \( v \in G'_{\text{ind}} \).

Repeat a similar argument with Lemma 4.4 as with the case of \( d = 1 \):

1. For every \( v \in G'_{\text{ind}} \) and integral \( w \), there exists a Zariski-dense subset \( A_{v,w} \subset \mathbb{R}^{k_d} \) of odd prime \( u \) (in the desired arithmetic progressions modulo powers of 2) such that every \( u \in A_{v,w} \), \( F(y) = F((u, v, w)) \) is 2-coprime to \( m \).

2. For every \( v \in G'_{\text{ind}} \), integral \( w \) and \( u \in A_{v,w} \), there exists a Zariski-dense subset \( A_{u,v,w} \subset \mathbb{R}^{k_d} \) of odd prime \( \tilde{u} \) (in the desired arithmetic progressions modulo powers of 2) such that every \( \tilde{u} \in A_{u,v,w} \), \( \tilde{F}(y) = \tilde{F}(\tilde{u}, v, w) \) is 2-coprime to both \( m \) and \( F(y) \).

By Lemma 2.6, the set

\[
G' = \left\{ y = (u, \tilde{u}, v, w) \mid \begin{array}{l}
(v_i), (w_i) \text{ are odd primes congruent to } q_i \mod{2^s_i} \text{ for every } i, \\
u \in A_{v,w} \text{ and } \tilde{u} \in A_{u,v,w}
\end{array} \right\} \subset \mathbb{Z}^{|\mathcal{Y}|}
\]

is Zariski-dense in \( \mathbb{R}^{|\mathcal{Y}|} \).
5 Congruence conditions on the coefficients: Main Theorem

In Section 1 we have presented several examples of varieties $X_m$ to which our method will be shown to apply, and therefore prime points are Zariski-dense in $X_m$ if and only if $m$ is such that there exists an odd point in $X_m$. Observe that in all of these examples—the varieties $D_m$, $Q_m$, $F_m$, $R_m$ and $P_m$—the necessary and sufficient condition on $m$ is a congruence condition modulo a power of 2. This is not a coincidence: substituting an odd point $x$ imposes such conditions on $m = \Delta (x)$, as well as on the polynomials $F_i(y)$. For example, consider the case of the variety $D_m$ defined by $\det (x) = m$. The determinant of an odd $x \in \text{Mat}_n$ is divisible by $2^n - 1$, since we can add the first row to the $n - 1$ remaining rows and obtain $n - 1$ even rows, without changing the determinant.

Note that this parity condition is the strongest that is shared by all the odd $n \times n$ matrices, since modulo $2^n$, $\det (x)$ can be congruent to either 0 or $2^n - 1$ (depending on $x$).

Let us now return to the notation set in equation (2.2). Recall that $x = (\xi, y, z)$ and

$$\Delta (x) = F_1(y) \xi_1 + \cdots + F_n(y) \xi_n + G(z)$$

where $F_i(y)$ are non-zero polynomials and $G(z)$ are polynomials with integer coefficients.

**Notation.** For an integral $y = \{y_1, \ldots, y_N\}$, let $\varepsilon (y)$ be the maximal positive integer such that $2^{\varepsilon(y) - 1} \mid F_i(y)$ for every $i = 1, \ldots, n$.

**Theorem E.** Let $x$ and $\Delta (x)$ as above, and let $m \in \mathbb{Z}$. Assume that $n \geq 3$, that $G(z)$ is not identically equal to $m$, and that there exist two polynomials in $\{F_1, \ldots, F_n\}$ that are intertwined. Then, the prime points are Zariski-dense in $X_m = \{x : \Delta (x) = m\}$ if and only if there exists an odd point $x^* = (\xi^*, y^*, z^*)$ in $\mathbb{Z}^{n+N+k}$ such that $m \equiv \Delta (x^*) \pmod{2^{\varepsilon(y^*)}}$.

**Proof.** The condition on $m$ is necessary; if prime points are Zariski-dense in $X_m$, then there exists an odd point $x^*$ in $X_m$. Otherwise, every prime point $x = (x_i)_{i=1}^{n+N+k} \in X_m$ has 2 as one of its entries. In particular, the prime points in $X_m$ are contained in $\mathbb{Z} \left( \prod_{i=1}^{n+N+k} (x_i - 2) \right)$, and are therefore not Zariski-dense in $X_m$, unless

$$X_m = \mathbb{Z} (\Delta - m) \subseteq \mathbb{Z} \left( \prod_{i=1}^{N+n+k} (x_i - 2) \right).$$

However, the latter is impossible, since $\Delta - m$ is irreducible and therefore the above implies that

$$(\Delta - m) \mid \prod_{i=1}^{n+N+k} (x_i - 2),$$

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\[ \Delta - m = x_i - 2 \text{ for some } i = 1, \ldots, n + N + k, \text{ a contradiction.} \]

We conclude that there exists an odd point \( x^* \) in \( X_m \), and in particular \( m \equiv \Delta (x^*) \pmod{2^\varepsilon(y^*)} \).

For sufficiency, we apply Theorem 2.3. Assume that there exists an odd \( x^* \) in \( \mathbb{Z}^{n+N+k} \) such that \( m \equiv \Delta (x^*) \pmod{2^\varepsilon(y^*)} \), and let \( \varepsilon = \varepsilon(y^*) \). We show that there exists a Zariski-dense set of odd prime \((y, z)\) such that

1. \( \sum_{i=1}^{n} F_i (y) + G(z) \equiv m \pmod{2^\varepsilon}, \)

2. the gcd of every \( n \)-sized subset of \( \{G(z) - m, F_1 (y), \ldots, F_n (y)\} \) equals \( 2^{\varepsilon - 1} \).

Hence the conditions of Theorem 2.3 are satisfied, and prime points are Zariski-dense in \( X_m \). We begin by showing that the odd point \( x^* = (\xi^*, y^*, z^*) \) (which is not necessarily in the variety \( X_m \)!) satisfies the first condition, and the following weakened form of the second condition:

**2.** the gcd of every \( n \)-sized subset of \( \{G(z) - m, F_1 (y^*), \ldots, F_n (y^*)\} \) is congruent to \( 2^{\varepsilon - 1} \pmod{2^\varepsilon} \).

By assumption,

\[ m \equiv \sum_{i=1}^{n} F_i (y^*) \xi_i^* + G(z^*) \pmod{2^\varepsilon}, \]

where \( 2^{\varepsilon - 1} \) divides every \( F_i (y^*) \), and therefore divides \( m - G(z^*) \) as well. Hence,

\[ \frac{m - G(z^*)}{2^{\varepsilon - 1}} \equiv \sum_{i=1}^{n} \frac{F_i (y^*)}{2^{\varepsilon - 1}} \cdot \xi_i^* \pmod{2}. \]

Since \( \{\xi_i^*\} \) are odd, then in particular

\[ \frac{m - G(z^*)}{2^{\varepsilon - 1}} \equiv \sum_{i=1}^{n} \frac{F_i (y^*)}{2^{\varepsilon - 1}} \pmod{2}. \]

Multiply by \( 2^{\varepsilon - 1} \) to obtain

\[ m - G(z^*) \equiv \sum_{i=1}^{n} F_i (y^*) \pmod{2^\varepsilon}, \quad (5.1) \]

which means that \( x^* = (\xi^*, y^*, z^*) \) indeed satisfies the first condition.

For condition 2*, observe that equation \(5.1\) implies

\[ G(z^*) - m + \sum_{i=1}^{n} F_i (y^*) \equiv 0 \pmod{2^\varepsilon}. \quad (5.2) \]
By the choice of $\varepsilon$, there exists some $i_1 \in \{1, \ldots, n\}$ such that $F_{i_1}(y^*) \equiv 2^{\varepsilon - 1} \pmod{2^\varepsilon}$. Hence equation 5.2 implies that there exists another summand among

$$\{G(z^*) - m, F_1(y^*), \ldots, F_n(y^*)\}$$

which is also congruent to $2^{\varepsilon - 1} \pmod{2^\varepsilon}$. Thus, the gcd of every $n$-sized subset of $\{G(z^*) - m, F_1(y^*), \ldots, F_n(y^*)\}$ is congruent to $2^{\varepsilon - 1} \pmod{2^\varepsilon}$.

Assume that $\{F, \tilde{F}\} \subset \{F_i\}_{i=1}^n$ are intertwined, and let $G'$ be a Zariski-dense set of odd prime $y$ such that $y \equiv y^* \pmod{2^\varepsilon}$ and such that every two integers in $\{m - G(z^*), F(y), \tilde{F}(y)\}$ are 2-coprime (the existence of $G'$ was establish in Theorem 4.7).

Finally, note that $y \equiv y^* \pmod{2^\varepsilon}$ implies that $P(y) \equiv P(y^*) \pmod{2^\varepsilon}$ for any polynomial $P$. There is a Zariski dense subset of odd prime $z$ satisfying the congruence condition $z \equiv z^* \pmod{2^\varepsilon}$, and they also satisfy $m - P(z) \equiv m - P(z^*) \pmod{2^\varepsilon}$ (for any $m$). Therefore the properties established above for $y^*, z^*$ imply that every $y \in G'$ and odd prime $z \equiv z^* \pmod{2^\varepsilon}$ satisfy conditions 1 and 2 above.

In all the examples we consider (Theorems 1.1, A, B, C and D), the polynomials $\Delta(x)$ and $\{F_j(y)\}_{j=1}^n$ have additional symmetry, yielding a situation where the congruence conditions on $m$ required in Theorem $\mathcal{E}$ for some odd point $x^* = (\xi^*, y^*, z^*)$ in $\mathbb{Z}^{n+N+k}$, are actually satisfied by all the odd points in $\mathbb{Z}^{n+N+k}$. Thus, in the proofs of the above mentioned theorems, we shall use the following special case of Theorem $\mathcal{E}$

**Theorem F.** In the setting of Theorem $\mathcal{E}$ let $\varepsilon$ be the maximal positive integer such that $2^{\varepsilon - 1} \mid F_i(y)$ for every $i = 1, \ldots, n$ and every odd $y = \{y_1, \ldots, y_N\}$ and assume that for any odd $x^*, x^{**}$ it holds that

$$\Delta(x^*) \equiv \Delta(x^{**}) \pmod{2^\varepsilon}.$$ 

Then prime points are Zariski-dense in $X_m$ if and only if $m \equiv \Delta(x) \pmod{2^\varepsilon}$ for some (and actually, every) odd $x$.

**Proof.** As in the proof of Theorem $\mathcal{E}$ if prime points are Zariski-dense in $X_m$, then there exists an odd point $x^*$ in $X_m$, and in particular $m \equiv \Delta(x^*) \pmod{2^{\varepsilon(y^*)}}$. Since $\varepsilon \leq \varepsilon(y^*)$, $m \equiv \Delta(x^*) \pmod{2^\varepsilon}$.

Conversely, assume $m \equiv \Delta(x) \pmod{2^\varepsilon}$ for every odd $x$. Let $y^* \in \mathbb{Z}^N$ be odd such that $2^{\varepsilon} \mid F_i(y^*)$ for some $i \in \{1, \ldots, n\}$; then $\varepsilon(y^*) = \varepsilon$. By assumption, for any odd $\xi^*$ and $z^*$, $m \equiv \Delta(x^*) \pmod{2^\varepsilon}$ where $x^* = (\xi^*, y^*, z^*)$. Then the conditions of Theorem $\mathcal{E}$ are met, and prime points are Zariski-dense in $X_m$. 

**Example 5.1.** We note that Theorem 1.1 for Zariski-density of prime points in $\mathcal{D}_m = \{x \in \text{Mat}_n : \det(x) = m\}$ is a consequence of Theorem $\mathcal{E}$. For $x \in \text{Mat}_n(\mathbb{Z})$ denote the
first row of $x$ by $(\xi_1, \ldots, \xi_n)$, and the matrix obtained by removing the first row of $x$ by $y \in \text{Mat}_{(n-1) \times n}(\mathbb{Z})$. An expansion of $\det(x)$ along the first row yields

$$D_1(y) \xi_1 + \cdots + D_n(y) \xi_n = m,$$

where $\{D_i(y)\}_{i=1}^n$ are polynomials in the entries of $y$, and more specifically, determinants of $(n-1) \times (n-1)$ submatrices of $y$ (with alternating signs). By Example 5.2 any pair of coefficients $D_{i_1}(y), D_{i_2}(y)$ is intertwined. As explained in the beginning of this section, for every odd $x^*$ and $i \in \{1, \ldots, n\}$ it holds that $\det(x^*) \equiv 0 \pmod{2^{n-1}}$ and $D_i(y^*) \equiv 0 \pmod{2^{n-2}}$, and it can be shown that these powers are the maximal that hold for every odd point; then $\varepsilon = n - 1$ and by Theorem $A$ prime points are Zariski-dense in $\mathcal{D}_m$ if and only if $m \equiv 0 \pmod{2^{n-1}}$.

**Example 5.2.** Theorem $A$ for Zariski density of prime points in $\mathcal{Q}_m$, defined as a level set of the quadratic form $Q_{n,k}(\xi, y, z) = \sum_{i=1}^n \xi_i y_i + \sum_{l=1}^k z_i^2$, is also a consequence of Theorem $F$. With $F_i(y) = y_i$ for all $i = i, \ldots, n$ and $G(z) = \sum_{l=1}^k z_i^2$ (as in the proof of Theorem $A$ Section 4), it is clearly the case that $\varepsilon = 1$ and $\Delta(x) = Q_{n,k}(\xi, y, z) \equiv m \equiv 0 \pmod{2^1}$ if and only if $n + k \equiv 0 \pmod{2}$.

We now proceed to prove Theorems $B$, $C$ and $D$ by verifying that the varieties in question satisfy the conditions of Theorem $F$.

### 6 Variety of anti-symmetric matrices of fixed Pfaffian

Denote by $\mathcal{A}_{2n}(\mathbb{R})$ the space of anti-symmetric matrices of order $2n \times 2n$ over $\mathbb{R}$, and recall $M_{i_1, \ldots, i_k}^j(a)$ denotes the matrix obtained from a matrix $a$ by deleting the rows indexed $i_1, \ldots, i_k$ and the columns indexed $j_1, \ldots, j_l$. The Pfaffian of a matrix in $\mathcal{A}_{2n}(\mathbb{R})$ is a polynomial of degree $n$ in the matrix entries that can be defined recursively. By convention, the Pfaffian of the $0 \times 0$ matrix is defined to be $1$. For $n \geq 1$ let $x \in \mathcal{A}_{2n}(\mathbb{R})$:

$$x = \begin{bmatrix}
0 & x_{12} & \cdots & x_{12n} \\
x_{12} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
-x_{12n} & \cdots & 0
\end{bmatrix},$$

and observe that $M_{i,j}^x(x)$ is a $2(n-1) \times 2(n-1)$ anti-symmetric matrix; then

$$\text{Pf}(x) = \sum_{j=2}^{2n} (-1)^j x_{1j} \text{Pf} \left( M_{1,j}^x(x) \right). \quad (6.1)$$

For example,

$$\text{Pf} \left( \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \right) = a.$$
The group $\text{GL}_{2n}(\mathbb{R})$ acts on $\mathcal{A}_{2n}(\mathbb{R})$ by matrix congruence: $g \cdot x = gxg^T$, and the non-singular matrices in $\mathcal{A}_{2n}(\mathbb{R})$ are a single orbit. To see that, observe that every non-singular $2n \times 2n$ anti-symmetric matrix is congruent to $\Omega_n = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$. A theorem by Cayley (\cite{4}, \cite{14}) states that for every $x \in \mathcal{A}_{2n}(\mathbb{R})$,

$$\det(x) = (\text{Pf}(x))^2. \quad (6.2)$$

It follows that for every $x \in \mathcal{A}_{2n}(\mathbb{R})$ and $g \in \text{GL}_{2n}(\mathbb{R})$:

$$\text{Pf}(g^t xg) = \det(g) \text{Pf}(x). \quad (6.3)$$

In particular, the Pfaffian is an invariant for the action of $\text{SL}_{2n}(\mathbb{R})$ on $\mathcal{A}_{2n}(\mathbb{R})$, and the orbits of $\text{SL}_{2n}(\mathbb{R})$ on the non-singular matrices in $\mathcal{A}_{2n}(\mathbb{R})$ are the level sets of the Pfaffian:

$$\mathcal{F}_m = \{ x \in \mathcal{A}_{2n}(\mathbb{R}) \mid \text{Pf}(x) = m \}$$

with $m \neq 0$. Call a matrix in $\mathcal{F}_m$ prime if all its non-diagonal entries are primes ($\neq 1$) in $\mathbb{Z}$. The goal of this section is to prove the following:

**Theorem B.** For $n \geq 2$, prime matrices are Zariski-dense in $\mathcal{F}_m \subset \mathcal{A}_{2n}$ if and only if $m$ is an odd integer.

If prime matrices are Zariski-dense in $\mathcal{F}_m$, then there exists an odd point $x \in \mathcal{F}_m$, which is a matrix $x$ whose non-diagonal entries are odd integers. The necessity of the condition on $m$ is then a consequence of the following.

**Lemma 6.1.** The Pfaffian of an odd anti-symmetric matrix, which is a matrix whose non-diagonal entries are odd integers, is an odd integer.

**Proof.** Let

$$x = \begin{bmatrix} 0 & x_{12} & \cdots & x_{12n} \\ -x_{12} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -x_{12n} & \cdots & 0 \end{bmatrix}$$

where $n \geq 1$ and $x_{ij}$ is odd for all $i \neq j$. We prove by induction on $n$.

If $n = 1$ then $x$ is of the form

$$x = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$$
where \( q \) is an odd integer, and \( \text{Pf}(x) = q \).

Let \( n > 1 \), and assume the claim holds for \( n - 1 \). For \( i \neq j \), the matrix \( M_{i,j}^j(x) \) lies in \( \mathcal{A}_{2(n-1)}(\mathbb{R}) \), and therefore \( \text{Pf}\left(M_{i,j}^j(x)\right) \) is odd. It follows that

\[
\text{Pf}(x) = \sum_{j=2}^{2n} (-1)^j x_{1j} \text{Pf}\left(M_{1,j}^1(x)\right)
\]

is odd, since each summand is odd and there is an odd number \( 2n - 1 \) of summands. \( \square \)

We now turn to prove Theorem B.

**Proof of Theorem B**. If \( x \) is a matrix in \( \mathcal{A}_{2n}(\mathbb{R}) \), write

\[
x = \begin{bmatrix}
0 & \xi_2 & \cdots & \xi_{2n} \\
-\xi_2 & \ddots & & \\
\vdots & & \ddots & \\
-\xi_{2n} & & & y
\end{bmatrix}.
\]

Denote by \( P_j(y) \) the Pfaffian of the \((2n - 2) \times (2n - 2)\) anti-symmetric matrix obtained from \( y \) by deleting its \( j \)-th row and column, namely, \( P_j(y) = \text{Pf}\left(M_j^j(y)\right) = \text{Pf}\left(M_{1,j+1}^1(x)\right) \). Then,

\[
\text{Pf}(x) = \sum_{j=2}^{2n} (-1)^j \xi_j P_{j-1}(y) = P_1(y) \xi_2 - P_2(y) \xi_3 + \ldots + (-1)^{2n} P_{2n-1}(y) \xi_{2n}
\]

and in particular \( \text{Pf}(x) = \Delta(x) \) is of the form 2.2.

Formula 6.1 for the Pfaffian and the fact that it is defined recursively imply that the Pfaffians of two anti-symmetric matrices that differ only in their first row and column are intertwined. Hence \( P_1(y), P_2(y) \) are intertwined.

By Lemma 6.1, when \( x \) is odd, then so are \( \text{Pf}(x) \) and the \( P_i(y) \)'s; the conditions of Theorem F are therefore satisfied with \( \varepsilon = 1 \). In particular, prime points are Zariski-dense in \( F_m \) if and only if \( m \equiv \Delta(x) \) (mod 2) for every odd \( x \), namely if and only if \( m \equiv 1 \) (mod 2). \( \square \)

**Remark 6.2.** An analog for the Pfaffian of anti-symmetric matrices of even order is defined for symmetric matrices of even order whose main diagonal is identically zero; it
is called the hafnian, and is defined as follows. For
\[
x = \begin{bmatrix}
0 & x_{1,2} & \cdots & x_{1,2n} \\
x_{1,2} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
x_{1,2n} & \cdots & 0
\end{bmatrix},
\]
the hafnian of \( x \) is
\[
\text{hf}(x) = \sum_{j=2}^{2n} x_{1j} \cdot \text{hf}(M_{1,j}^{1,j}(x)),
\]
where
\[
\text{hf}\left(\begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix}\right) = q.
\]
In other words, the hafnian polynomial is obtained from the Pfaffian by switching all the negative signs to positive ones. In this sense, it is analogous to the permanent of a square matrix (see Section 8). An identical proof to the one of Theorem B yields that prime matrices are Zariski-dense in the variety of fixed hafnian \( m \neq 0 \) if and only if \( m \) is odd. This variety is not invariant under a group action, namely it is non-homogeneous.

7 Variety of rectangular matrices

7.1 Motivation from pre-homogeneous vector spaces

In all the examples we considered so far (except for the variety \( \mathcal{P}_m \) and the hafnian variety, which are non-homogeneous), the varieties \( \mathcal{X}_m(\mathbb{R}) \) for \( 0 \neq m \in \mathbb{R} \) are \( \text{SL}_n(\mathbb{R}) \) orbits, and foliate an open orbit of \( \text{GL}_n(\mathbb{R}) \):

1. The varieties \( \mathcal{P}_m \) foliate the open \( \text{GL}_n(\mathbb{R}) \)-orbit \( \{ x \in \text{Mat}_n(\mathbb{R}) \mid \det(x) \neq 0 \} \);
2. the varieties \( \mathcal{F}_m \) foliate the open \( \text{GL}_{2n}(\mathbb{R}) \)-orbit \( \{ x \in \mathcal{A}_{2n}(\mathbb{R}) \mid \text{Pf}(x) \neq 0 \} \);
3. the varieties \( \mathcal{Q}_m \) foliate the open \( G_{n,k} \times \text{GL}_1(\mathbb{R}) \)-orbit \( \{ x \in \mathbb{R}^{2n+k} \mid Q_{n,k}(x) \neq 0 \} \).

The pairs \( (\text{GL}_n(\mathbb{R}), \text{Mat}_n(\mathbb{R})) \), \( (\text{GL}_{2n}(\mathbb{R}), \mathcal{A}_{2n}(\mathbb{R})) \), and \( (G_{n,k} \times \text{GL}_1(\mathbb{R}), \mathbb{R}^{2n+k}) \) are therefore examples of pre-homogeneous vector spaces:

**Definition 7.1** ([12]; see also [20], [21]). Let \( G \) be a connected linear algebraic group over an algebraically closed field \( \mathbb{K} \) and let \( V \) be a finite-dimensional vector space over \( \mathbb{K} \) which affords a rational representation of \( G \). The pair \( (G, V) \) is called a pre-homogeneous vector space (or P.V., for short) if \( G \) has a Zariski-open (and therefore Zariski-dense) orbit in \( V \).
Sato and Kimura have classified the irreducible pre-homogeneous vector spaces in [19]. According to this classification, there are only five infinite families of regular irreducible P.V.s, and the remaining P.V.s are exceptional cases. Two of these families are \((\text{GL}_{2n}(\mathbb{R}), \mathcal{A}_{2n}(\mathbb{R}))\) and \((\text{GL}_n(\mathbb{R}), \text{Mat}_n(\mathbb{R}))\); the pair \((G_{n,k} \times \text{GL}_1(\mathbb{R}), \mathbb{R}^{2n+k})\) is a sub-family of a third family. In this section we consider a fourth family, and provide a necessary and sufficient condition for Zariski density of prime points in the level sets defined by the associated invariant polynomial.

7.2 Exposition of the example

For \(\ell \geq n \geq 1\), we consider the action of \(\text{SP}_\ell(\mathbb{R}) \times \text{GL}_{2n}(\mathbb{R})\) on the space \(\text{Mat}_{2\ell \times 2n}(\mathbb{R})\) given by

\[(g, h) \cdot x = gxh^t\]

where \(x \in \text{Mat}_{2\ell \times 2n}(\mathbb{R})\), \(g \in \text{SP}_\ell(\mathbb{R})\) and \(h \in \text{GL}_{2n}(\mathbb{R})\). Define the polynomial

\[P(x) = Pf\left(x^t\Omega_\ell x\right),\]

where \(\Omega_\ell\) is as defined in 1.2. The set of matrices \(x\) for which \(P(x) \neq 0\) (equivalently, \(\det(x^t x) \neq 0\)) is an open orbit of \(\text{SP}_\ell(\mathbb{R}) \times \text{GL}_{2n}(\mathbb{R})\), and it is foliated by orbits of \(\text{SP}_\ell(\mathbb{R}) \times \text{SL}_{2n}(\mathbb{R})\), given by the level sets of \(P\):

\[R_m = \{x \in \text{Mat}_{2\ell \times 2n}(\mathbb{R}) \mid P(x) = m\}\]

with \(m \neq 0\). Indeed, \(P\) is invariant under the action of \(\text{SP}_\ell(\mathbb{R}) \times \text{SL}_{2n}(\mathbb{R})\), since, by 6.3:

\[P((g, h) \cdot x) = P(x)\]

(see [19], [10]). A necessary and sufficient condition for Zariski density of prime matrices in \(R_m\) is as follows:

**Theorem C** For \(n \geq 1\), \(\ell \geq 2\), \(\ell \geq n\), and \(0 \neq m \in \mathbb{Z}\), prime matrices are Zariski-dense in \(R_m \subset \text{Mat}_{2\ell \times 2n}(\mathbb{R})\) if and only if \(m \equiv 0 \pmod{2^{2n-1}}\).

Theorem C is also a consequence of Theorem F; in particular, the proof relies on the fact that the polynomial \(P(x)\) is of the form (2.2) with coefficients that are intertwined, as detailed below.

7.3 Proof of Theorem C

Throughout this section, for \(i \in \{1, \ldots, 2\ell\}\) we denote \(\hat{i} := (i + \ell) \pmod{2\ell}\). For \(x \in \text{Mat}_{2\ell \times 2n}(\mathbb{R})\), let \((\xi_1, \ldots, \xi_{2\ell})^t\) denote the last column of \(x\) and let \(y \in \text{Mat}_{2\ell \times (2n-1)}(\mathbb{R})\) denote the matrix obtained from \(x\) by deleting its last column. A key ingredient in the
proof of Theorem \[\text{C}\] is that the form \(\text{Pf} \left(x^\top \Omega x\right)\) can be expanded along every column of \(x\), and in particular along the last column:

\[
P(x) = \sum_{i=1}^{2\ell} B_i(y) \xi_i = B_1(y) \xi_1 + \ldots + B_{2\ell}(y) \xi_{2\ell}, \tag{7.1}
\]

where the coefficients \(B_i(y)\) are polynomials in the entries of \(y\) given by:

\[
B_i(y) = \sum_{k=1}^{2n-1} (-1)^k \cdot x_{i,k} \cdot \text{Pf} \left(M_{i,i}^{k,2n}(x)\right) = \sum_{k=1}^{2n-1} (-1)^k \cdot x_{i,k} \cdot \text{Pf} \left(M_{i,i}^k(y)\right) \tag{7.2}
\]

(recall the notation \(M_{i_1,\ldots,i_k}^{j_1,\ldots,j_l}\) introduced at the end of Section \[\text{I}\].) This, along with some further facts that we shall utilize on the structure of \(B_i(y)\), is proved in the short note \[\text{II}\].

For a matrix \(x'\) with \(2\ell\) rows (such as \(x\) and \(y\)) and an integer \(1 \leq k \leq \ell\), we define the following matrix:

\[
Z_{t_1,\ldots,t_k}(x') := \left(\begin{array}{c}
\cdots \\
- R_{t_1}(x') & - R_{t_1}^\top(x') \\
\vdots \\
- R_{t_k}(x') & - R_{t_k}^\top(x')
\end{array}\right),
\]

where \(\{t_1,\ldots,t_k\} \subset \{1,\ldots,\ell\}\) are such that \(t_1 < \ldots < t_k\). Then \(Z_{t_1,\ldots,t_k}(x')\) has \(2k\) rows, and the same number of columns as \(x'\).

In order to apply Theorem \[\text{F}\] we begin by establishing parity conditions on the coefficients \(B_i(y)\).

**Lemma 7.2.** Let \(x \in \text{Mat}_{2\ell \times 2n}(\mathbb{Z})\) be odd, and \(y = M_{2n}(x)\). Fix \(i \in \{1,\ldots,2\ell\}\) and let \(B_i(y)\) be as in \[7.7\]. Then \(B_i(y) \equiv 0 \pmod{2^{2n-2}}\), and it can be either \(0 \pmod{2^{2n-1}}\) or \(2^{2n-2} \pmod{2^{2n-1}}\); namely, there exist odd \(y^0, y^1 \in \text{Mat}_{2\ell \times (2n-1)}(\mathbb{Z})\) for which \(B_i(y^0) \equiv 0 \pmod{2^{2n-1}}\) and \(B_i(y^1) \equiv 2^{2n-2} \pmod{2^{2n-1}}\).

**Proof.** The polynomial \(B_i(y)\) can be presented as the sum of determinants of \((2n-1) \times (2n-1)\) sub-matrices of \(y\) as follows (\[7.7\]):

\[
B_i(y) = \sum_{t_1 < \ldots < t_{n-1} \in \{1,\ldots,n\} \setminus \{i\}} \det \left(Z_{t_1,\ldots,t_{n-1}}(y)\right). \tag{7.3}
\]

Each determinant in this sum is of an odd matrix, and is therefore divisible by \(2^{2n-2}\); thus, \(B_i(y) \equiv 0 \pmod{2^{2n-2}}\).
Fix $t_1 < \ldots < t_{n-1} \in \{1, \ldots, \ell\} \setminus \{i\}$ and define $y^1 \in \text{Mat}_{2\ell \times (2n-1)}(\mathbb{Z})$ with the following two properties. Firstly,

$$R_j (y^1) \equiv R_j (y^1) \pmod{2^{2n-2}} \quad \text{for every } j \notin \{t_1, \ldots, t_{n-1}\}.$$  

In particular, for every $\{s_1, \ldots, s_{n-1}\} \neq \{t_1, \ldots, t_{n-1}\}$,

$$\det \left( \begin{array}{c} R_i (y^1) \\ Z_{t_1, \ldots, t_{n-1}} (y^1) \end{array} \right) \equiv 0 \pmod{2^{2n-2}},$$

since at least two of the rows are equivalent modulo $2^{2n-2}$. Secondly,

$$\left( \begin{array}{c} R_i (y^1) \\ Z_{t_1, \ldots, t_{n-1}} (y^1) \end{array} \right) \equiv \left( \begin{array}{c} R_i (y^1) \\ Z_{t_1, \ldots, t_{n-1}} (y^1) \end{array} \right) \equiv \left( \begin{array}{ccc} 3 & 1 & \cdots & 1 \\ 1 & 3 & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 1 & \cdots & \cdots & 3 \end{array} \right) \pmod{2^{2n-2}}.$$

Since the eigenvalues of the matrix on the right-hand side are $2n + 1$ (of multiplicity 1) and 2 (of multiplicity $2n - 2$), its determinant equals $(2n + 1) \cdot 2^{2n-2}$. In particular,

$$\det \left( \begin{array}{c} R_i (y^1) \\ Z_{t_1, \ldots, t_{n-1}} (y^1) \end{array} \right) \equiv (2n + 1) \cdot 2^{2n-2} \equiv 2^{2n-2} \equiv 2^{2n-2} \pmod{2^{2n-1}}.$$

In formula (7.3) for $B_i (y^1)$, one summand is equivalent to $2^{2n-2} \pmod{2^{2n-1}}$, and the remaining summands are equivalent to $0 \pmod{2^{2n-1}}$; thus, $B_i (y^1) \equiv 2^{2n-2} \pmod{2^{2n-1}}$.

Finally, let $y^0 \in \text{Mat}_{2\ell \times (2n-1)}(\mathbb{Z})$ be such that $R_i (y^0) \equiv R_i (y^0) \pmod{2^{2n-2}}$ for every $i \in \{1, \ldots, 2\ell\}$. Then for every $i$ and every $t_1 < \ldots < t_{n-1} \in \{1, \ldots, \ell\} \setminus \{i\}$,

$$\det \left( \begin{array}{c} R_i (y^0) \\ Z_{t_1, \ldots, t_{n-1}} (y^0) \end{array} \right) \equiv 0 \pmod{2^{2n-2}};$$

thus, $B_i (y^0) \equiv 0 \pmod{2^{2n-1}}$ for every $i \in \{1, \ldots, 2\ell\}$. \hfill \qed

**Proposition 7.3.** For every $i \in \{1, \ldots, \ell\}$, $B_i (x)$ and $B_i (x)$ are intertwined.

**Proof.** Assume the claim holds for $x' \in \text{Mat}_{2(\ell-1) \times 2(n-1)}$. Thus, according to Formulas (7.1) if $x', x'' \in \text{Mat}_{2(\ell-1) \times 2(n-1)}$ differ only by their last column, then $P (x')$ and $P (x'')$ are intertwined through the set $\{B_i\}_{1 \leq t \leq 2\ell-2}$. Since $P \left( M_{i,i}^{2n-1} (y) \right)$ and $P \left( M_{i,i}^{2n-2} (y) \right)$ differ only by their last column, they are therefore intertwined through the set $\{B_i \left( M_{i,i}^{2n-2} (y) \right)\}_{t \in \{1, \ldots, 2\ell\} \setminus \{i\}}$. By Formula (7.2) we conclude that $B_i (x)$ and $B_i (x)$ are intertwined through the polynomials $\left\{ P \left( M_{i,i}^{k} (y) \right) \right\}_{k=1}^{2n-1}$. \hfill \qed
Proof of Theorem C. According to formula 7.1, $\Delta (x) = P(x)$ is of the form 2.2, where by Proposition 7.3 two of the coefficient-polynomials are intertwined. Lemma 7.2 asserts that the maximal $\varepsilon \in \mathbb{N}$ such that $2^{\varepsilon - 1}$ divides every $B_i(y)$ for every odd $y$ is $\varepsilon = 2n - 1$. Finally, we claim that $P(x) \equiv 0 \pmod{2^{2n-1}}$ for every odd $x$ in $\text{Mat}_{2n} (\mathbb{Z})$. To this end, we consider the following formula for $P(x)$ (10):

$$P(x) = \sum_{(t_1, \ldots, t_n) \in \{1, \ldots, \ell\}^n \atop t_1 < \ldots < t_n} \det (Z_{t_1, \ldots, t_n}(x));$$

it asserts that $P(x)$ is the sum of determinants of odd $2n \times 2n$ matrices, and is therefore divisible by $2^{2n-1}$. By Theorem E, prime matrices are Zariski-dense in $\mathcal{R}_m$ if and only if $m \equiv 0 \pmod{2^{2n-1}}$.

8 The Permanent Variety

Observe that Theorem E relies purely on the combinatorial properties of the defining polynomial $\Delta$ for the variety $\mathcal{X}_m$, and in particular does not assume homogeneity of the variety under a group action. This gives rise to examples of Zariski-density of prime points in varieties which are not necessarily homogeneous, such as the hafnian variety mentioned in Remark 6.2. Another such example is the permanent variety.

Definition 8.1. The permanent of an $n \times n$ matrix $x = (x_{ij})$ is defined as

$$\text{perm}(x) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

The permanent of a matrix can be expanded along any row or column; e.g., an expansion along the $i$-th row is given by

$$\text{perm}(x) = \sum_{j=1}^{n} x_{i,j} \cdot \text{perm}(M_{i}^{j}(x)).$$

The variety of matrices with fixed permanent

$$\mathcal{P}_m = \{ x \in \text{Mat}_n(\mathbb{R}) \mid \text{perm}(x) = m \}$$

is not invariant under a group action.

We note that the permanent is to the determinant as the hafnian is to the Pfaffian: it is obtained form switching all the negative signs in the determinant polynomial to positive signs. However, while the congruence condition on the hafnian variety for Zariski-density of prime points was identical to the one of the Pfaffian, the situation with the permanent is different from the determinant case.
**Theorem D.** Let \( n \geq 3 \) and \( 0 \neq m \in \mathbb{Z} \). Write
\[
2s - 1 \leq n < 2^{s+1} - 1
\]
for a unique integer \( s \geq 2 \). Then prime matrices are Zariski-dense in \( P_m \) if and only if
\[
m \equiv \begin{cases} 
2^{n-s} \pmod{2^{n-s+1}} & \text{when } n = 2^s - 1 \\
0 \pmod{2^n} & \text{when } 2^s - 1 < n < 2^{s+1} - 1.
\end{cases}
\]

The necessity part of Theorem D is slightly more involved than it was in the previous examples, due to the fact that the permanent is not invariant under linear actions on the rows of the matrix. We shall require the following Lemma, whose proof has been suggested in [6].

**Lemma 8.2.** Let \( n, s \geq 1 \) be integers, and let \( x \in \text{Mat}_n(\mathbb{Z}) \) with odd entries.

1. The permanent of \( x \) satisfies
\[
\text{perm}(x) \equiv \begin{cases} 
2^{n-s} \pmod{2^{n-s+1}} & n = 2^s - 1 \\
0 \pmod{2^n} & 2^s \leq n < 2^{s+1} - 1.
\end{cases} \tag{8.1}
\]

2. Furthermore, when \( 2^s \leq n < 2^{s+1} - 1 \), \( \text{perm}(x) \) can be congruent to either 0 or \( 2^{n-s} \pmod{2^{n-s+1}} \). Both cases occur: there exist odd matrices \( x^0, x^1 \in \text{Mat}_n(\mathbb{Z}) \) such that \( \text{perm}(x^1) \equiv 2^{n-s} \pmod{2^{n-s+1}} \) and \( \text{perm}(x^0) \equiv 0 \pmod{2^{n-s+1}} \).

The following fact is instrumental in the proof of Lemma 8.2.

**Fact 8.3.** Let \( \phi_2(n) \) denote the highest power of 2 that divides \( n! \), and let \( s \geq 1 \) be an integer.

1. If \( n = 2^s - 1 \) then \( \phi_2(n) = n - s \).
2. If \( 2^s \leq n < 2^{s+1} - 1 \), then \( \phi_2(n) \geq n - s \).

This fact is a direct consequence the Legendre Formula, which states that \( \phi_2(n) = n - s_2(n) \), where \( s_2(n) \) is the number of 1’s in the binary representation of \( n \).

**Proof of lemma 8.2.**

**Part 1.** We prove 8.1 by induction on \( n \). For \( n = 1 \) we have \( n = 2^1 - 1 \), i.e. \( s = 1 \), and for every odd integer \( x \): \( \text{perm}(x) = x \equiv 1 \pmod{2} \). Let \( J \) denote the \( n \times n \) matrix whose all entries are 1’s. Since \( \text{perm}(J) = n! \), the claim holds for \( J \) according to Fact 8.3. Every other odd \( n \times n \) matrix is obtained from \( J \) by a finite number of steps of the form “add/subtract 2 from a given entry of the matrix”, and it is therefore sufficient to prove that if an odd matrix \( x' \) satisfies 8.1 then a matrix \( x \) obtained from \( x' \) by adding
\[ \pm 2 \] to the \( (i, j) \) entry of \( x' \), also satisfies (8.1) Recall that \( M_{ij}'(x') \) denotes the matrix obtained from \( x' \) by deleting its \( i \)-th row and \( j \)-th column, and observe that:

\[
\text{perm}(x) = \text{perm}(x') \pm 2 \cdot \text{perm}
(M_{ij}'(x')). \quad (8.2)
\]

By the induction hypothesis, \( M_{ij}'(x') \) satisfies (8.1) We distinguish between three different cases.

- If \( n = 2^s - 1 \), then \( n - 1 = 2^s - 2 \in [2^{s-1}, 2^s - 1) \). Since \( x' \) and \( M_{ij}'(x') \) satisfy (8.1) we have

\[
\begin{align*}
\text{perm}(x') &\equiv 2^{n-s} \pmod{2^{n-s+1}} \\
\text{perm}
(M_{ij}'(x')) &\equiv 0 \pmod{2^{n-s+1}} \equiv 0 \pmod{2^{n-s}}.
\end{align*}
\]

In particular, by (8.2)

\[
\text{perm}(x) \equiv 2^{n-s} \pmod{2^{n-s+1}} + 0 \pmod{2^{n-s+1}} \equiv 2^{n-s} \pmod{2^{n-s+1}},
\]

as desired.

- If \( n = 2^s \), then \( n - 1 = 2^s - 1 \). Since \( x' \) and \( M_{ij}'(x') \) satisfy (8.1) we have

\[
\begin{align*}
\text{perm}(x') &\equiv 0 \pmod{2^{n-s}} \\
\text{perm}
(M_{ij}'(x')) &\equiv 2^{n-1-s} \pmod{2^{n-s}}.
\end{align*}
\]

By (8.2)

\[
\text{perm}(x) \equiv 0 \pmod{2^{n-s}} + 0 \pmod{2^{n-s}} \equiv 0 \pmod{2^{n-s}},
\]

as desired.

- If \( 2^s + 1 \leq n \leq 2^{s+1} - 2 \), then both \( n \) and \( n - 1 \) are in \([2^s, 2^{s+1} - 2]\), and since \( x' \) and \( M_{ij}'(x') \) satisfy (8.1) we have

\[
\begin{align*}
\text{perm}(x') &\equiv 0 \pmod{2^{n-s}} \\
\text{perm}
(M_{ij}'(x')) &\equiv 0 \pmod{2^{n-s}}.
\end{align*}
\]

In particular, by (8.1)

\[
\text{perm}(x) \equiv 0 \pmod{2^{n-s}} + 0 \pmod{2^{n-s}} \equiv 0 \pmod{2^{n-s}},
\]

which concludes the proof of part II
Part 2. Assume first that \( n = 2^s \). Let \( y \in \text{Mat}_{n-1}(\mathbb{Z}) \) be any odd matrix; by part 1, \( \text{perm} (y) \equiv 2^{n-s-1} \pmod{2^n} \). Consider the \( n \times n \) matrices

\[
x = \begin{bmatrix}
3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \ddots \\
\end{bmatrix}, \quad x' = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \ddots \\
\end{bmatrix}.
\]

Both \( \text{perm}(x) \) and \( \text{perm}(x') \) are congruent to \( 0 \pmod{2^n} \), by part 1 we claim that one of them is congruent to \( 2^{n-s} \pmod{2^{n-s+1}} \), and the other is congruent to \( 0 \pmod{2^{n-s+1}} \). This is due to the fact that

\[
\text{perm}(x) = \text{perm}(x') + 2 \cdot \text{perm}(y) \equiv \text{perm}(x') + 2^{n-s} \pmod{2^{n-s+1}}.
\]

This proves the claim of part 2 for \( n = 2^s \), and we proceed by induction on \( n \) in the interval \([2^s, 2^{s+1} - 1]\). Let \( 2^s + 1 \leq n < 2^{s+1} - 1 \). By the induction hypothesis, there exists \( \tilde{x}^1 \in \text{Mat}_{n-1}(\mathbb{Z}) \) such that \( \text{perm}(\tilde{x}^1) \equiv 2^{n-1-s} \pmod{2^n} \). Let

\[
x = \begin{bmatrix}
3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \ddots \\
\end{bmatrix}, \quad x' = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \ddots \\
\end{bmatrix}.
\]

As before, \( \text{perm}(x) \equiv \text{perm}(x') \equiv 0 \pmod{2^n} \) and

\[
\text{perm}(x) = \text{perm}(x') + 2 \cdot \text{perm}(\tilde{x}^1) \equiv \text{perm}(x') + 2^{n-s} \pmod{2^{n-s+1}};
\]

hence one one of the matrices \( x, x' \) is congruent to \( 2^{n-s} \pmod{2^{n-s+1}} \), and the other is congruent to \( 0 \pmod{2^{n-s+1}} \). \( \square \)

Proof of Theorem 12. For an \( n \times n \) matrix of variables \( x \), we let \((\xi_1, \ldots, \xi_n)\) denote the first row of \( x \), and let \( y \) denote the \((n-1) \times n\) matrix obtained by deleting the first row of \( x \). We write \( K_j(y) \) for the permanent of the \((n-1) \times (n-1)\) matrix obtained by deleting the \( j \)-th column of \( y \), i.e. \( K_j(y) = \text{perm}(M_{ij}(y)) \). Then, the polynomial

\[
\Delta(x) = \text{perm}(x) \text{ is of the form } 2^{s+2} \text{ considered in Theorems } 12 \text{ and } 13
\]

\[
\text{perm}(x) = K_1(y) x_1 + K_2(y) x_2 + \cdots + K_n(y) x_n.
\]

Since the permanent polynomial differs from the determinant only by the signs, it also has the property that the permanents of two matrices that differ only by a single row or column are intertwined. Hence every pair \((K_i(y), K_j(y))\) with \( i \neq j \) is intertwined.
We are left to verify the parity conditions on \( \text{perm}(x) \) and the coefficients \( K_i(y) \). We apply Lemma 8.2 for three different cases.

If \( n = 2^s - 1 \), then every \( M^j(y) \) is an odd square matrix of order \( n - 1 = 2^s - 2 \in [2^{s-1}, 2^s - 1) \) and in particular
\[
\text{perm}(x) \equiv 2^{n-s} \left( \text{mod } 2^{n-s+1} \right) \\
K_j(y) = \text{perm}(M^j(y)) \equiv 0 \left( \frac{2^{(n-1)-(s-1)}}{2^{n-s}} \right) \equiv 0 \left( \text{mod } 2^{n-s} \right).
\]

The conditions of Theorem F are met with \( \varepsilon = n - s, m \equiv 2^{n-s} \left( \text{mod } 2^{n-s+1} \right) \).

If \( n = 2^s \), then every \( M^j(y) \) is an odd square matrix of order \( n - 1 = 2^s - 1 \) and in particular
\[
\text{perm}(x) \equiv 0 \left( \text{mod } 2^{n-s} \right) \\
K_j(y) = \text{perm}(M^j(y)) \equiv 2^{n-s-1} \left( 2^{n-s} \right).
\]

The conditions of Theorem F are met with \( \varepsilon = n - s, m \equiv 0 \left( \text{mod } 2^{n-s} \right) \).

If \( 2^s + 1 \leq n < 2^{s+1} - 1 \), every \( M^j(y) \) is an odd square matrix of order \( n - 1 \in [2^s, 2^{s+1} - 2) \) and in particular
\[
\text{perm}(x) \equiv 0 \left( \text{mod } 2^{n-s} \right) \\
K_j(y) = \text{perm}(M^j(y)) \equiv 0 \left( 2^{n-1-s} \right).
\]

The conditions of Theorem F are met with \( \varepsilon = n - s, m \equiv 0 \left( \text{mod } 2^{n-s} \right) \).

\[\square\]

\section*{Appendix: Proof of Lemma 4.4}

The goal of this section is to prove Lemma 4.4. We start by recalling Dirichlet’s Theorem.

\textbf{Theorem A.1} (Dirichlet theorem on arithmetic progressions). \textit{Let \( \alpha \) and \( \beta \) be co-prime integers. Then there are infinitely many primes in the arithmetic progression \( \{\alpha + \beta \mathbb{Z}\} \). In other words, there are infinitely many primes that are congruent to \( \alpha \) modulo \( \beta \).}

The following is a simple consequence of Dirichlet’s Theorem and the Chinese Remainder Theorem:

\textbf{Fact A.2.} \textit{Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z} \) be such that}
\[
\gcd(\beta_1, \beta_2) = \gcd(\alpha_1, \beta_1) = \gcd(\alpha_2, \beta_2) = 1.
\]

\textit{Then there are infinitely many primes in the intersection of the arithmetic progressions \( \{\alpha_1 + \beta_1 \mathbb{Z}\} \) and \( \{\alpha_2 + \beta_2 \mathbb{Z}\} \).}
Proof. By the Chinese Remainder Theorem there exists \( x \in \mathbb{Z} \) in the intersection of the arithmetic progressions \( \{\alpha_1 + \beta_1\mathbb{Z}\} \) and \( \{\alpha_2 + \beta_2\mathbb{Z}\} \), since \( \gcd(\beta_1, \beta_2) = 1 \). Write
\[
x = \alpha_1 + k_1\beta_1 = \alpha_2 + k_2\beta_2.
\]
If \( \ell = \text{lcm}(\beta_1, \beta_2) \), then every element in the arithmetic progression \( \{x + \ell\mathbb{Z}\} \) is contained in \( \{\alpha_1 + \beta_1\mathbb{Z}\} \cap \{\alpha_2 + \beta_2\mathbb{Z}\} \). By Dirichlet’s theorem, there are infinitely many primes in \( \{x + \ell\mathbb{Z}\} \) if \( \gcd(x, \ell) = 1 \) — which is indeed the case, since:
\[
\gcd(x, \beta_1) = \gcd(\alpha_1 + k_1\beta_1, \beta_1) = \gcd(\alpha_1, \beta_1) = 1
\]
and
\[
\gcd(x, \beta_2) = \gcd(\alpha_1 + k_2\beta_2, \beta_2) = \gcd(\alpha_2, \beta_2) = 1.
\]

It is well known that the gcd of a finite set of integers \( \alpha_1, \ldots, \alpha_r \) can be presented as an integral combination of \( \alpha_1, \ldots, \alpha_r \); the content of the following claim is that the integral coefficients in this combination can be chosen to satisfy some desired congruence.

Claim A.3. Let \( \alpha_1, \ldots, \alpha_r \in \mathbb{Z} \) such that \( \gcd(\alpha_1, \ldots, \alpha_r) = d \), and let \( p \neq 2 \) a prime that does not divide \( d \). Then there exist \( t_1, \ldots, t_r \in \mathbb{Z} \) such that \( p \mid t_1, \ldots, p \mid t_r \) and \( t_1\alpha_1 + \ldots + t_r\alpha_r = d \).

Proof. Let \( \mu_1, \ldots, \mu_r \in \mathbb{Z} \) such that
\[
\mu_1\alpha_1 + \ldots + \mu_r\alpha_r = d.
\]
Since \( p \nmid d \), there exists \( i \in \{1, \ldots, r\} \) such that \( p \nmid \mu_i\alpha_i \). Assume \( p \nmid \mu_r\alpha_r \), namely \( p \nmid \mu_r \) and \( p \nmid \alpha_r \). If \( p \nmid \mu_i \) for all \( i = 1, \ldots, r - 1 \), we are done. Otherwise, rearrange the indexes such that \( p \mid \mu_i \) for \( i = 1, \ldots, k \), for \( 1 \leq k < r \). Consider the following presentation of \( d \) is an integral combination of \( \alpha_1, \ldots, \alpha_r \):
\[
(\mu_1 + \alpha_r)\alpha_1 + \ldots + (\mu_k + \alpha_r)\alpha_k + \mu_{k+1}\alpha_{k+1} + \ldots + \mu_{r-1}\alpha_{r-1} + (\mu_r - \alpha_1 - \ldots - \alpha_k)\alpha_r = d.
\]
Note that \( p \nmid \mu_i + \alpha_r \) for all \( i = 1, \ldots, k \), since \( p \mid \mu_i \) and \( p \nmid \alpha_r \). By assumption \( p \nmid \mu_{k+1}, \ldots, p \nmid \mu_{r-1} \). If \( p \nmid \mu_r - \alpha_1 - \ldots - \alpha_k \), we are done. Otherwise, consider the following presentation of \( d \) is an integral combination of \( \alpha_1, \ldots, \alpha_r \):
\[
(\mu_1 + 2\alpha_r)\alpha_1 + \ldots + (\mu_k + 2\alpha_r)\alpha_k + \mu_{k+1}\alpha_{k+1} + \ldots + \mu_{r-1}\alpha_{r-1} + (\mu_r - 2\alpha_1 - \ldots - 2\alpha_k)\alpha_r = d.
\]
Note that \( p \mid \mu_i + 2\alpha_r \) for all \( i = 1, \ldots, k \), since \( p \mid \mu_i \) and \( p \nmid 2\alpha_r \) (as \( p \neq 2 \) and \( p \nmid \alpha_r \)). By assumption \( p \nmid \mu_{k+1}, \ldots, p \nmid \mu_{r-1} \). Finally, \( p \nmid \mu_r - 2\alpha_1 - \ldots - 2\alpha_k \); indeed, if \( p \) divides both \( \mu_r - 2\alpha_1 - \ldots - 2\alpha_k \) and \( \mu_r - \alpha_1 - \ldots - \alpha_k \), then it must divide \( \alpha_1 + \ldots + \alpha_k \), and therefore \( p \mid \mu_r \), a contradiction. \[\square\]
Proof of Lemma 4.4. If $\gamma$ is a power of 2, the claim is trivial; indeed, since $\gcd(q_i, 2^s) = 1$, the arithmetic progression $\{q_i + 2^sZ\}$ contains infinitely many primes (by Dirichlet’s Theorem).

Otherwise, let $p$ be an odd prime factor of $\gamma$, and $\nu_p$ be an integer satisfying

$$
\begin{align*}
p &\nmid \nu_p, \\
p &\nmid (\nu_p \cdot \gcd(\alpha_1, \ldots, \alpha_n) + \beta).
\end{align*}
$$

Such $\nu_p$ exists, because for any integer $\delta_p$ for which $\delta_p \equiv \beta \pmod{\gcd(\alpha_1, \ldots, \alpha_n)}$, we can choose

$$
\nu_p \equiv \frac{\delta_p - \beta}{\gcd(\alpha_1, \ldots, \alpha_n)} \pmod{p}
$$

(since $\gcd(\alpha_1, \ldots, \alpha_n) \not\equiv 0 \pmod{p}$), and then

$$
\nu_p \not\equiv 0 \pmod{p}
$$

and

$$
\nu_p \cdot \gcd(\alpha_1, \ldots, \alpha_n) + \beta \equiv \delta_p \not\equiv 0 \pmod{p}.
$$

By Claim A.3 there exist $n$ integers $t_1^{(p)}, t_2^{(p)}, \ldots, t_n^{(p)}$ co-prime to $p$ such that

$$
t_1^{(p)} \alpha_1 + t_2^{(p)} \alpha_2 + \ldots + t_n^{(p)} \alpha_n = \gcd(\alpha_1, \ldots, \alpha_n).
$$

Then,

$$
\nu_p t_1^{(p)} \alpha_1 + \nu_p t_2^{(p)} \alpha_2 + \ldots + \nu_p t_n^{(p)} \alpha_n + \beta = \nu_p \cdot \gcd(\alpha_1, \ldots, \alpha_n) + \beta 
\equiv 0 \pmod{p};
$$

in particular, if

$$
(y_1, \ldots, y_n) \equiv \left(\nu_p t_1^{(p)}, \ldots, \nu_p t_n^{(p)}\right) \pmod{p},
$$

then

$$
\alpha_1 y_1 + \ldots + \alpha_n y_n + \beta 
\equiv \alpha_1 \cdot \nu_p t_1^{(p)} + \alpha_2 \cdot \nu_p t_2^{(p)} + \ldots + \alpha_n \cdot \nu_p t_n^{(p)} + \beta \pmod{p} \tag{A.1}
$$

Finally, consider the following set in $\mathbb{Z}^n$:

$$
\left\{ (y_1, \ldots, y_n) \mid \begin{array}{l}
y_i \text{ is prime for all } i \\
y_i \equiv q_i \pmod{2^s} \text{ for all } i \\
(y_1, \ldots, y_n) \equiv \left(\nu_p t_1^{(p)}, \ldots, \nu_p t_n^{(p)}\right) \pmod{p} \text{ for all } 2 \not\equiv p | \gamma
\end{array} \right\}. \tag{A.2}
$$
Observe that every $y_i$ assumes prime values in the following finite intersection of arithmetic progressions:

$$\left(\bigcap_{2 \neq p \mid \gamma} \left\{ \nu_{p, t_i}^{(p)} + N \cdot p \right\}\right) \cap \{q_i + N \cdot 2^{s_i}\},$$

and this intersection contains infinitely many primes, by Fact \ref{fact:A.2} indeed, $\gcd\left(\nu_{p, t_i}^{(p)}, p\right) = 1$ for every $p$ and every $i = 1, \ldots, n$, $\gcd(q_i, 2^{s_i}) = 1$, and the elements of $\{\{p : 2 \neq p \mid \gamma\}, \{2^{s_i}\}\}$ are pairwise co-prime. Hence, every $y_i$ assumes infinitely many values, and the set \ref{set:A.2} is Zariski-dense in $\mathbb{R}^n$.

We conclude the proof by observing that the set \ref{set:A.2} is contained in the set \ref{set:4.2}. First of all, if $y_i \equiv q_i \pmod{2^{s_i}}$ and $q_i$ is odd, then in particular $y_i$ is odd. By equation \ref{eq:A.1}, every odd prime factor $p$ of $\gamma$ does not divide $f(y)$. \hfill \Box

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