ON STABILIZATION OF $E_k$ CHAINS

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Abstract. We study special subgroups of infinite groups that generalize double centralizers. We analyze sufficient conditions for descending chains of such subgroups to stop after finitely many steps. We discuss whether this phenomenon can happen in the class of groups satisfying chain condition on centralizers.

1. Introduction

In modern infinite group theory, chain conditions have played an important role. A natural chain condition is the one on centralizers. A group is said to satisfy the descending chain condition on centralizers if every proper descending chain of centralizers stabilizes after finitely many steps. Such groups are denoted $M_c$-groups. By elementary properties of centralizers, the descending chain condition on centralizers is equivalent to the ascending one. Since many natural classes of groups such as linear groups and finitely generated abelian-by-nilpotent groups enjoy this property, it has been separately analyzed in such fundamental papers as [2] and [4].

$M_c$-groups are frequently encountered in model theory since the class of stable groups, a class of groups of fundamental importance in model theory, have this property. The model-theoretic analysis of mathematical structures frequently uses the notion of first-order definability in the sense of mathematical logic. When these structures are groups, a frequent question is whether algebraic properties (e.g. nilpotency, solvability) of subgroups are inherited by sufficiently small definable supergroups containing them (envelopes). In [4], Altınel and Baginski showed that in an $M_c$-group every nilpotent subgroup is contained in a definable nilpotent subgroup of the same nilpotency class. This generalizes a similar property of the Zariski closure of nilpotent subgroups of algebraic groups over algebraically closed fields.

In their proof, Altınel and Baginski start with a nilpotent subgroup $H$ of an $M_C$-group $G$, and construct a descending chain of supergroups of $H$ denoted $E_k(H)$ ($k \in \mathbb{N}$) reminiscent of double centralizers. The definability of the envelope constructed depends intimately on the ambient chain condition on centralizers as well as on the nature of the subgroups $E_k(H)$, while its nilpotency is related to this very nature and the nilpotency of $H$.

Our initial motivation was to measure to what extent these techniques would help prove similar definability results for other classes of subgroups (e.g. solvable subgroups of $M_C$-groups) as well as for classes of ambient groups satisfying weaker chain conditions. Along the way, we proved Theorem 1 (section 3.1) that states that

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the nilpotency of the envelope is a consequence of the nilpotency of \( H \), and from this it follows through a simple induction argument that when \( H \) is \( k \)-nilpotent, the descending chain \( (E_k(H))_i \) stabilizes after at most \( k \) steps. These refinements of part of Altınel and Baginski’s proof led us to investigating more this stabilization property, especially under the assumption that the ambient group is an \( M_C \)-group. Indeed, as proven in section 3.2, in groups enjoying various topological properties and satisfying Noetherianity conditions on closed subgroups, such as linear groups, the \( E_k \)-chains of envelopes of arbitrary subgroups stabilize. We interpret these affirmative answers to the stabilization problem of the \( E_k \)-chains as potentially useful for our initial purposes.

Nevertheless, the stabilization of \( E_k \)-chains is a strong property. One suspects that it is false in general, and understanding the conditions under which a counterexample would arise is likely to yield further information. In Section 4, where we construct an example in \( \text{Sym}(\mathbb{N}) \) with an infinite \( E_k \)-chain of envelopes. Somewhat to our surprise, the counterexample to the stabilization is rather involved. It should also be noted that \( \text{Sym}(\mathbb{N}) \) is far from satisfying the descending chain condition on centralizers.

We have not been able to prove the stabilization property for envelopes of arbitrary subgroups of \( M_C \)-groups. Potentially, this could have led to further results similar to those in [1], and it remains to be done. In a more positive vein, we are able to extend the results of Section 3.1 to hypercentral subgroups (of \( M_C \)-groups). These extensions necessitate introducing transfinite versions of iterated centralizers as well as of the \( E_k \)-chains, and are done in [5].

2. Key Facts

Our group-theoretic notation is standard. We write \( H \leq G \) to denote that \( H \) is a subgroup of \( G \) and \( H < G \) to denote \( H \) is normal in \( G \). If \( H \subseteq G \), then \( \langle H \rangle \) denotes the subgroup generated by \( H \). For any subset \( H \) of \( G \), the centralizer of \( H \) is \( C_G(H) = \{g \in G \mid \forall h \in H \ gh = hg\} \), while the normalizer of \( H \) is \( N_G(H) = \{g \in G \mid \forall h \in H \ g^{-1}hg \in H\} \). Given \( g, h \in G \) and the commutator of these elements is \([g, h] := g^{-1}h^{-1}gh\).

The key notion of this article is a special enveloping supergroup of a subgroup \( H \) of an arbitrary group \( G \), that will be denoted \( E_k(H) \) for every \( k \in \mathbb{N} \). To define it, we need to recall the notion of iterated centralizer introduced in [2]:

**Definition 2.1** (Bryant, 1979). Let \( G \) be a group and \( A \) any subset of \( G \). Set \( C_G^0(A) = 1 \) and for \( k \geq 1 \), the iterated centralizer of \( A \) in \( G \) is

\[
C^k_G(A) = \left\{ x \in \bigcap_{n<k} N_G(C^n_G(A)) \mid [x, A] \subseteq C^{k-1}_G(A) \right\}.
\]

One can show by induction that the iterated centralizers \( C^k_G(H) \) form an ascending sequence: \( 1 = C^0_G(H) \leq C^1_G(H) \leq ... \leq G \).
Some of the basic properties of iterated centralizers are stated in the following lemma. In particular, one sees that the notion of iterated centralizer generalizes the more commonly known notion of the $k$th center of a group.

**Lemma 2.2** (Bryant, 1979). Let $G$ be a group and $H$ a subgroup and $k \geq 0$. For the $k$th iterated centralizer of $H$ in $G$, the following relations hold:

(i) $C_G^k(H) \leq G$
(ii) $C_G^k(H) \cap H = Z_k(H)$
(iii) When $H = G$, $C_G^k(G) = Z_k(G)$
(iv) If $H$ is a nilpotent subgroup of class $k$, then $H \leq C_G^k(H)$.

Let us recall that if $G$ is a group then $Z_0(G) = \{1\}$ and inductively, for every $k \in \mathbb{N}$, $Z_{k+1}(G) = \{g \in G \mid [g, G] \subseteq Z_k(G)\}$. The series $(Z_k(G))_{k \in \mathbb{N}}$ is known as the upper central series; a group is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $Z_k(G) = G$. The least such $k$ is the nilpotency class of $G$.

We now define the central notion and tool of this article:

**Definition 2.3** (Altınel-Baginski, 2014, Definition 3.5). Let $G$ be a group and $H$ a subgroup. For $k \in \mathbb{N}$, a sequence of subgroups $E_k(H)$ of $G$ is defined

$$E_{k+1}(H) = \left\{ g \in E_k(H) \mid \left[ g, C_{E_k(H)}^{k+1}(H) \right] \leq C_{E_k(H)}^k(H) \right\}$$

where $E_0(H) = G$.

By definition, these subgroups of $G$ form a descending sequence such as

$$G = E_0(H) \geq E_1(H) \geq \ldots \geq H.$$ 

It follows easily from this definition that $E_1(H) = C_G(C_G(H))$.

Before finishing this section, we will recall some basic facts from [1] and draw some corollaries.

**Lemma 2.4** (Altınel-Baginski, 2014, Lemma 2.5). Let $A \leq B \leq C$ be groups and suppose that for all $j \leq k$ we have $C_C^j(A) = Z_j(C)$. Then

(i) $C_C^j(A) = C_C^j(B) = Z_j(C), \forall j \leq k$
(ii) $C_B^j(A) = Z_j(B) = Z_j(C) \cap B, \forall j \leq k$
(iii) $C_B^{k+1}(A) = C_C^{k+1}(A) \cap B, \forall j \leq k$.

**Corollary 2.5.** Let $i, j \in \mathbb{N}$ such that $i \leq j$. Then $Z_i(E_i(H)) \leq Z_j(E_j(H))$.

*Proof.* Since the iterated centers form an ascending chain $Z_i(E_i(H)) \leq Z_j(E_j(H))$ and inductively $Z_i(E_i(H)) \leq E_j(H)$, applying the Lemma 2.4 (ii) to $H \leq E_j(H) \leq E_i(H)$ subgroups,

$$Z_j(E_j(H)) \geq Z_i(E_i(H)) = Z_i(E_i(H)) \cap E_j(H) = Z_i(E_i(H))$$

is obtained. \qed
Lemma 2.6 (Altınel-Baginski, 2014). Let $G$ be an arbitrary group and $H$ a subgroup of $G$. Then
\[ C_{E_k(H)}^j(H) = Z_j(E_k(H)) \]
for all $j \leq k$.

A practical conclusion of this lemma is the following corollary. Throughout the paper, it will be used without mention.

Corollary 2.7. Let $G$ be a group and $H \leq G$. Then the iterated centralizer is
\[ C_{E_k(H)}^{i+1}(H) = \{ x \in E_k \mid [x, H] \subseteq Z_i(E_k(H)) \} \]
for all $i \leq k$.

Proof. By definition $C_{E_k}(H) = \left\{ x \in \bigcap_{j \leq i} N_{E_k}\left(C_{E_k}^j(H)\right) \mid [x, H] \subseteq C_{E_k}^i(H) \right\}$. By the previous lemma $C_{E_k(H)}^i(H) = Z_i(E_k(H))$ for all $i \leq k$
\[ C_{E_k}^{i+1}(H) = \left\{ x \in \bigcap_{j \leq i} N_{E_k}(C_{E_k}^j(H)) \mid [x, H] \subseteq C_{E_k}^i(H) \right\} \]
\[ = \left\{ x \in \bigcap_{j \leq i} N_{E_k}(Z_j(E_k(H))) \mid [x, H] \subseteq Z_i(E_k(H)) \right\} \]
\[ = \left\{ x \in E_k \mid [x, H] \subseteq Z_i(E_k(H)) \right\} \]
is obtained. □

3. Affirmative Answers

In this section, we show that under some conditions the chains of $E_k$-envelopes of some specific subgroups stabilize. First, we shall show that the chain of $E_k$-envelopes of any nilpotent subgroup of an arbitrary group $G$ stabilizes. Then, we will analyze topological conditions that yield the stabilization property.

3.1. Nilpotent Subgroups. In this subsection, unless otherwise mentioned, $H$ will stand for a nilpotent subgroup of a fixed arbitrary ambient group $G$. For simplicity, we will denote the various $E_k(H)$ by $E_k$.

Lemma 3.1.1. Let $G$ be a group and $H$ an abelian subgroup of $G$. Then $C_G(C_G(H))$ is abelian.

Proof. Since $H$ is abelian, $H \leq C_G(H)$, and then $C_G(H) \geq C_G(C_G(H))$. Thus
\[ Z(C_G(H)) = C_G(C_G(H)) \cap C_G(H) = C_G(C_G(H)) \]. □

Using this lemma we can prove the first main result of this subsection:

Theorem 1. Let $G$ be a group and $H \leq G$. If $H$ is $k$-nilpotent subgroup, then the envelope $E_k$ is also $k$-nilpotent.
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Proof. From the second isomorphism theorem it can be written

(3.1.1) \[ HZ_{k-1}(E_{k-1})/Z_{k-1}(E_{k-1}) \cong H/H \cap Z_{k-1}(E_{k-1}). \]

Considering Lemma 2.4 we have

(3.1.2) \[ Z_{k-1}(H) = Z_{k-1}(E_{k-1}) \cap H. \]

If equation 3.1.2 is used in equation 3.1.1 then

\[ HZ_{k-1}(E_{k-1})/Z_{k-1}(E_{k-1}) \cong H/Z_{k-1}(H). \]

Since $H$ is a $k$-nilpotent subgroup, $H/Z_{k-1}(H)$ is abelian. In a similar way, from the second isomorphism theorem we have

\[ E_kZ_{k-1}(E_{k-1})/Z_{k-1}(E_{k-1}) = E_k/E_k \cap Z_{k-1}(E_{k-1}) \]

\[ = E_k/Z_{k-1}(E_{k-1}). \]

From the definitions of $E_k$ and $C_{E_{k-1}}^k(H)$ we write

\[ E_kZ_{k-1}(E_{k-1})/Z_{k-1}(E_{k-1}) = C_{E_{k-1}}(H) \cap C_{E_{k-1}}^k(H) \]

\[ = E_k/Z_{k-1}(E_{k-1}). \]

It means $E_k/Z_{k-1}(E_{k-1})$ is abelian. Thus $E_k$ is a $k$-nilpotent group. \[\Box\]

As a result of Theorem we verify the following corollary which guarantees that the descending chain of envelopes stabilizes at most at step the nilpotency class of $H$.

**Corollary 3.1.2.** Let $G$ be a group and $H \leq G$. If $H$ is a $k$-nilpotent subgroup, $E_l = E_k$ for all $l \geq k$ natural numbers.

Proof. By Theorem we know that $E_k$ is a $k$-nilpotent subgroup. We will argue by induction on $l$. For $l = k + 1$,

\[ C_{E_k}^{k+1}(H) = \{ x \in E_k \mid [x, H] \subseteq C_{E_k}^k(H) \} \]

\[ = \{ x \in E_k \mid [x, H] \subseteq Z_k(E_k) \} \]

\[ = \{ x \in E_k \mid [x, H] \subseteq E_k \} = E_k \]

where Lemma and $k$-nilpotence of $E_k$ are used. Since

\[ E_{k+1} = \{ g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \subseteq C_{E_k}^k(H) \} \]

\[ = \{ g \in E_k \mid [g, E_k] \subseteq E_k \} = E_k \]

claim is true for $l = k + 1$. Suppose that our claim is true for $l = k + n$, i.e. $E_{k+n} = E_k$, we shall verify the same equality for $E_{k+n+1}$. Since

\[ E_k = Z_k(E_k) \leq C_{E_k}^k(H) \leq C_{E_k}^{k+n}(H) \leq E_k \]
$C_{E_k}^{k+n}(H) = E_k$. Then, by induction we have

$$
C_{E_k}^{k+n+1}(H) = \left\{ x \in \cap_{i \leq k+n} N^1_{E_k} \left( C_{E_k}^i(H) \right) \mid [x, H] \subseteq C_{E_k}^{k+n}(H) \right\}
$$

$$
= \left\{ x \in \cap_{i \leq k+n} N^1_{E_k} \left( C_{E_k}^i(H) \right) \mid [x, H] \subseteq C_{E_k}^{k+n}(H) \right\}
$$

$$
= \{ x \in E_k \mid [x, H] \subseteq E_k \} = E_k.
$$

Then, the following equation is obtained:

$$
E_{k+n+1} = \left\{ g \in E_{k+n} \mid [g, \, C_{E_k}^{k+n+1}(H)] \leq C_{E_k}^{k+n}(H) \right\}
$$

$$
= \{ g \in E_k \mid [g, E_k] \leq C_{E_k}^{k+n}(H) \} = \{ g \in E_k \mid [g, E_k] \leq E_k \} = E_k.
$$

Thus the claim holds for all $l \geq k$ natural numbers. \hfill \Box

In a separate preprint, we generalize the results of this subsection to hypercentral subgroups by defining transfinite versions of the $E_k$-chains ([3]).

### 3.2. Topological Results.

In this subsection, we will be working in a group $G$ endowed with a topology where singletons are closed and the following functions are continuous

$$
x \to x^{-1}, \quad x \to ax, \quad x \to xa, \quad x \to x^{-1}ax
$$

for all $a \in G$. We will show that the presence of such a topology on $G$ is sufficient to ensure that the $E_k$-envelopes of arbitrary subgroups are closed. In classes of groups that satisfy noetherianity properties of closed subgroups, this result suffices to conclude that the $E_k$-envelopes stabilize. An example of such class is that of groups satisfying the chain condition on closed subgroups (see ([2]).

**Lemma 3.2.1.** Let $G$ be a group satisfying the standing topological hypothesis and $X \subseteq G$. Then, for every $i \in \mathbb{N}$, the subgroup $C_G^i(X)$ is closed.

**Proof.** We proceed by induction on $i$. When $i = 0$, $C_G^0(X) = \{ e \}$, and since each single element subset of $G$ is closed, the claim holds. We now assume that $C_G^j(X)$ iterated centralizers are closed for all $j$ natural numbers such that $j < i$. By definition,

$$
C_G^i(X) = \left\{ g \in \cap_{j < i} N_G \left( C_G^j(X) \right) \mid [g, X] \leq C_G^{i-1}(X) \right\}.
$$

Each $C_G^j(X)$ is closed by the inductive assumption for $j < i$. Then by ([6] Lemma 5.4], $N_G \left( C_G^j(X) \right)$ is closed. Hence $\cap_{j < i} N_G \left( C_G^j(X) \right)$ is closed. On the other hand, the following function is continuous in $G$

$$
k_x : G \to G, \quad g \mapsto [g, x]
$$

where $x$ is a fixed element of $X$. The inverse image of $C_G^{i-1}(X) \leq G$ with respect to function $k_x$ is

$$
k_x^{-1} \left( C_G^{i-1}(X) \right) = \{ g \in G \mid k_x(g) \in C_G^{i-1}(X) \} = \{ g \in G \mid [g, x] \in C_G^{i-1}(X) \}.
$$
As the intersection \( \cap_{x \in X} k^{-1}_x (C_G^{i-1}(X)) \) is also closed, and
\[
g \in \cap_{x \in X} k^{-1}_x (C_G^{i-1}(X))
\]
\[\Leftrightarrow [g, x] \in C_G^{i-1}(X), \text{ for all } x \in X
\]
\[\Leftrightarrow [g, X] \subset C_G^{i-1}(X),
\]
it follows that \( C_G^i(X) \) is closed. \( \square \)

**Lemma 3.2.2.** Let \( G \) be a group satisfying the topological hypothesis of this subsection and \( H \) a subgroup of \( G \). Then the \( E_k(H) \) are closed.

**Proof.** Since \( G \) is the ambient space, our claim is trivial for \( k = 0 \). We now assume that \( E_k \) is closed. For \( k + 1 \), we have
\[
E_{k+1} = \{ g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \leq C_{E_k}^k(H) \}
\]
\[= \{ g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \leq Z_k(E_k) \}.
\]
For iterated centers \( Z_k(E_k) = C_G^k(E_k) \cap E_k \). By the induction hypothesis and Lemma 3.2.1, \( E_k \) and \( C_G^k(E_k) \) are closed. Thus, \( Z_k(E_k) \) is also closed. Let \( x \in C_{E_k(H)}^{k+1}(H) \). By the properties of the topology on \( G \), the following function is continuous:
\[k_x : G \to G, \quad g \mapsto [g, x].
\]
The inverse image of \( Z_k(E_k) \) with respect to \( k_x \) is
\[k^{-1}_x(Z_k(E_k)) = \{ g \in G \mid k_x(g) \in Z_k(E_k) \} = \{ g \in G \mid [g, x] \in Z_k(E_k) \}
\]
and closed. Moreover,
\[g \in \cap k^{-1}_x(Z_k(E_k)) \Leftrightarrow g \in k^{-1}_x(Z_k(E_k)), \text{ for all } x \in C_{E_k(H)}^{k+1}(H)
\]
\[\Leftrightarrow [g, x] \in Z_k(E_k), \text{ for all } x \in C_{E_k(H)}^{k+1}(H)
\]
\[\Leftrightarrow [g, C_{E_k(H)}^{k+1}(H)] \subset Z_k(E_k)
\]
It follows that \( E_{k+1} \) is closed. \( \square \)

**Corollary 3.2.3.** Let \( G \) be a group which satisfies the standing topological hypothesis of the subsection and \( H \leq G \). If \( G \) satisfies the minimal condition on closed subgroups then \( E_k(H) \)-envelopes stabilize.

**Corollary 3.2.4.** Let \( G \) be a linear group and \( H \leq G \). Then \( E_k(H) \)-envelopes stabilize in \( G \).

**Proof.** By Theorem 3.5 of [3], \( G \) satisfies the chain condition on closed subgroups. \( \square \)
4. Counter-Example

In this section we will construct a counterexample to the stabilization problem of the $E_k$-envelopes. We will be working in the symmetric group on the natural numbers $\text{Sym}(\mathbb{N})$ that we will denote $G$. The subgroup $H$ whose $E_k$-envelopes form an infinite descending chain is defined as follows. Let $K = \bigoplus_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle$.

Note that $K$ is a normal subgroup of $G$. Now we define a special permutation by the action $(2x \ 2x + 1) \mapsto (2f(x) \ 2f(x) + 1)$ for every $x \in \mathbb{N}$, where

$$f(x) = \begin{cases} x \mapsto x + 2, & \text{if } x \text{ is even,} \\ x \mapsto x - 2, & \text{if } x \text{ is odd and } x \neq 1, \\ 1 \mapsto 0, & \text{if } x = 1 \end{cases}.$$ 

We define $H = K \rtimes \langle f \rangle$. We will show the $E_k(H)$ form an infinite descending chain.

We emphasize that $G$ is not an $\mathfrak{MC}$-group. Indeed, one can easily show the existence of infinite descending chains of centralizers.

From now until the end of the section we will detail the determination of the nature of $E_k(H)$. The main step will be to show that the iterated centralizers of $H$ are finite subgroups of $G$ whose nontrivial elements are of infinite support and that form an infinite ascending chain.

We use a special notation for the elements of the group $\Pi_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle$:

$$g \in \Pi_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \Rightarrow g = \Pi_{x \in \mathbb{N}} (2x \ 2x + 1)^{j_g(x)}$$

where $j_g(x) \in \{0, 1\}$. The arguments about the function $j_g$ involve elementary arithmetic. This will always be modulo 2.

The actions on $\mathbb{N}$ will be on the left.

First, we shall give a technical lemma which include two known results:

**Lemma 4.1.** Let $G_0$ satisfy the property

$$G \geq G_0 \geq \prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle.$$ 

Then the following equalities hold:

(i) $C_G\left( \bigoplus_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \right) = \prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle$, 

(ii) $C_{G_0}(H) = \left\langle \prod_{x \in \mathbb{N}} (2x \ 2x + 1) \right\rangle$.

**Proof.** (i) Since $C_G\left( \bigoplus_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \right) = C_G(\{ (2x \ 2x + 1) \mid x \in \mathbb{N} \})$, $g \in C_G\left( \bigoplus_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \right)$ if and only if $g \langle (2x \ 2x + 1) \rangle g^{-1} = \langle (2x \ 2x + 1) \rangle$ for every $x \in \mathbb{N}$ if and only if $g \{2x, 2x + 1\} = \{2x, 2x + 1\}$ for every $x \in \mathbb{N}$.
(ii) Since the centralizer of $K$ in $G_0$ is known by (i), computing the centralizer $C_{C_{G_0}(K)}(\langle f \rangle)$ is enough. Let $g \in C_{G_0}(K)$. Then by part (i), $g$ is of the form $\prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_y(x)}$, $j_y(x) \in \{0, 1\}$. According to this

$$[g, f] = \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_y(x), f}$$

$$= \prod_{x \in \mathbb{N}} \left( (2x \cdot 2x + 1)^{j_y(x)} f^{-1}(2x \cdot 2x + 1)^{j_y(x) f} \right)$$

$$= \prod_{x \in \mathbb{N}} \left( (2x \cdot 2x + 1)^{j_y(x)} (2 f^{-1}(x) \cdot 2 f^{-1}(x) + 1)^{j_y(x)} \right)$$

$$= \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_y(x) + j_y(f(x))}$$

is obtained. Then we have

$$[g, f] = 1 \iff \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_y(x) + j_y(f(x))} = 1$$

$$\iff j_y(x) + j_y(f(x)) = 0 \quad \text{for all } x \in \mathbb{N}$$

$$\iff j_y(x) = j_y(f(x)) \quad \text{for all } x \in \mathbb{N}.$$

It follows that

$$C_{C_{G_0}(K)}(\langle f \rangle) = \left\{ 1, \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1) \right\} = \left\langle \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1) \right\rangle,$$

and $C_{G_0}(H) = \left\langle \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1) \right\rangle$.

The following result will be used in computing the iterated centralizers. It will first be proven under a specific hypothesis which will be eliminated in Corollary 4.3.

**Proposition 4.2.** Suppose that for every $k \in \mathbb{N}$, $\prod_{x \in \mathbb{N}} \langle (2x \cdot 2x + 1) \rangle \leq E_k(H)$. Then $C_{E_k(H)}^{i+1}(H) \leq \prod_{x \in \mathbb{N}} \langle (2x \cdot 2x + 1) \rangle$ for all $k \in \mathbb{N}$ and $i \leq k$, and for every $h \in C_{E_k(H)}^{i+1}(H)$, for all $x \in \mathbb{N}$ the following relation holds:

$$j_h(x) = j_h\left(x + 2^{(i+1)}\right).$$

In particular, $C_{E_k(H)}^{i+1}(H)$ is finite for every $k \in \mathbb{N}$ and $i \leq k + 1$.

**Proof.** Let $k \in \mathbb{N}$. We proceed by induction on $i \leq k$. For $i = 0$, we want to show that claim holds in the centralizer $C_{E_k(H)}(H)$. It is enough to compute the centralizers $C_{E_k(H)}(K)$ and $C_{E_k(H)}(\langle f \rangle)$. By the hypothesis of the proposition, since the group $\prod_{x \in \mathbb{N}} \langle (2x \cdot 2x + 1) \rangle$ is abelian, $\left\langle \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1) \right\rangle \leq C_{E_k(H)}(H)$. Now we show that this inclusion in fact is an equality.
By Lemma 4.1 (i) we have $C_{E_k(H)}(K) = \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$. We now compute $C_{C_{E_k(H)}(K)}(f)$. Let $g \in \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$. Thus $g$ is the form $\prod_{x \in \mathbb{N}} (2x \ 2x+1)^{j_g(x)}$. We have already proven in Lemma 4.1 that $[g, f] = 1$ if and only if $j_g(x) = j_g(f(x))$ for every $x \in \mathbb{N}$.

Taking into consideration the function $f$,

$$j_g(x) = j_g(f(x)) \iff \begin{cases} j_g(2x) = j_g(2x + 2), & x \in \mathbb{N} \\ j_g(2x + 1) = j_g(2x - 1), & 1 \leq x, \ x \in \mathbb{N} \\ j_g(1) = j_g(0) \end{cases}$$

is obtained. So,

$$C_{C_{E_k(H)}(K)}(f) = \left\{ 1, \prod_{x \in \mathbb{N}} (2x \ 2x+1) \right\}.$$ 

Therefore,

$$C_{E_k(H)}(H) = \left\langle \prod_{x \in \mathbb{N}} (2x \ 2x+1) \right\rangle.$$ 

The equation $j_g(x) = j_g(x + 2)$ holds directly since the equation $j_g(x) = j_g(x + 1)$ is true in $C_{E_k(H)}(H)$. The claim follows for $i = 0$.

Assume it is satisfied for $(i - 1)$. So by induction $C^i_{E_k(H)}(H) \leq \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$ and for every $h \in C^i_{E_k(H)}(H)$, the equality $j_h(x) = j_h(x + 2^i)$ holds for every $x \in \mathbb{N}$. Note that it follows from the induction hypothesis and the equality $j_h(x) = j_h(x + 2^i)$ that the elements of $C^i_{E_k(H)}(H)$ apart from the identity element have infinite support. For $i \in \mathbb{N}$, since $C^{i+1}_{E_k(H)}(H) = \left\{ g \in E_k(H) \mid [g, H] \subseteq C^i_{E_k(H)}(H) \right\}$ and $H = K \times \langle f \rangle$, every $g \in C^{i+1}_{E_k(H)}(H)$ has no satisfy the following conditions:

- $[g, K] \subseteq C^i_{E_k(H)}(H),$ 
- $[g, f] \subseteq C^i_{E_k(H)}(H).$

Our standing hypothesis on $\prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$ yields two cases:

a: $g \in \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle,$

b: $g \in E_k(H) \setminus \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle.$

We will describe the form of $g \in E_k(H)$ satisfying these conditions.

- We start with the commutator condition on $[g, K]$.

a: For $g \in \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$, $[g, K] = 1 \subseteq C^i_{E_k(H)}(H)$.

b: For $g \in E_k(H) \setminus \prod_{x \in \mathbb{N}} \langle (2x \ 2x+1) \rangle$, there exists $x \in \mathbb{N}$ such that $g^{-1} \{2x, 2x+1\} \neq \{2x, 2x+1\}.$

Thus $[g, (2x \ 2x+1)] \neq 1$. But $(2x \ 2x+1) \in K$ and one can easily compute that $[g, (2x \ 2x+1)]$ has finite support. Hence we have found an element of $K$, namely $(2x \ 2x+1)$, such that $[g, (2x \ 2x+1)] \notin C^i_{E_k(H)}(H)$; indeed, as it was mentioned at the beginning of the inductive step of the
proof, all nontrivial elements of $C_{E_k(H)}^i(H)$ are of infinite support. Thus, $C_{E_k(H)}^{i+1}(H) \leq \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)$.

- Using the previous step, we analyze $g \in \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)$ such that $[g, f] \subseteq C_{E_k(H)}^i(H)$. Hence, $g$ is of the form $\prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x)}$. We have already verified that $[g, f] = \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x) + j_h(f(x))}$ for every $x \in \mathbb{N}$. Since $g$ satisfies the condition

$$[g, f] = \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x) + j_h(f(x))} \in C_{E_k(H)}^i(H),$$

there exists $h = \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x)} \subset C_{E_k(H)}^i(H)$ such that

$$\prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x) + j_h(f(x))} = \prod_{x \in \mathbb{N}} (2x \cdot 2x + 1)^{j_h(x)}$$

Thus $j_h(x) + j_h(f(x)) = j_h(x)$ for every $x \in \mathbb{N}$. By the definition of $f$, for $a \in \mathbb{N}$, the following relations hold:

$$j_g(2a) + j_g(2a + 2) = j_h(2a)$$

$$j_g(2a + 3) + j_g(2a + 1) = j_h(2a + 3)$$

$$j_g(1) + j_g(0) = j_h(1)$$

(4.2.1)

The powers $j_g(x)$ will be determined separately but by using similar methods according to the parity of $x$. By the induction hypothesis the following system of equations is obtained for even $x$:

$$j_g(2a) + j_g(2a + 2) = j_h(2a), \quad 0 \leq a < 2^{i-1}.$$  

Summing up the two sides of these equations, we get

$$j_g(0) + 2j_g(2) + \ldots + j_g(2^i) = j_h(0) + \ldots + j_h(2^i) - 2^i$$,

and

$$j_g(0) + j_g(2^i) = \sum_{a=0}^{2^{i-1}-1} j_h(2a).$$

For the sum $\sum_{a=0}^{2^{i-1}-1} j_h(2a)$, there are two possibilities:

- **a:** $\sum_{a=0}^{2^{i-1}-1} j_h(2a) = 0$
- **b:** $\sum_{a=0}^{2^{i-1}-1} j_h(2a) = 1$.

In case (a), $j_g(0) + j_g(2^i) = 0$ implies $j_g(0) = j_g(2^i)$. Thus, the equation $j_g(x) = j_g(x + 2^i)$ holds for $x = 0$. By changing the initial value of $a$ between 0 and $2^{i-1} - 1$, we verify that $j_g(x) = j_g(x + 2^i)$ is true when $x$ is even.

In case (b), the periodicity of the powers is not complete. Thus we continue:

$$j_g(2a) + j_g(2a + 2) = j_h(2a), \quad 2^{i-1} \leq a < 2^i - 1.$$  

Summing up side by side, we obtain

$$j_g(2^i) + 2j_g(2^i + 2) + \ldots + j_g(2^{i+1}) = j_h(2^i) + \ldots + j_h(2^{i+1} - 2).$$
and
\[ j_g(2^i) + j_g(2^{i+1}) = \sum_{a=2^i}^{2^i-1} j_h(2a) = 1. \]

Finally, we put the two systems together:
\[
(j_g(0) + j_g(2^i)) + (j_g(2^i) + j_g(2^{i+1})) = \sum_{a=0}^{2^{i-1}-1} j_h(2a) + \sum_{a=2^i}^{2^i-1} j_h(2a),
\]
equivalently,
\[
j_g(0) + 2j_g(2^i) + j_g(2^{i+1}) = \sum_{a=0}^{2^i-1} j_h(2a).
\]

Using the \(2^{i-1}\)-periodicity of the permutation representation of \(h\), we have
\[
j_g(0) + j_g(2^{i+1}) = \sum_{a=0}^{2^i-1} j_h(2a) = 0.
\]

Therefore,
\[ j_g(0) = j_g(2^{i+1}). \]
Again, by changing the initial value of \(a\), we can verify that \(j_g(x) = j_g(x + 2^{i+1})\) for \(x\) even.

Similar computations using
\[ j_g(2a + 3) + j_g(2a + 1) = j_h(2a + 3) \]
and induction yield the equality 4.2.1 for odd \(x\). \(\square\)

Now we proceed to eliminate the assumption \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \leq E_k(H)\) for all \(k \in \mathbb{N}\), from the statement of Proposition 4.2.

**Corollary 4.3.** For every \(k \in \mathbb{N}\), \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \leq E_k(H)\); in particular, the assumption \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \leq E_k(H)\) is superfluous in Proposition 4.2.

**Proof.** The statement is trivially true for \(k = 0\). We assume that it holds for \(k\). For \(k + 1\), by definition,
\[ E_{k+1}(H) = \left\{ g \in E_k(H) \mid \left[ g ; C_{E_k(H)}^{k+1}(H) \right] \leq C_{E_k(H)}^k(H) \right\}. \]

By the inductive hypothesis \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle \leq E_k(H)\). Since \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle\) is abelian, and \(C_{E_k(H)}^{k+1}(H)\) is a subgroup of \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle\) by Proposition 1.2 we have
\[
\left[ g , C_{E_k(H)}^{k+1}(H) \right] = 1 \leq C_{E_k(H)}^k(H)
\]
for every \(g \in \prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle\). Thus, \(\prod_{x \in \mathbb{N}} \langle (2x \ 2x + 1) \rangle\) is a subgroup of \(E_{k+1}(H)\). \(\square\)
In the following two propositions, we will show that the iterated centralizers \( C^i_{E_k(H)}(H) \) form a specific ascending chain. In Proposition 4.3 we show that this ascendance is strict.

**Proposition 4.4.** For all \( k \in \mathbb{N} \), \( C^{k+1}_{E_{k+1}(H)}(H) = C^{k+1}_{E_k(H)}(H) \).

**Proof.** We will apply Lemma 2.4 (iii), to the triple \( H \leq E_{k+1}(H) \leq E_k(H) \). By Lemma 2.6 since \( C^i_{E_k(H)}(H) = Z_i(E_k(H)) \) for \( i, k \in \mathbb{N} \) and \( i \leq k \), the hypotheses of Lemma 2.4 is satisfied. Thus
\[
C^{k+1}_{E_{k+1}(H)}(H) = C^{k+1}_{E_k(H)}(H) \cap E_{k+1}(H).
\]
The following inclusion resulting from Proposition 4.2 and Corollary 4.3 yields
\[
C^{k+1}_{E_k(H)}(H) = C^{k+1}_{E_{k+1}(H)}(H).
\]
\[\square\]

**Proposition 4.5.** For all \( k \in \mathbb{N} \) and \( i \leq k \), \( C^i_{E_k(H)}(H) < C^{i+1}_{E_k(H)}(H) \).

**Proof.** We have shown in the proof of Proposition 4.2 that
\[
C_{E_k(H)}(H) = \left( \prod_{x \in \mathbb{N}} (2x \ 2x+1) \right).
\]
Thus, each \( C^i_{E_k(H)}(H) \) is nontrivial for every \( k \in \mathbb{N} \) and \( 1 \leq i \leq k + 1 \). For every nontrivial \( h \in C^i_{E_k(H)}(H) \), there is a first even (resp. odd) place \( x_0 \) such that \( j_h(x_0) = 1 \) and \( j_h(x) = 0 \) for every even \( x \) (resp. odd) such that \( x < x_0 \). We fix \( h \in C^i_{E_k(H)}(H) \) such that \( x_0 \) is maximal. Such an \( h \) exists because \( C^i_{E_k(H)}(H) \) is finite by Proposition 4.2. We will show that there exists \( g \in C^{i+1}_{E_k(H)}(H) \setminus C^i_{E_k(H)}(H) \) such that \( [g, f] = h \). We will note \( g = \prod_{x \in \mathbb{N}} (2x \ 2x+1)^{j_g(x)} \). We first analyze the case \( x_0 \) is even.

By equation 4.2 (i) in Proposition 4.2 applied to even places, we set up the following system of equations for \( g \) and \( h \):
\[
\begin{align*}
j_g(0) + j_g(2) &= j_h(0) = 0 \\
j_g(x_0 - 2) + j_g(x_0) &= j_h(x_0 - 2) = 0 \\
j_g(x_0) + j_g(x_0 + 2) &= j_h(x_0) = 1 \\
& \vdots \\
j_g(2^{i+1} - 2) + j_g(2^{i+1}) &= j_h(2^{i+1} - 2).
\end{align*}
\]
This system of equations is then repeated since the places of \( g \) are \( 2^{i+1} \)-periodic (Proposition 4.2).
We define \( g \) coherently with this periodicity condition. We set \( j_g (0) = 0 \). Then \( j_g (2a) = 0 \) for \( 2a \leq x_0 \) and \( j_g (x_0 + 2) = 1 \). The values of \( j_g (2a) \) for \( x + 2 \leq 2a \leq 2^{i+1} \) are then computed inductively using the equation system. We have to show that \( j_g (0) = j_g (2^{i+1}) \). Summing up the two sides of the equation system we obtain

\[
j_g (0) + j_g (2^{i+1}) = \sum_{x=0}^{2^{i+1}-2} j_h (x) = 0.
\]

The second equality follows from the \( 2^i \)-periodicity of \( j_h \).

It remains to define \( j_g (2a + 1) \) for \( a \in \mathbb{N} \) coherently with the \( 2^{i+1} \)-periodicity of \( j_g \). From the equality \( j_g (0) + j_g (1) = j_h (1) \) (equation 4.2.1), one deduces the value of \( j_g (1) \). The rest of the odd places of \( j_g \) is inductively computed using \( j_g (2a + 3) + j_g (2a + 1) = j_h (2a + 3) \) for \( a \in \mathbb{N} \) (equation 4.2.1). The values of \( j_g (2a + 1) \) are \( 2^{i+1} \)-periodic because the right hand side of the equation \( 2^i \)-periodic.

When \( x_0 \) is odd we use the equation

\[
j_g (0) + j_g (1) = j_h (1)
\]

in 4.2.1 to set \( j_g (1) = 0 \) and \( j_g (0) = 1 \). After this initial step, the rest of the values of \( j_g \) are determined coherently using the two recursive equalities of 4.2.1 in a similar way to the case when \( x_0 \) is even.

Finally, we remark that \( g \in C_{E_k (H)}^{i+1} (H) \setminus C_{E_k (H)}^i (H) \). Indeed, by construction \([g, f] \in C_{E_k (H)}^i (H)\) and trivially \([g, K] = 1\), hence \( g \in C_{E_k (H)}^{i+1} (H) \). Moreover, by the maximal choice of \( x_0 \), \( g \) is not in \( C_{E_k (H)}^i (H) \).

The two preceding propositions yield the following corollary:

**Corollary 4.6.** The following relations hold between the iterated centralizers such that \( j < j' \), \( k \leq k' \), \( j \leq k \), \( j' \leq k' \) for all natural numbers:

(i) \( C_{E_k (H)}^j (H) = C_{E_{k'} (H)}^{j'} (H) \)

(ii) \( C_{E_k (H)}^j (H) \triangleleft C_{E_{k'} (H)}^{j'} (H) \).

In particular, the chain \( \left( C_{E_k (H)}^j (H) \right) \) is strictly increasing, the indices \((k, j)\) being ordered lexicographically.

**Proof.** We know \( C_{E_k (H)}^j (H) \leq C_{E_{k'} (H)}^k (H) \) for \( j \leq k \) by properties of iterated centralizers. By Lemma 2.4, \( C_{E_k (H)}^j (H) = Z_k (E_k (H)) \). Since the inclusion \( Z_k (E_k (H)) \leq Z_k' (E_{k'} (H)) \) holds for \( k \leq k' \) by Corollary 2.5, we get

\[
C_{E_k (H)}^j (H) \leq C_{E_{k'} (H)}^k (H) = Z_k (E_k (H)) \leq Z_k' (E_{k'} (H)) = E_{k'} (H).
\]

Since \( C_{E_k (H)}^j (H) = Z_j (E_k (H)) \) for \( j \leq k \), we can apply Lemma 2.4 to \( H \leq E_{k'} (H) \leq E_k (H) \). Thus \( C_{E_k (H)}^j (H) = C_{E_{k'} (H)}^j (H) \cap E_{k'} (H) \) for all \( j \leq k \).

Since we have already shown \( C_{E_k (H)}^j (H) \leq E_{k'} (H) \), we get \( C_{E_k (H)}^j (H) = C_{E_{k'} (H)}^j (H) \); (i) follows. Conclusion (ii) is a consequence of Proposition 4.5 and conclusion (i).
We start the final part of our proof that the \( E_k (H) \) form an infinite decreasing chain. We first prove a technical lemma.

**Lemma 4.7.** Let \( k \in \mathbb{N} \). Let \( l \in \mathbb{N} \) be minimal such that for every \( g \in C_{E_k}^{k+1} (H) \) and for every \( x \in \mathbb{N} \), \( j_g (x) = j_g (x + 2^l) \). Then
\[
G_{x,l} \leq E_k
\]
where
\[
G_{x,l} = \left\{ \left( 2 \left( x + 2^i \right) \ 2 \left( x + 2^i \right) \right) \left( (2 \left( x + 2^i \right) + 1) \left( 2 \left( x + 2^i \right) + 1 \right) \right) \mid i, i' \in \mathbb{N} \right\}
\]
and \( 0 \leq x < 2^l \).

**Proof.** We proceed by induction on \( k \). For \( k = 0 \), the conclusion is trivial. We inductively assume that the claim holds for \( k \). In particular, \( G_{x,l} \leq E_k \). It is known from Proposition 4.2 that for every \( h \in C_{E_k}^{k+2} (H) \) there exist \( l' \leq k + 2 \) such that for every \( 0 \leq x < 2^{l'} \), \( j_h (x) = j_h (x + 2^{l'}) \). We want to show the inclusion \( G_{x,l'} \leq E_{k+1} \). Let \( g \in G_{x,l'} \). In order to verify \( g \in E_{k+1} \), there are two conditions to check:

- a: \( g \in E_k \),
- b: \( [g, C_{E_k}^{k+1} (H)] \leq C_{E_k}^k (H) \).

a: By induction, \( G_{x,l} \leq E_k \) for \( 0 \leq x < 2^l \). Since by Corollary 4.6 \( C_{E_k}^{k+1} (H) \leq C_{E_{k+1}}^{k+2} (H) \), we have \( l \leq l' \). Therefore \( 2^l \) divides \( 2^{l'} \) and we get \( G_{x,l'} \leq G_{x,l} \leq E_k \) for \( 0 \leq x < 2^l \).

b: By definition, \( g \in G_{x,l'} \) for a certain \( x \) such that \( 0 \leq x < 2^{l'} \). Since \( 2^l \) divides \( 2^{l'} \), \( [g, C_{E_k}^{k+1} (H)] = 1 \). Hence \( [g, C_{E_k}^{k+1} (H)] \subseteq C_{E_k}^k (H) \).

\( \square \)

**Theorem 2.** For every \( k \in \mathbb{N} \), there is at least one \( k' > k \) such that \( E_{k'+1} < E_{k+1} \).

**Proof.** We start by fixing \( k \in \mathbb{N} \). Let \( x,l \) and \( G_{x,l} \leq E_{k+1} \) be as in Lemma 4.7.

By Proposition 4.2 for every \( h \in C_{E_k}^{k+1} (H) \), \( j_h (x) = j_h (x + 2^l) \). Since \( C_{E_k}^k (H) \) is finite and by Corollary 4.6 the chain of such iterated centralizers is strictly increasing, there exist \( k' > k \), \( h \in C_{E_{k'}}^{k+1} (H) \) and \( x_o \), \( 0 \leq x_o < 2^l \), such that \( j_h (x_o) \neq j_h (x_o + 2^l) \). We define \( g \) to be \( (2x_o 2(x_o + 2^l))(2x_o + 1 2(x_o + 2^l) + 1) \). By Lemma 4.7, \( g \in E_k \). In fact, \( g \in E_{k+1} (H) \) since \( [g, C_{E_k}^{k+1} (H)] = 1 \subseteq C_{E_k}^k (H) \).

But \( g \notin E_{k'+1} \). Indeed, one computes
\[
[g, h] = h^2 h
\]
\[
= (2x_o 2x_o + 1) \left( 2 \left( x_o + 2^l \right) \ 2 \left( x_o + 2^l \right) \right) \left( \prod_{i \neq x_o, x_o + 2^l} (2i 2i + 1)^{j_h (i)} \right)^2
\]
\[
= (2x_o 2x_o + 1) \left( 2 \left( x_o + 2^l \right) \ 2 \left( x_o + 2^l \right) \right) .
\]

for \( i \in \mathbb{N} \). Since \( [g, h] \neq 1 \) and is of finite support, \( [g, h] \notin C_{E_{k'}}^{k+1} (H) \). Hence \( g \notin E_{k'+1} (H) \).
5. Concluding Remarks

Our final theorem shows that the subgroup $H \leq \text{Sym} (\mathbb{N})$ is a counterexample to the stabilization problem. Indeed, in the proof of the main theorem $k$ was arbitrary, and as a result, the descending chain $(E_k(H))_k$ is infinite. We recall that $\text{Sym} (\mathbb{N})$ is not an $\mathfrak{M}_C$-group. Whether the stabilization problem has an affirmative answer in groups satisfying the chain condition on centralizers remains open.

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