Solidity of liquids: How Holography knows it

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Abstract: Recently, it has been realized that liquids are able to support solid-like transverse modes with an interesting gap in momentum space developing in the dispersion relation. We show that this gap is also present in simple holographic bottom-up models, and it is strikingly similar to the gap in liquids in several respects. Firstly, the appropriately defined relaxation time in the holographic models decreases with temperature in the same way. More importantly, the holographic $k$-gap increases with temperature and with the inverse of the relaxation time. Our results suggest that the Maxwell-Frenkel approach to liquids, involving the additivity of liquid hydrodynamic and solid-like elastic responses, can be applicable to a much wider class of physical systems and effects than thought previously, including relativistic models and strongly-coupled quantum field theories.
1 Introduction

"There are no accidents in the universe"

Master Oogway

Is it a liquid...? Is it a solid...?

This may sound a naive question with a deceitfully simple answer: the ability to flow from a container means the system is a liquid. However, a simple demonstration in fig.1 shows that this test is not always definitive. As theory and experiments developed in recent years \(^1\), this question has been appreciated to be far from trivial, and the distinctions between liquids and solids appear to be more subtle than expected. Furthermore, it is not clear if such a question is sensible and meaningful without specifying a timescale and/or lengthscale, as discussed below in detail.

Liquids are traditionally discussed in the hydrodynamic regime \(\omega / T \ll 1, k / T \ll 1\) where no shear waves propagate [1]. From a theoretical point of view, the main traditional distinction between liquids and solids has been their ability to support propagating shear waves: solids do support propagating transverse phonons while liquids

\(^1\)Together with millions of Youtube views regarding funny uses of Non-Newtonian fluids https://youtu.be/RkLn2gR7SyE, https://youtu.be/RIUEZ3AhrVE.
do not. This property is strongly connected to the existence of a finite elastic shear modulus which indeed give solids the “rigidity” properties we are used to. Indeed, it is very well-known [2, 3] that transverse phonons in solids obey the simple dispersion relation:

$$\omega = v k, \quad \text{with} \quad v^2 = \frac{G}{\chi_{PP}}$$

where $G$ is the shear modulus, $\chi_{PP}$ the momentum susceptibility and $v$ the speed of shear sound.

However, this picture is not the complete story and it was challenged by Frenkel long time ago, in 1932 [4]. Accordingly to his ideas, above a certain frequency and away from the hydrodynamic regime where:

$$\omega > \omega_F \equiv \frac{1}{\tau}$$

liquids, in fact, can support propagating shear waves and therefore behave like solids. Here, $\tau$ is liquid relaxation time introduced by Frenkel, the time between consecutive particle jumps in the liquid. With this in mind, (1.2) has a simple interpretation: at times shorter than $\tau$, the system is a solid and therefore supports all three phonon modes, one longitudinal and two transverse. At times longer than $\tau$, the system is hydrodynamic and supports one longitudinal mode only.

Interestingly, Frenkel’s proposal implied that the difference between a liquid and an isotropic solid (glass) is only quantitative (because $\tau$ grows continuously with reducing
temperature) but not qualitative, sparking an interesting debate between him and Landau (see Ref. [5]) for details). The predictions of Frenkel theory have been confirmed in several simulations and experiments [5], supporting the idea that a distinction between liquids and solids does not hold if the frequency is large enough.

Very recently, more challenges from rheology experiments appeared. In particular, new results seem to suggest that liquids can support propagating shear waves and behave like solids even at low frequencies [6–9]. This behaviour does not seem to be consistent with the Frenkel’s proposal although, as we will see below, Frenkel did have another equation in his book capable of explaining the low-frequency shear resistance of liquids. This equation is in a different part of the book and, although Frenkel wrote the equation, he did not attempt to solve it.

Two proposals have been put forward to explain the low-frequency behavior of liquids:

1. The existence of a k-gap in the transverse spectrum of liquids [5, 10–12]

2. The role of non-affine deformations in viscoelastic materials and amorphous solids [13–15]

According to the first proposal, transverse phonons in a liquid have dispersion relation of the type:

\[ Re(\omega) = \sqrt{c^2 k^2 - \frac{1}{4 \tau^2}} \]  

(1.3)

As discussed below, this result can be derived from the Navier-Stokes equation using the so-called Maxwell interpolation [5]. This result implies that at large enough momenta

\[ k > k_{\text{gap}} \equiv \frac{1}{2 c \tau} \]  

(1.4)

liquids can support propagating shear waves even at low frequency \( \omega \approx 0 \). In other words, liquids could behave like solids even at very low frequencies. This observation may provide an explanation for the unusual experimental results obtained in [6–9]. We will analyze in more details the theoretical background of both Frenkel ideas and the k-gap proposal in section 2.

In a very different area of physics, the holographic correspondence has been very useful in the understanding hydrodynamics of strongly coupled fluids. Several important results have been obtained using such a technique [16, 17]. Moreover, the collective excitations of the dual field theory are easily addressed in the AdS-CFT framework. These excitations are simply encoded in the QNM frequencies of specific gravitational
black hole solution [18–20]. Finally, in the last years, holography has been extended to the identification of gravity duals for solids and viscoelastic materials which exhibit elastic properties and propagating transverse shear waves [21–32].

In this work we will ask the following simple question:

Does holography know that liquids are quite solid?
Does holography know about the k-gap?
And if it does, what is the resulting timescale $\tau$?

As we discuss below, the answer is positive to the first two questions, and moreover we will be able to identify precisely the corresponding relaxation time in the problem.

In this paper, we analyze two different holographic models [33],[32] which, similarly to liquids, exhibit the presence of a $k$-gap in the spectrum of transverse collective modes. Importantly and unexpectedly, the temperature dependence of the gap is the same as in the condensed matter system, liquids.

The discussed features go beyond the usual hydrodynamic limit$^2$ and need a detailed numerical study at large momenta $k \gg T$. Despite this difficulty we obtain an analytic formula for the liquid relaxation time which can be defined just in terms of hydrodynamic quantities and it is in perfect agreement with our numerical data. En passant we study the thermodynamic properties of the holographic models and we emphasize their similarities with the behaviour of amorphous solids, glasses and viscoelastic materials. We believe this work can constructively contribute to answering our initial question:

Is it a liquid or a solid? Or both?

Additionally, our results provide a new unexpected and extraordinary link between condensed and soft matter physics on one hand, and black hole physics on the other.

The paper is organized as follows: in section 2, we discuss the theoretical problems of the liquid description, Maxwell-Frenkel approach to liquids involving the additivity of hydrodynamic and solid-like elastic response and show how this perspective results in the emergence of the gap in the liquid transverse spectrum in $k$-space; in sections 3, 4 we present two simple existing holographic models exhibiting a k-gap in accordance to the Condensed Matter theories; finally in section 5 we conclude discussing the results and some possible future questions.

$^2$By hydrodynamic we mean the limit of small frequencies and momenta compared to the thermal energy scale:

$$\omega \ll T, \quad k \ll T.$$  

(1.5)
Note added: In a companion paper [34] we will explain in more details the condensed matter implications and the importance of our results with respect to the recent experiments and theoretical proposals with particular emphasis on the identification and field theory derivation of the relaxation time $\tau$.

## 2 Condensed matter prologue

A theory of weakly-interacting gases can be developed using perturbation theory. On the other hand, interactions in a liquid are strong. Therefore, the liquid energy and other properties are strongly system-dependent. For this reason, a theory of liquids was believed to be impossible to construct in general form [35].

Perturbation theories do not apply to liquids because inter-atomic interactions are strong as already noted. Solid-based approaches seemingly do not apply to liquids either: its unclear how to apply the traditional harmonic expansion around equilibrium positions because the equilibrium lattice does not exist to begin with, due to particle rearrangements that enable liquids to flow. This combination of strong interactions and large particle displacements has proved to be the ultimate problem in understanding liquids theoretically, and is known as the ”absence of a small parameter”.

The absence of traditional simplifying features in the liquid description does not mean that the problem can not be solved in some other way, including attempting to solve the problem from first-principles using the equations of motion. Interestingly, this involves solving a large number of non-linear equations which is exponentially complex and therefore is not currently tractable [5].

Recent progress in understanding liquids followed from considering what kind of collective modes (phonons) can propagate in liquids and supercritical fluids [5]. It has been ascertained that, (a) the $k$-gap $k_g$ increases with the inverse of liquid relaxation time in a wide range of temperature and pressure for different liquids and supercritical fluids and (b) $k_g$ increases with temperature. The first result directly agrees with [10]; the second result agrees with [10] noting that $\tau$ decreases with temperature.

Below we briefly review the origin of this result. This program starts with Maxwell interpolation:

\[
\frac{ds}{dt} = \frac{P}{\eta} + \frac{1}{G} \frac{dP}{dt} \tag{2.1}
\]

where $s$ is the shear strain, $\eta$ is viscosity, $G$ is shear modulus and $P$ is the shear stress.

The relation (2.1) reflects Maxwell’s proposal [36] that the shear response in a liquid is the sum of viscous and elastic responses given by the first and second right-hand side terms. Importantly (we will come to this point later), the dissipative term
containing the viscosity is not introduced as a small perturbation: both elastic and viscous deformations are treated in (2.1) on equal footing.

Frenkel proposed [4] to represent the Maxwell interpolation by introducing the operator $A$ as $A = 1 + \tau \frac{d}{dt}$ so that Eq. (2.1) can be written as $\frac{ds}{dt} = \frac{1}{\eta} A P$. Here, $\tau$ is Maxwell relaxation time.

\begin{equation}
\tau_M = \frac{\eta}{G}
\end{equation}

At the microscopic level, Frenkel’s theory approximately identifies $\tau_M$ with the time between consecutive diffusive jumps in the liquid [4]. This has become an accepted view since [37]. Frenkel’s idea was to generalize $\eta$ to account for liquid’s short-time elasticity as

\begin{equation}
\frac{1}{\eta} \rightarrow \frac{1}{\eta} \left( 1 + \tau \frac{d}{dt} \right)
\end{equation}

and subsequently use this $\eta$ in the Navier-Stokes equation

\begin{equation}
\nabla^2 v = \frac{1}{\eta} \left( \rho \frac{dv}{dt} + \nabla p \right)
\end{equation}

where $v$ is velocity field, $\eta$ is viscosity, $\rho$ is density, $p$ is pressure and $d/dt = \partial_t + v \cdot \nabla$.

This gives

\begin{equation}
\eta \frac{\partial^2 v}{\partial x^2} = \left( 1 + \tau \frac{d}{dt} \right) \left( \rho \frac{dv}{dt} + \nabla p \right)
\end{equation}

Having written the equation above, Frenkel did not attempt to solve it. While reviewing the field recently, we have solved this equation and assumed, for simplicity, the absence of any external force $p = 0$ and a slowly flowing fluid $d/dt = \partial_t$ [5]:

\begin{equation}
\eta \frac{\partial^2 v}{\partial x^2} = \rho \tau \frac{\partial^2 v}{\partial t^2} + \rho \frac{\partial v}{\partial t}
\end{equation}

which, using $\eta = G\tau = \rho c^2 \tau$, can be re-written as:

\begin{equation}
c^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2} + \frac{1}{\tau} \frac{\partial v}{\partial t}
\end{equation}

We note that this equation can also be obtained by starting with the solid-like elastic equation for the non-decaying wave and, using Maxwell interpolation (2.1), generalizing the shear modulus to include the viscous response [11]. This shows that the hydrodynamic approach commonly applied to liquids [38] is not a unique starting point.
and that the solid-like elastic approach is equally legitimate, implying an interesting symmetry of the liquid description.

Let’s now Fourier decompose the velocity field as \( v = v_0 e^{ikx - i\omega t} \) in order to get:

\[
\omega^2 + \frac{i}{\tau} \omega - c^2 k^2 = 0
\]  

(2.8)

The dispersion relation of the shear modes obtained from the previous equation reads:

\[
\omega = -\frac{i}{2\tau} \pm \sqrt{c^2 k^2 - \frac{1}{4\tau^2}}
\]  

(2.9)

and it gives what is known as the k-gap\(^3\).

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**Figure 2.** Molecular dynamics simulations for supercritical Ar and CO2 between 200K and 500 K. The simulations show the presence of a k-gap in the transverse phonons dispersion relation. Figures taken from [10].

We observe that the appearance of the k-gap is a direct consequence of the unification of hydrodynamic and elasticity theories via Maxwell interpolation. In other words, the system in question (e.g. liquid) is capable of both hydrodynamic and solid-like elastic response.

According to equation (2.9), liquids can support propagating shear waves and therefore behave like solids even at low frequencies \( \omega \approx 0 \) as long as:

\[
k > k_{\text{gap}} = \frac{1}{2c\tau}
\]  

(2.10)

\(^3\)Notice that at this stage momentum is a perfectly conserved quantity. We will analyze later how to modify this picture in presence of slow momentum relaxation.
Figure 3. Possible dispersion relations and dependencies of energy $E$ on momentum $p$. Top curve shows the dispersion relation for a massive particle. Middle curve shows gapless dispersion relation for a massless particle (photon) or a phonon in solids. Bottom curve shows the dispersion relation (1.3) with the gap in $k$-space, illustrating the results of Ref. [10].

We note that at large momenta $k \gg 1/c \tau$ the dispersion relation approaches the linear one:

$$\omega = c k$$

which is typical of common solids, i.e. massless transverse phonons. The same solid-like relationship follows when $\tau$ in (2.9) becomes large, corresponding to the solid (recall that $\tau$ is the time between molecular jumps in the liquids).

Recently [10], detailed evidence for the $k$-gap was presented on the basis of molecular dynamics simulations (see Fig. 2). It has been ascertained that $k_y$ increases with the inverse of liquid relaxation time in a wide range of temperature and pressure for different liquids and supercritical fluids, as (2.10) predicts.

The gap in $k$-space, or momentum space is interesting. Indeed, the two commonly discussed types of dispersion relations are either gapless as for photons and phonons, $E = p \ (c = 1)$, or have the energy (frequency) gap for massive particles, $E = \sqrt{p^2 + m^2}$, where the gap is along the Y-axis. On the other hand, (1.3) implies that the gap is in momentum space and along the X-axis, similar to the hypothesized tachyon particles with imaginary mass [39]. Figure 3 illustrates this point.

We can advance our analysis further and expand the expression (2.9) in the hydrodynamic limit $k \ll k_{gap}$:

$$\omega_1 = -i c^2 \tau k^2 + \ldots, \quad \omega_2 = -\frac{i}{\tau} + i c^2 \tau k^2 + \ldots$$

(2.12)
From the latter expansion we can immediately realize that the relaxation time $\tau$ is related to the hydrodynamic diffusion constant\footnote{We assume the usual definition for a diffusive hydrodynamic mode $\omega = -iDk^2$.}

\[ D = c^2 \tau \] \hspace{1cm} (2.13)

Notably, if we assume the familiar expression for the diffusion constant in a relativistic fluid and the speed of sound in solids:

\[ D = \frac{\eta}{\chi_{PP}} \quad \text{and} \quad c^2 = \frac{G}{\chi_{PP}} \] \hspace{1cm} (2.14)

we find

\[ \tau = \frac{\eta}{G} \] \hspace{1cm} (2.15)

and, therefore, recover the result (2.2) originating from the Maxwell interpolation.

The presence of a k-gap in the shear spectrum of relativistic hydrodynamic and an implicit version of formula (2.13) have already appeared in [40] within the discussion of the stability and causality of second order relativistic hydro.

We will generalize formula (2.13) for slow momentum relaxation in the next sections.

We note that a relation between the relaxation time and the diffusion constant is natural in condensed matter system\footnote{For example for Brownian motion that relation reads $D \sim a^2/\tau$ whith $a$ the lattice spacing.}. The main difference compared to gravity and holographic models is that in this case, due to the relativistic symmetries, the diffusion constant $D$ is directly, and not inversely, proportional to the viscosity of the system $\eta$ (see for more details, particularly related to Heavy Ions physics [41–43]).

In this work and the following sections we will provide evidence that the spectrum of liquids is strikingly similar to that following from simple bottom-up holographic models. The similarities are both general and detailed. Inspired by Master Oogway’s words, we do not believe it can be a coincidence. To the contrary, we believe that the same general mechanism is at operation in these two very disparate areas and physical objects.

## 3 A holographic relative

In this section we present a simple holographic model which was introduced to realize momentum dissipation in the dual field theory. The model does not exhibit any shear
elastic property at zero frequency and momentum, nor any propagating transverse phonons. In the hydrodynamic limit, it behaves like a fluid with a finite momentum relaxation time $\tau_{\text{rel}}$ where the dominant excitations are just damped diffusive modes. Despite numerous studies related to this model a deep analysis of the transverse modes away from the hydrodynamic limit was missing. In the following we will perform such a task and we will prove the existence of a k-gap in the dispersion relation for shear modes.

3.1 The model

We start considering a more generic class of holographic massive gravity models [21, 23]:

$$S = M_P^2 \int d^4x \sqrt{-g} \left[ \frac{R}{2} + \frac{3}{\ell^2} - m^2 V(X) \right] \quad (3.1)$$

with $X \equiv \frac{1}{2} g^{\mu \nu} \partial_\mu \phi^I \partial_\nu \phi^I$. These setups have been examined in various directions in the last years [22, 27, 28, 44–46] because they are particularly interesting from the condensed matter perspective. We study 4D AdS black brane geometries of the form:

$$ds^2 = \frac{\ell^2}{u^2} \left[ \frac{du^2}{f(u)} - f(u) \, dt^2 + dx^2 + dy^2 \right] \quad (3.2)$$

where $u \in [0, u_h]$ is the radial holographic direction spanning from the boundary to the horizon, defined through $f(u_h) = 0$, and $\ell$ is the AdS radius.

The $\phi^I$ fields are the St"uckelberg scalars and they admit a radially constant profile $\phi^I = x^I$ with $I = x, y$. This is an exact solution of the system due to the shift of symmetry. In the dual picture these fields represent scalar operators which break translational invariance because of their explicit dependence on the spatial coordinates. Within these theories the helicity-2 metric perturbations acquire a mass term given by $m^2_S(u) = 2m^2X V'$ with the background value for $X = u^2/\ell^2$. This is the reason why we refer to them as massive gravity theories.

The solution for the emblackening factor $f$ for generic potentials $V(X)$ is then given by

$$f(u) = u^3 \int_u^{u_h} dv \left[ \frac{3}{v^4} - \frac{m^2}{v^4} V(v^2) \right], \quad (3.3)$$

where $u_h$ stands for the location of the black brane horizon. The temperature and the entropy density are defined as

$$T = -\frac{f'(u_h)}{4\pi} = \frac{6 - 2m^2V(u_h^2)}{8\pi u_h}, \quad s = \frac{2\pi}{u^2_h}. \quad (3.4)$$
Figure 4. The heat capacity in function of the dimensionless parameter $T/m$. The low temperature behaviour is $c_v \propto T$ while the high temperature limit is $c_v \propto T^d$ where in our case $d = 2$. The crossover happens approximately at the value of $T/m \approx 0.148$ which coincides with the onset of the incoherent-coherent transition [47].

Finally the energy density of the dual field theory reads

$$
\epsilon = \frac{1}{u^3} \left[ 1 + \frac{m^2 u^2}{2n-3} \right], \quad V(X) = X^n.
$$

and can be easily derived from (3.3) or from the renormalized boundary stress tensor. The transverse or shear perturbations are then encoded in the fluctuations $a_x$, $h_{tx}$, $h_{xy} \equiv u^2 \delta g_{tx}$, $\delta \phi_x$, $\delta g_{xu}$. Assuming the radial gauge, i.e. $\delta g_{xu} = 0$, and using the ingoing Eddington-Finkelstein coordinates

$$
ds^2 = \frac{1}{u^2} \left[ -f(u) dt^2 - 2dt du + dx^2 + dy^2 \right]
$$

the remaining equations read$^6$

$$
-2(1 - u^2 V''/V') h_{tx} + uh_{tx}' - i k u h_{xy} - (k^2 u + 2i\omega(1 - u^2 V''/V')) \delta \phi_x + u f \delta \phi_x'' + (-2(1 - u^2 V''/V') f + u (2i\omega + f')) \delta \phi_x' = 0; \\
2i m^2 u^2 \omega V' \delta \phi_x + u^2 k \omega h_{xy} + (6 + k^2 u^2 - 2m^2(V - u^2 V')) - 6f + 2uf') h_{tx}
$$

$^6$Here we set the momentum $k$ along the $y$ direction. This freedom comes from the fact that the model enjoy SO(2) symmetry in the $(x, y)$ subspace.
\[
+ (2 u f - i u^2 \omega) h'_{tx} - u^2 f h''_{tx} = 0 ;
\]
\[
2 i k u h_{tx} - i k u^2 h'_{tx} - 2 i k m^2 u^2 V' \delta phi_x + 2 h_{xyy} (3 + i u \omega - 3 f + u f' - m^2 (V - u^2 V'))
- (2 i u^2 \omega - 2 u f + u^2 f') h'_{xyy} - u^2 f h''_{xyy} = 0 ;
\]
\[
2 h'_{tx} - u h''_{tx} - 2 m^2 u V' \delta phi_x + i k u h'_{xyy} = 0 .
\] (3.7)

and they can be solved numerically. The asymptotics of the various bulk fields close to the UV boundary \( u = 0 \) are:
\[
\delta phi_x = \phi_x (l) (1 + \ldots) + \phi_x (s) u^{5-2n} (1 + \ldots),
\]
\[
h_{tx} = h_{tx} (l) (1 + \ldots) + h_{tx} (s) u^3 (1 + \ldots),
\]
\[
h_{xy} = h_{xy} (l) (1 + \ldots) + h_{xy} (s) u^3 (1 + \ldots).
\] (3.8)

In these coordinates the ingoing boundary conditions at the horizon are automatically satisfied by regularity at the horizon. It follows that the various retarded Green’s functions can be defined as:
\[
G^{(R)}_{T_{tx}T_{tx}} = \frac{2 \Delta - d}{2} \frac{h_{tx} (s)}{h_{tx} (l)} = \frac{3 h_{tx} (s)}{2 h_{tx} (l)} ,
\]
\[
G^{(R)}_{T_{xy}T_{xy}} = \frac{2 \Delta - d}{2} \frac{h_{xy} (s)}{h_{xy} (l)} = \frac{3 h_{xy} (s)}{2 h_{xy} (l)} .
\] (3.9)

where spacetime dependences are omitted for simplicity. From the poles of the Green functions, defined as the zero of the leading terms in the UV expansions we can read off the QNMs frequency at finite momentum.

In this work we shall consider the simplest model
\[
V(X) = X .
\] (3.10)

which is erroneously known in the literature as ”linear axions” model [33]. The literature about this model is very vast and we do not intend to review it here. We will comment through the text about the main and relevant features of the model and the corresponding references.

For completeness let us remind the reader that this holographic model is thought to be an effective description for homogeneous momentum relaxation, where the momentum relaxation time is controlled by the bulk graviton mass. From the Condensed Matter perspective it might represent an averaged description for a disordered system whose features are very similar to the properties of amorphous solids.

Before proceeding let us just discuss some thermodynamic properties of the model. In particular we can derive the heat capacity as (see [22] for previous discussions):
\[
c_v = T \frac{ds}{dT} = \frac{4 \pi (3 - m^2 u_h^2)}{u_h^2 (m^2 u_h^4 + 3)}
\] (3.11)
Figure 5. The QNMs spectrum in the shear channel for $m^2 = 0$. The dashed black lines are the analytic expression for the diffusion mode at $O(k^2), O(k^4)$ from [55].

The result is shown in fig.4. The heat capacity displays a crossover behaviour between a low temperature and a high temperature scaling. In particular for low $T$ we have $c_v \propto T$ while at high $T$ we have $c_v \propto T^d$ where in our case $d = 2$. The crossover point happens around $T/m \approx 0.148$ and coincides exactly with the incoherent-coherent transition. The observed behaviour is typical of amorphous solids [37, 48]. The high temperature contribution $T^d$ is the usual Debye contribution. On the contrary, the linear in $T$ contribution at low temperature appears in glasses, polymers and amorphous solids; it was originally explained in [49, 50] and it is due to the existence of a new type of low-energy states [37, 48]. In the mentioned incoherent regime, all the modes are overdamped and they share a lot of similarities with the instantaneous normal modes [51–54]. We plan to come back to this interesting correspondence in the future.

3.2 The shear channel

In this section we focus on the dynamical modes in the shear sector; we leave the longitudinal channel aside. For some complete discussion about Hydrodynamics, QNMs and BH fluctuations see [18, 19, 56, 57].

In absence of any momentum relaxation time, i.e. $m^2 = 0$, we are left with a pure Schwarzschild solution which corresponds to a relativistic hydrodynamic system [41]. In the hydro limit $\omega/T \ll 1, k/T \ll 1$ the shear mode is purely diffusive:

$$\omega = -i D k^2 - i \# k^4 + \ldots$$  \hspace{1cm} (3.12)
Figure 6. **Left:** The relaxation time $\Gamma = \tau_{\text{rel}}^{-1}$ in function of temperature. The dots are the numerical values extracted from the QNMs while the purple curve is the approximated formula (3.16). A similar plot can be found in [58, 59]. **Right:** The momentum diffusion constant $D$ extracted from the shear QNMs. The green curve is the expression $\eta/\chi_{PP}$ which appears to be in good agreement with the numerics. The dashed line is the relativistic expression $D = 1/4\pi T$ which turns out to be correct only at large temperature.

where $\#$ is a specific combination of third order hydro coefficients [55] and $D$ is the momentum diffusion constant for a relativistic fluid:

$$D = \frac{\eta}{sT} = \frac{1}{4\pi T}$$  \hspace{1cm} (3.13)

Our interest is to study the spectrum of excitations of the system with $m^2 \neq 0$ and beyond the hydrodynamic limit. We proceed with analyzing in more details the feature of the QNMs spectrum in the shear channel.

The zero mass case $m^2 = 0$ has been studied in details in various papers. For completeness we just repeat some salient features which will be useful for the future. The results are summarized in fig.5. The only hydrodynamic mode in the shear channel is the diffusion mode (3.12) with exactly zero real part. Additional non-hydro modes are present in the spectrum. At a certain critical momentum $k^*$ one of the initially overdamped mode crosses the diffusive mode and becomes the dominant one at large momentum $k \gg k^*$. Moreover such a mode at large momentum asymptotes a relativistic dispersion relation $\omega = k$, forced by the UV relativistic fixed point, and in particular it becomes a propagating "sound" mode with $Re[\omega] \gg Im[\omega]$. The large momentum
limit can be analyzed using the WKB methods \cite{60, 61}. Nevertheless, for \( m^2 = 0 \) no k-gap appears in the spectrum and the phenomenology is far from the one discussed in the introduction.

The situation changes drastically for non-zero mass. A new dimensionful scale, \( i.e. \) the momentum relaxation time \( \tau_{\text{rel}}^{-1} \propto m^2 \), appears in the system and it competes with the original diffusion constant \( D \propto \eta \). The diffusion mode becomes damped and acquires a finite imaginary part at zero momentum which sets the rate of dissipation for the momentum operator. Moreover, at zero momentum and a precise value of \( m/T \) a coherent-incoherent transition appears; see \cite{47} for more details.

Let us start with analyzing the system within the hydrodynamic limit. The dispersion relation of the shear mode is now modified as:

\[
\omega = -i \Gamma - i D k^2 + \ldots \tag{3.14}
\]

where \( \Gamma = \tau_{\text{rel}}^{-1} \) controls the momentum relaxation timescale and \( D \) is still the diffusion constant now \( D \neq \eta/(sT) \). In order for the hydrodynamic expansion to be sensible we need the relaxation time to be not too short compared with the natural thermal timescale:

\[
\tau_{\text{rel}} T \gg 1 \tag{3.15}
\]

Notice how the modified dispersion relation (3.14) is not exactly the same of the k-gap equation (2.9) because of the presence of a finite relaxation time \( \Gamma = \tau_{\text{rel}}^{-1} \) which is not included in the original Navier-Stokes equation in section 2. We will show later how to fix this issue and get full agreement with the numerical results.

For small \( m/T \) we can write down the relaxation time as \cite{58}:

\[
\Gamma = \tau_{\text{rel}}^{-1} = \frac{m^2}{2 \pi T} + \ldots \tag{3.16}
\]

which implies that the hydro approximation is valid as far as \( m/T \ll 1 \). In the same limit we can have an approximate formula for the diffusion constant \( D \) \cite{62} and for the viscosity of the system \cite{24, 63, 64} which we define as:

\[
\eta \equiv \lim \omega \to 0 \frac{1}{\omega} Im \left[ G_{T_{xy} T_{xy}}^{R} (\omega, k = 0) \right] \tag{3.17}
\]

In particular, the results of \cite{62} show that:

\[
D_A = \frac{1}{4 \pi T} \left[ 1 + \frac{1}{24} \left( 9 + \sqrt{3} \pi - 9 \log 3 \right) \frac{m^2}{8 \pi^2 T^2} \right] + \ldots \tag{3.18}
\]

in the limit of large temperatures, \( i.e. T/m \gg 1 \). We plot the diffusion constant and the relaxation time for our system in fig.6. We notice that the approximate formulae
work quite well in the expected regime. We also notice that the following formula seems to be a very good approximation for the diffusion constant at all temperatures:

\[
D \approx \frac{\eta}{\chi_{PP}} \neq \frac{\eta}{sT}
\]

where \(\chi_{PP} = \epsilon + p \neq sT\) and in particular:

\[
\chi_{PP} = \frac{3}{2} \epsilon = sT + 2K
\]

where \(K\) is the elastic bulk modulus. This failure of the usual relation \(D = \eta/sT\) (see right panel of fig.6) is a consequence of the fact that the mechanical pressure and the thermodynamic one (extracted from the free energy) do not coincide for \(m^2 \neq 0\). This phenomenon has already been discussed and applied in [27, 28]. A more detailed fluid gravity study is necessary to solve this puzzle [65].

Moreover, decreasing the temperature \(T\) the diffusion constant grows and the momentum relaxation time becomes smaller. Notice that in non-relativistic liquids the diffusion constant usually increases with temperature and is inversely proportional to the viscosity \(\eta\). The difference hereby is caused by the relativistic symmetries of our system. Despite this difference, we will see that the temperature dependence of the \(k\)-gap and the dependence of the gap on relaxation time are the same in the two pictures: holographic gravity models and liquids.

At finite momentum \(k \neq 0\) the situation is more complicated and depends crucially on \(T/m\). Three different situations can arise and are summarized in fig.7 (notice, once
again, the surprising similarities with fig. 3 and the analysis of [11]). For small $T/m$ the lowest hydrodynamic mode is gapped and shows a dispersion relation typical of a massive particle $\omega^2 = k^2 + m^2$; moreover the mode is overdamped with an approximately constant imaginary part. Such modes share several features with the instantaneous normal modes observed and discussed in liquids [54]. Nevertheless, it is not clear to us if this regime is sensible and this kind of dispersion relation is observed in any realistic situation. Notice also that within such a regime the naive energy density $\epsilon$ is negative.

Increasing $T/m$ there is a particular value at which the dispersion relation becomes exactly linear $\omega = k$. Such a point signals the onset of incoherent-coherent transition at zero momentum which has been studied in the past literature in various contexts (see [47, 66]). Finally, increasing the temperature further, the physics is dominated by the hydro diffusive mode at low momenta $k < k_{gap}$. At a certain critical momentum $k \equiv k_{gap}$ the diffusive mode collides with the first non-hydro mode and a gap in momentum space opens. These features have been already studied from the QFT perspective in [11] and will be of interest in this work.

For a specific value of $T/m \approx 0.159$ the energy density vanishes and the system enjoys an enhancement of the symmetries which allow to compute the correlators and their poles analytically as shown in [47]. Hereby we notice that the self-dual point can be also defined by the relation $\Gamma = c^2/D$. For such a value the poles satisfy:

$$\omega = -\frac{3}{2}i \pm \sqrt{k^2 - \frac{1}{4}}$$

(3.21)

which clearly shows the presence of a gap in momentum space $k_{gap} = 1/2$. This results is confirmed by our numerical results in fig. 8. We notice that at the self-dual point the $k$-gap is given by (fixing $c = 1$):

$$k_{gap} = \Gamma$$

(3.22)
where $\Gamma$ is the momentum relaxation time defined in 3.16\textsuperscript{7}. We will come back on the nature and origin of the k-gap in the next section.

### 3.3 The k-gap and Maxwell interpolation

Our main interest is the region in the parameter space where the energy density is positive and a gap in momentum space is present. In particular we emphasize that the presence of the $k$-gap is limited to the so-called coherent region. In this region, the system lies in a coherent regime displaying a sharp Drude peak in the electric conductivity. In order to study this feature better, we collected more data for $T/m > 0.159$. Some examples are shown in fig.9. Our main purpose is to understand the origin of the emergence of the $k$-gap and make contact with the theories proposed in section 2.

The first comment, which simply comes from looking at fig.9 is that the k-gap moves towards higher momenta increasing the temperature of the system. This feature is exactly the same as that discussed in liquids earlier.

We now fit the curves to the functional form:

\footnote{This is just a consequence of the particular property of the self-dual point $c^2/D = \Gamma$ and it will not be true for generic values of $T/m$. It is indeed immediate to see that the temperature dependence of $\tau$ and $\tau_{rel}$ are exactly the opposite: $\tau_{rel}$ increases with $T$ while $\tau$ decreases.}
Figure 10. The timescale $\tau$ governing the k-gap in function of temperature. The red dot is the analytic result at the ”self-dual” temperature $T^\star$.

\[ Re(\omega) = \sqrt{c^2 k^2 - \frac{1}{4 \tau^2}} \]  \hspace{1cm} (3.23)

which has been proposed in [10] and discussed further in [5]. The agreement between the fits and the data is excellent (see fig.9). From our data we observe that the asymptotic speed is always relativistic:

\[ \omega = k \text{ for } k \to \infty \quad \leftrightarrow \quad c = 1 \]  \hspace{1cm} (3.24)

Such a feature is intimately connected to the presence of a relativistic and conformal UV fixed point. In CM language, our setup lacks the existence of a UV cutoff which can be identified with the lattice spacing; as a consequence the behaviour of our system is not cut by any UV scale and it differs from the condensed matter systems.

From the numerical fit we can obtain the behaviour of the timescale $\tau$ in function of the temperature of the system, which is shown in fig.10. The timescale $\tau$ decreases with temperature and it shows a behaviour consistent with real liquids (see [5]). Additionally we observe that, defining the critical ”self-dual” temperature $T^\star$, there is a clear change in the behaviour of $\tau$ below and above such a temperature scale as it is clear from the LogPlot in fig.10. The main question which remains to be addressed is the following:

**Which physical quantities set the timescale $\tau$??**

In order to answer this question within this model we have to take into account that hereby we have a finite relaxation time for the momentum operator $\tau_{rel} \neq 0$ and therefore the simple Maxwell’s relativistic result is expected to work just at high temperature.
(where momentum relaxation is negligible) as shown in fig.11. Due to the aforementioned complication we have to modify our setup and consider that our total relaxation time appearing in (3.23) contains also a contribution from the momentum relaxation time $\tau_{rel}$. In particular we have:

$$\frac{1}{\tau} = \frac{1}{\tau_G} + \frac{1}{\tau_{rel}} = \frac{c^2}{D} + \Gamma$$

(3.25)

where we still assume $\tau_G = D/c^2$ as in the original results presented earlier. Notice this is nothing else than adding another dissipative mechanism in the system and it can be thought as a generalized maxwell model, i.e. Maxwell-Wiechert model [67]. From the previous argument it follows immediately that the relaxation time defined in the fit function (3.23) is now:

$$\tau = \frac{D}{c^2 + D\Gamma} < \frac{D}{c^2}$$

(3.26)

and the k-gap:

$$k_{gap} = \frac{1}{2c} \frac{c^2 + D\Gamma}{D}$$

(3.27)

Notice that in the "hydro" limit $\Gamma \tau \ll 1$ (the shaded region in fig.11) the corrections coming from momentum dissipation are negligible and therefore we recover the original formula $\tau \approx D/c^2$.

This is an important result of this section and represents a generalization of Maxwell’s interpolation theory in the presence of small momentum dissipation $\Gamma \neq 0$. The expression (3.26) agrees perfectly with the data extracted from the QNMs (see fig.11). Clearly in absence of momentum dissipation, i.e. $\Gamma = 0$, we recover the results presented in the previous sections. We believe this result can be formally derived appropriately deforming the Navier-Stokes equation and computing the resulting damped shear modes; we will provide more details in the companion paper [34].

Finally, we address the behaviour of the k-gap as a function of the inverse of the relaxation time $1/\tau$ where we fix $c = 1$. The result is shown in fig.12. The curve shows a linear behaviour, in agreement with what Eq. (2.10) predicts and what was found in liquids and supercritical fluids [10]. This is another important result highlighting similarities between liquids and holographic models.

### 4 A second holographic relative

In this section we will consider a different holographic model, introduced in [32], which displays interesting viscoelastic properties and the presence of a k-gap. Most of the computations presented here are already derived in the original paper. Nevertheless we
Figure 11. A comparison between the relaxation time $\tau$ numerically computed from the QNMs (filled bullets) and several hydrodynamic quantities. (I) the diffusion constant $D$ (yellow), (II) the formula for the generalized liquid relaxation time $\tau$ (3.26) (purple) (III) the approximated formula for the diffusion constant (3.18) (dashed). Notice that for $\Gamma \tau \ll 1$, i.e. the shaded region, the corrections from $\tau_{rel}$ are small and $\tau \approx D/c^2$. In general the agreement between the data and the theory is extremely good. The red bullet indicates the self-dual point.

will discuss all the phenomenological properties that have been overlooked and we will show several interesting physical aspects of the model. Let us discuss the most relevant features of the setup.

4.1 The model

The model represents the holographic dual of a finite number of dynamical elastic lines defects embedded in a fluid state. The dynamical defects are described by two dynamical bulk two forms $B_{ab}^I$ whose field strengths $H^I = dB^I$ are chosen to be:

$$H^I_{txr} = H^I_{tyr} = m$$  \hspace{1cm} (4.1)

(not to be confused with the parameter $m$ in the previous section).

The gravitational bulk action is defined as:

$$S = \frac{1}{2 \kappa^4_4} \int d^4x \sqrt{g} \left( R + \frac{6}{L^2} - \frac{1}{12} H^I_{abc} H_I^{abc} \right)$$  \hspace{1cm} (4.2)
The setup at hand differs from the previous one for two specific and important features:

1. No explicit breaking of translational invariance is present in the system. As a consequence the momentum relaxation time is infinite $\tau_{\text{rel}} = \infty$ and the shear mode is not damped but purely diffusive. As a consequence the dispersion relation of the collective shear modes is exactly identical to the one obtained from Frenkel reduction (2.8). In this case there is no need of using expression (3.26) but we will simply have $\tau = D/c^2$.

2. The model contains a UV cutoff mass scale $\mathcal{M}$ which technically arises from the holographic renormalization procedure. This may correspond physically to the height of the interaction potential energy barrier $U$ present in liquids. See [68] for more details.
The dispersion relation (4.9) for $M/\rho h = 5, 8, 100$ (from green to blue) in terms of the dimensionless frequency and momentum $\bar{\omega} = \omega/\rho h$, $\bar{k} = k/\rho h$. At large cutoff the $k$-gap closes and the dispersion relation becomes $\bar{\omega} = \bar{k}$.

The thermodynamic of this model is very similar to the one presented in the previous section. Indeed, the heat capacity assumes a very similar form:

$$c_v = \frac{8 \pi \rho^2 h (6 \rho^2 - m^2)}{6 \rho^2 + m^2}$$

(4.4)

and, as before, shows the crossover:

$$c_v \propto T \text{ for } T \ll 1, \quad c_v \propto T^d \text{ for } T \gg 1 \text{ with } d = 2.$$

(4.5)

which is characteristic of glassy materials and amorphous solids.

### 4.2 The shear spectrum and the k-gap

Our main focus is the Green function of the current operators dual to the bulk two forms:

$$G_{JJ}^R(\omega, k) \equiv \langle J^{I_\mu} J^{I_\sigma}_\rho \rangle_R(\omega, k)$$

(4.6)

which has not to be confused with the Green function of the electric current. In particular we are interested in the transverse part of such correlator that we will define $G_{TT, JJ}^R$ and in its poles, 

i.e. transverse QNMs excitations.

Where no explicitly stated, we focus on the case $m = 0$ where no propagating transverse phonon is present at low frequency. In this limit the density of defects is zero and the system lies in a fluid (or better viscoelastic) phase. In the hydrodynamic limit
\( \omega/T \ll 1, k/T \ll 1 \) and in the limit of large enough cutoff \( \mathcal{M}/T \gg 1 \) the transverse QNMs are obtained solving the following equation\(^8\)

\[
\left(1 - i \frac{\omega}{\omega_g}\right) \omega + i \frac{\mathcal{M} - 1}{r_h} k^2 = 0 \tag{4.7}
\]

where \( \mathcal{M} \equiv \mathcal{M}/r_h \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14.png}
\caption{The shear modes speed (4.12) in function of the cutoff \( \mathcal{M} \).}
\end{figure}

Strikingly, the above equation for the dispersion relation of the shear modes is exactly what found in [5] which follows from Maxwell interpolation results (2.8). Within this approximation we can derive the perturbative expression:

\[
\omega_g = \frac{r_h}{\bar{M} - 1 + \frac{1}{2} (\ln 3 - \frac{\pi}{3\sqrt{3}})} \tag{4.8}
\]

Moreover, we can easily solve equation (4.7) and get the two dominating modes:

\[
\omega_{\pm} = -\frac{\omega_g}{2} \pm i \sqrt{\frac{(\mathcal{M} - 1) \omega_g}{r_h} k^2 - \frac{\omega_g^2}{4}} \tag{4.9}
\]

The dispersion relation of such modes is of the type:

\[
\omega = -\frac{i}{2\tau} \pm \sqrt{c^2 k^2 - \frac{1}{4 \tau^2}} \tag{4.10}
\]

and exhibits a \( k \)-gap at:

\[
k_{g}^2 = \frac{\omega_g r_h}{4 (\mathcal{M} - 1)} \tag{4.11}
\]

\(^8\)This assumes that \( m = 0 \) and therefore the equations for the perturbations are not coupled.
Notice how the lowest mode has $\text{Im}[\omega] = 0$ at zero momentum $k = 0$ as a consequence of the absence of momentum dissipation. This feature is clearly different from the previous model but as we shall see it is not substantial for the existence of a k-gap. An example of the dispersion relation is shown in fig.13. Notice that for large cutoff $\tilde{M} = M/T, \gg 1$ the k-gap closes and the dispersion relation becomes exactly linear and relativistic $\omega = k$. Let us first analyze the speed $c$ which appears at large momentum. Differently from the previous case, the speed now depends non trivially on the UV cutoff:

$$c^2 = \frac{(\tilde{M} - 1)\omega_g}{r_h} = \frac{9(4\pi - 3\tilde{M})}{2\pi (18 + \sqrt{3}\pi - 9 \log(3)) - 27\tilde{M}}$$

(4.12)

where we have used $\tilde{M} \equiv \mathcal{M}/T$. The results are presented in fig.14. As expected, if we send the UV cutoff to infinity $\tilde{M} \to \infty$, we recover the results of the previous model where the speed is always relativistic $c = 1$ and temperature independent. This confirms indeed the picture that such a feature is due to the absence of any UV cutoff. We now turn to studying the behaviour of the k-gap (4.11) and the relaxation time $\tau$ in this model. It can be immediately seen that:

$$\tau \equiv \frac{1}{\omega_g} = \frac{27\tilde{M} - 2\pi (18 + \sqrt{3}\pi - 9 \log(3))}{48\pi^2 T}$$

(4.13)

where $\tilde{M} \equiv M/T$. The first important feature to notice is that in the limit $\tilde{M} \to \infty$,
the k-gap completely disappears. In another words, the existence of the k-gap in this model is fully dependent on the existence of a UV cutoff. To this extent this model appears very different from the previous one which shows a k-gap also in absence of a UV cutoff. Fixing $\tilde{M}$, the relaxation time has a simple $1/T$ dependence which is shown in fig.15 for various values of the cutoff.

*Can this relaxation time be explained using Maxwell Interpolation?*

Let us discuss the physics of the transverse spectrum further. At low momenta the hydro poles in the $JJ$ correlator and in the $TT$ (stress tensor) correlator are purely diffusive:

$$\omega = -i D_T k^2 = -i \frac{1}{4\pi T} k^2 + \ldots \quad \text{TT correlator}$$  \hspace{0.5cm} (4.14)

$$\omega = -i D_J k^2 = -i \frac{3}{4\pi T} \left( \frac{3\tilde{M}}{4\pi T} - 1 \right) k^2 + \ldots \quad \text{JJ correlator}$$  \hspace{0.5cm} (4.15)

Reintroducing the parameter $m$, counting the density of defects, we can easily obtain the momentum susceptibility and the shear elastic modulus of the system by:

$$G_0 = 2K = (\tilde{M} - 1) m^2 r_h$$  \hspace{0.5cm} (4.16)

$$\chi_{PP} = \epsilon + p - 2K = m^2 \left( \tilde{M} - \frac{3}{2} \right) r_h + 3r_h^3$$  \hspace{0.5cm} (4.17)

where with $G_0$ we mean the static shear elastic modulus and $K$ is the elastic bulk modulus. Perhaps surprisingly, the shear elastic modulus grows with temperature. Using the previous expression we can also defines the $T, J$ ”viscosities” as:

$$\eta_T \equiv D_T \chi_{PP}, \quad \eta_J \equiv D_J \chi_{PP}.$$  \hspace{0.5cm} (4.18)

Let us remark that the validity of the Einstein relations above has not been proved at finite $m \neq 0$.

Now we can finally collect all our results and get back to our initial question regarding the nature of $\tau$. In particular we are interested in comparing the behaviour of $\tau$ with:

$$\frac{\eta_T}{G_0} = \frac{3\tilde{m}^2 \left( \sqrt{6 \tilde{m}^2 + 16 \pi^2} - 6 \tilde{M} \right)}{12 \pi \tilde{m}^2 T \left( \sqrt{6 \tilde{m}^2 + 16 \pi^2} - 6 \tilde{M} + 4\pi \right)}$$  \hspace{0.5cm} (4.19)

$$\frac{\eta_J}{G_0} = \frac{(4\pi - 3\tilde{M}) \left( 16\pi^2 \left( \sqrt{6 \tilde{M}^2 + 16 \pi^2} + 4\pi \right) - 3\tilde{M}^2 \left( \sqrt{6 \tilde{M}^2 + 16 \pi^2} - 6 \tilde{M} \right) \right)}{16\pi^2 \tilde{M}^2 T \left( \sqrt{6 \tilde{M}^2 + 16 \pi^2} - 6 \tilde{M} + 4\pi \right)}$$  \hspace{0.5cm} (4.20)

---

9Despite the $\eta_T$ has the usual meaning of shear viscosity, it is not clear to us what is the field theory interpretation of the parameter $\eta_J$. 

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Figure 16. The relaxation time \( \tau \) and the ratio \( \eta_J/G_0 \) for \( \tilde{m} = 1, \tilde{M} = 10^3 \). The inset shows the result for \( \tilde{M} = 10^5 \), the two curves are almost indistinguishable.

Notice that here we have finite \( m \) because we want to compare with the rigidity modulus that would be present in the solid phase \( m \neq 0 \). The temperature dependence is the same as observed in the first model and is reasonable, however it is evident (see fig.16) that for generic values of \( \tilde{M}, \tilde{m} \) the quantities do not match. Nevertheless, in the limit of large cutoff \( \tilde{M} \gg 1 \) and then zero mass \( m = 0 \) the expressions completely match:

\[
\tau = \frac{\eta_J}{G_0} = \frac{9 \tilde{M}}{16 \pi^2 T}
\]

and Maxwell’s interpolation is obeyed.

Let us analyze the diffusion constants in the \( m = 0 \) limit. In particular in that limit we can write:

\[
\frac{D_T}{c^2} = \frac{2 \pi \left( 18 + \sqrt{3} \pi - 9 \log(3) \right) - 27 \tilde{M}}{36 \pi T (4 \pi - 3 \tilde{M})}
\]

\[
\frac{D_J}{c^2} = \frac{27 \tilde{M} - 2 \pi \left( 18 + \sqrt{3} \pi - 9 \log(3) \right)}{48 \pi^2 T}
\]

where again \( \tilde{M} = M/T \). We first notice that:

\[
\frac{D_J}{c^2} = \frac{\eta_J}{G_0}, \quad \text{in the limit} \quad \left( \tilde{M} \to \infty, \tilde{m} = 0 \right)
\]
Figure 17. The $k_{\text{gap}}$ (4.11) in function of the inverse relaxation time $\tau$ for various cutoff values $\tilde{M} = 5, 10, 50, 1000$ (from blue to red).

This turns out to be not true at finite values of $\tilde{M}$ and $\tilde{m}$ and it deserves further investigation.

More importantly, in the fluid phase $m = 0$, we obtain the simple, and valid for any value of the cutoff $\tilde{M}$, relation (see (4.22) and (4.13)):

$$\tau = \frac{D_J}{c^2}$$

(4.24)

The latter is in perfect agreement with the results obtained from Maxwell interpolation in the relativistic limit and it proves that indeed the expected theoretical picture is correct.

Finally, we compare the $k_{\text{gap}}$ (4.11) with the relaxation time $\tau = 1/\omega_g$ in fig.17 for various cutoff values. At large relaxation times $1/\tau \ll 1$ the curve is quite linear. Decreasing the value of the relaxation time we observe a divergence from the linear behaviour which is more evident at small cutoff. The results are compatible with what observed in real liquids (see for example [5]).
5 Discussion

In this work we performed a detailed study of the shear collective modes beyond the hydrodynamic limit in two simple holographic bottom-up models exhibiting viscoelastic features. Our main result is that both models show the presence of k-gap in the transverse spectrum whose temperature dependence is in perfect agreement with what is seen in condensed matter systems, liquids. Since the gap and its properties in liquids are derived from the Maxwell interpolation, our results imply that relativistic, and strongly coupled, systems and systems with slow momentum relaxation also obey Maxwell interpolation. The corresponding generalized expression for the liquid relaxation time reads:

$$\tau = \frac{D}{c^2 - D \Gamma}$$  \hspace{1cm} (5.1)

where $D$ is the hydrodynamic diffusion constant, $\Gamma = \tau_{rel}^{-1}$ the momentum relaxation rate and $c$ the asymptotic transverse sound speed. Let us notice that, at zero momentum dissipation $\Gamma = 0$, substituting into the previous formula the expression for the diffusion constant in a relativistic fluid $D = \eta/\chi_{PP}$ and the formula for the transverse sound speed $c^2 = G/\chi_{PP}$ we immediately re-derive the Maxwell interpolation result $\tau = \eta/G$.

We summarize our findings in the following:

- The thermodynamic properties of the considered models are in agreement with the expectations from realistic viscoelastic materials [37, 48]. In particular the temperature dependence of the heat capacity displays a crossover between the Debye model $c_v \sim T^d$ at large temperature and a new emergent linear behaviour at small temperature $c_v \sim T$. This is known to happen in amorphous solids and glasses below the glass transition temperature $T_G$. Our results indicate that this effect can be associated to the onset of the incoherent-coherent transition [47].

- Both holographic models show the existence of a k-gap in the transverse spectrum of excitations (beyond the hydrodynamic limit) as is seen in liquids and supercritical fluids [10]. The k-gap in the holographic models is strikingly similar to the gap in liquids: it (a) increases with temperature and (b) linearly increases with the inverse of relaxation time.

- In the first model considered in section 3, the temperature dependence of relaxation time decrease with temperature and is similar to what is widely observed in liquids [5] (see fig.10).
We propose a generalized formula for the liquid relaxation time $\tau$ which is also valid in relativistic systems and in the presence of slow momentum relaxation. We examined the formula in both the holographic bottom-up models, and the agreement is perfect.

Our results strongly suggest that Maxwell interpolation has a much wider scope of application than envisaged originally and discussed until now. Namely, the ability of the system to support both hydrodynamic viscous and solid-like elastic response and, importantly, the additivity of both responses in the equation of motion (see 2.1) are the two properties which we believe are applicable in other systems. Importantly, this approach applies to strongly-coupled systems where neither elastic nor viscous components are small and where perturbation theory and other existing theories do not apply. For this reason, we believe that the approaches based on Maxwell interpolation will be fruitful in addressing strongly-coupled field theories which describe a number of important physical systems and phenomena. A more detailed discussion of this point will be given in a companion paper [34].

Several questions and future directions remain to be investigated:

- The k-gap phenomenon appears to be very general within the holographic constructions. Its presence has already been observed, but not really appreciated, in several contexts: in the dispersion relation of chiral magnetic waves within the
anomalous magneto response [69], in plasmas with finite magnetic field [70, 71], in the study of electromagnetism using global symmetries [72], in the QNMs spectrum of P-wave superfluids [73] and in holographic fermionic spectral functions\(^\text{10}\). It is extremely interesting to understand better the connection between the physics of the k-gap in liquids, the emergence of propagating shear waves and the features of Alfvén waves and emergent photons. Finally, we just want to leave here a simple but deep question: *Why the k-gap is so general and what does that mean?*

- The physics of the k-gap seems to be another interesting example of how the hydrodynamic expansion knows more than what it should. We find very surprising that we can defined the k-gap just using hydrodynamic observables such as the diffusion constant. Moreover, *can the k-gap a new and valuable method to measure the diffusion constant in relativistic fluids?*

- The physics of the k-gap is hiddenly connected with the so-called *zero sound* in D-branes setups which follow from a DBI-type action [74, 75]. Moreover, our equation (2.8) already appeared in this context in [76] and it might also be relevant for the hydrodynamic description of condensed matter systems like graphene [77]. Is there a unified picture of the k-gap phenomenon and the physics behind it?

- Can we obtain an independent derivation of the liquid relaxation time formula (3.26)? Can large-D methods [78] be useful in this direction? How generic is the presence of a k-gap? Can we re-derive our formula from field theory methods and appropriately deforming the hydrodynamic equations? We expect that our holographic results would agree with that analysis.

- The similarities between the thermodynamic and the transverse spectrum features of our holographic models and those of realistic liquids are suggestive. It is interesting to study this further and the relationships with the instantaneous normal modes present in real liquids [54].

\(^{10}\) *Private communication with Sang-Jin Sin and Yunseok Seo.*
Finally, we believe this work extends the physical description of an area erroneously believed to be largely understood: the phases of matter. The naive question presented in fig.18

*Is it a solid ...? Is it a liquid ...?*

emerges to have interesting and non-trivial physical consequences, with the answer that in many systems of interest, the system response is governed by Maxwell interpolation, i.e. by both hydrodynamic and solid-like elastic responses. This points to a new avenue to be investigated in regard to the cherished link between the holographic methods and experimental low energy results.

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