Will small particles exhibit Brownian motion in the quantum vacuum?

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Abstract

The Brownian motion of small particles interacting with a field at a finite temperature is a well-known and well-understood phenomenon. At zero temperature, even though the thermal fluctuations are absent, quantum fields still possess vacuum fluctuations. It is then interesting to ask whether a small particle that is interacting with a quantum field will exhibit Brownian motion when the quantum field is assumed to be in the vacuum state. In this paper, we study the cases of a small charge and an imperfect mirror interacting with a quantum scalar field in (1 + 1) dimensions. Treating the quantum field as a classical stochastic variable, we write down a Langevin equation for the particles. We show that the results we obtain from such an approach agree with the results obtained from the fluctuation-dissipation theorem. Unlike the finite temperature case, there exists no special frame of reference at zero temperature and hence it is essential that the particles do not break Lorentz invariance. We find that the scalar charge breaks Lorentz invariance, whereas the imperfect mirror does not. We conclude that small particles such as the imperfect mirror will exhibit Brownian motion even in the quantum vacuum, but this effect can be so small that it may prove to be difficult to observe it experimentally.

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1 Introduction and motivation

The random motion of a small particle that is immersed in a fluid in thermal equilibrium is a phenomenon that has been known to us for a long time now. No elaborate set up is required to observe this phenomenon. In fact, this phenomenon was first noticed by Brown, a botanist, early last century, when he observed, under a microscope, the motion of tiny pollen grains immersed in water at room temperature. This random motion of small test particles in fluids, which has come to be known as Brownian motion, was originally understood on the basis of molecular or kinetic theory of fluids. According to the kinetic theory, fluids consist of molecules which are in incessant and random motion because of intrinsic thermal fluctuations. Hence, if there is an external particle (which we shall hereafter refer to as the Brownian particle) present in a fluid, then the molecules of the fluid, apart from constantly colliding with each other, also collide with the Brownian particle, thereby imparting the particle with the observed random motion. (For a good discussion on Brownian motion at finite temperature, see Pathria [1].) Thus, Brownian motion reveals very clearly the statistical fluctuations that occur in a system in thermal equilibrium. Historically, this phenomenon proved to be important in helping to gain the acceptance for the atomic theory of all matter and for the validity of the statistical description thereof.

Obviously, a Brownian particle cannot gain energy from the surrounding medium indefinitely. Therefore, there should exist a mechanism for the particle to dissipate its energy in some form so that it reaches equilibrium with the environment. Early this century, it was Langevin who suggested that the force exerted on the Brownian particle by the surrounding medium can effectively be written as a sum of two parts: (i) an ‘averaged out’ part which represents a frictional force experienced by the particle and (ii) a ‘rapidly fluctuating’ part (see, for e.g., Reif [2]). The ‘rapidly fluctuating’ part is responsible for the random motion of the Brownian particle and the presence of the frictional force implies the existence of processes whereby the energy associated with the Brownian particle is dissipated in course of time to the degrees of freedom corresponding to the surrounding medium. Clearly, fluctuations and dissipation have to go hand in hand if the complete system has to stay in equilibrium.

A good example to illustrate the phenomenon we have described in the last two paragraphs is the case of a test charge interacting with the electromagnetic field at a finite temperature. According to quantum field theory, photons are the quanta of the electromagnetic field. The charge interacts with the photons present in the thermal bath and the recurrent collisions of the photons with the charge imparts the Brownian motion to the charge. But, as we have pointed out in the last paragraph, a Brownian particle cannot keep accruing energy from the thermal fluctuations present in the surrounding environment. The charge, when in non-uniform motion, radiates photons and this radiation reacts back on the charge (see, for instance, Jackson [3], Chap. 17) with the result that it achieves
the required equilibrium conditions.

Therefore, it is clear that dissipation of energy by a Brownian particle is necessary to attain equilibrium in the presence of fluctuations. In quantum statistical mechanics, the relation between fluctuation and dissipation is embodied in the fluctuation-dissipation theorem \cite{4}. (For a detailed account of the fluctuation-dissipation theorem, see Kubo \cite{5}.) We had mentioned above that according to quantum field theory, photons are the quanta of the electromagnetic field. An important lesson we learn in quantum field theory is that the electromagnetic field has a non-zero energy, called the zero-point energy, even at zero temperature (see, for e.g., Milonni \cite{6}, Sec. 2.5.). In quantum statistical mechanics, we have come to understand that if the fluctuation-dissipation theorem has to be satisfied, we have to take the zero-point energy of the electromagnetic field into account (see, for e.g., Landau and Lifshitz \cite{7}, Sec. 124).

The presence of the zero-point energy implies that fluctuations are present in the field even in the vacuum state (see, for instance, Milonni \cite{6}, Sec. 2.5.). We had mentioned earlier that it is the presence of the fluctuations in the surrounding medium that is responsible for the random motion of the Brownian particles. The question we are interested in addressing in this paper is as follows: If fluctuations are present in a quantum field even in the vacuum state, then will a small particle that is interacting with the field exhibit Brownian motion in the quantum vacuum? For the sake of simplicity, we shall study Brownian particles that are interacting with a quantized massless scalar field in \((1+1)\) dimensions. We shall consider two kinds of Brownian particles: (i) a small scalar charge that is coupled to the field through a monopole interaction and (ii) a mirror that is imperfect in the sense that it does not reflect modes higher than a certain frequency which we shall refer to as the plasma frequency.

This paper is organized as follows. In Sec. 2, treating the quantum field as a classical stochastic variable we write down a Langevin equation for the Brownian particles. We shall consider the cases of a small charge and an imperfect mirror. From the Langevin equation, we evaluate the mean-square velocities of these Brownian particles when they are in equilibrium with the quantum field. In Sec. 3, we compare the equilibrium values of the mean-square velocities we obtain from the Langevin equation with those obtained from the fluctuation-dissipation theorem. In Sec. 4, we evaluate the mean-square displacements of the small particles from the Langevin equation and examine whether the particles we consider will exhibit Brownian motion or not. Finally, in Sec. 5, we shall briefly summarize the main results of our analysis. (Unless we mention otherwise, we shall work with units such that \(\hbar = c = 1\).)
2 The Langevin equation

The systems we shall consider in this section are described by the action

\[ S = S_{\text{par}} + S_{\text{fld}} + S_{\text{int}}, \tag{1} \]

where \( S_{\text{par}} \) represents the action corresponding to the Brownian particles, \( S_{\text{fld}} \) denotes the action of the field that the Brownian particles are interacting with and \( S_{\text{int}} \) is the action that describes the interaction between the Brownian particles and the field. In the introductory section, we had mentioned that, for the sake of simplicity, we shall assume that the Brownian particles are interacting with a massless scalar field in \((1 + 1)\) dimensions. In such a case, the action \( S_{\text{fld}} \) is described by the integral

\[ S_{\text{fld}} = \int dt \int dx \left\{ \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi \right\}, \tag{2} \]

where \( \Phi \) denotes the massless scalar field. Also, we shall assume that the Brownian particles are moving non-relativistically. Then, the action \( S_{\text{par}} \) describing a non-relativistic Brownian particle of mass \( m \) that is in motion (in 1-dimension, say along the \( x \)-axis) on a trajectory \( z(t) \) is given by

\[ S_{\text{par}} = \int dt \frac{m}{2} \dot{z}^2, \tag{3} \]

where \( \dot{z} \equiv (dz/dt) \).

In the following two subsections, we shall study two kinds of Brownian particles interacting with the massless, quantum scalar field. Treating the quantum field as a classical stochastic variable, we shall write down a Langevin equation for the Brownian particles. In Subsec. 2.1, we shall discuss the case of a small charge and, in Subsec. 2.2, we shall consider the case of an imperfect mirror. The explicit form of the interaction \( S_{\text{int}} \) between the two kinds of Brownian particles and the massless scalar field will be given in the relevant subsection below.

2.1 For a small scalar charge

In this subsection, we shall consider the case of a small charge that is interacting with the massless scalar field through a monopole interaction. For such a case, the action \( S_{\text{int}} \) is given by

\[ S_{\text{int}} = \int dt \int dx \rho \Phi, \tag{4} \]

where \( \rho \) is the charge density corresponding to the scalar charge. As mentioned earlier, we shall assume that the Brownian particles are moving non-relativistically. The charge density \( \rho \) corresponding to a non-relativistic charge moving along a trajectory \( z(t) \) is given by

\[ \rho(t, x) = q \delta^{(1)} [x - z(t)], \tag{5} \]
where $q$ is the strength of the scalar charge. So, the complete system is now described by the action (1) with $S_{\text{int}}$ being given by Eqs. (4) and (5). Varying the action (1) with respect to the scalar field $\Phi$ and the trajectory $z(t)$ of the charge, we find that the equations of motion satisfied by the field and the charge are given by

$$
\Box \Phi \equiv \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \Phi = q \, \delta^{(1)}[x - z(t)]
$$

and

$$
m \ddot{z} = q \left( \partial_{z(t)} \Phi [t, z(t)] \right),
$$

where

$$
\ddot{z} \equiv \left( \frac{d^2 z}{dt^2} \right) \quad \text{and} \quad \partial_{z(t)} \Phi [t, z(t)] \equiv \left( \frac{\partial \Phi [t, z(t)]}{\partial z(t)} \right).
$$

In what follows, we shall first solve Eq. (6) for the scalar field and then substitute the resulting expression for $\Phi$ in Eq. (7) to obtain the final equation of motion for the charge.

The scalar field $\Phi$ satisfying Eq. (6) above can be decomposed as follows:

$$
\Phi(t, x) = \Phi_{\text{free}}(t, x) + \Phi_{\text{ret}}(t, x),
$$

where, as is obvious from the subscripts, $\Phi_{\text{free}}$ and $\Phi_{\text{ret}}$ denote the free and the retarded components of the scalar field, respectively. The free component of the scalar field $\Phi_{\text{free}}$ satisfies the homogeneous wave equation and hence is independent of the charge density $\rho$. Its most general solution can be written as a superposition of plane waves modes. The retarded component of the field $\Phi_{\text{ret}}$ is a solution of the inhomogeneous wave equation. It is related to the charge density $\rho$ by the following integral (see, for e.g., Roman [8], Sec. 3.1):

$$
\Phi_{\text{ret}}(t, x) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dx' \, D_{\text{ret}}(t, x; t', x') \, \rho(t', x'),
$$

where $D_{\text{ret}}$ is the retarded Green’s function corresponding to the massless scalar field. In $(1 + 1)$ dimensions, the retarded Green’s function $D_{\text{ret}}$ can be easily evaluated to be (see, for instance, Birrell and Davies [9], Sec. 2.7)

$$
D_{\text{ret}}(t, x; t', x') = \theta(t - t') \int_{-\infty}^{\infty} \frac{dk}{2\pi \omega} \sin[\omega(t - t')] \, e^{ik(x - x')},
$$

where $\omega = |k|$. Substituting the expressions (5) and (11) for the charge density $\rho$ and the retarded Green’s function $D_{\text{ret}}$ in Eq. (10), we find that the retarded component of the scalar field reduces to the following integral:

$$
\Phi_{\text{ret}}(t, x) = q \int_{-\infty}^{\infty} dt' \, \theta(t - t') \int_{-\infty}^{\infty} \frac{dk}{2\pi \omega} \sin[\omega(t - t')] \, e^{ik[x - z(t')]}.
$$
The decomposition of the scalar field $\Phi$ into the free and the retarded components as in Eq. (9) leads to the following equation of motion for the charge:

$$m \ddot{z} = F_{rr} + \mathcal{F},$$

where

$$F_{rr} = q \left( \partial_{z(t)} \Phi_{\text{ret}}[t, z(t)] \right)$$
and

$$\mathcal{F} = q \left( \partial_{z(t)} \Phi_{\text{free}}[t, z(t)] \right).$$

The explicit form of $F_{rr}$ can now be obtained by substituting $\Phi_{\text{ret}}$ from Eq. (12) in the above expression. We find that $F_{rr}$ is described by the integral

$$F_{rr} = q \left( \partial_{z(t)} \Phi_{\text{ret}}[t, z(t)] \right) = q^2 \int_{-\infty}^{\infty} dt' \theta(t - t') \int_{-\infty}^{\infty} \frac{dk}{2\pi\omega} (ik) \sin [\omega(t - t')] e^{ik[z(t) - z(t')]}.$$  

The integral over $k$ can be expressed in terms of $\delta$-functions and as a result the integral over $t'$ can be easily evaluated. We obtain that

$$F_{rr} = -\left( \frac{q^2}{2} \right) \left( \frac{\dot{z}}{1 - \dot{z}^2} \right).$$

Recall that we had assumed that the charge is moving non-relativistically. In the non-relativistic limit (i.e. when $|\dot{z}| \ll 1$), $F_{rr}$ above reduces to $(-q^2 \dot{z}/2)$ with the result that the equation of motion for the charge is now given by

$$\frac{dv}{dt} + \left( \frac{q^2}{2m} \right) v = \left( \frac{1}{m} \right) \mathcal{F},$$

where we have set $v = \dot{z}$. It is clear from this equation that the term $F_{rr}$ leads to dissipation. $F_{rr}$, which arises from the retarded component of the field, is in fact the radiation reaction force on the scalar charge. Classically, it is possible to choose initial conditions such that the free component of the scalar field is identically zero. In such a case, $\mathcal{F} = 0$ and the velocity of the charge decays to zero as the charge radiates when in motion.

Until now we have worked in the completely classical domain. We have obtained an equation of motion for the charge assuming that the charge as well as the scalar field are classical quantities. Our original motivation was to study the behavior of a Brownian particle that is interacting with a quantum field. If we now assume that $\Phi$ is a quantum field, then the retarded component of the field, viz. $\Phi_{\text{ret}}$, can still be regarded as a classical quantity, but the free component $\Phi_{\text{free}}$ should be treated as an operator (see, for e.g., Roman [3], Sec. 3.1). Therefore, on quantization of the scalar field, we would obtain an equation of motion for
the charge that is similar in form to Eq. (17), but the term \( F \) will now be an operator instead of a \( c \)-number. In such a case, we will have an equation wherein the left hand side is a \( c \)-number whereas the right hand side is an operator and we need to devise an approach to make sense of such an equation.

One possible way out of this situation would be to replace the operator on right hand side by its expectation value. As we are interested in studying motion in the quantum vacuum, the expectation value can be evaluated in the vacuum state of the quantum field. In \((1 + 1)\) dimensions, the free component of the quantum scalar field can be decomposed in terms of its normal modes as follows (cf. Birrell and Davies [9], Secs. 2.1 and 2.2):

\[
\hat{\Phi}_{\text{free}}(t, x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega}} \left( \hat{a}_k e^{-i(\omega t - kx)} + \hat{a}_k^\dagger e^{i(\omega t - kx)} \right),
\]

where \( \omega = |k| \) and \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) are the annihilation and the creation operators corresponding to the mode \( k \) of the quantum field. Imposing the canonical commutation relations on the field and its conjugate momentum would lead to the standard commutation relation between the operators \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) (see, for e.g., Birrell and Davies [9], Sec. 2.2). The vacuum state \(|0\rangle\) of the quantum scalar field is then defined as follows:

\[
\hat{a}_k |0\rangle = 0 \quad \forall k.
\]

Substituting Eq. (18) in the expression for \( F \) in Eq. (14), we find that the corresponding operator is given by

\[
\hat{F} [t, z(t)] = q \left( \partial_z(t) \hat{\Phi}_{\text{free}} [t, z(t)] \right)
= q \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi\omega}} \left( i k \left( \hat{a}_k e^{-i[\omega t - kz(t)]} - \hat{a}_k^\dagger e^{i[\omega t - kz(t)]} \right) \right).
\]

From this expression, it is easy to see that its expectation value is zero in the vacuum state of the quantum field. Therefore, replacing the operator \( \hat{F} \) by its expectation value would simply lead us to the classical result and we will miss out the effects arising due to the quantum nature of the scalar field.

The main feature of a quantum field is that it always exhibits fluctuations. This fluctuating nature of a quantum field induces fluctuations in the motion of the Brownian particles that are interacting with it. Therefore, to study the effects of a quantum field on Brownian particles we need to formulate an approach wherein we are able to take into account the fluctuations that arise in the quantum field. The approach we shall adopt here is to treat the force arising due to the quantum component of the scalar field as a classical stochastic force, say, \( \beta(t) \), whose moments are related to the symmetrized \( n \)-point correlation functions of
the operator $\hat{F}$. In other words, we shall assume that the charge interacting with
the quantum scalar field is described by a Langevin equation of the following form:
\[
\frac{dv}{dt} + \gamma_c v = \left( \frac{1}{m} \right) \beta(t),
\]
where
\[
\gamma_c = \left( \frac{q^2}{2m} \right)
\]
and $\beta(t)$ is a classical stochastic force whose first and second moments are given by
\[
\langle \beta(t) \rangle = \langle \hat{F} [t, z(t)] \rangle_{|\dot{z}| \ll 1},
\]
\[
\langle \beta(t) \beta(t') \rangle = \left( \frac{1}{2} \right) \langle \hat{F} [t, z(t)] \hat{F} [t', z(t')] + \hat{F} [t', z(t')] \hat{F} [t, z(t)] \rangle_{|\dot{z}| \ll 1}.
\]
In these equations, the quantities on the left hand sides are to be considered
as ensemble averages and the expectation values of the operators on the right
hand sides are to be evaluated in the vacuum state of the quantum scalar field.
(In App. A, we show that the correlation functions we have defined here satisfy
the required properties of a classical stochastic force.) Moreover, since we have
assumed that the charge is moving non-relativistically, it is necessary that we
consider the $|\dot{z}| \ll 1$ limit of these expectation values.

The presence of the stochastic force $\beta(t)$ in the Langevin equation we have
obtained above implies that quantities such as $v(t)$ and $z(t)$ that describe the
motion of the charge exhibit fluctuations. Therefore, $v(t)$ and $z(t)$ should be
treated as stochastic variables. We shall now solve Eq. (21) for $v(t)$ and then go
on to evaluate $\langle v(t) \rangle$ and $\langle v^2(t) \rangle$ by relating these quantities to the first and the
second moments of $\beta(t)$. We shall assume that the quantum scalar field is in the
vacuum state.

Integrating the Langevin equation (21), we obtain that
\[
v(t) = v(0) e^{-\gamma_c t} + \left( \frac{1}{m} \right) e^{-\gamma_c t} \int_0^t dt' e^{\gamma_c t'} \beta(t'),
\]
where we have set $v(t = 0) = v(0)$. The expectation value of $v(t)$ is then given by
\[
\langle v(t) \rangle = \langle v(0) \rangle e^{-\gamma_c t} + \left( \frac{1}{m} \right) e^{-\gamma_c t} \int_0^t dt' e^{\gamma_c t'} \langle \beta(t') \rangle
\]
\[= v(0) e^{-\gamma_c t},
\]
where we have used the fact that the first moment of the stochastic force $\beta(t)$
is zero (cf. App. A). This is just the classical result we had discussed earlier.
The initial velocity $v(0)$ of the charge decays to zero over a time scale of the order of $\gamma^{-1}$. It is then clear that the equilibrium value $\langle v \rangle$ (i.e. $\langle v(t) \rangle_{\gamma c \gg 1}$) is zero and the relaxation time of the system is $\gamma^{-1}c$. It is important to notice that there exists no special frame of reference at zero temperature. Therefore, the fact that the equilibrium value of the velocity of the charge is zero implies that the scalar charge breaks Lorentz invariance [10, 11]. (For a detailed discussion on this aspect, see App. B.)

Let us now go on to evaluate the quantity $\langle v^2(t) \rangle$. Using Eq. (25), we obtain

$$\langle v^2(t) \rangle = \langle v^2(0) \rangle e^{-2\gamma c t} + \left( \frac{2}{m} \right) e^{-\gamma c t} \int_0^t dt' e^{-\gamma c (t-t')} \langle v(0) \beta(t') \rangle$$

$$+ \left( \frac{1}{m^2} \right) e^{-2\gamma c t} \int_0^t dt' \int_0^t dt'' e^{\gamma c (t'+t'')} \langle \beta(t') \beta(t'') \rangle. \quad (27)$$

If we now assume that $t$ is large enough so that $(\gamma c t) \gg 1$, then in such a limit the charge would be in equilibrium with the quantum scalar field. In this limit, the first term in Eq. (27) clearly goes to zero and it can be shown that the second term reduces to zero as well. Therefore, we obtain that

$$\langle v^2 \rangle \equiv \langle v^2(t) \rangle_{\gamma c t \gg 1} = \left( \frac{\gamma c}{\pi m} \right) \int_0^\infty \frac{d\omega \omega}{(\gamma_c^2 + \omega^2)}, \quad (28)$$

where we have used the result (25) for $\langle \beta(t') \beta(t'') \rangle$. The expression for $\langle v^2 \rangle$ we have obtained above diverges in the upper limit of the integral. Such divergences are common in quantum field theory and it is standard practice to regularize these divergences by introducing a finite upper limit to the integral, i.e. by introducing an ultra-violet cut-off. In the presence of the scalar charge, there exists a good reason to introduce such a cut-off. When the scalar charge is present, the quantum scalar field is obviously not a free field, but is interacting with the charge. If we now assume that the scalar charge has a finite size, say, $\Lambda^{-1}$, then the modes

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1 At a first glance, one would think that $\langle v(0) \beta(t) \rangle$ is the same as $(v(0) \langle \beta(t) \rangle)$ and hence it should be identically zero. That is not the case. As we had pointed out earlier, the stochastic nature of $\beta(t)$ implies that $v(t)$ is a stochastic quantity as well. In general, there will exist non-zero correlations between two stochastic variables. But, in the limit of large $t$, a correlation such as $\langle v(0) \beta(t) \rangle$ will decay exponentially (cf. Landau and Lifshitz [6], Secs. 118 and 119). Therefore, the integral in the second term is a finite quantity with the result that the coefficient $e^{-\gamma c t}$ kills this term completely in the large $t$ limit.

2 Actually, the charge density as given in Eq. (5) corresponds to that of a point charge. The charge density of a charge that has a finite size will be described by a distribution with a finite width rather than the $\delta$-function. The radiation reaction force on a charge of a finite size can be expressed as a power series in the size of the charge (see, for e.g., Jackson [3], Sec. 17.3). Therefore, by assuming that the radiation reaction force on the finite sized charge is still given by Eq. (16), we are working under the approximation wherein we have retained only the leading order term.
of the quantum field with frequencies greater than $\Lambda$ will not affect the charge. The finite size of the charge will then provide us with a natural cut-off. (It is for this reason that we have repeatedly mentioned that the scalar charge we are considering here is a small particle rather than a point particle.) Carrying out the integral in Eq. (28) up to $\omega = \Lambda$, we obtain that

$$\langle v^2 \rangle = \left( \frac{\gamma_c}{\pi m} \right) \int_0^\Lambda \frac{d\omega \omega}{(\gamma_c^2 + \omega^2)} = \left( \frac{\gamma_c}{2\pi m} \right) \ln \left[ 1 + \left( \frac{\Lambda^2}{\gamma_c^2} \right) \right].$$

(29)

The only length scale available in the problem is $\gamma_c^{-1}$. (In fact, $\gamma_c^{-1}$ is the equivalent of the “classical electron radius” for the case of the scalar charge we are considering here.) Setting $\Lambda = \gamma_c$, we finally obtain that

$$\langle v^2 \rangle = \left( \frac{\gamma_c \ln 2}{2\pi m} \right).$$

(30)

This result can be expressed as

$$\frac{m}{2} \langle v^2 \rangle = \left( \frac{\gamma_c \ln 2}{4\pi} \right)$$

(31)

which is the average energy of the scalar charge when it is in equilibrium with the quantum scalar field.

### 2.2 For an imperfect mirror

We shall now study the case of a mirror that is interacting with the massless scalar field. In the last subsection, we had assumed that the charge was coupled to the scalar field through a monopole interaction as described by the interaction term (4). In the case of the mirror, there exists no such explicit interaction term in the action (i.e. the quantity $S_{\text{int}}$ is identically zero), but the mirror interacts with the field through a boundary condition. We shall impose the boundary condition that the scalar field vanishes on the surface of the mirror. In other words, we shall assume that

$$\Phi [t, z(t)] = 0,$$

(32)

where $z(t)$ is the trajectory of the mirror. Since $S_{\text{int}} = 0$, varying the action (4) with respect to $\Phi$ leads to the following the equation of motion for the scalar field

$$\Box \Phi = \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \Phi = 0.$$

(33)

The presence of the mirror implies that an incoming wave would be reflected by the mirror into an outgoing wave. If we assume that the scalar field $\Phi$ is a classical field, then it is possible to choose initial conditions such that there are no
incoming waves. Even if there is an occasional incoming wave, scattering of such a wave by the mirror will just result in an impulsive change in the momentum of the mirror\(^3\). Moreover, this change in momentum (and hence the shift in the frequency of the reflected wave) will be very small if the mirror is assumed to be relatively heavy. Apart from this effect, there would be no systematic effect of a classical scalar field on the motion of the mirror. On the other hand, if we consider \(\Phi\) to be a quantum field, it is well-known that a mirror which is interacting with a quantum field can radiate even when the field is in the vacuum state \([12, 13]\). Such a radiation will then lead to a radiation reaction force on the mirror \([13, 14]\). This proves to be a feature that distinguishes mirrors from charges. Unlike a charge which will radiate even when it is interacting with a classical field, a mirror will radiate only when it is interacting with a quantum field.

In the absence of an explicit interaction term between the mirror and the scalar field, varying the action \(I\) with respect to the trajectory \(z(t)\) of the mirror will just lead us to the equation of motion of a free particle. In the last paragraph, we had pointed out that a mirror interacting with a quantum field will radiate when it is in motion. This radiation will then lead to a non-zero radiation reaction force on the mirror, thereby affecting its motion. Assuming \(\Phi\) to be a quantum field, we shall now obtain an equation of motion for the mirror by demanding that the total energy of the system consisting of the mirror and the quantum field be conserved.

Conservation of energy implies that \((dH/dt) = 0\), where \(H\) is the Hamiltonian of the complete system. We had noted above that the radiation reaction force on a moving mirror has a quantum origin. If so, the radiation reaction force will exhibit fluctuations. In order to take these fluctuations into account, we shall write the Hamiltonian of the complete system consisting of the mirror and the quantum scalar field as follows:

\[
H = H_{\text{mir}} + \hat{H}_{\text{fld}},
\]

where \(H_{\text{mir}}\) denotes the Hamiltonian of the mirror and \(\hat{H}_{\text{fld}}\) is the Hamiltonian operator of the scalar field. The quantities \(H_{\text{mir}}\) and \(\hat{H}_{\text{fld}}\) are given by

\[
H_{\text{mir}} = \left(\frac{m}{2}\right) \dot{z}^2 \quad \text{and} \quad \hat{H}_{\text{fld}} \equiv \int_{-\infty}^{\infty} dx \hat{T}_{00},
\]

where, for the case of a massless scalar field in \((1+1)\) dimensions, the operator \(\hat{T}_{00}\) is given by (cf. Fulling and Davies \([13]\))

\[
\hat{T}_{00} = \left(\frac{1}{2}\right) \left\{ \left(\frac{\partial \hat{\Phi}}{\partial t}\right)^2 + \left(\frac{\partial \hat{\Phi}}{\partial x}\right)^2 \right\}.
\]

\(^3\)Such a scattering would also result, due to momentum conservation, in a shift in the frequency of the scalar wave. This frequency shift will depend on the mass of the mirror and also on its velocity at the point when it was hit by the incoming wave.
Demanding conservation of energy then leads to the following equation of motion for the mirror:

\[ m \ddot{z} \dddot{z} = -\left( d\hat{H}_{\text{fld}}[t, z(t)]/dt \right). \quad (37) \]

Adding \( \left( d\langle \hat{H}_{\text{fld}} \rangle/dt \right) \) to either side of this equation, we obtain that

\[ m \ddot{z} - F_{rr} = \hat{F}. \quad (38) \]

The quantities \( F_{rr} \) and \( \hat{F} \) in the above equation are given by the expressions

\[ F_{rr} = -[\dot{z}(t)]^{-1} \left( d\langle \hat{H}_{\text{fld}}[t, z(t)] \rangle/dt \right) \quad (39) \]

and

\[ \hat{F}[t, z(t)] = -[\dot{z}(t)]^{-1} \left( d\hat{H}[t, z(t)]/dt \right), \quad (40) \]

where

\[ \hat{H}[t, z(t)] \equiv \left( \hat{H}_{\text{fld}}[t, z(t)] - \langle \hat{H}_{\text{fld}}[t, z(t)] \rangle \right). \quad (41) \]

From the form of Eq. (38) it is easy to see that \( F_{rr} \) is the radiation reaction force on the mirror and \( \hat{F} \) represents the deviations of the radiation reaction force from its mean value. (It should now be clear as to why we had considered the operator \( \hat{H}_{\text{fld}} \) rather than its expectation value in the total Hamiltonian of the system as given by Eq. (34). Had we considered the expectation value, we would have only obtained the term \( F_{rr} \) and would have missed out the fluctuations arising due to the term \( \hat{F} \).) As in the case of the charge, we shall treat the force on the mirror arising due to the term \( \hat{F} \) as a classical stochastic force. We shall now calculate the radiation reaction force on the mirror, viz. \( F_{rr} \), from Eq. (39). We shall assume that the quantum scalar field is in the vacuum state.

The boundary condition (32) implies that the mirror separates the spacetime into two regions which are independent of each other. An incoming wave in either of these regions is reflected by the mirror into an outgoing wave in the same region. Let us now quantize the scalar field \( \Phi \) on either side of the mirror. On the right hand side of the mirror, a positive frequency mode that satisfies the boundary condition (32) is given by (cf. Birrell and Davies [9], Eq. (4.43))

\[ \phi_\omega(t, x) = \frac{i}{\sqrt{4\pi \omega}} \left[ e^{-i\omega u} - e^{-i\omega (2\tau_u - u)} \right], \quad (42) \]

whereas such a mode on the left hand side of the mirror is given by

\[ \tilde{\phi}_\omega(t, x) = \frac{i}{\sqrt{4\pi \omega}} \left[ e^{-i\omega u} - e^{-i\omega (2\tau_v - v)} \right], \quad (43) \]

where \( \omega \geq 0 \) and we have assumed that \( z(t) = 0 \) for \( t < 0 \). The quantities \( u, v, \tau_u \) and \( \tau_v \) are defined by the relations

\[ u = (t - x), \quad v = (t + x), \quad [\tau_u - z(\tau_u)] = u \quad \text{and} \quad [\tau_v + z(\tau_v)] = v. \quad (44) \]
(Note that the quantities \( \tau_u \) and \( \tau_v \) correspond to the time at which the incoming null waves \( u \) and \( v \) intersect the mirror, respectively. Hence, \( \tau_u \) depends only on \( u \) and \( \tau_v \) only on \( v \).) The quantum scalar field can now be decomposed in terms of the modes (42) and (43) as follows:

\[
\hat{\Phi}^R(t, x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \left( \hat{a}_\omega \phi_\omega(t, x) + \hat{a}^\dagger_\omega \phi^*_\omega(t, x) \right)
\]

and

\[
\hat{\Phi}^L(t, x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \left( \hat{b}_\omega \tilde{\phi}_\omega(t, x) + \hat{b}^\dagger_\omega \tilde{\phi}^*_\omega(t, x) \right),
\]

where the superscripts R and L denote the right and the left hand sides of the mirror, respectively. The operators \( \hat{a}_\omega \) and \( \hat{a}^\dagger_\omega \) (\( \hat{b}_\omega \) and \( \hat{b}^\dagger_\omega \)) are the annihilation and the creation operators corresponding to the mode \( \omega \) on the right (left) hand side of the mirror. \( (\hat{a}_\omega, \hat{a}^\dagger_\omega) \) and \( (\hat{b}_\omega, \hat{b}^\dagger_\omega) \) form two independent sets of operators and operators within the same set satisfy the standard commutation relations.

Let us now evaluate the quantity \( \langle d\langle H_{\text{fdr}} \rangle/dt \rangle \) in the vacuum state of the quantum scalar field. The vacuum states on either side of the mirror are defined as the states that are annihilated by the operators \( \hat{a}_\omega \) and \( \hat{b}_\omega \). Substituting the scalar field in Eqs. (45) and (46) in the expression (36) for \( \hat{T}_{00} \), we find that its expectation value in the vacuum state on either side of the mirror are given by

\[
\langle \hat{T}^R_{00} \rangle = \frac{1}{4\pi} \int_0^\infty d\omega \left[ 1 + (2\dot{\tau}_u - 1)^2 \right] \quad \text{and} \quad \langle \hat{T}^L_{00} \rangle = \frac{1}{4\pi} \int_0^\infty d\omega \left[ 1 + (2\dot{\tau}_v - 1)^2 \right],
\]

where \( \dot{\tau}_u \equiv (\partial\tau_u/\partial t) \) and \( \dot{\tau}_v \equiv (\partial\tau_v/\partial t) \) and these two quantities are related to the velocity of the mirror \( \dot{z}(t) \) by the following relations:

\[
\dot{\tau}_u = \left( \frac{1}{1 - \dot{z}(\tau_u)} \right) \quad \text{and} \quad \dot{\tau}_v = \left( \frac{1}{1 + \dot{z}(\tau_v)} \right).
\]

The integrals describing \( \langle \hat{T}^R_{00} \rangle \) and \( \langle \hat{T}^L_{00} \rangle \) in Eq. (47) above exhibit the characteristic ultra-violet divergence of quantum field theory.

The following few comments are in order at this stage of our discussion. Earlier, in the case of the scalar charge, we had encountered an ultra-violet divergence when evaluating the equilibrium value of its mean-squared velocity, viz. the quantity \( \langle v^2 \rangle \) (cf. Eq. (28)). We can expect that such a divergence will arise for the case of the mirror as well. However, as we have mentioned before, there exists an important difference between the case of the charge we had considered in the last subsection and the case of the mirror we are considering here. The radiation reaction force on the scalar charge is classical in nature (cf. Eq. (14)), whereas the radiation reaction force on the mirror has a quantum origin (cf. Eq. (39)).
It is due to this reason that we encounter divergences even when we evaluate the radiation reaction force on the mirror. In the case of the scalar charge, we had regularized the divergence in the mean-squared velocity by assuming that the charge has a finite size and then treating the finite size of the charge as an ultra-violet cut-off. We can introduce such a cut-off for the mirror by assuming that it is imperfect in the sense that it does not reflect modes higher than a certain frequency which we shall refer to as the plasma frequency. Moreover, in order to be consistent, it is essential that we evaluate not only the mean-squared velocity, but also the radiation reaction force on the mirror with the assumption that the mirror has a finite plasma frequency.

In what follows, we shall carry out our calculations assuming that the mirror has a finite plasma frequency $\omega_p$. In such a case, modes of the quantum field with frequencies $\omega > \omega_p$ will be unaffected by the mirror and these modes would correspond to the standard Minkowski plane wave modes. On subtracting the contribution to $\langle \hat{T}_{00}^R \rangle$ and $\langle \hat{T}_{00}^L \rangle$ due to the Minkowski vacuum for all modes, we obtain that

$$\langle \hat{T}_{00}^R \rangle = \int_0^{\omega_p} \frac{d\omega}{\pi} \omega \left[ \dot{\tau}_u (\dot{\tau}_u - 1) \right] = \left( \frac{\omega_p^2}{2\pi} \right) [\dot{\tau}_u (\dot{\tau}_u - 1)]$$

(49)

and

$$\langle \hat{T}_{00}^L \rangle = \int_0^{\omega_p} \frac{d\omega}{\pi} \omega \left[ \dot{\tau}_v (\dot{\tau}_v - 1) \right] = \left( \frac{\omega_p^2}{2\pi} \right) [\dot{\tau}_v (\dot{\tau}_v - 1)] .$$

(50)

We had pointed out earlier that $\tau_u$ depends only on $u$ and $\tau_v$ only $v$. Therefore, the quantities $\langle \hat{T}_{00}^R \rangle$ and $\langle \hat{T}_{00}^L \rangle$ depend only on $u$ and $v$, respectively. The expectation value of operator $\hat{H}_{\text{fld}}$ in the vacuum state of the quantum scalar field is then given by

$$\langle \hat{H}_{\text{fld}} \rangle = \int_{-\infty}^{\infty} dx \langle \hat{T}_{00}^R (u) \rangle + \int_{-\infty}^{\infty} dx \langle \hat{T}_{00}^L (v) \rangle$$

$$= \int_{-\infty}^{\infty} du \langle \hat{T}_{00}^R (u) \rangle + \int_{-\infty}^{\infty} dv \langle \hat{T}_{00}^L (v) \rangle ,$$

(51)

where in the last equation we have changed the variables of integration from $x$ to $u$ and $v$. From this expression it is easy to obtain that

$$\left( \frac{d\langle \hat{H}_{\text{fld}} \rangle}{dt} \right) = \left[ (1 - \dot{z}) \langle \hat{T}_{00}^R (u) \rangle_{[u=t-z(t)]} \right] + \left[ (1 + \dot{z}) \langle \hat{T}_{00}^L (v) \rangle_{[v=t+z(t)]} \right]$$

$$= \left( \frac{\omega_p^2}{\pi} \right) \left( \frac{\dot{z}^2}{1 - \dot{z}^2} \right) .$$

(52)
In the non-relativistic limit, the radiation reaction force \( F_{rr} \) as given by Eq. (39) then reduces to
\[
F_{rr} = -\left( \frac{\omega_p^2 \dot{z}}{\pi} \right).
\] (53)

Substituting this expression in Eq. (38) and treating the force arising due to the term \( \hat{\mathcal{F}} \) as a classical stochastic force \( \beta(t) \), we find that the motion of the mirror is described by the Langevin equation (21) with \( \gamma_m \) instead of \( \gamma_c \), where \( \gamma_m \) is given by
\[
\gamma_m = \left( \frac{\omega_p^2}{\pi m} \right).
\] (54)
The first and the second moments of \( \beta(t) \) are described by Eqs. (23) and (24) with the operator \( \hat{\mathcal{F}} \) now given by Eq. (40).

The first and the second moments of \( v(t) \) for the mirror can now be evaluated from the Langevin equation in the same fashion as in the case of the charge. Just as in the case of the charge, the equilibrium velocity \( \langle v \rangle \) (i.e. \( \langle v(t) \rangle_{\gamma_m t \ll 1} \)) of the mirror is zero. But, unlike the case of the charge, such a behavior on the part of the mirror does not break Lorentz invariance. The reason being that in obtaining this result we have assumed that the initial velocity of the mirror was zero. (Recall that we had assumed that \( z(t) = 0 \) for \( t < 0 \).) Also, it can be shown that if the mirror was moving with a non-zero velocity initially then the equilibrium value \( \langle v \rangle \) will be the same as the initial velocity. (For details, see App. [3].) The equilibrium value of \( \langle v^2(t) \rangle \) for the imperfect mirror can now be obtained by substituting the second moment of the stochastic force as given by (105) in Eq. (27). (The first term in Eq. (27) vanishes since the first moment of the stochastic force is zero for the mirror as well (see App. [4]) and the second term reduces to zero in the limit \( (\gamma_m t) \gg 1 \) for the same reasons we had mentioned earlier.) We find that
\[
\langle v^2 \rangle \equiv \langle v^2 \rangle_{\gamma_m t \gg 1} = \left( \frac{1}{2 \pi^2 m^2} \right) \int_0^{\omega_p} d\omega \int_0^{\omega_p} d\omega' \left( \frac{\omega \omega'}{(\omega + \omega')^2 + \gamma_m^2} \right). \tag{55}
\]
This integral can be evaluated by changing variables to \( \Omega = [(\omega + \omega')/2] \) and \( \Omega' = [(\omega - \omega')/2] \) and carrying out the integrals over \( \Omega \) and \( \Omega' \). We finally obtain that
\[
\langle v^2 \rangle = \left( \frac{\Gamma^2}{12} \right) \left\{ 3 \ln \left( \frac{4 + \Gamma^2}{1 + \Gamma^2} \right) + \Gamma^2 \ln \left( \frac{4 \Gamma^4 + 4 \Gamma^2}{1 + \Gamma^2} \right) - 1 \right\}
\]
\[
- \left( \frac{\Gamma}{3} \right) \left[ \arctan(2\Gamma^{-1}) - \arctan(\Gamma^{-1}) \right]. \tag{56}
\]
where \( \Gamma = (\gamma_m/\omega_p) = (\omega_p/\pi m) \). In terms of \( \hbar \) and \( c \), the dimensionless quantity \( \Gamma \) is given by
\[
\Gamma = \left( \frac{\hbar \omega_p}{\pi mc^2} \right). \tag{57}
\]
A typical value for the plasma frequency $\omega_p$ of the mirror would be $10^{16}$ sec$^{-1}$ (cf. Jackson [3], p. 321). For a mirror of mass $10^{-3}$ kg, $\Gamma \approx 10^{-31}$. Clearly, $\Gamma \ll 1$.

In this limit, we find that (56) reduces to

$$\langle v^2 \rangle \simeq \left( \frac{\Gamma^2}{4} \right) (\ln 4 - 1) \simeq \left( \frac{\gamma_m}{4\pi m} \right).$$

(58)

This expression can be rewritten as

$$\frac{m}{2} \langle v^2 \rangle = \left( \frac{\gamma_m}{8\pi} \right)$$

(59)

which is the average energy of the mirror when it is in equilibrium with the quantum scalar field.

### 3 Comparison with the fluctuation-dissipation theorem

It is the surrounding medium that leads to the dissipative and the random forces on a Brownian particle. Since these two forces have a common origin, they must be related in some fashion. Fluctuation-dissipation theorem is a statement about a general relationship between the response of a given system to an external disturbance and the internal fluctuation of the system in the absence of this disturbance [4].

The fluctuation-dissipation theorem can be used to predict the fluctuations of physical quantities from the known dissipative properties of the system when it is subject to an external interaction [5]. It is in this form that we shall use the theorem.

In this section, we shall first gather together the basic definitions and the essential results of the fluctuation-dissipation theorem. Applying the fluctuation-dissipation theorem to the cases of the small charge and the imperfect mirror we evaluate the equilibrium values of the quantity $\langle v^2(t) \rangle$. We shall then compare the results we obtain from the fluctuation-dissipation theorem with those we have obtained in the last section.

The fluctuations of physical quantities can be related, in many cases, to quantities that describe the behavior of the body under certain external interactions. The fluctuating physical quantities can be either classical variables or quantum operators. The external interactions appear in the Hamiltonian of the body as a perturbation term of the following form (cf. Landau and Lifshitz [7], Sec. 123):

$$\hat{V} = -\hat{x} f(t),$$

(60)

where $\hat{x}$ represents the physical quantity concerned and the perturbing generalized force $f(t)$ is a given function of time. The mean value of the quantity $\hat{x}(t)$
can be written as
\[ \langle \dot{x}(t) \rangle = \int_0^\infty dt' \alpha(t') f(t - t'), \] (61)

where \( \alpha(t') \) is a function of time which depends on the properties of the body. It is clear from the above expression that the value of \( \langle \dot{x} \rangle \) at a time \( t \) depends only on the value of the perturbing force \( f \) at earlier times. The quantity \( \langle \dot{x} \rangle \) is the response of the system to the perturbation. Decomposing the quantities \( \langle \dot{x}(t) \rangle \) and \( f(t) \) in Eq. (61) in terms of their Fourier components \( \langle \dot{x}_\omega \rangle \) and \( f_\omega \), we find that they are related by
\[ \langle \dot{x}_\omega \rangle = \alpha(\omega) f_\omega, \] (62)

where the function \( \alpha(\omega) \) is described by the following integral:
\[ \alpha(\omega) = \int_0^\infty dt \alpha(t) e^{i\omega t}. \] (63)

Once the function \( \alpha(\omega) \) is known, the behavior of the body under the external perturbation is completely determined. The quantity \( \alpha(\omega) \) is called the generalized susceptibility and is, in general, a complex quantity. We shall write \( \alpha(\omega) \) as
\[ \alpha(\omega) = \alpha'(\omega) + i \alpha''(\omega) \] (64)
and the imaginary part \( \alpha''(\omega) \) characterises the dissipative properties of the system (see Landau and Lifshitz [7], p. 379). Fluctuation-dissipation theorems essentially relates the equilibrium values of the fluctuations in the physical quantity \( \dot{x}(t) \) and the dissipative properties of the system effectively represented by \( \alpha''(\omega) \). The actual relationship between these quantities is given by the following expression (cf. Landau and Lifshitz [4], Eq. (124.10)):
\[ \langle \dot{x}^2 \rangle = \left( \frac{1}{\pi} \right) \int_0^\infty d\omega \alpha''(\omega) \coth(\omega/2T), \] (65)

where \( T \) is the temperature of the surrounding medium. At zero temperature, which is the case we are interested in, the above relation reduces to
\[ \langle \dot{x}^2 \rangle = \left( \frac{1}{\pi} \right) \int_0^\infty d\omega \alpha''(\omega). \] (66)

*It should be emphasised here that the quantity \( \dot{x} \) can correspond to either a classical variable or a quantum operator.*

In the last section, we had obtained a Langevin to describe the motion of a small scalar charge and an imperfect mirror that are interacting with a quantum
scalar field. The Langevin equation we had obtained was of the following form (cf. Eq. (21)):

$$\frac{dv}{dt} + \gamma v = \left(\frac{1}{m}\right)\beta(t),$$

(67)

where $\gamma$ is the relaxation time of the system and $\beta(t)$ is a classical stochastic force. We shall now identify the velocity $v$ of the Brownian particles to be the physical quantity $\hat{x}$ that appears in our discussion on the fluctuation-dissipation theorem above. Also, since $\beta(t)$ is the force that induces fluctuations in the velocity $v(t)$ of the Brownian particles, we shall identify the perturbing generalized force $f(t)$ (as defined in Eq. (60)) with the stochastic force $\beta(t)$. Comparing the dimensions of $f(t)$ and $\beta(t)$, we find that

$$[f(t)] = [\beta(t)] \cdot [\text{sec}],$$

(68)

where the square brackets denote the dimensions of the quantities inside them. The only time scale that is available in the problem is $\gamma^{-1}$. Therefore, we shall assume that

$$\beta(t) \equiv \gamma f(t).$$

(69)

We shall now calculate the generalized susceptibility $\alpha(\omega)$ for the systems described by the Langevin equation (67). Expressing the quantities $v(t)$ and $\beta(t)$ in Eq. (67) in terms of their Fourier components, we obtain that

$$m (\gamma - i\omega) v_\omega = \beta_\omega = \gamma f_\omega.$$  

(70)

Comparing Eqns. (62) and (70), we find that $\alpha(\omega)$ is given by

$$\alpha(\omega) = \left(\frac{\gamma}{m}\right) \left(\frac{1}{\gamma - i\omega}\right) = \left(\frac{\gamma}{m}\right) \left(\frac{\gamma + i\omega}{\gamma^2 + \omega^2}\right).$$

(71)

It is now easy to identify that

$$\alpha''(\omega) = \left(\frac{\gamma}{m}\right) \left(\frac{\omega}{\gamma^2 + \omega^2}\right).$$

(72)

Substituting this expression for $\alpha''(\omega)$ in Eq. (66), we obtain that

$$\langle v^2 \rangle = \frac{1}{\pi} \int_0^\infty d\omega \alpha''(\omega) = \left(\frac{\gamma}{m\pi}\right) \int_0^\infty d\omega \frac{\omega}{(\gamma^2 + \omega^2)^{3/2}}.$$  

(73)

This is exactly the integral (28) we had encountered in the case of the scalar charge. This integral diverges in the upper limit and, as we have discussed earlier, we can introduce a cut-off parameter $\Lambda$ if we assume that the Brownian particles have a finite size $\Lambda^{-1}$. Integrating up to $\Lambda$, we find that

$$\langle v^2 \rangle = \left(\frac{\gamma}{2\pi m}\right) \ln \left[1 + \left(\frac{\Lambda}{\gamma}\right)^2\right].$$

(74)
Setting $\Lambda$ to be $\gamma_c$ in the case of the charge and $\omega_p$ in the case of the mirror, we find that

$$\langle v^2 \rangle_{\text{chr}} = \left( \frac{\gamma_c}{2\pi m} \ln 2 \right)$$

and

$$\langle v^2 \rangle_{\text{mir}} = \left( \frac{\gamma_m}{2\pi m} \right) \ln \left[ 1 + \left( \frac{\omega_p}{\gamma_m} \right)^2 \right] \simeq \left( \frac{\gamma_m}{\pi m} \right) \ln(1/\Gamma),$$

where in the final expression we have used the fact that $\Gamma \ll 1$.

Let us now compare the results we have obtained from the fluctuation dissipation theorem with the results we had obtained in the last section. Comparing Eq. (30) with (75) and Eq. (58) with (76), it is easy to see that the expressions match exactly in the case of the charge but agree only up to the leading order (in $\Gamma^2$) in the case of the mirror (for a detailed discussion on this issue, see Gour [15]). The difference that arises in the case of the mirror has a simple explanation. The derivation of the fluctuation-dissipation theorem is crucially based on regarding the external interaction (60) as a small perturbation which ensures that the response of the system is linear (see Landau and Lifshitz [7], p. 387). The interaction between the charge and the scalar field as given by Eq. (4) is clearly linear. But, the mirror interacts with the scalar field through a boundary condition and such an interaction is a complex one. Therefore, it does not come as a surprise that the result we have obtained in the last section for the case of the mirror agrees only up to the leading order with the result from the fluctuation-dissipation theorem.

### 4 Will the small particles exhibit Brownian motion?

It is the time dependence of the mean-square displacement of the small particles that reflects whether these particles will exhibit Brownian motion or not. Earlier, we had obtained the Langevin equation (67) to describe the motion of a small scalar charge and an imperfect mirror interacting with a quantum scalar field. The mean-square displacement of the Brownian particles can be easily derived from the Langevin equation satisfied by them (see, for e.g., Reif [2]). We shall briefly discuss this derivation here in order to emphasize an assumption that will prove to be important for our discussion later on.

Multiplying the Langevin equation (67) by $z(t)$ and taking its mean value, we obtain that

$$\left\langle z \frac{dv}{dt} \right\rangle + \gamma \left\langle z v \right\rangle = \left( \frac{d\left\langle z v \right\rangle}{dt} \right) - \left\langle v^2 \right\rangle + \gamma \left\langle z v \right\rangle = \left( \frac{1}{m} \right) \left\langle z \beta(t) \right\rangle,$$

where $\beta(t)$ is the fluctuating part of the field and $\gamma$ is the friction coefficient. The expression for the mean-square displacement is then given by

$$\langle z^2 \rangle = \frac{1}{m} \gamma \left\langle z \beta(t) \right\rangle.$$

This equation shows that the mean-square displacement of the particles is proportional to the friction coefficient and the fluctuating part of the field. This is a fundamental result in the theory of Brownian motion and is known as the Einstein relation.

The friction coefficient $\gamma$ depends on the properties of the medium through which the particles are moving and can be measured experimentally. The fluctuating part of the field $\beta(t)$ represents the random fluctuations of the quantum field and is related to the vacuum fluctuations.

In summary, the mean-square displacement of the Brownian particles is governed by the friction coefficient and the quantum fluctuations of the field. This provides a direct link between the microscopic properties of the field and the macroscopic motion of the particles, which is a key feature of quantum Brownian motion.
where $\langle v^2 \rangle$ is the mean-square velocity of the Brownian particles when they are in equilibrium with the quantum field. The stochastic force is completely independent of the position of the Brownian particle. Therefore, we can set $\langle z(t) \beta(t) \rangle = 0$. On integrating the above differential equation twice and evaluating the constants of integration by assuming that $z(t = 0) = 0$, we find that the mean-square displacement of the Brownian particles is given by

$$\langle z^2(t) \rangle = 2\gamma^{-1}\langle v^2 \rangle \left[ t - \gamma^{-1} \left( 1 - e^{-\gamma t} \right) \right].$$

(78)

The two limits of interest are $(\gamma t) \ll 1$ and $(\gamma t) \gg 1$. When $(\gamma t) \ll 1$, the Brownian particle is yet to reach equilibrium with its environment. In this limit, the mean-square displacement of the particle is given by

$$\langle z^2(t) \rangle_{\gamma t \ll 1} \approx \langle v^2 \rangle t^2,$$

(79)
i.e. the particle behaves as a free particle moving with an average velocity $\sqrt{\langle v^2 \rangle}$.

Whereas, when $(\gamma t) \gg 1$, the particle is in equilibrium with the field and, in this limit, we find that

$$\langle z^2(t) \rangle_{\gamma t \gg 1} \approx 2\gamma^{-1} \langle v^2 \rangle t.$$

(80)

In other words, in an equilibrium situation the Brownian particle diffuses through the surrounding medium. Obviously, we need to know the typical value of the relaxation time of the system before we can say whether a Brownian particle will exhibit diffusion or not.

Until now, we have been treating the scalar charge and the mirror as classical Brownian particles. If we now assume that these Brownian particles are quantum objects, then we can consider the Langevin equation we have obtained as a Heisenberg equation of motion of the following form:

$$\frac{d\hat{v}}{dt} + \gamma_c \hat{v} = \left( \frac{1}{m} \right) \hat{F},$$

(81)

where $\hat{v}$ is the velocity operator corresponding to the Brownian particles. (Recall that, in the last section, we had emphasised that the quantity $\hat{x}$ can be treated as either a classical variable or a quantum operator. Therefore, the results we have obtained and the conclusions we have drawn in the last two sections will hold good even if we treat the quantities describing the motion of the Brownian particles as operators rather than as classical variables.) Earlier, when calculating the mean-square displacement of the classical Brownian particle we had assumed that $z(t = 0) = 0$. If we now treat the Brownian particle as a quantum mechanical object we cannot set its position operator $\hat{z}(t)$ to be identically zero at any instant of time because the position of the particle will always exhibit fluctuations. Instead, we shall demand that

$$\langle \hat{z}(0) \hat{v}(0) + \hat{v}(0) \hat{z}(0) \rangle = 0,$$

(82)
where the expectation value is evaluated in the vacuum state of the scalar field. (The Heisenberg equation of motion relates the operators describing the motion of the Brownian particle to those of the quantum scalar field. The expectation values of the operators corresponding to the Brownian particles we consider here are evaluated in the vacuum state of the quantum scalar field.) Also, in the case of a quantum Brownian particle the initial uncertainty associated with the particle’s position has to be taken into account. Therefore, its mean-square displacement will be given by

$$\langle \hat{z}^2(t) \rangle = 2\gamma^{-1} \langle \hat{v}^2 \rangle \left[ t - \gamma^{-1} \left( 1 - e^{-\gamma t} \right) \right] + \langle \hat{z}^2(0) \rangle,$$

where \( \langle \hat{z}^2(0) \rangle \) is the uncertainty in the position of the particle at \( t = 0 \).

It is now instructive to compare the above result for the quantum Brownian particles with the mean-square displacement of a free quantum particle. A free quantum particle satisfies the following Heisenberg equations of motion (cf. Sakurai [16]):

$$\frac{d\hat{z}}{dt} = \left( \frac{1}{m} \right) \hat{p} \quad \text{and} \quad \frac{d\hat{p}}{dt} = 0.$$  \( \tag{84} \)

These equations can be easily integrated to obtain the solution

$$\hat{z}(t) = \hat{v}(0)t + \hat{z}(0),$$  \( \tag{85} \)

where \( \hat{v}(0) \equiv (\hat{p}(0)/m) \). Then, the mean-square displacement of the free particle is given by

$$\langle \hat{z}^2(t) \rangle = \langle \hat{v}^2(0) \rangle t^2 + \langle \hat{z}^2(0) \rangle,$$

where, as in the case of the quantum Brownian particle, we have assumed that \( \langle \hat{v}(0) \hat{v}(0) + \hat{v}(0) \hat{z}(0) \rangle = 0 \). (The expectation values in the case of the free particle are assumed to be evaluated in a given state.) In the limit of \( \gamma \rightarrow 0 \), we find that the mean-square displacement of a quantum Brownian particle as given by Eq. (83) reduces to

$$\langle \hat{z}^2(t) \rangle \xrightarrow{\gamma \rightarrow 0} \langle \hat{v}^2(0) \rangle t^2 + \langle \hat{z}^2(0) \rangle.$$  \( \tag{87} \)

which is the same as Eq. (86). This means that in the limit \( (\gamma t) \ll 1 \) the intrinsic quantum nature of the Brownian particle dominates its motion and the particle behaves essentially as a free particle. By contrast, in the limit \( (\gamma t) \gg 1 \), the quantum nature of the field dominates the motion of the Brownian particle (as it is in equilibrium with the field) and the particle exhibits diffusion. (Note that, in the limit of large \( t \), the initial uncertainty in the position of the Brownian particle, viz. the quantity \( \langle \hat{z}^2(0) \rangle \) in Eq. (83), can be neglected.)

Let us now examine whether the Brownian particles we have considered in this paper will exhibit diffusion or not. In order to do so, we need to evaluate the numerical values of the relaxation time for the small charge and the imperfect
mirror. Introducing \( h \) and \( c \) in the expressions (22) and (54) for \( \gamma_c \) and \( \gamma_m \), we find that they are given by

\[
\gamma_c = \left( \frac{q^2}{2mc^2} \right) \quad \text{and} \quad \gamma_m = \left( \frac{\hbar \omega_p^2}{\pi mc^2} \right).
\]  

(88)

In the case of the scalar charge, if we now assume that the magnitude of \( q \) is the same as that of the electronic charge \( e \) and \( m \) to be the mass of an electron, we find that \( \gamma_c^{-1} \approx 10^{25} \text{ sec} \). On the other hand, the age of the universe \( \tau \) is of the order of \( 10^{17} \text{ sec} \). Clearly, \( \gamma_c^{-1} \gg \tau \). However, there exist two reasons that suggest that these estimates for the scalar charge should not be taken seriously. Firstly, the “classical electron radius” of the scalar charge, viz. \( (c \gamma_c^{-1}) \), corresponding to the above values turns out to be \( \approx 10^{33} \text{ m} \)! Secondly, the “classical electron radius” for a collection of these scalar charges turns out to be smaller than that of a single charge\(^5\)\(^!\). Furthermore, as we have discussed earlier, the scalar charge breaks Lorentz invariance in the quantum vacuum. For the case of the mirror, we had mentioned before that a typical value for its plasma frequency \( \omega_p \) would be \( 10^{16} \text{ sec}^{-1} \). If we now assume that the mass of the mirror is very small, say, \( 10^{-5} \text{ kg} \), then \( \gamma_m^{-1} \approx 10^{13} \text{ sec} \), which is smaller than \( \tau \). More importantly, unlike the case of the scalar charge, Lorentz invariance is preserved when the mirror is in motion in the quantum vacuum. Therefore, we can conclude that small particles such as the imperfect mirror we have considered here will exhibit Brownian motion in the quantum vacuum.

\(^4\)It should be pointed out here that, unlike the electromagnetic charge which is a dimensionless quantity, the scalar charge \( q \) we are considering here has dimensions of inverse time (in units such that \( \hbar = c = 1 \)). Therefore, the charge strength \( q \) and the electronic charge \( e \) should actually be related by a parameter, say, \( D \), which has the same dimensions as \( q \). We have assumed here that the magnitude of \( D \) is order unity. The exact value of \( D \) can swing the value of \( \gamma_c \) either way.

\(^5\)These features of the scalar charge may be counter-intuitive, but the origin of these features can be traced back to the fact that the charge strength \( q \) has non-zero dimensions. It is first useful to note that the “classical electron radius” of a unit charge interacting with the electromagnetic field in \((3 + 1)\) dimensions would be given by a quantity such as \( \gamma_c \) (see, for instance, Jackson \(^3\), p. 790). Such a quantity would be directly proportional to the square of the electronic charge and inversely proportional to the mass of the charge. So, for a collection of \( n \) such electromagnetic charges, the “classical electron radius” would go as \( n \). However, due to the fact that the charge strength \( q \) has non-zero dimensions, the “classical electron radius” of the scalar charge we are considering here is given by \( (c \gamma_c^{-1}) \) rather than \( \gamma_c \). Since \( (c \gamma_c^{-1}) \) is directly proportional to \( m \) and inversely proportional to \( q^2 \), for a collection of \( n \) such scalar charges, \( (c \gamma_c^{-1}) \) goes as \((1/n)\). This “inverse dependence” (when compared with the electromagnetic case in \((3 + 1)\) dimensions) on the mass and the charge strength is also quite likely to be the reason for the extraordinarily large value of the “classical electron radius” of a single scalar charge.
5 Summary

In this concluding section, we shall briefly summarize the main results we have obtained in this paper.

First and foremost, we would like to emphasize again the crucial difference between motion at a finite temperature and motion in the quantum vacuum. At a finite temperature, the thermal bath provides a special reference frame, but no such frame exists at zero temperature. Therefore, for a realistic system to exhibit Brownian motion in the quantum vacuum, it is absolutely essential that the system preserves Lorentz invariance. Of the two systems we have considered in this paper, since the scalar charge breaks Lorentz invariance whereas the mirror does not, we would like to emphasize here that the mirror is a more realistic example than that of the charge.

Secondly, we would like to stress that our answer to the title of this paper is in the affirmative. Small particles, such as the imperfect mirror we have considered in this paper, will, in principle, exhibit Brownian motion in the quantum vacuum. Since we find that the typical energy of the Brownian particles is of the order of $\gamma$, we can, in fact, expect the following universal behavior of small particles in the quantum vacuum:

$$\langle \hat{z}^2(t) \rangle_{\gamma \gg 1} \approx \left( \frac{\hbar}{m} \right) t,$$

where we have written $\hbar$ explicitly. From this expression it is easy to see that it will take an object of mass $10^{-3}$ kg about $10^{27}$ sec to move through a distance of $10^{-2}$ m. Obviously, observing such a behavior experimentally will prove to be a difficult task. On the other hand, if we assume that a mirror can be constructed out of, say, $10^3$ atoms or so, then the mass of such a mirror would be about $10^{-24}$ kg. This mirror would diffuse through a distance of $10^{-2}$ m within a rather short time scale of $10^6$ sec (which is about a month long). Possibly, such an effect can be observed in the laboratory.

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A Properties of the classical stochastic force

In studying Brownian motion, it is usually assumed that the stochastic force that is responsible for the random motion of Brownian particles satisfies the following
two properties (see Kubo [3], Sec. 3 or Saslaw [18]). (i) A positive stochastic force is as probable as a negative one and therefore the first moment of a stochastic force should be identically zero. (ii) The correlation time of the perturbations is very short and therefore the second moment of the stochastic force should be a sharply peaked function of the time interval between the two perturbations. In this appendix, we shall show that the first and the second moments of the stochastic force \( \beta(t) \) as we have defined in Eqs. (23) and (24) satisfy these two properties. Note that the operator \( \hat{F} \) is given by Eq. (20) in the case of the charge and by Eq. (40) in the case of the mirror. We shall discuss the case of the charge in App. A.1 and the case of the mirror in App. A.2.

A.1 In the case of the scalar charge

In the discussion following Eq. (20), we had mentioned that the expectation value of the operator \( \hat{F} \) vanishes in the vacuum state of the quantum field. This implies that \( \beta(t) \) satisfies property (i).

Let us now evaluate the second moment \( \langle \beta(t) \beta(t') \rangle \) in the vacuum state of the scalar field. We shall first evaluate the two point function \( \langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle \), take the \( |\dot{z}| \ll 1 \) limit and then symmetrize the resulting quantity to finally obtain \( \langle \beta(t) \beta(t') \rangle \). Using the expression (20) for \( \hat{F} \), it is easy to show that

\[
\langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle = q^2 \int_0^\infty \frac{dk}{4\pi} k \left( e^{-ik[t-t'-z(t)+z(t')]} + e^{-ik[t-t'+z(t)-z(t')]} \right)
\]

\[
= - \left( \frac{q^2}{2\pi(t-t')^2} \right) \left\{ \left( 1 + \frac{[z(t) - z(t')]^2}{(t-t')^2} \right) \left( 1 - \frac{[z(t) - z(t')]^2}{(t-t')^2} \right)^{-2} \right\}. \tag{90}
\]

The quantity \( ([z(t) - z(t')]/(t-t')) \) appearing in the above expression can be considered to be the average velocity \( \dot{z} \) of the Brownian particle between the two instants \( t \) and \( t' \). Then, in terms of \( \dot{z} \)

\[
\langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle = - \left( \frac{q^2}{2\pi(t-t')^2} \right) \left( 1 + \dot{z}^2 \right) \left( 1 - \dot{z}^2 \right)^{-2}. \tag{91}
\]

If we now consider the \( |\dot{z}| \ll 1 \) limit, this expression reduces to

\[
\langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle \bigg|_{|\dot{z}| \ll 1} = - \left( \frac{q^2}{2\pi(t-t')^2} \right). \tag{92}
\]

This correlation function can be written in its integral form as follows:

\[
\langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle \bigg|_{|\dot{z}| \ll 1} = \left( \frac{q^2}{2\pi} \right) \int_0^\infty d\omega e^{-i\omega(t-t')} \tag{93}
\]
(Compare this expression with the zero temperature limit of Eq. (1.2') in Caldeira and Legget [19].) On symmetrizing this quantity with respect to \( t \) and \( t' \) and substituting the resulting expression in Eq. (24) we finally obtain that

\[
\langle \beta(t) \beta(t') \rangle = \left( \frac{q^2}{4\pi} \right) \int_{0}^{\infty} d\omega \omega \left( e^{-i\omega(t-t')} + e^{i\omega(t-t')} \right)
\]

(94)

\[
= \left( \frac{q^2}{2\pi} \right) \int_{0}^{\infty} d\omega \omega \cos[\omega(t-t')].
\]

(95)

This integral can be easily carried out with the result

\[
\langle \beta(t) \beta(t') \rangle = -\left( \frac{q^2}{2\pi(t-t')^2} \right).
\]

(96)

As required by property (ii), this is clearly a sharply peaked function of the time interval between the two perturbations\(^6\).

**A.2 In the case of the mirror**

It is easy to see from Eq. (40) that the expectation value of \( F \) is identically zero. In other words, the first moment of the stochastic force \( \beta(t) \) vanishes thereby satisfying property (i) trivially.

In what follows, we shall first evaluate the second moment of the stochastic force \( \beta(t) \) for a perfect mirror. Then, in the final expression, we shall restrict the upper limit in the integrals over \( \omega \) to \( \omega_p \) in order to obtain the results for the imperfect mirror. We had mentioned earlier that a perfect mirror divides the spacetime into two independent regions. Therefore, the fluctuations on either side of the mirror are completely independent. We shall now evaluate the correlation function \( \langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \rangle \) in the vacuum state on the right side of the mirror.

On substituting the expression for the scalar field (45) in Eq. (41) and regularizing the expectation values by subtracting the contribution due to the Minkowski vacuum, we find that

\[
\langle \hat{H}^R[t, z(t)] \hat{H}^R[t', z(t')] \rangle
\]

\(^6\)The reader may be puzzled by the overall minus sign that appears in the second moment of a classical stochastic force. The root cause of the minus sign are the integrals in Eq. (94) which exhibit ultra-violet divergence. In field theory, ultra-violet divergences as in Eq. (95) are handled by considering the quantity \( (t - t') \) to be given by \( (t - t') \pm i\epsilon \), where \( \epsilon \to 0^+ \). Therefore, the second moment of the stochastic force should actually be written as

\[
\langle \beta(t) \beta(t') \rangle = -\left( \frac{q^2}{2\pi(t-t')^2 \pm i\epsilon} \right).
\]

(97)

This is a positive definite and infinite quantity in the limit \( t \to t' \) provided we take this limit \textit{before} setting \( \epsilon \) to be zero.
adding the two correlation functions (99) and (100) and treating the quantity correlation function is a sum of the correlation functions on either side. On since the two sides of the mirror are independent of each other, the complete
quantity $\dot{z}$

Calculating in a similar fashion for the left hand side of the mirror, we obtain

$$= \left( \frac{1}{16\pi^2} \right) \int_z^\infty \int_z^\infty \left\{ \int_0^\infty d\omega \omega \exp -i\omega (t - t' + x - x') \right\}^2 + (2\dot{\tau}_u - 1)^2 \left[ \int_0^\infty d\omega \omega \exp -i\omega (t - t' - x - x' + 2z(\tau_u)) \right]^2 + (2\dot{\tau}_u - 1)^2 \left[ \int_0^\infty d\omega \omega \exp -i\omega (t - t' + x + x' - 2z(\tau_u)) \right]^2 + (2\dot{\tau}_u - 1)^2 (2\dot{\tau}_u' - 1)^2 \times \left[ \int_0^\infty d\omega \omega \exp -i\omega (t - t' - x + x' + 2[z(\tau_u) - z(\tau_u')]) \right]^2 \right\} 

On differentiating this expression with respect to $t$ and $t'$ and dividing by the quantity $[\ddot{z}(t) \dot{z}(t')]$, we obtain that

$$\left\langle \hat{F}^R [t, z(t)] \hat{F}^R [t', z(t')] \right\rangle = \left( \frac{1}{16\pi^2} \right) \left( \frac{[1 + \dot{z}(t)] [1 + \dot{z}(t')]}{[1 - \dot{z}(t)] [1 - \dot{z}(t')]} \right) \times \left( \frac{1}{t - t' + z(t) - z(t')} \right). \hspace{1cm} (99)$$

Calculating in a similar fashion for the left hand side of the mirror, we obtain that

$$\left\langle \hat{F}^L [t, z(t)] \hat{F}^L [t', z(t')] \right\rangle = \left( \frac{1}{16\pi^2} \right) \left( \frac{[1 - \dot{z}(t)] [1 - \dot{z}(t')]}{[1 + \dot{z}(t)] [1 + \dot{z}(t')]} \right) \times \left( \frac{1}{t - t' - z(t) + z(t')} \right). \hspace{1cm} (100)$$

Since the two sides of the mirror are independent of each other, the complete correlation function is a sum of the correlation functions on either side. On adding the two correlation functions (99) and (100) and treating the quantity $([z(t) - z(t')]/(t - t'))$ as the average velocity $\dot{z}$ of the mirror and finally taking the limit $|\dot{z}| \ll 1$, we obtain that

$$\left\langle \hat{F} [t, z(t)] \hat{F} [t', z(t')] \right\rangle_{|\dot{z}| \ll 1} = \left( \frac{1}{2\pi^2(t - t')^4} \right). \hspace{1cm} (101)$$
This correlation function can be represented by the following integral expression:

\[
\left\langle \hat{F}[t, z(t)] \hat{F}[t', z(t')] \right\rangle \bigg|_{|\hat{z}| < 1} = \left( \frac{1}{2\pi^2} \right) \int_0^\infty \omega \int_0^\infty \omega' e^{-i(\omega+\omega')(t-t')} \, d\omega \, d\omega'.
\]

(102)

On symmetrizing this quantity with respect to \( t \) and \( t' \) and substituting the resulting expression in Eq. (24), we find that

\[
\langle \beta(t) \beta(t') \rangle = \left( \frac{1}{2\pi^2} \right) \int_0^\infty \omega \int_0^\infty \omega' \cos \left[ (\omega + \omega') (t - t') \right].
\]

(103)

On integration, we get that

\[
\langle \beta(t) \beta(t') \rangle = \left( \frac{1}{2\pi^2 (t - t')^2} \right),
\]

(104)

which is a very sharply peaked function of the time interval between the two perturbations as required by property (ii). Now, restricting the upper limit of the integrals over \( \omega \) and \( \omega' \) to \( \omega_p \), we obtain that

\[
\langle \beta(t) \beta(t') \rangle = \left( \frac{1}{2\pi^2 (t - t')^2} \right) \int_0^{\omega_p} \omega \int_0^{\omega_p} \omega' \cos \left[ (\omega + \omega') (t - t') \right]
\]

(105)

which is the second moment of the stochastic force for the case of the imperfect mirror.

B Is Lorentz invariance preserved?

In this appendix, we shall first show as to how the mirror preserves Lorentz invariance whereas the scalar charge does not. We shall then attempt to understand the origin of this difference in the motion of these two Brownian particles.

In Subsec. 2.1, we had found that the initial velocity \( v(0) \) of the charge decays to zero over a period of time \( \gamma_c^{-1} \). This implies that, when in equilibrium, the mean velocity of the charge is zero. Had we been working at a finite temperature, the thermal bath of quanta corresponding to the field would offer a special frame of reference and we can say that the mean velocity of the charge is zero with respect to this reference frame. But, at zero temperature, no such frame of reference exists and the fact that a charge moving with a uniform velocity radiates implies that Lorentz invariance is broken \([10, 11]\).

However, as we had mentioned in Subsec. 2.2, the fact that the mean velocity of the mirror when in equilibrium is zero does not break Lorentz invariance because in obtaining this result we had assumed the initial condition that \( z(t) = 0 \) for \( t < 0 \). By demanding that the total momentum of the system be conserved,
we shall now obtain the radiation reaction force on the mirror when the initial velocity of the mirror is assumed to be $v_0$. The momentum of the complete system is given by

$$P = m \dot{z} + P_{\text{fld}}, \quad \text{where} \quad P_{\text{fld}} = \int_{-\infty}^{\infty} dx \langle \hat{T}^{10} \rangle \quad (106)$$

and $\hat{T}^{10}$ is given by (cf. Fulling and Davies [13])

$$\hat{T}^{10} = -\left( \frac{1}{2} \right) \left\{ \left( \frac{\partial \hat{\Phi}}{\partial t} \right) \left( \frac{\partial \hat{\Phi}}{\partial x} \right) + \left( \frac{\partial \hat{\Phi}}{\partial x} \right) \left( \frac{\partial \hat{\Phi}}{\partial t} \right) \right\}. \quad (107)$$

(Since we are interested here only in the average velocity of the mirror, we shall not bother about the fluctuations that arise in the radiation reaction term.) Momentum conservation then implies that the radiation reaction force on the mirror is given by $F_{rr} = -(dP_{\text{fld}}/dt)$.

The momentum-density of the field on the right hand side of the mirror, viz. $\langle \hat{T}^{10}_R \rangle$, can now be evaluated by substituting the scalar field (45) in the expression for $\hat{T}^{10}$ above. On assuming that the mirror has a finite plasma frequency $\omega_p$ and then subtracting the contribution due to the Minkowski vacuum for all modes, we obtain that

$$\langle \hat{T}^{10}_R \rangle = \left( \frac{\omega_p^2}{2\pi} \right) \left[ \dot{\tau}_u (\dot{\tau}_u - 1) \right]. \quad (108)$$

The total momentum of the field to the right of the mirror is then given by

$$P^{\text{R}}_{\text{fld}} = \left( \frac{\omega_p^2}{2\pi} \right) \int_{z(t)}^{\infty} dx \left[ \dot{\tau}_u (\dot{\tau}_u - 1) \right] \quad (109)$$

The reason for dividing this expression for $P^{\text{R}}_{\text{fld}}$ into two integrals is due to the fact that $\tau_u > 0$ for $u > 0$. On changing the variable of integration from $x$ to $u$ in the first integral in the above expression, we obtain that

$$P^{\text{R}}_{\text{fld}} = \left( \frac{\omega_p^2}{2\pi} \right) \left\{ \int_{z(t)}^{t} du \left[ \dot{\tau}_u (\dot{\tau}_u - 1) \right] + \int_{t}^{\infty} dx \left[ \dot{\tau}_u (\dot{\tau}_u - 1) \right] \right\}. \quad (110)$$

It is now easy to see from this expression that the first term corresponds to the case $v_0 = 0$ (the case for which we had evaluated the radiation reaction force
earlier). Also, since \( \dot{z}(\tau_u) = v_0 \) for \( u < 0 \), we can set \( \tau_u = (1-v_0)^{-1} \) in the second integral. On differentiating \( P_{\text{R fld}}^R \) above with respect to \( t \), we obtain that

\[
\left( \frac{dP_{\text{R fld}}^R}{dt} \right) = \left( \frac{\omega_p^2}{2\pi} \right) \left\{ \left( \frac{\dot{z}}{(1-z)} \right) - \left( \frac{v_0}{(1-v_0)^2} \right) \right\}.
\] (111)

The quantity \( \left( \frac{dP_{\text{L fld}}^L}{dt} \right) \) on the left hand side of the mirror can be evaluated in a similar fashion. We find that

\[
\left( \frac{dP_{\text{L fld}}^L}{dt} \right) = \left( \frac{\omega_p^2}{2\pi} \right) \left\{ \left( \frac{\dot{z}}{(1+z)} \right) - \left( \frac{v_0}{(1+v_0)^2} \right) \right\}.
\] (112)

On adding these two quantities and neglecting terms of order \( \dot{z}^2 \) and \( v_0^2 \), we finally obtain that

\[
F_{rr} = -\gamma_m (\dot{z} - v_0).
\] (113)

Such a radiation reaction force ensures that the average velocity of the mirror (viz. \( \langle v(t) \rangle_{\gamma_m t \gg 1} \)) remains \( v_0 \) at any later time with the result that Lorentz invariance is preserved. In other words, the radiation reaction force does not affect the average velocity of the mirror, but only affects the fluctuations in the velocity.

We shall now attempt to understand as to why the scalar charge breaks Lorentz invariance, whereas the mirror does not. Using dimensionality arguments, let us now construct the radiation reaction force on a charge when the strength of the charge is a dimensionless quantity. It is reasonable to assume that the radiation reaction force on the charge will not depend on the trajectory \( z(t) \), but only on its velocity \( \dot{z}(t) \) and its derivatives, say, for instance, \( \ddot{z}(t) \) and \( \dddot{z}(t) \).

It is then easy to show that the radiation reaction force on such a charge will be of the following form:

\[
F_{rr} = a(\dot{z}) \dddot{z}^2 + b(\dot{z}) \dddot{z},
\] (114)

where \( a(\dot{z}) \) and \( b(\dot{z}) \) are functions of the velocity \( \dot{z} \). (In the non-relativistic limit, we expect the functions \( a(\dot{z}) \) and \( b(\dot{z}) \) to reduce to constants of order unity.) Clearly, such a radiation reaction force will preserve Lorentz invariance. However, as we have discussed earlier, the scalar charge we have considered possesses non-zero dimensions (see footnote 4). Evidently, the motion of the scalar charge breaks Lorentz invariance due to the fact that the charge strength \( q \) has non-zero dimensions.

Let us now compare the cases of the charge and the mirror. In the case of the scalar charge, the charge strength \( q \) (which is basically a coupling constant) appears explicitly in the interaction term in the action describing the complete system. Whereas, in the case of the mirror, there is no interaction term in the
action, but the mirror interacts with the field through a boundary condition. Moreover, the plasma frequency $\omega_p$ (which acts as the coupling constant for the system) appears *only* when we introduce it as an ultra-violet cut-off in order to regularize the divergent expressions and we do not expect the regularization procedure to change the physics involved. These arguments suggest that Lorentz invariance may not be preserved whenever a coupling constant that possesses non-zero dimensions appears *explicitly* in the interaction term in the action describing the complete system.
References

[1] R. K. Pathria, *Statistical Mechanics* (Pergamon Press, Oxford, 1972), Secs. 13.3 and 13.4.

[2] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), pp. 560–567.

[3] J. D. Jackson, *Classical Electrodynamics*, Second Edition (Wiley, New York, 1962).

[4] H. B. Callen and T. A. Welton, *Phys. Rev.* **83**, 34 (1951).

[5] R. Kubo, *Rep. Prog. Phys.* **29**, 255 (1966).

[6] P. W. Milonni, *The Quantum Vacuum* (Academic Press, Boston, 1994).

[7] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part I (Course of Theoretical Physics, Vol. 5), Third Edition (Pergamon Press, Oxford, 1980).

[8] P. Roman, *Quantum Field Theory* (Wiley, New York, 1969).

[9] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[10] W. H. Zurek, *Ann. N. Y. Acad. Sci.* **480**, 89 (1986).

[11] W. G. Unruh and W. H. Zurek, *Phys. Rev. D* **40**, 1071 (1989).

[12] B. DeWitt, *Phys. Reps.* **19C**, 297 (1975).

[13] S. A. Fulling and P. C. W. Davies, *Proc. Roy. Soc. Lond. A* **348**, 393 (1976).

[14] L. H. Ford and A. Vilenkin, *Phys. Rev. D* **25**, 2569 (1982).

[15] Gilad Gour, *Motion in the Quantum Vacuum*, M. Sc. Thesis, Hebrew University, Jerusalem, Israel (1998).

[16] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, Massachusetts, 1994), pp. 84–87.

[17] P. W. Milonni, *Phys. Letts. A* **82**, 225 (1981).

[18] W. C. Saslaw, *Gravitational Physics of Stellar and Galactic Systems* (Cambridge University Press, Cambridge, England, 1985), Chap. 3.

[19] A. O. Caldeira and A. J. Legget, *Physica* **121 A**, 587 (1983).