On group automorphisms in universal algebraic geometry

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Abstract

In this paper we study group equations with occurrences of automorphisms. We describe equational domains in this class of equations. Moreover, we solve a number of open problem posed in universal algebraic geometry.

1 Introduction

In the classic approach to algebraic geometry over groups we are dealing with equations over a group $G$ as expressions $w(X) = 1$, where $w(X)$ is an element of $G * F(X)$ ($F(X)$ is the free group generated by a set of variables $X$). This class of equations was studied in many papers (see [1, 2] and the survey [4] for more details).

In the current paper we consider a different class of equations over a group $G$: now $w(X)$ may contain the occurrences of symbols $\{\phi | \phi \in \text{Aut}(G)\}$. Any equation of this type is called below an equation with automorphisms. The study of such equations is justified by many important problems in group theory. For example, the twisted conjugacy problem for a group $G$ is equivalent to the solution of the following equation $\phi(x)u = vx$ for given $u, v \in G, \phi \in \text{Aut}(G)$ (also, see this problem in [5] for equations with endomorphisms).

There is a connection between the “standard” group equations and equations with automorphisms. Indeed, for an equation $c_0x_1c_1x_2c_2\ldots c_{k-1}x_kc_k = 1$ ($c_i \in G$) we have

\[
c_0x_1c_1x_2c_2\ldots c_{k-1}x_kc_k = (c_0x_1e_0^{-1})(c_0c_1x_2c_1^{-1}e_0^{-1})\ldots
\]

\[
(c_0c_1c_2\ldots c_{k-1}x_kc_{k-1}^{-1}\ldots c_2^{-1}c_1^{-1}e_0^{-1})c_0c_1c_2\ldots c_k = x_1^{c_0}x_2^{(c_0c_1)^{-1}}\ldots x_k^{(c_0c_1c_2\ldots c_{k-1})^{-1}} \prod c_i
\]

\[
= \phi_1(x_1)\phi_2(x_2)\ldots \phi_k(x_k) \prod c_i,
\]

where each $\phi_i$ is an inner automorphism of a group $G$. Thus, the equation above is equivalent to the following equation with automorphisms

\[
\phi_1(x_1)\phi_2(x_2)\ldots \phi_k(x_k)c = 1 (c \in G).
\]

This correspondence allows us to study equations with automorphisms by methods developed for group equations. For instance, in [2] equational domains for group equations were described. In Section 3 we solve the similar problem for equations.
with automorphisms. The results of Section 3 provide various examples of equational domains, so it allows to solve a number of open problems posed in universal algebraic geometry (Section 4). Namely, we solve Problem 4.4.7 from [2]. Notice that our solution implies negative answers for Problems 5.3.1-4 in [2] (the reduction of Problem 4.4.7 to Problems 5.3.1-4 was shown in [2]).

2 Definitions

All definitions below are derived from [2], where all notions of algebraic geometry were formulated for algebraic structures of arbitrary languages.

Denote by \( \mathcal{L} = \{ \cdot, -1, 1 \} \) the standard language of group theory. Let us fix a group \( G \) and consider the extended language \( \mathcal{L}(A) = \{ \cdot, -1, 1 \} \cup \{ \phi^{(1)} \mid \phi^{(1)} \in A \} \), where the unary functional symbols \( \phi^{(1)} \) correspond to a group of automorphisms \( A \subseteq \text{Aut}(G) \). Any group \( G \) of the language \( \mathcal{L}(A) \) is called an \( \mathcal{L}(A) \)-group (implicitly we fix an interpretation of the symbols \( \phi \) to the elements of the group \( A \subseteq \text{Aut}(G) \)).

Using the properties of automorphisms, any \( \mathcal{L}(A) \)-term in variables \( X = \{ x_1, x_2, \ldots, x_n \} \) is equivalent to a product

\[
\phi_1(x_{i_1}^{\varepsilon_1}) \phi_2(x_{i_2}^{\varepsilon_2}) \ldots \phi_k(x_{i_k}^{\varepsilon_k}),
\]

where \( \phi_j \in A, x_{i_j} \in X, \varepsilon_j \in \{-1, 1\} \).

An \( \mathcal{L}(A) \)-equation is an expression \( t(X) = 1 \), where \( t(X) \) is an \( \mathcal{L}(A) \)-term. An \( \mathcal{L}(A) \)-system is an arbitrary set of \( \mathcal{L}(A) \)-equations. The set of all solutions of an \( \mathcal{L}(A) \)-system \( S \) in \( G \) is denoted by \( V_G(S) \). A set \( Y \subseteq G^n \) is called an \( \mathcal{L}(A) \)-algebraic if there exists an \( \mathcal{L}(A) \)-system \( S \) in variables \( X = \{ x_1, x_2, \ldots, x_n \} \) with \( Y = V_G(S) \). An \( \mathcal{L}(A) \)-group \( G \) is an \( \mathcal{L}(A) \)-equational domain if for any \( n \) and arbitrary \( \mathcal{L}(A) \)-algebraic sets \( Y_1, Y_2 \subseteq G^n \) the union \( Y = Y_1 \cup Y_2 \) is also \( \mathcal{L}(A) \)-algebraic.

Theorem 2.1. (2) An \( \mathcal{L}(A) \)-group \( G \) is an \( \mathcal{L}(A) \)-equational domain iff there exists an \( \mathcal{L}(A) \)-system \( S \) in variables \( x, y \) such that

\[
V_G(S) = \{ (x, y) \mid x = 1 \text{ or } y = 1 \}.
\]

Remark 2.2. Actually, Theorem 2.1 was proved for group languages with constants, but its proof is valid for arbitrary group languages.

Let us recall the results of [2] related to group equations with no automorphisms.

Let \( H \) be a fixed subgroup of a group \( G \). We pick elements of \( H \) as constants in the language \( \mathcal{L}(H) = \mathcal{L} \cup \{ h \mid h \in H \} \). Any \( \mathcal{L}(H) \)-term in variables \( X \) is actually an element of the free product \( H * F(X) \), where \( F(X) \) is the free group generated by the set \( X \). An \( \mathcal{L}(H) \)-equation is an expression \( t(X) = 1 \), where \( t(X) \) is an \( \mathcal{L}(H) \)-term. Naturally, one can define the notions of algebraic sets and equational domains in the language \( \mathcal{L}(H) \). As we mentioned in Remark 2.2, Theorem 2.1 holds for \( \mathcal{L}(H) \)-equational domains.

It was found in [2] the complete description of \( \mathcal{L}(H) \)-equational domains.

Theorem 2.3. (2) An \( \mathcal{L}(H) \)-group \( G \) is an \( \mathcal{L}(H) \)-equational domain iff there are not \( a, b \in G \), \( a, b \neq 1 \) such that

\[
[a, h^{-1}bh] = 1 \text{ for any } h \in H
\]

(see [2]) few examples of \( \mathcal{L}(H) \)-equational domains:
1. the free group $F_2$ of rank 2 (for $H = F_2$),
2. the alternating group $A_5$ (for $H = A_5$).
Both examples will be used below in the paper.

3 Equational domains

Let us study equations with automorphisms, and the following theorem describes equational domains in the class of $\mathcal{L}(A)$-groups. Its proof is similar to Theorem 2.3 from [2].

**Theorem 3.1.** An $\mathcal{L}(A)$-group $G$ is an $\mathcal{L}(A)$-equational domain iff there are not $a, b \in G$, $a, b \neq 1$ such that

$$[a, \phi(b)] = 1 \text{ for any } \phi \in A. \quad (3)$$

**Proof.** Let us prove the “if” statement. We have that any solution $(x, y)$ of the $\mathcal{L}(A)$-system $S = \{[x, \phi(y)] = 1 \mid \phi \in A\}$ satisfies $x = 1$ or $y = 1$. Thus, $V_G(S) = \{(x, y) \mid x = 1 \text{ or } y = 1\}$, and Theorem 2.1 concludes the proof.

Now, we prove the “only if” part of the theorem. By Theorem 2.1 there exists an $\mathcal{L}(A)$-system $S$ with the solution set $\{(x, y) \mid x = 1 \text{ or } y = 1\}$. Let $w(x, y) = 1$ be an arbitrary $\mathcal{L}(A)$-equation of $S$. Using the following commutator identities,

$$[s, t]^{-1} = [t, s], \ [sp, t] = [s, t][p, t], \ [s^{-1}, t] = [t, s]^{-1}$$

one can equivalently rewrite $w(x, y)$ as a product

$$w(x, y) = u(x)v(y)\prod_i[\phi_i(x), \psi_i(y)]^{w_i(x, y)}$$

where $\phi_i, \psi_i \in A$, and $w_i(x, y), u(x), v(y)$ are $\mathcal{L}(A)$-terms.

Since $(1, y), (x, 1) \in V_G(S)$ for any $x, y \in G$, then $u(x) = 1$ and $v(y) = 1$ for all $x, y \in G$. Hence, one can assume that any equation $w(x, y) = 1 \in S$ is of the form

$$\prod_i[\phi_i(x), \psi_i(y)]^{w_i(x, y)} = 1.$$

Assume there exist $a, b \in G$, $a, b \neq 1$ with (3). We have

$$[a, \phi_i^{-1}(\psi_i(b))] = 1 \iff a\phi_i^{-1}(\psi_i(b)) = \phi_i^{-1}(\psi_i(b))a \iff$$

$$\phi_i(a\phi_i^{-1}(\psi_i(b))) = \phi_i(\phi_i^{-1}(\psi_i(b)))a \iff$$

$$\phi_i(a)\psi_i(b) = \psi_i(b)\phi_i(a) \iff [\phi_i(a), \psi_i(b)] = 1$$

and $(a, b) \in V_G(w(x, y) = 1)$. Thus, the point $(a, b)$ satisfies any equation of $S$, and we obtain a contradiction $V_G(S) \neq \{(x, y) \mid x = 1 \text{ or } y = 1\}$. 

Let us compare Theorem 3.1 and Theorem 2.3. Obviously, Theorem 2.3 follows from Theorem 3.1 for $A = \text{Inn}_H(G)$, where $\text{Inn}_H(G) = \{\phi_h(g) = h^{-1}gh \mid h \in H\}$ is a subgroup of the group $\text{Inn}(G)$ of inner automorphisms.

Moreover, if a group $G$ is an $\mathcal{L}(H)$-equational domain, then $G$ is an $\mathcal{L}(A)$-equational domain for any $A \supseteq \text{Inn}_H(G)$. Therefore, the alternating group $A_5$ is an $\mathcal{L}(A)$-equational domain for $A = \text{Aut}(A_5)$. The free group $F_2$ of rank 2 is also
an \( \mathcal{L}(A) \)-equational domain for \( A = \text{Aut}(F_2) \). However, the following statement provides \( F_2 \) to be an equational domain with a cyclic group of automorphisms.

**Example 3.2.** Let \( F_2 \) be the free group of rank 2, and \( a, b \) be free generators. Let \( \phi \) denote the automorphism \( \phi(a) = b, \phi(b) = a \). Then Theorem 3.1 states that \( F_2 \) is an \( \mathcal{L}(A) \)-equational domain for \( A = \langle \phi \rangle \).

The following two statements also follow from Theorem 3.1.

**Corollary 3.3.** If a group \( G \) has a nontrivial center \( Z(G) \), then \( G \) is not an \( \mathcal{L}(A) \)-equational domain for any \( A \subseteq \text{Aut}(G) \).

**Proof.** Let \( a \in Z(G) \setminus \{1\} \) be a central element. Hence, \( a \) commute with any \( \phi(a) \), and the pair \((a,a)\) satisfies (3) for all \( \phi \).

**Corollary 3.4.** Let \( G \) be an \( \mathcal{L}(A_0) \)-equational domain for some \( A_0 \subseteq \text{Aut}(G) \), and \( H = \Pi G \) be a direct power of \( G \) indexed by a set \( I \). In other words, any element of \( H \) is an ordered tuple \((g_i \mid i \in I)\). Let \( P \) be a set of permutations of \( I \) such that \( P \) is transitive on \( I \) (i.e. for any pair \( i, j \in I \) there exists \( \pi \in P \) with \( \pi(i) = j \)). Let us define automorphisms of \( H \) as follows:

\[
\tilde{f}_\phi((g_i \mid i \in I)) = (\phi(g_i) \mid i \in I),
\]

\[
\tilde{\sigma}_\pi((g_i \mid i \in I)) = (g_{\pi(i)} \mid i \in I),
\]

where \( \phi \in A_0, \pi \in P \). Let \( A \subseteq \text{Aut}(H) \) denote the group generated by \( \{f_\phi, \sigma_\pi \mid \phi \in A_0, \pi \in P\} \). Then the \( \mathcal{L}(A) \)-group \( H \) is an \( \mathcal{L}(A) \)-equational domain.

**Proof.** Let us take \( a = (a_i \mid i \in I), b = (b_i \mid i \in I) \in H, a, b \neq 1 \). Since \( P \) transitively acts on \( I \), there exists \( \psi \in A \) and an index \( i \in I \) such that \( a_i \neq 1, c_i \neq 1 \) where \( \psi(b) = c = (c_i \mid i \in I) \).

Since \( G \) is an \( \mathcal{L}(A_0) \)-equational domain, there exists \( \phi \in A_0 \) with

\[
[a_i, \phi(c_i)] \neq 1.
\]

Therefore,

\[
[a, f_\phi(\psi(b))] \neq 1,
\]

and Theorem 3.1 completes the proof.

\[
\square
\]

### 4 One problem from universal algebraic geometry

The book [2] contains an open problem (Problem 4.4.7), which can be equivalently formulated as follows: is there an algebraic structure \( A \) of an appropriate language \( L \) such that

1. \( A \) is an \( L \)-equational domain;
2. \( A \) is \( q_\omega \)-compact;
3. \( A \) is not \( u_\omega \)-compact.
We solve this problem in the class of $\mathcal{L}(A)$-groups. Let us give all necessary definitions.

Let $A \subseteq \text{Aut}(H)$ be a subgroup of automorphisms of a group $H$. An $\mathcal{L}(A)$-group $H$ is $q_\omega$-compact if for any $\mathcal{L}(A)$-system $S$ and an $\mathcal{L}(A)$-equation $w(X) = 1$ such that
\[ V_H(S) \subseteq V_H(w(X) = 1) \]
there exists a finite subsystem $S' \subseteq S$ with
\[ V_H(S') \subseteq V_H(w(X) = 1). \]

An $\mathcal{L}(A)$-group $H$ is $u_\omega$-compact if for any $\mathcal{L}(A)$-system $S$ and $\mathcal{L}(A)$-equations $w_i(X) = 1$ ($1 \leq i \leq m$) such that
\[ V_H(S) \subseteq \bigcup_{i=1}^{m} V_H(w_i(X) = 1) \]
there exists a finite subsystem $S' \subseteq S$ with
\[ V_H(S') \subseteq \bigcup_{i=1}^{m} V_H(w_i(X) = 1). \]

Let us define a group solving the problem above. Let $G$ be a finite group such that $G$ is an $\mathcal{L}(A_0)$-equational domain for $A_0 = \text{Aut}(G)$ (for example, one may take $G = A_5$). Following Corollary 3.4, we define the $\mathcal{L}(A)$-group $H = \Pi_G$ for $I = \mathbb{Z}$, $P = \{\pi\}$ (where $\pi$ is a permutation $\pi(n) = n + 1$ over $\mathbb{Z}$), and $A$ is generated by the automorphisms $f_\phi, \sigma_\pi$.

We denote the subgroup generated by $\{f_\phi \mid \phi \in \text{Aut}(G)\} \subseteq A$ by $A_G$. The automorphism $\sigma_\pi$ is denoted by $\sigma$ below. By the definition, $\sigma$ acts on an element $(g_i \mid i \in \mathbb{Z})$ by
\[ \sigma(g_i) = g_{i+1}. \]

Thus, we should prove that $H$ is
1. an $\mathcal{L}(A)$-equational domain (it immediately follows from Corollary 3.4);
2. $q_\omega$-compact (Lemma 4.7);
3. not $u_\omega$-compact (Lemma 4.1).

Below we will use the following denotation
\[ \sigma_k(x) = \begin{cases} 
\sigma(\underbrace{\sigma(\ldots \sigma(x) \ldots)}_{k \text{ times}}) & \text{for } k > 0, \\
\underbrace{\sigma^{-1}(\ldots \sigma^{-1}(x) \ldots)}_{k \text{ times}} & \text{for } k < 0, \\
x & \text{for } k = 0
\end{cases} \]

The automorphism $\sigma$ commute with any $f_\phi$, i.e. $\sigma(f_\phi(h)) = f_\phi(\sigma(h))$ for all $h \in H$. Hence any equation over the $\mathcal{L}(A)$-group $H$ can be written in the following form
\[ \sigma_{k_1}(f_1(x_{j_1}^{\varepsilon_1}))\sigma_{k_2}(f_2(x_{j_2}^{\varepsilon_2})) \ldots \sigma_{k_l}(f_l(x_{j_l}^{\varepsilon_l})) = 1, \]
where $f_i \in A_G$, $\varepsilon_i \in \{-1, 1\}$, $k_j \in \mathbb{Z}$.

**Lemma 4.1.** The $\mathcal{L}(A)$-group $H$ is not $u_\omega$-compact.
Proof. Since \( H \) is an \( \mathcal{L}(A) \)-equational domain, there are no \( a, b \in H \) such that \( a, b \neq 1 \) and \([a, \phi(b)] = 1\) for any \( \phi \in A \). Hence, any solution of the \( \mathcal{L}(A) \)-system \( S = \{[x, \phi(y)] = 1 \mid \phi \in A\} \) satisfies either \( x = 1 \) or \( y = 1 \). Thus, the following inclusion
\[
V_H(S) \subseteq V_H(x = 1) \cup V_H(y = 1).
\]
holds.

Let \( S' \) be a finite subsystem of \( S \) and \( n = \max\{|k| \mid \sigma_k \text{ occurs in } S'\} \). Define \( a = (a_i \mid i \in \mathbb{Z}) \), \( b = (b_i \mid i \in \mathbb{Z}) \) such that
\[
a_i = \begin{cases} g, & \text{if } i = 0 \\ 1, & \text{otherwise} \end{cases} \quad b_i = \begin{cases} g, & \text{if } i = n + 1 \\ 1, & \text{otherwise} \end{cases}
\]
where \( g \in G \setminus \{1\} \).

Let \( A' = \{\phi \mid [x, \phi(y)] = 1 \in S'\} \) (i.e. \( A' \) is the set of all \( \phi \) such that the equation \( [x, \phi(y)] = 1 \) belongs to \( S' \)) be a finite set of automorphisms. By the choice of \( b \), the element \( \phi(b) \) has 1 at the 0-th coordinate for each \( \phi \in A' \). Therefore, \( \phi(b) \) commutes with \( a \) and we obtain \( (a, b) \in V_H(S') \). Since \( a \neq 1, b \neq 1 \), then the inclusion
\[
V_H(S') \subseteq V_H(x = 1) \cup V_H(y = 1)
\]
fails. Thus, \( H \) is not \( u_w \)-compact. \( \square \)

There is a correspondence between \( \mathcal{L}(A) \)-systems over \( H \) and \( \mathcal{L}(A_0) \)-systems over \( G \). Let \( S \) be an \( \mathcal{L}(A) \)-system in variables \( X = \{x_1, x_2, \ldots, x_n\} \). The system \( S \) defines an \( \mathcal{L}(A_0) \)-system \( \gamma(S) \) over \( G \) in infinite number of variables \( Y = \{y_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\} \) (below \( \varepsilon_i \in \{-1, 1\} \)):
\[
\sigma_{k_1}(f_1(x_{j_1}^{\varepsilon_1})) \sigma_{k_2}(f_2(x_{j_2}^{\varepsilon_2})) \cdots \sigma_{k_l}(f_l(x_{j_l}^{\varepsilon_l})) = 1 \iff \prod f_i(y_{k_1+k, j_1}^{\varepsilon_1}) y_{k_2+k, j_2}^{\varepsilon_2} \cdots f_l(y_{k_l+k, j_l}^{\varepsilon_l}) = 1 \in \gamma(S) \text{ for all } k \in \mathbb{Z}. \quad (11)
\]

In other words, \( \gamma(S) \) is the coordinate-wise version of \( S \) over the direct power \( H = \Pi G \).

Example 4.2. If \( S = \{\sigma(x_1)x_2 = 1\} \) then
\[
\gamma(S) = \{\ldots, y_{-11} y_{-22} = 1, y_{01} y_{-12} = 1, y_{11} y_{02} = 1, y_{21} y_{12} = 1, y_{31} y_{22} = 1, \ldots\} = \{y_{k1} y_{(k-1)2} = 1 \mid k \in \mathbb{Z}\}.
\]

By the definition, any \( \mathcal{L}(A_0) \)-equation \( u(Y) = 1 \in \gamma(S) \) may come from several \( \mathcal{L}(A) \)-equations \( W = \{w_i(X) = 1\} \) of the system \( S \). Let us take an arbitrary equation \( w_i(X) = 1 \) from \( W \) and denote this correspondence by \( \gamma^{-1}(u(Y) = 1) = \{w_i(X) = 1\} \).

Remark 4.3. Below we will omit brackets in map compositions, i.e. we will write \( \alpha \beta(x) \) instead of \( \alpha(\beta(x)) \).

Lemma 4.4. For any \( \mathcal{L}(A_0) \)-equation
\[
f_1(y_{i_1 j_1}^{\varepsilon_1}) f_2(y_{i_2 j_2}^{\varepsilon_2}) \cdots f_l(y_{i_l j_l}^{\varepsilon_l}) = 1 \quad (12)
\]
and any number \( k \in \mathbb{Z} \) the equation
\[
f_1(y_{i_1+k j_1}^{\varepsilon_1}) f_2(y_{i_2+k j_2}^{\varepsilon_2}) \cdots f_l(y_{i_l+k j_l}^{\varepsilon_l}) = 1 \quad (13)
\]
also belongs to \( \gamma(S) \).

Further, if \( P = \{p_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\} \in V_G(\gamma(S)) \) then any shift \( \sigma_k(P) = (s_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n), s_{ij} = p_{i+k j} \) (\( k \in \mathbb{Z} \)) is also a solution of \( \gamma(S) \).
Proof. Observe that the system $\mathbf{S}$ from Example 4.2 clearly satisfies the statements of this lemma.

The first statement directly follows from the definition of the system $\gamma(\mathbf{S})$. Let us prove the second one.

Assume there exists an $L(A_0)$-equation \[ f_1(s_{i_1 j_1}^1), f_2(s_{i_2 j_2}^2), \ldots, f_l(s_{i_l j_l}^l) \neq 1 \] or, equivalently,
\[
 f_1(p_{i_1+k j_1}^1), f_2(p_{i_2+k j_2}^2), \ldots, f_l(p_{i_l+k j_l}^l) \neq 1
\] (14)

However, $\gamma(\mathbf{S})$ contains the equation $u(Y) = 1$ (13), and, by (14), we have $u(P) \neq 1 \Rightarrow P \notin V_G(\gamma(\mathbf{S})).$ □

Let $\mathbf{S}_0, \mathbf{S}_1$ be $L(A_0)$-systems in variables $Y = \{y_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\}$. We say that $\mathbf{S}_0, \mathbf{S}_1$ are $Z$-equivalent for a given $Z \subseteq Y$ if the projections of $V_G(\mathbf{S}_0)$ and $V_G(\mathbf{S}_1)$ onto the coordinates $Z$ are the same (in other words, for each $P = (p_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in V_G(\mathbf{S}_k)$ there exists $Q = (q_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in V_G(\mathbf{S}_{1-k})$ with $p_{ij} = q_{ij}$ for each $y_{ij} \in Z$, $k \in \{0, 1\}$.

**Lemma 4.5.** Let $\mathbf{S}_0$ be an $L(A_0)$-system in variables $Y = \{y_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\}$ over a finite group $G$. Then for any finite $Z \subseteq Y$ there exists a finite $Z$-equivalent subsystem $\mathbf{S}_1 \subseteq \mathbf{S}_0$.

**Proof.** The statement immediately follows from the finiteness of the group $G$. □

Let us denote a subsystem of an $L(A_0)$-system $\gamma(\mathbf{S})$ by $\gamma_Z(\mathbf{S})$, if $\gamma_Z(\mathbf{S})$ is $Z$-equivalent to $\gamma(\mathbf{S})$.

Let $Z$ be a set of variables occurring in an $L(A_0)$-system $\gamma(\mathbf{S})$. The system $\gamma(\mathbf{S})$ may contain subsystems which are $Z$-equivalent to $\gamma(\mathbf{S})$ (as it proved above, for finite $Z$ such subsystems always exist). Let us denote the class of such systems by $Z(\gamma(\mathbf{S}))$. We pick an arbitrary system from $Z(\gamma(\mathbf{S}))$ and denote it by $\gamma_Z(\mathbf{S})$.

Suppose an $L(A_0)$-system $\gamma_Z(\mathbf{S})$ was constructed by an $L(A)$-system $\mathbf{S}$ and a finite set $Z$. By the definition, $\gamma^{-1}_Z(\mathbf{S}) \subseteq \mathbf{S}$ is the set of equations from $\mathbf{S}$ which were essentially used in the construction of $\gamma_Z(\mathbf{S})$. One can apply the operator $\gamma$ to $\gamma^{-1}_Z(\mathbf{S})$ and obtain a new $L(A_0)$-system $\gamma^{-1}_Z \gamma Z(\mathbf{S})$.

Let us summarize all simple properties of the systems $\mathbf{S}, \gamma(\mathbf{S}), \gamma_Z(\mathbf{S}), \gamma^{-1}_Z(\mathbf{S}), \gamma^{-1}_Z \gamma Z(\mathbf{S})$:

1. $\mathbf{S}, \gamma^{-1}_Z(\mathbf{S})$ are $L(A)$-systems and their solutions belong to $H^n$;
2. $\gamma(\mathbf{S}), \gamma_Z(\mathbf{S}), \gamma^{-1}_Z \gamma Z(\mathbf{S})$ are $L(A_0)$-systems and coordinates of their solutions belong to $G$;
3. the systems $\gamma_Z(\mathbf{S}), \gamma^{-1}_Z(\mathbf{S})$ are finite for finite $Z$;
4. we have the inclusions $\gamma^{-1}_Z(\mathbf{S}) \subseteq \mathbf{S}, \gamma_Z(\mathbf{S}) \subseteq \gamma^{-1}_Z \gamma Z(\mathbf{S}) \subseteq \gamma(\mathbf{S})$;
5. the $L(A_0)$-systems $\gamma^{-1}_Z \gamma Z(\mathbf{S}), \gamma_Z(\mathbf{S}), \gamma(\mathbf{S})$ are $Z$-equivalent.

**Lemma 4.6.** Let $C$ be a finite set of pairs $(i, j), i \in \mathbb{Z}, 1 \leq j \leq n$. Then $\gamma^{-1}_Z \gamma Z(\mathbf{S})$ is $Z_k$-equivalent to $\gamma(\mathbf{S})$ for any set $Z_k = \{y_{i+j} \mid (i, j) \in C\}, k \in \mathbb{Z}$.

**Proof.** Let us take a point $P = (p_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in V_G(\gamma^{-1}_Z \gamma Z(\mathbf{S}))$ and consider the shift $R = \sigma_k(P) = (r_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n), r_{ij} = p_{i+k j}$. According to Lemma 4.1, $R$ is a solution of $V_G(\gamma^{-1}_Z \gamma Z(\mathbf{S}))$. By the $Z$-equivalence, there exists a point $R'' = (s_{ij}'' \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in V_G(\gamma(\mathbf{S}))$ with $s_{ij}'' = r_{ij}$ for any $(i, j) \in C$. By Lemma 4.4, the point $R'' = \sigma_{-k}(R'), R'' = (s_{ij}''' \mid i \in \mathbb{Z}, 1 \leq j \leq n)$ is a solution of $\gamma(\mathbf{S})$. By the definition of $R''$, for each $(i, j) \in C$ we have $s_{i+k j}''' = s_{ij}'' = r_{ij} = p_{i+k j}$, and, therefore, $\gamma^{-1}_Z \gamma Z(\mathbf{S})$ is $Z_k$-equivalent to $\gamma(\mathbf{S})$. □
Lemma 4.7. The $\mathcal{L}(A)$-group $H$ is $q_\omega$-compact.

Proof. Suppose an $\mathcal{L}(A)$-system $S$ and an $\mathcal{L}(A)$-equation $w(X) = 1$ satisfy (6).

The $\mathcal{L}(A)$-term $w(X)$ defines the set of pairs $C = \{(k_1, j_1), (k_2, j_2), \ldots, (k_l, j_l)\}$.

Let us put $S' = \gamma^{-1}Z(S)$ for $Z = \{y_{ij} \mid (i, j) \in C\}$ and prove (7). Assume there exists a point $(h_1, h_2, \ldots, h_n) \in V_H(S') \setminus V_H(w(X) = 1)$. In other words, there exists $P = (p_{ij} \mid i \in Z, 1 \leq j \leq n) \in V_G(\gamma^{-1}Z(S)) \setminus V_G(\gamma(w(X)) = 1)$.

We have $\gamma(w(X) = 1) = \{f_1(y_{k_1}^{k_1} j_1), f_2(y_{k_2}^{k_2} j_2) \ldots f_l(y_{k_l}^{k_l} j_l) = 1 \mid k \in Z\}$ and there exists $k \in Z$ such that $f_1(p_{k_1}^{k_1} j_1) f_2(p_{k_2}^{k_2} j_2) \ldots f_l(p_{k_l}^{k_l} j_l) \neq 1$.

By Lemma 4.6 there exists a point $Q = (q_{ij} \mid i \in Z, 1 \leq j \leq n) \in V_G(\gamma(S))$ with $q_{i+k} j = p_{i+k} j$ for any $(i, j) \in C$. Therefore, $w(Q) \neq 1$.

Thus, $Q \in V_G(\gamma(S)) \setminus V_G(\gamma(w(X) = 1))$. The point $Q$ defines $R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \in H^n$, $\mathbf{r}_j = (q_{ij} \mid i \in Z)$ such that $R \in V_G(S) \setminus V_G(w(X) = 1)$, and we obtain a contradiction with (6).

5 Conclusions

The construction of the group $H$ from Corollary 3.4 is close to the notion of wreath product. In particular, the group $H$ from Section 4 is structurally similar to the wreath product $G \wr \mathbb{Z}$.

This correspondence allows us to remind an important problem of universal algebraic geometry posed by B. Plotkin [3].

Problem (B. Plotkin [3]). Let $H = A \wr B$ be the wreath product of the groups $A$ and $B$.

1. When $H$ is $q_\omega$-compact?
2. When $H$ is $q_\omega$-compact but not equationally Noetherian (a group is equationally Noetherian if the subsystem $S' \subseteq S$ in (7) does not depend on an equation $w(X) = 1$)?
3. Is $H$ necessarily $q_\omega$-compact if both $A, B$ $q_\omega$-compact?

Let us explain the assertion of the problem above. Originally, B. Plotkin posed it for group equations in the “standard” language $\mathcal{L} = \{\cdot, ^{-1}, 1\}$. However, in [6] the problem was partially solved for languages with constants.

Theorem 5.1. [6] If a group $A$ is not abelian and $B$ is infinite, then $H$ is not $q_\omega$-compact in the language with constants $\mathcal{L}(H) = \mathcal{L} \cup \{h \mid h \in H\}$.

Thus, for the language $\mathcal{L}(H)$ the following problem remains open.

Problem. Let us consider the class of $\mathcal{L}(H)$-equations. Is $H$ $q_\omega$-compact (equationally Noetherian) for abelian $A$?

In the conclusion of the whole paper, we should discuss other ways to solve Problems 4.4.7, 5.3.1-4 from [2]. Usually (see [2]), the negative solution of a problem in universal algebraic geometry may be found in structures of pure relational languages, since such languages admit a very simple view of equations.
However, we cannot solve Problems 5.3.1-4 in relational languages. For this reason, we had to develop the algebraic geometry over equations with automorphisms. Thus, one can formulate a problem.

**Problem.** Is there an algebraic structure $\mathcal{A}$ of pure relational language $L$ such that

1. $\mathcal{A}$ is an $L$-equational domain,
2. $\mathcal{A}$ is $q_\omega$-compact,
3. $\mathcal{A}$ is not $u_\omega$-compact?

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