Rationally connected non-Fano type varieties

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Abstract

We study the relation between classes of rationally connected varieties and varieties of Fano type. We give examples of rationally connected varieties of dimension \( \geq 6 \) which are not birationally equivalent to Fano type varieties, thereby answering a question of Cascini and Gongyo.

1 Introduction

The log minimal model program (MMP) is one of the key notions in birational geometry. Finding when can we run the MMP for a pair \((X, D)\) \((D\text{-MMP})\) is one of the central subjects and is still being developed. If \(X\) is a variety of Fano type then we can run the \(D\text{-MMP}\) on it for any divisor \(D\) on \(X\) [11, Corollary 2.7]. We say that a normal projective variety \(X\) is a Fano type if there is a \(Q\)-divisor \(H\) on \(X\) such that a pair \((X, H)\) is Kawamata log terminal and \(- (K_X + H)\) is ample. Varieties of Fano type have been introduced by Shokurov and Prokhorov in [11]. The Fano type property is preserved under flips and contractions (11, Lemma 2.8) therefore at any stage of the minimal model program on a variety of Fano type we have a Fano type variety. In particular the result of running the MMP on a variety of Fano type is a variety of Fano type. It would be nice if we could say if in a given birational class there is a variety which behaves well under \(D\text{-MMP}\) for any divisor \(D\). For example a variety of Fano type.

It is known that varieties of Fano type are rationally connected [12, Theorem 1]. Therefore it is natural to ask the following question:

**Question 1** ([11, Question 5.2]). Let \(X\) be a rationally connected variety. Is \(X\) birationally equivalent to a variety of Fano type?

In dimension \( \leq 2 \) the answer is positive since every rationally connected surface is rational. We aim to prove in this paper that the answer is negative in dimension \( \geq 6 \).
Theorem 1.1. Let $W$ be a generic smooth divisor of degree $(2M, 2l)$ on $\mathbb{P}^M \times \mathbb{P}^1$, $M \geq 5$, $l \geq 3$. Let $V$ be a double cover ramified over $W$. Let $\pi : V \dashrightarrow X$ be a birational map to a Mori fiber space $f : X \to Z$. Then $Z = \mathbb{P}^1$ and $\pi$ is an isomorphism.

\[ \begin{array}{c} V \xrightarrow{\pi} X \\ \mathbb{P}^1 \xrightarrow{g} Z \end{array} \]

The variety $V$ is a Fano fibration over $\mathbb{P}^1$. We say that a Fano fibration $V$ over $\mathbb{P}^1$ is birationally superrigid if any birational map $f : V \to V'$ to a Mori fiber space is a fiberwise map. First, we use the result of Pukhlikov (Theorem 3.5) to show that $V$ is birationally superrigid. Then we use Tian's $\alpha$-invariant to conclude that any map to a Mori fiber space is an isomorphism.

We prove that any variety of Fano type is birationally equivalent to a Mori fiber space with big anticanonical divisor (Proposition 3.1). Since $-K_V$ is not big (Lemma 3.7) we get the following statement:

Corollary 1.2. The variety $V$, described in Theorem 1.1, is not birationally equivalent to a variety of Fano type.

Since $V$ is rationally connected we conclude that the answer to Question 1 is negative. We also expect a negative answer in the dimension $\geq 3$.

Conjecture 1.3. Let $W$ be a generic smooth divisor of degree $(2M, 2l)$ on $\mathbb{P}^M \times \mathbb{P}^1$, $M \geq 2$, $l \geq 3$. Let $V$ be a double cover ramified over $W$. Then $V$ is not birational to a variety of Fano type.

The same approach should work for $M$ equal to 3 and 4. For $M = 2$ the proof should be different because we may have fiberwise transformations \cite{21}.

Note that the Question 1 can be asked for geometrically rational surfaces defined over non-closed fields. The answer is not known there even in dimension 2.

2 Preliminaries

All varieties in this paper are considered to be projective and defined over $\mathbb{C}$ unless stated otherwise. We write $\sim_Q$ for linear equivalence of $\mathbb{Q}$-divisors.
Definition 2.1 ([10] p. 6]). Let $X$ be a normal variety and $D$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\pi : \tilde{X} \to X$ be a birational morphism from a normal variety $\tilde{X}$ and let $\tilde{D} = \pi^{-1}(D)$ be a proper transform of $D$. Then we can write

$$K_{\tilde{X}} + \tilde{D} \sim_{\mathbb{Q}} \pi^*(K_X + D) + \sum_E a(E, X, D)E,$$

where $E$ runs through all the distinct exceptional divisors of $\pi$ on $\tilde{X}$ and $a(E, X, D)$ is a rational number. We say that the pair $(X, D)$ is terminal (resp. canonical, log terminal, log canonical) if $a(E, X, D) > 0$ (resp. $a(E, X, D) \geq 0$, $a(E, X, D) > -1$, $a(E, X, D) \geq -1$) for every prime $\pi$-exceptional divisor $E$ on any normal variety $\tilde{X}$ such that there is a birational morphism from $\tilde{X}$ to $X$. If $D = 0$, we simply say that $X$ has only terminal (resp. canonical, log terminal, log canonical) singularities. The numbers $a(E, X) = a(E, X, 0)$ are called discrepancies of the divisor $E$. We say that the pair $(X, D)$ is klt if it is log terminal and coefficients $a_i$ of all prime divisors $D_i$ in $D = \sum a_iD_i$ are less than 1.

Let $\mathcal{M}$ be a linear system on $X$. We say that $(X, \lambda\mathcal{M})$ has terminal (resp. canonical, klt, log canonical) singularities if for a generic divisor $D \in \mathcal{M}$ the pair $(X, \lambda D)$ is terminal (resp. canonical, klt, log canonical).

Note that $E$ also runs through components of the proper transform $\pi^{-1}(D)$ of the divisor $D$. Thus if $(X, D)$ is klt then all coefficients of prime divisors in $D$ are $< 1$.

Definition 2.2 ([11] Lemma-Definition 2.6]). Let $X$ be a normal variety and $D$ an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. We say that $(X, D)$ is a log Fano pair if $-(K_X + D)$ is ample and $(X, D)$ is klt. We say that $X$ is of Fano type if there exists an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $(X, D)$ is a log Fano pair.

Example 2.3. The quotient $Y = X/G$ of a Fano variety by a finite group is a variety of Fano type. Since $K_X = f^*(K_Y + \frac{R}{|G|})$, where $R$ is a ramification divisor, $-(K_Y + \frac{R}{|G|})$ is ample. The pair $(Y, \frac{R}{|G|})$ is klt [20, Proposition 3.16] and hence $Y$ is a variety of Fano type.

Fano type varieties also appear as exceptional divisors of extremal contractions.

Definition 2.4 ([8] Definition 1]). We say that a Fano variety $V$ with $\text{Pic}(V) = \mathbb{Z}$ is birationally superrigid if $\text{Bir}(V) = \text{Aut}(V)$. We say that a Fano fibration $f : V \to \mathbb{P}^1$ is birationally superrigid if for any Mori fiber
space \( f' : V' \to S \) and any birational map \( \pi : V \dashrightarrow V' \) the base \( S \) is \( \mathbb{P}^1 \), there is a commutative diagram

\[
\begin{array}{ccc}
V \xrightarrow{f} & V' \\
\downarrow{\pi} & \downarrow{f'} \\
\mathbb{P}^1 & \xrightarrow{g} & \mathbb{P}^1 \\
\end{array}
\]

and \( \pi \) is an isomorphism on a generic fiber of \( f \).

It follows from the definition that, if \( V \) is a birationally superrigid fibration over \( \mathbb{P}^1 \), then \( \pi \) is a fiberwise map.

**Definition 2.5** ([5, Definition 1.2]). Let \( X \) be a \( Q \)-factorial Fano variety with at worst log terminal singularities. Define **Tian’s \( \alpha \)-invariant** of \( X \) by the number

\[
\alpha(X) = \sup \{ c : (X, cD) \text{ is log canonical} \}
\]

for every effective \( Q \)-divisor \( D, D \sim Q - K_X \} \).

Tian has introduced the \( \alpha \)-invariant for Kähler manifolds in [16] in a different way but it coincides with the one defined above ([17, Theorem A.3]).

**Proposition 2.6** ([18, Proposition 2.7]). Let \( X, X' \) be varieties with \( Q \)-factorial singularities, and \( f : X \dashrightarrow X' \) a birational map which is an isomorphism in codimension one. Let \( H' \) be an ample divisor on \( X \). If \( H = f^{-1}H' \) is ample then \( \Phi \) is an isomorphism.

**Proof taken from [18].** Let \( Y \) be a resolution of \( f \) and let \( p : Y \to X \) and \( q : Y \to X' \) be corresponding morphisms. Let \( E_i \) be the \( p \)-exceptional divisors. Since \( f \) is an isomorphism in codimension one, \( q \) has the same exceptional divisors. For suitable \( a_i \) we have

\[
\sum a_i E_i \sim_Q -p^*H + q^*H'.
\]

It follows from the negativity lemma [9, Lemma 2.19] that \( a_i = 0 \) and \( p^*H = q^*H' \). Let \( C \subset Y \) be a curve such that \( q(C) \subset X' \) is a point. By the projection formula

\[
0 = q(C) \cdot H' = C \cdot q^*H' = C \cdot p^*H = p_*(C) \cdot H
\]

hence \( p(C) \) is a point since \( H \) is ample. Thus \( f \) is an isomorphism. \( \square \)
Proposition 2.7 (Noether-Fano inequality, [18, Theorem 4.2]). Let $V$ be a terminal Fano variety with $\text{Pic} V = \mathbb{Z}$. Then $V$ is birationally superrigid if and only if there are no movable linear systems $\mathcal{M}$ on $V$ such that $\lambda \mathcal{M} + K_V \sim_\mathbb{Q} 0$ and the pair $(V, \lambda \mathcal{M})$ is not canonical for some $\lambda \in \mathbb{Q}$.

Proof. Let $V$ be birationally superrigid and let $\mathcal{M}$ be a movable linear system on $V$. Suppose that $\lambda \mathcal{M} + K_V \sim_\mathbb{Q} 0$ and the pair $(V, K_V + \lambda \mathcal{M})$ is not canonical for some $\lambda \in \mathbb{Q}$. Consider a resolution of singularities $h : \overline{V} \to V$ of $\mathcal{M}$ and the generic divisor $D \in h^{-1}(\mathcal{M})$. Let $E_i$ denote the exceptional divisors of $h$. We can run the $\lambda D$-MMP on $\overline{V}$. Let a terminal pair $(V', \lambda D')$ be a result of the $\lambda D$-MMP. Since $K_{\overline{V}} + \lambda D \sim_\mathbb{Q} \sum a_i E_i$, $K_{V'} + \lambda D'$ is not nef and thus $V'$ is a Mori fiber space. Let $g : V \dasharrow V'$ be the composition of $h^{-1}$ and the MMP. It follows from superrigidity of $V$ that $g$ is an isomorphism. The pair $(V', \lambda D')$ is terminal but $(V, \lambda g^{-1}D')$ is not canonical thus we have a contradiction.

Suppose $g : V \dasharrow X$ is a birational map to a Mori fiber space $X$. First, let $X$ have a non trivial Fano fibration $\pi : X \to S$. Take a pullback $H_X$ of a very ample divisor on $S$ and consider the linear system $\mathcal{M}_X = [H_X]$. Put $\mathcal{M} = g^{-1}\mathcal{M}_X$. There is a $\lambda \in \mathbb{Q}$ such that $\lambda \mathcal{M} + K_V \sim_\mathbb{Q} 0$. Let $W$ be a resolution of $g$ and let $\psi : W \to V$, $\varphi : W \to X$ be the corresponding morphisms.

We can write on $W$

$$\psi^*(K_V + \lambda \mathcal{M}) + \sum a_i E_i \sim_\mathbb{Q} K_W + \lambda \mathcal{M}_W \sim_\mathbb{Q} \varphi^*(K_X + \lambda \mathcal{M}_X) + \sum b_i E_i. \quad (1)$$

Note that the generic fiber $f$ of $\pi$ does not pass through centers of exceptional divisors of $\varphi$ and $f \cdot H_X = 0$. Intersecting $\varphi^* f$ with (1) we get

$$\sum a_i (\varphi^* f \cdot E_i) \sim_\mathbb{Q} \varphi^* (f \cdot K_X).$$

Since $K_X$ is anti-ample on $f$ and $\varphi^* f \cdot E_i$ is effective, there is some $i$ such that $a_i < 0$.

Now suppose that $X$ is a Fano variety. Without loss of generality we may assume that $\text{Pic}(X) = \mathbb{Z}$ — otherwise either $X$ is not minimal and we can contract some divisor or $X$ is a nontrivial Fano fibration so we consider the previous case. Take a very ample divisor $H \in (-nK_X)$ on $X$, $n > 1$. Define $\mathcal{M}, \mathcal{M}_X, W, \varphi, \psi$ as above. We can write (1) on $W$ again. The linear system $\mathcal{M}_X$ is free, therefore for all $i$, $b_i > 0$. It is easy to see that $K_X + \lambda \mathcal{M}_X \sim_\mathbb{Q} (1 - \lambda n)K_X$. Note that $\psi^*(K_V + \lambda \mathcal{M}) \sim_\mathbb{Q} 0$. Suppose that
$1 - \lambda n > 0$ then by intersecting (1) with the generic divisor $D \in \varphi^*(\mathcal{M})$ we have
\[
\sum a_i(D \cdot E_i) \sim_Q \varphi^*((1 - \lambda n)D \cdot K_X).
\]
On the right-hand side we have the intersection of an ample and an antiample class, therefore there is an $i$ such that $a_i < 0$.

Suppose that $1 - \lambda n \leq 0$ and $a_i \geq 0$ for all $i$. Rewriting (1) we have
\[
\varphi^*((1 - \lambda n)K_X) \sim_Q \sum a_iE_i - \sum b_iE_i.
\]
Applying the negativity lemma for $\varphi$ we see that $a_i \geq b_i$ for all $i$ such that $E_i$ is an exceptional divisor of $\varphi$. If there is an $i$ such that $E_i$ is not exceptional for $\varphi$, then $-b_i \geq 0$. Thus, all exceptional divisors of $\psi$ are also exceptional for $\varphi$. Since the Picard numbers of $V$ and $X$ are equal to 1, morphisms $\psi$ and $\varphi$ have the same number of exceptional divisors. Thus, $\psi$ and $\varphi$ have the same exceptional divisors hence $g$ is an isomorphism in codimension 1.

By Proposition 2.7, $g$ is an isomorphism.

**Theorem 2.8** (Inversion of adjunction, [9, Theorem 17.7]). Let $X$ be a normal variety and let $S \subset X$ be an irreducible divisor with singularities of codimension $\geq 3$. Let $B$ be an effective $\mathbb{Q}$-Cartier divisor. Assume that $K_X + S$ is $\mathbb{Q}$-Cartier and is terminal. Then $K_X + S + B$ is log canonical in a neighborhood of $S$ if and only if $K_S + B$ is log canonical.

We use the inversion of adjunction for the study of singularities of linear systems as follows. Let $F$ be a prime divisor with isolated quadratic nondegenerate singularities on a variety of dimension $\geq 4$. Suppose there is a divisor $D$ such that $K_X + D$ is not canonical along $B \subset F$. Then $K_X + D + F$ is not log canonical along $B$ and by inversion of adjunction $K_F + D|_F$ is not log canonical.

**Theorem 2.9** ([5, Theorem 1.5]). Let $Z$ be a smooth curve. Suppose that there is a commutative diagram
\[
\begin{array}{ccc}
V & \xrightarrow{\rho} & V' \\
\downarrow{\pi} & & \downarrow{\bar{\pi}} \\
Z & \xrightarrow{\rho' V \setminus X} & \bar{V} \setminus \bar{X}
\end{array}
\]
such that $\pi$ and $\bar{\pi}$ are flat morphisms, and $\rho$ is a birational map that induces an isomorphism
\[
\rho|_{V \setminus X} : V \setminus X \to \bar{V} \setminus \bar{X}.
\]
where $X$ and $\bar{X}$ are scheme fibers of $\pi$ and $\bar{\pi}$ over a point $O \in Z$, respectively. Suppose that the varieties $V$ and $\bar{V}$ have terminal $\mathbb{Q}$-factorial singularities, the divisors $-K_V$ and $-K_{\bar{V}}$ are $\pi$-ample and $\bar{\pi}$-ample respectively, the fibers $X$ and $\bar{X}$ are irreducible, the variety $X$ has log terminal singularities, and $\alpha(X) \geq 1$. Then $\rho$ is an isomorphism.

Proof. Suppose that the variety $X$ has log terminal singularities, the inequality $\alpha(X) \geq 1$ holds, and $\rho$ is not an isomorphism. Let $D$ be a generic very ample divisor on $Z$. Put

$$\Lambda = \left| -nK_V + \pi^*(nD) \right|, \quad \Gamma = \left| -nK_{\bar{V}} + \bar{\pi}^*(nD) \right|, \quad \bar{\Lambda} = \rho(\Lambda), \quad \bar{\Gamma} = \rho^{-1}(\Gamma),$$

where $n$ is a natural number such that $\Lambda$ and $\Gamma$ have no base points. Put

$$M_V = \frac{2\varepsilon}{n} \Lambda + \frac{1 - \varepsilon}{n} \Gamma, \quad M_{\bar{V}} = \frac{2\varepsilon}{n} \bar{\Lambda} + \frac{1 - \varepsilon}{n} \bar{\Gamma},$$

where $\varepsilon$ is a positive rational number.

The log pairs $(V, M_V)$ and $(\bar{V}, M_{\bar{V}})$ are birationally equivalent, and for $\varepsilon$ small enough $K_V + M_V$ and $K_{\bar{V}} + M_{\bar{V}}$ are ample. We claim that the map $\rho$ is biregular if the singularities of both log pairs $(V, M_V)$ and $(\bar{V}, M_{\bar{V}})$ are canonical. Consider the resolution $(V', M')$ of the map $\rho$, we can write

$$K_{V'} + M' \sim_{\mathbb{Q}} K_V + M_V + \sum a_i E_i \sim_{\mathbb{Q}} K_{\bar{V}} + M_{\bar{V}} + \sum b_i E_i.$$

Since the pairs are canonical the coefficients $a_i$ and $b_i$ are nonnegative for all $i$. Thus, applying the negativity lemma we see that the maps from $V'$ to $V$ and $\bar{V}$ have the same exceptional divisors. Hence $\rho$ is an isomorphism in codimension 1 and by Proposition 2.6 it is an isomorphism.

The linear system $\Gamma$ does not have base points. Thus, there is a rational number $\varepsilon$ such that the log pair $(\bar{V}, M_{\bar{V}})$ is canonical. So the log pair $(V, M_V)$ is not canonical. The log pair

$$\left( V, \ X + \frac{1 - \varepsilon}{n} \bar{\Gamma} \right)$$

is not log canonical, because $\Lambda$ does not have not base points, and $\bar{\Gamma}$ does not have base points outside of the fiber $X$, which is a Cartier divisor on the variety $V$. The log pair

$$\left( X, \ \frac{1 - \varepsilon}{n} \bar{\Gamma}|_X \right)$$

is not log canonical by inversion of adjunction, which is impossible because $\alpha(X) \geq 1$. \qed
It follows from Theorem 2.5 that if \( \alpha(F) \geq 1 \) for every fiber \( F \) of Mori fiber space \( \pi : V \to S \) then \( V \) does not have fiberwise maps. We expect that the converse of this theorem holds. That is, if there exists fiber \( F \) with \( \alpha(F) < 1 \), then there exists nontrivial fiberwise map \( V \to V' \).

**Conjecture 2.10.** Let \( \pi : V \to Z \) be a fibration over a smooth curve. Assume that there exists a fiber \( F \) of \( \pi \) such that \( \alpha(F) < 1 \). Then there exists a fibration \( \pi' : V' \to Z \) and fiberwise map \( f : V \to V' \) such that \( f|_F \) is not an isomorphism.

\[
\begin{array}{ccc}
V & \xrightarrow{f} & V' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
Z & \xrightarrow{} & Z
\end{array}
\]

We recall some results we use to show rational connectedness of the variety \( V \) from Theorem 1.1.

**Theorem 2.11** ([2, Theorem 0.1]). Any smooth Fano variety is rationally connected.

**Theorem 2.12** ([1, Corollary 1.3]). Let \( X \to B \) be a dominant morphism. If \( B \) is rationally connected and the generic fiber is rationally connected then \( X \) is rationally connected.

## 3 Fibrations over the projective line

In this section we prove Theorem 1.1 and Corollary 1.2.

**Proposition 3.1.** Let \( X \) be a variety of Fano type. Then there is a Mori fiber space \( V \) birationally equivalent to \( X \) such that \( -K_V \) is big.

**Proof.** There is an effective divisor \( D \) on \( X \) such that \( -(K_X + D) \) is ample and \((X, D)\) is klt. Since \(-K_X\) is a sum of an effective and an ample divisor it is big. If \( X \) is not Q-factorial we may replace it by its Q-factorization \( Y \) ([3, Corollary 1.4.3]). Note that there is a morphism \( \pi : Y \to X \) which is an isomorphism in codimension 1, therefore \(-K_Y\) is big and \((Y, D_Y)\) is klt. If \( X \) is Q-factorial, set \( Y = X \). Since \(-K_Y\) is Q-Cartier and \((Y, D_Y)\) is klt, \( Y \) is log terminal. There is morphism \( f : Z \to Y \) such that \( Z \) is terminal and all exceptional divisors \( E_i \) of \( f \) have \( a(E, X) \leq 0 \) ([3, Corollary 1.4.3]). The anticanonical class of \( Z \) is big since

\[-K_Z = -f^*K_Y - \sum a(E_i, Y)E_i.\]
Suppose $V$ is a result of running the MMP on $Z$. For a divisorial contraction $h: W \to U$ with the exceptional divisor $E$ we can write

$$-h^*K_U = -K_W + aE,$$  

hence $-K_V$ is big. Clearly $K_V$ is not numerically effective so $V$ is a Mori fiber space.

Recall that $V$ described in Theorem 1.1 is the fibration over $\mathbb{P}^1$. Here we describe the conditions which we require in order for the fiber to be superrigid. They have been introduced in [8].

Let $W \subset \mathbb{P}^M$ be a hypersurface of degree $2M$, $M \geq 5$. For a nonsingular point $x \in W$ fix a system of affine coordinates $z_1, \ldots, z_M$ on $\mathbb{P}^M$ with origin at $x$ and set

$$w = q_1 + q_2 + \cdots + q_{2M}$$

to be the equation of the hypersurface $W$, where $q_i = q_i(z)$ are homogeneous polynomials of degree $\deg q_i = i$. We can assume that $q_1 = z_1$. Denote

$$\bar{q}_i = \bar{q}_i(z_2, \ldots, z_M) = q_i|_{z_1=0} = q_i(0, z_2, \ldots, z_M).$$

We say that $W$ satisfies the regularity conditions at a smooth point $x$ if the rank of the quadratic form $\bar{q}_2$ is at least 2.

**Theorem 3.2** ([6, Theorem 4]). Let $\sigma: F \to \mathbb{P}^M$, $M \geq 5$, be a double cover branched over the hypersurface $W \subset \mathbb{P}^M$ of degree $2M$ with isolated non-degenerate quadratic singularities, satisfying the regularity conditions at every smooth point. Then for every divisor $D$ and $\lambda$ such that $\lambda D + K_F \sim_\mathbb{Q} 0$ the pair $(F, D)$ is canonical.

It immediately follows from this theorem and Proposition 2.8 that any smooth regular double space $F$ of dimension $\geq 5$ is birationally superrigid.

**Corollary 3.3.** Let $F$ be a regular double space of dimension $\geq 5$. Then $\alpha(F) \geq 1$.

**Proposition 3.4** ([8, Proposition 5]). Let $W$ be the space of all hypersurfaces of degree $2M$ in $\mathbb{P}^M$. Denote the space of all regular hypersurfaces with isolated quadratic singularities by $W_{\text{reg}} \subset W$. If $M \geq 5$ then the codimension of $W \setminus W_{\text{reg}}$ is 2 or more.
Proof. Consider a variety $V$ and an incidence hypersurface $I$ in it defined by

$$V = \mathbb{P}^M \times H^0(\mathbb{P}^M, \mathcal{O}(2M)),$$

$$I = \{(x, F) \in V | F(x) = 0\}.$$ 

Let $p$ and $q$ be natural projections $p : I \to \mathbb{P}^M$ and $q : I \to H^0(\mathbb{P}^M, \mathcal{O}(2M))$. Let $Y$ be the "bad" pairs

$$Y = \{(x, F) \in I | F \text{ is not regular at } x \text{ if } x \text{ is smooth, or } x \text{ is not quadratic or is not isolated or is a degenerate quadratic singularity}\}.$$ 

To prove the proposition it is enough to show that $\text{codim } q(Y) \geq 2$. To show this it is sufficient to prove that $\text{codim } p^{-1}(x) \cap Y \geq M + 1$.

Consider the equation of $W$ in affine coordinates in the neighborhood of a point $x$

$$q_1 + q_2 + \cdots + q_{2M} = 0$$

If $q_1 = 0$ then $W$ is singular at $x$. Let $Q$ be a matrix corresponding to $q_2$. If $\det Q \neq 0$ then $x$ is a quadratic non isolated nondegenerate point of $W$. Thus, the locus of hypersurfaces of degree $2M$ for which $x$ is not a quadratic nondegenerate isolated singularity is of codimension $M + 1$ in $p^{-1}(x)$. If $q_1 \neq 0$, we may assume that $q_1 = z_1$ and take $q_2 = q_2(0, z_2, \ldots, z_M)$. The set of quadratic forms of rank $\leq 1$ in the variables $z_2, \ldots, z_M$ is of codimension

$$c(M) = \frac{(M - 1)(M - 2)}{2}.$$ 

When $M \geq 5$ we have $c(M) \geq M + 1$. \( \square \)

We provide the specific case of the following theorem for the sake of simplicity.

**Theorem 3.5 (\cite{Theorem 2.1}).** Let $\pi : V \to \mathbb{P}^1$ be a Fano fibration with $\text{Pic}(V) = K_V \cdot \mathbb{Z} \oplus F \cdot \mathbb{Z}$, where $F$ is a fiber of $\pi$. Let $V$ satisfy the $K$-condition, that is, $|nK_V|$ is not movable for any $n$. Suppose the generic fiber of $\pi$ is birationally superrigid and assume that $\alpha(F) \geq 1$ for every fiber $F$ of $\pi$. Then $V$ is birationally superrigid.

**Proof.** Let $f$ be a birational map from $V$ to a Mori fiber space $X$ and assume $f$ is not fiberwise. Consider a linear system $\mathcal{M}_X$ which defines the Fano fibration $g : X \to S$. Let $\mathcal{M}$ be the proper transform $f^{-1}\mathcal{M}_X$ of $\mathcal{M}_X$, then

$$\mathcal{M} \subset \left| -nK_V + sF \right|. \quad (2)$$
If $n = 0$ then $f$ is fiberwise and clearly $n \geq 0$ hence $n > 0$. Since $\mathcal{M}$ is a movable system it follows from the $K$-condition that $s > 0$. We can rewrite (2) as

$$K_X + \lambda \mathcal{M} \sim Q lF.$$  \hspace{1cm} (3)

Consider the resolution $W$ of the birational map $f$. Let $\psi : W \rightarrow V$ and $\varphi : W \rightarrow X$ be birational morphisms corresponding to the resolution. Denote $\mathcal{M}_W = \varphi^{-1} \mathcal{M}$, then we can write

$$\psi^*(K_V + \lambda \mathcal{M}) + \sum a_i E_i \sim_Q K_W + \mathcal{M}_W \sim_Q \varphi^*(K_X + \lambda \mathcal{M}_X) + \sum b_i E_i. $$

Intersecting this equality with a generic divisor $D \in \varphi^* \mathcal{M}$ and applying (2) we get

$$D \cdot \psi^*(lF) + \sum a_i D \cdot E_i \sim_Q D \cdot \varphi^*(K_X).$$

Since $K_X|_D$ is anti-ample $a_i < 0$ for some $i$, therefore $(V, \lambda D)$ is not a canonical pair. Since this is true for a generic divisor $D$, $(V, \lambda \mathcal{M})$ is not canonical. Denote the centre of this non-canonical singularity by $Z = \psi(E_i)$. Suppose that $Z$ does not lie on any fiber of $\pi$ and consider a generic fiber $F$. The pair $(F, \lambda \mathcal{M}|_F)$ is not canonical at $Z|_F$ and

$$\lambda \mathcal{M} \sim_Q (-K_V + lF)|_F = -K_F,$$

which contradicts the superrigidity of a generic fiber $F$. Therefore $Z$ lies on some fiber $F$. The pair $(V, F)$ is terminal since $F$ has only isolated nondegenerate quadratic singularities, so we can apply the reverse of adjunction. Hence the pair $(F, \mathcal{M}|_F)$ is not log canonical at $Z$ which contradicts the bound on the $\alpha$-invariant. \hfill \Box

**Corollary 3.6.** Let $\pi : V \rightarrow \mathbb{P}^1$ be a Fano fibration with $\text{Pic}(V) = K_V \cdot Z \oplus F \cdot Z$, where $F$ is a fiber of $\pi$. Let $V$ satisfy the $K$-condition, that is, $|nK_V|$ is not movable for any $n$. Suppose the generic fiber of $\pi$ is birationally superrigid and assume that $\alpha(F) \geq 1$ for every fiber $F$ of $\pi$. Then all birational maps from $V$ to any Mori fiber space are automorphisms $V \rightarrow V$.

**Proof.** By Theorem 3.5 $V$ is birationally superrigid therefore every map from $V$ to a Mori fiber space is a fiberwise map. By Theorem 2.9 every fiberwise map is an isomorphism. \hfill \Box

**Proof of Theorem 1.1.** It is easy to compute that

$$-K_V = g^*(L),$$

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where $L$ is a divisor of bidegree $(1, 2 - l)$. For $l \geq 3$ the $K$-condition holds. The fiber of the natural morphism $\pi : V \to \mathbb{P}^1$ is a double cover ramified over a hypersurface of degree $2M$. The fibration $\pi$ can be considered as a map $\mathbb{P}^1 \to W$ where $W$ is a space of hypersurfaces of degree $2M$. By Proposition 3.5, the codimension of the set of non-regular hypersurfaces is 2 and therefore a generic rational curve of degree $2l$ does not intersect this set. Therefore every fiber of $\pi$ is regular for generic $W$. Theorem 3.2 implies that a generic fiber is birationally rigid and $\alpha(F) \geq 1$ for every fiber $F$.

Thus, $V$ satisfies conditions of the Corollary 3.6 and we conclude that any birational map from $V$ to a Mori fiber space is an isomorphism.

**Lemma 3.7.** The anticanonical class of the variety $V$ described in Theorem 1.1 is not big.

*Proof.* We see that $-K_V = g^*(L)$, where $L$ is a divisor of bidegree $(1, 2 - l)$. Thus $-K_V$ is not big since $l \geq 3$. 

*Proof of Corollary 1.2.* It follows from Theorem 1.1 and Lemma 3.6 that there are no maps from $V$ to a Mori fiber space with big anticanonical class. Therefore, by Proposition 3.1, $V$ is not birationally equivalent to a variety of Fano type.

**Corollary 3.8.** The variety $V$ described in Theorem 1.1 is rationally connected.

*Proof.* Since $V$ is a Fano fibration over $\mathbb{P}^1$ with smooth generic fiber, it follows from Theorem 2.3 and Theorem 2.4, that $V$ is rationally connected.

**Remark 3.9.** We could also construct an example by using fibrations onto Fano hypersurfaces of index one. But it is more tiresome, a little bit harder, and provides an example only for dimension $\geq 9$.

**Remark 3.10.** We can use similar ideas in dimension 2 over non-closed fields. A minimal conic bundle with negative self intersection index is birationally rigid. We can find a birationally rigid conic bundle $X$ with $-K_X$ not big, but we do not know if that property will hold under fiberwise maps. If it could be shown that there are no minimal conic bundles with big anticanonical class and negative $K^2$ then the answer to Question 1 would be negative in dimension 2 for non-closed fields.

**Remark 3.11.** We can run $D$-MMP on a Mori dream space for any divisor $D$ which is not nef [19, Proposition 1.11]. Thus, we have another class of varieties which behave very well under $D$-MMP. It has been proven in [3, Corollary 1.3.2] that every $Q$-factorial variety of Fano type is a Mori dream.
space. The converse is not true even for smooth Mori dream spaces: there exists a smooth rational Mori dream space of dimension 2 which is not of Fano type [15, Section 3]. In fact, a Q-factorial normal projective variety is of Fano type if and only if it is a Mori dream space and spectrum of its Cox ring has only log terminal singularities [14, Theorem 1.1]. One could ask Question 1 for Mori dream spaces instead of varieties of Fano type.

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