Perfectly Secure Index Coding

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Abstract

In this paper, we investigate the index coding problem in the presence of an eavesdropper. Messages are to be sent from one transmitter to a number of legitimate receivers who have side information about the messages, and share a set of secret keys with the transmitter. We assume perfect secrecy, meaning that the eavesdropper should not be able to retrieve any information about the message set. This problem is a generalization of the Shannon’s cipher system. We study the minimum key lengths for zero-error and perfectly secure index coding problems.

1 Introduction

An index coding problem comprises of a server, \( u \) clients and a set of distinct messages \( M = \{M_1, M_2, \ldots, M_t\} \). Each client has a subset of \( M \) as its side information, and wants to learn another subset of the message set which it has not. The goal is to find the minimum number of information bits that should be broadcast by the server so that each client can recover its desired messages with zero-error probability. This minimum required bits of information is called the optimal index code length. The index coding problem was originally introduced by Birk and Kol [1] in a satellite communication scenario. Consider a satellite that broadcasts a set of messages to a number of clients. Each receiver may miss some of the messages due to limited storage capacity, lack of interest, interrupted reception, or any other reason. The clients then inform the server about the messages they desire but are missing, as well as their side information via a feedback channel, and the server attempts to deliver their requested information by broadcasting information to all the clients. Index coding studies the efficient way of satisfying the needs of clients with minimum transmission from the satellite. To illustrate the significance of index coding, consider a communication scenario with one server, two clients and a message set \( \{M_1, M_2\} \) of binary random variables. The first client has \( M_2 \) as side information and wants \( M_1 \), yet the second one has \( M_1 \) and wants \( M_2 \). The server can send the XOR of \( M_1 \) and \( M_2 \), instead of broadcasting each of them individually.

An index coding problem, in its most general case, can be represented by a directed bipartite graph [2] or a hypergraph [3]. However, it admits a simple graphical representation.
on a directed graph if each message is desired by only one client. In this case, without loss of generality one can assume that the number of receivers and messages are the same (a client that desires two different messages can be replaced with two identical clients that desire a message each). Many of the known results in the literature are for this special case, which we also adopt in this paper.

Several upper and lower bounds are known for the optimal index code length $\ell^*(G)$ [1–10]. Most of proposed bounds are graph-theoretic based, but [9] considers this problem from an information-theoretic viewpoint and computes the capacity region of index coding problem with up to five messages. When we restrict ourselves to linear operations, the optimal linear index code is equal to a graph parameter called min-rank [5, 11]. However, the computation of min-rank is NP-hard [12]. Furthermore, linear index coding can be suboptimal in general [4]. Index coding is a special case of the network coding problem. On the other hand, [13, 14] show that any network coding problem can be reduced to an index coding problem.

Security aspects of network coding has been studied in [15–18]. In particular, secure throughput of a network coding problem in the presence of an active adversary who can eavesdrop and corrupt some links are studied. A similar problem with active adversaries has been studied in [19] for the linear index coding problem.

In this paper, we study secrecy in index coding from a different perspective. Our approach is similar to that of Shannon in his seminal paper [20]. He analyzed the cipher system shown in Fig. 1 comprising of a message $M$, a cipher text $C$, and a key $K$ - a secret common randomness shared between the sender and the legitimate receiver. The sender wishes to transmit $M$ to the legitimate receiver while keeping it secret from the eavesdropper. To this end, the sender transmits $C$ (a function of $M$ and $K$) on a public noiseless channel. By receiving $C$, the eavesdropper should not be able to attain any information about $M$. Shannon adopted the notion of perfect secrecy, of statistical independence between the message and the cipher text, i.e., $I(M; C) = 0$. Moreover, Shannon assumed zero error recovery of the message: the legitimate receiver should be able to retrieve the message from $C$ and $K$, imposing the constraint $H(M|K, C) = 0$. Shannon proved that the cipher system of Fig. 1 is perfectly secure, if the following inequality is satisfied:

$$H(K) \geq H(M).$$  \hspace{1cm} (1)

Roughly speaking, perfect secrecy is possible if and only if the key length is greater than or equal to the message length. Achievability follows from the one-time pad scheme.

The goal of this paper is to derive a condition similar to inequality (1) for a general zero error and perfectly secure index coding problem (observe that Shannon’s cipher system is a special index coding problem with one receiver). Consider a scenario with $t$ legitimate receivers, an eavesdropper, and a set of keys $K$ shared between the sender and the legitimate receivers. The question is to find the minimum entropy of keys required for perfect secrecy. Moreover, the effect of perfect secrecy condition on the optimal index code length is studied.

The rest of this paper is organized as follows. In Section 2, some notation is fixed, and the system model is defined. Section 3 lays out the main results. We state the proofs in Section 4. Section 5 concludes this paper. Proof of one of the main results is moved to the Appendix.
Figure 1: Shannon cipher system.

Notation. Random variables are shown in capital letters, whereas their realizations are shown in lowercase letters. Bold letters are used to denote sets or vectors. Alphabet set of random variables are shown in calligraphic font. We use \([t]\) to denote \(\{1, 2, \ldots, t\}\) and \(X_S\) for some subset \(S\) of indices to denote the collection of \((X_s : s \in S)\). We use \([a]_+\) to denote \(a\) if it is non-negative and zero otherwise. We use the term “conventional index code” to denote a classical index coding problem with no adversary and secret keys.

2 System Model

Conventional index coding is the problem of sending a set of \(t\) messages \(M = \{M_1, M_2, \ldots, M_t\}\) to \(t\) receivers. The \(i\)-th receiver wants the message \(M_i\), having a subset of remaining messages \(M \setminus M_i = \{M_1, M_2, \ldots, M_{i-1}, M_{i+1}, \ldots, M_t\}\) as side information. The side information set of \(i\)-th receiver is shown by \(S_i\). The goal is to minimize the amount of information that should be broadcast to the receivers for decoding their desired messages without any error.

Now, assume that an eavesdropper coexists with the legitimate receivers. Just like legitimate receivers, the eavesdropper receives the index code \(C\). However, we require that the eavesdropper should not be able to obtain any information about message set \(M\) from index code \(C\) (perfect secrecy). From an information theoretic perspective, the mutual information of \(M\) and \(C\) should be zero. To accomplish this, we assume that the transmitter and the legitimate receivers share common and private secret keys. The common key \(K\) is shared among the sender and all of the legitimate receivers, and the private key \(K_i, i \in [t]\) is shared between the sender and the \(i\)-th receiver. We are interested in the minimum entropy of the keys needed for perfect secrecy.

Below, we formally define a secure index code.

Definition 1 (Secure Index Code). Consider the scenario of Fig. 1 consisting of a sender (who broadcasts data), \(t\) legitimate receivers, and an illegal receiver named as the eavesdropper. Also, assume a key set \(K = \{K, K_1, K_2, \ldots, K_t\}\) of common and private keys. A secure index coding scheme consists of an encoder and \(t\) decoders satisfying the perfect secrecy condition, defined as follows:

1- Encoder: An encoder \(f\) maps the message set \(M\) and the key set \(K\) to a code symbol \(C \in \mathcal{C}\),

\[
f : \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_t \times \mathcal{K} \times K_1 \times \cdots \times K_t \rightarrow \mathcal{C}.
\]
where $M_i$, $K$, $K_i$, and $C$ are the alphabet sets of $M_i$, $K$, $K_i$, and $C$, respectively.

2- Decoder: A decoder $g_i$, $i = 1, \cdots, t$ recovers $M_i$ from code symbol $C$, its side information $S_i$, as well as the keys $K$ and $K_i$,

$$g_i : C \times S_i \times K \times K_i \to M_i.$$  \hspace{1cm} \text{(2)}

The recovery is exact: $g_i(c, s_i, k, k_i) = m_i$. Thus, for any $i$ and arbitrary input distribution on the message set $M$, we should have:

$$H(M_i|C, S_i, K, K_i) = 0.$$  \hspace{1cm} \text{(3)}

It means that each receiver should be able to retrieve its desired message from its side information, the code $C$, as well as the keys $K$ and $K_i$ with error probability zero.

3- Perfect secrecy condition: assuming that $K$ and $K_i$ are mutually independent and uniform over their alphabet sets, the conditional pmf $p(C = c | M = m)$ should not depend on the value of $m$, for any given $c$. Equivalently, for any distribution on input message $M$, we should have

$$I(M; C) = 0, \hspace{1cm} \forall_{PM}(m)$$  \hspace{1cm} \text{(4)}

as long as the message set $M$ and the key set $K$ are mutually independent.

4- Rate vector: corresponding to a secure index code, a rate vector

$$r = (r_1, r_2, \cdots, r_t, r_k, r_k_1, \cdots, r_k_t)$$  \hspace{1cm} \text{(5)}

is defined, where

$$r_i = \frac{\log |M_i|}{\log |C|}, \quad r_k = \frac{\log |K|}{\log |C|}, \quad r_k_i = \frac{\log |K_i|}{\log |C|}.$$  \hspace{1cm} \text{(6)}

\textbf{Remark 1.} Throughout, we reserve the notation “$r_k$” for the rate of common key. It should not be confused with $r_1, r_2, \cdots, r_t$ which are message rates. When we write $r_i$ for a variable $i \in [t]$, we mean one of $r_1, r_2, \cdots, r_t$, and not $r_k$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{The schematic of secure index coding scenario.}
\end{figure}
Remark 2. A secure index code is a generalization of the conventional index code with no adversary. If we consider a zero-error index code that does not necessarily satisfy the perfect secrecy constraint, and has a rate vector of the following form,

\[ r = (r_1, r_2, \cdots, r_t, 0, 0, \cdots, 0), \]  

i.e., no secret keys exist \( r_k = r_{k_i} = 0 \), then we get a conventional zero-error index code with rate vector

\[ (r_1, r_2, \cdots, r_t). \]

Linear index codes form a subclass of the general problem, in which both encoder and decoders are linear functions.

**Definition 2 (Linear Index Code).** A linear index code includes a linear encoder and linear decoders so that:

1- Encoder: A linear function \( f \) mapping the message set \( M \) and secret keys \( K \) to a code symbol \( C \in \mathbb{F}^l \),

\[ f : \mathbb{F}^{l_1} \times \mathbb{F}^{l_2} \times \cdots \times \mathbb{F}^{l_t} \times \mathbb{F}^{l_k} \times \mathbb{F}^{l_{k_1}} \times \mathbb{F}^{l_{k_2}} \times \cdots \times \mathbb{F}^{l_{k_t}} \rightarrow \mathbb{F}^l, \]

where \( \mathbb{F} \) is a finite field, \( l_i, l_k, l_{k_i} \) and \( l \) are respectively the length of message \( M_i \), the length of the common key \( K \), the length of private key \( K_i \) and the length of index code \( C \). In other words, \( M_i, K, K_i \) and \( C \) are sequences of length \( l_i, l_k, l_{k_i} \) and \( l \) in the field \( \mathbb{F} \).

2- Decoder: A linear function \( g_i \) for \( i \in [t] \) that acts on code symbol \( C \), side information \( S_i \) and secret keys \( K, K_i \) to recover the message \( M_i \)

\[ g_i : \mathbb{F}^{l_i} \times S_i \times \mathbb{F}^{l_k} \times \mathbb{F}^{l_{k_i}} \rightarrow \mathbb{F}^{l_i}. \]

3- Rate vector: the rate vector of linear index coding is defined as follows:

\[ r = (r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_i}) \]

where

\[ r_i = \frac{l_i}{l}, \quad r_k = \frac{l_k}{l}, \quad r_{k_i} = \frac{l_{k_i}}{l}. \]

Each code symbol is a linear function of the components of \( M_i, K \) and \( K_i \), i.e.,

\[ C_i = \sum_{p=1}^{l_k} \alpha_p^i K(p) + \sum_{p=1}^{l_k} \beta_p^i K_i(p) + \sum_{j=1}^{t} \sum_{p=1}^{l_i} \gamma_{jp}^i M_j(p) \]

for some coefficients \( \alpha_p^i, \beta_p^i \) and \( \gamma_{jp}^i \) in \( \mathbb{F} \). Here, \( M_i = (M_i(1), M_i(2), \cdots, M_i(l_i)) \), \( K = (K(1), K(2), \cdots, K(l_k)) \), and \( K_i = (K_i(1), K_i(2), \cdots, K_i(l_{k_i})) \) are strings of symbols in...
Thus, the encoding scheme in linear index coding problem has the following matrix representation

\[
C = \begin{pmatrix}
(C_1) \\
(C_2) \\
\vdots \\
(C_t)
\end{pmatrix} = \begin{pmatrix}
\alpha^1_1 & \beta^1_1 & \cdots & \beta^1_t & \gamma^1_1 & \cdots & \gamma^1_t \\
\alpha^2_1 & \beta^2_1 & \cdots & \beta^2_t & \gamma^2_1 & \cdots & \gamma^2_t \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha^t_1 & \beta^t_1 & \cdots & \beta^t_t & \gamma^t_1 & \cdots & \gamma^t_t
\end{pmatrix} \begin{pmatrix}
K_1 \\
K_2 \\
\vdots \\
K_t \\
M_1 \\
\vdots \\
M_t
\end{pmatrix},
\]

where

\[
\alpha^i_1 = (\alpha^i_1, \alpha^i_2, \cdots, \alpha^i_k), \\
\beta^j_1 = (\beta^j_1, \beta^j_2, \cdots, \beta^j_{j_k}), \\
\gamma^j_1 = (\gamma^j_1, \gamma^j_2, \cdots, \gamma^j_{j_k}),
\]

which construct the code generation matrix shown by $\Pi$ throughout this paper.

**Definition 3** (One-Shot and Asymptotic Index Coding). In the one-shot case, a single use of the index coding problem is considered. In other words, there are fixed message alphabet sets $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_t$, and the goal is to find an index code with minimum amount of keys and public communication that would ensure zero-error perfect secrecy. In other words, we are looking for the set of all possible minimal rate vectors

\[
r = (r_1, r_2, \cdots, r_t, r_{k_1}, \cdots, r_{k_t}),
\]

as in (1) for fixed alphabet sets $\mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_t$.

On the other hand, the asymptotic case asks for the set of all possible rate vectors $r$ that are asymptotically achievable, i.e. there exist a sequence of zero-error and perfectly secure index codes whose rate vectors converge to $r$.

**Definition 4.** The asymptotic secure index coding region, $\mathcal{R}_{\text{Secure}}$, is defined to be the set of all asymptotically achievable tuples

\[
r = (r_1, r_2, \cdots, r_t, r_{k_1}, \cdots, r_{k_t}).
\]

The conventional asymptotic index coding region is defined similarly using the achievable rate vectors as in equation (6). We denote this regions by $\mathcal{R}$.

**Remark 3.** In spite of the fact that the asymptotic case is commonly related to vanishing instead of zero probability of error, it has been shown in [21] that in the conventional index coding (with no adversary or secret keys), zero and asymptotic error capacities are the same.

**Remark 4.** Clearly, were a rate vector $r$ one-shot achievable, it is also asymptotically achievable. Also, if $(r_1, r_2, \cdots, r_t, r_{k_1}, \cdots, r_{k_t})$ is achievable, then so is $(r_1 - \alpha_1, r_2 - \alpha_2, \cdots, r_t - \alpha_t, r_{k_1} + \beta_{k_1}, \cdots, r_{k_t} + \beta_{k_t})$ for any non-negative values of $\alpha_i$ and $\beta_k$ and $\beta_{k_i}$.
3 Main Results

Theorem 1. Given non-negative values for \( r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_t} \), the following three statements are equivalent:

(a) \( \exists \alpha > 0 : \alpha \cdot (r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_t}) \in R_{\text{Secure}} \),

\( \iff \)

(b) \( \exists \alpha > 0 : \alpha \cdot ([r_1 - r_{k_1}]_+, [r_2 - r_{k_2}]_+, \cdots, [r_t - r_{k_t}]_+, r_k, 0, \cdots, 0) \in R_{\text{Secure}} \),

\( \iff \)

(c) \( \frac{[r_1 - r_{k_1}]_+}{r_k}, \frac{[r_2 - r_{k_2}]_+}{r_k}, \cdots, \frac{[r_t - r_{k_t}]_+}{r_k} \in R \).

Similarly,

(a) \( \exists \alpha > 0 : \alpha \cdot (r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_t}) \in R_{\text{Secure-Linear}} \),

\( \iff \)

(b) \( \exists \alpha > 0 : \alpha \cdot ([r_1 - r_{k_1}]_+, [r_2 - r_{k_2}]_+, \cdots, [r_t - r_{k_t}]_+, r_k, 0, \cdots, 0) \in R_{\text{Secure-Linear}} \),

\( \iff \)

(c) \( \frac{[r_1 - r_{k_1}]_+}{r_k}, \frac{[r_2 - r_{k_2}]_+}{r_k}, \cdots, \frac{[r_t - r_{k_t}]_+}{r_k} \in R_{\text{Linear}} \).

Here, to disambiguate the special case \( r_k = 0 \) showing up in the denominator, we define \( c/0 \) to be zero if \( c = 0 \), and infinity otherwise.

Corollary 1. In the case that only private keys \( K_i, i \in [t] \) are available, i.e. \( r_k = 0 \), perfect secrecy is possible if and only if

\[ r_{k_i} \geq r_i, i \in [t]. \]

This is because if \( r_{k_i} < r_i \) for some \( i \), then \( [r_i - r_{k_i}]_+ / r_k \) will be infinity. This is a contradiction since the rates in index coding are at most one.

Clearly, \( r_{k_i} \geq r_i \) implies that we can do separate one time pad on individual messages. With this strategy, the length of public communication \( l \) will be equal to \( \sum_{i=1}^t l_{k_i} \). It turns out that we cannot achieve zero-error perfect security with \( l < \sum_{i=1}^t l_{k_i} \) in this case.

Theorem 2. Suppose we are given message alphabet sets \( \mathcal{M}_1, \mathcal{M}_2, \cdots, \mathcal{M}_t \) where \( \mathcal{M}_i = \mathbb{F}^{l_i} \) for some finite field \( \mathbb{F} \). Then, there exists a linear zero-error perfectly secure index code with key lengths \( (l_k, l_{k_1}, \cdots, l_{k_t}) \) and code length \( l \), if and only if there exists a linear zero-error conventional index code (no secrecy) with code length \( l_k \) for message sets \( \tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}_2, \cdots, \tilde{\mathcal{M}}_t \) where \( \tilde{\mathcal{M}}_i = \mathbb{F}^{[l_i - l_{k_i}]} \) in which \( [a]_+ \) is a if it is non-negative, and is zero otherwise.

4 Proofs

4.1 Proof of Theorem 1

Proof of (c)→(b) for both linear and non-linear cases: Take a conventional index code \( \mathcal{C} \) and messages \( \mathcal{M}_i \) achieving rate tuple

\[ \left( \frac{[r_1 - r_{k_1}]_+}{r_k}, \frac{[r_2 - r_{k_2}]_+}{r_k}, \cdots, \frac{[r_t - r_{k_t}]_+}{r_k}, \epsilon \right). \]
We construct a new code on the same message sets, and a common keys $K$ on the same alphabet set as $C$, i.e. $|K| = |C|$. We use one-time pad and add $C$ with the common key $K$ and broadcast it. The receivers can uncover the original $C$ since they have access to $K$, but it remains hidden from the adversary. Observe that if the original index code was linear, the new index code is also linear.

The rates of the new code is:

\[
\frac{[r_1 - r_{k_1}] + \epsilon}{r_k}, \frac{[r_2 - r_{k_2}] + \epsilon}{r_k}, \ldots
\]

where $\alpha = 1/r_k$. Letting $\epsilon$ converge to zero, we get the desired result.

**Proof of (b)$\rightarrow$(a) for both linear and non-linear cases:** For the non-linear case, it suffices to show that if

\[
\alpha \cdot (r_1, r_2, \ldots, r_t, r_k, 0, 0, \ldots, 0) \in \mathcal{R}_{\text{Secure}},
\]

then for any non-negative $r_k$ one can find some $\alpha' > 0$ such that

\[
\alpha' \cdot (r_1 + r_{k_1}, r_2 + r_{k_2}, \ldots, r_t + r_{k_t}, r_k, 0, 0, \ldots, 0) \in \mathcal{R}_{\text{Secure}}.
\]

A similar statement is sufficient for the proof of the linear case. Roughly speaking, the idea is to take a code with messages $M_i$ and a common key $K$. Then we introduce private keys $K_i$ and expand the size of the message $M_i$ by the size of $K_i$. The new $K_i$ bits of $M_i$ are securely transmitted by taking their XOR with the symbols of the private key $K_i$. Again observe that if the original index code was linear, the new index code is also linear. For a rigorous argument, assume that we start with an index code with public communication $C$. We then have $\log |M_i| = \alpha r_i \log |C|$ and $\log |K| = \alpha r_k \log |C|$ in the original code. For the new code, we set the size of the messages to be $\log |M_i| = \alpha (r_i + r_{k_i}) \log |C|$; the size of the common key to be $\log |K| = \alpha r_k \log |C|$, and the size of private keys to be $\log |K_i| = \alpha r_k \log |C|$. The size of the public communication in the new code that we construct is $\log |C| + \sum_{i=1}^{t} \log |K_i|$, as we are sending $\sum_{i=1}^{t} \log |K_i|$ additional XORs. Therefore, the rate tuple of the new code is

\[
\alpha' \cdot (r_1 + r_{k_1}, r_2 + r_{k_2}, \ldots, r_t + r_{k_t}, r_k, 0, 0, \ldots, 0) \in \mathcal{R}_{\text{Secure}}
\]

where

\[
\alpha' = \frac{\alpha \log |C|}{\log |C| + \sum_{i=1}^{t} \log |K_i|} = \frac{\alpha}{1 + \sum_{i=1}^{t} r_{k_i}}.
\]

**Proof of (b)$\rightarrow$(c) for both linear and non-linear cases:** The linear case is immediate from Theorem 2. For the non-linear case, we need to show that if

\[
\exists \alpha > 0 : \quad \alpha \cdot (r_1, r_2, \ldots, r_t, r_k, 0, \ldots, 0) \in \mathcal{R}_{\text{Secure}}
\]
Then
\[
\left(\frac{r_1}{r_k}, \frac{r_2}{r_k}, \ldots, \frac{r_t}{r_k}\right) \in \mathcal{R}.
\]

Take a secure index code with messages \(M_i\) for \(i \in [t]\) and common key \(K\) whose rate vector is close to \((r_1, r_2, \ldots, r_t, r_k, 0, \ldots, 0)\). Let \(C\) be the public communication of this code. Then \(\log |K|/\log |C|\) is close to \(r_k\) and \(\log |M_i|/\log |C|\) is close to \(r_i\). Hence, \(\log |M_i|/\log |K|\) is close to \(r_i/r_k\).

Assuming that the messages \(M_i\) for \(i \in [t]\) and common key \(K\) are uniform and mutually independent of each other, we have

\[
H(M) = H(M|C) + I(M;C) \\
= H(M|C) \\
\leq H(M,K|C) \\
= H(M|K,C) + H(K|C) \\
\leq H(M|K,C) + H(K),
\]

where equality (8) comes from perfect secrecy condition. Hence,

\[
H(K) \geq I(M;K,C) \\
= I(M;C|K) + I(M;K) \\
= I(M;C|K) \\
= H(C|K) \\
\geq \min_k H(C|K = k),
\]

where equality (9) is due to independence of \(M\) and \(K\), and inequality (10) follows from the fact that \(C\) is a function of \((M,K)\).

If we fix a value of \(K = k\), we get a zero-error index code. Therefore, there exists a zero-error index code whose public communication has length less than or equal to \(H(K) = \log |K|\). The rate vector corresponding to this index code is coordinatewise greater than or equal to

\[
\left(\frac{r_1}{r_k}, \frac{r_2}{r_k}, \ldots, \frac{r_t}{r_k}\right).
\]

Remember that \(\log |M_i|/\log |K|\) could be made as close as we desire to \(r_i/r_k\). This completes the proof.

**Proof of (a)→(b):**

We begin with the linear case, i.e.

\[
\exists \alpha > 0 : \quad \alpha \cdot (r_1, r_2, \ldots, r_t, r_k, r_{k_1}, \ldots, r_{k_t}) \in \mathcal{R}_{\text{Secure-Linear}},
\]

implies that

\[
\exists \alpha > 0 : \quad \alpha \cdot ([r_1 - r_{k_1}]_+, [r_2 - r_{k_2}]_+, \ldots, [r_t - r_{k_t}]_+, r_k, 0, \ldots, 0) \in \mathcal{R}_{\text{Secure-Linear}}.
\]
Take a sequence of linear secure zero-error index codes with rate vectors approaching
\[ \alpha \cdot (r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_t}) \]
for some \( \alpha > 0 \). Let \((l_i, l, l_k, l_{k_i})\) for \( i \in [t] \) be a code from this sequence. Then we can apply Theorem 2 to this code to construct a conventional zero-error linear index code with messages of size \([l_i - l_{k_i}]_+\) and \(l_k\) symbols of public communication. If we have a secret key of size \(l_k\), we can use one-time pad and XOR it with the \(l_k\) symbols of public communication. This implies that we can find a secure zero-error index code with messages of size \([l_i - l_{k_i}]_+\), public communication and common key of size \(l_k\). This corresponds to the following rate vector
\[
\frac{1}{l_k} \cdot ([l_1 - l_{k_1}]_+, [l_2 - l_{k_2}]_+, \cdots, [l_t - l_{k_t}]_+, l_k, 0, \cdots, 0) = \\
\frac{l}{l_k} \cdot (\frac{[l_1 - l_{k_1}]}{l}, \frac{[l_2 - l_{k_2}]}{l}, \cdots, \frac{[l_t - l_{k_t}]}{l}, \frac{l_k}{l}, 0, \cdots, 0)
\]
which tends to
\[
\frac{1}{r_k} \cdot ([r_1 - r_{k_1}]_+, [r_2 - r_{k_2}]_+, \cdots, [r_t - r_{k_t}]_+, r_k, 0, \cdots, 0).
\]

This completes the proof for the linear case. Next, we consider the general non-linear case. We need to show that
\[ \exists \alpha > 0 : \alpha \cdot (r_1, r_2, \cdots, r_t, r_k, r_{k_1}, \cdots, r_{k_t}) \in \mathcal{R}_{\text{Secure}}, \]
implies that
\[ \exists \alpha > 0 : \alpha \cdot ([r_1 - r_{k_1}]_+, [r_2 - r_{k_2}]_+, \cdots, [r_t - r_{k_t}]_+, r_k, 0, \cdots, 0) \in \mathcal{R}_{\text{Secure}}. \]

Take an arbitrary index code \(C, K, M_i\) and \(K_i\) for \( i \in [t] \). We create a new secure index code that does not have private keys and is able to securely and reliably achieve message rates \((\log |M_i| - \log |K_i|) / \log |C|\) for \( i \in [t] \) and the same common key rate \(\log |K| / \log |C|\). This would conclude the proof.

In the original code, we assume that \(M_i\)'s, \(K\) and \(K_i\)'s are mutually independent. Let us now consider a different scenario where the receivers do not have access to \(K_i\)'s. In other words, \(K_i\) for \( i \in [t] \) is simply treated as a private randomness of the transmitter. Thus, only the common key is shared with the legitimate receivers and the private keys, \(K_i\), are not available at the receivers. Fig. 3 illustrates the secure index coding scheme by ignoring the private keys in the receivers. In the figure we use \(\hat{Y}_i\) to denote the total information available at the receiver \(i\) when \(K_i\)'s are not available. Here, the adversary cannot learn anything about the messages. However, the problem is that the legitimate receivers cannot decode their intended messages.

We construct a \(t\)-input, \(t\)-output interference channel as follows: the input of the \(i\)-th transmitter is \(M_i\), and the output of the \(i\)-th receiver is \(\hat{Y}_i\). Using the result of [22] p.
133] by treating interference as noise, rates \((R_1, \ldots, R_t)\) is asymptotically achievable with repeated use of this interference channel, if \(R_i \leq I(M_i; \hat{Y}_i)\). Observe that

\[
I(M_i; \hat{Y}_i) = I(M_i; \hat{Y}_i, K_i) - I(M_i; K_i|\hat{Y}_i)
\]

\[
= I(M_i; Y_i) - I(M_i; K_i|\hat{Y}_i)
\]

\[
\overset{(a)}{=} H(M_i) - I(M_i; K_i|\hat{Y}_i)
\]

\[
\geq H(M_i) - H(K_i),
\]

where \((a)\) follows from the fact that \(M_i\) is a function of \(Y_i\) as the receiver \(i\) can recover \(M_i\).

Therefore, messages of rates \(H(M_i) - H(K_i)\) can be sent with \(N\) uses of the original code. The input distribution on \(M_i^N\) will be uniform over the codewords, which is no longer uniform. However, the adversary would not learn anything about the messages since perfect security constraint holds as long as the common key is uniform and mutually independent of the messages; the marginal distribution of the messages is not important (see equation (3) and the justification given for it).

The more serious difficulty is that we only constructed a code with asymptotically zero probability of error, not exactly zero probability of error as required in our model. The following lemma resolves this difficulty:

**Lemma 1.** Assume that \((r_1, r_2, \cdots, r_t, r_k, 0, \cdots, 0)\) is achievable by a sequence of perfectly secure codes whose probabilities of error converge to zero asymptotically. We also allow the transmitter to use private randomization in these codes. Then there is some \(\alpha > 0\) such that \(\alpha \cdot (r_1, r_2, \cdots, r_t, r_k, 0, \cdots, 0)\) is achievable by a sequence of perfectly secure and zero-error codes, without using private randomization at the transmitter.

**Proof of Lemma 1.** Take an \(\epsilon\)-error code with corresponding variables \(K, C,\) and \(M_i\) for \(i \in [t]\) where \(M_i\) and \(K\) are uniform and mutually independent random variables. Also let \(\hat{M}_i\) to be the reconstruction by receiver \(i\). Since private randomization at the transmitter is allowed, \(C\) is not necessarily a deterministic function of \((K, M)\).
As before, we have
\[
H(M) = H(M|C) + I(M;C)
\]
\[
= H(M|C)
\]
\[
\leq H(M,K|C) = H(M|K,C) + H(K|C)
\]
\[
\leq H(M|K,C) + H(K), \quad (11)
\]
where equality (11) comes from perfect secrecy condition. Hence,
\[
H(K) = I(M;K)
\]
\[
= I(M;C|K) + I(M;K)
\]
\[
= I(M;C|K) \quad (12)
\]
where equality (13) is due to independence of \(M\) and \(K\). Hence \(H(K) \geq I(M;C|K)\).

Thus, the rate vector of the code is
\[
\left( \frac{H(M_1)}{\log |C|}, \frac{H(M_2)}{\log |C|}, \ldots, \frac{H(M_t)}{\log |C|}, \frac{H(K)}{\log |C|}, 0, 0, \ldots, 0 \right) = \frac{I(M;C|K)}{\log |C|} \left( \frac{H(M_1)}{I(M;C|K)}, \frac{H(M_2)}{I(M;C|K)}, \ldots, \frac{H(M_t)}{I(M;C|K)}, \frac{H(K)}{I(M;C|K)}, 0, 0, \ldots, 0 \right)
\]
Since \(H(K)/I(M;C|K) \geq 1\), to show that we can reach the rate vector
\[
\left( \frac{H(M_1)}{I(M;C|K)}, \frac{H(M_2)}{I(M;C|K)}, \ldots, \frac{H(M_t)}{I(M;C|K)}, \frac{H(K)}{I(M;C|K)}, 0, 0, \ldots, 0 \right)
\]
with perfectly secure zero-error codes, it suffices to show that there is a sequence of perfectly secure zero-error codes whose rate vectors converge to
\[
\frac{H(M_1)}{I(M;C|K)}, \frac{H(M_2)}{I(M;C|K)}, \ldots, \frac{H(M_t)}{I(M;C|K)}, 1, 0, 0, \ldots, 0
\]
But the rate of \(r_k = 1\) means that the size of common key and public communication are the same. Therefore one can always use one-time pad to ensure perfect security. It only remains to show that there is a sequence of conventional zero-error codes whose rate vectors converge to
\[
\left( \frac{H(M_1)}{I(M;C|K)}, \frac{H(M_2)}{I(M;C|K)}, \ldots, \frac{H(M_t)}{I(M;C|K)} \right)
\]
It has been shown in [21] that in the conventional index coding, zero and asymptotic error capacities are exactly the same. Therefore, it remains to show that the following claim:

**Claim:** There is a sequence of conventional codes with vanishing error probabilities whose rate vectors converge to

\[
\left( \frac{H(M_1)}{I(M;C|K)} , \frac{H(M_2)}{I(M;C|K)}, \ldots , \frac{H(M_t)}{I(M;C|K)} \right).
\]

Remember that we had started with a secure \( \epsilon \)-error code with corresponding variables \( K, C, \) and \( M_i \) for \( i \in [t] \) where \( M_i \) and \( K \) are uniform and mutually independent random variables. From the perspective of the legitimate parties and a shared common randomness \( K \) independent of the messages. We assume that the receiver \( i \) uses decoding function, as in equation (2),

\[ g_i : C \times S_i \times K \rightarrow M_i, \]

to produce \( \hat{M}_i \).

The above code induces a joint distribution \( p(M, C, K, \hat{M}) \). Let us take \( n \) i.i.d. repetitions of \( (M, K) \). We would like to use the covering lemma [22, Lemma 3.3]. If \( R = I(M;C|K) + \epsilon' \), there is a codebook \( \hat{C}_k^n(1), \hat{C}_k^n(2), \ldots , \hat{C}_k^n(2^{nR}) \) of sequences in \( C^n \) for each \( k^n \), such that with high probability, given \( k^n, m^n \), one can find an index \( j \) where \( (\hat{C}_k^n(j), k^n, m^n) \) are jointly typical according to \( p(C, K, M) \).

Now, let us construct a conventional index code (no secrecy) with messages \( M_i^n \) for \( i \in [t] \) and a shared common randomness \( K^n \) among all the parties. Having observed \( (k^n, m^n) \), the transmitter finds an index \( j \) where \( (\hat{C}_k^n(j), k^n, m^n) \) are jointly typical. Index \( j \) is sent over the public channel. Sending this index requires only \( I(M;C|K) + \epsilon' \) bits on average.

Let us denote \( \hat{C}_k^n(j) \) by \( c^n \). Now, receiver \( i \) gets a sequence \( c^n \), the common randomness \( K^n \) and its side information about other user’s messages. The decoder applies \( n \) copies of the same decoding function \( g_k(\cdot) \) to the sequences \( c^n, k^n \) and its side information about the messages (as if we were running \( n \) identical copies of the original code and \( c^n \) was \( n \) copies of the message from the \( n \) instances of the code). This results in reconstructions \( \hat{m}^n \) that is jointly typical with \( (c^n, k^n, m^n) \) with high probability according to \( p(M, C, K, \hat{M}) \). This implies that in particular, \((\hat{m}^n, m^n)\) will be jointly typical according to \( p(M, \hat{M}) \) with high probability. But since in the pmf induced by the code, error probability \( P(M \neq \hat{M}) \leq \epsilon \), \((\hat{m}^n, m^n)\) are jointly typical only if \( \hat{m}(j) = m(j) \) for \((1-\epsilon)n \) values of \( j \in [n] \).

Therefore, we have shown so far that with transmission of \( R = n(I(M;C|K) + \epsilon') \) bits, we can ensure that with high probability, \( M^n \) matches \( \hat{M}^n \) on \((1-\epsilon)\) fraction of its entries. However, we need the whole \( M^n \) to be equal to \( \hat{M}^n \) with high probability. We resolve this below, but observe that since the length of the messages are \( H(M_i^n) = nH(M_i) \), we have indeed reached the index code rate

\[
\left( \frac{H(M_1)}{I(M;C|K) + \epsilon'} , \frac{H(M_2)}{I(M;C|K) + \epsilon'}, \ldots , \frac{H(M_t)}{I(M;C|K) + \epsilon'} \right).
\]

Let us go back to the fact that with high probability \( 1 - \delta \), we have that \( M^n \) matches \( \hat{M}^n \) on \((1-\epsilon)\) fraction of its entries, and not entirely. We show that this can be fixed with a
negligible decrease in index coding rates. The idea is that by Fano's inequality

\[ \frac{1}{n} H(M^n | \hat{M}^n) \leq \frac{1}{n} + \delta H(M) + (1 - \delta) \epsilon H(M) \]

can be made as close as we want to zero. Thus, using Slepian-Wolf theorem, conveying \( M \) with side information \( \hat{M}^n \) at the decoder will require negligible amount of communication. To achieve this, one has to take \( N \) i.i.d. repetitions of \( M^n \) and \( \hat{M}^n \), and then use the Slepian-Wolf theorem to ensure that repetitions of \( M^n \) are recovered with high probability.

4.2 Proof of Theorem \[ \square \]

Assume that there exists a zero-error secure linear index code with key lengths \( l_k, l_{k_i} (i \in [t]) \).

Without loss of generality, we can assume that there is no zero-error secure index code with key lengths \((l'_k, l'_{k_1}, \ldots, l'_{k_t}) \neq (l_k, l_{k_1}, \ldots, l_{k_t})\) satisfying \( l'_k \leq l_k \) and \( l'_{k_i} \leq l_{k_i} \). This assumption then implies that the code matrix \( \Pi \) given in equation (7) has no all-zero column. Otherwise, there exist a key bit or a message bit which has not been used in producing the index code.

Our goal is to use elementary row and other valid operations to convert the code matrix \( \Pi \) to the following form, while preserving decodability and security of the code.

\[
\begin{pmatrix}
    \Lambda^{(0)} & 0 & 0 & \ldots & 0 \\
    0 & \Lambda^{(1)} & 0 & \ldots & 0 \\
    0 & 0 & \Lambda^{(2)} & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & \Lambda^{(t)}
\end{pmatrix}
\]

(14)

where \( \Lambda^{(0)} = I_{l_k \times l_k}, \Lambda^{(i)} = I_{l_{k_i} \times l_{k_i}} \) are identity matrices, and \( \Gamma \) is a \( l \times (\sum_{i=1}^{t} l_i) \) submatrix, which gets multiplied by the message vector.

Let us start with the first row and first column of the code matrix \( \Pi \). With elementary row operations, we switch the row whose first entry is non-zero with the first row. If there exist more than one row with this property, we arbitrarily choose one of them. By normalizing the first row, we make the first element of it to be one. We then set the first component of all the remaining rows to be zero, with elementary row operations. The first row corresponds to a linear equation of the form:

\[
K(1) + \sum_{p=2}^{l_k} \alpha_p^1 K(p) + \sum_{p=1}^{l_{k_1}} \beta^1_{p} K_{i}(p) + \sum_{j=1}^{t} \sum_{p=1}^{l_{k_i}} \gamma^1_{jp} M_{j}(p),
\]

(15)

where \( \alpha_p^1, \beta^1_{p} \) and \( \gamma^1_{jp} \) are coordinates of the first row. We claim that setting the coefficients \( \alpha_p^1 \) to zero, while keeping \( \beta^1_{p} \) and \( \gamma^1_{jp} \) unchanged, results in a valid code matrix, i.e. we
change the equation for the first row to

\[ K(1) + \sum_{p=1}^{l_k} \beta_p K_i(p) + \sum_{j=1}^{t} \sum_{p=1}^{l_k} \gamma^j_{2p} M_j(p). \]

(16)

First of all, the new code is secure since \( K(1) \) appears only in the first row of the code matrix and its XOR ensures that nothing will be leaked to the adversary from the first row. Furthermore, since \( K \) was a common secret key and all the receivers can compute \( \sum_{p=2}^{l_k} \alpha^i_p K(p) \), the legitimate receiver can add \( \sum_{p=2}^{l_k} \alpha^i_p K(p) \) to (16) to get back (15). Hence, the new code is both secure and reliable.

By continuing the same approach for \( K(2), K(3), \ldots, K(l_k) \), we can create the first diagonal matrix \( \Lambda^{(0)} = I_{l_k \times l_k} \) and make the matrix look as follows:

\[
\begin{pmatrix}
\Lambda^{(0)} & \ast & \ast & \ldots & \ast \\
0 & \ast & \ast & \ldots & \ast \\
0 & \ast & \ast & \ldots & \ast \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ast & \ldots & \ast \\
\end{pmatrix}
\begin{pmatrix}
\Gamma_1 \\
\end{pmatrix},
\]

(17)

In the second step, we consider the coefficients of \( K_1(1) \), and similarly put one row whose \( l_k + 1 \) entry is non-zero as the \( l_k + 1 \) row, and using the elementary row operations, we set the \( l_k + 1 \)-th component of all the remaining rows to be zero. Note that the row we are looking for already exists in \( i \)-th row for some \( i \geq l_k + 1 \), and the submatrix \( \Lambda^{(0)} \) remains unchanged. Assume this is not the case, \( i.e., \) all the rows from \( l_k + 1 \) onward have a zero coordinate at the \( l_k + 1 \)-th location. Then, we claim that we can shorten the length of the private key \( K_1 \) by one, while keeping the code both reliable and secure. This would contradict the assumption regarding the minimality of the keys. To shorten \( K_1 \), let us fix \( K_1(1) = 0 \) and reveal it to all parties (including the adversary). This clearly does not affect decodability of the receivers, since we are giving extra partial information about \( K_1 \) to the receivers 2 to \( t \) who do not know \( K_1 \). It also does not compromise the security of the code since equations in the first \( l_k \) rows were already made secure by the \( l_k \) bits of the common key \( K \). Since each coordinate of \( K \) appears in only one of the first \( l_k \) rows, it is not possible for the adversary to use rows \( l_k + 1 \) onwards to cancel out these symbols of \( K \) and obtain some information about the messages.

By continuing the same approach, we may convert the coding matrix to the following form

\[
\begin{pmatrix}
\Lambda^{(0)} & 0 & \ast & \ldots & \ast \\
0 & \Lambda^{(1)} & \ast & \ldots & \ast \\
0 & 0 & \ast & \ldots & \ast \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ast & \ldots & \ast \\
\end{pmatrix}
\begin{pmatrix}
\Gamma_2 \\
\end{pmatrix},
\]

(18)

We continue this procedure with \( K_2 \). For \( K_2(1) \) we look for a row \( i \) whose \( l_k + l_{k_1} + 1 \) entry is non-zero. Such a row exists because if column \( l_k + l_{k_1} + 1 \) is all zero, it will contradict
the minimality assumption. But we need the row index \( i \) to be greater than or equal to \( l_k + l_{k_1} + 1 \), so that elementary row operations do not affect submatrices \( \Lambda^{(0)} \) and \( \Lambda^{(1)} \). But if there is no \( i \geq l_k + l_{k_1} + 1 \), we can shorten the key length of \( K_2 \) by one: we set \( K_2(1) = 0 \) and reveal it to all the parties. It does not make the index code unsecure since the first \( l_k + l_{k_1} \) equations were already made secure by the keys \( K \) and \( K_1 \).

By continuing this procedure, we can convert the code matrix to one of the form \( (14) \). Therefore, if we are given message alphabet sets \( M_1, M_2, \ldots, M_t \) where \( M_i = \mathbb{F}^{l_i} \) for some finite field \( \mathbb{F} \), and a linear zero-error perfectly secure index code with key lengths \((l_k, l_{k_1}, \cdots, l_{k_t}) \) and code length \( l \) for sending the messages; Then, there exists another linear zero-error perfectly secure index code for the same message sets that uses secret keys of lengths \((l_k, l_{k_1}, \cdots, l_{k_t}) \) with the following property: each of the \( l \) symbols of the public message are of the form

\[
C_i = K(p) + \sum_{j=1}^{t} \sum_{p=1}^{l_i} \gamma_{jp}^i M_j(p)
\] (19)

for some \( p \in [l_k] \), or

\[
C_i = K_i(p) + \sum_{j=1}^{t} \sum_{p=1}^{l_i} \gamma_{jp}^i M_j(p)
\] (20)

for some \( i \in [t] \) and \( p \in [l_{k_1}] \). In other words, the expression of each of the code symbols \( C_i \) contains only one symbol from one of the secret keys.

Consider the first receiver. It has access to \( l \) linear equations of the form given in \( (19) \) (as it has \( K \)), and \( l_1 \) linear equations of the form given in \( (20) \) (as it has \( K_1 \)). Therefore, we call the \( l \) equations as public to all receiver, and the \( l_1 \) equations as private to receiver one.

We now use Lemma 2 with \( X = M_1 \) and \( Y = (M_2, M_3, \cdots, M_t) \), \( AX + BY \) being equations of the form given in \( (19) \), and \( CX + DY \) being the equations of the form given in \( (20) \). This lemma then implies that there is a subset of entries of \( M_1 \) of size at most \( l_1 \) such that from the values of these entries and the \( l \) public equations, receiver one can recover \( M_1 \). Let us fix \( M_1 \) on these \( l_1 \) locations and reveal its value to all the receivers. The number of free entries of \( M_1 \), i.e., the new length of the message of \( M_1 \), would then be greater than or equal to \( l - l_1 \). This message can be decoded by the first receiver using the \( l \) public linear equations of the form given in \( (19) \). The fact that we have fixed some of entries of \( M_1 \) and given it to other receivers can only help them recover their messages (because if they did not know \( M_1 \) we are giving them some partial information about \( M_1 \)). A similar procedure can be done for other receivers. This would imply that with \( l \) linear equations, it is possible for receiver \( i \) to recover \( l - l_i \) symbols using \( l \) public symbols of message. This is the claim we wanted to prove. The proof is complete.

**Lemma 2.** Let \( X_{1 \times n} \) and \( Y_{1 \times m} \) be two arbitrary column vectors in a field \( \mathbb{F} \). Assume that matrices \( A_{1 \times n}, B_{1 \times m}, C_{1 \times n} \) and \( D_{1 \times m} \) are such that the vector \( X \) can be recovered from the values of \( AX + BY \) and \( CX + DY \). Then, there is a subset of indices \( S \subset [n] \) with \( |S| \leq l_1 \), such that it is possible to find \( X \) from \( AX + BY \) and \( X(i), i \in S \). Here \( X(i) \) is used to denote the \( i \)-th entry of vector \( X \).
Proof. Consider the first row of $CX + DY$, which is a linear equation in terms of the entries of $X$ and $Y$, say $\sum \alpha_i X(i) + \sum \beta_j Y(j)$. If we can find $X$ without having access to this row, we discard it and proceed to the second row. Otherwise, there is an entry of $X$, say $i_1$ that cannot be decoded without the linear equation $\sum \alpha_i X(i) + \sum \beta_j Y(j)$. In other words, $X(i_1)$ is a linear combination of the linear equations that we have, with the equation $\sum \alpha_i X(i) + \sum \beta_j Y(j)$ being given a non-zero weight. Then if we put $i_1$ in the set $S$ of the entries that we know, we can conversely use it to recover the linear equation $\sum \alpha_i X(i) + \sum \beta_j Y(j)$. Therefore, having $X(i_1)$ is equivalent to having $\sum \alpha_i X(i) + \sum \beta_j Y(j)$. Continuing with this procedure, we can construct the set $S$ and its size will be less than or equal to the number of rows of $CX + DY$, which is $l_1$. □

5 Conclusion

In this paper, we studied the index coding problem in the presence of an eavesdropper. Assuming that a common as well as a set of dedicated private keys are shared between the transmitter and legitimate receivers, we obtained a condition on keys’ entropies by which the index code could be transmitted securely. As a future work, one can study the effect of adversary’s side information and/or capability of corrupting the public communication.

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