DOI-HOPF MODULES AND YETTER-DRINFELD MODULES FOR QUASI-HOPF ALGEBRAS

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Abstract. For a quasi-Hopf algebra $H$, a left $H$-comodule algebra $B$ and a right $H$-module coalgebra $C$ we will characterize the category of Doi-Hopf modules $C\mathcal{M}(H)_B$ in terms of modules. We will also show that for an $H$-bicomodule algebra $A$ and an $H$-bimodule coalgebra $C$ the category of generalized Yetter-Drinfeld modules $\mathcal{YD}(H)^C_A$ is isomorphic to a certain category of Doi-Hopf modules. Using this isomorphism we will transport the properties from the category of Doi-Hopf modules to the category of generalized Yetter-Drinfeld modules.

Introduction

Recall that the defining axioms for a quasi-bialgebra $H$ are the same as for a bialgebra, with the coassociativity of the comultiplication replaced by a weaker property, called quasi-coassociativity: the comultiplication is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$, called the reassociator. There are important differences with ordinary quasi-bialgebras: the definition of a quasi-bialgebra is not selfdual, and we cannot consider comodules over quasi-bialgebras, since they are not coassociative coalgebras. However, the category of (left or right) modules over a quasi-bialgebra is a monoidal category. Using this categorical point of view, the category of relative Hopf modules has been introduced and studied in [6]. A right $H$-module coalgebra $C$ is a coalgebra in the monoidal category $\mathcal{M}_H$, and a left $[C, H]$-Hopf module is a left $C$-comodule in the monoidal category $\mathcal{M}_H$. A generalization of this concept was presented in [3]: replacing the right $H$-action by an action of a left $H$-comodule algebra, we can define the notion of Doi-Hopf module over a quasi-bialgebra. At this point, we have to mention that there is a philosophical problem with the introduction of $H$-comodule algebras: we cannot introduce them as algebras in the category of $H$-comodules, since this category does not exist, as we mentioned above. However, a formal definition of $H$-comodule (and $H$-bicomodule) algebra was given by Hausser and Nill in [14]. A more conceptual definition has been proposed in [3]. If $\mathfrak{A}$ is an associative algebra then the category of $(\mathfrak{A} \otimes H, \mathfrak{A})$-bimodules is monoidal. Moreover, $\mathfrak{A} \otimes H$ has a coalgebra structure within this monoidal category “compatible” with the unit element $1_{\mathfrak{A}} \otimes 1_H$ if and only if $\mathfrak{A}$ is a right $H$-comodule algebra (for the complete statement see Proposition 1.1 below). Of course, a similar result holds

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for a left $H$-comodule algebra (see Proposition 1.3). Moreover, if $H$ is a quasi-Hopf algebra then any $H$-bicomaodule algebra can be viewed in two different (but twist equivalent) ways as a left (right) $H$-comodule algebra.

The aim of this paper is to study the category of Doi-Hopf modules over a quasi-Hopf algebra $H$, and its connections to the category of Yetter-Drinfeld modules.

If $H$ is a quasi-bialgebra, $\mathcal{B}$ a left $H$-comodule algebra and $C$ a right $H$-module coalgebra then $\mathcal{B} \otimes C$ is a $\mathcal{B}$-coring (this means a coalgebra in the monoidal category of $\mathcal{B}$-bimodules) and the category of right-left $(H, \mathcal{B}, C)$-Hopf modules, denoted by $C \mathcal{M}(H)_{\mathcal{B}}$, is isomorphic to the category of right comodules over the coring $\mathcal{B} \otimes C$, conform [3] Theorem 5.4]. In particular, since $\mathcal{B} \otimes C$ is flat as a left $\mathcal{B}$-module we obtain that $C \mathcal{M}(H)_{\mathcal{B}}$ is a Grothendieck category.

It was shown in [3, Proposition 5.2] that the category $C \mathcal{M}(H)_{\mathcal{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^* \triangleright \mathcal{B}$, if $C$ is finite dimensional. In Section 2 we look at the case where $C$ is infinite dimensional. Following the methods developed in [9, 11, 19], we will present two characterizations of the category of Doi-Hopf modules $C \mathcal{M}(H)_{\mathcal{B}}$. We first introduce the notion of rational (right) $C^* \triangleright \mathcal{B}$-module, and then we will show that the category $C \mathcal{M}(H)_{\mathcal{B}}$ is isomorphic to $\text{Rat}(\mathcal{M}_{C^* \triangleright \mathcal{B}})$, the category of rational (right) $C^* \triangleright \mathcal{B}$-modules.

We notice that, in the coassociative case, the notion of rational (right) $C^* \triangleright \mathcal{B}$-module reduces to the notion of right $C^* \triangleright \mathcal{B}$-module which is rational as a right $C^*$-module. Secondly, we will show that $\text{Rat}(\mathcal{M}_{C^* \triangleright \mathcal{B}})$ is equal to $\sigma_{C^* \triangleright \mathcal{B}}(C \otimes \mathcal{B})$, the smallest closed subcategory of $\mathcal{M}_{C^* \triangleright \mathcal{B}}$ containing $C \otimes \mathcal{B}$ (see Theorem 2.6). In this way we recover that $C \mathcal{M}(H)_{\mathcal{B}}$ is a Grothendieck category, a fortiori with enough injective objects. We will also introduce a generalized version of Koppinen’s smash product [10], relate it to the generalized smash product, and characterize the category of Doi-Hopf modules as the full subcategory of modules over the Koppinen smash product, consisting of rational modules.

In Section 3, we will generalize a result from [8]. If $H$ is a quasi-Hopf algebra, then an $H$-bicomaodule algebra $\mathcal{A}$ can be viewed in two different, but twist equivalent, ways as a right $H^{op} \otimes H$-comodule algebra. To this end, we first prove that any left $H$-comodule algebra $\mathcal{B}$ can be turned into a right $H^{op}$-comodule algebra. So, by this correspondence, the two (twist equivalent) left $H \otimes H^{op}$-comodule algebra structures on $\mathcal{A}$ obtained in [7] provide two (twist equivalent) right $H^{op} \otimes H$-comodule algebra structures on $\mathcal{A}$, which we will denote by $\mathcal{A}^1$ and $\mathcal{A}^2$. If $C$ is an $H$-bimodule coalgebra (that is, a coalgebra in the monoidal category of $H$-bimodules), then $C$ becomes in a natural way a left $H^{op} \otimes H$-module coalgebra, thus it makes sense to consider the Hopf module category $\mathcal{A}^2 \mathcal{M}(H^{op} \otimes H)^C$. The main result of Section 3 asserts that the category of generalized left-right Yetter-Drinfeld modules $\mathcal{A}^2 \mathcal{Y}D(H)^C$ is isomorphic to $\mathcal{A}^1 \mathcal{M}(H^{op} \otimes H)^C$, and also to $\mathcal{A}^1 \mathcal{M}(H^{op} \otimes H)^C$. Using the first isomorphism, we will characterize $\mathcal{A}^2 \mathcal{Y}D(H)^C$ as a category of comodules over a coring. In Section 3.3, we will characterize the category of Yetter-Drinfeld modules as a category of modules.

1. Preliminary results

1.1. Quasi-Hopf algebras. We work over a commutative field $k$. All algebras, linear spaces etc. will be over $k$; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [12], a quasi-bialgebra is a fourtuple $(H, \Delta, \varepsilon, \Phi)$, where $H$ is an associative algebra with unit, $\Phi$ is an invertible element in $H \otimes H \otimes H$, and $\Delta: H \rightarrow H \otimes H$ and $\varepsilon: H \rightarrow k$
are algebra homomorphisms satisfying the identities

\begin{align}
(1.1) & \quad (id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1}, \\
(1.2) & \quad (id \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes id)(\Delta(h)) = h,
\end{align}

for all \( h \in H; \Phi \) has to be a normalized 3-cocycle, in the sense that

\begin{align}
(1.3) & \quad (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi), \\
(1.4) & \quad (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1.
\end{align}

The map \( \Delta \) is called the coproduct or the comultiplication, \( \varepsilon \) the counit and \( \Phi \) the reassociator. As for Hopf algebras \cite{20} we denote \( \Delta(h) = h_1 \otimes h_2 \), but since \( \Delta \) is only quasi-coassociative we adopt the further convention (summation implicitly understood):

\begin{equation}
(\Delta \otimes id)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},
\end{equation}

for all \( h \in H \). We will denote the tensor components of \( \Phi \) by capital letters, and those of \( \Phi^{-1} \) by small letters, namely

\begin{align*}
\Phi &= X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \cdots, \\
\Phi^{-1} &= x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \cdots.
\end{align*}

A quasi-Hopf algebra is a quasi-bialgebra \( H \) equipped with an anti-automorphism \( S \) of the algebra \( H \) and elements \( \alpha, \beta \in H \) such that

\begin{align}
(1.5) & \quad S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1 \beta S(h_2) = \varepsilon(h)\beta, \\
(1.6) & \quad X^1 \beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2 \beta S(x^3) = 1,
\end{align}

for all \( h \in H \).

The antipode of a quasi-Hopf algebra is determined uniquely up to a transformation \( \alpha \mapsto U\alpha, \beta \mapsto \beta U^{-1}, S(h) \mapsto US(h)U^{-1} \), with \( U \in H \) invertible. The axioms imply that \( \varepsilon(\alpha)\varepsilon(\beta) = 1 \), so, by rescaling \( \alpha \) and \( \beta \), we may assume without loss of generality that \( \varepsilon(\alpha) = \varepsilon(\beta) = 1 \) and \( \varepsilon \circ S = \varepsilon \). The identities \cite{12,14} also imply that

\begin{equation}
(\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1.
\end{equation}

If \( H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta) \) is a quasi-bialgebra or a quasi-Hopf algebra then \( H^{\text{cop}} \), \( H^{\text{cop}} \) and \( H^{\text{cop}} \) are also quasi-bialgebras (respectively quasi-Hopf algebras), where the superscript “op” means opposite multiplication and “cop” means opposite co-multiplication. The structure maps are obtained by putting \( \Phi_{\text{op}} = \Phi^{-1}, \Phi_{\text{cop}} = (\Phi^{-1})^{321}, \Phi_{\text{op,cop}} = \Phi^{321}, S_{\text{op}} = S_{\text{cop}} = (S_{\text{op,cop}})^{-1} = S^{-1}, \alpha_{\text{op}} = S^{-1}(\beta), \beta_{\text{op}} = S^{-1}(\alpha), \alpha_{\text{cop}} = S^{-1}(\beta), \beta_{\text{cop}} = S^{-1}(\alpha) \). The definition of a quasi-bialgebra \( H \) is designed in such a way that the categories of left and right representations over \( H \) are monoidal (see \cite{15,13} for the terminology). Let \( (H, \Delta, \varepsilon, \Phi) \) be a quasi-bialgebra. For are left (resp. right) \( H \)-modules \( U, V, W \), the associativity constraints \( a_{U,V,W} \) (resp. \( a_{U,V,W} \) : \( U \otimes V \otimes W \to U \otimes (V \otimes W) \) are given by the formulas

\begin{align*}
a_{U,V,W}((u \otimes v) \otimes w) &= \Phi \cdot (u \otimes (v \otimes w)); \\
a_{U,V,W}((u \otimes v) \otimes w) &= (u \otimes (v \otimes w)) \cdot \Phi^{-1}.
\end{align*}

Next we recall that the definition of a quasi-bialgebra or quasi-Hopf algebra is “twist covariant” in the following sense. An invertible element \( F \in H \otimes H \) is called a gauge transformation or twist if \( (\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1 \). If \( H \) is
a quasi-bialgebra or a quasi-Hopf algebra and $F = F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = G^1 \otimes G^2$, then we can define a new quasi-bialgebra (respectively quasi-Hopf algebra) $H_F$ by keeping the multiplication, unit, counit (and antipode in the case of a quasi-Hopf algebra) of $H$ and replacing the comultiplication, reassociator and the elements $\alpha$ and $\beta$ by

\begin{align}
\Delta_F(h) &= F \Delta(h) F^{-1}, \\
\Phi_F &= (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \\
\alpha_F &= S(G^1)\alpha G^2, \quad \beta_F = F^1 \beta S(F^2).
\end{align}

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

\begin{equation}
f \Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{cop}(h)), \quad \text{for all } h \in H.
\end{equation}

The element $f$ can be computed explicitly. First set

\begin{align}
A^1 \otimes A^2 \otimes A^3 \otimes A^4 &= (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \\
B^1 \otimes B^2 \otimes B^3 \otimes B^4 &= (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1),
\end{align}

and then define $\gamma, \delta \in H \otimes H$ by

\begin{equation}
\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = B^1 \beta S(B^4) \otimes B^2 \beta S(B^3).
\end{equation}

Then $f$ and $f^{-1}$ are given by the formulae

\begin{align}
f &= (S \otimes S)(\Delta^{cop}(x^1))\gamma \Delta(x^2 \beta S(x^3)), \\
f^{-1} &= \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{cop}(x^3)).
\end{align}

Furthermore the corresponding twisted reassociator (see (1.8)) is given by

\begin{equation}
\Phi_f = (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1).
\end{equation}

1.2. Comodule and bicomodule algebras. A formal definition of comodule algebras over a quasi-bialgebra was given by Hausser and Nill [14].

**Definition 1.1.** Let $H$ be a quasi-bialgebra. A right $H$-comodule algebra is a unital associative algebra $A$ together with an algebra morphism $\rho : A \to A \otimes H$ and an invertible element $\Phi_\rho \in A \otimes H \otimes H$ such that:

\begin{align}
\Phi_\rho(\rho \otimes id)(\rho(a)) &= (id \otimes \Delta)(\rho(a))\Phi_\rho, \quad \text{for all } a \in A, \\
(1_A \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) &= (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho), \\
(id \otimes \varepsilon) \circ \rho &= id, \\
(id \otimes \varepsilon \otimes id)(\Phi_\rho) &= (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_A \otimes 1_H.
\end{align}

In a similar way, a left $H$-comodule algebra is a unital associative algebra $B$ together with an algebra morphism $\lambda : B \to H \otimes B$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes B$.
such that the following relations hold:

\[(1.18) \quad (id \otimes \lambda)(\lambda(b))\Phi_\lambda = \Phi_\lambda(\Delta \otimes id)(\lambda(b)), \quad \forall \ b \in \mathcal{B}, \]
\[(1.19) \quad (1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes id)(\Phi_\lambda)(\Phi \otimes 1_\mathcal{B}) \]
\[(1.20) \quad (id \otimes \varepsilon \otimes \lambda)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_\lambda), \]
\[(1.21) \quad (id \otimes \varepsilon \otimes id)(\Phi_\lambda) = (\varepsilon \otimes id \otimes id)(\Phi_\lambda) = 1_H \otimes 1_\mathcal{B}. \]

Observe that \( H \) is a left and right \( H \)-comodule algebra: take \( \rho = \lambda = \Delta, \Phi_\rho = \Phi_\lambda = \Phi \).

If \((\mathcal{B}, \lambda, \Phi_\lambda)\) is a left \( H \)-comodule algebra then

- \((\mathcal{B}, \lambda \circ \tau_{H,\mathcal{B}}, (\Phi_\lambda^{-1})^{321})\) is a right \( H^{cop} \)-comodule algebra;
- \((\mathcal{B}^{cop}, \lambda \circ \tau_{H,\mathcal{B}}, \Phi_\lambda^{321})\) is a right \( H^{cop}_{\lambda} \)-comodule algebra;
- \((\mathcal{B}^{cop}, \lambda, \Phi_\lambda^{-1})\) is a left \( H^{cop}_{\lambda} \)-comodule algebra,

and vice versa. \( \tau_{X,Y} : X \otimes Y \to Y \otimes X \) is the switch map mapping \( x \otimes y \) to \( y \otimes x \).

From [14], we recall the notion of twist equivalence for the coaction on a right \( H \)-comodule algebra \((\mathfrak{A}, \rho, \Phi_\rho)\): if \( V \in \mathfrak{A} \otimes H \) is an invertible element such that

\[(1.22) \quad (id_{\mathfrak{A}} \otimes \varepsilon)(V) = 1_{\mathfrak{A}} \]

then we can construct a new right \( H \)-comodule algebra \((\mathfrak{A}, \rho', \Phi_{\rho'})\) with

\[(1.23) \quad \rho'(a) = \forall \rho(a)V^{-1} \]

and

\[(1.24) \quad \Phi_{\rho'} = (id_{\mathfrak{A}} \otimes \Delta)(V)\Phi_\rho(\rho \otimes id_H)(V^{-1})(V^{-1} \otimes 1_H). \]

We say that \((\mathfrak{A}, \rho, \Phi_\rho)\) and \((\mathfrak{A}, \rho', \Phi'_{\rho'})\) are twist equivalent right \( H \)-comodule algebras.

We obviously have a similar notion for left \( H \)-comodule algebras. More precisely, if \((\mathcal{B}, \lambda, \Phi_\lambda)\) is a left \( H \)-comodule algebra and \( U \in H \otimes \mathcal{B} \) is an invertible element such that \( (\varepsilon \otimes id_{\mathcal{B}})(U) = 1_{\mathcal{B}} \) then we have a new left \( H \)-comodule algebra \((\mathcal{B}, \lambda', \Phi_{\lambda'})\) with \( \lambda'(b) = U(b)\lambda(b)^{-1} \), for all \( b \in \mathcal{B} \), and

\[(\mathcal{B}, \lambda, \Phi_\lambda) \text{ and } (\mathcal{B}, \lambda', \Phi_{\lambda'}) \text{ are twist equivalent.} \]

For a right \( H \)-comodule algebra \((\mathfrak{A}, \rho, \Phi_\rho)\) we will use the following Sweedler-type notation, for any \( a \in \mathfrak{A} \).

\[\rho(a) = a_{(0)} \otimes a_{(1)}, \quad (\rho \otimes id)(\rho(a)) = a_{(0,0)} \otimes a_{(0,1)} \otimes a_{(1)} \text{ etc.}\]

Similarly, for a left \( H \)-comodule algebra \((\mathcal{B}, \lambda, \Phi_\lambda)\), if \( b \in \mathcal{B} \), we adopt the following notation:

\[\lambda(b) = b_{[-1]} \otimes b_{[0]}, \quad (id \otimes \lambda)(\lambda(b)) = b_{[-1]} \otimes b_{[0,-1]} \otimes b_{[0,0]} \text{ etc.}\]

In analogy with the notation for the reassociator \( \Phi \) of \( H \), we will write

\[\Phi_\rho = X_\rho^1 \otimes X_\rho^2 \otimes X_\rho^3 = Y_\rho^1 \otimes Y_\rho^2 \otimes Y_\rho^3 = \cdots\]

and

\[\Phi_\rho^{-1} = \tilde{X}_\rho^1 \otimes \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^3 = \tilde{Y}_\rho^1 \otimes \tilde{Y}_\rho^2 \otimes \tilde{Y}_\rho^3 = \cdots.\]

We use a similar notation for the element \( \Phi_\lambda \) of a left \( H \)-comodule algebra \( \mathcal{B} \).
If \( H \) is an associative bialgebra and \( \mathfrak{A} \) is an ordinary right \( H \)-comodule algebra, then \( \mathfrak{A} \otimes H \) is a coalgebra in the monoidal category of \( \mathfrak{A} \)-bimodules. The quasi-bialgebra analog of this property was given in [3]. Let \( H \) be a quasi-bialgebra and \( \mathfrak{A} \) a unital associative algebra. We define by \( \mathfrak{A} \otimes H \mathfrak{A} \) the category of \( \mathfrak{A} \)-bimodules and \((H, \mathfrak{A})\)-bimodules \( M \) such that \( h \cdot (a \cdot m) = a \cdot (h \cdot m) \), for all \( a \in \mathfrak{A}, h \in H \) and \( m \in M \). Morphisms are left \( H \)-linear maps which are also \( \mathfrak{A} \)-bimodule maps. It is not hard to see that \( \mathfrak{A} \otimes H \mathfrak{A} \) is a monoidal category. The tensor product is \( \otimes \mathfrak{A} \) and for any two objects \( M, N \in \mathfrak{A} \otimes H \mathfrak{A} \), \( M \otimes \mathfrak{A} N \) is an object of \( \mathfrak{A} \otimes H \mathfrak{A} \) via 

\[(a \otimes h) \cdot (m \otimes \mathfrak{A} n) \cdot a' = a \cdot (h_1 \cdot m) \otimes \mathfrak{A} h_2 \cdot n \cdot a'\]

for all \( m \in M, n \in N, a, a' \in \mathfrak{A} \), and \( h \in H \). The associativity constraints are given by 

\[
\Delta_{M,N,P} : (M \otimes \mathfrak{A} N) \otimes \mathfrak{A} P \to M \otimes \mathfrak{A} (N \otimes \mathfrak{A} P),
\]

\[
\Delta_{M,N,P}(m \otimes \mathfrak{A} n \otimes \mathfrak{A} p) = X^1 \cdot m \otimes \mathfrak{A} (X^2 \cdot n \otimes \mathfrak{A} X^3 \cdot p);
\]

the unit object is \( \mathfrak{A} \) viewed as a trivial \( H \)-module, and the left and right unit constraints are the usual ones. Now, the definition of a comodule algebra in terms of monoidal categories can be restated as follows.

**Proposition 1.2.** ([3 Proposition 3.8]) Let \( H \) be a quasi-bialgebra and \( \mathfrak{A} \) an algebra, and view \( \mathfrak{A} \otimes H \) in the canonical way as an object in \( \mathfrak{A} \otimes H \mathfrak{A} \). There is a bijective correspondence between coalgebra structures \((\mathfrak{A} \otimes H, \overline{\Delta}, \overline{\varepsilon})\) in the monoidal category \( \mathfrak{A} \otimes H \mathfrak{A} \) such that \( \overline{\Delta}(1_\mathfrak{A} \otimes 1_H) \) is invertible and \( \overline{\varepsilon}(1_\mathfrak{A} \otimes 1_H) = 1_\mathfrak{A} \), and right \( H \)-comodule algebra structures on \( \mathfrak{A} \).

A similar result holds for a left \( H \)-comodule algebra \( \mathfrak{B} \). Denote by \( \mathfrak{B} \mathfrak{M}_{\mathfrak{B} \otimes H} \) the category whose objects are \( \mathfrak{B} \)-bimodules and \((\mathfrak{B}, H)\)-bimodules \( M \) such that \((m \cdot b) \cdot h = (m \cdot h) \cdot b\) for all \( m \in M, b \in \mathfrak{B} \) and \( h \in H \). Morphisms are right \( H \)-linear maps which are also \( \mathfrak{B} \)-bimodule maps. We can easily check that \( \mathfrak{B} \mathfrak{M}_{\mathfrak{B} \otimes H} \) is a monoidal category with tensor product \( \otimes \mathfrak{B} \) given via \( \Delta \), this means 

\[b \cdot (m \otimes \mathfrak{B} n) \cdot (b' \otimes h) := b \cdot m \cdot h_1 \otimes \mathfrak{B} (n \cdot h_2) \cdot b'\]

for all \( M, N \in \mathfrak{B} \mathfrak{M}_{\mathfrak{B} \otimes H}, m \in M, n \in N, b, b' \in \mathfrak{B} \) and \( h \in H \). The associativity constraints are given by 

\[
\Delta'_{M,N,P} : (M \otimes \mathfrak{B} N) \otimes \mathfrak{B} P \to M \otimes \mathfrak{B} (N \otimes \mathfrak{B} P),
\]

\[
\Delta'_{M,N,P}(m \otimes \mathfrak{B} n \otimes \mathfrak{B} p) = m \cdot x^1 \otimes \mathfrak{B} (n \cdot x^2 \otimes \mathfrak{B} p \cdot x^3),
\]

the unit object is \( \mathfrak{B} \) viewed as a right \( H \)-module via \( \varepsilon \), and the left and right unit constraints are the usual ones.

**Proposition 1.3.** Let \( H \) be a quasi-bialgebra and \( \mathfrak{B} \) an algebra, and view \( \mathfrak{B} \otimes H \) in the canonical way as an object in \( \mathfrak{M}_{\mathfrak{B} \otimes H} \). There is a bijective correspondence between coalgebra structures \((\mathfrak{B} \otimes H, \overline{\Delta}, \overline{\varepsilon})\) in the monoidal category \( \mathfrak{B} \mathfrak{M}_{\mathfrak{B} \otimes H} \) such that \( \overline{\Delta}(1_\mathfrak{B} \otimes 1_H) \) is invertible and \( \overline{\varepsilon}(1_\mathfrak{B} \otimes 1_H) = 1_\mathfrak{B} \), and left \( H \)-comodule algebra structures on \( \mathfrak{B} \).

**Proof.** Since the proof is similar to the one given in [3 Proposition 3.8], we restrict ourselves to a description of the correspondence. Suppose that \((\mathfrak{B} \otimes H, \overline{\Delta}, \overline{\varepsilon})\) is a coalgebra in \( \mathfrak{B} \mathfrak{M}_{\mathfrak{B} \otimes H} \) such that \( \overline{\Delta}(1_\mathfrak{B} \otimes 1_H) \) is invertible and \( \overline{\varepsilon}(1_\mathfrak{B} \otimes 1_H) = 1_\mathfrak{B} \). Write 

\[
\overline{\Delta}(1_\mathfrak{B} \otimes 1_H) = (1_\mathfrak{B} \otimes \overline{X}_1^1) \otimes \mathfrak{B} (\overline{X}_1^2 \otimes \overline{X}_1^3),
\]
and consider \( \Phi_\lambda = \hat{\lambda}^1 \otimes \hat{\lambda}^2 \otimes \hat{\lambda}^3 \). Define \( \lambda : \rightarrow H \otimes B \) by
\[
\lambda(b) = \tau_{B,H}(b \cdot (1_B \otimes 1_H)),
\]
for all \( b \in B \). Then \( (B, \lambda, \Phi_\lambda) \) is a left \( H \)-comodule algebra.

Conversely, if \( (B, \lambda, \Phi_\lambda) \) is a left \( H \)-comodule algebra then \( B \otimes H \in B \mathcal{M}_{B \otimes H} \)
via
\[
b \cdot (b' \otimes h) : (b'' \otimes h') = b[0] b'' \cdot b[1] hh'.
\]

Moreover, \( B \otimes H \) is a coalgebra in \( B \mathcal{M}_{B \otimes H} \), with comultiplication and counit given by the formulas
\[
\Delta(b \otimes h) = (1_B \otimes \hat{X}_1^1 h_1) \otimes_B (\hat{X}_1^2 b \otimes \hat{X}_1^3 h_2),
\]
\[
\varepsilon(b \otimes h) = \varepsilon(h)b,
\]
for all \( b \in B \) and \( h \in H \).

Let \( B \) be a left \( H \)-comodule algebra, and consider the elements \( \tilde{\rho}_\lambda \) and \( \tilde{\eta}_\lambda \) in \( H \otimes B \) given by the following formulas:
(1.25) \[
\tilde{\rho}_\lambda = \tilde{\rho}_\lambda^1 \otimes \tilde{\rho}_\lambda^2 = \hat{\lambda}^2 S^{-1}(\hat{\lambda}^1 \beta) \otimes \hat{\lambda}^3, \quad \tilde{\eta}_\lambda = S(\hat{\lambda}^1) \Delta(\hat{\lambda}^3) \otimes \hat{\lambda}^3.
\]

Then we have the following formulas, for all \( b \in B \) (see [14]):
(1.26) \[
\lambda(b[0])\tilde{\rho}_\lambda[S^{-1}(b[-1]) \otimes 1_B] = \tilde{\rho}_\lambda[1_H \otimes b],
\]
(1.27) \[
[S(b[-1]) \otimes 1_H] \tilde{\eta}_\lambda(b[0]) = [1 \otimes b] \tilde{\eta}_\lambda,
\]
(1.28) \[
\lambda(\tilde{\rho}_\lambda^1) \tilde{\rho}_\lambda[S^{-1}(\tilde{\rho}_\lambda^2) \otimes 1_B] = 1_H \otimes 1_B,
\]
(1.29) \[
[S(\tilde{\rho}_\lambda^1) \otimes 1_B] \tilde{\eta}_\lambda(\tilde{\rho}_\lambda^3) = 1_H \otimes 1_B,
\]
(1.30) \[
\Phi^{-1}(id_B \otimes \lambda)(\tilde{\rho}_\lambda)(1_B \otimes \tilde{\rho}_\lambda)
\]
(1.31) \[
= [S(\tilde{\rho}_\lambda^1) \otimes S(\tilde{\rho}_\lambda^3) \otimes 1_B][f \otimes 1_B](\Delta \otimes id_B)(\tilde{\eta}_\lambda \lambda(\tilde{\rho}_\lambda^3)).
\]

Bicomodule algebras where introduced by Hausser and Nill in [14], under the name “quasi-commuting pair of \( H \)-coactions”.

**Definition 1.4.** Let \( H \) be a quasi-bialgebra. An \( H \)-bicomodule algebra \( \Lambda \) is a quintuple \((\Lambda, \lambda, \rho, \Phi_\lambda, \Phi_\rho)\), where \( \lambda \) and \( \rho \) are left and right \( H \)-coactions on \( \Lambda \), and where \( \Phi_\lambda \in H \otimes H \otimes \Lambda, \Phi_\rho \in \Lambda \otimes H \otimes H \) and \( \Phi_{\lambda,\rho} \in H \otimes \Lambda \otimes H \) are invertible elements, such that
- \((\Lambda, \lambda, \Phi_\lambda)\) is a left \( H \)-comodule algebra,
- \((\Lambda, \rho, \Phi_\rho)\) is a right \( H \)-comodule algebra,
- the following compatibility relations hold, for all \( u \in \Lambda \):
(1.32) \[
(\Phi_{\lambda,\rho}(\lambda \otimes id)(\rho(u)) = (id \otimes \rho)(\lambda(u))\Phi_{\lambda,\rho}
\]
(1.33) \[
(1_H \otimes \Phi_{\lambda,\rho})(id \otimes \lambda \otimes id)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H)
\]
(1.34) \[
(1_H \otimes \Phi_{\rho})(id \otimes \rho \otimes id)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H)
\]
It was pointed out in [14] that the following additional relations hold in an \( H \)-bicomodule algebra \( \Lambda \):
(1.35) \[
(id_H \otimes id_\Lambda \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_\Lambda, \quad (\varepsilon \otimes id_\Lambda \otimes id_H)(\Phi_{\lambda,\rho}) = 1_\Lambda \otimes 1_H.
\]
As a first example, take $\mathbb{A} = H$, $\lambda = \rho = \Delta$ and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi$.

Let $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ be an $H$-bicomodule algebra; it is not hard to show that

- $(\mathbb{A}, \rho \circ \tau, \lambda \circ \tau, (\Phi_{\rho}^{-1})^{321}, (\Phi_\lambda)^{321}, (\Phi_{\lambda,\rho})^{321})$ is an $H^{\text{cop}}$-bicomodule algebra,
- $(\mathbb{A}^{\text{op}}, \rho \circ \tau, \lambda \circ \tau, \Phi_\rho^{321}, \Phi_\lambda^{321}, \Phi_{\lambda,\rho}^{321})$ is an $H^{\text{cop}}$-bicomodule algebra,
- $(\mathbb{A}^{\text{op}}, \lambda, \Phi_\rho^{-1}, \Phi_\rho^{-1}, \Phi_{\lambda,\rho}^{-1})$ is an $H^{\text{op}}$-bicomodule algebra.

We will use the following notation:

$$
\Phi_{\lambda,\rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \overline{\Theta}^1 \otimes \overline{\Theta}^2 \otimes \overline{\Theta}^3;
$$

$$
\Phi_{\lambda,\rho}^{-1} = \theta^1 \otimes \theta^2 \otimes \theta^3 = \overline{\theta}^1 \otimes \overline{\theta}^2 \otimes \overline{\theta}^3.
$$

Let $H$ be a quasi-Hopf algebra and $\mathbb{A}$ an $H$-bicomodule algebra. We define two left $H \otimes H^{\text{op}}$-coactions $\lambda_1, \lambda_2 : \mathbb{A} \to (H \otimes H^{\text{op}}) \otimes \mathbb{A}$ on $\mathbb{A}$, as follows:

$$
\lambda_1(u) = (u_{<0>_1} \otimes S^{-1}(u_{<1>_2})) \otimes u_{<0>_2},
$$

$$
\lambda_2(u) = (u_{<1>_1} \otimes S^{-1}(u_{<0>_2})) \otimes u_{<0>_1},
$$

for all $u \in \mathbb{A}$. We also consider the following elements $\Phi_{\lambda_1}, \Phi_{\lambda_2} \in (H \otimes H^{\text{op}}) \otimes (H \otimes H^{\text{op}}) \otimes \mathbb{A}$:

$$
\Phi_{\lambda_1} = \left( \Theta^1 \overline{X}^1(\overline{x}^1_{\rho}), \otimes \Theta^2 \overline{X}^2(\overline{x}^2_{\rho} g) \right) \otimes \Theta^3 \overline{X}^3(\overline{x}^3_{\rho} g^3)
$$

$$
\otimes \left( \Theta^1_{\rho} \Theta^2_{\rho} \Theta^3_{\rho} \right)(x_{\rho} \otimes y_{\rho} \otimes z_{\rho}),
$$

$$
\Phi_{\lambda_2} = \left( \overline{Y}^1 \otimes S^{-1}(\theta^1 \overline{Y}^1 \overline{g}) \overline{Y}^3 \right)(x_{\rho} \otimes y_{\rho} \otimes z_{\rho})
$$

$$
\otimes \left( \theta^1 \overline{g} \otimes S^{-1}(\theta^2 \overline{g} \overline{Y}^2 \overline{Y}^3) \right)(x_{\rho} \otimes y_{\rho} \otimes z_{\rho}).
$$

It was proved in [7] that $(\mathbb{A}, \lambda_1, \Phi_{\lambda_1})$ and $(\mathbb{A}, \lambda_2, \Phi_2)$ are twist equivalent left $H \otimes H^{\text{op}}$-comodule algebras. In particular, if $H$ is a quasi-Hopf algebra then the notion of $H$-bicomodule algebra can be restated in terms of monoidal categories. In Section 3, we will see that $\mathbb{A}$ can be also viewed in two twist equivalent ways as a right $H^{\text{op}} \otimes H$-comodule algebra.

## 2. Doi-Hopf modules and rationality properties

### 2.1. Doi-Hopf modules.

The category of left (right) modules over a quasi-bialgebra is monoidal. A coalgebra in $\mathcal{M}$ (resp. $\mathcal{M}_H$) is called a left (right) $H$-module coalgebra. Thus a left $H$-module coalgebra is a left $H$-module $C$ together with a comultiplication $\Delta : C \to C \otimes C$ and a counit $\varepsilon : C \to k$ such that

\begin{align*}
(1.1) & \quad \Phi(\Delta \otimes id_C)(\Delta(c)) = (id_C \otimes \Delta)(\Delta(c)), \\
(1.2) & \quad \Delta(h \cdot c) = h_1 \otimes c_1 \otimes h_2 \cdot c_2, \\
(1.3) & \quad \varepsilon(h \cdot c) = \varepsilon(h)\varepsilon(c),
\end{align*}

for all $c \in C$ and $h \in H$. Similarly, a right $H$-module coalgebra $C$ is a right $H$-module together with a comultiplication $\Delta : C \to C \otimes C$ and a counit $\varepsilon : C \to k$, satisfying the following relations

\begin{align*}
(2.4) & \quad (\Delta \otimes id_C)(\Delta(c))\Phi^{-1} = (id_C \otimes \Delta)(\Delta(c)), \\
(2.5) & \quad \Delta(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \\
(2.6) & \quad \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h),
\end{align*}
for all $c \in C$ and $b \in H$. Here we used the Sweedler-type notation

$$\Delta(c) = c_1 \otimes c_2, \quad (\Delta \otimes id_C)(\Delta(c)) = c_{(1,1)} \otimes c_{(1,2)} \otimes c_2, \quad \text{etc.}$$

It is easy to see that a left $H$-module coalgebra $C$ is in a natural way a right $H^{op}$-module coalgebra (and vice versa).

Let $H$ be a quasi-bialgebra and $C$ a right $H$-module coalgebra. A left $[C,H]$-Hopf module is a left $C$-comodule in the monoidal category $\mathcal{M}_H$. This definition was generalized in $\mathfrak{B}$.

**Definition 2.1.** Let $H$ be a quasi-bialgebra over a field $k$, $C$ a right $H$-module coalgebra and $(\mathfrak{B}, \lambda, \Phi)$ a left $H$-comodule algebra. A right-left $(H, \mathfrak{B}, C)$-Hopf module (or Doi-Hopf module) is a $k$-module $M$, with the following additional structure: $M$ is right $\mathfrak{B}$-module (the right action of $\mathfrak{B}$ on $M$ is denoted by $m \cdot b$), and we have a $k$-linear map $\lambda_M : M \rightarrow C \otimes M$, such that the following relations hold, for all $m \in M$ and $b \in \mathfrak{B}$:

\begin{align}
(2.7) \quad (\Delta \otimes id_M)(\lambda_M(m)) &= (id_C \otimes \lambda_M)(\lambda_M(m))\Phi,
(2.8) \quad (\varepsilon \otimes id_M)(\lambda_M(m)) &= m,
(2.9) \quad \lambda_M(m \cdot b) &= m_{(-1)} \cdot b_{[-1]} \otimes m_{(0)} \cdot b_{[0]}.
\end{align}

As usual, we use the Sweedler-type notation $\lambda_M(m) = m_{(-1)} \otimes m_{(0)}$. $C\mathcal{M}(H)_{\mathfrak{B}}$ is the category of right-left $(H, \mathfrak{B}, C)$-Hopf modules and right $\mathfrak{B}$-linear, left $C$-colinear $k$-linear maps.

Let $M$ be a right $\mathfrak{B}$-module; then $C \otimes M$ is a right-left $(H, \mathfrak{B}, C)$-Hopf module, with structure maps given by the following formulas

\begin{align}
(2.10) \quad (c \otimes m) \cdot b &= c \cdot b_{[-1]} \otimes m \cdot b_{[0]},
(2.11) \quad \lambda_{C \otimes M}(c \otimes m) &= c_1 \cdot \hat{x}_1 \otimes c_2 \cdot \hat{x}_2 \otimes m \cdot \hat{x}_3,
\end{align}

for all $c \in C$, $b \in \mathfrak{B}$ and $m \in M$. We obtain a functor $F = C \otimes \cdot : \mathcal{M}_\mathfrak{B} \rightarrow C\mathcal{M}(H)_{\mathfrak{B}}$. The functor $F$ sends a morphism $\vartheta$ to $id_C \otimes \vartheta$. In particular, $C \otimes \mathfrak{B} \in C\mathcal{M}(H)_{\mathfrak{B}}$, via the structure maps

$$\lambda_{C \otimes \mathfrak{B}}(c \otimes b) = c_1 \cdot \hat{x}_1 \otimes c_2 \cdot \hat{x}_2 \otimes b \hat{x}_3,$$

for all $c \in C$ and $b, b' \in \mathfrak{B}$.

The functor $F$ has a left and a right adjoint, so it is an exact functor.

**Proposition 2.2.** Let $H$ be a quasi-bialgebra, $\mathfrak{B}$ a left $H$-comodule algebra and $C$ a right $H$-module coalgebra. Then the functor $F = C \otimes \cdot$ is a right adjoint of the forgetful functor

$$C\mathfrak{U} : C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_\mathfrak{B},$$

and a left adjoint of the functor

$$\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, \cdot) : C\mathcal{M}(H)_{\mathfrak{B}} \rightarrow \mathcal{M}_\mathfrak{B}$$

defined as follows. For $M \in C\mathcal{M}(H)_{\mathfrak{B}}$, $\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, M)$ is a right $\mathfrak{B}$-module via the formula

$$(\eta \cdot b)(c \otimes b') = \eta(c \otimes bb'),$$

for all $\eta \in \text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, M)$, $c \in C$ and $b, b' \in \mathfrak{B}$. For a morphism $\kappa : M \rightarrow N$ in $C\mathcal{M}(H)_{\mathfrak{B}}$, we let

$$\text{Hom}_{\mathfrak{B}}^C(C \otimes \mathfrak{B}, \kappa)(v) = \kappa \circ v,$$
for all \( v \in \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, M) \).

Proof. Let \( M \) be a right-left \((H, \mathcal{B}, C)\)-Hopf module and \( N \) a right \( \mathcal{B} \)-module. Define

\[
\xi_{M,N} : \text{Hom}_\mathcal{B}^C(M, N) \rightarrow \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N)), \quad \xi_{M,N}(\zeta)(m) = m_{(-1)} \otimes \zeta(m_{(0)}),
\]

for all \( \zeta \in \text{Hom}_\mathcal{B}^C(M, N) \) and \( m \in M \), and

\[
\zeta_{M,N} : \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N)) \rightarrow \text{Hom}_\mathcal{B}^C(M, N), \quad \zeta_{M,N}(\chi)(m) = (\zeta \otimes id_N)(\chi(m)),
\]

for all \( \chi \in \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N)) \) and \( m \in M \). It is not hard to see that \( \xi_{M,N} \) and \( \zeta_{M,N} \) are well-defined natural transformations that are inverse to each other. For \( M \in \mathcal{M}_\mathcal{B} \) and \( N \in C\mathcal{M}(H)_\mathcal{B} \), we define

\[
\xi'_{M,N} : \text{Hom}_\mathcal{B}^C(C \otimes M, N) \rightarrow \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N))
\]

by

\[
\xi'_{M,N}(\zeta')(m \otimes b) = \zeta'(c \otimes m \cdot b)
\]

and

\[
\zeta'_{M,N} : \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N)) \rightarrow \text{Hom}_\mathcal{B}^C(C \otimes M, N)
\]

by

\[
\zeta'_{M,N}(\chi')(c \otimes m) = \chi'(m)(c \otimes 1_\mathcal{B}),
\]

for all \( \zeta' \in \text{Hom}_\mathcal{B}^C(C \otimes M, N) \), \( \chi' \in \text{Hom}_\mathcal{B}^C(M, \text{Hom}_\mathcal{B}^C(C \otimes \mathcal{B}, N)) \), \( m \in M \), \( c \in C \) and \( b \in \mathcal{B} \). Then \( \xi' \) and \( \zeta' \) are well-defined natural transformations that are inverse to each other. \( \square \)

2.2. Doi-Hopf modules and comodules over a coring. It was proved in \([8]\) that the category of right-left Doi-Hopf modules is isomorphic to the category of right comodules over a suitable \( \mathcal{B} \)-coring. For a general treatment of the theory of corings, we refer to \([1, 2]\).

Let \( R \) be a ring with unit. An \( R \)-coring \( C \) is an \( R \)-bimodule together with two \( R \)-bimodule maps \( \Delta_C : C \rightarrow C \otimes R C \) and \( \varepsilon_C : C \rightarrow R \) such that the usual coassociativity and counit properties hold. A right \( C \)-comodule is a right \( R \)-module together with a right \( R \)-linear map \( \rho_M^r : M \rightarrow M \otimes R C \) such that

\[
(\rho_M^r \otimes R id_C) \circ \rho_M^r = (id_M \otimes_R \Delta_C) \circ \rho_M^r,
\]

\[
(id_M \otimes_R \varepsilon_C) \circ \rho_M^r = id_M.
\]

A map \( h : M \rightarrow N \) between two right \( C \)-comodules is called right \( C \)-colinear if \( h \) is right \( R \)-linear and \( \rho_N^r \circ h = (h \otimes_R id_C) \circ \rho_M^r \). \( \mathcal{M}^C \) will be the category of right \( C \)-comodules and right \( C \)-comodule maps. It is well-known that \( \mathcal{M}^C \) is a Grothendieck category (in particular with enough injective objects) if \( C \) is flat as a left \( R \)-module. The category \( \mathcal{C}_M \) of left \( C \)-comodules and left \( C \)-comodule maps can be introduced in a similar way.

Let \( H \) be a quasi-bialgebra, \((\mathcal{B}, \lambda, \Phi_\lambda)\) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra. It was proved in \([8]\) that \( C = \mathcal{B} \otimes C \) is a \( \mathcal{B} \)-coring. More precisely, \( C \) is a \( \mathcal{B} \)-bimodule via

\[
b \cdot (b' \otimes c) = bb' \otimes c \quad \text{and} \quad (b \otimes c) \cdot b' = bb'[0] \otimes c \cdot b'[1],
\]

and the comultiplication and counit are given by

\[
\Delta_C(b \otimes c) = (b\hat{x}^3_\lambda \otimes c_2 \cdot \hat{x}^2_\lambda) \otimes_{\mathcal{B}} (1_{\mathcal{B}} \otimes c_1 \cdot \hat{x}^1_\lambda)
\]
and
\[ \varepsilon_C(c \otimes b) = \varepsilon(c)b, \]
for all \( b, b' \in \mathcal{B} \) and \( c \in C \). Then we have an isomorphism of categories (see [3] Theorem 5.4)
\[ C^\mathcal{M}(H)_{\mathcal{B}} \cong \mathcal{M}^C. \]
\( \mathcal{C} \) is free, and therefore flat, as a left \( \mathcal{B} \)-module, and we conclude that \( C^\mathcal{M}(H)_{\mathcal{B}} \cong \mathcal{M}^C \) is a Grothendieck category, and has therefore enough injectives (see [13] Prop. 1.2] or [2] 18.14).

2.3. Doi-Hopf modules and the generalized smash product. We now want to discuss when the category of Doi-Hopf modules is isomorphic to a module category. In the case where \( C \) is finite dimensional, this was already done in [3] Proposition 5.2. Let us explain this more precisely.

Let \( H \) be a quasi-bialgebra. A left \( H \)-module algebra \( A \) is an algebra in the monoidal category \( H \mathcal{M} \). This means that \( A \) is a left \( H \)-module with a multiplication \( A \otimes A \to A \) and a unit element \( 1_A \) satisfying the following conditions:
\begin{align}
(2.12) \quad (aa')a'' &= (X^1 \cdot a)(X^2 \cdot a')(X^3 \cdot a''), \\
(2.13) \quad h \cdot (aa') &= (h_1 \cdot a)(h_2 \cdot a'), \\
(2.14) \quad h \cdot 1_A &= \varepsilon(h)1_A,
\end{align}
for all \( a, a', a'' \in A \) and \( h \in H \). Following [3], we can define the generalized smash product of a left \( H \)-module algebra \( A \) and a left \( H \)-comodule algebra \( \mathcal{B} \): \( A \triangleright \mathcal{B} = A \otimes B \) as a vector space, with multiplication
\[ (a \triangleright b)(a' \triangleright b') = (\tilde{x}_1 \cdot a)(\tilde{x}_2 b_{[-1]} \cdot a') \triangleright \tilde{x}_3 b_{[0]} b', \]
for all \( a, a' \in A \), \( b, b' \in \mathcal{B} \). \( A \triangleright \mathcal{B} \) is an associative algebra with unit \( 1_A \triangleright 1_B \).

The linear dual \( C^* \) of a right \( H \)-module coalgebra \( C \) is a left \( H \)-module algebra. The multiplication is the convolution, the unit is \( \varepsilon \) and the left \( H \)-action is given by the formula \( (h \cdot c')(c) = c'(c \cdot h) \), for all \( h \in H \), \( c' \in C^* \) and \( c \in C \). So we can consider the generalized smash product algebra \( C^* \triangleright \mathcal{B} \). Moreover, we have a functor
\[ \mathcal{G} : \ C^\mathcal{M}(H)_{\mathcal{B}} \to \mathcal{M} C^* \triangleright \mathcal{B} \quad \mathcal{G}(M) = M, \]
with right \( C^* \triangleright \mathcal{B} \)-action given by
\[ m \cdot (c^* \triangleright b) = c^*(m_{(-1)})m_{[0]} \cdot b, \]
for all \( m \in M \), \( c^* \in C^* \) and \( b \in \mathcal{B} \). If \( C \) is finite dimensional, then \( \mathcal{G} \) is an isomorphism of categories (see [3]).

Now let \( C \) be infinite dimensional. We will show that the category \( C^\mathcal{M}(H)_{\mathcal{B}} \) is isomorphic to the category \( \sigma_C \triangleright \mathcal{B} \)[C \otimes \mathcal{B}] \]. Recall that if \( \mathcal{A} \) is a Grothendieck category and \( M \) is an object of \( \mathcal{A} \) then \( \sigma_{\mathcal{A}}[M] \) is the class of all objects \( N \in \mathcal{A} \) which are subgenerated by \( M \), that is, \( N \) is a subobject of a quotient of direct sums of copies of \( M \). It is well-known that \( \sigma_{\mathcal{A}}[M] \) is the smallest closed subcategory of \( \mathcal{A} \) containing \( M \), and that for any closed subcategory \( \mathcal{D} \) of \( \mathcal{A} \) there exists an object \( M \) of \( \mathcal{A} \) such that \( \mathcal{D} = \sigma_{\mathcal{A}}[M] \). In particular, \( \sigma_{\mathcal{A}}[M] \) is a Grothendieck category, and has a fortiori enough injective objects. For more detail, the reader is invited to consult [10] 21.
Let \( H \) be a quasi-bialgebra, \( C \) a right \( H \)-module coalgebra and \( \mathcal{B} \) a left \( H \)-comodule coalgebra. For a right \( C^*\triangleright\mathcal{B} \)-module \( M \) we define the linear maps

\[
\mu_M : M \rightarrow \text{Hom}(C^* \otimes \mathcal{B}, M), \quad \nu_M : C \otimes M \rightarrow \text{Hom}(C^* \otimes \mathcal{B}, M),
\]

\[
\mu_M(m)(c^* \otimes b) = m \cdot (c^* \triangleright b),
\]

\[
\nu_M(c \otimes m)(c^* \otimes b) = c^* \cdot c \cdot (c \cdot m) \cdot (c \cdot b),
\]

for all \( m \in M \), \( c \in C \), \( c^* \in C^* \) and \( b \in \mathcal{B} \). It is easily verified that \( \mu_M \) and \( \nu_M \) are injective linear maps.

Inspired by [9, 11] we propose the following

**Definition 2.3.** Let \( M \) be a right \( C^*\triangleright\mathcal{B} \)-module. We say that \( M \) is rational if \( \text{Im}(\mu_M) \subseteq \text{Im}(\nu_M) \). \( \text{Rat}(M_{C^*\triangleright\mathcal{B}}) \) will be the full subcategory of \( M_{C^*\triangleright\mathcal{B}} \) consisting of rational \( C^*\triangleright\mathcal{B} \)-modules.

**Remark 2.4.** If \( H \) is a coassociative bialgebra, \( \mathcal{B} \) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra in the usual sense, then a rational \( C^*\triangleright\mathcal{B} \)-module \( M \) is nothing else that a \( C^*\triangleright\mathcal{B} \)-module which is rational as a \( C^* \)-module. Here \( M \) is viewed as a right \( C^* \)-module via the canonical algebra map \( C^* \hookrightarrow C^*\triangleright\mathcal{B} \).

It follows easily from Definition 2.3 that a right \( C^*\triangleright\mathcal{B} \)-module is rational if and only if for every \( m \in M \) there exist two finite sets \( \{c_i\}_i \subseteq C \) and \( \{m_i\}_i \subseteq M \) such that

\[
(2.15) \quad m \cdot (c^* \triangleright b) = \sum_i c_i \cdot c_i \cdot m_i \cdot (c^* \triangleright b),
\]

for all \( c^* \in C^* \) and \( b \in \mathcal{B} \). If \( \{c'_i\}_i \subseteq C \) and \( \{m'_i\}_i \subseteq M \) are two other finite sets satisfying (2.15), then \( \sum c_i \otimes m_i = \sum c'_i \otimes m'_i \), because of the injectivity of the map \( \nu_M \). So we have a well-defined map

\[
\lambda_M : M \rightarrow C \otimes M, \quad \lambda_M(m) = \sum_i c_i \otimes m_i,
\]

for all \( m \in M \). If \( C \) is finite dimensional, then any right \( C^*\triangleright\mathcal{B} \)-module is rational. Indeed, we take a finite dual basis \( \{(c_i, c^*)\}_i \) of \( C \) and then consider for each \( m \in M \) the finite sets \( \{c_i\}_i \subseteq C \) and \( \{m_i\}_i \subseteq M \.

We now summarize the properties of rational \( C^*\triangleright\mathcal{B} \)-modules.

**Proposition 2.5.** Let \( H \) be a quasi-bialgebra, \( C \) a right \( H \)-module coalgebra and \( \mathcal{B} \) a left \( H \)-comodule algebra. Then:

i) A cyclic submodule of a rational \( C^*\triangleright\mathcal{B} \)-module is finite dimensional.

ii) If \( M \) is a rational \( C^*\triangleright\mathcal{B} \)-module and \( N \) is a \( C^*\triangleright\mathcal{B} \)-submodule of \( M \), then \( N \) and \( M/N \) are rational \( C^*\triangleright\mathcal{B} \)-modules.

iii) If \( (M_i)_{i \in I} \) is a family of rational \( C^*\triangleright\mathcal{B} \)-modules, then the direct sum \( M = \bigoplus_{i \in I} M_i \) in \( M_{C^*\triangleright\mathcal{B}} \) is a rational \( C^*\triangleright\mathcal{B} \)-module.

iv) To any right \( C^*\triangleright\mathcal{B} \)-module \( M \) we can associate a unique maximal rational submodule \( M^{\text{rat}} \), which is equal to \( M_{\mu_M^{-1}(\text{Im}(\nu_M))} \). It is also equal to the sum of all rational \( C^*\triangleright\mathcal{B} \)-submodules of \( M \). We have a left exact functor

\[
\text{Rat} : M_{C^*\triangleright\mathcal{B}} \rightarrow M_{C^*\triangleright\mathcal{B}}, \quad \text{Rat}(M) = M^{\text{rat}}.
\]

**Proof.** In fact this is a straightforward generalization of [20] Theorem 2.1.3 and [10] Theorem 2.2.6. Consider an element \( m \) of a rational \( C^*\triangleright\mathcal{B} \)-module \( M \). In what follows, \( \{c_i\}_i \subseteq C \) and \( \{m_i\}_i \subseteq M \) will then be two finite sets satisfying (2.15).
i) Let \( m \cdot (C^* \triangleright \mathfrak{B}) \) be a cyclic submodule of a rational \( C^* \triangleright \mathfrak{B} \)-module \( M \). \( m \cdot (C^* \triangleright \mathfrak{B}) \) is generated by the finite set \( \{m_i\}_i \), so it is finite dimensional.

ii) Take \( m \in N \subseteq M \). Choose the \( c_i \) in such a way that they are linearly independent. Fix \( j \), and take \( c^* \in C^* \) such that \( c^*(c_i) = \delta_{i,j} \). Then \( N \) contains \( m \cdot (c^* \triangleright 1_{\mathfrak{B}}) = \sum_i c^*(c_i) m_i \cdot (\triangleright 1_{\mathfrak{B}}) = m_j \), as needed.

Let \( \overline{m} \) be the class in \( N/M \) represented by \( m \in M \). For all \( c^* \in C^* \) and \( b \in \mathfrak{B} \), we have that \( \overline{m} \cdot (c^* \triangleright b) = \sum_i c^*(c_i) \overline{m_i} \cdot (\triangleright b) \), and it follows that \( M/N \) is a rational \( C^* \triangleright \mathfrak{B} \)-module.

iii) Every \( m \in M \) can be written in a unique way as

\[
m = \sum_{j \in J} m_j,
\]

with \( J \subset I \) finite, and \( m_j \in M_j \). Since \( M_j \) is a rational \( C^* \triangleright \mathfrak{B} \)-module, there exist two finite sets \( \{c_j^i\}_k \subseteq C \) and \( \{m_j^i\}_k \subseteq M_j \) such that

\[
m_j \cdot (c^* \triangleright b) = \sum_k c^*(c_j^i) m_j^i \cdot (\triangleright b),
\]

for all \( c^* \in C^* \) and \( b \in \mathfrak{B} \). We therefore have that

\[
m \cdot (c^* \triangleright b) = \sum_{j \in J} m_j \cdot (c^* \triangleright b) = \sum_{j \in J, k} c^*(c_j^i) m_j^i \cdot (\triangleright b),
\]

for all \( c^* \in C^* \) and \( b \in \mathfrak{B} \), and it follows that \( M \) is a rational \( C^* \triangleright \mathfrak{B} \)-module.

iv) Let \( M \) be a right \( C^* \triangleright \mathfrak{B} \)-module. We define \( M^{\text{rat}} = \mu_M^{-1}(\text{Im}(\nu_M)) \).

We first prove that \( M^{\text{rat}} \) is a right \( C^* \triangleright \mathfrak{B} \)-module. Take \( m \in M^{\text{rat}} \). Then there exist two finite sets \( \{c_i\}_i \subseteq C \) and \( \{m_i\}_i \subseteq M \) such that \( m \cdot (c^* \triangleright b) = c^*(c_i) m_i \cdot (\triangleright b) \), for all \( c^* \in C^* \) and \( b \in \mathfrak{B} \). Therefore:

\[
(m \cdot (c^* \triangleright b)) \cdot (d^* \triangleright b') = m \cdot ((c^* \triangleright b)(d^* \triangleright b'))
= m \cdot ((\tilde{x}_1^* \cdot c^*) (\tilde{x}_2^* b_{[-1]} \cdot d^*) \triangleright \tilde{x}_3^* b_{[0]} b')
= c^*(c_i) m_i \cdot (\tilde{x}_1 c_i \cdot \tilde{x}_2^* b_{[-1]} \cdot \tilde{x}_3^* b_{[0]} b)),
\]

for all \( m \in M, c^*, d^* \in C^* \) and \( b, b' \in \mathfrak{B} \). Thus \( m \cdot (c^* \triangleright b) \in M^{\text{rat}} \), hence \( M^{\text{rat}} \) is a \( C^* \triangleright \mathfrak{B} \)-submodule of \( M \). Using an argument similar to the one in the first part of the proof of assertion ii), we can easily check that \( M^{\text{rat}} \) is a rational \( C^* \triangleright \mathfrak{B} \)-module.

Let \( N \) be a rational \( C^* \triangleright \mathfrak{B} \)-submodule of \( M \), this means \( \text{Im}(\mu_N) \subseteq \text{Im}(\nu_N) \). Then

\[
\mu_M(N) = \mu_N(N) \subseteq \text{Im}(\nu_N) \subseteq \text{Im}(\nu_M),
\]

hence \( N \subseteq \mu_M^{-1}(\text{Im}(\nu_M)) = M^{\text{rat}} \), and we conclude that \( M^{\text{rat}} \) is the unique maximal rational submodule of \( M \). Assertions ii) and iii) show that \( M^{\text{rat}} \) is also equal to the sum of all rational \( C^* \triangleright \mathfrak{B} \)-submodules of \( M \). The proof of the final assertion is identical to the proof of [10, Theorem 2.2.6 iv)]. \( \square \)

We are now able to prove the main result of this Section.

**Theorem 2.6.** Let \( H \) be a quasi-bialgebra, \( \mathfrak{B} \) a left \( H \)-comodule algebra and \( C \) a right \( H \)-module coalgebra. The categories \( \mathcal{CM}(H)_{\mathfrak{B}} \) and \( \text{Rat}(\mathcal{CM}C^*, \triangleright \mathfrak{B}) \) are isomorphic, and \( \text{Rat}(\mathcal{CM}C^*, \triangleright \mathfrak{B}) \) is equal to \( \sigma_{C^* \triangleright \mathfrak{B}}[C \otimes \mathfrak{B}] \).
Proof. Recall that we have a functor

$$\mathcal{G} : C \times \mathcal{M}(H)_\mathcal{B} \to \mathcal{M}(C \times \mathcal{B}), \quad \mathcal{G}(M) = M,$$

with right $C^* \times \mathcal{B}$-action given by the formula

$$m \cdot (c^* \triangleright \triangleright b) = c^* (m_{(-1)}) m_{[0]} \cdot b.$$

It is clear that $\mathcal{G}(M)$ is rational as a $C^* \times \mathcal{B}$-module.

Consider a rational $C^* \times \mathcal{B}$-module $M$. Then $M$ is a right $\mathcal{B}$-module through

$$m \cdot b = m \cdot (\varepsilon \triangleright \triangleright b).$$

We define a linear map $\lambda_M : M \to C \otimes M$ as follows:

$$\lambda_M(m) = \sum_i c_i \otimes m_i$$

if and only if

$$m \cdot (c^* \triangleright \triangleright b) = \sum_i c^*(c_i) m_i \cdot (\varepsilon \triangleright \triangleright b),$$

for all $c^* \in C^*$ and $b \in \mathcal{B}$. It is clear that $\lambda_M$ is well-defined.

Fix $i$, and assume that

$$\lambda_M(m_i) = \sum_j c_j^i \otimes m_j^i,$$

or, equivalently,

$$m_i \cdot (c^* \triangleright \triangleright b) = c^*(c_j^i) m_j^i \cdot (\varepsilon \triangleright \triangleright b),$$

for all $c^* \in C^*$ and $b \in \mathcal{B}$. Therefore we have, for all $c^*, d^* \in C^*$ and $m \in M$, that

$$(c^* \otimes d^* \otimes id_M, (id_C \otimes \lambda_M)(\lambda_M(m)) \cdot \Phi_X) = c^* (c_j \cdot X_j) d^* \cdot X_j = m \cdot (X_j \cdot c^* \triangleright \triangleright b) \cdot (\varepsilon \triangleright \triangleright b)$$

proving that (2.3) is satisfied. (2.3) is trivial since $m = m \cdot (\varepsilon \triangleright \triangleright b)$ for all $m \in M$. We will next prove that (2.4) holds. First observe that

$$(m \cdot b) \cdot (c^* \triangleright \triangleright b') = (m \cdot (\varepsilon \triangleright \triangleright b)) \cdot (c^* \triangleright \triangleright b') = m \cdot ((\varepsilon \triangleright \triangleright b) (c^* \triangleright \triangleright b'))$$

$$= m \cdot (b_{[-1]} \cdot c^* \triangleright \triangleright b_{[0]} b') = c^* (c_i \cdot b_{[-1]})(m_i \cdot b_{[0]}) \cdot (\varepsilon \triangleright \triangleright b'),$$

for all $m \in M$, $c^* \in C^*$ and $b, b' \in \mathcal{B}$. This shows that $\lambda_M(m \cdot b) = c_i \cdot b_{[-1]} \otimes m_i \cdot b_{[0]}$, as needed, and it follows that $M \in \mathcal{C} \mathcal{M}(H)_\mathcal{B}$.

Let $\eta : M \to N$ be a morphism of rational $C^* \times \mathcal{B}$-modules. Take $m \in M$, and assume that $\lambda_M(m) = \sum_i c_i \otimes m_i$. We compute that

$$\eta(m) \cdot (c^* \triangleright \triangleright b) = \eta(m \cdot (c^* \triangleright \triangleright b)) = \sum_i c^*(c_i) \eta(m_i) \cdot (\varepsilon \triangleright \triangleright b),$$

for all $c^* \in C^*$ and $b \in \mathcal{B}$. This is equivalent to $\lambda_N(\eta(m)) = c_i \otimes \eta(m_i) = (id_C \otimes \eta)(\lambda_M(m))$, hence $\eta$ is left $C$-colinear. It is clear that $\eta$ is right $\mathcal{B}$-linear, so $\eta$ is a morphism in $\mathcal{C} \mathcal{M}(H)_\mathcal{B}$, and we have a functor $\mathcal{G} : \mathcal{Rat}(\mathcal{M}(C \times \mathcal{B})) \to \mathcal{C} \mathcal{M}(H)_\mathcal{B}$, which is inverse to $\mathcal{G}$.

The proof of the fact that $\mathcal{Rat}(\mathcal{M}(C \times \mathcal{B}))$ and $\sigma_{C^* \times \mathcal{B}}[C \otimes \mathcal{B}]$ are equal is similar to the proof of [13] Lemma 3.9.

We first show that $M \in \sigma_{C^* \times \mathcal{B}}[C \otimes \mathcal{B}]$ for every $M \in \mathcal{Rat}(\mathcal{M}(C \times \mathcal{B}))$. We recall first that a right $C^* \times \mathcal{B}$-module belongs to $\sigma_{C^* \times \mathcal{B}}[C \otimes \mathcal{B}]$ if and only if there
exists a set \( I \), a right \( C^* \triangleright \mathfrak{B} \)-module \( N \), and two \( C^* \triangleright \mathfrak{B} \)-linear maps \( \iota : M \to N \) and \( \pi : (C \otimes \mathfrak{B})(I) \to N \) such that \( \iota \) is injective and \( \pi \) is surjective.

Let \( M \) be a rational \( C^* \triangleright \mathfrak{B} \)-module. Thus \( M \in C \mathcal{M}(H)_\mathfrak{B} \) and \( \mathfrak{G}(M) = M \) as right \( C^* \triangleright \mathfrak{B} \)-modules. It is easy to check that \( \iota = \lambda_M : M \to C \otimes M \) is an injective morphism in \( C \mathcal{M}(H)_\mathfrak{B} \); here \( C \otimes M \) has the right-left \( \text{Doi-Hopf} \) module structure given by \( (\mathfrak{A}, \mathfrak{A}) \). The map

\[
\pi_M : \mathfrak{B}(M) \to M, \quad \pi_M((b_m)_m) = m \cdot b_m,
\]

is a surjective right \( \mathfrak{B} \)-linear map. By Proposition \( \mathfrak{A} \), it provides a surjective morphism \( id_C \otimes \pi_M : C \otimes \mathfrak{B}(M) \to C \otimes M \) in \( C \mathcal{M}(H)_\mathfrak{B} \), and since \( C \otimes \mathfrak{B}(M) \cong (C \otimes \mathfrak{B})(M) \) in \( C \mathcal{M}(H)_\mathfrak{B} \), we conclude that there exists a surjective morphism \( \pi : (C \otimes \mathfrak{B})(M) \to C \otimes M \) in \( C \mathcal{M}(H)_\mathfrak{B} \), so \( M \in \sigma_{C^* \triangleright \mathfrak{B}}[C \otimes \mathfrak{B}] \).

Take \( M \in \sigma_{C^* \triangleright \mathfrak{B}}[C \otimes \mathfrak{B}] \). Then we have right \( C^* \triangleright \mathfrak{B} \)-morphisms \( \iota : M \to N \) and \( \pi : (C \otimes \mathfrak{B})(I) \to N \) such that \( \iota \) is injective and \( \pi \) is surjective. Since the right \( C^* \triangleright \mathfrak{B} \)-module \( C \otimes \mathfrak{B} \) lies in the image of \( \mathfrak{G} \), it follows that \( C \otimes \mathfrak{B} \) is a rational module. By Proposition \( \mathfrak{A} \), \( (C \otimes \mathfrak{B})(I) \) is a rational module too and since \( \pi \) is surjective we deduce from Proposition \( \mathfrak{A} \) ii) that \( N \) is rational. Finally, from Proposition \( \mathfrak{A} \) ii) and the fact that \( \iota \) is injective, it follows that \( M \) is a rational \( C^* \triangleright \mathfrak{B} \)-module. \( \square \)

**Remark 2.7.** It is well-known that the category \( \sigma_A[M] \) is a Grothendieck category, so it follows from Theorem \( \mathfrak{A} \) that \( C \mathcal{M}(H)_\mathfrak{B} \) is a Grothendieck category, an observation that we already made before.

**Corollary 2.8.** \( \mathfrak{A} \) Proposition 5.2 Let \( H \) be a quasi-bialgebra, \( \mathfrak{B} \) a left \( H \)-comodule algebra and \( C \) a finite dimensional right \( H \)-module coalgebra. Then the categories \( C \mathcal{M}(H)_\mathfrak{B} \) and \( \mathcal{M}_{C^* \triangleright \mathfrak{B}} \) are isomorphic.

**Corollary 2.9.** Let \( M \) be a right-left \((H, \mathfrak{B}, C)\)-Hopf module. Then the following assertions hold:

i) The right-left \((H, \mathfrak{B}, C)\)-Hopf submodule generated by an element of \( M \) is finite dimensional.

ii) \( M \) is the sum of its finite dimensional \((H, \mathfrak{B}, C)\)-Hopf submodules.

**Proof.** i) \( M \) is a rational \( C^* \triangleright \mathfrak{B} \)-module, so the right-left \((H, \mathfrak{B}, C)\)-Hopf submodule generated by an element \( m \in M \) coincides with the cyclic \( C^* \triangleright \mathfrak{B} \)-submodule generated by \( m \), by Theorem \( \mathfrak{A} \) We know from Proposition \( \mathfrak{A} \) i) that it is finite dimensional.

ii) View \( M \) as a rational \( C^* \triangleright \mathfrak{B} \)-module. Obviously, \( M \) is the sum of its cyclic \( C^* \triangleright \mathfrak{B} \)-submodules and all of these are finite dimensional right-left \((H, \mathfrak{B}, C)\)-Hopf submodules. \( \square \)

### 2.4. \textit{Doi-Hopf} modules and Koppinen’s smash product.

We begin this Section with some general results about corings, taken from \( \mathfrak{A} \) Sec. 19 and 20]. Let \( R \) be a ring, and \( C \) an \( R \)-coring. Then \( *C = \text{RHom}(C, R) \) is an \( R \)-ring, with multiplication

\[
(\varphi \# \psi)(c) = \psi(c_{(1)} \varphi(c_{(2)})�\),
\]

for all \( \varphi, \psi \in \*C \). We have a functor \( F : \mathcal{M}^C \to \mathcal{M}^C \), \( F(M) = M \), with \( m \cdot \varphi = m_{(0)} \varphi(m_{(1)}) \), for all \( m \in M \) and \( \varphi \in \*C \). If \( C \) is finitely generated and projective as a left \( R \)-module, then \( F \) is an isomorphism of categories.
Assume now that $C$ is locally projective as a left $R$-module. Then $\mathcal{M}^C$ is isomorphic to the category $\sigma_C[C]$, see [2, 19.3]; observe that our multiplication on $^*C$ is opposite to the one from [2], so that left $^*C$-modules in [2] are our right $^*C$-modules. Take a right $^*C$-module $M$. $m \in M$ is called rational if there exists $\sum_i m_i \otimes c_i \in M \otimes C$ such that $m \cdot \varphi = \sum_i m_i \varphi(c_i)$, for all $\varphi \in ^*C$. Then $\text{Rat}^C(M) = \{m \in M \mid m$ is rational$\}$ is a right $C$-comodule, and we obtain a functor $\text{Rat}^C : \mathcal{M}_C \rightarrow \mathcal{M}^C$, which is right adjoint to $F$. $M$ is called rational if $\text{Rat}^C(M) = M$. $\mathcal{M}^C$ is isomorphic to the full category of $\mathcal{M}_C$ consisting of rational right $^*C$-modules.

Now let $H$ be a quasi-bialgebra, $\mathcal{B}$ a left $H$-comodule algebra, and $C$ a right $H$-comodule algebra. We consider the $\mathcal{B}$-coring $C = \mathcal{B} \otimes C$ from Section 2.2. Since we work over a field $k$, $C$ is projective as a $k$-module, hence $C$ is projective (and a fortiori locally projective) as a left $\mathcal{B}$-module. Hence we can apply the above results to this situation. We have an isomorphism of vector spaces $^*C = \mathcal{B}\text{Hom}(\mathcal{B} \otimes C, \mathcal{B}) \cong \text{Hom}(C, \mathcal{B})$.

The multiplication on $^*C$ can be transported to a multiplication on $\text{Hom}(C, \mathcal{B})$. This multiplication makes $\text{Hom}(C, \mathcal{B})$ into a $B$-ring $\langle\langle C, \mathcal{B} \rangle\rangle$, which we will call the Koppinen smash product. The multiplication is given by the following formula:

\begin{equation}
(f \# g)(c) = \bar{x}_1^3 f(c_1 \cdot \bar{x}_1^1) g \left( c_2 \cdot \bar{x}_1^2 f(c_1 \cdot \bar{x}_1^1) \right).
\end{equation}

In the situation where $H$ is an associative bialgebra, we recover the smash product introduced first by Koppinen in [15]. The relation to the generalized smash product introduced in Section 2.2 is discussed in Proposition 2.10.

**Proposition 2.10.** The $k$-linear map

$$\alpha : C^* \triangleright \triangleright \mathcal{B} \rightarrow \langle\langle C, B \rangle\rangle, \quad \alpha(c^* \triangleright \triangleright b) = f,$$ 

with $f(c) = \langle c^*, c \rangle b$, for all $c \in C$, is a morphism of $\mathcal{B}$-rings. It is an isomorphism if $C$ is finite dimensional.

**Proof.** We have to show that $\alpha$ is multiplicative. Take $c^* \triangleright \triangleright b, d^* \triangleright \triangleright b' \in C^* \triangleright \triangleright \mathcal{B}$, and write $\alpha(c^* \triangleright \triangleright b) = f, \alpha(d^* \triangleright \triangleright b') = g$. Using (2.16), we compute that

$$f \# g)(c) = \bar{x}_1^3 \langle c^*, c_1 \cdot \bar{x}_1^1 \rangle b_{[0]} \langle d^*, c_2 \cdot \bar{x}_1^2 b_{[1]} \rangle b'.
\end{equation}

We also have that

$$(c^* \triangleright \triangleright b)(d^* \triangleright \triangleright b') = (\bar{x}_1^1 \cdot c^*)(\bar{x}_1^2 b_{[-1]} \cdot d^*) \triangleright \triangleright \bar{x}_1^3 b_{[0]} b',
\end{equation}

so

$$\alpha((c^* \triangleright \triangleright b)(d^* \triangleright \triangleright b'))(c) = \langle c^*, c_1 \cdot \bar{x}_1^1 \rangle \langle d^*, c_2 \cdot \bar{x}_1^2 b_{[1]} \rangle \bar{x}_1^3 b_{[0]} b',$$

as needed. \hfill $\square$

Take a right $\langle\langle C, B \rangle\rangle$-module $M$. $m \in M$ is rational if there exists $\sum_i m_i \otimes c_i \in M \otimes C$ such that

$$m \cdot f = \sum_i m_i f(c_i),$$

for all $f : C \rightarrow B$. $M$ is called rational if every $m \in M$ is rational.
Corollary 2.11. Now let $H$ be a quasi-bialgebra, $\mathfrak{A}$ a left $H$-comodule algebra, and $C$ a right $H$-comodule algebra. Then the category $\mathcal{CM}(H)_{\mathfrak{A}}$ is isomorphic to the full subcategory of $\mathcal{M}_{#(C,\mathfrak{B})}$, which is also equal to $\sigma_{#(C,\mathfrak{B})}(\mathfrak{B} \otimes C)$.

2.5. Left, right and right-left Doi-Hopf modules. For the sake of completeness, we also define the other Doi-Hopf module categories. In fact, we have four different types of Doi-Hopf modules. The first one was already studied, namely the right-left version. We also have the left-right, right-right and left-left versions.

Definition 2.12. Let $H$ be a quasi-bialgebra, $\mathfrak{A}$ a right $H$-comodule algebra and $\mathfrak{B}$ a left $H$-comodule algebra. In the statements below we assume in i) and iii) that $C$ is a right $H$-module coalgebra, and in ii) that $C$ is a right $H$-module coalgebra, respectively.

i) A left-right $(H, \mathfrak{A}, C)$-Hopf module (or Doi-Hopf module) is a left $\mathfrak{A}$-module $M$ together with a $k$-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that the following relations hold, for all $m \in M$ and $a \in \mathfrak{A}$:

\[
\Phi_\rho, (\rho_M \otimes id_C)(\rho_M(m)) = (id_M \otimes \Delta)(\rho_M(m)),
\]

\[
(id_M \otimes \varepsilon)(\rho_M(m)) = m,
\]

\[
\rho_M(a \cdot m) = a_{(0)} \cdot m_{(0)} \otimes a_{(1)} \cdot m_{(1)}.
\]

$\mathfrak{A}M(H)^C$ is the category of left-right $(H, \mathfrak{A}, C)$-Hopf modules and left $\mathfrak{A}$-linear, right $C$-colinear maps.

ii) A right-right $(H, \mathfrak{A}, C)$-Hopf module (or Doi-Hopf module) is a right $\mathfrak{A}$-module $M$ together with a $k$-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that the following relations hold, for all $m \in M$ and $a \in \mathfrak{A}$:

\[
(\rho_M \otimes id_M)(\rho_M(m)) = (id_M \otimes \Delta)(\rho_M(m)) \cdot \Phi_\rho,
\]

\[
(id_M \otimes \varepsilon)(\rho_M(m)) = m,
\]

\[
\rho_M(m \cdot a) = m_{(0)} \cdot a_{(0)} \otimes m_{(1)} \cdot a_{(1)}.
\]

$\mathcal{M}(H)^{\mathfrak{A}C}$ is the category of right-right $(H, \mathfrak{A}, C)$-Hopf modules and left $\mathfrak{A}$-linear, right $C$-colinear maps.

iii) A left-left $(H, \mathfrak{B}, C)$-Hopf module (or Doi-Hopf module) is a left $\mathfrak{B}$-module $M$ together with a left $k$-linear map $\lambda_M : M \to C \otimes M$, $\lambda_M(m) = m_{[-1]} \otimes m_{[0]}$, such that the following relations hold, for all $m \in M$ and $b \in \mathfrak{B}$:

\[
\Phi_\lambda, (\Delta \otimes id_M)(\lambda_M(m)) = (id_C \otimes \lambda_M)(\lambda_M(m)),
\]

\[
(\varepsilon \otimes id_M)(\lambda_M(m)) = m,
\]

\[
\lambda_M(b \cdot m) = b_{[-1]} \cdot m_{[-1]} \otimes b_{[0]} \cdot m_{[0]}.
\]

$\mathfrak{B}M(H)^C$ is the category of left-left $(H, \mathfrak{B}, C)$-Hopf modules and left $\mathfrak{B}$-linear, left $C$-colinear maps.

We remind that if $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right $H$-comodule algebra then $\mathfrak{A}^{op} = (\mathfrak{A}^{op}, \rho \circ \tau_{\mathfrak{A},H}, (\Phi_\rho)^{s21})$ is a left $H^{op}$-$\mathfrak{A}^{op}$-comodule algebra and $\mathfrak{A} = (\mathfrak{A}, \rho \circ \tau_{\mathfrak{A},H}, (\Phi_\rho^{-1})^{s21})$ is a left $H^{cop}$-comodule algebra, where $\tau_{\mathfrak{A},H} : \mathfrak{A} \otimes H \to H \otimes \mathfrak{A}$ is the switch map. Also, if $\mathfrak{B}$ is a left $H$-comodule algebra then $\mathfrak{B}^{op} = (\mathfrak{B}^{op}, \lambda, \Phi_\lambda^{op})$ is a left $H^{op}$-comodule algebra.
On the other hand, if $C$ is a left $H$-module coalgebra then $C$ is a right $H^{\text{op}}$-module coalgebra and $C^{\text{cop}}$ is a right $H^{\text{cop}, \text{op}}$-module coalgebra (and vice versa). So if $C$ is a right $H$-module coalgebra then $C^{\text{cop}}$ is a right $H^{\text{cop}}$-module coalgebra.

Having these correspondences one can easily see that
\[ \mathfrak{a} \mathcal{M}(H)^C \cong C^{\text{cop}} \mathcal{M}(H^{\text{op}, \text{cop}})_{\mathfrak{a}^{\text{op}}}, \quad \mathcal{M}(H)^C \cong C^{\text{cop}} \mathcal{M}(H^{\text{cop}})_{\mathfrak{a}^{\text{op}}}, \]
\[ \mathfrak{a} \mathcal{M}(H) \cong C \mathcal{M}(H^{\text{op}})_{\mathfrak{a}^{\text{op}}}. \]

It follows that the four different types of Doi-Hopf modules are isomorphic to categories of comodules over suitable corings, and they are Grothendieck categories with enough injective objects. On the other hand, if $C$ is finite dimensional then the above categories are isomorphic to categories of modules over certain generalized smash product algebras. More precisely, we have:

**Remarks 2.13.**

i) The category $\mathfrak{a} \mathcal{M}(H)^C$ is isomorphic to the category of right modules over the generalized smash product $(C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}}$ (over $H^{\text{op}, \text{cop}}$), and therefore also to the category of left modules over $((C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}})^{\text{op}}$. It is not hard to see that the multiplication rule in $((C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}})^{\text{op}}$ is
\[ (c^{\ast} \triangleright a)(d^{\ast} \triangleright a') = (c^{\ast} \cdot a')_{(1)} (d^{\ast} \cdot a'_{(0)}) a_{(0)}, \]
for all $c^{\ast}, d^{\ast} \in C^{\ast}$ and $a, a' \in \mathfrak{a}$, where $(c^{\ast} \cdot h)(c) = c^{\ast}(h \cdot c)$ for all $c^{\ast} \in C^{\ast}$, $h \in H$ and $c \in C$. Therefore, under the trivial permutation of tensor factors we have that $((C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}})^{\text{op}} = \mathfrak{a} \triangleright C^{\ast}$, the right generalized smash product between the right $H$-comodule algebra $\mathfrak{a}$ and the right $H$-module algebra $C^{\ast}$ (see [2] for more details). We conclude that $\mathfrak{a} \mathcal{M}(H)^C \cong \mathfrak{a} \triangleright C^{\ast} \mathcal{M}$ if $C$ is finite dimensional.

ii) The above arguments entail that
\[ \mathcal{M}(H)^C_{\mathfrak{a}^{\text{op}}} \cong C^{\text{cop}} \mathcal{M}(H^{\text{op}})_{\mathfrak{a}^{\text{op}}} \cong \mathcal{M}(C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}} \cong ((C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}})^{\text{op}} \mathcal{M}, \]
where the generalized smash product is over $H^{\text{cop}}$. The explicit formula for the multiplication $\circ$ on $((C^{\text{cop}})^{\triangleright} \mathfrak{a}^{\text{op}})^{\text{op}}$ is given by
\[ (c^{\ast} \triangleright a)(d^{\ast} \triangleright a') = (\tilde{x}^{2} a'_{(1)} \cdot c^{\ast})(\tilde{x}^{3} \cdot d^{\ast}) \triangleright a_{(0)} a', \]
for all $c^{\ast}, d^{\ast} \in C^{\ast}$ and $a, a' \in \mathfrak{a}$.

iii) Obviously, $\mathfrak{a} \mathcal{M}(H) \cong \mathcal{M}_{C^{\ast}}^{\triangleright} \mathfrak{a}^{\text{op}}$, where the generalized smash product is taken over $H^{\text{op}}$.

Let $H$ be a quasi-Hopf algebra and $C$ a finite dimensional right $H$-module coalgebra. It was proved in [4] Proposition 3.2] that the category $\mathcal{M}_{H}^{C} = \mathcal{M}(H)^C_{H}$ is isomorphic to the category of left modules over the smash product algebra $C^{\ast} \# H = C^{\ast} \triangleright H$. Now, by Remark 2.13(ii) the category $\mathcal{M}_{H}^{C}$ is isomorphic to the category of left modules over $((C^{\text{cop}})^{\#} H)^{\text{op}}$. The next result shows that the smash product algebras $C^{\ast} \# H$ and $((C^{\text{cop}})^{\#} H)^{\text{op}}$ are isomorphic.

**Proposition 2.14.** Let $H$ be a quasi-Hopf algebra and $C$ a right $H$-module coalgebra. Then the map
\[ \varphi : C^{\ast} \# H \to ((C^{\text{cop}})^{\#} H)^{\text{op}}, \quad \varphi(c^{\ast} \# h) = S^{-1}(q^{1} h_{1} q^{1}) \cdot c^{\ast} \# S^{-1}(q^{2} h_{2} q^{2}) \]
is an algebra isomorphism. Here $f^{-1} = g^{1} \otimes g^{2}$ is the element defined by \([1, 2]\) and $\tilde{q}_{\Delta} = q^{1} \otimes q^{2}$ is the element defined in \([1, 2]\), in the special case where $\mathfrak{B} = H$.\]
Proof. We first show that \( \varphi \) is an algebra map. Let \( f = f^1 \otimes f^2, f^{-1} = g^1 \otimes g^2 = G^1 \otimes G^2 \) and \( \hat{q}_\Delta = q^1 \otimes q^2 = \Omega^1 \otimes \Omega^2 \) be the elements defined by \( (\ref{X}) \), \( (\ref{Y}) \) and \( (\ref{Z}) \), respectively. We compute

\[
\varphi((c^* \# h)(d^* \# h')) = \varphi((x^1 \cdot c^*)(x^2 h_1 \cdot d^*) \# x^3 h_2 h')
\]

\[= \left[ S^{-1}(f^2 q^1_1 x^1_{1,2}(h_2 h')_{(1,2)} g^1_2 G^2) x^1 \cdot c^* \right]
\]

\[= \left[ S^{-1}(f^1 q^1_1 x^3_{(1,1)}(h_2 h')_{(1,1)} g^1_1 G^1) x^2 h_1 \cdot d^* \right] \# S^{-1}(q^2 x^3_2 (h_2 h') g^2)
\]

\[= \left[ S^{-1}(q^1 \Omega^2_1(h_2 h')_{(2,1)} x^2 g^1_2 G^2) \cdot c^* \right] \left[ S^{-1}(\Omega^1(h_2 h')_{1} X^1 g^1_1 G^1) x_1 \cdot d^* \right]
\]

\[= \# S^{-1}(q^2 \Omega^2_2(h_2 h')_{(2,2)} X^3 g^2)
\]

\[
= \left[ X^2 S^{-1}(q^1 h_1 \Omega^2_1 h_{(2,1)} g^1_2 G^1) \cdot c^* \right] \left[ X^3 S^{-1}(\Omega^1 h_1 g^1) \cdot d^* \right]
\]

\[= \# X^1 S^{-1}(q^2 h_2 \Omega^2_2 g^2_2 G^2)
\]

\[= \left[ X^2 S^{-1}(\Omega^2 h_2 g^2) \right] \# S^{-1}(q^1 h_1 g^1) \cdot c^* \left[ X^3 S^{-1}(\Omega^1 h_1 g^1) \cdot d^* \right]
\]

\[= \# X^1 S^{-1}(\Omega^2 h_2 g^2) \cdot S^{-1}(q^2 h_2 G^2)
\]

\[= \varphi(c^* \# h) \otimes \varphi(d^* \# h'),
\]

as needed. It is clear that \( \varphi(\xi \# 1_H) = \xi \# 1_H \), so it remains to be shown that \( \varphi \) is bijective.

First we introduce some notation. Let \( A \) be a right \( H \)-comodule algebra, and define the element \( \hat{q}_\rho \in A \otimes H \) as follows:

\[
(\ref{Z})
\]

\[\hat{q}_\rho = \tilde{q}_\rho \otimes \check{q}_\rho = X_\rho^1 \otimes S^{-1}(\alpha X_\rho^3) X_\rho^2.
\]

In the special situation where \( A = H \), the element \( \hat{q}_\Delta = q^1 \otimes q^2 \).

We now claim that \( \varphi^{-1} : ((C^{\text{cop}})^* \# H)^{\text{op}} \to C^* \# H \) is given by the formula

\[
\varphi^{-1}(c^* \# h) = g^1 S(q^2 h_2) \cdot c^* \# g^2 S(q^1 h_1),
\]

for all \( c^* \in C^* \) and \( h \in H \).

From \( (\ref{Z}) \), Lemma 2.6, we recall the following formula:

\[
(\ref{Z})
\]

\[g^2 \alpha S^{-1}(q^1) = S^{-1}(\beta).
\]

Let \( \hat{p}_\Delta = p^1 \otimes p^2 \) be the element \( \hat{p}_\lambda \) defined in \( (\ref{Z}) \), in the special case where \( B = H \). We then compute, for all \( c^* \in C^* \) and \( h \in H \):

\[
\varphi^{-1} \circ \varphi(c^* \# h) = \varphi^{-1}(S^{-1}(q^1 h_1 g^1) \cdot c^* \# S^{-1}(q^2 h_2 g^2))
\]

\[= q^2 h_{(2,1)} g^1 S(q^2) S^{-1}(q^1 h^1 g^1) \cdot c^* \# \check{q}^2_{(2,2)} g^2 S(q^1)
\]

\[= q^2 h_{(2,1)} X^2 g^2 \alpha S^{-1}(q^1 h_1 X^1 g^1) \cdot c^* \# \check{q}^2_{(2,2)} X^3 g^2
\]

\[= S^{-1}(q^1 h_1 S(q^2 h_{(2,1)} p^1)) \cdot c^* \# \check{q}^2_{(2,2)} X^3 g^2
\]

\[= S^{-1}(q^1 h_1 S(q^2 h_{(2,1)} p^1)) \cdot c^* \# \check{q}^2_{(2,2)} X^3 g^2
\]

\[= \hat{c}^* \# h.
\]

The proof of the fact that \( \varphi \circ \varphi^{-1} = id_{((C^{\text{cop}})^* \# H)^{\text{op}}} \) is based on similar computations. \( \square \)
Remark 2.15. The isomorphism \( \varphi \) in Proposition 2.14 can be defined more generally for a left \( H \)-module algebra \( A \) instead of a right \( H \)-module coalgebra \( C \). Observe that \( H \) cannot be replaced by an \( H \)-bicomodule algebra \( \mathcal{A} \), because of the appearance of the antipode \( S \) of \( H \) on the second position of the tensor product.

3. Yetter-Drinfeld modules are Doi-Hopf modules

In this Section, we will show that Yetter-Drinfeld modules are special case of Doi-Hopf modules. We will then apply the properties of Doi-Hopf modules to Yetter-Drinfeld modules.

3.1. Yetter-Drinfeld modules over quasi-bialgebras. The category of Yetter-Drinfeld modules over a quasi-Hopf algebra \( H \) was introduced by Majid, as the center of the monoidal category \( \mathcal{H} \). His aim was to define the quantum double by an implicit Tannaka-Krein reconstruction procedure, see [17]. From [7], we recall the following more general definition of Yetter-Drinfeld modules.

The category of \((H,H)\)-bimodules, \( \mathcal{H}_H \), is monoidal. The associativity constraints \( a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) \) are given by

\[
a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)) \cdot \Phi^{-1}
\]

for all \( U,V,W \in \mathcal{H}_H \), \( u \in U \), \( v \in V \) and \( w \in W \). A coalgebra in the category of \((H,H)\)-bimodules will be called an \( H \)-bimodule coalgebra. More precisely, an \( H \)-bimodule coalgebra \( C \) is an \((H,H)\)-bimodule (denote the actions by \( h \cdot c \) and \( c \cdot h \)) with a comultiplication \( \Delta : C \to C \otimes C \) and a counit \( \varepsilon : C \to k \) satisfying the following relations, for all \( c \in C \) and \( h \in H \):

\[
\begin{align*}
(3.2) \quad \Phi \cdot (\Delta \otimes \text{id}_C)(\Delta(c)) \cdot \Phi^{-1} &= (\text{id}_C \otimes \Delta)(\Delta(c)), \\
(3.3) \quad \Delta(h \cdot c) &= h_1 \cdot c_1 \otimes h_2 \cdot c_2, \quad \Delta(c \cdot h) = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \\
(3.4) \quad (\varepsilon \otimes \text{id}_C) \circ \Delta &= (\text{id}_C \otimes \varepsilon) \circ \Delta = \text{id}_C, \\
(3.5) \quad \varepsilon(h \cdot c) &= \varepsilon(h) \varepsilon(c), \quad \varepsilon(c \cdot h) = \varepsilon(c) \varepsilon(h),
\end{align*}
\]

where we used the same Sweedler-type notation as introduced before.

For further use we note that an \( H \)-bimodule coalgebra \( C \) can be always viewed as a left \( H^{\text{op}} \otimes H \)-module coalgebra via the left \( H^{\text{op}} \otimes H \)-action given for all \( c \in C \) and \( h, h' \in H \) by

\[
(3.6) \quad (h \otimes h') \cdot c = h' \cdot c \cdot h.
\]

Definition 3.1. (\[\texttt{2}]\) Let \( H \) be a quasi-bialgebra, \( C \) an \( H \)-bimodule coalgebra and \( \mathcal{A} \) an \( H \)-bicomodule algebra. A left-right \((H,A,C)\)-Yetter-Drinfeld module is a \( k \)-vector space \( M \) with the following additional structure:

- \( M \) is a left \( \mathcal{A} \)-module; we write \( \cdot \) for the left \( \mathcal{A} \)-action;
- we have a \( k \)-linear map \( \rho_M : M \to M \otimes C \), \( \rho_M(m) = m_0 \otimes m_{(1)} \), called the right \( C \)-coaction on \( M \), such that for all \( m \in M \), \( \varepsilon(m_{(1)})m_{(0)} = m \) and \( (\theta^2 \cdot m_{(0)})_{(0)} \otimes (\theta^2 \cdot m_{(0)})_{(1)} \cdot \theta^1 \otimes \theta^3 \cdot m_{(1)} \)

\[
= \hat{x}^1_\rho \cdot (\hat{x}^3_\lambda \cdot m)_{(0)} \otimes \hat{x}^2_\rho \cdot (\hat{x}^3_\lambda \cdot m)_{(1)}_{\rho} \cdot \hat{x}^1_\lambda \otimes \hat{x}^3_\lambda \cdot (\hat{x}^3_\lambda \cdot m)_{(1)}_{\rho} \cdot \hat{x}^2_\lambda,
\]

- for all \( u \in \mathcal{A} \) and \( m \in M \) the following compatibility relation holds:

\[
(3.8) \quad u_{<0>} \cdot m_{(0)} \otimes u_{<1>} \cdot m_{(1)} = (u_{[0]} \cdot m)_{(0)} \otimes (u_{[0]} \cdot m)_{(1)} \cdot u_{[-1]}.
\]

\( \mathcal{A} \mathcal{YD}(H)^C \) will be the category of left-right \((H,A,C)\)-Yetter-Drinfeld modules and maps preserving the \( \mathcal{A} \)-action \( C \)-coaction.
We have seen in Section 1.2 that $H$ is an $H$-bimodule algebra; it is also clear that $H$ is an $H$-bimodule coalgebra (take $\Delta = \Delta$ and $\varepsilon = \varepsilon$). If we take $H = \Lambda = C$ in Definition 5.1 then we recover the category of Yetter-Drinfeld modules introduced by Majid in [17], and studied in [14, 15]. It is remarkable that a quasi-bialgebra $H$ is a coalgebra in the category of $H$-bimodules, but not in the category of vector spaces, or in the category of left (or right) $H$-modules.

Let $H$ be a quasi-bialgebra, $\Lambda$ an $H$-bimodule algebra and $C$ a left $H$-module coalgebra. It is straightforward to check that $C$ with the right $H$-module structure given by $\varepsilon$ is an $H$-bimodule coalgebra. Then $\Phi$ and $\lambda$ reduce respectively to (2.17) and (2.19), in which only the right $H$-coaction on $\Lambda$ appears. So in this particular case the category $\mathcal{A}M(H)^C$ is just $\mathcal{A}YD(H)^C$.

In order to show that the Yetter-Drinfeld modules are special case of Doi-Hopf modules we need a Doi-Hopf datum. As we have already seen, an $H$-bimodule algebra $C$ can be viewed as a left $H^{op} \otimes H$-module via the structure defined in (3.6). In order to provide a right $H^{op} \otimes H$-comodule algebra structure on an $H$-bimodule algebra $\Lambda$, we need the following result.

**Lemma 3.2.** Let $H$ be a quasi-Hopf algebra and $(\mathcal{B}, \lambda, \Phi_\lambda)$ a left $H$-comodule algebra. Then $\mathcal{B}$ is a right $H^{op}$-comodule algebra via the structure
\begin{align}
\rho : \ & \mathcal{B} \to \mathcal{B} \otimes H, \quad \rho(b) = b_0 \otimes S^{-1}(b_{-1}), \\
\Phi_\rho = \ & \tilde{x}_3^2 \otimes S^{-1}(f^2 \tilde{x}_3^2) \otimes S^{-1}(f^1 \tilde{x}_3^2) \in \mathcal{B} \otimes H \otimes H,
\end{align}

where $f^1 \otimes f^2$ is the Drinfeld twist defined in (1.17). Moreover, if $(\mathcal{B}, \lambda, \Phi_\lambda)$ and $(\mathcal{B}, \lambda', \Phi_{\lambda'})$ are twist equivalent left $H$-comodule algebras then the corresponding right $H^{op}$-comodule algebras are also twist equivalent.

**Proof.** The relation (1.15) follows easily by applying (1.18) and (1.17), and the relations (1.16). Since $\Phi_{op} = \Phi^{-1}$ we have
\begin{align}
\tilde{x}_3^1 \tilde{y}_3^1 \otimes x^1 \otimes (\tilde{x}_3^2)^{\text{op}} \otimes x^2 \otimes (\tilde{x}_3^3)^{\text{op}} \\
&= \tilde{x}_3^1 \tilde{y}_3^1 \otimes S^{-1}(f^2 \tilde{y}_3^2) \otimes S^{-1}(f^1 \tilde{x}_3^2) \otimes S^{-1}(f^1 \tilde{x}_3^1) \\
&= \tilde{x}_3^1 \tilde{y}_3^1 \otimes S^{-1}(f^2 \tilde{y}_3^2) \otimes S^{-1}(f^1 \tilde{x}_3^2) \otimes S^{-1}(f^1 \tilde{x}_3^1)
\end{align}

Finally, it is not hard to see that if the invertible element $U = U^1 \otimes U^2 \in H \otimes \mathcal{B}$ provides a twist equivalence between the left $H$-comodule algebras $(\mathcal{B}, \lambda, \Phi_\lambda)$ and $(\mathcal{B}, \lambda', \Phi_{\lambda'})$ then the invertible element $V = U^2 \otimes S^{-1}(U^1) \in \mathcal{B} \otimes H$ provides a twist equivalence between the associated right $H^{op}$-comodule algebras $(\mathcal{B}, \rho, \Phi_\rho)$ and $(\mathcal{B}, \rho', \Phi_{\rho'})$, respectively.
Proposition 3.3. Let $H$ be a quasi-Hopf algebra and $(\Lambda, \lambda, \rho, \Phi, \Phi_\rho, \Phi_{\lambda, \rho})$ an $H$-bicomodule algebra. We define two right $H^{\text{op}} \otimes H$-coactions
\[ \rho_1, \rho_2 : \Lambda \to \Lambda \otimes (H^{\text{op}} \otimes H) \]
on $\Lambda$, and corresponding elements $\Phi_{\rho_1}, \Phi_{\rho_2} \in \Lambda \otimes (H^{\text{op}} \otimes H) \otimes (H^{\text{op}} \otimes H)$ as follows:
\[ \rho_1(u) = u_{(0)} \otimes \left( S^{-1}(u_{(0)}\cdot 1) \otimes u_{(1)} \right), \]
\[ \Phi_{\rho_1} = (\tilde{X}_\rho^1(0) \tilde{X}_\rho^3 \tilde{\theta}_\rho^2) \otimes \left( S^{-1}(f_2^2(\tilde{X}_\rho^2\cdot 1) \cdot \tilde{\theta}_\rho^2) \otimes \tilde{X}_\rho^3 \right), \]
\[ \rho_2(u) = u_{(0)} \otimes \left( S^{-1}(u_{-1}) \otimes u_{(1)} \right), \]
\[ \Phi_{\rho_2} = (\tilde{X}_\rho^1)_{(0)} \tilde{X}_\rho^3 \tilde{\theta}_\rho^2 \otimes \left( S^{-1}(f_2^2(\tilde{X}_\rho^2(1) \cdot 1) \cdot \tilde{\theta}_\rho^2) \otimes \tilde{X}_\rho^3(1) \right), \]
\[ \rho_2(\Phi_{\rho_1}) = \rho_2(\Phi_{\rho_2}) \]

Then $(\Lambda, \rho_1, \Phi_{\rho_1})$ and $(\Lambda, \rho_2, \Phi_{\rho_2})$ are twist equivalent right $H^{\text{op}} \otimes H$-comodule algebras.

Proof. The statement follows from Lemma 3.2. Indeed, we have seen at the end of Section II that $\Lambda$ has two twist equivalent left $H \otimes H^{\text{op}}$-comodule algebra structures. Identifying $(H \otimes H^{\text{op}})^{\text{op}}$ and $H^{\text{op}} \otimes H$, and computing the induced right coactions we obtain the structures defined in (3.11-3.14). We point out that the reassociator, the antipode and the Drinfeld twist corresponding to $H \otimes H^{\text{op}}$ are given by

\[ \Phi_{H \otimes H^{\text{op}}} = (X_1 \otimes x_1) \otimes (X_2 \otimes x_2) \otimes (X_3 \otimes x_3), \]
\[ S_{H \otimes H^{\text{op}}} = S \otimes S^{-1}, \]
\[ f_{H \otimes H^{\text{op}}} = (f^1 \otimes S^{-1}(g^2)) \otimes (f^2 \otimes S^{-1}(g^1)), \]
where, as usual, $g^1 \otimes g^2$ is the inverse of $f = f^1 \otimes f^2$. \hfill \Box

Let $H$ be a quasi-Hopf algebra, $C$ an $H$-bimodule coalgebra and $\mathfrak{A}$ an $H$-bicomodule algebra. In the sequel, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ will be our notation for the right $H^{\text{op}} \otimes H$-comodule algebras $(\mathfrak{A}, \rho_1, \Phi_{\rho_1})$ and $(\mathfrak{A}, \rho_2, \Phi_{\rho_2})$. By the above arguments, it make sense to consider the left-right Doi-Hopf module categories $\mathfrak{A}_1 \mathcal{M}(H^{\text{op}} \otimes H)^C$ and $\mathfrak{A}_2 \mathcal{M}(H^{\text{op}} \otimes H)^C$. It will follow from Proposition 3.3 that these two categories are isomorphic.

Proposition 3.4. Let $H$ be a quasi-bialgebra, $C$ a left $H$-module coalgebra and $\mathfrak{A}^1 = (\mathfrak{A}, \rho, \Phi_{\rho})$ and $\mathfrak{A}^2 = (\mathfrak{A}, \rho', \Phi_{\rho'})$ two twist equivalent right $H$-comodule algebras. Then the categories $\mathfrak{A}^1 \mathcal{M}(H)^C$ and $\mathfrak{A}^2 \mathcal{M}(H)^C$ are isomorphic.

Proof. If $\mathfrak{A}^1$ and $\mathfrak{A}^2$ are twist equivalent, then there exists $\Psi \in \mathfrak{A} \otimes H$ satisfying (1.22-1.24). Take $M \in \mathfrak{A}_1 \mathcal{M}(H)^C$; $M$ becomes an object in $\mathfrak{A}_2 \mathcal{M}(H)^C$ by keeping the same left $\mathfrak{A}$-module structure and defining
\[ \rho'_M : M \to M \otimes C, \quad \rho'(m) = \Psi \cdot \rho_C(m). \]

Conversely, take $M \in \mathfrak{A}_2 \mathcal{M}(H)^C$ via the structures $\cdot$ and $\rho'_M$. Then $M$ can be viewed as a left-right $(H, \mathfrak{A}^1, C)$-Hopf module via the same left $\mathfrak{A}$-action $\cdot$ and the right $C$-coaction $\rho_M$ defined by
\[ \rho_M : M \to M \otimes C, \quad \rho_M(m) = \Psi^{-1} \cdot \rho'_M(m). \]
These correspondences define two functors which act as the identity on morphisms and produce inverse isomorphisms.

Remark 3.5. Let $H$ be a quasi-bialgebra, and $F \in H \otimes H$ a gauge transformation. We can consider the twisted quasi-bialgebra $H_F$ (see \ref{def:twisted-quasi-bialgebra}), and it is well-known that the categories of left $H$-modules and left $H_F$-modules are isomorphic. We have a similar property for Doi-Hopf modules.

Let $(\mathfrak{A}, \rho, \Phi)$ be a right $H$-comodule algebra, and let $\Phi_{\rho_F} = (1_\mathfrak{A} \otimes F)\Phi$. Then $\mathfrak{A}_F := (\mathfrak{A}, \rho, \Phi_{\rho_F})$ is a right $H_F$-comodule algebra, see \ref{def:comodule-algebra}.

Now let $C$ be a left $H$-module coalgebra, and define a new comultiplication $\Delta_F$ as follows: $\Delta_F(c) = F\Delta(c)$, for all $c \in C$. Then straightforward computations show that $(C, \Delta_F, \varepsilon)$ is a left $H_F$-module coalgebra and that the categories $\mathcal{M}(H)^C$ and $\mathcal{M}(H_F)^C$ are isomorphic. Of course, a similar result holds for left comodule algebras.

3.2. Yetter-Drinfeld modules and Doi-Hopf modules. Our next aim is to show that the category of left-right Yetter-Drinfeld modules $\mathcal{YD}(H)^C$ is isomorphic to the category of Doi-Hopf modules $\mathcal{M}(H^{op} \otimes H)^C$ and, a fortiori, to $\mathcal{M}(H^{op} \otimes H)^C$, by Proposition \ref{prop:isomorphism-categories}. We have divided the proof over a few lemmas.

Lemma 3.6. Let $H$ be a quasi-Hopf algebra, $\mathfrak{A}$ an $H$-bicomodule algebra and $C$ an $H$-bimodule coalgebra. We have a functor

$$F : \mathcal{YD}(H)^C \to \mathcal{M}(H^{op} \otimes H)^C$$

which acts as the identity on objects and morphisms. If $M$ is a left-right $(H, \mathfrak{A}, C)$-Yetter-Drinfeld module then $F(M) = M$ as a left $\mathfrak{A}$-module, and with the newly defined right $C$-coaction

$$(3.15) \quad \rho'_M(m) = m_{(0)} \otimes m_{(1)} = (\tilde{p}_\lambda \cdot m)_{(0)} \otimes (\tilde{p}_\lambda \cdot m)_{(1)} \cdot \tilde{p}_\lambda,$$

for all $m \in M$. Here $\tilde{p}_\lambda = \tilde{p}_\lambda \otimes \tilde{p}_\lambda$ is the element defined by \ref{def:tilde-p-lambda}.

Proof. It is not hard to see that \ref{def:tilde-p-lambda} and \ref{def:tilde-p-lambda} imply

$$(3.16) \quad \Theta^2_{[0]} \tilde{p}_\lambda \otimes \Theta^2_{[-1]} \tilde{p}_\lambda^{-1} S^{-1}(\Theta^1) \otimes \Theta^3 = \theta^2(\tilde{p}_\lambda(0)) \otimes \theta^1 \tilde{p}_\lambda^{-1} \otimes \theta^3(\tilde{p}_\lambda(1)).$$

We now show that $F(M)$ satisfies the relations \ref{def:tilde-p-lambda} and \ref{def:tilde-p-lambda}. Let $\tilde{P}_\lambda \otimes \tilde{P}_\lambda$ be another copy of $\tilde{p}_\lambda$, and compute

$$\Phi_{\rho_2} \cdot (\rho'_M \otimes id_C)(\rho_M(m))$$

$$= (\tilde{x}^3(0) \tilde{X}^4_\lambda \Theta^2(0) \cdot (\tilde{P}_\lambda \cdot (\tilde{p}_\lambda \cdot m)(0))(0) \otimes (\tilde{x}^3(0) \tilde{X}^2_\lambda \Theta^2(1) \cdot (\tilde{p}_\lambda \cdot m)(1))(0)) \cdot \tilde{P}_\lambda S^{-1}(f^2 \tilde{x}^2 \Theta^1)$$

$$\otimes (\tilde{x}^3(1) \tilde{X}^3_\lambda \Theta^3 \cdot (\tilde{p}_\lambda \cdot m)(1)) \cdot \tilde{p}_\lambda S^{-1}(f^1 \tilde{x}^1)$$

$$= (\tilde{x}^3(0) \tilde{X}^4_\lambda \cdot (\theta^2(\tilde{P}_\lambda(0)) \otimes (\tilde{p}_\lambda \cdot m)(0))(0) \otimes (\tilde{x}^3(1) \tilde{X}^2_\lambda \cdot (\theta^2(\tilde{P}_\lambda(0)) \cdot (\tilde{p}_\lambda \cdot m)(0))(1)) \cdot \theta^1 \tilde{P}_\lambda S^{-1}(f^2 \tilde{x}^2)$$

$$\otimes (\tilde{x}^3(1) \tilde{X}^3_\lambda \cdot (\theta^2(\tilde{P}_\lambda(1)) \cdot (\tilde{p}_\lambda \cdot m)(1))(1)) \cdot \tilde{P}_\lambda S^{-1}(f^1 \tilde{x})$$
\[ \begin{align*}
(\tilde{x}_3^1)_{(0)} & \cdot (\tilde{y}_A^3(\tilde{P}_2^2)_{(0)}\tilde{P}_2^2 \cdot m)_{(0)} \\
\otimes (\tilde{x}_3^1)_{(1)} & \cdot (\tilde{y}_A^3(\tilde{P}_2^2)_{(0)}\tilde{P}_2^2 \cdot m)_{(1)} \cdot \tilde{y}_A^3\tilde{P}_2^1 S^{-1}(f^2\tilde{x}_3^1) \\
\otimes (\tilde{x}_3^1)_{(1)_2} & \cdot (\tilde{y}_A^3(\tilde{P}_2^2)_{(0)}\tilde{P}_2^2 \cdot m)_{(1)_2} \cdot \tilde{y}_A^3(\tilde{P}_2^2)_{(-1)}\tilde{P}_2^1 S^{-1}(f^1\tilde{x}_3^1) \\
\otimes \tilde{P}_2^2 \cdot m)_{(0)} & \otimes (\tilde{P}_2^2 \cdot m)_{(1)} \cdot (\tilde{P}_2^3)_{(0)} \otimes (\tilde{P}_2^3)_{(1)} \cdot (\tilde{P}_2^3)_{(2)} \\
m_{(0)'} & \otimes m_{(1)'} \otimes m_2 = (id_M \otimes \Delta)(\rho'_M(m)),
\end{align*} \]
for all \( m \in M \), as needed. The relation (2.18) is trivial and (2.19) follows from (1.26), (3.18) and (3.19).

Lemma 3.7. Let \( H \) be a quasi-Hopf algebra, \( C \) an \( H \)-bimodule coalgebra and \( A \) an \( H \)-bimodule algebra. Then we have a functor

\[ G : \mathcal{H}_2(M(H^\text{op} \otimes H))^C \to \mathcal{A}_2(D(H))^C \]

which acts as the identity on objects and morphisms. Let \( M \) be a left-right \((H^\text{op} \otimes H, A^2, C)\)-Hopf module, with left \( A \)-action \( \cdot \) and right \( C \)-coaction \( \rho_M \), \( \rho'_M(m) = m_{(0)'} \otimes m_{(1)'} \in M \otimes C \). Then \( G(M) = M \) as a left \( A \)-module, with new right \( C \)-coaction \( \overline{\rho}_M : M \to M \otimes C \), given by the formula

\[ \overline{\rho}_M(m) = m_{(0)} \otimes m_{(1)} = (\tilde{q}_A^3)_{(0)} \cdot m_{(0)'} \otimes (\tilde{q}_A^3)_{(1)} \cdot m_{(1)'} \cdot S^{-1}(\tilde{q}_A^3), \]

for all \( m \in M \). Here \( \tilde{q}_A = \tilde{q}_A^3 \otimes \tilde{q}_A^3 \) is the element defined in (1.26).

Proof. The most difficult part is that we show that \( G(M) \) satisfies the relations (3.17) and \( \overline{\rho}_M \). \( M \) is a left-right \((H, A^2, C)\)-Hopf module, so we have by (2.47), (2.48) and (2.49):

\[ \begin{align*}
\tilde{X}_\rho^1((\tilde{x}_3^2)_{(0)} \Theta^2)_{(0)} \cdot m_{(0)'} \otimes X_\rho^2((\tilde{x}_3^2)_{(0)} \Theta^2)_{(1)} \cdot m_{(1)'} \cdot S^{-1}(f^2\tilde{x}_3^2 \Theta^1) \\
\otimes \tilde{X}_\rho^1((\tilde{x}_3^1)_{(1)} \Theta^3 \cdot m_{(1)'} \cdot S^{-1}(f^1\tilde{x}_3^1) = m_{(0)'} \otimes m_{(1)'} \otimes m_{(1)'} \otimes m_{(1)'} \otimes m_{(1)'}.
\end{align*} \]

(3.19) \((u \cdot m)_{(0)'} \otimes (u \cdot m)_{(1)'} = u_{(0)'} \otimes u_{(0)'} \otimes u_{(0)'} \otimes u_{(0)'} \otimes u_{(0)'} \cdot S^{-1}(u_{(-1)}), \)

for all \( m \in M \) and \( u \in A \). Also, (2.10) and (2.14) imply that

(3.20) \[ S(\Theta^1)\tilde{q}_A^3(\tilde{x}_3^2)_{(0)} \otimes \tilde{q}_A^3(\tilde{x}_3^2)_{(0)} \otimes \Theta^3 = \tilde{q}_A^3 \Theta^1 \otimes (\tilde{q}_A^3)_{(0)} \Theta^2 \otimes (\tilde{q}_A^3)_{(1)} \Theta^3. \]

Let \( \tilde{Q}_A^3 \otimes \tilde{Q}_A^3 \) be another copy of \( \tilde{q}_A \); for all \( m \in M \), we compute that

\[ \begin{align*}
(\Theta^2 \cdot m_{(0)})_{(0)} & \otimes (\Theta^2 \cdot m_{(0)})_{(1)} \otimes \Theta^3 \cdot m_{(1)} \\
\quad = (\tilde{Q}_A^3 \Theta^1)_{(0)} \cdot m_{(0)'} \otimes (\tilde{Q}_A^3 \Theta^1)_{(1)} \cdot m_{(1)'} \cdot S^{-1}(\tilde{Q}_A^3) \\
\quad \otimes S^{-1}(\tilde{Q}_A^3)_{(0)'} \otimes ((\tilde{Q}_A^3)_{(0)} \Theta^3)_{(0)} \cdot m_{(0)} \otimes ((\tilde{Q}_A^3)_{(0)} \Theta^3)_{(1)} \cdot m_{(1)} \cdot S^{-1}(\tilde{Q}_A^3) \\
\quad \otimes S^{-1}(\tilde{Q}_A^3)_{(1)} \otimes (\tilde{Q}_A^3)_{(0)} \Theta^3 \cdot m_{(1)} \cdot S^{-1}(\tilde{Q}_A^3)
\end{align*} \]
Since \( F \) is obtained using first Lemma 3.7 and then Lemma 3.6 we have the quasi-Hopf algebra setting.

An \( \rho \) -bimodule coalgebra. Then the categories \( A \) \( \rho \) -coaction. If \( A \) \( \rho \) -modules we have only as needed. \( \text{S8} \) also holds since

\[
\left( q^k_{\lambda} \right) (u_{(0)} \cdot \left( q^k_{\lambda} \right) (u_{(1)}) \cdot \left( q^k_{\lambda} \right) (u_{(2)}), \right. \\
\left( q^k_{\lambda} \right) (u_{(0)} \cdot \left( q^k_{\lambda} \right) (u_{(1)}) \cdot \left( q^k_{\lambda} \right) (u_{(2)}), \right.
\]

for all \( u \in A \) and \( m \in M \). The remaining details are left to the reader. \( \square \)

**Theorem 3.8.** Let \( H \) be a quasi-Hopf algebra, \( A \) an \( H \)-bicomodule algebra and \( C \) an \( H \)-bimodule coalgebra. Then the categories \( \Lambda \mathcal{YD}(H)^C \) and \( \Lambda \mathcal{M}(H^{\text{op}} \otimes H)^C \) are isomorphic. In particular \( \Lambda \mathcal{YD}(H)^C \) is a Grothendieck category, and therefore it has enough injective objects.

**Proof.** We show that the functors \( F \) and \( G \) from Lemmas 3.6 and 3.7 are inverses. Since \( F \) and \( G \) act as the identity functor at the level of \( A \)-modules we have only to show that \( F \) and \( G \) are inverses at the level of \( C \)-coactions.

Let \( M \in \Lambda \mathcal{YD}(H)^C \) and \( \rho_M(m) = m_{(0)} \otimes m_{(1)} \) its right \( C \)-coaction. We denote by \( \overline{\rho}_M(m) = m_{(0)} \otimes m_{(1)} \) the right \( C \)-coaction of \( G(F(M)) \) obtained using first Lemma 3.6 and then Lemma 3.7.

For all \( m \in M \) we then have

\[
(\overline{\rho}_M(m) = m_{(0)} \otimes m_{(1)}) = \rho_M(m). \]

Conversely, let \( M \in \Lambda \mathcal{M}(H^{\text{op}} \otimes H)^C \) and denote by \( \rho'_M(m) = m_{(0)} \otimes m_{(1)} \) its right \( C \)-coaction. If \( \rho_M(m) = m_{(0)} \otimes m_{(1)} \) is the right \( C \)-coaction on \( F(G(M)) \) obtained using first Lemma 3.6 and then Lemma 3.7.

We have

\[
\rho_M(m) = \rho'_M(m), \]

for all \( m \in M \) and \( \rho_M(m) \).
for all \( m \in M \), so the proof is finished. \( \square \)

Let \( H \) be a quasi-bialgebra, \( \mathfrak{A} \) a right \( H \)-comodule algebra and \( C \) a left \( H \)-module coalgebra. Identifying the category of left-right \((H, \mathfrak{A}, C)\)-Hopf modules to the category of right-left \((H^{\text{op}}, \mathfrak{A}^{\text{op}}, C^{\text{co}})\)-Hopf modules using the construction preceding Proposition 222 we obtain the functor, after permuting the tensor factors:

\[
\mathcal{F}' = \bullet \otimes C : \mathfrak{A} \mathcal{M} \to \mathfrak{A} \mathcal{M} (H)^C.
\]

If \( M \) is a left \( \mathfrak{A} \)-module then \( \mathcal{F}'(M) = M \otimes C \) with structure maps

\[
\begin{align*}
\alpha \cdot (m \otimes c) &= a_{(0)} \cdot m \otimes a_{(1)} \cdot c, \\
\rho_M (m \otimes c) &= \tilde{x}_1^1 \cdot m \otimes \tilde{x}_2^2 \cdot \eta_{\mathfrak{A}}^1 \otimes \tilde{x}_3^3 \cdot \eta_{\mathfrak{A}}^2,
\end{align*}
\]

for all \( \alpha \in \mathfrak{A} \), \( m \in M \) and \( c \in C \). For a morphism \( \nu \) in \( \mathfrak{A} \mathcal{M} \), we have that \( \mathcal{F}'(\nu) = \nu \otimes \text{id}_C \). In particular, we obtain that \( \mathfrak{A} \otimes C \) is a left-right \((H, \mathfrak{A}, C)\)-Hopf module.

Moreover, \( \mathcal{F}' \) is a right adjoint of the forgetful functor \( \mathcal{U}^C : \mathfrak{A} \mathcal{M} (H)^C \to \mathfrak{A} \mathcal{M} \), and it is a left adjoint of the functor \( \text{Hom}^C_G (\mathfrak{A} \otimes C, \bullet) : \mathfrak{A} \mathcal{M} (H)^C \to \mathfrak{A} \mathcal{M} \) defined as follows. For \( M \in \mathfrak{A} \mathcal{M} (H)^C \), \( \text{Hom}^C_G (\mathfrak{A} \otimes C, \bullet)(M) = \text{Hom}^C_G (\mathfrak{A} \otimes C, M) \), the set of morphisms in \( \mathfrak{A} \mathcal{M} (H)^C \) between \( \mathfrak{A} \otimes C \) and \( M \), viewed as a left \( \mathfrak{A} \)-module via

\[
(a \cdot \eta) (a' \otimes c) = \eta(a' \alpha \otimes c),
\]

for all \( \eta \in \text{Hom}^C_G (\mathfrak{A} \otimes C, M), a, a' \in \mathfrak{A} \) and \( c \in C \). The functor \( \text{Hom}^C_G (\mathfrak{A} \otimes C, \bullet) \) sends a morphism \( \kappa \) from \( \mathfrak{A} \mathcal{M} (H)^C \) to the morphism \( \vartheta \mapsto \kappa \circ \vartheta \).

**Corollary 3.9.** Let \( H \) be a quasi-Hopf algebra, \( \mathfrak{A} \) an \( H \)-bicomodule algebra and \( C \) an \( H \)-bimodule coalgebra. We have a functor \( \mathfrak{F} = \bullet \otimes C : \mathfrak{A} \mathcal{M} \to \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C \). The structure maps on \( \mathfrak{F}(M) = M \otimes C \) are the following:

\[
\begin{align*}
\alpha \cdot (m \otimes c) &= u_{(0)} \cdot m \otimes u_{(1)} \cdot c \cdot S^{-1} (u_{-1}), \\
\rho_M \otimes C (m \otimes c) &= \psi^{2,1}_{(2), (1)} (\tilde{x}_1^1) (\tilde{x}_2^2) (\tilde{x}_3^3) \cdot m \otimes \theta^{2,1}_{(1)} \tilde{x}_2^2 \cdot \rho_M (m) \cdot a_{(1)} \cdot c_{(2)} \cdot S^{-1} (\tilde{x}_1^1),
\end{align*}
\]

for all \( u \in \mathfrak{A} \), \( m \in M \) and \( c \in C \). In particular, \( \mathfrak{A} \otimes C \) is a left-right \((H, \mathfrak{A}, C)\)-Yetter-Drinfeld module. Moreover, the following assertions hold:

i) \( \mathfrak{F} \) is right adjoint to the forgetful functor \( \mathfrak{U} : \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C \to \mathfrak{A} \mathcal{M} \).

ii) \( \mathfrak{F} \) is left adjoint to the functor \( \text{Hom}^C_G (\mathfrak{A} \otimes C, \bullet) : \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C \to \mathfrak{A} \mathcal{M} \). If \( M \in \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C \) then \( \text{Hom}^C_G (\mathfrak{A} \otimes C, \bullet)(M) = \text{Hom}^C_G (\mathfrak{A} \otimes C, M) \), the set of Yetter-Drinfeld morphisms from \( \mathfrak{A} \otimes C \) to \( M \), viewed as a right \( \mathfrak{A} \)-module via the action

\[
(u \cdot \eta) (u' \otimes c) = \eta(u' u \otimes c),
\]

for all \( \eta \in \text{Hom}^C_G (\mathfrak{A} \otimes C, M), u, u' \in \mathfrak{A} \) and \( c \in C \).

**Proof.** The functor \( \mathfrak{F} \) is well defined because it is the composition

\[
\mathfrak{F} : \mathfrak{A} \mathcal{M} = \mathfrak{A} \mathcal{M} \xrightarrow{\mathcal{F}'} \mathfrak{A} \mathcal{M} (H^{\text{op}} \otimes H)^C \xrightarrow{G} \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C,
\]

where \( \mathcal{F}' \) is the functor described above but now for the context given by \( \mathfrak{A}, \mathfrak{A} \mathcal{M}, \mathfrak{A} \mathcal{Y} \mathcal{D}(H)^C \), and \( G \) is the functor from Lemma 3.7. All the other details are left to the reader. \( \square \)
Let $H$ be a quasi-bialgebra, $\mathfrak{A}$ a right $H$-comodule algebra and $C$ a left $H$-module coalgebra. Since the category $\mathfrak{A}\mathcal{M}(H)^C$ can be identified to a category of right-left $\text{Doi-Hopf}$ modules it follows that $\mathfrak{A}\mathcal{M}(H)^C$ is isomorphic to a category of comodules over the coring $C' = C \otimes \mathfrak{A}$, with $\mathfrak{A}$-bimodule structure given by

$$a \cdot (c \otimes a') \cdot a'' = a_{(1)} \cdot (c \otimes a_{(0)} a' a'')$$

and comultiplication and counit given by

$$\Delta_C (c \otimes a) = (x_3^3 \cdot c_2 \otimes 1_{\mathfrak{A}}) \otimes A (x_3^3 \cdot c_2 \otimes x_3^1 a), \quad \varepsilon_C (c \otimes a) = \varepsilon(c) a$$

for all $a, a', a'' \in \mathfrak{A}$ and $c \in C$. Using arguments similar to the ones given in Theorem 5.4, one can easily check that $\mathfrak{A}\mathcal{M}(H)^C$ is isomorphic to $C^*\mathcal{M}$.

**Corollary 3.10.** Let $H$ be a quasi-$\text{Hopf}$ algebra, $\mathfrak{A}$ an $H$-bicomodule algebra and $C$ an $H$-bimodule coalgebra. Then the category of left-right Yetter-Drinfeld modules $\mathfrak{A}\mathcal{YD}(H)^C$ is isomorphic to the category of left comodules over the coring $C' = C \otimes \mathfrak{A}$, with the following structure maps. The $\mathfrak{A}$-bimodule is given by

$$u \cdot (c \otimes u') \cdot u'' = u_{[0]}(u_{(1)} \cdot c \cdot S^{-1}(u_{[-1]}) \otimes u_{[0]}) u' u''$$

and the comultiplication and counit are defined by the formulas

$$\Delta_C (c \otimes u) = \left(\theta^3 \tilde{x}_3^3 (\tilde{X}_3^3)_{(1)} \otimes 1_{\mathfrak{A}} \right)$$

$$\otimes A \left(\theta^3 \tilde{x}_3^3 (\tilde{X}_3^3)_{(1)} \cdot c_2 \cdot S^{-1}(\theta^3 \tilde{x}_3^2 (\tilde{X}_3^2) \otimes \theta^2 (\tilde{x}_2^1 (\tilde{X}_2^1) u)), \quad \varepsilon_C (c \otimes u) = \varepsilon(c) u,$$

for all $u, u', u'' \in \mathfrak{A}$ and $c \in C$.

**Proof.** This is a direct consequence of the above comments and Theorem 3.8. \(\square\)

### 3.3. The category of Yetter-Drinfeld modules as a module category.

Our next aim is to describe the category of Yetter-Drinfeld modules as a category of modules. We will need the (right) generalized diagonal crossed product construction, as introduced in [14, 17].

Let $H$ be a quasi-bialgebra and $\mathfrak{A}$ an $H$-bicomodule algebra. In the sequel, $A$ will be an $H$-bimodule algebra. This means that $A$ is an $H$-bimodule which has a multiplication and a usual unit $1_A$ such that for all $\varphi, \psi, \xi \in \mathfrak{A}$ and $h \in H$ the following relations hold:

$$\langle \varphi \psi \rangle \xi = \langle X^1 \cdot \varphi \cdot x^1 \rangle \langle \langle X^2 \cdot \psi \cdot x^2 \rangle \langle X^3 \cdot \xi \cdot x^3 \rangle \rangle,$$

$$h \cdot (\varphi \psi) = (h_1 \cdot \varphi)(h_2 \cdot \psi), \quad (\varphi \psi) \cdot h = (\varphi \cdot h_1)(\psi \cdot h_2),$$

$$h \cdot 1_A = \varepsilon(h) 1_A, \quad 1_A \cdot h = \varepsilon(h) 1_A.$$

If $H$ is a quasi-bialgebra, then $H^*$, the linear dual of $H$, is an $H$-bimodule via the $H$-actions

$$\langle h \rightsquigarrow \varphi, h' \rangle = \varphi(h' h), \quad \langle \varphi \leftarrow h, h' \rangle = \varphi(h h'),$$

for all $\varphi \in H^*$, $h, h' \in H$. The convolution $\langle \varphi \psi, h \rangle = \varphi(h_1) \psi(h_2)$, $\varphi, \psi \in H^*$, $h \in H$, is a multiplication on $H^*$ which is not associative in general, but with this multiplication $H^*$ becomes an $H$-bimodule algebra.

Let $H$ be a quasi-bialgebra, $A$ an $H$-bimodule algebra and $(\Lambda, \lambda, \rho, \Phi_\Lambda, \Phi_\rho, \Phi_{\lambda, \rho})$ an $H$-bicomodule algebra. In the sequel $(\delta, \Psi)$ will be the pair

$$\delta_1 = (\lambda \otimes id_H) \circ \rho,$$

$$\Psi = (id_H \otimes \lambda \otimes id_H^2) \left((\Phi_{\lambda, \rho} \otimes 1_H)(\lambda \otimes id_H^2)(\Phi_{\rho}^{-1})\right)[\Phi_\Lambda \otimes 1_H^2].$$
or

\[ \delta_r = (id_H \otimes \rho) \circ \lambda, \]
\[ \Psi_r = (id_H^\otimes \otimes \rho \otimes id_H) \left( (1_H \otimes \Phi_\lambda^{-1}) id_H^\otimes (\Phi_\lambda) \right) \cdot [1_H^\otimes \otimes \Phi_\rho^{-1}]. \]

\( \Omega_{L/r}, \Omega_{R/r} \in H^\otimes A \otimes H^\otimes \) are defined by the following formulas

\[ \Omega_{L/r} = (id_H^\otimes \otimes \lambda \otimes S^{-1} \otimes S^{-1}) (\Psi^{-1}_r \cdot [1_H^\otimes \otimes \lambda \otimes S^{-1} (f^1) \otimes S^{-1} (f^2)], \]
\[ \Omega_{R/r} = [S^{-1} (g^1) \otimes S^{-1} (g^2) \cdot 1_H \otimes 1_H^\otimes] \cdot (S^{-1} \otimes S^{-1} \otimes \lambda \otimes \lambda^\otimes) (\Psi^{-1}_r). \]

Here \( f^1 \otimes f^2 \) is the Drinfeld twist and \( f^{-1} = g^1 \otimes g^2 \) is its inverse. We will use the notation

\[ \Omega_{L/r} = \Omega_{L/r}^1 \otimes \cdots \otimes \Omega_{L/r}^5. \]

Let \( A \bowtie_\delta A \) and \( A \bowtie_{\delta, A} A \) be the vector space \( A \otimes A \) furnished with the multiplication given respectively by the following formulas:

\[ (\varphi \bowtie u) (\psi \bowtie u') = (\Omega_{L_1}^3 \varphi \cdot \Omega_{L_2}^3) (\Omega_{L_1}^2 u_{(0)} \cdot \psi \cdot S^{-1} (u_{(1)}) \Omega_{L_2}^2 \cdot u_{(0)}' \cdot u', \]
\[ (\varphi \bowtie u) (\psi \bowtie u') = (\Omega_{L_1}^3 \varphi \cdot \omega_{L_2}^3) (\Omega_{L_1}^2 u_{(0)} \cdot \psi \cdot S^{-1} (u_{(1)}) \Omega_{L_2}^2 \cdot u_{(0)}' \cdot u', \]

for all \( \varphi, \psi \in A \) and \( u, u' \in A \). We write \( \varphi \bowtie u \) (respectively \( \varphi \bowtie_{\delta} u \)) for \( \varphi \bowtie u \) considered as an element of \( A \bowtie_\delta A \) (respectively \( A \bowtie_{\delta, A} A \)). \( A \bowtie_\delta A \) and \( A \bowtie_{\delta, A} A \) are isomorphic associative algebras with unit \( 1_A \bowtie 1_A \bowtie 1_A \), containing \( A \bowtie_\delta A \bowtie_\delta A \) as unital subalgebra. These algebras are called the left generalized diagonal crossed products.

The right generalized diagonal crossed products are introduced in a similar way: denote

\[ \Omega_{R/r} = \Omega_{R/r}^1 \otimes \cdots \otimes \Omega_{R/r}^5, \]

and let \( A \bowtie_{\delta} A \) and \( A \bowtie_{\delta, A} A \) be the vector space \( A \otimes A \) with the following product:

\[ (u \otimes \varphi)(u' \otimes \psi) = uu_{(0)} \bowtie_\delta \Omega_{R_1}^3 \bowtie (\Omega_{R_2}^2 S^{-1} (u_{(0)}' \cdot \varphi \cdot u_{(1)}') \cdot \Omega_{R_3}^2 \cdot \psi \cdot \Omega_{R_4}^2), \]
\[ (u \otimes \varphi)(u' \otimes \psi) = uu_{(0)} \bowtie_{\delta, A} \Omega_{R_1}^3 \bowtie (\Omega_{R_2}^2 S^{-1} (u_{(0)}' \cdot \varphi \cdot u_{(1)}') \cdot \Omega_{R_3}^2 \cdot \psi \cdot \Omega_{R_4}^2), \]

for all \( u, u' \in A \), \( \varphi, \psi \in A \). \( A \bowtie_{\delta} A \) and \( A \bowtie_{\delta, A} A \) are isomorphic associative algebras with unit \( 1_A \bowtie 1_A \bowtie 1_A \) and \( 1_A \bowtie_{\delta, A} 1_A \), containing \( A \) as a unital subalgebra.

As algebras, the left and right generalized crossed product algebras are isomorphic, see [14]. If \( H \) is a quasi-Hopf algebra then \( A = H^* \) is an \( H \)-bimodule algebra. In this particular case the left and right generalized diagonal crossed products are exactly the left and the right diagonal crossed products constructed in [14]. In this way Hauser and Nill gave four explicit realizations of \( D(H) \), the quantum double of a finite dimensional quasi-Hopf algebra \( H \). Two of them are build on \( H^* \otimes H \) and the other two on \( H \otimes H^* \). All these are, as algebras, diagonal crossed products. The first two realizations built on \( H^* \otimes H \) coincide to \( H \bowtie_{\delta} H \) and \( H^* \bowtie_{\delta} H \), and the last two realizations built on \( H \otimes H^* \) coincide to \( H \bowtie_{\delta} H \) and \( H \bowtie_{\delta} H^* \).
Proposition 3.11. Let $H$ be a quasi-Hopf algebra, $\mathbb{A}$ an $H$-bicoreflective algebra and $C$ an $H$-bicomodule coalgebra. Then $\mathbb{A}^1 \bowtie C^* \equiv \mathbb{A} \bowtie_{\delta} C^*$ and $\mathbb{A}^2 \bowtie C^* \equiv \mathbb{A} \bowtie C^*$, as algebras. In particular, the algebras $\mathbb{A}^1 \bowtie C^*$ and $\mathbb{A}^2 \bowtie C^*$ are isomorphic to each other, and also to the four generalized diagonal crossed products.

Proof. Keeping the above concepts and notations it is not hard to see that for an $H$-bicomodule algebra $A$ the reassociators $\Phi_{\rho_1}$ and $\Phi_{\rho_2}$ defined in (3.12) and (3.14), respectively, can be rewritten as

$$\Phi_{\rho_1} = \Omega_2^3 \otimes (\tilde{\Omega}_2^2 \otimes \Omega_1^1) \otimes (\tilde{\Omega}_1^1 \otimes \tilde{\Omega}_2^1),$$

$$\Phi_{\rho_2} = \Omega_3^3 \otimes (\tilde{\Omega}_3^2 \otimes \tilde{\Omega}_1^1) \otimes (\tilde{\Omega}_1^1 \otimes \tilde{\Omega}_3^1),$$

where we used the notation

$$\Omega^{-1}_{r_1/r} = \Omega_{R_1/r}^1 \otimes \cdots \otimes \Omega_{R_1/r}^{r_i}.$$ 

Now, if $C$ is an $H$-bimodule coalgebra viewed as a left $H^\text{op} \otimes H$-module coalgebra via the structure defined in (3.10) then $C^*$, the linear dual space of $C$, is a right $H^\text{op} \otimes H$-module algebra. The multiplication of $C^*$ is the convolution, that is $(c^* \cdot d^*)(c) = c^*(c_1) d^*(c_2)$, the unit is $\mathbb{E}$ and the right $H^\text{op} \otimes H$-module action is given by the formula $(c^* \cdot (h \otimes h'))(c) = c^*(h' \cdot c \cdot h) = (h \cdot c^* \cdot h')$, for all $h, h' \in H$, $c^*, d^* \in C^*, c \in C$.

Since $\mathbb{A}^{1/2}$ are right $H^\text{op} \otimes H$-comodule algebras and $C^*$ is a right $H^\text{op} \otimes H$-module coalgebra, it makes sense to consider the right generalized smash product algebras $\mathbb{A}^{1/2} \bowtie C^*$ (cf. Remark 2.13)). It also follows easily from Remark 2.13) that the multiplication on $\mathbb{A}^1 \bowtie C^*$ is given by

$$(u \bowtie c^*)(u' \bowtie d^*) = uu'c^* \cdot d^*$$

$$= uu'c^* \cdot (\tilde{\Omega}_2^2 \otimes \tilde{\Omega}_1^1) \otimes (\tilde{\Omega}_1^1 \otimes \tilde{\Omega}_2^1) \otimes (\tilde{\Omega}_1^1 \otimes \tilde{\Omega}_2^1)).$$

On the other hand, it is easy to see that the linear dual space $C^*$ of an $H$-bicomodule coalgebra $C$ is an $H$-bimodule algebra. The multiplication of $C^*$ is the convolution, the unit is $\mathbb{E}$ and the $H$-bimodule structure is given by the formula $(h \cdot c^* \cdot h') = c^*(h' \cdot c \cdot h)$, for all $h, h' \in H$, $c^* \in C^*, c \in C$. So we can consider the generalized right diagonal crossed product $\mathbb{A} \bowtie_{\delta} C^*$. From (3.23) it follows that the multiplication rule on $\mathbb{A} \bowtie_{\delta} C^*$ coincides with the multiplication of $\mathbb{A}^1 \bowtie C^*$, hence the algebras $\mathbb{A} \bowtie_{\delta} C^*$ and $\mathbb{A}^1 \bowtie C^*$ are equal. In a similar way we can show the equality of the $k$-algebras $\mathbb{A}^2 \bowtie C^*$ and $\mathbb{A} \bowtie_{\delta} C^*$.

Remark 3.12. It was shown in [7] that the left generalized crossed product algebras $\mathbb{A} \bowtie_{\delta} \mathbb{A}$ and $\mathbb{A} \bowtie_{\delta} \mathbb{A}$ coincide with the left generalized smash product algebras $\mathbb{A} \bowtie \mathbb{A}$ and $\mathbb{A} \bowtie \mathbb{A}$, respectively. The generalized smash products are made over $H \otimes H^\text{op}$ and by $\mathbb{A}^1$ and $\mathbb{A}^2$ we denote the left $H \otimes H^\text{op}$-comodule algebra structures on $\mathbb{A}$ defined at the end of Section II.

Let $H$ be a quasi-bialgebra, $\mathfrak{M}$ a right $H$-comodule algebra and $C$ a left $H$-module algebra. Viewing the category $\mathfrak{M}M(H)^C$ as a category of right-left Doi-Hopf modules, we deduce from Theorem 3.8 that $\mathfrak{M}M(H)^C$ is isomorphic to the category of rational $\mathfrak{M} \bowtie C^*$-modules, $\text{Rat}(\mathfrak{M} \bowtie C^*) = \sigma_{\mathfrak{M} \bowtie C^*} \cdot (\mathfrak{M} \otimes C)$. 

A rational $A \triangleright \triangleright C^\ast$-module is a left $A \triangleright \triangleright C^\ast$-module $M$ such that for any $m \in M$ there exist two finite families $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ such that
\[(a \triangleright \triangleright c^\ast) \cdot m = c^\ast(c_i)(a \triangleright \triangleright \varepsilon) \cdot m_i,\]
for all $a \in A$ and $c^\ast \in C^\ast$.

**Corollary 3.13.** Let $H$ be a quasi-Hopf algebra, $A$ an $H$-bicomodule algebra and $C$ an $H$-bimodule coalgebra. Then the following assertions hold:

i) The category of left-right Yetter-Drinfeld modules $\mathcal{YD}(H)_C$ is isomorphic to the category of rational $A \triangleright \triangleright \delta_r C^\ast$-modules $\mathcal{Rat}(A \triangleright \triangleright \delta_r C^\ast M)$, which is also equal to the category $\sigma_{A \triangleright \triangleright \delta_r C^\ast}[A \otimes C]$.

ii) If $C$ is finite dimensional then $\mathcal{YD}(H)_C$ is isomorphic to the category of left $C^\ast \bowtie \delta_l A$-modules.

**Proof.** The assertion i) follows easily from the above comments and Theorem 3.8.

ii) If $C$ is finite dimensional then $\mathcal{Rat}(A \triangleright \triangleright \delta_r C^\ast M) = A \triangleright \triangleright \delta_r C^\ast M$. Moreover, $A \triangleright \triangleright \delta_r C^\ast$ is always isomorphic to $C^\ast \bowtie \delta_l A$ as an algebra. We note that another proof of this result can be found in [7]. □

**References**

[1] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois properties, *Algebr. Represent. Theory* 5 (2002), 389–410.
[2] T. Brzeziński and R. Wisbauer, “Corings and comodules”, *London Math. Soc. Lect. Note Ser. 309*, Cambridge University Press, Cambridge, 2003.
[3] D. Bulacu and S. Caenepeel, Two-sided two-cosided Hopf modules and Doi-Hopf modules for quasi-Hopf algebras, *J. Algebra* 270 (2003), no. 1, 55–95.
[4] D. Bulacu, S. Caenepeel and F. Panaite, Yetter-Drinfeld categories for quasi-Hopf algebras, *Comm. Algebra*, to appear.
[5] D. Bulacu, S. Caenepeel and F. Panaite, More properties of Yetter-Drinfeld modules over quasi-Hopf algebras, in “Hopf algebras in non-commutative geometry and physics”, S. Caenepeel and F. Van Oystaeyen (eds.), Lecture Notes Pure Appl. Math. 239, Dekker, New York, 2004.
[6] D. Bulacu and E. Nauwelaerts, Relative Hopf modules for (dual) quasi Hopf algebras, *J. Algebra* 229 (2000), 632–659.
[7] D. Bulacu, F. Panaite and F. Van Oystaeyen, Generalized diagonal crossed products and smash products for (quasi) Hopf algebras, preprint 2004.
[8] S. Caenepeel, G. Militaru and S. Zhu, Crossed modules and Doi-Hopf modules, *Israel J. Math.* 100 (1997), 221–247.
[9] S. Dăscălescu, C. Năstăsescu and B. Torrecillas, Co-Frobenius Hopf Algebras: Integrals, Doi-Koppinen Modules and Injective Objects, *J. Algebra* 220 (1999), 542–560.
[10] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, Hopf Algebras: An Introduction, in: Monographs Textbooks in Pure Appl. Math., Vol. 235, Dekker, New York, 2001.
[11] Y. Doi, Generalized smash products and Morita contexts for arbitrary Hopf algebras, in “Advances in Hopf algebras”, J. Bergen and S. Montgomery, eds., *Lecture Notes Pure Appl. Math. 158*, Dekker, New York, 1994.
[12] V. G. Drinfeld, Quasi-Hopf algebras, *Leningrad Math. J.* 1 (1990), 1419–1457.
[13] L. El Kaoutit, J. Gómez-Torrecillas, F. Lobillo, Semisimple corings, *Algebra Colloquium*, to appear.
[14] F. Hausser and F. Nill, Diagonal crossed products by duals of quasi-quantum groups, *Rev. Math. Phys.* 11 (1999), 553–629.
[15] C. Kassel, “Quantum Groups”, *Graduate Texts in Mathematics 155*, Springer Verlag, Berlin, 1995.
[16] M. Koppinen, Variations on the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* 104 (1994), 61–80.
[17] S. Majid, Quantum double for quasi-Hopf algebras, *Lett. Math. Phys.* 45 (1998), 1–9.
[18] S. Majid, “Foundations of quantum group theory”, Cambridge Univ. Press, 1995.
[19] C. Menini and M. Zucconi, Equivalence theorems and Hopf-Galois extensions, J. Algebra 194 (1997), 245–274.
[20] M. E. Sweedler, “Hopf algebras”, Benjamin, New York, 1969.
[21] R. Wisbauer, ”Foundations of module and ring theory”, Gordon and Breach, Philadelphia, 1991.

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