Projective model structures on diffeological spaces
and smooth sets and the smooth Oka principle

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Abstract. We prove that the category of diffeological spaces does not admit a model structure transferred via the smooth singular complex functor from simplicial sets, resolving in the negative a conjecture of Christensen and Wu. Embedding diffeological spaces into sheaves of sets (not necessarily concrete) on the site of smooth manifolds, we then prove the existence of a proper combinatorial model structure on such sheaves transferred via the smooth singular complex functor from simplicial sets. We show the resulting model category to be Quillen equivalent to the model category of simplicial sets. We then show that this model structure is cartesian, all smooth manifolds are cofibrant, and establish the existence of model structures on categories of algebras over operads. We use these results to establish analogous model structures on simplicial presheaves on smooth manifolds, as well as presheaves valued in left proper combinatorial model categories, and prove a generalization of the smooth Oka principle established in arXiv:1912.10544. We finish by establishing classification theorems for differential-geometric objects like closed differential forms, principal bundles with connection, and higher bundle gerbes with connection on arbitrary cofibrant diffeological spaces.

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1 Introduction

Diffeological spaces were defined by Souriau [1980], with some closely related work previously done by Chen [1973.a]. The category \textbf{Diffeo} of diffeological spaces is a Grothendieck quasitopos (Remark 2.3), which is a particularly nice type of a category: it is complete and cocomplete, cartesian closed and locally cartesian closed category. Furthermore, \textbf{Diffeo} contains the category of smooth manifolds as a full subcategory. This makes \textbf{Diffeo} a convenient category to work with infinite-dimensional mapping spaces of manifolds and other smooth spaces more general than manifolds. A book-length treatment by Iglesias-Zemmour [2013.a] contains many examples illustrating the power of this formalism.

A closely related notion is that of \textbf{smooth sets}. A \textbf{smooth set} (Definition 2.4) is a sheaf of sets on the site of smooth manifolds and open covers. A \textbf{diffeological space} (Definition 2.7) is a smooth set $F$ that is a \textbf{concrete sheaf} (Remark 2.2): if two sections $s, t \in F(M)$ ($M \in \text{Man}$) coincide on every point $p : R^0 \to M$ (meaning $p^*s = p^*t$, where $p^* : F(M) \to F(R^0)$), then $s = t$. Morphisms of \textbf{smooth sets} and \textbf{diffeological spaces} are simply morphisms of sheaves. Thus, \textbf{smooth sets} contain \textbf{diffeological spaces} as a full subcategory.
The category of smooth sets is a Grothendieck topos, so it inherits all the nice properties of diffeological spaces and, in addition, it is a balanced category: if a morphism is a monomorphism and epimorphism, then it is an isomorphism. This last property is essential for showing that the category of abelian group objects in smooth sets is a Grothendieck abelian category, which immediately allows us to do homological algebra in this setting. In contrast, the category of abelian group objects in diffeological spaces is not an abelian category.

In complete analogy to topological spaces, one can define a (smooth) singular complex functor \( \text{SmSing} \) (Definition 3.5), which endows the categories of smooth sets and diffeological spaces with a relative category structure: a morphism \( f \) of smooth sets is a weak equivalence if \( \text{SmSing} f \) is a weak equivalence of simplicial sets. Continuing the analogy to topological spaces, one can then inquire whether the resulting relative categories of smooth sets and diffeological spaces can be promoted to model categories, by creating the class of fibrations using the functor \( \text{SmSing} \) and whether this turns \( \text{SmSing} \) into a right Quillen equivalence of model categories.

The first main result of this paper is that the answer in the case of diffeological spaces is mostly negative, whereas for smooth sets we do indeed get a Quillen equivalence of model categories.

**Theorem 1.1.** The category \( \text{Diffeo} \) of diffeological spaces (Definition 2.7) does not admit a model structure that is transferred (Definition 5.2) along the right adjoint functor \( \text{SmSing} : \text{Diffeo} \to \mathbf{sSet} \) (Definition 3.5), meaning its weak equivalences and fibrations are created by the functor \( \text{SmSing} \). However, the functor \( \text{SmSing} \) is a Dwyer–Kan equivalence of relative categories.

**Proof.** Combine Theorem 6.3 with Corollary 7.9.

**Theorem 1.2.** The category \( \text{SmSet} \) of smooth sets (Definition 2.4) admits a model structure transferred (Definition 5.2) along the right adjoint functor \( \text{SmSing} : \text{SmSet} \to \mathbf{sSet} \) (Definition 3.5), meaning its weak equivalences and fibrations are created by the functor \( \text{SmSing} \). Smooth boundary inclusions and smooth horn inclusions form a set of generating cofibrations respectively generating acyclic cofibrations. This model structure is left and right proper, combinatorial, cartesian (Definition 5.7), h-monoidal, symmetric h-monoidal, and flat (Proposition 11.1). All smooth manifolds \( M \) are cofibrant in this model structure and for every smooth manifold \( M \) the internal hom functor \( \text{Hom}(M, -) \) preserves weak equivalences. The functor \( \text{SmSing} \) is a right Quillen equivalence. Operads in these model categories and algebras over them enjoy a good set of properties, as described in Proposition 11.2, Proposition 11.3, Proposition 11.4. Analogous results hold for the category \( \text{PreSmSet} \) of presheaves of sets. Used in \( \text{2.1, 2.2, 2.3} \).

**Proof.** Combine Theorem 7.4, Proposition 8.11, Proposition 9.2, Proposition 10.1, Proposition 11.1, Proposition 11.2, Proposition 11.3, Proposition 11.4.

In 1999, Hovey [1999b, Problem 2] already inquired whether sheaves on a manifold admit a model structure, and the model structure constructed in this paper can be seen as one possible answer to this question: for a fixed manifold \( M \) we can take the slice model category of smooth sets over \( M \).

Finally, we extend Theorem 1.2 to the case of sheaves and presheaves valued in a left proper combinatorial model category \( V \), such as simplicial sets or chain complexes. This is relevant for applications, since many differential-geometric structures of interest such as the moduli stack of principal \( G \)-bundles with connection or the moduli stack of higher bundle gerbes with connection are encoded by such presheaves.

**Theorem 1.3.** Suppose \( V \) is a left proper combinatorial model category. The category \( \text{Sm}_V \) of \( V \)-valued sheaves and the category \( \text{PreSm}_V \) of \( V \)-valued presheaves admit a model structure with weak equivalences created by the \( \text{shape} \) functor (Definition 13.7) and generating cofibrations analogous to those of Theorem 1.2. This model structure is left proper, combinatorial, and inherits from \( V \) properties like being monoidal, h-monoidal, symmetric h-monoidal, and flat (Theorem 12.7). All smooth manifolds \( M \) are cofibrant in this model structure and the internal hom functor \( \text{Hom}(M, -) \) preserves weak equivalences. This model structure is Quillen equivalent to \( V \) via a zigzag of Quillen equivalences. Operads in these model categories and algebras over them enjoy a good set of properties analogous to those of Theorem 1.2.

**Proof.** Combine Theorem 12.7, Theorem 12.9, Theorem 12.11, and Theorem 13.8.
The closest in spirit to our paper is the work of Christensen–Wu [2013.c], who develop the homotopy theory of diffeological spaces using the functor SmSing (Definition 3.5). In particular, we settle several of the conjectures stated in their paper, including the nonexistence of a transferred model structure on diffeological spaces (Theorem 6.3) and a positive answer for the enlarged category of smooth sets in Theorem 7.4, cofibrancy of smooth manifolds (Proposition 9.2), cartesianness of the model structure on smooth sets (Proposition 8.10), in addition to the conjecture on the coincidence of smooth homotopy groups of diffeological spaces with the simplicial homotopy groups of their smooth singular simplicial sets (Corollary 10.2), which was already resolved in Berwick-Evans–Boavida de Brito–Pavlov [2019.b, Proposition 2.18].

1.4. Previous work

Kihara [2016] constructs a cosimplicial object in diffeological spaces by introducing a nonstandard diffeology on (nonextended) smooth simplices that turns smooth horn inclusions into deformation retracts, proves that the category of diffeological spaces admits a model structure transferred along the singular complex functor associated to this cosimplicial object, and shows that the resulting Quillen adjunction between simplicial sets and diffeological spaces is a Quillen equivalence. In the resulting model structure all diffeological spaces are fibrant and by Corollary 10.2 combined with Kihara [2016, Theorem 1.4] its weak equivalences coincide with the weak equivalences of Christensen–Wu [2013.c, Definition 4.8], which we also use in this paper (Definition 3.8). In particular, Kihara’s model structure is connected to the model structure of Theorem 1.2 by a chain of Quillen equivalences.

Kihara [2020.a, Theorem 1.11] proves that the class of diffeological spaces that are smoothly homotopy equivalent to cofibrant diffeological spaces is closed under gluing of D-numerable covers. In particular, this class contains a large class of infinite-dimensional manifolds (Kihara [2020.a, Theorem 11.1]). It would be interesting to see whether cofibrancy in the Kihara model structure could be established for differential-geometric objects like smooth manifolds.

We also point out the ongoing work of Haraguchi–Shimakawa [2020.c] on a different model structure on diffeological spaces, which is not cofibrantly generated.

Cisinski [2002.b, Théorème 3.9] proves a general result that constructs a model structure on smooth sets with the class of weak equivalences of Definition 3.8 and monomorphisms as cofibrations. In the closely related subject of simplicial smooth sets (i.e., simplicial presheaves on the site of cartesian spaces or the site of smooth manifolds), we point out the work of Sati–Schreiber–Stasheff [2008.b, §5.1] and [2009, §3], Ayala–Francis–Rozenblyum [2015.a, §2], Berwick-Evans–Boavida de Brito–Pavlov [2019.b] (which proves the smooth Oka principle), Bunk [2020.b]. Clough (preliminary manuscript available as 2021.a) constructs several Kihara-type noncartesian model structures on various types of presheaves, including both simplicial presheaves and sheaves of sets. Additional applications of the smooth Oka principle can be found in Sati–Schreiber [2021.c].

1.5. Acknowledgments

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2. Review of diffeological spaces and smooth sets

Definition 2.1. The category Cart of cartesian spaces has finite-dimensional real vector spaces as objects and smooth (i.e., infinitely differentiable) maps as morphisms. To make Cart a small category, we take the full subcategory on vector spaces $\mathbb{R}^m$ for all $m \geq 0$. We turn Cart into a site by equipping it with the Grothendieck topology generated by the coverage of all open covers whose finite intersections are empty or diffeomorphic to an object in Cart. Used in 2.1, 2.2, 2.4, 2.5, 2.7, 2.8, 2.9, 2.12, 3.2, 3.3, 3.4, 3.5, 4.1*, 6.1, 12.0*, 12.2, 12.6, 12.10, 12.10*, 13.2.

Remark 2.2. The site Cart (Definition 2.1) is a concrete site (Dubuc [1979, Definition 1.4]) meaning it has a terminal object $1 = \mathbb{R}^0$ such that $\text{hom}(1, -) : \text{Cart} \to \text{Set}$ is a faithful functor and for any covering family $\{ f_i : U_i \to V \}_{i \in I}$ the induced map of sets

$$\prod_{i \in I} \text{hom}(1, f_i) : \prod_{i \in I} \text{hom}(1, U_i) \to \text{hom}(1, V)$$
is surjective. On any concrete site one can define a concrete quasitopos (Dubuc [1979, Definition 1.3]) of concrete sheaves (Dubuc [1979, Definition 1.5]), where a presheaf

\[ F: \text{Cart}^\text{op} \rightarrow \text{Set} \]

is concrete if the canonical map

\[ F(X) \rightarrow \text{hom}(\text{hom}(1, X), F(1)) \]

adjoint to the map

\[ F(X) \times \text{hom}(1, X) \rightarrow F(1) \]

induced by the structure maps of the presheaf \( F \) is an injection of sets. Used in 1.0*, 2.3, 2.7, 2.8, 2.10, 2.12.

**Remark 2.3.** The category of concrete sheaves on any small concrete site (Remark 2.2) is a Grothendieck quasitopos (Penon [1973,b, 1977], Dubuc [1974, Theorem 1.7], Baez-Hoffnung [2008,a, Theorem 52], Johnstone [2002,a, Theorem C2.2.13]). Any Grothendieck quasitopos is a locally presentable category that is locally cartesian closed. Used in 1.0*.

We now introduce the main categories of this paper.

**Definition 2.4.** The Grothendieck topos \( \text{SmSet} \) of smooth sets is the category of sheaves of sets on the site \( \text{Cart} \) (Definition 2.1). Used in 1.0*, 1.2, 1.4*, 2.6, 2.9, 2.10, 2.13, 3.5, 3.6, 3.7, 3.8, 4.0*, 4.2, 6.0*, 7.0*, 7.2, 7.2*, 7.3, 7.3*, 7.4, 7.4*, 7.5, 7.6, 7.8, 7.8*, 8.7, 8.8, 8.9, 8.11, 9.0*, 9.2, 10.1, 10.2, 11.1, 11.1*, 11.2, 11.3, 11.4, 12.0*, 12.2, 14.0*, 14.1.

**Definition 2.5.** The Grothendieck topos \( \text{PreSmSet} \) of presmooth sets is the category of presheaves of sets on the site \( \text{Cart} \) (Definition 2.1). Used in 1.2, 2.6, 2.9, 2.10, 2.13, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 4.0*, 4.1, 4.1*, 4.2, 6.0*, 7.0*, 7.2, 7.2*, 7.3, 7.3*, 7.4, 7.4*, 7.5, 7.8, 7.8*, 8.7, 8.8, 8.9, 8.11, 11.1, 11.1*, 11.2, 11.3, 12.0*, 12.7*, 12.11*.

**Remark 2.6.** The inclusion \( \text{SmSet} \rightarrow \text{PreSmSet} \) (Definition 2.4, Definition 2.5) is a reflective subcategory. In particular, we have a left adjoint reflection functor \( \text{PreSmSet} \rightarrow \text{SmSet} \) known as the associated sheaf functor. Used in 2.9, 2.10, 4.1, 6.0*, 7.4, 7.4*, 11.1*, 12.2, 12.7*.

A precursor for the following definition can be found in Chen [1973.a], the modern definition first appeared in Souriau [1980], and a book-length treatment is given by Iglesias-Zemmour [2013.a].

**Definition 2.7.** The Grothendieck quasitopos \( \text{Diffeo} \) of diffeological spaces is the category of concrete sheaves of sets (Remark 2.2) on the site \( \text{Cart} \) (Definition 2.1). Used in 1.0*, 1.1, 2.9, 2.10, 2.13, 3.4, 3.5, 3.7, 3.8, 6.0*, 6.2, 6.3, 6.3*, 7.4*, 7.8*, 7.9.

**Definition 2.8.** The Grothendieck quasitopos \( \text{PreDiffeo} \) of prediffeological spaces is the category of concrete presheaves of sets (Remark 2.2) on the site \( \text{Cart} \) (Definition 2.1). Used in 2.9, 2.10, 2.13, 3.5, 3.7, 3.8, 6.0*, 7.4*, 7.9.

**Remark 2.9.** The inclusion \( \text{PreDiffeo} \rightarrow \text{PreSmSet} \) (Definition 2.8, Definition 2.5) is a reflective subcategory. In particular, we have a left adjoint reflection functor \( \text{PreSmSet} \rightarrow \text{PreDiffeo} \) known as the concretization functor. Concretely, the reflection map \( F \rightarrow G \) is the quotient map of presheaves that identifies two sections \( s, t \in F(U) \) \( U \in \text{Cart} \) if for all \( w: \Delta^0 \rightarrow U \) we have \( su = tu \), i.e., \( s \) and \( t \) induce the same maps on the underlying sets of points. The inclusion \( \text{Diffeo} \rightarrow \text{SmSet} \) (Definition 2.7, Definition 2.4) is also a reflective
subcategory, with the reflection functor computed as the concretization functor followed by the associated sheaf functor (Remark 2.6). Used in 2.9, 3.2, 3.3, 3.4, 3.5, 3.9, 3.10*

**Remark 2.10.** Limits in the categories \(\text{PreSmSet}, \text{SmSet}, \text{PreDiffeo}, \text{and Diffeo}\) are computed objectwise, since the sheaf property and concrete presheaf property are preserved under limits. Colimits in these categories are computed as follows:

- In \(\text{PreSmSet}\), colimits are computed objectwise.
- In \(\text{SmSet}\), colimits are computed by applying the associated sheaf functor (Remark 2.6) to the colimit in \(\text{PreSmSet}\).
- In \(\text{PreDiffeo}\), colimits are computed by applying the concretization functor (Remark 2.9) to the colimit in \(\text{PreSmSet}\).
- In \(\text{Diffeo}\), colimits are computed by applying the associated sheaf functor (Remark 2.6) to the colimit in \(\text{PreDiffeo}\). Since the latter is always a separated presheaf, the associated sheaf can be computed using the plus construction.

Used in 4.2*

**Definition 2.11.** The category \(\text{Man}\) of smooth manifolds has smooth manifolds as objects and smooth maps as morphisms. To make \(\text{Man}\) a small category, we take the full subcategory on smooth manifolds whose underlying set is a subset of the underlying set of \(\mathbb{R}\) (ignoring its topology). We turn \(\text{Man}\) into a small site by equipping it with the Grothendieck topology generated by the coverage of all open covers. Used in 1.0*, 2.11, 2.12, 2.13, 12.6, 13.1, 13.2, 13.8.

**Remark 2.12.** The restriction along the inclusion of sites \(\text{Cart} \to \text{Man}\) (Definition 2.1, Definition 2.11) induces equivalences of categories of sheaves of sets, as well as concrete sheaves of sets (Remark 2.2).

**Remark 2.13.** The (restricted) Yoneda embedding construction induces fully faithful functors (denoted by \(y\))

\[\begin{align*}
\text{Man} & \to \text{Diffeo}, \\
\text{Man} & \to \text{PreDiffeo}, \\
\text{Man} & \to \text{SmSet}, \\
\text{Man} & \to \text{PreSmSet}
\end{align*}\]

Used in 2.9, 3.2, 3.3, 3.4, 3.5, 3.6, 3.9, 3.10*, 4.1*, 4.3, 8.7*, 9.0*, 9.1, 9.2*, 10.2, 10.2*, 10.3, 10.3*, 12.4, 12.10, 12.10*, 13.7, 14.1.

3 Smooth singular complex and realization

**Definition 3.1.** The category \(\Delta\) of simplices is the category of finite nonempty totally ordered sets and order-preserving maps. To make \(\Delta\) small, we restrict to the full subcategory of objects given by standard simplices \([m] = \{0 < \cdots < m\}\) for all \(m \geq 0\). The category \(\text{SSet}\) of simplicial sets is defined as the category of presheaves of sets on \(\Delta\). Used in 2.7, 3.2, 3.3, 3.4, 3.5, 3.9, 3.10*, 4.1*, 4.3, 8.7*, 9.0*, 9.1, 9.2*, 10.2, 10.2*, 10.3, 10.3*, 12.4, 12.10, 12.10*, 13.7, 14.1.

**Definition 3.2.** The functor

\[\Delta \cdot \Delta \to \text{Cart}\]

(Definition 2.1) sends a simplex \([m]\) to the extended smooth simplex

\[\Delta^m = \left\{ x \in \mathbb{R}^{[m]} \left| \sum_{i \in [m]} x_i = 1 \right. \right\}\]

and a map of simplices \(f: [m] \to [n]\) to the smooth map

\[\Delta^f: \Delta^m \to \Delta^n, \quad x \mapsto \left( j \mapsto \sum_{i: f(i) = j} x_i \right).\]

Used in 2.7, 3.2, 3.3, 3.4, 3.5, 3.9, 3.10*, 4.1*, 4.3, 8.7*, 9.0*, 9.1, 9.2*, 10.2, 10.2*, 10.3, 10.3*, 12.4, 12.10, 12.10*, 13.7, 14.1.

**Definition 3.3.** The functor

\[\Delta: \Delta \to \text{Cart}\]

(Definition 2.1)
(Definition 2.1) sends a simplex $[m]$ to the *interior smooth simplex* 

$$\Delta^n = \left\{ x \in (0, \infty)^{[m]} \left| \sum_{i \in [m]} x_i = 1 \right. \right\}$$

and a map of simplices $f: [m] \to [n]$ to the restriction of $\Delta^n: \Delta^n \to \Delta^n$. Used in 9.2.

**Definition 3.4.** Consider the composition of functors 

$$\Delta \to \text{Cart} \to \text{Diffeo}$$

(Definition 3.2, Remark 2.13). The functor 

$$\Delta_\ast: \Delta \to \text{Diffeo}$$

sends a simplex $[m]$ to the *closed smooth simplex* 

$$\Delta^n = \left\{ U \mapsto \left\{ x \in \text{hom}(U, R)^{[m]} \left| (\forall i: x_i \geq 0) \land \sum_{i \in [m]} x_i = 1 \right. \right\} \right\},$$

which is not a representable sheaf on $\text{Cart}$, but still defines an object of $\text{Diffeo}$. Used in 9.1, 9.2.

**Definition 3.5.** The adjunctions 

$$\| - \|: \text{sSet} \to \text{PreSmSet} \quad \vdash \quad \text{SmSing}: \text{PreSmSet} \to \text{sSet},$$

$$\| - \|: \text{sSet} \to \text{PreDiffeo} \quad \vdash \quad \text{SmSing}: \text{PreDiffeo} \to \text{sSet},$$

$$| - |: \text{sSet} \to \text{SmSet} \quad \vdash \quad \text{SmSing}: \text{SmSet} \to \text{sSet},$$

$$| - |: \text{sSet} \to \text{Diffeo} \quad \vdash \quad \text{SmSing}: \text{Diffeo} \to \text{sSet},$$

are defined as follows. The right adjoints are given by (the restrictions of) the *smooth singular simplicial set* (alias *smooth singular complex*) functor 

$$\text{SmSing}: \text{PreSmSet} \to \text{sSet}$$

(with $\text{PreSmSet}$ as in Definition 2.4), which is the nerve functor associated to the cosimplicial object 

$$\Delta: \Delta \to \text{Cart} \to \text{PreSmSet}$$

(Definition 3.3, Remark 2.13). That is, for any $F \in \text{PreSmSet}$ and $n \geq 0$ we have 

$$\text{SmSing}(F)_n = \text{hom}(\Delta^n, F) = F(\Delta^n)$$

and likewise for simplicial structure maps. The left adjoints are the corresponding realization functors. For $\text{PreSmSet}$ and $\text{PreDiffeo}$, we get the *presmooth realization* functor 

$$| - |: \text{sSet} \to \text{PreSmSet}$$

For $\text{SmSet}$ and $\text{Diffeo}$, we get the *smooth realization* functor 

$$| - |: \text{sSet} \to \text{SmSet}$$

**Remark 3.6.** Concretely, $| - |$ sends a simplicial set $X$ to the colimit 

$$\text{colim}_{x \in X} U(x),$$

where $\Delta/X$ is the category of simplices of $X$ (objects are pairs ($[m] \in \Delta, x \in X_m$), morphisms ($[m], x) \to ([n], y)$ are maps of simplices $f: [m] \to [n]$ such that $X_f(y) = x$) and $U: \Delta/X \to \Delta \to \text{SmSet}$ denotes the
Consider the model category $\mathbf{L}$. The functor $\mathbf{L} \to \mathbf{F}$ sends the Čech nerve of a good open cover to a weak equivalence of simplicial presheaves. Consider the functor $\mathbf{L}$ and the same functor $\mathbf{L} \to \mathbf{F}$ (appropriately restricted) works for $\mathbf{SmSet}$ and $\mathbf{Diffeo}$.

**Definition 3.8.** The category $\mathbf{PreSmSet}$ (Definition 2.5) is turned into a relative category by postulating that its weak equivalences are precisely those morphisms whose image under $\mathbf{SmSing}$ (Definition 3.3) is a weak equivalence of simplicial sets. The categories $\mathbf{SmSet}$ (Definition 2.4), $\mathbf{Diffeo}$ (Definition 2.7), and $\mathbf{PreDiffeo}$ (Definition 2.8) are turned into relative categories in the same way.

**Definition 3.9.** A *smooth homotopy* between morphisms $f, g: A \to B$ in $\mathbf{PreSmSet}$ is a morphism of transfinite presheaves

$$h: \Delta \times A \to B,$$

where

$$\iota_k: A \to \{k\} \times A \to \Delta \times A$$

is the corresponding inclusion map. A *smooth homotopy equivalence* is a map $f: A \to B$ in $\mathbf{PreSmSet}$ such that there is a map $g: B \to A$ with a smooth homotopy from $\iota_A$ to $gf$ and a smooth homotopy from $f\iota$ to $id_B$. A *smooth deformation retraction* is a map $f: A \to B$ in $\mathbf{PreSmSet}$ that can be made into a smooth homotopy equivalence in such a way that $\iota_A = gf$.

**Proposition 3.10.** The functor $\mathbf{SmSing}$ sends smoothly homotopic maps in $\mathbf{PreSmSet}$ to simplicially homotopic maps in $\mathbf{sSet}$, smooth homotopy equivalences in $\mathbf{PreSmSet}$ to simplicial homotopy equivalences in $\mathbf{sSet}$, and smooth deformation retractions in $\mathbf{PreSmSet}$ to simplicial deformation retractions in $\mathbf{sSet}$.

**Proof.** (See also Christensen–Wu [2013, Lemma 4.10].) This follows immediately from the fact that $\mathbf{SmSing}$ is a right adjoint, in particular, it preserves small limits such as products used in the definition of a smooth homotopy. The canonical map $\Delta^1 \to \mathbf{SmSing}(\Delta)$ can be used to extract simplicial homotopies from $\mathbf{SmSing}$ evaluated on smooth homotopies.

### 4 The associated sheaf and concretization

The following result provides a powerful tool to work with colimits of smooth sets, by allowing us to replace them with colimits of presmooth sets, which are much easier to work with because colimits of presheaves are computed objectwise.

**Proposition 4.1.** Suppose $F \in \mathbf{PreSmSet}$ (Definition 2.3) and $s: F \to G$ is the canonical morphism from $F$ to its associated sheaf $G$ (Remark 2.6). Then the map $F \to G$ is a weak equivalence in the relative category $\mathbf{PreSmSet}$ (Definition 3.8). More generally, any local isomorphism of presheaves is a weak equivalence in $\mathbf{SmSing}$.

**Proof.** Consider the model category $M$ of simplicial presheaves on the site $\mathbf{Cart}$ equipped with its injective model structure left Bousfield localized at Čech nerves of good open covers. Consider the functor $L$ from $M$ to simplicial sets that sends a simplicial presheaf $F$ to the diagonal of the bisimplicial set $n \mapsto F(\Delta^n)$. The functor $L$ is a left Quillen functor that preserves monomorphisms and objectwise weak equivalences. Furthermore, the functor $L$ sends the Čech nerve of a good open cover to a weak equivalence of simplicial sets. Thus, $L$ is a left Quillen functor that preserves weak equivalences. In the model category $M$, the map $F \to G$ is a weak equivalence, hence so is $L(F) \to L(G)$. It remains to observe that for presheaves of sets we have $L = \mathbf{SmSing}$, so $\mathbf{SmSing}$ sends a weak equivalence in $\mathbf{PreSmSet}$ to an equivalent of simplicial sets and $F \to G$ is a weak equivalence in $\mathbf{PreSmSet}$. The following special case of Proposition 4.1 is important enough to be stated separately.
Proposition 4.2. Suppose $D: I \to \text{SmSet}$ is a diagram of smooth sets (Definition 2.4), $G$ its colimit, and $F$ its colimit in the category of presheaves (Definition 2.5). Then the canonical map $F \to G$ is a weak equivalence (Definition 3.8). Used in $7.3^\ast, 7.4^\ast$.

Proof. The map $F \to G$ maps $F$ to its associated sheaf $G$, since colimits of sheaves can be computed as associated sheaves of colimits in presheaves (Remark 2.10). We conclude by Proposition 4.1.

Remark 4.3. The analogous result for the concretization functor (Remark 2.9) is false. Consider the sheaf $F$ of closed differential $n$-forms, where $n > 0$. This sheaf is not concrete and its concretization (Remark 2.9) is $\Delta^0$ because $F(\Delta^0)$ is a single point. However, the map $F \to \Delta^0$ is not a weak equivalence because $\pi_n(\text{SmSing} F) \cong \mathbb{R}$, with the isomorphism given by integrating a closed differential $n$-form along $n$-dimensional singular simplices; the Stokes formula then shows that homotopic pointed spheres map to the same real number. Used in $6.0^\ast$.

5 Model categories

In this section, we recall some well-known facts about model categories. We provide proofs for those statements for which we could not locate a proof in the published literature.

Recall the Kan–Quillen model structure on the category of simplicial sets.

Proposition 5.1. (The Kan–Quillen model structure on simplicial sets.) The category $s\text{Set}$ admits a cartesian combinatorial proper model structure whose generating cofibrations are boundary inclusions $\delta_n: \partial \Delta^n \to \Delta^n \quad (n \geq 0)$ and generating acyclic cofibrations are horn inclusions $\lambda_{n,k}: \Lambda^n_k \to \Delta^n \quad (n > 0, \ 0 \leq k \leq n)$.

This model structure is proper, cartesian, its weak equivalences are closed under filtered colimits, and all objects are cofibrant. Used in $7.3^\ast, 7.4^\ast, 8.8, 8.9, 12.11^\ast$.

Recall the definition of a transferred model structure.

Definition 5.2. (Crans [1993, Theorem 3.3], Hirschhorn [2003, Theorem 11.3.2].) Suppose $C$ is a model category and $R: D \to C$ is a right adjoint functor. The transferred model structure (Definition 5.2) on $D$ (if it exists) is the unique model structure whose weak equivalences and fibrations are created by the functor $R$.

Recall the following transfer theorem for model structures, which we formulate for the specific case of adjunctions between locally presentable categories.

Proposition 5.3. (Crans [1993, Theorem 3.3], Hirschhorn [2003, Theorem 11.3.2].) Suppose $L \dashv R: C \rightleftarrows D$ is an adjunction between locally presentable categories and $C$ is equipped with a cofibrantly generated (hence combinatorial) model structure. The transferred model structure (Definition 5.2) on $D$ exists if and only if the functor $R$ sends transfinite compositions of cobase changes of elements of $L(J)$ to weak equivalences in $C$, where $J$ denotes any generating set of acyclic cofibrations in $C$. Given a set $I$ of generating (acyclic) cofibrations of $C$, the set $L(I)$ is a set of generating (acyclic) cofibrations of $D$. Used in $7.4^\ast$.

Recall the following variant of the Smith recognition theorem.

Proposition 5.4. (Lurie [2017b, Proposition A.2.6.15].) Suppose $C$ is a locally presentable category and $W$ is a class of morphisms in $C$ that is closed under the 2-out-of-3 property and is given by the closure under filtered colimits of a set of objects in the category of morphisms and commutative squares in $C$. Suppose $I$ is a set of $W$-cofibrations (Definition 7.1) in the relative category $(C,W)$ such that morphisms with the right lifting property with respect to $I$ necessarily belong to $W$. Then $C$ admits a left proper combinatorial model structure whose class of weak equivalences is given by $W$ and $I$ is its set of generating cofibrations. Used in $7.4^\ast, 2.7^\ast$.

We also need the following consequence of Proposition 5.4 which allows one to arbitrarily enlarge the set of generating cofibrations to a larger set of $W$-cofibrations.
Corollary 5.5. Suppose $C$ is a left proper combinatorial model category and $I$ is a set of $h$-cofibrations [Definition 7.1] in the relative category $(C, W)$ such that $I$ contains some set of generating cofibrations for $C$. Then $C$ admits a left proper combinatorial model structure $M$ whose class of weak equivalences is given by $W$ and $I$ is its set of generating cofibrations. The identity functor $C \to M$ is a left Quillen equivalence. 

Recall the following well-known fact about transferred model structures.

Proposition 5.6. Suppose $L \dashv R : C \rightleftarrows D$ is a Quillen adjunction such that the functor $R$ reflects weak equivalences (e.g., if the model structure on $D$ is transferred along $R$). If the derived unit of $L \dashv R$ is a weak equivalence on any object of $C$, then $L \dashv R$ is a Quillen equivalence.

Proof. It suffices to show that the derived counit $L(Q(RX)) \to X$ of any fibrant object $X$ in $D$ is a weak equivalence, where $Q(RX) \to RX$ is a cofibrant replacement of $RX$. Since the functor $R$ reflects weak equivalences, it suffices to show that the map

$$R(L(Q(RX))) \to RX$$

is a weak equivalence. Precomposing with the derived unit

$$Q(RX) \to RL(Q(RX))$$

of the object $Q(RX)$ yields the cofibrant replacement map

$$Q(RX) \to RX,$$

which is a weak equivalence. Since the derived unit is a weak equivalence by assumption, we conclude by the 2-out-of-3 property.

Recall the following definition, which is a special case of the definition of a monoidal model structure for the case of cartesian monoidal structures.

Definition 5.7. A model category $C$ is cartesian if its underlying category is cartesian closed (meaning the functor $A \times - : C \to C$ has a right adjoint functor $\text{Hom}(A, -) : C \to C$ for any object $A \in C$), the terminal object is cofibrant, and the pushout product

$$A \times D \sqcup_{A \times C} B \times C \to B \times D$$

of a cofibration $A \to B$ and an (acyclic) cofibration $C \to D$ is an (acyclic) cofibration. 

Used in 5.9, 12.3, 12.11.

We need the following notion of an equivalence between monoidal model categories. (We simplify the unit condition in the following definition since in our case all units are cofibrant.)

Definition 5.8. A weak monoidal Quillen adjunction weak monoidal Quillen equivalence (Schwede–Shipley [2002, Definition 3.6]) is a Quillen adjunction $L : C \rightleftarrows D : R$ between monoidal model categories such that the right adjoint functor $R$ is a lax monoidal functor, for any cofibrant objects $A, B \in C$ the comonoidal map

$$L(A \otimes B) \to LA \otimes LB$$

defined as the adjoint of the composition

$$A \otimes B \overset{\eta A \otimes \eta B}{\to} RL A \otimes RL B \longrightarrow R(LA \otimes LB)$$

is a weak equivalence, and the map

$$L1C \to 1_D$$

adjoint to the map $1_C \to R1D$ is a weak equivalence. 

We need the following criterion for showing that a Quillen equivalence between cartesian model categories is a weak monoidal Quillen equivalence.
Proposition 5.9. If $L: C ⇸ D: R$ is a Quillen equivalence between cartesian model categories (Definition 5.7), more generally, monoidal model categories such that the lax structure maps of $R$ are weak equivalences, the functors $L$ and $R$ preserve weak equivalences, and weak equivalences in $C$ are closed under monoidal products (more generally, monoidal products of unit maps of cofibrant objects are weak equivalences), then $L ⊸ R$ is a weak monoidal Quillen equivalence (Definition 5.8).

Proof. Since the functors $L$ and $R$ preserve weak equivalences, the unit and counit natural transformations are weak equivalences. Given $A, B ∈ C$, the comonoidal map

$$L(A ⊗ B) → LA ⊗ LB$$

can be computed by applying the left adjoint functor $L$ to the composition of the monoidal product of the unit maps

$$A ⊗ B → RLA ⊗ RLB$$

(which is a weak equivalence) with the lax structure map

$$RLA ⊗ RLB → R(LA ⊗ LB),$$

which is a weak equivalence by assumption, and then composing the resulting morphism

$$L(A ⊗ B) → LR(LA ⊗ LB),$$

which is a weak equivalence because $L$ preserves weak equivalences, with the counit map

$$LR(LA ⊗ LB) → LA ⊗ LB,$$

which is a weak equivalence. We conclude by the 2-out-of-3 property for weak equivalences. 

6 The nonexistence of the projective model structure on diffeological spaces

In this section we prove that the Kan–Quillen model structure on $\text{Set}$ does not transfer along the right adjoint functor $\text{SmSingDiffeo} \rightarrow \text{Set}$ (Definition 3.7). This is caused by the pathological behavior of colimits in $\text{Diffeo}$: colimits in $\text{PreSingSmSet}$ and colimits in $\text{PreDiffeo}$ are computed as the concretizations (Remark 2.9) of colimits in $\text{PreSingSmSet}$ and colimits in $\text{Diffeo}$ are computed as the associated sheaves of colimits in $\text{PreSingSmSet}$. The concretization functor (Remark 2.9) can change the homotopy type dramatically, as shown in Remark 4.3.

In particular, the concretization functor can interact in a wild way with cobase changes of smooth horn inclusions $|Λ^n_0| → |Δ^3|$, and this section exploits this behavior to construct a cobase change of the smooth 3-horn that is not a weak equivalence, which disproves the existence of a transferred model structure.

As shown in the next section, enlarging the category $\text{Diffeo}$ to $\text{SmSet}$ allows us to prove the existence of the transferred model structure.

The content of this section is not used anywhere else in the paper. Its only purpose is to motivate the enlargement of the category of diffeological spaces to the category of smooth sets.

Definition 6.1. Denote by $F$ the diffeological space given by the intersection of all diffeological subspaces of $|Δ^3|$ that contain $|Λ^n_0|$ together with the following section $S: S^1 \rightarrow |Δ^3|$, whose factorization through $F$ is denoted by $s: S^1 \rightarrow F$. (Since $S^1$ is not an object of $\text{Cart}$, we pull back along the covering map $R \rightarrow S^1$ first, then add the resulting section.) We parametrize $|Δ^3| = \{(x, y, z) ∈ R^3\}$, with the four faces of $|Δ^3|$ being $x + y + z = 1$, $x = 0$, $y = 0$, $z = 0$. We identify $|Λ^n_0|$ with the colimit of the diagram of hyperplanes $x = 0$, $y = 0$, $z = 0$, and their intersections.

We concentrate our attention on one of the three corner lines of $|Λ^n_0|$, given by the line with directional vector $(1, 0, 0)$, where the faces $y = 0$ and $z = 0$ of $|Λ^n_0|$ intersect. Choose countably many smooth curves $b_i: [0, 1] → |Λ^n_0|$ such that $b_i(0) = 0$, $b_i(0) = (1, 0, 0)$, all derivatives of $b_i$ at 0 and 1 vanish, and the images of $(0, 1) ⊂ [0, 1)$ under $b_i$ are disjoint subsets of $\{(x, y, z) ∈ |Λ^n_0|, y = 0, z > 0\}$. Likewise define $c_i: [0, 1] → |Λ^n_0|$, which land in the face $z = 0$, $y > 0$.

We construct a smooth map $f: [0, 1] → |Δ^3|$ whose underlying map of sets factors through the underlying map of sets of $|Λ^n_0| → |Δ^3|$. We do it by rescaling the arguments of the functions $b_i$ and $c_i$, placing them inside certain subintervals of $[0, 1]$ using a process that resembles the construction of the Cantor set $C$. Start by placing $b_0$ inside $[1/3, 2/3] ⊂ [0, 1]$. Place $c_0$ and $c_1$ inside $[1/9, 2/9]$ and $[7/9, 8/9]$. Place $b_1$, $b_2$, $b_3$,
and \(b_4\) inside the intervals \([1/27, 2/27], [7/27, 8/27], [19/27, 20/27]\), and \([25/27, 26/27]\). The process repeats indefinitely, placing the next elements of \(b\) (if \(k\) is even) or \(c\) (if \(k\) is odd) inside the \(2^k\) intervals of length \(3^{-k-1}\) at step \(k \geq 0\). The first three stages of the construction can be schematically depicted by the following graph, where the horizontal axis is the argument of \(f\) and the vertical axis is the normal coordinate with respect to the line \((1, 0, 0)\); the part above the horizontal line depicts the coordinate \(z \geq 0\) in the plane \(y = 0\), whereas the bottom part depicts the coordinate \(y \geq 0\) in the plane \(z = 0\):

Since we have control over the choices of \(b\) and \(c\), we can ensure that the resulting function \(f: [0, 1] \to |\Delta^3|\) is smooth. The idea behind \(f\) is that it oscillates countably many times between the different faces of \(|\Delta^3|\), while not factoring through \(|A_3| \to |\Delta^3|\) because at points \(p\) in the Cantor set \(C\), the restriction of \(f\) to any neighborhood of \(p\) straddles both faces \(z = 0\) of \(|\Delta^3|\).

We construct such a function \(f: [0, 1] \to |\Delta^3|\) for each of the three corner lines (with directional vectors \((1,0,0), (0,1,0), (0,0,1)\)) and glue them together into a single function \(S: S^1 \to |\Delta^3|\) (joining the three parts using straight line segments modified to have vanishing derivatives at endpoints). Thus, \(S\) wraps a circle around the three faces of \(|\Delta^3|\) other than the 0th face, while oscillating around the three corner lines as described above. Used in 6.3*

**Proposition 6.2.** The cobase change of the smooth horn inclusion \(|A_3| \to |\Delta^3|\) along the inclusion \(|A_3| \to F\) in the category \(\text{Diffeo}\) is not a weak equivalence. Used in 6.3*

**Proof.** The cobase change is the inclusion \(F \to |\Delta^3|\), since the concretization functor (Remark 2.9) identifies the section \(s: S^1 \to F\) that was manually added to \(|A_3|\) to form \(F\) with the section \(S: S^1 \to |\Delta^3|\). Since \(\text{SmSing}|\Delta^3|\) is contractible, we have to show that \(\text{SmSing}|\Delta^3|\) is not contractible. It suffices to show that the morphism \(s: S^1 \to F\) does not extend to a map \(D^2 \to F\), where \(D^2 = \mathbb{R}^2\) is the (extended) 2-dimensional disk such that \(S^1 \subset D^2\) is a circle of radius 1.

Suppose \(d: D^2 \to F\) is an extension of \(s: S^1 \to F\). Since \(s\) defines a noncontractible loop in \(F \setminus \{0\}\), where \(0 \in |A_3|\) is the apex of \(|A_3|\), the point \(0 \in |A_3|\) must be in the image of the map \(d: D^2 \to F\). Thus, without loss of generality we may assume \(d(0) = 0\).

Denote by \(c_r: S^1 \to D^2\) the map that sends \(S^1\) to the circle of radius \(r \geq 0\) inside \(D^2\). Every section of factors locally through the map \(|A_3| \to F\) or the map \(s: S^1 \to F\). For any \(r \in [0,1]\), the image of \(c_r\) can be covered by finitely many open sets such that the restriction of \(d: D^2 \to F\) to these open sets factors through \(|A_3| \to F\) or through \(s: S^1 \to F\). We may assume these open sets to be products of an open interval in \(S^1\) (i.e., the angle) and an open interval \(R \subset [0,1]\) (i.e., the radius). Furthermore, we can make \(R\) the same for all intervals for a given fixed \(r \geq 0\). By shrinking and refining the intervals in \(S^1\) as necessary, we can get a cyclically ordered set of open intervals \(W_i\) in \(S^1\) where nonconsecutive intervals have disjoint closures, the restriction of \(d\) to \(W_{2i+1} \times R\) factors through \(|A_3| \to F\), and the restriction of \(d\) to \(W_{2i} \times R\) factors through \(s: S^1 \to F\) as a map \(w_{2i}: W_{2i} \times R \to S^1\). Since the interval \([0,1]\) is compact, we can pick finitely many intervals \(R\) such that the open sets constructed above cover the compact unit ball inside \(D^2\).

(In the special case of radius 0, we simply pick an open ball around 0 \(\in D^2\) such that the restriction of \(d\) factors through \(|A_3|\).)

On the intersections of even and odd intervals, the map \(d\) must factor through both \(|A_3| \to F\) and \(s: S^1 \to F\). By construction of \(s\), such maps have the following local characterization: they are either constant maps landing in some point in the image of \(s\), which we can assume to be an irrational point \(q \in C\) (otherwise the next case applies), or factor through the closure \([u_i, v_i] \subset S^1\) of one of the open intervals in the complement of the Cantor set \(C \subset S^1\).
For every point $c \in C$, a generic point $p$ in an open neighborhood $U$ of $c$ is a regular value of the maps $w_2; W_2 \to S^1$ constructed above. In particular, the degree of $p$ is well-defined and can be computed by subtracting the number of points $b \in W_2$ such that $w_2(b) = p$ and $w'_2(b) > 0$ from the number of points $a \in W_2$ such that $w_2(a) = p$ and $w'_2(a) < 0$. With the exception of finitely many points given by the irrational points $q \in C$ defined above as well as the endpoints $u_i, v_i$, the degree at $c \in C$ is independent of the choice of $p$ once $U$ is sufficiently small and is equal to the sum of winding numbers of $w_i$ such that the image of $w_i$ hits $c$. This formula also implies that the index of $c \in C$ (other than at finitely many $c$) stays constant as the radius $r \in [0, 1]$ changes.

Thus, except for finitely many points, all points in $C$ have a constant index as the radius changes. For radius 1, the index of all points is 1 by construction of $s$. For radius 0 (or any radius close to 0), the index of all points is 0 because the restriction of $d$ factors through $|A_0^3|$. Thus, the map $d$ does not exist, which completes the proof.

**Theorem 6.3.** The category $\text{Diff}_\mathbb{R}$ does not admit a model structure transferred (Definition 5.2) along the right adjoint functor $\text{SmSing} \to s\text{Set}$ (Definition 3.5) from the Kan–Quillen model structure on the category $s\text{Set}$. Used in \textbf{1.1*, 1.3*, 7.8*}.

**Proof.** The simplicial horn inclusion $\Lambda^0_3 \to \Delta^3$ is an acyclic cofibration in $s\text{Set}$. Therefore, the smooth horn inclusion $|A_0^3| \to |\Delta^3|$ is an acyclic cofibration in the transferred model structure on $\text{Diff}_{\mathbb{R}}$, if it exists. Thus, any cobase change of $|A_0^3| \to |\Delta^3|$ must be a weak equivalence in $\text{Diff}_{\mathbb{R}}$. By Proposition 6.2, the cobase change of $|A_0^3| \to |\Delta^3|$ along the map $|A_0^3| \to F$ constructed in Definition 6.1 is not a weak equivalence in $\text{Diff}_{\mathbb{R}}$, contradicting the existence of the transferred model structure on $\text{Diff}_{\mathbb{R}}$.

## 7 Existence of the transferred model structure and its equivalence to simplicial sets

In this section we prove that the categories $\text{PreSmSet}$ and $\text{SmSing}$ admit model structures transferred along the right adjoint functor $\text{SmSing} \to s\text{Set}$ (Definition 3.5). We also prove that $\text{SmSing}$ is a right Quillen equivalence in both cases.

Recall the notion of an h-cofibration in a model category.

**Definition 7.1.** (Grothendieck; Batanin–Berger [2013.b, Definition 1.1].) A morphism $f: X \to Y$ in a relative category $C$ is an h-cofibration if the cobase change functor along $f$

$$f_! \colon X/C \to Y/C$$

preserves weak equivalences. Used in \textbf{7.4*, 7.5*, 11.1*}.

A model category is left proper if and only if all cofibrations are h-cofibrations and in a left proper model category, cobase changes along h-cofibrations are homotopy cobase changes. See, for example, Pavlov–Scholbach [2015.b, Definition 2.3] and references therein for more information.

**Proposition 7.2.** In the relative categories $\text{PreSmSet}$ and $\text{SmSing}$ (Definition 3.8), all monomorphisms are h-cofibrations. Furthermore, the functor $\text{SmSing}$ reflects h-cofibrations. Used in \textbf{7.4*, 7.5*, 11.1*}.

**Proof.** In the relative category $\text{PreSmSet}$, monomorphisms are h-cofibrations because the functor $\text{SmSing}$ preserves colimits, monomorphisms, and weak equivalences, so the image under $\text{SmSing}$ of the diagram of pushout squares

$$\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow f & & \downarrow w \\
Y & \longrightarrow & B
\end{array}$$

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}$$

where $f$ is a monomorphism and $w$ is a weak equivalence, is a diagram of pushout squares in $s\text{Set}$, where the image of $f$ is a monomorphism and the image of $w$ is a weak equivalence. Thus, the image of $w'$ is a weak equivalence of simplicial sets, hence the map $w'$ is a weak equivalence. Since the functor $\text{SmSing}$ preserves and reflects weak equivalences, it reflects h-cofibrations.

Applying Proposition 4.1, we deduce that in $\text{SmSing}$ all monomorphisms are h-cofibrations and $\text{SmSing}$ reflects h-cofibrations.
Proposition 7.3. Weak equivalences (Definition 3.8) in $\text{PreSmSet}$ (Definition 2.3) and $\text{SmSet}$ (Definition 2.4) are closed under filtered colimits, hence also transfinite compositions. Used in 7.4.

Proof. For $\text{PreSmSet}$ this holds because $\text{SmSing}$ preserves colimits and weak equivalences of simplicial sets are closed under filtered colimits (Proposition 5.1). For $\text{SmSet}$ we use Proposition 4.2 to reduce to the previous case. 

Theorem 7.4. The categories $\text{PreSmSet}$ (Definition 2.3) and $\text{SmSet}$ (Definition 2.4) admit left proper combinatorial model structures transferred (Definition 5.2) via the smooth singular simplicial set functor $\text{SmSing}$ (Definition 3.5) from the Kan–Quillen model structure on simplicial sets (Proposition 5.1). The associated sheaf functor $\text{PreSmSet} \to \text{SmSet}$ is a left Quillen equivalence. Used in 2.3, 3.3, 7.5, 7.7, 7.8, 8.10.

Proof. By Proposition 5.3, the transferred model structure on $\text{PreSmSet}$ exists if and only if the functor $\text{SmSing}$ sends transfinite compositions of cobase changes of elements of $\|J\|$ (Definition 3.5) to weak equivalences in $\text{PreSmSet}$, where $J$ denotes the set of simplicial horn inclusions (Definition 3.5). Cobase changes of elements of $\|J\|$ in $\text{PreSmSet}$ are weak equivalences because $\text{SmSing}$ preserves colimits and acyclic cofibrations, and the simplicial map $\text{SmSing}(\lambda_{n,k})$ is a simplicial homotopy equivalence. Cobase changes of elements of $\|J\|$ in $\text{SmSet}$ by Proposition 4.2 which reduces the problem to the case of $\text{PreSmSet}$ since the associated sheaf functor sends $\|\lambda_{n,k}\|$ to $|\lambda_{n,k}|$. By Proposition 7.5, weak equivalences in $\text{PreSmSet}$ are closed under transfinite compositions, which completes the proof in the case of $\text{PreSmSet}$. The same argument establishes the case of $\text{SmSet}$ using $|J|$ instead of $\|J\|$ and invoking Proposition 4.4. A model category is left proper if and only if all cofibrations are h-cofibrations. All cofibrations are monomorphisms by construction and all monomorphisms are h-cofibrations by Proposition 7.2. 

An alternative proof could be given using Proposition 5.4. The class of weak equivalences satisfies the desired properties by Proposition 7.3. Morphisms with the right lifting property with respect to $\|I\|$ (respectively $|I|$) are weak equivalences by adjunction $|-| \dashv |\text{SmSing}|$ (respectively $|-| \dashv |\text{SmSing}|$). Finally, elements of $\|I\|$ (respectively $|I|$) are h-cofibrations by Proposition 7.2.

Christensen–Wu [2013,d, Proposition 4.24] observed that the relative category $\text{Diffeo}$ is right proper for trivial reasons: the functor $\text{SmSing}$ preserves pullback squares, and the Kan–Quillen model structure on simplicial sets is right proper. The same argument shows the right properness of relative categories $\text{SmSet}$, $\text{PreSmSet}$ and $\text{PreDiffeo}$.

The associated sheaf functor sends the generating (acyclic) cofibrations of $\text{PreSmSet}$ to those of $\text{SmSet}$, therefore is a left Quillen functor. The associated sheaf functor and its right adjoint functor (the inclusion $\text{SmSet} \to \text{PreSmSet}$) both preserve and reflect weak equivalences. Furthermore, the unit map is a weak equivalence by Proposition 4.4 and the counit map is an isomorphism. Thus, the associated sheaf functor is a left Quillen equivalence.

Corollary 7.5. The categories $\text{PreSmSet}$ (Definition 2.3) and $\text{SmSet}$ (Definition 2.4) admit left proper combinatorial model structures whose weak equivalences coincide with that of Theorem 7.4 and the set of generating cofibrations is given by an arbitrary set of monomorphisms that contains the generating cofibrations of Theorem 7.4. The resulting model structures are Quillen equivalent to those of Theorem 7.4. Used in 8.12.

Proof. Combine Theorem 7.4 with Proposition 7.2 and Corollary 5.3. 

Remark 7.6. Using the class of monomorphisms as generating cofibrations (which is generated by a set), Corollary 7.5 implies the existence of a model structure on $\text{SmSet}$ with the same weak equivalences as in Theorem 7.4 and monomorphisms as cofibrations. This recovers the model structure on $\text{SmSet}$ constructed by Cisinski [2002,b, Théorème 3.9], taking the class of weak equivalences of Definition 3.8.

Remark 7.7. The proof of Corollary 7.5 also gives a new proof of the existence of Kihara’s model structure on diffeological spaces (Kihara [2016, Theorem 1.3]). Indeed, set the set $I$ of generating cofibrations to the set $\{\delta_{n,K} \mid n \geq 0\}$ of realizations of simplicial boundary inclusions with respect to Kihara’s cosimplicial object (Kihara [2016, Definition 1.2]). Since elements of $I$ are monomorphisms, to show that the transferred model structure exists, it suffices to prove that morphisms with the right lifting property with respect to $|I|_K$ are weak equivalences, which is shown in Kihara [2016, Lemma 9.6.(2)].

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Theorem 7.8. The Quillen adjunctions of \textit{Theorem 7.4} between \textit{sSet} (\textit{Proposition 5.1}) and the model categories \textit{PreSmSet} (\textit{Definition 2.3}), or \textit{SmSet} (\textit{Definition 2.4}) are Quillen equivalences, in fact, weak monoidal Quillen equivalences in the sense of Schwede–Shipley [2002.d, Definition 3.6]. Used in 7.9, 11.3, 11.4*

Proof. For \textit{PreSmSet}, the functor \textit{SmSing} preserves colimits. Thus, the derived unit natural transformation

\[
\begin{array}{c}
\Delta^n \longrightarrow X \\
\downarrow \quad \Rightarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
\partial \Delta^n \longrightarrow \text{SmSing} \parallel \partial \Delta^n \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
\Delta^n \longrightarrow Y \\
\downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{SmSing} \parallel \Delta^n \longrightarrow \text{SmSing} \parallel Y \\
\downarrow \quad \downarrow
\end{array}
\]

of corresponding pushout squares. The component

\[
X \rightarrow \text{SmSing} \parallel X
\]

is a weak equivalence by assumption. The component

\[
\Delta^n \rightarrow \text{SmSing} \parallel \Delta^n
\]

is a weak equivalence because its source and target are contractible. The component

\[
\partial \Delta^n \rightarrow \text{SmSing} \parallel \partial \Delta^n
\]

is a weak equivalence by inductive assumption (prove the claim by induction on the dimension of \(X\)). The left maps are monomorphisms, hence both squares are homotopy pushout squares in \textit{sSet} and the component

\[
Y \rightarrow \text{SmSing} \parallel Y
\]

is a weak equivalence.

For \textit{SmSet}, we combine the previous argument for \textit{PreSmSet} with \textit{Proposition 4.1}.

Finally, to show that the established Quillen equivalences are weak monoidal Quillen equivalences in the sense of Schwede–Shipley [2002.d, Definition 3.6], we invoke \textit{Proposition 5.9}, observing that the functors \(\parallel - \parallel\) (respectively \(|-|\)) and \textit{SmSing} preserve weak equivalences, and weak equivalences of simplicial sets are closed under finite products.

The following result can be seen as a surrogate for a transferred model structure on diffeological spaces, which was shown not to exist in \textit{Theorem 6.3}. We remark that Kihara [2017.a, Theorem 1.1.(1)] establishes a Quillen equivalence between simplicial sets and diffeological spaces equipped with the model structure constructed in Kihara [2016, Theorem 1.3], which shows that an analogue of \textit{Corollary 7.9} holds for the singular complex functor associated to Kihara’s cosimplicial diffeological space (Kihara [2016, Definition 1.2]). Kihara’s cosimplicial diffeological space embeds into the standard cosimplicial diffeological space (Kihara [2016, Lemma 3.1]), and this embedding induces a natural transformation between the corresponding smooth realization functors. A standard cube lemma argument then shows this natural transformation to be a weak equivalence. This provides an alternative proof of \textit{Corollary 7.9}.

Corollary 7.9. The functors

\[
\text{SmSing, Diffeo} \rightarrow \text{sSet} \\
\text{SmSing, PreDiffeo} \rightarrow \text{sSet}
\]

are Dwyer–Kan equivalences of relative categories, with the inverse functors given by

\[
\begin{array}{c}
|-|: \text{sSet} \rightarrow \text{Diffeo} \\
\parallel - \parallel: \text{sSet} \rightarrow \text{PreDiffeo}
\end{array}
\]

and the unit and counit natural weak equivalences inherited from \textit{Theorem 7.8}. Used in 1.1*, 7.8*.

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8 The projective model structure is cartesian

We start by recalling the notion of semisimplicial sets and their embedding into simplicial sets.

**Definition 8.1.** Denote by $\Delta_{\text{inj}}$ the subcategory of $\Delta$ given by the same objects and injective maps of finite nonempty ordered sets.

**Definition 8.2.** Denote by $sSet_{\text{inj}}$ the subcategory of $sSet$ given by the essential image of the left adjoint of the restriction functor

$$sSet = \text{Fun}(\Delta^{\text{op}}, \text{Set}) \to \text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}).$$

**Remark 8.3.** The left adjoint functor

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \to sSet$$

is faithful, so we have an equivalence of categories

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set}) \to sSet_{\text{inj}}.$$  

Objects and morphisms in $\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \text{Set})$ are known as semisimplicial sets and semisimplicial maps respectively. Objects in $sSet_{\text{inj}}$ are precisely those simplicial sets for which face maps preserve nondegenerate simplices. Morphisms in $sSet_{\text{inj}}$ are precisely those simplicial maps that preserve nondegenerate simplices.

**Remark 8.4.** If $D$ is a cocomplete category, the restriction functor along the Yoneda embedding

$$\text{Fun}(sSet_{\text{inj}}^{\text{op}}(D)) \to \text{Fun}(\Delta_{\text{inj}}^{\text{op}}(D))$$

becomes an equivalence of categories if we take the full subcategory of cocontinuous functors on the left side. Likewise, the restriction functor

$$\text{Fun}(sSet_{\text{inj}}^{\text{op}} \times sSet_{\text{inj}}^{\text{op}}(D)) \to \text{Fun}(\Delta_{\text{inj}}^{\text{op}} \times \Delta_{\text{inj}}^{\text{op}}(D))$$

becomes an equivalence of categories if on the left side we take the full subcategory of functors that are separately cocontinuous in each variable. We use this observation to construct functors of the form $sSet_{\text{inj}}^{\text{op}} \times sSet_{\text{inj}}^{\text{op}} \to D$ and natural transformations between them. Used in §8, 8.8.*

**Definition 8.5.** The functor

$$\circ: sSet_{\text{inj}} \times sSet_{\text{inj}} \to sSet_{\text{inj}}, \quad (K, L) \mapsto K \odot L$$

is the functor that is separately cocontinuous in each variable and extends the functor

$$\circ: \Delta_{\text{inj}} \times \Delta_{\text{inj}} \to sSet_{\text{inj}}, \quad ([m], [n]) \mapsto [\Delta[m] \times [n]],$$

where $\otimes$ denotes the ordinary product of finite sets equipped with the lexicographic order. This construction is manifestly functorial with respect to injective maps of simplices. Used in §8.

**Proposition 8.6.** There is a natural weak equivalence

$$\times \to \circ: sSet_{\text{inj}} \times sSet_{\text{inj}} \to sSet_{\text{inj}}, \quad (K, L) \mapsto (K \times L \to K \odot L).$$

**Proof.** All involved bifunctors are cocontinuous in each variable, so by Remark 8.4 it suffices to construct the natural transformation on $\Delta_{\text{inj}}$. The natural weak equivalence

$$\Delta^m \times \Delta^n \to \Delta^m \odot \Delta^n$$

is the nerve of the map of posets

$$[m] \times [n] \mapsto [m] \times [n]$$

given by the identity map on underlying sets.

The natural transformation $\times \to \circ$ is a weak equivalence on representable simplicial sets because both sides are weakly contractible. To show that $A \times B \to A \odot B$ is a weak equivalence for all $A$ and $B$, use induction on $A \in sSet_{\text{inj}}$. If $A = \emptyset$, the map is equal to the identity map. If $A \times B \to A \odot B$ is a weak equivalence, pick an arbitrary map $\partial \Delta^n \to A$ in $sSet_{\text{inj}}$ and consider the pushout $A' = A \sqcup_{\partial \Delta^n} \Delta^n$. The cube lemma (Hovey [1999, Lemma 5.2.6]) implies that $A' \times B \to A' \odot B$ is also a weak equivalence. Since simplicial weak equivalences are closed under transfinite compositions, this completes the proof. \[\blacksquare\]
Proposition 8.7. Denote by $C$ the category $\text{SmSet}$. Recall the functors $|−|$ and $|−|$ (Definition 3.5). The functor
\[ |−| \times |−|: \text{Set}_{\text{inj}} \times \text{Set}_{\text{inj}} \to C, \quad (K, L) \mapsto |K| \times |L| \]
is a retract of the functor
\[ |−| \circ |−|: \text{Set}_{\text{inj}} \times \text{Set}_{\text{inj}} \to C, \quad (K, L) \mapsto |K \circ L|. \]
The same is true for the category $C = \text{PreSmSet}$ with the functor $|−|$ replaced by $∥−∥$.

Proof. By Remark 8.4, it suffices to exhibit the functor
\[ |−| \times |−|: \Delta_{\text{inj}} \times \Delta_{\text{inj}} \to C, \quad (K, L) \mapsto |K| \times |L| \]
as a retract of the functor
\[ |−| \circ |−|: \Delta_{\text{inj}} \times \Delta_{\text{inj}} \to C, \quad (K, L) \mapsto |K \circ L|. \]
The natural inclusion
\[ l: \Delta^m \times \Delta^n \to \Delta^{m+n} \]
\[ \begin{array}{l}
(x_0, \ldots, x_m, y_0, \ldots, y_n) \mapsto (x_0 y_0, x_0 y_1, \ldots, x_0 y_n, x_1 y_0, \ldots, x_1 y_n, \ldots, x_m y_0, \ldots, x_m y_n).
\end{array} \]
The natural retraction
\[ \begin{array}{ll}
p: \Delta^m \times \Delta^n \to \Delta^m \times \Delta^n, \\
\rho: \Delta^m \times \Delta^n \to \Delta^m \times \Delta^n,
\end{array} \]
sends
\[ \begin{array}{l}
(z_0, \ldots, z_m, n) \mapsto (z_0, z_0 + \cdots + z_{m,0}, \ldots, z_{m,0} + \cdots + z_{m,n}, z_{0,0} + \cdots + z_{0,n}, \ldots, z_{0,n} + \cdots + z_{m,n}).
\end{array} \]
The composition $\rho$ is the identity map by construction, which completes the proof.

Proposition 8.8. Given $m \geq 0$, $n \geq 0$, the pushout product
\[ \begin{array}{ll}
p: \Delta^m \times \Delta^n \to |\Delta^m| \times |\Delta^n|,
\end{array} \]
of the maps
\[ \begin{array}{ll}
\delta_m: |\partial \Delta^m| \to |\Delta^m|, & \delta_n: |\partial \Delta^n| \to |\Delta^n|
\end{array} \]
(Proposition 5.1, Definition 3.3) is a cofibration in $\text{PreSmSet}$ (Theorem 7.4). Likewise, the pushout product
\[ \begin{array}{ll}
p: \Delta^m \times \Delta^n \to |\Delta^m| \times |\Delta^n|,
\end{array} \]
of the maps
\[ \begin{array}{ll}
\delta_m: |\partial \Delta^m| \to |\Delta^m|, & \delta_n: |\partial \Delta^n| \to |\Delta^n|
\end{array} \]
(Proposition 5.1, Definition 3.3) is a cofibration in $\text{SmSet}$ (Theorem 7.4). Used in 8.10.

Proof. This follows formally from Proposition 8.7, since the involved maps are morphisms in $\text{Set}_{\text{inj}}$. Consider the simplicial map $q$ given by the pushout product of $\partial \Delta^m \to \Delta^m$ and $\partial \Delta^n \to \Delta^n$ with respect to the operation $\circ$ of Definition 8.3. The natural retraction defined there exhibits $p$ as a retract of $∥q∥$ respectively $|q|$. Since $∥q∥$ respectively $|q|$ is a cofibration, so is $p$.

Proposition 8.9. Given $m > 0$, $0 \leq k \leq m$, $n \geq 0$, the pushout product of the maps
\[ \begin{array}{ll}
\lambda_{m,k}: |\Lambda^m_k| \to |\Delta^m|, & \delta_n: |\partial \Delta^n| \to |\Delta^n|
\end{array} \]
(Proposition 5.1, Definition 3.5) is a weak equivalence in $\text{SmSet}$ (Definition 3.8). The same is true for the category $\text{PreSmSet}$, with the functor $|−|$ replaced by $∥−∥$.

Proof. The inclusion of the apex $|\Delta^0| \to |\Delta^m|$ is a smooth homotopy equivalence (Definition 3.9). Therefore, its pushout product with $\delta_n: |\partial \Delta^n| \to |\Delta^n|$ is also a smooth homotopy equivalence. Smooth homotopy equivalences are weak equivalences, which completes the proof. The case of $∥−∥$ is treated in the same way.

The following result resolves in the affirmative a conjecture of Christensen–Wu [2013.c, Proposition 4.38].
Proposition 8.10. The model structures of Theorem 7.4 are cartesian model structures (Definition 5.7).

Proof. The same proof works for both model categories. By Proposition 8.8, the pushout product of generating cofibrations is a cofibration. Thus, the pushout product of cofibrations is a cofibration. By Proposition 8.4, the pushout product of a generating cofibration and a generating acyclic cofibration is a weak equivalence. Since it is also a cofibration, it must be an acyclic cofibration. Therefore, the pushout product of a cofibration and an acyclic cofibration is an acyclic cofibration. Finally, the terminal object (given by a point) is cofibrant, which completes the proof.

Proposition 8.11. The categories PreSmSet (Definition 2.5) and SmSet (Definition 2.4) admit cartesian left proper combinatorial model structures whose weak equivalences coincide with that of Theorem 7.4 and the set of generating cofibrations is given by an arbitrary set of monomorphisms that is closed under pushout products and contains the generating cofibrations of Theorem 7.4.

Proof. Combine Corollary 7.5 with the fact that the pushout product axiom can be checked on generating cofibrations.

9 Cofibrancy of manifolds

By Christensen–Wu [2013.4, Corollary 4.36], every manifold is fibrant in PreSmSet and SmSet. In this section, we show that every manifold is cofibrant in SmSet, resolving in the affirmative (Proposition 9.2) a conjecture of Christensen–Wu [2013.4, §4.2].

We start by recalling the following result of Boavida de Brito–Berwick-Evans–Pavlov [2019.1, Proposition 4.17]. (A stronger statement is asserted without proof in Madsen–Weiss [2002.4, Appendix A.1].) (The cited work assumes $U = \Delta^n$, but the proof makes no use of it.) The idea behind the following proposition is that a collection of sections over the simplices of $|K|$ that match on codimension 1 faces can be homotoped to a compatible collection of sections that can be glued using the sheaf property.

Proposition 9.1. (Boavida de Brito–Berwick-Evans–Pavlov [2019.1, Proposition 4.17].) Suppose $K$ is a simplicial set and $j: |K| \to U$ is a rectilinear (hence smooth) triangulation of an open subset $U \subset \mathbb{R}^n$ ($n \geq 0$). Then we can find a morphism $r: U \to |K|$ with the following properties.

- The map $r$ collapses an open neighborhood $U_\sigma$ of every closed simplex $\sigma$ (Definition 3.4) in the triangulation $j$ to $\sigma$.
- There is a smooth homotopy $h: \Delta^1 \times U \to U$ from the identity map on $U$ to $jr$. This homotopy preserves the image of every closed simplex in $|K|$.
- The smooth homotopy $h$ restricts to a smooth homotopy $\Delta^1 \times |K| \to |K|$ from the identity map on $|K|$ to $rj$. This homotopy preserves every closed simplex in $|K|$.

Proof. Coproducts of cofibrant objects are cofibrant, so we can assume the manifold to be connected, hence second countable. Any second countable Hausdorff manifold is a retract of a tubular neighborhood of the image of its embedding into some $\mathbb{R}^n$. Thus, it remains to treat the case when $M$ is an open subset of $\mathbb{R}^n$.

Pick smooth functions $f_1, \ldots, f_n: M \to (0, \infty)$ such that for every $i$ the vector field $e_i f_i$ has an everywhere defined flow $\alpha_i: \mathbb{R} \times M \to M$, where $e_i$ are elements of the standard basis of $\mathbb{R}^n$. We get a smooth map $a: \mathbb{R}^n \times M \to M$ that sends $(t, x)$ to

$$a_n(t, a_{n-1}(t_{n-1}, \ldots a_1(t_1, x), \ldots)).$$

For any $m \in M$ the map $b_m = a(\cdot, m): \mathbb{R}^n \to M$ is an open embedding that sends 0 to $m$. In particular, the map

$$b_m^{-1}: D_m \to \mathbb{R}^n$$

is well defined, with its domain $D_m$ being the open subset of $M$ given by the image of $b_m$, so that

$$a(b_m^{-1}(x), m) = x$$
for all \( x \in D_m \). The maps \( b_m \) combine into the smooth map
\[
c: D \to \mathbb{R}^n, \quad (x, m) \mapsto b_m^{-1}(x),
\]
where
\[
D = \{(x, m) \in M \times M \mid x \in D_m\}
\]
is an open subset of \( M \times M \). We have \( a(c(x, m), m) = x \) for all \( (x, m) \in D \).

Pick a rectilinear triangulation \( K \) of \( M \), with the induced map \( v: [K] \to M \). (Since \( M \) is an open subset of \( \mathbb{R}^n \), such a triangulation can be constructed in an elementary fashion without using the full strength of the triangulation theorem for smooth manifolds.) We now exhibit \( M \) as a retract of \( \Delta^k \times [K] \). The latter object is cofibrant by Proposition 8.4, which implies that \( M \) is also cofibrant.

Using Proposition 9.1, pick a map \( \alpha: M \to [K] \) with the following properties.

- Given a simplex \( \sigma \) in \( K \), consider its associated map \( v: \Delta_k \to M \). Denote by \( V_\sigma \subset M \) the \( \iota \)-image of the closed simplex \( \Delta_k \subset \Delta^k \) (Definition 3.3). We require that \( \alpha \) maps some open neighborhood \( U_\sigma \) of \( V_\sigma \) to the image of \( \Delta_k \to \Delta^k \to [K] \), where the map \( \Delta_k \to [K] \) is induced by \( \sigma \).
- Additionally, we require that for any \( m \in U_\sigma \) we have \( m \in D_{\iota(\alpha(m))} \). We can always shrink \( U_\sigma \) to a smaller open neighborhood of \( V_\sigma \) so that it satisfies this condition, since \( \iota(\alpha(m)) \in V_\sigma \) and \( V_\sigma \) is compact, so there is \( \varepsilon > 0 \) such that for any \( m \in V_\sigma \) and any \( x \in M \) with \( \|x - m\| < \varepsilon \) we have \( x \in D_m \), and for any \( \varepsilon > 0 \) we can choose \( \alpha \) so that for all \( m \in V_\sigma \) we have \( \|m - \iota(\alpha(m))\| < \varepsilon \).

The retraction \( r \) is given by the composition
\[
r: \Delta^k \times [K] \to \Delta^k \times M \to M.
\]
Consider the inclusion
\[
i: M \to \Delta^k \times [K], \quad m \mapsto (c(m, \alpha(m)), \iota^{-1}(\alpha(m))).
\]
By definition of \( \alpha \) we have \( m \in D_{\iota(\alpha(m))} \), so \( (m, \alpha(m)) \in D \) and the first component is well defined and smooth.

The point \( \alpha(m) \) belongs to the \( \iota \)-image of a unique interior simplex \( \Delta_k \subset [K] \) (Definition 3.3), so the second map is well defined on individual points. To show that it is induced by a (necessarily unique) morphism of sheaves, it suffices to observe that for any \( k \)-simplex \( \sigma \in K \) the restriction of \( i \) to \( U_\sigma \subset M \) is given by the composition of morphisms of sheaves
\[
f_\sigma: U_\sigma \xrightarrow{\text{diag}} U_\sigma \times U_\sigma \xrightarrow{\text{id} \times \text{c}} U_\sigma \times U_\sigma \xrightarrow{(c, \pi_2)} \Delta^k \times \Delta^k \xrightarrow{\text{id} \times \iota} \Delta^k \times [K].
\]
Furthermore, the collection \( \{U_\sigma\} \) is an open cover of \( M \) and the family \( \{f_\sigma\} \) is compatible because it is compatible on underlying sets by construction and the sheaf \( \Delta^k \times [K] \) is concrete. Thus, the compatible family \( \{f_\sigma\} \) can be glued to a morphism of sheaves \( i \).

The composition \( ri: M \to M \) sends \( m \in M \) to
\[
a(c(m, \alpha(m)), \alpha(m)) = m,
\]
so \( ri = \text{id}_M \) by concreteness of \( M \), which completes the proof. \( \blacksquare \)

10 The smooth Oka principle for smooth sets

The following result improves on the usual way of computing derived internal homs in cartesian model categories by eliminating the fibrant replacement functor. The proof of a more general result (discussed in §13 below) can be found in Berwick-Evans–Boavida de Brito–Pavlov [2019, Theorem 1.1].

The specific formulation using the internal hom functor from smooth manifolds originates in a question asked by Charles Rezk on November 17, 2015, and the name “smooth Oka principle” was suggested by Urs Schreiber.

**Proposition 10.1.** (The smooth Oka principle for smooth sets and diffeological spaces.) If \( X \) is a smooth manifold, the functor
\[
\text{Hom}(X, -): \text{SmSet} \to \text{SmSet}
\]
preserves weak equivalences (Definition 3.8) and therefore computes the derived internal hom in the model structure of \( \text{Hom}(X, -) \) used in [2012, 1.0.2].

The following result resolves in the affirmative a conjecture of Christensen–Wu [2013, §1], as already established in Berwick-Evans–Boavida de Brito–Pavlov [2019, Proposition 2.18]. We reproduce the proof here for the sake of completeness, adding a few more details.
Corollary 10.2. For every $X \in \text{SmSet}$, the canonical map from the $n$th smooth homotopy group of $X$ at point $x_0 \in X$ to the $n$th simplicial homotopy group of $\text{SmSing} X$ at point $x_0$ is an isomorphism. Here the $n$th smooth homotopy group of $X$ at $x_0 \in X$ is defined as the quotient of the set of morphisms $s: S^n \to X$ that send $* \in S^n$ to $x_0$ modulo the equivalence relation that identifies $s \sim s'$ if there is a morphism $h: \Delta^0 \times S^n \to X$ whose restriction to $\Delta^1 \times \{x_0\}$ is the constant map $\Delta^1 \to \Delta^0 \to X$.

Proof. Recall that the simplicial homotopy group $\pi_n (\text{SmSing} X, x_0)$ can be computed as the set of connected components of the homotopy fiber of the map of derived mapping simplicial sets

$$R \xrightarrow{\text{Hom}(\text{SmSing} S^n, \text{SmSing} X)} R \xrightarrow{\text{Hom}(\text{SmSing} \Delta^n, \text{SmSing} X)} \text{Hom}(S^n, X) \to \text{Hom}(\Delta^n, X).$$

By Proposition 10.1, the latter map is weakly equivalent to $\text{SmSing}$ applied to the map

$$\text{Hom}(S^n, X) \to \text{Hom}(\Delta^n, X).$$

The set of connected components of the homotopy fiber of the latter map can be computed as the following quotient. Elements are morphisms $S: S^n \to X$ together with a map $P: \Delta^n \to X$ that sends $1 \mapsto s(*)$ and $0 \mapsto x_0$. The pair $(S, P)$ can be encoded as a single map $S^n \cup \Delta^n \to X$. We identify $(S, P) \sim (S', P')$ if there is a smooth homotopy $\Delta^1 \times (S^n \cup \Delta^n) \to X$ between them. The canonical map $S^n \cup \Delta^n \to S^n$ that projects $\Delta^i$ to $* \in S^n$ is a smooth homotopy equivalence, the inverse map $S^n \to S^n \cup \Delta^n$ is constructed by projecting a disk of small radius $\varepsilon > 0$ around $*$ to the interval $[0, 1] \subset \Delta^1$ using the appropriately smoothened distance function from *. Since this smooth homotopy equivalence preserves the basepoint, this proves that the set of connected components of the homotopy fiber is isomorphic to the $n$th smooth homotopy group of $X$.

The following result answers a question by Sati–Schreiber [2021,c, Remark 2.2.9]. We remark that the extended simplex $\Delta^i$ can be replaced with the closed simplex $\Delta^i = [0, 1]$ in the statement below, since both simplices give rise to the same notion of concordance. The result is applicable when $X$ is a manifold, since these are cofibrant by Proposition 9.2.

Proposition 10.3. Suppose $P_0 \to X$ and $P_1 \to X$ are diffeological principal bundles over a cofibrant diffeological space $X$, e.g., a smooth manifold. Suppose $P_0 \to X$ and $P_1 \to X$ are concordant, meaning there is a diffeological principal bundle over $\Delta^1 \times X$ whose pullback to $\{i\} \times X$ is isomorphic to $P_i \to X$. Then $P_0 \to X$ and $P_1 \to X$ are isomorphic.

Proof. As pointed out in Sati–Schreiber [2021,c, Theorem 2.2.8 and Remark 2.2.9], it suffices to show that $X \to \Delta^1 \times X$ is an acyclic cofibration and every diffeological fiber bundle is a fibration. The former holds by Proposition 8.11 and the latter holds by Christensen–Wu [2013,c, Propositions 4.28 and 4.30].

11 Algebras over operads in smooth sets

In this section, we establish model structures on operads and algebras over operads in (pre)smooth sets and compare them to the existing constructions in the simplicial and quasicategorical settings.

Proposition 11.1. The model categories $\text{PreSmSet}$ and $\text{SmSet}$ of Theorem 7.4 are h-monooidal (Pavlov–Scholbach [2015,1, Definition 3.2.2]), symmetric h-monooidal (Pavlov–Scholbach [2015,1, Definition 4.2.4]), and flat (Pavlov–Scholbach [2015,1, Definition 3.2.4]).

Proof. For h-monoiality, since these model structures are cartesian by Proposition 8.10, it suffices to show that the product of any object and an (acyclic) cofibration is an (acyclic) h-cofibration. The nonacyclic part holds because cofibrations are monomorphisms, the product of an object and a monomorphism is a monomorphism, and monomorphisms are h-cofibrations by Proposition 7.2. The acyclic part holds because $\text{SmSing}$ preserves and reflects weak equivalences.

For symmetric h-monoiality, the argument is the same, using the fact that $\text{SmSing}$ preserves colimits in $\text{PreSmSet}$ and we need to further observe that the associated sheaf functor preserves monomorphisms and weak equivalences by Proposition 4.1.

Flatness in $\text{PreSmSet}$ follows from the fact that $\text{SmSing}$ preserves products and pushouts, and the model category $\text{PreSmSet}$ is flat (Pavlov–Scholbach [2015,1, §7.1]). Flatness in $\text{SmSet}$ then follows from Proposition 4.1.
Recall (Pavlov–Scholbach [2014.1, Definition 2.1]) that a map \( f : A \to B \) in a symmetric monoidal model category is flat if \( f \) is a weak equivalence and the pushout product \( f \square s \) is a weak equivalence for any cofibration \( s \). In \( \text{SmSet} \) and \( \text{PreSmSet} \) flat maps coincide with weak equivalences. Likewise, a \( \Sigma_n \)-equivariant map \( f \) is symmetric flat if \( f \square_{\Sigma_n} s^n \) is a weak equivalence for any multi-index \( n \) and finite family of cofibrations \( s \). A sufficient condition is given in Pavlov–Scholbach [2014.1, Lemma 7.6], essentially requiring the \( \Sigma_n \)-action to be projectively cofibrant.

**Proposition 11.2.** Suppose \( O \) is a colored (symmetric) operad in \( \text{PreSmSet} \) or \( \text{SmSet} \) (Theorem 7.4). The category of algebras over \( O \) admits a model structure transferred along the forgetful functor that extracts underlying objects. If \( f : O \to O' \) is a weak equivalence of colored (symmetric) operads, then it induces a Quillen equivalence of model categories of algebras over \( O \) and \( O' \) if and only if \( f \) is a (symmetric) flat map. (In the nonsymmetric case, flat maps coincide with weak equivalences.) Used in 1.2, 1.2*, 11.4, 11.5, 12.9.

**Proof.** Combine Proposition 11.1 together with Pavlov–Scholbach [2014.1, Theorem 5.11, Theorem 7.5, Theorem 7.11].

**Proposition 11.3.** Suppose \( O \) is a \( \Sigma \)-cofibrant colored symmetric operad in \( \text{PreSmSet} \) or \( \text{SmSet} \) (Theorem 7.4), where an operad \( O \) is \( \Sigma \)-cofibrant if the unit map \( 1 \to O(a,a) \) is a cofibration for every color \( a \) and every component of \( O \) is projectively cofibrant as an object in \( \text{PreSmSet} \) or \( \text{SmSet} \) with respect to the action of the appropriate symmetric group. Then the functor of quasicategories

\[ \text{Alg}_O(\text{SmSet})^c[W^{-1}_O] \to \text{Alg}_O(\text{SmSet})W^{-1} \]

is an equivalence of quasicategories. Here \( \text{Alg}_O \) on the left denotes the category of algebras over the operad \( O \), \( \text{Alg}_O \) on the right denotes the quasicategory of quasicategorical algebras over the operad \( O \), the brackets \([ - \]) denote quasicategorical localizations, superscript \( c \) denotes the full subcategory of cofibrant objects, and \( W_O \) and \( W \) denotes the weak equivalences with respect to the corresponding model structures. In particular, the quasicategory \( \text{SmSet}W^{-1} \) is equivalent to the underlying quasicategory of \( \text{SmSet} \) by [Theorem 7.8.], the right side is equivalent to the quasicategory of algebras over the operad \( \text{SmSing}(O) \) in spaces. All statements also hold if \( \text{SmSet} \) is replaced by \( \text{PreSmSet} \). Used in 1.2, 1.2*, 11.4, 11.5, 12.9.

**Proof.** Combine Proposition 11.1 and Haugseng [2019.a, Theorem 4.10].

**Proposition 11.4.** There is a Quillen equivalence

\[ L \dashv R : \text{PreSmSet} \leftrightarrow \text{SmSet} \]

of model categories of colored symmetric operads in \( \text{PreSmSet} \) and \( \text{SmSet} \) (constructed using Proposition 11.2). Here the right adjoint functor \( R \) applies the functor \( \text{SmSing} \) componentwise to a given operad in \( \text{PreSmSet} \). For any cofibrant (in the model category \( \text{PreSmSet} \)) colored symmetric simplicial operad \( O \), there is a Quillen equivalence

\[ L_O \dashv R_O : \text{Alg}_O(\text{SmSet}) \leftrightarrow \text{Alg}_O(\text{PreSmSet}), \]

where the right adjoint functor \( R_O \) applies \( \text{SmSing} \) to components of a given algebra over \( L_O \) and equips the result with an action of \( O \) using the unit map \( O \to R_O \). For any fibrant (in \( \text{PreSmSet} \)) operad \( P \), there is a Quillen equivalence

\[ L_P \dashv R_P : \text{Alg}_P(\text{SmSet}) \leftrightarrow \text{Alg}_P(\text{PreSmSet}), \]

where the right adjoint functor \( R_P \) applies \( \text{SmSing} \) to components of a given algebra over \( P \). All statements also hold if \( \text{PreSmSet} \) is replaced by \( \text{SmSet} \). Also, without (co)fibrancy conditions on \( O \) and \( P \) we still get Quillen adjunctions. Used in 1.2, 1.2*, 11.4, 11.5, 12.9.

**Proof.** Combine Proposition 11.1, Theorem 7.3, and Pavlov–Scholbach [2014.1, Theorem 8.10].

**Example 11.5.** As a special case, we see that (strict) monoids in smooth sets are Quillen equivalent to simplicial monoids. Likewise, \( E_\infty \)-monoids in smooth sets (where \( E_\infty \) denotes a \( \Sigma \)-cofibrant operad in smooth sets weakly equivalent to the terminal operad) are Quillen equivalent to \( \Gamma \)-spaces and \( E_\infty \)-monoids in simplicial sets, which can be seen by combining the second part of Proposition 11.4 with Proposition 11.2.
12 Model structures on simplicial smooth sets and enriched presheaves

In this section, we extend the results obtained so far to the case of simplicial presheaves on the site $\mathcal{Cart}$, i.e., simplicial objects in the categories $\mathbf{PreSmSet}$ and $\mathbf{SmSet}$. This is of crucial importance to applications, many of which involve objects that have higher homotopy groups, such as the stack of vector bundles with connections or the stack of bundle gerbes.

More generally, we construct a model structure on presheaves and sheaves on $\mathcal{Cart}$ valued in a left proper combinatorial model category $V$. Its weak equivalences are precisely those morphisms $F \to G$ of $V$-valued presheaves (or sheaves) on manifolds such that the induced map on shapes (Definition 13.7) is a weak equivalences in $V$.

Examples 12.1. We have the following principal examples of left proper combinatorial model categories $V$:

- $V = \mathbf{Set}$: suitable for encoding structures such as principal $G$-bundles and higher nonabelian bundles;
- $V = \mathbf{Ch}_{\geq 0}$: suitable for encoding abelian sheaf cohomology, e.g., bundle $n$-gerbes with connection;
- $V = \mathbf{Sp}$: suitable for encoding extraordinary differential cohomology, e.g., differential $K$-theory.

Definition 12.2. Suppose $V$ is a cocomplete and complete category. Denote by $\mathbf{PreSm}_V$ respectively $\mathbf{Sm}_V$ the category of presheaves respectively sheaves on the site $\mathcal{Cart}$ valued in $V$. In particular, for $V = \mathbf{Set}$ objects in $\mathbf{SmSet}$ are simplicial objects in smooth sets, i.e., simplicial smooth set. Denote by

$$\otimes : V \times \mathbf{Set} \to V,$$

the tensoring of $V$ over sets. Denote by

$$\otimes : V \times \mathbf{PreSm}_V \to \mathbf{PreSm}_V$$

the functor

$$(X,F) \mapsto (W \mapsto X \otimes F(W))$$

and by

$$\otimes : V \times \mathbf{Sm}_V \to \mathbf{Sm}_V$$

the functor that takes the associated sheaf (Remark 2.6) of the tensoring in $\mathbf{PreSm}_V$. Denote by

$$\boxtimes : \mathbf{PreSm}_V \to \mathbf{PreSm}_V, \quad \boxtimes : \mathbf{Sm}_V \to \mathbf{Sm}_V$$

the associated pushout product functors. Objects in $\mathbf{PreSm}_V$ (Definition 2.3) and $\mathbf{Sm}_V$ (Definition 2.4) can be (silently) promoted to objects in $\mathbf{PreSm}_V$ respectively $\mathbf{Sm}_V$ using the cocontinuous functor $1 \otimes - : \mathbf{Set} \to V$, where $1$ is the terminal object in $V$.

Definition 12.3. Suppose $V$ is a left proper combinatorial model category. Denote by $\mathcal{A}_V$ the category of simplicial objects in $V$. Turn $\mathcal{A}_V$ into a relative category by creating its weak equivalences using the homotopy colimit functor $\mathcal{A}_V \to V$. Turn $\mathcal{A}_V$ into a model category by equipping it with the left Bousfield localization of the projective model structure at maps of representable presheaves $\Delta^n \to \Delta^0$ tensored with an arbitrary object of $V$. It suffices to take the set of $\lambda$-small objects in $V$ for a sufficiently large regular cardinal $\lambda$. We also have a left Quillen equivalence $\operatorname{colim} : \mathcal{A}_V \to V$, which takes the colimit of a simplicial object. It is a weak monoidal Quillen equivalence (Definition 13.8).

Definition 12.4. Suppose $V$ is a left proper combinatorial model category. Denote by

$$||-|| : \mathcal{A}_V \to \mathbf{PreSm}_V, \quad \dashv \quad \operatorname{Sing}_V : \mathbf{PreSm}_V \to \mathcal{A}_V$$

$$\dashv - : \mathcal{A}_V \to \mathbf{Sm}_V, \quad \dashv \quad \operatorname{Sing}_V : \mathbf{Sm}_V \to \mathcal{A}_V$$

the adjunctions constructed as follows. The right adjoint $\operatorname{Sing}_V$ evaluates the given presheaf on smooth simplices $\Delta^n$. The left adjoints send $V \otimes \Delta^n$ to $V \otimes ||\Delta^n||$ respectively $V \otimes |\Delta^n|$. Equip $\mathbf{PreSm}_V$ and $\mathbf{Sm}_V$
with weak equivalences created by the functor $\operatorname{Sing}_V$, which turns them into relative categories. Used in 2.4, 3.1, 12.7, 12.11, 2.11, 3.5.

**Proposition 12.5.** Given a left proper combinatorial model category $V$, any Čech-local (equivalently, stalkwise) weak equivalence in $\operatorname{PreSm}_V$ is a weak equivalence in $\operatorname{PreSm}_V$. As a special case, the map $F \to LF$ that takes the associated sheaf of a presheaf $F$ is a weak equivalence in $\operatorname{PreSm}_V$. Used in 2.7, 2.11, 3.5.

**Proof.** (Compare Proposition 4.1.) Consider the model structure on the category $\operatorname{PreSm}_V$ given by the injective model structure left Bousfield localized at Čech nerves of good open covers. Consider the model structure on the category $\operatorname{PreSm}_V$ given by the injective model structure left Bousfield localized at maps of representable presheaves $\Delta^n \to \Delta^0$. (In both cases we tensor the representable presheaves with an arbitrary $\lambda$-small object of $V$, for a sufficiently large cardinal $\lambda$.) Consider the functor

$$\operatorname{Sing}_V: \operatorname{PreSm}_V \to V_{\lambda}.$$ 

The functor $\operatorname{Sing}_V$ is a left adjoint functor that preserves injective cofibrations and injective weak equivalences. Furthermore, by the nerve theorem, the functor $\operatorname{Sing}_V$ sends the Čech nerve of a good open cover to a weak equivalence in $V_{\lambda}$. Thus, $\operatorname{Sing}_V$ is a left Quillen functor that preserves weak equivalences.

The map $F \to G$ is a Čech-local weak equivalence by assumption. Thus, the map $\operatorname{Sing}_V F \to \operatorname{Sing}_V G$ is a weak equivalence in $V_{\lambda}$, therefore $F \to G$ is a weak equivalence in $\operatorname{PreSm}_V$.

**Remark 12.6.** Weak equivalences in $\operatorname{PreSm}_V$ (and $\operatorname{Sm}_V$) are precisely the weak equivalences in the $R$-local Čech-local projective (or injective) model structure on $V$-valued presheaves on $\operatorname{Cart}$, defined as the left Bousfield localization of the projective or injective model structure at Čech nerves of covers and maps $R^n \to R^0$. This continues to hold if $\operatorname{Cart}$ is replaced by $\operatorname{Man}$. In the case of $\operatorname{Cart}$ it suffices to localize at maps $R^n \to R^0$, since after such a localization Čech nerves of covers become weak equivalences by the nerve theorem.

**Theorem 12.7.** Given a left proper combinatorial model category $V$, the categories $\operatorname{PreSm}_V$ and $\operatorname{Sm}_V$ (Definition 12.2) admit left proper combinatorial model structures whose weak equivalences are as in Definition 12.2 and generating cofibrations are given by the maps $i \boxtimes \| \delta_n \|$ (respectively $i \boxtimes \| \delta_n \|$), where $i$ belongs to a fixed set of generating cofibrations in $V$, the map $\delta_n: \partial \Delta^n \to \Delta^n$ is a simplicial boundary inclusion ($n \geq 0$), and $\boxtimes$ is defined in Definition 12.2. Both model structures have the following properties.

- If weak equivalences in $V$ are closed under filtered colimits, then so are weak equivalences in $\operatorname{PreSm}_V$ and $\operatorname{Sm}_V$.
- Objectwise $h$-cofibrations are $h$-cofibrations in $\operatorname{PreSm}_V$ and $\operatorname{Sm}_V$, and the functor $\operatorname{Sing}_V$ reflects $h$-cofibrations.
- (Compare Proposition 11.1.) The model categories $\operatorname{PreSm}_V$ and $\operatorname{Sm}_V$ inherit from $V$ properties such as being monoidal (with respect to the objectwise monoidal product), tractable, $h$-monoidal, and flat (Pavlov–Scholbach 2015, Definitions 2.1, 3.2.2, and 3.2.4], symmetric $h$-monoidal (Pavlov–Scholbach 2015, Definition 4.2.4]).

**Proof.** If weak equivalences in $V$ are closed under filtered colimits, then filtered colimits in $V$ are also homotopy colimits. Therefore, weak equivalences in $\operatorname{PreSm}_V$ are closed under filtered colimits because filtered homotopy colimits commute with homotopy colimits of simplicial objects. Since the functor $\operatorname{Sing}_V$ preserves colimits, weak equivalences in $\operatorname{PreSm}_V$ are closed under filtered colimits. For $\operatorname{Sm}_V$ we use Proposition 12.7 to reduce to the case of $\operatorname{PreSm}_V$.

In the relative category $\operatorname{PreSm}_V$, objectwise $h$-cofibrations are $h$-cofibrations because the functor $\operatorname{Sing}_V$ preserves colimits, objectwise $h$-cofibrations, and weak equivalences, so the image under $\operatorname{Sing}_V$ of the diagram of pushout squares

$$
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow f & & \downarrow w \\
Y & \longrightarrow & A' \\
\end{array}
$$

where $f$ is an objectwise $h$-cofibration and $w$ is a weak equivalence, is a diagram of pushout squares in $\operatorname{PreSm}_V$ where the image of $f$ is an objectwise $h$-cofibration and the image of $w$ is a weak equivalence. Interpreting
the resulting pushout squares in $V \prod$ as a simplicial object in the category of diagrams of homotopy pushout squares in $V$, its homotopy colimit is also a diagram of homotopy pushout squares in $V$. Hence the map $w'$ is a weak equivalence. Since the functor \( \text{Sing}_V : \text{PreSm}_V \to V \prod \) preserves and reflects weak equivalences, it reflects h-cofibrations. Applying Proposition 12.7, we deduce that in $V \prod$ all objectwise h-cofibrations are h-cofibrations and $\text{Sing}_V$ reflects h-cofibrations.

All generating cofibrations $i \otimes \| \delta_n \|$ (respectively $i \Box \| \delta_n \|$) are objectwise (coproducts of) cofibrations, hence also objectwise h-cofibrations by left properness of $V$, therefore they are h-cofibrations.

To show the existence of the model structure on $\text{PreSm}_V$ (respectively $V \prod$), we apply Proposition 5.3 to the set of generating cofibrations $i \otimes \| \delta_n \|$ (respectively $i \Box \| \delta_n \|$). The class of weak equivalences satisfies the desired properties because the functor $\text{Sing}_V$ preserves filtered colimits and weak equivalences in $V \prod$ satisfy the desired properties. Morphisms $f$ with the right lifting property with respect to generating cofibrations $i \otimes \| \delta_n \|$ (respectively $i \Box \| \delta_n \|$) are weak equivalences by adjunction of Definition 12.4, which forces the Reedy matching maps of $\text{Sing}_V$ to have the right lifting property with respect to generating cofibrations $i$, making them into acyclic fibrations in $V$. This implies that $\text{Sing}_V f$ is a Reedy acyclic fibration, hence an objectwise weak equivalence, hence $f$ is a weak equivalence. Since the generating cofibrations are h-cofibrations, this proves the existence of the model structure.

To show that the model structures on $\text{PreSm}_V$ and $V \prod$ are monoidal (with respect to objectwise monoidal products of presheaves) whenever $V$ is a monoidal model category, observe that the pushout product of generating cofibrations can be rewritten as follows:

\[
(i \otimes \| \delta_n \|) \square (j \otimes \| \delta_n \|) = (i \square j) \otimes (\| \delta_m \| \Box \| \delta_n \|).
\]

The pushout product $\| \delta_m \| \Box \| \delta_n \|$ is a cofibration in $\text{PreSmSet}_V$ by Proposition 8.10 and the pushout product is a cofibration in the model category $V$ because the model structure on $V$ is monoidal. Thus, the pushout product of cofibrations in $\text{PreSm}_V$ is a cofibration, and likewise for $V \prod$. On $\text{PreSm}_V$, the functor $\text{Sing}_V$ preserves pushouts, monoidal products, and tensoring. The functor $\text{hocolim}_V : V \to V$ preserves homotopy pushout squares, and also preserves and reflects weak equivalences. The cocartesian square for the pushout product of a cofibration and acyclic cofibration in $\text{PreSm}_V$ is a homotopy pushout square. Therefore, its image under $\text{Sing}_V$ followed by $\text{hocolim}_V$ is a homotopy pushout square. Therefore, the pushout product is a weak equivalence by the 2-out-of-3 property. Thus, the pushout product of a cofibration and acyclic cofibration in $\text{PreSm}_V$ is a weak equivalence. By Proposition 12.5, the same holds for $V \prod$.

Assuming $V$ is tractable, h-monoidal, and flat, the model category $\text{PreSm}_V$ is tractable because $i \otimes \| \delta_n \|$ has a cofibrant domain since $i$ has cofibrant domain, and likewise for $V \prod$. The nonacyclic part of h-monoidality holds because cofibrations in $\text{PreSm}_V$ are objectwise h-cofibrations, the monoidal product of an object and an objectwise h-cofibration is an objectwise h-cofibration by h-monoidality of $V$, and objectwise h-cofibrations are h-cofibrations. Flatness in $\text{PreSm}_V$ follows from the same argument as the acyclic part of the pushout product axiom, using the fact that the cocartesian square for the pushout product of a cofibration and a weak equivalence is a homotopy pushout product square by the nonacyclic part of h-monoidality. Flatness in $V \prod$ then follows from Proposition 12.3. The acyclic part of h-monoidality holds by Pavlov–Scholbach [2015b, Theorem 3.2.6, Corollary 3.2.8]. (Pretty smallness in the cited results is only used to show that weak equivalences are closed under filtered colimits, which indeed holds in our case.)

For symmetric h-monoidality, the argument is the same, using Pavlov–Scholbach [2015b, Theorem 3.2.7] and the fact that $\text{Sing}_V$ preserves colimits in $\text{PreSm}_V$. For $V \prod$ we need to further observe that the associated hom functor preserves objectwise h-cofibrations and weak equivalences by Proposition 12.5.

Example 12.8. (Pavlov–Scholbach [2015b, §7]) The following model categories satisfy the properties that occur in the statement of Theorem 12.7, as well as Theorem 12.9 below.

- Simplicial sets with simplicial weak equivalences: all properties.
- Chain complexes (unbounded or nonnegatively graded): all properties other than the property of symmetric h-monoidality.
- Chain complexes in characteristic 0: all properties, and every quasi-isomorphism is symmetric flat.
- Simplicial modules: all properties. In characteristic 0 every weak equivalence is symmetric flat.
- Symmetric simplicial spectra: all properties, and every weak equivalence is symmetric flat.

Theorem 12.9. Suppose $V$ is a left proper combinatorial model category that is a tractable (meaning it admits a set of generating cofibrations with cofibrant domains) symmetric monoidal model category whose
weak equivalences are closed under filtered colimits. In the case of symmetric h-monoidal and in the case of nonsymmetric operads, we assume \( V \) to be h-monoidal. All operads are colored. All statements below are formulated for \( \text{PreSm} \), and an analogous version for \( \text{Sm} \) also holds.

- (Compare Proposition 11.3) The category of algebras over any operad \( O \) admits a model structure transferred along the forgetful functor that extracts underlying objects.
- If \( f: O \to O' \) is a weak equivalence of operads, then it induces a Quillen equivalence of model categories of algebras over \( O \) and \( O' \) if and only if \( f \) is a (symmetric) flat map. (In the nonsymmetric case, flat maps coincide with weak equivalences.)
- (Compare Proposition 11.3) For every operad \( O \) in \( \text{PreSm} \), the canonical comparison functor
  \[
  \text{Alg}_O(\text{PreSm})^\Delta[W_o^{-1}] \to \text{Alg}_O(\text{PreSm})[W_o^{-1}]
  \]
  is an equivalence of quasicategories.
- (Compare Proposition 11.4) There are Quillen equivalences
  \[
  L \dashv R: \text{Open} \leftrightarrow \text{Open}_{\text{PreSm}}, \quad L' \dashv R': \text{Open} \leftrightarrow \text{Open}_V
  \]
of model categories of operads in \( \text{PreSm} \), and \( V \).
- For any cofibrant operad \( O \in \text{Open}_{\text{PreSm}} \), there are Quillen equivalences
  \[
  L_O \dashv R_O: \text{Alg}_O(\text{PreSm}) \leftrightarrow \text{Alg}_O(\text{PreSm}) \text{,} \quad \text{Alg}_O(V) \leftrightarrow \text{Alg}_O(V)\]
- For any fibrant operad \( P \in \text{Open}_{\text{PreSm}} \) (respectively \( P' \in \text{Open}_V \)), there are Quillen equivalences
  \[
  L_P \dashv R_P: \text{Alg}_{RP}(\text{PreSm}) \leftrightarrow \text{Alg}_{RP}(\text{PreSm}) \text{,} \quad \text{Alg}_{RP}(V) \leftrightarrow \text{Alg}_{RP}(V).
  \]

\[\text{Proof.} \text{ Combine Theorem 12.7 with Pavlov–Scholbach 2014, Theorem 5.11, Theorem 7.5, Theorem 7.11, Haugseng 2019, Theorem 4.10, as well as Theorem 12.11 combined with Pavlov–Scholbach 2014, Theorem 8.10.}\]

**Proposition 12.10.** The functor \( \Delta \times \Delta \to \text{Card} \) (Definition 3.2) is an initial functor and a homotopy initial functor.

**Proof.** To show that \( \Delta \) is a homotopy initial functor (and hence an initial functor), we verify that for every \( V \in \text{Card} \), the comma category \( \Delta / V \) has a weakly contractible nerve. Objects of \( \Delta / V \) are pairs \( (m, \Delta^n \to V) \) and morphisms \( (m, \Delta^n \to V) \to (n, \Delta^m \to V) \) are maps of simplices \( f: [m] \to [n] \) that make the triangle with vertices \( \Delta^m, \Delta^n, V \), and \( V \) commute. By construction, \( \Delta / V \) is the category of simplices of the smooth singular simplicial set of \( V \). Therefore, the nerve of \( \Delta / V \) is weakly equivalent to the smooth singular simplicial set of \( V \), which is contractible.

**Theorem 12.11.** The right adjoint functors

\[
\text{Sing}_V: \text{PreSm}_V \to \text{V}_0, \quad \text{Sing}_V: \text{Sm}_V \to \text{V}_0
\]

are right Quillen equivalences, in fact, weak monoidal Quillen equivalences (Definition 5.8). Here \( \text{PreSm}_V \) and \( \text{Sm}_V \) are equipped with the model structure of Theorem 12.7 and \( \text{Sing}_V \) is equipped with the model structure of Definition 12.3.

**Proof.** We prove the claim for \( \text{PreSm}_V \) first. We denote the left adjoint of \( \text{Sing}_V \) by \( \| - \| \) (omitting \( V \)). The functor \( \| - \| \) sends a generating (acyclic) cofibration \( i \otimes \Delta^n \) in \( \text{V}_0 \) to an (acyclic) cofibration \( i \otimes \| \Delta^n \| \) in \( \text{PreSm}_V \). Furthermore, the left derived functor of \( \| - \| \) sends morphisms \( X \otimes \Delta^n \to X \otimes \Delta^0 \) to weak equivalences \( X' \otimes \| \Delta^n \| \to X' \otimes \| \Delta^0 \| \), where \( X' \to X \) is a cofibrant replacement of \( X \). Therefore, the functor \( \| - \| \) is a left Quillen functor by the universal property of left Bousfield localizations.

For \( \text{PreSm}_V \), the functor \( \text{Sing}_V \) preserves colimits. Thus, the derived unit natural transformation \( X \to \text{Sing}_V \| X \| \) is cocontinuous in \( X \in \text{V}_0 \). Since weak equivalences in \( \text{V}_0 \) are closed under filtered colimits, we can present \( X \) as a transfinite composition of cobase changes of morphisms \( i \otimes \delta_n: A \to B \), where \( \delta_n \) is a boundary inclusion (Proposition 5.1) and \( i: P \to Q \) is a generating cofibration of \( V \) and reduce the problem...
to the following elementary step: if $X \to Y$ is a cobase change of $i \cong \delta_n$ and $X \to \SmSing |Y|$ is a weak equivalence, then so is $Y \to \SmSing |Y|$. Indeed, we have a natural transformation
\begin{align*}
A \longrightarrow X & \quad \text{Sing}_V \| A \| \longrightarrow \text{Sing}_V \| X || \quad \text{Sing}_V \| i \circ \delta_n || \quad \text{Sing}_V \| B || \\
\downarrow \quad \downarrow & \quad \text{Sing}_V \| \quad \downarrow \\
B \longrightarrow Y & \quad \text{Sing}_V \| B || \longrightarrow \text{Sing}_V \| Y ||
\end{align*}
of corresponding pushout squares. The component
\[ X \to \text{Sing}_V \| X || \]
is a weak equivalence by assumption. The component
\[ Q \otimes \Delta^n = B \to \text{Sing}_V \| B || \cong Q \otimes \SmSing |\Delta^n || \]
is a weak equivalence because the map $\Delta^n \to \SmSing |\Delta^n ||$ has contractible source and target. The component
\[ A \to \text{Sing}_V \| A || \]
is a weak equivalence by inductive assumption (prove the claim by induction on $n$). The maps $i \cong \delta_n$ and $\text{Sing}_V \| i \circ \delta_n || \cong i \otimes \| \delta_n ||$ are cofibrations in $\mathbb{V}_A$, hence h-cofibrations, hence both squares are homotopy pushout squares in $\mathbb{V}_A$ and the component
\[ Y \to \text{Sing}_V \| Y || \]
is a weak equivalence.

For $\Sm_i$, we combine the previous argument for $\PreSm_i$ with Proposition 12.5.

Finally, to show that the established Quillen equivalences are weak monoidal Quillen equivalences in the sense of Schwede–Shipley [2002.c, Definition 3.6], we observe that the left adjoint functor $\| - ||$ (respectively $| - |$) preserves small colimits and commutes with tensoring over $V$. This allows us to prove that the comonoidal maps $L(A \otimes B) \to LA \otimes LB$ are weak equivalences for all cofibrant objects $A, B \in \PreSm_i$ by induction on $A$. If $A = 0$, then the comonoidal map is identity on $0$. Suppose the claim is true for $A$ and the map $A \to A'$ is given by the cobase change of a generating cofibration $i \cong \| \delta_n ||$. The natural transformation of left Quillen functors $L(- \otimes B) \to L(-) \otimes LB$ induces a natural transformation of the resulting cobase change squares. Since $\mathbb{V}_A$ and $\PreSm_i$ are left proper, cofibrations are h-cofibrations and the two cobase change squares are homotopy cobase change squares. This reduces the problem to showing that the three components of the natural transformation between squares are weak equivalences. This is true for $A$ by assumption, holds for the domain of $i \cong \| \delta_n ||$ by induction, and holds for the codomain of $i \cong \| \delta_n ||$ by the following argument. After performing a symmetric reduction for $B$, we reduce the problem to the case $A = P \otimes \| \Delta^m ||$, $B = Q \otimes \| \Delta^n ||$. The comonoidal map is $P \otimes Q \otimes \| \Delta^m \times \Delta^n || \to P \otimes Q \otimes \| \Delta^m || \times \| \Delta^n ||$, which is a weak equivalence because $\| \Delta^m \times \Delta^n || \to \| \Delta^m || \times \| \Delta^n ||$ is a map between weakly contractible objects in $\PreSmSet$. Thus, the cube lemma (Hovey [1999.a, Lemma 5.2.6]) implies that $L(A' \otimes B) \to LA' \otimes LB$ is also a weak equivalence.

To show weak monoidality in the case of $\Sm_i$, we combine the previous argument with Proposition 12.5.

13 The smooth Oka principle for enriched presheaves

The following result improves on the usual way of computing derived internal homs in cartesian model categories by eliminating the fibrant replacement functor. The proof can be found in Berwick-Evans–Boavida de Brito–Pavlov [2019.b, Theorem 1.1].

Proposition 13.1. (The smooth Oka principle for simplicial smooth sets.) If $X$ is a smooth manifold, the functor
\[ \text{Hom}(X, -) : \PreSmSet \to \PreSmSet \]
preserves weak equivalences (Definition 12.4) and therefore computes the derived internal hom in the model structure of Theorem 12.7. In particular, we have a weak equivalence of simplicial sets (natural in $X \in \text{Man}$ and $F \in \PreSmSet$)
\[ \SmSing \text{Hom}(X, F) \simeq \text{RHom} \SmSing X, \SmSing F), \]

25
The relevant theory was developed by Rosický [2005, 2014.a] and Lurie [2017.b, internal hom over $\text{Man}$, Examples 13.5. Used in 13.2*.

Examples 13.5. The following model categories are examples of model varieties:

- Simplicial sets, simplicial monoids, simplicial groups, simplicial rings. More generally, simplicial objects in any variety of algebras.
- Many models for connective spectra, e.g., $\Gamma$-spaces or simplicial symmetric spectra (appropriately left Bousfield localized to enforce connectivity).
- Nonnegatively graded chain complexes with quasi-isomorphisms.
- $E_n$-spaces ($0 \leq n \leq \infty$) and group-like $E_n$-spaces ($1 \leq n \leq \infty$) in simplicial sets.
- Connective $E_n$-ring spectra ($0 \leq n \leq \infty$) in simplicial sets.

Remark 13.4. The following equivalent definitions of a model variety connect it to other notions found in the literature that encode the same idea in different formalisms:

- A combinatorial model category whose underlying quasicategory is equivalent to a projectively generated $\infty$-category in the sense of Lurie [2017.b, Definition 5.5.8.23].
- A combinatorial model category whose underlying fibrant simplicial category (e.g., the fibrant replacement of the hammock localization) is a homotopy variety in the sense of Rosický [2005, Definition 4.10].
- A combinatorial model category connected by a chain of Quillen equivalences to the model category of algebras over a simplicial algebraic theory (Rosický 2005, Theorem 4.15, Corollary 4.18).

Examples 13.5. The following model categories are examples of model varieties:

- Simplicial sets, simplicial monoids, simplicial groups, simplicial rings. More generally, simplicial objects in any variety of algebras.
- Many models for connective spectra, e.g., $\Gamma$-spaces or simplicial symmetric spectra (appropriately left Bousfield localized to enforce connectivity).
- Nonnegatively graded chain complexes with quasi-isomorphisms.
- $E_n$-spaces ($0 \leq n \leq \infty$) and group-like $E_n$-spaces ($1 \leq n \leq \infty$) in simplicial sets.
- Connective $E_n$-ring spectra ($0 \leq n \leq \infty$) in simplicial sets.

Remark 13.6. Without loss of generality, we can assume a model variety $V$ to be left proper and simplicial, since any model variety is Quillen equivalent to a left proper simplicial model variety. This follows from Rezk [2000] Theorem $B_l$. Used in 13.3, 13.4.

Definition 13.7. Suppose $V$ is a left proper simplicial model variety (Definition 13.3, Remark 13.6). The functor

$$B_V: \text{PreSm}_V \rightarrow V, \quad F \mapsto \text{hocolim}_{\Delta \rightarrow F} F(\Delta')$$

is known as the path $\infty$-groupoid functor, or, abusing the language, simply as the shape functor. (Strictly speaking, the shape of $F$ is the locally constant sheaf on the path $\infty$-groupoid of $F$.) Used in 13.2, 13.3, 13.4, 13.5, 13.6.

Theorem 13.8. (The smooth Oka principle for model varieties.) Suppose $V$ is a left proper simplicial model variety (Definition 13.3, Remark 13.6). If $X$ is a smooth manifold, the functor

$$\text{Hom}(X, -): \text{PreSm}_V \rightarrow \text{PreSm}_V, \quad F \mapsto (M \mapsto F(M \times X))$$

where $\text{PreSm}_V(−)$ denotes the diagonal of the bisimplicial set $\text{Sing}_V(−)$ (Definition 12.4). Used in 13.3, 13.4.
preserves weak equivalences and therefore computes the derived hom. In particular, \( X \mapsto \text{Hom}(X, F) \)

\[ \text{is an } \infty\text{-sheaf of the form } \text{Map}^p \to \text{PreSm}_V. \]

- In particular, if \( F \) is Čech-local object in \( \text{PreSm}_V \), then the functor

\[ \text{B}_f(\text{Hom}(-, F)); \text{Map}^p \to V \]

is connected by a chain of natural weak equivalences to the functor

\[ \text{Hom}(\text{SmSing}(-), \text{B}_f F); \text{Map}^p \to V, \]

where the last \( \text{Hom} \) denotes the powering of \( V \) over simplicial sets.

**Proof.** Since \( V \) is a model variety, we can find a set \( G \) of objects in \( V \) with properties as described in Definition 13.3. In particular, for any \( X \in G \) the functors \( \text{Map}(X, -) : \text{PreSm}_V \to \text{sSet} \) (where \( \text{Map}(X, -) \) denotes the mapping simplicial set in the Dwyer–Kan hammock localization) have the following properties.

- The preserve all small homotopy limits and homotopy sifted homotopy colimits.
- In particular, they preserve the homotopy limits used for Čech descent objects and the homotopy colimit used in the definition of the functor \( \text{B}_f \).
- They jointly reflect weak equivalences: if \( \text{Map}(X, f) \) is a weak equivalence for all \( X \in G \), then \( f \) is a weak equivalence.

Together, these properties allow us to reduce the case of arbitrary model variety \( V \) to the case of \( \text{sSet} \) which holds by Proposition 13.1. \( \square \)

The following result answers a question posed to the author by Kiran Luecke.

**Proposition 13.9.** If \( V \) is a left proper model variety (Definition 13.3, Remark 13.6), \( F \in \text{PreSm}_V \), and \( G \in \text{PreSm}_V \) is the associated Čech-local object (i.e., the associated \( \infty \)-sheaf of \( F \)), with the localization map \( F \to G \), then the induced map on \( \text{shapes} \) (Definition 13.7) \( \text{B}_f F \to \text{B}_f G \) is a weak equivalence. That is to say, the shape of the associated \( \infty \)-sheaf of a presheaf can be computed without computing the associated \( \infty \)-sheaf.

**Proof.** This is a special case of Proposition 12.3. \( \square \)

14 Applications: classifying spaces for differential-geometric objects on diffeological spaces

We revisit the classical theorems on classifications of differential geometric objects such as closed differential forms (classified by real cohomology), bundle \((d−1)\)-gerbes with connection (classified by integral cohomology), principal \( G \)-bundles with connection (classified by the classifying space of \( G \)). In addition to recovering the classical versions of these results for smooth manifolds (thanks to Proposition 9.2), we also establish them in much larger generality for arbitrary cofibrant smooth sets or simplicial smooth sets.

**Example 14.1.** Working in the model category \( \text{SmSet} \), consider the internal hom \( \text{Hom}(M, \Omega^n_{\text{closed}}) \), where \( n \geq 0 \) and \( M \) is a smooth manifold, or, more generally, any cofibrant smooth set. This internal hom computes the derived internal hom because the source \( M \) is cofibrant and the target \( \Omega^n_{\text{closed}} \) is fibrant. Thus, the shape of \( \text{Hom}(M, \Omega^n_{\text{closed}}) \) can be computed as the derived mapping simplicial set from the shape \( \text{B}_f M \) of \( M \) to the shape of \( \Omega^n_{\text{closed}} \). The latter is simply \( K(\mathbb{R}, n) \), the \( n \)th Eilenberg–MacLane space of the reals. Thus, the smooth set \( \text{Hom}(M, \Omega^n_{\text{closed}}) \) can be seen as a smooth refinement of the simplicial set

\[ \text{Hom}(\text{SmSing} M, K(\mathbb{R}, n)). \]

In particular, connected components of \( \text{Hom}(M, \Omega^n_{\text{closed}}) \) are in bijection with \( H^n(M, \mathbb{R}) \), the \( n \)th de Rham (or real singular) cohomology of \( M \). This is well known when \( M \) is a manifold, but appears to be new when \( M \) is a cofibrant smooth set. In concrete terms, the chain complex

\[ \Omega^n_{\text{closed}}(M) \leftarrow \Omega^n_{\text{closed}}(M \times \Delta^1) \leftarrow \Omega^n_{\text{closed}}(M \times \Delta^2) \leftarrow \cdots, \]

where the differential in degree \( m \) is given by alternating sums of face maps of \( \Delta^n \), is quasi-isomorphic to the chain complex

\[ \Omega^n_{\text{closed}}(M) \leftarrow \Omega^{n−1}(M) \leftarrow \Omega^{n−2}(M) \leftarrow \cdots, \]
where the quasi-isomorphism can be implemented by fiberwise integration over the maps $M \times \Delta^n \to M$.

**Example 14.2.** Working in the model category $\mathbf{SmCh}_{\geq 0}$, consider the internal hom $\text{Hom}(M, D_n) = (\Omega^n \leftarrow \cdots \leftarrow \Omega^0 \leftarrow Z)$, where $n \geq 0$ and $M$ is a cofibrant simplicial smooth set. (Here we convert simplicial sets into chain complexes using the normalized chains functor.) The target $D_n$ is also known as the Deligne complex. This internal hom computes the derived internal hom because the source $M$ is cofibrant and the target is a fibrant object in $\mathbf{SmCh}_{\geq 0}$. The shape of $\text{Hom}(M, D_n)$ can be computed as the derived mapping chain complex from the shape $B_\int M$ of $M$ to the shape of $D_n$. The latter is simply $K(Z, n + 1)$, the $(n + 1)$st Eilenberg–MacLane space of the integers. In particular, this proves that concordance classes of bundle $(n - 1)$-gerbes with connections over $M$ are classified by the group $H^{n+1}(B_\int M, Z)$. This is well known when $M$ is a manifold, but appears to be new when $M$ is a cofibrant simplicial smooth set.

**Example 14.3.** Working in the model category $\mathbf{SmSet}$, consider the internal hom $\text{Hom}(M, B\mathcal{G})$, where $\mathcal{G}$ is a Lie group and $M$ is a cofibrant simplicial smooth set. The target $B\mathcal{G}$ is the delooping of the representable presheaf of $\mathcal{G}$. This internal hom computes the derived internal hom because the source $M$ is cofibrant and the target is a fibrant object in $\mathbf{SmSet}$. Thus, the shape of $\text{Hom}(M, B\mathcal{G})$ can be computed as the derived mapping chain complex from the shape $B_\int M$ of $M$ to the shape of $B\mathcal{G}$. The latter shape is simply $B\mathcal{G}$, the classifying space of $\mathcal{G}$ as a topological group, i.e., the delooping of the singular complex of $\mathcal{G}$. In particular, this proves that concordance classes of principal $\mathcal{G}$-bundles over $M$ are classified by the set $[B_\int M, B\mathcal{G}]$. This is well known when $M$ is a manifold, but appears to be new when $M$ is a cofibrant simplicial smooth set.

15 References

[1973.a] Kuo-Tsai Chen. Iterated integrals of differential forms and loop space homology. Annals of Mathematics 97:2 (1973), 217–246. doi:10.2307/1970846. 1.0*, 2.6 *

[1973.b] Jacques Penon. Quasi-topos. Comptes Rendus Hebdomadaires des Séances de l’Académie des Sciences 276 (1973), 237–240. https://gallica.bnf.fr/ark:/12148/bpt6k6217213f/f251.image

[1977] Jacques Penon. Sur le quasi-topos. Cahiers de topologie et géométrie différentielle catégoriques 18:2 (1977), 181–218. numdam:CTGDC_1977__18_2_1810_0

[1979] Eduardo J. Dubuc. Concrete quasitopoi. Lecture Notes in Mathematics 753 (1979), 239–254. doi:10.1007/BFb0061821

[1980] Jean-Marie Souriau. Groupes différentiels. In: Differential Geometrical Methods in Mathematical Physics. Lecture Notes in Mathematics 836 (1980), 91–128. doi:10.1007/BFb0069728

[1993] Sjoerd E. Crans. Quillen closed model structures for sheaves. Journal of Pure and Applied Algebra 101:1 (1995), 35–57. doi:10.1016/0022–4049(94)00033–f

[1999.a] Mark Hovey. Model categories. Mathematical Surveys and Monographs 63 (1999). doi:10.1090/surv/063

[1999.b] Mark Hovey. Algebraic Topology Problem List. Model categories. March 6, 1999. https://web.archive.org/web/20000830081819/http://claude.math.wesleyan.edu/~mhovey/problems/model.html

[2000] Charles Rezk. Every homotopy theory of simplicial algebras admits a proper model. Topology and its Applications 119:1 (2002), 65–94. arXiv:math/0003065v1

[2002.a] Peter T. Johnstone. Sketches of an Elephant. A Topos Theory Compendium. II. Oxford Logic Guides 44 (2002).

[2002.b] Denis-Charles Cisinski. Théories homotopiques dans les topos. Journal of Pure and Applied Algebra 174:1 (2002), 43–82. doi:10.1016/s0022–4049(01)00176 –1

[2002.c] Stefan Schwede, Brooke Shipley. Equivalences of monoidal model categories. Algebraic & Geometric Topology 3 (2003), 287–334. arXiv:math/0209342v2 doi:10.2140/agt.2003.3.287

[2002.d] Ib Madsen, Michael Weiss. The stable moduli space of Riemann surfaces: Mumford’s conjecture. Annals of Mathematics 165:3 (2007), 843–941. arXiv:math/0212321v3 doi:10.4007/annals.2007.165.843
[2003] Philip S. Hirschhorn. Model categories and their localizations. Mathematical Surveys and Monographs 99 (2003). doi:10.1090/surv/099

[2005] Jiří Rosický. On homotopy varieties. Advances in Mathematics 214 (2007), 525–550. arXiv:math/0509655v2

[2008.a] John C. Baez, Alexander E. Hoffnung. Convenient categories of smooth spaces. Transactions of the American Mathematical Society 363:11 (2011), 5789–5825. arXiv:0807.1704v4

[2008.b] Hisham Sati, Urs Schreiber, Jim Stasheff. L∞-algebra connections and applications to String- and Chern-Simons n-transport. (Quantum Field Theory. Competitive Models,) Birkhäuser (2009), 303–424. arXiv:0801.3480v2

[2009] Hisham Sati, Urs Schreiber, Jim Stasheff. Twisted differential String and Fivebrane structures. Communications in Mathematical Physics 315:1 (2012), 169–213. arXiv:0910.4001v2

[2013.a] Patrick Iglesias-Zemmour. Diffeology. Mathematical Surveys and Monographs 185 (2013).

[2013.b] Michael Batanin, Clemens Berger. Homotopy theory for algebras over polynomial monads. Theory and Applications of Categories 32:6 (2017), 148–253. arXiv:1305.0086v7

[2013.c] J. Daniel Christensen, Enxin Wu. The homotopy theory of diffeological spaces. New York Journal of Mathematics 20 (2014), 1269–1303. arXiv:1311.6394v4

[2014.a] Jiří Rosický. Corrigendum to “On homotopy varieties”. Advances in Mathematics 259 (2014), 841–842. doi:10.1016/j.aim.2014.02.032

[2014.b] Dmitri Pavlov, Jakob Scholbach. Admissibility and rectification of colored symmetric operads. Journal of Topology 11:3 (2018), 559–601. arXiv:1410.5675v4

[2015.a] David Ayala, John Francis, Nick Rozenblyum. A stratified homotopy hypothesis. Journal of the European Mathematical Society 21:4 (2019), 1071–1178. arXiv:1502.01713v4

[2015.b] Dmitri Pavlov, Jakob Scholbach. Homotopy theory of symmetric powers. Homology, Homotopy and Applications 20:1 (2018), 359–397. arXiv:1510.04969v3

[2016] Hiroshi Kihara. Model category of diffeological spaces. Journal of Homotopy and Related Structures 14:1 (2019), 51–90. arXiv:1605.06794v3

[2017.a] Hiroshi Kihara. Quillen equivalences between the model categories of smooth spaces, simplicial sets, and arc-generated spaces. arXiv:1702.04070v1

[2017.b] Jacob Lurie. Higher Topos Theory. April 9, 2017. https://www.math.ias.edu/~lurie/papers/HTT.pdf

[2019.a] Rune Haugseng. Algebras for enriched ∞-operads. arXiv:1909.10042v1

[2019.b] Daniel Berwick-Evans, Pedro Boavida de Brito, Dmitri Pavlov Classifying spaces of infinity-sheaves. arXiv:1912.10544v2

[2020.a] Hiroshi Kihara. Smooth homotopy of infinite-dimensional C∞-manifolds. arXiv:2002.03618v1

[2021.a] Christopher Adrain Clough. A convenient category for geometric topology. Ph.D. dissertation, The University of Texas at Austin, 2021. https://hdl.handle.net/2152/114981

[2021.b] Dmitri Pavlov. Combinatorial model categories are equivalent to presentable quasicategories. arXiv:2110.04679v3

[2021.c] Hisham Sati, Urs Schreiber. Equivariant principal ∞-bundles. arXiv:2112.13654v3