Universal local linear kernel estimators in nonparametric regression

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Abstract

New local linear estimators are proposed for a wide class of nonparametric regression models. The estimators are uniformly consistent regardless of satisfying traditional conditions of dependence of design elements. The estimators are the solutions of a specially weighted least-squares method. The design can be fixed or random and does not need to meet classical regularity or independence conditions. As an application, several estimators are constructed for the mean of dense functional data. The theoretical results of the study are illustrated by simulations. An example of processing real medical data from the epidemiological cross-sectional study ESSE-RF is included. We compare the new estimators with the estimators best known for such studies.

Keywords: nonparametric regression; kernel estimator; local linear estimator; uniform consistency; fixed design; random design; dependent design elements; mean of dense functional data; epidemiological research.

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1 Introduction

In this paper, we consider a nonparametric regression model, where bivariate observations

\[
\{(X_1, z_1), \ldots, (X_n, z_n)\}
\]

satisfy the following equations:

\[
X_i = f(z_i) + \varepsilon_i, \quad i = 1, \ldots, n,
\]

where \(f(t) \equiv f(\omega, t), \quad t \in [0, 1]\), is an unknown random function (process) which is continuous almost surely, the design \(\{z_i; \quad i = 1, \ldots, n\}\) consists of a set of observable random variables with possibly unknown distributions lying in \([0, 1]\), the design points are not necessarily independent or identically distributed. We will consider the design as a triangular array, i.e., the random variables \(\{z_i; \quad i = 1, \ldots, n\}\) may depend on \(n\). In particular, this scheme includes regression models with fixed design. The random regression function \(f(t)\) is not supposed to be design independent. We will give below some fairly standard conditions for the regression analysis on the random errors \(\{\varepsilon_i; \quad i = 1, \ldots, n\}\). In particular, they are supposed to be centered, not necessarily independent or identically distributed.

The paper is devoted to constructing uniformly consistent estimators for the regression function \(f(t)\) under minimal assumptions on the correlation of design points.

The most popular kernel estimation procedures in the classical case of nonrandom regression function are apparently related with the estimators of Nadaray–Watson, Priestley–Zhao, Gasser–Müller, local polynomial estimators, as well as their modifications (e.g., see \cite{14, 15, 18, 22, 47}). We are primarily interested in the dependence conditions of design elements \(\{z_i\}\). In this regard, a huge number of publications in the field of nonparametric regression can be conditionally...
divided into two groups. We will classify papers with a random design to the first one, and to the second one – with a fixed design.

In the papers dealing with random design, either independent and identically distributed quantities are considered or, as a rule, stationary sequences of observations that satisfy one or another known form of dependence. In particular, various types of mixing conditions, schemes of moving averages, associated random variables, Markov or martingale properties, and so on have been used. In this regard, we note, for example, the papers [10], [12], [14], [15], [20], [21], [24], [27], [30], [33], [41], [54]–[56], [57], [65]. In the recent papers [8], [28], [42], and [55], nonstationary sequences of design elements with one or another special type of dependence are considered (Markov chains, autoregression, partial sums of moving averages, etc.). In the case of fixed design, in the overwhelming majority of works, certain conditions for the regularity of the design are assumed (e.g., see [12], [13], [17], [20], [21], [26], [32], [33], [34], [43], [48], [53], [55], [64], and the references there).

In connection with studying the random regression function \( f(t) \), we note, for example, the papers [19], [29], [31], [35], [59]–[64] where the mean and covariance functions of the random regression function \( f \) are estimated in the case when, for \( N \) independent copies \( f_1, \ldots, f_N \) of the function \( f \), noisy values of each of these trajectories are observed for some collection of design points \( z_i \), are most often given by the formula \( z_i = g(i/n) + o(1/n) \) with some function \( g \) of bounded variation, where the error \( o(1/n) \) is uniform in all \( i = 1, \ldots, n \). If \( g \) is linear then we get a so-called equidistant design. Another version of the regularity condition is the relation \( \max_{i \leq n} (z_i - z_{i-1}) = O(1/n) \) (here it is assumed that the design elements ranged in increasing order).

The problem of uniform approximation of a regression function has been studied by many authors (e.g., see [12], [13], [17], [20], [21], [26], [32], [33], [43], [48], [53], [55], [64], and the references there).
2 Main results

We need a number of assumptions.

(D) The observations \(X_1, \ldots, X_n\) are represented in the form \([1]\), where the unknown random regression function \(f : [0, 1] \rightarrow \mathbb{R}\) is continuous almost surely. The design points \(\{z_i; i = 1, \ldots, n\}\) are a set of observable random variables with values in \([0, 1]\), having, generally speaking, unknown distributions, not necessarily independent or equally distributed. Moreover, the random variables \(\{z_i; i = 1, \ldots, n\}\) may depend on \(n\), i.e., can be considered as an array of design observations. The random function \(f(t)\) may be design dependent.

(E) For all \(n \geq 1\), the unobservable random errors \(\{\varepsilon_i; i = 1, \ldots, n\}\) satisfy with probability 1 the following conditions for all \(i, j \leq n\) and \(i \neq j\):

\[
E_{F_n} \varepsilon_i = 0, \quad \sup_{i \leq n} E_{F_n} \varepsilon_i^2 \leq \sigma^2, \quad E_{F_n} \varepsilon_i \varepsilon_j = 0,
\]

where the constant \(\sigma^2 > 0\) may be unknown and does not depend on \(n\), the symbol \(E_{F_n}\) stands for the conditional expectation given the \(\sigma\)-field generated both by the paths of the random process \(f(\cdot)\) and by the random variables \(\{z_i; i = 1, \ldots, n\}\).

(K) A kernel \(K(t), t \in \mathbb{R}\), is equal to zero outside the interval \([-1, 1]\) and is the density of a symmetric distribution with the support in \([-1, 1]\), i.e., \(K(t) \geq 0, K(t) = K(-t)\) for all \(t \in [-1, 1]\), and \(\int_{-1}^{1} K(t) dt = 1\). We assume that the function \(K(t)\) satisfies the Lipschitz condition with constant \(1 \leq L \leq \infty\) and \(K(\pm 1) = 0\).

In the future, we denote by \(\kappa_j, j = 0, 1, 2, 3\), the absolute \(j\)th moment of the distribution with density \(K(t)\), i.e., \(\kappa_j = \int_{-1}^{1} |u|^j K(u) du\). Put \(K_h(t) = h^{-1} K(h^{-1} t)\). It is clear that \(K_h(s)\) is a probability density with support lying in \([-h, h]\). We need also the notation

\[
\|K\|^2 = \int_{-1}^{1} K^2(u) du, \quad \kappa_j(\alpha) = \int_{-1}^{1} t^j K(t) dt, \quad \alpha \in [0, 1], \quad j = 0, 1, 2, 3.
\]

We emphasize that assumption (D) includes a fixed design situation. We consider the segment \([0, 1]\) as an area of design change solely for the sake of simplicity of exposition of the approach. In the general case, instead of the segment \([0, 1]\), one can consider an arbitrary Jordan measurable subset of \(\mathbb{R}\).

Further, we denote by \(z_{n,1} \leq \ldots \leq z_{n,n}\) the order statistics constructed by the sample \(\{z_i; i = 1, \ldots, n\}\). Put

\[
z_{n,0} := 0, \quad z_{n,n+1} := 1, \quad \Delta z_{ni} := z_{n,i} - z_{n,i-1}, \quad i = 1, \ldots, n + 1.
\]

The response variable \(X_i\) and the random error \(\varepsilon_i\) from \([1]\) corresponding to the order statistic \(z_{n,i}\), will be denoted by \(X_{ni}\) and \(\varepsilon_{ni}\), respectively. It is easy to see that the new errors \(\{\varepsilon_{ni}; i = 1, \ldots, n\}\) satisfy condition (E) as well. Next, by \(O_p(\eta_n)\) we denote a random variable \(\zeta_n\) such that, for all \(M > 0\), one has

\[
\limsup_{n \to \infty} P(|\zeta_n|/\eta_n > M) \leq \beta(M),
\]

where \(\lim_{M \to \infty} \beta(M) = 0\) and \(\{\eta_n\}\) are positive (maybe random or not) variables and the function \(\beta(M)\) that may depend on the kernel \(K\) and \(\sigma^2\). We agree that, throughout what follows, all limits, unless otherwise stated, are taken for \(n \to \infty\).

Let us introduce one more constraint, which is the crucial condition of the paper (in particular, the only condition on design points that guarantees the existence of a uniformly consistent estimator; see also the comments at the end of the section).

(D0) The following limit relation holds: \(\delta_n := \max_{1 \leq i \leq n+1} \Delta z_{ni} \mathop{\lesssim}_{D} 0\).

Finally, for any \(h \in (0, 1)\), we introduce into consideration the following class of estimators for the regression function \(f\):

\[
\hat{f}_{n,h}(t) := I(\delta_n \leq c_n h) \sum_{i=1}^{n} w_{n2}(t) - (t - z_{ni}) w_{n1}(t) w_{n0}(t) w_{n2}(t) - w_{n1}(t) X_{ni} K_h(t - z_{ni}) \Delta z_{ni}, \quad (3)
\]
where \( I(\cdot) \) is the indicator function,

\[
c_* \equiv c_*(K) := \frac{\kappa_2 - \kappa_1^2}{96L(6L + \kappa_2 + \kappa_1/2)} < \frac{1}{864L};
\]  

(4)

hereinafter, we use the notation

\[
w_{nj}(t) := \sum_{i=1}^{n} (t - z_{ni})^j K_h(t - z_{ni}) \Delta z_{ni}, \quad j = 0, 1, 2, 3.
\]

Remark 6. With the constant \( c_* \) hereinafter, we use the notation

\[
\text{Thus, the proposed class of estimators in a certain sense (in fact, by construction) is close to the classical local linear kernel estimators, but in the weighted least squares method (5) we use slightly different weights.}
\]

Remark 3. It is easy to verify that kernel estimator (3), without the indicator factor, is the same points \( z \) with their sample mean and leaving only one design point out of multiples in the new sample. In this case, the averaged observations will have less noise. So, despite the smaller size of the new sample, we do not lose the information contained in the original sample.

Let us further agree to denote by \( C_1, j \geq 1 \), absolute positive constants, and by \( C_1^* \), positive constants depending only on the kernel \( K \). The main result of this section is as follows.

Theorem 1. Let conditions (D), (E), and (K) be satisfied. Then, for any fixed \( h \in (0, 1/2) \), with probability 1 it is satisfied

\[
\sup_{t \in [0,1]} |\tilde{f}_{n,t}(t) - f(t)| \leq C_1^* \omega_f(h) + \zeta_n(h),
\]  

(6)

where \( \omega_f(h) := \sup_{u,v \in [0,1], |u-v| \leq h} |f(u) - f(v)| \) and the random variable \( \zeta_n(h) \) meets the relation

\[
P(\zeta_n(h) > y, \delta_n \leq c_*h) \leq C_2^* \sigma^2 \frac{\mathbb{E}\delta_n}{h^2 y^2},
\]  

(7)

with the constant \( c_* \) from \([4]\).

Remark 4. In the case when there are multiple design points, some spacings \( \Delta z_{ni} \) vanish, and we lose some of the sample information in the estimator \([3]\). In this case, it is proposed, before using the estimator \([3]\), to slightly reduce the sample by replacing the observations \( X_i \) with the same points \( z_i \) with their sample mean and leaving only one design point out of multiples in the new sample. In this case, the averaged observations will have less noise. So, despite the smaller size of the new sample, we do not lose the information contained in the original sample.

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\]  

(7)

with the constant \( c_* \) from \([4]\).

Remark 5. As follows from the proof of Theorem 1, the constants \( C_1^* \) and \( C_2^* \) have the following structure:

\[
C_1^* = C_1 \frac{L^2}{\kappa_2 - \kappa_1^2}, \quad C_2^* = C_2 \frac{L^4}{(\kappa_2 - \kappa_1^2)^2}.
\]

Remark 6. Since \( \delta_n \leq 1 \), then under condition (D0) the limit relation \( \mathbb{E}\delta_n \to 0 \) holds. Therefore, taking into account Theorem 1, we can assert that \( \zeta_n(h) = O_p(h^{-1}(\mathbb{E}\delta_n)^{1/2}) \). Thus, the bandwidth \( h \) can be determined, for example, by the relation

\[
h_n = \sup \left\{ h > 0 : \mathbb{P} \left( \omega_f(h) \geq h^{-1}(\mathbb{E}\delta_n)^{1/2} \right) \leq h^{-1}(\mathbb{E}\delta_n)^{1/2} \right\}.
\]  

(8)
It is easy to see that, when \((D_0)\) is satisfied, the limit relations

\[ h_n \to 0 \quad \text{and} \quad h_n^{-1}(\tilde{E}h_n)^{1/2} \to 0 \]

hold. In fact, the value of \(h_n\) equalizes in \(h\) the order of smallness in probability of both terms on the right-hand side of the relation \((\ref{eq:holder})\). Note also that, for nonrandom \(f\), one can choose \(h \equiv h_n\) as a solution to the equation

\[ h^{-1}(\tilde{E}h_n)^{1/2} = \omega_f(h). \]

It is clear that this solution tends to zero as \(n\) grows.

The relations \((\ref{eq:holder})\) and \((\ref{eq:holder})\) allow us to obtain the order of smallness of the optimal bandwidth \(h\), but not the optimal value of \(h\). In practice, \(h\) can be chosen, for example, by so-called cross-validation.

From Theorem \((\ref{thm:holder})\) and Remark \((\ref{rem:holder})\) it is easy to obtain the following corollary.

**Corollary 1.** Let the conditions \((D), (D_0), (K),\) and \((E)\) be satisfied, the regression function \(f(t)\) is nonrandom, and \(C\) is an arbitrary subset of equicontinuous functions in \(C[0,1]\) (for example, a precompact set). Then

\[ \gamma_n(C) = \sup_{f \in C} \sup_{t \in [0,1]} |\tilde{f}_{n,\hat{h}_n}(t) - f(t)| \to 0, \]

where \(\hat{h}_n\) is defined by equation \((\ref{eq:holder})\), in which the modulus of continuity \(\omega_f(h)\) is replaced with the universal modulus \(\omega_C(h) = \sup_{f \in C} \omega_f(h)\). Moreover, the asymptotic relation \(\gamma_n(C) = O_p(\omega_C(\hat{h}_n))\) hold.

**Remark 7.** It is easy to see that for a nonrandom \(f(t)\) the modulus of continuity in \((\ref{eq:holder})\) can be replaced by one or another upper bound for \(\omega_f(h)\), obtaining the corresponding upper bound for \(\gamma_n(C)\). Consider the case \(\tilde{E}h_n = O(1/n)\). If \(C\) consists of functions \(f(t)\) satisfying the Hölder condition with exponent \(\alpha \in (0,1]\) and a universal constant then \(\hat{h}_n = O( n^{-1/\alpha+\alpha} )\) and \(\omega_C(\hat{h}_n) = O( n^{-1+\alpha} )\). In particular, if the functions from \(C\) satisfy the Lipschitz condition \((\alpha = 1)\) with a universal constant then \(\gamma_n(C) = O_p(n^{-1/4})\).

From Theorem \((\ref{thm:holder})\) and Remark \((\ref{rem:holder})\) we obtain the following corollary.

**Corollary 2.** Let the conditions \((D), (D_0), (K),\) and \((E)\) be satisfied and let the modulus of continuity \(\omega_f(h)\) of the random regression function \(f(t)\) with probability 1 admit the upper bound \(\omega_f(h) \leq \zeta d(h)\), where \(\zeta > 0\) is a random variable and \(d(h)\) is a positive continuous nonrandom function such that \(d(h) \to 0\) as \(h \to 0\). Then

\[ \sup_{t \in [0,1]} |\tilde{f}_{n,\hat{h}(n)}(t) - f(t)| \to 0, \]

where the value \(\hat{h}_n\) is defined in \((\ref{eq:holder})\) after replacement \(d(h)\).

Let us discuss in more detail condition \((D_0)\). Obviously, condition \((D_0)\) is satisfied for any nonrandom regular design (this is the case of nonidentically distributed \(\{z_i\}\) depending on \(n\)). If \(\{z_i\}\) are independent and identically distributed and the interval \([0,1]\) is the support of distribution of \(z_1\) then condition \((D_0)\) is also satisfied. In particular, if the distribution density of \(z_1\) is separated from zero on \([0,1]\) then \(\delta_n = O(\log n / n)\) holds (see details in \((\ref{thm:regular})\)). If \(\{z_i; i \geq 1\}\) is a stationary sequence with a marginal distribution with the support \([0,1]\), satisfying an \(\alpha\)-mixing condition then condition \((D_0)\) is also satisfied (see Remark \((\ref{rem:holder})\) below). Note that the dependence of the random variables \(\{z_i\}\) satisfying condition \((D_0)\) can be much stronger, which is illustrated in the following example.

**Example 1.** Let the sequence of random variables \(\{z_i; i \geq 1\}\) be defined by the relation

\[ z_i = \nu_i u_i^1 + (1 - \nu_i) u_i^r, \]

where \(\{u_i^1\}\) and \(\{u_i^r\}\) are independent and uniformly distributed on \([0,1/2]\) and \([1/2,1]\), respectively, the sequence \(\nu_i\) does not depend on \(\{u_i^1\}\), \(\{u_i^r\}\) and consists of Bernoulli random variables etc.
with success probability $1/2$, i.e., the distribution of random variables $z_i$ is an equilibrium mixture of two uniform distributions on the corresponding intervals. The dependence between the random variables $\nu_i$ for any natural number $i$ is defined by the equalities $\nu_{2i-1} = \nu_1$ and $\nu_{2i} = 1 - \nu_1$. In this case, the random variables $\{z_i; i \geq 1\}$ form a stationary sequence of random variables uniformly distributed on the segment $[0, 1]$, satisfying condition $(D_0)$. On the other hand, for all natural numbers $m$ and $n$,

$$\Pr(2z_{2m} \leq 1/2, z_{2n-1} \leq 1/2) = 0.$$  

Thus, all the known conditions for the weak dependence of random variables (in particular, the mixing conditions) are not satisfied here.

According to the scheme of this example, it is possible to construct various sequences of dependent random variables uniformly distributed on $[0, 1]$ by choosing sequences of Bernoulli switches with the conditions $\nu_{j_k} = 1$ and $\nu_{k} = 0$ for infinite numbers of indices $\{j_k\}$ and $\{k\}$. In which case, condition $(D_0)$ will also be satisfied, but the corresponding sequence $\{z_i\}$ (not necessarily stationary) may not even satisfy the strong law of large numbers. For example, this is the case when $\nu_j = 1 - \nu_1$ for $j = 2^{2k-1}, \ldots, 2^{2k} - 1$, and $\nu_j = \nu_1$ for $j = 2^{2k}, \ldots, 2^{2k+1} - 1$, where $k = 1, 2, \ldots$ (i.e., we randomly choose one of the two segments $[0, 1/2]$ and $[1/2, 1]$), into which we randomly throw the first point, and then alternate the selection of one of the two segments by the following numbers of elements of the sequence: $1, 2, 2^2, 2^3, \text{etc.}$). Indeed, we can introduce the notation $n_k = 2^{2k} - 1$, $\tilde{n}_k = 2^{2k+1} - 1$, $S_m = \sum_{i=1}^m z_i$ and note that, for all elementary events from the event $\{\nu_1 = 1\}$, one has

$$\frac{S_{n_k}}{n_k} = \frac{1}{n_k} \sum_{i \in N_{1,k}} u_i^l + \frac{1}{n_k} \sum_{i \in N_{2,k}} u_i^r,$$

where $N_{1,k}$ and $N_{2,k}$ are the sets of indices, for which the observations $\{z_i; i \leq n_k\}$ lie in the intervals $[0, 1/2]$ or $[1/2, 1]$, respectively. It is easy to see that $|N_{1,k}| = n_k/3$ and $|N_{2,k}| = 2|N_{1,k}|$. Hence, $S_{n_k}/n_k \to 7/12$ almost surely as $k \to \infty$ due to the strong law of large numbers for the sequences $\{u_i^l\}$ and $\{u_i^r\}$. On the other hand, as $k \to \infty$, for all elementary events from $\{\nu_1 = 0\}$ one has

$$\frac{S_{\tilde{n}_k}}{\tilde{n}_k} = \frac{1}{\tilde{n}_k} \sum_{i \in \tilde{N}_{1,k}} u_i^l + \frac{1}{\tilde{n}_k} \sum_{i \in \tilde{N}_{2,k}} u_i^r \to \frac{5}{12},$$  

(12)

where $\tilde{N}_{1,k}$ and $\tilde{N}_{2,k}$ are the sets of indices, for which the observations $\{z_i; i \leq \tilde{n}_k\}$ lie in the intervals $[0, 1/2]$ or $[1/2, 1]$, respectively. Proving the convergence in (12), we took into account that $|N_{1,k}| = (2^{2k+2} - 1)/3$ and $|\tilde{N}_{2,k}| = 2n_k/3$, i.e., $|N_{1,k}| = 2|\tilde{N}_{2,k}| + 1$.

Similar arguments are valid for all elementary events from $\{\nu_1 = 0\}$.  

Re m a r k 8. In the case of i.i.d. random variables $\{z_i\}$, condition $(D_0)$ will be fulfilled if, for all $\delta \in (0, 1)$,

$$p_n(\delta) \equiv \sup_{|\Delta| = \delta} \Pr(\bigcap_{i \leq n} \{z_i \notin \Delta\}) \to 0,$$

(13)

where the supremum is taken over all intervals $\Delta \subset [0, 1]$ of length $\delta$. Indeed, for any natural $N > 1$, we divide the interval $[0, 1]$ into $N$ subintervals $\Delta_k$, $k = 1, \ldots, N$, of length $1/N$. Then one has

$$\Pr\left(\max_{1 \leq i \leq n+1} \Delta z_{ni} > \frac{2}{N}\right) \leq \sum_{k=1}^{N} \Pr\left(\bigcap_{i \leq n} \{z_i \notin \Delta_k\}\right) \leq N \max_k \Pr\left(\bigcap_{i \leq n} \{z_i \notin \Delta_k\}\right) \leq Np_n(1/N),$$

since the event $\{\max_{1 \leq i \leq n+1} \Delta z_{ni} > 2/N\}$ implies the existence of an interval $\Delta_k$ of length $1/N$ that does not contain any points from the collection $\{z_i\}$. Thereby, condition (13) implies the limit relation $\max_{1 \leq i \leq n+1} \Delta z_{ni} \overset{p}{\to} 0$, which is equivalent to convergence with probability 1 due to
the monotonicity of the sequence \( \max_{i \leq n+1} \Delta z_{ni} \). In particular, if \( \{z_i\} \) are independent then \( p_n(\delta) = e^{-c(\delta)n} \) and \( c(\delta) > 0 \), i.e., as \( n \to \infty \), the finite collection \( \{z_i\} \) with probability 1 form a refining partition of the finite segment \([0, 1]\). It is easy to show that if \( \{z_i; i \geq 1\} \) is a stationary sequence satisfying an \( \alpha \)-mixing condition and having a marginal distribution with the support \([0, 1]\) then \([13]\) will be valid.

\[ \square \]

### 3 Estimating the mean function of a stochastic process

Consider the following statement of the problem of estimating the expectation of an almost surely continuous stochastic process \( f(t) \). There are \( N \) independent copies of the regression equation \([1]\):

\[ X_{i,j} = f_j(z_{i,j}) + \varepsilon_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, N, \tag{14} \]

where \( f(t), f_1(t), \ldots, f_N(t), t \in [0, 1] \), are independent identically distributed almost surely continuous unknown random processes, the set \( \{\varepsilon_{i,j}; i = 1, \ldots, n\} \) satisfies condition \((E)\) for any \( j \), the set \( \{z_{i,j}; i = 1, \ldots, n\} \) meets conditions \((D)\) and \((D_0)\) for any \( j \) (here and below the index \( j \) for the considered random variables means the number of copy of Model \([1]\)). In particular, under the assumption that condition \((K)\) is valid, by \( \hat{f}_{n,h,j}(t), j = 1, \ldots, N, \) we denote the estimator given by the relation \([3]\) when replacing the values from \([1]\) with the corresponding characteristics from \([14]\). Finally, an estimator for the mean-function is determined by the equality

\[ \hat{f}_{N,n,h}(t) = \frac{1}{N} \sum_{j=1}^{N} \hat{f}_{n,h,j}(t). \tag{15} \]

As a consequence of Theorem \([1]\) we obtain the following assertion.

**Theorem 2.** Let Model \([14]\) satisfy the above-mentioned conditions and, moreover,

\[ \mathbb{E} \sup_{t \in [0,1]} |f(t)| < \infty, \tag{16} \]

while the sequences \( h \equiv h_n \to 0 \) and \( N \equiv N_n \to \infty \) meet the restrictions

\[ h^{-2} \mathbb{E} \delta_n \to 0 \quad \text{and} \quad N \mathbb{P}(\delta_n > c_n h) \to 0. \tag{17} \]

Then

\[ \sup_{t \in [0,1]} \left| \hat{f}_{N,n,h}(t) - \mathbb{E} f(t) \right| \overset{P}{\to} 0. \tag{18} \]

\( R \ e \ m \ a \ r \ k \ 9 \). If condition \([16]\) is replaced with a slightly stronger constraint

\[ \mathbb{E} \sup_{t \in [0,1]} f^2(t) < \infty \]

then, under conditions similar to \([17]\), one can prove the uniform consistency of the estimator

\[ \hat{M}_{N,n,h}(t_1, t_2) = \frac{1}{N} \sum_{j=1}^{N} \hat{f}_{n,h,j}(t_1) \hat{f}_{n,h,j}(t_2), \quad t_1, t_2 \in [0,1], \]

for the unknown mixed second moment \( \mathbb{E} f(t_1) f(t_2) \) where \( h \equiv h_n \) and \( N \equiv N_n \) satisfy \([17]\). The arguments in proving this fact are quite similar to those in proving Theorem \([2]\) and they are omitted. In other words, under the above-mentioned restrictions, the estimator

\[ \hat{\text{Cov}}_{N,n,h}(t_1, t_2) = \hat{M}_{N,n,h}(t_1, t_2) - \hat{f}_{N,n,h}(t_1) \hat{f}_{N,n,h}(t_2) \]

uniformly consistent for the covariance of the random regression function \( f(t) \).
The problem of estimating the mean and covariance functions plays a fundamental role in the so-called functional data analysis (see, for example, [23], [29], [31], [46]). The property of uniform consistency of certain estimates of the mean function, which is important in the context of the problem under consideration, was considered, for example, in [23], [31], [60], [62], [64]. For a random design, as a rule, it is assumed that all its elements are independent identically distributed random variables (see, for example, [4], [19], [31], [58], [61]). In the case where the design is deterministic, certain regularity conditions discussed above in Introduction are usually used. Moreover, in the problem of estimating the mean function, it is customary to subdivide design elements into certain types depending on the density of filling with the design points the regression function domain. The literature focuses on two types of data: or the design is in some sense “sparse” (for example, the number of design elements in each series is uniformly limited [5], [19], [31], [58], [64]), or the design is somewhat “dense” (the number of elements in each series grows with the number of series [6], [31], [58], [64]). Theorem 2 considers the second of the specified types of design under condition $(D_0)$ in each of the independent series. Note that our formulation of the problem of estimating the mean function also includes the situation of a general deterministic design.

Note that the methodologies for estimating the mean function used for dense or sparse data are often different (see, for example, [46], [56]). In the situation of a growing number of observations in each series, it is natural to preliminarily estimate trajectories of a random regression function in each series, and then average over all series (e.g., see [5], [19], [61]). This is exactly what we do in [15] following this conventional approach.

4 Comparison with some known approaches

In [4], under the conditions of the present paper, the following estimators were studied:

$$f_{n,h}^*(t) = \frac{\sum_{i=1}^n X_{ni} K_h(t - z_{ni}) \Delta z_{ni}}{\sum_{i=1}^n K_h(t - z_{ni}) \Delta z_{ni}}.$$  \hspace{1cm} (19)

Notice that

$$f_{n,h}^*(t) = \arg\min_a \sum_{i=1}^n (X_{ni} - a)^2 K_h(t - z_{ni}) \Delta z_{ni}. \hspace{1cm} (20)$$

It is interesting to compare the new estimators $\hat{f}_{n,h}(t)$ with the estimators $f_{n,h}^*(t)$ from [4] as well as with other estimators (for example, the Nadaraya–Watson estimators $\hat{f}_{NW}(t)$ and classical local linear estimators $\hat{f}_{LL}(t)$). Throughout this section, we assume that conditions $(D)$, $(K)$, and $(E)$ are satisfied and the regression function $f(t)$ is nonrandom. Moreover, we need the following constraint.

(IID) The regression function $f(t)$ in Model (1) twice continuously differentiable, the errors $\{\varepsilon_i\}$ are independent, identically distributed, centered, and independent of the design $\{z_i\}$, whose elements are independent and identically distributed. In addition, the distribution function of the random variable $z_1$ has a strictly positive density $p(t)$ continuously differentiable on $(0, 1)$.

Such severe restrictions on the parameters of the regression model are explained both by problems in calculating the asymptotic representation for the variances of the estimators $\hat{f}_{n,h}(t)$ and $f_{n,h}^*(t)$ as well as by properties of the Nadaraya–Watson estimators, which are very sensitive to the nature of the correlation of design elements.

For any statistical estimator $f_n(t)$ of the regression function $f(t)$, we will use the notation $\text{Bias}_n(f_n(t))$ for its bias, i.e., $	ext{Bias}_n(f_n(t)) := \mathbb{E} f_n(t) - f(t)$. Put $\overline{f} = \sup_{t \in [0,1]} |f(t)|$ and for $j = 0, 1, 2, 3$, introduce the notation

$$w_j(t) = \int_0^1 (t - z)^j K_h(t - z)dz = \int_{z \in [0,1]: |t - z| \leq h} (t - z)^j K_h(t - z)dz, \hspace{1cm} t \in [0, 1]. \hspace{1cm} (21)$$
The following asymptotic representation for the bias and variance of the estimator \( f_{n,h}^* (t) \) was obtained in [3].

**Proposition 1.** Let condition \((IID)\) be fulfilled and \( \inf_{t \in [0,1]} p(t) > 0 \). If \( n \to \infty \) and \( h \to 0 \) so that \((\log n)^{-1} h \sqrt{n} \to \infty\), \( h^{-2} \mathbb{E} \delta_n \to 0 \) and \( h^{-3} \mathbb{E} \delta_n^2 \to 0 \) then, for any \( t \in (0,1) \), the following asymptotic relations are valid:

\[
\text{Bias} f_{n,h}^*(t) = \frac{h^2 \kappa_2}{2} f''(t) + o(h^2), \quad \text{Var} f_{n,h}^*(t) \sim \frac{2 \sigma^2}{hnp(t)} \|K\|^2.
\]

Note that the first statement concerning the asymptotic behavior of the bias in Proposition 1 was actually proved for arbitrarily dependent design elements when condition \((D_0)\) is met. The following two propositions and corollaries are also obtained without any assumptions about correlation of design elements, only conditional centering and conditional orthogonality of the errors from condition \((E)\) are used.

**Proposition 2.** Let \( h < 1/2 \). Then, for any fixed \( t \in [h, 1-h] \),

\[
\text{Bias} \hat{f}_{n,h}(t) = \text{Bias} f_{n,h}^*(t) + \gamma_{n,h}(t), \quad \text{Var} \hat{f}_{n,h}(t) = \text{Var} f_{n,h}^*(t) + \rho_{n,h}(t),
\]

where

\[
|\gamma_{n,h}(t)| \leq C_n \gamma h^{-1} \mathbb{E} \delta_n, \quad |\rho_{n,h}(t)| \leq C_n \sigma^2 + h^{-1} \mathbb{E} \delta_n.
\]

**Proposition 3.** Let the regression function \( f(t) \) be twice continuously differentiable. Then, for any fixed \( t \in (0,1) \),

\[
\text{Bias} \hat{f}_{n,h}(t) = \frac{f''(t)}{2} B_0(t) + O(\mathbb{E} \delta_n/h) + o(h^2),
\]

where

\[
B_0(t) = \frac{w_2(t) - w_3(t) w_1(t)}{w_0(t) w_2(t) - w_1^2(t)}.
\]

Moreover,

\[
\text{Bias} f_{n,h}^*(t) = -f'(t) \frac{w_1(t)}{w_0(t)} + \frac{f''(t)}{2} \frac{w_2(t)}{w_0(t)} + O(\mathbb{E} \delta_n) + o(h^2),
\]

besides, the error terms \( o(h^2) \) and \( O(\cdot) \) in (22) and (24) are uniform in \( t \).

**Corollary 3.** Let the regression function \( f(t) \) be twice continuously differentiable, \( h \to 0 \), and \( h^{-3} \mathbb{E} \delta_n \to 0 \). Then, for each fixed \( t \in (0,1) \) such that \( f''(t) \neq 0 \), the following asymptotic relations are valid:

\[
\text{Bias} \hat{f}_{n,h}(t) \sim \text{Bias} f_{n,h}^*(t) \sim \frac{f''(t)}{2} \kappa_2 h^2.
\]

**Corollary 4.** Suppose that, under the conditions of the previous corollary, \( f \) has nonzero first and second derivatives in a neighborhood of zero. Then for any fixed positive \( \alpha < 1 \) such that \( \kappa_1(\alpha) < 0 \), the following asymptotic relations hold:

\[
\text{Bias} f_{n,h}(\alpha h) \sim \frac{1}{2} h^2 D(\alpha) f''(0+), \quad \text{Bias} f_{n,h}^*(\alpha h) \sim -h^{\kappa_3(\alpha)} \frac{\kappa_1(\alpha)}{\kappa_0(\alpha)} f'(0+),
\]

where

\[
D(\alpha) = \frac{\kappa_2^2(\alpha) - \kappa_3(\alpha) \kappa_1(\alpha)}{\kappa_0(\alpha) \kappa_2(\alpha) - \kappa_1^2(\alpha)}.
\]

Note that, due to the Cauchy–Bunyakovsky inequality and the properties of the density \( K(\cdot) \), the strict inequality \( \kappa_0(\alpha) \kappa_2(\alpha) - \kappa_1^2(\alpha) > 0 \) holds for any \( \alpha \in [0,1] \).
Remark 12. If condition (IID) is satisfied then, for the bias and variance of estimators \( \hat{f}_{NW}(t) \) and \( \hat{f}_{LL}(t) \), the following asymptotic representations are well known (see, for example, [14]), which are valid for any \( t \in (0, 1) \) under broad conditions on the parameters of the model under consideration:

\[
\text{Bias} \hat{f}_{NW}(t) = \frac{h^2\kappa_2}{2p(t)} (f''(t)p(t) + 2f'(t)p'(t)) + o(h^2), \quad \forall \text{var} \hat{f}_{NW}(t) \sim \frac{\sigma^2}{hnp(t)} \|K\|^2,
\]

\[
\text{Bias} \hat{f}_{LL}(t) = \frac{h^2\kappa_2}{2} f''(t) + o(h^2), \quad \forall \text{var} \hat{f}_{LL}(t) \sim \frac{\sigma^2}{hnp(t)} \|K\|^2.
\]

The above asymptotic representations show that if the assumptions (IID) are valid then the variance of the Nadaraya–Watson estimator \( \hat{f}_{NW}(t) \) and the locally linear estimator \( \hat{f}_{LL}(t) \) under broad conditions is asymptotically half the variance of the estimators \( f_{n,h}^*(t) \) and \( f_{n,h}(t) \), respectively. But the mean-square error of any estimator is equal to the sum of the variance and squared bias, which for the compared estimators is asymptotically determined by the quantities \( f''(t)p(t) + 2f'(t)p'(t) \) or \( f''(t)p(t) \), respectively. In other words, if the standard deviation \( \sigma \) of the errors is not very large and

\[
|f''(t)p(t) + 2f'(t)p'(t)| > |f''(t)p(t)|, \quad (25)
\]

then the estimator \( f_{n,h}^*(t) \) or \( \hat{f}_{n,h}(t) \) may be more accurate than \( \hat{f}_{NW}(t) \). The indicated effect for the estimator \( f_{n,h}^*(t) \) is confirmed by the results of computer simulations in [4].

Note also that in order to choose in a certain sense the optimal bandwidth \( h \), the orders of the smallness of the bias and the standard deviation of the estimator are usually equated. In other words, if the assumptions (IID) are fulfilled, for all four types of estimators considered here, we need to solve the equation \( h^2 \approx (nh)^{-1/2} \). Thus the optimal bandwidth has the standard order \( h \approx n^{-1/5} \).

Remark 13. Estimators of the form \( \hat{f}_{n,h}(t) \) and \( f_{n,h}^*(t) \) given in [3] and [19] can define a little differently, depending on the choice of one or another partition with highlighted points \( \{z_i; i = 1, \ldots, n\} \) of the domain of the regression function underlying these estimators. For example, using the Voronoi partition of the segment \([0, 1] \), an estimator of the form \( f_{n,h}^*(t) \) can be given by the equality

\[
\hat{f}_{n,h}(t) = \frac{\sum_{i=1}^{n} X_i K_h(t - z_{ni}) \Delta z_{ni}}{\sum_{i=1}^{n} K_h(t - z_{ni}) \Delta z_{ni}}, \quad (26)
\]

where \( \Delta z_{ni} = \Delta z_{n1} + \Delta z_{n2}/2, \quad \Delta z_{nn} = \Delta z_{nn}/2 + \Delta z_{n+1}, \quad \Delta z_{ni} = (\Delta z_{ni} + \Delta z_{ni+1})/2 \) for \( i = 2, \ldots, n-1 \). Looking through the proofs from [4] it is easy to see that in this case all properties of the estimator \( \hat{f}_{n,h}(t) \) are preserved, except for the asymptotic representation of the variance. Repeating with obvious changes the arguments in proving Proposition 1 in [4], we have

\[
\forall \text{var} \hat{f}_{n,h}(t) \sim \frac{1.5\sigma^2}{hnp(t)} \|K\|^2.
\]
Thus, in the case of independent and identically distributed design points, the asymptotic variance of the estimator can be somewhat reduced by choosing one or another partition.

Similarly, in the definition (3), the estimators $\hat{f}_{n,h}(t)$, the quantities $\{\Delta z_{ni}\}$ can be replaced by the Voronoi tiling $\{\tilde{\Delta} z_{ni}\}$. It is also worth noting that the indicator factor involved in the determination (3) of the estimator $\hat{f}_{n,h}(t)$, does not affect the asymptotic properties of the estimator given in Theorem 1 and we only needed it to calculate the exact asymptotic behavior of the estimator bias.

5 Simulations

In the following computer simulations, instead of estimator (3), we used the equivalent estimator $\hat{f}_{n,h}(t)$ of the weighted least-squares method defined by the relation

$$\hat{f}_{n,h}(t) \approx \min_a \sum X_{ni}^2 K_h(t - z_{ni}),$$

where the quantities $\tilde{\Delta} z_{ni}$ are defined in (13) above. Estimator (27) differs from estimator (3) by excluding the indicator factor and replacing $\Delta z_{ni}$ with $\tilde{\Delta} z_{ni}$, which is not essential (see Remark 13). Besides, if we had several observations at one design point, then the observations were replaced by one observation presenting their arithmetic mean (see Remark 4 above). Although the notation $\hat{f}_{n,h}(t)$ in (27) is somewhat different from the same notation in (3), we retained the notation $\hat{f}_{n,h}(t)$, which will not lead to ambiguity.

In the simulations below, we will also consider the local constant estimator $\tilde{f}_{n,h}^*(t)$ from (26), which can be defined by the equality

$$\tilde{f}_{n,h}^*(t) \equiv \min_a \sum X_{ni}^2 K_h(t - z_{ni}),$$

Here we also replace the observations corresponding to one design point by their arithmetic mean.

Recall that the Nadaraya-Watson estimator differs from (28) by the absence of the factors $\tilde{\Delta} z_{ni}$ in the weighting coefficients:

$$\hat{f}_{NW}(t) = \frac{\sum_i X_{ni} K_h(t - z_{ni})}{\sum_i K_h(t - z_{ni})},$$

The Nadaraya-Watson estimators are also weighted least-squares estimators:

$$\hat{f}_{NW}(t) \equiv \min_a \sum X_{ni}^2 K_h(t - z_{ni}).$$

In the following examples, estimators (27) and (28), which will be called universal local linear (ULL) and universal local constant (ULC), respectively, will be compared with the estimator of linear regression (LR), the Nadaraya-Watson (NW) estimator, LOESS of order 1, as well as with estimators of generalized additive models (GAM) and of random forest (RF). For LOESS estimators, the \texttt{R loess()} function was used.

It is worth noting that, in the examples below, the best results were obtained by the new estimators (27) and (28), LOESS estimator of order 1, and the Nadaraya-Watson estimator.

With regard to the simulation examples, the main difference between estimators (27) and (28), and the Nadaraya–Watson and LOESS ones is that estimators (27) and (28) are “more local”. This means that if a function $f(z)$ is evaluated on a design interval $A$ with a “small” number of observations adjacent to a design interval $B$ with a “large” number of observations, the Nadaraya-Watson and LOESS estimators will primarily seek to adjust to the “large” cluster of observations.
on the interval $B$. At the same time, estimators (27) and (28) will equally consider observations on intervals of equal lengths, regardless of the distribution of design points on the intervals.

In the examples below, for all of the kernel estimators which are the Nadaraya-Watson ones, LOESS, (27), and (28), we used the tricubic kernel

$$K(t) = \frac{70}{81} \max\{0, (1 - |t|^3)^3\}.$$ 

We chose the tricubic kernel because that kernel is employed in the R function `loess()` which was used in the simulations.

The accuracy of the models was estimated with respect to the maximum error and the mean squared error. In all the examples below, except Example 3, the maximum error was estimated on the uniform grid of 1001 points on the segment $[0, 10]$ by the formula

$$\max_{j=1, \ldots, 1001} |\hat{f}(t_j) - f(t_j)|,$$

where $t_j$ are the grid points of segment $[0, 10]$, $t_1 = 0$, $t_{1001} = 10$, $\hat{f}(t_j)$ are the values of the constructed estimator at the points of the partition grid, $f(t_j)$ are the true values of the estimated function. In Example 3, a grid of 1001 points was taken on the interval from the minimum to the maximum point of the design. That was done in order to avoid assessing the quality of extrapolation, since, in that example, the minimum design point could fall far from 0.

The mean squared error was calculated for one random splitting of the whole sample into training and validation samples in proportion of 80% to 20%, according to the formula

$$\frac{1}{m} \sum_{j=1}^{m} (\hat{f}(z_j) - X_j)^2,$$

where $m$ is the validation sample size, $z_j$ are the validation sample design points, $X_j$ are the noisy observations of the predicted function in the validation sample, $\hat{f}$ is the estimate calculated by the training sample. The splittings into training and validation samples were identical for all models.

For each of the kernel estimators, the parameter $h$ of the kernel $K_h$ was determined using cross-validation minimizing the mean squared error, where the set of observations was partitioned into 10 folds randomly. The same partitions were taken for all the kernel estimators.

When calculating the root mean square error, the cross-validation for choosing $h$ was carried out on the training set. To calculate the maximum error, the cross-validation was performed on the whole sample. For the Nadaraya-Watson models as well as for estimators (27) and (28), the parameter $h$ was selected from 20 values located on the logarithmic grid from $\max\{0.0001, 1.1 \max_i \Delta z_i\}$ to 0.9. For LOESS, the parameter span was chosen in the same way from 20 values located on the logarithmic grid from 0.0001 to 0.9.

The simulations also included testing basic statistical learning algorithms: linear regression without regularization, generalized additive model, and random forest [23]. The training of the generalized additive model was carried out using the R library `mgcv`. Thin-plate splines were used, the optimal form of which was selected using generalized cross-validation. Random forest training was done using the R library `randomForest`. The number of trees is chosen to be 1000 based on the out-of-bag error plot for a random forest with five observations per leaf. The optimal number of observations in a random forest leafs was chosen using 10-fold cross-validation on a logarithmic grid out of 20 values from 5 to 2000.

In each example, 1000 realizations of different train and validation sets were performed, for each of which the errors were calculated. In each of train and validation sets realizations, 5000 observations were generated. The results of the calculations are presented below in the boxplots, where every box represents the median and the 1st and 3rd quartiles. The plots do not show the results of linear regression, since in the examples, the results appeared to be significantly worse than those of the other models. The mean squared and maximum errors of estimator (27) were compared with the errors of LOESS estimator by the paired Wilcoxon test. The summaries of the
errors on the 1000 realizations of different train and validation sets are reported as median (1st quartile, 3rd quartile).

The examples of this section were constructed so that the distribution of design points is “highly nonuniform”. Potentially, this could demonstrate the advantage of the new estimator (27) over known estimation approaches.

**Example 2.** Let us set the target function

\[ f(z) = (z - 5)^2 + 10, \quad 0 \leq z \leq 10 \quad (31) \]

and let the noise be centered Gaussian with standard deviation \( \sigma = 2 \) (Fig. 1). In each realization, we draw 4500 independent design points uniformly distributed on the segment \( z \in [0, 5] \), and 500 independent design points uniformly distributed on the segment \( z \in [5, 10] \).

![Figure 1: Example 2. Sample observations, target function, and two estimators.](image)

The results are presented in Fig. 2. For the maximum error, the advantage of the estimators of order 1 (LOESS and (27)) over the estimators of order 0 (the Nadaraya-Watson and (28)) is noticeable, while the estimator (27) turns out to be the best of all considered estimators, in particular, the estimator (27) performs better than LOESS: 0.6357 (0.4993, 0.8224) vs. 0.6582 (0.5205, 0.8508), \( p = 0.019 \).

For the mean squared error, all models, except random forest and linear regression, show similar results. Besides, the estimator (27) turns out to be the best of the considered ones, although the difference between estimators (27) and LOESS is not statistically significant: 4.017 (3.896, 4.139) vs. 4.030 (3.906, 4.154), \( p = 0.11 \).
Figure 2: The maximum (left) and mean squared (right) errors in Example 2. For the mean squared error, the random forest model performed worse \((10.97 \, (10.55, \, 11.39))\) than the GAM model and the kernel estimators, so the results of the random forest model “did not fit” into the plot.

**Example 3.** The piecewise linear target function is shown in Fig. 3. For the sake of simplicity of presentation, we do not present the formula for the definition of this function. Here the centered Gaussian noise has the standard deviation \(\sigma = 2\). The design points are independent and identically distributed with density proportional to the function \((z - 5)^2 + 2\), \(0 \leq z \leq 10\).

The results are presented in Fig. 4. The Nadaraya-Watson estimator appears to be the best model both for the maximum error and for the mean squared error. For the both errors, estimator \((27)\) is better than LOESS \((p < 0.0001\) for the maximum error, \(p = 0.0030\) for the mean squared error).

**Example 4.** In this example, the design points are strongly dependent. We will define them as follows: \(z_i := s(A_i), \ i = 1, \ldots, n\), where \(A\) is a positive number such that \(A/\pi\) is irrational (we chose \(A = 0.0002\) in this example),

\[
\begin{align*}
s(t) := 10 \left| \sum_{k=1}^{100} \eta_k \cos(tk) \right| \quad \text{with} \quad \eta_k := k^{-1} \psi_k \left( \sum_{j=1}^{100} j^{-1} \psi_j \right)^{-1},
\end{align*}
\]

and \(\psi_j\) are independent uniformly distributed on \([0, 1]\) random variables independent of the noise. It was shown \([4]\) that the random sequence \(s(A_i)\) is asymptotically everywhere dense on \([0, 10]\) with probability 1.

The target function is

\[
f(z) = 0.2 \left( ((z - 5)^2 + 25) \ \cos((z - 5)^2/2) + 60 \right),
\]

shown in Fig. 5.

For maximum error, estimate \((27)\) turns out to be the best of all the considered estimators. In particular, estimator \((27)\) is better than LOESS: \(1.757 \, (1.491, \, 2.053)\) vs. \(2.538 \, (2.216, \, 2.886)\), \(p < 0.0001\).
The median mean squared error for estimator (27) also turns out to be the smallest of those considered. In that sense, estimator (27) is better than LOESS, but the difference is not significant: 4.166 (4.025, 4.751) vs. 4.219 (4.096, 4.338), $p = 0.92$.

**Example 5.** In this example, the target function was the same as in Example 4. The difference from the previous example is that 50,000 design points were generated by the same technique, and then 5,000 points of the 50,000 ones were selected. This allowed us to fill the domain of $f$ with design elements “more uniformly” than in the previous example, while preserving the clusters of design points.

For maximum error, estimator (27) turns out to be the best of all the considered estimators. In particular, estimator (27) is better than LOESS: 2.872 (2.369, 3.488) vs. 9.435 (5.719, 10.9), $p < 0.0001$.

For the mean squared error, the best estimator is LOESS. Estimator (27) is worse than LOESS: 5.108 (4.535, 6.597) vs. 4.378 (4.229, 4.541), $p < 0.0001$, but it is better than the other estimators considered.

### 6 Example of processing real medical data

In this section, we consider an application of the models considered in the previous section to the data collected in the multicenter study “Epidemiology of cardiovascular diseases in the regions of the Russian Federation”. In that study, representative samples of unorganized male and female populations aged 25–64 years from 13 regions of the Russian Federation were studied. The study was approved by the Ethics Committees of the three federal centers: State Research Center for Preventive Medicine, Russian Cardiology Research and Production Complex, Almazov Federal Medical Research Center. Each participant has written informed consent for the study. The study was described in detail in [51].
Figure 4: The maximum (left) and mean squared (right) errors in Example 3. For the mean-squared error, the random forest model performed worse (6.699 (6.412, 7.046)) than the GAM model and the kernel estimators, so the results of the random forest model “did not fit” into the plot.

One of the urgent problems of modern medicine is to study the relationship between heart rate (HR) and systolic arterial blood pressure (SBP), especially for low values of the observations. Therefore we will choose SBP as the outcome, and HR as the predictor. The association between these variables was previously estimated to be nonlinear [52]. The general analysis included 6597 participants from 4 regions of the Russian Federation. The levels of SBP and HR were statistically significantly pairwise different between the selected regions. Thus, the hypothesis of the independence of design points was violated.

In this section, the maximum error cannot be calculated because the exact form of the relationship is unknown, so only the mean squared error is reported. The mean squared error was calculated for 1000 random partitions of the entire set of observations into training (80%) and validation (20%) samples.

The results are presented in Fig. 4. Here the GAM estimator and the kernel estimators showed similar results, which were better than the results of both the linear regression and random forest.

The best estimator turned out to be (28), although its difference from the Nadaraya-Watson estimator was not statistically significant: 220.2 (215.4, 225.9) vs. 220.4 (215.4, 225.8), \( p = 0.91 \).

The difference between estimator (27) and LOESS was not significant too: 220.4 (215.4, 225.9) vs. 220.6 (215.6, 226.1), \( p = 0.52 \).

7 Conclusion

In this paper, for a wide class of nonparametric regression models with a random design, universal uniformly consistent kernel estimators are proposed for an unknown random regression function of a scalar argument. These estimators belong to the class of local linear estimators. But in contrast to the vast majority of previously known results, traditional conditions of dependence of design elements are not needed for the consistency of the new estimators. The design can be either fixed and not necessarily regular, or random and not necessarily consisting of independent or weakly
dependent random variables. With regard to design elements, the only condition that is required
is the dense filling of the regression function domain with the design points.

Explicit upper bounds are found for the rate of uniform convergence in probability of the
new estimators to an unknown random regression function. The only characteristic explicitly
included in these estimators is the maximum spacing statistic of the variational series of design
elements, which requires only the convergence to zero in probability of the maximum spacing as
the sample size tends to infinity. The advantage of this condition over the classical ones is that
it is insensitive to the forms of dependence of the design observations. Note that this condition
is, in fact, necessary, since only when the design densely fills the regression function domain, it
is possible to reconstruct the regression function with some accuracy. As a corollary of the main
result, we obtain consistent estimators for the mean function of continuous random processes.

In the simulation examples of Section 5, the new estimators were compared with known kernel
estimators. In some of the examples, the new estimators proved to be the most accurate. In the
application to real medical data considered in Section 6, the accuracy of new estimators was also
comparable with that of the best-known kernel estimators.

8 Proofs

In this Section, we will prove the assertions stated in Sections 2–4. Denote

$$\beta_{n,i}(t) := \frac{w_{n2}(t) - (t - z_{n,i})w_{n1}(t)}{w_{n0}(t)w_{n2}(t) - w_{n1}^2(t)} \tag{32}$$

Taking into account the relations $X_{ni} = f(z_{n,i}) + \varepsilon_{ni}$, $i = 1, \ldots, n$, and the identity

$$\sum_{i=1}^{n} \beta_{n,i}(t)K_h(t - z_{n,i})\Delta z_{ni} \equiv 1, \tag{33}$$
Figure 6: The maximum (left) and mean squared (right) errors in Example 4. As before, for the mean squared error, the results of the random forest model (13.95 (11.69, 16.18)) not shown in full on the graph. In addition, the outliers for the GAM, NW, ULC, and ULL estimators are “cut off” in this graph.

Figure 7: The maximum (left) and mean-squared (right) errors in Example 5. As before, for the mean-squared error, the results of the random forest model not shown in full on the graph. In addition, the outliers for the NW, ULC, and ULL estimators are “cut off” in this graph.
we obtain the representation

$$\hat{f}_{n,h}(t) = f(t) + f(t)I(\delta_n > c_\ast h) + \hat{r}_{n,h}(f,t) + \hat{\nu}_{n,h}(t), \quad (34)$$

where

$$\hat{r}_{n,h}(f,t) = I(\delta_n \leq c_\ast h) \sum_{i=1}^{n} \beta_{n,i}(t) (f(z_{n,i}) - f(t)) K_h(t - z_{n,i}) \Delta z_{ni},$$

$$\hat{\nu}_{n,h}(t) = I(\delta_n \leq c_\ast h) \sum_{i=1}^{n} \beta_{n,i}(t) K_h(t - z_{n,i}) \Delta z_{ni} \epsilon_{ni}.$$ 

We emphasize that, in view of the properties of the density $K_h(\cdot)$, the domain of summation in the last two sums as well as in all sums defining the quantities $w_{nj}(t)$ from (4) coincides with the set $A_{n,h}(t) = \{i : |t - z_{n,i}| \leq h, 1 \leq i \leq n\}$, which is a crucial point for further analysis.

**Lemma 1.** For $h < 1/2$, the following equalities are valid:

$$\inf_{t \in [0,1]} (w_0(t)w_2(t) - w_1^2(t)) = \frac{1}{4} (\kappa_2 - \kappa_1^2) h^2, \quad \inf_{t \in [0,1]} w_0(t) = 1/2, \quad (35)$$

$$\sup_{t \in [0,1]} |w_j(t)| = \left(\frac{1}{2}\right)^{\frac{j-2(j/2)}{2}} \kappa_j h^j, \quad j = 0, 1, 2, 3. \quad (36)$$

Moreover, on the set of elementary events such that $\delta_n \leq c_\ast h$, the following inequalities hold:

$$\sup_{t \in [0,1]} |w_{nj}(t)| \leq 3Lh^j, \quad \sup_{t \in [0,1]} |w_{nj}(t) - w_j(t)| \leq 12L \delta_n h^{j-1}, \quad j = 0, 1, 2, 3, \quad (37)$$

$$\inf_{t \in [0,1]} (w_n0(t)w_n2(t) - w_n^2(t)) \geq \frac{1}{8} (\kappa_2 - \kappa_1^2) h^2, \quad \inf_{t \in [0,1]} w_n0(t) \geq 1/4, \quad (38)$$

$$\forall t_1, t_2 \in [0,1] \quad |w_{nj}(t_2) - w_{nj}(t_1)| \leq 18Lh^{j-1} |t_2 - t_1|, \quad j = 0, 1, 2. \quad (39)$$

**Proof.** Let us prove (35) and (36). First of all, note that, due to the Cauchy–Bunyakovsky–Schwartz inequality, $w_0(t)w_2(t) - w_1^2(t) \geq 0$ for all $t \in [0,1]$ and this difference is continuous in $t$. 

Figure 8: Mean-squared prediction error of the dependence of BP from HR.
First, consider the simplest case where $h \leq t \leq 1 - h$. For such $t$, after changing the integration variable in the definition \(21\) of the quantities $w_j(t)$ we have
\[
 w_j(t) = \int_{t-h}^{t+h} (t-z)^j K_h(t-z)dz = h^j \int_{-1}^{1} v^j K(v)dv, \tag{40}
\]
i.e., $w_0(t) \equiv 1$, $w_1(t) \equiv 0$, and $w_2(t) \equiv h^2 \kappa_2$. In other words, on the segment \([h, 1-h]\), the following identity is valid:
\[
w_0(t)w_2(t) - w_1^2(t) \equiv h^2 \kappa_2. \tag{41}
\]

We now consider the case $t = \alpha h$ for all $\alpha \in [0,1]$. Then
\[
w_j(\alpha h) = \int_{0}^{(1+\alpha)h} (\alpha h - z)^j K_h(\alpha h - z)dz = h^j \kappa_j(\alpha). \tag{42}
\]

Next, by \(42\), we obtain
\[
\frac{d}{d\alpha} h^{-2}(w_0(\alpha h)w_2(\alpha h) - w_1^2(\alpha h)) = \frac{d}{d\alpha} (\kappa_0(\alpha) \kappa_2(\alpha) - \kappa_1^2(\alpha))
= K(\alpha) \left( \alpha^2 \int_{-1}^{\alpha} K(v)dv + \int_{-1}^{\alpha} v^2 K(v)dv - 2\alpha \int_{-1}^{\alpha} vK(v)dv \right) \geq 0
\]
in view of the relation $\int_{-1}^{\alpha} vK(v)dv \leq 0$ since $K(v)$ is an even function. Similarly we study the symmetrical case where $t = 1 - \alpha h$ for all $\alpha \in [0,1]$. From here and \(41\) we obtain the first relation in \(35\):
\[
\inf_{\tau \in [0,1]} \{w_0(\tau)w_2(\tau) - w_1^2(\tau)\} = w_0(0)w_2(0) - w_1^2(0) = \frac{1}{4} h^2(\kappa_2 - \kappa_1^2).
\]

The second relation in \(35\) directly follows from \(42\). Moreover, the above-mentioned arguments and the representations \(40\) and \(42\) imply \(36\).

Further, the first estimator in \(37\) is obvious by the above remark about the domain of summation in the definition of functions $w_{nj}(t)$, and the relations
\[
\sup_{s \in [0,1]} K(s) \leq L, \quad \sum_{i \in A_{n,1}(t)} \Delta z_{ni} \leq 2h + \delta_n \leq 3h. \tag{43}
\]

The second estimator in \(37\) immediately follows from the well-known estimate of the error of approximation by Riemann integral sums of the corresponding integrals of smooth functions on a finite closed interval:
\[
\left| \sum_{i \in A_{n,1}(t)} g_{t,j}(z_{ni}) \Delta z_{ni} - \int_{z \in [0,1]} g_{t,j}(z)dz \right| \leq (2h + \delta_n) \delta_n L_{g_{t,j}}, \tag{44}
\]
where the functions $g_{t,j}(z) = (t-z)^j K_h(t-z)$, $j = 0, 1, 2, 3$, are defined for all $z \in [0 \vee t-h, 1 \land t+h]$, and $L_{g_{t,j}}$ is the Lipschitz constant of the function $g_{t,j}(z)$; it easy to verify that $\sup_{t \in [0,1]} L_{g_{t,j}} \leq 4Lh^{j-2}$ for all $h \in (0, 1/2)$ and $j = 0, 1, 2, 3$. So, on the set of elementary events such that $\{\delta_n \leq c_n h\}$ (recall that $c_n < 1$), the right-hand side in \(44\) can be replaced with $12L\delta_n h^{-1}$.

In addition, taking \(36\) and \(37\) into account, we obtain
\[
|w_{n0}(t)w_{n2}(t) - w_0(t)w_2(t)|
\leq w_{n0}(t)|w_{n2}(t) - w_2(t)| + w_2(t)|w_{n0}(t) - w_0(t)| \leq 9L\delta_n (3L + \kappa_2)h,
\]

\[20\]
\[ |w_{n1}^2(t) - w_{n2}^2(t)| \leq |w_{n1}(t) - w_{n2}(t)| (|w_{n1}(t)| + |w_{n2}(t)|) \leq 9L\delta_n(3L + \kappa_1/2)h. \]

Hence follows the estimate

\[ |w_{n0}(t)w_{n2}(t) - w_{n1}^2(t) - w_{n0}(t)w_2(t) + w_2^2(t)| \leq 9L\delta_n(6L + \kappa_2 + \kappa_1/2)h. \]  

(45)

The inequalities in (38) follow from (35), (45), and the definition of the constant \(c_\ast\).

To prove (39), note that

\[
w_{nj}(t_2) - w_{nj}(t_1) = \sum_{i=1}^{n} \left\{ (t_2 - z_{n,i})^j K_h(t_2 - z_{n,i}) - (t_1 - z_{n,i})^j K_h(t_1 - z_{n,i}) \right\} \Delta z_{ni}
\]

\[ = \sum_{i \in A_{n,h(t_1),i} \cup A_{n,h(t_2)}} \left\{ (t_2 - z_{n,i})^j K_h(t_2 - z_{n,i}) - (t_1 - z_{n,i})^j K_h(t_1 - z_{n,i}) \right\} \Delta z_{ni}\]

where we can use the estimates \(|(t_2 - z_{n,i})^j - (t_1 - z_{n,i})^j| \leq 2h^{j-1}|t_2 - t_1|\) for \(j = 0, 1, 2, |t_k - z_{n,i}| \leq h\) for \(k = 1, 2,\) and also the inequalities

\[ |K_h(t_2 - z_{n,i}) - K_h(t_1 - z_{n,i})| \leq Lh^{-2}|t_2 - t_1|,\]

\[ \sum_{i \in A_{n,h(t_1),i} \cup A_{n,h(t_2)}} \Delta z_{ni} \leq 4h + 2\delta_n \leq 6h.\]

Thus, Lemma 3 is proved.

\[\square\]

**Lemma 2.** For any positive \(h < 1/2\), the following estimate is valid:

\[ \sup_{t \in [0,1]} |\tilde{\sigma}_{n,h}(f,t)| \leq C_1^* \omega_f(h), \quad \text{with} \quad C_1^* = C_1 \frac{L^2}{\kappa_2 - \kappa_1^2}. \]

**Proof.** Without loss of generality, the required estimate can be derived on the set of elementary events determined by the condition \(\delta_n \leq c_\ast h\). Then the assertion of the lemma follows from the inequality

\[ |\tilde{\sigma}_{n,h}(f,t)| \leq \frac{\omega_f(h)w_{n2}(t)}{w_{n0}(t)w_{n2}(t) - w_{n1}^2(t)} \sum_{i \in A_{n,h(t)}} K_h(t - z_{n,i}) \Delta z_{ni} + \frac{\omega_f(h)|w_{n1}(t)|}{w_{n0}(t)w_{n2}(t) - w_{n1}^2(t)} \sum_{i \in A_{n,h(t)}} |t - z_{n,i}|^2 K_h(t - z_{n,i}) \Delta z_{ni},\]

(47)

the estimates from (43), and Lemma 1.

\[\square\]

**Lemma 3.** For any \(y > 0\) and \(h < 1/2\), on the set of elementary events such that \(\delta_n \leq c_\ast h\), the following estimate is valid:

\[ \mathbb{P}_{F_n} \left( \sup_{t \in [0,1]} |\tilde{\sigma}_{n,h}(t)| > y \right) \leq C_2^* \sigma^2 \frac{\delta_n}{h^{1/2}} y^2, \quad \text{with} \quad C_2^* = C_2 \frac{L^4}{(\kappa_2 - \kappa_1^2)^2}, \]

where the symbol \(\mathbb{P}_{F_n}\) denotes the conditional probability given the \(\sigma\)-field \(F_n\).

**Proof.** Put

\[ \mu_{n,h}(t) = \sum_{i \in A_{n,h}(t)} h^{-2} \alpha_{n,i}(t) K_h(t - z_{n,i}) \Delta z_{ni},\]

(48)

where \(\alpha_{n,i}(t) = w_{n2}(t) - (t - z_{n,i})w_{n1}(t)\), and note that from Lemma 1 and the conditions of Lemma 3 it follows that, firstly, \(h^{-2} |\tilde{\alpha}_{n,i}(t)| \leq 6L\) if only \(i \in A_{n,h(t)}\), and secondly,

\[ |\tilde{\sigma}_{n,h}(t)| \leq 8(\kappa_2 - \kappa_1^2)^{-1} |\mu_{n,h}(t)|.\]

(49)
The distribution tail of the random variable \( \sup_{t \in [0,1]} |\mu_{n,h}(t)| \) will be estimated by the so-called chaining proposed by A.N. Kolmogorov to estimate the distribution tail of the supremum norm of a stochastic process with almost surely continuous trajectories (see \([9]\)). First of all, note that the set \([0,1]\) under the supremum sign can be replaced by the set of dyadic rational points
\[
\mathcal{R} = \{ j/2^k; \ j = 1, \ldots, 2^k - 1; k \geq 1 \}.
\]
Thus,
\[
\sup_{t \in [0,1]} |\mu_{n,h}(t)| = \sup_{t \in \mathcal{R}} |\mu_{n,h}(t)| \leq \max_{j=1,\ldots,2^{m-1}} |\mu_{n,h}(j2^{-m})| + \sum_{k=m+1}^{\infty} \max_{j=1,\ldots,2^{k-2}} |\mu_{n,h}((j+1)2^{-k}) - \mu_{n,h}(j2^{-k})|,
\]
where the natural number \( m \) is defined by the equality \( m = \lceil |\log_2 h| \rceil \) (here \( \lceil a \rceil \) is the minimal natural number greater than or equal to \( a \). One has
\[
\mathbb{P}_{\mathcal{F}_n}\left( \sup_{t \in [0,1]} |\mu_{n,h}(t)| > y \right) \leq \mathbb{P}_{\mathcal{F}_n}\left( \max_{j=1,\ldots,2^{m-1}} |\mu_{n,h}(j2^{-m})| > a_my \right) + \sum_{k=m+1}^{\infty} \mathbb{P}_{\mathcal{F}_n}\left( \max_{j=1,\ldots,2^{k-2}} |\mu_{n,h}((j+1)2^{-k}) - \mu_{n,h}(j2^{-k})| > a_ky \right) + \sum_{k=m+1}^{\infty} \sum_{j=1}^{2^{k-2}} \mathbb{P}_{\mathcal{F}_n}\left( |\mu_{n,h}((j+1)2^{-k}) - \mu_{n,h}(j2^{-k})| > a_ky \right),
\]
where \( a_m, a_{m+1}, \ldots \) is a sequence of positive numbers such that \( a_m + a_{m+1} + \ldots = 1 \).

Let us now estimate each of the terms on the right-hand side of (50). Using Markov’s inequality for the second moment and the estimates (43), we obtain
\[
\mathbb{P}_{\mathcal{F}_n}(|\mu_{n,h}(j2^{-m})| > a_my) \leq \frac{(6L)^2}{(a_my)^2} \sum_{i \in A_{n,h}(j2^{-m})} K_h^2(j2^{-m} - z_{ni})(\Delta z_{ni})^2 \sigma^2 \\
\leq (6L)^2 \sigma^2 (a_my)^{-2} \delta_n(2h + \delta_n)h^{-2} \leq C_3L^2 \sigma^2 (a_my)^{-2} \delta_n h^{-1}.
\]
Further,
\[
\mathbb{P}_{\mathcal{F}_n}(|\mu_{n,h}((j+1)2^{-k}) - \mu_{n,h}(j2^{-k})| > a_ky) \leq (a_ky)^{-2} h^{-4} \\
\times \sum_{i=1}^{n} \mathbb{E}_{\mathcal{F}} \left( (\alpha_{n,i}((j+1)2^{-k})K_h((j+1)2^{-k} - z_{ni}) - \alpha_{n,i}(j2^{-k})K_h(j2^{-k} - z_{ni})) \Delta z_{ni}^{\epsilon_{ni}} \right)^2 \\
\leq \sigma^2 (a_ky)^{-2} h^{-4} \\
\times \sum_{i=1}^{n} \left( (\alpha_{n,i}((j+1)2^{-k})K_h((j+1)2^{-k} - z_{ni}) - \alpha_{n,i}(j2^{-k})K_h(j2^{-k} - z_{ni})) \Delta z_{ni}^{2} \right)^2 \\
\leq Lh^{-2} |u - v| \leq C_4 \sigma^2 L^4(a_ky)^{-2} 2^{-2k} \delta_n(4h + 2\delta_n)h^{-4} \leq C_5 \sigma^2 L^4(a_ky)^{-2} 2^{-2k} \delta_n h^{-3}.
\]
Here we took into account that the summation range in (52) coincides with the set
\[
\{ i : i \in A_{n,h}((j+1)2^{-k}) \cup A_{n,h}(j2^{-k}) \},
\]
and hence, due to the relation \(|(j+1)2^{-k} - j2^{-k}| = 2^{-k} \leq h \) for \( k > m \), the estimate (46) is valid for \( t_1 = j2^{-k} \) and \( t_2 = (j+1)2^{-k} \). Moreover, we used the estimates
\[
\sup_{t} K_h(t) \leq Lh^{-1}, \quad |K_h(u) - K_h(v)| \leq Lh^{-2} |u - v|,
\]

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and took into account the following inequalities in the above range of parameter changes (see Lemma 1):

\[|\alpha_{n,i}(j+1)2^{-k} - \alpha_{n,i}(j2^{-k})| \leq C_{\alpha}Lh2^{-k}, \quad |\alpha_{n,i}(j2^{-k})| \leq C_{\alpha}Lh^2,\]

\[|\alpha_{n,i}(j+1)2^{-k}K_h(j+1)2^{-k} - z_{n,i}) - \alpha_{n,i}(j2^{-k})K_h(j2^{-k} - z_{n,i})| \leq C_{\alpha}L2^{-k}.\]

We now obtain from [50, 52] that

\[\mathbb{P}_{F_n} \left( \sup_{t \in [0,1]} |\mu_{n,h}(t)| > y \right) \leq C_{10}y^{-2}2^{-1}a_{k}h^{-1} \left( 2^{-m/3} + h^{-2/3}2^{-m/3}(2 + 2^{1/3} + 2^{2/3}) \right)^3 \leq C_{11}y^{-2}2^{-1}a_{k}h^{-2}.\]

The optimal sequence \(a_k\) minimizing the right-hand side of this inequality is \(a_m = c2^{m/3}\) and \(a_k = ch^{-2/3}2^{-(k+1)/3}\) for \(k = m+1, m+2, \ldots\), where \(c\) is defined by the relation \(a_m + a_{m+1} + \ldots = 1\). For the indicated sequence, we conclude that

\[\mathbb{P}_{F_n} \left( \sup_{t \in [0,1]} |\mu_{n,h}(t)| > y \right) < \]

\[
\leq C_{10}y^{-2}2^{-1}a_{k}h^{-1} \left( 2^{-m/3} + h^{-2/3}2^{-m/3}(2 + 2^{1/3} + 2^{2/3}) \right)^3 \leq C_{11}y^{-2}2^{-1}a_{k}h^{-2}.\]

The assertion of the lemma follows from (49).

**Proof of Theorem 2** The assertion follows from Lemmas 2 and 3 if we set

\[\zeta_n(h) = \sup_{t \in [0,1]} |\hat{\mu}_{n,h}(t)| + \sup_{t \in [0,1]} |f(t)|I(\delta_n > c_h)\]

and take into account the relation

\[\mathbb{P}(\zeta_n(h) > y, \delta_n \leq c_h) = \mathbb{E}I(\delta_n \leq c_h)\mathbb{P}_{F_n}(\zeta_n(h) > y),\]

which was required.

To prove Theorem 2 we need the two auxiliary assertions below.

**Lemma 4.** If the condition (16) is fulfilled then \(\lim_{\varepsilon \to 0} \mathbb{E}\omega_f(\varepsilon) = 0\) and for independent copies of the a.s. continuous random process \(f(t)\) the following strong law of large numbers is valid: As \(N \to \infty\), then

\[\sup_{t \in [0,1]} |\mathcal{J}_N(t) - EF(t)| \overset{P}{\to} 0, \quad \text{where} \quad \mathcal{J}_N(t) = N^{-1}\sum_{j=1}^{N} f_j(t). \quad (53)\]

**Proof.** The first assertion of the lemma follows from (16) and Lebesgue’s dominated convergence theorem. We put

\[\omega_l^N(\varepsilon) = \sup_{t,s:|t-s| \leq \varepsilon} |\mathcal{J}_N(t) - \mathcal{J}_N(s)|, \quad \omega_E^f(\varepsilon) = \sup_{t,s:|t-s| \leq \varepsilon} |EF(t) - EF(s)|.\]

For any fixed \(k > 0\) and \(i = 0, \ldots, k\), one has

\[\sup_{t \in [0,1]} |\mathcal{J}_N(t) - EF(t)| \leq \max_{0 \leq i \leq k} |\mathcal{J}_N(i/k) - EF(i/k)| +\]

\[+ \max_{1 \leq i \leq k} \sup_{(i-1)/k \leq i/k \leq i/k} |\mathcal{J}_N(t) - \mathcal{J}_N(i/k)| + \max_{1 \leq i \leq k} \sup_{(i-1)/k \leq i/k} \sup_{(i-1)/k \leq i/k} |EF(t) - EF(i/k)| \leq\]

\[\leq \max_{0 \leq i \leq k} |\mathcal{J}_N(i/k) - EF(i/k)| + \omega_l^N(k) + \omega_E^f(k). \quad (54)\]

Put \(\omega_{f_j}(\varepsilon) = \sup_{t,s:|t-s| \leq \varepsilon} |f_j(t) - f_j(s)|\) and note that \(\omega_{E}f(\varepsilon) \leq \omega_{E}f(\varepsilon)\), and as \(N \to \infty\),

\[\mathcal{J}_N(i/k) \overset{P}{\to} EF(i/k), \quad \omega_l^N(\varepsilon) \leq \frac{1}{N} \sum_{j=1}^{N} \omega_{f_j}(\varepsilon) \overset{P}{\to} \mathbb{E}\omega_f(\varepsilon).\]
Therefore, the right-hand side in (54) does not exceed \( \mathbb{E}_{\omega_f}(1/k) + o_n(1) \) and by the arbitrariness of \( k \) and the first statement of the lemma, the relation (53) is proved. \( \square \)

**Lemma 5.** Under the conditions of Theorem 2 the following limit relation holds:

\[
\frac{1}{N} \sum_{j=1}^{N} \Delta_{n,h,j} \xrightarrow{p} 0, \quad \text{where} \quad \Delta_{n,h,j} = \sup_{t \in [0,1]} |f_{n,h,j}^*(t) - f_j(t)|. \tag{55}
\]

**Proof.** Let the sequences \( h = h_n \to 0 \) and \( N = N_n \to \infty \) be such that condition (17). Introduce the event \( B_{n,h,j} = \{ \delta_{n,j} \leq c_n h \} \), where \( j = 1, \ldots, N \). For any positive \( \nu \) one has

\[
P \left\{ \frac{1}{N} \sum_{j=1}^{N} \Delta_{n,h,j} > \nu \right\} \leq P \left\{ \frac{1}{N} \sum_{j=1}^{N} \Delta_{n,h,j} I(B_{n,h,j}) > \nu \right\} + N \mathbb{P}(\overline{B_{n,h,1}}). \tag{56}
\]

Next, from Theorem 1 we obtain

\[
\mathbb{E}\Delta_{n,h,j} I(B_{n,h,j}) \leq C_1^* \mathbb{E}_{\omega_f}(h) + \int_{0}^{\infty} \mathbb{P}(\zeta_n(h) > y, \delta_n \leq c_n h) dy \leq \tag{57}
\]

\[
\leq C_1^* \mathbb{E}_{\omega_f}(h) + h^{-1} (\mathbb{E}\delta_n)^{1/2} + \int_{h^{-1} (\mathbb{E}\delta_n)^{1/2}}^{\infty} \mathbb{P}(\zeta_n(h) > y, \delta_n \leq c_n h) dy \leq \tag{58}
\]

\[
\leq C_1^* \mathbb{E}_{\omega_f}(h) + (1 + C_1^* \sigma^2) h^{-1} (\mathbb{E}\delta_n)^{1/2}.
\]

To complete the proof of the lemma, it remains for the first probability on the right-hand side of (55) to apply Markov’s inequality, use the last estimate, limit relations (17), and the first statement of Lemma 4. \( \square \)

**The proof of Theorem 2** follows from Lemmas 4 and 5.

**Proof of Proposition 2.** For the estimator \( f_{n,h}^*(t) \) defined in (19), we need the following representation:

\[
f_{n,h}^*(t) = f(t) + r_{n,h}^*(f,t) + \nu_{n,h}^*(t), \tag{59}
\]

where

\[
r_{n,h}^*(f,t) = w_{n0}^{-1}(t) \sum_{i=1}^{n} (f(z_{ni}) - f(t)) K_h(t - z_{ni}) \Delta z_{ni},
\]

\[
\nu_{n,h}^*(t) = w_{n0}^{-1}(t) \sum_{i=1}^{n} K_h(t - z_{ni}) \Delta z_{ni} \varepsilon_{ni}.
\]

In view of the representations (34) and (57), we obtain

\[
\text{Bias} \hat{f}_{n,h}(t) = \mathbb{E} \hat{r}_{n,h}(f,t) + f(t) \mathbb{P}(\delta_n > c_n h)
\]

\[
= \sum_{i=1}^{n} \mathbb{E} \{ I(\delta_n \leq c_n h) \beta_{n,i}(t)(f(z_{ni}) - f(t)) K_h(t - z_{ni}) \Delta z_{ni} \} + f(t) \mathbb{P}(\delta_n > c_n h), \tag{58}
\]

\[
\text{Bias} f_{n,h}^*(t) = \mathbb{E} r_{n,h}^*(f,t)
\]

\[
= \sum_{i=1}^{n} \mathbb{E} \{ I(\delta_n \leq c_n h) w_{n0}^{-1}(t)(f(z_{ni}) - f(t)) K_h(t - z_{ni}) \Delta z_{ni} \} + \tau_n, \tag{59}
\]

where \( |\tau_n| \leq \omega_f(h) \mathbb{P}(\delta_n > c_n h) \). Further, it follows from Lemma 1 that, under the condition \( \delta_n \leq c_n h \), for any point \( t \in [h, 1 - h] \) one has

\[
\sup_{t \in A_{n,i}(t)} |\beta_{n,i}(t) - w_{n0}^{-1}(t)| \leq C_5^* \delta_n h^{-1}. \tag{60}
\]
When deriving the relation (60), we also took into account that $w_0(t) = 1$ and $w_1(t) = 0$ for all $t \in [h, 1 - h]$ (see the proof of Lemma 1). Now, reckoning with the relations (43), (58), (59), (60), and Lemma 1, it is easy to derive the first assertion of the lemma since

$$|\text{Bias}_n \hat{f} - \text{Bias}_n\hat{f}| \leq C_n h^{-1} \omega(f) \mathbb{E} \left\{ \delta_i I(\delta_i \leq c_h) \sum_{i=1}^n K_h(t - z_{ni}) \Delta z_{ni} \right\}$$

$$+ (|f(t)| + \omega(f)) \mathbb{P}(\delta_i > c_h) \leq C_n h^{-1} \omega(f) \mathbb{E} \delta_i + (|f(t)| + \omega(f)) \mathbb{P}(\delta_i > c_h). \quad (61)$$

To prove the second assertion, first of all, note that

$$\text{Var} \hat{r}_{n,h}(t) = \text{Var} \hat{r}_{n,h}(t) + \text{Var} \hat{r}_{n,h}(f, t) + f(t) \mathbb{P}(\delta_i > c_h) \mathbb{P}(\delta_i \leq c_h),$$

$$\text{Var} r_{n,h}(t) = \text{Var} r_{n,h}(t) + \text{Var} r_{n,h}(f, t),$$

Thus, we need to compare the two variances on the right-hand side of the first equality with the corresponding variances of the second one. Using (43) and (60), we get

$$\text{Var} \hat{r}_{n,h}(t) \leq \sigma^2 \mathbb{E} \sum_{i=1}^n \left| I(\delta_i \leq c_h) (\beta_{ni}^2(t) - w_{n0}^2(t)) K_h^2(t - z_{ni}) (\Delta z_{ni})^2 \right|$$

$$+ \sigma^2 \mathbb{P}(\delta_i > c_h) \leq C_n h^{-1} \mathbb{E} \delta_i \sum_{i=1}^n h K_h^2(t - z_{ni}) (\Delta z_{ni}) \left| < c_h \right| \leq C_n \sigma h^{-1} \mathbb{E} \delta_i,$$

when deriving this estimate, we took into account that

$$\sum_{i=1}^n w_{n0}^2(t) K_h^2(t - z_{ni}) (\Delta z_{ni})^2 \leq 1.$$

To estimate the difference $|\text{Var} \hat{r}_{n,h}(t) - \text{Var} r_{n,h}(t)|$, note that the bound $C_n h^{-1} \mathbb{E} \delta_i$ for the modulus of the difference between the squares of the displacements of the random variables $\hat{r}_{n,h}(f, t)$ and $r_{n,h}(f, t)$ is essentially contained in (47) and (61). Estimation of the difference of the second moments of the specified random variables is done similarly with (43), (60), and (61):

$$|\mathbb{E} r_{n,h}(f, t) - \mathbb{E} r_{n,h}(f, t)| \leq \mathbb{E} |\hat{r}_{n,h}(f, t) - \hat{r}_{n,h}(f, t)| \mathbb{E} r_{n,h}(f, t) + r_{n,h}(f, t)| \leq C_{10} h^{-1} \mathbb{E} \delta_i,$$

which completes the proof.

Proof of Proposition 3. From the definition of $\beta_{ni}(t)$ in (32), it follows that, for any $t \in [0, 1],\n
$$\sum_{i=1}^n \beta_{ni}(t) (z_{ni} - z_{ni}) K_h(t - z_{ni}) \Delta z_{ni} = 0,$$

$$\sum_{i=1}^n \beta_{ni}(t) (z_{ni} - z_{ni}) K_h(t - z_{ni}) \Delta z_{ni} = D_n^{-1}(t) (w_{n2}(t) - w_{n3}(t)) =: B_n(t),$$

where $D_n(t) := w_{n0}(t) w_{n2}(t) - w_{n1}^2(t)$. Expanding the function $f(\cdot)$ by the Taylor formula in a neighborhood of the point $t$ (up to the second derivative), from the above identities we obtain, using (32), (58), and Lemma 1 that for any point $t$ we have

$$\text{Bias}_n \hat{f}_{n,h}(t) = \mathbb{E} I(\delta_i \leq c_h) \sum_{i=1}^n \beta_{ni}(t) (f(z_{ni}) - f(t)) K_h(t - z_{ni}) \Delta z_{ni} + f(t) \mathbb{P}(\delta_i > c_h)$$

$$= \frac{f''(t)}{2} \mathbb{E} I(\delta_i \leq c_h) B_n(t) + f(t) \mathbb{P}(\delta_i > c_h) + o(h^2)$$

$$= \frac{f''(t)}{2} B_n(t) + O(\mathbb{E} \delta_i / h) + o(h^2); \quad (62)$$

25
moreover, the $O$- and $o$-symbols on the right-hand side of (62) are uniform in $t$. Note that $B_0(t) = O(h^2)$ holds for any $t$.

Next, since for $j = 1, 2$ we have $|w_j(t)/w_0^{-1}(t)| \leq h^j$ and $|w_{nj}(t)/w_{n0}^{-1}(t)| \leq h^j$ for all natural $n$, the following asymptotic representation holds:

$$\text{Bias}_{n,h}^f(t) = \sum_{i=1}^{n} E w_{n1}^{-1}(t)(f(z_{ni}) - f(t))K_h(t - z_{ni}) \Delta z_{ni}$$

$$= -f'(t)E w_{n1}(t) I(\delta_n \leq c,h) + \frac{f''(t)}{2}E w_{n2}(t) w_{n0}(t) I(\delta_n \leq c,h) + O(hP(\delta_n > c,h)) + o(h^2)$$

$$= -f'(t)w_1(t)/w_0(t) + \frac{f''(t)}{2}w_2(t)/w_0(t) + O(E\delta_n) + o(h^2). \quad (63)$$

**Proof of Corollary 3** Without loss of generality, we can assume that $t \in [h, 1-h]$. Then, as noted in the proof of Lemma 1, for the indicated $t$, one has $w_0(t) = 1$, $w_1(t) = 0$, and $w_2(t) = \kappa_2 h^2$, i.e., $B_0(t) = \kappa_2 h^2$. □

**Proof of Corollary 4** This assertion follows from Proposition 3 and (42). □

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