We study the optimization problem for decomposing $d$ dimensional fourth-order Tensors with $k$ non-orthogonal components. We derive deterministic conditions under which such a problem does not have spurious local minima. In particular, we show that if $\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} < \frac{5}{4}$, and incoherence coefficient is of the order $O(\sqrt{d})$, then all the local minima are globally optimal. Using standard techniques, these conditions could be easily transformed into conditions that would hold with high probability in high dimensions when the components are generated randomly. Finally, we prove that the tensor power method with deflation and restarts could efficiently extract all the components within a tolerance level $O(\kappa \sqrt{k\tau^3})$ that seems to be the noise floor of non-orthogonal tensor decomposition.

1 Introduction

Tensor Decomposition approaches are demonstrated to be effective tools for modeling and solving a wide range of problems in the context of signal processing, statistical inference, and machine learning. In particular, many unsupervised learning problems such as Gaussian Mixture Models [Ge et al., 2015], Latent Dirichlet Allocation [Anandkumar et al., 2014b], Topic Modeling [Cheng et al., 2015, Anandkumar et al., 2012], Hidden Markov Models [Azizzadenesheli et al., 2016], Latent Graphical Models [Song et al., 2013, Chaganty and Liang, 2014], and Community Detection [Al-Sharoa et al., 2017, Anandkumar et al., 2014d] can be modeled as a canonical decomposition (CANDDECOMP) problem which is also known as Parallel Factorization (PARAFAC).

It has been shown that under mild assumptions such as $2R \leq k_A + k_B + k_C$ where $R$ is the number of components, and $k_A, k_B,$ and $k_C$ are the k-rank of the component matrices $A, B,$ and $C$ respectively, the CANDDECOMP/PARAFAC (CP) decomposition exists and it is unique [Harshman, 1970, Kruskal, 1977]. Finding such a decomposition is an NP-hard problem in general [Håstad, 1990, Hillar and Lim, 2013]. Despite the hardness results, many of the proposed algorithms in the literature work well for practical problems [Leurgans et al., 1993, Anandkumar et al., 2014R, Kolda and Bader, 2009]. In fact, for a wide range of these algorithms there are theoretical local and global guarantees under some realistic assumptions [Uschmajew, 2012, Anandkumar et al., 2014b]. One of the important cases where the problem of finding the decomposition has been very well-studied is the case where the components are orthogonal. [Anandkumar et al., 2014b] demonstrates that many problems in practice can be reduced to an orthogonal tensor decomposition with a pre-processing phase known as data whitening. While data whitening helps us transform the problem to the orthogonal case, it is computationally expensive especially in high-dimensional settings. Besides, it can affect the performance of the model for problems such as Independent Component Analysis [Le et al., 2011].

Practical drawbacks of data whitening, alongside with theoretical concerns such as instability in high-dimensional cases [Anandkumar et al., 2014a], have motivated researchers to investigate the CP tensor decomposition problem in the non-orthogonal scenario. [Anandkumar et al., 2014d] provide local and global guarantees for recovering the components of CP under mild non-orthogonality assumptions. However, their result requires on the proper initialization...
of the algorithm close to the global optimum. [Ge and Ma, 2017] analyze the non-convex landscape of the non-orthogonal tensor decomposition problem, and characterize the local minima of the problem under the over-complete regime (rank of the tensor is much higher than the dimension of the components). [Sharan and Valiant, 2017] show that the orthogonalized alternating least square approach can globally recover the components of the tensor decomposition problem when \( k = \mathcal{O}(d^{0.5}) \) where \( d \) is the dimension. In this paper, we aim to show the global convergence of a variation of Tensor Power Method (TPM) augmented by the deflation and restart of the algorithm for fourth-order tensors. This algorithm can recover all components of a given non-orthogonal decomposition problem when \( k = \mathcal{O}(d^{0.35}) \).

Before proceeding to the main results, let us define some notations. Let \( u(i) \) be the \( i \)-th coordinate of vector \( u \), the Kronecker product of \( n \) vectors \( u_1, u_2, \ldots, u_n \) denoted by \( T = u_1 \otimes u_2 \otimes \ldots \otimes u_n \), is defined as an \( n \)-th order tensor \( T \), such that \( T(i_1, i_2, \ldots, i_n) = u_1(i_1)u_2(i_2)\ldots u_n(i_n) \). Moreover,

\[
u \otimes^n \triangleq u \otimes u \otimes \ldots \otimes u \quad \text{\( n \) times}
\]

A fourth-order tensor \( T \), can be seen as a multi-linear transformation, defined for given \( d \)-dimensional vectors \( x, y, z, \) and \( t \) as

\[
T(x, y, z, t) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} T(i, j, k, \ell)x(i)y(j)z(k)t(\ell).
\]

(1)

To understand the ideas of the proposed algorithm, let us first start by considering the tensor decomposition of a fourth-order tensor in the orthogonal scenario. Suppose that the tensor of interest, \( T \), is decomposed as

\[
T = \sum_{i=1}^{k} u_i \otimes^4,
\]

(2)

where \( u_i \)'s are \( d \)-dimensional orthogonal vectors, i.e., \( (u_i, u_j) = 0 \), \( i \neq j \). Here, for simplicity of presentation, we assumed that different components have the same weight. For this case, the aim of tensor decomposition is to find the orthogonal decomposition vectors \( \{u_i\}_{i=1}^{k} \) efficiently given the tensor \( T \). [Anandkumar et al., 2014b] proves that in this case the non-convex optimization problem

\[
\min_w \quad f(w) = -\frac{1}{4} \sum_{i=1}^{k} (u_i^T w)^4 = -\frac{1}{4} T(w, w, w, w),
\]

(3)

s.t. \( \|w\|^2 = 1 \).

does not have any spurious local minima. Consequently, all the local minima of this problem correspond to \( \pm u_i \), \( i = 1, \ldots, k \). This property implies that most of the simple first order methods, such as randomly initialized manifold gradient descent, which are proven to converge to local minima [Lee et al., 2016] would be able to find the components, almost surely. However, in practice, algorithms such as gradient descent are shown to be slow due to their conservative and static step-size choices. On the other hand, algorithms such as tensor power method (TPM) [Anandkumar et al., 2014b] are shown to be practically faster. Moreover, [Anandkumar et al., 2014b] shows that TPM with multiple random restarts and deflation is capable of finding all components \( u_i \) with high probability. Unfortunately, the results in [Anandkumar et al., 2014b] assume the orthogonality of the components.

In this work, we extend the results of [Anandkumar et al., 2014b] to the non-orthogonal case. To establish our result, we first analyze the optimization landscape of problem (3). In Section 5 under incoherence condition, restricted isometry property (RIP), and a certain upper-bound on the ratio of the weight of different components. We show that any local minimizer of problem (3) is close to one of the actual components. In other words, for any local minimizer \( u^* \) of (3), there exists an index \( i \), such that \( \|u_i - u^*\| \) is small (up to sign ambiguity in \( u^* \)). In Section 5 we show that Tensor Power Method (TPM) with deflation and restarting can recover all components of a given tensor with high probability.

2 Optimality Conditions for the Tensor Decomposition Problem

In order to solve problem (3), we can use the manifold gradient descent method. It is well-known that manifold gradient descent with random initialization converges to local minima [Lee et al., 2016], almost surely. Thus, if we
prove any local minima of the above problem is close to one of the components $u_i$, we can use manifold gradient descent for recovering the components. This gradient descent method for solving tensor decomposition has been used before in the case of orthogonal tensors; see, e.g. [Anandkumar et al., 2014b]. In fact, in the orthogonal case, the gradient descent method coincides with Tensor Power Method [Anandkumar et al., 2014b].

To study the landscape of (3), let us first present the first- and second-order optimality conditions. Let the projection matrix to the manifold $\mathcal{M} = \{ w \mid \| w \| = 1 \}$ at point $w$ be $P_w = I - w w^T$. Then, for the optimization problem (3), any local minimizer point $w$ has to satisfy the following two optimality conditions:

- **First-order optimality condition:**
  \[
  \nabla_{\mathcal{M}} f(w) = -P_w \sum_{i=1}^k (w^T u_i)^3 u_i = 0
  \]  

- **Second-order optimality condition:**
  \[
  \nabla^2_{\mathcal{M}} f(w) = \left( \sum_{i=1}^k (u_i^T w)^4 \right) P_w - 3 \sum_{i=1}^k (u_i^T w)^2 P_w u_i u_i^T P_w \succeq 0
  \]

The first-order optimality condition implies that for any local optimal point $w$, we have

\[
\left( \sum_{i} (w^T u_i)^4 \right) w = \sum_{i} (w^T u_i)^3 u_i.
\]

Consequently, $w$ has to be in the span of $u_i$’s if $\lambda \neq 0$. Based on these optimality conditions, we study the landscape of the tensor decomposition problem (3) in two steps: In the first step, we show that if a local minimum exists close to one of the components, the local minimum should be in fact very close to the true component. In other words, within a region around any true component, the local minima are all very close to the true component. Thus, the landscape is locally well-behaved around the true components. In the second step, we make our result global by showing that any local minimizer of the tensor decomposition problem (3) is relatively close to one of the true components (up to sign ambiguity).

To proceed, let us make the following standard assumptions:

**Assumption 2.1.** $u_i$’s are all norm 1 and satisfy the following incoherence condition with constant $\tau$, i.e.,

\[
|u_i^T u_j| < \tau,
\]  

and $\tau = \mathcal{O}(\frac{1}{\sqrt{d}})$. Moreover, we assume that for any vector $w$ in the span of $\{u_i\}_{i=1}^k$, we have

\[
(1 - \delta)\|w\|^2 \leq \|U^T w\|^2 \leq (1 + \delta)\|w\|^2,
\]

where $U = [u_1, \ldots, u_k] \in \mathbb{R}^{d \times k}$ and $\delta = \mathcal{O}\left(\sqrt{k/d}\right)$. This condition is known in the literature as Restricted Isometric Property (RIP) that is usually satisfied with high probability when $d$ is large for many forms of random matrices.

Furthermore, for general matrices, it is easy to prove that $\delta \leq (k - 1)\tau$.

### 3 Geometric Analysis

Throughout this section, for any given point $w$, we define $c_i(w) = |w^T u_i|$. For simplicity of notations and since it is clear from the context, we use $c_i$ instead of $c_i(w)$. We also, without loss of generality, assume that $|c_1| \geq |c_2| \geq \cdots \geq |c_k|$. Using this definition, the following lemma shows that there is always a gap between $c_1$ and the rest of the components $c_j, j \neq 1$.

---

The subscript $\mathcal{M}$ refers to the fact that the corresponding derivative is calculated while projecting the directions on the manifold $\mathcal{M}$.
Lemma 3.1. Let \( w \) be a local minimizer of (3), then \( |c_1| > \sqrt{2}|c_2| \).

Proof. We prove by contradiction. Assume the contrary that \( |c_1| \leq \sqrt{2}|c_2| \). Take a unit vector \( v \) in the span of \( u_1 \) and \( u_2 \) which is orthogonal to \( w \). First of all, we have that

\[
v^T \nabla^2_{A^2} f(w)v = \left( \sum_i (w^T u_i)^4 \right) \|v\|^2 - 3 \sum_i (u_i^T w)^2 (u_i^T v)^2 \leq c_1^2 \left( \sum_i (w^T u_i)^2 \right) - 3 \min_{i=1,2} \|u_i\|_2 \|u_1, u_2\|_2 v^T \|v\|^2 \leq (1 + \delta) \|v\|^2 = (1 + \delta)
\]

Based on the RIP assumption and since \( \|v\| = 1 \), we have \( \|u_1, u_2\|_2^2 \geq (1 - \delta) \). Furthermore, \( \min_{i=1,2} (u_i^T w)^2 \geq \frac{1}{2} c_1^2 \) based on the contrary assumption we made. Thus, when \( \delta \leq k \tau \leq 0.01 \) we have

\[
v^T \nabla^2_{A^2} f(w)v \leq c_1^2 (1 + \delta) - \frac{3}{2} c_1^2 (1 - \delta) \leq c_1^2 (1 + \delta - \frac{3}{2} (1 - \delta)) < 0.
\]

On the other hand, since \( w \) is a local minimizer of (3), the second-order optimality condition implies that \( v^T \nabla^2_{A^2} f(w)v \geq 0 \), which contradicts (9). \( \square \)

The following lemma shows that any local minimizer \( w \) of problem (3), is very close one of the to the component \( u_1 \), with the highest value of \( u_i^T w \) for \( i \in \{1, \ldots, k\} \). In other words, the projection of \( w \), onto space spanned by the rest of components is very small compared to its projection onto \( u_1 \).

Lemma 3.2. If \( w \) is a local minimizer of (3) and \( k \tau \leq 0.05 \) then

\[
\frac{\|P_{u_1^+} (w)\|}{\|u_1^T w\|} \leq O(\sqrt{k} \tau^3)
\]

Proof. Assume that \( z = \sum_i (w^T u_i)^3 u_i \). Based on (6), we have \( z = \lambda w \). Thus:

\[
\frac{\|P_{u_1^+} (w)\|}{\|u_1^T w\|} = \frac{\|P_{u_1^+} (z)\|}{\|z^T u_1\|}
\]

According to the previous lemma, \( |c_1| > \sqrt{2}|c_i| \) for any \( i \in \{2, 3, \ldots, k\} \). Therefore,

\[
|z^T u_1| = \left| c_1^3 + \sum_i c_i^3 u_i^T u_1 \right| \geq |c_1|^3 - \frac{k \tau}{2 \sqrt{2}} |c_1|^3 \geq 0.98 |c_1|^3.
\]

Moreover,

\[
\|P_{u_1^+} (z)\|^2 = \left\| P_{u_1^+} \left( \sum_{i \neq 1} (w^T u_i)^3 u_i \right) \right\|^2 \leq (1 + \delta) \sum_{i \neq 1} c_i^6
\]

Note that \( c_i = c_1 u_i^T u_i + P_{u_1^+} (w^T u_i) \). To find an upper-bound for \( c_i^6 \), we use the following lemma.

Lemma 3.3. For any two real numbers \( a \) and \( b \), we have \( (a + b)^6 \leq 1.01 a^6 + O(b^6) \).

\[
\left\| P_{u_1^+} (z) \right\|^2 \leq 1.01 (1 + \delta) \sum_{i \neq 1} (u_i^T P_{u_1^+} (w))^6 + O \left( \sum_{i \neq 1} c_i^6 \tau^6 \right)
\]

(13)
Thus, Theorem 3.4.

If

which completes the proof.

Thus, the optimization problem (3) turns to:

Now we have

Now note that

Corollary 3.5.

Note that in the case where the components are randomly generated with dimension $d$ our result shows that when $k \leq O(\sqrt{d})$ there are no spurious local minima.

4 Extension to the Non-equally Weighted Scenario

In this section, we extend the result of the previous section to the case when the tensor decomposition components have not equal weights. In this scenario, the tensor of interest $T$ is in the form of:

Thus, the optimization problem (3) turns to:

\[
\min_w f(w) = -\frac{1}{4} \sum_{i=1}^{k} \lambda_i (u_i^T w)^4 = -\frac{1}{4} T(w, w, w, w),
\]

s.t. $\|w\|^2 = 1$. 
Let $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ be the maximum and minimum values among $\{\lambda_i\}_{i=1}^t$, respectively. The following theorem demonstrates that under an additional assumption on the ratio of $\lambda_{\text{max}}$ to $\lambda_{\text{min}}$, any local minimizer of the optimization problem (17) is very close to one of the actual components $u_i$.

**Theorem 4.1.** If $k\tau \leq 0.05$ and $\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \leq \frac{5}{4}$ then for any local minimizer $w$ of (17) there is an index $i$ such that $\|w - u_i\| \leq O(\kappa \sqrt{k\tau^3})$.

We leave the optimality conditions of problem (17), the extension of lemma 3.1, lemma 3.2, and the proof of the above theorem to appendix 3.

### 5 Recovery via Tensor Power Method and Deflation

The landscape analysis in Section 3 demonstrates that to recover at least one of the true components $u_i$ of a given tensor $T$, with the error of at most $O(\kappa \sqrt{k\tau^3})$, it suffices to find any local optimum of problem (17). To recover the other components of $T$, we can deflate the obtained component from $T$, and find a local optimizer of problem (17) for the deflated tensor. However, the introduced deflation error could potentially make the problem of finding the remaining $u_i$ components difficult. In Section 5.1 we show that TPM can tolerate error residuals from deflation if it is well-initialized. In other words, TPM can recover all $u_i$’s within a noise floor of $O(\kappa \sqrt{k\tau^3})$. Similar to the geometric result, our convergence guarantee for TPM with “good” initialization is deterministic.

Finally, in Section 5.2 we provide a probabilistic argument to show that when $d$ is large after restarting TPM randomly for $L = \text{poly}(k, d)$ times, with high probability, we obtain a “good” initialization. This concludes that TPM with restarts and deflation can efficiently find all $u_i$’s within some noise floor $O(\sqrt{T\tau^3})$.

#### 5.1 Convergence of Well-initialized TPM with Deflation

Tensor Power Method is one of the most widely used algorithms for solving tensor decomposition problems. For a given tensor $T$, and vectors $x, y,$ and $z$, consider the following mapping:

$$T(I, x, y, z) = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d \sum_{\ell=1}^d T(i, j, k, \ell)x(j)y(k)z(\ell)e_i,$$

where $e_i$ is the $i$-th standard unit vector.

Assume that a tensor $T$ and initialization $w_0$ is given. At each iteration, TPM applies vector-valued mapping (18) to current $w$, which is initialized to $w_0$. Then, it normalizes the resulted vector, and update $\lambda$, by applying mapping (1) to $w$. The details of Tensor Power Method are summarized in Algorithm 1.

**Algorithm 1 TPM(T, w_0, T)**

```plaintext
for $t = 1, \cdots, \bar{T}$ do
    $w_t = T(I, w_{t-1}, w_{t-1}, w_{t-1}) / \|T(I, w_{t-1}, w_{t-1}, w_{t-1})\|
    \lambda_t = T(w_t, w_t, w_t, w_t)
end for
return ($w_T$, $\lambda_T$)
```

The following theorem shows that when $k\tau$ is small enough, applying algorithm 1 to a deflated version of tensor $T$ can recover all $u_i$ components.

**Theorem 5.1.** Fix $1 \leq i \leq k$. Assume that we have access to $\hat{u}_j \in \mathbb{R}^d$ and $\hat{\lambda}_j \in \mathbb{R}$, $j = 1, \cdots, i - 1$ such that there are appropriate absolute constants $c_1$ and $c_2$ where:

- $k\tau \leq c_1$
- $\|\hat{u}_j - u_j\| \leq \epsilon = c_2 \sqrt{k\tau^3}$ and $|1 - \hat{\lambda}_j| \leq 5\epsilon$.

Moreover, assume that we have an initial unit vector $w_0$, for which $|w_0^T u_j| \geq \tau$ and $|w_0^T u_i| \geq 2|w_0^T u_j|$, $\forall j \neq i$. Define the deflated tensor $T_i = T + \sum_{j=1}^{i-1} -\hat{\lambda}_j \hat{u}_j \otimes 4$. Suppose that we apply TPM for appropriate number of iterations $T = O(1) + O(\log(1/\sqrt{k\tau^4}))$ to obtain $(\hat{u}_i, \hat{\lambda}_i) = TPM(T_i, w_0, T)$. Then, we have

$$\|\hat{u}_i - u_i\| \leq \epsilon \text{ and } |1 - \hat{\lambda}_i| \leq 5\epsilon.$$
Corollary 5.2. A similar result can be obtained when the original tensor $T$ is in the form of $T = \sum_{i=1}^{k} \lambda_i u_i \otimes^4$, for $\lambda_1 \geq \cdots \geq \lambda_k > 0$.

5.2 Obtaining good initialization by random restarts

In Section 5.1 we proved that TPM is effective in finding the components of $T$ when applied on deflated tensors sequentially and with good enough initialization. In this section we prove that we can obtain such a good initialization by doing multiple random restart in each iteration. Algorithm 2 describes Tensor Power Method augmented by multiple random restarts at each iteration (TPMR).

Algorithm 2 $\text{TPMR}(T, T_i, L, k)$

for $i = 1, \cdots, k$

if $i=1$

else

end if

Generate $v_1, \cdots, v_L$ uniformly on $\mathcal{M}$

Find $(w_\ell, \lambda_\ell) = \text{TPM}(T, v_\ell, T)$, $\ell = 1, \cdots, L$

Find $\ell^* = \arg \max_\ell \lambda_\ell$

Set $\hat{u}_i = w_{\ell^*}$ & $\hat{\lambda}_i = \lambda_{\ell^*}$

end for

return $(\hat{u}_1, \cdots, \hat{u}_k)$

TPMR calls TPM with multiple restarts on its inside loop. Thus, to prove the effectiveness of TMPR algorithm, it suffices to find good initialization points for TPM using random restarts. The following theorem states such a result.

Theorem 5.3. For any small threshold $\eta > 0$, if we sample $L$ uniform vectors from $\mathcal{M}$ such that $L$ satisfies

\[
A_1(L, \eta) = 0.5 \sqrt{\log L} - \sqrt{2 \log \frac{12}{\eta}} \geq 2 \times B_1(L, \eta, \tau),
\]

\[
\frac{A_1(L, \eta)}{C_1(\eta, d)} \geq \tau
\]

where

\[
B_1(L, \eta, \tau) = \sqrt{2(1 + \tau^2) \log(2k)} + \tau \left( \sqrt{2 \log(2L)} + \sqrt{2 \log \frac{12}{\eta}} \right) + \sqrt{2(1 + \tau^2) \log \frac{3}{\eta}}
\]

\[
C_1(\eta, d) = \sqrt{3 \log(3/\eta)} d + 2 \log(3/\eta)
\]

then, with probability $1 - \eta$, at least one of the samples $v_\ell \in \{v_1, \cdots, v_L\}$ will satisfy:

\[
|v_\ell^T u_1| \geq 2 |v_\ell^T u_i|, \forall i \neq 1
\]

\[
|v_\ell^T u_1| \geq \tau
\]

Corollary 5.4. To make sure that the good initialization condition is satisfied throughout the for loop in TPMR Algorithm 2 with probability $\eta_0$, we need to plug in $\eta = \eta_0/k$ in Theorem 5.3.

Corollary 5.5. It is easy to verify that one can find $L = \text{Poly}(\exp(d\tau^2), k)$ that satisfies (20) and (21).

Remark 5.6. Note that in the case where $u_i$’s are randomly generated $d$ dimensional vectors, then $\tau \leq O(1/\sqrt{d})$. Thus, $\exp(d\tau^2) = O(d)$. Thus, $L = \text{Poly}(d, k)$.  

7
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A Helper Lemmas for Proving TPM Convergence

Lemma A.1. If \( \|u_i - \bar{u}_i\| \leq \epsilon \) and \( |1 - \tilde{\lambda}_i| \leq 5\epsilon \leq 1 \), then for any \( w \) such that \( \|w\| = 1 \),
\[
E_i(I, w, w, w) = (w^T u_i)^3 u_i - \tilde{\lambda}_i (w^T \bar{u}_i)^3 \bar{u}_i = A_i u_i - B_i \hat{u}_i^4 / \|\hat{u}_i^4\|,
\]
where \( \hat{u}_i^4 = \bar{u}_i - (u_i^T \bar{u}_i/u_i) \) and
\[
|A_i| \leq 7|c_i|^3 \epsilon + 10|c_i|\epsilon^2 + 2\epsilon^2 \quad \& \quad |B_i| \leq 8|c_i|^3 \epsilon + 8\epsilon^4,
\]
where \( c_i = w^T u_i \).

Proof. Let us use the following definitions: \( c_i = w^T u_i \), \( a_i = u_i^T \bar{u}_i \) and \( b_i = w^T(\hat{u}_i^4 / \|\hat{u}_i^4\|) \). Then, we have
\[
E_i(I, w, w, w) = c_i^3 u_i - \tilde{\lambda}_i (a_i c_i + \|\hat{u}_i^4\| b_i)^3 (a_i u_i + \hat{u}_i^4)
\]
\[
= c_i^3 - \tilde{\lambda}_i a_i (c_i + \|\hat{u}_i^4\| b_i)^3 u_i - \tilde{\lambda}_i \|\hat{u}_i^4\| ((a_i c_i + \|\hat{u}_i^4\| b_i)^3 \|\hat{u}_i^4\|)
\]
(27)

Also, note that \( 1 - a_i = \|u_i - \bar{u}_i\|^2 \leq \epsilon^2 / 2 \). In addition, \( \|\hat{u}_i^4\|^2 \leq \|u_i - \bar{u}_i\|^2 \leq \epsilon^2 \) and \( |a_i|, |b_i| \leq 1 \). Note that we can easily prove that \( 1 - a_i^4 \leq 2\epsilon^2 \). Now for \( A_i \), we have
\[
|A_i| = |c_i|^3 (1 - \tilde{\lambda}_i^4) - \tilde{\lambda}_i a_i (|\hat{u}_i^4|^3 b_i^3 + 3a_i c_i |\hat{u}_i^4|^2 b_i^2 + 3a_i^2 c_i^2 |\hat{u}_i^4| b_i)
\]
\[
\leq |c_i|^3 ((1 - \tilde{\lambda}_i^4) + |\tilde{\lambda}_i| (1 - a_i^4)) + (1 + 5\epsilon) (\epsilon^3 + 3\epsilon^2 |c_i| + 3\epsilon c_i^2)
\]
\[
\leq |c_i|^3 (5\epsilon + 4\epsilon^2) + 6c_i^2 \epsilon + 6|c_i|\epsilon^2 + 2\epsilon^3
\]
\[
\leq 11|c_i|^2 \epsilon + 10|c_i|\epsilon^2 + 2\epsilon^3,
\]
(28)

where the last step follows from \( |c_i| \leq 1 \).

For bounding \( B_i \) we use the fact that for \( \alpha, \beta \geq 0 \) \( (\alpha + \beta)^3 \leq 4\alpha^3 + 4\beta^3 \).
\[
|B_i| \leq (1 + 5\epsilon) \epsilon (|c_i| + \epsilon) \leq 8\epsilon (|c_i|^3 + \epsilon^3)
\]
(29)

Lemma A.2. If \( \|u_i - \bar{u}_i\| \leq \epsilon \) and \( |1 - \tilde{\lambda}_i| \leq 5\epsilon \leq 1 \) for any \( i \), then for any \( w \) such that \( \|w\| = 1 \), we have:
\[
\left\| \tilde{E}_i(I, w, w, w) \right\|^2 = \left\| \sum_{j=1}^i E_j(I, w, w, w) \right\|^2 \leq 2(1 + \delta) \sum_{j=1}^i A_j^2 + 2\left( \sum_{j=1}^i |B_j| \right)^2.
\]
(30)

Thus, we can conclude that
\[
\left\| \tilde{E}_i(I, w, w, w) \right\| \leq \left( 67\epsilon \sqrt{\sum_{j=1}^i c_j^4 + 40\epsilon^2} \right) \left( \sqrt{\sum_{j=1}^i c_j^2 + 16\epsilon} \sum_{j=1}^i |c_j|^3 + 8\sqrt{4\epsilon^3} + 16\epsilon^4 \right),
\]
(31)

where if we further have \( \epsilon \) small enough, i.e. \( \epsilon \leq 350\sqrt{\epsilon^3} \) and \( \delta \leq k\tau \leq 0.01 \), we will have
\[
\left\| \tilde{E}_i(I, w, w, w) \right\| \leq 100\epsilon.
\]
(32)

Proof. The proof of the first part is very simple and only uses the the results of Lemma A.1 and the fact that \( u_i \)'s satisfy the RIP condition.

The rest of the proof is also simple arithmetic.

Lemma A.3. If \( w \) is a unit vector, \( |c_i| \geq |c_j|, \forall j \), and moreover \( |c_i| \geq \tau \), and \( \epsilon \leq 350\sqrt{\epsilon^3} \), then if \( k\tau \leq 0.01 \)
\[
\left\| \tilde{E}_{i-1}(I, w, w, w) \right\| \leq 0.09|c_i|^3
\]
(33)
Proof. We assume that $\tau \leq k\tau \leq 0.01$. Let us look at each term in the right hand side of (31). For the first term

$$67\epsilon \sum_{j=1}^{i+1} c_{j}^{4} \leq 67|c_{i}|^{3} \times k\tau^{2} \leq 2.35 \times 10^{-2}|c_{i}|^{3}$$

(34)

For the second term

$$40\epsilon^{2} \sum_{j=1}^{i} c_{j}^{2} \leq 4.9 \times 10^{6}k\sqrt{k\tau^{4}}|c_{1}|^{3} \leq 4.9 \times 10^{-2}|c_{1}|^{3}.$$  

(35)

For the third term we have

$$16\epsilon \sum_{j=1}^{i} |c_{j}|^{3} \leq 5.6 \times 10^{-3}|c_{i}|^{3}$$

(36)

For the fourth term

$$8\sqrt{k\epsilon^{3}} \leq 9.8 \times 10^{-7}|c_{1}|^{3}$$

(37)

And for the last term we have

$$16\epsilon k^{4} \leq 2.5 \times 10^{-7}|c_{1}|^{3}$$

(38)

Finally the bound could be obtained by adding all these bounds.

$\square$

Now let us prove a recursive bound that we can use for the proving our final result.

**Lemma A.4.** Assume that for a norm 1 vector $w, |c_{i}| \geq 2|c_{j}|, \forall j \neq i$ and moreover $|c_{i}| \geq \tau$. Also assume that the conditions of Lemma A.3 is satisfied and $\epsilon \leq 350\sqrt{k\tau^{3}}$. Then for $w_{+} = \sum_{j=1}^{k} c_{j}^{3}u_{j} + \hat{E}_{i-1}(I, w, w, w)$ we have

$$\frac{||P_{u_{+}^{i}}(w_{+})||}{|w_{+}^{T}u_{i}|} \leq 0.95 \frac{||P_{u_{+}^{i}}(w)||}{|w_{+}^{T}u_{i}|} + 3||\hat{E}_{i-1}(I, w_{+}, w, w)|| + 15\sqrt{k\tau^{3}} \leq 0.95 \frac{||P_{u_{+}^{i}}(w)||}{|w_{+}^{T}u_{i}|} + O(\sqrt{k\tau^{3}}).$$

Moreover, $|w_{+}^{T}u_{i}| \geq 2|w_{+}^{T}u_{j}|, \forall j \neq i$.

**Proof.** Let us first lower bound $|w_{+}^{T}u_{i}|$ using Lemma A.3 and the fact that $k\tau \leq 0.01$ we have:

$$|w_{+}^{T}u_{i}| \geq |c_{i}|^{3} - \sum_{j=1}^{k} |c_{j}|^{3} - 0.09|c_{i}|^{3} \geq 0.9|c_{i}|^{3}. \quad (40)$$

Let us define $z = \sum_{j=1}^{k} c_{j}^{3}u_{j}$. Then, $||P_{u_{+}^{i}}(w_{+})|| \leq ||P_{u_{+}^{i}}(z)|| + ||\hat{E}_{i-1}(I, w_{+}, w, w)||$.

$$||P_{u_{+}^{i}}(z)||^{2} = \left(\sum_{j=1}^{k} c_{j}^{3}u_{j}\right)^{2} \leq (1 + \delta) \sum_{j=1+1}^{k} c_{j}^{6} \quad (41)$$

Note that $c_{j} = c_{i}u_{i}^{T}u_{j} + P_{u_{+}^{i}}(w)^{T}u_{j}$. And $c_{j}^{3} \leq 200c_{j}^{6} + 9(P_{u_{+}^{i}}(w)^{T}u_{j})^{6}$. Also note that $|u_{j}^{T}P_{u_{+}^{i}}(w)| \leq |c_{j}| + |c_{i}| \tau \leq |c_{i}|(0.5 + \tau)$ for $j > i$. Thus,

$$||P_{u_{+}^{i}}(z)||^{2} \leq (1 + \delta) c_{i}^{4} \left(9(0.5 + \tau)^{4} \sum_{i \neq i}^{k} (v_{j}^{T}P_{u_{+}^{i}}(u))^{2} + 200c_{j}^{6}O(k\tau^{5})\right)^{\frac{1}{2}} \leq (1 + \delta) ||P_{u_{+}^{i}}(u)||^{2} \leq c_{i}^{4}(202k\tau^{5}c_{i}^{2}) + O(k\tau^{5}(202k\tau^{5}c_{i}^{2}))$$

$$\leq c_{i}^{4}(202k\tau^{5}c_{i}^{2} + 63||P_{u_{+}^{i}}(w)||^{2}) \quad (42)$$

Therefore,

$$\frac{||P_{u_{+}^{i}}(z)||}{|w_{+}^{T}u_{i}|} \leq 0.8 \frac{||P_{u_{+}^{i}}(w)||}{|c_{i}|} + 15\sqrt{k\tau^{3}} \quad (43)$$

To bound the other part, i.e. $\frac{||\hat{E}_{i-1}(I, w_{+}, w, w)||}{|w_{+}^{T}u_{i}|}$, we use the following lemma.
Lemma A.5. Under the assumptions of Lemma A.4, we have
\[ \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{|u_i^T w_+|} \leq 0.15 \frac{\| P_{u_i^+}(w) \|}{|c_i|} + 3 \| \bar{E}_{i-1}(I, w, w, w) \| \leq 0.15 \frac{\| P_{u_i^+}(w) \|}{|c_i|} + O(\sqrt{k\tau^3}). \] (44)

Proof. Let us consider two cases:

- If $|c_i| \leq 0.8$, then $\| P_{u_i^+}(w) \| = \sqrt{1 - c_i^2} \geq 0.6$, thus
  \[ \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{|u_i^T w_+|} \leq \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{0.9|c_i|^3} \leq \frac{\| P_{u_i^+}(w) \|}{0.9|c_i|^3} \leq 0.9 \times 0.6 \frac{\| P_{u_i^+}(w) \|}{|c_i|} \leq 0.15 \frac{\| P_{u_i^+}(w) \|}{|c_i|}, \] (45)
  using Lemma A.3.

- If $|c_i| \geq 0.8$, then
  \[ \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{|u_i^T w_+|} \leq \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{0.9|c_i|^3} \leq 3 \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{0.9|c_i|^3} \leq \frac{\| \bar{E}_{i-1}(I, w, w, w) \|}{0.9|c_i|^3} \leq O(\sqrt{k\tau^3}), \] (46)
  where the last inequality is due to (31).

Combining these two cases, the result is obvious. \[ \square \]

Now we can combine the result of this lemma with our bound in (43):
\[ \frac{\| P_{u_i^+}(w) \|}{|w_i^T u_i|} \leq 0.95 \frac{\| P_{u_i^+}(w) \|}{|w_i^T u_i|} + 15 \sqrt{k\tau^3} + 3 \| \bar{E}_{i-1}(I, w, w, w) \| \leq 0.95 \frac{\| P_{u_i^+}(w) \|}{|w_i^T u_i|} + O(k\tau^3). \] (47)

Now we need to make sure the initial conditions of the recursion also hold for normalized $\bar{w}_+ = w_+/\|w_+\|$. First of all as absolute value of the angle between the $w_+$ and $u_i$ is decreasing it is obvious that $|\bar{w}_+^T u_i| \geq |w_i^T u_i| \geq \tau$. So, we only need to prove the gap condition
\[ \frac{|w_i^T u_i|}{|w_j^T u_j|} \geq 2, \quad \forall j \neq i. \] (48)

From (48) we know $|w_i^T u_i| \geq 0.9|c_i|^3$. Note that $|w_i^T u_j| \leq |w_i^T u_i| + \| \bar{E}_{i-1}(I, w, w, w) \|$. Let us consider two cases from here:

- $j > i$: Then, $|z^T u_j| \leq \sum_{\ell=1}^k |c_\ell|^3 |u_j^T u_\ell| \leq (\frac{1}{8} + \kappa^2) |c_i|^3 \leq \frac{1}{8} |c_i|^3$. Now we can use (33), i.e., \[ |\bar{E}_{i-1}(I, w, w, w)\| \leq 0.1 |c_i|^3. \] Combining these facts it is obvious that $\frac{|w_i^T u_i|}{|w_j^T u_j|} \geq \frac{0.9}{0.1} \geq 2$.

- $j < i$: In this case, $|z^T u_j| \leq \sum_{\ell=1}^k |c_\ell|^3 |u_j^T u_\ell| \leq (\kappa^2) |c_i|^3 \leq 0.01 |c_i|^3$. Now, if we use (33) we get $\frac{|w_i^T u_i|}{|w_j^T u_j|} \geq \frac{0.9}{0.01} \geq 2$.

Thus, we have proved that if the initial conditions hold, then $\bar{w}_+$ gets closer to $u_i$, unless it is already very $\epsilon = O(\sqrt{k\tau^3})$ close to $u_i$, see (39), and also $\bar{w}_+$ satisfies the initial conditions. Before proving our final convergence result, we need to follow the smallest lemma which is also a simple lemma to control the final error.

Lemma A.6. Assume that $|u_j - \bar{u}_j| \leq \epsilon$ and $|1 - \lambda_j| \leq 5\epsilon$, where $\epsilon \leq 350/\sqrt{k\tau^3}$ for all $j < i$. If for some norm 1 vector $w$, $|c_j| \leq 4\tau$ for all $j \neq i$ and $k\tau \leq 0.01$, then we have
\[ \| \bar{E}_i(I, w, w, w) \| \leq 0.25 \sqrt{k\tau^3}. \] (49)

Proof. The proof can be easily obtained by plugging in the assumptions in (31). \[ \square \]
Proof. Throughout the proof we assume \( c_1 = 10^{-3} \) and \( c_2 = 350 \). We also note that \( \delta \leq k \tau \). As \( \epsilon \leq 350 \sqrt{k \tau^3} \), we know that \( \| \bar{E}_i (I, w, w, w) \| \leq 3500 \sqrt{k \tau^3} \) based on Lemma A.2. Now define the ratio \( r_t = \frac{\| P_u(w_t) \|}{|w^T_t u_t|} \). First of all based on our assumptions on the initialization, \( r_0 \leq 1/\tau \). Moreover, based on the recursion that we proved in Lemma A.4, we know

\[
r_{t+1} \leq 0.95 r_t + 3\| \bar{E}_{i-1} (I, w, w, w) \| + 15 \sqrt{k \tau^2} \leq 0.95 r_t + 1.06 \times 10^5 \sqrt{k \tau^3}
\]

Now if we open up this recursion and use \( \epsilon \) to denote \( \epsilon = 1.06 \times 10^5 \sqrt{k \tau^3} \) we have

\[
r_t \leq \frac{0.95^t}{\tau} + 20 \epsilon.
\]

So, if we run tensor power method with initialization \( w_0 \) for \( t_0 = \frac{\log(0.95 e)}{\log(0.95 \epsilon)} \), we would have \( r_{t_0} \leq 20.5 \epsilon \) and thus, \( \| w_{t_0} - u_i \| \leq 21 \epsilon \leq 3 \tau \) as \( k \tau \leq 0.001 \). As a result, for any \( j \neq i \), \( | w^T_{t_0} u_j | \leq 4 \tau \). It is obvious that these conditions would hold for any \( t \geq t_0 \). Therefore, using Lemma A.6 we have

\[
\| \bar{E}_i (I, w_t, w_t, w_t) \| \leq 0.25 \sqrt{k \tau^3}, \forall t \geq t_0.
\]

In the light of this new bound for the error, we can re-write the unrolled version of the recursion in (39) for \( t \geq t_0 \) as

\[
r_t \leq 21 \epsilon \times 0.95^{t-t_0} + 20 \epsilon, \forall t \geq t_0
\]

where \( \dot{e} = 15.75 \sqrt{k \tau^3} \). Now after \( T = t_0 + \frac{\log(0.95 e)}{\log(0.95 \epsilon)} \), we would have \( r_T \leq 20.5 \epsilon \leq 325 \sqrt{k \tau^3} \). Thus, \( \| w_T - u_i \| \leq \epsilon = 350 \sqrt{k \tau^3} \). Now by plugging in the \( w_T \) in tensor \( T_i = T + \bar{E}_i \) we get

\[
\dot{\lambda}_i = T_i(w_T, w_T, w_T, w_T).
\]

It would be easy to check that \( | 1 - \dot{\lambda}_i | \leq 5 \epsilon \). Finally, note that \( \dot{e}/e \) is constant and thus the number of required iterations \( T \) is of the order \( O(1) + O(\log(\sqrt{k \tau^3})) \).

\( \square \)

C Proof of Theorem 5.3

Proof. First define \( Z_{\ell,i} = v_{\ell}^T u_i \). It is obvious that \( Z_{\ell,i} \)'s are Gaussian. Let us define the following probability events.

\[
\Xi_1 = \left\{ Z : \max_{\ell} |Z_{\ell,i}| \geq A_1(L, \eta) \right\}
\]

\[
\max_{\ell} |Z_{\ell,1}| \leq \sqrt{2 \log 2L + \sqrt{2 \log \frac{12}{\eta}}} \}
\]

\[
\Xi_2,\ell = \left\{ Z_{\ell,i} : \max_{i \neq 1} |Z_{\ell,i}| \leq B_1(L, \eta, \tau) \right\}
\]

\[
\Xi_3,\ell = \left\{ v_k : \| P(u_1, \ldots, u_k) v_k \| \leq C_1(\eta, d)^2 \right\}
\]

It would be easy to see that if \( \ell^* = \arg \max_{\ell} |Z_{\ell,i}| \) and \( \Xi_1 \cap \Xi_2,\ell \cap \Xi_3,\ell \) happens, then the final result would be true when \( L = O(k^2) \). So, we just need to prove that \( P(\Xi_1 \cap \Xi_2,\ell \cap \Xi_3,\ell) \geq 1 - \eta \). To do so, note that \( P(\Xi_1 \cap \Xi_2,\ell \cap \Xi_3,\ell) = P(\Xi_1)P(\Xi_2,\ell | \Xi_1)P(\Xi_3,\ell | \Xi_2,\ell \cap \Xi_1) = P(\Xi_1)P(\Xi_2,\ell | \Xi_1)P(\Xi_3,\ell) \), where the last inequality is due to the independence of \( P(u_1, \ldots, u_k) v_k \) with respect to \( Z \)'s. Note that for any \( \ell \), it can be easily proved that \( P(\Xi_3,\ell) \geq 1 - \eta/3 \).

As the initializations are independent, it is well known that

\[
P \left( |Z_{\ell,1}| \leq 0.5 \sqrt{\log L} - t \right) \leq 2 \exp -t^2/2
\]

\[
P \left( |Z_{\ell,1}| \geq 2 \log(2L) + t \right) \leq 2 \exp -t^2/2
\]
Using these concentrations, it is clear that with choice of \( t = \sqrt{2 \log(12/\eta)} \), we get \( P(\Xi_1) \geq 1 - \eta/3 \).
The only remaining part to prove is to show \( P(\Xi_2, t^* | \Xi_1) \geq 1 - \eta/3 \). To do that, note that given \( Z_{\epsilon \Xi_2}^*, Z_{t^*,-1} = [Z_{t^*}, \cdots, Z_{t^*}]^T \) could be written as Gaussian random variable

\[
Z_{t^*,-1} | Z_{t^*,-1} \sim \mathcal{N}(\rho Z_{t^*,-1}, C - \rho \rho^T),
\]

where \( C = \mathbb{E}(Z_{t^*,-1} Z_{t^*,-1}^T) \) and \( \rho \) is the correlation vector, i.e. \( \rho = \mathbb{E}(Z_{t^*,-1} Z_{t^*,-1}) \). With some abuse of notation, we can write that \( Z_{t^*,-1} | Z_{t^*,-1} = \rho Z_{t^*,-1} + e_i \), where \( e_i \) is Gaussian, zero mean with variance \( C_{ii} - \rho_i^2 \leq \sigma^2 = 1 + \tau^2 \leq 1.1 \).

Thus, we can write

\[
P\left( \|Z_{t^*,-1}\|_\infty \geq \sqrt{2 \sigma^2 \log(2k)} + \tau \left( \sqrt{2 \log(2L)} + \sqrt{2 \log \frac{12}{\eta}} \right) + \sqrt{2 \sigma^2 \log \frac{3}{\eta}} \left| Z_{t^*,-1} \right| \right) \leq \sum_i P\left( |e_i| \geq \sqrt{2 \sigma^2 \log(2k)} + \sqrt{2 \sigma^2 \log \frac{3}{\eta}} \right) \leq \eta/3.
\]

This completes the proof. \( \square \)

## D Generalizing The Geometrical Results to the Non-equally Weighted Scenario

In this appendix, we extend the geometrical results demonstrated in Section 3 to the case where the weights of components are not equal to one necessarily. Suppose that a given tensor \( T \) can be decomposed as \( T = \sum_{i=1}^k \lambda_i u_i \otimes u_i \) where \( u_i \)'s are unit and \( \lambda_i \)'s are non-negative. Optimization problem \( \mathcal{M} \) can be generalized to:

\[
\begin{align*}
\min_w & \quad f(w) = -\frac{1}{4} \sum_{i=1}^k \lambda_i (u_i^T w)^4 = -\frac{1}{4} T(w, w, w, w), \\
\text{s.t.} & \quad ||w||^2 = 1.
\end{align*}
\]

The first, and second order conditions for this problem are as follows:

- **First-order optimality condition:**

\[
\nabla_{\mathcal{M}} f(w) = -P_w \sum_{i=1}^k \lambda_i (u_i^T w)^3 u_i = 0
\]

(61)

- **Second-order optimality condition:**

\[
\nabla^2_{\mathcal{M}} f(w) = \left( \sum_{i=1}^k \lambda_i (u_i^T w)^4 \right) P_w - 3 \sum_{i=1}^k \lambda_i (u_i^T w)^2 P_w u_i u_i^T P_w \geq 0
\]

(62)

As a consequence of the first-order optimality condition, for any stationary point of problem (61) we have:

\[
\sum_i \lambda_i (w^T u_i)^4 \geq \sum_i \lambda_i (w^T u_i)^3 u_i.
\]

(63)

**Lemma D.1.** Let \( w \) be a local minimizer of (61), then \( |e_1| > \sqrt{3} |c_2| \).
Proof. To arrive in a contradiction assume that $|c_1| \leq \sqrt{2}|c_2|$ / Take a unit vector $v$ in the span of $u_1$ and $u_2$ which is orthogonal to $w$.

$$v^T \nabla^2_{\mathcal{M}} f(w)v = \left( \sum_i \lambda_i (w^T u_i)^4 \right) \|v\|^2 - 3 \sum_i \lambda_i (u_i^T w)^2 (u_i^T v)^2 \leq c_1^2 \lambda_{\max} \left( \sum_i (w^T u_i)^2 \right) - 3 \min_{i=1,2} \lambda_i ((u_i^T w)^2) \|[u_1, u_2]^T v\|^2_2 \leq (1+\delta)\|w\|^2$$

(64)

Based on our assumptions $\|u_1, u_2\|^2_2 > (1 - \delta)$ as $\|v\| = 1$. Also note that $\min_{i=1,2}((u_i^T w)^2) \geq \frac{1}{2} c_1^2$ based on our assumption. Thus, when $\delta \leq k\tau \leq 0.01$ we have

$$v^T \nabla^2_{\mathcal{M}} f(w)v \leq c_1^2 \lambda_{\max} (1 + \delta) - \frac{3}{2} c_1^2 \lambda_{\min} (1 - \delta) \leq c_1^2 (\lambda_{\max} - \frac{3}{2} \lambda_{\min}) + \delta (\lambda_{\max} + \frac{3}{2} \lambda_{\min}) < 0,$$

(65)

(66)

which is a contradiction with the second-order optimality condition for $w$ that states $v^T \nabla^2_{\mathcal{M}} f(w)v \geq 0$. Note that the last inequality holds when $\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} < \frac{3}{2}$ and $\delta < \frac{\lambda_{\min} - \lambda_{\max}}{\lambda_{\max} + \frac{3}{2} \lambda_{\min}}$.

□

Lemma D.2. If $w$ is a local minimizer of $(60)$, $k\tau \leq 0.05$, and $\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{3}{4}$, then

$$\frac{\|P_{u_1^+}(w)\|}{\|u_1^+ w\|} \leq O(\kappa \sqrt{k\tau})$$

(67)

Proof. First of all note that based on the above Lemma we know that $|c_1| > \sqrt{2}|c_2|$. Moreover, due to the first order optimality condition $(6)$ we know that $w \propto z = \sum_i \lambda_i (w^T u_i)^3 u_i$. Thus,

$$\frac{\|P_{u_1^+}(w)\|}{\|u_1^+ w\|} = \frac{\|P_{u_1^+}(z)\|}{\|z^T u_1\|}$$

(68)

Note that

$$\|z^T u_1\| = \left| \lambda_1 c_1^4 + \sum_{i \neq 1} \lambda_i c_i^3 u_i^T u_1 \right| \geq \lambda_{\min} (|c_1|^3 - \frac{k\tau}{2\sqrt{2}} |c_1|^3) \geq 0.98 \lambda_{\min} |c_1|^3.$$

(69)

Moreover,

$$\|P_{u_1^+}(z)\|^2 = \left\| P_{u_1^+} \left( \sum_{i \neq 1} \lambda_i (w^T u_i)^3 u_i \right) \right\|^2 \leq (1 + \delta) \sum_{i \neq 1} \lambda_i^2 c_i^6 \leq \lambda_{\max}^2 (1 + \delta) \sum_{i \neq 1} c_i^6$$

(70)

Note that $c_i = c_1 u_1^T u_i + P_{u_1^+}(w) u_i$. Now we can use lemma D.3 to find an upper-bound for $c_i^6$.

$$\|P_{u_1^+}(z)\|^2 \leq \lambda_{\max}^2 (1.01 \sum_{i \neq 1} (u_i^T P_{u_1^+}(w))^6 + O(\sum_{i \neq 1} c_i^6 \tau^6)) \leq c_1^6 O(k\tau^6)$$

(71)

Note that $|u_i^T P_{u_1^+}(w)| \leq |c_i| + |c_1| \tau \leq |c_1| \left( \frac{1}{\sqrt{2}} + \tau \right)$ for $i \neq 1$. Thus,

$$\|P_{u_1^+}(z)\|^2 \leq \lambda_{\max}^2 c_1^4 \left( \frac{1}{\sqrt{2}} + \tau \right)^4 \sum_{i \neq 1} (u_i^T P_{u_1^+}(w))^2 + \lambda_{\max}^2 c_1^6 \tau^6 \leq (1 + \delta) \|P_{u_1^+}(w)\|^2 \leq \lambda_{\max}^2 2c_1^4 \max \left( \left( \frac{1}{\sqrt{2}} + \tau \right)^4 (1 + \delta) \|P_{u_1^+}(w)\|^2, c_1^6 \right)$$

(72)
Therefore,
\[
\frac{\|P_{u_1^\perp}(w)\|}{|w^T u_1|} = \frac{\|P_{u_1^\perp}(z)\|}{|z^T u_1|}
\leq \frac{1.01 \lambda_{\text{max}}}{0.98 \lambda_{\text{min}}} \max \left( \sqrt{2} \left( \frac{1}{\sqrt{2}} + \tau \right)^2 \sqrt{1 + \delta} \frac{\|P_{u_1^\perp}(w)\|}{c_1} , O(\sqrt{\kappa \tau^3}) \right).
\]
(73)

Now note that if \( \kappa \leq \frac{5}{4} \), and \( \delta \) is small enough, then:
\[
t = \frac{1.01}{0.98} \kappa \sqrt{2} \left( \frac{1}{\sqrt{2}} + \tau \right)^2 \sqrt{1 + \delta} < 1
\]

Thus,
\[
\frac{\|P_{u_1^\perp}(w)\|}{|w^T u_1|} \leq \max \left( t \frac{\|P_{u_1^\perp}(w)\|}{|w^T u_1|} , O(\kappa \sqrt{\kappa \tau^3}) \right),
\]
(74)
which completes the proof.

\[ \square \]

**Theorem D.3.** If \( k\tau \leq 0.05 \) and \( \kappa \leq \frac{5}{4} \) then for any local minimizer \( w \) of (3) there is an index \( i \) such that \( \|w - u_i\| \leq O(\kappa \sqrt{\kappa \tau^3}) \).

**Proof.** Based on the above lemma and without loss of generality assume that \( \frac{\|P_{u_1^\perp}(w)\|}{|w^T u_1|} \leq O(\kappa \sqrt{\kappa \tau^3}) \). Thus, \( \|P_{u_1^\perp}(w)\|^2 = 1 - c_1^2 \leq c_1^2 O(\kappa^2 k\tau^6) \). Therefore,
\[
c_1 \geq \frac{1}{\sqrt{1 + O(\kappa^2 k\tau^6)}} \geq 1 - O(\kappa^2 k\tau^6).
\]
Now we have \( \|w - u_1\|^2 = 2(1 - c_1) \leq O(\kappa^2 k\tau^6) \). \[ \square \]