Statistical Physics of 3D Hairy Black Holes

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Abstract

We investigate the statistical behaviors of 3D hairy black holes in the presence of a scalar field. The present study is made in terms of two relevant parameters: rotation parameter $a$ and $B$ parameter related to the scalar field. More precisely, we compute various statistical quantities including the partition function for non-charged and charged black hole solutions. Using a partition function calculation, we show that the probability is independent of $a$ and $B$ parameters.
Recently, many effects have been devoted to study thermodynamic behaviors of black holes in lower and higher dimensions \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\]. For certain systems, the equation of states have been worked out sharing similarities with Van der Waals P-V systems. In four dimensions for instance, RN-AdS black holes with spherical geometries have been extensively investigated \[17, 18\]. More precisely, it has been remarked that there is a nice interplay between the behaviors of the RN-AdS black hole systems which has been explored in many works. The P-V criticality, the Gibbs free energy, the first order phase transition and the behavior near the critical points are associated with the statistical liquid-gas systems. In particular, the critical behaviors of charged RN-AdS black holes in arbitrary dimensions of the spacetime have been investigated \[7\].

On the other hand, a particular interest has been put on the three dimensional case corresponding to the BTZ black hole whose critical behaviors are associated with the ideal gas ones \[7, 9\]. More recently, a novel exact rotating black hole solution in (2+1)-dimensional gravity with a non-minimally coupled scalar field has been studied using an appropriate metric ansatz \[19, 20\]. In this way, critical behaviors of a class of such black holes has been investigated. Interpreting the cosmological constant as a thermodynamic pressure and its conjugate quantity as a volume, the corresponding equation of state has been established. In a generic region of the corresponding moduli space, these black holes behave like a Van der Waals system \[9\].

The aim of this work is to contribute to these activities by studying the statistical behaviors of 3D hairy black holes. In particular, we compute various statistical quantities including the partition function for non-charged and charged solutions. This study is made in terms of two parameters \(B\) and \(a\). These parameters are associated with the scalar field and the angular momentum respectively. Using a partition function calculation, we reveal that the probability is independent of such parameters.

To start we reconsider the study of the statistical physics of 3D-dimensional gravity with a non-minimally coupled scalar field. This black hole solution is known as hairy black hole in three dimensions. In the absence of the Maxwell gauge fields, this model can be described by the following action \[19\]

\[
I_R = \frac{1}{2} \int d^3x \sqrt{-g} \left[ R - g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi) \right]
\]

where \(\phi\) is the dynamical scalar field. For simplicity reason, we consider a particular situation where the coupling and the gravitational constants are fixed to \(\xi = \frac{1}{8}\) and
\[ \kappa = 8\pi G = 1 \] respectively. In this way, the solution takes the following form

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 (d\theta + \omega(r)dt)^2. \]  

(2)

It is noted that the functions \( f \) and \( \omega \), controlling such a solution, read as

\[ f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r + 2B)^2a^2}{r^4} + \frac{r^2}{\ell^2} \]  

(3)

\[ \omega(r) = -\frac{(3r + 2B)a}{r^3} \]  

(4)

where \( \beta \) is identified with \(-\frac{M}{4}\). In these equations, \( a \) describes the rotating angular momentum parameter. While, the parameter \( B \) is linked to the dynamical scalar field.

For later use, an explicit form of the potential will be needed. In the present study, we explore the potential \( V(\phi) \) proposed in [20] given by the following equation

\[ V(\phi) = \frac{1}{512} \left( a^2 \left( \frac{\phi^6 - 40\phi^4 + 640\phi^2 - 4608}{B^4 (\phi^2 - 8)^5} \right) \phi^{10} + \phi^6 \left( \frac{\beta}{B^2} + \frac{1}{\ell^2} \right) + \frac{1024}{\ell^2} \right). \]  

(5)

Having fixed the metric backgrounds, we move now to study thermodynamical behaviors of such black hole solutions. More precisely, we compute the entropy considered as the most important thermodynamical quantity. In 3D, the calculation produces the following entropy function

\[ s = 4\pi r_+ \]  

(6)

where \( r_+ \) is the horizon radius obtained by solving \( f(r) = 0 \). Taking this entropy function, we can derive many statistical quantities including the temperature, the momentum and the specific heat capacity. In order to obtain such quantities, we should first determine the black hole mass. Indeed, the equations (6) and (2) give the entropy-dependent mass

\[ M = \frac{48\pi^2 a^2 (8\pi B + 3s)}{s^3} + \frac{3s^3}{16\pi^2 \ell^2 (8\pi B + 3s)}. \]  

(7)

The corresponding numerical behaviors are presented in figure 1.

It follows from this figure that the scalar charge \( B \) decreases the mass of the 3D black hole. More precisely, the rotation parameter modifies the thermodynamical behavior of the mass and the temperature in terms of the entropy. In such a solution,
the mass function shows a minimum. The corresponding moduli space contains two relevant regions associated with the asymptotic behaviors of the entropy. Indeed, in the first region corresponding to small values, the mass of the 3D black holes increases with the $B$ parameter. However, in the large limit values associated with second region, the mass decreases with the $B$ parameter. Similar behaviors occur in the corresponding black hole temperature. It is recalled that the later can be obtained by using the following relation

$$ T = \left( \frac{\partial M}{\partial s} \right) . \quad (8) $$

Exploring eq. (8), the computation can give the black hole temperature in terms of the entropy function. The obtained relation reads as

$$ T = \frac{9s^2(4\pi B + s)}{8\pi^2\ell^2(8\pi B + 3s)^2} - \frac{288\pi^2a^2(4\pi B + s)}{s^4}. \quad (9) $$

This temperature function is illustrated in figure 2.

From this figure, it has been observed similar behaviors appearing in the mass term of the black hole. However, the novelty is associated with the negative value of the temperature.

To complete the statistically investigation of the such 3D black holes, we compute
the specific heat. This quantity can be obtained from the following equation

$$C = T \frac{\partial s}{\partial T}. \quad (10)$$

Using Eq. (9), we get the specific heat function

$$C = \frac{s(4\pi B + s)(8\pi B + 3s)\left(s^6 - 256\pi^4 a^2 \ell^2 (8\pi B + 3s)^2\right)}{256\pi^4 a^2 \ell^2 (8\pi B + 3s)^3(16\pi B + 3s) + s^6 (64\pi^2 B^2 + 24\pi Bs + 3s^2)}. \quad (11)$$

This function is presented in figure 3.

![Figure 3: Plots of black hole heat capacity for $\ell = 1$.](image)

It is recalled that the sign of the specific heat can determine the thermodynamical stability of the black hole. The positivity of such a quantity ensures the stable equilibrium [16]. It follows from the figure 3 that the rotating parameter leads to an instability solution. Indeed, it has been observed that similar behaviors appear in non rotating solution. This means that the heat capacity decreases with scalar charge $B$. In fact, it is still always positive explaining that the black hole solution is stable. However, the introduction of the rotation parameter modifies the black hole behaviors and perturbs its the stability. Moreover, the heat capacity shows a minima and takes negative values in the first region associated with small values of the entropy.

It is observed that in the vanishing limit of $a$ with $\beta = -\frac{a^2}{\ell}$, the specific heat reduces to

$$C = 8\pi \ell \sqrt{\frac{M}{3}} = \hat{s} \quad (12)$$

recovering the result reported in [21].

The obtained quantities will be useful for discussing the most important function used in statistical physics. In particular, we compute the partition function. The
calculation gives the following form

\[ Z = \exp \left( -\frac{F}{T} \right) \]

\[ = \exp \left[ \frac{2\pi r^4 \left( \frac{16B^3(8B+9r)}{(2B+3r)^2} + \frac{27a^2(8B+9r)}{r^2} \right)}{81(B + r) \left( \frac{r^6}{(2B+3r)^2} - a^2 \right)} \right] \]

where \( F \) is the free energy expressed as follows

\[ F = -\int sdT \]

\[ = \frac{24a^2B}{r^3} + \frac{27a^2}{r^2} - \frac{16B^4}{9\ell^2(2B + 3r)^2} - \frac{16B^3r}{3\ell^2(2B + 3r)^2} - \frac{r^2}{\ell^2}. \]

The probability can be derived from the partition function. The computation leads to

\[ p = e^{-e^{-4\pi r^+}} = e^{-e^s}. \]

It is observed that this quantity does not depend neither on the scalar charge \( B \) nor the rotation parameter \( a \).

Having discussed the non charged 3D black hole solution, now we investigate the case of the charged back holes. Introducing the Maxwell gauge field, the corresponding action, as given in \([19]\), takes the following form

\[ I_{RQ} = \frac{1}{2} \int d^3x \sqrt{-g} \left[ R - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \]

For simplicity reason, the coupling constant is fixed to \( \frac{1}{8} \) and the self coupling potential is given by

\[ V(\phi) = \frac{2}{\ell^2} + \frac{1}{512} \left[ \frac{1}{\ell^2} + \frac{\beta}{B^2} + \frac{Q^2}{9B^2} \left( 1 - \frac{3}{2} \ln \left( \frac{8B}{\phi^2} \right) \right) \right] \phi^6 + \mathcal{O}(Q^2a^2\phi^8) \]

where \( Q \) is the infinitesimal electric charge. The parameter \( \beta \) takes the following form

\[ \beta = \frac{1}{3} \left( \frac{Q^2}{4} - M \right). \]
For this solution, the metric background (3) get modified and becomes as follows

\[ f(r) = 3\beta - \frac{Q^2}{4} + \left(2\beta - \frac{Q^2}{9}\right) \frac{B}{r} - Q^2 \left(\frac{1}{2} + \frac{B}{3r}\right) \ln(r) + \frac{(3r + 2B)^2 a^2}{r^4} + \frac{r^2}{r^2} + \mathcal{O}(a^2 Q^2) \]

(21)

where \( \omega \) is still given by (4). Using a similar analysis, as used in the previous section, we can obtain the mass in terms of the entropy. The calculation produces the following mass equation

\[ M_Q = \frac{32\pi^3 \ell^2 (72\pi a^2 (8\pi B + 3s)^2 + BQ^2 s^3) + 24\pi^2 Q^2 s^3 \ell^2 (8\pi B + 3s) (\log(4\pi) - \log(s)) + 9s^6}{48\pi^2 s^2 \ell^2 (8\pi B + 3s)}. \]

(22)

This thermodynamical quantity is plotted in figure 4.

![Figure 4: Plots of black hole mass for \( \ell = Q = 1 \).](image)

It is observed that the introduction of the rotating parameter leads to certain convergences in the curves. This can be seen from the presence of the \( a^2 \) term in the mass equation. Using Eq. (8) and (22), the temperature, for the charged solution, can take the following form

\[ T_Q = \frac{32\pi^3 \ell^2 (72\pi a^2 (8\pi B + 3s)^2 + BQ^2 s^3) + 24\pi^2 Q^2 s^3 \ell^2 (8\pi B + 3s) (\log(4\pi) - \log(s)) + 9s^6}{48\pi^2 s^2 \ell^2 (8\pi B + 3s)}. \]

(23)

Eq. (10) and (23) can be used to compute the heat capacity. Indeed, it becomes

\[ C_Q = \frac{a(4\pi B + s)(8\pi B + 3s) \left(9s^6 - 4\pi^2 s^3 \left(576\pi^2 a^2 (8\pi B + 3s)^2 + Q^2 s^3 (16\pi B + 9s) \right) \right)}{4\pi^2 \ell^2 (576\pi^2 a^2 (8\pi B + 3s)^2 (16\pi B + 3s) + Q^2 s^3 (576\pi^2 B^3 + 576\pi^2 B s + 240\pi B s^2 + 2s^3)) + 9s^6 (64\pi^2 B^2 + 24\pi B s + 3s^2)}. \]

(24)

From all these figures, it is observed that the charged black hole presents an instability. However, the rotation parameter and the scalar charge can be used to reduce such an instability.
To compute the corresponding statistical quantities, we keep the same analysis used for the uncharged case. Indeed, the free energy function reads

\[ F_Q = -\frac{1}{2\ell^2} \left( -\frac{48a^2B\ell^2}{r^3} - \frac{54a^2\ell^2}{r^2} + \frac{2B(B+3r)(16B^2 - 3Q^2\ell^2)}{9(2B+3r)^2} + Q^2\ell^2\log(r) + 2r^2 \right). \]

(25)

In this case, the probability is also independent of the charge of the black hole and is given by the equation [17].

In this work, we have investigated the statistical behaviors of 3D hairy black holes in the presence of a scalar field. The study has been made in terms of two relevant parameters: the rotation parameter \( a \) and \( B \) parameter related to the scalar field. In particular, we have computed various statistical quantities including the partition function for non-charged and charged black hole solutions. It has been shown, using a partition function computation, that the probability is independent of the relevant parameters.

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