Natural Wave Numbers, Natural Wave Co-numbers, and the Computation of the Primes.

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Abstract

The paper exploits an isomorphism between the natural numbers \( \mathbb{N} \) and a space \( \mathcal{U} \) of periodic sequences of the roots of unity in constructing a recursive procedure that represents the prime numbers. The \( n \)-th natural wave number \( u_n \) is the countable sequence of the \( n \)-th roots of unity having frequencies \( k/n \) for all integer phases \( k \). The space \( \mathcal{U} \) of natural wave numbers is closed under a commutative and associative binary operation \( u_m \odot u_n = u_{mn} \), termed the circular product, and is isomorphic with \( \mathbb{N} \) under their respective product operators. Functions are defined on \( \mathcal{U} \) that partition wave numbers into two complementary sequences, of which the co-number \( u^*_n \) is a function of a wave number in which zeros replace its positive roots of unity. It is shown, using circular products \( \mathcal{U}_i \) of the first \( i \) prime co-numbers, that if \( p_1, ..., p_{i+1} \) are the first \( i+1 \) prime phases, then the phases in the range \( p_{i+1} \leq k < p_{i+1}^2 \) that are associated with the non-zero terms of \( \mathcal{U}_i \) are, together with \( p_1, ..., p_i \), all of the prime phases less than \( p_{i+1}^2 \). The recursive procedure \( \mathcal{U}_{N+1} = \mathcal{U}_N \odot \mathcal{U}_{N+1} \) therefore represents prime numbers explicitly in terms of preceding prime numbers, starting with the first prime number \( p_1 = 2 \), and is shown never to terminate. When applied with all of the primes identified at the previous step, the recursive procedure identifies approximately \( 7^{2(N-1)}/(2(N-1)\ln 7) \) primes at each iteration for \( N > 1 \). When the phases of wave numbers are represented in modular arithmetic, the prime phases are representable in terms of sums of reciprocals of the initial set of prime phases and have a relation with the \( \zeta \)-function.

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1. Introduction

Elementary sieve methods for identifying prime numbers (see, for example, Friedlander and Iwaniec (2010)) may be characterized as procedures which, at every stage, select a periodic waveform propagating outward from the origin of an axis labelled with the integers. Beginning with a waveform of wavelenth two, the wavelength of the next waveform selected is equal to the integer of smallest magnitude through which previous waveforms have yet to propagate. The wavelengths selected are the prime numbers.

The procedure implies that the identification of the primes is an initial value problem that is consistent with their negative definition as natural numbers that are not divisible by preceding prime numbers. It leads to the problem of identifying the primes as numbers that do not belong to countable sets of multiples of the initial prime numbers. Hence the initial value problem involves representing countable sequences of numbers that have a wavelike structure and numbers that are defined in terms of what they are not. Although there are many solutions to this initial value problem (see, for example, Montgomery and Vaughan (2012)), it has proven difficult to represent solutions as functions of the initial prime values.

These difficulties suggest finding representations of the problem in which wave-like sequences of numbers are represented as numbers possessing an appropriate algebra and in which there are numbers that are, in some useful sense, complements of other numbers. It is shown that such a representation may be based on defining a number system whose \( n \)th element is a countable, periodic sequence of the \( n \)th roots of unity. Such numbers, termed the natural wave numbers, are shown to possess a simple algebra and to be capable of representing, in a usefully restricted sense, the complement of a wave number. Products of these complements are shown to be of use in identifying increasingly large sets of prime numbers. When the products are represented in terms of modular arithmetic, one may derive representations, formed from sets of initial prime numbers, of following sets of prime numbers.
2. The Natural Wave Numbers and their Operators

The one-to-one correspondence between the natural numbers $\mathbb{N}$ and countable, periodic sequences of the $n$th roots of unity motivates

**Definition** A natural wave number of wavelength $n$ is a sequence

\[
\mathbf{u}_n = \left( e^{\frac{2\pi i k}{n}} \mid k \in \mathbb{Z}, \ n \in \mathbb{N} \right),
\]

in which $\mathbf{u}_n$ represents the countable, periodic sequence of the $n$th roots of unity. The number $k$, corresponding to a term $e^{\frac{2\pi i k}{n}}$ of $\mathbf{u}_n$, is termed a phase of the wave number and the number $\frac{k}{n}$ is termed a frequency.

It is convenient to adopt the notation

\[
m\mathbf{u}_n = \left( e^{\frac{2\pi i k}{n}} \mid 1 \leq k \leq mn, \ n \in \mathbb{N} \right)
\]

for the $nm$ elements of a sequence for which $1 \leq k \leq mn$. In particular, $\mathbf{1}_n$ is termed the principle part of the sequence $\mathbf{u}_n$ and $k = n$ is termed its principle phase. The principal part of $\mathbf{u}_n$ may be visualized as a unit circle centered on the origin of the complex plane, with a circumference that is marked, in an anticlockwise direction, with $n$ equi-spaced points corresponding to the first $n$ of the $n$th roots of unity with positive phases. The space of wave numbers is denoted $\mathbb{U}$ and the wave numbers may be linearly ordered in terms of $n \in \mathbb{N}$.

2.1. Operators and Functions on the Space of Natural Wave Numbers

Operators and functions defined on wave numbers are assumed to act in an element-wise manner on the exponential terms defining a wave number.

2.1.1. The Unary Translation Operator

A unary operator on $\mathbf{u}_n \in \mathbb{U}$ circulates the roots of unity around the circumference of the unit circle. In particular, the operator

\[
R_j(\mathbf{u}_n) = \mathbf{u}_n^j = \left( e^{\frac{2\pi i j k}{n}} \mid j, k \in \mathbb{Z} \right)
\]

translates the roots of unity around the circumference by raising them to a power $j \in \mathbb{Z}$.
2.1.2. The Circular Product Binary Operator

The action of the circular product operator is specified in

**Definition** The application of the circular product operator \(\odot\) to wave numbers \(u_m\) and \(u_n\) results in a sequence whose principal part is

\[
1 (u_m \odot u_n) = (n u_m \cdot m u_n)^{1/(n+m)}
\]

in which the operator \(\cdot\) represents the arithmetic, element-wise product operator and the exponent \(1/(n+m)\) represents an element-wise power operator.

In the case of \(u_2\) and \(u_3\), for example,

\[
1 (u_2 \odot u_3) = (3 u_2 \cdot 2 u_3)^{1/5} = u_6.
\]

and more generally one has

**Theorem 2.1.** The space \(U\) of natural wave numbers is closed under the direct circular product \(\odot\).

**Proof**

\[
(n u_m \cdot m u_n)^{1/(n+m)} = 1 ((e^{2\pi i k/m} | k \in \mathbb{Z}) \cdot (e^{2\pi i l/n} | l \in \mathbb{Z}))^{1/(n+m)}
\]

\[
= 1 (e^{2\pi i k/m} \cdot e^{2\pi i l/n} | k \in \mathbb{Z})^{1/(n+m)}
\]

\[
= 1 (e^{2\pi i (kn+m) / mn} | k \in \mathbb{Z})^{1/(n+m)}
\]

\[
= 1 (e^{2\pi i k / mn} | k = k \in \mathbb{Z})
\]

\[
= 1 u_{mn}
\]

and it follows that \(u_m \odot u_n = u_{mn}\).

The circular product has an interpretation in terms of the rotations of three unit circles whose centers coincide with the origin of the complex plane. The circles may be viewed as rotating in a clockwise direction at rates that are inversely proportional, with the same constant of proportionality, to the numbers \(n, m, mn\) of their roots of unity. They commence their rotation with each of their first roots of unity \((e^{2\pi i k/m}, e^{2\pi i l/n}, e^{2\pi i k / mn})\) aligned at a point \(p\) on the circumference of a fourth coincident unit circle that does not rotate. As a result of their speeds of rotation, each of their successive roots of unity
coincide at the point \( p \), because the time to rotate from one root to the next is the same for all three circles.

The circular product is commutative and associative as a result of its element-wise definition and the commutativity and associativity of the arithmetic addition and multiplication operators. One may therefore generalize its definition to products of \( N \) wave numbers

\[
\bigotimes_{i=1}^{N} u_{n_i} = \left( \frac{p}{n_1} u_{n_1} \cdot \frac{p}{n_2} u_{n_2} \cdot \ldots \cdot \frac{p}{n_N} u_{n_N} \right) \sum_{i=1}^{N} \left( \frac{p}{n_i} \right) = 1 u_{P} \tag{7}
\]

in which \( P = \prod_{i=1}^{N} n_i \).

It is clear that an invertible mapping \( M : \mathbb{N} \to \mathbb{U} \) exists between the natural numbers and the natural wave numbers. Furthermore, one has

**Theorem 2.2.** The space \( \mathbb{N} \) of natural numbers and the space \( \mathbb{U} \) of natural wave numbers are isomorphic under their product operators \( \cdot \) and \( \odot \).

**Proof** Let \( M : \mathbb{N} \to \mathbb{U} \) be the invertible mapping from \( \mathbb{N} \) onto \( \mathbb{U} \), then

\[
M(m) \odot M(n) = u_m \odot u_n = u_{n \cdot m} = M(n \cdot m).
\]

2.2. Functions of a Wave Number

It is useful to specify functions of a wave number in terms of

**Definition** A function of a wave number \( u_n \in \mathbb{U} \) is a periodic sequence

\[
f(u_n) = \left( f\left( e^{2\pi i \frac{k}{n}} \right) \mid k \in \mathbb{Z}, \ n \in \mathbb{N} \right) \tag{8}
\]

defined by the element-wise application of a function \( f : \mathbb{C} \to \mathbb{C} \).

A function may be viewed as a one-to-one mapping between the phase \( k \) of the wave number and the value \( f\left( e^{2\pi i \frac{k}{n}} \right) \). Since the arithmetic product and addition operators, \( \cdot \) and \( + \), are applicable to pairs of function values, the circular product operator is applicable to functions of wave numbers.

2.2.1. Decomposition Functions and the Co-numbers

It is useful to define functions of a natural wave number \( u_n \) that, together, form a partition of the number. Of particular value are the functions \( \hat{u}_n \) and \( \check{u}_n \) having
Definition The principal parts of the \( \star \) and \( \circ \) functions are
\[
\begin{align*}
\dot{1}u_n & = \left( e^{2\pi i \frac{1}{n}}, e^{2\pi i \frac{2}{n}}, \ldots, e^{2\pi i \frac{(n-1)}{n}}, 0 \right) \\
\ddot{1}u_n & = \left( 0, 0, \ldots, 0, e^{2\pi i \frac{n}{n}} \right) = \left( 0, 0, \ldots, 0, 1 \right),
\end{align*}
\]
which may be viewed, respectively, as mapping the positive real elements of \( u_n \) into zero and the complex and negative real values of \( u_n \) into zero.

The sum of the functions is
\[
\ddot{u}_n + \dot{u}_n = u_n
\]
in which \( + \) is the arithmetic addition operator applied in an element-wise manner. Since the principal part \( \dot{1}u_n = (0, \ldots, 0, 1) \) may be viewed as a minimal representation of \( u_n \), and since \( \ddot{u}_n \circ \dot{u}_n = 0 \), it is reasonable to view \( \ddot{u}_n \) as a complement of \( u_n \) and to make

Definition The function
\[
\dddot{u}_n = \left( (1 - \delta_{k,jn})e^{2\pi i \frac{k}{n}} \mid j \in \mathbb{Z}, k \in \mathbb{Z}, n \in \mathbb{N} \right),
\]
in which \( \delta_{k,jn} \) is the Kronecker delta, is termed the co-number of \( u_n \).

The partition (10) leads to partitions of circular products of wave numbers
\[
\begin{align*}
1(u_m \odot u_n) & = \left( n \left( \ddot{u}_m \circ \dddot{u}_m \right) \cdot m \left( \dddot{u}_n \circ \dot{u}_n \right) \right) \frac{1}{n+m} \\
& = \left( \ddot{u}_m \odot \dddot{u}_n \right) + 1\left( \dddot{u}_m \circ \dot{u}_n \right) \\
& + 1\left( \dddot{u}_m \odot \ddot{u}_n \right) + 1\left( \ddot{u}_m \circ \dot{u}_n \right)
\end{align*}
\]
in which the term \( 1\left( \ddot{u}_m \circ \dddot{u}_n \right) \) is of particular interest since it involves all phases of \( u_m \) and \( u_n \) except \( k = m \) and \( k = n \) as, for example, in the case
\[
1\left( \dddot{u}_2 \odot \dddot{u}_3 \right) = \left( e^{2\pi i \frac{1}{6}}, 0, 0, e^{2\pi i \frac{3}{6}}, 0 \right)
\]
which, one notes, is not equal to \( \dot{u}_6 = \left( e^{2\pi i \frac{1}{6}}, e^{2\pi i \frac{2}{6}}, e^{2\pi i \frac{3}{6}}, e^{2\pi i \frac{1}{6}}, e^{2\pi i \frac{2}{6}}, 0 \right) \).

3. The Prime Natural Wave Numbers

One may formulate the primality of natural wave numbers in terms of
Definition A natural wave number \( u_n \) is prime iff the number \( n \) representing its wavelength and principal phase is a prime number.

Hence there is a one-to-one correspondence between the prime wave numbers and the prime numbers. An analogue of the Fundamental Theorem of Arithmetic for natural wave numbers is stated in

**Theorem 3.1.** If the natural number \( n = p_1^{\alpha_1} \cdots p_N^{\alpha_N} \) for some \( N \), then the natural wave number

\[
u_n = \nu_{p_1 p_2 \cdots p_N} = N \odot \left( \bigotimes_{i=1}^{N} \nu_{p_i}^{\alpha_i} \right) = N \odot \left( \bigotimes_{j=1}^{N} \nu_{p_j} \right)
\]

(13)

**Proof** By the definition of the circular product.

4. Co-Numbers and the Recusive Identification and Representation of the Primes

The cumulative circular products of the \( N-1 \) natural wave co-numbers \( \ast u_2, \ldots, \ast u_N \) are of value in identifying the prime numbers. The circular product

\[
\ast U_N(k) = \left( \bigotimes_{n=2}^{N} \ast u_n \right)(k)
\]

is a function of the phases \( k \) of the circular product \( U_N(k) = \bigotimes_{n=2}^{N} u_n(k) \) of the first \( N-1 \) wave numbers. The principal part of the domain of the function may be visualized as the unit circle in the complex plane defined by the \( P_N = \prod_{n=2}^{N} n \) roots of unity, with each phase \( k \) corresponding to an exponential term \( e^{2\pi ik/P_N} \) of the circular product \( U_N(k) \).

It follows immediately from the definitions of the natural wave co-numbers \( \ast u_2, \ldots, \ast u_N \), and of their circular product (14), that one has

**Theorem 4.1.** The function \( \ast U_N(k) = \left( \bigotimes_{n=2}^{N} \ast u_n \right)(k) = 0 \) iff the phase \( k \) is a multiple of any of the phases \( k = 2, \ldots, N \).

**Proof** The value of the function for any phase \( k \) is an arithmetic product of exponential terms and zeros. Such a product is zero iff it contains at least one zero which occurs iff \( k \) is a multiple of one of the first \( N \) phases.
The function values are therefore non-zero for any phase \( k \) iff it is prime relative to the phases \( k = 2, \ldots, N \). In the case of \( N = 3 \), for example, the domain of the function is \( U_3(k) = u_6(k) \) and the function values for the phases of the first six wavelengths of the product are

\[
U_3^*(k) = 6(u_2 \odot u_3^*)(k) = 6((u_2 \cdot u_3^*)^1)(k)
\]

\[
= (e^{2\pi i \frac{1}{6}}, 0, 0, e^{2\pi i \frac{2}{6}}, 0, e^{2\pi i \frac{3}{6}}, 0, 0, e^{2\pi i \frac{4}{6}}, 0, e^{2\pi i \frac{5}{6}}, 0, e^{2\pi i \frac{6}{6}}, 0)
\]

in which the first non-prime phase is \( 5^2 = 25 \) which, one notes, is the square of the first prime number following 2 and 3.

### 4.1. The Circular Product of Prime Co-numbers

In identifying the phases associated with the zeros and non-zeros of \( \hat{U}_N(k) \), there is no loss of generality in using circular products of the prime wave co-numbers instead of the natural wave co-numbers as long as they are defined over consistent ranges. Denoting by \( \lceil N \rceil_p \) the largest prime number less than or equal to \( N \) and by \( \pi_p = \pi(\lceil N \rceil_p) \) the value of the prime counting function, the circular product of prime wave co-numbers

\[
\hat{U}_{\pi_p}(k) = \bigodot_{i=1}^{\pi_p} u_{p_i}(k)
\]

is equivalent to \( \hat{U}_N(k) = \bigodot_{n=2}^{N} u_n(k) \) in terms of the phases of its zero values.

**Lemma 4.2.** The phases associated with the zeros of the circular product of prime wave co-numbers \( \bigodot_{i=1}^{\pi_p} u_{p_i}(k) \) and the phases associated with the zeros of the product \( \bigodot_{n=2}^{N} u_n(k) \) of natural wave co-numbers are identical.

**Proof** By the definition of the circular product of natural wave co-numbers \( \bigodot_{n=2}^{N} u_n(k) \), the co-number for any multiple of a prime phase is preceded by the co-number for the prime phase. Furthermore the phases of the zeros of a co-number whose principal phase is a multiple of a preceding prime phase form a subset of the phases of the zeros of the preceding prime co-number as, for example, in the case \( 2u_2 = (e^{2\pi i \frac{1}{2}}, 0, e^{2\pi i \frac{1}{2}}, 0) \) which precedes...
\[ \mathbf{U}_4 = (e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{2}{4}}, e^{2\pi i \frac{3}{4}}, 0). \] Hence the occurrence of any non-prime co-number in \( \bigcirc \mathbf{u}_n(k) \) following the occurrence of a prime factor co-number has no effect on which phases are associated with zeros.

It follows that one may employ circular products of prime numbers in identifying both the prime wave numbers and the prime natural numbers. Before stating and proving a theorem concerning their identification for general values of \( N \), it is helpful to examine special cases of \( N \). Employing the notation \( \mathbf{U}_N(k) = \bigcirc_{n=2}^{N} \mathbf{u}_{p_n}(k) \) for circular products of prime co-numbers, one obtains for the first six wavelengths of the product for \( N = 1 \)

\[ \mathbf{U}_1(k) = \mathbf{u}_2(k) = (e^{2\pi i \frac{1}{2}}, 0, e^{2\pi i \frac{3}{2}}, 0, e^{2\pi i \frac{5}{2}}, 0, e^{2\pi i \frac{7}{2}}, 0, e^{2\pi i \frac{9}{2}}, 0), \]

whose function values may be viewed as alternately occurring at the two roots of unity \((-1, 0)\) and \((1, 0)\) on the product circle of \( \mathbf{u}_2 \) in the complex plane. The values include all prime phases \( 3 \leq k < 3^2 \), with the first non-prime phase occurring at \( k = 3^2 \).

The function values for the case \( N = 2 \), corresponding to \( \mathbf{U}_2(k) = \mathbf{u}_2 \bigcirc \mathbf{u}_3 \), are displayed in equation (15). They include all prime phases \( 5 \leq k < 5^2 \), with the first non-prime phase occurring at \( k = 5^2 \). It may be noted, however, that the identification of three prime phases in addition to the prime phase 2 for the case \( N = 1 \) suggests that one might have proceeded directly to the case \( N = 3 \) and to the circular product \( \mathbf{U}_3(k) = \mathbf{u}_2 \bigcirc \mathbf{u}_3 \bigcirc \mathbf{u}_5 \) while using the knowledge that the square of the next prime phase is \( 7^2 = 49 \) to limit the sequence to phases that are prime. It is straightforward to show that this leads to the identification of the prime phases \( 7 \leq k < 7^2 \), with \( k = 49 \) being the first non-prime phase.

4.2. Recursive Identification of Increasingly-Large Sets of Primes

The preceding special cases suggest

**Theorem 4.3.** If \( p_1, ..., p_{N+1} \) are the first \( N + 1 \) prime phases, then the phases in the range \( p_{N+1} \leq k < p_{N+1}^2 \) that are associated with the non-zero terms of the circular product of the first \( N \) prime co-numbers

\[ \mathbf{U}_{p_N} = \bigcirc_{n=1}^{N} \mathbf{u}_{p_n}, \]

9
are, together with \( p_1, ..., p_N \), all of the prime phases less than \( p^2_{N+1} \).

**Proof** Let \( \hat{U}_{P_N}(k) \) be the circular product of the first \( N \) prime wave numbers and let \( \hat{\mathbf{u}}_{P_N}(k) = \bigcirc_{j=1}^{N} \hat{u}_{p_j}(k) \) be function values defined on the phases of \( U_{P_N}(k) \). By Lemma 4.1, \( \hat{\mathbf{u}}_{P_N}(k) = 0 \) iff the phase \( k \) is a multiple of any of the prime numbers \( p_1, ..., p_N \). The phase \( k = p^2_{N+1} \) is not prime and is associated with a non-zero value since it does not satisfy this condition. Any phase smaller than \( k = p^2_{N+1} \) may be represented by the Fundamental Theorem of Arithmetic

\[
k = p_1^{a_1} \cdot p_2^{a_2} \cdots p_N^{a_N} \cdot p_{N+1}^{a_{N+1}} \cdots (\lfloor p^2_{N+1} \rfloor_p)^{a_m}
\]

for some \( m \), in which \( \lfloor n \rfloor_p \) represents the largest prime phase that is less than \( n \). Since the function values of phases that contain any prime factors less than \( p_{N+1} \) are zero and since permissible phases are less than \( p^2_{N+1} \), it follows that phases associated with non-zero function values must have representations

\[
k = p_{N+1}^{s_1} \cdots \lfloor p^2_{N+1} \rfloor_p^{s_m}
\]

for some \( m \) in which at most one of the \( s_i \)'s can take on the value one. It follows that any non-zero phase that is less than \( p^2_{m+1} \) must be a prime number in the range \( p_{N+1}, ..., \lfloor p^2_{N+1} \rfloor_p \).

Theorem 5.3 implies a recursive procedure for determining the primes, since

\[
\hat{U}_{P_{N+1}}(k) = \hat{U}_{P_N}(k) \odot \hat{\mathbf{u}}_{P_{N+1}}(k),
\]

which is defined entirely in terms of the initial condition \( \hat{U}_{1}(k) = \hat{\mathbf{u}}_2 \). Furthermore, this procedure leads to the identification of every prime by

**Corollary 4.4.** The recursive procedure that is defined by applying Theorem 5.3 sequentially for \( N = 1, 2, 3, ... \) does not terminate.

**Proof** Theorem 5.3 implies that in order to identify prime phases at step \( N + 1 \), it is necessary at step \( N \) to identify \( p_{N+1} \) in order to construct \( \hat{U}_{N_{P+1}} \) and to identify \( p_{N+2} \) in order to ensure the primality of the numbers generated by \( \hat{U}_{N_{P+1}} \). As shown above, this is the case for the first three steps \( N \leq 3 \). Assume it to be true for the \( N \)th step and that \( p_{N+1} \) and \( p_{N+2} \) are known.
Theorem 5.3 also implies that all prime phases $k < p_{N+2}^2$ are identified by $\hat{U}_{Np+1}$, and hence $p_{N+2}$ is identified at the $(N+1)^{st}$ iteration. By Bertrand’s Theorem, $p_{N+3} \leq 2p_{N+2}$, and since $2p_{N+2} \leq p_{N+2}^2$ for $p_{N+2} > 2$, $p_{N+3}$ is determined.

Theorem 5.3 implies that one may identify increasingly-large sets of maximum size of prime numbers in an iterative process by which, at the $N^{th}$ step, one employs the maximum number of correct primes determined in the previous steps to compute the following largest, error-free set of primes. In particular, one has

**Corollary 4.5.** The largest prime phase in the set of prime phases that may be correctly identified at each application of Theorem 5.3 is

\[
\lfloor 7 \rfloor_p, \lfloor 7^2 \rfloor_p, \lfloor \lfloor 7^2 \rfloor_p \rfloor_p, \lfloor \lfloor \lfloor 7^2 \rfloor_p \rfloor_p \rfloor_p, \ldots \tag{22}
\]

**Proof** In the first step at $N = 1$, $\hat{U}_1 = \hat{u}_2$ and it follows from equation (21) that the largest prime phase found is 7. From Theorem 5.3 it follows that the largest correctly identifiable prime phase at each iteration for $N \geq 2$ is the largest prime phase that is less than the square of the next known prime phase. Hence for $N = 2$, the next prime phase is defined to be $7 = \lfloor 7 \rfloor_p$, and largest prime phase that is correctly identifiable is $\lfloor 7^2 \rfloor_p$, which is the next prime phase at the $N = 3^{rd}$ iteration. Hence the largest prime phase identifiable at $N = 3$ is $\lfloor \lfloor 7^2 \rfloor_p \rfloor_p$, which is the next prime phase at the step $N = 4$. The same argument applies at each iteration $N \geq 4$.

The length of the domain of phases over which all prime phases are correctly inferred therefore increases at each iteration as approximately the square of the length of the previous domain, and is representable as a function of 7 at each iteration. Recalling that 7 is the largest of the prime numbers derived at the first iteration and that the derivation of the prime numbers is an initial value problem, this reflects the dependence of the solution on the initial conditions.

When applied to the results of Corollary 5.5, the Gauss/Legendre approximation $\pi(N) \approx N/\ln(N)$ to the prime counting function, together with the approximation $\lfloor N \rfloor_p \approx N$, implies

**Corollary 4.6.** The maximum number of prime phases identifiable at each iteration of Theorem 5.3 is approximately $7^{2(N-1)}/(2(N-1)\ln7)$ for $N > 1$. 

11
There are computationally more-efficient procedures than the recursive application of $\mathbf{U}_{N_p}(k)$ for identifying prime phases. One has, for example,

**Corollary 4.7.** If $p_1, \ldots, p_{N+1}$ are the first $N + 1$ prime phases then $p_N < k < p_{N+1}^2$ is prime iff $\sum_{j=1}^{N} \mathbf{u}(j)(k) = 0$.

**Proof** By Theorem 5.3, the definition (2) of $\mathbf{u}_p$, and the definition of the arithmetic plus operator.

5. Representing Prime Phases as Functions of the Initial Values

While Theorem 5.3 identifies the prime phases associated with the circular product $\mathbf{U}_{N_p}(k)$ of prime co-numbers, the representation of the phases does not explicitly involve the initial set of prime phases. A more explicit representation may be obtained by employing modular representations of the phases of the wave numbers.

5.1. The Modular Representation Functions

It is useful to define the modular phase function $m_{\mathbf{u}_n}$ of a wave number $\mathbf{u}_n$ as a function that maps the frequency $k/n$ of non-zero terms to $(k \mod n)/n$:

**Definition** The principal sequence of values of the modular phase function $m_{\mathbf{u}_n}$ of a wave number $\mathbf{u}_n$, represented as frequencies of exponential terms, is

$$m_{\mathbf{u}_n} \sim \left( \frac{1 \mod n}{n}, \frac{2 \mod n}{n}, \ldots, \frac{n \mod n}{n} \right) = \left( \frac{1}{n}, \frac{2}{n}, \ldots, \frac{0}{n} \right)$$

in which $\sim$ is to be interpreted as "is represented by".

The principal sequence of function values for the modular co-number is represented in terms of frequencies by $1^m_{\mathbf{u}_n} \sim (1/n, \ldots, (n-1)/n, \square)$, in which $\square$ represents that the corresponding function value is zero rather than an exponential term.

The definition of the circular product (1) must be modified for application to the values of modular phase functions of natural wave numbers to
Definition  The modular circular product operation \( \mathbf{u}_j \odot \mathbf{u}_k \) results in a sequence in which the frequency of the \( k \)th term is
\[
\left( \frac{P}{n} \left( k \mod n \right) + \frac{P}{m} \left( k \mod m \right) \right) \mod P.
\]
(24)
in which \( P = mn \)

For co-numbers \( \mathbf{u}_2, \mathbf{u}_3 \) in modular form, for example
\[
\begin{align*}
1 \left( \mathbf{u}_2 \odot \mathbf{u}_3 \right) & \sim (1/2, 1/2, 1/2, 1/2, \square) \odot (1/3, 2/3, 1/3, 2/3, \square) \\
& \equiv ((5 \mod 6)/6, \square, \square, (7 \mod 6)/6, \square) \\
& \equiv ((5/6, \square, \square, \square, 1/6, \square)
\end{align*}
\]
(25)

5.2. Modular Representations of the Prime Phases

The modular wave co-numbers and modular circular product may be used in defining the circular product
\[
\mathbf{u}_N^m(k) = \odot \mathbf{u}_{p_n}, \text{ for } N = 1, 2, 3, ...
\]
(26)
which may be employed, rather than \( \mathbf{U}_N^*(k) \), in deriving representations of prime phases in terms of the initial prime phases from which they are derived. In the case for \( N = 3 \) the function values for each \( k \) are determined by
\[
\mathbf{u}_3^m(k) = \frac{p_2p_3(k \mod p_1) + p_1p_3(k \mod p_2) + p_1p_2(k \mod p_3)) \mod p_1p_2p_3}{p_1p_2p_3}
\]
and are illustrated for \( k = 1, 30 \) in Table I, from which it is clear that equation (27) correctly represents the value and phase of each of the prime numbers in this range in terms of the prime numbers (2, 3, 5) from which they are derived. It is the case that they are correctly represented for \( k < 7^2 \).

Analogous computations of \( \mathbf{U}_4^*(k) \) and \( \mathbf{U}_5^*(k) \) lead to the identification of all prime phases less than \( p_{N+1}^2 \) on the product circles, although they do not occur in ascending order. Hence one has the

**Theorem 9** If \( p_1, \ldots, p_{N+1} \) are the first \( N+1 \) prime phases and \( P_N = \prod_{j=1}^{N} p_j \), then the set of phases \( k = 1, P_N \) that satisfy the conditions
\[
\frac{k}{P_N} = \frac{1}{P_N} \left( \frac{P_N \left( k \mod p_1 \right)}{p_1} + \frac{k \mod p_2}{p_2} + \ldots + \frac{k \mod p_N}{p_N} \right) \mod P_N,
\]
(28)

\( k < p_{N+1}^2 \)
Table 1: Results for the computation of the circular products of the first three modular prime co-numbers ($\ast m^2$, $\ast m^3$, $\ast m^5$). The symbol $\Box$ represents a frequency corresponding to a zero occurring in the principal phase of the prime wave co-numbers. The columns with only non-zero frequencies represent prime wave numbers whose phases are derived from the addition of the frequencies.

| $\ast m$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\ast m^2$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ |
| $\ast m^3$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ |
| $\ast m^5$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{4}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{4}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{4}{7}$ |
| $\ast m^N(k)$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ |

are all of the prime phases $p_{N+1} \leq k < p_{N+1}^2$.

One notes possible connections between the expressions representing the prime phases and the zeta function $1/\zeta(1)$ for $s = 1$. For example, since the total number of phases satisfying the first condition of Theorem 6.1 is $\prod_{j=1}^N (p_j - 1)$, the proportion of phases that must be considered as possible prime numbers is

$$\frac{\prod_{n=1}^N \left( \frac{p_n - 1}{p_n} \right)}{1/\zeta_N(1)} = 1/\zeta_N(1) \quad (29)$$

in which $1/\zeta_N(1)$ represents the first $N$ terms in the prime representation of the zeta function.

6. Conclusions

The natural wave numbers $u_n = (e^{2\pi i(k/n)} \mid k \in \mathbb{Z}, n \in \mathbb{N})$ and their complements, the co-numbers $\tilde{u}_n = ((1 - \delta_{k,j})e^{2\pi i(k/n)} \mid j \in \mathbb{Z}, k \in \mathbb{Z}, n \in \mathbb{N})$, provide a useful representation of numbers that may be used for investigating the distribution of the primes. In particular, they facilitate the identification
of sets of numbers that are defined in terms of not belonging to countable sets of multiples of numbers that are specified in an initial set of numbers.

The application of wave co-numbers to the identification and representation of prime numbers leads to error-free predictions of increasingly large sets of prime numbers that extend to all finite prime numbers. These results, as presented in Theorem 5.3 and its corollaries, are conservative in predicting the next sequence of prime numbers whose end-point is determined by where the first error occurs for an initial sequence of primes. Sequences of primes that follow the square of the next known prime have endpoints that may be determined by more refined analyses.

The application also leads, when the wave numbers are represented in modular form, to representations of prime numbers in terms of partial sums of the reciprocals of the prime numbers that form the data for the initial value problem of identifying the primes.

Wavelike patterns in representations of the primes that have been observed by many investigators, and recently for example by Wang (2021, 2022), follow naturally from a representation of numbers in terms of sequences of the roots of unity. It is reasonable to assume that the natural wave numbers have applications in the physical sciences, since a physical interpretation of the circular product $u_m \odot u_n = u_{mn}$ is of three unit circles rotating at rates that are inversely proportional to the number of their roots of unity.
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