Charged rotating black holes in dilaton gravity

A. Sheykhi 1,2 * and N. Riazi 1 †

1 Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran
2 Department of Physics, Shahid Bahonar University, Kerman, Iran

We consider charged black holes with curved horizons, in five dimensional dilaton gravity in the presence of Liouville-type potential for the dilaton field. We show how, by solving a pair of coupled differential equations, infinitesimally small angular momentum can be added to these static solutions to obtain charged rotating dilaton black hole solutions. In the absence of dilaton field, the non-rotating version of the solution reduces to the five dimensional Reissner-Nordström black hole, and the rotating version reproduces the five dimensional Kerr-Newman modification thereof for small rotation parameter. We also compute the angular momentum and the angular velocity of the rotating black holes which appear at the first order.

I. INTRODUCTION

Inspired by the string theory that gravity is not given by the Einstein action, at least at sufficiently high energies, a lot of investigations have been done in the literature in recent years. The low-energy limit of the string theory leads to the Einstein gravity, coupled nonminimally to a scalar dilaton field [1]. When a dilaton is coupled to Einstein-Maxwell theory, it has profound consequences for the black hole solutions. This fact may be seen in the case of rotating Einstein-Maxwell-dilaton (EMD) black holes of Kaluza-Klein theory with coupling constant \( \alpha = \sqrt{3} \) which does not possess the gyromagnetic ratio \( g = 2 \) of Kerr-Newman black hole [2, 3, 4]. Thus it is worth finding black hole solutions of EMD gravity for an arbitrary value of dilaton coupling constant and investigate how the properties of black holes are modified when a dilaton is present.

Exact charged dilaton black hole solutions in the absence of dilaton potential have been constructed by many authors [5, 6, 7, 8, 9, 10, 11]. The dilaton changes the causal structure of the black hole and leads to curvature singularities at finite radii. These black holes are asymptotically flat. In recent years, non-asymptotically flat black hole spacetimes are attracting much interest. A motivation to investigate non-asymptotically flat, nonasymptotically AdS solutions of Einstein gravity is that these might lead to possible extensions of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [12]. Another motivation is that such solutions may be used to extend the range of validity of methods and tools originally developed for, and tested in the case of, asymptotically flat or asymptotically AdS black holes. Black hole spacetimes which are neither asymptotically flat nor (anti)-de Sitter (A)dS have been found and investigated by many authors. The uncharged solutions have been found in [13, 14, 15], while the charged solutions have been considered in [16, 17, 18]. In the presence of Liouville-type potential, static charged solutions of EMD gravity have been discovered with positive, zero or negative constant curvature horizons [19, 20, 21, 22, 23, 24, 25, 26]. Recently, the properties of these black hole solutions which are neither asymptotically flat nor (A)dS have been disclosed in [27, 28].

These exact solutions [6, 7, 8, 9, 10, 11] and [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] are all static. Charged rotating dilaton black holes with curved horizon have not been constructed in four or higher dimensions for an arbitrary coupling constant and arbitrary rotation parameter. Indeed, exact magnetic rotating solutions have been considered in three dimensions [29], while exact rotating solutions of EMD gravity have been obtained only for some limited values of the coupling constant [30, 31, 32, 33]. For general dilaton coupling constant, the properties of charged dilaton black holes only in four dimensions with small angular momentum [4, 34, 35, 36] or small charge [37] have been investigated. When the horizons are flat, magnetic and electric rotating solutions in four-dimensional EMD gravity have also been constructed in [38] and [39], respectively. Recently, this solutions have been generalized to the \( (n + 1) \)-dimensional EMD gravity [40, 41]. These solutions [40, 41] are not black holes and describe charged rotating black branes with flat horizons. Until now, charged rotating dilaton black holes solutions with curved horizons for an arbitrary value of dilaton coupling constant in more than four dimensions have not been constructed. The motivation for studying higher dimensional black holes comes from developments in string/M-theory, which is believed to be the most consistent approach to quantum theory of gravity in higher dimensions. It was argued that black holes may play a crucial role in the analysis of dynamics in higher dimensions as well as in the compactification mechanisms. In particular, to test novel predictions of string/M-theory microscopic black holes may serve as good theoretical laboratories. It has been thought that the statistical-mechanical calculation of the Bekenstein-Hawking entropy for a class of supersymmetric black holes in five dimensions is one of

*sheykhi@mail.uk.ac.ir
†riazi@physics.susc.ac.ir
the remarkable results in string theory [42, 43]. Another motivation on studying higher dimensional black holes originates from the braneworld scenarios, as a new fundamental scale of quantum gravity. An interesting consequence of these models is the possibility of mini black hole production at future colliders [44]. These serve as our main motivation to explore of the effects of dilaton field on the properties of charged rotating black holes in higher dimension. In this regard, as a new step to shed some light on this issue for further investigation, we report a new class of solution of the Einstein-Maxwell gravity coupled to a dilaton field which describes an electrically charged, slowly rotating black hole with curved horizon in five dimensions with arbitrary value of coupling constant $\alpha$. We shall also investigate the effects of dilaton field as well as rotation parameter on the physical quantities such as temperature, entropy, angular momentum and the angular velocity of these rotating black holes.

II. FIELD EQUATIONS AND SOLUTIONS

Our starting point is the following action

$$ S = -\frac{1}{16\pi} \int_{\mathcal{M}} d^n x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-2} (\nabla \Phi)^2 - V(\Phi) - e^{-4\alpha \Phi/(n-2)} F_{\mu \nu} F^{\mu \nu} \right) - \frac{1}{8\pi} \int_{\partial \mathcal{M}} d^{n-1} x \sqrt{-\gamma} \Theta(\gamma), $$

where $\mathcal{R}$ is the Ricci scalar curvature, $\Phi$ is the dilaton field and $V(\Phi)$ is a potential for $\Phi$. $\alpha$ is a constant determining the strength of coupling of the scalar and electromagnetic field, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $A_\mu$ is the electromagnetic potential. The last term in Eq. (1) is the Gibbons-Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold $\mathcal{M}$ has metric $g_{\mu \nu}$ and covariant derivative $\nabla_\mu$. $\Theta$ is the trace of the extrinsic curvature $\Theta^{ab}$ of any boundary(ies) $\partial \mathcal{M}$ of the manifold $\mathcal{M}$, with induced metric(s) $\gamma_{ab}$. In this paper, we consider the action (1) with a Liouville type potential,

$$ V(\Phi) = 2\Lambda e^{2\beta \Phi}, $$

where $\Lambda$ and $\beta$ are arbitrary constants. One may refer to $\Lambda$ as the cosmological constant, since in the absence of the dilaton field ($\Phi = 0$) the action (1) reduces to the action of Einstein-Maxwell gravity with cosmological constant. The equations of motion can be obtained by varying the action (1) with respect to the gravitational field $g_{\mu \nu}$, the dilaton field $\Phi$ and the gauge field $A_\mu$ which yields the following field equations

$$ \mathcal{R}_{\mu \nu} = \frac{4}{n-2} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu \nu} V(\Phi) \right) + 2e^{-4\alpha \Phi/(n-2)} \left( F_{\eta \mu} F^{\eta \nu} - \frac{g_{\mu \nu}}{2(n-2)} F_{\lambda \eta} F^{\lambda \eta} \right), $$

$$ \nabla^2 \Phi = \frac{n-2}{8} \frac{\partial V}{\partial \Phi} \alpha - \frac{\alpha}{2} e^{-4\alpha \Phi/(n-2)} F_{\lambda \eta} F^{\lambda \eta}, $$

$$ \nabla_\mu \left( e^{-4\alpha \Phi/(n-2)} F^{\mu \nu} \right) = 0. $$

We wish to find five dimensional rotating solutions of the above field equations, thus we set $n = 5$. To this end, we first study the non-rotating black hole for arbitrary $\alpha$, then, we consider the effect of adding a small amount of rotation parameter $a$ to the black hole. We will discard any terms involving $a^2$ or higher power in $a$. For infinitesimal rotation, we can solve Eqs. (2)-(5) to first order in the angular momentum parameter $a$. This is because most of the metric components depend only on $a^2$. In fact, many of the interesting physical quantities also depend only on $a^2$, however we can still extract some useful information from the first-order solutions. Inspection of the five dimensional Kerr-Newman solutions shows that the only term in the metric changes to $O(a)$ is $g_{\phi \phi}$. Similarly, the dilaton field does not change to $O(a)$ and $A_\phi$ is the only component of the vector potential that changes. Therefore, for infinitesimal angular momentum up to $O(a)$, we can take the following form of the metric

$$ ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} - 2af(r) \sin^2 \theta dtd\phi + R^2(\gamma + \sin^2 \theta d\theta^2 + \cos^2 \theta d\psi^2). $$

The unknown functions $U(r)$, $R(r)$ and $f(r)$ should be determined. In the particular case $a = 0$, this metric reduces to the static and spherically symmetric cases. For small $a$, we can expect to have solutions with $U(r)$ still a function of $r$ alone.

The $t$ component of the Maxwell equations can be integrated immediately to give

$$ F_{tr} = \frac{q e^{4\alpha \Phi/3}}{R^3(r)}, $$

where $q$ is an integration constant related to the electric charge of the solutions. Defining the electric charge via

$$ Q = \frac{1}{4\pi} \int_{S^3} e^{-4\alpha \Phi/3} F d\Omega, $$

where * is the Hodge dual and $S^3$ is any 3-sphere defined at spatial infinity, with its volume element denoted by $d\Omega$. Then the electric charge of the black hole will be

$$ Q = \frac{q w_3}{4\pi}, $$

where $w_3$ represents the volume of the unit 3-sphere. In general, in the presence of rotation, there is also a vector potential in the form

$$ A_\phi = aqh(r) \sin^2 \theta. $$
It is worth noting that for infinitesimal rotation parameter, the electric field does not change from the static case.

With the metric, and the Maxwell fields, the field equations reduce to the following system of coupled ordinary differential equations

\[ R^6 \frac{d^2 U}{d r^2} + R^3 \frac{d R}{d r} \frac{d U}{d r} - 4 R^4 U \left( \frac{d R}{d r} \right)^2 = -2 R^5 U \left( \frac{d^2 R}{d r^2} \right) = -4 R^4 + 4 q^2 e^{4 \Phi / 3}, \quad (11) \]

\[ \frac{1}{R^3} \frac{d}{d r} \left( U \frac{d R}{d r} \right) = \frac{6}{R^2} - V(\Phi) - 2 e^{4 \Phi / 3} q^2 R^6, \quad (12) \]

\[ \frac{1}{R^3} \frac{d}{d r} \left( R^3 U \frac{d \Phi}{d r} \right) = \frac{3}{8} \frac{d V}{d \Phi} + \alpha e^{4 \Phi / 3} q^2 R^6, \quad (13) \]

\[ \frac{1}{R} \frac{d^2 R}{d r^2} = - \frac{4}{9} \left( \frac{d \Phi}{d r} \right)^2. \quad (14) \]

In addition, we have two coupled differential equations for functions \( f(r) \) and \( h(r) \).

\[ R_3 \frac{d^2 f}{d r^2} + R^2 \frac{d f}{d r} - 4 f R \frac{d f}{d r}^2 = -2 R^2 \frac{d R^2}{d r^2} - 4 q^2 \frac{d h}{d r} = 0, \quad (15) \]

\[ R^2 \frac{d}{d r} \left( U e^{-4 \alpha \Phi / 3} \frac{d h}{d r} \right) - R \frac{d}{d r} \left( \frac{f}{R^2} \right) + \frac{1}{2} \left( U \frac{d h}{d r} \frac{d R^2}{d r} - 8 h \right) e^{-4 \alpha \Phi / 3} = 0. \quad (16) \]

These two equations which arise from the presence of \( A_\Phi \), appear only when \( a \neq 0 \), while the other equations were there also in the static, spherically symmetric case.

In order to solve eqs. (11)-(14), we make the ansatz

\[ R(r) = r^N, \quad (17) \]

where \( N \) is a constant. Using (17), one can easily show that Eqs. (11)-(14), have solutions of the form

\[ U(r) = \frac{2(1 + \alpha^2)^2}{(2 + \alpha^2)(1 - \alpha^2)} \frac{4 M(1 + \alpha^2)}{3 r^{(\alpha^2 + 2)}} + \frac{2 q^2(1 + \alpha^2)^2 e^{4 \alpha h / 3}}{3(2 + \alpha^2)} \frac{r^{2(\alpha^2 + 2)}}{r_+^{\alpha^2 + 1}}, \quad (18) \]

\[ \Phi(r) = b - \frac{3 \alpha}{2(1 + \alpha^2)} \ln(r), \quad (19) \]

\[ R(r) = r^{1/(\alpha^2 + 1)}. \quad (20) \]

with \( b \) a constant and \( M \) is the quasi-local mass of the black hole [19, 45]. In order to fully satisfy the system of equations, we must have \( \beta = 2/3 \alpha \). The constant \( b \) is related to the \( \Lambda \) parameter via

\[ \lambda = \frac{3 \alpha^2 e^{-4 b / 3 \alpha}}{\alpha^2 - 1}. \quad (21) \]

It is worth noting that the solution does not exist for string case where \( \alpha = 1 \). The \( \alpha \to \infty \) limit produces (anti)-de Sitter behavior for zero \( M \) and \( q \). On the other hand, in the absence of dilaton field \( (\alpha = 0) \), the solution becomes

\[ U(r) = 1 - \frac{4 M}{3r^2} + \frac{q^2}{3r^4}, \quad R(r) = r, \quad (22) \]

which is the the five dimensional Reissner-Nordström black hole solutions for vanishing rotation parameter \( \alpha \). Horizons are located at

\[ r_h^{\alpha^2 + 2} = \left( 1 - \alpha^2 \right) \left( 2 + \alpha^2 \right) M \quad 3 \left( 1 + \alpha^2 \right) \]

\[ \times \left( 1 \pm \sqrt{1 - \frac{3 q^2 e^{4 \alpha h} (1 + \alpha^2)^2}{M^2 (1 - \alpha^2) (2 + \alpha^2)^2}} \right). \quad (23) \]

If \( \alpha^2 < 1 \), then there are two horizons, while we have a single horizon for \( \alpha^2 > 1 \). There is an extremal limit for the electric charge

\[ q^2_{ext} = \frac{(1 - \alpha^2)(2 + \alpha^2)^2 e^{4 b / \alpha}}{3(1 + \alpha^2)^2} M^2. \quad (24) \]

In the case \( q^2 > q^2_{ext} \), we have naked singularity. The metric corresponding to (18) is neither asymptotically flat nor (anti)-de Sitter. In order to study the general structure of these solutions, we first look for the curvature singularities in the presence of dilaton gravity up to \( O(\alpha) \). It is easy to show that the Kretschmann scalar diverges at \( r = 0 \), it is finite for \( r \neq 0 \) and goes to zero as \( r \to \infty \). Also, it is notable to mention that the Ricci scaler is finite everywhere except at \( r = 0 \), and goes to zero as \( r \to \infty \). Therefore \( r = r_h \) is a regular horizon and we have an essential singularity located at \( r = 0 \). Note that the dilaton field is regular on the horizons, too.

Black hole entropy typically satisfies the so called area law of the entropy [46, 47, 48] which states that the entropy is a quarter of the event horizon area. This near universal law applies to almost all kinds of black holes and black holes in Einstein gravity [49, 51, 52]. Since the surface gravity and area of the event horizon do not change to \( O(\alpha) \), one can easily show that the temperature and the entropy of black hole on the outer event horizon...
can be written as
\[ T = \frac{1}{4\pi} \frac{dU}{dr} (r_h) = \left( \frac{2 + \alpha^2}{3\pi} - \frac{2q^2}{3} \right) \frac{r_{+1}}{r_h} \times \left( \frac{r_{+1}^2}{r_h^2} + \frac{1}{2q^2r_{+1}^2} \right) \],
\[ S = \frac{\pi^2}{2} \frac{r_{+1}^3}{(\alpha^2 + 1)}. \]

Note that the temperature vanishes in the extremal limit when \( q = q_{ext} \).

Here, we are interested in finding the rotating version of this static solution, that is to say, in solving the corresponding coupled equations for two unknown functions \( f(r) \) and \( h(r) \). For arbitrary value of the dilaton coupling constant \( \alpha \), we could obtain the following exact solution
\[ f(r) = \frac{4M(\alpha^2 + 2)e^{\frac{1}{3}\frac{\alpha^2}{r_{+1}}}}{3} - 2q^2(\alpha^2 + 1) \frac{r - \frac{1}{3}}{r_{+1}^2}, \]
\[ h(r) = r - \frac{\alpha^2}{r_{+1}^2}. \]

Note that for \( \alpha^2 < 2 \), this solution decreases with increase of \( r \). For \( \alpha = 0 \), the non-rotating version of the solution reduces to the five dimensional Reissner-Nordström black hole \([22]\), and the rotating version \([27]\) reproduces the five dimensional Kerr-Newman modification thereof for small rotation parameter \( \alpha \) \([53]\) (see also \([54]\))
\[ f(r) = \frac{8M}{3r_{+1}^2} - \frac{2q^2(\alpha^2 + 1)}{3r_{+1}^4}, \]
\[ h(r) = \frac{1}{r_{+1}^2}. \]

Finally, we study the physical properties of these solutions, by computing the angular velocity of the solutions at the horizons and the value of the angular momentum in the general situation when \( \alpha \neq 0 \). The angular velocity at the horizon \( r = r_h \) is given in the leading order by
\[ \Omega_h = \frac{g_{\theta\phi}(r = r_h, \theta = \pi/2)}{R^2(r_h)} = \left( -\frac{4M(a^2 + 2)e^{\frac{2q}{3\pi}r_{+1}^2}}{3} \right) \frac{r_{+1}^2}{r_h} \]
\[ + \frac{2q^2(a^2 + 1)}{3} \left( 1 - \frac{\alpha^2}{r_h^2} \right). \]

while the angular momentum of the black hole can be calculated through the use of the quasi-local formalism of the Brown and York \([43]\). According to the quasi-local formalism, the quantities can be constructed from the information that exists on the boundary of a gravitating system alone. Such quasi-local quantities will represent information about the spacetime contained within the system boundary, just like the Gauss’s law. In our case the finite stress-energy tensor can be written as
\[ T^{ab} = \frac{1}{8\pi} \left( G^{ab} - \Theta^{ab} \right), \]

which is obtained by variation of the action \([1] \) with respect to the boundary metric \( \gamma_{ab} \). To compute the angular momentum of the spacetime, one should choose a spacelike surface \( \partial M \) with metric \( \sigma_{ij} \), and write the boundary metric in ADM (Arnowitt-Deser-Misner) form:
\[ \gamma_{ab}dx^a dx^b = -N^2dt^2 + \sigma_{ij} \left( d\phi^i + V^i dt \right) \left( d\phi^j + V^j dt \right), \]

where \( \sigma \) is the determinant of the metric \( \sigma_{ij} \), \( \xi \) and \( n^a \) are the Killing vector field and the unit normal vector on the boundary \( B \). For boundaries with rotational \( \zeta = \partial / \partial \phi \) Killing vector fields, one obtains the quasilocal angular momentum
\[ J = \int_B d^{n-2} \varphi \sqrt{\sigma} T_{\alpha\beta} n^\alpha n^\beta, \]

provided the surface \( B \) contains the orbits of \( \zeta \). Finally, the angular momentum of the black holes can be calculated through the use of Eq. \([32]\). We find
\[ J = \frac{(a^2 + 2)(4 - \alpha^2)e^{-4\phi/3}}{12\pi(a^2 + 1)} M\alpha. \]

For \( a = 0 \), the angular momentum vanishes, and therefore \( a \) is the rotational parameter of the black hole. It is worth noting that \( J \propto Ma \), as one expected, and reduces to the case of slowly rotating five dimensional Kerr solution in the absence of dilaton field \( (\alpha = 0) \).

### III. SUMMARY AND CONCLUSION

As noted in the introduction, exact, charged rotating dilaton black hole solutions with curved horizons for an arbitrary value of dilaton coupling constant in more than four dimensions have not been constructed. In this paper we studied charged black hole solutions in five dimensional Einstein-Maxwell-dilaton gravity. These black holes have unusual asymptotics. They are neither asymptotically flat nor (anti-) de Sitter. Then, we considered the effect of adding a small amount of rotation parameter \( a \) to the black hole. We discarded any terms involving \( \alpha^2 \) or higher power in \( \alpha \). Inspection of the Kerr-Newman solutions shows that the only term in the metric changes to \( O(a) \) is \( g_{\theta\theta} \). Similarly, the dilaton does not change to \( O(a) \) and \( A_{\phi} \) is the only component of the vector potential that change to \( O(a) \). For small angular momentum, the field equations led to the coupled differential
equations (13) and (16) for two unknown functions $f(r)$ and $h(r)$, for which we find a class of solutions in the presence of Liouville-type potential. We showed that in the absence of dilaton field ($\alpha = 0$), the non-rotating version of the solution reduces to the five dimensional Reissner-Nordström black hole, and the rotating version reproduces the five dimensional Kerr-Newman modification for small $\alpha$. We computed temperature and entropy of black hole, which did not change to $O(\alpha)$ from the static case. We also obtained the angular momentum and the angular velocity of these rotating black holes which appear at the first order of rotation parameter $\alpha$. It is notable to mention that the five dimensional charged rotating dilaton black hole solutions obtained here have small angular momentum. Thus, it would be interesting if one can construct charged rotating dilaton black holes with arbitrary rotation parameter. One can also attempt to extend these solutions to higher dimensional ($n > 5$) rotating dilaton black holes with curved horizons.

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