Some New Tempered Fractional Pólya-Szegö and Chebyshev-Type Inequalities with Respect to Another Function

Gauhar Rahman, Kottakkaran Soopy Nisar, Thabet Abdeljawad, and Muhammad Samraiz

1Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal 18000, Upper Dir, Pakistan
2Department of Mathematics, College of Arts and Sciences, Prince Sattam Bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia
3Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, KSA, Saudi Arabia
4Department of Medical Research, China Medical University, Taichung 40402, Taiwan
5Department of Computer Science and Information Engineering, Asia University, Taichung 40402, Taiwan
6Department of Mathematics, University of Sargodha, Sargodha, Pakistan

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

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In this present article, we establish certain new Pólya–Szegö-type tempered fractional integral inequalities by considering the generalized tempered fractional integral concerning another function Ψ in the kernel. We then prove certain new Chebyshev-type tempered fractional integral inequalities for the said operator with the help of newly established Pólya–Szegö-type tempered fractional integral inequalities. Also, some new particular cases in the sense of classical tempered fractional integrals are discussed. Additionally, examples of constructing bounded functions are considered. Furthermore, one can easily form new inequalities for Katugampola fractional integrals, generalized Riemann–Liouville fractional integral concerning another function Ψ in the kernel, and generalized fractional conformable integral by applying different conditions.

1. Introduction

The well-known Chebyshev functional [1] is defined by

$$\mathcal{T}(f_1, f_2) = \frac{1}{x_2-x_1} \int_{x_1}^{x_2} f_1(\theta)f_2(\theta)d\theta - \frac{1}{x_2-x_1} \left( \int_{x_1}^{x_2} f_1(\theta)d\theta \right) \frac{1}{x_2-x_1} \left( \int_{x_1}^{x_2} f_2(\theta)d\theta \right), \quad (1)$$

where the functions $f_1$ and $f_2$ are integrable on $[x_1, x_2]$. If the functions $f_1$ and $f_2$ are synchronous on $[x_1, x_2]$, i.e.,

$$(f_1(\theta) - f_1(\zeta))(f_2(\theta) - f_2(\zeta)) \geq 0, \quad (2)$$

for any $\theta, \zeta \in [x_1, x_2]$, then $\mathcal{T}(f_1, f_2) \geq 0$. Functional (1) has gained more recognition due to its diverse applications in the fields of transform theory, numerical quadrature, probability, and statistical problems. Additionally, the researchers have established a large number of integral
inequalities by utilizing functional (1). The interesting readers may consult [2–5]. In [6], Tassaddiq et al. recently established certain inequalities via fractional conformable integrals by considering functional (1).

In [7], Grüss introduced the following inequality:

\[
|\mathcal{F}(f_1, f_2)| \leq \frac{(M_1 - m_1)(N_1 - n_1)}{4},
\]

(3)

where the functions \(f_1\) and \(f_2\) are integrable on \([x_1, x_2]\) such that \(f_1\) and \(f_2\) satisfy the inequalities \(m_1 \leq f_1(\theta) \leq M_1\) and \(n_1 \leq f_2(\zeta) \leq N_1\), for all \(\theta, \zeta \in [x_1, x_2]\) and for some constant \(m_1, n_1, M_1, N_1 \in \mathbb{R}\).

In [8], Pólya–Szegő presented the following inequality:

\[
\frac{\int_{x_1}^{x_2} f_1^2(\theta) d\theta}{\frac{1}{4} \left( \frac{M_1 N_1}{m_1 n_1} + \frac{m_1 n_1}{M_1 N_1} \right)^2} \leq \frac{1}{4} \left( \frac{M_1 N_1}{m_1 n_1} + \frac{m_1 n_1}{M_1 N_1} \right)^2.
\]

(4)

In [9], Dragomir and Diamond presented the following inequality with the help of Pólya–Szegő inequality:

\[
|\mathcal{F}(f_1, f_2)| \leq \frac{(M_1 - m_1)(N_1 - n_1)}{4} \left( \frac{1}{\sqrt{x_2 - x_1}} \right)^2 \int_{x_1}^{x_2} f_1(\theta) d\theta \\
\int_{x_1}^{x_2} f_2(\theta) d\theta,
\]

(5)

where the functions \(f_1\) and \(f_2\) are positive and integrable on \([x_1, x_2]\) such that \(f_1\) and \(f_2\) satisfy the inequalities \(m_1 \leq f_1(\theta) \leq M_1\) and \(n_1 \leq f_2(\zeta) \leq N_1\), for all \(\theta, \zeta \in [x_1, x_2]\) and for some constant \(m_1, n_1, M_1, N_1 \in \mathbb{R}\).

In the last few decades, the researchers have considered that fractional integral inequalities are the most powerful tools for the development of both applied and pure mathematics. In [10], the authors presented some Grüss-type integral inequalities by considering fractional integrals. Some new integral inequalities in sense of Riemann–Liouville fractional integrals can be found in the work of Dahmani [11].

In [12], Sarikaya et al. gave the idea of generalized \((k, s)\)-fractional integrals with applications. Set et al. [13] investigated some Grüss-type inequalities by considering generalized \(k\)-fractional integrals.

Very recently, the idea of fractional conformable and proportional fractional integral operators was proposed by Jarad et al. [14, 15]. Later on, Huang et al. [16] presented generalized Hermite–Hadamard-type inequalities by considering generalized fractional conformable integrals. In [17], Qi et al. established Chebyshev-type inequalities for generalized fractional conformable integrals.

In [18], Ntouyas et al. investigated some new Pólya–Szegő and Chebyshev-type inequalities by considering Riemann–Liouville fractional integrals. The tempered fractional integral was first studied by Buschman [19], but Li et al. [20] and Meerschaert et al. [21] have described the associated tempered fractional calculus more explicitly. Fernandez and Ustaoglu [22] investigated several analytic properties of the tempered fractional integral. In [23], Fahad et al. proposed the general form of the generalized tempered fractional integral concerning another function. In this paper, we investigate the said inequalities for the so-called tempered fractional integrals containing another function in the kernel.

The structure of the paper as follows.

In Section 2, some basic definitions are presented. Some new Pólya–Szegő-type for the so-called generalized tempered fractional integral in the sense of another function is presented in Section 3. In Section 4, we present some new generalized Chebyshev-type tempered fractional integral inequalities. In Section 5, certain new particular cases in terms of classical tempered fractional integrals are discussed. An example of constructing bounding functions is considered in Section 6. Finally, the concluding remarks are discussed in Section 7.

2. Preliminaries

In this section, we consider some well-known definitions and mathematical preliminaries.

**Definition 1** (see [7]). Suppose that the functions \(f_1, f_2 : [x_1, x_1] \to \mathbb{R}\) are positive with \(\mathcal{A} \leq f_1(\theta) \leq \mathcal{B}\) and \(\mathcal{C} \leq f_2(\theta) \leq \mathcal{D}\), for all \(\theta \in [x_1, x_1]\), then the following inequality holds:

\[
\left| \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_1(\theta) d\theta - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_2(\theta) d\theta \right| \leq \frac{1}{4} (\mathcal{B} - \mathcal{A})(\mathcal{D} - \mathcal{C}),
\]

(6)

where the constants \(\mathcal{B}, \mathcal{A}, \mathcal{C}, \mathcal{D} \in \mathbb{R}\) and \(1/4\) is the sharp of inequality (6).

**Definition 2** (see [24, 25]). The function \(f_1\) will be in the space \(L_{p, r} [0, \infty] \{ \text{if} \)

\[
L_{p, r} [0, \infty] = \left\{ f_1 : \left\| f_1 \right\|_{L_{p, r}[0, \infty]} = \left( \int_{0}^{\infty} \left| f_1(\theta) \right|^p \theta^r d\theta \right)^{1/p} \right\},
\]

(7)
If we apply $r = 0$, then (7) gives

$$L_p \left[ 0, \infty \right] = \left\{ f_1 : \| f_1 \|_{L_p(0, \infty)} = \left( \int_0^\infty \| f_1 (\theta) \|^p d\theta \right)^{1/p} < \infty, 1 \leq p < \infty \right\}. $$

**Definition 3** (see [26]). Suppose that the function $f_1 \in L_1 \left[ 0, \infty \right]$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$ and having continuous derivative $\Psi'$ on $[0, \infty]$ with $\Psi(0) = 0$. Then, the Lebesgue real-valued measurable function $f_1$ defined on $[0, \infty]$ is said to be in the space $X_p \left( 0, \infty \right) \left( 1 \leq p < \infty \right)$ if

$$\| f_1 \|_{X_p} = \left( \int_0^\infty \| f_1 (\theta) \|^p \Psi' (\theta) d\theta \right)^{1/p} < \infty, 1 \leq p < \infty. $$

When $p = \infty$, then

$$\| f_1 \|_{X_\infty} = \text{ess sup}_{0 \leq \theta < \infty} \left[ \Psi' (\theta) f_1 (\theta) \right].$$

Remark 1. By setting $\tau = 0$ in (11) yields the following Riemann–Liouville fractional integral, which is defined by

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta. $$

where $\tau \geq 0, \kappa, \lambda \in \mathbb{C}$ with $\Re (\kappa) > 0$, and $\Gamma (\cdot)$ is the well-known gamma function.

Remark 2. The following results can be obtained:

(i) Applying Definition 6 for $\Psi (\theta) = \theta$, we get (11).

(ii) Applying Definition 6 for $\tau = 0$, then it will reduce to the left-sided generalized Riemann–Liouville fractional integral operator [27].

(iii) Applying Definition 6 for $\Psi (\theta) = \ln \theta$, then it will reduce to the following left-sided Hadamard tempered integral defined by [23]

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta.$$

The following results for (11) hold:

$$\left[ x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right](\theta) = \left[ x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right](\theta) = \left[ x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right](\theta). \quad (13)$$

We define the following one-sided tempered fractional integral.

**Definition 5.** The one-sided tempered fractional integral of order $\kappa > 0$, $\tau \geq 0$ is defined by

$$\left( \mathcal{R}_0^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_0^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda. $$

**Definition 6** (see [23]). Let the function $f_1$ be an integrable in the space $X_p \left( 0, \infty \right)$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$, and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Then, the left-sided generalized tempered fractional integral of the function $f_1$ concerning another function $\Psi$ in the kernel is defined by

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta.$$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$

$$\left( x_1, \mathcal{R}_\tau^{\kappa \tau} f_1 \right)(\theta) = \frac{1}{\Gamma (\kappa \tau)} \int_{x_1}^\theta e^{-\tau (\theta - \lambda)} (\theta - \lambda)^{\kappa - 1} f_1 (\lambda) d\lambda, x_1 < \theta, $$
Lemma 1. Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$ and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Then, the one-sided generalized tempered fractional integral of the function $f_1$ concerning another function $\Psi$ in the kernel is defined by

$$
\left( \Psi \mathcal{R}_0^{\kappa, \tau} f_1 \right) (\theta) = \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(\theta) - \Psi(\theta^\tau))} \left( \Psi(X) - \Psi(\theta) \right)^{\kappa-1} \Psi'(\theta) f_1(\theta) d\theta.
$$

Remark 3. If we set $\Psi(X) = X$ and $\tau = 0$, then (18) will reduce to the subintegrals of Riemann–Liouville fractional integral (12) will be obtained.

Definition 7. Let the function $f_1$ be integrable in the space $L^1(0, \infty)$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$, and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Then, the one-sided generalized tempered fractional integral of the function $f_1$ concerning another function $\Psi$ in the kernel is defined by

$$
\left( \Psi \mathcal{R}_0^{\kappa, \tau} f_1 \right) (\theta) = \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-\tau(\Psi(X) - \Psi(\theta))} \left( \Psi(X) - \Psi(\theta) \right)^{\kappa-1} \Psi'(\theta) f_1(\theta) d\theta.
$$

Note that

$$
\left( \Psi \mathcal{R}_0^{\kappa, \tau} f_1 \right) (X) = \sum_{i=0}^p \Psi \mathcal{R}_{x_i}^{\kappa, \tau} \left( f_1 \right) (X) = \frac{1}{\Gamma(\kappa)} \int_{x_i}^X e^{-\tau(\Psi(X) - \Psi(\theta))} \left( \Psi(X) - \Psi(\theta) \right)^{\kappa-1} \Psi'(\theta) f_1(\theta) d\theta
$$

$$
+ \frac{1}{\Gamma(\kappa)} \int_{x_i}^X \left( \Psi(X) - \Psi(\theta) \right)^{\kappa-1} \Psi'(\theta) f_1(\theta) d\theta
$$

$$
+ \cdots + \frac{1}{\Gamma(\kappa)} \int_{x_p}^X \left( \Psi(X) - \Psi(\theta) \right)^{\kappa-1} \Psi'(\theta) f_1(\theta) d\theta.
$$

3. Pólya–Szegő-Type Tempered Fractional Integral Inequalities

In this section, we provide some new Pólya–Szegő-type tempered fractional integral inequalities for positive and integrable functions via tempered fractional integral (17) containing another function $\Psi$ in the kernel.

Lemma 1. Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$, and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ such that

$$(H_1) 0 < \mathcal{V}_1(\theta) \leq f_1(\theta) \leq \mathcal{V}_2(\theta), 0 < \mathcal{V}_1'(\theta) \leq f_2'(\theta) \leq \mathcal{V}_2'(\theta), 0 < \mathcal{V}_1'(\theta) \leq f_2'(\theta)$$

$$
\leq \mathcal{V}_2'(\theta), \theta \in [0, \theta], \theta > 0.
$$

Then, for $\kappa > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$
\left( \Psi \mathcal{R}_0^{\kappa, \tau} \left( \mathcal{V}_1 f_1 + \mathcal{V}_2 f_2 \right) \right) (\theta) \leq \frac{1}{4}
$$
Proof. From the given hypothesis, we have
\[
\left( \frac{\mathcal{U}_2(\theta) - f_1(\theta)}{\mathcal{V}_1(\theta)} - \frac{f_1(\theta)}{f_2(\theta)} \right) \geq 0. \tag{22}
\]
Taking product of (22) and (23), we get
\[
\left( \frac{\mathcal{U}_2(\theta) - f_1(\theta)}{\mathcal{V}_1(\theta)} - \frac{f_1(\theta)}{f_2(\theta)} \right) \left( \frac{f_1(\theta)}{f_2(\theta)} - \frac{\mathcal{U}_1(\theta)}{\mathcal{V}_2(\theta)} \right) \geq 0. \tag{24}
\]
From (24), it can be written as
\[
(\mathcal{U}_1(\theta) \mathcal{V}_1(\theta) + \mathcal{U}_2(\theta) \mathcal{V}_2(\theta)) f_1(\theta) f_2(\theta) \geq \mathcal{V}_1(\theta) \mathcal{V}_2(\theta) (f_1(\theta))^2(\theta) + \mathcal{U}_1(\theta) \mathcal{U}_2(\theta) f_2^2(\theta). \tag{25}
\]
Now, taking product of (25) with $e^{-\tau (\Psi(\theta) - \Psi(\theta)) (\Psi(\theta) - \Psi(\theta))^{-1} \Psi'(\theta) / \Gamma(\kappa)}$ and integrating the resultant identity with respect to $\theta$ over $(0, \theta)$, we have
\[
\frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-r(\Psi(\theta) - \Psi(\theta))} (\Psi(\theta) - \Psi(\theta))^{-1} \Psi'(\theta) (\mathcal{U}_1(\theta) \mathcal{V}_1(\theta) + \mathcal{U}_2(\theta) \mathcal{V}_2(\theta)) f_1(\theta) f_2(\theta) d\theta \\
\geq \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-r(\Psi(\theta) - \Psi(\theta))} (\Psi(\theta) - \Psi(\theta))^{-1} \Psi'(\theta) \mathcal{V}_1(\theta) \mathcal{V}_2(\theta) f_1^2(\theta) d\theta \\
+ \frac{1}{\Gamma(\kappa)} \int_0^\theta e^{-r(\Psi(\theta) - \Psi(\theta))} (\Psi(\theta) - \Psi(\theta))^{-1} \Psi'(\theta) \mathcal{U}_1(\theta) \mathcal{U}_2(\theta) f_2^2(\theta) d\theta. \tag{26}
\]
With the aid of Definition 8, we can write
\[
\mathbb{R}_0^{\kappa,r} \left((\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2 \right)(\theta) \geq \mathbb{R}_0^{\kappa,r} \left(\mathcal{V}_1 \mathcal{V}_2 f_1^2 \right)(\theta) + \mathbb{R}_0^{\kappa,r} \left(\mathcal{U}_1 \mathcal{U}_2 f_2^2 \right)(\theta). \tag{27}
\]
By applying AM-GM inequality, i.e., $x_1 + x_2 \geq 2\sqrt{x_1 x_2}, x_1, x_2 \in \mathbb{R}^+$, we get
\[
\mathbb{R}_0^{\kappa,r} \left((\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2 \right)(\theta) \geq 2 \mathbb{R}_0^{\kappa,r} \left(\mathcal{V}_1 \mathcal{V}_2 f_1^2 \right)(\theta) \mathbb{R}_0^{\kappa,r} \left(\mathcal{U}_1 \mathcal{U}_2 f_2^2 \right)(\theta). \tag{28}
\]
It follows that
\[
\mathbb{R}_0^{\kappa,r} \left(\mathcal{V}_1 \mathcal{V}_2 f_1^2 \right)(\theta) \mathbb{R}_0^{\kappa,r} \left(\mathcal{U}_1 \mathcal{U}_2 f_2^2 \right)(\theta) \\
\leq \frac{1}{4} \mathbb{R}_0^{\kappa,r} \left((\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2 \right)(\theta)^2, \tag{29}
\]
which gives the desired assertion (21). \qed

Corollary 1. Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$ and assume that the function $\Psi$ is positive, monotone, and increasing on $[0, \infty]$, and its derivative $\Psi'$ is continuous on $[0, \infty]$ with $\Psi(0) = 0$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1,$ and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ such that

\[
(\mathcal{H}_2) 0 < m_1 \leq f_1(\theta) \leq M_1 < \infty, 0 < n_1 \leq f_2(\theta) \leq N_1 < \infty, \theta \in [0, \theta], \theta > 0. \tag{30}
\]
Then, for $\kappa > 0, \tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:
\[
\frac{\mathbb{R}_0^{\kappa,r} \left(\mathcal{V}_1 \mathcal{V}_2 f_1^2 \right)(\theta) \mathbb{R}_0^{\kappa,r} \left(\mathcal{U}_1 \mathcal{U}_2 f_2^2 \right)(\theta)}{\mathbb{R}_0^{\kappa,r} \left(f_1 f_2 \right)(\theta)^2} \\
\leq \frac{1}{4} \left( \frac{m_1 n_1}{M_1 N_1} + \frac{M_1 N_1}{m_1 n_1} \right)^2. \tag{31}
\]
Lemma 2. Let all the conditions of Lemma 1 hold. Then, for $\kappa, \lambda > 0$, $\tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$
\left( \frac{\frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (f^2) (\theta) \frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (f_2) (\theta) \frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (\mathcal{U}_1 \mathcal{U}_2) (\theta)}{\frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (f_1) (\theta) \frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (f_1) (\theta) + \frac{\partial^\kappa \mathcal{H}}{\partial \tau^\kappa} (\mathcal{U}_1 \mathcal{U}_2) (\theta)} \right)^{2} \geq \frac{1}{4}
$$

(32)

Proof. From the hypothesis $(H_1)$ defined by (20), we have

$$
\left( \frac{\mathcal{U}_2 (\theta)}{\mathcal{V}_2 (\theta)} - \frac{f_1 (\theta)}{f_2 (\theta)} \right) \geq 0,
$$

(33)

$$
\left( \frac{f_1 (\theta)}{f_2 (\theta)} - \frac{\mathcal{U}_1 (\theta)}{\mathcal{V}_2 (\theta)} \right) \geq 0,
$$

(34)

which follows that

$$
\left( \frac{\mathcal{U}_2 (\theta)}{\mathcal{V}_2 (\theta)} + \frac{\mathcal{U}_2 (\theta)}{\mathcal{V}_2 (\theta)} \right) \frac{f_1 (\theta)}{f_2 (\theta)} \geq \frac{f_2 (\theta)}{f_2 (\theta)} + \frac{\mathcal{U}_1 (\theta) \mathcal{U}_2 (\theta)}{\mathcal{U}_1 (\theta) \mathcal{U}_2 (\theta)}
$$

(35)

Taking product on both sides of (18) by $\mathcal{V}_1 (\xi) \mathcal{V}_2 (\xi) f_2 (\xi)$, we obtain

$$
\mathcal{U}_1 (\theta) f_1 (\theta) \mathcal{V}_1 (\xi) f_2 (\xi) + \mathcal{U}_2 (\theta) f_1 (\theta) \mathcal{V}_2 (\xi) f_2 (\xi) \geq \mathcal{V}_1 (\xi) \mathcal{V}_2 (\xi) f_1 (\theta) \mathcal{U}_2 (\theta) f_2 (\xi).
$$

(36)

Taking product of both sides of (36) with $e^{-\tau(\Psi (\theta) - \Psi (\xi))}$, we get

$$
\mathcal{U}_1 (\theta) f_1 (\theta) \mathcal{V}_1 (\xi) f_2 (\xi) + \mathcal{U}_2 (\theta) f_1 (\theta) \mathcal{V}_2 (\xi) f_2 (\xi) \geq \mathcal{V}_1 (\xi) \mathcal{V}_2 (\xi) f_1 (\theta) \mathcal{U}_2 (\theta) f_2 (\xi).
$$

(37)

Again, taking product of both sides of (38) with $e^{-\tau(\Psi (\theta) - \Psi (\xi))}$, we get

$$
\mathcal{U}_1 (\theta) f_1 (\theta) \mathcal{V}_1 (\xi) f_2 (\xi) + \mathcal{U}_2 (\theta) f_1 (\theta) \mathcal{V}_2 (\xi) f_2 (\xi) \geq \mathcal{V}_1 (\xi) \mathcal{V}_2 (\xi) f_1 (\theta) \mathcal{U}_2 (\theta) f_2 (\xi).
$$

(39)

By using AM-GM inequality, we get

$$
\mathcal{U}_1 (\theta) f_1 (\theta) \mathcal{V}_1 (\xi) f_2 (\xi) + \mathcal{U}_2 (\theta) f_1 (\theta) \mathcal{V}_2 (\xi) f_2 (\xi) \geq 2 \sqrt{\mathcal{U}_1 (\theta) f_1 (\theta) \mathcal{V}_1 (\xi) f_2 (\xi) \mathcal{U}_2 (\theta) f_2 (\xi) \mathcal{V}_2 (\xi) f_2 (\xi) f_1 (\theta) \mathcal{U}_2 (\theta) f_2 (\xi)}.
$$

(40)
It follows that
\[
\Psi R_{0}^{\kappa, \tau}(f_{1}, f_{2})(\theta) R_{0}^{\kappa, \tau}(U_{1}U_{2})(\theta) \leq \frac{1}{4} \left( \Psi R_{0}^{\kappa, \tau}(f_{1})(\theta) + \Psi R_{0}^{\kappa, \tau}(U_{1}f_{1})(\theta) + \Psi R_{0}^{\kappa, \tau}(U_{2}f_{1})(\theta) \right)^{2},
\]
which completes the desired assertion (17).

Corollary 2. Let the functions \( f_{1} \) and \( f_{2} \) be positive and integrable on \([0, \infty)\) satisfying the hypothesis \((H_{1})\) defined by (16) and assume that the function \( \Psi \) is positive, monotone, and increasing on \([0, \infty)\), and its derivative \( \Psi' \) is continuous on \([0, \infty)\) with \( \Psi(0) = 0 \). Then, for \( \kappa, \lambda > 0 \), \( \tau \geq 0 \), and \( \theta > 0 \), the following tempered fractional integral inequality holds:
\[
\gamma(\kappa, \tau \Psi(\theta)) \gamma(\kappa, \tau \Psi(\theta)) \left\{ \frac{\Psi R_{0}^{\kappa, \tau}(f_{1})(\theta)}{\tau^{\kappa+1} \Gamma(\kappa)} \right\}^{2} \leq \frac{1}{4} \left( \frac{M_{1}N_{1}}{m_{1}n_{1}} \right)^{2},
\]
where
\[
\gamma(\kappa, \tau) = \int_{0}^{\tau} e^{-\kappa u^{\kappa-1}du}
\]
is the well-known incomplete gamma function (see [22]).

Lemma 3. Suppose that all the conditions of Lemma 1 hold and assume that the function \( \Psi \) is positive, monotone, and increasing on \([0, \infty)\), and its derivative \( \Psi' \) is continuous on \([0, \infty)\) with \( \Psi(0) = 0 \). Then, for \( \kappa, \lambda > 0 \), \( \tau \geq 0 \), and \( \theta > 0 \), the following tempered fractional integral inequality holds:
\[
\Psi R_{0}^{\kappa, \tau}(f_{1}, f_{2})(\theta) \leq \Psi R_{0}^{\kappa, \tau}(U_{1}f_{1})(\theta),
\]
Proof. From the hypothesis \((H_{1})\) defined by (20), we have
\[
\frac{1}{\Gamma(\kappa)} \int_{0}^{\theta} e^{-\tau(\Psi(\theta) - \Psi(\theta))}(\Psi(\theta) - \Psi(\theta))^{\kappa-1} \Psi'(\theta) f_{1}(\theta) d\theta \leq \frac{1}{\Gamma(\kappa)} \int_{0}^{\theta} e^{-\tau(\Psi(\theta) - \Psi(\theta))}(\Psi(\theta) - \Psi(\theta))^{\kappa-1} \Psi'(\theta) f_{2}(\theta) d\theta \leq \frac{1}{\Gamma(\kappa)} \int_{0}^{\theta} e^{-\tau(\Psi(\theta) - \Psi(\theta))}(\Psi(\theta) - \Psi(\theta))^{\kappa-1} \Psi'(\theta) \frac{U_{1} f_{1}}{U_{1}}(\theta) d\theta
\]
which in view of (17) yields
\[\frac{\mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(f_1\right)}{\theta} \leq \frac{\mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(\frac{U_2 f_1 f_2}{\psi_1}\right)}{\theta}.\] (46)

Similarly, one can obtain
\[\frac{\mathcal{R}_0^\lambda \mathcal{R}_1^\tau \left(f_2\right)}{\theta} \leq \frac{\mathcal{R}_0^\lambda \mathcal{R}_1^\tau \left(\frac{U_2 f_1 f_2}{\psi_1}\right)}{\theta}.\] (47)

Hence, the product of (46) and (47) yields the desired assertion (44). \(\square\)

**Corollary 3.** Let the functions \(f_1\) and \(f_2\) be positive and integrable on \([0,\infty)\) satisfying the hypothesis (H2) defined by (30) and assume that the function \(\Psi\) is positive, monotone, and increasing on \([0,\infty]\), and its derivative \(\Psi'\) is continuous on \([0,\infty]\) with \(\Psi(0) = 0\). Then, for \(k, \lambda > 0\), \(\tau \geq 0\), and \(\theta > 0\), the following tempered fractional integral inequality holds:
\[\frac{\mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(f_1 f_2\right)}{\theta} \leq \frac{M_{11}}{m_{11}}.\] (48)

4. **Chebyshev-Type Tempered Fractional Integral Inequalities**

In this section, certain Chebyshev-type inequalities via tempered fractional integral (20) are presented with the help of Pólya–Szegő integral inequality given by Lemma 1.

**Theorem 1.** Let the functions \(f_1\) and \(f_2\) be positive and integrable on \([0,\infty)\) and assume that the function \(\Psi\) is positive, monotone, and increasing on \([0,\infty]\), and its derivative \(\Psi'\) is continuous on \([0,\infty]\) with \(\Psi(0) = 0\). Suppose that \(\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1,\) and \(\mathcal{V}_2\) are four positive and integrable functions on \([0,\infty)\) satisfying the hypothesis (H1) defined by (10). Then, for \(k, \lambda > 0\), \(\tau \geq 0\), and \(\theta > 0\), the following tempered fractional integral inequality holds:

\[
\left|\frac{\gamma(k, r \Psi'(\theta))}{r^\Gamma(k)} \mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(f_1 f_2\right) + \frac{\gamma(\lambda, r \Psi(\theta))}{r^\Gamma(\lambda)} \mathcal{R}_0^\lambda \mathcal{R}_1^\tau \left(f_1 f_2\right) \right| \\
\leq \left|F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) + F_1(f_1, \mathcal{U}_1, \mathcal{V}_2)(\theta) + F_2(f_1, \mathcal{V}_1, \mathcal{V}_2)(\theta) \right|^{1/2},
\] (49)

where

\[
F_1(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\lambda, r \Psi(\theta))}{4r^\Gamma(\lambda)} \mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(\frac{(\mathcal{U}_1 + \mathcal{U}_2) f_1^2}{2}\right) - \frac{\gamma(\kappa, r \Psi(\theta))}{4r^\Gamma(\kappa)} \mathcal{R}_0^\lambda \mathcal{R}_1^\tau \left(\mathcal{U}_1 \mathcal{U}_2 f_1^2\right),
\] (50)

\[
F_2(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\kappa, r \Psi(\theta))}{4r^\Gamma(\kappa)} \mathcal{R}_0^\kappa \mathcal{R}_1^\tau \left(\mathcal{U}_1 \mathcal{U}_2 f_1^2\right) - \frac{\gamma(\lambda, r \Psi(\theta))}{4r^\Gamma(\lambda)} \mathcal{R}_0^\lambda \mathcal{R}_1^\tau \left(\frac{(\mathcal{U}_1 + \mathcal{U}_2) f_1^2}{2}\right),
\] (51)

**Proof.** By the given hypothesis, both the functions \(f_1\) and \(f_2\) are positive and integrable functions on \([0,\infty)\). Therefore, for \(\delta, \xi \in (0, \theta)\) with \(\theta > 0\), we define \(\mathcal{A}(\theta, \zeta)\) by
\[
\mathcal{A}(\theta, \zeta) = \left(f_1(\theta) - f_1(\xi)\right) f_2(\theta) - f_1(\xi) f_2(\theta).
\] (52)

Multiplying (52) by \((1/\Gamma(\xi))\) and integrating with respect to \(\theta\) and \(\zeta\) over \((0, \theta)\) and then using (17), we obtain

\[
\left|\int_0^\theta \frac{1}{\Gamma(\xi)} \left(\int_0^\delta \exp[-r(\Psi(\theta) - \Psi(\xi))] \exp[-r(\Psi'(\theta) - \Psi'(\xi))] \Psi(\theta) - \Psi(\xi)\right)^{\kappa-1} \Psi(\theta)\Psi'(\xi)
\] (53)

By applying Cauchy–Schwartz inequality for double integrals, we have
In view of (17) and (43), we get

\[
\left| \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right|
\]

\[
\leq \left[ \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right]
\]

\[
\times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) f_1^2 (\theta) \right|^\frac{1}{2}
\]

\[
\times \left[ \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right]
\]

\[
\times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) f_2^2 (\theta) \right|^\frac{1}{2}
\]

(54)

In view of (17) and (43), we get

\[
\left| \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right|
\]

\[
\leq \left[ \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right]
\]

\[
\times \left[ \frac{1}{\Gamma(k) \Gamma(\lambda)} \int_0^\theta \int_0^\theta \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \exp[-\tau (\Psi(\theta) - \Psi(\zeta))] \times (\Psi(\theta) - \Psi(\zeta))^{\lambda - 1} \Psi(\theta) \Psi(\zeta) \mathcal{A} (\theta, \zeta) d\theta d\zeta \right]^\frac{1}{2}
\]

(55)
Applying Lemma 1 for $\mathcal{V}_1(\theta) = \mathcal{V}_2(\theta) = f_2(\theta) = 1$, we get

$$\frac{\gamma(\lambda, r\psi(\theta))}{\Gamma(\lambda)} \mathcal{R}_0^{\kappa, r}(f_1^2)(\theta) \leq \frac{\gamma(\lambda, r\psi(\theta))}{4\Gamma(\lambda)} \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2.$$

(56)

It follows that

$$\gamma(\lambda, r\psi(\theta)) \mathcal{R}_0^{\kappa, r}(f_1^2)(\theta) \leq \gamma(\lambda, r\psi(\theta)) \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2.$$

Similarly, one can get

$$\gamma(\kappa, r\psi(\theta)) \mathcal{R}_0^{\kappa, r}(f_1^2)(\theta) \leq \gamma(\kappa, r\psi(\theta)) \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2.$$

(57)

(58)

Again applying Lemma 1 for $\mathcal{U}_1(\theta) = \mathcal{U}_2(\theta) = f_1(\theta) = 1$, we get

$$\gamma(\lambda, r\psi(\theta)) \mathcal{R}_0^{\kappa, r}(f_1^2)(\theta) \leq \gamma(\lambda, r\psi(\theta)) \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2.$$

Thus, by considering (53) to (60), we arrive at the desired assertion (49). This is the desired proof of Theorem 1.

**Theorem 2.** Suppose that all the conditions of Theorem 1 are satisfied. Then, for $\kappa > 0$, $r > 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{\gamma(\kappa, r\psi(\theta))}{\Gamma(\kappa)} \mathcal{R}_0^{\kappa, r}(f_1 f_2)(\theta) \leq \frac{\gamma(\lambda, r\psi(\theta))}{\Gamma(\lambda)} \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2,$$

where

$$F(f_1, \mathcal{U}_1, \mathcal{U}_2)(\theta) = \frac{\gamma(\kappa, r\psi(\theta))}{\Gamma(\kappa)} \mathcal{R}_0^{\kappa, r}(f_1)(\theta),$$

(61)

Proof. Applying Theorem 1 for $\kappa = \lambda$, we get the desired result in (61).

**Remark 4.** If we consider $\mathcal{U}_1 = m_1$, $\mathcal{U}_2 = M_1$, $\mathcal{V}_1 = n_1$, and $\mathcal{V}_2 = N_1$, then we have

$$F(f_1, m_1, M_1)(\theta) = \frac{(M_1 - m_1)^2}{4M_1 m_1} \left( \mathcal{R}_0^{\kappa, r}(f_1)(\theta)\right)^2,$$

(63)

$$F(f_2, m_1, M_1)(\theta) = \frac{(N_1 - n_1)^2}{4N_1 m_1} \left( \mathcal{R}_0^{\kappa, r}(f_2)(\theta)\right)^2.$$

(64)

**Corollary 4.** Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$ and satisfying the hypothesis $(H_2)$ given by (30). Then, for $\kappa > 0$, $r > 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{\gamma(\kappa, r\psi(\theta))}{\Gamma(\kappa)} \mathcal{R}_0^{\kappa, r}(f_1 f_2)(\theta) \leq \frac{\gamma(\lambda, r\psi(\theta))}{\Gamma(\lambda)} \left( \mathcal{R}_0^{\kappa, r}[(\mathcal{U}_1 + \mathcal{U}_2)f_1]\right)^2.$$
5. Particular Cases

The following new Pólya–Szegö- and Chebyshev-type inequalities for classical tempered fractional integral (14) can be easily established.

Lemma 4. Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ satisfying the hypothesis $(H_1)$ defined by (20). Then, for $\kappa > 0, \tau \geq 0$, and $\theta > 0$, the following tempered fractional integral inequality holds:

$$\frac{\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta)}{(\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta))^{\frac{1}{2}}} \leq \frac{1}{4}.$$  (66)

Proof. Applying Lemma 1 for $\Psi (\theta) = \theta$, we get Lemma 4. □

Lemma 5. Let all the conditions of Lemma 4 are satisfied. Then, for $\kappa, \lambda > 0, \tau \geq 0, \text{and } \theta > 0$, the following tempered fractional integral inequality holds:

$$\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta) \leq \frac{1}{4}.$$  (67)

Theorem 3. Let the functions $f_1$ and $f_2$ be positive and integrable on $[0, \infty)$. Suppose that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$, and $\mathcal{V}_2$ are four positive and integrable functions on $[0, \infty)$ satisfying the hypothesis $(H_1)$ defined by (20). Then, for $\kappa, \lambda > 0, \tau \geq 0, \text{and } \theta > 0$, the following tempered fractional integral inequality holds:

$$\left| \mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta) \right|^{\frac{1}{2}} \leq \left| \mathcal{F}_1 (\mathcal{U}_1, \mathcal{U}_2) (\theta) + \mathcal{F}_2 (\mathcal{U}_1, \mathcal{U}_2) (\theta) \right|^{\frac{1}{2}} \times \left| \mathcal{F}_1 (\mathcal{V}_1, \mathcal{V}_2) (\theta) + \mathcal{F}_2 (\mathcal{V}_1, \mathcal{V}_2) (\theta) \right|^{\frac{1}{2}},$$  (68)

where

$$\mathcal{F}_1 (\mathcal{U}_1, \mathcal{U}_2) (\theta) = \frac{\gamma (\kappa, \tau \theta)}{r^\Gamma (\kappa)} \frac{\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta) - \mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta)}{\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta)}$$  (69)

$$\mathcal{F}_2 (\mathcal{U}_1, \mathcal{U}_2) (\theta) = \frac{\gamma (\kappa, \tau \theta)}{r^\Gamma (\kappa)} \frac{\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta) - \mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta)}{\mathcal{R}_0^{\kappa, r} (\mathcal{U}_1 f_1) (\mathcal{V}_1 f_2) (\theta) \mathcal{R}_0^{\kappa, r} (\mathcal{U}_2 f_1) (\mathcal{V}_2 f_2) (\theta)}$$  (70)

Proof. Applying Theorem 1 for $\Psi (\theta) = \theta$, then we get the desired Theorem 3. □

Similarly, we can derive particular result of Theorem 2.

6. Applications

In this section, we define a way for constructing four bounded functions and then utilize them to present certain estimates of Chebyshev-type tempered fractional integral inequalities of two unknown functions.

Let the unit function $h_1 (\theta)$ be defined by

$$h_1 (\theta) = \begin{cases} 1, & \theta > 0, \\ 0, & \theta \leq 0. \end{cases}$$  (71)

and let the Heaviside unit step function $h_a (\theta)$ be defined by

$$h_a (\theta) = \begin{cases} 1, & \theta > a, \\ 0, & \theta \leq a. \end{cases}$$  (72)

Suppose that the function $\mathcal{U}_1$ is piecewise continuous function on $[0, X]$ defined by
where \( m_{i_0} = 0 \) and \( 0 = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = X \).

Similarly, we define
\[
\mathcal{U}_2(x) = \sum_{i=0}^{p} (M_{i+1} - M_i) h_{x_i}(x),
\]
\[
\mathcal{V}_1(x) = \sum_{i=0}^{p} (n_{i+1} - n_i) h_{x_i}(x),
\]
\[
\mathcal{V}_2(x) = \sum_{i=0}^{p} (N_{i+1} - N_i) h_{x_i}(x),
\]
where the constants \( n_0 = N_0 = M_0 = 0 \). If there exists an integrable function \( f_1 \) on \([0, X] \)
satisfying the hypothesis \((H_1)\), then we have \( m_{i+1} \leq f_1(x) \leq M_i \) for each \( x \in (x_i, x_{i+1}], \ i = 0, 1, 2, \ldots, p \).

**Proposition 1.** Let the functions \( f_1 \) and \( f_2 \) be two positive and integrable on \([0, X] \). Assume that the functions \( \mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \) and \( \mathcal{V}_2 \) are defined by (73)–(76), respectively, and satisfying the hypothesis \((H_1)\) defined by (30). Then, for \( \kappa > 0 \), the following inequality for tempered fractional integral holds:

\[
\left( \sum_{i=0}^{p} \left[ n_{i+1} N_{i+1} \right] \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1^2)(X) \right)^{\frac{1}{2}} \leq \frac{1}{4} \sum_{i=0}^{p} \left( n_{i+1} N_{i+1} + m_{i+1} M_i \right) \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1 f_2)(X).
\]

**Proof.** By applying the Definition 8, we have

\[
\mathcal{R}_{\kappa, X}^{0,1} (\mathcal{V}_1 \mathcal{V}_2 f_1^2)(X) = \sum_{i=0}^{p} n_{i+1} N_{i+1} \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1^2)(X),
\]
\[
\mathcal{R}_{\kappa, X}^{0,2} (\mathcal{U}_1 \mathcal{U}_2 f_2^2)(X) = \sum_{i=0}^{p} m_{i+1} M_i \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_2^2)(X),
\]
\[
\mathcal{R}_{\kappa, X}^{1,2} ((\mathcal{U}_1 \mathcal{V}_1 + \mathcal{U}_2 \mathcal{V}_2) f_1 f_2)(X) = \sum_{i=0}^{p} (m_{i+1} n_{i+1} + M_{i+1} N_i) \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1 f_2)(X).
\]

Hence, by applying Lemma 1, we get the desired assertion (77). \( \square \)

**Proposition 2.** By setting \( \Psi(\theta) = \theta \) in Proposition 1, then we arrive to the following result in terms of classical tempered fractional integral:

\[
\left( \sum_{i=0}^{p} \left[ n_{i+1} N_{i+1} \right] \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1^2)(X) \right)^{\frac{1}{2}} \leq \frac{1}{4} \sum_{i=0}^{p} \left( n_{i+1} N_{i+1} + m_{i+1} M_i \right) \mathcal{R}_{\kappa, X}^{x_i, x_{i+1}} (f_1 f_2)(X).
\]
Remark 5. Throughout the paper, if we apply $\Psi(\theta) = 0$ and $\tau = 0$, then all the newly presented inequalities will be reduced to the work derived earlier by Ntouyas et al. [18].

7. Concluding Remarks

Certain new Pólya–Szegö- and Chebyshev-type inequalities by utilizing tempered fractional integral are presented in this paper. These inequalities generalized the existing inequalities. We can easily get the said Pólya–Szegö- and Chebyshev-type inequalities for Katugampola, generalized Riemann–Liouville, classical Riemann–Liouville, generalized conformable, and conformable fractional integrals by applying different conditions on function $\Psi$ given in Remark 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All the authors contributed equally, and they have read and approved the final manuscript for publication.

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