SUPERCRITICAL PROBLEMS IN DOMAINS WITH THIN TOROIDAL HOLES

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Abstract. In this paper we study the Lane-Emden-Fowler equation

\[ (P)_\epsilon \begin{cases} \Delta u + |u|^{q-2}u = 0 & \text{in } D_\epsilon, \\ u = 0 & \text{on } \partial D_\epsilon. \end{cases} \]

Here \( D_\epsilon = D \{ x \in D : \text{dist}(x, \Gamma_\ell) \leq \epsilon \} \), \( D \) is a smooth bounded domain in \( \mathbb{R}^N \), \( \Gamma_\ell \) is an \( \ell \)-dimensional closed manifold such that \( \Gamma_\ell \subset D \) with \( 1 \leq \ell \leq N - 3 \) and \( q = \frac{2(N-\ell)}{N-\ell-2} \). We prove that, under some symmetry assumptions, the number of sign changing solutions to \( (P)_\epsilon \) increases as \( \epsilon \) goes to zero.

1. Introduction

The paper deals with the Lane-Emden-Fowler equation

\[ \begin{cases} \Delta u + |u|^{q-2}u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \]

where \( D \) is a smooth bounded domain in \( \mathbb{R}^N \) and \( q > 2 \).

A main characteristic of problem \( (1.1) \) is the role played by the \((\ell + 1)\)-critical exponent \( 2^*_{N,\ell} \) in the solvability question. Here \( 2^*_{N,\ell} := \frac{2(N-\ell)}{N-\ell-2} \) where \( \ell \) is an integer between 0 and \( N - 3 \). \( 2^*_{N,\ell} \) is nothing but the critical Sobolev exponent in dimension \( N - \ell \), i.e. \( 2^*_{N,\ell} = 2^*_{N-\ell,0} \).

If \( q < 2^*_{N,0} \) problem \( (1.1) \) has one positive solution and infinitely many sign changing solutions. The proof relies on standard variational arguments and uses the compactness of the embedding \( H^1_0(D) \hookrightarrow L^q(D) \). When \( q \geq 2^*_{N,0} \) the compactness of the embedding is not true anymore and so existence of solutions becomes a delicate issue. Pohozaev \[20\] discovered that no solution exists when the domain is star-shaped. On the other hand Kazdan-Warner \[15\] proved that if \( D \) is an annulus the compactness is restored in the class of radial functions and so problem \( (1.1) \) has one positive radial solution and infinitely many sign changing radial solutions for any \( q \). If \( q = 2^*_{N,0} \) Bahri and Coron \[2\] established the existence of at least one positive solution to problem \( (1.1) \) in every domain \( D \) having nontrivial reduced homology with \( \mathbb{Z}/2 \)-coefficients. However, the topology in the supercritical case is not enough to guarantee existence. In fact, for each \( 1 \leq \ell \leq N - 3 \), Passaseo \[18, 19\] exhibited domains having the homotopy type of an \( \ell \)-dimensional sphere in which problem \( (1.1) \) does not have a nontrivial solution for \( q \geq 2^*_{N,\ell} \). Existence may fail even in domains with richer topology, as shown by Clapp-Faya-Pistoia \[5\].

Many results about solvability of \( (1.1) \) have been obtained when the exponent \( q \) is close to \( 2^*_{N,\ell} \) for some integer \( \ell \). In particular, in this case it is possible to find positive and sign changing solutions which blows-up at \( \ell \)-dimensional manifolds as \( q \) approaches \( 2^*_{N,\ell} \). In the easiest case \( \ell = 0 \) many authors have constructed positive and sign changing solutions which blows-up at one or more points in

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$\mathcal{D}$ as $q$ approaches the usual critical Sobolev exponent $2^*_{N,0}$ (see for example Bahri-Li-Rey [3], Bartsch-Micheletti-Pistoia [4], Musso-Pistoia [17], Pistoia-Weth [21], Del Pino-Felmer-Musso [8]). We note that we could think at a point like a 0−dimensional manifold! When $\ell = 1$ Del Pino-Musso-Pacard [9] built a positive solution to (1.1) which blows-up at a suitable geodesic (i.e. 1−dimensional manifold) of the boundary of $\mathcal{D}$ as $q$ approaches $2^*_{N,1}$. Recently, positive and sign changing solutions blowing-up at $\ell$−dimensional manifolds as $q$ approaches $2^*_{N,\ell}$ have been built in domains involving symmetries by Ackermann-Clapp-Pistoia [1] and Kim-Pistoia [14, 13].

There are a few results about existence and multiplicity of solutions to problem (1.1) in the pure supercritical case, i.e. $q = 2^*_{N,\ell}$. In particular, Yan-Wei [22] exhibited a torus-like domain in which $\mathcal{D}$ is an annulus in $\mathbb{R}^{2N}$ with a think spherical hole.

In this paper we consider the supercritical problem (1.1) in a domain with $\ell$−dimensional hole, namely

$$\begin{cases}
\Delta u + |u|^{q-2} u = 0 & \text{in } \mathcal{D}_\epsilon, \\
u = 0 & \text{on } \partial \mathcal{D}_\epsilon,
\end{cases}$$

where $\mathcal{D}_\epsilon := \mathcal{D} \setminus \{ x \in \mathcal{D} : \text{dist}(x, \Gamma_\ell) \leq \epsilon \}$, $\mathcal{D}$ is a smooth bounded domain in $\mathbb{R}^N$, $\Gamma_\ell$ is a closed $\ell$−dimensional manifold such that $\Gamma_\ell \subset \mathcal{D}$, $\epsilon$ is small enough and $q \geq 2^*_{N,\ell}$.

If $\ell = 0$, the set $\Gamma_\ell$ reduces to a point $\xi_0 \in \mathcal{D}$ and $\mathcal{D}_\epsilon := \mathcal{D} \setminus B(\xi_0, \epsilon)$ is the Coron’s type domain. In this case, it is known that problem (1.2) has one positive solution and an arbitrary large number of sign-changing solutions whose number increases as $\epsilon$ goes to zero, for almost all the exponents $q's$. The critical case $q = 2^*_{N,0}$ has been studied by Coron [7], Musso-Pistoia [16] and Ge-Musso-Pistoia [12]. When $q > 2^*_{N,0}$ and $q$ is different from a resonant sequence $q_j \nearrow +\infty$, the result has been obtained by Del Pino-Wei [10] and Dancer-Wei [11].

A question naturally arises: if $1 \leq \ell \leq N - 3$ and $q = 2^*_{N,\ell}$ or $q > 2^*_{N,\ell}$ (possibly different from a resonant sequence $q_j \nearrow +\infty$) does problem (1.2) have one positive solutions and an arbitrary large number of sign changing solutions whose number increases as $\epsilon$ goes to zero?

In this paper we give a positive answer in the pure supercritical case $q = 2^*_{N,\ell}$ provided the domain $\mathcal{D}$ satisfies some symmetry assumptions. In particular, for any integer $1 \leq \ell \leq N - 3$ we build torus-like domains $\mathcal{D}$ and torus-like manifolds $\Gamma_\ell$ for which the number of sign-changing solutions to problem (1.2) with $q = 2^*_{N,\ell}$ increases as $\epsilon$ goes to zero. These solutions have an arbitrary large number of alternate positive and negative layers which concentrate with different rates along the $\ell$−dimensional manifold $\Gamma_\ell$. More precisely, let us state our main results.

Fix $\ell_1, \ldots, \ell_m \in \mathbb{N}$ with $\ell := \ell_1 + \cdots + \ell_m \leq N - 3$ and a bounded smooth domain $\Omega$ in $\mathbb{R}^n$ with $n := N - \ell$ such that

$$\Omega \subset \{(x_1, \ldots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{N-\ell-m} : x_i > 0, i = 1, \ldots, m\}. \quad (1.3)$$

Let $\xi_0 \in \Omega$ be fixed and set $\Omega_\epsilon := \Omega \setminus B(\xi_0, \epsilon)$ for $\epsilon$ small enough. Set

$$\mathcal{D} := \{(y^1, \ldots, y^m, z) \in \mathbb{R}^{\ell_1+1} \times \cdots \times \mathbb{R}^{\ell_m+1} \times \mathbb{R}^{N-\ell-m} : (|y^1|, \ldots, |y^m|, z) \in \Omega\} \quad (1.4)$$

and

$$\Gamma_\ell := \{(y^1, \ldots, y^m, z) \in \mathbb{R}^{\ell_1+1} \times \cdots \times \mathbb{R}^{\ell_m+1} \times \mathbb{R}^{N-\ell-m} : (|y^1|, \ldots, |y^m|, z) = \xi_0\}. \quad (1.5)$$

$\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$ and $\Gamma_\ell$ is an $\ell$−dimensional manifold in $\mathcal{D}$ which is diffeomorphic to $S^2 \times \cdots \times S^{\ell_m}$. Here $S^d$ is the unit sphere in $\mathbb{R}^{d+1}$. Set $\mathcal{D}_\epsilon := \mathcal{D} \setminus \{ x \in \mathcal{D} : \text{dist}(x, \Gamma_\ell) \leq \epsilon \}$. Note
that $\mathcal{D}$, $\Gamma_\ell$ and $\mathcal{D}_\ell$ are invariant under the action of the group $\Theta := O(\ell_1 + 1) \times \cdots \times O(\ell_m + 1)$ on $\mathbb{R}^N$ given by
\[
(g_1, \ldots, g_m)(y^1, \ldots, y^m, z) := (g_1 y^1, \ldots, g_m y^m, z).
\]
for every $g_i \in O(\ell_i + 1)$, $y^i \in \mathbb{R}^{\ell_i+1}$, $z \in \mathbb{R}^{N-\ell_m}$. Here, as usual, $O(d)$ denotes the group of linear isometries of $\mathbb{R}^d$. Let $q = 2^*_N,\ell$ and note that $2^*_N,\ell = 2^*_n,0$. We shall look for $\Theta$-invariant solutions to problem (1.2), i.e. solutions $v$ of the form
\[
v(y^1, \ldots, y^m, z) = u(|y^1|, \ldots, |y^m|, z).
\]
A simple calculation shows that $v$ solves problem (1.2) if and only if $u$ solves
\[
-\Delta u - \sum_{i=1}^m \frac{\ell_i}{x_i} \frac{\partial u}{\partial x_i} = |u|^{2^*_N,\ell-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]
This problem can be rewritten as
\[
-\text{div}(a(x)\nabla u) = a(x)|u|^{2^*_N,0-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]
where $a(x_1, \ldots, x_m) := x_1^{\ell_1} \cdots x_m^{\ell_m}$. Our goal is to construct solutions $u_\epsilon$ to problem (1.7) with an arbitrary large number of alternate positive and negative bubbles which accumulate with different rates at the same point $\xi_0$ as $\epsilon \to 0$. They correspond, via (1.6), to $\Theta$-invariant solutions $v_\epsilon$ of problem (1.2) with positive and negative layers which accumulate with different rates along the $\ell$-dimensional manifold $\Gamma_\ell$ defined in (1.3).

Thus, we are lead to study the more general anisotropic critical problem
\[
\begin{cases}
-\text{div}(a(x)\nabla u) = a(x)|u|^{2^*_{n,0}-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $\xi_0 \in \Omega$, $\Omega_\epsilon := \Omega \setminus B(\xi_0, \epsilon)$ for $\epsilon$ small enough and $a \in C^2(\Omega)$ satisfies $\min_{x \in \Omega} a(x) > 0$.

First of all, we construct sign-changing solutions of (1.8) whose shape resemble a tower of bubbles with alternate sign centered at the point $\xi_0$. We recall that the bubbles
\[
U_{\delta,\xi}(x) := \alpha_n \left(\frac{\delta}{\delta^2 + |x-\xi|^2}\right)^{-\frac{n-2}{2}} \quad \text{for some } \delta > 0, \xi \in \mathbb{R}^n,
\]
where $\alpha_n = (n(n-2))^{-\frac{n-2}{2}}$ are the positive solution for the problem
\[
-\Delta u = u^{\frac{4n}{n-2}} \quad \text{in } \mathbb{R}^n, \quad u \in H^{1,2}(\mathbb{R}^n).
\]

Our main result concerning problem (1.8) reads as follows.

**Theorem 1.1.** Suppose $n \geq 4$. For any integer $k \geq 1$, there is $\epsilon_k > 0$ such that for each $\epsilon \in (0, \epsilon_k)$, problem (1.8) has a sign-changing solution $u_\epsilon$ which satisfies
\[
u_\epsilon(x) = \sum_{i=1}^k (-1)^i U_{\delta_i(\epsilon),\xi_i(\epsilon)}(x) + o(1) \quad \text{in } H^1(\Omega),
\]
where the weights $\delta_i(\epsilon)$ and the centers $\xi_i(\epsilon)$ satisfy for any $i = 1, \ldots, k$
\[
\epsilon^{-\frac{(n-2)+2i(k-i)}{2}} \delta_i(\epsilon) \to d_\epsilon > 0 \text{ and } \xi_i(\epsilon) \to \xi_0 \text{ as } \epsilon \to 0.
\]

It is clear that according to the previous discussion, by Theorem 1.1 we immediately deduce the following result concerning problem (1.2).
Theorem 1.2. Assume $1 \leq \ell \leq N - 4$. Then for any integer $k$ there exists $\epsilon_k > 0$ such that for any $\epsilon \in (0, \epsilon_k)$, problem (1.2) has a $\Theta$-invariant solution $v_\epsilon$ which satisfies

$$v_\epsilon(y) = \sum_{i=1}^{k} (-1)^i \tilde{U}_{\delta_i(\epsilon), \xi_i(\epsilon)}(y) + o(1) \quad \text{in } H^1(D_\epsilon),$$

where

$$\tilde{U}_{\delta, \xi}(y^1, \ldots, y^m, z) := U_{\delta, \xi}(|y^1|, \ldots, |y^m|, z),$$

the weights $\delta_i(\epsilon)$ and the centers $\xi_i(\epsilon)$ satisfy

$$\epsilon^{-\frac{(n-2)+2(n-1)}{2}} \delta_i(\epsilon) \to d_i > 0 \text{ and } \xi_i(\epsilon) \to \xi_0 \quad \text{as } \epsilon \to 0.$$

The proof of Theorem 1.1 relies on a very well known Lyapunov-Schmidt reduction. In particular, we will follow the arguments used in [16, 12]. We shall omit many details of the proof because they can be found, up to some minor modifications, in [16, 12]. We only compute what cannot be deduced from known results. The paper is arranged as follows. Section 2 contains the main steps of the proof of Theorem 1.1. In Section 3 and in Section 4 we study the reduced energy. Appendix is devoted to prove some technical results which are necessary to perform the finite dimensional reduction.

2. The scheme of the proof of Theorem 1.1

Let $H^1_0(\Omega_\epsilon)$ be the Sobolev space endowed with the inner product $(v_1, v_2)_\epsilon := \int_{\Omega_\epsilon} a(\nabla v_1 \cdot \nabla v_2)$ for $v_1, v_2 \in H^1_0(\Omega_\epsilon)$. Also, denote $\|v\|_\epsilon = ((v, v)_\epsilon)^{\frac{1}{2}}$ for all $v \in H^1_0(\Omega_\epsilon)$. Let $i_* : H^1_0(\Omega_\epsilon) \to L^{\frac{2n}{n-2}}(\Omega_\epsilon)$ be the Sobolev embedding and let $i_{\epsilon}^* : L^{\frac{2n}{n-2}}(\Omega_\epsilon) \to H^1_0(\Omega_\epsilon)$ be its adjoint so that $i_{\epsilon}^*(v) = u$ if and only if $-\text{div}(a\nabla u) = av$ in $\Omega_\epsilon$ and $u = 0$ on $\partial \Omega_\epsilon$. Note that there exists a constant $C > 0$ which depends only on the dimension $n$ such that $\|i_{\epsilon}^*(v)\|_\epsilon \leq C\|v\|_{L^{\frac{2n}{n-2}}(\Omega_\epsilon)}$ for any $v \in L^{\frac{2n}{n-2}}(\Omega_\epsilon)$.

For any given $w \in D^{1,2}(\mathbb{R}^n)$, we denote by $P_\epsilon w \in H^1_0(\Omega_\epsilon)$ the unique solution of the Dirichlet problem $\Delta P_\epsilon w = \Delta w$ in $\Omega_\epsilon$ and $P_\epsilon w = 0$ on $\partial \Omega_\epsilon$.

The solutions we are looking for have the form

$$u_\epsilon = V_\epsilon + \phi, \quad V_\epsilon = \sum_{i=1}^{k} (-1)^{i+1} P_\epsilon U_i,$$

where the bubble $U_i := U_{\delta_i, \xi_i}$ is given in (1.9) with

$$\delta_i := \epsilon^{\frac{(n-2)+2(n-1)}{2}} d_i, \quad \xi_i := \xi_0 + \delta_i \sigma_i \quad \text{for some } d_i > 0, \sigma_i \in \mathbb{R}^n \quad (i = 1, \ldots, k)$$

and $\phi$ is a remainder term which belongs to a suitable space defined as follows.

It is well known that the space of solutions of the linearized equation

$$-\Delta \psi = pU_{\delta, \xi}^{-1}\psi \quad \text{in } \mathbb{R}^n, \quad \|\psi\|_{L^{\infty}(\mathbb{R}^n)} < \infty$$

is spanned by $(n+1)$ functions

$$\psi^0_{\delta, \xi}(x) := \frac{\partial U_{\delta, \xi}}{\partial \delta} = \alpha_n \left( \frac{n - 2}{2} \right) \delta^{-\frac{n-4}{2}} \frac{|x - \xi|^2 - \delta^2}{\delta^2 + |x - \xi|^2}$$

and

$$\psi^j_{\delta, \xi}(x) := \frac{\partial U_{\delta, \xi}}{\partial \xi_j} = \alpha_n (n - 2) \delta^{\frac{n-2}{2}} \frac{x_j - \xi_j}{\delta^2 + |x - \xi|^2}$$

for $j = 1, \ldots, n$. 
We set \( \psi_i^j = \psi_{\delta, k}^j \), with parameters as in (2.2) and we define the subspaces of \( H^1_0(\Omega_\epsilon) \)

\[
K_\epsilon = \text{span}\{ P_\epsilon \psi_i^j : i = 1, \ldots, k, \ j = 0, 1, \ldots, n \}
\]

and

\[
K_\epsilon^+ = \left\{ \phi \in H^1_0(\Omega_\epsilon) : (\phi, P_\epsilon \psi_i^j)_\epsilon = 0 \text{ for } i = 1, \ldots, k, \ j = 0, 1, \ldots, n \right\}.
\]

We also introduce the projections \( \Pi_\epsilon : H^1_0(\Omega_\epsilon) \to K_\epsilon \) and \( \Pi_\epsilon^+ : H^1_0(\Omega_\epsilon) \to K_\epsilon^+ \). As it is usual in the Lyapunov-Schmidt procedure, solving problem (1.8) is equivalent to finding a function \( \phi \in K_\epsilon^+ \)

Firstly we solve equation (2.7).

\[
\Pi_\epsilon \left( V_\epsilon + \phi - i^*_\epsilon \left( |V_\epsilon + \phi|^{\frac{4}{n-2}}(V_\epsilon + \phi) \right) \right) = 0
\]

and

\[
\Pi_\epsilon \left( V_\epsilon + \phi - i^*_\epsilon \left( |V_\epsilon + \phi|^{\frac{4}{n-2}}(V_\epsilon + \phi) \right) \right) = 0
\]

where \( V_\epsilon \) is a function given in (2.1) which depends on \( \epsilon, d \) and \( \sigma \).

Proposition 2.1. For any compact set \( \mathcal{C} \subset (0, \infty)^k \times (\mathbb{R}^n)^k \), there exists \( \epsilon_0 > 0 \) such that for each \( \epsilon \in (0, \epsilon_0) \) and \( (d, \sigma) \in \mathcal{C} \), equation (2.4) possesses a unique solution \( \phi_\epsilon^{d, \sigma} \in K_\epsilon \) satisfying

\[
\| \phi_\epsilon^{d, \sigma} \|^2 = o \left( \epsilon^{\frac{n-2}{2(n+1)}} \right).
\]

Moreover, \( (d, \sigma) \mapsto \phi_\epsilon^{d, \sigma} \) is a \( C^1 \)-map.

The proof is postponed in Appendix A.

Then, the problem reduces to find \((d, \sigma)\) which solves (2.8). Notice that equation (1.8) has a variational structure, namely, its solutions are critical points of the energy functional

\[
I_\epsilon(u) = \frac{1}{2} \int_{\Omega_\epsilon} a(x)|\nabla u|^2 dx - \frac{1}{p+1} \int_{\Omega_\epsilon} a(x)|u|^{p+1} dx \quad \text{for } u \in H^1_0(\Omega_\epsilon).
\]

We introduce the reduced energy

\[
J_\epsilon(d, \sigma) := I_\epsilon(V_\epsilon^{d, \sigma} + \phi_\epsilon^{d, \sigma}) \quad \text{for } (d, \sigma) \in (0, \infty)^k \times (\mathbb{R}^n)^k
\]

where the superscript of \( V_\epsilon^{d, \sigma} := V_\epsilon \) (see (2.1)) emphasizes its dependence on \( d \) and \( \sigma \). Next, we reduce the problem to a finite dimensional one.

Proposition 2.2. If \((d, \sigma)\) is a critical point of \( J_\epsilon \), then \( V_\epsilon^{d, \sigma} + \phi_\epsilon^{d, \sigma} \) is a solution of (1.8).

The proof is postponed in Appendix A.

Therefore, since the problem is reduced to looking for a critical point of the reduced energy \( J_\epsilon \), we need its asymptotic expansion.

Proposition 2.3. Assume \( n \geq 4 \). It holds true that

\[
J_\epsilon(d, \sigma) = c_1 \kappa(\xi_0) + \Psi(d, \sigma) \epsilon^{\frac{n-2}{2(n+1)}} \left( a_0 + a \left( \epsilon^{\frac{n-2}{2(n+1)}} \right) \right)
\]

where \( \Psi(d, \sigma) = c_2 (\nabla a(\xi_0), \sigma_1) d_1 + \frac{c_3 \phi(\xi_0)}{1 + |\sigma_k|^2} \frac{\epsilon^{\frac{n-2}{2(n+1)}}}{d_k^{n-2}} + \frac{c_4 a(\xi_0)}{(1 + |\sigma_i|^2)^{\frac{n-2}{2}}} \left( \frac{d_{i+1}}{d_i} \right)^{\frac{n-2}{2}} \)

and \( c_1, \ldots, c_4 \) are positive constants.
The proof is postponed in Section 3.

The last step consists in showing that the leading term of the reduced energy has a critical point which is stable under $C^1$-perturbation.

**Proposition 2.4.** $\Psi$ has a nondegenerated critical point.

The proof is postponed in Section 4.

Finally, we have all the ingredients to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 2.4 and Proposition 2.3 it follows that $J_\epsilon$ has a critical point provided $\epsilon$ is small enough. By Proposition 2.2 the claim immediately follows. □

3. Expansion of the energy of approximate solutions

The purpose of this section is to provide the proof of Proposition 2.3. We start this section by recalling the lemma [12, Lemma 3.1], which describes the difference between $U_{\delta, \xi}$ and its projection $P_\epsilon U_{\delta, \xi}$ onto $H_0^1(\Omega)$. Denote by $G(x, y)$ the Green function associated to $-\Delta$ with Dirichlet boundary condition and $H(x, y)$ its regular part, that is, let $G(x, y)$ and $H(x, y)$ be functions defined by

$$
\begin{align*}
\begin{cases}
-\Delta_x G(x, y) = \delta_0(x) & \text{for } x \in \Omega, \\
G(x, y) = 0 & \text{for } x \in \partial \Omega,
\end{cases}
\end{align*}
$$

and

$$
G(x, y) = \gamma_n \left( \frac{1}{|x - y|^{n-2}} - H(x, y) \right)
$$

where $\gamma_n = \frac{1}{(n-2)|S^{n-1}|}$ and $|S^{n-1}| = (2\pi^{n/2})/\Gamma(n/2)$ is the Lebesgue measure of the $(n-1)$-dimensional unit sphere.

**Lemma 3.1.** Given $\delta > 0$ and $\xi \in \Omega$, such that $\epsilon = o(\delta)$ as $\epsilon \to 0$ and $|\xi - \xi_0| \leq c\delta$ for some $c > 0$ fixed, the following expansions are valid.

$$
U_{\delta, \xi}(x) = P_\epsilon U_{\delta, \xi}(x) + \alpha_\epsilon \delta^{\frac{n-2}{2}} H(x, \xi) + \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |\xi - \xi_0|^2)^{\frac{n-2}{2}}} |x - \xi_0|^{n-2}
$$

$$
+ \delta^{\frac{n-2}{2}} \cdot O \left( \epsilon + \left( \frac{\epsilon}{\delta} \right)^{n-1} \right)
$$

$$
\psi_{\delta, \xi}^0(x) = P_\epsilon \psi_{\delta, \xi}^0(x) + \alpha_\epsilon \left( \frac{n-2}{2} \right)^{\frac{n-2}{2}} H(x, \xi) + \alpha_n \left( \frac{n-2}{2} \right)^{\frac{n-2}{2}} \frac{|\xi - \xi_0|^2}{(\delta^2 + |\xi - \xi_0|^2)^{\frac{n-2}{2}}} |x - \xi_0|^{n-2}
$$

$$
+ \delta^{\frac{n-4}{2}} \cdot O \left( \epsilon + \left( \frac{\epsilon}{\delta} \right)^{n-1} \right)
$$

and

$$
\psi_{\delta, \xi}^j(x) = P_\epsilon \psi_{\delta, \xi}^j(x) + \alpha_\epsilon \delta^{\frac{n-2}{2}} \frac{\partial H}{\partial \xi_j}(x, \xi) - \alpha_n(n-2) \delta^{\frac{n-2}{2}} \frac{(\xi - \xi_0)_j}{(\delta^2 + |\xi - \xi_0|^2)^{\frac{n-2}{2}}} |x - \xi_0|^{n-2}
$$

$$
+ \delta^{\frac{n-2}{2}} \cdot O \left( \epsilon + \left( \frac{\epsilon}{\delta} \right)^{n-1} \right)
$$

for any $x \in \Omega$ and $j = 1, \ldots, n$. In particular,

$$
|\psi_{\delta, \xi}^0 - P_\epsilon \psi_{\delta, \xi}^0| = O \left( \delta^{\frac{n-2}{2}} + \frac{\epsilon^{n-2}}{\delta^2 |x - \xi_0|^{n-2}} \right),
$$

$$
|\psi_{\delta, \xi}^j - P_\epsilon \psi_{\delta, \xi}^j| = O \left( \delta^{\frac{n-2}{2}} + \frac{\epsilon^{n-2}}{\delta^2 |x - \xi_0|^{n-2}} \right)
$$

(3.1)
As a consequence of Lemma 3.1, we can obtain
\[
\delta = \frac{\delta_{l-1}}{\delta_{l+1}}^2 (x - \xi_0)^{n-2}, \quad x \in \Omega.
\]

Let us choose \( \rho > 0 \) so small that \( B(\xi_0, \rho) \subset \Omega \). Also, introduce the annulus \( A_l \) whose definition is given by
\[
A_l = B(\xi_0, \sqrt{\delta_{l-1}}) \setminus B(\xi_0, \sqrt{\delta_l \delta_{l+1}}) \quad \text{with} \quad \delta_0 = \frac{\rho^2}{\delta_1} \text{ and } \delta_{k+1} = \frac{\epsilon^2}{\delta_k} \quad (3.2)
\]
for \( l = 1, \cdots, k \). Then clearly
\[
\Omega_\epsilon = \left( \bigcup_{l=1}^k A_l \right) \cup (\Omega \setminus B(\xi_0, \rho)).
\]

As a consequence of Lemma 3.1, we can obtain

**Lemma 3.2.** For any \( i, j = 1, \cdots, k, \; i \neq l \), the following estimations are satisfied.

(i) \(
\int_{A_l} aU_l^{\frac{2n}{n-2}} = a(\xi_0) \int_{A_l} U_l^{\frac{2n}{n-2}} + \int_{A_l} (a - a(\xi_0))U_l^{\frac{2n}{n-2}}
\)
\[
= (n(n-2))^{\frac{1}{2}} a(\xi_0) \delta_l \left( \int_{\Omega} \frac{dy}{1 + |y|^2} \right)^n + o \left( \epsilon^{(n-1)+2(n-1)} \right);
\]

(ii) \(
\int_{A_l} U_l^{\frac{2n}{n-2}} (PU_l - U_l)
\)
\[
= -(n(n-2))^{\frac{1}{2}} |B_n| \left( \frac{\delta_{l+1}^2}{(1 + |\sigma_l|^2)^2} \epsilon^{(n-1)+2(n-1)} + o \left( \epsilon^{(n-1)+2(n-1)} \right) \right),
\]

(iii) \(
\int_{A_l} U_l^{\frac{n+2}{n-2}} U_i
\)
\[
= (n(n-2))^{\frac{1}{2}} |B_n| \left( \frac{\delta_{l+1}^2}{(1 + |\sigma_l|^2)^2} \epsilon^{(n-1)+2(n-1)} + o \left( \epsilon^{(n-1)+2(n-1)} \right) \right);
\]

(iv) \(
\int_{A_l} |PU_l - U_l|^{\frac{2n}{n-2}} + \int_{A_l} U_i^{\frac{2n}{n-2}} = O \left( \epsilon^{(n-1)+2(n-1)} \right);
\]

(v) \(
\int_{\Omega \setminus A_l} aU_l^{p+1}, \; \int_{\Omega} aU_l^p (PU_l - U_l), \; \int_{\Omega \setminus A_l} aU_l^p U_i = o \left( \epsilon^{(n-1)+2(n-1)} \right);
\]

(vi) \(
\int_{\Omega} (a - a(\xi_0))U_l^{\frac{2n}{n-2}} (PU_l - U_l), \; \int_{\Omega} (a - a(\xi_0))U_l^{\frac{n+2}{n-2}} U_i = o \left( \epsilon^{(n-1)+2(n-1)} \right)
\)
where \( \delta_{lk} \) is the Kronecker delta and \( |B_n| = \pi^{n/2} / \Gamma(n/2 + 1) \) denotes the volume of the \( n \)-dimensional unit ball.

and

**Lemma 3.3.** Assume \( i, \; l = 1, \cdots, k \). If \( n \geq 4 \), we have
\[
\int_{\Omega} (\nabla a \cdot \nabla P_i U_l) P_i U_l = o \left( \epsilon^{(n-1)+2(n-1)} \right) \quad (3.3)
\]
and
\[
\int_{\Omega} (\nabla a \cdot \nabla P_i U_i) \cdot \left( \epsilon^{(n-2)+2(k-1)} P_i \psi_i^2 \right) = \left\{ \begin{array}{ll}
\delta_{11} \delta_{11} \frac{\partial a}{\partial x_1} \epsilon_{2} c_2 \epsilon^{(n-1)+2(k-1)} + o \left( \epsilon^{(n-1)+2(k-1)} \right) & \text{for } j = 1, \ldots, n \\
o \left( \epsilon^{(n-1)+2(k-1)} \right) & \text{for } j = 0.
\end{array} \right.
\] (3.4)

**Remark 3.4.** In [1] Lemma A.2 (see also [13] Lemma A.9) and (A.1) below, the authors proved that
\[
\int_{\Omega} (\nabla a \cdot \nabla P_i U_i) P_i U_i, \quad \int_{\Omega} (\nabla a \cdot \nabla P_i U_i) \cdot (\delta_i P_i \psi_i^2) = O \left( \delta_i^{-\eta} \right) \quad \text{for some small } \eta > 0
\]
by utilizing Young’s inequality. However, this estimation is insufficient in our situation so we will pursue another approach making the use of Lemma 3.1 in a direct way, which turns out to be more complicated.

**Proof of Lemma 3.2.** The computations follow as an application of Lemma 3.1 or by the direct computation using the definition of \( U_i \) in (3.9). For the detailed exposition in similar settings, see [12] Section 3 and [13] Lemma 4.2.

**Proof of Lemma 3.3.** We prove (3.3) first. To do this, we decompose the left-hand side of (3.3) into
\[
\int_{\Omega} (\nabla a \cdot \nabla P_i U_i) P_i U_i = \int_{\Omega} (\nabla a \cdot \nabla (P_i U_i - U_i)) P_i U_i + \int_{\Omega} (\nabla a \cdot \nabla U_i) (P_i U_i - U_i) + \int_{\Omega} (\nabla a \cdot \nabla U_i) U_i
\] (3.5)
and estimate each of terms in the right-hand side. For brevity, we will use \( P_i U_i = P_i U_i \).

First, we compute the first term. Since \( 0 \leq P_i U_i \leq U_i \) in \( \Omega \) for all \( i = 1, \ldots, k \), it holds
\[
\int_{\Omega} |\nabla a| \cdot |\nabla (P_i U_i - U_i)| P_i U_i \leq C \|P_i U_i - U_i\|_{H^1(\Omega)} \cdot \|U_i\|_{L^2(\Omega)}
\]
\[
\leq \begin{cases}
C \delta_i \cdot \|P_i U_i - U_i\|_{H^1(\Omega)} & \text{if } n \geq 5, \\
C \delta_i |\log \delta_i| \cdot \|P_i U_i - U_i\|_{H^1(\Omega)} & \text{if } n = 4.
\end{cases}
\]
Moreover, since
\[
\int_{\Omega} |\nabla P_i U_i|^2 = \int_{\Omega} \nabla U_i \cdot \nabla P_i U_i = \int_{\Omega} U_i^2 \delta_i^{n+2} \quad \text{by applying (2.3)},
\]
we obtain that
\[
\|P_i U_i - U_i\|_{H^1(\Omega)}^2 = -\alpha_n \frac{2n}{\delta_i^n} \int_{\Omega} \frac{\delta_i^n}{\delta_i^2 + |x - \xi_i|^2} + \alpha_n^2 (n-2)^2 \delta_i^{n-2} \int_{\Omega} \frac{|x - \xi_i|^2}{\delta_i^2 + |x - \xi_i|^2} + \int_{\Omega} U_i^{\alpha_n^2} (U_i - P_i U_i)
\]
\[
= -\left[ \alpha_n \frac{2n}{\delta_i^n} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} + O \left( \delta_i^n + \left( \frac{\epsilon}{\delta_i} \right)^n \right) \right]
\]
\[
+ \begin{cases}
\alpha_n^2 (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} + O \left( \delta_i^{n-2} + \left( \frac{\epsilon}{\delta_i} \right)^n \right) \quad \text{if } n \geq 5,

\alpha_n^2 (n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} + O \left( \delta_i^{n-2} + \left( \frac{\epsilon}{\delta_i} \right)^n \right),
\end{cases}
\]

namely,
\[
\|P_i U_i - U_i\|_{H^1(\Omega)} = O \left( \delta_i^{n/2} + \left( \frac{\epsilon}{\delta_i} \right)^{n/2} \right).
\]
Thus the first term is \( o(\delta_1) \) provided \( n \geq 4 \). Note that our argument gives only \( \int_{\Omega} |\nabla a \cdot \nabla(P_i U_i - U_i)| \leq O(\sqrt{\delta_i}) \neq o(\delta_1) \) if \( n = 3 \).

Next, we consider the second term. From Lemma 3.1 we deduce

\[
\int_{\Omega_i} |\nabla a| |\nabla U_i| |P U_i - U_i| \leq C \int_{\Omega_i} |\nabla U_i| \left( \frac{\delta_i^{n-2}}{\delta_i} + \frac{\epsilon^{n-2}}{\delta_i^{n-2} |x - \xi_0|^{n-2}} \right).
\]

Denoting \( \overline{\Omega}_e = (\Omega - \xi)/\delta_i \), we get from (2.5) that

\[
\frac{\epsilon^{n-2}}{\delta_i^{n-2}} \int_{\Omega_i} |\nabla U_i| dx = \frac{\epsilon^{n-2}}{\delta_i^{n-2}} \int_{\Omega} \frac{|y - \sigma_i|}{(1 + |y - \sigma_i|^2)^{n-2}} \frac{1}{|\nabla U|} dy = O \left( (\delta_i \delta_i)^{n-2} \right) = o(\delta_i).
\]

Moreover there holds

\[
\frac{\epsilon^{n-2}}{\delta_i^{n-2}} \cdot \int_{\Omega_i} |\nabla U_i| \frac{1}{|x - \xi_0|^{n-2}} dx = C \delta_i \cdot \left( \frac{\epsilon}{\delta_i} \right)^{n-2} \left( \frac{\epsilon}{\delta_i} \right)^{n-2} \int_{\Omega_i} \frac{1}{(1 + |y - \sigma_i|^2)^{n-2}} \frac{1}{|\nabla U|} dy = o(\delta_i),
\]

from which we conclude that the second term is \( o(\delta_i) \).

As a result, it suffices to show that the third term is also \( o(\delta_1) \). We consider when \( i = l \) first. To estimate the term for this case, we will divide the domain \( \Omega_e \) into two disjoint sets \( B(\xi_0, \sqrt{\delta_i}) \setminus B(\xi_0, \epsilon) \) and \( \Omega \setminus B(\xi_0, \sqrt{\delta_i}) \) and then deal with each of the integrations of \( (\nabla a \cdot \nabla U_i) U_i \) over these domains. Employing the dimension assumption \( n \geq 4 \), we find that

\[
\int_{\Omega \setminus B(\xi_0, \sqrt{\delta_i})} |\nabla a \cdot \nabla U_i| U_i \leq C \int_{R^n \setminus B(\xi_0, \sqrt{\delta_i})} \frac{\delta_i^{n-2} |x - \xi_0|}{(\delta_i^2 + |x - \xi_0|^2)^{n-1} dx}
\]

\[
= C \delta_i \int_{R^n \setminus B(0,1/\sqrt{\delta_i})} \frac{|y - \sigma_i|}{(1 + |y - \sigma_i|^2)^{n-1}} dy = o(\delta_i).
\]

On the other hand, we see

\[
\left| \int_{B(\xi_0, \sqrt{\delta_i})} (\nabla a \cdot \nabla U_i) U_i \right|
\]

\[
= \sum_{j=1}^{n} \left| \frac{\partial a}{\partial x_j}(\xi_0) \right| \left| \int_{B(\xi_0, \sqrt{\delta_i}) \setminus B(\xi_0, \epsilon)} \frac{\partial U_i}{\partial x_j} U_i \right| + \left| \int_{B(\xi_0, \sqrt{\delta_i}) \setminus B(\xi_0, \epsilon)} ((\nabla a - \nabla a(\xi_0)) \cdot \nabla U_i) U_i \right|
\]

\[
\leq C \delta_i \sum_{j=1}^{n} \left| \frac{\partial a}{\partial x_j}(\xi_0) \right| \left| \int_{B(0,1/\sqrt{\delta_i}) \setminus B(0, \epsilon/\delta_i)} \frac{(y - \sigma_i)_j}{(1 + |y - \sigma_i|^2)^{n-1}} dy \right|
\]

\[
+ \delta_i \int_{B(0,1/\sqrt{\delta_i}) \setminus B(0, \epsilon/\delta_i)} |\nabla a(\xi_0 + \delta_i y) - \nabla a(\xi_0)| \frac{|y - \sigma_i|}{(1 + |y - \sigma_i|^2)^{n-1}} dy.
\]

Since

\[
\int_{B(0,1/\sqrt{\delta_i}) \setminus B(0, \epsilon/\delta_i)} (y - \sigma_i)_j \frac{1}{(1 + |y - \sigma_i|^2)^{n-1}} dy = - \left( \int_{B(0, \epsilon/\delta_i)} + \int_{B(0,1/\sqrt{\delta_i})^c} \right) \frac{(y - \sigma_i)_j}{(1 + |y - \sigma_i|^2)^{n-1}} dy = o(1)
\]

and

\[
|\nabla a(\xi_0 + \delta_i y) - \nabla a(\xi_0)| \leq \|D^2 a\|_{L^\infty(\Omega)} |\delta_i| \leq C \sqrt{\delta_i} \text{ for any } y \in B(0,1/\sqrt{\delta_i}),
\]

we get

\[
\int_{\Omega \setminus B(\xi_0, \sqrt{\delta_i})} |\nabla a \cdot \nabla U_i| U_i \leq o(\delta_i).
\]
we arrive at $\int_{B(\xi_0, \sqrt{c}) \setminus B(\xi_0, c)} (\nabla a \cdot \nabla U_i) = o(\delta_1)$. This implies that the third term of the right-hand side in (3.3) is $o(\delta_1)$ if $i = l$. For $i \neq l$, we have

$$
\int_{\Omega_c} |\nabla U_i| U_i = C \int_{\Omega_c} \frac{\delta_i^{n-2}}{(\delta_i^2 + |x - \xi_i|^2)^{\frac{n-2}{2}}} \frac{\delta_i^{n-2}}{(\delta_i^2 + |x - \xi_i|^2)^{\frac{n-2}{2}}} dx
$$

$$
\begin{cases}
C\delta_i \left( \frac{\delta_i}{\delta_1} \right)^{n-2} \int_{\mathbb{R}^n} \frac{|y - \sigma_i|}{(1 + |y - \sigma_i|^2)^{\frac{n-2}{2}}} \cdot |y - (\delta_1/\delta_i)\sigma_i|^{n-2} dy = o(\delta_1) = o(\delta_1) & \text{if } i < l, \\
\left( \frac{\delta_i}{\delta_1} \right)^{n-2} \int_{B(\xi_0, \sqrt{c}) \setminus B(\xi_0, c)} (\delta_i^2 + |x - \xi_i|^2)^{\frac{n-2}{2}} + ||\nabla U_i||_{L^2(\Omega, B(\xi_0, \sqrt{c}))} ||U_i||_{L^2(\Omega_c)}
\end{cases}
$$

$$
\begin{cases}
= o(\delta_1) \int_{B(0, \sqrt{c}) \setminus B(0, \sigma)} (\delta_i^2 + |x - \xi_i|^2)^{\frac{n-2}{2}} dy + O(\delta_1 |\log \delta_1|) \cdot O \left( \frac{\delta_i}{\sqrt{\delta_i}} \right)^{n-2} & \text{if } i > l = 1, \\
= o(\delta_1) \int_{B(0, \sqrt{c}) \setminus B(0, \sigma)} \frac{|\delta_i|^{n-2}}{(1 + |y - \sigma_i|^2)^{\frac{n-2}{2}}} dy + \int_{B(\xi_0, \sqrt{c}) \setminus B(\xi_0, c)} (\delta_i^2 + |x - \xi_i|^2)^{\frac{n-2}{2}} dy + O(\delta_1 |\log \delta_1|) \cdot O \left( \frac{\delta_i}{\sqrt{\delta_i}} \right)^{n-2} & \text{if } i > l \geq 2.
\end{cases}
$$

Hence (3.3) is true.

The derivation of (3.4) goes along the same way as the above except the part that corresponds to (3.5). In this case, instead of (3.5), we have that

$$
\int_{B(\xi_0, \sqrt{c}) \setminus B(\xi_0, c)} (\nabla a \cdot \nabla U_i) (\delta_i \psi_i^j) = \sum_{m=1}^n \frac{\partial a}{\partial x_m}(\xi_0) \int_{B(\xi_0, \sqrt{c}) \setminus B(\xi_0, c)} \frac{\partial U_i}{\partial x_m} \cdot (\delta_i \psi_i^j) + o(\delta_1)
$$

$$
= \delta_i \frac{\partial a}{\partial x_j}(\xi_0) c_2 \delta_1 + o(\delta_1)
$$

for $i = 1, \ldots, k$ where $c_2$ is a constant defined in (3.8) below. Thus (3.4) follows and the proof of Lemma 3.3 is completed. \hfill \square

Using Proposition 2.1, we can also check that

Lemma 3.5. It holds that

$$
I_{\epsilon}(V^{d, \sigma}_e + \phi^{d, \sigma}_e) - I_{\epsilon}(V^{d, \sigma}_e) = o \left( \epsilon^{(n-1)+\frac{n-2}{2-2(k-1)}} \right).
$$

Proof. We refer to the proof of [13] Lemma 4.1. \hfill \square

Now we are ready to prove that (2.10) holds $C^1$-uniformly on compact sets of $(0, \infty)^k \times (\mathbb{R}^n)^k$.

Proof of Proposition 2.3 By the previous lemma, it is sufficient to show that

$$
I_{\epsilon}(V^{d, \sigma}_e) = c_1 k + \Psi(d, \sigma) \epsilon^{\frac{n-2}{(n-1)+2(k-1)}} + o \left( \epsilon^{\frac{n-2}{(n-1)+2(k-1)}} \right)
$$

where $\Psi$ is the function given in (2.11) and $c_1 > 0$ is a fixed quantity. For simplicity, we set $p = (n+2)/(n-2)$ and omit the subscripts and superscripts of $V^{d, \sigma}_e$. 

As in (4.6) and (4.7) in [13], we write
\[
\frac{1}{2} \int_{\Omega} a|\nabla V|^2 = \frac{1}{2} \sum_{i=1}^{k} \left[ \int_{A_i} aU_i^{p+1} + \int_{\Omega \setminus A_i} aU_i^p (PU_i - U_i) + \int_{A_i} aU_i^{p+1} - \int_{\Omega \setminus A_i} (\nabla a \cdot \nabla PU_i) PU_i \right]
\]
\[+ \sum_{l<i} (-1)^{i+l} \left[ \int_{A_i} aU_i^{p+1}U_i + \int_{\Omega \setminus A_i} aU_i^p (PU_i - U_i) + \int_{A_i} aU_i^{p+1}U_i - \int_{\Omega \setminus A_i} (\nabla a \cdot \nabla PU_i) PU_i \right]
\]
and
\[
\frac{1}{p+1} \int_{\Omega} a|V|^{p+1} dx = \sum_{i=1}^{k} \left[ \frac{1}{p+1} \int_{A_i} aU_i^{p+1} + \int_{\Omega \setminus A_i} aU_i^p (PU_i - U_i) \right] + \sum_{i \neq l} (-1)^{i+l} \left[ \int_{A_i} aU_i^{p+1}U_i + \int_{A_i} aU_i^p (PU_i - U_i) \right]
\]
\[+ p \int_{0}^{1} (1-\theta) \int_{A_i} a \left| (-1)^{i+1}U_i + \theta (-1)^{i+1}(PU_i - U_i) + \sum_{i \neq l} (-1)^{i+1}PU_i \right|^{p-1}
\]
\[\left( (-1)^{i+1}(PU_i - U_i) + \sum_{i \neq l} (-1)^{i+1}PU_i \right)^2 dA + O(\delta_1^p).
\]
(3.7)

Following the computations which were conducted to obtain (4.8) in [13], we see that
\[\int_{A_i} aU_i^{p+1}U_i = \int_{A_i} aU_i^p U_i + o(\delta_1) \quad \text{for any pair} \ (i,l) \ \text{such that} \ i < l.
\]

Furthermore, (ii)-(iv) in Lemma 3.2 lead us to observe that the last term \( p \int_{0}^{1} (1-\theta) \int_{A_i} \cdots dx d\theta \) in the right-hand side in (3.7) is \( o(\delta_1) \) (see (4.14) in [13]). As a result, (v) in Lemma 3.2 and Lemma 3.3 yield
\[
\frac{1}{2} \int_{\Omega} a|\nabla V|^2 = \frac{1}{p+1} \int_{\Omega} a|V|^{p+1} dx
\]
\[= \frac{1}{n} \sum_{i=1}^{k} \int_{A_i} aU_i^{p+1} - \frac{1}{2} \sum_{l=1}^{k} \int_{\Omega \setminus A_i} aU_i^p (PU_i - U_i) - \sum_{l<i} (-1)^{i+l} \int_{A_i} aU_i^{p+1}U_i + o(\delta_1).
\]

However, (i)-(iii) and (vi) in Lemma 3.2 then give us (2.10) and (2.11) with
\[
c_1 = c_2 = \frac{1}{n} \cdot (n(n-2))^{\frac{p}{2}} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^n} \quad \text{and} \quad 2c_3 = c_4 = (n(n-2))^\frac{p}{2} |B_n|,
\]
(3.8)
as we desired.

Having (3.8) in mind, we can perform the \( C^1 \)-estimate in a similar way to [13, Subsection 5.1], which we omit.

4. Existence of critical points for \( \Psi \)

Here we will give the proof of Proposition 2.4. If \( k = 1 \) or 2, then the proof is relatively simple so we will assume here \( k \geq 3 \).
Setting \( d_1 = t_1, \ d_2 = t_1 t_2, \cdots, \ d_k = t_1 t_2 \cdots t_k \) and writing \( \mathbf{t} = (t_1, \cdots, t_k) \), let us define

\[
\tilde{\Psi}(\mathbf{t}, \mathbf{\sigma}) = \Psi(d(\mathbf{t}), \mathbf{\sigma}) = c_2 \langle \nabla a(\xi_0), \sigma_1 \rangle t_1 + \sum_{i=1}^{k-1} \frac{c_4 a(\xi_0)}{(1 + |\sigma_i|^2)^{\frac{n-2}{2}}} t_i^{\frac{n-2}{2}} + \frac{c_3 a(\xi_0)}{(1 + |\sigma_k|^2)^{n-2}} \frac{1}{(t_1 \cdots t_k)^{n-2}}
\]

for any

\((\mathbf{t}, \mathbf{\sigma}) \in (0, \infty)^k \times \Xi \) where \( \Xi := \{ (\sigma_1, \cdots, \sigma_k) : \langle \nabla a(\xi_0), \sigma_1 \rangle > 0, \sigma_2, \cdots, \sigma_k \in \mathbb{R}^n \} \).

Then for each fixed \( \mathbf{\sigma} \in \Xi \), there exists the unique point \( \mathbf{t} = \mathbf{t}(\mathbf{\sigma}) \) such that \( \frac{\partial \tilde{\Psi}}{\partial t_i} (\mathbf{t}, \mathbf{\sigma}) = 0 \) \((i = 1, \cdots, k)\).

In fact, after some computations, one can show that \( \frac{\partial \tilde{\Psi}}{\partial t_1} (\mathbf{t}, \mathbf{\sigma}) = 0 \) and \( \frac{\partial \tilde{\Psi}}{\partial t_i} (\mathbf{t}, \mathbf{\sigma}) = 0 \) \((i = 2, \cdots, k)\) are equivalent to

\[
c_2 \langle \nabla a(\xi_0), \sigma_1 \rangle t_1 = \frac{c_4 (n-2) a(\xi_0)}{2 (1 + |\sigma_{i-1}|)^{\frac{n-2}{2}}}, \quad t_2^{\frac{n-2}{2}} = \frac{c_3 (n-2) a(\xi_0)}{(1 + |\sigma_k|^2)^{n-2}} \frac{1}{(t_1 \cdots t_k)^{n-2}}.
\]

This system is uniquely solvable and the solution is given by

\[
t_1 = \frac{c_4 (n-2) a(\xi_0)}{2 c_2 \langle \nabla a(\xi_0), \sigma_1 \rangle (1 + |\sigma_1|^2)^{\frac{n-2}{2}}} t_2^{\frac{n-2}{2}}, \quad t_i = \frac{1 + |\sigma_i|^2}{1 + |\sigma_1|^2}, \quad t_2 \ (i = 3, \cdots, k)
\]

and

\[
t_2 = \left[ \frac{c_2 \langle \nabla a(\xi_0), \sigma_1 \rangle}{c_3 (n-2) a(\xi_0)} \cdot (1 + |\sigma_1|^2)^{\frac{n+2k-3}{2}} \cdot \prod_{i=2}^{k} \frac{1}{1 + |\sigma_i|^2} \right]^{\frac{2}{n+2k-3}}.
\]

Also, \( 4.\text{11} \) and the relation \( 2c_3 = c_4 \) (see \( 3.8 \)) ensure that

\[
\tilde{\Psi}(\mathbf{\sigma}) := \tilde{\Psi}(\mathbf{t}(\mathbf{\sigma}), \mathbf{\sigma}) = \frac{n + 2k - 3}{n - 2} \left[ c_2^{n-2} (c_3 a(\xi_0) (n-2))^{2k-1} \right]^{\frac{n-2}{n+2k-3}} \left[ \frac{\langle \nabla a(\xi_0), \sigma_1 \rangle}{1 + |\sigma_1|^2} \cdot \prod_{i=2}^{k} \frac{1}{1 + |\sigma_i|^2} \right]^{\frac{n-2}{n+2k-3}}.
\]

By inspection, we can see that \( \tilde{\Psi} \) has a maximum point \( \mathbf{\sigma}_0 = (\nabla a(\xi)/|\nabla a(\xi)|, 0, \cdots, 0) \) in \( \Xi \). Hence \((\mathbf{t}(\mathbf{\sigma}_0), \mathbf{\sigma}_0)\) is a critical point of \( \tilde{\Psi} \).

We claim that \((\mathbf{t}(\mathbf{\sigma}_0), \mathbf{\sigma}_0)\) is a nondegenerate critical point of \( \tilde{\Psi} \). Without loss of generality, we may assume that \( \nabla a(\xi_0) = (|\nabla a(\xi)|, 0, \cdots, 0) \). Then the determinant of the Hessian matrix \( D^2 \tilde{\Psi}(\mathbf{t}(\mathbf{\sigma}), \mathbf{\sigma}) \) is

\[
det \left( D^2 \tilde{\Psi}(\mathbf{t}(\mathbf{\sigma}), \mathbf{\sigma}) \right) = C \det \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}
\]
where $C > 0$ is some constant and $\mathcal{A} := A_1 + A_2$, $\mathcal{B}$ and $\mathcal{D}$ are $k \times k$, $k \times n$ and $n \times n$ matrices, respectively, defined by

$$
\mathcal{A}_1 = b_2 f(t) \begin{pmatrix}
\frac{n-1}{t_1^2} & \frac{n-2}{t_1 t_2} & \frac{n-2}{t_1 t_3} & \cdots & \frac{n-2}{t_1 t_k} \\
\frac{n-2}{t_1 t_2} & \frac{n-1}{t_2^2} & \frac{n-2}{t_2 t_3} & \cdots & \frac{n-2}{t_2 t_k} \\
\frac{n-2}{t_1 t_3} & \frac{n-2}{t_2 t_3} & \frac{n-1}{t_3^2} & \cdots & \frac{n-2}{t_3 t_k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{n-2}{t_1 t_k} & \frac{n-2}{t_2 t_k} & \frac{n-2}{t_3 t_k} & \cdots & \frac{n-1}{t_k^2}
\end{pmatrix}, \quad
\mathcal{A}_2 = b_2(n-4) \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & \frac{n-6}{2n^2/2} & 0 & \cdots & 0 \\
0 & 0 & \frac{n-6}{2n^2/2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{n-6}{2n^2/2}
\end{pmatrix}
$$

$$
\mathcal{B} = \begin{pmatrix}
b_1 & 0 & \cdots & 0 \\
-(n-2)b_2 \frac{n-4}{2n^2/2} & 0 & \cdots & 0 \\
0 & -b_2 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$

and

$$
\mathcal{D} = \begin{pmatrix}
(n-2)b_2 \frac{n-4}{2n^2/2} & 0 & \cdots & 0 \\
0 & \frac{n-6}{2n^2/2} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
$$

Note that here we used the notations $b_1$, $b_2$ and $f(t)$ to denote $b_1 := c_2|\nabla a(\xi_0)|$, $b_2 := c_2a(\xi_0)(n-2)$ and $f(t) := (t_1 \cdots t_k)^{2-n}$. Also, we applied the fact that the components of $\boldsymbol{\sigma}_0$ are $\sigma_i = (1, 0, \cdots, 0)$ and $\sigma_i = 0$ for $i \geq 2$ so that

$$
t(\boldsymbol{\sigma}_0) = \left( \frac{b_2}{b_1}, \frac{t_2}{2}, \frac{t_3}{2}, \cdots, \frac{t_2}{2} \right) \quad \text{and} \quad t_2 = 2 \frac{n+2k-5}{n+2k-3} \cdot \left( \frac{b_1}{b_2} \right)^{\frac{n+2k-5}{n+2k-3}}.
$$

On the other hand, $\det(\mathcal{A} - \mathcal{B} \mathcal{D}^{-1} \mathcal{B}^t) \neq 0$ guarantees the nondegeneracy of the matrix $D^2 \bar{\Psi}(t(\boldsymbol{\sigma}_0), \boldsymbol{\sigma}_0)$ and so it is sufficient to prove it. We see

$$
\mathcal{A} - \mathcal{B} \mathcal{D}^{-1} \mathcal{B}^t = \begin{pmatrix}
[(n-1) - \frac{2}{n-2}] \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & (n-1) \lambda^{-\beta_i} \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & (n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & \cdots & (n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 \\
(n-1) \lambda^{-\beta_i} \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & (n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & 2(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & \cdots & 2(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 \\
(n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & 2(n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & 6(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & \cdots & 4(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & 2(n-2) \lambda^2 \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & 4(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2 & \cdots & 6(n-2) \lambda^\beta \tilde{b}_i^2 \tilde{b}_j^2 \tilde{b}_k^2
\end{pmatrix}
$$

where $\lambda := 2 \frac{n+2k-6}{n+2k-3}$, $\tilde{b}_i := b_i^{-\beta_i^{-1}}$ for $i = 1, 2$, $\beta_1 := n-2$, $\beta_2 := n+4k-4$, $\beta_3 := -2k+1$, $\beta_4 := n+2k-5$, $\beta_5 := -3n+4k+12$, $\beta_6 := n-6$ and $\beta_7 := 2k+3$. Therefore

$$
\det(\mathcal{A} - \mathcal{B} \mathcal{D}^{-1} \mathcal{B}^t) = C \det(\begin{pmatrix}
(n-1) - \frac{2}{n-2} & n-1 & 1 & 1 & \cdots & 1 \\
n-1 & n-2 & 1 & 1 & \cdots & 1 \\
2(n-2) & 2(n-2) & 3 & 2 & \cdots & 2 \\
2(n-2) & 2(n-2) & 2 & 3 & \cdots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2(n-2) & 2(n-2) & 2 & 2 & \cdots & 3
\end{pmatrix}) = -C(n+2k-3) \neq 0
$$

for some $C > 0$. Here the second equality can be derived from the induction on $k$. This concludes the proof.
APPENDIX A. SKETCH OF PROOFS OF PROPOSITION 2.1 AND PROPOSITION 2.2

Here we sketch the proofs of Proposition 2.1 and Proposition 2.2. We omit many details which can be found in the literature. We only highlight the steps where the effect of the anisotropic coefficient $a$ leads to new estimates. Set $p = (n + 2)/(n - 2)$.

The first step is the estimate of the error term.

Lemma A.1. Let $R_\epsilon := \Pi_1^\perp \left( \Psi_\epsilon \left( \left| V_{\epsilon} \right| \frac{\nabla}{\nabla} \nabla V_{\epsilon} \right) - V_{\epsilon} \right)$. Then
\[
\| R_\epsilon \| = O \left( \epsilon^{\frac{n+6}{(n-1)+2(1+\sigma)}} \right) = o \left( \epsilon^{\frac{n-2}{(n-1)+2(1+\sigma)}} \right)
\]
where $o \left( \epsilon^{\frac{n-2}{(n-1)+2(1+\sigma)}} \right) \rightarrow 0$ as $\epsilon \rightarrow 0$ uniformly in $(d, \sigma) \in \Lambda_{d_0,s_0}$.

Proof. Write
\[
R_\epsilon = \Pi_1^\perp \left( \left| V_{\epsilon} \right| \nabla V_{\epsilon} - \sum_{i=1}^{k} \left| P_{\epsilon} U_{i} \right| \nabla \left| V_{\epsilon} \right| \right) + \sum_{i=1}^{k} \left| \nabla \log a \cdot \nabla PU_{i} \right| = R_1 + R_2 + R_3.
\]

In a similar way to [13, Section 3], we can check that $R_1, R_2 = O \left( \delta_1^{-1-\sigma} \right)$ and $R_3 = O(\delta_1^{-1})$ for any small $\delta_1 > 0$. Hence the lemma follows.

The following lemma is crucial in the proof of Propositions 2.1 and 2.2.

Lemma A.2. Assume $i \leq l$. Then it holds
\[
(P\psi_{l}^{m}, P\psi_{l}^{m})_{\epsilon} = \delta_{i0}\delta_{jm}a(\xi_0)\tilde{c}_{j}\delta_{i}^{-2} + o(\delta_{i}^{-2})
\]
for some $\tilde{c}_{0}, \tilde{c}_{1} = \cdots = \tilde{c}_{n} > 0$. Here $\delta_{jm} = 1$ if $j = m$ and $\delta_{jm} = 0$ otherwise.

Proof. By (2.3), we have
\[
(P\psi_{l}^{m}, P\psi_{l}^{m})_{\epsilon} = p \int_{\Omega_{\epsilon}} aU_{i}^{p-1}\psi_{l}^{m} \psi_{l}^{m} + p \int_{\Omega_{\epsilon}} aU_{i}^{p-1}\psi_{l}^{m} \left( P\psi_{l}^{m} - \psi_{l}^{m} \right) - \int_{\Omega_{\epsilon}} (\nabla a \cdot \nabla P\psi_{l}^{m})P\psi_{l}^{m}.
\]
The first term in the right-hand side in the above equality can be estimated as in [17, Lemma A.5], showing that
\[
p \int_{\Omega_{\epsilon}} aU_{i}^{p-1}\psi_{l}^{m} = \delta_{i0}\delta_{jm}a(\xi_0)\tilde{c}_{j}\delta_{i}^{-2} + o(\delta_{i}^{-2}).
\]
Also, the third term, which arises due to the anisotropy of $a$, can be handled by Green’s representation formula of $\nabla P\psi_{l}^{m}$ and Young’s inequality (see [13, Lemma A.9]). Indeed,
\[
\int_{\Omega_{\epsilon}} \| \nabla P\psi_{l}^{m} \| \| P\psi_{l}^{m} \| \leq \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{|P\psi_{l}^{m}(x)||U_{i}^{p}(y)|}{|x-y|^{n-1}} \, dx \, dy 
\leq C \| P\psi_{l}^{m} \|_{L^{q}(\Omega_{\epsilon})} \| U_{i}^{p-1}\psi_{l}^{m} \|_{L^{r}(\Omega_{\epsilon})} = O \left( \delta_{i}^{-\frac{n-q}{2}} \right)
\]
for some $C > 0$ and any $q, r > 1$ such that $q^{-1} + r^{-1} < (n + 1)/n$. Thus choosing $q^{-1} = r^{-1} = (n + 1)/(2n - \eta)$ for some $\eta > 0$ sufficiently small, we conclude that the third term is $o(\delta_{i}^{-2})$. On the
other hand, from (3.1) in Lemma 3.1 we deduce that the second term satisfies
\[
p \int_{\Omega} aU_i^{p-1} \psi'(P \psi^m_i - \psi^m_i) = \int_{\Omega} \frac{\delta_i^2}{\delta_i^{n-4} \delta_i^{n-4}} \cdot O \left( \frac{\epsilon^{n-2}}{\delta_i^{n-2}} + \frac{\epsilon^{n-2}}{|x - \xi|^n} \right) = O \left( \frac{\epsilon^{n-2}}{\delta_i^{n-2}} \right)
\]
if \( j = 0 \) and \( m = 0 \). The other cases (either \( j \geq 1 \) or \( m \geq 1 \)) can be handled similarly. This completes the proof. □

**Proof of Proposition 2.2.** Let \( L_\alpha \phi = \phi - p \cdot \Pi^+ (i \ast \{V[p^{-1} \phi]\}) \) for \( \phi \in (K_{\alpha}^{d,t})^\perp \). (Note that \( K_{\alpha}^{d,t} \) in (2.6) depends on the choice of \((d, t)\). To emphasize it, we used the notation \((K_{\alpha}^{d,t})^\perp\). The main step for the proof is to show that there exist \( \epsilon_0 > 0 \) and \( c > 0 \) such that for each \( \epsilon \in (0, \epsilon_0) \) and \((d, t) \in \Lambda_{d_0, \sigma_0}\), the operator \( L_\alpha \) satisfies \( \|L_\alpha \phi\| \geq c \|\phi\|_\epsilon \) for all \( \phi \in (K_{\alpha}^{d,t})^\perp \).

On the contrary, suppose that there are sequences of positive numbers \( \{s_\epsilon\}_\epsilon \) and functions \( \phi_n \) such that \( \phi_n \in (K_{\alpha}^{d,t})^\perp \), \( \|\phi_n\|_{\epsilon_n} = 1 \) for all \( n \in \mathbb{N} \), and \( \|L_n \phi_n\|_{\epsilon_n} \to 0 \) as \( n \to \infty \). If we denote \( \psi_n = L_\alpha \phi_n \) and drop superscripts and subscripts, we obtain
\[
div(a \nabla \phi + p|V|^{p-1} \phi) = \div(a \nabla \psi) + \div(a \nabla \tau)
\]
for some \( \tau \in K \). Then by writing \( \tau = \sum_{i=1}^k \sum_{m=0}^n c_{im} P \psi^m_i \) with some \( c_{im} \in \mathbb{R} \) (see (2.6) and (2.10)) and multiplying (A.2) by \( p|V|^{p-1} \phi \phi^j_i \) \((i = 1, \ldots, k, \; j = 0, \ldots, n)\), we see that
\[
p \int_{\Omega} a|V|^{p-1} \phi \phi^j_i = - \sum_{i,m} c_{im} (P \psi^m_i, P \psi^m_i)_\epsilon.
\]
On the other hand, from (2.6), (12) (5.7) (see also (17) (3.7)) and the Young’s inequality argument, we get
\[
p \int_{\Omega} a|V|^{p-1} \phi \phi^j_i = \int_{\Omega} \|\nabla a \cdot \nabla P \psi^j_i \| + o (\delta_i^{-1}) = O \left( \|\nabla P \psi^j_i \|_{L^\infty(\Omega)} \right) + O (\delta_i^{-1}) = o (\delta_i^{-1}).
\]
Thus by considering Lemma A.2 we conclude \( c_{im} = o(\delta_i) \) for each \( l \) and \( m \), or equivalently, \( \|\tau\|_\epsilon = o(1) \). Then by testing (A.2) with \( u := \phi - (\psi + \tau) \), we obtain \( \int_{\Omega} a|V|^{p-1} u^2 \to 1/p \) as \( \epsilon \to 0 \). However, the nondegeneracy of (1.10) implies \( \int_{\Omega} a|V|^{p-1} u^2 \to 0 \) as \( \epsilon \to 0 \) as shown in Step 3 and 4 of the proof of [12] Lemma 5.1 (again all the terms involved with \( \nabla a \) are negligible). Therefore a contradiction arises, which proves the main step.

Now, the Fredholm alternative implies that the inverse \( L^{-1} \) of \( L \) exists, and by employing Lemma A.1 and the contraction mapping principle on the set \( \left\{ \phi \in K^+: \|\phi\|_\epsilon \leq c \epsilon^{n+1-2} \right\} \) for some \( c > 0 \) small, we can deduce that the operator \( T(\phi) := L^{-1}(N(\phi) + R) \) where \( N(\phi) := \Pi^+ \left( i^* \left( |V + \phi|^{p-1} (V + \phi) - |V|^{p-1} V - p \cdot |V|^{p-1} \phi \right) \right) \) has a fixed point, which satisfies (2.10) and is a solution of (2.7). Furthermore, the standard argument taking advantage of the implicit function theorem shows that the map \((d, \sigma) \mapsto \phi^{d, \sigma}_\epsilon \) is \( C^1 \). For a detailed treatment of these claims, we refer to [12] Proposition 2.1. □

**Proof of Proposition 2.2.** Given \((d, \sigma) = ((d_1, \cdots, d_k), (\sigma_1, \cdots, \sigma_k)) \in \Lambda_{d_0, \sigma_0}\), let \( s \) be any of \( d_1, \cdots, d_k, \sigma_1, \cdots, \sigma_k \) where \( \sigma_i = (\sigma_{i1}, \cdots, \sigma_{in}) \) \( \in \mathbb{R}^n \) for each \( i = 1, \cdots, k \).
Suppose $J'_\epsilon(d,\sigma) = 0$. Then, if we write $V = V^{d,\sigma}_\epsilon$ and $\phi = \phi^{d,\sigma}_\epsilon$, we get
\[ 0 = I'_\epsilon(V + \phi)(\partial_s V + \partial_s \phi) = \sum_{i,j} c_{ij} \left( (P\psi^j_i, \partial_s V)_\epsilon - (\partial_s P\psi^j_i, \phi)_\epsilon \right) \]
for some $c_{ij} \in \mathbb{R}$ ($i = 1, \cdots, k$ and $j = 0, \cdots, n$) where $\partial_s$ denotes $\frac{\partial}{\partial s}$. Hence the proof is done once we show that all $c_{ij}$’s are equal to 0.

By virtue of Lemma A.2, we have
\[ (P\psi^j_i, \partial_s V)_\epsilon = \begin{cases} (-1)^{i+1} \left[ \delta_{il} \delta_{ij} \xi_0 c_j d_l^{-1} \sigma_{ij} \delta_l^{-1} + o(\delta_l^{-1}) \right] & \text{if } s = d_l \text{ for } l = 1, \cdots, k, \\ (-1)^{i+1} \left[ \delta_{il} \delta_{jm} \sigma_0 \xi_0 c_j d_l^{-1} + o(\delta_l^{-1}) \right] & \text{if } s = \sigma_{lm} \text{ for } l = 1, \cdots, k, \ m = 0, \cdots, n \end{cases} \]
where $\sigma_{i0} := 1$. Besides one can deduce $\|\partial_s P\psi^j_i\|_\epsilon = O(\delta_l^{-1})$ as in the proof of [13, Lemma A.8], resulting $(\partial_s P\psi^j_i, \phi)_\epsilon = o(\|\partial_s P\psi^j_i\|_\epsilon) = o(\delta_l^{-1})$. These estimates are enough to draw that $c_{ij} = 0$, so our assertion is true. \qed

**References**

[1] N. Ackermann, M. Clápp, and A. Pistoia. Boundary clustered layers near the higher critical exponents. J. Differential Equations, 254:4168–4193, 2013.

[2] A. Bahri and J. M. Coron. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Comm. Pure Appl. Math., 41:255–294, 1988.

[3] A. Bahri, Y.-Y. Li, and O. Rey. On a variational problem with lack of compactness: the topological effect of the critical points at infinity. Calc. Var. Partial Diff. Eq., 3:67–93, 1995.

[4] T. Bartsch, A. M. Micheletti, and A. Pistoia. On the existence and the profile of nodal solutions of elliptic equations involving critical growth. Calc. Var. Partial Diff. Eq., 26:265–282, 2006.

[5] M. Clapp, J. Faya, and A. Pistoia. Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents. Calc. Var. Partial Diff. Eq., (to appear).

[6] M. Clapp, J. Faya, and A. Pistoia. Positive solutions to a supercritical elliptic problem which concentrate along a thin spherical hole. arXiv:1304.1907

[7] J.M. Coron. Topologie et cas limite des injections de sobolev. C. R. Acad. Sci. Paris Ser. I Math., 299:209–212, 1984.

[8] M. del Pino, P. Felmer, and M. Musso. Two-bubble solutions in the super-critical Bahri-Coron’s problem. Calc. Var. Partial Diff. Eq., 16:113–145, 2003.

[9] M. del Pino, M. Musso, and F. Pacard. Bubbling along boundary geodesics near the second critical exponent. J. Eur. Math. Soc., 12:1553–1605, 2010.

[10] M. del Pino and J. Wei. Supercritical elliptic problems in domains with small holes. Ann. Inst. H. Poincaré Anal. Non Linéaire, 24:507–520, 2007.

[11] E. N. Dancer and J. Wei. Sign-changing solutions for supercritical elliptic problems in domains with small holes. Manuscripta Math., 123:493–511, 2007.

[12] Y. Ge, M. Musso, and A. Pistoia. Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains. Commun. Partial Differ. Equ., 35:1419–1457, 2010.

[13] S. Kim and A. Pistoia. Boundary towers of layers for some supercritical problems. Preprint.

[14] S. Kim and A. Pistoia. Clustered boundary layer sign changing solutions for a supercritical problem. J. London Math. Soc., (to appear).

[15] J. Kazdan and F. Warner. Remarks on some quasilinear elliptic equations. Comm. Pure Appl. Math., 28:567–597, 1975.

[16] M. Musso and A. Pistoia. Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains. J. Math. Pures Appl., 86:510–528, 2006.

[17] M. Musso and A. Pistoia. Tower of bubbles for almost critical problems in general domains. J. Math. Pures Appl., 93:1–40, 2010.

[18] D. Passaseo. Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains. J. Func. Anal., 114:97–105, 1993.

[19] D. Passaseo. New nonexistence results for elliptic equations with supercritical nonlinearity. Diff. Int. Equat., 8:577–586, 1995.
[20] S. I. Pohožaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Dokl.*, 6:1408–1411, 1965.
[21] A. Pistoia and T. Weth. Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24:325–340, 2007.
[22] S. Yan and J. Wei. Infinitely many positive solutions for an elliptic problem with critical or supercritical growth. *J. Math Pures Appl.*, 96:307–333, 2011.

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