THE HIGHER DIMENSIONAL PROPOSITIONAL CALCULUS

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Abstract. In recent research, some of the present authors introduced the concept of an \( n \)-dimensional Boolean algebra and its corresponding propositional logic \( n \text{CL} \), generalising the Boolean propositional calculus to \( n \geq 2 \) perfectly symmetric truth values. This paper presents a sound and complete sequent calculus for \( n \text{CL} \), named \( n \text{LK} \). We provide two proofs of completeness: one syntactic and one semantic. The former implies as a corollary that \( n \text{LK} \) enjoys the cut admissibility property. The latter relies on the generalisation to the \( n \)-ary case of the classical proof based on the Lindenbaum algebra of formulas and Boolean ultrafilters.

1. Introduction

A research programme developed over the last few years [19, 12, 18, 5, 17, 4] has attempted to single out the features of Boolean algebras that let them encompass virtually all the “desirable” properties one can expect from a variety of algebras.

In the first part of this introduction, we summarise the main concepts and results obtained in this research. A Church algebra is an algebra \( A \) with term definable operations \( q \) (ternary) and 0, 1 (nullary), such that for every \( a, b \in A \), 
\[ q(1, a, b) = a \quad \text{and} \quad q(0, a, b) = b. \]

The ternary operation \( q \) intends to formalise the “if-then-else” construct, widely employed in computer science. An element \( e \) of a Church algebra \( A \) is called 2-central if \( A \) can be decomposed as the product 
\[ A/\theta(e, 0) \times A/\theta(e, 1), \]
where \( \theta(e, 0) (\theta(e, 1)) \) is the smallest congruence on \( A \) that collapses \( e \) and 0 (\( e \) and 1). According to a well-known result in the elementary structure theory of Boolean algebras, all elements of a Boolean algebra are 2-central. More generally, Church algebras, where every element is 2-central, were called Boolean-like algebras in [19], since the variety of all such algebras in the language \( (q, 0, 1) \) is term-equivalent to the variety of Boolean algebras. In [17, 4] and in the present paper, on the other hand, they are called Boolean algebras of dimension 2.

This approach can be generalised to algebras \( A \) having \( n \) term definable elements \( e_1, \ldots, e_n \) and one \((n+1)\)-ary term definable operation \( q \) satisfying 
\[ q(e_i, x_1, \ldots, x_n) = \ldots \]

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Accordingly, the notion of a 2-central element can be extended to that of an \( n \)-central element, that induces a decomposition of \( A \) into \( n \), rather than just 2, factors. In \([17, 4]\), naturally enough, these algebras were called \textit{Church algebras of dimension} \( n \) \((nCHs)\). Free \( \mathcal{V} \)-algebras \((\text{for } \mathcal{V} \text{ a variety})\), rings with unit, semimodules over semirings — hence, in particular, Boolean vector spaces — give rise to \( nCHs \), where \( n \) is generally greater than 2. Church algebras of dimension \( n \), all of whose elements are \( n \)-central, were given the name of \textit{Boolean algebras of dimension} \( n \) \((nBAs)\). Since \( n \)-central elements are equationally definable, the \( nBAs \) of a specified type form a variety. Those \( nBAs \) whose type includes just the \( q \) operation and the \( n \) constants are called \textit{pure}. Varieties of \( nBAs \) happen to share many remarkable properties with the variety of Boolean algebras. In particular:

- all subdirectly irreducible \( nBAs \) of type \( \tau \) have cardinality \( n \); moreover, any \( nBA \) of type \( \tau \) is a subdirect product of algebras of cardinality \( n \);
- any pure \( nBA \) is a subdirect power of the unique pure \( nBA \) \( n \) of cardinality \( n \), whose elements are considered as “generalised truth values”;
- for every \( n \geq 2 \) and type \( \tau \), all \( nBAs \) of type \( \tau \) having cardinality \( n \) are primal. Moreover, every variety generated by an \( n \)-element primal algebra is a variety of \( nBAs \).

In \([4]\) the theory of \( n \)-central elements was put to good use to yield an extension to arbitrary semirings of the technique of \textit{Boolean powers}. We defined the semiring power \( A[R] \) of an algebra \( A \) by a semiring \( R \), and showed that any pure \( nBA \) \( A \) is a retract of the semiring power \( A[B_{A}] \) of \( A \) by what we called the inner Boolean algebra \( B_{A} \) of \( A \). Foster's celebrated theorem on primal algebras follows as a corollary from this result.

In \([5]\) the connection between a noncommutative version of Boolean algebras, called skew Boolean algebras \([13]\), and pure \( nBAs \) is explored. Moreover, the notion of a multideal, which plays an analogous role as filters and ideals do for Boolean algebras, is introduced and studied.

In \([17]\) we focused on an application to logic. Just like Boolean algebras are the algebraic counterpart of classical propositional logic \( CL \), for every \( n \geq 2 \) we defined a logic \( nCL \) whose algebraic counterpart are \( nBAs \). We also proved that every tabular logic with a single designated value is a sublogic of some \( nCL \). Although we provided Hilbert-style calculi for each \( nCL \), the proof theory of these logics was not investigated in detail.

In the present paper, we intend to develop more perspicuous calculi for the logics \( nCLs \), which may afford better insights into the behaviour of higher-dimensional logical connectives. The proof-theoretic framework of Gentzen's \textit{sequent calculi} appears as a promising candidate to achieve such a goal. While it is relatively easy to transform the sequent calculus for classical logic into a calculus for the
equivalent logic 2CL, logics with a dimension greater than 2 present trickier challenges.

To face these issues, we introduce the framework of higher-dimensional calculi. Although our approach appears to be new, it presents similarities with other proof-theoretic methods available in the literature. In particular:

- A successful generalisation of Gentzen’s sequent calculi to finite-valued logics, dating back to Rousseau [16], consists in replacing 2-sided sequents \( \Gamma \Rightarrow \Delta \) by \( n \)-sided sequents \( \Gamma_1 | ... | \Gamma_n \), one for each different truth value. The rules for each connective can be directly read off the truth table of the connective itself, which dictates what side of the conclusion-sequent should host the principal formula depending on the whereabouts of the auxiliary formulas in the premise-sequents. Clearly, this process can be entirely automated: the programme Multlog (https://www.logic.at/multlog/) generates an \( n \)-sided calculus for any finite-valued logic whose truth tables are given as input [3]. Here, instead, we keep Gentzen’s 2-sided sequents but we split the turnstile \( \vdash \) into \( n \) different turnstiles \( \vdash_i \) (\( 1 \leq i \leq n \)), one for each truth value \( e_i \) of the calculus.

- Ordinary sequent calculi for a logic \( L \) aim at generating the \( L \)-valid sequents and, in particular, the tautologies of \( L \); refutation systems (see e.g. [10] for a survey) aim at generating the \( L \)-antivalent sequents and, in particular, the contradictions of \( L \). In the literature, a number of hybrid deduction-refutation systems have been proposed [9, 15, 11], characterised by the presence of two different turnstiles \( \vdash \) and \( \dashv \) for deduction and refutation, respectively. The rules of these calculi, generally speaking, admit the simultaneous presence, as premises or conclusions, of sequents formed with both turnstiles. In our calculus this idea is further generalised in so far as each truth value has its own associated turnstile.

- Hypersequents [1] are finite multisets, or sequences, of ordinary sequents, each of which is said to be a component of the hypersequent. While a sequent can be intuitively read as a claim to the effect that some formula in the consequent can be derived from the formulas in the antecedent, a hypersequent can be viewed as a disjunction of such derivability claims. Hypersequents are employed to obtain analytic calculi for several fuzzy, intermediate, relevant, and modal logics, for example (but not only) logics whose characteristic axioms have a disjunction as principal connective. Relational hypersequents (see e.g. [2, 7, 8]) generalise hypersequents in so far as different types of turnstiles can be used in different components. On some versions of the idea, components are expressions built by applying to formulas certain predicates of a first-order meta-language. It would be interesting to probe the affinities between this approach and the present one.
In this paper, for all \( n \geq 2 \), we define a sequent calculus \( n \)LK for the logic \( n \)CL, and we prove that \( n \)LK is sound and complete with respect to the natural \( n \)-valued semantics of \( n \)LK formulas. Soundness and completeness hold for all dimensions: we prove that the sequent \( \Gamma \vdash_i \Delta \) is provable if and only if it is valid, meaning that whenever an environment assigns the truth value \( i \) to all the formulas in \( \Gamma \), then it assigns the truth value \( i \) to at least one formula in \( \Delta \). Our first proof of completeness is based on fact that the Lindenbaum algebra of \( n \)LK is an \( n \)BA. This semantic proof relies on the cut rule of \( n \)LK, which is needed in particular to show that equiprovability in all dimensions is an equivalence relation.

We also provide another proof of completeness, based on syntactic properties of \( n \)LK, in particular on the invertibility of some of the rules of the sequent calculus. This syntactic completeness result states in particular that, whenever a sequent \( \Gamma \vdash_i \Delta \) is valid, then it is provable without using the cut rule, thus ensuring that \( n \)LK enjoys the cut admissibility property.

**Outline of the paper.** In Section 2 we recapitulate a few notions to make the present work reasonably self-contained. In particular, we give the definitions of Church algebras and of Boolean algebras of finite dimension, summarising the main results thus far obtained about them. In Section 3 we introduce the sequent calculus \( n \)LK. Section 3.2 and its companion Section 6.3 present the case \( n = 2 \). In section 4 we study some syntactic properties of \( n \)LK, and we provide examples of higher dimensional proofs. Section 5 presents the semantics of \( n \)LK, and the proof of soundness of the calculus. Section 6 contains the semantic proof of completeness. Some work is required, in particular, to show that the set of formulas of \( n \)LK, quotiented by equiprovability, is the universe of a Boolean algebra of dimension \( n \). In Section 7 we prove that \( n \)LK is sound and complete for \( n \)CL, the propositional logic of the \( n \)BAs. As a corollary, we have that \( n \)LK is equivalent to the Hilbert-style axiomatisation of \( n \)CL presented in [17]. In Section 8 we provide a syntactic proof of completeness, yielding the cut admissibility property.

2. Preliminaries

The proof of completeness of the \( n \)-dimensional propositional calculus presented is Section 6 relies on the fact that its Lindenbaum algebra, whose universe is the set of formulas quotiented by equiprovability, is a Boolean algebra of dimension \( n \). In this preliminary section we recall the main definitions and facts about \( n \)BAs, making this paper reasonably self-contained. For a start, let us introduce the concept of an \( n \)-dimensional Church algebra, from which the notion of an \( n \)-dimensional Boolean algebra has emerged. The proofs of the results stated in this section can be found in [17], [4] and [5].

2.1. Church algebras of finite dimension. In [17] we introduced Church algebras of dimension \( n \), algebras having \( n \) designated elements \( e_1, \ldots, e_n \) (\( n \geq 2 \))
and an operation $q$ of arity $n + 1$ satisfying $q(e_i, x_1, \ldots, x_n) = x_i$. The operator $q$ induces, through the so-called $n$-central elements, a decomposition of the algebra into $n$ factors.

**Definition 2.1.** An algebra $A$ of type $\tau$ is a Church algebra of dimension $n$ (an $n$CH, for short) if there are term definable elements $e_1^A, e_2^A, \ldots, e_n^A \in A$ and a term operation $q^A$ of arity $n + 1$ such that, for all $b_1, \ldots, b_n \in A$ and $1 \leq i \leq n$, $q^A(e_i^A, b_1, \ldots, b_n) = b_i$. A variety $V$ of type $\tau$ is a variety of algebras of dimension $n$ if every member of $V$ is an $n$CH with respect to the same terms $q, e_1, \ldots, e_n$.

If $A$ is an $n$CH, then $A_0 = (A, q^A, e_1^A, \ldots, e_n^A)$ is the pure reduct of $A$.

Church algebras of dimension 2 were introduced as Church algebras in [14] and studied in [19]. Examples of Church algebras of dimension 2 are Boolean algebras (with $q(x, y, z) = (x \land z) \lor (\neg x \land y)$) or rings with unit (with $q(x, y, z) = xz + y - xy$). Next, we provide one example of Church algebras having dimension greater than 2.

**Example 2.2.** ($n$-Subsets) Let $X$ be a set. An $n$-subset of $X$ is a sequence $(Y_1, \ldots, Y_n)$ of subsets of $X$. We denote by $\text{Set}_n(X)$ the family of all $n$-subsets of $X$. $\text{Set}_n(X)$ can be viewed as the universe of a Church algebra of dimension $n$, where $e_1 = (X, \emptyset, \ldots, \emptyset), e_2 = (\emptyset, X, \emptyset, \ldots, \emptyset), \ldots, e_n = (\emptyset, \ldots, \emptyset, X)$ and for all $y^0, \ldots, y^n$:

$$q(y^0, y^1, \ldots, y^n) = (\bigcup_{i=1}^{n} Y_i^{0} \cap Y_i^{1}, \ldots, \bigcup_{i=1}^{n} Y_i^{0} \cap Y_i^{n}).$$

In [20], Vaggione introduced the notion of a central element to study algebras whose complementary factor congruences can be replaced by certain elements of their universes. If a neat description of such elements is available, one usually gets important insights into the structure theories of the algebras at issue. To list a few examples, central elements coincide with central idempotents in rings with unit, with complemented elements in $\text{FL}_{ew}$-algebras, which form the equivalent algebraic semantics of the full Lambek calculus with exchange and weakening, and with members of the centre in ortholattices. In [19], T. Kowalski and three of the present authors investigated central elements in Church algebras of dimension 2. In [17], the idea was generalised to Church algebras of arbitrary finite dimension.

**Definition 2.3.** If $A$ is an $n$CH, then $c \in A$ is called $n$-central if the sequence of congruences $(\theta(c, e_1), \ldots, \theta(c, e_n))$ is an $n$-tuple of complementary factor congruences of $A$.

The following characterisation of $n$-central elements, as well as the subsequent elementary result about them, were also proven in [17].

**Theorem 2.4.** If $A$ is an $n$CH of type $\tau$ and $c \in A$, then the following conditions are equivalent:
(1) \( c \) is \( n \)-central;
(2) \( \bigcap_{i \leq n} \theta(c, e_i) = \Delta; \)
(3) for all \( a_1, \ldots, a_n \in A \), \( q(c, a_1, \ldots, a_n) \) is the unique element such that \( a_i \theta(c, e_i) q(c, a_1, \ldots, a_n) \), for all \( 1 \leq i \leq n; \)
(4) The following conditions are satisfied:

\[ \text{B1: } q(c, e_1, \ldots, e_n) = c. \]
\[ \text{B2: } q(c, x, x, \ldots, x) = x \text{ for every } x \in A. \]
\[ \text{B3: If } \sigma \in \tau \text{ has arity } k \text{ and } x \text{ is an } n \times k \text{ matrix of elements of } A, \]

of rows \( x_1, \ldots, x_n \) and columns \( x^1, \ldots, x^k \) then \( q(c, \sigma(x_1), \ldots, \sigma(x_n)) = \sigma(q(c, x^1), \ldots, q(c, x^k)) \).

For any \( n \)-central element \( c \) and any \( n \times n \) matrix \( x \) of elements of \( A \), a direct consequence of (B1)-(B3) gives

\[ \text{B4: } q(c, q(c, x_1), \ldots, q(c, x_n)) = q(c, x_1^1, x_2^2, \ldots, x_n^n). \]

**Proposition 2.5.** Let \( A \) be an \( n \)CH. Then the set of all \( n \)-central elements of \( A \) is a subalgebra of the pure reduct of \( A \).

Hereafter, we denote by \( \text{Ce}_n(A) \) the algebra \((\text{Ce}_n(A), q, e_1, \ldots, e_n)\) of all \( n \)-central elements of an \( n \)CH \( A \).

### 2.2. Boolean algebras of finite dimension.

Boolean algebras are Church algebras of dimension 2 all of whose elements are 2-central. It turns out that, among the \( n \)-dimensional Church algebras, those algebras all of whose elements are \( n \)-central inherit many of the remarkable properties that distinguish Boolean algebras.

**Definition 2.6.** An \( n \)CH \( A \) is called a Boolean algebra of dimension \( n \) (nBA, for short) if every element of \( A \) is \( n \)-central.

By Proposition 2.5, the algebra \( \text{Ce}(A) \) of all \( n \)-central elements of an \( n \)CH \( A \) is a canonical example of nBA. The class of all nBAs of type \( \tau \) is a variety of nCHs axiomatised by the identities B1-B3 in Theorem 2.4.

The following examples present two paradigmatic nBAs, playing the role of the Boolean algebras of universe \( \{0, 1\} \) and \( 2^X \) for some set \( X \), respectively, in the \( n \)-ary case.

**Example 2.7.** The algebra \( n \) of universe \( \{1, \ldots, n\} \) in the type of pure nBAs such that \( e_i^n = i \) and \( q^n(i, x_1, \ldots, x_n) = x_i \) for every \( i \leq n \) is a pure nBA.

**Example 2.8.** (\( n \)-Partitions: Example 2.2 continued) Let \( X \) be a set. An \( n \)-partition of \( X \) is an \( n \)-subset \((Y_1, \ldots, Y_n) \) of \( X \) such that \( \bigcup_{i=1}^n Y_i = X \) and \( Y_i \cap Y_j = \emptyset \) for all \( i \neq j \). The set of \( n \)-partitions of \( X \) is closed under the \( q \)-operator defined in Example 2.2 and constitutes the algebra of all \( n \)-central elements of the \( n \)CH \( \text{Set}_n(X) \) of all \( n \)-subsets of \( X \). Hence it is a pure nBA. Notice that the
algebra of \( n \)-partitions of \( X \), denoted by \( \text{Par}_n(X) \), is isomorphic to the \( n \)BA \( n^X \). In particular, the set of 2-partitions of a set \( X \) is nothing but the powerset of \( X \) in disguise.

Several remarkable properties of Boolean algebras find some analogue in the structure theory of \( n \)BAs.

**Theorem 2.9.** (1) An \( n \)BA \( A \) is subdirectly irreducible if and only if \( |A| = n \).

(2) Every \( n \)BA \( A \) is isomorphic to a subdirect product of \( B_1^i \times \cdots \times B_k^j \) for some sets \( I_1, \ldots, I_k \) and some \( n \)BAs \( B_1, \ldots, B_k \) of cardinality \( n \).

(3) Every pure \( n \)BA \( A \) is isomorphic to a subdirect power of \( n^I \), for some set \( I \).

A subalgebra of the \( n \)BA \( \text{Par}_n(X) \) of the \( n \)-partitions on a set \( X \), defined in Example 2.8, is called a field of \( n \)-partitions on \( X \). The Stone representation theorem for \( n \)BAs follows.

**Corollary 2.10.** Any pure \( n \)BA is isomorphic to a field of \( n \)-partitions on a suitable set \( X \).

### 2.3. Multideals and ultramultideals

In this section we recall from [5] some definitions and facts about the notion of multideal, whose role in the present context is analogous to the one played by filters and ideals in Boolean algebras. As a matter of fact, Proposition 2.16, needed in the proof of completeness of the \( n \)LK, is new.

**Definition 2.11.** Let \( A \) be an \( n \)BA. A multideal is an \( n \)-subset \((I_1, \ldots, I_n)\) of \( A \) such that:

- (m1) \( e_k \in I_k \), for \( k = 1, \ldots, n \);
- (m2) If \( a \in I_j \), \( b \in I_k \) and \( c_1, \ldots, c_n \in A \) then \( q(a, c_1, \ldots, c_{j-1}, b, c_{j+1}, \ldots, c_n) \in I_k \), for all \( 1 \leq j, k \leq n \);
- (m3) If \( a \in A \) and \( c_1, \ldots, c_n \in I_k \) then \( q(a, c_1, \ldots, c_n) \in I_k \), for all \( 1 \leq k \leq n \).

The set \( I = I_1 \cup \ldots \cup I_n \) is called the carrier of the multideal. An ultramultideal of \( A \) is a multideal whose carrier is \( A \).

The multideal \((I_1, \ldots, I_n)\) is proper if it is a \( n \)-partition of its carrier. The unique multideal of \( A \) that is not proper is the tuple \((A, \ldots, A)\).

As a matter of notation, if \( A \) is an \( n \)BA and \( x, y, z \in A \) let us write \( t_i(x, y, z) \) for \( q(x, y/\overline{i}, z/i) = q(x, y, \ldots, y, z, y, \ldots y) \), \( z \) being the \((i + 1)\)-th argument of \( q \).

**Definition 2.12.** Let \( A \) be an \( n \)BA and \( 1 \leq i \neq j \leq n \).

- The Boolean centre of \( A \) with respect to \( i, j \), denoted by \( A_{ij} \), is the Boolean algebra of 2-central elements of the 2CH \((A, t_i, e_i, e_j)\).
- The \((i, j)\)-coordinates of \( a \in A \) are the elements \( a_{(k)} = t_k(a, e_i, e_j) \in A_{ij} \), for \( 1 \leq k \leq n \).
The following lemma relates multideals of \( A \) to ideals/filters of \( A_{ij} \).

**Lemma 2.13.** Let \((I_1, \ldots, I_n)\) be a proper multideal of \( A \) and \( 1 \leq i \neq j \leq n \). Then \( I_i = A_{ij} \cap I_i \) is a Boolean ideal of \( A_{ij} \) and \( I^* = A_{ij} \cap I_j \) is the Boolean filter complement of \( I^* \).

The following lemma characterises multideals in terms of coordinates.

**Lemma 2.14.** Let \((I_1, \ldots, I_n)\) be a proper multideal of a \( nBA \), \( 1 \leq i \neq j \leq n \) and \( b \in A \). Then we have, for all \( 1 \leq r \leq n \), \( b \in I_r \) if and only if the \((i, j)\)-coordinate \( b(r) \) of \( b \) belongs to \( I_j \).

The main lemma toward Proposition 2.16 is the following one.

**Lemma 2.15.** Let \((I_1, \ldots, I_n)\) be a proper multideal, \( 1 \leq i \neq j \leq n \) and \( U \) be a Boolean ultrafilter of \( A_{ij} \) that extends \( I^* = A_{ij} \cap I_j \). Then we have:

(i) For all \( a \in A \), there exists a unique \( k \) such that \( a(k) \in U \).

(ii) For \( 1 \leq k \leq n \) let \( G_k = \{ a \in A : a(k) \in U \} \). Then \((G_1, \ldots, G_n)\) is a proper ultramultideal which extends \((I_1, \ldots, I_n)\).

The proof of the following proposition relies on the corresponding result for Boolean algebras and on the connections between multideals and ideals/filters established above.

**Proposition 2.16.** Let \((I_1, \ldots, I_n)\) be a multideal of \( A \), \( 1 \leq k \leq n \) and \( a \in A \) be such that \( a \notin I_k \). Then there exists a proper ultramultideal \((G_1, \ldots, G_n)\) extending \((I_1, \ldots, I_n)\) such that \( a \notin G_k \).

**Proof.** Let us choose \( 1 \leq i \neq j \leq n \). Since \( a \notin I_k \), then by Lemma 2.14 \( a(k) \notin I_j \). Since the Boolean filter \( I^* = A_{ij} \cap I_j \) is included in \( I_j \), then \( a(k) \notin I^* \). Let \( U \) be an ultrafilter of \( A_{ij} \) containing \( I^* \) such that \( a(k) \notin U \). By Lemma 2.15 (ii), the sequence \( G_r = \{ b \in A : b(r) \in U \} \) \((1 \leq r \leq n)\) is a proper ultramultideal extending \((I_1, \ldots, I_n)\) such that \( a \notin G_k \). \(\square\)

3. The \( n \)-dimensional Propositional Calculus

In this section we introduce the \( n \)-dimensional propositional calculus, \( nLK \) in what follows, whose deduction rules are given in Figure 1.

The formulas of \( nLK \) are built up starting from \( n \) different constants \( e_1, \ldots, e_n \), denoting the generalised truth values \( 1, \ldots, n \). Hence the denomination “higher-dimensional calculus”.

A peculiar characteristic of \( nLK \) is that it has a unique connective, \( q \), of arity \( n + 1 \). The intended meaning of this connective is the following: the truth value of the formula \( q(F, G_1, \ldots, G_n) \) is that of \( G_k, \) \( k \) being the truth value of \( F \). When

\footnotesize
[1] For the existence of such an ultrafilter, see for instance Theorem 3.17 in Chapter IV of [6].
$n = 2$, this is the usual interpretation of if\_then\_else($F, G_1, G_2$), by renaming the truth values: true $= 1$, false $= 2$.

Another distinctive feature of this deductive system is that each dimension $i \in \hat{n} = \{1, 2, \ldots, n\}$ has its own turnstile $\vdash_i$. In the 2-dimensional case, this gives rise to the turnstiles $\vdash_1$ and $\vdash_2$, the former deriving tautologies, and the latter deriving contradictions. Entailments in the various dimensions are symmetric, in the sense expressed by the rule (Sym) of Figure 1, that can be instantiated as follows, for $n = 2$ and for a given propositional formula $F$:

\[
\begin{align*}
\vdash_1 F & \quad \vdash_2 F^{(12)}
\end{align*}
\]

where $F^{(12)}$ is obtained by switching $e_1$ and $e_2$ in $F$ (see Definition 3.2). Intuitively $F^{(12)} \equiv q(F, e_2, e_1)$, whereas $F \equiv q(F, e_1, e_2)$. Notice that in the instantiation of the rule (Sym) above we have used the fact that $(F^{(12)})^{(12)} = F$, shown in Lemma 3.4.

In the case $n = 2$ there is a unique way of switching the constants $e_1$ and $e_2$, corresponding to the classical negation, whereas in general there are $\binom{n}{2}$ possible “negations”, and more generally there are $n! - 1$ ways of perturbing a formula by permuting the constants $e_1, \ldots, e_n$ in it. The permutations, and in particular the exchanges, are primitive in our syntax for the formulas of $n$LK.

3.1. Syntax of $n$LK. Let $S_n$ denote the group of permutations of $\hat{n}$, ranged over by $\pi, \rho, \sigma, \ldots$, and let $V = \{X, Y, Z, X', \ldots\}$ be a countable set of propositional variables.

**Definition 3.1.** A formula of $n$LK is either:
- a decorated propositional variable $X^\pi$, or
- one of the constants $e_1, \ldots, e_n$, or
- a compound formula $q(F, G_1, \ldots, G_n)$, where $F, G_1, \ldots, G_n$ are formulas.

We write $\mathcal{F}_n$ for the set of formulas.

The choice of decorating propositional variables by permutations deserves some explanations. As a matter of fact, $X^\pi$ will be proven to be logically equivalent to $q(X, e_{\pi(1)}, \ldots, e_{\pi(n)})$ (see Lemma 3.4) and we could have stipulated that atomic formulas are either propositional variables or constants. Nevertheless, the choice of decorating variables by permutations eases the task of defining an involutive form of negation.

**Definition 3.2.** Given $\rho \in S_n$ and a formula $F \in \mathcal{F}_n$, let $F^\rho$ be the formula inductively defined by:

\[
F^\rho = \begin{cases} 
X^{\rho \circ \pi} & \text{if } F = X^\pi \\
 e_{\rho(k)} & \text{if } F = e_k \\
 q(F, G^\rho_1, \ldots, G^\rho_n) & \text{if } F = q(F, G_1, \ldots, G_n).
\end{cases}
\]
Lemma 3.3. For each \( F \in \mathcal{F}_n \), \( \pi, \rho \in S_n \): \( (F^\pi)^\rho = F^{\rho \circ \pi} \) and \( F^{id} = F \).

Proof. Easy induction on \( F \). \qed

The simplest non-trivial permutations are the exchanges, noted \((ij)\). Given \( i, j \in \hat{n} \), the formula \( F^{(ij)} \) is the negation of \( F \) relatively to the dimensions \( i \) and \( j \). This kind of negation is involutive.

Lemma 3.4. For each \( F \in \mathcal{F}_n \), \( i, j \in \hat{n} \), we have that \( F^{(ij)(ij)} = F \).

Proof. By Lemma 3.3 since \((ij) \circ (ij) = id\). \qed

Thus, in \( n\mathrm{LK} \), the negations are strongly involutive in the sense that \( F^{(ij)(ij)} \) and \( F \) are the same formula.

Hence, in particular, replacing \( F^{(ij)(ij)} \) by \( F \) is not a semantic shortcut. Those are nothing but two different ways of writing the same formula. Throughout the paper, we will write \( F \) for \( F^{(ij)(ij)} \).

It is worth noticing that, by defining the size of a formula as the maximal nesting level of compound formulas in it, \( F \) and \( F^{(ij)} \) have the same size, for all formulas. In an inductive proof, when dealing with \( q(F, G_1, \ldots, G_n) \), the inductive hypothesis may be applied to \( F, G_1, \ldots, G_n \), and to \( F^{(ij)}, G_1^{(ij)}, \ldots, G_n^{(ij)} \) as well.

Contexts ranged over by \( \Gamma, \Delta, \ldots \) are finite multisets of formulas, written as sequences. If \( \Gamma = F_1, \ldots, F_n \), the notation \( \Gamma^{(ij)} \) stands for \( F_1^{(ij)}, \ldots, F_n^{(ij)} \). Also, in some premises of the deduction rules of Figure 1, \( \{ \Gamma_i \vdash_i \Delta_i \}_{i \in I} \), \( I \subseteq \hat{n} \), stands for a sequence of \( |I| \) premises, one for each \( i \in I \).

The notation

\[ \triangleright \Gamma \vdash_i \Delta \]

means that the sequent \( \Gamma \vdash_i \Delta \) is provable, using the rules of Figure 1.

The deduction rules of the systems may be justified using the notion of \( n \)-partition presented in Example 2.8.

If \( G^1 = (G^1_1, \ldots, G^1_n), G^2, \ldots, G^r, F^1, \ldots F^s \) are \( n \)-partitions of a set \( X \) and \( 1 \leq i \leq n \) then, intuitively, the sequent \( G^1, \ldots, G^r \vdash_i F^1, \ldots F^s \) states that

\[ \bigcap_{l=1}^r G^l_i \subseteq \bigcup_{l=1}^s F^l_i \]

This remark provides a handy guideline to verify the validity of the rules of figure 1 by considering that:

- \( e_i \) stands for the \( n \)-partition \((\emptyset, \ldots, \emptyset, X, \emptyset, \ldots, \emptyset)\), \( X \) being at the \( i \)-th position.
- If \( Y = (Y_1, \ldots, Y_n) \) is an \( n \)-partition and \( \sigma \) is a permutation, then \( Y^\sigma = (Y_{\sigma(1)}, \ldots, Y_{\sigma(n)}) \).
- The operator \( q_n \) acts on \( n \)-partitions as defined in Example 2.8.
Let us look now at rule (NegL), for instance: the premise states that the intersection of the i-th component of the elements of \( \Gamma \) is included in the union of the i-th component of \( F \) and of the i-th components of the elements of \( \Delta \). Let us write this as \( \bigcap \Gamma_i \subseteq F_i \bigcup \Delta_i \). Now, \( F_i \) and \( F_j = F^{(ij)}_j \) are disjoint, since \( i \neq j \). This ensures that \( \bigcap \Gamma_i \cap F^{(ij)}_i \subseteq \bigcup \Delta_i \), which is what is stated in the conclusion of the rule. This kind of argument applies to all the rules of nLK.

3.2. The classical case: syntax. Before going further, we compare the usual propositional calculus PC and the 2-dimensional propositional calculus 2LK. Formulas of PC are built with the constants 0, 1 (written in the same way as the corresponding truth values), variables \( X \in V \), and the connectives \( \neg \), \( \wedge \) and \( \vee \).

The sequent calculus is given in Figure 2. It is sound and complete: a sequent \( \Gamma \vdash \Delta \) is provable iff it is valid. We have chosen a Ketonen-style formulation which is close to the 2LK sequent calculus.

In order to compare PC and 2LK, we provide in Figure 3 translations from PC formulas to 2LK formulas and conversely. As a matter of notation, \( P, Q, R \) denote formulas of PC, and \( F, G, H \) formulas of 2LK, and we write \( \Gamma^o \) and \( \Gamma^* \) to denote the translations of sequences of formulas. Notice that, for \( n = 2 \) there exist two permutations of \( \hat{n} = \{1, \ldots, n\} \): the identity \( id \) and the exchange \( (12) \).

In Section 6.3 we will show that PC and 2LK are equivalent in the sense that:
Figure 2. The Propositional Calculus.

\[
\begin{align*}
\Gamma \vdash P \quad (\text{Id}) \\
\Gamma, P, Q \vdash \Delta \quad (\wedge L) \\
\Gamma, P \wedge Q \vdash \Delta \quad (\wedge R) \\
\Gamma, P \vdash \Delta \quad (\neg L) \\
\Gamma \vdash P, \Delta \quad (\neg R) \\
\Gamma, F \vdash \Delta \quad (\text{WeakL}) \\
\Gamma \vdash F, \Delta \quad (\text{WeakR}) \\
\Gamma, F, F \vdash \Delta \quad (\text{ConL}) \\
\Gamma \vdash F, F, \Delta \quad (\text{ConR})
\end{align*}
\]

Since both calculi are sound and complete with respect to suitable notions of validity of sequents, the easiest way to prove this equivalence is through the semantics.

We conclude this section by presenting two examples of simulations of PC by 2LK and vice-versa. When translating sequents of the form \( \Gamma \vdash_2 \Delta \) from 2LK to PC, we use the following property, that is an easy consequence of the second item above, by using the rule \text{Sym} of 2LK:

\[(3.1) \quad \forall \Gamma, \Delta \text{ sequences of 2LK formulas, we have } \triangleright \Gamma \vdash_2 \Delta \text{ iff } \triangleright \Gamma^o \vdash \Delta^o;\]

Since both calculi are sound and complete with respect to suitable notions of validity of sequents, the easiest way to prove this equivalence is through the semantics.

We conclude this section by presenting two examples of simulations of PC by 2LK and vice-versa. When translating sequents of the form \( \Gamma \vdash_2 \Delta \) from 2LK to PC, we use the following property, that is an easy consequence of the second item above, by using the rule \text{Sym} of 2LK:

\[(3.1) \quad \forall \Gamma, \Delta \text{ sequences of 2LK formulas, we have } \triangleright \Gamma \vdash_2 \Delta \text{ iff } \triangleright \Gamma^{(12)} \vdash \Delta^{(12)};\]

Figure 3. The translations 2LK↔PC.
Example 3.5. • Consider the following instance of the rule $\land R$ of PC:

$$
\begin{align*}
\vdash X & \quad \vdash Y \\
\vdash X \land Y
\end{align*}
$$

where $X$ and $Y$ are propositional variables. The 2LK formula $(X \land Y)^\circ = q(X^{id}, Y^{id}, e_2)$ may be proven as follows, under the hypothesis $\vdash_1 X^{id}$ and $\vdash_1 Y^{id}$

$$
\begin{align*}
\vdash_1 X^{id} & \\
\vdash_2 X^{id} & (\text{NegL}) \\
\vdash_1 Y^{id} & (\text{WeakL}) \\
\vdash_1 q(X^{id}, Y^{id}, e_2) & (\text{qR})
\end{align*}
$$

• The left rule for $\land$

$$
\begin{align*}
X, Y \vdash \\
X \land Y \vdash (\land L)
\end{align*}
$$

is translated as follows:

$$
\begin{align*}
\vdash_1 e_1 & (\text{Const}) \\
\vdash_1 e_2 & (\text{NegL}) \\
\vdash_1 (\text{Id}) & (\text{Sym}) \\
\vdash_2 e_1 & (\text{WeakL}) \\
\vdash_1 q(X^{id}, Y^{id}, e_2) & (\text{qL})
\end{align*}
$$

Example 3.6. • Consider the following instance of the rule $qR$ of 2LK:

$$
\begin{align*}
\vdash_1 Y^{id} & \\
\vdash_1 X^{id} \quad \vdash_2 Z^{(12)} & (\text{WeakR, } \lor R) \\
\vdash_1 q(X^{id}, Y^{id}, Z^{id}) & (\text{qR})
\end{align*}
$$

The PC formula $q(X^{id}, Y^{id}, Z^{id}) = (X \land Y) \lor (\neg X \land Z)$ may be proven as follows, under the hypothesis $X \vdash Y$ and $\neg X \vdash Z$ (notice that the latter hypothesis is obtained by $X^{id} \vdash Z^{(12)}$, by applying $[7,7]$):

$$
\begin{align*}
\vdash X & (\text{Id}) \\
\vdash X \land Y & (\land R) \\
\vdash X \land Y & (\text{WeakR, } \lor R) \\
\vdash (X \land Y) \lor (\neg X \land Z) & (\neg R) \\
\vdash (X \land Y) \lor (\neg X \land Z) & (\text{Cut})
\end{align*}
$$
• The rule \( (qL) \):

\[
\frac{X^{id}, Y^{id} \vdash_1 X^{id}, Z^{(12)} \vdash_2}{q(X^{id}, Y^{id}, Z^{id}) \vdash_1} \quad (qL)
\]

is translated as follows (notice the use of \( 3.1 \) in the translation of the rightmost premise \( X^{id}, Z^{(12)} \vdash_2 \)):

\[
\frac{X, Y \vdash (\land L)}{X \land Y \vdash (\land L)} \quad \frac{\neg X, Z \vdash (\land L)}{\neg X \land Z \vdash (\land L)} \quad \frac{(X \land Y) \lor (\neg X \land Z) \vdash}{(X \land Y) \lor (\neg X \land Z) \vdash} (\lor L)
\]

4. A closer look at the syntax of \( nLK \)

In this section, ended by two examples of proofs in \( 3LK \), we study some syntactic properties of \( nLK \), and in particular we introduce two derivable rules that will be used in the semantic completeness result of Section 6.

We show first that the property expressed in the axiom \( \text{Id} \) for decorated propositional variables may be proved for all the formulas. This is proved in Corollary 4.2.

**Lemma 4.1.** Let \( \sigma, \rho \in S_n \) and \( i \) be such that \( \rho^{-1}(i) \neq \sigma^{-1}(i) \). Then \( \vdash F^\sigma, F^\rho \vdash_1 \), for every formula \( F \).

**Proof.** By induction on \( F \).

(i) If \( F \) is a decorated variable \( X^\pi \), let us choose \( k = \rho(\sigma^{-1}(i)) \). From the hypothesis \( \rho^{-1}(i) \neq \sigma^{-1}(i) \) it follows immediately that \( k \neq i \). Consider

\[
\frac{(\sigma \circ \pi)^{-1}(i) = ((ik) \circ \rho \circ \pi)^{-1}(i)}{X^{\sigma \circ \pi} \vdash (ik) \circ \rho \circ \pi \vdash i \neq k} \quad \text{Id}
\]

\[
\frac{X^{\sigma \circ \pi}, X^{(ik) \circ \rho \circ \pi} \vdash_1}{X^{\sigma \circ \pi}, X^{(ik) \circ \rho \circ \pi} \vdash_1} \quad \text{NegL}
\]

The axiom \( \text{Id} \) can be applied since \( ((ik) \circ \rho \circ \pi)^{-1}(i) = \pi^{-1}(\rho^{-1}((ik)^{-1}(i))) = \pi^{-1}(\rho^{-1}(\rho(\sigma^{-1}(i)))) = \pi^{-1}(\sigma^{-1}(i)) = (\sigma \circ \pi)^{-1}(i) \). The conclusion of this proof is \( F^\sigma, F^\rho \vdash_1 \), as expected.

(ii) If \( F = e_j \) then either \( \sigma(j) \neq i \) or \( \rho(j) \neq i \). Let us suppose without loss of generality that \( \sigma(j) \neq i \). Consider

\[
\frac{\vdash e_{\sigma^{-1}(i)}, (i, \sigma(j))}{e_{\sigma^{-1}(i)}, (i, \sigma(j)) \vdash i \neq \sigma(j)} \quad \text{Const}
\]

\[
\frac{e_{\sigma^{-1}(i)}, (i, \sigma(j)) \vdash}{e_{\sigma^{-1}(i)} \vdash i} \quad \text{NegL}
\]

\[
\frac{e_{\sigma^{-1}(i)} \vdash i}{e_{\sigma^{-1}(i)} \vdash i} \quad \text{WeakL}
\]

The conclusion of this proof is \( F^\sigma, F^\rho \vdash_1 \), as expected.
(iii) If $F = q(G, H_1, \ldots, H_n)$ then proceeding as follows:

\[
\begin{align*}
\{ & G^{(j,l)}, H_j^{(j,l)}(i) \circ \rho, G, H_l^{(j,l)}(i) \circ \sigma \vdash_1 \} \subset \mathcal{L} \\
\{ & G, H_j^{(ij)} \circ q(G, H_1^\sigma, \ldots, H_n^\sigma)(ij) \vdash_j \} \subset \mathcal{L} \\
q(G, H_1^\sigma, \ldots, H_n^\sigma), q(G, H_1^\sigma, \ldots, H_n^\sigma) & \vdash_i
\end{align*}
\]

we obtain $n^2$ branches, for $1 \leq j, l \leq n$, whose leaves are of the form

\[
G^{(j,l)}, H_j^{(j,l)}(i) \circ \rho, G, H_l^{(j,l)}(i) \circ \sigma \vdash_1
\]

If $j = l$ then $H_j^{(j,l)}(i) \circ \rho, H_l^{(j,l)}(i) \circ \sigma \vdash_1$ is provable by the inductive hypothesis, and hence the sequent above may be proved by using the rule (WeakL), whereas if $j \neq l$ then $G^{(j,l)}, G \vdash_1$, is provable by the inductive hypothesis, and again the sequent above may be proved by weakening. This provides a proof of $F^\sigma, F^\rho \vdash_i$, as expected.

Lemma 4.1 allows us to generalise the axiom 1d to all the formulas.

**Corollary 4.2.** For all formulas $F$, $1 \leq i \leq n$ and permutations $\sigma, \rho$, if $\sigma^{-1}(i) = \rho^{-1}(i)$ then $\triangleright F^\sigma \vdash_i F^\rho$.

**Proof.**

\[
\begin{align*}
\{ & F^{(ij)} \circ \sigma, F^\rho \vdash_j \} \subset \mathcal{L} \\
F^\sigma & \vdash_i F^\rho
\end{align*}
\]

Each of the $n - 1$ premises is provable by Lemma 4.1 since $((ij) \circ \sigma)^{-1}(j) = \sigma^{-1}(i) = \rho^{-1}(i) \neq \rho^{-1}(j)$, for all $j \neq i$. □

Notice that Corollary 4.2 shows in particular that $\triangleright F \vdash_i F$ holds for every formula $F$ and dimension $i$. The next lemma introduces two derivable rules that we use in the rest of the paper.

**Lemma 4.3.** The rules

\[
\begin{align*}
\Gamma^{(ij)} & \vdash_i F, \Delta^{(ij)} \quad i \neq k \quad \text{Neg1} \\
\Gamma, F^{(jk)} & \vdash_j \Delta \\
\Gamma^{(ij)} & \vdash_i F, \Delta^{(ij)} \quad j \neq k \quad \text{Neg2} \\
\Gamma, F^{(ik)} & \vdash_j \Delta
\end{align*}
\]

are derivable.
Proof. Concerning Neg1 we have: if \( j = i \), then Neg1 is an instance of NegL. Otherwise:

\[
\Gamma^{(ij)} \vdash F, \Delta^{(ij)} \quad i \neq k
\]

\[
\frac{\Gamma^{(ij)}, F^{(ik)} \vdash \Delta^{(ij)}}{\Gamma^{(ij)} \vdash F^{(ik)} \circ (ik) \vdash \Delta^{(ij)}} \text{NegL}
\]

\[
\frac{\Gamma, F^{(ij)} \circ (ik), F^{(jk)} \vdash j \Delta}{\Gamma, F^{(ij)} \vdash j \Delta} \quad \text{Cut}
\]

where the rightmost premise of the cut rule is given by Corollary 4.2, plus weakenings, since \((jk)^{-1}(j) = ((ij) \circ (ik))^{-1}(j) = k\).

Concerning Neg2 we have: if \( j = i \), then Neg2 is an instance of NegL. Otherwise:

\[
\Gamma^{(ij)} \vdash F, \Delta^{(ij)} \quad i \neq j
\]

\[
\frac{\Gamma^{(ij)}, F^{(ij)} \vdash \Delta^{(ij)}}{\Gamma, F \vdash \Delta} \text{Sym}
\]

\[
\frac{\Gamma, F^{(ij)} \circ (ik), F^{(jk)} \vdash j \Delta}{\Gamma, F^{(jk)} \vdash j \Delta} \quad \text{WeakL}
\]

\[
\frac{\Gamma, F^{(ij)} \vdash \Delta}{\Gamma, F^{(ij)} \vdash \Delta} \quad \text{Cut}
\]

where the rightmost premise of the cut rule is given by Corollary 4.2, plus weakenings, since \((ik)^{-1}(j) = \text{id}^{-1}(j) = j\), and \(F = F^{id}\). □

An expected property of nLK is stated in the following lemma.

**Lemma 4.4.** For all \( i \neq j \in \hat{n}, \vdash e_i \vdash j \).

**Proof.**

\[
\frac{\vdash e_i (\text{Const}) \quad i \neq j}{e_i \vdash j} \quad \text{(Neg1)}
\]

□

We conclude this section with two examples of derivations in 3LK.

**Example 4.5.** (The exclusion of the \((n + 1)\)th) For each formula \( F \in \mathcal{F}_n \) and dimension \( i \in \{1, \ldots, n\} \), the sequent \( \vdash F^{(1i)}, F^{(2i)}, \ldots, F^{(ni)} \) is provable. Let us see the case \( n = 3, i = 1 \)

\[
F \vdash_2 F^{(12)} \quad F \vdash_3 F^{(13)}
\]

\[
\frac{\vdash_1 F, F^{(12)}, F^{(13)}}{F^{(13)}} \quad \text{NegR}
\]

Both premises are provable by Corollary 4.2, since \( F^{(12)} = F^{(13)} = F \).
Example 4.6. In 2LK $q(x, y, z)$ reads “if $x$ then $y$ else $z$”. In general, $q(x, y_1, \ldots, y_n)$ reads “if $x = e_1$ then $y_1$ else if $x = e_2$ then $y_2$ else ... else if $x = e_n$ then $y_n$”. The following 3LK derivation of the sequent $\vdash q(1, q(e_2, F, e_1, G), H, K)$ illustrates how the first argument of the $q$ connective acts as a projector. This sequent is provable since $q(e_1, q(e_2, F, e_1, G), H, K)$ is equivalent to $q(e_2, F, e_1, G)$, which in turn is equivalent to $e_1$, which is provable in dimension 1.

\[
\begin{array}{c}
\Gamma_1 q(e_2, F, e_1, G) \\
\vdash q(e_2, F, e_1, G) \\
\Gamma_1 q(e_1, q(e_2, F, e_1, G), H, K)
\end{array}
\]

The pending premises are all provable by Lemma 4.4 and weakenings.

5. Semantics of nLK

Environments, ranged over by $v, u, v_1, \ldots$ are functions from the set $V$ of propositional variables to the set $\hat{n}$ of dimensions, that should be considered here as generalised truth values.

Definition 5.1. Given an environment $v$, nLK formulas are interpreted into $\hat{n}$ as follows:

- $[X^\pi]_v = \pi(v(X))$;
- $[e_i]_v = i$, for all $i \in \hat{n}$;
- $[q(F, G_1, \ldots, G_n)]_v = [G_i]_v$ if $[F]_v = i$.

In this setting the notions of tautology, satisfiability etc. are relative to dimensions. For instance, $F$ is an $i$-tautology if $[F]_v = i$, for all $v$.

The following generalisation of the notion of logical consequence is used in order to prove the soundness of the calculus.

Definition 5.2. Given two sequences of formulas $\Gamma, \Delta$, and a dimension $i \in \hat{n}$ we say that $\Delta$ is an $i$-logical consequence of $\Gamma$, written $\Gamma \models_i \Delta$, if for all environments $v$, if $[G]_v = i$ for all $G$ in $\Gamma$, then there exists $F$ in $\Delta$ such that $[F]_v = i$.

If $\Gamma \models_i \Delta$, then we say that the sequent $\Gamma \vdash \Delta$ is valid.

Lemma 5.3. For all $F \in F_n$, environment $v$, dimensions $i$ and $j$, we have $[F]_v = i$ if and only if $[F^{(ij)}]_v = j$.

Proof. The proof is an easy induction on $F$. We describe case $F = q(G, H_1, \ldots, H_n)$:

$[q(G, H_1, \ldots, H_n)]_v = [H_{[G]_v}]_v = i$ iff by induction hypothesis

$[H_{[G]_v}]_v = [q(G, H_1^{(ij)}, \ldots, H_n^{(ij)})]_v = [q(G, H_1, \ldots, H_n)^{(ij)}]_v$. \hfill $\square$

Proposition 5.4 (Soundness). For all $i \in \hat{n}$, $\Gamma$ and $\Delta$, if $\Gamma \vdash \Delta$ then $\Gamma \models_i \Delta$. 

Proof. By induction on the proof of $\Gamma \vdash_i \Delta$. If the last rule is an instance of Const, Id, WeakL, WeakR, ConL, ConR or Cut, the conclusion is immediate. If it is Sym, NegL or NegR, the conclusion follows from Lemma 5.3.

Let us consider the case $qL$, using the same notation as in Figure 1: given $v$ such that

- for all $R$ in $\Gamma$, $[R]_v = i$.
- $[q(F, G_1, \ldots, G_n)]_v = [G_{iF}]_v = i$.

we have to show that there exists $H$ in $\Delta$ such that $[H]_v = i$. Let $j = [F]_v$, and consider the $j$-th premise: $\Gamma^{(ij)} F, G^{(ij)} \vdash_j \Delta^{(ij)}$. By Lemma 5.3 for all $R \in \Gamma$, we have $[R^{(ij)}]_v = j$, and $[G^{(ij)}]_v = [(G_{[F]}^{(ij)})]_v = j$. Moreover $[F]_v = j$, and by ind. hyp. $\Gamma^{(ij)} F, G^{(ij)} \vdash_j \Delta^{(ij)}$. Then there exists $H$ in $\Delta$ such that $[H^{(ij)}]_v = j$, and we conclude $[H]_v = i$ by Lemma 5.3.

The case $qR$ is similar, and omitted. □

6. Completeness

In this section we adapt to $nLK$ the completeness proof for the classical propositional calculus based on the use of the Lindenbaum algebra of formulas.

6.1. The Lindenbaum algebra $L_n$.

Definition 6.1. Two formulas $F$ and $G$ of $nLK$ are equivalent, written $F \sim G$, if, for all $i \in \hat{n}$, both $F \vdash_i G$ and $G \vdash_i F$ are provable.

Lemma 6.2. The relation $\sim$ is an equivalence relation on the set of formulas of $nLK$.

Proof. Symmetry is immediate. Reflexivity is shown in Corollary 4.2. Transitivity follows from the cut rule. □

Lemma 6.3. If $F_i \sim G_i$ for $0 \leq i \leq n$, then $q(F_0, \ldots, F_n) \sim q(G_0, \ldots, G_n)$.

Proof. We can apply $qL$: $q(F_0, \ldots, F_n) \vdash_i q(G_0, \ldots, G_n)$

\[ \left\{ F_0, F_j^{(ij)} \vdash_j q(G_0, \ldots, G_n)^{(ij)} \right\}_{j \in \hat{n}} (qL) \]

Therefore, we have to prove the sequents $F_0, F_j^{(ij)} \vdash_j q(G_0, \ldots, G_n)^{(ij)}$, for $1 \leq j \leq n$. Let us consider first the case $j = i$:

$\ldots \quad F_i \vdash_i G_i \quad F_0, F_i, G_0 \vdash_i G_i \quad F_0 \vdash_i G_0 \quad (Neg1) \quad F_0^{(ik)}; G_0 \vdash_k \quad F_0^{(ik)}; F_i^{(ik)}; G_0 \vdash_k G_k^{(ik)} (Weak^+) \quad F_0, F_i \vdash q(G_0, \ldots, G_n) (qR)$
In the above proof the $i$-th and $k$-th premisses are developed, for some $k \neq i$. All the other premisses are similar to the $k$-th. The conclusion follows.

Let us consider now the case $j \neq i$:

\[
\begin{array}{c}
\frac{F_j \vdash G_j}{F_j^{(ij)}, G_0 \vdash G_j^{(ij)}} \text{(Sym)} \\
\frac{F_0 \vdash G_0}{F_0^{(jk)}, G_0 \vdash G_k^{(ij)(jk)}} \text{(Neg1)} \\
\frac{F_0, F_j^{(ij)}, G_0 \vdash G_j^{(ij)}}{F_0, F_j^{(ij)}, G_0 \vdash q(G_0, \ldots, G_n)^{(ij)}} \text{(qR)}
\end{array}
\]

In the above proof the $j$-th and $k$-th premisses are developed, for some $k \neq j$. All the other premisses are similar to the $k$-th. The conclusion follows.  

By Lemmas [6.2] and [6.3] the relation $\sim$ is a congruence on $F_n$.

The next lemma explains the meaning of a decorated formula.

**Lemma 6.4.** For each $H \in F_n$ and $\pi \in S_n$, $H^{\pi} \sim q(H, e_{\pi(1)}, \ldots, e_{\pi(n)})$.

**Proof.** Given $i$, let us begin by proving that $\vdash q(H, e_{\pi(1)}, \ldots, e_{\pi(n)}) \vdash_i H^{\pi}$:

\[
\begin{array}{c}
\frac{\{H, e_{\pi(j)}^{(ij)} \vdash H^{(ij)}_{\pi}\}_{j \in \hat{n}}}{q(H, e_{\pi(1)}, \ldots, e_{\pi(n)}) \vdash_i H^{\pi} \text{(qL)}}
\end{array}
\]

We reason by case analysis on $j$: if $\pi(j) \neq i$, then $e_{\pi(j)}^{(ij)} = e_{(ij)\circ \pi(j)} \neq e_j$. Since the sequent $e_{\pi(j)}^{(ij)} \vdash_j$ is provable by Lemma [1.4], then by applying WeakR and WeakL we get $H, e_{\pi(j)}^{(ij)} \vdash H^{(ij)_{\circ \pi}}$. If $\pi(j) = i$, then we apply Corollary [1.2] to prove $H \vdash_j H^{(ij)_{\circ \pi}}$, because $((ij) \circ \pi)^{-1}(j) = j = i\sigma^{-1}(j)$. We obtain the conclusion $H, e_j \vdash_j H^{(ij)_{\circ \pi}}$ by applying WeakL.

Concerning $\vdash H^{\pi} \vdash_i q(H, e_{\pi(1)}, \ldots, e_{\pi(n)})$ we apply qR:

\[
\begin{array}{c}
\frac{\{H^{(ij)_{\circ \pi}}, H \vdash e_{\pi(j)}^{(ij)}\}_{j \in \hat{n}}}{H^{\pi} \vdash_i q(H, e_{\pi(1)}, \ldots, e_{\pi(n)}) \text{(qR)}}
\end{array}
\]

If $\pi(j) = i$, then $e_{\pi(j)}^{(ij)} = e_j$ and we are done by Const and weakening. If $\pi(j) \neq i$, then $((ij) \circ \pi)(j) \neq j$. In this case, renaming $\sigma$ the permutation $(ij) \circ \pi$ and putting $r = \sigma(j)$, we proceed by applying Neg1 with $\Gamma := H, F := H^{\sigma}$ and $k := j$:

\[
\begin{array}{c}
\frac{H^{(jr)} \vdash_r H^{\sigma}}{H^{\sigma}, H \vdash_j (\text{Neg1})} \\
\frac{H^{\sigma}, H \vdash e_{\pi(j)}^{(ij)}}{H^{\sigma}, H \vdash_j (\text{WeakR})}
\end{array}
\]

Since $\sigma^{-1}(r) = j = (jr)^{-1}(r)$, the sequent $H^{(jr)} \vdash_r H^{\sigma}$ is provable by Corollary [4.2].
Theorem 6.5. The quotient set \( \mathcal{F}_n/\sim \) is the universe of an nBA, denoted by \( L_n \), and called the Lindenbaum algebra of nLK.

Proof. We have to prove that:

(B0) \( q(e_k, H_1, \ldots, H_n) \sim H_k \):

\[
\frac{H_k^{(ik)} \vdash_k H_k^{(ik)}}{e_k, H_k^{(ik)} \vdash_k H_k^{(ik)} (\text{WeakL})} \quad \{ \frac{e_k \vdash_j}{e_k, H_j^{(ij)} \vdash_j H_k^{(ij)} (\text{Weak}^+)} \}_{j \in \hat{n} \setminus \{k\}}
\]

\[
q(e_k, H_1, \ldots, H_n) \vdash_i H_k
\]

(B1) \( q(H, e_1, \ldots, e_n) \sim H \):

\[
\frac{H \vdash_i H}{H, e_i \vdash_i H (\text{WeakL})} \quad \{ \frac{e_i \vdash_j}{H, e_j^{(ij)} \vdash_j H^{(ij)} (\text{Weak}^+)} \}_{j \in \hat{n} \setminus \{i\}}
\]

\[
q(H, e_1, \ldots, e_n) \vdash_i H
\]

Lemma 4.1

\[
\frac{\vdash_i e_i}{H, H \vdash_i e_i (\text{WeakL})} \quad \{ \frac{H^{(ij)} \vdash_j}{H^{(ij)}, H \vdash_j e_j^{(ij)} (\text{WeakR})} \}_{j \in \hat{n} \setminus \{i\}}
\]

\[
q(H, e_1, \ldots, e_n)
\]

(B2) \( q(H, F, \ldots, F) \sim F \):

\[
\frac{F^{(ij)} \vdash_j F^{(ij)}}{H, F^{(ij)} \vdash_j F^{(ij)} (\text{WeakL})} \quad \frac{F^{(ij)} \vdash_j F^{(ij)}}{F^{(ij)}, H \vdash_j F^{(ij)} (\text{WeakL})}
\]

\[
q(H, F, \ldots, F) \vdash_i F
\]

\[
F \vdash_i q(H, F, \ldots, F)
\]

(B3) \( q(H, q(F_0^1, \ldots, F_n^1), \ldots, q(F_0^n, \ldots, F_n^n)) \sim q(H, F_0^1, \ldots, F_n^1, \ldots, q(H, F_0^n, \ldots, F_n^n)) \):

We have to show that, for all \( H, F_k^r \ (1 \leq r \leq n \) and \( 0 \leq k \leq n \), the formulas

\[
H_1 = q(H, q(F_0^1, \ldots, F_1^1), \ldots, q(F_n^m, \ldots, F_n^n))
\]

and

\[
H_2 = q(H, F_0^1, \ldots, F_n^1, \ldots, q(H, F_0^n, \ldots, F_n^n))
\]

are equivalent. For all \( i \), the proofs of \( H_1 \vdash_i H_2 \) and of \( H_2 \vdash_i H_1 \) are trees of branching factor \( n \) and depth 5. Each leaf of those proof trees is identified by a sequence of 5 integers \( j, k, l, h, m \) between 1 and \( n \), its
branch. We are going to construct one of such branches, and argue that the corresponding leaf is always a provable sequent, by case analysis on the sequence of integers associated to it. Five different exchanges come into play in this derivation, that we rename for typographical reasons: $\pi := (ij), \rho := (jk), \sigma := (kl), \tau := (lh), \psi := (hm)$. In the following proof the principal formula of each rule application is depicted in bold.

\[
\frac{H^{\psi \sigma \tau}, F_0^{k \psi \tau}, F_l^{k \psi \sigma \rho \sigma \rho \pi}}{H^{\sigma \rho}} \quad \frac{H^{\tau \sigma}, F_0^{k \tau}, F_l^{k \tau \sigma \rho \sigma \rho \pi}, H, F_0^{h \tau \sigma \rho \sigma \rho \pi}}{H^{\sigma \rho}, F_0^{k \sigma \rho}, F_l^{k \sigma \rho \sigma \rho \pi}, H, F_0^{h \tau \sigma \rho \sigma \rho \pi}} \quad \frac{H^{\psi \sigma \tau}, F_0^{h \psi \sigma \rho}, H \vdash_m F_j^m \psi \sigma \rho \sigma \rho \pi}}{(qR)}
\]

We denote by $S$ the uppermost sequent of the above branch of the proof. We are going to show that $S$ is provable, by case analysis on $j, k, l, h, m$.

First of all, if $h \neq m$, then $H^{\psi \sigma \tau}, H \vdash_m$ is provable by Lemma 4.1 and $S$ is provable by weakening. Hence we are left with the case $h = m$; by applying the rule (ConL), the uppermost sequent $S$ becomes:

\[
H^{\tau \sigma}, F_0^{k \tau}, F_l^{k \tau \sigma \rho \sigma \rho \pi}, H, F_0^{h \tau \sigma \rho \sigma \rho \pi} \vdash_h F_j^{h \tau \sigma \rho \sigma \rho \pi}.
\]

We analyse the cases $k \neq h$ and $k = h$, both splitting in two other cases.

($k \neq h$): The sequent $H^{\tau \sigma}, H \vdash_h$ is provable as follows:

($l \neq h$): Since in this case $(lh)^{-1}(l) = h = (\tau \circ \sigma)^{-1}(l)$, then we have:

\[
\text{Corollary 4.2}
\]

\[
\frac{H^{(lh)} \vdash_l H^{\tau \sigma} \quad l \neq h}{H, H^{\tau \sigma} \vdash_h} \quad \text{(Neg1)}
\]

($l = h$): $H^{\sigma}, H \vdash_l$ follows from Lemma 4.1

($k = h$): In this case $\tau = \sigma$ and $\sigma \circ \tau = \tau \circ \sigma = id$. The uppermost sequent $S$ becomes:

\[
H, F_0^{k \tau}, F_l^{k \tau \rho \pi}, H, F_0^{k \rho \pi} \vdash_k F_j^{h \rho \pi}.
\]

($j \neq l$): $F_0^{k \tau}, F_0^{k \rho \pi} \vdash_k$ is provable as follows, recalling that $k = h$:

\[
\frac{F_0^{k (jk) \circ (lk)}, F_0^{k \rho \pi}}{F_0^{k (jk) \circ (lk)}} \quad \frac{F_0^{k (lk)}, F_0^{k (jk) \circ (lk)}}{(Sym)}
\]

Now the proof of $F_0^{k (jk) \circ (lk)}, F_0^{k \rho \pi} \vdash_k$ is exactly like that of $H^{\tau \sigma}, H \vdash_h$ above, by considering that $[(jk) \circ (lk)]^{-1}(j) = l \neq j = id^{-1}(j)$.
(j = l): Both side of the uppermost sequent contain $F_{i}^{k_{\rho \pi}}$, and we are done.

The proof of $H_{2} \vdash _{i} H_{1}$ is similar. \hfill \square

6.2. Completeness of $n$LK. We start by showing that all valid sequents of the form $\Gamma \vdash _{i} F$ are provable, then we generalise to all the valid sequents of the form $\Gamma \vdash _{i} \Delta$. Let $\Gamma, F, i$ be such that $\Gamma \models _{i} F$. To begin with, we define a multideal in the Lindenbaum algebra of $n$LK (see Definition 2.11) depending on $\Gamma$ and $i$.

Recall that if $\gg \Gamma \vdash _{i} F$ and $F \sim G$ then, by a simple application of rule (Cut), we get $\gg \Gamma \vdash _{i} G$.

We denote by $[F]$ the equivalence class of the formula $F$ with respect to the equivalence $\sim$.

Definition 6.6. Given $i \in \hat{n}$ and a set of formulas $\Gamma$, we define $\mathcal{I}_{j}^{(\Gamma, i)} = \{ [F] : \gg \Gamma \vdash _{i} F^{(ij)} \}$, for $j = 1, \ldots, n$.

Lemma 6.7. $\mathcal{I}_{j}^{(\Gamma, i)}$ is a multideal of the Lindenbaum algebra $L_{n}$ of $n$LK.

Proof. We show that $(\mathcal{I}_{j}^{(\Gamma, i)})_{j \in \hat{n}}$ satisfies the conditions of Definition 2.11

- $e_{j} \in \mathcal{I}_{j}^{(\Gamma, i)}$ is trivially true.
- if $F \in \mathcal{I}_{j}^{(\Gamma, i)}$ and $G \in \mathcal{I}_{k}^{(\Gamma, i)}$ then $q(G, H_{1}, \ldots, H_{k-1}, F, H_{k+1}, \ldots, H_{n}) \in \mathcal{I}_{j}^{(\Gamma, i)}$, for all formulas $H_{1}, \ldots, H_{n}$:

$$\Gamma \vdash _{i} q(G, H_{1}^{(ij)}, \ldots, H_{k-1}^{(ij)}, F^{(ij)}, H_{k+1}^{(ij)}, \ldots, H_{n}^{(ij)}) \quad (qR)$$

The rightmost antecedent follows from $\Gamma \vdash _{i} F^{(ij)}$ by rules (Weak) + (Sym).

Concerning the leftmost antececdents, we show that they are provable for $l \neq k$.

$$\begin{align*}
\Gamma \vdash _{i} G^{(ik)} & \quad (\text{Neg2}) \\
\Gamma^{(il)}, G \vdash _{l} F^{(ij)} & \quad (\text{WeakR}) \\
\Gamma^{(il)}, G \vdash _{l} (H_{1}^{(ij)})^{(il)} & \quad (\text{WeakR})
\end{align*}$$

- if $F_{1}, \ldots, F_{n} \in \mathcal{I}_{j}^{(\Gamma, i)}$ then, for all $G, q(G, F_{1}, \ldots, F_{n}) \in \mathcal{I}_{j}^{(\Gamma, i)}$:

$$\Gamma \vdash _{i} q(G, F_{1}^{(ij)}, \ldots, F_{n}^{(ij)}) \quad (qR)$$

\hfill \square
Given a proper ultramultideal $U = (U_1, \ldots, U_n)$ on $L_n$, let $v^U$ be the environment such that $v^U(X_i) = i$ if $[X_i^{id}] \in U_i$. The environment $v^U$ is well-defined since $\bigcup_{i=1}^n U_i = L_n$ and $U_i \cap U_j = \emptyset$ for $i \neq j$.

**Lemma 6.8.** For all formulas $F$, $\llbracket F \rrbracket_{v^U} = i$ if and only if $F \in U_i$.

*Proof.* The proof is by induction on $F$.

$(\Leftarrow)$ We analyse only the case $F = X^\rho_i$. Let $[X^\rho_i] = [q(X^{id}_i, e_{\rho(1)}, \ldots, e_{\rho(n)})) \in U_i$. Since $U$ is a ultramultideal, there exists $j$ such that $[X^{id}_j] \in U_j$. Then, by definition of multideal, $e_{\rho(j)} \in U_i$. This implies $\rho(j) = i$ since $U$ is proper. We conclude that $\llbracket X^\rho_i \rrbracket_{v^U} = \rho(v^U(x)) = \rho(j) = i$.

$(\Rightarrow)$ Again, we analyse only the case $F = X^\rho_i$. Let $\llbracket X^\rho_i \rrbracket_{v^U} = \rho(v^U(X)) = i$ and $[X^{id}_i] \in U_m$, for some $m$, so that $v^U(X) = m$ and $\rho(m) = i$. Since $[X^\rho_i] = [q(X^{id}_i, e_{\rho(1)}, \ldots, e_{\rho(n)}))$ and $[X^{id}_i] \in U_m$, then $\llbracket X^\rho_i \rrbracket \in U_{\rho(m)} = U_i$, by definition of multideal. \hfill $\Box$

**Proposition 6.9 (Completeness).** If $\Gamma \models_i F$ then $\Gamma \vdash_i F$.

*Proof.* Let us suppose that $\Gamma \not\models_i F$, so that $[F] \not\in \mathcal{T}_i^{(\Gamma, i)}$ and the multideal $\mathcal{T}_i^{(\Gamma, i)}$ is proper. Then by Proposition 2.16 there exists a proper ultramultideal $U$ extending $\mathcal{T}_i^{(\Gamma, i)}$ such $[F] \not\in U_i$. Remark that for all $G \in \Gamma$, $[G] \in \mathcal{T}_i^{(\Gamma, i)} \subseteq U_i$, by Corollary 4.2. Hence by Lemma 6.8 $\llbracket F \rrbracket_{v^U} \neq i$ and, for all $G \in \Gamma$, $\llbracket G \rrbracket_{v^U} = i$, proving that $\Gamma \not\models_i F$. \hfill $\Box$

In the final part of this section we generalise the completeness to sequents with several formulas in the right-hand side.

Let $\Delta \equiv G_1, \ldots, G_k$ be a sequence of $\mathcal{F}_n$-formulas and $i \in \hat{n}$. We define a formula $F^i(\Delta)$ by induction on the length $k$ of the sequence. As a matter of notation, $\Delta \setminus G_1$ denotes the sequence $G_2, \ldots, G_k$. If $k = 0$, then we define $F^i(\Delta) = e_j$ with $j \neq i$. If $k = 1$, then $F^i(\Delta) = G_1$. If $k > 1$, then $F^i(\Delta) = q(G_1, F^i(\Delta \setminus G_1), \ldots, F^i(\Delta \setminus G_1), G_1, F^i(\Delta \setminus G_1), \ldots, F^i(\Delta \setminus G_1))$, where $G_1$ is at position $i + 1$.

**Lemma 6.10.** If $\Delta$ contains at least one formula and $i, j \in \hat{n}$, then $F^i(\Delta)^{(ij)} \sim (F^j(\Delta)^{(ij)})$.

*Proof.* By induction on $|\Delta| \geq 1$. If $\Delta = G$, then $(F^i(\Delta))^{(ij)} = (F^j(\Delta)^{(ij)}) = G^{(ij)}$.

If $\Delta = G, \Delta'$ then

$$(F^i(\Delta))^{(ij)} = q(G, F^i(\Delta')^{(ij)}, \ldots, F^i(\Delta')^{(ij)}, G^{(ij)}, F^i(\Delta')^{(ij)}, \ldots, F^i(\Delta')^{(ij)}),$$

where $G^{(ij)}$ is at position $i + 1$, and

$$(F^j(\Delta)^{(ij)}) = q(G^{(ij)}, F^j(\Delta^{(ij)}), \ldots, F^j(\Delta^{(ij)}), G^{(ij)}, F^j(\Delta^{(ij)}), \ldots, F^j(\Delta^{(ij)}))$$

\footnote{Every $e_j$ with $j \neq i$ is suitable. We can choose $j$ minimum such that $j \neq i$, for instance.}
where \( G^{(ij)} \) is at position \( j+1 \). It is clear that for all environments \( v \), \( [(F^i(\Delta))^{(ij)}]_v = [(F^j(\Delta^{(ij)}))]_v \), so that \((F^i(\Delta))^{(ij)} = k \ (F^j(\Delta^{(ij)}))\) for all \( k \), and conversely. Hence \((F^i(\Delta))^{(ij)} \sim (F^j(\Delta^{(ij)}))\) follows by Proposition 6.9.

**Lemma 6.11.** For all sequences \( \Gamma \) and \( \Delta \) of \( F_n \)-formulas, and for all \( i \in \hat{n} \), we have:

\[ \vdash \Gamma \vdash_i \Delta \iff \vdash \Gamma \vdash_i F^i(\Delta) \]

**Proof.** (\( \Rightarrow \)) By Proposition 5.4 we have that \( \vdash \Gamma \vdash_i \Delta \) implies \( \Gamma \models_i \Delta \). It is an easy exercise to verify that \( \Gamma \models_i \Delta \) if and only if \( \Gamma \models_i F^i(\Delta) \). Therefore, by Proposition 6.9 we get the conclusion.

(\( \Leftarrow \)) Let \( k \) be the size of \( \Delta \). The case \( k = 0 \) is settled by the following proof, where the leftmost sequent is provable by hypothesis, and the rightmost one by Lemma 4.4:

\[
\frac{\Gamma \vdash e_j \quad e_j \vdash_i}{\Gamma \vdash_i} \quad \text{(Cut)}
\]

Then, we proceed by induction on \( k > 0 \). The case \( k = 1 \) is trivial since if \( \Delta = G \) then \( F^i(\Delta) = G \). Let \( \Delta = G, \Delta' \) be a sequence of length \( k > 1 \). By induction hypothesis we have that \( \vdash \Gamma'' \vdash_k \Delta'' \iff \vdash \Gamma'' \vdash_j F^k(\Delta'') \), for all \( j \in \hat{n} \) and all sequences \( \Gamma'', \Delta'' \) with \( |\Delta''| < k \). The first step in the proof of \( \Gamma \vdash_i G, \Delta' \) is (NegR):

\[
\frac{\{\Gamma^{(ij)}, G \vdash_j \Delta''^{(ij)}\}}{\Gamma \vdash_i G, \Delta'} \quad \text{(NegR)}
\]

We are left with the problem of proving \( \Gamma^{(ij)}, G \vdash_j \Delta''^{(ij)} \) for all \( j \neq i \). Since by hypothesis \( \vdash \Gamma \vdash_i q(G, F^i(\Delta'), \ldots, F^i(\Delta'), \ldots, F^i(\Delta'), G, F^i(\Delta'), \ldots, F^i(\Delta')) \), we can use the soundness of \( nLK \), the invertibility of the rule (qR) (see Lemma 8.1), and the completeness result of Proposition 6.9 to get \( \vdash \Gamma^{(ij)}, G \vdash_j F^i(\Delta'')^{(ij)} \), for all \( j \neq i \). Using Lemma 6.10 we get \( \vdash \Gamma^{(ij)}, G \vdash_j F^j(\Delta''^{(ij)}) \). Now, the inductive hypothesis allows us to conclude that \( \vdash \Gamma^{(ij)}, G \vdash_j \Delta''^{(ij)} \), and we are done.

**Theorem 6.12.** For all sequences \( \Gamma \) and \( \Delta \) of \( F_n \)-formulas and for all \( i \in \hat{n} \), we have that \( \Gamma \models_i \Delta \) iff \( \vdash \Gamma \vdash_i \Delta \).

**Proof.** By Lemma 6.11 and by Propositions 5.4 and 6.9 

\[ \text{6.3. The classical case: semantics.} \]

We are now able to prove the equivalence of \( 2LK \) and \( PC \), through the translations \( (\_)^{\circ} \) and \( (\_)^{\bullet} \) defined in Section 3.2.

Each environment of \( 2LK \) may be seen as an environment of \( PC \), and conversely, simply by exchanging the truth values 2 and 0. In the sequel, we will keep this exchange implicit.
Lemma 6.13. Let $F$ be a formula of 2LK, $P$ a formula of PC and $v$ an environment. Then $[F]_v = 1$ iff $[F^\bullet]_v = 1$ and $[P]_v = 1$ iff $[P^\circ]_v = 1$.

Proof. Both statements are proven by straightforward inductions, on $F$ and $P$ respectively. \hfill \Box

Corollary 6.14. 

1. Let $\Gamma, \Delta$ be sequences of 2LK formulas and $\Gamma_1, \Delta_1$ sequences of PC formulas. Then $\Gamma \vdash \Delta \text{ iff } \Gamma^\bullet \vdash \Delta^\bullet$, and $\Gamma_1 \models \Delta_1 \text{ iff } \Gamma_1^\circ \models \Delta_1^\circ$.

2. Let $\Gamma, \Delta$ be sequences of 2LK formulas and $\Gamma_1, \Delta_1$ sequences of PC formulas. Then $\triangleright \Gamma \vdash \Delta \text{ iff } \triangleright \Gamma^\bullet \vdash \Delta^\bullet$, and $\triangleright \Gamma_1 \models \Delta_1 \text{ iff } \triangleright \Gamma_1^\circ \models \Delta_1^\circ$.

Proof. (1) follows from Lemma 6.13.

(2) Since the calculus of Figure 2 is sound and complete for the Propositional Calculus, the result follows from Theorem 6.12 and Corollary 6.14(1). \hfill \Box

7. Equivalence of nCL and nLK

Let $\tau$ be the algebraic type of the pure $n$BAs, $T_\tau(V)$ be the absolutely free term $\tau$-algebra over the countable set $V$ of generators, whose universe is the set of the nCL formulas, and $n$ be the pure $n$BA of universe $\{1, \ldots, n\}$ of Example 2.7. More explicitly, a nCL formula is either one of the constants $e_1, \ldots, e_n$, or a variable $X \in V$, or it is of the form $q(F_0, \ldots, F_n)$ for some a nCL formulas $F_1, \ldots, F_n$. A $\tau$-matrix is a pair $(A, F)$ where $A$ is a $\tau$-algebra and $F \subseteq A$ is a set of designated values. A $\tau$-matrix $A = (A, F)$ defines a logic $L = (\tau, \vdash_A)$ as follows: $\Gamma \vdash_A \phi$ if for any homomorphism $h : T_\tau(V) \rightarrow A$, if $h(\psi) \in F$ for all $\psi \in \Gamma$, then $h(\phi) \in F$.

In order to compare the logics nCL defined in [17] and the sequent calculus nLK, we focus on the logics of the form $(\tau, \vdash_{(n,i)})$, for $i = 1, \ldots, n$. The homomorphisms from $T_\tau(V)$ to $n$ and the environments defined in Section 5 are in bijective correspondence: given a homomorphism $h : T_\tau(V) \rightarrow n$, let $v_h$ be the environment defined by $v_h(X) = h(X)$, for all $X \in V$; symmetrically, given $v$, let $h_v$ be the unique homomorphism extending $v$. The map $v \mapsto v_h = (h \mapsto h_v)^{-1}$ is a bijection, such that $h_{v_h} = h$ and $v_{h_v} = v$.

In order to compare the two systems, we need to translate nCL formulas into nLK formulas.

Definition 7.1. The translation $(\_)^\dagger$ from the set of nLK formulas to the set of nCL formulas is defined as follows:

1. $e_i^\dagger = e_i$;
2. $(X^\rho)^\dagger = q(X, e_{\rho(1)}, \ldots, e_{\rho(n)})$;
3. $q(F, G_1, \ldots, G_n)^\dagger = q(F^\dagger, G_1^\dagger, \ldots, G_n^\dagger)$.

The proof of the following lemma, omitted, is an easy induction on the size of nLK formulas. Notice that the item (2) below is obtained by (1) applied to the environment $v_h$.
Lemma 7.2. For every nLK formula \( F \), environment \( v \) and homomorphism \( h : T_\tau(V) \to n \), we have that:

1. \([ F ]_v = h_v(F^\dagger)\), and
2. \( h(F^\dagger) = [ F ]_{vh} \).

Lemma 7.3. For every nLK sequent \( \Gamma \vdash_i F \), we have: \( \Gamma \vdash_i F \) iff \( \Gamma^\dagger \vdash_{(n,\{i\})} F^\dagger \).

Proof. Immediate by Lemma 7.2.

Proposition 7.4. For every nLK sequent \( \Gamma \vdash_i F \), we have that \( \Gamma \vdash_i F \) iff \( \Gamma^\dagger \vdash_{(n,\{i\})} F^\dagger \).

Proof. By Lemma 7.3 and Theorem 6.12.

For the sake of completeness, we present below the Hilbert-style system, defined in [17], axiomatising the logic \((\tau, \vdash_{(n,\{i\})})\). By Proposition 7.4, this axiomatic system is equivalent to \( \vdash_i \), up to the \((\_\dagger)\) translation of formulas. We write \( \vdash e_i \) for the derivations in the Hilbert-style system.

Given \( \varphi, \psi \in T_\tau(V) \) and \( j, k, l \leq n \), let

- \( e_j[e_k/l] = e_j, e_j, \ldots, e_k, \ldots, e_j \) (\( e_k \) in \( l \)-th position).
- \( \varphi \iff^j \psi = q(\varphi, q(\psi, e_j[1]), \ldots, q(\psi, e_j[n])) \)

Fixing \( j \neq i \), the axioms and rules of the axiomatisation of the logic \((\tau, \vdash_{(n,\{i\})})\) are the following (17):

A1 \( \varphi \iff^i j \varphi \)
A2 \( q(\varphi, e_1, \ldots, e_n) \iff^i j \varphi \)
A3 \( q(\varphi, \psi) \iff^i j \psi \)
A4 \( q(\varphi, q(\psi_1, \chi_{11}, \ldots, \chi_{1n}), \ldots, q(\psi_n, \chi_{n1}, \ldots, \chi_{nn})) \)
\( \iff^i j \psi(q(\varphi, \psi_1, \ldots, \psi_n), \ldots, q(\varphi, \chi_{11}, \ldots, \chi_{nn})) \)
R1 \( \varphi, \varphi \iff^i j \psi \vdash e_i \psi \)
R2 \( \varphi \iff^i j \psi \vdash e_i \psi \iff^i j \varphi \)
R3 \( \varphi_1 \iff^i j \psi_1, \ldots, \varphi_{n+1} \iff^i j \psi_{n+1} \)
\( \vdash e_i, q(\varphi_1, \ldots, \varphi_{n+1}) \iff^i j q(\psi_1, \ldots, \psi_{n+1}) \)
R4 \( \varphi \vdash e_i, \varphi \iff^i j e_i \)
R5 \( \varphi \iff^i j e_i \vdash e_i \varphi \)

8. Cut Admissibility

We prove in this section that all the valid sequents admit a cut-free proof, thus obtaining an alternative proof of completeness, and a cut admissibility result. Canonical cut-free proofs are obtained by repeatedly applying the invertible rules \( qR \) and \( qL \) until we attain sequents that contain no occurrence of the \( q \) operator. Subsequently, invertible instances of weakening are applied until we reach sequents that contain only variables and constants. If constants are present, these sequents are easily provable without cuts, while for those containing only
variables a slightly more complex analysis is required. The first observation is that the rules \( q_L \) and \( q_R \) are invertible, meaning that if their conclusion is a valid sequent then all their premises are valid sequents.

**Lemma 8.1.** The rules \( q_L \) and \( q_R \) are invertible.

**Proof.** Concerning the rule \( q_L \), let us suppose that \( \Gamma, q(F, G_1, \ldots, G_n) \vdash_i \Delta \) is valid. We have to prove that \( \Gamma^{(i,j)}, F, G_j^{(i,j)} \vdash_j \Delta^{(i,j)} \) is valid, for all \( j \). Let us suppose that an environment \( v \) is such that \([H]_v = j \) for all \( H \in \Gamma^{(i,j)} \), \([F]_v = j \) and \([G_j^{(i,j)}]_v = j \). Then by Lemma 5.3 \([K]_v = i \) for all \( K \in \Gamma \), and \([q(F,G_1,\ldots,G_n)]_v = i \). From the validity of \( \Gamma, q(F, G_1, \ldots, G_n) \vdash_i \Delta \) it follows that there exists \( H \in \Delta \) such that \([H]_v = i \). Hence \([H^{(i,j)}]_v = j \), and we are done.

The case of \( q_R \) is similar, and omitted. \( \square \)

Hence, all valid sequents admit a cut-free proof iff all valid sequents containing only constants and decorated variables do. In fact, given a valid sequent \( \Gamma \vdash_i \Delta \), one can apply the rules \( q_R \) and \( q_L \) as long as possible, in whatever order. By Lemma 8.1 this produces a \( n \)-branching tree whose leaves are valid sequents containing only constants and decorated variables.

A second simple observation that allows us to restrict a bit more the set of sequents to be considered is that some instances of the weakening rules are invertible, namely:

**Fact 1.** Weakenings of the form \( \Gamma \vdash_i \Delta \) WeakL and \( \Gamma \vdash_i \Delta \) WeakR, with \( j \neq i \), are invertible.

Let us call atomic sequents those containing only constants and decorated variables, and such that if they are of the form \( \Gamma, e_i \vdash_i \Delta \) (resp. \( \Gamma \vdash_i e_j, \Delta \)) then \( j \neq i \) (resp. \( j = i \)).

In order to prove that all valid sequents admit a cut-free proof it is enough to prove that all valid atomic sequents do. The following easy lemma allows us to get rid of constants.

**Lemma 8.2.** All atomic sequents containing at least a constant are valid and admit a cut-free proof.

**Proof.** The validity is immediate. Concerning the provability, sequents of the form \( \Gamma \vdash_i e_i, \Delta \) are proved by using the rule \( \text{Const} \) followed by weakenings, and those of the form \( \Gamma, e_j \vdash_i \Delta \) where \( i \neq j \) are proved by using the rule \( \text{Const} \) followed by \( \text{NegL} \) and weakenings. \( \square \)

We are left with the problem of showing that, if a sequent containing only decorated variables is valid, then it admits a cut-free proof. Any such sequent is
of the form:

\[ X_1^{r_1}, \ldots, X_s^{r_s} \vdash_i X_1^{\sigma_1}, \ldots, X_s^{\sigma_s} \]

where, for all \( 1 \leq r \leq s \), \( k_r + l_r > 0 \). This means that each variable \( X_r \) may appear only on the left hand side of the sequent, or only on the right hand side, or in both.

The following lemma provides a characterisation of the valid sequents containing only decorated variables.

**Lemma 8.3.** A sequent of the form

\[ X_1^{r_1}, \ldots, X_s^{r_s}, \ldots, \vdash_i X_1^{\sigma_1}, \ldots, X_s^{\sigma_s} \]

is valid iff

\[ \exists r \leq s \forall \epsilon \in \hat{n} \left[ (\forall m \leq k_r \rho_m^r(j) = i) \Rightarrow (\exists m \leq l_r \sigma_m^r(j) = i) \right]. \]

**Proof.** It is immediate to prove the contrapositive: the given sequent is not valid iff there exists an environment assigning \( i \) to all the decorated variables in the left-hand side, and to none of those in the right-hand side. Such an environment \( v \) exists iff for every \( r \in \{1, \ldots, s\} \) there exists \( v(X_r) \in \{1, \ldots, n\} \) such that \( \rho_m^r(v(X_r)) = i \) for all \( m \in \{1, \ldots, k_r\} \) and \( \sigma_m^r(v(X_r)) \neq i \) for all \( m \in \{1, \ldots, l_r\} \).

Let us call pure a sequent of the form \( X_1^{r_1}, \ldots, X_s^{r_s} \vdash_i X_1^{\sigma_1}, \ldots, X_s^{\sigma_s} \).

Lemma 8.3 says that a sequent containing only decorated variables as above is valid iff there exists \( r \in \{1, \ldots, s\} \) such that \( X_1^{\sigma_1}, \ldots, X_s^{\sigma_s}, \vdash_i X_1^{\sigma_1}, \ldots, X_s^{\sigma_s} \) is a valid pure sequent. If all the valid pure sequents admit a cut-free proof, then also all the valid sequents containing only decorated variables admit a cut-free proof, obtained by adding the suitable weakenings to the proof of the pure valid sequent they contain. Hence, we are left with the following bit:

**Lemma 8.4.** If a pure sequent is valid then it admits a cut-free proof.

**Proof.** Let \( S = X_1^{r_1}, \ldots, X_s^{r_s} \vdash_i X_1^{\sigma_1}, \ldots, X_s^{\sigma_s} \) be a valid pure sequent. For the sake of this proof, let us abbreviate \( \exists 1 \leq m \leq k \rho_m(j) \neq i \) by \( A(j) \) and \( \exists 1 \leq m \leq l \sigma_m(j) = i \) by \( B(j) \). By Lemma 8.3 we know that \( \forall 1 \leq j \leq n A(j) \lor B(j) \).

We consider two cases:

1. \( A(j) \) holds for at least one \( j \in \hat{n} \). Then there exists \( m \in \{1, \ldots, k\} \) such that \( \rho_m(j) \neq i \). Let \( h = \rho_m^{-1}(i) \neq j \). There are two cases:
   a. \( A(h) \) holds. Then there exists \( m' \in \{1, \ldots, k\} \), such that \( \rho_{m'}(h) \neq i \). So \( \rho_m^{-1}(i) \neq h = \rho_m^{-1}(i) \). In this case the sequent \( X^r, X^{r'} \vdash_i \) is cut-free provable as we have shown in part (i) of Lemma 4.1. Then we conclude by weakening. Hence, \( S \) admits a cut-free proof.
(b) \( B(h) \) holds. Then there exists \( m' \in \{1, \ldots, l\} \), such that \( \sigma_{m'}(h) = i \). So \( \sigma_{m}^{-1}(i) = \rho_{m}^{-1}(i) \). Then \( X^{\rho_{m}} \vdash_i X^{\sigma_{m'}} \) is the conclusion of an instance of \( \text{Id} \), so that \( \vdash X^{\rho_{m}} \vdash_i X^{\sigma_{m'}} \), and we conclude by weakening.

(2) \( B(j) \) holds for all \( j \in \hat{n} \). Let \( f : \hat{n} \to \{1, \ldots, l\} \) be such that \( \sigma_f(j) = i \), for \( j \in \hat{n} \). If we show that \( \vdash X^{\sigma_f(1)} , \ldots , X^{\sigma_f(n)} \) is cut-free provable we are done, since the provability of the whole pure sequent follows by weakening.

Now consider

\[
\frac{\{X^{\sigma_f(1)} \vdash_j X^{(ij) \circ \sigma_f(1)}, \ldots , X^{(ij) \circ \sigma_f(i-1)}, X^{(ij) \circ \sigma_f(i+1)}, \ldots , X^{(ij) \circ \sigma_f(n)}\}}{\vdash_i X^{\sigma_f(1)} , \ldots , X^{\sigma_f(n)}}_{j \neq i} \text{NegR}
\]

Each of the \( n-1 \) premises of this instance of \( \text{NegR} \) is cut-free provable: let \( j \neq i \) and \( r \in \hat{n} \) be the unique truth value such that \( \sigma_f(i) = r \). Then the sequent \( X^{\sigma_f(i)} \vdash_j X^{(ij) \circ \sigma_f(r)} \) is the conclusion of an instance of \( \text{Id} \), because \( \sigma_f^{-1}(j) = r = ((ij) \circ \sigma_f(r))^{-1}(j) \). The whole premise is obtained by weakening.

\[\square\]

Summing up:

**Theorem 8.5.** All valid sequents of the \( n \)-dimensional propositional calculus admit a cut-free proof.

**Proof.** Given a valid sequent \( \Gamma \vdash_i \Delta \), apply \( qR \) and \( qL \) as long as possible, in whatever order. By Lemma \([8.1]\) this produces a \( n \)-branching tree whose leaves are valid sequents containing only constants and decorated variables. Let \( \Gamma' \vdash_j \Delta' \) be one of these leaves. Apply the invertible instances of \( \text{WeakL} \) and \( \text{WeakR} \) given in Fact \([1]\) as long as possible, in whatever order. The result is a valid atomic sequent \( \Gamma'' \vdash_j \Delta'' \). If it contains at least a constant, then it admits a cut-free proof by Lemma \([8.2]\), otherwise it is a valid sequent containing only decorated variables. In this last case, by Lemma \([8.3]\) \( \Gamma'' \vdash_j \Delta'' \) may be obtained by suitably weakening a valid pure sequent \( \Gamma''' \vdash_j \Delta''' \). By Lemma \([8.4]\) \( \Gamma''' \vdash_j \Delta''' \) admits a cut free proof. The tree rooted in \( \Gamma \vdash_i \Delta \) whose definition is sketched above is a cut-free \( nLk \) proof, and we are done. \[\square\]

9. Conclusion

The sequent calculus introduced in this paper exemplifies a new proof-theoretic format for finite-valued logics with \( n \) truth values, where (i) every truth-value \( e_i \) has its own turnstile \( \vdash_i \), and (ii) propositional variables are decorated with elements of the symmetric group \( S_n \). The former feature generalises what happens in hybrid deduction-refutation systems: in the 2-valued case, indeed, proofs of \( \vdash_2 F \) should be regarded as refutations of \( F \). The latter feature, on the other hand, allows us to identify a formula \( F \) and its double-negation not only as regards their semantic interpretation, but even as syntactic objects. Our calculus
is sound and complete with respect to the canonical notion of semantic validity of \(n\)-dimensional propositional formulas. We proved that all valid sequent admit a cut-free proof, as a corollary to our syntactic proof of completeness. This result falls short of a full-blooded cut elimination theorem, in that no algorithm is provided to transform proofs containing cuts into cut-free proofs. The main difficulty in obtaining it is due to the fact that the propositional variables are decorated. A constructive proof of cut elimination remains as a valuable goal for further investigations.

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References

[1] Arnon Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52:939–951, 1987.
[2] Matthias Baaz and Christian G. Fermüller. Analytic calculi for projective logics. In *Tableaux ’99*, LNAI, vol. 1617, pages 36–50. Springer, 1999.
[3] Matthias Baaz, Christian G. Fermüller, and Richard Zach. Dual systems of sequents and tableaux for many-valued logics. *Bull. EATCS*, 51:192–197, 1993.
[4] Antonio Bucciarelli, Antonio Ledda, Francesco Paoli, and Antonino Salibra. Boolean-like algebras of finite dimension: from Boolean products to semiring product. In Jacek Malinowski and Rafał Palczewski, editors, *Janusz Czelakowski on Logical Consequence*, Outstanding Contributions to Logic, pages 377–400. Springer, 2024.
[5] Antonio Bucciarelli and Antonino Salibra. On noncommutative generalisations of Boolean algebras. *The Art of Discrete and Applied Mathematics*, 2(2), 2019.
[6] Stanley Burris and Hanamantagouda P. Sankappanavar. A course in universal algebra, volume 78 of *Graduate texts in mathematics*. Springer, 1981.
[7] Agata Ciabattoni, Christian G. Fermüller, and George Metcalfe. Uniform rules and dialogue games for fuzzy logics. In *LPAR 2004*, LNCS, vol. 3452, pages 496–510. Springer, 2004.
[8] Agata Ciabattoni and Franco Montagna. Proof theory for locally finite many-valued logics: Semi-projective logics. *Theoretical Computer Science*, 480:26–42, 2013.
[9] Valentin Goranko. Hybrid deduction-refutation systems. *Axioms*, 8(4), 2019.
[10] Valentin Goranko, Gabriele Pulcini, and Tomasz F. Skura. Refutation systems: An overview and some applications to philosophical logics. In F. Liu, H. Ono, and J. Yu, editors, *Knowledge, Proof and Dynamics*, Outstanding Contributions to Logic, pages 173–197. Springer, 2020.
[11] Rajeev Gore and Linda Postniece. Combining derivations and refutations for cut-free completeness in bi-intuitionistic logic. *Journal of Logic and Computation*, 20(1):233–260, 2010.
[12] Antonio Ledda, Francesco Paoli, and Antonino Salibra. On semi-boolean-like algebras. *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, 52, 2013.
[13] Jonathan Leech. Skew Boolean algebras. *Algebra Universalis*, 27:497–506, 1990.

[14] Giulio Manzonetto and Antonino Salibra. From lambda-calculus to universal algebra and back. In Edward Ochmanski and Jerzy Tyszkiewicz, editors, *Mathematical Foundations of Computer Science 2008, 33rd International Symposium, MFCS 2008, Torun, Poland, August 25-29, 2008, Proceedings*, volume 5162 of *Lecture Notes in Computer Science*, pages 479–490. Springer, 2008.

[15] Sara Negri. On the duality of proofs and countermodels in labelled sequent calculi. In Didier Galmiche and Dominique Larchey-Wendling, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 22th International Conference, TABLEAUX 2013, Nancy, France, September 16-19, 2013. Proceedings*, volume 8123 of *Lecture Notes in Computer Science*, pages 5–9. Springer, 2013.

[16] G. Rousseau. Sequents in many-valued logic I. *Fundamenta Mathematicae*, 60:23–131, 1967.

[17] Antonino Salibra, Antonio Bucciarelli, Antonio Ledda, and Francesco Paoli. Classical logic with n truth values as a symmetric many-valued logic. *Foundations of Science*, 28:115–142, 2023.

[18] Antonino Salibra, Antonio Ledda, and Francesco Paoli. Boolean product representations of algebras via binary polynomials. In J. Czelakowski, editor, *Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science*, volume 16 of *Outstanding Contributions to Logic*, pages 297–321. Springer, 2018.

[19] Antonino Salibra, Antonio Ledda, Francesco Paoli, and Tomasz Kowalski. Boolean like algebras. *Algebra Universalis*, 69:113–138, 2013.

[20] Diego Vaggione. Varieties in which the Pierce stalks are directly indecomposable. *Journal of Algebra*, 184:424–434, 1996.

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