MODELS OF SOME SIMPLE MODULAR LIE SUPERAＬGEBRAS

ALBERTO ELDUQUE

Abstract. Models of the exceptional simple modular Lie superalgebras in characteristic $p \geq 3$, that have appeared in the classification due to Bouarroudj, Grozman and Leites [BGLb] of the Lie superalgebras with indecomposable symmetrizable Cartan matrices, are provided. The models relate these exceptional Lie superalgebras to some low dimensional nonassociative algebraic systems.

Introduction

The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices over algebraically closed fields are classified in [BGLb] under some extra technical hypotheses. Their results assert that, for characteristic $\geq 3$, apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0 [Kac77], by reducing the Cartan matrices modulo $p$, there are the following exceptions that have to be added to the list of known simple Lie superalgebras:

(i) Two exceptions in characteristic 5: $\mathfrak{br}(2; 5)$ and $\mathfrak{cl}(5; 5)$. (The superalgebra $\mathfrak{cl}(5; 5)$ first appeared in [Eld07b].)

(ii) A family of exceptions given by the Lie superalgebras that appear in the Supermagic Square in characteristic 3 considered in [CE07a, CE07b]. With the exception of $\mathfrak{g}(3, 6) = \mathfrak{g}(S_{1}, 2, S_{4}, 2)$ these Lie superalgebras first appeared in [Eld06b] and [Eld07b].

(iii) Another two exceptions in characteristic 3, similar to the ones in characteristic 5: $\mathfrak{br}(2; 3)$ and $\mathfrak{el}(5; 3)$.

The Lie superalgebra $\mathfrak{el}(5; 5)$ was shown in [Eld07b] to be related to Kac’s 10-dimensional exceptional Jordan superalgebra, by means of the Tits construction of Lie algebras in terms of alternative and Jordan algebras [Tits66].

The purpose of this paper is to provide models of the other three exceptions: $\mathfrak{br}(2; 3)$ and $\mathfrak{el}(5; 3)$ in characteristic 3, and $\mathfrak{br}(2; 5)$ in characteristic 5.

Actually, the superalgebra $\mathfrak{br}(2; 3)$ already appeared in [Eld06b] Theorem 3.2(i)] related to a symplectic triple system of dimension 8. Here it will be shown to be related to a nice five dimensional orthosymplectic triple system.

The Lie superalgebra $\mathfrak{el}(5; 3)$ will be shown to be a maximal subalgebra of the Lie superalgebra $\mathfrak{g}(8, 3) = \mathfrak{g}(S_{8}, S_{1}, 2)$ in the Supermagic Square. Furthermore, it will be shown to be related to an orthogonal triple system defined on the direct sum of two copies of the octonions and, finally, it will be proved to be the Lie superalgebra of derivations of a specific orthosymplectic triple system, and this

Date: May 9, 2008.

2000 Mathematics Subject Classification. Primary 17B50; Secondary 17B60, 17B25.

Key words and phrases. Lie superalgebra, Cartan matrix, simple, modular, exceptional, orthosymplectic triple system.

* Supported by the Spanish Ministerio de Educaci´on y Ciencia and FEDER (MTM 2007-67884-C04-02) and by the Diputaci´on General de Arag´on (Grupo de Investigaci´on de ´Algebra).
latter result will relate \( \mathfrak{e}(5; 3) \) to the Lie superalgebra \( \mathfrak{g}(6,6) = \mathfrak{g}(S_{4,2}, S_{4,2}) \) in the Supermagic Square.

Finally, a very explicit model of the Lie superalgebra \( \mathfrak{br}(2; 5) \) will be constructed.

The paper is organized as follows. The construction of the Extended Magic Square (or Supermagic Square) in characteristic 3 in terms of composition superalgebras is recalled in §1. Then, in §2, the Lie superalgebra \( \mathfrak{e}(5; 3) \) (in characteristic 3) is shown to be a maximal subalgebra of the Lie superalgebra \( \mathfrak{g}(S_8, S_{1,2}) \) in the Supermagic Square. This gives a very concrete realization of \( \mathfrak{e}(5; 3) \) in terms of simple components: copies of the three dimensional simple Lie algebra \( \mathfrak{sl}_2 \) and of its natural two-dimensional module. Orthogonal triple systems are reviewed in §3 and the Lie superalgebra \( \mathfrak{e}(5; 3) \) is shown to be isomorphic to the Lie superalgebra of an orthogonal triple system defined on the direct sum of two copies of the split Cayley algebra. Then the orthosymplectic triple systems, which extend both the orthogonal and symplectic triple systems, are recalled in §4. A very simple such system is defined on the set of trace zero elements of the \( 4|2 \) dimensional composition superalgebra \( \mathcal{B}(4,2) \). (The dimension being \( 4|2 \) means that the even part has dimension 4 and the odd part dimension 2.) The Lie superalgebra naturally attached to this orthosymplectic triple system is shown to be isomorphic to the Lie superalgebra \( \mathfrak{br}(2; 3) \). §5 deals with another distinguished orthosymplectic triple system, which lives inside the Lie superalgebra \( \mathfrak{g}(S_8, S_{1,2}) \) in the Supermagic Square. It turns out that the Lie superalgebra \( \mathfrak{e}(5; 3) \) is isomorphic to the Lie superalgebra of derivations of this system. This shows also how \( \mathfrak{e}(5; 3) \) embeds in the Lie superalgebra \( \mathfrak{g}(S_{4,2}, S_{4,2}) \) of the Supermagic Square. Finally, §6 is devoted to give an explicit model of the Lie superalgebra \( \mathfrak{br}(2; 5) \) (in characteristic 5) in terms of two copies of \( \mathfrak{sl}_2 \) and of their natural modules.

All the vector spaces and superspaces considered in this paper will be assumed to be finite dimensional over a ground field \( k \) of characteristic \( \neq 2 \). In dealing with elements of a superspace \( V = V_0 \oplus V_1 \), an expression like \( (-1)^{uv} \), for homogeneous elements \( u, v \), is a shorthand for \( (-1)^{q(u)p(v)} \), where \( p \) is the parity function.

1. The Supermagic Square in characteristic 3

Recall that an algebra \( C \) over a field \( k \) is said to be a composition algebra if it is endowed with a regular quadratic form \( q \) (that is, its polar form \( b(x, y) = q(x+y) - q(x) - q(y) \) is a nondegenerate symmetric bilinear form) such that \( q(xy) = q(x)q(y) \) for any \( x, y \in C \). The unital composition algebras will be termed Hurwitz algebras. On the other hand, a composition algebra is said to be symmetric in case the polar form is associative: \( b(xy, z) = b(x, yz) \).

Hurwitz algebras are the well-known algebras that generalize the classical real division algebras of the real and complex numbers, quaternions and octonions. Over any algebraically closed field \( k \), there are exactly four of them: \( k, k \times k, \text{Mat}_2(k) \) and \( C(k) \) (the split Cayley algebra), with dimensions 1, 2, 4 and 8.

Let us superize the above concepts.

A quadratic superform on a \( \mathbb{Z}_2 \)-graded vector space \( U = U_0 \oplus U_1 \) over a field \( k \) is a pair \( q = (q_0, b) \) where \( q_0 : U_0 \to k \) is a quadratic form, and \( b : U \times U \to k \) is a supersymmetric even bilinear form such that \( b|_{U_0 \times U_0} \) is the polar of \( q_0 \):

\[
b(x_0, y_0) = q_0(x_0 + y_0) - q_0(x_0) - q_0(y_0)
\]

for any \( x_0, y_0 \in U_0 \).

The quadratic superform \( q = (q_0, b) \) is said to be regular if the bilinear form \( b \) is nondegenerate.
Then a superalgebra $C = C_0 \oplus C_1$ over $k$, endowed with a regular quadratic superform $q = (q_0, b)$, called the \textit{norm}, is said to be a \textit{composition superalgebra} (see [EO02]) in case

$$q_0(x_0y_0) = q_0(x_0)q_0(y_0), \quad (1.1a)$$

$$b(xy, z) = q_0(x_0)b(yz, z) = b(yx_0, zx_0), \quad (1.1b)$$

$$b(xy, zt) + (-1)^{x+y+z+y}b(zx, yt) = (-1)^{y+z}b(x, z)b(y, t), \quad (1.1c)$$

for any $x_0, y_0 \in C_0$ and homogeneous elements $x, y, z, t \in C$. Since the characteristic of the ground field is assumed to be not $2$, equation (1.1c) already implies (1.1a) and (1.1b).

The unital composition superalgebras are termed \textit{Hurwitz superalgebras}, while a composition superalgebra is said to be \textit{symmetric} in case its bilinear form is associative, that is,

$$b(xy, z) = b(x, yz),$$

for any $x, y, z$.

Only over fields of characteristic $3$ there appear nontrivial Hurwitz superalgebras (see [EO02]):

- Let $V$ be a two dimensional vector space over a field $k$, endowed with a nonzero alternating bilinear form $(\cdot, \cdot)$ (that is $(v|v) = 0$ for any $v \in V$). Consider the superspace $B(1, 2)$ (see [She97]) with

$$B(1, 2)_0 = k1, \quad \text{and} \quad B(1, 2)_1 = V, \quad (1.2)$$

endowed with the supercommutative multiplication given by

$$1x = x1 = x \quad \text{and} \quad uv = (u|v)1$$

for any $x \in B(1, 2)$ and $u, v \in V$, and with the quadratic superform $q = (q_0, b)$ given by:

$$q_0(1) = 1, \quad b(u, v) = (u|v), \quad (1.3)$$

for any $u, v \in V$. If the characteristic of $k$ is equal to $3$, then $B(1, 2)$ is a Hurwitz superalgebra ([EO02 Proposition 2.7]).

- Moreover, with $V$ as before, let $f \mapsto \bar{f}$ be the associated symplectic involution on $\text{End}_k(V)$ (so $(f(u)|v) = (u|\bar{f}(v))$ for any $u, v \in V$ and $f \in \text{End}_k(V)$). Consider the superspace $B(4, 2)$ (see [She97]) with

$$B(4, 2)_0 = \text{End}_k(V), \quad \text{and} \quad B(4, 2)_1 = V, \quad (1.4)$$

with multiplication given by the usual one (composition of maps) in $\text{End}_k(V)$, and by

$$v \cdot f = f(v) = \bar{f} \cdot v \in V,$$

$$u \cdot v = (\langle u|v \rangle \in \text{End}_k(V)$$

for any $f \in \text{End}_k(V)$ and $u, v \in V$, where $(\langle u|v \rangle$ denotes the endomorphism $w \mapsto (u|w)v$; and with quadratic superform $q = (q_0, b)$ such that

$$q_0(f) = \det(f), \quad b(u, v) = (u|v),$$

for any $f \in \text{End}_k(V)$ and $u, v \in V$. If the characteristic is equal to $3$, $B(4, 2)$ is a Hurwitz superalgebra ([EO02 Proposition 2.7]).

Given any Hurwitz superalgebra $C$ with norm $q = (q_0, b)$, its standard involution is given by

$$x \mapsto \bar{x} = b(x, 1)1 - x.$$

A new product can be defined on $C$ by means of

$$x \bullet y = \bar{x}y.$$
The resulting superalgebra, denoted by \( \overline{C} \), is called the \textit{para-Hurwitz superalgebra} attached to \( C \), and it turns out to be a symmetric composition superalgebra.

Given a symmetric composition superalgebra \( S \), its \textit{triality Lie superalgebra} \( \text{tri}(S) = \text{tri}(S)_0 \oplus \text{tri}(S)_1 \) is defined by:

\[
\text{tri}(S)_i = \{ (d_0, d_1, d_2) \in \mathfrak{osp}(S, q)^3 : \]
\[
d_0(x \cdot y) = d_1(x) \cdot y + (-1)^{x} x \cdot d_2(y) \quad \forall x, y \in S_0 \cup S_1, \]

where \( \tilde{i} = 0, 1 \), and \( \mathfrak{osp}(S, q) \) denotes the associated orthosymplectic Lie superalgebra. The bracket in \( \text{tri}(S) \) is given componentwise.

Now, given two symmetric composition superalgebras \( S \) and \( S' \), one can form (see [CE07a] §3, or [Eld04] for the non-super situation) the Lie superalgebra:

\[
g = g(S, S') = (\text{tri}(S) \oplus \text{tri}(S')) \oplus ( \bigoplus_{i=0}^2 \text{tri}(S \otimes S') ),
\]

where \( \iota_i(S \otimes S') \) is just a copy of \( S \otimes S' \) (i = 0, 1, 2), with bracket given by:

- the Lie bracket in \( \text{tri}(S) \oplus \text{tri}(S') \), which thus becomes a Lie subalgebra of \( g \),
- \([[(d_0, d_1, d_2), \iota_i(x \otimes x')], \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x') \),
- \([[(d_0', d_1', d_2'), \iota_i(x \otimes x')], \iota_i(x \otimes x')] = (-1)^{d_i} d_i(x \otimes d_i'(x')) \),
- \([\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = (-1)^{x} \iota_i(x \otimes (x' \cdot y')) \) (indices modulo 3),
- \([\iota_i(x \otimes x'), \iota_i(y \otimes y')] = (-1)^{xx' + yy' + yy'} b'(x', y') \Theta^i(t_{x,y}) + (-1)^{yy'} b(x, y) \Theta^i(t_{x',y'}) \),

for any \( i = 0, 1, 2 \) and homogeneous \( x, y \in S, x', y' \in S' \), \( (d_0, d_1, d_2) \in \text{tri}(S) \), and \( (d_0', d_1', d_2') \in \text{tri}(S') \). Here \( \Theta \) denotes the natural automorphism \( \Theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1) \) in \( \text{tri}(S) \), while \( t_{x,y} \) is defined by

\[
t_{x,y} = (\sigma_{x,y}, \frac{1}{2} b(x, y) 1 - r_x l_y, \frac{1}{2} b(x, y) 1 - l_x r_y) \quad (1.6)
\]

with \( l_x(y) = x \cdot y, r_x(y) = (-1)^{x} y \cdot x, \) and \( \sigma_{x,y}(z) = (-1)^{yz} b(x, z) y - (-1)^{y+x} b(y, z) x \)

(1.7) for homogeneous \( x, y, z \in S \). Also \( \theta' \) and \( t_{x',y'} \) denote the analogous elements for \( \text{tri}(S') \).

Over a field \( k \) of characteristic 3, let \( S_r \) (\( r = 1, 2, 4, 8 \) or 4 or 8) denote the para-Hurwitz algebra attached to the split Hurwitz algebra of dimension \( r \) (this latter algebra being either \( k, k \times k, \text{Mat}_2(k) \) or \( C(k) \)). Also, denote by \( S_{1,2} \) the para-Hurwitz superalgebra \( B(1,2) \), and by \( S_{1,4} \) the para-Hurwitz superalgebra \( B(4,2) \). Then the Lie superalgebras \( g(S, S') \), where \( S, S' \) run over \( \{ S_1, S_2, S_4, S_8, S_{1,2}, S_{1,4} \} \), appear in Table 1 which has been obtained in [CE07a].

Since the construction of \( g(S, S') \) is symmetric, only the entries above the diagonal are needed. In Table 1 \( f_4, e_6, e_7, e_8 \) denote the simple exceptional classical Lie algebras, \( \tilde{c}_6 \) denotes a 78 dimensional Lie algebra whose derived Lie algebra is the 77 dimensional simple Lie algebra \( e_6 \) in characteristic 3. The even and odd parts of the nontrivial superalgebras in the table which have no counterpart in the classification in characteristic 0 ([Kac77]) are displayed, \textit{spin} denotes the spin module for the corresponding orthogonal Lie algebra, while \( (n) \) denotes a module of dimension \( n \), whose precise description is given in [CE07a]. Thus, for example, \( g(S_4, S_{1,2}) \) is a Lie superalgebra whose even part is (isomorphic to) the direct sum of the symplectic Lie algebra \( \mathfrak{sp}_6 \) and of \( \mathfrak{sl}_2 \), while its odd part is the tensor product of a 13 dimensional module for \( \mathfrak{sp}_6 \) and the natural 2 dimensional module for \( \mathfrak{sl}_2 \).

In Table 2 a more precise description of the Lie superalgebras that appear in the Supermagic Square is given. This table displays the even parts and the highest
weights of the odd parts. The numbering of the roots follows Bourbaki’s conventions [Bour02]. The fundamental dominant weight for \( \mathfrak{sl}_2 \) will be denoted by \( \omega \), while the fundamental dominant weights for a Lie algebra with a Cartan matrix of order \( n \) will be denoted by \( \omega_1, \ldots, \omega_n \). Below each entry, there appears the result in [CE07a] where the result can be found.

| \( S_1 \)   | \( S_2 \)       | \( S_4 \)       | \( S_8 \)       | \( S_{1,2} \)       | \( S_{4,2} \)       |
|-------------|----------------|----------------|----------------|----------------|----------------|
| \( \mathfrak{sl}_2 \) | \( \mathfrak{pgl}_3 \) | \( \mathfrak{sp}_6 \) | \( \mathfrak{f}_4 \) | \( \mathfrak{psl}_{2,2} \) | \( \mathfrak{sp}_6 \oplus (14) \) |
| \( \mathfrak{pgl}_3 \oplus \mathfrak{sl}_2 \oplus \mathfrak{pgl}_6 \oplus \mathfrak{sl}_2 \oplus (\mathfrak{psl}_3 \oplus (2)) \) | \( \mathfrak{pgl}_6 \oplus (20) \) | \( \mathfrak{so}_{12} \oplus \mathfrak{sl}_2 \oplus (\mathfrak{psl}_6 \oplus (13) \oplus (2)) \) | \( \mathfrak{so}_{12} \oplus \mathfrak{spin}_{12} \) | \( \mathfrak{e}_8 \oplus (56) \) |
| \( \mathfrak{so}_{7} \oplus 2\mathfrak{spin}_7 \) | \( \mathfrak{sp}_8 \oplus (40) \) | \( \mathfrak{so}_{13} \oplus \mathfrak{spin}_{13} \) |

Table 2. Even and odd parts in the Supermagic Square
2. The Lie superalgebra \( \mathfrak{e}(5; 3) \)

The aim of this section is to show how the Lie superalgebra \( \mathfrak{e}(5; 3) \) embeds in a nice way as a maximal subalgebra in the simple Lie superalgebra \( \mathfrak{g}(S_8, S_{1,2}) \) of the Supermagic Square.

Throughout this section the characteristic of the ground field \( k \) will be assumed to be 3.

The para-Hurwitz superalgebra \( S_{1,2} = \overline{B(1, 2)} \) is described as \( S_{1,2} = k1 \oplus V \) (see (1.2) and (1.3)), where \( (S_{1,2})_0 = k1 \) is a copy of the ground field, and \( (S_{1,2})_1 = V \) is a two dimensional vector space equipped with a nonzero alternating bilinear form \( \langle , \rangle \). The multiplication is given by:

\[
1 \bullet 1 = 1, \quad 1 \bullet u = -u = u \bullet 1, \quad u \bullet v = \langle u | v \rangle 1, \quad (1.1)
\]

for any \( u, v \in V \), and the norm \( q = (q_0, b) \) is given by

\[
q_0(1) = 1, \quad b(u, v) = \langle u | v \rangle, \quad (1.2)
\]

for any \( u, v \in V \).

Recall from [EO02] or [CE07a, Corollary 2.12] that the triviality Lie superalgebra of \( S_{1,2} \) is given by:

\[
\text{tri}(S_{1,2}) = \{(d, d, d) : d \in \mathfrak{osp}(S_{1,2}, q)\}, \quad (2.3)
\]

and thus \( \text{tri}(S_{1,2}) \) can (and will) be identified with the Lie superalgebra \( \mathfrak{b}_{0,1} = \mathfrak{sp}(V) \oplus V \) (see [EO07a (2.18)]), with even part \( \mathfrak{sp}(V) \cong \mathfrak{sl}_2 \), odd part \( V \), where \([\rho, v] = \rho(v) \) and \( [u, v] = \gamma_{u, v} \) for any \( \rho \in \mathfrak{sp}(V) \) and \( u, v \in V \), with \( \gamma_{u, v} = \langle u | v \rangle + \langle v | u \rangle \).

Besides, the action of \( \mathfrak{b}_{0,1} \) on \( S_{1,2} \) is given by:

\[
\rho : 1 \mapsto 0, \quad u \mapsto \rho(u),
\]

\[
u : 1 \mapsto -u, \quad v \mapsto -\langle u | v \rangle 1,
\]

for any \( \rho \in \mathfrak{sp}(V) \) and \( u, v \in V \) (see [CE07a (2.16)]).

Consider now the Lie superalgebra \( \mathfrak{g}(S_8, S_{1,2}) \) in the Supermagic Square:

\[
\mathfrak{g}(S_8, S_{1,2}) = \left( \text{tri}(S_8) \oplus \text{tri}(S_{1,2}) \right) \oplus \left( \oplus_{i=0}^{2} t_i (S_8 \otimes S_{1,2}) \right).
\]

This is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded with

\[
\mathfrak{g}(S_8, S_{1,2})_{(0,0)} = \text{tri}(S_8) \oplus \text{tri}(S_{1,2}),
\]

\[
\mathfrak{g}(S_8, S_{1,2})_{(1,0)} = \mathfrak{t}_0(S_8 \otimes S_{1,2}),
\]

\[
\mathfrak{g}(S_8, S_{1,2})_{(0,1)} = \mathfrak{t}_1(S_8 \otimes S_{1,2}),
\]

\[
\mathfrak{g}(S_8, S_{1,2})_{(1,1)} = \mathfrak{t}_2(S_8 \otimes S_{1,2}),
\]

and, therefore, the linear map \( \tau \), defined by

\[
\tau = \begin{cases} id & \text{on } \mathfrak{g}(S_8, S_{1,2})_{(0,0)} \oplus \mathfrak{g}(S_8, S_{1,2})_{(1,0)}, \\ -id & \text{on } \mathfrak{g}(S_8, S_{1,2})_{(0,1)} \oplus \mathfrak{g}(S_8, S_{1,2})_{(1,1)}, \end{cases}
\]

is a Lie superalgebra automorphism. On the other hand, the grading automorphism \( \sigma \):

\[
\sigma = \begin{cases} id & \text{on } \mathfrak{g}(S_8, S_{1,2})_{\bar{0}}, \\ -id & \text{on } \mathfrak{g}(S_8, S_{1,2})_{\bar{1}}, \end{cases}
\]

commutes with \( \tau \). Consider the order two automorphism \( \xi = \sigma \tau = \tau \sigma \), which provides a \( \mathbb{Z}_2 \)-grading of \( \mathfrak{g}(S_8, S_{1,2}) \) with even and odd components given by:

\[
\mathfrak{g}(S_8, S_{1,2})_+ = (\text{tri}(S_8) \oplus \mathfrak{sp}(V)) \oplus \mathfrak{t}_0(S_8 \otimes 1) \oplus \mathfrak{t}_1(S_8 \otimes V) \oplus \mathfrak{t}_2(S_8 \otimes V),
\]

\[
\mathfrak{g}(S_8, S_{1,2})_- = V \oplus \mathfrak{t}_0(S_8 \otimes V) \oplus \mathfrak{t}_1(S_8 \otimes 1) \oplus \mathfrak{t}_2(S_8 \otimes 1).
\]

(2.4)
Theorem 2.5. In the situation above, the subalgebra \( \mathfrak{g}(S_8, S_{1,2})_+ \) of \( \mathfrak{g}(S_8, S_{1,2}) \) fixed by the automorphism \( \xi \) is a maximal subalgebra of \( \mathfrak{g}(S_8, S_{1,2}) \) isomorphic to the Lie superalgebra \( \mathfrak{e}(5;3) \).

Proof. As a module for the subalgebra \( \mathfrak{tri}(S_8) \oplus \mathfrak{sp}(V) \) of \( \mathfrak{g}(S_8, S_{1,2})_+ \), the odd component \( \mathfrak{g}(S_8, S_{1,2})_- \) relative to the \( \mathbb{Z}_2 \)-grading given by \( \xi \) decomposes as the direct sum of the nonisomorphic irreducible modules:

\[
V, \quad \iota_0(S_8 \otimes V), \quad \iota_1(S_8 \otimes 1), \quad \iota_2(S_8 \otimes 1).
\]

Actually, identifying \( \mathfrak{tri}(S_8) \) to the orthogonal Lie algebra \( \mathfrak{s} \mathfrak{o}_8 \) through the projection onto the first component (this is possible because of the Local Principle of Triality [KMT98, §35]), \( \iota_1(S_8 \otimes 1) \) and \( \iota_2(S_8 \otimes 1) \) are the two half-spin representations of \( \mathfrak{s} \mathfrak{o}_8 \), while \( \iota_0(S_8 \otimes V) \) is the tensor product of the natural modules for \( \mathfrak{s} \mathfrak{o}_8 \) and for \( \mathfrak{sp}(V) \), so these four modules are indeed nonisomorphic. Therefore, any \( \mathfrak{g}(S_8, S_{1,2})_+ \)-submodule of \( \mathfrak{g}(S_8, S_{1,2})_- \) is a direct sum of some of them. But the definition of the Lie bracket in \( \mathfrak{g}(S_8, S_{1,2}) \) shows that any of these spaces generates \( \mathfrak{g}(S_8, S_{1,2})_- \) as a module over \( \mathfrak{g}(S_8, S_{1,2})_+ \). Hence \( \mathfrak{g}(S_8, S_{1,2})_- \) is an irreducible module for \( \mathfrak{g}(S_8, S_{1,2})_+ \) and, therefore, \( \mathfrak{g}(S_8, S_{1,2})_+ \) is a maximal subalgebra of \( \mathfrak{g}(S_8, S_{1,2}) \).

From now on, the proof relies heavily on the description of \( \mathfrak{g}(S_8, S_{1,2}) \) given in [CE07a, §5.10] (which follows the ideas in [Eld07a]). This description is obtained in terms of five vector spaces of dimension 2: \( V_1, \ldots, V_5 \), endowed with nonzero alternating bilinear forms:

\[
\mathfrak{g}(S_8, S_{1,2}) = \bigoplus_{\sigma \in \tilde{S}_{8,3}} V(\sigma),
\]

with

\[
\tilde{S}_{8,3} = \{ \emptyset, \{1,2,3,4\}, \{5\}, \{1,2\}, \{3,4\}, \{1,2,5\}, \{3,4,5\}, \{2,3\}, \{1,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3\}, \{2,4\}, \{1,3,5\}, \{2,4,5\} \}.
\]

Here \( V(\emptyset) = \bigoplus_{r=1}^5 \mathfrak{sp}(V_r) \), while for \( \emptyset \neq \sigma = \{i_1, \ldots, i_r\}, V(\sigma) = V_{i_1} \otimes \cdots \otimes V_{i_r} \). Also, any \( \sigma \subseteq \{1,2,3,4,5\} \) can be thought of as an element in \( \mathbb{Z}_2^n \) (for instance, \( \{1,3,5\} = (1,0,1,0,1) \in \mathbb{Z}_2^5 \)), so it makes sense to consider \( \sigma + \tau \) for \( \sigma, \tau \subseteq \{1,2,3,4,5\} \).

The brackets \( V(\sigma) \times V(\tau) \rightarrow V(\sigma + \tau) \) are nonzero scalar multiples of the ‘contraction maps’ \( \varphi_{\sigma,\tau} \) in [CE07a (4.9)]. Under this description,

\[
\mathfrak{g}(S_8, S_{1,2})_+ = \bigoplus_{\sigma \in \tilde{S}_{8,3}} V(\sigma),
\]

with

\[
\tilde{S}_{8,3} = \{ \emptyset, \{1,2,3,4\}, \{1,2\}, \{3,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3,5\}, \{2,4,5\} \}.
\]

Thus, the even and odd degrees (same notation as in [CE07a §5]) are:

\[
\Phi_0 = \{ \pm \epsilon_i : 1 \leq i \leq 5 \} \cup \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \} \cup \{ \pm \epsilon_i \pm \epsilon_j : (i,j) \in \{(1,2),(3,4)\} \},
\]

\[
\Phi_1 = \{ \pm \epsilon_i \} \cup \{ \pm \epsilon_i \pm \epsilon_j : (i,j) \in \{(2,3),(1,4),(1,3),(2,4)\} \}.
\]

With the lexicographic order given by \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 < \epsilon_5 \) in [CE07a §5.10], the set of irreducible degrees is

\[
\Pi = \{ \alpha_1 = \epsilon_5 - \epsilon_2 - \epsilon_4, \alpha_2 = \epsilon_2 - \epsilon_1, \alpha_3 = 2\epsilon_1, \alpha_4 = \epsilon_4 - \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_5 = 2\epsilon_3 \},
\]

which is a \( \mathbb{Z} \)-linearly independent set with \( \Phi = \Phi_0 \cup \Phi_1 \subseteq \mathbb{Z}\Pi \). The associated Cartan matrix is:

\[
\begin{pmatrix}
0 & -2 & 0 & 0 & 0 \\
-1 & 2 & -2 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]
which is equal to the second matrix in [BGL] §13.2 for \(\mathfrak{e}(5; 3)\) with the third and fourth rows and columns permuted. This shows that \(\mathfrak{g}(S_8, S_1, 2)\) is isomorphic to \(\mathfrak{e}(5; 3)\). Note that the \(4 \times 4\) submatrix on the lower right corner is the Cartan matrix of type \(B_4\), and indeed it corresponds to the subalgebra \(\mathfrak{tri}(S_8) \oplus \mathfrak{t}_0(S_8 \otimes 1)\), which is isomorphic to the orthogonal Lie algebra \(\mathfrak{so}_8\) (\(\mathfrak{tri}(S_8)\) being isomorphic to \(\mathfrak{so}_8\) and \(S_8\) to its natural module).

\[\square\]

3. Orthogonal triple systems and the Lie superalgebra \(\mathfrak{e}(5; 3)\)

This section is devoted to the proof of the fact that the Lie superalgebra \(\mathfrak{e}(5; 3)\) is the Lie superalgebra associated to a particular orthogonal triple system defined on the direct sum of two copies of the split octonions.

Orthogonal triple systems were introduced in [Oku93]:

**Definition 3.1.** Let \(T\) be a vector space over a field \(k\) endowed with a nonzero symmetric bilinear form \((\cdot,\cdot) : T \times T \to k\), and a triple product \(T \times T \times T \to T:\ (x,y,z) \mapsto [xyz]\). Then \((T,\{\cdot,\cdot\},(\cdot,\cdot))\) is said to be an orthogonal triple system if it satisfies the following identities:

\[
\begin{align*}
[xy] &= 0 \quad (3.2a) \\
[xy] &= (x|y)y - (y|y)x \quad (3.2b) \\
[xy][uvw] &= [[xy][vw] + [u[xyv]w] + [uw[xyv]] \\
([xy][v] + [u][xyv]) &= 0 \quad (3.2d)
\end{align*}
\]

for any elements \(x, y, u, v, w \in T\).

Equation (3.2a) shows that \(\text{ind}_{T} T = \text{span} \{[xy] : x, y \in T\}\) is a subalgebra (actually an ideal) of the Lie algebra \(\text{der} T\) of derivations of \(T\). The elements in \(\text{ind}_{T} T\) are called inner derivations. Because of (3.2a), if \(\dim T \geq 2\), then \(\text{der} T\) is contained in the orthogonal Lie algebra \(\mathfrak{so}(T, (\cdot,\cdot))\). Also note that (3.2a) is a consequence of (3.2b) and (3.2c) (see the comments in [Eld06b] after Definition 4.1).

An ideal of an orthogonal triple system \((T,\{\cdot,\cdot\},(\cdot,\cdot))\) is a subspace \(I\) such that \([ITT] + [TTI] + [TTI]\) is contained in \(I\). The orthogonal triple system is said to be simple if it does not contain any proper ideal.

Some of the main properties of these systems are summarized in the next result, taken from [Eld06b] Proposition 4.4, Theorem 4.5 and Theorem 5.1 (see also [CE07b] Theorem 4.3):

**Proposition 3.3.** Let \((T,\{\cdot,\cdot\},(\cdot,\cdot))\) be an orthogonal triple system of dimension \(\geq 2\). Then:

1. \((T,\{\cdot,\cdot\},(\cdot,\cdot))\) is simple if and only if \((\cdot,\cdot)\) is nondegenerate.
2. Let \((V, (\cdot,\cdot))\) be a two dimensional vector space endowed with a nonzero alternating bilinear form. Let \(\mathfrak{s}\) be a Lie subalgebra of \(\text{der} T\) containing \(\text{ind}_{T} T\). Define the superalgebra \(\mathfrak{g} = \mathfrak{g}(T, \mathfrak{s}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) with

\[
\begin{align*}
\mathfrak{g}_0 &= \text{sp}(V) \oplus \mathfrak{s} \\
\mathfrak{g}_1 &= V \otimes T,
\end{align*}
\]

and superanticommutative multiplication given by:

- the multiplication on \(\mathfrak{g}_0\) coincides with its bracket as a Lie algebra (the direct sum of the ideals \(\text{sp}(V)\) and \(\mathfrak{s}\));
- \(\mathfrak{g}_0\) acts naturally on \(\mathfrak{g}_1\), that is,

\[
[s, v \otimes x] = s(v) \otimes x, \quad [d, v \otimes x] = v \otimes d(x),
\]

for any \(s \in \text{sp}(V), d \in \mathfrak{s}, v \in V, \text{ and } x \in T\);
• for any \( u, v \in V \) and \( x, y \in T \):
\[
[u \otimes x, v \otimes y] = -(x|y)\gamma_{u,v} + (u|v)d_{x,y}
\]
(3.4)
where \( \gamma_{u,v} = (u|.)v + (v|.)u \) and \( d_{x,y} = [xy] \).

Then \( g(T, s) \) is a Lie superalgebra. Moreover, \( g(T, s) \) is simple if and only if \( s \) coincides with \( \text{ind}r T \) and \( T \) is a simple orthogonal triple system.

Conversely, given a Lie superalgebra \( g = g_0 \oplus g_1 \) with
\[
\begin{cases}
g_0 = \mathfrak{sp}(V) \oplus s & \text{(direct sum of ideals)},
g_1 = V \otimes T & \text{(as a module for } g_0),
\end{cases}
\]
where \( T \) is a module over \( s \), by \( \mathfrak{sp}(V) \)-invariance of the Lie bracket, equation (3.4) is satisfied for a symmetric bilinear form \( (., .) : T \times T \to k \) and an antisymmetric bilinear map \( d_{., .} : T \times T \to s \). Then, if \( (., .) \) is not 0 and a triple product on \( T \) is defined by means of \([xyz] = d_{x,y}(z)\), \( T \) becomes and orthogonal triple system and the image of \( s \) in \( g_l(T) \) under the given representation is a subalgebra of \( \det T \) containing \( \text{ind}r T \).

(3) If the characteristic of the ground field \( k \) is equal to 3, define the \( \mathbb{Z}_2 \)-graded algebra \( \tilde{g} = g(T) = g_{0} \oplus g_{1} \), with:
\[
\tilde{g}_{0} = \text{ind}r(T), \quad \tilde{g}_{1} = T,
\]
and anticommutative multiplication given by:
• the multiplication on \( \tilde{g}_0 \) coincides with its bracket as a Lie algebra;
• \( \tilde{g}_0 \) acts naturally on \( \tilde{g}_1 \), that is, \([d, x] = d(x)\) for any \( d \in \text{ind}r(T) \) and \( x \in T \);
• \([x, y] = d_{x,y} = [xy] \), for any \( x, y \in T \).

Then \( \tilde{g}(T) \) is a Lie algebra. Moreover, \( T \) is a simple orthogonal triple system if and only if \( \tilde{g}(T) \) is a simple \( \mathbb{Z}_2 \)-graded Lie algebra.

The Lie superalgebra \( g(T) = g(T, \text{ind}r(T)) \) in item 2) above will be called the Lie superalgebra of the orthogonal triple system \( T \) and, if the characteristic is 3, the Lie algebra \( \tilde{g}(T) \) will be called the Lie algebra of the orthogonal triple system \( T \).

The classification of the simple finite dimensional orthogonal triple systems in characteristic 0 appears in [Eld06b, Theorem 4.7]. In characteristic 3, there appears at least one new family of simple orthogonal triple systems, which are attached to degree 3 Jordan algebras (see [Eld06b, Examples 4.20]):

Let \( J = \text{Ford}(n, 1) \) be the Jordan algebra of a nondegenerate cubic form \( n \) with basepoint 1, over a field \( k \) of characteristic 3, and assume that \( \dim_k J \geq 3 \). Then any \( x \in J \) satisfies a cubic equation [McC04, I.4]
\[
x^3 - t(x)x^{o2} + s(x)x - n(x)1 = 0,
\]
(3.5)
where \( t \) is its trace linear form, \( s(x) \) is the spur quadratic form and the multiplication in \( J \) is denoted by \( \circ \). For our purposes it is enough to consider the Jordan algebras in (3.7) below.

Let \( J_0 = \{ x \in J : t(x) = 0 \} \) be the subspace of trace zero elements. Since \( \text{char} k = 3 \), \( t(1) = 0 \), so that \( k1 \in J_0 \). Consider the quotient space \( \tilde{J} = J_0/k1 \). For any \( x \in J_0 \), we have \( s(x) = -\frac{1}{3}t(x^{o2}) \) and, by linearization of (3.5), we get that for any \( x, y \in J_0 \):
\[
y^{o2} \circ x - (x \circ y) \circ y \equiv -2t(x, y)y - t(y, y)x \mod k1,
y \equiv t(x, y)y - t(y, y)x \mod k1.
\]
(3.6)
Let us denote by \( \tilde{x} \) the class of \( x \) modulo \( k1 \). Since \( J_0 \) is the orthogonal complement of \( k1 \) relative to the trace bilinear form \( t(a, b) = t(a \circ b) \), \( t \) induces a nondegenerate symmetric bilinear form on \( \tilde{J} \) defined by \( t(\tilde{x}, \tilde{y}) = t(x, y) \) for any \( x, y \in J_0 \). Now,
for any \( x, y \in J_0 \) consider the inner derivation of \( J \) given by \( D_{x,y} : z \mapsto x \circ (y \circ z) \) (see [Jac68]). Since the trace form is invariant under the Lie algebra of derivations, \( D_{x,y} \) leaves \( J_0 \) invariant, and obviously satisfies \( D_{x,y}(1) = 0 \), so it induces a map \( d_{x,y} : \hat{J} \to \hat{J} \), \( \hat{z} \mapsto D_{x,y}(z) \) and a well defined bilinear map \( \hat{J} \times \hat{J} \to \mathfrak{gl}(\hat{J}), (\hat{x}, \hat{y}) \mapsto d_{x,y} \). Consider now the triple product \([\ldots]\) on \( J \) defined by

\[
[x \hat{y} \hat{z}] = d_{x,y}(\hat{z})
\]

for any \( x, y, z \in J_0 \). This is well defined and satisfies \([3.2a]\), because of the anti-symmetry of \( d_{\ldots} \). Also, \([3.6]\) implies that

\[
[x \hat{y} \hat{y}] = d_{x,y}(\hat{y}) = t(x,y)\hat{y} - t(y,y)\hat{x} = t(\hat{x},\hat{y})\hat{y} - t(\hat{y},\hat{y})\hat{x},
\]

so \([3.2b]\) is satisfied too, relative to the trace bilinear form. Since \( D_{x,y} \) is a derivation of \( J \) for any \( x, y \in J \), \([3.2a]\) follows immediately, while \([3.2b]\) is a consequence of \( D_{x,y} \) being a derivation and the trace \( t \) being associative.

Therefore, by nondegeneracy of the trace form, \((J,\ldots,t(\ldots))\) is a simple orthogonal system over \( k \) [Ed06, Examples 4.20].

Now, let \( e \neq 0,1 \) be an idempotent \((e^{\circ 2} = e)\) of such a Jordan algebra. Changing \( e \) by \( 1-e \) if necessary, it can be assumed that \( t(e) = 1 \). Consider the Peirce 1-space:

\[
J_1(e) = \{ x \in J : e \circ x = \frac{1}{2} x \}
\]

Note that \( J_1(e) \) is contained in \( J_0 \), because for any \( x \in J_1(e), \) we have

\[
t(x) = 2t(e \circ x) = 2t((e \circ e) \circ x) = 2t(e \circ (e \circ x)) = \frac{1}{2} t(x),
\]

so \( t(x) = 0 \), and since \( 1 \in J_0(e) \oplus J_2(e) \), \( J_1(e) \) embeds in \( \hat{J} = J_0/k1 \). Besides, since \( J_1(e) \circ J_1(e) \subseteq J_0(e) \oplus J_2(e) \), and \((J_0(e) \oplus J_2(e)) \circ J_1(e) \subseteq J_1(e) \) (see [McC04 II.8]), it follows that \( J_1(e) \) is an orthogonal triple subsystem of the orthogonal triple system \( \hat{J} \) above.

In particular, let \( C \) be a Hurwitz algebra over the field \( k \) of characteristic 3 with norm \( q \) and polar form \( b \), and consider the Jordan algebra \( J = H_3(C,\ast) \) of hermitian \( 3 \times 3 \) matrices (where \((a_{ij})^\ast = (\bar{a}_{ji})\)) under the symmetrized product \( x \circ y = \frac{1}{2}(xy + yx) \). Let \( S \) be the associated para-Hurwitz algebra. Then,

\[
J = H_3(C,\ast) = \left\{ \begin{pmatrix} a_0 & \bar{a}_2 & a_1 \\ a_2 & a_1 & \bar{a}_0 \\ \bar{a}_1 & a_0 & a_2 \end{pmatrix} : a_0, a_1, a_2 \in k, \ a_0, a_1, a_2 \in S \right\}
\]

\[
= (\oplus_{i=0}^2 k e_i) \oplus (\oplus_{i=0}^2 t_i(S))
\]

(3.7)

where

\[
e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\iota_0(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & \bar{a} & 0 \end{pmatrix}, \quad \iota_1(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

(3.8)

for any \( a \in S \). Then \( J \) is the Jordan algebra of the nondegenerate cubic form \( n \) with basepoint 1, where

\[
n(x) = a_0 a_1 a_2 + b(a_0 a_1 a_2, 1) - \sum_{i=0}^2 a_i q(a_i),
\]
for $x = \sum_{i=0}^{2} \alpha_i e_i + \sum_{i=0}^{2} \iota_i(a_i)$. Here the trace form $t$ is the usual trace: $t(x) = \sum_{i=0}^{2} \alpha_i$.

Identify $ke_0 \oplus ke_1 \oplus ke_2$ with $k^3$ by means of $\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 \cong (\alpha_0, \alpha_1, \alpha_2)$. Then the Jordan product becomes:

$$
\begin{align*}
(\alpha_0, \alpha_1, \alpha_2) \circ \iota_i(a) &= \frac{1}{2}(\alpha_{i+1} + \alpha_{i+2}) \iota_i(a), \\
\iota_i(a) \circ \iota_{i+1}(b) &= \iota_{i+2}(a \bullet b), \\
\iota_i(a) \circ \iota_{i+2}(b) &= 2b(a, b) (e_{i+1} + e_{i+2}),
\end{align*}
$$

(3.9)

for any $\alpha_i, \beta_i \in k, a, b \in S$, where $i = 0, 1, 2$, and indices are taken modulo 3.

Now, $e = e_0$ is an idempotent of trace 1 and the Peirce 1-space is $\iota_1(S) \oplus \iota_2(S)$. Denote by $T_{2S}$ this orthogonal triple system. Then, in case $S = S_8$, $T_{2S}$ is an orthogonal triple system defined on the direct sum of two copies of the split octonions, and we obtain:

**Theorem 3.10.** Let $k$ be a field of characteristic 3. Then the Lie superalgebra $\mathfrak{g}(T_{2S})$ of the orthogonal triple system $T_{2S}$ is isomorphic to $\mathfrak{sl}(5; 3)$. 

Proof. Let $C$ be the split Cayley algebra over $k$, whose associated para-Hurwitz algebra is $S_8$, and let $J$ be the degree three simple Jordan algebra $H_3(C, *)$ considered above. Then, as vector spaces, $T_{2S}$ coincides with the Peirce 1-space $J_1(e_0)$. The decomposition in (3.7) is a grading over $Z_2 \times Z_2$ of the Jordan algebra $J$, and thus the Lie algebra of derivations of $J$ is also $Z_2 \times Z_2$-graded as follows (see [CE07b (3.12)]):

$$
\text{der } J = D_{\tri(S_8)} \oplus (\oplus_{i=0}^{2} D_i(S_8)),
$$

where, for $(d_0, d_1, d_2) \in \tri(S_8),

$$
\begin{align*}
D_{(d_0, d_1, d_2)}(e_i) &= 0, \\
D_{(d_0, d_1, d_2)}(\iota_i(a)) &= \iota_i(d_i(a))
\end{align*}
$$

for any $i = 0, 1, 2$ and $a \in S_8$ (see [CE07b (3.6)]), while

$$
D_i(a) = 2[L_{\iota_i(a)}, L_{e_{i+1}}]
$$

(indices modulo 3) for $0 \leq i \leq 2$ and $a \in S_8$, where $L_x$ denotes the left multiplication by $x$.

Note that $D_{\tri(S_8)} \oplus D_0(S_8)$ leaves $J_1(e_0) = \iota_1(S_8) \oplus \iota_2(S_8)$ invariant, and therefore embeds naturally in $\text{der } T_{2S}$.

Besides, the Lie superalgebra of the orthogonal triple system $\hat{J}$ is (see [CE07b §4]):

$$
\mathfrak{g}(J) = (\mathfrak{sp}(V) \oplus \text{der } J) \oplus (V \otimes \hat{J}),
$$

which is shown in [CE07b Theorem 4.9] to be isomorphic to $\mathfrak{g}(S_8, S_{1,2})$. Under this isomorphism $V \otimes T_{2S}$ corresponds to $\iota_1(S_8 \otimes V) \oplus \iota_2(S_8 \otimes V)$ inside $\mathfrak{g}(S_8, S_{1,2})$ which, under the isomorphism in Theorem 2.3 corresponds to the odd part of $\mathfrak{sl}(5; 3)$, and this odd part generates $\mathfrak{sl}(5; 3)$ as a Lie superalgebra. Therefore, the Lie superalgebra generated by $V \otimes T_{2S}$ corresponds to the subalgebra $\mathfrak{g}(S_8, S_{1,2})^+$ (isomorphic to $\mathfrak{sl}(5; 3)$). Using the isomorphism in [CE07b Theorem 4.9], this proves that the subalgebra generated by $V \otimes T_{2S}$ in $\mathfrak{g}(J)$ is

$$
(\mathfrak{sp}(V) \oplus (D_{\tri(S_8)} \oplus D_0(S_8))) \oplus (V \otimes T_{2S}).
$$

Since this is a simple Lie superalgebra, by Proposition 3.3(2) it follows that it is isomorphic to the Lie superalgebra of the orthogonal triple system $T_{2S}$. \qed
Remark 3.11. Proposition 3.3(3) shows that \( \mathfrak{g}(T_{2S_8}) \) is a simple Lie algebra. By the proof above, it is \( \mathbb{Z}_2 \)-graded with even component isomorphic to \( D_{\text{tril}(S_8)} \oplus D_0(S_8) \), which is isomorphic to the orthogonal Lie algebra \( \mathfrak{so}_9 \), and with odd component (in the \( \mathbb{Z}_2 \)-grading) given by \( T_{2S_8} \), which is the spin module for the even component. It follows that \( \mathfrak{g}(T_{2S_8}) \) is the exceptional Lie algebra of type \( F_4 \).

4. Orthosymplectic triple systems and the Lie superalgebra \( \mathfrak{br}(2,3) \)

Orthosymplectic triple systems are the supervision of the orthogonal triple systems. They unify both orthogonal and symplectic triple systems. The definition was given in [CE07b, Definition 6.2]:

**Definition 4.1.** Let \( T = T_0 \oplus T_1 \) be a vector superspace endowed with an even nonzero supersymmetric bilinear form \((.,.) : T \times T \to k \) (that is, \( (T_0|T_1) = 0 \), \((.,.)\) is symmetric on \( T_0 \) and alternating on \( T_1 \)) and a triple product \( T \times T \times T \to T \): \( (x, y, z) \mapsto [xyz] = [x,y,z] \in T_{i+j+k} \) for any \( x \in T_i, \ y \in T_j, \ z \in T_k \), where \( i, j, k = 0 \) or \( 1 \). Then \( T \) is said to be an orthosymplectic triple system if it satisfies the following identities:

\[
\begin{align*}
[xyz] + (-1)^{\nu(x)(\nu(y)+\nu(z))}[yzx] &= 0 \quad (4.2a) \\
[xyz] + (-1)^{\nu(x)(\nu(y)+\nu(z))}[zxy] &= (x|y)z + (-1)^{\nu(x)}(x|z)y - 2(y|z)x \quad (4.2b) \\
[xy][uvw] &= [(xy)[uv]] + (-1)^{(\nu(x)+\nu(y))u}[u[xy][v]] + (-1)^{(\nu(x)+\nu(y))v}[uv[xy]] \quad (4.2c) \\
vido[xy][uv] &= (x|y)u + (-1)^{\nu(x)(\nu(y)+\nu(u))}[u[xy]] = 0 \quad (4.2d)
\end{align*}
\]

for any homogeneous elements \( x, y, u, v, w \in T \).

**Remark 4.3.** If \( T_1 = 0 \), this is just the definition of an orthogonal triple system, while if \( T_0 = 0 \), then it reduces to a symplectic triple system.

As for orthogonal triple systems, the subspace \( \mathfrak{ind} T = \text{span} \{[xy] : x, y \in T \} \) is a subalgebra (actually an ideal) of the Lie superalgebra \( \mathfrak{der} T \) of derivations of \( T \), whose elements are called inner derivations.

**Proposition 4.4.** Let \( T \) be a simple orthosymplectic triple system. Then its supersymmetric bilinear form \((.,.)\) is nondegenerate. The converse is valid unless the characteristic of \( k \) is 3, \( T = T_1 \) and \( \dim T = 2 \).

**Proof.** Given an orthosymplectic triple system, the kernel of its supersymmetric bilinear form: \( T^\perp = \{ x \in T : (x|T) = 0 \} \), satisfies \( [TT^\perp] \subseteq T^\perp \) because of (4.2d), while equations (4.2a) and (4.2b) show that \( [TT^\perp T] = [T^\perp TT] \subseteq [TT^\perp] + T^\perp \), so \( T^\perp \) is an ideal of \( T \). Thus, if \( T \) is simple, then \((.,.)\) is nondegenerate.

Conversely, assume \( T^\perp = 0 \) and let \( I = I_0 \oplus I_1 \) be a proper ideal of \( T \). For homogeneous elements \( x, y, z \in T \), (4.2b) shows that the element

\[
(x|y)z + (-1)^{\nu(x)(\nu(y)+\nu(z))}[y]z - 2(y|z)x
\]

belongs to \( I \) if at least one of \( x, y, z \) is in \( I \). For \( x \in I \) we obtain

\[
(x|y)z + (-1)^{\nu(x)(\nu(y)+\nu(z))}[y]z \in I,
\]

while for \( y \in I \), after permuting \( x \) and \( y \),

\[
(x|y)z - 2(-1)^{\nu(x)(\nu(y)+\nu(z))}[y]z \in I,
\]

for homogeneous \( x \in I, \ y, z \in T \). If the characteristic of \( k \) is not 3, it follows that \( (I|T)T \subseteq I \), so \( I = T \), a contradiction. But, even if the characteristic is 3, it follows that the codimension 1 subspace \( (kx)^\perp = \{ y \in T : (x|y) = 0 \} \) is contained in \( I \) for any homogeneous element \( x \in I \), and \( I = T \) unless \( \dim T = 2 \). In the latter case,
either $T = T_0$ or $T = T_1$. But for $T = T_0$, $(x|y)y \in I$ for any homogeneous $x \in I$ and $y \in T$, and hence also $\{y \in T : (x|y) \neq 0\}$ is contained in $I$, so $I = T$. Thus $T = T_1$. \hfill \Box

Remark 4.5. The two dimensional symplectic triple system in [El06] Proposition 2.7(i) shows that there are indeed nonsimple orthosymplectic triple systems of superdimension 0/2 (that is, $\dim T_0 = 0$, $\dim T_1 = 2$).

As for orthogonal triple systems, the following result (see [CE07] Theorem 6.3) holds:

**Proposition 4.6.** Let $(T, [\cdot, \cdot, \cdot], (\cdot, \cdot))$ be an orthosymplectic triple system and let $(V, \langle \cdot, \cdot \rangle)$ be a two dimensional vector space endowed with a nonzero alternating bilinear form. Let $s$ be a Lie subsuperalgebra of $\text{der} T$ containing $\text{indeg} T$. Define the $\mathbb{Z}_2$-graded superalgebra $g = g(T, s) = g(0) \oplus g(1)$ with

\[
\begin{align*}
g(0) &= \mathfrak{sp}(V) \oplus s \quad \text{(so $g(0)_0 = \mathfrak{sp}(V) \oplus s_0$, $g(0)_1 = s_1$)}, \\
g(1) &= V \otimes T \quad \text{(with $g(1)_0 = V \otimes T_1$, $g(1)_1 = V \otimes T_0$, $V$ is odd)},
\end{align*}
\]

and superantisymmetric multiplication given by:
- the multiplication on $g(0)$ coincides with its bracket as a Lie superalgebra;
- $g(0)$ acts naturally on $g(1)$:
  \[
  [s, v \otimes x] = s(v) \otimes x, \quad [\langle d, v \otimes x] = (-1)^{d}v \otimes d(x),
  \]
  for any $s \in \mathfrak{sp}(V)$, $v \in V$, and homogeneous elements $d \in s$ and $x \in T$;
- for any $u, v \in V$ and homogeneous $x, y \in T$:
  \[
  [u \otimes x, v \otimes y] = (-1)^{\gamma_{u,v}}(-x|y)\gamma_{u,v} + \langle u|v \rangle d_{x,y}
  \]
  where $\gamma_{u,v} = \langle u|v \rangle v + \langle v|u \rangle u d_{x,y} = [x|y]$. (4.7)

Then $g(T, s)$ is a $\mathbb{Z}_2$-graded Lie superalgebra. Moreover, $g(T, s)$ is simple if and only if $s$ coincides with $\text{indeg} T$ and $(\cdot, \cdot)$ is nondegenerate.

Conversely, given a $\mathbb{Z}_2$-graded Lie superalgebra $g = g(0) \oplus g(1)$ with

\[
\begin{align*}
g(0) &= \mathfrak{sp}(V) \oplus s, \\
g(1) &= V \otimes T,
\end{align*}
\]

where $T$ is an $s$-module and $V$ is considered as an odd vector space, by $\mathfrak{sp}(V)$-invariance of the bracket, equation (4.7) is satisfied for an even supersymmetric bilinear form $(\cdot, \cdot) : T \times T \to k$ and a superantisymmetric bilinear map $d_{\cdot, \cdot} : T \times T \to s$. Then, if $(\cdot, \cdot)$ is not 0 and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $T$ becomes and orthosymplectic triple system and the image of $s$ in $\mathfrak{gl}(T)$ under the given representation is a subalgebra of $\text{der} T$ containing $\text{indeg} T$.

The Lie superalgebra $g(T) = g(T, \text{indeg}(T))$ is called the Lie superalgebra of the orthosymplectic triple system $T$.

If the characteristic of the ground field $k$ is equal to 3, then for any homogeneous elements $x, y, z$ in an orthosymplectic triple system, we have:

\[
\begin{align*}
[xyz] + (-1)^{x+y+z}[yzx] + (-1)^{x+y+z}[zxy] &= [xyz] + (-1)^{x+y+z}[yzx] - 2(-1)^{x+y+z}[zxy] \\
&= ([xyz] + (-1)^{y+z}[yzx]) - (-1)^{xy+xz+yz}([xyz] + (-1)^{xy}[zxy]) \quad \text{(by (4.2a))} \\
&= ([xy]z + (-1)^{y+z}[zxy])y - 2(y|z)x \\
&= (-1)^{xy+xz+yz}((z|y) + (-1)^{xy}z|y - 2(y|z)x) \quad \text{(by (4.2b))} \\
&= 3([x|y]z - (y|z)x) = 0,
\end{align*}
\]
so that, as in [Eld06b, Theorem 5.1]:

**Proposition 4.8.** Let \((T, \lbrack\cdot, \cdot\rbrack, \langle\cdot,\cdot\rangle)\) be an orthosymplectic triple system over a field \(k\) of characteristic 3. Define the \(\mathbb{Z}_2\)-graded superalgebra \(\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T) = \tilde{\mathfrak{g}}(T)^+ \oplus \tilde{\mathfrak{g}}(T)^-\), with

\[
\tilde{\mathfrak{g}}^+ = \text{ind} \mathfrak{r}(T), \quad \tilde{\mathfrak{g}}^- = T,
\]

and superanticommutative multiplication given by:

- the multiplication on \(\tilde{\mathfrak{g}}^+\) coincides with its bracket as a Lie superalgebra;
- \(\tilde{\mathfrak{g}}^+\) acts naturally on \(\tilde{\mathfrak{g}}^-\), that is, \([d, x] = d(x)\) for any \(d \in \text{ind} \mathfrak{r}(T)\) and \(x \in T\);
- \([x, y] = d_{x,y} = [xy], for any x, y \in \tilde{\mathfrak{g}}^- = T.\)

Then \(\tilde{\mathfrak{g}}\) is a \(\mathbb{Z}_2\)-graded Lie superalgebra, with the even part \(\tilde{\mathfrak{g}}^0 = \text{ind} \mathfrak{r}(T)^0 \oplus T^0\) and the odd part \(\tilde{\mathfrak{g}}^1 = \text{ind} \mathfrak{r}(T)^1 \oplus T^1\). Moreover, \(T\) is a simple orthosymplectic triple system if and only if \(\tilde{\mathfrak{g}}\) is simple as a \(\mathbb{Z}_2\)-graded Lie superalgebra.

Now, let \(C\) be a Hurwitz superalgebra of dimension \(\geq 1\) over a field \(k\) of characteristic \(\neq 2\), with norm \(q = (q_0, b)\), and standard involution \(x \mapsto \bar{x}\). For any homogeneous elements \(x, y, z\), the following holds (see [EO02]):

\[
\begin{align*}
 b(xy, z) &= (-1)^{xy}b(y, \bar{x}z) = (-1)^{yz}b(x, z\bar{y}), \\
 xy + (-1)^{xy}y\bar{x} &= b(x, y)1 = \bar{x}y + (-1)^{xy}\bar{y}x, \\
 \bar{x}(yz) + (-1)^{xy}\bar{y}(xz) &= b(x, y)z = (xz)\bar{y} + (-1)^{xy}(zy)\bar{x}.
\end{align*}
\]

Consider the subspace of trace zero elements, \(C^0 = \{x \in C : b(1, x) = 0\} = \{x \in C : \bar{x} = -x\}\).

Then, for any homogeneous elements \(x, y \in C^0\), we have

\[
xy + (-1)^{xy}yx = -(xy + (-1)^{xy}y\bar{x}) = -b(x, y)1,
\]

while \(xy - (-1)^{xy}yx = [x, y]\). Thus

\[
xy = \frac{1}{2}(-b(x, y)1 + [x, y]). \tag{4.9}
\]

Also, for any homogeneous elements \(x, y, z \in C^0\), we have

\[
b([x, y], z) = b(xy - (-1)^{xy}yx, z) = b(x, (-1)^{yz}z\bar{y} - \bar{y}z) = b(x, yz - (-1)^{yz}zy) = b(x, [y, z]),
\]

so

\[
b([x, y], z) = b(x, [y, z]) \tag{4.10}
\]

for any \(x, y, z \in C^0\). Using (4.9) and (4.10) we obtain:

\[
[x, y, z] + (-1)^{yz}[[x, z], y] = b([x, y], z)1 + 2[x, y]z + (-1)^{yz}(b([x, z], y)1 + 2[x, z]y) = 2(b(x, y)z + (-1)^{yz}[x, z]y) = 2(b(x, y)z + 2(xy)zy + (-1)^{yz}(b(x, z)y + 2(xz)y)) = 2(b(x, y)z + (-1)^{yz}b(x, z)y) - 4((xy)\bar{z} + (-1)^{yz}(xz)\bar{y}) = 2b(x, y)z + 2(-1)^{yz}b(x, z)y - 4b(y, z)x,
\]

for any homogeneous \(x, y, z \in C^0\). Therefore, with \((x|y) = 2b(x, y)\), for any \(x, y, z \in C^0\) we have:

\[
[x, y, z] + (-1)^{yz}[[x, z], y] = (x|y)z + (-1)^{yz}(x|z)y - 2(y|z)x. \tag{4.11}
\]


Now, if the characteristic of the ground field $k$ is equal to 3, for any homogeneous $x, y, z \in C^0$ we have:

\[
[[x, y], z] + (-1)^{x+y+z}[[y, z], x] + (-1)^{(x+y)z}[[z, x], y]
= [[x, y], z] + (-1)^{x+y+z}[[y, z], x] - 2(-1)^{(x+y)z}[[z, x], y]
= ([[x, y], z] + (-1)^{yz}[[x, z], y])
- (-1)^{xy+x+y}((z, y) + (-1)^{y}[[z, x], y])
= ((x[y]z + (-1)^{y}(x[z])y - 2(y[z])x)
- (-1)^{xy+x+y}((z[y] + (-1)^{y}[(z)[x]y - 2(y[x])z).
= 3((x[y]z - (y[z])z) = 0.

Thus, $(C^0, [.,.])$ is a Lie superalgebra, and then equations (4.10) and (4.11) show the validity of the first assertion in the following result:

**Theorem 4.12.** Let $C$ be a Hurwitz superalgebra of dimension $\geq 2$ over a field $k$ of characteristic 3 with norm $q = (q_0, b)$. Then, with the triple product $[xyz] = [[x, y], z]$ and the supersymmetric bilinear form $(.,.) = 2b(.,.)$, $C^0$ becomes an orthosymplectic triple system. Moreover, if the dimension of $C$ is $\leq 3$, then the triple product is trivial, otherwise the inner derivation algebra $\text{ind}(C^0)$ equals $\text{ad}_{C^0}$, the linear span of the adjoint maps $\text{ad}_z (\cdot) = [x, y]$ for any $x \in C^0$.

**Proof.** Only the last assertion needs to be verified. If the dimension of $C$ is at most 3, then $C$ is supercommutative, so $[C^0, C^0] = 0$. However, if the dimension of $C$ is at least 4 (hence either 4, 6 or 8) then $[C^0, C^0] = C^0$.

**Corollary 4.13.** Let $C$ be a Hurwitz superalgebra of dimension $\geq 4$ over a field $k$ of characteristic 3. Let $V$ be a two-dimensional vector space endowed with a nonzero alternating bilinear form $(.,.)$. Consider the anticommutative superalgebra

\[
g = (\mathfrak{sp}(V) \oplus C^0) \oplus (V \otimes C^0),
\]

with $g_0 = (\mathfrak{sp}(V) \oplus (C^0)_0) \oplus (V \otimes (C^0)_1)$ and $g_1 = (C^0)_1 \oplus (V \otimes (C^0)_0)$, and multiplication given by:

- the usual Lie bracket in the direct sum of the Lie algebra $\mathfrak{sp}(V)$ and the Lie superalgebra $C^0$,
- $[\gamma, v \otimes x] = \gamma(v) \otimes x$, for any $\gamma \in \mathfrak{sp}(V)$, $v \in V$ and $x \in C^0$,
- \[ [x, v \otimes y] = (-1)^x v \otimes [x, y], \text{ for any homogeneous } x, y \in C^0 \text{ and } v \in V, \]
- \[ [u \otimes x, v \otimes y] = (-1)^x (-v[v]x, y) \gamma_{uv} + (u[v])[x, y]) \in \mathfrak{sp}(V) \oplus C^0, \text{ for any } u, v \in V \text{ and homogeneous } x, y \in C^0 (\text{where, as before, } (x[y] = 2b(x, y) \text{ and } \gamma_{uv} = (u[v]v + (v), u)).

Then $g$ is a Lie superalgebra.

**Proof.** It suffices to note that the Lie superalgebra $g$ is just the Lie superalgebra $g(C^0, \text{ind}(C^0))$ in Proposition 4.10 of the orthosymplectic triple system $(C^0, [.,.], (.,.)$ after the natural identification of $\text{ind}(C^0) = \text{ad}_{C^0}$ with $C^0$.

If the dimension of $C$ in Corollary 4.13 is 4 (and hence $C$ is a quaternion algebra), it is not difficult to see that the Lie superalgebra $g$ is a form of the orthosymplectic Lie superalgebra $\mathfrak{sp}_{3,2}$. Also, if the dimension of $C$ is 8, so that $C$ is an algebra of octonions, then $g$ is a form of the Lie superalgebra that appears in [EL06b, Theorem 4.22(i)], which is the derived subalgebra of the Lie superalgebra $g(S_2, S_1, 2)$ in the Supermanig Square (see [CE07b, Corollary 4.10(ii)] and [ELd §3]). Also note that if the characteristic is not 3, then $C^0$ is still an orthogonal triple system, but its
associated Lie superalgebra is a simple Lie superalgebra of type $G(3)$ (see [Eld06b, Theorem 4.7 (G-type)]).

We are left with the $4|2$ dimensional Hurwitz superalgebra $C = B(4, 2)$ in (1.4) over a field $k$ of characteristic 3. The Lie bracket of elements in $C^0$ is given by:

- The usual bracket $[f, g] = fg - gf$ in $\mathfrak{sl}(V) = \mathfrak{sp}(V)$.
- $[f, u] = f \cdot u - u \cdot f = -2f(u) = f(u)$ for any $f \in \mathfrak{sp}(V)$ and $u \in V$.
- $[u, v] = u \cdot v + (-1)^{|u||v|}v \cdot u = u \cdot v + v \cdot u = b(\langle u \rangle v) + b(v)u = (\langle u \rangle)v + (\langle v \rangle)u$ for any $u, v \in V$ (recall that $\langle \cdot, \cdot \rangle = 2b(\langle \cdot, \cdot \rangle) = -b(\langle \cdot, \cdot \rangle)$).

**Proposition 4.14.** The Lie superalgebra $B(4, 2)^0$ is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1,2}$.

**Proof.** The orthosymplectic Lie superalgebra $\mathfrak{osp}_{1,2}$ is the subalgebra of the general Lie superalgebra $\mathfrak{gl}(1, 2)$ given by:

$$\mathfrak{osp}_{1,2} = \left\{ \begin{pmatrix} 0 & -\nu & \mu \\ \mu & \alpha & \beta \\ \nu & \gamma & -\alpha \end{pmatrix} : \alpha, \beta, \gamma, \mu, \nu \in k \right\}.$$

Fix a basis $\{u, v\}$ of $V$ with $\langle u | v \rangle = 1$, and consider the linear map:

$$C^0 = \mathfrak{sp}(V) \otimes V \longrightarrow \mathfrak{osp}_{1,2},$$

$$f \in \mathfrak{sp}(V) \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & -\alpha \end{pmatrix}$$

with $\begin{cases} f(u) = \alpha u + \gamma v, \\ f(v) = \beta u - \alpha v, \end{cases}$

$$\mu u + \nu v \in V \mapsto \begin{pmatrix} 0 & -\nu & \mu \\ \mu & 0 & 0 \\ \nu & 0 & 0 \end{pmatrix}.$$  

This is checked to be an isomorphism of Lie algebras by straightforward computations. \hfill \Box

Also note that for $f \in \mathfrak{sp}(V)$, $f^2 = -\det(f)1$ (by the Cayley-Hamilton equation) and $q_0(f) = \det(f)$ and $tr(f^2) = -2\det(f) = \det(f)$, so $q_0(f) = tr(f^2)$, $b(f, g) = 2tr(fg)$, and $(f|g) = tr(fg)$ for any $f, g \in \mathfrak{sp}(V) = (C^0)^0$. (Here $tr$ denotes the usual trace in $\text{End}_k(V) = B(4, 2)^0$).

**Theorem 4.15.** The Lie superalgebra of the orthosymplectic triple system $B(4, 2)^0$ is isomorphic to the Lie superalgebra $\mathfrak{br}(2; 3)$.

**Proof.** Since there are two vector spaces of dimension 2 involved here, let us denote them by $V_1$ and $V_2$, whose nonzero alternating bilinear forms will be both denoted by $\langle \cdot, \cdot \rangle$. Then consider the Hurwitz superalgebra $C = B(4, 2) = \text{End}_k(V_2) \oplus V_2$, as defined in (1.4). The Lie superalgebra associated to the orthosymplectic triple system $C^0$ is given, up to isomorphism, in Corollary (1.13)

$$\mathfrak{g} = (\mathfrak{sp}(V_1) \oplus C^0) \oplus (V_1 \otimes C^0).$$

Its even part is

$$\mathfrak{g}_0 = (\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)) \oplus (V_1 \otimes V_2),$$

with multiplication given by the natural Lie bracket in the direct sum $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$, the natural action of this subalgebra on $V_1 \otimes V_2$, and by

$$[a \otimes u, b \otimes v] = \langle u | v \rangle \gamma_{a,b} - \langle a | b \rangle \gamma_{u,v},$$

for any $a, b \in V_1$ and $u, v \in V_2$, where $\langle \cdot, \cdot \rangle = 2b(\langle \cdot, \cdot \rangle)$. Here $\gamma_{a,b} = \langle a | b \rangle + \langle b | a \rangle$, while $\gamma_{u,v} = \langle u | v \rangle + \langle v | u \rangle$. This Lie algebra is precisely the Lie algebra $L(1)$ of Kostrikin (see [Kos70] or [Eld06b, Proposition 2.12]).
On the other hand, its odd part is
\[ \mathfrak{g}_1 = V_2 \oplus (V_1 \otimes \mathfrak{sp}(V_2)). \]

Since \( C^0 \) is a simple orthosymplectic triple system, the Lie superalgebra \( \mathfrak{g} \) is simple (Proposition 4.6). Fix bases \( \{a_i, b_i\} \) of \( V_i \) \((i = 1, 2)\) with \( \langle a_1 | b_1 \rangle = 1 = \langle a_2 | b_2 \rangle \), and let \( h_i, e_i, f_i \in \mathfrak{sp}(V_i) \) be given by
\[
\begin{align*}
 h_i(a_i) &= a_i, & h_i(b_i) &= -b_i, \\
 e_i(a_i) &= 0, & e_i(b_i) &= a_i, \\
 f_i(a_i) &= b_i, & f_i(b_i) &= 0.
\end{align*}
\]

Then span \{\( h_1, h_2 \)\} is a Cartan subalgebra of \( \mathfrak{g} \), and it is the \((0,0)\)-component of the \( \mathbb{Z} \times \mathbb{Z} \)-grading of \( \mathfrak{g} \) obtained by assigning \( \deg(a_i) = \epsilon_i, \deg(b_i) = -\epsilon_i, i = 1, 2 \), where \( \{\epsilon_1, \epsilon_2\} \) is the canonical basis of \( \mathbb{Z} \times \mathbb{Z} \). The set of nonzero degrees is
\[ \Phi = \{ \pm 2\epsilon_1, \pm 2\epsilon_2, \pm \epsilon_1 \pm \epsilon_2, \pm \epsilon_1 \pm 2\epsilon_2 \} \].

Consider the elements
\[
\begin{align*}
 E_1 &= a_1 \otimes f_2, & F_1 &= b_1 \otimes e_2, & H_1 &= [E_1, F_1] = h_1 - h_2, \\
 E_2 &= a_2, & F_2 &= -b_2, & H_2 &= [E_2, F_2] = h_2.
\end{align*}
\]

Then we have that the subspace span \{\( H_1, H_2 \)\} = span \{\( h_1, h_2 \)\} is the previous Cartan subalgebra of \( \mathfrak{g} \), \( E_1 \) belongs to the homogeneous component \( \mathfrak{g}_{-2}\epsilon_2 \) in the \( \mathbb{Z} \times \mathbb{Z} \)-grading, and similarly \( F_1 \in \mathfrak{g}_{-\epsilon_1 - 2\epsilon_2}, E_2 \in \mathfrak{g}_{\epsilon_2}, \) and \( F_2 \in \mathfrak{g}_{-\epsilon_2} \). Also, the elements \( E_1, E_2, F_1, F_2 \) generate the Lie superalgebra \( \mathfrak{g} \). Besides,
\[
\begin{align*}
 [H_1, E_1] &= h_1(a_1) \otimes f_2 - a_1 \otimes [h_2, f_2] = a_1 \otimes f_2 + 2a_1 \otimes f_2 = 0, \\
 [H_1, E_2] &= (h_1 - h_2)(a_2) = -a_2, \\
 [H_2, E_1] &= a_1 \otimes [h_2, f_2] = -2a_1 \otimes f_2, \\
 [H_2, E_2] &= h_2(a_2) = a_2,
\end{align*}
\]

and similarly for the action of the \( H_i \)’s on the \( F_j \)’s. It follows, with the same arguments as in [CE07a] §4, that \( \mathfrak{g} \) is the Lie superalgebra with Cartan matrix \( \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} \), which is the first Cartan matrix of the Lie superalgebra \( \mathfrak{br}(2;3) \) given in [BGLB] §10.1.

In this way, the Lie superalgebra \( \mathfrak{br}(2;3) \), of superdimension 10\/8 is completely determined by the orthosymplectic triple system \( B(4,2)^0 \) (that is, by the orthosymplectic triple system obtained naturally from the Lie superalgebra \( \mathfrak{osp}_{1,2} \) of superdimension 3\/2).

5. Orthosymplectic triple systems and the Lie superalgebra \( \mathfrak{e}(5;3) \)

In this section, the characteristic of the ground field \( k \) will always be assumed to be 3, since we will be dealing with the superalgebras \( S_{1,2} \) and \( \mathfrak{e}(5;3) \), which only make sense in this characteristic.

Equation (2.4), together with Theorem 2.8, which allows us to identify the Lie superalgebra \( \mathfrak{e}(5;3) \) with the maximal subalgebra \( \mathfrak{g}(S_8, S_{1,2})_+ \), show that there is a decomposition of \( \mathfrak{g}(S_8, S_{1,2}) \) into the direct sum (\( \mathbb{Z}_2 \)-grading) \( \mathfrak{g}(S_8, S_{1,2}) = \mathfrak{e}(5;3) \oplus T \), where:
\[
\begin{align*}
 \mathfrak{e}(5;3) &= (\mathfrak{tr}(S_8) \oplus \mathfrak{sp}(V)) \oplus \iota_0(S_8 \otimes 1) \oplus (\iota_1(S_8 \otimes V) \oplus \iota_2(S_8 \otimes V)), \\
 T &= (\iota_1(S_8 \otimes 1) \oplus \iota_2(S_8 \otimes 1)) \oplus (V \oplus \iota_0(S_8 \otimes V)),
\end{align*}
\]

where \( V \) is a two dimensional vector space endowed with a nonzero alternating bilinear form.
Lemma 5.2. There exists a unique supersymmetric associative bilinear form

\[ B : \mathfrak{g}(S_8, S_{1,2}) \times \mathfrak{g}(S_8, S_{1,2}) \to k \]

such that

\[ B(\epsilon_i(x \otimes u), \epsilon_j(y \otimes v)) = \delta_{ij} b(x, y)b(u, v), \tag{5.3} \]

for any \( i, j = 0, 1, 2, x, y \in S_8 \) and \( u, v \in S_{1,2} \). (Here \( \delta_{ij} \) is the usual Kronecker delta and \( b \) denotes the polar form of the norm in both \( S_8 \) and \( S_{1,2} \).)

Proof. This is proved as in [Eld06a, Corollary 4.9]. First there is a unique invariant supersymmetric bilinear form \( B_{1,2} \) on the orthosymplectic Lie superalgebra \( \mathfrak{osp}(S_{1,2}, q) \) such that:

\[ B_{1,2}(d, \sigma_{u,v}) = b(d(u), v) \]

for any \( u, v \in S_{1,2} \) and \( d \in \mathfrak{osp}(S_{1,2}, q) \), where \( \sigma_{u,v} \) is defined in (1.7). Actually, \( B_{1,2} \) is given by \( B_{1,2}(d, d') = -\frac{1}{2} \text{str}(dd') \), where \( \text{str} \) denotes the supertrace. (Note that

\[ \text{str}(\sigma_{u,v} \sigma_{u,v}) = -2((-1)^{|u|}b(x, u)b(y, v) - (-1)^{|y+u|}b(x, v)b(y, u)). \]

Also, in [Eld06a] it is proved that there is a unique invariant symmetric bilinear form \( B_8 \) on \( \text{tri}(S_8) \) such that

\[ B_8((d_0, d_1, d_2), \theta(t_{x,y})) = b(d_i(x), y) \]

for any \( x, y \in S_8 \) and \( (d_0, d_1, d_2) \in \text{tri}(S_8) \).

Then the supersymmetric invariant bilinear form \( B \) required is defined by imposing the following conditions:

- The restriction of \( B \) to \( \text{tri}(S_{1,2}) \) is given by \( B_{1,2} \) (after identifying \( \text{tri}(S_{1,2}) \) with \( \mathfrak{osp}(S_{1,2}, q) \) because of (2.3)).
- The restriction of \( B \) to \( \text{tri}(S_8) \) is given by \( B_8 \).
- The restriction of \( B \) to \( \bigoplus_{i=0}^{2} t_i(S_8 \otimes S_{1,2}) \) is given by \( B_{0,0} \).

Note that \( \mathfrak{g}(S_8, S_{1,2}) \) is then the orthogonal direct sum, relative to \( B \), of the subspaces \( \text{tri}(S_8) \), \( \text{tri}(S_{1,2}) \) and \( t_i(S_8 \otimes S_{1,2}), i = 0, 1, 2 \).

Now, the description of \( \mathfrak{g}(S_8, S_{1,2}) \) in the proof of Theorem 2.5 becomes quite useful in the proof of the next result:

Lemma 5.4. Any derivation of the Lie superalgebra \( \mathfrak{g}(S_8, S_{1,2}) \) is inner.

Proof. As in [CF07a], take five two-dimensional vector spaces \( V_1, \ldots, V_5 \) endowed with nonzero alternating bilinear forms \( (\cdot, \cdot) \). Take symplectic bases \( \{v_i, w_i\} \) of \( V_i \) for any \( i = 1, \ldots, 5 \) \( (v_i|w_i) = 1 \) and the basis \( \{h_i, e_i, f_i\} \) of \( \mathfrak{sp}(V_i) \) given by

\[ h_i = \gamma_{v_i, w_i}, \quad e_i = \gamma_{w_i, v_i}, \quad f_i = -\gamma_{v_i, v_i}, \]

which satisfy \( [h_i, e_i] = 2e_i, [h_i, f_i] = -2f_i, \) and \( [e_i, f_i] = h_i \).

Consider the description of \( \mathfrak{g}(S_8, S_{1,2}) \) in (2.6): \( \mathfrak{g}(S_8, S_{1,2}) = \bigoplus_{\sigma \in S_3} V(\sigma) \). This shows that \( \mathfrak{g}(S_8, S_{1,2}) \) is \( \mathbb{Z}^5 \)-graded, by assigning deg \( w_i = e_i \), deg \( v_i = -e_i \), where \( \{\epsilon_1, \ldots, \epsilon_5\} \) is the canonical basis of \( \mathbb{Z}^5 \). The vector subspace \( \mathfrak{h} = \langle h_1, \ldots, h_5 \rangle \) is a Cartan subalgebra of \( \mathfrak{g}(S_8, S_{1,2}) \). Consider the \( \mathbb{Z} \)-linear map:

\[ R : \mathbb{Z}^5 \to \mathfrak{h}^* \]

\[ \epsilon_i \mapsto R(\epsilon_i) : h_j \mapsto \delta_{ij}. \]
The set of nonzero degrees of $\mathfrak{g}(S_8, S_{1,2})$ in the $\mathbb{Z}^5$-grading is given by

$$\Phi = \{ \pm 2e_1 : i = 1, \ldots, 5 \}$$

$$\cup \{ \pm \epsilon_i : 1 \leq i < \cdots < i_r \leq 5, \{ i_1, \ldots, i_r \} \in S_{8,3} \setminus \{ \emptyset \} \}.$$  

The set $R(\Phi)$ is the set of $R$-roots of $g(S_8, S_{1,2})$ relative to the Cartan subalgebra $\mathfrak{h}$. Note that the restriction of $R$ to $\Phi$ fails to be one-to-one only because $\{ \pm 2\epsilon_5, \pm \epsilon_5 \}$ is contained in $\Phi$, and $R(\pm 2\epsilon_5) = R(\mp \epsilon_5)$, as the characteristic is equal to 3.

The Lie superalgebra of derivations of $\mathfrak{g} = \mathfrak{g}(S_8, S_{1,2})$ inherits the $\mathbb{Z}^5$-grading, so in order to prove the Lemma it is enough to prove that homogeneous derivations (in this $\mathbb{Z}^5$-grading) are inner. Thus, assume that $d \in \text{der}(\mathfrak{g}_{\nu})$, with $\nu \in \mathbb{Z}^5$:

1. If $\nu \neq 0$ and $d(\mathfrak{h}) = 0$ (note that the Cartan subalgebra $\mathfrak{h}$ is just the 0-component in this grading), then $d$ preserves the eigenspaces (root spaces) of $\mathfrak{h}$, and hence $d(\mathfrak{g}_\nu) = 0$ for any $\mu \in \Phi \setminus \{ \pm 2\epsilon_5, \pm \epsilon_5 \}$, as $d(\mathfrak{g}_\mu)$ must simultaneously be contained in $\mathfrak{g}_{\mu + \nu}$ and in the root space of root $R(\mu)$. But the subspaces $\mathfrak{g}_\mu$, with $\mu \in \Phi \setminus \{ \pm 2\epsilon_5, \pm \epsilon_5 \}$ generate the Lie superalgebra $\mathfrak{g}$. (This can be checked easily, but it also follows from [CE07a, Proposition 5.25].) Hence $d = 0$, which is trivially inner.

2. If $\nu \neq 0$ and $d(\mathfrak{h}) \neq 0$, then $d(\mathfrak{h})$ is contained in $\mathfrak{g}_\nu$, which has dimension at most 1. Thus $\mathfrak{g}_\nu = kx_\nu$ for some $x_\nu$. Then for any $h \in \mathfrak{h}$, $d(h) = f(h)x_\nu$ for some $f \in \mathfrak{h}^*$. Then, for any $h, h' \in \mathfrak{h}$,

$$0 = d([h, h']) = [d(h), h'] + [h, d(h')] = \langle -R(\nu)(h')f(h) + R(\nu)(h')f(h') \rangle x_\nu.$$  

As $R(\nu) \neq 0$, it follows that there is a scalar $\alpha \in k$ with $f = \alpha R(\nu)$, and $d' = d - \alpha d x_\nu$ is another derivation in $\text{der}(\mathfrak{g}_\nu)$ with $d'(\mathfrak{h}) = 0$, so $d'$ must be 0 by the previous case, and hence $d$ is inner.

3. Finally, if $\nu = 0$, then $d(e_i) \in \mathfrak{g}_{2e_i} = ke_i$, so $d(e_i) = \alpha_i e_i$ for any $i$. Also, $d(f_i) = \beta_i f_i$ for any $i (\alpha_i, \beta_i \in k)$. As $ke_i + kf_i + kh_i$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2$, it follows at once that $\alpha_i + \beta_i = 0$. Then the derivation $d' = d - \frac{1}{2}\text{ad}_{\alpha_1 h_1 + \cdots + \alpha_n h_n}$ satisfies $d'(e_i) = 0 = d'(f_i)$ for any $i$, so $d'(h_i) = 0$, and hence $d'(\mathfrak{sp}(V_i)) = 0$ for any $i$. As $d'$ preserves degrees, it preserves each subspace $V(\sigma)$, for $\emptyset \neq \sigma \in S_{8,3}$, which is an irreducible module for $\mathfrak{sp}(V_i)$. By Schur’s Lemma, there is a scalar $\alpha_\sigma \in k$ such that the restriction of $d'$ to any $V(\sigma)$ is $\alpha_\sigma id$. But $0 \neq [V(\sigma), V(\sigma)] \subseteq \oplus_{i=1}^5 \mathfrak{sp}(V_i)$, so $2\alpha_\sigma = 0$ for any such $\sigma$ and $d' = 0$. Thus $d$ is inner in this case, too.

Consider now the triple product on the subspace $T$ (the odd component in the $\mathbb{Z}_2$-grading of $\mathfrak{g}(S_8, S_{1,2})$ considered so far) inherited from the Lie bracket in $\mathfrak{g}(S_8, S_{1,2})$:

$$T \otimes T \otimes T \rightarrow T$$

$$X \otimes Y \otimes Z \rightarrow [XYZ] = [[X, Y], Z],$$

As $T$ is the odd component of $\mathfrak{g}(S_8, S_{1,2})$, it is a Lie triple supersystem. Therefore $(T, [\ ])$ satisfies equations (4.2a) and (4.2c).

Also, if we consider the supersymmetric bilinear form $(\cdot, \cdot)$ on $T$ given by the restriction of the bilinear form $B$ given in Lemma 5.2 the invariance of $B$ immediately shows that $(T, [\ ]), (\cdot, \cdot)$ also satisfies equation (4.2d).

**Theorem 5.5.** $(T, [\ ], (\cdot, \cdot))$ is an orthosymplectic triple system whose Lie superalgebra of derivations is isomorphic to $\mathfrak{sl}(5; 3)$. Moreover, the associated Lie superalgebra $\mathfrak{g}(T)$ is isomorphic to the Lie superalgebra $\mathfrak{g}(S_{1,2}, S_{4,2})$ in the Supermagic Square.
Proof. It is enough to check equation (4.2b).

Take a symplectic basis \(\{a, b\}\) of the two dimensional vector space \(V\) in (5.1) (that is, \(\langle a | b \rangle = 1\)), then \(T\) is generated, as a module over \(\text{tl}(5;3)\) by \(i_0(S_8 \otimes a)\) or by \(i_0(S_8 \otimes b)\). Also, \(T \otimes T\) is generated by \(i_0(S_8 \otimes a) \otimes i_0(S_8 \otimes b)\). Both the left and right sides of equation (4.2b) are given by \(\text{tl}(5;3)\)-invariant trilinear maps \(T \otimes T \otimes T \to T\). Therefore, it is enough to prove that the condition

\[
[X \ i_0(y \otimes a) \ i_0(z \otimes b)] - [X \ i_0(z \otimes b) \ i_0(y \otimes a)]
= (X i_0(y \otimes a)) i_0(z \otimes b) - (X i_0(z \otimes b)) i_0(y \otimes a) + b(y, z) X
\]

holds for any \(X \in T\) and \(y, z \in S_8\).

- For \(X = u \in V \simeq \text{tr}(S_1,2)\), since \(\{a, b\}\) is a symplectic basis, \(u = \langle u | a \rangle a - \langle u | a \rangle b\), so:

\[
[u \ i_0(y \otimes a) \ i_0(z \otimes b)] = -\langle u | a \rangle [i_0(y \otimes 1), i_0(z \otimes b)] = -\langle u | a \rangle b(y, z) i_1, b \]

where, as before, \(V\) is identified with \(\text{tr}(S_1,2)\). Thus,

\[
[X \ i_0(y \otimes a) \ i_0(z \otimes b)] - [X \ i_0(z \otimes b) \ i_0(y \otimes a)] = -\langle u | a \rangle b(y, z) + \langle u | b \rangle b(y, z) a = b(y, z) u = b(y, z) X.
\]

Since \((X i_0(y \otimes a)) = 0 = (X i_0(z \otimes b))\), the result follows in this case.

- For \(X = i_0(x \otimes u), x \in S_8, u \in V\), we have

\[
i_0(x \otimes u) i_0(y \otimes a) i_0(z \otimes b)
= [(\langle u | a \rangle x, y + b(x, y) i_{u,a}, i_0(z \otimes b)]
= \langle u | a \rangle i_0(x \otimes y - b(x, y) i_0(z \otimes \sigma_{u,a}(b))
= \langle u | a \rangle i_0(b \otimes (b(x, z) y - b(y, z) x)) - b(x, y) i_0(z \otimes ((\langle u | b \rangle a + \langle a | b \rangle u))
\]

Thus,

\[
[X \ i_0(y \otimes a) \ i_0(z \otimes b)] - [X \ i_0(z \otimes b) \ i_0(y \otimes a)]
= -b(x, y) i_0(z \otimes ((\langle u | b \rangle a + \langle a | b \rangle u + \langle u | a \rangle b))
+ b(x, z) i_0(y \otimes ((\langle u | a \rangle b + \langle b | a \rangle u + \langle u | a \rangle b))
+ b(y, z) i_0(z \otimes (-\langle u | a \rangle b + \langle u | b \rangle a))
= b(x, y) i_0(z \otimes b) - b(x, z) i_0(y \otimes a) + b(y, z) i_0(x \otimes u)
= (X i_0(y \otimes a)) i_0(z \otimes b) - (X i_0(z \otimes b)) i_0(y \otimes a) + b(y, z) X.
\]

- For \(X = i_1(x \otimes 1)\), we have,

\[
[i_1(x \otimes 1) i_0(y \otimes a) i_0(z \otimes b)]
= [i_2(y \bullet x \otimes a), i_0(z \otimes b)]
= i_1((y \bullet x) \bullet x \otimes 1) \quad (\text{as } a \bullet 1 = -a \text{ and } a \bullet b = 1),
\]

\[
[i_1(x \otimes 1) i_0(z \otimes b) i_0(y \otimes a)]
= [i_2(z \bullet x \otimes b), i_0(y \otimes a)]
= -i_1((z \bullet x) \bullet y \otimes 1) \quad (\text{as } b \bullet 1 = -b, b \bullet a = -1).
\]

Thus,

\[
[X \ i_0(y \otimes a) i_0(z \otimes b)] - [X \ i_0(z \otimes b) i_0(y \otimes a)]
= i_1(((y \bullet x) \bullet z + (z \bullet x) \bullet y) \otimes 1)
= b(y, z) X,
\]
because the associativity of the bilinear form \( b \) in a symmetric composition algebra is equivalent to the condition \((x \bullet y) \bullet z = q(x)y = x \bullet (y \bullet x)\) (see [KMR1998 (34.1)]) and hence it follows that \((y \bullet x) \bullet z + (z \bullet x) \bullet y = b(y, z)x\) by linearization.

- For \( X = \mathfrak{t}_2(x \otimes 1) \) the situation is similar.

Therefore, \((T, [\cdot, \cdot], \langle \cdot, \cdot \rangle)\) is an orthosymplectic triple system and, by its own construction, its Lie superalgebra of inner derivations is isomorphic to \( \mathfrak{e}(5; 3) \), as \([TT.] = \text{ad}_{T,T} = \text{ad}_{\mathfrak{e}(5; 3)}\). Thus, the Lie superalgebra \( \tilde{g}(T) \) in Proposition 5.8 is isomorphic to the Lie superalgebra \( g(S_8, S_{1,2}) \).

But any derivation \( d \in \text{der} T \) extends to a derivation of \( \tilde{g}(T) \) which, by Lemma 5.4, is inner. It follows that \( \text{der} T = \text{ind der} T \) is isomorphic to \( \mathfrak{e}(5; 3) \), as required.

The associated Lie superalgebra (see Proposition 5.6) is

\[
\mathfrak{g} = (\mathfrak{sp}(V) \oplus \mathfrak{e}(5; 3)) \oplus (V \otimes T).
\]

Consider again the description of \( g(S_8, S_{1,2}) \) in (2.4):

\[
\mathfrak{g}(S_8, S_{1,2}) = \oplus_{\sigma \in S_8, 3} V(\sigma).
\]

Then, as in (2.7),

\[
\mathfrak{e}(5, 3) = \oplus_{\sigma \in S_+} V(\sigma), \quad T = \oplus_{\sigma \in S_-} V(\sigma),
\]

with

\[
S_+ = \emptyset, \{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{2, 3, 5\}, \{1, 4, 5\}, \{1, 3, 5\}, \{2, 4, 5\},
\]

\[
S_- = \{\{5\}, \{1, 2, 5\}, \{3, 4, 5\}, \{2, 3\}, \{1, 4\}, \{1, 3\}, \{2, 4\}\}.
\]

Now, assign the index 6 to the new copy of \( V \) in (5.6). Then,

\[
\mathfrak{g} = \oplus_{\sigma \in \tilde{S}} V(\sigma),
\]

with \( \tilde{S} \subseteq 2^{\{1,2,3,4,5,6\}} \) given by:

\[
\tilde{S} = \emptyset, \{1, 2, 3, 4\}, \{1, 2\}, \{3, 4\}, \{2, 3, 5\}, \{1, 4, 5\}, \{1, 3, 5\}, \{2, 4, 5\}
\]

\[
\{5, 6\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{2, 3, 6\}, \{1, 4, 6\}, \{1, 3, 6\}, \{2, 4, 6\}\}.
\]

Write now

\[
1 = 1, \quad 2 = 3, \quad 3 = 5, \quad 4 = 2, \quad 5 = 4, \quad 6 = 6.
\]

Then, we obtain,

\[
\tilde{S} = \emptyset, \{1, 2, 3, 5\}, \{1, 4\}, \{2, 5\}, \{2, 3, 4\}, \{1, 3, 5\}, \{1, 2, 3\}, \{3, 4, 5\}
\]

\[
\{3, 6\}, \{1, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 6\}, \{1, 5, 6\}, \{1, 2, 6\}, \{4, 5, 6\}\},
\]

and this coincides with \( S_{4,2}, S_{4,2} \) in [CE07a §5.4]. Hence this superalgebra is a Lie superalgebra with the same Cartan matrix \( A_{S_{4,2}, S_{4,2}} \) in [CE07a §5.4], thus proving that \( \mathfrak{g} \) is isomorphic to the Lie superalgebra \( g(S_{4,2}, S_{4,2}) \) in the Supermagic Square. \( \square \)

**Remark 5.7.** The previous Theorem shows that the Lie superalgebra \( \mathfrak{e}(5; 3) \) lives inside \( g(S_{4,2}, S_{4,2}) \), and that, in fact, \( g(S_{4,2}, S_{4,2}) \) contains a maximal subalgebra isomorphic to \( \mathfrak{sl}_2 \oplus \mathfrak{e}(5; 3) \).
6. The Lie superalgebra $\mathfrak{br}(2;5)$

In this section a model of the simple Lie superalgebra $\mathfrak{br}(2;5)$ is explicitly built. To this aim, consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)} \oplus \mathfrak{g}_{(1,1)},$$

with

$$\mathfrak{g}_{(0,0)} = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2),$$
$$\mathfrak{g}_{(1,0)} = \mathfrak{sp}(V_1) \otimes V_2,$$
$$\mathfrak{g}_{(0,1)} = V_1 \otimes \mathfrak{sp}(V_2),$$
$$\mathfrak{g}_{(1,1)} = V_1 \otimes V_2,$$

where, as usual, $V_1$ and $V_2$ are two-dimensional vector spaces endowed with nonzero alternating bilinear forms denoted by $\langle \cdot, \cdot \rangle$.

This vector space becomes a superspace with

$$\mathfrak{g}_0 = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}, \quad \mathfrak{g}_1 = \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}.$$

Now, define a superanticommutative product on $\mathfrak{g}$ by means of the natural Lie bracket on $\mathfrak{g}_{(0,0)}$, the natural action of $\mathfrak{g}_{(0,0)}$ on each $\mathfrak{g}_{(i,j)}$ ($V_i$ is the natural module for $\mathfrak{sp}(V_i)$, while $\mathfrak{sp}(V_i)$ is its adjoint module), and by:

$$[f \otimes u, g \otimes v] = \langle u|v \rangle [f, g] + 2 \text{tr}(fg) \gamma_{u,v},$$
$$[a \otimes p, b \otimes q] = -2 \text{tr}(pq) \gamma_{a,b} + \langle a|b \rangle \langle p|q \rangle,$$
$$[a \otimes u, b \otimes v] = \langle u|v \rangle \gamma_{a,b} + \langle a|b \rangle \gamma_{u,v},$$
$$[f \otimes u, a \otimes p] = f(a) \otimes p(u),$$
$$[f \otimes u, a \otimes v] = f(a) \otimes \gamma_{u,v},$$
$$[a \otimes p, b \otimes v] = -\gamma_{a,b} \otimes p(v),$$

for any $a, b \in V_1$, $u, v \in V_2$, $f, g \in \mathfrak{sp}(V_1) = \mathfrak{sl}(V_1)$ and $p, q \in \mathfrak{sp}(V_2) = \mathfrak{sl}(V_2)$. Here, as before, $\gamma_{a,b} = \langle a| \rangle + \langle b| \rangle a$ and similarly for $\gamma_{u,v}$.

This multiplication converts $\mathfrak{g}$ into a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded anticommutative superalgebra.

**Theorem 6.2.** Let $k$ be a field of characteristic 5. Then the superalgebra $\mathfrak{g}$ above is a Lie superalgebra isomorphic to $\mathfrak{br}(2;5)$.

**Proof.** It is clear that all the products in (6.1) are invariant under the action of $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$. Several instances of the Jacobi identity have to be checked. To do so, it is harmless to assume that the ground field $k$ is infinite (extend scalars otherwise) and hence, Zariski topology can be used.

First, for elements $a, b, c \in V_1$ and $u, v, w \in V_2$, to check that the Jacobian

$$J(a \otimes u, b \otimes v, c \otimes w) = [[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v]$$

is 0, it can be assumed, by Zariski density, that $\langle a|b \rangle \neq 0$ and $\langle u|v \rangle \neq 0$. (Note that the set $\{a, b \in V \times V : \langle a|b \rangle \neq 0\}$ is a nonempty open set in the Zariski topology of $V \times V$, and hence it is dense.) Moreover, scaling now $b$ and $v$ if necessary, it can be assumed that $\langle a|b \rangle = 1 = \langle u|v \rangle$; that is, $\{a, b\}$ is a symplectic basis of $V_1$ and $\{u, v\}$ is a symplectic basis of $V_2$. Now $c = \alpha a + \beta b$ and $w = \mu u + \nu v$ for some

$$\alpha, \beta, \mu, \nu \in k.$$
\( \alpha, \beta, \mu, \nu \in k \). Then:

\[
J(a \otimes u, b \otimes v, c \otimes w) = \left[ [a \otimes u, b \otimes v], [c \otimes w, a \otimes u], [b \otimes v, c \otimes w], a \otimes u \right] + [c \otimes w, a \otimes u], b \otimes v
\]

\[
= (u|v)_\gamma_{a,b}(c) \otimes w + (a|b)c \otimes \gamma_{a,w}(u)
+ (v|w)_\gamma_{b,c}(a) \otimes u + (b|c)a \otimes \gamma_{v,w}(u)
+ (w|u)_\gamma_{c,a}(b) \otimes v + (c|a)b \otimes \gamma_{w,u}(v)
\]

\[
= (\beta b - \alpha a) \otimes (\mu u + \nu v) + (\alpha a + \beta b) \otimes (\nu v - \mu u)
+ \mu(\alpha a + 2\beta b) \otimes u + \alpha a \otimes (\mu u + 2\nu v)
- \nu(2\alpha a + \beta b) \otimes v - \beta b \otimes (2\mu u + \nu v)
\]

\[
= 0.
\]

Hence, \( \mathfrak{g}_0 = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)} \) is a Lie algebra, which can be easily checked to be isomorphic to the symplectic Lie algebra \( \mathfrak{sp}(V_1 \perp V_2) \simeq \mathfrak{sp}_4 \).

Now, for elements \( f, g, h \in \mathfrak{sp}(V_1) \) and \( u, v, w \in V_2 \), it can be assumed as before that \( (u|v) = 1 \) and that \( w = \mu u + \nu v \). Then,

\[
[[f \otimes u, g \otimes v], h \otimes w] = [(u|v)[f, g] + 2 \text{tr}(fg)\gamma_{u,v}, h \otimes w]
= (u|v)[[f, g], h] \otimes w + 2 \text{tr}(fg)h \otimes \gamma_{u,v}(w),
\]

so that

\[
J(f \otimes u, g \otimes v, h \otimes w)
= [[f \otimes u, g \otimes v], h \otimes w] + [[g \otimes v, h \otimes w], f \otimes u] + [[h \otimes w, f \otimes u], g \otimes w]
\]

\[
= (u|v)[[f, g], h] \otimes w + 2 \text{tr}(fg)h \otimes \gamma_{u,v}(w)
+ (v|w)[[g, h], f] \otimes u + 2 \text{tr}(gh)f \otimes \gamma_{v,w}(u)
+ (w|u)[[h, f], g] \otimes v + 2 \text{tr}(hf)g \otimes \gamma_{w,u}(v)
\]

\[
= \mu([[f, g], h] - 2 \text{tr}(fg)h - [[g, h], f] - 2 \text{tr}(gh)f + 4 \text{tr}(hf)g) \otimes u
+ \nu([[f, g], h] + 2 \text{tr}(fg)h - 2 \text{tr}(gh)f - [[h, f], g] + 2 \text{tr}(hf)g) \otimes v.
\]

(6.3)

But for any \( f \in \mathfrak{sl}(V_1) \), \( f^2 = -\det(f) = \frac{1}{2} \text{tr}(f^2)id \). Hence \( fg + gf = \text{tr}(fg)id \) for any \( f, g \in \mathfrak{sl}(V_1) \), and thus

\[
fgf = \text{tr}(fg)g - gfg = \text{tr}(fg)f - \frac{1}{2} \text{tr}(f^2)g.
\]

Hence,

\[
[[g, f], h] = gf^2 + f^2g - 2fgf = 2\text{tr}(f^2)g - 2\text{tr}(fg)f
\]

and

\[
[[g, h], f] + [[g, h], f] = 4\text{tr}(fh)g - 2\text{tr}(fg)h - 2\text{tr}(gh)f.
\]

This shows that the Jacobian in (6.3) is trivial. Therefore, the subspace \( \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,0)} \) is a Lie superalgebra. The same happens to the subspace \( \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)} \).

Take now elements \( a, b \in V_1, f \in \mathfrak{sp}(V_1) \) and \( u, v, w \in V_2 \). Then it can be assumed that \( (a|b) = 1 = (u|v) \), and \( w = \mu u + \nu v \). In this situation:

\[
J(a \otimes u, b \otimes v, f \otimes w)
= [[a \otimes u, b \otimes v], f \otimes w] + [[b \otimes v, f \otimes w], a \otimes u] + [[f \otimes w, a \otimes u], b \otimes v]
\]

\[
= (u|v)[[g_{a,b}, f] \otimes w + (a|b)f \otimes \gamma_{a,w}(u)
+ \gamma_{a,f}(b) \otimes \gamma_{v,w}(u) - \gamma_{f(a),b} \otimes \gamma_{u,w}(v)
\]

\[
= \mu([[g_{a,b}, f] - f - \gamma_{a,f}(b) - 2\gamma_{f(a),b}) \otimes u
+ \nu([\gamma_{a,b}, f] + f - 2\gamma_{a,f}(b) - \gamma_{f(a),b}) \otimes v.
\]
But, since the bilinear map $(c, d) \mapsto \gamma_{c,d}$ is $\mathfrak{sp}(V_1)$-invariant, $[f, \gamma_{a,b}] = \gamma_{f(a),b} + \gamma_{a,f(b)}$. Hence,

$$J(a \otimes u, b \otimes v, f \otimes w) = -\mu(f + 3\gamma_{f(a),b} + 2\gamma_{a,f(b)}) \otimes u + \nu(f - 2\gamma_{f(a),b} - 3\gamma_{a,f(b)}) . \quad (6.4)$$

Also, by taking the coordinate matrix of $f$ in the symplectic basis $\{a, b\}$, it is checked at once that $f = -\frac{1}{2} \gamma_{f(a),b} + \frac{3}{2} \gamma_{a,f(b)}$. Since the characteristic of $k$ is equal to 5, this proves that the Jacobian in (6.4) is trivial.

The other instances of the Jacobi identity are checked in a similar way.

Finally, fix symplectic bases $\{a_i, b_i\}$ of $V_i$ ($i = 1, 2$). Then $\mathfrak{g}$ is $\mathbb{Z} \times \mathbb{Z}$-graded by assigning $\deg(a_i) = \epsilon_i$, $\deg(b_i) = -\epsilon_i$, where $\{\epsilon_1, \epsilon_2\}$ denotes the canonical $\mathbb{Z}$-basis of $\mathbb{Z} \times \mathbb{Z}$. Let $\{h_i, e_i, f_i\}$ be the basis of $\mathfrak{sp}(V_i)$ defined as in (4.16). Then $\text{span}\{h_1, h_2\}$ is a Cartan subalgebra of $\mathfrak{g}$, and coincides with the $(0,0)$-component in the $\mathbb{Z} \times \mathbb{Z}$-grading. The set of nonzero degrees is

$$\Phi = \{\pm \epsilon_1, \pm \epsilon_2, \pm \epsilon_1 \pm \epsilon_2, \pm 2\epsilon_1 \pm \epsilon_2, \pm \epsilon_1 \pm 2\epsilon_2\} .$$

Consider the elements

$$E_1 = a_1 \otimes f_2, \quad F_1 = -b_1 \otimes e_2, \quad H_1 = [E_1, F_1] = -2h_1 - h_2 , \quad E_2 = h_1 \otimes a_2 , \quad F_2 = h_1 \otimes b_2 , \quad H_2 = [E_2, F_2] = h_2 .$$

Then, $\text{span}\{H_1, H_2\}$ coincides with the previous Cartan subalgebra $\text{span}\{h_1, h_2\}$ of $\mathfrak{g}$. $E_1$ belongs to the homogeneous component $\mathfrak{g}_{\epsilon_1 - 2\epsilon_2}$ in the $\mathbb{Z} \times \mathbb{Z}$-grading, and similarly $F_1 \in \mathfrak{g}_{-\epsilon_1 + 2\epsilon_2}$, $E_2 \in \mathfrak{g}_{\epsilon_2}$, and $F_2 \in \mathfrak{g}_{-\epsilon_2}$. The elements $E_1, E_2, F_1, F_2$ generate the Lie superalgebra $\mathfrak{g}$. Besides,

$$[H_1, E_1] = -2h_1(a_1) \otimes f_2 - a_1 \otimes [h_2, f_2] = -2a_1 \otimes f_2 + 2a_1 \otimes f_2 = 0 ,$$

$$[H_1, E_2] = h_1 \otimes (-h_2)(a_2) = -h_1 \otimes a_2 ,$$

$$[H_2, E_1] = a_1 \otimes [h_2, f_2] = -2a_1 \otimes f_2 ,$$

$$[H_2, E_2] = h_1 \otimes h_2(a_2) = h_1 \otimes a_2 ,$$

and similarly for the action of the $H_i$’s on the $F_j$’s. It follows, with the same arguments as in [CE07a §4], that $\mathfrak{g}$ is the Lie superalgebra with Cartan matrix $\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$, which is the first Cartan matrix of the Lie superalgebra $\mathfrak{br}(2; 5)$ given in [BGLb §12].

---

**References**

[BGLa] Sofiane Bouarroudj, Pavel Grozman and Dmitry Leites, *Cartan matrices and presentations of Cunha and Elduque Superalgebras*, arXiv [math.RT/0611391].

[BGLb] ________, *Classification of simple finite dimensional modular Lie superalgebras with Cartan matrix*, arXiv [math.RT/0710.0319].

[Bour02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6* (Translated from the 1968 French original by Andrew Pressley), Springer-Verlag, Berlin, 2002.

[CE07a] Isabel Cunha and Alberto Elduque, *An extended Freudenthal Magic Square in characteristic 3*, J. Algebra 317 (2007), 471–509.

[CE07b] ________, *The extended Freudenthal magic square and Jordan algebras*, Manuscripta Math. 123 (2007), no. 3, 325–351.

[Eld04] Alberto Elduque, *The magic square and symmetric compositions*, Rev. Mat. Iberoam. 20 (2004), no. 2, 475–491.

[Eld06a] ________, *A new look at Freudenthal’s magic square*, Non-associative algebra and its applications, L. Sabinin, L.V. Sbitneva, and I. P. Shestakov, eds., Lect. Notes Pure Appl. Math., vol. 246, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 149–165.

[Eld06b] ________, *New simple Lie superalgebras in characteristic 3*, J. Algebra 296 (2006), no. 1, 196–233.

[Eld07a] ________, *The Magic Square and Symmetric Compositions II*, Rev. Mat. Iberoamericana 23 (2007), 57–84.
[Eld07b] _______, Some new simple modular Lie superalgebras, Pacific J. Math. 231 (2007), no. 2, 337–359.

[Eld] _______, The Tits construction and some simple Lie superalgebras in characteristic 3, arXiv:math.RA/0703784.

[EO02] Alberto Elduque and Susumu Okubo, Composition superalgebras, Comm. Algebra 30 (2002), no. 11, 5447–5471.

[Jac68] Nathan Jacobson, Structure and representations of Jordan algebras, American Mathematical Society Colloquium Publications, Vol. XXXIX, American Mathematical Society, Providence, R.I., 1968.

[Kac77] Victor G. Kac, Lie superalgebras, Advances in Math. 26 (1977), no. 1, 8–96.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, The book of involutions, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998.

[Kos70] Alexei I. Kostrikin, A parametric family of simple Lie algebras, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 744–756.

[McC04] Kevin McCrimmon, A taste of Jordan algebras, Universitext, Springer-Verlag, New York, 2004.

[Oku93] Susumu Okubo, Triple products and Yang-Baxter equation. I. Octonionic and quaternionic triple systems, J. Math. Phys. 34 (1993), no. 7, 3273–3291.

[She97] Ivan P. Shestakov, Prime alternative superalgebras of arbitrary characteristic, Algebra i Logika 36 (1997), no. 6, 675–716, 722.

[Tit66] Jacques Tits, Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles. I: Construction, Nederl. Akad. Wet., Proc., Ser. A 69 (1966), 223–237.

Departamento de Matemáticas e Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address: elduque@unizar.es