Fractal Descriptors of Texture Images Based on the Triangular Prism Dimension

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Abstract
This work presents a novel descriptor for texture images based on fractal geometry and its application to image analysis. The descriptors are provided by estimating the triangular prism fractal dimension under different scales with a weight exponential parameter, followed by dimensionality reduction using Karhunen–Loève transform. The efficiency of the proposed descriptors is tested on four well-known texture data sets, that is, Brodatz, Vistex, UIUC and KTH-TIPS2b, both for classification and image retrieval. The novel method is also tested concerning invariances in situations when the textures are rotated or affected by Gaussian noise. The obtained results outperform other classical and state-of-the-art descriptors in the literature and demonstrate the power of the triangular descriptors in these tasks, suggesting their use in practical applications of image analysis based on texture features.

Keywords Pattern recognition · Texture analysis · Fractal descriptors · Triangular prism

1 Introduction
Fractal theory has presented a growing interest in many applied areas in the last decades, for instance, in Medicine [1,22,26], Physics [17,40,44], Computer Science [7,15,30], Engineering [25,41,42], among many other fields.

Most of these applications employ the fractal dimension to describe objects that should be classified or simply described in some manner. Fractal dimension provides important information about the object. While in a mathematical fractal, the dimension measures the “fractality,” in a real-world object, it expresses the spatial occupation of the structure. More practically, this implies that fractal dimension is capable of capturing important physical and visual attributes, like roughness, luminance or the repetition of geometrical patterns.

Despite its importance as a powerful descriptor, fractal dimension is still limited in the representation of more complex structures. This is true mainly in the analysis of real-world objects, when the level of “fractality” varies along the same object. The literature shows some approaches to solve this problem, such as multifractals [3,9,20], multiscale fractal dimension [11,29] and fractal descriptors [5,13,14,36]. This work is focused on the fractal descriptors solution.

Here, we propose a novel fractal descriptor based on the triangular prism fractal dimension [8] and apply it to the discrimination and retrieval of texture images. In this approach, the fractal dimension is estimated at different scales of observation and a weight parameter is introduced as an exponent in the sum of the areas, changing the influence of each scale in the final result and ensuring a more complete and flexible description of the image.

The proposed technique is tested over four well-known texture data sets used for benchmark purposes. The results of
the classification and retrieval of such data sets are compared to other classical and recent texture analysis methods. The results confirm that the proposed descriptor is a valuable tool for image analysis tasks.

2 Related Works

Texture analysis is a paradigm where the image is described in terms of statistical patterns formed by spatial arrangements of pixel intensities. The first known systematic study on this topic was carried out by Haralick [19] and his co-occurrence matrices. Since then, a large number of methods on texture analysis have been proposed in the literature. Among the most successful approaches, one can mention local binary patterns [18,32,33], bag-of-features [39], scale invariant feature transform [23], spatial pyramid matching [24], invariants of scattering transforms [37], fast features invariant to rotation and scale of texture [38], and others.

During the last decades, another family of methods that have presented interesting results in texture analysis, especially on natural images, are those based on fractal geometry, particularly multifractals [43], multiscale fractal dimension [29] and fractal descriptors [5]. In this context, this work proposes the study and application of fractal descriptors based on the estimation of the fractal dimension using a tessellation of triangular prisms [8].

Our proposal has some particular characteristics that distinguish it from other approaches in the literature. First, rather than preselecting preferable regions in the image as in [23,24], here all pixels and scales are equally important a priori, which simplifies the modeling and interpretation of the texture descriptors. Another difference from methods such as those in [23,37,38] is that image invariances are not treated explicitly, although the underlying model and multiscale process ensures that such effect is attenuated in practice. This is confirmed here in the experimental analysis and avoids the use of cumbersome strategies when in many cases invariances are not a critical issue or even when, for example, a rotated texture should be interpreted as a different object. Finally, an important distinction should be done from approaches such as those in [18,19,32,33,39] where direct relations are established based on the pixel values. Here, there is a complete and well-defined physical model behind the statistics extracted from the image, making it more precise in most cases and more robust to deformations usually found in natural structures.

3 Fractal Theory

Fractal geometry has been applied to diverse areas [7,22,41,44]. This is motivated mainly by the flexibility of fractal theory in modeling natural objects, which usually cannot be precisely represented by the conventional Euclidean geometry.

Fractal theory also presents a concise and powerful framework to describe and identify a natural object, based primarily on the fractal dimension concept. Roughly speaking, fractal dimension measures the complexity of a structure. In this case, complexity is related to the property of presenting details at different scales of observation. In this way, fractal dimension is of particular importance because it is strongly related to fundamental physical and visual features of the object, such as roughness, luminance, distribution of colors and others.

The following section describes in a few words some important aspects of fractal geometry and its application to texture analysis.

3.1 Fractal Dimension

Fractal dimension is formally defined as being the Hausdorff–Besicovitch dimension of a geometrical set of points \( X \) that composes the fractal object.

In order to define the Hausdorff–Besicovitch dimension, initially the Hausdorff measure \( \mathcal{H}^s(X) \) should be defined:

\[
H^s(X) = \lim_{\delta \to 0} \inf \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } X, \quad (1)
\]

where \( |U_i| \) states for the diameter of \( U_i \) and \( \{U_i\} \) is a \( \delta \)-cover of \( X \), if \( X \subset \bigcup_{i=1}^{\infty} U_i \) with \( |U_i| \leq \delta \), for all \( i \).

Analyzing the behavior of \( H^s(X) \) against \( s \), one observes that \( H^s \) jumps from \( \infty \) to 0 at a particular real nonnegative value of \( s \). This value is the Hausdorff–Besicovitch or fractal dimension of \( X \).

Although the above definition is consistent and can be, at least theoretically, applied to any set of points immersed in the Euclidean space, it shows to be difficult or even impracticable in many situations where the fractal dimension of a real-world object should be estimated. In fact, as detailed in [12], the Hausdorff–Besicovitch dimension of any countable set of points is zero, which makes such formalism useless in our finite world. With the aim of simplifying the computation in such situations, an approximate discrete version of the Hausdorff–Besicovitch dimension, known as similarity dimension \( D_s \), can be defined by:

\[
D_s = -\lim_{\epsilon \to 0} \frac{\log(N)}{\log(\epsilon)}, \quad (2)
\]

where \( N \) is the number of “rulers” with length \( \epsilon \) used to cover the fractal object. Details on the connection between (1) and (2) involve advanced concepts of measure theory and can be found for example in [12]. Actually, \( N \) is a metric that can be generalized to a large sort of measures, in both spatial
and frequency domains. This gives rise to a lot of methods for estimation of fractal dimension [12], like Bouligand–Minkowski, box-counting, Fourier, etc. The triangular prism dimension employed in this work is an example of method derived from the similarity dimension.

### 3.2 Triangular Dimension

This method for the estimation of the fractal dimension of objects represented in a gray-level image, proposed in [8], is based on the relation between the surface area of a triangular tessellation of the gray-level map and the area of the base of each triangle.

The image is divided into a grid of squares with side-length $\epsilon$. For each square, a triangular prism is constructed using the pixel intensities in each corner as the heights and a central point whose height is given by the average of the corner heights. Thus in a gray-level image $I$, let $a$, $b$, $c$ and $d$ be the pixel intensities delimiting the grid square, such that:

$$a = I(i, j); \quad b = I(i + \epsilon, j); \quad c = I(i, j + \epsilon); \quad d = I(i + \epsilon, j + \epsilon).$$

The center of the prism has height $\epsilon$ given by the simple average:

$$\epsilon = (a + b + c + d)/4. \quad (3)$$

Figure 1 depicts a scheme of the prism construction.

The set of prisms composes a tessellation surface, such that the total area of this surface can be computed by using some geometrical procedures (Heron formulas). Thus, for each prism, the semi-perimeter of each face $A$, $B$, $C$ and $D$ is given by:

$$s_a = 1/2(w + p + o); \quad s_b = 1/2(x + p + q);$$
$$s_c = 1/2(y + q + r); \quad s_d = 1/2(z + o + r). \quad (5)$$

where $w, x, y, z, o, p, q$ and $r$ are the segments as labeled in Fig. 1c:

$$w = \sqrt{(b - a)^2 + \epsilon^2}; \quad x = \sqrt{(c - b)^2 + \epsilon^2};$$
$$y = \sqrt{(d - c)^2 + \epsilon^2}; \quad z = \sqrt{(a - d)^2 + \epsilon^2}$$

$$o = \sqrt{(a - e)^2 + \left(\frac{\sqrt{2}}{2} \epsilon\right)^2};$$
$$p = \sqrt{(b - e)^2 + \left(\frac{\sqrt{2}}{2} \epsilon\right)^2};$$
$$q = \sqrt{(c - e)^2 + \left(\frac{\sqrt{2}}{2} \epsilon\right)^2};$$

$$r = \sqrt{(d - e)^2 + \left(\frac{\sqrt{2}}{2} \epsilon\right)^2}. \quad (6)$$

The total area of each triangular prism is provided by the sum of the areas of the four faces:

$$S_{ij,\epsilon} = A + B + C + D. \quad (7)$$

where the area of each face is given by:

$$A = \sqrt{s_a(s_a - w)(s_a - p)(s_a - o)};$$
$$B = \sqrt{s_b(s_b - x)(s_b - p)(s_b - q)};$$
$$C = \sqrt{s_c(s_c - y)(s_c - q)(s_c - r)};$$
$$D = \sqrt{s_d(s_d - z)(s_d - o)(s_d - r)}. \quad (8)$$

The total area of the surface is computed by summing the area of each prism in the grid with step $\epsilon$:

$$S(\epsilon) = \sum_{i', j' \in \mathcal{G}_\epsilon} S_{i', j', \epsilon}. \quad (9)$$

where $\mathcal{G}_\epsilon$ is the set of points in the grid with step $\epsilon$.

This procedure is repeated for a range of values of $\epsilon$, and in each step the area $S(\epsilon)$ is estimated. The fractal dimension $D$ is extracted from the log–log relation between $\epsilon$ and $S$ such that, $D = 2 - \alpha$, where $\alpha$ is the slope of a straight line fit to the curve $\log(S(\epsilon))$.

### 3.3 Fractal Descriptors

Although fractal dimension is an important descriptor, it is still insufficient to represent more complex systems. We can easily observe distinct fractals with the same fractal dimension despite their completely different appearance. Such situation is even more complicated when we deal with objects from the real world, which are not exact fractals. In these structures, we find different levels of “fractality” according to the observed scale or even to the spatial region analyzed. In this context, we need a tool capable of modeling the object in all its extension.

Fractal descriptors [5,13,14,36] constitute a solution to fill this gap, making possible a richer analysis of fractal characteristics present in the object. Figure 2 illustrates two distinct texture images whose fractal dimensions (FD) are similar. Hence, using only the FD estimation is not enough to distinguish the images. For the same images, Fig. 2 shows the normalized fractal descriptor curves, which demonstrates to be capable of discriminating the textures in a straightforward manner.

Essentially, the purpose of fractal descriptors is to estimate the fractal dimension under different scales, providing information from different patterns and arrangements present
Fig. 1 Triangular prisms. a Gray-level image, b a sample prism, c projected prism

(a) (b) (c)

FD = 1.383

FD = 1.383

Fig. 2 On top, two distinct textures with similar fractal dimension (in this case, estimated by the Bouligand–Minkowski approach). At bottom, curves of fractal descriptors corresponding to each image and illustrating their discrimination power.

in the structure. A natural candidate to allow this analysis is the power-law relation intrinsic to the fractal dimension \( D \). From the similarity dimension, we can write:

\[
D = - \lim_{\epsilon \to 0} \frac{\log(\mathcal{M}(\epsilon))}{\log(\epsilon)},
\]

where \( \mathcal{M} \) is any measure related to the spatial or frequency distribution of the object and \( \epsilon \) is the scale parameter. The power-law relation may be stated in a quite simple fashion:

\[
\mathcal{M} \propto \epsilon^{-D}.
\]

Fractal descriptors consist of extracting features from the following function:

\[
u : \log(\epsilon) \rightarrow \log(\mathcal{M}).
\]

The function \( u \) may be used in different manners, either directly, as in [2], or after a multiscale transform [5] or by the application of principal component analysis or still functional data analysis as in [14], among many other possibilities.

4 Proposed Method

This work proposes a novel fractal descriptor based on the fractal dimension estimated by triangular prisms. In this case, the fractality function \( \mathcal{M}(\epsilon) \) corresponds to the area function \( S(\epsilon) \). Following the general idea of fractal descriptors, the proposed features are provided by \( \log(S(\epsilon)) \).

Furthermore, to improve the ability of identifying multiscale patterns along the texture, the proposed method introduces an exponent weight \( \alpha \) to the area sum \( S(\epsilon) \), which now has the following expression:

\[
S^\alpha(\epsilon) = \sum_{i',j'\in\Theta_{\epsilon}} S^{\alpha}_{i',j',\epsilon},
\]

where \( S^{\alpha}_{i',j',\epsilon} \) is defined as in Eq. 7. Here, the rationale for the exponent \( \alpha \) is that it implicitly allows for correlations within groups of area values to be expressed in the resultant feature vector. Such correlations also contribute for a richer description of the original image. More details are presented in Sect. 4.1.2.

For image analysis purposes, a range of values \( \alpha \) is chosen empirically and the following values are taken into account to compose the descriptors \( D \):

\[
D = \bigcup_{1 \leq \epsilon \leq \epsilon_{\text{max}}, \alpha_{\text{min}} \leq \alpha \leq \alpha_{\text{max}}} S^\alpha(\epsilon).
\]

More specifically, we tested three parameters: \( \alpha_{\text{min}}, \alpha_{\text{max}} \) and \( \Delta_\alpha \) (the step between two successive values). For those tests, we employed a collection of images randomly selected from the databases studied here. The parameter values were chosen as those providing the highest accuracies in both image
retrieval and classification. In particular, we obtained in this way the values \(a_{\text{min}} = -5\), \(a_{\text{max}} = 5\) and \(\Delta_a = 0.1\).

Finally, since the above expression generates a large set of features, a reduction in dimensionality is necessary. Such procedure is carried out by a Karhunen–Loève (KL) transform. Thus, let \(M_D\) be the feature matrix, that is, a matrix where each row corresponds to the vector \(D\) of a sample and each column is a descriptor. The covariance matrix \(M_C\) is defined by:

\[
M_C(i, j) = \frac{\sum_{i=1}^{n} (M(:, i) - \overline{M(:, i)})(M(:, j) - \overline{M(:, j)})}{n - 1},
\]

where \(n\) is the number of descriptors in \(D\), \(M(i, .)\) is the \(i\)th column of \(M\), and \(\overline{M(:, i)}\) is the average of the vector. In the following, a second matrix \(U\) is defined as having in each column the eigenvectors of \(M_C\), sorted according to decreasing values of eigenvalues. Finally, the KL transform \(\tilde{M}_D\) of \(M_D\) is obtained by

\[
\tilde{M}_D = U^T M_D.
\]

Now, \(\tilde{M}_D\) is the new matrix of features. It has the same dimensions of \(M_D\), and each row in \(\tilde{M}_D\) contains the final descriptors of the respective sample. Based on the KL transform theory, the first descriptors are the most meaningful for the analysis. Here, the descriptors are considered in this order and, as it will be described in Sect. 7, the number of such descriptors is varied between 1 and a predefined maximum, to find out the best configuration empirically.

Figure 3 summarizes the described steps in a visual diagram.

The proposed method combines two types of information to provide a rich and reliable image descriptor. The first is the complexity, measured by the fractal dimension. Such property is strongly correlated with physical characteristics of the object. The second one is the multiscale analysis that is accomplished by the parameter \(\epsilon\). Using the curve of \(S(\epsilon)\) ensures that the complexity information is expressed along a range of scales. Finally, the exponent \(\alpha\) provides an empirical weight for each scale, making the proposed descriptors more flexible to address the description of so diverse objects. All this combination yields a method capable of identifying and discriminating objects even in severe situations as when there is high variability among elements of a same class and/or high similarity among samples of different classes. Figure 4 visually illustrates the discrimination of two classes of textures by the proposed method.

### 4.1 Motivation

#### 4.1.1 Fractal Modeling

To better understand how and why triangular prisms are effective in providing image descriptors, we should look at its mathematical interpretation. The classical analysis of the relation between triangular prisms and fractal theory is that adopted in [8]. It relies on the idea of extending the “walking-dividers” approach [27] to a two-dimensional manifold.

Here, we propose a second interpretation based on the theory of fractional Brownian motion (fBm) [12].

Our first objective is to establish the relation between the prism areas and the pixel gray levels in the image. We illustrate the calculation in face \(A\). As \(\epsilon\) can be written as a function of \((a, b, c, d)\), we can rewrite \(\epsilon\) and \(p\):

\[
o = \sqrt[4]{\left(\frac{3a-b-c-d}{4}\right)^2 + \frac{1}{2} \epsilon^2},
p = \sqrt[4]{\left(\frac{3a-b-c-d}{4}\right)^2 + \frac{1}{2} \epsilon^2}.
\]

(17)

We can also substitute the definition of \(s_{n}\) in the area \(A\):

\[
A = \sqrt[4]{w + o + p} \left(\frac{w + o + p}{2}\right) \left(\frac{w + o - p}{2}\right) \left(\frac{w - o + p}{2}\right)
\]

\[
= \sqrt[4]{w^2 + (o^2 + p^2 - w^2)},
\]

\[
= \frac{4}{4} \sqrt{4 p^2 w^2 - (o^2 + p^2 - w^2)^2}.
\]

(18)

Replacing \(w\), \(p\) and \(o\) with their respective representation in terms of \(a, b, c\) and \(d\) we end up with the following expression:

\[
A = \sqrt[4]{4 \left(\frac{3a-b-c-d}{4}\right)^2 + \frac{1}{2} \epsilon^2} \left(h - a\right)^2 + \epsilon^2 - \left[\left(\frac{3a-b-c-d}{4}\right)^2 + \frac{1}{2} \epsilon^2\right] - \left(\frac{3a-b-c-d}{4}\right)^2 + \frac{1}{2} \epsilon^2 - \left((b-a)^2 + \epsilon^2\right)^2
\]

\[
= \frac{\epsilon}{1.25 a^2 + 0.25 b^2 + 0.25 c^2 + 0.25 d^2 - 1.5 ab - 0.5 ac - 0.5 ad - 0.5 bc - 0.5 bd + 0.5 cd + \epsilon^2}
\]

\[
= \frac{\epsilon}{8} \sqrt{5 a^2 + 5 b^2 + 5 c^2 + 5 d^2 - 6 ab - 2 ac - 2 ad - 2 bc - 2 bd + 2 cd + 4 \epsilon^2}
\]

(19)
Rearranging terms, we can write the above expression as

\[
A = \frac{\epsilon}{4} \left[ \left( \frac{a + b + c + d}{2} \right)^2 + (a - b)^2 - (a + b)(c + d) + \epsilon^2 \right]^{1/2}
\]

\[
= \frac{\epsilon}{4} \left[ \frac{(a + b - (c + d))^2}{4} + (a - b)^2 + \epsilon^2 \right]^{1/2}
\]

\[
= \frac{\epsilon}{8} \sqrt{4(a - b)^2 + ((a + b) - (c + d))^2 + 4\epsilon^2}
\]  \hspace{1cm} (20)

The expression enclosed by the square root contains three squared terms and the leading one is \((a - b)^2\). The points in the image where the intensities are \(a\) and \(b\) are separated by a distance \(\epsilon\). Therefore, the distribution of this term is classically related to the fBm process [12].

An fBm process \(B_H(t)\) is a Gaussian non-stationary stochastic process, with mean zero and variance \(\sigma\), whose covariance statistics satisfies

\[
E(B_H(t)B_H(s)) = \frac{\sigma^2}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right).
\]  \hspace{1cm} (21)

Fig. 3 A diagram illustrating the composition of the proposed descriptors. From top to bottom, the texture analyzed, the log–log curves of triangular dimension with three exponents (0.1, 0.3 and 0.5), the concatenated descriptors and the final descriptors after the KL transform.
where $E$ is the expected value and $H$ is a parameter named Hurst exponent. Another way of defining the same process is to write

$$E \left( (B_H(t_1) - B_H(t_2))^2 \right) \propto |t_1 - t_2|^{2H}. \quad (22)$$

Seminal works on fractal geometry in images, such as those of Mandelbrot [27] and Pentland [35], already demonstrated that fractal characteristics in images represent fractality in the physical process originating the pictured object. Pentland even carried out a survey to discover that such relation can be perceptually confirmed by psychological tests. Since then, a number of studies have been based on this assumption and fBm has been considered the canonical representation for such physical processes giving rise to fractal images.

In this context, the expression (22) has been paramount. If the image obeys the statistics in (22) with an acceptable $p$-value, the image can be analyzed as a statistical fractal. Furthermore, as demonstrated in [12], the fBm has fractal dimension (in the sense of s-Besicovitch) equals to $3 - H$ with probability 1.
In practice, it is usual to verify the statistics (22) in an image $I$ by checking the curve of ln ε versus ln $<(I(x) - I(y))^2>$ where < ... > stands for the average and $x$ and $y$ are points separated by $\epsilon$. This is in essence what is expressed by the term $(a - b)^2$ in (20). The second squared term $(((a + b) - (c + d))^2)$ can be seen as a correction taking into account the influence of the neighbor pixels in the fBm covariance. Similarly, the parameter $\epsilon$ appears as a weight expressing the scale size.

We can conclude from this that the triangular prisms are capable of representing two classical viewpoints in fractal geometry, to know, a geometrical interpretation such as “walking-dividers” and a statistical approach derived from fBm. Back to the context of fractal descriptors, rather than stating that the statistics (22) perfectly fits the image, what we expect to find is how the image is more or less close to a fractal in each scale. And this analysis highly depends on the richness of the measure provided by the method employed to estimate the fractal dimension. In this way, a method with a more complete description of the fractal process, in both a geometrical sense and a statistical sense, tends to be more appropriate than other classical methods in the literature, hence explaining the success achieved by the proposed approach.

### 4.1.2 Role of exponent $\alpha$

Now we turn our attention to the exponent $\alpha$ in (13). Although a generalized version of the multinomial theorem with real exponents exists, it is not straightforward to be interpreted. Therefore, we opted for employing MacLaurin series up to the second order. The general expression for a real multivariable function is:

$$f(x_1, x_2, \ldots, x_n) \approx f(0, 0, \ldots, 0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0, 0, \ldots, 0)x_i + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(0, 0, \ldots, 0)x_i x_j + \frac{1}{3!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(0, 0, \ldots, 0)x_i x_j x_k + \ldots$$

(23)

We will replace the first term in the summation (13) by $(1 + x_1)$ and all the others by $x_2, x_3, x_4, \ldots$ resulting in the function $(1 + x_1 + x_2 + \cdots + x_n)^\alpha$. By including the power function derivatives:

$$(1 + x_1 + x_2 + \cdots + x_n)^\alpha \approx 1 + (\alpha - 1) \sum_{i=1}^{n} x_i + \frac{(\alpha - 1)^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j$$

(24)

whereas the first summation term is related to the mean of $x_i$ the following ones relate to the correlation between groups of $2, 3, 4, \ldots$ variables. The exponent $\alpha$ gives weight to these terms and the higher the exponent the larger the influence of the correlation within larger groups of variables. Correlation is a well-established statistics in texture images as it is related to Haralick second-order statistics [19] and $\alpha$ quantify such property in terms of the prism areas and indirectly on the local fractality of the image.

Altogether the combination of an accurate fractal analysis with an intrinsic correlation is what allows for the analysis proposed here to be powerful even in the most challenging scenarios such as when we have high variability among images of the same group or when images from different groups can be confused by more traditional approaches in the literature. This may happen for example when the modeling is not sufficiently solid and flexible at the same time or when the metrics employed cannot express the parameters of the underlying model with the necessary accuracy.

### 5 Algorithm

The following pseudocode contains the algorithm for the triangular prism descriptors. Essentially, it is a translation of the mathematical expressions in Sect. 3.2. For practical purposes, we used exponents between $-5$ and $5$ (with step size 0.1) and considered both the area and its logarithm to compose descriptors.

### 6 Experiments

The proposed method is tested over four data sets, that is, Brodatz [4], Vistex [31], UIUC [23] and KTH-TIPS2b [6]. Brodatz is a gray-level texture database composed by 111 texture images divided each one into 16 windows with 128×128 pixels without overlapping. Vistex is composed by 54 color texture images, each one divided into 16 128×128 windows. UIUC contains 40 groups of textures with 25 images in each one. Finally, KTH-TIPS2b is a database of images acquired under distinct conditions of illumination, scale and pose, containing 11 materials (groups), each one with 432 samples. The color images are converted into gray-level ones before the descriptor analysis.

The classification is performed by a linear discriminant analysis (LDA) method [10]. The classification of each data set is achieved by inputting the proposed descriptors and other well-known texture descriptors to the clas-
7 Results

7.1 Image Retrieval

The proposed method is applied to practical problems in image analysis, and its performance is compared to other texture descriptors proposed in the literature. The first task discussed is image retrieval, when the user inputs a sample and tries to recall similar images (from the same class). Each attempt to recall the expected sample is named “query,” and the aim is to obtain the maximum ratio of expected samples with the minimum possible of queries. A complete description of this trade-off is given by the precision/recall curve. Precision is the ratio between the number of expected samples returned and the number of queries. Recall is the ratio between the number of right guesses and the total number of expected samples in the database.

Originally, precision/recall is defined only for two-classes problems. As there are more classes here, the curve is computed for each class, assuming the current class as the positive prediction and all the remaining as the negative prediction. Thus, an average precision/recall curve is presented for discussion. Figure 5 shows the curves for each database. A good method is supposed to have a curve the closest possible of a constant curve with precision always being 1. On average, this profile is satisfied by the proposed descriptors either for lower or higher recall. Such behavior implies that the triangular descriptors are appropriate in simple situations and do even better in more complicated problems where more queries are required to obtain the correct result. In all cases, the proposed method behaves more regularly than the other approaches, making it a potential candidate to be applied in practical situations of image retrieval problems.

To quantify the performance on image retrieval, a global measure may also be necessary and the most commonly used is the area under the curve of precision/recall. Table 1 compares this measure for different methods on each set of textures. The proposed method has the largest computed area confirming its efficiency for image retrieval. Actually, the great precision when the number of queries increases is rather influential in this result.

Generally speaking, the proposed method presents better results than their counterparts after a number of retrieval attempts. In particular, it outperformed state-of-the-art approaches such as LBP versions and BM fractal descriptors. This is a consequence of combining different weights given by the exponents to each scale in the triangular descriptors. Although the empirical combination may not be enough for an initial guess, it ensures more robustness when more queries are requested.

7.2 Image Classification

In this section, the textures from the benchmark data sets are classified. Since the actual classes are known previously, the accuracy of each approach can be measured and compared by statistical metrics.

Figure 6 shows in a plot the relation between the number of descriptors employed in the classification and the rate of images correctly classified (success rate). Even though the
Recall

0.7

0.75

0.8

0.85

0.9

0.95

1

Precision

Proposed

RI-LBP

CLBP_M/C

BM

Recall

0.75

0.8

0.85

0.9

0.95

1

Precision

Proposed

LBP+VAR

CLBP_M/C

BM

Brodatz Vistex

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Recall

0.4

0.5

0.6

0.7

0.8

0.9

1

Precision

Proposed

CLBP_M/C

LBP+VAR

Multifractal

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Recall

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Recall

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Recall

0.4

0.5

0.6

0.7

0.8

0.9

1

Precision

Proposed

CLBP_M/C

Gabor

Fourier

UIUC KTHTIPS2b

Fig. 5 Precision/recall curves

Table 1 Areas under the curves of precision/recall for each compared descriptor

| Method              | Brodatz | Vistex | UIUC | KTHTIPS2b |
|---------------------|---------|--------|------|-----------|
| LBP [33]            | 0.84    | 0.82   | 0.79 | 0.76      |
| GLCM [19]           | 0.82    | 0.79   | 0.57 | 0.69      |
| Multifractal [43]   | 0.81    | 0.79   | 0.83 | 0.75      |
| Gabor [28]          | 0.81    | 0.81   | 0.68 | 0.81      |
| Fourier [16]        | 0.74    | 0.80   | 0.62 | 0.79      |
| RI-LBP [33]         | 0.86    | 0.83   | 0.79 | 0.76      |
| LBP+VAR [34]        | 0.83    | 0.83   | 0.87 | 0.74      |
| CLBP_M/C [18]       | 0.85    | 0.90   | 0.88 | 0.83      |
| BM descriptors [2]  | 0.85    | 0.83   | 0.81 | 0.77      |
| Proposed            | 0.90    | 0.92   | 0.89 | 0.85      |

state-of-the-art variants of the LBP method are the best ones when using a reduced number of descriptors, the triangular fractal descriptors ensures a more accurate classification using more features, which is reasonable given that the error is kept within an acceptable range. The general shape of each curve is similar in all the databases, confirming the reproducibility of the experiment. The highest success rates in Brodatz and Vistex are caused by the smaller number of classes.

Table 2 shows the success rates, the respective errors and number of descriptors for each compared texture descriptors. The proposed descriptors achieved an advantage of at least about 2% over the other approaches. Such gain is relevant given the size and number of classes in those databases, which makes such rates close to the limit of best possible results. Moreover, if on the one hand more descriptors are necessary,
Fig. 6  Success rates according to the number of descriptors

Table 2  Success rates (with respective errors) and number of descriptors for each compared method

| Method                | Brodatz       | Vistex        | UIUC          | KTHTips       |
|-----------------------|---------------|---------------|---------------|---------------|
| LBP [33]              | 87.33 ± 0.02(15) | 91.55 ± 0.03(13) | 77.40 ± 0.05(31) | 70.16 ± 0.01(31) |
| GLCM [19]             | 86.48 ± 0.02(70) | 88.21 ± 0.03(70) | 58.70 ± 0.03(13) | 65.05 ± 0.01(16) |
| Multifractal [43]     | 85.64 ± 0.03(70) | 88.31 ± 0.03(76) | 82.40 ± 0.03(70) | 69.95 ± 0.01(75) |
| Gabor [28]            | 85.42 ± 0.02(19) | 90.39 ± 0.01(17) | 69.10 ± 0.02(18) | 74.94 ± 0.01(16) |
| Fourier [16]          | 78.71 ± 0.03(15) | 84.49 ± 0.02(15) | 64.00 ± 0.03(31) | 72.50 ± 0.01(64) |
| RILBP [33]            | 89.92 ± 0.02(28) | 91.78 ± 0.02(22) | 77.40 ± 0.05(31) | 70.16 ± 0.01(31) |
| LBP+VAR OPM02         | 85.87 ± 0.01(14) | 94.91 ± 0.01(26) | 83.90 ± 0.02(21) | 70.73 ± 0.02(81) |
| CLBP_M/C [18]         | 88.34 ± 0.02(19) | 94.56 ± 0.02(19) | 85.90 ± 0.04(19) | 76.33 ± 0.02(20) |
| BM descriptors [2]    | 88.85 ± 0.02(49) | 92.94 ± 0.02(56) | 80.50 ± 0.04(56) | 69.97 ± 0.01(69) |
| Proposed              | 94.37 ± 0.02(90) | 96.30 ± 0.01(99) | 87.00 ± 0.02(91) | 78.83 ± 0.02(193) |
on the other the error in the cross-validation is lower, showing that no dimensionality curse is detected.

Figure 7 shows the confusion matrices for the respective descriptors on Brodatz data set. Such matrices expressing the
number of images assigned to each class are represented in a gray-scale image. In these diagrams, an ideal method should present a black diagonal with a clear white background. Any light point in the diagonal and gray point outside corresponds to errors in the classification. The primary goal of this type of expression is to show the behavior of the descriptors in each class. For instance, both triangular and LBP methods fail in the classes 42/43. Those classes do not have enough texture information, but they are more suitable for a shape-based analysis. On the other hand, they completely differ in classes like 42 and 106. Those classes are characterized by local illumination changes associated with the preservation of multiscale patterns, which constitute the ideal scenario for the application of fractal-based methods.

Figure 8 shows the confusion matrices for Vistex database. Though the best descriptors have matrices somewhat similar, the behavior in each class differs among the approaches. For example, the class 34 has different global patterns that can be confused by methods like LBP + VAR and Triangular, while the class 5 has large homogeneous regions that improves the performance of both approaches.

Figure 9 shows the confusion matrices for UIUC textures. Now the matrices are visually more similar confirming the reduced difference in the classification performance. It can be however observed that the proposed method was more precise in classes such as 3 and LBP was the most adequate solution for class 1. The difference between these classes is basically the most significant presence of relevant information at multiple scales in class 3, which makes the use of fractal descriptors more appropriate. The triangular descriptors were not as precise as LBP in classes 23 and 24. The images in those groups are artificial and strongly periodic, presenting well-defined geometrical patterns. This is a situation where multiscale information is not the most relevant attribute and the information expressed by the texture image may be insufficient for the good discrimination by fractal-based approaches.

Finally, Fig. 10 shows the confusion matrices for KTH-TIPS2b database. Now the small number of classes facilitates the visual distinction among the descriptors. Particularly relevant is the great performance of the proposed approach in class 6 and the smaller success rate in class 3. Again, the advantage of LBP in class 3 is expected given the influence of local patches to discriminate those images. Triangular descriptors also performed slightly worse than other approaches in class 9. Again, this is an atypical case in which the multiscale analysis may be not as precise as it might be expected. We have here the presence of visually different
structures at different scales. Those different structures somewhat resemble parts of images in classes 2 and 4 and that is an explanation for the confusion observed and detected by the confusion matrix.

### 7.3 Invariances

Another important test to assess the robustness of any image descriptor is to verify its invariance to changes like rotation and addition of noise. In these experiments, we employed the same settings adopted for the compared methods in the basic classification test.

The first evaluated invariance is to noise following Gaussian distribution, a situation commonly found in practice. Here, this is simulated by an artificial additive process where an amount \( \eta \) sampled from the Gaussian distribution with mean zero is added to a pixel with gray value \( g \). Such noise is calibrated according to the variance \( \sigma \) of the Gaussian, and thus, \( \eta \) is a random variable whose probability density function is

\[
P_\eta(g) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{g^2}{2\sigma^2}}.
\]  

One can find the relation between this parameter and classical quality metrics like peak-signal-to-noise ratio (PSNR) or structural similarity index measure (SSIM) in [21]. Here, \( \sigma \) ranges between 0.01 and 0.05. Figure 11 shows examples of
such images affected by the different levels of noise in our tests.

Tables 3, 4, 5 and 6 present results comparing with other descriptors in Brodatz, Vistex, UIUC, and KTHTips data sets, respectively. A similar behavior is expected for other textures. Although Gabor obtained the best results for more severe noises, the proposed method performs better than the state-of-the-art LBP method. Furthermore, for a small amount of noise it has a performance significantly better than Gabor in Brodatz, Vistex and UIUC. Another issue that is worth to be commented here is that the intrinsic random nature of the noisy process makes possible the rising of some unexpected behavior such as the slight increasing in the classification accuracy with more noise, as observed in some atypical cases in the tables (for instance, for levels 0.01 and 0.02 in Brodatz using Gabor). This is possible, for example, when two images from different classes but looking very similar can be more easily distinguished after the application of random noise.

A second invariance experiment evaluates textures rotated by predefined angles. To ensure the same dimension for all textures, a procedure to extract only the central region of each image is accomplished, as illustrated in Fig. 12. In this way, if the original texture image has dimensions $d \times d$, the extracted region after the rotation is the central part with dimension $d/\sqrt{2} \times d/\sqrt{2}$.

Figure 13 shows the example of a texture rotated by the angles considered in this experiment, that is, $0^\circ$, $30^\circ$, $60^\circ$, $90^\circ$, $120^\circ$, $150^\circ$ and $180^\circ$.
Tables 7, 8, 9 and 10 present the correctness rates for the classification of Brodatz, Vistex, UIUC, and KTHTips databases, respectively, after the rotation of the textures. The lower rates compared to Table 2 are caused by the use of a cropped region of the original texture, leading to an obvious loss of information. The proposed approach outperforms the other methods with higher success rates and a low variability in the results in most cases when the angle is changed. The maximum difference is less than 2% in all databases, while LBP_M/C, for example, shows a difference of more than 5% in the Brodatz result when the textures are rotated by 60° and 150°, respectively, and 10% in Vistex when the angle goes from 180° to 120°. In terms of accuracy, the only exception is KTHTips, where triangular descriptors are outperformed for multiples of 90°. This is a consequence of the rectangular dimensions of many samples in that database, which makes the rotated image significantly smaller after the cropping process, impairing the multiscale approach followed the proposed method.

### 7.4 Computational Time

Table 11 lists the computational time for running the algorithm of the proposed method compared with the other texture descriptors. All the algorithms were coded in MATLAB® and run on a laptop equipped with a CPU Intel core-i5 5200u and 8GB of RAM. We adopted the image size used in most databases analyzed here, namely 128 × 128, and the time is an average over 100 images, where each descriptor is computed for only one image each time. The number of descriptors is that obtained when one runs the algorithm described in the respective literature and is the same number used in the experimental tests for retrieval and classification. Even though the proposed method becomes slower than most LBP variants, mainly when a high number of exponents are employed, the difference is not excessive especially when those methods are coded in more optimized environments, such as using C programming language for example. It is also interesting to notice that the computational time is reduced more than 10 times in comparison with the BM fractal descriptors. Finally, there are also plans for future optimizations of the algorithm.

### 7.4.1 Discussion

The proposed method differs from other approaches to texture analysis in the literature in that the physical process underlying the generation of the image is taken into account to a high extent, as we demonstrated in its relation with fractional Brownian processes. These processes are known to be widely presented in nature and its association with the way that our brain perceives materials and scenarios around us is well established. Parallel to this, the triangular prisms also encompass a geometrical analysis similar to “walking-dividers,” intimately related to the primitive idea of fractals and also commonly observed in natural structures.

---

**Table 7** Success rates for different angles of rotation applied to Brodatz data

| Method               | Rotation angle (°) |
|----------------------|--------------------|
|                      | 0  | 30  | 60  | 90  | 120 | 150 | 180  |
| LBP [33]             |    | 81.98 | 73.70 | 72.86 | 81.98 | 73.98 | 72.75 | 82.15 |
| GLCM [19]            |    | 77.70 | 75.28 | 75.50 | 77.93 | 75.50 | 76.23 | 77.53 |
| Multifractal [43]    |    | 76.63 | 76.24 | 73.99 | 77.25 | 76.12 | 76.18 | 76.63 |
| Gabor [28]           |    | 80.13 | 80.41 | 79.06 | 80.01 | 80.13 | 79.11 | 80.29 |
| Fourier [16]         |    | 70.32 | 72.07 | 71.56 | 69.98 | 72.07 | 71.78 | 70.21 |
| RI-LBP [33]          |    | 82.99 | 72.86 | 74.77 | 82.94 | 73.19 | 74.60 | 82.83 |
| LBP + VAR [34]       |    | 79.56 | 75.90 | 66.33 | 79.61 | 76.29 | 67.90 | 79.11 |
| CLBP_M/C [18]        |    | 82.04 | 78.27 | 77.93 | 82.21 | 77.82 | 77.59 | 82.32 |
| BM descriptors [2]   |    | 82.09 | 78.38 | 77.87 | 82.21 | 78.72 | 77.93 | 82.15 |
| Proposed             |    | 89.58 | 89.42 | 88.91 | 89.41 | 89.70 | 89.02 | 89.58 |
itself, such complementary and dual viewpoint of fractality is an important contribution and novelty with regard to other fractal approaches such as multifractals [43], Bouligand–Minkowski [2] and others.

At this point, a comparison with BM fractal descriptors is particularly valid as both are based on the idea of using the power-law curve associated with some numerical method employed to estimate the fractal dimension. The proposed method presents two main novelties in comparison with the previous fractal descriptors. The first one is the use of a more direct approach to estimate the dimension, as the interpretation of methods like “walking-dividers” is more intuitive in the context of self-similar fractals than methods like BM. The second is the introduction of a parameter ($\alpha$) into the
process. Actually, this was facilitated in particular by the way that the fractal measure (in this case the prism area) is constructed, by summing up individual area values. This represented an enhancement over the original fractal descriptors and demonstrated that more elaborated schemes than merely using the log–log curve may capture more information about the texture image and hence provide more precise and robust descriptors.

The consideration of the physical process also showed to be more advantageous than simply quantifying relations between neighbor pixels without accounting for the semantic involved in the represented object, as in LBP and GLCM descriptors. Even though the interpretation of these approaches can be considered more straightforward, the lack of a more realistic model makes them insufficient in more complex cases with larger and more heterogeneous databases as those presented here.

Finally, we also compare to other methods where the image is observed beyond the simple pixel values, such as in Fourier and Gabor descriptors. These descriptors, however, are ultimately based on linear filtering and are not the ideal solution to explain the nonlinearities present in many natural images, mostly caused by the chaotic behavior associated with fBm processes.

Generally speaking, the great performance of triangular fractal descriptors in the presented results confirms what is expected from the fractal analysis of natural texture images. Indeed, fractal geometry constitutes a reliable tool to model those structures, which cannot be well described in an Euclidean framework. This analysis is especially complete here as two fractal viewpoints (statistical and geometrical) are merged to provide an even more accurate description of the image. Furthermore, fractal descriptors approach enhances the conventional fractal geometry analysis by extracting relevant information of complexity under different scales, by the multiscale analysis implicit when the dimension is estimated for different values of $\epsilon$. The combination of a solid modeling with a multiscale context also explains the robustness to noise and rotation, as noise is more active in local scales and deformations like the image rotation do not alter model parameters at a substantial level. In this way, these descriptors are capable of providing features representing precise and rich measures of the spatial structure of complex textures. This type of image commonly appears not only in benchmark data, as used here, but also in practical applications. This is the case in problems involving the identification and discrimination of intricate patterns in objects and scenarios represented in a digital image.

### 8 Conclusion

This work proposed a novel gray-level texture descriptor based on a particular method of fractal dimension estimation, named triangular prism method. The descriptors were obtained by combining values of area of a triangular tessellation of the texture, using different steps for the tessellation grid as well as different values for an exponential parameter used as a weight for the area within each cell in the grid.

Our study demonstrated that the triangular prisms are capable of extracting fractal characteristics of the image under two complementary perspectives: geometrical (walking-dividers) and statistical (fractional Brownian motion).

These descriptors were compared to classical and state-of-the-art texture descriptors in tasks of classification and image retrieval of four well-known benchmark texture data sets, e.g., Brodatz, Vistex, UIUC, and KTH-TIPS2b. In image retrieval, the proposed method achieved high accuracy and was particularly efficient when the number of retrieval attempts was higher, which simulates more complicated cases in practice. In the classification test, the triangular descriptors also outperformed other texture features, with greater advantage in textures containing relevant information at multiple scales. We also carried out tests on the robustness of the proposed descriptors to noise and rotation. The triangular descriptors achieved higher accuracy than state-of-the-art approaches like LBP variations in this test as well. Finally, we checked the computational time and the proposed method presented a competitive performance in this respect when compared to other texture descriptors in the literature.

In general, the proposal was more robust than all the other compared approaches in all the scenarios, including other state-of-the-art approaches such as LBP and the original fractal descriptors. The results illustrated that the novel approach is a valuable descriptor for different types of recognition tasks, what suggests its application to a wide range of problems comprising tasks of pattern recognition in digital images.
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