An indecomposable $PD_3$-complex: II

Jonathan A. Hillman

Abstract We show that there are two homotopy types of $PD_3$-complexes with fundamental group $S_3 \ast \mathbb{Z}/2\mathbb{Z} S_3$, and give explicit constructions for each, which differ only in the attachment of the top cell.

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In [3] we showed that $\pi = S_3 \ast \mathbb{Z}/2\mathbb{Z} S_3$ satisfies the criterion of [5] and thus is the fundamental group of a $PD_3$-complex. As $\pi$ has infinitely many ends but is indecomposable, this illustrates a divergence from the known properties of 3-manifolds, and provides a counter-example to an old question of Wall [6]. In particular, the Sphere Theorem does not extend to all $PD_3$-complexes.

Here we shall give an explicit description of a finite $PD_3$-complex $Y$ realizing this group. The construction is modelled on a similar construction for a $PD_3$-complex $X$ with fundamental group $S_3$. In each case the cellular chain complex of the universal cover has the striking property that it is self-dual. In §2 we show a $PD_3$-complex with fundamental group $\pi$ must be orientable, and we use Turaev’s work to show there are two homotopy types of such $PD_3$-complexes. The 2-fold cover of $Y$ is homotopy equivalent to $L(3,1)\sharp L(3,1)$, while a simple modification of our construction (suggested by the referee) gives a $PD_3$-complex with 2-fold cover homotopy equivalent to $L(3,1)\sharp -L(3,1)$. (This group was first suggested as a test case in [2].)

1 A finite complex with group $S_3 \ast \mathbb{Z}/2\mathbb{Z} S_3$

Let $G$ be a group and let $\Gamma = \mathbb{Z}[G]$, $\varepsilon : C_1 = \Gamma \to \mathbb{Z}$ and $I(G) = \text{Ker}(\varepsilon)$ be the integral group ring, the augmentation homomorphism and the augmentation ideal, respectively. If $M$ is a left $\Gamma$-module $\overline{M}$ shall denote the conjugate right module, with $G$-action given by $m.g = g^{-1}m$ for all $g \in G$ and $m \in M$, and similarly $\overline{N}$ shall denote the conjugate left module structure on a right $\Gamma$-module $N$. If $C_\ast$ is a chain complex over $\Gamma$ with an augmentation $\varepsilon : C_0 \to \mathbb{Z}$
a diagonal approximation is a chain homomorphism $\Delta : C_* \to C_* \otimes_{\mathbb{Z}} C_*$ (with diagonal $G$-action) such that $(\varepsilon \otimes 1)\Delta = id_{C_*} = (1 \otimes \varepsilon)\Delta$.

The cellular chain complex $C_*(\tilde{K})$ for the universal covering space of a finite 2-complex $K$ determined by a presentation for a group is isomorphic to the Fox-Lyndon complex of the presentation, via an isomorphism carrying generators corresponding to based lifts of cells of $K$ to the standard generators.

The symmetric group $S_3$ has a presentation $\langle a, b \mid a^2, abab^{-2} \rangle$. Let $\pi = S_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, with presentation $\langle a, b, c \mid r, s, t \rangle$, where $r = a^2$, $s = abab^{-2}$ and $t = acac^{-2}$. The two obvious embeddings of $S_3$ into $\pi$ admit retractions, as $\pi/\langle b \rangle \cong \pi/\langle c \rangle \cong S_3$. Let $A$, $B$ and $C$ be the cyclic subgroups generated by the images of $a$, $b$ and $c$, respectively. The inclusions of $A$ into $S_3$ and $\pi$ induce isomorphisms on abelianization, while the commutator subgroups are $S_3' = B$ and $\pi' = B \ast C$. Thus these groups are semidirect products: $S_3 \cong B \times (\mathbb{Z}/2\mathbb{Z})$ and $\pi \cong (B \ast C) \times \mathbb{Z}/2\mathbb{Z}$. In particular, $\pi$ is virtually free, and so has infinitely many ends. However it follows easily from the Grushko-Neumann Theorem that $\pi$ is indecomposable. (See [3]).

The above presentations determine finite 2-complexes $K$ and $L$, with fundamental groups $S_3$ and $\pi$, respectively. There are two obvious embeddings of $K$ as a retract in $L$, with retractions $r_b, r_c : L \to K$ given by collapsing the pair of cells $\{c, t\}$ and $\{b, s\}$, respectively.

The chain complex $C_*(\tilde{K})$ has the form

$$\mathbb{Z}[S_3]^2 \to \mathbb{Z}[S_3]^2 \to \mathbb{Z}[S_3],$$

where $\partial_1(1,0) = a - 1$, $\partial_1(0,1) = b - 1$, $\partial_2(1,0) = (a + 1, 0)$ and $\partial_2(0,1) = (b^2a + 1, a - b - 1)$. The 2-chain $\psi = (a - 1, -ba + a + b^2 - b)$ is a 2-cycle, and so determines an element of $\pi_2(K) = H_2(\tilde{K}; \mathbb{Z})$, by the Hurewicz Theorem. Let $X = K \cup_\psi e^3$, and let $C_*$ be the cellular chain complex for the universal cover $\tilde{X}$. (Thus $C_i = C_i(\tilde{K})$ for $i \leq 2$ and $C_3 \cong \mathbb{Z}[S_3]$). The dual cochain complex $C^* = \text{Hom}_\Gamma(C_*, \mathbb{Z}[S_3])$ is a complex of right $\mathbb{Z}[S_3]$-modules.

We shall define new bases which display the structure of $C_*$ to better advantage, as follows. Let $e_1 = (1, 0)$ and $e_2 = (-ba - b^2, 1)$ in $C_1$ and $f_1 = (1, 0)$ and $f_2 = (0, -a)$ in $C_2$, and let $g$ be the generator of $C_3$ corresponding to the top cell. Then $\partial_1e_1 = a - 1$, $\partial_1e_2 = -b^2a + ba + b^2 - 1$, $\partial_2f_1 = (a + 1)e_1$, $\partial_2f_2 = (b^2a + a - 1)e_2$, and $\partial_3g = \psi = (a - 1)f_1 + (-b^2a + ba + b - 1)f_2$.

The matrix for $\partial_2$ with respect to the bases $\{\tilde{e}_i\}$ and $\{\tilde{f}_j\}$ is diagonal, and is hermitian with respect to the canonical involution of $\mathbb{Z}[S_3]$, while the matrix for $\partial_3$ is the conjugate transpose for that of $\partial_1$. Hence the chain complex $C^{3-*}$
obtained by conjugating and reindexing the cochain complex $C^*$ is isomorphic to $C_*$.

Let $\beta = b^2 + b + 1$ and $\nu = \Sigma_{s \in S_3} s = \beta(a + 1)$.

**Lemma 1** The complex $X$ is a PD$_3$-complex with $\tilde{X} \simeq S^3$.

**Proof** Since $C_*$ is the cellular chain complex of a 1-connected cell complex $H_0(C_*) \cong \mathbb{Z}$ and $H_1(C_*) = 0$. If $\partial_2(rf_1 + sf_2) = 0$ then $r(a + 1) = 0$ and $s(b^2a + a - 1) = 0$. Now the left annihilator ideals of $a + 1$ and $b^2a + a - 1$ in $\mathbb{Z}[S_3]$ are principal left ideals, generated by $a - 1$ and $(b - 1)(ba - 1)$, respectively. Hence $r = p(a - 1)$ and $s = q(b - 1)(ba - 1)$ for some $p, q \in \mathbb{Z}[B]$. A simple calculation gives $\partial_3((p(ba + b + 1) + q(ba + b))g) = rf_1 + sf_2$ and so $H_2(C_*) = 0$.

If $\partial_3hg = 0$ then $h(a - 1) = 0$, so $h = h_1(a + 1)$ for some $h_1 \in \mathbb{Z}[B]$, and $h(b^2a - ba - 1) = 0$. Now $h(b^2a - ba - 1) = h_1(1 - b)(a + b + 1)$, so $h_1(1 - b) = 0$. Therefore $h_1 = m\beta$ for some $m \in \mathbb{Z}$, so $h = m\nu$ and $H_3(C_*) = \mathbb{Z}[S_3][\nu]g \cong \mathbb{Z}$. Hence $\tilde{X} \simeq S^3$. Now $H_3(X; \mathbb{Z}) = H_3(\mathbb{Z} \otimes \mathbb{Z}[S_3] C_*) = \mathbb{Z}[1 \otimes g]$ and $tr([1 \otimes g]) = \nu g$, where $tr : H_3(X; \mathbb{Z}) \to H_3(\tilde{X}; \mathbb{Z})$ is the transfer homomorphism. The homomorphisms from $H^q(C^*)$ to $H_3-\nu(C_*)$ determined by cap product with $[X] = [1 \otimes g]$ may be identified with the Poincaré duality isomorphisms for $\tilde{X}$, and so $X$ is a PD$_3$-complex.

The verification that $\tilde{X} \simeq S^3$ is essentially due to [2] and the fact that $X$ is a PD$_3$-complex is due to [6]. The only novelty here is the diagonalization of $\partial_2$, which was a guiding feature in the study of $\pi = \mathbb{S} \ast \mathbb{Z}/2\mathbb{Z} \mathcal{S}_3$.

Let $\Pi = \mathbb{Z}[\pi]$. The cellular chain complex for the universal covering space $\tilde{L}$ has the form

$$\Pi^3 \xrightarrow{\partial_2} \Pi^3 \xrightarrow{\partial_1} \Pi.$$  

The differentials are given by $\partial_1(1,0,0) = a - 1$, $\partial_1(0,1,0) = b - 1$ and $\partial_1(0,0,1) = c - 1$, $\partial_2(1,0,0) = (a + 1,0,0)$, $\partial_2(0,1,0) = (b^2a + 1,a - b - 1,0)$ and $\partial_2(0,0,1) = (c^2a + 1,0,a - c - 1)$. In particular, $H_2(\tilde{L}; \mathbb{Z}) = \text{Ker}(\partial_2)$.

Let $\theta = (a - 1, -ba + a + b^2 - b, -ca + a + c^2 - c)$. Then $\partial_2(\theta) = 0$, and so $\theta$ determines an element of $\pi_2(L) = H_2(\tilde{L}; \mathbb{Z})$, by the Hurewicz Theorem. Let $Y = L \cup_b e^3$ and let $D_*$ be the cellular chain complex for the universal covering space $\tilde{Y}$.

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Let 
\[ \tilde{e}_1 = (1, 0, 0), \quad \tilde{e}_2 = (1, -a, 0), \quad \tilde{e}_3 = (0, 1, 0). \]
Therefore 
\[ \partial_1 \tilde{e}_1 = a - 1, \quad \partial_1 \tilde{e}_2 = b - b^2 a + b^2 - 1, \quad \partial_1 \tilde{e}_3 = c - c^2 a + c^2 - 1. \]
Moreover \( \theta = (a - 1) \tilde{f}_1 + (b - a + b - 1) \tilde{f}_2 + (-c^2 a + c + c - 1) \tilde{f}_3. \) Let 
\[ D^* = \text{Hom}_\Gamma(D_\ast, \Pi) \]
be the cochain complex dual to \( D_\ast. \) Then it is easily seen that \( \overline{D^*} \cong D_{3-\ast}. \)

**Theorem 2** The complex \( Y \) is a PD\(_3\)-complex.

**Proof** Clearly \( H_0(D_\ast) \cong \mathbb{Z} \) and \( H_1(D_\ast) = 0. \) The argument of the first part of Lemma 1 extends immediately to show that the kernel of \( \partial_2 \) is generated by \( (a - 1) \tilde{f}_1, \ (b - 1)(ba - 1) \tilde{f}_2 \) and \( (c - 1)(ca - 1) \tilde{f}_3. \) Hence these elements represent generators for \( H_2(D_\ast). \) Let \( \tilde{g} \) be the generator for \( D_3 \) corresponding to the top cell, so that \( \partial_3 \tilde{g} = \theta. \) Note that the image of \( g \) in \( \mathbb{Z} \otimes \mathfrak{e} D_3 \) is a cycle, and represents a generator for \( H_3(Y; \mathbb{Z}) = H_3(\mathbb{Z} \otimes \mathfrak{e} D_\ast). \) If \( h \theta = 0 \) then (as in Lemma 1) \( h = h_1(a + 1) \) for some \( h_1 \in \mathbb{Z}[B \ast C] \) such that \( h_1(b - 1) = h_1(c - 1) = 0. \) It follows that \( h_1 = 0. \) Hence \( \partial_3 \) is injective and so \( H_3(D_\ast) = 0. \)

Let \( \hat{1}, \hat{e}^*, \hat{f}^* \) and \( \hat{g} \) denote the bases of \( D^* \) dual to the above bases for \( D_\ast. \) Let \( \Delta \) be a diagonal approximation for \( D_\ast \) and suppose that \( \Delta(\hat{g}) = \Sigma_{0 \leq q \leq 3} \Sigma_{i \in I(3)} x_i \otimes y_i, \) where \( x_i \in D_q \) and \( y_i \in D_{3-q}, \) for all \( i \in I(3) \) and \( 0 \leq q \leq 3. \) Then \( \Sigma_{i \in I(3)} x_i = \hat{g}. \) Let \( r_i = \hat{g}(x_i) \) for \( i \in I(3) \) and let \( \hat{f} \) denote the image of \( \hat{g} \) in \( H_3(Y; \mathbb{Z}) = \mathbb{Z} \otimes \mathfrak{e} D_3. \) Then \( \varepsilon(\hat{g} \cap \hat{f}) = \varepsilon(\Sigma_{i \in I(3)} y_i) = 1, \) and so \( \hat{g} \cap \hat{f} \) generates \( H_0(D_\ast). \) Since \( H_1(D_\ast) = H_3(D_\ast) = H^0(\overline{D^*}) = H^2(\overline{D^*}) = 0, \) \( \partial \cap \hat{f} \) induces isomorphisms \( H^q(\overline{D^*}) \cong H_{3-q}(D_\ast) \) for all \( q \neq 1. \) The remaining case follows as in [5] from the facts that \( \overline{D^*} \cong D_{3-\ast} \) and \( \Delta \) is chain homotopic to \( \tau \Delta, \) where \( \tau : D_\ast \otimes D_\ast \to D_\ast \otimes D_\ast \)

Can the last step of this argument be made more explicit? The work of Handel on diagonal approximations for dihedral groups may be adapted to give the following formulae for a diagonal approximation for the truncation to degrees \( \leq 2 \) of \( D_\ast \) which is compatible with the above two embeddings of \( K \) as a retract in \( L: \)

\[ \text{Algebraic & Geometric Topology, Volume 4 (2004)} \]
\[ \Delta(1) = 1 \otimes 1 \]
\[ \Delta(\tilde{e}_1) = \tilde{e}_1 \otimes a + 1 \otimes \tilde{e}_1, \]
\[ \Delta(\tilde{e}_2) = \tilde{e}_2 \otimes 1 - ba\tilde{e}_1 \otimes (b - 1) - b^2\tilde{e}_1 \otimes (b^2a - 1) - (ba - b) \otimes ba\tilde{e}_1 \]
\[- (b^2 - b) \otimes b^2\tilde{e}_1 + b \otimes \tilde{e}_2, \]
\[ \Delta(\tilde{e}_3) = \tilde{e}_3 \otimes 1 - ca\tilde{e}_1 \otimes (c - 1) - c^2\tilde{e}_1 \otimes (c^2a - 1) - (ca - c) \otimes ca\tilde{e}_1 \]
\[- (c^2 - c) \otimes c^2\tilde{e}_1 + c \otimes \tilde{e}_3, \]
\[ \Delta(\tilde{f}_1) = \tilde{f}_1 \otimes 1 + \tilde{e}_1 \otimes a\tilde{e}_1 + 1 \otimes \tilde{f}_1, \]
\[ \Delta(\tilde{f}_2) = \tilde{f}_2 \otimes a + (b^2 + b)\tilde{f}_1 \otimes (a - ba) + (b^2a + b^2)\tilde{f}_2 \otimes (a - ba) \]
\[ + ((ba + b^2 - 1)\tilde{e}_1 + \tilde{e}_2) \otimes ((b^2a)\tilde{e}_1 + ba\tilde{e}_2) \]
\[- ((b^2a + 1)\tilde{e}_1 + ba\tilde{e}_2) \otimes ((ba + a + b^2 + b)\tilde{e}_1 + (b^2a + a)\tilde{e}_2) \]
\[- ((a + b)\tilde{e}_1 + b^2a\tilde{e}_2) \otimes ((ba + b^2)\tilde{e}_1 + a\tilde{e}_2) - (a + 1)\tilde{e}_1 \otimes \tilde{e}_1 \]
\[ + (a - b) \otimes (b^2 + b)\tilde{f}_1 + (a - b) \otimes (b^2a + b^2)\tilde{f}_2 + a \otimes \tilde{f}_2 \quad \text{and} \]
\[ \Delta(\tilde{f}_3) = \tilde{f}_3 \otimes a + (c^2 + c)\tilde{f}_1 \otimes (a - ca) + (c^2a + c^2)\tilde{f}_3 \otimes (a - ca) \]
\[ + ((ca + c^2 - 1)\tilde{e}_1 + \tilde{e}_3) \otimes ((c^2a)\tilde{e}_1 + ca\tilde{e}_3) \]
\[- ((c^2a + 1)\tilde{e}_1 + ca\tilde{e}_3) \otimes ((ca + a + c^2 + c)\tilde{e}_1 + (c^2a + a)\tilde{e}_3) \]
\[- ((a + c)\tilde{e}_1 + c^2a\tilde{e}_3) \otimes ((ca + c^2)\tilde{e}_1 + a\tilde{e}_3) - (a + 1)\tilde{e}_1 \otimes \tilde{e}_1 \]
\[ + (a - c) \otimes (c^2 + c)\tilde{f}_1 + (a - c) \otimes (c^2a + c^2)\tilde{f}_3 + a \otimes \tilde{f}_3 \]

These formulae were derived from the work of Handel by using the canonical involution of \( \mathbb{Z}[S_3] \) to switch right and left module structures and showing that \( C_* \) is a direct summand of a truncation of the Wall-Hamada resolution for \( S_3 \). (In Handel’s notation \( a = y, b = x, e_1 = c^2, e_2 = -c_1 - c_1(x + xy), f_1 = c^2, f_2 = -c_1^2y + c^2_2x^2 - c^2_2y \) and \( g = -(c^2_3 + c^2_3)(x + y) - c^2_3y \). Handel’s work also leads to a formula for \( \Delta(g) \), but it is not clear what \( \Delta(\tilde{g}) \) should be.

## 2 Other PD₃-complexes with this group

Having constructed one PD₃-complex with group \( \pi \) one may ask how many there are. Any such PD₃-complex must be orientable. For let \( w_1 : \pi \to \{\pm 1\} \) be a homomorphism and define an involution on \( \Gamma \) by \( \tilde{g} = w_1(g)g^{-1} \), for all \( g \in \pi \). Let \( w = w_1(a) \) and \( R = \mathbb{Z}[\pi/\pi'] = \mathbb{Z}[a]/(a^2 - 1) \). Let \( J = \text{Coker}(\partial_2^{tr}) \),
where $\partial_2 : \Pi^3 \rightarrow \Pi^3$ is the presentation matrix for $I(\pi)$ given in §1. Then $R \otimes_{\Gamma} I(\pi) \cong R/(a + 1) \oplus (R/(a + 1, 3))^2$, while $R \otimes_{\Gamma} J \cong R/(a + w) \oplus (R/(a + w, 3))^2$. If the pair $(\pi, w_1)$ is realized by a $PD_3$-complex then $I(\pi)$ and $J$ are projective homotopy equivalent [4]. But then $R \otimes_{\Gamma} I(\pi)$ and $R \otimes_{\Gamma} J$ are projective homotopy equivalent $R$-modules, and so we must have $w = 1$.

If $W$ is an oriented $PD_3$-complex with fundamental group $G$ and $c_W : W \rightarrow K(G, 1)$ is a classifying map let $\mu(W) = c_{w*}[W] \in H_3(W; \mathbb{Z})$. Two such $PD_3$-complexes $W_1$ and $W_2$ are homotopy equivalent if and only if $\mu(W_1)$ and $\mu(W_2)$ agree up to sign and the action of $Out(G)$. Turaev constructed an isomorphism $\nu_C$ from $H_3(G; \mathbb{Z})$ to a group $[F^3(C), I(G)]$ of projective homotopy classes of module homomorphisms and showed that $\mu \in H_3(G; \mathbb{Z})$ is the image of the orientation class of a $PD_3$-complex if and only if $\nu_C(\mu)$ is the class of a self-homotopy equivalence [5].

When $G = \pi = S_3 * \mathbb{Z}/2\mathbb{Z} \ast_3 \mathbb{Z}$ we have $F^2(C) \cong I(\pi)$, and $H_3(\pi; \mathbb{Z}) \cong H_3(\pi'; \mathbb{Z}) \oplus H_3(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})$. Let $W'$ be the double cover of $W$, with fundamental group $\pi' \cong (\mathbb{Z}/3\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$. Then $W'$ is a connected sum, by Theorem 1 of [5], and so it is homotopy equivalent to one of the 3-manifolds $L(3, 1) \sharp L(3, 1)$ and $L(3, 1) \sharp L(3, 1)$. (These may be distinguished by the torsion linking forms on their first homology groups). In particular, $\mu(W')$ has nonzero entries in each summand. Since $\mu(W')$ is the image of $\mu(W)$ under the transfer to $H_3(\pi'; \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2$ the image of $\mu(W)$ in each $\mathbb{Z}/3\mathbb{Z}$-summand must be nonzero. Let $u \in H^1(W'; \mathbb{F}_2)$ correspond to the abelianization homomorphism. Since $\beta_2(W'; \mathbb{F}_2) = \beta_1(W; \mathbb{F}_2) = 1 = \beta_2(\pi; \mathbb{F}_2)$ we have $u^2 \neq 0$, and so $u^3 \neq 0$, by Poincaré duality. It follows easily that the image of $\mu(W)$ in the $\mathbb{Z}/2\mathbb{Z}$-summand must be nonzero also. (Note that $W'$ is $\mathbb{Z}(2)$-homology equivalent to $S^3$ and so $W$ is $\mathbb{Z}(2)[\mathbb{Z}/2\mathbb{Z}]$-homology equivalent to $RP^3$). Since reversing the orientation of $W$ reverses that of $W'$, we may conclude that there are at most two distinct homotopy types of $PD_3$-complexes with fundamental group $\pi$, and that they may be detected by their double covers.

The retraction $r_b$ and $r_c$ of $L$ onto $K$ extend to maps $r_b, r_c : Y \rightarrow X$. These maps induce the same isomorphism $H_3(Y; \mathbb{Z}) \cong H_3(X; \mathbb{Z})$, and so their lifts to the double covers induce the same isomorphism $H_3(Y'; \mathbb{Z}) \rightarrow H_3(X'; \mathbb{Z})$. Hence $Y' \simeq L(3, 1) \sharp L(3, 1)$, rather than $L(3, 1) \sharp L(3, 1)$. The referee has pointed out that if we use $\xi = (a - 1)f_1 + (-b^2a + ba + b - 1)f_2 - (-c^2a + ca + c - 1)f_3$ instead of $\theta$ (changing only the sign of the final term) then $Z = L \cup_\xi e^3$ is another $PD_3$-complex with $\pi_1(Z) \cong \pi$, and a similar argument shows that the double cover is now $Z' \simeq L(3, 1) \sharp L(3, 1)$.

The question of whether every aspherical $PD_3$-complex is homotopy equivalent to a 3-manifold remains open. The recent article [7] gives a comprehensive
survey of Poincaré duality in dimension 3, emphasizing the role of the JSJ decomposition in relation to this question.

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