INVARIANT SYSTEMS OF REPRESENTATIVES,
OR
THE COST OF SYMMETRY
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Suppose that one can destroy all 100-gons in a graph by removing 2019 edges. How many edges must be removed
to destroy all 100-gons in such a way that the set of removed edges is invariant with respect to all automorphisms
the initial graph? This paper contains solutions to such kind of problems. Several open questions are raised.

0. Introduction
Consider the following “applied” problem.

We are recruiting a team for the Mars expedition, and we want to satisfy (for example) the following
compatibility requirement: among any five participants, there should be at least two, each of which
respects at least three members of this five. Our dossiers show that the expulsion of ten particular can-
didates would make this requirement satisfied. The problem is that we want to be fair and impartial,
i.e., we want the set of expelled candidates to be invariant under all permutations (of all candidates)
preserving the relation “respects”. How many candidates do we need to expel (in the worst case)?

What is the cost of fairness? The question is not very trivial. For instance, if we try to expel all candidates obtained
from the initial ten “bad” candidates by the action of all permutations preserving the relation “respects”, then we
can end up with expelling all candidates, even if there are infinitely many of them. In fact, the optimal set of fairly
expelled candidates is always finite and needs neither contain the given set of ten “bad” candidates nor be contained
in it.

In algebra, there are many theorems of this kind, e.g.,
- if a group $G$ contains an abelian subgroup of finite index, then $G$ contains a characteristic (i.e. invariant with
  respect to all automorphisms) abelian subgroup of finite index [KaM82];
- if a group $G$ contains a nilpotent subgroup of finite index, then $G$ contains a characteristic nilpotent (of the same
class) subgroup of finite index [BrNa04];
- if a group $G$ contains a solvable subgroup of finite index, then $G$ contains a characteristic solvable (of the same
derived length) subgroup of finite index [KhM07a];
- if a group $G$ contains a central metabelian subgroup of finite index, then $G$ contains a characteristic central
metabelian subgroup of finite index [KhM07a];
- if a group $G$ contains a paranilpotent subgroup of finite index, then $G$ contains a characteristic paranilpotent
subgroup of finite index [dGT19b];
- if a group $G$ contains a finite-index subgroup whose commutator subgroup is finite, then $G$ contains a characteristic
finite-index subgroup whose commutator subgroup is finite [KMi15];
- if a group $G$ of finite exponent contains a finite normal subgroup $N$, then $G$ contains a characteristic finite
subgroup $H$ such that the spectrum (i.e. set of orders of all elements) of the quotient group $G/H$ is contained in
the spectrum of $G/N$ [KMi15];
- if an algebra $G$ (associative or Lie) over a field contains a solvable ideal of finite codimension, then $G$ contains an
invariant with respect to all automorphisms solvable (of the same derived length) ideal of finite codimension
[KhM08].

The list can be extended, see, e.g., [Vd00], [KhM08], [KhM09], [KIMe09], [KhKMM09], [MSh12], [KIMe15], [Fr18],
[dGT18a], [dGT18b], [dGT19a], [dGT19b], and references therein. The assertions of this kind are called sometimes
([KIMe15], [Fr18]) Khukhro–Makarenko type theorems after one of such results [KhM07a] (see also [KIMe09]) including
as special cases the first four of the listed facts.

The first of these facts (on abelian subgroups) is simple in the sense that there is a short and elementary proof.
However, quite naïve approaches do not work. How can one make a given abelian finite-index subgroup $N \subset G$
characteristic?
- One can take the intersection of all automorphic images of $N$. The obtained subgroup $\bigcap_{\varphi \in \text{Aut} G} \varphi(N)$ is characteristic
  and abelian (because it is contained in $N$), but, unfortunately, the index can be infinite.

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One can take the subgroup generated by all automorphic images of $N$. The obtained subgroup $\left< \bigcup_{\varphi \in \text{Aut } G} \varphi(N) \right>$ is characteristic and of finite index (because it contains $N$), but, unfortunately, it can be non-abelian. Actually, it may happen that the sought characteristic abelian finite-index subgroup neither contain nor is contained in $N$. The same is true for other Khukhro–Makarenko type theorems (and for the Mars-expedition problem too).

Almost all the listed Khukhro–Makarenko type theorems and the Khukhro–Makarenko theorem itself are special cases of a very general fact [KImi15] referred hereafter as the \textit{multilinear-property theorem}. This general assertion gives also a universal estimate of the corresponding parameter (the index of the characteristic subgroup, the codimension of the automorphism-invariant ideal and so on). This estimate can however be far from sharp in particular cases. For instance, the multilinear-property theorem gives the following quantitative refinement of the first listed fact:

\begin{quote}
\textit{if a group $G$ contains an abelian subgroup of a finite index $n$, then $G$ contains a characteristic abelian subgroup of index at most $(n!)^{\log_2(n!)+1}$.}
\end{quote}

While the sharp estimate here is $n^2$, see [PSz02].\(^*\) In principle, the multilinear-property theorem is applicable also to purely combinatorial questions (see [KImi15]); e.g., for the Mars-expedition problem, this theorem gives the following practically useless answer:

\begin{quote}
it is sufficient to expel at most $22229709804712410 = f(f(f(10)))$ candidates, where $f(x) = x(x + 1)$.
\end{quote}

This bound (greatly exceeding the Earth’s population) is also very coarse. The following theorem says that the actual answer is 50 (and this estimate is sharp).

\textbf{Main theorem.} Suppose that a group $G$ acts on a set $U$, and $\mathcal{F}$ is a $G$-invariant family of finite subsets of $U$ of uniformly bounded cardinality (i.e. $\max |F| < \infty$). Let $X \subseteq U$ be a finite system of representatives for this family (i.e. $X \cap F \neq \emptyset$ for any $F \in \mathcal{F}$). Then there exists a $G$-invariant system of representatives $Y$ such that $|Y| \leq |X| \cdot \max_{F \in \mathcal{F}} |F|$.

Here, the word \textit{family} means an unordered family, i.e., $\mathcal{F}$ is a set of subsets of $U$. The \textit{invariance} of $\mathcal{F}$ means that $g\mathcal{F} = \mathcal{F}$ for all $g \in G$, i.e., $g\mathcal{F} \equiv \{gf \mid f \in \mathcal{F}\} \in \mathcal{F}$ for all $g \in G$ and $F \in \mathcal{F}$.

The proof of the main theorem is elementary, except that we use a theorem of B. Neumann [Neu54] on covering groups by cosets (see the last section). The main theorem immediately implies the following fact about graphs.

\textbf{Corollary 1.} Let $\Gamma$ be a graph and let $K$ be a finite graph. Then

1) if $\Gamma$ contains a finite set of vertices $X$ such that each subgraph of $\Gamma$ isomorphic to $K$ has at least one vertex from $X$, then $\Gamma$ contains a finite set of vertices $Y$ invariant with respect to all automorphisms of $\Gamma$ and such that again each subgraph of $\Gamma$ isomorphic to $K$ has at least one vertex from $Y$; moreover, $|Y| \leq |X| \cdot$ (the number of vertices of $K$);

2) if $\Gamma$ contains a finite set of edges $X$ such that each subgraph of $\Gamma$ isomorphic to $K$ has at least one edge from $X$, then $\Gamma$ contains a finite set of edges $Y$ invariant with respect to all automorphisms of $\Gamma$ and such that again each subgraph of $\Gamma$ isomorphic to $K$ has at least one edge from $Y$; moreover, $|Y| \leq |X| \cdot$ (the number of edges of $K$);

The word \textit{graph} hereafter can be understood in any reasonable sense:

- a graph can be directed, undirected, or mixed,
- multiple edges and/or loops can be allowed or not allowed.

Of course, the word “isomorphism” (and “automorphism”) should be understood correspondingly, i.e. an isomorphism must preserve the direction of edges for directed graphs.

\textbf{Remark.} An analogue of Corollary 1 holds when there are finitely many “forbidden” finite graphs $K_1, \ldots, K_n$ (instead the single graph $K$). In this case, the inequality is $|Y| \leq |X| \cdot \max_i (\text{the number of vertices [edges] of } K_i)$.

The proof is easy: to obtain the assertion (about vertices) we apply the main theorem putting

$$U = \{\text{vertices of } \Gamma\}, \quad G = \text{Aut } (\Gamma), \quad \mathcal{F} = \left\{ \{\text{vertices of } S\} \mid S \text{ is a subgraph of } \Gamma \text{ isomorphic to one of } K_i \right\}.$$ 

In Sections 1 and 2, we discuss the sharpness of estimates from Corollary 1; the situation is the following:

- the both estimates from Corollary 1 are sharp in the sense that we cannot replace the functions $|X| \cdot \text{(number vertices [edges] graph } K)$ with smaller functions of $|X|$ and the number vertices [edges] of $K$;

\(^*\) See also paper [dT18], whose authors reproved the estimate $n^2$ apparently being unaware of [PSz02]. For finite groups $G$, this estimate was known earlier ([ChD89], see also [Is08], theorem 1.41). While the qualitative fact – if a group contains an abelian finite-index subgroup, then it contains a characteristic abelian finite-index subgroup – was known even earlier, see [KaM82].
- if we fix a graph $K$ and ask about the sharpness of the estimates for this given $K$, then the problem becomes more interesting:
- the class of graphs $K$ such that the estimate from Corollary 1(1) (about vertices) is sharp have a simple description (but some interesting questions remain open nevertheless), see the next section;
- as for Corollary 1(2) (about edges), we have more questions than answers; see Section 2.

In conclusion, note that there are situations where no Khukhro–Makarenko type theorems can be proved. For example, the existence of a finite-index subgroup (in a group) with the law $x^{2019} = 1$ does not imply the existence of a characteristic finite-index subgroup satisfying this law [KhKMM09].

Paper [KMi15] contains an amusing example of a boundary situation: on the one hand,

- if a graph can be made planar by removing finitely many edges, then such a finite set of edges can be chosen invariant with respect to all automorphisms of the graph;

on the other hand, no estimate is possible (i.e. there exists $n$ such that, for any $m$, there exists a graph that can be made planar by removing $n$ edges, but cannot be made planar by removing a set of edges consisting of less than $m$ edges and invariant with respect to all automorphisms of the graph).

Our Notation and conventions are mainly standard. Note only that the word graph have, except where otherwise indicated, six meanings specified above. The index of a subgroup $H$ of a group $G$ is denoted as $|G:H|$. The letter $\mathbb{Z}$ denotes the set of integers. The symbol $|X|$ means the cardinality of a set $X$.

1. Vertex representativeness

We say that the vertex representativeness $\Upsilon_v(K, \Gamma)$ of a graph $K$ in a graph $\Gamma$ is the minimal integer $n$ such that $\Gamma$ contains a set $X$ of $n$ vertices satisfying the following condition:

$$\text{each subgraph of } \Gamma \text{ isomorphic to } K \text{ contains a vertex from } X.$$  

Let us define the symmetric vertex representativeness $\Upsilon_v^\text{sym}(K, \Gamma)$ of a graph $K$ in a graph $\Gamma$ as the minimal integer $n$ such that $\Gamma$ contains an (Aut $\Gamma$)-invariant set $X$ of $n$ vertices satisfying (*)).

For instance, for the faces of the tetrahedron and cube, we have:

$$\Upsilon_v(K, \Gamma) = 2, \quad \Upsilon_v^\text{sym}(K, \Gamma) = 4$$

Clearly, $\Upsilon_v(K, \Gamma) \leq \Upsilon_v^\text{sym}(K, \Gamma)$. Corollary 1 says that $\Upsilon_v^\text{sym}(K, \Gamma) \leq \Upsilon_v(K, \Gamma) \cdot (\text{the number of vertices of } K)$.

The following assertion shows that, for a connected graphs $K$, this estimate is sharp.

A graph $K$ is called costly in the sense of symmetric vertex representativeness or simply costly (or vertex-costly) if $\forall m \in \mathbb{Z}$ there exists a graph $\Gamma_m$ such that $\Upsilon_v^\text{sym}(K, \Gamma_m) = \Upsilon_v(K, \Gamma_m) \cdot (\text{the number of vertices of } K)$.

Thus, $K$ is costly if the estimate from corollary 1(1) is sharp for $K$. We call a costly graph $K$ (vertex-)costly in a class of graphs $\mathcal{K}$ if the graphs $\Gamma_m$ in (***) can chosen from the class $\mathcal{K}$.

Theorem 1. A finite graph $K$ is costly if and only if it is connected. Moreover, any connected graph $K$ without hanging edges is costly in the class of connected graphs.

Proof. Note that

$$\text{any graph } K \text{ embeds into a vertex-transitive graph } \tilde{K} \text{ with the same number of vertices.}$$

Indeed, if the word graph means an undirected graph without multiple edges and loops, then we can take the complete graph as $\tilde{K}$; in other cases, this fact remains valid (we leave it to readers as an exercise, see graphs $K$ and $\tilde{K}$ in Figure 1).

If $K$ is connected, then we can take the disjoint union of $m$ copies of $\tilde{K}$ as the graph $\Gamma_m$. Indeed, to represent all subgraphs isomorphic to $K$ it is sufficient (and necessary) to take one vertex from each copy of $\tilde{K}$. Thus, $\Upsilon_v(K, \Gamma_m) = m$. The graph $\Gamma_m$ is vertex-transitive, therefore, $\Upsilon_v^\text{sym}(K, \Gamma_m) = mk$, where $k$ is the number of vertices of $K$.

To prove the assertion “Moreover”, we add to this graph $\Gamma_m$ chains of length $N$ joining each vertex of the $i$-th copy of $\tilde{K}$ with the corresponding vertex of the $(i+1)$-th copy of $\tilde{K}$, where $i \in \{1, \ldots, m-1\}$ and the integer $N$ (the same for all chains) is larger than the number of vertices of $K$, see Figure 1.
Again, \( \Upsilon_v(K, \Gamma_m) = m \), because to represent all subgraphs isomorphic to \( K \), we should mark one vertex from each copy of \( \bar{K} \) (because the graph \( K \) has no hanging edges and the number \( N \) is sufficiently large); \( \Upsilon^\text{sym}(K, \Gamma_m) = mk \), because the group \( \text{Aut} \Gamma_m \) acts transitively on vertices of each copy of \( \bar{K} \).

It remains to show that no disconnected graph \( K \) is costly. Let us prove slightly more:

\[
\Upsilon^\text{sym}_v(K, \Gamma) \leq k_1(\Upsilon_v(K, \Gamma) + k_2)
\]

if \( K = K_1 \sqcup K_2 \) where \( k_1 \) is the number of vertices of \( K_1 \), and \( k_2 \leq k_1 \) is the number vertices of \( K_2 \).

Let us mark \( \Upsilon_v(K, \Gamma) \) vertices in graph \( \Gamma \) in such a way that each subgraph isomorphic to \( K = K_1 \sqcup K_2 \) have a marked vertex. If \( \Gamma \) has a subgraph \( \bar{K}_2 \simeq K_2 \) without marked vertices, then it must intersect each subgraph isomorphic to \( K_1 \) without marked vertices. Therefore, when we mark all vertices of \( \bar{K}_2 \), we obtain that any subgraph of \( \Gamma \) isomorphic to \( K_1 \) has a marked vertex. Thus, \( \Upsilon_v(K_1, \Gamma) \leq \Upsilon_v(K, \Gamma) + k_2 \) and

\[
\Upsilon^\text{sym}_v(K, \Gamma) \leq \Upsilon^\text{sym}_v(K_1, \Gamma) \leq k_1 \Upsilon_v(K_1, \Gamma) \leq k_1(\Upsilon_v(K, \Gamma) + k_2) \quad \text{(where the next to last inequality is Corollary 1)}
\]
as required. If \( \Gamma \) has no subgraphs isomorphic to \( K_2 \) without marked vertices, then we obtain the inequality

\[
\Upsilon_v(K_2, \Gamma) \leq \Upsilon_v(K, \Gamma)
\]

(which actually is an equality) and

\[
\Upsilon^\text{sym}_v(K, \Gamma) \leq \Upsilon^\text{sym}_v(K_2, \Gamma) \leq k_2 \Upsilon_v(K_2, \Gamma) \leq k_2 \Upsilon_v(K, \Gamma) \leq k_1 \Upsilon_v(K, \Gamma) < k_1(\Upsilon_v(K, \Gamma) + k_2)
\]
as required. This completes the proof.

**Question 1.** Is any finite connected graph costly in the class of connected graphs?

According to Theorem 1, all finite connected graphs without vertices of degree one are costly in the class of connected graphs. Chains are also costly in the class of connected graphs; we can take polygons (cycles) as \( \Gamma_m \) in this case.

The four-vertex graphs *tailed triangle* and *claw* shown in Figure 2 on the left are also costly in the class of connected graphs; the corresponding graphs \( \Gamma_m \) are shown in Figure 2 on the right. To be more precise, this figure shows an infinite vertex-transitive graph where one-fourth of the vertices are marked in such a way that each tailed triangle and each claw have a marked vertex; this pattern on the plane is twice periodic, hence, we can obtain arbitrarily large finite patterns on the torus with the same property (i.e. vertex-transitive graphs in which one quarter of the vertices are marked, and each tailed triangle and each claw has a marked vertex).
Thus,

\textit{all connected graphs with at most four vertices are costly in the class of connected graphs.}

(Strictly speaking, Figure 2 proves this assertion only if the word graph means an undirected graph without loops and multiple edges; but an obvious modification of this figure makes it possible to prove the assertion in other cases.)

Nevertheless, we conjecture that the answer to Question 1 is negative, and a counter-example is probably the five-vertex graph \( D_5 \) shown on Figure 3 (i.e., hypothetically \( \Upsilon^\text{sym}_v(D_5, \Gamma) < 5\Upsilon_v(D_5, \Gamma) \) for any connected graph \( \Gamma \)

with sufficiently large representativeness \( \Upsilon_v(D_5, \Gamma) \)).
A partial confirmation of this conjecture is the following fact.

**Theorem 2.** The graph \( D_5 \) is not costly in the class vertex-transitive connected graphs. More precisely, if \( \Gamma \supseteq D_5 \) is a vertex-transitive undirected connected graph with more than five vertices, and the representativeness \( \Upsilon_v(D_5, \Gamma) \) is finite, then \( \Upsilon_{\sym}(D_5, \Gamma) < 5\Upsilon_v(D_5, \Gamma) \).

(A graph \( \Gamma \) is called vertex-transitive if its automorphism group acts transitively on the set of vertices, i.e., for any two vertices \( u \) and \( v \), there exists an automorphism \( \phi \) of the graph such that \( \phi(u) = v \).)

**Proof.** First note that we can (and shall) assume that \( \Gamma \) has no loops and multiple edges. Indeed, if we remove all loops, and replace every bunch of multiple edges with a single edge, then the conditions of the theorem remain fulfilled, and the assertion for the obtained graph implies the assertion for the initial graph.

Note also that \( \Gamma \) must be finite, because by Corollary 1 the finiteness of \( \Upsilon_v(D_5, \Gamma) \) implies the finiteness of \( \Upsilon_{\sym}(D_5, \Gamma) \), which is equal to the number of vertices of the graph by virtue of transitivity.

Suppose that the degree of each vertex of \( \Gamma \) is \( k \geq 3 \) (if \( k < 3 \), then we have nothing to prove). Let us mark a finite set \( X \) of vertices of \( \Gamma \) (where \( |X| = \Upsilon_v(D_5, \Gamma) \)) in such a way that each subgraph isomorphic to \( D_5 \) contains a marked vertex. This means that the graph \( \Gamma' \) obtained from \( \Gamma \) by deleting all marked vertices and edges incident to them contains no subgraphs isomorphic to \( D_5 \). It is easy to classify such graphs.

**Lemma 1.** A finite connected undirected graph without loops, multiple edges, and subgraphs isomorphic to \( D_5 \) is either
- an \( l \)-gon (cycle) (with \( l \geq 3 \)),
- a chain (with \( l \geq 0 \) edges),
- a star \( K_{1,l} \) (with \( l \geq 3 \) edges),
- or a connected four-vertex graph.

Figure 4 shows (representatives of) the three infinite series; Figure 5 shows three remaining four-vertex graph (do not pay attention to black vertices and edges incident to them for now).

**Proof.** If a connected finite graph contains no vertices of degree higher than two, then this graph is either a polygon or a chain. If there is a vertex \( v \) of degree three or higher, then the neighbouring vertices can be joined by edges only between themselves (and \( v \)), because otherwise we obtain \( D_5 \) as a subgraph. Thus, all vertices of the graph, except \( v \), are neighbours of \( v \) (by virtue of connectedness).
If no neighbours of $v$ are joined by edges, then the graph is a star.

If an edge joins two neighbours $u$ and $w$ of $v$, then at most one additional vertex (except $v$, $u$, and $w$) can exist, because otherwise we again obtain a subgraph isomorphic to $D_5$. Thus, our graph has at most four vertices, that completes the proof of the lemma.

Proceeding with the proof of the theorem, let us calculate the number $p$ of edges joining marked vertices with non-marked ones. On the one hand, $p \leq k|X|$ (because each marked vertex has degree $k$), and the equality is achieved only when no two marked vertices are joined by an edge. On the other hand, $p \geq (k - 3)|Z|$, where $Z$ is the set of non-marked vertices, because

\[
\text{the number of edges joining a component of } \Gamma' \text{ with marked vertices} \geq k - 3
\]

\[
\text{the number of vertices in this component}
\]

(see Figs. 4 and 5, where marked vertices are black and $k = 4$), and the equality is achieved only on components which are complete graphs with four vertices.

Thus, $|X| \geq p \geq (k - 3)|Z|$, i.e. $|X| \geq (1 - \frac{3}{k})|Z|$. This means that either

1) $|X| > \frac{1}{2}|Z|$,  
2) $k = 3$,  
3) or $k = 4$ and $|X| = \frac{1}{2}|Z|$; as was mentioned, this is possible only when all components of $\Gamma'$ are complete graphs on four vertices, and no two marked vertices of $\Gamma$ are joined by an edge.

Consider these cases.

1) In this case, we have $\Upsilon^*(D_5, \Gamma) \leq |X| + |Z| < |X| + 4|X| = 5|X| = 5\Upsilon_v(D_5, \Gamma)$ as required.

2) In this case, a component of $\Gamma'$ cannot be a complete graph on four vertices shown in Figure 5 on the left (because the degree of each vertex of $\Gamma$ is three). Neither can a component of $\Gamma'$ be a diamond, shown on Figure 5 in the centre; indeed, the neighbourhood of the vertex $u$ (in $\Gamma$) is a chain (in this case), while the neighbourhood of the vertex $v$ is a disconnected graph; this is impossible in a vertex-transitive graph. Recall that the neighbourhood of a vertex $v$ is the graph consisting of vertices neighbouring to $v$ and all edges between them. Note that, in the case under consideration, the degree of each vertex is three (not four, as on Figure 5).

Thus, the argument that have led us to the inequality $k|X| \geq (k - 3)|Z|$ are modified as follows:

\[
\text{the number of edges joining a component of } \Gamma' \text{ with marked vertices} \geq 1
\]

\[
\text{the number of vertices in this component}
\]

(there the equality is achieved on components shown in Figure 5 on the right and on cycles, Figure 4). This implies that $|X| \geq |Z|$, and we come to case 1).

3) The neighbourhood of a marked vertex is a disjoint union of several cliques (consisting of non-marked vertices). By virtue of transitivity, the neighbourhood of a non-marked vertex has the same form. Since the neighbourhood of a non-marked vertex must contain a triangle consisting of non-marked vertices, we come to the conclusion that either

a) the neighbourhood of each vertex is a complete graph on four vertices

b) no triangle contains a marked vertex.

In Case a) the graph $\Gamma$ must be the complete graph with five vertices that completes the proof. Case b) is impossible in a vertex-transitive graph containing a cliques of order four.

2. Edge representativeness

We say that the edge representativeness $\Upsilon_v(K, \Gamma)$ of a graph $K$ in a graph $\Gamma$ is the minimal integer $n$ such that $\Gamma$ contains a set $X$ of $n$ edges satisfying the following condition:

\[
\text{each subgraph of } \Gamma \text{ isomorphic to } K \text{ contains an edge from } X. \quad (***)
\]

Let us define the symmetric edge representativeness $\Upsilon^*_v(K, \Gamma)$ of a graph $K$ in a graph $\Gamma$ as the minimal integer $n$ such that $\Gamma$ contains an $(\Aut \Gamma)$-invariant set $X$ of $n$ edges satisfying $(***)$.

Clearly, $\Upsilon_v(K, \Gamma) \leq \Upsilon^*_v(K, \Gamma)$. Corollary 1 says that $\Upsilon^*_v(K, \Gamma) \leq \Upsilon_v(K, \Gamma) \cdot (\text{the number of edges of } K)$. We call a graph $K$ costly in the sense of symmetric edge representativeness or simply edge-costly if

\[
\forall m \in \mathbb{Z} \text{ there exists a graph } \Gamma_m \text{ such that } \Upsilon^*_v(K, \Gamma_m) = \Upsilon_v(K, \Gamma_m) \cdot (\text{the number of edges of } K) \geq m. \quad (****)
\]

Thus, a graph $K$ is edge-costly if the estimate from Corollary 1(2) is sharp for $K$. We call an edge-costly graph $K$ edge-costly in the class of graphs $\mathcal{K}$ if the graphs $\Gamma_m$ in $(****)$ can chosen from the class $\mathcal{K}$.  

7
Proposition 1. Any finite edge-transitive connected graph $K$ is edge-costly in the class of connected graphs.

Proof. If $K$ has no hanging edges (i.e. no vertices of degree one), then we can do pretty much the same as in the vertex case. Take as the graph $\Gamma_m$ the disjoint union of $m$ copies of $K$. This shows that $K$ is edge-costly (in the class of all graphs yet). Indeed, to destroy all subgraphs isomorphic to $K$, it is sufficient (and necessary) to remove one edge from each copy of $K$. Thus, $\Upsilon_e(K, \Gamma_m) = m$. The graph $\Gamma_m$ is edge-transitive, therefore, $\Upsilon_{e sym}(K, \Gamma_m) = mk$, where $k$ is the number of edges of $K$.

To make the graph $\Gamma_m$ connected, we add to $\Gamma_m$ chains of length $N$ joining each vertex of the $i$-th copy of $K$ with the corresponding vertex of the $(i+1)$-th copy of $K$, where $i \in \{1, \ldots, m-1\}$, and the integer $N$ (the same for all chains) is larger than the number of vertices of $K$.

Again, $\Upsilon_e(K, \Gamma_m) = m$, because to destroy the subgraphs isomorphic to $K$ we can remove one vertex from each copy of $K$ (as $K$ has no hanging edges, and the number $N$ is large enough). Now, $\Upsilon_{e sym}(K, \Gamma_m) = mk$, because the group $\text{Aut} \Gamma_m$ acts transitively on the edges of each copy of $K$.

If the edge-transitive connected graph $K$ has hanging edges, then all edges are hanging and $K$ is a star.

If the edges are directed, then (by the edge-transitivity) the star $K = K_{1,l}$ has one source and $l$ sinks or vice versa. Assuming that there is one source, take the complete bipartite graph $K_{m,l} = \Gamma_m$ in which all edge are directed from the first part (consisting of $m$ vertices) to the second part (consisting of $l$ vertices). Clearly, $\Upsilon_e(K, \Gamma_m) = m$ and $\Upsilon_{e sym}(K, \Gamma_m) = ml$ (because the graph $\Gamma_m$ is edge-transitive).

It remains to consider the case, where $K = K_{1,l}$ is an undirected star.

If $l$ equals one or two, then we can take the cycle of length $2m$ as $\Gamma_m$.

If $l = 3$, then we can take “honeycombs” as $\Gamma_m$. “Honeycombs” (Fig. 6) is an infinite graph, in which one third of the edges are marked (vertical edge on Figure 6), and each subgraph isomorphic to the claw $K = K_{1,3}$ has a marked edge. To make this graph finite, we note that the “honeycombs” is a doubly periodic pattern on the plane; therefore, we can draw an arbitrarily large finite graph on the torus with the same properties (i.e. an edge-transitive graph, in which one third of the edges are marked, and each claw has a marked edge).

If $l > 3$, then we can act as shown in Figure 6 on the right: $\Gamma_m$ consists of $m$ copies of the star $K$ joined by edges; $\Upsilon_e(K, \Gamma_m) = m$ and $\Upsilon_{e sym}(K, \Gamma_m) = ml$. This completes the proof.

![Fig. 6](image)

The exact analogue of Theorem 1 is not true:

there exist disconnected edge-costly graphs and even disconnected graphs edge-costly in the class of connected graphs.

Indeed, if take the disjoint union of two edges as $K$, and the complete biparite graph $K_{2,m}$ as $\Gamma$, then $\Upsilon_e(K, \Gamma) = m$ (because, to destroy all subgraphs of $\Gamma$ isomorphic to $K$, we can remove all edges incident to a vertex of degree $m$, see Figure 7), and $\Upsilon_{e sym}(K, \Gamma) = 2m$ (because the graph is edge-transitive).
Are all graphs edge-costly? No, but we have only trivial examples: if the word “graph” means a directed graph without multiple edges but loops are allowed, then the graph $K$ shown in Figure 8 is not edge-costly by a trivial reason: a common edge of two subgraphs of this form in any graph $\Gamma$ must be a loop, therefore $Y_{e}^{\text{sym}}(K,\Gamma) = Y_e(K,\Gamma)$ (because, if we want to destroy all subgraphs of this form, it is more efficient to remove only loops).

We do not know, e.g., an answer to the following question.

**Question 2.** Let the word *graph* mean an undirected graph without loops and multiple edges. Does there exist a finite non-edge-costly graph? Can such a graph be connected? Does there exist a finite graph which is not edge-costly in the class of connected graphs? Can such a graph be connected?

If the word *graph* means a directed graph without loops and multiple edges, the similar questions are also open.

### 3. Proof of the main theorem

Put $m = \max_{F \in \mathcal{F}} |F|$ and consider the following set $Y = \left\{ y \in U \mid |Gy \cap X| \geq \frac{1}{m}|Gy| \right\}$ (in particular, $Y$ contains no points with infinite orbits). Clearly, this set is $G$-invariant, and $|Y| \leq m |X|$ (because, for each orbit $Gu$, we have $|Gu \cap Y| \leq m |Gu \cap X|$).

It remains to show, that $Y$ is a system of representatives for $\mathcal{F}$. Take a set $F \in \mathcal{F}$. Each set $gF$ (where $g \in G$) belongs to $\mathcal{F}$ (because the family $\mathcal{F}$ is $G$-invariant) and, hence, intersects $X$. Therefore,

$$G = \bigcup_{f \in F} \{g \in G \mid gf \in X\}.$$

Each set $\{g \in G \mid gf \in X\}$ is either empty or a finite union of left cosets of stabilisers $\text{St}(f)$ of points:

$$\{g \in G \mid gf \in X\} = \bigcup_{x \in X} \{g \in G \mid gf = x\} = \bigcup_{x \in X \cap Gf} g_x \cdot \text{St}(f), \quad \text{where } g_x \in G \text{ are fixed such that } g_x f = x.$$

Thus, we obtain a decomposition of the group $G$ into a finite union of left cosets of some subgroups. B. Neumann’s theorem ([Neu54], Proposition 4.5) says that

if a group $G$ is covered by finitely many cosets of subgroups: $G = g_1 G_1 \cup \ldots \cup g_n G_n$, then

$$\sum \frac{1}{|G \setminus G_i|} \geq 1 \quad (\text{where the inverse of an infinite cardinal is zero}).$$
Therefore, (taking into account that the index of a stabiliser equals the length of the corresponding orbit) we obtain

\[
1 \leq \sum_{f \in F} \frac{1}{|G : \text{St}(f)|} \cdot |Gf \cap X| = \sum_{f \in F} \frac{|Gf \cap X|}{|Gf|}.
\]

Since the number of terms equals $|F| \leq m$, some term is at least $1/m$, i.e., $|Gf \cap X|/|Gf| \geq 1/m$, and, hence, $f \in Y$ (by the definition of $Y$). This completes the proof.

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