THE CLASSIFICATION PROBLEM FOR EXTENSIONS OF TORSION ABELIAN GROUPS

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Abstract. Given countable abelian groups \( C, A \), with \( C \) torsion, we compute the potential Borel complexity class of the classification problem \( \mathcal{R}_{\text{Ext}(C, A)} \) for extensions of \( C \) by \( A \). In particular, we show that such a problem can have arbitrarily high potential complexity. Furthermore, if \( \Gamma \) is one of the possible potential Borel complexity classes of the isomorphism relation for a Borel class of countable first-order structures, then there exist countable torsion groups \( C, A \) such that \( \mathcal{R}_{\text{Ext}(C, A)} \) has potential Borel complexity class \( \Gamma \).

1. Introduction

In this paper we study the complexity of classifying extensions of two given countable abelian groups up to equivalence. (In what follows, we assume all the groups to be abelian and additively denoted.) Suppose that \( A \) and \( C \) are countable groups. An extension of \( C \) by \( A \) is a short exact sequence

\[
0 \to A \to X \to C \to 0
\]

in the abelian category of abelian groups. Two such extensions

\[
0 \to A \to X \to C \to 0
\]

and

\[
0 \to A \to X' \to C \to 0
\]

are equivalent if there exists a group isomorphism \( \psi : X \to X' \) that makes the following diagram commute.

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \psi \\
A & \longrightarrow & X'
\end{array}
\]

We will explain below that one can regard the space \( \text{Ext}(C, A) \) of extensions of \( C \) by \( A \) as a Polish space. Thus, the relation of equivalence of extensions of \( C \) by \( A \) is an equivalence relation \( \mathcal{R}_{\text{Ext}(C, A)} \) on \( \text{Ext}(C, A) \).

We will study the relations \( \mathcal{R}_{\text{Ext}(C, A)} \) from the perspective of Borel complexity theory. This framework allows one to compare the complexity of different classification problems in mathematics. A classification problem is identified with a pair \((X, \mathcal{R})\) where \( X \) is a Polish space and \( \mathcal{R} \) is an equivalence relation on \( X \), which we will assume to be Borel as a subset of \( X \times X \). The notion of Borel reducibility captures the idea that a classification problem is at most as complicated as another one.

Definition 1.1. Suppose that \((X, \mathcal{R})\) and \((Y, \mathcal{S})\) are Borel equivalence relations on Polish spaces. A Borel reduction from \( \mathcal{R} \) to \( \mathcal{S} \) is a Borel function \( f : X \to Y \) satisfying \( f(x) \mathcal{S}\, f(y) \iff x \mathcal{R} \, y \) for every \( x, y \in X \). We say that \( \mathcal{R} \) is Borel reducible to \( \mathcal{S} \), and write \( \mathcal{R} \leq \mathcal{S} \), if there is a Borel reduction from \( \mathcal{R} \) to \( \mathcal{S} \). We say that \( \mathcal{R} \) is Borel bireducible to \( \mathcal{S} \) if \( \mathcal{R} \leq \mathcal{S} \) and \( \mathcal{S} \leq \mathcal{R} \).

While Borel reducibility provides a way to compare different classification problems, some important Borel equivalence relations serve as benchmarks to calibrate the complexity of other classification problems. Particularly, an equivalence relation \( \mathcal{R} \) is:

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smooth if it is Borel reducible to the relation of equality on some Polish space;
• essentially hyperfinite if it is Borel reducible to the relation $E_0$ of tail equivalence of binary sequences;
• essentially countable if it is Borel reducible to a Borel equivalence relation whose equivalence classes are countable.

The relation $E_0$ is by definition the Borel equivalence relation on $\mathcal{C} := \{0, 1\}^\omega$ defined by setting, for $(x_i), (y_i) \in \mathcal{C}$, $(x_i) E_0(y_i) \iff \exists n \forall i \geq n, x_i = y_i$. We then let $E_0^\omega$ to be the equivalence relation on $\mathcal{C}^\omega$ defined by setting, for $(x_i), (y_i) \in \mathcal{C}^\omega$, $(x_i) E_0^\omega(y_i) \iff \forall i \in \omega \exists n (x_i E_0 y_i)$.

As $\mathcal{R}_{\text{Ext}(C,A)}$ is a Borel equivalence relation, by the Glimm–Effros Dichotomy [HKL90], $\mathcal{R}_{\text{Ext}(C,A)}$ is not smooth if and only if $E_0 \leq \mathcal{R}_{\text{Ext}(C,A)}$. Furthermore, $\mathcal{R}_{\text{Ext}(C,A)}$ is the orbit equivalence relation induced by a continuous action of a non-archimedean abelian Polish group. Therefore, by [DG17, Theorem 6.1, Corollary 6.3] and [HK97, Theorem 8.1], $\mathcal{R}_{\text{Ext}(C,A)}$ is not essentially hyperfinite if and only if it is not essentially countable if and only if $E_0^\omega \leq \mathcal{R}_{\text{Ext}(C,A)}$.

Recall that an abelian group $A$ has a largest divisible subgroup $D(A)$, which is a direct summand of $A$. We say that $A$ is reduced if $D(A) = 0$, and bounded if $nA = 0$ for some $n \geq 1$, where $nA = \{nx : x \in A\}$. For a prime $p$, the $p$-primary subgroup of $A$ is the subgroup consisting of elements of order $p^n$ for some $n \in \omega$. The Ulm subgroups $u_n(A) = A^{\alpha}$ for $\alpha < \omega_1$ are defined recursively by setting $A^0 = A$ and $A^{\alpha} = \bigcap_{p < \alpha} \bigcap_{n \in \omega} nA^{\beta}$ for $0 < \alpha < \omega_1$.

**Theorem 1.2.** Suppose that $C, A$ are countable abelian groups, with torsion. For a prime number $p$, we let $A_p \subseteq A / D(A)$ be the $p$-primary subgroup of $A / D(A)$ and $C_p \subseteq C$ be the $p$-primary subgroup of $C$.

1. $\mathcal{R}_{\text{Ext}(C,A)}$ is smooth if and only if, for every prime number $p$, either $u_1(C_p) = 0$ or $A_p$ is bounded;
2. $\mathcal{R}_{\text{Ext}(C,A)}$ is essentially hyperfinite if and only if for every prime number $p$, either $u_2(C_p) = 0$ or $u_1(A_p) = 0$, and the sum of $|u_1(C_p)|$ ranging over all prime numbers $p$ such that $A_p$ is unbounded is finite;
3. $\mathcal{R}_{\text{Ext}(C,A)} \leq E_0^\omega$ if and only if, for every prime number $p$, one of the following holds: (a) $u_2(C_p) = 0$; (b) $u_1(A_p) = 0$; (c) $u_3(C_p) = 0$ and $u_1(A_p)$ is bounded.

We will obtain a generalization of Theorem 1.2 that completely determines the potential complexity of the relation $\mathcal{R}_{\text{Ext}(C,A)}$. The notion of potential complexity of a Borel equivalence relation on a Polish space was introduced by Louveau in [Lou94]; see also [Kec02, HK96, HKL98] and [Gao09, Definition 12.5.1]. A complexity class of sets $\Gamma$ is a function $X \mapsto \Gamma(X)$ that assigns to each Polish space $X$ a collection $\Gamma(X)$ of Borel subsets of $X$, such that if $f : X \to Y$ is a continuous function and $A \in \Gamma(Y)$ then $f^{-1}(A) \in \Gamma(X)$. If $A \in \Gamma(X)$ we also say that $A$ is $\Gamma$ in $X$.

Following [HKL98], for a complexity class $\Gamma$, we let $D(\Gamma)$ be the complexity class consisting of differences between sets in $\Gamma$, and $\Gamma$ be the dual complexity class consisting of complements of elements of $\Gamma$. We denote by $D(\Gamma)$ the dual class of $D(\Gamma)$. We will mainly be interested in the complexity classes $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $D(\Pi^0_\alpha)$ for $\alpha \in \omega_1$; see [Kec95, Section 11.1B].

**Definition 1.3.** Let $\Gamma$ be a complexity class. Suppose that $(X, \mathcal{R})$ is a Borel equivalence relation on a Polish space. Then $\mathcal{R}$ is potentially $\Gamma$ if there exists a Borel equivalence relation $(Y, \mathcal{S})$ such that $\mathcal{S} \in \Gamma(Y \times Y)$ and $\mathcal{R}$ is Borel reducible to $\mathcal{S}$.

The concept of potential complexity affords us to measure the complexity of an equivalence relation. The benchmarks considered above can be expressed in terms of potential complexity as follows. Let $(X, \mathcal{R})$ be a Borel equivalence relation on a Polish space. Then $\mathcal{R}$ is smooth if and only if it is potentially $\Pi^0_3$ [Gao09, Lemma 12.5.3]. Suppose that $\mathcal{R}$ is the orbit equivalence relation associated with a continuous action of a non-archimedean abelian Polish group. Then $\mathcal{R}$ is essentially hyperfinite if and only if $\mathcal{R}$ is potentially $\Sigma^0_3$ and only if $\mathcal{R}$ is potentially $\Sigma^0_3$ by [HK96, Theorem 3.8], [HKL98, Theorem 8.3], and [DG17, Theorem 6.1]; see also [Gao09, Theorem 12.5.7]. Furthermore, $\mathcal{R} \leq E_0^\omega$ if and only if it is potentially $\Pi^0_3$ [All20, Corollary 6.11].

**Definition 1.4.** Let $\Gamma$ be one of the classes $\Pi^0_\mu$, $\Sigma^0_\mu$, $D(\Pi^0_\mu)$, $D(\Sigma^0_\mu)$, $\bar{D}(\Pi^0_\mu)$, $\bar{D}(\Sigma^0_\mu)$ for some countable ordinal $\mu \geq 1$. Let $(X, \mathcal{R})$ be a Borel equivalence relation on a Polish space. Then $\Gamma$ is the potential class of $\mathcal{R}$ if $\mathcal{R}$ is potentially $\Gamma$ and $\mathcal{R}$ is not potentially $\Gamma$. 


The following result imposes restrictions on the possible values of the potential class of an orbit equivalence relation associated with a continuous action of a non-Archimedean Polish group on a Polish space; see [HKL98, Theorem 1].

**Proposition 1.5** (Hjorth–Kechris–Louveau). Let \( \mathcal{R} \) be the orbit equivalence relation associated with a continuous action of a non-Archimedean Polish group on a Polish space. Then the potential complexity class of \( \mathcal{R} \) is one of the following: \( \Pi^0_{1 + \lambda}, \Sigma^0_{1 + \lambda + 1}, D(\Pi^0_{1 + \lambda + n + 2}), \) and \( \Pi^0_{1 + \lambda + n + 2} \) for \( \lambda < \omega_1 \) either zero or a limit ordinal, and \( n < \omega \).

In view of the above remarks, the following result can be seen as an extension of Theorem 1.2. In particular, Proposition 1.5 shows that each one of the complexity classes in Proposition 1.5 arises the potential complexity class of \( \mathcal{R}_{\text{Ext}(C,A)} \) for some countable torsion groups \( A \) and \( C \).

To simplify the statement, we will assume that, for some prime number \( p \), \( u_1(C_p) \) is nonzero and \( A_p \) is unbounded. If this fails, then Theorem 1.2 applies. We denote by \( \mathbb{P} \) the set of primes.

**Theorem 1.6.** Suppose that \( C,A \) are countable abelian groups, where \( C \) is a torsion group. For \( p \in \mathbb{P} \), let \( A_p \subseteq A/D(A) \) be the \( p \)-primary subgroup of \( A/D(A) \) and \( C_p \subseteq C \) be the \( p \)-primary subgroup of \( C \). Assume that, for some \( p \in \mathbb{P} \), \( u_1(C_p) \) is nonzero and \( A_p \) is unbounded. For every \( p \in \mathbb{P} \), let \( \mu_p \) be the least countable ordinal such that either \( C^\mu_p = 0 \) or \( \mu_p \) is the successor of \( \mu_p - 1 \) and \( A^{\mu_p - 1}_p = 0 \). Set \( \mu := \sup_{p \in \mathbb{P}} \mu_p \). If \( \mu \) is the successor of \( \mu - 1 \), set

\[
\mathbb{P}_\mu = \{ p \in \mathbb{P} : \mu_p = \mu \}
\]

and

\[
W := \sum \{|C^\mu_p - 1| : p \in \mathbb{P}_\mu \}.
\]

If furthermore \( \mu - 1 \) is the successor of \( \mu - 2 \), set

\[
w := \sum \{|C^\mu_p - 1| : p \in \mathbb{P}_\mu, A^{\mu_p - 2}_p \text{ is unbounded} \}.
\]

Then we have that:

1. \( \Pi^0_0 \) is the complexity class of \( \mathcal{R}_{\text{Ext}(C,A)} \) if and only if either (a) \( \mu \) is a limit ordinal, or (b) \( \mu = 1 + \lambda + n \) where \( \lambda \) is either zero or limit, \( 2 \leq n < \omega, \) and \( w = 0 \);
2. \( \Sigma^0_0 \) is the complexity class of \( \mathcal{R}_{\text{Ext}(C,A)} \) if and only if \( \mu = 1 + \lambda + 1 \) where \( \lambda \) is either zero or limit and \( W < \infty \);
3. \( D(\Pi^0_\mu) \) is the complexity class of \( \mathcal{R}_{\text{Ext}(C,A)} \) if and only if \( \mu = 1 + \lambda + n \) where \( \lambda \) is either zero or limit, \( 2 \leq n < \omega, \) and \( 0 < w < \infty \);
4. \( \Pi^0_{\mu + 1} \) is the complexity class of \( \mathcal{R}_{\text{Ext}(C,A)} \) if and only if either (a) \( \mu = 1 + \lambda + 1 \) where \( \lambda \) is either zero or limit and \( W = \infty \), or (b) \( \mu = 1 + \lambda + n \) where \( \lambda \) is either zero or limit and \( 2 \leq n < \omega \) and \( w = \infty \).

In particular, \( \mathcal{R}_{\text{Ext}(C,A)} \) is \( \Pi^0_{\mu + 1} \) and not \( \Pi^0_\alpha \) for \( \alpha < \mu \).

**Corollary 1.7.** Let \( \mathbb{Z}(p^{\infty}) \) be the Prüfer \( p \)-group. Suppose that \( A \) is a countable unbounded reduced \( p \)-group. Let \( \beta \) be the least countable ordinal such that \( A^\beta \) is bounded. Then \( \Pi^0_{\beta + 2} \) is the complexity class of \( \mathcal{R}_{\text{Ext}(\mathbb{Z}(p^{\infty}), A)} \).

For countable abelian groups \( C,A \), the group \( \text{Ext}(C,A) \) classifying extensions of \( C \) by \( A \) can be seen as a group with a (non-Archimedean) Polish cover in the sense of [BLP20]. In the proof of Theorem 1.6 we will apply tools from the theory of groups with a Polish cover and their (non-Archimedean) Polishable subgroups as developed in [BLP20, Lup22a, Lup22b]. In particular, we will use the fact that groups with a (non-Archimedean) Polish cover form an abelian category [Lup22b, Section 11], whose morphisms are the Borel-definable group homomorphisms. (Recall that we are assuming all the groups to be abelian.) Furthermore, each group with a Polish cover is endowed with a canonical chain of Polishable subgroups, called Solecki subgroups [Lup22b, Section 6]. It is meaningful to define the Borel complexity class of a Polishable subgroup [Lup22b, Proposition 5.5], and it is possible to compute the Borel complexity class of \( \{ 0 \} \) in terms of the Solecki subgroups [Lup22b, Theorem 6.4].

The main ingredient in the proof of Theorem 1.6 and Theorem 1.2 is that, for groups \( C \) and \( A \) as in the statement, the \( (1 + \alpha) \)-th Ulm subgroup of \( \text{Ext}(C,A) \) is equal to the \( \alpha \)-th Solecki subgroup of \( \text{Ext}(C,A) \) for \( \alpha < \omega_1 \). This is obtained by applying homological results about the functors \( C \mapsto C^{\alpha} \) due to Dünkel [Nun67].

The rest of this paper is divided into three sections. Section 2 presents the notions of Borel-definable set and Borel-definable group, group with a Polish cover, and recalls in this context the notions of Polishable subgroup, Solecki subgroup, and their complexity, as well as other notions and results from [Lup22a, Lup22b]. Section 3 recalls
the definition of the group Ext and how it can be regarded as a group with a Polish cover, and presents in this context the Borel-definable content of the results about cotorsion functors from [Num67]. These results are used in Section 4 to describe the Solecki subgroups of Ext in terms of the Ulm subgroups, and thus establish Theorem 1.6 and Theorem 1.2.

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2. Groups with a Polish cover and their Solecki subgroups

In this section, we recall the notion of Borel-definable set and Borel-definable group from [Lup20b, Lup20a], and the notion of group with a Polish cover from [BLP20].

2.1. Definable sets and groups. A Borel-definable set is a pair $(\hat{X}, E)$ where $\hat{X}$ is a standard Borel space and $E$ is a “well-behaved” equivalence relation on $\hat{X}$. Precisely, we require that $E$ be Borel (as a subset of $\hat{X} \times \hat{X}$) and idealistic, in the sense of the following definition.

Definition 2.1. Suppose that $X$ is a standard Borel space, and $E$ is an equivalence relation on $X$. We say $E$ is idealistic if there exist a Borel function $s : X \to X$ and an assignment $C \to \mathcal{F}_C$ mapping each $E$-class $C$ to a $\sigma$-filter $\mathcal{F}_C$ on $C$ such that $s(x) E x$ for every $x \in X$, and for every Borel subset $A \subseteq X \times X$,

$$A_{s,F} := \{ x \in X : \{ x' \in [x]_E : (s(x), x') \in A \} \in \mathcal{F}_{[x]_E} \}$$

is a Borel subset of $X$.

The notion of idealistic equivalence relation as in Definition 2.1 is slightly more generous than the definition usually considered in the literature, where the function $s : X \to X$ is required to be the identity.

We denote a Borel-definable set $(\hat{X}, E)$ by $X = \hat{X}/E$, as we think of it as an explicit presentation of the set $X$ as a quotient of the Polish space $\hat{X}$ by the well-behaved equivalence relation $E$. A standard Borel space $\hat{X}$ is, in particular, a Borel-definable set where $E$ is the identity relation.

If $\hat{X}$ is a standard Borel space and $E$ is the orbit equivalence relation associated with a Borel action of a Polish group on $\hat{X}$, then $E$ is idealistic [Gao09, Proposition 5.4.10]. Thus, if $E$ is furthermore Borel, then $\hat{X}/E$ is a definable set.

A function $f : X \to Y$ between Borel-definable sets $X = \hat{X}/E$ and $Y = \hat{Y}/F$ is Borel-definable if its lifted graph

$$\hat{f} := \{ (x, y) \in \hat{X} \times \hat{Y} : f([x]_E) = [y]_F \}$$

is a Borel subset of $\hat{X} \times \hat{Y}$ or, equivalently, if there exists a Borel function $\hat{f} : \hat{X} \to \hat{Y}$ that is a lift of $f$, in the sense that $f([x]_E) = [\hat{f}(x)]_F$ for every $x \in X$. The fact that these two definitions are equivalent follows from the version of the selection theorem [Kec95, Theorem 18.6] presented in the proof of [KM16, Lemma 3.7]. We regard Borel-definable sets as the objects of a category with Borel-definable functions as morphisms. Notice that standard Borel spaces form a full subcategory of the category of Borel-definable sets. It follows from results in [KM16, MR12] that many fundamental facts concerning standard Borel spaces admit generalizations to Borel-definable sets; see [Lup20a, Lup22b] for more details.

It is sometimes necessary to consider the more generous notion of Borel-semidefinable set $X = \hat{X}/E$, where $E$ is an analytic equivalence relation that is not necessarily Borel or idealistic. An useful fact is that if $X = \hat{X}/E$ is a Borel-definable set, $Y = \hat{Y}/F$ is a semidefinable set, and $f : X \to Y$ is a Borel-definable bijection, then $Y$ is a Borel-definable set, and the inverse function $f^{-1} : Y \to X$ is Borel-definable; see [Lup20a, Corollary 1.14].

2.2. Groups with a Polish cover. We now present the notion of group with a Polish cover introduced in [BLP20]. Recall that we are assuming all the groups to be abelian. A group with a Polish cover is a group $G = \hat{G}/N$ explicitly presented as the quotient of a Polish group $\hat{G}$ by a subgroup $N$ of $\hat{G}$ that is Polishable. This means that $N$ is the image of a continuous group homomorphism $H \to \hat{G}$ for some Polish group $H$ or, equivalently, $N$ is a Polish group with respect to some topology whose open sets are Borel in $\hat{G}$. In particular, this implies that $N$ is Borel in $\hat{G}$.

A Polish group $G$ is a particular instance of a group with a Polish cover $\hat{G}/N$ where $G = \hat{G}$ and $N = \{0\}$. A group with a Polish cover is, in particular, a Borel-definable set by [Gao09, Proposition 5.4.10]. Thus, a group
homomorphism \( \varphi : G \to H \) between groups with a Polish cover \( G = \hat{G}/N \) and \( H = \hat{H}/M \) is Borel-definable if it is induced by a Borel function \( \hat{\varphi} : \hat{G} \to \hat{H} \). Notice that \( \hat{\varphi} \) is not required to be a group homomorphism.

We regard groups with a Polish cover as objects of a category \( \mathcal{M} \) with Borel-definable group homomorphisms as morphisms. It is proved in [Lup22b, Section 11] that \( \mathcal{M} \) is an abelian category, which is in fact the left heart of (the derived category of) the quasi-abelian category \( \mathcal{A} \) of Polish groups. (Recall that we are assuming all the groups to be abelian.) In particular, \( \mathcal{A} \) is a full subcategory of \( \mathcal{M} \).

The subobjects in \( \mathcal{M} \) of a group with a Polish cover \( G = \hat{G}/N \) are the subgroups \( H \) of \( G \) that are Polishable, in the sense that are of the form \( \hat{H}/N \) where \( \hat{H} \) is a Polishable subgroup of \( \hat{G} \). The kernel in \( \mathcal{M} \) of a Borel-definable group homomorphism \( \varphi : G \to H \) between groups with a Polish cover \( G = \hat{G}/N \) and \( H = \hat{H}/M \) is the Polishable subgroup \( \ker(\varphi) = \{ g \in G : \varphi(g) = 0 \} \) of \( G \). The image of \( \varphi \) in \( \mathcal{M} \) is the Polishable subgroup \( L = \text{im}(\varphi) = \{ \varphi(g) : g \in G \} \) of \( H \), and the cokernel of \( \varphi \) is the quotient map \( \hat{H}/M \to \hat{H}/L = \hat{L} \) where \( \hat{L} \) is the Polishable subgroup \( \{ h \in \hat{H} : h + M \in L \} \) of \( H \). In particular, a Borel-definable group homomorphism is epic in \( \mathcal{M} \) if and only if it is surjective, and monic in \( \mathcal{M} \) if and only if it is injective. The zero object is the trivial group, and the biproduct of \( G = \hat{G}/N \) and \( H = \hat{H}/M \) is the group with a Polish cover \( (G \oplus H)/(N \oplus M) \).

A group with a Polish cover \( G = \hat{G}/N \) is non-Archimedean if both \( G \) and \( N \) are non-Archimedean Polish groups. A Polishable subgroup \( H = \hat{H}/N \) of \( G \) is non-Archimedean if \( \hat{H} \) is a non-Archimedean Polish group. We have that groups with a non-Archimedean Polish cover form a full abelian subcategory \( \mathcal{M}_{\text{mA}} \) of the category of groups with a Polish cover, which is the left heart of the quasi-abelian category of non-Archimedean Polish groups. The subobjects in \( \mathcal{M}_{\text{mA}} \) of a Polish group with a non-Archimedean Polish cover are precisely the non-Archimedean Polishable subgroups, which are also the Polishable subgroups that are Borel-definably isomorphic to a group with a non-Archimedean Polish cover. In particular, the kernel and the image of a Borel-definable group homomorphism between groups with a non-Archimedean Polish groups are non-Archimedean Polishable subgroups.

Since the category of groups with a Polish cover is an abelian category, one can consider the notion of exact sequence in this category. This is simply an exact sequence in the category of groups that consists of groups with a Polish cover and Borel-definable homomorphisms. We refer to such an exact sequence as a Borel-definable exact sequence of groups with a Polish cover.

2.3. Complexity of Polishable subgroups. It is observed in [Lup22b] that it is meaningful to define the Borel complexity class of a Polishable subgroup of a group with a Polish cover. Suppose that \( G = \hat{G}/N \) is a group with a Polish cover and \( H = \hat{H}/N \) is a Polishable subgroup of \( \hat{G} \). Let also \( \Gamma \) be a Borel complexity class. Then we say that \( H \) is \( \Gamma \) in \( G \) or \( H \in \Gamma(G) \) if \( \hat{H} \in \Gamma(\hat{G}) \), and that \( \Gamma \) is the complexity class of \( H \) in \( G \) if \( \Gamma \) is the complexity class of \( H \) in \( \hat{G} \). In particular, we have that \( H \) is \( \Pi^0_1 \) in \( G \) if and only if \( H \) is closed in \( G \) (with respect to the quotient topology). We let \( \text{clo}^G \) be the closure of \( H \) in \( G \).

From [Lup22b, Proposition 5.2 and Proposition 5.3] we have that the following is a complete list of the possible complexity classes of Polishable subgroups of groups with a Polish cover: \( \Pi^0_1 \), \( \Sigma^0_2 \), \( D(\Pi^0_{1+\lambda+1}) \), and \( \Pi^0_{1+\lambda+n+2} \) for \( \lambda < \omega_1 \) either zero or limit and \( n < \omega \). Furthermore, in the case of non-Archimedean Polishable subgroups the case \( D(\Pi^0_{1+\lambda+1}) \) is excluded. From [Lup22b, Proposition 5.5] we have the following.

Proposition 2.2. Let \( G \) and \( H \) be groups with a Polish cover, \( H_0 \) be a Polishable subgroup of \( H \), and let \( f : G \to H \) be Borel-definable group homomorphism.

- Suppose that \( \Gamma \) is one of the possible Borel complexity classes of Polishable subgroups other than \( \Sigma^0_2 \). If \( H_0 \) is \( \Gamma \) in \( H \), then \( f^{-1}(H_0) \) is \( \Gamma \) in \( G \). The converse holds if \( f \) is surjective.
- Suppose that \( G, H, H_0 \) are groups with a non-Archimedean Polish cover, and that \( \Gamma \) is one of the possible Borel complexity classes of non-Archimedean Polishable subgroups. If \( H_0 \) is \( \Gamma \) in \( H \), then \( f^{-1}(H_0) \) is \( \Gamma \) in \( G \). The converse holds if \( f \) is surjective.

Let use denote by \( =^G \) for a group with a Polish cover \( G = \hat{G}/N \) the coset equivalence relation of \( N \in \hat{G} \). We have the following characterization of the Borel complexity class \( \Gamma \) of a non-Archimedean Polish subgroup in terms of Borel reducibility when \( \Gamma \subseteq \Pi^0_1 \); see [Lup22b, Proposition 5.6]. Recall that \( E_0 \) denotes the \( \Sigma^0_2 \) equivalence relation on the space \( C := \{0, 1\}^\omega \) of infinite binary sequences obtained by setting \( (x_i)E_0(y_i) \iff \exists n \in \omega \forall i \geq n, x_i = y_i \). The \( \Pi^0_2 \) equivalence relation \( E_0^\omega \) on \( C^\omega \) is defined by setting \( (x_i)E_0^\omega(y_i) \iff \forall i, x_iE_0y_i \).
Proposition 2.3. Suppose that $G = \hat{G}/N$ is a group with a Polish cover, and that $H$ is a non-Archimedean Polishable subgroup of $G$. Then:

1. $\mathit{eq}_{G/H}$ is smooth if and only if $H$ is $\Pi^0_2$ in $G$;
2. $\mathit{eq}_{G/H}$ is Borel reducible to $E_0$ if and only if $H$ is $\Sigma^0_2$ in $G$, and Borel bireducible with $E_0$ if and only if $\Sigma^0_2$ is the complexity class of $H$ in $G$;
3. $\mathit{eq}_{G/H}$ is Borel reducible to $E^0_\omega$ if and only if $H$ is $\Pi^0_3$ in $G$, and Borel bireducible with $E^0_\omega$ if and only if $\Pi^0_3$ is the complexity class of $H$ in $G$.

2.4. The Solecki subgroups. A canonical chain of Polishable subgroups of a given group with a Polish cover is described in [Lup22b, Section 6] relying on previous work of Solecki [Sol99]; see also [FS06, Sol09, Lup22a].

The first Solecki subgroup $s_1(G)$ of a group with a Polish cover $G = \hat{G}/N$ is the $\Pi^0_3$ Polishable subgroup $s_1^N(\hat{G})/N$, where $s_1^N(\hat{G})$ is the $\Pi_3^0$ Polishable subgroup of $\hat{G}$ consisting $z \in \hat{G}$ such that for every open neighborhood $V$ of 0 in $N$ there exists $w \in N$ such that $z + w$ belongs to the closure $\overline{V}\hat{G}$ of $V$ in $G$. It is proved in [Sol99, Lemma 2.3] that $s_1(G)$ is the smallest $\Pi^0_3$ Polishable subgroup of $G$, and that $\{0\}$ is dense in $G$. We have the following characterization of $s_1^N(\hat{G})$; see [Lup22a, Lemma 4.2].

Lemma 2.4. Suppose that $G = \hat{G}/N$ is a group with a Polish cover. Let $H = \hat{H}/N$ be a Polishable subgroup of $G$ such that:

1. $\{0\}$ is dense in $H$;
2. for every open neighborhood $V$ of zero in $N$, $\overline{V}\hat{G} \cap \hat{H}$ contains an open neighborhood of zero in $\hat{H}$.

If $A \subseteq G$ is $\Pi^0_3$ and contains $N$, then $A \cap \hat{H}$ is comeager in $H$. In particular, $H \subseteq s_1(G)$. If $H$ is furthermore $\Pi^0_3$, then $H = s_1(G)$.

The sequence of Solecki subgroups $s_\alpha(G) = s_\alpha^N(\hat{G})/N$ for $\alpha < \omega_1$ of the group with a Polish cover $G$ is defined recursively by setting:

- $s_0(G) = \{0\}$;
- $s_{\alpha+1}(G) = s_1(s_\alpha(G))$ for $\alpha < \omega_1$;
- $s_\lambda(G) = \bigcap_{\beta < \lambda} s_\beta(G)$ for a limit ordinal $\lambda < \omega_1$.

We have that, for every $\alpha < \omega_1$, $s_\alpha(G)$ is the smallest $\Pi_{\alpha+1}^0$ Polishable subgroup of $G$. Furthermore, $G \rightarrow s_\alpha(G)$ is a subfunctor of the identity on the category of groups with a Polish cover: if $\varphi : G \rightarrow H$ is a Borel-definable group homomorphism between groups with a Polish cover, then $\varphi$ restricts to a Borel-definable group homomorphism $s_\alpha(G) \rightarrow s_\alpha(H)$ for every $\alpha < \omega_1$. Finally, if $G$ is a group with a non-Archimedean Polish cover, then $s_\alpha(G)$ is non-Archimedean for every $\alpha < \omega_1$; see [Lup22b, Section 6]. The following result allows one to compute the complexity class of $\{0\}$ in $G$ in terms of the Solecki subgroups; see [Lup22b, Theorem 6.4].

Theorem 2.5. Suppose that $G = \hat{G}/N$ is a group with a Polish cover. Let $\alpha = \lambda + n$ be the Solecki rank of $G$, where $\lambda < \omega_1$ is either zero or a limit ordinal and $n < \omega$.

1. Suppose that $n = 0$. Then $\Pi^0_{\lambda+n}$ is the complexity class of $\{0\}$ in $G$;
2. Suppose that $n \geq 1$. Then:
   (a) if $\{0\} \in \Pi^0_{\lambda+n}(s_{\lambda+n-1}(G))$ and $\{0\} \notin D(\Pi^0_{\lambda+n-1}(G))$, then $\Pi^0_{\lambda+n+1}$ is the complexity class of $\{0\}$ in $G$;
   (b) if $n \geq 2$ and $\{0\} \in D(\Pi^0_{\lambda+n-1}(G))$, then $D(\Pi^0_{\lambda+n})$ is the complexity class of $\{0\}$ in $G$;
   (c) if $n = 1$, $\{0\} \in D(\Pi^0_{\lambda+1}(s_\lambda(G))$, and $\{0\} \notin \Sigma^0_2(s_\lambda(G))$, then $D(\Pi^0_{\lambda+1})$ is the complexity class of $\{0\}$ in $G$;
   (d) if $n = 1$ and $\{0\} \in \Sigma^0_2(s_\lambda(G))$, then $\Sigma^0_{\lambda+2}$ is the complexity class of $\{0\}$ in $G$.

Furthermore, if $\{0\}$ is a non-Archimedean Polishable subgroup of $G$, then the case (2c) is excluded.

We also have the following reformulation of [Lup22a, Lemma 5.6 and Lemma 5.7].

Lemma 2.6. Suppose that, for every $k \in \omega$, $G_k$ is a group with a Polish cover, and $\alpha < \omega_1$. Define $G = \prod_{k \in \omega} G_k$.

1. If, for every $k \in \omega$, $\{0\}$ is $\Pi^0_\alpha$ in $G_k$, and for every $\beta < \alpha$ there exist infinitely many $k \in \omega$ such that $\{0\}$ is not $\Pi^0_\beta$ in $G_k$, then $\Pi^0_\alpha$ is the complexity class of $\{0\}$ in $G$;
2.5. Ulm subgroups. Suppose that $G = \hat{G}/N$ is a group with a Polish cover. The first Ulm subgroup $u_1(G) = G^1$ is the subgroup $\bigcap_{n>0} nG$ of $G$. One then defines the sequence $u_\alpha(G)$ of Ulm subgroups by recursion on $\alpha < \omega_1$ by setting:

1. $u_0(G) = G$;
2. $u_{\alpha+1}(G) = u_1(u_\alpha(G))$;
3. $u_\lambda(G) = \bigcap_{\alpha < \lambda} u_\alpha(G)$ for $\lambda$ limit.

We also let $D(G)$ be the largest divisible subgroup of $G$. A group $G$ is reduced if $D(G) = 0$. It is remarked in [Lup22b, Section 7] that $u_\alpha(G)$ for every $\alpha < \omega_1$ and $D(G)$ are Polishable subgroups of $G$, which are non-Archimedean when $G$ is non-Archimedean. Furthermore, there exists $\alpha < \omega_1$ such that $u_\alpha(G) = D(G)$. The least such $\alpha < \omega_1$ is the Ulm rank of $G$.

A related sequence can be defined for a given prime number $p$. One defines by recursion:

- $p^0G = G$;
- $p^{\alpha+1}G = p(p^\alpha G)$;
- $p^\alpha G = \bigcap_{\beta < \alpha} p^\beta G$ for $\alpha$ limit.

The $p$-length of $G$ is the least $\alpha$ such that $p^\alpha G$ is $p$-divisible, in which case $p^\alpha G$ is the largest $p$-divisible subgroup of $G$. (A group $A$ is $p$-divisible if the function $A \to A$, $x \mapsto px$ is surjective.) If $G$ is $p$-local, then one can easily prove by induction that, for every ordinal $\beta$, $G^\beta = p^{\omega^\beta}G$. (An abelian group $G$ is $p$-local if, for every prime $q$ other than $p$, the function $x \mapsto qx$ is an automorphism of $G$.)

**Definition 2.7.** If $G, H$ are groups with a Polish cover, and $\varphi : G \to H$ is an Borel-definable injective homomorphism, then $\varphi$ is isotype if $\varphi^{-1}(u_\alpha(H)) = u_\alpha(G)$ for every ordinal $\alpha$. If $G \subseteq H$ is a Polishable subgroup, then $G$ is isotype if the inclusion $G \to H$ is isotype.

Suppose that $\alpha$ is a countable ordinal and $(G_\beta)_{\beta < \alpha}$ is an inverse sequence of groups with a Polish cover $G_\beta$ and Borel-definable homomorphisms $p^{(\beta,\beta')} : G_\beta \to G_{\beta'}$ for $\beta \leq \beta' < \alpha$. The limit $\lim_{\beta < \alpha} G_\beta$ in the category of groups with a Polish cover is

$$\left\{ (x_\beta) \in \prod_{\beta < \alpha} G_\beta : \forall \beta \leq \beta', x_\beta = p^{(\beta,\beta')} (x_{\beta'}) \right\},$$

which is a Polishable subgroup of $\prod_{\beta < \alpha} G_\beta$.

Let $G$ be a group with a Polish cover. The $\alpha$-topology on $G$ is the group topology that has $(G^\beta)_{\beta < \alpha}$ as neighborhoods of 0. Let $L_\alpha(G) := \lim_{\beta < \alpha} G / G^\beta$ be the corresponding Hausdorff completion, which is a Polish group. We have a canonical Borel-definable exact sequence of groups with a Polish cover

$$0 \to G^\alpha \to G \xrightarrow{\alpha} L_\alpha(G) \to E_\alpha(G) \to 0.$$

The map $\kappa_\alpha : G \to L_\alpha(G)$ is the homomorphism induced by the quotient maps $G \to G / G^\beta$ for $\beta < \alpha$, while $E_\alpha(G) = \text{coker}(\kappa_\alpha) = L_\alpha(G) / \kappa_\alpha(G)$. Notice that the $\alpha$-topology on $G$ is Hausdorff if and only if $G^\alpha = 0$, and it is Hausdorff and complete if and only if $G^\alpha = 0$ and $E_\alpha(G) = 0$, in which case $G \to L_\alpha(G)$ is a Borel-definable isomorphism. If $\varphi : G \to H$ is an isotype Borel-definable homomorphism, then $\varphi$ induces a Borel-definable homomorphism $L_\alpha(G) \to L_\alpha(H)$ that makes the diagram

$$
\begin{array}{ccc}
G & \to & H \\
\downarrow & & \downarrow \\
L_\alpha(G) & \to & L_\alpha(H)
\end{array}
$$

commute. In turn, this induces a Borel-definable homomorphism $E_\alpha(G) \to E_\alpha(H)$. 

(2) If, for every $k \in \omega$, $G_k$ has Solecki rank $\alpha$, then $\Pi^0_{\omega+1+\alpha}$ is the complexity class of $\{0\}$ in $G$ if $\alpha$ is a successor ordinal, and $\Pi^0_{1+\alpha}$ is the complexity class of $\{0\}$ in $G$ if $\alpha$ is either zero or a limit ordinal.
3. Ext groups

In this section, we recall how Ext and PExt groups from abelian group theory and homological algebra can be regarded as groups with a Polish cover as in [BLP20]. We formulate in this context a description of Ext groups from [EM42], and some homological algebra results about cotorsion functors from [Nun67]. Recall that we assume all the groups to be abelian and additively denoted.

3.1. Groups of homomorphisms. We begin by describing how the group Hom of homomorphisms from a countable group to a group with a Polish cover can be regarded as a group with a Polish cover. Suppose that \( A \) is a countable group, and \( G = \hat{G}/N \) is a group with a Polish cover. We regard \( A \) as a group with a Polish cover as in [BLP20]. We formulate in this context a description of Ext groups.

We regard \( \text{Hom}(A,G) \) as a group with a Polish cover, and describe how it can be regarded as a group with a Polish cover. We present two equivalent descriptions of it, one in terms of cocycles, and the other in terms of projective resolutions. Suppose that \( A \) is a countable group and \( G = \hat{G}/N \) is a group with a Polish cover. A (normalized 2-)cocycle on \( A \) with values in \( G \) is a function \( c : A \times A \to G \) such that, for every \( x, y, z \in A \):

- \( c(x,0) = 0; \)
- \( c(x,y) = c(y,x); \)
- \( c(y,z) - c(x+y,z) + c(x,y+z) - c(y,z) = 0. \)

We let \( Z(A,G) \) be the group of cocycles on \( A \) with values in \( G \), where the operation is defined pointwise. We regard \( Z(A,G) \) as a group with a Polish cover \( \hat{H}/N_H \), as follows. Let \( \hat{H} \) be the Polish group of functions \( c : A \times A \to \hat{G} \) such that, for every \( x, y, z \in A \):

- \( c(x,0) = 0; \)
- \( c(x,y) = c(y,x); \)
- \( c(y,z) - c(x+y,z) + c(x,y+z) - c(y,z) \equiv 0 \mod N. \)

The operations in \( \hat{H} \) are defined pointwise, and we declare a net \((c_i)\) in \( H \) to converge to \( c \) if and only if, for every \( x, y, z \in A \),

\[
(c_i - c)(x,y) \to 0
\]
in \( \hat{G} \), and

\[
(c_i - c)(y,z) - (c_i - c)(x+y,z) + (c_i - c)(x,y+z) - (c_i - c)(y,z) \to 0
\]
in \( N \). We let \( N_H \) be the Polishable subgroup of \( \hat{H} \) consisting of the functions \( c \) in \( \hat{H} \) such that \( c(x,y) \equiv 0 \mod N \) for every \( x, y \in A \). Then we can identify \( Z(A,G) \) with the group with a Polish cover \( \hat{H}/N_H \). Notice that \( Z(A,G) \) is non-Archimedean when \( G \) is non-Archimedean, and a Polish group when \( G \) is a Polish group.

A cocycle \( c \) on \( A \) with values in \( G \) is a coboundary if there exists a function \( \phi : A \to G \) such that \( \phi(0) = 0 \) and

\[
c(x,y) = \phi(y) - \phi(x+y) + \phi(x)
\]
for every $x, y \in A$. Coboundaries form a Polishable subgroup $B(A, G)$ of the group with a Polish cover $Z(A, G)$. One then defines $\text{Ext}(A, G)$ to be the group with a Polish cover $Z(A, G)/B(A, G)$, which is non-Archimedean when $G$ is non-Archimedean.

Following [EM42, Section 5], one can give an alternative description of $\text{Ext}(A, G)$, as follows. Suppose that $F$ is a free countable abelian group, and $R$ is a subgroup. Let $\text{Hom}(F|R, G)$ be the Polishable subgroup of $\text{Hom}(R, G)$ consisting of group homomorphisms $R \rightarrow G$ that extend to a group homomorphism $F \rightarrow G$.

**Proposition 3.1** (Eilenberg–Mac Lane [EM42]). Suppose that $G$ is a group with a Polish cover, and $A$ is a countable group. Write $A = F/R$ where $F$ is a countable free group, and $R \subseteq F$ is a subgroup. Fix a right inverse $t : A \rightarrow F$ for the quotient map $F \rightarrow A$ such that $t(0) = 0$. Define then the cocycle $\zeta$ on $A$ with values in $R$ by setting $\zeta(x, y) = t(y) - t(x + y) + t(x)$ for $x, y \in A$. The Borel-definable homomorphism $\text{Hom}(R, G) \rightarrow Z(A, G)$, $\theta \mapsto \theta \circ \zeta$ induces a natural Borel-definable isomorphism

$$\frac{\text{Hom}(R, G)}{\text{Hom}(F|R, G)} \cong \text{Ext}(A, G).$$

Fix a countable abelian group $A$. Then $\text{Ext}(A, \_)$ is an additive functor from the category of groups with a Polish cover to itself. Similarly, for a fixed group with a Polish cover $G$, $\text{Ext}(-, G)$ is an additive functor from the category of countable groups to the category of groups with a Polish cover. If $(A_i)_{i \in \omega}$ is a sequence of countable groups, and $(G_i)_{i \in \omega}$ is a sequence of groups with a Polish cover, then we have, for every countable group $A$ and group with a Polish cover $G$, natural isomorphisms

$$\text{Ext}(\bigoplus_{i \in \omega} A_i, G) \cong \prod_{i \in \omega} \text{Ext}(A_i, G)$$

and

$$\text{Ext}(A, \prod_{i \in \omega} G_i) \cong \prod_{i \in \omega} \text{Ext}(A, G_i)$$

in the category of groups with a Polish cover; see [Fuc70, Theorem 52.2].

If $D$ is a divisible abelian group, then $\text{Ext}(A, D) = 0$ [Fuc70, page 222]. If $G$ is an arbitrary countable group, then one can write $G = D \oplus G'$ where $D$ is divisible and $G'$ is reduced (i.e., it has no nonzero divisible subgroup). Thus, we have that $\text{Ext}(A, G)$ and $\text{Ext}(A, G')$ are Borel-definably isomorphic.

3.3. Groups of pure extensions. An important subgroup of $\text{Ext}$ is the group $P\text{Ext}$ corresponding to pure extensions. We recall the fundamental observation that $P\text{Ext}$ is the first Ulm subgroup of $\text{Ext}$, as well as the closure of the $(0)$ in $\text{Ext}$. Suppose that $A$ is a countable group, and $G$ is a group with a Polish cover. Define the group $B_\omega(A, G)$ of weak coboundaries to be the Polishable subgroup of $Z(A, G)$ consisting of $c$ such that $c|_{S \times S} \in B(S, G)$ for every finite subgroup $S \subseteq A$. One defines the pure (or phantom) $\text{Ext}$ group $P\text{Ext}(A, G)$ to be the group with a Polish cover $B_\omega(A, G)/B(A, G)$ [Sch03, CS98]. We also define the weak $\text{Ext}$ group $\text{Ext}_w(A, G)$ to be the group with a Polish cover $Z(A, G)/B_\omega(A, G)$. One has that $P\text{Ext}(A, G)$ is equal to the first Ulm subgroup $u_1(\text{Ext}(A, G))$ of the group with a Polish cover $\text{Ext}(A, G)$; see [Fuc70, Proposition 53.4].

Suppose that $A = F/R$ for some countable free group $F$ and subgroup $R \subseteq F$. Define $\text{Hom}_f(F|R, G)$ to be the Polishable subgroup of $\text{Hom}(R, G)$ consisting of all group homomorphisms $R \rightarrow G$ that extend to a group homomorphism $F_0 \rightarrow R$ for every subgroup $F_0$ of $F$ containing $R$ as a finite index subgroup. The natural Borel-definable isomorphism

$$\frac{\text{Hom}(R, G)}{\text{Hom}(F|R, G)} \cong \text{Ext}(F/R, G)$$

as in Proposition 3.1 induces a Borel-definable isomorphism

$$\frac{\text{Hom}_f(R, G)}{\text{Hom}(F|R, G)} \cong P\text{Ext}(A, G)$$

see [EM42, Section 5].

**Lemma 3.2.** Suppose that $A, G$ are countable groups. Then $P\text{Ext}(A, G)$ is the closure of $(0)$ in $\text{Ext}(A, G)$, and $\text{Ext}_w(A, G)$ is a Polish group.
Proof. Write \( A = F/R \) for some countable free group \( F \) and subgroup \( R \subseteq F \). In view of the above discussion, it suffices to prove that \( \text{Hom}_{t}(R,G) \) is the closure of \( \text{Hom}(F|R,G) \) inside \( \text{Hom}(R,G) \).

For \( \varphi \in \text{Hom}(R,G) \), one has that \( \varphi \in \text{Hom}_{t}(F|R,G) \) if and only if, for every prime \( p \), \( x \in F \) and \( m \in \mathbb{Z} \) satisfying \( p^{m}x \in R \), one has that \( \varphi(p^{m}x) \in p^{m}G \); see [EM42, Lemma 5.1 and Lemma 5.2]. Thus, it follows that \( \text{Hom}_{t}(F|R,G) \) is a closed subgroup of the Polish group \( \text{Hom}(R,G) \). The proof of [EM42, Lemma 5.3] shows that, in fact, \( \text{Hom}_{t}(F|R,G) \) is the closure of \( \text{Hom}(F|R,G) \) inside \( \text{Hom}(R,G) \). \( \square \)

3.4. Ext groups and extensions. Fix countable groups \( A,C \). We now describe how one can regard the relation of equivalence of extensions of \( C \) by \( A \) as a Borel equivalence relation on a Polish space, and how the elements of \( \text{Ext}(C,A) \) parametrize the corresponding equivalence classes.

An extension of \( C \) by \( A \) is a short exact sequence \( A \rightarrow B \rightarrow C \) in the abelian category of countable groups. (Recall that all the groups are assumed to be abelian.) One can regard such an extension as a first-order structure in the countable language \( \mathcal{L} \) obtained by adding to the language of groups constant symbols \( c_{a} \) for \( a \in A \) and unitary relation symbols \( R_{c} \) for \( c \in C \). The extension \( A \rightarrow B \rightarrow C \) corresponds to the \( \mathcal{L} \)-structure given by the group \( B \), where \( c_{a} \) is the interpreted as the image of \( a \in A \) under the injective homomorphism \( A \rightarrow B \), while the relation symbol \( R_{c} \) is interpreted as the preimage of \( c \in C \) under the surjective homomorphism \( B \rightarrow C \). It is easy to see that the set \( \text{Ext}(C,A) \) of such \( \mathcal{L} \)-structures is a \( G_{δ} \) subset of the space \( \text{Mod}(\mathcal{L}) \) of \( \mathcal{L} \)-structures, whence it is a Polish space with the induced topology. The relation \( \mathcal{R}_{\text{Ext}(C,A)} \) of isomorphism of extensions is simply the restriction to \( \text{Ext}(C,A) \) of the relation of isomorphism of \( \mathcal{L} \)-structures.

Given an extension \( A \xrightarrow{h} B \xrightarrow{\iota} C \) of \( C \) by \( A \), one can define a corresponding cocycle \( c \) on \( C \) with coefficients in \( A \), as follows. Fix a right inverse \( t : C \rightarrow B \) for the map \( h : B \rightarrow C \), and set \( c(x,y) := g^{-1}(t(y) - t(x+y) + t(x)) \). This defines a Borel function from \( \text{Ext}(C,A) \) to the Polish group \( \mathbb{Z}(A,G) \), which is a Borel reduction from \( \mathcal{R}_{\text{Ext}(C,A)} \) to the coset relation \( =_{\text{Ext}(C,A)} \) of \( \mathbb{B}(A,G) \) inside \( \mathbb{Z}(A,G) \). In fact, the induced Borel-definable function

\[
\frac{\text{Ext}(C,A)}{\mathcal{R}_{\text{Ext}(C,A)}} \rightarrow \text{Ext}(C,A)
\]

is a bijection with Borel-definable inverse; see [Fuc70, Section 50]. In particular, we have that \( \text{Ext}(C,A)/\mathcal{R}_{\text{Ext}(C,A)} \) is a Borel-definable set, and \( \mathcal{R}_{\text{Ext}(C,A)} \) is Borel bireducible with \( =_{\text{Ext}(C,A)} \).

An extension \( A \rightarrow B \rightarrow C \) of \( C \) by \( A \) is pure if (the image of) \( A \) is a pure subgroup of \( B \), namely \( nB \cap A = nA \) for every \( n \geq 1 \). The pure extensions of \( C \) by \( A \) form a \( G_{δ} \) subset \( \text{PExt}(C,A) \) of \( \text{Ext}(C,A) \). Letting \( \mathcal{R}_{\text{PExt}(C,A)} \) be the restriction of \( \mathcal{R}_{\text{Ext}(C,A)} \) to \( \text{PExt}(C,A) \), we have that \( \text{PExt}(C,A)/\mathcal{R}_{\text{PExt}(C,A)} \) is also a Borel-definable set, and the Borel-definable bijection

\[
\frac{\text{Ext}(C,A)}{\mathcal{R}_{\text{Ext}(C,A)}} \sim \text{Ext}(C,A)
\]

restricts to a Borel-definable bijection

\[
\frac{\text{PExt}(C,A)}{\mathcal{R}_{\text{PExt}(C,A)}} \sim \text{PExt}(C,A)
\]

with Borel-definable inverse.

3.5. Ext and Hom. The functor \( \text{Ext} \) can be seen as the first derived functor of the functor \( \text{Hom} \) from the perspective of homological algebra [Wei94, Chapter 2]. For our purposes, it will be sufficient to recall some exact sequences relating \( \text{Ext} \) and \( \text{Hom} \).

Suppose that \( A \) is a countable group and \( G_{0} \rightarrow G_{1} \rightarrow G_{2} \) is a short exact sequence of groups with a Polish cover. Then we have a Borel-definable exact sequence

\[
0 \rightarrow \text{Hom}(A,G_{0}) \rightarrow \text{Hom}(A,G_{1}) \rightarrow \text{Hom}(A,G_{2}) \xrightarrow{\partial} \text{Ext}(A,G_{0}) \rightarrow \text{Ext}(A,G_{1}) \rightarrow \text{Ext}(A,G_{2}) \rightarrow 0;
\]

see [Fuc70, Theorem 51.3]. To describe the Borel-definable homomorphism

\[
\partial : \text{Hom}(A,G_{2}) \rightarrow \text{Ext}(A,G_{0})
\]

one can consider a short exact sequence \( R \rightarrow F \rightarrow A \), where \( F \) is free. Then we have natural Borel-definable isomorphisms

\[
\text{Hom}(A,G_{2}) \cong \frac{\text{Hom}((F,R),(G_{1},G_{0}))}{\text{Hom}(F,G_{0})}
\]
and

$$\text{Ext}(A, G_0) \cong \frac{\text{Hom}(R, G_0)}{\text{Hom}(\mathcal{F}R, G_0)}.$$ 

Here, \(\text{Hom}((F, R), (G_1, G_0))\) is the Polishable subgroup of \(\text{Hom}(F, G_1)\) consisting of homomorphisms \(F \to G_1\) mapping \(R\) to \(G_0\) (where we identify \(G_0\) with a Polishable subgroup of \(G_1\)). Via these Borel-definable isomorphisms, \(\partial\) correspond to the Borel-definable homomorphism

\[
\frac{\text{Hom}((F, R), (G_1, G_0))}{\text{Hom}(F, G_0)} \to \frac{\text{Hom}(R, G_0)}{\text{Hom}(\mathcal{F}R, G_0)}
\]

induced by the Borel-definable homomorphism

\[
\text{Hom}((F, R), (G_1, G_0)) \to \text{Hom}(R, G_0), \ \varphi \mapsto \varphi_R.
\]

If furthermore \(G_0 \to G_1 \to G_2\) is a pure Borel-definable short exact sequence, namely \(G_0\) is a pure subgroup of \(G_1\), then the Borel-definable exact sequence above restricts to a Borel-definable exact sequence

\[0 \to \text{Hom}(A, G_0) \to \text{Hom}(A, G_1) \to \text{Hom}(A, G_2) \to \text{PExt}(A, G_0) \to \text{PExt}(A, G_1) \to \text{PExt}(A, G_2) \to 0;
\]

see [Fuc70, Theorem 53.7].

Suppose now that \(A \to B \to C\) is a short exact sequence of countable groups and \(G\) is a group with a Polish cover. Then we have a Borel-definable exact sequence

\[0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to \text{Ext}(C, G) \to \text{Ext}(B, G) \to \text{Ext}(A, G) \to 0
\]

of groups with a Polish cover; see [Fuc70, Theorem 51.3]. The Borel-definable homomorphism \(\delta : \text{Hom}(A, G) \to \text{Ext}(C, G)\) is defined by setting, for \(\varphi \in \text{Hom}(A, G)\), \(\delta(\varphi)\) to be the element of \(\text{Ext}(C, G)\) associated with the cocycle \(C \times C \to G, (x, y) \mapsto \varphi(t(y) - t(x + y) + t(x))\), where \(t : C \to B\) is a right inverse for the quotient map \(B \to C\), and we identify \(A\) with a subgroup of \(B\). If furthermore the short exact sequence \(A \to B \to C\) is pure, then the exact sequence above restricts to a Borel-definable exact sequence

\[0 \to \text{Hom}(C, G) \to \text{Hom}(B, G) \to \text{Hom}(A, G) \to \text{PExt}(C, G) \to \text{PExt}(B, G) \to \text{PExt}(A, G) \to 0;
\]

see [Fuc70, Theorem 53.7].

3.6. Radicals and cotorsion functors. We now recall some terminology and results of homological nature about \(\text{Ext}\) from [Nun67]. These will be used in Section 4 to describe the Solecki subgroups of \(\text{Ext}\) for countable \(p\)-groups.

A preradical or subfunctor of the identity \(S\) is a function \(A \to SA\) assigning to each group \(A\) a subgroup \(SA \subseteq A\) such that, if \(f : A \to B\) is a group homomorphism, then \(f\) maps \(SA\) to \(SB\). This gives a functor from the category of abelian groups to itself, where one defines for a homomorphism \(f : A \to B\), \(Sf := f|_{SA} : SA \to SB\). A radical is a preradical \(S\) such that \(S(A/SA) = 0\) for every group \(A\).

An extension \(Z \to G \to H\) of a countable group \(H\) by \(Z\) defines a preradical \(S\), by setting

\[SA = \text{Ran}(\text{Hom}(G, A) \to \text{Hom}(Z, A) = A) = \text{Ker}(A = \text{Hom}(Z, A) \to \text{Ext}(H, A)).
\]

In this case, one says that \(S\) is the preradical represented by the extension \(Z \to G \to H\). A cotorsion functor is a preradical that is represented by an extension \(Z \to G \to H\) such that \(H\) is a countable torsion group. Notice that, as \(H\) is countable, this implies that \(SA\) is a Polishable subgroup of \(A\) whenever \(A\) is a group with a Polish cover.

Suppose that \(S\) is a cotorsion functor. An extension \(C \to E \to A\) is \(S\)-pure if it defines an element of \(S\text{Ext}(A, C)\). In this case, the map \(C \to E\) is an \(S\)-pure injective homomorphism and the map \(E \to A\) is an \(S\)-pure surjective homomorphism, and the image of \(C\) inside of \(E\) is an \(S\)-pure subgroup. A group \(A\) is \(S\)-projective if \(S\text{Ext}(A, C) = 0\) for every group \(C\), and \(S\)-injective if \(S\text{Ext}(C, A) = 0\) for every group \(C\). The cotorsion functor \(S\) has enough projectives if for every group \(A\) there exists an \(S\)-pure extension \(M \to P \to A\) where \(P\) is \(S\)-projective. This is equivalent to the assertion that \(S\) is represented by an extension \(Z \to G \to H\) where \(H\) is an \(S\)-projective torsion group [Nun63, Theorem 4.8]. A cotorsion functor with enough projectives is necessarily a radical.

The following lemma is a consequence of [Nun67, Lemma 1.1].

**Lemma 3.3.** Suppose that \(S\) is a cotorsion functor, \(A,C\) are countable group, and \(B\) is a subgroup of \(C\) contained in \(SC\). Then the exact sequence \(B \to C \to C/B\) induces a Borel-definable exact sequence

\[0 \to \text{Hom}(A, B) \to \text{Hom}(A, C) \to \text{Hom}(A, C/B) \to \text{Ext}(A, B) \to S\text{Ext}(A, C) \to S\text{Ext}(A, C/B) \to 0\]
of groups with a Polish cover. Furthermore, $S\text{Ext}(A,C)$ is the preimage of $S\text{Ext}(A,C/B)$ under the Borel-definable surjective homomorphism $\text{Ext}(A,C) \to \text{Ext}(A,C/B)$.

Suppose that $S$ is a cotorsion functor represented by the exact sequence $\mathbb{Z} \to G_S \to H_S$, where $H_S$ is a countable $S$-projective torsion group. If $C$ is a countable group, then we have a corresponding Borel-definable exact sequence of groups with a Polish cover

$$0 \to SC \to C \to \text{Ext}(H_S,C) \to \text{Ext}(G_S,C) \to 0.$$  

The proofs of [Nun67, Lemma 1.3, Lemma 1.4, and Theorem 1.11] give the following lemma.

**Lemma 3.4.** Suppose that $S$ is a cotorsion functor represented by an extension $\mathbb{Z} \to G_S \to H_S$ where $H_S$ is a countable $S$-projective torsion group. Let $A,C$ be a countable groups such that $A/SA$ is $S$-projective. Then:

1. The quotient map $C \to C/SC$ induces Borel-definable isomorphisms $\text{Ext}(G_S,C) \to \text{Ext}(G_S,C/SC)$ and $\text{Ext}(H_S,C) \to \text{Ext}(H_S,C/SC)$;
2. If $SC = 0$ then $S\text{Ext}(A,C)$ and $\text{Hom}(SA, \text{Ext}(G_S,C)) = \text{Hom}(SA, S\text{Ext}(G_S,C))$ are Borel-definably isomorphic;
3. The short exact sequences $SA \to A \to A/SA$ and $SC \to C \to C/SC$ induce Borel-definable homomorphisms $\text{Hom}(A,C/SC) \to \text{Ext}(A,SC)$

and

$$\text{Ext}(A,SC) \to \text{Ext}(SA,SC),$$

which induce a Borel-definable isomorphism

$$\eta_{A,C} : \frac{\text{Ext}(A,SC)}{\text{Ran}(\text{Hom}(A,C/SC) \to \text{Ext}(A,SC))} \to \text{Ext}(SA,SC);$$

4. The short exact sequence $SC \to C \to C/SC$ induces a Borel-definable exact sequence

$$0 \to \text{Hom}(A,C/SC) \to \text{Ext}(A,SC) \to S\text{Ext}(A,C) \to S\text{Ext}(A,C/SC) \to 0$$

which induces a pure Borel-definable exact sequence

$$0 \to \frac{\text{Ext}(A,SC)}{\text{Ran}(\text{Hom}(A,C/SC) \to \text{Ext}(A,SC))} \stackrel{\rho_{A,C}}{\to} S\text{Ext}(A,C) \to S\text{Ext}(A,C/SC) \to 0;$$

5. The Borel-definable injective homomorphism

$$r_{A,C} := \rho_{A,C} \circ \eta_{A,C}^{-1} : \text{Ext}(SA,SC) \to S\text{Ext}(A,C)$$

restricts to a Borel-definable isomorphism

$$\gamma_{A,C} : \text{PExt}(SA,SC) \to u_1(S\text{Ext}(A,C)).$$

4. **Solecki subgroups of Ext groups**

This section is dedicated to the proof of Theorem 1.6 and Theorem 1.2. We begin by showing that one can reduce to the case of $p$-groups for some prime number $p$. We then show that the Solecki and Ulm subgroups of $\text{PExt}$ coincide. We conclude the proof of Theorem 1.6 and Theorem 1.2 by using the results concerning the complexity of Solecki subgroups from [Lup22a], and particularly Theorem 2.5. As before, we assume all the groups to abelian and additively denoted.

4.1. Reduction to the $p$-group case. Towards the proof of Theorem 1.6 and Theorem 1.2, we begin with showing how one can reduce to the case of $p$-groups for a given prime number $p$. This is the content of the following proposition. Recall that we denote by $\mathbb{P}$ the set of prime numbers.

**Proposition 4.1.** Suppose that $C$ is a countable torsion group and $A$ is a countable group. For a prime number $p$, let $C_p \subseteq C$ be the $p$-primary component of $C$, and $A_p \subseteq A/D(A)$ be the $p$-primary component of $A/D(A)$. Let $\Gamma$ be one of the classes $\Sigma^0_\alpha$, $\Pi^0_\alpha$, or $D(\Pi^0_\alpha)$ for $2 \leq \alpha < \omega_1$.

1. $\text{Ext}(C,A)$ is a Polish group if and only if $\text{Ext}(C_p,A_p)$ is a Polish group for every prime number $p$;
2. $\{0\}$ is $\Gamma$ in $\text{Ext}(C,A)$ if and only if $\{0\}$ is $\Gamma$ in $\prod_{p \in \mathbb{P}} \text{Ext}(C_p,A_p)$. 

The rest of this section is dedicated to the proof of Proposition 4.1. We begin with a lemma from [Fuc70, page 223], whose short proof we recall for convenience.

**Lemma 4.2.** Fix a prime number $p$. Suppose that $A, C$ are countable groups such that $C$ is a $p$-group and $A$ is $p$-divisible. Then $\text{Ext} (C, A) = 0$.

**Proof.** Let $D$ be the divisible hull of $A$ [Fuc70, Section 24]. Then $D/A$ is a torsion group with trivial $p$-primary component. In particular, $\text{Hom} (C, D/A) = 0$. As $D$ is divisible, $\text{Ext} (C, D) = 0$. Considering the Borel-definable exact sequence

$$\text{Hom} (C, D/A) \to \text{Ext} (C, A) \to \text{Ext} (C, D)$$

induced by the exact sequence $A \to D \to D/A$, we conclude that $\text{Ext} (C, A) = 0$. □

The following lemma is established in [EM42]; see also [Fuc70, Theorem 52.3].

**Lemma 4.3.** If $A$ is a countable torsion-free group and $C$ is a countable torsion group, then $\text{Ext} (C, A)$ is Borel-definably isomorphic to the Polish group $\text{Hom} (C, D/A)$ where $D$ is the divisible hull of $A$. In particular, $\text{PExt} (C, A) = 0$.

**Proof.** The exact sequence $A \to D \to D/A$ induces a Borel-definable exact sequence

$$\text{Hom} (C, D) \to \text{Hom} (C, D/A) \to \text{Ext} (C, A) \to \text{Ext} (C, D).$$

Since $C$ is torsion and $D$ is torsion-free, $\text{Hom} (C, D) = 0$. Furthermore, since $D$ is divisible, $\text{Ext} (C, D) = 0$. This concludes the proof. □

As an immediate consequence of the previous lemma, one obtains the following.

**Lemma 4.4.** Suppose that $C$ is a countable torsion group and $A$ is a countable group. Let $tA \subseteq A$ be its torsion subgroup. Then $\text{PExt} (C, tA)$ and $\text{PExt} (C, A)$ are Borel-definably isomorphic.

**Proof.** We have that $J := A/tA$ is torsion-free. Since $tA$ is a pure subgroup of $A$, we have a Borel-definable exact sequence

$$\text{Hom} (C, J) \to \text{PExt} (C, tA) \to \text{PExt} (C, A) \to \text{PExt} (C, J).$$

Since $C$ is torsion and $J$ is torsion-free, $\text{Hom} (C, J) = 0$ and $\text{PExt} (C, J) = 0$ by Lemma 4.3. This concludes the proof. □

**Lemma 4.5.** Suppose that $C$ is a countable $p$-group and $A$ is a countable group. Let $A_p \subseteq A$ be the $p$-primary subgroup of $A$. Then $\text{PExt} (C, A)$ and $\text{PExt} (C, A_p)$ are Borel-definably isomorphic.

**Proof.** By Lemma 4.4 we can assume without loss of generality that $A$ is a torsion group. Thus, we can write $A = A_p \oplus B$ where $B$ is $p$-divisible. By Lemma 4.2, $\text{PExt} (C, B) = 0$. Therefore, $\text{PExt} (C, A)$ is Borel-definably isomorphic to $\text{PExt} (C, A_p) \oplus \text{PExt} (C, B) = \text{PExt} (C, A_p)$. □

**Lemma 4.6.** Suppose that $C$ is a countable torsion group and $A$ is a countable group. Let $A_p \subseteq A$ be the $p$-primary subgroup of $A$ and $C_p \subseteq C$ be the $p$-primary subgroup of $C$. Then $\text{PExt} (C, A)$ and $\prod_p \text{PExt} (C_p, A_p)$ are Borel-definably isomorphic.

**Proof.** Since $C$ is a countable torsion group, we have that $C \cong \bigoplus_p C_p$. Therefore, $\text{PExt} (C, A)$ is Borel-definably isomorphic to $\prod_p \text{PExt} (C_p, A)$. By Lemma 4.5, for every prime $p$ we have that $\text{PExt} (C_p, A)$ is Borel-definably isomorphic to $\text{PExt} (C_p, A_p)$. This concludes the proof. □

We now present the proof of Proposition 4.1.

**Proof of Proposition 4.1.** By Lemma 3.2 and Lemma 4.6 we have Borel-definable isomorphisms

$$s_0 (\text{Ext} (C, A)) = \text{PExt} (C, A) \cong \prod_p \text{PExt} (C_p, A_p) = s_0 \left( \prod_p \text{Ext} (C_p, A_p) \right).$$
Lemma 4.7. Suppose that \( s_0 \) (Ext \(( C, A)\)) = 0 if and only if \( \prod_{p \in \mathbb{P}} \text{PExt} \( (C_p, A_p)\)\), and if and only if \( \text{PExt} \( (C_p, A_p)\)\) is a Polish group for every \( p \in \mathbb{P}\).

This follows from the above remarks, after observing that if \( G \) is a group with a Polish cover, then \( \{0\} \in \Gamma(\mathbb{G})\).

\[\square\]

4.2. Ulm subgroups and cotorsion functors. We now fix a prime \( p \), and assume all the groups to be \( p \)-local.

(A group \( C \) is \( p \)-local if the function \( x \mapsto qx \) is an isomorphism of \( C \) for every prime \( q \) other than \( p \).) Notice that, in particular, every \( p \)-group is \( p \)-local. Recall the notion of cotorsion functor from Section 3.6. For every ordinal \( \alpha < \omega_1 \), the assignment \( A \mapsto p^\alpha A \) is a cotorsion functor with enough projectives [Gri70, Chapter V]. For example, \( A \mapsto p^\alpha A \) is the cotorsion functor represented by the extension \( \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}(p^\infty) \). Here, we let \( \mathbb{Z}[1/p] \) be the subgroup of \( \mathbb{Q} \) consisting of rationals of the form \( ap^{-n} \) for \( a \in \mathbb{Z} \) and \( n \in \omega \), and \( \mathbb{Z}(p^\infty) \) is the Prüfer \( p \)-group \( \mathbb{Z}[1/p]/\mathbb{Z} \) [Fuc70, Section 3]. More generally, for \( \alpha < \omega_1 \), \( A \mapsto A^\alpha \) is represented by the extension \( \mathbb{Z} \to G_\alpha \to H_\alpha \) as in [Gri70, Theorem 69], where \( H_\alpha \) is a countable \( p \)-group of Ulm length \( \alpha \). A \( p \)-group \( A \) is called totally injective if and only if \( A/p^\alpha A \) is \( p^\alpha \)-projective for every ordinal \( \alpha \) [Fuc73, Section 82]. Every countable \( p \)-group is totally projective [Fuc73, Theorem 82.4]. For a countable \( p \)-group \( C \) and limit ordinal \( \alpha < \omega_1 \), define as in Section 2.5 \( L_\alpha(C) = \lim_{\alpha<\omega} C/p^\alpha C \), and consider the Borel-definable exact sequence

\[0 \to p^\alpha C \to C \to L_\alpha(C) \to E_\alpha(C) \to 0.\]

Lemma 4.7. Suppose that \( A, C \) are countable \( p \)-groups, and \( \alpha < \omega_1 \) is a limit ordinal. Then \( \text{Hom}(A, E_\alpha(C)) \) and \( \text{Hom}(A, p^\alpha \text{Ext}(G_\alpha(C))) \) are Borel-definably isomorphic with a Polish cover.

Proof. By Lemma 3.4(1), after replacing \( C \) with \( C/p^\alpha C \), we can assume that \( p^\alpha C = 0 \). Consider the Borel-definable short exact sequence \( C \to \text{Ext}(H_\alpha(C)) \to \text{Ext}(G_\alpha(C)) \) induced by \( \mathbb{Z} \to G_\alpha \to H_\alpha \). As in the proof of [Kee94], \( \text{Ext}(H_\alpha(C)) \) is complete in the \( \alpha \)-topology. Furthermore, the monomorphism \( C \to \text{Ext}(H_\alpha(C)) \) is isotope (see Definition 2.7), and hence it induces a Borel-definable monomorphism \( L_\alpha(C) \to \text{Ext}(H_\alpha(C)) \cong L_\alpha \text{Ext}(H_\alpha(C)). \) In turn, this induces a Borel-definable injective homomorphism \( E_\alpha(C) \to \text{Ext}(G_\alpha(C)) \). It is proved in [Kee94] that this Borel-definable injective homomorphism induces an isomorphism between the torsion subgroups of \( E_\alpha(C) \) and \( p^\alpha \text{Ext}(G_\alpha(C)) \). As \( A \) is a torsion group, this induces a Borel-definable isomorphism \( \text{Hom}(A, E_\alpha(C)) \to \text{Hom}(A, p^\alpha \text{Ext}(G_\alpha(C))). \)

\[\square\]

The following corollary is an immediate consequence of Lemma 4.7 and Lemma 3.4, considering that every countable \( p \)-group is totally projective. Recall that, for a \( p \)-local group and \( \alpha < \omega_1 \), \( A^\alpha = p^\alpha A \).

Corollary 4.8. Suppose that \( A, C \) are countable \( p \)-groups, and \( \alpha < \omega_1 \) is an ordinal.

- The quotient map \( C \to C/C^\alpha \) induces Borel-definable isomorphisms \( \text{Ext}(G_\alpha(C)) \to \text{Ext}(G_\alpha(C/C^\alpha)) \) and \( \text{Ext}(H_\alpha(C)) \to \text{Ext}(H_\alpha(C/C^\alpha)); \)
- If \( C^\alpha = 0 \) then \( \text{Ext}(A, C^\alpha) \) and \( \text{Hom}(A, E_\alpha(C)) \) are Borel-definably isomorphic;
- The short exact sequences \( A^\alpha \to A \to A/A^\alpha \) and \( C^\alpha \to C \to C/C^\alpha \) induce Borel-definable homomorphisms \( \text{Ext}(A, C^\alpha) \to \text{Ext}(A^\alpha, C^\alpha) \)

and

\[\text{Hom}(A, C/C^\alpha) \to \text{Ext}(A, C^\alpha),\]

which induce a Borel-definable isomorphism

\[\eta^\alpha_{A,C} : \frac{\text{Ext}(A, C^\alpha)}{\text{Ran}(\text{Hom}(A, C/C^\alpha) \to \text{Ext}(A, C^\alpha))} \to \text{Ext}(A^\alpha, C^\alpha);\]

- The short exact sequence \( C^\alpha \to C \to C/C^\alpha \) induces a Borel-definable exact sequence

\[0 \to \text{Hom}(A, C/C^\alpha) \to \text{Ext}(A, C^\alpha) \to \text{Ext}(A, C/C^\alpha)^\alpha \to 0\]

which induces a Borel-definable exact sequence

\[0 \to \frac{\text{Ext}(A, C^\alpha)}{\text{Ran}(\text{Hom}(A, C/C^\alpha) \to \text{Ext}(A, C^\alpha))} \to \text{Ext}(A, C) \to \text{Ext}(A, C/C^\alpha)^\alpha \to 0;\]

and
Theorem 17.3, a countable reduced $p$-group of Ulm rank $\sigma$ is a sum of cyclic $p$-groups.

(2) Suppose that $T$ is a reduced countable $p$-group of Ulm rank $\sigma$. Then $\sigma = \beta + 1$ is a successor ordinal, and $T^\beta$ is infinite.

Proof. Recall that, by [Fuc70, Theorem 17.3], a countable reduced $p$-group of Ulm rank $1$ is a sum of cyclic $p$-groups.

(2) Suppose that, for every $n \in \omega$, $T_n$ is a reduced countable $p$-group of Ulm rank $\sigma$. If $\sigma = \beta + 1$ is a successor ordinal, then we have that $T^\beta = \bigoplus_{n \in \omega} T_n^\beta$. For $n \in \omega$, $T_n$ has Ulm length $\beta + 1$, and hence $T_n^\beta$ is nonzero. Therefore, $T^\beta$ is infinite.

(1) Suppose that $T^\beta = \bigoplus_{n \in \omega} T_n^\beta$. For every $n \in \omega$, $S_n$ is a countable reduced $p$-group of Ulm rank $1$, and hence $S_n^\alpha$ is finite if $\alpha + 1 < \sigma$. By Ulm’s classification of countable reduced $p$-groups [Fuc73, Theorem 76.1, Corollary 76.2], for every $n \in \omega$ there exists a

\[ \eta_{A,C} : \text{Ext}(A, C^\alpha) \to \text{Ext}(A, C^\beta) \]

$$ \rho_{A,C}^\alpha : \text{Ext}(A, C^\alpha) \to \text{Ext}(A, C^\beta) $$

$$ \varphi : \text{Ext}(A, C^\alpha) \to \text{Hom}(A^\alpha, E_{\omega\alpha}(C)) $$

where $f$ is the Borel-definable homomorphism induced by the inclusion $C^\alpha \to C$. As $\text{Hom}(A^\alpha, C^\beta) = 0$, $f$ is injective. Since $\eta_{A,C}$ is an isomorphism, while $\rho_{A,C}^\alpha$ and $f$ are injective, we have that $\varphi|_{\text{Ker}(\psi)}$ is injective, concluding the proof.

4.3 Complexity of Hom. In this section, we continue to assume all the groups to be $p$-local for a fixed prime $p$.

We will compute the complexity class of $\{0\}$ in $\text{Hom}(T, L_\alpha(A))$ for countable $p$-groups $T$ and $A$ and $\alpha < \omega_1$.

Lemma 4.10. Suppose that $T$ is a reduced countable $p$-group of Ulm rank $\sigma \geq 1$. The following assertion are equivalent:

(1) either $\alpha$ is a limit ordinal, or $\alpha = \beta + 1$ is a successor ordinal and $T^\beta$ is infinite;

(2) $T = \bigoplus_{n \in \omega} T_n$ where, for every $n \in \omega$, $T_n$ is a reduced countable $p$-group of Ulm rank $\sigma$. If furthermore $\sigma = \beta + 1$ is a successor ordinal, then one can choose the $T_n$’s such that $T^\beta_n$ is finite for every $n \in \omega$.

Proof. Recall that, by [Fuc70, Theorem 17.3], a countable reduced $p$-group of Ulm rank $1$ is a sum of cyclic $p$-groups. For every $\alpha < \sigma$, $T^\alpha / T^{\alpha+1}$ is a countable reduced $p$-group of Ulm rank $1$, which is in fact unbounded if $\alpha + 1 < \sigma$. Thus, one can write $T^\alpha / T^{\alpha+1} = \bigoplus_{n \in \omega} S_{n,n}$ where, for every $n \in \omega$, $S_{n,n}$ is a countable reduced $p$-group of Ulm rank $1$, such that $S_{n,n}$ is unbounded if $\alpha + 1 < \sigma$, and $S_{n,n}$ is finite and nonzero if $\alpha + 1 = \sigma$. By Ulm’s classification of countable reduced $p$-groups [Fuc73, Theorem 76.1, Corollary 76.2], for every $n \in \omega$ there exists a

$$ \rho_{A,C}^\alpha : \text{Ext}(A, C^\alpha) \to \text{Ext}(A, C^\beta) $$

$$ \varphi : \text{Ext}(A, C^\alpha) \to \text{Hom}(A^\alpha, E_{\omega\alpha}(C)) $$

where $f$ is the Borel-definable homomorphism induced by the inclusion $C^\alpha \to C$. As $\text{Hom}(A^\alpha, C^\beta) = 0$, $f$ is injective. Since $\eta_{A,C}$ is an isomorphism, while $\rho_{A,C}^\alpha$ and $f$ are injective, we have that $\varphi|_{\text{Ker}(\psi)}$ is injective, concluding the proof.
Lemma 4.11. Suppose that $A$ is a countable group, and $G$ is a group with a Polish cover. Then there is a Borel-definable injective homomorphism $\text{Hom}(A,G) \to G^\omega$. If $A$ is generated by $n$ elements, then there is a Borel-definable injective homomorphism $A \to G^n$.

Proof. Consider a short exact sequence $R \to F \to A$, where $F$ is a countable free group. This induces a Borel-definable injective homomorphism $\text{Hom}(A,G) \to \text{Hom}(F,G)$. Notice now that, since $F$ is free, $\text{Hom}(F,G)$ is Borel-definably isomorphic to a product of copies of $G$ indexed by the elements of a $\mathbb{Z}$-basis for $F$. \hfill $\square$

Lemma 4.12. Suppose that $A$ is a countable reduced $p$-group, $d \in \omega$, and $\alpha < \omega_1$ is a limit ordinal such that $p^\beta A \neq 0$ for every $\beta < \alpha$. Then:

- $E_0(A)$ is divisible;
- $E_0(A)[p^d] = \{x \in E_0(A) : p^d x = 0\}$ is a Polishable subgroup of $E_0(A)$;
- $\Sigma_0^2$ is the complexity class of $\{0\}$ in $E_0(A)[p^d]$ and in $E_0(A)$.

Proof. After replacing $A$ with $A/p^\omega A$ we can assume that $p^\omega A = 0$. Thus, the canonical map $A \to L_\alpha(A)$ is a monomorphism, and $E_0(A) = L_\alpha(A)/A$. Notice that $p L_\alpha(A)$ is an open subgroup of $L_\alpha(A)$, and $A$ is a dense subgroup $L_\alpha(A)$. Thus, $p L_\alpha(A) + A = L_\alpha(A)$. Hence, the map $p L_\alpha(A) \to E_0(A)$, $a \mapsto a + A$ is surjective, and $E_0(A)$ is divisible.

Since $p^\beta A \neq 0$ for every $\beta < \alpha$, we have that $L_\alpha(A)$ is uncountable. In particular, $E_0(A)$ and $E_0(A)[p^d]$ are nontrivial.

Fix a strictly increasing sequence $(\alpha_n)_{n \in \omega}$ of countable ordinals such that $\alpha_0 = 0$ and $\sup_n \alpha_n = \alpha$. We have that $E_0(A)[p^d] = \hat{G}/A$ where $\hat{G} = \{x \in L_\alpha(A) : p^d x \in A\} \subseteq L_\alpha(A)$. The Polish group topology on $\hat{G}$ is defined by letting $(x_k)$ converge to $0$ if and only if $x_k \to x$ in $L_\alpha(A)$ and $p^d x_k = p^d x$ eventually. Thus, $E_0(A)[p^d]$ is a Polishable subgroup of $E_0(A)$. This also follows from the fact that $E_0(A)[p^d]$ is the kernel of the Borel-definable homomorphism $E_0(A) \to E_0(A)$, $x \mapsto p^d x$.

It is clear that $\{0\} = \Sigma_0^2$ in $E_0(A)$, since $A$ is countable. We now show that $\{0\}$ is dense in $E_0(A)[p^d]$.

Consider an element $x$ of $E_0(A)[p^d]$. Then $x$ can be written as $a + A$ where $a \in L_\alpha(A)$. In turn, one can write $a$ as $\sum_{n \in \omega} \alpha_n$ for $a_n \in p^{\alpha_n} A$. Since $p^d x = 0$ we have that $\sum_n p^d a_n \in A \cap p^d L_\alpha(A) = p^d A$. Thus, we can find $b \in A$ such that $\sum_n p^d a_n = p^d b$. After replacing $a_0$ with $a_0 - b$ we can assume that $\sum_n p^d a_n = 0$. Thus, for every $k \in \omega$, we have that $\sum_{n=0}^k p^d a_n \in p^{\alpha_{k+1}} A$ and hence we can find $b_k \in p^{\alpha_{k+1}} A$ such that $\sum_{n=0}^k p^d a_n = p^d b_k$. Thus, we have that $(a_0 + a_1 + \cdots + a_k - b_k)_{k \in \omega}$ is a sequence in $\hat{G}$ that converges to $a$. This shows that $\{0\}$ is dense in $E_0(A)[p^d]$, and hence not $\Pi_0^1$ in $E_0(A)[p^d]$. This concludes the proof that $\Sigma_0^2$ is the complexity class of $\{0\}$ in $E_0(A)[p^d]$ and in $E_0(A)$.

$\square$

Lemma 4.13. Suppose that $T = \mathbb{Z}(p^\omega)$ is the Prüfer $p$-group, and $A$ is a countable unbounded reduced $p$-group. Then $\Pi_0^1$ is the complexity class of $\{0\}$ in $\text{Hom}(T,E_0(A))$.

Proof. By Lemma 4.11 and Lemma 4.12, $\{0\}$ is $\Pi_0^1$ in $\text{Hom}(T,A)$. Thus, it remains to prove that $\{0\}$ is not $\Sigma_0^1$ in $\text{Hom}(T,A)$. After replacing $A$ with $A/A^2$, we can assume that $A$ has Ulm length $1$. Thus $A$ is a direct sum of cyclic groups $[\text{Fuc70, Theorem 17.2}]$. We can write $A = \bigoplus_k A_k$, where $A_k$ is a nonzero direct sum of cyclic groups of order $p^k$ for some strictly increasing sequence $(\ell_k)$ with $\ell_0 \geq 1$. (Notice that $\ell_k \geq k + 1$ for every $k \in \omega$.) Then we have that

$$L_\omega(A) = \left\{ (x_k) \in \prod_k A_k : \lim_{k \to \infty} h^{A_k}_{p^k}(x_k) = \omega \right\}$$

where $h^{A_k}_{p^k}(x) = \max \{k < \omega : x \in p^k A_k\}$ for $k \in \omega$ and $x \in A_k$. Hence, $\text{Hom}(T,E_0(A))$ is Borel-definably isomorphic to $\hat{G}/A^\omega$, where

$$\hat{G} = \{ (x_n) \in L_\omega(A) : px_0 \equiv 0 \mod A, \forall n \in \omega, x_n \equiv px_{n+1} \mod A \}.$$

By Proposition 2.3, it suffices to prove that the equivalence relation $E_0^\omega$ is Borel reducible to the coset relation of $A^\omega$ inside $\hat{G}$. After replacing $A$ with a direct summand, we can assume that $\ell_k \geq 2k + 1$ for every $k \in \omega$. 

\hfill $\square$
Define $\hat{X} := \{0, 1, \ldots, p-1\}^{\omega \times \omega}$, and let $E$ to be the equivalence relation on $\hat{X}$ defined by setting $(\varepsilon_{i,k})E(\varepsilon'_{i,k}) \iff \forall i \exists k (\varepsilon) \forall k \geq k(i), \varepsilon, \varepsilon'_{i,k} = \varepsilon'_{i,k}$. We claim that there is a Borel-definable injective function $\hat{X}/E \to \hat{G}/A^\omega$, namely $E$ is Borel reducible to the coset relation of $A^\omega$ inside $\hat{G}$. Since $E$ is Borel bireducible with $E_0^\omega$, this will conclude the proof.

Define recursively for $k \in \omega$ and $0 \leq i \leq k$ pairwise distinct elements $a_{i,k}(0), a_{i,k}(1), \ldots, a_{i,k}(p-1) \in p^{2k-1}A_k$ of order $p^{i+1}$ such that $p a_{i,k}(\varepsilon) = a_{i,k}(\varepsilon)$ for $\varepsilon \in \{0, 1, \ldots, p-1\}$. (Recall that $A_k$ is a sum of cyclic groups of order $p^k$ where $\ell_k \geq 2k+1$.) Given a sequence $(\varepsilon_{i,k}) \in \hat{X} = \{0, 1, \ldots, p-1\}^{\omega \times \omega}$ define the element $(x_i)_{i \in \omega} \in \hat{G}$ where, for $i \in \omega$, $x_i = (b_{i,k})_{k \in \omega} \in L_\omega(A) \subseteq \prod_k A_k$ is defined by setting $b_{i,k} = 0 \in A_k$ for $k < i$, and

$$b_{i,k} = a_{0,k}(\varepsilon_{i,k}) + a_{1,k}(\varepsilon_{i-1,k}) + \cdots + a_{i,k}(\varepsilon_{0,k})$$

for $k \geq i$. Notice that, in particular, we have that

$$b_{0,k} = a_{0,k}(\varepsilon_{0,k}) \in p^k A_k$$

has order $p$, and

$$pb_{i+1,k} = p(a_{0,k}(\varepsilon_{i+1,k}) + a_{1,k}(\varepsilon_{i,k}) + \cdots + a_{i,k}(\varepsilon_{1,k}) + a_{i+1,k}(\varepsilon_{0,k}))$$

$$= p(a_{1,k}(\varepsilon_{i,k}) + \cdots + a_{i,k}(\varepsilon_{1,k}) + a_{i+1,k}(\varepsilon_{0,k}))$$

$$= a_{0,k}(\varepsilon_{i,k}) + \cdots + a_{i-1,k}(\varepsilon_{1,k}) + a_{i,k}(\varepsilon_{0,k})$$

$$= b_{i,k}.$$}

Thus, $(x_i)_{i \in \omega}$ indeed defines an element of $\hat{G}$. The function $\hat{X} \to \hat{G}$, $(\varepsilon_{i,k}) \mapsto (x_i)$ is continuous. We claim that it induces an injection $\hat{X}/E \to \hat{G}/A^\omega$. Suppose that $(\varepsilon_{i,k})$ and $(\varepsilon'_{i,k})$ are elements of $\hat{X}$. Define as above $x_i = (b_{i,k}) \in L_\omega(A) \subseteq \prod_k A_k$ and $x'_i = (b'_{i,k}) \in L_\omega(A) \subseteq \prod_k A_k$ for $i \in \omega$ such that, for $k < i$, $b_{i,k} = b'_{i,k} = 0$, and for $k \geq i$,

$$b_{i,k} = a_{0,k}(\varepsilon_{i,k}) + a_{1,k}(\varepsilon_{i-1,k}) + \cdots + a_{i,k}(\varepsilon_{0,k})$$

$$b'_{i,k} = a_{0,k}(\varepsilon'_{i,k}) + a_{1,k}(\varepsilon'_{i-1,k}) + \cdots + a_{i,k}(\varepsilon'_{0,k}).$$

Suppose that, for some increasing sequence $(k(i))_{i \in \omega}$ in $\omega$, for every $i \in \omega$ and $k \geq k(i)$ one has that $\varepsilon_{i,k} = \varepsilon'_{i,k}$. We claim that, for every $i \in \omega$, $x_i - x'_i \in A = \bigoplus_k A_k$. To this purpose, it suffices to prove that for every $i \in \omega$ and for every $k \geq k(i)$ one has that $b_{i,k} = b'_{i,k}$. Indeed, we have that, for $k \geq k(i)$,

$$b_{i,k} = a_{0,k}(\varepsilon_{i,k}) + a_{1,k}(\varepsilon_{i-1,k}) + \cdots + a_{i,k}(\varepsilon_{0,k})$$

$$= a_{0,k}(\varepsilon'_{i,k}) + a_{1,k}(\varepsilon'_{i-1,k}) + \cdots + a_{i,k}(\varepsilon'_{0,k})$$

$$= b'_{i,k}.$$}

Conversely, suppose that $x_i - x'_i \in A = \bigoplus_k A_k$ for every $i \in \omega$. This implies that there exists an increasing sequence $(k(i))_{i \in \omega}$ in $\omega$ such that for every $i \in \omega$ and $k \geq k(i)$ one has that $b_{i,k} = b'_{i,k}$. We now prove by induction on $i \in \omega$ that for every $k \geq k(i)$ one has that $\varepsilon_{i,k} = \varepsilon'_{i,k}$. For $i = 0$ and $k \geq k(0)$ one has that

$$a_{0,k}(\varepsilon_{0,k}) = b_{0,k} = b'_{0,k} = a_{0,k}(\varepsilon'_{0,k})$$

and hence $\varepsilon_{0,k} = \varepsilon'_{0,k}$. Suppose that the conclusion holds for $i$. We prove it for $i + 1$. We have that, for $k \geq k(i),

$$b_{i+1,k} = a_{0,k}(\varepsilon_{i+1,k}) + a_{1,k}(\varepsilon_{i,k}) + \cdots + a_{i+1,k}(\varepsilon_{0,k})$$

and

$$b'_{i+1,k} = a_{0,k}(\varepsilon'_{i+1,k}) + a_{1,k}(\varepsilon'_{i,k}) + \cdots + a_{i+1,k}(\varepsilon'_{0,k})$$

and hence

$$a_{0,k}(\varepsilon_{i+1,k}) = b_{i+1,k} - (a_{1,k}(\varepsilon_{i,k}) + \cdots + a_{i+1,k}(\varepsilon_{0,k}))$$

$$= b'_{i+1,k} - (a_{1,k}(\varepsilon'_{i,k}) + \cdots + a_{i+1,k}(\varepsilon'_{0,k}))$$

$$= a_{0,k}(\varepsilon'_{i+1,k})$$

and hence $\varepsilon_{i+1,k} = \varepsilon'_{i+1,k}$. This concludes the proof. □
Lemma 4.14. Suppose that $T = \mathbb{Z}(p^\infty)$ is the Prüfer $p$-group, $\alpha$ is a countable ordinal, and $A$ is a countable reduced $p$-group satisfying $p^\beta A \neq 0$ for every $\beta < \omega_\alpha$. Then $\Pi^0_3$ is the complexity of $\{0\}$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$.

Proof. After replacing $A$ with $A/A^\alpha$, we can assume that $A^\alpha = 0$. By Lemma 4.11 and Lemma 4.12, $\{0\}$ is $\Pi^0_3$ in $\text{Hom}(T, A)$. It remains to prove that $\{0\}$ is not $\Sigma^0_3$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$. If $\alpha = \beta + 1$ for some $\beta < \omega_\alpha$, then we have a Borel-definable injective homomorphism $E_{\omega_\alpha}(A^\beta) \to E_{\omega_\alpha}(A)$, which induces an Borel-definable injective homomorphism $\text{Hom}(T, E_{\omega_\alpha}(A^\beta)) \to \text{Hom}(T, E_{\omega_\alpha}(A))$. By Lemma 4.13 applied to $A^\beta$, $\{0\}$ is not $\Sigma^0_3$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$.

Suppose now that $\alpha$ is a limit ordinal. Thus, there is a strictly increasing sequence $(\alpha_k)$ of countable ordinals such that $\alpha = \sup_k \alpha_k$. As in the proof of Lemma 4.10, by Ulm’s classification of countable $p$-groups, one can write $A = \bigoplus_k A_k$ where, for every $k \in \omega$, $A_k$ is a countable reduced $p$-group of Ulm length $\alpha_k$. In this case, one has that

$$L_{\omega_\alpha}(A) = \left\{ (x_k) \in \prod_{k \in \omega} A_k : \lim_{k \to \infty} h^t_{\beta}(x_k) = \omega \alpha \right\}$$

where $h^t_{\beta}(x_k) = \sup \{ \beta < \omega_\alpha : x \in p^\beta A_k \}$ for $k \in \omega$ and $x \in A_k$. One can then proceed as in the proof of Lemma 4.13. \hfill \Box

Proposition 4.15. Suppose that $T$ is a nonzero countable $p$-group, and $A$ is a countable reduced $p$-group, and $\alpha < \omega_1$ is an ordinal such that $p^\beta A \neq 0$ for every $\beta < \omega_\alpha$.

1. If $T$ is finite, then $\Sigma^0_3$ is the complexity class of $\{0\}$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$.
2. If $T$ is infinite, then $\Pi^0_3$ is the complexity class of $\{0\}$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$.

Proof. Suppose initially that $T$ is finite. In this case, $T$ is a finite sum of cyclic $p$-groups. Thus, without loss of generality, we can assume that $T$ is cyclic of order $p^d$ for some $d \geq 1$. In this case, we have that $\text{Hom}(T, E_{\omega_\alpha}(A))$ is Borel-definably isomorphic to $E_{\omega_\alpha}(A) [p^d]$, and the conclusion follows from Lemma 4.12.

Suppose now $T$ is infinite. By Lemma 4.11 and Lemma 4.12, $\{0\}$ is $\Pi^0_3$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$. We now prove that $\{0\}$ is not $\Sigma^0_3$ in $\text{Hom}(T, E_{\omega_\alpha}(A))$. It suffices to consider the case when $T$ is either reduced or divisible. Consider initially the case when $T$ is reduced. The quotient map $T \to T/T^1$ induces a Borel-definable injective homomorphism

$$\text{Hom}(T/T^1, E_{\omega_\alpha}(A)) \to \text{Hom}(T, E_{\omega_\alpha}(A)).$$

Then after replacing $T$ with $T/T^1$, we can assume that $T$ has Ulm rank 1, and hence $T \cong \bigoplus_n T_n$ where, for every $n \in \omega$, $T_n$ is a nonzero cyclic $p$-group. In this case, we have that

$$\text{Hom}(T, E_{\omega_\alpha}(A)) \cong \prod_{n \in \omega} \text{Hom}(T_n, E_{\omega_\alpha}(A)).$$

In this case, the conclusion follows from Lemma 2.6 and the case when $T$ is finite.

Suppose now that $T$ is divisible. In this case, by the classification theorem of divisible groups [Fuc70, Theorem 23.1], $T$ is a direct sum of copies of $\mathbb{Z}(p^\infty)$. Thus, by Lemma 2.6 it suffices to consider the case when $T = \mathbb{Z}(p^\infty)$. In this case, the conclusion follows from Lemma 4.14. \hfill \Box

4.4. The Solecki subgroups of Ext. In this section, we continue to assume all the groups to be $p$-local. We will prove that, for countable $p$-groups $T$ and $A$, the Solecki and Ulm subgroups of $\text{PExt}(T, A)$ coincide; see Theorem 4.19. In view of Lemma 4.6, the same conclusion holds when $T$ and $A$ are arbitrary countable groups, with $T$ torsion.

Lemma 4.16. Suppose that $A, T$ are countable $p$-groups. Then there exist Borel-definable isomorphisms $\tau^\alpha_{T,A} : \text{PExt}(T^\alpha, A^\alpha) \to \text{Ext}(T, A)^{\alpha+1}$ for $\alpha < \omega_1$ that make the diagrams

$$\begin{array}{c}
\text{PExt}(T^\alpha, A^\alpha) \xrightarrow{\tau^\alpha_{T,A}} \text{Ext}(T, A)^{\alpha+1} \\
\gamma_{\text{PExt}}^{T^\alpha, A^\alpha} \uparrow \\
\text{PExt}(T^{\alpha+1}, A^{\alpha+1}) \xrightarrow{\tau^{\alpha+1}_{T,A}} \text{Ext}(T, A)^{\alpha+2}
\end{array}$$

commute for every $\alpha < \omega_1$. 
Proof. We define \( \gamma^\alpha_{T,A} \) by recursion on \( \alpha < \omega_1 \). For \( \alpha = 0 \) one lets \( \gamma^0_{T,A} \) to be the identity map. Suppose that \( \gamma^\beta_{T,A} \) has been defined. The Borel-definable monomorphism \( r^T_{T,A} : \text{Ext} \left( T^{\alpha+1}, A^{\alpha+1} \right) \to \text{PExt} \left( T^\alpha, A^\alpha \right) \) as in Corollary 4.8 restricts to a Borel-definable isomorphism

\[
\gamma^\alpha_{T,A} : \text{PExt} \left( T^{\alpha+1}, A^{\alpha+1} \right) \to \text{PExt} \left( T^\alpha, A^\alpha \right)
\]

Thus, \( \gamma^{\alpha+1}_{T,A} := \gamma^\alpha_{T,A} \circ \gamma^1_{T,A} \) is a Borel-definable isomorphism

\[
\text{PExt} \left( T^{\alpha+1}, A^{\alpha+1} \right) \to \text{Ext} \left( T, A \right)^{\alpha+2}
\]

Suppose now that \( \alpha \) is a limit ordinal such that \( \gamma^\beta_{T,A} \) has been defined for every \( \beta < \alpha \). Consider the Borel-definable injective homomorphism

\[\gamma^\alpha_{T,A} : \text{Ext} \left( T^\alpha, A^\alpha \right) \to \text{Ext} \left( T, A \right)^\alpha\]

This restricts to a Borel-definable isomorphism

\[\gamma^\alpha_{T,A} : \text{PExt} \left( T^\alpha, A^\alpha \right) \to \text{Ext} \left( T, A \right)^{\alpha+1}\]

Define then \( \gamma^\alpha_{T,A} = \gamma^\alpha_{T,A} \).

\[\square\]

Lemma 4.17. Suppose that \( A, T \) are countable \( p \)-groups. Then \( s_1 \left( \text{Ext} \left( T, A \right) \right) = s_1 \left( \text{PExt} \left( T, A \right) \right) = \text{PExt} \left( T, A \right)^1 \).

Proof. Consider the Borel-definable exact sequence

\[0 \to \text{Ext} \left( T^1, A^1 \right) \xrightarrow{\gamma^1_{T,A}} \text{PExt} \left( T, A \right) \to \text{Hom} \left( T^1, E_\omega \left( A \right) \right) \to 0\]

as in Corollary 4.8(6), where \( r^T_{T,A} \) restricts to a Borel-definable isomorphism

\[\gamma^1_{T,A} : \text{PExt} \left( T^1, A^1 \right) \to \text{PExt} \left( T, A \right)^1\]

Since \( \{0\} \) is \( \Pi^0_3 \) in \( \text{Hom} \left( T^1, E_\omega \left( A \right) \right) \) by Proposition 4.15, we have that \( r^T_{T,A} \left( \text{Ext} \left( T^1, A^1 \right) \right) \) is \( \Pi^0_3 \) in \( \text{PExt} \left( T, A \right) \). Thus, since \( \{0\} \) is dense in \( s_1 \left( \text{PExt} \left( T, A \right) \right) \) and \( s_1 \left( \text{PExt} \left( T, A \right) \right) \) is the smallest \( \Pi^0_3 \) Polishable subgroup of \( \text{Hom} \left( T^1, E_\omega \left( A \right) \right) \), we have that \( s_1 \left( \text{PExt} \left( T, A \right) \right) \) is contained in

\[\left\{ 0 \right\} \gamma^1_{T,A} \left( \text{Ext} \left( T^1, A^1 \right) \right) = \gamma^1_{T,A} \left( \text{PExt} \left( T^1, A^1 \right) \right) = \text{PExt} \left( T, A \right)^1\]

It remains to prove that

\[\text{PExt} \left( T, A \right)^1 = \gamma^1_{T,A} \left( \text{PExt} \left( T^1, A^1 \right) \right) \subseteq s_1 \left( \text{PExt} \left( T, A \right) \right) = s_1 \left( \text{Ext} \left( T, A \right) \right)\]

We can write \( T = F/R \) where \( F \) is a countable free group and \( R \subseteq F \) is a subgroup. Let \( \mathbb{N} \) denote the set of strictly positive integers. For \( n \in \mathbb{N} \), define \( T_n = \{x \in T : p^n x = 0\} \). Then \( (T_n)_{n \in \mathbb{N}} \) is an increasing sequence of subgroups of \( T \) such that, for every \( n \in \mathbb{N} \), \( T_n \) is a direct sum of cyclic \( p \)-groups. We can fix a free basis \( (x_{n,j})_{n,j \in \mathbb{N}} \) for \( F \) and \( S_n \subseteq \mathbb{N} \) for \( n \in \mathbb{N} \) such that:

1. For every \( n \in \mathbb{N} \), \( (x_{n,j} + R)_{j \in S_n} \) is a \( p \)-basis of \( T_n \) [Fuc70, Section 32], \( x_{n,j} + R \) generates a cyclic subgroup \( T_{n,j} \) of \( T_n \) of order \( d_{n,j} \in \mathbb{N} \), and \( T_n = \bigoplus_{j \in S_n} T_{n,j} \);
2. \( R \) is generated by elements of the form

\[px_{n,t} - \sum_{j \in S_{n-1}} k_j x_{n-1,j}\]

for \( n \in \mathbb{N} \), \( t \in S_n \), and \( k_j \in \{0, 1, \ldots, d_{n-1,j} - 1\} \), where we set \( S_0 = \emptyset \) and by convention the empty sum is equal to 0.

Let \( R_t \) be the set of such generators with \( n \in \{1, 2, \ldots, \ell \} \). We identify \( \text{Ext} \left( C, A \right) \) with

\[\frac{\text{Hom} \left( R, A \right)}{\text{Hom} \left( F|R, A \right)}\]

via the Borel-definable isomorphisms as in Proposition 3.1. Under this isomorphism, \( \text{PExt} \left( T, A \right)^1 = \gamma^1_{T,A} \left( \text{PExt} \left( T^1, A^1 \right) \right) \) corresponds to the subgroup

\[\frac{\text{Hom}_f \left( R, A^1 \right) + \text{Hom} \left( F|R, A \right)}{\text{Hom} \left( F|R, A \right)}\].
By Lemma 2.4, it suffices to prove that if $V$ is an open neighborhood of 0 in $\text{Hom}(F|R,A)$, then the closure $V^{\text{Hom}(R,A)}$ of $V$ in $\text{Hom}(R,A)$ contains a neighborhood of 0 in $\text{Hom}_f(R,A)$.  

Since $V$ is an open neighborhood of 0 in $\text{Hom}(F|R,A)$, there exists $n_0 \in \mathbb{N}$ such that

$$\{\rho|_{R} : \rho \in \text{Hom}(F,A), \forall i < n_0, \forall j \in S_i, \rho(x_{i,j}) = 0\} \subseteq V.$$ 

Consider the open neighborhood

$$W := \{\rho \in \text{Hom}_f(R,A^1) : \rho|_{R_{n_0}} = 0\}$$

of 0 in $\text{Hom}(R,A^1)$. We now prove that $W \subseteq V^{\text{Hom}(R,A)}$. To this purpose, fix $\rho \in W$, and let $U$ be an open neighborhood of $\rho$ in $\text{Hom}(R,A)$. Thus, there exists $n_1 \geq n_0$ such that

$$\{\eta \in \text{Hom}(R,A) : \eta|_{R_{n_1}} = \rho|_{R_{n_1}}\} \subseteq U.$$ 

Thus, to conclude the proof it suffices to show that there exists $\tilde{\rho} \in \text{Hom}(F,A)$ satisfying:

1. $\tilde{\rho}|_{R_{n_1}} = \rho|_{R_{n_1}}$;
2. $\tilde{\rho}(x_{i,j}) = 0$ for $i \leq n_0$ and $j \in S_i$.

We define a $\tilde{\rho} : F \to A$ by setting $\tilde{\rho}(x_{i,j}) = 0$ for $i < n_0$ and for $i > n_1$. We now define $\tilde{\rho}(x_{\ell,j}) \in p^{n_1-\ell}A$ for $n_0 \leq \ell \leq n_1$ and every $j \in S_\ell$ by recursion on $\ell$ as follows. Suppose that $\tilde{\rho}(x_{i,j}) \in p^{n_1-i}A$ has been defined for $i < \ell$ and $j \in S_i$, for some $\ell \in \{n_0, \ldots, n_1 - 1\}$. For $t \in S_\ell$, there exists a unique expression

$$r_{\ell,t} := px_{\ell,t} - \sum_{j \in S_{\ell-1}} k_jx_{\ell-1,j} \in R$$

where $k_j \in \{0, 1, \ldots, d_{\ell-1,j} - 1\}$. By the inductive hypothesis, we have that

$$\rho(r_{\ell,t}) + \sum_{j \in S_{\ell-1}} k_j\tilde{\rho}(x_{i,j}) \in A^{1} + p^{n_1-\ell+1}A = p^{n_1-\ell+1}A.$$

Thus, there exists $\tilde{\rho}(x_{\ell,j}) \in p^{n_1-\ell}A$ such that

$$\tilde{\rho}(r_{\ell,t}) = p\tilde{\rho}(x_{\ell,j}) - \sum_{j \in S_{\ell-1}} k_j\tilde{\rho}(x_{i,j}) = \rho(r_{\ell,t}).$$

This concludes the recursive definition of $\tilde{\rho}(x_{\ell,j})$. By construction, we have that $\tilde{\rho}$ has the required properties. □

**Lemma 4.18.** Suppose that $A, T$ are countable $p$-groups, and $n \in \omega$. Then $s_n(\text{Ext}(T,A)) = s_n(P\text{Ext}(T,A)) = P\text{Ext}(T,A)^n$.

**Proof.** We prove the statement by induction on $n \in \omega$. For $n = 0$, this follows from the fact that $P\text{Ext}(T,A)$ is the closure of $\{0\}$ in $\text{Ext}(T,A)$ by Lemma 3.2. Suppose that the conclusion holds for $n$. We will show that it holds for $n + 1$. We need to show that $s_1(P\text{Ext}(T,A)^n) = P\text{Ext}(T,A)^{n+1}$. By Lemma 4.16 there exists a Borel-definable isomorphisms $\tau_{T,A}^n : P\text{Ext}(T^n, A^n) \to P\text{Ext}(T,A)^n$ for $n \in \omega$ that make the diagrams

$$\begin{array}{ccc}
P\text{Ext}(T^n, A^n) & \xrightarrow{\tau_{T,A}^n} & P\text{Ext}(T,A)^n \\
\gamma_{T^n,A^n}^1 \uparrow & & \uparrow \\
P\text{Ext}(T^{n+1}, A^{n+1}) & \xrightarrow{\tau_{T,A}^{n+1}} & P\text{Ext}(T,A)^{n+1}
\end{array}$$

commute for every $n < \omega$. Thus, it suffices to prove that

$$s_1(P\text{Ext}(T^n, A^n)) = \gamma_{T^n,A^n}(P\text{Ext}(T^{n+1}, A^{n+1})) = P\text{Ext}(T^n, A^n)^1.$$ 

This follows from Lemma 4.17 applied to $T^n$ and $A^n$. □

**Theorem 4.19.** Suppose that $A, T$ are countable $p$-groups, and $\alpha < \omega_1$ is an ordinal. Then $s_\alpha(\text{Ext}(T,A)) = s_\alpha(P\text{Ext}(T,A)) = P\text{Ext}(T,A)^\alpha = \text{Ext}(T,A)^{1+\alpha}$. In particular, the Solecki rank and the Ulm rank of $P\text{Ext}(T,A)$ are equal.
Proof. We prove this holds by induction on $\alpha < \omega_1$. The case $\alpha < \omega$ has already been considered in Lemma 4.18.

If $\alpha$ is a limit ordinal, and the conclusion holds for $\beta < \alpha$, then we have that

$$s_{\alpha}(\text{Ext}(T, A)) = \bigcap_{\beta < \alpha} s_{\beta}(\text{Ext}(T, A)) = \bigcap_{\beta < \alpha} \text{Ext}(T, A)^{\beta} = \text{Ext}(T, A)^{\alpha}.$$ 

Suppose now that the conclusion holds for $\alpha$. We will show that it holds for $\alpha + 1$. By Lemma 4.18 we can assume that $\alpha \geq \omega$, in which case we have that $1 + \alpha = \alpha$ and hence $\text{PExt}(T, A)^{\alpha} = \text{Ext}(T, A)^{\alpha}$ and $\text{PExt}(T, A)^{\alpha+1} = \text{Ext}(T, A)^{\alpha+1}$. Thus, in this case we need to show that $s_1(\text{Ext}(T, A)^{\alpha}) = \text{Ext}(T, A)^{\alpha+1}$.

By Lemma 4.16 there exist Borel-definable isomorphisms $\tau^\beta_{T,A} : \text{PExt}(T^\beta, A^\beta) \to \text{PExt}(T, A)^{\beta+1}$ for $\beta < \omega_1$ for such that the diagrams

$$
\begin{array}{c}
\text{PExt}(T^\beta, A^\beta) \\
\downarrow \tau^\beta_{T,A} \\
\text{Ext}(T, A)^{\beta+1} \\
\end{array}
\begin{array}{c}
\gamma_{T,A}^{\beta,\alpha} \\
\text{PExt}(T^{\beta+1}, A^{\beta+1}) \\
\uparrow \\
\text{Ext}(T, A)^{\beta+2} \\
\end{array}
$$

commute for every $\beta < \omega_1$. Consider initially the case when $\alpha = \beta + 1$ is a successor ordinal. Thus, it suffices to prove that

$$s_1(\text{PExt}(T^\beta, A^\beta)) = \gamma_{T,A}^{1,\alpha}(\text{PExt}(T^{\beta+1}, A^{\beta+1})).$$

This follows from Lemma 4.17 applied to $T^\beta$ and $A^\beta$.

Suppose now that $\alpha$ is a limit ordinal. Consider the Borel-definable exact sequence

$$0 \to \text{Ext}(T^\alpha, A^\alpha) \xrightarrow{r^\alpha_{\beta,T,A}} \text{Ext}(T, A)^{\alpha} \to \text{Hom}(T^\alpha, E_{\omega_1}(A)) \to 0$$

as in Corollary 4.9(6). Since $\{0\}$ is $\Pi_1^0$ in $\text{Hom}(T^\alpha, E_{\omega_1}(A))$ by Proposition 4.15, we have that $r^\alpha_{\beta,T,A}(\text{Ext}(T^\alpha, A^\alpha))$ is $\Pi_1^0$ in $\text{Ext}(T, A)^{\alpha}$. Since $s_1(\text{Ext}(T, A)^{\alpha})$ is the smallest $\Pi_1^0$ subgroup of $\text{Ext}(T, A)^{\alpha}$, this implies that $s_1(\text{Ext}(T, A)^{\alpha}) \subseteq r^\alpha_{\beta,T,A}(\text{Ext}(T^\alpha, A^\alpha))$. Since $r^\alpha_{\beta,T,A}(\text{PExt}(T^\alpha, A^\alpha)) = \text{Ext}(T, A)^{\alpha+1}$ is the closure of $\{0\}$ in $r^\alpha_{\beta,T,A}(\text{Ext}(T^\alpha, A^\alpha))$ by Lemma 3.2 and $\{0\}$ is dense in $s_1(\text{Ext}(T, A)^{\alpha})$, we have that $s_1(\text{Ext}(T, A)^{\alpha}) \subseteq \text{Ext}(T, A)^{\alpha+1}$. Thus, it remains to prove that $\text{Ext}(T, A)^{\alpha+1} = \gamma_{T,A}^{\alpha}(\text{PExt}(T^\alpha, A^\alpha)) \subseteq s_1(\text{Ext}(T, A)^{\alpha})$.

We now proceed as in the proof of Lemma 4.17, also adopting the same notation. We identify $\text{Ext}(T, A)$ with

$$\frac{\text{Hom}(R, A)}{\text{Hom}(F|R, A)}$$

via the Borel-definable isomorphisms as in Proposition 3.1. Under this isomorphism, for $\beta \leq \alpha$, the image of $\text{Ext}(T, A)^{\beta+1} = \gamma_{T,A}^{\beta}(\text{PExt}(T^\beta, A^\beta))$ corresponds to

$$\frac{\text{Hom}_f(R, A^\beta) + \text{Hom}(F|R, A)}{\text{Hom}(F|R, A)}.$$ 

Thus, by the inductive hypothesis we have that $s_\alpha(\text{Ext}(T, A))$ corresponds to

$$\frac{\tilde{H}}{\text{Hom}(F|R)}$$

where

$$\tilde{H} := \bigcap_{\beta < \alpha} (\text{Hom}_f(R, A^\beta) + \text{Hom}(F|R, A))$$

Thus, by Lemma 2.4, it suffices to prove that if $V$ is an open neighborhood of 0 in $\text{Hom}(F|R, A)$, then the closure $\overline{\nabla^H}$ of $V$ in $\tilde{H}$ contains an open neighborhood of 0 in $\text{Hom}_f(R, A^\alpha)$.

There exists $n_0 \in \mathbb{N}$ such that

$$\{\rho_R : \rho \in \text{Hom}(F, A), \forall i \leq n_0, \forall j \in S_i, \rho(x_{i,j}) = 0\} \subseteq V.$$ 

Let $W$ be the open neighborhood

$$\{\rho \in \text{Hom}_f(R, A^\alpha) : \rho|_{R_{n_0}} = 0\}$$
of 0 in $\text{Hom}_f(R, A)$. We claim that $W \subseteq \mathbb{T}^B$. To show this, fix $\rho \in W$. Let $U$ be an open neighborhood of $\rho$ in $\hat{H}$. Thus, there exist $\beta < \alpha$ and $n_1 \geq n_0$ such that
$$\{\eta \in \text{Hom}_f(R, A^\beta) : \eta|_{R_{n_1}} = \rho|_{R_{n_1}}\} \subseteq U.$$  
In order to conclude the proof, it suffices to define a homomorphism $\tilde{\rho} : F \to A$ such that:
- $\tilde{\rho}|_{R_{n_1}} = \rho|_{R_{n_1}}$;
- $\tilde{\rho}(x_i, j) = 0$ for $i \leq n_0$ and $j \in S_i$.

Indeed, such a $\tilde{\rho}$ would then belong to $U \cap V$. Using the fact that $\beta + 1 < \alpha$ and $\rho \in \text{Hom}_f(R, A^\alpha) \subseteq \text{Hom}_f(R, A^{\beta+1})$, one can define $\tilde{\rho}$ as in the proof of Lemma 4.17.

\textbf{4.5. Complexity of Ext.} The goal of this section is to compute the complexity class of $\{0\}$ in $\text{Ext}(T, A)$ given countable $p$-groups $T$ and $A$. This is the main ingredient in the proof of Theorem 1.6.

\textbf{Lemma 4.20.} Suppose that $A, T$ are countable $p$-groups, where $A$ is reduced and nonzero. Then $\text{Ext}(T, A)$ is countable if and only if (a) $T/D(T)$ is finite, and (b) $D(T) = 0$ if $A$ is unbounded.

\textit{Proof.} Recall the definition of $\text{Ext}_w$ from Section 3.3. Notice that, by Lemma 3.2, $\text{Ext}(T, A)$ is countable if and only if $\text{Ext}_w(T, A)$ is countable, in which case $\text{Ext}(T, A) = \text{Ext}_w(T, A)$. By Lemma 3.3 applied to the cotorsion functor $C \mapsto C^1 = p^aC$ for $p$-local groups, we have that the quotient map $A \to A/A^3$ induces a Borel-definable epimorphism $\text{Ext}(T, A) \to \text{Ext}(T, A/A^1)$ such that $\text{PExt}(T, A)$ is the preimage of $\text{PExt}(T, A/A^1)$. Thus, we have an isomorphism of Polish groups
$$\text{Ext}_w(T, A) \cong \text{Ext}_w(T, A/A^1).$$

Hence, without loss of generality, we can assume that $A^1 = 0$ and $A$ has Ulm rank 1.

If $C$ is a cyclic $p$-group of order $p^B$, then
$$\text{Ext}(C, A) \cong \frac{A}{p^B A}$$
is countable, while
$$\text{Ext}_w(\mathbb{Z}(p^\infty), A)$$
is isomorphic to the $p$-adic completion of $A$, which is countable if and only if $A$ is bounded, in which case $\text{Ext}_w(\mathbb{Z}(p^\infty), A) \cong A$ [Fuc70, Lemma 56.6].

Suppose that $\text{Ext}(T, A)$ is countable. This implies that $\text{PExt}(T, A) = 0$ and $\text{Ext}(T, A) = \text{Ext}_w(T, A)$. By the above, $T$ must be reduced if $A$ is unbounded. After replacing $T$ with $T/D(T)$, we can assume that $T$ is reduced.

The short exact sequence $T^1 \to T \to T/T^1$ induces a Borel-definable short exact sequence
$$0 = \text{Hom}(T^1, A) \to \text{Ext}(T/T^1, A) \to \text{Ext}(T, A) \to \text{Ext}(T^1, A) \to 0$$
Thus, we have that $\text{Ext}(T/T^1, A)$ is countable. Since $T/T^1$ has Ulm rank 1, we have that $T/T^1 \cong \bigoplus_n C_n$ where, for every $n \in \omega$, $C_n$ is cyclic of order $p^{d_n}$. Thus, we have that
$$\text{Ext}(T/T^1, A) \cong \prod_{n \in \omega} \frac{A}{p^{d_n} A}.$$This implies that $\{n \in \omega : p^{d_n} > 1\}$ is finite. Therefore, $T/T^1$ is finite, whence $T^1 = 0$ and $T = T/T^1$ is finite.

The converse implication is immediate from the above remarks. \hfill $\square$

\textbf{Lemma 4.21.} Suppose that $A, T$ are countable $p$-groups, where $A$ is reduced. Then $\text{Ext}(T, A)$ is a Polish group if and only if either $T^1 = 0$ or $A$ is bounded.

\textit{Proof.} By Corollary 4.8(6) we have a Borel-definable short exact sequence
$$0 \to \text{Ext}(T^1, A^1) \to \text{PExt}(T, A) \to \text{Hom}(T^1, E_\omega(A)) \to 0.$$If $T^1 = 0$ then $\text{Ext}(T^1, A^1) = \text{Hom}(T^1, E_\omega(A)) = 0$ and hence $\text{PExt}(T, A) = 0$. Similarly, if $A$ is bounded, then $A^1 = 0$ and $E_\omega(A) = 0$ and hence $\text{PExt}(T, A) = 0$. Conversely, if $T^1$ is nonzero and $A$ is unbounded, then by Proposition 4.15 we have that $\text{Hom}(T^1, E_\omega(A))$ is nonzero and hence $\text{PExt}(T, A)$ is nonzero. \hfill $\square$
Theorem 4.22. Suppose that $A, T$ are countable $p$-groups, where $T^1$ is nonzero and $A$ is reduced and unbounded. Define $\mu$ to be the least countable ordinal such that either $T^\mu = 0$ or $\mu$ is the successor of $\mu - 1$ and $A^{\mu-1} = 0$. Then:

1. If $\mu$ is a limit ordinal, then $\Pi_\mu^0$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$;
2. If $\mu = 1 + \lambda + 1$ where $\lambda$ is either zero or limit and $T^{1+\lambda}$ is finite, then $\Sigma^0_\lambda$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$;
3. If $\mu = 1 + \lambda + n$ where $\lambda$ either zero or limit and $2 \leq n < \omega$, and $A^{1+\lambda+n-2}$ is bounded, then $\Pi^0_\mu$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$;
4. If $\mu = 1 + \lambda + n$ where $\lambda$ is either zero or limit, $2 \leq n < \omega$, $T^{1+\lambda+n-1}$ is finite, and $A^{1+\lambda+n-2}$ is unbounded, then $D(\Pi^0_\mu)$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$;
5. If $\mu = 1 + \lambda + n$ where $\lambda$ is either zero or limit, $1 \leq n < \omega$, $T^{1+\lambda+n-1}$ is infinite, and $A^{1+\lambda+n-2}$ is unbounded if $n \geq 2$, then $\Pi^0_{\mu+1}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$.

In particular, $\{0\}$ is $\Pi^0_{\mu+1}$ and not $\Pi^0_\alpha$ for $\alpha < \mu$ in $\text{Ext}(T, A)$.

Proof. Recall that, by Corollary 4.8 and Corollary 4.9, for every $\alpha < \omega_1$ we have a Borel-definable short exact sequence

$$0 \to \text{Ext}(T^\alpha, A^\alpha) \to \text{Ext}(T, A)^\alpha \to \text{Hom}(T^\alpha, E_{\omega\alpha}(A)) \to 0$$

and a Borel-definable injective homomorphism

$$\text{Ext}(T, A)^\alpha \to \text{Hom}(T^\alpha, E_{\omega\alpha}(A)) \oplus \text{Ext}(T^\alpha, A).$$

By Theorem 4.19, if $1 + \alpha$ is the Ulm rank of $\text{Ext}(T, A)$, then the Solecki rank of $\text{Ext}(T, A)$ is $\alpha$.

1. Suppose that $\mu$ is a limit ordinal. In this case, $T$ is reduced and $\mu$ is the Ulm rank of $T$. For $\alpha < \mu$ and $\delta < \omega_\alpha$ we have $T^\alpha \neq 0$ and $p^\delta A \neq 0$. Therefore, by Proposition 4.15, for $\alpha < \mu$, $\text{Ext}(T^\alpha, E_{\omega\alpha}(A)) \neq 0$, and hence $\text{Ext}(T, A)^\alpha \neq 0$. Similarly, for $\mu \leq \alpha < \omega_1$ we have that $T^\alpha = 0$ and hence $\text{Ext}(T, A)^\alpha = 0$. This shows that the Ulm rank and the Solecki rank of $\text{Ext}(T, A)$ are $\mu$. By Lemma 4.10 we have that $T \cong \bigoplus_{n \in \omega} T_n$ where, for every $n \in \omega$, $T_n$ has Ulm rank $\mu$. Therefore, $\text{Ext}(T, A) \cong \prod_{n \in \omega} \text{Ext}(T_n, A)$ where, for every $n \in \omega$, $\text{Ext}(T_n, A)$ has Solecki rank $\mu$. The conclusion thus follows from Lemma 2.6.

2. Suppose that $\mu = 1 + \lambda + 1$ where $\lambda$ is either zero or a limit ordinal, and $T^{1+\lambda}$ is finite, in which case $T$ is reduced. Thus, we have that $T^{1+\lambda+1} = 0$ and $\text{Ext}(T, A)^{1+\lambda+1} = 0$. Furthermore, we have that $\text{Ext}(T^{1+\lambda}, A)$ is countable by Lemma 4.20, and $\Sigma^0_\lambda$ is the complexity class of $\{0\}$ in $\text{Hom}(T^{1+\lambda}, E_{\omega(1+\lambda)}(A))$ by Proposition 4.15, using the assumption that $A$ is unbounded in the case when $\lambda = 0$. Considering the Borel-definable surjective homomorphism

$$\text{Ext}(T, A)^{1+\lambda} \to \text{Hom}(T^{1+\lambda}, E_{\omega(1+\lambda)}(A)) \neq 0$$

and the Borel-definable injective homomorphism

$$\text{Ext}(T, A)^{1+\lambda} \to \text{Hom}(T^{1+\lambda}, E_{\omega(1+\lambda)}(A)) \oplus \text{Ext}(T^{1+\lambda}, A),$$

we obtain that $\text{Ext}(T, A)^{1+\lambda} = s_\lambda(\text{Ext}(T, A)) \neq 0$ and $\{0\}$ is $\Sigma^0_\lambda$ in $\text{Ext}(T, A)^{1+\lambda}$. Hence, the Solecki rank of $\text{Ext}(T, A)$ is $\lambda + 1$, and $\Sigma^0_{1+\lambda+1}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$ by Theorem 2.5.

3. Suppose that $\mu = 1 + \lambda + n$ for $2 \leq n < \omega$ and $\lambda$ either zero or limit, and that $A^{1+\lambda+n-2}$ is bounded. Then we have that $A^{1+\lambda+n-1} = 0$ and $E_{\omega(1+\lambda+n-1)}(A) = 0$. Thus,

$$\text{Ext}(T^{1+\lambda+n-1}, A^{1+\lambda+n-1}) = 0$$

and

$$\text{Hom}(T^{1+\lambda+n-1}, E_{\omega(1+\lambda+n-1)}(A)) = 0,$$

and hence $\text{Ext}(T, A)^{1+\lambda+n-1} = 0$. Since $p^n A \neq 0$ for $\beta < \omega (1 + \lambda + n - 2)$ and $T^{1+\lambda+n-2}$ is infinite, we have that $\Pi^0_\mu$ is the complexity class of $\{0\}$ in $\text{Hom}(T^{1+\lambda+n-2}, E_{\omega(1+\lambda+n-2)}(A))$ by Proposition 4.15. Considering the Borel-definable short exact sequence

$$0 \to \text{Ext}(T^{1+\lambda+n-2}, A^{1+\lambda+n-2}) \to \text{Ext}(T, A)^{1+\lambda+n-2} \to \text{Hom}(T^{1+\lambda+n-2}, E_{\omega(1+\lambda+n-2)}(A)) \to 0,$$

we obtain that $\Pi^0_\mu$ is the complexity class of $\text{Ext}(T^{1+\lambda+n-2}, A^{1+\lambda+n-2})$ in $\text{Ext}(T, A)^{1+\lambda+n-2}$, the Ulm rank of $\text{Ext}(T, A)$ is $1 + \lambda + n - 1$, and the Solecki rank of $\text{Ext}(T, A)$ is $\lambda + n - 1$. 
We claim that $\Pi^0_{\lambda+n}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$. Notice that

$$\text{Ext}(T, A)^{1+\lambda+n-2} = s_{\lambda+n-2} (\text{Ext}(T, A)).$$

Thus, by Theorem 2.5, it suffices to prove that $\{0\}$ is not $\Sigma^0_2$ in $\text{Ext}(T, A)^{1+\lambda+n-2}$. It also suffices to consider the case when $T$ is either divisible or reduced.

When $T$ is divisible, we have that $\text{Ext}(T^{1+\lambda+n-2}, A^{1+\lambda+n-2}) = \text{Ext}(T, A^{1+\lambda+n-2})$ is countable by Lemma 4.20. Since $\text{Ext}(T^{1+\lambda+n-2}, A^{1+\lambda+n-2})$ is not $\Sigma^0_2$ in $\text{Ext}(T, A)^{1+\lambda+n-2}$, this implies that $\{0\}$ is not $\Sigma^0_2$ in $\text{Ext}(T, A)^{1+\lambda+n-2}$.

Consider now the case when $T$ is reduced. Since $T^{1+\lambda+n-2}$ is infinite, then as in the proof of Lemma 4.10 one can write $T = \bigoplus_{k \in \omega} T_k$ where, for every $n \in \omega$, $T_k$ is a countable $p$-group such that $T^{1+\lambda+n-2}_k$ is nonzero. Fix $k \in \omega$. Then, as above, $\text{Hom}(T^{1+\lambda+n-2}_k, E^{(1+\lambda+n-2)}_\omega(A)) \neq 0$ by Proposition 4.15. Considering the Borel-definable surjective homomorphism

$$\text{Ext}(T_k, A)^{1+\lambda+n-2} \to \text{Hom}(T^{1+\lambda+n-2}_k, E^{(1+\lambda+n-2)}_\omega(A)) \neq 0$$

and the Borel-definable short exact sequence

$$0 = \text{Ext}(T^{1+\lambda+n-2}_k, A^{1+\lambda+n-2}) \to \text{Ext}(T_k, A)^{1+\lambda+n-2} \to \text{Hom}(T^{1+\lambda+n-2}_k, E^{(1+\lambda+n-2)}_\omega(A)) = 0$$

we have that $\text{Ext}(T_k, A)^{1+\lambda+n-2} \neq 0$ and $\text{Ext}(T_k, A)^{1+\lambda+n-1} = 0$. Since $\{0\}$ is dense in $\text{Ext}(T_k, A)^{1+\lambda+n-2} = s_{\lambda+n-2} (\text{Ext}(T_k, A))$, we have that $\{0\}$ is $\Pi^0_3$ and not $\Pi^0_2$ in $\text{Ext}(T_k, A)^{1+\lambda+n-2}$. Since $\text{Ext}(T, A)^{1+\lambda+n-2}$ is Borel-definably isomorphic to $\prod_{k \in \omega} \text{Ext}(T_k, A)^{1+\lambda+n-2}$, it follows from Lemma 2.6 that $\Pi^0_3$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)^{1+\lambda+n-2}$.

(4) Suppose that $\mu = 1 + \lambda + n$ where $\lambda$ is either zero or a limit ordinal, $2 \leq n < \omega$, and $T^{1+\lambda+n-1}$ is finite, in which case $T$ is reduced, and $A^{1+\lambda+n-2}$ is unbounded. Thus, we have that $T^{1+\lambda+n} = 0$ and $\text{Ext}(T, A)^{1+\lambda+n-1} = 0$. Furthermore, $\text{Ext}(T^{1+\lambda+n-1}, A)$ is countable by Lemma 4.20, and $\Sigma^0_2$ is the complexity class of $\{0\}$ in $\text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A))$ by Proposition 4.15, since $A^{1+\lambda+n-2}$ is unbounded. Therefore, considering the Borel-definable surjective homomorphism

$$\text{Ext}(T, A)^{1+\lambda+n-1} \to \text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A)) \neq 0$$

and the Borel-definable injective homomorphism

$$\text{Ext}(T, A)^{1+\lambda+n-1} \to \text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A)) \oplus \text{Ext}(T^{1+\lambda+n-1}, A).$$

we have that $\text{Ext}(T, A)^{1+\lambda+n-1} = s_{\lambda+n-1} (\text{Ext}(T, A)) \neq 0$ and $\{0\}$ is $\Sigma^0_2$ in $\text{Ext}(T, A)^{1+\lambda+n-1}$. Hence, the Solecki rank of $\text{Ext}(T, A)$ is $\lambda + n$, and $D(\Pi^0_{\lambda+n})$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$ by Theorem 2.5.

(5) Suppose now that $\mu = 1 + \lambda + n$ where $\lambda$ is either zero or limit, $1 \leq n < \omega$, $T^{1+\lambda+n-1}$ is infinite, and $A^{1+\lambda+n-2}$ is unbounded if $n \geq 2$. Then we have that either $T^{1+\lambda+n} = 0$, or $A^{1+\lambda+n-1} = 0$. Since $T^{1} \neq 0$, this implies that $n > 1$ if $\lambda = 0$. In either case, we have that $\text{Ext}(T, A)^{1+\lambda+n} = 0$. Since $A^{1+\lambda+n-2}$ is unbounded if $n \geq 2$, by Proposition 4.15 we have that $\text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A)) \neq 0$ and hence $\text{Ext}(T, A)^{1+\lambda+n-1} \neq 0$. Therefore, we have that the Solecki rank of $\text{Ext}(T, A)$ is $\lambda + n$.

We now show that $\Pi^0_{\mu+1}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$. We consider initially the case when $A^{1+\lambda+n-1} = 0$. In this case, we have that $\text{Ext}(T, A)^{1+\lambda+n-1}$ is Borel-definably isomorphic to $\text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A))$. By Proposition 4.15, $\Pi^0_{\lambda+n}$ is the complexity class of $\{0\}$ in $\text{Hom}(T^{1+\lambda+n-1}, E^{(1+\lambda+n-1)}_\omega(A))$. Thus, $\Pi^0_{\lambda+n+1}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)^{1+\lambda+n-1} = s_{\lambda+n-1} (\text{Ext}(T, A))$. Thus, by Theorem 2.5, $\Pi^0_{\lambda+n+1}$ is the complexity class of $\{0\}$ in $\text{Ext}(T, A)$.

Consider now the case when $T^{1+\lambda+n} = 0$, whence $T$ is reduced. In this case, by Lemma 4.10, we have that $T \cong \bigoplus_{k \in \omega} T_k$ where, for every $k \in \omega$, $T_k$ is a reduced countable $p$-group of Ulm rank $1 + \lambda + n$ such that $T^{1+\lambda+n-1}_k$ is infinite. By the above discussion, for every $k \in \omega$, the Solecki rank of $\text{Ext}(T_k, A)$ is $\lambda + n$. Thus, by Lemma 2.6, $\Pi^0_{\lambda+n+1}$ is the complexity class of $\{0\}$ in $\prod_{k \in \omega} \text{Ext}(T_k, A)$. As $\prod_{k \in \omega} \text{Ext}(T_k, A)$ is Borel-definably isomorphic to $\text{Ext}(T, A)$, this concludes the proof.
Proof of Theorem 1.2. (1) By Proposition 2.3 and the remarks in Section 3.4, we have that \( R_{\text{Ext}(C,A)} \) is smooth if and only if \( \text{Ext}(C,A) \) is a Polish group. By Proposition 4.1, this holds if and only if, for every \( p \in \mathbb{P} \), \( \text{Ext}(C_p,A_p) \) is a Polish group. Finally, by Lemma 4.21, this holds if and only if, for every \( p \in \mathbb{P} \), either \( u_1(C_p) = 0 \) or \( A_p \) is bounded.

(2) As above, we have that \( R_{\text{Ext}(C,A)} \) is essentially hyperfinite if and only if \( \{0\} = \Sigma^0_2 \) in \( \text{Ext}(C,A) \). By Proposition 4.1 and Lemma 2.6, this holds if and only if, for every \( p \in \mathbb{P} \), \( \{0\} = \Sigma^0_2 \) in \( \text{Ext}(C_p,A_p) \), and the number of \( p \in \mathbb{P} \) such that \( \Sigma^0_2 \) is the complexity class of \( \{0\} \) in \( \text{Ext}(C_p,A_p) \) is finite. By Theorem 4.22, this is equivalent to the assertion that: (a) for every \( p \in \mathbb{P} \), either \( u_2(C_p) = 0 \) or \( u_1(A_p) = 0 \); and (b) for every \( p \in \mathbb{P} \) such that \( A_p \) is unbounded, \( u_1(C_p) \) is finite; and (c) the set of \( p \in \mathbb{P} \) such that \( A_p \) is nonzero is finite.

(3) Again, we have that \( R_{\text{Ext}(C,A)} \) is Borel reducible to \( E^0_3 \) if and only if \( \{0\} = \Pi^0_3 \) in \( \text{Ext}(C,A) \), if and only if for every prime number \( p \), \( \{0\} = \Pi^0_3 \) in \( \text{Ext}(C_p,A_p) \). By Theorem 4.22, for every prime number \( p \), \( \{0\} = \Pi^0_3 \) in \( \text{Ext}(C_p,A_p) \) if and only if one of the following holds: (a) \( u_2(C_p) = 0 \); or (b) \( u_1(A_p) = 0 \); or (c) \( u_3(C_p) = 0 \) and \( u_1(A_p) \) is bounded.

\( \square \)

Proof of Theorem 1.6. Let \( \Gamma \) be one of the complexity classes \( \Sigma^0_\alpha, \Pi^0_\alpha, D(\Pi^0_\alpha) \) for \( 2 \leq \alpha < \omega_1 \). By Proposition 2.2, Proposition 4.1, and the remarks in Section 3.4, we have that \( \Gamma \) is the potential complexity class of \( R_{\text{Ext}(C,A)} \) if and only if \( \Gamma \) is the complexity class of \( \{0\} \) in \( \prod_{p \in \mathbb{P}} \text{Ext}(C_p,A_p) \). The conclusion then follows from Lemma 2.6 and Theorem 4.22.  

\( \square \)

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