Canonical Quantization of the Liouville Theory,
Quantum Group Structures, and Correlation
Functions

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Abstract

We describe a self-consistent canonical quantization of Liouville theory in terms of canonical free fields. In order to keep the non-linear Liouville dynamics, we use the solution of the Liouville equation as a canonical transformation. This also defines a Liouville vertex operator. We show, in particular, that a canonical quantized conformal and local quantum Liouville theory has a quantum group structure, and we discuss correlation functions for non-critical strings.

1. Introduction

Liouville theory arises in mathematical and physical situations which involve surfaces. Although its classical dynamics was solved in terms of free fields a long time ago [1], there are still many open problems especially in the quantum case. Recent results of this theory [2], which follow from the matrix model approach or by performing path integrals, remain obscure from the point of view of an exactly solved quantum Liouville theory.

However, it is tempting to assume that the quantum structure of the Liouville theory is also completely given by free fields, as in the classical case [3-5]. Then canonical quantization might be instrumental in understanding the quantum Liouville dynamics. To simplify the calculations, we shall take the regular solution of the Liouville equation as the canonical transformation excluding the interesting singular part [6] which can be treated by regular methods [7], too. But in any case locality plays an important role [5,8].

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We are going to discuss the non-critical bosonic string theory as a particularly simple and interesting example. It induces the non-linear Liouville dynamics as an anomaly [9] and describes for space-time dimension $d = 0$ pure 2-dimensional gravity, for $d = 25$ the critical string, and in between matter coupled to gravity.

We shall follow in this talk mainly references [5,10]. First we look at the classical structure of Liouville theory and search for a suitable free field which allows us to construct a Liouville vertex operator. But canonical quantization requires deformations in order to satisfy the conformal and local properties which characterize the classical Liouville theory, otherwise unwanted anomalies would appear. These deformations indicate an internal $SL(2)_q$ quantum group structure.

We calculate the algebraic properties of the quantum Liouville theory and discuss correlation functions of non-critical strings for the particularly interesting space-time dimensions $d = 1$ and $d = 25$, where as a function of $d$ the Liouville field behaves either as an euclidean ‘coordinate’ or as a minkowskian ‘time’ respectively. Finally, some conclusions are drawn.

### 2. Canonical quantization of Liouville theory

#### 2.1 The classical structure

We choose canonical quantization of Liouville theory as an alternative to handle the still unsolved path integration over the exponential Liouville action ($D_{t}\phi$ is the translation invariant measure)

$$< \prod_{k} O_k(\phi) > \equiv \int D_{t}\phi \prod_{k} O_k(\phi) e^{\frac{d-25}{8\pi} \int ((\partial\phi)^2 + \mu^2 e^{\phi})}$$  \hspace{1cm} (1)

The calculations will be carried out in the conformal gauge

$$g_{ab} = e^{\phi(\tau, \sigma)} \delta_{ab}$$  \hspace{1cm} (2)

for the simple genus zero case and periodic bounderies

$$\phi(\tau, \sigma) = \phi(\tau, \sigma + 2\pi)$$  \hspace{1cm} (3)

In order that the quantization feels the non-linear Liouville dynamics, we will understand the solution of the Liouville equation ($\xi_{\pm} = \tau \pm \sigma$)

$$e^{\lambda \phi(\tau, \sigma)} = \left( \frac{1}{\sqrt{A'(\xi_{+})}} \frac{B(\xi_{-})}{\sqrt{B'(\xi_{-})}} - \frac{A(\xi_{+})}{\sqrt{A'(\xi_{+})}} \frac{1}{\sqrt{B'(\xi_{-})}} \right)^{-2\lambda}$$  \hspace{1cm} (4)
as a canonical transformation between the Liouville field \( \varphi(\tau, \sigma) \) and a canonical free field \( \psi(\tau, \sigma) \)

\[
\psi(\tau, \sigma) = \psi^+(\xi_+) + \psi^-(\xi_-)
\]  

Indeed, the chiral functions

\[
\psi_{\pm \frac{1}{2}}(\xi_+) = (A'(\xi_+))^{-\frac{1}{2}}, \quad \psi_{\pm \frac{1}{2}}(\xi_+) = A(\xi_+)(A'(\xi_+))^{-\frac{1}{2}}
\]

respectively the anti-chiral ones

\[
\chi_{- \frac{1}{2}}(\xi_-) = (B'(\xi_-))^{-\frac{1}{2}}, \quad \chi_{+ \frac{1}{2}}(\xi_-) = B(\xi_-)(B'(\xi_-))^{-\frac{1}{2}}
\]

can easily be expressed by free fields. We remind the reader [3] that \( \psi_{\pm \frac{1}{2}}(\chi_{\pm \frac{1}{2}}) \) behave under \( SL(2, R) \) transformations

\[
A(\xi_+), \quad B(\xi_-) \rightarrow T[A] = \frac{aA + b}{cA + d}, \quad T[B]
\]

as fundamental representations \((-\frac{1}{2}, 0)((0, -\frac{1}{2})\)), which satisfy the Fuchsian differential equation

\[
\left( \frac{d^2}{d\xi^2_+} + \frac{1}{2}T(\xi_+) \right) \psi_{\pm \frac{1}{2}}(\xi_+) = 0
\]

(and correspondingly for \( \chi_{\pm \frac{1}{2}}(\xi_-) \)). \( \varphi(\tau, \sigma) \) remains invariant. \( T(\xi_+) \) (\( T(\xi_-) \)) is the chiral (anti-chiral) component of the energy-momentum-tensor given by the Schwarzian derivative of \( A \) \((B)\).

An exponential free field representation of these chiral fields

\[
\psi_{\pm \frac{1}{2}} \sim e^{-\frac{1}{2}\psi^+}, \quad \chi_{\pm \frac{1}{2}} \sim e^{-\frac{1}{2}\psi^-}
\]

could be the solution of the problem since \( T(\xi_\pm) \) becomes the improved free field energy-momentum tensor

\[
T(\xi_\pm) = -\frac{1}{2}(\partial \psi^\pm)^2 + \partial^2 \psi^\pm
\]

However this is not the canonical transformation we are looking for. In this case we would have to interpret (4) as a \( SL(2, R) \) decomposition of a tensor power of the fundamental representation \((-\frac{1}{2}, -\frac{1}{2})\) by infinite-dimensional representation theory. But this is not an easy and straightforward task [11].

We can avoid this difficulty if we assume the exponential form (10) for \( \psi_{- \frac{1}{2}}(\chi_{- \frac{1}{2}}) \) only and find \( \psi_{+ \frac{1}{2}}(\chi_{+ \frac{1}{2}}) \), the second solutions of the Fuchsian differential equations, by integration using the monodromy properties for closed strings
\begin{align*}
A(\tau + \sigma + 2\pi) &= e^{\gamma p/2}A(\tau + \sigma) \\
B(\tau - (\sigma + 2\pi)) &= e^{\gamma p/2}B(\tau - \sigma)
\end{align*}
\quad (12)

We obtain
\begin{equation}
A(\tau + \sigma) = \frac{1}{2} \sinh^{-1} \left( \frac{1}{4} \gamma p \right) \int_0^{2\pi} d\sigma' e^{\frac{1}{4} \gamma p \epsilon(\sigma - \sigma')} e^{\psi^+(\tau + \sigma')}
\end{equation}
(13)

(and correspondingly for \(B(\xi_-)\)). The Liouville solution (4) then becomes for \(\lambda = -1/2\)
the well-known integrated canonical Bäcklund transformation [4]
\begin{equation}
e^{-\frac{1}{2} \varphi(\tau, \sigma)} = e^{-\frac{1}{2} (\psi^+(\tau + \sigma) - \psi^- (\tau - \sigma))} \sinh^{-1} \left( \frac{1}{4} \gamma p \right) \int_0^{2\pi} d\sigma' e^{\frac{1}{4} \gamma p \epsilon(\sigma - \sigma')} \cdot e^{\frac{1}{2} (\psi^+(\tau + \sigma') - \psi^- (\tau - \sigma'))} \cosh \left( \frac{1}{2} \psi(\tau, \sigma') \right)
\end{equation}
\quad (14)

Although this compact expression allows to some extent exact quantum-mechanical calculations [4], it is equally useless for defining a Liouville vertex operator \(e^{\lambda \varphi}\).

Motivated by ref.[12], we found a more suitable canonical transformation of the Liouville field in terms of the free field
\begin{equation}
\psi(\tau, \sigma) = \ln \left( \varphi'(\xi_+)(-\frac{1}{B(\xi_-)})' \right)
\end{equation}
\quad (15)

It exponentiates \((B(\xi_-) is here replaced by \(-1/B(\xi_-) in a formula like (13) !)
\begin{equation}
\psi_{-\frac{1}{2}}(\xi_+) = e^{-\frac{1}{2} \psi^+(\xi_+)} , \quad \chi_{+\frac{1}{2}} = e^{-\frac{1}{2} \psi^- (\xi_-)}
\end{equation}
\quad (16)

and allows to factorize the Liouville solution (4) into the form
\begin{equation}
e^{\lambda \varphi(\tau, \sigma)} = e^{\lambda \psi(\tau, \sigma)} (1 - A(\xi_+)/B(\xi_-))^{-2\lambda}
\end{equation}
\quad (17)

The free field factor \(e^{\lambda \psi}\) now carries the total conformal weight of \(e^{\lambda \varphi}\), whereas the bracket of (17) has conformal weight zero. The infinite-dimensional representation theory of \(SL(2,R)\) does not operate here, and the canonical transformation
\begin{equation}
e^{\lambda \varphi(\tau, \sigma)} = e^{\lambda \psi(\tau, \sigma)} (1 + \mu^2 X(\tau, \sigma))^{-2\lambda}
\end{equation}
\quad (18)

can, at least formally, be expanded as
\begin{equation}
e^{\lambda \varphi(\tau, \sigma)} = \sum_{n=0}^{\infty} \frac{(-\mu^2)^n}{n!} \frac{\Gamma(2\lambda + n)}{\Gamma(2\lambda)} Z^{(\lambda,n)}(\tau, \sigma)
\end{equation}
\quad (19)

where we used the notations
\[ Z^{(\lambda,n)}(\tau, \sigma) = e^{\lambda \psi(\tau, \sigma)} (X(\tau, \sigma))^n \] (20)

and
\[ X(\tau, \sigma) = \frac{1}{4} \sinh^{-2}(\frac{1}{4} \gamma p) \int_0^{2\pi} d\sigma' d\sigma'' e^{\frac{1}{4} \gamma p(\varepsilon(\sigma-\sigma') - \varepsilon(\sigma-\sigma''))} e^{\psi^+(\tau+\sigma')} e^{\psi^-(\tau-\sigma')} \] (21)

In the following we shall construct a Liouville vertex operator by means of the expansion (19).

2.2 The quantum structure

This section mainly follows reference [5]. Once the Liouville fields are expressed in terms of free fields
\[ \psi^\pm(\xi^\pm) = \frac{1}{2} \gamma q + \frac{\gamma}{4\pi} p \xi^\pm + \frac{i\gamma}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{1}{n} a_n^\pm e^{-in\xi^\pm} \] (22)

we define the corresponding Liouville operators by canonically quantizing these free fields, as usual, by means of the commutation relations
\[ [q, p] = i, \quad [a_n^+, a_m^+] = n\delta_{n+m,0}, \quad ... \] (23)

and take into account the normal-ordering prescription. But we have to pay attention that the conformal covariance of the classical Liouville theory
\[ [L_n, e^{\lambda \varphi(\sigma)}] = -e^{in\sigma}(i\partial_\sigma + n\Delta(\lambda)) e^{\lambda \varphi(\sigma)} \] (24)

and locality
\[ [e^{\lambda \varphi(\sigma)}, e^{\delta \varphi(\sigma')} ] = 0 \] (25)

remain valid quantum-mechanically, too. Here \( L_n \) is the Liouville-Virasoro operator and \( \Delta(\lambda) \) the conformal weight of \( e^{\lambda \varphi} \).

Naive calculations would provide anomalous contributions to (24, 25). To get rid of them we have to accept quantum deformations [3-5]. Let us briefly illustrate the situation by writing, without proof, the most simple explicit example. It is given by the chiral fields \( \psi^\pm_{\pm \frac{1}{2}}(\chi^\pm \frac{1}{2}) \) themselves which quantum-mechanically become
\[ \psi_{-\frac{1}{2}}(\sigma) = :e^{-\frac{1}{2} \gamma \psi^+(\sigma)} : \]
\[ \psi_{\frac{1}{2}}(\sigma) = \frac{1}{2} :\sinh^{-1} \left( \frac{\gamma \eta p}{4} + 2\pi \hbar \eta^2 \right) \psi_{-\frac{1}{2}}(\sigma) \]
\[ \cdot \int_0^{2\pi} d\sigma' e^{\frac{1}{4} \gamma p(\varepsilon(\sigma-\sigma'))} \left[ 4 \sin^2 \left( \frac{\sigma-\sigma'}{2} \right) \right]^2 \hbar \eta^2 \psi_{-\frac{1}{2}}(\sigma') : . \] (26)
We see that the Liouville coupling $\gamma$ is renormalized

$$\gamma \to \gamma \eta ,$$

find the sinh factors quantum-mechanically shifted in the momentum zero-mode with respect to the classical expression (13), and observe short-distance factors that were eliminated by the normal-ordering prescription reintroduced here as a conformal correction.

In case that the renormalization parameter $\eta$ obeys the quadratic equation

$$2\hbar \eta^2 - \eta + 1 = 0, \quad \hbar = \gamma^2/(16\pi) = 3/(25 - d)$$

we could show that these quantum deformed chiral operators (26) satisfy the quantum Fuchsian differential equation [3]

$$\left( \frac{d^2}{d\sigma^2} - \frac{1}{4} \hbar \eta^2 \right) \psi_{\pm \frac{1}{2}} = 2\hbar \eta^2 \left[ \sum_{n=1}^{\infty} e^{in\sigma} L_{-n} \psi_{\pm \frac{1}{2}} L + \frac{1}{2} L_0 \psi_{\pm \frac{1}{2}} \right.
\left. + \frac{1}{2} \psi_{\pm \frac{1}{2}} L_0 + \psi_{\pm \frac{1}{2}} \sum_{n=1}^{\infty} e^{-in\sigma} L_n \right]$$

(29)

(the same holds for $\chi_{\pm \frac{1}{2}}$), and that all these operator fields have the same conformal weight

$$\Delta \left( -\frac{1}{2} \right) = -\frac{1}{2} \left( \eta + \hbar \eta^2 \right) \xrightarrow{\hbar \to 0} -\frac{1}{2}$$

(30)

We should also mention here that $\psi_{\pm \frac{1}{2}}$ and $\chi_{- \frac{1}{2}}$ are defined in a reparametrization invariant way by the same condition (28), which means that the expression under the integral of (26) has conformal weight one

$$\Delta(1) = \eta - 2\hbar \eta^2 = 1$$

(31)

This quadratic equation determines $\eta$ as a function of the space-time dimension $d$

$$g = 2\hbar \eta^2 \eta_{\mp} = \eta_{\mp} - 1 = \frac{1}{12} \left( 13 - d \mp \sqrt{(1 - d)(25 - d)} \right)$$

(32)

The parameter $g$ controls any quantum correction in the string-induced quantum Liouville theory. It also determines the KPZ critical exponents and the anomalous Kac-Moody central charge of SL(2,R). But $g$ underlies here several serious limitations, for instance, reality of the conformal weight of physical operators restricts the space-time dimension to

$$d \leq 1 \quad \text{or} \quad d \geq 25$$

(33)

excluding the physical interesting region $1 < d < 25$ where $g$ becomes complex.
Now we are ready to discuss the Liouville vertex operator $e^{\lambda \varphi(\sigma)}$. Although possessing the operators $\psi_{\pm \frac{1}{2}}(\sigma) \left( \chi_{\pm \frac{1}{2}}(\sigma) \right)$, we cannot simply calculate the vertex operator from them as in the classical case. The composite operator $e^{\lambda \varphi(\sigma)}$ has to be constructed separately by means of the whole machinery of quantization rules which we have discussed before. We obtain, finally, for the Liouville vertex operator a remarkable result (from here on $\eta$ is absorbed in $\psi$!)

$$e^{\lambda \varphi(\sigma)} = :e^{\lambda \psi(\sigma)}:\sum_{n=0}^{\infty} \frac{(-\mu^2)^n \Gamma_q(2\lambda + n)}{[n]_q \Gamma_q(2\lambda)} \prod_{j=1}^{n} F\left(\lambda + j - 1, \frac{1}{4} \gamma \eta p - 2\pi i(\lambda - n)\hbar \eta^2\right) \left[ :S(\sigma) : \right]^n$$

We recognize the special normal-ordering as a consequence of the requirement that the conformal covariance condition (24) should be anomaly-free. It hides the reintroduced short-distance singularities

$$\left[ \frac{4 \sin^2 \frac{\sigma'_i - \sigma'_j}{2} \right]^{-g}$$

which become integrable by restricting the Liouville coupling $\gamma$ respectively the parameter $g$. The $n$-th power of the normal-ordered ‘screening charge’ operator in (34)

$$S(\sigma) = \frac{1}{4} \int_0^{2\pi} d\sigma' d\sigma'' e^{\frac{i}{\hbar} \gamma \eta p (\epsilon(\sigma - \sigma') - \epsilon(\sigma - \sigma''))} e^{\psi^+(\sigma')} e^{\psi^-(\sigma'') \right)}$$

provides no other singularities, in particular there do not arise singularities in the variable $\sigma$ which is an argument of the sign-function $\epsilon(\sigma - \sigma')$ only.

Most interestingly, the additional requirement of locality (25) does not only shift the zero-mode arguments of the sinh factors

$$F(\lambda, P) = \sinh^{-1}(P + i\pi \lambda g) \sinh^{-1}(P - i\pi \lambda g)$$

as expected. Here, in addition, numbers become q-numbers

$$n! \rightarrow [n]_q = \prod_{j=1}^{n} \frac{\sin(2\hbar \eta^2 j \pi)}{\sin(2\hbar \eta^2 \pi)}$$

$$\Gamma(x) \rightarrow \Gamma_q(x)$$

if we turn the classical expansion (19) into its operator form (34). This indicates a quantum group structure as a consequence of both the conformal and local properties of the quantum Liouville theory.

The locality condition (25) which operates iteratively for each order $n$ of the expansion (34)
\[
\frac{\Gamma(2\lambda + n)}{\Gamma(2\lambda)} \left[ : Z_q^{(\lambda,n)}(\sigma) : , : e^{\delta\psi(\rho)} : \right] - ((\lambda, \sigma) \leftrightarrow (\delta, \rho)) \\
= \sum_{k=1}^{n-1} \binom{n}{k} \frac{\Gamma(2\lambda + n - k)}{\Gamma(2\lambda)} \frac{\Gamma(2\delta + k)}{\Gamma(2\delta)} \left[ : Z_q^{(\lambda,n-k)} : , : Z_q^{(\delta,k)} : \right]
\]

(39)
determines the deformations (37,38) altogether. \( Z_q \) is the deformed \( Z \) of eq (20).

It also seems to be worth mentioning that for the special value \( \lambda = \frac{1}{2} \) the q-deformed expansion (34) can be resummed again (the q-numbers disappear!), and we obtain

\[
e^\frac{1}{2}\varphi(\sigma) = : e^\frac{1}{2}\varphi(\sigma) : \left( 1 - : F(\frac{1}{2}, \frac{1}{4} \gamma \eta \rho - \pi i \bar{\eta} \rho) S(\sigma) : \right)^{-1}
\]

(40)
The short-distance singularities of the expansion (34) which are contained in \([ : S(\sigma) ]^n\) surprisingly disappear in the analytic result (40). This may indicate that expansions like (34) have only a very formal nature.

Finally, we make reference to the conformal weight of the operator (34)

\[
\Delta(\lambda) = \lambda \eta (1 - 2\bar{\eta} \lambda) = \bar{\lambda} \eta (1 - 2\bar{\eta} \bar{\lambda})
\]

(41)
which defines a ‘background charge’ \( 2\beta_0 \)

\[
\bar{\lambda} = 2\beta_0 - \lambda, \quad 2\beta_0 = \frac{1}{2\eta}, \quad \Delta(2\beta_0) = 0
\]

(42)
The Liouville vertex operator so has the expected short distance behaviour \( (z = e^{i\sigma}) \)

\[
e^{\lambda\varphi(z)} e^{\bar{\lambda}\varphi(z')} \sim | z - z' |^{-2\Delta(\lambda)}
\]

(43)
It satisfies, furthermore, the operator Liouville equation and represents, indeed, a canonical transformation between \( \varphi \) and \( \psi \), classically as well as quantum-mechanically [5].

### 2.3 The exchange algebra

It is obvious in this canonical quantization that the exchange algebra of the chiral (anti-chiral) fields \( \psi_{\pm \frac{1}{2}}(\chi_{\pm \frac{1}{2}}) \)

\[
\psi_+(\sigma) \psi_k(\rho) = S_{ik}^+ \psi_+(\rho) \psi_k(\sigma), \ldots
\]

(44)
can be calculated by direct operator algebra. We obtain for the exchange matrix \( S \) a result as in ref. [3], which shows that both the canonical quantization and the quantization by the quantum group are equivalent.

The matrix \( S \) unfortunately depends on the momentum zero-mode and local phases
\[
S^{nn}_{nn} = e^{-i\pi\hbar\eta^2\varepsilon(\sigma-\rho)}, \quad n = 1, 2
\]
\[
S^{12}_{12} = ie^{-\frac{1}{4}\gamma\eta p + i\pi\hbar\eta^2}\frac{\sin(\frac{\pi\hbar\eta^2}{2})}{\sinh(\frac{1}{4}\gamma\eta p)}e^{-i\pi\hbar\eta^2\varepsilon(\sigma-\rho)}
\]
\[
S^{21}_{12} = (1 - S^{12}_{12})\frac{\sinh(\frac{1}{4}\gamma\eta p + 2i\pi\hbar\eta^2)}{\sinh(\frac{1}{4}\gamma\eta p)}e^{-i\pi\hbar\eta^2\varepsilon(\sigma-\rho)}, \ldots
\]

But in a different basis [13]

\[
\xi_k = \sum_{i=\pm 1/2} u^i_k \psi_i, \quad \zeta_k = \sum_{i=\pm 1/2} v^i_k \chi_i, \quad k = \pm \frac{1}{2}
\]

the new exchange matrix S will become a $SL(2)_q$ R-matrix, showing more directly the internal quantum group structure of Liouville theory.

The local phases in (45) arise because of the symmetric distribution of the zero modes among the left and right moving fields in $\psi, \chi$ (compare eq. (21)) and would disappear in the exchange algebra of the fields $\pi_\pm =: \psi_\pm \chi_\mp \ldots$.

3. Correlation functions for non-critical strings

The calculation of correlation functions is the most difficult and likewise uncertain problem in the quantum Liouville theory. There does neither exist a Möbius invariant vacuum nor ‘screening charge’ conservation, tools which often determine the concrete calculations and so the results. It is also not desirable to put to zero the cosmological term of the Liouville action ($\mu^2 = 0!$) and replace the Liouville theory by a free field theory with a background charge [14,15].

Correlation functions can be calculated from reparametrization invariant operators. For non-critical strings scattering of physical states is described by integrated vertex operators

\[
V(\lambda_k, p_k) = \int d^2 z e^{\lambda_k \varphi(z)} e^{ip_k \cdot x(z)}
\]

Such operators are defined in a reparametrization invariant way if the total conformal weight of the Liouville vertex $e^{\lambda_k \varphi}$ and the string vertex $e^{ip_k \cdot x}$ is equal to one

\[
\Delta(e^{\lambda_k \varphi}) + \Delta(e^{ip_k \cdot x}) = 1
\]

This relates the parameters $\lambda_k, p_k$ to the spacetime dimension $d$
\( \lambda_k = \frac{\left( \sqrt{25-d} - \sqrt{1-d+12p_k^2} \right) \sqrt{25-d} - \sqrt{1-d} \right)}{\left( \sqrt{25-d} - \sqrt{1-d} \right)} \) \hspace{1cm} (49)

As a simple example, we discuss a 4-point correlation function for special ‘charges’

\[ V_4 = \langle P' | \prod_{k=1}^4 V(\lambda_k, p_k) | P \rangle \] \hspace{1cm} (50)

\[ \lambda_i = \lambda, \ i = 1, 2, 3; \ \lambda_4 = \bar{\lambda} \ \text{with} \ \ 3\lambda + \bar{\lambda} = 2\lambda + 2\beta_0 \] \hspace{1cm} (51)

The ground state is chosen to be Möbius non-invariant with a non-vanishing momentum

\[ | P \rangle = e^{i\eta \gamma} | 0 \rangle \] \hspace{1cm} (52)

We will use both solutions \( \eta_{\pm} \) of the quadratic equation (28) with the properties

\[ \eta_{+} + \eta_{-} = \eta_{+} \eta_{-} = \frac{1}{2\hbar} \] \hspace{1cm} (53)

For a while let us assume ‘screening charge conservation’. Then the Liouville vertex operator provides its own counting of ‘screening charges’ with the neutrality condition

\[ 2\lambda + 2\beta_0 + \sum_{k=1}^4 n_k = 0 \] \hspace{1cm} (54)

where the \( n_k \) is the corresponding expansion parameter in (34). This neutrality condition has a non-trivial solution only if

\[ 2\lambda + 2\beta_0 = (1-m)\eta_{+} + (1-n)\eta_{-} \] \hspace{1cm} (55)

is a negative integer. There are two interesting solutions. For \( d=25 \) the eq (32) yields \( \eta_{\pm} = 0 \) and we obtain

\[ \gamma \eta = 2i\sqrt{2\pi}, \ \ \lambda_k^2 = 1 - \frac{1}{2}p_k^2 \] \hspace{1cm} (56)

Now the neutrality condition (55) does not allow ‘screening charges’ \( (n = 0 \text{ in } (34)!) \) and the Liouville field \( \varphi(\tau, \sigma) \) becomes the free field \( \psi(\tau, \sigma) \) with a central charge \( c=1 \) and a pure imaginary coupling \( \gamma \eta \). Since the kinetic term of the Liouville theory then gets a negative sign, one could interprete the Liouville field for \( d=25 \) as a time coordinate of a 26-dimensional Minkowski space-time [16]. For the 4-point-function we obtain then the expected critical string result, a Virasoro-Shapiro amplitude in 26-dimensional Minkowski space-time (\( \lambda_k, p_k \) form here a new vector \( p_k \) !)

\[ V_4 = \int dz \left| z \right|^\frac{1}{2}p_1 \cdot p_2 \left| 1 - z \right|^\frac{1}{2}p_2 \cdot p_3 \] \hspace{1cm} (57)

The second interesting case is \( d = 1 \), with
Now, the Liouville field acts like a ‘coordinate’, and the ‘screening charges’ contribute to the integrand of (57) a complicated factor $K(z)$ which is partially due to the conformal corrections discussed in the chapter before. The integrals of these correlation functions are in general not of the Dotsenko-Fateev type [17,10].

With such calculations we are able, for instance, to rederive results of refs [18,19]. But we started this calculation with wrong assumptions. Without ‘screening charge’ conservation each term of (34) should contribute and we should be able to sum up the whole series in order to get analytical results.

For 3-point functions, at least, one can obtain order by order closed results of the Dotsenko-Fateev type [20].

\begin{equation}
\eta_\pm = 2, \quad \gamma_\eta = 2\sqrt{2\pi}, \quad \lambda = 1 - \frac{1}{\sqrt{2}p}
\end{equation}

4. Conclusion

We have given an operator representation of the string induced quantum Liouville theory. It is based on a canonical free field quantization which preserves the non-linear Liouville dynamics and takes into account locality and the conformal properties of Liouville theory in a self-consistent manner.

From the algebraic point of view, this operator formulation fits rather well into the general scheme of a 2-dimensional conformal quantum field theory, in particular, it allows to construct a Liouville vertex operator, and it shows beyond it an interesting internal $SL(2)_q$ quantum group structure. At least formally the algebraic calculations are not sensitive to the restrictions of space-time (33).

But the calculation of correlation functions is less convincing, except that the critical string is reproduced by interpreting the Liouville field as a time coordinate. For these calculations we have used rather formal expansions. It might be that in this way extra difficulties arise, as the experience with the example (40) seems to tell us.

Finally, there also remains the challenge to understand the structure of the Hilbert space of Liouville theory. The question arises if the internal quantum group may be important in this respect.

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