On freedom and independence
in hypergraphs of models of theories∗

B.Sh. Kulpeshov, S.V. Sudoplatov

Abstract

Notions of freedom and independence for hypergraphs of models of a theory are defined. Properties of these notions and their applications to some natural classes of theories are studied.

Keywords: hypergraph of models, elementary theory, free set, independent sets, complete union of hypergraphs.

Hypergraphs of models of a theory are related to derived objects, allowing to obtain an essential structural information on both theories themselves and related semantical objects including graph ones [1, 2, 3, 4, 5, 6, 7, 8, 9].

In the present paper notions of freedom and independence for hypergraphs of models of a theory are defined. Properties of these notions and their applications to some natural classes of theories are studied.

1 Preliminaries

Recall that a hypergraph is any pair of sets \((X, Y)\), where \(Y\) is some subset of the Boolean \(\mathcal{P}(X)\) of a set \(X\). The set \(X\) is called the universe of the hypergraph \((X, Y)\), and elements of \(Y\) are edges of the hypergraph \((X, Y)\).

Let \(\mathcal{M}\) be some model of a complete theory \(T\). Following [5] we denote by \(H(\mathcal{M})\) the family of all subsets \(N\) of the universe \(\mathcal{M}\) of the structure \(\mathcal{M}\), which are universes of elementary submodels \(\mathcal{N}\) of the model \(\mathcal{M}\): \(H(\mathcal{M}) = \{N \mid \mathcal{N} \preceq \mathcal{M}\}\). The pair \((\mathcal{M}, H(\mathcal{M}))\) is called the hypergraph of elementary submodels of the model \(\mathcal{M}\) and it is denoted by \(\mathcal{H}(\mathcal{M})\).

For a cardinality \(\lambda\) we denote by \(H_\lambda(\mathcal{M})\) and \(\mathcal{H}_\lambda(\mathcal{M})\), respectively, the restrictions of \(H(\mathcal{M})\) and \(\mathcal{H}(\mathcal{M})\) on the class of elementary submodels \(\mathcal{N}\) of \(\mathcal{M}\) such that \(|N| < \lambda\).

By \(\mathcal{H}_p(\mathcal{M}), \mathcal{H}_l(\mathcal{M}), \mathcal{H}_{\text{npl}}(\mathcal{M}), \mathcal{H}_h(\mathcal{M}), \mathcal{H}_s(\mathcal{M})\) we denote the restrictions of the hypergraph \(\mathcal{H}_{\omega_1}(\mathcal{M})\) on the class of elementary submodels \(\mathcal{N}\) of the model \(\mathcal{M}\), that are prime over finite sets, limit, non-prime and non-limit, homogeneous, saturated respectively. Similarly, by \(\mathcal{H}_p(\mathcal{M}), \mathcal{H}_l(\mathcal{M}), \mathcal{H}_{\text{npl}}(\mathcal{M}), \mathcal{H}_h(\mathcal{M}), \mathcal{H}_s(\mathcal{M})\) are denoted the correspondent restrictions for \(\mathcal{H}_{\omega_1}(\mathcal{M})\).

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Definition 1.1 [5]. Let \( (X, Y) \) be a hypergraph, \( x_1, x_2 \) be distinct elements of \( X \). We say that the element \( x_1 \) is separated or separable from the element \( x_2 \), or \( T_0 \)-separable if there is \( y \in Y \) such that \( x_1 \in y \) and \( x_2 \notin y \). The elements \( x_1 \) and \( x_2 \) are called separable, \( T_2 \)-separable, or Hausdorff separable if there are disjoint \( y_1, y_2 \in Y \) such that \( x_1 \in y_1 \) and \( x_2 \in y_2 \).

Theorem 1.2 [5]. Let \( M \) be an \( \omega \)-saturated model of a countable complete theory \( T \), \( a \) and \( b \) be elements of \( M \). The following are equivalent:

1. the element \( a \) is separable from the element \( b \) in \( H(M) \);
2. the element \( a \) is separable from the element \( b \) in \( H_{\omega_1}(M) \);
3. \( b \notin \text{acl}(a) \).

Theorem 1.3 [5]. Let \( M \) be an \( \omega \)-saturated model of a countable complete theory \( T \), \( a \) and \( b \) be elements of \( M \). The following are equivalent:

1. the elements \( a \) and \( b \) are separable in \( H(M) \);
2. the elements \( a \) and \( b \) are separable in \( H_{\omega_1}(M) \);
3. \( \text{acl}(a) \cap \text{acl}(b) = \emptyset \).

Corollary 1.4 [5]. Let \( M \) be an \( \omega \)-saturated model of a countable complete theory \( T \), \( a \) and \( b \) be elements of \( M \), and there exists the prime model over \( a \). The following are equivalent:

1. the element \( a \) is separable from the element \( b \) in \( H(M) \);
2. the element \( a \) is separable from the element \( b \) in \( H_{\omega_1}(M) \);
3. the element \( a \) is separable from the element \( b \) in \( H_p(M) \);
4. \( b \notin \text{acl}(a) \).

Corollary 1.5 [5]. Let \( M \) be an \( \omega \)-saturated model of a countable complete theory \( T \), \( a \) and \( b \) be elements of \( M \), and there exist the prime models over \( a \) and \( b \) respectively. The following are equivalent:

1. the elements \( a \) and \( b \) are separable in \( H(M) \);
2. the elements \( a \) and \( b \) are separable in \( H_{\omega_1}(M) \);
3. the elements \( a \) and \( b \) are separable in \( H_p(M) \);
4. \( \text{acl}(a) \cap \text{acl}(b) = \emptyset \).

Definition 1.6 [5]. Let \( (X, Y) \) be a hypergraph, \( X_1, X_2 \) be disjoint nonempty subsets of the set \( X \). We say that the set \( X_1 \) is separated or separable from the set \( X_2 \), or \( T_0 \)-separable if there is \( y \in Y \) such that \( X_1 \subseteq y \) and \( X_2 \cap y = \emptyset \). The sets \( X_1 \) and \( X_2 \) are called separable, \( T_2 \)-separable, or Hausdorff separable if there are disjunct \( y_1, y_2 \in Y \) such that \( X_1 \subseteq y_1 \) and \( X_2 \subseteq y_2 \).

By using proofs of theorems 1.2 and 1.3 the following generalizations of these theorems are established.

Theorem 1.7 [5] Let \( M \) be a \( \lambda \)-saturated model of a complete theory \( T \), \( \lambda \geq \max\{ |\Sigma(T)|, \omega \} \), \( A \) and \( B \) be nonempty sets in \( M \) having the cardinalities \( < \lambda \). The following are equivalent:

1. the set \( A \) is separable from the set \( B \) in \( H(M) \);
2. the set \( A \) is separable from the set \( B \) in \( H_\lambda(M) \);
3. \( \text{acl}(A) \cap B = \emptyset \).
Theorem 1.8 [5]. Let $M$ be a $\lambda$-saturated model of a complete theory $T$, $\lambda \geq \text{max}\{|\Sigma(T)|, \omega\}$, $A$, $B$ be nonempty sets in $M$ having the cardinalities $< \lambda$. The following are equivalent:

1. the sets $A$ and $B$ are separable in $H(M)$;
2. the sets $A$ and $B$ are separable in $H_{\lambda}(M)$;
3. $\text{acl}(A) \cap \text{acl}(B) = \emptyset$.

We obtain by analogy with corollaries 1.4 and 1.5

Corollary 1.9 [5]. Let $M$ be an $\omega$-saturated model of a small theory $T$, $A$ and $B$ be finite nonempty sets in $M$. The following are equivalent:

1. the set $A$ is separable from the set $B$ in $H(M)$;
2. the set $A$ is separable from the set $B$ in $H_{\omega_1}(M)$;
3. the set $A$ is separable from the set $B$ in $H_{p}(M)$;
4. $\text{acl}(A) \cap B = \emptyset$.

Corollary 1.10 [5]. Let $M$ be an $\omega$-saturated model of a small theory $T$, $A$ and $B$ be finite nonempty sets in $M$. The following are equivalent:

1. the sets $A$ and $B$ are separable in $H(M)$;
2. the sets $A$ and $B$ are separable in $H_{\omega_1}(M)$;
3. the sets $A$ and $B$ are separable in $H_{p}(M)$;
4. $\text{acl}(A) \cap B = \emptyset$.

Definition 1.11 [8]. Let $M$ be a model of a theory $T$ with a hypergraph $H = (M, H(M))$ of elementary submodels, $A$ be an infinite definable set in $M$, of arity $n$: $A \subseteq M^n$. The set $A$ is called $H$-free if for any infinite set $A' \subseteq A$, $A' = A \cap Z^n$ for some $Z \in H(M)$ containing parameters for $A$. Two $H$-free sets $A$ and $B$ of arities $m$ and $n$ respectively are called $H$-independent if for any infinite $A' \subseteq A$ and $B' \subseteq B$ there is $Z \in H(M)$ containing parameters for $A$ and $B$ and such that $A' = A \cap Z^m$ and $B' = B \cap Z^n$.

Note the following properties [8].

1. Any two tuples of a $H$-free set $A$, whose distinct tuples do not have common coordinates, have same type.

Indeed, if there are tuples $\bar{a}, \bar{b} \in A$ with $\text{tp}(\bar{a}) \neq \text{tp}(\bar{b})$ then for some formula $\varphi(\bar{x})$ the sets of solutions of that formula and of the formula $\neg \varphi(\bar{x})$ divide the set $A$ into two nonempty parts $A_1$ and $A_2$, where at least one part, say $A_1$, is infinite. Taking $A_1$ for $A'$ we have $A_1 = A \cap Z^n$ for appropriate $Z \in H(M)$ and $n$. Then by the condition for tuples in $A$ we have $A_2 \cap Z^n = \emptyset$ that is impossible since $Z$ is the universe of an elementary submodel of $M$.

Thus the formula $\varphi(\bar{x})$, defining $A$, implies some complete type in $S^n(\emptyset)$, and if $A$ is $\emptyset$-definable then $\varphi(\bar{x})$ is a principal formula.

In particular, if the set $A$ is $H$-free and $A \subseteq M$, then the formula, defining $A$, implies some complete type in $S^1(\emptyset)$.

2. If $A \subseteq M$ is a $H$-free set, then $A$ does not have nontrivial definable subsets, with parameters in $A$, i.e., subsets distinct to subsets defined by equalities and inequalities with elements in $A$.

Indeed, if $B \subseteq A$ is a nontrivial definable subset then $B$ is defined by a tuple $\bar{a}$ of parameters in $A$, forming a finite set $A_0 \subseteq A$, and $B$ is distinct to subsets of $A_0$ and to $A \setminus C$, where $C \subseteq A_0$. Then removing from $A$ a set $B \setminus A_0$ or $(A \setminus B) \setminus A_0$, we obtain some
Z ∈ H(M) violating the satisfiability for B or its complement. It contradicts the condition that Z is the universe of an elementary submode of M.

3. If A and B are two H-independent sets, where A ∪ B does not have distinct tuples with common coordinates, then A ∩ B = ∅.

Indeed, if A ∩ B contains a tuple a, then, choosing infinite sets A' ⊆ A and B' ⊆ B with a ∈ A' and a /∈ B', we obtain a ∈ A' = A ∩ Z^n for appropriate Z ∈ H(M) and n, as so a ∈ B ∩ Z^n = B'. This contradiction means that A ∩ B = ∅.

**Definition 1.12** [9]. The complete union of hypergraphs (X_i, Y_i), i ∈ I, is the hypergraph \( \bigcup_{i \in I} X_i, Y \), where Y = \( \bigcup_{i \in I} Z_i | Z_i \in Y_i \). If the sets X_i are disjoint, the complete union is called disjoint too. If the set X_i form a \( \subseteq \)-chain, then the complete union is called chain.

By Property 3 we have the following theorem on decomposition of restrictions of hypergraphs H, representable by unions of families of H-independent sets.

**Theorem 1.13** [8]. A restriction of hypergraph \( H = (M, H(M)) \) to a union of a family of H-free H-independent sets A_i ⊆ M is represented as a disjoint complete union of restrictions \( H_i \) of the hypergraph H to the sets A_i.

Proof. Consider a family of H-independent sets A_i ⊆ M. By Property 3 these sets are disjoint, and using the definition of H-independence we immediately obtain that the union of restrictions \( H_i \) of H to the sets A_i is complete. \( \square \)

**Definition 1.14** [10]. Let M be some model of a complete theory T, \( (M, H(M)) \) be a hypergraph of elementary submodels of the model M. Sets N ∈ H(M) are called elementarily submodel or elementarily substructural in M.

**Proposition 1.15** [10] Let A be a definable set in an \( \omega_1 \)-saturated model M of a countable complete theory T. Then exactly one of the following conditions holds:

1. The set A is finite and is contained in any elementarily substructural set in M;
2. The set A is infinite, has infinitely many different intersections with elementarily substructural sets in M, and all these intersections are infinite; and the indicated intersections can be chosen so as to form an infinite chain/antichain by inclusion.

**Proposition 1.16** [10] Let A be a definable set in the countable saturated model M of a small theory T. Then exactly of the following conditions holds:

1. The set A is finite and is contained in any elementarily substructural set in M;
2. The set A is infinite, has infinitely many different intersections with elementarily substructural sets in M, and all these intersections are infinite; and the indicated intersections can be chosen so as to form an infinite chain / antichain by inclusion.

Note that the above concepts and statements by a natural manner are transferred to hypergraphs \( H_\lambda(M), H_p(M), H_t(M), H_{apl}(M), H_b(M), H_s(M) \).

Recall that a subset A of a linearly ordered structure M is called convex if for any a, b ∈ A and c ∈ M whenever a < c < b we have c ∈ A. A weakly o-minimal structure is a linearly
ordered structure \( \mathcal{M} = \langle M, =, <, \ldots \rangle \) such that any definable (with parameters) subset of the structure \( \mathcal{M} \) is a union of finitely many convex sets in \( \mathcal{M} \).

In the following definitions \( \mathcal{M} \) is a weakly o-minimal structure, \( A, B \subseteq M, \mathcal{M} \) be \( |A|^\ast \)-saturated, \( p, q \in S_1(A) \) be non-algebraic types.

**Definition 1.17 [14]** We say that \( p \) is not weakly orthogonal to \( q \) (\( p \not\Perp^w q \)) if there exist an \( A \)-definable formula \( H(x, y), \alpha \in p(\mathcal{M}) \) and \( \beta_1, \beta_2 \in q(\mathcal{M}) \) such that \( \beta_1 \in H(\mathcal{M}, \alpha) \) and \( \beta_2 \notin H(\mathcal{M}, \alpha) \).

**Definition 1.18 [15]** We say that \( p \) is not quite orthogonal to \( q \) (\( p \not\Perp^q q \)) if there exists an \( A \)-definable bijection \( f : p(\mathcal{M}) \to q(\mathcal{M}) \). We say that a weakly o-minimal theory is quite o-minimal if the notions of weak and quite orthogonality of 1-types coincide.

In the paper [16] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

**Theorem 1.19** Let \( T \) be a quite o-minimal theory with non-maximum many countable models. Then \( T \) has exactly \( 3^k \cdot 6^s \) countable models, where \( k \) and \( s \) are natural numbers. Moreover, for any \( k, s \in \omega \) there exists a quite o-minimal theory \( T \) having exactly \( 3^k \cdot 6^s \) countable models.

Realizations of these theories with a finitely many countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of Ehrenfeucht type. Moreover, these realizations are representative examples for hypergraphs of prime models [18][3][5]. We consider operators for hypergraphs allowing on one hand to describe the decomposition of hypergraphs of prime models for quite o-minimal theories with few countable models, and on the other hand pointing out constructions leading to the building of required hypergraphs by some simplest ones.

Denote by \((M, H_{dlo}(M))\) a hypergraph of (prime) elementary submodels of a countable model \( M \) of the theory of dense linear order without endpoints.

**Remark 1.20** The class of hypergraphs \((M, H_{dlo}(M))\) is closed under countable chain complete unions, modulo density and having an encompassing dense linear order without endpoints. Thus, any hypergraph \((M, H_{dlo}(M))\) is represented as a countable chain complete, modulo density, union of some of its proper subhypergraphs.

Any countable model of a theory of Ehrenfeucht type is a disjoint union of some intervals, which are ordered both themselves and between them, and of some singletons. Dense subsets of the intervals form universes of elementary substructures. So, in view of Remark 1.20 we have:

**Theorem 1.21 [6]** A hypergraph of prime models of a countable model of a theory of Ehrenfeucht type is represented as a disjoint complete, modulo density, union of some hypergraphs in the form \((M, H_{dlo}(M))\) as well as singleton hypergraphs of the form \((\{c\}, \{\{c\}\})\).

**Remark 1.22** Taking into consideration links between sets of realizations of 1-types, which are not weakly orthogonal, as well as definable equivalence relations, the construction for the proof of Theorem 1.21 admits a natural generalization for an arbitrary quite o-minimal theory with few countable models. Here conditional complete unions should be additionally coordinated, i.e., considering definable bijections between sets of realizations of 1-types, which are not quite orthogonal.
2 On relative freedom and independence in hypergraphs of models of theories

As shown in Section 1, $\mathcal{H}$-free sets does not have non-trivial definable subsets. By this note at studying subsets $A'$ of definable sets $A \subseteq M^n$ in structures $\mathcal{M}$ of a non-empty signature, where $A' = A \cap (M_1)^n$ for some $\mathcal{M}_1 < \mathcal{M}$, it is naturally instead of “absolute” $\mathcal{H}$-freedom to consider relative $\mathcal{H}$-freedom taking into account, as for dense linear orders, the specifics of subsets $A'$ by some syntactical information taken from the complete diagram $D^*(\mathcal{M})$ of the system $\mathcal{M}$. In the following section we take into account this specific for ordered theories, and in this section we introduce general notions of relative $\mathcal{H}$-freedom and $\mathcal{H}$-independence, and we also establish links between distinct types of relativity.

**Definition 2.1** Let $\mathcal{M}$ be some model of a theory $T$ with a hypergraph of elementary submodels $\mathcal{H} = (\mathcal{M}, H(\mathcal{M}))$, $D^*(\mathcal{M})$ be the complete diagram of the model $\mathcal{M}$, $\mathbf{D}$ be some set of diagrams $\Phi(A_0) \subseteq D^*(\mathcal{M})$ such that for some language $\Sigma \subseteq \Sigma(\mathcal{M})$ if $\varphi(\bar{a})$ is a quantifier-free formula of the language $\Sigma$, $\bar{a} \in A_0$, then $\varphi(\bar{a}) \in \Phi(A_0)$ or $\neg \varphi(\bar{a}) \in \Phi(A_0)$. Here, the set $A_0$ called the *universe* of the diagram $\Phi(A_0)$. We say that a set $A \subseteq M^n$ satisfies a diagram $\Psi \in \mathbf{D}$ if $\Psi = \Phi(A_0)$ for the set $A_0$ consisting of all the coordinates of tuples from $A$. The set $A \subseteq M^n$ is called relatively $\mathcal{H}$-free, $\mathcal{H}$-free modulo $\mathbf{D}$, or $(\mathcal{H}, \mathbf{D})$-free if for any set $A' \subseteq A$ satisfying some diagram of $\mathbf{D}$ the equality $A' = A \cap Z^n$ holds for some $Z \in H(\mathcal{M})$ containing parameters for $A$. Two $(\mathcal{H}, \mathbf{D})$-free sets $A$ and $B$ of arity $m$ and $n$ respectively are called relatively $\mathcal{H}$-independent, $\mathcal{H}$-independent modulo $\mathbf{D}$, or $(\mathcal{H}, \mathbf{D})$-independent if for any sets $A' \subseteq A$ and $B' \subseteq B$ satisfying some diagrams of $\mathbf{D}$ there exists $Z \in H(\mathcal{M})$ containing parameters for $A$ and $B$ and such that $A' = A \cap Z^n$ and $B' = B \cap Z^n$.

Note that at defining “absolute” $\mathcal{H}$-freedom and $\mathcal{H}$-independence as $\Sigma$ it is considered the empty language, a set $A$ is definable and infinite, and diagrams $\Phi$ are taken either quantifier-free on all infinite sets $A' \subseteq A$ or as result of adding to these quantifier-free diagrams schemes of infinity of sets for $A'$.

Unlike definability in the case of type definability or non-definability of a set $A$ under consideration of relative freedom and independence both a scheme of infinity and an infinity itself of sets $A'$ may not be required. Indeed, for a theory $T$ of unary predicates $P_i$ with $P_{i+1} \subseteq P_i$, $i \in \omega$, the non-isolated type $p(x) = \{P_i(x) \mid i \in \omega\}$ can have the set of realizations of any, finite or infinite, cardinality. Thus, the set $A = p(\mathcal{M})$, $\mathcal{M} \models T$, non-having non-trivial connections is $(\mathcal{H}, \mathbf{D}_p)$-free for a set of diagrams $\mathbf{D}_p$ describing realizability of the type $p(x)$ by elements of an arbitrary set $A' \subseteq A$.

If the theory $T$ is expanded by unary predicates $Q_i$ with conditions $Q_{i+1} \subseteq Q_i \subseteq \overline{\mathbf{T}}_0$, $i \in \omega$, then the set $B = q(\mathcal{M})$, where $q(x) = \{Q_i(x) \mid i \in \omega\}$ is free relatively a set of diagrams $\mathbf{D}_q$ describing realizability of the type $q(x)$ by elements of an arbitrary set $B' \subseteq B$, will be $(\mathcal{H}, \mathbf{D}_p \cup \mathbf{D}_q)$-independent.

Since at extending diagrams of $\mathbf{D}$ a family of considered sets can only decrease the following hold:

**Monotonicity properties.** 1. If $\mathbf{D} \subseteq \mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$ and a set $A$ is $(\mathcal{H}, \mathbf{D}')$-free then $A$ is $(\mathcal{H}, \mathbf{D})$-free.
2. If $\mathbf{D} \subseteq \mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$, sets $A$ and $B$ are $(\mathcal{H}, \mathbf{D}')$-independent then $A$ and $B$ are $(\mathcal{H}, \mathbf{D})$-independent.
3. If diagrams of \( D \) are some restrictions/extensions of suitable diagrams from the set \( D' \subseteq \mathcal{P}(D^*(M)) \), with preservation of their universes, and the set \( A \) is \( (\mathcal{H}, D) \)-free, then \( A \) is \( (\mathcal{H}, D') \)-free.

4. If diagrams of \( D \) are some restrictions/extensions of suitable diagrams of the set \( D' \subseteq \mathcal{P}(D^*(M)) \), with preservation of their universes, the sets \( A \) and \( B \) are \( (\mathcal{H}, D) \)-independent, then \( A \) and \( B \) are \( (\mathcal{H}, D') \)-independent.

Inverse implications in monotonicity properties are not true. Indeed, if an infinite definable set \( A \subseteq M \) is partitioned by a unary predicate \( P \) into two non-empty parts then \( A \) is not \( \mathcal{H} \)-free, although this set can be considered as \( (\mathcal{H}, D') \)-free, for the language \( \{P^{(1)}\} \), where \( D' \) consists of diagrams describing cardinalities \( |A \cap P| \) and \( |A \cap \overline{P}| \). The considered effect, at which two disjoint infinite definable sets \( A \) and \( B \) are partitioned by the predicate \( P \) into non-empty disjoint parts, shows that an independence of the sets \( A \) and \( B \) can be failed at transition from \( (\mathcal{H}, D') \) to \( (\mathcal{H}, D) \).

Further for simplicity we will mostly consider the notions of relative freedom and independence for sets \( A \subseteq M \), although these considerations can be adapted, for example, by the operation \( M_{eq} \), for arbitrary sets \( A \subseteq M^n \).

In connection with the introduced concepts, a series of natural questions and problems arises.

1. Is an arbitrary set in the given structure free relative to some set of diagrams \( D \)?
2. Characterize the condition of \( (\mathcal{H}, D) \)-freedom of a set.
3. Characterize the condition of \( (\mathcal{H}, D) \)-independence of sets.
4. Is there and if yes then which, a condition on sets of diagrams \( D' \) such that sets \( A \) are \( (\mathcal{H}, D') \)-free, but not \( (\mathcal{H}, D) \)-free when \( D \subset D' \)?
5. Is there and if yes then which, a condition on sets of diagrams \( D' \) such that sets \( A \) and \( B \) are \( (\mathcal{H}, D') \)-independent, but not \( (\mathcal{H}, D) \)-independent when \( D \subset D' \)?

One of the ways of answering these questions is the considered below choice for sets \( A \) of diagrams \( \Phi(A_0) \) with suitable sets \( A_0 \). However, this approach does not take into account the structural specificity of sets \( A \) and thus not in the full extent it reflects the real freedom of these sets, as well as their independence. This specificity is taken into account in the following sections, for some specific classes of theories.

**Proposition 2.2** For any set \( A \subseteq M \) in a model \( M \) of a theory \( T \) there exists the set of diagrams \( D \) such that \( A \) is \( (\mathcal{H}, D) \)-free.

Proof. It is sufficiently to take as \( D \) an arbitrary set of diagrams \( \Phi \subseteq D^*(M) \) of which the universes contain the set \( A \). \( \square \)

**Proposition 2.3** For any sets \( A, B \subseteq M \) in a model \( M \) of a theory \( T \) there exists the set of diagrams \( D \) such that the sets \( A \) and \( B \) are \( (\mathcal{H}, D) \)-independent.

Proof. It is sufficiently to take as \( D \) an arbitrary set of diagrams \( \Phi \subseteq D^*(M) \) of which the universes contain the set \( A \cup B \). \( \square \)

By Propositions 1.15 and 1.16 the following statement holds.
Proposition 2.4 For any set $A \subseteq M$ in an $\omega_1$-saturated (countable saturated) model $M$ of a countable (small) theory $T$ exactly one of the following conditions holds:

1. $A$ is finite and $(\mathcal{H}, \mathcal{D})$-free only relative to the set of diagrams $\mathcal{D}$ whose universes contain the set $A$;

2. $A$ is infinite and $(\mathcal{H}, \mathcal{D})$-free relative to some infinite set of diagrams $\mathcal{D}$ whose universes have infinite distinct intersections with $A$, and these intersections can be chosen so that they form an infinite chain/antichain by inclusion.

3 On freedom and independence in hypergraphs of models of theories with unary predicates and theories with equivalence relations

In this section we describe decompositions of hypergraphs $\mathcal{H}(M)$ for theories with unary predicates and some theories with equivalence relations.

Firstly we consider the theory $T$ with unary predicates $P_i$, $i \in I$, and an equivalence relation coinciding with the equality relation. Since all language connections between elements are bounded by the condition of their uniformity, i.e., coincidence of 1-types, a description of a hypergraph $\mathcal{H}(M)$ is reduced to description of its restrictions on the sets of realizations of complete 1-types.

Due to the lack of connections between 1-types, based on Proposition 1.15 the following assertion holds for definable sets consisting of realizations of isolated types.

Proposition 3.1 For any definable set $A$ consisting of realizations in a model $M$ of some isolated 1-type $p$, either $|\mathcal{H}(M)| = 1$ when $|A| < \omega$ or $|\mathcal{H}(M)| = 2^\lambda$ when $|A| = \lambda \geq \omega$. Here, infinite sets $A$ are $\mathcal{H}$-free and $\mathcal{H}$-independent.

By Theorem 1.13 and Proposition 3.1 the following holds

Corollary 3.2 The restriction of a hypergraph $\mathcal{H} = (M, \mathcal{H}(M))$ on a union of any family of sets $A_j \subseteq M$, each of which is the set of realizations of some isolated 1-type $p_j$, is represented in the form of disjoint complete union $\mathcal{H}_{\text{isol}}$ of restrictions $\mathcal{H}_j$ of the hypergraph $\mathcal{H}$ on sets $A_j$.

Now we consider restrictions of a hypergraph $\mathcal{H}(M)$ on the sets $B_k$ of realizations of non-isolated types $q_k$. Since these types can be both omitted and realized by an arbitrary quantity of realizations, restrictions $\mathcal{H}(M) \upharpoonright B_k$ are represented in the form of atomic Boolean lattices $L_k$.

If considered types $q_k$ can be omitted in aggregate (for example, if the theory has a prime model) then the family of lattices $L_k$ compose their complete union also forming an atomic Boolean lattice:

Proposition 3.3 Restrictions of a hypergraph $\mathcal{H} = (M, \mathcal{H}(M))$ on a union of any family of sets $B_k \subseteq M$, each of which is the set of realizations of some non-isolated 1-type $q_k$, where the types $q_k$ can be omitted in aggregate, are represented in the form of disjoint complete union $\mathcal{H}_{\text{n-isol}}$ of restrictions $\mathcal{H}_k$ of the hypergraph $\mathcal{H}$ on sets $B_k$. This disjoint complete union forms an atomic Boolean lattice.
At consideration of restriction of a hypergraph $H = (M, H(M))$ on a union of a family of sets $A_j$ and $B_k$, if types $q_k$ can be omitted, a representation of this restriction in the form of disjoint complete union of restrictions $H_j$ and $H_k$, and also in the form of disjoint complete union $H_{\text{isol}}$ and $H_{n\text{-isol}}$ holds:

**Proposition 3.4** Restriction of a hypergraph $H = (M, H(M))$ on a union of any family of sets $A_j \subseteq M$, each of which is the set of realizations of some isolated 1-type $p_j$, and also any family of sets $B_k \subseteq M$, each of which is the set of realizations of some non-isolated 1-type $q_k$, where the types $q_k$ can be omitted in aggregate, is represented in the form of disjoint complete union $H_j$ of restrictions $H_j$ of the hypergraph $H$ on sets $A_j$ and restrictions $H_k$ of the hypergraph $H$ on sets $B_k$, i.e. in the form of disjoint complete union $H_{\text{isol}}$ and $H_{n\text{-isol}}$. This disjoint complete union forms an atomic Boolean lattice modulo $H_{\text{isol}}$.

To complete a description of decomposition of a hypergraph $H(M)$ it remains to consider its restrictions on the sets $C_l$ of realizations of non-isolated types $r_l$, non-omitted in aggregate. Such families of types arise, for example, in the theory of independent unary predicates, non-having isolated 1-types. In this case subsets $C_l$ also can be varied arbitrarily, but with condition $C$ of satisfaction of all consistent formulas by elements from $A_j, B_k, C_l$. Thus, a *conditional* complete union or *C-union* of restrictions $H_l$ of the hypergraph $H$ on sets $C_l$ arises:

**Proposition 3.5** Restriction of a hypergraph $H = (M, H(M))$ on a union of any family of sets $A_j \subseteq M$, each of which is the set of realizations of some isolated 1-type $p_j$, any family of sets $B_k \subseteq M$, each of which is the set of realizations of some non-isolated 1-type $q_k$, where the types $q_k$ can be omitted in aggregate, and also any family of sets $C_l \subseteq M$, each of which is the set of realizations of some non-isolated 1-type $r_l$, where the types $r_l$ cannot be omitted in aggregate, is represented in the form of C-union of restrictions $H_j$ of the hypergraph $H$ on sets $A_j$, restrictions $H_k$ of the hypergraph $H$ on sets $B_k$, and also restrictions $H_l$ of the hypergraph $H$ on sets $C_l$.

On the base of statements 3.1, 3.2, 3.3, 3.4, 3.5 the following theorem holds describing the decomposition of a hypergraph $H(M)$ by means of four types of hypergraphs holds.

**Theorem 3.6** For any model $M$ of a theory $T$ of some unary predicates the hypergraph $H(M) = (M, H(M))$ is represented in the form of disjoint complete union of some of the following hypergraphs:

1) a hypergraph with the universe $M_0$ consisting of realizations of all algebraic 1-types, and having the only edge coinciding with $M_0$;

2) a disjoint complete union of $H$-free hypergraphs of which the universes consist of realizations of non-algebraic isolated 1-types;

3) a disjoint complete union of hypergraphs forming atomic Boolean lattices, the universes of which consist of realizations of non-isolated 1-types omitted in aggregate;

4) a C-union of hypergraphs the universes of which consist of realizations of non-isolated 1-types non-omitted in aggregate.

**Example 3.7** As examples including all the types of hypergraphs 1)–4), described in Theorem 3.6 it can be considered hypergraphs of disjoint unions of the following structures:

i) structures consisting of unique non-empty finite unary predicates;
ii) structures consisting of unique non-empty infinite unary predicates;
iii) structures consisting of countably many disjoint non-empty unary predicates;
iv) structures consisting of countably many independent unary predicates.

Now we consider theories $T$ with equivalence relations $E_i$, $i \in I$.

If the relation $E_i$ is unique then in the theory there is an information on the number and cardinality of equivalence classes. If the number of these classes is finite then all of them are presented in any elementary submodel $N$ of a model $M \models T$ and a hypergraph $H = (M, H(M))$ is represented in the form of disjoint complete union of its restrictions on $E_i$-classes. The same is related to finite $E_i$-classes with a finite number for given cardinality $n$. If the number of such $E_i$-classes is infinite then in elementary submodels $N$ of the model $M$ are included arbitrary families of $n$-element $E_i$-classes, i.e., the hypergraph $H$ is free relative to $E_i$-classes.

At consideration of infinite $E_i$-classes each of which is $H$-free and distinct $E_i$-classes are $H$-independent. If the number of infinite $E_i$-classes is finite then each of which is presented in models $N$, and if the number of infinite $E_i$-classes is infinite then a $H$-freedom for $E_i$-classes holds, i.e. any infinite subset of these $E_i$-classes together with given finite $E_i$-classes form an elementary submodel with the universe of $H(M)$.

The indicated $H$-freedom and $H$-independence is extended to theories with successively embedded equivalence relations under the condition of uniformity $E_i$-classes. If in uniform equivalence classes there exist structures with unary predicates then at representation of a hypergraph $H$ in the form of disjoint complete union of restrictions on these classes also provides a decomposition described in Theorem \[4.6\]

In general case the problem of describing hypergraphs for theories with equivalence relations remains open.

4 On freedom and independence in hypergraphs of models of ordered theories

Note that if we consider an arbitrary non-algebraic isolated type $p \in S_1(\emptyset)$ in an arbitrary almost $\omega$-categorical quite $o$-minimal theory $T$ then in any model $M \models T$ the set $p(M)$ will not be $H$-free, since if we take as $A'$ some closed interval $[a, b] \subset p(M)$, where $a < b$, then there is no $M_1 \prec M$ such that $A' = p(M) \cap M_1$.

Another reason for the violation of $H$-freedom is the possibility of taking an infinite set $A' \subset p(M)$ that is not dense, while the sets $p(M) \cap M_1$ for models of the theory $T$ must be dense.

An arbitrary open interval containing an element $b$ is said to be a *neighbourhood* of the element $b$. Recall that an arbitrary subset $A$ of a linearly ordered structure $M$ is open if for any $b \in A$ there is a neighbourhood of the element $b$ that is contained in $A$.

**Definition 4.1** Let $T$ be a weakly $o$-minimal theory, $M \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type. We say that $p(M)$ is relatively $H$-free, $H$-free relative to convex sets, or $(H, cs)$-free if for any open convex set $A' \subseteq p(M)$ the equality $A' = p(M) \cap M_1$ holds for some $M_1 \in H(M)$.

We note that, in addition to the fact that $H$ hypergraphs allow to select all infinite subsets of $H$-free sets, the corresponding hypergraphs for $(H, cs)$-free sets in addition to convex sets...
without endpoints make it possible to isolate dense sets without endpoints. For example, for a theory of dense linear order without endpoints any dense subset without endpoints is distinguished by such way.

We also introduce the following necessary definitions.

**Definition 4.2** \[\text{Let } p_1(x_1), \ldots, p_n(x_n) \in S_1(T). \text{ A type } q(x_1, \ldots, x_n) \in S(T) \text{ is called } (p_1, \ldots, p_n)\text{-type if } q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^n p_i(x_i). \text{ The set of all } (p_1, \ldots, p_n)\text{-types of the theory } T \text{ is denoted by } S_{p_1, \ldots, p_n}(T). \text{ A countable theory } T \text{ is called almost } \omega\text{-categorical if for any types } p_1(x_1), \ldots, p_n(x_n) \in S(T) \text{ there is only finitely many types } q(x_1, \ldots, x_n) \in S_{p_1, \ldots, p_n}(T).\]

**Definition 4.3** \[\text{Let } M \text{ be a weakly o-minimal structure, } A \subseteq M, M \text{ be } |A|^+\text{-saturated, } p \in S_1(A) \text{ be a non-algebraic type.}

(1) An } A\text{-definable formula } F(x, y) \text{ is } p\text{-preserving or } p\text{-stable if there are } \alpha, \gamma_1, \gamma_2 \in p(M) \text{ such that } F(M, \alpha) \setminus \{\alpha\} \neq \emptyset \text{ and } \gamma_1 < F(M, \alpha) < \gamma_2.

(2) A } p\text{-preserving formula } F(x, y) \text{ is convex to right (left) if there is } \alpha \in p(M) \text{ such that } F(M, \alpha) \text{ is convex, } \alpha \text{ is the left (right) endpoint of the set } F(M, \alpha) \text{ and } \alpha \in F(M, \alpha).

**Definition 4.4** \[\text{We say that a } p\text{-preserving convex to right (left) formula } F(x, y) \text{ is equivalence-generating if for any } \alpha, \beta \in p(M) \text{ such that } M \models F(\beta, \alpha), \text{ the following holds:}

\[M \models \forall x [x \geq \beta \rightarrow [F(x, \alpha) \leftrightarrow F(x, \beta)]] \quad (M \models \forall x [x \leq \beta \rightarrow [F(x, \alpha) \leftrightarrow F(x, \beta)])\]

**Definition 4.5** \[\text{Let } T \text{ be a weakly o-minimal theory, } M \text{ be a sufficiently saturated model of } T, \text{ and let } \phi(x) \text{ be an arbitrary } M\text{-definable formula with one free variable. The convexity rank of the formula } \phi(x) (RC(\phi(x))) \text{ is defined as follows:}

1) } RC(\phi(x)) \geq 1 \text{ if } \phi(M) \text{ is infinite.}

2) } RC(\phi(x)) \geq \alpha + 1 \text{ if there are a parametrically definable equivalence relation } E(x, y) \text{ and infinitely many elements } b_i, i \in \omega, \text{ such that:}

- For any } i, j \in \omega \text{ whenever } i \neq j \text{ we have } M \models \neg E(b_i, b_j);

- For each } i \in \omega \text{ } RC(E(x, b_i)) \geq \alpha \text{ and } E(M, b_i) \text{ is a convex subset of } \phi(M).

3) } RC(\phi(x)) \geq \delta \text{ if } RC(\phi(x)) \geq \alpha \text{ for all } \alpha < \delta (\delta \text{ is limit}).

\text{If } RC(\phi(x)) = \alpha \text{ for some } \alpha \text{ then we say that } RC(\phi(x)) \text{ is defined. Otherwise (i.e. if } RC(\phi(x)) \geq \alpha \text{ for all } \alpha), \text{ we put } RC(\phi(x)) = \infty.

The convexity rank of an } 1\text{-type } p \text{ (} RC(p) \text{) is called the infimum of the set } \{RC(\phi(x)) \mid \phi(x) \in p\}, \text{ i.e. } RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}.

**Lemma 4.6** \[\text{Let } T \text{ be an almost } \omega\text{-categorical quite o-minimal theory, } M \models T, p \in S_1(\emptyset) \text{ be a non-algebraic isolated type. Then } p(M) \text{ is relatively } \mathcal{H}\text{-free } \iff RC(p) = 1.\]

Proof of Lemma 4.6 \(\Rightarrow\) \[\text{Let } p(M) \text{ is relatively } \mathcal{H}\text{-free. Assume the contrary: } RC(p) > 1. \text{ By binarity of } T \text{ there is an } \emptyset\text{-definable equivalence relation } E(x, y) \text{ partitioning } p(M) \text{ into infinitely many infinite convex sets. Take an arbitrary } a \in p(M) \text{ and consider } E(a, M).

\text{Obviously, } E(a, M) \text{ is open convex set, and there is no an elementary submodel } M_1 \text{ of } M \text{ such that } E(a, M) = p(M) \cap M_1.\]
Let \( RC(p) = 1 \). We argue to show that \( p(\mathcal{M}) \) is indiscernible over \( \emptyset \). By binarity of \( T \) it is sufficiently to prove that \( p(\mathcal{M}) \) is 2-indiscernible over \( \emptyset \). Assume the contrary: there are \( \langle a_1, a_2 \rangle, \langle a'_1, a'_2 \rangle \in \langle p(\mathcal{M}) \rangle^2 \) such that \( a_1 < a_2, a'_1 < a'_2 \) and \( tp((a_1, a_2)/\emptyset) \neq tp((a'_1, a'_2)/\emptyset) \).

Then there exists \( a'_{2''} \in p(\mathcal{M}) \) such that \( a_1 < a'_{2''} \) and \( tp((a_1, a_2)/\emptyset) \neq tp((a_1, a'_{2''})/\emptyset) \). Consequently, there is a \( 0 \)-definable formula \( \phi(x, y) \) such that \( \mathcal{M} \models \phi(a_1, a_2) \land \neg \phi(a_1, a'_{2''}) \). By weak o-minimality we can assume that \( \phi(a_1, \mathcal{M}) \) is convex. Without loss of generality, we will also assume that \( a_2 < a'_{2''} \). Then consider the following formula:

\[
F(x, a_1) := x \geq a_1 \land \exists y[\phi(a_1, y) \land x \leq y].
\]

It is easy to see that \( F(x, y) \) is a \( p \)-preserving convex to right. If \( F(x, y) \) is equivalence-generating, we have a contradiction with \( RC(p) = 1 \). If \( F(x, y) \) is not equivalence-generating, it contradicts to almost \( \omega \)-categoricity of \( T \). Thus, \( p(\mathcal{M}) \) is indiscernible over \( \emptyset \), whence for any open convex set \( A' \subseteq p(\mathcal{M}) \) there is an elementary submodel \( \mathcal{M}_1 \) of \( \mathcal{M} \) such that \( A' = p(\mathcal{M}) \cap \mathcal{M}_1 \).

Example 4.7 Let \( \mathcal{M} = (\mathbb{Q}, <, f^1) \) be a linearly ordered structure, where \( \mathbb{Q} \) is the set of rational numbers, \( f(x) = x + 1 \) is an unary function on \( \mathbb{Q} \).

It is easily seen that \( \mathcal{M} \) is an o-minimal structure, and \( Th(\mathcal{M}) \) is not almost \( \omega \)-categorical. Note also that \( p(x) := \{x = x \} \in S_1(\emptyset) \) is a non-algebraic isolated type, \( RC(p) = 1 \), but \( p(\mathcal{M}) \) is not relatively \( H \)-free.

Definition 4.8 Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \models T, p, q \in S_1(\emptyset) \) be non-algebraic isolated types, \( RC(p) = RC(q) = 1 \). We say that \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) are relatively \( H \)-independent, \( H \)-independent with regard to convex sets, or \( (H, cs) \)-independent if for any open convex sets \( A' \subseteq p(\mathcal{M}) \) and \( B' \subseteq q(\mathcal{M}) \) there is \( M_1 \in H(M) \) such that \( A' = p(\mathcal{M}) \cap M_1 \) and \( B' = q(\mathcal{M}) \cap M_1 \).

Let \( p_1, p_2, \ldots, p_s \in S_1(\emptyset) \) be non-algebraic types. We say that the family of types \( \{p_1, \ldots, p_s\} \) is orthogonal over \( \emptyset \) if for any sequence \( (n_1, \ldots, n_s) \in \omega^s \), for any increasing tuples \( \tilde{a}_1, \tilde{a}'_1 \in [p_1(\mathcal{M})]^{n_1}, \ldots, \tilde{a}_s, \tilde{a}'_s \in [p_s(\mathcal{M})]^{n_s} \) such that \( tp(\tilde{a}_1/\emptyset) = tp(\tilde{a}'_1/\emptyset), \ldots, tp(\tilde{a}_s/\emptyset) = tp(\tilde{a}'_s/\emptyset) \) we have \( tp(\tilde{a}_1, \ldots, \tilde{a}_s)/\emptyset = tp(\tilde{a}'_1, \ldots, \tilde{a}'_s)/\emptyset) \).

Lemma 4.9 Let \( T \) be an almost \( \omega \)-categorical quite o-minimal theory, \( \mathcal{M} \models T, p, q \in S_1(\emptyset) \) be non-algebraic isolated types, \( RC(p) = RC(q) = 1 \). Then \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) are relatively \( H \)-independent \( \iff \) \( p \perp^w q \).

Proof of Lemma 4.9 \((\Rightarrow)\) Let \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) are relatively \( H \)-independent. Assume the contrary: \( p \not\perp^w q \). By quite o-minimality there is a \( 0 \)-definable bijection \( f : p(\mathcal{M}) \to q(\mathcal{M}) \).

Since \( RC(p) = RC(q) = 1 \), this bijection is strictly monotonic. Take an arbitrary open convex set \( A' \subseteq p(\mathcal{M}) \) and consider \( f(A') \). By strict monotonicity of \( f \) the image \( f(A') \) is also open convex set. Take arbitrary \( a, b \in f(A') \) with the condition \( a < b \). Then let \( B' := \{c \in q(\mathcal{M}) \mid a < c < b\} \). Then there is no \( M_1 \prec \mathcal{M} \) such that \( A' = p(\mathcal{M}) \cap M_1 \) and \( B' = q(\mathcal{M}) \cap M_1 \).

\((\Leftarrow)\) Let \( p \perp^w q \). Then by almost \( \omega \)-categoricity of \( T \) the family \( \{p, q\} \) is orthogonal over \( \emptyset \), whence \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) are relatively \( H \)-independent.
Corollary 4.10 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type, $RC(p) = n$, where $n > 1$. Suppose that $E_1(x, y)$, $E_2(x, y)$, \ldots, $E_{n-1}(x, y)$ are $\emptyset$-definable equivalence relations partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes so that $E_1(a, \mathcal{M}) \subset E_2(a, \mathcal{M}) \subset \cdots \subset E_{n-1}(a, \mathcal{M})$ for any $a \in p(\mathcal{M})$. Then the following holds:

1) Every $E_1$-class is relatively $\mathcal{H}$-free;
2) Any two $E_1$-classes are relatively $\mathcal{H}$-independent;
3) For any $2 \leq i \leq n-1$ every $E_i$-class is not relatively $\mathcal{H}$-free.

Example 4.11 Let $\mathcal{M} = (\mathbb{Q} \times \mathbb{Q}; <, E^2, f^1)$ be a linearly ordered structure, where $\mathbb{Q} \times \mathbb{Q}$ is lexicographically ordered. The symbol $E$ is interpreted by a binary relation defined as follows: $E(a, b) \leftrightarrow n_1 = n_2$ for any $a = (n_1, m_1), b = (n_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$. The symbol $f$ is interpreted by a unary function defined by the equality $f((n, m)) = (n + 1, m)$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

Obviously, $E(x, y)$ is an equivalence relation partitioning $M$ into infinitely many infinite convex classes.

It can be established that $Th(\mathcal{M})$ is a quite o-minimal theory, and it is not almost $\omega$-categorical. Note that $p(x) := \{x = x\} \in S_1(\emptyset)$ is a non-algebraic isolated type, $RC(p) = 2$, every $E$-class is relatively $\mathcal{H}$-free, however $E(a, \mathcal{M})$ and $E(f(a), \mathcal{M})$ are not relatively $\mathcal{H}$-independent for each $a \in M$.

Definition 4.12 Let $T$ be a weakly o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type. Let $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. If $A \subseteq p(\mathcal{M})$ then we denote by $A/E$ the set of representatives of $E$-classes having a nonempty intersection with $A$. We say that $p(\mathcal{M})$ is relatively $(\mathcal{H}, E)$-free if for any convex $A' \subseteq p(\mathcal{M})$ such that $A'/E$ is a open set the equality $A' = p(\mathcal{M}) \cap M_1$ holds for some $M_1 \in H(\mathcal{M})$.

Note that in the latter definition in case of dense ordering of $p(\mathcal{M})$ the convexity of $A'$ is essential. Indeed, let $p(x) := \{U(x)\}$, $a_1, a_2 \in p(\mathcal{M})$ such that $\mathcal{M} \models E(a_1, a_2) \land a_1 < a_2$. Consider the following formula:

$$\phi(x, a_1, a_2) := U(x) \land [x \leq a_1 \lor x \geq a_2].$$

Let $A' = \phi(\mathcal{M}, a_1, a_2)$. Obviously, $A' \subseteq p(\mathcal{M})$, $A'$ is not convex, $A'/E$ is open convex set, but there is no $M_1 \preceq \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$.

Proposition 4.13 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type, $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. Then $p(\mathcal{M})$ is relatively $(\mathcal{H}, E)$-free $\iff$ for any $\emptyset$-definable equivalence relation $E'(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes we have $E'(a, \mathcal{M}) \subseteq E(a, \mathcal{M})$ for some $a \in p(\mathcal{M})$.

Proof of Proposition 4.13 ($\Rightarrow$) Let $p(\mathcal{M})$ be relatively $(\mathcal{H}, E)$-free. By almost $\omega$-categoricity of $T$ $RC(p) < \omega$, i.e. there is an $\emptyset$-definable equivalence relation $E^*(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many convex classes, and for any $\emptyset$-definable equivalence relation $E'(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many convex classes we have $E'(a, \mathcal{M}) \subseteq E^*(a, \mathcal{M})$ for some $a \in p(\mathcal{M})$. Then we assert that $E(x, y) \equiv E^*(x, y)$ on $p(\mathcal{M})$. If this is not true then either $E(a, \mathcal{M}) \subset E^*(a, \mathcal{M})$ or $E^*(a, \mathcal{M}) \subset E(a, \mathcal{M})$. In the first case we have a contradiction
with relative \((\mathcal{H}, E)\)-freedom of \(p(\mathcal{M})\). In the second case we have a contradiction with that \(E^*(x, y)\) is the greatest.

\((\iff)\) Let \(E(x, y)\) be the greatest \(\emptyset\)-definable equivalence relation partitioning \(p(\mathcal{M})\) into infinitely many convex classes. Then if as \(A'\) we take an arbitrary convex subset of \(p(\mathcal{M})\) so that \(A'/E\) is open then we easily find \(\mathcal{M}_1 \prec \mathcal{M}\) with the condition \(A' = p(\mathcal{M}) \cap M_1\). 

\(\square\)

**Example 4.14** Let \(\mathcal{M} = (M, <, E_1^{\omega})_{i \in \omega}\) be a linearly ordered structure, where for each \(i \in \omega\) \(E_i(x, y)\) defines an equivalence relation partitioning \(M\) into infinitely many convex classes, and \(E_i\) partitions every \(E_{i+1}\)-class into infinitely many \(E_i\)-classes, every \(E_i\)-class is convex and open so that \(E_i\)-subclasses of each \(E_{i+1}\)-class are densely ordered without endpoints.

It can be established that \(Th(\mathcal{M})\) is a quite o-minimal theory non-being almost \(\omega\)-categorical. Obviously, \(\mathcal{M}\) is 1-indiscernible, i.e. \(p(x) := \{x = x\} \in S_1(\emptyset)\). It is not difficult to see that \(p(\mathcal{M})\) is relatively \((\mathcal{H}, E_i)\)-free for any \(i \in \omega\).

**Definition 4.15** Let \(T\) be a weakly o-minimal theory, \(\mathcal{M} \models T\), \(p_1, p_2 \in S_1(\emptyset)\) be non-algebraic isolated types. Let \(E_1(x, y), E_2(x, y)\) be \(\emptyset\)-definable equivalence relations partitioning \(p_1(\mathcal{M})\) and \(p_2(\mathcal{M})\) respectively into infinitely many convex classes. Suppose that \(p_1(\mathcal{M})\) is relatively \((\mathcal{H}, E_1)\)-free and \(p_2(\mathcal{M})\) is relatively \((\mathcal{H}, E_2)\)-free. We say that \(p_1(\mathcal{M})\) and \(p_2(\mathcal{M})\) are relatively \((\mathcal{H}, E_1, E_2)\)-independent if for any convex \(A' \subseteq p_1(\mathcal{M})\) and \(B' \subseteq p_2(\mathcal{M})\) such that \(A'/E_1\) and \(B'/E_2\) are open sets there is \(M_1 \in H(\mathcal{M})\) such that \(A' = p_1(\mathcal{M}) \cap M_1\) and \(B' = p_2(\mathcal{M}) \cap M_1\).

**Proposition 4.16** Let \(T\) be an almost \(\omega\)-categorical quite o-minimal theory, \(\mathcal{M} \models T\), \(p_1, p_2 \in S_1(\emptyset)\) be non-algebraic isolated types. Let \(E_1(x, y), E_2(x, y)\) be \(\emptyset\)-definable equivalence relations partitioning \(p_1(\mathcal{M})\) and \(p_2(\mathcal{M})\) respectively into infinitely many convex classes. Suppose that \(p_1(\mathcal{M})\) is relatively \((\mathcal{H}, E_1)\)-free, \(p_2(\mathcal{M})\) is relatively \((\mathcal{H}, E_2)\)-free. Then \(p_1(\mathcal{M})\) and \(p_2(\mathcal{M})\) are relatively \((\mathcal{H}, E_1, E_2)\)-independent \(\iff p_1 \perp^w p_2\).

**Proof** of Proposition 4.16 Let \(p_1(\mathcal{M})\) and \(p_2(\mathcal{M})\) be relatively \((\mathcal{H}, E_1, E_2)\)-independent. Assume the contrary: \(p_1 \not\perp^w p_2\). By quite o-minimality there is an \(\emptyset\)-definable bijection \(f : p_1(\mathcal{M}) \to p_2(\mathcal{M})\), whence \(RC(p_1) = RC(p_2)\) and \(f(E_1(a, \mathcal{M})) = E_2(f(a), \mathcal{M})\) for any \(a \in p_1(\mathcal{M})\). Take an arbitrary convex set \(A' \subseteq p_1(\mathcal{M})\) with open \(A'/E_1\) and consider \(f(A')\). Obviously, \(f(A')\) is convex and \(f(A')/E_2\) is open. Take arbitrary \(E_2\)-classes \(C = E_2(a, \mathcal{M})\) and \(D = E_2(b, \mathcal{M})\) for some \(a, b \in p_2(\mathcal{M})\) with the condition \(C < D\) lying in \(f(A')\). Then let \(B' := \{e \in p_2(\mathcal{M}) \mid C < e < D\}\). Obviously, \(B'\) will be also convex, and \(B'/E_2\) will be open. It is easily to see that there is no \(\mathcal{M}_1 \prec \mathcal{M}\) such that \(A' = p_1(\mathcal{M}) \cap M_1\) and \(B' = p_2(\mathcal{M}) \cap M_1\). 

\(\square\)

**Corollary 4.17** Let \(T\) be an almost \(\omega\)-categorical quite o-minimal theory, \(\mathcal{M} \models T\), \(p_1, p_2 \in S_1(\emptyset)\) be non-algebraic isolated types, and suppose that there exists an \(\emptyset\)-definable bijection \(f : p_1(\mathcal{M}) \to p_2(\mathcal{M})\). Let \(E_1(x, y)\) be an \(\emptyset\)-definable equivalence relation partitioning \(p_1(\mathcal{M})\) into infinitely many convex classes. Define on the set \(p_2(\mathcal{M})\) the relation \(E_2(x, y)\) as follows:

\[
\text{for any } a, b \in p_2(\mathcal{M}) \quad E_2(a, b) \iff E_1(f^{-1}(a), f^{-1}(b)).
\]

Then \(p_1(\mathcal{M})\) is relatively \((\mathcal{H}, E_1)\)-free \(\iff p_2(\mathcal{M})\) is relatively \((\mathcal{H}, E_2)\)-free.

Further we extend definitions of relative \(\mathcal{H}\)-freedom, relative \(\mathcal{H}\)-independence, relative \((\mathcal{H}, E)\)-freedom and relative \((\mathcal{H}, E_1, E_2)\)-independence on non-isolated 1-types.
Recall that if \( A \) is an arbitrary subset of a linearly ordered structure \( \mathcal{M} \) then we denote by \( A^+ \) (and respectively by \( A^- \)) the sets of elements \( b \) of the considered structure with the condition \( A < b \) (\( b < A \)).

**Definition 4.18** [14] Let \( \mathcal{M} \) be a weakly o-minimal structure, \( A \subseteq M, p \in S_1(A) \) be a non-algebraic type. We say that \( p \) is quasirational to right (left) if there is an \( A \)-definable convex formula \( U_p(x) \in p \) such that for any sufficiently saturated model \( \mathcal{N} \succ \mathcal{M} \), \( U_p(\mathcal{N})^+ = p(\mathcal{N})^+ \) \((U_p(\mathcal{N})^- = p(\mathcal{N})^-)\). A non-isolated 1-type is called quasirational if it is either quasirational to right or quasirational to left. A non-quasirational non-isolated 1-type is called irrational.

Obviously, an 1-type being simultaneously quasirational to right and quasirational to left is isolated.

We say that a convex set \( A \) is open to right (left) if there is \( a \in A \) such that for any \( b > a \) \((b < a)\) there exists a neighbourhood of the element \( b \) containing in \( A \). Obviously, a set being simultaneously open to right and open to left is open.

**Definition 4.19** Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p \in S_1(\emptyset) \) be a non-isolated type, \( RC(p) = 1 \). If \( p \) is quasirational to right (left) then we say \( p(\mathcal{M}) \) is relatively \( \mathcal{H} \)-free if for any open to right (left) convex \( A' \subseteq p(\mathcal{M}) \) the equality \( A' = p(\mathcal{M}) \cap M_1 \) holds for some \( M_1 \in H(\mathcal{M}) \). If \( p \) is irrational then it is sufficiently to take an arbitrary convex set as \( A' \).

**Lemma 4.20** Let \( T \) be an almost \( \omega \)-categorical quite o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p \in S_1(\emptyset) \) be a non-isolated type. Then \( p(\mathcal{M}) \) is relatively \( \mathcal{H} \)-free \( \Rightarrow \) \( RC(p) = 1 \).

Proof of Lemma 4.20 \((\Rightarrow)\) Indeed, if \( RC(p) > 1 \) then there is an \( \emptyset \)-definable equivalence relation \( E(x, y) \) partitioning \( p(\mathcal{M}) \) into infinitely many infinite convex classes. Obviously, there is no \( M_1 \prec M \) such that \( E(a, \mathcal{M}) = p(\mathcal{M}) \cap M_1 \) for some \( a \in p(\mathcal{M}) \).

\((\Leftarrow)\) If \( RC(p) = 1 \) then by analogy with proof of Lemma 4.13 it is established that \( p(\mathcal{M}) \) is indiscernible over \( \emptyset \). Then if \( p \) is quasirational to right (left) then for any open to right (left) convex set \( A' \subseteq p(\mathcal{M}) \) there is \( M_1 \prec M \) such that \( A' = p(\mathcal{M}) \cap M_1 \). If \( p \) is irrational then for any convex set \( A' \subseteq p(\mathcal{M}) \) (including the case when \( A' = \{a\} \) for some \( a \in p(\mathcal{M}) \)) there exists \( M_1 \prec M \) with \( A' = p(\mathcal{M}) \cap M_1 \).

**Definition 4.21** Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p, q \in S_1(\emptyset) \) be non-isolated types, \( RC(p) = RC(q) = 1 \). We say that \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) relatively \( \mathcal{H} \)-independent if for any convex sets \( A' \subseteq p(\mathcal{M}) \) and \( B' \subseteq q(\mathcal{M}) \) corresponding to \( p \) and \( q \) (as in Definition 4.19) there exists \( M_1 \in H(\mathcal{M}) \) such that \( A' = p(\mathcal{M}) \cap M_1 \) and \( B' = q(\mathcal{M}) \cap M_1 \).

**Proposition 4.22** [14] Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \models T, A \subseteq M, p, q \in S_1(A) \) be non-algebraic types, \( p \perp^w q \). Then:

\(1\) \( p \) is irrational \( \iff \) \( q \) is irrational;

\(2\) \( p \) is quasirational \( \iff \) \( q \) is quasirational.

**Lemma 4.23** Let \( T \) be an almost \( \omega \)-categorical quite o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p, q \in S_1(\emptyset) \) be non-isolated types, \( RC(p) = RC(q) = 1 \). Then \( p(\mathcal{M}) \) and \( q(\mathcal{M}) \) are relatively \( \mathcal{H} \)-independent \( \iff \) \( p \perp^w q \).
Proof of Lemma \textit{[4.23]} If \( p \not\perp^w q \) then by Proposition \textit{[4.22]} \( p \) and \( q \) are simultaneously either quasirational or irrational. Without loss of generality, suppose that \( p \) and \( q \) are quasirational. By quite o-minimality there is an \( \emptyset \)-definable bijection \( f : p(\mathcal{M}) \to q(\mathcal{M}) \). Since the convexity ranks of the types are equal to 1 then this bijection is strictly monotonic. For definiteness, suppose that \( p \) is quasirational to right. Then if \( f \) is strictly increasing (decreasing) then \( q \) will be quasirational to right (left). Take an arbitrary open to right convex set \( A' \subseteq p(\mathcal{M}) \) and consider \( f(A') \). If \( f \) is strictly increasing (decreasing) then \( f(A') \) will also be open to right (left) convex set. Take arbitrary \( a, b \in f(A') \) with \( a < b \). Then let \( B' := \{ c \in q(\mathcal{M}) \mid a < c < b \} \). Then there is no \( \mathcal{M}_1 < \mathcal{M} \) such that \( A' = p(\mathcal{M}) \cap M_1 \) and \( B' = q(\mathcal{M}) \cap M_1 \). \( \square \)

\textbf{Definition 4.24} Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p \in S_1(\emptyset) \) be a non-isolated type. Let \( E(x, y) \) be an \( \emptyset \)-definable equivalence relation partitioning \( p(\mathcal{M}) \) into infinitely many infinite convex classes. If \( p \) is quasirational to right (left) then we say \( p(\mathcal{M}) \) is relatively \(( \mathcal{H}, E)\)-free if for any convex \( A' \subseteq p(\mathcal{M}) \) such that \( A'/E \) is open to right (left) set the equality \( A' = p(\mathcal{M}) \cap M_1 \) holds for some \( M_1 \in H(\mathcal{M}) \). If \( p \) is irrational then it is sufficiently to take any open convex subset of \( p(\mathcal{M}) \) as \( A' \), leaving the type of the set \( A'/E \) for an arbitrariness.

\textbf{Proposition 4.25} Let \( T \) be an almost \( \omega \)-categorical quite o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p \in S_1(\emptyset) \) be a non-isolated type, \( E(x, y) \) be an \( \emptyset \)-definable equivalence relation partitioning \( p(\mathcal{M}) \) into infinitely many infinite convex classes. Then \( p(\mathcal{M}) \) is relatively \(( \mathcal{H}, E)\)-free \( \iff \) \( E(x, y) \) is the greatest \( \emptyset \)-definable equivalence relation partitioning \( p(\mathcal{M}) \) into infinitely many convex classes.

Proof of Proposition \textit{[4.25]} \( (\Rightarrow) \) is proved similarly Proposition \textit{[4.13]}

\( (\Leftarrow) \) If \( p \) is quasirational to right (left) then taking as \( A' \) an arbitrary convex subset of \( p(\mathcal{M}) \) with the condition that \( A'/E \) is open to right (left) we easily find \( \mathcal{M}_1 < \mathcal{M} \) such that \( A' = p(\mathcal{M}) \cap M_1 \). If \( p \) is irrational then take as \( A' \) an arbitrary open convex subset of \( p(\mathcal{M}) \).

\( \square \)

\textbf{Definition 4.26} Let \( T \) be a weakly o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p_1, p_2 \in S_1(\emptyset) \) be non-isolated types. Let \( E_1(x, y), E_2(x, y) \) be \( \emptyset \)-definable equivalence relations partitioning \( p_1(\mathcal{M}) \) and \( p_2(\mathcal{M}) \) respectively into infinitely many infinite convex classes. Suppose that \( p_1(\mathcal{M}) \) is relatively \(( \mathcal{H}, E_1)\)-free and \( p_2(\mathcal{M}) \) is relatively \(( \mathcal{H}, E_2)\)-free. We say that \( p_1(\mathcal{M}) \) and \( p_2(\mathcal{M}) \) are relatively \(( \mathcal{H}, E_1, E_2)\)-independent if for any convex \( A' \subseteq p_1(\mathcal{M}) \) and \( B' \subseteq p_2(\mathcal{M}) \) corresponding to \( p_1 \) and \( p_2 \) (as in Definition \textit{[4.24]}) there is \( M_1 \in H(\mathcal{M}) \) such that \( A' = p_1(\mathcal{M}) \cap M_1 \) and \( B' = p_2(\mathcal{M}) \cap M_1 \).

\textbf{Proposition 4.27} Let \( T \) be an almost \( \omega \)-categorical quite o-minimal theory, \( \mathcal{M} \) be a sufficiently saturated model for \( T \), \( p_1, p_2 \in S_1(\emptyset) \) be non-isolated types. Suppose that \( p_1(\mathcal{M}) \) is relatively \(( \mathcal{H}, E_1)\)-free and \( p_2(\mathcal{M}) \) is relatively \(( \mathcal{H}, E_2)\)-free, where \( E_1(x, y), E_2(x, y) \) are \( \emptyset \)-definable equivalence relations partitioning \( p_1(\mathcal{M}) \) and \( p_2(\mathcal{M}) \) respectively into infinitely many infinite convex classes. Then \( p_1(\mathcal{M}) \) and \( p_2(\mathcal{M}) \) are relatively \(( \mathcal{H}, E_1, E_2)\)-independent \( \iff \) \( p_1 \perp^w p_2 \).

Proof of Proposition \textit{[4.27]} If \( p_1 \not\perp^w p_2 \) then by Proposition \textit{[4.22]} \( p_1 \) and \( p_2 \) are simultaneously either quasirational or irrational. Without loss of generality, suppose that \( p_1 \) and \( p_2 \) are quasirational. By quite o-minimality there is an \( \emptyset \)-definable bijection \( f : p_1(\mathcal{M}) \to p_2(\mathcal{M}) \).
For definiteness, let \( p_1 \) be quasirational to right. Then take an arbitrary convex \( A' \subseteq p_1(\mathcal{M}) \) with the condition that \( A'/E_1 \) is open to right. Obviously, \( f(A') \) will be convex. If \( f \) is strictly increasing (decreasing) on \( p_1(\mathcal{M})/E_1 \) then \( f(A')/E_2 \) will be open to right (left). Taking arbitrary \( E_2 \)-classes \( E_2(a, \mathcal{M}) \) and \( E_2(b, \mathcal{M}) \) for some \( a, b \in p_2(\mathcal{M}) \) with \( E_2(a, \mathcal{M}) < E_2(b, \mathcal{M}) \) lying in \( f(A') \), and considering \( B' := \{ h \in p_2(\mathcal{M}) \mid E_2(a, \mathcal{M}) < h < E_2(b, \mathcal{M}) \} \), we see that \( B' \) is convex, and \( B'/E_2 \) is open Obviously, there is no \( \mathcal{M}_1 < \mathcal{M} \) such that \( A' = p_1(\mathcal{M}) \cap M_1 \) and \( B' = p_2(\mathcal{M}) \cap M_1 \). \( \square \)

5 On freedom and independence in hypergraphs of models of theories of unars

In the case of unary theories, if sets \( A_i = f^{-k_i}(a_i) \), \( k_i > 0 \), are \( \mathcal{H} \)-independent, then the restriction of \( \mathcal{H} \) on \( \bigcup_i A_i \) is represented in the form of disjoint complete union of restrictions \( \mathcal{H}_i = \mathcal{H} \upharpoonright A_i \).

Example 5.1 Consider a connected free unar \( \mathcal{M} = \langle M, f \rangle \), i.e. a connected unar non-having cycles and such that every element has infinitely many \( f \)-preimages. Consider also a hypergraph \( \mathcal{H} \) of elementary subsystems of \( \mathcal{M} \). Then for every element \( a \in M \) and pairwise distinct elements \( a_i \in f^{-k}(a) \), \( k > 0 \), the sets \( A_i = f^{-k_i}(a_i) \) are \( \mathcal{H} \)-independent for any \( k_i > 0 \). The restriction of \( \mathcal{H} \) on \( \bigcup_i A_i \) is represented in the form of disjoint complete union of restrictions \( \mathcal{H}_i = \mathcal{H} \upharpoonright A_i \).

On the other hand, the sets \( f^{-k}(a) \) and \( f^{-m}(b) \) for \( k, m > 0 \) and \( b \in f^{-k}(a) \) are not \( \mathcal{H} \)-independent, since \( b \not\in Z \in H(\mathcal{M}) \) implies \( f^{-m}(b) \cap Z = \emptyset \).

Thus, \( \mathcal{H} \)-independence of the sets \( f^{-k}(a) \) and \( f^{-m}(b) \) is equivalent to their disjointness, and also the conditions \( b \not\in \triangle_f(a) \) and \( a \not\in \triangle_f(b) \), where \( \triangle_f(a) = \bigcup_{n \in \omega} f^{-n}(a) \) is a lower cone of the root \( a \). The indicated description of \( \mathcal{H} \)-independence is naturally is extended on an arbitrary family of the sets \( f^{-k_i}(a_i) \).

By Properties 1 and 3, of \( \mathcal{H} \)-freedom and \( \mathcal{H} \)-independence, in general case the \( \mathcal{H} \)-independence of the sets \( A_i = f^{-k_i}(a_i) \) for \( a_i \in f^{-k}(a) \), allowing to conduct an indicated decomposition, implies their infinity and presence of completeness of the types \( p_i(x) \) over \( \{a_i\} \) isolated by the formulas \( f^{k_i}(x) \approx a_i \).

If the set \( A_i = f^{-k_i}(a_i) \) is finite then its inclusion in \( Z \in H(\mathcal{M}) \) is equivalent to inclusion of \( a_i \) in \( Z \).

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