Asymptotic Properties of Coupled Forward-Backward Stochastic Differential Equations

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Abstract

In this paper, we consider coupled forward-backward stochastic differential equations (FBSDEs in short) with parameter $\varepsilon > 0$, of the following type

$$
\begin{cases}
X^{\varepsilon,t,x}(s) = x + \int_t^s f(r, X^{\varepsilon,t,x}(r), Y^{\varepsilon,t,x}(r)) \, dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X^{\varepsilon,t,x}(r), Y^{\varepsilon,t,x}(r)) \, dW(r), \\
Y^{\varepsilon,t,x}(s) = h(X^{\varepsilon,t,x}(T)) + \int_s^T g(r, X^{\varepsilon,t,x}(r), Y^{\varepsilon,t,x}(r), Z^{\varepsilon,t,x}(r)) \, dr - \int_s^T Z^{\varepsilon,t,x}(r) \, dW(r), \quad 0 \leq t \leq s \leq T.
\end{cases}
$$

We study the asymptotic behavior of its solutions and establish a large deviation principle for the corresponding processes.

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1 Introduction

Non-linear backward stochastic differential equations (BSDEs in short) were first introduced in Stochastic
Optimal Control Theory with the pioneering work of Bismut [6] and then developed by Pardoux and Peng
[26]. Since then, they have become a powerful tool in many fields, such as mathematics finance, optimal
control, stochastic games, partial differential equations and homogenization etc. Simultaneously, it is
well known that the Hamiltonian system associated with the maximum principle for stochastic optimal
control problems corresponds to certain fully coupled forward backward stochastic differential equations
(FBSDEs in short) (see [31]). In mathematical finance, fully coupled FBSDEs can be encountered when
one studies the problems of hedging options involved in a large investor in financial market (see [10, 16]).
Besides, fully coupled FBSDEs can provide probabilistic interpretations for the solutions of a class of
quasilinear parabolic and elliptic PDEs (see [28, 29, 30, 31]).

There are three main approaches to solve FBSDEs. The first one, the Method of Contraction Mapping,
which was considered by Antonelli [2] and later developed by Pardoux-Tang [30], works well when the
duration $T$ is relatively small. Second, the Four Step Scheme. It removed the restriction on the time
duration for Markovian FBSDEs and was initiated by Ma-Protter-Yong [24]. Third, the Method of
Continuation. This method can treat non-Markovian FBSDEs with arbitrary time duration; it was
initiated by Hu-Peng [19] and Peng-Wu [31] and later developed by Ma et al in [23]. It is worth noting
that these three methods do not cover each other.

In our paper, on one hand, we obtain two kind of asymptotic results. The first employs the assump-
tions established by Peng et al in [19, 31] and provides the convergence in distribution of the associated
processes (Theorem 3.1). The second, more closely related to the PDE point of view but still using
strictly probabilistic methods, provides almost-sure convergence with values in some $L^2$ type spaces (The-
orem 3.2). In this last result the problem of convergence of classical/viscosity solutions of the quasilinear
parabolic system of PDE’s associated to the backward equation is naturally addressed. Notice that when
this PDE takes the form of the backward Burgers equation, the problem becomes the convergence of
solutions for vanishing viscosity hydrodynamical parameter. We mention the work [18] where the authors
do the asymptotic studies of FBSDEs with generalized Burgers type nonlinearities.

The large deviation principle (LDP) characterizes the limiting behavior of probability measure in
applied probability and is largely used in rare events simulation. Recently, there has been a growing
literature on studying the applications of LDP in finance (see [32]).

We now consider the following small perturbation of FBSDEs (1.1)

\[
\begin{align*}
X^{\varepsilon,t,x} (s) &= x + \int_t^s f (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r)) \, dr \\
&\quad + \sqrt{\varepsilon} \int_t^s \sigma (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r)) \, dW (r), \\
Y^{\varepsilon,t,x} (s) &= h (X^{\varepsilon,t,x} (T)) + \int_s^T g (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r), Z^{\varepsilon,t,x} (r)) \, dr \\
&\quad - \int_s^T Z^{\varepsilon,t,x} (r) \, dW (r), \quad 0 \leq t \leq s \leq T.
\end{align*}
\]

The solution of this equation is denoted by

\[
(X^{\varepsilon,t,x} (s), Y^{\varepsilon,t,x} (s), Z^{\varepsilon,t,x} (s), t \leq s \leq T).
\]

We want to establish the large deviation principle of the law of $Y^{\varepsilon,t,x}$ in the space of $C ([0, T] ; \mathbb{R}^n)$, namely
the asymptotic estimates of probabilities $P (Y^{\varepsilon,t,x} \in \Gamma)$, where $\Gamma \in \mathcal{B} (C ([0, T] ; \mathbb{R}^n))$.

Ma et al in [25] first considered the sample path large deviation principle for the adapted solutions
to the FBSDEs in the case where the drift term contains $Z$ and under appropriate conditions in terms of
a certain type of convergence of solutions for the associated quasilinear PDEs. While, with probability methods, by the contraction principle, the same small random perturbation for BSDEs and the Freidlin-Wentzell’s large deviation estimates in $C([0,T];\mathbb{R}^n)$ are also obtained by [13, 17, 18, 33, 20] (see references therein). In [11] a large deviation principle of Freidlin and Wentzell type under nonlinear probability for diffusion processes with a small diffusion coefficient was obtained.

Our aim in this paper extends the previous work [13, 17, 18, 33, 20] to the coupled case. Namely, when the drift and diffusion terms contain $Y$, by probability methods and under some suitable assumptions. Note that in [11], the authors considered the fully coupled BSDEs via the corresponding PDE techniques.

Our work is organized as follows. In Section 2, we give the framework of our paper. Some estimates and regularity results are established for the solutions of FBSDEs (1.1) in Section 2. Then in Section 3 we show our main results Theorem 3.1 and Theorem 3.2. Section 4 is devoted to establishing the large deviation results for (1.1). Some proofs of technique lemmas are given in the Appendix.

2 Preliminaries

Let us begin by introducing the setting for the stochastic differential systems we want to investigate. Consider as Brownian motion $W$ the $d$-dimensional coordinate process on the classical Wiener space $(\Omega, \mathcal{F}, P)$, i.e., $\Omega$ is the set of continuous functions from $[0,T]$ to $\mathbb{R}^d$ starting from $0$, $\Omega = C([0,T];\mathbb{R}^d, 0)$, $\mathcal{F}$ the completed Borel $\sigma$-algebra over $\Omega$, $P$ the Wiener measure and $W$ the canonical process:

$$W_s(\omega) = \omega_s, s \in [0,T], \omega \in \Omega.$$  

By $\{\mathcal{F}_s, 0 \leq s < T\}$ we denote the natural filtration generated by $\{W_s\}_{0 \leq s < T}$ and augmented by all $P$-null sets, i.e.,

$$\mathcal{F}_s = \sigma\{W(r), r \leq s\} \vee \mathcal{N}_P, \ s \in [0,T],$$

where $\mathcal{N}_P$ is the set of all $P$-null subsets. For each $t > 0$, we denote by $\{\mathcal{F}_s^t, t \leq s \leq T\}$ the natural filtration of the Brownian motion $\{W(s) - W(t), t \leq s \leq T\}$, augmented by the $P$-null set of $\mathcal{F}$.

We denote by $\mathcal{M}^2(t,T;\mathbb{R}^p)$ the set of all $\mathbb{R}^p$-valued $\mathcal{F}_t$-adapted process $v(\cdot)$ such that

$$\mathbb{E}\left[\int_t^T |v(s)|^2 \, ds\right] < +\infty,$$

and $\mathcal{S}^2(t,T;\mathbb{R}^n)$, the set of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process $v(\cdot)$ such that

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |v(s)|^2\right] < +\infty.$$  

Clearly, $\mathcal{N}_t = \mathcal{S}^2(t,T;\mathbb{R}^n) \times \mathcal{S}^2(t,T;\mathbb{R}^n) \times \mathcal{M}^2(t,T;\mathbb{R}^p)$ forms a Banach space.

The space $\mathcal{N}_t$ is a Banach space for the natural norm associated with the product topology structure.

We use the usual inner product and Euclidean norm in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^{m \times d}$.

Definition 2.1. A triple of processes

$$(X^{\varepsilon,t,x}, Y^{\varepsilon,t,x}, Z^{\varepsilon,t,x}) : [0,T] \times \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d},$$

is called an adapted solution of the Eqs. (1.1), if $(X^{\varepsilon,t,x}, Y^{\varepsilon,t,x}, Z^{\varepsilon,t,x}) \in \mathcal{N}_t$, and it satisfies (1.1) P-a.s.
It is clear that the above Eq. (1.1) is a stochastic two point boundary value problem. Especially, it contains a deterministic two point boundary value problem as a special case when $\epsilon \to 0$.

We are given an $m \times n$ full-rank matrix $G$. We use the notations
\[
u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A^\epsilon(t, u) = \begin{pmatrix} -G^T g \\ Gf \\ \sqrt{\epsilon} G\sigma \end{pmatrix}(t, u).
\]

We now give the first assumptions of our paper:

(A1) Let $m \geq n$. The processes $g(\cdot, x, y, z)$, $f(\cdot, x, y)$ and $\sigma(\cdot, x, y)$ are $\mathcal{F}_t$-adapted, and the random variable $h(x)$ is $\mathcal{F}_T$-measurable, for all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$. $f$, $\sigma$ and $g$ are continuous in $t$, $P$-a.s.. Moreover, the following holds:
\[\mathbb{E} \left[ |h(0)|^2 \right] < +\infty.\]

There exists a constant $C_1 > 0$, such that, $u^i = (x^i, y^i, z^i) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, $i = 1, 2$,
\[
\begin{align*}
|g(t, x^1, y^1, z^1) - g(t, x^2, y^2, z^2)| &\leq C_1 \left( |x^1 - x^2| + |y^1 - y^2| + |z^1 - z^2| \right), \\
|f(t, x^1, y^1) - f(t, x^2, y^2)| &\leq C_1 \left( |x^1 - x^2| + |y^1 - y^2| \right), \\
|\sigma(t, x^1, y^1) - \sigma(t, x^2, y^2)| &\leq C_1 \left( |x^1 - x^2| + |y^1 - y^2| \right), \\
|h(x^1) - h(x^2)| &\leq C_1 |x^1 - x^2|, \\
P\text{a.s., a.e. } t \in \mathbb{R}^+.
\end{align*}
\]

(A2) There exists a constant $C_2 > 0$, such that for any fixed $\epsilon \in (0, 1]$,
\[
\begin{align*}
\langle A^\epsilon(t, u^1) - A^\epsilon(t, u^2), u^1 - u^2 \rangle &\leq -\left( C_2 + \sqrt{\epsilon} \right) \left( |G(x^1 - x^2)|^2 + |G^T(y^1 - y^2)|^2 \right), \\
P\text{a.s., a.e. } t \in \mathbb{R}^+,
\end{align*}
\]
and
\[
\langle h(x^1) - h(x^2), x^1 - x^2 \rangle \geq C_2 |G(x^1 - x^2)|^2, \\
P\text{a.s., } \forall (x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

We have the following:

**Proposition 2.1.** Assume that (A1) and (A2) hold, then there exists a unique adapted solution
\[
(X^{\epsilon, t, x}, Y^{\epsilon, t, x}, Z^{\epsilon, t, x})
\]
for Eqs. (1.1).

The proof can be seen in Theorem 2.6, Remark 2.8 in [31].

**Remark 2.1.** If $m = n$, then $G = I_n$. If $m < n$, according to Theorem 2.6 in [31], the first condition of (A2) should be
\[
\begin{align*}
\langle A^\epsilon(t, u^1) - A^\epsilon(t, u^2), u^1 - u^2 \rangle &\leq -\left( C_2 + \sqrt{\epsilon} \right) \left( |G(z^1 - z^2)|^2 + |G^T(y^1 - y^2)|^2 \right), \\
P\text{a.s., a.e. } t \in \mathbb{R}^+.
\end{align*}
\]
However, if setting $x^1 = x^2$, $y^1 = y^2$, there exists a contradiction, because $\sigma$ does not contain $z$. 

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From now on, let us suppose that

the coefficients \( f, \sigma, g \) and \( h \) are deterministic, i.e., independent of \( \omega \in \Omega \).

Now consider

\[
    u^\varepsilon(t, x) = Y^\varepsilon,t,x(t), \quad (t, x) \in [0, T] \times \mathbb{R}, \quad \varepsilon \in (0, 1),
\]

which is a deterministic vector since it is \( \mathcal{F}_t^\varepsilon \) measurable (Blumenthal’s 0-1 Law—see Remark 1.2 of [14]). In [29] it is shown that \( u \) is a viscosity solution of the associated quasilinear parabolic partial differential equation

\[
\begin{align*}
    \frac{\partial (u^\varepsilon)^l}{\partial t}(t, x) &+ g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, x)\sigma^\varepsilon(t, x, u^\varepsilon(t, x))) \\
    &+ \sum_{i=1}^{n} f_i^l(t, x, u^\varepsilon(t, x)) \frac{\partial (u^\varepsilon)^l}{\partial x_i}(t, x) + \varepsilon \sum_{i=1}^{n} a_{ij}(t, x, u^\varepsilon(t, x)) \frac{\partial^2 (u^\varepsilon)^l}{\partial x_i \partial x_j}(t, x) = 0, \\
    u^\varepsilon(T, x) &\equiv h(x), \quad x \in \mathbb{R}^n; \quad t \in [0, T]; \quad l = 1, \ldots, n,
\end{align*}
\]

where \( a_{ij} = (\sigma \sigma^T)_{i,j} \).

We define below the notion of viscosity solution for the parabolic system of Partial Differential Equations (PDEs for short) (2.2). For each \( \varepsilon > 0 \), consider the following differential operator,

\[
    (L^\varepsilon_i \varphi)(t, x, y, z) = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(t, x, y) \frac{\partial^2 \varphi^l}{\partial x_i \partial x_j}(t, x) + \langle f(t, x, y), \nabla \varphi^l(t, x) \rangle
\]

\[ l = 1, \ldots, n; \quad \forall \varphi \in C^{1,2}([0, T] \times \mathbb{R}^n, \mathbb{R}^n), \quad t \in [0, T], \quad (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}. \]

The space \( C^{1,2}([0, T] \times \mathbb{R}^n, \mathbb{R}^n) \) is the space of the functions \( \phi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), which are \( C^1 \) with respect to the first variable and \( C^2 \) with respect to the second variable.

The system (2.2) reads

\[
\begin{align*}
    \frac{\partial (u^\varepsilon)^l}{\partial t}(t, x) &+ (L^\varepsilon_i u^\varepsilon)(t, x, u^\varepsilon(t, x), \nabla_x u^\varepsilon(t, x)\sigma(t, x, u^\varepsilon(t, x))) + \\
    &g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, x)\sigma(t, x, u^\varepsilon(t, x))) = 0, \\
    u^\varepsilon(T, x) &\equiv h(x), \quad x \in \mathbb{R}^d; \quad t \in [0, T]; \quad l = 1, \ldots, n.
\end{align*}
\]

**Definition 2.2 (Viscosity Solutions).** Let \( u^\varepsilon \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n) \). The function \( u^\varepsilon \) is said to be a viscosity sub-solution (resp. super-solution) of the system (2.2) if

\[
(u^\varepsilon)^l(T, x) \leq h^l(x); \quad \forall l = 1, \ldots, n; \quad x \in \mathbb{R}^n
\]

(resp. \( (u^\varepsilon)^l(T, x) \geq h^l(x); \quad \forall l = 1, \ldots, n; \quad x \in \mathbb{R}^n \))

and for each \( l = 1, \ldots, n; (t, x) \in [0, T] \times \mathbb{R}^n \), \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}^n, \mathbb{R}^n) \) such that \( (t, x) \) is a local minimum (resp. maximum) point of \( \varphi - u^\varepsilon \), we have

\[
\frac{\partial \varphi}{\partial t}(t, x) + (L^\varepsilon_i \varphi)(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x \varphi(t, x)\sigma(t, x, u^\varepsilon(t, x))) + \\
g^l(t, x, u^\varepsilon(t, x), \sqrt{\varepsilon} \nabla_x \varphi(t, x)\sigma(t, x, u^\varepsilon(t, x))) \geq 0; \quad l = 1, \ldots, n
\]

(resp. \( \leq 0 \)).
The function $u^\varepsilon$ is said to be a viscosity solution of the system (2.2) if $u^\varepsilon$ is both a viscosity sub-solution and a viscosity super-solution of this system.

Under more restrictive assumptions (that we present in Section 3), we shall have

$$u^\varepsilon(t, x) = Y^{t, \varepsilon, x}(t)$$

(2.4)

and $u^\varepsilon$ will be actually a classical solution of (2.2). This can be proved using the Four Step Scheme Methodology of Ma-Protter-Yong [24] with the help of Ladyzhenskaja’s work in quasilinear parabolic PDEs [21].

For simplicity, we only consider the case where both $X^{\varepsilon,t,x}$ and $Y^{\varepsilon,t,x}$ are $n$-dimensional, that is $m = n$, then $G = I_n$. The result is analogous for the case $m > n$.

Next we introduce another set of assumptions which is slightly different from (A1)-(A2).

(A3) We say $f$, $g$, $\sigma$ and $h$ satisfy (A3) if there exist two constants $C_1$, $\Lambda > 0$ such that: (A1) hold as well as:

(A.3.1) $\forall t \in [0, T], \forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$:

$$\langle x_1 - x_2, f(t, x_1, y_1) - f(t, x_2, y_1) \rangle \leq C_1 |x_1 - x_2|^2,$$

$$\langle y_1 - y_2, f(t, x_1, y_1) - f(t, x_1, y_2) \rangle \leq C_1 |y_1 - y_2|^2.$$

(A.3.2) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$:

$$|f(t, x, y)| \leq \Lambda(1 + |x| + |y|),$$

$$|g(t, x, y, z)| \leq \Lambda(1 + |x| + |y| + |z|),$$

$$|\sigma(t, x, y)| \leq \Lambda(1 + |x| + |y|),$$

$$|h(x)| \leq \Lambda(1 + |x|).$$

(A.3.3) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$:

$$u \mapsto f(t, u, y),$$

$$v \mapsto g(t, x, v, z)$$

are continuous mappings.

Under this set of hypothesis, Theorem 1.1 of [14] ensures that there exists a constant $C = C(L) > 0$, depending only on $C_1$, such that for every $T \leq C$, (1.1) admits a unique solution in $\mathcal{N}'_t$.

Moreover, using Theorem 2.6 of [14], we have

$$\mathbb{P} \left( \forall s \in [t, T] : u^\varepsilon(s, X_s^{t, \varepsilon}) = Y_s^{t, \varepsilon} \right) = 1.$$  

(2.5)

$$\mathbb{P} \otimes \mu\left( \{ (\omega, s) \in \Omega \times [t, T] : |Z_s^{t, \varepsilon, x}(\omega)| \geq \Gamma_1 \} \right) = 0.$$  

(2.6)

where $\mu$ stands for the Lebesgue measure in the real line and $\Gamma_1$ is a constant which only depends on $C_1$, $\Lambda$, $n$, $d$, $T$.

By Remark 2.7 of [14] we have that, for each $\varepsilon > 0$, $u^\varepsilon$ only depends on the coefficients of the system (2.1), $f$, $g$, $\sqrt{\varepsilon}\sigma$, $h$. The fact that the dependence of $\Gamma_1 = \Gamma_1(C_1, \Lambda, n, T)$ determines that the properties (2.5), (2.6) above hold uniformly in $\varepsilon$; in particular, there exist continuous versions of $(Y_s^{t, \varepsilon, x}, Z_s^{t, \varepsilon, x})_{t \leq s \leq T}$ which are uniformly bounded, also in $\varepsilon$. 


We now assume more regularity on the coefficients of (1.1) in the next set of assumptions.

(A4) For $T \leq C$, we say that $f, g, h, \sigma$ satisfy (A4) if there exists three constants $\lambda, \Lambda, \gamma > 0$

(A.4.1) $\forall t \in [0, T], \forall (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$:

\[
|f(t, x, y)| \leq \Lambda (1 + |y|), \\
|g(t, x, y, z)| \leq \Lambda (1 + |y| + |z|), \\
|\sigma(t, x, y)| \leq \Lambda, \\
|h(x)| \leq \Lambda.
\]

(A.4.2) $\forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$: $\langle \xi, a(t, x, y) \xi \rangle \geq \lambda |\xi|^2, \forall \xi \in \mathbb{R}^n$, where $a(t, x, y) = \sigma \sigma^T(t, x, y)$.

(A.4.3) The function $\sigma$ is continuous.

(A.4.4) The function $\sigma$ is differentiable with respect to $x$ and $y$ and its derivatives with respect to $x$ and $y$ are $\gamma$-Hölder in $x$ and $y$, uniformly in $t$.

Under the set of assumptions (A.3) and (A.4), using Proposition 2.4 and Proposition B.6 of [1] and Theorem 2.9 of [15], one can prove that there exist two constants $\kappa, \kappa_1$, only depending on $L, \Lambda, n, T$ (independent of $\varepsilon$) such that:

\[
|u^\varepsilon(t, x)| \leq \kappa, \quad (2.7) \\
u^\varepsilon \in C^{1,2}_b([0, T] \times \mathbb{R}^n), \quad (2.8) \\
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |\nabla_x u^\varepsilon(t, x)| \leq \kappa_1, \quad (2.9)
\]

and

\[
Z_{s,t}^{s,t,\varepsilon,x} = \sqrt{\varepsilon} \nabla_x u^\varepsilon(t, X_s^{s,t,\varepsilon,x}) \sigma(s, X_s^{s,t,\varepsilon,x}, Y_s^{s,t,\varepsilon,x}), \quad (2.10)
\]

where $u^\varepsilon$ solves uniquely (2.2) in $C^{1,2}_b([0, T] \times \mathbb{R}^n)$, the space of $C^1$ functions with respect to the first variable and $C^2$ with respect to the second variable, with bounded derivatives. All these facts can be proven probabilistically. The last claim and the properties (2.7), (2.8) and (2.10) are proved in [14]. Delarue delivers in the appendices of [14] probabilistic methods to obtain these regularity results, under assumptions (A.3) and (A.4) and over a small time enough duration, using Malliavin Calculus techniques. The estimate of the gradient (2.9) is established by a probabilistic scheme in [15], using a variant of the Malliavin-Bismut integration by parts formula proposed by Thalmaier [34] and applied in [35] to establish a gradient estimate of interior type for the solutions of a linear elliptic equation on a manifold.

To start with, let us fix $x, y \in \mathbb{R}^n$, $\varepsilon \in (0, 1]$. We establish second order moment estimates for the solution of FBSDEs (1.1), which will be essential in Section 3. For convenience, we use the following notations in this paper:

\[
\begin{align*}
    f^{s,t,\varepsilon,x}(r) &= f(r, X^{s,t,\varepsilon,x}(r), Y^{s,t,\varepsilon,x}(r)), \\
    \sigma^{s,t,\varepsilon,x}(r) &= \sigma(r, X^{s,t,\varepsilon,x}(r), Y^{s,t,\varepsilon,x}(r)), \\
    g^{s,t,\varepsilon,x}(r) &= g(r, X^{s,t,\varepsilon,x}(r), Y^{s,t,\varepsilon,x}(r), Z^{s,t,\varepsilon,x}(r)), \\
    h^{s,t,\varepsilon,x}(T) &= h(X^{s,t,\varepsilon,x}(T)), \\
    &\vdots \\
    &\text{etc.}
\end{align*}
\]
The proof of the following results can be found in the Appendix.

**Lemma 2.1.** Assume that (A1) and (A2) hold. Then we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s) - X^{\varepsilon,t,y}(s)|^2 \right] & \leq C_1 |x - y|^2, \\
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^{\varepsilon,t,x}(s) - Y^{\varepsilon,t,y}(s)|^2 \right] & \leq C_1 |x - y|^2, \\
\int_t^T |Z^{\varepsilon,t,x}(s) - Z^{\varepsilon,t,y}(s)|^2 \, ds & \leq C_1 |x - y|^2,
\end{align*}
\]  

(2.11)

where \( C_1 \) is a positive constant independent of \( \varepsilon \) and \( t \).

**Lemma 2.2.** Assume that (A1) and (A2) hold. Then we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s)|^2 \right] & \leq C_2 \left( 1 + |x|^2 \right), \\
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^{\varepsilon,t,x}(s)|^2 \right] & \leq C_2 \left( 1 + |x|^2 \right), \\
\int_t^T |Z^{\varepsilon,t,x}(s)|^2 \, ds & \leq C_2 \left( 1 + |x|^2 \right),
\end{align*}
\]  

(2.12)

where \( C_2 \) is a positive constant independent of \( \varepsilon \) and \( t \).

The following Lemma shows the continuity on \( t \).

**Lemma 2.3.** Assume that (A1) and (A2) hold, then we have.

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t_1 \vee t_2 \leq s \leq T} |X^{\varepsilon,t_1,x}(s) - X^{\varepsilon,t_2,x}(s)|^2 \right] & \leq C_3 |t_1 - t_2| \left( 1 + |x|^2 \right), \\
\mathbb{E} \left[ \sup_{t_1 \vee t_2 \leq s \leq T} |Y^{\varepsilon,t_1,x}(s) - Y^{\varepsilon,t_2,x}(s)|^2 \right] & \leq C_3 |t_1 - t_2| \left( 1 + |x|^2 \right), \\
\int_t^T \left[ Z^{\varepsilon,t_1,x}(s) - Z^{\varepsilon,t_2,x}(s) \right]^2 \, ds & \leq C_3 |t_1 - t_2| \left( 1 + |x|^2 \right),
\end{align*}
\]  

(2.13)

where \( C_3 \) is a positive constant independent of \( \varepsilon \).

**Lemma 2.4.** Assume that (A1) and (A2) hold. Pick \( 0 < \varepsilon_2 < \varepsilon_1 < 1 \). Then we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{\varepsilon_1,t,x}(s) - X^{\varepsilon_2,t,x}(s)|^2 \right] & \leq C_4 \left( \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} \right), \\
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^{\varepsilon_1,t,x}(s) - Y^{\varepsilon_2,t,x}(s)|^2 \right] & \leq C_4 \left( \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} \right), \\
\int_t^T \left[ Z^{\varepsilon_1,t,x}(s) - Z^{\varepsilon_2,t,x}(s) \right]^2 \, ds & \leq C_4 \left( \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} \right),
\end{align*}
\]  

(2.14)

where \( C_3 \) is a positive constant independent of \( \varepsilon \) and \( t \).

Now consider the following deterministic equations

\[
\begin{align*}
\mathcal{X}^{t,x}(s) & = x + \int_t^s f \left( r, \mathcal{X}^{t,x}(r), \mathcal{Y}^{t,x}(r), 0 \right) \, dr, \\
\mathcal{Y}^{t,x}(s) & = h \left( \mathcal{X}^{t,x}(T) \right) + \int_t^T g \left( r, \mathcal{X}^{t,x}(r), \mathcal{Y}^{t,x}(r), 0 \right) \, dr.
\end{align*}
\]  

(2.15)
By the regularity of $f$, $g$, $h$ we have the following

**Lemma 2.5.** Assume that $(A1)$ and $(A2)$ hold, then there exists a unique solution $(X^{t,x}, Y^{t,x})$ for Eqs. (2.15).

Clearly, the estimates in Lemma 2.4 shows that the triple of

$$(X^{\varepsilon,t,x}(s), Y^{\varepsilon,t,x}(s), Z^{\varepsilon,t,x}(s))_{s \in [t,T]}$$

are Cauchy sequences and therefore converge in

$$\left(S^2(t,T; \mathbb{R}) \times S^2(t,T; \mathbb{R}) \times \mathcal{M}^2(t,T; \mathbb{R}^d)\right).$$

Denote the limit by $(X^0,t,x(s), Y^0,t,x(s), Z^0,t,x(s))_{s \in [t,T]}$, as $\varepsilon \to 0$. By uniqueness and existence of Eqs. (2.15), we know that the limit $(X^0,t,x(s), Y^0,t,x(s), Z^0,t,x(s))_{s \in [t,T]}$ is the unique solution of Eqs. (2.15). Therefore, the conclusions in Lemma 2.1, Lemma 2.2, and Lemma 2.3 also hold when $\varepsilon \to 0$.

3 Main Results

3.1 Convergence of Distributions

In this subsection we first study FBSDEs (1.1) with small noise intensity. To conclude this subsection we introduce the notions of pseudo-path topology and quasimartingales (Meyer-Zheng [22]), adjusted to our setting. For that, let us introduce the following notations.

Let $T > 0$ be a real constant (the terminal time). For any natural number $l > 0$, by $D(\mathbb{R}^l) \doteq D([0,T]; \mathbb{R}^l)$ we denote the Skorohod space of right continuous with left-hand limits functions $x$ on $[0,T]$ with values in $\mathbb{R}^l$ such that

$$x(T-) \doteq \lim_{t \uparrow T} x(t) = x(T),$$

and by convention, $x(0-) = 0$. Additionally, we introduce the following metric $\rho$ on $D(\mathbb{R}^l)$:

$$\rho(x, y) = \int_0^T (|x(s) - y(s)| \wedge 1) \, ds, \quad x, y \in D(\mathbb{R}^l).$$

The topology induced by this metric is the Meyer-Zheng topology introduced below on $D(\mathbb{R}^l)$. Let $\zeta$ be the coordinate mapping on $D(\mathbb{R}^l)$ defined by

$$\zeta_t(x) = x(t), \quad t \in [0,T], \quad x \in D(\mathbb{R}^n),$$

and introduce the $\sigma$-algebras of subsets of $D(\mathbb{R}^l)$,

$$D^t_s \doteq D^t_s(\mathbb{R}^l) \doteq \sigma(\{\zeta_u : u \in [t,s]\}), \quad 0 \leq t \leq s \leq T,$$

$$D \doteq D(\mathbb{R}^l) \doteq D^0_T(\mathbb{R}^l).$$

**(Meyer-Zheng topology)** Let $\lambda(dt)$ the measure $e^{-t}dt$ on $\mathbb{R}^+$. Let $w(t)$ be a real valued Borel function on $\mathbb{R}^+$. The pseudo-path of $w$ is a probability law on $[0, \infty) \times \mathbb{R}^l$ : the image measure of $\lambda$ under the
mapping \( t \to (t, w(t)) \). We denote by \( \psi \) the mapping which associates to a path \( w \) its pseudo-path: it is clear that \( \psi \) identifies two paths if and only if they are equal a.e. in Lebesgue’s sense. In particular, \( \psi \) is 1–1 on \( D(\mathbb{R}) \), and provides an imbedding of \( D(\mathbb{R}) \) into the compact space \( \overline{P} \) of all probability laws on the compact space \([0, \infty] \times \mathbb{R}^d\). We give to the induced topology on \( D(\mathbb{R}) \) the name of pseudo-path topology or Meyer-Zheng topology. Let us introduce some intermediate sets between \( D(\mathbb{R}) \), \( \overline{P} \) and \( \Psi \). Let \( \Psi \) be the set of all pseudo-paths. We have inclusions

\[
D(\mathbb{R}) \subset \Psi \subset \overline{P}.
\]

The following characterization of the Meyer-Zheng topology is worth noting.

**Lemma 3.1.** The pseudo-path topology on \( \Psi \) is equivalent to the convergence in measure.

Furthermore, it is known that \( \Psi \) is a Polish space; and \( D(\mathbb{R}) \) is a Borel set in \( \overline{P} \).

**Lemma 3.2.** Let \( \mathcal{B}(D(\mathbb{R})) \) be the \( \sigma \)-algebra of Borel subsets of \( D(\mathbb{R}) \) in the Meyer-Zheng topology. Then \( \mathcal{B}(D(\mathbb{R})) = \mathcal{D}(\mathbb{R}) \).

The proof can be seen in [22]. The most important application of the Meyer-Zheng topology is a tightness result for quasimartingales. We give the definition here

**Definition 3.1.** Let \( X \) be an \( \mathcal{F} \)-adapted, cadlag process defined on \([0, T]\), such that \( \mathbb{E}|X(t)| < \infty \) for all \( t \geq 0 \). For any partition \( \pi : 0 = t_0 < t_1 < \cdots < t_n \leq T \), let us define

\[
V^\pi_T(X) := \sum_{0 \leq i < n} \mathbb{E}\{|X(t_{i+1}) - X(t_i)| | \mathcal{F}_{t_i} \} + \mathbb{E}|X(t_n)|, \tag{3.1}
\]

and define the conditional variation of \( X \) by \( V_T(X) := \sup_\pi V^\pi_T(X) \). If \( V_T(X) \) is finite, then \( X \) is called a quasimartingale.

The following result holds.

**Proposition 3.1.** Let \( P_n \) be a sequence of probability laws on \( D(\mathbb{R}) \) such that under each \( P_n \) the coordinate process \( X(\cdot) \) is a quasimartingale with conditional variation \( V_n(X(\cdot)) \) uniformly bounded in \( n \). Then there exists a subsequence \( (P_{n_k}) \) which converges weakly on \( D(\mathbb{R}) \) to a law \( P \), and \( X(\cdot) \) is a quasimartingale under \( P \).

The proof can be seen in [22].

Now we turn back to Eqs. (1.1). Define

\[
\begin{align*}
   u^\varepsilon(t, x) &= Y^{\varepsilon, t, x}(t), \\
v^\varepsilon(t, x) &= Z^{\varepsilon, t, x}(t),
\end{align*}
\tag{3.2}
\]

From the existence and uniqueness of solution for Eq. (1.1), we have the following Markov property

\[
\begin{align*}
   u^\varepsilon(s, X^{\varepsilon, t, x}(s)) &= Y^{\varepsilon, s, X^{\varepsilon, t, x}(s)}(s) = Y^{\varepsilon, t, x}(s) \quad \text{a.e., a.e. \( s \in [t, T] \).}
\end{align*}
\tag{3.3}
\]

\(^1\)Note that the quasimartingale in [22] is defined on \([0, +\infty)\). However, it is fairly easy to check that if \( X \) is a quasimartingale on \([0, T]\) as is defined above, then the process \( X(t) = X(t) \mathbf{1}_{[0,T]}(t) + X(T) \mathbf{1}_{[T, +\infty)}(t), t \in [0, +\infty] \) is a quasimartingale in the sense of [22]. Furthermore, the conditional variation \( V_T(X) \) defined here, is exactly the same as \( V(X) \) defined in [22]. In other words, our quasimartingale is a “local” version of the one in [22].
In Lemma 2.1 we have already seen that the function $u^\varepsilon$ is Lipschitz continuous in $x$, uniformly in $t$. With the help of Lemma 2.3 we also know the continuity property for $u^\varepsilon$ in $t$. We now set
\begin{align*}
  f^\varepsilon(s,x) &= f(s,x,u^\varepsilon(s,x)), \\
  \sigma^\varepsilon(s,x) &= \sigma(s,x,u^\varepsilon(s,x)), \\
  g^\varepsilon(s,x) &= g(s,x,u^\varepsilon(s,x),u^\varepsilon(s,x)).
\end{align*}
(3.4)

We study the properties of $f^\varepsilon$, $\sigma^\varepsilon$, $f^0$, and $\sigma^0$ as follows:

**Lemma 3.3.** Assume that (A1) and (A2) hold. Then $f^\varepsilon$, $\sigma^\varepsilon$, $f^0$, and $\sigma^0$ satisfy uniformly Lipschitz continuous and $f^\varepsilon$, $\sigma^\varepsilon$ converge uniformly $f^0$, $\sigma^0$, respectively. Moreover, $f^\varepsilon$, $\sigma^\varepsilon$, $f^0$ and $\sigma^0$ satisfy sublinear growth.

**Proof.** For any $x, y \in \mathbb{R}^n$, $s \in [t,T]$, we have
\begin{align*}
 |f^\varepsilon(s,x) - f^\varepsilon(s,y)| &= |f(s,x,u^\varepsilon(s,x)) - f(s,y,u^\varepsilon(s,y))| \\
 &\leq C_1(|x-y| + |u^\varepsilon(s,x) - u^\varepsilon(s,y)|) \\
 &= C_1|x-y| + C_1C_1 |x-y| \\
 &\leq \max\{C_1,C_1C_1\} |x-y|,
\end{align*}
by Lemma 2.1 and (A1). The same properties for $\sigma^\varepsilon f^0$, and $f^0$ are proved similarly. Next, we show the uniform convergence,
\begin{align*}
 |f^\varepsilon(s,x) - f^0(s,x)| &= |f(s,x,u^\varepsilon(s,x)) - f(s,x,u^0(s,x))| \\
 &\leq C_1|u^\varepsilon(s,x) - u^0(s,x)| \leq C_1C_3\sqrt{\varepsilon},
\end{align*}
by Lemma 2.4 and (A1). Once again, the same properties for $\sigma^\varepsilon$ are proved similarly. We are going to prove $b^\varepsilon$ satisfies sublinear growth. We have
\begin{align*}
 |f^\varepsilon(s,x) - f(s,0,0)| &= |f(s,x,u^\varepsilon(s,x)) - f(s,0,0)| \\
 &\leq C_1(|x| + |u^\varepsilon(s,x)|) \\
 &\leq C_1(|x| + \sqrt{C_2}(1+|x|)) \\
 &= (C_1 + C_1\sqrt{C_2}) |x| + C_1\sqrt{C_2}.
\end{align*}
The assumption (A1) yields
\begin{align*}
 |f^\varepsilon(s,x)| &\leq (C_1 + C_1\sqrt{C_2}) |x| + C_1\sqrt{C_2} + \sup_{0\leq r\leq T} |f(r,0,0)|.
\end{align*}
Once again, the same properties for $\sigma^\varepsilon$, $f^0$, $\sigma^0$ are proved similarly. \(\square\)

Then it is easy to check that $(X^{\varepsilon,t,x}(\cdot),Y^{\varepsilon,t,x}(\cdot),Z^{\varepsilon,t,x}(\cdot))$ solves the following decoupled FBSDEs
\begin{align*}
  X^{\varepsilon,t,x}(s) &= x + \int_s^t f^\varepsilon(r,X^{\varepsilon,t,x}(r)) \, dr + \sqrt{\varepsilon} \int_s^t \sigma^\varepsilon(r,X^{\varepsilon,t,x}(r)) \, dW(r), \\
  Y^{\varepsilon,t,x}(s) &= h(X^{\varepsilon,t,x}(T)) + \int_s^T g^\varepsilon(r,Y^{\varepsilon,t,x}(r)) \, dr - \int_s^T Z^{\varepsilon,t,x}(r) \, dW(r), \\
  0 &\leq t \leq s \leq T.
\end{align*}
(3.5)

From now on, we are concerned on the behavior laws of $(X^{\varepsilon,t,x},Y^{\varepsilon,t,x})$ when $\varepsilon \to 0.$
Theorem 3.1. Under the assumptions (A1) and (A2), we can conclude the following results:

i) For all $\delta > 0$,

$$\lim_{\varepsilon \to 0} P \left\{ \sup_{t \leq s \leq T} \left| X^{\varepsilon,t,x} (s) - \lambda^{t,x} (s) \right| > \delta \right\} = 0. \quad (3.6)$$

ii) Let $Q^\varepsilon = P \left( Y^{\varepsilon,t,x} (\cdot) \right)^{-1}$ be the probability measure on $D (\mathbb{R}^n)$. Then there exists a subsequence $Q^{\varepsilon_n}$ of $Q^\varepsilon$ and a probability law $Q$ on $D (\mathbb{R}^n)$ such that $Q^{\varepsilon_n}$ converges weakly in the Meyer-Zheng topology to $Q$ as $n \to +\infty$.

Proof. It follows from the definition of $X^{\varepsilon,t,x}$ and $\lambda^{t,x}$ that

$$\sup_{t \leq s \leq T} \left| X^{\varepsilon,t,x} (s) - \lambda^{t,x} (s) \right| \leq \int_t^s \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^0 (r, \lambda^{t,x} (r)) \right| \, dr + \sqrt{\varepsilon} \sup_{t \leq s \leq T} \left| \int_t^s \sigma (r, X^{\varepsilon,t,x} (r)) \, dW (r) \right|. \quad (3.7)$$

From Chebyshev's inequality and the first assertion of the Lemma 2.1, we obtain an estimate of the first term of the right side of (3.7):

$$P \left\{ \int_t^s \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^0 (r, \lambda^{t,x} (r)) \right| \, dr > \frac{\delta}{2} \right\}$$

$$\leq 4\delta^{-2} E \left[ \int_t^s \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^0 (r, \lambda^{t,x} (r)) \right| \, dr \right]^2$$

$$\leq 4\delta^{-2} T E \left[ \int_t^s \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^0 (r, \lambda^{t,x} (r)) \right|^2 \, dr \right]$$

$$= 4\delta^{-2} T E \left[ \int_t^s \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^\varepsilon (r, \lambda^{t,x} (r)) + f^\varepsilon (r, \lambda^{t,x} (r)) - f^0 (r, \lambda^{t,x} (r)) \right|^2 \, dr \right]$$

$$\leq 8\delta^{-2} T E \left[ \int_t^T \left| f^\varepsilon (r, X^{\varepsilon,t,x} (r)) - f^\varepsilon (r, \lambda^{t,x} (r)) \right|^2 \, dr + TC_3 \varepsilon \right]$$

$$\leq 8\delta^{-2} T (\max \{ C_1, C_1 C_1 \}^2 E \left[ \int_t^T \left| X^{\varepsilon,t,x} (r) - \lambda^{t,x} (r) \right|^2 \, dr + T (C_1 C_3)^2 \varepsilon \right]$$

$$\leq 8\delta^{-2} T (\max \{ C_1, C_1 C_1 \}^2 T E \left[ C_3 \sqrt{\varepsilon} + (C_1 C_3)^2 \varepsilon \right]. \quad (3.8)$$

The estimation of the second term in (3.7) can be accomplished with the use of the generalized Kolmogorov inequality for stochastic integrals:

$$P \left\{ \sqrt{\varepsilon} \sup_{t \leq s \leq T} \left| \int_t^s \sigma^\varepsilon (r, X^{\varepsilon,t,x} (r)) \, dW (r) \right| > \frac{\delta}{2} \right\}$$

$$\leq 4\delta^{-2} \varepsilon E \left[ \int_t^T \left| \sigma^\varepsilon (r, X^{\varepsilon,t,x} (r)) \right|^2 \, dr \right]$$

$$= 4\delta^{-2} \varepsilon E \left[ \int_t^T \left| \sigma (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r)) - \sigma (r, 0, 0) + \sigma (r, 0, 0) \right|^2 \, dr \right]$$

$$\leq 4\delta^{-2} \varepsilon E \left[ 4C_1^2 \int_t^T \left( \left| X^{\varepsilon,t,x} (r) \right|^2 + \left| Y^{\varepsilon,t,x} (r) \right|^2 + 2 \left| \sigma (r, 0, 0) \right|^2 \right) \, dr \right]$$
\[ \leq 4 \delta^{-2} \varepsilon \left[ 8TC^2 \frac{1}{\varepsilon} \left( 1 + |x|^2 \right) + \int_0^T 2 | \sigma (r, 0, 0) |^2 \, dr \right], \]  

(3.9)

where we have used Lemma 2.2. Estimates (3.8), (3.9) imply the assertion of the theorem (i).

We are going to prove the second one. First, we establish the connection of solution of BSDE (3.5) to quasimartingales. Given a subdivision \( \pi : 0 = t_0 < t_1 < \cdots t_n = T \), we get

\[
V_T^\pi (Y^{\varepsilon,t,x}) = \mathbb{E} \left[ h (X^{\varepsilon,t,x} (T)) + \sum_{k=0}^{n-1} \mathbb{E} \left[ \left| \mathbb{E} \left[ Y^{\varepsilon,t,x} (t_{k+1}) - Y^{\varepsilon,t,x} (t_k) \right| \mathcal{F}_t \right] \right] \right]
\]

\[
= \mathbb{E} \left[ h (X^{\varepsilon,t,x} (T)) + \sum_{k=0}^{n-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} g (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r), Z^{\varepsilon,t,x} (r)) \, dr \right] \mathcal{F}_t \right] \]

\[
\leq \mathbb{E} \left[ h (X^{\varepsilon,t,x} (T)) + \mathbb{E} \left[ \int_0^T g (r, X^{\varepsilon,t,x} (r), Y^{\varepsilon,t,x} (r), Z^{\varepsilon,t,x} (r)) \, dr \right] \right]
\]

\[
\leq \mathbb{E} \left[ h (X^{\varepsilon,t,x} (T)) + \mathbb{E} \left[ \int_0^T \left| g^{\varepsilon,t,x} (r) - g (r, 0, 0, 0) + g (r, 0, 0, 0) \right| \, dr \right] \right]
\]

\[
\leq C_1 \mathbb{E} \left[ |X^{\varepsilon,t,x} (T)| \right] + C_1 \mathbb{E} \left[ \int_0^T \left( |X^{\varepsilon,t,x} (r)| + |Y^{\varepsilon,t,x} (r)| + |Z^{\varepsilon,t,x} (r)| \right) \, dr \right]
\]

\[
+ \int_0^T |g (r, 0, 0, 0)| \, dr + h (0).
\]

(3.10)

By Jensen’s inequality, it follows from Lemma 2.2

\[ \mathbb{E} \left[ |X^{\varepsilon,t,x} (T)| \right] \leq \sqrt{C_2 \left( 1 + |x|^2 \right)}, \]

and

\[ \mathbb{E} \left[ \int_0^T \left( |X^{\varepsilon,t,x} (r)| + |Y^{\varepsilon,t,x} (r)| + |Z^{\varepsilon,t,x} (r)| \right) \, dr \right]
\]

\[
\leq \sqrt{T} \mathbb{E} \left[ \int_0^T \left( \int_0^r |X^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^r |Y^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^r |Z^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right]
\]

\[
\leq \sqrt{T} \left( \mathbb{E} \left[ \int_0^T \left( \int_0^r |X^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \int_0^T \left( \int_0^r |Y^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \int_0^T \left( \int_0^r |Z^{\varepsilon,t,x} (s)| \, ds \right)^{\frac{1}{2}} \, dr \right] \right)^{\frac{1}{2}}
\]

\[
\leq 3 \sqrt{T} \left( TC_2 \left( 1 + |x|^2 \right) \right)^{\frac{1}{2}},
\]

by Hölder inequality and Jensen’s inequality for concave functions. So we have

\[ V_T^\pi (Y^{\varepsilon,t,x}) \leq C_1 \sqrt{C_2 \left( 1 + |x|^2 \right)} + 3 \sqrt{T} \left( C_2 \left( 1 + |x|^2 \right) \right)^{\frac{1}{2}} + h (0) + T \sup_{0 \leq r \leq T} |g (r, 0, 0, 0)|.
\]

Hence, by (A1), noting that \( V_T (Y^{\varepsilon,t,x}) = \sup_{\pi} V_T^\pi (Y^{\varepsilon,t,x}) < + \infty \), the result follows.

Now since \( D (\mathbb{R}^n) \) is a separable metric space, there exists a compact metric space \( K \) such that \( D (\mathbb{R}^n) \) is a subset of \( K \). Note that \( D (\mathbb{R}^n) \) is a Lusin space: for every embedding in a compact metric space \( K \), \( D (\mathbb{R}^n) \) is a Borel set in \( K \): \( D (\mathbb{R}^n) \in \mathcal{B} (K) \). On the compact metric space \( K \) we define

\[ \tilde{Q}^\varepsilon (A) = Q^\varepsilon (A \cap D (\mathbb{R}^n)), \quad A \in \mathcal{B} (K).
\]
Clearly, $A \cap D(\mathbb{R}^n)$ belongs to $\mathcal{B}(D(\mathbb{R}^n)) = \mathcal{D}(\mathbb{R}^n)$, the last equality being true in view of Lemma 3.2. The set of probability measures on the compact metric space $K$ is compact for the weak convergence. Hence, we can choose a subsequence also denoted $(\varepsilon_n)_{n \geq 1}$, and a probability measure $Q^\varepsilon$ on $K$ such that

$$Q^\varepsilon_n \xrightarrow{w} Q^\varepsilon$$
onumber

on $K$.

We now show that

$$Q^\varepsilon(D(\mathbb{R}^n)) = 1.$$  

We notice that $Q^\varepsilon$ is the distribution of $Y^{\varepsilon,\varepsilon}$ considered as a random variable with values in $(K, \mathcal{B}(K))$. Furthermore, by Proposition 3.1, we know that, possibly along a subsequence, $Q^\varepsilon_n$ converges weakly to a probability law $Q^* \in \mathcal{M}(D(\mathbb{R}^n))$. The uniqueness of the weak limit implies that

$$Q^* (A) = \tilde{Q} (A), \quad \forall A \in \mathcal{B}(D(\mathbb{R}^n)),$$  

(3.11)

In particular,

$$1 = Q^* (D(\mathbb{R}^n)) = \tilde{Q} (D(\mathbb{R}^n)).$$

The proof is complete.

\[ \square \]

3.2 Almost Sure Convergence

Considering assumptions (A3)-(A4), we have the following

**Theorem 3.2.** Under the assumptions (A3) and (A4), supposing further $T \leq C$, with $C = C(C_1, T)$ depending on the Lipschitz constant $C_1$ and on $T$, we have

1. For each $s, t \in [0, T]$, $t \leq s$, the solution of (3.1), $(X^{\varepsilon,t,x}(s), Y^{\varepsilon,t,x}(s), Z^{\varepsilon,t,x}(s))_{t \leq s \leq T}$ converges in $\mathcal{N}_t$, when $\varepsilon \to 0$ to $(X(s), Y(s), 0)_{t \leq s \leq T}$, where $(X(s), Y(s))_{t \leq s \leq T}$ solves the coupled system of differential equations:

$$\begin{aligned}
\dot{X}(s) &= f(s, X(s), Y(s)), \\
\dot{Y}(s) &= -g(s, X(s), Y(s), 0), \quad t \leq s \leq T, \\
X(t) &= x, \quad Y(T) = h(X(T)).
\end{aligned}$$

(3.12)

2. Denoting $u(t, x) = Y^{t,x}(t)$ the limit in $\varepsilon \to 0$ of $Y^{\varepsilon,t,x}(t)$, the function $u$ is a viscosity solution of

$$\begin{aligned}
\frac{\partial u}{\partial t} + \sum_{i=1}^d f_i(t, x, u(t, x)) \frac{\partial u}{\partial x_i}(t, x) + g(t, x, u(t, x), 0) &= 0, \\
u^\varepsilon(t, x) &= h(x); \quad x \in \mathbb{R}^d; \quad t \in [0, T]; \quad l = 1, \ldots, n.
\end{aligned}$$

(3.13)

3. The function $u$ is bounded, continuous Lipschitz in $x$ and uniformly continuous in time.

4. Furthermore, if $u \in C^{1,1}([0, T] \times \mathbb{R}^n)$, since (3.12) has a unique continuous solution, the function $u$ is a classical solution of (3.13).

**Proof.** Given $\varepsilon, \varepsilon_1 > 0$, $x \in \mathbb{R}^n$, for $T \leq C$, if $(X^{\varepsilon,t,x}_s, Y^{\varepsilon,t,x}_s, Z^{\varepsilon,t,x}_s)_{t \leq s \leq T}$ is the unique solution in $\mathcal{N}_t$ of

$$\begin{aligned}
X^{t,\varepsilon,x}(s) &= x + \int_t^s f(r, X^{t,\varepsilon,x}(r), Y^{t,\varepsilon,x}(r))dr + \sqrt{\varepsilon} \int_t^s \sigma(r, X^{t,\varepsilon,x}(r), Y^{t,\varepsilon,x}(r))dW(r), \\
Y^{t,\varepsilon,x}(s) &= h(X^{t,\varepsilon,x}(T)) + \int_s^T g(r, X^{t,\varepsilon,x}(r), Y^{t,\varepsilon,x}(r), Z^{t,\varepsilon,x}(r))dr - \int_s^T Z^{t,\varepsilon,x}(r) dW(r), \quad x \in \mathbb{R}^d; \quad 0 \leq t \leq s \leq T,
\end{aligned}$$

(3.14)
and \((X^{\varepsilon_1,t,x}_s, Y^{\varepsilon_1,t,x}_s, Z^{\varepsilon_1,t,x}_s)_{t \leq s \leq T}\) is the unique solution in \(\mathcal{N}_t\) of
\[
\begin{align*}
X^{\varepsilon_1,t,x}(s) &= x + \int_t^s f(r, X^{\varepsilon_1,t,x}(r), Y^{\varepsilon_1,t,x}(r))dr + \sqrt{\varepsilon_1} \int_t^s \sigma(r, X^{\varepsilon_1,t,x}(r), Y^{\varepsilon_1,t,x}(r))dW(r), \\
Y^{\varepsilon_1,t,x}(s) &= h(X^{\varepsilon_1,t,x}(T)) + \int_T^s g(r, X^{\varepsilon_1,t,x}(r), Y^{\varepsilon_1,t,x}(r), Z^{\varepsilon_1,t,x}(r))dr - \int_s^T Z^{\varepsilon_1,t,x}(r)dW(r), \quad x \in \mathbb{R}^d; \quad 0 \leq t \leq s \leq T.
\end{align*}
\] (3.15)

Write, for simplicity of the notations, with \(\delta = \varepsilon, \varepsilon_1\)
\[
\begin{align*}
f^\delta(s) &= f(s, X^{\delta,t,x}(s), Y^{\delta,t,x}(s)), \\
g^\delta(s) &= g(s, X^{\delta,t,x}(s), Y^{\delta,t,x}(s), Z^{\delta,t,x}(s)), \\
\sigma^\delta(s) &= \sqrt{\varepsilon} \sigma(s, X^{\delta,t,x}(s), Y^{\delta,t,x}(s)), \\
h^\delta &= h(X^{\delta,t,x}(T)).
\end{align*}
\]

As in the proof of Theorem 1.2 of [14], using Itô’s formula:
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 \right] &\leq \mathbb{E} \left[ \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \right] \\
+ 2\mathbb{E} \left[ \sup_{t \leq s \leq T} \int_t^s \langle X^\varepsilon(r) - X^{\varepsilon_1}(r), f^\varepsilon(r) - f^{\varepsilon_1}(r) \rangle dr \right] \\
+ 2\mathbb{E} \left[ \sup_{t \leq s \leq T} \int_t^s \langle X^\varepsilon(r) - X^{\varepsilon_1}(r), (\sigma^\varepsilon(r) - \sigma^{\varepsilon_1}(r))dW(r) \rangle \right].
\end{align*}
\] (3.16)

Using Burkholder-Davis-Gundy’s inequalities, there exists \(\gamma > 0\) such that:
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 \right] &\leq \mathbb{E} \left[ \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \right] \\
+ 2\mathbb{E} \left[ \sup_{t \leq s \leq T} \int_t^s \langle X^\varepsilon(r) - X^{\varepsilon_1}(r), f^\varepsilon(r) - f^{\varepsilon_1}(r) \rangle dr \right] \\
+ 2\gamma \mathbb{E} \left[ \int_t^T |X^\varepsilon(r) - X^{\varepsilon_1}(r)|^2 |\sigma^\varepsilon(r) - \sigma^{\varepsilon_1}(r)|^2 dr \right]^{1/2}.
\end{align*}
\] (3.17)

where \(\gamma > 0\) only depends on the Lipschitz constant \(C_1\) and \(T\). Using the Lipschitz property (A.1), there exists \(\gamma > 0\), eventually different, but only dependent on \(C_1, T\) such that:
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 \right] \\
&\leq \gamma \left\{ \mathbb{E} \int_t^T (|X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 + |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2)ds \\
+ \mathbb{E} \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + 2|\varepsilon - \sqrt{\varepsilon_1}|^2 \\
+ 2\sqrt{T} \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 \right\}.
\end{align*}
\] (3.18)
Then, assuming that $1 - 2\gamma\sqrt{T} \geq 0$, we have

$$(1 - 2\gamma\sqrt{T})E \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 \right]$$

$$\leq \gamma \left\{ E \left[ \int_t^T (|X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 + |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2) ds \right] + 2|\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 + \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds \right\}. \tag{3.19}$$

As before, we get

$$E \left[ \sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 + \int_t^T |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 ds \right]$$

$$\leq \gamma \left\{ E|h^\varepsilon - h^{\varepsilon_1}|^2 + E \left[ \sup_{t \leq s \leq T} \int_t^T \langle X^\varepsilon(s) - X^{\varepsilon_1}(s), Y^\varepsilon(s) - Y^{\varepsilon_1}(s) \rangle ds \right] \right\}. \tag{3.18}$$

Using (3.18) and the assumption (A.3) modifying $\gamma$ eventually:

$$E \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 + \sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 + \int_t^T |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 ds \right]$$

$$\leq \gamma \left\{ E|h^\varepsilon - h^{\varepsilon_1}|^2 + E \int_t^T |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 ds + E \int_t^T |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 ds \right.$$  

$$+ E \left( \int_t^T |f^\varepsilon(s) - f^{\varepsilon_1}(s)| + |g^\varepsilon(s) - g^{\varepsilon_1}(s)| \right) ds \right)^2 \right.$$  

$$+ E \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \right.$$  

$$+ E \int_t^T |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)| \left( |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)| + |X^\varepsilon(s) - X^{\varepsilon_1}(s)| \right) ds \right\}. \tag{3.20}$$

So there exist $C^* \leq C$ and $\Gamma$, only depending on $C_1$ and $\gamma_1$, such that, for $T - t \leq C^*$

$$E \left[ \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 + \sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 + \int_t^T |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 ds \right]$$

$$\leq \gamma \left\{ E|h^\varepsilon - h^{\varepsilon_1}|^2 + E \int_t^T |\sigma^\varepsilon(s) - \sigma^{\varepsilon_1}(s)|^2 ds + |\sqrt{\varepsilon} - \sqrt{\varepsilon_1}|^2 \right.$$  

$$+ E \int_t^T |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)| + |g^\varepsilon(s) - g^{\varepsilon_1}(s)| ds \right)^2 \right\}. \tag{3.20}$$

We can repeat the argument in $[T - 2C^*; T - C^*]$ and recurrently we get this important a-priori estimate \((3.20)\) valid for all $[t, T]$, with some (possibly different) constant $\gamma > 0$ only depending on $C_1$. 

Now, from (A.3), the following estimate holds:
\[
\mathbb{E}\left[|h^\varepsilon - h^{1\varepsilon}|^2\right] \leq L^2 \mathbb{E}\left[\sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{1\varepsilon}(s)|^2\right].
\]

(3.21)

One can also derive the estimate
\[
\mathbb{E}\left[\int_t^T \left(\sqrt{\varepsilon}\sigma^\varepsilon(s) - \sqrt{1}\sigma^{1\varepsilon}(s)\right)^2 ds\right] \\
\leq c_1 |\sqrt{\varepsilon} - \sqrt{1}|^2 + c_2 \mathbb{E}\left[\int_t^T \left(|X^\varepsilon(s) - X^{1\varepsilon}(s)|^2 + |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2\right) ds\right].
\]

(3.22)

where \(c_1, c_2 > 0\) only depending on \(T, L\) and on the bound of \(|\sigma|\).

Furthermore, using Jensen’s inequality and the Lipschitz property of \(g\), for some \(c_3\) modified eventually along the various steps,
\[
\mathbb{E}\left[\int_t^T |Y^\varepsilon(s) - Y^{1\varepsilon}(s)| + |g^\varepsilon(s) - g^{1\varepsilon}(s)| ds\right]^2 \\
\leq \mathbb{E}\left[\int_t^T |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2 + |g^\varepsilon(s) - g^{1\varepsilon}(s)|^2 + 2 |Y^\varepsilon(s) - Y^{1\varepsilon}(s)||g^\varepsilon(s) - g^{1\varepsilon}(s)| ds\right] \\
\leq c_3 \mathbb{E}\left[\int_t^T |X^\varepsilon(s) - X^{1\varepsilon}(s)|^2 + |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2 + |Z^\varepsilon(s) - Z^{1\varepsilon}(s)|^2 ds\right] \\
\leq c_3 \mathbb{E}\left[\sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{1\varepsilon}(s)|^2 \right] + \mathbb{E}\left[\sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2 \right] \\
+ \mathbb{E}\left[\sup_{t \leq s \leq T} |Z^\varepsilon(s) - Z^{1\varepsilon}(s)|^2 \right].
\]

(3.23)

Using (2.10) and the boundedness of \(|\sigma|\), we have \(\mathbb{E}\left[\int_t^T |Z^\varepsilon(s)|^2 ds\right] \to 0\) as \(\varepsilon \to 0\) and
\[
\lim_{|\varepsilon - 1| \to 0} \mathbb{E}\left[\int_t^T |Z^\varepsilon(s) - Z^{1\varepsilon}(s)|^2 ds\right] = 0.
\]

(3.24)

Furthermore, by Burkholder-Davis-Gundy’s inequalities, there exists a constant \(\gamma > 0\), eventually new such that:
\[
\mathbb{E}\left[\sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{1\varepsilon}(s)|^2 + \sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2 + \int_t^T |Z^\varepsilon(s) - Z^{1\varepsilon}(s)|^2 ds\right] \\
\leq \gamma \mathbb{E}\left[\int_t^T |X^\varepsilon(s) - X^{1\varepsilon}(s)|^2 + |Y^\varepsilon(s) - Y^{1\varepsilon}(s)|^2 + |Z^\varepsilon(s) - Z^{1\varepsilon}(s)|^2 ds\right] \\
\leq \gamma \mathbb{E}\left[\int_t^T \sup_{t \leq r \leq s} |X^\varepsilon(r) - X^{1\varepsilon}(r)|^2 + \sup_{t \leq r \leq s} |Y^\varepsilon(r) - Y^{1\varepsilon}(r)|^2 \\
+ \sup_{t \leq r \leq s} |Z^\varepsilon(r) - Z^{1\varepsilon}(r)|^2 ds\right].
\]

(3.25)

Moreover, for some \(\gamma_1, \gamma_2 > 0\), using (3.21)-(3.25), we have
In conclusion, 

\[ \lim_{\varepsilon \to 0} \left( \sup_{t \leq s \leq T} |X^\varepsilon(s) - X^{\varepsilon_1}(s)|^2 + \sup_{t \leq s \leq T} |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 + \int_t^T |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 \, ds \right) \]

\[ \leq \gamma_1 \sqrt{\varepsilon - \sqrt{\varepsilon_1}}^2 + \gamma_2 \mathbb{E} \left[ \int_t^T \sup_{t \leq r \leq s} |X^\varepsilon(r) - X^{\varepsilon_1}(r)|^2 \, dr \right. \\
\left. + \sup_{t \leq r \leq s} |Y^\varepsilon(r) - Y^{\varepsilon_1}(r)|^2 + \sup_{t \leq r \leq s} |Z^\varepsilon(r) - Z^{\varepsilon_1}(r)|^2 \, ds \right]. \tag{3.26} \]

Using Gronwall's inequality, we get

\[ \mathbb{E} \left[ \int_t^T \sup_{t \leq r \leq s} |X^\varepsilon(r) - X^{\varepsilon_1}(r)|^2 + \sup_{t \leq r \leq s} |Y^\varepsilon(s) - Y^{\varepsilon_1}(s)|^2 \, dr \right] \]

\[ \leq C_2 \sqrt{\varepsilon - \sqrt{\varepsilon_1}}^2 \mathbb{E} \left[ \int_t^T \sup_{t \leq s \leq T} |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 \, ds \right] \to 0. \tag{3.27} \]

for some \( C_2 > 0 \) only depending on \( C_1 \) (independent of \( \varepsilon \)), since \( \mathbb{E} \left[ \sup_{t \leq s \leq T} |Z^\varepsilon(s) - Z^{\varepsilon_1}(s)|^2 \right] \) is bounded, by the results \( (2.9) \) and \( (2.10) \).

We conclude, by the previous estimates, that the pair \( (X^{t,\varepsilon}(s), Y^{t,\varepsilon}(s))_{t \leq s \leq T} \) form a Cauchy sequence and therefore converges in \( S^2(t, T; \mathbb{R}^d) \times S^2(t, T; \mathbb{R}^k) \).

Denote \( (X(s), Y(s))_{t \leq s \leq T} \) its limit. \( (X(s), Y(s), 0)_{t \leq s \leq T} \) is the limit in \( \mathcal{N}_t \) of

\[ (X^{t,\varepsilon}(s), Y^{t,\varepsilon}(s), Z^{t,\varepsilon,r}(s))_{t \leq s \leq T} \]

when \( \varepsilon \to 0 \).

If we consider the forward equation in \( (1.1) \) and if we take the limit pointwise when \( \varepsilon \to 0 \), we have

\[ X^t(s) = x + \int_t^s f(r, X(r), Y(r))\,dr, \tag{3.28} \]

where we have used the boundedness of \( \sigma \) and the continuity of \( f \).

Similarly, we can take the limit on the backward equation when \( \varepsilon \to 0 \). Using the continuity of the functions \( h, g \) and

\[ \mathbb{E} \left[ \int_s^T Z^{t,\varepsilon}(r) \, dW(r) \right]^2 = \mathbb{E} \left[ \int_s^T |Z^{t,\varepsilon}(r)|^2 \, dr \right] \to 0, \text{ as } \varepsilon \to 0, \]

which implies \( \int_s^T Z^{t,\varepsilon}(r) \, dW(r) \to 0, \, P\text{-a.s.} \), we have

\[ Y^t_s = h(X(T)) + \int_s^T g(r, X(r), Y(r), 0) \, dr. \tag{3.29} \]

In conclusion, \( (X(s), Y(s))_{t \leq s \leq T} \) solves the following deterministic problem of ordinary (coupled) differ-
ential equations, almost surely,

\[
\begin{align*}
\begin{cases}
\dot{X}(s) &= f(s, X(s), Y(s)), \\
\dot{Y}(s) &= -g(s, X(s), Y(s), 0), \\
X(t) &= x, Y(T) = h(X(T)).
\end{cases}
\end{align*}
\tag{3.30}
\]

Given \( t, t' \in [0, T], x, x' \in \mathbb{R}^n \), consider \((X_{s}^{\epsilon,t,x}, Y_{s}^{\epsilon,t,x}, Z_{s}^{\epsilon,t,x})_{t \leq s \leq T}\) the unique solution in \( \mathcal{N}_t \)

\[
\begin{align*}
\begin{cases}
X_{s}^{\epsilon,t,x}(s) &= x + \int_{s}^{t} f(r, X_{s}^{\epsilon,t,x}(r), Y_{s}^{\epsilon,t,x}(r))dr \\
&\quad + \sqrt{2} \int_{s}^{t} \sigma(r, X_{s}^{\epsilon,t,x}(r), Y_{s}^{\epsilon,t,x}(r))dW(r), \\
Y_{s}^{\epsilon,t,x}(s) &= h(X_{s}^{\epsilon,t,x}(T)) + \int_{s}^{T} g(r, X_{s}^{\epsilon,t,x}(r), Y_{s}^{\epsilon,t,x}(r), Z_{s}^{\epsilon,t,x}(r))dr \\
&\quad - \int_{s}^{T} Z_{s}^{\epsilon,t,x}(r) dW(r), \\
&\quad x \in \mathbb{R}^n, t \leq s \leq T,
\end{cases}
\end{align*}
\tag{3.31}
\]

extended to the whole interval \([0, T]\), putting

\[
\forall 0 \leq s \leq t \quad X_{s}^{\epsilon,t,x}(s) = x; \quad Y_{s}^{\epsilon,t,x}(s) = Y_{s}^{\epsilon,t,x}(t); \quad Z_{s}^{\epsilon,t,x}(s) = 0,
\tag{3.32}
\]

and consider

\[
(X_{s}^{\epsilon,t',x'}(s), Y_{s}^{\epsilon,t',x'}(s), Z_{s}^{\epsilon,t',x'}(s))_{t' \leq s \leq T}
\]

the unique solution in \( \mathcal{N}_{t'} \)

\[
\begin{align*}
\begin{cases}
X_{s}^{\epsilon,t',x'}(s) &= x' + \int_{s}^{t'} f(r, X_{s}^{\epsilon,t',x'}(r), Y_{s}^{\epsilon,t',x'}(r))dr \\
&\quad + \sqrt{2} \int_{s}^{t'} \sigma(r, X_{s}^{\epsilon,t',x'}(r), Y_{s}^{\epsilon,t',x'}(r))dW(r), \\
Y_{s}^{\epsilon,t',x'}(s) &= h(X_{s}^{\epsilon,t',x'}(T)) + \int_{s}^{T} g(r, X_{s}^{\epsilon,t',x'}(r), Y_{s}^{\epsilon,t',x'}(r), Z_{s}^{\epsilon,t',x'}(r))dr \\
&\quad - \int_{s}^{T} Z_{s}^{\epsilon,t',x'}(r) dW(r), \\
&\quad x' \in \mathbb{R}^n, t' \leq s \leq T,
\end{cases}
\end{align*}
\tag{3.33}
\]

extended to the whole interval \([0, T]\), putting

\[
\forall 0 \leq s \leq t' \quad X_{s}^{\epsilon,t',x'}(s) = x; \quad Y_{s}^{\epsilon,t',x'}(s) = Y_{s}^{\epsilon,t',x'}(t'); \quad Z_{s}^{\epsilon,t',x'}(s) = 0.
\tag{3.34}
\]

Using the estimate (3.20), such as in the Corollary 1.4 of [14], we are lead to

\[
\mathbb{E}\left[\sup_{0 \leq s \leq T}|X_{s}^{\epsilon,t,x}(s) - X_{s}^{\epsilon,t',x'}(s)|^2 + \sup_{0 \leq s \leq T}|Y_{s}^{\epsilon,t,x}(s) - Y_{s}^{\epsilon,t',x'}(s)|^2
\right.
\]
\[
+ \int_{0}^{T}|Z_{s}^{\epsilon,t,x}(s) - Z_{s}^{\epsilon,t',x'}(s)|^2 \right]
\leq \alpha|t - t'|^2 + \beta(1 + |x|^2)|t - t'|^2,
\tag{3.35}
\]

where \( \alpha, \beta > 0 \) are constants only depending on \( C_1, \Lambda \).

Finally,

\[
|u^{\epsilon}(t, x) - u^{\epsilon}(t', x')|^2 \leq \alpha|x - x'|^2 + \beta(1 + |x|^2)|t - t'|^2,
\tag{3.36}
\]

which proves that \((u^{\epsilon})_{\epsilon > 0}\) is a family of equicontinuous maps on every compact set of \([0, T] \times \mathbb{R}^n\). We can apply Arzela’s Ascoli Theorem and conclude that the convergence of \(u^{\epsilon}\) to \(u\), where \(u(t, x) = Y_{t}^{t,x}\), is uniform in \([0, T] \times K\), for every \(K\) compact subset of \(\mathbb{R}^d\).
Taking the limit in $\varepsilon \to 0$ in (3.36), we get
\[ |u(t', x') - u(t, x)|^2 \leq \alpha |x - x'|^2 + \beta (1 + |x|^2) |t - t'|^2, \tag{3.37} \]
for all $(t, x), (t', x') \in [0, T] \times \mathbb{R}^n$, which proves that the function $u$, limit in $\varepsilon \to 0$, is Lipschitz continuous in $x$ and uniformly continuous in $t$. The boundedness of $u$ is given by (2.7), since this bound is uniform in $\varepsilon$. Using Theorem 5.1 of [29] we deduce that $u^\varepsilon$ is a viscosity solution in $[0, T] \times \mathbb{R}^d$ of (2.2).

Since the coefficients of the quasilinear parabolic system are Lipschitz continuous, by the property of the compact uniform convergence of viscosity solutions for quasilinear parabolic equations (see [9] for details), we conclude that $u$ is a viscosity solution of (3.13).

Moreover, let $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a $C_b^{1,1}([0, T], \mathbb{R}^n)$ solution, continuous Lipschitz in $x$ and uniformly continuous in $t$ for (3.13). Fixing $(t, x) \in [0, T] \times \mathbb{R}^n$, we take the following function:
\[ \psi : [t, T] \to \mathbb{R}^n, \]
\[ \psi(s) : = v(s, X^{t,x}(s)). \]

Computing its time derivative:
\[
\frac{d\psi}{ds}(s) = \frac{\partial v}{\partial s}(s, X^{t,x}(s)) + \sum_{i=1}^{d} \frac{\partial v}{\partial x_i}(s, X^{t,x}(s)) \frac{\partial (X^{t,x}(s))}{\partial t}
= \frac{\partial v}{\partial s}(s, X^{t,x}(s)) + \sum_{i=1}^{d} \frac{\partial v}{\partial x_i}(s, X^{t,x}(s)) f(s, X^{t,x}(s), Y^{t,x}(s))
= -g(s, X^{t,x}(s), v(s, X^{t,x}(s)), 0),
\]
\[ \psi(T) = v(T, X^{t,x}(T)) = h(x). \]

As a consequence, $v(t, x) = v(t, X^{t,x}(t)) = u(t, x)$, under the hypothesis of uniqueness of solution for the system of ordinary differential equations (3.12). So, under these hypothesis, we have a uniqueness property of solution for (3.13) in the class of $C_b^{1,1}([0, T] \times \mathbb{R}^n)$, which are Lipschitz continuous in $x$ and uniformly continuous in $t$.

**Remark 3.1.** The result presented above (Theorem 3.2) is stated under the assumptions (A.3) and (A.4), which ensure existence and uniqueness of solution of (3.14) for a local time $T \leq C$, where $C$ is a certain constant only depending on the Lipschitz constant $C_1$. Under these assumptions, our results and the existence and uniqueness of solution for the FBSDEs ([14, 24]) do not depend on results for PDE’s but only on probabilistic arguments. However, it is possible to extend Theorem 3.2 to global time. In order to do it and to prove the important properties in [14], results of deterministic quasilinear parabolic partial differential equations [21] are used. If we maintain the assumption of smoothness on the coefficients of the FBSDEs (A.3) and (A.4), the Four Step Scheme Methodology ensures existence and uniqueness of solution for (1.1), and if we remove this requirement of smoothness on the coefficients of the system (1.1), a regularization argument in [14] is used in order to prove it. In this setting, we can also conclude the claims (2) and (3) of Theorem 3.2.
4 Large Deviation Principle

In this section, we study the Freidlin-Wentzell’s large deviation principle for the laws of the family of processes

$$\{(X^{\varepsilon,t,x} (\cdot), Y^{\varepsilon,t,x} (\cdot))\}_{\varepsilon \in (0,1]} \in \mathcal{S}^2 (t,T; \mathbb{R}^n) \times \mathcal{S}^2 (t,T; \mathbb{R}^n)$$

as $\varepsilon \to 0$.

To begin, let us recall the following definitions mainly from [12].

**Definition 4.1.** If $E$ is a complete separable metric space, then a function $I$ defined on $E$ is called a rate function if it has the following properties:

\begin{align}
(\text{a}) & \quad I : E \to [0, +\infty], \ I \text{ is lower semicontinuous.} \\
(\text{b}) & \quad \text{If } 0 \leq a \leq \infty, \ \text{then } C_I (a) = \{ x \in E : I (x) \leq a \} \text{ is compact.} \quad (4.1)
\end{align}

**Definition 4.2 (Large deviation principle).** If $E$ is a complete separable metric space, $\mathcal{B}$ is the Borel $\sigma$-field on $E$, $\{\mu_\varepsilon : \varepsilon > 0\}$ is a family of probability measures on $(E, \mathcal{B})$, and $I$ is a function defined on $E$ and satisfying (4.1), then we say that $\{\mu_\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle with rate $I$ if:

\begin{align}
(\text{a}) & \quad \text{For every open subset } A \text{ of } E, \ \lim inf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon (A) \geq - \inf_{g \in A} I (g) . \\
(\text{b}) & \quad \text{For every closed subset } A \text{ of } E, \ \lim sup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon (A) \leq - \inf_{g \in A} I (g) , \quad (4.2)
\end{align}

here the infimum over the empty set is defined to be $+\infty$.

For our purpose recall the following results from [1, 3, 5] and suppose that $f$ and $\sigma$ are independent of $t$.

**Proposition 4.1.** Consider

\begin{align}
\begin{cases}
   f^0 : \mathbb{R}^n \to \mathbb{R}^n, \\
   \sigma^0 : \mathbb{R}^n \to \mathbb{R}^{n \times d}.
\end{cases} \quad (4.3)
\end{align}

Assume that $f^0, \sigma^0$ satisfy Lipschitz continuous functions, with sublinear growth.

Consider

\begin{align}
\begin{cases}
   f^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n, \\
   \sigma^\varepsilon : \mathbb{R}^n \to \mathbb{R}^{n \times d}.
\end{cases} \quad (4.4)
\end{align}

Assume that $f^\varepsilon, \sigma^\varepsilon$ are uniformly Lipschitz continuous, have sublinear growth and converge uniformly to $f := f^0, \ \sigma := \sigma^0$, respectively. Then, the family $\{X^{\varepsilon,t,x} (\cdot) : 0 < \varepsilon < 1\}$ of random variables of the solutions of the following perturbed stochastic differential equations

\begin{align}
\begin{cases}
   dX^{\varepsilon,t,x} (r) = f^\varepsilon (X^{\varepsilon,t,x} (r)) \, dr + \sqrt{\varepsilon} \sigma^\varepsilon (X^{\varepsilon,t,x} (r)) \, dW (r), \\
   X^{\varepsilon,t,x} (t) = x,
\end{cases} \quad (4.5)
\end{align}

obey a large deviations principle in $C ([t,T]; \mathbb{R}^n)$, the space of the continuous functions $f : [t,T] \to \mathbb{R}^n$, with the good rate function $I$ defined as

$$I (\phi) = \inf \left\{ \frac{1}{2} \| \varphi \|^2_{\mathcal{H}_1 (t)} : \varphi \in \mathcal{H}_1 (t), \ \phi' (s) = f (\phi (s)) + \sigma (\phi (s)) \varphi' (s) \right\} .$$
Moreover, the level sets of $\mathcal{I}$ are compact and for every Borel subset $A$ of $C ([t, T] ; \mathbb{R}^n)$, we have

$$ - \inf_{g \in A} \mathcal{I} (g) \leq \liminf_{\varepsilon \to 0} \varepsilon \log P \left( X^{\varepsilon,t,x} \in A \right) \leq \limsup_{\varepsilon \to 0} \varepsilon \log P \left( X^{\varepsilon,t,x} \in A \right) \leq - \inf_{g \in A} \mathcal{I} (g) .$$

The proof can be seen in [5]. Next we introduce a very important result in Large deviation theory, used to transfer a LDP from one topology space to another one.

**Lemma 4.1 (Contraction Principle Theorem 4.2.23 Page 133 in [12])**. Let $\{ \mu_\varepsilon \}$ be a family of probability measures that satisfies the Large Deviation Principle with a good rate function $I$ on a Hausdorff topological space $S$, and for $\varepsilon \in (0, 1)$, let $f_\varepsilon : S \to Q$ be continuous functions, with $(Q, d)$ a metric space. Assume that there exists a measurable map $f : S \to Q$ such that for every $\alpha < +\infty$,

$$ \limsup_{\varepsilon \to 0} \sup_{x : I(x) \leq \alpha} d \left( f_\varepsilon (x) , f (x) \right) = 0 .$$

Then the family of probability measures $\{ \mu_\varepsilon \circ f_\varepsilon^{-1} \}$ satisfy the LDP in $Q$ with the good rate function

$$ I^* (y) = \inf \{ I (x) : y = f (x) \} .$$

For our purposes, we give the following

**Definition 4.3**. Let $t \in [0, T]$. We define the mapping $F^\varepsilon : C ([t, T] ; \mathbb{R}^n) \to C ([t, T] ; \mathbb{R}^n)$ by

$$ F^\varepsilon (\psi) = \left[ s \to u^\varepsilon \left( s, \psi (s) \right) \right] , \quad 0 \leq t \leq s \leq T , \ \psi \in C ([t, T] ; \mathbb{R}^n) ,$$

where $u^\varepsilon$ is given by (2.7).

As we have seen before

$$ Y^{\varepsilon,t,x} (t) = F^\varepsilon \left( X^{\varepsilon,t,x} \right) (t) .$$

From now on, for $\varepsilon = 0$, $u$ and $F$ stand for $u^0$ and $F^0$.

In order to prove the uniform convergence of the mapping $F^\varepsilon$, we need the following formula

$$ \| F^\varepsilon (\varphi) - F (\varphi) \| = \sup_{t \in [t, T]} \left| u^\varepsilon \left( s, \varphi (s) \right) - u \left( s, \varphi (s) \right) \right| , \quad \varphi \in C ([t, T] ; \mathbb{R}^n) $$

or

$$ \| F^\varepsilon (\varphi) - F (\varphi) \| = \sup_{s \in [t, T]} \left| Y_{s,\varphi(s)} (s) - Y_{s,\varphi(s)} (s) \right| .$$

We have the following:

**Theorem 4.1**. Assume (A1) and (A2) hold. Then,

1) The family $\left( X^{\varepsilon,t,x} (\cdot) \right)_{\varepsilon \in (0, 1]}$ satisfy, as $\varepsilon$ goes to 0, a large deviation principle with a rate function

$$ \mathcal{I}_1 (\phi) = \inf_{\{ \varphi \in \mathcal{H}_1 (t) : \phi' (s) = f(\phi (s), u(s,\phi(s)))+\sigma(\phi (s), u(s,\phi(s)))\varphi' (s), \ s \in [t,T] \} } \frac{1}{2} \| \varphi \|_{\mathcal{H}_1 (t)}^2 ,$$

for $g \in C ([t, T] ; \mathbb{R}^n)$.
ii) The family \((Y^{\varepsilon,t,x} (\cdot))_{\varepsilon \in (0,1]}\) satisfy, as \(\varepsilon\) goes to 0, a large deviation principle with a rate function

\[ I_2 (\phi) = \inf \{ I_1 (\varphi) : F (\varphi) (s) = \phi (s) = u (s, \varphi (s)), s \in [t, T] \}. \]

Proof. Indeed, noting (A1) and Lemma 3.3 the first assertion has been obtained in Proposition 4.1. We are going to prove the second assertion. By virtue of the contraction principle Lemma 4.1, we just need to show that \(F^{\varepsilon}, \varepsilon \in (0,1]\) are continuous and \(\{F^{\varepsilon}\}\) converges uniformly to \(F\) on every compact of \(C ([0,T]; \mathbb{R}^n)\), as \(\varepsilon\) tends to zero.

Proving the continuity of \(F^{\varepsilon}\):

Let \(\varepsilon > 0\) and \(x \in C([t,T], \mathbb{R}^d)\). Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(C([t,T], \mathbb{R}^d)\) converging to \(x\) in the uniform norm. Fix \(\delta > 0\). Since \(\|x_n - x\|_\infty \to 0\), there exists \(M > 0\) such that \(\|x\|_\infty, \|x_n\|_\infty \leq M\).

Due to Lemma 2.1 and Lemma 3.2 we know that \(u^{\varepsilon}\) is a continuous function in \([0,T] \times \mathbb{R}^d\) and \(u^{\varepsilon}\) is uniformly continuous in \([t,T] \times K\) where \(K = \overline{B}(0, M) \subset \mathbb{R}^d\).

There exists \(\eta > 0\) such that for \(s, s_1 \in [t,T]\) and \(z, z_1 \in K\), \(|s-s_1| < \eta\) and \(|z-z_1| < \eta\), we have

\[ |u^{\varepsilon}(s,z) - u^{\varepsilon}(s_1,z_1)| < \delta. \]

Since \(x_n \to x\) in \(C([t,T], \mathbb{R}^d)\), fixing \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) we have \(\|x_n - x\|_\infty < \eta\).

For all \(r \in [t,T]\) and for all \(n \geq n_0\), \(x_n (r), x (r) \in K\) and

\[ |u^{\varepsilon}(r,x (r)) - u^{\varepsilon}(r,x_n (r))| < \delta. \]

So we conclude that \(F^{\varepsilon}(x_n) \to F^{\varepsilon}(x)\), which proves the continuity of \(F^{\varepsilon}\) in the point \(x \in C([t,T], \mathbb{R}^d)\).

Next let us show the uniform convergence of the mapping \(F^{\varepsilon}\). Consider a compact set \(K\) of \(C ([0,T]; \mathbb{R}^n)\) and let

\[ \mathcal{L} = \{ \varphi (s), \varphi \in K, s \in [t,T] \}. \]

Clearly, \(\mathcal{L}\) is a compact set of \(\mathbb{R}^n\). By Lemma 6, there exists a positive constant \(C_3\) such that

\[ \sup_{\varphi \in \mathcal{K}} \|F^{\varepsilon} (\varphi) - F (\varphi)\|^2 = \sup_{\varphi \in \mathcal{K}} \sup_{s \in [t,T]} \left| Y^{\varepsilon,s,\varphi(s)} (s) - \mathcal{Y}^{s,\varphi(s)} (s) \right|^2 \]

\[ \leq \sup_{x \in \mathcal{L}} \sup_{s \in [t,T]} \left| Y^{\varepsilon,s,x} (s) - \mathcal{Y}^{s,x} (s) \right|^2 \]

\[ \leq C_3 \varepsilon. \]

The proof is complete.

\[ \square \]

Remark 4.1. Under the conditions (A3) and (A4), we have the same conclusion as in Theorem 4.1. The proof is similar we omit it.

Remark 4.2. For fully coupled FBSDEs, that is,

\[
\begin{align*}
X^\varepsilon,t,x (s) &= x + \int_s^t f \left( X^\varepsilon,t,x (r), Y^\varepsilon,t,x (r), Z^\varepsilon,t,x (r) \right) \, dr \\
&\quad + \sqrt{\varepsilon} \int_s^t \sigma \left( X^\varepsilon,t,x (r), Y^\varepsilon,t,x (r), Z^\varepsilon,t,x (r) \right) \, dW (r), \\
Y^\varepsilon,t,x (s) &= h \left( X^\varepsilon,t,x (T) \right) + \int_s^T g \left( r, X^\varepsilon,t,x (r), Y^\varepsilon,t,x (r), Z^\varepsilon,t,x (r) \right) \, dr \\
&\quad - \int_s^T Z^\varepsilon,t,x (r) \, dW (r), \quad 0 \leq t \leq s \leq T,
\end{align*}
\]

first of all, note that trajectories of the process \((Z^\varepsilon,t,x (s))_{t \leq s \leq T}\) are not continuous in general. As a
matter of fact, under the assumptions of Theorem 2.6 in [31], we know only that $Z_{t,x}^\epsilon$ belongs to the space $M^2 (0, T; \mathbb{R}^{m \times d})$, which allows us to arbitrarily change the values of the process $Z_{t,x}^\epsilon$ in any $P$-null set. In particular, $\{t\} \times \Omega$ is a $P$-null set, which means that $Z_{t,x}^\epsilon$ can be any $m \times d$ matrix. Hence, we can not get Lipschitz property for $v^\epsilon$. This issue will be carried out in our future publications.

A Appendix

The Proofs of Lemma 2.1-Lemma 2.4

Proof of Lemma 2.1

Proof. Applying Itô’s formula to $|Y_{t,x}^\epsilon (\cdot) - Y_{t,y}^\epsilon (\cdot)|^2$ on $[s, T]$, we have

$$
|Y_{t,x}^\epsilon (s) - Y_{t,y}^\epsilon (s)|^2 + \mathbb{E}^{F_s} \left[ \int_s^T \left( |Z_{t,x}^\epsilon (r) - Z_{t,y}^\epsilon (r)|^2 \right) dr \right]
$$

$$
= \mathbb{E}^{F_s} \left[ |h (X_{t,x}^\epsilon (T)) - h (X_{t,y}^\epsilon (T))|^2 \right]
$$

$$
+ 2\mathbb{E}^{F_s} \left[ \int_s^T \left( Y_{t,x}^\epsilon (r) - Y_{t,y}^\epsilon (r) \right) \left( f_{t,x}^\epsilon (r) - f_{t,y}^\epsilon (r) \right) dr \right].
$$

(A.1)

which yields

$$
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_{t,x}^\epsilon (s) - Y_{t,y}^\epsilon (s)|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T \left( |Z_{t,x}^\epsilon (r) - Z_{t,y}^\epsilon (r)|^2 \right) dr \right]
$$

$$
\leq \mathbb{E} \left[ |h (X_{t,x}^\epsilon (T)) - h (X_{t,y}^\epsilon (T))|^2 \right]
$$

$$
+ \mathbb{E} \left[ \int_t^T C_1 |X_{t,x}^\epsilon (r) - X_{t,y}^\epsilon (r)|^2 + (3C_1 + 2C_1^2) |Y_{t,x}^\epsilon (r) - Y_{t,y}^\epsilon (r)|^2 dr \right]
$$

$$
\leq C_1^2 \mathbb{E} \left[ |X_{t,x}^\epsilon (T) - X_{t,y}^\epsilon (T)|^2 \right]
$$

$$
+ \mathbb{E} \left[ \int_t^T (C_1 |X_{t,x}^\epsilon (r) - X_{t,y}^\epsilon (r)|^2 + (3C_1 + 2C_1^2) |Y_{t,x}^\epsilon (r) - Y_{t,y}^\epsilon (r)|^2 \right) dr \right]
$$

$$
\leq (3C_1 + 2C_1^2) \left\{ \mathbb{E} \left[ |X_{t,x}^\epsilon (T) - X_{t,y}^\epsilon (T)|^2 \right]
$$

$$
+ \mathbb{E} \left[ \int_t^T \left( |X_{t,x}^\epsilon (r) - X_{t,y}^\epsilon (r)|^2 + |Y_{t,x}^\epsilon (r) - Y_{t,y}^\epsilon (r)|^2 \right) dr \right] \right\},
$$

(A.2)

since $2ab \leq a^2 + b^2$.

Applying Itô’s formula to $X_{t,x}^\epsilon (s) - X_{t,y}^\epsilon (s)$ yields that

$$
|X_{t,x}^\epsilon (s) - X_{t,y}^\epsilon (s)|^2
$$

$$
= \int_t^s 2 \left( X_{t,x}^\epsilon (r) - X_{t,y}^\epsilon (r) \right) \left( f_{t,x}^\epsilon (r) - f_{t,y}^\epsilon (r) \right) dr + \varepsilon \int_t^s |\sigma_{t,x}^\epsilon (r) - \sigma_{t,y}^\epsilon (r)|^2 dr
$$
Combining (A.5) and (A.6), we have
\[ \sqrt{\epsilon} \int_t^T 2 (X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)) \left ( \sigma_{\epsilon,t,x}^r (r) - \sigma_{\epsilon,t,y}^r (r) \right ) dW (r). \] (A.3)

By Burkholder-Davis-Gundy’s inequality, there is a constant \( C_3 > 0 \) such that
\[
\mathbb{E} \left [ \sup_{t \leq s \leq T} |X_{\epsilon,t,x}^r (s) - X_{\epsilon,t,y}^r (s)|^2 \right ] \\
\leq C_3 \left \{ (1 + \sqrt{\epsilon}) \mathbb{E} \left [ \int_t^T \left ( |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ] + \mathbb{E} \left [ \int_t^T \left ( |f_{\epsilon,t,x}^r (r) - f_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ] \\
+ (\sqrt{\epsilon} + \epsilon) \mathbb{E} \left [ \int_t^T \left ( |\sigma_{\epsilon,t,x}^r (r) - \sigma_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ] \right \}
\]
\[
\leq C_3 \mathbb{E} \left [ \int_t^T \left ( (6C_1^2 + 2) |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)|^2 + 6C_1^2 |Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ]
\]
\[
\leq C_3 \mathbb{E} \left [ \int_t^T \left ( |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)|^2 + |Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ] + 6TC_1^2 \sup_{t \leq s \leq T} |Y_{\epsilon,t,x}^r (s) - Y_{\epsilon,t,y}^r (s)|^2. \] (A.4)

Applying Itô’s formula to \((X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)) (Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r))\), and by virtue of assumption (A2), we have
\[
C_2 \mathbb{E} \left [ \left |X_{\epsilon,t,x}^r (T) - X_{\epsilon,t,y}^r (T) \right |^2 \right ] - \mathbb{E} \left [(x - y) (Y_{\epsilon,t,x}^r (t) - Y_{\epsilon,t,y}^r (t)) \right ]
\leq \mathbb{E} \left [ h \left (X_{\epsilon,t,x}^r (T) \right ) - h \left (X_{\epsilon,t,y}^r (T) \right ) \right ] (X_{\epsilon,t,x}^r (T) - X_{\epsilon,t,y}^r (T))
- \mathbb{E} \left [(x - y) \left (Y_{\epsilon,t,x}^r (t) - Y_{\epsilon,t,y}^r (t) \right ) \right ]
\leq -C_2 (1 + \sqrt{\epsilon}) \mathbb{E} \left [ \int_t^T \left ( |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)|^2 + |Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ]
\leq -C_2 \mathbb{E} \left [ \int_t^T \left ( |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r)|^2 + |Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r)|^2 \right ) dr \right ]. \] (A.5)

The inequality (A.5) can be rewritten as follows
\[
C_2 \mathbb{E} \left [ \left |X_{\epsilon,t,x}^r (T) - X_{\epsilon,t,y}^r (T) \right |^2 \right ]
+ C_2 \mathbb{E} \left [ \int_t^T \left |X_{\epsilon,t,x}^r (r) - X_{\epsilon,t,y}^r (r) \right |^2 + \left |Y_{\epsilon,t,x}^r (r) - Y_{\epsilon,t,y}^r (r) \right |^2 \right ] dr
\leq \mathbb{E} \left [(x - y) \left (Y_{\epsilon,t,x}^r (t) - Y_{\epsilon,t,y}^r (t) \right ) \right ]. \] (A.6)

Combining (A.5) and (A.6), we have
\[
\mathbb{E} \left [ \sup_{t \leq s \leq T} \left |Y_{\epsilon,t,x}^r (s) - Y_{\epsilon,t,y}^r (s) \right |^2 \right ] + \frac{1}{2} \mathbb{E} \left [ \int_t^T \left |Z_{\epsilon,t,x}^r (r) - Z_{\epsilon,t,y}^r (r) \right |^2 \right ] dr
\leq \frac{(3C_1^2 + 2C_1^2)}{C_2} \mathbb{E} \left [(x - y) \left (Y_{\epsilon,t,x}^r (t) - Y_{\epsilon,t,y}^r (t) \right ) \right ]
\leq \frac{1}{2} \left |Y_{\epsilon,t,x}^r (t) - Y_{\epsilon,t,y}^r (t) \right |^2 + \frac{1}{2} \left (\frac{3C_1^2 + 2C_1^2}{C_2} \right )^2 |x - y|^2. \] (A.7)
Therefore,
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} \left| Y^{\varepsilon,t,x}(s) - Y^{\varepsilon,t,y}(s) \right|^2 \right] \leq \left( \frac{3C_1 + 2C_2^2}{C_2} \right)^2 |x - y|^2 .
\] (A.8)

By Gronwall’s inequality, there exists a positive constant \( C_4 \), such that
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} \left| X^{\varepsilon,t,x}(s) - X^{\varepsilon,t,y}(s) \right|^2 \right] \leq C_4 |x - y|^2 ,
\] (A.9)

where \( C_4 \) depends on \( C_1, C_2, C_3, \) and \( T \). Finally, taking \( C_1 = \max \left\{ \left( \frac{3C_1 + 2C_2^2}{C_2} \right)^2, C_4 \right\} \), we get the desired result. \[\square\]

**Proof of Lemma 2.2**

**Proof.** Applying Itô’s formula to \( |X^{\varepsilon,t,x}(s)|^2 \) yields that
\[
|X^{\varepsilon,t,x}(s)|^2 = 2 \int_t^s \left( X^{\varepsilon,t,x}(r) f^{\varepsilon,t,x}(r) \right) \, dr + \varepsilon \int_t^s \left| \sigma^{\varepsilon,t,x}(r) \right|^2 \, dr
\]
\[
+ \sqrt{\varepsilon} \int_t^s 2X^{\varepsilon,t,x}(r) \sigma^{\varepsilon,t,x}(r) \, dB(r) .
\] (A.10)

By Burkholder-Davis-Gundy’s inequality, there is a constant \( C_5 > 0 \) such that
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s)|^2 \right]
\]
\[
\leq C_5 \left\{ \mathbb{E} \left[ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |f^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] \right. + \varepsilon \mathbb{E} \left[ \int_t^T \left( |\sigma^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] \right. \\
\]
\[
\left. + \sqrt{\varepsilon} \mathbb{E} \left[ \int_t^T \left( 2X^{\varepsilon,t,x}(r) \sigma^{\varepsilon,t,x}(r) \right) \, dB(r) \right] \right\}
\]
\[
\leq C_5 \left\{ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |f^{\varepsilon,t,x}(r)|^2 \right) \, dr \right. + 6C_1^2 \varepsilon \mathbb{E} \left[ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |Y^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] \right. \\
\]
\[
\left. + 2 |f(r,0,0)|^2 \right\}
\]
\[
+ 4\varepsilon C_1^2 \mathbb{E} \left[ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |Y^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] + 2 \int_t^T \left( |\sigma(r,0,0)|^2 \right) \, dr \right\}
\]
\[
\leq C_5 \left\{ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |f(r,0,0)|^2 + |\sigma(r,0,0)|^2 \right) \, dr \right\}
\]
\[
+ \varepsilon C_1^2 \mathbb{E} \left[ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |Y^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] + 2 \int_t^T \left( |\sigma(r,0,0)|^2 \right) \, dr \right\}
\]
\[
\leq C_5 \left\{ 2 + 8C_2^2 \right\} \mathbb{E} \left[ \int_t^T \left( |X^{\varepsilon,t,x}(r)|^2 + |Y^{\varepsilon,t,x}(r)|^2 \right) \, dr \right] + \sup_{t \leq s \leq T} |Y^{\varepsilon,t,x}(s)|^2
\]
\[
+ \int_0^T \left( |f(r,0,0)|^2 + |\sigma(r,0,0)|^2 \right) \, dr \right\} .
\] (A.11)
Second applying Itô’s formula to \( |Y^{ε,t,x}(\cdot)|^2 \) on \([s,T]\), we have

\[
\mathbb{E} \left| Y^{ε,t,x}(s) \right|^2 + \mathbb{E} \mathbb{F}_s \left[ \int_s^T \left( |Z^{ε,t,x}(r)|^2 \right) dr \right]
= \mathbb{E} \mathbb{F}_s \left[ |h(X^{ε,t,x}(T))|^2 \right] + 2\mathbb{E} \mathbb{F}_s \left[ \int_s^T Y^{ε,t,x}(r) f^{ε,t,x}(r) dr \right],
\]

(A.12)

which yields

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} \left| Y^{ε,t,x}(s) \right|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T \left( |Z^{ε,t,x}(r)|^2 \right) dr \right]
\leq \mathbb{E} \left[ |h(X^{ε,t,x}(T))|^2 \right] + \int_t^T \left( |Y^{ε,t,x}(r)|^2 + |f^{ε,t,x}(r)|^2 \right) dr
\leq \mathbb{E} \left[ |h(X^{ε,t,x}(T)) - h(0) + h(0)|^2 \right]
+ \mathbb{E} \left[ \int_t^T \left( |Y^{ε,t,x}(r)|^2 + |f^{ε,t,x}(r) - f(r,0,0,0) + f(r,0,0,0)|^2 \right) dr \right]
\leq \mathbb{E} \left[ 2C_1^2 \left| X^{ε,t,x}(T) \right|^2 + 2|h(0)|^2 \right]
+ \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} + 3C_1^2 \right) |Y^{ε,t,x}(r)|^2 + \frac{1}{2} |X^{ε,t,x}(r)|^2 + \frac{|f(r,0,0,0)|^2}{6C_1^2} \right] dr
\leq \left( 3C_1^2 + 3 + \frac{1}{6C_1^2} \right) \left( |h(0)|^2 + \int_0^T \left( |g(r,0,0,0)|^2 \right) dr \right) + \int_0^T \left( |f(r,0,0,0)|^2 \right) dr + |h(0)|^2 \right)
+ \int_0^T \left( |f(r,0,0,0)|^2 \right) dr + |h(0)|^2 \right).
\]

(A.13)

Set

\[ C_6 = \left( 3C_1^2 + 3 + \frac{1}{6C_1^2} \right) \left( |h(0)|^2 + \int_0^T \left( |g(r,0,0,0)|^2 \right) dr \right) + \int_0^T \left( |f(r,0,0,0)|^2 \right) dr + |h(0)|^2 \right).
\]

Applying Itô’s formula to \( X^{ε,t,x}(r) Y^{ε,t,x}(r) \), and by virtue of assumption (A2), we have

\[
\mathbb{E} \left[ X^{ε,t,x}(T) f^{ε,t,x}(X^{ε,t,x}(T)) \right] - \mathbb{E} \left[ x Y^{ε,t,x}(t) \right]
\leq -C_2 \left( 1 + \sqrt{\varepsilon} \right) \mathbb{E} \left[ \int_t^T \left( |X^{ε,t,x}(r)|^2 + |Y^{ε,t,x}(r)|^2 \right) dr \right]
+ \mathbb{E} \left[ \int_t^T \left( X^{ε,t,x}(r) g(r,0,0,0) + Y^{ε,t,x}(r) f(r,0,0) + \sqrt{\varepsilon} Z^{ε,t,x}(r) \tau(r,0,0) \right) dr \right]
\leq -\frac{C_2}{2} \mathbb{E} \left[ \int_t^T \left( |X^{ε,t,x}(r)|^2 + |Y^{ε,t,x}(r)|^2 \right) dr \right]
+ \frac{1}{2C_2} \left( \int_0^T \left( |g(r,0,0,0)|^2 + |f(r,0,0)|^2 \right) dr \right)
+ \mathbb{E} \left[ \int_t^T \sqrt{\varepsilon} Z^{ε,t,x}(r) \tau(r,0,0) dr \right]
\leq -\frac{C_2}{2} \mathbb{E} \left[ \int_t^T \left( |X^{ε,t,x}(r)|^2 + |Y^{ε,t,x}(r)|^2 \right) dr \right] + M_1,
\]

(A.14)

where

\[
M_1 = \frac{1}{2} \int_0^T \left( |g(r,0,0,0)|^2 + |f(r,0,0)|^2 \right) dr + \mathbb{E} \left[ \int_t^T \sqrt{\varepsilon} Z^{ε,t,x}(r) \tau(r,0,0) dr \right].
\]
On the other hand,
\[
\begin{align*}
\mathbb{E} \left[ X_{\varepsilon,t,x} (T) h_{\varepsilon,t,x} (X_{\varepsilon,t,x} (T)) \right] \\
= \mathbb{E} \left[ (X_{\varepsilon,t,x} (T) - 0) (h_{\varepsilon,t,x} (X_{\varepsilon,t,x} (T)) - h_{\varepsilon,t,x} (0) + h_{\varepsilon,t,x} (0)) \right] \\
\geq \mathbb{E} \left[ C_2 |X_{\varepsilon,t,x} (T)|^2 + X_{\varepsilon,t,x} (T) h_{\varepsilon,t,x} (0) \right] \\
\geq \mathbb{E} \left[ C_2 |X_{\varepsilon,t,x} (T)|^2 - \frac{|X_{\varepsilon,t,x} (T)|^2}{2 \alpha} - \frac{|h_{\varepsilon,t,x} (0)|^2}{2} \right],
\end{align*}
\]  
(A.15)

where \(\alpha > 0\) large enough such that \(C_2 - \frac{1}{2\alpha} > 0\).

Then, we have
\[
\begin{align*}
\left( C_2 - \frac{1}{2\alpha} \right) \mathbb{E} \left[ |X_{\varepsilon,t,x} (T)|^2 \right] + \frac{C_2}{2} \mathbb{E} \left[ \int_t^T \left( |X_{\varepsilon,t,x} (r)|^2 + |Y_{\varepsilon,t,x} (r)|^2 \right) dr \right] \\
\leq \mathbb{E} \left[ xY_{\varepsilon,t,x} (t) \right] + M_1 + \frac{\alpha |h_{\varepsilon,t,x} (0)|^2}{2}. 
\end{align*}
\]  
(A.16)

Setting
\[
\begin{align*}
\left\{ \begin{array}{l}
M_2 = M_1 + \alpha |h_{\varepsilon,t,x} (0)|^2, \\
\tilde{C} = \min \left\{ C_2 - \frac{1}{2\alpha}, C_2 \right\},
\end{array} \right.
\]

we have
\[
\begin{align*}
\mathbb{E} \left[ |X_{\varepsilon,t,x} (T)|^2 + \int_t^T \left( |X_{\varepsilon,t,x} (r)|^2 + |Y_{\varepsilon,t,x} (r)|^2 \right) dr \right] \\
\leq \frac{\mathbb{E} \left[ xY_{\varepsilon,t,x} (t) \right] + M_2}{\tilde{C}}. 
\end{align*}
\]  
(A.17)

Noting (A.13), we obtain that
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_{\varepsilon,t,x} (s)|^2 + \frac{1}{2} \int_t^T |Z_{\varepsilon,t,x} (r)|^2 dr \right] \\
\leq \left( 3C_1^2 + 3 + \frac{1}{6C_1^2} \right) \left( \frac{\mathbb{E} \left[ xY_{\varepsilon,t,x} (t) \right] + M_2}{\tilde{C}} \right) + C_6 \\
\leq \frac{|Y_{\varepsilon,t,x} (t)|^2}{2} + \frac{|x|^2 \left( 3C_1^2 + 3 + \frac{1}{6C_1^2} \right)^2}{2\tilde{C}^2} \\
+ \frac{\left( 3C_1^2 + 3 + \frac{1}{6C_1^2} \right) M_2}{\tilde{C}} + C_6. 
\end{align*}
\]  
(A.18)

Define \(\tilde{M} = \left( \frac{3C_1^2 + 3 + \frac{1}{6C_1^2}}{\tilde{C}} \right)\). The expression (A.18) can be rewritten as
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_{\varepsilon,t,x} (s)|^2 + \frac{1}{2} \int_t^T |Z_{\varepsilon,t,x} (r)|^2 dr \right] \\
\leq \frac{|Y_{\varepsilon,t,x} (t)|^2}{2} + \frac{|x|^2 \tilde{M}^2}{2}
\end{align*}
\]  
(A.19)
By Gronwall’s inequality, we derive
\[ + \bar{M} \left( \alpha \frac{|h^{\varepsilon,t,x}(0)|^2}{2} + \int_0^T \left( \frac{|g(r,0,0,0)|^2 + |f(r,0,0)|^2}{2C_2} \right) dr \right) \]
\[ + \bar{M}^2 \int_0^T |\sigma(r,0,0)|^2 dr + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z^{\varepsilon,t,x}(r)|^2 dr \right] + C_6. \] (A.19)

Consequently, we get
\[ \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y^{\varepsilon,t,x}(s)|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z^{\varepsilon,t,x}(r)|^2 dr \right] \leq |x|^2 \bar{M}^2 + \bar{M} \left( \alpha \frac{|h^{\varepsilon,t,x}(0)|^2}{2} + \int_0^T \left( \frac{|g(r,0,0,0)|^2 + |f(r,0,0)|^2}{2C_2} \right) dr \right) \]
\[ + 2\bar{M}^2 \int_0^T |\sigma(r,0,0)|^2 dr + 2C_6 \]
\[ \leq \max \left\{ \Sigma, \bar{M}^2 \right\} \left( 1 + |x|^2 \right), \] (A.20)

where
\[ \Sigma = \bar{M} \left( \alpha \frac{|h^{\varepsilon,t,x}(0)|^2}{2} + \int_0^T \left( \frac{|g(r,0,0,0)|^2 + |f(r,0,0)|^2}{2C_2} \right) dr \right) \]
\[ + 2\bar{M} \int_0^T |\sigma(r,0,0)|^2 dr \] + 2C_6.

By Gronwall’s inequality, we derive
\[ \mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{\varepsilon,t,x}(s)|^2 \right] \leq C_7 \left( 1 + |x|^2 \right), \] (A.21)

where \( C_7 \) is independent of \( \varepsilon \). Taking \( C_2 = \max \left\{ \max \left\{ \Sigma, \bar{M}^2 \right\}, C_7 \right\} \), we get the desired result. \( \square \)

The proof of Lemma 2.3.

**Proof.** Suppose that \( t_1 > t_2 \). By standard estimates and Itô isometry we have
\[ \mathbb{E} \left[ \sup_{t_1 \leq s \leq T} |X^{\varepsilon,t_1,x}(s) - X^{\varepsilon,t_2,x}(s)|^2 \right] \leq 2(t_1 - t_2) \mathbb{E} \left[ \int_{t_2}^{t_1} |f(r,X^{\varepsilon,t_2,x}(r),Y^{\varepsilon,t_2,x}(r))|^2 dr \right] \]
\[ + 2\mathbb{E} \left[ \int_{t_2}^{t_1} |\sigma(r,X^{\varepsilon,t_2,x}(r),Y^{\varepsilon,t_2,x}(r))|^2 dr \right] \]
\[ \leq (t_1 - t_2) \mathbb{E} \left[ 16TC_1^2 \left( \sup_{t_2 \leq r \leq t_1} |X^{\varepsilon,t_2,x}(r)|^2 + \sup_{t_2 \leq r \leq t_1} |Y^{\varepsilon,t_2,x}(r)|^2 \right) \right] \]

where \( T \) is a fixed constant. Taking \( t_2 = r \), we have
\[ \mathbb{E} \left[ \sup_{t_1 \leq s \leq T} |X^{\varepsilon,t_1,x}(s) - X^{\varepsilon,r,x}(r)|^2 \right] \leq (t_1 - r) \mathbb{E} \left[ 16TC_1^2 \left( \sup_{r \leq x \leq t_2} |X^{\varepsilon,r,x}(x)|^2 + \sup_{r \leq x \leq t_2} |Y^{\varepsilon,r,x}(x)|^2 \right) \right] \]
+2\int_0^T |f (r, 0, 0)|^2 \, dr + 2 \int_0^T |\sigma (r, 0, 0)|^2 \, dr \right].

It follows from Lemma 2.1 that

\[ \mathbb{E} \left[ \sup_{t_1 \leq s \leq T} \left| X^{\varepsilon, t_1, x} (s) - X^{\varepsilon, t_2, x} (s) \right|^2 \right] \leq (t_1 - t_2) \Theta, \]

where

\[ \Theta = \left[ 16T C_1^2 C_2 \left( 1 + |x|^2 \right) + 2 \int_0^T |f (r, 0, 0)|^2 \, dr + 2 \int_0^T |\sigma (r, 0, 0)|^2 \, dr \right] \]

Similarly as in Lemma 2.1 we have

\[ \mathbb{E} \left[ \sup_{t_1 \leq s \leq T} \left| Y^{\varepsilon, t_1, x} (s) - Y^{\varepsilon, t_2, x} (s) \right|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_{t_1}^T \left| Z^{\varepsilon, t_1, x} (r) - Z^{\varepsilon, t_2, x} (r) \right|^2 \, dr \right] \]

\[ \leq \left( 3C_1 + 2C_2^2 \right) \left[ \mathbb{E} \left[ \left( X^{\varepsilon, t_1, x} (T) - X^{\varepsilon, t_2, x} (T) \right)^2 \right] \right. \]

\[ + \mathbb{E} \int_{t_1}^T \left| X^{\varepsilon, t_1, x} (r) - X^{\varepsilon, t_2, x} (r) \right|^2 + \left| Y^{\varepsilon, t_1, x} (r) - Y^{\varepsilon, t_2, x} (r) \right|^2 \, dr \right]. \]

By Gronwall’s inequality, we obtain

\[ \mathbb{E} \left[ \sup_{t_1 \leq s \leq T} \left| Y^{\varepsilon, t_1, x} (s) - Y^{\varepsilon, t_2, x} (s) \right|^2 \right] \leq (t_1 - t_2) e^T \left( 3C_1 + 2C_1^2 \right) (1 + T) \Theta, \]

and

\[ \mathbb{E} \left[ \int_{t_1}^T \left| Z^{\varepsilon, t_1, x} (r) - Z^{\varepsilon, t_2, x} (r) \right|^2 \, dr \right] \]

\[ \leq 2 (t_1 - t_2) \left( 3C_1 + 2C_1^2 \right) \Theta \left( 1 + T + e^T \left( 3C_1 + 2C_1^2 \right) (1 + T) \right). \]

Set

\[ \begin{cases} C_5 = e^T \left( 3C_1 + 2C_1^2 \right) (1 + T) \Theta \\ C_6 = 2 \left( 3C_1 + 2C_1^2 \right) \Theta \left( 1 + T + e^T \left( 3C_1 + 2C_1^2 \right) (1 + T) \right) \end{cases} \]

and \( C_3 = \max \{ \Theta, C_5, C_6 \} \). We get the desired result. \( \square \)

The proof of Lemma 2.4.

Proof. Analogously, applying Itô’s formula to \( |Y^{\varepsilon_1, t, x} (\cdot) - Y^{\varepsilon_2, t, x} (\cdot)|^2 \) on \([t, T]\), by the method used above, we have

\[ \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| Y^{\varepsilon_1, t, x} (s) - Y^{\varepsilon_2, t, x} (s) \right|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \int_t^T \left| Z^{\varepsilon_1, t, x} (r) - Z^{\varepsilon_2, t, x} (r) \right|^2 \, dr \right] \]

\[ \leq \mathbb{E} \left[ \left| \varepsilon \left( X^{\varepsilon_1, t, x} (T) \right) - \varepsilon \left( X^{\varepsilon_2, t, x} (T) \right) \right|^2 \right] \]

\[ + \mathbb{E} \int_t^T \left( C_1 \left| X^{\varepsilon_1, t, x} (r) - X^{\varepsilon_2, t, x} (r) \right|^2 + (3C_1 + 2C_1^2) \left| Y^{\varepsilon_1, t, x} (r) - Y^{\varepsilon_2, t, x} (r) \right|^2 \right) \, dr \]
Applying Itô's formula to $X^{\varepsilon_1,t,x}(s) - X^{\varepsilon_2,t,x}(s)$ yields that

$$
|X^{\varepsilon_1,t,x}(s) - X^{\varepsilon_2,t,x}(s)|^2
= \int_t^s 2 (X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)) \left( f^{\varepsilon_1,t,x}(r) - f^{\varepsilon_2,t,x}(r) \right) dr + \varepsilon \int_t^s \left| \sigma^{\varepsilon_1,t,x}(r) - \sigma^{\varepsilon_2,t,x}(r) \right|^2 dr
+ \sqrt{\varepsilon} \int_t^s 2 (X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)) \left( \sigma^{\varepsilon_1,t,x}(r) - \sigma^{\varepsilon_2,t,x}(r) \right) dW(r).$$

By Burkholder-Davis-Gundy's inequality, there is a constant $C_8 > 0$ such that

$$
E \left[ \sup_{t \leq s \leq T} |X^{\varepsilon_1,t,x}(s) - X^{\varepsilon_2,t,x}(s)|^2 \right]
\leq C_8 \left\{ E \left[ \int_t^T (1 + \sqrt{\varepsilon}) |X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)|^2 dr \right] + E \left[ \int_t^T |f^{\varepsilon_1,t,x}(r) - f^{\varepsilon_2,t,x}(r)|^2 dr \right]
+ (\sqrt{\varepsilon} + \varepsilon) E \left[ \int_t^T \left| \sigma^{\varepsilon_1,t,x}(r) - \sigma^{\varepsilon_2,t,x}(r) \right|^2 dr \right] \right\}
\leq C_8 \left[ \int_t^T \left( (6C_1^2 + 2) |X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)|^2 + 6C_1^2 \left| Y^{\varepsilon_1,t,x}(r) - Y^{\varepsilon_2,t,x}(r) \right|^2 \right) dr \right]
\leq C_8 \left[ \int_t^T \left( 6C_1^2 \int_t^s |X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)|^2 dr + 6C_1^2 \sup_{t \leq s \leq T} |Y^{\varepsilon_1,t,x}(s) - Y^{\varepsilon_2,t,x}(s)|^2 \right) \right].
$$

Once again applying Itô's formula to $(X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r))(Y^{\varepsilon_1,t,x}(r) - Y^{\varepsilon_2,t,x}(r))$, and by virtue of assumption (A2), we have

$$
C_2 E \left[ |X^{\varepsilon_1,t,x}(T) - X^{\varepsilon_2,t,x}(T)|^2 \right]
\leq E \left[ h \left( X^{\varepsilon_1,t,x}(T) \right) - h \left( X^{\varepsilon_2,t,x}(T) \right) \right]
\leq -C_2 \left( 1 + \sqrt{\varepsilon_1} \right) E \left[ \int_t^T \left( |X^{\varepsilon_1,t,x}(r) - X^{\varepsilon_2,t,x}(r)|^2 + |Y^{\varepsilon_1,t,x}(r) - Y^{\varepsilon_2,t,x}(r)|^2 \right) dr \right]
+ (\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}) E \left[ \int_t^T \sigma^{\varepsilon_2,t,x}(r) (Z^{\varepsilon_1,t,x}(r) - Z^{\varepsilon_2,t,x}(r)) dr \right].
$$
Hence, combining (A.22), (A.25), (A1) and (A2), we have

\[ C \]
By Gronwall’s inequality, we also obtain

\[ \text{From Lemma 2.2, we know that} \]

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