\( \mathcal{N}=2 \) String Geometry and the Heavenly Equations

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This paper is dedicated to Professor Jerzy F. Plebański on the occasion of his 75th birthday

Abstract
In this paper we survey some of the relations between Plebański description of self-dual gravity through the heavenly equations and the physics (and mathematics) of \( \mathcal{N}=2 \) Strings. In particular we focus on the correspondence between the infinite hierarchy in the ground ring structure of BRST operators and its associated Boyer-Plebański construction of infinite conserved quantities in self-dual gravity. We comment on “Mirror Symmetry” in these models and the large-\( N \) duality between topological \( \mathcal{N}=4 \) gauge theories in two dimensions and topological gravity in four dimensions. Finally D-branes in this context are briefly outlined.
1. Introduction

The description of the gravitational field at the quantum level is nowadays one of the biggest problems in theoretical physics. At the present time, there are two contending approaches to quantum gravity: *Loop Quantum Gravity* and *String Theory*. In the former one, the basic degrees of freedom of the quantum gravitational field are represented by tiny loops, while in the latter one, they are the fundamental strings or D-branes. These approaches are quite different and a relation between them does not exist at the moment. In the present paper we focus on the latter one.

On the other hand, two of the great insights made by Plebański were two papers published in the middle of seventies [1,2]. In the first paper he proposed a SL(2, C) chiral Lagrangian for the classical degrees of freedom of self-dual gravity. Nowadays, it is well known that this Lagrangian is related through a Legendre transformation to a Hamiltonian approach [3] which is written in terms of the Ashtekar’s variables [4]. The second paper [2] is perhaps less known among the loop quantum gravity people, however it is very well known in the community of mathematical physicists working in exact solutions of Einstein equations, in the theory of integrable systems and in twistor theory. In this paper, Plebański found a description of self-dual gravity degrees of freedom in terms of the so called first heavenly equation in the “weak heavens” gauge or \( B \)-gauge

\[
\Omega_{,pr}(x)\Omega_{,qs}(x) - \Omega_{,ps}(x)\Omega_{,qr}(x) = 1, \tag{1.1}
\]

where \( x \equiv (p,q,r,s) \) and \( \Omega_{,pr} \equiv \frac{\partial \Omega(x)}{\partial p^r} \) etc. This is a second order nonlinear differential equation satisfied by a holomorphic function \( \Omega = \Omega(p,q,r,s) \) in a particular coordinate system \( \{ x \} = \{ p,q,r,s \} \) of a complex four-dimensional manifold \( M \). This equation expresses the condition of Ricci-flatness of the self-dual manifold \( M \) (or \( \mathcal{H} \)-space) and it is an equation for the Kähler potential \( \Omega(x) \). A solution of this equation determines a self-dual

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2 Actually both approaches are conceptually different. In the string theory approach, perturbative nonrenormalizability of general relativity suggest that it is an effective field theory and it doesn’t represent a fundamental theory. Thus, new physical degrees of freedom can be found at higher energies in an extended theory whose low energy limit gives precisely Einstein theory or some generalizations of it. Loop Quantum Gravity suggest that the problem can be faced in a strongest way by finding, at the nonperturbative level, a suitable quantum theory for a Lagrangian which couples general relativity and matter.
metric on $M$ which coincides precisely with the Kähler metric obtained from the Kähler potential.

In the same reference [2] it is given another description of self-dual gravity in a different gauge. This is given by the Plebański second heavenly equation

$$\Theta_{,xx}(\tilde{x})\Theta_{,yy}(\tilde{x}) - (\Theta_{,xy})^2(\tilde{x}) + \Theta_{,xp}(\tilde{x}) + \Theta_{,yq}(\tilde{x}) = 0,$$

where $\tilde{x} \equiv (x, y, p, q)$. This is also a nonlinear second order partial differential equations for a holomorphic function $\Theta = \Theta(x, y, p, q)$ with $\{\tilde{x}\} = \{p, q, x, y\}$ being another coordinate system in the $A$-gauge, where $x \equiv \Omega_{,p}$ and $y \equiv \Omega_{,q}$.

Both are descriptions of the $\mathcal{H}$-space in different “gauges”, thus classically both descriptions are equivalent but they might give rise to two inequivalent quantum descriptions. Various results about $\mathcal{H}$ and some generalizations of them involving null strings called Hyperheavenly spaces or $\mathcal{HH}$-spaces for short, are collected in the very nice survey paper [4]. Such a description has been proved to be equivalent to the Penrose nonlinear graviton [6] and that of the Newman approach reviewed in Ref. [7].

In 1990 in a very important paper, Ooguri and Vafa [8] found that the first heavenly equation was in fact, encoded in the description of string theory with local $\mathcal{N} = 2$ supersymmetry in the worldsheet. Soon after, in Ref. [9], they were able to find a string theory (called the $\mathcal{N} = 2$ heterotic string theory) such that the target space effective field theory consists of self-dual gravity coupled to self-dual Yang-Mills theory, just as in the ordinary heterotic string theory in ten dimensions. A different description of $\mathcal{N} = 2$ strings in terms of a large $N$ limit of a WZW field theory was proposed in Ref. [10]. The connection between $\mathcal{N} = 2$ self-dual supergravity and $\mathcal{N} = 2$ super-WZW was worked out in Ref. [11]. $\mathcal{N} = 2$ strings have been also studied in relation to $\mathcal{W}$ gravity and SU($\infty$) Yang-Mills instantons [12].

The present paper is dedicated to Professor Jerzy F. Plebański on the occasion of his 75th birthday. Although exact solutions of the Einstein equations have been his constant preoccupation in mathematical-physics, he has also made some of the leading contributions to self-dual gravity whose formalism was shown to be relevant to describe the geometry of $\mathcal{N} = 2$ strings. I hope that this modest contribution to understand these systems presented here seems to be appropriate for this occasion.
1.1. $\mathcal{N} = 2$ Strings and Scattering Amplitudes

Set up

The purpose of this section will be to survey some geometry of the complex spacetime Einstein self-dual geometry ($\mathcal{H}$-spaces) viewed as the target space geometry in the context of the worldsheet theory of $\mathcal{N} = 2$ strings. This theory is, in fact, a $\mathcal{N} = 2$ supersymmetric nonlinear-sigma model with flat target space and it was proposed by the first time in Ref. [13]. We will follows mainly in Ref. [8], however some very nice expository articles can be found in Refs. [14,15]. We will focus mainly in closed strings, however some comments for open strings will be given.

Closed string theory with $\mathcal{N} = 2$ worldsheet local supersymmetry is given by $\mathcal{N} = 2$ supergravity in two dimensions coupled to $\mathcal{N} = 2$ superconformal matter. This theory is based on an action which has $\mathcal{N} = 2$ superconformal symmetry. For closed strings there are two infinite algebras acting independently on left- and right-sectors whose generators are $(G,G^*,T,J)_L$ and $(\overline{G},\overline{G}^*,\overline{T},\overline{J})_R$, respectively. The quantization of $\mathcal{N} = 2$ string theory with these infinite symmetries and the appropriate counting of ghost fields leads to an anomaly free theory (total central charge vanishing) whose critical dimension $D = 2\mathbb{Q}$. That means that target space $M^D$ has two complex dimensions and target space can have signature $(2,2)$ or $(0,4)$. In the present paper we will focus in backgrounds with signature $(2,2)$, however, occasionally we make some comments on the theory with $(0,4)$ signature. The spectrum for this theory consists of only one complex scalar field on the target space $M^{2,2}$. All oscillator excitations are vanishing and there is not spacetime supersymmetry.

After gauge fixing all supergravity fields we keep with a theory for a $\mathcal{N} = 2$ superfield $X : S \rightarrow M^D$ which is the $\mathcal{N} = 2$ supermatter and $S$ is a $\mathcal{N} = 2$ super-Riemann surface. This theory is based on the action

$$S_0 = \int_S \frac{d^2z}{\pi} d^2\theta d^2\overline{\theta}K_0(X,\overline{X}),$$

(1.3)

where $X^i \ (i = 1,2)$ are two chiral superfields given by $X^i(z,\overline{z};\theta,\overline{\theta}) = x^i(z,\overline{z}) + \psi^i_R(z,\overline{z})\theta^- + \psi^i_L(z,\overline{z})\overline{\theta^-} + F^i(z,\overline{z})\theta^-\overline{\theta^-}$, where $z = x - \theta^+\theta^-$ and $K_0(X,\overline{X}) = X^1\overline{X}^1 - X^2\overline{X}^2$ is the Kähler potential.

For $\mathcal{N} = 2$ matter we have two possibilities: (i) Two chiral superfields $X^i$ (and $\overline{X}^i$) ($i = 1,2$ and $\overline{i} = \overline{1},\overline{2}$) where the fermions $\psi^i_L(z,\overline{z})$ and $\psi^i_R(z,\overline{z})$ are charged under the $\mathbb{U}(1)$ group. The chiral superfields satisfy $D_+X^i = 0$, and $D_+\overline{X}^i = 0$ and (ii) two chiral
superfields $X^i$ ($i = 1, 2$) and two *twisted* chiral superfields $\Sigma^a$ ($a = 1, 2$), satisfying:

$$\overline{D}_\mp X^i = 0, \ D_\mp \overline{X}^i = 0 \text{ and } D_+ \Sigma^a = 0, \ D_- \overline{\Sigma}^a = 0, \ \overline{D}_+ \overline{\Sigma}^a = 0.$$  

The first selection leads to the description of the usual self-dual geometry \[8\]. The second possibility leads to the description of a self-dual geometry with hermitian (non-Kahlerian) structure and torsion \[17,18\].

**Scattering Amplitudes of Closed, Open and Open/Closed $\mathcal{N} = 2$ Strings**

For the closed string the only non-vanishing scattering amplitudes is the three-point function on the sphere (fig. 1) \[8\]. This is given by:

$$A_{ccc} = \left\langle V_c|_{\theta = 0}(0) \cdot \int_S d^2\theta d^2\overline{\theta} V_c(1) \cdot V_c|_{\theta = 0}(\infty) \right\rangle = \kappa c_{12}^2 \neq 0, \quad (1.4)$$

where $c_{12} \equiv k_1 \overline{k}_2 - \overline{k}_1 k_2$ and $V_c = \frac{\pi}{i} \exp \left( k \cdot \overline{X} + \overline{k} \cdot X \right)$ is the vertex operator for the complex scalar field $X$.

**Fig. 1:** Three-point scattering amplitude on the sphere is the only non-vanishing amplitude.

Four point function is given by

$$A_{cccc} = \left\langle V_c|_{\theta = 0}(0) \cdot \int_S d^2\theta d^2\overline{\theta} V_c(z) \cdot \int_S d^2\theta d^2\overline{\theta} V_c(1) \cdot V_c|_{\theta = 0}(\infty) \right\rangle, \quad (1.5)$$

which is vanishing on-shell. Higher order amplitudes for higher point and higher genus correlation functions are vanishing as well \[19,20\].

For open strings the story is quite similar. Three-point scattering amplitude at the tree level (on the disk) is given by \[21\]

\[3\] The existence of twisted chiral superfields was pointed out by the first time in Ref. \[16\]. The name of “twisted chiral multiplets” was first introduced in Ref. \[17\].

\[4\]
where \(a, b, c\) stand for the Chan-Paton indices and \(V_0 \equiv g \exp(k \cdot X + \bar{k} \cdot X)\) is the corresponding vertex operator. Four-point scattering amplitude \(A_{oooo}\) vanishes on-shell as well for the same reasons.

For the case of mixing open/closed scattering amplitudes we have for the case of three-point function \([22]\)

\[
A_{ooc} \propto \int_{-\infty}^{+\infty} dx \left\langle V_0|_{\theta=0}(x) \cdot \int d^2 \theta d^2 \bar{\theta} V_c(z = i) \cdot V_0|_{\theta=0}(\infty) \right\rangle = \kappa \epsilon_{12}^a \delta^{ab} \neq 0. \tag{1.7}
\]

While that of \(A_{oooc}\) also vanishes on-shell. So, \(S\)-matrix is almost trivial and therefore the theory is almost topological.

1.2. Target Space Low Energy Effective Action

Non-vanishing three-point function on the sphere determines the low energy effective action in the three cases previously considered.

For closed strings we have that Eq. (1.4) gives rise to an effective field theory for a complex scalar field \(\Omega\) whose dynamics is given by the Plebański action \([5]\)

\[
S_P = \int_{M^{2,2}} \left( \partial \Omega \overline{\partial} \Omega + \frac{1}{3} \partial \overline{\partial} \Omega \wedge \partial \overline{\partial} \Omega \right). \tag{1.8}
\]

Equation of motion for this action is precisely the first heavenly equation written in a slightly different form

\[
\partial^i \overline{\partial}_i \Omega - 2\kappa \partial \overline{\partial} \Omega \wedge \partial \overline{\partial} \Omega = 0. \tag{1.9}
\]

Here, the complex scalar field \(\Omega\) can be regarded as a perturbation of the Kähler potential around the flat space \(M^{2,2}\). The metric of \(M^{2,2}\) is given by \(g_{ij} = \partial_i \overline{\partial}_j(x_k \overline{x}^k + 4\kappa \Omega)\). This metric satisfies the condition \(\det(g_{ij}) = -1\). Therefore the corresponding Ricci tensor is given by \(R_{ij} = \partial_i \overline{\partial}_j(\log|\det g_{ij}|) = 0\). Thus the first heavenly equation is equivalent to the Ricci-flatness condition. By Atiyah-Hitchin-Singer theorem \([23]\), a four-dimensional Riemannian space is self-dual if and only if it is Kähler and Ricci-flat. Then, the space \(M^{2,2}\) should be a Kähler and Ricci-flat complex manifold.

For the open string case, the low energy effective action is given by

\[
A_{oooo} = \left\langle V_0|_{\theta=0}(0) \cdot \int d^2 \theta V_0(1) \cdot V_0|_{\theta=0}(\infty) \right\rangle = gc_{12}(-if^{abc}), \tag{1.6}
\]
\[ S^o_{\text{eff}} = \tilde{S}^o_{\text{eff}} + \int d^4x \left(-\frac{g^2}{6} f^{abc} f^{x} \partial^i \phi^a \cdot \phi^b \cdot \nabla^c \phi^c \cdot \phi^d \right), \] (1.10)

where

\[ \tilde{S}^o_{\text{eff}} = \int d^4x \left( \frac{1}{2} \partial^i \phi^a \cdot \nabla^i \phi^a - \frac{i g}{3} f^{abc} \phi^a \cdot \partial^i \phi^b \cdot \nabla^i \phi^c \right) + O(\phi^4). \] (1.11)

The equation of motion of this action is precisely the Yang equation [24]

\[ \partial^i \left( \exp(-2ig\phi) \partial^i \exp(2ig\phi) \right) = 0, \] (1.12)

where \( \phi \) is a hermitian matrix. Yang equation is equivalent to self-dual Yang-Mills equation \( \tilde{F}_{ij} = +F_{ij} \).

Up to here we have selected the option (i) which consist of two \( \mathcal{N} = 2 \) chiral supermultiplets which give rise to a self-dual geometry of the target space. In the case we select the option (ii), that is, one chiral superfield \( X \) and one twisted chiral superfield \( \Sigma \), we get a non-Kählerian target space and therefore a hermitian self-dual manifold with torsion [17,18]. The torsion is generated by the anti-symmetric \( B \)-field. The case when this field vanishes leads to a free torsion hermitian self-dual manifold. In this case the perturbative analysis of the scattering amplitudes for the action \( S = \int \frac{d^2z}{\pi} d^4K_0(X, \Sigma) \) might lead us to the new version of the heavenly equation found recently by Plebański and Przanowski [25]

\[ \Omega_{,pr} \Omega_{,qs} - \Omega_{,ps} \Omega_{,qr} - \frac{1}{(1 - iqs)^2} (2\Omega \Omega_{,pr} - \Omega_{,p} \Omega_{,r}) = 0, \] (1.13)

where \( \Omega = \Omega(p, q, r, s) \) with \( x^1 = p, x^2 = q, \sigma^1 = r \) and \( \sigma^2 = s \).

[Remark\(^4\): The computation of the effective action arising from a world-sheet action, given by \( \mathcal{N} = 2 \) sigma model with chiral and twisted chiral superfields, was done by Gluck, Oz and Sakai in Ref. [26]. The result is the Laplace equation.]

### 2. Hierarchy of Conserved Quantities

All properties of the target space have a worldsheet interpretation. For instance, the infinite hierarchy of conserved quantities in self-dual gravity are interpreted in terms of the spectral flow of the a ground ring of ghost number zero operators in the chiral BRST cohomology of the closed \( \mathcal{N} = 2 \) string theory [27].

\(^4\) I would like to thank Yaron Oz for pointing out to me these results.
It was found in Ref. [28] a deep interrelation between the hidden symmetries of self-dual gravity generated by abelian symmetries and the ground ring of ghost number zero operators coming from the BRST-cohomology. These operators are, in fact, related to the picture changing operators and to the spectral flow. In this subsection we briefly review that construction (for details see Ref. [28]).

Within the Superconformal Algebra (SCA) structure one can construct a BRST current \( j_{BRST} \) for the left-moving SCA and \( \bar{j}_{BRST} \) for the right-moving one. These currents are written in terms of the generators \((T, J, G)_L\) and \((\bar{T}, \bar{J}, \bar{G})_R\) of the corresponding SCA. Thus, one can define the BRST charge \( Q_{BRST} \) as \( Q_{BRST} = \oint (j_{BRST} + \bar{j}_{BRST}) \). Each \( \mathcal{N} = 2 \) SCA has two sets of picture families of charges \((\pi_+, \pi_-)\) and \((\bar{\pi}_+, \bar{\pi}_-))\) connecting the families of Fock spaces and giving rise to different pictures. The first systematic study of the BRST cohomology of \( \mathcal{N} = 2 \) strings with picture changing was done in Ref. [29].

Another necessary ingredient is the spectral flow operator \( S \) which interpolate between the \( NR \) and \( R \) sectors of the theory acting independently on left- and right-movers. The \( \mathcal{N} = 2 \) spectral flow operator was first constructed explicitly in Ref. [30].

Closed \( \mathcal{N} = 2 \) strings needs from subsidiary constraints \((b_0 - \bar{b}_0)|\psi\rangle = 0\) and \(b'_0|\psi\rangle = 0 = \bar{b}'_0|\psi\rangle\) with \((\pi_+, \pi_-) \in \mathbb{Z} \times \mathbb{Z}\). Then, physical states are organized into cohomology classes called relative chiral BRST-cohomology \( H_{BRST}^*(g, \pi) \), where \( g \) is the ghost number and \( \pi \) is the picture changing charge. The special case when the ghost number is zero \( H_{BRST}^*(g = 0, \pi) \) has the structure of a ring and it is called the ground ring [31]. This ring structure is due to the fact that the multiplication of cohomology classes preserves the ghost number and the picture gradings. This product of cohomology classes is translated into the OPE of operators. In this product a typical element of the the ground ring is given by

\[
\mathcal{O}_{m,n}^\ell = (X_+)^{j+m} \cdot (Y_-)^{j-m} \cdot H^\ell, \tag{2.1}
\]

where \( \ell = 0, 1, \ldots, 2j, m, n-m = -j, j+1, \ldots = j \) and \( j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \). Here \( X_+, X_- \) and \( Y_- \) are suitable picture-raising operators satisfying \( X_+ \cdot H = X_- \cdot S \) and \( Y_- = X_+ \cdot S^{-1} = X_- \cdot H^{-1} \) with \( H \) being a picture-neutral formal operator and \( S \) being the spectral flow operator. The changing picture operators \( \Pi \pm \) are playing the role of derivations in the polynomial algebra of two-variable functions \( x, y \). Thus defining
\[ x \equiv X_+ = O_0^0 \cdot \frac{1}{\ell} + \frac{1}{\ell}, \quad y \equiv X_- = Y_- \cdot H = O_0^0 \cdot \frac{1}{\ell} \cdot H. \] (2.2)

In terms of these variables we have

\[ O_{m,n}^\ell = x^{j+m} \cdot y^{-m} \cdot h^{-\ell} \cdot \beta, \quad \Pi_+ + 1 = x \frac{\partial}{\partial x}, \quad \Pi_- + 1 = y \frac{\partial}{\partial y}. \] (2.3)

**Geometric Structure of the Ground Ring Manifolds**

As we have seen before the physical states of the theory generate the ground ring structure which encodes all relevant information about the symmetries of the theory. Moreover, much of the power lies in that actually there is a geometrical structure underlying the BRST approach. This is precisely the symplectic geometry \[32\] and the theory of homotopy Lie algebras \[33\]. Let \( C^\infty(A_L) \) be the chiral ground ring with the operators \( O_{u,n} = h \cdot x^{u+n} \cdot y^{u-n} \) are precisely the polynomial functions on the \( x - y \)-plane \( A_L \). This is a two-dimensional symplectic manifold with symplectic two-form \( \omega = dx \wedge dy \). Thus, the pair \( (A_L, \omega) \) is a two-dimensional symplectic manifold with the symplectic two-form \( \omega \). Similarly we can say the same about the right-movers, i.e., \( (A_R, \tilde{\omega}) \), where \( \tilde{\omega} = d\bar{x} \wedge d\bar{y} \).

Now we review briefly the result of \[34\]. Given \( A_L \) and \( A_R \) we define the full quantum ground ring manifold \( W \) to be the product \( W = A_L \times A_R \). Define \( \rho : A_L \times A_R \to A_L \) and \( \tilde{\rho} : A_L \times A_R \to A_R \) to be the corresponding projections. Operators \( \Pi_\pm \) can be interpreted as Hamiltonian (or volume-preserving) vector fields on the the full quantum ground ring manifold \( W \).

**2.2. Boyer-Plebański Construction**

In this subsection, we re-derive the same structure in a different way making emphasis only on the fact that the full quantum ground ring manifold \( W \) is merely the product manifold \( A_L \times A_R \), and that the chiral ground ring manifolds are symplectic manifolds with symplectic two-forms \( \omega = dx \wedge dy \) and \( \tilde{\omega} = d\bar{x} \wedge d\bar{y} \) respectively. For this end we use the construction of \[35\] and \[36\].

**The Construction**

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Consider the flat chiral complexified ground ring manifolds \( \mathcal{A}^\mathcal{C}_L \) and \( \mathcal{A}^\mathcal{C}_R \) and the complexified full ground ring manifold \( \mathcal{W}^\mathcal{C} = \mathcal{A}^\mathcal{C}_L \times \mathcal{A}^\mathcal{C}_R \). Since \( (\mathcal{A}^\mathcal{C}_L, \omega) \) and \( (\mathcal{A}^\mathcal{C}_R, \tilde{\omega}) \) are symplectic manifolds one can show that \( \mathcal{W}^\mathcal{C} \) is also a symplectic manifold with symplectic form given by \( \rho^*\omega - \tilde{\rho}^*\tilde{\omega} \) where \( \rho : \mathcal{W}^\mathcal{C} \to \mathcal{A}^\mathcal{C}_L \) and \( \tilde{\rho} : \mathcal{W}^\mathcal{C} \to \mathcal{A}^\mathcal{C}_R \) are the corresponding projections.

Now, let \( T^r\mathcal{A}^\mathcal{C}_L \), \( T^r\mathcal{A}^\mathcal{C}_R \) and \( T^r\mathcal{W}^\mathcal{C} \) be the \( r \)-th order holomorphic tangent bundles of \( \mathcal{A}^\mathcal{C}_L \), \( \mathcal{A}^\mathcal{C}_R \) and \( \mathcal{W}^\mathcal{C} \), respectively. Then we have the following sequence of projections for \( \mathcal{A}^\mathcal{C}_L \) as follows:

\[
\ldots \to T^r\mathcal{A}^\mathcal{C}_L \to T^{r-1}\mathcal{A}^\mathcal{C}_L \to \ldots \to T^1\mathcal{A}^\mathcal{C}_L \to T^0\mathcal{A}^\mathcal{C}_L \cong \mathcal{A}^\mathcal{C}_L
\]

(2.4)

and similarly for \( \mathcal{A}^\mathcal{C}_R \) and \( \mathcal{W}^\mathcal{C} \).

Following [36] one can define functions \( O^{(\lambda)}_{u,n} \), \( \overline{O}^{(\lambda)}_{u,n} \) and \( V^{(\lambda)}_{u,n,n'} \) vector fields \( Y^{(+\lambda)}_{s,n} \), \( J^{(+\lambda)}_{s,n} \), as well as \( J^{(\lambda)}_{u,n,n'} \), \( J^{(\lambda)}_{n,n'} \), and differential forms \( \omega^{(\lambda)} \), \( \tilde{\omega}^{(\lambda)} \), (with \( \lambda = 0, 1, 2, \ldots, r \)) on the \( r \)-order holomorphic tangent (or cotangent) bundles \( T^r\mathcal{A}^\mathcal{C}_L \), \( T^r\mathcal{A}^\mathcal{C}_R \) and \( T^r\mathcal{W}^\mathcal{C} \). For example \( O^{(\lambda)}_{u,n}(j_r \circ \psi(0)) = \frac{1}{\lambda!} \frac{d^\lambda (O_{u,n} \circ \psi)}{dt^\lambda} |_{t=0} \) and \( V^{(\lambda)}_{u,n,n'}(j_r \circ \psi(0)) = \frac{1}{\lambda!} \frac{d^\lambda (V_{u,n,n'} \circ \psi)}{dt^\lambda} |_{t=0} \), where \( j_r(\psi) \), is the \( r \)-jet of the holomorphic curve \( \psi \). Then we can define the OPE for these objects as

\[
O^{(\lambda)}_{u,n}(j_r \circ \psi(0)) \cdot O^{(\lambda')}_{u,n'}(j_r \circ \psi(z)) = \frac{1}{\lambda!} \frac{\partial^\lambda (O_{u,n} \circ \psi)}{\partial s^\lambda} |_{s=0} \cdot \frac{1}{\lambda'!} \frac{\partial^{\lambda'} (O_{u,n'} \circ \psi)}{\partial t^{\lambda'}} |_{t=z} .
\]

(2.5)

From the theorem 2 of [36] one can see that \( (T^r\mathcal{A}^\mathcal{C}_L, \omega^{(\lambda)}) \) and \( (T^r\mathcal{A}^\mathcal{C}_R, \tilde{\omega}^{(\lambda)}) \) are symplectic manifolds. Therefore, we can define another symplectic manifold \( T^r\mathcal{W}^\mathcal{C} \) with symplectic two form given by \( \rho^*\omega^{(r)} - \tilde{\rho}^*\tilde{\omega}^{(r)} \). One can also establish the bundle diffeomorphism \( \Phi : T^r\mathcal{W}^\mathcal{C} \to \rho^*T^r\mathcal{A}^\mathcal{C}_L \oplus \tilde{\rho}^*T^r\mathcal{A}^\mathcal{C}_R \).

The following diagram summarizes our construction:

\[
\begin{array}{ccc}
T^2\mathcal{W}^\mathcal{C} & \to & \rho^*T^2\mathcal{A}^\mathcal{C}_L \oplus \tilde{\rho}^*T^2\mathcal{A}^\mathcal{C}_R \\
\pi_1^2 & \downarrow & \rho_1^2 \\
T\mathcal{W}^\mathcal{C} & \to & \rho^*T^2\mathcal{A}^\mathcal{C}_L \\
\pi & \downarrow & \tilde{\rho}^2 \\
\mathcal{W}^\mathcal{C} & \to & \tilde{\rho}^*T^2\mathcal{A}^\mathcal{C}_R
\end{array}
\]

In order to show the existence of self-dual gravity structures on the full quantum ground ring manifold we restrict ourselves to the case \( r = 2 \). As it is mentioned in [36] \( (\rho^*T^2\mathcal{A}^\mathcal{C}_L, \rho^*\omega^{(2)} - \tilde{\rho}^*\tilde{\omega}^{(0)}) \) and \( (\rho^*T^2\mathcal{A}^\mathcal{C}_R, \rho^*\omega^{(0)} - \tilde{\rho}^*\tilde{\omega}^{(2)}) \) are also symplectic manifolds,
where \( \omega^{(0)} = \omega = dx \wedge dy, \) \( \tilde{\omega}^{(0)} = \tilde{\omega} = d\tilde{x} \wedge d\tilde{y}, \) \( \omega^{(2)} = dx \wedge dy^{(2)} + dx^{(1)} \wedge dy^{(1)} + dx^{(2)} \wedge dx, \) and \( \tilde{\omega}^{(2)} = d\tilde{x} \wedge d\tilde{y}^{(2)} + d\tilde{x}^{(1)} \wedge d\tilde{y}^{(1)} + d\tilde{x}^{(2)} \wedge d\tilde{x}. \) We need only to consider the manifold \( \rho^*T^2\mathcal{A}_L^C. \)

Consider the following sequences

\[
\mathcal{L} \xrightarrow{i} T^2\mathcal{W}^C \xrightarrow{\pi^2} T\mathcal{W}^C \xrightarrow{\pi^1} \mathcal{W}^C,
\]

and

\[
\mathcal{L} \xrightarrow{i} T^2\mathcal{W}^C \xrightarrow{\rho^2} \rho^*T^2\mathcal{A}_L^C \xrightarrow{\rho^1} \mathcal{W}^C,
\]

where \( \mathcal{L} \) is a horizontal Lagrangian submanifold of both \( T\mathcal{W}^C \) and \( \rho^*T^2\mathcal{A}_L^C. \)

Let \( \sigma : W \to T^2\mathcal{W}^C, W \subset \mathcal{W}^C, \) be a holomorphic section such that \( i(\mathcal{L}) = \sigma(W), \) where \( i : \mathcal{L} \to T^2\mathcal{W}^C \) is an injection.

Let \( \Gamma(W) \) be the set of all holomorphic sections of \( T^2\mathcal{W}^C \to \mathcal{W}^C \) such that \( \pi^2 \circ i(\mathcal{L}) \) and \( \rho^2 \circ i(\mathcal{L}) \) are horizontal Lagrangian submanifolds of \( T\mathcal{W}^C \) and \( \rho^*T^2\mathcal{A}_L^C, \) respectively.

Thus \( \Gamma(W) \) is defined by the equations

\[
\sigma^*_1(\omega^{(1)} - \tilde{\omega}^{(1)}) = 0, \quad \text{on} \quad W \subset \mathcal{W}^C,
\]

\[
\sigma^*_2(\omega^{(2)} - \tilde{\omega}^{(0)}) = 0, \quad \text{on} \quad W \subset \mathcal{W}^C,
\]

[Notice that we have omitted the pull-back \( \rho^* \) and \( \tilde{\rho}^* \) in these formulas.]

where \( \omega^{(1)} = dx \wedge dy^{(1)} + dx^{(1)} \wedge dy, \) \( \tilde{\omega}^{(1)} = d\tilde{x} \wedge d\tilde{y}^{(1)} + d\tilde{x}^{(1)} \wedge d\tilde{y}, \) \( \sigma_1 = \pi^2 \circ \sigma \) and \( \sigma_2 = \rho^2 \circ \sigma. \)

Now the problem arises of how \( \sigma \in \Gamma(W) \) determines a self-dual gravity structure on the open sets \( W \) of the full quantum ground ring manifold \( \mathcal{W}^C. \)

The solution of this problem is given by the following

**Theorem [36]:** Let \( \sigma : W \to T^2\mathcal{W}^C, W \subset \mathcal{W}^C, \) be a holomorphic section. The triplet \( (\omega, \tilde{\omega}, \Omega_0) = (\sigma^*\omega^{(0)}, \sigma^*\tilde{\omega}^{(0)}, \sigma^*\omega^{(1)}) \) defines a self-dual structure on \( W \) if and only if there exist a choice of a holomorphic section \( \sigma \) such that \( \sigma^*(\omega^{(1)} - \tilde{\omega}^{(1)}) = 0 \) and \( \sigma^*(\omega^{(2)} - \tilde{\omega}^{(0)}) = 0. \)

(For the proof see [36]). Thus, the desired self-dual gravity structures arise in a natural manner from the mathematical structure of the quantum states in \( \mathcal{N} = 2 \) string theory.

Taking \( \Omega_0 = \sigma^*[dx \wedge dy^{(1)} + dx^{(1)} \wedge dy] \) we arrive at the first heavenly equation

\[ \text{heavenly equation} \]
\[ \Omega_0 \wedge \Omega_0 + 2\omega \wedge \bar{\omega} = 0. \quad (2.10) \]

This can be extended to the cases with \( r \geq 3 \). Then, by using the projective limit one can formulate the problem in terms of the infinite-dimensional tangent bundle \( T^\infty \mathcal{W}^L = \rho^* T^\infty \mathcal{A}_L^C \oplus \bar{\rho}^* T^\infty \mathcal{A}_R^C \) (for details see \[36\]). It can be proved that given \( (\mathcal{W}^C, \rho^* \omega - \bar{\rho}^* \bar{\omega}) \) a symplectic manifold \( (T^\infty \mathcal{A}_L^C, \omega_2(t)) \) turns out to be a formal symplectic manifold, where \( \omega_2(t) = \sum_{k=0}^{\infty} \pi_k \omega(k) t^k; \ t \in \mathbb{C} \), and \( \pi_k : T^\infty \mathcal{A}_L^C \rightarrow T^k \mathcal{A}_L^C \) is the natural projection.

By the Proposition 2 of \[36\] we observe that \( (T^\infty \mathcal{W}^C, \omega(t)) \) is a formal symplectic manifold with

\[ \omega(t) = t^{-1} \rho^* \omega_2(t) - t\bar{\omega}_2(t^{-1}), \quad (2.11) \]

where \( t \in \mathbb{C}^* = \mathbb{C} - \{0\} \) and \( T^\infty \mathcal{W}^C = T^\infty \mathcal{A}_L^C \times T^\infty \mathcal{A}_R^C = T^\infty (\mathcal{A}_L^C \times \mathcal{A}_R^C) = \rho^* T^\infty \mathcal{A}_L^C \times \bar{\rho}^* T^\infty \mathcal{A}_R^C \).

**Curved Twistor Construction on Full Quantum Ground Ring Manifolds**

Consider the formal symplectic manifold \( (T^\infty \mathcal{W}^C, \omega(t)) \). Since \( \mathcal{A}_L^C \) and \( \mathcal{A}_R^C \) are diffeomorphic, we have \( T^\infty \mathcal{W}^C = T^\infty \mathcal{A}_L^C \times T^\infty \mathcal{A}_R^C \). Define the holomorphic maps

\[ \hat{D} = (D, I) : T^\infty \mathcal{A}_L^C \times \mathbb{C}^* \rightarrow T^\infty \mathcal{A}_L^C \times \mathbb{C}^*, \quad (2.12) \]

where \( I(t) = t^{-1} \) and the graph of the diffeomorphism \( D, grD \), can be identified with some local section \( grD = \sigma' : W \rightarrow T^\infty \mathcal{W}^C \) such that \( \sigma'^* \omega(t) = 0 \). From Eq. \( (2.11) \), this last relation holds if and only if

\[ D^* \omega_2(t^{-1}) = t^{-2} \omega_2(t). \quad (2.13) \]

Consider now a local section \( \sigma'' \) of the formal tangent bundle \( T^\infty \mathcal{W}^C \rightarrow \mathcal{W}^C \) on an open set \( W \subset \mathcal{W}^C \) such that \( \sigma''^* \omega(t) = 0 \). For \( t \in \mathbb{C}^* \) we have \( \sigma'' = (\Psi^A(t), \bar{\Psi}^B(t^{-1})) \).

Now assume that \( \Psi^A(t) \) and \( \bar{\Psi}^B(t^{-1}) \) converge in some open disks \( \mathcal{U}_0 \) and \( \mathcal{U}_\infty \) (\( 0 \in \mathcal{U}_0 \) and \( \infty \in \mathcal{U}_\infty \)) respectively, such that \( \mathcal{U}_0 \cap \mathcal{U}_\infty \neq \phi \). Consequently, the functions \( \Psi^A : t \mapsto \Psi^A(t) \) and \( \bar{\Psi}^B : s \mapsto \bar{\Psi}^B(s) \) define local holomorphic sections of the twistor space \( \mathcal{T} \). Due to the condition \( (2.13) \) defining the self-dual structure on the quantum ground ring manifold \( \mathcal{W}^C \) we get the transition functions for a global holomorphic section \( \Psi \in \tilde{\Gamma}(\mathcal{T}) \). Thus one can recover the Penrose twistor construction \[3\]. Of course the inverse process is also possible.
3. Mirror Symmetry

The theory of $\mathcal{N} = 2$ strings also presents many features of Calabi-Yau three-folds $X$. One of them is mirror symmetry. However this symmetry is realized in a different form than in Calabi-Yau three-folds $X$. For CY-3 fold mirror pair $(X,Y)$ the fact that the Betti number $h^{2,0}(X) = 0$ leads to a local factorization of the moduli space of Kähler $\mathcal{M}_{h^{1,1}}(X)$ and complex structures $\mathcal{M}_{h^{2,1}}(Y)$ i.e., $\mathcal{M} = \mathcal{M}_{h^{1,1}}(X) \times \mathcal{M}_{h^{2,1}}(Y)$. Moreover, in topological sigma models on a CY 3-fold there are two possible ways to twist these class of models. These are the $A(X)$ and $B(X)$ models [37]. Mirror symmetry is realized through the interchanging of $A(X)$ and $B(X)$ models.

In $(0,4)$ real four-dimensional mirror pairs $(\tilde{X}, \tilde{Y})$, for instance, for the $\tilde{X} = K3$ surface this is not true, since $h^{2,0}(\tilde{X}) = 1$, the moduli spaces are mixed and it cannot be factorized [38]. However the mirror map still interchanges the moduli of Kähler and complex structures. At the level of the metric, two metrics $g(\tilde{X})$ and $g^*(\tilde{Y})$, under the mirror map, are related by $g^* = \phi^*(g)$, with $\phi$ a non-trivial automorphism of the moduli space of metrics. For the present case, i.e., in the non-compact case with signature $(2,2)$, this automorphism corresponds to a change of gauge which gives the two different metrics which are solutions of the first and the second heavenly equations (1.1) and (1.2). Thus first and second heavenly equations are related by a mirror map of a non-compact CY two-fold or self-dual space i.e., $(\tilde{X}, g_\Omega) \to (\tilde{Y}, g_\Theta)$. Thus, we have

$$\mathcal{A}(\tilde{X}, g_\Omega) \to \mathcal{B}(\tilde{Y}, g_\Theta), \quad (3.1)$$

where $\Omega$ and $\Theta$ are solutions of the first and the second heavenly equations (1.1) and (1.2) respectively. These considerations are confirmed both for closed and open strings by the explicit computation of the worldsheet instantons in Ref. [39].

For open strings in the $\mathcal{A}$ model it gives the Leznov-Parkes equations [40] (instead of the Yang’s equation (1.12)) and for the closed string it gives rise to the second heavenly equation (1.2). In what follows we will consider only the $\mathcal{A}$-twist and we will give some explicit examples of the so called large-$N$ duality which is a manifestation of the open/closed duality [41,42] (see [43], for a recent review). Thus, we only will consider issues related to the second heavenly equation (1.2).
4. Large $N$ Duality and Open/Closed Duality

For CY 3-folds also there are nice correspondences besides of the mirror symmetry. For $\mathcal{A}$ and $\mathcal{B}$ models there is an internal correspondence between open and closed topological strings. This is a topological open/closed duality which is also known as large $N$-duality. For instance, in $\mathcal{A}$ models it interchanges (from the open string sector) the Chern-Simons gauge theory with the Kähler gravity (in the closed string sector) \[^4\text{4}^\]. For $\mathcal{B}$ models it interchanges holomorphic Chern-Simons gauge theory from the open sector with Kodaira-Spencer gravity from the closed one. This duality has refinements once that one incorporates topological D-branes. Gopakumar and Vafa \[^4^2\] showed that the open sector can be regarded as topological D-branes wrapping a deformed conifold and this sector is dual to a topological closed string (without D-branes) propagating on the resolved conifold.

For non-compact CY 2-folds this duality is also true. One example is precisely, the $\mathcal{N} = 4$ topological open string. We start with the Park-Husain heavenly equation for the holomorphic function $\Theta = \Theta(x, y, p, q)$, describing self-dual spacetime \[^4^3\]

$$\partial_x^2 \Theta + \partial_y^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_P = 0,$$

(4.1)

where the Poisson bracket is given by $\{F, G\}_P = (\frac{\partial F}{\partial p} \cdot \frac{\partial G}{\partial q} - \frac{\partial F}{\partial q} \cdot \frac{\partial G}{\partial p})$. On the self-dual space $M^{2,2} = \mathcal{X} \times \mathcal{Y}$ with $(x, y)$ and $(p, q)$ the local coordinates on $\mathcal{X}$ and $\mathcal{Y}$ respectively
and $\Theta(x,y,p,q) = \Theta(x^i,y^i)$. Equation (4.1) can be derived from the Lagrangian $S_\infty = \int_{\mathcal{X} \times \mathcal{Y}} dx dy dp dq \mathcal{L}_\infty$ where the Lagrangian is given by

$$L_\infty = \frac{1}{2} (\partial_x \Theta)^2 + (\partial_y \Theta)^2 - \frac{1}{3} \Theta \{\partial_x \Theta, \partial_y \Theta\} P.$$  

Equation of motion (4.1) are classically equivalent to the following system

$$F_{xy} = \partial_x A_y - \partial_y A_x + \{A_x, A_y\} P = 0,$$

$$\partial_x A_x + \partial_y A_y = 0, \quad A_x = -\partial_y \Theta, \quad A_y = \partial_x \Theta.$$  

Equations (4.1) are classically equivalent to the following system

We can identify a sdiff(\mathcal{Y})-valued flat connection. As a Lie algebra sdiff(\mathcal{Y}) is isomorphic to su(∞). Thus we have a large-$N$ limit of a two-dimensional gauge theory corresponding to the flat connection. At the classical level we may have equivalently the system

$$\partial_x (g^{-1} \partial_x g) + \partial_y (g^{-1} \partial_y g) = 0,$$

$$F_{xy} = 0, \quad A_x = g^{-1} \partial_x g, \quad A_y = g^{-1} \partial_y g.$$  

Equations (4.4) can be obtained from the Lagrangian

$$I = \int_{\mathcal{X}} \text{Tr}(g^{-1} dg)^2,$$  

where $\text{Tr}(\cdot) = -\int_{\mathcal{Y}} dp dq (\cdot)$. For finite group, this result can be interpreted as the effective field theory on a topological D1 brane wrapped in the two-sphere $\mathcal{X} = S^2$ in the target space $\mathcal{X} \times \mathcal{Y} = T^* S^2$. This is precisely the deformed conifold.

Something similar can be carried over to the six-dimensional version of the heavenly equation [47] for the holomorphic function $\Theta(x,y,\tilde{x},\tilde{y},p,q)$

$$\partial_x \partial_x \Theta + \partial_y \partial_y \Theta + \{\partial_x \Theta, \partial_y \Theta\} P = 0.$$  

These systems are classically equivalent but at the quantum level they have very different properties [10].
5. D-branes in $\mathcal{N} = 2$ Strings

In this section we briefly survey the problem of having D-branes in $\mathcal{N} = 2$ strings. This is a very interesting and important topic nowadays.

It is known that $\mathcal{N} = 2$ strings has not space-time supersymmetry. Thus, the usual definition of D-branes as BPS states is not longer valid. However we still have the notion of space-time self-duality which, in many respects, plays the role of a BPS condition.

In Ref. [48], it was argued that the effective field theory on a “D-brane” of $\mathcal{N} = 2$ string theory is determined by dimensional reductions of the self-duality Yang-Mills theory on $M^2$. The different reduced theories give rise to an spectrum of integrable systems in various dimensions. For instance in dimension three it is given by the Bogomolny equation. In two dimensions it is given by the Hitchin system, in one dimension by the Nahm equation and in zero dimensions by the ADHM equation. In the literature it was discovered that distinct $\mathcal{N} = 2$ must arise from the existence of models that use mixtures of chiral and twisted chiral multiplets [49].

Very recently, in Refs. [50,26], this fact was rediscovered in the context of the conformal field theory formalism. Thus $\mathcal{N} = 2$ theories were classified by families of theories according whether the left and right SCFA has different (inequivalent) complex structures. This give rise to $\alpha$-strings and $\beta$-strings respectively. It was proved there that both families are related by T-duality.

In Ref. [26,51], the D-brane-D-brane scattering amplitudes were studied in more detail, as well as D-brane scattering with open and closed strings in different cases and then the brane effective field theory on the brane of different dimensions was computed giving rise precisely to the field theories argued in [48]. Also in [50] the coupling of D-branes to the bulk theory for $\alpha$ and $\beta$-strings was computed.

6. Final Remarks

In this contribution we have discussed the target space/worldsheet correspondence of $\mathcal{N} = 2$ strings. These theories are of topological nature and they are completely determined by its target space geometry, which is a self-dual geometry. Thus self-dual geometry can be regarded as an avenue to study string theory and vice-verse. Thus self-dual geometry would be destined to shed some light of what really string theory is. For this reason, many results of those obtained by Plebański and his co-workers would be important to continue exploring this target space/worldsheet correspondence. One of these generalizations is the
extension of self-dual geometry to all algebraically degenerated $\mathcal{H}$-spaces and in general,
to know what is the stringy version of the Penrose-Petrov classification. Also of particular
interest is the stringy description of $\mathcal{HH}$-spaces. It is known that $\mathcal{HH}$-spaces under certain
conditions can give rise to real spacetimes \[52\] and it would be also a bridge to connect
$\mathcal{N} = 2$ String theory to real space-time physics. $\mathcal{HH}$-spaces are not self-dual spaces but
they are so close to them that the methods pursued by $\mathcal{N} = 2$ string theory surely will
be useful. Recently a new $\mathcal{N} = 2$ supersymmetric extension of the heavenly equation was
found in Ref. \[53\]. It would be very interesting whether this equation can be related to
$\mathcal{N} = 2$ strings.

Finally it would be interesting to address, in the context of the present paper, the
correspondence between YM amplitudes and topological string in twistor space according
to Refs. \[54,55\].

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