Monopole free realization of easy plane deconfined critical point

Disha Hou,1 Yuhai Liu,2,3 Toshihiro Sato,4 Wenan Guo,1,2 Fakher F. Assaad,4,5,6 and Zhenjiu Wang6

1Department of Physics, Beijing Normal University, Beijing 100875, China
2Beijing Computational Science Research Center, 10 East Xibeiwang Road, Beijing 100193, China
3School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
4Institut für Theoretische Physik und Astrophysik, Universität Würzburg, 97074 Würzburg, Germany
5Würzburg-Dresden Cluster of Excellence ct.qmat, Am Hubland, 97074 Würzburg, Germany
6Max-Planck-Institut für Physik komplexer Systeme, Dresden 01187, Germany

The quantum spin Hall State can be understood in terms of spontaneous O(3) symmetry breaking. Topological skyrmion configurations of the O(3) order parameter vector carry charge 2e, and, as show previously, when they condense, generate a superconducting state. We show that this topological route to superconductivity survives easy plane anisotropy. Upon reducing the O(3) symmetry to O(2) × Z2, skyrmions give way to monopoles that carry unit charge. On the basis of large scale auxiliary field quantum Monte Carlo simulations, we show that at the particle-hole symmetric point, we can trigger a continuous and direct transition between the quantum spin Hall state and s-wave superconductor by condensing pairs of monopoles. This statement is valid in the strong and weak anisotropy limits. The transition is an instance of an easy-plane deconfined quantum critical point. However, in contrast to the previous studies in quantum spin models, our realization of this quantum critical point, conserves U(1) charge.

I. INTRODUCTION

Topology is a key factor for understanding phase transition. In the Kosterlitz-Thouless transition an O(2) local order parameter in two dimensional space allows for the definition of the vortex, the proliferation of which drives the transition. Let us stay in two dimensional space, x = (x, y), but consider an O(3) local order parameter n(x) with unit norm. This combination of space and order parameter defines a winding number

\[ \frac{1}{4\pi} \int d^2x \cdot \partial_x n \times \partial_y n. \]  \hspace{1cm} (1)

For smooth configurations, this quantity is quantized and counts the winding of the unit vector on the unit sphere: a skyrmion. We can now reduce the O(3) symmetry to O(2) × Z2. In the context of spin systems, this would correspond to restricting the O(3) symmetry to O(2) transformations around say the z-axis and change of sign of the third component of the n-vector. Assuming that energetics favors the n-vector to be in-plane (i.e. vanishing z-component) then the topological excitations will correspond to vortices in the x-y plane. Due to the normalization condition, the n-vector at the core of the vortex will have to point along the z-direction. Since the n-vector lies in the x-y plane at infinity, the integrand of Eq. 1 vanishes at infinity, and the integral takes half-integer values: a monopole.

The above considerations acquire different interpretations depending on the specifics of the local order parameter. Here, it will correspond to the order-parameter of the quantum-spin Hall state. In particular, let \( \hat{H}_0 = -v_F \sum_{p,\sigma=1,2} \Psi_\sigma^\dagger(p) \gamma_0 \gamma_\sigma \Psi_\sigma(p) \) be the Hamiltonian akin to graphene in the absence of interactions. Here, we use the notation of Ref. 4. Inclusion of the quantum spin Hall mass term leads to:

\[ \hat{H} = \hat{H}_0 + \int_V d^2x N(x) \cdot \Psi_\sigma^\dagger(x) \gamma_0 \gamma_3 \sigma,\sigma \Psi_\sigma(x). \]  \hspace{1cm} (2)

Here, the order parameter N can be normalized to unity \( N = N/|N| \) if, as will be the case in our model, the single-particle gap, that is proportional to the length of N, does not vanish. Furthermore, \( n \) is odd under charge conjugation such that the expression in Eq. 1 carries the same quantum numbers as the charge density \( \rho \) measured with respect to half-filling. In particular, in Ref. 5 it is shown that

\[ \rho(x) = \frac{2e}{4\pi} n \cdot \partial_x n \times \partial_y n \]  \hspace{1cm} (3)

such that the skyrmion (meron pair) carries charge 2e. Since topological defects of one phase carry the charge of the other, their proliferation will lead to a broken symmetry state. In the above discussion, we have considered the quantum spin Hall (QSH) state such that the skyrmions (or pairs of monopoles) carry charge 2e and their proliferation will lead to a superconducting state. Alternatively, one could consider the three spin-density wave mass terms. In this case, the skyrmion creation/annihilation process would acquire a phase under spatial rotation of the lattice. This proliferation of skyrmions is the essence of so called ‘deconfined’ quantum critical points (DQCPs).

The above discussion takes place in the continuum and some type of regularization is needed to carry out numerical simulations. Starting with magnetic phases, where the topological defects carry a U(1) rotational
charge, a lattice regularization leads to additional symmetry breaking terms that may lead to subtleties since they do not exist in the expected IR field theory. For instance for a square lattice regularization scheme, U(1) rotational symmetry gives way to a $Z_4$ invariance of the valence bond solid (VBS) state. This regularization induced symmetry reduction introduces novel operators that have to be argued to be irrelevant at the critical point. In particular, in the framework of the CP theory induced symmetry reduction introduces novel operators which are related to the strength of the anisotropy continuously tunes the energy gap of meron configurations of a QSH order parameter. Regardless on the strength of the anisotropy and in contrast to lattice spin models, our model displays no obvious signs of first order transitions. We argue that this transition flows to the easy-plane DQCP.

Although the $Z_4$ lattice symmetry breaking field is relevant in the VBS state, a necessary condition for the continuous nature of the transition into an xy-antiferromagnetic (AFM) phase is that this symmetry breaking field is irrelevant at the critical point. Numerically, AFM-VBS phase transitions in the easy-plane case have clear first order signatures in most cases. Among numerical works, Desai et. al. especially emphasize the absence of the continuous transition in any easy plane spin system: the authors generally claimed the absence of the easy-plane deconfined fixed point without considering the effect of quadruple monopoles. However, could it be that the $Z_4$ symmetry breaking field introduces a runaway flow, leading to a first order transition?

Instead of encoding the U(1) symmetry as a rotational invariance, one can encode it in terms of charge conservation. Importantly, charge conservation will not be broken by the lattice regularization. Following the work of Ref. and we set up a set of designer Hamiltonians which have the potential to realize an easy-plane DQCP without quadruple monopoles. A dynamically generated QSH insulating state which spontaneously breaks the O(2) spin rotational symmetry emerges from a Dirac semi-metal via a spin-orbital interaction. Our particular interest is the phase transition between the QSH and s-wave superconducting (SSC) states.

The fermion basis introduces a simple but much more provoking picture: meron defects of the QSH order parameter which carry unit electron charge is the fundamental excitation at the critical point; on the other side of the transition, condensation of meron-pair creation/annihilation operators forms the superconducting state. Crucially, the U(1) charge conservation broken by the SSC phase is an exact symmetry of our lattice Hamiltonian: quadruple monopoles are absent by definition.

The aim of this work is to systematically search the existence of an easy-plane DQCP without monopoles. The continuity of phase transitions does not only depend on the symmetry in our lattice model, a model parameter which is related to the strength of the easy-plane anisotropy continuously tunes the energy gap of meron configurations of a QSH order parameter. Regardless on the strength of the anisotropy and in contrast to lattice spin models, our model displays no obvious signs of first order transitions. We argue that this transition flows to the easy-plane DQCP.

The paper is organized as follows. In Sec. II we introduce our lattice Hamiltonian, the quantum Monte Carlo algorithm, as well as basic observables. The numerical results are shown in Sec. III, beginning with the ground state phase diagram, and followed by a detail investigation of the nature of the phase transitions. Finally, we draw conclusions and give an outlook in Sec. IV.

II. MODEL AND METHODS

We consider a model of Dirac fermions in $2 + 1$ dimensions on the honeycomb lattice with Hamiltonian

$$\hat{H}_t = -t \sum_{\langle i,j \rangle} (\hat{c}_i \hat{c}_j^\dagger + H.c.).$$

(4)

The spinor $\hat{c}_i^\dagger = (\hat{c}_{i,\uparrow}^\dagger, \hat{c}_{i,\downarrow}^\dagger)$ where $\hat{c}_{i,\sigma}^\dagger$ creates an electron in a Wannier state centered around lattice site $i$ with $\sigma$-component of spin $\sigma$. This term accounts for nearest-neighbor hopping. The interaction term that we consider reads:

$$\hat{H}_\lambda = -\lambda \sum_{\langle\langle ij \rangle\rangle} \left[ \left( \sum_{\langle\langle ij \rangle\rangle} \hat{J}_{ij}^z \right)^2 + \left( \sum_{\langle\langle ij \rangle\rangle} \hat{J}_{ij}^\tau \right)^2 \right] + \Delta \left( \sum_{\langle\langle ij \rangle\rangle} \hat{J}_{ij}^z \right)^2 \right]$$

(5)

where $\hat{J}_{ij} \equiv i\nu_{ij} \hat{c}_{i,\sigma}^\dagger \sigma \hat{c}_j + H.c.$ The components of $\sigma = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli spin-1/2 matrices. This term is a plaquette interaction involving next-nearest-neighbor pairs $\langle\langle ij \rangle\rangle$ of sites and phase factors $\nu_{ij} = \pm 1$ identical to the Kane-Mele model see also Ref. 18.

The Hamiltonian $\hat{H} = \hat{H}_t + \hat{H}_\lambda$ with $\Delta = 1$ has been studied in Ref. 18. A dynamically generated QSH insulator that breaks SU(2) spin rotational symmetry spontaneously is found at intermediate interacting strength ($\lambda$), separating a Dirac semi-metal(DSM) state at small $\lambda$ and an SSC state at large $\lambda$. The DSM-QSH transition is in the Gross-Neveu-Heisenberg universality class whereas the QSH-SSC transition falls into the class of DQCP. In the current work, we focus on the case of $\Delta \in [0,1]$ where the SU(2) spin rotational symmetry is lowered to $U(1) \times Z_2$.

We used the ALF (Algorithms for Lattice Fermions) implementation of the well-established auxiliary-field quantum Monte Carlo (QMC) method. Because $\lambda > 0$ and $\Delta > 0$, we can use a real Hubbard-Stratonovich decomposition for the perfect square term. We set the imaginary interval $\Delta t = 0.2$ and choose a symmetric Trotter decomposition to ensure the hermiticity of the imaginary time propagation in the Monte Carlo simulations. Additionally, a checkerboard decomposition is applied to the exponential of hopping matrix $\hat{H}_t$. For each field configuration, time-reversal symmetry
and charge are conserved. Hence the eigenvalues of the fermion determinant occur in complex conjugate pairs and we do not suffer from the negative sign problem.\cite{footnote:15}

We simulated lattices with $L \times L$ unit cells (each containing two Dirac fermions) and periodic boundary conditions. Following our previous work\cite{19} we used a projective version of the algorithm (PQMC)\cite{20,21,22} This algorithm is based on the form:

$$
\langle \hat{O} \rangle = \lim_{\theta \to \infty} \frac{\langle \Psi_T e^{-\theta \hat{H}} \otimes e^{-\theta \bar{H}} | \Psi_T \rangle}{\langle \Psi_T e^{-2\theta \hat{H}} | \Psi_T \rangle}.
$$

(6)

Provided that the trial wave function $| \Psi_T \rangle$ is not orthogonal to the ground state, $\langle \hat{O} \rangle$ corresponds to the ground state expectation value of the observable $\hat{O}$. To avoid the negative sign problem, we consider the same time-reversal symmetric trial wave function as the one used in Ref.\cite{19} We explicitly checked the projection convergence to the ground state at each system size and each $\Delta$: simulations are performed at $L = 6, 9, 12, 15, 18$ and the values of projection length $\theta$ for ground state calculation are listed in Tab.\ref{tab:1}.

The basic measurements in our QMC simulations are equal time correlation functions in real space:

$$
S^O(r) = \frac{1}{L^2} \sum_{n} \sum_{r'} \langle \hat{O}^+_r \hat{O}^+_{r+n} \rangle,
$$

(7)

and the structure factor:

$$
S^{O}_{m,n}(q) = \frac{1}{L^2} \sum_{r,r'} e^{iq \cdot (r-r')} \langle \hat{O}^+_r \hat{O}^+_{r-m} \rangle,
$$

(8)

where $\hat{O}_{r,n}$ is a local operator with $r$ denoting the unit cell and $n$ denoting the intra unit-cell dependence that we will refer to as orbital.

For instance, the spin-orbit coupling operators correspond to $\hat{O}_{r,n} = \hat{J}_{r+\delta_n,r+\eta_n}$. Here $n$ runs over the six next-nearest neighbor bonds of the corresponding hexagon with legs $r+\delta_n$ and $r+\eta_n$ ($n = 1, 2, \ldots, 6$), see Fig.\ref{fig:1}. To detect QSH ordering which breaks the $O(2)$ spin rotational symmetry, we calculate structure factor matrix associated with the $X, Y$ components of $\hat{O}_{r,n}$:

$$
S^{\text{QSH}}_{m,n}(q) = \frac{1}{L^2} \sum_{r,r'} e^{iq \cdot (r-r')} \langle \hat{O}^+_r \hat{O}^+_{r-m} \rangle,
$$

(9)

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$$

(9)

TABLE I. Projection length $\theta$ at different values of $\Delta$ and $L$.

| lattice size | $\Delta = 0.1$ | $\Delta = 0.5$ | $\Delta = 0.75$ |
|-------------|--------------|--------------|--------------|
| $L = 6$     | 15           | 15           | 15           |
| $L = 9$     | 21           | 21           | 21           |
| $L = 12$    | 24           | 24           | 24           |
| $L = 15$    | 42           | 36           | 36           |
| $L = 18$    | 42           | 36           | 36           |

The physical meaning of this quantity will be discussed in Sec.\ref{sec:3}.

To detect SSC ordering which breaks $U(1)$ charge conservation, we consider the following structure factor matrix:

$$
S_{a,b}^{\text{SSC}}(q) = \frac{1}{L^2} \sum_{r,r'} e^{iq \cdot (r-r')} \langle \hat{O}_{r,a} \hat{O}_{r',b} \rangle + \langle \hat{O}_{r,b} \hat{O}_{r',a} \rangle,
$$

(11)

with $a, b = 1, 2$, denoting the A(B) sublattices, where the $s$-wave pairing operator is defined as

$$
\hat{\eta}_{r,\delta_a}^+ = \hat{c}_{r+\delta_a} \hat{c}_{r+\delta_a}^+.
$$

(12)

Here $\delta_a$ runs over the two orbitals in unit cell $r$, see Fig.\ref{fig:1}.

We use Eq.\ref{eq:9} and Eq.\ref{eq:11} to calculate the order parameter:

$$
m^O = \sqrt{\Lambda_1(S^O(Q))}
$$

(13)

Here, $\Lambda_1(\cdot)$ indicates the largest eigenvalue of the corresponding matrix in orbital space, $O$ denotes QSH and SSC, $Q = (0, 0)$. The corresponding eigenvector will determine the orbital structure. This is of particular importance for the QSH state since we expect it to reflect the sign structure $\nu_{ij}$ of the Kane-Mele model.

To locate the critical points and study the critical properties, after diagonalizing the corresponding structure factors, we calculated the renormalization-group invariant correlation ratio

$$
R^O = 1 - \frac{S^O(Q + \Delta q)}{S^O(Q)}
$$

(14)
using the largest eigenvalue \( S^O \) \((O = \text{QSH}, \text{QSH}_x, \text{SSC})\); \( Q = (0,0) \) is the ordering wave vector and \( Q + \Delta q \) is a neighboring wave vector with \( |\Delta q| = \frac{4\pi}{\sqrt{3}L} \). By definition, \( R^O \to 1 \) for \( L \to \infty \) in the corresponding ordered state, whereas \( R^O \to 0 \) in the disordered state. At the critical point, \( R^O \) is scale-invariant for sufficiently large \( L \) so that results for different system sizes cross.

### III. QUANTUM MONTE CARLO RESULTS

In this section we will first provide the ground state phase diagram and then will proceed in investigating the nature of the phase transitions.

#### A. Ground state phase diagram

As mentioned previously, we are interested in the parameter range of \( \Delta \in [0,1] \) where the spin rotational symmetry of the Hamiltonian \( \hat{H} = \hat{H}_1 + \hat{H}_\lambda \) is lowered to \( U(1) \times Z_2 \). The DSM and SSC states that were found in the \( SU(2) \) symmetric case \cite{18} \(( \Delta = 1 \) \) are naturally stable against weak easy plane anisotropy \( \Delta \approx 1 \) since both states are spin rotational invariant. Furthermore, since time reversal symmetry is not broken by our symmetry reduction, we equally expect the QSH phase to be stable. To confirm the above, we can use the mean field approach introduced in Ref. \cite{19} that carries over to the anisotropic case. Due to the Dirac nature of the kinetic term in Eq. \ref{eq:1}, we foresee the robustness of the DSM phase in the weakly interacting case. On the other hand, the attractive nature of \( \hat{H}_\lambda \) term \(( \text{for } \Delta \geq 0, \lambda > 0 )\) suggests that the mean field picture in the large \( \lambda \) case \cite{19} will still favor a SSC instability. Finally, the dynamically generated QSH state at intermediate values of \( \lambda \) will be restricted to the \( U(1) \) plane in the current case. We present the mean field phase diagram in Fig. \ref{fig:2}. The details of the calculations are summarized in App. \ref{app:A}.

As shown in Fig. \ref{fig:2}, Fig. \ref{fig:3}, and Fig. \ref{fig:6}, the phase boundaries of \( SU(2) \) symmetric Hamiltonian \(( \text{the overline of } \Delta = 1 )\) are from Ref. \cite{13}.

We summarize the exact ground state phase diagram in \(( \Delta, \lambda )\) plane based on QMC results in Fig. \ref{fig:5} where the overline \( \Delta = 1 \) corresponds to the \( SU(2) \) symmetric model that was studied in Ref. \cite{18}. Generally speaking, we found a DSM state at weak interaction (small \( \lambda \) region), an SSC state at strong interaction (large \( \lambda \) region), as well as a \( U(1) \) broken QSH state at an intermediate region.

The numerical simulations that we performed cover three horizontal lines as a function of \( \lambda \): \( \Delta = 0.1, 0.5 \) and 0.75. As shown in Fig. \ref{fig:5}a, Fig. \ref{fig:5}b, and Fig. \ref{fig:6}b, the QSH correlation ratio \( R^{\text{QSH}} \) increases toward 1 at intermediate values of \( \lambda \), indicating a robust QSH state for all three cases. We have checked explicitly that the eigenvalue corresponding to the largest eigenvector matches the sign structure \( \nu_{ij} \) of the Kane-Mele model. Fur-
Furthermore, in Appendix D we have used the flux-insertion scheme presented in Ref. 31 to probe for the topological invariant. The DSM-QSH and QSH-SSC phase boundaries are estimated by fitting the equal-time correlation ratios of the two biggest lattice sizes $L = 15$ and $L = 18$ according to the following function

$$f(L, \lambda) = R_c + a_1(\lambda - \lambda_c)L^{1/\nu} + a_2(\lambda - \lambda_c)^2L^{2/\nu}. \quad (16)$$

The details of the fit are listed in Tab. II and Tab. III. Remarkably, the critical values of $\lambda$ where superconducting order develops, as shown by the crossing points of $R_{SSC}$ in Fig. 4(a), Fig. 5(a), and Fig. 6(a), match the values of $\lambda$ where the QSH order vanishes, which indicates that, regardless of the strength of anisotropy, direct phase transitions exist between the QSH and SSC phases. The order parameters give the consistent results, as shown in Fig 7.

Comparison of the mean-field, Fig. 2, and QMC, Fig. 3, phase diagrams are very instructive. The transition from the DSM to QSH insulator at $\Delta < 1$ belongs to the $U(1)$ Gross Neveu universality class. The essence of this transition, a symmetry breaking induced electronic mass generation, is captured at the mean-field level. In fact, an $\epsilon$-expansion around the upper critical dimension accounts rather well for this transition for the SO(3) and $U(1)$ cases. It is hence not unexpected that the comparison between the mean-field, Fig. 2, and QMC, Fig. 3, phase diagrams is good for this transition. In contrast, the competition and interplay between the QSH and SSC phases are radically different at the mean-field and QMC levels. We interpret this mismatch as a hint that topology – not accounted for at the mean-field level – is crucial for the understanding of the intertwinement of the QSH and SSC phases. Of particular importance is that the QMC phase diagram does not show a coexistence of the QSH and SSC phases. The nature of the transition will be discussed in the next section.

| Table II. DSM-QSH crossing points $\lambda_c$ | anisotropy $\Delta$ | $\lambda_c$ | $\chi^2$ | $O$ |
|---------------------------------------------|-----------------|---------|------|-----|
| $\Delta = 0.1$                              | 0.0243(2)       | 0.14    | QSH  |
| $\Delta = 0.5$                              | 0.0225(2)       | 2.7     | QSH  |
| $\Delta = 0.75$                             | 0.0208(3)       | 0.81    | QSH  |

| Table III. QSH-SSC crossing points $\lambda_c$ | anisotropy $\Delta$ | $\lambda_c$ | $\chi^2$ | $O$ |
|-----------------------------------------------|-----------------|---------|------|-----|
| $\Delta = 0.1$                               | 0.06006(8)      | 0.08    | QSH  |
|                                               | 0.05988(2)      | 2.41    | SSC  |
| $\Delta = 0.5$                               | 0.0448(2)       | 19.87   | QSH  |
|                                               | 0.04444(4)      | 3.77    | SSC  |
| $\Delta = 0.75$                              | 0.03829(4)      | 1.67    | QSH  |
|                                               | 0.03788(3)      | 3.44    | SSC  |

Our QSH insulator at zero temperature is also a gapless phase reflecting the emergence of Goldstone modes upon breaking of the global XY symmetry. Hence both the spin current operators $J^z$ and the corresponding angular momentum operator $\hat{S}^z$ reveal gapless excitations around zero momentum. On the other hand, merons in this phase are another low energy excitation with finite gap. Roughly speaking, the binding energy of pairs of merons, is higher in the case of strong anisotropy. The numerical evidence that meron bind can be deduced from the spectral functions presented in Appendix C. Comparison between the single particle and superconducting spectral functions shows that the cost for adding a pair is less than twice the single particle gap. This binding energy can be tuned to zero by increasing the interaction, thus triggering a direct transition to a superconducting state.

FIG. 4. Equal-time correlation ratio $R_{SSC}$ (a) and $R_{QSH}$ (b) as a function of $\lambda$ for $\Delta = 0.1$.

B. Nature of the QSH-SSC phase transition

Our most important result is the seemingly continuous nature of the QSH-SSC transition. According to the Ginzburg-Landau paradigm observing, two independent 3D XY phase transitions that happen at the same point, requires fine-tuning. Instead, this transition fits into the
category of DQCP with easy-plane anisotropy, in the sense that fractionalized objects instead of the XY order parameters are the elementary excitations. In our specific case this corresponds to meron excitations of the spin current operator that carry unit charge and the spinons that emerge at the core of superconducting vortices.

The QSH-SSC transition has only bosonic excitations at low energy. To characterize this, we extrapolate the fermionic single particle gap $\Delta_{sp}$ from the Green’s function:

$$
\sum_{\sigma} \langle c_{k,\sigma}(\tau) c_{k,\sigma}^\dagger(0) \rangle \propto e^{-\Delta_{sp} \tau},
$$

at $k = M \equiv (0, \frac{2\pi}{\sqrt{3}})$. Figure 8 demonstrates that $\Delta_{sp}$ remains nonzero across the QSH-SSC transition at $\lambda_c$ for all three considered values of the anisotropy.

In the following step, we inquire if the QSH-SSC transition is discontinuous, as suggested by the Ginzburg-Landau paradigm. Considering that the computational cost of the AFQMC scales as $\theta L^3$ with $\theta$ being the projection length, we do not apply the commonly used method for detecting first-order transitions, e.g., analyzing the finite-size behavior of the histogram of order parameters or/and the behavior of the Binder cumulant, etc. Instead, we study the correlation length to reflect the nature of phase transitions. A continuous phase transition is characterized by a diverging correlation length in the thermodynamic limit: $\xi \propto |\lambda - \lambda_c|^{-\nu}$. On the other hand, for a first-order transition, the correlation length saturates to a finite value. We use real-space, equal-time correlation functions $S^O(\mathbf{r})$ of the order parameter to define the correlation length

$$
\xi^O \equiv \sqrt{\frac{\sum_{\mathbf{r}} |\mathbf{r}|^2 S^O(\mathbf{r})}{\sum_{\mathbf{r}} S^O(\mathbf{r})}},
$$

where $S^O(\mathbf{r})$ is defined in Eq. 7. An issue with this definition is that it includes the information of correlation lengths along both the longitudinal and the transverse directions in a continuous symmetry broken state. Hence without a specific symmetry-breaking pinning field to resolve the longitudinal direction, the correlation length $\xi$ in Eq. (18) is well defined only in the disordered state or at the critical point. Hence we discard the data of the QSH(SSC) correlation length inside the QSH(SSC) state.

As depicted in Fig. 9(a), Fig. 10(a) and Fig. 11(a), around transition points at different values of $\Delta$, the correlation length $\xi$ of both the QSH and SSC order parameters grows with system size $L$, without any tendency of

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**FIG. 5.** Same as Fig. 4 for $\Delta = 0.5$.

**FIG. 6.** Same as Fig. 4 for $\Delta = 0.75$. 
saturation. We define the scaled correlation length $\xi/L$ as the ratio between the correlation length and system size. As seen in Fig. 9(b), Fig. 10(b) and Fig. 11(b), for both the QSH and SSC correlation lengths, the ratios for different system sizes cross at the same point, suggesting that the correlation lengths diverge with $L$. The divergence of correlation lengths indicates the continuous nature of the phase transition.

It is worth mentioning that we observe an amazing match of the value of the scaled correlation length $\xi/L$ between different anisotropy strengths $\Delta$. The same value of $\xi/L$ at the three transition points implies that all three QSH-SSC phase transition points correspond to the same easy-plane DQCP fixed point. $\xi/L$ is a dimensionless quantity which is a renormalization-group invariant at the critical point. On the other hand, this number is claimed to be universal in $2+1$ dimensional system with conformal invariance. On a lattice system, this universal number is not only pinned by the scaling dimension of the order parameter, but also by the microscopic couplings in different directions, the boundary conditions, and the shape of the system (e.g. the aspect ratio).[8,38]

In our case, the only difference between the three different $\Delta$ is the intrinsic spin anisotropy which is not related to the lattice geometry, such that $\xi/L$ should be universal if all three transitions at $\Delta = 0.1, 0.5$ and $0.75$ belong to same universality class. We also notice the interesting behavior that QSH and SSC operators cross at the same value of $\xi/L$ at the transition point: this also indicates the identical value of the anomalous dimension $\eta$ between the two order parameters. This is a significant evidence of the emergent $O(4)$ symmetry at the easy plane DQCP [8,38].

We also calculate the first-order partial derivative of the free energy density with respect to the coupling

FIG. 7. QSH and SSC order parameter as a function of $\lambda$, for $\Delta = 0.1$ ((a) and (b)), $0.5$ ((c) and (d)) and $0.75$ ((e) and (f)).

FIG. 8. Single particle gap $\Delta_{sp}$ across the QSH-SSC transition for $\Delta = 0.1$ (a), $\Delta = 0.5$ (b), and $\Delta = 0.75$ (c), respectively.
strength $\lambda$,

$$\frac{\partial f}{\partial \lambda} = \frac{1}{L^2} \lambda \langle \bar{H}_\lambda \rangle,$$

(19)

to study the nature of phase transition. This approach requires no information of the order parameter (and the associated symmetry breaking). In the case of a first order transition, when the system size is much larger than correlation length $\xi$, one expects that this derivative has a discontinuity at the transition point. Figure 12 shows $\partial f/\partial \lambda$ as a function of $\lambda$ in the vicinity of the QSH-SSC transition point for $\Delta = 0.1, 0.5$ and 0.75. Our data reveals no clear signs of a jump for the accessible system sizes. This is consistent with the aforementioned correlation length analyses.

However, in the strong anisotropic case $\Delta = 0.1$, the slope of the curve scales up moving towards the transition point when increasing the system size. This result may be interpreted as the signature of a ‘weakly first order’ transition. Around a continuous transition point, the derivative of the free energy scales as

$$\frac{\partial f}{\partial \lambda} \propto |\lambda - \lambda_c|^{(d+z)\nu - 1}$$

(20)
in the thermodynamic limit. For a ‘weakly first order’ transition, one expects a ‘pseudo critical’ phenomenon that the $\nu(L)$ estimated from finite sizes would approach $1/(d+z)$ as $L$ reaches $\xi$ such that $\partial f/\partial \lambda$ asymptotically shows a jump. However, we won’t be able to conclude the nature of transition at $\Delta = 0.1$ since it is not clear if the finite size slope diverges or saturates upon approaching the thermodynamic limit. On the other hand, the robustness of the slope in the case of $\Delta = 0.5$ and $\Delta = 0.75$ as shown in Fig. 12(b) and (c) indicates clear continuous phase transitions, unless there exists a non-diverging $\xi$ that is significantly larger than $L$.

Fractionalized spinons are not directly measurable at the DQCP point, since they do not directly correspond to any local second quantized operators. The easy plane anisotropy gives a natural playground for investigating deconfinement at criticality. Consider the two-component complex field $z \equiv (z_1, z_2)$ describing the spinons that are deconfined at the critical point. A gauge-invariant object that is gapped in the two ordered phases but gapless at the critical point would correspond to $\bar{z} \sigma_z z$. This maps onto the $z$-component of our spin current operator $\hat{J}_Z$. Although correlation functions of $\hat{J}_Z$ should also be characterized by an algebraic decay at criticality, a large anomalous dimension, compared to
that of the $O(2)$ order parameter, is predicted by previous numerically studies:\cite{40}

\begin{equation}
S^{QSH}(Q) \propto L^{1-\eta_{XY}}
\end{equation}

and

\begin{equation}
S^{QSH \parallel}(Q) \propto L^{1-\eta_{Z}}
\end{equation}

where $Q \equiv (0,0)$. One expects $\eta_{XY} \approx 0.13$ and $\eta_{Z} \approx 0.91$ from Ref.\cite{40}. In order to detect the algebraic behavior of $\hat{J}_{Z}$, it is in principle more efficient to consider the susceptibility as defined in Eq.\cite{31} since the equal time structure factor would suffer from an effective scaling correction term with a power of around 0.1. However, the correlation ratio based on the imaginary time correlation function of $\hat{J}_{Z}$ operator has larger fluctuation. Therefore here we show the data $R^{QSH \parallel}$ based on equal time structure factor.

Remarkably, at the QSH-SSC critical points for $\Delta = 0.5$ and 0.75, $R^{QSH \parallel}$ have the tendency of converging to a finite value in the large size limit, as shown in Fig.\cite{13} (b) and (c). This indicates an algebraic decay of $\hat{J}_{Z}$ at the easy plane DQCP and we claim that this is a signature of deconfined spinons. On the other hand, the divergence of the static structure factor of the $J_{Z}$ operator would be very slow due to its large anomalous dimension (around 0.91), and this leads to a relative large scaling correction from the non-diverging part of free energy. This could explain the gradually decreasing behavior of $R^{QSH \parallel}$ within our achievable system sizes. In contrast, the correlation ratio of $\hat{J}_{Z}$, defined via integrating over imaginary time correlation functions in the PQMC, suffers less from this scaling correction and thus displays a better saturation behavior around the critical point. Numerical data to support this statement is found in Fig.\cite{19} (b), and Ap-
Fig. 13. Equal-time correlation ratio $R_{QSHz}$ for $\Delta = 0.1$ (a), $\Delta = 0.5$ (b), and $\Delta = 0.75$ (c), respectively.

Fig. 14. QSH (a), SSC (b) and QSHz (c) dynamical spectrum at QSH-SSC critical point ($\lambda = 0.038$) for $\Delta = 0.75$. Here, $L = 18$. Blue lines in (a) and (b) are the momentum dependence of the extrapolated excitation gap of $\hat{J}$ and $\hat{\eta}^+ (\hat{\eta}^-)$ operators. Blue line in (c) is copied from SSC dispersion relation in (b) to guide the eyes.

Appendix B defines the notion of susceptibilities within the PQMC.

On the other hand, around the QSH-SSC transition point at $\Delta = 0.1$, we seemingly observe an absence of critical fluctuation of $J^z$ from Fig. 13 (a): the corresponding correlation ratio $R_{QSHz}$ decays quickly at $\lambda_c \approx 0.06$. Whether or not $R_{QSHz}$ will saturate to a small finite value in the thermodynamic limit remains an open question.

The dynamical structure factor of the two order parameters is also interesting. As shown in Fig. 14 both the spin-orbit, $\hat{J}^{XY}$, and the pairing, $\hat{\eta}^+ (\hat{\eta}^-)$, correlations display gapless excitation with linear dispersing behavior, as expected from emergent Lorentz invariance at the deconfined critical point. In particular, the QSH and SSC excitations share the same bosonic velocity. It is notable that the $J^z$ operator also shows gapless signature around the $\Gamma$ point as shown in Fig. 14 (c), although its spectrum weight at low frequency is relatively weak. This is also a smoking gun evidence of deconfined spinons at this critical point. The definition of the spectral function and the analytical continuation of imaginary time for all three values of $\Delta$ are detailed in Appendix C.
IV. DISCUSSIONS AND OUTLOOK

The model Hamiltonian studied in this article supports the picture of a superconducting state with topological origins, that emerges from a U(1) spontaneous symmetry broken QSH insulator. Our numerical observation of a direct QSH-SSC phase transition without phase-coexistence region requires a physical picture that lies beyond mean field theories. In particular we understand the transition from the QSH to SSC in terms of a condensation of monopoles of the U(1) QSH order parameter. Moreover, upon tuning the parameter that controls the easy plane anisotropy, no obvious discontinuous signature is observed at the QSH-SSC transition. This implies that our QSH-SSC transitions flow to the easy plane deconfined quantum critical point irrespective of the anisotropy parameter. This is also supported numerically by the observed universal value of the scaled correlation length upon changing the anisotropy. The scaled correlation length takes the same value for the QSH and SSC fluctuations at the easy-plane DQCP thus supporting an emergent O(4) symmetry. Our model summarizes the very first calculations of this critical point in a lattice Hamiltonian where the lattice regularization does not break the IR symmetries of the putative field theory. As a consequence our lattice regularization does not introduce quadruple monopoles as in the easy plane QJ model.

To place our results in a broader perspective, we can ask the question if designer Hamiltonians with higher symmetry can impact criticality. Although it is well known that phase transitions numerically observed in easy-plane lattice spin models have a higher tendency to a run away flow, explaining the first order nature of transition: 2. Even if symmetry breaking terms due to lattice regularization (e.g. $O(4)$ down to $U(1) \times U(1)$ or $U(1) \times Z_2$) are imposed to be zero, the easy plane DQCP may not even exist in any unitary conformal field theory. Unfortunately, our simulation that are limited to small sizes can not provide a definite answer to this question. Nevertheless, our high symmetry regularization and the fact that our data can be consistently explained in terms of a continuous transition provides support the first hypothesis. It certainly motivates future computations on larger clusters.

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Appendix A: Mean field calculation

In this appendix we present the mean field calculation. Expanding Eq. D2 of the main text as

\[ H_{\lambda} = -\lambda \sum_{\langle (ij) \rangle \in \Omega} \left( \sum_{\langle (ij) \rangle \in \Omega} \tilde{J}_{x}^{i,j} \right)^{2} + \left( \sum_{\langle (ij) \rangle \in \Omega} \tilde{J}_{y}^{i,j} \right)^{2} \]

\[ + \Delta \left( \sum_{\langle (ij) \rangle \in \Omega} \tilde{J}_{z}^{i,j} \right)^{2} \]

\[ = -\lambda \sum_{\langle (ij) \rangle \in \Omega} \sum_{\langle (i'j') \rangle \neq \langle (ij) \rangle} \tilde{J}_{x}^{i,j} \cdot \tilde{J}_{x}^{i',j'} + \tilde{J}_{y}^{i,j} \cdot \tilde{J}_{y}^{i',j'} + \Delta \tilde{J}_{z}^{i,j} \cdot \tilde{J}_{z}^{i',j'} \]

\[ - \lambda \sum_{\langle (ij) \rangle \in \Omega} \left[ +(4 + 2\Delta) \tilde{n}_{i}^{\dagger} \tilde{n}_{j} + h.c. + ... \right] \]

(A1)

where

\[ \tilde{J}_{x}^{\langle (i,j) \rangle} = i \nu_{ij} \hat{c}_{i}^{\dagger} \sigma_{x} \hat{c}_{j} + H.c., \]

\[ \tilde{n}_{i} = \hat{c}_{i}^{\dagger} \hat{c}_{i}, \quad \tilde{n}_{i}^{\dagger} = \hat{c}_{i}^{\dagger} \hat{c}_{i}^{\dagger}. \]

(A2)

The ellipsis denotes terms that have no contribution to the SSC or QSH ordering within the mean field decomposition.

The mean-field calculation is based on selecting a polarization direction for the two components of the QSH and SSC order parameters. The calculation is done by numerically minimizing the free energy in the space of the two order parameters.

The two order parameters as a function of \( \lambda \) for the half-filled case are shown in Fig. 15. For all three different values of anisotropy, we observe a semimetal (\( \phi_{QSH} = \phi_{SSC} = 0 \)), a pure QSH state (\( \phi_{QSH} \neq 0, \phi_{SSC} = 0 \)) as well as a coexistence of QSH and SSC phases (\( \phi_{QSH} \neq 0, \phi_{SSC} \neq 0 \)).

The mean field phase diagram of Fig. 2 shows a larger stability of the QSH phase at stronger anisotropy. The reason for this becomes transparent when taking a glimpse at Eq. A1. Here \( \Delta \) modulates the magnitude of the pair-hopping \( \tilde{n}_{i}^{\dagger} \tilde{n}_{j} + h.c. \) but not of the in-plane spin-orbit interactions \( \tilde{J}_{x}^{\langle (i,j) \rangle} \cdot \tilde{J}_{\langle (i',j') \rangle}^{x} + x \leftrightarrow y \).
Appendix B: Time displaced observables

To define susceptibilities in the realm of the zero temperature projective QMC algorithm used this article, we will distinguish between observables \( \hat{O}_q = \frac{1}{N} \sum_{\tau,q} e^{\theta \tau} \hat{O}_\tau \) that commute or not with the Hamiltonian. Here, \( N = L^2 \). The key point is that for \( \hat{O}_q, \hat{H} \) is not degenerate then for a tight-binding model when the boundary conditions vanish. A canonical example is the spin susceptibility.

In the above, \( \langle \Theta(\tau) \rangle_T = \frac{1}{Z} \text{Tr} \left( e^{-\beta \hat{H}} \Theta \right) \) and \( \hat{O}_q(\tau) = e^{\tau \hat{H}} \hat{O} e^{-\tau \hat{H}} \). In particular if the ground state is not degenerate then for the ground state algorithm where we first take the limit \( \lim_{\beta \to \infty} \chi^{FT}(q, \beta, L) \neq \lim_{\beta \to \infty} \lim_{L \to \infty} \chi^{FT}(q, \beta, L) \) (B1)

\[
\lim_{\beta \to \infty} \lim_{L \to \infty} \chi^{FT}(q, \beta, L) 
= \lim_{L \to \infty} \lim_{\beta \to \infty} \chi^{FT}(q, \beta, L) = 0
\]

where \( \chi^{FT}(q, \beta, L) = \int_0^\beta d\tau \left( \langle \hat{O}_q(\tau) \hat{O}_{-q}(0) \rangle - \langle \hat{O}_q(\tau) \rangle \langle \hat{O}_{-q}(0) \rangle \right) \) (B2)

In the above, \( \langle \Theta(\tau) \rangle_T \) is the divergence of the dynamical exponent.

Let us now consider the case of finite momentum \( q \neq 0 \) for a momentum conserving Hamiltonian. Hence, \( \hat{O}_q, \hat{H} \) is not degenerate. Provided that the ground state is unique, we will show that the limits can be interchanged. Our starting point is the Lehmann representation:

\[
\chi^{FT}(q, \beta, L) = \frac{\beta}{Z} \sum_n e^{-\beta E_n} \left| \langle n | \hat{O}_q | n \rangle \right|^2 + \frac{1}{Z} \sum_{n \neq m} e^{-\beta E_n} \frac{E_n}{E_m - E_n} \left| \langle n | \hat{O}_q | m \rangle \right|^2
\]

with \( \hat{H}|n\rangle = E_n|n\rangle \) and \( n \in \mathbb{N} \). Since we have assumed that \( q \neq 0 \) the first term of the right hand side of the above equation vanishes. Defining the density of state as, \( N(E) = \lim_{L \to \infty} \sum_n \delta(E_n - E) \) we obtain:

\[
\lim_{L \to \infty} \lim_{\beta \to \infty} \chi^{FT}(q, \beta, L) = \int dE N(E) \frac{1}{E - E_0} \times \left( \left| \langle E | \hat{O}_q | E_0 \rangle \right|^2 + \left| \langle E_0 | \hat{O}_q | E \rangle \right|^2 \right)
\]

and

\[
\lim_{\beta \to \infty} \lim_{L \to \infty} \chi^{FT}(q, \beta, L) = N(E_0) \lim_{L \to \infty} \lim_{\beta \to \infty} \chi^{FT}(q, \beta, L)
\]

For a unique ground state, \( N(E_0) = 1 \). Hence, under the aforementioned assumptions we can interchange the limits and it makes sense to define susceptibilities within the ground state algorithm where we first take the limit of zero temperature and then consider larger and larger lattices.

For practical purposes, we compute:

\[
\chi(q) = \int_0^\beta d\tau \langle \hat{O}_q(\tau) \hat{O}_{-q}(0) \rangle - \langle \hat{O}_q(\tau) \rangle \langle \hat{O}_{-q}(0) \rangle \) (B6)

where

\[
\langle \hat{O}_q(\tau) \hat{O}_{-q}(0) \rangle = \frac{\langle \Psi_T | e^{-\theta \hat{H}} e^{-i(\beta - \tau) \hat{H}} \hat{O}_q e^{-i\theta \hat{H}} \hat{O}_{-q} e^{-\theta \hat{H}} | \Psi_T \rangle}{\langle \Psi_T | e^{-i(2\theta + \beta) \hat{H}} | \Psi_T \rangle}
\]

and \( |\Psi_T\rangle \) is the trial wave function. We consider \( \beta = L \) for all three values of \( \Delta \). As shown in the main text, we implicitly checked that for the considered size \( L, \theta \) is chosen to be large enough to converge to the ground state.

Since the zero temperature approach to susceptibilities matches the result obtained with the traditional calculations, the scaling behaviours are identical. In particular in the vicinity of a Lorentz invariant (\( z=1 \)) critical point we expect:

\[
\chi(q) \propto \xi^{-2\Delta+d+1} \]

(B8)

Hence, as for the finite temperature case, we expect at the critical point that \( \chi \) suppresses background contributions of the non-singular part of free energy by an additional power (of the dynamical exponent). Assuming Lorentz invariance at the easy-plane DQCP, this additional power is unity.

We define the susceptibility of the QSH, SSC and QSHz in [B10], [B11] and [B12]:

\[
\chi_{m,n}^{SSC}(q) \equiv \frac{1}{L^2} \sum_{r,r'} \int_0^\beta d \tau \epsilon(q - \tau) \langle \hat{O}_{r,m}(\tau) \hat{O}_{r',n}(0) + \hat{O}_{r,m}(0) \hat{O}_{r',n}(\tau) \rangle,
\]

(B10)

and

\[
\chi_{m,n}^{QSHZ}(q) \equiv \frac{1}{L^2} \sum_{r,r'} \int_0^\beta d \tau \epsilon(q - \tau) \langle \hat{O}_{r,m}(\tau) \hat{O}_{r',n}(0) \rangle.
\]

(B11)

And

\[
\chi_{a,b}^{SSC}(q) \equiv \frac{1}{L^2} \sum_{r,r'} \int_0^\beta d \tau \epsilon(q - \tau) \left[ \langle \hat{O}_{r,a}(\tau) \hat{O}_{r',b}(0) + \hat{O}_{r,a}(0) \hat{O}_{r',b}(\tau) \rangle \right],
\]

(B12)
For all the three susceptibilities, we also consider the largest eigenvalue of $\chi_{m,n}^{QSH}(q)$, $\chi_{m,n}^{QSHz}(q)$ and $\chi_{a,b}^{SSC}(q)$ for each momentum, which we denote by $\chi^{QSH}(q)$, $\chi^{QSHz}(q)$ and $\chi^{SSC}(q)$. The correlation ratio for these three quantities then reads:

$$R^O_x \equiv 1 - \frac{\chi^O(Q + \Delta q)}{\chi^O(Q)}$$  \hspace{1cm} (B13)

with $O$ referring to SSC, QSH, and QSHz, respectively. The ordering wave vector is $Q = (0, 0)$, and $|\Delta q| = \frac{4\pi}{\sqrt{3}L}$.

We note that since the superconducting order parameter breaks U(1) global charge symmetry and the QSH order parameter breaks inversion symmetry so that the conditions for inter-changing the limits of zero temperature and infinite size are satisfied.

The finite size behavior of $R^Q_{x}$ and $R^S_{x}$ show no significant differences as compared to the corresponding equal time correlations considered in the main text. As shown in Fig. 16 [17 and 18] the crossing points are consistent with the estimation of critical points from main text. On the other hand, the scaling of $R^Q_{x}$ remains ambiguous: at the QSH-SSC transition point this quantity decays toward zero at strong anisotropy ($\Delta = 0.1$). In the case of $\Delta = 0.5$ and $0.75$ it could converge to a finite constant in thermodynamic limit; upon increasing system sizes, its decreasing tendency is similar to the one from equal time correlation ratio in main text.
FIG. 18. Same as Fig. 16 for $\Delta = 0.75$.

FIG. 19. Displaced-time correlation ratio $R^{QSHZ}$ for $\Delta = 0.1$ (a), $\Delta = 0.5$ (b), and $\Delta = 0.75$ (c), respectively.
Appendix C: Spectrum

The spectral function for a given operator at zero temperature reads:

\[ A(k, \omega) = \pi \sum_n |\langle n | \hat{O} | 0 \rangle|^2 \delta(E_n - E_0 - \omega). \quad (C1) \]

Here, \(|0\) in Eq. (C1) is the ground state and \(|n\rangle\) are the eigenstates of the Hamiltonian with energy \(E_n\). Given the imaginary-time Monte Carlo data, the spectrum is obtained by using the stochastic analytical continuation approach\(^{22}\) to solve for \(A(k, \omega)\) given \(|\langle 0 | \hat{O}(\tau) | 0 \rangle|\):

\[ \langle 0 | \hat{O}(\tau) | 0 \rangle| = \int d\omega e^{-\tau \omega} A(k, \omega). \quad (C2) \]

Here, \(\hat{O}\) stands for the momentum space operators defined in the main text: \(J^{XY}, \hat{n}, \hat{J}^z\) and \(\hat{c}\). \(A(k, \omega)\) is the corresponding spectral function, denoted as \(A^{QSH}, A^{SSC}, A^{QSH^z}\) and \(A^{sp}\), respectively.

As expected, the single particle spectrum \(A^{sp}\) is clearly gapped in both the QSH and SSC ordered states, as well as across the QSH-SSC transition points. As one can observe from Fig. 20, 21 and 22, \(A^{sp}\) shows no fundamental differences between three considered values of \(\Delta\).

Deep inside the ordered QSH and SSC phases, and irrespective of the anisotropy parameter, \(\Delta\), the order parameter excitations exhibit Goldstone modes stemming from global U(1) symmetry breaking. In particular in Fig. 20 for the case \(\Delta = 0.1\), a linear mode is observed for QSH operator at \(\lambda = 0.04\) and for SSC operator at \(\lambda = 0.065\). The same behavior is visible at \(\Delta = 0.5\) and 0.75 in Fig. 21 and 22.

On the other hand, the excitation of both QSH and SSC order parameters at the critical point show linear dispersion relation. Near the \(\Gamma\) point, the Goldstone mode is expected to give rise to a branch cut reflecting the anomalous dimension:

\[ A_{QSH}(k, \omega) \propto (v^2|k|^2 - \omega^2)^{1 - n_{QSH}/2}, \]
\[ A_{SSC}(k, \omega) \propto (v^2|k|^2 - \omega^2)^{1 - n_{SSC}/2}. \quad (C3) \]

Though the anomalous dimension of two order parameters can in general be different, the velocity \(v\) is uniquely defined at a Lorentz invariant critical point. We mark the velocity of these two excitations in Fig. 20, 21 and 22.

The spectrum of the \(Z\) component of spin current operator \((\hat{J}^z)\) is controversial at the transition point. \(A_{QSH^z}(k, \omega)\) shows a clear gap around the \(\Gamma\) point in the cases of \(\Delta = 0.1\) and 0.5. On the other hand, we observe a that the gap decreases upon reducing the anisotropy. As shown in Fig. 22 and in Fig. 14 for \(\Delta = 0.75\), the value of gap at the \(\Gamma\) is comparable to the finite size gap of the superconducting fluctuations. A consistent picture in terms of easy plane DQCP with deconfined spinons at criticality requires that \(\Delta_{QSH^z}\) to scale to zero in the thermodynamic limit.

Appendix D: Local detection of \(Z_2\) topology using \(\pi\) flux insertion

To detect the topology of our QSH insulator, we employ the magnetic flux insertion approach\(^{33}\) that has successfully been tested in Ref. 31. When \(\pi\) fluxes are pumped locally in a QSH insulator, mid-gap states that carry nontrivial spin quantum numbers are exponentially localized around the flux insertion points. This approach directly probes the \(Z_2\) topological invariant, and we refer the reader to Ref. 31 for a detailed discussion.

Consider the following kinetic Hamiltonian:

\[ \hat{H}_K = -\lambda \sum_{\langle ij \rangle} \left( \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^x e^{iA_{ij}} + \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^y e^{iA_{ij}} \right)^2 + \Delta \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^z e^{iA_{ij}} \]

\[ \hat{H}_A = - \lambda \sum_{\langle \langle ij \rangle \rangle \in \Omega} \left( \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^x e^{iA_{ij}} + \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^y e^{iA_{ij}} \right)^2 \]

\[ + \Delta \sum_{\langle \langle ij \rangle \rangle \in \Omega} \hat{J}_{ij}^z e^{iA_{ij}} \]

\[ \text{where } A_{ij} \text{ is the vector potential that accounts for the pair of } \pi\text{-fluxes. Practically, we consider an arbitrary string connecting the centres of the two hexagons in which we will insert a } \pi\text{-flux. Each time an electron crosses this string it acquires an } e^{i\pi} \text{ phase factor.} \]

Due to the easy plane anisotropy, the dynamical generation of the QSH insulator is associated with long range order of spin currents in the \(U(1)\) plane. Thus, the mid-gap objects localized around the \(\pi\) fluxes are Kramers pairs of spin ‘up’ and ‘down’ states rotating in the \(x - y\) plane, as well as doublets of charge fluxons. The presence of localized spin and charge fluxons can be captured by the low energy spectral weight of \(c_i^\dagger \sigma^x c_i \) \((c_i^\dagger \sigma^y c_i)\) operator:

\[ S^{xy}_\Omega(i) = \int_0^\Omega d\omega S^{xy}(i, \omega) \]
\[ S^{char}_\Omega(i) = \int_0^\Omega d\omega S^{char}(i, \omega) \quad \Omega = 0.25 \]

where

\[ S^{xy}(i, \omega) = \sum_n \int \left| \langle n | c_i^\dagger \sigma^x c_i | 0 \rangle \right|^2 \delta(\omega - E_n - E_0) \]

\[ S^{char}(i, \omega) = \sum_n \int \left| \langle n | c_i^\dagger c_i | 0 \rangle \right|^2 \delta(\omega - E_n - E_0) \]

where \(|n\rangle\) is an energy eigenstate with energy \(E_n\). The energy window of \(\Omega = 0.25\) is well below twice single particle gap (\(\Delta \approx 1.0\)).

The enhanced spectral weight in \(S^{xy}_\Omega(i)\) and \(S^{char}_\Omega(i)\) around \(\pi\) fluxes depicted in Fig. 23 clearly demonstrates
the existence of spin and charge fluxons. This numerically proves that the insulating state that we observe at intermediate values of $\lambda$ is $Z_2$ nontrivial.

We note that our model shows no quantized spin Hall conductivity since there is no $U(1)$ spin conservation at low energy in our system. In other words the topological invariant is the $Z_2$ index and not the so called spin Chern number.

FIG. 20. QSH, SSC, QSHz and single particle spectrum inside two (QSH, SSC) phases and near the critical point, for $\Delta = 0.1$. Blue lines are the momentum dependence of the extrapolated excitation gap of $\hat{J}$ and $\hat{\eta}^+ (\hat{\eta}^-)$ operators. We took $L = 18$.

FIG. 21. Same as Fig. 20 for $\Delta = 0.5$. 
FIG. 22. Same as Fig. 20 for $\Delta = 0.75$. 
FIG. 23. $S_{\Omega}^{xy}(i)$ (a) and $S_{\Omega}^{\text{charge}}(i)$ (b) distribution on a $L = 12$ honeycomb lattice. The simulation is performed deep inside QSH state ($\Delta = 0.1, \lambda = 0.045$). We took $\beta = 24$. 