ANSWER TO THE QUESTIONS OF YANYAN LI AND LUC NGUYEN IN ARXIV:1302.1603

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ABSTRACT. In this note we answer the two questions raised by Y.Y Li and L. Nguyen in their note [LN2] below.

In the note [LN2], Y.Y Li and L. Nguyen raised two questions.

Q1. Whether the maximal radial function is super-harmonic.

Q2. A proof of the property $h \to 0$ as $x \to 0$ for bounded $h$, where $h(x) = w(x) - 2 \log |x|$. 

Answer to Q1: Given a lower semi-continuous function $v$ in $B_R(x_0)$, the maximal radial function of $v$ is defined by

$$\tilde{v}(x) = \inf\{v(y) : y \in \partial B_r(x_0), \ r = d(x, x_0)\},$$

where $B_r(x_0)$ is the geodesic ball of radius $r$ centered at $x_0$. For any $r \in (0, R)$, there is a point $x_r \in \partial B_r(x_0)$ such that $\tilde{v}(x_r) = v(x_r)$.

In page 2445, line -9, the paper [TW] contains the statement “If $v$ is superharmonic, then $\tilde{v}$ is also superharmonic.” This statement should be changed to “If $v$ is superharmonic, then $\tilde{v}$ is also superharmonic with respect to a rotationally symmetric linear operator in $B_r(x_0)$. At any point $x \in B_R(x_0)$, the coefficients of the operator are equal to those of the Laplacian at $x_r$. Note that by the exponential map, the Laplacian operator on a manifold in local coordinates is a linear elliptic operator with variable coefficients.

In [TW], we used a $W^{1,p}$ estimate for super-solutions. This estimate holds for any linear elliptic equations. We would like to thank Y.Y. Li and L. Nguyen for pointing out this inaccuracy in our paper.

Answer to Q2: This question was already answered in my email of November 14, 2012 to Y.Y. Li, which was included at the beginning of Section 4 in [W]. “with the convergence in $W^{1,p}$, if the function $h$ (h is the function in your note) is locally
uniformly bounded, then the interior gradient estimate or the Harnack inequality (for locally bounded solutions) implies the convergence is locally uniform”.

I think if one can understand the proof of $|h| \leq C$ in page 2456, then one should see immediately $h(x) \to 0$ as $x \to 0$, by repeating the proof in page 2456 and using the interior gradient estimate. Let me give the details here.

For any sequence $x_m \to 0$, as in [TW] one makes the rescaling: $x \to x/r_m$ (with $r_m = |x_m|$) such that $\text{dist}(0, x_m) = 1$. Denote $A_r = \{x \mid 1 - r < \text{dist}(0, x) < 1 + r\}$ the annulus. We have shown in Lemma 3.4 [TW] that

$$(i) \quad \int_{A_{7/8}} |h| \to 0 \quad \text{as} \quad m \to \infty.$$

From the proof of Theorem 1.3 (page 2456),

$$(ii) \quad |h| \leq C \quad \text{in} \quad A_{3/4},$$

uniformly in $m$. By the interior gradient estimate, we have

$$(iii) \quad |Dh| \leq C \quad \text{in} \quad A_{1/2},$$

uniformly in $m$. From (i), (ii), and (iii), we conclude that $h \to 0$ in $A_{1/4}$, uniformly. Scaling back, we obtain $h(x) \to 0$ as $x \to 0$.

Let me pointed out that the main body of the paper [TW] is to prove (i). From (i), one easily obtains (ii). The interior gradient estimate (iii) was proved in other papers.

**Remark 1.** When Y.Y. Li asked me Q2 in December 2012, I thought the answer was already given in [W] and didn’t bother to write more. I just simply said “there is no need for further correspondence of this mathematics”. For Q1, I am sure Y.Y. Li also knew the answer above.

**Remark 2.** I didn’t know that Y.Y. Li and L. Nguyen posted their note [LN2] on arXiv until Wednesday last week. I sent the above explanation to them last Friday but have not yet received their response for five days. So I assume my explanation is clear to them.
Based on the questions raised by Y.Y. Li in his emails and in his note [LN2], we need to make the following clarifications for the paper [TW].

(1) (This one is copied from **Answer to Q1** above).
In page 2445, line -9, the statement “If $v$ is superharmonic, then $\tilde{v}$ is also superharmonic.” This statement should be changed to

“If $v$ is superharmonic, then $\tilde{v}$ is also superharmonic with respect to a rotationally symmetric linear operator in $B_r(x_0)$. At any point $x \in B_R(x_0)$, the coefficients of the operator are equal to those of the Laplacian at $x_r$. Note that by the exponential map, the Laplacian operator on a manifold in local coordinates is a linear elliptic operator with variable coefficients.”

Accordingly, Line 1, page 2454, the sentence “Noticing that $\tilde{v}_j$ is superharmonic with respect to the conformal Laplace operator (1.16),” should be changed to “Noticing that $\tilde{v}_j$ is superharmonic with respect to a rotationally symmetric linear elliptic operator,”

(2) (This one is copied from the note [W] below).
After formula (3.29) in page 2455, add

“where $h(x) := w(x) - 2 \log |x| = o(1)$ is in the sense

$$\lim_{r \to 0} r^{-n} \int_{\{r < |x| < 2r\}} |h(x)|dx = 0,”$$

(3) (This is from **Answer to Q2** above).
In page 2456, line 10 after “This is a contradiction” add the new paragraph

“This argument also implies that $h(x) \to 0$ as $x \to 0$. Indeed, $\forall x_m \to 0$, make the above rescaling and denote $A_r = \{x \mid 1 - r < \text{dist}(0, x) < 1 + r\}$. By Lemma 3.4, we have $\int_{A_{7/8}} |h| \to 0$ as $m \to \infty$. The above paragraph tells that $|h| \leq C$. By the interior gradient estimate, $|Dh| \leq C$ in $A_{1/2}$. Hence $h \to 0$ in $A_{1/4}$ uniformly as $m \to \infty$. Scaling back, we obtain $h(x) \to 0$ as $x \to 0$.”

**Acknowledgement.** I would like to thank Y.Y. Li and L. Nguyen for giving me the opportunity to make the above clarifications for the paper [TW]. I am sorry that some parts of the paper was not well written and have caused difficulties for some readers to understand the proof.
References

[LN1] Y.Y. Li and L. Nguyen, A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound, arXiv:1212.0460, Dec. 3, 2012.

[LN2] Y.Y. Li and L. Nguyen, Response to an article of Xu-Jia Wang, arXiv:1302.1603, Feb. 6, 2013.

[TW] N. Trudinger and X.-J. Wang, The intermediate case of the Yamabe problem for higher order curvatures, IMRN, Vol 2010, no.13, pp 2437-2458.

[W] X.-J. Wang, Response to a question of Yanyan Li and Luc Nguyen in their paper “A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound”, arXiv:1212.0460, arXiv:1212.3130, Dec. 13, 2012.

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