HYPERBOLIC GRAPHS OF SURFACE GROUPS

Abstract

We give a sufficient condition under which the fundamental group of a reglued graph of surfaces is hyperbolic. A reglued graph of surfaces is constructed by cutting a fixed graph of surfaces along the edge surfaces, then regluing by pseudo-Anosov homeomorphisms of the edge surfaces. By carefully choosing the regluing homeomorphism, we construct an example of such a reglued graph of surfaces, whose fundamental group is not abstractly commensurate to any surface-by-free group, i.e., which is different from all the examples given in the paper [Mos97].
1 Introduction

The fundamental group of the mapping torus of a pseudo-Anosov homeomorphism of an oriented closed hyperbolic surface is hyperbolic. This was first proved by Thurston. A direct proof was given by Bestvina and Feighn [BF92]. Using their idea, Mosher [Mos97] proved the following theorem.

Consider an oriented closed hyperbolic surface $S$. Let $\Phi_1, \cdots, \Phi_m \in MCG(S)$ be an independent set of pseudo-Anosov mapping classes of $S$, and let $\phi_1, \cdots, \phi_m \in \text{Homeo}(S)$ be pseudo-Anosov representatives of $\Phi_1, \cdots, \Phi_m$ respectively. If $i_1, \cdots, i_m$ are large enough positive integers, then the fundamental group of the graph of spaces $G$, as shown in Figure 1, is a hyperbolic group. In the statement of this theorem, by saying a set $B$ of pseudo-Anosov mapping classes is \emph{independent}, we mean the sets $\text{Fix}(\Phi)$ are pairwise disjoint for $\Phi \in B$, where $\text{Fix}(\Phi)$ consists of the attractor and the repeller of $\Phi$ on the space of projective measured foliations $PMF(S)$.

A graph of surfaces $ST$ consists of an oriented connected finite underlying graph $\Gamma$, a function which assigns to each vertex a closed hyperbolic surface or orbifold, to each edge a closed hyperbolic surface, and another function which assigns to each oriented edge a covering map from the edge surface to the vertex surface of the origin of the edge. In the cases studied in this paper, we change the canonical graph of surfaces by cutting along the edge surfaces, then choosing pseudo-Anosov homeomorphisms of the edge surfaces, then regluing. We call it a \emph{graph of surfaces with pseudo-Anosov regluing}. Thus the mapping torus of a pseudo-Anosov homeomorphism can be considered as this type of space whose underlying graph consists of only one vertex and one edge, and the vertex and edge spaces are the same hyperbolic surface. The case studied by Mosher is another reglued graph of surfaces with the underlying graph consists of only one vertex, in addition the vertex and edge spaces are the same hyperbolic surface $S$.

We shall extend Mosher’s theorem to the general graphs of surfaces with pseudo-Anosov regluing. Theorem 1 says that if the pseudo-Anosov homeomorphisms are
chosen to satisfy an appropriate independence condition, then the fundamental group of the reglued graph of surfaces is word hyperbolic, when these homeomorphisms are replaced with sufficiently high powers of themselves.

We shall describe this cutting and regluing process with more details. Let \( ST \) be a graph of surfaces with the underlying graph \( \Gamma \), let \( E \) be the set of oriented edges of \( \Gamma \), and let \( V \) be the set of vertices of \( \Gamma \). For each \( e \in E \), let \( S_e \) be the corresponding edge surface. For each oriented edge \( e \), there is a finite covering map \( p_e : S_e \to F_{o(e)} \), where \( F_{o(e)} \) is the vertex surface of the origin \( o(e) \) of the edge \( e \). For each inverse pair of oriented edges \( e, e' \), there is an inverse pair of homeomorphisms \( g_e : S_e \to S_e, g_e^{-1} : S_e \to S_e \). Let \( \phi = \{ \phi_e | e \in E \} \), where \( \phi_e : S_e \to S_e \) is a pseudo-Anosov homeomorphism of \( S_e \). Let \( S_{\Gamma}^{\phi} \) be the graph of surfaces with pseudo-Anosov regluing obtained from \( S_{\Gamma} \) by cutting along each \( S_e \) and regluing using \( \phi_e \), i.e., in the reglued graph of surfaces, the effect is to replace the map \( g_e : S_e \to S_e \) by the map \( g_e \circ \phi_e \), for \( e \in E \). Moreover, let \( m = \{ m_e | e \in E \} \), where \( m_e \) are positive integers, and let \( S_{\Gamma}^{\phi^m} \) be the graph of surfaces obtained from \( S_{\Gamma} \) by regluing using \( \phi_e^m \) for each \( e \in E \).

Given a vertex \( v \) of the underlying graph \( \Gamma \), let \( F_v \) be the corresponding vertex surface (or orbifold). For each \( v \in V \), denote \( I_v = \{ i | e_i \text{ is an oriented edge such that the origin of } e_i \text{ is } v \} \). For each \( v \in V \) and each \( i \in I_v \), there is a finite index covering map \( p_i : S_i \to F_v \), where \( S_i \) is a shorthand notation of \( S_{e_i} \). For an oriented edge \( e_i \) has the vertex \( v \) as both of its origin and terminal, the covering maps \( S_i \xrightarrow{p_i} F_v \) and \( S_i \xrightarrow{g_i} S_{\tilde{v}} \xrightarrow{p_{\tilde{v}}} F_v \) might be different, where \( g_i \) is a shorthand notation of \( g_{e_i} \). The portion of \( S_{\Gamma}^{\phi^m} \) around a vertex space \( F_v \) could look like in Figure 2

\[ S_{\tilde{v}} \xrightarrow{p_{\tilde{v}}} F_v \]

For the purpose of Theorem 1, fix a hyperbolic structure on each vertex surface \( F_v \). For each \( v \in V \) and each \( i \in I_v \), suppose \( S_i \) equipped with the pullback metric by the covering map \( p_i : S_i \to F_v \). Hence for each covering map \( p_i \), there is the derivative map \( Dp_i : PS_i \to PF_v \), where \( PS_i, PF_v \) are the projective tangent bundles of \( S_i \) and
For an oriented edge \( e_j \), let \( \phi^{m_j}_j : S_j \to S_j \) be the pseudo-Anosov homeomorphism for the edge \( e_j \), with the stable geodesic lamination \( \Lambda^s_j \subset S_j \); the stable geodesic lamination \( \Lambda^s_j \subset S_j \) of \( (\phi^{m_j}_j) = g_j \phi^{-m_j}_j g_j^{-1} \) is homotopic to the image under \( g_j \) of the unstable geodesic lamination of \( \phi^{m_j}_j \). The geodesic laminations \( \Lambda^s_j \) and \( \Lambda^s_j \) are independent of the choice of the exponent \( m_j \). In the following, let \( T\Lambda^s_i \) denote the unit tangent vector space of \( \Lambda^s_i \).

The main theorem of this paper is

**Theorem 1.** Let \( \Sigma \Gamma \phi^m \) be a graph of surfaces with pseudo-Anosov regluing. Let \( \Gamma \) be its underlying graph. If for each vertex \( v \in \Gamma \), and for each \( i \in I_v \), the derivative maps \( Dp_i|T\Lambda^s_i \) are injections, and their images are disjoint compact subsets of \( PF_v \), then the fundamental group of \( \Sigma \Gamma \phi^m \) is hyperbolic, when \( m_i \in m \) are sufficiently large.

The proof of the hyperbolicity of \( \Sigma \Gamma \phi^m \) depends ultimately on the Combination Theorem of [BF92]. The Combination Theorem says that if the quasi-isometrically embedded condition (which is automatically satisfied in the cases studied in this paper) and the hallways flare condition (which is much more difficult to check) both hold, then \( \Sigma \Gamma \phi^m \) is a hyperbolic space. In order to check the satisfaction of the hallways flare condition, we need to extend the parallel corresponds lemma [Mos97], the key in that paper, to a new version of the parallel corresponds lemma.

The idea of the proof of Theorem 1 is: by applying the new version of parallel corresponds lemma, if the hypothesis of Theorem 1 is satisfied, then the hallways flare condition is satisfied. Therefore the fundamental group of \( \Sigma \Gamma \phi^m \) is hyperbolic.

Here are some applications of this theorem.

First: let \( S \) be a closed hyperbolic surface, let \( G, H \) be finite subgroups of the mapping class group \( \text{MCG}(S) \), and let \( \Phi \in \text{MCG}(S) \) be a pseudo-Anosov mapping class. Suppose \( G, H \) each have trivial intersection with the virtual centralizer of \( \langle \Phi \rangle \) in \( \text{MCG}(S) \), then for sufficiently large \( n \), the subgroup \( A \) of \( \text{MCG}(S) \) generated by \( G, \Phi^n H \Phi^{-n} \) is isomorphic to the free product of these subgroups. Even more, \( A \) is a virtual Schottky subgroup of \( \text{MCG}(S) \), in the sense of [FM02a].

Second: let \( \mathcal{G}_{\phi^m} \) be a graph of surfaces with regluing as in Figure 3, where \( S, F \)

![Figure 3:](image)

are genus 3 and 2 tori, \( \phi : S \to S \) is a pseudo-Anosov homeomorphism. Suppose there exist simple closed curves \( a \subset F \) and \( c \subset S \), as shown in Figure 4, such that
$p^{-1}(a) = c$, $c \subset q^{-1}(a)$, and $q^{-1}(a)$ is disconnected. In addition, suppose that in the group $\text{MCG}(S)$, the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with the deck transformation groups of $p$ and $q$, where $\Phi$ is the mapping class of $\phi$. Then $\pi_1(G_{\phi^m})$ is hyperbolic when $m$ is sufficiently large.

![Figure 4:](image.png)

More interesting, we will see that there exists a pseudo-Anosov homeomorphism $\phi$ of $S$, such that $\pi_1(G_{\phi^m})$ is not commensurate to $\pi_1(S') \rtimes K$, for any oriented, closed hyperbolic surface $S'$, and for any free group $K$, where $G_{\phi^m}$ as the above. More than that, $\pi_1(G_{\phi^m})$ is not even quasi-isometric to any surface-by-free group. Therefore $\pi_1(G_{\phi^m})$ is different from all the hyperbolic groups constructed in [Mos97].

**Problems.** Do there exist some reducible homeomorphisms of the edge surfaces, such that the graph of surfaces with reducible homeomorphism regluing are hyperbolic?

Is Theorem 1 still true when the vertex and edge groups are free groups?

### 2 Preliminaries

In this section, we recall some preliminaries about combinatorial and geometric group theory, and some facts of hyperbolic geometry which will be used later.

**Graphs of surfaces** (The material in this subsection can be found in [SW79] and [Ser80])

Let $\Gamma$ be a connected finite graph, let $e$ be an oriented edge of $\Gamma$, and let $\overline{e}$ be the inverse edge of $e$. The vertex $o(e)$ is called the origin of $e$ and the vertex $t(e)$ is called the terminal of $e$, obviously $o(e) = t(\overline{e})$.

A graph of surfaces $S\Gamma$ consists of a connected finite graph $\Gamma$ and a function which assigns to each vertex $v \in \Gamma$ a closed hyperbolic surface or orbifold $F_v$, to each pair of oriented edges $e$, $\overline{e}$ closed hyperbolic surfaces $S_e$, $S_{\overline{e}}$ and an inverse pair of homeomorphisms $S_e \to S_{\overline{e}}$, $S_{\overline{e}} \to S_e$, and to each edge $e$ a continuous map $p_e : S_e \to F_{o(e)}$, such that $p_e$ induces an injection on the fundamental groups. In most of our cases $p_e$ are covering maps for every edge $e$ of $\Gamma$.

Given a graph of surfaces $S\Gamma$, we can define the total space $S\Gamma$ as the quotient of the disjoint union $(\cup \{F_v | v \in V(\Gamma)\}) \cup (\cup \{S_e \times I | e \in E(\Gamma)\})$ by identifying the equivalent classes: $(s, 0) \sim p_e(s)$ for $(s, 0) \in S_e \times 0$, $p_e(s) \in F_{o(e)}$; $(s, 1) \sim p_{\overline{e}}(s)$ for...
\((s, 1) \in S_e \times 1, \, p_e(s) \in F_{\ell(e)}\). The fundamental group of the graph of surfaces \(\pi_1(S\Gamma)\) is defined to be the fundamental group of the total space \(S\Gamma\). There is a projection map \(\pi : S\Gamma \to \Gamma\) such that each vertex surface \(F_v\) maps to the vertex \(v\) and each \(S_e \times I\) maps to the edge \(e\). \(\pi\) is an onto map.

The universal cover \(\wt{S}\Gamma\) of \(\Gamma\) is a union of copies of the universal covers \(\wt{S}_e \times I\) and \(\wt{F}_v\). In \(\wt{S}\Gamma\), if we identify each copy of \(\wt{F}_v\) to a point and each copy of \(\wt{S}_e \times I\) to a copy of \(I\), then we obtain a graph \(t\) and there is a canonical projection map \(\wt{\pi} : \wt{S}\Gamma \to t\). It is not hard to see that \(t\) is a tree, called the Bass-Serre tree. The action of \(\pi_1(S\Gamma)\) on \(\wt{S}\Gamma\) descends to an action of \(\pi_1(\Gamma)\) on \(t\), where the quotient graph coincides with the original graph \(\Gamma\), and the stabilizers of each vertex and each edge of \(t\) are conjugates of corresponding fundamental groups of \(F_v\) and \(S_e\).

**The Bestvina – Feighn Combination Theorem** (The material in this subsection can be found in [BF92])

For the purpose of this paper, instead of the original combination theorem, we shall state the tailored Bestvina-Feighn Combination Theorem in the context of the graphs of surfaces only.

Let \(\Gamma\) be a graph of surfaces with the underlying graph \(\Gamma\), and let \(\pi : \Gamma \to \Gamma\) be the projection map. Denote the preimage of the midpoint of an edge \(e \in \Gamma\) under \(p\) by \(S_e\). For a vertex \(v \in \Gamma\), we consider the component containing \(v\) of \(\Gamma\) cut open along the midpoints of edges. Let \(X_v\) denote the preimage of this component under \(p\), called vertex space. For any vertex \(v \in \Gamma\), the vertex surface \(F_v\) is a deformation retract of the vertex space \(X_v\).

For each edge \(e\) of \(\Gamma\), the lift to the universal covers of the finite index covering map \(p_e : S_e \to F_{\ell(e)}\) is a quasi-isometry. This is precisely the 'quasi-isometrically embedded condition' in [BF92]. We may omit this condition from the hypothesis of the combination theorem for the cases of the graphs of surfaces.

Define a continuous function \(\Delta : [-k, k] \times I \to S\Gamma\) to be a hallway of length \(2k\), if for any \(i\) from \(-k\) to \(k\), \(\Delta(\{i\} \times I)\) is a geodesic in \(\wt{S}_e(\{i\})\), and \(\Delta((i, i + 1) \times I)\) stays in the interior of \(\wt{X}_e(\{i\})\). Suppose \(\Delta([i, i + 1] \times I)\) stays in the closure of \(\wt{X}_e(\{i\})\). \(\Delta\) is \(\rho\)-thin if \(d_{\wt{X}_e(\{i\})}(\Delta((i, t)), \Delta((i + 1, t))) \leq \rho\) for \(i \in \{-k, -k + 1, \ldots, k - 1\}\) and \(t \in I\). The hallway \(\Delta\) is essential if the edge path \(e(-k) * \cdots * e(k)\) never backtracks in the Bass-Serre tree \(t\), i.e., \(e(i) \neq \ell(i + 1)\) for \(i \in \{-k, \ldots, k - 1\}\). The girth of \(\Delta\) is the length of \(\Delta(\{0\} \times I)\). Let \(\lambda > 1\). The hallway \(\Delta\) is \(\lambda\)-hyperbolic if \(\lambda l(\Delta(\{0\} \times I)) \leq \max\{l(\Delta([-k] \times I)), l(\Delta(\{k\} \times I)))\}\). The graph of surfaces \(\Gamma\) is said to satisfy the hallways flare condition if there exist numbers \(\lambda > 1\) and \(k \geq 1\) such that for any \(\rho\) there exists a constant \(H(\rho)\), such that any \(\rho\)-thin essential hallway of length \(2k\) and girth at least \(H(\rho)\) is \(\lambda\)-hyperbolic.

**Theorem 2.** (Combination Theorem) Let \(\Gamma\) be a graph of surfaces. Suppose that \(\Gamma\) satisfies the hallways flare condition, then the fundamental group of \(\Gamma\) is hyperbolic.

**Remarks:** 1. Notice that in the Bestvina-Feighn’s combination theorem, the vertex spaces are used in defining the hallways; but in the proof of Theorem 1, we use the vertex surfaces instead. We are allowed to do so, because the vertex surface is a
retraction of the corresponding vertex space, and their universal covers are quasi-isometric to each other.

2. The hallways in the combination theorem are 'edge hallways', i.e., the rungs $\Delta(i) \times I$ of the hallway $\Delta$ are geodesics in the edge surfaces. In Theorem 1, the hallways are 'vertex hallways', i.e., $\Delta(i) \times I$ are geodesics in the vertex surfaces. Since the covering map from an edge surface to a vertex space is a quasi-isometry, and the vertex spaces and the vertex surfaces are quasi-isometric, if the vertex hallways flare condition is satisfied then the edge hallways flare condition is satisfied.

3. For the cases studied in this paper, we will prove the hallways flares condition for length 2 hallways only.

**Construction of pseudo-Anosov homeomorphisms** (The materials is covered by [Pen88])

In a surface $S$, $C$ is an essential curve system, if $C = \{c_1, \cdots, c_n\}$, where $c_1, \cdots, c_n$ are non-trivial simple closed curves on $S$ which are pairwise disjoint and pairwise non-homotopy.

Let $C$ and $D$ be two disjoint essential curve systems, $C$ hits $D$ efficiently if $C$ intersect $D$ transversely, and no component on $S \setminus (C \cup D)$ is a bigon, an interior of a disc whose boundary consists of one arc of $C \in C$ and one arc of $D \in D$. We say that $C \cup D$ fills $S$ if the components of the complement of $(C \cup D)$ are disks.

The following shows how to construct pseudo-Anosov homeomorphisms.

**Theorem 3.** ([Pen88]) Suppose that $C$ and $D$ are essential curve systems in an oriented surface $F$ so that $C$ hits $D$ efficiently and $C \cup D$ fills $F$. Let $R(C^+, D^-)$ be the free semigroup generated by the Dehn twists $\{\tau_c^+ : c \in C\} \cup \{\tau_d^- : d \in D\}$. Each component map of the isotopy class of $\omega \in R(C^+, D^-)$ is either the identity or pseudo-Anosov, and the isotopy class of $\omega$ is itself pseudo-Anosov if each $\tau_c^+$ and $\tau_d^-$ occur at least once in $\omega$.

**Surface group extensions** (The materials is covered by [Mos] and [FM02c])

A surface group extension is a short exact sequence of the form

\[1 \rightarrow \pi_1(S, x) \rightarrow \Gamma \rightarrow G \rightarrow 1 \quad (1)\]

where $S$ is a closed, oriented surface of genus $g \geq 2$. The canonical example is the sequence

\[1 \rightarrow \pi_1(S, x) \overset{i}{\rightarrow} \text{MCG}(S, x) \overset{q}{\rightarrow} \text{MCG}(S) \rightarrow 1 \quad (2)\]

where $\text{MCG}(S)$ is the mapping class group of $S$, $\text{MCG}(S, x)$ is the mapping class group of $S$ punctured at $x$. This short exact sequence is universal for surface group extension, meaning that for any extension as in (1), there exists a commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(S, x) & \rightarrow & \Gamma & \rightarrow \rightarrow & G & \rightarrow \rightarrow & 1 \\
\downarrow & & \downarrow & & \alpha & \downarrow & & \\
1 & \rightarrow & \pi_1(S, x) & \overset{i}{\rightarrow} & \text{MCG}(S, x) & \overset{q}{\rightarrow} & \text{MCG}(S) & \rightarrow \rightarrow & 1
\end{array}
\quad (3)
\]
where $\Gamma$ is identified with the pushout group
\[ \Gamma_\alpha = \{(\phi, \gamma) \in MCG(S, x) \times G | q(\phi) = \alpha(\gamma)\}, \tag{4} \]
$\alpha$ is a homomorphism from $G$ to $MCG(S)$, and the homomorphisms $\Gamma \to G$ and $\Gamma \to MCG(S, x)$ are the projection homomorphisms of the pushout group. We are more interested in the case where $\alpha$ is an inclusion.

**Virtual centralizer of $\Phi$** (The material is covered by [Mos]

Given a subgroup $H$ of a group $G$, the virtual centralizer $VC(H)$ of $H$ in $G$ is the subgroup of all $g \in G$ which commute with a finite index subgroup of $H$. The virtual centralizer of an infinite cyclic pseudo-Anosov subgroup has a nice geometric description. Let $PML(S)$ denote the space of projective measured laminations of the surface $S$. Let $\Lambda^s, \Lambda^u \subset PML$ be the fixed points of a pseudo-Anosov mapping class $\Phi$, and let $Fix\{\Lambda^s, \Lambda^u\}$ denote the subgroup in $MCG(S)$ whose elements fix $\Lambda^s, \Lambda^u$ point wise. [Mos] shows that $Fix\{\Lambda^s, \Lambda^u\} = VC(\langle \Phi \rangle)$.

**Facts of hyperbolic geometry** (The material is covered by [BH99] and [CB]

Our proofs make heavy use of the following facts of hyperbolic space, $H^2$, geometry:

**Fact 1.** For any $0 < \delta < 1$, and $D > 0$, there exists $l(\delta, D)$, such that if $\gamma, \alpha$ are geodesic segments of length at least $l(\delta, D)$, and the end points $x, y$ of $\gamma$ have distance at most $D$ from the end points $x', y'$ of $\alpha$ respectively, then there exist subsegments $\gamma' \subset \gamma, \alpha' \subset \alpha$ of lengths at least $(1 - \delta)Length(\gamma)$ and $(1 - \delta)Length(\alpha)$ respectively, such that the Hausdorff distance between $\gamma'$ and $\alpha'$ is less than $\delta$.

Roughly speaking, for any two geodesic segments, if their end points have bounded distances from each other, then most part of them can be arbitrarily close to each other as long as the segments are long enough.

**Fact 2.** Given $k \geq 1, c \geq 0$, there exists a constant $N_0(k, c)$, such that any $(k, c)$ quasi-geodesics line or segment in $H^2$ has Hausdorff distance at most $N_0(k, c)$ from a geodesic line or segment with the same end points.

**Fact 3.** Let $\Lambda_1$ and $\Lambda_2$ be two minimal geodesic laminations filling a hyperbolic surface $S$. If their lifts $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ on the universal cover of $\tilde{S}$ have at least one end point in common, then $\Lambda_1 = \Lambda_2$. A geodesic lamination $\Lambda$ is minimal if every leaf $L$ is dense, that is, $\overline{L} = \Lambda$. A geodesic lamination $\Lambda \subset S$ is a filling lamination if no simple closed curve in $S$ is disjoint from $\Lambda$.

The reason is that two minimal filling surface geodesic laminations either transversely intersect with each other or are equal to each other.

From [FLP+79], we know that the stable and unstable geodesic laminations of a pseudo-Anosov homeomorphism are minimal and filling.

### 3 Main Theorem

We will give a new version of Mosher’s parallel corresponds lemma and use it to prove Theorem 1. Moreover we will reformulate the hypothesis of Theorem 1. The original corresponds lemma of Mosher is in [Mos97].
3.1 New version of the parallel corresponds lemma

Consider a pseudo-Anosov mapping class $\Phi \subset MCG(S)$, let $\phi \in Homeo(S)$ be a pseudo-Anosov representative with the stable and unstable measured foliations $f^s_\phi$, $f^u_\phi$. Recall that the transverse measures on $f^s_\phi$ and $f^u_\phi$ define a singular Euclidean structure on $S$, with isolated cone singularities. We call the leaves of $f^s_\phi$ the horizontal leaves and the leaves of $f^u_\phi$ the vertical leaves. The singular Euclidean structure determines a metric $d_\phi$ on $S$ for which each path can be homotopic to a unique geodesic rel end points. The lifts to the universal covers of the hyperbolic metric and the singular Euclidean metric are quasi-isometric equivalent.

In the following, for a homotopy class $\gamma$ of a curve rel. end points, let $\gamma^h$ denote the hyperbolic geodesic segment in the homotopy class of $\gamma$, and let $\gamma^E$ denote the singular Euclidean geodesic segment in the same homotopy class. Let $|\cdot|$ denote the hyperbolic metric, and let $|\cdot|_E$ denote the singular Euclidean metric. For a homotopy class $\gamma$, let $|\gamma|$ denote the hyperbolic length of $\gamma^h$, let $|\gamma|_E$ denotes the singular Euclidean length of $\gamma^E$.

Given $0 < \eta < 1$, define $\text{slope}^\eta_\phi$ to be the set of all homotopy classes $\gamma$, such that the (unsigned) Euclidean angle between $\gamma^E$ and $f^s_\phi$ is at least $\eta$, on a subset of $\gamma^E$ of length at least $\eta \cdot \text{Length} \gamma^E$. Given $\lambda > 1$, let $\text{stretch}^\lambda_\phi = \{ \gamma \mid |(\phi(\gamma))| > \lambda |\gamma| \}$. Let $n$ be a large enough integer, such that if the vector $v \in E^2$ has an angle at least $\eta$ with the horizontal axis, then the matrix

$$
\begin{pmatrix}
\lambda^{-n}_\phi & 0 \\
0 & \lambda^n_\phi
\end{pmatrix}
$$

stretches $v$ by a factor of at least $\lambda/\eta$, where $\lambda_\phi = \lim_{i \to -\infty} |\phi^i(\alpha)|^{1/i}$ is the stretching factor of $\phi$, $\alpha$ is a simple closed geodesic on $S$. Since the singular Euclidean metric is quasi-isometric to the hyperbolic metric, it follows that given $\phi$, $0 < \eta < 1$, and $\lambda > 1$, there exists $N$ such that if $n \geq N$, then $\text{slope}^\eta_\phi \subset \text{stretch}^\lambda_\phi$.

An $\eta$-lever is a homotopy from a singular Euclidean geodesic segment $\alpha$ to a horizontal segment $\beta$, where $\beta$ is a segment of a nonsingular leaf of the horizontal foliation $f^s_\phi$, such that each track of the homotopy is a vertical geodesic segment, maybe degenerate, and each point of $\text{int}(\alpha)$ is disjoint from singularities during the homotopy, and $\text{int}(\alpha)$ makes an angle at most $\eta$ with the horizontal leaves. In [Mos97], $\beta$ is not necessary to be a segment of a nonsingular leaf. But we can always make $\beta$ be a segment of a nonsingular leaf, because there exist nonsingular leafs which are arbitrary close to a singular leaf. Notice that the angle between a singular Euclidean geodesic and the horizontal leaves changes only when the singular Euclidean geodesic passes a singularity. Therefore the interior of $\alpha$ has a constant angle with the horizontal leaf.

A lever is denoted by $(\alpha, \beta)$, where $\alpha$ is called the inclined edge of the lever, and $\beta$ is called the horizontal edge of the lever. A lever is maximal if and only if a singularity contained in the track of each end point of $\alpha$. The length of the lever is $|\alpha|_E$, the height of the lever is the maximum length of the tracks of the points of $\alpha$, which is achieved at the endpoints.

Proposition 4. For any $l$, $H > 0$, there exists $\eta(l, H) > 0$, so that every maximal $\eta$-lever has length at least $l$ and height at most $H$. 
The proof is given in the first seven paragraphs of the proof of the sublemma on page 3451 in [Mos97]. This proposition will be used in the proof of the following lemma.

In the proof of the following lemma, we need some facts. It is well known that the measured foliations $f^s_\phi$, $f^u_\phi$ can be straightened to measured geodesic laminations $l^s_\phi$, $l^u_\phi$. Actually, there is a 1-1 correspondence between leaves of $l^s_\phi$ and smooth leaves of $f^s_\phi$, where a smooth leaf is either a nonsingular leaf or the union of two singular half-leaves meeting at a singularity with angle $180^\circ$. Similarly for $f^u_\phi$. The singularities are discrete, so the length of any geodesic between them has a positive lower bound.

**Lemma 5. (New version of Parallel Corresponds lemma)** Given any pseudo-Anosov homeomorphism $\phi$ and $0 < \epsilon < 1$, there exist $0 < \eta < 1$ and $L > 0$ such that for any homotopy class $\gamma$, if $\gamma \notin \text{slope}_0^n$ and $|\gamma|_E \geq L$, then on a subset of $\gamma^h$ of length at least $(1 - \epsilon)\text{Length}(\gamma^h)$, the distance between the tangent line of $\gamma^h$ and the set $l^s_\phi$, measured in $PS$, is at most $\epsilon$.

The differences between the Parallel Corresponds lemma in [Mos97] and this new version are as follows. In [Mos97], the Parallel corresponds lemma works only for closed based geodesics, and the word metric is used to define the stretching factor; in this paper, the new version of the parallel corresponds lemma works for non closed geodesics as well, and the hyperbolic metric is used to define the stretching factor.

**Proof.** The first step is to find long subsegments $\alpha_i \subset \gamma^E$ and segments $\beta_i$ of leaves of $f^s_\phi$, such that $\alpha_i$ is homotopic to $\beta_i$ by homotoping through short paths. Then we shall project $\alpha_i$ to a subsegment of $\gamma^h$ and project $\beta_i$ to a segment of a leaf $B^h_i$ of $l^s_\phi$, and show that a big portion of these projections are very close to each other. Finally we shall prove most part of $\gamma^h$ are covered by big portion of these projections.

For $\gamma \notin \text{Slope}_0^n$, let $\{(\alpha_i, \beta_i)\}$ be the set of all maximal $\eta$-levers of $\gamma^E$, where the inclined edge $\alpha_i$ is a subsegment of $\gamma^E$ and the horizontal edge $\beta_i$ is a segment of some non-singular leaf $B^E_i$ of $f^s_\phi$.

Step 1: first, let $H = 1$, by proposition 4, we know that for any $l > 0$, there exists $\eta > 0$ such that every maximal $\eta$-lever $\{(\alpha_i, \beta_i)\}$ has length at least $l$ and height at most $H = 1$. The first step is proven.

Step 2: we shall construct long subsegments of $\gamma^h$ from the inclined edges $\alpha_i \subset \gamma^E$ of the maximal levers, such that these long subsegments of $\gamma^h$ have small distance with $l^s_\phi$ measured in $PS$. In the rest of this lemma, the distance and length mean hyperbolic distance and length, otherwise we will use the notations Euclidean distance and length. I will use the notation "$E$" to represent the Euclidean distance and length.

We know that any non-singular leaf $B^E_i$ of $f^s_\phi$ is a $k,c$ quasi-geodesic under the hyperbolic metric, and it can be straightened to a unique leaf $B^h_i$ of $l^s_\phi$. Let $\delta_i \subset \gamma^h$ and $\sigma_i \subset B^h_i$ denote the closest point projections from $\alpha_i \subset \gamma^E$ to $\gamma^h$, and from $\beta_i \subset B^E_i$ to $B^h_i$ respectively. We shall see that most portion of $\delta_i$ has small distance with $\sigma_i$, for all $i$.

Since $\gamma^E$ is a $k,c$ quasi-geodesic segment contained in the $N_0(k,c)$ neighborhood of $\gamma^h$, and $\delta_i$, $\alpha_i$ are subsegments of $\gamma^h$, $\gamma^E$ respectively, it follows that the distances
between the end points of $\delta_i$ and $\alpha_i$ are not greater than $N_0$. For the same reason, the distance between the end points of $\sigma_i$ and $\beta_i$ are not greater than $N_0$. The singular Euclidean distances between the end points of $\beta_i$ and $\alpha_i$ is less than the height $H = 1$. The hyperbolic distances between the end point of them are at most $mk$ for some $m > 0$, because the singular Euclidean and hyperbolic metric are $k,c$ quasi-isometric to each other. Therefore the distances between the end points of $\delta_i$ and $\sigma_i$ are less than $2N_0 + mk$. According to Fact 1, for any $\epsilon_1 > 0$, there exists $L_1$ depending on $2N_0 + mk$ and $\epsilon_1$, if the length of $\delta_i$ is greater than $L_1$, then more than $(1 - \epsilon_1)|\delta_i|$ part of $\delta_i$ has distance less than $\epsilon_1$ with $\sigma_i$.

The condition on the length of $\delta_i$ greater than $L_1$ is easy to satisfy. Since $\alpha_i$ is a quasi-geodesic segment whose end points have distances less than $N_0$ with the end points of $\delta_i$, there exists $l_1 > 0$, such that if the Euclidean length of $\alpha_i$ is greater than $l_1$, then the length of $\delta_i$ is greater than $L_1$. By applying the step 1, we may now choose $\eta$ small enough, so that the Euclidean length of $\alpha_i$ is greater than $l_1$ for any $i$. Therefore more than $(1 - \epsilon_1)|\delta_i|$ part of $\delta_i$ has distance less than $\epsilon_1$ with $\sigma_i$.

So far, we have proved that for any $\epsilon_1$, there exists $\eta$, such that if $\gamma \notin \text{slope}^{\eta}_\phi$, then we can locate long subsegments $\delta_i$ of $\gamma^h$, such that more than $(1 - \epsilon_1)$ of the length of $\delta_i$ has distance less than $\epsilon_1$ with $\sigma_i$. 

Step 3: we will prove that $(1 - \epsilon_1)\sum_i |\delta_i|$ part of $\cup_i(\delta_i)$ covers most part of $\gamma^h$. We call this $(1 - \epsilon_1)\sum_i |\delta_i|$ part of $\cup_i(\delta_i)$ the 'good' part of $\gamma^h$.

Since $\gamma \notin \text{slope}^{\eta}_\phi$, on a subset of $\gamma^E$ of length at least $(1 - \eta)|\gamma|_E$, the angle between $\gamma^E$ and $f_\phi^{s_i}$ is less than $\eta$, i.e., the $\eta$-levers cover more than $(1 - \eta)$ part of $\gamma^E$. The worst situation is that the two end subsegments of $\gamma^E$ are covered by $\eta$-levers with lengths less than $l_1$. In this case, after straightening, the end subsegments of $\gamma^h$ may not have distances less than $\epsilon_1$ with $B^h$. We will only prove this lemma for the worst situation, i.e., more than $(1 - \epsilon_1)|\gamma|_E$ part of $\gamma^E$ is covered by the union of the maximal $\eta$-levers $(\alpha_i, \beta_i)$ and two end $\eta$-levers which cover the two end segments of $\gamma^E$ respectively and with lengths less than $l_1$.

In the following the quasi-isometries will be replaced by bi-Lipschitz maps when dealing with long segments. In the rest of this proof, let $|\alpha_i|$ denote the length of the hyperbolic geodesic which is homotopic to $\alpha_i$ rel. end points, and let $|\alpha_i|_E$ denote the Euclidean length of the singular Euclidean geodesic $\alpha_i$. Keep in mind that none of the following $\delta_i$ is the projection of an end subsegment of $\gamma^E$.

\[
(1 - \epsilon_1)\sum_i |\delta_i| \geq (1 - \epsilon_1)(\sum_i (|\alpha_i| - 2N_0))
\]

\[
\geq (1 - \epsilon_1)(\sum_i (|\alpha_i|_E/k - 2N_0))
\]

According to Proposition 4, we can take $\eta$ to be small enough, so that $|\alpha_i|_E \geq l_2 = 4kN_0$ for any $i$

\[
\geq (1 - \epsilon_1)\frac{\sum_i |\alpha_i|_E}{2k}
\]
Since the union of the $\eta$-levers—the maximal $\eta$-levers and the two end $\eta$-levers, covers more than $(1 - \epsilon_1)|\gamma|_E$ part of $\gamma^E$, and we suppose that the two end $\eta$-levers have lengths less than $l_1$,

$$\geq (1 - \epsilon_1)(1 - \epsilon_1)|\gamma|_E - 2l_1$$

Take $|\gamma|_E$ to be long enough, so that $|\gamma|_E \geq L_2 = \frac{2l_1}{\epsilon_1}$

$$\geq (1 - \epsilon_1)(1 - 2\epsilon_1)|\gamma|_E \geq (1 - 2\epsilon_1)^2|\gamma|_E$$

Hence, $(1 - \epsilon_1)\sum_i |\delta_i| \geq \frac{(1 - 2\epsilon_1)^2|\gamma|_E}{2k}.$

The ‘bad’ parts of $\gamma^h$ are of three kinds. The first kind of bad part is the two end subsegments of $\gamma^h$ which have lengths less than $L_1$. The sum of the lengths of the end subsegments of $\gamma^h$ is at most $2L_1$. We can take $|\gamma|_E$ to be big enough such that $2L_1 \leq \epsilon_1|\gamma|_E$.

The second kind of bad part of $\gamma^h$ is the $\epsilon_1|\delta_i|$ part of $\delta_i$’s which may be out of the $\epsilon_1$ neighborhood of $\sigma_i$. Since the projection map can not prolong length, and the distances between the ends of $\alpha_i$ and $\delta_i$ are not greater than $N_0$,

$$\sum_i \epsilon_1|\delta_i| \leq \epsilon_1 \sum_i (|\alpha_i| + 2N_0)$$

We can take $\eta$ to be small enough, so that $|\alpha_i|_E \geq l_2 = 4kN_0$ for any $i$. The singular Euclidean metric and the hyperbolic metric are $k$ bi-Lipschitz shows that $|\alpha_i|_E \leq k|\alpha_i|$. Therefore $2N_0 \leq 2kN_0 \leq \frac{|\alpha_i|}{2}$

$$\leq \epsilon_1 \frac{3}{2} \sum_i |\alpha_i| \leq \epsilon_1 \frac{3}{2} k \sum_i |\alpha_i|_E \leq \epsilon_1 2k|\gamma|_E$$

The third kind of bad part of $\gamma^h$ are the projections of $\epsilon_1 |\gamma|_E$ part of $\gamma^E$ which has slope greater than $\epsilon_1$ with $f_*$. Let $\xi_i$ denote this kind of subsegment of $\gamma^E$. There is a lower bound $b$ of the Euclidean lengths of $\xi_i$ for all $i$, which equals the minimum of the Euclidean distances between singularities. The sum of the lengths of the projections from $\xi_i$ to $\gamma^h$ is at most $\sum_i (k|\xi_i|_E + c) \leq \sum_i (k|\xi_i|_E + (n - 1)kb) \leq n \sum_i (k|\xi_i|_E) \leq nk\epsilon_1|\gamma|_E$, for some $n$ satisfies $c \leq (n - 1)kb$. 

Therefore, the length of the ‘bad’ part of $\gamma^h$ is at most the sum of the above three kinds, which is $(2k + 1 + nk)e_1|\gamma|_E$. Hence the ratio of the ‘good’ part of $\gamma^h$ to the ‘bad’ part of $\gamma^h$ is at least $\frac{(1-2\epsilon_1)^2}{2k(1+2k+nk)e_1}$. It is easy to see, for any constant $\epsilon$ there exists a small enough $\epsilon_1$, such that the ratio of the ‘good’ part of $\gamma^h$ to $\gamma^h$ is at least $(1-\epsilon)$.

To recap: for any $\epsilon > 0$, we can choose small enough $\epsilon_1$, so that $\frac{(1-2\epsilon_1)^2}{2k(1+2k+nk)e_1}$ is greater than $1-\epsilon$, therefore the ‘good’ part of $\gamma^h$ covers more than $(1-\epsilon)$ of the total length of $\gamma^h$. Then choose $\eta$ small enough so that if $\gamma^E \notin \text{slope}^n_{\phi}$, then more than $(1-\epsilon)|\delta_i|$ part of $\delta_i$ has distance less than $\epsilon_1$ with $\sigma_i$. In addition, take $|\gamma|_E$ to be at least $L$, where $L = \max\{L_2, 2L_1/\epsilon_1\}$. Hence if $\eta$ is small enough, $\gamma \notin \text{slope}^n_{\phi}$ and $|\gamma|_E \geq L$, then most part of $\gamma^h$ has distance at most $\epsilon$ to $l^h_s$, measured in $PS$.

Given a geodesic lamination $\Lambda$ and $0 < \epsilon < 1$, let $WN(\Lambda)$ denote the set of all the homotopy class $\gamma$, so that on a subset of $\gamma^h$ of length at least $(1-\epsilon)\text{Length}(\gamma^h)$, the distance from the tangent line of $\gamma^h$ to the set $\Lambda$, measured in $PS$, is at most $\epsilon$. Using this notation, the parallel corresponds lemma says that for any $0 < \epsilon < 1$, there exists $0 < \eta < 1$ and $L > 0$, such that if $\gamma \notin \text{slope}^n_{\phi}$ and $|\gamma|_E \geq L$, then $\gamma \in WN(\Lambda^s)$, where $\Lambda^s$ is the measured stable geodesic lamination of $\phi$.

3.2 Proof of the main theorem

Proof of Theorem 1. We shall prove that there exist $\lambda > 1$ and $C > 0$, so that for any vertex $w \in \Gamma$, if a based geodesic segment $\gamma^h_w \subset F_w$ has length at least $C$, then all but at most one preimages of it are stretched by corresponding $\phi^m_i$ by a factor of at least $\lambda$, for any $i \in I_w$, where $I_w = \{i | e_i$ is an oriented edge such that the origin of $e_i$ is $w\}$. Hence the hallways flare condition is satisfied. Therefore $ST\Gamma_{|\varphi^m}$ is a hyperbolic surface.

Let $v$ be a vertex of $\Gamma$, let $\gamma^h_v \subset F_v$ be a based geodesic segment. Consider the set $\Sigma = \bigcup_{i \in I_v} p^{-1}_i(\gamma^h_v)$, where $p^{-1}_i(\gamma^h_v)$ is the set of all preimages of $\gamma^h_v$ under the map $p_i$. Notice that all the elements of $\Sigma$ are based geodesics, since the edge surfaces of $ST\Gamma_{|\varphi^m}$ equipped with the pullback metrics.

First, we claim that there exist $0 < \epsilon_0 < 1$ and $H_0 > 0$, such that if the length of $\gamma^h_v$ is greater than $H_0$, then at most one of the elements of $\Sigma$, say $\beta \in p^{-1}_i(\gamma^h_v)$, such that $\beta \in WN(\Lambda^s)$, for some $i_0 \in I_v$; all other elements of $\Sigma$ are not contained in $WN(\Lambda^s)$ for corresponding $\Lambda^s$. Second, according to Lemma 5, for this $\epsilon_0$, there exist $0 < \eta(\epsilon_0) < 1$ and $L(\epsilon_0) > 0$, such that any $\alpha \in \Sigma$ with length $|\alpha| = |\gamma^h_v|$ greater than $L(\epsilon_0)$, if $\alpha \notin WN(\Lambda^s)$, then $\alpha \in \text{slope}^n_{\phi_i}$. Therefore $\alpha$ is stretched by $\phi^m_i$ by a factor of at least $\lambda$ for sufficiently large $m_j$. Combining these, we know that for any $\gamma^h_v$ with length greater $C = \max\{H, L(\epsilon_0)\}$, all but at most one preimages of $\gamma^h_v$ are stretched by corresponding $\phi^m_i$ by at least a factor $\lambda$.

Suppose the claim is not true. Namely for any $\epsilon_n \rightarrow 0$, and any $H_n \rightarrow \infty$, there exist based geodesic segments $\gamma^h_n \subset F_v$ with lengths at least $H_n$, by passing to a subsequence, without loss of generality, suppose $A^h_n \in p^{-1}_1(\gamma^h_n)$ and $B^h_n \in p^{-1}_2(\gamma^h_n)$,
such that $A_n^h \in WN_{\kappa}(\Lambda_i^s)$ and $B_n^h \in WN_{\kappa}(\Lambda_2^s)$. Project $A_n^h$ and $B_n^h$ to $\Lambda_i^s$ and $\Lambda_2^s$ respectively, there exist long subsegments $\nu_n \subset \Lambda_i^s$ and $\omega_n \subset \Lambda_2^s$, such that $|\nu_n|, |\omega_n| \to \infty$, and the distance between $Dp_1[T\nu_n]$ and $Dp_2[T\omega_n]$ converges to zero. This conflicts with the fact that $Dp_1[T\Lambda_i^s]$ and $Dp_2[T\Lambda_2^s]$ are disjoint.

\[ \square \]

### 3.3 Reformulation of Theorem 1

Notations here are the same as in the introduction. The only difference is the edge surfaces are not necessary equipped with the pullback metrics here.

Let $v$ be a vertex of $\Gamma$, let $y$ be the base point of $F_v$, and let $I_v$ be as defined before. Consider the set $p_i^{-1}(y) \subset S_i$ of all the points of $S_i$ that cover $y$ via the map $p_i$, for $i \in I_v$. Denote $X = \cup_{i \in I_v} p_i^{-1}(y)$.

Suppose $a \in p_i^{-1}(y)$, choose a lift $\tilde{p}_a : (\tilde{S}_i, \tilde{a}) \to (\tilde{F}_v, \tilde{y})$, where $\tilde{S}_i$ and $\tilde{F}_v$ are the universal covers of $S_i$ and $F_v$ respectively. Let $\Lambda_i^s \subset S_i$ be the stable lamination of $\phi_i$, and let $\Lambda_i^s \subset \tilde{S}_i$ be the lift of $\Lambda_i^s$. Notice that $\partial \tilde{p}_a(\Lambda_i^s) \subset \partial \tilde{F}_v$ is well defined independent of the choice of $\tilde{y}, \tilde{a}$. If for any $a \neq b \in X$, $\partial \tilde{p}_a(\Lambda_i^s) \cap \partial \tilde{p}_b(\Lambda_j^s) = \emptyset$, where $a \in p_i^{-1}(y), b \in p_j^{-1}(y)$, then we say $v$ satisfies the disjointness condition. We only ask $a \neq b$, but $i$ may equal to $j$. The reformulation of Theorem 1 is the following.

**Theorem 6.** Let $\Sigma_{\varphi \smash{\kappa}}$ be a finite graph of surfaces with underlying graph $\Gamma$. If for any vertex $v \in \Gamma$, the disjointness condition is satisfied, then $\pi_1(\Sigma_{\varphi \smash{\kappa}})$ is a hyperbolic group, when $m_i \in \mathbf{m}$ are sufficiently large.

We shall show the equivalence of the hypothesis of Theorem 1 and Theorem 6.

First, suppose $Dp_i(T\Lambda_i^s)$ is disjoint from $Dp_j(T\Lambda_j^s)$, for $i \neq j$. Then the images of the leaves $\Lambda_i^s$ under the map $p_i$ must transversely intersect the images of the leaves $\Lambda_j^s$ under the map $p_j$. Thus the end points of their lifts in $\tilde{F}$ are disjoint.

Second, suppose $Dp_i(T\Lambda_i^s)$ is injection for all $i$. If $\partial \tilde{p}_{a_1}(\tilde{\Lambda}_i^s) \cap \partial \tilde{p}_{a_2}(\tilde{\Lambda}_i^s) \neq \emptyset$, for some $a_1, a_2 \in p_i^{-1}(y)$, then there exist leaves $\tilde{L}_1, \tilde{L}_2 \subset \tilde{\Lambda}_i^s$, such that $\tilde{p}_{a_1}(\tilde{L}_1) = \tilde{p}_{a_2}(\tilde{L}_2)$. It contradicts with the injectiveness of $Dp_i(T\Lambda_i^s)$. We have finished the proof of one direction.

Suppose $Dp_i(T\Lambda_i^s)$ is not disjoint with $Dp_j(T\Lambda_j^s)$, i.e., there exist leaves $L \subset \Lambda_i^s$ and $J \subset \Lambda_j^s$, such that $Dp_i(L) = Dp_j(J)$. Therefore there exist a lift $\tilde{L}$ of $L$, a lift $\tilde{J}$ of $J$, such that $\tilde{p}_{a_1}(\tilde{L}) = \tilde{p}_{a_2}(\tilde{J})$ for some $a_1 \in p_i^{-1}(y)$ and some $b \in p_j^{-1}(y)$. It conflicts with the hypothesis of Theorem 6. Similar proof for the injections of $Dp_i(T\Lambda_i^s)$ for all $i$.

### 4 Applications

The theorem below will be used to prove Corollary 8.
Corollary 8. Let $G$, $H$ be finite subgroups of $\text{MCG}(S)$, and let $\Phi \in \text{MCG}(S)$ be a pseudo-Anosov mapping class. If the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with $G$ and $H$, then $\langle G, \Phi^M H\Phi^{-M} \rangle$ is a free product in $\text{MCG}(S)$, i.e., $\langle G, \Phi^M H\Phi^{-M} \rangle \cong G \ast \Phi^M H\Phi^{-M}$, and its extension group is hyperbolic, for sufficiently large $M$.

Remark: if $G$ is a finite subgroup of $\text{MCG}(S)$, then $G$ has a faithful representation, still called $G \subset \text{Homeo}(S)$. The quotient $S/G$, called $F_0$, is a hyperbolic surface or orbifold. There exists a canonical embedding $i: \mathcal{PML}(F_0) \hookrightarrow \mathcal{PML}(S)$, where $\mathcal{PML}$ is the projective measured geodesic laminations space. Given a pseudo-Anosov mapping class $\Phi \in \text{MCG}(S)$, if the stable and unstable geodesic laminations $\Lambda^s, \Lambda^u \notin i(\mathcal{PML}(F_0))$, then the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with $G$. Therefore, it is very easy to find $\Phi \subset \text{MCG}(S)$ which satisfies the hypothesis of this corollary.

Proof. Let the symbols $G$, $H$ denote both the finite groups of $\text{MCG}(S)$ and their faithful representations in $\text{Homeo}(S)$. Let $F_0 = S/G$, $F_1 = S/H$. Let $p : S \to F_0$, $q : S \to F_1$ denote the corresponding covering maps, and let $p_* : \pi_1(S) \to \pi_1(F_0)$, $q_* : \pi_1(S) \to \pi_1(F_1)$ denote the induced maps on fundamental groups.

Let $G\Gamma$ be the graph of groups:

$$
\begin{align*}
\pi_1(F_0) \xrightarrow{p_*} \pi_1(S) \xrightarrow{\Phi^M} \pi_1(S) \xrightarrow{q_*} \pi_1(F_1)
\end{align*}
$$

$\pi_1(G\Gamma)$ is the fundamental group of the graph of surfaces $ST$:

$$
\begin{align*}
F_0 \xrightarrow{p} S \xrightarrow{\Phi^M} S \xrightarrow{q} F_1
\end{align*}
$$

where $\phi \in \text{Homeo}(S)$ is a pseudo-Anosov representative homeomorphism of $\Phi$.

There exists a short exact sequence

$$
1 \to \pi_1(S, x) \to \Gamma_{G \ast \Phi^M H\Phi^{-M}} \to G \ast \Phi^M H\Phi^{-M} \to 1
$$

It is not hard to see that $\Gamma_{G \ast \Phi^M H\Phi^{-M}}$ is isomorphic to $G \ast \pi_1(S) \Gamma_{\Phi^M H\Phi^{-M}}$, and $\Gamma_{G \ast \pi_1(S) \Gamma_{\Phi^M H\Phi^{-M}}}$ is isomorphic to $\pi_1(G\Gamma)$.

According to Theorem 7, if $\pi_1(G\Gamma)$ is a word hyperbolic group, then $\delta : G \ast \Phi^M H\Phi^{-M} \to \text{MCG}(S)$ has finite kernel. Since $G$ and $\Phi^M H\Phi^{-M}$ are finite groups, by applying Theorem 3.11 of Scott and Wall [SW79], a normal subgroup of $G \ast \Phi^M H\Phi^{-M}$ must be trivial or finite index. Therefore $\delta$ is an injection, which tells us that $\langle G, \Phi^M H\Phi^{-M} \rangle \cong G \ast \Phi^M H\Phi^{-M}$.

In order to prove $\pi_1(G\Gamma)$ is word hyperbolic, we only need to show that $ST$ is a hyperbolic graph of surfaces.
Let \( y \in F_0 \) be the base point, let \( \{x_1, \ldots, x_r\} = p^{-1}(y) \) denote the preimages of \( y \) under the covering map \( p \), and let \( \tilde{x}_i \in \tilde{S} \) be a covering point of \( x_i \) for \( i \in \{1, \ldots, r\} \). Let \( \tilde{p}_i : (\tilde{S}, \tilde{x}_i) \to (F_0, \tilde{y}) \) be a lift of \( p \), let \( D_{ik} : (\tilde{S}, \tilde{x}_i) \to (S, x_k) \) be a deck transformation of covering map \( p \), and let \( \tilde{D}_{ik} : (\tilde{S}, \tilde{x}_i) \to (\tilde{S}, \tilde{x}_k) \) be a lift of \( D_{ik} \).

According to Theorem [6] if \( \partial \tilde{p}_1(\tilde{\Lambda}^s) \subset \partial \tilde{F}_0 \) are pairwise disjoint on \( \partial \tilde{F}_0 \), and the similar condition holds on \( \partial \tilde{F}_1 \), then \( \Delta \) is hyperbolic.

In the following, we only prove that \( \partial \tilde{p}_1(\tilde{\Lambda}^s) \) and \( \partial \tilde{p}_2(\tilde{\Lambda}^s) \) are disjoint; a similar argument holds for the pairwise disjointness of \( \{\partial \tilde{p}_i(\tilde{\Lambda}^s)\} \) for all \( i \in \{1, \ldots, r\} \), and the pairwise disjointness of \( \{\partial \tilde{q}_j(\tilde{\Lambda}^u)\} \) for all \( j \).

Since \( \tilde{p}_1 = \tilde{p}_2 \tilde{D}_{12}, \tilde{p}_1(\tilde{\Lambda}^s) = \tilde{p}_2 \tilde{D}_{12}(\tilde{\Lambda}^s) \). Hence if the boundary points of the images of \( \tilde{\Lambda}^s \) under \( \tilde{p}_1 \) and \( \tilde{p}_2 \) have one point in common, then \( \tilde{D}_{12}(\tilde{\Lambda}^s) \) and \( \tilde{\Lambda}^s \) have one end point in common. Since \( \tilde{D}_{12}(\tilde{\Lambda}^s) \) and \( \tilde{\Lambda}^s \) are the lift of the geodesic laminations \( D_{12}(\Lambda^s) \) and \( \Lambda^s \) respectively, by Fact 3, we know \( D_{12}(\Lambda^s) = \Lambda^s \), where \( D_{12} \) considered as an element of \( G \subset MCG(S) \). Applying Theorem 3.5 in [Mos], if \( D_{12}(\Lambda^s) = \Lambda^s \), then \( D_{12} \) is contained in the virtual centralizer of \( \langle \Phi \rangle \). This contradicts with the hypothesis that the virtual centralizer of \( \langle \Phi \rangle \) has trivial intersection with \( G \).

Let \( G_{\phi^m} \) as in Figure 3, where \( S, F \) are genus 3 and 2 tori. Let \( p : S \to F \) and \( q : S \to F \) be covering maps, and let \( \phi \) be a pseudo-Anosov homeomorphism of the mapping class \( \Phi \). Abusing of notations, we use \( D_p, D_q \) for both the deck transformations of \( p, q \) and the mapping classes of the deck transformations. It is easy to see that the deck transformation group \( GD_p \) of \( p \) contains only two elements, \( D_p \) and the identity, the same is true for the deck transformation group of \( q \). Abusing of notations, we let \( GD_p \) denote both the deck transformation group of \( p \) and its image in \( MCG(S) \).

**Corollary 9.** Suppose \( a : S^1 \to F \) and \( c : S^1 \to S \) are simple closed curves such that \( p^{-1}(a(S^1)) = c(S^1) \), \( c(S^1) \subset q^{-1}(a(S^1)) \), and \( q^{-1}(a(S^1)) \) is disconnected, as in Figure 4. In addition, suppose the virtual centralizer of \( \langle \Phi \rangle \) has trivial intersection with the images of the deck transformation groups of \( p \) and \( q \) in \( MCG(S) \). Then \( \pi_1(G_{\phi^m}) \) is a hyperbolic group, when \( m \) is sufficiently large.

**Proof.** Let \( z \) be the base point of \( F \), let \( x_1, x_2 \) be the covering points of \( z \) through the covering map \( p \), and let \( y_1, y_2 \) be the covering points of \( z \) through the covering map \( q \). Let \( \tilde{p}_1 : (\tilde{S}, \tilde{x}_1) \to (\tilde{F}, \tilde{z}) \) and \( \tilde{p}_2 : (\tilde{S}, \tilde{x}_2) \to (\tilde{F}, \tilde{z}) \) be the lifts of \( p \), and let \( \tilde{D}_p : (\tilde{S}, \tilde{x}_1) \to (\tilde{S}, \tilde{x}_2) \) be the lift of \( D_p \). Similar notations hold for \( q \).

According to Theorem [6] we only need to show that \( \{\partial \tilde{p}_1(\tilde{\Lambda}^s), \partial \tilde{p}_2(\tilde{\Lambda}^s), \partial \tilde{q}_1(\tilde{\Lambda}^u), \partial \tilde{q}_2(\tilde{\Lambda}^u)\} \) is a pairwise disjoint set.

First, we shall prove that \( \partial \tilde{p}_1(\tilde{\Lambda}^s) \cap \partial \tilde{p}_2(\tilde{\Lambda}^s) = \emptyset \), \( \partial \tilde{q}_1(\tilde{\Lambda}^u) \cap \partial \tilde{q}_2(\tilde{\Lambda}^u) = \emptyset \).

We know that \( \tilde{p}_1(\tilde{\Lambda}^s) = \tilde{p}_2 \tilde{D}_p(\tilde{\Lambda}^s) \). If \( \partial \tilde{p}_1(\tilde{\Lambda}^s) \) and \( \partial \tilde{p}_2(\tilde{\Lambda}^s) \) are not disjoint, then \( \tilde{\Lambda}^s = \tilde{D}_p(\tilde{\Lambda}^s) \), as discussed in Corollary 8. It conflicts with the hypothesis that the virtual centralizer of \( \langle \Phi \rangle \) has trivial intersection with \( GD_p \) and \( GD_q \).

Therefore \( \partial \tilde{p}_1(\tilde{\Lambda}^s) \) and \( \partial \tilde{p}_2(\tilde{\Lambda}^s) \) are disjoint, the same holds for \( \partial \tilde{q}_1(\tilde{\Lambda}^u) \) and \( \partial \tilde{q}_2(\tilde{\Lambda}^u) \).
Second, we claim that if there exist \( \partial \tilde{p}_* (\Lambda^s) \) and \( \partial \tilde{q}_* (\Lambda^u) \) are not disjoint, for some \( r, t \in \{1, 2\} \), then \( p(\Lambda^s) = q(\Lambda^u) \) is a geodesic lamination on \( F \). It follows that \( \Lambda^s \) is a fixed point of \( GD_p \subset MCG(S) \). Therefore the virtual centralizer of \( \langle \Phi \rangle \) and the deck transformation group have non-trivial intersection. A contradiction.

In the following, we will prove the above claim.

Since \( p_*(\pi_1(S)) \neq q_*(\pi_1(S)) \), and they are both index two subgroups of \( \pi_1(F) \), \( p_*(\pi_1(S)) \cap q_*(\pi_1(S)) \) is an index 4 subgroup of \( \pi_1(S) \). By calculating the Euler characteristic, we know there is a genus 5 surface \( G \), and covering maps \( i \) and \( j \), such that the diagram below commutes, i.e.,

\[
p i = q j
\]

![Diagram](attachment:image.png)

After straightening, the preimages of \( i^{-1}(\Lambda^s) \) and \( j^{-1}(\Lambda^u) \) are geodesic laminations, called \( \mathcal{L}^s \) and \( \mathcal{L}^u \), on \( G \).

Without loss of generality, suppose \( \tilde{p}_1(\Lambda^s) \) and \( \tilde{q}_1(\Lambda^u) \) have one end point in common, then \( \tilde{p}_1(i(\Lambda^s)) \) and \( \tilde{q}_1(j(\Lambda^u)) \) have one end point in common. It follows that \( \tilde{\mathcal{L}}^s \) and \( \tilde{\mathcal{L}}^u \) have one common end point. We claim that \( \mathcal{L}^s \) and \( \mathcal{L}^u \) are minimal geodesic laminations and fill the surface \( G \). Therefore if they have one common end point in the universal cover of \( G \), then \( \mathcal{L}^s = \mathcal{L}^u \). It is not hard to see that \( \mathcal{L}^s \) is connected and without isolated leaves, thereby \( \mathcal{L}^s \) is minimal according to Corollary 4.7.2 in [BC88]. \( \mathcal{L}^s \) and \( \mathcal{L}^u \) fill \( G \) because they are lifts of filling laminations \( \Lambda^s \) and \( \Lambda^u \).

There exists some \( m \), such that \( \phi^m : S \to S \) is lifted by \( i \) and \( j \) to homeomorphisms of \( G \) respectively. Denote the lift of \( \phi^m : S \to S \) through \( i \) as \( \zeta : G \to G \), and the lift of \( \phi^{-m} \) through \( j \) as \( \sigma : G \to G \). Notice that \( \mathcal{L}^s \) is the stable geodesic lamination of \( \zeta \), \( \mathcal{L}^u \) is the stable geodesic lamination of \( \sigma \). Since \( \mathcal{L}^s = \mathcal{L}^u \), there exist positive integers \( k_1, k_2 \), such that \( \zeta^{k_1} \) is homotopic to \( \sigma^{k_2} \).

Since \( \zeta^{k_1} \) is homotopic to \( \sigma^{k_2} \) and \( pi = qj \), we know: \( pi \zeta^{k_1} \) is homotopic to \( qj \sigma^{k_2} \). \( p \phi^{k_1} i \) is homotopic to \( q \phi^{-k_2} j \), because \( \phi^{k_1} i = i \zeta^{k_1} \) and \( \phi^{-k_2} j = j \sigma^{k_2} \).

\( p(c) \) is a notation for the closed curve \( p(c) : S^1 \to F \) which is the composition of \( c : S^1 \to S \) with the covering map \( p : S \to F \). Similar notations are used for other compositions of closed curves with covering maps. \( c^2 : S^1 \to S \) is defined to be the composition of the map \( z \to z^2 \) on the unit circle \( S^1 \) with map \( c : S^1 \to S \). Let \([a],[c]\] denote the conjugacy classes in the fundamental group of \( F \) which represented by the simple closed curve \( a, c \).

Since \( p(c) \) is homotopic to \( a^2 \) and \( q(c) \) is homotopic to \( a \), it tells us that \([a] \notin p_*(\pi_1(S))\), \([a] \in q_*(\pi_1(S))\), and \([a]^2 \in p_*(\pi_1(S)) \cap q_*(\pi_1(S))\). Hence there exists \( \gamma : S^1 \to G \) which is homotopic to a simple closed curve, such that \( i(\gamma) \) is homotopic to \( c \) and \( j(\gamma) \) is homotopic to \( c^2 \). Therefore \( p\phi^{k_1} (c), p\phi^{k_1} i(\gamma), q\phi^{-k_2} j(\gamma) \) and \( q\phi^{-k_2} (c^2) \) are homotopic to each other.
We claim that \( q\phi^{-k_2m}(c) \) is homotopic to a simple closed curve on \( F \). Let \( \beta \) be the closed geodesic on \( F \) which is homotopic to \( q\phi^{-k_2m}(c) \). If \( \beta \) is not simple, then there exists a point \( z \in \beta(S^1) \), and a simple closed curve \( \alpha : S^1 \to S \) which is homotopic to \( \phi^{-k_2m}(c) \), such that \( q(\alpha) = \beta \), and there exit two points \( x_1 \neq x_2 \in \alpha(S^1) \) such that \( q(x_1) = q(x_2) = z \). Since \( p\phi^{k_1m}(c) \) is homotopic to \( q\phi^{-k_2m}(c^2) \), there exists a simple closed curve \( \eta \) from \( S^1 \) to \( S \) which is homotopic to \( \phi^{k_1m}(c) \), and whose image under the map \( p \) goes around \( \beta \) twice, to be more precise, \( p(\eta) = \beta^2 \). It follows that there are four different points \( y_1, y_2, y_3, y_4 \in \eta(S^1) \) such that \( p(y_1) = p(y_2) = p(y_3) = p(y_4) = z \), which conflicts with the fact that \( p : S \to F \) is an index 2 covering map.

By iterating, we have:

\[
\begin{align*}
& p\iota^{nk_1} \text{ is homotopic to } qj\sigma^{nk_2}, \text{ for all } n \in N \\
& p\phi^{nk_1m_1}(\gamma) \text{ is homotopic to } q\phi^{-nk_2m_1}(\gamma), \text{ for all } n \in N \\
& p\phi^{nk_1m}(c) \text{ is homotopic to } q\phi^{-nk_2m}(c^2), \text{ for all } n \in N
\end{align*}
\]

By using the same argument, we know \( q\phi^{-nk_2m}(c) \) is homotopic to a simple closed curve on \( F \), for all \( n \in N \). Let \( \alpha_n \) denote the geodesics in the free homotopy class of \( \phi^{-nk_2m}(c) \), there exists a subsequence of \( \alpha_n \), without loss of generality, still call it \( \alpha_n \), such that \( \alpha_n \to \Lambda^u \) as \( n \to \infty \). Since \( q\phi^{-nk_2m}(c) \) is homotopic to a simple closed curve on \( F \) for all \( n \), the geodesics in the free homotopy classes of \( q\phi^{-nk_2m}(c) \) converge to a geodesic lamination \( \Theta \subset F \), by passing to a subsequence. It follows that \( q(\Lambda^u) \) is a geodesic lamination.

Notice that in the proof, we can only lift \( \phi^m : S \to S \) by \( i \) and \( j \) to homeomorphisms of \( G \) for some \( m \in N \), but the end points of \( \partial\tilde p_i(\Lambda^s) \) and \( \partial q_j(\Lambda^u) \) for any \( i, j \in \{1, 2\} \) do not depend on \( m \). Therefore we have proved that \( \{\partial\tilde p_1(\Lambda^s), \partial\tilde p_2(\Lambda^s), \partial q_1(\Lambda^u), \partial q_2(\Lambda^u)\} \) is a pairwise disjoint set. According to Theorem 6, we know \( \pi_1(G_{\phi^m}) \) is hyperbolic for sufficiently large \( m \).

\[\square\]

5 An example which is not abstractly commensurate to a surface-by-free group

In this section, we will show that there exist a graph of surfaces whose fundamental group is hyperbolic, but is not abstractly commensurate to any surface-by-free group, for any closed hyperbolic surface or orbifold \( S' \) and any free group \( K \). Therefore this group is different from all the groups constructed in [Mos97]. By applying Theorem 1.1 in [FM02b], it follows that the example constructed here is not even quasi-isometric to any surface-by-free group.

Recall that, groups \( G \) and \( H \) are called abstractly commensurate, if there exist finite index subgroups \( G_1 < G \) and \( H_1 < H \), so that \( G_1 \) is isomorphic to \( H_1 \). A group \( G \) is called a surface-by-free group, if there is a hyperbolic surface or a hyperbolic orbifold \( S \), and a free group \( K \), such that there exists a short exact sequence:
1 \to \pi_1(S) \to G \to K \to 1

First, we shall give a necessary and sufficient condition for a group to be abstractly commensurate to a surface-by-free group. Second, we shall construct a non-hyperbolic graph of surfaces \( \mathcal{G} \), by applying the condition, whose fundamental group is not abstract commensurate to any surface-by-free group. Finally, we shall construct a hyperbolic graph of surfaces \( \mathcal{G}_{\rho,m} \) from \( \mathcal{G} \) such that \( \pi_1(\mathcal{G}_{\rho,m}) \) is not abstractly commensurate to any surface-by-free group.

Let \( t \) denote the Bass-Serre tree of a graph of surfaces \( S\Gamma \), and let \( V, E \) denote the set of all the vertices and edges of \( t \) respectively. \( \pi_1(\mathcal{G}) \) acts on \( t \) with subgroups \( \text{stab}(v) \) and \( \text{stab}(e) \), which stabilize the vertex \( v \in V \) and the edge \( e \in E \) respectively.

**Lemma 10.** The fundamental group of a graph of surfaces \( S\Gamma \) is abstractly commensurate to a surface-by-free group if and only if \( [\text{stab}(v) : \cap_{w \in V} \text{stab}(w)] < \infty \), for any \( v \in V \).

*Proof.* According to [FM02], a finite index subgroup of a surface-by-free group is a surface-by-free group. If \( \pi_1(S\Gamma) \) is abstractly commensurate to a surface-by-free group, then there exists a finite index subgroup of \( H \) of \( \pi_1(S\Gamma) \) which is isomorphic to a surface-by-free group.

\( H \) acts on \( t \), and \( [\text{stab}(v) : H \cap \text{stab}(v)] \leq [\pi_1(S\Gamma) : H] \) is finite. \( H \) acts on \( t \) with compact quotient, \( t \) may be identified with the Bass-Serre tree of \( H \). Since \( H \) is isomorphic to a surface-by-free group \( \pi_1(S') \ltimes F \), where \( S' \) is a hyperbolic surface, \( F \) is a finite rank free group, there exists a normal subgroup \( N \) of \( H \) which is isomorphic to \( \pi_1(S') \), such that \( N \) acts trivially on \( t \).

Let \( N \) denote \( \cap_{w \in V} (\text{stab}(w) \cap H) \) which is a finite index subgroup of \( \text{stab}(v) \cap H \) for any vertex \( v \in t \), i.e., \( [\text{stab}(v) \cap H : \cap_{w \in V} (\text{stab}(w) \cap H)] < \infty \). Therefore:

\[
[\text{stab}(v) : \cap_{w \in V} \text{stab}(w)] < [\text{stab}(v) : \cap_{w \in V} (\text{stab}(w) \cap H)] = [\text{stab}(v) : H \cap \text{stab}(v)][H \cap \text{stab}(v) : \cap_{w \in V} (\text{stab}(w) \cap H)] < \infty.
\]

We have finished the proof for one direction.

Now we will prove the other direction. The action of \( \pi_1(S\Gamma) \) on \( t \) induces a homomorphism \( \sigma : \pi_1(S\Gamma) \to \text{Aut}(t) \). Let \( K = \cap_{w \in V} \text{stab}(w) \), \( K = \ker(\sigma) \). Since \( K \) is a finite index subgroup of \( \text{stab}(v) \) for any \( v \in V \), \( \pi_1(S\Gamma)/K \) acts on \( t \) with finite edge and vertex stabilizers. In addition \( \pi_1(S\Gamma)/K \) acts on \( t \) cocompactly. Therefore \( t/(\pi_1(S\Gamma)/K) \) is a finite graph of finite groups. Applying Theorem 7.3 in [SW79], it follows that \( \pi_1(S\Gamma)/K \) is virtually free. Hence \( \pi_1(S\Gamma) \) is abstractly commensurate to a surface-by-free group.

In the rest of this paper, let \( \mathcal{G} \) denote a graph of surfaces as in Figure 5, where \( S, F, p, q \) and the simple closed curves \( c \subset S, a \subset F \) are as described in Corollary 9.

The conclusion of the following lemma that \( \pi_1(\mathcal{G}) \) is not commensurate to a surface-by-free group was discovered and proved independently by Chris Odden in his thesis, and by Lee Mosher. I will give a different proof which will generalize to my later examples.
Suppose the edge group of Lemma 11. 

If isomorphic to some edge or vertex group of the graph of groups . Let If isomorphic to some edge or vertex group of the graph of groups . Let be an edge of the Bass-Serre tree . Let be the generator of the underlying graph of the graph of spaces ; says that if is identified with , then . There exists a unique edge

Define subgroups , , , and subgroups of by induction as follows:

- let , , 
- let , , 
- let , , 
- let , , 

From , we know , but . Similarly, from , we know , , , but . Inductively, we have . Hence we get two sequences , , and .

Lemma 11. Suppose the edge group of contains two nested sequences of finite index normal subgroups , which are constructed inductively as follows:

1. 
2. 
3. 

If for all , then is not commensurate to a surface-by-free group.

Proof. It is known that every edge or vertex stabilizer in the Bass-Serre tree is isomorphic to some edge or vertex group of the graph of groups. Let be an edge of the Bass-Serre tree such that the stabilizer . Let be the generator of the underlying graph of the graph of spaces ; says that if is identified with , then . There exists a unique edge.
Lemma 14. Let $e_2 \in t$, such that $e_2 = ge_1$. It is easy to see that $R_1 = q_1^{-1}(p_1(S) \cap q_1(\pi_1(S))) = stab(e_1) \cap stab(e_2)$. Let $e_j = ge_{j-1}$ for a positive integer $j$, let $\alpha_i$ be the oriented path $e_1 * \cdots * e_i$ in the Bass-Serre tree $t$. $\cap_{e \in \alpha_i} stab(e) = \cap_{j=1}^i stab(e_j) = R_i$. Similarly, there exists another sequence of oriented paths $\{\beta_k\}$ in $t$ such that $\cap_{e \in \beta_k} stab(e) = L_k$. Therefore $[\pi_1(S) : L_i] \to \infty$ and $[\pi_1(S) : R_i] \to \infty$ imply $[stab(e) : \cap_{e \in E} stab(e)] = \infty$. For the case studied here, every edge stabilizer is a finite index subgroup of some vertex stabilizers, if the vertex is an end point of that edge. So $[stab(e) : \cap_{e \in E} stab(e)] = \infty$ implies $[stab(v) : \cap_{w \in V} stab(w)] = \infty$. According to Lemma 10, $\pi_1(G)$ is not commensurate to a surface-by-free group.

In order to construct a group which is not abstractly commensurate to a surface-by-free group, our first strategy is to find a pseudo-Anosov mapping class $\Phi$ which fixes all the finite index normal subgroups of $\pi_1(S)$. But unfortunately, the theorem below tells us that there does not exist such a pseudo-Anosov mapping class.

**Theorem 12.** Let $S_n$ be a closed surface with genus $n$, where $n \geq 2$. For any $\Phi \in Aut(\pi_1(S_n))$, if $\Phi$ fixes all the finite index normal subgroups of $\pi_1(S_n)$, then $\Phi \in Inn(\pi_1(S_n))$.

Before proving this theorem, we introduce some related history and preliminaries first.

In [Lub80], Lubotzky proved that for any free group $F_n$, $n \geq 2$, if $\Psi \in Aut(F_n)$ fixes all the finite index normal subgroups of $F_n$, then $\Psi \in Inn(F_n)$. In particular, every normal automorphism of $F_n$ is inner. Bogopolski, Kudryavtseva and Zieschang in [BKZ04] proved that for any closed hyperbolic surface $S_n$ with genus $n$ not less than 2, if $\Phi \in Aut(\pi_1(S_n))$ fixes all the normal subgroups of $\pi_1(S_n)$, then $\Phi \in Inn(\pi_1(S_n))$. The main theorem in that paper says for any non-separating simple closed curve $\alpha$ on $S$, up to conjugate equivalent, $\alpha$ is the only non-separating simple closed curve in its normal closure. The theorem in [BKZ04] says:

**Theorem 13.** Let $S$ be a closed orientable surface and $g, h$ are non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves $\gamma$ and $\kappa$, resp. The group element $h$ belongs to the normal closure of $g$ if and only if $h$ is conjugate to $g^\varepsilon$ or to $(gug^{-1}u^{-1})^\varepsilon$, $\varepsilon \in \{1, -1\}$; here $u$ is a homotopy class containing a simple closed curve $\mu$ which properly intersects $\gamma$ exactly once.

I would like to thank Jason DeBlois for help with Lemma 14.

**Lemma 14.** For any two non-trivial, non freely homotopic, non-separating simple closed curves $a$ and $b$ on $S$, let $[a], [b]$ denote the homotopy class of them in $\pi_1(S)$. There exists a finite index normal subgroup $N \in \pi_1(S)$, such that $[a] \in N$ and $[b] \notin N$.

A group $G$ is said to be residually finite, if for any element $g \in G$, $g \neq 1$, there exists a finite group $K$ and a homomorphism $h : G \to K$, such that $h(g) \neq 1$.

A *Haken manifold* is a compact, orientable, irreducible 3-manifold which contains a 2-sided incompressible surface.
Proof. : Let $M = S \times I$, where $I$ is the interval $[0,1]$. $\pi_1(M)$ is isomorphic to $\pi_1(S)$. Since $a$ is a simple closed curve on $S$, attach a 2-handle $B$ to $M$ along $a \times \{0\} \cup a \times \{1\}$ obtain a Haken manifold $M'$. This attachment gives a surjective homomorphism $\epsilon : \pi_1(M) \to \pi_1(M')$, and the kernel is the normal closure of $[a]$. Since $a$ is the only non-separating simple closed curve in the normal closure of $[a]$, by applying Theorem 13, it follows that $[b]$ does not belong to the kernel of $\epsilon$.

According to Theorem 1.1 in [Hem72], $\pi_1(M')$ is residually finite. So for $[b] \in \pi_1(M)$, there exist a finite group $K$ and a homomorphism $\delta : \pi_1(M') \to K$, such that $[b] \notin \ker(\delta)$.

Let $N$ denote the kernel $\ker(\delta \circ \epsilon)$. Obviously, $N$ is a finite index normal subgroup of $\pi_1(S)$, and $[a] \in N$, but $[b] \notin N$.

Proof of Theorem 12: Let $\Phi$ be an element of $\text{Aut}(\pi_1(S))$, and let $\phi$ be a representative of it in $\text{Homeo}(S)$. According to [BKZ04], if $\Phi \notin \text{Inn}(\pi_1(S))$, then there exists a non-separating simple closed curve $a$ on $S$, such that $a$ and $\phi(a)$ are not freely homotopic to each other. According to Lemma 14, there exist a finite index normal subgroup $N \triangleleft \pi_1(S)$, such that $[a] \in N$ and $[\phi(a)] \notin N$. It follows that $\Phi(N) \neq N$.

In the following, we shall construct a pseudo-Anosov mapping class which does not fix all the finite index normal subgroups of $\pi_1(S)$, but fixes $L_i$ and $R_i$ as in Lemma 11.

In the following, let $\mathcal{G}_\phi$ be a graph of surfaces as in Figure 3, where $F$, $S$, $p$, $q$ as described in Corollary 9.

**Theorem 15.** There exists a pseudo-Anosov homeomorphism $\phi \in \text{Homeo}(S)$, so that $\pi_1(\mathcal{G}_\phi)$ is hyperbolic but is not commensurate to a surface-by-free group.

![Figure 6](image-url)

Proof. If there exists a pseudo-Anosov homeomorphism $\phi$, such that $\phi_*(L_i) = L_i$ and $\phi_*(R_i) = R_i$, according to Lemma 11, then $[\text{stab}(e) : \cap \text{stab}_{e \in E}(e)] = \infty$. Therefore $\pi_1(\mathcal{G}_\phi)$ is not commensurate to a surface-by-free group.
Curves mentioned in this theorem are shown in Figure 6 and Figure 7, these two figures are a refinement of Figure 4.

First, we will describe the covering maps $p$ and $q$ with more details. Let $p^{-1}(a^2) = c$, $q^{-1}(a) = c \cup d$. It is easy to see that $p(\alpha)$ is homotopic to $q(\alpha) \subset F$ and $p(\beta)$ is homotopic to $q(\beta) \subset F$, where $\alpha$, $\beta \subset S$ as in Figure 6. Therefore $[\alpha], [\beta] \in L_i \cap R_i$ for all $i$.

Second, we claim that if $\gamma$ is a simple closed curve in $S$, such that $[\gamma] \subset L_i$ for some $i$, then $(\tau_\gamma)_*$ of the Dehn-twist $\tau_\gamma$ fixes $L_i$. Note that $L_i$ is a finite index normal subgroup of $\pi_1(S)$ if and only if there exists a finite group $K$ and a homomorphism $f : \pi_1(S) \to K$ such that $L_i = \ker(f)$. We shall see that $\tau_\gamma$ maps every element in the kernel of $f$ to an element in the kernel of $f$, i.e., $(\tau_\gamma)_*$ fixes $L_i$. Let $[g] \in \pi_1(S)$ be an element in $L_i$. Decompose $[g] = [h_1] \cdot \cdots [h_n]$, so that $[h_j] \in \pi_1(S)$ is represented by a closed loop in $S$ which has only one transverse intersection point with $\gamma$, for all $j \in \{1, \cdots, n\}$. Depending on how $h_i$ intersects with $\gamma$, $[\tau_\gamma(h_i)]$ is one of the following four kinds: $[h_i\gamma]$, $[\gamma h_i]$, $[\gamma h_i \gamma^{-1}]$, $[\gamma^{-1} h_i \gamma]$. The trivial case is $g \cap \gamma = \emptyset$, so $f([\tau_\gamma(g)]) = [g]$. Otherwise,

$$f([\tau_\gamma(g)]) = f([\tau_\gamma(h_1)] \cdot \cdots [\tau_\gamma(h_n)])$$
$$= f([\tau_\gamma(h_1)]) \cdot \cdots f([\tau_\gamma(h_n)])$$
$$= f([h_1]) \cdot \cdots f([h_n]) = f([h_1 \cdots h_n]) = f([g]) = I_k$$

where $I_k$ is the identity of $K$. It shows that $(\tau_\gamma)_*$ fixes $L_i$.

If we can find disjointly essential curve systems $C$ and $D$ which satisfy the conditions in Theorem 3, and if all the homotopy classes of the elements of $C$ and $D$ belong to $L_i$ and $R_i$ for all $i$, then we can construct a pseudo-Anosov homeomorphism $\phi$ as described in Theorem 3, such that $\phi_*$ fixes $L_i$ and $R_i$ for all $i$.

In the following, we will prove that there exist disjointly essential curve systems $C = \alpha \cup \hat{\alpha}$, and $D = \beta \cup \hat{\beta}$, such that $C \cup D$ fills $S$, where $\alpha$, $\beta$ as in Figure 6. In addition, $[\alpha], [\hat{\alpha}], [\beta]$ and $[\hat{\beta}] \in \cap_i(L_i \cap R_i)$.

In order to find a simple closed curve $\hat{\alpha}$ satisfying the above conditions, first, we will show that there exists a simple closed curve $\alpha'$ such that $[\alpha'] \in \cap_i(L_i \cap R_i)$. Since $L_i$ and $R_i$ are finite index normal subgroups of $\pi_1(S)$, and $[\alpha] \in \cap_i(L_i \cap R_i)$,
normal closure $N_\alpha$ of $[\alpha]$ is a subgroup of $\cap_i(L_i \cap R_i)$. Recall that the normal closure $N_\alpha$ of $[\alpha]$ is the smallest normal subgroup of $\pi_1(S)$ contains $[\alpha]$. Applying Theorem 13, we only need the easy direction of this theorem, the separating curve $\alpha'$ as in Figure 6 represents an element in $N_\alpha$.

Second, we shall construct a simple closed curve $\widehat{\alpha}$ on $S$ from the simple closed curve $\alpha'$.

From [Mos03], we know there exists a short exact sequence:

$$1 \to \langle T_\alpha \rangle \to \text{stab}(\alpha) \to \text{MCG}(S - \alpha) \to 1$$

where $\langle T_\alpha \rangle$ is a cyclic subgroup of $\text{MCG}(S)$ generated by the mapping class $T_\alpha$ of the Dehn-twist $\tau_\alpha$ around $\alpha$, $\text{stab}(\alpha)$ is a subgroup of $\text{MCG}(S)$ which fixes $\alpha$, $S - \alpha$ is a surface by cutting $S$ along $\alpha$. The homomorphism $\iota : \text{stab}(\alpha) \to \text{MCG}(S - \alpha)$ is defined as $\Phi \to \Phi|_{S - \alpha}$, for $\Phi \in \text{stab}(\alpha)$.

Choose a pseudo-Anosov homeomorphism $\psi \in \text{Homgeo}(S - \alpha)$, maybe need pass to a high enough power of $\psi$, such that $\widehat{\alpha} = \psi(\alpha')$ is very close to the stable geodesic lamination $\Lambda^S_\psi$ of $\psi$, therefore $\widehat{\alpha} \cup \beta$ fills $S - \alpha$. Also $\widehat{\alpha}$ is disjoint with $\alpha$ because $\alpha'$ is disjoint with $\alpha$.

Using the same method, we can find a simple closed curve $\widehat{\beta}$ which is disjoint with $\beta$ and $\widehat{\alpha} \cup \alpha$ fills $S - \beta$.

Let $C = \{\alpha, \widehat{\alpha}\}$, $D = \{\beta, \widehat{\beta}\}$, it is easy to see that $C \cup D$ fills $S$. According to Theorem 3, if $\phi_0$ is a homeomorphism of $S$, such that $\tau^+_\widehat{\alpha}$, $\tau^-\widehat{\alpha}$, $\tau^-\widehat{\beta}$ and $\tau^+\widehat{\beta}$ appear at least once in $\phi_0$, then $\phi_0$ is a pseudo-Anosov homeomorphism. Since $[\widehat{\alpha}]$, $[\widehat{\beta}] \in \cap_i(L_i \cap R_i)$, $[\phi_0]$ fixes $L_i$ and $R_i$ for all $i$.

In order to finish the proof of this theorem, according to Corollary 3 we only need to show that there exists some pseudo-Anosov homeomorphism $\phi$ constructed as above, so that the virtual centralizer $VC\langle \Phi \rangle$ of $\langle \Phi \rangle$ has trivial intersection with the mapping classes of the deck transformation groups of the covering maps $p$ and $q$ respectively, where $\Phi \in \text{MCG}(S)$ is the mapping class of $\phi$. Abusing of notation, denote both the deck transformations and the mapping classes of the deck transformations by $D_p$ and $D_q$. The deck transformation group of $p$ has only two elements $D_p$ and the identity.

Let $\phi_0$ be a pseudo-Anosov homeomorphism of $S$ constructed above, and let $\Phi_0$ be its mapping class. Let $\Lambda^s_{\phi_0}$ and $\Lambda^u_{\phi_0}$ be the stable and unstable geodesic laminations of $\phi_0$ respectively. It is known that $\Phi_0$ fixes $L_i$ and $R_i$ for all $i$.

Suppose the deck transformation group of $p$ has nontrivial intersection with the virtual centralizer of $\langle \Phi_0 \rangle$, i.e., $D_p(\Lambda^s_{\phi_0}) = \Lambda^s_{\phi_0}$, we claim that $D_p(T_\alpha(\Lambda^s_{\phi_0})) \neq T_\alpha(\Lambda^s_{\phi_0})$, where $T_\alpha$ is the mapping class of the Dehn-twist $\tau_\alpha$. Notice that $T_\alpha(\Lambda^s_{\phi_0})$ is the stable geodesic lamination of the pseudo-Anosov mapping class $T_\alpha \Phi_0 T_\alpha^{-1}$, and $T_\alpha \Phi_0 T_\alpha^{-1}$ fixes $L_i$ and $R_i$ for all $i$. If the claim is true, let $\Phi_1 = T_\alpha \Phi_0 T_\alpha^{-1}$, then $VC\langle \Phi_1 \rangle$ has trivial intersection with $D_p$.

We shall prove the claim. Notice that there exists a simple closed curve $\gamma$ on $S$ is disjoint with $\alpha$, such that $D_p(\alpha) = \gamma$. According to Lemma 4.1.C in [Iva02], $D_p T_\alpha D_p^{-1} = T_{D_p(\alpha)} = T_\gamma$. 
Suppose $D_p T_{\alpha}(\Lambda_{\phi_0}^s) = T_{\alpha}(\Lambda_{\phi_0}^s)$, then:
\[
D_p T_{\alpha}(\Lambda_{\phi_0}^s) = D_p T_{\alpha} \Phi_0 T^{-1}_{\alpha}(T_{\alpha}(\Lambda_{\phi_0}^s)) \\
= T_{\gamma} D_p \Phi_0 T^{-1}_{\alpha}(T_{\alpha}(\Lambda_{\phi_0}^s)) \\
= T_{\gamma} D_p \Phi_0 (\Lambda_{\phi_0}^s) = T_{\gamma}(\Lambda_{\phi_0}^s)
\]

Therefore $T_{\alpha}(\Lambda_{\phi_0}^s) = T_{\gamma}(\Lambda_{\phi_0}^s)$.

It follows that $T^{-1}_{\alpha} T_{\gamma} \in VC(\Phi_0)$, but from Theorem 3.5 in [Mos], we know that $VC(\Phi_0)$ has $(\Phi_0)$ as a finite index subgroup. Hence up to some power $m$, $(T^{-1}_{\alpha} T_{\gamma})^m \in (\Phi_0)$, but obviously $(T^{-1}_{\alpha} T_{\gamma})^m$ is neither pseudo-Anosov nor the identity, so it is not an element of $(\Phi_0)$. Therefore $D_p T_{\alpha}(\Lambda_{\phi_0}^s) \neq T_{\alpha}(\Lambda_{\phi_0}^s)$.

If in addition $D_q T_{\alpha}(\Lambda_{\phi_0}^u) \neq T_{\alpha}(\Lambda_{\phi_0}^u)$, then take $\Phi = \Phi_1$, this theorem is proved.

If $D_q T_{\alpha}(\Lambda_{\phi_0}^u) = T_{\alpha}(\Lambda_{\phi_0}^u)$, then we claim $D_q T_{2\alpha}(\Lambda_{\phi_0}^u) \neq T_{2\alpha}(\Lambda_{\phi_0}^u)$. If the claim is not true, then
\[
D_q T_{2\alpha}(\Lambda_{\phi_0}^u) = T_{2\alpha}(\Lambda_{\phi_0}^u) \\
= T_{\alpha}(D_q T_{\alpha}(\Lambda_{\phi_0}^u)) \\
= D_q T_{\alpha}(T_{\alpha}(\Lambda_{\phi_0}^u)),
\]

where $\theta = D_q(\alpha)$ is a simple closed curve on $S$ disjoint from $\alpha$. Therefore
\[
T^{-1}_{\alpha} T^{-1}_{\theta} D_q T_{2\alpha}(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u
\]

It follows that $T^{-1}_{\alpha} T^{-1}_{\theta} T_{2\alpha}(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u$. Since $\theta, \alpha$ are disjoint simple closed curves, $T_{\alpha} T_{\theta} = T_{\theta} T_{\alpha}$. Hence $T_{\alpha} T_{2\alpha}(\Lambda_{\phi_0}^u) = T_{\theta} T_{\alpha}(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u$. By the same reason in the above argument, it is impossible.

Replacing $T_{\alpha}$, $T_{\gamma}$ by $T_{2\alpha}$, $T_{2\gamma}$ in the above proof of $D_p T_{\alpha}(\Lambda_{\phi_0}^s) \neq T_{\alpha}(\Lambda_{\phi_0}^s)$, we can see $D_p T_{2\alpha}(\Lambda_{\phi_0}^s) \neq T_{2\alpha}(\Lambda_{\phi_0}^s)$. Take $\Phi = T_{2\alpha} \Phi_0 T^{-2}_{\alpha}$, then this theorem is proved. □


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