Multivariate Feedback Particle Filter via F-divergence and the Well-posedness of Its Admissible Control Input

Xue Luo, senior member, IEEE.

Abstract—In this paper, we shall first derive the admissible control input of the multivariate feedback particle filter (FPF) by minimizing the f-divergence of the posterior conditional density function and the empirical conditional density of the controlled particles. On the contrast, in the original derivation [26], a special f-divergence, Kullback-Leibler (K-L) divergence, is used in the 1-dimensional nonlinear filtering problems. We show that the control input is invariant under the f-divergence class. That is, the control input satisfies exactly the same equations as those obtained by minimizing K-L divergence, no matter what f-divergence in use. In the latter half of this paper, we show the existence and uniqueness of the control input under suitable regular conditions. We confirm that the explicit expression of the control input given in [27] is the only admissible one.

I. INTRODUCTION

Nonlinear filtering (NLF) or called nonlinear estimation is to give the state/signal $X_t$ a “proper” estimation based on the observation history $\{Z_s, 0 \leq s \leq t\}$ in some sense, say the conditional expectation $\mathbb{E}[X_t|Z_t]$, $Z_t := \sigma(\{Z_s, 0 \leq s \leq t\})$. The most famous Kalman filter [11], Kalman-Bucy filter [12] yields the optimal estimation if the problem is linear and the initial density is Gaussian. Unfortunately, as far as we know, there is no such effective algorithms for NLF problems. There are lots of Kalman filter’s derivatives which can obtain suboptimal estimations for NLF problems, but far from satisfactory. We refer the approaches that only aim to obtain the approximation of interested statistical quantities, say expectation, variance etc., as local approaches, while those consider to compute/approximate the posterior distribution of the hidden Markov process $X_t$, given the history of observation is called global approaches. The author of this paper wrote a survey on the approaches for NLF problems and carefully discussed their advantage/disadvantage, see [13].

One global approach is to numerically solving the Kushner-Stratonovich equation [13] or Duncan-Mortensen-Zakai equation [7]. In this direction, there are wide range of literatures including the splitting-up method [3], $S^3$-algorithm [17], on- and off-line algorithm [23], [19], [20], etc. Nevertheless, the computation load is extremely heavy if the state/signal is high-dimensional. Besides these partial differential equation based algorithm, the most popular global approach is the so-called particle filter (PF) [2], [4], [6]. The PF is a simulation-based algorithm, which approximates the posterior distribution by the empirical distribution of the particle population $\{X^i_t\} = 1, \ldots, N$. A common remedy to avoid particle impoverish and degeneracy in the traditional PF is to vigor the particles by resampling according to the importance weight at each time step. After the proper reampling strategy, the PF can propagate the posterior distribution with accuracy improving by increasing the number of the particles [4]. Nevertheless, the choice of the importance weight is crucial, problem-dependent and with no universal guidelines.

Recently, Yang et. al. [26] introduced a control-oriented PF, called feedback particle filter (FPF), for the scalar case, i.e. the dimensional of the state/signal and observation process both are one. Later, [27] extends this algorithm to multivariate case without detailed derivation. The FPF is motivated by mean-field optimal control techniques [4], [10]. Let us consider the NLF problem in the form:

\begin{align}
\label{equation:1}
dX_t &= a(X_t)dt + \sigma_B dB_t \\
\label{equation:2}
dZ_t &= h(X_t)dt + dW_t,
\end{align}

where $X_t \in \mathbb{R}^d$ is the state at time $t$, $Z_t \in \mathbb{R}^m$ is the observation process, $a(\cdot)$, $h(\cdot)$ are functions of $X_t$ and $\{B_t\}$, $\{W_t\}$ are mutually independent Wiener processes of appropriate dimensions. In FPF, they model the $i$th particle evolves according the controlled system

\begin{align}
\label{equation:3}
dX^i_t &= a(X^i_t)dt + \sigma_B dB^i_t + dU^i_t,
\end{align}

where $dU^i_t$ is the control input of the $i$th particle, and $\{B^i_t\}$ are also mutually independent standard Wiener process. The aim of the FPF is to choose the appropriate control input for each particle such that the empirical distribution approximates the conditional distribution of $X^i_t$ for large number of particles. In [26], [27], the optimal control input is obtained by minimizing the Kullback-Leibler (K-L) divergence as the cost function.

In this paper, we shall discuss two nature questions related to the multivariate FPF. On the one hand, in probability theory the K-L divergence is only one member in the category called f-divergence, which measures the difference between two probability distributions. These f-divergences were introduced and studied independently by Csiszár [5], Morimoto [22] and Ali, et. al. [11] and are sometimes known as Csiszár f-divergences, Csiszár-Morimoto divergences or Ali-Silvey distances. Thus, a natural question is raised: if the other f-divergences are used as the cost function in FPF, shall we obtain different control input from that obtained by using the K-L divergence? We answered this question in the scalar case, i.e. $d = m = 1$, in [21] that the control input is independent of the choice of f-divergence. When it comes to the multivariate case, it is not trivial. As one will see in this paper, the trivial identity (20) for $d = 1$, which has to be shown rigorously for $d \geq 2$, see Proposition [3] in section III.A.

On the other hand, the derivation of the multivariate FPF in [27] is too informal for the readers to suspect that the
control input is just an analogue of the one in the scalar case, without any detailed derivation, let alone the discussion of the existence and uniqueness of the control input in what sense. In section III.B, we patch the detailed derivation for the equations which the control input should satisfy for the case $d \geq 1$, $m = 1$. Consequently, the consistency can be shown rigorously based on the equations derived, rather than on the analogous control input “guessed” in [27]. For the most general case $m \geq 1$, we point out that our derivation should also work but with more involved computations and notations. In section IV, we established the well-posedness of the control input in appropriate function space under certain conditions. Thus, the control input given in [27] has been checked to be admissible (Definition 2.1), so as to be unique in the suitable function space. The conclusions are arrived in the end.

II. PRELIMINARIES

The precise formulation begins with continuous time model with sampled observations:

$$Y_{tn} = h(X_{tn}) + W_{tn},$$

(4)

where $\Delta t := t_{n+1} - t_n$ and $\{W_{tn}\}$ is i.i.d. and drawn from $N(0, \frac{1}{\Delta t})$, the Gaussian with zero mean, $\frac{1}{\Delta t}$ variance. The observation history is denoted as $Y_n := \{Y_{tn} : k \leq n, k \in \mathbb{N}\}$. We follow the same notations as in [26], [27]. Let us denote the conditional distributions:

1) $p_n^s$: the conditional distribution of $X_{tn}$ given $Y_n$ and $Y_{n-1}$, respectively.

2) $p_n$ and $p_n^-$: the conditional distribution of the $i$th particle $X^i_{tn}$ given $Y_n$ and $Y_{n-1}$, respectively.

These distributions evolve according to the recursion

$$p_n^s = \mathcal{P}^s(p_n^{s-1}, Y_n), \quad p_n = \mathcal{P}(p_n^{s-1}, Y_n).$$

The mappings $\mathcal{P}^s$ and $\mathcal{P}$ can be decomposed into two parts. The first part is identical for each of these mappings: the transformation that takes $p_n^{s-1}$ to $p_n^s$ coincides with the mapping from $p_n^-_{i-1}$ to $p_n^+_{i-1}$. In each case it is defined by the Kolmogorov forward equation (KFE) associated with the diffusion on $[t_{n-1}, t_n]$.

The second part of the mapping is the updating that takes $p_n^-_{i-1}$ to $p_n^+_{i-1}$ by synchronizing the observation data $Y_{tn}$, which is obtained according to the Bayes’ rule. Given the observation $Y_{tn}$ made at time $t = t_n$

$$p_n(s) = \frac{p_n^-(s) \cdot p_{Y|X}(Y_{tn} | s)}{p_Y(Y_{tn})}, \quad s \in \mathbb{R}^d,$$

(5)

where $p_Y$ denotes the probability density function (pdf) for $Y_{tn}$, and $p_{Y|X}(Y_{tn} | s)$ denotes the conditional distribution of $Y_{tn}$ given $X_{tn} = s$. In the case that the observation noise is Gaussian, we have

$$p_{Y|X}(Y_{tn} | s) = \frac{1}{\sqrt{\frac{2\pi}{\Delta t}}} \exp \left( - \frac{(Y_{tn} - h(s))^2}{2 \Delta t} \right).$$

(6)

The operator $\mathcal{P}^s$ is the composition of KFE and $\mathcal{P}$.

The updating from $p_n^-$ to $p_n$ is not due to the Bayes’ rule, but depends on the control input $dU_{tn}^i$ in (3). In the

discrete setting, at time $t = t_n$, we seek the control input $v_n^i = K(X^i_{tn}, t) \Delta z + u(X^i_{tn}, dZ_t)$, which is the discrete counterpart of $dU_{tn}^i = K(X^i_t, t) dZ_t + u(X^i_t, t) dt$ at time $t = t_n$. We shall restrict ourselves to find the control input in the admissible class. We will suppress the superscript $i$ and the subscript $n$ in $v_n^i$, and write $K = K(X^i_{n}, t_n), u = u(X^i_{n}, t_n)$ for short, if there is no confusion.

**Definition 2.1 (Admissible Input):** The control sequence $v_n = K \Delta z + u \Delta t$ is called admissible, if for each $n$

1) $K \in H^l(\mathbb{R}^d; p), l \geq \frac{d}{2} + 1$ and $u \in L^2(\mathbb{R}^d; p)$, where $H^l(\mathbb{R}^d; p)$ is the weighted Sobolev space with its norm defined as

$$|| \delta \mathcal{O} ||_{H^l(\mathbb{R}^d; p)}^2 = \sum_{i=0}^l || \nabla^i \delta \mathcal{O} ||_2^2 L^2(\mathbb{R}^d; p),$$

with

$$\nabla_{\sigma_1} \delta \mathcal{O} := \frac{\partial \delta \mathcal{O}}{\partial x_{\sigma_1 (1)} \cdots \partial x_{\sigma_1 (i)}},$$

(7)

and the convention that $\nabla^0 = Id$, the identity mapping, where the norm is defined as

$$|| \nabla \delta \mathcal{O} ||_2^2 L^2(\mathbb{R}^d; p) = \sum_{j=1}^d \sum_{\sigma \in \{1, \cdots, d\}} \int_{\mathbb{R}^d} |\nabla_{\sigma_j} \delta \mathcal{O} |^2 p dx,$$

(8)

for $\delta \mathcal{O} = (\delta_1, \cdots, \delta_d) \in \mathbb{R}^d$.

2) $I + (\nabla v_n)^T$ is invertible for all $x$, where $I$ is the identity matrix, $\mathcal{O}^T$ is the transpose of the matrix $\mathcal{O}$ and $\nabla \mathcal{O}^T = \left( \frac{\partial \delta_1}{\partial x_1}, \cdots, \frac{\partial \delta_d}{\partial x_1}, \cdots, \frac{\partial \delta_1}{\partial x_d}, \cdots, \frac{\partial \delta_d}{\partial x_d} \right)$, which is the transpose of the Jacobian matrix $\frac{\partial \mathcal{O}}{\partial x}$.

Under the assumption that $I + (\nabla v_n)^T$ is invertible for all $x$, the updating from $p_n^-$ to $p_n$ is

$$p_n(s) = \frac{p_n^-(x)}{I + (\nabla v_n)^T(x)},$$

(9)

where $s = x + u(x)$, and $| \delta \mathcal{O} |$ represents the determinant of the matrix $\mathcal{O}$.

Let us denote $\hat{p}^s := \mathcal{P}^s(p_n^- - 1, Y_{tn})$. We should choose the control input $v_n^i$ such that the difference between $\hat{p}^s_n$ and $p_n$ as small as possible at every time step $t_n$.

In 1960s, the evolution equation for $p^s$ has been derived by Kushner [13]:

$$dp^s = \mathcal{L}^s p^s dt + (h - \hat{h}_t)(dZ_t - \hat{h}_tdt)p^s,$$

(10)

where $\hat{h}_t = \int h \nu^* dx$ and

$$\mathcal{L}^s p^s := -\nabla^T(p^s \mathcal{O}) + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} (p^s | \sigma_B \sigma_B^T)_{ij}.$$
This is the so-called the Kushner-Stratonovich equation. Moreover, the propagation of the particle’s conditional distribution is described by the Kolmogorov forward equation (KFE) [24]:

\[ dp = \mathcal{L}^* p dt - \nabla^{T} (p K) dz_t - \nabla^{T} (p u) dt + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (p [K K^T]_{ij}) dt, \]

where \( \mathcal{L}^* \) is defined in (11). The derivation of this KFE in scalar case can be found in Proposition 3.1, [26]. The multivariate case is straightforward.

III. MULTIVARIATE FEEDBACK PARTICLE FILTER

A. Invariance of control input via F-divergence

In this section, we measure the difference from distribution \( p_1 \) to \( p_2 \) by \( f \)-divergence defined as

\[ D_f(p_1 || p_2) = \int_{\mathbb{R}^d} p_2(s) f \left( \frac{p_1(s)}{p_2(s)} \right) ds. \]

With different choice of \( f \), \( f \)-divergence can become Kullback-Leibler (K-L) divergence, total variation distance, Hellinger distance, etc. The K-L divergence is a special case of \( f \)-divergence with \( f(s) = s \log s \). In [26], [27], the control input is obtained by minimizing the K-L divergence from \( p_1 \) to \( \tilde{p}_n^* \).

In this section, we will determine the control input \( v \) by minimizing the \( f \)-divergence from \( \tilde{p}_n^* \) and \( p_n \). Although it is well-known that the \( f \)-divergence is not symmetric, i.e. \( D_f(p_1||p_2) \neq D_f(p_2||p_1) \), the control input \( v \) obtained by minimizing one is exactly the same. Nevertheless, the computation of \( D_f(\tilde{p}_n^* || p_n) \) is much easier. Thus, we use \( D_f(\tilde{p}_n^* || p_n) \) in the derivation:

\[ \int_{\mathbb{R}^d} p_n(x) f \left( \frac{p_n(x)}{p_n^*(x)} \right) dx, \]

where \( s = x + v(x) \). Let us denote the integrand in (13) as

\[ \mathcal{L}(x, v, \nabla v^T) := p_n^*(x) f \left( \frac{p_n(x) p_Y|X(Y_{tn}|x + v) |I + \nabla v^T|}{p_n^*(x) p_Y(Y_{tn})} \right). \]

For simplicity of notation, let us denote the argument of \( f \) in (13) as

\[ \xi = \frac{p_n^*(x) p_Y|X(Y_{tn}|x + v) |I + \nabla v^T|}{p_n(x) p_Y(Y_{tn})}. \]

It is well-known that the minimizer of the functional \( D_f(\tilde{p}_n^* || p_n) \) can be obtained by solving the corresponding Euler-Lagrange (E-L) equation:

\[ \left( \frac{\partial \mathcal{L}}{\partial v} \right)^T = \nabla^{T} \left( \frac{\partial \mathcal{L}}{\partial (\nabla v^T)} \right), \]

where taking the derivative with respect to a vector or a matrix has been defined properly in matrix calculus, see Chapter 9, [16]. The notation \( \nabla_x^T(A) \) in (16) is \( \nabla_x^T(A) = \left( \sum_{j=1}^{d} \frac{\partial A_{ij}}{\partial x_j}, \ldots, \sum_{j=1}^{d} \frac{\partial A_{ij}}{\partial x_j} \right) \), if \( A \) is a \( d \times d \) matrix and \( A_{ij} \) is the \( ij \)th entry of \( A \). The left-hand side of (16) equals:

\[ \left( \frac{\partial \mathcal{L}}{\partial v} \right)^T = \frac{f'(\xi)|I + \nabla v^T|}{p_Y(Y_{tn})} \nabla^{T} \left( p_n(x + v)p_Y|X(Y_{tn}|x + v) \right) \]

\[ = \frac{f'(\xi)|I + \nabla v^T|}{p_Y(Y_{tn})} \cdot \nabla^{T} \left( p_n(x + v)p_Y|X(Y_{tn}|x + v) \right) (I + \nabla v^T)^{-T}, \]

where the notation \( A^{-T} = (A^{-1})^T \), while its right-hand side is

\[ \nabla_x^T \left( \frac{\partial \mathcal{L}}{\partial (\nabla v^T)} \right) = \nabla_x^T \left( f'(\xi) \frac{|I + \nabla v^T|}{p_Y(Y_{tn})} \right) \cdot \nabla_x^{T} \left( p_n^*(x + v)p_Y|X(Y_{tn}|x + v) \right) \cdot \left( I + \nabla v^T \right)^{-T} \]

\[ = p_n^*(x + v)p_Y(1|X(Y_{tn}|x + v) \cdot \left( I + \nabla v^T \right)^{-T} \]

\[ + f'(\xi) \frac{|I + \nabla v^T|}{p_Y(Y_{tn})} \nabla_x^{T} \left[ p_n^*(x + v)p_Y|X(Y_{tn}|x + v) \right] \cdot \left( I + \nabla v^T \right)^{-T} \]

\[ + p_n^*(x + v)p_Y|X(Y_{tn}|x + v) f'(\xi) \frac{|I + \nabla v^T|}{p_Y(Y_{tn})} \cdot \left( I + \nabla v^T \right)^{-T}, \]

where the second equality in (18) follows from the fact that \( \frac{\partial A^{-1}}{\partial x} = -A^{-1} \frac{\partial A}{\partial x} A^{-1} \). Notice that the second term on the right-hand side of (18) cancels out with (17). Therefore, we obtain from (16) that

\[ 0 = \nabla_x^T \left( f'(\xi)|I + \nabla v^T| \right) \cdot \left( I + \nabla v^T \right)^{-T} \]

\[ + f'(\xi)|I + \nabla v^T| \nabla_x^T \left[ f'(\xi)|I + \nabla v^T| \right] \cdot \left( I + \nabla v^T \right)^{-T} \]

\[ + f'(\xi) \nabla_x^T \left[ |I + \nabla v^T| \right] \cdot \left( I + \nabla v^T \right)^{-T} \]

after dividing by \( p_n^*(x + v)p_Y|X(Y_{tn}|x + v) \) throughout.

We can show that actually the second term on the right-hand side of (19) vanishes.

Proposition 3.1: For \( x \in \mathbb{R}^d \), \( d \geq 1 \), the identity

\[ \nabla_x^T \left[ |I + \nabla v^T| \right] \cdot \left( I + \nabla v^T \right)^{-T} = 0 \]

holds.

The detailed proof has been appended in Appendix A for interested readers.

Hence, the control input \( v \) satisfies

\[ 0 = \nabla_x^T \left[ f'(\xi)|I + \nabla v^T| \right] \cdot \left( I + \nabla v^T \right)^{-T}, \]

(21)
where $\xi$ is defined in \eqref{eq:15}. Let us take a look at the term $\nabla^T_x f'(\xi)$:

\[
\nabla_x^T f'(\xi) = f''(\xi) \nabla^T_x \left[ \frac{p_n^-(x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T|}{p_n^-(x)p_Y(Y_t)} \right] = \frac{f''(\xi)}{p_Y(Y_t)p_n^-(x)} \cdot \{ \nabla^T_x \left[ p_n^-(x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T|p_n^-(x) \right] + p_n^-(x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T| \}
\]

\[
\cdot \left[ \nabla^T_x \nabla^{T - 1} \left( \nabla v^T \right) \right]_{i = 1, \ldots, d} \cdot p_n^-(x) - p_n^-(x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T| \nabla^{T}_x p_n^-(x) \right),
\]

where $\text{tr}()$ means the trace of the matrix $\circ$ and the second term on the right-hand side follows from the fact that $\frac{dA(x)}{dx} = |A|\text{tr} \left( A^{-1} \frac{\partial A(x)}{\partial x} \right)$, with $A$ being a matrix-valued function of $x$.

If the terms in the brace on the right-hand side of \eqref{eq:22} equals zero, then the control input is independent of what $f$-divergence we are using. We summarize this invariance in the following theorem:

**Theorem 3.1:** The control input $v$ obtained by minimizing the $f$-divergence from $p_n^-$ to $p_n$ is independent of the choice of $f$-divergence in use. No matter what $f$ is, the control input $v$ always satisfies the following equation:

\[
0 = \nabla^T_x \left[ p_n^- (x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T|p_n^- (x) \right] + p_n^- (x + v)p_{Y|X}(Y_t|x + v) \cdot \left[ \nabla^T_x \nabla^{T - 1} \left( \nabla v^T \right) \right]_{i = 1, \ldots, d} \cdot p_n^- (x) - p_n^- (x + v)p_{Y|X}(Y_t|x + v)|I + \nabla v^T| \nabla^{T}_x p_n^- (x).
\]

\[
\tag{23}
\]

**Remark 3.1:** Equation \eqref{eq:23} holds for any $p_{Y|X}$. That is, the observation noise can be other type beyond the Gaussian. For example, the author of this paper and her co-work investigate the FPF for observation noise with Laplace distribution \cite{21}.

### B. Consistency

Let us back to the situation that the observation noise is Gaussian as in \cite{22}, i.e.

\[
\nabla p_{Y|X}(Y_t|x + v) = p_{Y|X}(Y_t|x + v)(\nabla h(x + v) + \Delta z) \nabla h(x + v),
\]

where $\Delta z = \frac{Y_t - h(x + v)}{\sqrt{2} \Delta t}$. Thus, \eqref{eq:23} becomes

\[
0 = \nabla^T_x p_n^- (x + v)p_n^- (x)
\]

\[
+ p_n^- (x + v)(\Delta z - h(x + v) + \Delta t) \nabla^T_x h(x + v)p_n^- (x)
\]

\[
+ p_n^- (x + v)
\]

\[
\cdot \left[ \nabla^T_x \nabla^{T - 1} \left( \nabla v^T \right) \right]_{i = 1, \ldots, d} \cdot p_n^- (x)
\]

\[
- p_n^- (x + v) \nabla^T_x p_n^- (x) =: II_1 + II_2 + II_3 + II_4.
\]

\[
\tag{24}
\]

after dividing by $p_{Y|X}(Y_t|x + v)$ throughout. For the conciseness of the notation, we shall suppress $p_n^- (x)$ as $p$ below, if no confusion will arise. We shall seek for the control input in the form $v = K \Delta z + u \Delta t$. The Taylor expansion around $x$ is applied to $II_1 + II_4$ one-by-one:

\[
II_1 = \nabla^T p(x + v)p(I + \nabla v^T)p
\]

\[
= p \left[ \nabla^T p + v^T \nabla^2 p + \frac{1}{2} v^T [\nabla \otimes (\nabla p)] (I \otimes v) \right] \cdot (I + \nabla v^T)\]

\[
= p \left[ \nabla^T + K^T \nabla^2 p \Delta z \right.
\]

\[
\cdot \left[ u^T \nabla^2 p + \frac{1}{2} K^T [\nabla \otimes (\nabla p)] (I \otimes K) \right] \Delta t \}
\]

\[
= p \left[ \nabla^T + pK^T \nabla^2 p \Delta z \right.
\]

\[
+ p \left[ u^T \nabla^2 p + \frac{1}{2} K^T [\nabla \otimes (\nabla p)] (I \otimes K) \right]
\]

\[
+ p \nabla^T p(u^T) + pK^T \nabla^2 p \Delta t, t
\]
and
\[
II_4 = - \left[ p + (\nabla^T p) K \Delta z \right. \\
+ \left. \left( (\nabla^T p) u + \frac{1}{2} K^T (\nabla^2 p) K \right) \Delta t \right] \nabla^T p \\
= - p \nabla^T p - (\nabla^T p) K \nabla^T p \Delta z \\
- \left( (\nabla^T p) u + \frac{1}{2} K^T (\nabla^2 p) K \right) \nabla^T p \Delta t,
\]
respectively, where \( \nabla^2 p \) is the Hessian matrix of \( p \) and \( \otimes \) is the Kronecker product. Collecting the \( O(\Delta z) \) and \( O(\Delta t) \) terms in \( II_1 - II_4 \), we obtain two identities:
\[
O(\Delta z) : \quad 0 = p K^T \nabla^2 p + p \nabla^T p (\nabla K)^T + p^2 \nabla^T h \\
+ p^2 \left[ \text{tr} \left( \frac{\partial (\nabla K)}{\partial x_i} \right) \right]_{i=1, \ldots, d} \\
- (\nabla^T p) K \nabla^T p,
\]
and
\[
O(\Delta t) : \quad 0 = p u^T \nabla^2 p + \frac{1}{2} p K^T (\nabla T \otimes (\nabla^2 p)) (I \otimes K) \\
+ p \nabla^T p (\nabla u)^T + p K^T \nabla^2 p (\nabla K)^T \\
+ p^2 h \nabla^T h + p (\nabla^T p) K \nabla^T h + p K^T \nabla^2 h \\
+ p^2 \nabla^T h (\nabla K)^T \\
+ p (\nabla^T p) \left[ \text{tr} \left( \frac{\partial (\nabla u)}{\partial x_i} \right) \right]_{i=1, \ldots, d} \\
+ p^2 \left[ \text{tr} \left( \frac{\partial (\nabla u)}{\partial x_i} - \nabla K \frac{\partial (\nabla K)}{\partial x_i} \right) \right]_{i=1, \ldots, d} \\
- (\nabla^T p) u \nabla^T p - \frac{1}{2} K^T (\nabla^2 p) K \nabla^T p.
\]

**Proposition 3.2:** The control input pair \((K, u)\) satisfies the equations:
\[
\nabla^T \left( \frac{1}{p} \nabla^T (pK) \right) = - \nabla^T h,
\]
and
\[
- \nabla^T \left( \frac{1}{p} \nabla^T (pu) \right) \\
= \nabla^T \left\{ - \frac{1}{2} \left( h^2 - \frac{1}{2} p (K T \nabla^2 p K) + \frac{1}{2} K^T (\nabla^2 p) K \right) \\
+ \frac{1}{2} K^T (\nabla^2 p) K \\
- K^T (\nabla^2 p) K \\
- \nabla^T (\nabla^T K) K - \frac{1}{2} \text{tr} \left( (\nabla^2 K) (\nabla^T K)^T \right) \right\},
\]
respectively.

The proof of this proposition is extremely long and involved. To avoid distraction, we postpone this tedious computation to Appendix B.

**Theorem 3.2:** The admissible control input pair \((K, u)\) satisfies the equations:
\[
\nabla^T (pK) = -(h - \hat{h})p,
\]
and
\[
\nabla^T (pu) = (h - \hat{h})p + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p (K K^T)_{ij} \right],
\]
where \( \hat{h} = \int h \, dx \).

**Proof:** Starting from (27), integrating over \( \mathbb{R} \) for each \( x_i \), \( i = 1, \ldots, d \), yields that
\[
- \nabla^T (pK) = (h - c_i) p,
\]
\( i = 1, \ldots, d \), where \( c_i \) is a constant with respect to \( x_i \). Thus, \( c_i = c \), for all \( i \), where \( c \) is a constant to be determined. Recall that \( K \) is admissible, i.e. \( K \in L^2(\mathbb{R}^d; p) \), then \( \int_{\mathbb{R}^d} |K|^2 p \, dx \leq (\int_{\mathbb{R}^d} |K|^2 p \, dx)^{1/2} < \infty \), thus, \( \lim_{|x| \to \infty} p K(x) = 0 \). Equation (29) is obtained by integrating (31) over \( \mathbb{R}^d \), so that
\[
0 = \int (h - c) p \, dx \quad \Rightarrow \quad c = \hat{h}.
\]
To show (30), integrating (28) once, one obtain that
\[
\nabla^T (pu) = \frac{1}{2} h^2 p + \frac{1}{2} (K T \nabla^2 p K) \\
- \frac{p}{2} K^T (\nabla^2 p) (\nabla^T K) K \\
+ \frac{p}{2} \text{tr} \left( (\nabla^2 K) (\nabla^T K)^T \right) + C_1 p,
\]
where \( C_1 \) is a constant to be determined later. Let us take a look at the third term on the right-hand side of (33):
\[
- \frac{p}{2} K^T (\nabla^2 p) (\nabla^T K) K \\
- \frac{p}{2} \left[ (h - \hat{h}) + \nabla^T K \right] \right)^2 \\
= - \frac{p}{2} h^2 + (h - \hat{h}) \hat{h} p + \frac{p}{2} \hat{h}^2 + p \nabla^T (log p) K \nabla^T K \\
+ \frac{p}{2} (\nabla^T K)^2.
\]
Substituting (34) back to (33), we have
\[
\nabla^T (pu) = \frac{1}{2} (K T \nabla^2 p K) + (h - \hat{h}) \hat{h} p + \frac{p}{2} \hat{h}^2 + p \nabla^T (log p) K \nabla^T K \\
+ \frac{p}{2} (\nabla^T K)^2 + p K^T (\nabla^2 p) K \nabla^T K \\
+ \frac{p}{2} \text{tr} \left( (\nabla^2 K) (\nabla^T K)^T \right) + C_1 p
\]
\[
= (h - \hat{h}) \hat{h} p + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left[ p (K K^T)_{ij} \right] + \frac{\hat{h}^2}{2} p + C_1 p,
\]

where the last equality follows by direct computation:

\[
\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} [p(KK^T)_{ij}] = \sum_{i,j=1}^{d} \left\{ \frac{\partial^2 p}{\partial x_i \partial x_j} K_i K_j + 2 \frac{\partial p}{\partial x_i} \frac{\partial K_i}{\partial x_j} K_j + 2 \frac{\partial p}{\partial x_i} K_i \frac{\partial K_j}{\partial x_j} \right\} + 2p \frac{\partial K_j}{\partial x_i} K_j + p \frac{\partial K_i}{\partial x_i} K_j \partial x_j \right\} = K^T \nabla^2 p K + 2 \nabla^T p K \nabla K + 2K^T (\nabla K^T) \nabla p + p(\nabla K) + 2p(\nabla K) K + \text{ptr}[(\nabla K^T)(\nabla K)^T].
\]

Recall that \( u \) is admissible, i.e. \( u \in L^2(\mathbb{R}^d; \mathbb{R}) \), then \( \int_{\mathbb{R}^d} |u|^2 dx \leq \left( \int_{\mathbb{R}^d} |u|^2 dx \right)^{\frac{1}{2}} < \infty \), thus \( \lim_{|x| \to \infty} pu(x) = 0 \). Additionally,

\[
\int_{\mathbb{R}^d} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (pK_i K_j) dx = \sum_{i,j} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} (pK_i K_j) dx d(x \setminus x_j) = \sum_{i,j} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} \left[ (pK_i K_j) \right]_{x_i \to \infty} d(x \setminus x_j) \to 0,
\]

if \( ||K||_{\infty} < \infty \). This boundedness of \( K \) is followed by the admissible condition, since \( K \in H^l(\mathbb{R}^d; \mathbb{R}) \subset L^\infty(\mathbb{R}^d), l > \left[ \frac{d}{2} \right] + 1 \), by Sobolev embedding theorem (Theorem C.1). The constant \( C_1 = -\frac{k^2}{\gamma} \) is obtained by integrating (35) over \( \mathbb{R}^d \), similarly as the procedure in obtaining \( c \) in (32).

**Theorem 3.3 (Consistency):** Suppose the admissible control input \( (K, u) \) are obtained according to (29) and (30), respectively, then provided \( p(x, 0) = p^*(x, 0) \), we have for all \( t > 0 \),

\[
p(x, t) = p^*(x, t).
\]

**Proof:** Notice that the KFE of \( X_t \) given the filtration \( \mathcal{Z}_t \) is

\[
dp \mathcal{L}^* p dt - \nabla^T (pK) dZ_t - \nabla^T (pu) dt + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} [p(KK^T)_{ij}] dt = \mathcal{L}^* p dt + (h - \hat{h}) dZ_t - (h - \hat{h}) h dt p dt,
\]

where \( \mathcal{L}^* \) is defined in (11), which is exactly the Kushner-Stratonovich equation (10) of \( X_t \).

**IV. Existence and uniqueness of the control input in suitable space**

In this section, we shall discuss the existence and uniqueness of the solution in the suitable function space to (29) and (30) under certain conditions.

We first investigate the weak solution to (29) with \( K = \nabla \phi \) such that

\[
\int \nabla^T \psi \nabla \phi p dx = \int (h - \hat{h}) \psi p dx,
\]

for all \( \psi \in H_0^1(\mathbb{R}^d, p) \), which has the norm defined

\[
||\psi||_{H_0^1(\mathbb{R}^d, p)}^2 = ||\psi||_{L^2(\mathbb{R}^d, p)}^2 + ||\nabla \psi||_{L^2(\mathbb{R}^d, p)}^2,
\]

with \( \lim_{|x| \to \infty} \psi(x) = 0 \).

Assume that

(A-1) \( h \in H^k(\mathbb{R}^d; p) (:= W^{k,2}(\mathbb{R}^d; p)) \), for some \( k \geq 0 \);

(A-2) Poincaré-type inequality: there exists a constant \( C > 0 \) such that for any \( \phi \in H_0^1(\mathbb{R}^d; p) \),

\[
||\phi||_{L^2(\mathbb{R}^d; p)} \leq C||\nabla \phi||_{L^2(\mathbb{R}^d; p)}
\]

(A-3) \( \nabla^l(\log p) \in L^\infty(\mathbb{R}^d) \), for \( 2 \leq l \leq k + 1 \). Here, \( k \) is the one in (A-1), with the convention that \( l = 0 \), if \( k < 1 \).

In (29), Yang et. al. gave the similar assumptions as above to guarantee the existence and uniqueness of the solution \( \hat{u} \) to (29), but they have no discussion on the uniqueness of the control input \( u \). They gave the explicit expression (59) of \( u \) and verified that \( u \in L^1(\mathbb{R}^d; p) \).

We point out that Assumption (A-2) may not hold in \( \mathbb{R}^d \). This has been proven in Lemma 10.2(ii), (25) with the unweighted norm. In the following lemma, we show the similar result in the weighted Sobolev space \( W^{1,q}_0(\Omega; p) \), \( 1 \leq q < \infty \):

**Lemma 4.1:** If for some small \( \epsilon < 1 \), \( ||\nabla(\log p)||_{L^\infty(\Omega)} \leq q(1 - \epsilon) \) and \( p(x) > 0 \) in \( \Omega \), then Poincaré-type inequality

\[
||u||_{L^q(\Omega; p)} \leq C||\nabla u||_{L^q(\Omega; p)}
\]

do not hold for \( u \in W^{1,q}_0(\Omega; p) \), if \( \Omega \) contains arbitrarily large balls, i.e., if there exists a sequence \( r_n \to \infty \) and points \( x_n \to \Omega \) such that the ball centered at \( x_n \) with radius \( r_n \) is in \( \Omega \), i.e., \( B(x_n, r_n) \subset \Omega \). Here, \( W^{1,q}(\Omega; p) \) is the Sobolev space with

\[
||\phi||_{W^{1,q}_0(\Omega; p)} = ||\phi||_{L^q(\Omega; p)} + ||\nabla \phi||_{L^q(\Omega; p)}
\]

and \( \phi(x) \equiv 0 \) on \( \partial \Omega \).

**Proof:** (By contradiction) Let \( \gamma(x) \in C_c^\infty(\mathbb{R}^d) \) with \( \gamma \neq 0 \) and support(\( \gamma \)) \( \subset B(0, 1) \), then one defines \( u_n(x) = \gamma(\frac{x - x_n}{r_n}) \frac{1}{p^*(x)} \) is also compactly supported in \( \Omega \). Thus, one has

\[
||u_n||_{L^q(\Omega; p)} \leq \left[ \int_\Omega \gamma \left( \frac{x - x_n}{r_n} \right)^q dx \right]^{1/q} = r_n^{d/q} \left[ \int_{B(0, 1)} |\gamma(y)|^q dy \right]^{1/q} = r_n^{d/q} ||\gamma||_{L^q(B(0, 1))},
\]
while

\[ ||\nabla u_n||_{L^q(\Omega;p)} \leq \frac{1}{r_n} \left[ \int_\Omega \left| \nabla \left( \frac{x - x_n}{r_n} \right) \right|^q dx \right]^{1/q} + \frac{1}{q} \left[ \int_\Omega \left| \nabla \left( \frac{x - x_n}{r_n} \right) \right|^q |\nabla (\log p)|^q dx \right]^{1/q} \]

\[ = r_n^{-1 + \frac{q}{2}} \left[ \int_{B(0,1)} |\nabla \phi(y)|^q dy \right] \frac{1}{q} \]

\[ + \frac{r_n^q}{q} \left[ \int_{B(0,1)} |\nabla \phi(y)|^q (\nabla \log p)^q dy \right] \frac{1}{q} \]

\[ \leq r_n^{-1 + \frac{q}{2}} ||\nabla \phi||_{L^q(B(0,1))} \frac{r_n^q}{q} \left[ \int_{B(0,1)} |\nabla \phi(y)|^q (\nabla \log p)^q dy \right] \frac{1}{q} \]

\[ \leq r_n^{-1 + \frac{q}{2}} ||\nabla \phi||_{L^q(B(0,1))} \frac{r_n^q}{q} \left[ \int_{B(0,1)} |\nabla \phi(y)|^q (\nabla \log p)^q dy \right] \frac{1}{q} \]

\[ \leq r_n^{-1 + \frac{q}{2}} ||\nabla \phi||_{L^q(B(0,1))} \frac{r_n^q}{q} \left[ \int_{B(0,1)} |\nabla \phi(y)|^q (\nabla \log p)^q dy \right] \frac{1}{q} \]

where the last inequality follows from the assumption on 
\[ ||\nabla (\log p)||_{\infty}. \] Suppose Poincaré-type inequality (42) holds for all \( u_n \in W^{1,q}_0(\Omega;p) \), then there exists a constant \( C \) independent of \( n \) such that

\[ 0 < ||\nabla \phi||_{L^q(B(0,1))} \leq \frac{C}{r_n} ||\nabla \phi||_{L^q(B(0,1))} \rightarrow 0, \]

as \( r_n \rightarrow \infty \). The contradiction is arrived.

However, it has been verified that Assumption (As-2) does hold for the nonlinear non-Gaussian case with a constant signal model in [15]. It is an interesting question that under what conditions, the Poincaré-type inequality is guaranteed for \( W^{1,q}_0(\mathbb{R}^d;p) \). This is our future project.

A. The existence and uniqueness of \( \phi \in H^{k+1}_0(\mathbb{R}^d;p) \)

Theorem 4.4: Under Assumptions (As-1)-(As-3), equation (29) has a unique weak solution \( \phi \in H^{k+1}_0(\mathbb{R}^d;p) \), where \( k \) is the one in (As-1).

It follows easily from Lax-Milgram theorem that there exists a unique solution \( \phi \in H^{k+1}_0(\mathbb{R}^d;p) \). Or one can consult the proof of Theorem 2, section A.2. [27]. We shall show that \( \phi \in H^{k+1}_0(\mathbb{R}^d;p), k \geq 1 \), in Lemma [4.3] later. But before that, we need the following lemma:

Lemma 4.2: If \( \phi \in C^\infty(\mathbb{R}^d) \) satisfies (29) with \( K = \nabla\phi \), we have

\[ -\nabla^T \left[ p \nabla \left( \nabla^l_{\sigma_1} \phi \right) \right] = G^l_{\sigma_1} \phi, \]

for \( l \geq 1 \), where \( \nabla^l_{\sigma_1} \phi \) is defined in (7), \( \sigma_1 \in \{1, \cdots, d\}^l \) and

\[ G^0_{\sigma_1} := \nabla \left( \frac{\partial (\log p)}{\partial x_{\sigma_1(l)}} \right) \phi + \frac{\partial h}{\partial x_{\sigma_1(l)}} \]

\[ G^l_{\sigma_1} := \nabla \left( \frac{\partial (\log p)}{\partial x_{\sigma_1(l)}} \right) \left[ \nabla \left( \nabla^{l-1}_{\sigma_1(i)} \phi \right) + \frac{\partial G^{l-1}_{\sigma_1(i+1)}}{\partial x_{\sigma_1(l)}} \right]. \]

for \( l \geq 2 \), with \( \sigma_{l+1} = (\sigma_l(1), \cdots, \sigma_l(l-1)) \in \{1, \cdots, d\}^{l-1} \) is the first \( l-1 \) component of \( \sigma_l \), where

\[ \frac{\partial G^{l-1}_{\sigma_l(i+1)}}{\partial x_{\sigma_l(i)}} \]

\[ = \sum_{i=2}^{l} \nabla^{l-1}_{\sigma_l(i)} \left[ \nabla^T \left( \frac{\partial (\log p)}{\partial x_{\sigma_l(i-1)}} \right) \nabla \left( \nabla^{l-2}_{\sigma_l(i-2)} \phi \right) \right] + \nabla^l_{\sigma_l} h, \]

with the convention that \( \nabla^0 = I_d \), where \( I_d \) represents the identity mapping.

To avoid the distraction, we append the proof of this lemma in Appendix D.

Lemma 4.3: Under Assumption (As-1)-(As-3), if the weak solution of (29) with \( K = \nabla\phi, \phi \in H^{k+1}_0(\mathbb{R}^d;p), \) then \( \phi \) is actually is in \( H^{k+1}_0(\mathbb{R}^d;p) \), for \( k \) in (As-1). Moreover, it satisfies that for any \( l \geq 1 \) and \( \sigma_l \in \{1, \cdots, d\}^l \) such that

\[ ||\nabla^{l+1}_{\sigma_l} \phi||_{L^2(\mathbb{R}^d;p)}^2 \]

\[ \leq ||\nabla^{l}_{\sigma_l} \phi||_{L^2(\mathbb{R}^d;p)}^2 + \sum_{\sigma_{l-1} \in \{1, \cdots, d\}^{l-1}} ||G^l_{\sigma_l} \phi||_{L^2(\mathbb{R}^d;p)}^2, \]

with

\[ ||G^l_{\sigma_l} \phi||_{L^2(\mathbb{R}^d;p)}^2 \]

\[ \leq \sum_{i=2}^{l} \sum_{m=0}^{l-1} \sum_{\sigma_{l-1} \in \sigma_{l-1}} ||\nabla^T \left[ \nabla^{m+1}_{(\sigma_{l-1}, \sigma_{l-1})} (\log p) \right] \||_{\infty} \]

\[ \cdot \left[ ||\nabla \left( \nabla^{l-m-1}_{\sigma_{l-1} \sigma_{l-1}} \phi \right) \phi \right]_{L^2(\mathbb{R}^d;p)}^2 + ||\nabla^l_{\sigma_l} h||_{L^2(\mathbb{R}^d;p)}^2, \]

where

\[ ||\nabla^l_{\sigma_l} \phi||_{L^2(\mathbb{R}^d;p)}^2 \]

\[ = \sum_{\sigma_{l-1} \in \{1, \cdots, d\}^{l-1}} \int |\nabla^{l}_{\sigma_l} \phi|^2 p dx. \]

Proof: We first claim that for \( l \geq 1, \sigma_l \in \{1, \cdots, d\}^l \),

\[ \int |\nabla (\nabla^{l}_{\sigma_l} \phi)|^2 p dx \leq \int |\nabla^{l}_{\sigma_l} \phi| G^l_{\sigma_l} p dx, \]

holds if \( \phi \in H^{l+1}_0(\mathbb{R}^d;p) \).

Let \( \beta(x) \geq 0 \) be a smooth, compactly supported, radially decreasing function in \( \mathbb{R}^d \), with \( \beta(0) = 1 \). Let \( s > 0 \). For any \( l \geq 1 \) and \( \sigma_l \in \{1, \cdots, d\}^l \), multiplying (42) with \( \beta^2(sx) \nabla^l_{\sigma_l} \phi(x) \) and integrating it over \( \mathbb{R}^d \), yields that

\[ \int \beta^2(sx) \nabla^{l}_{\sigma_l} \phi G^l_{\sigma_l} p dx \]

\[ = \nabla^T \left[ \beta^2(sx) \nabla^{l}_{\sigma_l} \phi \right] \nabla (\nabla^{l}_{\sigma_l} \phi) p dx \]

\[ = 2s \beta(sx) \nabla^T \beta(sx) \nabla^{l}_{\sigma_l} \phi \nabla (\nabla^{l}_{\sigma_l} \phi) p dx \]

\[ + \int \beta^2(sx) |\nabla (\nabla^{l}_{\sigma_l} \phi)|^2 p dx, \]
where the first equality follows by integration by parts, and $\nabla^l_{\sigma_1}$ is defined as (7). The first term on the right-hand side of (51) can be controlled as
\[
\left| \int 2s\beta(sx)\nabla^T \beta(sx)(\nabla^l_{\sigma_1}\phi)(\nabla^l_{\sigma_1}\phi) dx \right|
\leq 2s||\nabla^T \beta||_{\infty} \int \beta(sx) |\nabla^l_{\sigma_1}\phi| |\nabla(\nabla^l_{\sigma_1}\phi)| \, dx
\leq s||\nabla^T \beta||_{\infty} \left( \int |\nabla^l_{\sigma_1}\phi|^2 \, dx + \int \beta^2(sx)|\nabla(\nabla^l_{\sigma_1}\phi)|^2 \, dx \right),
\]
where the last inequality follows by Cauchy-Schwarz inequality. Substituting (52) back to (51), we obtain that
\[
\left| \int G^l_{\sigma_1} |\nabla^l_{\sigma_1}\phi| \, dx \right|
\geq \int \beta^2(sx)G^l_{\sigma_1}(\nabla^l_{\sigma_1}\phi) \, dx
\geq (1-s||\nabla^T \beta||_{\infty}) \int \beta^2(sx)|\nabla(\nabla^l_{\sigma_1}\phi)|^2 \, dx
- s||\nabla^T \beta||_{\infty} \left( \int |\nabla^l_{\sigma_1}\phi|^2 \, dx + \int |\nabla^l_{\sigma_1}\phi|^2 \, dx \right),
\]
where the last inequality is due to $\beta(0) = 1$ and $\beta$ is radially decreasing. Thus, (50) follows immediately by letting $s \to 0$ and dominated convergence theorem.

In the sequel, we show that (47) holds, for $1 \leq l \leq k$, here $k$ is the one in (As-1). In fact, the direct computation yields that
\[
||\nabla^{l+1}\phi||^2_{L^2(\mathbb{R}^d;\mathbb{P})}
= \sum_{\sigma_{l+1} \in \{1, \ldots, d\}^{l+1}} \left| \int |\nabla^{l+1}_{\sigma_{l+1}}\phi|^2 \, dx \right|
= \sum_{\sigma_{l+1} \in \{1, \ldots, d\}^{l+1}} \left[ \int |\nabla^{l}_{\sigma_{l+1}|l-1, i}\phi|^2 \, dx \right]^2
\leq \sum_{\sigma_{l+1} \in \{1, \ldots, d\}^{l+1}} \left[ \int |G^l_{\sigma_{l+1}|l-1, i}\phi|^2 \, dx \right]^2
+ \sum_{\sigma_{l+1} \in \{1, \ldots, d\}^{l+1}} \int |G^l_{\sigma_{l+1}|l-1, i}\phi|^2 \, dx,
\]
where the last two inequalities follow from Hölder’s inequality and Cauchy-Schwarz inequality, respectively. Thus, the right-hand side of (47) is obtained by recalling the definition of $L^2$-norm (49).

From (47), we claim by induction that $\nabla^{l+1}\phi, G^l_{\sigma_{l+1}} \in L^2(\mathbb{R}^d;\mathbb{P})$, for any $l \geq 1$, provided $G^l_{\sigma_{l+1}}, \nabla \phi \in L^2(\mathbb{R}^d;\mathbb{P})$.

We begin with examining $||\nabla \phi||_{L^2(\mathbb{R}^d;\mathbb{P})} < \infty$ and $||G^l_{\sigma_{l+1}}||_{L^2(\mathbb{R}^d;\mathbb{P})} < \infty$. Multiplying (29) with $K = \nabla \phi$ by $\phi$ and integrating over $\mathbb{R}^d$:
\[
||\nabla \phi||^2_{L^2(\mathbb{R}^d;\mathbb{P})} = - \int \nabla^T(p\nabla \phi) \, dx \int (h - \hat{h}) \phi \, dx
\leq \frac{1}{2} \int \nabla^T \phi \, \nabla \phi \, dx + |\hat{h}| \|\phi\|_{L^2(\mathbb{R}^d;\mathbb{P})}
\leq C \left( \|h\|_{L^2(\mathbb{R}^d;\mathbb{P})} + |\hat{h}| \right) \|\nabla \phi\|_{L^2(\mathbb{R}^d;\mathbb{P})},
\]
Therefore, we obtain that $||\nabla \phi||_{L^2(\mathbb{R}^d;\mathbb{P})} < \infty$, provided $h \in L^2(\mathbb{R}^d;\mathbb{P})$. Now, we check whether $G^l_{\sigma_{l+1}} \in L^2(\mathbb{R}^d;\mathbb{P})$:
\[
\left| \int G^l_{\sigma_{l+1}} |\nabla^l_{\sigma_{l+1}}\phi| \, dx \right|
\leq \left| \int \nabla^T \frac{\partial(\log p)}{\partial x_{\sigma_{l+1}/l}} \nabla \phi \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})}
+ \left| \frac{\partial h}{\partial x_{\sigma_{l+1}/l}} \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})}
\]
provided that $\nabla \phi, \nabla h \in L^2(\mathbb{R}^d;\mathbb{P})$ and (As-3).

By induction, suppose that for all $\sigma_{l-1} \in \{1, \ldots, d\}^{l-1}$, $G^{l-1}_{\sigma_{l-1}}, \nabla^{l-1}_{\sigma_{l-1}} \phi \in L^2(\mathbb{R}^d;\mathbb{P})$, then we have $\nabla^l \phi \in L^2(\mathbb{R}^d;\mathbb{P})$, $l \geq 2$, by (47), and
\[
\left| \int G^l_{\sigma_{l+1}} |\nabla^l_{\sigma_{l+1}}\phi| \, dx \right|
\leq \left| \int \nabla^T \frac{\partial(\log p)}{\partial x_{\sigma_{l+1}/l}} \nabla \left( \nabla^{l-1}_{\sigma_{l-1}} \phi \right) \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})}
+ \left| \frac{\partial G^{l-1}_{\sigma_{l+1}/l-1}}{\partial x_{\sigma_{l+1}/l}} \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})},
\]
since
\[
\left| \int \nabla^T \frac{\partial(\log p)}{\partial x_{\sigma_{l+1}/l}} \nabla \left( \nabla^{l-1}_{\sigma_{l-1}} \phi \right) \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})}
\leq \sum_{i=2}^l \sum_{m=0}^i \left| \nabla^T \left( \nabla^{i-2}_{\sigma_{l-1}|l-2, i} \phi \right) \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})}
+ \left| \frac{\partial G^{l-1}_{\sigma_{l+1}/l-1}}{\partial x_{\sigma_{l+1}/l}} \right|^2_{L^2(\mathbb{R}^d;\mathbb{P})},
\]
where $\sigma_m$ is the sub-vector of $\sigma$ of size $m$, while $\sigma \setminus \sigma_m$ is the rest of the vector $\sigma$ after removing $\sigma_m$, with the convention that $\sigma_0 = \emptyset$. Therefore, (48) follows by substituting (55) back to (54). Consequently, $G^{l}_{\sigma_{l+1}} \in L^2(\mathbb{R}^d;\mathbb{P})$, provided $\nabla^m \phi, \nabla^m h \in L^2(\mathbb{R}^d;\mathbb{P})$ and $||\nabla^m (\log p)||_{\infty} < \infty$, for all $m \leq l$.

Under Assumption (As-1) and (As-3), we conclude that $\phi \in H^l_{0,1}(\mathbb{R}^d;\mathbb{P})$, if $\phi \in H^l_{0,1}(\mathbb{R}^d;\mathbb{P})$. Here, $k$ is the one in (As-1).
B. The existence and uniqueness of the solution $\varphi \in H^1_0(\mathbb{R}^d; p)$

In [27], the expression of the solution $u$ is directly given without any derivation. There is NO uniqueness result of this solution $u$. In this subsection, we confirm that the explicit solution given in [27] is the unique one in the space $L^2(\mathbb{R}^d; p)$ under Assumption (As-1)-(As-3), with $k \geq \lfloor \frac{d+2}{4} \rfloor$ in (As-1).

Similarly as we did for [29], we seek the weak solution to (50) with $u = \nabla \varphi$ such that

$$\int \nabla \varphi \nabla \psi dx = h \int h \nabla \psi dx - \hat{h}^2 \int \psi dx$$

(56)

$$+ \int \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} (pK_i K_j) \psi dx,$$

for $\psi \in H^1_0(\mathbb{R}^d; p)$.

**Theorem 4.5:** Under Assumption (As-1)-(As-3) with $k \geq \lfloor \frac{d+2}{4} \rfloor$ in (As-1), equation (50) has a unique solution $u = \nabla \varphi$, with $\varphi \in H^1_0(\mathbb{R}^d)$.

**Proof:** The existence and uniqueness of $\varphi \in H^1_0(\mathbb{R}^d; p)$ is guaranteed by Riesz representation theorem. In fact, the inner product of $(\varphi, \psi)$ is defined as the integral on the left-hand side of (56). We only need to check its boundedness. Let us first look at the last term on the right-hand side of (56):

$$\int \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} (pK_i K_j) \psi dx$$

$$= - \int \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (pK_i K_j) \frac{\partial \psi}{\partial x_j} dx$$

$$= - \int \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (pK_i) K_j \frac{\partial \psi}{\partial x_j} dx - \int \sum_{i,j=1}^d pK_i \frac{\partial K_j}{\partial x_j} \frac{\partial \psi}{\partial x_i} dx$$

$$= - \int \nabla^T (p \nabla \psi) \nabla^T \nabla \psi dx - \int \nabla^T \nabla \psi \nabla^T \nabla \psi dx$$

(57)

By Theorem 4.4, we have $\phi \in H^{k+1}_0(\mathbb{R}^d; p)$. Thus, the inner product can be bounded

$$|\langle \varphi, \psi \rangle|$$

$$\leq \hat{h} ||h||_{L^2(\mathbb{R}^d; p)} ||\psi||_{L^2(\mathbb{R}^d; p)} + ||h||_{L^r(\mathbb{R}^d; p)} ||\nabla \psi||_{L^r(\mathbb{R}^d; p)}$$

$$+ ||\nabla \psi||_{L^r(\mathbb{R}^d; p)} ||\nabla \varphi||_{L^r(\mathbb{R}^d; p)}$$

(58, 59)

where $\frac{1}{r} + \frac{1}{s} = \frac{1}{2} + \frac{1}{r^*} = \frac{1}{2}$. By Sobolev embedding theorem (Theorem C.1), it is easy to deduce that for the second term on the right-hand side to be bounded:

$$||h||_{L^r(\mathbb{R}^d; p)} : \frac{1}{r} \geq \frac{1}{2} - \frac{k}{d}$$

$$||\nabla \psi||_{L^r(\mathbb{R}^d; p)} : \frac{1}{r} \geq \frac{1}{2} - \frac{k+1}{d}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{r} + \frac{1}{r^*} \geq 1 - \frac{2k}{d},$$

thus, $k \geq \frac{d}{4}$, and for the third term:

$$||\nabla \varphi||_{L^r(\mathbb{R}^d; p)} :$$

$$\frac{1}{r} \geq \frac{1}{2} - \frac{k+1}{d}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{r} + \frac{1}{r^*} \geq 1 - \frac{2k+1}{d},$$

$$\Rightarrow \frac{1}{2} = \frac{1}{r} + \frac{1}{r^*} \geq 1 - \frac{2k+1}{d}.$$
where \( \sigma_{l_i} \) is the sub-vector of \( \sigma \) of size \( l_1 \) and \( \sigma \setminus \sigma_{l_i} \) is the same notation as before. Taking \( L^2(\mathbb{R}^d; p) \) norm on both sides of (60), we have

\[
\| \nabla^l V_l \|_{L^2(\mathbb{R}^d; p)}^2 \leq \sum_{l_1=0}^l \sum_{\sigma_{l_1} \subseteq \sigma} \| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 \leq \sum_{l_1=0}^l \sum_{\sigma_{l_1} \subseteq \sigma} \| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 + \| \nabla_{\sigma_{l_1}} \phi \|_{L^2(\mathbb{R}^d; p)}^2.
\]

Next, we carefully derived the equations that the \( \sigma_{l_1} \) are bounded. For \( \sigma_{l_1} \in \sigma \), \( \sigma_{l_1} \in \{ 1, \ldots, d \} \).

For \( V_1 \), let us check that how large \( l \) could be so that \( V_1 - V_3 \) are bounded. For \( V_1 \):

\[
\| \nabla (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 = \sum_{l_1=0}^l \sum_{\sigma_{l_1} \subseteq \sigma} \| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 \leq \sum_{l_1=0}^l \sum_{\sigma_{l_1} \subseteq \sigma} \| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 + \| \nabla_{\sigma_{l_1}} \phi \|_{L^2(\mathbb{R}^d; p)}^2.
\]

\[
\| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 \leq \sum_{l_1=0}^l \sum_{\sigma_{l_1} \subseteq \sigma} \| \nabla^l (\nabla_{\sigma_{l_1}} \phi) \|_{L^2(\mathbb{R}^d; p)}^2 + \| \nabla_{\sigma_{l_1}} \phi \|_{L^2(\mathbb{R}^d; p)}^2.
\]

\[
\Rightarrow 1 + \frac{1}{r_{\sigma_{l_1}}} + \frac{1}{r^*_{\sigma_{l_1}}} \geq 2 - \frac{4k - 2l}{d} \Rightarrow l \leq 2k - \frac{d}{2}.
\]

Therefore, \( l = \min \{ k, \lfloor 2k - \frac{d}{2} \rfloor \} \).

V. Conclusion

In this paper, we re-investigated the multivariate feedback particle filter. We first show that the control input can be obtained by any f-divergence, not just the Kullback-Leibler divergence. Next, we carefully derived the equations that the control input satisfies, which has not been done in [27]. The derivation is for \( d \geq 1 \) and \( m \geq 1 \), but can be extended to the most general case \( m \geq 1 \), with more involved notations and computations. We re-defined admissible for the control input, so that it can be shown that with this new definition the admissible control input exists and is unique in appropriate function spaces. Furthermore, we show that the explicit expression given in [27] for the control input is actually the only admissible one.

APPENDIX

A. Proof of Proposition 3.1

Proof: It is clear to see that when \( d = 1 \),

\[
\frac{1}{d} \left| \frac{d}{dx} \left[ 1 + v' \right] - \left| \frac{d}{dx} \left[ 1 + v' \right] \right| - 1 \right| = \frac{d}{dx} [1] = 0.
\]

By induction, we shall validate (20) for \( d \geq 2 \). Let us denote \( V = I + \nabla v^T \), then \( \nabla v^T [V^{-T}] = \nabla v^T [V^*]^T \), where \( V^* \) is the adjugate matrix of \( V \) with the element \( (V^*)_{ij} = (-1)^{i+j} M_{j,i} \), and \( M_{j,i} \) is the \( (j,i) \)th minor matrix of \( V \). Therefore, we have

\[
\nabla v^T [V^*]^T = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right) \left( \begin{array}{cccc}
(V^*)_{11} & \cdots & (V^*)_{1d} \\
\vdots & \ddots & \vdots \\
(V^*)_{d1} & \cdots & (V^*)_{dd}
\end{array} \right) = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right) \left( \begin{array}{cccc}
M_{1,1} & \cdots & (-1)^{1+d} M_{1,d} \\
\vdots & \ddots & \vdots \\
(-1)^{d+1} M_{d,1} & \cdots & M_{d,d}
\end{array} \right) = \sum_{i=1}^d (-1)^{i+j} \frac{\partial}{\partial x_i} M_{i,j}].
\]

Actually, we can show more general statement than (20). For any \( d \geq 2 \), \( i = 1, \ldots, d \) and a vector-valued function of \( x = (v_1, \ldots, v_d) \in \mathbb{R}^d \), we have

\[
\left( \begin{array}{cccc}
\frac{\partial \tilde{v}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{v}_1}{\partial x_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial \tilde{v}_d}{\partial x_1} & \cdots & \frac{\partial \tilde{v}_d}{\partial x_d}
\end{array} \right) = \left( \begin{array}{cccc}
\frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial v_d}{\partial x_1} & \cdots & \frac{\partial v_d}{\partial x_d}
\end{array} \right) \left( \begin{array}{cccc}
\frac{\partial \tilde{v}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{v}_1}{\partial x_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial \tilde{v}_d}{\partial x_1} & \cdots & \frac{\partial \tilde{v}_d}{\partial x_d}
\end{array} \right) \equiv (20),
\]

where the determinant notation on the left-hand side of (62) is defined as the minor expansion along the \( j \)-th column.

Equality (20) is just a special case of this statement by letting \( \tilde{v}_k(x) = x_k + v_k(x) \) in (62). The right-hand side of (62) gives exactly the right-hand side of (61).

Now, we shall validate (62) for \( d \geq 2 \) by induction. When \( d = 2 \), direct computation yields that

\[
\frac{\partial^2 \tilde{v}_2}{\partial x_1 \partial x_2} = \frac{\partial^2 \tilde{v}_2}{\partial x_2 \partial x_1} = 0,
\]

and so does \( \frac{\partial \tilde{v}_1}{\partial x_1} \).

By induction, suppose (62) holds for \( d - 1 \). That is, for any fixed \( l, k = 1, \ldots, d \), we have

\[
0 = \sum_{i \in \{1, \ldots, d \}} (-1)^{i+j} \frac{\partial}{\partial x_1} (M_{1,k})_{i,j} + \sum_{i \in \{1, \ldots, d \}} (-1)^{i+j+1} \frac{\partial}{\partial x_1} (M_{1,k})_{i,j},
\]

if \( k > j \), and

\[
0 = \sum_{i \in \{1, \ldots, d \}} (-1)^{(j+1)-i} \frac{\partial}{\partial x_1} (M_{1,k})_{i,j} + \sum_{i \in \{1, \ldots, d \}} (-1)^{(i+1)+j-1} \frac{\partial}{\partial x_1} (M_{1,k})_{i,j} = 0,
\]
if $k < j$, where $M_{i,k}^\circ$ is the $(i,k)$-th minor matrix of $\nabla v^T$, and $(M_{i,j}^\circ)_{i,j}$ is the $(i,j)$th minor of $M_{i,j}^\circ$.

For any fixed $j = 1, \ldots, d$, let us expand $M_{i,j}^\circ$ along the $k$th column, $k \in \{1, \ldots, d\} \setminus \{j\}$:

$$
\sum_{i=1}^{d} (-1)^{i+j} \frac{\partial M_{i,j}^\circ}{\partial x_i}
$$

where the first equality follows by interchanging the order of the two summations. With the fact that $(M_{i,j}^\circ)_{i,k} = (M_{i,j}^\circ)_{i,k}$, we have the two summations in the bracket in $I_1$ cancelled out, i.e., $I_1 = 0$. Meanwhile, by interchanging the order of the summations in $I_2$, one obtain that

$$
I_2 = \sum_{i=1}^{d} (-1)^{l-1} \frac{\partial \bar{v}_k}{\partial x_i}
$$

since the terms in the bracket equal zero, by induction hypothesis. The case $k < j$ can be argued similarly to verify that $I_2$ holds.

$\blacksquare$

B. Proof of Proposition 3.2

Proof: We move the term with $h$ in (25) to the left-hand side and divide by $p^2$ throughout:

$$
-\nabla^T h = \frac{1}{p} K^T \nabla^2 p + \frac{1}{p} \nabla^T (p \nabla K^T)^T
$$

$$
+ \left[ \text{tr} \left( \frac{\partial (\nabla K^T)}{\partial x_i} \right) \right]_{i=1, \ldots, d} - \frac{1}{p^2} (\nabla^T p) K \nabla^T p
$$

$$
=: III_1 + III_2 + III_3 + III_4.
$$

Notice that

$$
\nabla^2 (\log p) = \nabla \left( \nabla^T (\log p) \right) = \nabla \left( \frac{1}{p} \nabla^T p \right)
$$

$$
= - \frac{1}{p^2} \nabla p \nabla^T p - \frac{1}{p} \nabla^2 p
$$

$$
= - \nabla (\log p) \nabla^T (\log p) + \frac{1}{p} \nabla^2 p,
$$

then

$$
III_1 = K^T \left[ \nabla^2 (\log p) + \nabla (\log p) \nabla^T (\log p) \right].
$$

With the fact that for any $i = 1, \ldots, d$,

$$
\text{tr} \left( \frac{\partial (\nabla K^T)}{\partial x_i} \right) = \text{tr} \left[ \frac{\partial K_i}{\partial x_1} \ldots \frac{\partial K_i}{\partial x_T} \right]
$$

$$
= \frac{\partial}{\partial x_i} (\nabla^T K),
$$

we have

$$
III_3 = \nabla^T (\nabla K).
$$

Lastly, it is easy to see that

$$
III_2 = \nabla^T (\log p) (\nabla K^T)^T,
$$

$$
III_4 = - \nabla^T (\log p) K \nabla^T (\log p).
$$

Substituting (69), (71)-(73) back to (67), we obtain that

$$
-\nabla^T h = K^T \nabla^2 (\log p) + \nabla^T (\log p) (\nabla K^T)^T + \nabla^T (\nabla K),
$$
since the second term of \( III_1 \) cancels out with \( III_4 \). Let us verify that the right-hand side of (74) is \( \nabla^T \left( \frac{1}{p} \nabla^T (pK) \right) \) by direct computation:

\[
\nabla^T \left( \frac{1}{p} \nabla^T (pK) \right) = \nabla^T \left( \frac{\nabla^T pK}{p} + \nabla^T K \right) = \nabla^T \left( (\nabla^T (log p)) K + \nabla^T K \right) = \nabla^T (\nabla^T (log p))(\nabla K)^T + \nabla^T (\nabla^T K),
\]

where \( \nabla^T (\nabla^T (log p))^T = \nabla^2 (log p) \) due to its symmetry and the third equality follows from the fact that, for any two column vectors \( a, b \in \mathbb{R}^d \),

\[
\nabla^T (a^T b) = b^T (\nabla a)^T + a^T (\nabla b)^T.
\]

To derive (28), we look at the terms in (26). It contains the same terms as those \( III_1 - III_4 \) in (77) with \( K \) replacing by \( u \). Therefore, by moving all these terms with \( u \) to the left-hand side and dividing by \( p^2 \) throughout, we have

\[
- \nabla^T \left( \frac{1}{p} \nabla^T (pu) \right) - h\nabla^T h + K^T \nabla^2 h
+ \frac{1}{2p} K^T (\nabla^T \otimes (\nabla^2 p)) \quad (I_d \otimes K)
+ \frac{1}{2p} \nabla^2 p (\nabla K)^T + \frac{1}{p}(\nabla^T p) K \nabla^T h
+ \nabla^T h (\nabla K)^T + \frac{1}{p}(\nabla^T p) K \left[ \frac{\partial (\nabla K^T)}{\partial x_i} \right]_{i=1,\ldots,d}
- \left[ \frac{\partial (\nabla K^T)}{\partial x_i} \right]_{i=1,\ldots,d}
- \frac{1}{2p^2} K^T (\nabla^2 p) K \nabla^T p
= - h\nabla^T h + IV_1 + IV_2 + \cdots + IV_8.
\]

In the sequel, we shall simplify the terms \( IV_1 - IV_8 \) one-by-one into the derivatives of \( log p \) and \( K \):

\[
IV_1 = K^T \nabla^T (\nabla^T h) = K^T \nabla^T \left\{ [K^T \nabla^2 (log p) + \nabla^T (log p)] (\nabla K)^T \right\} + \nabla^T (\nabla^T K)
\]

\[
= - K^T (\nabla K)^T \nabla^2 (log p) - K^T (I_d \otimes K^T) (\nabla \otimes (\nabla^2 (log p))) - K^T \nabla^2 (log p) (\nabla K)^T - K^T (I_d \otimes (\nabla K)^T) (\nabla \otimes (\nabla K)^T)
- K^T \nabla^2 (\nabla K)^T
=: IV_{1,1} + IV_{1,2} + IV_{1,3} + IV_{1,4} + IV_{1,5},
\]

where the second equality follows from the fact that for a column vector \( a \in \mathbb{R}^d \) and a \( d \times d \) matrix \( A \),

\[
\nabla (a^T A) = (\nabla a^T) A + (I_d \otimes a^T) (\nabla \otimes A).
\]
In fact, for fixed \( l = 1, \ldots, d \):

\[
\text{tr} \left[ \nabla K^T \frac{\partial (\nabla K^T)}{\partial x_l} \right]
\]

\[
= \sum_{i,j=1}^{d} \frac{\partial K_{ij}}{\partial x_l} \frac{\partial^2 K_{ij}}{\partial x_i \partial x_j} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_l} \left( \frac{\partial K_{ij}}{\partial x_i} \right) \frac{\partial K_{ij}}{\partial x_j}
\]

\[
= \frac{\partial}{\partial x_l} \text{tr} \left[ (\nabla K^T)(\nabla K^T)^T \right] - \text{tr} \left[ \nabla K^T \frac{\partial}{\partial x_l} (\nabla K^T) \right].
\]

Consequently, \( IV_7 \) follows by combining the similar terms in (79) and dividing by 2.

\[
IV_8 = -\frac{1}{2} K^T \left( \frac{1}{p} \nabla^2 p \right) K \nabla^T (\log p)
\]

\[
= -\frac{1}{2} K^T \left[ \nabla^2 (\log p) + \nabla (\log p) \nabla^T (\log p) \right] K \nabla^T (\log p)
\]

\[
= -\frac{1}{2} K^T \nabla^2 (\log p) K \nabla^T (\log p)
\]

\[
- \frac{1}{2} K^T \nabla (\log p) \nabla^T (\log p) K \nabla^2 (\log p)
\]

\[
=: IV_{8,1} + IV_{8,2}.
\]

After close examination to the terms in \( IV_1 - IV_8 \), we obtain the following identities:

1. \( IV_{1,1} = -IV_{1,1} \);
2. \( IV_{1,2} = -IV_{1,2} \), due to the fact that \( K^T \nabla (\log p) = \nabla^T (\log p) K \);
3. \( IV_{3,3} = -IV_3 \);
4. \( IV_{2,1} = -IV_{8,1} \) and \( IV_{2,2} = -IV_{8,2} \);
5. \( IV_{2,4} + IV_{2,5} + IV_{1,1} = 0 \);
6. \( IV_{2,1} + IV_{2,3} + IV_{5,1} = -\frac{1}{2} \nabla^T \left[ \frac{1}{p} (K^T \nabla^2 p K) \right] + \frac{1}{p} \nabla^T \left[ K^T \nabla^2 (\log p) \nabla^T (\log p) K \right] \);  
7. \( IV_{1,1} + IV_{1,4} + IV_{5,3} = -\nabla^T \left[ K^T (\nabla K^T) (\nabla^T (\log p)) K \right] \);
8. \( IV_{1,5} + IV_{5,3} = -\nabla^T \left[ \nabla^T (\nabla K^T) K \right] \).

The necessary proofs of these identities are postponed to the end of the proof. Equation (28) follows by substituting these identity 1)-8) back to \( IV_1 - IV_8 \).

In the rest of this subsection, we shall detail the proof of identity 4)-8).

**Proof of 4):** which holds due to the following identities

\[
K^T \left[ \nabla^2 (\log p) \nabla^T (\log p) K \right] (I_d \otimes K)
\]

\[
= (1 \otimes K) \left\{ \nabla^T (\log p) \otimes \nabla^2 (\log p) K \right\}
\]

\[
= \nabla^T (\log p) \otimes [K^T \nabla (\log p) K] = K^T \nabla^2 (\log p) K \nabla^T (\log p),
\]

and

\[
K^T \left\{ \nabla^T (\log p) \otimes \nabla (\log p) \nabla^T (\log p) K \right\} (I_d \otimes K)
\]

\[
= (1 \otimes K) \left\{ \nabla^T (\log p) \otimes \nabla (\log p) \nabla^T (\log p) K \right\}
\]

\[
= \nabla^T (\log p) \otimes [K^T \nabla (\log p) \nabla^T (\log p) K] = [K^T \nabla (\log p) \nabla^T (\log p)] K \nabla^T (\log p),
\]

where the last equalities in both identities are obtained by noting that \( K^T \nabla^2 (\log p) K \) and \( K^T \nabla (\log p) \nabla^T (\log p) K \) are scalars.

**Proof of 5):** Notice that \( K^T \nabla (\log p) = \nabla^T (\log p) K \) is a scalar. Divide by this scalar throughout these terms, it yields that

\[
\frac{1}{K^T \nabla (\log p)} (IV_{2,4} + IV_{2,5} + IV_{1,1})
\]

\[
= \frac{1}{2} K^T \nabla^2 (\log p) + \frac{1}{2} (\nabla^T \otimes \nabla^T (\log p))(I_d \otimes K)
\]

\[
= K^T \nabla^2 (\log p)
\]

\[
= \frac{1}{2} (\nabla^T \otimes \nabla^T (\log p))(I_d \otimes K) - K^T \nabla^2 (\log p) = 0,
\]

since

\[
\left[ (\nabla^T \otimes \nabla^T (\log p))(I_d \otimes K) \right]_{j} = \left[ \frac{\partial \nabla^T (\log p)}{\partial x_j} K \right]_{j},
\]

for \( j = 1, \ldots, d \).

**Proof of 6):** Let us begin with the direct computation

\[
\nabla^T \left[ \frac{1}{p} (K^T \nabla^2 p K) \right]:
\]

\[
= \nabla^T \left[ \frac{1}{p} (K^T \nabla^2 p K) \right] + \nabla^T \left[ \frac{1}{p} K^T (\nabla^2 (\log p) K) \right] + \nabla^T \left[ \frac{1}{p} K^T \nabla^2 (\log p) K \right],
\]

\[
= \nabla^T \left[ K^T (\nabla (\nabla^2 (\log p) K)) \right]) = \nabla^T \left[ K^T (\nabla (\nabla^2 (\log p) K)) \right]) + \nabla^T \left[ K^T (\nabla^2 (\log p) K) \right]
\]

\[
\nabla^T \left[ K^T (\nabla (\nabla^2 (\log p) K)) \right]) + \nabla^T \left[ K^T (\nabla^2 (\log p) K) \right]
\]

\[
= \nabla^T \left[ K^T (\nabla (\nabla^2 (\log p) K)) \right]) + \nabla^T \left[ K^T (\nabla^2 (\log p) K) \right]
\]

\[
= \nabla^T \left[ K^T (\nabla (\nabla^2 (\log p) K)) \right]) + \nabla^T \left[ K^T (\nabla^2 (\log p) K) \right]
\]

Moreover, we notice that the terms \( IV_{1,2} \) and \( IV_{2,3} \) can be combined, since

\[
(I_d \otimes K^T)(\nabla \otimes \nabla^2 (\log p)) = (\nabla^T \otimes \nabla^2 (\log p))(I_d \otimes K).
\]

In fact, the left-hand side of (81) is

\[
(I_d \otimes K^T)(\nabla \otimes \nabla^2 (\log p)) = \left[ K^T \frac{\partial \nabla^2 (\log p)}{\partial x_1} \right],
\]

\[
\vdots ,
\]

\[
K^T \frac{\partial \nabla^2 (\log p)}{\partial x_d},
\]

where the \( (l,j) \)th entry of

\[
\left[ (I_d \otimes K^T)(\nabla \otimes \nabla^2 (\log p)) \right]_{lj} = \sum_{i=1}^{d} K_{ij} \frac{\partial^3 (\log p)}{\partial x_i \partial x_j},
\]

while the right-hand side of (81) is

\[
(\nabla^T \otimes \nabla^2 (\log p))(I_d \otimes K) = \left[ \frac{\partial \nabla^2 (\log p)}{\partial x_1} K, \ldots, \frac{\partial \nabla^2 (\log p)}{\partial x_d} K \right],
\]
where the \((i,l)\)th entry of 
\[ \left[(\nabla^T \otimes \nabla^2 (\log p))(I_d \otimes K)\right]_{il} = \sum_{j=1}^d \frac{\partial^3 (\log p)}{\partial x_l \partial x_i \partial x_j} K_j. \tag{83} \]

Equality (81) immediately follows by replacing \((i,j,l)\) in (83) into \((l,i,j)\) in (82).

Combining (80) and (81), we finished the proof of (6).

**Proof of 7:** Through direct computations of \(\nabla^T [K^T \nabla K^T \nabla (\log p)]\), we have
\[
\begin{align*}
\nabla^T [K^T \nabla K^T \nabla (\log p)] &= \nabla^T (\log p) [\nabla (K^T \nabla K^T)]^T + K^T \nabla K^T [\nabla^2 (\log p)]^T \\
&= \nabla^T (\log p) [\nabla (K^T \nabla K^T)]^T \\
&= \sum_{i=1}^d K_i \left\{ [I_d \otimes \nabla^T (\log p)] [\nabla \otimes (\nabla K^T)^T] \right\}_{ij} \\
&= \sum_{i=1}^d K_i \sum_{l=1}^d \frac{\partial \log p}{\partial x_l} \frac{\partial (\nabla K^T)^T}{\partial x_i} \\
&= \sum_{i=1}^d K_i \sum_{l=1}^d \frac{\partial \log p}{\partial x_l} \frac{\partial^2 K_j}{\partial x_i \partial x_j},
\end{align*}
\] (84)

It is easy to check that
\[
\begin{align*}
IV_{1,4} &= K^T (I_d \otimes \nabla^T (\log p)) \left[ \nabla \otimes (\nabla K^T)^T \right] \\
&= \nabla^T (\log p) \left[ \nabla \otimes (\nabla K^T)^T \right] (I_d \otimes K),
\end{align*}
\] (85)

since the \(j\)th component of the middle term of (85) is
\[
 \begin{align*}
\{ K^T (I_d \otimes \nabla^T (\log p)) \left[ \nabla \otimes (\nabla K^T)^T \right] \}_{ij} \\
&= \sum_{i=1}^d K_i \left\{ [I_d \otimes \nabla^T (\log p)] \left[ \nabla \otimes (\nabla K^T)^T \right] \right\}_{ij} \\
&= \sum_{i=1}^d K_i \sum_{l=1}^d \frac{\partial \log p}{\partial x_l} \frac{\partial (\nabla K^T)^T}{\partial x_i} \\
&= \sum_{i=1}^d K_i \sum_{l=1}^d \frac{\partial \log p}{\partial x_l} \frac{\partial^2 K_j}{\partial x_i \partial x_j}.
\end{align*}
\]

while the \(k\)th component of the right-hand side is
\[
\begin{align*}
\nabla^T (\log p) \left[ \nabla \otimes (\nabla K^T)^T \right] (I_d \otimes K) \\
&= \sum_{i=1}^d \frac{\partial \log p}{\partial x_i} \left\{ \left[ \nabla \otimes (\nabla K^T)^T \right] (I_d \otimes K) \right\}_{ik} \\
&= \sum_{i=1}^d \frac{\partial \log p}{\partial x_i} \sum_{j=1}^d \frac{\partial [\nabla K^T]^T}{\partial x_k} K_j \\
&= \sum_{i=1}^d \frac{\partial \log p}{\partial x_i} \sum_{j=1}^d \frac{\partial^2 K_j}{\partial x_j \partial x_k} K_j.
\end{align*}
\]

Item (7) follows immediately from (84) and (85).

**Proof of 8:** Item (8) follows immediately by directly computing \(\nabla^T \left[ \nabla^T (\nabla^T K) \right] \):
\[
\begin{align*}
\nabla^T \left[ \nabla^T (\nabla^T K) \right] &= \nabla^{T 2} (\nabla^T K) + \nabla^T (\nabla^T K)(\nabla K)^T. \\
&= K^T \nabla^{T 2} (\nabla^T K) + \nabla^T (\nabla^T K)(\nabla K)^T.
\end{align*}
\]

**C. Sobolev embedding theorem**

Let \(W^{k,p}(\mathbb{R}^d; w)\) denote the Sobolev space consisting of all real-valued functions on \(\mathbb{R}^d\) whose first \(k\) weak derivatives are functions in \(L^p(\mathbb{R}^d; w)\). Here \(k\) is a non-negative integer and \(1 \leq p < \infty\).

**Theorem C.1** (88, 89): If \(k > l\) and \(1 \leq p < q < \infty\) are two real numbers such that \(((k-l)p < n\) and \(\frac{1}{p} - \frac{k}{q} = \frac{1}{q} - \frac{l}{p}\), then \(W^{k,p}(\mathbb{R}^d; w) \subset W^{l,q}(\mathbb{R}^d; w)\).

**D. Proof of Lemma 4.2**

**Proof:** We proceed by induction. For the case \(k = 1\), it can be easily verified by differentiating (29) with respect to \(x_{\sigma_1}(l)\), for any \(\sigma_1 \in \{1, \ldots, d\}\):
\[
LHS = -\nabla^T \left[ p \nabla \left( \frac{\partial \phi}{\partial x_{\sigma_1}(l)} \right) \right] - \nabla^T \left[ \frac{\partial p}{\partial x_{\sigma_1}(l)} \nabla \phi \right]
\]
\[
= -\nabla^T \left[ p \nabla \left( \frac{\partial \phi}{\partial x_{\sigma_1}(l)} \right) \right] - \nabla^T \left[ \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} \right] p \nabla \phi
\]
\[
+ \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} (h \dot{h}) p,
\]
while
\[
RHS = \frac{\partial h}{\partial x_{\sigma_1}(l)} p + (h \dot{h}) \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} p.
\]

The last terms on the both sides are cancelled out with each other. Thus, we have
\[
\begin{align*}
-\nabla^T \left[ p \nabla \left( \frac{\partial \phi}{\partial x_{\sigma_1}(l)} \right) \right] &= \nabla^T \left[ \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} \right] p \nabla \phi + \frac{\partial h}{\partial x_{\sigma_1}(l)} p \\
&=: G_{\sigma_1}^1 p,
\end{align*}
\] (86)

with \(G_{\sigma_1}^1\) defined in (44). Suppose (43) holds for \(k = l - 1\), i.e.
\[
-\nabla^T \left[ p \nabla \left( \nabla_{\sigma_1}^{l-1} \phi \right) \right] = G_{\sigma_1}^{l-1} p,
\] (87)

where \(\sigma_1 \in \{1, \ldots, d\}^{l-1}\), then we shall validate it for \(k = l\). Differentiating (86) with respect to \(x_{\sigma_1}(l)\), \(\sigma(l) \in \{1, \ldots, d\}\) yields that
\[
\begin{align*}
LHS &= -\nabla^T \left[ p \nabla \left( \nabla_{\sigma_1}^{l-1} \phi \right) \right] \\
&= \nabla^T \left[ \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} \right] p \nabla \left( \nabla_{\sigma_1}^{l-1} \phi \right) + \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} G_{\sigma_1}^{l-1} p
\end{align*}
\]
while
\[
RHS = \frac{\partial G_{\sigma_1}^{l-1}}{\partial x_{\sigma_1}(l)} p + G_{\sigma_1}^{l-1} \frac{\partial p}{\partial x_{\sigma_1}(l)}.
\]

Similarly, the last terms on both sides are cancelled out. Thus, we have
\[
\begin{align*}
-\nabla^T \left[ p \nabla \left( \nabla_{\sigma_1}^{l-1} \phi \right) \right] &= \nabla^T \left[ \frac{\partial (\log p)}{\partial x_{\sigma_1}(l)} \right] p \nabla \left( \nabla_{\sigma_1}^{l-1} \phi \right) + \frac{\partial G_{\sigma_1}^{l-1}}{\partial x_{\sigma_1}(l)} p,
\end{align*}
\]
which is exactly (43) with $G_{\sigma_k}^k$ defined in (45) with $\sigma_k = (\sigma_{k-1}, \sigma(l))$. Next, we compute $\frac{\partial G_{\sigma_k}^l}{\partial x_{\sigma(l)}}$, for $l \geq 2$:

$$\frac{\partial G_{\sigma_k}^{l-1}}{\partial x_{\sigma(l)}} = \nabla_{\sigma(l)} \left( \nabla_{\sigma(l)} \frac{\partial (\log p)}{\partial x_{\sigma_{l-1}(k-1)}} \phi \right)$$

Equation (46) follows immediately by denoting $\sigma_l = (\sigma_{l-1}, \sigma(l)) \in \{1, \cdots, d\}$.

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