ON LINEAR SYSTEMS OF $\mathbb{P}^3$ THROUGH MULTIPLE POINTS

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Abstract. In this paper we prove a conjecture about the dimension of linear systems of surfaces of degree $d$ in $\mathbb{P}^3$ through at most eight multiple points in general position.

1. Introduction

In this paper we assume the ground field is algebraically closed of characteristic 0. The aim of this paper is to evaluate the dimension of linear systems of surfaces of degree $d$ of $\mathbb{P}^3$ through at most eight multiple points in general position, i.e. $\mathcal{O}_{\mathbb{P}^3}(d) - \sum_{i=1}^{8} m_i p_i$. The virtual dimension of the system is the dimension of the projective space of polynomials of degree $d$ minus the number of conditions imposed by each point evaluated independently. It is possible that these conditions are actually dependent, giving place to the existence of a special linear system. Recently a conjecture on the structure of special systems of $\mathbb{P}^3$ has been formulated in [2]. In this paper we provide a proof of this conjecture for these systems. The main idea is an extension of a procedure introduced in [1] for the study of linear systems on $\mathbb{P}^2$ through at most nine multiple points. By using cubic Cremona transformations of $\mathbb{P}^3$ it is possible to transform a linear system $\mathcal{L}$ into another one which is in “standard form”. The dimensions of the two systems are the same, while the virtual dimensions may be different and this difference is measured by proposition 3.2. This is a completely new phenomenon which does not occur in $\mathbb{P}^2$. Once the system $\mathcal{L}$ is in standard form its dimension is related with that of $\mathcal{L}|_{Q_i}$, where $Q_i = -\frac{1}{2}K_{X_i}$ is half of the anti-canonical bundle of the blow-up of $\mathbb{P}^3$ along $i$ points. In this way it is possible to evaluate the speciality of $\mathcal{L}$ (theorem 5.3).

The paper is organized as follows: in the first section we recall some preliminary definitions. In Section 2 we give a description of Cremona transformations of $\mathbb{P}^3$ and their action on linear systems while Section 3 deals with special linear systems produced by $(-1)$-curves. In Section 4 we prove the main theorem and section 5 is an appendix on a birational transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$.

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2. Preliminaries

We start by fixing some definitions and notations.

Let \( Z = \sum m_i p_i \) be a zero-dimensional scheme of general fat points of \( \mathbb{P}^3 \); with \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) we will denote the linear system associated to the sheaf \( \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z \). Given a linear system \( \mathcal{L} \), by abuse of notation we will denote by \( v(\mathcal{L}) \) the virtual dimension of the associated sheaf:

\[
v(\mathcal{L}) := \left( \frac{d+n}{n} \right) - \sum_{i=1}^r \left( \frac{m_i + n - 1}{n} \right) - 1.
\]

In the same way we will denote by \( H^i(\mathcal{L}) \) the \( i \)-th cohomology group of the sheaf associated to \( \mathcal{L} \).

Let \( X \to \mathbb{P}^3 \) be the blow-up of \( \mathbb{P}^3 \) along \( \{p_1, \ldots, p_r\} \); with abuse of notation we will denote by \( \mathcal{L} \) the linear system associated to \( \mathcal{L} = dH - \sum m_i E_i \), where \( H \) is the pull-back of an hyperplane of \( \mathbb{P}^3 \) and \( E_i = \pi^{-1}(p_i) \). With \( \{h, e_1, \ldots, e_r\} \) we denote a basis of \( \mathbb{P}^3(X) \) where \( h \) is the pull-back of a class of a general line in \( \mathbb{P}^3 \) and \( e_i \) is the class of a line of \( E_i \). The notation \( \ell = \ell_3(\delta, \mu_1, \ldots, \mu_r) \) indicates a curve in \( \mathbb{P}^3 \) of degree \( \delta \) through \( r \) points of multiplicity \( \mu_1, \ldots, \mu_r \) or equivalently a curve \( \delta h - \sum_{i=1}^r \mu_i e_i \) of \( X \). The intersection product \( \ell \mathcal{L} \) is to be intended always on \( X \), i.e.

\[
\ell_3(\delta, \mu_1, \ldots, \mu_r) \mathcal{L}_3(d, m_1, \ldots, m_r) = \delta d - \sum_{i=1}^r \mu_i m_i.
\]

In what follows we give some definitions in the same spirit of [1].

**Definition 2.1.** A linear system \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) is in standard form if \( 2d \geq \sum_{i=1}^r m_i \) and \( m_1 \geq \ldots \geq m_r \).

Let \( S_i = \mathcal{L}_3(2, 1^i) \) be a linear system of quadrics through \( 4 \leq i \leq 9 \) simple points. We call \( S_i \) a standard class.

**Proposition 2.2.** A linear system in standard form may be always written in the following way \( \mathcal{L} = S + \sum_{i=4}^a c_i S_i \), where the \( c_i \)'s are non negative integers and \( S = \mathcal{L}_3(d - 2m_4, m_1 - m_4, m_2 - m_4, m_3 - m_4) \).

**Proof.** Let \( a = \max\{i \mid m_i > 0\} \); if \( a \leq 3 \) there is nothing to prove, otherwise consider the system \( \mathcal{L}' = \mathcal{L} - S_a \). Since \( \mathcal{L}' = \mathcal{L}_3(d - 2, m_1 - 1, \ldots, m_a - 1) \) it follows that also \( \mathcal{L}' \) is in standard form. Proceeding in the same way on \( \mathcal{L}' \), after a finite number of steps one obtains a linear system \( S \) through at most three points. In this way one obtains that \( c_a = m_a \) and \( c_i = m_i - m_{i+1} \) for \( i < a \) while \( \mathcal{L} = \mathcal{L} - \sum_{j=1}^a c_j S_j \) is given by \( \mathcal{L}_3(d - 2m_4, m_1 - m_4, \ldots, m_{i+1} - m_i) \). \( \square \)

Let \( Q \) be a quadric, with \( \mathcal{L}_Q(a, b, m_1, \ldots, m_r) \) we will mean the system \( |\mathcal{O}_Q(a, b)| \) through \( r \) general points of multiplicities \( m_1, \ldots, m_r \).

3. Cubic Cremona transformations of \( \mathbb{P}^3 \)

In this section we focus our attention on a class of cubic Cremona transformations of \( \mathbb{P}^3 \). Consider the system \( \mathcal{L}_3(3, 2^4) \), by putting the four double points in the
fundamental ones, the associated rational map is given by:

\[ \text{Cr} : (x_0 : x_1 : x_2 : x_3) \mapsto (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}). \] (3.1)

This birational map induces an action on the picard group of the blow-up \( X \) of \( \mathbb{P}^3 \) along the four points which can be described in the following way:

**Proposition 3.1** ([2]). The action of transformation \( [7.1] \) on \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) is given by:

\[ \text{Cr} (\mathcal{L}) := \mathcal{L}_3(d + k, m_1 + k, \ldots, m_4 + k, m_5, \ldots, m_r), \] (3.2)

where \( k = 2d - \sum_{i=1}^4 m_i \).

Observe that \( \dim \text{Cr} (\mathcal{L}) = \dim \mathcal{L} \) but in general the virtual dimensions of the two systems may be different.

**Proposition 3.2** ([2]). Let \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) be a linear system such that \( 2d \geq m_i + m_j + m_k \) for any choice of \( \{i, j, k\} \subset \{1, 2, 3, 4\} \) then

\[ v(\text{Cr} (\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \left( \frac{1 + t_{ij}}{3} \right) - \sum_{t_{ij} \leq -2} \left( \frac{1 - t_{ij}}{3} \right), \] (3.3)

where \( t_{ij} = m_i + m_j - d \).

**Corollary 3.3.** Under the same assumptions of Proposition [2.4] if the degree of \( \text{Cr} (\mathcal{L}) \) is smaller than that of \( \mathcal{L} \), then \( v(\text{Cr} (\mathcal{L})) \geq v(\mathcal{L}) \).

**Proof.** The difference between the degree of \( \text{Cr} (\mathcal{L}) \) and that of \( \mathcal{L} \) is equal to \( k = 2d - \sum_{i=1}^4 m_i \). From \( 2d < \sum_{i=1}^4 m_i \) we deduce that, if \( t_{12} \geq 2 \) then \( d - m_3 - m_4 < m_1 + m_2 - d \) which is equivalent to \( -t_{34} < t_{12} \). The same holds for each \( t_{ij} \) such that \( t_{ij} \geq 2 \), hence the right side of equation [2.3] is nonnegative. \( \square \)

4. \((-1)\)-CURVES AND SPECIAL SYSTEMS

Starting from the results of the preceding section it is possible to define a class of special linear systems.

**Definition 4.1.** A curve \( C \in \mathcal{L} \) is called \((-1)\)-curve if \( \mathcal{L} \) is obtained by applying a finite set of Cremona transformations on the system \( \mathcal{L}_3(1, 1^2) \).

**Example 4.2.** Given six general points of \( \mathbb{P}^3 \) there exists a unique rational normal curve through them. This curve is an element of \( \ell_3(3, 1^6) \) which may be obtained as \( \text{Cr}(\ell_3(1, 0^4, 1^2)) \), i.e. as the Cremona transformation of a line through two points.

**Proposition 4.3.** Let \( \mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_r) \) be a linear system and \( \ell_1, \ldots, \ell_n \) be a set of \((-1)\)-curves such that \( t_i \mathcal{L} = -t_i \leq -2 \) for \( i = 1, \ldots, n \). Then

\[ \dim \mathcal{L} - v(\mathcal{L}) \geq \sum_{i=1}^n \left( \frac{t_i + 1}{3} \right) - h^2(\mathcal{L} \otimes \mathcal{I}_{\ell_1}^{t_1} \otimes \cdots \otimes \mathcal{I}_{\ell_n}^{t_n}), \]

where with \( \mathcal{L} \otimes \mathcal{I}_{\ell_i}^{t_i} \) we mean the sheaf \( \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_X^{t_i} \).
Idea of the proof. For a complete proof of this proposition see [2].

By definition, each one of the $\ell_i$ is given by a smooth curve $C_i \subset X$ such that $N_{C_i|X} \cong \mathcal{O}_{P_i}(-1) \oplus \mathcal{O}_{P_i}(-1)$. Consider the blow-up $Y \xrightarrow{p} X$ along the curves $C_1, \ldots, C_n$, the exceptional divisors are quadrics $Q_1, \ldots, Q_n$. From the evaluation of the tautological line bundle associated to the blow-up of $C_i$ and from the intersection $C_i\mathcal{L} = -t_i$ one obtains:

\[
Q_i|_{Q_i} \cong \mathcal{O}(-1, -1)
\]

\[
p^*\mathcal{L}|_{Q_i} \cong \mathcal{O}(-t_i, 0).
\]

these formulas imply that $t_iQ_i \subseteq \text{Bs}(p^*\mathcal{L})$ and that

\[
\chi(p^*\mathcal{L}) = \chi(p^*\mathcal{L} - t_iQ_i) + \sum_{k=0}^{t_i} \chi(\mathcal{O}(k - t_i, k)).
\]

An easy calculation shows that the last sum is equal to $\binom{t_i+1}{3}$, hence applying this procedure to each one of the $C_i$ one obtains:

\[
\chi(p^*\mathcal{L}) = \chi(p^*\mathcal{L} - \sum_{i=1}^{n} t_iQ_i) + \sum_{i=1}^{n} \binom{t_i+1}{3}.
\]

This, together with the fact that $h^0(p^*\mathcal{L}) = h^0(p^*\mathcal{L} - \sum_{i=1}^{n} t_iQ_i)$ proves the thesis. \hfill \Box

Example 4.4. The system $\mathcal{L} = \mathcal{L}_3(3, 3^3)$ has $v(\mathcal{L}) = -11$ while $\dim \mathcal{L} = 0$ since it consists of three times the plane $\mathcal{L}_3(1, 1^3)$. For each line $\ell_i$ through two of the three points, we have $\mathcal{L} \cdot \ell_i = -3$, hence the speciality is greater then or equal to $3\binom{3}{3} - h^2(\mathcal{L} \otimes \mathcal{I}_{\ell_1} \otimes \mathcal{I}_{\ell_2} \otimes \mathcal{I}_{\ell_3})$. So the $h^2$ is equal to 1.

The preceding proposition allow us to give an estimate of the speciality of a given linear system $\mathcal{L}$. In particular consider a system in standard form, then we have the following:

Proposition 4.5. Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_8)$ be a linear system in standard form, let $t_i := m_i + m_i - d$ for $i \geq 2$ and $t_1 := m_2 + m_3 - d$, then the following holds:

\[
h^1(\mathcal{L}) \geq \sum_{t_i \geq 2} \binom{t_i+1}{3}.
\]

Proof. Let $l$ be a line through two multiple points of $\mathcal{L}$, we say that the 1-dimensional scheme $\Gamma = \sum t_il_i$ is associated to $\mathcal{L}$ if $l_i\mathcal{L} = -t_i < 0$. By proposition 4.3 it is sufficient to prove that $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) = 0$. Consider the system given by the plane through the first three points $\mathcal{H} = \mathcal{L}_3(1, 1^3)$ and let $\Gamma = \Gamma' + \Gamma''$ where $\Gamma' = l_2 + l_3$, then one has the exact sequence

\[
0 \xrightarrow{} (\mathcal{L} - \mathcal{H}) \otimes \mathcal{I}_{\Gamma'} \xrightarrow{} \mathcal{L} \otimes \mathcal{I}_\Gamma \xrightarrow{} (\mathcal{L} \otimes \mathcal{I}_\Gamma)|_W \xrightarrow{} 0,
\]

where $W \in \mathcal{H} \otimes \mathcal{I}_{\Gamma'}$ is the plane through the first three points.

Claim 1. In the preceding sequence $h^2((\mathcal{L} \otimes \mathcal{I}_\Gamma)|_W) = 0$ and $\mathcal{L} - \mathcal{H}$ is a system in standard form unless $\mathcal{L} = \mathcal{L}_3(2m + t, m + 2t, m_5, m_7, m_8)$, with $1 \leq t \leq m$. 

Observe that $\Gamma''$ is associated to $L - H$, since this system is equal to $\mathcal{L}_3(d - 1, m_1 - 1, m_2 - 1, m_3, \ldots, m_8)$. So, if $L - H$ is still in standard form, then after sorting the new multiplicities we can proceed by induction on the length of $\Gamma$; once we obtain a system $\tilde{L}$ for which the associated $\tilde{\Gamma}$ is empty, we know that $h^2(\tilde{L}) = 0$.

Suppose now that $L - H$ is not in standard form, then the system must be the one of the claim. The worst possibility is that $m_7 = m_8 = m$, so we fix our attention on the system $\mathcal{L}_3(2m + t, m + 2t, m^2)$. By abuse of notation, we still call this system $L$. Applying sequence (1) one time we obtain the system $L - H = \mathcal{L}_3(2m + t - 1, m + 2t - 1, (m - 1)^2, m^3)$. Now, let $\mathcal{H}_1 = \mathcal{L}_3(2, 2, 1^3)$ be the quadric cone through the point of multiplicity $m + 2t - 1$ and the five points of multiplicity $m$ and let $W_1 \in \mathcal{H}_1$, $\mathcal{L}_1 = L - H$, and $\Gamma_1 = \Gamma'_1 + \Gamma''_1$ with $\Gamma'_1 = \sum_{i=4}^{8} l_i$ and $\Gamma''_1 = (t - 1) \sum_{i=2}^{8} l_i$. Observe that $\Gamma_1$ is associated to $\mathcal{L}_1$, while $\Gamma''_1$ is associated to $\mathcal{L}_1 - \mathcal{H}_1$. The last system is equal to $\mathcal{L}_3(2m' + t', m' + 2t', m''_1)$ where $m' = m - 1$ and $t' = t - 1$, hence by using sequence (1) with $\mathcal{H}_1$ instead of $\mathcal{H}$ and by the following

**Claim 2.** With the preceding notation $h^2((\mathcal{L}_1 \otimes \mathcal{I}_{\Gamma'_1})|_{W_1}) = 0$,

we can make induction on $t$ and proving that $h^2(L \otimes \mathcal{I}_{\Gamma'}) = 0$. \hfill $\square$

**Proof of Claim 1** First of all observe that

$$(L \otimes \mathcal{I}_{\Gamma'})|_{W} = \mathcal{L}_2(d - t_2 - t_3, m_1 - t_2 - t_3, m_2 - t_2, m_3 - t_3).$$

Since $d - t_2 - t_3 = (d - m_1) + (2d - m_1 - m_2 - m_3) \geq 0$ then the $H^2$ of this linear system vanishes. The sequence of multiplicities of $L - H$ is

$m_1 - 1, m_2 - 1, m_3 - 1, m_4, \ldots, m_8$

and the degree is $d - 1$. This means that the system is still stable if at least $m_1 - 1$ and $m_2 - 1$ are between the biggest four. If this is not the case then $m_2 - 1 \leq m_6$ and this implies that $m_2 = \cdots = m_6$, call this number $m$. Since $L - H$ is not in standard form we have that $2(d - 1) < m_1 - 1 + 3m$, but we also know that $2d \geq m_1 + 3m$ and this implies that $2d = m_1 + 3m$. Let $t = t_2 = \cdots = t_6$, then the preceding equation gives $m_1 = m + 2t$ and $d = 2m + t$. \hfill $\square$

**Proof of Claim 2** The blow-up of a quadric cone along the vertex is an $\mathbb{F}_2$ surface, hence the strict transform $\tilde{W}_1$ of $W_1$ of the blow-up of $\mathbb{P}^3$ along the six points is a blow-up of an $\mathbb{F}_2$ surface along five points. The vanishing of $h^2((\mathcal{L}_1 \otimes \mathcal{I}_{\Gamma'_1})|_{W_1})$ is equivalent to the vanishing of $h^2((\tilde{\mathcal{L}}_1 \otimes \mathcal{I}_{\tilde{\Gamma}'_1})|_{\tilde{W}_1})$ where $\tilde{\mathcal{L}}_1$ is the strict transform of $\mathcal{L}_1$ and $\tilde{\Gamma}'_1$ is a 1-dimensional subscheme of $\tilde{W}_1$ corresponding to the strict transform of $\Gamma'_1$. A basis for the picard group of $W_1$ may be written as $(f, c, e_1, \ldots, e_5)$ where $f^2 = 0$, $fc = 1$, $c^2 = -2$ and the $e_i$ are $(-1)$-curves of the blow-up. With this notation, the hyperplane section of $\tilde{W}_1$ is given by $c + 2f$ (it is very ample outside $c$ which is contracted to the vertex of the cone). Instead of the system $\tilde{\mathcal{L}}_1$ we can consider $|(2m + t - 1)H - (m + 2t - 1)E - \sum_{i=1}^{5} mE_i|$, since the two exceptional
divisors of multiplicity \((m - 1)\) have no intersection with \(\tilde{W}_1\). In this way we have:

\[
(\tilde{L}_1 \otimes I_{\tilde{P}}_{1})|_{\tilde{W}_1} = (2m + t - 1)(c + 2f) - (m + 2t - 1)c
- \sum_{i=1}^{5} me_i - \sum_{i=1}^{5} t(f - e_i)
= (m - t)c + (4m - 3t - 2)f - \sum_{i=1}^{5} (m - 1)e_i.
\]

Since \(t \leq m\) this implies that both the coefficients of \(c\) and \(f\) are non-negative and this, by adjunction, implies the vanishing of the \(H^2\) of this divisor. \(\Box\)

5. Linear systems through at most 8 points

In what follows we will denote by \(L\) a linear system of type \(L_3(d, m_1, \ldots, m_8)\) in standard form and let \(S + \sum_{i=4}^{a} c_i S_i\) be its decomposition.

Lemma 5.1. A linear system \(L = L_3(d, m_1, m_2, m_3)\) is empty if and only if and only if there exists \(i \in \{1, 2, 3\}\) such that \(d < m_i\).

Proof. One part of the proof is trivial, since there are no surfaces of degree \(d\) with a singularity of multiplicity greater than \(d\). On the other hand, if all the \(m_i\) are equal to \(d\), then the system \(L_3(d, d^3)\) is non-empty since it contains \(d\) times the plane \(L_3(1,1^3)\).

Lemma 5.2. If \(d \geq m_1\) then \(h^1(L_{Q_a}) = 0\) and \(h^0(L_{|Q_a}) > 0\) where \(Q_a \in S_a\).

Proof. The system on the quadric \(L_{Q_a} = L_{Q_a}(d, d, m_1, \ldots, m_a)\) is equivalent (by \([A]\)) to \(L_2(2d - m_1, (d - m_1)^2, m_2, \ldots, m_a)\). This is a plane system through at most 9 points. First of all we want to see if it is possible to put it in standard form (i.e. a plane Cremona transformation can not decrease its degree). Observe that the three bigger multiplicities may be: \(\{d - m_1, d - m_1, m_2\}, \{m_2, m_3, d - m_1\}, \{m_2, m_3, m_4\}\). In the first and third case it is obvious that \(2d - m_1\) is greater then or equal to the sum of these multiplicities (since the system \(L\) is in standard form). In the second case the inequality \(2d - m_1 \geq d - m_1 + m_2 + m_3\) is true only if \(d \geq m_2 + m_3\), so we may assume that \(d = m_2 + m_3 - t\) where \(t \geq 1\). After applying a Cremona transformation to this system we obtain the following \(L_2(d', m_1', m_2', m_3', d - m_1, m_4, \ldots, m_a)\) where \(d' = 3d - m_1 - m_2 - m_3, m_1' = d - m_3, m_2' = d - m_2, m_3' = 2d - m_1 - m_2 - m_3\). The bigger multiplicities of this system are \(\{d - m_3, d - m_2, d - m_1\}\) since \(d - m_1 \geq m_4\) by assumption and \(2d - m_1 - m_2 - m_3 = d - m_1 - t < d - m_1\). This implies that the system is in standard form \((d' = d - m_3 + d - m_2 + d - m_1)\). So we proved that after a quadratic transformation of \(\mathbb{P}^2\), the system \(L_2(2d - m_1, (d - m_1)^2, m_2, \ldots, m_a)\) becomes \(M\) which is in standard form. By \([B]\) this implies that \(h^0(M) > 0\). The intersection \(M.K_{\mathbb{P}^2_a} = -4d + m_1 \ldots + m_a\), where \(\mathbb{P}_a^2\) is the blow-up of \(\mathbb{P}^2\) along the \(a + 1\) points, is non-positive, so \(h^1(M) = 0\) (by \([J]\)). \(\Box\)
Theorem 5.3. A linear system $\mathcal{L}(d, m_1, \ldots, m_a)$ (with $a \leq 8$) in standard form is special if and only if $d \leq m_1 + m_2 - 2$ and its dimension is given by

$$\dim \mathcal{L} = v(\mathcal{L}) + \sum_{i \geq 2} \left( t_i + 1 \right),$$

where $t_1 := m_2 + m_3 - d$ and $t_i := m_1 + m_i - d$ for $i \geq 2$.

Proof. Since $\mathcal{L}$ is in standard form, by 2.2 it can be written as $\mathcal{L} = \mathcal{S} + \sum_{i=4}^{a} c_i \mathcal{S}_i$. We will distinguish two cases:

If $\mathcal{S} \neq \emptyset$ it follows immediately that $h^0(\mathcal{L}) > 0$ since the system may be written as a sum of effective ones. In order to see that $h^1(\mathcal{L}) = h^1(\mathcal{S})$, consider the exact sequence:

$$0 \longrightarrow \mathcal{L}' - \mathcal{S}_i \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L}'_{|Q_i} \longrightarrow 0,$$  \hspace{1cm} (5.1)

where $\mathcal{L}'$ is obtained from $\mathcal{L}$ by subtracting some of the $\mathcal{S}_j$. The degree of $\mathcal{L}'$ is greater than or equal to its first multiplicity, otherwise it would be empty but this is not possible by the assumption on $\mathcal{S}$. By lemma 5.2 $h^2(\mathcal{L}'_{|Q_i}) = 0$, which implies that $h^1(\mathcal{L}') \leq h^1(\mathcal{L}' - \mathcal{S}_i)$. This gives the following

$$h^1(\mathcal{L}) \leq h^1(\mathcal{L} - \mathcal{S}_a) \leq \ldots \leq h^1(\mathcal{S}).$$

The speciality of $\mathcal{S}$ may be given only by the lines $\langle p_i, p_j \rangle$ with $1 \leq i < j \leq 3$ if $\delta \leq \mu_i + \mu_j - 2$. From the equality $d - m_i - m_j = \delta - \mu_i - \mu_j$ one has the same speciality also for $\mathcal{L}$ and this implies that $h^1(\mathcal{L}) \geq h^1(\mathcal{S})$.

From proposition 5.1 we know that $\mathcal{S} = \emptyset$ if and only if the degree of the system is less then one of its multiplicities. By proposition 2.2 this means $d - 2m_4 < m_1 - m_4$. Recall that $t_4 = m_1 + m_4 - d$, since $\mathcal{L}$ is in standard form the inequality $2d = d + m_1 + m_4 - t_4 \geq m_1 + m_2 + m_3 + m_4$ gives $d \geq m_2 + m_3 + t_4$. This implies that $d \geq m_i + m_j$ for each $i \geq 2$, $j \geq 3$, $i \neq j$. So the speciality of $\mathcal{L}$ coming from lines is due only to $\langle p_i, p_i \rangle$ with $i \geq 2$. Recall that $t_i = m_1 + m_j - d$ and let $b = \max\{i \mid t_i \geq 1\}$. Observe that by definition $t_i - t_j = m_i - m_j$ and that $b \geq 4$.

By proposition 3.3 we have the following inequality:

$$h^1(\mathcal{L}) \geq \sum_{i=2}^{b} \left( t_i + 1 \right).$$  \hspace{1cm} (5.2)

Consider the system

$$\mathcal{N}_b = \mathcal{S} + \sum_{i=4}^{b-1} c_i \mathcal{S}_i + (t_b - 1) \mathcal{S}_b,$$

We now use the following:

Claim 3. Under these assumptions, $h^0(\mathcal{N}_b) = 0$ and $h^1(\mathcal{N}_b) = \sum_{i=2}^{b} \left( t_i + 1 \right)$. Furthermore, if $d''$ and $m_1''$ are, respectively, the degree and the first multiplicity of the system $\mathcal{N}_b + \mathcal{S}_b$ then $d'' = m_1''$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{N}_b \longrightarrow \mathcal{N}_b + \mathcal{S}_b \longrightarrow \mathcal{N}_b + \mathcal{S}_b|_{Q_b} \longrightarrow 0,$$
by claim \ref{claim:irrelevant} and lemma \ref{lemma:vanishing}, we know that \(h^1(\mathcal{N}_b + \mathcal{S}_b(Q_0)) = 0\), this implies that

\[ h^1(\mathcal{L}) \leq h^1(\mathcal{L} - \mathcal{S}_a) \leq \ldots \leq h^1(\mathcal{N}_b). \]

This together with lemma \ref{lemma:vanishing} and inequality \ref{inequality:vanishing} implies that \(h^1(\mathcal{L}) = \sum_{i=2}^{b} \binom{t_i+1}{3}\).

\[ \square \]

**Proof of Claim** \ref{claim:irrelevant} The system \(\mathcal{N}_b\) is given by \(\mathcal{L}_3(d - 2m_b + 2t_b - 2, m_1 - m_b + t_b - 1, \ldots, m_{b-1} - m_b + t_b - 1, t_b - 1)\). So \(d' = d - 2m_b + 2t_b - 2 = m_1 - m_b + t_b - 2 = m'_1 - 1\) implies that \(h^0(\mathcal{N}_b) = 0\) and \(h^1(\mathcal{N}_b) = -\chi(\mathcal{N}_b)\). The last quantity is

\[ \chi(\mathcal{N}_b) = \binom{d' + 3}{3} - \binom{m'_1 + 2}{3} - \sum_{i=2}^{b} \binom{m'_i + 2}{3}. \]

Since \(d' = m'_1 - 1\) the first two terms vanish and from \(m'_1 = m_1 - m_b + t_b - 1 = t_i - 1\) one obtains \(\chi(\mathcal{N}_b) = \sum_{i=2}^{b} \binom{t_i + 1}{3}\) which proves the first part of the claim. For the second part observe that \(d'' = d' + 2\) and \(m''_1 = m'_1 + 1\).

\[ \square \]

All these results may be summarized in the following procedure which allows us to evaluate the dimension of a linear system and its speciality.

**Remark 5.4.** Take a linear system \(\mathcal{L} = \mathcal{L}_3(d, m_1, \ldots, m_8)\) and let \(v := v(\mathcal{L})\).

1 - Sort the multiplicities in descending order.
2 - If \(2d - m_1 - m_2 - m_3 < 0\) remove the plane \(\mathcal{H}\) through the first three points, redefine \(\mathcal{L}\) as \(\mathcal{L} - \mathcal{H}\) and goto step 1.
3 - If \(2d - m_1 - m_2 - m_3 - m_4 < 0\) make a cubic Cremona transformation, redefine \(\mathcal{L}\) as \(\text{Cr}(\mathcal{L})\) and goto step 1.
4 - Evaluate \(d = \dim \mathcal{L}\) with theorem \ref{theorem:vanishing}.

6. **Appendix on a birational map** \(\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2\)

In this section we consider a birational map \(\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2\) given by blowing up a point on the quadric and contracting the strict transforms of the two lines through it. In this way it is possible to give a correspondence between linear systems through fat points on the quadric and those on the projective plane. Let us consider a linear system \(\mathcal{L}_Q(a, b, m)\), i.e. a system of curves on the quadric \(Q\) of kind \(\mathcal{O}(a, b)\) through one point \(p\) of multiplicity \(m\). Blowing up the quadric at \(p\), the strict transform of the preceding system is given by \(af_1 + bf_2 - me_p\), where \(f_1, f_2\) are the pull-back of the two rulings of \(Q\) and \(e_p\) is the exceptional divisor of the blow-up \(Q\). A base change in \(\text{Pic}(Q)\) allows us to write this divisor as \((a + b - m)(f_1 + f_2 - e_p) - (b - m)(f_1 - e_p) - (a - m)(f_2 - e_p)\). Since the divisors \(f_i - e_p\) are \((-1)\)-curves, they can be contracted giving a linear system on \(\mathbb{P}^2\) of degree \(a + b - m\) through two points of multiplicity \(b - m\) and \(a - m\). This implies that the map \(\varphi\) induces the following correspondence:

\[ \mathcal{L}_Q(a, b, m, m_1, \ldots, m_r) \leftrightarrow \mathcal{L}_2(a + b - m, b - m, a - m, m_1, \ldots, m_r). \]  

(6.1)

It is an easy computation to verify that the virtual dimensions of the two systems are the same.
References

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