Numerical solution of the direct and inverse problem of electrical exploration using the finite element method

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Abstract. This article discusses the results of solving the problem of electrical exploration using direct current. The developed methods and algorithms for solving the direct and inverse exploration tasks are tested on several inhomogeneous models of the environment. The task is considered in three-dimensional approximation. To solve using the finite element method. Based on direct problem algorithm, a method for solving the inverse problem was implemented, which consists in finding the minimum of the deviation functional, which in turn leads to the multiple solution of the direct problem. The results were analyzed in order to identify disadvantages and advantages. An analysis was also made of the search time for optimal solutions, and the dependence of the accuracy of the solution on the thickening of the mesh when solving the problem by the finite element method was discussed.

1. Introduction

The theory of electrical prospecting is based on solving direct and inverse electrical exploration problems. The direct task is the search for field elements on the surface or inside a given model of the medium with a known location of the field sources. [1,2] The form of the model is chosen such that it most closely matches the typical geological models of the environment and at the same time allows for a rigorous mathematical solution, on the basis of which it would be possible to make quantitative calculations. For example, the contact of two media, a vertical or inclined layer, a horizontally layered half-space, a ball, and other bodies of regular geometric shape [3].

The inverse problem is the reconstruction of the internal structure of the medium model by the found distribution of field elements on its surface or inside the medium. For example, the observed anomalies of the potential of the natural electric field determine the location, shape and depth of the ore body, which is the natural source of the observed anomaly.

Thus, direct and inverse problems together constitute the physical and mathematical support of interpretation techniques [4,5].

In the case when electromagnetic fields are used, the problem can be interpreted as the classical inverse problem, which consists in finding the coefficients of the system of Maxwell equations. The problems of electrical exploration were studied in [1–20]. The mathematical model of this problem was considered by us in [7].
In this article, we consider the results of a numerical solution using the example of a three-layer medium with different parameters. We consider both the results of solving the direct and inverse problems. The methods considered in [7] are used.

It should be noted that these statements in the geophysical sense are formulated as problems of interpreting the results of field measurements, and mathematically lead to conditionally correct problems related to the need to solve very poorly conditioned systems of equations in the conditions of inaccuracy of the initial information. Here, to improve the reliability of the final results of modeling, a promising combination of fields of different nature — galvanic, induction, seismic, NMR, etc. — is presented, which leads to multicriteria optimization and essentially interdisciplinary (cooperative) problems [6].

The algorithms used for the calculation are based on the approximation of multidimensional mixed boundary value problems by the finite element method (FEM) of high accuracy, on fast iterative processes for solving generated systems of linear algebraic equations (SLAE) with sparse matrices of large order, and also on effective optimization methods [7].

2. Mathematical statement of the problem.

Recall the main points of the mathematical model described in [7].

There are two electrodes on the surface of the earth: a positively charged A and a Negatively charged B. The current on these electrodes matters \( I_+ \) and \( -I_- \) accordingly. It is required to find the distribution of the electrostatic potential in a given region [7].

The task of solving the direct problem is to find the distribution of the electrostatic potential in a given area [7].

The electromagnetic fields described by the system of Maxwell equations were studied in [6], [7], [20], [16]. In this problem, electromagnetic fields do not change in time, and volume sources are absent, therefore, the electrostatic potential in a given region is described by the following equation [7, 17]

\[
L \varphi \equiv \nabla \sigma \nabla \varphi(\vec{x}) = 0, \quad \vec{x} \in \Omega
\]  

In this equation the differential operator \( L \) is specified in cylindrical coordinates for axisymmetric problems. In the general case of three-dimensional problems, this operator is specified in Cartesian coordinates [17]. In the case of an inhomogeneous medium with conductivities \( \sigma_x, \sigma_y, \sigma_z \) in the directions of the Cartesian axes \( x, y, z \), we have

\[
L \varphi = \frac{\partial}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial y} \sigma_y \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial z} \sigma_z \frac{\partial \varphi}{\partial z} = 0
\]  

At different parts of the outer boundary of the computational domain, for physical reasons, boundary conditions of the first or second kind are set:

\[
\varphi|_{\Gamma_i} = \varphi^e|_{\Gamma_i}, \quad \frac{\partial \varphi}{\partial n}|_{\Gamma_i} = 0, \quad \sigma \frac{\partial \varphi}{\partial n}|_{\Gamma_i} = I_k, \quad \sigma \frac{\partial \varphi}{\partial n}|_{\Gamma_i} = -I_k, \quad \frac{\partial \varphi}{\partial n}|_{\Gamma_i} = \frac{\partial \varphi}{\partial n}|_{\Gamma_i}.
\]

Further, to solve the problem, we use the finite element method. For convenience, we rewrite our equation in the following form

\[
\int_{\Omega} [N]^T \left( \frac{\partial}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial y} \sigma_y \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial z} \sigma_z \frac{\partial \varphi}{\partial z} \right) d\Omega = 0
\]  

First, we transform the equation containing only the first derivatives with respect to \( x \) and \( z \). We do this with the term with derivative with respect to \( x \). To do this, use the first Green formula

\[
\frac{\partial}{\partial x} (\sigma_x [N]^T \frac{\partial \varphi}{\partial x}) = [N]^T \frac{\partial}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x} [N]^T \sigma_x \frac{\partial \varphi}{\partial x}\]

Using the expression, we can write

\[
[N]^T \frac{\partial}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left( [N]^T \sigma_x \frac{\partial \varphi}{\partial x} \right) - \frac{\partial}{\partial x} [N]^T \sigma_x \frac{\partial \varphi}{\partial x}
\]
After that, the first term is converted to the following expression

\[
\int_{\partial \Omega} [N]^T \frac{\partial}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} \, d\Omega = \int_{\partial \Omega} \frac{\partial}{\partial x} \left( [N]^T \sigma_x \frac{\partial \varphi}{\partial x} \right) \, d\Omega - \int_{\partial \Omega} \frac{\partial [N]^T}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} \, d\Omega
\]  
(7)

Then we use the Ostrogradsky-Gauss formula and get

\[
\int_{\partial \Omega} \frac{\partial}{\partial x} \left( [N]^T \sigma_x \frac{\partial \varphi}{\partial x} \right) \, d\Omega = \int_{s} [N]^T \sigma_x \frac{\partial \varphi}{\partial x} \, l_x \, dS
\]  
(8)

Next, we carry out similar transformations over the remaining terms of the initial equation. We get the expression.

\[
\int_{s} [N]^T \left( \sigma_x \frac{\partial \varphi}{\partial x} l_x + \sigma_y \frac{\partial \varphi}{\partial y} l_y + \sigma_z \frac{\partial \varphi}{\partial z} l_z \right) \, dS - \int_{\partial \Omega} \frac{\partial [N]^T}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} + \frac{\partial [N]^T}{\partial y} \sigma_y \frac{\partial \varphi}{\partial y} + \frac{\partial [N]^T}{\partial z} \sigma_z \frac{\partial \varphi}{\partial z} \, d\Omega = 0
\]

In this expression, the surface integral we express through the normal derivative \( \frac{\partial \varphi}{\partial n} \)

\[
\int_{\partial \Omega} \left( \frac{\partial [N]^T}{\partial x} \sigma_x \frac{\partial \varphi}{\partial x} + \frac{\partial [N]^T}{\partial y} \sigma_y \frac{\partial \varphi}{\partial y} + \frac{\partial [N]^T}{\partial z} \sigma_z \frac{\partial \varphi}{\partial z} \right) \, d\Omega - \int_{s} [N]^T \frac{\partial \varphi}{\partial n} \, dS = 0
\]  
(9)

Next, we make SLAU. The first integral in the resulting equation contributes to the stiffness matrix, the second is a column of free terms. We will look for an unknown function \( \varphi \) as a solution to SLAE.

\[
\varphi = [N][\Phi]
\]

To solve the problem, we divide the original domain into tetrahedral finite elements. [7,17]

3. Calculations of the direct problem

The developed algorithm for the solution of the problem of electrical exploration by the finite element method was tested by the model of earth consisting of three layers of Fig. 1.
Based on the results of solving the problem, the lines of the level of equal potential, obtained in the solution of Figure 2, were noted. The obtained SLAE was solved by the method of biconjugate gradients.

**Figure 2. Lines of equal medium potential**

4. **Inverse task**

As mentioned earlier, the inverse problem is an optimization problem. The objective functionality of this task can be represented as

\[
F(\sigma_1, \ldots, \sigma_n) = \sqrt{\sum_{i=1}^{n} (\hat{\varphi}_i - \varphi_i(\sigma_1, \ldots, \sigma_n))^2} \rightarrow \min
\]

To solve the inverse problem, optimization methods with similar objective functions are often used [7], [17], [20]. We will use a method based on dividing a given area by an arbitrary number of media of various shapes. First, the assumption is made that the studied area contains only one layer with a certain resistance, we obtain a solution to the direct problem for this area. In the second step, this area is divided into two areas with resistance \(\sigma_1, \sigma_2\), after that we get the solution of the inverse problem for the new region, taking into account the fact that it consists of two layers. At the next steps, we continue to arbitrarily break the studied region into subregions with resistivities \(\sigma_1, \sigma_2, \ldots, \sigma_n\) [5,7,8,9].

At each of the steps we have to solve the optimization problem, we will do this by the Hook-Jeeves method. The condition for stopping will be the achievement of a given difference at adjacent steps \(\delta = 0.001\). The condition for stopping the algorithm when searching for a solution to the inverse problem is the difference between the theoretical and experimental values of the potential at the nodes. If the difference is less \(\varepsilon = 0.0001\), when this condition is satisfied, the algorithm stops. If this value does not change over the course of 20 steps, then the search for a solution stops and begins anew, the area is divided in a different way. All partitions of the region will be stored in memory. The algorithm is based on existing methods for solving inverse problems [7,17,20]. To implement the algorithm, the methods described in the following works [6], [7] were used.
5. Calculations of the inverse problem

As an illustration of the results of the inverse problem, let us represent the arrangement of the strata obtained in the solution. For the experimental data, we take the values obtained in solving a direct problem for a given model of the medium.

Figure 3 shows the model of the medium obtained in solving the inverse problem. The error of the solution in all parameters does not exceed 7 percent, the most significant deviation in solving the problem was found in the thickness of the first layer.

![Diagram of the medium](image)

**Figure 3.** The model of the medium obtained in solving the inverse problem

When comparing the obtained solution with the original model, the following deviations were observed:

1. The resistivity of the second layer differs by 5 ohms.
2. The resistivity of the third layer differs by 0.0003 Ω.
3. At the boundaries of the separation of layers in some nodes of the computational domain, the resistivity values differ more significantly.
4. The values of resistivity at nodes close to the origin of the reference are equal to the resistivity of the second layer, although it should be equal to the resistivity of the second layer. This is explained by the fact that when solving the direct problem, the angle and the boundaries of the separation of layers were accurately given. In solving the inverse problem, these data were not taken into account.

6. Conclusions

The paper tested algorithms for the numerical solution of direct and inverse problems by the finite element method.

The methods developed for solving the inverse problem showed good convergence. The algorithms used to solve the inverse problem allow us to significantly reduce the time for finding a solution.

In comparing the results obtained when solving the inverse problem with the true values of the parameters of the layered medium, the most important discrepancies were determined. Significant deviations are given a physical justification.

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