Superdiffusion in random two dimensional system with long-range hopping $V(r) \propto r^{-2}$.

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Although it is recognized that Anderson localization always takes place for a dimension $d$ less or equal $d = 2$, while it is not possible for hopping $V(r)$ decreasing with the distance slower or as $r^{-d}$, the localization problem in the crossover regime for the dimension $d = 2$ and hopping $V(r) \propto r^{-2}$ is not resolved yet. Following earlier suggestions we show that for the hopping determined by two-dimensional anisotropic dipole-dipole or RKKY interactions there exist two distinguishable phases at weak and strong disorder. The first phase is characterized by ergodic dynamics and superdiffusive transport, while the second phase is characterized by diffusive transport and delocalized eigenstates with fractal dimension less than 2. The crossover between phases is resolved analytically using the extension of scaling theory of localization and verified using an exact numerical diagonalization.

**Introduction.** Low dimensional systems with a dimension $d \leq 2$ possessing the time reversal symmetry are critical in Anderson localization problem \[1\]. All states there must be exponentially localized at arbitrarily small critical in Anderson localization problem \[1\]. Analysis of conductivity \[3\] and extensive numerical simulations \[4, 5\] (see also reviews \[6–8\]). This localization is originated from the singular backscattering due to random potential dramatically enhanced in low dimension $d \leq 2$ where random paths inevitably return to the origin \[5\].

The scaling theory of localization suggests the single parameter scaling for the dimensionless conductance $C = G/\left(e^2/h\right)$ dependence on the size $L$ in the form \[2, 3\] ($G$ is the conductance)

$$\frac{d \ln(C)}{d \ln(L)} = \beta(C),$$

$$\beta(C) = - \frac{c_{\text{loc}}}{C} + O(C^{-4}), \quad c_{\text{loc}} = \frac{1}{2\pi^2}, \quad (1)$$

where the $\beta$-function has been evaluated using expansion of the equivalent two-dimensional $\sigma$-model \[8, 10\] valid at $C \gg c_{\text{loc}}$. Eq. \[10\] predicts the reduction of conductance with the system size $L$ as $C(L) = C_0 - c_{\text{loc}} \ln(L)$ where $C_0$ is the conductance at the lower cutoff $L = 1$. Logarithmic reduction of conductance with the system size results in the inevitable localization at large sizes $L$.

This universal scaling is limited to systems with a short-range hopping, while the hopping decreasing with the distance as $V(r) \propto r^{-d}$ leads to delocalization \[11–12\] except for the marginal case of constant sign hopping with possible delocalization at the tail \[13–15\]. If disorder is strong, eigenstates of the problem are multifractals and the time dependent displacement of the particle $r$ obeys the law $r \propto t^{1/d}$ which is subdiffusive in 3D, diffusive in 2D and superdiffusive in 1D \[11\]. The long-range hopping $V(r) \propto r^{-2}$ is ubiquitous in pure two-dimensional systems, where it can be originated from the virtual exchange by two dimensional photons leading to the 2D dipole-dipole interaction \[13\] or indirect exchange by 2D electron-hole pairs leading to 2D RKKY interaction \[13\] (cf. Ref. \[21\]).

If disorder gets weaker there is the transition in 3D to the standard delocalized phase characterized by the diffusive transport and ergodic dynamics \[21\], while in 1D eigenstates turns out to be multifractals with the dimension smaller than 1 \[12\]. 2D systems are more complicated, because for the hopping $V(r) \propto r^{-d}$ the dimensionless Drude conductance diverges logarithmically with the system size as $C_0(L) = c_{\text{loc}} \ln(L)$ \[13\]. Considering the balance of this logarithmic rise of conductance and its logarithmic suppression by coherent back scattering $c_{\text{loc}} \ln(L)$ it was suggested in Ref. \[18\] that two delocalized phases can exist including the superdiffusive (fast) phase at $c_{\text{s}} > c_{\text{loc}}$ and the slow phase with diffusive transport at $c_{\text{s}} < c_{\text{loc}}$ and the phase boundary realized at $c_{\text{s}} = c_{\text{loc}}$. Yet it turns out that for the isotropic dipole-dipole interaction considered in Ref. \[13\] $c_{\text{s}} < c_{\text{loc}}$ for arbitrarily disordering so the fast phase does not exist.

These achievements motivated us to search for the superdiffusive, fast phase using different hopping interaction including anisotropic dipole-dipole interaction with identically oriented dipoles and RKKY interaction in 2D. These interactions differ from the isotropic dipole-dipole hopping model of Ref. \[13\] by the presence of dispersive modes with mean free path increasing unlimitedly with decreasing disorder with similar increase of the logarithmic growth parameter $c_{\text{s}}$. This makes the appearance of the fast phase with $c_{\text{s}} > c_{\text{loc}}$ unavoidable. Recent experimental realizations of 2D Anderson localization \[22\] and long-range hopping \[23\] represent the steps towards generating the settings targeted in the present work. Consequently, we believe that its experimental realization is possible and strongly encouraged.

We investigate two phases for two-dimensional Anderson model with a long-range hopping, formulated below, using the extension of scaling theory of localization for the long-range hopping and exact numerical diagonalization. The phase boundary $c_{\text{s}} = c_{\text{loc}}$ is identified analytically and verified numerically (see Fig. \[1\]). Both phases are characterized by level statistics, that is of Wigner-Dyson for the fast phase and intermediate between Poisson and Wigner-Dyson otherwise (see Fig. \[2\]), eigenstate dimension (2 for the fast phase and less than 2 otherwise,
as depicted in Fig. 3 and particle transport (superdiffusive or diffusive, see Fig. 4).

**Model and Drude conductance** Anderson model in 2D is investigated. The Hamiltonian of the model can be expressed as

$$\hat{H} = \sum_{i,j} V_{ij} c_i^\dagger c_j + \sum_{i} \phi_i c_i^\dagger c_i,$$  \hspace{1cm} (2)

where the summation is performed over \(N = L^2\) lattice sites enumerated by indices \(i\) with coordinates \(r_i = (x_i, y_i)\) occupying the periodic square lattice placed onto the surface of torus characterized by the radii \(R = L/(2\pi)\). Independent random energies \(\phi_i\) obey the Lorentzian distribution with the width \(W\) characterizing disordering. For the Lorentzian distribution the Green’s functions can be evaluated exactly as \(G(E, q) = (E - V(q) - iW)^{-1}\) in the momentum representation [24]. We consider hopping due to the two dimensional dipole-dipole interaction of dipoles oriented along the \(x\) axis and two dimensional RKKY interaction [19].

The Fourier transform of hopping amplitudes \(V(q)\) can be well approximated by their continuum limits as justified in Appendix, Sec. A. These limits are given by \(V_1(q) \approx 2\pi V_0 (a^2_q - q^2)/q^2\) for the dipole-dipole interaction [15] and \(V_2(q) \approx 2\pi V_0 \left(\theta(q - 2p_F)\sqrt{1 - 4p_F^2/q^2} - z_F\right)\) for the 2D RKKY interaction [19], where \(z_F\) is defined by the condition \(\int dq V_{RRKKY}^2\), where \(p_F\) is the Fermi momentum of electrons responsible for the RKKY interaction chosen using the golden ratio \(p_F = (1 + \sqrt{5})/4\) that is incompatible with lattice periods. For this specific choice \(z_F = 0.5963\).

Further consideration is given for isotropic transport regime realized at zero energy only for the dipole-dipole interaction and for any energy for the RKKY interaction, while the straightforward but tedious generalization to anisotropic regime is given in Appendix, Sec. C2. A zeroth order (Drude) dimensionless conductance is defined as [3, 4, 18]

$$C_0 = \frac{1}{2\pi} \int \frac{dq}{(2\pi)^2} \left(\frac{\partial V(q)}{\partial q}\right)^2 \left|\text{Im}(G(E, q))\right|^2.$$

The integral diverges logarithmically at \(q \to 0\) for the dipole-dipole interaction or \(q \to 2p_F\) for the RKKY interaction. For a finite size \(L\) this divergence should be cutoff at \(q \sim 1/L\) or \(q \sim 2p_F \sim 1/L\), respectively. The logarithmic growth rate of the diverging contribution can be evaluated as \(c_{s1} = V_0^2/(2W \sqrt{\pi V_0^2 + W^2})\) or \(c_{s2} = V_0^2 Z(2p_F)/(\text{Im}(G(E, 2p_F))^2/2\) for dipole-dipole or RKKY interactions, respectively.

**Generalized scaling theory of localization.** The transition between two regimes and finite size asymptotic behaviors for isotropic conductance is determined by the generalized scaling equation derived in Appendix, Sec. C2 in the form

$$\frac{d \ln(C)}{d \ln(L)} = \frac{c_s - c_{loc}}{C} + \frac{c_s c_{loc}}{C^2} + O(C^{-3}).$$  \hspace{1cm} (4)

This equation is similar to that proposed in Ref. 18 except for the second term in the R. H. S. that is twice bigger, which is our contribution to this equation. This term was not derived in Ref. 18 and, according to our consideration (see Appendix, Sec. C1) it is originated from the wavevector dependence of the zeroth order conductance rather than the two loop order expansion announced in Ref. 18.

**Phase transition and phase diagram.** According to Eq. 1 the conductance diverges logarithmically for \(L \to \infty\) under the condition \(c_s > c_{loc}\), while it remains finite otherwise. In the fast phase \(c_s > c_{loc}\). Setting \(c_s = c_{loc}\) one can find critical disorder separating slow and fast phases. For the dipole-dipole interaction the isotropic regime is realized only for the band center \(E = 0\) where the critical disorder is given by \(W_c = \pi V_0/\sqrt{(1 + \sqrt{5})/2} \approx 2.47V_0\). The fast phase emerges first at that energy. For the anisotropic regime realized at \(E \neq 0\) two distinguishable logarithmic growth rates for conductances \(C^{z\pi}_{0}\) and \(C^{yy}_{0}\) can be introduced as \(c_2^z = dC^{z\pi}_{0}/d\ln(L)\) and \(c_2^y = dC^{yy}_{0}/d\ln(L)\) and the transition takes place at \(c_2^z c_2^y = c_{loc}^2\) (see Appendix, Sec. C2). These criteria determine the phase diagrams depicted in Fig. 1 (a) where solid lines indicate boundaries between slow and fast phases.

For the RKKY interaction the superdiffusive phase is realized at \(\pi V_0/2 - \pi \sqrt{V_0^2/4 - 4(E + V_0 z_F)^2} \leq W \leq \pi V_0/2 - \pi \sqrt{V_0^2/4 - 4(E + V_0 z_F)^2} \leq W \leq \pi V_0/2 - \pi \sqrt{V_0^2/4 - 4(E + V_0 z_F)^2}\).
The transitions are reentrant with increasing the disorder strength for most of energies except for \( E = -V_0 z_F \) because of the isotropic character of particle spectrum. The superdiffusive phase emerges first at \( E = -V_0 z_F \) and \( W = \pi V_0 \).

\[
\pi V_0/2 + \sqrt{V_0^2/4 - 4(E + V_0 z_F)^2} \text{ as illustrated in Fig. 1(b).}
\]

The level statistics is also shown together with the phase diagram for the RKKY interaction in Fig. 1(b). Here the consistency is quite good for the bottom side of the phase diagram, but it is worse for the top side especially for the small disorder in contrast with the case of dipole dipole interaction. This is because the localization length for that energy domain is exponentially large for small \( W \) and energies belonging to the energy band, so it exceeds the system size there. The analogous domain of the phase diagram for the dipole-dipole interaction corresponds to the band edge. Therefore we use the system with the dipole-dipole interaction for further considerations of fractal dimension and transport.

**Fractal dimension.** We define the fractal dimension using the informational dimension [27, 28] that can be expressed in terms of the average eigenstate wavefunction amplitude logarithm \( \xi(L) = \sum_k |c_k|^2 \ln(|c_k|^2) \). Then the fractal dimension is defined as \( D = -d \xi/d \ln(L) \), cf. Ref. [28]. We used this definition for numerical estimates of size-dependent fractal dimension calculating the functions \( \xi \) for the sequence of lengths \( L_1, L_2, \ldots, L_n \) arranged in ascending order and then numerically differentiating them. This yields \( n - 1 \) estimates for fractal dimensions \( D(k) = -\ln(L_{k+1}/L_k)/\ln(L_{k+1}/L_k) \) at sizes \( L_k = \sqrt{L_{k-1}L_{k+1}} \) and \( k = 1, 2, \ldots, n - 1 \). Numerical results should be compared with analytical estimates for fractal dimensions obtained using the generalized scaling theory Eq. 4.

According to the sigma model for the short-range hopping [8, 30–32] the fractal dimension depends on the dimensionless conductance Eq. 4 as \( D = 2 - \eta_d C_{\text{loc}}/C \) with \( \eta_d = 1 \). Integrating Eq. 4 we obtain the transcendental equation that expresses size dependent conductance as \( C + C_\infty \ln(1 - C/C_\infty) = (c_s - \eta_d C_{\text{loc}}) \ln(L/L_0) \), where \( C_\infty = c_s C_{\text{loc}}/(c_{\text{loc}} - c_s) \) represents the infinite size limit of conductance in the slow phase or its analytical continuation for the fast phase and \( L_0 \) is the unknown integration constant. We define these constants shown in Fig. 3 minimizing the deviation of analytical \((2 - \eta_d C_{\text{loc}}/C)\) and numerical estimates of fractal dimensions.

We were unable to fit the numerical data reasonably well using the analytical expression with \( \eta_d = 1 \), but obtained an excellent agreement between two approaches setting \( \eta_d \approx 1.3 \) (see Appendix, Sec. 1). In Fig. 4 we present analytical results for the fractal dimension (solid line) together with its numerical estimate for the zero energy states of the system with the dipole-dipole interaction. The failure of the exact expression of Ref. 32 can be due to the long-range character of hopping.

**Particle transport.** Finally, we consider the particle transport that is the main distinction of two phases, namely, superdiffusive in the fast phase and diffusive in the slow phase. Since conductance and, correspondingly diffusion coefficient depend weakly on the size at rela-
tively short time $t \ll L^2/D(L)$ the characteristic particle displacement can be characterized by the self-consistent diffusion law $r(t) \sim \sqrt{D(r)t}$ where $r$ stands for the particle displacement during the time $t$ and $D(r) = C/g$ is the diffusion coefficient given by the ratio of the dimensionless conductance and the density of states $g \sim 1/V_0^2$ in accord with the Einstein relationship. Then in the fast phase one can expect the super linear increase of the particle squared displacement with the time as $r^2 \propto t \ln(tV_0)$, while the linear dependence is expected in the slow phase.

To verify this expectation we investigate the time evolution of the system wavefunction originally localized in the origin where we set a random potential to zero. Different strengths of random potentials were investigated including $W = 0.5$ and $1.5$ for the fast phase, $W = 2.5$ for the transition point and $W = 3.5$ for the slow phase. The particle transport has been characterized using the average logarithm of the displacement (excluding the origin). We did not consider the most often used root mean square displacement because of the power-law tails of the wavefunction that can lead to the misrepresentation of the actual displacement.

Fig. 4 shows the logarithm of squared radius vs. the logarithm of the time fitted with the power law $r^2 \propto t^\alpha$. It is clear that the transport in the fast phase is characterized by the exponent $\alpha > 1$ corresponding to the superdiffusive behavior. Moreover in the superdiffusive regime the particle occupies nearly the whole system ($W = 0.5, 1.5$) while at a higher disorder only a small fraction of the volume is occupied. This is consistent with our expectations for the fractal dimension below 2 in the slow phase.

**Conclusions.** We show the emergence of a superdiffusive fast phase in two-dimensional system Anderson model with long-range hopping $V(r) \propto r^{-2}$ and sufficiently small disorder. The fast phase is characterized by delocalized eigenstates occupying the whole space and characterized by ergodic dynamics and superdiffusive transport. The complementary slow phase is characterized by delocalized eigenstates with fractal dimension less than 2 characterized by diffusive transport restricted to the maximum displacement substantially smaller than the system size. The boundary between two phases is determined analytically which is unprecedented for Anderson localization problem with the only exception of the celebrated self-consistent theory of localization valid for the Bethe lattice.\[33\]

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Appendix A: Model and Green functions

Here we introduce the models used in the main text in greater detail. The system Hamiltonian can be expressed as

$$\hat{H} = \sum_{i,j} V_{ij} c_i^\dagger c_j + \sum_i \phi_i c_i^\dagger c_i,$$

(A1)

where the summation is performed over $N = L^2$ lattice sites enumerated by indices $i$ with coordinates $r_i = (x_i, y_i)$ occupying the periodic square lattice placed onto the surface of the torus characterized by the radii $R = L/(2\pi)$.

Vectors connecting two points $i$ and $j$ located in the torus are defined as $r_{ij} = \left(\frac{L}{\pi}(\cos(\pi (x_i - x_j)/L), \sin(\pi (y_i - y_j)/L))\right)$ and hopping interactions $V_{ij}$ are defined as the function of those vectors. Here we consider two long-range hopping interactions including the anisotropic two-dimensional dipole-dipole interaction with dipoles oriented along the $x$ axis and the RKKY interaction. The dipole-dipole interaction is defined as

$$V(r) = V_0 \frac{x^2 - y^2}{r^4},$$

(A2)

while the RKKY interaction in two dimensions reads

$$V(r) = \pi V_0 p_F^2 (J_0(p_F r) N_0(p_F r) + J_1(p_F r) N_1(p_F r)),$$

(A3)

where $J_{0,1}(p_F r)$ and $N_{0,1}(p_F r)$ are Bessel and Neumann functions, respectively. With this definition of interaction constant the asymptotic behavior of the RKKY interaction at long distances can be expressed as $V(r) \approx V_0 \sin(2p_F r)/r^2$ [34]. Here $p_F$ is the Fermi momentum for the electrons responsible for the RKKY interaction. It is chosen as $p_F = (1 + \sqrt{5})/4$ to be incompatible with the lattice period $a = 1$.

Random potentials $\phi$ are non-correlated with each other and characterized by the normalized by one probability density $P(\phi/W)/W$. The function $P(x)$ has a unit width and the parameter $W$ is the width of random potential distribution expressing the strength of disordering. The case $W > V_0$ corresponds to strong disordering while the opposite regime $W < V_0$ corresponds to weak disordering.

![Graph](image1.png)

FIG. 5: Comparison of analytical and numerical Fourier transforms for (a) dipole-dipole interaction (at fixed ratios $k_y/k_x$) and (b) RKKY interaction. Analytical approximation is shown by dashed lines and numerical by solid lines.

In our calculations we use the Lorentzian distribution of random energy, suggesting

$$P(x) = \frac{1}{\pi(x^2 + 1)}.$$

(A4)

For the Lorentzian distribution the Green’s functions of the problem can be evaluated exactly as

$$G(E, \mathbf{q}) = \frac{1}{E - V(\mathbf{q}) - iW}.$$  

(A5)
in the momentum representation $[24]$. $V(q)$ stands for the Fourier transforms of the hopping amplitudes $V(q) = \sum_j V_j e^{iqr_j}$. The Green functions are needed to evaluate the zeroth order dimensionless conductance needed to resolve the phase transition analytically.

It turns out that both dipole-dipole and RKKY interaction Fourier transforms can be represented by their continuous limits as shown below. For the dipole-dipole interaction one can express its Fourier transform as

$$V(q) = \pi V_0 \frac{-q_x^2 + q_y^2}{q^2},$$  \hspace{1cm} (A6)$$

This approximation works reasonably well for the periodic square lattice with the size $L = 100$ as illustrated in Fig. 3a. In the limit of interest $q \to 0$, where the logarithmic infrared divergence of the zeroth order conductance emerges, this approach becomes exact for $L \to \infty$. Indeed, since the diverging contribution to the conductance is determined by $q \to 0$ in this limit the regular correction to Eq. (A6) disappears because the sum of all dipole-dipole interactions is zero.

For the RKKY interaction one can express the continuous limit Fourier transform as (cf. [19])

$$V(q) \approx \pi V_0 \left( \theta(q - 2p_F) \sqrt{1 - \frac{4p_F^2}{q^2} - z_F} \right), \quad z_F = \int \frac{dq}{(2\pi)^2} \theta(q - 2p_F) \sqrt{1 - \frac{4p_F^2}{q^2}}.$$  \hspace{1cm} (A7)$$

This approximation agrees reasonably well with the Fourier transform evaluated numerically for the periodic square lattice with the size $L = 100$ as shown in Fig. 3b. The agreement is getting better at small Fermi momentum $p_F \ll 1$ while at $p_f \geq 1$ the significant interaction anisotropy in the wavevector dependence emerges. In the present work we investigate the case of relatively small Fermi momentum, where the transition to the super-diffusive regime shows a number of interesting features including for instance the reentrant behavior. For a large Fermi momentum this behavior will probably disappear. For our specific choice of fermi-momentum $p_F = \frac{1 + \sqrt{5}}{4}$ one has $z_F = 0.5963$.

### Appendix B: Zeroth order conductance

One can express the zeroth order dimensionless conductance tensor at given energy $E$ using the Kubo formula as

$$C_{ab}^{(0)} = \frac{1}{\pi (2\pi)^d} \int dq \left( \frac{\partial V(q)}{\partial q_a} \right) \left( \frac{\partial V(q)}{\partial q_b} \right) |\text{Im}G(E, q)|^2,$$  \hspace{1cm} (B1)$$

where $G(E,q)$ is the retarded Green function at energy $E$ and wavevector $q$ defined in Eq. [A5]. It diverges logarithmically at long distances. Below we evaluate its diverging part needed for the characterization of the phase transition both for the dipole-dipole (Sec. B1) and RKKY (Sec. B2) interactions.

#### 1. Dipole-dipole interaction

For the dipole-dipole hopping Eqs. (A2), (A6) the integral for the zeroth order conductance Eq. (B1) diverges logarithmically at $q = 0$. For the finite size $L$ the integral should be cutoff at the minimum value $q = \eta/L$, with the parameter $\eta \sim 1$ depending on the specific boundary conditions. Then the conductance tensor components in Eq. (B1) can be evaluated with the logarithmic accuracy as $C_{xx} = c_x \ln(L/L_0), C_{yy} = c_y \ln(L/L_0), c_{xy} = c_{yx} = 0$, where

$$c_x = \frac{\partial C_{xx}^{(0)}}{\partial \ln(L)} = \frac{4V_0^2}{\pi} \int_0^{2\pi} d\phi \frac{\cos(\phi)^2 \sin(\phi)^4 W^2}{[\sqrt{(E - \pi V_0 \cos(2\phi))^2 + W^2}]^3},$$

$$c_y = \frac{\partial C_{yy}^{(0)}}{\partial \ln(L)} = \frac{4V_0^2}{\pi} \int_0^{2\pi} d\phi \frac{\cos(\phi)^4 \sin(\phi)^2 W^2}{[\sqrt{(E - \pi V_0 \cos(2\phi))^2 + W^2}]^3},$$

$$c_{xy} = c_{yx} = 0,$$  \hspace{1cm} (B2)$$
In the middle of the band $E = 0$ the conductance is isotropic, $C_{xx} = C_{yy} = c_\ast \ln(L/L_0)$, and the logarithmic growth rate $c_\ast$ is given by

$$c_\ast = \frac{V_0^2}{2W\sqrt{\pi^2 V_0^2 + W^2}}.$$  

It is noticeable, that in the weak disorder limit ($W \to 0$) the rate parameter $c_\ast$ in Eq. \ref{b3} approaches infinity, so the transition to the superdiffusive regime should take place at a finite disorder strength $W$ where $c_\ast(W) = c_{\text{loc}} = 1/(2\pi^2)$ in contrast with the case of isotropic dipole-dipole hopping \cite{18}.

The generalization to the arbitrary distribution of random potentials can be made by the replacements $E \to E - \Re \Sigma(E, 0)$ and $W \to -\Im \Sigma(E, 0)$ in Eqs. \ref{b2}, \ref{b3}, where $\Sigma(E, 0)$ is the self-energy evaluated at energy $E$ and wavevector $q = 0$. One should notice that $\Sigma(0, 0) = 0$. Finding the self-energy for arbitrary distribution of random potentials remains a challenge; yet this problem is much easier compared to the localization problem itself.

2. RKKY interaction

For the RKKY hopping Eqs. \ref{A3}, \ref{A7} the integral for the zeroth order conductance Eq. \ref{B1} diverges logarithmically at $q = 2p_F$. For the finite size $L$ the integral should be cutoff at the minimum value $q = \eta/L + 2p_F$, with the parameter $\eta \sim 1$. Then the conductance Eq. \ref{B1} is isotropic and can be evaluated with the logarithmic accuracy as $C_{xx} = C_{yy} = c_\ast \ln(L/L_0)$, $c_{xy} = c_{yx} = 0$, where (integration over angles is straightforward since the spectrum is isotropic)

$$c_\ast = \frac{V_0^2}{2} \frac{Z(2p_F)}{Z(q)} \frac{W^2}{(E + V_0 z_F)^2 + W^2}, \quad Z(q) = \begin{cases} 1 & q < \pi, \\ \frac{2(\sin^{-1}(\pi/q) - \sin^{-1}(\sqrt{q^2 - \pi^2/q}))}{\pi} & q > \pi. \end{cases}$$  \hspace{1cm} \text{(B4)}

The generalization to the arbitrary distribution of random potentials is straightforward if the Green function is known. Then one gets

$$c_\ast = \frac{V_0^2}{4\pi} Z(2p_F) \int d\mathbf{n} \left(\Im G(E, 2p_F \mathbf{n})\right)^2,$$

where integration is performed over directions of the unit vector $\mathbf{n}$.

Appendix C: Derivation of the $\beta$-function.

We begin our consideration with the isotropic regime approximately valid for the dipolar system at zero energy and for RKKY hopping that is considered in Sec. \ref{C1}. The results for anisotropic regime are outlined in Sec. \ref{C2}. For the sake of simplicity we use the scalar field. Yet the generalization to the vector field is straightforward since the form of the expressions for the one loop order corrections does not change.

1. Isotropic conductance.

We investigate the renormalization of conductance $C(q, p_1)$ for the orthogonal sigma model within the single loop order. Here $q$ is the current momentum and $p$ is the maximum momentum $q$ reduced during renormalization procedure. The renormalization of the conductance associated with the reduction of the maximum momentum by the factor of $\mu$ can be expressed as \ref{B3} ($q \ll p$)

$$q^2(C(q, p_1/\mu) - C(q, p_1)) \approx -\int' \frac{d\mathbf{p}}{(2\pi i)^2} \frac{C(|\mathbf{p} + \mathbf{q}|, p_1)(p + q)^2 - C(p, p_1)p^2}{C(p, p_1)p^2}.$$  \hspace{1cm} \text{(C1)}

where integration is taken over the domain of momenta $p_1/\mu < p < p_1$ that is getting excluded from the consideration. The subtracted term in the numerator is originated from the field normalization term. The initial conditions at large $p_1 \sim 1$ are set using the zeroth order conductance as

$$C(q, 1) = c_\ast \ln(1/(qL_0)),$$  \hspace{1cm} \text{(C2)}
where the inverse wavevector $q$ serves as the cutoff radius in the definition of the conductance. The long-range interaction enters into consideration through this initial condition.

In the limit $q \ll p_1$ one can expand the expression in the numerator to the second order in $q$ as (the first order disappears because of the integration over angles)

$$q^2(C(q,p_1/\mu) - C(q,p_1)) \approx - \sum_{\alpha,\beta=x,y} q_\alpha q_\beta \int' \frac{dp}{\pi(2\pi)^2} \frac{\partial^2(C(p,p_1)p^2)}{\partial p_\alpha \partial p_\beta} \frac{1}{C(p,p_1)p^2}. \quad (C3)$$

Evaluating derivatives and averaging over angles of vector $p \ (< p_\alpha p_\beta >= p^2\delta_{\alpha\beta}/2)$ Eq. (C3) we got

$$C(q,p_1/\mu) - C(q,p_1) = -\int' \frac{dp}{\pi(2\pi)^2} \left[ C(p,p_1) + \frac{\partial C(p,p_1)}{\partial \ln(p)} + \frac{1}{4} \frac{\partial^2 C(p,p_1)}{\partial \ln(p)^2} \right] \frac{1}{C(p,p_1)p^2}. \quad (C4)$$

Assuming that the logarithmic derivatives of the conductance are smooth functions one can perform logarithmic integration in the right hand side of Eq. (C4) and express this equation in the standard differential form as

$$\frac{\partial C(q,p)}{\partial \ln(p)} = \frac{1}{2\pi^2} \left[ 1 + \frac{1}{C(p,p)} \frac{\partial C(p_1,p)}{\partial \ln(p_1)} \right]_{p_1=p} + \frac{1}{4C(p,p)} \frac{\partial^2 C(p_1,p)}{\partial \ln(p_1)^2} \right]_{p_1=p}. \quad (C5)$$

Since the right hand side of Eq. (C5) is independent of the wavevector $q$ one can evaluate logarithmic derivatives using the initial condition Eq. (C2) as

$$\left. \frac{\partial C(p_1,p)}{\partial \ln(p_1)} \right|_{p_1=p} = -c_\ast, \quad \left. \frac{\partial^2 C(p_1,p)}{\partial \ln(p_1)^2} \right|_{p_1=p} = 0.$$  

Then Eq. (C5) takes the form

$$\frac{\partial C(q,p)}{\partial \ln(p)} = \frac{1}{2\pi^2} \left[ 1 - \frac{c_\ast}{C(p,p)} \right]. \quad (C6)$$

The renormalized conductance at the given momentum $p$ can be determined with the logarithmic accuracy as $C(p,p)$ and it can be denoted as $C(p)$ for the convenience. Using the initial condition Eq. (C2) for the derivative with respect to the first argument we end up with the renormalization group equation in the form

$$\frac{dC(p)}{d\ln(p)} = -c_\ast + \frac{1}{2\pi^2} \left[ 1 - \frac{c_\ast}{C(p)} \right]. \quad (C7)$$

For the size $L$ dependent conductance one can express the relevant wavevector $p$ as $\eta/L$ for $\eta \sim 1$. This leads to the renormalization group equation for the size dependent conductance in the form

$$\frac{dC}{d\ln(L)} = \beta(C), \quad \beta(C) = c_\ast - c_{\text{loc}} + \frac{c_\ast c_{\text{loc}}}{C} + O(C^{-2}), \ c_{\text{loc}} = \frac{1}{2\pi^2}. \quad (C8)$$

This equation is given in the main text. Assuming that $C \gg c_{\text{loc}}$ we can ignore higher order terms. Then for $c_{\text{loc}} > c_\ast$ the steady state solution reads

$$C = \frac{c_\ast c_{\text{loc}}}{c_{\text{loc}} - c_\ast}, \quad (C9)$$

It is a stable fixed point. This solution is applicable in the infinite size limit and for $c_{\text{loc}} - c_\ast \ll c_{\text{loc}}$ where higher order terms in $1/C$ can be neglected. In the opposite case $c_\ast > c_{\text{loc}}$ the solution approaches infinity for $L \to \infty$.

For the short-range hopping there is no contribution to the $\beta$-function in the two loop order. The logarithmic dependence of the zeroth order conductance Eq. (C2) can modify this result. Yet similarly to the case of the one loop order it is natural to expect that the associated corrections will be of higher order in $1/C$ so they can be neglected in Eq. (C8). The evaluation of the two loop order is beyond the scope of the present work, so we leave this expectation as a reasonable hypothesis that is qualitatively consistent with the numerical results, as it is pointed out in the main text.
2. Anisotropic conductance.

The renormalization group equation for the anisotropic conductance can be derived similarly to the isotropic regime. In the one loop order we got

\[
\begin{align*}
\frac{dC_x}{d\ln(L)} &= c_x^* - c_{\text{loc}} \frac{C_x}{\sqrt{C_x C_y}} + \frac{2c_x^*}{\sqrt{C_x (\sqrt{C_x} + \sqrt{C_y})}} + \frac{C_y c_x^* - C_x c_y^*}{4\sqrt{C_x C_y} (\sqrt{C_x} + \sqrt{C_y})^2} + O(C^{-2}), \\
\frac{dC_y}{d\ln(L)} &= c_y^* - c_{\text{loc}} \frac{C_y}{\sqrt{C_x C_y}} + \frac{2c_y^*}{\sqrt{C_y (\sqrt{C_x} + \sqrt{C_y})}} + \frac{C_x c_y^* - C_y c_x^*}{4\sqrt{C_x C_y} (\sqrt{C_x} + \sqrt{C_y})^2} + O(C^{-2}).
\end{align*}
\tag{C10}
\]

Similarly to the isotropic case Eq. (C9) this equation has a stable fixed point at \( c_x^* c_y^* < c_{\text{loc}}^2 \). In the infinite size limit the steady state solution for conductance at that point can be approximated by

\[
\left( \frac{C_x}{C_y} \right) = \frac{2c_{\text{loc}}^2}{c_x^2 - c_x^* c_y^*} \left( \frac{c_y^*}{c_x^*} \right)
\tag{C11}
\]

Conductance approaches infinity in the infinite size limit for \( c_x^* c_y^* \geq c_{\text{loc}}^2 \). Consequently the transition to the superdiffusive regime is defined as

\[
c_{\text{loc}}^2 = c_x^* c_y^*.
\tag{C12}
\]

This criterion is quoted within the main text where it is used to construct the phase diagram for the system with hopping defined by the dipole-dipole interaction.

Appendix D: Connection of conductance and fractal dimension.

In the present work we described conductance using the single parameter scaling equation in the form

\[
\frac{d\ln(C)}{d\ln(L)} = \frac{c_* - c_{\text{loc}}}{C} + \eta \frac{c_* c_{\text{loc}}}{C^2},
\tag{D1}
\]

and with the parameter \( \eta = 1 \) (see Sec. C) that contrasts to the estimate \( \eta = 1/2 \) in Ref. [18]. We studied the only system property directly related to the conductance that is the fractal dimension that can be expressed as

\[
D = 2 - \eta_d c_{\text{loc}}/C,
\]

with the dimensionless parameter \( \eta_d = 1 \).

In Fig. we show the comparison of the numerical data and analytical theory for two choices of the parameter \( \eta \) using the best fit minimizing the deviation of analytical and numerical data sets choosing the optimum parameter \( L_0 \) in the integral form of Eq. (D1)

\[
C + C_\infty \ln(1 - C/C_\infty) = (c_* - \eta_d c_{\text{loc}}) \ln(L/L_0),
\]

where \( C_\infty = c_* c_{\text{loc}}/(c_{\text{loc}} - c_*) \). It is clear that in both cases theory does not provide the acceptable fit of the data. However, if we set \( \eta_d = 1.3 \), then we get an almost perfect agreement of numerical and analytical data as reported in the main text. The reasonable data fit can also be obtained for \( \eta_d = 1.4 \) and 1.5. Yet in those case an anomalously small fitting parameter \( L_0 < 0.1 \) is required for the maximum disorder (\( W = 3 \)).
FIG. 6: Comparison of analytical and numerical estimates for the fractal dimension obtained comparing analytical result of Ref. [32] with conductance evaluated solving single parametric scaling equation following Ref. [18] (a) or the present work (b). In the latter case we always have $L_0 = 10$. 