GEOMETRIC CONSTRUCTION OF GENERALIZED LOCAL SYMBOLS ON ALGEBRAIC CURVES

FERNANDO PABLOS ROMO
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE SALAMANCA

Abstract. The aim of this work is to provide a construction of generalized local symbols on algebraic curves as morphisms of group schemes. From a closed point of a complete, irreducible and non-singular curve $C$ over a perfect field $k$ as the only data, using theta groups over Picard schemes of curves, we offer a geometric construction that allows us to define generalizations of the tame symbol and the Hilbert norm residue symbol.

1. Introduction

In 1959 J.-P. Serre [22] introduced local symbols on curves in order to study the factorization of a rational morphism of complete curves through a generalized Jacobian of Rosenlicht ([19], [20]). The additive local symbol (the classic residue on curves) and the multiplicative local symbol are examples of these symbols.

Given a complete, irreducible and non-singular curve $C$ over an algebraically closed field $k$ and a closed point $p \in C$, for each $f, g \in \Sigma^*_C$, (the multiplicative group of the function field $\Sigma_C$ of $C$), the expression of the multiplicative local symbol is:

$$(f, g)_p = (-1)^{v_p(f) - v_p(g)} \frac{f^{v_p(g)}(p)}{g^{v_p(f)}(p)} \in k^*.$$  

A few years later, in 1971, J. Milnor [11] defined a generalization of the multiplicative local symbol, the tame symbol $d_v$, associated with a discrete valuation $v$ on a field $F$. Explicitly, if $A_v$ is the valuation ring, $p_v$ is the unique maximal ideal and $k_v = A_v/p_v$ is the residue class field, Milnor defined $d_v: F^* \times F^* \to k_v^*$ by

$$d_v(x, y) = (-1)^{v(x)-v(y)} \frac{x^{v(y)} y^{v(x)}}{y^v(x) \pmod{p_v}}.$$  

Moreover, when $C$ is defined over a finite field that contains the $m^{th}$-roots of unity, the formula of the Hilbert norm residue symbol is:

$$(f, g)_p = \left( N_{k(p)/k} \left[ (-1)^{v_p(f) - v_p(g)} \frac{f^{v_p(g)}(p)}{g^{v_p(f)}(p)} \right] \right)^{\frac{q-1}{m}} \in \mu_m,$$  

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where \( \#k = q \). This expression was offered by H.L. Schmid [21], and we can obtain it from the multiplicative local symbol by using the norm morphism.

In recent years both symbols, the multiplicative local symbol and the Hilbert norm residue symbol, have been studied by several authors from different points or view (1, 3, 4, 5, 10, 15 or 17).

The goal of this paper is to give an algebraic construction that will allow us to define generalized local symbols as morphisms of schemes, and by using a closed point of a complete, irreducible and non-singular curve \( C \) over a perfect field \( k \), as the unique datum to make the construction.

As results, we offer morphisms of group schemes that generalize the multiplicative local symbol and several symbols that we can obtain from it, among them the Hilbert norm residue symbol.

We should remark that generalizations of the tame symbol have been made by C. Contou-Carrère [4] as a morphism of functors and by the author as a morphism of schemes ([14]). This paper gives the relation between the classic datum used to define a local symbol (a closed point of a curve) and the formal scheme that appears in [14]. As far as we know, generalized local symbols, different from the symbols referred to previously, have not been stated explicitly in the literature. Following the method of [2], we also offer a reciprocity law for these generalized symbols when we consider Spec \( B \)-valued points, \( B \) being an artinian local finite \( k \)-algebra.

The organization of the paper is as follows:

In Section 2 definitions and basic results are introduced. Thus, Witt’s schemes are defined from a ring scheme, the Jacobian of a cuspidal curve \( C^n \), constructed from a smooth curve \( C \), is characterized, and the group scheme \( K[\mathcal{L}] \), associated with an invertible sheaf on the Jacobian \( J_C \), are also defined.

Section 3 contains the properties of the group schemes \( K[\mathcal{L}] \) over Picard schemes of curves, which are basic to the geometric construction offered. Group schemes associated with invertible sheaves over Pic \( C \) and Pic \( C^n \) are defined, and a detailed study of the group \( K[(\mathcal{L})^n] \), associated with the invertible sheaf \( (\mathcal{L})^n \) over Pic \( C^n \) and constructed from the principal polarization of \( J_C \), is made.

Finally, Section 4 is devoted to constructing, from the scheme \( K[[(\mathcal{L})^n]] \), a formal scheme that allows us to define morphisms of schemes that generalize the local symbols referred to above. Similar to [14], we provide two generalizations of the multiplicative local symbol: one is associated with a rational point and the other, which is more general, is associated with an arbitrary closed point. The first generalization coincides, when we consider rational points, with the expression of J.P. Serre and, if we consider \( S \)-valued points with \( S \) a connected \( k \)-scheme, with the formula of C. Contou-Carrère. The second generalization offered allows us to provide new morphisms of schemes, from morphisms of groups schemes \( \mathbb{G}_m \rightarrow G \) and, in particular, from the characters of \( \mathbb{G}_m \). Hence, the Hilbert norm residue symbol is generalized and the classic formula is again recovered by considering rational points. To conclude, we offer reciprocity laws for these generalized symbols when one considers Spec \( B \)-valued points, \( B \) being an artinian local finite \( k \)-algebra.
2. Preliminaries

Firstly, we shall recall the concept of $S$-valued points in the theory of schemes: if $Y$ and $S$ are $k$-schemes, an $S$-valued point of $Y$ is a morphism of schemes from $S$ to $Y$. The set of all such is denoted

$$Y^\bullet(S) = \text{Hom}_{k-\text{sch.}}(S, Y).$$

2.1. Witt's Schemes. Let $A$ be a ring $k$-scheme: i.e., it is a scheme whose functor of points takes values in the category of commutative rings with unit element.

Let us assume that $A$ is unipotent a group scheme under addition (there exists an isomorphism of group schemes between $A$ and a unipotent group scheme). We shall now define the Witt schemes associated with $A$.

Recall from ([22], page 168) that given two commutative group schemes, $A$ and $B$, $H^2(A, B)$ is the group of classes of 2-cocycles (mod. 2-coboundaries), and that $H^2_{\text{reg}}(A, B)_s$ is the subgroup of $H^2(A, B)$ consisting of classes of regular and symmetric 2-cocycles. It is also known that each regular and symmetric 2-cocycle determines an extension between unipotent groups, i.e. $H^2_{\text{reg}}(A, B)_s \subseteq \text{Ext}(A, B)$. Moreover, if the ground field is algebraically closed, each extension of unipotent groups is induced by a regular 2-cocycle ([22], page 171).

Let us now consider the regular morphism of schemes $f_2: A \times A \to A$, which defines the multiplication in $A$. One has that $f_2$ satisfies the condition of a 2-cocycle:

$$f_2(y, z) - f_2(x + y, z) + f_2(x, y + z) - f_2(x, y) = 0,$$

where $x, y, z \in A^\bullet(S)$, $S$ being an arbitrary $k$-scheme.

Hence, $f_2 \in H^2_{\text{reg}}(A, A)_s$ and it induces an extension

$$0 \to A \to W^2_A \to A \to 0$$

such that $W^2_A \simeq A \times A$ as $k$-schemes, and its group structure is defined by the map

$$m([(x, y), (\bar{x}, \bar{y})] = (x + \bar{x}, y + \bar{y} + f_2(x, \bar{x})), $$

for all $x, \bar{x} \in A^\bullet(S)$, with $S$ a $k$-scheme.

Since $W^2_A$ is a unipotent group, by induction we obtain the extensions

$$0 \to A \to W^h_A \to W^{h-1}_A \to 0,$$

determined by the morphisms of schemes

$$f_h: W^{h-1}_A \times W^{h-1}_A \to A,$$

where

$$f_h(x, y) = \sum_{i + j = h} x_i \cdot y_j$$

for all $x = (x_i), y = (y_i) \in (W^{h-1}_A)^\bullet(S)$.

If $z = (z_i) \in (W^{h-1}_A)^\bullet(S)$, it follows from the equality

$$f_h(y, z) - f_h(x + y, z) + f_h(x, y + z) - f_h(x, y) = 0$$

that $f_h$ is a 2-cocycle and that it determines a factor system that defines the group structure of $W^h_A$. 

Example 1. Let $K$ be a finite extension of the field $k$.

From the multiplicative law of $K$, $m: K \times K \to K$, we have an induced map $K^\vee \to K^\vee \oplus K^\vee$, where $K^\vee = \text{Hom}_k(K, k)$. Thus, by computing the symmetric algebra, we get a morphism of $k$-algebras

$$
S^i(K^\vee) \to S^i(K^\vee) \otimes S^j(K^\vee),
$$

which induces a map $\text{Spec } S^i(K^\vee) \times \text{Spec } S^j(K^\vee) \to \text{Spec } S^i(K^\vee)$, which, together with the morphism of schemes induced over $\text{Spec } S^i(K^\vee)$ by the addition map of $K$, determines the structure of a ring scheme on $\text{Spec } S^i(K^\vee)$.

Furthermore, if $d = \dim_k K$, there exists an isomorphism of additive groups $K \simeq k^d$, and therefore the ring scheme $\text{Spec } S^i(K^\vee)$ is isomorphic as a group scheme to $\mathbb{G}_a \times \mathbb{G}_a \times \ldots \times \mathbb{G}_a$. However, both schemes are not isomorphic as ring schemes.

Let us now consider a $k$-algebra $A$, and let us set $W^n_A$ to denote the group $A^n$ together with the operation $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (c_1, \ldots, c_n)$, where $c_i = a_i + b_i + \sum h+k=1 a_h \cdot b_k$.

Hence, we can now define the Witt scheme $W^h_{\text{Spec } S^i(K^\vee)}$, whose functor of points is

$$[W^h_{\text{Spec } S^i(K^\vee)}]^*(S) = W^h_{\Gamma(S, \mathcal{O}_S \otimes_k K)},$$

for each $k$-scheme $S$.

Moreover, if $W(A)^n_+$ is the $n^{\text{th}}$-group defined in (9), one has that $W(A)^n_+$ and $W_A^n$ are isomorphic through the map

$$W(A)^n_+ \to W_A^n$$

$$(a_i) \to (b_j),$$

with

$$b_j = \sum_{i_1 < i_2 < \cdots < i_k \atop i_1 + \cdots + i_k = j} (-1)^k \bar{a}_{i_1} \cdots \bar{a}_{i_k}.$$

Furthermore, both groups are also isomorphic to $\Gamma^n_+(A)$, which is the multiplicative subgroup of $(A[t]/t^{n+1})^*$ consisting of elements of the form

$$\{1 + a_1 t + \cdots + a_n t^n\}.$$

2.2. Cuspidal Curve associated with a Smooth Curve. Let us now consider a complete, irreducible and non-singular curve $C$ over a perfect field $k$ and with a rational point. If $p \in C$ is a closed point, $\mathcal{O}_p$ is its local ring, and $k(p)$ is the residue field at $p$, for each positive integer $n$ one has that there exists the scheme

$$C^n = C \coprod_{\text{Spec } \mathcal{O}_p/m_p^n} \text{Spec } k(p),$$

such that:

- The topological spaces of $C$ and $C^n$ are homeomorphic.
- If $\{U_i\}_{i \in I}$ is an open covering of $C$ such that $p$ is only included in $U_1$, then $\mathcal{O}_{C^n}(U_i) = \mathcal{O}_C(U_i)$ if $i \neq 1$ and

$$\mathcal{O}_{C^n}(U_1) = \mathcal{O}_C(U_1) \times k(p),$$

$$\mathcal{O}_p/m_p^n.$$
k(p) \rightarrow \mathcal{O}_p/m^n_p \text{ being the morphism induced by the separable elements of } \mathcal{O}_p/m^n_p \text{ over } k(p), \text{ that is well-defined because } k \text{ is perfect.}

C \text{ and } C^{(n)} \text{ are birationally equivalent curves, and } C^{(n)} \text{ is a singular curve, with a unique cuspidal singularity of multiplicity } n \text{ [16]).}

Let \( \pi_n : C \to C^{(n)} \) be the normalization morphism. If \( \bar{p}_n \) is the singular point of \( C^{(n)} \), one has that \( \pi_n(p) = \bar{p}_n \). We also denote by \( D_n \) the fiber of \( \bar{p}_n \) by \( \pi_n \), and \( \mathcal{O}_{D_n} \simeq \mathcal{O}_p/m^n_p \). Moreover, \( k(\bar{p}_n) \simeq k(p) \) and \( \mathcal{O}_{\bar{p}_n} \to \mathcal{O}_p \).

**Lemma 2.1.** Let \( B \) be a \( k \)-algebra. One has that

\[
(\mathcal{O}_{D_n} \otimes B)^*/(k(p) \otimes B)^* \simeq W_{k(p)\otimes B}^{n-1},
\]

and, when the characteristic of \( k \) is 0, there exists an isomorphism of groups

\[
(\mathcal{O}_{D_n} \otimes B)^*/(k(p) \otimes B)^* \simeq (m_p/m^n_p) \otimes B
\]

**Proof.** It follows from the group structure of \( W_{k(p)\otimes B}^{n-1} \) that the map

\[
W_{k(p)\otimes B}^{n-1} \to (\mathcal{O}_{D_n} \otimes B)^*/(k(p) \otimes B)^*
\]

\[
(\sum_{j_h=1}^{\lambda_h} x_{j_h} \otimes b_{j_h})_h \mapsto 1 \otimes 1 + \sum_{h=1}^{n-1} ((\bar{t}_p^h \otimes 1) \cdot (\sum_{j_h=1}^{\lambda_h} x_{j_h} \otimes b_{j_h}))
\]

is an isomorphism of groups, where \( t_p \) is a local parameter at \( p \), \( \bar{t}_p \in \mathcal{O}_p/m^n_p \), \( x_{j_h} \in k(p) \) and \( b_{j_h} \in B \).

Furthermore, if the characteristic of \( k \) is 0, the exponential map induces an isomorphism of groups between \( (m_p/m^n_p) \otimes B \) and \( (\mathcal{O}_{D_n} \otimes B)^*/(k(p) \otimes B)^* \),

which is the second part of the statement. \( \square \)

**Proposition 2.2.** Let \( \pi'_{n,S} : D_n \times S \to \bar{p}_n \times S \) be the morphism induced by the desingularization map and let \( \phi'_{n,2} : \bar{p}_n \times S \to S \) be the natural projection. If \( \overline{\xi}_S \) is the cokernel sheaf of the injective morphism of sheaves of abelian groups on \( \bar{p}_n \times S \) induced by \( \pi'_{n,S} \):

\[
\mathcal{O}^*_{\bar{p}_n \times S} \to (\pi'_{n,S})_* \mathcal{O}^*_{D_n \times S},
\]

and one defines \( W_{k(p)\otimes S}^{n-1} \) as the sheaf of groups on a \( k \)-scheme \( S \) such that it assigns

\[
W_{\Gamma(U,k(p)\otimes S)}^{n-1} \to \mathcal{O}^*_{\bar{p}_n \times S} \to (\pi'_{n,S})_* \mathcal{O}^*_{D_n \times S} \to \overline{\xi}_S \to 0
\]

to each open set \( U \) of \( S \), one has an isomorphism of sheaves of abelian groups on \( S \)

\[
(\phi'_{n,2})_* \overline{\xi}_S \simeq W_{k(p)\otimes S}^{n-1}.
\]

**Proof.** Since \( \phi'_{n,2} \) is a finite morphism, one has that \( R^1(\phi'_{n,2})_* \mathcal{O}^*_{\bar{p}_n \times S} = 0 \) ([17], Proposition 5.5) and, thus, from the exact sequence of sheaves of groups on \( \bar{p}_n \times S \)

\[0 \to \mathcal{O}^*_{\bar{p}_n \times S} \to (\pi'_{n,S})_* \mathcal{O}^*_{D_n \times S} \to \overline{\xi}_S \to 0\]

we have the exact sequence of sheaves of groups on \( S \)

\[0 \to (\phi'_{n,2})_* \mathcal{O}^*_{\bar{p}_n \times S} \to (\phi'_{n,2})_* (\pi'_{n,S})_* \mathcal{O}^*_{D_n \times S} \to (\phi'_{n,2})_* \overline{\xi}_S \to 0.\]
Bearing in mind that \((\phi_{n,2})_*\mathcal{O}_p^* \simeq (k(p) \otimes \mathcal{O}_S)^*\) and
\[(\phi_{n,2})_* \pi_{n,S}^* \mathcal{O}_{D_n \times S}^* \simeq (\mathcal{O}_p/m_p^n \otimes \mathcal{O}_S)^*,\]
one has that \((\phi_{n,2})_* \xi_S^*\) is the sheaf associated with the presheaf:
\[U \rightarrow (\mathcal{O}_p/m_p^n \otimes \mathcal{O}_S(U))^*/(k(p) \otimes \mathcal{O}_S(U))^*,\]
where \(U\) is an open set of a \(k\)-scheme \(S\). Hence, it follows from the previous lemma that there exists a morphism of sheaves between \((\phi_{n,2})_* \xi_S^*\) and \(W^{n-1}_{k(p) \otimes \mathcal{O}_S}\) that is clearly an isomorphism.

**Theorem 2.3.** The Jacobian of the singular curve \(C^n\), \(J_{C^n}\), is an extension of the abelian variety \(J_C\) by a unipotent, connected and commutative group scheme. Moreover, if we denote this group scheme by \(K[n]\), we have that \(K[n]\) is isomorphic to the Witt scheme \(W^{n-1}_{\text{Spec } k(p)}\) (Example 7) and, when the characteristic of \(k\) is 0, there exists an isomorphism of group schemes \(K[n] \simeq \mathbb{G}_a^{d(n-1)}\), where \(d = \dim_k k(p)\).

**Proof.** Let \(S\) be an arbitrary \(k\)-scheme. We use \(\phi_{n,2} : C^n \times S \rightarrow S\) to denote the natural projection and \(\pi_{n,S} : C \times S \rightarrow C^n \times S\) to denote the morphism induced by the normalization.

Thus, from the exact sequence of sheaves of groups on \(C^n \times S\)
\[0 \rightarrow \mathcal{O}^*_{C^n \times S} \rightarrow (\pi_{n,S})_* \mathcal{O}_{C \times S}^* \rightarrow \xi_S \rightarrow 0,\]
where \(\xi_S\) is the corresponding cokernel sheaf, and bearing in mind that
\[R^1(\phi_{n,2})_* \xi_S = 0\]
because \(\xi_S\) is a concentrated sheaf at \(\bar{p}_n \times S\) which is a closed subscheme of \(C^n \times S\), we obtain the morphisms of groups
\[(2.1)\]
\[\begin{array}{c}
0 \rightarrow \Gamma(S, (\phi_{n,2})_* \xi_S) \\
\Gamma(S, R^1(\phi_{n,2})_* \mathcal{O}_{C^n \times S}^*) \rightarrow \Gamma(S, R^1(\phi_{n,2})_* (\pi_{n,S})_* \mathcal{O}_{C \times S}^*) \rightarrow \Gamma(S, \mathcal{O}_{C^n \times S}^*) \\
\end{array}
\]
with \(\text{Coker} \pi_{n,S}^* \hookrightarrow H^1(S, (\phi_{n,2})_* \xi_S)\).

Hence, if \(\xi_S^*\) is the sheaf on \(\bar{p}_n \times S\) defined above, from the commutative diagram of morphisms of schemes
\[
\begin{array}{ccc}
C \times S & \xrightarrow{\pi_{n,S}} & C^n \times S \\
\downarrow & & \downarrow \\
D_n \times S & \xrightarrow{\pi_{n,S}} & \bar{p}_n \times S \\
\end{array}
\]
one has that \((\phi_{n,2})_* \xi_S^* \simeq (\phi'_{n,2})_* \xi_S^*\), and it follows from Proposition 2.2 that \((\phi_{n,2})_* \xi_S^* \simeq W^{n-1}_{k(p) \otimes \mathcal{O}_S}\).
Furthermore, if Spec $B$ is an affine $k$-scheme, the sheaf $W_{k(p)\otimes \mathcal{O}_{\text{Spec } B}}^{n-1}$ is acyclic, and therefore from (2.1) there exists an exact sequence of groups:

$$0 \rightarrow \Gamma(\text{Spec } B, W_{k(p)\otimes \mathcal{O}_{\text{Spec } B}}^{n-1}) \rightarrow \text{Pic}(\mathcal{C}^n)^* (\text{Spec } B) \rightarrow \text{Pic}(\mathcal{C})^* (\text{Spec } B) \rightarrow 0$$

that shows that the induced morphism between the Picard schemes, which is denoted by $\bar{\pi}_n$, is surjective.

Accordingly, by considering the restriction of the morphism $\bar{\pi}_n$ to the Jacobian $J_{\mathcal{C}^n}$, we obtain the morphisms of group schemes

$$0 \rightarrow K[n] \rightarrow J_{\mathcal{C}^n} \rightarrow J_C \rightarrow 0,$$

where $K[n]^*(S) = \Gamma(S, W_{k(p)\otimes \mathcal{O}_{\text{Spec } S}}^{n-1})$. Finally, when the characteristic of $k$ is 0, since a unipotent, connected, and commutative group is isomorphic to a direct product of additive groups, one concludes that

$$K[n] \simeq \mathbb{G}_a^{d(n-1)}.$$

\hspace{1cm}$\square$

**Corollary 2.4.** In arbitrary characteristic, one has that $K[n] \simeq (\Gamma_+^{\infty})_{k(p)}$, with

$$[(\Gamma_+^{\infty})_{k(p)}]^*(S) = \left\{ 1 + a_1 t + \cdots + a_n t^{n-1} \text{ where } a_h \in \Gamma(S, \mathcal{O}_S) \otimes k(p) \right\}$$

for an arbitrary $k$-scheme $S$.

**Corollary 2.5.** One has that $K[\infty] = \varprojlim_n K[n] \simeq (\Gamma_+)_{k(p)}$, with

$$[(\Gamma_+)_{k(p)}]^*(S) = \left\{ 1 + a_1 t + a_2 t^2 + \cdots \text{ where } a_h \in \Gamma(S, \mathcal{O}_S) \otimes k(p) \right\}$$

for an arbitrary $k$-scheme $S$.

### 2.3. Group Schemes $K[\mathcal{L}^n]$ over $J_{\mathcal{C}^n}$

For each positive integer $n \in \mathbb{N}$, let us again consider the morphism of group schemes $\bar{\pi}_n: J_{\mathcal{C}^n} \rightarrow J_C$ induced by the normalization morphism $\pi_n: C \rightarrow \mathcal{C}^n$.

Recall from [13] that if $X$ is a commutative group $k$-scheme, $S$ is an arbitrary $k$-scheme and $f: S \rightarrow X$ is an $S$-valued point of $X$, then $T_f$, translation by $f$, denotes the $S$-isomorphism of schemes

$$(m \circ (1_X \times f), q_2): X \times S \rightarrow X \times S,$$

where $m$ is the multiplication on $X$ and $q_2: X \times S \rightarrow S$ is the natural projection.

Let $\mathcal{L}$ be an invertible sheaf over $J_C$. If we consider the invertible sheaf $\mathcal{L}^n$ over $J_{\mathcal{C}^n}$, $\mathcal{L}^n = (\bar{\pi}_n)^* \mathcal{L}$, from [13] we know of the existence of a group scheme $K[\mathcal{L}^n]$ such that its functor of points is

$$K[\mathcal{L}^n]^*(S) = \left\{ f \in J_{\mathcal{C}^n}^*(S) \text{ such that } T_f^* \mathcal{L}_S^n \simeq \mathcal{L}_S^n \otimes \bar{\phi}_2^* \mathcal{M} \right\},$$

where $S$ is an arbitrary $k$-scheme, $\mathcal{L}_S^n$ is the pullback of the sheaf $\mathcal{L}^n$ to the $k$-scheme $J_{\mathcal{C}^n} \times S$, the map $T_f: J_{\mathcal{C}^n} \times S \rightarrow J_{\mathcal{C}^n} \times S$ is the translation by the $S$-valued point $f$, $\bar{\phi}_2: J_{\mathcal{C}^n} \times S \rightarrow S$ is the natural projection, and $\mathcal{M}$ is an invertible sheaf over $S$. 
Furthermore, if \( K[\mathcal{L}] \) is the group scheme defined by D. Mumford in (13), there exists an exact sequence of group schemes:

\[
0 \to K[n] \to K[\mathcal{L}^n] \to K[\mathcal{L}] \to 0.
\]

3. Group Schemes \( K[\mathcal{L}] \) over Picard Schemes of Curves

Let us again consider a complete, irreducible and non-singular curve \( C \) over a perfect field \( k \) and with a rational point. If \( p \in C \) is an arbitrary closed point, we can construct the cuspidal curve \( C_n \) associated with \( p \) that was considered in the previous section.

The aim of this section is to construct group schemes \( K[\mathcal{L}] \), associated with invertible sheaves over Picard schemes of curves, in order to generalize the construction made by D. Mumford for abelian varieties. Indeed, we shall provide generalized theta groups for the schemes Pic(\( C \)) and Pic(\( C_n \)), and we shall study the relation between them.

**Remark 3.1.** Let \( X \) be a \( k \)-scheme such that \( X = \coprod \lambda V_\lambda \), \( V_\lambda \) being open subschemes of \( X \). With these conditions, one has an isomorphism between the Picard groups

\[
\text{Pic}(X) \xrightarrow{\sim} \prod \lambda \text{Pic}(V_\lambda) \quad \mathcal{L} \mapsto (\mathcal{L}|_{V_\lambda})
\]

Hence, an invertible sheaf over \( X \) is determined by its restrictions to the open subschemes \( V_\lambda \).

3.1. Group Schemes \( K[\mathcal{L}] \) over Pic \( C \). Let \( \mathcal{O}(\alpha p) \) be the invertible sheaf over \( C \) associated with the Weil divisor \( D = \alpha p \) for all \( \alpha \in \mathbb{Z} \).

If we consider the sheaf \( \mathcal{O}(\alpha p) \) as a rational point of the group scheme Pic \( C \), we have an automorphism of group schemes \( T_{\mathcal{O}(\alpha p)} : \text{Pic} C \xrightarrow{\sim} \text{Pic} C \), and, in particular, we have that \( T_{\mathcal{O}(\alpha p)} : J_{-ad} \xrightarrow{\sim} J_0 \) with \( d = \deg(p) = \dim_k k(p) \), and \( J_{-ad} \) being the Jacobian that parametrizes invertible sheaves of degree \( -ad \) on \( C \). Let us denote by \( \mu_\alpha \) the isomorphism \( T_{\mathcal{O}(\alpha p)} : J_{ad} \xrightarrow{\sim} J_0 \).

Moreover, since \( J_0 = J_C \hookrightarrow \text{Pic}(C) \) is an open subgroup (8, page 2), one has that \( \{J_\beta\}_{\beta \in \mathbb{Z}} \) is an open covering of \( \text{Pic}(C) \). Hence, an invertible sheaf over Pic \( C \) is determined by its restrictions to the varieties \( J_\beta \).

**Definition 3.2.** We shall use \( Y_{\{p\}} = \bigsqcup_{\alpha \in \mathbb{Z}} J_{ad} \) to refer to the group subscheme of the Picard scheme Pic \( C \), obtained by translation of the Jacobian \( J_0 \) by all the points \( \mathcal{O}(\alpha p) \) with \( \alpha \) an integer.

If \( p \) is a rational point, it is clear that \( Y_{\{p\}} = \text{Pic}(C) \).

Let \( \mathcal{L} \) be an invertible sheaf over \( J_C \). From \( \mathcal{L} \) we can construct the invertible sheaves \( \mathcal{L}^\alpha = \mu_\alpha^* \mathcal{L} \) over \( J_{ad} \), and we can consider the unique invertible sheaf over \( Y_{\{p\}} \), \( \tilde{\mathcal{L}} \), whose restriction to each \( J_{ad} \) coincides with \( \mathcal{L}^\alpha \). Indeed, from the canonical immersion \( i_0 : J_0 \hookrightarrow Y_{\{p\}} \), we have that \( i_0^* \tilde{\mathcal{L}} \simeq \mathcal{L} \) and, if we set \( i_\alpha : J_{ad} \hookrightarrow Y_{\{p\}} \) to denote the corresponding immersion, we have that \( i_\alpha^* \tilde{\mathcal{L}} \simeq \mathcal{L}^\alpha \).
Let $C_{k-sch.}$ be the category of algebraic $k$-schemes and let $C_{gr.}$ be the category of groups. If $S$ is an arbitrary $k$-scheme, we can consider the functor

$$\tilde{F}: C_{k-sch.} \rightarrow C_{gr.},$$

where $\tilde{F}(S) = \{g \in Y_{(p)}(S) \text{ such that } T_g^*\bar{\mathcal{L}}_S \simeq \bar{\mathcal{L}}_S \otimes \phi_S^*\mathcal{M}\}$, $\mathcal{M}$ being an invertible sheaf over $S$, $\phi_S: Y_{(p)} \times S \rightarrow S$ and $\phi_1: Y_{(p)} \times S \rightarrow Y_{(p)}$ being the respective projections, and $\bar{\mathcal{L}}_S = \phi_1^*\bar{\mathcal{L}}$.

**Theorem 3.3.** The functor $\tilde{F}$ is representable in the category of $k$-schemes and its representant, $K[\bar{\mathcal{L}}]$, is a group subscheme of $\text{Pic}(C)$.

**Proof.** If $\{J_{\beta d}\}_{\beta \in \mathbb{Z}}$ is the covering of $Y_{(p)}$ by the open subschemes defined above, to prove the representability of $\tilde{F}$ is sufficient to see that the functors $\tilde{F} \times J_{\beta d}^*$ are representable.

Moreover, from the isomorphisms of varieties $\mu_{\beta}: J_{\beta d} \sim J_0$, one has that

$$\tilde{F} \times J_{\beta d}^* \simeq \left( \tilde{F} \times J_0^* \right) \times J_{\beta d}^*.$$

Hence, to prove the representability of $\tilde{F}$, one only has to see that $\tilde{F} \times J_0^*$ is representable.

Furthermore, since $\mathcal{L}$ is an invertible sheaf over $J_0$, which is an abelian variety, there exists a group scheme $K[\mathcal{L}] \subseteq J_0$ [13], where

$$K[\mathcal{L}]^*(S) = \{f \in J_0^*(S) \text{ such that } T_f^*\mathcal{L}_S \simeq \mathcal{L}_S \otimes \phi_2^*\mathcal{M}\}$$

for a certain invertible sheaf $\mathcal{M}$ over each $k$-scheme $S$, $\phi_2: J_0 \times S \rightarrow S$ being the natural projection.

Thus, to see the representability of $\tilde{F}$, we only need to prove that

$$\tilde{F} \times J_0^* \simeq K[\mathcal{L}]^*.$$

Accordingly, given an arbitrary $k$-scheme $S$, and setting

$$(i_0)_S: J_0 \times S \leftrightarrow Y_{(p)} \times S$$

to denote the morphism of schemes induced by $i_0$, one has that $(i_0)_S^*\bar{\mathcal{L}}_S \simeq \mathcal{L}_S$. If we also consider $f \in (\tilde{F} \times J_0^*)(S)$, we deduce that $(i_0)_S \circ T_f = T_f \circ (i_0)_S$, with $\tilde{f} \in \tilde{F}(S)$, and it follows from this that there exists a natural injective morphism $\tilde{F} \times J_0^* \xrightarrow{j} K[\mathcal{L}]^*$.

To conclude, we have to see that the map $j$ is surjective. Let us now consider $g \in K[\mathcal{L}]^*(S)$ for a certain $k$-scheme $S$. One has that

$$T_g^*\mathcal{L}_S \simeq \mathcal{L}_S \otimes \phi_2^*\mathcal{M}.$$

If $(\mu_{\beta})_S: J_{\beta d} \sim J_0 \times S$ are the morphisms induced by $\mu_{\beta}$, it follows from the isomorphism of sheaves

$$T_g^*(\mu_{\beta})_S^*\mathcal{L}_S \simeq (\mu_{\beta})_S^*T_g^*\mathcal{L}_S$$

...
that $T^*_g \mathcal{L}^\beta_S \simeq \mathcal{L}^\beta_S \otimes (\phi^\beta_2)^* \mathcal{M}$ with $\mathcal{L}^\beta_S = (\mu^\beta)_S^* \mathcal{L}_S$, $\phi^\beta_2: J_{\beta d} \times S \to S$ being the natural projection.

Finally, since $\tilde{\mathcal{L}}_S = \coprod_{\beta \in \mathbb{Z}} \mathcal{L}^\beta_S$, from the commutative diagrams of morphisms of schemes

$$
\begin{array}{c}
Y_{\{p\}} \times S \downarrow \phi_S \quad \quad \quad \quad J_{\beta d} \times S \xrightarrow{\phi^\beta_2} S,
\end{array}
$$

one has that $T^*_g \tilde{\mathcal{L}}_S \simeq \tilde{\mathcal{L}}_S \otimes \tilde{\phi}^*_S \mathcal{M}$, and, therefore, $g \in (\tilde{\Phi} \times J^*_0)(S)$.

Hence $\tilde{\Phi}$ is representable in the category of $k$-schemes. Moreover, its representant $K[\tilde{\mathcal{L}}]$ is a group subscheme of $Y_{\{p\}}$ and is hence a group subscheme of $\text{Pic}(C)$.

\begin{proof}
The statement is a direct consequence of the fact that the connected component of the identity element of $K[\tilde{\mathcal{L}}]$ is $K[\mathcal{L}]$ which, with the hypothesis of the corollary, is a reduced $k$-scheme.
\end{proof}

\textbf{Corollary 3.4.} If $\mathcal{L}$ is an invertible sheaf over $J_C$ such that $K[\mathcal{L}]$ is reduced, then the group scheme $K[\tilde{\mathcal{L}}]$ is reduced.

\begin{proof}
The statement is a direct consequence of the fact that the connected component of the identity element of $K[\tilde{\mathcal{L}}]$ is $K[\mathcal{L}]$ which, with the hypothesis of the corollary, is a reduced $k$-scheme.
\end{proof}

\subsection{Group Schemes $K[\tilde{\mathcal{L}}^n]$ over Pic($C^n$)}

In the previous section we set $Y_{\{p\}} = \coprod_{\alpha \in \mathbb{Z}} J_{\alpha d}$ to denote the group subscheme of $\text{Pic}(C)$ obtained by translation of the Jacobian $J_0$ by all the points $\mathcal{O}(\alpha p)$.

For each positive integer $n \in \mathbb{N}$, let us again consider the morphism of group schemes $\tilde{\pi}_n: \text{Pic}(C^n) \to \text{Pic} C$ induced by the normalization morphism $\pi_n: C \to C^n$.

\begin{definition}
We shall use $Y^n_{\{p\}}$ to denote the group subscheme of $\text{Pic}(C^n)$ which is the fiber of $Y_{\{p\}}$ by $\tilde{\pi}_n$; i.e. $Y^n_{\{p\}} = \tilde{\pi}^{-1}_n(Y_{\{p\}})$.
\end{definition}

If we set $(Y^n_{\{p\}})_\beta = \tilde{\pi}^{-1}_n(J_{\beta d})$, and $d = \deg(p) = \dim_k k(p)$, we have that

$$
Y^n_{\{p\}} = \prod_{\alpha \in \mathbb{Z}} (Y^n_{\{p\}})_{\alpha d},
$$

and, similar to above, when $p$ is a rational point, one has that $Y^n_{\{p\}} = \text{Pic}(C^n)$.

In particular, we know that $(Y^n_{\{p\}})_0 = \tilde{\pi}^{-1}_n(J_C) = J_{C^n}$ is a group scheme that is an extension of $J_C$ by a unipotent, connected and commutative group scheme $K[n] = \text{Ker } \tilde{\pi}_n$.

Moreover, it follows from translating by a fixed rational point of $\text{Pic}(C^n)$ in the fiber of each invertible sheaf $\mathcal{O}(\alpha p)$, that there exist isomorphisms of schemes

$$
\mu^n_\beta: (Y^n_{\{p\}})_{\beta d} \xrightarrow{\sim} J_{C^n}.
$$

Let us again consider an invertible sheaf $\mathcal{L}$ over $J_C$, and let us denote by $\tilde{\mathcal{L}}$ its extension to $Y_{\{p\}}$, defined previously. If we set $\tilde{\mathcal{L}}^n = (\tilde{\pi}_n)^* \tilde{\mathcal{L}}$ and $\mathcal{L}^n = (\tilde{\pi}_n)^* \mathcal{L}$, keeping the above notations, by construction we have that $\mathcal{L}^n$
is an invertible sheaf over $J_{C^n}$. Furthermore, if $(i^n_\alpha): (Y^n_{\{\alpha\}})_{\text{ad}} \rightarrow \text{Pic}(C^n)$ is the natural immersion, one has that $(i^n_\alpha)^* \tilde{L}^n \simeq (\mathcal{L}^n)^\alpha$ for each $\alpha \in \mathbb{Z}$, where

$$(\mathcal{L}^n)^\alpha = (\pi^n_\alpha)^* \mathcal{L}^n = (\mu^n_\alpha)^* L^n.$$  

Recall from Subsection [23] that, according to the results of [15], there exists a group scheme $K[\mathcal{L}^n] \subseteq J_{C^n}$, whose functor of points is

$$K[\mathcal{L}^n] \cdot (S) = \{f \in J_{C^n}(S) \text{ such that } T_f^* \mathcal{L}^n \simeq \mathcal{L}^n_S \otimes \tilde{\phi}^*_S \mathcal{M}\}.$$  

Let $S$ be an arbitrary $k$-scheme. Analogously to the previous subsection, we define the functor

$$\widetilde{F}_n: \mathcal{C}_{k-sch.} \rightarrow \mathcal{C}_{gr.},$$

where $\widetilde{F}_n(S) = \{f \in (Y^n_{\{p\}})^* \cdot (S) \text{ such that } T_f^* \tilde{L}^n_S \simeq \tilde{L}^n_S \otimes \tilde{\phi}^*_n \mathcal{M}, \mathcal{M} \text{ is an invertible sheaf over } S, \tilde{\phi}_n: Y^n_{\{p\}} \times S \rightarrow S \text{ and } \tilde{\phi}_n: Y^n_{\{p\}} \times S \rightarrow Y^n_{\{\alpha\}} \text{ are the respective projections, and } \tilde{L}^n_S = \tilde{\phi}^*_n \tilde{L}^n\}$.

**Theorem 3.6.** For each integer $n > 1$, the functors $\widetilde{F}_n$ are representable in the category of $k$-schemes.

Their respective representants are group schemes $K[\tilde{\mathcal{L}}^n] \rightarrow \text{Pic}(C^n)$.

**Proof.** Fixing a positive integer $n > 1$, let us consider the covering of $Y^n_{\{p\}}$ by the open subschemes $\{Y^n_{\{p\}}\}_{\beta \in \mathbb{Z}}$, such that $(Y^n_{\{p\}})_{\beta \in \mathbb{Z}}$ is the fiber of the Jacobian $J_{\beta \in \mathbb{Z}}$ by the morphism $\pi_n$.

From the isomorphisms $\mu^n_\beta: (Y^n_{\{p\}})_{\beta \in \mathbb{Z}} \rightarrow J_{C^n}$, and with similar arguments to the proof of Theorem 3.3, one has that

$$\widetilde{F}_n \times (Y^n_{\{p\}}) \cdot J_{C^n} \simeq K[\mathcal{L}^n] \cdot$$

and, hence,

$$\widetilde{F}_n \times (Y^n_{\{p\}}) \cdot J_{C^n} \simeq K[\mathcal{L}^n] \cdot (Y^n_{\{p\}}) \cdot$$

Therefore $\widetilde{F}_n$ is representable because the functors $\widetilde{F}_n \times (Y^n_{\{p\}}) \cdot$ are representable and $\{(Y^n_{\{p\}})_{\beta \in \mathbb{Z}} \}$ is an open covering of $Y^n_{\{p\}}$ (3). We shall use $K[\tilde{\mathcal{L}}^n]$ to denote its representant, which is a group subscheme of Pic$(C^n)$.

**Corollary 3.7.** If $\mathcal{L}$ is an invertible sheaf over $J_C$ such that $K[\mathcal{L}]$ is reduced, one has that the group scheme $K[\tilde{\mathcal{L}}^n]$ is reduced for all $n > 1$.

**Proof.** It is known that the group scheme $K[\mathcal{L}^n]$ is reduced ([15]) and therefore we are done, bearing in mind that this group scheme is the connected component of the identity element of the $k$-scheme $K[\tilde{\mathcal{L}}^n]$.

Our purpose now is to determine the relation between the group scheme $K[\tilde{\mathcal{L}}]$, defined in the above subsection, and the group schemes $K[\tilde{\mathcal{L}}^n]$. 

Lemma 3.8. For each separated $k$-scheme $S$, if $\overline{\pi}_{n,S} : Y_{\{p\}}^n \times S \to Y_{\{p\}} \times S$ is the morphism of group schemes induced by $\overline{\pi}_n$, then the natural morphism of groups

$$\overline{\pi}_{n,S}^* : \text{Pic}(Y_{\{p\}} \times S) \to \text{Pic}(Y_{\{p\}}^n \times S)$$

is injective.

Proof. With the above notations, one has that

$$Y_{\{p\}} \times S = \bigsqcup_{\alpha \in \mathbb{Z}} (Y_{\{p\}})_{\alpha d} \times S \quad \text{and} \quad Y_{\{p\}}^n \times S = \bigsqcup_{\beta \in \mathbb{Z}} (Y_{\{p\}}^n)_{\beta d} \times S,$$

and there exist isomorphisms of schemes

$$\mu_\alpha : (Y_{\{p\}})_{\alpha d} \xrightarrow{\sim} J_C \quad \text{and} \quad \mu_\beta : (Y_{\{p\}}^n)_{\beta d} \xrightarrow{\sim} J_{C^n}.$$

Moreover, from Lemma 3.8 one has that the natural morphism of groups

$$\text{Pic}(J_C \times S) \to \text{Pic}(J_{C^n} \times S)$$

is injective for each separated $k$-scheme $S$.

Thus, the morphism of groups $\overline{\pi}_{n,S}^* : \text{Pic}(Y_{\{p\}} \times S) \to \text{Pic}(Y_{\{p\}}^n \times S)$ is injective because it is in each component.

Proposition 3.9. If $S$ is a separated $k$-scheme, one has that

$$[K[\overline{\mathcal{L}}^\bullet \times Y_{\{p\}}^n]^*](S) \simeq K[\overline{\mathcal{L}}^n]^*(S).$$

Proof. Given an arbitrary $k$-scheme $S$, one has that

$$[K[\overline{\mathcal{L}}^\bullet \times Y_{\{p\}}^n]^*](S) \xrightarrow{i} K[\overline{\mathcal{L}}^n]^*(S)$$

because if $f \in [K[\overline{\mathcal{L}}^\bullet \times Y_{\{p\}}^n]^*](S) \subseteq (Y_{\{p\}}^n)^* \subseteq (Y_{\{p\}}^n)^*(S)$, from the commutative diagram of morphisms of schemes

$$\begin{array}{ccc}
Y_{\{p\}}^n \times S & \xrightarrow{\overline{\pi}_{n,S}^*} & Y_{\{p\}}^n \times S \\
\downarrow{\overline{\pi}_{n,S}} & & \downarrow{\overline{\pi}_{n,S}} \\
Y_{\{p\}} \times S & \xrightarrow{\overline{T}_n(f)} & S,
\end{array}$$

there exists the sequence of isomorphisms

$$T_f^* \overline{\mathcal{L}}_S^n \simeq \overline{\pi}_{n,S}^*(T_{\overline{T}_n(f)}^* \overline{\mathcal{L}}_S) \simeq \overline{\pi}_{n,S}^* (\overline{\mathcal{L}}_S \otimes \overline{\phi}_S^* \mathcal{N}) \simeq \overline{\mathcal{L}}_S^n \otimes \overline{\phi}_{n,S}^* \mathcal{N}$$

for a certain invertible sheaf $\mathcal{N}$ over $S$, and one deduces that $f \in K[\overline{\mathcal{L}}^n]^*(S)$.

Thus, to conclude we must see that the map $i$ is surjective when $S$ is a separated $k$-scheme.

Let us consider $g \in K[\overline{\mathcal{L}}^n]^*(S)$, such that $T_g^* \overline{\mathcal{L}}_S^n \simeq \overline{\mathcal{L}}_S^n \otimes \overline{\phi}_{n,S}^* \mathcal{M}$, $\mathcal{M}$ being an invertible sheaf over $S$.

Bearing in mind that $T_g^* (\overline{\pi}_{n,S}^* \overline{\mathcal{L}}_S) \simeq \overline{\pi}_{n,S}^* (T_{\overline{T}_n(g)}^* \overline{\mathcal{L}}_S)$, from Lemma 3.8 one has that

$$T_{\overline{T}_n(g)}^* (\overline{\mathcal{L}}_S) \simeq (\overline{\mathcal{L}}_S \otimes \overline{\phi}_S^* \mathcal{M}) \otimes \overline{\phi}_{n,S}^* \mathcal{M}' \simeq \overline{\mathcal{L}}_S \otimes \overline{\phi}_S^* \mathcal{M},$$
with \( N \cong M \otimes M' \), and \( M' \) being an invertible sheaf over \( S \).

Then, \( \bar{\pi}_n(g) \in K[\bar{L}]^\bullet(S) \), i.e. \( g \in [K[\bar{L}]^\bullet \times (Y_{\{p\}}^n)^\bullet](S) \).

**Proposition 3.10.** For each \( n > 1 \), if \( K[n] = \text{Ker} \bar{\pi}_n \), there exists an exact sequence of group schemes

\[
0 \to K[n] \to K[\bar{L}^n] \xrightarrow{\bar{\varpi}_n} K[\bar{L}] \to 0.
\]

Hence the natural morphism \( K[\bar{L}^n] \xrightarrow{\bar{\varpi}_n} K[\bar{L}] \) is a geometric quotient of group schemes by the action of \( K[n] \), which is a unipotent, connected and commutative group scheme.

**Proof.** From the morphisms of schemes

\[
0 \to K[n] \to Y_{\{p\}}^n \xrightarrow{\bar{\varpi}_n} Y_{\{p\}} \to 0,
\]

and bearing in mind that functors of points are sheaves for the faithfully flat and quasi-compact topology, the claim is deduced immediately from the previous proposition. \( \square \)

### 3.3. Group Schemes \( K[(\bar{L}_\Theta)^n] \).

If \( g \) is the genus of \( C \), let us now consider the invertible sheaf \( \mathcal{L}_\Theta^{g-1} \) associated with the principal polarization of the Jacobian \( J_{g-1} \subseteq \text{Pic} C \), which is defined canonically. Fixing a rational point in the curve \( C \), we can translate \( \mathcal{L}_\Theta^{g-1} \) to the Jacobian \( J_C \), and we shall use \( \mathcal{L}_\Theta \) to denote this principal polarization.

Moreover, if \( d = \deg(p) \), and \( \mu_\alpha : J_{ad} \xrightarrow{\sim} J_C \) is again the isomorphism of schemes obtained by translation by \( \mathcal{O}(-\alpha p) \), we shall use \( \bar{\mathcal{L}}_\Theta \) to denote the unique invertible sheaf over \( Y_{\{p\}} \) which coincides with \( \mathcal{L}_\Theta^\alpha = \mu_\alpha^* \mathcal{L}_\Theta \) when we restrict to \( J_{ad} \).

If \( (\mathcal{L}_\Theta)^n) = \pi_n^* \bar{\mathcal{L}}_\Theta \), bearing in mind the previous subsection, there exist group schemes \( K[(\bar{L}_\Theta)^n] \subseteq Y_{\{p\}}^n \), whose structure will be determined in this subsection.

**Lemma 3.11.** One has that:

\[
K[\bar{L}_\Theta]^\bullet(\text{Spec } k) \simeq \{ \mathcal{O}(\alpha p) \}_{\alpha \in \mathbb{Z}} \subseteq \text{Pic}(C)^\bullet(\text{Spec } k),
\]

i.e., \( K[\bar{L}_\Theta]^\bullet(\text{Spec } k) \simeq \mathbb{Z} \).

**Proof.** From the definition of \( K[\bar{L}_\Theta] \), we have that

\[
K[\bar{L}_\Theta]^\bullet(\text{Spec } k) = \{ x \in Y_{\{p\}}^\bullet(\text{Spec } k) \text{ such that } T_x^* (\bar{L}_\Theta) \simeq (\bar{L}_\Theta) \}
\]

\[
= \{ x \in Y_{\{p\}}^\bullet(\text{Spec } k) \text{ such that } T_x^* (\mathcal{L}_\Theta^\alpha) \simeq \mathcal{L}_\Theta \text{ for a certain } \alpha \in \mathbb{Z} \}
\]

\[
= \{ x \in Y_{\{p\}}^\bullet(\text{Spec } k) \text{ such that } T_x^* (\mathcal{L}_\Theta^\alpha (\mathcal{O}(-\alpha p) \mathcal{L}_\Theta)) \simeq \mathcal{L}_\Theta \text{ for a certain } \alpha \in \mathbb{Z} \},
\]

where \( \mathcal{L}_\Theta \) is the sheaf associated with the principal polarization of \( J_C \).

Thus, since \( K[\mathcal{L}_\Theta]^\bullet(\text{Spec } k) = \{ e \} \), one concludes that

\[
K[\bar{L}_\Theta]^\bullet(\text{Spec } k) \simeq \{ \mathcal{O}(\alpha p) \}_{\alpha \in \mathbb{Z}}.
\]

\( \square \)
Proposition 3.12. If we consider the exact sequence of groups

\[ 0 \to K[n]^\cdot \text{Spec } k \to K[(\widehat{\mathcal{L}_0})^n]^\cdot \text{Spec } k \xrightarrow{\pi_n} K[\widehat{\mathcal{L}_0}]^\cdot \text{Spec } k \to 0, \]

there exists a natural section \( \sigma: K[\widehat{\mathcal{L}_0}]^\cdot \text{Spec } k \to K[(\widehat{\mathcal{L}_0})^n]^\cdot \text{Spec } k \) of the morphism \( \pi_n \).

Proof. If \( \pi_n: C \to C^n \) is the normalization morphism, one has that \( \pi_n \) coincides with \( \pi_n^* \) over the invertible sheaves of the singular curve \( C^n \).

It follows from Lemma 3.11 that \( K[\widehat{\mathcal{L}_0}]^\cdot \text{Spec } k = \{ \mathcal{O}(\alpha p) \}_{\alpha \in \mathbb{Z}} \), where \( \mathcal{O}(\alpha p) \) is the invertible sheaf associated with the Cartier divisor

\[ D_{\alpha} = \{(t_{\alpha}^p, U_1), (1, U_i) \}_{i \neq 1}, \]

taking \( t_p \) being the generator of the maximal ideal \( m_p \subseteq \mathcal{O}_p \), and \( \{ U_i \}_{i \in I} \) being a covering of \( C \) such that \( p \) is only included in \( U_1 \); hence, we can identify \( \mathcal{O}(\alpha p) \) with \( t_{\alpha}^p \).

Moreover, since \( \Sigma^C = \Sigma^{C_0} \), the Cartier divisor \( D_{\alpha} = \{(t_{\alpha}^p, U_1), (1, U_i) \} \) determines over \( C^n \) an invertible sheaf \( \mathcal{L}_{\alpha p}^C \).

Furthermore, \( \mathcal{L}_{\alpha p}^C(U_1) \simeq t_{\alpha}^C \cdot \mathcal{O}_{C_0}(U_1), \mathcal{O}(\alpha p)(U_1) \simeq t_{\alpha}^p \cdot \mathcal{O}_{C}(U_1) \), and for \( i \neq 1 \), there also exist isomorphisms \( \mathcal{L}_{\alpha p}^C(U_i) \simeq \mathcal{O}_{C_0}(U_i) \) and \( \mathcal{O}(\alpha p)(U_i) \simeq \mathcal{O}_{C}(U_i) \).

By construction, it is clear that \( \mathcal{O}(\alpha p) \simeq \pi_n^* \mathcal{L}_{\alpha p}^C \), and hence there exists a section of the morphism \( \pi_n \). \( \square \)

By considering the category of locally noetherian schemes, we shall now prove that \( K[(\widehat{\mathcal{L}_0})^n] \) is a direct product, where one of the factors is the connected component of its identity element.

Let us set \( \mathbb{Z}_* = \coprod_{\alpha \in \mathbb{Z}} \text{Spec } k \). For each locally noetherian scheme \( S \), one has that

\[ \mathbb{Z}_*(S) = \text{Map}_{\text{cont.}}(S, \mathbb{Z}), \]

where \( \mathbb{Z} \) is a discrete topological space. Moreover, we have a structure of group scheme in \( \mathbb{Z}_* \) that coincides, when we consider points on connected \( k \)-scheme, with the group structure of \( \mathbb{Z} \).

Theorem 3.13. For each \( n \in \mathbb{N} \), one has an isomorphism of group schemes

\[ K[(\widehat{\mathcal{L}_0})^n] \simeq K[(\widehat{\mathcal{L}_0})^n]_0 \times \mathbb{Z}_*, \]

where \( K[(\widehat{\mathcal{L}_0})^n]_0 \) is the connected component of the identity element of the scheme \( K[(\widehat{\mathcal{L}_0})^n] \).

Proof. Let us use \( \lambda_{\alpha}^n : K[(\widehat{\mathcal{L}_0})^n] \xrightarrow{\sim} K[(\widehat{\mathcal{L}_0})^n] \) to denote the isomorphism of schemes induced by translation by the rational point \( \mathcal{L}_{\alpha p}^C \).

Moreover, since the schemes of the statement are locally noetherian, the sets of their connected components are locally finite, and they are therefore locally connected (\( \mathbb{F} \), page 50). Hence, their connected components are open subschemes, and to prove the theorem it is sufficient to see that there exist isomorphisms when we consider \( S \)-valued points, \( S \) being a connected \( k \)-scheme.
For a connected $k$-scheme $S$, we consider the map

$$
\varphi: (K[([\widetilde{L}_0)^n]_0 \times \mathbb{Z}_s)^\bullet(S) \rightarrow K[([\widetilde{L}_0)^n]_0^\bullet(S),
$$

defined by $\varphi(f, \alpha) = \lambda^{\alpha}_n \circ f$.

Since each $S$-valued point of $K[([\widetilde{L}_0)^n]$, $S$ being a connected $k$-scheme, takes values in a unique connected component, which is determined by an integer number $\alpha$, one has that the map $\varphi$ is bijective.

Furthermore, it follows from the isomorphism of invertible sheaves $L_n^\alpha = L_n^{(\alpha + \beta)}$, that $\varphi$ is a morphism of groups for each locally noetherian $k$-scheme $S$, and one concludes that $K[([\widetilde{L}_0)^n]_0 \times \mathbb{Z}_s$ are isomorphic as group schemes.

**Corollary 3.14.** One has that

$$
K[([\widetilde{L}_0)^n]_0 \times \mathbb{Z}_s 
$$

Proof. Bearing in mind that $K[([\widetilde{L}_0)^n]_0 = K[([L_0)^n] = \{e\}$, from Proposition 3.10 we have the existence of an isomorphism of groups $K[([\widetilde{L}_0)^n]_0 \simeq K[n]$, and the claim is deduced. \qed

**4. Generalized Local Symbols**

4.1. The Contou-Carrère symbol associated with a rational point.

Let $C$ be a complete, irreducible and non-singular curve over a perfect field $k$ and let $p \in C$ be a rational point. We set:

$$
K_\{p\} = \lim_{\leftarrow} K[([\widetilde{L}_0)^n],
$$

$K[([\widetilde{L}_0)^n])$ being the group scheme constructed in the previous section.

Bearing in mind that $K[([\widetilde{L}_0)^n]_0 \simeq K[n] \times \mathbb{Z}_s$ (Corollary 3.14), where $K[n]$ is the kernel of the natural morphism of group schemes Pic$(C^n) \rightarrow$ Pic$(C)$, from Corollary 2.5 one has that

$$
K_\{p\} \simeq \mathbb{Z}_s \times K[\infty] \simeq \mathbb{Z}_s \times \Gamma_+,
$$

$\Gamma_+$ being the representant of the functor on groups:

$$
S \rightsquigarrow \Gamma_+(S) = \left\{\text{series } 1 + a_1 z + a_2 z^2 + \ldots \right\},
$$

where $a_i \in H^0(S, \mathcal{O}_S)$.

Moreover, the groups schemes $\mathbb{Z}_s$ and $\mathbb{G}_m$ are autodual and the universal character coincides, for each connected $k$-scheme $S$, with the morphism of groups

$$
\mathbb{Z}_s^\bullet(S) \times \mathbb{G}_m^\bullet(S) \rightarrow \mathbb{G}_m^\bullet(S)
$$

$(\alpha, \lambda) \mapsto \lambda^\alpha$.

Let us also consider the formal group scheme $\Gamma_-$, whose functor of points is

$$
S \rightsquigarrow \Gamma_-(S) = \left\{\text{series } a_n z^{-n} + \cdots + a_1 z^{-1} + 1 \right\},
$$

where $a_i \in H^0(S, \mathcal{O}_S)$ are nilpotents and $n$ is arbitrary.
It follows from the duality between the group schemes $\Gamma_+$ and $\Gamma_-$ ([9], page 500) that $\hat{K}_p \simeq \mathbb{G}_m \times \Gamma_-$, $K_\Theta^{-}(\mathbb{Z}, \mathbb{Z})$ being the Cartier dual of the group scheme $K_\Theta^\Theta$.

Let us now set $\Gamma_p = K_\Theta^\Theta \times \hat{K}_p^\Theta$. We have that

$$\Gamma_p \simeq \mathbb{Z}^* \times \Gamma_+ \times \mathbb{G}_m \times \Gamma_-,$$

and therefore $\Gamma_p$ is a locally connected group scheme.

If $\hat{O}_p$ is the completion of the local ring $O_p$ and $(\hat{O}_p)^{(0)}$ is the field of fractions of $\hat{O}_p$, for each connected $k$-scheme $S$ one has that

$$\Gamma_p(S) \simeq H^0(S, O_S)((z))^*,$$

and hence $\Gamma_p^*(\text{Spec } k) \simeq k((z))^* \simeq (\hat{O}_p)^{(0)}$, which implies that

$$\Sigma^*_C \hookrightarrow \Gamma_p^*(\text{Spec } k).$$

**Definition 4.1.** We shall use the term Heisenberg group scheme associated with $\Gamma_p$, denoting this by $H(\Gamma_p)$, to refer to the scheme

$$H(\Gamma_p) = \mathbb{G}_m \times K_\Theta^\Theta \times \hat{K}_p^\Theta,$$

together with the group law

$$(\alpha, x, l) \cdot (\alpha', x', l') = (\alpha \cdot \alpha' \cdot l(x'), x \cdot x', l \cdot l')$$

for $S$-valued points, $S$ being a $k$-scheme.

If $e_{H(\Gamma_p)}(x, y)$ is the commutator in the Heisenberg group scheme associated with $\Gamma_p$, and since $\Gamma_p \simeq \hat{\Gamma}_p$, the map

$$\varphi: \Gamma_p \rightarrow \hat{\Gamma}_p,$$

$$x \mapsto e_{H(\Gamma_p)}(x, )$$

is an isomorphism of groups, because if $x = (a, b) \in K_\Theta^\Theta \times \hat{K}_p^\Theta$, then

$$\varphi(a, b) = (a^{-1}, b).$$

Therefore, $H(\Gamma_p)$ satisfies the characterization of a Heisenberg group of an extension by the multiplicative group ([12], page 2).

Furthermore, since $e_{H(\Gamma_p)}$ is a 2-cocycle, it determines an element of the cohomology group $H^2_{\text{reg}}(\Gamma_p, \mathbb{G}_m)$, which contains the classes of 2-cocycles that are morphisms of schemes. We shall now recall from [14] the definition, from this cohomology class, of the Contou-Carrère symbol as a morphism of schemes.

**Definition 4.2.** If $S$ is a connected $k$-scheme and $f \in \Gamma_p^*(S) \simeq \mathbb{Z}^*(S) \times (\Gamma_+ \times \mathbb{G}_m \times \Gamma_-)^*(S)$, we shall call its component in $\mathbb{Z}^*(S)$, which is an integer number, $v(f)$, “the valuation of $f$.”

Bearing in mind that $\Gamma_p = \hat{\Gamma}$, $\hat{\Gamma}$ being the formal group scheme studied in [13], it follows from [14], Theorem 3.5, that:

**Theorem 4.3.** There exists a unique element $(f, g)$ in the cohomology class $[e_{H(\Gamma_p)}] \in H^2_{\text{reg}}(\Gamma_p, \mathbb{G}_m)$ satisfying the conditions:

- $(f, g \cdot g')(p) = (f, g)(p) \cdot (f, g')(p)$.
\( (f, g)_p = e_{\hat{H}(\mathbb{F}_p)}(f, g) \) if \( v(f) = 0 \);
\( (f, -f)_p = 1 \)

for \( f, g, g' \in \Gamma_p^*(S) \), with \( S \) a connected \( k \)-scheme.

**Corollary 4.4.** If \( S \) is a connected \( k \)-scheme and \( u, w \in \Gamma_p^*(S) \) with
\[
\begin{aligned}
  u &= \lambda z^n \prod_{i=1}^h (1 - a_i z^{-i}) \prod_{j=1}^\infty (1 - a_i z^{-i}) \\
  w &= \mu z^m \prod_{j=1}^h (1 - b_j z^{-j}) \prod_{i=1}^\infty (1 - b_j z^{-j}),
\end{aligned}
\]
where \( \lambda, \mu \in H^0(S, \mathcal{O}_S)^* \), \( a_i, b_j \in H^0(S, \mathcal{O}_S) \), \( v(u) = n \), \( v(w) = m \), and \( a_i, b_j \) are nilpotent elements of \( H^0(S, \mathcal{O}_S) \), one has that
\[
(u, w)_p = (-1)^{n \cdot m} \left( \frac{\lambda^m \prod_{i=1}^\infty \prod_{j=1}^h (1 - a_i^{j/(i,j)}) b_j^{(i,j)} (i,j)}{\mu^n \prod_{j=1}^\infty \prod_{i=1}^h (1 - b_j^{j/(i,j)}) a_i^{(i,j)} (i,j)} \right),
\]
where, finitely, many of the terms appearing in the products differ from 1 in \( \mathbb{G}_m^*(S) = H^0(S, \mathcal{O}_S)^* \). This symbol is a generalization of the Contou-Carrére symbol associated with the rational point \( p \).

**Corollary 4.5.** If we add the condition \( \text{char}(k) = 0 \) to the hypothesis of Corollary 4.4, we have that
\[
(u, w)_p = (-1)^{n \cdot m} \left( \frac{\lambda^m \cdot \exp(\sum_{i>0} (\delta_i(u) \cdot \delta_{-i}(w)/i))}{\mu^n \cdot \exp(\sum_{i>0} (\delta_{-i}(u) \cdot \delta_i(w)/i))} \right),
\]
where \( \delta_i(f) = \text{Res}(z^i \cdot \frac{df}{dz}) \). This expression was offered by C. Contou-Carrère in \( \text{[4]} \), Corollary 4.5.

**Remark 4.6.** By considering \( \text{Spec} k \)-valued points in the above morphism of schemes, if \( f, g \in \Sigma_C \) one has that
\[
(f, g)_p = (-1)^{v_p(f) \cdot v_p(g)} \frac{f_{\mathbb{F}_p}(g)}{g_{\mathbb{F}_p}(f)}(p),
\]
which is the expression of the multiplicative local symbol defined by J.-P. Serre \( \text{[22]} \). We should note that the definition offered is “local” because we do not need to use the reciprocity law \( \prod_{p \in C} (f, g)_p = 1 \) to determine the uniqueness of the symbol. Indeed, we only need a closed point \( p \in C \) of a complete, irreducible and non-singular curve to define it. Moreover, the conditions that appear in the definition are natural in the theory of local symbols.

**4.2. Generalization of the Contou-Carrére symbol associated with a closed point.** Let us now consider a complete, irreducible and non-singular curve \( C \) over a perfect field \( k \) and with a rational point, and let \( p \in C \) an arbitrary closed point. Analogously to the previous section, we set
\[
K_{\Theta(p)} = \lim_n K[(\mathcal{O}_C)^n].
\]
If $k(p)$ is the residue class field of the closed point $p$, it follows from Theorem 3.13 and Corollary 2.5 that $K^\Theta_{\{p\}} \simeq \mathbb{Z}_s \times (\Gamma_+)(k(p))$, with

$$(\Gamma_+)^{\cdot}_{k(p)}(S) = \Gamma_+^\cdot(S \times \text{Spec } k(p))$$

for each $k$-scheme $S$.

Let $(\Gamma_-)_{k(p)}$ and $(\mathbb{G}_m)_{k(p)}$ be the group schemes constructed in [14], whose functors of points are

$$(\Gamma_-)^{\cdot}_{k(p)}(S) = \Gamma_-^\cdot(S \times \text{Spec } k(p))$$

$$(\mathbb{G}_m)^{\cdot}_{k(p)}(S) = \mathbb{G}_m^\cdot(S \times \text{Spec } k(p))$$

for each $k$-scheme $S$. We can thus consider the group scheme $\tilde{K}^\Theta_{\{p\}} = (\mathbb{G}_m)_{k(p)} \times (\Gamma_-)_{k(p)}$ and set $\tilde{\Gamma}_p = K^\Theta_{\{p\}} \times \tilde{K}^\Theta_{\{p\}}$, which is a locally connected group scheme. Also, for each $k$-scheme $S$ one has that

$$\tilde{\Gamma}_p(S) \simeq (H^\Theta_k(S, \mathcal{O}_S) \otimes k(p))(\langle z \rangle)^\cdot,$$

and therefore $\tilde{\Gamma}_p(\text{Spec } k) \simeq k(p)(\langle z \rangle)^\cdot \simeq (\hat{\mathcal{O}}_p)^\cdot(\langle y \rangle)$.

Furthermore, from the natural immersion of functors of groups

$$\mathbb{Z}_s^\cdot(S) \hookrightarrow (\mathbb{Z}_s)^{\cdot}_{k(p)}(S) = \mathbb{Z}_s^\cdot(S \times \text{Spec } k(p)),$$

and the duality referred to in Subsection 4.1, one has defined a morphism of schemes:

$$\chi_p : K^\Theta_{\{p\}} \times \tilde{K}^\Theta_{\{p\}} \longrightarrow (\mathbb{G}_m)_{k(p)}.$$

If we now denote by $N_{k(p)/k} : (\mathbb{G}_m)_{k(p)} \rightarrow \mathbb{G}_m$ the morphism of group schemes induced by the norm of the extension of fields $k \hookrightarrow k(p)$, we have the morphism:

$$\tilde{\chi} : K^\Theta_{\{p\}} \times \tilde{K}^\Theta_{\{p\}} \longrightarrow \mathbb{G}_m,$$

defined by

$$\tilde{\chi}(f, g) = N_{k(p)/k}(\chi_p(f, g)).$$

Using $\tilde{H}(\tilde{\Gamma}_p)$ to denote the group scheme $\mathbb{G}_m \times K^\Theta_{\{p\}} \times \tilde{K}^\Theta_{\{p\}}$ characterized by the group law

$$(\alpha, f, g) \cdot (\alpha', f', g') = (\alpha \cdot \alpha', \tilde{\chi}(f', g), f \cdot f', g \cdot g')$$

for $S$-valued points, $S$ being a $k$-scheme, keeping the notations of the above section and bearing in mind that $\tilde{\Gamma}_p = \tilde{\Gamma}_{k(p)}$, with $\tilde{\Gamma}_{k(p)}$ the group scheme constructed in [14], Section 2.B, and with a equivalent definition of $v$ to the offered above, one has that:

**Theorem 4.7.** There exists a unique element $\langle \ , \ , \rangle_p \in H^\cdot_2(\tilde{\Gamma}_p, \mathbb{G}_m)$ satisfying the conditions:

- $(f, g, g')_p = (f, g)_p \cdot (f, g')_p$;
- $(f, g)_p = e_{\tilde{H}(\tilde{\Gamma}_p)}(f, g)$ if $v(f) = 0$;
- $(f, -f)_p = 1$

for $f, g, g' \in \Gamma_p^\cdot(S)$, with $S$ a connected $k$-scheme. This element is a generalization of the Contou-Carrère symbol associated with the closed point $p$. 

Corollary 4.8. If $S$ is a connected $k$-scheme and $u, w \in \Gamma_p^*(S)$ with

$$u = \lambda z^n \prod_{i=1}^t (1 - a_i z^{-i}) \prod_{i=1}^\infty (1 - a_i z^i)$$

$$w = \mu z^m \prod_{j=1}^h (1 - b_j z^{-j}) \prod_{j=1}^\infty (1 - b_j z^j),$$

where $\lambda, \mu \in [H^0(S, \mathcal{O}_S) \otimes_k k(p)]^*$, $a_i, b_j \in H^0(S, \mathcal{O}_S) \otimes_k k(p)$, $v(u) = n$, $v(w) = m$, and $a_i, b_j$ are nilpotent elements of $H^0(S, \mathcal{O}_S) \otimes_k k(p)$, one has that

$$(u, w)_p = (-1)^{n \cdot m \cdot \deg(p)} N_k(p/k) \left( \frac{\lambda^n \prod_{i=1}^\infty \prod_{j=1}^h (1 - a_i^{j/(i,j)} b_j^{-j/(i,j)})}{\mu^m \prod_{j=1}^\infty \prod_{i=1}^h (1 - b_j^{j/(i,j)} a_i^{-j/(i,j)})} \right),$$

where, finitely, many of the terms appearing in the products differ from 1 in $G_m^*(S) = H^0(S, \mathcal{O}_S)^*$.

Corollary 4.9. If we add the condition $\text{char}(k) = 0$ to the hypothesis of the previous corollary, we have that

$$(u, w)_p = (-1)^{n \cdot m \cdot \deg(p)} N_k(p/k) \left( \frac{\lambda^n \cdot \exp(\sum_{i>0} (\delta_i(u) \cdot \delta_i(w)/i))}{\mu^m \cdot \exp(\sum_{i>0} (\delta_{-i}(u) \cdot \delta_i(w)/i))} \right),$$

where $\delta_i(f) = \text{Res}(z^i \cdot f')$.

4.3. Generalized Contou-Carrère symbol associated with a morphism. Keeping the notations of the previous subsection, let $\varphi: \mathbb{G}_m \to G$ be a morphism of group schemes, $G$ being an algebraic group scheme.

Let us now consider the group scheme $\mathcal{H}_\varphi(\Gamma_p) = G \times K^\Theta_{\{p\}} \times K^\Theta_{\{p\}}$, characterized by the group law

$$(a, f, g) \cdot (a', f', g') = (a \cdot a' \cdot \bar{\chi}_\varphi(f', g), f \cdot f', g \cdot g')$$

for $S$-valued points on a connected $k$-scheme, and with $\bar{\chi}_\varphi = \varphi \circ \bar{\chi}$.

If we denote by $e_{\mathcal{H}_\varphi(\Gamma_p)}$ the commutator of the induced extension of group schemes

$$0 \to G \to \mathcal{H}_\varphi(\Gamma_p) \to \Gamma_p \to 0,$$

similar to the above cases, one has that:

**Theorem 4.10.** There exists a unique element $(\ , \ )_p^\varphi$ in the cohomology class $[e_{\mathcal{H}_\varphi(\Gamma_p)}] \in H^2_{\text{reg}}(\Gamma_p, G)$ satisfying the conditions:

- $(f, g, g')_p^\varphi = (f, g)_p^\varphi \cdot ((f, g')_p^\varphi)$;
- $(f, g)_p^\varphi = e_{\mathcal{H}_\varphi(\Gamma_p)}(f, g)$ if $v(f) = 0$;
- $(f, -f)_p^\varphi = 1$

for $f, g, g' \in \Gamma_p^*(S)$, with $S$ a connected $k$-scheme. This element is a generalization of the Contou-Carrère symbol associated with the morphism $\varphi: \mathbb{G}_m \to G$.

Corollary 4.11. If $S$ is a connected $k$-scheme and $u, w \in \Gamma_p^*(S)$ with

$$u = \lambda z^n \prod_{i=1}^t (1 - a_i z^{-i}) \prod_{i=1}^\infty (1 - a_i z^i)$$

$$w = \mu z^m \prod_{j=1}^h (1 - b_j z^{-j}) \prod_{j=1}^\infty (1 - b_j z^j),$$

being an algebraic group scheme.
where \( \lambda, \mu \in [H^0(S, \mathcal{O}_S) \otimes_k k(p)]^* \), \( a_i, b_j \in H^0(S, \mathcal{O}_S) \otimes_k k(p) \), \( v(u) = n \), \( v(w) = m \), and \( a_{-i}, b_{-j} \) are nilpotent elements of \( H^0(S, \mathcal{O}_S) \otimes_k k(p) \), one has that

\[
(u, w)^\varphi_p = (g_{-1})^{n - \deg(p)} \varphi \left( \frac{N_{k(p)/k}}{\mu^n \prod_{i=1}^{\infty} \prod_{j=1}^{h} \left( 1 - \frac{a_{j(i,j)} b_{j(i,j)}^{j(i,j)}}{a_{-i}} \right)} \right),
\]

where, finitely, many of the terms appearing in the products differ from 1 in \( G^* \). (Reciprocity Law)

**Corollary 4.12.** Let \( C \) be a complete, irreducible, non-singular, and with a rational point curve over a perfect field \( k \). For each morphism of group schemes \( \varphi : G_m \to G \), and each artinian local finite \( k \)-algebra \( B \), one has that

\[
\prod_{p \in C} \left( f, g \right)^\varphi_p = 1 \text{ for all } f, g \in (\Sigma_C \otimes_k B)^*,
\]

where, finitely, many of the terms appearing in the products differ from 1 in \( G^*(\text{Spec } B) \).

**Proof.** From the natural immersion \((\Sigma_C \otimes_k B)^* \hookrightarrow \tilde{\Gamma}_p^*(\text{Spec } B)\), and following the method of [2], the claim is a direct consequence of the statements proved in [13]. \( \square \)

Henceforth, \( C \) will be an irreducible, complete, non-singular and with a rational point curve over a finite field \( k \) that contains the \( m \)-th roots of unity, with \( \# k = q \).

Let us now consider \( \phi_N \), the character associated with the integer number \( N \in \mathbb{Z} \),

\[
\phi_N : G_m \to G_m \quad \lambda \mapsto \lambda^N
\]

where \( \lambda \in G_m^*(S) \), with \( S \) a connected \( k \)-scheme.

**Lemma 4.13.** For each closed point \( p \in C \) and each artinian local finite \( k \)-algebra \( B \), there exists a positive integer number \( N \) such that the morphism of group schemes

\[
(\ , )_p^\phi : \Gamma_p \times \Gamma_p \to G_m
\]

satisfies a factorization

\[
\tilde{\Gamma}_p^*(\text{Spec } B) \times \tilde{\Gamma}_p^*(\text{Spec } B) \xrightarrow{(\ , )_p^\phi} G_m^*(\text{Spec } B) \xrightarrow{\mu_m^*} \mu_m^*(\text{Spec } B)
\]

where \( \mu_m^* \) is the \( k \)-scheme of \( m \)-th roots of unity and \( i_B \) is the natural immersion.

**Proof.** If \( \alpha = \sharp B^* \), since \( \mu_m^*(\text{Spec } B) = \{ b \in B^* \text{ such that } b^m = 1 \} \), and since \( m \) divides \( \alpha \) because \( k^* \) is a subgroup of \( B^* \), the statement follows from considering the integer number \( N = \frac{\alpha}{m} \). \( \square \)

One has that:
Proposition 4.14. \( (\cdot,\cdot)_{p,B}^{\phi N} \) is the unique element in the cohomology class \([e_{H_{\phi N}(\tilde{\Gamma}^\bullet_p(Spec\ B))}] \in H^2(\tilde{\Gamma}^\bullet_p(Spec\ B),\mu^\bullet_m(Spec\ B))\) satisfying the conditions:

- \( (f,g \cdot g')_{p,B}^{\phi N} = (f,g)_{p,B}^{\phi N} \cdot (f,g')_{p,B}^{\phi N} \)
- \( (f,g)_{p,B}^{\phi N} = e_{H_{\phi N}(\tilde{\Gamma}^\bullet_p(Spec\ B))}(f,g) \) if \( v(f) = 0 \)
- \( (f,-f)_{p,B}^{\phi N} = 1 \)

for \( f, g, g' \in \tilde{\Gamma}^\bullet_p(Spec\ B) \).

Proof. The claim follows from the definition of \( (\cdot,\cdot)_{p,B}^{\phi N} \) and the above factorization. \( \square \)

Remark 4.15. If \( B = k \), and we consider the immersion of groups

\[ \Sigma^*_{C} \hookrightarrow \tilde{\Gamma}^\bullet_p(Spec\ k), \]

the symbol \( (\cdot,\cdot)_{p,k}^{\phi N} \) is the Hilbert norm residue symbol for the closed point \( p \in C \). Hence the morphism of schemes \( (\cdot,\cdot)^{\phi N}_{p} \) is a generalization of the construction of the Hilbert norm residue symbol made in [15].

Corollary 4.16. If \( f, g \in (\Sigma_C \otimes_k B)^* \), one has that

\[ \prod_{p \in C} (f,g)_{p,B}^{\phi N} = 1, \]

where, finitely, many of the terms appearing in the products differ from 1 in \( \mu^\bullet_m(Spec\ B) \).

Proof. This reciprocity law is a direct consequence of the factorization referred to in Lemma 4.13 and the reciprocity law of Corollary 4.12. \( \square \)

Remark 4.17. If \( B = k \), the formula

\[ \prod_{p \in C} (f,g)_{p,k}^{\phi (q-1) m} = 1 \]

is the reciprocity law of the Hilbert norm residue symbol.

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