SUSY Adjoint $SU(5)$ Grand Unified Model with $S_4$ Flavor Symmetry

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Abstract

We construct a supersymmetric (SUSY) $SU(5)$ model with the flavor symmetry $S_4 \times Z_3 \times Z_4$. Three generations of adjoint matter fields are introduced to generate the neutrino masses via the combined type I and type III see-saw mechanism. The first two generations of the the $10$ dimensional representation in $SU(5)$ are assigned to be a doublet of $S_4$, the second family $10$ is chose as the first component of the doublet, and the first family as the second component. Tri-bimaximal mixing in the neutrino sector is predicted exactly at leading order, charged lepton mixing leads to small deviation from the tri-bimaximal mixing pattern. Subleading contributions introduce corrections of order $\lambda^2$ to all three lepton mixing angles. The model also reproduces a realistic pattern of quark and charged lepton masses and quark mixings. The phenomenological implications of the model are analyzed in detail.
1 Introduction

So far there is convincing evidence that the so-called solar and atmospheric anomaly can be well explained through the neutrino oscillation. The mass square differences $\Delta m_{\odot}^2$, $\Delta m_{\text{atm}}^2$ and the mixing angles have been measured with good accuracy [1–3]. Global fit to the current neutrino oscillation data demonstrates that the observed lepton mixing matrix is remarkably compatible with the tri-bimaximal (TB) mixing pattern [4], which suggests the following values of the mixing angles

$$\sin^2 \theta_{12} = \frac{1}{3}, \quad \sin^2 \theta_{23} = \frac{1}{2}, \quad \sin \theta_{13} = 0$$

This simple structure of the mixing matrix suggests that there may be some symmetry underlying the lepton sector. Recent study [5–19] (also in the context of grand unified theories [20,21]) showed that the discrete group $A_4$ is particularly suitable to produce the TB mixing at leading order (LO), if it is properly managed to be broken differently in the neutrino and the charged lepton sector. In the $A_4$ based models, it seems very difficult and unnatural to generate the correct mass hierarchies and mixing for quarks. An interesting solution is to enlarge the symmetry group $A_4$, two non-Abelian discrete group $T'$ [22] and $S_4$ [23–30] have been investigated. We note that both groups have a doublet representation, which can be utilized to give the 2+1 representation assignments for the quarks. In the context of $U(2)$ flavor group, this assignment has been known to give realistic quark mixing matrix and mass hierarchy [31]. The irreducible representations of $T'$ are those of $A_4$ plus three two dimensional representations $2$, $2'$ and $2''$ with the multiplication rules $2 \otimes 2 = 2' \otimes 2'' = 3 \oplus 1$, $2 \otimes 2' = 2'' \otimes 2' = 3 \oplus 1'$ and $2 \otimes 2'' = 2' \otimes 2' = 3 \oplus 1''$, these ingredients allow us to reproduce the successful $U(2)$ predictions in the quark sector.

By working only with the triplet and singlet representations, $T'$ is indistinguishable from $A_4$, thus we can replicate with $T'$ the successful construction realized within $A_4$ in the lepton sector. $S_4$ is claimed to be the minimal group which can predict the TB mixing in a natural way, namely without ad hoc assumptions, from the group theory point of view [32]. Actually the exact TB mixing can be realized in the $S_4$ flavor model [23–24]. Moreover, the group $S_4$ as a flavor symmetry, as is shown for example in Ref. [27–30], can also give a successful description of the quark and lepton masses and mixing angles within the framework of $SU(5)$ or $SO(10)$ grand unified theory (GUT). For a review of discrete flavor symmetry models, please see the Refs. [34,35].

The $SU(5)$ GUT is the simplest grand unified theory [33], in this case each generation of the standard model fields resides in $\overline{5}$ and $10$ dimensional representations. To be specific, one family of right-handed down quarks and left-handed leptons are unified in a $\overline{5}$ and the rest fields of the family are in a $10$. It is well-known that neutrino masses are zero at renormalizable level in the minimal $SU(5)$. In GUT neutrino masses come naturally through the see-saw mechanism, where integrating out large masses leads to the appearance of small masses. However, this requires some extra matter fields or Higgs to be added below the GUT scale. A popular choice is to add at least two right-handed neutrinos which are $SU(5)$ singlets, neutrino masses are generated through type I see-saw mechanism. The second choice is to introduce a $15$-plet of Higgs, this is the $SU(5)$ implementation of the so-called type II see-saw mechanism. The third choice is to generate neutrino masses
through the combination of type I see-saw with type III see-saw mechanism by adding at least one matter multiplet in the adjoint 24 representation [36, 38].

In this work, we shall construct a supersymmetric SU(5) model, and the flavor symmetry group is $S_4 \times Z_3 \times Z_4$, where the auxiliary symmetry $Z_3 \times Z_4$ plays an important role in eliminating unwanted couplings, ensuring the needed vacuum alignment and reproducing the observed charged fermion mass hierarchies. We will introduce three generations of adjoint matter fields to generate the neutrino masses. We remark that some variants of $SU(5) \times S_4$ flavor models have been proposed in Refs. [28, 29], where three right handed neutrinos are introduced and the neutrino masses are generated via the type I see-saw mechanism.

The paper is organized as follows. In section 2 we discuss the structure of the model, and the leading order (LO) results for fermion masses and mixings are presented. In section 3 we justify the choice of the vacuum configuration assumed in the previous section, by minimizing the scalar potential of the theory in the supersymmetric limit. In section 4 the subleading corrections to the vacuum alignment and the LO results of fermion masses and flavor mixings are discussed. In section 5 we study the phenomenological predictions of the model in detail. Finally, section 6 is devoted to our conclusion.

2 The structure of the model

In the following we present our $SU(5)$ model in the framework of supersymmetry, which simplifies the minimization of the scalar potential greatly. The $S_4$ group acts as a flavor symmetry of our model, the group $S_4$ has already been studied in literature [39, 40], but with different aims and different results. $S_4$ is the permutation group of four objects, it has five irreducible representations $1_1$, $1_2$, $2$, $3_1$ and $3_2$, the group theory of $S_4$ is presented in Appendix A. In addition to 5 matter fields denoted by $F$ and the tenplet 10 dimensional matter fields denoted by $T_{1,2,3}$, we introduce the chiral superfields $A$ in the adjoint 24 representation [37]. In the Higgs sector we introduce $H_{24}$, $H_5$ and $H_{5\overline{5}}$ in order to break the gauge symmetry $SU(5)$ into the standard model symmetry and subsequently into the residual $SU(3)_c \times U(1)_{em}$. Moreover, $H_{45}$ and $H_{\overline{45}}$ are introduced to avoid the wrong predictions $M^T_d = M_\ell$, where $M_d$ and $M_\ell$ represent the mass matrix of down type quark and charged lepton respectively. As usual, the flavon fields are introduced to spontaneously break the $S_4$ flavor symmetry properly. The transformation rules of the matter fields, Higgs fields and the flavon fields under $SU(5)$, $S_4$, $Z_3$ and $Z_4$ are summarized in Table 1. The first and the second generation of the 10 dimensional representations are assigned to be a doublet of $S_4$, and the third generation of 10 to $1_1$ of $S_4$. This assignment is indicated by the heavi ness of the top quark. There is the freedom of choosing the first family or the second family as the first component of the $S_4$ doublet. In this work, the second family 10 is taken to be the first component of the doublet, and the first family as the second component. If we assign the first family 10 as the first component of the doublet and the second family as the second component as usual, the down quark and strange quark
masses would be of the same order without fine tuning unless some special mechanisms are introduced such as Ref. [28]. Three generation of $\Phi$ fields $F$ are assigned to $3_1$ of $S_4$, and three generations of adjoint matter fields $A$ are also assigned to be $3_1$ of $S_4$. Fermion masses and mixings arise from the spontaneous breaking of the flavor symmetry by means of the flavon fields. In the following, we shall discuss the LO predictions for fermion masses and flavor mixings. For the time being we assume that the scalar components of the flavon fields acquire vacuum expectation values (VEV) according to the following scheme

$$
\langle \chi \rangle = \begin{pmatrix} v_\chi \\ v_\chi \\ v_\chi \end{pmatrix}, \quad \langle \varphi \rangle = \begin{pmatrix} v_\varphi \\ v_\varphi \end{pmatrix}, \quad \langle \xi \rangle = 0
$$

$$
\langle \phi \rangle = \begin{pmatrix} 0 \\ v_\phi \\ 0 \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ v_\eta \end{pmatrix}
$$

$$
\langle \Delta \rangle = \begin{pmatrix} v_\Delta \\ 0 \\ 0 \end{pmatrix}, \quad \langle \xi \rangle = v_\xi
$$

We will prove it to be a natural solution of the minimization of the scalar potential in section 3. Furthermore we take the VEVs (scaled by the cutoff $\Lambda$) $v_\chi/\Lambda$, $v_\varphi/\Lambda$, $v_\phi/\Lambda$, $v_\eta/\Lambda$, $v_\Delta/\Lambda$ and $v_\xi/\Lambda$ to be of the same order of magnitude about $O(\lambda_c^2)$ with $\lambda_c \simeq 0.22$ being the Cabibbo angle, and we will parameterize the ratio $\text{VEV}/\Lambda$ by the parameter $\varepsilon$. This order of magnitude is indicated by the observed ratios of up quarks and down quarks and charged lepton masses, by the scale of the light neutrino masses and is also compatible with the current bounds on the deviations from TB mixing for the leptons. We note that the assumed size of the VEVs can be partially explained by the minimization of the scalar potential, as it will be clearer in the following.

2.1 Neutrino sector

The LO superpotential which contributes to the neutrino masses is given by

$$
\nu = y_\nu (FA)_1 H_5 + \lambda_1 (AA)_3 \chi + \lambda_2 (AA)_2 \varphi
$$

If we take $T_1$ and $T_2$ as the first and the second components of the $S_4$ doublet, i.e., $(T_1, T_2)^T \sim 2$, then $(TF)_{3_1} \sim (T_1 F_2 + T_2 F_3, T_1 F_3 + T_2 F_1, T_1 F_1 + T_2 F_2)^T$ and $(TF)_{3_2} \sim (T_1 F_2 - T_2 F_3, T_1 F_3 - T_2 F_1, T_1 F_1 - T_2 F_2)^T$. We see that $T_1 F_1$ and $T_2 F_2$, which are related to the down and strange quark masses respectively, appear simultaneously as the third component of both the combinations $(TF)_{3_1}$ and $(TF)_{3_2}$. The operators $TFH_\Phi$ and $TFH_{\Phi}$ combining with the flavon fields or the composition of flavons, which transform as $3_1$ or $3_2$, contribute to the first two families down quark and charged lepton masses after the $S_4$ and GUT symmetry breaking. As a result, the down and strange quark masses would be of the same order except for the case that the $(TF)_{3_1}$ and $(TF)_{3_2}$ relevant contributions to down quark or strange quark cancel with each other. We notice that the same view has been put forward by Altarelli et al. [34]. Whereas if we choose $(T_2, T_1)^T \sim 2$ as we proposed, then $(TF)_{3_1} \sim (T_2 F_2 + T_1 F_3, T_2 F_3 + T_1 F_1, T_2 F_1 + T_1 F_2)^T$ and $(TF)_{3_2} \sim (T_2 F_2 - T_1 F_3, T_2 F_3 - T_1 F_1, T_2 F_1 - T_1 F_2)^T$, the degeneracy between first two families down quark (charged lepton) masses is dissolved.
The last two terms in Eq.(3) lead to the Majorana mass matrices of \( \rho_3 \) and singlet \( \rho_0 \) with hypercharge \( Y = 0 \) in the model. The neutrino masses are generated through the type I (mediated by the SU(2) singlet \( \rho_0 \) of \( A \)) and type III (mediated by the SU(2) triplet \( \rho_3 \) of \( A \)) see-saw mechanism. The Dirac mass matrices is obtained from the first term in Eq.(3),

\[
M^{D}_{\rho_3} = \frac{1}{2} y_{\nu} v_5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^{D}_{\rho_0} = \frac{\sqrt{15}}{10} y_{\nu} v_5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

(4)

where \( M^{D}_{\rho_3} \) and \( M^{D}_{\rho_0} \) are the Dirac mass matrices associated with \( \rho_3 \) and \( \rho_0 \) respectively. The last two terms in Eq.(3) lead to the Majorana mass matrices of \( \rho_3 \) and \( \rho_0 \)

\[
M^{M}_{\rho_3} = \begin{pmatrix} 2\lambda_1 v_\chi & -\lambda_1 v_\chi + \lambda_2 v_\varphi & -\lambda_1 v_\chi + \lambda_2 v_\varphi \\ -\lambda_1 v_\chi + \lambda_2 v_\varphi & 2\lambda_1 v_\chi + \lambda_2 v_\varphi & -\lambda_1 v_\chi \\ -\lambda_1 v_\chi + \lambda_2 v_\varphi & -\lambda_1 v_\chi & 2\lambda_1 v_\chi + \lambda_2 v_\varphi \end{pmatrix}
\]

\[
M^{M}_{\rho_0} = M^{M}_{\rho_3}
\]

(5)

It is notable that the Majorana mass matrices of \( \rho_3 \) and \( \rho_0 \) are exactly the same. As a result, the mass spectrums of \( \rho_3 \) and \( \rho_0 \) are degenerate. This degeneracy is violated at NLO by the Higgs \( H_{24} \). We note the VEVs \( \langle \chi \rangle \) and \( \langle \varphi \rangle \) are invariant under the action of \( S_4 \) elements \( TSTS^2 \), \( TST \) and \( S^2 \), consequently the \( S_4 \) flavor symmetry is broken down to the Klein four subgroup in the neutrino sector. The light neutrino mass matrix is the sum of type I and type III see-saw contributions

\[
M_\nu = - (M^{D}_{\rho_3})^T (M^{M}_{\rho_3})^{-1} M^{D}_{\rho_3} - (M^{D}_{\rho_0})^T (M^{M}_{\rho_0})^{-1} M^{D}_{\rho_0}
\]

\[
= \begin{pmatrix} -a-b & -a+b & -a+b \\ -\frac{a}{3} & -\frac{a}{3} & -\frac{a}{3} \\ -\frac{a}{3} & -\frac{a}{3} & -\frac{a}{3} \end{pmatrix} y_{\nu}^2 v_5^2
\]

(6)

where

\[
a \equiv \lambda_1 v_\chi, \quad b \equiv \lambda_2 v_\varphi
\]

(7)

The above light neutrino mass matrix \( M_\nu \) is 2 \( \leftrightarrow \) 3 invariant and it satisfies the magic symmetry \((M_\nu)_{11} + (M_\nu)_{13} = (M_\nu)_{22} + (M_\nu)_{32}\). Therefore it is exactly diagonalized by the

\[
| 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 |
\]
TB mixing matrix

\[ U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3) \]  

(8)

where \( m_{1,2,3} \) are the light neutrino masses, in unit of \( \frac{2}{3} y_\nu^2 v_\nu^2 \) they are

\[ m_1 = \frac{1}{|3a - b|} \]

\[ m_2 = \frac{1}{2|b|} \]

\[ m_3 = \frac{1}{|3a + b|} \]  

(9)

The neutrino mass spectrum can be normal hierarchy (NH) or inverted hierarchy (IH). The unitary matrix \( U_\nu \) is given by

\[ U_\nu = U_{TB} \text{diag}(e^{-i\alpha_1/2}, e^{-i\alpha_2/2}, e^{-i\alpha_3/2}) \]  

(10)

\( U_{TB} \) is the well-known TB mixing matrix

\[
U_{TB} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & 1 & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}
\end{pmatrix}
\]  

(11)

The phases \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are

\[ \alpha_1 = \text{arg}(-y_\nu^2 v_\nu^2/(3a - b)) \]

\[ \alpha_2 = \text{arg}(-y_\nu^2 v_\nu^2/b) \]

\[ \alpha_3 = \text{arg}(-y_\nu^2 v_\nu^2/(3a + b)) \]  

(12)

According to Eq. (9), the light neutrino mass spectrum is directly related to the heavy neutrino (\( \rho_3 \) or \( \rho_0 \)) masses, it is determined by only two complex parameters \( a \) and \( b \). In the following, we shall follow the method of Ref. [17] to analyze the light neutrino mass spectrum in detail. For the sake of convenience, we define

\[ \frac{a}{b} = re^{i\theta} \]  

(13)

Note that the parameter \( r \) is real and positive and the phase \( \theta \) is between 0 and 2\( \pi \). We can express \( r \) and the phase \( \theta \) in term of light neutrino masses as follows

\[ r = \frac{1}{3} \sqrt{\frac{2m_3^2 - m_1^2}{m_1^2} + \frac{2m_2^2}{m_3^2} - 1} \]  

(14)

\[ \cos \theta = \frac{\frac{m_3^2}{m_1^2} - \frac{m_1^2}{m_2^3}}{\sqrt{\frac{2m_3^2}{m_1^2} + \frac{2m_2^2}{m_3^2} - 1}} \]  

(15)

Experimentally, only the mass square differences have been measured. For normal (inverted) hierarchy they are

\[ \Delta m^2_{\text{sol}} = m_2^2 - m_1^2 = (7.67^{+0.22}_{-0.21}) \times 10^{-5} \text{eV}^2 \]

\[ \Delta m^2_{\text{atm}} = |m_3^2 - m_1^2(m_2^2)| = (2.46(2.45) \pm 0.15) \times 10^{-3} \text{eV}^2 \]  

(16)
By expressing the neutrino masses in terms of the lightest neutrino mass \( m_l \) (\( m_l = m_1 \) for NH and \( m_3 \) for IH), \( \Delta m^2_{\text{sol}} \) and \( \Delta m^2_{\text{atm}} \), \( \cos \theta \) becomes a function of the lightest neutrino mass \( m_l \). We display \( \cos \theta \) versus \( m_l \) in Fig. 1 for both normal hierarchy and inverted hierarchy spectrum. Imposing the condition \( |\cos \theta| \leq 1 \), we obtain the following constraints on the lightest neutrino mass

\[
\begin{align*}
m_1 & \geq 0.011 \text{eV}, \quad \text{for normal hierarchy} \\
m_3 & \geq 0.028 \text{eV}, \quad \text{for inverted hierarchy}
\end{align*}
\]

(17)

The results presented so far are of course approximate since the model gets corrections when higher dimensional operators are included in the Lagrangian. If the mixing of the left handed charged lepton is neglected, we can straightforwardly derive the neutrinoless double decay parameters \( |m_{ee}| \)

\[
|m_{ee}| = \frac{m_1}{3} \sqrt{2 - \frac{m_2^2}{m_1^2} + 2 \frac{m_2^2}{m_3^2}}
\]

(18)

2.2 Up quarks sector

The LO superpotential invariant under symmetry group \( SU(5) \times S_4 \times Z_3 \times Z_4 \), which gives rise to the masses of the up type quarks after \( S_4 \) and \( SU(5) \) symmetry breaking, is given by

\[
w_u = y_t T_3 T_3 H_5 + \sum_{i=1}^{4} \frac{y_{ci}}{\Lambda^2} TT O^{(1)}_i H_5 + \frac{y_{ult}}{\Lambda^2} TT_3 (\phi \chi) T_2 H_5 + \frac{y_{ult}}{\Lambda^2} TT_3 (\eta \varphi) T_2 H_5 \\
+ \frac{y_{ult}}{\Lambda^2} TT_3 (\eta \zeta) H_5 + \frac{y_{ult}}{\Lambda} TT_3 (\eta \eta) H_{45}
\]

(19)

where

\[
O^{(1)} = \{ (\phi \phi)_{11}, (\phi \phi)_2, (\eta \eta)_{11}, (\eta \eta)_2 \} 
\]
With the vacuum alignment in Eq. [2], it is immediate to derive the mass matrix as follows

$$M_u = \begin{pmatrix} 0 & 0 & 4(y_{ul1} \frac{v_\phi v_\psi}{\Lambda^2} + y_{ul2} \frac{v_\chi v_\psi}{\Lambda^2})v_5 \\ 0 & 8(y_{ul2} \frac{v_\chi^2}{\Lambda^2} + y_{ul3} \frac{v_\psi^2}{\Lambda^2})v_5 & 8y_{uct} \frac{v_\eta}{\Lambda}v_{45} + 4y_{ut1} \frac{v_\phi v_\psi}{\Lambda^2}v_5 \\ 4(y_{ul1} \frac{v_\phi v_\psi}{\Lambda^2} + y_{ul2} \frac{v_\chi v_\psi}{\Lambda^2})v_5 & -8y_{uct} \frac{v_\eta}{\Lambda}v_{45} + 4y_{ut1} \frac{v_\phi v_\psi}{\Lambda^2}v_5 & 8y_{ut} v_5 \end{pmatrix}$$

(20)

We note that the Higgs field $H_5 (H_{45})$ induces a symmetric (antisymmetric) contribution to $M_u$. Since the product decompositions $(TT)_1 \sim T_1T_2 + T_2T_1$ and $(TT)_2 \sim \left( \frac{T_1^2}{T_2^2} \right)$

are both symmetric under the exchange of the two tenplets, the combinations of $T_3T_3H_{45}$ and $TTH_{45}$ with arbitrary number of flavon fields don’t contribute to up quarks masses. Therefore the corresponding terms are omitted from the beginning. The mass matrix $M_u$ is diagonalized by bi-unitary transformation

$$V_R^{u\dagger} M_u V_L^u = \text{diag}(m_u, m_c, m_t)$$

(21)

The up type quark masses are given by

$$m_u \simeq \left| \frac{2(y_{ul2} v_\chi^2 + y_{ul3} v_\psi^2)(y_{ul1} v_\phi v_\chi/\Lambda^2 + y_{ul2} v_\eta v_\phi/\Lambda^2)^2}{y_t(y_{ul2} v_\chi^2 + y_{ul3} v_\psi^2)v_5^2 + y_{ct} v_\eta^2 v_{45}^2} \right| v_5$$

$$m_c \simeq \left| 8(y_{ul2} \frac{v_\chi^2}{\Lambda^2} + y_{ul3} \frac{v_\psi^2}{\Lambda^2})v_5 + 8y_{ct} v_\eta^2 v_{45}^2 \right|$$

$$m_t \simeq \left| 8y_{ut} v_5 \right|$$

(22)

The mixing matrix $V_L^u$ is

$$V_L^u \simeq \begin{pmatrix} 1 & s_{12}^u & s_{13}^u \\ -s_{12}^u & 1 & s_{23}^u \\ -s_{13}^u & -s_{23}^u & 1 \end{pmatrix}$$

(23)

where

$$s_{12}^u = -\frac{1}{2} \left[ \frac{y_{ct}(y_{ul1} v_\phi v_\chi + y_{ul2} v_\eta v_\phi)v_5 v_{45} v_\eta}{y_t(y_{ul2} v_\chi^2 + y_{ul3} v_\psi^2)v_5^2 + y_{ct} v_\eta^2 v_{45}^2 \Lambda} \right]^*$$

$$s_{23}^u = \left( \frac{y_{ct} v_{45} v_\eta}{y_t v_5 \Lambda} + \frac{y_{ul1} v_\phi v_\chi}{y_t \Lambda^2} \right)^*$$

$$s_{13}^u = \frac{1}{2} \left[ \frac{y_{ul1} v_\phi v_\chi + y_{ul2} v_\eta v_\phi}{y_t \Lambda^2} \right]^*$$

(24)

We note that there is a mixing of order $\lambda_2^2$ between the first and the second family, although the (12) and (21) elements of $M_u$ vanish at LO. The top quark mass is generated at tree level, and the mass hierarchies among the up quarks are reproduced naturally given the VEVs $v_\chi$, $v_\phi$, $v_\psi$ and $v_\eta$ of order $\lambda_2^2 \Lambda$.

2.3 Down type quarks and charged leptons sector

The superpotential generating the masses of down quarks and charged lepton is

$$w_d = \frac{y_{bl}}{\Lambda} T_3F \phi H_5 + \frac{y_{s1}}{\Lambda^2} (TF)_{31}(\Delta \Delta)_{31} H_{45} + \frac{y_{s2}}{\Lambda^2} (TF)_{31} \Delta \xi H_{45} + \sum_{i=1}^{9} \frac{y_{di}}{\Lambda^3} T_3F \mathcal{O}^{(i)}_i H_5$$

7
\[ + \sum_{i=1}^{6} \frac{x_{i}}{\Lambda^{3}} T_{2} F O_{i}^{(3)} H F_{i}^{(3)} + \sum_{i=1}^{7} \frac{z_{i}}{\Lambda^{3}} T F O_{i}^{(4)} H F + \ldots \]  

(25)

where dots stand for higher dimensional operators.

\[ O^{(2)} = \{ \chi^{2} \phi, \chi^{2} \eta, \varphi \chi \phi, \varphi \chi \eta, \varphi \chi \zeta, \chi \eta \zeta, \varphi \phi \zeta, \varphi \phi \zeta, \varphi \phi \zeta, \phi \zeta \} \]

\[ O^{(3)} = \{ \phi^{3}, \phi^{2} \eta, \phi \eta^{2}, \Delta^{3}, \Delta^{2} \xi, \Delta \xi^{2} \} \]

\[ O^{(4)} = \{ \phi^{2} \chi, \phi^{2} \varphi, \phi^{2} \zeta, \eta \phi \chi, \eta \phi \varphi, \eta \phi \zeta, \eta \xi \phi, \eta \xi \zeta, \eta \xi \zeta, \phi \xi \zeta \} \]

(26)

For the last three terms in Eq. (25), one operator frequently induces several different contractions, we should take into account all possible independent contractions for each operator. Note that the auxiliary symmetry $Z_{3} \times Z_{4}$ imposes different powers of the flavon fields for the bottom (tau), strange (muon) and down quark (electron) mass relevant terms. In the expansion in powers of $1/\Lambda$, the bottom (tau) mass is generated at order $1/\Lambda$, the strange (muon) and down quark (electron) masses are generated at order $1/\Lambda^{2}$ and $1/\Lambda^{3}$ respectively. As a result, if we only consider the LO operators suppressed by $1/\Lambda$ and $1/\Lambda^{2}$, the down quark and electron would be massless.

Recalling the vacuum configuration in Eq. (2), we can write down the mass matrix for down quarks and charged leptons as follows

\[ M_{d} = \begin{pmatrix} y_{11}^{d} \varepsilon^{3} v_{e}^{3} & y_{12}^{d} \varepsilon^{3} v_{e}^{3} & y_{13}^{d} \varepsilon^{3} v_{e}^{3} + 2y_{13}^{d} \varepsilon^{3} v_{e}^{3} \\ y_{21}^{d} \varepsilon^{3} v_{e}^{3} & 2y_{22}^{d} \varepsilon^{3} v_{e}^{3} + y_{22}^{d} \varepsilon^{3} v_{e}^{3} & y_{23}^{d} \varepsilon^{3} v_{e}^{3} \\ 2y_{22}^{d} \varepsilon^{3} v_{e}^{3} + y_{22}^{d} \varepsilon^{3} v_{e}^{3} & y_{32}^{d} \varepsilon^{3} v_{e}^{3} & y_{33}^{d} \varepsilon^{3} v_{e}^{3} \end{pmatrix} \]

\[ M_{\ell} = \begin{pmatrix} y_{11}^{d} \varepsilon^{3} v_{e}^{3} & y_{12}^{d} \varepsilon^{3} v_{e}^{3} & y_{13}^{d} \varepsilon^{3} v_{e}^{3} + 2y_{13}^{d} \varepsilon^{3} v_{e}^{3} \\ y_{12}^{d} \varepsilon^{3} v_{e}^{3} & 2y_{22}^{d} \varepsilon^{3} v_{e}^{3} + y_{22}^{d} \varepsilon^{3} v_{e}^{3} & y_{23}^{d} \varepsilon^{3} v_{e}^{3} + 6y_{22}^{d} \varepsilon^{3} v_{e}^{3} \\ y_{13}^{d} \varepsilon^{3} v_{e}^{3} + 6y_{13}^{d} \varepsilon^{3} v_{e}^{3} & y_{23}^{d} \varepsilon^{3} v_{e}^{3} + 6y_{22}^{d} \varepsilon^{3} v_{e}^{3} & y_{33}^{d} \varepsilon^{3} v_{e}^{3} \end{pmatrix} \]

(27)

(28)

where the coefficients $y_{ij}^{d}$, $y_{ij}^{s}$, $y_{ij}^{u}$ and $y_{ij}^{d}$ are linear combinations of the leading order coefficients, $y_{13}^{d}$ coincides with the LO parameter $y_{u}$ up to corrections of order $\varepsilon^{2}$ which origins from the operators $T_{2} F O^{(3)} H_{F}$. The factor of 3 difference in the (13), (22) and (31) elements between $M_{d}$ and $M_{\ell}$ is the so-called Georgi-Jarlskog factor $\text{[11]}$, which is induced by the Higgs $H_{F}$. Similar to the up type quarks, the mass matrices $M_{d}$ and $M_{\ell}$ can be diagonalized by the following transformations

\[ V_{R}^{d} M_{d} V_{L}^{d} = \text{diag}(m_{d}, m_{s}, m_{b}), \quad V_{R}^{\ell} M_{\ell} V_{L}^{\ell} = \text{diag}(m_{e}, m_{\mu}, m_{\tau}) \]

(29)

The mass eigenvalues are given by

\[ m_{d} \simeq |y_{11}^{d} \varepsilon^{3} v_{e}^{3} - y_{12}^{d} y_{21}^{d} \varepsilon^{4} v_{e}^{2} / (2y_{22}^{d} v_{e}^{3}) - 2y_{13}^{d} y_{22}^{d} \varepsilon^{4} v_{e}^{3} / y_{33}^{d} - 4y_{13}^{d} y_{22}^{d} \varepsilon^{4} v_{e}^{2} / y_{33}^{d} v_{e}^{3} | \]

\[ m_{s} \simeq 2y_{22}^{d} \varepsilon^{2} v_{e}^{3} + y_{22}^{d} \varepsilon^{3} v_{e}^{3} | \]

\[ m_{b} \simeq |y_{33}^{d} \varepsilon^{3} v_{e}^{3} | \]

(30)

and

\[ m_{e} \simeq |y_{11}^{d} \varepsilon^{3} v_{e}^{3} + y_{12}^{d} y_{21}^{d} \varepsilon^{4} v_{e}^{2} / (6y_{22}^{d} v_{e}^{3}) + 6y_{13}^{d} y_{22}^{d} \varepsilon^{4} v_{e}^{3} / y_{33}^{d} - 36y_{13}^{d} y_{22}^{d} \varepsilon^{4} v_{e}^{2} / y_{33}^{d} v_{e}^{3} | \]

\[ m_{\mu} \simeq -6y_{22}^{d} \varepsilon^{2} v_{e}^{3} + y_{22}^{d} \varepsilon^{3} v_{e}^{3} | \]

\[ m_{\tau} \simeq |y_{33}^{d} \varepsilon^{3} v_{e}^{3} | \]

(31)
The diagonalization matrices \(V^d_L\) and \(V^e_L\) are

\[
V^d_L \simeq \begin{pmatrix}
1 & \frac{(y^d_{31} y^d_{33} x - y^d_{33} y^d_{22} x)}{y^d_{23} v_\tau} \varepsilon & \frac{(2 y^d_{32} y^d_{33} v_\tau x + y^d_{33} y^d_{33} v_\tau \varepsilon)}{y^d_{23} v_\tau} \\
\frac{-(y^d_{31} y^d_{33} x - y^d_{33} y^d_{22} x)}{y^d_{23} v_\tau} \varepsilon - y^d_{33} y^d_{33} v_\tau \varepsilon^2 & 1 & \frac{(y^d_{33} x^2 - 6 y^d_{33} y^d_{33} v_\tau \varepsilon^2)}{y^d_{23} v_\tau} \\
\frac{y^d_{31} y^d_{33} v_\tau x}{y^d_{23} v_\tau} - y^d_{33} y^d_{33} v_\tau \varepsilon^2 & \frac{y^d_{31} y^d_{33} v_\tau x}{y^d_{23} v_\tau} - y^d_{33} y^d_{33} v_\tau \varepsilon^2 & 1
\end{pmatrix}
\] (32)

\[
V^e_L \simeq \begin{pmatrix}
1 & \frac{(y^e_{31} y^e_{33} x - y^e_{33} y^e_{22} x)}{2 y^e_{33} v_\tau} \varepsilon & \frac{(y^e_{33} x^2 - 6 y^e_{33} y^e_{33} v_\tau \varepsilon^2)}{2 y^e_{33} v_\tau} \\
\frac{-(y^e_{31} y^e_{33} x - y^e_{33} y^e_{22} x)}{2 y^e_{33} v_\tau} \varepsilon - y^e_{33} y^e_{33} v_\tau \varepsilon^2 & 1 & \frac{(y^e_{33} x^2 - 6 y^e_{33} y^e_{33} v_\tau \varepsilon^2)}{2 y^e_{33} v_\tau} \\
\frac{y^e_{31} y^e_{33} v_\tau x}{2 y^e_{33} v_\tau} - y^e_{33} y^e_{33} v_\tau \varepsilon^2 & \frac{y^e_{31} y^e_{33} v_\tau x}{2 y^e_{33} v_\tau} - y^e_{33} y^e_{33} v_\tau \varepsilon^2 & 1
\end{pmatrix}
\] (33)

As is shown in Eq. (30) and Eq. (31), obviously we have

\[
m_\tau \simeq m_b, \quad m_\mu \simeq 3m_s
\] (34)

The well-known bottom-tau unification and the Georgi-Jarlskog relation \[41\] between the down type quark and the charged lepton masses for the second generation are produced in the present model. We note that generally the vanishing of the (11) elements of both the down quark and charged lepton mass matrices is required, to obtain the Georgi-Jarlskog relation for both the first and second generations simultaneously. The mass difference of electron and down quark is induced by the Higgs field \(H_{35}\), acceptable values of the masses for electron and down quark can be accomplished due to the Georgi-Jarlskog factor. It is well-known that the CKM mixing between the first and the second family is exactly described by the Cabibbo angle. In order to satisfy this phenomenological constraint, for the parameters \(y^d_{21}\) and \(y^d_{22}\) of order \(O(1)\) we could choose

\[
v_{35} \sim \lambda c v_\tau
\] (35)

Furthermore we can see from Eq. (32) and Eq. (33) that the mixing angle between the first and the second family charged leptons approximately is \(\lambda c / 3\). The resulting quark mixing matrix is given by

\[
V_{CKM} = V^d_L V^d_L
\] (36)

We can straightforwardly read the CKM matrix elements as follows

\[
V_{ud} \simeq V_{cs} \simeq V_{tb} \simeq 1
\]

\[
V_{us}^* \simeq -V_{cd} \simeq \frac{y^d_{21} v_\tau}{2 y^d_{22} v_\tau} \varepsilon + \frac{1}{2 y^d(2 y^d_{22} v_\tau^2 + y^d_{33} v_\tau^2)} \varepsilon^2 - \frac{1}{2 y^d(2 y^d_{22} v_\tau^2 + y^d_{33} v_\tau^2)} \varepsilon^2 = \frac{1}{2 y^d(2 y^d_{22} v_\tau^2 + y^d_{33} v_\tau^2)} \varepsilon^2
\]

\[
V_{ub}^* = \frac{2 y^d_{22} v_\tau}{y^d_{33} v_\tau} \varepsilon + \frac{y^d_{33} v_\tau^2}{y^d_{33} v_\tau} \varepsilon^2 - \frac{2 y^d_{22} v_\tau v_\rho}{y^d_{33} v_\tau} \varepsilon^2 - \frac{1}{y^d_{33} v_\tau} \varepsilon^2 = \frac{1}{y^d_{33} v_\tau} \varepsilon^2
\]

\[
V_{cb}^* \simeq -V_{ts} \simeq \frac{y^d_{21} v_\tau}{y^d_{33} v_\tau} \varepsilon, \quad \frac{y^d_{22} v_\tau}{y^d_{33} v_\tau} \varepsilon = \frac{y^d_{33} v_\tau^2}{y^d_{33} v_\tau} \varepsilon^2
\]

\[
V_{td} = \frac{-2 y^d_{22} v_\tau}{y^d_{33} v_\tau} \varepsilon + \frac{y^d_{33} v_\tau^2}{y^d_{33} v_\tau} \varepsilon^2 = \frac{y^d_{21} v_\tau v_\rho}{y^d_{33} v_\tau} \varepsilon^2 + \frac{y^d_{22} v_\tau v_\rho}{y^d_{33} v_\tau} \varepsilon^2
\]

\[
V_{ut} = \frac{-2 y^d_{22} v_\tau}{y^d_{33} v_\tau} \varepsilon - \frac{y^d_{33} v_\tau^2}{y^d_{33} v_\tau} \varepsilon^2 = \frac{y^d_{21} v_\tau v_\rho}{y^d_{33} v_\tau} \varepsilon^2 + \frac{y^d_{22} v_\tau v_\rho}{y^d_{33} v_\tau} \varepsilon^2
\]

We note that the CKM elements \(V_{us}, V_{cd}, V_{ub}\) and \(V_{td}\) are dominantly determined by the mixing in the down type quark sector, \(V_{cb}\) and \(V_{ts}\) origin from the left handed up quarks
mixing. Considering \( \nu_\mu/\nu_\tau \sim \lambda_c \), we find that \( V_{us} \) and \( V_{cd} \) are of order \( \lambda_c \), \( V_{ub} \) and \( V_{td} \) are of order \( \lambda_c^2 \). \( V_{cb} \) and \( V_{ts} \) are of order \( \lambda_c^2 \). The correct pattern of CKM mixing matrix is reproduced.

Due to the non-trivial mixing \( V_L \) present in the charged lepton sector, we note that the lepton mixing is not the TB mixing, although the light neutrino mass matrix is exactly diagonalized by the TB mixing matrix. The lepton mixing matrix (PMNS matrix) is given by

\[
U_{PMNS} = V_L^d U_\nu
\]

Consequently the lepton mixing angles are

\[
\sin \theta_{13} = |(U_{PMNS})_{e3}| \approx \left| \frac{y_{12}^d \nu_\tau}{6\sqrt{2}y_{22}^d \nu_{45}} \right| \\
\sin^2 \theta_{12} \approx \frac{1}{2} + \frac{1}{18} \left[ \frac{y_{12}^d v_{45}}{y_{22}^d v_{45}} \varepsilon + \left( \frac{y_{12}^d v_{45}}{y_{22}^d v_{45}} \varepsilon \right)^* \right] \\
\sin \theta_{23} \approx \frac{1}{2} \left( \frac{y_{12}^d v_{45}}{y_{22}^d v_{45}} \varepsilon \right)^2
\]

Taking into account the results for quark mixing shown in Eq.(37), we have \( |V_{us}| \approx |\frac{y_{12}^d \nu_\tau}{2y_{22}^d \nu_{45}}| \sim \lambda_c \). As a result, the model predicts the deviation of the lepton mixing from the TB pattern as follows

\[
\sin \theta_{13} \sim \frac{\lambda_c}{3\sqrt{2}} \approx 2.97^\circ \\
|\sin^2 \theta_{12} - \frac{1}{3}| \sim \frac{2}{9} \lambda_c \\
|\sin^2 \theta_{23} - \frac{1}{2}| \sim \frac{\lambda_c^2}{36}
\]

The lepton mixing angles are predicted to be in agreement at 3\( \sigma \) error with the experimental data \[1\]–\[3\]. It is remarkable that Eq.(40) belongs to a set of well-known leptonic mixing sum rules \[42\]–\[43\], and the same results have been obtained in Ref. \[28\].

### 2.4 High dimensional Weinberg operators

In the previous section, the neutrinos acquire masses via the see-saw mechanism. It is interesting to note that the higher dimensional Weinberg operator could also contribute to the neutrino masses directly. In the present model, these effective light neutrino mass operators are\[3\]

\[
w^\text{eff}_\nu = \frac{y_{\nu 1}}{\Lambda^2} (FF)_{31} \chi H_5 H_5 + \frac{y_{\nu 2}}{\Lambda^2} (FF)_{22} \chi H_5 H_5 + \frac{y_{\nu 3}}{\Lambda^2} (FF)_{31} \chi H_{45} H_{45} + \frac{y_{\nu 4}}{\Lambda^2} (FF)_{22} \chi H_{45} H_{45}
\]

\[2\]Concretely the operator \((FF)_{31} \chi H_5 H_5\) denotes \(y_{abc}(F_a)_\alpha(F_b)_\beta(H_5)_{\gamma\alpha}(H_5)^{\delta\beta}_{\gamma}\), where the Greek indices are contracted in the \(SU(5)\) space, and the Latin indices are contracted in the \(S_4\) space, the coefficient with three \(S_4\) indices \(y_{abc}\) is the Clebsch-Gordon coefficient of the \(S_4\) group, its value can be read directly from the product decomposition rules shown in Appendix A, so that the effective operator is invariant under the flavor group \(S_4\). The contractions of the remaining operators in Eq.(41) can be read out similarly.
Table 2: Driving fields and their transformation rules under the symmetry group $S_4 \times Z_3 \times Z_4$.

Taking into account the vacuum alignment $\langle \chi \rangle = v_\chi (1, 1, 1)^T$ and $\langle \phi \rangle = v_\phi (1, 1)^T$, the Weinberg operators in $w_\nu^{\text{eff}}$ lead to the following effective light neutrino mass matrix

$$M_\nu^{\text{eff}} = \begin{pmatrix}
2\alpha & -\alpha + \beta & -\alpha + \beta \\
-\alpha + \beta & 2\alpha + \beta & -\alpha \\
-\alpha + \beta & -\alpha & 2\alpha + \beta
\end{pmatrix}$$

where

$$\alpha = (2y_{\nu 1} \frac{v^2_\chi}{\Lambda} + 24y_{\nu 3} \frac{v^2_3}{\Lambda}) \frac{v_\chi}{\Lambda}$$

$$\beta = (2y_{\nu 2} \frac{v^2_\phi}{\Lambda} + 24y_{\nu 4} \frac{v^2_4}{\Lambda}) \frac{v_\phi}{\Lambda}$$

The mass matrix $M_\nu^{\text{eff}}$ is exactly diagonalized by the TB mixing matrix

$$U_T^{TB} M_\nu^{\text{eff}} U_T^{TB} = \text{diag}(m_1^{\text{eff}}, m_2^{\text{eff}}, m_3^{\text{eff}})$$

As a result, the lepton mixing angles displayed in Eq.(39) are not corrected even if the Weinberg operators are taken into account. The mass eigenvalues $m_i^{\text{eff}}$ are given by

$$m_1^{\text{eff}} = 3\alpha - \beta$$
$$m_2^{\text{eff}} = 2\beta$$
$$m_3^{\text{eff}} = 3\alpha + \beta$$

Comparing with the see-saw mechanism induced masses in Eq.(9), we have

$$\frac{m_i^{\text{eff}}}{m_i} \sim \frac{v^2_\chi}{\Lambda^2}$$

It is obvious that the contributions of the Weinberg operator are highly suppressed relative to those induced by the see-saw mechanism, so that they can be completely negligible.

### 3 Vacuum alignment

In this section we discuss the minimization of the scalar potential in order to justify the vacuum alignment used in the previous section. As usual we introduce a global continuous $U(1)_R$ symmetry which contain the discrete $R$–parity as a subgroup. The flavon and
GUT Higgs fields are uncharged under $U(1)_R$, the supermultiplets containing the standard model matter fields and the adjoint field $A$ carry $U(1)_R$ charge +1. Moreover, we include additional gauge singlets, the so called driving fields $\chi^0$, $\varphi^0$, $\rho^0$ and $\Delta^0$ with $U(1)_R$ charge +2. They transform in a non-trivial way under the flavor symmetry $S_4 \times Z_3 \times Z_4$, as is presented in Table 2. Since the driving fields carry +2 unit $U(1)_R$ charge, they enter linearly into the superpotential. The LO superpotential depending on the driving fields, which is invariant under the flavor symmetry, is given by

$$w_v = f_1 \chi^0 (\chi \varphi)_{32} + f_2 \chi^0 \chi \zeta + f_3 \varphi^0 (\chi \chi)_2 + f_4 \varphi^0 (\varphi \varphi)_2 + f_5 \varphi^0 \varphi \zeta + g_1 \varphi^0 (\phi \phi)_{31} + g_2 \varphi^0 (\eta \varphi)_{31} + g_3 \rho^0 (\phi \phi)_{11} + g_4 \rho^0 (\eta \eta)_{11} + h_1 \Delta^0 (\Delta \Delta)_{31} + h_2 \Delta^0 \Delta \zeta$$

(47)

In the SUSY limit, the equations for the minimum of the scalar potential are obtained by deriving $w_v$ with respect to each component of the driving fields

$$\frac{\partial w_v}{\partial \chi^0_1} = f_1 (\varphi_1 \chi_2 - \varphi_2 \chi_3) + f_2 \chi_1 \zeta = 0$$
$$\frac{\partial w_v}{\partial \chi^0_2} = f_1 (\varphi_1 \chi_1 - \varphi_2 \chi_2) + f_2 \chi_3 \zeta = 0$$
$$\frac{\partial w_v}{\partial \chi^0_3} = f_1 (\varphi_1 \chi_3 - \varphi_2 \chi_1) + f_2 \chi_2 \zeta = 0$$
$$\frac{\partial w_v}{\partial \varphi^0_1} = f_3 (\chi^2_3 + 2 \chi_1 \chi_2) + f_4 \varphi^2_1 + f_5 \varphi_2 \zeta = 0$$
$$\frac{\partial w_v}{\partial \varphi^0_2} = f_3 (\chi^2_2 + 2 \chi_1 \chi_3) + f_4 \varphi^2_2 - f_5 \varphi_1 \zeta = 0$$

(48)

This set of equations are satisfied by three types of vacuum alignment

$$\langle \chi \rangle = v_{\chi} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \quad \langle \varphi \rangle = v_{\varphi} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \quad \langle \zeta \rangle = 0$$

(49)

with the conditions

$$v_{\chi}^2 = -\frac{f_2}{3f_3} v_{\varphi}^2, \quad v_{\varphi} \text{ undetermined}$$

(50)

The second is

$$\langle \chi \rangle = v_{\chi} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \quad \langle \varphi \rangle = v_{\varphi} \left( \begin{array}{c} 1 \\ -1 \\ -1 \end{array} \right), \quad \langle \zeta \rangle = v_{\zeta}$$

(51)

with the relations

$$v_{\chi}^2 = -\frac{f_2^2 f_4 + 2f_1 f_2 f_5}{12 f_1^2 f_3} v_{\zeta}^2, \quad v_{\varphi} = -\frac{f_2}{2 f_1} v_{\zeta}$$

(52)

where $v_{\zeta}$ is undetermined. The third solution is

$$\langle \chi \rangle = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right), \quad \langle \varphi \rangle = v_{\varphi} \left( \begin{array}{c} 1 \\ -1 \\ -1 \end{array} \right), \quad \langle \zeta \rangle = v_{\zeta}$$

(53)
with
\[
v_\varphi = f_5 \frac{f_6}{f_4} v_\zeta, \quad v_\zeta \text{ undetermined} \quad (54)
\]

Thus, without assuming any fine-tuning among the parameters \( f_i (1 = 1 - 5) \), the VEVs \( v_\chi \) and \( v_\varphi \) are expected to be of the same order of magnitude for the three cases,
\[
v_\chi \sim v_\varphi \quad (55)
\]

Only the first alignment can produce the results in the previous sections, we need of some soft masses in order to discriminate it as the lowest minimum of the scalar potential, since the values of the scalar potential for the three solutions are exactly the same in the SUSY limit. It is well-known that the soft mass usually is of order TeV, consequently the difference of the scalar potential for different vacuum solutions is marginal comparing with the flavon VEVs. As has been shown in the previous section, at LO the \( S_4 \) flavor symmetry is broken by the VEV of \( \chi \) and \( \varphi \) in the neutrino sector. The flavon fields \( \phi, \eta, \Delta \) and \( \xi \) are involved in generating the quark and charged lepton masses, their vacuum configurations are determined by

\[
\begin{align*}
\frac{\partial w_v}{\partial \phi_1^0} &= 2g_1 (\phi_1^2 - \phi_2 \phi_3) + g_2 (\eta_1 \phi_2 + \eta_2 \phi_3) = 0 \\
\frac{\partial w_v}{\partial \phi_2^0} &= 2g_1 (\phi_2^2 - \phi_1 \phi_3) + g_2 (\eta_1 \phi_1 + \eta_2 \phi_2) = 0 \\
\frac{\partial w_v}{\partial \phi_3^0} &= 2g_1 (\phi_3^2 - \phi_1 \phi_2) + g_2 (\eta_1 \phi_3 + \eta_2 \phi_1) = 0 \\
\frac{\partial w_v}{\partial \rho^0} &= g_3 (\phi_1^2 + 2\phi_2 \phi_3) + 2g_4 \eta_1 \eta_2 = 0 \quad (56)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w_v}{\partial \Delta_1^0} &= 2h_1 (\Delta_1^2 - \Delta_2 \Delta_3) + h_2 \Delta_1 \xi = 0 \\
\frac{\partial w_v}{\partial \Delta_2^0} &= 2h_1 (\Delta_2^2 - \Delta_1 \Delta_3) + h_2 \Delta_2 \xi = 0 \\
\frac{\partial w_v}{\partial \Delta_3^0} &= 2h_1 (\Delta_3^2 - \Delta_1 \Delta_2) + h_2 \Delta_3 \xi = 0 \quad (57)
\end{align*}
\]

The equations in Eq. (56) lead to two different vacuum configurations, the first is

\[
\langle \phi \rangle = \begin{pmatrix} 0 \\ v_\phi \\ 0 \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ v_\eta \end{pmatrix} \quad (58)
\]

with
\[
v_\phi = \frac{g_2}{2g_1} v_\eta, \quad v_\eta \text{ undetermined} \quad (59)
\]

The second is

\[
\langle \phi \rangle = v_\phi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \eta \rangle = v_\eta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (60)
\]
with $v_\phi^2 = \frac{2a_1}{3a_3} v_\eta^2$ and $v_\eta$ undetermined. Obviously $v_\phi$ and $v_\eta$ are expected to be of the same order for both solutions. As before, we select the first vacuum configuration. The equations in Eq. (57) admit three unequivalent solutions, the first is

$$\langle \Delta \rangle = \begin{pmatrix} v_\Delta \\ 0 \\ 0 \end{pmatrix}, \quad \langle \xi \rangle = v_\xi$$

with $v_\Delta = -\frac{h_2}{2m_1} v_\xi$ and $v_\xi$ undetermined. In a similar way, the VEVs $v_\Delta$ and $v_\xi$ should be of the same order. The second solution is

$$\langle \Delta \rangle = v_\Delta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \xi \rangle = 0$$

with $v_\Delta$ undetermined. The third vacuum configuration is

$$\langle \Delta \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \xi \rangle = v_\xi$$

and $v_\xi$ remaining undetermined. The first vacuum solution is chosen in the present model. Acting on the vacuum configurations of Eq. (49), Eq. (58) and Eq. (61) with the elements of the flavor symmetry group $S_4$, we can generate other minima of the scalar potential. These new minima are physically equivalent to the original set, and they all lead to the same physics, i.e. fermion masses and flavor mixings. As a result, without loss of generality we can analyze the model by choosing exactly the vacuum in Eqs. (49, 58, 61) as local minimum, and the different scenarios are related by field redefinitions. Since no superpotential couplings of positive mass dimension are involved in the flavon superpotential, the trivial solution with all flavon VEVs vanishing can not be excluded. However, by taking into account the contribution of the soft mass terms, we can discriminate the configuration in Eqs. (49, 58, 61) as the lowest minimum of the scalar potential.\footnote{We consider the soft terms involving $\chi$ and $\varphi$, which is generally written as $m_\chi^2 |\chi|^2 + m_\varphi^2 |\varphi|^2 + \tilde{m}_\chi^2 \chi^2 + \tilde{m}_\varphi^2 \varphi^2$. By choosing $m_\chi^2$, $m_\varphi^2$, $\tilde{m}_\chi^2$ and $\tilde{m}_\varphi^2 < 0$, the vacuum shown in Eqs. (49, 58, 61) are more stable than the vanishing VEVs configuration.} Regarding the size of the flavon VEVs, we have the relations $v_\chi \sim v_\varphi$, $v_\phi \sim v_\eta$ and $v_\Delta \sim v_\xi$, as have been demonstrated above.\footnote{In the absence of specific dynamical tricks (see Ref. \[1\] for a model in which such a trick is implemented), the uncorrelated VEVs naturally have the same order of magnitude, this is consistent with the the constraints from the measured mass hierarchies and flavor mixing. Moreover, we note that the correlation of scales of more flavon VEVs can be achieved by adding further driving fields, whereas this procedure would introduce more fields and free parameters.} Furthermore, The magnitudes of the flavon VEVs are determined by the patterns of fermion mass hierarchy and mixing. From the LO predictions presented in the previous sections, we find that in order to reproduce the correct patterns of fermion masses and flavor mixings, a common order of magnitude for the VEVs scaled by the cutoff $\Lambda$ is expected

$$\frac{v_\chi}{\Lambda} \sim \frac{v_\varphi}{\Lambda} \sim \frac{v_\phi}{\Lambda} \sim \frac{v_\eta}{\Lambda} \sim \frac{v_\Delta}{\Lambda} \sim \frac{v_\xi}{\Lambda} \sim \lambda_c^2.$$  

\[64\]
Moreover, we will show in the following that the successful LO predictions are not destroyed by the subleading corrections when the vacuum alignment is chosen as has been stated above. Similar to the fact that the F-terms of the driving fields are the origin of the alignment of the flavon VEVs, we can derive the vacuum structure of the driving fields from the F-terms of the flavon fields. As all terms in the flavon superpotential are linear in the driving fields, the configuration in which all these fields have vanishing VEVs is in any case a solution. Moreover, we have checked that this is unique vacuum configuration of driving fields in our model.

4 Subleading corrections

At the next level of expansion in $1/\Lambda$, the superpotentials $w_\nu$, $w_u$, $w_d$ and $w_v$ are corrected by higher dimensional operators whose contributions are suppressed by at least one power of $1/\Lambda$. The corrections to $w_v$ result in small deviations from the LO vacuum alignment thus affect the results for fermion masses and mixings. In addition the fermion mass and mixing matrices are corrected by the subleading operators in $w_\nu$, $w_u$ and $w_d$. In the following we shall first present the analysis for the subleading corrections to the vacuum alignment, then move to the corrections to fermion mass matrices.

4.1 Corrections to the vacuum alignment

We detail the discussion of this issue in Appendix B, here we only present the results. The vacuum configuration is shifted into

$$
\langle \chi \rangle = \begin{pmatrix} v_\chi + \delta v_{\chi 1} \\ v_\chi + \delta v_{\chi 2} \\ v_\chi + \delta v_{\chi 3} \end{pmatrix}, \quad \langle \varphi \rangle = \begin{pmatrix} v_\varphi \\ v_\varphi + \delta v_{\varphi 2} \end{pmatrix}, \quad \langle \zeta \rangle = \delta v_\zeta
$$

$$
\langle \phi \rangle = \begin{pmatrix} \delta v_{\phi 1} \\ v_\phi + \delta v_{\phi 2} \\ \delta v_{\phi 3} \end{pmatrix}, \quad \langle \eta \rangle = \begin{pmatrix} \delta v_\eta 1 \\ v_\eta \end{pmatrix}
$$

$$
\langle \Delta \rangle = \begin{pmatrix} v_\Delta + \delta v_{\Delta 1} \\ \delta v_{\Delta 2} \\ \delta v_{\Delta 3} \end{pmatrix}, \quad \langle \xi \rangle = v_\xi
$$

(65)

where $v_\varphi$, $v_\eta$ and $v_\xi$ remain undetermined. Since we typically have $VEV/\Lambda \sim \lambda_c^2$ at LO, we expect the order of magnitude of the shifts as follows

$$
\delta v_{\chi 1}/v_\chi \sim \lambda_c^2, \quad \delta v_{\chi 2}/v_\chi \sim \lambda_c^2, \quad \delta v_{\chi 3}/v_\chi \sim \lambda_c^2, \quad \delta v_{\varphi 2}/v_\varphi \sim \lambda_c^2, \quad \delta v_\xi/v_\chi \sim \lambda_c^2
$$

$$
\delta v_{\phi 1}/v_\phi \sim \lambda_c^2, \quad \delta v_{\phi 2}/v_\phi \sim \lambda_c^4, \quad \delta v_{\phi 3}/v_\phi \sim \lambda_c^4, \quad \delta v_{\eta 1}/v_\eta \sim \lambda_c^4
$$

$$
\delta v_{\Delta 1}/v_\Delta \sim \lambda_c^4, \quad \delta v_{\Delta 2}/v_\Delta \sim \lambda_c^4, \quad \delta v_{\Delta 3}/v_\Delta \sim \lambda_c^4
$$

(66)

From Appendix B, we can see that the subleading operators linear in $\chi^0$ and $\varphi^0$ are suppressed by $\Lambda$, while the subleading operators linear in $\phi^0$, $\rho^0$ and $\Delta^0$ are suppressed by $\Lambda^2$, due to the constraint of the flavor symmetry $S_4 \times Z_3 \times Z_4$. Consequently the shifts are of order $\lambda_c^2$ or $\lambda_c^4$ with respect to the LO VEVs, as is displayed in Eq. (66).
4.2 Corrections to fermion masses and mixings

The fermion masses and mixings receive corrections from both the shifted VEVs and the higher order terms in the superpotentials $w$, $w_u$ and $w_d$. We can obtain the mass matrix by inserting the modified VEV into the LO operators plus the contribution of the higher dimensional operators evaluated with the LO VEVs. For convenience, we denote $\chi$, $\varphi$ and $\zeta$ with $\Phi_\nu$, $\Delta$ and $\xi$ with $\Phi_{f1}$, $\phi$ and $\eta$ with $\Phi_{f2}$ in the following.

4.2.1 Corrections to up quark sector

First we discuss the corrections to the up type quark mass matrix coming from the modified vacuum alignment. Plugging the shifted vacuum configuration shown in Eq. (65) into the LO superpotential in Eq. (19), we find that the (12) element receives a correction of order $\epsilon^4$ from the operators $(TT)_{11}(\phi\phi)_{11}H_5$ and $(TT)_{11}(\eta\eta)_{11}H_5$. The operator $(TT)_{2}(\phi\phi)_{2}H_5$ induces corrections of order $\epsilon^4$ to both (11) and (22) elements. The corrections to the (13) element due to modified VEVs are of order $\epsilon^3$, they arise from the contractions $TT_3(\phi\chi)_{2}H_5$, $TT_3(\eta\varphi)_{2}H_5$ and $TT_3\eta H_{45}$. Meanwhile, $TT_3(\phi\chi)_{2}H_5$ introduces correction of order $\epsilon^3$ to the (23) element.

Then we come to discuss the corrections caused by the higher dimensional operators in the matter superpotential $w_u$. We note that all corrections to the (33) element can be absorbed into the coupling $y_i$ of the LO operator $T_3T_3H_5$, and any operator comprising the superfields $T_3T_3H_{45}$ ($TTH_{45}$) and arbitrary number of flavon fields gives a vanishing contribution to the up quark mass matrix. Due to the auxiliary symmetry $Z_3 \times Z_4$, the subleading corrections to (11), (12), (21) and (22) elements arise at order $1/\Lambda^4$, they come from the following contraction

$$\frac{1}{\Lambda^4}TT\Phi_\nu\Phi_\nu\Phi_{f2}\Phi_{f2}H_5$$

Consequently the corrections to the $(ij)(i,j = 1,2)$ element from the high dimensional operators are of the same order as those from the shifted vacuum. In the same way, we find the (13) and (23) elements are corrected by the following contractions

$$\frac{1}{\Lambda^3}TT_3\Phi_{f1}\Phi_{f2}\Phi_{f2}H_5, \quad \frac{1}{\Lambda^3}TT_3\Phi_\nu\Phi_\nu\Phi_{f2}H_{45}$$

Substituting the LO VEVs into the above contractions, we notice that the corrections to the (23) and (32) elements originate from the latter contraction, whereas both operators contribute to (13) and (31) elements. In short summary, the up type quark mass matrix are corrected by both the deviations from the LO VEV alignment and the higher dimensional operators allowed by the flavor symmetry. We can parameterize the up quark mass matrix as

$$M_u = \begin{pmatrix}
8y_{11}^u\epsilon^4v_5 & 8y_{12}^u\epsilon^4v_5 & 4y_{13}^u\epsilon^2v_5 + 8y_{13}'^u\epsilon^3v_{45} \\
8y_{12}^u\epsilon^4v_5 & 8y_{22}^u\epsilon^2v_5 & 8y_{23}^u\epsilon v_45 + 4y_{23}'^u\epsilon^2v_5 \\
4y_{13}^u\epsilon^2v_5 - 8y_{13}'^u\epsilon^3v_{45} & -8y_{23}^u\epsilon v_45 + 4y_{23}'^u\epsilon^2v_5 & 8y_{33}^u v_5
\end{pmatrix}$$

(69)

where $y_{ij}^u (i,j = 1,2,3)$ are complex numbers with absolute value of order one, and they are linear combinations of the leading and subleading coefficients. We note that $y_{ij}^u$ coincides...
with the LO parameter $y_t$ up to higher order corrections which are due to two flavons and three flavons insertions in the operator $T_3H_T$. Similarly the parameters $y_{13}^u$, $y_{22}^u$, $y_{23}^u$ and $y_{23}^d$ are determined by the LO couplings in Eq. (19) up to small corrections of relative order $\varepsilon$ or $\varepsilon^2$. The mass matrix $M_u$ in Eq. (69) leads to the up type quark masses

$$m_u \approx \left[ 8y_{11}^u v_5 - 2\left(\frac{y_{13}^u}{y_{33}^u}\right)^2 v_5 + \frac{2\left(\frac{y_{13}^u}{y_{33}^u}\right)^2 y_{23}^u v_5 v_{45}}{y_{22}^u y_{33}^u v_5^2 + y_{23}^u y_{23}^u v_5^2} \right] \varepsilon^4$$

$$m_c \approx 8\left[ \frac{y_{22}^u}{y_{33}^u} \frac{v_{45}^2}{v_5^2} \right]$$

$$m_t \approx \left[ 8y_{33}^u v_5 \right]$$

(70)

The correct hierarchies among the up quark masses are obtained.

### 4.2.2 Corrections to down quark and charged lepton sector

Plugging the shifted vacuum of $\phi$ into the LO operator $T_3F\phi H_T$ leads to corrections to the (13), (23) and (33) elements of $M_d$ of order $\varepsilon^4 v_5$, this amounts to a rescaling of the parameters $y_{13}^d$, $y_{23}^d$ and $y_{23}^d$. If the non-zero shifts $\delta v_{\Delta, 23}$ are taken into account, the LO operators $T F \Delta H_T$ and $T F \Delta \xi H_T$ introduces corrections of order $\varepsilon^4 v_5$ to the (22), (31), (32), (11), (12) and (21) elements which also receive corrections of order $\varepsilon^4 v_5$ from the operators $T F O^{(5)} H_T$. At LO the superpotential $w_d$ is expanded to $1/\Lambda^3$, it is corrected by the following subleading operators

$$\frac{1}{\Lambda^4} T_3F\Phi_\nu\Phi_{f1}\Phi_{f2}\Phi_{f2}H_T,$$

$$\frac{1}{\Lambda^4} T F \Phi_{f1}\Phi_{f1}\Phi_{f1}H_T,$$

$$\frac{1}{\Lambda^4} T F \Phi_{f1}\Phi_{f1}\Phi_{f1}H_T$$

(71)

With the LO VEVs, the above high dimensional operators lead to corrections of order $\varepsilon^4 v_5$ or $\varepsilon^4 v_5$ in each entry of $M_d$. Note that the subleading terms involving the fields combination $T_3F H_T$ arise at order $1/\Lambda^3$ with the insertion of five flavon fields. Therefore we conclude that the NLO corrections to the down quark and charged lepton mass matrices can be reabsorbed into a redefinition of the LO parameters whose order of magnitudes are not changed. As a result, we can parameterize the down quark and charged lepton mass matrices in the same way as Eq. (27) and Eq. (28), and the parameters $y_{ij}^d(i, j = 1, 2, 3)$ are still used to avoid introducing extra unnecessary parameters. However, we should keep in mind that the value of $y_{ij}^d$ is different from the corresponding LO one due to the subleading corrections of relative order $\varepsilon$ or $\varepsilon^2$. Since the down quark and charged lepton mass matrices remain the same form, the down quark and charged lepton masses are still given by Eq. (30) and Eq. (31) respectively. Starting from the quark mass matrices $M_d$ in Eq. (27) and $M_u$ in Eq. (69), we can straightforwardly find the CKM mixing matrix as follows

$$V_{ud} \approx V_{cs} \approx V_{tb} \approx 1$$

$$V_{us}^* \approx -V_{cd} \approx \frac{y_{21}^d v_5}{2y_{22}^d v_{45}^2} \varepsilon + \frac{1}{2} \frac{y_{13}^u y_{24}^u v_5 v_{45}}{y_{22}^u y_{33}^u v_5^2 + (y_{23}^u)^2 v_{45}^2} \varepsilon$$

17
where $\delta \chi$ to the Majorana mass matrices. As a result, the next to leading order corrections to the LO results, the remaining operators give rise to three independent additional contributions to the Majorana mass matrices have the same structure as the LO neutrino Dirac couplings are

$$y_{\nu 1} \Lambda (FA)_3 \chi H_{45} + y_{\nu 2} \Lambda (FA)_2 \varphi H_{45}$$

(73)

Using the alignment of $\chi$ and $\varphi$ given in Eq. (49), the NLO corrections to the Dirac mass matrices read

$$\delta M_{\rho_3}^D = -\frac{3}{2} v_{45} \left( \begin{array}{ccc} -2 y_{\nu 1} v_3 \delta \chi & -y_{\nu 1} v_5 & -y_{\nu 1} v_3 \\ -y_{\nu 1} v_5 & 2 y_{\nu 1} v_3 & -y_{\nu 1} v_3 \\ -y_{\nu 1} v_3 & -y_{\nu 1} v_3 & -y_{\nu 1} v_3 \end{array} \right)$$

$$\delta M_{\rho_0}^D = -\frac{\sqrt{15}}{3} \delta M_{\rho_3}^D$$

(74)

We see that the CKM parameters are determined by the LO results up to small corrections, which is absorbed in the redefinition of the parameters. The successful LO predictions for the order of magnitudes of the CKM matrix elements are not spoiled by the subleading corrections.

4.2.3 Corrections to neutrino sector

The superpotential $w_\nu$ in Eq. (5) is corrected by terms with more flavons insertion. The VEV shifts in the LO operator do not affect the Dirac mass, the NLO corrections to the neutrino Dirac couplings are

$$\delta M_{\rho_3}^D = \left( \begin{array}{ccc} -6 x_A e \epsilon v_\chi & 3 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi & 3 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi \\ 3 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi & -6 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi & 3 x_A e \epsilon v_\chi \\ 3 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi & 3 x_A e \epsilon v_\chi & -6 x_A e \epsilon v_\chi - 3 x_B e \epsilon v_\varphi \end{array} \right)$$

(75)

We note that the subleading corrections $\delta M_{\rho_3}^D$ and $\delta M_{\rho_0}^D$ are still compatible with the TB mixing. The Majorana mass matrices of $\rho_3$ and $\rho_0$ are modified by the terms

$$\delta \chi + \delta \varphi$$

where $\delta \chi$ and $\delta \varphi$ denote the shifted vacua of the flavons $\chi$ and $\varphi$. The operators $A^2 \chi H_{24}$ and $A^2 \varphi H_{24}$ lead to mass splitting between the triplet $\rho_3$ and the singlet $\rho_0$, and the resulting contributions to the Majorana mass matrices have the same structure as the LO predictions. Taking into account the possibility of absorbing the corrections partly into the LO results, the remaining operators give rise to three independent additional contributions to the Majorana mass matrices. As a result, the next to leading order corrections to $M_{\rho_3}^M$ and $M_{\rho_0}^M$ can be parameterized as
Eqs. (4,5) and the NLO corrections in Eqs. (74,76,77), the light neutrino mass matrix can be written as

\[ M^M_{\nu_\odot} = \left( \begin{array}{ccc} 0 & -\bar{x}_C & \bar{x}_D - \bar{x}_E \\ -\bar{x}_C & \bar{x}_D + 2\bar{x}_E & 0 \\ \bar{x}_D - \bar{x}_E & 0 & 2\bar{x}_C \end{array} \right) \frac{v_\chi^2}{\Lambda} \]  

where \( \epsilon = \frac{1}{\sqrt{30} A^2} \). The first term in both Eq. (76) and Eq. (77) represents the contribution of \( \frac{\epsilon}{\Lambda} A^2 \chi H_{24} \) and \( \frac{\epsilon}{\Lambda} A^2 \varphi H_{24} \), the second term denotes the effects of the modified vacuum configuration and the subleading terms with the form \( AA\Phi_1 \Phi_2 \), which leads to deviations from the TB mixing pattern in the neutrino sector. With the LO contributions shown in Eqs. (4,5) and the NLO corrections in Eqs. (74,76,77), the light neutrino mass matrix can be obtained straightforwardly via the see-saw formula. To first order in small parameters \( v_\chi/\Lambda, v_\varphi/\Lambda \) and \( \epsilon \), we find that the light neutrino masses are given by

\[
m_1 \approx \frac{2y_{12}^2 v_\Sigma^2}{5(-3\lambda_1 v_\chi + \lambda_2 v_\varphi)} + \frac{9(-3x_A v_\chi + x_B v_\varphi) v_\Sigma^2}{10(-3\lambda_1 v_\chi + \lambda_2 v_\varphi)^2} \epsilon + \frac{(2\bar{x}_C - \bar{x}_D + 2\bar{x}_E) v_\chi^2 v_\varphi^2}{5(-3\lambda_1 v_\chi + \lambda_2 v_\varphi)^2} \Lambda
\]

\[
m_2 \approx -\frac{y_{12}^2 v_\Sigma^2}{5\lambda_2 v_\varphi} - \frac{9x_B y_{12}^2 v_\Sigma^2}{20\lambda_2 v_\varphi} \epsilon + \frac{\bar{x}_D v_\chi^2 v_\varphi^2}{10\lambda_2 v_\varphi^2} \Lambda
\]

\[
m_3 \approx -\frac{2y_{12}^2 v_\Sigma^2}{5(3\lambda_1 v_\chi + \lambda_2 v_\varphi)} - \frac{9(3x_A v_\chi + x_B v_\varphi) v_\Sigma^2}{10(3\lambda_1 v_\chi + \lambda_2 v_\varphi)^2} \epsilon + \frac{(2\bar{x}_C + \bar{x}_D + 2\bar{x}_E) v_\chi^2 v_\varphi^2}{5(3\lambda_1 v_\chi + \lambda_2 v_\varphi)^2} \Lambda
\]

The lepton mixing angles are modified as

\[
\sin \theta_{13} = |U_{e3}| \approx \left| \left( \frac{y_{12}^d}{6\sqrt{2} y_{12}^{d*} v_\Sigma} \right)^* - \frac{\bar{x}_D v_\chi^2 (3\lambda_1 v_\chi^{*2} + \lambda_2 v_\varphi^* + 3\lambda_1 v_\chi - \lambda_2 v_\varphi) \bar{x}_D (v_\chi^{*2})^2}{\sqrt{2} [3\lambda_1 v_\chi + \lambda_2 v_\varphi]^2 - [3\lambda_1 v_\chi + \lambda_2 v_\varphi]^2] \Lambda} \right|
\]

\[
\sin^2 \theta_{12} \approx \frac{1}{3} + \frac{y_{12}^d v_\Sigma^*}{18 \sqrt{2} y_{12}^{d*} v_\Sigma} + \frac{1}{3} \left[ 3\lambda_1 v_\chi + \lambda_2 v_\varphi^2 - 4 \lambda_2 v_\varphi^2 \right] \Lambda
\]

\[
\sin^2 \theta_{23} \approx \frac{1}{2} - \frac{3\bar{x}_D v_\chi^2 \lambda_1^2 v_\chi^2 + 3\bar{x}_D v_\chi^2 \lambda_1 v_\chi^2}{2[3\lambda_1 v_\chi + \lambda_2 v_\varphi^2 - 3\lambda_1 v_\chi + \lambda_2 v_\varphi^2] \Lambda} - \frac{3\bar{x}_D v_\chi^2 \lambda_1 v_\chi^2}{2[4 \lambda_2 v_\varphi^2 - 3 \lambda_1 v_\chi + \lambda_2 v_\varphi^2] \Lambda}
\]

We note that the subleading terms in the neutrino sector induce corrections of order \( \epsilon \) to all three mixing angles. and the deviation of \( \theta_{23} \) from its TB value is mainly determined by the NLO contributions in the neutrino sector. The lepton mixing angles are still compatible with the current experimental data. In particular, the reactor angle \( \theta_{13} \) is within the reach of next generation neutrino oscillation experiments.

\[ v_\Sigma = v_{24}/\sqrt{30} \text{ diag}(2, 2, 2, -3, -3). \]
5 Phenomenological consequences

In this section, we shall present the predictions for some phenomenologically interesting observables in our model. Here we are particularly interested in the neutrino sector, both the LO and the NLO contributions are taken into account in the following.

5.1 Lepton mixing angles

The analytic expressions for the mixing angles are shown in Eq. (79), which allows us to estimate the deviations from the TB mixing pattern qualitatively. In order to see clearly the allowed ranges of the mixing parameters, as well as for cross-checking the reliability of the analytical estimates, we shall perform a numerical analysis. All the involved LO and NLO coefficients are taken to random complex number with absolute value in the interval \([1/3, 3]\), the expansion parameters ɛ and ɛ have been fixed at the representative value of 0.04 and 0.001 respectively \(^6\) and the VEV ratio \(v_{\text{at}}/v_{\text{at}}\) is set to the indicative value 0.22. Furthermore, we require the oscillation parameters 2, 2, and 2, sin 2 and sin 2 to lie in their 3σ interval. The allowed regions of sin 2 − sin 2 and sin 2 − sin 2 for both normal hierarchy and inverted hierarchy are showed in Fig. 2. It is obvious that rather small 2 is favored for both NH and IH spectrum, which is

\(^{6}\text{For other indicative values of the small parameters ɛ and ɛ, the resulting numerical results don’t change qualitatively.}\)
consistent with our theoretical analysis.

5.2 Neutrinoless double beta decay

The discovery of neutrinoless double beta decay $0\nu2\beta$ is very important because it could directly establish lepton number violation and the Majorana nature of neutrino. The $0\nu2\beta$ decay amplitude is proportional to the effective mass $|m_{ee}|$, which is (11) element of the neutrino mass matrix in the basis where the charged lepton mass matrix is real and diagonal. The allowed region for $m_{ee}$ is displayed in Fig. 3. The horizontal lines denote the future sensitivity of some $0\nu2\beta$ decay experiments, which are 15 meV and 20 meV respectively of CUORE [45], Majorana [46]/GERDA III [47] experiments.

![Figure 3: Scatter plot of the effective mass $|m_{ee}|$ with respect to the lightest neutrino mass. The blue corresponds to the normal hierarchy neutrino spectrum and the red to the inverted hierarchy spectrum.](image)

We see that $|m_{ee}|$ is predicted be below the present bound from the Heidelberg-Moscow experiment [44]. For the NH spectrum, $|m_{ee}|$ can be so small to be close to zero, while the scatter plot indicates a lower bound for $|m_{ee}|$ of about 14 meV in the case of IH spectrum. It is quite close to the future experimental sensitivity so that $0\nu2\beta$ decay should be observed by future experiments for IH spectrum. We note that a partial cancellation between the LO and NLO contributions takes place, so that the lightest neutrino mass can be very small of order $10^{-4}$ eV in contrast with the LO constraints shown in Eq. (17). This however requires an additional fine tuning of the parameters which has been reproduced in our numerical analysis only partially and by very few points.

In Fig. 4 we plot the effective mass $m_\beta = \left[ \sum_k |(U_{PMNS})_{ek}|^2 m_k^2 \right]^{1/2}$ in $\beta$ decay experiments, which could measure the non-zero neutrino masses. We clearly see that the effective mass $m_\beta$ is predicted to be below the prospective sensitivity 0.2 eV of the KATRIN experiment for both NH and IH neutrino mass spectrum [48].
Figure 4: $m_\beta$ as a function of the lightest neutrino mass. The blue corresponds to the normal hierarchy neutrino spectrum and the red to the inverted hierarchy spectrum.

5.3 Sum of neutrino masses

The prediction for the sum of neutrino mass is presented in Fig. 5. The horizontal line is the cosmological bound at 0.19 eV, which is obtained by combining the data from the Cosmic Microwave Background (CMB) anisotropy (from WMAP 5y [49], Arcminute Cosmology Bolometer Array Receiver (ACBAR) [50], Very Small Array (VSA) [51], Cosmic Background Imager (CBI) [52] and BOOMERANG [53] experiments) plus the large-scale structure (LSS) information on galaxy clustering (from the Luminous Red Galaxies Sloan Digital Sky Survey (SDSS) [54]) plus the Hubble Space Telescope (HST) plus the luminosity distance SN-Ia data of [55] plus the BAO data from [56] and finally plus the small scale primordial spectrum from Lyman-alpha (Ly$\alpha$) forest clouds [57]. We see that our model predicts $\sum m_k$ too similar for both hierarchies to be distinguished using the current cosmological information on the sum of the neutrino masses.

The present model has rich phenomenological implications, we only study few interesting observables here. Especially the predictions for lepton flavor violation branching ratios and leptogenesis deserve to be studied carefully, which are important to test the model and distinguish this model from other discrete flavor symmetry models. These topics will be discussed in future work [58].

6 Conclusion

In this work, we have built a SUSY $SU(5)$ model based on the flavor symmetry $S_4 \times Z_3 \times Z_4$. Three generations of adjoint matter superfields are introduced, and they are assigned to transform as $3_1$ of $S_4$. The neutrino masses are generated via the combination of type I and type III see-saw mechanism in the model. To describe quarks, we make use of the doublet representation of $S_4$, we accommodate the first two generations of tenplets.
Figure 5: The sum of neutrino masses $\sum_{k} m_k$ as a function of the lightest neutrino mass. The blue corresponds to the normal hierarchy neutrino spectrum and the red to the inverted hierarchy spectrum.

10 in doublet under $S_4$. In particular, the first generation 10 is assigned to the second component of the doublet, and the second generation as the first component, whereas the third generation of 10 is kept invariant. The observed mass hierarchies of quarks are reproduced naturally via the spontaneous breaking of the flavor symmetry without invoking the Froggatt-Nielsen mechanism. In order to generate the CKM mixing between the first two generations, we require a moderate fine tuning $\frac{v_4}{v_5} \sim \lambda_c$, then the model generates the observed pattern of the CKM mixing matrix.

In the neutrino sector, the flavor symmetry $S_4$ is broken down to the Klein four subgroup by the VEV of the flavon fields $\chi$ and $\varphi$ at LO. The resulting light neutrino mass matrix is exactly diagonalized by the TB mixing matrix, and the neutrino mass spectrum can be normal hierarchy or inverted hierarchy. There are only three independent parameters in the neutrino sector at LO, the model is rather predictive so that the lightest neutrino mass is constrained to be larger than 0.011 eV and 0.028 eV for normal hierarchy and inverted hierarchy respectively. The mixing of the left-handed charged leptons results in corrections to the TB mixing pattern, which is described in terms of a well-known lepton mixing sum rule, and the reactor mixing angle is predicted to be close to three degrees.

The subleading corrections to the flavon alignment and the fermion mass matrices have been analyzed carefully. We show that the successful LO predictions for the pattern of quark masses and CKM mixing angles are not spoiled by the subleading contributions, and all the three lepton mixing angles receive corrections of order $\lambda_c^2$. The phenomenological implications of the model are investigated in details, we find that future neutrinoless double beta decay experiment with high precision is an important test of the model, it allow us to distinguish the NH spectrum from the IH one. Finally we note that the predictions presented in the work are valid just below the GUT scale. To determine the fermion masses and mixings and phenomenologically interesting observables at the electroweak scale, we should study the renormalization group running carefully. These issues will be studied
elsewhere in future [58].

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Appendix A: The group $S_4$ and its representation

The group $S_4$ is the group of the permutations of four objects, it has $4! = 24$ elements. Let a generic permutation be denoted by $(1, 2, 3, 4) \to (n_1, n_2, n_3, n_4) \equiv (n_1 n_2 n_3 n_4)$. $S_4$ can be generated by the two basic permutations $S$ and $T$ where $S = (2341)$ and $T = (2314)$. We can check that $S^4 = T^3 = 1$, $ST^2S = T$. $S_4$ has five conjugate classes as follows

$C_1 : 1$
$C_2 : STS^2 = (2134), TSTS^2 = (3214), ST^2 = (4231), S^2TS = (1324), TST = (1432), T^2S = (1243)$
$C_3 : T^2S^2T^2 = (2143), S^2 = (3412), T^2S^2T = (3241)$
$C_4 : T = (2314), T^2 = (3124), T^2S^2 = (2431), S^2T = (4132), S^2TS^2 = (3241), STS = (4213), S^2T^2 = (1342), T^2S = (1423)$
$C_5 : S = (2341), T^2ST = (2413), ST = (3421), TS = (3142), TST^2 = (4312), S^3 = (1234)$

The structure of the group $S_4$ is rather rich, it has thirty proper subgroups of orders 1, 2, 3, 4, 6, 8, 12 or 24. Concretely, the details about the subgroups of $S_4$ can be found in Ref. [24]. For a finite group the number of irreducible representation is equal to the number of conjugate class. Consequently the $S_4$ group have five irreducible representation: $1_1$, $1_2$, $2$, $3_1$ and $3_2$, which are all real. Concretely the representation matrix can be chosen as

\[
S = 1, \\
S = -1, \\
S = \frac{1}{3} \begin{pmatrix}
-1 & 2\omega & 2\omega^2 \\
2\omega & 2\omega^2 & -1 \\
2\omega^2 & -1 & 2\omega
\end{pmatrix}, \\
T = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}
\]

for $1_1$

for $1_2$

for $2$

for $3_1$

for $3_2$

The characters, i.e. the trace of the representation matrix, are given in the character table (please see Table 3). From the character table of the $S_4$ group, we can straightforwardly obtain the multiplication rules between the various representations

\[
1_1 \otimes 1_1 = 1_{(i+j) \text{mod } 2+1}, \quad 1_1 \otimes 1_2 = 2, \quad 1_1 \otimes 3_j = 3_{((i+j) \text{mod } 2+1} \\
2 \otimes 2 = 1_1 \oplus 1_2 \oplus 2, \quad 2 \otimes 3_i = 3_1 \oplus 3_2, \quad 3_i \otimes 3_i = 1_1 \oplus 2 \oplus 3_1 \oplus 3_2, \\
3_1 \otimes 3_2 = 1_2 \oplus 2 \oplus 3_1 \oplus 3_2, \quad \text{with } i, j = 1, 2
\]

(80)

Starting from the explicit matrix representation, we get the product decomposition rules of the $S_4$ group. In the following we use $\alpha_i$ to denote the elements of the first representation of the product and $\beta_i$ to indicate those of the second representation.
| Classes |  
|---------|
| $C_1$  | $C_2$  | $C_3$  | $C_4$  | $C_5$  |
| $1_1$   | 1      | 1      | 1      | 1      |
| $1_2$   | 1      | -1     | 1      | 1      |
| $2$     | 2      | 0      | 2      | -1     |
| $3_1$   | 3      | 1      | -1     | 0      |
| $3_2$   | 3      | -1     | -1     | 0      |

Table 3: Character table of the $S_4$ group.

- $1_2 \otimes 1_2 = 1_1$
  
  $1_1 \sim \alpha \beta$

- $1_2 \otimes 2 = 2$
  
  $2 \sim \begin{pmatrix} \alpha \beta_1 \\ -\alpha \beta_2 \end{pmatrix}$

- $1_2 \otimes 3_1 = 3_2$
  
  $3_2 \sim \begin{pmatrix} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{pmatrix}$

- $1_2 \otimes 3_2 = 3_1$
  
  $3_1 \sim \begin{pmatrix} \alpha \beta_1 \\ \alpha \beta_2 \\ \alpha \beta_3 \end{pmatrix}$

- $2 \otimes 2 = 1_1 \oplus 1_2 \oplus 2$
  
  $1_1 \sim \alpha_1 \beta_2 + \alpha_2 \beta_1$
  
  $1_2 \sim \alpha_1 \beta_2 - \alpha_2 \beta_1$

  $2 \sim \begin{pmatrix} \alpha_2 \beta_2 \\ \alpha_1 \beta_1 \end{pmatrix}$

- $3_1 \otimes 3_1 = 3_2 \otimes 3_2 = 1_1 \oplus 2 \oplus 3_1 \oplus 3_2$
  
  $1_1 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2$

  $2 \sim \begin{pmatrix} \alpha_2 \beta_2 + \alpha_1 \beta_3 + \alpha_3 \beta_1 \\ \alpha_3 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1 \end{pmatrix}$

  $3_1 \sim \begin{pmatrix} 2\alpha_1 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ 2\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ 2\alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1 \end{pmatrix}$

  $3_2 \sim \begin{pmatrix} \alpha_2 \beta_3 - \alpha_3 \beta_2 \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ \alpha_3 \beta_1 - \alpha_1 \beta_3 \end{pmatrix}$
\[ \bullet 3_1 \otimes 3_2 = 1_2 \oplus 2 \oplus 3_1 \oplus 3_2 \]

\[ 1_2 \sim \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 \quad (92) \]

\[ 2 \sim \left( \begin{array}{c}
\alpha_2 \beta_2 + \alpha_1 \beta_3 + \alpha_3 \beta_1 \\
-\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1
\end{array} \right) \quad (93) \]

\[ 3_1 \sim \left( \begin{array}{c}
\alpha_2 \beta_3 - \alpha_3 \beta_2 \\
\alpha_1 \beta_2 - \alpha_2 \beta_1 \\
\alpha_3 \beta_1 - \alpha_1 \beta_3
\end{array} \right) \quad (94) \]

\[ 3_2 \sim \left( \begin{array}{c}
2\alpha_1 \beta_1 - \alpha_2 \beta_3 - \alpha_3 \beta_2 \\
2\alpha_3 \beta_3 - \alpha_1 \beta_2 - \alpha_2 \beta_1 \\
2\alpha_2 \beta_2 - \alpha_1 \beta_3 - \alpha_3 \beta_1
\end{array} \right) \quad (95) \]

\[ \bullet 2 \otimes 3_1 = 3_1 \oplus 3_2 \]

\[ 3_1 \sim \left( \begin{array}{c}
\alpha_1 \beta_2 + \alpha_2 \beta_3 \\
\alpha_1 \beta_3 + \alpha_2 \beta_1 \\
\alpha_1 \beta_1 + \alpha_2 \beta_2
\end{array} \right) \quad (96) \]

\[ 3_2 \sim \left( \begin{array}{c}
\alpha_1 \beta_2 - \alpha_2 \beta_3 \\
\alpha_1 \beta_3 - \alpha_2 \beta_1 \\
\alpha_1 \beta_1 - \alpha_2 \beta_2
\end{array} \right) \quad (97) \]

\[ \bullet 2 \otimes 3_2 = 3_1 \oplus 3_2 \]

\[ 3_1 \sim \left( \begin{array}{c}
\alpha_1 \beta_2 - \alpha_2 \beta_3 \\
\alpha_1 \beta_3 - \alpha_2 \beta_1 \\
\alpha_1 \beta_1 - \alpha_2 \beta_2
\end{array} \right) \quad (98) \]

\[ 3_2 \sim \left( \begin{array}{c}
\alpha_1 \beta_2 + \alpha_2 \beta_3 \\
\alpha_1 \beta_3 + \alpha_2 \beta_1 \\
\alpha_1 \beta_1 + \alpha_2 \beta_2
\end{array} \right) \quad (99) \]

We note that the multiplication rules presented above are in accordance with the results in Ref. [23].

**Appendix B: Vacuum alignment beyond the leading order**

In this appendix we discuss the subleading terms of the driving superpotential \( w_v \) and the resulting corrections to the LO vacuum alignment. At the next level of approximation the LO driving superpotential is corrected by operators of higher dimension whose contributions are suppressed by at least one power of \( 1/\Lambda \). As a result, the superpotential depending on the driving fields becomes

\[ w_v = w_v^0 + \delta w_v \quad (100) \]
The leading order term $w^0_v$ reads

\[
    w^0_v = f_1 \chi^0(\chi \varphi)_{3_2} + f_2 \chi^0 \chi \zeta + f_3 \varphi^0(\chi \chi)_{2} + f_4 \varphi^0(\varphi \varphi)_2 + f_5 \varphi^0 \varphi \zeta + g_1 \phi^0(\phi \phi)_{3_1} + g_2 \phi^0(\eta \phi)_{3_1} + g_3 \phi^0(\phi \eta)_{1_1} + g_4 \phi^0(\eta \eta)_{1_1} + h_1 \Delta^0(\Delta \Delta)_{3_1} + h_2 \Delta^0 \Delta \zeta
\]  

(101)

In the SUSY limit, $w^0_v$ leads to the following vacuum configuration

\[
    \langle \chi \rangle = \left( \begin{array}{c} v_X \\ v_X \\ v_X \end{array} \right), \quad \langle \varphi \rangle = \left( \begin{array}{c} v_\varphi \\ v_\varphi \\ v_\varphi \end{array} \right), \quad \langle \zeta \rangle = 0, \quad \text{with} \quad v_\chi^2 = -\frac{f_4}{3f_3} v_\varphi^2
\]

\[
    \langle \phi \rangle = \left( \begin{array}{c} 0 \\ v_\phi \\ 0 \end{array} \right), \quad \langle \eta \rangle = \left( \begin{array}{c} 0 \\ v_\eta \\ 0 \end{array} \right), \quad \text{with} \quad v_\phi = -\frac{g_2}{2g_1} v_\eta
\]

\[
    \langle \Delta \rangle = \left( \begin{array}{c} v_\Delta \\ 0 \\ 0 \end{array} \right), \quad \langle \xi \rangle = v_\xi, \quad \text{with} \quad v_\Delta = -\frac{h_2}{2h_1} v_\xi
\]  

(102)

The correction term $\delta w_v$ consists of the most general subleading operators linear in the driving fields, and they should be invariant under the flavor symmetry $S_4 \times Z_3 \times Z_4$.

\[
    \delta w_v = \frac{1}{\Lambda} \sum_{i=1}^{15} k_i \mathcal{O}_i^{\zeta_0} + \frac{1}{\Lambda} \sum_{i=1}^{10} w_i \mathcal{O}_i^{\zeta_0} + \frac{1}{\Lambda^2} \sum_{i=1}^{39} s_i \mathcal{O}_i^{\zeta_0} + \frac{1}{\Lambda^2} \sum_{i=1}^{18} r_i \mathcal{O}_i^{\zeta_0} + \frac{1}{\Lambda^2} \sum_{i=1}^{28} t_i \mathcal{O}_i^{\Delta_0}
\]  

(103)

where $k_i$, $w_i$, $s_i$, $r_i$ and $t_i$ are order one coefficients, their specific values are not determined by the flavor symmetry. \{\mathcal{O}_i^{\zeta_0}, \mathcal{O}_i^{\zeta_0}, \mathcal{O}_i^{\zeta_0}, \mathcal{O}_i^{\zeta_0}, \mathcal{O}_i^{\Delta_0}\} denote the complete set of subleading contractions invariant under $S_4 \times Z_3 \times Z_4$.

\[
    \mathcal{O}_1^{\zeta_0} = (\chi^0 \chi)_{2}(\phi \Delta)_{3_2}, \quad \mathcal{O}_2^{\zeta_0} = (\chi^0 \chi)_{3_1}(\phi \Delta)_{3_1}, \quad \mathcal{O}_3^{\zeta_0} = (\chi^0 \chi)_{3_2}(\phi \Delta)_{3_2},
\]

\[
    \mathcal{O}_4^{\zeta_0} = (\chi^0 \varphi)_{3_1}(\phi \Delta)_{3_1}, \quad \mathcal{O}_5^{\zeta_0} = (\chi^0 \varphi)_{3_2}(\phi \Delta)_{3_2}, \quad \mathcal{O}_6^{\zeta_0} = (\chi^0 \chi)_{3_1} \phi \xi,
\]

\[
    \mathcal{O}_7^{\zeta_0} = (\chi^0 \varphi)_{3_1} \phi \xi, \quad \mathcal{O}_8^{\zeta_0} = (\chi^0 \chi)_{3_2}(\eta \Delta)_{3_1}, \quad \mathcal{O}_9^{\zeta_0} = (\chi^0 \chi)_{3_2}(\eta \Delta)_{3_2},
\]

\[
    \mathcal{O}_{10}^{\zeta_0} = (\chi^0 \varphi)_{3_1}(\Delta \eta)_{3_1}, \quad \mathcal{O}_{11}^{\zeta_0} = (\chi^0 \varphi)_{3_2}(\eta \Delta)_{3_2}, \quad \mathcal{O}_{12}^{\zeta_0} = (\chi^0 \chi)_{2} \eta \xi,
\]

\[
    \mathcal{O}_{13}^{\zeta_0} = \chi^0 \xi(\phi \Delta)_{3_1}, \quad \mathcal{O}_{14}^{\zeta_0} = \chi^0 \xi(\phi \Delta)_{3_1}, \quad \mathcal{O}_{15}^{\zeta_0} = \chi^0 \xi(\eta \Delta)_{3_1}
\]  

(104)

\[
    \mathcal{O}_1^{\zeta_0} = (\varphi^0 \chi)_{3_1}(\phi \Delta)_{3_1}, \quad \mathcal{O}_2^{\zeta_0} = (\varphi^0 \chi)_{3_2}(\phi \Delta)_{3_2}, \quad \mathcal{O}_3^{\zeta_0} = (\varphi^0 \varphi)_{1_1}(\phi \Delta)_{1_1},
\]

\[
    \mathcal{O}_4^{\zeta_0} = (\varphi^0 \varphi)_{2}(\phi \Delta)_{2}, \quad \mathcal{O}_5^{\zeta_0} = (\varphi^0 \chi)_{3_1} \phi \xi, \quad \mathcal{O}_6^{\zeta_0} = (\varphi^0 \chi)_{3_2}(\eta \Delta)_{3_1},
\]

\[
    \mathcal{O}_7^{\zeta_0} = (\varphi^0 \chi)_{3_2}(\eta \Delta)_{3_2}, \quad \mathcal{O}_8^{\zeta_0} = (\varphi^0 \varphi)_{2} \eta \xi, \quad \mathcal{O}_9^{\zeta_0} = \varphi^0 \xi(\phi \Delta)_{2}
\]

\[
    \mathcal{O}_{10}^{\zeta_0} = \varphi^0 \xi(\eta \xi)
\]  

(105)
The subleading contribution $\delta w_v$ induces shifts in the LO VEVs shown above, then the new vacuum configuration can be parameterized as

$$
\langle \chi \rangle = \left( \begin{array}{c} v_\chi + \delta v_{\chi_1} \\ v_\chi + \delta v_{\chi_2} \\ v_\chi + \delta v_{\chi_3} \end{array} \right), \quad \langle \varphi \rangle = \left( \begin{array}{c} v_\varphi \\ v_\varphi + \delta v_{\varphi_2} \end{array} \right), \quad \langle \zeta \rangle = \delta v_\zeta
$$

$$
\langle \phi \rangle = \left( \begin{array}{c} \delta v_{\phi_1} \\ \delta v_{\phi_2} \\ \delta v_{\phi_3} \end{array} \right), \quad \langle \eta \rangle = \left( \begin{array}{c} \delta v_{\eta_1} \\ \delta v_{\eta_2} \end{array} \right)
$$

$$
\langle \Delta \rangle = \left( \begin{array}{c} \delta v_{\Delta_1} \\ \delta v_{\Delta_2} \\ \delta v_{\Delta_3} \end{array} \right), \quad \langle \xi \rangle = v_\xi
$$
where the shifts $\delta v_{\varphi 1}$, $\delta v_{\varphi 2}$ and $\delta v_{\xi}$ have been absorbed into the redefinition of the undetermined parameters $v_{\varphi}$, $v_{\eta}$ and $v_{\xi}$ respectively. The new vacua is obtained by searching for the zeros of the F-terms, i.e., the first derivative of $w_{\nu} + \delta w_{\nu}$ with respect to the driving fields $\chi^0$, $\varphi^0$, $\phi^0$, $\rho^0$ and $\Delta^0$. By keeping only the terms linear in the shift $\delta v$ and neglecting the terms proportional to $\delta v/\Lambda$, the minimization equations become
\begin{align*}
 f_1[v_{\varphi}(\delta v_{\chi 2} - \delta v_{\chi 3}) - v_{\chi} \delta v_{\varphi 2}] + f_2 v_{\chi} \delta v_{\zeta} + a_1 v_{\chi} v_{\varphi} v_{\Delta}/\Lambda = 0 \\
 f_1[v_{\varphi}(\delta v_{\chi 2} - \delta v_{\chi 3}) - v_{\chi} \delta v_{\varphi 2}] + f_2 v_{\chi} \delta v_{\zeta} + a_2 v_{\chi} v_{\varphi} v_{\Delta}/\Lambda = 0 \\
 f_1[v_{\varphi}(\delta v_{\chi 3} - \delta v_{\chi 1}) - v_{\chi} \delta v_{\varphi 2}] + f_2 v_{\chi} \delta v_{\zeta} + a_3 v_{\chi} v_{\varphi} v_{\Delta}/\Lambda = 0 \\
 2f_3 v_{\chi}(\delta v_{\chi 1} + \delta v_{\chi 2} + \delta v_{\chi 3}) + f_5 v_{\varphi} \delta v_{\zeta} + b_1 v_{\chi} v_{\varphi} v_{\Delta}/\Lambda = 0 \\
 2f_3 v_{\chi}(\delta v_{\chi 1} + \delta v_{\chi 2} + \delta v_{\chi 3}) + 2f_4 v_{\varphi} \delta v_{\varphi 2} - f_5 v_{\varphi} \delta v_{\zeta} + b_2 v_{\chi} v_{\varphi} v_{\Delta}/\Lambda = 0
\end{align*}
(110)
where the coefficients $a_{1,2,3}$ and $b_{1,2}$ are linear combinations of the subleading coefficients
\begin{align*}
a_1 &= k_1 + k_2 + k_3 - (k_4 + k_5) v_{\varphi}/v_{\chi} - k_6 v_{\xi}/v_{\Delta} + k_7 v_{\varphi}/(v_{\chi} v_{\Delta}) + (-k_8 + k_9) v_{\eta}/v_{\phi} \\
&+ (k_{10} - k_{11}) v_{\nu}/v_{\phi} + k_{12} v_{\eta} v_{\nu}/(v_{\phi} v_{\Delta}) \\
b_1 &= -w_1 - w_2 + w_5 v_{\xi}/v_{\Delta} + (w_6 - w_7) v_{\eta}/v_{\phi} \\
b_2 &= -w_1 + w_2 + w_4 v_{\varphi}/v_{\chi} + w_5 v_{\xi}/v_{\Delta} + (w_6 + w_7) v_{\eta}/v_{\phi} + w_8 v_{\varphi} v_{\eta} v_{\xi}/(v_{\chi} v_{\phi} v_{\Delta})
\end{align*}
(111)
The solution to the linear equations Eq. (110) is
\begin{align*}
\delta v_{\chi 1} &= \frac{a_3 - a_2}{3f_1} v_{\chi} v_{\varphi} v_{\Delta} - \frac{a_1 + a_2 + a_3}{18(f_1 f_5 - f_2 f_4) f_3} v_{\chi} v_{\varphi} v_{\Delta} - \frac{(b_1 + b_2) f_1 f_5 - 2b_1 f_2 f_4}{12(f_1 f_5 - f_2 f_4) f_3} v_{\varphi} v_{\Delta} \\
\delta v_{\chi 2} &= \frac{a_2 - a_1}{3f_1} v_{\chi} v_{\varphi} v_{\Delta} - \frac{a_1 + a_2 + a_3}{18(f_1 f_5 - f_2 f_4) f_3} v_{\chi} v_{\varphi} v_{\Delta} - \frac{(b_1 + b_2) f_1 f_5 - 2b_1 f_2 f_4}{12(f_1 f_5 - f_2 f_4) f_3} v_{\varphi} v_{\Delta} \\
\delta v_{\chi 3} &= \frac{a_1 - a_3}{3f_1} v_{\chi} v_{\varphi} v_{\Delta} - \frac{a_1 + a_2 + a_3}{18(f_1 f_5 - f_2 f_4) f_3} v_{\chi} v_{\varphi} v_{\Delta} - \frac{(b_1 + b_2) f_1 f_5 - 2b_1 f_2 f_4}{12(f_1 f_5 - f_2 f_4) f_3} v_{\varphi} v_{\Delta} \\
\delta v_{\varphi 2} &= \frac{(a_1 + a_2 + a_3) f_5}{3(f_1 f_5 - f_2 f_4)} v_{\varphi} v_{\Delta} - \frac{(b_1 - b_2) f_1}{2(f_1 f_5 - f_2 f_4)} v_{\chi} v_{\varphi} v_{\Delta} \\
\delta v_{\xi} &= \frac{(a_1 + a_2 + a_3) f_4}{3(f_1 f_5 - f_2 f_4)} v_{\varphi} v_{\Delta} - \frac{(b_1 - b_2) f_1}{2(f_1 f_5 - f_2 f_4)} v_{\chi} v_{\varphi} v_{\Delta}
\end{align*}
(112)
In the same way, we obtain the minimization equations for the shifts $\delta v_{\phi 1,2,3}$ and $\delta v_{\eta 1}$
\begin{align*}
(-2g_1 v_{\varphi} + g_2 v_{\eta}) \delta v_{\phi 3} + g_2 v_{\varphi} \delta v_{\eta 1} + c_1 v_{\chi}^2 v_{\varphi}^2/\Lambda^2 &= 0 \\
(4g_1 v_{\varphi} + g_2 v_{\eta}) \delta v_{\phi 2} + c_2 v_{\chi}^2 v_{\varphi}^2/\Lambda^2 &= 0 \\
(-2g_1 v_{\varphi} + g_2 v_{\eta}) \delta v_{\phi 1} + c_3 v_{\chi}^2 v_{\varphi}^2/\Lambda^2 &= 0 \\
2g_3 v_{\varphi} \delta v_{\phi 3} + 2g_4 v_{\eta} \delta v_{\eta 1} + c_4 v_{\chi}^2 v_{\varphi}^2/\Lambda^2 &= 0
\end{align*}
(113)
where the parameters $c_{1,2,3,4}$ are given by
\begin{align*}
c_1 &= 6s_2 + 3(s_7 + s_8) v_{\eta}/v_{\phi} + 2s_{14} v_{\varphi}^2/v_{\chi}^2 + (s_{16} + s_{17}) v_{\varphi}^2 v_{\eta}/(v_{\chi}^2 v_{\varphi}) + (2s_{19} - 4s_{20}) v_{\varphi}/v_{\chi}
\end{align*}
Obviously Eq. (113) admits the solutions

\[
\begin{align*}
\delta v_1 &= \frac{c_3 v^2_{\phi}}{4g_1 \Lambda^2} \\
\delta v_2 &= -\frac{c_2 v^2_{\phi}}{2g_1 \Lambda^2} \\
\delta v_3 &= \frac{4c_1 g_1 g_4 + c_4 g_2^2 v^2_{\phi}}{16g_1^2 g_4 - 2g_2 g_3 \Lambda^2} \\
\delta v_4 &= \frac{c_1 g_2 g_3 + 2c_4 g_1 g_2 v^2_{\phi}}{8g_1 g_4 - g_2 g_3 \Lambda^2}
\end{align*}
\]

(115)

From the above equations Eq. (115), we can clearly see that all the shifts \(\delta v_{61}/v_\phi, \delta v_{62}/v_\phi, \delta v_{63}/v_\phi\) and \(\delta v_{61}/v_\eta\) are of order \(\lambda^4\). Finally the equations for the corrections \(\delta v_{61,2,3}\) are

\[
\begin{align*}
(4h_1 v_\Delta + h_2 v_\xi)\delta v_{61} + d_1 v^2_\phi v^2_\Delta/\Lambda^2 &= 0 \\
(-2h_1 v_\Delta + h_2 v_\xi)\delta v_{63} + d_2 v^2_\phi v^2_\Delta/\Lambda^2 &= 0 \\
(-2h_1 v_\Delta + h_2 v_\xi)\delta v_{62} + d_3 v^2_\phi v^2_\Delta/\Lambda^2 &= 0
\end{align*}
\]

(116)

where the coefficients \(d_{1,2,3}\) are

\[
\begin{align*}
d_1 &= 6t_1 + 3t_6 v_\xi/v_\Delta + (2t_{10} + 8t_{12})v_\phi/v_\chi + 4t_{15}v_\phi v_\xi/(v_\chi v_\Delta) + 2t_{17}v_\phi v^2_\xi/(v_\chi v^2_\Delta) \\
&\quad + 4t_{18}v^2_\phi/v^2_\Delta + 2t_{20}v^2_\phi v_\xi/(v^2_\chi v_\Delta) \\
d_2 &= 6t_2 + 3t_7 v_\xi/v_\Delta + (2t_{10} - 4t_{12})v_\phi/v_\chi - 2t_{15}v_\phi v_\xi/(v_\chi v_\Delta) + 2t_{17}v_\phi v^2_\xi/(v_\chi v^2_\Delta) \\
&\quad + 2t_{19}v^2_\phi/v^2_\chi + t_{21}v^2_\phi v_\xi/(v^2_\chi v_\Delta) \\
d_3 &= 6t_2 + 3t_7 v_\xi/v_\Delta + (2t_{10} - 4t_{12})v_\phi/v_\chi - 2t_{15}v_\phi v_\xi/(v_\chi v_\Delta) + 2t_{17}v_\phi v^2_\xi/(v_\chi v^2_\Delta) \\
&\quad + 2t_{19}v^2_\phi/v^2_\chi + t_{21}v^2_\phi v_\xi/(v^2_\chi v_\Delta)
\end{align*}
\]

(117)

The solutions to the above equations Eq. (116) are given by

\[
\begin{align*}
\delta v_{61} &= -\frac{d_1 v^2_\phi v_\Delta}{2h_1 \Lambda^2} \\
\delta v_{62} &= \frac{d_3 v^2_\phi v_\Delta}{4h_1 \Lambda^2} \\
\delta v_{63} &= \frac{d_2 v^2_\phi v_\Delta}{4h_1 \Lambda^2}
\end{align*}
\]

(118)

It is obvious that \(\delta v_{61,2,3}/v_\Delta\) are of order \(\lambda^4\), this is because the corrections to the vacuum alignment of \(\Delta\) and \(\xi\) arise at the next to next leading order. In short summary, the
modified vacuum configuration of the flavon fields can be parameterized by Eq. (109), the shifts $\delta v_{\varphi_1}$, $\delta v_{\varphi_2}$ and $\delta v_\xi$ have been reabsorbed into the redefinition of $v_\varphi$, $v_\eta$ and $v_\xi$ respectively, which remain undetermined. The subleading corrections are suppressed by at least one power of $1/\Lambda$ with respect to the LO results, concretely $\delta v_{\chi_{1,2,3}}/v_\chi$, $\delta v_{\varphi_{2,3}}/v_\varphi$ and $\delta v_\zeta/v_\zeta$ are of order $\lambda^2$, while $\delta v_{\varphi_{1,2,3}}/v_\varphi$, $\delta v_{\varphi_1}/v_\varphi$ and $\delta v_{\Delta_{1,2,3}}/v_\Delta$ are of order $\lambda^4$. These order of magnitudes can be clearly seen from Eqs. (112, 115, 118), we note that the different suppressions of the shifts are due to the constraint of the flavor symmetry $S_4 \times Z_3 \times Z_4$.

**Appendix C: GUT symmetry breaking**

In the following, we shall briefly discuss the GUT Higgs sector of the model in the present effective theory. Our Higgs sector is composed of $H_5$, $H_{45}$, $H_3$, $H_{15}$ and $H_{24}$, the LO $S_4 \times Z_3 \times Z_4$ invariant interactions between the different Higgs chiral superfields in the model are

\[
w_H = m_{24}H_{24}H_{24} + \lambda_{24}H_{24}H_{24}H_{24} + \sum_i^3 f_{H_i} \frac{1}{\Lambda^2} H_\tau H_5 H_5 O_i^{(5)} + \sum_i^3 \lambda_{H_i} \frac{1}{\Lambda^3} H_\tau H_5 H_{24} O_i^{(5)} + \sum_i^2 c_{H_i} \frac{1}{\Lambda^2} H_\tau H_{45} H_{45} O_i^{(6)} + \sum_i^3 b_{H_i} \frac{1}{\Lambda^3} H_{35} H_{24} H_5 O_i^{(7)} + \sum_i^3 m_{45} H_{45} H_{45} + a_{H} H_{45} H_{45} H_{24}
\]

(119)

where

\[
O^{(5)} = \{\Delta^2 \chi, \Delta^2 \varphi, \Delta \chi \xi\}
\]

\[
O^{(6)} = \{\Delta^2, \xi^2\}
\]

\[
O^{(7)} = \{\chi^3, \chi^2 \varphi, \varphi^3\}
\]

(120)

Using the vacuum alignment shown in Eq. (102) we can immediately obtain that

\[
w_H = m_{24}H_{24}H_{24} + \lambda_{24}H_{24}H_{24}H_{24} + f_{H} H_{\tau} H_{5} + \lambda_{H} H_{\tau} H_{5} H_{24} + c_{H} H_{\tau} H_{24} H_{45} + b_{H} H_{35} H_{24} H_{5} + m_{45} H_{35} H_{45} + a_{H} H_{35} H_{45} H_{24}
\]

(121)

with

\[
f_{H} = 2 f_{H_1} \frac{v_\chi^2 v_\chi}{\Lambda^2} + f_{H_3} \frac{v_\Delta v_\chi v_\xi}{\Lambda^2}
\]

\[
\lambda_{H} = 2 \lambda_{H_1} \frac{v_\chi^2 v_\chi}{\Lambda^3} + \lambda_{H_3} \frac{v_\Delta v_\chi v_\xi}{\Lambda^3}
\]

\[
c_{H} = c_{H_1} \frac{v_\chi^2}{\Lambda^2} + c_{H_2} \frac{v_\xi^2}{\Lambda^2}
\]

\[
b_{H} = 6 b_{H_2} \frac{v_\chi^2 v_\varphi}{\Lambda^3} + 2 b_{H_3} \frac{v_\varphi^3}{\Lambda^3}
\]

(122)

Since all the Higgs fields are neutral under the continuous $U(1)_R$ symmetry, the superpotential Eq. (119) explicitly break $U(1)_R$, while preserve the usual R-parity. Certainly we
can construct invariant operators comprising the driving fields, the Higgs fields and an arbitrary number of flavon fields, however, these operators don’t contribute to the scalar potential due to the vanishing VEVs of the driving fields. Consequently, To completely understand the GUT symmetry breaking, maybe we should go beyond the effective theory framework and consider the ultraviolet completion.

We note that the effective superpotential in Eq. (119) could help us to qualitatively understand the GUT symmetry breaking, although this approach is not so satisfactory because of the $U(1)_R$ symmetry breaking. In the context of the ultraviolet completion of the effective model, the terms in Eq. (119) could be generated from a $U(1)_R$ conserving superpotential in which the breaking is mediated by additional fields which carry $U(1)_R$ charge. The ultraviolet completion of the model deserves considerable theoretical work (please see Ref. [59] for an example of the ultraviolet completion of the $A_4$ model), it is beyond the scope of the present work.

The scalar potential of the model is determined by the SUSY $F$ terms, $D$ terms and soft terms contributions. We notice that the first two terms in Eq. (122) is the interactions for $H_{24}$, they are exactly the same as those in the conventional GUT theory, this is because that the Higgs $H_{24}$ is neutral under the flavor symmetries, consequently the $SU(5)$ GUT symmetry is broken into the standard model one as usual. Subsequently the VEVs of $H_5$, $H_{45}$, $H_{55}$ and $H_{45}$ break the standard model symmetry into the residual $SU(3)_c \times U(1)_{em}$. Recalling that the parameter $\tan \beta$ could be small or large in the minimal supersymmetric standard model, this means that a hierarchy between the Higgs VEVs $v_u$ and $v_d$ can be accommodated. In exactly the same way, the minor hierarchy between $v_5$ and $v_{45}$ in Eq. (35) can be achieved by moderately fine-tuning the parameters in the superpotential $w_H$.

---

7The same is true for a large class of models with discrete flavor symmetry, where a continuous $U(1)_R$ symmetry is used to solve the vacuum alignment problem.
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