ON THE INTEGRAL INEQUALITIES FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE CONVEX AND APPLICATIONS

MEHMET ZEKI SARIKAYA*, ERHAN. SET, AND M. EMIN OZDEMIR

Abstract. In this paper, we establish several new inequalities for some twice differentiable mappings. Then, we apply these inequalities to obtain new midpoint, trapezoid and perturbed trapezoid rules. Finally, some applications for special means of real numbers are provided.

1. Introduction

In 1938 Ostrowski obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is well known in the literature as Ostrowski’s integral inequality [14]:

Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative \( f' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \|f'\|_{\infty} = \sup_{t \in (a, b)} |f'(t)| < \infty \). Then, the inequality holds:

\[
|f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) dt| \leq \left[ \frac{1}{4} + \left( \frac{x-a}{b-a} \right)^2 \right] \frac{(b-a)}{2} \|f'\|_{\infty}
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is the best possible.

In 1976, Milovanovic and Pecaric proved a generalization of the Ostrowski inequality for \( n \)-times differentiable mappings (see for example [13, p.468]). Dragomir and Wang ([10], [11]) extended the result (1.1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means. Also, Sofo and Dragomir [18] extended the result (1.1) in the \( L_p \) norm. Dragomir ([6]-[8]) further extended the (1.1) to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings. For recent results and generalizations concerning Ostrowski’s integral inequality see [1]-[13], [18], [19], and the references therein.

In [4], Cerone and Dragomir find the following perturbed trapezoid inequalities:
Theorem 2. Let $f : [a, b] \to \mathbb{R}$ be such that the derivative $f'$ is absolutely continuous on $[a, b]$. Then, the inequality holds:

$$\left| \int_{a}^{b} f(t)dt - \frac{b-a}{2} [f(b) + f(a)] + \frac{(b-a)^2}{8} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{24} \|f''\|_{\infty} \quad \text{if } f'' \in L_{\infty}[a, b]$$

$$\leq \begin{cases} 
\frac{(b-a)^{2+\frac{1}{p}}}{8(2q+1)^{\frac{q}{p}}} \|f''\|_p & \text{if } f'' \in L_p[a, b], \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1 \\
\frac{(b-a)^2}{8} \|f''\|_1 & \text{if } f'' \in L_1[a, b]
\end{cases}$$

for all $t \in [a, b]$.

In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas. They used an approach from the inequalities point of view. For example, the midpoint quadrature rule is considered in [4],[15],[17], the trapezoid rule is considered in [4],[16],[20]. In most cases estimations of errors for these quadrature rules are obtained by means of derivatives and integrands.

In this article, we first derive a general integral identity for twice derivatives functions. Then, we apply this identity to obtain our results and using functions whose twice derivatives in absolute value at certain powers are convex, we obtained new inequalities related to the Ostrowski’s type inequality. Finally, we gave some applications for special means of real numbers.

2. Main Results

In order to prove our main results, we need the following Lemma (see, [12]):

Lemma 1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I'$ with $f'' \in L_1[a, b]$, then

$$\frac{1}{b-a} \int_{a}^{b} f(u)du - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2}(x - \frac{a + 3b}{4}) [f'(x) - f'(a + b - x)]$$

$$= \frac{(b-a)^2}{2} \int_{0}^{1} k(t) f''(ta + (1-t)b)dt$$

where

$$k(t) := \begin{cases} 
t^2, & 0 \leq t < \frac{b-x}{b-a} \\
(t-\frac{1}{2})^2, & \frac{b-x}{b-a} \leq t < \frac{b-a}{6-a} \\
(t-1)^2, & \frac{b-a}{6-a} \leq t \leq 1
\end{cases}$$

for any $x \in [\frac{a+b}{2}, b]$.
Proof. It suffices to note that
\[
I = \int_0^1 k(t) f''(ta + (1 - t)b) dt
\]
\[
= \int_0^{b-a} t^2 f''(ta + (1 - t)b) dt + \int_{b-a}^{b-x} \left( t - \frac{1}{2} \right)^2 f''(ta + (1 - t)b) dt
\]
\[
+ \int_{b-x}^{b-a} \left( t - 1 \right)^2 f''(ta + (1 - t)b) dt
\]
\[
= I_1 + I_2 + I_3.
\]
By integration by parts, we have the following identity
\[
I_1 = \int_0^{b-x} t^2 f''(ta + (1 - t)b) dt
\]
\[
= \frac{t^2}{(a-b)} f'(ta + (1 - t)b) \bigg|_0^{b-x} - \frac{2}{a-b} \int_0^{b-x} t f'(ta + (1 - t)b) dt
\]
\[
= \frac{1}{(a-b)} \left( b-x \right)^2 f'(x) - \frac{2}{a-b} \left[ \int_0^{b-x} \frac{t}{a-b} f'(ta + (1 - t)b) dt \right]
\]
\[
= \frac{(b-x)^2}{(b-a)^3} f'(x) - \frac{2(b-x)}{(b-a)^3} f(x) + \frac{2}{(b-a)^2} \int_0^{b-x} f'(ta + (1 - t)b) dt.
\]
Similarly, we observe that
\[
I_2 = \int_{b-x}^{e-x} \left( t - \frac{1}{2} \right)^2 f''(ta + (1 - t)b) dt
\]
\[
= \frac{(a+b-2x)^2}{4(b-a)^3} [f'(x) - f'(a+b-x)] + \frac{(a+b-2x)}{(b-a)^3} [f(x) + f(a+b-x)]
\]
\[
+ \frac{2}{(b-a)^2} \int_{b-x}^{e-x} f'(ta + (1 - t)b) dt.
\]
Thus, we can write

\[ I_3 = \int_{\frac{a}{b-a}}^1 (t-1)^2 f''(ta + (1-t)b)dt \]

\[ = \frac{(b-x)^2}{(b-a)^3} f'(a+b-x) - \frac{2(b-x)}{(b-a)^3} f(a+b-x) + \frac{2}{(b-a)^2} \int_{\frac{a}{b-a}}^1 f(ta + (1-t)b)dt. \]

Using the change of the variable \( u = ta + (1-t)b \) for \( t \in [0,1] \) and by multiplying the both sides by \( (b-a)^2/2 \) which gives the required identity (2.1).

Now, by using the above lemma, we prove our main theorems:

**Theorem 3.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^0 \) such that \( f'' \in L_1[a,b] \) where \( a,b \in I, a < b \). If \( |f''| \) is convex on \( [a,b] \), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x - \frac{a+3b}{4}) [f'(x) - f'(a+b-x)] \right| \\
\leq \frac{1}{(b-a)} \left[ (b-x)^3 + \left(x - \frac{a+b}{2}\right)^3 \right] \left( \frac{|f''(a)| + |f''(b)|}{6} \right) \\
\] 

for any \( x \in \left[ \frac{a+b}{2}, b \right] \).
Proof. From Lemma 1 and by the definition $k(t)$, we get

\[
\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2} (x - \frac{a + 3b}{4}) [f'(x) - f'(a + b - x)] \right|
\leq \frac{(b-a)^2}{2} \int_0^1 |k(t)| |f''(ta + (1-t)b)| \, dt
\end{aligned}
\]

We rewrite this inequality as

\[
\begin{aligned}
&\leq \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{b-a}{b-a}} t^2 |f''(ta + (1-t)b)| \, dt + \int_{\frac{b-a}{b-a}}^{b-a} (t - \frac{1}{2})^2 |f''(ta + (1-t)b)| \, dt \\
&\quad + \int_{\frac{b-a}{b-a}}^{1} (t-1)^2 |f''(ta + (1-t)b)| \, dt \right\}
\end{aligned}
\]

(2.3)

Investigating the three separate integrals, we may evaluate as follows:

By the convexity of $|f''|$, we arrive at

\[
\begin{aligned}
J_1 &\leq \int_0^{\frac{b-a}{b-a}} (t^3 |f''(a)| + (t^2 - t^3) |f''(b)|) \, dt \\
&= \frac{(b-x)^4}{4(b-a)^4} |f''(a)| + \left( \frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4} \right) |f''(b)|,
\end{aligned}
\]

\[
\begin{aligned}
J_2 &\leq \int_{\frac{b-a}{b-a}}^{b-a} \left[ \left( t - \frac{1}{2} \right)^2 t |f''(a)| + \left( t - \frac{1}{2} \right)^2 (1-t) |f''(b)| \right] \, dt \\
&= \frac{1}{3(b-a)^3} \left( x - \frac{a+b}{2} \right)^3 |f''(a)| + \frac{1}{3(b-a)^3} \left( x - \frac{a+b}{2} \right)^3 |f''(b)|,
\end{aligned}
\]

\[
\begin{aligned}
J_3 &\leq \int_{\frac{b-a}{b-a}}^{1} \left[ (t-1)^2 t |f''(a)| + (1-t)^3 |f''(b)| \right] \, dt \\
&= \left( \frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4} \right) |f''(a)| + \frac{(b-x)^4}{4(b-a)^4} |f''(b)|.
\end{aligned}
\]

By rewrite $J_1$, $J_2$, $J_3$ in (2.3), we obtain (2.2) which completes the proof. \qed
Theorem 4. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I$ such that $f'' \in L^1[a, b]$ where $a, b \in I, a < b$. If $|f''|^q$ is convex on $[a, b], q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(b) + f(a)] + \frac{1}{2} (x - \frac{a+3b}{4}) [f'(x) - f'(a+b-x)] \right| \leq \frac{(b-a)^2}{2}\int_0^1 |k(t)| |f''(ta + (1-t)b)| dt
\]

(2.6)

Remark 1. We choose $|f''(x)| \leq M, M > 0$ in Corollary 2 then we recapture the first part of the inequality (1.2).

Corollary 2 (Trapezoid inequality). Under the assumptions Theorem 3 with $x = b$ and $f'(a) = f'(b)$ in Theorem 3 we have
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f\left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{48} (|f''(a)| + |f''(b)|).
\]

(2.4)

Corollary 3 (Midpoint inequality). Under the assumptions Theorem 3 with $x = \frac{a+b}{2}$ in Theorem 3 we have
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f\left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{48} (|f''(a)| + |f''(b)|).
\]

(2.5)

Another similar result may be extended in the following theorem

Theorem 4. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I$ such that $f'' \in L^1[a, b]$ where $a, b \in I, a < b$. If $|f''|^q$ is convex on $[a, b], q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x - \frac{a+3b}{4}) [f'(x) - f'(a+b-x)] \right| \leq \frac{(b-a)^2}{2}\int_0^1 |k(t)| |f''(ta + (1-t)b)| dt
\]

(2.6)

Proof. From Lemma 1 by the definition $k(t)$ and using by Hölder’s inequality, it follows that
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x - \frac{a+3b}{4}) [f'(x) - f'(a+b-x)] \right| \leq \frac{(b-a)^2}{2}\int_0^1 |k(t)| |f''(ta + (1-t)b)| dt
\]

(2.7)

\[
\leq \frac{(b-a)^2}{2} \left( \int_0^1 |k(t)|^p dt \right)^\frac{1}{p} \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^\frac{1}{q}.
\]
Theorem 5. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a, b]$ and $q \geq 1$, then

$$
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2}(x - \frac{a + b}{4}) [f'(x) - f'(a + b - x)] \right|
$$

$$
\leq \frac{1}{3 (b-a)} \left[ (b-x)^3 + \left( x - \frac{a + b}{2} \right)^3 \right] \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}
$$

for any $x \in \left[ \frac{a+b}{4}, b \right]$. 

Corollary 4 (Perturbed Trapezoid inequality). Under the assumptions Theorem 4 with $x = b$, we have

$$
\left| \int_a^b f(u)du - \frac{b-a}{2} [f(b) + f(a)] + \frac{(b-a)^2}{8} [f'(b) - f'(a)] \right| \leq \frac{(b-a)^3}{8 (2p + 1)^3} \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}.
$$

Corollary 5 (Trapezoid inequality). Under the assumptions Theorem 4 with $x = a$ in Theorem 4, we have

$$
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{8 (2p + 1)^3} \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}.
$$

Corollary 6 (Midpoint inequality). Under the assumptions Theorem 4 with $x = \frac{a+b}{2}$ in Theorem 4, we have

$$
\left| \frac{1}{b-a} \int_a^b f(u)du - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{8 (2p + 1)^3} \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}.
$$

Using (2.8) and (2.9) in (2.7), we obtain (2.6). □

Theorem 5. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a, b]$ and $q \geq 1$, then

$$
\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2}(x - \frac{a + b}{4}) [f'(x) - f'(a + b - x)] \right|
$$

$$
\leq \frac{1}{3 (b-a)} \left[ (b-x)^3 + \left( x - \frac{a + b}{2} \right)^3 \right] \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}
$$

for any $x \in \left[ \frac{a+b}{4}, b \right]$. 

Since $|f''|^q$ is convex on $[a, b]$, we know that for $t \in [0, 1]$

$$
|f''(ta + (1-t)b)|^q \leq t |f''(a)|^q + (1-t) |f''(b)|^q,
$$

hence, a simple computation shows that

$$
\int_0^1 |f''(ta + (1-t)b)|^q dt \leq \frac{|f''(a)|^q + |f''(b)|^q}{2}
$$

also,

$$
\int_0^1 \frac{(b-x)^2}{2} dt + \int_0^1 (1-t)^2 p dt
$$

$$
\leq \frac{2}{(2p + 1)(b-a)^{2p+1}} \left[ (b-x)^{2p+1} + \left( x - \frac{a + b}{2} \right)^{2p+1} \right].
$$

Using (2.8) and (2.9) in (2.7), we obtain (2.6). □
Proof. From Lemma 1 by the definition \( k(t) \) and using by power mean inequality, it follows that

\[
\left| \frac{1}{b-a} \int_a^b f(u) du - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} (x - \frac{a+3b}{4}) [f'(x) - f'(a+b-x)] \right| \\
\leq \frac{(b-a)^2}{2} \int_0^1 |k(t)| \left| f''(ta + (1-t)b) \right| dt \\
\leq \frac{(b-a)^2}{2} \left( \int_0^1 |k(t)| dt \right)^{1-\frac{1}{q}} \left( \int_0^1 |k(t)| \left| f''(ta + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}. 
\]

(2.13)

Since \( |f''|^{\frac{q}{q}} \) is convex on \([a,b]\), we know that for \( t \in [0,1] \)

\[
|f''(ta + (1-t)b)|^{\frac{q}{q}} \leq t |f''(a)|^{\frac{q}{q}} + (1-t) |f''(b)|^{\frac{q}{q}},
\]

hence, by simple computation

\[
\int_0^1 |k(t)| dt = \int_0^{\frac{b-a}{2}} t^2 dt + \int_{\frac{b-a}{2}}^1 t - \frac{1}{2} \left| t - \frac{1}{2} \right|^2 dt + \int_{\frac{b-a}{2}}^1 (1-t)^2 dt
\]

(2.14)

\[
= \frac{2}{3} \left( \frac{b-a}{2} \right)^3 \left[ (b-x)^3 + \left( x - \frac{a+b}{2} \right)^3 \right],
\]
and
\[
\int_0^1 |k(t)| |f''(ta + (1 - t)b)|^q \, dt
\leq \int_0^1 |k(t)| \left( t |f''(a)|^q + (1 - t) |f''(b)|^q \right) \, dt
= \int_0^1 (t^3 |f''(a)|^q + (t^2 - t^3) |f''(b)|^q) \, dt
\]
\[
+ \int \left[ \left( t - \frac{1}{2} \right)^2 |f''(a)|^q + \left( t - \frac{1}{2} \right)^2 \left( 1 - t \right) |f''(b)|^q \right] \, dt
\]
(2.15)
\[
= \frac{(b - x)^4}{4(b - a)^3} |f''(a)|^q + \frac{(b - x)^3}{3(b - a)^3} \left( \frac{b - x}{4(b - a)^4} \right) |f''(b)|^q
+ \frac{1}{3(b - a)^3} \left( x - \frac{a + b}{2} \right)^3 |f''(a)|^q + \frac{1}{3(b - a)^3} \left( x - \frac{a + b}{2} \right)^3 |f''(b)|^q
+ \left( \frac{b - x}{3(b - a)^3} - \frac{(b - x)^4}{4(b - a)^4} \right) |f''(a)|^q + \frac{(b - x)^4}{4(b - a)^4} |f''(b)|^q
= \frac{1}{3(b - a)^3} \left( (b - x)^3 + \left( x - \frac{a + b}{2} \right)^3 \right) \left( |f''(a)|^q + |f''(b)|^q \right).
\]
Using (2.14) and (2.15) in (2.13), we obtain (2.12). □

**Corollary 7.** Under the assumptions Theorem 3 with \( x = b \), we have
\[
\left| \int_a^b f(a) \, du - \frac{b - a}{2} \left[ f(b) + f(a) \right] + \frac{(b - a)^2}{8} \left[ f'(b) - f'(a) \right] \right| \leq \frac{(b - a)^3}{24} \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{4}}.
\]

**Corollary 8.** Under the assumptions Theorem 3 with \( x = b \) and \( f'(a) = f'(b) \) in Theorem 3, we have
\[
\left| \frac{1}{b - a} \int_a^b f(u) \, du - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)^2}{24} \left( \frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{4}}.
\]
Corollary 9. Under the assumptions Theorem 5 with \(x = \frac{a+b}{2}\) in Theorem 5 we have

\[
\frac{1}{b-a} \int_a^b f(u) \, du - f\left(\frac{a+b}{2}\right) \leq \frac{(b-a)^2}{24} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\]

3. Applications for special means

Recall the following means:

(a) The arithmetic mean
\[A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;\]

(b) The geometric mean
\[G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;\]

(c) The harmonic mean
\[H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0;\]

(d) The logarithmic mean
\[L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0;\]

(e) The identric mean
\[I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e}\left(\frac{b^p}{a^p}\right)^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad a, b > 0;\]

(f) The \(p\)-logarithmic mean:
\[L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.
\]

It is also known that \(L_p\) is monotonically nondecreasing in \(p \in \mathbb{R}\) with \(L_{-1} := L\) and \(L_0 := I\). The following simple relationships are known in the literature

\[H \leq G \leq L \leq I \leq A.\]

Now, using the results of Section 2, some new inequalities is derived for the above means.

**Proposition 1.** Let \(p > 1\) and \(0 \leq a < b\). Then we have the inequality:

\[|L_p(a, b) - A(a^p, b^p)| \leq p(p-1) \frac{(b-a)^2}{24} A(a^{p-2}, b^{p-2}).\]

**Proof.** The assertion follows from (2.4) applied for \(f(x) = x^p, \quad x \in [a, b]\). We omitted the details. \(\square\)
Proposition 2. Let \( p > 1 \) and \( 0 \leq a < b \). Then we have the inequality:

\[
\left| L^{-1}(a,b) - A^{-1}(a,b) \right| \leq \frac{(b-a)^2}{12} A \left( a^{-3}, b^{-3} \right).
\]

Proof. The assertion follows from (2.10) applied for \( f(x) = \frac{1}{x}, \ x \in [a, b] \). We omitted the details.

Proposition 3. Let \( p > 1 \) and \( 0 \leq a < b \). Then we have the inequality:

\[
| \ln I(a,b) - \ln G(a,b) | \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}} \left[ A \left( a^{-2q}, b^{-2q} \right) \right]^{1/q}.
\]

Proof. The assertion follows from (2.10) applied for \( f(x) = -\ln x, \ x \in [a, b] \).

Proposition 4. Let \( p > 1 \) and \( 0 \leq a < b \). Then we have the inequality:

\[
| L^p(a,b) - A^p(a,b) | \leq p(p-1) \frac{(b-a)^2}{8(2p+1)^{1/p}} \left[ A \left( a^{q(p-2)}, b^{q(p-2)} \right) \right]^{1/q}.
\]

Proof. The assertion follows from (2.11) applied for \( f(x) = x^p, \ x \in [a, b] \).

Proposition 5. Let \( p > 1 \) and \( 0 \leq a < b \). Then we have the inequality:

\[
| L^{-1}(a,b) - H^{-1}(a,b) | \leq \frac{(b-a)^2}{12} \left[ A \left( a^{-3q}, b^{-3q} \right) \right]^{1/q}.
\]

Proof. The assertion follows from (2.10) applied for \( f(x) = \frac{1}{x}, \ x \in [a, b] \).

Proposition 6. Let \( p > 1 \) and \( 0 \leq a < b \). Then we have the inequality:

\[
| \ln I(a,b) - \ln A(a,b) | \leq \frac{(b-a)^2}{24} \left[ A \left( a^{-2q}, b^{-2q} \right) \right]^{1/q}.
\]

Proof. The assertion follows from (2.17) applied for \( f(x) = -\ln x, \ x \in [a, b] \).

4. Applications for composite quadrature formula

Let \( d \) be a division \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) of the interval \( [a, b] \) and \( \xi = (\xi_0, \ldots, \xi_{n-1}) \) a sequence of intermediate points, \( \xi_i \in [x_i, x_{i+1}], \ i = 0, n - 1 \).

Then the following result holds:

Theorem 6. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a twice differentiable function on \( I^o \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I, \ a < b \). If \( |f''| \) is convex on \( [a, b] \) then we have

\[
\int_a^b f(u) \, du = A(f, f', d, \xi) + R(f, f', d, \xi)
\]

where

\[
A(f, f', d, \xi) = \sum_{i=0}^{n-1} \frac{h_i}{2} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i)]
\]

\[
- \sum_{i=0}^{n-1} \frac{h_i}{2} \left( \xi_i - \frac{x_i + 3x_{i+1} - 4}{4} \right) [f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)].
\]
The remainder $R(f, f', d, \xi)$ satisfies the estimation:

\[(4.1) \quad |R(f, f', d, \xi)| \leq n^{-1} \sum_{i=0}^{n-1} \left[ (x_{i+1} - \xi_i)^3 + \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^3 \right] \left( |f''(x_i)| + |f''(x_{i+1})| \right)
\]

for any choice $\xi$ of the intermediate points.

**Proof.** Apply Theorem 3 on the interval $[x_i, x_{i+1}]$, $i = 0, n - 1$ to get

\[
\left| \frac{h_i}{2} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i)] - \frac{h_i}{2} \left( \xi_i - \frac{x_i + 3x_{i+1}}{4} \right) \right|
\]

\[
\times |f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)| - \int_a^b f(u) du \leq \left[ (x_{i+1} - \xi_i)^3 + \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right)^3 \right] \left( |f''(x_i)| + |f''(x_{i+1})| \right).
\]

\[
R_T(f, f', d) \leq n^{-1} \sum_{i=0}^{n-1} \left( h_i \right)^3 \left( |f''(x_i)| + |f''(x_{i+1})| \right).
\]

Summing the above inequalities over $i$ from 0 to $n - 1$ and using the generalized triangle inequality, we get the desired estimation (4.1).

**Corollary 10.** The following perturbed trapezoid rule holds:

\[
\int_a^b f(u) du = T(f, f', d) + R_T(f, f', d)
\]

where

\[
T(f, f', d) := \sum_{i=0}^{n-1} \frac{h_i}{2} [f(x_i) + f(x_{i+1})] - \frac{1}{8} \sum_{i=0}^{n-1} (h_i)^2 |f'(x_{i+1}) - f'(x_i)|
\]

and the remainder term $R_T(f, f', d)$ satisfies the estimation,

\[
R_T(f, f', d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^3}{48} (|f''(x_i)| + |f''(x_{i+1})|).
\]

**Corollary 11.** The following midpoint rule holds:

\[
\int_a^b f(u) du = M(f, d) + R_M(f, d)
\]

where

\[
M(f, d) := \sum_{i=0}^{n-1} h_i \left[ f \left( \frac{x_i + x_{i+1}}{2} \right) \right]
\]

and the remainder term $R_M(f, d)$ satisfies the estimation,

\[
R_M(f, d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^3}{48} (|f''(x_i)| + |f''(x_{i+1})|).
\]
ON THE INTEGRAL INEQUALITIES

References

[1] M. Alomari and M. Darus, Some Ostrowski’s type inequalities for convex functions with applications, RGMIA, 13(1) (2010), Article 3.[ONLINE: http://ajmaa.org/RGMIA/v13n1.php]

[2] N.S. Barnett, P. Cerone, S.S. Dragomir, M.R. Pinheiro and A. Sofo, Ostrowski type inequalities for functions whose modulus of derivatives are convex and applications, RGMIA Res. Rep. Coll., 5(2) (2002), Article 1.[ONLINE: http://rgmia.vu.edu.au/v5n2.html]

[3] P. Cerone, S.S. Dragomir and J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, RGMIA Research Report Collection, 1(1) (1998), Article 4.

[4] P. Cerone and S.S. Dragomir, Trapezoidal type rules from an inequalities point of view, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press N.Y. (2000).

[5] S.S. Dragomir and N. S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, RGMIA Research Report Collection, 1(2) (1998), Article 9.

[6] S.S. Dragomir, On the Ostrowski’s integral inequality for mappings with bounded variation and applications, Math. Ineq. &Appl., 1(2) (1998).

[7] S.S. Dragomir, Ostrowski’s inequality for monotonous mappings and applications, J. KSIAM, 3(1) (1999), 127-135.

[8] S.S. Dragomir, The Ostrowski integral inequality for Lipschitzian mappings and applications, Comp. and Math. with Appl., 38 (1999), 33-37.

[9] S.S. Dragomir and A. Sofo, Ostrowski type inequalities for functions whose derivatives are convex, Proceedings of the 4th International Conference on Modelling and Simulation, November 11-13, 2002. Victoria University, Melbourne, Australia. RGMIA Res. Rep. Coll., 5 (2002), Supplement, Article 30. [ONLINE: http://rgmia.vu.edu.au/v5(E).html]

[10] S. S. Dragomir, S. Wang, A new inequality of Ostrowski’s type in L1-norm and applications to some special means and to some numerical quadrature rules. Tamkang J. of Math., 28 (1997), 239–244.

[11] S. S. Dragomir, S. Wang, A new inequality of Ostrowski’s type in Lp-norm and applications to some special means and to some numerical quadrature rules. Indian J. of Math., 40 (3) (1998), 245–304.

[12] Z. Liu, Some companions of an Ostrowski type inequality and application, J. Inequal. in Pure and Appl. Math, 10(2), 2009, Art. 52, 12 pp.

[13] G.V. Milovanovic and J.E. Pecaric , On generalizations of the inequality of A. Ostrowski and related applications, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz., No. 544-576 (1976), 155–158.

[14] A. M. Ostrowski, Über die absolutabweichung einer differentiabaren funktion von ihrem integraalmittelwert, Comment. Math. Helv. 10(1938), 226-227.

[15] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.

[16] U.S. Kirmaci, M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comp., 153 (2004), 361-368.

[17] C.E.M Pearce, J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula. Appl. Math. Lett. 13 (2000) 51-55.

[18] A. Sofo and S.S. Dragomir, An inequality of Ostrowski type for twice differentiable mappings in term of the Lp norm and applications, Soochow J. of Math. 27(1), 2001, 97-111.

[19] M.Z. Sarıkaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae, Vol. LXXIX, 1 (2010), 129-134.

[20] M.Z. Sarıkaya, E. Set and M.E. Özdemir, On New Inequalities of Simpson’s Type for s-Convex Functions, RGMIA, Res. Rep. Coll., 13 (2) (2010), Article 2.
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail address: sarikayamz@gmail.com

Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey
E-mail address: erhanset@yahoo.com

Graduate School of Natural and Applied Sciences, Ağrı İbrahim Çeçen University, Ağrı, Turkey
E-mail address: emos@atauni.edu.tr