Entanglement is Sometimes Enough

Xiao-Feng Qian and J.H. Eberly

Rochester Theory Center and the Department of Physics & Astronomy
University of Rochester, Rochester, New York 14627

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For many decades the word “entanglement” has been firmly attached to the world of quantum mechanics. So is the phrase “Bell violation”. Here we show, without contradicting quantum mechanics, that classical non-deterministic fields also provide a natural basis for entanglement and Bell analyses. Surprisingly, such fields are not eliminated by the Clauser-Horne-Shimony-Holt Bell violation test as viable alternatives to quantum theory. An experimental setup for verification is proposed.

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Without contradicting quantum mechanics, we show that some classical field theories can provide entanglement and predict Bell violation while also complying with all three of Shimony’s criteria [1] for a viable theory of Nature. No viable field theory candidates other than quantum mechanics have previously survived the Bell violation test. The incentive to demonstrate that classical fields can have all the desired properties is provided by a logical gap in the following remark made by John Bell [2] in 1972: “It can indeed be shown that the quantum mechanical correlations cannot be reproduced by a hidden variables theory even if one allows a ‘local’ sort of indeterminism. ... This would not work.”

But it does work and what Bell did not anticipate, now well accepted, was that classical fields allow intrinsic entanglement. The remaining gap has been filled by the recognition [3, 4] that examples exist among quantum mechanics have previously survived the Bell violation test. The incentive to demonstrate that classical fields can have all the desired properties is provided by a logical gap in the following remark made by John Bell [2] in 1972: “It can indeed be shown that the quantum mechanical correlations cannot be reproduced by a hidden variables theory even if one allows a ‘local’ sort of indeterminism. ... This would not work.”

By eventualities Shimony just means measurement outcomes.

Bell’s motivations were clear and are an important guide. We will exhibit a classical physical theory that is intrinsically non-deterministic and allows, contrary to Bell’s assertion, both derivations and violations of the same Bell Inequalities. This apparently self-contradictory possibility, which arises when addressing field (wave) rather than particle aspects of natural phenomena, will be resolved below. The category of theories open for study has a potentially large number of members, and at least one very well known member, namely optical partial coherence theory [5, 6]. We use this as our example. It is compatible with the conclusions of the Bell violation experiments because Bell violation occurs within it in exactly the same way and to exactly the same degree as in quantum theory. We demonstrate this in the context of a two-party CHSH Bell Inequality [7] that embraces entanglement (without quantization), and we show that it is subject to experimental verification.

We are apparently dealing with a domain where characteristics labeled classical and quantum have not yet been definitively separated.

Attending to Shimony’s validity conditions (I)-(III), the need for indefinite truth values is automatically met in the indeterministic theories that concerned Bell. In the ordinary classical optical coherence theory of partial polarization [3, 4] one interprets a light field’s physical variables indeterministically. As a concrete system one can think of a beam of thermal light, in which the value of the “optical field” is unpredictable. The real and in principle observable optical field of such a light beam is simply the space-time dependent electric field vector

$$\vec{E}(r,t) = \hat{h} E_h(r,t) + \hat{v} E_v(r,t),$$  

(1)

where $\hat{h}$, $\hat{v}$ are orthogonal and arbitrarily oriented “horizontal” and “vertical” polarization directions. The corresponding components $E_h(r,t) = \sum_n h_n(t) \phi_n(r)$ and $E_v(r,t) = \sum_n v_n(t) \phi_n(r)$ of the optical field are elements in an abstract stochastic function space [8]. Here the $\phi_n(r)$ are orthonormal spatial mode functions with $\int dr \phi_n^*(r) \phi_l(r) = \delta_{n,l}$, and $h_n(t)$ and $v_l(t)$ represent stochastic random coefficients whose origin we can assign to distant and unknowable dipole radiators.

In thermal the value of $\vec{E}$ is unpredictable at any time, so criterion (I) about indefinite truth values is satisfied. When the field is appropriately detected, only one value is recorded, and so a definite truth value is then obtained, satisfying criterion (II). No one would think that this picture of the optical field [1] implies that it is quantized, and it is not, but a non-deterministic viewpoint is employed to extract predictions from the randomly unknown ensemble of field potentialities. This is conven-
tionally done via observable field correlation functions.

Criterion III, the need for entanglement, can be thought difficult. When he introduced the word “entanglement” in 1935, Schrödinger unfortunately created the nearly indelible impression that entanglement is identified exclusively with quantum mechanics. However, the truth is, as Schrödinger clearly knew, that entanglement had already been part of an exhaustive study of integral equations in Hilbert space in 1907 by Schmidt when Schrödinger was just a teenager, two decades before quantum mechanics even existed.

As it turns out, condition III, the existence of entanglement, is readily associated with partial polarization in optics. This has been recently emphasized by Simon, et al. and Qian and Eberly for non-deterministic optical fields. Entanglement requires sums of tensor products of vectors from at least two vector spaces, and optical field is clearly such a sum, where the vectors are associated with the “lab space” of and on the one hand and on the other hand with the statistical infinite-dimensional continuous Hilbert space of the components .

In very special cases there is a direction (a particular linear superposition of and ) that turns into . This allows the two spaces to be separated (factored), and such a separable field is obviously completely polarized (in the direction ). Any non-factorable form for represents partial polarization, and a finite degree (of fully) classical entanglement. We will identify as the light intensity. Here the angle bracket (...) denotes a combination of an ensemble average for the random coefficients, and a spatial average for the mode functions because we are considering the character of an entire light beam instead of specific modes or photons.

Thermal light is understood completely both classically and quantum mechanically and in both domains is probabilistically characterized as completely uncorrelated, Gaussian, and statistically stationary, with the consequence that and have zero statistical overlap: . The classical (and quantum) degree of polarization of such a field is zero and the degree of entanglement is maximal. In fact a purely thermal field is a classical Bell state of the two “parties” in lab space and function space.

To encourage the picture of the optical field as a state in a product tensor space, which it clearly is, we will adopt Dirac-type notation for the vectors: , etc., where we use boldface to emphasize the “bi-vector” two-space character of the field:

\[ |\mathbf{E} \rangle = |h\rangle \otimes |E_h\rangle + |v\rangle \otimes |E_v\rangle. \tag{2} \]

The theory is thus built on superposable states (wave fields), but without any implication that quantization has been imposed. For the general partially coherent case, then, we can write the normalized field as:

\[ |\mathbf{e}\rangle = \kappa_1 |u_1\rangle \otimes |f_1\rangle + \kappa_2 |u_2\rangle \otimes |f_2\rangle, \tag{3} \]

where \( \langle u_j | u_k \rangle = \delta_{jk} \), and \( \kappa_1 \) and \( \kappa_2 \) are normalization coefficients, both equal to \( 1/\sqrt{2} \) in the exactly thermal case. For an arbitrary field state such a decomposition is guaranteed by the Schmidt theorem (see [3]).

Bell’s agenda was to ascertain measurement probabilities and correlations between vectors in separate vector spaces, and polarization vectors of different photons have been used most frequently for the observations. Under obvious conditions on observability, and independent of the possible existence of hidden control parameters, various well-known “Bell Inequalities” serve to constrain the range of these correlations. The CHSH (Clauser-Horne-Shimony-Holt) Inequality is the most useful of these inequalities and has been employed repeatedly. As Gisin has observed, it can be straightforwardly violated in correlation measurements made on any quantum mechanical pair of vectors in a state of the same form as [3].

The new result presented here is that the same violation should also be expected classically. Any classical violation contradicts Bell’s conclusion that quantum theory must be the explanation for any violation. The fact is that in all cases, quantum and classical, it is entanglement that provides the violation.

We replay a derivation of the CHSH inequality in the supplemental material, and only sketch the main features here. A standard Bell-CHSH approach is based on correlations observed in various rotated basis frames. For the case of light beam [3], rotations such as shown in Fig. 1 for lab and function spaces can be denoted as

\[ |u_k^a\rangle = \cos a |u_1\rangle - \sin a |u_2\rangle \quad \text{and} \quad |u_k^b\rangle = \sin a |u_1\rangle + \cos a |u_2\rangle. \]

To characterize the beam in lab space, one can make a projection in any rotated basis \( |u_k^a\rangle \), \( k = 1, 2 \), so that one has \( |\mathbf{e}^a\rangle \equiv |u_k^a\rangle (u_k^a|\mathbf{e}\rangle \) which leaves the field only in the \( |u_k^a\rangle \) lab direction. Then one can always obtain the fraction of intensity in this component as

\[ P_k(a) = |\langle u_k^a|\mathbf{e}\rangle|^2 = \langle |u_k^a\rangle |u_k^a|\mathbf{e}\rangle \equiv |\mathbf{e}| \langle P_k^a \rangle |\mathbf{e}\rangle. \]

The dimensionless fractions \( P_k(a) \) can obviously be interpreted as probabilities and they satisfy the natural relation \( P_1(a) + P_2(a) = 1 \). To capture the individual
projections $P_1(a)$ and $P_2(a)$ in lab space we define a new outer product
\[ \hat{A}_a = \hat{p}_1^a - \hat{p}_2^a, \] (4)
which fully characterizes the beam $|\vec{3}\rangle$ in lab space. Its average $A(a) = \langle e | \hat{A}_a | e \rangle$ is a real number between $-1$ and 1, and it fully determines both $P_1(a)$ and $P_2(a)$ at the same time, through use of $P_1(a) + P_2(a) = 1$. Similarly one can also characterize the beam in the function space by defining the outer product $\hat{B}_k = \hat{p}_1^k - \hat{p}_2^k$, where $\hat{p}_l^k = |f^k_l\rangle \langle f^k_l|$ with $l = 1, 2$. Then its average value $B(b) = \langle e | \hat{B}_b | e \rangle$ is also bounded by $-1$ and 1. Finally, the measurement outcome correlation between the lab and function spaces can be written
\[ C(a, b) = \langle e | \hat{A}_a \otimes \hat{B}_b | e \rangle, \] (5)
which is a combination of 4 joint probabilities
\[ P_{kl}(a, b) = \langle e | \left| u^a_k \right\rangle \langle f^b_l | \langle f^b_l | \left| u^b_k \right\rangle \rangle | e \rangle, \] (6)
with $k, l = 1, 2$. If one defines $S = C(a, b) - C(a, b') + C(a', b) + C(a', b')$, where $a, a', b, b'$ are arbitrary rotation angles, then one can obtain the CHSH result
\[ -2 \leq S \leq 2. \] (7)
We remark that the lab and function spaces, based on which all the correlations are analyzed, belong to the same entity (i.e., the thermal light beam), so the usual locality assumption is replaced here by the statistical independence assumption in the derivation of the above inequality (see supplemental material). This replacement is also made in quantum analyses of Bell inequalities for hybrid entanglement (see for example 13,21 and references therein).

Note that, for an ideal thermal beam, classical optics gives $C(a, b) = \cos 2(a - b)$. This special case paradoxically tells us that there must be a violation of the CHSH Inequality for classically non-deterministic beams, even though only classical results are used to obtain the inequality. Obviously, this requires comment below.

Going further, for the general field $|\vec{3}\rangle$, the joint probabilities in terms of rotated angles $a$ and $b$ can all be calculated and have familiar values in classical statistical optics. The result is an explicit formula for any classical indeterministic $S$:
\[ S = 2\kappa_1\kappa_2[\sin 2a(\sin 2b - \sin 2b')] + \sin 2a'(\sin 2b + \sin 2b') + \cos 2a(\cos 2b - \cos 2b') + \cos 2a'(\cos 2b + \cos 2b'). \] (8)
One can choose the angles freely, and a useful choice is $a = 0$, $a' = \pi/4$, $b = \pi/8, b' = 3\pi/8$. Then one quickly finds that $S$ takes its maximum value for $\kappa_1 = \kappa_2 = 1/\sqrt{2}$, and then $S = \sqrt{2}(1 + 2\kappa_1\kappa_2) \to 2\sqrt{2}$, which obviously violates the inequality $|S| \leq 2$. The only difference to a familiar quantum derivation is that $A(a)$ and $B(b)$ both lie anywhere in the continuum between $-1$ and $+1$, rather than taking discrete values such as $\pm 1$. Here we have no quantum particles to be detected or counted, but a statistical light beam with variable intensity between its components.

We now describe the experiment sketched in Fig. 2 to test these theoretical predictions with a non-deterministic classical light beam. The experiment is designed to determine the correlation function $C(a, b)$ through the joint probabilities (5) by measuring various intensities. Measurements and projections in lab space can be achieved by passing the light beam through polarizers placed at desired angles. Photon detection is clearly not called for and even a calorimeter could be used to measure the intensities of the input and output beams, and thus determine the outcome probabilities.

It is obvious that the two sub-beams $|E\rangle$ and $|E'\rangle$ inherit the statistical properties of the master beam and thus both can be expressed as Eq. (3), with corresponding intensities $I$ and $I'$. An unimportant $i$ phase comes from the beam splitter. To determine the joint probabilities $P_{kl}(a, b)$ of the test beam $|E\rangle$, the first natural step is to project the field in the lab space onto the basis $|u^a_k\rangle$. This can be realized by a polarizer $\hat{u}^a_k$, as shown in the figure, allowing only the $|u^a_k\rangle$ component to pass. Then the corresponding transmitted beam becomes
\[ |E^a_k\rangle = \sqrt{I^a_k}|u^a_k\rangle(A_{k1}|f^h_1\rangle + A_{k2}|f^h_2\rangle), \] (9)
where $I^a_k$ is the intensity, and $A_{kl}$ with $k, l = 1, 2$, are
normalized amplitude coefficients which relate to joint probabilities in an obvious way: $P_k(a, b) = I_k^a |A_k|^2 / I$. One sees that the intensities $I$ and $I_k^a$ can be measured directly but not the coefficients $A_k$.

The statistically identical auxiliary beam $|E'|$ is used to help determine these coefficients. One needs to produce a beam that carries only one of the two components $|f_1^{ab}|$, $|f_2^{ab}|$. However, there is no technology for rotation in function space, i.e., for projecting a non-deterministic field onto an arbitrary direction in the continuously infinite-dimensional function space. We have solved this problem with an indirect measurement setup by passing the beam through a polarizer $i\hat{a}_k^s$ that passes only the special lab space vector $|u_k^s\rangle$, rotated by angle $s$ from the initial basis $|u_1\rangle$, where the angle $s$ is chosen so that the other statistical component $|f_2^{ab}|$ is stripped off. The transmitted beam $|E_k|_1$ has only the $|f_1^{ab}|$ component in the function space, i.e., $|E_k|_1 = i\sqrt{I_1^a}|u_k^a\rangle |f_1^{ab}|$. Here $I_1^a$ is the corresponding intensity and the special stripping angle $s$ can be determined by the relation

$$\tan s = (\kappa_1/\kappa_2) \tan b. \quad (10)$$

The function-space-oriented beam $|E_k^b\rangle$ is then sent through polarizer $i\hat{a}_k^b$ that transmits only the $|u_k^b\rangle$ component in the lab space and becomes $|E_k^{ab}\rangle = |u_k^a\rangle |E_k^b\rangle = i\sqrt{I_k^{ab}} |u_k^a\rangle |f_1^{ab}|$, where $I_k^{ab}$ is the corresponding intensity.

Finally, as shown in Fig. 2 the beams $|E_k^b\rangle$ and $|E_k^{ab}\rangle$ are combined by a 50:50 beam splitter which yields the outcome beam as $|E_k^{ab}\rangle = (|E_k^b\rangle + i|E_k^{ab}\rangle)/\sqrt{2}$. Then the intensity $I_k^{ab}$ of this outcome beam can be easily expressed in terms of the coefficients $A_k$. Some simple arithmetic will immediately provide the joint probabilities $P_k(a, b)$ in terms of various intensities

$$P_{k1}(a, b) = (2I_k^{ab} - I_k^b - I_k^a)^2/4I_k^{ab},$$

$$P_{k2}(a, b) = I_k^a I_k^b - (2I_k^{ab} - I_k^b - I_k^a)^2/4I_k^{ab}, \quad (11)$$

where $k = 1, 2$. Therefore the joint probabilities can be obtained by measuring four different intensities, $I$, $I_k^a$, $I_k^b$, and $I_k^{ab}$. One can immediately achieve the correlation function $C(a, b)$ as defined in (5), and consequently the value of $S$ by carrying out three more sets of experiments with different combinations of polarizer direction setups to achieve the remaining three correlations $C(a, b')$, $C(a', b)$, and $C(a', b')$.

In deriving an inequality and then showing that it can be violated within the same framework as its derivation, one is certain to be making a mis-step. The mis-step is of course the main point. What we have done that leads to the apparent contradiction is to make the statistical independence assumption contained in all CHSH Inequality derivations. Because of the independence of methods used to register the vectors in the two distinct vector spaces, it is always assumed that all possible connections, beyond those originating with the $\{\lambda\}$ distribution, have been excluded. But the presence of entanglement supplies another connection, one that by its nature bridges the two vector spaces. This exposes the fact that what is normally regarded as quantum behavior (indeterminism plus Bell violation) is not restricted to quantum contexts. Our results allow one to say that Bell violation can be explained without quantum mechanics, but not without entanglement.

In conclusion we have provided a test of the uniqueness of quantum theory. We have shown that the statistical theory of optical coherence embodies the three “viable theory” criteria of Shimony, and it allows both the standard derivation of the CHSH Bell Inequality and the violation of it. This contradiction is naturally explained, as in the preceding sentences, and is interesting because it refutes the understanding by Bell [2], accepted generally since, that correlations violating a Bell Inequality cannot originate in other than quantum states. Our results thus clarify two long-standing and widely held mis-impressions: that indeterministic entanglement is unique to quantum mechanics, and that quantum mechanics is unique to violate Bell inequalities. Although our discussion was based on the example of non-deterministic light fields, it seems clear that the analysis can be extended to general stochastic wave theories.

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* Electronic address: xfqian@pas.rochester.edu

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Supplemental Material

Review of CHSH derivation. Due to the non-deterministic characteristics (of the thermal light field), Bell analysis allows that the average measurement outcomes of one space could be influenced by some unknown parameters as well as hidden control variables. We follow tradition and label these contextual unknowns collectively by a single multi-dimensional parameter $\lambda$ with an overall unspecified distribution $\rho(\lambda)$. Therefore the measurement results can be rewritten in terms of probabilities admitting such cross-dependent conditional possibilities:

$$A(a) \rightarrow A(a, \{\lambda\}|B) = P_1(a, \{\lambda\}|B) - P_2(a, \{\lambda\}|B),$$

and similarly

$$B(b) \rightarrow B(b, \{\lambda\}|A) = P_1(b, \{\lambda\}|A) - P_2(b, \{\lambda\}|A).$$

Probability measurement setups in one space are independent of measurement setups in the other space. This allows a measurement outcome statistical independence assumption, so the probabilities reduce to the simpler forms

$$P_k(a, \{\lambda\}|B) = P_k(a, \{\lambda\}), \quad P_l(b, \{\lambda\}|A) = P_l(b, \{\lambda\}).$$

This does not exclude correlations between the measurements in the two spaces, i.e., the outcomes in both spaces may still be related because of $\{\lambda\}$. However, as usual, it says that joint $a - b$ probabilities become products of individual probabilities:

$$P_{kl}(a, b, \{\lambda\}) = P_k(a, \{\lambda\})P_l(b, \{\lambda\}).$$

Consequently the correlation function (5) in the Letter, for arbitrary angles $a$ and $b$, is equivalent in the usual way to

$$C(a, b) = \int A(a, \{\lambda\})B(b, \{\lambda\})\rho(\{\lambda\})d\{\lambda\}.$$

Then one can follow the standard CHSH procedure and find

$$S = C(a, b) - C(a, b') + C(a', b) + C(a', b')$$

$$= \int A(a, \{\lambda\})[B(b, \{\lambda\}) - B(b', \{\lambda\})]$$

$$+ A(a', \{\lambda\})[B(b, \{\lambda\}) + B(b', \{\lambda\})]\rho(\{\lambda\})d\{\lambda\}.$$

From the fact that any measurement results $A(a, \{\lambda\})$ and $B(a, \{\lambda\})$ lie between the values $-1$ and $1$, the above expression of $S$ straightforwardly obeys the inequality $|S| \leq 2$ for any $a, a', b, b'$. This concludes the derivation of CHSH inequality.