SATellite KNOTS oVER LORENZ KNOTS WHICH ARE NOT LORENZ KNOTS

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Abstract. We find infinitely many satellite knots with companions and patterns being Lorenz knots which are not Lorenz knots giving an answer to the question whether a satellite knot having Lorenz pattern and companion is also a Lorenz knot, originally addressed by Birman and Williams in a special case in the 1980s.

1. Introduction

The meteorologist Edward Lorenz has described a three-dimensional system of ordinary differential equations to predict weather patterns [10]. The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, electric circuits, chemical reactions, etc. And for some choices of parameters, it has knotted periodic orbits, which are called Lorenz links.

Lorenz links have been given special attention, specially Lorenz knots which are satellites.

To obtain a satellite knot we start with a knot $P$ inside the solid torus $S^1 \times D^2$ that is not isotopic into a ball and to $S^1 \times 0$ in $S^1 \times D^2$. Next, we consider a non-trivial knot $C$ and a homeomorphism $f : D^2 \times S^1 \rightarrow N(C)$ that takes $D^2 \times S^1$ to the tubular neighbourhood $N(C)$ of $C$. Then, the image $K = f(P)$ is the satellite knot with pattern $P$ and companion $C$. However, there are infinitely many homeomorphisms taking $D^2 \times S^1$ to $N(C)$. Hence we need to give $f$ an additional condition to make $K$ well-defined. With this in mind, the convention is that $f$ must send the standard longitude $S^1 \times \{1\}$ of $S^1 \times D^2$ to the standard longitude of $N(C)$. In other words, $f$ is not allowed to introduce additional twisting.

There is a conjecture about satellite Lorenz knots, called Lorenz satellite conjecture, which claims that satellite Lorenz knots have companions and patterns equivalent to Lorenz knots (see [7, Conjecture 1.2]).

The converse of Lorenz satellite conjecture is a question that naturally arises. See [7, Question 1.4]. More precisely, it wonders:

Question 1.1. Let $K_1$ and $K_2$ be Lorenz knots. Is the satellite knot with pattern $K_1$ and companion $K_2$ also a Lorenz knot?

The first results addressing this question gave some partial positive answers. The first one was given by Birman and Williams in 1980 when they proved that if $K$ is a Lorenz knot with crossing number $c$ and $a, b'$ are arbitrary
coprime positive integers, then the satellite knot with companion the Lorenz knot $K$ and pattern the torus knot $T(a, b + ac)$ is a Lorenz knot [2, Theorem 6.2] (this result is valid, but not as general as it is claimed to be if we fix the standard longitudes as we will see in Remark 3.6). The next results that address Question 1.1 were given by de Paiva and Purcell when they proved that the answer to Question 1.1 is yes to large families of Lorenz knots [7, Theorem 1.3, Theorem 6.1, and Theorem 6.2].

In this paper we find infinitely many satellite knots which are not Lorenz knots but have Lorenz knots as companions and patterns. Therefore, we give the first negative answer to Question 1.1.

Our main theorem is the following.

**Theorem 1.2.** There are infinitely many satellite knots with companions and patterns being Lorenz knots which are not Lorenz knots.

In this paper we work with T-links as Birman and Kofman showed that they are equivalent to non-trivial Lorenz links [1, Theorem 1]. T-links are defined as follows: For $2 \leq r_1 < \cdots < r_k$, and all $s_i > 0$, the T-link $T((r_1, s_1), \ldots, (r_k, s_k))$ is defined to be the closure of the following braid, which we call the standard braid,

$$(\sigma_1 \sigma_2 \ldots \sigma_{r_1-1})^{s_1} (\sigma_1 \sigma_2 \ldots \sigma_{r_2-1})^{s_2} \cdots (\sigma_1 \sigma_2 \ldots \sigma_{r_k-1})^{s_k},$$

where $\sigma_1, \ldots, \sigma_{r_k-1}$ are the standard generators of the braid group $B_{r_k}$. In particular, we conclude that torus knots are T-links from these representations for T-links (or [2, Theorem 6.1]).

To prove Theorem 1.2, we first prove that every T-link has a positive braid with at least one positive full twist, Theorem 2.7. Then, we find some satellite knots having companions and patterns being Lorenz but they can’t be represented by a positive braid with at least one positive full twist. And so, we complete the proof.

2. Minimal Braids for T-links

In this section we prove that every T-link has a positive braid with at least one positive full twist.

**Definition 2.1.** Let $i, j, r$ be positive integers with $i < j$. The $(i, j, r)$-torus braid, denoted by $B_{i,j}^r$, is defined by the braid

$$(\sigma_i \ldots \sigma_{j-1})^r.$$
Figure 5 of [3] illustrates a positive half twist along four strands.

The following Lemma was proved by de Paiva in [4, Lemma 3.2].

Lemma 2.3. The torus braid $B_{i,j}$ is obtained by full and/or half twists along three circles with each one encircling at most $j - i + 1$ strands.

Proposition 2.4. Consider $p, q$ positive integers with $p > q > 1$. Let $K$ be a link given by a positive braid $B$ with $p$ strands of the form

$$B_{a_1,b_1}^{r_1} B_{a_2,b_2}^{r_2} \ldots B_{a_{m-1},b_{m-1}}^{r_{m-1}} B_{a_m,b_m}^{q} B_{a_{m+1},b_{m+1}}^{r_{m+1}} \ldots B_{a_n,b_n}^{r_n}$$

such that if $a_i = 0$ then $b_i \leq q$ or if $a_i \neq 0$ then $b_i - a_i + 1 \leq q$. Then, the link $K$ has a positive braid with at least one positive full twist.

Proof. Just for convenience, we push all torus braids $B_{a_i,b_i}$ which are above the biggest torus braid $B_{0,q}^{p}$ around the braid closure to be below $B_{0,p}^{q}$.

Each torus braid $B_{a_i,b_i}$ can be obtained by full and/or half twists along three circles where each one is encircling at most $q$ strands from Lemma 2.3. Thus, $B$ is obtained from the torus braid $B_{0,q}^{p}$ together with all these circles by full and/or half twists along them.

We deform the torus braid $B_{0,q}^{p}$ to the torus braid $B_{0,q}^{p}$ using the isotopy of the first and second drawings of [5, Figure 1], where this isotopy can be described as a rotation around a diagonal of the square that gives the torus in which the $(p,q)$-torus knot lies. After this, the circles are now encircling the meridional (horizontal) lines of the torus braid $B_{0,q}^{p}$. Furthermore, they enclose the same amount of strands as before, which are at most $q$ strands. We push all these circles to lie below the torus braid $B_{0,q}^{p}$ and then apply the full and/or half twists as before. This gives a link equivalent to $K$ because these isotopes preserve half twists as the tangles (bunches of crossings) of the torus braids travel in the same direction all the time. The link $K$ is now given by a positive braid with the torus braid $B_{0,q}^{p}$ on the top. Moreover, this braid has at least one full twist as $p > q$. □

Next proposition generalizes the isopoty of [6, Lemma 2.1], which was based on [9, Figure 6].

Proposition 2.5. Consider $B$ a braid with $r$ strands and $p,q$ integers such that $0 < q \leq r < p$. Then, there is an isotopy that takes the closure of the braid $B(\sigma_{1} \ldots \sigma_{p-1})^{q}$ to the braid

$$(\sigma_{r-1} \ldots \sigma_{r-q+1})^{p-r} B(\sigma_{1} \ldots \sigma_{r-1})^{q}$$

if $q > 1$ and

$$B(\sigma_{1} \ldots \sigma_{r-1})^{q}$$

if $q = 1$. Moreover, this isotopy happens in the complement of the braid axis of $B$.

Proof. The case $q = 1$ immediately follows by shrinking (or applying some destabilization moves to) the last $p - r$ strands of $B(\sigma_{1} \ldots \sigma_{p-1})^{q}$. Hence we assume that $q > 1$. 

Figure 1. This series of drawings illustrates how to reduce one strand from the closure of the leftmost braid.

The \((r - q + 1)\)-st strand anticlockwise goes around the braid closure to pass through the sub-braid \((\sigma_1 \ldots \sigma_{p-1})^q\) and finally ends in the \((r + 1)\)-st strand as \(q \leq r\). Thus, the \((r - q + 1)\)-st strand is connected to the \((r + 1)\)-st strand by an under strand that goes one time around the braid closure as illustrated by the blue strand in Figure 1. We push this under strand down and shrink it to reduce one strand from the last braid, as shown in Figure 1. After that, this strand becomes an under strand between the \((r - q + 1)\) and the \(r\)-st strand yielding the braid

\[
(\sigma_{r-1} \ldots \sigma_{r-q+1})B(\sigma_1 \ldots \sigma_{p-2})^q.
\]

We see that this isotopy doesn’t change the sub-braid \(B\). So, it happens in the complement of the braid axis of \(B\).

As the sub-braid \((\sigma_{r-1} \ldots \sigma_{r-q+1})B\) of the last braid is a braid on the leftmost \(r\) strands, the \((r - q + 1)\)-st strand is still connected to the \((r + 1)\)-st strand by an under strand that goes one time around the braid closure. So, we can apply the last isotopy again to obtain the braid

\[
(\sigma_{r-1} \ldots \sigma_{r-q+1})^2B(\sigma_1 \ldots \sigma_{p-3})^q.
\]

We continue applying this procedure and then after doing it \(p - r - 2\) times, we obtain the braid

\[
(\sigma_{r-1} \ldots \sigma_{r-q+1})^{p-r}B(\sigma_1 \ldots \sigma_{p-1})^q.
\]

Proposition 2.6. Consider \(p > q > 1\) and \(r \geq q\). Then, the T-link

\[
K = T((r_1, s_1), \ldots, (r_n, s_n), (p, q))
\]

has a positive braid with at least one positive full twist.

Proof. By Proposition 2.5, there is an isotopy that takes the standard braid of \(K\) to the braid

\[
B_1 = (\sigma_{r_n-1} \ldots \sigma_{r_{n-q+1}})^{p-r_n}(\sigma_1 \ldots \sigma_{r_1-1})^{s_1} \ldots (\sigma_1 \ldots \sigma_{r_{n-1}-1})^{s_{n-1}}(\sigma_1 \ldots \sigma_{r_{n-1}})^{s_n+q}.
\]
If \( s_n + q \geq r_n \), then \( B_1 \) is itself a positive braid with at least one positive full twist. If \( s_n + q < r_n \) and \( s_n + q \geq r_{n-1} \) (possibly \( r_{n-1} = 0 \)), the result follows from Proposition 2.4.

Consider next that \( s_n + q < r_{n-1}, r_n \) and \( r_{n-1} \neq 0 \). Then, we anticlockwise push the sub-braid \((\sigma_{r_{n-1}} \ldots \sigma_{r_n-q+1})^{p-r_n} \) of \( B_1 \) to be between the \((r_{n-1}, s_{n-1})\) and the \((r_n, s_n + q)\)-torus braid. Then, we apply the isotopy of Proposition 2.5 and push the \((\sigma_{r_{n-1}} \ldots \sigma_{r_n-q+1})^{p-r_n} \) back to the top. After that, we obtain the braid

\[
B_2 = (\sigma_{r_{n-1}-1} \ldots \sigma_{r_{n-1}-q+1})^{p-r_n}(\sigma_{r_{n-1}-1} \ldots \sigma_{r_{n-1}-s_n-q+1})^{r_n-r_{n-1}} \\
(\sigma_1 \ldots \sigma_{r_2-1})^{s_1} \ldots (\sigma_1 \ldots \sigma_{r_{n-2}-1})^{s_{n-2}}(\sigma_1 \ldots \sigma_{r_{n-1}-1})^{s_n+q}.
\]

If \( s_{n-1} + s_n + q \geq r_{n-1} \), then \( B_2 \) is itself a positive braid with at least one positive full twist. If \( s_{n-1} + s_n + q < r_{n-1} \) and \( s_{n-1} + s_n + q \geq r_{n-2} \) (possibly \( r_{n-2} = 0 \)), then the result follows from Proposition 2.4. Otherwise, \( s_{n-1} + s_n + q < r_{n-2}, r_{n-1} \) and \( r_{n-2} \neq 0 \). Then, we continue likewise until we obtain a positive braid with at least one full twist for \( K \). \( \square \)

**Theorem 2.7.** Every T-link has a positive braid with at least one positive full twist.

**Proof.** Consider \( K \) the T-link \( T((r_1, s_1), \ldots, (r_n, s_n), (p, q)) \).

If \( q = 1 \), then \( K \) is equivalent to the T-link \( T((r_1, s_1), \ldots, (r_n, s_n + 1)) \) by Proposition 2.5. So, we can assume that \( q > 1 \).

If \( q \geq p \), then the standard braid of \( K \) is itself a positive braid with at least one positive full twist.

If \( p > q \geq r_n \) (possibly \( r_n = 0 \)), then \( K \) has a positive braid with at least one positive full twist by Proposition 2.4.

Finally, if \( r_n > q \), then \( K \) also has a positive braid with at least one positive full twist from Proposition 2.6. \( \square \)

### 3. Satellite knots and Positive braids

In this section we prove Theorem 1.2.

**Lemma 3.1.** Let \( B_1, B_2 \) be two positive braids with the same amount of strands representing the same knot \( K \). Then, \( B_1 \) and \( B_2 \) have the same number of crossings.

**Proof.** The genus of \( K \) is given by the formula

\[
\frac{c_i - b_i + 1}{2},
\]

where \( c_i, b_i \) is the number of crossings, strands, respectively, of \( B_i \) with \( i = 1 \) or \( 2 \) [11]. As the genus is a knot invariant and \( B_1 \) and \( B_2 \) have the same amount of strands, it follows that \( B_1 \) and \( B_2 \) have the same number of crossings. \( \square \)
Lemma 3.2. Suppose that $B$ is a positive braid with $p$ strands and at least one full twist on $p$ strands and represents a knot. Then, $B$ must have more than
\[ p(p - 1) + p - 2 \]
crossings.

Proof. By contradiction, consider that $B$ has less than or equal to $p(p - 1) + p - 2$ crossings.

The braid $B$ has at least $p(p - 1)$ crossings because $B$ has at least one positive full twist. In addition to these crossings, $B$ has at most more $p - 2$ crossings. Hence the braid $B$ is formed by adding at most $p - 2$ crossings to the braid $F$ given by a positive full twist on $p$ strands.

The full twist $F$ is obtained by applying a full twist on $p$ unknotted link components. As full twists are homeomorphisms, the closure of $F$ still has $p$ link components.

When we add a crossing to $F$, we obtain a braid $F_1$ whose closure has $p - 1$ link components. Now if we add a crossing to $F_1$, we obtain a braid $F_2$ whose closure has at least $p - 2$ link components. Similarly, the closure of the braid formed by adding a new crossings to $F_2$ has at least $p - 3$ link components.

We conclude that the closure of $B$ has at least 2 link components as $B$ is obtained by adding at most $p - 2$ crossings to $F$. However, the closure of $B$ should be a knot, a contradiction.

\[ \square \]

Theorem 3.3. Consider $c_1, d_1, \ldots, c_n, d_n, a, b, k$ positive integers such that the $T$-link $T((c_1, d_1), \ldots, (c_n, d_n), (b, (a - 1)(a + 1)b + k))$ is a knot and
\[ (ab - b)b(a + 1) + k(b - 1) + d_n(c_n - 1) + \cdots + d_1(c_1 - 1) \leq ab(ab - 1) + (ab - 2). \]
Then, the satellite knot $K$ with pattern the $T$-knot
\[ T((c_1, d_1), \ldots, (c_n, d_n), (b, (a - 1)(a + 1)b + k)) \]
and companion the torus knot $T(a, a + 1)$ can not be represented as a positive braid with at least one positive full twist.

Proof. It is well known that the torus knot $T(a, a + 1)$ has braid index equal to $a$ (or also see [8, Corollary 2.4]).

The knot $K$ has braid index equal to $ab$ by [12, Theorem 1].

Suppose that $K$ can be represented by a positive braid $B$ with $n$ strands and at least one positive full twist on $n$ strands.

By [8, Corollary 2.4], the number $n$ is equal to braid index of $B$. As $B$ represents $K$, we conclude that $n$ is equal to $ab$.

The minimal positive braid for $T(a, a + 1)$ induces a minimal positive braid $B'$ for $K$. Furthermore, since the minimal positive braid for $T(a, a + 1)$ has $(a - 1)(a + 1)$ crossings, then after tangling the $T$-knot
\[ T((c_1, d_1), \ldots, (c_n, d_n), (b, (a - 1)(a + 1)b + k)) \]
along the torus knot $T(a, a + 1)$ preserving the standard longitudes to obtain $K$, the knot inside the companion receives $(a - 1)(a + 1)$ negative full twists.
on the $b$ strands. So, $B'$ has

$$(ab - b)b(a + 1) + k(b - 1) + d_n(c_n - 1) + \cdots + d_1(c_1 - 1)$$
crossings.

By Lemma 3.1, $B$ has the same amount of crossings as $B'$. However, this
contradicts Lemma 3.2 as $(ab - b)b(a + 1) + k(b - 1) + d_n(c_n - 1) + \cdots + d_1(c_1 - 1)$
is less than or equal to $ab(ab - 1) + (ab - 2)$. □

**Corollary 3.4.** Consider $c_1, d_1, \ldots, c_n, d_n, a, b, k$ positive integers such that
the $T$-link $T((c_1, d_1), \ldots, (c_n, d_n), (b, (a - 1)(a + 1)b + k))$ is a knot and
$(ab - b)b(a + 1) + k(b - 1) + d_n(c_n - 1) + \cdots + d_1(c_1 - 1) \leq ab(ab - 1) + (ab - 2)$.
Then, the satellite knot $K$ with pattern the $T$-knot

$$T((c_1, d_1), \ldots, (c_n, d_n), (b, (a - 1)(a + 1)b + k))$$
and companion the torus knot $T(a, a + 1)$ is not a $T$-knot.

*Proof.* It follows from Theorem 2.7 and the last theorem. □

**Corollary 3.5.** Consider $a, b$ integers greater than one. The satellite knot
with pattern the torus knot $T(b, (a - 1)(a + 1)b + 1)$ and companion the torus
knot $T(a, a + 1)$ is not a $T$-knot.

*Proof.* As $b > 1$, then $b^2 > 1 + b$, or $-b^2 + b < -1$, or $-b^2 + b - 1 < -2$. This
implies that $(ab)^2 - b^2a + b^2a - b^2 + b - 1 < (ab)^2 - ab + ab - 2$. Hence

$$(ab - b)b(a + 1) + b - 1 < ab(ab - 1) + (ab - 2).$$

Now the result follows from the last corollary. □

Therefore, we have proved Theorem 1.2.

**Remark 3.6.** From the last corollary, the satellite knot with pattern the
torus knot $T(b, (a - 1)(a + 1)b + 1)$ and companion the torus knot $T(a, a + 1)$
is not a $T$-knot. However, Theorem 6.2 of [2] states that they are $T$-knots.
Therefore, we conclude that the authors were not considering any fixed
longitudes, as discussed in the introduction, in this result.

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