On multiple Einstein rings

M. C. Werner,1⋆ J. An2,3⋆ and N. W. Evans1⋆

1Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA
2Dark Cosmology Centre, Niels Bohr Institute, University of Copenhagen, Juliane Maries Vej 30, 2100 Copenhagen Ø, Denmark
3Niels Bohr International Academy, Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

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ABSTRACT

A number of recent surveys for gravitational lenses have found examples of double Einstein rings. Here, we analytically investigate the occurrence of multiple Einstein rings. We prove, under very general assumptions, that at the most one Einstein ring can arise from a mass distribution in a single plane lensing a single background source. Two or more Einstein rings can therefore only occur in multiplane lensing. Surprisingly, we show that it is possible for a single source to produce more than one Einstein ring. If two point masses, or two isothermal spheres, in different planes are aligned with observer and source on the optical axis, we show that there are up to three Einstein rings. We also discuss the image morphologies for these two models if axisymmetry is broken, and give the first instances of magnification invariants in the case of two-lens planes.

Key words: gravitational lensing.

1 INTRODUCTION

In his seminal article on the gravitational lensing effect, Einstein (1936) discussed the circular image of a point-like source, noting that ‘there is no hope of observing this phenomenon directly’. But, thanks to advances in instrumentation since then, arcs of partial and even complete Einstein rings are now being routinely found. Recently, surveys have found the first instances of multiple Einstein rings. This includes the partial double Einstein rings of SDSS J0924+0219 discovered from Hubble Space Telescope (HST) images by the Cosmological Monitoring of Gravitational Lenses (COSMOGRAIL) team (Eigenbrod et al. 2006), and of SDSS J0946+1006 found by the Sloan Lens ACS Survey (Gavazzi et al. 2008). The Cambridge Sloan Survey of Wide Arcs in the Sky (CASSOWARY; Belokurov et al. 2008) has also uncovered a number of examples of multiple ring systems, such as CASSOWARY 2. Such systems may offer valuable insights into the mass distribution of the lensing galaxies.

From a theoretical point of view, the possibility of forming multiple Einstein rings has long been known in the case of the strong deflection limit near the photon sphere of a black hole (for a review, see e.g. Nemiroff 1993, and references therein). However, a systematic investigation for the weak deflection limit appears to be lacking so far, and we present results to this end for the case of one- and two-lens planes in this paper. After an outline of the general lensing setup, the condition for Einstein rings is derived in Section 2. In the single-lens plane case considered in Section 3, we prove that, under rather general assumptions, multiple Einstein rings cannot arise. We therefore proceed to two-lens planes in Section 4 and consider two singular isothermal spheres and two-point lenses as simple models where multiple Einstein rings of a single source do, in fact, occur. In the first example, there are up to two Einstein rings due to the singular isothermal spheres in different planes, and another ring if the second lens is also luminous. Similarly, two-point lenses can give rise to up to three Einstein rings overall. Therefore, the usual supposition that arcs of multiple Einstein rings indicate the presence of as many sources at different distances is not necessarily correct. We briefly discuss the image configuration in these cases if axisymmetry is broken. For models with two-lens planes, we also find that analogues of the invariants of the signed magnification sum (see e.g. Witt & Mao 1995; Hunter & Evans 2001) hold in the domains of maximal image multiplicity.

Regarding notation, we write $\nabla_\nu$ and $\Delta_\nu$ for the gradient operator and Laplacian, respectively, expressed in the same coordinate system as the vector $\nu$. Furthermore, $||\nu||$ is the vector norm with respect to the Euclidean metric, and square brackets $[\nu]$ denote functional dependence on the variable $u$. The set $\mathbb{R}_+^d = \{ x \in \mathbb{R} : x \geq 0 \}$ denotes the set of all non-negative reals. Subscripts can label vector components or images according to context. The universal gravitational constant and the speed of light in vacuum are represented by $G$ and $c$, respectively, as usual.

2 LENSING FRAMEWORK

2.1 General setup

Gravitational lensing in the weak deflection limit is conveniently described in terms of the impulse approximation, with piecewise
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While $\xi^{(1)}/D_1$ would correspond to the usual angular coordinate in $L^{(1)}$, we will find the space of parameters more convenient if the following normalization is used (Erdl & Schneider 1993):

$$x^{(1)} = \frac{\xi^{(1)}}{\xi_0}, \quad x^{(2)} = \frac{\xi^{(2)}/D_2}{\xi_0/D_1}, \quad y = \frac{\eta/D_1}{\xi_0/D_1},$$

with parameters

$$\xi_0 = D_1 \left( \frac{\mu^{(1)} + \mu^{(2)}}{D_1} \right)^{1/2},$$

where $\mu^{(1)} = M^{(1)}(D_1/D_1)$, and $\mu^{(2)} = M^{(2)}(D_2/D_1)$ and $\beta = D_{12}D_1/D_{12} = 1 - D_{12}D_1/D_{12}$. Here, $0 \leq \beta \leq 1$. In particular, $\beta = 0$ if and only if $D_{12} = 0$ and $\beta = 1$ if and only if $D_{12} = 0$, provided that $D_1 \neq 0$, $D_2 \neq 0$ and $S \neq 0$. Then, the lens equations (1) and (2) can be rewritten, thus, as

$$x^{(2)} = x^{(1)} - \beta m^{(1)} \nabla_{x^{(1)}} f^{(1)}(x^{(1)}),$$

$$y = x^{(1)} - m^{(2)} \nabla_{x^{(2)}} f^{(1)}(x^{(1)}) - m^{(2)} \nabla_{x^{(2)}} f^{(2)}(x^{2}) - x^{(2)},$$

introducing the normalized mass parameters

$$m^{(1)} = \frac{\mu^{(1)}}{\mu^{(1)} + \mu^{(2)}},$$

$$m^{(2)} = \frac{\mu^{(2)}}{\mu^{(1)} + \mu^{(2)}},$$

such that $m^{(1)} + m^{(2)} = 1$.

### 2.2 Einstein rings

Einstein rings, within the geometrical optics approximation of the standard lensing framework, are infinitely magnified, circularly symmetric images. We therefore stipulate circular symmetry about the optical axis such that $x^{(2)} = 0$, and

$$f^{(1)}(x^{(1)}) = f^{(2)}(x^{(2)}) = 1, \quad i \in \{1, 2\}, \text{ and use } \|x^{(i)}\| = x^{(i)}$$

for notational simplicity. But, then the lens equation (10) implies that $x^{(2)} = x^{(2)}(x^{(1)})$ and (11) becomes

$$y = x^{(1)} - x^{(1)} m^{(1)} \frac{df^{(1)}}{dx^{(1)}}$$

$$-x^{(2)} m^{(2)} (1 - \beta m^{(2)} \frac{df^{(1)}}{dx^{(1)}} \frac{df^{(2)}}{dx^{(2)}}) \frac{df^{(2)}}{dx^{(2)}} = x^{(1)} F(x^{(1)}).$$

Introducing plane polar coordinates $(x^{(1)}, \phi)$, the Jacobian determinant of the lensing map (14) is therefore

$$\det \mathbf{J} = \frac{1}{x^{(1)}} \left[ \frac{\partial F}{\partial x^{(1)}} \frac{\partial}{\partial x^{(1)}} - \frac{\partial F}{\partial x^{(1)}} \frac{\partial}{\partial x^{(1)}} \right] = F + x^{(1)} \frac{dF}{dx^{(1)}}.$$

The condition for critical curves $\det \mathbf{J} = 0$ gives rise to two classes of critical curves. The first, given by $F(x^{(1)}) = 0$, defines tangential critical circles in $L^{(1)}$ which are solutions of the lens equation (14), and hence infinitely magnified images, and map to the caustic point $y = 0$. The other solution $F + x^{(1)} \frac{dF}{dx^{(1)}} = 0$ gives radial critical circles in $L^{(1)}$ which map to caustics and hence define domains of constant image multiplicity in $S$ (Schneider et al. 1999, p. 233).
Definition 2.1. An Einstein ring is a circular, critical image of a point source at $\mathbf{y} = 0$ whose radius is a solution of $F(x^{(1)}) = 0$, $x^{(1)} > 0$.

Finally, we note that if $\mathbf{y} \neq 0$ so that the axisymmetry is broken, discrete images are formed at some $x^{(1)}$ which have finite signed magnification (Schneider et al. 1999, p. 162)

$$\mu [x^{(1)}] = \frac{1}{\det J [x^{(1)}]}.$$  (16)

3 ONE-LENS PLANE

We continue by specializing the previous discussion to the simpler case of a single-lens plane $L^{(1)} = x^{(2)}$ such that $m^{(2)} = 0$, $\beta = 0$. Note, that then, that (5), (7) and (8) yield

$$\nabla x^{(1)} f^{(1)} = \frac{D_1 D_2 M^{(1)}}{d_0} \nabla x^{(1)} f^{(1)} = \frac{D_1 D_2}{d_0} \bar{\alpha}^{(1)} = \alpha,$$

which is the normalized deflection angle in the standard form (Schneider et al. 1999, p. 158). Similarly, the Poisson equation (6) becomes

$$M^{(1)} \nabla x^{(1)} f^{(1)} = \frac{8\pi G}{c^2} \Sigma^{(1)} \Rightarrow \nabla x^{(1)} f^{(1)} = 2 \frac{\Sigma^{(1)}}{\Sigma_{\text{crit}}} = 2 \kappa$$

with the usual definition of the critical surface density $\Sigma_{\text{crit}} = c^2 D_L/(4\pi G D_S D_A)$. Using Definition 2.1 and writing $||\alpha|| = \alpha = d\kappa^{(1)}/dx^{(1)}$, the problem of finding Einstein rings therefore reduces to a fixed point equation

$$\alpha [x^{(1)}] = x^{(1)}, \quad x^{(1)} > 0,$$

subject to the Poisson equation for the given mass distribution of the lens,

$$\Delta x^{(1)} f^{(1)} [x^{(1)}] = 2 \kappa [x^{(1)}].$$

(18)

Hence, we need to define general, yet astrophysically sensible lens models $\kappa[x^{(1)}]$ that allow for Einstein rings. Apart from the circular symmetry inherent in the problem, it would be plausible to stipulate that $\kappa$ decreases monotonically with $x^{(1)}$. However, we will use a condition even weaker than monotonicity, namely that, at every radius, the surface density be smaller than the average density of the mass enclosed. This is natural for self-gravitating and hence centrally condensed systems, and a more specific model for the mass distribution need not be assumed.

Definition 3.1. The gravitational lens is defined by a normalized surface density $\kappa$ as mass model such that the following conditions are fulfilled:

(i) Continuity: $\kappa : L^{(1)} \rightarrow \mathbb{R}_0^+$ be a continuous function except at $0 \in L^{(1)}$ for singular lenses.

(ii) Circular symmetry: $\kappa = \kappa [x^{(1)}]$ only.

(iii) Finiteness: $\kappa < \infty$ for $x^{(1)} > 0$, $\kappa [0] = 1/C_1$ with constant $C_1 \geq 0$, and $\lim_{x^{(1)} \rightarrow \infty} \kappa x^{(1)} = C_2$ with constant $0 \leq C_2 < \infty$.

(iv) Self-gravitation: $\kappa [x^{(1)}] < \bar{\kappa} [x^{(1)}] \text{ where } \bar{\kappa} [x^{(1)}] = \int_0^{2\pi} \int_0^{x^{(1)}} \kappa [x] x \, dx.$

For example, the point lens has $C_1 = 0$, $C_2 = 0$, and the isothermal sphere $C_1 = 0$, $C_2 > 0$ in our notation. Both are usually regarded as singular, since their projected surface densities diverge at the centre. Moreover, the total mass is infinite in the isothermal case, and we should expect $C_2 = 0$ for more realistic lenses. However, it turns out that we do not need to require this for our purposes and hence define singular lenses simply as follows.

Definition 3.2. A gravitational lens is called singular if, and only if, $C_1 = 0$, and is called non-singular otherwise.

With these definitions, one can now prove that there is at the most one Einstein ring. This result follows from existence, whose necessary and sufficient condition is established in Theorem 3.1, and uniqueness, shown in Theorem 3.2. First of all, however, we will need a lemma regarding the surface density, and a lemma concerning the deflection angle.

Lemma 3.1. Given a gravitational lens in the sense of Definitions 3.1 and 3.2, the normalized surface density fulfils $\kappa [x^{(1)}] < \kappa [0]$ $\forall x^{(1)} > 0$.

Proof. If the lens is singular, then the lemma follows immediately from Definition 3.1(iii). Otherwise, by Definition 3.1(iv), we have $\kappa [x^{(1)}] < \bar{\kappa} [x^{(1)}] < \max_{0 \leq x^{(1)} \leq x^{(1)}_0} \kappa [x] \forall x^{(1)}.$

Suppose that this maximum is attained at some $x' < x^{(1)}$. Then, the previous inequalities hold for $\kappa [x']$ as well, with maximum at $x'' < x'$, say. Repeating this argument sufficiently often shows that $\kappa [x^{(1)}] < \kappa [0]$, as required. □

Lemma 3.2. Given a gravitational lens in the sense of Definitions 3.1 and 3.2, the normalized deflection angle $\alpha$ has the following properties:

(i) Smoothness: $\alpha : L^{(1)} \rightarrow \mathbb{R}_0^+$ is a smooth function except at $0 \in L^{(1)}$ for singular lenses.

(ii) Circular symmetry: $\alpha = \alpha [x^{(1)}]$ only.

(iii) Properties at the centre: for non-singular lenses, $\alpha [0] = 0$ and the derivative $d\alpha/dx^{(1)} [0] = 0$, for singular lenses $\alpha [0] > 0$.

(iv) Finiteness: $\lim_{x^{(1)} \rightarrow \infty} \alpha = A$ with constant $A < \infty$.

Proof. Definition 3.1(i) implies property (i), and property (ii) follows immediately from Definition 3.1(ii) and the uniqueness of solutions of Poisson’s equation. Circular symmetry and smoothness at the centre imply that $\alpha [0] = 0$ for non-singular lenses, and $\alpha [0]$ is some positive, possibly infinite, value for singular lenses. Now integrate Poisson’s equation (18) to find

$$2\kappa = \frac{d}{x^{(1)} dx^{(1)}} (\alpha x^{(1)}) \quad \Rightarrow$$

$$\alpha [x^{(1)}] = \frac{2}{x^{(1)}} \int_0^{x^{(1)}} \kappa [x] x \, dx.$$

(19)

(20)

using $\alpha [0] = 0$. Then, (19) implies the second property of non-singular lenses in (iii).

$$\kappa [0] = \frac{1}{2} \lim_{x^{(1)} \rightarrow 0} \left( \frac{d\alpha}{dx^{(1)}} + \alpha + \frac{d\alpha}{dx^{(1)}} x^{(1)} + \mathcal{O} \left[ (x^{(1)})^2 \right] \right)$$

$$= \frac{dx}{dx^{(1)}} [0]$$

using again $\alpha [0] = 0$, and property (iv) follows from equation (20) and Definition 3.1(iii) since

$$\lim_{x^{(1)} \rightarrow \infty} \alpha = 2 \lim_{x^{(1)} \rightarrow \infty} \kappa x^{(1)} = 2C_2,$$

so $A = 2C_2 < \infty$, as required. This is also true for singular lenses where the integration of (19) has to start at some $x > 0$, so that the limit for $\alpha$ as $x^{(1)} \rightarrow \infty$ is the same as before plus some finite integration constant. □

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Theorem 3.1. (Existence) Given a gravitational lens in the sense of Definitions 3.1 and 3.2, at least one Einstein ring exists if, and only if, the condition \( \kappa[0] > 1 \) holds.

Proof. First, we prove the sufficiency of the condition: \( \kappa[0] > 1 \Rightarrow \exists \) Einstein ring. Let us consider the graph of the deflection angle
\[
\alpha = (x^{(1)}, \alpha(x^{(1)})) \in \mathbb{R}_+^n \times \mathbb{R}_+^n.
\]
Then, according to equation (17), the existence of an Einstein ring is equivalent to the existence of a fixed point and hence an intersection of graph \( \alpha \) with the diagonal \( (x^{(1)}, x^{(1)}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \) at some \( x^{(1)} > 0 \). Let \( A_1 = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : y > x^{(1)}\} \) and \( A_2 = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : y < x^{(1)}\} \) be two domains to the left and right of the diagonal, respectively. For non-singular lenses, Lemma 3.2(iii) implies that \( x^{(1)} = 0 \) is a fixed point but no Einstein ring, and \( \kappa[0] > 1 \Rightarrow \) graph \( \alpha \in A_1 \) for \( x^{(1)} \rightarrow 0 \). This is also true for singular lenses because here \( \alpha[0] = 0 \) holds. On the other hand, Lemma 3.2(iv) implies that graph \( \alpha \in A_2 \) for \( x^{(1)} \rightarrow \infty \). This ensures the existence of at least one fixed point with \( x^{(1)} > 0 \) and hence Einstein ring.

Now we show necessity: \( \exists \) Einstein ring \( \Rightarrow \kappa[0] > 1 \). This statement is equivalent to its contraposition, \( \kappa[0] \leq 1 \Rightarrow \nexists \) Einstein ring, which we prove by contradiction. So, suppose \( \kappa[0] \leq 1 \) and \( \exists \) Einstein ring, then there is some \( x^{(1)} > 0 \) such that \( \alpha[x^{(1)}] = x^{(1)} \) by equation (17). By recasting Poisson’s equation (18) and using Lemma 1.3.1, we have
\[
\frac{d\alpha}{dx^{(1)}}[x^{(1)}] = 2\kappa [x^{(1)}] - 1 < 2\kappa[0] - 1 \Rightarrow \frac{d\alpha}{dx^{(1)}}[x^{(1)}] < 1,
\]

since by assumption \( \kappa[0] \leq 1 \). But, on the other hand, for a fixed point \( x^{(1)} > 0 \) to exist, we must require \( d\alpha/dx^{(1)}[x^{(1)}] \geq 1 \). To see this, first of all note that the assumption \( \kappa[0] \leq 1 \) also means, by Definitions 3.1(iii) and 3.2, that we only need to consider non-singular lenses here. Now, by Lemma 3.2(iii), we can write
\[
\frac{d\alpha}{dx^{(1)}}[x^{(1)}] = \frac{2}{3} \frac{d\alpha}{dx^{(1)}}[x^{(1)}] < 0 \text{ by Lemma 3.1},
\]
so even if \( \kappa[0] = 1 \), graph \( \alpha \in A_2 \) for small \( x^{(1)} \), and the result follows. This completes the proof by contradiction and hence the proof of the theorem.

Theorem 3.2. (Uniqueness) Given a gravitational lens in the sense of Definitions 3.1 and 3.2, then if an Einstein ring exists, there is exactly one.

Proof. The existence of an Einstein ring \( \alpha \) implies that \( \kappa[0] > 1 \) by Theorem 3.1, and that there is some \( x^{(1)} > 0 \) such that \( \alpha[x^{(1)}] = x^{(1)} \) by equation (17). Using the integral equation (20) and Definition 3.1(iv), we obtain
\[
1 = \frac{\alpha(x^{(1)})}{x^{(1)}} = \frac{2}{(x^{(1)})^2} \int_0^{x^{(1)}} \kappa(x)dx \Rightarrow \kappa[x^{(1)}] = \kappa[x^{(1)}] = \kappa[x^{(1)}].
\]
Hence, by the Poisson equation (19) and Definition 3.1(iv),
\[
\frac{d\alpha}{dx^{(1)}}[x^{(1)}] = 2\kappa [x^{(1)}] - 1 < 2\kappa [x^{(1)}] - 1 \Rightarrow \frac{d\alpha}{dx^{(1)}}[x^{(1)}] < 1.
\]
Suppose there is another Einstein ring \( \alpha \) at \( x^{(1)} \). But by the same token used in the sufficiency proof of the previous theorem, graph \( \alpha \in A_1 \) for small \( x^{(1)} \) and graph \( \alpha \in A_2 \) for \( x^{(1)} \rightarrow \infty \), we need \( d\alpha/dx^{(1)}[x^{(1)}] \geq 1 \) for the fixed point of \( \alpha \) to exist. This cannot be according to (21), and the result follows.

Since multiple Einstein rings cannot occur in a one-lens plane setting by Theorems 3.1 and 3.2, we now discuss two simple models in the two-lens plane case.

4 TWO-LENS PLANES

4.1 Singular isothermal spheres

4.1.1 Lens equation

The first system of two lenses discussed here consists of a singular isothermal sphere both in \( L^1 \) and \( L^2 \). This is hence a special case of the cored, spherically symmetric lenses in two-lens planes considered by Kochanek & Apostolakis (1988) in the context of a numerical study of lensing cross-sections. However, one can easily extend this model, which has two identical lenses, to include the effect of different surface densities. Because of the circular symmetry required for Einstein rings, we will assume the two singular isothermal spheres to be centred on the optical axis so that \( \xi_{i,1} = 0 \) again, and consider projected surface densities (Schneider et al. 1999, p. 243)
\[
\Sigma^{(i)}[\xi^{(i)}] = \frac{\sigma^{(i)}(\xi^{(i)})}{2G\xi^{(i)}}, \quad i \in \{1, 2\},
\]
where \( \xi^{(i)} = \|\xi^{(i)}\| \) and \( \sigma^{(2)i} \) is line-of-sight velocity dispersion of the isothermal sphere in the \( i \)th lens plane. Now, according to Poisson’s equation (6), the deflection potentials (3), (4) become
\[
\Psi^{(i)}[\xi^{(i)}] = M^{(i)}[\xi^{(i)}], \quad M^{(i)} = \frac{4\pi\sigma^{(2)i}}{c^2}, \quad i \in \{1, 2\},
\]
and hence the lens equation (14)
\[
y = x^{(i)} \left( 1 - \frac{m_{\pm}}{x^{(i)}} \right),
\]
in which \( m^{(1)} \), \( m^{(2)} \) from equations (12) and (13) have been combined to define a new mass parameter in this case,
\[
m_{\pm} = \xi_{0} \left( m^{(1)} \pm \frac{D_{2}}{D_{1}} m^{(2)} \right),
\]
where the positive sign is valid for \( x^{(i)} \geq \beta\xi_{0}m^{(i)} \) and the negative sign for \( x^{(i)} < \beta\xi_{0}m^{(i)} \).

4.1.2 Einstein rings

Now if \( y = 0, F[x^{(i)}] = 1 - m_{\pm}/x^{(i)} \), no radial critical circles exist. Definition 2.1 and (21) imply that the radii of Einstein rings are given by
\[
x^{(i)} = m_{\pm},
\]
subject to the two domains of the mass parameter (22). It turns out then that the point source in \( S \) lensed by both isothermal spheres in \( L^1 \) and \( L^2 \) always produces one Einstein ring of radius
\[
x^{(i)} = \frac{1}{D_{1}^{2/3}(\mu^{(i)} + m^{(2)})(\mu^{(2)} + m^{(1)})^{1/2}},
\]

and, in addition, provided \( M^{(1)}/M^{(2)} < D_2/D_1 \), another one of radius
\[
\lambda_{E_1} = 1 + \frac{M^{(1)}D_{1u} - M^{(2)}D_{2u}}{(\mu^{(1)} + \mu^{(2)})^{1/2}}.
\]

(25)

Furthermore, there is a third Einstein ring of the isothermal sphere in \( L^{(2)} \) lensed by the one in \( L^{(1)} \), assuming, of course, that at least the former is luminous. Because the lens in \( L^{(2)} \) is extended, this second Einstein ring is not a circle but has some radial width. We therefore define the radius of this third Einstein ring \( \lambda_{E_3} \) in \( L^{(1)} \) to be that of the circular image produced by the centre of the lens (the most luminous part) in \( L^{(2)} \). Its radius can be read off immediately from the previous results by letting \( M^{(2)} = 0 \) and replacing \( D_1, D_{1u}, D_2, D_{2u} \), respectively, so that there are two different definitions of \( \xi_0 \), one for the two rings due to the point source and one for the ring due to the isothermal sphere in \( L^{(2)} \). Hence, using (8) and (22),
\[
\lambda_{E_3} = \left( \frac{D_1D_{1u}}{D_2} M^{(1)} \right)^{1/2}.
\]

In view of observational applications, it is more convenient to express these Einstein ring radii in terms of angular coordinates in \( L^{(1)} \). Given the small angle approximation, one can take the angular radius to be \( \vartheta = \xi^{(1)}/D_1 = x^{(1)}\xi_0/D_1 \) using (7). Hence, for the three Einstein rings,
\[
\vartheta_{E_1} = \frac{M^{(1)}D_{1u} + M^{(2)}D_{2u}}{D_1},
\]
\[
\vartheta_{E_2} = \frac{M^{(1)}D_{1u} - M^{(2)}D_{2u}}{D_1} \text{ if } M^{(1)} < \frac{D_2}{D_1},
\]
\[
\vartheta_{E_3} = \frac{D_{1u} D_{2u}}{D_2} M^{(1)}.
\]

(26)

(27)

(28)

4.1.3 Image configuration

According to the lens equation (21), this system of two singular isothermal spheres at different distances is a modification of the well-known single-lens plane case in the sense that the two instances \( m_\pm \) have to be distinguished here. It turns out, then, that up to four images can be obtained in the present case. To be more precise, consider the image configurations if axisymmetry is broken such that \( y \neq 0 \). Without the loss of generality, one may take \( y_1 \equiv y > 0, y_2 = 0 \) so that \( y = (y, 0) \). Images are hence colinear with the centre of \( L^{(1)} \) and the source such that \( x^{(1)} = (x, 0) \), say. Therefore, (21) and (22) imply the following image positions \( x_i, i \in \{1, \ldots, 4\} \), for given domains of \( y \in S \),
\[
x_1 = y + m_+ > 0 \text{ for } y \geq -m_+ + \beta \xi_0 m^{(1)} = -\frac{\xi_0 D_2}{D_1} \frac{M^{(1)}D_1 + M^{(2)}D_2}{\mu^{(1)} + \mu^{(2)}},
\]
\[
x_2 = y + m_- > 0 \text{ for } y < -m_- + \beta \xi_0 m^{(1)} = -\frac{\xi_0 D_2}{D_1} \frac{M^{(1)}D_1 - M^{(2)}D_2}{\mu^{(1)} + \mu^{(2)}},
\]
\[
x_3 = y - m_+ < 0 \text{ for } y \geq -m_+ - \beta \xi_0 m^{(1)} = \frac{\xi_0 D_2}{D_1} \frac{M^{(1)}D_1 + M^{(2)}D_2}{\mu^{(1)} + \mu^{(2)}},
\]
\[
x_4 = y - m_- < 0 \text{ for } y > m_- - \beta \xi_0 m^{(1)} = \frac{\xi_0 D_2}{D_1} \frac{M^{(1)}D_1 - M^{(2)}D_2}{\mu^{(1)} + \mu^{(2)}}.
\]

(29)

(30)

(31)

(32)

(33)

The condition on \( y \) for (29) to hold is clearly always fulfilled, and the condition for (31) defines a cut circle in \( S \) within which another image occurs. Hence, the images given by \( x_1 \) and \( x_3 \) correspond to the single-lens plane case if \( \beta = 0 \) (see e.g. Schneider et al. 1999, p. 244). Similarly, the condition on \( y \) in (30) defines a second cut circle. But because we need to keep \( y \) positive by setup, this cut circle only exists if \( M^{(1)} < M^{(2)}D_2/D_1 \), the same condition as for the second Einstein ring (27) discussed in the previous section. Furthermore, if the source is outside of the second cut circle, then one image \( x_2 \) now occurs on the opposite (i.e. negative) side of \( L^{(1)} \). If the source is inside the second cut circle, another image \( x_2 \) appears on the same side of \( L^{(1)} \) as the source. Finally, if \( y \to 0 \), then all image positions approximate the Einstein rings given by (23), as expected. However, it is interesting that, unlike in the single-lens plane case, the cut circles and the Einstein rings do not coincide here.

Now assume that \( y \) is in a maximal domain in \( S \) such that all four possible images \( x_i \) (29–32) are present. Then, given that \( \Delta = 1 - m_+/x^{(1)} \) by (15), we can directly evaluate the sum of the signed magnifications (16) of the images to find

\[
\sum_{i=1}^{4} \mu(x_i) = 4.
\]

This is therefore a magnification invariant in the two-lens plane case.

4.2 Point lenses

4.2.1 Lens equation

We now turn to two-point lenses. A thorough study of the caustic structure for two-point lenses in general positions within three-dimensional space was done by Erdl & Schneider (1993). However, the aspect of multiple Einstein rings was not studied there explicitly, so we present results to this end below. Using a setup analogous to the one in the previous section, consider two-point masses \( M_1, M_2 \) on the optical axis in \( L^{(1)}, L^{(2)} \), respectively. In this case, then, the surface densities are given by (Schneider et al. 1999, p. 239)

\[
\Sigma^{(i)}(\xi^{(i)}) = M_i \delta^{(2i)}(\xi^{(i)}), \quad i \in \{1, 2\},
\]

where \( \delta^{(2i)} \) denotes the two-dimensional delta function of the \( i \)th lens plane. Hence, the corresponding deflection potentials

\[
\psi^{(i)}(\xi^{(i)}) = M_i \ln \frac{\xi^{(i)}}{\xi_0}, \quad M_i = \frac{4GM_i}{c^2}, \quad i \in \{1, 2\},
\]

follow from (6), and the lens equation (14) becomes

\[
y = x^{(1)} \left( 1 - \frac{m^{(1)}}{x^{(1)}} \right) = \frac{m^{(2)}}{x^{(1)} - \beta m^{(1)}},
\]

using (9), (12) and (13) (Erdl & Schneider 1993). Recall also that the range of the parameters used in (34) is \( 0 \leq \beta, m^{(1)}, m^{(2)} \leq 1 \) by definition.

4.2.2 Einstein rings

Again, according to Definition 2.1, Einstein rings are given by positive solutions of \( F(x^{(1)}) = 0 \) for \( y = 0 \), which is a quartic in \( x^{(1)} \).
by lens equation (34). Hence, letting \( (x^{(1)})^2 = r \), we seek solutions  
\[
F(r) = 1 - \frac{m^{(1)}}{r} - \frac{m^{(2)}}{r - \beta m^{(1)}} = 0, \quad r > 0,  
\]
and obtain the squares of Einstein ring radii  
\[
r_e = \frac{1}{2} \left( 1 + \beta m^{(1)} \pm \sqrt{(1 + \beta m^{(1)})^2 - 4\beta (m^{(1)})^2} \right) 
\]  
using (12) and (13). Note that  
\[
(1 + \beta m^{(1)})^2 > 4\beta (m^{(1)})^2 
\]
unless \( \beta = 1 \) or \( m^{(1)} = 1 \) and  
\[
1 + \beta m^{(1)} > \sqrt{(1 + \beta m^{(1)})^2 - 4\beta (m^{(1)})^2} 
\]  
unless \( \beta = 0 \) or \( m^{(1)} = 0 \).

Therefore, in the general case \( 0 < \beta < 1, \) \( 0 < m^{(1)} < 1 \), there are exactly two Einstein rings with radii \( r_{e1,2} \) from (35) due to a point source in \( S \) and the point lenses in \( L^{(1)} \) and \( L^{(2)} \). Otherwise, one ring becomes a critical point on the optical axis, and we recover the case of a single Einstein ring, as expected.

If, in addition, the point lens in \( L^{(2)} \) is also luminous, it produces a third Einstein ring at radius \( r_{e3} \) due to the point lens in \( L^{(1)} \). As before, this radius can be obtained simply by setting \( m^{(2)} = 0 \) and identifying \( D_s, D_b \), with \( D_s, D_b \), respectively. The corresponding angular radii are again given by \( \theta = \xi^{(1)}/D_1 = x^{(1)}\xi_0/D_1 \), and we find  
\[
\theta_{e1,2} = \sqrt{\frac{D_s}{D_1 D_2}} \left( p + \sqrt{p^2 - \beta (M^{(1)})^2} \right), 
\]
\[
\theta_{e3} = \sqrt{\frac{D_s}{D_1 D_2}} M^{(1)}, \quad \text{with mass parameter} 
\]
\[
p = \frac{1 + \beta}{2} M^{(1)} + \frac{1 - \beta}{2} M^{(2)} 
\]
using (8), (9), (12) and (13).

### 4.2.3 Image configuration

Now consider the case when \( y \neq 0 \). As before, we can choose without loss of generality \( y = (y, 0), y > 0 \). The images are collinear with the source so that \( x^{(1)} = (x, 0) \). Hence, solutions of the lens equation (34) are defined by a quintic in \( x \),  
\[
0 = x^5 - xy^4 - (1 + \beta m^{(1)}) x^3 + y \beta m^{(1)} x^2 + \beta (m^{(1)})^2 x. 
\]

We immediately have a solution on the optical axis, \( x_s = 0 \), but this image is infinitely demagnified, and therefore invisible, since its signed magnification is given by  
\[
\mu[x_s] = \frac{1}{\det[J_{xs}]} = 0 \quad \text{because} \quad \lim_{x \to x_s} F[(x, 0)] = -\infty 
\]
using (16). The \{remaining roots \( x_i, i \in \{1, \ldots, 4\} \}, \} of solutions of the quartic equation  
\[
0 = x^4 - xy^3 - (1 + \beta m^{(1)}) x^3 + y \beta m^{(1)} x^2 + \beta (m^{(1)})^2. 
\]

We note that this immediately implies that the roots satisfy  
\[
\sum_{i=1}^4 x_i = y. 
\]

Now Descartes’ rule of signs shows that (40) has two positive and two negative real roots, two positive real roots and a complex conjugate pair, two negative real roots and a complex conjugate pair, or two complex conjugate pairs of roots. So, in the first case there are four images, in the second and third case two images each and in the last case none. As usual, these cases can be distinguished by means of the cubic resolvent of (40). With the standard substitution \( z \equiv x - y/4 \), we can eliminate the cubic term in (40) to obtain a reduced quartic in \( z \),  
\[
0 = z^4 + P z^2 + Q z + R, \quad \text{where} 
\]
\[
P = -\frac{3y^2}{8} - (1 + \beta m^{(1)}), 
\]
\[
Q = -\frac{y^3}{8} - \frac{y}{2} (1 - \beta m^{(1)}), 
\]
\[
R = \frac{3y^4}{256} - \frac{y^2}{16} (1 - 3 \beta m^{(1)}) + \beta (m^{(1)})^2. 
\]

Hence, the resolvent of (40) is given by the cubic equation (Bronshtein & Semendyayev 1985, p. 121)  
\[
0 = z^3 + Az^2 + Bz + C, \quad \text{(41)} 
\]
where  
\[
A = 2P, \quad B = P^2 - 4R, \quad C = -Q^2. 
\]

Note then that \( A < 0 \) and  
\[
B = \frac{3y^4}{16} + y^2 + (1 + \beta m^{(1)})^2 - 4\beta (m^{(1)})^2 > 0 
\]
using (36). Applying Descartes’ rule of signs to (41) with \( z \) replaced by \( -z \) shows that the cubic resolvent has no negative real roots, and so the quartic (40) does not have two complex conjugate pairs. Thus, there are always images present. Furthermore, one can consider the discriminant \( D_{res} \) (Bronshtein & Semendyayev 1985, p. 120) of the cubic resolvent to find  
\[
D_{res} = \left( \frac{3B - A^2}{9} \right)^3 + \left( \frac{2A^3 - 9AB + 27C}{54} \right)^2 \leq 0. 
\]

Therefore, the cubic resolvent has three positive real roots and hence the quartic (40) has four real roots corresponding to four images which are, in general, distinct. In the limiting case \( D_{res} = 0 \), however, double roots occur and we recover the single-lens plane case with two images. This result is in agreement with Erdl & Schneider (1993) who have shown, using catastrophe theory, that two-point lenses can produce either four or six images. This has also been proven by Petters (1995) using the Morse theory. It is clear then that the caustic domain giving rise to six images does not occur in our axisymmetrical case. The limiting behaviour of these four images can be read off directly from the quartic equation (40). For \( y \to 0 \), there are two images with positive \( x \) and two images with negative \( x \) approaching the two Einstein circles. For \( y \to \infty \), the four images \( x_i \) satisfy  
\[
x_1 \to y, \quad x_2 \to \sqrt{\beta m^{(1)}}, \quad x_3 \to -\sqrt{\beta m^{(1)}}, \quad x_4 \to -m^{(1)}/y. 
\]

Using (16) in these cases, we find that \( x_2, x_3, x_4 \) become infinitely demagnified so that only \( x_1 \) with signed magnification \( \mu[x_1] \to 1 \) remains as \( y \to \infty \), as expected.
Furthermore, it turns out that the images obey an invariant of the signed magnification sum. To see this, we can regard the present case as an example of a more general argument (Werner 2007). First, note that all five possible roots of the lens equation (34) for \( y > 0 \) are real by the previous discussion such that all of \( S \setminus \emptyset \) is a maximal domain. Also, the deflection angle tends to zero as \( y \to \infty \). Then, the lens equation can be complexified such that images correspond to fixed points of a complex rational lensing map. This in turn induces a map on complex projective space which is holomorphic almost everywhere and, in particular, at the fixed points. Then, the holomorphic Lefschetz fixed point formula can be applied to yield
\[
\sum_{i=1}^{5} \mu[x_i] = 1. \tag{42}
\]
In fact, this statement is also true for the four visible images because \( x_5 \) is infinitely demagnified as noted above.

5 CONCLUSIONS

In this article, we have presented an analytical study of multiple Einstein rings in the weak deflection limit of strong lensing, and now summarize the three main results. First, it was proven generally that at the most one Einstein ring can occur for one-lens plane. A natural and weak assumption was used here, namely that the normalized surface density always be smaller than the average surface density.

Accordingly, we turned to models with lenses in two planes, and in which source, observer and both lenses are exactly aligned along the optical axis. As a second lens is introduced on the optic axis, the existing Einstein ring generally bifurcates. If the more distant of the two lenses is also luminous, then it produces a third Einstein ring. In the case of two singular isothermal sphere lenses, the angular radii of the Einstein rings are given by equations (26), (27) and (28). In the case of two-point mass lenses, three Einstein rings arise whose angular radii are given by (37), (38) and (39). These expressions are our second result, the main point being that a single source can, in fact, give rise to more than one Einstein ring.

We also discussed briefly the image configurations for these two models if axial symmetry is broken. In the case of the two singular isothermal lenses, up to two cut circles and four images can arise. In the case of the two-point lenses, we find that there are always four images. In both instances, the images turn out to possess lensing invariants – for example, the magnification invariants (33) and (42). This has not been established before for multiplane lensing and is our third result.

Finally, we make some critical remarks about possible extensions of this work. After noting the existence of multiple Einstein rings due to a single source, the next obvious question is to ask how many can there be. But counting Einstein rings is clearly more difficult than counting discrete images. This is because discrete images are non-degenerate stationary points of the time delay surface, so theorems of the Morse theory can be used, whereas Einstein rings are degenerate stationary curves and the Morse theory does not apply. Alternatively, the problem of counting Einstein rings can be set up as a one-dimensional fixed-point problem according to equation (17) within the framework of intersection theory in algebraic geometry and topology. However, there is a degeneracy problem here, too, because a non-transverse intersection occurs if the Einstein ring is degenerate in the sense that, in one-lens plane, \( \kappa = 1 \) at this radius (Schneider et al. 1999, p. 234). Given the large number of free parameters in the case of multiple-lens planes, it will be difficult to ensure the absence of degenerate Einstein rings.

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