DOUBLY STOCHASTIC QUADRATIC OPERATORS AND
BIRKHOFF’S PROBLEM

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Abstract. In the present we introduce a concept of doubly stochastic quadratic operator. We prove necessary and sufficient conditions for doubly stochasticity of operator. Besides, we prove that the set of all doubly stochastic operators forms convex polytope. Finally, we study analogue of Birkhoff’s theorem for the class of doubly stochastic operators.

Mathematics Subject Classification: 15A51, 15A63, 46T99, 46A55.

Key words: Quadratic stochastic operator, doubly stochastic operator, extremal point.

1. Introduction

Throughout the paper we will consider the simplex

\[ S^{m-1} = \{ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \forall i \}^{\frac{1}{m}} \sum_{i=1}^{m} x_i = 1 \} . \]

For any \( x = (x_1, x_2, \cdots, x_m) \in S^{m-1} \) due to [9], we define \( x_{\downarrow} = (x_{[1]}, x_{[2]}, \cdots, x_{[m]}) \), where \( x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[m]} \) nonincreasing rearrangement of \( x \). The point \( x_{\downarrow} \) is called rearrangement of \( x \) by nonincreasing. Recall that for two elements \( x, y \) taken from the simplex \( S^{m-1} \) we say that an element \( x \) majorized by \( y \) (or \( y \) majorates \( x \)), and write \( x \prec y \) (or \( y \succeq x \)) if the following hold

\[ \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \text{ for any } k = \frac{1}{m}, m-1. \]

Note that such a term and notation was introduced by Hardy, Littlewood and Polya in [7]. It is easy to see that for any \( x \in S^{m-1} \) we have

\[ \left( \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m} \right) \prec x \prec (1, 0, \cdots, 0). \]

A matrix \( P = (p_{ij})_{i,j=1,m} \) is called doubly stochastic (sometimes bistochastic), if

\[ p_{ij} \geq 0, \forall i, j = \frac{1}{m}, \sum_{i=1}^{m} p_{ij} = 1, \forall j = \frac{1}{m}, \sum_{j=1}^{m} p_{ij} = 1, \forall i = \frac{1}{m}. \]

It is known [9] that doubly stochasticity of a matrix \( P \) is equivalent to \( Px \prec x \) for all \( x \in S^{m-1} \). It is clear that the set of all doubly stochastic matrices is convex and compact. Therefore, it was a problem due to Birkhoff
concerning description of the set of all extremal points of such a set. A solution of that problem was given in [2], and it states that the extremal points consist of only permutations matrices.

The paper is devoted to the same problem mentioned above, but in a class of nonlinear operators. To state the problem let us recall some notions.

An operator $V: S^{m-1} \to S^{m-1}$ given by

$$ (Vx)_k = \sum_{i,j=1}^{m} p_{ij,k}x_i x_j. \quad (1) $$

is called quadratic stochastic operator (q.s.o.), here the coefficients $p_{ij,k}$ satisfy the following conditions

$$ p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1. $$

One can see that q.s.o. is well defined, i.e. it maps the simplex into itself. Note that such operators arise in many models of physics, biology and so on. A lot of papers were devoted to investigations of such operators (see, for example [1, 3, 8, 10, 11, 12]). We mention that in those papers authors studied a central problem in the theory of q.s.o., namely limit behavior of the trajectory of q.s.o.

We say that a q.s.o. (1) is called doubly stochastic (here we saved the same terminology as above) if

$$ Vx < x \quad (2) $$

for all $x \in S^{m-1}.1$

The main object of the paper is the set of all doubly stochastic quadratic operators (d.s.q.o.). The goal is to study necessary and sufficient conditions for doubly stochasticity of quadratic stochastic operators and extremal points of such the set of all doubly stochastic quadratic operators. To do it, first in sec. 2 we study some properties of doubly stochastic q.s.o. Then we give necessary conditions for doubly stochasticity. In section 3 we, somehow, describe the set of doubly doubly stochastic operators. Finally, in section 4 first of all we study a sufficient condition for q.s.o. to be doubly stochastic and then study extreme points of the set of such operators. Note that a part of the results was announced in [4],[14].

2. NECESSARY CONDITIONS FOR DOUBLY STOCHASTICITY OF OPERATORS

This section is devoted to some properties of the doubly stochastic q.s.o.

**Theorem 2.1.** Let $V: S^{m-1} \to S^{m-1}$ be a d.s.q.o.. Then the coefficients $p_{ij,k}$ satisfy the following conditions

\[
1^\text{Note that in mathematical economics such an operator is usually called the operator of prosperity.}
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\[ a) \sum_{i,j=1}^{m} p_{ij,k} = m, \quad \forall k = 1, m. \quad (3) \]

\[ b) \sum_{j=1}^{m} p_{ij,k} \geq \frac{1}{2}, \quad \forall i, k = 1, m. \quad (4) \]

\[ c) \sum_{i,j \in \alpha} p_{ij,k} \leq |\alpha|, \quad \forall \alpha \subset I, \quad k = 1, m. \quad (5) \]

here \( I = \{1, 2, \ldots, m\}, |\alpha| - \text{cardinality of } \alpha. \)

Proof. a) Let \( C = \left( \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m} \right). \) From the definition we have \( V(C) \prec C, \) on the other hand \( C \prec x, \forall x \in S^{m-1}. \) Therefore \( V(C)_1 = C_1 \) and since \( C_1 = C \) one gets \( V(C) = C \) which implies (3).

c) Let \( Vx < x. \) Then according to [Ma] there exists a doubly stochastic matrix \( P(x) = (p_{ij}(x)) \) (which depends on \( x \)) such that \( Vx = P(x)x. \) Put \( x^0(\alpha) = (x^0_1, x^0_2, \ldots, x^0_m), \) here

\[
\begin{cases}
  x^0_i = \frac{1}{|\alpha|}, & i \in \alpha \\
  x^0_i = 0, & i \notin \alpha,
\end{cases}
\]

where \( \alpha \) is an arbitrary subset of \( I. \)

It is evident that \( |\alpha| \leq m, \) therefore \( x^0 \in S^{m-1} \) and we have

\[
(Vx^0)_k = \sum_{i,j \in \alpha} p_{ij,k} \frac{1}{|\alpha|^2} = \sum_{i=1}^{m} p_{i,j,k}(x^0) \frac{1}{|\alpha|} \leq \sum_{i=1}^{m} p_{i,j,k}(x^0) \frac{1}{|\alpha|} = \frac{1}{|\alpha|}.
\]

Since \( \alpha \) is an arbitrary set, then we infer that

\[
\sum_{i,j \in \alpha} p_{ij,k} \leq |\alpha|, \quad \forall \alpha \subset I, \quad \forall k = 1, m.
\]

Using c) we are going to prove b).

b) We will prove \( \sum_{j=1}^{m} p_{i_0,j,k} \geq \frac{1}{2} \) for some fixed \( i_0. \)

From the equality (see (3))

\[
m = \sum_{i,j=1}^{m} p_{ij,k} = \sum_{i,j=1, \ i, j \neq i_0}^{m} p_{ij,k} + p_{i_0,i_0,k} + 2 \sum_{j=1, \ j \neq i_0}^{m} p_{i_0,j,k}
\]

and

\[
\sum_{i,j=1, \ i, j \neq i_0}^{m} p_{ij,k} \leq m - 1
\]

we obtain

\[
p_{i_0,i_0,k} + 2 \sum_{j=1, \ j \neq i_0}^{m} p_{i_0,j,k} \geq 1
\]

or
Therefore, $b)$ is satisfied. \hfill \Box

3. **Description of the class of doubly stochastic operators**

In this section we are going to describe a class of doubly stochastic operators.

Recall that a matrix $T = (t_{ij})$, $i, j = 0, 1, \ldots, m$ is said to be **stochastic** if $t_{ij} \geq 0$ and $\sum_{j=1}^{m} t_{ij} = 1$.

Let $A = (a_{ij})$, $i, j = 0, 1, \ldots, m$ be a symmetric matrix, with $a_{ij} \geq 0$.

Consider the following equation with respect to $T$:

$$A = \frac{1}{2} (T + T'),$$

(6)

here $T'$ is the transposed matrix.

Below we are going to study conditions for solvability of equation (6) in the class of all stochastic matrices.

Let $G_m$ be the group of permutations of $m$ elements. For $g \in G_m$ by $A_g$ we denote a matrix $A_g = (a_{g(i)g(j)})$, $i, j = 0, 1, \ldots, m$, which is called row and column permutation of $A$.

The following assertions are evident:

i) $$(A_{g^{-1}})_{g} = A$$

for any $g \in G_m$

ii) If $A$ symmetric, then $A_g$ is also symmetric for any $g \in G_m$

iii) If $A$ stochastic, then $A_g$ is also stochastic for any $g \in G_m$

iv) $$(A + B)_g = A_g + B_g; \quad (\lambda A)_g = \lambda A_g.$$  

From i) -- iv) we conclude that if $T$ is a solution of (6), then $T_g$ is a solution of the equation

$$A_g = \frac{1}{2} (T + T').$$

At first we will study the following set

$$U_1 = \{ A = (a_{ij}) : a_{ij} = a_{ji} \geq 0, \sum_{i,j \in \alpha} a_{ij} \leq |\alpha|, \sum_{i,j \in I} a_{ij} = m \}.$$  

It is easy to see that the above set is convex and compact. Below we will study its extremal points. Let us shortly recall some necessary notations. Let $A \subset X$ be convex set and $X$ be some vector space. A point $x \in A$ is called extremal, if from $2x = x_1 + x_2$, where $x_1, x_2 \in A$ and $x_1 \neq x_2$, it follows that $x = x_1 = x_2$. The set of all extremal points of a given set $A$ is denoted
The Krein-Milman theorem asserts that any convex compact set on some topological vector space is a completion of the convex hull of its extremal points (see, for review [?]).

**Theorem 3.1.** If $A = (a_{ij}) \in \text{extr} \, U_1$ then $a_{ii} = 0 \lor 1, \ a_{ij} = 0 \lor \frac{1}{2} \lor 1$

Here and henceforth $c = a \lor b$ means that $c$ is either $a$ or $b$.

**Proof.** We prove this by induction with respect to the order of the set $U_1$. Let $m = 2$.

First we prove that $a_{11} = 0 \lor 1$ and $a_{22} = 0 \lor 1$. Let $0 < a_{11} < 1$, then from $a_{11} + 2a_{12} + a_{22} = 2$ we get $0 < a_{12} < 1$. Let us consider the following matrices

$$A_1 = \begin{pmatrix} a_{11} + 2\varepsilon & a_{12} - \varepsilon \\ a_{12} - \varepsilon & a_{22} \end{pmatrix} \quad \text{and}$$

$$A_2 = \begin{pmatrix} a_{11} - 2\varepsilon & a_{12} + \varepsilon \\ a_{12} + \varepsilon & a_{22} \end{pmatrix}$$

Since $0 < a_{11} < 1$ and $0 < a_{12} < 1$, then one can choose $\varepsilon$ such that $A_1, A_2 \in U_1$. Now we obtain that $2A = A_1 + A_2$. Therefore $A \notin \text{extr} \, U_1$.

The last contradiction shows that $a_{11} = 0 \lor 1$. By this analogy one can prove that $a_{22} = 0 \lor 1$. Since $a_{11} = 0 \lor 1$ and $a_{22} = 0 \lor 1$ it follows from $A \in U_1$ that $a_{ij} = 0 \lor \frac{1}{2} \lor 1$. One can easily see that the number of such matrices is 4. That is the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

And all of them are extremal. Thus, for $A \in \text{extr} \, U_1$ it is necessary and sufficient to be $a_{ii} = 0 \lor 1, \ a_{ij} = 0 \lor \frac{1}{2} \lor 1$. However for $m \geq 3$ it is not true at all.

Let us suppose that the conditions of the theorem is valid for all matrices of order less than $m$ and prove it for matrix of order $m$.

First we prove that if $A \in \text{extr} \, U_1$ then $a_{ii} = 0 \lor 1$. Let us assume that it is not so, that is there exists some diagonal entry, which neither 0 nor 1. Without loss of generality we may assume that $0 < a_{11} < 1$.

We call a set $\alpha \subseteq I$ is said to be saturated, if $\sum_{i,j \in \alpha} a_{ij} = |\alpha|$. Corresponding minor to the saturated set we call saturated minor. For instance, $I$ itself is saturated, but it is a proper set. Further, we mean only non proper saturated sets.

If there is a saturated set, such that $1 \in \alpha$ then by the assumption of the induction we infer that $a_{11} = 0 \lor 1$. Now remains the case when $1 \notin \alpha$ for any saturated set $\alpha$.

Now we will prove that if $0 < a_{11} < 1$ then $a_{jj} = 0 \lor 1$ for all $j$. Indeed, if there is $j_0$ such that $0 < a_{j_0j_0} < 1$, then we put

$$A' = \{ a'_{11} = a_{11} + \varepsilon, \ a'_{j_0j_0} = a_{j_0j_0} - \varepsilon, \ a'_{ij} = a_{ij} \ \text{for all other values of} \ i,j \}$$
and
\[ A'' = \{ a''_{11} = a_{11} - \varepsilon, \ a''_{i,j_{0}} = a_{i,j_{0}} + \varepsilon, \ a''_{i,j} = a_{i,j} \text{ for all other values of } i, j \} \]

If \( j_0 \) is contained in some saturated set, then from the assumption of the induction it directly follows that \( a_{i,j_{0}} = 0 \lor 1 \). If it is not so, then \( A', A'' \in U_1 \) and consequently \( 2A = A' + A'' \), that is \( A \notin U_1 \), which is a contradiction.

So if \( 0 < a_{11} < 1 \), then \( a_{jj} = 0 \lor 1 \) for all \( j \). Furthermore, since \( I \) is a saturated set, then there is \( a_{i_0,j_0} \neq 0 \lor \frac{1}{2} \lor 1(i_0 \neq j_0) \). Now we put

\[ A' = \{ a'_{11} = a_{11} + 2\varepsilon, \ a'_{i,j_{0}} = a_{i,j_{0}} - \varepsilon, \ a'_{i,j} = a_{i,j} \text{ for all other values of } i, j \} \]

and

\[ A'' = \{ a''_{11} = a_{11} - 2\varepsilon, \ a''_{i,j_{0}} = a_{i,j_{0}} + \varepsilon, \ a''_{i,j} = a_{i,j} \text{ for all other values of } i, j \} \]

If \( \{i_0, j_0\} \) is not contained in any saturated set, then \( A', A'' \in U_1 \) and \( 2A = A' + A'' \).

If \( \{i_0, j_0\} \) is contained in some saturated set. Then by the assumption of the induction we get \( a_{i,j_{0}} 0 = 0 \lor \frac{1}{2} \lor 1(i_0 \neq j_0) \), which contradicts to \( a_{i,j_{0}} \neq 0 \lor \frac{1}{2} \lor 1(i_0 \neq j_0) \).

Thus, all cases come to a contradiction. Therefore, \( a_{11} = 0 \lor 1 \). Now we have to prove that if \( A \in extr U_1 \) then \( a_{i,j} = 0 \lor \frac{1}{2} \lor 1(i \neq j) \).

Let \( a_{i,j_{0}} \neq 0 \lor \frac{1}{2} \lor 1(i_0 \neq j_0) \). If there is a saturated set \( \alpha \) such that \( \{i_0, j_0\} \subset \alpha \) then by the assumption it follows that \( a_{i,j} = 0 \lor \frac{1}{2} \lor 1(i \neq j) \). Since \( I \) is saturated, one can choose \( \{i_1, j_1\} \) such that \( a_{i_1,j_1} \neq 0 \lor \frac{1}{2} \lor 1(i_0 \neq j_0) \). Also it follows that \( \{i_1, j_1\} \) is not contained in any saturated set.

Let us put

\[ A' = \{ a'_{i,j_{0}} = a_{i,j_{0}} + \varepsilon, \ a'_{i_1,j_1} = a_{i_1,j_1} - \varepsilon, \ a'_{i,j} = a_{i,j} \text{ for all other values of } i, j \} \]

and

\[ A'' = \{ a''_{i,j_{0}} = a_{i,j_{0}} - \varepsilon, \ a''_{i_1,j_1} = a_{i_1,j_1} + \varepsilon, \ a''_{i,j} = a_{i,j} \text{ for all other values of } i, j \} \]

The sets \( \{i_0, j_0\} \) and \( \{i_1, j_1\} \) is not contained in any saturated set. Therefore \( A', A'' \in U_1 \) and \( 2A = A' + A'' \).

**Corollary 3.2.** If \( m = 3 \), then \( A \in extr U_1 \) if and only if \( a_{ii} = 0 \lor 1 \), \( a_{ij} = 0 \lor \frac{1}{2} \lor 1 \) and \( A \neq M \). Here \( M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \)

Moreover, \( |extr U_1| = 25 \)

**Corollary 3.3.** If \( A \in extr U_1 \) then either \( A \) is stochastic or has a saturated minor of order \( m - 1 \).
Proof. Let \( A \in \text{extr} \mathbf{U}_1 \). Let us assume that the matrix \( A \) has no saturated minors of order \( m - 1 \). Then we have \( \sum_{i,j \in \alpha} a_{ij} < |\alpha| \) for any \( \alpha \) such that \( |\alpha| = m - 1 \). By taking into account \( \sum_{i,j=1}^{m} a_{ij} = m \), we obtain
\[
a_{ii} + \sum_{j \neq i} a_{ij} > 1.
\]

By theorem it follows that \( a_{ii} = 0 \lor 1, \quad a_{ij} = 0 \lor \frac{1}{2} \lor 1 \). Therefore an expression \( a_{ii} + \sum_{j \neq i} a_{ij} \) is an integer. Therefore
\[
a_{ii} + \sum_{j \neq i} a_{ij} \geq 2.
\]

By taking summation in the above inequality from \( i = 1 \) to \( i = m \) we find
\[
\sum_{i=1}^{m} \left( a_{ii} + \sum_{j \neq i} a_{ij} \right) = 2 \sum_{i,j=1}^{m} a_{ij} - \sum_{i=1}^{m} a_{ii} \geq 2m.
\]

Hence
\[
\sum_{i=1}^{m} a_{ii} \leq 0.
\]

and \( a_{ii} = 0 \) for all \( i = 1, \ldots, m \). Further, \( a_{ii} + \sum_{j \neq i} a_{ij} \geq 2 \) and \( a_{ii} = 0 \) imply
\[
\sum_{j=1}^{m} a_{ij} \geq 1 \quad \text{for all } i = 1, \ldots, m.
\]

Since \( \sum_{i,j=1}^{m} a_{ij} = m \) then \( \sum_{j=1}^{m} a_{ij} = 1 \). The last means that \( A \) is stochastic.

\( \square \)

**Theorem 3.4.** Let \( A = (a_{ij}) \) be a symmetric and nonnegative matrix. For the existence of a stochastic matrix \( T = (t_{ij}) \) satisfying the equation (6) necessary and sufficient condition is
\[
\sum_{i,j \in \alpha} a_{ij} \leq |\alpha|
\]
for all \( \alpha \subset I = \{1, 2, \ldots, m\} \).

**Proof.** Necessity. If \( T = (t_{ij}) \) is a stochastic matrix and (6) is satisfied, then
\[
\sum_{i,j \in \alpha} a_{ij} = \frac{1}{2} (\sum_{i,j \in \alpha} t_{ij} + \sum_{i,j \in \alpha} t_{ji}) = \sum_{i,j \in \alpha} t_{ij} = \sum_{i \in \alpha} \sum_{j \in \alpha} t_{ij} \leq \sum_{i,j \in \alpha} 1 = |\alpha|
\]

Sufficiency. Let \( A \in \mathbf{U}_1 \). We have to show that existence of a stochastic matrix for which (6) is satisfied.

First, we prove it for extremal points of \( \mathbf{U}_1 \). Now let \( A \in \text{extr} \mathbf{U}_1 \). Then from corollary 3.3 it follows that either \( A \) is stochastic or has a saturated
minor of order \( m - 1 \). If \( A \) is stochastic, then we take \( T = A \) since \( A \) is symmetric then (6) is satisfied.

Now, let us suppose that \( A \) is not stochastic, then it follows that it has a saturated minor order \( m - 1 \). This case we prove by induction relatively order of the matrix. Let \( m = 2 \). In this case we have the following matrices

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \ A_4 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.
\]

For \( A_1 \) or \( A_2 \) we take \( T = A_1 \) and \( T = A_2 \), respectively.

For \( A_3 \) or \( A_4 \) we take \( T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \) or \( T = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), respectively.

Let us assume that assumption of the theorem is true for all matrices of order less that \( m \).

Since \( A \) has a saturated set of order \( m - 1 \), without loss of generality we may assume that this minor is \((a_{ij})_{i,j=1,m-1}\). From the assumption of the induction there is a stochastic matrix for this minor which satisfies (6). Let it be \( V = (v_{ij})_{i,j=1,m-1} \).

Since \( \sum_{i,j=1}^{m-1} a_{ij} = m - 1 \) and \( \sum_{i,j=1}^{m} a_{ij} = m \) it follows that \( a_{mm} + 2 \sum_{i=1}^{m-1} a_{im} = 1 \). From theorem it follows either \( a_{mm} = 1 \) and \( a_{im} = 0 \) for \( i = 1, m-1 \) or \( a_{ii} = 0, a_{iom} = \frac{1}{2} \) \( a_{im} = 0 \) for \( i = 1, m-1 \setminus \{i_0\} \).

For the first case we define \( T \) by

\[
T\{(t_{ij}) : t_{ij} = v_{ij}, \ i, j = 1, m-1 \ t_{mm} = 1, \ t_{im} = t_{mi} = 0, \ i = 1, m-1 \setminus \{i_0\}\}
\]

and it is easy to see that above matrix is stochastic and satisfies (6).

For the second case we define \( T \) by

\[
T\{(t_{ij}) : t_{ij} = v_{ij}, \ i, j = 1, m-1 \ t_{mi} = 1, \ t_{im} = t_{mi} = 0, \ i = 1, m \}\}
\]

and it is easy to see that above matrix is stochastic and satisfies (6). So, for extremal points of \( \mathbf{U}_1 \) the theorem has been proved.

Now let \( A \in \mathbf{U}_1 \) From Krein-Milman(see [?]) it follows that \( A \) can be produced as the convex hull of its extremal points.

Let

\[
A = \sum_{i=1}^{k} \lambda_i A_i.
\]

Here \( 0 \leq \lambda \leq 1, \ \sum_{i=1}^{k} \lambda_i = 1 \) and \( A_i \in extr \mathbf{U}_1, \ i = 1, \ldots, k \).

Let for matrices \( A_i \) correspond stochastic matrices \( T_i \) that satisfy (6). Then

\[
T = \sum_{i=1}^{k} \lambda_i T_i
\]

is stochastic and satisfies (6).
Remark. It should be mentioned that stochastic matrix $T$ (which is a solution of (6)) exists but not unique.

Example. For the matrix

$$A = \begin{pmatrix}
0 & 1 & 0.3 & 0.4 \\
0.3 & 0.1 & 0.5 \\
0.4 & 0.5 & 0.4
\end{pmatrix}$$

the set of the solutions of $A = \frac{1}{2}(T + T')$ in the class of stochastic matrices is

$$T_\alpha = \begin{pmatrix} 0,1 & \alpha & 0.9 - \alpha \\
0,6 - \alpha & 0,1 & 0,3 + \alpha \\
\alpha - 0,1 & 0,7 - \alpha & 0,4
\end{pmatrix}$$

here $\alpha \in [0,1; 0,6]$.

Recall that a set $A$ in $R^k$ is said to be a polytope, if it is nonempty, bounded and consists of intersection of a finite number of semispaces.

Proposition 3.5. The set of all solutions of the equation

$$A = \frac{1}{2}(T + T')$$

with respect to $T$ forms a polytope.

Proof. Let us rewrite the solutions of the equation in the following form:

$$0 \leq t_{ij} \leq 1,$$

$$t_{ij} + t_{ji} \leq a_{ij}, \quad t_{ij} + t_{ji} \geq a_{ij}$$

$$\sum_{j=1}^{m} t_{ij} \leq 1, \quad \sum_{j=1}^{m} t_{ij} \geq 1.$$  

From above given relations one can see that the set of all solutions of the equation is nonempty, bounded and it is intersection of semispaces.

From (6) it follows that $(Ax, x) = (Tx, x)$, here $(\cdot, \cdot)$ is the inner product in $R^m$. Indeed

$$(Ax, x) = \frac{1}{2}((T + T')x, x) = \frac{1}{2}[(Tx, x) + (T'x, x)] = (Tx, x). \quad (7)$$

Now let’s return to the study of d.s.q.o. According to the theorem 2.1., if $V : S^{m-1} \to S^{m-1}$ – d.s.q.o., then

$$\sum_{i,j \in \alpha} p_{ij,k} \leq |\alpha|, \quad \forall \alpha \in I, \quad k = 1,m$$

moreover, for $\alpha = I$ an equality is held.
By putting
\[ A_k = (p_{ij,k})_{i,j=1,m}, \]
and using the theorem one can find a stochastic matrix \( T_k \) such that
\[ A_k = \frac{1}{2}(T_k + T'_k), \quad k = 1, m. \]

From (7) one gets
\[ \sum_{i,j=1}^m p_{ij,k} x_i x_j = (A_k x, x) = (T_k x, x). \]
Hence, if \( V \) is a d.s.q.o., then there are stochastic matrices \( T_1, T_2, \ldots, T_m \) such that
\[ V x = ((T_1 x, x), (T_2 x, x), \ldots, (T_m x, x)). \]

Now we prove the following inequality:
\[ \min_{1 \leq i \leq m} x_i \leq (T x, x) \leq \max_{1 \leq i \leq m} x_i, \quad \forall x \in S^{m-1}, \]
or in accepted notations:
\[ x[m] \leq (T x, x) \leq x[1], \quad \forall x \in S^{m-1}, \quad (8) \]
here \( T- \) is a stochastic matrix.

Indeed, if \( t_{ij} \geq 0 \) and \( \sum_{j=1}^m t_{ij} = 1 \), then
\[ x[m] \leq \sum_{j=1}^m t_{ij} x_j \leq x[1] \]
for all \( x \in R^m \). In particular, for \( x \in S^{m-1} \) we get
\[ (T x, x) = \sum_{i,j=1}^m t_{ij} x_i x_j = \sum_{i=1}^m x_i \sum_{j=1}^m t_{ij} x_j. \]

By \( x_i \geq 0, \sum_{i=1}^m x_i = 1 \) and above inequality we get the required inequality (8).

**Theorem 3.6.** Let \( A = (a_{ij}) \) be a nonnegative symmetric matrix. Then for the fulfillment of the inequality
\[ x[m] \leq (A x, x) \leq x[1], \quad \forall x \in S^{m-1}, \]
it is necessary and sufficient that
\[ \sum_{i,j \in \alpha} a_{ij} \leq |\alpha| \quad \forall \alpha \in I, \quad k = 1, m \]
moreover, for \( \alpha = I \) an equality is held.
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Proof. Sufficiency straightforwardly follows from Theorem 3.4 and the inequality (8).

Necessity. Consider a set \( \alpha \subset I \) and put \( x^0 = (x_1^0, x_2^0, \ldots, x_m^0) \), with
\[
x_i^0 = \frac{1}{|\alpha|}, \quad i \in \alpha, \quad x_i^0 = 0, \quad i \notin \alpha.
\]
Then for \( |\alpha| < m \) we get \( x_{[m]} = 0, \ x_{[1]} = \frac{1}{|\alpha|} \). Therefore, one has
\[
0 \leq (Ax, x) = \sum_{i,j=1}^{m} a_{ij} x_i x_j = \frac{1}{|\alpha|^2} \sum_{i,j \in \alpha} a_{ij} \leq \frac{1}{|\alpha|}
\]
hence
\[
\sum_{i,j \in \alpha} a_{ij} \leq |\alpha|
\]
If \( |\alpha| = m \), then \( x_{[m]} = x_{[1]} = \frac{1}{m} \). Consequently
\[
\frac{1}{m} \leq \frac{1}{m^2} \sum_{i,j \in \alpha} a_{ij} \leq \frac{1}{m},
\]
that is \( \sum_{i,j=1}^{m} a_{ij} = m \). \( \square \)

By Theorems 3.4 and 3.5 one can deduce that an inequality
\[
x_{[m]} \leq (Ax, x) \leq x_{[1]} \quad \forall x \in S^{m-1}
\]
here \( A = (a_{ij}) \) is a nonnegative matrix, is equivalent to the existence of stochastic matrix \( T = (t_{ij}) \), such that
\[
A + A^T = T + T^T.
\]

Since we are going to study sufficient conditions for doubly stochasticity of q.s.o. from the proved theorem one arises the following

Problem 3.7. For which symmetric matrices \( A \) the following inequality holds
\[
x_{[m]} + x_{[m-1]} + \cdots + x_{[m-k+1]} \leq (Ax, x) \leq x_{[1]} + \cdots + x_{[k]},
\]
for all \( x \in S^{m-1} \), here \( k \) – natural number \( 1 \leq k \leq m \)?

For \( k = 1 \) an answer is given by theorem 3.3. To solve the problem we give some notations:

\[
T_k = \{ T = (t_{ij}), \ i, j = 1, m : 0 \leq t_{ij} \leq 1, \ \sum_{j=1}^{m} t_{ij} = k \}, \quad 1 \leq k \leq m.
\]

\[
U_k = \{ A = (a_{ij}) : a_{ij} = a_{ji}, \ A = \frac{1}{2}(T + T^T), \ T \in T_k \}, \quad 1 \leq k \leq m.
\]
The set $U_k$ is called a symmetrization of $T_k$. Evidently, $T_1$— the set of all stochastic matrices.

**Theorem 3.8.** For $A \in U_k$ the inequality (10) holds.

**Proof.** Let $x = (x_1, \ldots, x_m)$ be a nonincreasing rearrangement of $x$ and $\lambda_1, \ldots, \lambda_m$ be an arbitrary numbers such that $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{m} \lambda_i = k$.

Consider the following sum
$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m.$$ (11)

Let, $i < j$. Replacement of the coefficients $\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_m$ into $\lambda_1, \ldots, \lambda_i + \varepsilon, \ldots, \lambda_j - \varepsilon, \ldots, \lambda_m$ we will call backward shift, if $\varepsilon > 0$ and forward shift if $\varepsilon < 0$. Shift is called admissible, if it preserves the condition $0 \leq \lambda'_i$, here $\lambda'_i$ are the coefficients, obtained in the result of shift.

Since $x_i \geq x_j$, then it is clear that under the admissible backward shifts, the sum (11) does not decrease, and respectively, does not increase under the admissible forward shifts.

It is easy to see that an admissible backward shift is possible up to obtaining the collection $1, 1, \ldots, 1, 0, 0, \ldots, 0$, and to the right up to obtaining the collection $0, 0, \ldots, 0, 1, 1, \ldots, 1$, here in both cases the number of ones is $k$. That’s why

$$x_{[m-k+1]} + x_{[m-k+2]} + \cdots + x_{[m]} \leq \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m \leq x_1 + \cdots + x_k.$$ (12)

for an arbitrary $x \in R^m$ and $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^{m} \lambda_i = k$.

Now let $x \in S^{m-1}$ and $A \in U_k$.

Choose $T \in T_k$, such that
$$A = \frac{1}{2} (T + T').$$

Then
$$(Tx, x) = \sum_{i,j=1}^{m} t_{ij} x_i x_j = \sum_{i=1}^{m} x_i (\sum_{j=1}^{m} t_{ij} x_j)$$

Since $0 \leq t_{ij} \leq 1$ and $\sum_{j=1}^{m} t_{ij} = k$, from (12) one gets the following
$$\sum_{i=m-k+1}^{m} x_i \leq \sum_{j=1}^{m} t_{ij} \leq \sum_{i=1}^{k} x_i$$

According to $x_i \geq 0$, $\sum_{i=1}^{m} x_i = 1$ we obtain
$$\sum_{i=m-k+1}^{m} x_i \leq (Tx, x) \leq \sum_{i=1}^{k} x_i$$
or

\[ \sum_{i=m-k+1}^{m} x[i] \leq (Ax, x) \leq \sum_{i=1}^{k} x[i] \]

So if \( A \in U_k \) then (10) holds. \( \square \)

Let \( B \) be the set of all d.s.q.o. By putting \( A_k = \{p_{ij,k}\}_{(i,j=1,m)} \) we rewrite an operator \( V \) in the following form

\[ V = (A_1|A_2|\cdots|A_m). \] (13)

Then theorems 2.1, 3.4, 3.5 and 3.6 imply that for \( V \in B \) the conditions

\[ A_k \in U_1, \ k = \overline{1,m} \] (14)

are necessary and the conditions

\[ \forall \alpha \subset I, \sum_{k \in \alpha} A_k \in U_{|\alpha|} \] (15)

are sufficient.

Whether the conditions (14) is sufficient to be \( V \in B \)? We give a positive answer for the question in next section.

Now we show some simple properties of the \( U_k \) which will be helpful. Let \( E \) be the matrix of \( m \times m \) with all entries equal to unit.

**Theorem 3.9.** The following assertions hold

\[ i) A \in U_k \iff E - A \in U_{m-k}; \]

\[ ii) U_k \cap U_l = \emptyset, \ k \neq l; \]

\[ iii) U_m = \{E\}; \]

\[ iv) A \in U_k \Rightarrow \frac{p}{k}A \in U_p, 1 \leq p \leq k; \]

\[ v) U_k + U_l \supset U_{k+l}, \ k+l \leq m. \]

**Proof.**

\[ i) \] Let \( 1 \leq k \leq m-1 \) and \( A \in U_k \). From

\[ A = \frac{1}{2}(T + T') \]

here \( T = (t_{ij}), 0 \leq t_{ij} \leq 1, \sum_{j=1}^{m} t_{ij} = k \) one gets

\[ E - A = \frac{1}{2}[(E - T) + (E - T)'] \]

It is obvious that \( E - T \in T_{m-k} \), therefore \( E - A \in U_{m-k}. \)

\[ ii) \] It follows from the definition of \( U_k \).

\[ iii) \] Since \( A \in U_k \) then \( 0 \leq a_{ij} \leq 1 \) for all \( k = \overline{1,m} \). If \( A \in U_m \) then

\[ \sum_{i,j=1}^{m} a_{ij} = m^2 \] and therefore \( a_{ij} = 1 \) that is \( U_m = \{E\} \)
iv) Let $A \in U_k$ and $A = \frac{1}{2}(T + T')$. Then
$$p_k A = \frac{1}{2} [p_k T + p_k T']$$
moreover $\frac{p_k}{2} T \in T_p$. Consequently, $\frac{p_k}{2} A \in U_p$.

v) Let $k + l \leq m$, $A \in U_{k+l}$ and $A = \frac{1}{2}(T + T')$ By denoting
$$A_1 = \frac{k}{2(k+l)}(T + T'), \quad A_2 = \frac{l}{2(k+l)}(T + T')$$
we obtain $A = A_1 + A_2$, furthermore $A_1 \in U_k, A_2 \in U_l$. The last implies
$$U_k \cup U_l \supset U_{k+l}, \quad k + l \leq m.$$ 
It should be mentioned that the last inclusion is strict for all $k, l \geq 1$ Indeed, let $I$ be the identity matrix. Clearly that $I \in U_1$, but $I + I / \in U_2$. \qed

4. Extreme point of the set of doubly stochastic operators and Birkhoff’s problem

According to the classic result of Birkhoff (see [2]) extreme points of the set of doubly stochastic matrices are permutation matrices, i.e. matrices having exactly one unit entry in each row and exactly one unit entry in each column, all other entries being equal to zero. It is interesting to know an answer for the similar problem about the set of all doubly stochastic nonlinear operators. Therefore, to investigate such a problem it is better first to start with the set of all d.s.q.o.’s, since such a set contains as a subset of the set of all doubly stochastic matrices. In this section we are going to describe extreme points of the set of d.s.q.o. At first, we prove necessary and sufficient conditions for q.s.o. to be d.s.q.o.(it was proved in the last section but in this section we show that necessary and sufficient will coincide).

The set of all d.s.q.o. we denoted by $B$. Directly from definition it follows that
$$V = (A_1 | A_2 | \cdots | A_m) \in B \iff V = (A_{\pi(1)} | A_{\pi(2)} | \cdots | A_{\pi(m)}) \in B$$
here $\pi$ is an arbitrary permutation of the index set $I = \{1, 2, \cdots, m\}$

**Theorem 4.1.** Assume that $A_1 \in U_1$. Then one can choose $A_2, \cdots, A_m \in U_1$ such that $V = (A_1 | A_2 | \cdots | A_m) \in B$.

**Proof.** Take
$$A_2 = A_3 = \cdots = A_m = \frac{1}{m-1}(E - A_1)$$
here $E$ is the matrix with all entries equal to unit. According to the theorem 3.8 we have
$$E - A_1 \in U_{m-1}$$
and

\[
\frac{1}{m-1}(E - A_1) \in U_1
\]

Let a stochastic matrix \(T_1 \in T_1\) be a solution of the equation

\[
A_1 = \frac{1}{2}(T + T')
\]

Now we put

\[
T_2 = T_3 = \cdots = T_m = \frac{1}{m-1}(E - T_1)
\]

we show that the sum of any \(k\) of the matrices \(T_1, T_2, \ldots, T_m\) belongs to \(T_k\). Indeed, let \(T_1 = \{t_{ij}\}\) such that \(0 \leq t_{ij} \leq 1\). Then

\[
T_2 = T_3 = \cdots = T_m = \frac{1}{m-1}\{1 - t_{ij}\}
\]

From \(0 \leq t_{ij} \leq 1\) one gets

\[
0 \leq \frac{k}{m-1}(1 - t_{ij}) \leq 1
\]

and

\[
0 \leq t_{ij} + \frac{k - 1}{m-1}(1 - t_{ij}) \leq 1.
\]

If the sum contains \(T_1\) then our assertion follows from the inequality (17), otherwise from the inequality (16). So the sum of any \(k\) matrices of \(T_1, \ldots, T_m\) belongs to \(T_k\).

Now we return to \(A_1, \ldots, A_m\) and obtain that the sum of any of them belongs to \(U_k\). Therefore we get that

\[
V = (A_1|A_2|\cdots|A_m) \in B
\]

It is clear that if \(T_1, \ldots, T_p \in T_1(p \leq m)\) and \(\sum_{i=1}^{p} T_i \in T_p\) then for any \(k \leq p\) the sum of any \(k\) matrices of \(T_1, \ldots, T_p\) belongs to \(T_k\). This implies that from \(A_1, \ldots, A_p \in U_1, \sum_{i=1}^{p} A_i \in U_p(p < m)\) it follows that the sum of any \(k(k \leq p)\) matrices belongs to \(U_k\). Using the last assertion we get the following

**Corollary 4.2.** If \(A_1, \ldots, A_p \in U_1, \sum_{i=1}^{p} A_i \in U_p(p < m)\) then one can choose \(A_{p+1}, \ldots A_m \in U_1\) such that

\[
V = (A_1|A_2|\cdots|A_m) \in B
\]

So conditions (15) can be changed to conditions:

\[
A_i \in U_1, \quad i = 1, m, \quad \sum_{i=1}^{m} A_i = E
\]

since \(U_m\) is \(E\). Therefore
Corollary 4.3. The conditions (18) are necessary and sufficient for q.s.o. to be d.s.q.o.

Corollary 4.4. The set of all d.s.q.o. forms a convex polytope.

Proof. We recall that finite intersection of nonempty, bounded and closed semispaces is called convex polytope (see [6]). For any q.s.o. there is a cubic matrix which can be embedded in the space $R^{m^3}$. One can embed it this cubic matrix in $R^{m^2(m-1)/2}$. Furthermore, each of the condition (18) defines closed semispace in $R^{m^2(m-1)/2}$. If we denote the set of all d.s.q.o. by $B$, from the corollary 4.3 it follows that consists of intersection of these semispaces. It is clear that this intersection is nonempty and bounded set in $R^{m^2(m-1)/2}$, since it lies in positive ortant and in hyperplane $\sum_{i,j,k=1}^{m} p_{ij,k} = m^2$. Due to Grunbaum [6] we conclude that $B$ is convex polytope. □

In [3] it was conjectured that an operator $V = (A_1|A_2|\cdots|A_m)$ is extremal point of $B$ if and only if $A_i \in extrU$, $\forall i = \overline{1,m}$. We show that in general it is not true.

Theorem 4.5. Let $V = (A_1|\cdots|A_m) \in extrB$. Then $V_\pi = (A_{\pi(1)}|\cdots|A_{\pi(m)}) \in extrU$ for any permutation $\pi$ of the index set $\{1,2,\cdots,m\}$.

Proof. Let $V_\pi \notin extrB$. Then $\exists V', V'' \in B$, $V' \neq V''$, such that $2V_\pi = V' + V''$. Let $V' = (A'_1|\cdots|A'_m)$, $V'' = (A''_1|\cdots|A''_m)$. Then $(2A_{\pi(1)} - A'_1 - A''_1|\cdots|2A_{\pi(m)} - A'_m - A''_m) = 0$. Since the matrices $2A_{\pi(i)} - A'_i - A''_i$ are symmetric, then $2A'_{\pi(i)} = A'_i + A''_i$, therefore $2A_i = A'_{\pi^{-1}(i)} + A''_{\pi^{-1}(i)}$. Since

$$\sum_{i=1}^{m} A'_i = \sum_{i=1}^{m} A''_i = E$$

then

$$\sum_{i=1}^{m} A'_{\pi^{-1}(i)} = \sum_{i=1}^{m} A''_{\pi^{-1}(i)} = E.$$  

Denote $W' = (A'_{\pi^{-1}(1)}|\cdots|A'_{\pi^{-1}(m)})$ and $W'' = (A''_{\pi^{-1}(1)}|\cdots|A''_{\pi^{-1}(m)})$. Then we get $W', W'' \in B$, $W' \neq W''$ 2V = W' + W'' which contradicts to $V \notin extrB$. □

Theorem 4.6. Let $V = (A_1|\cdots|A_m)$. If for any permutation $\pi$ of any $m-1$ elements of $\{1,2,\cdots,m\}$ we have $A_{\pi(k)} \in extrU$, then $V \in extrB$.

Proof. Using the above theorem it is enough to show that if $A_1, A_2, \cdots, A_{m-1} \in extrU$, then $V \in extrB$. Let us assume that $V \notin extrB$. Then $\exists V', V'' \in B$, $V' \neq V''$ such that $2V = V' + V''$. Let $V' = (A'_1|\cdots|A'_m)$, $V'' = (A''_1|\cdots|A''_m)$. Then

$$(2A_1 - A'_1 - A''_1|\cdots|2A_m - A'_m - A''_m) = 0.$$
So \(2A_i - A'_i - A''_i = 0 \forall i = \overline{1, m}\). Since \(A_i \in \text{extr} U, \forall i \geq 2\), then \(A'_i = A''_i, \forall i \geq 2\). From the equality \(\sum_{i=1}^{m} A'_i = \sum_{i=1}^{m} A''_i = E\) we get \(A'_i = A''_i\). That’s why \(V' = V''\), which contradicts to \(V \notin \text{extr} B\). \(\square\)

**Corollary 4.7.** If \(A_i \in \text{extr} U, \forall i = \overline{1, m}\), then \(V \in \text{extr} B\). However, the converse case is not true

**Example:** Consider an operator

\[
\begin{align*}
x'_1 &= x_1 x_2 + x_1 x_3 + x_2 x_3 \\
x'_2 &= x_2^2 + x_1 x_3 \\
x'_3 &= x_1^2 + x_1 x_2 + x_3 x_2
\end{align*}
\]

Corresponding matrices have the following view

\[
\begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{1}{2} \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0
\end{pmatrix}
\]

It is easy to see that two of them are extremal. Then from the above theorem it follows that \(V \notin \text{extr} B\).

Whether the conditions of the theorem are necessary and sufficient? In two dimensional simplex the problem is solved positively.

**Theorem 4.8.** Let \(V = (A_1 | A_2 | A_3) \in B\). \(V \in \text{extr} B\) if the only if at least 2 of the matrices \(A_1, A_2, A_3\) are extremal of \(U_1\).

**Proof.** For the proof of the given theorem we refer to the theorem 5 from [4], which says that if \(V \in \text{extr} B\), then entries of the matrices \(A_1, A_2, A_3\) either 0 or \(\frac{1}{2}\) or 1. Therefore, if \(V \in \text{extr} B\), then either \(A_1, A_2, A_3 \in \text{extr} U\) or \(A_1, A_2, A_3 = M\).

The case \(A_1 = M, A_2 = M, A_3 = M\) is impossible because of \(A_1 + A_2 + A_3 = E\).

It can be easily shown that if two of the matrices \(A_1, A_2, A_3\) is \(M\) then \(V\) is not extremal. Therefore we can conclude that at least two of the matrices \(A_1, A_2, A_3\) are extremal. \(\square\)

**Corollary 4.9.** For \(m = 3\) we have \(|\text{extr} B| = 222\).

**Proof.** Let \(V \in \text{extr} B, \ V = (A_1 | A_2 | A_3)\). From the theorem 4.8 it follows that \(A_2, A_3 \in \text{extr} U\) up to permutation. From \(A_1 + A_2 + A_3 = E\) one gets that either \(A_1 \in \text{extr} U\) or \(A_1 = M\).

Let \(A_1 \in \text{extr} U\). From corollary 3.2 we know all extreme points of \(U\). Therefore we can choose those triples of extreme points, sum of which is \(E\). The number of such triples is 31. For the case of \(A_1 = M\), the number of such triples is 6.

Consequently, \(|\text{extr} B| = 37 \times 3! = 222\). \(\square\)
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