Abstract

High-dimensional vector autoregressive (VAR) models are important tools for the analysis of multivariate time series. This paper focuses on high-dimensional time series and on the different regularized estimation procedures proposed for fitting sparse VAR models to such time series. Attention is paid to the different sparsity assumptions imposed on the VAR parameters and how these sparsity assumptions are related to the particular consistency properties of the estimators established. A sparsity scheme for high-dimensional VAR models is proposed which is found to be more appropriate for the time series setting considered. Furthermore, it is shown that, under this sparsity setting, thresholding extents the consistency properties of regularized estimators to a wide range of matrix norms. Among other things, this enables application of the VAR parameters estimators to different inference problems, like forecasting or estimating the second-order characteristics of the underlying VAR process. Extensive simulations compare the finite sample behavior of the different regularized estimators proposed using a variety of performance criteria.

Keywords: Dantzig Selector, Lasso, Sparsity, Vector Autoregression, Yule-Walker Estimators, Thresholding

1 Introduction

The vector autoregressive (VAR) model is one of the most prominent and frequently used models for analyzing multivariate time series; see among others the textbooks by Brockwell and Davis (1991), Reinsel (2003), Lütkepohl (2007), Tsay (2013) and Kilian and Lütkepohl (2017). Due to the increasing availability of time series data, high-dimensional VAR models has attracted the interest of many researchers during the last two decades. Initiated by developments in the i.i.d. setup, statistical methods have been proposed for inferring properties of high-dimensional VAR models. However, to make statistical inference for such models possible, the model complexity has to be reduced. One way to achieve this, is to limit the allowed direct influences between the variables (time series) involved in the high-dimensional VAR system. A common strategy toward this, is to impose some kind of sparsity or of approximately sparsity assumptions. For sparse VAR models, the dimension of the unknown parameters is considerably reduced by assuming that a large number of these parameters is zero. That is, only few “variables” out of a large set of potential “explanatory variables” are allowed to have a direct influence on the other variables of the system. In an approximately sparse setting, it is allowed that a large number of parameters is not
exactly zero but rather small in magnitude. As we will see, most of the sparsity patterns used in the literature are motivated by and related to the particular estimation method used. Apart from reviewing the different sparsity settings used in the literature and the corresponding estimation methods proposed, one of the aims of this paper is to introduce a sparsity pattern for high-dimensional VAR models which is more appropriate for VAR models. Furthermore, for the introduced sparsity setting, we use thresholding of regularized estimators as a tool to extent their consistency to a wide range of matrix norms.

The main estimation strategy followed in the high-dimensional setting is to use some kind of regularized estimator. In this context, three different procedures have been proposed: Regularized least squares (LASSO), see Basu and Michailidis (2015); Kock and Callot (2015); regularized maximum-likelihood estimators, see among others Basu and Michailidis (2015); Davis et al. (2016), and regularized Yule-Walker estimators using the CLIME approach or the Dantzig estimator, respectively; see Han et al. (2015); Wu et al. (2016). We refer to Cai et al. (2011) for the CLIME method, and to Candes et al. (2007) for the Dantzig estimator. Different sparsity patterns are used in the aforementioned papers and an overview of these different patterns as well as on their impact on the consistency properties of the estimators used, is given in the next section.

We first fix some notation. For a vector \( x \in \mathbb{R}^d \), \( \|x\|_0 = \sum_{j=1}^d 1(x_j \neq 0) \) denotes the number of non-zero coefficients, \( \|x\|_1 = \sum_{j=1}^d |x_j| \), \( \|x\|_2 = \sum_{j=1}^d |x_j|^2 \), and \( \|x\|_\infty = \max_j |x_j| \) denote the \( l_1 \), \( l_2 \) and \( l_\infty \) norm, respectively. Furthermore, for an \( r \times s \) matrix \( B = (b_{i,j})_{i=1,\ldots,r; j=1,\ldots,s} \), \( \|B\|_l = \max_{x \in \mathbb{R}^s : \|x\|_l = 1} \|Bx\|_l \), \( l \in [1,\infty] \) with \( \|B\|_1 = \max_{1 \leq j \leq s} \sum_{i=1}^r |b_{i,j}| \), \( \|B\|_\infty = \max_{1 \leq i \leq r} \sum_{j=1}^s |b_{i,j}| \), \( \|B\|_\max = \max_{i,j} |e_i^T B e_j| \), where \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \) denotes a vector of appropriate dimension with the one appearing in the \( j \)th position and zero elsewhere. Denote the largest absolute eigenvalue of a square matrix \( B \) by \( \rho(B) \) and note that \( \|B\|_2^2 = \rho(BB^T) \). The \( d \)-dimensional identity matrix is denoted by \( I_d \). Furthermore, for the Kronecker product of two matrices \( A \) and \( B \) we write \( A \otimes B \), and for a matrix \( A \), \( \text{vec}(A) \) denotes its vectorization obtained by stacking the columns to a vector; see among others Appendix A.11 in Lütkepohl (2007). For a vector \( x \in \mathbb{R}^d \), \( \text{diag}(x) \) is a diagonal matrix of size \( d \) the entries of which on the main diagonal are given by the elements of the vector \( x \).

Let \( \{X_t, t \in \mathbb{Z}\} \) be a \( d \)-dimensional vector autoregressive process of order \( p \), in short VAR(\( p \)), given by

\[
X_t = \sum_{j=1}^p A_j X_{t-j} + \varepsilon_t, \tag{1}
\]

where \( \varepsilon_t \) is a white noise process with covariance \( \Sigma_\varepsilon \). We first summarize some results for VAR(\( p \)) processes which will be used later on and which can be found in many standard textbooks for time series analysis; see among others, Lütkepohl (2007) and Tsay (2013). A VAR(\( p \)) model can be stacked to a
VAR(1) model as follows. Let \( W_t = (X_t^T, X_{t-1}^T, \ldots, X_{t-p+1}^T)^T \in \mathbb{R}^{dp \times 1}, \)

\[
A = \begin{pmatrix}
A_1 & A_2 & \ldots & A_p \\
I_d & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \ldots & I_d & 0
\end{pmatrix} \in \mathbb{R}^{dp \times dp}, \quad E = e_1 \otimes I_d = \begin{pmatrix} I_d \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{dp \times d} \quad \text{and} \quad U_t = E \varepsilon_t.
\]

Then,

\[
W_t = AW_{t-1} + U_t
\]

is the VAR(1) stacked form of (1) and \( X_t = E^TW_t. \) It is well known that a VAR(p) process is stable if \( \det(I - \sum_{s=1}^{p} A_s z^s) \neq 0 \) for all \( |z| \leq 1 \) or equivalently, if \( \rho(A) < 1. \) In what follows, we assume that this stability condition is satisfied. A stable VAR(p) process has the moving average representation \( X_t = E^T \sum_{j=0}^{\infty} A^j U_{t-j} \) and its autocovariance matrix function \( \Gamma : \mathbb{Z} \rightarrow \mathbb{R}^{d \times d}, \) can be expressed as

\[
\Gamma(h) = \begin{dcases}
E^T A^h \sum_{j=0}^{\infty} A^j E \Sigma_e E^T (A^\top)^j E & \text{for } h \geq 0, \\
\Gamma(-h)^\top & \text{for } h < 0.
\end{dcases}
\]

The spectral density matrix of the VAR(p) process at frequency \( \omega \in [-\pi, \pi] \) is given by

\[
f(\omega) = \frac{1}{2\pi} A^{-1}(\exp(-i\omega)) \Sigma_e (A^{-1}(\exp(i\omega)))^\top,
\]

and its inverse by

\[
f^{-1}(\omega) = 2\pi A(\exp(i\omega))^\top \Sigma_e^{-1} A(\exp(-i\omega)),
\]

where \( A(z) = I_d - \sum_{s=1}^{p} A_s z^s, z \in \mathbb{C}. \)

Given observation \( X_1, \ldots, X_n, \) we can write (1) in a compact form as the regression equation,

\[
\mathcal{Y} = \mathcal{X} B + \mathcal{E},
\]

where \( \mathcal{Y} = (X_n, \ldots, X_{n+p+1})^\top \) and \( \mathcal{E} = (\varepsilon_n, \ldots, \varepsilon_{n+p+1})^\top \) are \((n-p) \times d\) dimensional matrices, \( \mathcal{X} = (W_{n-1}, \ldots, W_p)^\top \) is \((n-p) \times dp\) dimensional matrix and \( B = (A_1^\top, \ldots, A_p^\top) \in \mathbb{R}^{dp \times d}. \)

Let \( \hat{A}_1, \ldots, \hat{A}_p \) be an estimator of the VAR parameter matrices \( A_1, \ldots, A_p \) and let \( \hat{A} \) be the corresponding stacked matrix version of these estimators. The innovations can be estimated by \( \hat{\varepsilon}_t = X_t - \sum_{s=1}^{p} \hat{A}_s X_{t-s}, t = p+1, \ldots, n, \) and can be used to construct estimators for the covariance matrix \( \Sigma_e. \) Successful applications of the estimated VAR model require consistency of the estimators of \( A_1, \ldots, A_p \) and \( \Sigma_e \) used. Since for a fixed dimension \( d, \) the commonly used matrix norms are equivalent, it is not important in a low-dimensional setting, with respect to which matrix norm consistency of the estimators is established. This however, changes in the high-dimensional setting. As we will see in Section 2 some of the estimators proposed in the literature, are consistent with respect to some matrix norms only. Notice that the consistency requirements on the estimators \( \hat{A}_s \) and \( \hat{\Sigma}_e, \) also depend on the applications of the VAR model one has in mind. For instance, for consistency of the one step ahead
forecast $\hat{X}_{n+1} = \sum_{s=1}^{d} \hat{A}_s X_{n+1-s}$, it suffices to have consistency of the estimators $\hat{A}$ with respect to the $\| \cdot \|_{\infty}$ norm, that is $\sum_{s=1}^{d} \| A_s - \hat{A}_s \|_{\infty} = \| A - \hat{A} \|_{\infty} = o_P(1)$. However, if interest is directed towards estimating the autocovariance matrix function $\Gamma(h)$, then, additionally, $\| A - \hat{A} \|_{1} = \| A^T - \hat{A}^T \|_{\infty} = o_P(1)$ is required; see also (2). If one wants to to consistently estimate the spectral density matrix $\hat{\Sigma} = \hat{\Sigma}(\hat{\beta})$, then additionally consistency with respect to the $1$-norm, that is $\| \hat{\Sigma} - \Sigma \|_{1} = \| \hat{\Sigma}^T - \Sigma^T \|_{1} = o_P(1)$ is required. If one wants to to consistently estimate the aforementioned second-order characteristics of the VAR model arises, for instance, in the context of bootstrap-based inference for high-dimensional VAR models; see Krampe et al. (2019). Observe that $\| A - \hat{A} \|_{\infty} = o_P(1)$ and $\| A - \hat{A} \|_{1} = o_P(1)$ implies that $\| A - \hat{A} \|_{l} = o_P(1)$ for any $l \in [1, \infty]$. 

### 2 Different Sparsity Patterns for VAR Models

In this section we discuss in more detail some of the sparsity patterns that have been used in the literature. We first note that there is a strong connection between the particular sparsity assumptions made and the error bounds obtained for the corresponding parameter estimators. Moreover, the sparsity patterns imposed are often motivated by the particular estimation procedure used. Hence, and for a better comparison of the different sparsity patterns used, we also include in our discussion the estimators developed and the error bounds obtained. We focus in the following on $\ell_1$ penalized estimators with tuning parameters always denoted by $\lambda_n$.

We begin with the sparsity pattern used in Basu and Michailidis (2015). The authors use a vectorized version of (1), i.e., $\text{vec}(\mathcal{Y}) = (I_d \otimes \mathcal{X}) \text{vec}(B) + \text{vec}(\mathcal{E})$ and formulate the following $\ell_1$-penalized estimator

$$\hat{\beta}^{(L_1)} = \arg\min_{\beta \in \mathbb{R}^{d^2}} \frac{1}{n-p}(\text{vec}(\mathcal{Y}) - (I_d \otimes \mathcal{X})\beta)^\top W(\text{vec}(\mathcal{Y}) - (I_d \otimes \mathcal{X})\beta) + \lambda_n \| \beta \|_1,$$  

(6)

where $W$ is a weighting matrix. $W = I_{d(n-p)}$ leads to an $\ell_1$-penalized least squares estimator and $W = (\Sigma^{-1}_x \otimes I_{n-p})$ to a $\ell_1$-penalized maximum likelihood estimator. Weighting is helpful if $\Sigma_x$ is not well approximated by a diagonal matrix $\sigma^2 I_d$ for some $\sigma^2 > 0$. Note Basu and Michailidis (2015) considered Gaussian innovations. Furthermore, they assumed that $\text{vec}(B)$ is a sparse vector in the sense that $\| \text{vec}(B) \|_0 = k$. Recall that the parameter matrices $A_1, \ldots, A_p$ have $d^2 p$ unknown coefficients. Hence, by this sparsity assumption, the number of non-zero coefficients of the VAR system is limited by $k$. For the estimator (6), Basu and Michailidis (2015) obtained on a set having high probability, the error bound

$$\| \hat{\beta}^{(L_1)} - \text{vec}(B) \|_1 \leq C k \sqrt{\log(pd^2)/(n-p)},$$  

(7)

where $C$ is some constant depending on properties of the process $\{X_t\}$ but not on $n, p$ and $k$. Since $\| \text{vec}(\cdot) \|_1$ is the sum of all component-wise absolute errors, an error bound for $\| \text{vec}(\cdot) \|_1$ implies an error bound with respect to most matrix norms. Let $\hat{A}_1^{(L_1)}, \ldots, \hat{A}_p^{(L_1)}$ be the estimators of the parameter matrices corresponding to $\hat{\beta}^{(L_1)}$. We have $\sum_{s=1}^{p} \| \hat{A}_s^{(L_1)} - A_s \|_1 \leq \| \hat{\beta}^{(L_1)} - \text{vec}(B) \|_1$ and $\sum_{s=1}^{p} \| \hat{A}_s^{(L_1)} - A_s \|_{\infty} \leq \| \hat{\beta}^{(L_1)} - \text{vec}(B) \|_1$. Limiting the total number of non-zero coefficients by $k$, has, however, a major impact on the growth rate allowed for the dimension $d$ of the VAR system. To elaborate, consider the
case $p = 1$ and assume that the VAR(1) process solely consists of $d$ univariate AR(1) processes, i.e., that $A$ is a diagonal matrix. Then, without further sparsity restrictions we have $\| \text{vec}(A) \|_0 = d$. Thus, and by ignoring the log-term, the estimator $\hat{\beta}^{(\text{RoLa})}_j$ of Basu and Michailidis (2015) is consistent if the dimension $d$ increases slower than $\sqrt{n}$, that is if $d = o_P(\sqrt{n})$. This implies that the sparsity pattern $\| \text{vec}(A) \|_0 \leq k$ used in Basu and Michailidis (2015), can be satisfied in the high-dimensional setting considered, if the majority of the time series included in the VAR(1) system are essentially white noise series. This seems, however, to be rather restrictive for time series. This is so since it is not uncommon to assume that some kind of interactions between the components of the $d$-dimensional process $\{X_t\}$ exists. If this is the case and the time series included in the VAR(1) system are not white noises, then $\| \text{vec}(A) \|_0 \geq d$ which contradicts the sparsity condition $\| \text{vec}(A) \|_0 = k$. Therefore, a price for obtaining consistency results with respect to the strong $\| \text{vec}(\cdot) \|_1$ norm seems to be paid by the rather restrictive sparsity assumptions one has to impose on the parameters of the VAR system.

Instead of using a vectorized version of (5), Kock and Callot (2015) partitioned the same regression equation into single equations by formulating one equation for each time series, i.e., $Ye_j = XB e_j + \varepsilon e_j$ where $j = 1, \ldots, d$. They then propose the following $\ell_1$-penalized least squares estimator of $\hat{\beta}_j = Be_j$,

\[
\hat{\beta}^{(\text{RoLa})}_j = \arg\min_{\beta \in \mathbb{R}^{pd}} \frac{1}{n - p} \| Ye_j - X \beta \|_2 + \lambda_n \| \beta \|_1.
\]  

(8)

Notice that the tuning parameter $\lambda_n$ may differ from equation to equation, i.e., $\lambda_n$ may depend on $j \in \{1, 2, \ldots, d\}$. In terms of the original VAR($p$) equation (1), it is clear that this estimator is formulated row-wise, that is, each time series is modelled individually by fitting a $l_1$-regularized regression. This has the drawback that a weighting as in (6) can not be easily implemented. However, one advantage is that, now, the sparsity assumptions can be formulated row-wise, that is $e^\top_j (A_1, \ldots, A_p)$ can be assumed to be a sparse vector. Kock and Callot (2015) assume that $\sum_{s=1}^d \| e^\top_j A_s \|_0 = \| B e_j \|_0 \leq k_j$. Under the assumption of Gaussian innovations, they obtain on a set with high probability, that

\[
\| \hat{\beta}^{(\text{RoLa})}_j - Be_j \|_1 \leq C k_j \sqrt{\log(pd)/(n - p)},
\]  

(9)

where is the above bound some additional log-terms have been omitted for simplicity. Let $\hat{A}_1^{(\text{RoLa})}, \ldots, \hat{A}_p^{(\text{RoLa})}$ be the estimators of the matrices $A_s$ corresponding to $(\hat{\beta}_1, \ldots, \hat{\beta}_d)$. Then, the bound in (9) expressed with respect to the rows of $\hat{A}_s - A_s$, implies that $\sum_{s=1}^p \| \hat{A}_s^{(\text{RoLa})} - A_s \|_\infty \leq C \max_j k_j \sqrt{\log(pd)/(n - p)}$. Hence, and according to this approach, the parameters of the VAR system can consistently be estimated with respect to the $\| \cdot \|_\infty$ matrix norm. Furthermore, the corresponding sparsity pattern only requires that the number of non-zero coefficients within the $j$th row is limited by $\max_j k_j$. Thus, each time series at time point $t$ can be directly affected by $\max_j k_j$ other lagged variables. Therefore, this sparsity pattern is, more flexible than the one considered in Basu and Michailidis (2015) in which the total number of non-zero coefficients is limited, i.e., $\| \text{vec}(B) \|_0 = k$ is assumed. To further clarify the differences, consider again the example of a VAR(1) process which solely consists of $d$ univariate AR(1) processes. Then, $\max_j \| e^\top_j A \|_0 = 1$ and $\hat{A}^{(\text{RoLa})}$ is consistent if $d = o_P(\exp(n))$. Hence, the dimension $d$ of the system can grow much faster compared to what is allowed for the sparsity pattern used in Basu and Michailidis.
and which, as we have seen, $d = o_P(\sqrt{n})$.

However, since the estimators $\hat{A}_1^{(RoLa)}, \ldots, \hat{A}_p^{(RoLa)}$ are constructed row-wise by fitting a regularized regression to each time series, an error bound with respect to the $\| \cdot \|_1$ norm, cannot easily be obtained. It may even not be possible without imposing further assumptions on the VAR process. Furthermore, within this sparsity framework, it is possible that there exists a time series which affects all others, that is, for some $j$ the coefficient matrix $A_j$ has a dense column. Regarding the bound with respect to $\| \cdot \|_1$ of the estimator $[\hat{S}]$, we have

$$\| \text{vec}(\hat{A}_1^{(RoLa)}, \ldots, \hat{A}_p^{(RoLa)}) - \text{vec}(B)\|_1 = \sum_{j=1}^d \|\hat{\beta}_j^{(RoLa)} - \beta_j\|_1 \leq C \sum_{j=1}^d k_j \sqrt{\log(pd)/(n-p)},$$

which leads to the following bound with respect to the matrix norm $\| \cdot \|_1$:

$$\| \hat{A}^{(RoLa)} - A \|_1 \leq \sum_{s=1}^p \|\hat{A}_s^{(RoLa)} - A_s\|_1 \leq C \sum_{j=1}^d k_j \sqrt{\log(pd)/(n-p)}.$$ 

It is not clear if this bound can be improved to $Cd \sqrt{\log(pd)/n}$. Moreover, the considered sparsity pattern, which is more flexible than the one in Basu and Michailidis (2015), is possible only if consistency of the estimators with respect to the $\| \cdot \|_\infty$ norm is required. This implies that this estimator can be used in a high-dimensional setting for forecasting purposes but it may be of limited value if one is interested in estimating the second-order properties of the VAR model, like the autocovariance matrix $\Gamma(h)$ or the spectral density matrix $f(\lambda)$.

The approaches of Basu and Michailidis (2015) and Kock and Callot (2015) are inspired by the i.i.d regression setup. In contrast to this, the approach of Han et al. (2015) is inspired by the setup of high-dimensional covariance estimation. In this setup, CLIME (constrained $\ell_1$-minimization for inverse matrix estimation), see Cai et al. (2011), provides an approach to estimate the inverse covariance matrix and it is based on the Dantzig estimator, see Candes et al. (2007). The corresponding estimator of the precision matrix $\Sigma^{-1}_{\epsilon}$ is obtained as the solution of the following optimization problem,

$$\min_{\Omega \in \mathbb{R}^{d \times d}} \sum_{i,j=1}^d |e_i^\top \Omega e_j| \text{ s.t. } \|\Sigma_{\epsilon,n} \Omega - I_d\|_\infty \leq \lambda_n, \quad (10)$$

where $\Sigma_{\epsilon,n}$ is the sample covariance matrix and $\lambda_n$ a tuning parameter. The above optimization problem can be split into sub-problems, that is, $\hat{\beta}_j = \arg\min_{\beta \in \mathbb{R}^d} \|\beta\|_1$ s.t. $\|\Sigma_{\epsilon,n} \beta - e_j\|_\infty \leq \lambda_n$. This sub-problem strategy enables the derivation of error bounds with respect to the $\| \cdot \|_1$ norm without the need for any additional thresholding of the estimators obtained; see Cai et al. (2011) for details. Han et al. (2015) focus on VAR(1) model and use the Yule-Walker equation $\Gamma(-1) = \Gamma(0)A^\top$ to formulate an optimization problem similar to (10). They derive the following estimator of $\hat{\beta}_j = A^\top e_j$, $j \in \{1, 2, \ldots, d\}$,

$$\hat{\beta}_j^{(YW)} = \arg\min_{\beta \in \mathbb{R}^d} \|\beta\|_1 \text{ s.t. } S_0 \beta - S_1 e_j\|_{\max} \leq \lambda_n, \quad (11)$$

where $S_0 = 1/n \sum_{t=1}^n X_t X_t^\top$ and $S_1 = 1/(n-1) \sum_{t=1}^{n-1} X_t X_{t+1}^\top$ are sample autocovariances at lag zero and lag minus one, respectively. Let $(\hat{A}^{(YW)})^\top = (\hat{\beta}_1^{(YW)} : \cdots : \hat{\beta}_d^{(YW)})$ be the estimator of $A^\top$ in matrix
form. Han et al. (2015) follow Cai et al. (2011) regarding the sparsity assumptions they impose on the VAR(1) system. In particular, they assume that

\[
A \in \mathcal{M}(q, s, M) = \left\{ B \in \mathbb{R}^{d \times d} : \max_{1 \leq j \leq d} \sum_{i=1}^{d} |B_{j,i}|^q \leq s, \|B\|_\infty \leq M \right\}. \tag{12}
\]

This means, the rows of \( A \), i.e., the columns of \( A^\top \), are considered as approximately sparse and bounded in \( \ell_1 \)-norm by the positive constant \( M \). Under the sparsity assumption \( \text{(12)} \), they obtain for Gaussian innovations the following error bound, on a set with high probability,

\[
\|A^\top - \hat{A}^{(YW)}\|_1 = \|A - \hat{A}^{(YW)}\|_\infty \leq CM\|\Gamma(0)^{-1}\|_1 s \sqrt{\log(d)/n}. \tag{13}
\]

As mentioned, the norm \( \| \cdot \|_1 \) arises canonically for the Dantzig estimator since the optimization sub-problems are build up column-wise for the matrix \( A^\top \). However, an error bound for \( \|A^\top - \hat{A}^{(YW)}\|_\infty = \|A - \hat{A}^{(YW)}\|_1 \) cannot be derived without further assumptions. In fact, only the following naive bounds hold true: \( \|A - \hat{A}^{(YW)}\|_\infty \leq CM\|\Gamma(0)^{-1}\|_1 d \sqrt{\log(d)/n}. \) Wu et al. (2016) extended the approach of using the CLIME method for VAR parameter estimation to general V AR\((p)\) processes and to possible non-Gaussian innovations. They focus on error bounds with respect to the \( \| \cdot \|_{\max} \) norm. For this, they do not need to specify a particular sparsity pattern. Using the same approximately sparsity setting \( \text{(12)}, \) Masini et al. (2019) showed that the row-wise Lasso \( \text{(8)} \) possesses with high-probability and under some restrictions on the growth rates of \( n, q, \) and \( s \), the following error bound with respect to the \( \| \cdot \|_2 \) norm,

\[
\|\hat{\beta}^{(RoLa)} - \beta_j\|_2^2 \leq C_\tau s\|\Gamma^{(s)}(0)^{-1}\|_2^{(2-q)} \left[ (d^2 pn)^{2/\tau}/\sqrt{n} \right]^{2-q}. \tag{14}
\]

Here \( \tau \) denotes the number of finite moments of the innovations \( \varepsilon_i \), i.e., \( \max_{\|\epsilon\|_1 \leq 1} (E|\epsilon^\top \varepsilon_1|^{\tau})^{1/\tau} \leq C_\tau < \infty, \tau > 4, \) and \( C_\tau \) denotes a particular constant depending on \( C_\tau \) and on \( \tau \) only. For sub-Gaussian innovations, they obtained sharper error bounds which allow for a larger value of the dimension \( d \) and which are similar to the rates given in \( \text{(9)} \).

3 A Sparsity Setting for VAR Time Series

Our aim in this section is twofold. First, and based on the discussion of the previous section, we introduce a sparsity setting for VAR models which is appropriate for the high-dimensional time series setup considered in this paper. Second, for the sparsity setting introduced, we then derive estimators of the VAR model parameters, which are consistent with respect to all matrix norms \( \| \cdot \|_l \), for \( l \in [1, \infty] \).

As already mentioned, the aim of any sparsity pattern is to reduce the complexity of the model such that consistent estimation becomes possible even in a high-dimensional setup. Towards developing a sparsity setting which is appropriate for high-dimensional time series, and in particular for VAR models, it is worth to first recall the meaning of the coefficients of the parameter matrices \( A_s, s = 1, 2, \ldots, p \). The coefficients \( e_j^\top (A_1, \ldots, A_p) \), i.e., those in the \( j \)th row of the autoregressive matrices, describe the direct linear influence of all time series (in lagged form), that is of \( X_{t-1}, \ldots, X_{t-p} \), onto the \( j \)th component.
at time \( t \), that is onto \( X_{t+j} \). Furthermore, the coefficients on the \( j \)th column of the matrix \( A_k \), i.e., the coefficients \( A_k e_j \), describe the direct linear influence of the \( j \)th component at lag \( k \), that is of \( X_{t-k+j} \), onto all time series of the system at time \( t \), that is onto \( X_t \). Imposing a sparsity pattern on the coefficient matrices \((A_1, \ldots, A_p)\) means that the direct influences between the different time series are restricted. A reasonable sparsity pattern will be one in which it is assumed that a single time series (including all its past values up to lag \( p \)) can affect directly and can be directly affected only by a limited number of other time series (including their lagged versions). This requirement leads to the need of imposing sparsity assumptions in the rows and in the columns of the matrices \( A_s, s = 1, 2, \ldots, p \). Row-wise sparsity in the form \( \max_j \sum_{k=1}^p \| A_k e_j \|_0 \leq s \), means that a time series can be influenced directly only by \( s \) other time series (including their lagged values). However, for column-wise sparsity, two reasonable options exist. The first is the column-wise analog to the aforementioned row-wise sparsity, which leads to the requirement that \( \max_j \sum_{k=1}^p \| A_k e_j \|_0 \leq s \). This means that a single time series \( j \) and all its past values, e.g., \( X_{t-k+j}, k = 1, \ldots, p \), has at most \( s \) direct channels to affect the elements of the vector \( X_t \). The second option one has is the requirement \( \max_j \max_{1 \leq k \leq p} \| A_k e_j \|_0 \leq s \). This means that a single time series in one of its lagged versions, for instance \( X_{t-p+j} \), can affect at most \( s \) other time series, that is at most \( s \) of the components of the vector \( X_t \). Hence, in the second option a single time series with all its lagged values has at most \( s \times p \) channels to affect directly the components of \( X_t \). We mention here that if there is a (near to) full interaction among the time series of the system, then it may be more reasonable to consider alternative approaches for inferring properties of high-dimensional time series. Factor models and more specifically, dynamic factor models could be a possible alternative in such a case. Such an approach will avoid the imposition of (unrealistic) sparsity assumptions on the interaction between the time series considered. However, other assumptions are required in this case, like for instance, that the evolution of the entire high dimensional system of time series is driven by few non observable components. We refer here to the surveys Stock and Watson (2005, 2011, 2016); Bai et al. (2008). See also Chapter 16 in Kilian and Lütkepohl (2017).

The above discussion regarding an appropriate sparsity pattern for high-dimensional VAR models, was devoted to the case of the so called strict sparsity. This is the case where coefficients are counted only if they are different from zero. Nevertheless we may also consider the case of so called approximately sparsity. Towards this, we adopt the approximately sparsity settings used for high-dimensional covariance matrices; see among others Bickel and Levina (2008); Rothman et al. (2009). Since in contrast to covariance matrices, the parameter matrices \( A_1, \ldots, A_p \) are in general not symmetric, we state the following two classes of approximately sparse VAR(p) matrices, where each one of them refers to the two different column-wise sparsity options we have discussed before.

\[
\mathcal{M}^{(1)}(q, s, M, p) = \left\{ (M_1, \ldots, M_p), M_i \in \mathbb{R}^{d \times d} : \max_{1 \leq j \leq d} \max_{1 \leq k \leq p} \sum_{i=1}^d |A_{k,i,j}|^q \leq s, \max_{1 \leq k \leq p} \| A_k \|_1 \leq M, \right. \\
\left. \max_{1 \leq i \leq d} \sum_{k=1}^p \sum_{j=1}^d |A_{k,i,j}|^q \leq s, \sum_{k=1}^p \| A_k \|_{\infty} \leq M \right\}, \quad (15)
\]
\[ M^{(2)}(q, s, M, p) = \left\{ (M_1, \ldots, M_p) \in \mathbb{R}^{d \times d} : \max_{1 \leq i \leq d} \sum_{k=1}^{p} \sum_{j=1}^{d} |A_{k;i,j}|^q \leq s, \sum_{k=1}^{p} \|A_k\|_1 \leq M, \right. \\
\left. \max_{1 \leq i \leq d} \sum_{k=1}^{p} \sum_{j=1}^{d} |A_{k;i,j}|^q \leq s, \sum_{k=1}^{p} \|A_k\|_\infty \leq M \right\}, \quad (16) \\
\]

where \( q \in [0, 1) \). Notice that \( q = 0 \) refers to the case of strict sparsity whereas \( q > 0 \) to that of approximately sparsity. In the remaining of this paper, we focus on the pattern \( M(q, s, M, p) = M^{(1)}(q, s, M, p) \) only. This sparsity pattern is a generalization of the one used in Krampe et al. (2019) and a subset of the sparsity pattern used in Han et al. (2015) and Masini et al. (2019). However, Han et al. (2015) obtained consistency only with respect to the \( \| \cdot \|_\infty \) norm, i.e., \( \|A - \hat{A}^{(YW)}\|_\infty \), whereas our aim is to obtain consistency with respect to \( \| \cdot \|_l \) for all values of \( l \in [1, \infty] \). As mentioned, this will enable the use of the estimators obtained in several applications, like forecasting or estimating the second-order structure of the VAR process. Since \( M^{(2)}(q, s, M, p) \subseteq M(q, s, M, p) \), all results presented here also hold true for the other sparsity pattern \( M^{(2)}(q, s, M, p) \) stated in (16). If \( (A_1, \ldots, A_p) \in M^{(2)}(q, s, M, p) \), then, additionally to \( \|A - \hat{A}\|_l = o_P(1), \sum_{k=1}^{p} \|A_k - \hat{A}_k\|_l = o_P(1), \) for all \( l \in [1, \infty] \), can be established. This is important if one wants to obtain a consistent estimator of the inverse of the spectral density matrix of the VAR model; see Theorem 6 below for details. Notice that the two sparsity patterns coincide for VAR(1) models, i.e., \( M^{(2)}(q, s, M, 1) = M(q, s, M, 1) \).

As we have seen, regularization is an important tool for obtaining consistent estimates in a high-dimensional setting. In the context of covariance matrix estimation, one approach is thresholding the sample covariance matrix; see Bickel and Levina (2008), Rothman et al. (2009), Cai and Liu (2011). Since the sample covariance matrix is (under certain assumptions) consistent with respect to the \( \| \cdot \|_{\max} \) norm, thresholding helps to transmit the component-wise consistency to consistency with respect to a matrix norm. The CLIME method, see Cai et al. (2011), achieves consistency with respect to the \( \| \cdot \|_{\max} \) also for the precision matrix, i.e., the inverse of the covariance matrix. As mentioned, Han et al. (2015) use the CLIME method to estimate the parameter matrix \( A \) of a VAR(1) model and they established \( \|A - \hat{A}\|_{\max} = o_P(1) \). Cai et al. (2011) pointed out that the optimization problem of the CLIME method can be split into sub-problems which lead to error bounds with respect to the \( \| \cdot \|_1 \) norm without the use of thresholding. Han et al. (2015) followed this idea for constructing an estimator of the transposed matrix, leading to the error bound \( \|A_1, \ldots, A_p\|_{\max} = o_P(1) \). Wu et al. (2016) generalized the approach of Han et al. (2015) to VAR(p) processes and to possible non-Gaussian time series. In this context, Wu et al. (2016) obtained the result \( \|A_1, \ldots, A_p\|_{\max} = o_P(1) \). The same result also can be established for the Lasso. Hence, in order to obtain a consistent estimator for the sparsity pattern \( M^{(2)} \) adopted in this paper, we propose to threshold an estimator which fulfills \( \|A_1, \ldots, A_p\|_{\max} = o_P(1) \). Toward this, we use the class of thresholding functions given by Cai and Liu (2011). In particular, we require that a thresholding function \( \text{THR}_{\lambda_n} : \mathbb{R} \rightarrow \mathbb{R} \) at threshold level \( \lambda_n \) satisfies the following three conditions:
1. THR\textsubscript{\text{\(\lambda\)}}(z) \leq c|y|$ for all $z, y$ satisfying $|z - y| \leq \lambda_n$ and some $c \in (0, \infty)$.

2. THR\textsubscript{\text{\(\lambda\)}}(z) = 0 for $|z| \leq \lambda_n$.

3. $|\text{THR}_{\text{\(\lambda\)}}(z) - z| \leq \lambda_n$, for all $z \in \mathbb{R}$.

For a matrix $A$ with elements $a_{i,j}$, we set $\text{THR}_{\text{\(\lambda\)}}(A) := (\text{THR}_{\text{\(\lambda\)}}(a_{i,j}))_{i,j}$, which means that thresholding is applied component-wise. These conditions are satisfied among others by the soft thresholding, $\text{THR}^{S}_{\text{\(\lambda\)}}(z) := \text{sgn}(z)(|z| - \lambda_n)\_+$ and the adaptive Lasso thresholding, $\text{THR}^{A}_{\text{\(\lambda\)}}(z) = z \max(0, 1 - |\lambda_n/z|^{\nu}$, with $\nu \geq 1$; see Cai and Liu (2011) and Figure 1 in Rothman et al. (2009) for an illustration of the different thresholding operations. Note that the hard thresholding, $\text{THR}^{H}_{\text{\(\lambda\)}}(z) := z\text{1}_(|z| > \lambda_n)$, does not fulfill condition 1) above. The following theorem shows that this thresholding strategy succeeds in obtaining estimators which are consistent with respect to all matrix norms $\| \cdot \|$ for all $l \in [1, +\infty]$.

**Theorem 1.** Let $(A_1, \ldots, A_p) \in \mathcal{M}(q, s, M, p)$ and assume that $(\hat{A}_1, \ldots, \hat{A}_p)$ is an estimator which fulfills on a subset $\Omega_n$ of the sample space, the following condition

$$\max_{1 \leq s \leq p} \| A_s - \hat{A}_s \|_{\text{max}} \leq C_1 t_n.$$  

Then it holds true on the same subset $\Omega_n$ with thresholding parameter $\lambda_n = C_1 t_n$, that

$$\| A - \text{THR}_{\text{\(\lambda\)}}(\hat{A}) \|_l \leq (4 + c)C_1^{1-q} s t_n^{1-q},$$

for all $l \in [1, \infty]$. In the above expression, $c$ is a constant which depends on the particular thresholding function used.

Since $\| \cdot \|_{\text{max}} \leq \| \cdot \|_2 \leq \| \cdot \|_1$, the error bounds given in Section 2 are (not necessarily sharp) error bounds for the element-wise error based on the $\| \cdot \|_{\text{max}}$ norm. Furthermore, the aforementioned relation between the matrix norms implies that the row-wise Lasso, which is obtained as

$$\hat{\beta}^{(\text{RoLa})}_j = \text{argmin}_{\beta \in \mathbb{R}^p} 1/(n - p)\| Ye_j - X\beta \|_2 + \lambda_n \| \beta \|_1,$$

as well as the Dantzig estimator for VAR($p$) models, that both estimators fulfill the assumptions of Theorem 1. That is, both estimators can be used to obtain via thresholding, a row- and column-wise consistent estimator of the VAR parameter matrices $A_s$, $s = 1, 2, \ldots, p$.

For the Lasso estimator, we can use the results of Masini et al. (2019), since the sparsity setting described in (13) is covered by the sparsity setting used by these authors. Their results lead to the following error bound for the row-wise lasso, on a set with high probability,

$$\| \hat{\beta}^{(\text{RoLa})}_j - \beta_j \|_{\text{max}} \leq C_\tau \| Y^{(\text{at})} (0) - 1 \|_2^{(2-q)/2} \sqrt{s} \sqrt{g(p, d, n, \tau)/\sqrt{n}} \sqrt{\tau/2},$$

where $g(p, d, n, \tau) = (d^2pn)^{2/\tau}$, $\tau$ denotes the number of finite moments of the innovations, i.e.,

$$\max_{1 \leq \| \nu \|_1 \leq 1} (E|\nu|^{\tau})^{1/\tau} \leq c_\tau \leq \infty, \tau > 4,$$

and $C_\tau$ denotes a constant depending on $c_\tau$ and $\tau$. Notice that $g(p, d, n, \tau) = \log(dp)$ in the case of sub-Gaussian innovations where all moments exist. In both
cases $C_\tau$ depends among others on $\|\Sigma_\tau\|_2$. Thus, Theorem 4 leads for $l \in [1, \infty]$ to the following bound for the thresholded, row-wise lasso estimator,

$$\|A - \text{THR}_{\lambda_n}(\hat{A}^{(\text{RoLa})})\|_l = O_P\left(\|\Sigma_\tau\|^{1-q}_2\|\Gamma(0)\|^{-1}_2\|\epsilon^{(2-q)}_t(1-q)/2\cdot s_{1+(1-q)/2}^\tau\sqrt{n}\right)^{1-q}/2.$$  

For the Dantzig estimator, an error bound with respect to the $\| \cdot \|_{\text{max}}$ norm can be derived directly and without imposing any sparsity constrains. The Dantzig estimator, $\hat{B}^{(\text{Dantzig})}$, is given by

$$\hat{B}^{(\text{Dantzig})} = \arg\min_{B \in \mathbb{R}^{d \times q}} \sum_{j=1}^d \|B\|_1 \text{ s.t. } \|X^\top X/(n-p)B - X^\top Y/(n-p)\|_{\text{max}} \leq \lambda_n,$$  

with $X$ and $Y$ defined as in (11). Notice that (20) or (21), respectively, is a VAR($p$) version of the estimator given in [11]. Cai et al. (2011) pointed out that this optimization problem can be split into sub-problems such that parallel-processing can be used to speed up computation. Hence, an estimator also is given by $\hat{B}^{(\text{Dantzig})} = (\hat{\beta}_1^{(\text{Dantzig})}, \ldots, \hat{\beta}_d^{(\text{Dantzig})})$, where

$$\hat{\beta}_j^{(\text{Dantzig})} = \arg\min_{\beta \in \mathbb{R}^d} \|\beta\|_1 \text{ s.t. } \|X^\top X/(n-p)\beta - X^\top Y\epsilon_j/(n-p)\|_{\text{max}} \leq \lambda_n, \quad j = 1, 2, \ldots, d.$$  

To discuss the bounds obtained for different estimators, we fix the following notation. Let

$$D_{1,n} = \sqrt{\log(dp)} \bigg/ \sqrt{n-p} + \frac{(dp)^{4/\tau}}{(n-p)^{1-2/\tau}},$$  

and

$$D_{2,n} = \sqrt{\log(dp)} \bigg/ \sqrt{n-p} + \frac{(dp)^{1/\tau}}{(n-p)^{1-1/\tau}},$$  

where $\tau > 0$ is some constant depending on the moments of the innovations $\epsilon_t$.

If $\{\epsilon_t\}$ is an i.i.d. sequence with max$_{\|v\|_2 \leq 1} E(v^\top \epsilon_0)^\tau =: C_{\epsilon, \tau} < \infty$ for $\tau > 2$, then Wu et al. (2016) showed for the estimator (21) the following error bound

$$P\left(\|B\epsilon_j - \hat{\beta}_j^{(\text{Dantzig})}\|_{\text{max}} \leq 2\|\Gamma^{(s)}(0)^{-1}\|_1\left(\sum_{j=0}^\infty \|A^1\|_2^2C_{\epsilon, \tau}\right)^2(D_{1,n}M + D_{2,n})\right)$$  

$$\geq 1 - \frac{dp(n-p)^{1-\tau}}{D_{1,n}^2} - dpC_{\epsilon, \tau}^{(s)}(n-p)D_{1,n}^2$$  

$$= p^{(\text{Dantzig})_n}.$$  

Here $C_{\epsilon, \tau}^{(s)}$ and $C_{\epsilon, \tau}^{(s)}^{(s)}$ are constants depending on $\tau$ only and $\Gamma^{(s)}(0) = \text{Var}(X_p, \ldots, X_1) = \text{Var}(W_1)$, is the lag zero autocovariance of the stacked VAR(1) model. Notice that the error bound (24) refers to the case in which the innovations possess only a finite number of moments and a key ingredient in its derivation is Nageev’s inequality, which Wu et al. (2016) generalized for dependent sequences of random variables. The same authors also obtain an error bound if all moments of the innovations $\epsilon_t$ are finite. In this case a sharper bound can be obtained where polynomial terms do not occur and the exponential term depends on the tail behavior of the distribution of the innovations. For the sake of an easy presentation, we

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do not discuss this case here but we will come back to it later on. As already mentioned, \cite{wu2016estimation} derived this bound without imposing specific assumptions on the underlying sparsity setting. For the sparsity setting used in this section, this bound can be improved. The new bound which is also obtained using Theorem 1 is stated in the following Corollary 2. This corollary states that a stable VAR(p) model can be estimated consistently, with respect to all norms \( \| \cdot \|, l \in [1, \infty] \), by a thresholded Dantzig estimator in a row- and column-wise approximately sparsity setting and with (possibly) non-Gaussian innovations.

**Corollary 2.** Let \((A_1, \ldots, A_p) \in \mathcal{M}(q, s, M, p)\) and \(\{\varepsilon_t\}\) be an i.i.d. sequence with finite \(\tau > 2\) moments, i.e., \(\max_{\|v\|_2 = 1} (E(v^\top \varepsilon_0)^q)^{1/\tau} =: C_{\varepsilon, \tau} < \infty\). Furthermore, let \(\lambda_n = C(\sum_{j=0}^{\infty} \|A^j\|_2 C_{\varepsilon, \tau})^2 D_{2, n}\) for some constant \(C > 0\), be the tuning parameter for \(\hat{B}^{(\text{Dantzig})}\) and denote its thresholded version by \((\hat{A}_1^{(\text{TD})}, \ldots, \hat{A}_p^{(\text{TD})}) = \text{THR}_{\lambda_n}(\hat{B}^{(\text{Dantzig})})^\top\). Then it holds true on a set with probability equal or higher to \(p_n^{(\text{Dantzig})}\), where

\[
p_n^{(\text{Dantzig})} := 1 - \frac{d^2 p^2 (n-p)^{1-\tau}}{D_{2, n}^2} - d^2 p^2 e^{C^{(\text{w})} (n-p) D_{2, n}^2} - \frac{d^2 p^2 (n-p)^{1-\tau/2}}{D_{1, n}^2} - \frac{d^2 p^2 e^{C^{(\text{w})} (n-p) D_{1, n}^2}}{D_{1, n}^2},
\]

we have that

\[
\|\Gamma^{(st)}(0) - \mathcal{X}^\top \mathcal{X}/N\|_{\text{max}} \leq \left( \sum_{j=0}^{\infty} \|A^j\|_2 C_{\varepsilon, \tau} \right) D_{1, n} \leq \lambda_n^{2^{3/\tau}} p^{3/\tau} (n-p)^{1/\tau},
\]

\[
\|\mathcal{X}^\top \mathcal{E}/N\|_{\text{max}} \leq \left( \sum_{j=0}^{\infty} \|A^j\|_2 C_{\varepsilon, \tau} \right) D_{2, n} \leq \lambda_n,
\]

\[
\|B - B^{(\text{Dantzig})}\|_{\text{max}} \leq \|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n \left( D^3 N \right)^{1/\tau} \|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n \left( 1 + M(D^3 N)^{1/\tau} \right)^{1-q} \times \left( 1 + 2^{1-q} + 3^{1-q} s + 2 \right),
\]

\[
= O_P \left( \|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n \left( D^3 N \right)^{1/\tau} \right),
\]

if \(\|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n M(D^3 N)^{1/\tau} = o_P(1)\), \(D = dp\) and \(N = n-p\). Furthermore, using the same notation, we have that it holds true for all \(l \in [1, \infty]\) and on the same set as above, that

\[
\|A_T - \hat{A}_T^{(TD)}\|_1 \leq (4 + c) s \|B - B^{(\text{Dantzig})}\|_{\text{max}}^{1-q} \leq (4 + c) s \left( \|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n \left( D^3 N \right)^{1/\tau} \|\Gamma^{(st)}(0)^{-1} \|_1 \lambda_n \left( 1 + M(D^3 N)^{1/\tau} \right)^{1-q} \times \left( 1 + 2^{1-q} + 3^{1-q} s + 2 \right) \right)^{1-q},
\]

where \(c\) is a constant which depends on the thresholding operation used.

In some applications of VAR models, estimation of the covariance matrix of the innovations is also required. Given some estimators \((\hat{A}_1, \ldots, \hat{A}_p)\), estimates of the innovations can be obtained as \(\hat{\varepsilon}_t = X_t - \sum_{s=1}^p \hat{A}_s X_{t-s}, t = p + 1, \ldots, n\). For simplicity, we omit the centering of the residuals \(\hat{\varepsilon}_t\), but we recommend to use it in practise. To obtain an estimator of the innovations covariance matrix, several
Corollary 3. The CLIME estimator of $\Sigma^{-1}$ with tuning parameter $\lambda_n$ is given by

$$
\hat{\Sigma}^{(CLIME)} = \arg\min_{\beta \in \mathbb{R}^d} \|\beta\|_1 \text{ s.t. } \|1/(n-p) \sum_{t=p+1}^n \xi_t \beta - \xi_t\|_{\max} \leq \lambda_n, \quad j = 1, \ldots, p.
$$

Since the estimated innovations $\hat{\xi}_t$ are used instead of the true ones, an additional estimation error may occur which depends on the behavior of the particular estimators ($\hat{A}_1, \ldots, \hat{A}_p$) used. In particular we have

$$
\|1/(n-p) \sum_{t=p+1}^n \xi_t \beta - \xi_t\|_{\max} \leq \|\hat{A}_n - \hat{A}\|_1 \left(2\|X^T\mathcal{E}/(n-p)\|_{\max} + \|\mathcal{E}\|_{1}(0) - \mathcal{X}^T\mathcal{X}/(n-p)\|_{\max}\right).
$$

Corollary 3 below gives the error bound obtained when the estimator ($\hat{A}^{(TD)}_1, \ldots, \hat{A}^{(TD)}_p$) discussed in Corollary 2 is used. Notice that for Gaussian innovations we have $\|\Sigma - 1/(n-p) \sum_{t=p+1}^n \xi_t \beta\|_{\max} = O_p(\sqrt{\log(d)/n})$. For non-Gaussian innovations and using the results already presented, we have on a set with probability of at least $p_n^{(Dantzig)}$ and using (25), that the bound

$$
\|\hat{\Sigma} - 1/(n-p) \sum_{t=p+1}^n \xi_t \beta\|_{\max} \leq C_{1,\tau}^2 \left(\frac{\log(d)}{n-p} + \frac{d^{1/\tau}}{(n-p)^{1-2/\tau}}\right).
$$

Note that $1/(n-p) \sum_{t=p+1}^n \xi_t \beta = \mathcal{E} \mathcal{E}^T/(n-p)$ and $\mathcal{E}$ takes the role of $\mathcal{X}$ for $\hat{A}_n \equiv 0$. This means that the fact that estimated residuals are used instead of the true innovations, affects the corresponding convergence rate only if the bound in (32) is larger than the bound in (33). For $q < 1/2$, (33) is usually larger. It also depends, therefore, on the underlying sparsity setting; see Corollary 3 below.

**Corollary 3.** Under the assumptions of Corollary 2, we have on a set with probability of at least $p_n^{(Dantzig)}$, that for $\hat{\xi}_t = X_t - \sum_{s=1}^p \hat{A}^{(TD)}_{t-s} X_{t-s}, t = p + 1, \ldots, n,

\begin{align*}
\|1/(n-p) \sum_{t=p+1}^n \hat{\xi}_t \beta^T - 1/(n-p) \sum_{t=p+1}^n \xi_t \beta^T\|_{\max} & \leq 2(4 + c)^2\|\mathcal{E}(0)\|_{\max}s^2 \\
& \times \left(2\|\mathcal{E}(0)\|_{1}(\sum_{j=0}^{\infty} \|\hat{A}_j\|_2C_{1,\tau})^2 [D_{1,n} M + D_{2,n}]^{-1}\right)^{2(1-q)}.
\end{align*}

**Theorem 4.** Under the assumptions of Corollary 3 and if $\Sigma \in M(q^2, s, M, 1)$, we have on a set with probability of at least $p_n^{(Dantzig)}$,

\begin{align*}
\|\text{THR}_{\lambda_n}(1/(n-p) \sum_{t=p+1}^n \hat{\xi}_t) - \Sigma\| & \leq (4 + c)s \left(C_{1,\tau}^2 \left(\frac{\log(d)}{n-p} + \frac{d^{1/\tau}}{(n-p)^{1-2/\tau}}\right) + 2(4 + c)^2\|\mathcal{E}(0)\|_{\max}s^2\right) \left(2\|\mathcal{E}(0)\|_{1}(\sum_{j=0}^{\infty} \|\hat{A}_j\|_2^2 [D_{1,n} M + D_{2,n}]^{-1}\right)^{2(1-q)}
\end{align*}

for all $l \in [1, \infty]$, where $\hat{\xi}_t = X_t - \sum_{s=1}^p \hat{A}^{(TD)}_{t-s} X_{t-s}, t = p + 1, \ldots, n$. 


Theorem 4 follows directly from Theorem 1, equation (33), and Corollary 3.

In the remaining of this section, we propose estimators of the autocovariance function (2) and of the spectral density, more precisely of the inverse of the spectral density matrix of the VAR model; see (3). Regarding the autocovariance function the following expression is useful for our derivations,

\[
\Gamma(h)^{(st)} = \begin{cases} 
\sum_{j=0}^{\infty} A^j \Sigma_U (A^j)^\top & \text{for } h \geq 0, \\
(\Gamma(-h)^{(st)})^\top & \text{for } h < 0,
\end{cases}
\]

where \( \Sigma_U = E \Sigma E^\top \). Since \( \Gamma(h) = E^\top \Gamma(h)^{(st)} E \), an error bound for \( \Gamma^{(st)}(h) \) leads to an error bound for \( \Gamma(h) \).

**Theorem 5.** Let \( \hat{A} \) be some estimator of \( A \), \( \hat{\Sigma}_e \) some estimator of \( \Sigma_e \), and \( \hat{\Gamma}(h)^{(st)} \) the analogue of \( \Gamma(h)^{(st)} \) with \( A \) and \( \Sigma_e \) replaced by \( \hat{A} \) and \( \hat{\Sigma}_e \). Furthermore, for any sub-multiplicative matrix norm \( \| \cdot \| \), let \( \sum_{j=0}^{\infty} \| A^j \|^2 = C_{\gamma,A} \sum_{j=0}^{\infty} \| (A^j)^\top \|^2 = C_{\gamma,A^\top} \sum_{j=0}^{\infty} \| (A^j)^\top \|^2 = C_{\gamma,A^\top} \) and \( \| \Sigma_e \| = C_{\gamma,\Sigma_e} \). Then, for \( h \geq 0 \)

\[
\| \hat{\Gamma}(h) - \Gamma(h) \| \leq \| \hat{A} - A \| (C_{\gamma,A} + C_{\gamma,A^\top})\| \Gamma^{(st)}(0) \| \\
+ \| \hat{\Sigma}_e - \Sigma_e \| (C_{\gamma,A} + C_{\gamma,A^\top})/2 + \| \hat{\Sigma}_e - \Sigma_e \| (C_{\gamma,A} + C_{\gamma,A^\top})/4.
\]

For \( \| \hat{A} - A \| \) small, we have \( \sum_{j=0}^{\infty} \| A^j \| \leq \sum_{j=0}^{\infty} \| A^j \|/(1 - \| \hat{A} - A \| \sum_{j=0}^{\infty} \| A^j \|) \). This means that \( C_{\gamma,A} \) and \( C_{\gamma,A^\top} \) can be bounded by \( C_{\gamma,A} \) and \( C_{\gamma,A^\top} \), respectively, and Theorem 5 implies that \( \hat{\Gamma}(h) \) is a consistent estimator for \( \Gamma(h) \) and that

\[
\| \hat{\Gamma}(h) - \Gamma(h) \|_\infty = O_P \left( \left( \sum_{j=0}^{\infty} \| A^j \|_1 + \sum_{j=0}^{\infty} \| A^j \|_\infty^2 \right) \| \Sigma_e \|_1 \left( \| \hat{A} - A \|_1 + \| \hat{A} - A \|_\infty + \| \Sigma_e - \Sigma_e \|_1 \right) \right).
\]

Notice that the term \( \left( \sum_{j=0}^{\infty} \| A^j \|_1 + \sum_{j=0}^{\infty} \| A^j \|_\infty^2 \right) \| \Sigma_e \|_1 \) depends on the VAR process and that this term can be large. If \( (A_1, \ldots, A_p) \in M(0, s, M, p) \) and \( \Sigma_e \in M(0, s_{\varepsilon}, M_{\varepsilon}, 1) \), this term is at least of the order \( s^2 s_{\varepsilon} \). Consequently, the sparsity setting enabling a consistent autocovariance estimator with respect to the \( \| \cdot \|_\infty \) norm, is more restrictive than the sparsity setting enabling a consistent parameter estimator with respect to the same norm. In particular, if we recall the results of the Lasso estimator with Gaussian innovations and focus on sparsity and on the dimension of the system only, then we have

\[
\| \hat{\Gamma}(h) - \Gamma(h) \|_\infty = O_P \left( s^{3/5} s_{\varepsilon} \sqrt{\log(dp)/(n - p)} \right) \text{ in contrast to } \| \hat{A} - A \|_\infty = O_P \left( s \sqrt{\log(dp)/(n - p)} \right).
\]

We conclude this section with a result related to the estimation of the inverse of the spectral density matrix of the high dimensional VAR model considered.

**Theorem 6.** Let \( \hat{A} \) be some estimator of \( A \) and \( \hat{\Sigma}_e^{-1} \) some estimator of \( \Sigma_e^{-1} \). If \( (A_1, \ldots, A_p) \in M^{(2)}(q, s, M, p), \Sigma_e^{-1} \in M(q_{\varepsilon}^{-1}, s_{\varepsilon}^{-1}, M_{\varepsilon}^{-1}, 1) \) and for \( l \in [1, \infty], \sum_{s=1}^{p} \| \hat{A}_s - A_s \|_l \leq t_{n,1} \) and \( \| \Sigma_e^{-1} - \hat{\Sigma}_e^{-1} \|_l \leq t_{n,2} \), then

\[
\| f^{-1}(\omega) - \hat{f}^{-1}(\omega) \|_l \leq 2 M M_{\varepsilon}^{-1} t_{n,1} + M^2 t_{n,2} + 2 M t_{n,1} t_{n,2} + t_{n,1}^2 M_{\varepsilon}^{-1} + t_{n,1}^2 t_{n,2},
\]

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where
\[
\hat{f}^{-1}(\omega) = (I_d - \sum_{s=1}^{p} \hat{A}_s \exp(is\omega))^\top \Sigma_e^{-1} (I_d - \sum_{s=1}^{p} \hat{A}_s \exp(-is\omega))
= \hat{A}(\exp(i\omega))^\top \Sigma_e^{-1} \hat{A}(\exp(-i\omega)).
\] (37)

We stress here the fact that for consistency of the inverse of the spectral density matrix, the more restrictive sparsity setting \([10]\) is used. If we consider again the results of the Lasso estimator with Gaussian innovations and focus on sparsity and on dimension only, we get the bound \(\|f^{-1}(\omega) - \hat{f}^{-1}(\omega)\|_\infty = O_P(s^{2.5} \epsilon^{-1} \sqrt{\log(dp)/(n - p)})\).

### 3.1 The effects of the VAR process parameters on the error bounds

In the following, we discuss the effects of the parameters \(\Sigma_e\) and \(A\) on the estimation performance of the Lasso and of the Dantzig estimator. For the vectorized Lasso estimator with Gaussian innovations, we refer to Proposition 4.3 in \([15]\). We focus here only on terms which depend on \(\Sigma_e\) and \(A\) while the effect of all other terms is summarized using the notation \(g(p, d, s, n, \tau)\), where this term may differ from equation to equation. The following error bounds are obtained. For the Lasso we have
\[
\|A - \hat{A}^{(\text{RoLa})}\|_{\text{max}} \leq C^{(\text{RoLa})}_{\text{DEP}} \|\Gamma^{(st)}(0)^{-1}\|_2^{(2-\eta)/2} g(p, d, s, n, \tau)
\]
and for the Dantzig estimator
\[
\|A - \hat{A}^{(\text{Dantzig})}\|_{\text{max}} \leq C^{(\text{Dantzig})}_{\text{DEP}} \|\Gamma^{(st)}(0)^{-1}\|_2 g(p, d, s, n, \tau).
\]

Notice that the terms \(C^{(\text{Dantzig})}_{\text{DEP}}\) and \(C^{(\text{RoLa})}_{\text{DEP}}\) appearing in these expressions may differ since \([19, 16]\) use different dependence conditions to derive their concentration inequalities and the corresponding error bounds. If the error bounds for the Lasso are derived under the dependence conditions used in \([16]\), i.e., under physical dependence, then \(C^{(\text{RoLa})}_{\text{DEP}}\) can be chosen such that it is identical up to constants to \(C^{(\text{Dantzig})}_{\text{DEP}}\). Here we focus on the bound derived under the physical dependence condition. In this case we get that
\[
C^{(\text{Dantzig})}_{\text{DEP}} \leq C(\sum_{j=0}^{\infty} \|A_j\|_2^2) \|\Sigma_e\|_2 \max_{\|v\|_2 = 1} (E|v^\top \Sigma_e^{-1/2} \epsilon_1\|_2^\top|_2^{2/\tau}).
\]

Furthermore, we have by Proposition 2.3 in \([15]\),
\[
\|\Gamma^{(st)}(0)^{-1}\|_2 \leq \sup_{\omega \in [-\pi, \pi]} \|f(\omega)^{-1}\|_2 = \sup_{\omega \in [-\pi, \pi]} \|\Gamma(0) - \sum_{s=1}^{p} A_s \exp(is\omega))^\top \Sigma_e^{-1} (I_d - \sum_{s=1}^{p} A_s \exp(-is\omega))\|_2
\leq (1 + \sum_{s=1}^{p} \|A_s\|_2^2) \|\Sigma_e^{-1}\|_2.
\]

Hence,
\[
\|A - \hat{A}\|_{\text{max}} \leq C \max_{\|v\|_2 = 1} (E|v^\top \Sigma_e^{-1/2} \epsilon_1\|_2^{\top})^{2/\tau} g(p, d, s, n, \tau)
\]
\[
\times \left( \sum_{j=0}^{\infty} \| A_j \|_2 \right)^2 (1 + \sum_{a=1}^{P} \| A_a \|_2 \| \Sigma_\varepsilon \|_2 \| \Sigma^{-1}_\varepsilon \|_2^{(2-\tilde{q})/2}),
\]
where \( \tilde{q} = q \) for the Lasso and \( \tilde{q} = 0 \) for the Dantzig estimator.

The term \( \max_{|v|=1}(E|v^\top \Sigma^{-1}_\varepsilon e_1\|_2^{2/\tau}) \) depends on the distribution of the innovations. If \( \varepsilon \) is Gaussian, this quantity is not affected by \( \Sigma_\varepsilon \). Furthermore, we have \( \| \Sigma_\varepsilon \|_2 \| \Sigma^{-1}_\varepsilon \|_2 \geq \max_i e_i^\top \Sigma_\varepsilon e_i / \min_i e_i^\top \Sigma_\varepsilon e_i \).

This means that the dependence among the innovations as well as different variances between the components of the innovations vector, could have a negative effect on the behavior of the estimators.

Regarding the influence of \( A \), recall that the decay rate of \( A_j \) depends on the largest absolute eigenvalue of \( A \). Hence, if the VAR system is highly persistent, i.e., the largest absolute eigenvalue of \( A \) is close to one, then the constant \( \sum_{j=0}^{\infty} \| A_j \|_2 \) can be large.

## 4 Numerical results

In this section, we investigate by means of simulations, the finite sample performance of the estimation procedures discussed, i.e., of the lasso estimator (6), of the row wise lasso (8) and of the Dantzig estimator (21). We denote the estimator (6) by Vec-Lasso, the estimator (19) by Row-Lasso, and the estimator (21) by Row-Dantzig. All results presented are based on implementations in R [R Core Team, 2019].

To compute the Lasso estimators (6) and (19), we use the package glmnet [Simon et al., 2011]. Vec-Lasso uses as weighting matrix the inverse of the estimated innovation variance based on the estimated residuals of the Row-Lasso estimator. For the Dantzig estimator (21), we use the package fastclime [Pang et al., 2016]. It is worth mentioning here that the estimation procedures using the aforementioned implementations highly differ with respect to computing time. For instance, in order to estimate a VAR(1) model of dimension \( d = 100 \) using \( n = 100 \) observations and without parallel computing on a personal computer, Row-Lasso requires approximately 5 seconds, Vec-Lasso approximately 4.3 minutes, and Row-Dantzig approximately 16 minutes. More advanced techniques in linear programming may speed up the computation of the Dantzig estimator; see for instance Mazumder et al. (2019).

The three estimators considered are used plain as well as with the following three modifications:

**S**: We standardize all input time series, i.e., we insert a weighting matrix \( W \) in (5) where \( W \) is a diagonal matrix with diagonal entries given by the estimated standard deviations of each time series. We then apply each estimation procedure on the transformed data \( \tilde{Y}W^{-1} = X(I_p \otimes W)^{-1}(I_p \otimes W)BW^{-1} + \tilde{E}W^{-1} = \tilde{X}\tilde{B} + \tilde{\varepsilon} \) and transform the obtained estimates \( \tilde{B} \) back using \( \tilde{B} = (I_p \otimes W)^{-1}\tilde{B}W \).

**A**: We apply a second adaptive step, i.e., we run the estimators twice and in the second run we insert penalty weights. For the coefficient \( B_{i,j} \), we use in a second round the penalty \( 1/(|\hat{B}_{i,j}^{(1)}| + 1/\sqrt{n}) \), where \( \hat{B}_{i,j}^{(1)} \) denotes the estimated coefficients obtained in the first round.

**T**: We threshold the estimates, i.e., the final estimate is obtained by \( \text{THR}_{\lambda_n}(\hat{B}) \). Here, we use the adaptive thresholding, that is, \( \text{THR}_{\lambda_n}(z) = z(1 - |\lambda_n/z|^\nu)_+ \) for \( \nu > 3 \).
A combination of the three aforementioned modifications also can be used and the particular modification applied is denoted by capitalized letters. For instance, the notation Row-Lasso SA means that Row-Lasso is used with standardized time series and a second adaptive step. Notice that Vec-Lasso possesses one tuning parameter, while Row-Lasso and Row-Dantzig possess \( d \) tuning parameters, i.e., one for each row. These tuning parameters are selected using the Bayesian Information Criterion (BIC). Additionally, the Extended Regularized Information Criterion (ERIC), see Hui et al. (2015), is used as a second option to select the tuning parameters.

To evaluate the performance of the different estimation procedures compared, we use the following quantities:

i) \( \| A - \hat{A} \|_\infty \), i.e., the estimation error for the parameter matrix \( A \) with respect to the \( \| \cdot \|_\infty \) matrix norm.

ii) \( \| \hat{\Gamma}^{(st)}(0) - \Gamma^{(st)}(0) \|_\infty / \| \Gamma^{(st)}(0) \|_\infty \), i.e., the relative estimation error for the lag zero autocovariance with respect to the \( \| \cdot \|_\infty \) matrix norm.

iii) \( \int \| f(\omega) - \hat{f}(\omega) \|_\infty d\omega / \int \| f(\omega) \|_\infty d\omega \), i.e., the relative integrated estimation error for the spectral density matrix with respect to the \( \| \cdot \|_\infty \) matrix norm. In our calculations, integrals are approximated by sums over the corresponding Fourier frequencies.

iv) \( 1/d \sum_{j=1}^d MSE(\hat{X}_{n+h,j}) / \sigma_j^2 \), where \( \sigma_j = \sqrt{\text{Var}(\hat{\epsilon}_{1,j})} \) and \( \hat{X}_{n+h,j} \) denotes the forecast of the \( j \)th element of \( X_{n+h} \) using \( \hat{A} \) and \( X_1, \ldots, X_n \). That is, the averaged forecast error is computed which is measured by the mean squared error for the forecasting horizon \( h \). The mean squared error is estimated using 1000 Monte Carlo runs.

In order to estimate the second-order characteristics, i.e., \( \Gamma(0) \) and \( f \), we need to estimate the innovations variance \( \Sigma_e \). For this we use the estimator (30) and the implementation given in the package FinCovRegularization (Yan and Lin, 2016), which uses cross-validation to select the threshold parameter.

Additionally to the comparison of the different estimators, we also investigate the influence of the data generating process on the performance of the estimators. For this, we consider two groups of examples. In the first group, we vary the variance matrix of the innovations and keep everything else fixed. In the second group, we vary the dimension, the sparsity and the persistence of the processes but keep the variance matrix of the innovations fixed.

The data generating processes in the first group of examples are different VAR(4) processes. These processes are of dimension \( d = 14 \) and the same parameter matrix \( A \in \mathcal{M}^{(2)}(0, 5, 17, 4) \) is used, with largest absolute eigenvalue equal to 0.8. The innovations are Gaussian and four different variance matrices are considered: a diagonal matrix with homogeneous variances among the components (denoted as DM), i.e., \( \Sigma_e = I_d \), a diagonal matrix with heterogeneous variances among the components (denoted as DT), \( \Sigma_e = \text{diag}(1.88 \times 10^{-02}, 2.61 \times 10^{-03}, 4.40 \times 10^{-03}, 3.04 \times 10^{-06}, 1.58 \times 10^{-06}, 3.99 \times 10^{-03}, 1.51 \times 10^{-05}, 2.51 \times 10^{-05}, 1.34 \times 10^{-06}, 1.03 \times 10^{-02}, 4.32 \times 10^{-03}, 9.77 \times 10^{-06}, 3.93 \times 10^{-05}, 2.03 \times 10^{-06}) \), a non-sparse variance matrix with homogeneous variances among the components and largest eigenvalue of
2.5 and smallest of 0.21 (denoted as FM), and the same non-sparse variance matrix but now with heterogeneous variances among the components as given in the second matrix leading to a largest eigenvalue of $1.92 \times 10^{-2}$ and a smallest eigenvalue of $4.45 \times 10^{-7}$ (denoted as FT).

The data generating processes in the second group of examples are VAR(1) processes. The processes are of different dimensions $d = 10, 25, 50$, and 100, the innovations are Gaussian with $\Sigma_\varepsilon = I_d$ and the parameter matrix $A$ is generated randomly with row- and column-wise maximal sparsity given by $s = 1, 3, 5, 10$, i.e., $A \in \mathcal{M}(0, s, \tilde{M}, 1)$, where $\tilde{M}$ may differ from matrix to matrix, and the largest absolute eigenvalue takes the values $\rho = 0.6, 0.8, 0.9, 0.95$. The random generation of $A$ is done in four steps. First, a random matrix with largest absolute eigenvalue less than one is generated. Second, the $d - s$ smallest coefficients in absolute value within each row are set equal to zero. Third, the $d - s$ smallest coefficients in absolute value within each column are set equal to zero. Finally, the obtained sparse matrix is rescaled so that its largest absolute eigenvalue equals one. The $d - s$ coefficients in absolute value within each row are set equal to zero. If in the fourth step no scaling is possible, i.e., the eigenvalue of the obtained sparse matrix is zero, then we set $e_1^T A e_1 = \rho$ and we rescale the matrix.

A summary of the results obtained are shown in Table 1 to Table 5. Table 1 presents results for Example 1 and for all four performance criteria used. Table 2 to Table 5 present the results for Example 2 and for each one of the four different performance criteria separately. For all three estimators considered, standardizing the input leads in most cases to a better performance of the estimator. Furthermore, including a second adaptive step also improves the performance of the estimators in most of the cases considered. Additional thresholding has in most cases no negative effect on the performance of the estimators. For this reason we present for Example 1 only the estimates obtained after applying all modifications discussed while for Example 2 we focus on the estimates with standardized inputs, a second adaptive step, and (an optional) thresholding. Furthermore, Vec-Lasso performs much better with ERIC than with BIC and for this reason we focus on this selection rule only for this estimator applied in Example 2. In the following we discuss the results obtained separately for Example 1 and for Example 2.

### 4.1 Example 1

As mentioned, the underlying processes of this example are VAR(4) processes with four different variance matrices for the Gaussian innovations. Two of them have very heterogeneous variances among the components. The parameter matrix $B = (A_1, A_2, A_3, A_4) \in \mathbb{R}^{14 \times 56}$ has a row- and column-wise sparsity of 5 and has in total 25, non-zero coefficients. A sample size of $n = 100$ observations is used in the example. Given this sample size, 25 non-zero coefficients may be considered to be too many for the Vec-Lasso estimator to handle. More specifically, if we plug this parameter design into the corresponding error bounds, we get for Vec-Lasso the bound $\|B - \hat{B}\|_\infty \leq \|\text{vec}(B) - \text{vec}(\hat{B})\|_1 \leq 25\sqrt{\log(dp)/nC} \approx 6.5 \times C$, compared to the bound $\|B - \hat{B}\|_\infty \leq 5\sqrt{\log(d^p)/nC} \approx 1 \times C$, for Row-Lasso and Row-Dantzig. Notice that the error bounds for Vec-Lasso are derived, using $\|\text{vec}(B) - \text{vec}(\hat{B})\|_1$ and that $\|B - \hat{B}\|_\infty$ could be
substantially smaller.

In Section 3.1 we mentioned that the estimation error of the parameter matrix $A$ can be bounded among others by $\|\Sigma \varepsilon\|_2 \|\Sigma^{-1} \varepsilon\|_2$. This differs highly between the heterogeneous cases, in which we have $\|\Sigma \varepsilon\|_2 \|\Sigma^{-1} \varepsilon\|_2 > 10^4$, and the homogeneous cases, in which we have $\|\Sigma \varepsilon\|_2 \|\Sigma^{-1} \varepsilon\|_2 < 10^2$. This means that $\|\Sigma \varepsilon\|_2 \|\Sigma^{-1} \varepsilon\|_2$ is at least 100 times higher for the heterogeneous cases than for the homogeneous ones, and we see this difference also in the performance of the estimators. For all estimation procedures considered, the estimation error of the parameter matrix $A$, i.e., criterion i), is considerably higher (up to factor 10) for the heterogeneous cases than for the homogeneous ones. Interestingly, we observe this only for the estimation error $\|A - \hat{A}\|_\ell$. For the second-order properties as well as for forecasting, the corresponding errors are affected much less by the heterogeneity of the variance of the innovations.

For Vec-Lasso we observe, that standardizing the input (S) greatly improves its performance. Furthermore, a second adaptive step (A) is also very beneficial. However, additional thresholding (T) has almost no effect. Furthermore, Vec-Lasso performs better with ERIC than with BIC. For the heterogeneous cases, i.e., FT and DT, Vec-Lasso SA ERIC is among the best ones with respect to all four evaluation criteria [i] to [iv], previously discussed. For the homogeneous case, Vec-Lasso SA ERIC performs good in forecasting, i.e., with respect to criterion [iv], but considerably worse in estimating the second-order characteristics of the VAR process, i.e., with respect to criteria [ii] and [iii].

For Row-Lasso, a second adaptive step (A) improves the performance. Regarding the estimation of the second-order properties, criteria [ii] and [iii], standardizing the input (S) is beneficial for all cases. When it comes to forecasting, standardizing is only beneficial for the heterogeneous cases. Row-Lasso performs better with BIC than with ERIC. Additional thresholding (T) seems to affect the performance only slightly with no clear tendency. For the estimation of the second-order properties, Row-Lasso SA BIC performs close to the best one in all cases. For forecasting, Row-Lasso SA BIC is close to the best one in the heterogeneous cases and Row-Lasso A BIC performs close to the best one in the homogeneous cases.

The combination of standardizing the input (S) and a second adaptive step (A) greatly improves the performance of Row-Dantzig. Again, BIC is here the better option than ERIC. Additional thresholding (T) has almost no effect but in some cases it brings some improvements. Row-Dantzig TSA BIC is not among the best estimates of the parameter matrix $A$ itself, i.e, for criterion [ii], but it is best or close to the best one in all cases for the estimation of the second-order properties as well as for forecasting, i.e., for criteria [iii], [iii], and [iv].

4.2 Example 2

We focus on the results of the thresholded estimators with standardized input, a second adaptive step, i.e., on estimators denoted by TSA. Notice that the results presented in Table 2 to Table 5 give the relative performances of the different estimates. That is, for each case considered, the results of each estimators are divided with those of the best performing estimator. Hence the closer to one is the corresponding
entry in the tables, the closer to the best performing estimator is the particular estimator. Additionally, and in order to also have the information related to the level of its performance, we denote for the best performing estimator and instead of 1.00, the absolute value of its score in brackets.

For the estimation or the parameter matrix $A$ itself and over all considered settings, Vec-Lasso TSA ERIC is best with Row-Lasso BIC TSA performing very close to the best. Overall the performance of Row-Lasso BIC TSA is no more that of 3% worse than that of the best performing estimator. When the dimension of the VAR model is small, additional thresholding could have a negative effect on the performance whereas for large dimensions it is the other way around. Here Row-Dantzig and Row-Lasso perform much better with BIC than with ERIC. The estimation error of the best performing procedures increases with dimension and decreases with increasing persistence. A change in the sparsity levels seems to have a minor affect on performance.

For the estimation of the second-order properties, i.e., for criteria ii) and iii), Row-Lasso TSA BIC performs very good for all persistence, all sparsity levels and for all dimensions considered. For Row-Lasso and Row-Dantzig using ERIC seems to lead to worse results compared to those obtained using BIC. The best Vec-Lasso and Row-Dantzig estimates perform usually more than 10% to 20% worse than the estimates Row-Lasso TSA BIC. In the case of a strong persistence level ($\rho = 0.95$), Row-Dantzig may lead to unstable results, that is, the modulus of the largest absolute eigenvalue of the estimated parameter matrix of VAR model may be greater than one. No correction to stability is used here and therefore, these results lead to estimates of the second-order characteristics which are not satisfactory. The performance of Row-Dantzig seems to get worse with increasing dimension whereas no clear tendency can be observed with respect to the sparsity and to the different persistence levels considered.

Regarding forecasting, both selection options, i.e., BIC and ERIC, lead for Row-Lasso and Row-Dantzig to good results. The performance of the three estimation methods considered differs only slightly. Row-Lasso SA ERIC performs best with Vec-Lasso SA ERIC and Row-Dantzig TSA ERIC being very close to the best performance. The difference between Vec-Lasso and Row-Lasso increases with increasing dimension and persistence level, whereas no clear tendency can be identified for the differences between Row-Dantzig and Row-Lasso. Here, thresholding has a slight negative effect on the performance. Note, however, that in contrast to estimating second-order characteristics, thresholding is not necessary. Therefore, the available theoretical results for the estimators considered, justify their application to forecasting without the need for the use of an additional thresholding step. The forecast error of the best procedures increases slightly with dimension and with persistence. A change in the sparsity level seems to have a rather minor affect on the results obtained.

4.3 Conclusions

If one is interested in estimating the second-order characteristics of a VAR system, Row-Dantzig seems to be a good choice for the first example, while Row-Lasso performs much better for the same estimation problem and for the second example considered. For this reason we suggest to use Row-Lasso
for this objective. Furthermore, we suggest to use Row-Lasso with the modifications TSA, i.e., with thresholding, standardizing the time series and incorporating a second adaptive step. A second adaptive step improves considerably the performance of this estimator and standardizing the time series helps to develop some robustness in the sense that the performance of this estimator is not largely affected by the particular second order characteristics of the underlying processes. As we have seen in Theorem 1, thresholding gives the theoretical justification for using Row-Lasso in order to consistently estimate the second-order characteristics of the underlying VAR process. In the examples considered, thresholding does not necessarily improve the performance of the estimator but it enlarges the range of its applicability by gaining consistency with respect to a much larger set of matrix norms. To select the tuning parameter for estimating second-order characteristics, our simulation study suggests that BIC is the better option for Row-Lasso, that is Row-Lasso TSA BIC is the suggested estimation method to use for estimating second-order properties.

If the main interest is forecasting, all three estimators perform well and there is no one which clearly outperforms the others. Since valid forecasts can be obtained with less consistency requirements on the estimators compare to those needed for consistent estimation of second order characteristics, an additional thresholding may be omitted in this case. Based on the first example, Vec-Lasso SA ERIC and Row-Dantzig TSA BIC seems to be slightly more robust than Row-Lasso TSA BIC. Our findings also suggest that if Vec-Lasso is used, then ERIC should be preferred to BIC for selecting the tuning parameter. Notice, however, that the existing theory for Vec-Lasso does not cover all sparsity settings considered in our simulation study.
|                | $\| \hat{A} - A \|_\infty$ | $\| \hat{\Gamma}(0) - \Gamma(0) \|_1 / \| \hat{\Gamma}(0) \|_1$ | $f / f(\omega) - f(\omega) \|_1 d\omega / f(\omega) \|_1 d\omega$ | $1/d \sum_{j=1}^d MSE(\hat{X}_{n+1:j})/\sigma_j^2$ |
|----------------|----------------------|----------------------|----------------------------------|----------------------------------|
| **Vec-Lasso**  |                      |                      |                                 |                                  |
| Vec-Lasso      | 14.47 14.50 14.40 14.52 | 1.07 0.67 3.91 0.65 | 0.91 0.70 0.91 0.69             | 70.96 2.35 63.31 2.47           |
| Vec-Lasso S    | 2.16 9.99 2.05 10.10   | 0.77 0.65 0.72 0.61  | 0.76 0.67 0.71 0.60             | 1.42 1.21 1.42 1.21             |
| Vec-Lasso A    | 14.49 14.47 14.40 14.47 | 0.80 0.66 0.80 0.60  | 0.82 0.69 0.82 0.64             | 38.45 2.02 35.12 2.11           |
| Vec-Lasso SA   | 1.90 9.09 1.86 9.59    | 0.74 0.56 0.67 0.50  | 0.73 0.57 0.66 0.48             | 1.27 1.11 1.25 1.12             |
| Vec-Lasso TA   | 14.49 14.47 14.40 14.47 | 0.81 1.07 0.80 0.66  | 0.83 0.68 0.82 0.63             | 39.03 1.89 35.67 1.93           |
| Vec-Lasso TSA  | 1.84 10.58 1.84 12.12  | 0.74 0.56 0.67 0.50  | 0.74 0.57 0.66 0.48             | 1.28 1.13 1.26 1.14             |
| Vec-Lasso S ERIC | 1.98 11.10 1.90 11.12  | 0.72 0.63 0.67 0.60  | 0.71 0.64 0.67 0.60             | 1.30 1.17 1.31 1.17             |
| Vec-Lasso SA ERIC | 1.62 9.29 1.59 9.54   | 0.69 0.55 0.63 0.51  | 0.68 0.56 0.63 0.50             | 1.20 1.10 1.20 1.10             |
| Vec-Lasso TSA ERIC | 1.53 8.86 1.53 9.56   | 0.70 0.55 0.64 0.48  | 0.69 0.56 0.63 0.46             | 1.20 1.11 1.20 1.11             |
| **Row-Lasso**  |                      |                      |                                 |                                  |
| Row-Lasso      | 1.94 14.53 1.74 14.54  | 0.66 0.62 0.69 0.63  | 0.69 0.67 0.71 0.68             | 1.42 2.11 1.41 2.15             |
| Row-Lasso S    | 2.12 12.33 2.05 12.15  | 0.71 0.66 0.72 0.63  | 0.71 0.68 0.72 0.63             | 1.54 1.23 1.48 1.23             |
| Row-Lasso A    | **1.24** 14.50 **1.22** 14.50 | 0.51 0.57 0.51 0.56 | 0.59 0.62 0.59 0.61             | 1.18 1.86 1.18 1.89             |
| Row-Lasso SA   | 1.49 11.46 1.39 11.65  | 0.47 0.55 0.46 0.52  | 0.48 0.58 0.47 0.53             | 1.35 1.11 1.30 1.11             |
| Row-Lasso TA   | 1.52 14.50 1.57 14.50  | 0.87 0.57 0.87 0.56  | 1.25 0.62 1.24 0.61             | 578.12 1.86 458.47 1.89         |
| Row-Lasso TSA  | 1.42 11.07 1.33 11.05  | 0.48 0.55 0.47 0.50  | 0.49 0.57 0.48 0.50             | 1.35 1.11 1.30 1.11             |
| Row-Lasso S ERIC | 2.22 56.90 2.20 60.73  | 0.68 0.76 0.69 0.82  | 0.68 0.95 0.69 1.07             | 1.57 1.97 1.51 2.03             |
| Row-Lasso SA ERIC | 1.77 60.39 1.76 62.45 | 0.49 0.95 0.48 0.94  | 0.52 1.30 0.51 1.38             | 1.43 2.00 1.39 2.06             |
| Row-Lasso TSA ERIC | 1.72 59.88 1.71 61.62 | 0.52 0.97 0.50 0.93  | 0.54 1.31 0.52 1.36             | 1.42 1.99 1.38 2.06             |
| **Row-Dantzig** |                      |                      |                                 |                                  |
| Row-Dantzig    | 2.53 14.49 2.20 14.53  | 0.60 0.53 0.63 0.47  | 0.66 0.62 0.67 0.57             | 1.49 6.08 1.45 5.47             |
| Row-Dantzig S  | 2.59 13.74 2.23 13.49  | 0.61 0.64 0.62 0.62  | 0.61 0.65 0.62 0.63             | 1.27 1.20 1.24 1.20             |
| Row-Dantzig A  | 1.69 14.49 1.56 14.52  | 0.48 **0.52** 0.48 0.45 | 0.60 0.61 0.61 0.54             | 1.22 6.08 1.19 5.47             |
| Row-Dantzig SA | 2.01 12.00 1.77 11.88  | **0.42** 0.54 **0.42** 0.50 | **0.44** 0.57 0.43 0.52 | **1.14** 1.11 **1.12** 1.10 |
| Row-Dantzig TA | 2.37 14.49 2.10 14.52  | 0.81 **0.52** 0.80 0.45 | 1.15 0.61 1.11 0.54             | 500.57 6.08 394.63 5.47         |
| Row-Dantzig TSA | 1.99 11.65 1.76 11.57  | **0.42** 0.54 **0.42** 0.48 | **0.44** 0.56 **0.42** 0.49 | **1.14** 1.10 **1.12** 1.10 |
| Row-Dantzig S ERIC | 2.67 50.81 2.33 57.45  | 0.56 0.91 0.57 0.92  | 0.57 1.20 0.58 1.32             | 1.30 1.71 1.27 1.79             |
| Row-Dantzig SA ERIC | 2.17 50.37 1.96 55.83 | 0.43 1.11 0.43 1.07  | 0.48 1.50 0.48 1.62             | 1.20 1.68 1.19 1.78             |
| Row-Dantzig TSA ERIC | 2.16 50.02 1.95 55.42 | 0.43 1.13 0.43 1.07  | 0.48 1.52 0.47 1.62             | 1.20 1.68 1.19 1.77             |

Table 1: Example 1 – VAR(4), $d = 14$, $p = 0.8$, $s = 5$, $n = 100$
| $s$ | $d$ | $ρ$ | 1    | 0.6 | 0.8 | 0.9 | 0.95 | 3    | 0.6 | 0.8 | 0.9 | 0.95 | 5    | 0.6 | 0.8 | 0.9 | 0.95 | 10   | 0.6 | 0.8 | 0.9 | 0.95 |
|-----|-----|-----|------|-----|-----|-----|------|------|-----|-----|-----|------|------|-----|-----|-----|------|------|-----|-----|-----|------|
|     |     |     |      |     |     |     |      |      |     |     |     |      |      |     |     |     |      |      |     |     |     |      |
| Vec-Lasso SA ERIC | 10  | 0.52 | 1.02 | 1.04 | 1.04 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 |
| Vec-Lasso TSA ERIC | 25  | 0.52 | 0.82 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 | 0.91 |
| Row-Lasso SA BIC  | 50  | 1.15 | 1.09 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 |
| Row-Lasso TSA ERIC | 100 | 1.23 | 1.28 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 |
| Row-Lasso TSA BIC  |     | 1.17 | 1.12 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 |
| Row-Dantzig TSA ERIC| 124 | 1.21 | 1.30 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 |
| Row-Dantzig TSA BIC | 120 | 1.17 | 1.14 | 1.11 | 1.09 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 |
| Vec-Lasso SA ERIC | 25  | 0.52 | 1.03 | 1.03 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 | 1.16 |
| Vec-Lasso TSA ERIC | 50  | 1.09 | 1.04 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 |
| Row-Lasso SA BIC  | 100 | 1.23 | 1.28 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 | 1.15 |
| Row-Lasso TSA ERIC |     | 1.17 | 1.12 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 |
| Row-Lasso TSA BIC  |     | 1.17 | 1.12 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 | 1.02 |
| Row-Dantzig TSA ERIC| 124 | 1.21 | 1.30 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 | 1.22 |
| Row-Dantzig TSA BIC | 120 | 1.17 | 1.14 | 1.11 | 1.09 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 | 1.10 |

Table 2: Example 2 – VAR(1), $\|\hat{A} - A\|_\infty$, n=100
|                  | s   | 1     | 3     | 5     | 10    |
|------------------|-----|-------|-------|-------|-------|
|                  | d   | 0.6   | 0.8   | 0.9   | 0.95  |
|                  |     | 0.6   | 0.8   | 0.9   | 0.95  |
|                  |     | 0.6   | 0.8   | 0.9   | 0.95  |
|                  |     | 0.6   | 0.8   | 0.9   | 0.95  |
|                  |     | 0.6   | 0.8   | 0.9   | 0.95  |
| Vec-Lasso SA ERIC| 10  | 1.02  | 1.05  | 1.10  | 1.07  |
|                  |     | 1.02  | 1.05  | 1.10  | 1.07  |
|                  |     | 1.02  | 1.05  | 1.10  | 1.07  |
|                  |     | 1.02  | 1.05  | 1.10  | 1.07  |
|                  |     | 1.02  | 1.05  | 1.10  | 1.07  |
|                  |     | 1.02  | 1.05  | 1.10  | 1.07  |
| Row-Lasso SA BIC |     | 0.46  | 0.47  | 0.46  | 0.47  |
|                  |     | 1.02  | 1.05  | 1.11  | 1.12  |
|                  |     | 0.56  | 0.49  | 0.49  | 0.47  |
|                  |     | 0.56  | 0.49  | 0.49  | 0.47  |
|                  |     | 0.56  | 0.49  | 0.49  | 0.47  |
|                  |     | 0.56  | 0.49  | 0.49  | 0.47  |
|                  |     | 0.56  | 0.49  | 0.49  | 0.47  |
| Row-Dantzig TSA ERIC | 25  | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
| Row-Lasso TSA BIC |     | 1.02  | 1.03  | 1.02  | 1.02  |
|                  |     | 0.56  | 0.46  | 0.44  | 0.46  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
| Row-Dantzig TSA ERIC |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
| Row-Lasso TSA BIC |     | 1.02  | 1.03  | 1.03  | 1.02  |
|                  |     | 0.56  | 0.46  | 0.44  | 0.46  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
| Row-Dantzig TSA ERIC |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
| Row-Lasso TSA BIC |     | 1.02  | 1.03  | 1.03  | 1.02  |
|                  |     | 0.56  | 0.46  | 0.44  | 0.46  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
| Row-Dantzig TSA ERIC |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
|                  |     | 1.04  | 1.04  | 1.02  | 1.04  |
| Row-Lasso TSA BIC |     | 1.02  | 1.03  | 1.03  | 1.02  |
|                  |     | 0.56  | 0.46  | 0.44  | 0.46  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
|                  |     | 0.49  | 0.52  | 0.52  | 0.52  |
| Row-Dantzig TSA ERIC |     | 0.59  | 0.50  | 0.50  | 0.50  |
|                  |     | 1.04  | 1.08  | 0.50  | 1.04  |
|                  |     | 1.02  | 0.50  | 0.50  | 0.50  |
|                  |     | 1.02  | 0.50  | 0.50  | 0.50  |
|                  |     | 1.02  | 0.50  | 0.50  | 0.50  |
| Row-Dantzig TSA BIC |     | 1.09  | 1.06  | 1.06  | 1.06  |
|                  |     | 1.05  | 1.10  | 1.10  | 1.10  |
|                  |     | 1.05  | 1.10  | 1.10  | 1.10  |
|                  |     | 1.05  | 1.10  | 1.10  | 1.10  |
|                  |     | 1.05  | 1.10  | 1.10  | 1.10  |

Table 3: Example 2 – VAR(1), $\|\hat{\Gamma}^{(st)}(0) - \Gamma^{(st)}(0)\|_\infty /\|\Gamma^{(st)}(0)\|_\infty$, n=100
|                | $s$  | 1   | 3   | 5   | 10  |
|----------------|------|-----|-----|-----|-----|
|                | $\rho$ | 0.6 | 0.8 | 0.9 | 0.95 | 0.6 | 0.8 | 0.9 | 0.95 | 0.6 | 0.8 | 0.9 | 0.95 |
|                |      | 0.6 | 0.8 | 0.9 | 0.95 | 0.6 | 0.8 | 0.9 | 0.95 | 0.6 | 0.8 | 0.9 | 0.95 |
| Vec-Lasso SA ERIC |      | 0.49 | 1.05 | 1.07 | 3.10 | 1.07 | 1.11 | 1.13 | 1.07 | 1.07 | 1.05 | 1.73 | 1.07 | 1.10 | 1.13 | 1.12 |
| Vec-Lasso TSA ERIC |      | 1.02 | 1.05 | 1.10 | 2.25 | 1.11 | 1.11 | 1.16 | 1.09 | 1.02 | 1.10 | 1.73 | 1.11 | 1.12 | 1.13 | 1.14 |
| Row-Lasso SA BIC |      | 0.49 | 0.49 | 0.49 | 0.49 | 1.04 | 0.94 | 0.94 | 0.94 | 1.04 | 1.4 | 1.4 | 1.4 | 1.4 |
| Row-Lasso TSA ERIC | 10   | 1.04 | 1.05 | 1.05 | 0.51 | 1.04 | 1.05 | 1.05 | 1.49 | 0.46 | 1.02 | 1.03 | 1.02 |
| Row-Lasso TSA BIC |      | 1.02 | 1.04 | 1.04 | 1.04 | 1.04 | 1.04 | 1.04 | 1.04 | 1.02 | 1.02 | 1.02 | 1.02 |
| Row-Dantzig TSA ERIC |      | 1.06 | 1.12 | 1.12 | 2.78 | 1.04 | 1.05 | 1.08 | 1.07 | 1.04 | 1.10 | 1.17 | 2.33 |
| Row-Dantzig TSA BIC |      | 1.00 | 1.10 | 1.14 | 3.04 | 0.98 | 1.08 | 1.10 | 1.17 | 1.06 | 1.10 | 1.10 | 1.17 |
|                |      | 1.02 | 1.17 | 1.22 | 1.14 | 1.02 | 1.17 | 1.24 | 1.19 | 1.03 | 1.16 | 1.27 | 1.21 |
|                |      | 1.03 | 1.15 | 1.22 | 1.16 | 1.03 | 1.17 | 1.22 | 1.21 | 1.05 | 1.16 | 1.27 | 1.23 |
|                |      | 0.61 | 0.48 | 0.45 | 0.51 | 0.61 | 0.48 | 1.02 | 1.06 | 0.60 | 0.49 | 1.02 | 1.02 |
|                |      | 1.03 | 1.04 | 1.07 | 1.04 | 1.03 | 1.06 | 1.07 | 1.04 | 1.03 | 1.04 | 1.09 | 1.04 |
|                |      | 0.61 | 0.48 | 0.45 | 0.51 | 0.61 | 0.48 | 1.02 | 1.06 | 0.60 | 0.49 | 1.02 | 1.02 |
|                |      | 1.05 | 1.08 | 1.18 | 1.90 | 1.05 | 1.10 | 1.13 | 2.25 | 1.03 | 1.10 | 1.16 | 1.86 |
|                |      | 1.07 | 1.10 | 1.44 | 1.04 | 1.07 | 1.13 | 1.20 | 2566.75 | 1.08 | 1.10 | 1.16 | 3.44 |
|                |      | 1.06 | 1.22 | 1.33 | 1.33 | 1.06 | 1.24 | 1.33 | 1.33 | 1.08 | 1.24 | 1.33 | 1.29 |
|                |      | 1.09 | 1.24 | 1.33 | 1.35 | 1.12 | 1.25 | 1.35 | 1.33 | 1.13 | 1.24 | 1.33 | 1.29 |
|                |      | 0.67 | 1.02 | 0.51 | 0.49 | 0.65 | 1.02 | 0.51 | 1.02 | 0.64 | 0.54 | 0.54 | 0.54 |
|                |      | 1.10 | 1.06 | 1.02 | 1.06 | 1.17 | 1.10 | 1.02 | 1.08 | 1.16 | 1.06 | 1.02 | 1.06 |
|                |      | 0.67 | 1.02 | 0.51 | 0.49 | 0.65 | 1.02 | 0.51 | 1.02 | 0.64 | 0.54 | 0.54 | 0.54 |
|                |      | 1.10 | 1.13 | 1.08 | 1.63 | 1.17 | 1.16 | 1.08 | 1.82 | 1.17 | 1.13 | 1.10 | 2.59 |
|                |      | 1.07 | 1.17 | 1.14 | 1.20 | 1.11 | 1.20 | 1.16 | 1.88 | 1.09 | 1.17 | 1.35 | 1.29 |
|                |      | 1.10 | 1.23 | 1.28 | 1.26 | 1.11 | 1.22 | 1.24 | 1.73 | 1.08 | 1.22 | 1.24 | 1.20 |
|                |      | 1.12 | 1.20 | 1.27 | 1.26 | 1.11 | 1.20 | 1.24 | 1.73 | 1.11 | 1.20 | 1.24 | 1.20 |
|                |      | 0.73 | 0.64 | 0.60 | 0.62 | 0.74 | 1.02 | 0.62 | 0.63 | 0.75 | 0.65 | 0.68 | 0.64 |
|                |      | 1.71 | 1.95 | 1.62 | 1.35 | 1.72 | 1.95 | 1.58 | 1.33 | 1.65 | 1.85 | 1.56 | 1.42 |
|                |      | 0.73 | 0.64 | 0.60 | 0.62 | 0.74 | 1.02 | 0.62 | 0.63 | 0.75 | 0.65 | 0.68 | 0.64 |
|                |      | 1.49 | 1.72 | 1.60 | 46.39 | 1.49 | 1.78 | 1.52 | 1.38 | 1.44 | 1.71 | 1.52 | 1.73 |
|                |      | 1.08 | 1.12 | 1.13 | 29.52 | 1.07 | 1.12 | 1.13 | 1.14 | 1.07 | 1.14 | 1.11 | 2.55 |

Table 4: Example 2 – VAR(1), $\int \|f(\omega) - f(\omega)\|_\infty d\omega/ \int \|f(\omega)\|_\infty d\omega$, n=100
| $s$  | 1     | 3     | 5     | 10    |
|------|-------|-------|-------|-------|
| $\rho$ | 0.6   | 0.8   | 0.9   | 0.95  |
| 1    | 0.6   | 0.8   | 0.9   | 0.95  |
| 2    | 0.6   | 0.8   | 0.9   | 0.95  |
| 10   | 0.6   | 0.8   | 0.9   | 0.95  |
| 25   | 0.6   | 0.8   | 0.9   | 0.95  |
| 50   | 0.6   | 0.8   | 0.9   | 0.95  |
| 100  | 0.6   | 0.8   | 0.9   | 0.95  |

| Method            | $d$ = 10 | $d$ = 25 | $d$ = 50 | $d$ = 100 |
|-------------------|----------|----------|----------|-----------|
| Vec-Lasso SA ERIC | [1.06]   | [1.06]   | [1.06]   | [1.06]    |
| Row-Lasso SA BIC  | [1.01]   | [1.01]   | [1.01]   | [1.01]    |
| Row-Lasso TSA BIC | [1.02]   | [1.03]   | [1.01]   | [1.01]    |
| Row-Dantzig TSA BIC| [1.02] | [1.01] | [1.01] | [1.01] |
| Vec-Lasso SA ERIC | [1.13]   | [1.13]   | [1.13]   | [1.13]    |
| Row-Lasso SA BIC  | [1.04]   | [1.03]   | [1.03]   | [1.03]    |
| Row-Lasso TSA BIC | [1.02]   | [1.11]   | [1.11]   | [1.11]    |
| Row-Dantzig TSA BIC| [1.02] | [1.01] | [1.01] | [1.01] |
| Vec-Lasso SA ERIC | [1.01]   | [1.01]   | [1.01]   | [1.01]    |
| Row-Lasso SA BIC  | [1.01]   | [1.01]   | [1.01]   | [1.01]    |
| Row-Lasso TSA BIC | [1.03]   | [1.03]   | [1.03]   | [1.03]    |
| Row-Dantzig TSA BIC| [1.05] | [1.05] | [1.05] | [1.05] |

Table 5: Example 2 – VAR(1), $1/d\sum_{j=1}^{d}MSE(X_{n+1,j})/\sigma_j^2$, n=100
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5 Proofs

Proof of Theorem 1. Since for a matrix \( A \) we have \( \| A \|_\infty = \max_j \| e_j^T A \|_1 \) and \( \| A \|_1 = \max_j \| A e_j \|_1 \), it is sufficient to show that for all \( j = 1, \ldots, p \), \( \| A - \text{THR}_{\lambda_n}(\hat{A}) \|_\infty = \sum_{k=1}^d \| e_j^T (A_k - \text{THR}_{\lambda_n}(\hat{A}_k)) \|_1 \leq C_2 t_n \) and \( \| A - \text{THR}_{\lambda_n}(\hat{A}) \|_1 = \max_{1 \leq k \leq p} \| (A_k - \text{THR}_{\lambda_n}(\hat{A}_k)) e_j \|_1 \leq C_2 t_n \). In order to bound \( \sum_{k=1}^d \| e_j^T (A_k - \text{THR}_{\lambda_n}(\hat{A}_k)) \|_1 \) we can mainly follow the arguments used in the proof of Theorem 1 in Cai and Liu (2011). For some \( j \) and by the conditions imposed on the thresholding operation and because \( (A_1, \ldots, A_p) \in \mathcal{M}(q, s, M, p) \) and \( \max_k \| A_k - \hat{A}_k \|_{\infty} \leq \lambda_n \), we have that

\[
\sum_{k=1}^p \| e_j^T (A_k - \text{THR}_{\lambda_n}(\hat{A}_k)) \|_1 = \sum_{k=1}^p \max_j \sum_{i=1}^d |A_{k,j} - \text{THR}_{\lambda_n}(\hat{A}_{k,j})|
\]

\[
= \sum_{k=1}^p \max_j \sum_{i=1}^d |\text{THR}_{\lambda_n}(\hat{A}_{k,j}) - A_{k,j}| \mathbb{I}(|\hat{A}_{k,j}| > \lambda_n, |A_{k,j}| > \lambda_n)
\]

\[
+ \sum_{k=1}^p \max_j \sum_{i=1}^d |A_{k,j}| \mathbb{I}(|\hat{A}_{k,j}| \leq \lambda_n, |A_{k,j}| > \lambda_n)
\]

\[
+ \sum_{k=1}^p \max_j \sum_{i=1}^d |\text{THR}_{\lambda_n}(\hat{A}_{k,j}) - A_{k,j}| \mathbb{I}(|\hat{A}_{k,j}| > \lambda_n, |A_{k,j}| \leq \lambda_n)
\]

\[
+ \sum_{k=1}^p \max_j \sum_{i=1}^d |A_{k,j}| q |A_{k,j}|^{1-q} \frac{\lambda_n^{1-q} \lambda_n^{(1-q)}}{(1-q)} \mathbb{I}(|\hat{A}_{k,j}| \leq \lambda_n, |A_{k,j}| \leq \lambda_n)
\]

\[
\leq 2 \sum_{k=1}^p \max_j \sum_{i=1}^d \lambda_n |A_{k,j}|^{q} |A_{k,j}|^{1-q} \lambda_n^{1-q} \mathbb{I}(|A_{k,j}| > \lambda_n)
\]

\[
+ (1 + c) \sum_{k=1}^p \max_j \sum_{i=1}^d |A_{k,j}| \mathbb{I}(|\hat{A}_{k,j}| > \lambda_n, |A_{k,j}| \leq \lambda_n)
\]

\[
+ \lambda_n^{1-q} \sum_{k=1}^p \max_j \sum_{i=1}^d |A_{k,j}| q \mathbb{I}(|\hat{A}_{k,j}| \leq \lambda_n, |A_{k,j}| \leq \lambda_n)
\]

\[
\leq \lambda_n^{1-q} (4 + c) = (4 + c) C_1^{1-q} s t_n^{1-q}.
\]

This implies \( \| A - \text{THR}_{\lambda_n}(\hat{A}) \|_\infty = \sum_{k=1}^p \| A_k - \text{THR}_{\lambda_n}(\hat{A}_k) \|_\infty \leq (4 + c) C_1^{1-q} s t_n^{1-q} \). \( \| A - \text{THR}_{\lambda_n}(\hat{A}) \|_1 = \max_{1 \leq k \leq p} \| A_k - \text{THR}_{\lambda_n}(\hat{A}_k) \|_1 \) can be bounded by the same arguments. \( \blacksquare \)

Proof of Corollary 2. We have \( (\hat{A}_1^{(TD)}, \ldots, \hat{A}_p^{(TD)}) = \text{THR}_{\lambda_n}(\hat{\beta}_1^{(Dantzig)}, \ldots, \hat{\beta}_d^{(Dantzig)})^T \). For each \( \hat{\beta}_j^{(Dantzig)} \), \( j = 1, \ldots, d \), an error bound with respect to the \( \| \cdot \|_{\infty} \) norm is obtained by Theorem 4 in Wu et al. (2016) on a set with probability of at least \( p_0^{(Dantzig)} \) as defined in (24). Since all \( \hat{\beta}_j^{(Dantzig)} \) share the same regressors, only the event denoted by \( B \) in the proof of Theorem 4 in Wu et al. (2016) differs among the \( \hat{\beta}_j^{(Dantzig)} \). Hence, the probability of the intersection of all these \( p \) events, where the corresponding set is denoted by \( B \), and of the event denoted by \( A \) in the proof of Theorem 4 in Wu et al. (2016), is at
least $p_n^{(Dantzig)}$. Assertion (25) and (26) follow directly by arguments used in the proof of Theorem 4 in (Wu et al. (2016) with

$$a = \left(\frac{\sqrt{\log (d)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-2/\tau}}\right) \left(\sum_{j=0}^{\infty} \|A^j\|_2 \|C_e,\tau\|^2\right)^2$$

and

$$b = \left(\frac{\sqrt{\log (d)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-1/\tau}}\right) \left(\sum_{j=0}^{\infty} \|A^j\|_2 \|C_e,\tau\|^2\right)^2.$$ 

See also Example 1 and 4, and Remark 6 in (Wu et al. (2016)).

On a set with probability of at least $p_n^{(Dantzig)}$, Theorem 4 in (Wu et al. (2016)) leads with the above choice of $a$ and $b$, to the bound

$$\|B - \hat{B}^{(Dantzig)}\|_{\max} \leq 2\|\Gamma^{(st)}(0)\|^{-1} \left(\sum_{j=0}^{\infty} \|A^j\|_2 \|C_e,\tau\|^2\right)^2 \times \left[\left(\frac{\sqrt{\log (dp)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-2/\tau}}\right) M \left(\frac{\sqrt{\log (dp)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-1/\tau}}\right)\right]^{1-q}.$$ 

Following the proof of Theorem 6 in (Cai et al. (2011)), i.e. the arguments leading to equation (27) in the aforesaid paper, we obtain $\|B - \hat{B}^{(Dantzig)}\|_{\max} \leq (1+2^{1-q}+3^{1-q})s(2\|\Gamma^{(st)}(0)\|^{-1} \left(\sum_{j=0}^{\infty} \|A^j\|_2 \|C_e,\tau\|^2\right)^2 \left[\left(\frac{\sqrt{\log (dp)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-2/\tau}}\right) M \left(\frac{\sqrt{\log (dp)}}{\sqrt{N}} + \frac{d^{1/\tau}}{N^{1-1/\tau}}\right)\right]^{1-q}$. Furthermore, we have

$$\|B - \hat{B}^{(Dantzig)}\|_{\max} \leq \left\|\Gamma^{(st)}(0)\right\|^{-1} \left(\|\Gamma^{(st)}(0) - \mathcal{X}^\top \mathcal{X}/N\|_{\max} \|B - \hat{B}^{(Dantzig)}\|\right) \leq \left\|\Gamma^{(st)}(0)\right\|^{-1} \left\|\mathcal{X}^\top \mathcal{X}/N\|_{\max} \|B - \hat{B}^{(Dantzig)}\|\right.$$ 

which gives expression (28). (29) follows then directly by Theorem 11.

Proof of Theorem 12. Notice that $\Gamma(h)^{(st)} = A^h \Gamma^{(st)}(0)$ which implies $\hat{\Gamma}(h)^{(st)} - \Gamma(h)^{(st)} = \hat{A}^h (\hat{\Gamma}^{(st)}(0) - \Gamma^{(st)}(0)) + (\hat{A}^h - A^h) \Gamma^{(st)}(0)$. Furthermore, we have $\hat{\Gamma}(0)^{(st)} - \Gamma(0)^{(st)} = \sum_{s=1}^{\infty} (\hat{A}^s - A^s) \sum_{j=1}^{s-1} \hat{A}^j (\hat{A}^j - \mathcal{A}^j)^{s-1-j} + \sum_{s=1}^{\infty} \hat{A}^s \hat{\Sigma}_U - \hat{\Sigma}_U (\mathcal{A}^s - \mathcal{A}^s)^\top$. Following the proof of Lemma 8 in (Krampe et al. (2019)) we have for $s \geq 1$, $\hat{A}^s - A^s = \sum_{j=0}^{s-1} \hat{A}^j (\hat{A}^j - \mathcal{A}^j)^{s-1-j}$. Hence, $\hat{\Gamma}(0)^{(st)} - \Gamma(0)^{(st)} = \sum_{s=1}^{\infty} \sum_{j=0}^{s-1} \hat{A}^j (\hat{A}^j - \mathcal{A}^j)^{s-1-j} + \sum_{s=0}^{\infty} \hat{A}^s (\hat{\Sigma}_U - \Sigma_U) (\mathcal{A}^s - \mathcal{A}^s)^\top + \sum_{s=1}^{\infty} \hat{A}^s \hat{\Sigma}_U (\sum_{j=0}^{s-1} \hat{A}^j (\hat{A}^j - \mathcal{A}^j)^{s-1-j})$. For some sub-multiplicative matrix norm $\|\cdot\|$ and since $ab \leq (a^2 + b^2)/2$, we further have

$$\|\hat{\Gamma}(0)^{(st)} - \Gamma(0)^{(st)}\| \leq \|\hat{A} - A\| \|\Sigma_e\| \sum_{j=0}^{\infty} \|\hat{A}^j\| \|\mathcal{A}^j\|^\top \sum_{s=0}^{\infty} \|\mathcal{A}^s\| \|\mathcal{A}^s\|^\top \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{j=0}^{\infty} \|\hat{A}^j\| \|\mathcal{A}^j\|^\top \sum_{s=0}^{\infty} \|\mathcal{A}^s\| \|\mathcal{A}^s\|^\top$$

$$\leq \|\hat{A} - A\| \|C_{\gamma,\mathcal{A}} + C_{\gamma,\mathcal{A}}\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{s=1}^{\infty} \|\hat{A}^s\| \|\mathcal{A}^s\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{s=1}^{\infty} \|\hat{A}^s\| \|\mathcal{A}^s\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{s=1}^{\infty} \|\hat{A}^s\| \|\mathcal{A}^s\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{s=1}^{\infty} \|\hat{A}^s\| \|\mathcal{A}^s\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\| \sum_{s=1}^{\infty} \|\hat{A}^s\| \|\mathcal{A}^s\| \|\hat{\Sigma}_e - \hat{\Sigma}_e\|$$

$28$
Proof of Theorem. We first write
\[
\begin{align*}
  f^{-1}(\omega) - \hat{f}^{-1}(\omega) &= (\hat{A}(\exp(i\omega)) - A(\exp(i\omega)))^T \Sigma^{-1}_e \Sigma^{-1}_e (\hat{A}(\exp(-i\omega))) \\
  &+ A(\exp(i\omega))^T (\Sigma^{-1}_e - \Sigma^{-1}_e) A(\exp(-i\omega)) \\
  &+ A(\exp(i\omega))^T \Sigma^{-1}_e (\hat{A}(\exp(-i\omega)) - A(\exp(-i\omega))) \\
  &+ (\hat{A}(\exp(i\omega)) - A(\exp(i\omega)))^T (\Sigma^{-1}_e - \Sigma^{-1}_e) A(\exp(-i\omega)) \\
  &+ A(\exp(i\omega))^T (\Sigma^{-1}_e - \Sigma^{-1}_e) (\hat{A}(\exp(-i\omega)) - A(\exp(-i\omega))).
\end{align*}
\]

Observe that
\[
\| (\hat{A}(\exp(i\omega)) - A(\exp(i\omega)))^T \|_1 \leq \sum_{s=1}^p \| \hat{A}_s - A_s \|_1 \\
= \sum_{s=1}^p \| \hat{A}_s - A_s \|_{\infty} \leq t_{n,1}
\]
and that
\[
\| (\hat{A}(\exp(i\omega)) - A(\exp(i\omega)))^T \|_{\infty} \leq \sum_{s=1}^p \| \hat{A}_s - A_s \|_1 \leq t_{n,1}.
\]
Hence, \(\| (\hat{A}(\exp(i\omega)) - A(\exp(i\omega)))^T \|_l \leq t_{n,1}\) and \(\| \hat{A}(\exp(i\omega)) - A(\exp(i\omega)) \|_l \leq t_{n,1}\) for all \(l \in [1, \infty]\).

Since \(\| \cdot \|_l\) is sub-multiplicative and \((A_1, \ldots, A_p) \in M(q, s, M, p), \Sigma^{-1}_e \in M(q_{e-1}, s_{e-1}, M_{e-1}, 1)\), the assertion follows.

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