Pitchfork bifurcation at line solitons for nonlinear Schrödinger equations on the product space $\mathbb{R} \times \mathbb{T}$

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Abstract

In this paper, we study the bifurcation problem from a line soliton for a stationary nonlinear Schrödinger equation on the product space $\mathbb{R} \times \mathbb{T}$. We extend earlier results to a larger class of the nonlinearity in the equation. The salient point of our analysis relies on a lower bound of solution to the “auxiliary equation” and then on the application of the Crandall-Rabinowitz argument.

1 Introduction

In this paper, we study the bifurcation problem from line solitons for the following stationary nonlinear Schrödinger equation posed on the product space $\mathbb{R} \times \mathbb{T}$:

$$-\partial_{x}^{2} u - \partial_{y}^{2} u + \omega u - |u|^{p-1} u = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T},$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the one dimensional torus, $\omega > 0$, $p > 1$ and $u$ is the unknown real-valued function on $\mathbb{R} \times \mathbb{T}$. Introducing the function $F: (0, \infty) \times H^{2}(\mathbb{R} \times \mathbb{T}) \to L^{2}(\mathbb{R} \times \mathbb{T})$ defined by

$$F(\omega, u) := -\partial_{x}^{2} u - \partial_{y}^{2} u + \omega u - |u|^{p-1} u,$$

equation (1.1) can now be written as $F(\omega, u) = 0$. We will use $\partial_{u} F$ to denote the derivative of $F = F(\omega, u)$ with respect to the second variable $u$.

For each $\omega > 0$, equation (1.1) admits a positive solution in $H^{1}(\mathbb{R} \times \mathbb{T})$ which is independent of the variable $y \in \mathbb{T}$. Such a solution is called a line soliton to (1.1). In other words, a line soliton to (1.1) is defined as a positive $H^{1}$-solution to the following ordinary differential equation:

$$-\frac{d^2 R}{dx^2} + \omega R - R^{p} = 0, \quad x \in \mathbb{R}.$$  

It is known that for any $\omega > 0$ and $p > 1$, a unique positive $H^{1}$-solution to (1.3) exists; we use $R_{\omega}$ to denote the line soliton to (1.1). The line soliton $R_{\omega}$ is explicitly given as

$$R_{\omega}(x, y) = R_{\omega}(x) := \omega^{\frac{p+1}{2}} \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \text{sech}^{\frac{2}{p-1}} \left(\frac{p-1}{2} \sqrt{\omega} x\right).$$

Note that the function $\omega \in (0, \infty) \mapsto R_{\omega} \in H^{2}(\mathbb{R})$ is $C^{\infty}$. Furthermore, let $L_{\omega, +, 0}$ be the linearized operator around $R_{\omega}$ for the ODE (1.3), namely,

$$L_{\omega, +, 0} := -\frac{d^2}{dx^2} + \omega - pR_{\omega}^{p-1}.$$  

Then, the following are known (see, e.g., Section 3 of [2]):
1. \[ \sigma(L_{\omega,+0}) = \left\{ -\frac{\omega}{\omega_p} \right\} \cup \{0\} \cup [0, \infty) \quad \text{with} \quad \omega_p := \frac{4}{(p-1)(p+3)}. \quad (1.6) \]

2. All eigenvalues of \( L_{\omega,+0} \) are simple, and

\[ L_{\omega,+0}^\pm\omega = -\frac{\omega}{\omega_p} R_{\omega_p}^{\pm1}, \quad L_{\omega,+0} \frac{dR_\omega}{dx} = 0. \quad (1.7) \]

Note that \( R_{\omega_p}^{\pm1} \) is even and \( \frac{dR_\omega}{dx} \) is odd. By the Fourier series expansion with respect to the second variable \( y \), we see that for any \( \omega > 0 \) and \( f \in H^2(\mathbb{R} \times \mathbb{T}) \),

\[ \partial_x \mathcal{F}(\omega, R_\omega)f = \sum_{n \in \mathbb{Z}} (L_{\omega,+0} + n^2) f_n(x)e^{iny} \quad \text{with} \quad f_n(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{-iny} f(x,y) \, dy. \quad (1.8) \]

Thus, we find from (1.6) and (1.8) that

\[ \sigma(\partial_x \mathcal{F}(\omega, R_\omega)) = \bigcup_{n \in \mathbb{Z}} \sigma(L_{\omega,+0} + n^2). \quad (1.9) \]

The aim of this paper is to show that for any \( p > 1 \), a “pitchfork bifurcation” occurs at the line soliton \( R_{\omega_p} \) (see Theorem 1.1). Let us recall that the pair \((\omega_p, R_{\omega_p})\) is called a bifurcation point for the equation \( \mathcal{F} = 0 \) with respect to the curve \( \omega \mapsto (\omega, R_\omega) \) if every neighborhood of \((\omega_p, R_{\omega_p})\) contains zeros of \( \mathcal{F} \) not lying on the curve (see [3]). The implicit function theorem shows that if \((\omega_p, R_{\omega_p})\) is the bifurcation point, then zero is an eigenvalue of \( \partial_x \mathcal{F}(\omega_p, R_{\omega_p}) \). Since \( \partial_x \mathcal{F}(\omega_p, R_{\omega_p}) \) is just the linearized operator around \( R_{\omega_p} \) for (1.7), we can say that a necessary condition for \((\omega_p, R_{\omega_p})\) to be the bifurcation point is that \( R_{\omega_p} \) is degenerate. In addition, by (1.6), (1.7) and (1.9), we see that if \( \omega = \omega_p \), then the kernel of \( \partial_x \mathcal{F}(\omega_p, R_{\omega_p}) \) contains the real-valued functions \( \partial_x R_{\omega_p}, R_{\omega_p}^{\pm1} \cos y \) and \( R_{\omega_p}^{\pm1} \sin y \). We introduce the spaces \( L^2_{\text{sym}} \) and \( H^2_{\text{sym}} \) as

\[ L^2_{\text{sym}} := \left\{ u \in L^2(\mathbb{R} \times \mathbb{T}) : u(x,y) = u(-x,y) = u(x,-y) \right\}, \]

\[ H^2_{\text{sym}} := H^2(\mathbb{R} \times \mathbb{T}) \cap L^2_{\text{sym}}. \]

Note that \( R_{\omega_p}^{\pm1} \cos y \in H^2_{\text{sym}} \), whereas \( \partial_x R_{\omega_p} \) and \( R_{\omega_p}^{\pm1} \sin y \) fail to lie in \( H^2_{\text{sym}} \). Thus, \((\omega_p, R_{\omega_p})\) is still a candidate for the the bifurcation point in the setting \( H^2_{\text{sym}} \). Here, we remark that the information about branches bifurcating from \((\omega_p, R_{\omega_p})\) is useful in the study of the stability of the degenerate line soliton \( R_{\omega_p} \) (see [3]).

Now, we state the main result of this paper.

**Theorem 1.1.** For any \( p > 1 \), the pair \((\omega_p, R_{\omega_p})\) is a pitchfork bifurcation point for \( \mathcal{F} = 0 \) in \((0, \infty) \times H^2_{\text{sym}}\) with respect to the curve \( \omega \mapsto (\omega, R_\omega) \); Precisely, there exist \( a_\ast > 0, \delta_\ast > 0 \) and a \( C^2\)-curve \( a \in (-a_\ast, a_\ast) \mapsto (\omega(a), Q(a)) \in (\omega_p - \delta_\ast, \omega_p + \delta_\ast) \times H^2_{\text{sym}} \) with the following properties:

1. The set of zeros of \( \mathcal{F} \) in \((\omega_p - \delta_\ast, \omega_p + \delta_\ast) \times H^2_{\text{sym}} \) consists of two curves \( \omega \in (0, \infty) \mapsto (\omega, R_\omega) \) and \( a \in (-a_\ast, a_\ast) \mapsto (\omega(a), Q(a)) \).
2. The following hold for all \( a \in (-a_*, a_*) \):

\[
\frac{d^2 \omega}{da^2}(0) = \omega_0, \quad \frac{d \omega}{da}(0) = 0, \quad (1.10)
\]

\[
\frac{d^2 \omega}{da^2}(0) = \omega_p (p(p-1))^2 \left( T(\omega_p, 0)^{-1} \left( R_{\omega_p}^{-2} \{ \psi_{\omega_p} \cos y \}^2 \right), \right.
\]

\[
+ \frac{p(p-1)(p-2)\omega_p}{3} \left( R_{\omega_p}^{-3} \{ \psi_{\omega_p} \cos y \}^4 \right),
\]

where \( T(\omega_p, 0) := P \partial_a F(\omega_p, R_{\omega_p})|_{X_2} \); Note that \( T(\omega_p, 0) : X_2 \to Y_2 \) is a bijection.

3. For any \( a \in (-a_*, a_*) \), \( Q(a) \) is positive and written as

\[
Q(a) = R_{\omega_p} + a C_p R_{\omega_p} \frac{x}{\omega_p} \cos y + O(a^2) \quad \text{in } H^2(\mathbb{R} \times T),
\]

where \( C_p > 0 \) is a normalizing constant chosen such that

\[
\| C_p R_{\omega_p} \frac{x}{\omega_p} \cos y \|_{L^2(\mathbb{R} \times T)} = 1.
\]

In particular, \( Q(0) = R_{\omega_p} \). Furthermore, the mass of \( Q(a) \) is written as

\[
\| Q(a) \|^2_{L^2(\mathbb{R} \times T)} = \pi \| R_{\omega_p} \|^2_{L^2(\mathbb{R})} + \frac{a^2}{\omega_p} \left( \frac{d^2 \omega}{da^2}(0) \right) \frac{5-p}{4(p-1)} \| R_{\omega_p} \|^2_{L^2(\mathbb{R})} - 1 \bigg) + o(a^2).
\]

**Remark 1.1.** When \( p \geq 2 \), the same result as in Theorem 1.1 had been obtained in [\( C \)]. The salient point of our result is that we can treat all \( p > 1 \); When \( 1 < p < 2 \), the twice differentiability of the curve \( (\omega(a), Q(a)) \) is not obvious because the nonlinearity is not twice differentiable.

We give a proof of Theorem 1.1 in Section 5. In order to prove the theorem, we derive a lower bound of the solution to the “auxiliary equation” (see (2.20)) first, and then apply the Crandall-Rabinowitz argument [\( H \)]. Such an approach enables us to treat the case \( 1 < p < 2 \).

At the end of this section, we give a basic properties of line solitons to (1.11). It follows from the definition of the line solitons (see (1.14)) that

\[
R_{\omega}(x) = \omega \frac{1}{\omega} R_1(\sqrt{\omega}x),
\]

\[
\partial_{\omega} R_{\omega} = \frac{1}{p-1} \omega^{-1} R_{\omega} + \frac{1}{2} \omega^{-1} x \frac{d R_{\omega}}{dx},
\]

\[
\omega \frac{1}{\omega} e^{-\sqrt{\omega}|x|} \leq R_{\omega}(x) \leq \{2(p+1)\omega\}^{\frac{1}{p-1}} e^{-\sqrt{\omega}|x|}.
\]

Differentiating both sides of \( F(\omega, R_{\omega}) = 0 \) with respect to \( \omega \), we see that

\[
\partial_{\omega} F(\omega, R_{\omega}) \partial_{\omega} R_{\omega} = -R_{\omega}.
\]

Furthermore, by (1.16) and the integration by parts, we can verify that

\[
\int_{\mathbb{R}} R_{\omega}^q \partial_{\omega} R_{\omega} = \frac{2q-p+3}{2(p-1)(q+1)} \omega^{-1} \int_{\mathbb{R}} R_{\omega}^{q+1} \quad \text{for all } q \geq 1.
\]
We can also derive the following equation (see Lemma 2.2 of [6]):

\[ \int_{\mathbb{R}} R_{\omega}^{p+r} \frac{(p + 1)(r + 1)}{2r + p + 1} \omega \int_{\mathbb{R}} R_{\omega}^{r+1} \quad \text{for all } r > 1. \] (1.20)

The rest of this paper is organized as follows. In Section 2, we apply the Lyapunov-Schmidt method and introduce the auxiliary and bifurcation equations (see (2.20) and (2.21)). In Section 3, we derive a lower bound and decay estimates for solutions to the auxiliary equation. In Section 4, we show the three times differentiability of solutions to the auxiliary equation with respect to certain parameters. In Section 5, we give a proof of Theorem 1.1.

2 Lyapunov-Schmidt method

In order to construct a bifurcation branch of \( F = 0 \) from \((\omega, R_\omega)\) in \((0, \infty) \times H^2_{\text{sym}}(\mathbb{R} \times \mathbb{T})\), we employ the Lyapunov-Schmidt method. Let us begin by introducing a few symbols:

\textbf{Notation 2.1.} 
1. For \( \omega > 0 \), define \( \psi_\omega \) as

\[ \psi_\omega := \frac{R_{\omega}^{p+1}}{\| R_{\omega}^2 \cos y \|_{L^2(\mathbb{R} \times \mathbb{T})}} = \frac{R_{\omega}^{p+1}}{\sqrt{\pi} \| R_{\omega}^2 \|_{L^2(\mathbb{R})}}. \] (2.1)

Observe from (1.17) that

\[ |\psi_{\omega,p}(x)| \lesssim e^{-\frac{p+1}{2} \sqrt{\omega}|x|} \quad \text{for all } x \in \mathbb{R}, \] (2.2)

where the implicit constant depends only on \( p \).

2. For \( p > 1 \) and \( \omega > 0 \), let \( \lambda(\omega) \) denote the second eigenvalue of the operator \( \partial_u F(\omega, R_\omega) \) restricted to \( H^2_{\text{sym}} \). Note that \( \partial_u F(\omega, R_\omega) \big|_{H^2_{\text{sym}}} \) is a self-adjoint operator on \( L^2(\mathbb{R} \times \mathbb{T}) \) with the values in \( L^2_{\text{sym}} \). By (1.11), (1.10), (1.11) and (2.1), we see that

\[ \lambda(\omega) = 1 - \frac{\omega}{\omega_p}, \quad \partial_u F(\omega, R_\omega) \big|_{H^2_{\text{sym}}} \psi_\omega \cos y = \lambda(\omega) \psi_\omega \cos y \quad \text{for all } \omega > 0. \] (2.3)

Furthermore, we see that

\[ \text{Ker} \partial_u F(\omega_p, R_{\omega_p}) \big|_{H^2_{\text{sym}}} = \text{span} \{ \psi_{\omega_p} \cos y \}, \] (2.4)

\[ \frac{d \lambda}{d \omega}(\omega) = -\frac{1}{\omega_p} \quad \text{for all } \omega > 0. \] (2.5)

3. We use \( L^2_{\text{real}}(\mathbb{R} \times \mathbb{T}) \) to denote the real Hilbert space of square integrable functions on \( \mathbb{R} \times \mathbb{T} \) equipped with the inner product

\[ \langle u, v \rangle := \int_{\mathbb{R} \times [0,2\pi]} u(x,y)v(x,y) \, dx \, dy. \]
4. We define the spaces $X_2$ and $Y_2$ as

$$X_2 := \{ u \in H^2_{\text{sym}}(\mathbb{R} \times T): \langle u, \psi_p \cos y \rangle = 0 \},$$

$$Y_2 := \{ u \in L^2_{\text{sym}}(\mathbb{R} \times T): \langle u, \psi_p \cos y \rangle = 0 \}.$$

Note that $X_2 \subset Y_2$. Since the line solitons are independent of $y$, it is easy to verify that

$$R_\omega, \partial_\omega R_\omega \in X_2 \quad \text{for all } p > 1 \text{ and } \omega > 0. \quad (2.6)$$

We look for solutions to the equation $F = 0$ of the form $(\omega, u) = (\omega_p + \delta, R_{\omega_p} + a\psi_{\omega_p} \cos y + h)$ with $\delta > 0$, $a \in \mathbb{R}$ and $h \in H^2_{\text{sym}}$. To this end, we consider the orthogonal projection $P_\perp$ from $L^2(\mathbb{R} \times T)$ onto $Y_2$:

$$P_\perp u := u - \langle u, \psi_p \cos y \rangle \psi_p \cos y, \quad (2.7)$$

Then, we define the function $F_\perp: (0, \infty) \times (-1, 1) \times H^2_{\text{sym}} \rightarrow Y_2$ as

$$F_\perp(\omega, a, h) := P_\perp F(\omega, R_{\omega_p} + a\psi_{\omega_p} \cos y + h). \quad (2.8)$$

Note that

$$F_\perp(\omega, 0, R_\omega - R_{\omega_p}) = P_\perp F(\omega, R_\omega) = 0. \quad (2.9)$$

Note that [2.4] shows that

$$\text{Ker } \partial_\omega F(\omega_p, R_{\omega_p})|_{X_2} = \{0\}. \quad (2.10)$$

Furthermore, by the decay of $R_\omega$ (see (1.17)), (2.10) and the Fredholm alternative theorem, we see that

$$\text{Ran } \partial_\omega F(\omega_p, R_{\omega_p})|_{X_2} = Y_2. \quad (2.11)$$

By (2.10) and (2.11), we see that $\partial_\omega F(\omega_p, R_{\omega_p}): X_2 \rightarrow Y_2$ is bijective.

Using the implicit function theorem, we can find solutions to $F_\perp(\omega, a, h) = 0$:

**Lemma 2.1.** Assume $p > 1$. Then, there exist $a_0 > 0$, $\delta_0 > 0$, $r_0 > 0$ and a $C^1$-function $\eta: (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \rightarrow X_2$ such that the following hold for all $(\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)$:

1. Let $h \in X_2$. Then, $F_\perp(\omega, a, h) = 0$ if and only if $h = \eta(\omega, a)$.
2. $\|\eta(\omega, a)\|_{H^2(\mathbb{R} \times T)} < r_0$.
3. Define $\varphi(\omega, a)$ as

$$\varphi(\omega, a) := R_{\omega_p} + a\psi_{\omega_p} \cos y + \eta(\omega, a). \quad (2.12)$$

Then, the following hold:

$$\partial_\omega \eta(\omega, a) = -(P_\perp \partial_\omega F(\omega, \varphi(\omega, a))|_{X_2})^{-1} P_\perp \varphi(\omega, a), \quad (2.13)$$

$$\partial_a \eta(\omega, a) = -(P_\perp \partial_a F(\omega, \varphi(\omega, a))|_{X_2})^{-1} P_\perp \partial_a F(\omega, \varphi(\omega, a)) \psi_{\omega_p} \cos y. \quad (2.14)$$

**Remark 2.1.** 1. Lemma [2.7] only states that the function $\eta$ is $C^1$ since the assumption includes the case $1 < p < 2$. When $p \geq 2$, we can prove that $\eta$ is $C^2$. 



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2. Since \( \eta \) is \( C^1 \) with respect to \((\omega,a)\) in \( H^2(\mathbb{R} \times T) \), replacing \( \delta_0 \) by \( \frac{\delta_0}{2} \) and \( a_0 \) with \( \frac{a_0}{2} \) if necessary, we may assume that

\[
\sup_{\omega \in (\omega_p-\delta_0, \omega_p+\delta_0)} \sup_{a \in (-a_0, a_0)} \{ \| \partial_\omega \eta(\omega,a) \|_{H^2(\mathbb{R} \times T)} + \| \partial_a \eta(\omega,a) \|_{H^2(\mathbb{R} \times T)} \} \lesssim 1, \tag{2.15}
\]

where the implicit constant depends only on \( p \).

Let \( a_0 > 0 \), \( \delta_0 > 0 \), \( r_0 > 0 \), \( \eta \) and \( \varphi \) be the same as in Lemma 2.1. By (2.16), (2.19), Lemma 2.1 we see that

\[
\eta(\omega,0) = R_{\omega} - R_{\omega_p} \quad \text{for all} \quad \omega \in (\omega_p - \delta_0, \omega_p + \delta_0). \tag{2.16}
\]

In particular, we see that

\[
\eta(\omega_p,0) = 0, \quad \varphi(\omega,0) = R_{\omega}, \quad \partial_\omega \eta(\omega,0) = \partial_\omega \varphi(\omega,0) = \partial_\omega R_{\omega}. \tag{2.17}
\]

We can also verify that

\[
\partial_a \eta(\omega_p,0) = 0, \quad \partial_a \varphi(\omega_p,0) = \psi_{\omega_p} \cos y. \tag{2.18}
\]

Now, we introduce the function \( \mathcal{F} : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \to \mathbb{R} \) as

\[
\mathcal{F}_1(\omega,a) := \langle \mathcal{F}(\omega, \varphi(\omega,a)) \rangle, \quad \psi_{\omega_p} \cos y. \tag{2.19}
\]

Then, we may write the equation \( \mathcal{F}(\varphi, R_{\omega_p} + a\psi_{\omega_p} \cos y + h) = 0 \) as follows:

\[
\begin{align*}
\mathcal{F}_1(\omega,a,h) &= 0, \tag{2.20} \\
\mathcal{F}_1(\omega,a) &= 0. \tag{2.21}
\end{align*}
\]

The first equation (2.20) is called the auxiliary equation and the second one (2.21) the bifurcation equation. Lemma 2.1 shows that \((\omega,a, \eta(\omega,a))\) is a solution to the auxiliary equation (2.20) for all \((\omega,a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\). Observe from (2.17), (2.19) and \( \mathcal{F}(\omega, R_{\omega}) = 0 \) that

\[
\mathcal{F}_1(\omega,0) = 0. \tag{2.22}
\]

3 Lower bound and decay of solution to the auxiliary equation

Throughout this section, for a given \( p > 1 \), we use \( a_0 \), \( \delta_0 \) and \( \eta : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \to X_2 \) provided by Lemma 2.1. Furthermore, let \( \varphi(\omega,a) \) be the function defined by (2.12), namely, \( \varphi(\omega,a) = R_{\omega_p} + a\psi_{\omega_p} \cos y + \eta(\omega,a) \).

Our aim in this section is to show the following propositions:

**Proposition 3.1.** Assume \( p > 1 \). Then, there exists \( \varepsilon_1 > 0 \) depending only on \( p \) with the following property: for any \( 0 < \varepsilon < \varepsilon_1 \), there exist \( 0 < a_\varepsilon < a_0 \) and \( C_1(\varepsilon) > 1 \) depending only on \( p \) and \( \varepsilon \) such that for any \((\omega,a,x,y) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \times \mathbb{R} \times T \),

\[
\frac{1}{C_1(\varepsilon)} e^{-|\sqrt{\varepsilon}+\varepsilon|x|} \leq \varphi(\omega,a,x,y) \leq C_1(\varepsilon) e^{-|\sqrt{\varepsilon}+\varepsilon|x|}. \tag{3.1}
\]

In particular, \( \varphi(\omega,a) \) is positive.
The maximal principle plays an important role in the proof. We first show a uniform
following the argument of Berestycki and Nirenberg [1], we shall prove Proposition 3.1.

Lemma 3.3. Assume \( p > 1 \). Then, for any \( 0 < \varepsilon < \varepsilon_1 \), there exist \( 0 < a_\varepsilon < a_0 \), \( C_2(\varepsilon) > 0 \) and \( C_3(\varepsilon) > 0 \) depending only on \( p \) and \( \varepsilon \) such that for any \( (\omega, a, x, y) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \times \mathbb{R} \times \mathbb{T} \),

\[
|\partial_x \varphi(\omega, a, x, y)| \leq C_2(\varepsilon)e^{-(\sqrt{\varepsilon}-2\varepsilon)|x|}, \quad (3.2)
\]

\[
|\partial_y \varphi(\omega, a, x, y)| \leq C_3(\varepsilon)e^{-(\sqrt{\varepsilon}-\varepsilon)|x|}. \quad (3.3)
\]

We use these propositions to show the differentiability of \( \varphi(\omega, a) \) with respect to \( \omega \) and \( a \) (see Section 4 below in details).

We will introduce symbols used in the rest of this paper:

Notation 3.1. Let \( p > 1 \) and \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \).

1. Define the operators \( L_-(\omega, a) \) and \( L_+(\omega, a) \) on \( L^2_{\text{rad}}(\mathbb{R} \times \mathbb{T}) \) with the domain \( H^2(\mathbb{R} \times \mathbb{T}) \) as

\[
L_-(\omega, a) := -\partial_x^2 - \partial_y^2 + \omega - \varphi(\omega, a)^{p-1}, \quad (3.4)
\]

\[
L_+(\omega, a) := \partial_\omega \mathcal{F}(\omega, \varphi(\omega, a)) = -\partial_x^2 - \partial_y^2 + \omega - p\varphi(\omega, a)^{p-1}. \quad (3.5)
\]

Note that both \( L_-(\omega, a) \) and \( L_+(\omega, a) \) are self-adjoint operators on \( L^2(\mathbb{R} \times \mathbb{T}) \) with domain \( H^2(\mathbb{R} \times \mathbb{T}) \). Observe from (2.17) that

\[
L_+(\omega, 0) = \partial_\omega \mathcal{F}(\omega, R_\omega). \quad (3.6)
\]

Furthermore, by (3.6), (2.17) and (1.18), we see that

\[
L_+(\omega, 0)\partial_\omega \varphi(\omega, 0) = \partial_\omega \mathcal{F}(\omega, R_\omega)\partial_\omega R_\omega = -R_\omega. \quad (3.7)
\]

2. Define

\[
E(\omega, a) := \mathcal{F}_\parallel(\omega, a)\psi_{\omega_p} \cos y. \quad (3.8)
\]

Lemma 2.1 together with (2.12) shows that

\[
L_-(\omega, a)\varphi(\omega, a) = \mathcal{F}_\perp(\omega, a, \eta(\omega, a)) + \mathcal{F}_\parallel(\omega, a)\psi_{\omega_p} \cos y = E(\omega, a). \quad (3.9)
\]

3.1 Proof of Proposition 3.1

Following the argument of Berestycki and Nirenberg [1], we shall prove Proposition 3.1. The maximal principle plays an important role in the proof. We first show a uniform decay of \( \varphi(\omega, a) \).

Lemma 3.3. Assume \( p > 1 \). Then, for any \( \omega \in (\omega_p - \delta_0, \omega_p + \delta_0) \), the following holds:

\[
\lim_{|x| \to \infty} \sup_{a \in [-\frac{a_0}{2}, \frac{a_0}{2}]} \sup_{y \in \mathbb{T}} \varphi(\omega, a, x, y) = 0. \quad (3.10)
\]

Proof. If the claim were false, then there existed \( \omega_* \in (\omega_p - \delta_0, \omega_p + \delta_0) \), \( C_* > 0 \) and a sequence \( \{x_n\} \) with \( \lim_{n \to \infty} |x_n| = \infty \) such that

\[
\sup_{a \in [-\frac{a_0}{2}, \frac{a_0}{2}]} \sup_{y \in \mathbb{T}} |\varphi(\omega_*, a, x_n, y)| \geq C_*, \quad \text{for all } n \in \mathbb{N}.
\]
Furthermore, for each \( n \in \mathbb{N} \), there exist \( a_n \in [-\frac{a_0}{2}, \frac{a_0}{2}] \) and \( y_n \in \mathbb{T} \) such that

\[
|\varphi(\omega, a_n, x_n, y_n)| \geq \frac{C_*}{2}.
\]  
(3.11)

Since \([-\frac{a_0}{2}, \frac{a_0}{2}]\) is compact, we may assume that there exists \( a_\infty \in [-\frac{a_0}{2}, \frac{a_0}{2}] \) such that \( \lim_{n \to \infty} a_n = a_\infty \). Note that \( \bar{\partial}_a \varphi(\omega, a) \) is continuous with respect to \( a \) in \( H^2(\mathbb{R} \times \mathbb{T}) \).

Then, by the Sobolev inequality and the mean-value theorem, we see that

\[
|\varphi(\omega, a_n, x, y_n) - \varphi(\omega, a, x, y_n)| \leq \|\varphi(\omega, a_n) - \varphi(\omega, a_\infty)\|_{H^2(\mathbb{R} \times \mathbb{T})} \leq |a_n - a_\infty|,
\]

where the implicit constant is independent of \( n \). We find from (3.12) that there exists a number \( n_* \geq 1 \) such that

\[
|\varphi(\omega, a_n, x, y_n) - \varphi(\omega, a_\infty, x, y_n)| \leq \frac{C_*}{4} \quad \text{for all } n \geq n_*.
\]  
(3.13)

Furthermore, it follows from (3.11) and (3.13) that

\[
|\varphi(\omega, a_\infty, x, y_n)| \geq \frac{C_*}{8} \quad \text{for all } n \geq n_*,
\]

which is absurd because \( \varphi(\omega, a_\infty) \in H^2(\mathbb{R} \times \mathbb{T}) \) implies that

\[
\lim_{|x| \to \infty} \sup_{y \in \mathbb{T}} \varphi(\omega, a_\infty, x, y) = 0.
\]

Thus, the claim of the lemma must be true. \( \square \)

Now, we give a proof of Proposition 3.1.

**Proof of Proposition 3.1.** Let \( \omega \in (\omega_p - \delta_0, \omega_p + \delta_0) \) and \( \varepsilon > 0 \) be a small constant to be specified later (see (3.19)), and define

\[
W_1(x, y) = W_1(x) := e^{-(\sqrt{\omega} - \varepsilon)|x|}, \quad W_2(x, y) = W_2(x) := e^{-(\sqrt{\omega} + \varepsilon)|x|}.
\]

Furthermore, let \( a \in [-\frac{a_0}{2}, \frac{a_0}{2}] \) be a constant to be specified later. It is easy to verify that

\[
L_-(\omega, a)W_1 = (2\sqrt{\omega} - \varepsilon^2 - |\varphi(\omega, a)|^{p-1}) W_1, \quad L_-(\omega, a)W_2 = (-2\sqrt{\omega} - \varepsilon^2 - |\varphi(\omega, a)|^{p-1}) W_2.
\]  
(3.14)  
(3.15)

Furthermore, we define

\[
v_{1,a} := W_1 - \varphi(\omega, a), \quad v_{2,a} := \varphi(\omega, a) - W_2.
\]

Then, by (3.14), (3.15) and (3.9), we see that

\[
L_-(\omega, a)v_{1,a} = (2\sqrt{\omega} - \varepsilon^2 - |\varphi(\omega, a)|^{p-1}) W_1 - E(\omega, a),
\]  
(3.16)

\[
L_-(\omega, a)v_{2,a} = (2\sqrt{\omega} + \varepsilon^2 + |\varphi(\omega, a)|^{p-1}) W_2 + E(\omega, a).
\]  
(3.17)
Define 
\[ B_{r_0} := \{ u \in X_2 : \| u \|_{H^2(\mathbb{R} \times T)} < r_0 \}. \]

Then, by Lemma 2.1 there exists a constants \( C_* > 0 \) depending only on \( p \) such that for any \( x \in \mathbb{R} \),
\[
\sup_{y \in \mathbb{T}} |E(\omega, a, x, y)| \leq \| F(\omega, \varphi(\omega, a)) \|_{L^2(\mathbb{R} \times T)} \| \psi_{\omega p} \cos y \|_{L^2(\mathbb{R} \times T)} |\psi_{\omega p}(x)| 
\leq C_* e^{-\frac{p}{p-1} \sqrt{\omega |x|}}. \tag{3.18}
\]

Now, we impose the smallness condition on \( \varepsilon \) by
\[
0 < \varepsilon < \frac{\min\{p - 1, 1\}}{4} \sqrt{\omega_p - \delta_0}. \tag{3.19}
\]

Let \( \rho(\varepsilon) > 0 \) be a number such that
\[
C_* e^{-\varepsilon \rho(\varepsilon)} \leq \varepsilon^2. \tag{3.20}
\]

Then, by (3.18), we see that if \( |x| \geq \rho(\varepsilon) \), then
\[
\sup_{y \in \mathbb{T}} |E(\omega, a, x, y)| \leq C_* e^{-\frac{p}{p-1} \sqrt{\omega_p - \delta_0} |x|} W_2 \leq C_* e^{-\varepsilon \rho(\varepsilon)} W_2 \leq \varepsilon^2 W_2 \leq \varepsilon^2 W_1. \tag{3.21}
\]

Furthermore, by Lemma 3.3 we may assume that \( \rho(\varepsilon) > 0 \) is so large that
\[
\sup_{a \in [-a_\varepsilon, a_\varepsilon]} \sup_{y \in \mathbb{T}} |\varphi(\omega, a, x, y)|^{p-1} \leq \varepsilon^2 \quad \text{for all } |x| \geq \rho(\varepsilon).
\]

Then, it follows from (3.16), (3.17), (3.21) and (3.19) that
\[
L_-(\omega, a)v_{1,a} \geq 0, \quad L_-(\omega, a)v_{2,a} \geq 0 \quad \text{for all } |x| \geq \rho(\varepsilon) \text{ and } y \in \mathbb{T}. \tag{3.22}
\]

On the other hand, Since \( \lim_{a \to 0} \| \varphi(\omega, a) - \varphi(\omega, 0) \|_{L^\infty(\mathbb{R} \times T)} = 0 \) and \( \varphi(\omega, 0) = R_{\omega} \geq \omega \frac{1}{2} e^{-\sqrt{\omega_p + \delta_0} |x|} \) (see (2.17) and (1.17)), we can take \( 0 < a_\varepsilon \leq \frac{\omega}{2} \) depending only on \( p \) and \( \varepsilon \) such that if \( a \in (-a_\varepsilon, a_\varepsilon) \), then
\[
\frac{1}{2} R_{\omega}(x) \leq \varphi(\omega, a, x, y) \leq 2 R_{\omega}(x) \quad \text{for all } |x| \leq 2 \rho(\varepsilon) \text{ and } y \in \mathbb{T}. \tag{3.23}
\]

Furthermore, by (3.20), (3.21) and (1.17), we see that for any \( y \in \mathbb{T} \),
\[
v_{1,a}(\rho(\varepsilon), y) \geq \varepsilon^{-2} C_* e^{-\varepsilon \sqrt{\rho(\varepsilon)}} - 2 \{ 2(p + 1) \sqrt{\omega_p + \delta_0} \} \frac{1}{p-1} e^{-\sqrt{\rho(\varepsilon)}}. \tag{3.24}
\]

Similarly, we see that for any \( y \in \mathbb{T} \),
\[
v_{2,a}(\rho(\varepsilon), y) \geq \frac{1}{2} (\omega_p - \delta_0) \frac{1}{p-1} e^{-\sqrt{\rho(\varepsilon)}} - C_*^{-1} \varepsilon^2 e^{-\sqrt{\rho(\varepsilon)}}. \tag{3.25}
\]

Thus, we find from (3.24) and (3.25) that if \( \varepsilon \) is sufficiently small dependently only on \( p \), then
\[
v_{1,a}(\rho(\varepsilon), y) \geq 0, \quad v_{2,a}(\rho(\varepsilon), y) \geq 0 \quad \text{for all } y \in \mathbb{T}. \tag{3.26}
\]

Define
\[
S_\varepsilon := \{ (x, y) \in \mathbb{R} \times \mathbb{T} : |x| > \rho(\varepsilon) \}.
\]
We claim that
\[ v_{1,a}, v_{2,a} \geq 0 \quad \text{for all } (x, y) \in S_\varepsilon. \] (3.27)

Note that (3.27) together with (3.23) proves Proposition 3.1. We prove (3.27) by contradiction. Suppose that (3.27) fails. Then, we see that there exists \((x_{\min}, y_{\min}) \in \mathbb{R} \times [-\pi, \pi]\) such that \(v_{1,a}(x_{\min}, y_{\min}) = \min_{(x,y) \in S_\varepsilon} v_{1,a}(x, y) < 0\). By (3.22), (3.26), and \(\omega - |\varphi(\omega, a)|^{p-1} \geq 0\) on \(S_\varepsilon\), we can apply the maximum principle and find that \(y_{\min} \not\in (-\pi, \pi)\), that is, \(y_{\min} = \pi\) or \(-\pi\). However, by the Hopf lemma and \(\partial_y v_{1,a}(\pi) = \partial_y v_{1,a}(-\pi) = 0\), we can also find that \(y_{\min} \not= \pm\pi\), which is absurd. Therefore, \(v_{1,a} \geq 0\) for all \((x, y) \in S_\varepsilon\). Similarly, we can prove that \(v_{2,a} \geq 0\) on \(S_\varepsilon\). Thus, we have completed the proof. \(\square\)

### 3.2 Exponential decay of the derivatives of \(\varphi(\omega, a)\)

In this subsection, we give a proof of Proposition 3.2. To this end, for \((\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\) and a given function \(G\) on \(\mathbb{R} \times \mathbb{T}\), we consider the following equation:
\[
L_{+}(\omega, a)v = G \quad \text{in } \mathbb{R} \times \mathbb{T}. \tag{3.28}
\]

By a standard Fourier analysis (see also Theorem 6.23 of [5]), we may write (3.28) as
\[
v(x, y) = C \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} e^{imy} \int_{\mathbb{R}} \frac{e^{i\xi(z-x)}}{\xi^2 + m^2 + \omega} \int_{\mathbb{T}} e^{-im\omega} \{p|\varphi(\omega, a, z, w)|^{p-1}v(z, w) + G(z, w)\} \, dz \, d\xi \, dw,
\]
where \(C\) is some constant. A key in proving Proposition 3.2 is the following:

**Proposition 3.4.** Assume \(p > 1\). Let \(\varepsilon_1 > 0\) be the constant given in Proposition 3.1, \(0 < \varepsilon < \varepsilon_1\), and let \(a_\varepsilon\) denote the same constant as in Proposition 3.1. Furthermore, let \((\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon)\), \(v\) be a solution to (3.28), \(A > 0\), \(B > 0\) and \(0 < \alpha < \sqrt{\omega}\). Assume that the function \(G\) on the right-hand side of (3.28) obeys that
\[
|G(x, y)| \leq Ae^{-\alpha|x|} \quad \text{for all } (x, y) \in \mathbb{R} \times \mathbb{T}. \tag{3.30}
\]

Furthermore, assume that
\[
\|v\|_{L^\infty(\mathbb{R} \times \mathbb{T})} \leq B. \tag{3.31}
\]

Then, the following holds:
\[
|v(x, y)| \leq C_\varepsilon e^{-(\alpha-\varepsilon)|x|} \quad \text{for all } (x, y) \in \mathbb{R} \times \mathbb{T}, \tag{3.32}
\]
where the implicit constant depends only on \(p, A, B, \alpha\) and \(\varepsilon\).

**Proof of Proposition 3.4.** By (3.29), we see that
\[
|v(x, y)| \sim \left| \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} e^{imy} \frac{e^{-\sqrt{m^2 + \omega}|x-z|}}{\sqrt{m^2 + \omega}} \int_{\mathbb{T}} e^{-im\omega} \{p|\varphi(\omega, a, z, w)|^{p-1}v(z, w) + G(z, w)\} \right|.
\]

Put \(\nu_m := \sqrt{m^2 + \omega}\).
Then, by (3.30), (3.31) and the assumptions (3.30) and (3.31), we see that
\[ |v(x, y)| \lesssim B \int_\mathbb{R} \sum_{m \in \mathbb{Z}} e^{-\nu_m |x-z|} e^{-(p-1)(\sqrt{\omega} - \varepsilon)|z|} \, dz + A \int_\mathbb{R} \sum_{m \in \mathbb{Z}} \frac{e^{-\nu_m |x-z|}}{\nu_m} e^{-\alpha|z|} \, dz, \quad (3.35) \]
where the implicit constant depends only on \( p \).

Consider the second term on the right-hand side of (3.35). Let \( x \in \mathbb{R} \). Then, for any \( m \in \mathbb{Z} \setminus \{0\} \), a direct computation together with \( 0 < \alpha < \sqrt{\omega} \leq \nu_m \) and \( \omega < \omega_p \) shows that
\[ \int_\mathbb{R} e^{-\nu_m |x-z|} e^{-\alpha|z|} \, dz \leq \int_\mathbb{R} e^{-(\nu_m - \alpha)|x-z|} e^{-(\nu_m - \alpha)|z|} \, dz \leq \frac{2e^{-\alpha|x|}}{\nu_m - \alpha} \lesssim \frac{e^{-\alpha|x|}}{m}. \quad (3.36) \]
where the implicit constant depends only on \( p \). Thus, by (3.36) and \( \omega > \omega_p - \delta_0 \), we see that
\[ \int_\mathbb{R} e^{-\nu_m |x-z|} e^{-\alpha|z|} \, dz \lesssim \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{m^2} e^{-\alpha|x|} + \frac{1}{\sqrt{\omega}(\sqrt{\omega} - \alpha)} e^{-\alpha|x|} \lesssim e^{-\alpha|x|}, \quad (3.37) \]
where the implicit constants depend only on \( p \) and \( \alpha \).

Consider the first term on the right-hand side of (3.35). Then, a computation similar to (3.30) together with \( \omega \geq \omega_p - \delta_0 \) shows that
\[ \int_\mathbb{R} \sum_{m \in \mathbb{Z}} \frac{e^{-\nu_m |x-z|}}{\nu_m} e^{-(p-1)(\sqrt{\omega} - \varepsilon)|z|} \, dz \lesssim \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{m^2} e^{-\min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon)|x|} + e^{-\min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon)|x|} \]
\[ \lesssim e^{-\min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon)|x|}, \quad (3.38) \]
where the implicit constants depend only on \( p \) and \( \varepsilon \). Putting (3.35), (3.37) and (3.38) together, we find that
\[ |v(x, y)| \lesssim Be^{-\min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon)|x|} + Ae^{-(\alpha-\varepsilon)|x|}, \quad (3.39) \]
where the implicit constant depends only on \( p, \alpha \) and \( \varepsilon \).

When \( \min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon) \leq \alpha - \varepsilon \), (3.39) implies the desired estimate (3.32). On the other hand, when \( \min\{1, (p-1)\}(\sqrt{\omega} - \varepsilon) > \alpha - \varepsilon \), using (3.39) in the computation (3.39)–(3.35) instead of the assumption (3.31), we can verify that
\[ |v(x, y)| \lesssim \max \left\{ e^{-\min\{1, 2(p-1)\}(\sqrt{\omega} - \varepsilon)|x|}, e^{-(\alpha-\varepsilon)|x|} \right\}, \quad (3.40) \]
where the implicit constant depends only on \( p, A, B, \alpha, \varepsilon \) and \( k \). Updating the bound of \( |v| \) in (3.39)–(3.39) one after another, we see that for any integer \( k \geq 1 \),
\[ |v(x, y)| \lesssim \max \left\{ e^{-\min\{1, k(p-1)\}(\sqrt{\omega} - \varepsilon)|x|}, e^{-(\alpha-\varepsilon)|x|} \right\}, \quad (3.41) \]
where the implicit constant depends only on \( p, A, B, \alpha, \varepsilon \) and \( k \). This implies (3.32).

Now, we are in a position to prove Proposition 3.2.
Proof of Proposition 3.2. Note that \( \partial_\omega \varphi(\omega, a) \) and \( \partial_a \varphi(\omega, a) \) satisfy
\[
L_+ (\omega, a) \partial_\omega \varphi(\omega, a) = -\varphi(\omega, a) + \tilde{E}(\omega, a), \quad L_+ (\omega, a) \partial_a \varphi(\omega, a) = \tilde{E}(\omega, a),
\]
where
\[
\tilde{E}(\omega, a) := \langle L_+ (\omega, a) \partial_\omega \varphi(\omega, a) + \varphi(\omega, a), \psi_{\omega_p} \cos y \rangle \psi_{\omega_p} \cos y,
\]
\[
\tilde{E}(\omega, a) := \langle L_+ (\omega, a) \partial_a \varphi(\omega, a), \psi_{\omega_p} \cos y \rangle \psi_{\omega_p} \cos y.
\]
Then, by Proposition 3.1 (2.2), Lemma 2.1 and (2.15), we see that \( \varphi(\omega, a), \tilde{E}(\omega, a) \) and \( \tilde{E}(\omega, a) \) have exponential decay with respect to \( x \):
\[
|\varphi(\omega, a)| \lesssim e^{-\sqrt{\omega-\varepsilon}|x|}, \quad |\tilde{E}(\omega, a)| + |\tilde{E}(\omega, a)| \lesssim e^{-\frac{p+1}{2} \sqrt{\omega-\varepsilon}} \lesssim e^{-\frac{p-1}{2} \sqrt{\omega-\varepsilon}|x|},
\]
where the implicit constants depend only on \( p \) and \( \varepsilon \). Applying Proposition 3.4 as
\( G = -\varphi(\omega, a) + \tilde{E}(\omega, a) \) and \( G = \tilde{E}(\omega, a) \), we find that Proposition 3.2 is true. \( \square \)

4 Computation of derivatives

The aim of this section is to compute the second and third derivatives of \( \varphi(\omega, a) \) and \( F_{\|} (\omega, a) \) with respect to \( a \) and \( \omega \), which are used in the application of the Crandall-Rabinowitz argument (3) (see Section 4). Note that when \( 1 < p < 2 \), even twice differentiability is not obvious, as the nonlinearity of (1.1) is not twice differentiable.

Throughout this section, let \( a_0 > 0 \) and \( \delta_0 > 0 \) be the constants given in Lemma 2.1 and let \( \varepsilon_1 > 0 \) be the constant given in Proposition 3.1. Furthermore, for \( 0 < \varepsilon < \varepsilon_1 \), we use \( a_0 \) to denote the same constant as in Proposition 3.1. We will assume that \( \delta_0 \) and \( \varepsilon_1 \) are sufficiently small dependently only on \( p \) without any notice.

Recall from Proposition 3.2 that if \( p > 1 \) and \( 0 < \varepsilon < \varepsilon_1 \), then the following holds for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a\varepsilon) \) and \((x, y) \in \mathbb{R} \times \mathbb{T} \):
\[
|\partial_a \varphi(\omega, a, x, y)| + |\partial_\omega \varphi(\omega, a, x, y)| \lesssim e^{-\sqrt{\omega-\varepsilon}|x| + 2\varepsilon |x|},
\]
where the implicit constant depends only on \( p \) and \( \varepsilon \).

We introduce symbols which are used in this and the next sections:

Notation.

1. For \( p > 1 \), \( k \geq 1 \) and \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \), we define \( V_k(\omega, a) \) as
\[
V_k(\omega, a) := \frac{d^{2k} x^p}{d x^k} \bigg|_{x=\varphi(\omega, a)} = p(p-1) \cdots (p-k+1) \varphi(\omega, a)^{p-k}.
\]
Note that
\[
L_+ (\omega, a) = -\partial_x^2 - \partial_y^2 + \omega - V_1(\omega, a).
\]
Since \( \varphi(\omega, a) \) is positive (see Proposition 3.1) and of class \( C^1 \) on \( (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \) in the \( H^2(\mathbb{R} \times \mathbb{T}) \)-topology (see Lemma 2.1 and 2.12), the following holds:
\[
\partial V_k(\omega, a) = V_{k+1}(\omega, a) \partial_\omega \varphi(\omega, a) \quad \text{everywhere in} \quad \mathbb{R} \times \mathbb{T},
\]
where \( \partial \) denotes either \( \partial_a \) or \( \partial_\omega \). Furthermore, Proposition 3.1 shows that if \( 0 < \varepsilon < \varepsilon_1 \) and \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a\varepsilon) \), then
\[
V_k(\omega, a, x, y) \lesssim \varphi(\omega, a, x, y)^{p-k} \lesssim e^{(k-p)\sqrt{\omega-\varepsilon}|x| + \varepsilon(k+p)|x|},
\]
where the implicit constants depend only on \( p, \varepsilon \) and \( k \).
2. For \( p > 1 \) and \((\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\), we define \( T(\omega, a) \) as

\[
T(\omega, a) := P_\perp L_+(\omega, a)|_{X_2}.
\]

By (3.6), we see that

\[
T(\omega_p, 0) = P_\perp \partial_a F(\omega_p, R_{\omega_p})|_{X_2},
\]

(4.6)

We have to pay attention to the difference between \( T(\omega, a) \) and \( P_\perp L_+(\omega, a) \); In particular, \( T(\omega, a) \) has the inverse, but \( P_\perp L_+(\omega, a) \) does not.

4.1 Basic results

It is easy to verify that for any \((\omega_1, a_1), (\omega_2, a_2) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\),

\[
\begin{align*}
T(\omega_1, a_1) - T(\omega_2, a_2) &= -P_\perp \{V_1(\omega_1, a_1) - V_1(\omega_2, a_2)\}, \\
T(\omega_1, a_1)^{-1} - T(\omega_2, a_2)^{-1} &= T(\omega_1, a_1)^{-1}\{T(\omega_2, a_2) - T(\omega_1, a_1)\}T(\omega_2, a_2)^{-1}.
\end{align*}
\]

(4.7) \hspace{1cm} (4.8)

Furthermore, it is known that for any \( p > 1 \) and \((\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\),

\[
\|T(\omega, a)^{-1}\|_{L^2(\mathbb{R} \times T) \rightarrow H^2(\mathbb{R} \times T)} \lesssim 1,
\]

(4.9)

where the implicit constant depends only on \( p \), which together with the linearity implies the continuity, namely if \( \lim_{n \rightarrow \infty} f_n = f \) in \( L^2(\mathbb{R} \times T) \), then

\[
\lim_{n \rightarrow \infty} T(\omega, a)^{-1} f_n = T(\omega, a)^{-1} f \quad \text{in} \quad H^2(\mathbb{R} \times T).
\]

(4.10)

By the boundedness of \( P_\perp \) in \( L^2(\mathbb{R} \times T) \), the continuity of \( \varphi(\omega, a) \) with respect to \((\omega, a)\) (see Lemma 2.1 and (2.12)), and (4.7) through (4.9), we can obtain the following lemma:

**Lemma 4.1.** Assume \( p > 1 \). Then, the operators \( T(\omega, a) : X_2 \rightarrow Y_2 \) and \( T(\omega, a)^{-1} : Y_2 \rightarrow X_2 \) are continuous with respect to \((\omega, a)\) on \((\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0)\), namely,

\[
\lim_{(\gamma_1, \gamma_2) \rightarrow (0, 0)} \|T(\omega + \gamma_1, a + \gamma_2) - T(\omega, a)\|_{H^2(\mathbb{R} \times T) \rightarrow L^2(\mathbb{R} \times T)} = 0,
\]

(4.11)

\[
\lim_{(\gamma_1, \gamma_2) \rightarrow (0, 0)} \|T(\omega + \gamma_1, a + \gamma_2)^{-1} - T(\omega, a)^{-1}\|_{L^2(\mathbb{R} \times T) \rightarrow H^2(\mathbb{R} \times T)} = 0.
\]

(4.12)

It is easy to verify that the following lemma holds:

**Lemma 4.2.** If \( f : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \rightarrow L^2(\mathbb{R} \times T) \) has the partial derivatives in \( L^2(\mathbb{R} \times T) \) \((\partial_\omega f(\omega, a), \partial_a f(\omega, a) \in L^2(\mathbb{R} \times T))\), then,

\[
\begin{align*}
\partial_\omega (f(\omega, a), \psi_{\omega_p} \cos y) &= \langle \partial_\omega f(\omega, a), \psi_{\omega_p} \cos y \rangle, \\
\partial_a (f(\omega, a), \psi_{\omega_p} \cos y) &= \langle \partial_a f(\omega, a), \psi_{\omega_p} \cos y \rangle.
\end{align*}
\]

In particular, the following holds in \( L^2(\mathbb{R} \times T) \):

\[
\begin{align*}
\partial_\omega \{P_\perp f(\omega, a)\} &= P_\perp \partial_\omega f(\omega, a), \\
\partial_a \{P_\perp f(\omega, a)\} &= P_\perp \partial_a f(\omega, a).
\end{align*}
\]

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Furthermore, by (4.16), (4.5), and (4.1), we see that together, we find that (4.13) holds. Similarly, we can prove (4.14).

Lemma 4.3. Let \( f : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \to H^2(\mathbb{R} \times \mathbb{T}) \) has the partial derivatives in \( H^2(\mathbb{R} \times \mathbb{T}) \) \( (\partial_\omega f(\omega, a), \partial_a f(\omega, a)) \in H^2(\mathbb{R} \times \mathbb{T}) \), then, the following hold in \( L^2(\mathbb{R} \times \mathbb{T}) \) for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \):

\[
\partial_\omega \{(-\partial_\omega^2 - \partial_a^2 + \omega) f(\omega, a)\} = (-\partial_\omega^2 - \partial_a^2 + \omega) \partial_\omega f(\omega, a),
\]

\[
\partial_a \{(-\partial_\omega^2 - \partial_a^2 + \omega) f(\omega, a)\} = (-\partial_\omega^2 - \partial_a^2 + \omega) \partial_a f(\omega, a) + f(\omega, a).
\]

Lemma 4.4. Assume \( p > 1 \), and let \( 0 < \varepsilon < \varepsilon_1 \). Then, the following hold in \( L^2(\mathbb{R} \times \mathbb{T}) \) for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \):

\[
\partial_\omega V_1(\omega, a) = V_2(\omega, a) \partial_\omega \varphi(\omega, a),
\]

(4.13)

\[
\partial_a V_1(\omega, a) = V_2(\omega, a) \partial_a \varphi(\omega, a).
\]

(4.14)

Proof of Lemma 4.4. We shall prove (4.13). It follows from (4.14) that

\[
\partial_\omega V_1(\omega, a) = V_2(\omega, a) \partial_\omega \varphi(\omega, a) \quad \text{everywhere in } \mathbb{R} \times \mathbb{T}.
\]

(4.15)

We may assume \( \sqrt{\omega} > \frac{1}{2} \sqrt{\omega_p} \gg \varepsilon_1 > \varepsilon \). Then, by the fundamental theorem of calculus, (4.5) and (4.1), we see that

\[
\left| V_1(\omega, a + \delta) - V_1(\omega, a) \right| \leq \int_0^1 \left| V_2(\omega, a + \theta \delta) \partial_\omega \varphi(\omega, a + \theta \delta) \right| d\theta
\]

\[
\leq e^{-\frac{\varepsilon_1}{2} \sqrt{\omega_p} |x| + \varepsilon (4+p)|x|} \quad \text{everywhere in } \mathbb{R} \times \mathbb{T},
\]

(4.16)

where the implicit constant depends only on \( p \) and \( \varepsilon \). Hence, Lebesgue’s dominated convergence theorem together with (4.15) shows that

\[
\partial_\omega V_1(\omega, a) = V_2(\omega, a) \partial_\omega \varphi(\omega, a) \quad \text{in } L^1(\mathbb{R} \times \mathbb{T}).
\]

(4.17)

Furthermore, by (4.16), (4.5), and (4.1), we see that

\[
\limsup_{\delta \to 0} \left\| \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} - V_2(\omega, a) \partial_\omega \varphi(\omega, a) \right\|_{L^\infty(\mathbb{R} \times \mathbb{T})} \leq 1,
\]

(4.18)

where the implicit constant depends only on \( p, \varepsilon \) and \( k \). Putting (4.17) and (4.18) together, we find that (4.13) holds. Similarly, we can prove (4.14).

By (4.5), (4.1), Lemma 4.4 and a direct computation, we can obtain the following lemma:

Lemma 4.5. Assume \( p > 1 \) and let \( 0 < \varepsilon < \varepsilon_1 \). If \( f : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \to H^2(\mathbb{R} \times \mathbb{T}) \) has the partial derivatives in \( H^2(\mathbb{R} \times \mathbb{T}) \) \( (\partial_\omega f(\omega, a), \partial_a f(\omega, a)) \in H^2(\mathbb{R} \times \mathbb{T}) \), then the following hold in \( L^2(\mathbb{R} \times \mathbb{T}) \) for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \):

\[
\partial_\omega \{V_1(\omega, a) f(\omega, a)\} = V_1(\omega, a) \partial_\omega f(\omega, a) + V_2(\omega, a) \partial_\omega \varphi(\omega, a) f(\omega, a),
\]

\[
\partial_a \{V_1(\omega, a) f(\omega, a)\} = V_1(\omega, a) \partial_a f(\omega, a) + V_2(\omega, a) \partial_a \varphi(\omega, a) f(\omega, a).
\]
Remark 4.1. Observe that we assume the differentiability of \( f(\omega, a) \) in \( H^2(\mathbb{R} \times T) \), but the derivatives of the product \( V_1(\omega, a)f(\omega, a) \) is taken in the \( L^2(\mathbb{R} \times T) \)-sense.

The following lemma immediately follows from Lemmas 4.3 and 4.5.

Lemma 4.6. Assume \( p > 1 \) and let \( 0 < \varepsilon < \varepsilon_1 \). If \( f: (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e) \to H^2(\mathbb{R} \times T) \) has the partial derivatives in \( H^2(\mathbb{R} \times T) \) (\( \partial_\omega f(\omega, a), \partial_x f(\omega, a) \in H^2(\mathbb{R} \times T) \)), then, the following hold in \( L^2(\mathbb{R} \times T) \) for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e) \):

\[
\begin{align*}
\partial_\omega \{ \mathbf{L}_+(\omega, a)f(\omega, a) \} &= \mathbf{L}_+ \{ \partial_\omega f(\omega, a) - V_2(\omega, a)\partial_\omega \varphi(\omega, a)f(\omega, a) \}, \\
\partial_x \{ \mathbf{L}_+(\omega, a)f(\omega, a) \} &= \mathbf{L}_+ \{ \partial_x f(\omega, a) + f(\omega, a) - V_2(\omega, a)\partial_x \varphi(\omega, a)f(\omega, a) \}.
\end{align*}
\]

Lemma 4.7. Assume \( p > 1 \) and \( 0 < \varepsilon < \varepsilon_1 \). If \( f: (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e) \to Y_2 \) has the partial derivatives in \( Y_2 \) (\( \partial_\omega f(\omega, a), \partial_x f(\omega, a) \in Y_2 \)), then, the following hold in \( H^2(\mathbb{R} \times T) \) for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e) \): If

\[
\tilde{f}(\omega, a) := T(\omega, a)^{-1}f(\omega, a),
\]

then

\[
\begin{align*}
\partial_\omega \tilde{f}(\omega, a) &= T(\omega, a)^{-1}\partial_\omega f(\omega, a) + T(\omega, a)^{-1}P_\perp \{ V_2(\omega, a)\partial_\omega \varphi(\omega, a)\tilde{f}(\omega, a) \}, \\
\partial_x \tilde{f}(\omega, a) &= T(\omega, a)^{-1}\partial_x f(\omega, a) - T(\omega, a)^{-1}\tilde{f}(\omega, a) + T(\omega, a)^{-1}P_\perp \{ V_2(\omega, a)\partial_x \varphi(\omega, a)\tilde{f}(\omega, a) \}.
\end{align*}
\]

Proof of Lemma 4.7. We shall prove \((4.19)\). We compute the derivative of \( \tilde{f}(\omega, a) := T(\omega, a)^{-1}f(\omega, a) \) with respect to \( a \) in accordance with the definition. Let \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e) \), and let \( \delta \) be a constant to be taken \( \delta \to 0 \). It is easy to see that the following identity holds everywhere in \( \mathbb{R} \times T \):

\[
\frac{\tilde{f}(\omega, a + \delta) - \tilde{f}(\omega, a)}{\delta} = \frac{T(\omega, a + \delta)^{-1}f(\omega, a + \delta) - T(\omega, a)^{-1}f(\omega, a)}{\delta}
\]

\[
= \delta^{-1}\{ T(\omega, a + \delta)^{-1} - T(\omega, a)^{-1} \} f(\omega, a + \delta)
\]

\[
+ T(\omega, a)^{-1}f(\omega, a + \delta) - f(\omega, a).
\]

Using \((4.18)\) and \((4.27)\), we can rewrite the first term on the right-hand side of \((4.21)\) as follows:

\[
\begin{align*}
&\delta^{-1}\{ T(\omega, a + \delta)^{-1} - T(\omega, a)^{-1} \} f(\omega, a + \delta) \\
&= -\delta^{-1}T(\omega, a)^{-1}\{ T(\omega, a + \delta) - T(\omega, a) \} T(\omega, a + \delta)^{-1}f(\omega, a + \delta) \\
&= T(\omega, a)^{-1}P_\perp \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} T(\omega, a + \delta)^{-1}f(\omega, a + \delta) \right) \\
&= T(\omega, a)^{-1}P_\perp \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} T(\omega, a)^{-1}f(\omega, a) \right) \\
&+ T(\omega, a)^{-1}P_\perp \left( \{ V_1(\omega, a + \delta) - V_1(\omega, a) \} T(\omega, a)^{-1}f(\omega, a) \right) \\
&+ T(\omega, a)^{-1}P_\perp \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} \{ T(\omega, a + \delta)^{-1} - T(\omega, a)^{-1} \} f(\omega, a + \delta) \right).
\end{align*}
\]
We consider the first term on the right-hand side of (4.22). It follows from the continuity of $T(\omega, a)^{-1} : L^2(\mathbb{R} \times T) \to H^2(\mathbb{R} \times T)$ (see (4.10)) and the differentiability of $V_1(\omega, a)$ in $L^2(\mathbb{R} \times T)$ (see Lemma 4.1) that

$$
\lim_{\delta \to 0} T(\omega, a)^{-1} P_{\perp} \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} T(\omega, a)^{-1} f(\omega, a) \right) = T(\omega, a)^{-1} P_{\perp} \{ V_2(\omega, a) \partial_a \varphi(\omega, a) T(\omega, a)^{-1} f(\omega, a) \} \quad \text{in } H^2(\mathbb{R} \times T).
$$

(4.23)

Next, we consider the second term on the right-hand side of (4.22). Let $1 < p \leq 2$ By (4.9), a convexity ($0 < p - 1 < 1$), the continuity of $\varphi(\omega, a)$ with respect to $a$ in $H^2(\mathbb{R} \times T)$, and the differentiability of $f(\omega, a)$, we see that

$$
\| T(\omega, a)^{-1} P_{\perp} \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} T(\omega, a)^{-1} f(\omega, a + \delta) - f(\omega, a) \right) \|_{H^2(\mathbb{R} \times T)}
\lesssim \| \{ V_1(\omega, a + \delta) - V_1(\omega, a) \} T(\omega, a)^{-1} \frac{f(\omega, a + \delta) - f(\omega, a)}{\delta} \|_{L^2(\mathbb{R} \times T)}
\leq \| \varphi(\omega, a + \delta) - \varphi(\omega, a) \|_{L^p(\mathbb{R} \times T)}^{p-1} \| \frac{f(\omega, a + \delta) - f(\omega, a)}{\delta} \|_{L^2(\mathbb{R} \times T)}
\to 0 \quad \text{as } \delta \to 0.
$$

(4.24)

We can prove the case of $p \geq 2$ similarly.

We consider the last term on the right-hand side of (4.22). By the fundamental theorem of calculus, Lemma 4.3 (4.5), and (4.11), we see that

$$
\| \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} \|_{L^\infty(\mathbb{R} \times T)} = \| \int_0^1 V_2(\omega, a + \theta \delta) \partial_a \varphi(\omega, a + \theta \delta) \, d\theta \|_{L^\infty(\mathbb{R} \times T)} \lesssim 1,
$$

(4.25)

where the implicit constant depends only on $p$ and $\varepsilon$. Then, by (4.9), (4.25) and Lemma 4.1 we see that

$$
\| T(\omega, a)^{-1} P_{\perp} \left( \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} \right) \|_{H^2(\mathbb{R} \times T)}
\lesssim \| \frac{V_1(\omega, a + \delta) - V_1(\omega, a)}{\delta} \|_{L^\infty(\mathbb{R} \times T)} \| T(\omega, a)^{-1} \|_{L^2(\mathbb{R} \times T)}
\to 0 \quad \text{as } \delta \to 0.
$$

(4.26)

It remains to consider the second term on the right-hand side of (4.21). By the continuity of $T(\omega, a)^{-1} : L^2(\mathbb{R} \times T) \to H^2(\mathbb{R} \times T)$ (see (4.10)) and the differentiability of $f(\omega, a)$, we see that

$$
\lim_{\delta \to 0} T(\omega, a)^{-1} \frac{f(\omega, a + \delta) - f(\omega, a)}{\delta} = T(\omega, a)^{-1} \partial_a f(\omega, a) \quad \text{in } H^2(\mathbb{R} \times T).
$$

(4.27)

Putting the above computations together, we find that (4.19) holds. Similarly, we can prove (4.20).
4.2 Second derivatives of \( \varphi(\omega, a) \)

In this section, we compute the second derivatives of \( \varphi(\omega, a) \) with respect to \( \omega \) and \( a \). We emphasize that when \( 1 < p < 2 \), the twice differentiability of \( \varphi(\omega, a) \) is not obvious, as the nonlinearity of the equation (1.1) is not \( C^2 \).

Recall that \( \eta: (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \rightarrow X_2 \) is \( C^1 \) and \( P_1 F(\omega, \varphi(\omega, a)) = 0 \) (see Lemma 2.1). Observe from a direct computation that the following holds in \( L^2(\mathbb{R} \times T) \):

\[
\begin{align*}
\partial_\omega \{ V_0(\omega, a) \} &= p \varphi(\omega, a)^{p-1} \partial_\omega \varphi(\omega, a) = V_1(\omega, a) \{ \psi_{\omega p} \cos y + \partial_\omega h(\omega, a) \}, \\
\partial_\omega \{ V_0(\omega, a) \} &= p \varphi(\omega, a)^{p-1} \partial_\omega \varphi(\omega, a) = V_1(\omega, a) \partial_\omega h(\omega, a).
\end{align*}
\]

(4.28)

Note that Lemma 2.1 shows that

\[
\partial_\omega \varphi(\omega, a) = -T(\omega, a)^{-1} \{ R_{\omega p} + \eta(\omega, a) \},
\]

(4.30)

\[
\partial_\omega \varphi(\omega, a) = \psi_{\omega p} \cos y - T(\omega, a)^{-1} P_1 L_+ (\omega, a) \psi_{\omega p} \cos y.
\]

(4.31)

In order to prove the continuity of the second and higher derivatives of \( \varphi(\omega, a) \) in \( H^2(\mathbb{R} \times T) \), we prepare the following lemma:

**Lemma 4.8.** Assume \( p > 1 \) and let \( k, j \) be integers satisfying \( k > p \) and \( j \geq 1 \). Then, there exists \( 0 < \varepsilon(k, j) < \varepsilon_1 \) depending only on \( p, k, j \) and \( \varepsilon \) with the following property: Let \( 0 < \varepsilon < \varepsilon(k, j) \), and let \( g \) be a function in \( C((\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0), L^2(\mathbb{R} \times T)) \). Assume that for any \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \) and \( (x, y) \in \mathbb{R} \times T \):

\[
|g(\omega, a, x, y)| \lesssim e^{-k \sqrt{\omega} |x| + j \varepsilon |x|},
\]

where the implicit constant depends only on \( p, k, j \) and \( \varepsilon \). Furthermore, assume that

\[
\lim_{(\gamma_1, \gamma_2) \rightarrow (0, 0)} \| e^{(k-1) \sqrt{\omega} |x| + k \varepsilon |x|} \{ g(\omega + \gamma_1, a + \gamma_2) - g(\omega, a) \} \|_{L^2(\mathbb{R} \times T)} = 0.
\]

Then, \( T(\omega, a)^{-1} P_1 \{ V_k(\omega, a) g(\omega, a) \} \) is continuous with respect to \( (\omega, a) \) on \( (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \) in the \( H^2(\mathbb{R} \times T) \)-topology.

**Proof of Lemma 4.8.** Let \( 0 < \varepsilon < \varepsilon_1 \), \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \), and let \( (\gamma_1, \gamma_2) \in \mathbb{R}^2 \). We will assume \( \sqrt{\omega} > \frac{1}{2} \sqrt{\omega_p} \) and \( \varepsilon \) being sufficiently small dependently only on \( p, k, j \), and \( \varepsilon \), without any notice. Furthermore, we will take \( (\gamma_1, \gamma_2) \rightarrow (0, 0) \), so that we may assume that

\[
(\omega + \gamma_1, a + \gamma_2) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0),
\]

(4.34)

\[
\sqrt{\omega} - \varepsilon \leq \sqrt{\omega - |\gamma_1|} \leq \sqrt{\omega + |\gamma_1|} \leq \sqrt{\omega} + \varepsilon.
\]

(4.35)

First, we shall show that

\[
\lim_{(\gamma_1, \gamma_2) \rightarrow (0, 0)} \| V_k(\omega + \gamma_1, a + \gamma_2) g(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a) g(\omega, a) \|_{L^2(\mathbb{R} \times T)} = 0.
\]

(4.36)

Observe that

\[
\begin{align*}
\| V_k(\omega + \gamma_1, a + \gamma_2) g(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a) g(\omega, a) \|_{L^2(\mathbb{R} \times T)} \\
\leq \| V_k(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a) \|_{L^2(\mathbb{R} \times T)} g(\omega + \gamma_1, a + \gamma_2) \|_{L^2(\mathbb{R} \times T)} \\
+ \| V_k(\omega, a) \|_{L^2(\mathbb{R} \times T)} \| g(\omega + \gamma_1, a + \gamma_2) - g(\omega, a) \|_{L^2(\mathbb{R} \times T)}.
\end{align*}
\]

(4.37)
Consider the first term on the right-hand side of (4.37). By the fundamental theorem of calculus, (4.32), \(4.31\), \(4.1\), \(4.35\) and \(p > 1\), the following holds everywhere in \(\mathbb{R} \times T\):

\[
\left| \{ V_k(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a) \} g(\omega + \gamma_1, a + \gamma_2) \right| \\
\lesssim |V_k(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a + \gamma_2)| e^{-k\sqrt{-\gamma_1|x| + jx|x|}} \\
+ |V_k(\omega, a + \gamma_2) - V_k(\omega, a)| e^{-k\sqrt{-\gamma_1|x| + jx|x|}} \\
\lesssim |\gamma_1| \int_0^1 |V_{k+1}(\omega + \theta \gamma_1, a + \gamma_2)| |\partial_\omega \varphi(\omega + \theta \gamma_1, a + \gamma_2)| d\theta e^{-k\sqrt{-\gamma_1|x| + jx|x|}} \\
+ |\gamma_2| \int_0^1 |V_{k+1}(\omega, a + \theta \gamma_2)| |\partial_a \varphi(\omega, a + \theta \gamma_2)| d\theta e^{-k\sqrt{-\gamma_1|x| + jx|x|}} \\
\lesssim (|\gamma_1| + |\gamma_2|) e^{-\frac{p}{2} \sqrt{-\gamma_1|x| + (3k + j + 5 - 2p)\varepsilon|x|}} \lesssim (|\gamma_1| + |\gamma_2|) e^{-\frac{p}{2} \sqrt{-\gamma_1|x|}},
\]

where the implicit constants depend only on \(p, k, j\) and \(\varepsilon\). Thus, we find that

\[
\lim_{(\gamma_1, \gamma_2) \to (0, 0)} \| \{ V_k(\omega + \gamma_1, a + \gamma_2) - V_k(\omega, a) \} g(\omega + \gamma_1, a + \gamma_2) \|_{L^2(\mathbb{R} \times T)} = 0.
\]

(4.39)

Move on to the second term on the right-hand side of (4.37). By (4.5) and \(p > 1\), we see that

\[
|V_k(\omega, a) e^{-(k-1)\sqrt{-\gamma_1|x|-k\varepsilon|x|}}| \lesssim e^{-\frac{(k-1)}{2} \sqrt{-\gamma_1|x|}},
\]

(4.40)

where the implicit constant depends only on \(p, k, j\) and \(\varepsilon\). Then, by (4.40) and (4.33), we see that

\[
\lim_{(\gamma_1, \gamma_2) \to (0, 0)} \| V_k(\omega, a) \{ g(\omega + \gamma_1, a + \gamma_2) - g(\omega, a) \} \|_{L^2(\mathbb{R} \times T)} = 0.
\]

(4.41)

Putting (4.37), (4.39) and (4.41) together, we find that (4.36) holds.

We shall finish the proof of the lemma.

Observe from (4.32), (4.31), \(p > 1\) and (4.33) that

\[
\| V_k(\omega + \gamma_1, a + \gamma_2) g(\omega + \gamma_1, a + \gamma_2) \|_{L^2(\mathbb{R} \times T)} \\
\lesssim \| e^{\frac{p}{2} \sqrt{-\gamma_1|x| + (3k - 2p)\varepsilon|x|}} \|_{L^2(\mathbb{R} \times T)} \lesssim \| e^{\frac{p}{2} \sqrt{-\gamma_1|x|}} \|_{L^2(\mathbb{R} \times T)} \lesssim 1,
\]

(4.42)

where the implicit constants depend only on \(p, k, j\) and \(\varepsilon\). Then, by (4.42), Lemma 4.1 (4.9) and (4.36), we can prove the continuity of \(T(\omega, a)^{-1} P_\perp \{ V_k(\omega, a) g(\omega, a) \} \) on \((\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_x, a_x)\) in the \(H^2(\mathbb{R} \times T)\)-topology. \(\square\)

Next, we give the second derivatives of \(\varphi(\omega, a)\):

**Proposition 4.9.** Assume \(p > 1\). Then, there exists \(0 < \varepsilon_2 < \varepsilon_1\) depending only on \(p\) such that if \(0 < \varepsilon < \varepsilon_2\), then \(\varphi\) is \(C^2\) on \((\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_x, a_x)\) in the \(H^2(\mathbb{R} \times T)\)-topology; and the following hold for all \((\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_x, a_x)\):

\[
\partial_\omega^2 \varphi(\omega, a) = T(\omega, a)^{-1} P_\perp \{ V_2(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^2 \},
\]

(4.43)

\[
\partial_a^2 \varphi(\omega, a) = T(\omega, a)^{-1} P_\perp \{ V_2(\omega, a) \{ \partial_a \varphi(\omega, a) \}^2 - \partial_\omega \varphi(\omega, a) \},
\]

(4.44)

\[
\partial_\omega \partial_a \varphi(\omega, a) = \partial_\omega \partial_a \varphi(\omega, a),
\]

(4.45)
Remark 4.2. Since $T(\omega, a)^{-1}$ maps $Y_2$ to $X_2$, Proposition 4.9 shows that
\[ \partial_{a}^2 \varphi(\omega, a), \partial_{a} \partial_{a} \varphi(\omega, a), \partial_{a} \partial_{\omega} \varphi(\omega, a), \partial_{a} \varphi(\omega, a) \in X_2. \]

Proof of Proposition 4.9. We shall prove (4.43). By Lemmas 4.2 and 4.6 the following holds in $L^2(\mathbb{R} \times T)$:
\[
\partial_{a} \left( P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) = -P_{\perp} \left( V_{2}(\omega, a) \partial_{a} \varphi(\omega, a) \psi_{\omega_p} \cos y \right), \quad (4.46) \\
\partial_{\omega} \left( P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) = P_{\perp} \{ \psi_{\omega_p} \cos y \} - P_{\perp} \left( V_{2}(\omega, a) \partial_{a} \varphi(\omega, a) \psi_{\omega_p} \cos y \right). \quad (4.47)
\]
Furthermore, by (4.31), Lemma 4.7 and (4.46), the following holds in $H^2(\mathbb{R} \times T)$:
\[
\partial_{a} \varphi(\omega, a) = -\partial_{a} \left( T(\omega, a)^{-1} P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) \\
= -T(\omega, a)^{-1} \partial_{a} \left( P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) \\
- T(\omega, a)^{-1} P_{\perp} \left( V_{2}(\omega, a) \partial_{a} \varphi(\omega, a) T(\omega, a)^{-1} P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) \\
= T(\omega, a)^{-1} P_{\perp} \left( V_{2}(\omega, a) \partial_{a} \varphi(\omega, a) \{ \psi_{\omega_p} \cos y - T(\omega, a)^{-1} P_{L+}(\omega, a) \{ \psi_{\omega_p} \cos y \} \right) \right). \quad (4.48)
\]
Plugging (4.31) into the right-hand side of (4.48), we obtain (4.43). Similarly, we can prove (4.44).

It remains to prove the continuity of the second derivatives. When $2 \leq p$, we see from the implicit function theorem that $\varphi(\omega, a)$ is $C^2$ with respect to $a$ and $\omega$. Hence, we may assume that $2 > p$. Observe from (4.1) that
\[
\| \{ \partial_{a} \varphi(\omega, a, x, y) \}^2 \| \lesssim e^{-2\sqrt{\omega}|x|+4|x|}, \quad (4.49)
\]
where the implicit constant depends only on $p$ and $\varepsilon$. Furthermore, by (4.1), the embedding $H^2(\mathbb{R} \times T) \hookrightarrow L^\infty(\mathbb{R} \times T)$, and the continuity of $\partial_{a} \varphi(\omega, a)$ in $H^2(\mathbb{R} \times T)$, we see that
\[
\| e^{\sqrt{\omega}|x|+2|x|} \left( (\partial_{a} \varphi(\omega + \gamma_1, a + \gamma_2))^2 - (\partial_{a} \varphi(\omega, a, x, y))^2 \right) \|_{L^2(\mathbb{R} \times T)} \leq \| e^{\sqrt{\omega}|x|+2|x|} e^{-\frac{1}{2}(\sqrt{\omega+|\gamma_1|^2+|\gamma_2|^2})|x|} \|_{L^2(\mathbb{R} \times T)} \| \varphi(\omega + \gamma_1, a + \gamma_2) - \partial_{a} \varphi(\omega, a, x, y) \|_{L^\infty(\mathbb{R} \times T)} \right. \\
\rightarrow 0 \quad \text{as} \ (\gamma_1, \gamma_2) \rightarrow (0, 0). \quad (4.50)
\]
Then, by (4.43), (4.49) and (4.50), we find that Lemma 4.8 can apply to $\partial_{a}^2 \varphi(\omega, a)$ as $k = 2, j = 4, g(\omega, a) = \{ \partial_{a} \varphi(\omega, a) \}^2$. Thus, we see that $\partial_{a}^2 \varphi(\omega, a)$ is continuous with respect to $(\omega, a)$ in the $H^2(\mathbb{R} \times T)$-topology. Similarly, we can prove the continuity of the other partial derivatives.

We state decay properties of the derivatives of $\varphi(\omega, a)$:
Lemma 4.10. Let $p > 1$. Then, there exists $\tilde{\varepsilon}_2 > 0$ depending only on $p$ such that for any $0 < \varepsilon < \tilde{\varepsilon}_2$ and $(\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e)$, the following holds:

$$|\partial^2_a \varphi(\omega, a, x, y)| + |\partial_x \partial_a \varphi(\omega, a, x, y)| + |\partial^2_x \varphi(\omega, a, x, y)| \lesssim e^{-\sqrt{\varepsilon} |x|}.$$  \hspace{1cm} (4.51)

where the implicit constants depend only on $p$ and $\varepsilon$.

Proof. We may write (4.43) through (4.49) in Proposition 4.9 as follows:

$$(4.52) \hspace{1cm} L_1(\omega, a)\partial^2_a \varphi(\omega, a) = V_2(\omega, a)\{\partial_a \varphi(\omega, a)\}^2,$$

$$(4.53) \hspace{1cm} L_1(\omega, a)\partial_x \partial_a \varphi(\omega, a) = V_2(\omega, a)\partial_x \varphi(\omega, a)\partial_a \varphi(\omega, a) - \partial_a \varphi(\omega, a),$$

$$(4.54) \hspace{1cm} L_1(\omega, a)\partial^2_x \varphi(\omega, a) = V_2(\omega, a)\{\partial_x \varphi(\omega, a)\}^2 - \partial_x \varphi(\omega, a).$$

Observe from (4.3) and (4.1) that if $\varepsilon$ is sufficiently small depending only on $p$, then

$$|V_2(\omega, a)\{\partial_a \varphi(\omega, a)\}^2| \lesssim e^{-\sqrt{\varepsilon} |x|},$$

$$|V_2(\omega, a)\partial_x \varphi(\omega, a)\partial_a \varphi(\omega, a)| + |\partial_a \varphi(\omega, a)| \lesssim e^{-\sqrt{\varepsilon} |x|},$$

$$|V_2(\omega, a)\{\partial_x \varphi(\omega, a)\}^2| + |\partial_x \varphi(\omega, a)| \lesssim e^{-\sqrt{\varepsilon} |x|},$$

where the implicit constants depend only on $p$ and $\varepsilon$. Then, applying Proposition 3.4 to (4.52) through (4.51) as $\alpha = \sqrt{\varepsilon} - 2\varepsilon$, we obtain (4.51). \hfill \Box

4.3 Third derivatives of $\varphi(\omega, a)$

In this section, we find the third derivatives of $\varphi(\omega, a)$ with respect to $\omega$ and $a$.

Let us begin with a generalization of Lemma 4.4.

Lemma 4.11. Assume $p > 1$, and let $k, j$ be integers satisfying $k > p$ and $j \geq 1$. Then, there exists $\varepsilon(k, j) > 0$ depending only on $p$, $k$, and $j$ with the following property: Let $0 < \varepsilon < \varepsilon(k, j)$, and let $g \in C^1((\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e), L^\infty(\mathbb{R} \times \mathbb{T}))$. Furthermore, assume that the following hold for all $(\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e)$ and $(x, y) \in \mathbb{R} \times \mathbb{T}$:

$$|g(\omega, a, x, y)| \lesssim e^{-k\sqrt{\varepsilon} |x| + j |x|}, \hspace{1cm} (4.55)$$

$$|\partial_x g(\omega, a, x, y)| + |\partial_a g(\omega, a, x, y)| \lesssim e^{-k\sqrt{\varepsilon} |x| + (j+2) |x|}, \hspace{1cm} (4.56)$$

where the implicit constant depends only on $p$, $j$, and $\varepsilon$. Then, the following holds in the $L^2(\mathbb{R} \times \mathbb{T})$-topology for all $(\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_e, a_e)$:

$$\partial \{V_k(\omega, a)g(\omega, a)\} = V_{k+1}(\omega, a)\partial \varphi(\omega, a)g(\omega, a) + V_k(\omega, a)\partial g(\omega, a),$$

where $\partial$ denotes either $\partial_a$ or $\partial_x$.

We omit the proof of Lemma 4.11 as the lemma can be proven in a way similar to Lemma 4.4.

Now, we give the third derivatives of $\varphi(\omega, a)$:
Proposition 4.12. Assume $p > 1$ and let $\varepsilon_2$ be the constant given in Proposition 4.9. Then, there exists $0 < \varepsilon_3 < \varepsilon_2$ depending only on $p$ such that if $0 < \varepsilon < \varepsilon_3$, then $\varphi$ is of class $C^3$ on $(\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_{\varepsilon}, a_{\varepsilon})$ in the $H^2(\mathbb{R} \times T)$-topology; and the following hold for all $(\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_{\varepsilon}, a_{\varepsilon})$:

$$
\partial^3_{\omega} \varphi(\omega, a) = T(\omega, a)^{-1} P_\perp \left( V_3(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^3 \right) + 3T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial_\omega \varphi(\omega, a) \partial^2_{\omega} \varphi(\omega, a) \right),
$$

(4.57)

$$
\partial^2_{\omega} \partial_\omega \varphi(\omega, a) = \partial_\omega \partial_\omega \partial_\omega \varphi = \partial_\omega^2 \partial_\omega \varphi
$$

$$
= T(\omega, a)^{-1} P_\perp \left( V_3(\omega, a) \partial_\omega \varphi(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^2 \right) + 2T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial_\omega \varphi(\omega, a) \partial_\omega \partial_\omega \varphi(\omega, a) \right) + T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial^2_{\omega} \varphi(\omega, a) \partial_\omega \varphi(\omega, a) \right) - T(\omega, a)^{-1} P_\perp \partial^2_{\omega} \varphi(\omega, a),
$$

(4.58)

$$
\partial_\omega \partial^2_{\omega} \varphi(\omega, a) = \partial_\omega \partial_\omega \partial_\omega \varphi = \partial^2_{\omega} \partial_\omega \varphi
$$

$$
= T(\omega, a)^{-1} P_\perp \left( V_3(\omega, a) \partial_\omega \varphi(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^2 \right) + 2T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial_\omega \varphi(\omega, a) \partial_\omega \partial_\omega \varphi(\omega, a) \right) + T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial^2_{\omega} \varphi(\omega, a) \partial_\omega \varphi(\omega, a) \right) - T(\omega, a)^{-1} P_\perp \partial^2_{\omega} \varphi(\omega, a),
$$

(4.59)

$$
\partial^3_{\omega} \varphi(\omega, a) = T(\omega, a)^{-1} P_\perp \left( V_3(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^3 \right) + 3T(\omega, a)^{-1} P_\perp \left( V_2(\omega, a) \partial^2_{\omega} \varphi(\omega, a) \partial_\omega \varphi(\omega, a) \right) - wT(\omega, a)^{-1} P_\perp \partial^2_{\omega} \varphi(\omega, a).
$$

(4.60)

Remark 4.3. Since $T(\omega, a)^{-1}$ maps $Y_2$ to $X_2$, Proposition 4.12 shows that $\partial^3_{\omega} \varphi(\omega, a), \partial^2_{\omega} \partial_\omega \varphi(\omega, a), \partial_\omega \partial^2_{\omega} \varphi(\omega, a), \partial^3_{\omega} \varphi(\omega, a) \in X_2$.

Proof of Proposition 4.12. We shall prove (4.57). By (4.43) in Proposition 4.9, Lemma 4.7 with $f = P_\perp (V_2(\omega, a) \{ \partial_\omega \varphi(\omega, a) \}^2)$, Lemma 4.2 and Lemma 4.11 with $k = 2, j = 4,$
4.4 Derivatives of \( F \)

In this section, we compute the derivatives of \( F \).

Then, the following result follows from Lemma 4.2, (4.31), (4.30) and (4.61):

Thus, we have proved (4.57). Similarly, we can prove (4.58) through (4.60).

It remains to prove the continuity of the third derivatives of \( \varphi \). When \( 3 \leq p \), we see from the implicit function theorem that \( \varphi(\omega,a) \) is \( C^3 \) with respect to \( a \) and \( \omega \). Hence, we may assume that \( p < 3 \). Then, we shall prove the continuity of \( \partial_3^2 \varphi \). Observe from the continuity of \( \partial_3 \varphi \) and (4.11) that Lemma 4.8 can apply to the first term on the right-hand side of (4.57) as \( k = 3 \), \( j = 6 \), \( g(\omega,a) = (\partial_3^2 \varphi(\omega,a))^3 \). Moreover, observe from the continuity of \( \partial_3 \varphi \) and \( \partial_3^2 \varphi \). (4.11) and Lemma 4.10 that we can apply Lemma 4.8 to the second term on the right-hand side of (4.57) as \( k = 2 \), \( j = 5 \), \( g = \partial_3 \varphi(\omega,a) \partial_3^2 \varphi(\omega,a) \).

Thus, we find that \( \partial_3^2 \varphi(\omega,a) \) is continuous with respect to \( (\omega,a) \) in the \( H^2(\mathbb{R} \times T) \)-topology. Similarly, we can prove the continuity of the other third order derivatives. \( \square \)

4.4 Derivatives of \( F_{||}(\omega, a) \)

In this section, we compute the derivatives of \( F_{||}(\omega, a) \) up to the third order.

Observe that

\[
(\varphi(\omega, a), \psi_{\omega_p} \cos y) = a. 
\]  

(4.61)

Then, the following result follows from Lemma 4.2, (4.31), (4.30) and (4.61):

**Proposition 4.13.** Assume \( p > 1 \). Then, \( F_{||} \) is \( C^1 \) on \( (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \), and the following hold for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \):

\[
\partial_3 F_{||}(\omega, a) = \langle L_+(\omega, a) \partial_3 \varphi(\omega, a), \psi_{\omega_p} \cos y \rangle, 
\]  

(4.62)

\[
\partial_3 F_{||}(\omega, a) = \langle L_+(\omega, a) \partial_3 \varphi(\omega, a), \psi_{\omega_p} \cos y \rangle + a. 
\]  

(4.63)

The following result follows from Proposition 4.12 Lemma 4.2 Lemma 4.6 and Proposition 4.13:  

**Proposition 4.14.** Assume \( p > 1 \). Let \( \varepsilon_2 \) be the constant given in Proposition 4.9 and \( 0 < \varepsilon < \varepsilon_2 \). Then, \( F_{||} \) is \( C^2 \) on \( (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \), and the following hold for all \( (\omega, a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \):

\[
\partial_3^2 F_{||}(\omega, a) = \langle L_+(\omega, a) \partial_3^2 \varphi(\omega, a) - V_2(\omega, a) \{ \partial_3 \varphi(\omega, a) \}^2, \psi_{\omega_p} \cos y \rangle, 
\]  

(4.64)

\[
\partial_3 \partial_3 F_{||}(\omega, a) = \partial_3 \partial_3 F_{||}(\omega, a) 
\]  

(4.65)

\[
= 1 + \langle L_+(\omega, a) \partial_3 \partial_3 \varphi(\omega, a) - V_2(\omega, a) \partial_3 \varphi(\omega, a) \partial_3 \varphi(\omega, a), \psi_{\omega_p} \cos y \rangle, 
\]  

\[
\partial_3^3 F_{||}(\omega, a) = \langle L_+(\omega, a) \partial_3^3 \varphi(\omega, a) - V_2(\omega, a) \{ \partial_3 \varphi(\omega, a) \}^2, \psi_{\omega_p} \cos y \rangle. 
\]  

(4.66)
The following result follows from Propositions 4.12 and 4.14, Lemmas 4.2, 4.6 and 4.11.

Proposition 4.15. Assume \( p > 1 \). Let \( \varepsilon_3 \) be the same constant given in Proposition main-lem-3 and \( 0 < \varepsilon < \varepsilon_3 \). Then, \( F_\parallel \) is \( C^3 \) on \( (\omega_0 - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \), and the following hold for all \( (\omega, a) \in (\omega_0 - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon) \):

\[
\delta_3^2 F_\parallel (\omega, a) = (L_+ (\omega, a) \delta_0^3 \varphi(\omega, a) - 3V_2(\omega, a) \delta_0 \partial_\omega \varphi(\omega, a) \delta_0^2 \varphi(\omega, a) - V_3(\omega,a) \{ \partial_0 \varphi(\omega, a) \}^3, \psi_{\omega_0} \cos y),
\]

\[
\partial_\omega \partial_\omega F_\parallel (\omega, a) = \partial_0 \partial_\omega \partial_\omega F_\parallel (\omega, a) = \partial_0^2 \partial_\omega^2 F_\parallel (\omega, a)
\]

\[
\partial_\omega \partial_\omega \partial_\omega F_\parallel (\omega, a) = (L_+ (\omega, a) \delta_0^2 \partial_\omega \varphi(\omega, a) - V_2(\omega, a) \delta_0 \partial_\omega \varphi(\omega, a) \partial_\omega^2 \varphi(\omega, a) - 2V_2(\omega, a) \partial_\omega \partial_\omega \varphi(\omega, a) \partial_\omega \partial_\omega \varphi(\omega, a) - V_3(\omega,a) \{ \partial_\omega \varphi(\omega, a) \}^2 \partial_\omega \varphi(\omega, a), \psi_{\omega_0} \cos y)
\]

\[
\delta_3^2 F_\parallel (\omega, a) = (L_+ (\omega, a) \delta_0^3 \varphi(\omega, a) - 3V_2(\omega, a) \delta_0 \partial_\omega \varphi(\omega, a) \delta_0^2 \varphi(\omega, a) - V_3(\omega,a) \{ \partial_\omega \varphi(\omega, a) \}^3, \psi_{\omega_0} \cos y).
\]

5 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Throughout this section, for a given \( p > 1 \), let \( a_0, \delta_0 \) and \( \eta : (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_0, a_0) \to X_2 \) be the same as in Lemma 2.1. \( \varphi(\omega, a) \) be the function defined by (2.12), and \( \varepsilon_3 \) be the constant given in Proposition 4.12. Furthermore, for \( 0 < \varepsilon < \varepsilon_3 \), we use \( a_\varepsilon \) to denote the same constant given in Proposition 3.1.

We will employ the argument developed by Crandall and Rabinowitz [3] (see also the proof of Theorem 4 (ii) in [4]). Let \( 0 < \varepsilon < \varepsilon_3 \), and define

\[
D_\varepsilon := (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_\varepsilon, a_\varepsilon).
\]

Furthermore, we introduce the function \( g : D_\varepsilon \to \mathbb{R} \) as

\[
g(\omega, a) := \begin{cases} \frac{F_\parallel (\omega, a)}{a} & \text{if } a \neq 0, \\ \partial_\omega F_\parallel (\omega, 0) & \text{if } a = 0. \end{cases}
\]

Lemma 5.1. Assume \( p > 1 \) and let \( 0 < \varepsilon < \varepsilon_3 \). Then, \( g \) is twice differentiable on \( D_\varepsilon \).
Furthermore, the following hold:

\[
\begin{align*}
\partial_\omega g(\omega,0) &= \partial_\omega \partial_a \mathcal{F}_\parallel(\omega,0) = \partial_a \partial_\omega \mathcal{F}_\parallel(\omega,0), \\
\partial_\omega^2 g(\omega,0) &= \partial_\omega^2 \partial_a \mathcal{F}_\parallel(\omega,0), \\
\partial_a g(\omega,0) &= \frac{1}{2} \partial_a^2 \mathcal{F}_\parallel(\omega,0),
\end{align*}
\]

(5.3)\hspace{1cm}(5.4)\hspace{1cm}(5.5)

Furthermore, we will use the Taylor expansion of \(\partial_a \mathcal{F}_\parallel\omega,a\) twice differentiable at \((\omega,a) \in \mathcal{E}_\c\).

**Proof of Lemma 5.1.** Since \(\mathcal{F}_\parallel\) is \(C^3\) on \(\mathcal{E}_\c\) (see Proposition 4.15), it is obvious that \(g\) is twice differentiable at \((\omega,a) \in \mathcal{E}_\c\) with \(a \neq 0\). Furthermore, we can easily verify (5.3) and (5.4).

It remains to prove the twice differentiability of \(g\) at \((\omega,0)\) and \((\omega,0)\) through \((\omega,0)\). The Taylor expansion together with \(\mathcal{F}_\parallel(\omega,0) = 0\) (see (2.22)) shows that

\[
\mathcal{F}_\parallel(\omega,a) = \partial_\omega \mathcal{F}_\parallel(\omega,0)a + \frac{1}{2} \partial_\omega^2 \mathcal{F}_\parallel(\omega,0)a^2 + \frac{1}{3!} \partial_\omega^3 \mathcal{F}_\parallel(\omega,0)a^3 + o(a^3). \tag{5.8}
\]

Furthermore, by Proposition 4.13 (5.7) and the integral of \(\cos y\) over \([-\pi,\pi]\) being zero, we see that

\[
\partial_\omega \mathcal{F}_\parallel(\omega,0) = \langle L_+(\omega,0) \partial_\omega \varphi(\omega,0), \psi_{\omega p} \cos y \rangle = \langle -R_\omega, \psi_{\omega p} \cos y \rangle = 0. \tag{5.9}
\]

Hence, the Taylor expansion together with (5.9) shows that

\[
\partial_\omega \mathcal{F}_\parallel(\omega,a) = \partial_a \partial_\omega \mathcal{F}_\parallel(\omega,0)a + \frac{1}{2} \partial_a^2 \partial_\omega \mathcal{F}_\parallel(\omega,0)a^2 + o(a^2). \tag{5.10}
\]

Moreover, we will use the Taylor expansion of \(\partial_a \mathcal{F}_\parallel(\omega,a)\):

\[
\partial_a \mathcal{F}_\parallel(\omega,a) = \partial_a \mathcal{F}_\parallel(\omega,0) + \partial_\omega^2 \mathcal{F}_\parallel(\omega,0)a + \frac{1}{2} \partial_a^3 \mathcal{F}_\parallel(\omega,0)a^2 + o(a^2). \tag{5.11}
\]

The claim (5.5) follows from (5.8):

\[
\partial_a g(\omega,0) = \lim_{a \to 0} \frac{g(\omega,a) - g(\omega,0)}{a} = \lim_{a \to 0} \frac{\mathcal{F}_\parallel(\omega,a) - a \partial_\omega \mathcal{F}_\parallel(\omega,0)}{a^2} = \frac{1}{2} \partial_a^2 \mathcal{F}_\parallel(\omega,0). \tag{5.12}
\]

Furthermore, (5.12) shows that

\[
\partial_c \partial_a g(\omega,0) = \lim_{\delta \to 0} \frac{\partial_a g(\omega,0) - \partial_a g(\omega,\delta)}{\delta} = \frac{1}{2} \partial_a^2 \partial_\omega \mathcal{F}_\parallel(\omega,0). \tag{5.13}
\]

By (5.3) and (5.11), we see that

\[
\partial_\omega \partial_a g(\omega,0) = \lim_{a \to 0} \frac{\partial_\omega \mathcal{F}_\parallel(\omega,a) - a \partial_\omega \partial_\omega \mathcal{F}_\parallel(\omega,0)}{a^2} = \frac{1}{2} \partial_a^2 \partial_\omega \mathcal{F}_\parallel(\omega,0). \tag{5.14}
\]

Then, (5.6) follows from (5.13), (5.14) and (4.68) in Proposition 4.15.
It remains to prove (5.7). Let \((\omega, a) \in D_\varepsilon\) with \(a \neq 0\). Then, by the differentiation of quotient and (5.8), we see that

\[
\partial_a g(\omega, a) = \frac{a \partial_a F(\omega, a) - F(\omega, a)}{a^2}
\]

\[
= \frac{\partial_a F(\omega, a) - \partial_a F(\omega, 0)}{a} - \frac{1}{2} \partial_a^2 F(\omega, 0) - \frac{1}{3!} \partial_a^3 F(\omega, 0) a + o(a)
\]

(5.15)

\[
= \int_0^1 \partial^2_a F(\omega, \theta a) d\theta - \frac{1}{2} \partial_a^2 F(\omega, 0) - \frac{1}{3!} \partial_a^3 F(\omega, 0) a + o(a).
\]

Furthermore, by (5.15) and (5.12), we see that

\[
\partial_a^2 g(\omega, 0) = \lim_{a \to 0} \frac{\partial_a g(\omega, a) - \partial_a g(\omega, 0)}{a}
\]

\[
= \lim_{a \to 0} \frac{1}{a} \left\{ \int_0^1 \partial^2_a F(\omega, \theta a) d\theta - \partial^2_a F(\omega, 0) - \frac{1}{3!} \partial^3_a F(\omega, 0) a + o(a) \right\}
\]

(5.16)

\[
= \lim_{a \to 0} \int_0^1 \int_0^1 \partial^3_a F(\omega, \theta \kappa a) d\kappa d\theta - \frac{1}{3!} \partial^3_a F(\omega, 0)
\]

\[
= \int_0^1 \int_0^1 \partial^3_a F(\omega, 0) d\kappa d\theta - \frac{1}{6} \partial^3_a F(\omega, 0) = \frac{1}{3} \partial^3_a F(\omega, 0).
\]

Thus, we have completed the proof. \(\square\)

**Lemma 5.2.** Assume \(p > 1\), and let \(0 < \varepsilon < \varepsilon_3\). Then, the function \(g\) is \(C^2\) on \(D_\varepsilon\). Furthermore, the following hold:

\[
g(\omega_p, 0) = 0, \quad \partial_a g(\omega_p, 0) = 0, \quad \partial_{\omega} g(\omega_p, 0) = -\frac{1}{\omega_p} < 0.
\]

(5.17)

**Proof of Lemma 5.2.** We shall show that \(g\) is \(C^2\) on \(D_\varepsilon\). Since \(F_\|\) is \(C^3\) on \(D_\varepsilon\) (see Proposition 4.19), it is obvious that \(g\) is twice continuously differentiable at \((\omega, a) \in D_\varepsilon\) with \(a \neq 0\). Furthermore, by Lemma 5.1 it suffices to prove the continuity of the second derivatives \(\partial^2_\omega g, \partial_\omega \partial_a g\) and \(\partial^2_a g\) at \((\omega, 0)\).

By (5.17), (5.6) and \(\varphi\) being \(C^2\) on \(D_\varepsilon\) (see Proposition 4.9), we see that both \(L_\| (\omega, 0) \partial^2_\omega \varphi(\omega, 0)\) and \(V_2(\omega, 0) \{\partial_\omega \varphi(\omega, 0)\}^2\) are independent of \(y\). Hence, (4.66) in Proposition 4.14 together with the integral of \(\cos y\) over \([-\pi, \pi]\) being zero shows that

\[
\partial^2_\omega F(\omega, 0) = \langle L_\| (\omega, 0) \partial^2_\omega \varphi(\omega, 0) - V_2(\omega, 0) \{\partial_\omega \varphi(\omega, 0)\}^2, \psi_\omega, \cos y \rangle = 0.
\]

(5.18)

Then, by the definition of \(g\), (5.18) and (5.4) in Lemma 5.1 we see that

\[
\lim_{a \to 0} \partial^2_a g(\omega, a) = \lim_{a \to 0} \frac{\partial^2_\omega F(\omega, a)}{a} = \partial_\omega \partial^2_\omega F(\omega, 0) = \partial^2_\omega g(\omega, 0).
\]

(5.19)

Next, we consider \(\partial_\omega \partial_a g\). By the differentiation of quotient, the Taylor expansion of

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\[ \partial_\omega F\] (see (5.11)), and (5.6) in Lemma 5.1, we see that
\[
\lim_{a \to 0} \partial_\omega \partial_a g(\omega, a) = \lim_{a \to 0} \partial_\omega \left\{ \frac{a \partial_a F(\omega, a) - F(\omega, a)}{a^2} \right\} \\
= \lim_{a \to 0} \left\{ \frac{a \partial_a \partial_\omega F(\omega, a) - a \partial_\omega \partial_a F(\omega, a)}{a^2} - \frac{1}{2} \frac{\partial_a^2 \partial_\omega F(\omega, a) + o_a(1)}{a^2} \right\} \\
= \frac{1}{2} \frac{\partial_a^2 \partial_\omega F(\omega, 0)}{a^2} = \partial_\omega \partial_a g(\omega, 0).
\]

Finally, we consider \( \partial_a^2 g \). By the definition of \( g \), (5.8) and (5.11), we can verify that for any \((\omega, a) \in D_\varepsilon \) with \( a \neq 0 \),
\[
\partial_a^2 g(\omega, a) = \frac{a^2 \partial_a^2 F(\omega, a) - 2a \partial_a F(\omega, a) + 2F(\omega, a)}{a^3} \\
= \frac{a^2 \partial_a^2 F(\omega, a) - 2a \partial_a F(\omega, a) + 2a \partial_a F(\omega, 0) + a^2 \partial_a^2 F(\omega, 0)}{a^3} \\
+ \frac{1}{a} \partial_a^3 F(\omega, 0) + o_a(1) \\
= a^{-1} \partial_a^2 F(\omega, a) - a^{-1} \partial_a^2 F(\omega, 0) - \frac{2}{3} \partial_a^3 F(\omega, 0) + o_a(1) \\
= \int_0^1 \partial_a^3 F(\omega, \theta a) d\theta - \frac{2}{3} \partial_a^3 F(\omega, 0) + o_a(1).
\]

Then, the continuity of \( \partial_a^2 g \) at \((\omega, 0)\) follows from (5.21) and (5.7) in Lemma 5.1,
\[
\lim_{a \to 0} \partial_a^2 g(\omega, a) = \int_0^1 \partial_a^3 F(\omega, 0) d\theta - \frac{2}{3} \partial_a^3 F(\omega, 0) = \partial_a^2 g(\omega, 0).
\]

Thus, we have proved that \( g \) is \( C^2 \) on \( D_\varepsilon \).

We shall prove (5.17). Note that the following follows from the self-adjointness of \( L_+(\omega, a) \) on \( L^2(\mathbb{R} \times \mathbb{T}) \), (5.6) and (2.4):
\[
\langle L_+(\omega, p) u, \psi_{\omega p} \rangle = 0 \quad \text{for all } u \in H^2(\mathbb{R} \times \mathbb{T}).
\]

Then, by (4.62) in Proposition 4.13 and (5.23), we see that
\[
g(\omega, p, 0) = \langle L_+(\omega, p) \partial_a \varphi(\omega, 0), \psi_{\omega p} \rangle = 0.
\]

By (5.5), (4.64) in Proposition 4.14, \( V_2(\omega, 0) = p(p-1)R_{\omega p}^{p-2} \) (see (4.2) and (2.17)), \( \partial_a \varphi(\omega, 0) = \psi_{\omega p} \cos y \) (see (2.18)), (5.23) and the integral of \( (\cos y)^3 \) over \([-\pi, \pi]\) being zero, we see that
\[
\partial_a g(\omega, p, 0) = \frac{1}{2} L_+(\omega, p) \partial_a^2 \varphi(\omega, 0) - p(p-1)R_{\omega p}^{p-2} \psi_{\omega p}^3 \cos y^2 \psi_{\omega p} \cos y \\
= - \frac{p(p-1)}{2} \int_\mathbb{R} R_{\omega p}^{p-2} \psi_{\omega p}^3 \, dx \int_\mathbb{T} (\cos y)^3 \, dy = 0.
\]
By (5.23), (5.26), (2.14), (2.18), the definition of \( \psi_{\omega_p} \) (see (2.1)), (1.19) and (1.20), we see that
\[
\partial_\omega g(\omega_p, 0) = \partial_\omega \partial_\alpha F|^\ast(\omega_p, 0)
\]
\[
= 1 + \langle L_+(\omega_p, 0) \partial_\omega \partial_\alpha \varphi(\omega_p, 0) - V_2(\omega_p, 0) \partial_\omega \varphi(\omega_p, 0), \psi_{\omega_p} \cos y \rangle
\]
\[
= 1 - p(p - 1) \langle R^{-2}_{\omega_p} (\psi_{\omega_p} \cos y) \partial_\omega R_{\omega_p} |_{\omega = \omega_p}, \psi_{\omega_p} \cos y \rangle
\]
\[
= 1 - \frac{p(p - 1)}{\| R_{\omega_p} \|_{p+1}^{p+1}(\mathbb{R})} \int_{\mathbb{R}} R_{\omega_p}^{p+1}(\omega) \partial_\omega R_{\omega_p} |_{\omega = \omega_p} dx
\]
\[
= 1 - \frac{1}{\| R_{\omega_p} \|_{p+1}^{p+1}(\mathbb{R})} \frac{(p + 1)^2}{4} \int_{\mathbb{R}} R_{\omega_p}^{p+1} = - \frac{(p - 1)(p + 3)}{4} = - \frac{1}{\omega_p}.
\]
Thus, we have completed the proof of the lemma.

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** The first step to prove Theorem 1.1 is to find a curve \( \omega(a) \) parametrized by \( a \) such that \( (\omega(a), a) \) satisfies the bifurcation equation (2.21):
\[
F|^\ast(\omega(a), a) = \langle F(\omega, \varphi(\omega, a)), \psi_{\omega_p} \cos y \rangle = 0.
\]
(5.27)
The implicit function theorem together with Lemma 5.2 shows that there exist \( a_*, \delta_0 > 0 \) and a \( C^2 \)-curve \( \omega(\cdot) : a \in (-a_*, a_*) \mapsto \omega(a) \in (\omega_p - \delta_0, \omega_p + \delta_0) \) with the following properties:

1. \( g(\omega(a), a) = 0, \omega(0) = \omega_p, \frac{d\omega}{da}(a) = -\frac{\partial_\alpha g(\omega(a), a)}{\partial_\omega g(\omega(a), a)}. \)

2. \( Z_p \) be the set of zeros of \( g = 0 \) in \( (\omega_p - \delta_0, \omega_p + \delta_0) \times (-a_*, a_*) \). Then,
\[
Z_p = \{(\omega(a), a) : a \in (-a_*, a_*)\}.
\]
(5.29)
Next, we define the function \( Q : (-a_*, a_*) \to H^2(\mathbb{R} \times T) \) by
\[
Q(a) := \varphi(\omega(a), a) = R_{\omega_p} + a \psi_{\omega_p} \cos y + \eta(\omega(a), a).
\]
(5.30)
Note that \( Q \) is \( C^2 \) on \( (-a_*, a_*) \) in \( H^2(\mathbb{R} \times T) \).
Since \( g(\omega, a) = 0 \) with \( a \neq 0 \) implies that \( F|^\ast(\omega, a) = 0 \), the first claim of Theorem 1.1 follows from (5.29) and Lemma 2.1. Furthermore, the claim (1.10) follows immediately from (5.28) and (5.17).

We shall prove (1.11), (1.12) and (1.14).
First, note that the integral of \( (\cos y)^3 \) over \([-\pi, \pi]\) being zero shows that
\[
R^{-2}_{\omega_p} \{\psi_{\omega_p} \cos y\}^2 \in X_2.
\]
(5.31)
Furthermore, by (4.34) in Proposition 4.3, \( V_2(\omega_p, 0) = p(p - 1)R_p^{-2} \) (see (4.2) and (2.11)), (2.13) and (5.31), we see that
\[
\partial_a^2 \varphi(\omega_p, 0) = p(p - 1)\{P_\perp \partial_a \mathcal{F}(\omega_p, R_\omega)|_{X_2}\}^{-1}(R_p^{e-2}\{\psi_{\omega_p} \cos y\}^2). \tag{5.32}
\]

Differentiating both sides of the last equation in (5.28), and using (1.10), (5.17) and (5.31), we see that
\[
\frac{d^2}{da^2} \varphi(\omega_p, 0) = \frac{-\omega_p}{3} \partial_a^3 \bar{F}(\omega_p, a). \tag{5.33}
\]

By (4.67) in Proposition 4.15, (5.23), \( V_2(\omega_p, 0) = p(p - 1)R_p^{e-2} \) and \( V_3(\omega_p, 0) = p(p - 1)(p - 2)R_p^{e-3} \) (see (1.2) and (2.17)), \( \partial_a \varphi(\omega_p, 0) = \psi_{\omega_p} \cos y \) (see (2.18)), (5.32) and (5.31), we see that
\[
\partial_a^3 \bar{F}(\omega_p, 0) = (-3V_2(\omega_p, 0)\partial_a \varphi(\omega_p, 0)\partial_a^2 \varphi(\omega_p, 0) - V_3(\omega_p, 0)\{\partial_a \varphi(\omega_p, 0)\}^3, \psi_{\omega_p} \cos y)
\]
\[
= -3(p(p - 1))^2(R_p^{e-2}\psi_{\omega_p} \cos y T(\omega_p, 0)^{-1}(R_p^{e-2}\{\psi_{\omega_p} \cos y\}^2), \psi_{\omega_p} \cos y)
\]
\[
- p(p - 1)(p - 2)\langle R_p^{e-3}\{\psi_{\omega_p} \cos y\}^3, \psi_{\omega_p} \cos y\rangle
\]  
\[
= -3(p(p - 1))^2(T(\omega_p, 0)^{-1}(R_p^{e-2}\{\psi_{\omega_p} \cos y\}^2), R_p^{e-2}\{\psi_{\omega_p} \cos y\}^2)
\]
\[
- p(p - 1)(p - 2)\langle R_p^{e-3}\{\psi_{\omega_p} \cos y\}^3, \psi_{\omega_p} \cos y\rangle.
\]

Plugging (5.34) into (5.33), we obtain (1.11).

Observe that
\[
\frac{dQ}{da}(a) = \partial_a \varphi(\omega(a), a) \frac{d\omega}{da}(a) + \partial_a^2 \varphi(a, a). \tag{5.35}
\]

Then, by (5.35), (1.10) and (2.18), we see that
\[
\frac{dQ}{da}(0) = \partial_a \varphi(\omega_p, 0) = \psi_{\omega_p} \cos y. \tag{5.36}
\]

Furthermore, differentiating both sides of (5.35), and using (1.10) and (2.17), we see that
\[
\frac{d^2 Q}{da^2}(0) = \partial_a^2 \varphi(\omega(0), 0)\left(\frac{d\omega}{da}(0)\right)^2 + \partial_a \varphi(\omega(0), 0)\frac{d^2 \omega}{da^2}(0)
\]
\[
+ \partial_a \partial_a \varphi(\omega(0), 0) \frac{d\omega}{da}(0) + \partial_a^2 \varphi(\omega(0), 0) \tag{5.37}
\]
\[
= \partial_a R_\omega|_{\omega = \omega_p} \frac{d^2 \omega}{da^2}(0) + \partial_a^2 \varphi(\omega_p, 0).
\]

Then, the Taylor expansion together with \( Q(0) = \varphi(\omega(0), 0) = R_\omega \) (see (2.18)), (5.36) and (5.37) shows that the following holds in \( H^2(\mathbb{R} \times T) \):
\[
Q(a) = R_{\omega_p} + a\psi_{\omega_p} \cos y + \frac{1}{2}a^2 \{\partial_a R_\omega|_{\omega = \omega_p} \frac{d^2 \omega}{da^2}(0) + \partial_a^2 \varphi(\omega_p, 0)\} + o(a^2). \tag{5.38}
\]

Clearly, this shows that (1.12) holds.
It remains to prove the last claim \((1.14)\). By \((5.38)\), \(\|\psi_{\omega p} \cos y\|_{L^2(\mathbb{R} \times T)} = 1\) and the integral of \(\cos y\) over \([-\pi, \pi]\) being zero, we see that

\[
\|Q(a)\|_{L^2(\mathbb{R} \times T)}^2 = \|R_{\omega p}\|_{L^2(\mathbb{R} \times T)}^2 + a^2 + a^2 \frac{d^2 \omega}{da^2}(0) \langle R_{\omega p}, \partial_\omega R_{\omega|\omega=\omega p} \rangle + a^2 \langle R_{\omega p}, \partial_\omega^2 \varphi(\omega_p, 0) \rangle + o(a^2) \tag{5.39}
\]

Recall that \(R_\omega, \partial_\omega R_\omega \in X_2\) (see \((2.6)\)) and \(\partial_\omega F(\omega_p, R_{\omega p}) : X_2 \to Y_2\) is bijective (see \((2.10)\) and \((2.11)\)). Then, by \((5.32)\), \((3.7)\) and the same computation as \((5.26)\), we see that

\[
\langle R_{\omega p}, \partial_\omega^2 \varphi(\omega_p, 0) \rangle = p(p-1) \langle \{ \partial_\omega \mathcal{F}(\omega_p, R_{\omega p}) \}^{-1} R_{\omega p}, R_{\omega p}^{p-2} \{ \psi_{\omega p} \cos y \}^2 \rangle - 1 - \frac{1}{\omega_p} \tag{5.40}
\]

Plugging \((5.40)\) into \((5.39)\), and using \((1.13)\), we obtain \((1.14)\). Thus, we have completed the proof of the theorem.

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