Compactons and kink-like solutions of BBM-like equations by means of factorization

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Abstract

In this work, we study the Benjamin-Bona-Mahony like equations with a fully nonlinear dispersive term by means of the factorization technique. In this way we find the travelling wave solutions of this equation in terms of the Weierstrass function and its degenerated trigonometric and hyperbolic forms. Then, we obtain the pattern of periodic, solitary, compacton and kink-like solutions. We give also the Lagrangian and the Hamiltonian, which are linked to the factorization, for the nonlinear second order ordinary differential equations associated to the travelling wave equations.

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1 Introduction

The nonlinear dispersive equations $K(m, n)$ (or generalized Korteweg-de Vries equation)

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3$$

have been dealt with by Rosenau P. and Hyman J.M. [1]. They built travelling wave solutions of (1) with compact support and called them compactons. Such compactons are solitary waves but, besides the compact support, they have some different features from the solitons. So, while the width of the soliton depends on its amplitude, the width and amplitude of a compacton are independent. However, compactons sometimes behave as solitons: they also collide elastically and their amplitudes depend on speed of the wave. In the literature there are many studies on $K(m, n)$ equations and their compacton solutions [1,2,3,4,5,6,7,8].

The Benjamin-Bona-Mahony (BBM) equation has a higher order nonlinearity of the form

$$u_t + u_x + a u^m u_x - u_{xxx} = 0, \quad n \geq 1$$

where $a$ is constant [9,10]. This equation is an alternative to the Korteweg-de Vries equation, and describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel, hydromagnetics waves in cold plasma and acoustic waves in anharmonic crystals. Thus, in this sense, it has some advantages compared with the Korteweg-de Vries equation.

Now, we will consider the BBM-like equations, $B(m, n)$, with a fully nonlinear dispersive term, similar to the $K(m, n)$ equations,

$$u_t + u_x + a (u^m)_x - (u^n)_{xxx} = 0, \quad m, n > 1.$$  

(3)

There are some studies related with these BBM-like equations [13,14,15,16], in general using the sine-cosine and the tanh method, in order to get the travelling wave solutions of this family. In this respect, let us remark that in the literature there are many methods to obtain exact or special travelling wave solutions (soliton solutions)
of nonlinear equations \cite{11}, but there is not a standard method. So, it is important to investigate new methods to get solutions in different ways. In this work, we deal with BBM-like equations systematically by means of the factorization technique \cite{10,17,18,19} to find the travelling wave solutions. When looking for the travelling wave solutions, the BBM-like equations will reduce to second order nonlinear ordinary differential equations (ODE). Then, we can apply straightforwardly the factorization to these ODE’s.

In section 2, we briefly introduce the factorization technique for a class of nonlinear second order ODE’s. Then, in section 3 we apply this method to the BBM-like equations. In section 4, we obtain the travelling wave solutions of this equation in terms of Weierstrass functions which cover the pattern of solitary wave and periodic solutions. We also construct compactons and kink-like solutions for this family and we give the graphics of these solutions in Fig. 1 (periodic), Fig. 2 (compactons), Fig. 3 (kink-like) and Fig. 4 (solitary wave pattern). We give the Lagrangian and the Hamiltonian corresponding to the second order nonlinear ODE and their relation with the factorization in section 5. Finally, in the last section we end this work with some conclusions.

2 Factorization of the nonlinear second order ODE

Let us consider, the nonlinear second order ODE of the special form

\[
d\frac{d^2W}{d\theta^2} - \beta \frac{dW}{d\theta} + F(W) = 0
\]

where \(\beta\) is constant and \(F(W)\) is an arbitrary function of \(W\). The factorized form of this equation can be written as

\[
\left[\frac{d}{d\theta} - f_2(W, \theta)\right] \left[\frac{d}{d\theta} - f_1(W, \theta)\right] W(\theta) = 0.
\]

Here, \(f_1\) and \(f_2\) are unknown functions that may depend explicitly on \(W\) and \(\theta\). Expanding \(\frac{d}{d\theta} - f_2(W, \theta)\) and comparing with \(\frac{d}{d\theta} - f_1(W, \theta)\), we obtain the following consistency conditions

\[
f_1 f_2 = \frac{F(W)}{W} + \frac{\partial f_1}{\partial \theta}, \quad f_2 + \frac{\partial(W f_1)}{\partial W} = \beta.
\]

If we solve \(\text{6}\) for \(f_1\) or \(f_2\), it will allow us to write a compatible first order ODE

\[
\left[\frac{d}{d\theta} - f_1(W, \theta)\right] W(\theta) = 0
\]

that provides a solution for the nonlinear ODE \(\text{4}\) \cite{10,17,18,19}. In the applications of this paper \(f_1\) and \(f_2\) will depend only on \(W\).

3 Factorization of the second order nonlinear ODE’s corresponding to the BBM-like equations

Let us assume that \(\text{3}\) has travelling wave solutions

\[
u(x, t) = \phi(\xi), \quad \xi = hx + wt
\]

where \(h\) and \(w\) are real constants. Substituting \(\text{8}\) into \(\text{3}\) with the condition \(m = n\) and after integrating, we obtain the second order nonlinear ODE

\[
(\phi^n)_{\xi\xi} - A \phi - B \phi^n + D = 0
\]
therefore, the consistency conditions take the form:

Next, we will apply the factorization technique to Eq. (12). In this case, 

where

the first order ODE (18) can be rewritten in terms of

variable

\[ \xi = \theta \]

then, Eq. (9) becomes

Substituting (14) into (15), we get only one first order consistency equation

In order to solve this equation for

heren

is integration constant. Then, we can write the first order ODE

\[ \frac{dW}{d\bar{\theta}} = -B W^2 - 2 D W + \frac{2 n A}{n + 1} W^{(n+1)/n} + C \]

where \( C \) is an integration constant. Then, we can write the first order ODE

In order to solve this equation for \( W \) in a more general way, let us take \( W \) in the form \( W = \varphi^p, p \neq 0, 1 \) so, the first order ODE (18) can be rewritten in terms of \( \varphi \) as

\[ \left( \frac{d\varphi}{d\bar{\theta}} \right)^2 = \frac{B}{p^2} \varphi^2 - 2 D \frac{2 n A}{(n + 1) p^2} \frac{1}{p} \varphi^p \left( \frac{1}{n} + 1 \right) + C \varphi^{2 - p}. \]

If we want to guarantee the integrability of (19), the powers of \( \varphi \) have to be integer numbers between 0 and 4 [20]. Having in mind the conditions on \( n (n > 1) \) and \( p (p \neq 0, 1) \), we have the following possible cases:

- If \( C = 0, \ D = 0 \), then we can choose \( p \) in the following way

\[ p \in \left\{ -\frac{2 n}{1 - n}, -\frac{n}{1 - n}, \frac{n}{1 - n}, \frac{2 n}{1 - n} \right\}. \]

Here, we deal with only the case \( p = -\frac{2 n}{1 - n} \), since the cases \( p = \frac{2 n}{1 - n} \) and \( p = \frac{n}{1 - n} \) give rise to the same solutions for (3). Thus, Eq. (19) takes the form for \( p = -\frac{n}{1 - n} \):

\[ \left( \frac{d\varphi}{d\bar{\theta}} \right)^2 = \frac{(n - 1)^2 B}{n^2} \varphi^2 + \frac{2 (n - 1)^2 A}{n (n + 1)} \varphi. \]

Notice that for the special values of \( n = 2, 3 \), we have the \( B(2, 2) \) and \( B(3, 3) \) equations and the obtained solutions can be read as particular cases of this situation.

- If \( C = 0 \) and \( D \neq 0 \), we have the special cases: \( (p = 2, n = 2) \) and \( (p = -2, n = 2) \) which correspond to the \( B(2, 2) \) equations. Since these two cases also give rise to the same solutions for (3), we will consider only one of them. Now, Eq. (19) has the following form for \( (p = 2, n = 2) \)

\[ \left( \frac{d\varphi}{d\bar{\theta}} \right)^2 = \frac{B}{4} \varphi^2 + \frac{A}{3} \varphi - \frac{D}{2}. \]
4 Travelling wave solutions for BBM-like equations

In this section we will obtain the solutions of the differential equations (21) and (22) which allow us to get the travelling wave solutions of $B(n, n)$ (3). To solve this type of differential equations, it will be useful to recall some properties of the Weierstrass functions [21, 22].

Let us consider a differential equation with a quartic polynomial

$$\left(\frac{d\varphi}{d\theta}\right)^2 = P(\varphi) = a_0 \varphi^4 + 4 a_1 \varphi^3 + 6 a_2 \varphi^2 + 4 a_3 \varphi + a_4.$$ (23)

The solution of this equation can be written in terms of the Weierstrass function $\wp(\theta; g_2, g_3)$ where the invariants $g_2$ and $g_3$ of (23) are

$$g_2 = a_0 a_4 - 4 a_1 a_3 + 3 a_2^2, \quad g_3 = a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_3^3 - a_0 a_3^2 - a_1^3 a_4.$$ (24)

Then, the solution $\varphi$ can be found as

$$\varphi(\theta) = \varphi_0 + \frac{1}{4} P'(\varphi_0) \left( \varphi(\theta; g_2, g_3) - \frac{1}{24} P''(\varphi_0) \right)^{-1}$$ (25)

where the prime (’) denotes the derivative with respect to $\varphi$, and $\varphi_0$ is one of the roots of the polynomial $P(\varphi)$ (23). The discriminant $\Delta = g_3^2 - 27 g_2^3 = 0$, allows us to express Weierstrass functions in terms of trigonometric and hyperbolic functions [21, 22]

$$\wp(\theta; 12 b^2, -8 b^3) = b + 3 b \sin^{-2}[((3 b)^{1/2})]$$ (26)

$$\wp(\theta; 12 b^2, 8 b^3) = -b + 3 b \sin^{-2}[(3 b)^{1/2}].$$ (27)

Now, we will examine by separate each of the two cases considered in the above section.

4.1 $C = 0, D = 0$

In this case we have second order polynomial

$$P(\varphi) = \frac{(n-1)^2 B}{n^2} \varphi^2 + \frac{2 (n-1)^2 A}{n (n+1)} \varphi$$ (28)

whose roots are

$$\varphi_0 = 0, \quad \varphi_0 = -\frac{2 n A}{(n+1) B}.$$ (29)

The invariants have the form

$$g_2 = \frac{(n-1)^4 B^2}{12 n^4}, \quad g_3 = -\frac{(n-1)^6 B^3}{216 n^6}$$ (30)

and discriminant $\Delta = 0$.

Then, we can find the solutions from (25) in terms of the Weierstrass functions: for $\varphi_0 = 0$

$$\varphi(\theta) = -\frac{6 n (n-1)^2 A}{(n+1) [(n-1)^2 B - 12 n^2 \wp(\theta; g_2, g_3)]}$$ (31)

and for $\varphi_0 = -\frac{2 n A}{(n+1) B}$

$$\varphi(\theta) = \frac{2 n A}{(n+1) B} \left[ \frac{2 (n-1)^2 B + 12 n^2 \wp(\theta; g_2, g_3)}{(n-1)^2 B - 12 n^2 \wp(\theta; g_2, g_3)} \right].$$ (32)
Since the discriminant $\Delta = 0$, we can substitute (26) and (27) into the solutions (31) and (32), leading to the following type solutions of (21): a) periodic solutions

$$\varphi(\theta) = -\frac{2nA}{(n+1)B} \sin^2 \left( \frac{\sqrt{-B}(n-1)\theta}{2n} \right)$$ (33)

$$\varphi(\theta) = -\frac{2nA}{(n+1)B} \cos^2 \left( \frac{\sqrt{-B}(n-1)\theta}{2n} \right)$$ (34)

for $B < 0$, and b) hyperbolic solutions

$$\varphi(\theta) = \frac{2nA}{(n+1)B} \sinh^2 \left( \frac{\sqrt{B}(n-1)\theta}{2n} \right)$$ (35)

$$\varphi(\theta) = -\frac{2nA}{(n+1)B} \cosh^2 \left( \frac{\sqrt{B}(n-1)\theta}{2n} \right)$$ (36)

for $B > 0$. Notice that the first two real solutions (33) and (34) are simply related by a transformation.

Now, taking into account (8), (11) and (36), and $W = \varphi^p$, $p = n/(n-1)$, the solution of (3) can be written as

$$u(x, t) = \phi(\xi) = W^{1/n}(\theta) = \varphi^{p/n}(\theta), \quad \theta = \xi = h x + w t.$$ (37)

Substituting (33) and (34) into (37), we obtain periodic solutions of the equation (3)

$$u(x, t) = \left[ -\frac{2nA}{(n+1)B} \sin^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}$$ (38)

$$u(x, t) = \left[ -\frac{2nA}{(n+1)B} \cos^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}$$ (39)

for $B < 0$. Then, combining the trivial solution $u(x, t) = 0$ with (38) and (39), we have the compact support solutions for (3) in the following way [1],

$$u(x, t) = \begin{cases} 
\left[ -\frac{2nA}{(n+1)B} \sin^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}, & 0 \leq \sqrt{-B}(h x + w t) \leq \frac{2n\pi}{(n-1)} \\
0, & \text{otherwise}
\end{cases}$$ (40)

$$u(x, t) = \begin{cases} 
\left[ -\frac{2nA}{(n+1)B} \cos^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}, & |\sqrt{-B}(h x + w t)| \leq \frac{2n\pi}{(n-1)} \\
0, & \text{otherwise}
\end{cases}$$ (41)

For the trivial solution $u(x, t) = \left( -\frac{2nA}{(n+1)B} \right)^{1/n-1}$, we also get the compactons:

$$u(x, t) = \begin{cases} 
\left[ -\frac{2nA}{(n+1)B} \sin^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}, & |\sqrt{-B}(h x + w t)| \leq \frac{2n\pi}{(n-1)} \\
\left( -\frac{2nA}{(n+1)B} \right)^{1/n-1}, & \text{otherwise}
\end{cases}$$ (42)

$$u(x, t) = \begin{cases} 
\left[ -\frac{2nA}{(n+1)B} \cos^2 \left( \frac{\sqrt{-B}(n-1)(h x + w t)}{2n} \right) \right]^{1/n-1}, & 0 \leq \sqrt{-B}(h x + w t) \leq \frac{2n\pi}{(n-1)} \\
\left( -\frac{2nA}{(n+1)B} \right)^{1/n-1}, & \text{otherwise}
\end{cases}$$ (43)
Now, we can construct kink-like solutions \[23\], combining the non trivial solutions (38) and (39) with both (different) constant solutions, \(u(x, t) = 0\) and \(u(x, t) = \left(-\frac{2nA}{(n+1)B}\right)^{1/n-1}\), as follows

\[
u(x, t) = \begin{cases} 
0, & \sqrt{B} (h x + w t) < 0 \\
\left(-\frac{2nA}{(n+1)B}\right)^{1/n-1}, & 0 \leq \sqrt{B} (h x + w t) \leq \frac{n\pi}{(n-1)} \\
\left(-\frac{2nA}{(n+1)B}\right)^{1/n-1}, & \sqrt{B} (h x + w t) > \frac{n\pi}{(n-1)}.
\end{cases}
\] (44)

When we put (35) and (36) in (37), we have the solitary wave pattern solutions of hyperbolic type for (3). In order to get real solutions, it is necessary to examine the solutions for \(n\) even and \(n\) odd. Then, for \(n\) even and \(B > 0\) we have the solutions

\[
u(x, t) = \left[-\frac{2nA}{(n+1)B} \sin^2 \left(\frac{\sqrt{B} (n-1)}{2n} (h x + w t)\right)\right]^{1/n-1}, \quad 0 \leq \sqrt{B} (h x + w t) \leq \frac{n\pi}{(n-1)}
\] (46)

\[
u(x, t) = \left[-\frac{2nA}{(n+1)B} \cos^2 \left(\frac{\sqrt{B} (n-1)}{2n} (h x + w t)\right)\right]^{1/n-1}, \quad 0 \leq \sqrt{B} (h x + w t) \leq \frac{n\pi}{(n-1)}
\] (47)

However, if \(n\) is odd, we have the solution (46) provided \(A > 0\), while the solution (47) is valid only when \(A < 0\).

The solutions for the \(B(2, 2)\) and the \(B(3, 3)\) can be easily read from the above solutions of (3).

**Figure 1**: Plot of trigonometric solutions (38) and (39) for \(h = 1, w = 1, a = -1, n = 2\).

### 4.2 \(C = 0, D \neq 0\)

The case \(n = 2\) corresponds to the \(B(2, 2)\) equation and, according to (22) we have another kind of solutions for this equation. In this case, the polynomial

\[
P(\varphi) = \frac{B}{4} \varphi^2 + \frac{A}{3} \varphi - \frac{D}{2}
\] (48)

has two roots

\[
\varphi_0^\pm = -A \pm \frac{\sqrt{4A^2 + 18BD}}{3B}.
\] (49)
From (24), the invariants for Weierstrass functions take the values
\[ g_2 = \frac{B^2}{192}, \quad g_3 = \frac{B^3}{13824} \] (50)
and the discriminant also vanishes \( \Delta = 0 \). Then, from (25) the solutions read
\[ \varphi^\pm = -\frac{2A}{3B} \pm \frac{\sqrt{4A^2 + 18BD^2}}{3B} \left( \frac{5B + 48\wp(\theta; g_2, g_3)}{B - 48\wp(\theta; g_2, g_3)} \right). \] (51)
Since the discriminant equals zero, we can express these solutions in terms of trigonometric and hyperbolic functions using the relations (26) and (27):
\[ \varphi^\pm = -\frac{2A}{3B} \pm \frac{\sqrt{4A^2 + 18BD}}{3B} \cos\left(\frac{\sqrt{-B}}{2} \theta\right) \] (52)
for \( B < 0 \) and
\[ \varphi^\pm = -\frac{2A}{3B} \pm \frac{\sqrt{4A^2 + 18BD}}{3B} \cosh\left(\frac{\sqrt{B}}{2} \theta\right) \] (53)
for \( B > 0 \). Hence, from (37) we have \( u(x,t) = \phi(\xi) = \varphi(\theta) \), \( \theta = \xi = h x + w t \), and the solitary wave pattern and trigonometric solutions of (3) can be found substituting (52) and (53) into this expression. Here, again combining the trivial solution \( u(x,t) = -\frac{2A}{3B} \) with the trigonometric solution, we can construct compactons as
\[ u^\pm(x,t) = \begin{cases} \left[ -\frac{2A}{3B} \pm \frac{\sqrt{4A^2 + 18BD}}{3B} \cos\left(\frac{\sqrt{-B}}{2} (h x + w t)\right) \right], & |\sqrt{-B} (h x + w t)| \leq \pi \\ -\frac{2A}{3B}, & \text{otherwise} \end{cases} \] (54)
Figure 4: Plot of solitary wave pattern solutions (46) and (47) for $h = 1$, $w = 1$, $a = 1$, $n = 2$.

Kink-like solutions can also be obtained by taking into account another trivial solution $u(x, t) = -\frac{2A}{3B}$ together with $u(x, t) = -\frac{2A}{3B}$:

$$u^\pm(x, t) = \begin{cases} -\frac{2A}{3B}, & \sqrt{-B} (h x + w t) < -\pi \\ -\frac{2A}{3B} + \frac{\sqrt{4A^2 + 18BD}}{3B} \cos\left(\frac{\sqrt{-B}}{2} (h x + w t)\right), & -\pi \leq \sqrt{-B} (h x + w t) \leq 0 \\ -\frac{2A}{3B} + \frac{\sqrt{4A^2 + 18BD}}{3B}, & \sqrt{-B} (h x + w t) > 0 \end{cases} \quad (55)$$

where $4A^2 + 18BD > 0$ and $B < 0$. The solitary wave (hyperbolic type) pattern solutions are

$$u^\pm(x, t) = -\frac{2A}{3B} + \frac{\sqrt{4A^2 + 18BD}}{3B} \cosh\left(\frac{\sqrt{-B}}{2} (h x + w t)\right) \quad (56)$$

with the conditions $4A^2 + 18BD > 0$ and $B > 0$.

5 Lagrangian and Hamiltonian

From the second order nonlinear ODE (12), the Lagrangian can be written as

$$L_W(W, W_\theta, \theta) = \frac{1}{2} \left(W_\theta^2 + \frac{2n}{n+1} A W^{\frac{n+1}{n}} + B W^2 - 2DW\right). \quad (57)$$

Then, the canonical momentum is $P_W = \frac{\partial L_W}{\partial W_\theta} = W_\theta$, and the Hamiltonian $H_W = W_\theta P_W - L_W$ has the form

$$H_W(W, P_W, \theta) = \frac{1}{2} \left(P_W^2 - \frac{2n}{n+1} A W^{\frac{n+1}{n}} - B W^2 + 2DW\right). \quad (58)$$

Since $H_W$ does not depend on the variable $\theta$, it is a constant of motion $H_W = E$

$$E = \frac{1}{2} \left(\left(\frac{dW}{d\theta}\right)^2 - \frac{2n}{n+1} A W^{\frac{n+1}{n}} - B W^2 + 2DW\right). \quad (59)$$

This equation also gives rise to the first order ODE (18) with the identification $C = 2E$. We can express this constant of motion $H_W = E$ as a product of two independent constants of motion

$$E = \frac{1}{2} I_1 I_2 \quad (60)$$

where

$$I_1 = (W_\theta - \sqrt{\frac{2n}{n+1} A W^{\frac{n+1}{n}} + B W^2 - 2DW}) e^{S(\theta)} \quad (61)$$
\[ I_2 = (W_\theta + \sqrt{\frac{2nA}{n+1} W^\frac{n+1}{n} + BW^2 - 2DW}) e^{-S(\theta)} \]  

(62)

and \( S(\theta) \) has the form such that \( I_1 \) and \( I_2 \) satisfy \( dI_j/d\theta = 0, \ j = 1, 2 \)

\[ S(\theta) = \int \frac{AW^\frac{n}{n+1} + BW - D}{\sqrt{\frac{2nA}{n+1} W^\frac{n+1}{n} + BW^2 - 2DW}} d\theta. \]

6 Conclusions

In this work, we have investigated the travelling wave solutions of the BMM-like equations by means of the factorization technique. We have factorized the nonlinear second order ODE’s corresponding to the BMM-like equations. Then, we have obtained the solutions in terms of Weierstrass functions giving rise trigonometric (periodic) and hyperbolic type solutions (solitary wave pattern) and we have constructed compactons and kink-like solutions for the BBM-like equations. In addition to these we give the Lagrangian and the Hamiltonian corresponding to the second order ODE. We have seen that the first order ODE which gives rise to the solutions, can also be obtained from the constant of motion \( E \) corresponding to the Hamiltonian. Then, we have written this constant of motion \( E \) as a product of two independent constants of motion. We note that this technique is more systematic than others previously used for the analysis of this type equations. In general, we have more general solutions and we have recovered all the solutions previously reported \([14][15][16]\). Finally, we must mention that this method can be easily implemented to the other nonlinear equations, in particular we plan to give some results on the \( B(m, n) \) equations with \( m \neq n \) in a future publication.

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References

[1] Rosenau P, Hyman JM. Compactons: Soliton with finite wavelength. Phys Rev Lett 1993;70:564-567.
[2] Rosenau P. Compact and noncompact dispersive patterns. Phys Lett A 2000;275:193-203.
[3] Wazwaz A-M, Taha T. Compact and noncompact structures in a class of nonlinearly dispersive equations. Math Comput Simul 2003;62:171-189.
[4] Wazwaz A-M. General solutions with solitary patterns for the defocusing branch of the nonlinear dispersive K(n,n) equations in higher dimensional spaces. Appl Math Comput 2002;133:229-244.
[5] Wazwaz A-M. An analytic study of compactons structures in a class of nonlinear dispersive equations. Math Comput Simul 2003;63:35-44.
[6] Wazwaz A-M. Compactons and solitary patterns structures for variants of the KdV and KP equations. Appl Math Comput 2003;139:37-54.
[7] Ismail MS, Taha TR. A numerical study of compactons. Math Comput Simul 1998;47:519-530.
[8] Ludu A, Draayer JP. Patterns on liquid surfaces: cnoidal waves, compactons and scaling. Physica D 1998;123:82-91.
[9] Benjamin TB, Bona JL, and Mahony JJ. Model equations for long waves in nonlinear dispersive systems. Philos Trans R Soc London, Ser A 1972;272:47-78.
[10] Estévez PG, Kuru Ş, Negro J, and Nieto LM. Travelling wave solutions of the generalized Benjamin-Bona-Mahony equation. Chaos, Solitons and Fractals 2007; accepted.

[11] Helal MA. Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics. Chaos, Solitons and Fractals 2002;13:1917-1929.

[12] He Ji-H, Wu Xu-H. Construction of solitary solution and compacton-like solution by variational iteration method. Chaos, Solitons and Fractals 2006;29:108-113.

[13] Wazwaz A-M, Helal MA. Nonlinear variants of the BBM equation with compact and noncompact physical structures. Chaos, Solitons and Fractals 2005;26:767-776.

[14] Wazwaz A-M. Two reliable methods for solving variants of the KdV equation with compact and noncompact structures. Chaos, Solitons and Fractals 2006;28:454-462.

[15] Wang L, Zhou J, Ren L. The exact solitary wave solutions for a family of BBM equation. Int J Nonlinear Sciences 2006;1:58-64.

[16] Yadong S. Explicit and exact solutions for BBM-like $B(m, n)$ equations with fully nonlinear dispersion. Chaos, Solitons and Fractals 2005;25:1083-1091.

[17] Estévez PG, Kuru Ş, Negro J, and Nieto LM. Travelling wave solutions of two-dimensional Korteweg-de Vries-Burgers and Kadomtsev-Petviashvili equations. J Phys A: Math Gen 2006;39:11441-11452.

[18] Cornejo-Pérez O, Negro J, Nieto LM, and Rosu HC. Travelling-wave solutions for Korteweg-de Vries-Burgers equations through factorization. Found Phys 2006;36:1587-1599.

[19] Estévez PG, Kuru Ş, Negro J, and Nieto LM. Factorization of a class of almost linear second-order differential equations. J Phys A: Math Theor 2007;40:9819-9824.

[20] Ince EL. Ordinary Differential Equations. New York: Dover; 1956.

[21] Erdelyi A et al (Eds.). The Bateman Manuscript Project. Higher Transcendental Functions Volume II. Malabar, FL: Krieger Publishing Co.; 1981.

[22] Whittaker ET and Watson G. A Course of Modern Analysis. Cambridge: Cambridge University Press; 1988.

[23] Dusuel S, Michaux, and Remoissenet. From kinks to compactonlike kinks. Phys Rev E 1998;57:2320-2326.