Abstract

It is shown that the probability density satisfies a hyperbolic equation of motion with the unique characteristic that in its many-particle form it contains derivatives acting at spatially remote regions. Based on this feature we explore inter-particle correlations and the relation between the quantum equilibrium condition and the permutation invariance of the probability density. Some remarks with respect to the quantum to classical transition are also presented.

1 Introduction

A major difference between quantum mechanics and classical physics is the violation of the spatial separability principle [2]. For some physicists like Einstein this even undermined the possibility of doing science and establishing physical laws. It must be noted nevertheless that this distinct quality of quantum physics is reflected only on the entangled form of the total wavefunction of a many-particle system. On the other hand the dynamical equation that governs it exhibits exactly the same kind of system separability as in the classical case. Indeed, from a mereological point of view the Hamiltonian of a many-particle quantum system is exactly as a classical one. In the present article without modifying the fundamental equation of motion for the wavefunction we deduce a wave equation for the probability density which will make apparent the holistic aspect of quantum theory. In the next section we present the single particle formalism and examine the conditions that may modify the quantum mechanical predictions. This is related to the validity of the quantum equilibrium condition $\rho = |\Psi|^2$. In section 3 we examine the two-particle formulation and the connection between the quantum equilibrium condition and the permutation invariance principle and we conclude with the final remarks.

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2 Single-particle formalism

It has been noted by Dyson [1] that in close analogy to Maxwell’s theory, the structure of quantum theory is two-fold, with two distinct layers of description. The more fundamental substrate constitutes the first layer which contains abstract wavefunctions, whereas the second one is more concrete and contains directly observable quantities like probabilities and intensities. In the present section we derive a hyperbolic equation that governs the probability density and also includes spatial and time derivatives of the velocity field.

Considering then the single-particle one-dimensional Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \Psi = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \right) \Psi, \]  

(1)

where \( V(x,t) \) the time-dependent external potential and \( \Psi \) a wavefunction in the coordinate representation, we will perform a Madelung transformation

\[ \Psi = \sqrt{\rho} e^{iS/\hbar}, \]  

(2)

where \( S/\hbar \) is the phase of the wavefunction and the velocity field is given by the guiding equation

\[ v = \frac{1}{m} \partial_x S. \]  

(3)

After separating real and imaginary parts two formulas are obtained. By equating the imaginary parts we arrive at local continuity equation for the probability density

\[ \partial_t \rho + \partial_x (\rho v) = 0. \]  

(4)

Consequently the Hamilton-Jacobi equation is obtained from (1) through (2) by equating its real parts,

\[ -\partial_t S - \frac{1}{2m} (\partial_x S)^2 = V + Q. \]  

(5)

In the above expression

\[ Q = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \sqrt{\rho} = -\frac{\hbar^2}{4m\rho} \left[ \frac{\partial^2 \rho}{\partial x^2} - \frac{(\partial_x \rho)^2}{2\rho} \right], \]  

(6)

is the Bohmian quantum potential. Taking the divergence gives the second hydrodynamic equation

\[ \partial_t v + v \partial_x v = -\frac{1}{m} \partial_x (V + Q), \]  

(7)

from which it follows \[ \partial_t (\rho v) = -\frac{1}{m} \partial_x \Pi - \frac{\rho}{m} \partial_x V, \]  

(8)
where the probability quantum stress tensor is written as
\[
\Pi = \rho v^2 + \frac{\hbar^2}{4m^2} \left( \frac{\partial_x \ln \rho}{\rho} - \partial_x^2 \rho \right).
\] (9)

Differentiating (4) over time and (8) over the spatial dimension we finally this
wave equation for the probability density
\[
\partial_t^2 \rho = \partial_x^2 \Pi + \partial_x \left( \rho \partial_x V \right).
\] (10)

An equation of this general form was derived by Lighthill in the study of aero-
coustics [7] and by substituting \( \rho = e^{\psi} \) in the above expression we can ensure its
positivity. We note that (10) is nonlinear even though we started from a linear
equation for \( \Psi \) and of course the local continuity equation for the probability
density is linear too. It must be also remembered that at least in the Bohmian
conceptual framework the probability density has only a contingent and not a
necessary relation to \(|\Psi|^2\). Even if it happens that these two quantities coincide
numerically they refer to two distinct realities in the quantum mechanical
formalism. As it is evident from the polar substitution (10) satisfies the quantum
equilibrium condition. In a way we have used Born’s rule as a heuristic tool in
order to deduce a second order equation in time. At the same time, (10) admits
also solutions that violate this condition, so \( \rho \neq |\Psi|^2 \). Our task is to examine if
we should accept or exclude such a family of solutions. Definitely, the ultimate
desideratum of a physical theory is agreement with observation and experiment
and this clearly dictates that the quantum equilibrium condition must be ful-
filled but as we will see in the following section there are additional fundamental
considerations for that related to permutation symmetry. So, we will assume
that the quantum equilibrium condition (or hypothesis) always holds which also
means that the Hamilton-Jacobi equation is valid by setting \( \rho = |\Psi|^2 \) and the
local velocity (3). So after repeating the above steps having substituted (5) for
the sum of the classical and quantum potentials we will find
\[
\partial_t^2 |\Psi|^2 = \partial_x^2 \left( |\Psi|^2 v^2 \right) - \partial_x \left[ |\Psi|^2 (\partial_t + v \partial_x) v \right],
\] (11)

where we have included the material derivative. The above is a linear equation
with respect to \(|\Psi|^2\), in contradistinction to the non-linear (10), satisfies
by construction the quantum equilibrium condition, the guiding equation for
the velocity field and the quantum Hamilton-Jacobi equation. As with the con-
tinuity equation (3) it does not depend on the applied potential but only on
the velocity field and its time derivative and gradient. It relates the second
derivative of the probability density with the velocity of the probability fluid
without any reference to the classical or quantum potential and is subject to
appropriate initial conditions for \(|\Psi(x,0)|^2\) and \(\partial_t |\Psi(x,0)|^2\). This is the for-
mail we will extend and investigate the consequences of the two-particle case
in the next section. As a side note we could observe that in three dimensions
the second spatial derivative term includes the dyad of the velocity and since
\(\frac{1}{2} \nabla v^2 = (v \cdot \nabla) v + v \times (\nabla \times v)\) it follows that the probability density displays
dependence on the velocity field even if the latter is irrotational and coupling to
a vector potential leads immediately to the Aharonov-Bohm effect. We should
also note that according to (11) the velocity field is uniquely defined so it is not
possible to add a divergence-less velocity field term to the guiding equation as
in [8].

3 Permutation invariance for a bi-particle quantum compound

The two-particle case is more interesting from a conceptual point of view. We
consider a system of two interacting spinless particles with equal masses in one
dimension. Following the derivation in [3] for the single particle case, we derive
the probability density local continuity equation

\[ \partial_t |\Psi_{12}|^2 + \partial_{x_1} (|\Psi_{12}|^2 v_1) + \partial_{x_2} (|\Psi_{12}|^2 v_2) = 0 \]  

(12)

where \( \Psi = \Psi_{12} \) the two-body probability amplitude. The corresponding
two-body Hamilton-Jacobi equation

\[ -\partial_t S_{12} - \frac{1}{2m_1} (\partial_{x_1} S_{12})^2 - \frac{1}{2m_2} (\partial_{x_2} S_{12})^2 = V + Q. \]  

(13)

where the \( V, Q \) the two-particle classical and quantum potentials. The two
corresponding probability field velocities are expressed as

\[ v_i = \frac{1}{m_i} \partial_{x_i} S_{12}, i = 1, 2, \]  

(14)

and we also note that

\[ \frac{1}{m_2} \partial_{x_2} v_1 = \frac{1}{m_1} \partial_{x_1} v_2. \]  

(15)

Taking the two spatial derivatives of (13) yields

\[ \partial_t v_1 = -v_1 \partial_{x_i} v_1 - v_2 \partial_{x_2} v_2 - \frac{1}{m_1} \partial_{x_1} (V + Q), \]  

(16)

\[ \partial_t v_2 = -v_2 \partial_{x_2} v_2 - v_1 \partial_{x_1} v_1 - \frac{1}{m_1} \partial_{x_2} (V + Q). \]  

(17)

Employing the above and the continuity equation we obtain these Navier-Stokes
equations for the momentum fields

\[ \partial_t (|\Psi|^2 v_1) = -\partial_{x_1} (|\Psi|^2 v_1^2) - |\Psi|^2 v_2 \partial_{x_2} v_1 \]

\[ - v_1 \partial_{x_2} (|\Psi|^2 v_2) - \frac{|\Psi|^2}{m_1} \partial_{x_1} (V + Q), \]  

(18)
\[
\frac{\partial_t}{\partial t}(\lvert \Psi \rvert^2 v_2) = - \frac{\partial_{x_2}}{\partial x_2}(\lvert \Psi \rvert^2 v_2) - \lvert \Psi \rvert^2 v_1 \partial_{x_1} v_2 \\
- v_2 \partial_{x_1} (\lvert \Psi \rvert^2 v_1) - \frac{\lvert \Psi \rvert^2}{m_2} \partial_{x_2} (V + Q).
\]  

(19)

As earlier we substitute the sum of the classical and quantum potentials from (13) and then differentiate (12) over time, (18) over \(x_1\) and (19) over \(x_2\) and subtract from the first the sum of the two latter and we obtain finally

\[
\frac{\partial^2}{\partial t^2} \lvert \Psi \rvert^2 = 2 \sum_{i=1}^{2} \left[ \partial^2_{x_i} (\lvert \Psi \rvert^2 v_i^2) - \partial_{x_i} \lvert \Psi \rvert^2 (\partial_t + v_i \partial_{x_i}) v_i \right] \\
+ \left[ \partial_{x_1} v_1 \partial_{x_2} (\lvert \Psi \rvert^2 v_2) + v_1 \partial_{x_1} \partial_{x_2} (\lvert \Psi \rvert^2 v_2) + (1 \leftrightarrow 2) \right].
\]

(20)

This is the two-body extension of the single particle wave-like equation of motion derived earlier. The last term in the above expression is highly important due to the inclusion of spatial derivatives acting simultaneously at two distant points which indicates the non-separability of the bi-particle compound. It exhibits particular conceptual interest as the gradient operators act at two distant sites and cannot be ascribed to one or the other individual particle and the inter-particle expressions for the many-body case follow the same pattern. Even though such kinds of derivatives also appear in the Pauli-Jordan field commutation relations [9] their physical significance is associated with limitations in the simultaneous measurability of field averages for space-time connected finite space-time regions. As it has been shown in particular by Bohr and Rosenfeld in their seminal paper, the disturbance caused by a scalar electric field induced by the test body is responsible for these spatial derivative terms. The startling difference is that in the present case the relevant points may be very well casually disconnected so no such influence is possible, which clearly undermines one of the basic premises of Bell’s premises which is separability and shows there is considerable tension between quantum mechanics and the field theory description. It should be added furthermore that the ontological status of particle trajectories, which has such a prominent place in Bohm’s interpretation, here is severely undermined since there is an immediate effect between distant non-crossing trajectories. Another point to consider is that the expression given in (20) depends inversely on the two interacting particle masses. When \(m_1 \neq m_2\) it is reasonable to assume that \(S_{12} \neq S_{21}\) from which it follows that when we interchange indices (20) does not remain invariant so \(\lvert \Psi_{12} \rvert^2 \neq \lvert \Psi_{21} \rvert^2\). In a sense it is the non-invariance of the unobservable phase function that determines the asymmetry in the observable probability density. It is clear, and we will return later to this point, that this is not the case for identical particles with equal masses. It is evident that an \(N\) particle compound corresponds to a sum of \(N(N-1)/2\) terms of that kind and when the mass of a particle is macroscopic its contribution will be negligible. What is clearly violated is the principle of spatio-temporal separability and the individuation of particles associated with
the bi-particle which seems to constitute a single irreducible entity. Accordingly, the measure of non-classicality may be quantified by exactly that kind of inter-particle terms. Those terms that include gradients acting independently at two distant points express the classicality of the system, but those rest of them involving inter-site derivatives illustrate its unique non-classical character. Finally, we can explore the implications following from the two-particle equation of motion we obtained and its relation to the permutation invariance principle. It follows after expanding the derivatives in (20) that

\[ \left( \partial_t^2 - \hat{\Lambda} \right) \left( |\Psi_{12}|^2 - |\Psi_{21}|^2 \right) = 0, \quad (21) \]

where \( \hat{\Lambda} \) a linear operator acting on the coincidence probability density which is invariant if we permute the two particle indices, so that \( \hat{\Lambda} = \hat{\Lambda}_{12} = \hat{\Lambda}_{21} \). It follows directly then from (21) that \( |\Psi_{12}|^2 = |\Psi_{21}|^2 \). We see then that Born’s rule dictates a linear equation for the probability density and consequently ensures its invariance under particle permutations, so it is not possible to accept stochastic fluctuations that may generate deviations from the guiding velocity law and in addition the equilibrium condition stated earlier. Aside from this kind of argumentation, we could also maintain that a possible variance from this rule would not automatically ensure the validity of the permutation invariance \[10\] of the coincidence probability density which is a conclusion not acceptable on physical grounds. Incidentally, this kind of reasoning also excludes the possibility of instantaneous signalling \[11\].

4 Concluding remarks

Without modifying Schrödinger’s equation we sought an equation of motion for the probability density with second order time derivatives that admits solutions which satisfy the condition \( \rho = |\Psi|^2 \). The desired wave equation contains some intriguing terms that signify non-classical inter-particle correlations and the interconnectedness of the quantum compound. It was suggested that a dual ontology is indicated by the presented formalism. On one hand the two sum terms in (20) correspond to the two distinct single particles while the inter-particle cross-terms signify the bi-particle presence and the unicity of the quantum aggregate. We have also proved that as a consequence of Born’s rule the probability density remains invariant under the permutation of the constituent particles of the quantal compound. This conclusion follows directly from the linearity of the operator \( \Lambda \), so it is based solely on theoretical considerations and reveals a deep underlying relationship between the quantum equilibrium hypothesis \[6, 11\] and permutation invariance \[10\]. In addition we have argued that a possible gradual transition to quantum equilibrium \( \rho \to |\Psi|^2 \), through stochastic or other relaxation processes \[4\] seems implausible.
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