The complement value problem for non-local operators

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Let $D$ be a bounded Lipschitz domain of $\mathbb{R}^d$. We consider the complement value problem

\[
\begin{aligned}
& (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0 \quad \text{in } D, \\
& u = g \quad \text{on } D^c.
\end{aligned}
\]

Under mild conditions, we show that there exists a unique bounded continuous weak solution. Moreover, we give an explicit probabilistic representation of the solution. The theory of semi-Dirichlet forms and heat kernel estimates play an important role in our approach.

Keywords: Complement value problem; non-local operator; probabilistic representation; semi-Dirichlet form; heat kernel estimate.

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1 Introduction and Main Result

Let $d \geq 1$ and $D$ be a bounded Lipschitz domain of $\mathbb{R}^d$. Suppose $0 < \alpha < 2$ and $p > d/2$. Let $a > 0$, $b = (b_1, \ldots, b_d)^*$ satisfying $|b| \in L^{2p}(D; dx)$ if $d \geq 2$ and $|b| \in L^\infty(D; dx)$ if $d = 1$, $c \in L^{p^*2}(D; dx)$, $f \in L^{2[p^*1]'}(D; dx)$ and $g \in B_b(D^c)$. We consider the complement value problem:

\[
\begin{aligned}
& (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0 \quad \text{in } D, \\
& u = g \quad \text{on } D^c.
\end{aligned}
\]

The fractional Laplacian operator $\Delta^{\alpha/2}$ can be written in the form

\[
\Delta^{\alpha/2} \phi(x) = \lim_{\varepsilon \to 0} A(d, -\alpha) \int_{\{|x-y| \geq \varepsilon\}} \frac{\phi(y) - \phi(x)}{|x-y|^{d+\alpha}} dy, \quad \phi \in C_c^\infty(\mathbb{R}^d),
\]
where $A(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$ and $C^\infty_c(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support.

The problem (1.1) is an analogue of the Dirichlet problem for second order elliptic integro-differential equations. For these non-local equations, as opposed to the classical local case, the function $g$ should be prescribed not only on the boundary $\partial D$ but also in the whole complement $D^c$. The complement value problem for non-local operators has many applications, for example, in peridynamics [11, 15, 31], particle systems with long range interactions [20], fluid dynamics [14] and image processing [21]. The problem has been widely studied by using different approaches from both probability and analysis. These include, in particular, the semi-group approach by Bony, Courrège and Priouret [8], the classical PDE approach by Garroni and Menaldi [19], the viscosity solution approach by Barles, Chasseigne and Imbert [3] and Arapostathisa, Biswasb and Bony, Courrège and Priouret [8], the classical PDE approach by Garonii and Menaldii [19], the

Different from [2, 3], $b, c, f$ and $g$ in (1.1) are not assumed to be continuous. Also, the second order elliptic integro-differential operator in (1.1) is not assumed to have the maximum principle. To overcome these complications, in this paper, we will use the theory of semi-Dirichlet forms to study both the existence and uniqueness of solutions to the problem (1.1). Our work is partially motivated by Guan and Ma [24], which uses the Dirichlet form approach to study the boundary value problem for regional fractional Laplacians. The heat kernel estimates recently obtained by Chen and Hu [11] play an important role in our work.

Denote $L := \Delta + a^2 \Delta^{\alpha/2} + b \cdot \nabla$. By setting $b = 0$ off $D$, we may assume that the operator $L$ is defined on $\mathbb{R}^d$. By [11, Theorem 1.4], the martingale problem for $(L, C^\infty_c(\mathbb{R}^d))$ is well-posed for every initial value $x \in \mathbb{R}^d$. We use $((X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ to denote the strong Markov process associated with $L$. Let $\rho > 0$. Define

$$q_\rho(t,z) = t^{-d/2} \exp\left(-\frac{\rho |z|^2}{t}\right) + t^{-d/2} \wedge \frac{t}{|z|^{d+\alpha}}, \quad t > 0, z \in \mathbb{R}^d.$$  

By [11, Theorems 1.2-1.4], $X$ has a jointly continuous transition density function $p(t,x,y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, and for every $T > 0$ there exist positive constants $C_i, i = 1, 2, 3, 4$ such that

$$C_1 q_{C_2} (t, x-y) \leq p(t,x,y) \leq C_3 q_{C_4} (t, x-y), \quad (t,x,y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (1.2)$$

Define

$$e(t) := e^{\int_0^t c(X_s) \, ds}, \quad t \geq 0,$$

and $\tau := \inf\{t > 0 : X_t \in D^c\}$. Denote $W^{1,2}(D) = \{u \in L^2(D; dx) : |\nabla u| \in L^2(D; dx)\}$, $W^{1,2}_0(D) = \{u \in W^{1,2}(D) : \exists \{u_n\}_{n \in \mathbb{N}} \subset C^\infty_c(D) \text{ such that } u_n \to u \text{ in } W^{1,2}(D)\}$, and $W^{1,2}_{loc}(D) := \{u : u\phi \in W^{1,2}_0(D) \text{ for any } \phi \in C^\infty_c(D)\}$.

The main result of this paper is the following theorem.
Theorem 1.1 There exists $M > 0$ such that if $\|c^+\|_{L^{p\vee 1}} \leq M$, then for any $f \in L^{2(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$, there exists a unique $u \in B_b(\mathbb{R}^d)$ satisfying $u|_D \in W^{1,2}_{loc}(D) \cap C(D)$ and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0 \text{ in } D, \\ u = g \text{ on } D^c. \end{cases}$$

Moreover, $u$ has the expression

$$u(x) = E_x \left[ e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right], \quad x \in \mathbb{R}^d. \quad (1.3)$$

In addition, if $g$ is continuous at $z \in \partial D$ then

$$\lim_{x \to z} u(x) = u(z).$$

Hereafter $(\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0$ is understood in the distribution sense: for any $\phi \in C^\infty_c(D)$,

$$\int_D \langle \nabla u, \nabla \phi \rangle dx + \frac{a^\alpha A(d,-\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} dx dy$$

$$- \int_D \langle b, \nabla u \rangle \phi dx - \int_D cu \phi dx - \int_D f \phi dx = 0. \quad (1.4)$$

Note that the double integral appearing in (1.4) is well-defined for any $u \in B_b(\mathbb{R}^d)$ with $u|_D \in W^{1,2}_{loc}(D)$ and $\phi \in C^\infty_c(D)$.

As a direct consequence of Theorem 1.1, we have the following corollary.

Corollary 1.2 If $c \leq 0$, then for any $f \in L^{2(p\vee 1)}(D; dx)$ and $g \in B_b(D^c)$ satisfying $g$ is continuous on $\partial D$, there exists a unique $u \in B_b(\mathbb{R}^d)$ such that $u$ is continuous on $\overline{D}$, $u|_D \in W^{1,2}_{loc}(D)$, and

$$\begin{cases} (\Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla + c)u + f = 0 \text{ in } D, \\ u = g \text{ on } D^c. \end{cases}$$

Moreover, $u$ has the expression

$$u(x) = E_x \left[ e(\tau)g(X_\tau) + \int_0^\tau e(s)f(X_s)ds \right], \quad x \in \mathbb{R}^d.$$
2 Some Lemmas

Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(\mathbb{R}^d; dx)$ and denote by $C$ a generic fixed strictly positive constant, whose value can change from line to line. Recall that a measurable function $\varphi$ on $\mathbb{R}^d$ is said to be in the Kato class if and only if

$$
\begin{align*}
\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\|y-x\| \leq r} \frac{|\varphi(y)|}{|x-y|^{d-2}} dy &= 0, \quad \text{if } d \geq 3, \\
\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\|y-x\| \leq r} (-\ln(|x-y|))|\varphi(y)| dy &= 0, \quad \text{if } d = 2, \\
\sup_{x \in \mathbb{R}^d} \int_{\|y-x\| \leq 1} |\varphi(y)| dy < \infty, \quad \text{if } d = 1.
\end{align*}
$$

Lemma 2.1 Define

$$
\begin{align*}
\mathcal{E}^a(\phi, \psi) &= \int_{\mathbb{R}^d} \langle \nabla \phi, \nabla \psi \rangle dx + \frac{a^\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x-y|^{d+\alpha}} dx dy \\
D(\mathcal{E}^0) &= W^{1,2}(\mathbb{R}^d).
\end{align*}
$$

Then, $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a regular lower-bounded semi-Dirichlet form on $L^2(\mathbb{R}^d; dx)$. Moreover, $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$ is the Hunt process associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$.

Proof. By the assumption on $b$ and Hölder's inequality, we find that $|b|^2$ belongs to the Kato class. Then, we obtain by [37, Chapter 7, Lemma 7.5] that there exists $\beta_0 > 0$ such that

$$
\int_{\mathbb{R}^d} |b|^2 \phi^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 dx + \beta_0 \int_{\mathbb{R}^d} |\phi|^2 dx, \quad \forall \phi \in W^{1,2}(\mathbb{R}^d). 
$$

Define

$$
\mathcal{E}^a_\beta(\phi, \psi) = \mathcal{E}^a(\phi, \psi) + \beta(\phi, \psi), \quad \phi, \psi \in D(\mathcal{E}^a).
$$

Then, $(\mathcal{E}^a_\beta, D(\mathcal{E}^a))$ is a coercive closed form on $L^2(\mathbb{R}^d; dx)$ for any $\beta > \beta_0$.

Denote by $C_0(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$ which vanish at infinity. If $\phi \in C_c(\mathbb{R}^d)$, then $\Delta^{\alpha/2} \phi \in C_0(\mathbb{R}^d)$ (cf. [36, Theorem 31.5]). Moreover, we have $\Delta^{\alpha/2} \phi \in L^2(\mathbb{R}^d; dx)$. In fact, suppose $\text{supp}[\phi] \subset B(0, N)$ for some $N \in \mathbb{N}$, then we get

$$
\begin{align*}
\int_{\{|x| > 2N\}} |\Delta^{\alpha/2} \phi|^2 dx &= \int_{\{|x| > 2N\}} \left( A(d, -\alpha) \int_{\mathbb{R}^d} \frac{\phi(x + y)}{|y|^{d+\alpha}} dy \right)^2 dx \\
&= \int_{\{|x| > 2N\}} \left( A(d, -\alpha) \int_{\{|y| \geq 1\}} \frac{\phi(x + y)}{|y|^{d+\alpha}} dy \right)^2 dx \\
&\leq C \int_{\{|x| > 2N\}} \int_{\{|y| \geq 1\}} \frac{\phi^2(x+y)}{|y|^{d+\alpha}} dy dx \\
&\leq C \int_{\mathbb{R}^d} \phi^2 dx \int_{\{|y| \geq 1\}} \frac{1}{|y|^{d+\alpha}} dy \\
&< \infty.
\end{align*}
$$
We have
\[ \mathcal{E}^0(\phi, \psi) = (-L\phi, \psi), \quad \forall \phi, \psi \in C_c^\infty(\mathbb{R}^d). \]

By [38] Theorem 3.1, \((\mathcal{E}^0, D(\mathcal{E}^0))\) is a regular lower-bounded semi-Dirichlet form on \(L^2(\mathbb{R}^d; dx)\).

We now show that \(((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})\) is the Hunt process associated with \((\mathcal{E}^0, D(\mathcal{E}^0))\). We will follow the method of [18, Section 4], which relates the Hunt process associated with a semi-Dirichlet form to a martingale problem. Since \(b\) in [11] is not assumed to be continuous, we cannot directly apply [18, Theorem 4.3]. We give the detailed argument below.

Let \(((X_t^\varepsilon)_{t \geq 0}, (P_x^\varepsilon)_{x \in \mathbb{R}^d})\) be a Hunt process associated with \((\mathcal{E}^0, D(\mathcal{E}^0))\). Suppose that \(\phi \in C_c^\infty(\mathbb{R}^d)\). Define
\[ M_t^\phi := \phi(X_t^\varepsilon) - \phi(X_0^\varepsilon) - \int_0^t L\phi(X_s^\varepsilon)ds. \]
Let \(\psi = (1-L)\phi\). Then, \(\psi \in L^2(\mathbb{R}^d; dx)\). Since \(\phi = G_1\psi\ \text{d}x\text{-a.e.}, we get \(\phi = R_1^\varepsilon\psi\ \text{q.e.}\), where \(G_1\) and \(R_1^\varepsilon\) are the 1-resolvents of \(\mathcal{E}^0\) and \(X^\varepsilon\), respectively. Hence
\[ M_t^\phi = R_1^\varepsilon\psi(X_t^\varepsilon) - R_1^\varepsilon\psi(X_0^\varepsilon) - \int_0^t (R_1^\varepsilon\psi - \psi)(X_s^\varepsilon)ds, \quad P_x^\varepsilon - \text{a.s., q.e.} \ x \in \mathbb{R}^d, \]
which implies that \(\{M_t^\phi\}\) is a martingale under \(P_x^\varepsilon\) for q.e. \(x \in \mathbb{R}^d\).

Let \(\Phi\) be a countable subset of \(C_c^\infty(\mathbb{R}^d)\) such that for any \(\phi \in C_c^\infty(\mathbb{R}^d)\) there exist \(\{\phi_n\} \subset \Phi\) satisfying \(\|\phi_n - \phi\|_\infty, \|\partial_i \phi_n - \partial_i \phi\|_\infty, \|\partial_i \partial_j \phi_n - \partial_i \partial_j \phi\|_\infty \to 0\) as \(n \to \infty\) for any \(i, j \in \{1, 2, \ldots, d\}\). Then, there is an \(\mathcal{E}^0\)-exceptional set of \(\mathbb{R}^d\), denoted by \(F\), such that \(\{M_t^\phi\}\) is a martingale under \(P_x^\varepsilon\) for any \(x \in F^c\). Note that
\[ E_x^\varepsilon \left[ \int_0^t |b \cdot \nabla \phi|(X_s^\varepsilon)ds \right] \leq e^t \|\nabla \phi\|_\infty R_1^\varepsilon |b|(x). \]
We obtain by taking limits that \(\{M_t^\phi\}\) is a martingale under \(P_x^\varepsilon\) for any \(\phi \in C_c^\infty(\mathbb{R}^d)\) and q.e. \(x \in \mathbb{R}^d\). Therefore, by the uniqueness of solutions to the martingale problem for \((L, C_c^\infty(\mathbb{R}^d))\) (see [11] Theorem 1.4), we find that \(((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})\) is the Hunt process associated with \((\mathcal{E}^0, D(\mathcal{E}^0))\). \(\square\)

**Lemma 2.2**
\[ \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} P_x \left( \sup_{0 \leq s \leq t} |X_s - x| > r \right) = 0, \quad \forall r > 0. \]

**Proof.** Let \(t, r > 0\). Define
\[ \iota_t := \sup_{x \in \mathbb{R}^d, 0 \leq s \leq t} P_x(|X_s - x| \geq r) = \sup_{x \in \mathbb{R}^d, 0 \leq s \leq t} \int_{B(x,r)^c} p(s, x, y)dy. \]
By [12], we get
\[ \lim_{t \to 0} \iota_t = 0. \quad (2.2) \]
Define
\[ S = \inf \{ t > 0 : |X_t - X_0| > 2r \} . \]

For \( x \in \mathbb{R}^d \), we have
\[
P_x \left( \sup_{0 \leq s \leq t} |X_s - x| > 2r \right) = P_x(S \leq t) 
\leq P_x(|X_t - x| \geq r) + P_x(S \leq t, X_t \in B(x, r)) 
\leq \iota_t + P_x(S \leq t \text{ and } |X_t - X_S| > r) 
\leq \iota_t + E_x[1_{\{S \leq t\}} P_{X_S}(|X_{t-S} - X_0| > r)] 
\leq 2\iota_t. \tag{2.3} \]

The proof is complete by (2.2) and (2.3).

Let \( U \) be an open set of \( \mathbb{R}^d \). Define
\[ \tau_U := \inf \{ t > 0 : X_t \in U^c \} . \]

Denote by \( p^U(t, x, y) \) the transition density function of the part process \( ((X^U_t)_{t \geq 0}, (P_x)_{x \in U}) \). Define \( G^U_\gamma(x, y) := \int_0^\infty e^{-\gamma t} p^U(t, x, y) dt \) for \( x, y \in U \) and \( \gamma \geq 0 \).

**Lemma 2.3** Let \( U \) be a bounded open set of \( \mathbb{R}^d \).

1. For any \( x \in U \),
\[
P_x(\tau_U < \infty) = 1. \tag{2.4} \]

2. There exist positive constants \( \theta_1 \) and \( \theta_2 \) such that
\[
p^U(t, x, y) \leq \theta_1 q_{\theta_2}(t, x - y), \quad (t, x, y) \in (0, \infty) \times U \times U. \tag{2.5} \]

3. For any \( t > 0 \), \( P_x(\tau_U = t) = 0 \) and the function \( x \mapsto P_x(\tau_U > t) \) is upper semi-continuous on \( \mathbb{R}^d \).

4. For any \( x, y \in U \), the function \( t \mapsto p^U(t, x, y) \) is continuous on \( (0, \infty) \).

**Proof.** By (1.2), similar to [29, Lemma 6.1], we can show that
\[
\sup_{x \in U} P_x(\tau_U > 1) < 1, \tag{2.6} \]

and there exist positive constants \( \theta_1^* \) and \( \theta_2^* \) such that
\[
p^U(t, x, y) \leq \theta_1^* e^{-\theta_2^*(t-1)}, \quad (t, x, y) \in (1, \infty) \times U \times U. \tag{2.7} \]

By (2.6) and the Markov property of \( X \), we conclude that (2.4) holds. By (1.2) and (2.7), we conclude that (2.5) holds.

The proof of (3) is the same as [34, Theorem 1.4.7 and Proposition 2.2.1]. We now prove (4). For \( x, y \in U \) and \( t > 0 \), we have
\[
p^U(t, x, y) = p(t, x, y) - E_x[p(t - \tau_U, X_{\tau_U}, y)1_{\{\tau_U \leq t\}}]. \tag{2.8} \]

Then, (4) follows from (2.8), the continuity of \( p(t, x, y) \), (1.2) and (3). \( \square \)
Lemma 2.4 Let $U$ be a bounded open set of $\mathbb{R}^d$. Suppose that $\varphi$ is a measurable function on $\mathbb{R}^d$ which belongs to the Kato class. Then, we have

$$ \limsup_{t \to 0} \sup_{x \in U} E_x \left[ \int_0^t |\varphi(X_s^U)| ds \right] = 0. $$

Proof. We have

$$ t^{-d/2} \wedge \frac{t}{|x-y|^{d+\alpha}} \leq t^{-d/2} \leq e^{\rho t-d/2} \exp \left( -\frac{\rho |x-y|^2}{t} \right) \quad \text{if} \quad |x-y|^2 < t, \quad (2.9) $$

and

$$ \int_0^{|x-y|^2} \left( t^{-d/2} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) dt \leq \int_0^{|x-y|^2} \frac{t}{|x-y|^{d+\alpha}} dt \leq \frac{1}{2|x-y|^{d+\alpha-1}}. \quad (2.10) $$

Then, we obtain by (2.5), (2.7), (2.9) and (2.10) that there exists $C > 0$ such that for any $x, y \in U$,

$$ G_0^U(x, y) \leq \begin{cases} \frac{C}{|x-y|^{d-2}}, & d \geq 3, \\ C \ln \left( 1 + \frac{1}{|x-y|} \right), & d = 2, \\ C, & d = 1. \end{cases} \quad (2.11) $$

The proof is complete by Lemma 2.2, (2.11) and [39, Theorem 1].

Lemma 2.5 There exists $C > 0$ such that

$$ \sup_{x \in D} E_x \left[ \int_0^\tau v(X_s) ds \right] \leq C \|v\|_{L^{p \vee 1}}, \quad \forall v \in L^{p \vee 1}_+(D). \quad (2.12) $$

Proof. We only prove (2.12) when $d \geq 3$. The cases that $d = 1, 2$ can be considered similarly. Let $v \in L^{p \vee 1}_+(D)$ and $x \in D$. Denote by $\varsigma(D)$ the diameter of $D$. By (2.11), we have

$$ E_x \left[ \int_0^\tau v(X_s) ds \right] \leq \int_D G_0^D(x, y) v(y) dy $$

$$ \leq C \int_D \frac{v(y)}{|x-y|^{d-2}} dy $$

$$ \leq C \left( \int_D v(y)^p dy \right)^{1/p} \left( \int_D |x-y|^{-q(d-2)} dy \right)^{1/q} $$

$$ = C' \|v\|_{L^p} \left( \int_{\varsigma(D)} r^{d-1-q(d-2)} dr \right)^{1/q} $$

$$ = C'' \|v\|_{L^p}, $$

7
where $C'$ and $C''$ are positive constants.

Suppose that $\overline{D} \subset B(0, N)$ for some $N \in \mathbb{N}$. Define

$$\Omega = B(0, N).$$

(2.13)

**Lemma 2.6** Let $\gamma \geq 0$. For any compact set $K$ of $\Omega$, there exist $\delta > 0$ and $\vartheta_1, \vartheta_2 \in (0, \infty)$ such that for any $x, y \in K$ satisfying $|x - y| < \delta$, we have

\[
\begin{cases}
\frac{\vartheta_1}{|x - y|^{d - 2}} \leq G^\Omega_\gamma(x, y) \leq \frac{\vartheta_2}{|x - y|^{d - 2}}, & \text{if } d \geq 3, \\
\vartheta_1 \ln \frac{1}{|x - y|} \leq G^\Omega_\gamma(x, y) \leq \vartheta_2 \ln \frac{1}{|x - y|}, & \text{if } d = 2.
\end{cases}
\]

(2.14)

**Proof.** We only prove (2.14) when $d \geq 3$. The case that $d = 2$ can be considered similarly. Similar to (2.11), we can prove that there exists $\vartheta_2 > 0$ such that for any $x, y \in K$,

$$G^\Omega_\gamma(x, y) \leq \frac{\vartheta_2}{|x - y|^{d - 2}}.$$

We obtain by (1.2) and (2.8) that there exist $C_1, C_2, C_3, \varepsilon > 0$ such that if $0 < t \leq \varepsilon$ and $x, y \in K$ satisfying $|x - y| < \varepsilon$ then

$$p^\Omega(t, x, y) \geq C_1 t^{-d/2} \exp \left( -\frac{C_2 |x - y|^2}{t} \right) - C_3.$$

Thus, for $x, y \in K$ satisfying $|x - y| < \varepsilon$, we have

\[
G^\Omega_\gamma(x, y) \geq e^{-\gamma\varepsilon} \int_0^\varepsilon p^\Omega(t, x, y) dt \\
\geq e^{-\gamma\varepsilon} \int_0^\varepsilon \left[ C_1 t^{-d/2} \exp \left( -\frac{C_2 |x - y|^2}{t} \right) - C_3 \right] dt \\
\geq e^{-\gamma\varepsilon} \left[ \int_0^\varepsilon C_1 t^{-d/2} \exp \left( -\frac{C_2 |x - y|^2}{t} \right) dt - \int_\varepsilon^\infty C_1 t^{-d/2} dt - C_3 \varepsilon \right] \\
= e^{-\gamma\varepsilon} \left[ \frac{C_4}{|x - y|^{d - 2}} - \frac{C_5}{\varepsilon^{d - 2}} - C_3 \varepsilon \right],
\]

where $C_4$ and $C_5$ are positive constants. Therefore, there exist $0 < \delta < \varepsilon$ and $\vartheta_1 > 0$ such that if $x, y \in K$ satisfying $|x - y| < \delta$ then

$$G^\Omega_\gamma(x, y) \geq \frac{\vartheta_1}{|x - y|^{d - 2}}.$$

\[
\]

**Lemma 2.7** Any point on $\partial D$ is a regular point of $D$ and $D^c$ for the process $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$. 8
Proof. We first consider the case that \( d \geq 2 \). Let \( \beta > \beta_0 \) (see (2.11) and \( \Omega \) be defined as in (2.13)). Denote by \( ((X^\Omega_t)_{t \geq 0}, (P^\beta_x)_{x \in \Omega}) \) the Markov process associated with \( (\mathcal{E}^0_\beta, \mathcal{W}^{1,2}(\Omega)) \). To prove the lemma, it is sufficient to show that any point on \( \partial D \) is a regular point of \( D \) and \( D^c \) for the process \( ((X^\Omega_t)_{t \geq 0}, (P^\beta_x)_{x \in \Omega}) \).

Let \( A \) be a Borel set of \( \Omega \) satisfying \( \overline{A} \subset \Omega \). Denote by \( e_A \) the 0-equilibrium measure of \( A \) w.r.t \( ((X^\Omega_t)_{t \geq 0}, (P^\beta_x)_{x \in \Omega}) \). Then, there exists a finite measure \( \mu_A \) concentrating on \( \overline{A} \) such that (cf. [32, page 58 and Theorem 3.5.1]),

\[
P^\beta_x(\sigma_A < \tau_\Omega) = e_A(x) = \int_A G^\Omega_\beta(x, y)\mu_A(dy) \quad \text{for q.e.} \ x \in \Omega,
\]

where \( \sigma_A \) is the first hitting time of \( A \). Since both \( \varphi(x) := P^\beta_x(\sigma_A < \tau_\Omega) \) and \( \psi(x) := \int_A G^\Omega_\beta(x, y)\mu_A(dy) \) are 0-excessive functions of \( ((X^\Omega_t)_{t \geq 0}, (P^\beta_x)_{x \in \Omega}) \), we have

\[
P^\beta_x(\sigma_A < \tau_\Omega) = \int_A G^\Omega_\beta(x, y)\mu_A(dy), \quad \forall x \in \Omega. \tag{2.15}
\]

Let \( z \in \partial D \). By the assumption on \( D \), we know that \( z \) is a regular point of \( D \) and \( D^c \) for the Brownian motion in \( \mathbb{R}^d \). Therefore, \( z \) is a regular point of \( D \) and \( D^c \) for \( ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d}) \) by Lemma 2.13 (2.15) and 2.7 Theorem 4.2.

We now consider the case that \( d = 1 \). To prove the lemma, it is sufficient to show that for any \( x \in \mathbb{R}^1 \), \( x \) is a regular point of both \( (-\infty, x) \) and \( (x, \infty) \). We assume without loss of generality that \( x = 0 \). We will use an idea from [26] to show below that 0 is a regular point of \( (0, \infty) \). Using the same method, we can show that 0 is also a regular point of \( (-\infty, 0) \).

Let \( B \) be a Brownian motion on \( \mathbb{R}^1 \) and \( Y \) be a rotationally symmetric \( \alpha \)-stable process on \( \mathbb{R}^1 \) that is independent of \( B \). Then, \( B + aY \) is the symmetric Lévy process associated with \( \Delta + a\alpha \Delta^{\alpha/2} \). Denote by \( \mathbb{P} \) and \( \mathbb{Q} \) the probability measures on \( D([0, \infty), \mathbb{R}^1) \) that are solutions to the martingale problems for \( (\Delta + a\alpha \Delta^{\alpha/2}, C^\infty_c(\mathbb{R}^1)) \) and \( (L, C^\infty_c(\mathbb{R}^1)) \) with initial value 0, respectively. Since \( \|b\| \in L^\infty(D; dx) \), \( \mathbb{P} \) and \( \mathbb{Q} \) are mutually locally absolutely continuous (cf. e.g. [13, Theorem 2.4]). Define

\[
\sigma(\omega) = \inf\{t > 0 : \omega(t) > 0\}, \quad \sigma'(\omega) = \inf\{t > 0 : \omega(t) < 0\} \quad \text{for} \ \omega \in D([0, \infty), \mathbb{R}^1),
\]

and

\[
S = \{\omega \in D([0, \infty), \mathbb{R}^1) : \sigma(\omega) = 0\}, \quad S' = \{\omega \in D([0, \infty), \mathbb{R}^1) : \sigma'(\omega) = 0\}.
\]

By the Blumenthal 0-1 law, we know that \( \mathbb{P}(S) = 0 \) or 1. If \( \mathbb{P}(S) = 0 \), then we obtain by the symmetry of \( B + aY \) that \( \mathbb{P}(S') = 0 \) also. We have a contradiction. Therefore,

\[
\mathbb{P}(S) = 1,
\]

which implies that

\[
\mathbb{P}(S^c) = 0. \tag{2.16}
\]

Define

\[
T_n = \{\omega \in D([0, \infty), \mathbb{R}^1) : 0 < \sigma(\omega) \leq n\} \quad \text{for} \ n \in \mathbb{N},
\]

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\[ T = \{ \omega \in D([0, \infty), \mathbb{R}^1) : 0 < \sigma(\omega) < \infty \}, \]
\[ R_n = \{ \omega \in D([0, \infty), \mathbb{R}^1) : \sigma(\omega) > n \} \text{ for } n \in \mathbb{N}, \]
and
\[ R = \{ \omega \in D([0, \infty), \mathbb{R}^1) : \sigma(\omega) = \infty \}. \]

Then, (2.16) implies that \( \mathbb{P}(T_n) = \mathbb{P}(R_n) = 0 \) for any \( n \in \mathbb{N} \). Since \( Q \) is locally absolutely continuous w.r.t. \( \mathbb{P} \), we have \( Q(T_n) = Q(R_n) = 0 \) for any \( n \in \mathbb{N} \). Then, \( Q(T) = Q(R) = 1 \), which implies that 0 is a regular point of \((0, \infty)\).

**Lemma 2.8** Define \( \xi(x) = \mathbb{E}_x[g(X_\tau)] \) for \( x \in \mathbb{R}^d \). If \( g \) is continuous at \( z \in \partial D \), then \( \lim_{x \to z} \xi(x) = \xi(z) \).

**Proof.** Suppose that \( g \) is continuous at \( z \in \partial D \). Let \( \delta > 0 \). We define
\[ A_\delta = \{ y \in \mathbb{R}^d : |y - z| < \delta \}, \quad T = \inf \{ t > 0 : X_t \in A_\delta \}. \]

For \( t > 0 \), we have
\[ \lim_{x \to z} \mathbb{P}_x(T \leq \tau) \leq \limsup_{x \to z} \mathbb{P}_x(\tau > t) + \limsup_{x \to z} \mathbb{P}_x(T \leq t). \]

Then, we obtain by Lemma 2.2, Lemma 2.3 (3) and Lemma 2.7 that
\[ \lim_{x \to z} \mathbb{P}_x(T \leq \tau) = 0. \quad (2.17) \]

By the strong Markov property of \( X \), we get
\[ \xi(x) = \mathbb{E}_x[g(X_\tau)1_{\{\tau < T\}}] + \mathbb{E}_x[\xi(X_T)1_{\{\tau \geq T\}}]. \]

Therefore, the proof is complete by the continuity of \( g \) at \( z \), the boundedness of \( g \) and (2.17).

**Lemma 2.9** For any \( t > 0 \) and \( z \in \partial D \), we have
\[ \lim_{x \to z} \left( \sup_{y \in D} p^D(t, x, y) \right) = 0. \quad (2.18) \]

**Proof.** By (1.2), for \( \varepsilon < t \), we have
\[ p^D(t, x, y) = \int_D p^D(\varepsilon, x, w)p^D(t - \varepsilon, w, y)dw \leq C(t - \varepsilon)^{-d/2}p_x(\tau > \varepsilon). \]

Therefore, we obtain (2.18) by Lemma 2.3 (3) and Lemma 2.7.
Lemma 2.10 Let $U$ be a bounded open set of $\mathbb{R}^d$ and $\varphi \in B_{b}(\mathbb{R}^d)$ with $\text{supp}[\varphi] \subset \overline{U}$. Then, for $dx$-a.e. $x \in U$, we have
\[
E_{x}[\varphi(X_{\tau_U})1_{\{\tau_U \leq t\}}] = a^\alpha A(d,-\alpha) \int_{0}^{t} \left( \int_{U} \int_{U} \frac{p^{U}(s,x,z)\varphi(y)}{|z-y|^{d+\alpha}} \, dy \, dz \right) ds.
\] (2.19)

Proof. Let $\varphi \in B_{b}(\mathbb{R}^d)$ with $\text{supp}[\varphi] \subset \overline{U}$ and $\psi \in B_{b}(\mathbb{R}^d)$ with $\text{supp}[\psi] \subset U$. By the quasi-left continuity of $((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d})$, we have
\[
E_{\psi \cdot dx}[\varphi(X_{\tau_U})1_{\{\tau_U \leq t\}}] = E_{\psi \cdot dx}[1_{\{X_{\tau_U} \in U\}} \varphi(X_{\tau_U})1_{\{\tau_U \leq t\}}].
\]

By Lemma 2.1, we know that $(\mathcal{E}^{0}, W^{1,2}_{0}(U))$ is a regular lower-bounded semi-Dirichlet form on $L^{2}(U;dx)$ and $X^{U}$ is the Hunt process associated with $(\mathcal{E}^{0}, W^{1,2}_{0}(U))$ (cf. [32, Theorem 3.5.7]). Let $(T^{U}_{t})_{t \geq 0}$ be the $L^{2}$-semigroup associated with $(\mathcal{E}^{0}, W^{1,2}_{0}(U))$. Denote by $(\dot{T}^{U}_{t})_{t \geq 0}$ the dual semigroup of $(T^{U}_{t})_{t \geq 0}$ on $L^{2}(U;dx)$. Similar to [17, Lemma 4.5.5], we can show that for any $\varrho \in B_{b}(\mathbb{R}^d)$ with $\text{supp}[\varrho] \subset U$,
\[
E_{\psi \cdot dx}[\varrho(X_{\tau_U^{-}})\varphi(X_{\tau_U})1_{\{\tau_U \leq t\}}] = a^\alpha A(d,-\alpha) \int_{0}^{t} \left( \int_{U} \int_{U} \frac{\dot{T}^{U}_{s} \psi(x)\varrho(x)\varphi(y)}{|x-y|^{d+\alpha}} \, dx \, dy \right) ds.
\]

Then,
\[
\begin{align*}
E_{\psi \cdot dx}[\varphi(X_{\tau_U})1_{\{\tau_U \leq t\}}] &= a^\alpha A(d,-\alpha) \int_{0}^{t} \left( \int_{U} \int_{U} \frac{\dot{T}^{U}_{s} \psi(x)\varphi(y)}{|x-y|^{d+\alpha}} \, dx \, dy \right) ds \\
&= a^\alpha A(d,-\alpha) \int_{\mathbb{R}^d} \psi(x) \int_{0}^{t} \left( \int_{U} \int_{U} \frac{p^{U}(s,x,z)\varphi(y)}{|z-y|^{d+\alpha}} \, dz \, dy \right) ds dx.
\end{align*}
\]

Since $\psi$ is arbitrary, (2.19) holds for $dx$-a.e. $x \in U$. \hfill \Box

3 Proof of Theorem 1.1

3.1 Boundedness and continuity of solutions

Let $u$ be defined by (1.3). In this subsection, we will show that $u \in B_{b}(\mathbb{R}^d)$, $u$ is continuous in $D$, and if $g$ is continuous at $z \in \partial D$ then $\lim_{x \to z} u(x) = u(z)$.

(1) By Khasminskii's inequality and (2.12), there exists $C > 0$ such that for any $v \in L^{p_{1}}_{+}(D)$ satisfying $\|v\|_{L^{p_{1}}_{+}} \leq C$, we have
\[
\sup_{x \in D} E_{x} \left[ e^{\int_{0}^{t} v(x_{s}) \, ds} \right] < \infty.
\] (3.1)

In particular, this implies that there exists $\delta > 0$ such that
\[
\sup_{x \in D} E_{x} \left[ e^{\delta r} \right] < \infty.
\] (3.2)
By (2.12), we get
\[ E_x \left[ \int_0^\tau e^{\int_0^s f(X_t)dt} f(X_s) ds \right] \leq \left( E_x \left[ \int_0^\tau e^{2\int_0^s f(X_t)dt} ds \right] \right)^{1/2} \left( E_x \left[ \int_0^\tau f^2(X_s) ds \right] \right)^{1/2} \leq C \left( E_x \left[ e^{2\int_0^\tau f(X_s)ds} \right] \right)^{1/2} \| f \|_{L_{p+1}}^{1/2} \]
\[ \leq C \left( E_x \left[ e^{4\int_0^\tau f(X_s)ds} \right] \right)^{1/4} \left( E_x \left[ \tau^2 \right] \right)^{1/4} \| f \|_{L_{p+1}}^{1/2}. \] (3.3)

By (3.1)–(3.3), we know that there exists \( M > 0 \) such that if \( \| c^+ \|_{L_{p+1}} \leq M \), then for any \( f \in L^{2(p+1)}(D; dx) \) and \( g \in B_0(D^c), u \in B_0(\mathbb{R}^d) \).

(2) For \( x \in D \) and \( t > 0 \), we have
\[ u(x) = E_x \left[ (e(t)g(X_{\tau})) 1_\{\tau \leq t\} + \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \]
\[ + E_x \left[ (e(t)g(X_{\tau})) 1_\{\tau > t\} + 1_\{\tau > t\} \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \]
\[ = E_x \left[ (e(t)g(X_{\tau})) 1_\{\tau \leq t\} + \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \]
\[ + E_x \left[ (e(t)g(X_{\tau})) 1_\{\tau > t\} E_{X_t} \left[ e(\tau)g(X_{\tau}) + \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \right] \]
\[ = E_x \left[ e(t)u(X_t) 1_\{\tau > t\} + e(\tau)g(X_{\tau}) 1_\{\tau \leq t\} + \int_0^{t \wedge \tau} e(s) f(X_s) ds \right]. \] (3.4)

Define
\[ u_t(x) = E_x \left[ u(X_t) \right], \]
and
\[ \varepsilon_t(x) = E_x \left[ -u(X_t) 1_\{\tau \leq t\} + (e(t) - 1) u(X_t) 1_\{\tau > t\} + e(\tau)g(X_{\tau}) 1_\{\tau \leq t\} + \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \]
\[ := \sum_{i=1}^4 \varepsilon_t^{(i)}. \]

Then, we have \( u = u_t + \varepsilon_t \). By (1.2) and the joint continuity of \( p(t, x, y) \) on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\), we obtain that \( u_t \) is continuous in \( D \). By Lemma 2.2 we find that
\[ \lim_{t \to 0} P_x(\tau \leq t) = 0 \quad \text{uniformly on any compact subset of } D. \] (3.5)

Then, we obtain by the boundedness of \( u \) and (3.5) that \( \varepsilon_t^{(1)} \) converges to 0 uniformly on any compact subset of \( D \).
Let \( \varphi = |c| + |f| \). By Lemma 2.4 and the assumptions on \( c \) and \( f \), we have

\[
\lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t \varphi(X^D_s) ds \right] = 0, \tag{3.6}
\]

and

\[
\sup_{x \in D} E_x \left[ e^{\int_0^t \varphi(X^D_s) ds} \right] < \infty, \quad \forall t, r > 0. \tag{3.7}
\]

Note that for \( t < \tau \), we have

\[
e(t) = 1 - \left( e^{\int_s^t c(X^D_w)dw} \right) \bigg|_{s=0}^t = 1 - \int_0^t d \left( e^{\int_s^t c(X^D_w)dw} \right) = 1 + \int_0^t e^{\int_s^t c(X^D_w)dw} c(X^D_s) ds. \tag{3.8}
\]

By (3.6), (3.7) and (3.8), we get

\[
\lim_{t \to 0} \sup_{x \in D} E_x \left[ 1_{\{\tau > t\}} |e(t) - 1| \right] \\
\leq \lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t e^{\int_s^t |c(X^D_w)| dw} |c(X^D_s)| ds \right] \\
= \lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t |c(X^D_s)| E_{X^D_s} \left[ e^{\int_0^{t-s} |c(X^D_w)| dw} \right] ds \right] \\
= 0.
\]

Then, \( \varepsilon_t^{(2)} \) converges to 0 uniformly on \( D \).

By (3.5), (3.7) and the boundedness of \( g \), we obtain that \( \varepsilon_t^{(3)} \) converges to 0 uniformly on any compact subset of \( D \). Similar to (3.8), we can show that for \( t < \tau \),

\[
e^{\int_0^t \varphi(X^D_s) ds} = 1 + \int_0^t e^{\int_s^t \varphi(X^D_w) dw} \varphi(X^D_s) ds. \tag{3.9}
\]

By (3.6), (3.7) and (3.9), we get

\[
\lim_{t \to 0} \sup_{x \in D} \left| E_x \left[ \int_0^{t \wedge \tau} e(s) f(X_s) ds \right] \right| \\
\leq \lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t e^{\int_0^s \varphi(X^D_w) dw} \varphi(X^D_s) ds \right] \\
= \lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t e^{\int_s^t \varphi(X^D_w) dw} \varphi(X^D_s) ds \right] \\
= \lim_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^t \varphi(X^D_s) E_{X^D_s} \left[ e^{\int_0^{t-s} \varphi(X^D_w) dw} \right] ds \right] \\
= 0.
\]
Then, \( \varepsilon_t^{(4)} \) converges to 0 uniformly on \( D \). Therefore, \( u \) is continuous in \( D \).

(3) Define
\[
\mathcal{M}_t = u(X_t)1_{\{\tau > t\}} + g(X_{\tau})1_{\{\tau \leq t\}} + \int_0^{\tau \wedge t} (f + cu)(X_s)ds,
\]
and
\[
\mathcal{N}_t = e(t)u(X_t)1_{\{\tau > t\}} + e(\tau)g(X_{\tau})1_{\{\tau \leq t\}} + \int_0^{\tau \wedge t} e(s)f(X_s)ds.
\]

Let \( 0 \leq s < t \). By (3.4), we get
\[
u(X_s) = E_{X_s}\left[ e(t-s)u(X_{t-s})1_{\{\tau \leq t-s\}} + e(\tau)g(X_{\tau})1_{\{\tau \leq t-s\}} + \int_{s}^{(t-s)\wedge \tau} e(w)f(X_w)dw \right],
\]
which together with the strong Markov property of \( X \) implies that
\[
E_x[\mathcal{N}_t - \mathcal{N}_s | \mathcal{F}_s] = 0.
\]
Then, \( (\mathcal{N}_t)_{t \geq 0} \) is a martingale under \( P_x \) for any \( x \in D \).

By (3.10) and (3.11), we get
\[
\mathcal{N}_t = e(t)u(X_t)1_{\{\tau > t\}} + e(t)g(X_{\tau})1_{\{\tau \leq t\}} - \int_0^t e(s)g(X_s)g(X_{\tau})1_{\{\tau \leq s\}}ds
\]
\[
+ \int_0^{\tau \wedge t} e(s)f(X_s)ds
\]
\[
= e(t)u(X_t)1_{\{\tau > t\}} + e(t)g(X_{\tau})1_{\{\tau \leq t\}} + e(t)\int_0^{\tau \wedge t} (f + cu)(X_s)ds
\]
\[
- \int_0^t e(s)g(X_s)\left( u(X_s)1_{\{\tau > s\}} + g(X_{\tau})1_{\{\tau \leq s\}} + \int_0^s f(X_w)1_{\{\tau \geq w\}}dw \right)ds
\]
\[
- \int_0^t c(X_w)u(X_w)1_{\{\tau \geq w\}}\left( \int_w^t e(s)c(X_s)ds \right)dw
\]
\[
= e(t)\mathcal{M}_t - \int_0^t \mathcal{M}_sde(s).
\]

By the integration by parts formula for semi-martingales, we have
\[
e(t)\mathcal{M}_t - u(x) = \int_0^t \mathcal{M}_sde(s) + \int_0^t e(s)d\mathcal{M}_s.
\]
Hence we obtain by (3.12) that \( (\mathcal{M}_t)_{t \geq 0} \) is a martingale under \( P_x \) for any \( x \in D \). Therefore, we have
\[
u(x) = E_x\left[ u(X_t)1_{\{\tau > t\}} + g(X_{\tau})1_{\{\tau \leq t\}} + \int_0^{\tau \wedge t} (f + cu)(X_s)ds \right], \quad x \in D.
\]

Define
\[
\xi(x) = E_x[\xi(X_{\tau})], \quad x \in \mathbb{R}^d,
\]
(3.14)
and
\[ w(x) = u(x) - \xi(x), \quad x \in \mathbb{R}^d. \] (3.15)

By (3.13), we get
\[ w(x) = E_x \left[ w(X_t)1_{\{r > t\}} + \int_0^{t \wedge r} (f + cu)(X_s)ds \right], \quad x \in D. \] (3.16)

By the assumptions on \( f \) and \( c \), the boundedness of \( u \) and Lemma 2.4, we have
\[ \limsup_{t \to 0} \sup_{x \in D} E_x \left[ \int_0^{t \wedge r} |f + cu|(X_s)ds \right] = 0. \] (3.17)

Therefore, we obtain by Lemma 2.8, Lemma 2.9 and (3.14)–(3.17) that if \( g \) is continuous at \( z \in \partial D \), then \( \lim_{x \to z} u(x) = u(z) \).

### 3.2 Existence of solutions

Let \( u \) be defined by (1.3), and \( \xi \) and \( w \) be defined by (3.14) and (3.15), respectively.

We will first show that \( \xi \in W^{1,2}_{\text{loc}}(D) \) and \( \mathcal{E}^0(\xi, \phi) = 0 \) for any \( \phi \in C_c^\infty(D) \). We assume without loss of generality that \( g \geq 0 \) on \( D^c \). Let \( \{D_n\}_{n \in \mathbb{N}} \) be a sequence of relatively compact open subsets of \( D \) such that \( D_n \subset D_{n+1} \) and \( D = \bigcup_{n=1}^\infty D_n \), and \( \{\chi_n\}_{n \in \mathbb{N}} \) be a sequence of functions in \( C_c^\infty(D) \) such that \( 0 \leq \chi_n \leq 1 \) and \( \chi_n|_{D_n} = 1 \). Suppose that \( \beta > 0 \) (see (2.1)). Let \( e_{D_n}^\beta \) be the \( \beta \)-equilibrium of \( D_n \) w.r.t. \( X^D \). By (3.18), \( e_{D_n}^\beta \in W^{1,2}_0(D) \) and \( e_{D_n}^\beta = 1 \) \( dx \)-a.e. on \( D_n \). Note that
\[ \xi(x) = E_x[\xi(X_t)], \quad x \in \mathbb{R}^d. \] (3.18)

We find that \( \xi|_{D_n} \) is a \( \beta \)-excessive function w.r.t. \( X^D \). Then, we get \( (\|\xi\|_{\infty} e_{D_n}^\beta) \wedge \xi \in W^{1,2}_0(D) \) (cf. (3.18)). Since \( (\|\xi\|_{\infty} e_{D_n}^\beta) \wedge \xi = \xi \) \( dx \)-a.e. on \( D_n \) and \( n \in \mathbb{N} \) is arbitrary, we have \( \xi \in W^{1,2}_{\text{loc}}(D) \).

Suppose \( \phi \in C_c^\infty(D_m) \) for some \( m \in \mathbb{N} \). By (3.18), we know that \( \langle \xi(X_{t \wedge \tau}) \rangle \geq t \) is a martingale under \( P_x \) for \( x \in D \). By the integration by parts formula for semi-martingales, we get
\[ E_x[e^{-\beta(t \wedge \tau_{D_m})} \xi(X_{t \wedge \tau_{D_m}})] = \xi(x) - \beta E_x \left[ \int_0^{t \wedge \tau_{D_m}} e^{-\beta s} \xi(X_s)ds \right]. \]

Then, we have
\[ \lim_{t \to 0} \int_{\mathbb{R}^d} \phi(x) \frac{\xi(x) - E_x[e^{-\beta(t \wedge \tau_{D_m})} \xi(X_{t \wedge \tau_{D_m}})]}{t} dx = \beta \int_{D_m} \xi \phi dx. \] (3.19)

For \( n > m \), define
\[ \eta_n(x) = E_x[e^{-\beta_{D_m}} \xi \chi_n(X_{\tau_{D_m}})], \quad x \in \mathbb{R}^d. \]
We have \( \eta_h(x) = E_x[e^{-\beta(t + \tau_{mD})} \eta_n(X_{t + \tau_{mD}})] \) for \( t \geq 0 \) and \( x \in D_m \), and \( \eta_h(x) = \xi \chi_n(x) \) for q.e.-\( x \in D_m^c \). By [22, Theorem 3.5.1], we get

\[
\mathcal{E}^0(\xi \chi_n, \phi) = \mathcal{E}^0(\xi \chi_n - \eta_n, \phi) = \lim_{t \to 0} \int_{D_m} \phi(x) \xi \chi_n - \eta_n - E_x[e^{-\beta(t + \tau_{mD})}(\xi \chi_n(X_{t + \tau_{mD}}) - \eta_n(X_{t + \tau_{mD}}))] \, dx
\]

\[
= \lim_{t \to 0} \int_{D_m} \phi(x) \xi \chi_n - \eta_n - E_x[e^{-\beta(t + \tau_{mD})}(\xi \chi_n(X_{t + \tau_{mD}}) - \eta_n(X_{t + \tau_{mD}}))] \, dx.
\]

By (3.19) and (3.20), we get

\[
\mathcal{E}^0(\xi \chi_n, \phi) = \lim_{t \to 0} \frac{1}{t} \int_{D_m} \phi(x) E_x[e^{-\beta(t + \tau_{mD})} \xi(X_{t + \tau_{mD}})(1 - \chi_n(X_{t + \tau_{mD}}))] \, dx
\]

\[
= \lim_{t \to 0} \frac{1}{t} \int_{D_m} \phi(x) E_x[1_{\{\tau_{mD} \leq t\}} e^{-\beta \tau_{mD}} \xi(X_{t + \tau_{mD}})(1 - \chi_n(X_{t + \tau_{mD}}))] \, dx. \tag{3.21}
\]

Let \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that for any \( 0 < t < \delta, 1 - e^{-\beta t} < \varepsilon \). Suppose that \( \bar{D} \subset B(0, N) \) for some \( N \in \mathbb{N} \). Then, we obtain by Lemma 2.10 that for \( 0 < t < \delta \),

\[
\left| \frac{1}{t} \int_{D_m} \phi(x) E_x[1_{\{\tau_{mD} \leq t\}} (1 - e^{-\beta \tau_{mD}}) \xi(X_{t + \tau_{mD}})(1 - \chi_n(X_{t + \tau_{mD}}))] \, dx \right| \\
\leq \frac{\varepsilon}{t} \int_{D_m} |\phi(x)| E_x[1_{\{\tau_{mD} \leq t\}} \xi(X_{t + \tau_{mD}})(1 - \chi_n(X_{t + \tau_{mD}}))] \, dx
\]

\[
= \frac{\varepsilon a^\alpha A(d, -\alpha)}{t} \int_{D_m} |\phi(x)| \left[ \int_0^t \int_{D_m} (\xi(y)(1 - \chi_n(y)) \int_{D_m} \frac{p_{mD}(s, x, z)}{|z - y|^{d+\alpha}} \, dz \right] \, dy \, ds \, dx
\]

\[
\leq \frac{\varepsilon a^\alpha A(d, -\alpha)}{t} \left( \int_{D_m} |\phi(x)| \left[ \int_0^t \int_{(B(0, 2N))^c} (\xi(y)(1 - \chi_n(y)) \int_{D_m} \frac{p_{mD}(s, x, z)}{|z - y|^{d+\alpha}} \, dz \right] \, dy \, ds \, dx
\]

\[
+ \int_{D_m} |\phi(x)| \left[ \int_0^t \int_{(B(0, 2N))^c \cap D_m^c} (\xi(y)(1 - \chi_n(y)) \int_{D_m} \frac{p_{mD}(s, x, z)}{|z - y|^{d+\alpha}} \, dz \right] \, dy \, ds \, dx
\]

\[
\leq \frac{\varepsilon a^\alpha A(d, -\alpha)}{t} \left( \int_{D_m} |\phi(x)| \left[ \int_0^t \int_{(B(0, 2N))^c} \frac{1}{|y|^{d+\alpha}} \, dy + \vartheta^{-(d+\alpha)}|B(0, 2N) \cap D_m^c| \right] \right), \tag{3.22}
\]

where \( \vartheta = \inf\{|x - y| : x \in D_m, y \in D_m^c\} \), and \( |D_m| \) and \( |B(0, 2N) \cap D_m^c| \) denote the Lebesgue measures of \( D_m \) and \( B(0, 2N) \cap D_m^c \), respectively. Since \( \varepsilon > 0 \) is arbitrary, we obtain by (3.21) and (3.22) that

\[
\mathcal{E}^0(\xi \chi_n, \phi) = \lim_{t \to 0} \frac{1}{t} \int_{D_m} \phi(x) E_x[1_{\{\tau_{mD} \leq t\}} \xi(X_{t + \tau_{mD}})(1 - \chi_n(X_{t + \tau_{mD}}))] \, dx. \tag{3.23}
\]

Define

\[
F_n(z) = \int_{D_m^c} \frac{\xi(y)(1 - \chi_n(y))}{|z - y|^{d+\alpha}} \, dy, \quad z \in D_m. \tag{3.24}
\]
Thus, we obtain by (3.25), Lemma 2.10 and (3.24), we get
\[
\mathcal{E}^0(\xi \chi_n, \phi) = \lim_{t \to 0} \frac{a^\alpha A(d, -\alpha)}{t} \int_{D_m} \phi(x) \left[ \int_0^t \int_{D_m} \left( \xi(y)(1 - \chi_n(y)) \int_{D_m} \frac{p^{D_m}(s, x, z)}{|z - y|^{d+\alpha}} dz \right) dy ds \right] dx
\]
\[
= \lim_{t \to 0} \frac{a^\alpha A(d, -\alpha)}{t} \int_{D_m} \phi(x) \left[ \int_0^t \int_{D_m^c} \left( \xi(y)(1 - \chi_n(y)) \int_{D_m} \frac{p^{D_m}(s, x, z)}{|z - y|^{d+\alpha}} dz \right) dy ds \right] dx
\]
\[
= \lim_{t \to 0} \frac{a^\alpha A(d, -\alpha)}{t} \int_0^t \int_{D_m} \phi(x)p_s^{D_m} F_n(x) dx ds
\]
\[
= a^\alpha A(d, -\alpha) \int_{D_m} \phi(x) F_n(x) dx
\]
\[
= a^\alpha A(d, -\alpha) \int_{D_m} \int_{D_m^c} \frac{\xi(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy \phi(x) dx. \quad (3.25)
\]

On the other hand, we have
\[
\mathcal{E}^0(\xi \chi_n, \phi) = \int_{\mathbb{R}^d} \langle \nabla(\xi \chi_n), \nabla \phi \rangle dx - \int_{\mathbb{R}^d} \langle b, \nabla(\xi \chi_n) \rangle \phi dx
\]
\[
+ \frac{a^\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{((\xi \chi_n)(x) - (\xi \chi_n)(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} dx dy
\]
\[
= \mathcal{E}^0(\xi, \phi) + a^\alpha A(d, -\alpha) \int_{D_m} \int_{D_m^c} \frac{\xi(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy \phi(x) dx. \quad (3.26)
\]

Thus, we obtain by (3.25) and (3.26) that \( \mathcal{E}^0(\xi, \phi) = 0 \).

By (3.16), we have
\[
\lim_{t \to 0} \int_D w(x) \frac{w(x) - p_t^D w(x)}{t} dx
\]
\[
\leq \lim_{t \to 0} \int_D |w|(x) E_x[t^{\lambda_\tau}] |f + cu|(X_s) ds dx
\]
\[
= \lim_{t \to 0} \frac{1}{t} \int_0^t (p_s^D |f + cu|, |w|) ds
\]
\[
= (|f + cu|, |w|)
\]
\[
< \infty.
\]

Then, \( w \in W^{1,2}_0(D) \) and hence \( u = \xi + w \in W^{1,2}_{loc}(D) \). For \( \phi \in C_c^\infty(D) \), we have
\[
\mathcal{E}^0(w, \phi) = \lim_{t \to 0} \int_D \phi(x) \frac{w(x) - p_t^D w(x)}{t} dx
\]
\[
= \lim_{t \to 0} \int_D \phi(x) E_x[t^{\lambda_\tau}] (f + cu)(X_s) ds dx
\]
\[
= \lim_{t \to 0} \frac{1}{t} \int_0^t (p_s^D (f + cu), \phi) ds
\]
\[
= (f + cu, \phi).
\]
Therefore,
\[ \mathcal{E}^0(u, \phi) = \mathcal{E}^0(\xi + w, \phi) = (f + cu, \phi), \]
which implies that (1.4) holds.

3.3 Uniqueness of solutions

In this subsection, we will prove the uniqueness of solutions. To this end, we will show that there exists \( M > 0 \) such that if \( \|c^+\|_{L^{p+1}} \leq M \), then \( v \equiv 0 \) is the unique function in \( B_b(\mathbb{R}^d) \) satisfying \( v|_D \in W^{1,2}_{\text{loc}}(D) \cap C(D) \) and
\[ \begin{cases} \mathcal{E}^0(v, \phi) = (cv, \phi), & \forall \phi \in C_c^\infty(D), \\ v = 0 & \text{on } D^c. \end{cases} \] (3.27)

Suppose that \( v \in B_b(\mathbb{R}^d) \) satisfying \( v|_D \in W^{1,2}_{\text{loc}}(D) \cap C(D) \) and (3.27). Let \( \{D_n\}_{n \in \mathbb{N}} \) be a sequence of relatively compact open subsets of \( D \) such that \( \overline{D}_n \subset D_{n+1} \) and \( D = \cup_{n=1}^\infty D_n \), and \( \{\chi_n\}_{n \in \mathbb{N}} \) be a sequence of functions in \( C_c^\infty(D) \) such that \( 0 \leq \chi_n \leq 1 \) and \( \chi_n|_{D_n} = 1 \). We have \( v\chi_n \in W^{1,2}_0(D) \). Note that
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(v(y) - v(x))(\chi_n(y) - \chi_n(x))|}{|x - y|^{d+\alpha}} \, dy \, dx < \infty. \] (3.28)

Let \( \beta > \beta_0 \) (see (2.1)) and \( \phi \in C_c^\infty(D) \). By (3.27) and (3.28), we get
\[ \mathcal{E}^0_{\beta}(v\chi_n, \phi) = \int_{\mathbb{R}^d} \langle \nabla (v\chi_n), \nabla \phi \rangle \, dx - \int_{\mathbb{R}^d} \langle b, \nabla (v\chi_n) \rangle \, dx \]
\[ + \frac{a\alpha A(d, -\alpha)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v\chi_n(x) - (v\chi_n)(y))(\phi(x) - \phi(y))}{|x - y|^{d+\alpha}} \, dx \, dy + (\beta, v\chi_n \phi) \]
\[ = \mathcal{E}^0(v, \chi_n \phi) - \int_{\mathbb{R}^d} (L\chi_n)v \phi \, dx - 2 \int_{\mathbb{R}^d} \langle \nabla v, \nabla \chi_n \rangle \phi \, dx \]
\[ - a\alpha A(d, -\alpha) \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{(v(y) - v(x))(\chi_n(y) - \chi_n(x))}{|x - y|^{d+\alpha}} \, dy \right] \phi(x) \, dx + (\beta, v\chi_n \phi) \]
\[ = (c + \beta)v \chi_n - (L\chi_n)v - 2\langle \nabla v, \nabla \chi_n \rangle - a\alpha A(d, -\alpha) \int_{\mathbb{R}^d} \frac{(v(y) - v(x))(\chi_n(y) - \chi_n(x))}{|y|^{d+\alpha}} \, dy, \phi \]
\[ := (\theta_n, \phi). \] (3.29)

Let \( n > m \) and \( \phi \in C_c^\infty(D_m) \). By (3.29), we get
\[ (\theta_n, \phi) = \mathcal{E}^0(v, \phi) + a\alpha A(d, -\alpha) \int_{D_m} \int_{D_n} \frac{v(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} \, dy \phi(x) \, dx + (\beta, v\phi) \]
\[ = \left( c + \beta \right)v + a\alpha A(d, -\alpha) \int_{D \cap D_n} \frac{v(y)(1 - \chi_n(y))}{|y|^{d+\alpha}} \, dy, \phi \right). \] (3.30)
Since $\phi \in C_+^\infty(D_m)$ is arbitrary, by (3.30), we find that for $n > m$,
\[
\theta_n(x) = (c(x) + \beta)v(x) + a^a A(d, -\alpha) \int_{D \cap D_n} \frac{v(y)(1 - \chi_n(y))}{|x - y|^{d+\alpha}} dy, \quad x \in D_m.
\]

Hence
\[
\theta_n \text{ converges to } (c + \beta)v \text{ uniformly on any compact subset of } D.
\] (3.31)

Denote by $((X_t)_{t \geq 0}, (P^\beta_x)_{x \in \mathbb{R}^d})$ the Markov process associated with $(\mathcal{E}^0_\beta, W^{1,2}(\mathbb{R}^d))$. For $n > m$, define
\[
A^m_n := \int_0^{t \wedge D_m} \theta_n(X_s) ds \quad \text{and} \quad c^{m,n}_t(x) := E^\beta_x[A^m_n] \quad t \geq 0, \quad x \in D_m.
\]

By the joint continuity of $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, we know that the function $t \mapsto c^{m,n}_t(x)$ is continuous for any $x \in D_m$. We have $c^{m,n}_t \in L^2(D_m; dx)$ for $t \geq 0$ and
\[
c^{m,n}_{t+s}(x) = c^{m,n}_t(x) + p^\beta_{t,s}(x) c^{m,n}_s(x), \quad t, s \geq 0,
\] (3.32)

where $(p^\beta_{t,s}(x))_{t \geq 0}$ is the transition semigroup of the part process $((X^D_{t|m})_{t \geq 0}, (P^\beta_x)_{x \in D_m})$. By (3.29), we get
\[
\lim_{t \to 0} \frac{1}{t} E^{\beta}_{\phi,dx}[A^m_n] = \lim_{t \to 0} \frac{1}{t} \int_0^t (p^\beta_{s,t}(\theta_n \phi)) ds = (\theta_n, \phi) = \mathcal{E}^0_\beta(v \chi_n, \phi), \quad \forall \phi \in W^{1,2}_0(D_m).
\] (3.33)

Define
\[
\eta_{m,n}(x) = E^\beta_x[(v \chi_n)(X_{\tau_{D_m}})], \quad x \in \mathbb{R}^d.
\]

We have
\[
\eta_{m,n}(x) = E^\beta_x[\eta_{m,n}(X_{\tau_{D_m}})], \quad t \geq 0, \quad x \in D_m,
\] (3.34)

and $\eta_{m,n}(x) = v \chi_n(x)$ for q.e.-$x \in D_m$. By [32, Theorem 3.5.1], we get
\[
\mathcal{E}^0_\beta(v \chi_n, \phi) = \mathcal{E}^0_\beta(v \chi_n - \eta_{m,n}, \phi), \quad \forall \phi \in W^{1,2}_0(D_m).
\] (3.35)

Let $(T^\beta_{t,D_m})_{t \geq 0}$ be the $L^2$-semigroup associated with $(\mathcal{E}^0_\beta, W^{1,2}(D_m))$. Denote by $(\hat{T}^\beta_{t,D_m})_{t \geq 0}$ the dual semigroup of $(T^\beta_{t,D_m})_{t \geq 0}$ on $L^2(D_m; dx)$. Define
\[
\hat{S}^m_t := \int_0^t \hat{T}^\beta_{s,D_m} ds, \quad t \geq 0.
\] (3.36)

Similar to [17] (1.5.5), page 39, we can show that
\[
\mathcal{E}^0_\beta(v, \hat{S}^m_t \rho) = (v, \rho - \hat{T}^\beta_{t,D_m} \rho), \quad \forall v \in W^{1,2}_0(D_m), \quad \rho \in L^2(D_m; dx).
\] (3.37)
Then, we obtain by (3.32), (3.33), (3.35), (3.36) and (3.37) that for \( \phi \in C_c^\infty(D_m) \) and \( t, r > 0 \),

\[
(c_{t+r}^{m,n}, \phi - \hat{T}_r^{\beta,D_m} \phi) = \\
\lim_{s \to 0} \frac{1}{s} (c_{t}^{m,n}, \hat{S}_r^m \phi - \hat{T}_s^{\beta,D_m} \hat{S}_r^m \phi) \\
= \lim_{s \to 0} \frac{1}{s} (c_{s}^{m,n}, \hat{S}_r^m \phi - \hat{T}_t^{\beta,D_m} \hat{S}_r^m \phi) \\
= \mathcal{E}_t^\beta(v \chi_n, \hat{S}_r^m \phi - \hat{T}_t^{\beta,D_m} \hat{X}_r \phi) \\
= \mathcal{E}_t^\beta(v \chi_n - \eta_{m,n}, \hat{S}_r^m \phi - \hat{T}_t^{\beta,D_m} \hat{S}_r^m \phi) \\
= (v \chi_n - \eta_{m,n}, \phi - \hat{T}_t^{\beta,D_m} \phi + \hat{T}_t^{\beta,D_m} \phi) \\
= (v \chi_n - \eta_{m,n}, \phi - \hat{T}_t^{\beta,D_m} \phi).
\]

Hence \( l_{t}^{m,n} := (c_{t}^{m,n} - (v \chi_n - \eta_{m,n}) + p_t^{\beta,D_m}(v \chi_n - \eta_{m,n}), \phi) \) satisfies the linear equation \( l_{t}^{m,n} = l_{t+r}^{m,n} - l_{r}^{m,n} \). By (3.33) and (3.35), we obtain \( \lim_{t \to 0} l_{t}^{m,n}/t = 0 \). Then, \( l_{t}^{m,n} = 0 \). Since \( \phi \in C_c^\infty(D_m) \) is arbitrary, we obtain by the continuity of the function \( t \mapsto p_t^{\beta,D_m}(t,x,y) \), which can be proved similar to Lemma 2.3 (4), and the continuity of the function \( t \mapsto c_{t}^{m,n}(x) \) that for \( dx \)-a.e. \( x \in D_m \),

\[
(v \chi_n - \eta_{m,n})(x) = E_x^\beta([v \chi_n - \eta_{m,n}(X_{t\wedge \tau_{D_m}})] + E_x^\beta \left[ \int_0^{t\wedge \tau_{D_m}} \theta_n(X_s) ds \right], \quad \forall t \geq 0.
\]

By (3.34), we obtain that for \( dx \)-a.e. \( x \in D_m \),

\[
(v \chi_n)(x) = E_x^\beta([v \chi_n(X_{t\wedge \tau_{D_m}})] + E_x^\beta \left[ \int_0^{t\wedge \tau_{D_m}} \theta_n(X_s) ds \right], \quad \forall t \geq 0. \tag{3.38}
\]

Note that \( v \in B_0(\mathbb{R}^d) \) and \( v = 0 \) on \( D_c \). Letting \( n \to \infty \), we obtain by (3.31) and (3.38) that for \( dx \)-a.e. \( x \in D_m \),

\[
v(x) = E_x^\beta[v(X_{t\wedge \tau_{D_m}})] + E_x^\beta \left[ \int_0^{t\wedge \tau_{D_m}} ((c + \beta)v(X_s) ds \right], \quad \forall t \geq 0. \tag{3.39}
\]

Letting \( m \to \infty \), we obtain by (3.39) that for \( dx \)-a.e. \( x \in D \),

\[
v(x) = E_x^\beta[v(X_t)1_{\{t > \tau\}}] + E_x^\beta \left[ \int_0^{t\wedge \tau} ((c + \beta)v(X_s) ds \right], \quad \forall t \geq 0. \tag{3.40}
\]

Define

\[
\mathcal{I}_t = v(X_t)1_{\{t > \tau\}} + \int_0^{t\wedge \tau} ((c + \beta)v(X_s) ds.
\]

By (3.40), we find that \( (\mathcal{I}_t)_{t \geq 0} \) is a martingale under \( P_x^\beta \) for \( dx \)-a.e. \( x \in D \). Define

\[
e_\beta(t) := e^{-\int_0^t (c+\beta)(X_s) ds}, \quad t \geq 0.
\]

The integration by parts formula for semi-martingales implies that

\[
e_\beta(t)\mathcal{I}_t - v(x) = \int_0^t \mathcal{I}_s d\mathcal{E}_\beta(s) + \int_0^t \mathcal{E}_\beta(s) d\mathcal{I}_s.
\]
By (3.41), we get
\[ e_\beta(t)I_t - \int_0^t I_s d e_\beta(s) \]
\[ = e_\beta(t)v(X_t)1_{\{\tau>t\}} + e_\beta(t) \int_0^{t\wedge \tau} ((c + \beta)v)(X_s)ds - \int_0^t e_\beta(s)((c + \beta)v)(X_s)1_{\{\tau>s\}}ds \]
\[ - \int_0^t ((c + \beta)v)(X_w)1_{\{\tau\geq w\}} \left( \int_w^t e_\beta(s)(c + \beta)(X_s)ds \right) dw \]
\[ = e_\beta(t)v(X_t)1_{\{\tau>t\}} := J_t. \]

Hence \((J_t)_{t \geq 0}\) is a martingale under \(P^\beta_x\) for \(dx\)-a.e. \(x \in D\). Then, we have
\begin{align*}
 v(x) &= E_x^\beta[e_\beta(t)v(X_t)1_{\{\tau>t\}}] \\
 &= E_x[v(t)v(X_t)1_{\{\tau>t\}}], \quad dx - \text{a.e. } x \in D. \quad (3.42)
\end{align*}

By (3.1), there exists \(M > 0\) such that if \(\|c^+\|_{L^{p\vee 1}} \leq M\) then
\[ \sup_{x \in D} E_x \left[ e_{0}^\tau c^+(X_s)ds \right] < \infty. \quad (3.43) \]

Therefore, by letting \(t \to \infty\), we obtain by (3.42), (3.43) and the dominated convergence theorem that \(v(x) = 0\) for \(dx\)-a.e. \(x \in D\). Since \(v|_D \in C(D)\), we obtain \(v \equiv 0\) on \(\mathbb{R}^d\). The proof is complete.

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References

[1] B. Aksoylu and T. Mengesha, Results on nonlocal boundary value problems, Numer. Funct. Anal. Optim. 31 (2010) 1301-1317.

[2] A. Arapostathisa, A. Biswasb and L. Caffarelli, The Dirichlet problem for stable-like operators and related probabilistic representations, Comm. Part. Diff. Equ. 41 (2016) 1472-1511.

[3] G. Barles, E. Chasseigne and C. Imbert, On the Dirichlet problem for second-order elliptic integro-differential equations, Indiana Univ. Math. J. 57 (2008) 213-246.

[4] G. Barles, E. Chasseigne and C. Imbert, Hölder continuity of solutions of second-order elliptic integro-differential equations, J. Eur. Math. Soc. 13 (2011) 1-26.

[5] R. Bass and D. Levin, Harnack inequalities for jump processes, Potential Anal. 17 (2002) 375-388.
[6] K. Bogdan, The boundary Harnack principle for the fractional Laplacian, *Studia Math.* **123** (1997) 43-80.

[7] K. Bogdan, T. Kumagai and M. Kwaśnicki, Boundary Harnack inequality for Markov processes with jumps, *Trans. Amer. Math. Soc.* **367** (2015) 477-517.

[8] J.-M. Bony, P. Courrége and P. Priouret, Semi-groupes de Feller sur une variété à bord compacte et problème aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum, *Ann. Inst. Fourier* **18** (1968) 369-521.

[9] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, *Comm. Pure Appl. Math.* **62** (2009) 597-638.

[10] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, *Arch. Rat. Mech. Anal.* **200** (2011) 59-88.

[11] Z. Q. Chen and E. Y. Hu, Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ under gradient perturbation, *Stoch. Proc. Appl.* **125** (2015) 2603-2642.

[12] Z. Q. Chen and R. Song, Estimates on Green functions and Poisson kernels for symmetric stable processes, *Math. Ann.* **312** (1998) 465-501.

[13] P. Cheridito, D. Filipovic and M. Yor, Equivalent and absolutely continuous measure changes for jump-diffusion processes, *Ann. Appl. Probab.* **15** (2005) 1713-1732.

[14] A.-L. Dalibard and D. Gérard-Varet, On shape optimization problems involving the fractional Laplacian, *ESAIM Control Optim. Calc. Var.* **19** (2013) 976-1013.

[15] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, *SIAM Rev.* **54** (2012) 667-696.

[16] M. Felsinger, M. Kassmann and P. Voigt, The Dirichlet problem for nonlocal operators, *Math. Z.* **279** (2015) 779-809.

[17] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, second and extended edition, De Gruyter, 2011.

[18] M. Fukushima and T. Uemura, Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms, *Ann. Probab.* **40** (2012) 858-889.

[19] M. G. Garroni and J. L. Menaldi, Second Order Elliptic Integro-differential Problems, Research Notes in Math. **430**, Chapman & Hall/CRC, 2002.

[20] G. Giacomin and J. L. Lebowitz, Phase segregation dynamics in particle systems with long range interaction I. Macroscopic limits, *J. Stat. Phys.* **87** (1997) 37-61.

[21] G. Gilboa and S. Osher, Nonlocal operators with applications to image processing, *Multiscale Model. Simul.* **7** (2008) 1005-1028.

[22] G. Grubb, Local and nonlocal boundary conditions for $\mu$-transmission and fractional elliptic pseudodifferential operators, *Anal. PDE* **7** (2014) 1649-1682.
[23] G. Grubb, Fractional Laplacians on domains, a development of Hörmander’s theory of \( \mu \)-transmission pseudodifferential operators, *Adv. Math.* 268 (2015) 478-528.

[24] Q. Y. Guan and Z. M. Ma, Boundary problems for fractional Laplacians, *Stoch. Dyn.* 5 (2005) 385-424.

[25] W. Hoh and N. Jacob, On the Dirichlet problem for pseudodifferential operators generating Feller semigroups, *J. Funct. Anal.* 137 (1996) 19-48.

[26] Z. C. Hu, W. Sun and L. F. Wang, Two theorems on Hunt’s hypothesis (H) for Markov processes, arXiv:1903.00050v3, 2019.

[27] M. Kanda, Regular points and Green functions in Markov processes, *J. Math. Soc. Japan* 19 (1967) 46-69.

[28] M. Kassmann, A priori estimates for integro-differential operators with measurable kernels, *Calc. Var. Part. Diff. Equ.* 34 (2009) 1-21.

[29] P. Kim and R. Song, Tow-sided estimates on the density of Brownian motion with singular drift, *Illinois J. Math.* 50 (2006) 635-688.

[30] Z. M. Ma, L. Overbeck and M. Röckner, Markov processes associated with semi-Dirichlet forms, *Osaka J. Math.* 32 (1995) 97-119.

[31] T. Mengesha and Q. Du, The bond-based peridynamic system with Dirichlet-type volume constraint, *Royal Proc. Soc. Edingburgh, Sec. A* 144 (2014) 161-186.

[32] Y. Oshima, Semi-Dirichlet Forms and Markov Processes, De Gruyter, 2013.

[33] S. Port and C. Stone, Brownian Motion and Classical Potential Theory, Academic Press, 1978.

[34] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.* 101 (2014) 275-302.

[35] X. Ros-Oton and J. Serra, Boundary regularity for fully nonlinear integro-differential equations, *Duke Math. J.* 165 (2016) 2079-2154.

[36] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.

[37] M. Schechter, Spectra of Partial Differential Operators, North-Holland Publishing Company, 1971.

[38] T. Uemura, On multidimensional diffusion processes with jumps, *Osaka J. Math.* 51 (2014) 969-992.

[39] Z. Zhao, A probabilistic principle and generalized Schrödinger perturbation, *J. Funct. Anal.* 101 (1991) 162-176.