Enhanced visibility of two-mode thermal squeezed states via degenerate parametric amplification and resonance

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Abstract

Two-mode squeezed states, generated via non-degenerate parametric down-conversion, are invariably revealed via their entangled vacuum or correlated thermal fluctuations. Here, two-mode thermal squeezed states, generated in an electromechanical system, are made bright by means of degenerate parametric amplification of their constituent modes to the point where they are almost perfect, even when seeded from low intensity non-degenerate parametric down-conversion. More dramatically, activating the degenerate parametric resonances of the underlying modes yields perfect correlations which can be resolved via the coordinated switching of their phase bi-stable vibrations, without recourse to monitoring their thermal fluctuations. This ability to enhance the two-mode squeezed states and to decipher them without needing to observe their intrinsic noise is supported by both analytical and numerical modelling and it suggests that the technical constraints to making this phenomenon more widely available can be dramatically relaxed.

1. Introduction

Two-mode squeezed states generated via non-degenerate parametric down-conversion are routinely observed in the lab via their vacuum noise principally due to the widespread availability of efficient detectors for optical photons [1–3]. On the other hand two-mode vacuum squeezed states realised with microwave photons in superconducting circuits are more challenging to observe due to their small energy thus requiring the development efficient amplifiers [4–6]. This challenge is further magnified when extended to phonons, usually localised in low frequency mechanical resonators, with the dual requirement of operation in their vacuum state combined with the ability to detect their miniscule fluctuations [7, 8]. Indeed only recently has it become possible to create entangled phonons via the two-mode squeezer interaction and even then in a hybrid system composed of photons and phonons [9, 10]. Alternatively two-mode squeezing with a system composed purely of phonons has also been realised but was limited to thermal populations yielding correlated states [11–13].

Here a protocol is developed to increase the visibility of the two-mode thermal squeezed states from phonons which concomitantly furnishes perfect correlations and it has the potential to be extended to vacuum squeezed states. It is based on degenerately parametrically amplifying the two-mode thermal squeezed states that are engendered from the non-degenerate parametric down-conversion. Specifically this is achieved by modulating the spring constants of the modes, underpinning the two-mode squeezed states, at twice their natural frequency [14]. On the other hand large gains from the degenerate parametric amplification results in the constituent modes parametrically resonating, with bi-stable phases, and in this regime the two-mode squeezed states can instead be identified from the correlated vibration phases of the underlying modes rather than from their more usual intrinsic noise [15]. Indeed this latter regime offers the tantalising prospect of more readily resolving the correlations between modes, from the two-mode squeezing, which could be harnessed for technological applications. In many respects these observations are analogous to the readout of superconducting quantum bits [16] via a Josephson parametric amplifier [17] when operated in the amplification [18, 19] or in the...
oscillation regime [20] thus suggesting the applicability of this protocol to two-mode vacuum squeezed states with phonons in an electromechanical system [21].

2. Experimental

The prototype electromechanical system in which these concepts are explored is shown in figure 1(a) and it consists of two GaAs based mechanical beams of length, width and thickness of 80 μm, 20 μm and 800 nm respectively which are strongly intercoupled via the exaggerated overhangs between them. Piezoelectric transducers are also incorporated into the clamping points of both mechanical elements, consisting of a doped GaAs layer and gold electrodes sandwiching an insulating GaAs/AlGaAs superlattice, and they enable the strain in the electromechanical system to be dynamically modulated [15]. The motion of the mechanical beams, at room temperature and in a high vacuum, is transduced via a laser Doppler interferometer and both the device and the measurement setup are detailed elsewhere [11].

The strongly coupled beams in the electromechanical system sustain two spectrally closely spaced vibration modes labelled symmetric (S) and asymmetric (A), as shown in figure 1(a), and they can be identified by their thermal Langevin force driven random motion with natural frequencies $\omega_p/2\pi \approx 244$ kHz and $\omega_p/2\pi \approx 257$ kHz and quality factors $Q_s = 1300$ and $Q_s = 2300$ respectively, as shown in figures 1(b)–(d) at 0 $V_{\text{rms}}$. Non-degenerate parametric amplification can be activated in the electromechanical system by modulating the spring constants of both modes at their sum frequency via piezoelectrically pumped ($P$) strain $V_p(\omega_p)$ which results in the simultaneous amplification of the random displacement fluctuations of both modes as shown in figure 1(b). The simultaneous generation of phonons in both modes from this process amplifies and correlates their displacement fluctuations resulting in a two-mode thermally squeezed state namely the classical analogue of entanglement from parametric down-conversion [11–13]. On the other hand degenerate parametric amplification can also be activated in the electromechanical system by piezoelectrically modulating the spring constant of either mode at twice their natural frequency via $V_p(2\omega_p)$ where $n = S, A$ as shown in figures 1(c) and (d). This process not only amplifies the thermally driven random displacement fluctuations of either mode [14, 22] but once the phonon generation rate from this process exceeds the rate of loss, from either mode, it results in a parametric resonance [15, 23].
3. Theoretical

The Hamiltonian of this electromechanical system can then be expressed as

\[ H = \sum_{n=1}^{N} \left( \frac{p_n^2}{2} + \omega_n^2 q_n^2/2 \right) - \gamma_n q_n \cos(\omega_n t), \]  

(1)

where the summation expresses the kinetic and potential energies from both modes in terms of their position \( q_n \) and momentum \( p_n \) [11, 15, 24]. The potential energy term contains two contributions, the second arising from the periodic modulation of the mechanical spring constant with amplitude \( \Gamma_n \) at twice the natural mode frequency, namely \( V_n(2\omega_n) \) experimentally, which yields degenerate parametric amplification and parametric resonance [14, 23]. The last term in equation (1) describes non-degenerate parametric down-conversion from the pump with amplitude \( \Lambda \) at \( \omega_p = \omega_s + \omega_a \), namely \( V_n(\omega_p) \) experimentally, which results in both modes being amplified and correlated yielding a two-mode squeezed state [11, 25].

This Hamiltonian can be translated into the rotating frame following standard procedure [26] with the introduction of canonically conjugate variables \( P \) and \( Q \) defined as

\[ q_n = (\sin(\omega_n t) + Q_n \cos(\omega_n t))/\sqrt{\omega_n} \quad \text{and} \quad p_n = \sqrt{\omega_n} (P_n \cos(\omega_n t) - Q_n \sin(\omega_n t)). \]

(2)

The effective Hamiltonian in these variables is then given by

\[ \mathcal{H}(P, Q) \approx \sum_{n=1}^{N} \left( \frac{\omega_n \Gamma_n}{4} \left( P_n^2 - Q_n^2 \right) \right) + \frac{\Lambda}{4} \sqrt{\omega_n \omega_s} (Q_s P_A - P_s Q_A), \]

(3)

where all the fast off-resonance terms have been neglected. In order to solve this equation with damping \( \gamma_n \), the corresponding equations of motion are first extracted from \( \dot{Q}_n = \partial \mathcal{H}/\partial P_n \) and \( \dot{P}_n = -\partial \mathcal{H}/\partial Q_n \) which yields

\[ \dot{Q}_s + \gamma_s Q_s - \eta_s P_s + \lambda P_A = -F_B, \]
\[ \dot{P}_s + \gamma_s P_s - \eta_s Q_s + \lambda Q_A = F_Q, \]
\[ \dot{Q}_A + \gamma_A Q_A - \eta_A P_A + \lambda P_s = -F_B, \]
\[ \dot{P}_A + \gamma_A P_A - \eta_A Q_A + \lambda Q_s = F_Q, \]

(4)

with the thermal Langevin force, decomposed into in-phase and quadrature components namely \( F_Q \) and \( F_B \), driving both modes within their bandwidths i.e. \( \gamma_n \ll \omega_s - \omega_a \), where

\[ \eta_n = \frac{\omega_n \Gamma_n}{2} \quad \text{and} \quad \lambda = \frac{\Lambda}{4 \sqrt{\omega_s \omega_a}} \quad \text{and} \quad \gamma_n = \frac{\omega_n}{Q_n}. \]

(5)

3.1. Numerical solutions

The equations of motion in (4) are first numerically solved using the Runge–Kutta method for the in-phase \( Q_n \) and quadrature components \( P_n \) where four independent random number generators are used to mimic \( F_Q \) and \( F_B \) in all four equations, driving both modes at 300 K. The equations are solved 6000 times, with \( \eta_0 = 0 \), as a function \( \Lambda \) and the two-mode squeezed states from this interaction can be unveiled by reconstructing the phase portrait from the cross-quadrature \( Q_A \) : \( P_A \) (the phase portrait from \( Q_s : P_s \) is similar and not shown throughout) and is shown in the inset to figure 2(a). This reveals a squashed distribution which implies that the vibrations in the two modes have become correlated from the non-degenerate parametric down-conversion as experimentally observed previously [11–13]. This assertion can be statistically confirmed by evaluating the corresponding correlation coefficient \( \text{cov}(Q_A, P_A)/\sigma_Q \sigma_P \) where the numerator describes the covariance and \( \sigma \) is the standard deviation. The results of this analysis as a function of \( \Lambda \) are shown in figure 2(a) which reveals that this metric tends to 1 indicating perfectly correlated vibrations between the symmetric and asymmetric modes.

Given the system Hamiltonian in equation (1), it is interesting to investigate how the two-mode squeezed states behave in the presence of degenerate parametric amplification namely \( \eta_0 = 0 \). To that end a value of \( \Lambda \) is selected that yields a weakly correlated state and then equations (4) are solved again but now as function of \( \eta_0 \) and \( \eta_1 \) which are both simultaneously increased to the threshold for parametric resonance i.e. \( \Gamma_n \leq Q_n \) [15]. Again the cross-quadratures from the resultant \( Q_A \) and \( P_A \) solutions are reconstructed in phase space and the corresponding correlation coefficient is also extracted with both being shown in figure 2(b). The numerical simulations reveal that even weak correlations generated between the two modes from \( \Lambda \) can be enhanced via \( \Gamma_n < Q_n \) namely in the amplification regime and even be made perfect when \( \Gamma_n \geq Q_n \) that is when both constituent modes parametrically resonate.
3.2. Analytical solutions

To confirm that the enhancement in the two-mode squeezing via degenerate parametric amplification is not an artefact of the numerical simulations, the equations of motion in (4) are also analytically solved with $\gamma = \gamma_S \equiv \gamma$ and $\eta = \eta_A \equiv \eta$. The Langevin equations in this form can then be solved by linear combinations of variables $M_\pm$ and $N_\pm$ which are defined as

$$M_\pm = (P_\pm \pm Q_\pm \pm \pm Q_,)/2,$$
$$N_\pm = (P_\pm \mp Q_\pm \pm \mp Q_,)/2. \tag{6}$$

From equations (4) and (6), four independent Langevin equations can then be derived as

$$M_\pm + (\gamma \mp \eta + \lambda)M_\pm = F_{M_\pm},$$
$$N_\pm + (\gamma \pm \eta - \lambda)N_\pm = F_{N_\pm}, \tag{7}$$

where the decomposed thermal Langevin forces are now given by

$$F_{M_\pm} = (\mp F_{x_\pm} + F_{Q_\pm} - F_{P_\pm} \pm F_{Q_A}/2,,$$
$$F_{N_\pm} = (\pm F_{x_\pm} + F_{Q_\pm} + F_{P_\pm} \pm F_{Q_A}/2). \tag{8}$$

Consequently the correlation functions for variables $M_\pm$ and $N_\pm$, driven by the thermal Langevin forces $F_{M_\pm}$ and $F_{N_\pm}$ respectively, can then readily be extracted as

$$\langle M_+ (t)^2 \rangle = \frac{k_B T}{\omega(1 + (\lambda - \eta)/\gamma)}, \quad \langle M_- (t)^2 \rangle = \frac{k_B T}{\omega(1 + (\lambda + \eta)/\gamma)},$$
$$\langle N_+ (t)^2 \rangle = \frac{k_B T}{\omega(1 - (\lambda - \eta)/\gamma)}, \quad \langle N_- (t)^2 \rangle = \frac{k_B T}{\omega(1 - (\lambda + \eta)/\gamma)}. \tag{9}$$

where $k_B$ is the Boltzmann constant, $T$ is the temperature and the frequency difference between the two modes is assumed to be small so that $\omega - \omega \equiv \omega$. The correlation coefficient between the cross-quadratures $Q_\pm : P_\pm$, for instance, can then be analytically derived using equations (6) and (9) which yields

$$\frac{\langle Q_+ (t)P_+ (t) \rangle}{\sqrt{\langle Q_+ (t)^2 \rangle \langle P_+ (t)^2 \rangle}} = \begin{vmatrix}
\frac{1}{\gamma + \lambda - \eta} + \frac{1}{\gamma + \lambda + \eta} & \frac{1}{\gamma - \lambda + \eta} & \frac{1}{\gamma - \lambda - \eta} \\
\frac{1}{\gamma + \lambda - \eta} & \frac{1}{\gamma + \lambda + \eta} + \frac{1}{\gamma - \lambda + \eta} & \frac{1}{\gamma - \lambda - \eta} \\
\frac{1}{\gamma + \lambda - \eta} & \frac{1}{\gamma + \lambda + \eta} & \frac{1}{\gamma - \lambda + \eta} + \frac{1}{\gamma - \lambda - \eta}
\end{vmatrix}. \tag{10}$$

The resultant correlation coefficient as a function of the non-degenerate parametric pump intensity $\lambda/\gamma$ for a range of degenerate parametric amplification amplitudes $\eta/\gamma$ is plotted in figure 2(c). This reveals that the analytical solutions confirm the numerical results namely even weak correlations from the two-mode squeezing can be enhanced and made perfect by the degenerate parametric amplification and resonance of the constituent modes.
3.3. Physical interpretation

The above numerical and analytical analysis can be more intuitively interpreted as follows: the non-degenerate parametric down-conversion not only amplifies the constituent modes, leading to larger circular distributions in phase-space for both modes (see red points in figures 3(a) and (b)), but it also creates a two-mode squeezed state which can be resolved in the cross-quadrature phase portrait (see red points in figure 3(c)). On the other hand the degenerate parametric amplification of both modes is phase sensitive and it amplifies the quadrature component while the in-phase component is nominally unaffected (see blue points in figures 3(a) and (b)). Note the degenerate parametric amplification can yield squeezing below the thermal noise in the in-phase components of both modes with more modest $\Gamma_p$ excitations. Alternatively the in-phase component can also be amplified while the quadrature component is squeezed, or any arbitrary orthogonal combination in phase space, by simply adjusting the phase of the degenerate parametric amplification. The cross-quadrature phase portrait reconstructed from the non-degenerate parametric down-conversion with degenerate parametric amplification exhibits further squeezing (see blue points in figure 3(c)). This effect is identified as enhanced correlations between the modes as quantified by the correlation coefficient in figure 2(b). This enhancement is a consequence of the squashed thermal fluctuation distributions of the constituent modes where the quadrature components are stretched and these underlying geometries naturally lead to narrower two-mode squeezed distributions.

4. Results

Before these expectations can be tested, the availability of thermomechanical two-mode squeezed states in this electromechanical system needs to be verified. To that end, the spring constant of both modes is pumped with voltage $V_p(\omega_p)$ which simultaneously amplifies the thermomechanical displacement fluctuations of both modes as shown in figure 1(b) [27]. In this configuration all four in-phase and quadrature components of the fluctuations from both modes are simultaneously acquired over a period of 300 s with a 50 ms sampling rate. From this measurement the phase portrait from the cross-quadrature $Q_x : P_y$ can be reconstructed and the corresponding correlation coefficient (from both sets of cross-quadratures) can be extracted, as a function of pump voltage, and is shown in figure 4(a). The phase portrait reveals a squashed distribution which implies that the motion from both modes is intertwined and this is statistically confirmed from the correlation coefficient which tends to 0.9 as $V_p(\omega_p)$ approaches 500 mV$_{\text{rms}}$. However when $V_p(\omega_p) > 500$ mV$_{\text{rms}}$, both modes begin to self-oscillate and amplification due to non-degenerate parametric down-conversion is no longer available [11, 28, 29]. The phase portrait shown in the inset to figure 4(a) also enables the noise squeezing below the thermal level to be quantified as detailed in [11]. In this case starting with thermal populations of approximately $10^7$ phonons in each mode at 300 K with $V_p(\omega_p) = 0$ V$_{\text{rms}}$ (black points in the inset to figure 4(a)), noise squeezing of $-1.55$ dB below the thermal level is achieved with $V_p(\omega_p) = 0.5$ V$_{\text{rms}}$ (red points in the inset to figure 4(a)).

Next the correlations between the two modes generated from the non-degenerate parametric down-conversion are experimentally investigated while undergoing degenerate parametric amplification. To that end

![Figure 3](image-url)
first a weakly correlated state is created with \( V_p(\omega_p) = 0.2 \sqrt{V_{\text{rms}}} \) and then \( V_l(2\omega_l) \) and \( V_l(2\omega_l) \) are both simultaneously increased from 0 to 0.35 \( V_{\text{rms}} \) in 0.05 \( V_{\text{rms}} \) increments. The cross-quadrature phase portrait from \( Q_\Lambda : P_\alpha \) reconstructed from this measurement reveals that the initially weakly correlated symmetric and asymmetric modes becoming more strongly entwined as evidenced by the increasingly squashed distribution and this is quantitatively confirmed via their correlation coefficient which tends to unity as shown in figure 4(b). This measurement also yields enhanced squeezing of -3 dB below the thermal level when \( V_p(2\omega_p) = 0.35 \sqrt{V_{\text{rms}}} \) (red points in the inset to figure 4(b)). These experimental observations confirm the theoretical expectation in figure 2(b) and they naturally suggest the possibility of perfect correlations, even with \( V_p(\omega_p) \rightarrow 0 \), if \( \Gamma_u \) (from \( V_p(2\omega_p) \)) can be increased sufficiently as shown in figure 2(c).

Experimentally a large \( \Gamma_u \) implies that the thermal fluctuations of both modes are being degenerately amplified to such an extent that they parametrically resonate as, shown in figures 1(c) and (d), when \( V_l(2\omega_l) > 0.65 \sqrt{V_{\text{rms}}} \) and \( V_l(2\omega_l) > 0.55 \sqrt{V_{\text{rms}}} \) respectively [15, 23]. However a unique feature of the parametric resonance is that it can only vibrate with one of two phases separated by \( \pi \) radians [30]. Consequently the cross-quadrature phase portraits and indeed the language of two-mode squeezing is no longer available and relevant respectively in this regime as the steady state parametric resonances of the constituent modes with a particular phase will simply yield a single point far from the origin (the opposite phase will yield point with a \( \pi \) phase rotation) with almost no distribution as this vibration amplitude is orders of magnitude larger than both the thermomechanical noise of the mechanics and the electrical noise in the detection circuit see figures 1(c) and (d). Additionally the statistics of the resultant correlation will also be unavailable in this configuration as it essentially constitutes a single sample.

To that end an alternative strategy is developed where the bi-stable phase of the parametric resonance from both modes is monitored via the in-phase component of their vibrations. To observe this phase bi-stability, both modes are simultaneously and repeatedly activated via \( V_p(2\omega_p) \) with a period of 0.3 Hz and the resultant in-phase components of their vibrations are monitored over a period of 1000 s and are shown in figure 5(a). This measurement confirms the phase bi-stability of the parametric resonance [15, 30] and it also indicates the absence of correlations between the modes as the choice of vibration phase in one of the modes is random and completely unrelated to the vibration phase in the other mode. On the other hand if this measurement is repeated in the presence of continuous-wave non-degenerate parametric down-conversion with \( V_p(\omega_p) = 600 \mu V_{\text{rms}} \) it results in the outputs shown in figure 5(b) which reveals that both modes always vibrate with same phase as the actuation of their degenerate parametric resonance is periodically modulated. This observation indicates that the two modes have become correlated from the non-degenerate parametric down-conversion and indeed the corresponding correlation coefficient extracted from this measurement yields a value of unity. Evaluating the correlation coefficient from this protocol as function of \( V_p(\omega_p) \) yields the data in figure 5(c) which indicates that correlations can be observed via the parametric resonances of both modes with a
amplitude that is three orders of magnitude smaller than the equivalent measurement utilising thermomechanical noise fluctuations as shown in figure 4(a).

An intuitive insight to these observations can be gleaned by interpreting the non-degenerate parametric down-conversion, which generates phonons in the underlying modes at their natural frequency, as breaking symmetry of the double-well potentials which underpin the phase bi-stable parametric resonances [24]. This in turn results in a given mode preferentially vibrating with one of the two available phases. However since the parametric down-conversion simultaneously generates a pair of phonons in both modes, which are intertwined, this naturally results in the symmetry breaking in the double-well potential of the counterpart mode correlating with the first mode and it results in it vibrating with a coordinated phase.

5. Summary

The enhancement of thermomechanical two-mode squeezing, via phonon ensembles in two vibration modes, in an electromechanical system is investigated. Both numerical and analytical analysis indicates that the resultant squeezing can not only be enhanced via degenerate parametric amplification of the constituent modes but the resultant correlated mechanical vibrations can also be made more visible via their parametric resonance. The experimental measurements based on this analysis confirm these expectations where the non-degenerate parametric amplification amplitude necessary to generate correlations can be reduced by three orders of magnitude while simultaneously yielding bright and perfectly correlated vibrations in the underlying mechanical modes.

The ability to not only enhance the thermomechanical two-mode squeezed states via the degenerate parametric resonances of the constituent modes but also to simultaneously increase their visibility suggests an alternative approach to identifying mechanical vacuum squeezed states which could prove pivotal in detecting a macroscopic all-mechanical entanglement [21]. Meanwhile these protocols make the two-mode squeezing phenomenon with phonons in electromechanical systems [11–13] more widely available by bringing the resultant bright correlations, namely the classical analogue of entanglement, a step closer to being exploited in technological applications.

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