Geodesics of projections in von Neumann algebras

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Abstract

Let \( A \) be a von Neumann algebra and \( \mathcal{P}_A \) the manifold of projections in \( A \). There is a natural linear connection in \( \mathcal{P}_A \), which in the finite dimensional case coincides with the the Levi-Civita connection of the Grassmann manifold of \( \mathbb{C}^n \). In this paper we show that two projections \( p, q \) can be joined by a geodesic, which has minimal length (with respect to the metric given by the usual norm of \( A \)), if and only if

\[
p \wedge q^\perp \sim p^\perp \wedge q,
\]

where \( \sim \) stands for the Murray-von Neumann equivalence of projections. It is shown that the minimal geodesic is unique if and only if \( p \wedge q^\perp = p^\perp \wedge q = 0 \). If \( A \) is a finite factor, any pair of projections in the same connected component of \( \mathcal{P}_A \) (i.e., with the same trace) can be joined by a minimal geodesic.

We explore certain relations with Jones’ index theory for subfactors. For instance, it is shown that if \( \mathcal{N} \subset \mathcal{M} \) are II\(_1\) factors with finite index \([\mathcal{M} : \mathcal{N}] = t^{-1}\), then the geodesic distance \( d(e_{\mathcal{N}}, e_{\mathcal{M}}) \) between the induced projections \( e_{\mathcal{N}} \) and \( e_{\mathcal{M}} \) is \( d(e_{\mathcal{N}}, e_{\mathcal{M}}) = \arccos(t^{1/2}) \).

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1 Introduction

If \( A \) is a C*-algebra, let \( \mathcal{P}_A \) denote the set of (selfadjoint) projections in \( A \). \( \mathcal{P}_A \) has a rich geometric structure, see for instance the papers [12] by H. Porta and L.Recht and [6] by G. Corach, H. Porta and L. Recht. In these works, it was shown that \( \mathcal{P}_A \) is a \( C^\infty \) complemented submanifold of \( \mathcal{A}_s \), the set of selfadjoint elements of \( A \), and has a natural linear connection, whose geodesics can be explicitly computed. A metric is introduced, called in this context a Finsler metric: since the tangent spaces of \( \mathcal{P}_A \) are closed and complemented linear subspaces of \( \mathcal{A}_s \), they can be endowed with the norm metric. With this Finsler metric, Porta and Recht [12] showed that two projections \( p, q \in \mathcal{P}_A \) which satisfy that \( \|p - q\| < 1 \) can be joined by a unique geodesic, which is minimal for the metric (i.e., it is shorter than any other smooth curve in \( \mathcal{P}_A \) joining the same endpoints).

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In general, two projections $p, q$ in $\mathcal{A}$ satisfy that $\|p - q\| \leq 1$, so that what remains to consider is what happens in the extremal case $\|p - q\| = 1$: under what conditions does there exist a geodesic, or a minimal geodesic, joining them.

In the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators in a Hilbert space $\mathcal{H}$, it is known (see for instance [4]) that there exists a geodesic joining $p$ and $q$ if and only if

$$\dim R(p) \cap N(q) = \dim N(p) \cap R(q).$$

The geodesic is unique if and only if these intersections are trivial.

The purpose of this note is to show that these facts remain valid if $\mathcal{A}$ is a von Neumann algebra, if we replace $\dim$ by the dimension relative to $\mathcal{A}$. Namely, it is shown that there exists a minimal geodesic joining $p$ and $q$ in $\mathcal{P}_A$, if and only if

$$p \wedge q^\perp \sim p^\perp \wedge q.$$

Here $\wedge$ denotes the infimum of two projections, $p^\perp = 1 - p$, and $\sim$ is the Murray-von Neumann equivalence of projections. Also, it is shown that there exists a unique minimal geodesic if and only if

$$p \wedge q^\perp = 0 = p^\perp \wedge q.$$

We show that if $\mathcal{A}$ is a finite factor, any pair of projections in $\mathcal{A}$ in the same connected component of $\mathcal{P}_A$ (i.e., with the same trace), can be joined by a minimal geodesic.

In the final section of this paper, we explore the relationship with the index theory of von Neumann factors, introduced by V. Jones in [10]. A pairing $\mathcal{N} \subset \mathcal{M}$ of factors of type $\mathcal{II}_1$, induces a sequence of projections, by means of the basic construction. We show that one recovers Jones index as a geodesic distance (minima of lengths of curves joining two given points): if $e, f$ are two consecutive terms in the sequence of projections, then

$$d(e, f) = \arccos(t^{1/2}),$$

where $t^{-1} = [\mathcal{M} : \mathcal{N}]$. Also we show that if $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$ with Jones’ projections $e_0, e_1$, satisfy that $\|e_0 - e_1\| < 1$, then the unique geodesic $\delta(t)$ induces a smooth path of conditional expectations between $\mathcal{M}$ and intermediate factors $\mathcal{N}_t$, and the parallel transport of this geodesic, induces a smooth path of normal $\ast$-isomorphisms between $\mathcal{N}_0$ and $\mathcal{N}_t$.

## 2 Preliminaries

The space $\mathcal{P}_A$ is sometimes called the Grassmann manifold of $\mathcal{A}$. The reason for this name is that in the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ parametrizes the set of closed subspaces of $\mathcal{H}$: to each closed subspace $\mathcal{S} \subset \mathcal{H}$ corresponds the orthogonal projection $P_S$ onto $\mathcal{S}$. Let us describe below the main features of the geometry of $\mathcal{P}_A$ in the general case.

### 2.1 Homogeneous structure

Denote by $\mathcal{U}_A = \{u \in \mathcal{A} : u^*u = uu^* = 1\}$ the unitary group of $\mathcal{A}$. It is a Banach-Lie group, whose Banach-Lie algebra is $\mathcal{A}_{\mathcal{A}} = \{x \in \mathcal{A} : x^* = -x\}$. This group acts on $\mathcal{P}_A$ by means of $u \cdot p = upu^*$, $u \in \mathcal{U}_A$, $p \in \mathcal{P}_A$. The action is smooth and locally transitive. It is known (see [12], [6]) that $\mathcal{P}_A$ is what in differential geometry is called a homogeneous space of the group
\( \mathcal{U}_A \). The local structure of \( \mathcal{P}_A \) is described using this action. For instance, the tangent space \( (T\mathcal{P}_A)_p \) of \( \mathcal{P}_A \) at \( p \) is given by \( (T\mathcal{P}_A)_p = \{ x \in \mathcal{A}_s : x = px + xp \} \).

The isotropy subgroup of the action at \( p \), i.e., the elements of \( \mathcal{U}_A \) which fix a given \( p \), is \( \mathcal{I}_p = \{ v \in \mathcal{U}_A : vp = pv \} \). The isotropy algebra \( \mathcal{I}_p \) at \( p \) is its Banach-Lie algebra \( \mathcal{I}_p = \{ y \in \mathcal{A}_s : yp = py \} \).

It is useful, in order to describe and understand the geometry of \( \mathcal{P}_A \), to consider the diagonal / co-diagonal decomposition of \( \mathcal{A} \) in terms of a fixed projection \( p_0 \in \mathcal{P}_A \). Elements \( x \in \mathcal{A} \) which commute with \( p_0 \), or equivalently, commute with the symmetry \( 2p_0 - 1 \), when written as \( 2 \times 2 \) in terms of \( p_0 \), have diagonal matrices. Co-diagonal matrices correspond with elements in \( \mathcal{A} \) which anti-commute with \( 2p_0 - 1 \).

Then, the isotropy subgroup and the isotropy algebra \( \mathcal{I}_{p_0}, \mathcal{I}_{p_0} \) at \( p_0 \), are respectively the sets of diagonal unitaries and diagonal anti-Hermitian elements of \( \mathcal{A} \). On the other side, the tangent space \( (T\mathcal{P}_A)_{p_0} \) is the set of diagonal selfadjoint elements of \( \mathcal{A} \).

### 2.2 Reductive structure

Given an homogeneous space, a reductive structure is a smooth distribution \( p \mapsto \mathcal{H}_p \subset \mathcal{A}_s \), \( p \in \mathcal{P}_A \), of supplements of \( \mathcal{I}_p \) in \( \mathcal{A}_s \), which is invariant under the action of \( \mathcal{I}_p \). That is, a distribution \( \mathcal{H}_p \) of closed linear subspaces of \( \mathcal{A}_s \) verifying that \( \mathcal{H}_p \oplus \mathcal{I}_p = \mathcal{A}_s \); \( v\mathcal{H}_p v^* = \mathcal{H}_p \) for all \( v \in \mathcal{I}_p \); and the map \( p \mapsto \mathcal{H}_p \) is smooth.

In the case of \( \mathcal{P}_A \), the choice of the (so called) horizontal subspaces \( \mathcal{H}_p \) is natural. The horizontal \( \mathcal{H}_p \) defined in [6] is \( \mathcal{H}_p = \{ \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} : z \in p\mathcal{A}p^\perp \} \), i.e., the set of co-diagonal anti-Hermitian elements of \( \mathcal{A} \).

As in classical differential geometry, a reductive structure on a homogeneous space defines a linear connection: if \( X(t) \) is a smooth curve of vectors tangent to a smooth curve \( p(t) \) in \( \mathcal{P}_A \), i.e., a smooth curve of selfadjoint elements of \( \mathcal{A} \), which are pointwise co-diagonal with respect to \( p(t) \), then the covariant derivative of the linear connection is given by

\[
\frac{D}{dt}X(t) := \text{diagonal part w.r.t. } p(t) \text{ of } \dot{X}(t) = p(t)\dot{X}(t)p(t) + p^\perp(t)\dot{X}(t)p^\perp(t).
\]

It is not difficult to deduce then that a geodesic starting at \( p_0 \in \mathcal{P}_A \) is given by the action of a one parameter group with horizontal (anti-Hermitian co-diagonal) velocity on \( p_0 \). Namely, given the base point \( p_0 \in \mathcal{P}_A \), and a tangent vector \( x = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (T\mathcal{P}_A)_{p_0} \), the unique geodesic \( \delta \) of \( \mathcal{P}_A \) with \( \delta(0) = p_0 \) and \( \dot{\delta}(0) = x \) is given by

\[
\delta(t) = e^{t\frac{1}{2}x}p_0 e^{-t\frac{1}{2}x},
\]

where \( z_x := \begin{pmatrix} 0 & -x \\ x^* & 0 \end{pmatrix} \). The horizontal element \( z_x \) is characterized as the unique horizontal element at \( p_0 \) such that \( [z_x, p_0] = x \).

### 2.3 Finsler metric

As we mentioned above, one endows each tangent space \( (T\mathcal{P}_A)_p \) with the usual norm of \( \mathcal{A} \). We emphasize that this (constant) distribution of norms is not a Riemannian metric (the C*-norm
is not given by an inner product), neither is it a Finsler metric in the classical sense (the map \( a \mapsto \|a\| \) is non differentiable). Therefore the minimality result which we describe below does not follow from general considerations. It was proved in [12] using ad-hoc techniques.

1. Given \( p \in \mathcal{P}_A \) and \( x \in (T\mathcal{P}_A)_p \), normalized so that \( \|x\| \leq \pi/2 \), then the geodesic \( \delta \) remains minimal for all \( t \) such that \( |t| \leq 1 \).

2. Given \( p, q \in \mathcal{P}_A \) such that \( \|p - q\| < 1 \), there exists a unique minimal geodesic \( \delta \) such that \( \delta(0) = p \) and \( \delta(1) = q \).

We shall call these geodesics (with initial speed \( \|x\| \leq \pi/2 \)) normalized geodesics.

3 Von Neumann algebras

In this paper we consider the case when \( A \) is a von Neumann algebra. We shall suppose \( A \) acting in a Hilbert space \( H \) (i.e., \( A \subset B(H) \)). As we shall see, this representation is auxiliary, and the results on the geometry of \( \mathcal{P}_A \) do not depend on the representation. The main assertion of this section is that the conditions of existence and uniqueness of minimal geodesics joining given projections \( p, q \in \mathcal{P}_A \) are the a natural generalization of the conditions valid in the case of \( B(H) \).

If \( p, q \in \mathcal{P}_A \), we denote by \( p^\perp = 1 - p \), and by \( p \wedge q \) the projection onto \( R(p) \cap R(q) \) (which belongs to \( \mathcal{P}_A \)); \( p \) and \( q \) are said to be Murray - von Neumann equivalent, in symbols \( p \sim q \), if there exists \( v \in A \) (a partial isometry) such that \( v^*v = p \) and \( vv^* = q \). Our main result follows:

**Theorem 3.1.** Let \( p, q \in \mathcal{P}_A \).

1. There exists a geodesic \( \delta \) of \( \mathcal{P}_A \) joining \( p \) and \( q \) if and only if
   \[ p \wedge q^\perp \sim p^\perp \wedge q. \]
   Moreover, the geodesic can be chosen minimal (i.e., normalized).

2. There is a unique normalized geodesic if and only if \( p \wedge q^\perp = p^\perp \wedge q = 0 \).

**Proof.** Existence: suppose first that \( p \wedge q^\perp \sim p^\perp \wedge q \). Consider following projections which sum 1 and commute both with \( p \) and \( q \):

\[
e_{11} = p \wedge q , \quad e_{00} = p^\perp \wedge q^\perp , \quad e_{10} = p \wedge q^\perp , \quad e_{11} = p^\perp \wedge q , \quad e_0 = 1 - \sum_{i,j=0,1} e_{i,j}.
\]

It is straightforward to verify that \( e_{ij} \) commute with \( p \) and \( q \), and thus \( e_0 \) also does. The decomposition of the Hilbert space induced by these projections is sometimes called the Halmos decomposition of the space, in the presence of two closed subspaces (\( R(p) \) and \( R(q) \)); the last subspace \( R(e_0) \), is called the generic part of \( p \) and \( q \). We shall construct the exponent \( x \) of the geodesic joining \( p \) and \( q \) as a sum of anti-Hermitian elements in \( A \),

\[
x = x' + x'' + x_0,
\]

where \( x' \) acts in the range of \( e_{11} + e_{00} \), \( x'' \) acts in the range of \( e_{10} + e_{01} \) and \( x_0 \) acts in the range of \( e_0 \). Moreover, each of these elements is co-diagonal with respect to the corresponding...
reduction of $p$ to these subspaces. First note that $pe_{ii} = qe_{ii}$ (on $e_{00}$ they are both zero, on $e_{11}$ they are both the identity). Thus the exponent $x'$ can be chosen 0.

Let us consider next the part in $e_0$. Here we make use of the representation $A \subset B(H)$. Denote by $H_0 = R(e_0)$, and by $p_0 = pe_0$, $q_0$ the reductions of $p, q$ to this subspace $H_0$. Then, it is clear that $p_0, q_0$ lie in generic position ([9], [8]): their ranges and nullspaces intersect trivially. Thus, by a result by P. Halmos [9], there exist a Hilbert space $L$, a positive operator $X \in B(L)$ ($\|X\| \leq \mathcal{P}/2$ and a unitary isomorphism $H_0 \to L \times L$ which carries

$$p_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } q_0 \to \begin{pmatrix} \cos^2(X) & \cos(X) \sin(X) \\ \cos(X) \sin(X) & \sin^2(X) \end{pmatrix}.$$ 

Between these operator matrices, one can find the (co-diagonal) exponent $Z = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$, which satisfies

$$e^Z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-Z} = \begin{pmatrix} \cos^2(X) & \cos(X) \sin(X) \\ \cos(X) \sin(X) & \sin^2(X) \end{pmatrix}.$$ 

These are straightforward verifications, and provide the exponent for a geodesic joining the two operator matrices. One loses track though of how elements of $A$ are changed by the Halmos isomorphism. The key fact to relate these matrices to the former projections $p_0, q_0$ is the following elementary identity proved in [4]

$$e^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V,$$

(1)

where $V$ is the unitary part in the polar decomposition of

$$B - 1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos^2(X) & \cos(X) \sin(X) \\ \cos(X) \sin(X) & \sin^2(X) \end{pmatrix} - 1,$$

i.e. $B - 1 = V|B - 1|$. Again, this is an elementary matrix computation. Let $b_0 = p_0 + q_0$, and let $v_0$ be the isometric part in the polar decomposition (recall that $e_0$ is the unit in this part of the algebra)

$$b_0 - e_0 = v_0|b_0 - e_0|.$$ 

Clearly $v_0 \in A$ and is carried by the Halmos isomorphism to $V$. Therefore, if one regards (1), it follows that the unitary element $v_0(2p_0 - 1)$ is carried by this isomorphism to $e^Z$, i.e. $e^Z$ corresponds to an element of $A$. Moreover,

$$\|Z\| = \|X\| \leq \pi/2,$$

which implies that $Z$ is the unique anti-Hermitian logarithm of $e^Z$ with spectrum in $(-i\pi, i\pi)$. It follows that there exists a unique element $x_0 \in A$ which corresponds to $Z$, and therefore satisfies

$$e^{x_0} p_0 e^{-x_0} = q_0.$$ 

It remains to construct the exponent $x''$ acting in $e_{10} + e_{01}$. Note that the reductions of $p$ and $q$ to this part are

$$p(e_{10} + e_{01}) = p(p \wedge q^\perp + p^\perp \wedge q) = p \wedge q^\perp,$$

and
ans similarly \(q(e_1 + e_0) = p^\perp \wedge q\). By hypothesis, there is a partial isometry \(w \in \mathcal{A}\) such that
\[
w^*w = p \wedge q^\perp \quad \text{and} \quad ww^* = p^\perp \wedge q.
\]
Then \(x'' = i\pi(p + w^*)\) does the feat: since \(p \wedge q^\perp \perp p^\perp \wedge q\), it follows that \(x''\) is \(p_{10} + p_{01}\) co-diagonal. Clearly \(\|x''\| = \pi/2\). Note also that \(w^2 = 0\), so that 
\[
e^{x''} = i(w + w^*).
\]
Finally, 
\[
e^{x''}(p \wedge q^\perp) = i(w + w^*)(w^*w) = iw^*w = iw^*(w + w^*) = (p^\perp \wedge q)e^{x''},
\]
i.e., \(e^{x''}\) intertwines the reductions of \(p\) and \(q\) to this part.

If we put together \(x = x' + x'' + x_0\), which is an orthogonal sum, we have a \(p\)-co-diagonal anti-Hermitian element of \(\mathcal{A}\), with \(\|x\| \leq \pi/2\) (note that \(x''\) might be zero, if \(p \wedge q^\perp = p^\perp \wedge q = 0\)), which satisfies  
\[
e^xpe^{-x} = q.
\]

Conversely, suppose that there exists a normalized geodesic which joins \(p\) and \(q\), i.e. there exists a \(p\)-co-diagonal anti-Hermitian element \(x \in \mathcal{A}\) with \(\|x\| \leq \pi/2\) such that \(e^xpe^{-x} = q\). We claim that \(e^x\) maps \(R(p \wedge q^\perp)\) onto \(R(p^\perp \wedge q)\). Clearly \(e^x\) maps \(R(p)\) onto \(R(q)\). Pick \(\xi \in R(p \wedge q^\perp) = R(p) \cap N(q)\). Then \(e^x\xi \in R(q)\). It was noted in [12], that the fact that \(x\) is \(p\)-co-diagonal means that \(x\) anti-commutes with \(2p - 1\). Thus,
\[
(2p - 1)e^x = e^{-x}(2p - 1).
\]
Then, since \((2p - 1)\xi = \xi\) and \((2q - 1)\xi = -\xi\),
\[
(2p - 1)e^x\xi = e^{-x}(2p - 1)\xi = e^{-x}(2q - 1)\xi = -(2p - 1)e^{-x}\xi = -e^x(2p - 1)\xi = -e^x\xi,
\]
i.e., \(e^x\xi \in N(p)\), and thus \(e^x(R(p) \cap N(q)) \subseteq R(q) \cap N(p)\). The other inclusion follows similarly (or by symmetry: in fact \(x\) is also \(q\)-co-diagonal, because \(-x\) is the initial velocity of the reversed geodesic which starts at \(q\)). It follows that \(w = e^x(p \wedge q^\perp) \in \mathcal{A}\) is a partial isometry with initial space \(p \wedge q^\perp\) and final space \(p^\perp \wedge q\).

Uniqueness: if \(p \wedge q^\perp = p^\perp \wedge q = 0\), then \(R(p) \cap N(q) = N(p) \cap R(q) = \{0\}\), and there exists a unique normalized geodesic in \(\mathcal{P}_{\mathcal{B}(H)}\) joining \(p\) and \(q\). By the first part of the proof, there is a normalized geodesic joining them in \(\mathcal{P}_{\mathcal{A}}\). Thus, it is unique.

Conversely, suppose that there exists a unique geodesic joining \(p\) and \(q\). Then necessarily \(p \wedge q^\perp \sim p^\perp \wedge q\). Suppose that these projections are non zero. Then, there are infinitely many different partial isometries \(w\) such that \(w^*w = p \wedge q^\perp\) and \(ww^* = p^\perp \wedge q\). As in the first part of the proof, any such \(w\) give rise to different exponents \(x''\), and thus different \(x\), i.e. different geodesics joining \(p\) and \(q\). \(\square\)

**Remark 3.2.** In the above result, it was shown in fact that the submanifold \(\mathcal{P}_{\mathcal{A}} \subset \mathcal{P}_{\mathcal{B}(H)}\) is totally geodesic: the geodesics of \(\mathcal{P}_{\mathcal{A}}\) are geodesics of the bigger manifold \(\mathcal{P}_{\mathcal{B}(H)}\); if \(p, q \in \mathcal{P}_{\mathcal{A}}\) are joined by a unique geodesic of \(\mathcal{P}_{\mathcal{B}(H)}\), then this geodesic remains inside \(\mathcal{P}_{\mathcal{A}}\).
4 Hopf-Rinow theorem in finite factors

Two subspaces of dimension \( k \) in \( \mathbb{C}^n \) can be joined by a minimal geodesic of the Levi-Civita connection in the Grassmann manifold. This fact can be proved using the projection formalism. That is, parametrizing subspaces with orthogonal projections in \( M_n(\mathbb{C}) \), by means of

\[
\mathbb{C}^n \supset S \leftrightarrow P_S \in M_n(\mathbb{C}),
\]

where \( P_S \) is the orthogonal projection onto \( S \). Two subspaces \( S, T \subset \mathbb{C}^n \) have the same dimension if and only if the corresponding projections \( P_S, P_T \) have the same rank, i.e.

\[
Tr(P_S - P_T) = 0.
\]

Let us see that in this case one has, automatically, that

\[
\dim(S \cap T^\perp) = \dim(S^\perp \cap T).
\]

This fact has an elementary proof. Let us prove it in a non totally elementary fashion, which will allow us to obtain a generalization. The operator \( A = P_S - P_T \) is a selfadjoint contraction, and if \( B = P_S + P_T \),

\[
N(B - 1) = S \cap T^\perp \oplus S^\perp \cap T = N(A - 1) \oplus N(A + 1).
\]

On the subspace \( N(B - 1)^\perp \), \( B - 1 \) is an invertible matrix, and the symmetry \( V \) in its polar decomposition \( B - 1 = V|B - 1| \) satisfies that \( VP_SV = P_T \) in \( N(B - 1)^\perp \). Then \( A \) is reduced by \( N(B - 1) \), and

\[
V(A\big|_{N(B - 1)^\perp})V = -A\big|_{N(B - 1)^\perp}.
\]

This implies that the spectrum of \( A\big|_{N(B - 1)^\perp} \) is symmetric with respect to the origin: if \( \lambda \) is an eigenvalue of \( A \) with \( |\lambda| < 1 \), then \( -\lambda \) is also an eigenevalue of \( A \), and they have the same multiplicity: \( \dim(N(A - \lambda)) = \dim(N(A + \lambda)) \). Then

\[
A = -P_{N(A + 1)} + P_{N(A - 1)} + A\big|_{N(B - 1)^\perp} = -P_{N(A + 1)} + P_{N(A - 1)} + \sum_{0 < \lambda < 1} \lambda(P_{N(A - \lambda)} - P_{N(A + \lambda)}).
\]

Thus, the fact that \( Tr(A) = 0 \), means that \( Tr(P_{N(A - 1)}) = Tr(P_{N(A + 1)}) \). Thus \( P_S \) and \( P_T \) can be joined by a normalized geodesic. Remarkably, this geodesic is minimal for the Levi-Civita connection of the Grassmann manifold, but also, using the projection formalism, for the operator norm of \( M_n(\mathbb{C}) \), the \( p \)-Schatten norms ([3]), or more generally, for unitary invariant norms (see [5]).

Let us suppose now that \( \mathcal{A} \) is a finite von Neumann factor, with trace \( \tau \). We shall see that the above argument holds (essentially unaltered):

**Theorem 4.1.** Let \( \mathcal{A} \) be a finite von Neumann factor with faithful normal trace \( \tau \). Two projections \( p, q \in \mathcal{P}_\mathcal{A} \) with \( p \sim q \) (i.e., unitarily equivalent, or equivalently, in the same connected component of \( \mathcal{P}_\mathcal{A} \)) can be joined by a normalized geodesic.
Proof. Let \( a = p - q \), and again note that \( N(a - 1) = R(p) \cap N(q) \). Following previous notations, \( P_{N(a-1)} = p \wedge q^\perp = e_{10} \). Similarly, \( P_{N(a+1)} = p^\perp \wedge q = e_{01} \). Therefore, \( P_{N(b-1)} = e_{10} + e_{01} := e'' \).

Again, since \( e_{10} \) and \( e_{01} \) are eigenspaces of \( a \), these projections reduce \( a \), let \( a_0 \) be the reduction of \( a \) to \( R(e'')^\perp \). Then, we have that

\[
a = e_{10} - e_{01} + a_0.
\]

The operator \( a_0 \) is a difference of projections: \( a_0 = p_0 - q_0 \), where \( p_0 \) and \( q_0 \) are the reductions of \( p \) and \( q \) to \( R(e'')^\perp \), with \( N(a_0 \pm e_\theta) = \{0\} \) (\( e_\theta \) is the identity in \( R(e'')^\perp \)). It was shown by Chandler Davis [7] that there exists a symmetry \( v_0 (v_0^* = v_0, v_0^2 = e_\theta \); namely, \( v_0 \) is the isometric part in the polar decomposition of \( b_0 - e_\theta \) such that

\[
v_0 a_0 v_0 = -a_0.
\]

Let \( \mu \) be the projection-valued spectral measure of \( a_0 \):

\[
a_0 = \int_{-1}^{1} \lambda d\mu(\lambda).
\]

As in the above argument in \( M_n(\mathbb{C}) \), the existence of the symmetry \( v_0 \) implies the symmetry of the spectral measure of \( a_0 \) with respect to the origin: if \( \Lambda \subset [-1,1] \) is a Borel subset, then

\[
\mu(-\Lambda) = v_0 \mu(\Lambda) v_0.
\]

Then

\[
\tau(a_0) = \int_{-1}^{1} \lambda d\tau(\mu(\lambda)) = 0,
\]

because the function \( f(\lambda) = \lambda \) is odd and the measure \( \tau \mu \) is symmetric with respect to the origin:

\[
\tau(\mu(-\Lambda)) = \tau(v_0 \mu(\Lambda) v_0) = \tau(e_\theta \mu(\Lambda)) = \tau(\mu(\Lambda)),
\]

because \( \mu \leq e_\theta \). Then, since \( p \sim q \),

\[
0 = \tau(p) - \tau(q) = \tau(a) = \tau(e_{10} - e_{01} + a_0) = \tau(e_{10}) - \tau(e_{01}),
\]

so that \( \tau(p \wedge q^\perp) = \tau(p^\perp \wedge q) \), i.e., \( p \wedge q^\perp \sim p^\perp \wedge q \). Therefore, by Theorem 3.1, there exists a (minimal) normalized geodesic joining \( p \) and \( q \) in \( \mathcal{P}_A \).

\[\square\]

Remark 4.2. In [1], it was shown that in a finite algebra with faithful trace \( \tau \), the geodesics have minimal length also when measured with the \( \rho \) norms \( \|\|_\rho \) of the trace, for \( \rho \geq 2 \left( \|x\|_\rho = (\tau(x^* x)^{\rho/2})^{1/\rho} \right) \). Namely, it was shown that if \( \delta(t) = e^{itx}pe^{-itx} \) is a normalized geodesic \( (\|z\| \leq \pi/2) \) with \( \delta(1) = q \), and \( \gamma \) is any other smooth curve in \( \mathcal{P}_A \) with \( \gamma(t_0) = p \) and \( \gamma(t_1) = q \), then

\[
\ell_\rho(\gamma) := \int_{t_0}^{t_1} \|\gamma(t)\|_\rho dt \geq \ell_\rho(\delta) = \|z\|_\rho.
\]

As we have seen, on finite factors, a version of the the Hopf-Rinow is valid in \( \mathcal{P}_A \), and the geodesics are minimal for the usual norm of \( A \) at every tangent space. However, as a consequence of the fact in the above remark, we have that for the \( p \)-norms in the tangent space, including the
pseudo-Riemannian case \( p = 2 \), there are no normal neighbourhoods if \( \mathcal{A} \) is a type II\(_1\) factor. Indeed, for \( 2 \leq \rho < \infty \), denote by

\[
d_\rho(p, q) = \inf \{ \ell_\rho(\gamma) : \gamma \text{ is smooth and joins } p \text{ and } q \text{ in } \mathcal{P}_\mathcal{A} \}
\]

the metric induced in \( \mathcal{P}_\mathcal{A} \) by the \( \rho \)-norm.

**Proposition 4.3.** Let \( \mathcal{A} \) a type II\(_1\) factor and \( 2 \leq \rho < \infty \). Then there exist pairs of projections in \( \mathcal{P}_\mathcal{A} \), which are arbitrarily close for the \( d_\rho \) metric, which can be joined by infinitely many geodesics.

**Proof.** Given \( 0 < r \leq \frac{1}{2} \), let \( p \in \mathcal{P}_\mathcal{A} \) such that \( \tau(p) = r \). Let \( q \in \mathcal{P}_\mathcal{A} \) such that \( q \leq p \perp \) and \( \tau(q) = r \) (consider the reduced factor \( p \perp A p \perp \), which is also of type II\(_1\)), and pick there a projection \( q \) with (renormalized) trace \( \frac{r}{\tau(r)} \). Then, the Halmos decomposition given by \( p \) and \( q \) yields (following the notation of the preceding section)

\[
e_{00} = 0, \quad e_{11} = 0, \quad e_{10} = p, \quad e_{10} = p, \quad e_{01} = q, \quad e_0 = 0.
\]

Since \( p \sim q \), there exist infinitely many \( v \in \mathcal{A} \) such that \( v^*v = p \) and \( vv^* = q \). Any of these \( v \) provides a geodesic joining \( p \) and \( q \), given by (see the last part of the proof of Theorem 3.1) the exponent \( x = \frac{\pi}{\tau}(v + v^*) \). The length of any of these geodesics is

\[
\|x\|_\rho = \tau ((x^*x)^{\rho/2})^{1/\rho} = \frac{\pi}{2} \tau ((v^*v + vv^*)^{\rho/2})^{1/\rho} = \frac{\pi}{2} 2^{1/\rho} \tau(p)^{1/\rho} = \pi 2^{1/\rho} r^{1/\rho}.
\]

\[\Box\]

5 **Applications to finite index subfactors**

V.F.R. Jones introduced the theory of index for subfactors of a II\(_1\) factor in [10]. An inclusion \( \mathcal{N} \subset \mathcal{M} \) of II\(_1\) factors is said to be of finite index if the relative dimension (or coupling constant)

\[\dim_{\mathcal{M}}(L^2(\mathcal{M}, \tau)) = t\]

is finite (see [11]). A sequence of projections arises in this circumstance, by means of Jones’ basic construction. Denote by \( \tau \) the normalized trace of \( \mathcal{M} \) (and of the subsequent finite extensions which will be considered). Let \( e_{\mathcal{N}} \) be the orthogonal projection of \( L^2(\mathcal{M}, \tau) \) onto \( L^2(\mathcal{N}, \tau) \). This projection, restricted to \( \mathcal{M} \subset L^2(\mathcal{M}, \tau) \), induces the unique trace invariant conditional expectation \( E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N} \). Jones proved that the von Neumann algebra \( \mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle \) generated in \( B(L^2(\mathcal{M}, \tau)) \) by \( \mathcal{M} \) and \( e_{\mathcal{N}} \) is again a II\(_1\) factor, and that the inclusion \( \mathcal{M} \subset \mathcal{M}_1 \) has finite index, with \( [\mathcal{M}_1 : \mathcal{M}] = [\mathcal{M} : \mathcal{N}] \). Thus, iterating the basic construction, a sequence of orthogonal projections arises: \( e_1 = e_{\mathcal{N}}, e_2 = e_{\mathcal{M}}, \ldots \). We shall be concerned only with the first two. These projections recover the index:

\[\tau(e_{\mathcal{N}}) = \tau(e_{\mathcal{M}}) = t = [\mathcal{M} : \mathcal{N}]^{-1} \]

In particular, they are unitarily equivalent in any factor of the tower of factors enabled by the basic construction, in which both lie. More precisely, Jones proved ([10], Proposition 3.4.1) that

\[e_{\mathcal{N}} e_{\mathcal{M}} = t e_{\mathcal{N}} \quad \text{and} \quad e_{\mathcal{N}}^\perp \wedge e_{\mathcal{M}} = e_{\mathcal{N}} \perp e_{\mathcal{M}}^\perp = 0.\]
It follows that $e_N$ and $e_M$ can be joined by a unique geodesic which lies in $\mathcal{M}_2 = \langle \mathcal{M}, e_N, e_M \rangle$. Thus, the finite index inclusion $\mathcal{N} \subset \mathcal{M}$ gives rise to a unique element $z_{\mathcal{M},\mathcal{N}} \in \mathcal{M}_2$, the exponent of this geodesic:

$$z_{\mathcal{M},\mathcal{N}}^* = -z_{\mathcal{M},\mathcal{N}}, \quad d(e_N, e_M) = \|z_{\mathcal{M},\mathcal{N}}\| \leq \pi/2, \quad z_{\mathcal{M},\mathcal{N}} \text{ is } e_N \text{ and } e_M \text{ co-diagonal},$$

and

$$e_{\mathcal{M},\mathcal{N}} e_N e_{\mathcal{M},\mathcal{N}}^* = e_M.$$

The index $[\mathcal{M} : \mathcal{N}] = t^{-1}$ is related to the geodesic distance between $e_N$ and $e_M$, measured with the usual norm of $\mathcal{M}_2$, or with the $\rho$-norms ($1 \leq \rho < \infty$):

**Theorem 5.1.** With the above notations,

$$d(e_N, e_M) = \|z_{\mathcal{M},\mathcal{N}}\| = \arccos(t^{1/2}) \quad \text{and} \quad d_\rho(e_N, e_M) = \|z_{\mathcal{M},\mathcal{N}}\|_\rho = t^{1/\rho} \arccos(t^{1/2}).$$

**Proof.** The projections $e_N$ and $e_M$ act in $L^2(\mathcal{M}_1, \tau)$. Denote by $e'_N$ and $e'_M$ the generic part of these projections, acting on the Hilbert space $\mathcal{H}' \subset L^2(\mathcal{M}_1, \tau)$. By Halmos’ theorem, there exists an isometric isomorphism between $\mathcal{H}'$ and $\mathcal{L} \times \mathcal{L}$ such that $e'_N$ and $e'_M$ are carried, respectively, onto

$$P_N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_M = \begin{pmatrix} \cos(X) & \cos(X) \sin(X) \\ \cos(X) \sin(X) & \sin^2(X) \end{pmatrix},$$

where $0 \leq X \leq \pi/2$. Note that since the only (non trivial) non generic part of $e_N$ and $e_M$ is $e_N^+ \wedge e_M^+$, on which both $e_N$ and $e_M$ act trivially, we have that $e'_Ne'_N = e_N e_M e_N = te_N$. Therefore,

$$t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = t P_N = P_N P_M P_N = \begin{pmatrix} \cos(X) & \cos(X) \sin(X) \\ \cos(X) \sin(X) & \sin^2(X) \end{pmatrix},$$

i.e., $\cos(X) = t^{1/2} 1_{\mathcal{L}}$, and therefore $X$ is a scalar multiple of the identity in $\mathcal{L}$: $X = \arccos(t^{1/2}) 1_{\mathcal{L}}$. The unique exponent $z = z_{\mathcal{M},\mathcal{N}}$ of the geodesic joining $e_N$ and $e_M$, is zero on the non generic part $e_N^+ \wedge e_M^+$, and in the generic part is related (via the Halmos’ isomorphism) to the operator

$$Z = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}.$$

Then $z^*z$ corresponds to

$$Z^* = \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix} = (\arccos(t^{1/2}))^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $z^*z = (\arccos(t^{1/2}))^2 e_N$. The geodesic distance induced by the usual operator norm is

$$d(e_N, e_M) = \|z\| = \|z^*z\|^{1/2} = \arccos(t^{1/2}),$$

and the one induced by the $\rho$ norm is

$$d_\rho(e_N, e_M) = \|z\|_\rho = \arccos(t^{1/2})(\tau(e_N))^{1/\rho} = t^{1/\rho} \arccos(t^{1/2}).$$

□
Next, we consider the case of two projections arising from two subfactors $N_0, N_1 \subset M$. These give rise to two orthogonal projections $e_0, e_1$ in $B(L^2(M, \tau))$. We make the assumption that both inclusions have finite index: $[M : N_0], [M : N_1] < \infty$. If both projections lie in the same II$_1$ factor $M_0 \supset M$ (with trace $\tau$ extending the trace of $M$), then a necessary and sufficient condition for the existence of a geodesic joining $e_0$ and $e_1$ is $\tau(e_0) = \tau(e_1)$.

**Lemma 5.2.** Let $N_0, N_1 \subset M$ be finite index subfactors. Then there exists a II$_1$ factor $M_0$ such that $M \subset M_0$ has finite index, and $e_0, e_1 \in M_0$.

**Proof.** Let $E_i : M \to N_i$, $i = 0, 1$, be the unique trace preserving conditional expectations, giving rise to the orthogonal projections $e_0, e_1$. Let $\mathcal{M}_1 = \langle M, e_0 \rangle$, and $F : \mathcal{M}_1 \to M$ the corresponding expectation. Note that $F_1 = E_1 F : \mathcal{M}_1 \to N_1$ is a conditional expectation, which is trace invariant (for the trace of $\mathcal{M}_1$), and which corresponds to the finite index inclusion $N_1 \subset \mathcal{M}_1$: $[\mathcal{M}_1 : N_1] = \langle [M : \mathcal{M}], [M, N_1] \rangle$. Let $f_1$ be the orthogonal projection in $B(L^2(\mathcal{M}_1))$ induced by this inclusion, and $M_0 = \langle \mathcal{M}_1, f_1 \rangle$, which is a finite factor with $[M_0 : \mathcal{M}_1] = \langle [M : \mathcal{M}], [M, N_1] \rangle < \infty$.

We claim that $f_1 = e_1$. Denote by $[x]$ the element $x \in \mathcal{M}_1$ regarded as a vector in $L^2(\mathcal{M}_1)$. Then, if $x \in \mathcal{M}$,

$$f_1([x]) = [F_1(x)] = [E_1(F(x))] = [E_1(x)] = e_1([x]).$$

If $x \in \mathcal{M}_1$, then $f_1(\xi) = e_1(\xi) = 0$. \hfill $\Box$

**Remark 5.3.** Note that $M_0 \subset \{M, e_0, e_1\}'' \subset B(L^2(\mathcal{M}_1))$. However, $M$, $e_0$ and $e_1$ act also on $L^2(M)$. Thus, the algebra $\{M, e_0, e_1\}'' \subset B(L^2(\mathcal{M}_1))$ is *-isomorphic (by means of a normal isomorphism, given by restriction to $L^2(\mathcal{M}_1)$) to the von Neumann II$_1$ factor $M_{1,2} = \langle M, e_0, e_1 \rangle < B(L^2(\mathcal{M}_1))$.

**Proposition 5.4.** Let $N_0, N_1 \subset M$ be finite index subfactors. Then there exists a geodesic joining $e_0$ and $e_1$ if and only if $[M : N_0] = [M : N_1]$.

**Proof.** $[M : N_0] = [M : N_1] = t^{-1}$ if and only if $\tau_{M_0}(e_0) = \tau_{M_0}(e_1) = t$. \hfill $\Box$

With the same notations as above, we have the following:

**Lemma 5.5.** Suppose that $\|e_0 - e_1\| < 1$, and let $\delta(t) = e^{itz}e_0e^{-itz}$, $t \in [0, 1]$, be the unique geodesic of $\mathcal{P}_{M_0}$ joining $\delta(0) = e_0$ and $\delta(1) = e_1$. Then

$$E_t = \delta_t|_M : M \to N_t := e^{itz}N_0e^{-itz} \subset M$$

is a pointwise smooth path of conditional expectations joining $E_{N_0}$ and $E_{N_1}$ (i.e., the map $[0, 1] \ni t \mapsto E_t(a) \in M$ is $C^1$ for all $a \in M$).

**Proof.** We shall use repeatedly the following argument. Suppose that $m \in M_0$ is normal, and satisfies that $m[M] \subset [M]$, i.e., $m$ as an operator acting in $L^2(M)$, leaves the dense linear manifold $[M] = \{[x] : x \in M\}$ invariant, and let $f$ be a continuous function in the spectrum $\sigma(m)$ of $m$. Then $f(m)$ also leaves $[M]$ invariant. Indeed, let $p_k(z, \bar{z})$ be polynomials in $z$ and $\bar{z}$ which converge uniformly to $f(z)$ in $\sigma(m)$. Clearly $p_k(m, m^*)$ leave $[M]$ invariant. Let $x \in M$.

Then, if we denote by $L_x$ the element $x$ acting by left multiplication on $L^2(M)$,

$$\|p_k(m,m^*)(x) - p_j(m,m^*)(x)\|_M = \|p_k(m,m^*) - p_j(m,m^*)\|L_x\|B(L^2(M))$$
for all leaves \([M] \parallel \delta\). Therefore, \(e\) again using the argument at the beginning of this proof, it follows that

\[
e^z = (2e_0 - 1)(e_0 + e_1 - 1)\}
\]

leaves \([M]\) invariant. On the other hand, as remarked before, the fact that \(\|e_0 - e_1\| < 1\) also implies that \(\|e^z - 1\| < \sqrt{2} < 2\) (or equivalently, that \(\|z\| < \pi/2\)). It follows that there is a continuous logarithm defined in the spectrum of \(e^z\), \(\operatorname{arg} : \sigma(e^z) \to (-\pi/2, \pi/2)\). Therefore, again using the argument at the beginning of this proof, it follows that \(z\) leaves \([M]\) invariant. Therefore, \(e^{tz}\) leave \([M]\) invariant for \(t \in [0, 1]\). It follows that \(\delta(t)\), restricted to \(M\), induce the linear mappings

\[\delta(t)|_M = e^{tz}e_0e^{-tz}|_M : M \to M.\]

The range of \(\delta(t)|_M\) is \(e^{tz}L^2(N_0)e^{-tz} \cap M = e^{tz}N_0e^{-tz} = N_t\). Clearly these maps are idempotents, \(*\)-preserving, normal, and contractive for the norm of \(M\). Thus, by the theorem of Tomiyama [13], they are normal conditional expectations, interpolating between \(E_0\) and \(E_1\). The fact that the path is strongly smooth is also clear.

**Remark 5.6.** Let us recall Theorem 2.6 of [2]:

Let \(A\) be a unital \(C^*\)-algebra and suppose that for \(t \in [0, 1]\) one has subalgebras \(1 \in B_t \subset A\) and conditional expectations \(E_t : A \to B_t\). Assume that for each \(a \in A\), the map \(t \mapsto E_t(a) \in A\) is continuously differentiable. Denote by \(dE_t : A \to A\) the derivative of \(E_t\): \(dE_t(a) = \frac{d}{dt}E_t(a)\). For each fixed \(t\), the operator \(dE_t : A \to A\) is bounded. Consider the differential equation, for \(a \in A\),

\[
\begin{align*}
\dot{\alpha}(t) &= [dE_t, E_t](\alpha(t)) \\
\alpha(0) &= a
\end{align*}
\]

We call this equation the **parallel transport equation**. Denote by \(\Gamma_t\) the propagator of this equation, i.e., the map \(\Gamma_t : A \to A\) given by the solutions: \(\Gamma_t(a) = \alpha(t)\) with \(\alpha(0) = a\). Then

- \(\Gamma_tE_0\Gamma_{-t} = E_t\), and
- \(\Gamma_t|_{B_0} : B_0 \to B_t\) is a \(C^*\)-algebra isomorphism.

**Corollary 5.7.** Let \(N_0, N_1 \subset M\) be subfactors with \(\|e_0 - e_1\| < 1\). Then the exponentials \(e^{tz}\) which induce the unique geodesic \(\delta(t) = e^{tz}e_0e^{-tz}\) joining \(e_0\) and \(e_1\) in \(PM_0\) (\(M_0 \triangleright \triangleright M, e_0, e_1 >\)) satisfy that

\[
\Gamma_t|_{N_0} : N_0 \to N_t = e^{tz}N_0e^{-tz}
\]

are normal \(*\)-isomorphisms. In particular, \(N_0\) and \(N_1\) are isomorphic.

**Proof.** It is straightforward to verify that \(e^{tz}\) are the propagators \(\Gamma_t\) of equation (2) in this case: since \(E_t = \delta(t)|_M = e^{tz}|_M E_0e^{-tz}|_M\), we have that (the maps below are restricted to \(M\))

\[
dE_t = ze^{tz}E_0e^{-tz} - e^{tz}E_0e^{-tz}z
\]
and after straightforward computations

\[ dE_t, E_t \] = \left[ z e^{t\varphi} E_0 e^{-t\varphi} - 2 e^{t\varphi} E_0 z E_0 e^{-t\varphi} + e^{t\varphi} E_0 e^{-t\varphi} z \right].

Since \( z \) is \( e_0 \)-co-diagonal, it maps \( L^2(N) \) into \( L^2(N) \perp \), and therefore \( E_0 z E_0 = 0 \). Thus,

\[ dE_t, E_t \] = \left[ z e^{t\varphi} E_0 + e^{t\varphi} E_0 z = e^{t\varphi}(z E_0 + E_0 z) \right].

The fact that the element \( z \) is \( e_0 \)-co-diagonal, also means that

\[ z = e_0 z (1 - e_0) + (1 - e_0) z e_0 = e_0 z - 2 e_0 z e_0 + z e_0 = z e_0 + e_0 z. \]

Restricted to \( \mathcal{M} \) gives \( z E_0 + E_0 z = z \). Therefore, for \( x \in \mathcal{M} \),

\[ dE_t, E_t \] = \left[ e^{t\varphi}(z E_0 + E_0 z) z x = e^{t\varphi} z x = (e^{t\varphi} x). \right.

That is, \( \alpha(t) = e^{t\varphi} x \) is the solution of (2) with \( \alpha(0) = x \), i.e. \( \Gamma_t = e^{t\varphi}|_{\mathcal{M}} \) is the propagator of this equation, and the proof follows using Theorem 2.6 of [2]

**Remark 5.8.** Suppose that \( e_0, e_1 \) as above satisfy the condition \( e_0 \wedge e_1 = 0 = e_0^+ \wedge e_1 \) (weaker that \( \|e_0 - e_1\| < 1 \)), then there exists a unique geodesic \( \delta(t) = e^{t\varphi} e_0 e^{-t\varphi} \) joining \( e_0 \) and \( e_1 \). We would like to know if also in this case, the propagators \( \Gamma_t \) of the parallel transport equation induce as in the above case, a curve of automorphisms. Following the same argument as above, it amounts to knowing if the projections \( \delta(t) \) induce conditional expectations onto the intermediate algebras \( e^{t\varphi} N_0 e^{-t\varphi}. \)

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