RATIONALLY ISOTROPIC EXCEPTIONAL PROJECTIVE HOMOGENEOUS VARIETIES ARE LOCALLY ISOTROPIC

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Abstract. Assume that $R$ is a local regular ring containing an infinite perfect field, or that $R$ is the local ring of a point on a smooth scheme over an infinite field. Let $K$ be the field of fractions of $R$ and $\text{char}(K) \neq 2$. Let $X$ be an exceptional projective homogeneous scheme over $R$. We prove that in most cases the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

1. Introduction

We prove the following theorem.

Theorem 1. Let $R$ be local regular ring containing an infinite perfect field, or the local ring of a point on a smooth scheme over an infinite field, $K$ be the fraction field of $R$ and $\text{char}(K) \neq 2$. Let $G$ be a split simple group of exceptional type (that is, $E_6$, $E_7$, $E_8$, $F_4$, or $G_2$), $P$ be a parabolic subgroup of $G$, $\xi$ be a class from $H^1(R, G)$, and $X = \xi(G/P)$ be the corresponding homogeneous space. Assume that $P \neq P_7$, $P_8$, $P_{7,8}$ in case $G = E_8$, $P \neq P_7$ in case $G = E_7$, and $P \neq P_1$ in case $G = E_7^{ad}$. Then the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

2. Purity of some $H^1$ functors

Let $A$ be a commutative noetherian domain of finite Krull dimension with a fraction field $F$. We say that a functor $F$ from the category of commutative $A$-algebras to the category of sets satisfies purity for $A$ if we have

$$\text{Im} [F(A) \to F(F)] = \bigcap_{ht p=1} \text{Im} [F(A_p) \to F(F)].$$

If $\mathcal{H}$ is an étale group sheaf we write $H^1(-, \mathcal{H})$ for $H^1_{\text{ét}}(-, \mathcal{H})$ below through the text. The following theorem is proven in the characteristic zero case [Pa2, Theorem 4.0.3]. We extend it here to an arbitrary characteristic case.

Theorem 2. Let $R$ be the regular local ring from Theorem 1 and $k \subset R$ be the subfield of $R$ mentioned in that Theorem. Let

$$(*) \quad 1 \to Z \to G \to G' \to 1$$

be an exact sequence of algebraic $k$-groups, where $G$ and $G'$ are reductive and $Z$ is a closed central subgroup scheme in $G$. If the functor $H^1(-, G')$ satisfies purity for $R$ then the functor $H^1(-, G)$ satisfies purity for $R$ as well.

Lemma 1. Consider the category of $R$-algebras. The functor

$$R' \mapsto F(R') = H^1_{\text{fppf}}(R', Z)/\text{Im} (\delta_{R'}),$$

where $\delta$ is the connecting homomorphism associated to sequence $\text{(*)}$, satisfies purity for $R$.

Proof. Similar to the proof of [Pa2, Theorem 12.0.36]\hfill \Box

Lemma 2. The map

$$H^2_{\text{fppf}}(R, Z) \to H^2_{\text{fppf}}(K, Z)$$

is injective.

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Proof. Similar to the proof of [Pa] Theorem 13.0.38.

Proof of Theorem 2 Reproduce the diagram chase from the proof of [Pa2] Theorem 4.0.3. For that consider the commutative diagram

\[
\begin{array}{cccccc}
\{1\} & \longrightarrow & \mathcal{F}(K) & \overset{\delta_K}{\longrightarrow} & H^1(K, G) & \overset{\pi_K}{\longrightarrow} & H^1(K, G') \\
& & \alpha & \downarrow & & \downarrow & \gamma \\
\{1\} & \longrightarrow & \mathcal{F}(R) & \overset{\delta}{\longrightarrow} & H^1(R, G) & \overset{\pi}{\longrightarrow} & H^1(R, G') \\
& & & & \Delta & \downarrow & \Delta \\
& & & & H^2_{fppf}(K, Z) & \overset{\alpha_1}{\longrightarrow} & H^2_{fppf}(R, Z)
\end{array}
\]

Let \( \xi \in H^1(K, G) \) be an \( R \)-unramified class and let \( \tilde{\xi} = \pi_K(\xi) \). Clearly, \( \tilde{\xi} \in H^1(K, G') \) is \( R \)-unramified. Thus there exists an element \( \tilde{\xi}' \in H^1(R, G') \) such that \( \tilde{\xi}' = \tilde{\xi} \). The map \( \alpha_1 \) is injective by Lemma 2. One has \( \Delta(\tilde{\xi}') = 0 \), since \( \Delta_K(\tilde{\xi}) = 0 \). Thus there exists \( \xi' \in H^1(R, G) \) such that \( \pi(\xi') = \xi' \).

The group \( \mathcal{F}(K) \) acts on the set \( H^1(K, G) \), since \( Z \) is a central subgroup of the group \( G \). If \( a \in \mathcal{F}(K) \) and \( \xi \in H^1(K, G) \), then we will write \( a \cdot \xi \) for the resulting element in \( H^1(K, G) \).

Further, for any two elements \( \zeta_1, \zeta_2 \in H^1(K, G) \), having the same images in \( H^1(K, G) \) there exists a unique element \( a \in \mathcal{F}(K) \) such that \( a \cdot \zeta_1 = \zeta_2 \). These remarks hold for the cohomology of the ring \( R \), and for the cohomology of the rings \( R_p \), where \( p \) runs over all height 1 prime ideals of \( R \). Since the images of \( \xi_K \) and \( \xi \) coincide in \( H^1(K, G') \), there exists a unique element \( a \in \mathcal{F}(K) \) such that \( a \cdot \xi_K = \xi \) in \( H^1(K, G) \).

Lemma 3. The above constructed element \( a \in \mathcal{F}(K) \) is an \( R \)-unramified.

Assuming Lemma 3 complete the proof of the Theorem. By Lemma 1 the functor \( \mathcal{F} \) satisfies the purity for regular local rings containing the field \( k \). Thus there exists an element \( a' \in \mathcal{F}(R) \) with \( a'_K = a \). It’s clear that \( \xi'' = a' \cdot \xi \in H^1(R, G) \) satisfies the equality \( \xi'' = \xi \). It remains to prove Lemma 3.

For that consider a height 1 prime ideal \( p \) in \( R \). Since \( \xi \) is an \( R \)-unramified there exists its lift up to an element \( \tilde{\xi} \) in \( H^1(R_p, G) \). Set \( \xi'_p \) to be the image of the \( \xi' \) in \( H^1(R_p, G) \). The classes \( \pi_p(\xi), \pi_p(\xi'_p) \in H^1(R_p, G') \) being regarded in \( H^1(K, G') \) coincide.

The map

\[ H^1(R_p, G') \to H^1(K, G') \]

is injective by Lemma 3 formulated and proven below. Thus \( \pi_p(\tilde{\xi}) = \pi_p(\xi'_p) \). Therefore there exists a unique class \( a_p \in \mathcal{F}(R_p) \) such that \( a_p \cdot \xi'_p = \xi \in H^1(R_p, G) \). So, \( a_{p, K} \cdot \xi_K = \xi \in H^1(K, G) \) and \( a_{p, K} \cdot \xi_K = \xi = a \cdot \xi_K \). Thus \( a = a_{p, K} \). Lemma 3 is proven.

To finish the proof of the theorem it remains to prove the following

Lemma 4. Let \( H \) be a reductive group scheme over a discrete valuation ring \( A \). Assume that for each \( A \)-algebra \( \Omega \) with an algebraically closed field \( \Omega \) the algebraic group \( H_\Omega \) is connected. Let \( K \) be the fraction field of \( A \). Then the map

\[ H^1(R, H) \to H^1(K, H) \]

is injective.

Proof. Let \( \xi_0, \xi_1 \in H^1(A, H) \). Let \( H_0 \) be a principal homogeneous \( H \)-bundle representing the class \( \xi_0 \). Let \( H_0 \) be the inner form of the group scheme \( H \), corresponding to \( H_0 \). Let \( X = \text{Spec}(A) \). For each \( X \)-scheme \( S \) there is a well-known bijection \( \phi_S : H^1(S, H) \to H^1(S, H_0) \) of non-pointed sets. That bijection takes the principal homogeneous \( H \)-bundle \( H_0 \times_X S \) to the trivial principal homogeneous \( H_0 \)-bundle \( H_0 \times_X S \). That bijection respects to morphisms of \( X \)-schemes.

Assume that \( \xi_{0, K} = \xi_{1, K} \). Then one has \( * = \phi_K(\xi_{0, K}) = \phi_K(\xi_{1, K}) \in H^1(K, H_0) \). The kernel of the map \( H^1(A, H_0) \to H^1(K, H_0) \) is trivial by Nisnevich theorem [N]. Thus \( \phi_A(\xi_1) = * = \).

□
Theorem 3. The functor $H^1(-, \text{PGL}_n)$ satisfies purity.

Proof. Let $\xi \in H^1(-, \text{PGL}_n)$ be an $R$-unramified element. Let $\delta : H^1(-, \text{PGL}_n) \to H^2(-, G_m)$ be the boundary map corresponding to the short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to \text{GL}_n \to \text{PGL}_n \to 1.$$ 

Let $D_\xi$ be a central simple $K$-algebra of degree $n$ corresponding to $\xi$. If $D_\xi \cong M_l(D')$ for a skew-field $D'$, then there exists $\xi' \in H^1(K, \text{PGL}_{n'})$ such that $D' = D_{\xi'}$. Then $\delta(\xi') = [D'] = [D] = \delta(\xi)$. Replacing $\xi$ by $\xi'$, we may assume that $D := D_\xi$ is a central skew-field over $K$ of degree $n$ and the class $[D]$ is $R$-unramified.

Clearly, the class $\delta(\xi)$ is $R$-unramified. Thus there exists an Azumaya $R$-algebra $A$ and an integer $d$ such that $A_K = M_d(D)$.

There exists a projective left $A$-module $P$ of finite rank such that each projective left $A$-module $Q$ of finite rank is isomorphic to the left $A$-module $P^m$ for an appropriative integer $m$ (see [?, Cor.2]). In particular, two projective left $A$-modules of finite rank are isomorphic if they have the same rank as $R$-modules. One has an isomorphism $A \cong P^s$ of left $A$-modules for an integer $s$. Thus one has $R$-algebra isomorphisms $A \cong \text{End}_A(P^s) \cong M_s(\text{End}_A(P))$. Set $B = \text{End}_A(P)$.

Observe, that $B_K = \text{End}_{A_K}(P_K)$, since $P$ is a finitely generated projective left $A$-module.

The class $[P_K]$ is a free generator of the group $K_0(A_K) = K_0(M_d(D)) \cong \mathbb{Z}$, since $[P]$ is a free generator of the group $K_0(A)$ and $K_0(A) = K_0(A_K)$. The $P_K$ is a simple $A_K$-module, since $[P_K]$ is a free generator of $K_0(A_K)$. Thus $\text{End}_{A_K}(P_K) = B_K$ is a skew-field.

We claim that the $K$-algebras $B_K$ and $D$ are isomorphic. In fact, $A_K = M_s(B_K)$ for an integer $r$, since $P_K$ is a simple $A_K$-module. From the other side $A_K = M_d(D)$. As $D$, so $B_K$ are skew-fields. Thus $r = d$ and $D$ is isomorphic to $B_K$ as $K$-algebras.

We claim further that $B$ is an Azumaya $R$-algebra. That claim is local with respect to the étale topology on $\text{Spec}(R)$. Thus it suffices to check the claim assuming that $\text{Spec}(R)$ is stickly henselian local ring. In that case $A = M_l(R)$ and $P = (R^l)^m$ as an $M_l(R)$-module. Thus $B = \text{End}_A(P) = M_m(R)$, which proves the claim.

Since $B_K$ is isomorphic to $D$, one has $m = n$. So, $B$ is an Azumaya $R$-algebra, and the $K$-algebra $B_K$ is isomorphic to $D$. Let $\zeta \in H^1(R, \text{PGL}_n)$ be class corresponding to $B$. Then $\zeta_K = \xi$, since $\delta(\zeta)_K = [B_K] = [D] = \delta(\xi) \in H^2(K, \mathbb{G}_m)$.

\[ \square \]

Theorem 4. The functor $H^1(-, \text{Sim}_n)$ satisfies purity.

Proof. Let $\varphi$ be a quadratic form over $K$ whose similarity class represents $\xi \in H^1(K, \text{Sim}_n)$. Diagonalizing $\varphi$ we may assume that $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ for certain non-zero elements $f_1, f_2, \ldots, f_n \in K$. For each $i$ write $f_i$ in the form $f_i = \frac{g_i}{h_i}$ with $g_i, h_i \in R$ and $h_i \neq 0$.

There are only finitely many height one prime ideals $q$ in $R$ such that there exists $0 \leq i \leq n$ with $f_i$ not in $R_q$. Let $q_1, q_2, \ldots, q_s$ be all height one prime ideals in $R$ with that property and let $q_i \neq q_j$ for $i \neq j$.

For all other height one prime ideals $p$ in $R$ each $f_i$ belongs to the group of units $R_p^\times$ of the ring $R_p$.

If $p$ is a height one prime ideal of $R$ which is not from the list $q_1, q_2, \ldots, q_s$, then $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ may be regarded as a quadratic space over $R_p$. We will write $p \varphi$ for that quadratic space over $R_p$. Clearly, one has $(p \varphi) \otimes_{R_p} K = \varphi$ as quadratic spaces over $K$.

For each $j \in \{1, 2, \ldots, s\}$ choose and fix a quadratic space $j \varphi$ over $R_{q_j}$ and a non-zero element $\lambda_j \in K$ such that the quadratic spaces $(j \varphi) \otimes_{R_{q_j}} K$ and $\lambda_j \cdot \varphi$ are isomorphic over $K$. The ring $R$ is factorial since it is regular and local. Thus for each $j \in \{1, 2, \ldots, s\}$ we may choose an element $\pi_j \in R$ such that firstly $\pi_j$ generates the only maximal ideal in $R_{q_j}$ and secondly $\pi_j$ is an invertible element in $R_n$ for each height one prime ideal $n$ different from the ideal $q_j$. 

\[ \square \]
Let \( v_j : K^\times \to \mathbb{Z} \) be the discrete valuation of \( K \) corresponding to the prime ideal \( q_j \). Set \( \lambda = \prod_{i=1}^s v_j(\lambda_j) \) and \( \varphi_{\text{new}} = \lambda \cdot \varphi \).

Claim. The quadratic space \( \varphi_{\text{new}} \) is \( R \)-unramified. In fact, if a high one prime ideal \( p \) is different from each of \( q_j \)'s, then \( v_p(\lambda) = 0 \). Thus, \( \lambda \in R_p^\times \). In that case \( \lambda \cdot (p \varphi) \) is a quadratic space over \( R_p \) and moreover one has isomorphisms of quadratic spaces \( (\lambda \cdot (p \varphi)) \otimes_{R_p} K = \lambda \cdot \varphi = \varphi_{\text{new}} \). If we take one of \( q_j \)'s, then \( \frac{\lambda}{\lambda_j} \in R_{q_j}^\times \). Thus, \( \frac{\lambda}{\lambda_j} \cdot (j \varphi) \) is a quadratic space over \( R_{q_j} \). Moreover, one has
\[
\frac{\lambda}{\lambda_j} \cdot (j \varphi) \otimes_{R_{q_j}} K = \frac{\lambda}{\lambda_j} \cdot \lambda_j \cdot \varphi = \varphi_{\text{new}}.
\]
The Claim is proven.

By [PP Corollary 3.1] there exists a quadratic space \( \tilde{\varphi} \) over \( R \) such that the quadratic spaces \( \tilde{\varphi} \otimes_{R} K \) and \( \varphi_{\text{new}} \) are isomorphic over \( K \). This shows that the similarity classes of the quadratic spaces \( \tilde{\varphi} \otimes_{R} K \) and \( \varphi \) coincide. The theorem is proven.

Theorem 5. The functor \( H^1(-, \text{Sim}^+_{\mathcal{R}}) \) satisfies purity.

Proof. Consider an element \( \xi \in H^1(K, \text{Sim}^+_{\mathcal{R}}) \) such that for any \( p \) of height 1 \( \xi \) comes from \( \xi_p \in H^1(R_p, \text{Sim}^+_{\mathcal{R}}) \). Then the image of \( \xi \) in \( H^1(K, \text{Sim}^+_{\mathcal{R}}) \) by Theorem 4 comes from some \( \zeta \in H^1(R, \text{Sim}^+_{\mathcal{R}}) \). We have a short exact sequence
\[
1 \to \text{Sim}^+_{\mathcal{R}} \to \text{Sim}_{\mathcal{R}} \to \mu_2 \to 1,
\]
and \( R^\times/(R^\times)^2 \) injects into \( K^\times/(K^\times)^2 \). Thus the element \( \zeta \) comes actually from some \( \zeta' \in H^1(R, \text{Sim}_{\mathcal{R}}) \). It remains to show that the map
\[
H^1(K, \text{Sim}^+_{\mathcal{R}}) \to H^1(K, \text{Sim}_{\mathcal{R}})
\]
is injective, or, by twisting, that the map
\[
H^1(K, \text{Sim}^+_{\mathcal{R}}(q)) \to H^1(K, \text{Sim}(q))
\]
has the trivial kernel. The latter follows from the fact that the map
\[
\text{Sim}(q)(K) \to \mu_2(K)
\]
is surjective (indeed, any reflection goes to \( -1 \in \mu_2(K) \)).

3. Proof of the Main Theorem

Let \( \xi \) be a class from \( H^1(R, G) \), and \( X = \xi(G/P) \) be the corresponding homogeneous space. Denote by \( L \) a Levi subgroup of \( P \).

Lemma 5. Let \( L \) modulo its center be isomorphic to \( \text{PGO}_{2m}^\times \) (resp., \( \text{PGO}_{2m+1}^\times \times \text{PGL}_2 \)). Denote by \( \Psi \) the closed subset in \( X^\times(T) \) of type \( D_m \) (resp. \( B_m \) or \( D_m + A_1 \)) corresponding to \( L \), \( T \) stands for a maximal split torus in \( L \). Assume that there is an element \( \lambda \in X^\times(T) \) such that \( \Psi \) and \( \lambda \) generate a closed subset of type \( D_{m+1} \) (resp. \( B_{m+1} \) or \( D_{m+1} + A_1 \)), and \( \Psi \) forms the standard subsystem of type \( D_m \) (resp. \( B_m \) or \( D_m + A_1 \)) therein. Then there is a surjective map \( L \to \text{Sim}_{2m}^+(\text{resp. } L \to \text{Sim}_{2m+1}^+ \times \text{PGL}_2) \) whose kernel is a central closed subgroup scheme in \( L \). In particular, the functor \( H^1(-, L) \) satisfies purity.

Proof. It is easy to check that \( \text{Sim}_{2m}^+ \) (resp. \( \text{Sim}_{2m+1}^+ \)) is a Levi subgroup in the split adjoint group of type \( D_{m+1} \) (resp. \( B_{m+1} \)). Now the first claim follows from [SGA] Exp. XXIII, Thm. 4.1, and the rest follows from Theorem 5 and Theorem 3.

Lemma 6. For any \( R \)-algebra \( S \) the map
\[
H^1(S, L) \to H^1(S, G)
\]
is injective. Moreover, \( X(S) \neq \emptyset \) if and only if \( \xi_S \) comes from \( H^1(S, L) \).

Proof. See [SGA] Exp. XXVI, Cor. 5.10. \( \square \)
Lemma 7. Assume that the functor $H^{1}(-, L)$ satisfies purity. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

Proof. By Lemma 6, $\xi_K$ comes from some $\zeta \in H^{1}(K, L)$, which is uniquely determined. Since $X$ is smooth projective, for any prime ideal $p$ of height 1 we have $X(R_p) \neq \emptyset$. By Lemma 5, $\xi_{R_p}$ comes from some $\zeta_p \in H^{1}(R_p, L)$. Now $(\zeta_p)_K = \zeta$, and so by the purity assumption there is $\zeta' \in H^{1}(R, L)$ such that $\zeta'_K = \zeta$.

Set $\xi'$ to be the image of $\zeta'$ in $H^{1}(R, G)$. We claim that $\xi' = \xi$. To prove this recall that by the construction $\xi'_K = \xi_K$. Further, there are natural in $R$-algebras bijections $\alpha_S : H^{1}(S, G) \to H^{1}(R, \xi G)$, which takes the $\xi'$ to the distinguished element $*_{R} \in H^{1}(R, \xi G)$. The $R$-group scheme $\xi G$ is isotropic and one has equalities

$$(\alpha_R(\xi))_K = \alpha_K(\xi_K) = \alpha_K(\xi'_K) = *_{K} \in H^{1}(K, \xi G).$$

Thus by [Pa, Theorem 1.0.1] one has

$$\alpha_R(\xi) = s' = \alpha_R(\xi') \in H^{1}(R, \xi G).$$

The $\alpha_R$ is a bijection, whence

$$\xi = \xi' \in H^{1}(R, G).$$

Lemma is proved. \hfill \Box

Lemma 8. Let $Q \leq P$ be another parabolic subgroup, $Y = \xi(G/Q)$. Assume that $X(K) \neq \emptyset$ implies $Y(K) \neq \emptyset$, and $Y(K) \neq \emptyset$ implies $Y(R) \neq \emptyset$. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

Proof. Indeed, there is a map $Y \to X$, so $Y(R) \neq \emptyset$ implies $X(R) \neq \emptyset$. \hfill \Box

Proof of Theorem 7. By Lemma 8 we may assume that $P$ corresponds to an item from the list of Tits [T, Table II]. We show case by case that $H^{1}(-, L)$ satisfies purity, hence we are in the situation of Lemma 7.

If $P = B$ is the Borel subgroup, obviously $H^{1}(-, L) = 1$. In the case of index $E^{9}_{7,4}$ (resp. $E^{16}_{6,2}$) $L$ modulo its center is isomorphic to $PGL_{2} \times PGL_{2} \times PGL_{2}$ (resp. $PGL_{3} \times PGL_{3}$), and we may apply Theorem 2 and Theorem 3. In the all other cases we provide an element $\lambda \in X^{*}(T)$ such that the assumption of Lemma 5 holds ($\bar{\alpha}$ stands for the maximal root, enumeration follows [B]). The indices $E^{78}_{7,1}, E^{133}_{8,1}$ and $E^{78}_{8,2}$ are not in the list below since in those cases the $L$ does not belong to one of the type $D_{m}, B_{m}, D_{m} \times A_{1}$. The index $E^{66}_{7,1}$ is not in the list below since in that case we need a weight $\lambda$ which is not in the root lattice. So, the indices $E^{78}_{7,1}, E^{133}_{8,1}, E^{78}_{8,2}$ and $E^{66}_{7,1}$ are the exceptions in the statement of the Theorem.

| Index  | $\lambda$ |
|--------|-----------|
| $E^{28}_{6,2}$ | $\alpha_1$ |
| $E^{66}_{7,1}$ | $-\omega_7$ |
| $E^{133}_{7,2}$ | $-\bar{\alpha}$ |
| $E^{78}_{8,1}$ | $\alpha_1$ |
| $E^{78}_{8,2}$ | $\alpha_8$ |
| $E^{78}_{8,3}$ | $\alpha_1$ |
| $F^{4,1}_{7,1}$ | $-\bar{\alpha}$ |

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