Modified gravity with $\ln R$ terms and cosmic acceleration

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Abstract

The modified gravity with $\ln R$ or $R^{-n} (\ln R)^m$ terms which grow at small curvature is discussed. It is shown that such a model which has well-defined newtonian limit may eliminate the need for dark energy and may provide the current cosmic acceleration. It is demonstrated that $R^2$ terms are important not only for early time inflation but also to avoid the instabilities and the linear growth of the gravitational force. It is very interesting that the condition of no linear growth for gravitational force coincides with the one for scalar mass in the equivalent scalar-tensor theory to be very large. Thus, modified gravity with $R^2$ term seems to be viable classical theory.

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1. **Introduction.** The astrophysical data from high redshift surveys of type Ia supernovae [1] and from the anisotropy power spectrum of CMB [2] change our image of current universe which seems to be accelerating. The theoretical foundation which is used to construct such universe is also quickly evolving. In particular, the popular explanation of current universe acceleration is based on the dominance of some mysterious, exotic matter called dark energy. There is still no the satisfactory theoretical explanation for the origin of this exotic matter which should appear precisely at current epoch.

Having in mind that more astrophysical data will be available soon, it seems the right time to search for alternative explanation of current cosmic speed-up. In recent papers [3, 4] (see also [5]) it has been suggested the gravitational alternative for dark energy. The key idea is to modify the Einstein action by $1/R$ term which dominates at low curvature. Remarkably, that such terms may be predicted by some compactifications of string/M-theory [6]. Unfortunately, it has been found [7, 8] that $1/R$ model contains unacceptable instabilities which does not appear in the Palatini version of the theory [9, 10, 11]. (It is known that in general case Palatini and standard metric versions are not the same, basically they lead to different physical systems [12]). Moreover, the scalar-tensor theory which seems to be non-realistic [13] in the standard formulation due to solar system observations [14] becomes viable in Palatini form.

Nevertheless, more complicated modification of Einstein gravity of the form $R^2 + R + 1/R$ [15] predicts the unification of the early time inflation and late time cosmic acceleration in the standard, metric formulation. Moreover, the instability found in ref.[7] is significantly suppressed by higher derivative (HD) $R^2$ term or other HD terms like $R^3$. The solar system test for equivalent scalar-tensor theory [13] may be passed because scalar has large mass induced again by HD terms. The consideration of such theory in Palatini form has been recently done [16]. It is shown that account of $R^2$ term makes the theory viable also in Palatini form.

In the present paper we continue the search for the realistic modified gravity which may provide the gravitational alternative for dark energy. As such a model, we suggest to account for the $\ln R$ terms in modified gravity. Such terms are basically induced by quantum effects in curved spacetime. Various versions of such modified gravity may eliminate the need for dark energy and may serve for the unification of the early time inflation and cosmic
acceleration. HD terms again suppress the instability and improve the solar system bounds so that the theory may be viable. The correction to the gravitational coupling constant may be small enough too.

2. General formulation and simplified model. One may start from rather arbitrary function \( F(a, b_\mu, c, g_{\mu\nu}) \) which depends on two scalars \( a, c \), one vector \( b_\mu \) and metric \( g_{\mu\nu} \). The general starting action is:

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} F(R, \partial_\mu R, \Box R, g_{\mu\nu}) \, .
\]

(1)

Here \( R \) is the scalar curvature. Introducing the auxiliary fields \( A \) and \( B \), one may rewrite the action (1) as following:

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \{ B (R - A) + F(A, \partial_\mu A, \Box A, g_{\mu\nu}) \} \, .
\]

(2)

By the variation over \( B \), \( A = R \) follows. Substituting it into (2), the action (1) can be reproduced. Making the variation with respect to \( A \) first, we obtain

\[
B = f(A) \equiv (\partial_a F(a, b_\mu, c, g_{\mu\nu}) - \nabla_\mu (\partial_{b_\mu} F(a, b_\mu, c, g_{\mu\nu})) + \Box (\partial_a F(a, b_\mu, c, g_{\mu\nu}))|_{a=A, b_\mu=\partial A, c=\Box A} \, ,
\]

(3)

which may be solved with respect to \( A \) as \( A = G(B) \). In general, \( A \) is solved non-locally as a function of \( B \). Eliminating \( A \) in (2) by using \( G(B) \), we obtain

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \{ B (R - G(B)) + F(G(B), \partial_\mu G(B), \Box G(B), g_{\mu\nu}) \} \, .
\]

(4)

Instead of \( A \), one may eliminate \( B \) and arrive at the equivalent action.

The scalar field \( \sigma \) may be defined \( \sigma = -\ln B = \ln f(A) \). One can scale the metric by \( g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu} \). Then the action (4) can be rewritten as

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right\}
\]

\[
V(\sigma) \equiv G(e^{-\sigma}) e^\sigma - e^{2\sigma} F(G(e^{-\sigma}), \partial_\mu G(e^{-\sigma}), \Box G(e^{-\sigma})) + g^{\mu\nu} \partial_\mu \sigma \partial_\nu (G(e^{-\sigma}), e^\sigma g_{\mu\nu}) \, .
\]

(5)

It is given in the Einstein frame. On the other hand, the (physical) action (4) is given in the Jordan frame.
As the simple example, we consider the following case:

\[ F(A) = A + \alpha \ln \frac{\Box A}{\mu^4}. \]  

(6)

One finds

\[ e^{-\sigma} = B = 1 + \alpha \Box \left( \frac{1}{\Box A} \right), \]

(7)

which can be solved non-locally with respect to \( A \). Then the complicated expression for the potential follows

\[ V(\sigma) = e^{\sigma \Box^{-1}} \left( \frac{\alpha}{\Box^{-1} (e^{-\sigma} - 1)} \right) - e^{2\sigma} \left[ \Box^{-1} \left( \frac{\alpha}{\Box^{-1} (e^{-\sigma} - 1)} \right) + \alpha \ln \left\{ \frac{\alpha}{\mu^4 \Box^{-1} (e^{-\sigma} - 1)} + g^{\mu \nu} \partial_{\mu} \sigma \partial_{\nu} \left( \Box^{-1} \left( \frac{\alpha}{\mu^4 \Box^{-1} (e^{-\sigma} - 1)} \right) \right) \right\} \right]. \]

(8)

We may consider more general example

\[ F(A) = A + \alpha' \ln \frac{\Box A}{\mu^4} + \alpha \ln \frac{A}{\mu^2} + \beta A^m. \]

(9)

This model is very complicated. If we consider the case \( R \) is almost constant, the second term turns to the (cosmological) constant. The natural starting model looks as follows:

\[ F(A) = A + \alpha' \ln \frac{A}{\mu^2} + \beta A^m. \]

(10)

Furthermore, \( m = 2 \) choice simplifies the model. The generalizations of this model will be considered at final section. The correspondent \( R^2 \) term at large curvature leads to well-known trace anomaly driven (Starobinsky) inflation. Assuming the scalar curvature is constant and the Ricci tensor is also covariantly constant, the equations of motion corresponding to the action (1) are:

\[ 0 = 2F(R) - RF'(R) = f(R) \equiv R + 2\alpha' \ln \frac{R}{\mu^2} - \alpha'. \]

(11)

\footnote{If the action contains only \( R^2 \) term, \( F(R) = \beta R^2 \), Eq.\( (11) \) is trivial, which means the arbitrary constant curvature space is a solution. Adding the Einstein-Hilbert term, \( F(R) = R + \beta R^2 \), the only solution of (11) is \( R = 0 \). Then if we start with large \( R \) solution, which may correspond to the inflation, due to the Einstein-Hilbert term, \( R \) decreases slowly and goes to zero. The inflation will be stopped.}
If $\alpha' > 0$, $f(R)$ is monotonically increasing function and we have $\lim_{R \to 0} f(R) = -\infty$ and $\lim_{R \to +\infty} f(R) = +\infty$. There is one and only one solution of (11). This solution may correspond to the inflation. On the other hand, if $\alpha' < 0$, $\lim_{R \to 0} f(R) = \infty$ and $\lim_{R \to +\infty} f(R) = +\infty$. Since $f'(R) = 1 + \frac{\alpha'}{R}$, the minimum of $f(R)$, where $f'(R) = 0$, is given by $R = -2\alpha'$. If $f(-2\alpha') > 0$, there is no solution of (11). If $f(-2\alpha') = 0$, there is only one solution and if $f(-2\alpha') < 0$, there are two solutions. Since $f(-2\alpha') = -2\alpha' \left(1 - \ln \frac{-2\alpha'}{\mu^2}\right)$, there are two solutions if $\frac{-2\alpha'}{\mu^2} > e$. Since the square root of the curvature $R$ corresponds to the rate of the expansion of the universe, the larger solution in two solutions might correspond to the inflation in the early universe and the smaller one to the present accelerating universe.

For $m = 2$ case, we have

$$e^{-\sigma} = 1 + \frac{\alpha'}{A} + 2\beta A, \quad (12)$$

which can be solved with respect to $A$:

$$A = \frac{-(1 - e^{-\sigma}) \pm \sqrt{(1 - e^{-\sigma})^2 - 8\beta\alpha'}}{4\beta}. \quad (13)$$

In the branch that the r.h.s. in (12) is negative, one may choose

$$e^{-\sigma} = -\left(1 + \frac{\alpha'}{A} + 2\beta A\right), \quad (14)$$

Then instead of (5), we obtain

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ -R + \frac{3}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \tilde{V}(\sigma) \right\}$$

$$\tilde{V}(\sigma) \equiv -G \left(e^{-\sigma}\right) e^{\sigma} - e^{2\sigma} F \left(G \left(e^{-\sigma}\right)\right). \quad (15)$$

As the sign in front of the scalar curvature $R$ is changed, this seems to be the indication to the anti-gravity. Of course, the anti-gravity should not be

\footnote{It is interesting that only $\ln R$ may lead to early time inflation and current acceleration. The AdS cosmologies [17] may be considered as well.}
real but apparent since the physical theory in the real spacetime should be
given by (1) with (10) ($m = 2$ case).

The potential is

$$V(A) = \tilde{V}(A) = \frac{\alpha' \left(1 - \ln \frac{A}{\mu^2}\right) + \beta A^2}{(1 + \frac{\alpha'}{\mu} + 2\beta A)^2}. \quad (16)$$

Then in terms of $\sigma$, $V(\sigma)$ can be expressed as

$$V(\sigma) = e^{2\sigma} \left\{ \alpha' \left(1 - \ln \frac{1 - e^{-\sigma}}{4\beta} \pm \sqrt{(1 - e^{-\sigma})^2 - 8\beta \alpha'} \right) + \beta \left(-1 + e^{-\sigma} \pm \sqrt{(1 - e^{-\sigma})^2 - 8\beta \alpha'} \right)^2 \right\}. \quad (17)$$

$\tilde{V}(A)$ can be expressed in a similar way. If $A$ is small, from (12) it follows $A \sim \pm \alpha' e^{\sigma}$. Here $+(-)$ sign corresponds to the case that the r.h.s. in (12) is positive (negative), which also corresponds to the case that $\alpha' > 0 \ (\alpha' < 0)$.

Then $A \rightarrow 0$ corresponds to $\sigma \rightarrow -\infty$ and $V(\sigma) \ (\tilde{V}(\sigma))$ behaves as

$$V(\sigma) = \tilde{V}(\sigma) \sim -\frac{A^2}{\alpha' \ln \frac{A}{\mu^2}}. \quad (18)$$

or

$$V(\sigma) = -\tilde{V}(\sigma) \sim -\alpha' e^{2\sigma} \sigma. \quad (19)$$

On the other hand, when $A$ is large, we find

$$V(A) \rightarrow \frac{1}{4\beta} \left(1 - \frac{2}{\beta A}\right). \quad (20)$$

Then $V(A)$ is monotonically increasing function for large $A$ if $\beta > 0$ and approaches to a constant $\frac{1}{4\beta}$. In order to find the extrema of $V(A)$ and $\tilde{V}(A)$ one differentiate them with respect to $A$:

$$V'(A) = \tilde{V}'(A) = \frac{(-\frac{\alpha'}{\mu^2} + 2\beta) \left(A - \alpha' + 2\alpha' \ln \frac{A}{\mu^2}\right)}{(1 + \frac{\alpha'}{\mu} + 2\beta A)^3}. \quad (21)$$
\[ V'(A) = 0 \text{ if } A - \alpha' + 2\alpha' \ln \frac{A}{\mu} = 0, \] which corresponds to (11). Furthermore if \( \alpha', \beta > 0 \) there appears a new extremum when \(-\frac{\alpha'}{A} + 2\beta = 0 \) or \( A = A_0 \equiv \sqrt{\frac{\alpha'}{2\beta}} \). As \( A_0 \) is not a solution of (11), this may be artificially caused by the rescaling. In fact, \( A_0 \) corresponds to the point that the mapping (12) is degenerate, that is, \( \frac{d\sigma}{dA} = 0 \) at \( A = A_0 \). Anyway we may discuss the (in)stability of the solution in (11) by using the potential \( V(A) \). When \( \alpha', \beta > 0 \), let the solution corresponding to (11) be \( A_1 \). Combining (18), (20), and (21), one finds that if \( A_0 < A_1 \) (\( A_0 > A_1 \)), the solution of (11) \( (A = A_1) \) is locally stable (instable). On the other hand, if \( \alpha' < 0 \) and \( \beta > 0 \), there is a singularity in \( V(A) \) when \( 1 + \frac{\alpha'}{A} + 2\beta A = 0 \). It corresponds to \( \sigma \to +\infty \). Let the two solutions (if exist) in (11) be \( A_\pm \) (\( A_- < A_+ \)). Combining (18), (20), and (21) again, we find that there are three cases: 1. When \( A_2 < A_- < A_+ \), \( A_+ \) \( (A_-) \) is locally stable (instable). 2. When \( A_- < A_2 < A_+ \), both of \( A_+ \) and \( A_- \) are locally stable. 3. When \( A_- < A_+ < A_2 \), \( A_- \) \( (A_+) \) is locally stable (instable).

When \( \alpha' > 0 \), a typical (conceptual) potential is given in Figure 1. There may be the following scenario of the inflation and of the present accelerating universe: If we start with large curvature \( A = R = R_{\text{initial}} \), the inflation occurs due to the large curvature. Since the potential is slowly increasing, the curvature rolls slowly down the potential and becomes smaller. If the curvature reaches the local minimum \( A = R = R_0 \), the curvature will stay there with non-trivial value \( R = R_0 \neq 0 \), which may correspond to the present acceleration of the universe. As the potential plays a role of the cosmological constant, the ratio \( w \) of the effective pressure with respect to the effective energy density becomes minus unity, \( w = -1 \). As we will show later in (46), by fine-tuning the parameters, the \( \sigma \)-field can become heavy and decouple.

3. Cosmic deceleration in ln R gravity. Let us assume the metric in the physical (Jordan) frame (4) is given in the FRW form:

\[ ds^2 = -dt^2 + \hat{a}(t)^2 \sum_{i,j=1}^{3} \hat{g}_{ij} dx^i dx^j . \] (22)

In the Einstein frame, the FRW equation looks like

\[ 3H_E^2 = \frac{3}{4} \dot{\sigma}^2 + \frac{1}{2} V(\sigma) \text{ or } 3H_E^2 = \frac{3}{4} \dot{\sigma}^2 - \frac{1}{2} \tilde{V}(\sigma) . \] (23)
Here we distinguish the quantities in the Einstein frame by the subscript $E$. The Hubble parameter $H_E$ is now defined by $H_E \equiv \frac{\dot{a}_E}{a_E}$. On the other hand, the equation derived by the variation over $\sigma$ is

$$0 = 3 (\ddot{\sigma} + 3 H_E \dot{\sigma}) + V'(\sigma) \quad \text{or} \quad 0 = 3 (\ddot{\sigma} + 3 H_E \dot{\sigma}) - \tilde{V}'(\sigma). \quad (24)$$

We now consider the case that $A$ or the scalar curvature in the Jordan (physical) frame is small and the potential is given by (18). Then the solution (in the leading order of $t_E$) of the combined equations (23) and (24) is given by

$$\sigma \sim - \ln \frac{t_E}{t_0} + \frac{1}{2} \ln \ln \frac{t_E}{t_0} \ldots, \quad a_E \propto t_E^{\frac{3}{2}} \ldots. \quad (25)$$

The time coordinate $t_E$ in the Euclidean frame is related with the time coordinate $t$ in the (physical) Jordan frame by $e^{\sigma/2} dt_E = dt$. As a result $t_0^{1/2} t_E \propto t$. The power law inflation occurs in the physical (Jordan) frame

$$\dot{a} = e^{\sigma/2} \dot{a}_E \propto t_E^{\frac{3}{2}} \propto t^{\frac{1}{2}}. \quad (26)$$

Since we have $\dot{a} > 0$ but $\ddot{a} < 0$, the decelerated expansion occurs. In case of the original $1/R$ model in [3], the solution is $\dot{a} \propto t^2$. This might suggest that for the model containing $R^n$ ($-1 < n < 0$), one may have more moderately accelerating universe: $\dot{a} \propto t^m$ ($\frac{1}{2} < m < 2$).

From (23) and (24), we may evaluate

$$\alpha' t_0^2 \sim O(1). \quad (27)$$

Eqs.(26) indicate $H = \frac{\dot{a}}{a} = \frac{1}{2t}$. Since $H^2 = \frac{\dot{a}^2}{6} \rho$, the energy density $\rho$ corresponding to $\sigma$ may be defined as $\rho = \frac{\rho_0}{t}$. Here $\rho_0$ is a constant. Denoting the pressure of $\sigma$ by $p$ and substituting the above expressions of $H$ and $\rho$ to the conservation law for the energy-momentum tensor one finds $w \equiv \frac{p}{\rho} = \frac{1}{3}$, which is nothing but that of the radiation.

One may account for the matter contribution to the energy-momentum tensor. When it is dominant compared with the one from $\sigma$, the obtained results are not changed from those in [3] where the possibility of cosmic acceleration in $1/R$ model was established.

4. Instabilities and corrections to gravitational coupling constant.

We now discuss the (in)stability of our model under the perturbations. In
[7], small gravitational object like the Earth or the Sun in the model [3] is considered. It has been shown that the system quickly becomes unstable.

Following to the idea in ref.[15] we start from the action (1), where $F$ is given by (10) with $m = 2$. If the Ricci tensor is not covariantly constant, the trace of equation of motion with matter is given by

$$\Box R + \frac{F^{(3)}(R)}{F^{(2)}(R)} \nabla_\rho R \nabla^\rho R + \frac{F'(R)R}{3F^{(2)}(R)} - \frac{2F(R)}{3F^{(2)}(R)} = \frac{\kappa^2}{6F^{(2)}(R)} T. \quad (28)$$

Here $T = T_{\rho}^{\rho}$. In case of the Einstein gravity, where $\alpha' = \beta = 0$, the solution of Eq.(28) is given by $R = R_0 \equiv -\frac{\kappa^2}{2} T$. The perturbation around the solution may be addressed

$$R = R_0 + R_1, \quad (|R_1| \ll |R_0|). \quad (29)$$

Then by linearizing (28), we obtain

$$0 = \Box R_0 + \frac{F^{(3)}(R_0)}{F^{(2)}(R_0)} \nabla_\rho R_0 \nabla^\rho R_0 + \frac{F'(R_0)R_0}{3F^{(2)}(R_0)} - \frac{2F(R_0)}{3F^{(2)}(R_0)} - \frac{R_0}{3F^{(2)}(R_0)}$$

$$+ \Box R_1 + 2 \frac{F^{(3)}(R_0)}{F^{(2)}(R_0)} \nabla_\rho R_0 \nabla^\rho R_1 + U(R_0) R_1. \quad (30)$$

Here

$$U(R_0) \equiv \left( \frac{F^{(4)}(R_0)}{F^{(2)}(R_0)^2} - \frac{F^{(3)}(R_0)^2}{F^{(2)}(R_0)^2} \right) \nabla_\rho R_0 \nabla^\rho R_0 + \frac{1}{3} R_0 \quad (31)$$

$$- \frac{F^{(1)}(R_0)F^{(3)}(R_0)R_0}{3F^{(2)}(R_0)^2} - \frac{F^{(1)}(R_0)}{F^{(2)}(R_0)} + \frac{2F(R_0)F^{(3)}(R_0)}{3F^{(2)}(R_0)^2} - \frac{R_0 F^{(3)}(R_0)}{F^{(2)}(R_0)^2}.$$

If $U(R_0)$ is negative, the perturbation $R_1$ grows up exponentially with time. The system becomes instable. By including the $R^2$-term, the time for instability to occur is significantly improved (by the order of $10^{29}$) [15], compared

5We should note that the convention of the spacetime signature here is different from those in [7].

6Due to the spherical symmetry and the structure of the lagrangian it is enough to consider only perturbation of the curvature. Equivalently, one can transform the analysis to Jordan (Brans-Dicke) theory. There are no problems with instabilities in the Einstein part, while resolution of the instabilities for sigma field which corresponds to curvature leads to the same bounds as in below analysis.
with the original $1/R$ model in [3]. If the coefficient of $R^2$ is already fixed by some other condition, one can use other HD terms (like $R^3$) to eliminate the instability completely.

In (28), we only considered the Ricci scalar. More general equation of motion, including all the metric components or Ricci tensor, has the following form:

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F(R) - g_{\mu\nu}\nabla_\rho\nabla_\sigma(F'(R))$$

$$+ \nabla_\mu\nabla_\nu(F'(R)) = -\frac{\kappa^2}{2}T_{\mu\nu}.$$  \( (32) \)

Let the perturbation of the metric be $\delta g_{\mu\nu}$. We may choose a gauge condition $\nabla^\mu\delta g_{\mu\nu} = 0$. Furthermore we decompose $\delta g_{\mu\nu}$ into the sum of the trace part $\delta G = g^{\mu\nu}\delta g$ and traceless part $\delta \tilde{g}_{\mu\nu}$ as

$$\delta g_{\mu\nu} = \delta \tilde{g}_{\mu\nu} + \frac{1}{4}g_{\mu\nu}\delta G.$$  \( (33) \)

For perturbation from the background with constant curvature ($R_{\mu\nu} = \frac{3}{l^2}g_{\mu\nu}$), one gets

$$\delta R = R_1 = -\frac{3}{l^2}\delta G - \nabla^2\delta G.$$  \( (34) \)

Then if $R_1$ is given, $\delta G$ is uniquely determined as in the usual Einstein gravity up to the homogeneous part $\delta G_h$ which satisfies $-\frac{3}{l^2}\delta G_h - \nabla^2\delta G_h = 0$. For the Ricci tensor $R_{\mu\nu}$, we have

$$\delta R_{\mu\nu} = \frac{2}{l^2}\delta \tilde{g}_{\mu\nu} - \frac{1}{2}\nabla^2\delta \tilde{g}_{\mu\nu} - \frac{1}{2l^2}g_{\mu\nu}\delta G - \frac{1}{8}g_{\mu\nu}\nabla^2\delta G - \frac{1}{2}\nabla_\mu\nabla_\nu\delta G.$$  \( (35) \)

Therefore if we use (32), the traceless part $\delta \tilde{g}_{\mu\nu}$ can be also uniquely determined up to the homogeneous part $\delta \tilde{g}_{h,\mu\nu}$ which satisfies $\frac{2}{l^2}\delta \tilde{g}_{h,\mu\nu} - \frac{1}{2}\nabla^2\delta \tilde{g}_{h,\mu\nu} = 0$, as in the usual Einstein gravity. Since the homogeneous parts $\delta G_h$ and $\delta \tilde{g}_{h,\mu\nu}$ appear in the usual perturbation from the background of the deSitter space, they are not related with the (in)stability. Then if $\delta R = R_1$ is stable, whole the metric perturbation $\delta g_{\mu\nu}$ is also stable.

In [8], it has been found that the linearly growing force appears in $1/R$ model due to a diffuse source in a locally deSitter background:

$$ds^2 = -dt^2 + e^{2Ht} \sum_{i=1,2,3} (dx^i)^2,$$  \( (36) \)
which is a solution, with a constant curvature \( R_0 = 12H^2 \), of the equation corresponding to (28) or (11) for the vacuum case. If we consider the perturbation (29) by assuming \( R_1 = R_1(y) \) (\( y \equiv e^{Ht}H \sqrt{\sum_{i=1,2,3} (x^i)^2} \)), the solution \( R_1 \) is a sum of two independent solutions \( f_0(y) \) and \( f_{-1}(y) \):

\[
R_1(y) = \beta_1 f_0(y) + \beta_2 f_{-1}(y).
\]

First, we consider the case that \( \beta \gg \alpha' R_0^2 \). Then \( U(R_0) = \frac{R_0}{3} \) and

\[
f_0(y) = 1 - \frac{2}{3}y + \mathcal{O}(y^2), \quad f_{-1}(y) = \frac{1}{y} \left( 1 - 3y + \mathcal{O}(y^2) \right), \quad (37)
\]

for the vacuum solution. If we assume there is a spherical matter source with mass \( M \) and the radius \( r_0 \) as in [8], the coefficients \( \beta_1 \) and \( \beta_2 \) are determined by

\[
\beta_1 = \frac{3MG}{r_0^3} \left\{ f_0(y_0) - \frac{f'_0(y_0)f_{-1}(y_0)}{f'_0(y_0)} \right\}^{-1} = \frac{3MG}{r_0^3} \left\{ 1 + \mathcal{O}(y_0^3) \right\},
\]

\[
\beta_2 = \frac{3MG}{r_0^3} \left\{ f_{-1}(y_0) - \frac{f'_{-1}(y_0)f_0(y_0)}{f'_{-1}(y_0)} \right\}^{-1} = -\frac{4MGy_0^3}{r_0^3} \left\{ 1 + \mathcal{O}(y_0^3) \right\}. \quad (38)
\]

Here \( 16\pi G = \kappa^2 \). It is assumed that the source exists in \( y \leq y_0 \). Since for the size of galaxies, \( y = 10^{-6} \) and for the typical distance between galaxies, \( y = 10^{-4} \), one may assume \( y_0 \ll y \ll 1 \) [8]. Then \( \beta_2 \ll \beta_1 \) and term with \( \beta_2 \) may be neglected. If we denote the trace part of the perturbation of the metric by \( h \), we find

\[
h'(y) = -\frac{2}{y^2 (1-y^2)} \int_0^y dy'y^2 \left( 1-y'^2 \right)^{1\over 2} \frac{R_1(y')}{H^2} \sim -\frac{2GM}{H^2 r_0^3} y + \mathcal{O}(y^3). \quad (39)
\]

Then there appears a linear growth as in [8], which might be a phenomenological disaster. However, that was the case with large \( \beta \). In more general case, the equation corresponding to (30) has the following form:

\[
0 = \left[ (1-y^2) \frac{d^2}{dy^2} + \frac{2}{y} (1-2y^2) \frac{d}{dy} + \frac{12U(R_0)}{R_0} \right] R_1(y), \quad (40)
\]

\[
U(R_0) = -\frac{8R_0^2\beta^2}{3} - \frac{4\alpha'\beta}{3R_0} - 2\beta + \frac{\alpha'^2}{3R_0} \left( 2 + 4 \ln \frac{R_0}{\mu^2} \right) - \frac{\alpha'}{3R_0^2} \left( -\frac{\alpha'}{R_0} + 2\beta \right)^2. \quad (41)
\]
As clear from Eq.(11), $R_0$ does not depend on $\beta$. Choosing $\beta \to \frac{\alpha'}{2t_0^2}$, $U(R_0)$ becomes very large and we find $R_1 \to 0$ in the vacuum, which is identical with the case of the Einstein gravity with cosmological constant. Then contrary to the case of ref.[8], there does not appear the linear growth of $h'(y)$ but $h'(y)$ behaves as $y^{-2}$, which does not conflict with the present cosmology. We should also note that the condition $\beta \to \frac{\alpha'}{2t_0^2}$ is identical with the condition that $\sigma$-field decouples (46), as will be shown below. Hence, modified gravity with terms growing at small curvature and with higher derivative terms important for early time inflation may be viable theory.

It has been mentioned in ref.[13] that $1/R$ model which is equivalent to some scalar-tensor gravity is ruled out as realistic theory due to the constraints to such theories. As the coupling of $\sigma$ with matter is not small [11], we now calculate the square of scalar mass, which is proportional to $V''(\sigma)$. We consider the fluctuation from the solution (25). Then

$$V''(\sigma) \sim \frac{\alpha' t_0^2}{t^2_E} \sim \frac{t_0^2}{t^4}. \tag{42}$$

Here we have used $\frac{t^2}{t^2_E} \propto t$ and (27) are used. The result (42) itself does not change from the original $1/R$ model [3]. Since the Hubble parameter is given by $H = \frac{\dot{a}}{a} = \frac{1}{2t}$, in the present universe, we have $\frac{1}{t} \sim H_0$. Here $H_0 \sim 10^{-33}\text{eV}$ is the Hubble parameter of the present universe. Then

$$V''(\sigma) \sim t_0^2 H_0^4. \tag{43}$$

Surely $H_0$ is very small but we have no restriction (or we have not found it) on $t_0$. Then if $t_0$ is very large, the mass of $\sigma$ can be large. Assuming the mass is larger than 1 TeV, we have $t_0 \sim 10^{78}\text{eV}^{-1}$. As $\alpha' t_0^2 \sim \mathcal{O}(1)$ (27), $\alpha' \sim (10^{-78}\text{eV})^2$. This indicates that such class of theories may still pass the solar system bounds for scalar-tensor gravity. Moreover, the account of the

\[\text{footnote:} \text{The parameter } t_0 \text{ should be determined from the initial condition but since there is unknown parameter } \alpha', \text{ which may differ from } (10^{-33}\text{eV})^2 \text{ as we have argued, } t_0 \text{ contains the ambiguity coming from } \alpha'. \text{ As the curvature is small, from (12), we have } \sigma \sim \ln \frac{A}{t_0}. \text{ On the other hand from (25), } \sigma \sim -\ln \frac{H_0 t_0^2}{t} \sim -\ln \frac{1}{t_0} \sim \ln (H_0^2 t_0^2). \text{ As } A \text{ corresponds to the real curvature, one may assume } A \sim H_0^2. \text{ Then we obtain } \alpha' t_0^2 \sim \mathcal{O}(1), \text{ which is identical with (27). In order to determine the value of the parameter } \alpha', \text{ one should use the information related with the inflation and the matter contents.}\]
terms with derivatives of the curvature \(^8\) may permit to pass the solar system tests even easier.

In [13], the PPN (Parametrized Post-Newtonian) parameters have been investigated in the Jordan (Brans-Dicke) frame and it has been found that the VLBI parameter is above the constraint due the solar system experiments. The analysis relied on the mass of the Brans-Dicke scalar, which corresponds to \(\sigma\) or \(A\) in this paper. In the Brans-Dicke form, we find \(\omega = 0\), that is, there is no kinetic term for \(\sigma\). In the current limit, \(\omega > 3500\) [14]. Then the Brans-Dicke scalar should be heavy to avoid the problem. In the Minkowski background, it has been found that the mass is too light and unnatural. In our case, the action corresponding to (10) can be rewritten as

\[
S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left[ \left( 1 + \frac{A'}{A} + 2\beta A \right) R + \alpha' \left( \ln \frac{A}{\mu^2} - 1 \right) - \beta A^2 \right], \tag{44}
\]

which is also the Brans-Dicke form with \(\omega = 0\). We should note, however, the present universe is not exactly Minkowski but accelerating. The mass of the BD scalar is given by (43). Then as discussed, the mass is determined by the parameter of the integration \(t_0\), which may be determined dynamically. Such a mass may be quite large and then such theory does not violate solar system test as in [13].

One may consider the case that the present universe corresponds to the solution (11). In such a case, by tuning the parameters \(\beta\) and \(\alpha'\), the mass of \(\sigma\) can be made large again. Let us write the solution(s) of (11) as \(A_1\) for \(\alpha' > 0\) case and \(A_\pm\) for \(\alpha' < 0\) case. Then

\[
\left. \frac{d^2V(\sigma)}{d\sigma^2} \right|_{A=A_1,A_\pm} = \left. \left\{ \frac{d\sigma}{dA} \right\}^{-2} \frac{d^2V(A)}{dA^2} \right|_{A=A_1,A_\pm} = \left. \frac{(1 + \frac{2\alpha'}{A})}{(1 + \frac{2\alpha'}{A} + 2\beta A) \left( -\frac{A'}{A^2} + 2\beta \right)} \right|_{A=A_1,A_\pm}. \tag{45}
\]

Choosing

\[
\beta \approx \frac{\alpha'}{2A^2} \bigg|_{A=A_1,A_\pm}, \tag{46}
\]

\(^8\)The correspondent scalar-tensor theory becomes higher derivative one as it follows from second section.
for $\alpha' > 0$ or
\[ \beta \sim -\left(\frac{1}{2A} + \frac{\alpha'}{2A^2}\right)|_{A=A_1, A_2}, \] (47)
for $\alpha' < 0$, the mass of $\sigma$ becomes large again. Thus, HD term may help to pass the solar system tests for $\ln R$ (or $1/R$) gravity.

5. Discussion. Finally, we calculate the corrections to gravitational coupling constant. The easiest way is to consider the perturbation around the constant curvature solution (11). We now write the metric as $g^{(0)}_{\mu\nu}$ corresponding the solution and the constant scalar curvature as $R^{(0)} = \frac{l^2}{4}$. The metric is splitted into the background part $g^{(0)}_{\mu\nu}$ and the perturbation $h_{\mu\nu}$:
\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}. \] (48)
The following gauge conditions are imposed:
\[ 0 = g^{(0)\mu\nu} h_{\mu\nu} = \nabla^{(0)\mu} h_{\mu\nu}. \] (49)
The first condition is chosen to simplify our discussion as graviton is spin 2 field. Then
\[ R = R^{(0)} + \frac{1}{l^2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h^{\mu\nu} \square^{(0)} h_{\mu\nu} + \square (h^{\mu\nu} h_{\mu\nu}) + O(h^3), \] (50)
Using the equation $0 = R^{(0)} + 2\alpha' \ln \frac{R^{(0)}}{\mu^2} - \alpha'$, the expanded action $\tilde{S}$ has the following form:
\[ \tilde{S} = \frac{1}{\kappa^2} \left(1 + \frac{\alpha' l^2}{12} + \frac{24\beta}{l^2}\right) \int d^4 x \sqrt{-g^{(0)}} \left(\frac{6}{l^2} - \frac{1}{2l^2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h^{\mu\nu} \square^{(0)} h_{\mu\nu}\right). \] (51)
Here the total derivative term is dropped. In case of the Einstein action with a positive cosmological constant
\[ S_E = \frac{1}{\kappa^2} \int \sqrt{-g} \left( R - \frac{6}{l^2} \right), \] (52)
which has deSitter solution, the corresponding linearized action is:
\[ \tilde{S}_E = \frac{1}{\kappa^2} \int d^4 x \sqrt{-g^{(0)}} \left(\frac{6}{l^2} - \frac{1}{2l^2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h^{\mu\nu} \square^{(0)} h_{\mu\nu}\right). \] (53)
Comparing (53) with (51), the gravitational constant $\kappa$ is renormalized as

$$\frac{1}{\kappa^2} \rightarrow \frac{1}{\kappa^2} \left( 1 + \frac{\alpha' l^2}{12} + \frac{24 \beta}{l^2} \right).$$

The Newton potential will be modified respectively. When $R_0 = \frac{12}{l^2}$ corresponds to the rate of the expansion of the present universe, one gets $R_0 \sim \frac{12}{l^2} \sim \mu^2 \sim 10^{-33}$eV. In (54), $\beta$-dependent term is of the same order term as second term when scalar has large mass. With the assumption $\alpha' \sim (10^{-33}$eV) the correction may be significant. We may, however, consider the case $\alpha'$ is much smaller than $10^{-33}$eV, since there seems to be no constraint for the value of $\alpha'$ itself. We only have a constraint (27) and $t_0$ can be very large. In such a case, the correction to the Newton constant can be very small and the starting theory has acceptable newtonian limit.

One may discuss further generalizations of modified gravity like

$$F(A) = A + \gamma A^{-n} \left( \ln \frac{A}{\mu^2} \right)^m.$$  

(55)

Here we restrict $n$ by $n > -1$ ($m$ is arbitrary) in order that the second term could be more dominant than the Einstein term when $A$ is small. $n$ and $m$ can be fractional or irrational numbers in general.

When the physical scalar curvature $A = R$ is small, we find

$$e^{-\sigma} \sim -\gamma n A^{-1} \left( \ln \frac{A}{\mu^2} \right) \gamma \left( \frac{\sigma}{n+1} \right)^{\frac{1}{n+1}},$$

$$V(\sigma) \sim \left( 1 + \frac{1}{n} \right) (-\gamma n)^{\frac{1}{n+1}} e^{\frac{n+2}{n+1} \sigma} \left( \frac{\sigma}{n+1} \right)^{-m},$$

when $n \neq 0$ and

$$e^{-\sigma} \sim \frac{m \gamma}{A} \left( \ln \frac{A}{\mu^2} \right)^{m-1}, \quad A \sim m \gamma e^{\sigma} \sigma^{m-1},$$

$$V(\sigma) \sim \gamma e^{2\sigma} \sigma^m,$$  

(57)

9The sum of such terms (whose coefficients could be constrained by the condition of avoiding the linear growing gravitational force) may be considered. The presence of $R^2$ at large curvature is supposed.
when \( n = 0 \). With the similar procedure as in the previous sections,

\[
\sigma \sim -\frac{2(n+1)}{(n+2)} \ln \frac{t}{t_0} + \frac{m(n+1)}{n+2} \ln \frac{t}{t_0} + \cdots \quad (n \neq 0),
\]

\[
\sigma \sim -\ln \frac{t}{t_0} - \frac{m}{2} \ln \frac{t}{t_0} + \cdots \quad (n = 0),
\]

\[
\frac{t}{t_0} \sim \mathcal{O}(1). \tag{58}
\]

As a result

\[
t \sim t_E^{\frac{1}{n+2}}, \quad a_E \sim t_E^{\frac{3(n+1)^2}{(n+2)^2}}, a \sim t^{\frac{(n+1)(2n+1)}{n+2}}. \tag{59}
\]

This does not depend on \( m \). The logarithmic factor is almost irrelevant. We also find the effective \( w \) for the \( \sigma \)-field is

\[
w = -\frac{6n^2 + 7n - 1}{3(n + 1)(2n + 1)}. \tag{60}
\]

Then \( w \) can be negative if

\[
-1 < n < -\frac{1}{2} \text{ or } n > -\frac{7 + \sqrt{73}}{12} = 0.1287 \cdots. \tag{61}
\]

From (59), the condition that the universe could accelerate is \( \frac{(n+1)(2n+1)}{n+2} > 1 \), that is:

\[
n > -\frac{1 + \sqrt{3}}{2} = 0.366 \cdots. \tag{62}
\]

Clearly, the effective dark energy \( w \) may be within the existing bounds.

Thus, we demonstrated that modified gravity with \( \ln R \) or \( R^{-n}(\ln R)^m \) terms may be responsible for the current acceleration of the universe. Hence, like the simplest \( 1/R \) modified gravity this provides the gravitational alternative for dark energy. Moreover, the presence of HD terms like \( R^2 \) (which may be responsible for early time inflation) helps to pass the existing arguments (instabilities, solar system tests) against such modification of the Einstein gravity. The theory may also have the well-acceptable newtonian limit. It is clear that much more work is required to (dis)prove that one of the versions of such modified gravity is currently realistic theory. Nevertheless, the fine-tuning of parameters of modified gravity to provide the effective gravitational dark energy looks more promising than the introduction by hands some mysterious fluid.
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Figure 1: A typical potential when $\alpha' > 0$. We may start with large curvature $A = R = R_{\text{initial}}$ (inflation). Then the curvature rolls down the potential slowly and stops at the small curvature $A = R = R_0$ (the present accelerating universe).