Phase dependence of phonon tunnelling in bosonic superfluid–insulator–superfluid junctions

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Abstract. We consider the tunnelling of phonon excitations across a potential barrier spatially separating two condensates with different macroscopic phases. We analyse the relation between the phase difference $\varphi$ of the two condensates and the transmission coefficient $T$ by solving the Bogoliubov equations. It is found that $T$ strongly depends on $\varphi$, and that the perfect transmission of low-energy excitations disappears when the phase difference reaches the critical value which gives the maximum supercurrent of the condensate. We also discuss the feasibility of observing the phase differences in experiments.

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1. Introduction

Since the first observation of Bose–Einstein condensation (BEC) in trapped dilute atomic gases [1, 2], developments of experimental and theoretical studies in this field have advanced rapidly [3, 4]. One of the greatest advantages of these systems is that experimentalists have useful tools to create various external potentials for atoms. Magnetic traps, which are standard equipments to confine atomic gases, create harmonic potentials for atoms. In the early stages of the study, many characteristics of BECs in harmonic confinement were vigorously investigated. Laser beams can also make periodic potentials for atoms, called optical lattices, by superposing the counter-propagating laser beams and preparing standing waves [5]. Moreover, blue-detuned laser beams act as potential barriers for atoms, and one can create double-well potentials by focusing the laser beam into the centre of a confining potential [6]. One can regard Bose–Einstein condensates in a double-well potential as a bosonic superfluid–insulator–superfluid (S-I-S) junction like a fermionic superconductor–insulator–superconductor junction.

The controllability of external potentials provides a suitable stage for study of quantum tunnelling phenomena. Actually, many theorists have investigated the Josephson-like effect, corresponding to a kind of quantum tunnelling [3, 4, 7], since the experimental realization of BECs in a double-well trap [6]. Recently, some types of the Josephson effect, called Josephson plasma oscillation and self-trapping in [7], were experimentally observed [8].

The study of elementary excitations is one of the main subjects to understand various properties of the Bose condensate, such as dynamics, thermodynamics and superfluidity. Several interesting properties of the elementary excitations have been probed experimentally in gaseous BECs, including the observation of collective modes [3, 9, 10] and the measurement of the Bogoliubov excitation spectrum with use of Bragg spectroscopy [3, 11, 12]. As for the tunnelling problem of the elementary excitations, it was predicted that low-energy excitations perfectly transmit through a potential barrier, and such behaviour was called anomalous tunnelling [13, 14]. A recent work shows that the anomalous tunnelling property is crucial to the phonon-like form of the excitation spectrum of BECs in a periodic potential, which is directly connected to the superfluidity [15]. In addition, Woo et al [16] theoretically presented a way to control tunnelling...
of Bogoliubov excitations through the Landau–Zener mechanism. Thus, the sophisticated experimental techniques in this field have been stimulating studies on tunnelling phenomena of Bogoliubov excitations.

On the other hand, the macroscopic phase difference $\varphi$ plays a key role in Bose condensed systems. Experiments on matter–wave interference [6] and the Josephson effect [8] have clearly demonstrated the existence of the phase difference between two spatially separated BECs. In addition, the problem of the tunnelling of quasi-particles in the fermionic superconductor–insulator–superconductor junction has been studied in detail, and it is well known that the phase difference between two superconductors essentially affects the scattering processes at the potential barrier [17]. It is also expected that the tunnelling of the elementary excitations in bosonic S-I-S junctions depends on the phase difference.

In the present paper, we study the tunnelling of the elementary excitations in a bosonic S-I-S junction and analyse the $\varphi$-dependence of the transmission coefficient $T$ by solving the Bogoliubov equations analytically. It is found that $T$ strongly depends on $\varphi$, with regard to the anomalous tunnelling. The peak width of the transmission coefficient decreases as $\varphi$ increases, and the peak vanishes when the phase difference reaches $\varphi_c \simeq \pi/2$ which gives the maximum supercurrent of the condensate. We will show that the anomalous tunnelling and its remarkable $\varphi$-dependence originate from the existence of the components localized near the potential barrier.

The outline of the paper is as follows. In section 2, we introduce a formulation of the problem using the mean field theory. We calculate the condensate wavefunction and the relation between the supercurrent and the phase difference analytically. In section 3, we analytically solve the Bogoliubov equations and obtain the transmission coefficient. In section 4, we discuss a mechanism of anomalous tunnelling. We also discuss the feasibility of observing anomalous tunnelling in real systems. We summarize our results in section 5.

2. Mean field theory and model potential

We consider a BEC at the absolute zero of temperature in a box-shaped trap which consists of a radial harmonic confinement and end caps in the axial direction. We assume that the frequency $\omega_\perp$ of the radial harmonic potential is large enough compared to the excitation energy for the axial direction. Then, one can justify a one-dimensional treatment of the problem. Such a configuration was realized in a recent experiment [18]. It is assumed that the axial size $L$ of the system is so large that the effect of the edge of the system can be neglected. Setting a potential barrier at the centre of the BEC, one can create a bosonic S-I-S junction. In order to treat the problem analytically, we adopt a $\delta$-function potential barrier as the potential barrier,

$$ V(x) = V_0\delta(x). $$

(1)

The schematic picture of the bosonic S-I-S junction is shown in figure 1.

Our formulation of the problem is based on the mean field theory, which consists of the time-independent Gross–Pitaevskii equation and the Bogoliubov equations. They are written in dimensionless form as

$$ \left[ -\frac{1}{2} \frac{d^2}{dx^2} - \tilde{\mu} + \tilde{V}(\tilde{x}) + |\tilde{\Psi}_0(\tilde{x})|^2 \right] \tilde{\Psi}_0(\tilde{x}) = 0, $$

(2)
Figure 1. Schematic picture of the bosonic S-I-S junction.

and

\[
\begin{pmatrix}
\tilde{H}_0 & -\tilde{\psi}_0(\tilde{x})^2 \\
\tilde{\psi}_0(\tilde{x})^2 & -\tilde{H}_0
\end{pmatrix}
\begin{pmatrix}
\tilde{u}(\tilde{x}) \\
\tilde{v}(\tilde{x})
\end{pmatrix}
= \tilde{\varepsilon}
\begin{pmatrix}
\tilde{u}(\tilde{x}) \\
\tilde{v}(\tilde{x})
\end{pmatrix},
\]

(3)

\[
\tilde{H}_0 = -\frac{1}{2} \frac{d^2}{d\tilde{x}^2} - \tilde{\mu} + \tilde{V}(\tilde{x}) + 2|\tilde{\psi}_0|^2.
\]

(4)

We have introduced the following notation:

\[
\tilde{x} = \frac{x}{\xi},
\]

(5)

\[
(\tilde{\psi}_0, \tilde{u}, \tilde{v}) = \sqrt{\frac{1}{n_0}} (\psi_0, u, v),
\]

(6)

\[
\tilde{\varepsilon} = \frac{\varepsilon}{gn_0},
\]

(7)

\[
\tilde{\mu} = \frac{\mu}{gn_0},
\]

(8)

\[
\tilde{V}(\tilde{x}) = \tilde{V}_0 \delta(\tilde{x}) = \frac{V_0}{gn_0} \delta\left(\frac{x}{\xi}\right),
\]

(9)

where \(\mu\) is the chemical potential, \(n_0\) is the density of the condensate fraction for \(x \gg \xi\) and \(\varepsilon\) is the excitation energy. The healing length \(\xi\) is expressed as \(\xi = \hbar/\sqrt{mgn_0}\). Here, \(\psi_0(x)\) is the wavefunction of the condensate and \((u(x), v(x))^T\) is the wavefunction of the excitation. The coupling constant \(g\) is affected by the harmonic oscillator length \(a_\perp\) of the radial confinement as \(g = 2\hbar^2 a_0/ma_\perp^2\) [19]. We shall omit the bars for all variables in equations (2)–(9) hereafter. It is clear in equation (3) that the density of the condensates acts as a kind of potential for excitations.

Since the purpose of our study is to investigate the relation between phase difference and the tunnelling of elementary excitations, we need to obtain the condensate wavefunction with phase
difference from equation (2). It corresponds to a solution of equation (2) which has stationary Josephson current. Let us first find such a solution assuming

\[ \Psi_0(x) = e^{i(qx + C_\pm)} \]  

(10)

far from the potential barrier \(|x| \gg 1\). The constant \(C_\pm\) expresses the phase of the condensate wavefunction at \(x \to \pm \infty\). If there is no interatomic interaction, or \(g = 0\), the solution of equation (2) satisfying the boundary condition (10) does not exist. This is because a finite fraction of the incident wave is inevitably reflected by the potential barrier. On the other hand, if there is repulsive interatomic interaction, the solution with the boundary condition (10) exists as shown in [20, 21]. That means that BECs with repulsive interaction go through the potential barrier without reflection, and this behaviour clearly exhibits the superfluidity of the BECs. In this case, the chemical potential is expressed as

\[ \mu = 1 + \frac{q^2}{2}. \]  

(11)

One can realize such a situation in experiments by moving the potential barrier at the velocity of \(-\hbar q/m\) and choosing the coordinate system with their origin at the potential barrier. A similar procedure has been conducted in experiments of BECs in an optical lattice [22], where current-carrying condensates have been prepared by moving the optical lattice potential itself. In these experiments, the BECs are trapped in a harmonic potential, and the systems are finite. Nevertheless, the experiments demonstrated the energetic and dynamical instabilities of the BECs which were predicted through theoretical calculations in uniform and infinite systems [23, 24]. This fact supports that our results in an infinite system, which will be obtained below, are safely applicable to real finite systems.

Substituting \(\Psi_0 = A(x)e^{i\tilde{S}(x)}\), we can rewrite equation (2) as

\[ -\frac{1}{2} \left( \frac{d^2 A}{dx^2} - q^2 A^{-3} \right) + \left[ V(x) - 1 - \frac{q^2}{2} \right] A + A^3 = 0, \]  

(12)

\[ A^2 \frac{dS}{dx} = q, \]  

(13)

where \(q\) expresses the current of the condensate fraction in dimensionless form and equation (13) corresponds to the equation of continuity. In dimensional form, the current is expressed as \(\hbar q n_0/m\). We do not explicitly write \(V(x)\) in equation (12) hereafter, because the effect of the \(\delta\)-function potential barrier appears only in the boundary condition at \(x = 0\). Multiplying equation (12) by \(dA/dx\), one can integrate the equation as

\[ -\frac{1}{4} \left( \frac{dA}{dx} \right)^2 - \frac{q^2}{4} A^{-2} - \left( \frac{1}{2} + \frac{q^2}{4} \right) A^2 + \frac{1}{4} A^4 = C, \]  

(14)

where \(C\) is an integration constant. The integration constant can be determined by the boundary conditions

\[ A(\pm \infty) = 1, \]  

(15)
\[
\left. \frac{dA}{dx} \right|_{\pm \infty} = 0. \tag{16}
\]

Then, equation (14) can be written as
\[
\left( A \frac{dA}{dx} \right)^2 = (1 - A^2)(A^2 - q^2). \tag{17}
\]

Integrating equation (17) again, one obtains
\[
A^2 = \gamma(x)^2 + q^2, \tag{18}
\]
where
\[
\gamma(x) \equiv \sqrt{1 - q^2 \tanh \left( \sqrt{1 - q^2} |x| + x_0 \right)} . \tag{19}
\]

Substituting equation (18) into equation (13), one can obtain the phase \( S(x) \) of the condensate wavefunction,
\[
S(x) - S(0) = \int_0^x dx \frac{q}{A^2} = qx + \text{sgn}(x) \left[ \tan^{-1} \left( \frac{\gamma(x)}{q} \right) - \tan^{-1} \left( \frac{\gamma(0)}{q} \right) \right]. \tag{20}
\]

One can easily find from equations (18) and (20) that the condensate wavefunction is expressed as
\[
\Psi_0(x) = e^{i[qx - \text{sgn}(x)x_0]} [\gamma(x) - \text{sgn}(x)iq], \tag{21}
\]
where
\[
e^{i\theta_0} \equiv \frac{\gamma(0) - iq}{\sqrt{\gamma(0)^2 + q^2}}. \tag{22}
\]

This solution is almost the same as that for a grey soliton [25], which is a kind of nonlinear excitations of BECs with repulsive interaction. The only difference from the solution for a grey soliton is the constant \( x_0 \) in \( \gamma(x) \) which depends on the potential strength \( V_0 \). The constant \( x_0 \) can be determined by the boundary conditions at \( x = 0 \),
\[
\Psi_0(+0) = \Psi_0(-0), \tag{23}
\]
\[
\left. \frac{d\Psi_0}{dx} \right|_{+0} = \left. \frac{d\Psi_0}{dx} \right|_{-0} + 2V_0 \Psi_0(0). \tag{24}
\]

Substituting equation (21) into equation (24), one obtains the equation to determine \( x_0 \),
\[
\gamma(0)^3 + V_0 \gamma(0)^2 - (1 - q^2)\gamma(0) + V_0q^2 = 0. \tag{25}
\]

Solving equation (25), one obtains \( x_0 \) as a function of \( q \) and \( V_0 \). The solution (21) has been obtained in [20, 21].
We shall next define the phase difference $\varphi$ and express the current $q$ and the constant $x_0$ (or $\gamma(0)$) by using $\varphi$. According to [26], the phase difference is given by

$$\varphi \equiv q \int_{-\infty}^{\infty} dx \left( \frac{1}{A^2} - 1 \right). \quad (26)$$

Substituting equation (20) into equation (26), one obtains

$$\varphi = 2 \left[ \tan^{-1} \left( \frac{\sqrt{1-q^2}}{q} \right) - \tan^{-1} \left( \frac{\gamma(0)}{q} \right) \right]. \quad (27)$$

Equations (25) and (27) yield $q$ and $\gamma(0)$ as functions of $V_0$ and $\varphi$. Assuming $V_0 \gg 1$, we expand equations (25) and (27) into power series of $1/V_0$, and obtain

$$q \simeq \frac{\sin \varphi}{2V_0} \left( 1 + \frac{\cos \varphi}{V_0} - \frac{2 + \cos \varphi - 3 \cos^2 \varphi}{2V_0^2} \right), \quad (28)$$

$$\gamma(0) \simeq \frac{1 + \cos \varphi}{2V_0} \left( 1 - \frac{1 - \cos \varphi}{V_0} - \frac{1 + 4 \cos \varphi - 3 \cos^2 \varphi}{2V_0^2} \right). \quad (29)$$

The leading term of equation (28) is the well-known relation between the Josephson current and the phase difference. It is obvious from equation (28) that there is the critical current,

$$q_c \simeq \frac{1}{2V_0} - \frac{1}{4V_0^3}. \quad (30)$$

at the critical phase difference,

$$\varphi_c \simeq \frac{\pi}{2} - \frac{1}{V_0} + \frac{1}{2V_0^2}. \quad (31)$$

As the potential strength increases, the critical current decreases. The leading term in equation (30) is consistent with the result in [20]. We can easily see from equations (9) and (30) that the critical current $q_c$ is equal to zero when $g = 0$.

3. Calculations and results

In this section, we shall solve the Bogoliubov equations with the condensate wavefunction of equation (21) which has the phase difference and discuss the tunnelling of the elementary excitations. Assuming that the system size is large enough compared to wavelength of the excitations, we neglect effects of the edges, corresponding to the optical endcaps in experiments, throughout this section.

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3.1. Definition of transmission and reflection coefficients

We need to clarify the general definition of transmission and reflection coefficients, before we discuss the tunnelling of the elementary excitations. In the case of the tunnelling of a single particle, it is well known that one can easily define the transmission and reflection coefficients by means of the equation of continuity for the probability current. When one discusses the tunnelling of a single particle through a potential barrier \( V_{pb}(x) \), one usually considers a situation in which the particle comes from the left (or right). In this situation, one assumes that the solution \( \phi_{sp}(x) \) of the Schrödinger equation takes the form

\[
\phi_{sp}(x) = \begin{cases} 
  e^{i k_- x} + a_{sp} e^{-i k_- x}, & x \to -\infty, \\
  b_{sp} e^{i k_+ x}, & x \to \infty,
\end{cases}
\]  

where the wave number \( k_\pm \) is related to the energy of the particle \( E \) as

\[
k_\pm \equiv \sqrt{\frac{2m}{\hbar^2} (E - V_{pb}(\pm \infty))}.
\]

The equation of continuity for the probability current gives the relation:

\[
|a_{sp}|^2 + \frac{k_+}{k_-} |b_{sp}|^2 = 1.
\]

This equation means that the transmission coefficient \( T_{sp} \) and the reflection coefficient \( R_{sp} \) are defined as

\[
T_{sp} = \frac{k_+}{k_-} |b_{sp}|^2, \quad R_{sp} = |a_{sp}|^2.
\]

When \( V_{sp}(\infty) = V_{sp}(-\infty) \), \( T_{sp} \) and \( R_{sp} \) coincide with \( |b_{sp}|^2 \) and \( |a_{sp}|^2 \), respectively.

We shall define the transmission coefficient \( T \) and the reflection coefficient \( R \) for the elementary excitations. Assuming that an excitation comes from the left, we write the asymptotic form of the wavefunction of the excitation as

\[
\psi'(x) = (u'(x)v'(x)) = \begin{cases} 
  (u_{k_1} v_{k_1}) e^{i k_1 x} + a' (u_{k_2} v_{k_2}) e^{i k_2 x}, & x \to -\infty, \\
  e^{i k_1 x}, & x \to \infty,
\end{cases}
\]

where \( k_1 \) and \( k_2 \) are real, and they satisfy

\[
\varepsilon = qk + \sqrt{\frac{k^2}{2} \left( \frac{k^2}{2} + 2 \right)}.
\]

Equation (37) is the Bogoliubov excitation spectrum in a uniform system where the condensate has supercurrent proportional to \( q \) [27]. Since equation (37) is a fourth order equation for \( k \), one can solve it and obtain \( k_1 \) and \( k_2 \) analytically. However, since we focus on low-energy regions
where a specific tunnelling behaviour appears, we only write the approximate forms of $k_1$ and $k_2$ for $\varepsilon \ll 1$ here,

\begin{align}
  k_1 &\simeq \frac{\varepsilon}{1 + q} - \frac{\varepsilon^3}{8(1 + q)^4}, \\
  k_2 &\simeq -\frac{\varepsilon}{1 - q} + \frac{\varepsilon^3}{8(1 - q)^4}.
\end{align}

The amplitudes $u_k$ and $v_k$ are given by

\begin{align}
  u_k &= \sqrt{\frac{1 + (k^2/2) + \varepsilon - qk}{2(\varepsilon - qk)}} e^{i[q + \text{sgn}(x)\xi]}, \\
  v_k &= \sqrt{\frac{1 + (k^2/2) - \varepsilon + qk}{2(\varepsilon - qk)}} e^{-i[q + \text{sgn}(x)\xi]}.
\end{align}

The Wronskian defined as

\[ W(\psi^j, \psi^i) = u^j \frac{d}{dx} u^i - u^i \frac{d}{dx} u^j + v^j \frac{d}{dx} v^i - v^i \frac{d}{dx} v^j \]

yields a relation between $a^j$ and $c^j$, which defines the transmission and reflection coefficients. One can easily prove from the Bogoliubov equations that $W$ is independent of $x$ when $\psi^j$ and $\psi^i$ have the same energy. By evaluating $W(\psi^j, \psi^i)$, one obtains

\[ -k_2(|u_k|^2 + |v_k|^2) - qk_1(|u_k|^2 + |v_k|^2) + q|a^j|^2 + |c^j|^2 = 1, \]

which expresses the conservation law of the energy flux [13]. It is obvious that the reflection coefficient $R$ and transmission coefficient $T$ are defined from equation (43) as

\begin{align}
  R &= -\frac{k_2(|u_k|^2 + |v_k|^2) - q|a^j|^2}{k_1(|u_k|^2 + |v_k|^2) + q|a^j|^2}, \\
  T &= |c^j|^2.
\end{align}

When there is no supercurrent, $R$ and $T$ are simply given by $|a^j|^2$ and $|c^j|^2$, respectively.

3.2. Calculation of transmission coefficient

The condensate wavefunction of equation (21) takes the same form as that for a grey soliton except for the constant $x_0$. Consequently, the solutions of the Bogoliubov equations are also the same as those for a grey soliton which was obtained in [25]. Substituting equation (21) into the Bogoliubov equations, one can analytically obtain four particular solutions.
They are

\[ u_n(x) = \Lambda_n e^{i(k_n x + \text{sgn}(x) \frac{k_n^2}{2\varepsilon})} \left\{ \left( 1 + \frac{k_n^2}{2\varepsilon} \right) \gamma(x) - \text{sgn}(x) \left[ q + \frac{k_n}{2\varepsilon} \left( 1 - q^2 - \gamma(x)^2 + \varepsilon \right) + \frac{k_n^3}{4\varepsilon} \right] \right\}, \]

\[ v_n(x) = \Lambda_n e^{i(k_n x - \text{sgn}(x) \frac{k_n^2}{2\varepsilon})} \left\{ \left( 1 - \frac{k_n^2}{2\varepsilon} \right) \gamma(x) + \text{sgn}(x) \left[ q + \frac{k_n}{2\varepsilon} \left( 1 - q^2 - \gamma(x)^2 - \varepsilon \right) + \frac{k_n^3}{4\varepsilon} \right] \right\}, \]

where \( k_n \) satisfies equation (37). Approximate forms of \( k_3 \) and \( k_4 \) for \( \varepsilon \ll 1 \) are given by

\[ k_{3,4} \approx \mp 2i \sqrt{1 - q^2} + \frac{q\varepsilon}{1 - q^2} \mp 2(1 + 2q^2)\varepsilon^2 - \frac{(q + q^3)\varepsilon^3}{2(1 - q^2)^{3/2}}. \]

Equations (38) and (39) show that \((u_1(x), v_1(x))^\dagger\) and \((u_2(x), v_2(x))^\dagger\) describe scattering components. It is noted that there exist the scattering components whose energy is lower than the condensate potential, because the excitation spectrum is gapless in contrast to the case of the superconductor. Equation (48) shows that \((u_3(x), v_3(x))^\dagger\) on the left-hand side of the barrier and \((u_4(x), v_4(x))^\dagger\) on the right-hand side describe the localized components around the potential barrier, and \((u_3(x), v_3(x))^\dagger\) on the left-hand side of the barrier and \((u_4(x), v_4(x))^\dagger\) on the right-hand side describe the divergent component far from the potential barrier. The normalization constant \( \Lambda_n \) is expressed as

\[ \Lambda_n = \begin{cases} \frac{e^{i\alpha_n}}{\sqrt{2(\varepsilon - qk_n)}}, & n = 1, 3, 4, \\ \frac{e^{i\alpha_2}}{\sqrt{2(\varepsilon - qk_2)}}, & n = 2, \end{cases} \]

where

\[ e^{i\alpha_n} = \frac{4\varepsilon + 2\varepsilon qk_n + 2(1 - q^2)k_n^2 + qk_n^3}{4\varepsilon \sqrt{1 + (k_n^2/2) + \varepsilon - qk_n}} + \frac{\sqrt{1 - q^2}k_n(2\varepsilon - 2qk_n + k_n^2)}{4\varepsilon \sqrt{1 + (k_n^2/2) + \varepsilon - qk_n}}. \]

The normalization constant of the scattering components is determined to satisfy equation (36). Two independent eigenfunctions of equation (3) corresponding to two types of scattering process are obtained by omitting divergent components. One is the process in which an excitation comes from the left-hand side (\( \psi_l(x) \)), and the other from the right-hand side (\( \psi_r(x) \)). Here, we consider the former written as

\[ \psi'(x) = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{cases} ( u_1 \ v_1 ) + a' ( u_2 \ v_2 ) + b' ( u_3 \ v_3 ) , & x < 0, \\ c' ( u_1 \ v_1 ) + d' ( u_4 \ v_4 ) , & x > 0. \end{cases} \]
The coefficients \(a', b', c'\) and \(d'\) are the amplitudes of the reflected, the left localized, the transmitted and the right localized components, respectively. They are functions of the energy \(\epsilon\), the potential strength \(V_0\) and the phase difference \(\phi\). The boundary conditions at \(x = 0\) yield four equations to determine all the coefficients

\[
\psi'(0^+)=\psi'(0^-), \tag{52}
\]

\[
\left. \frac{d\psi'}{dx} \right|_{0^+}=\left. \frac{d\psi'}{dx} \right|_{0^-} + 2V_0\psi'(0). \tag{53}
\]

These equations are linear simultaneous equations for the coefficients \(a', b', c'\) and \(d'\), and one can analytically solve them. Assuming \(\epsilon \ll 1\) and \(V_0 \gg 1\), we can obtain analytical solutions of them within the first order of \(\epsilon\) or \(1/V_0\),

\[
a' = \left[ V_0\epsilon + \frac{(3\cos \phi + \sin \phi + 4)\epsilon}{2} - iV_0\epsilon^2 \right] / z', \tag{54}
\]

\[
b' = \epsilon \left[ -\cos \phi + \frac{-4\cos^2 \phi + \sin \phi \cos \phi - 2\cos \phi + 2}{2V_0} + i(\cos \phi + \sin \phi)\epsilon \right] / (2z'), \tag{55}
\]

\[
c' = \left[ i\left( \cos \phi + \frac{3\cos^2 \phi + 2\cos \phi - 1}{V_0} \right) + 2\sin \phi \epsilon \right] / z', \tag{56}
\]

\[
d' = \epsilon \left[ \cos \phi + \frac{4\cos^2 \phi - \sin \phi \cos \phi + 2\cos \phi - 2}{2V_0} - i\cos \phi \epsilon \right] / (2z'), \tag{57}
\]

\[
z' = V_0\epsilon + \frac{(3\cos \phi + \sin \phi + 4)\epsilon}{2} + i\left( \cos \phi + \frac{3\cos^2 \phi + 2\cos \phi - 1}{V_0} \right). \tag{58}
\]

The relations between the coefficients in \(\psi'\) and those in \(\psi\) are

\[
a'(\epsilon, -\phi) = a'(\epsilon, \phi), \quad b'(\epsilon, -\phi) = b'(\epsilon, \phi), \tag{59}
\]

\[
c'(\epsilon, -\phi) = c'(\epsilon, \phi), \quad d'(\epsilon, -\phi) = d'(\epsilon, \phi).
\]

When \(\phi\) is not equal to \(\phi_c\), we easily obtain the transmission coefficient \(T = |c'|^2\) from equations (56) and (58). It is expressed as

\[
T = \frac{\Delta^2}{\Delta^2 + \epsilon^2}, \tag{60}
\]

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Figure 2. The transmission coefficient $T$ as a function of $\varepsilon$ and $\varphi$, where $V_0 = 10$. (a) $T$s at low energy with $\varphi = 0$, $\varphi = \pi/4$, $\varphi = \varphi_c \simeq \pi/2$, $\varphi = 3\pi/4$ and $\varphi = \pi$ are shown. (b) $T$s are shown up to a higher-energy region.

where

$$\Delta = \sqrt{\frac{\cos^2 \varphi}{V_0^2} + \frac{3 \cos^3 \varphi - 2 \cos \varphi}{V_0^3}}. \quad (61)$$

The transmission coefficient has a peak at $\varepsilon = 0$, and the peak has Lorentzian shape with half width $\Delta$. This means that the potential barrier is transparent for the low-energy excitations. This behaviour of the transmission coefficient, called anomalous tunnelling, has been predicted for a current-free condensate [13, 14]. We can see from equation (61) that the peak width decreases as $\varphi$ approaches $\varphi_c \simeq \pi/2$, and it becomes infinitesimal for $\varphi \rightarrow \varphi_c$. Thus, anomalous tunnelling is suppressed by the supercurrent. When $\varphi$ reaches $\varphi_c$, the transmission coefficient is expressed as

$$T \simeq \frac{4}{V_0^2}. \quad (62)$$

Obviously, the anomalous tunnelling behaviour does not exist any longer, and the transmission coefficient has only a small residual value.

We plot the transmission coefficient as a function of $\varepsilon$ at $\varphi = 0, \frac{\pi}{4}, \varphi_c (\simeq \frac{\pi}{2}), \frac{3\pi}{4}$ and $\pi$ are shown in figure 2, where we set $V_0 = 10$. In figure 2(a), we can clearly see the properties of the transmission coefficient mentioned above. In the region far from the anomalous tunnelling peak, $T$ increases with $\varepsilon$ in a conventional way and its $\varphi$-dependence disappears as shown in figure 2(b). This is because the condensate potential terms in the Bogoliubov equations, written as $|\Psi_0|^2, \Psi_0^* \Psi_0$ and $\Psi_0^2$, begin to become so small compared to the kinetic energy term that the Bogoliubov excitations behave as single-particle excitation.

While we adopted the $\delta$-function potential barrier of equation (1) in the above calculations, the width of a potential barrier is finite in real systems. However, the treatment of the problem using the $\delta$-function potential barrier gives us valuable qualitative insights, because it enables us to analytically calculate physical quantities such as the transmission coefficient. In fact, the result of the calculation of the transmission coefficient for the $\delta$-function potential barrier [14] is
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qualitatively the same as that for the rectangular potential barrier [13] in the case of current-free condensate. Moreover, comparing equation (28) to equation (A.16) in appendix, we can see that the relation between the supercurrent and the phase difference for the \( \delta \)-function potential is also qualitatively the same as that for the rectangular potential. Accordingly, we consider that the \( \varphi \)-dependence of the transmission coefficient is valid not only for the \( \delta \)-function potential barrier but also for a potential barrier with finite width.

4. Discussion

4.1. Origin of anomalous tunnelling

Since anomalous tunnelling exhibits behaviours specific to resonant tunnelling, it is expected that its origin can be attributed to the appearance of quasi-bound states with lifetime \( \tau_{qb} \sim \hbar / \Delta \). Kagan et al actually insisted that the quasi-bound states were induced by spatial changes of the condensate density. Their argument is as follows. The condensate density acts as a kind of potential for excitations. It inevitably reduces near the potential barrier, and consequently creates a potential well for excitations as shown in figure 1. Using an analogy from the resonant tunnelling of single particles, such a potential well induces the quasi-bound state, and this is the origin of anomalous tunnelling.

Here, we try to elucidate mechanisms of anomalous tunnelling from another viewpoint of the prominence of the localized components with imaginary momenta of equation (48). Calculation of the probability density of each component in the left-incident state \( \psi_l \) yields useful information to such viewpoints. Since the normalization condition for \((u(x), v(x))\) is

\[
\frac{1}{L} \int dx (|u(x)|^2 - |v(x)|^2) = 1,
\]

the probability density of each component is defined as \(|u_1(x)|^2 - |v_1(x)|^2\) for the incident component, \(|d'|^2(|u_2(x)|^2 - |v_2(x)|^2)\) for the reflected component, \(|b'|^2(|u_3(x)|^2 - |v_3(x)|^2)\) for the left-localized component, \(|c'|^2(|u_4(x)|^2 - |v_4(x)|^2)\) for the transmitted component or \(|d'|^2(|u_4(x)|^2 - |v_4(x)|^2)\) for the right-localized component. When \( V_0 \gg 1 \) and \( \varepsilon \ll 1 \), one obtains the probability density at \( x = 0 \) of each component,

\[
|u_1(-0)|^2 - |v_1(-0)|^2 \sim \frac{1}{2} + \frac{\sin \varphi}{2V_0},
\]

\[
|d'|^2(|u_2(-0)|^2 - |v_2(-0)|^2) \sim \frac{1}{2} - \frac{\sin \varphi}{2V_0},
\]

\[
|b'|^2(|u_3(-0)|^2 - |v_3(-0)|^2) \sim \frac{|b'|^2}{\varepsilon^2} \left( -2 - \frac{3\cos \varphi + \sin \varphi}{V_0} \right),
\]

\[
|c'|^2(|u_1(+0)|^2 - |v_1(+0)|^2) \sim \frac{|c'|^2}{2} \left( \frac{\sin \varphi}{2V_0} \right),
\]

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Figure 3. The probability densities at $x = 0$ of the transmitted component (TC) and the right localized component (LC) as a function of $\varepsilon$ and $\varphi$, where $V_0 = 10$. (a) They are shown in a low-energy region. (b) They are shown up to a higher-energy region.

\[ |d|^2 \left( |u_{+0}|^2 - |v_{+0}|^2 \right) \sim \frac{|d|^2}{\varepsilon^2} \left( -2 - \frac{3 + 3 \cos \varphi + \sin \varphi}{V_0} \right). \]  

(68)

We show the probability densities at $x = 0$ of the transmitted component (TC) and the localized component (LC) in figure 3. We can see from equation (67) and figure 3(a) that the maximum value at $\varepsilon = 0$ of the probability density of the transmitted component increases slightly as the phase difference increases. However, the maximum of the transmission coefficient does not change, because the probability density of the incident component also increases slightly as seen in equation (64). When $\varphi$ is not equal to $\varphi_c$, the localized components prominently appear around $\varepsilon = 0$ where anomalous tunnelling occurs. When $\varphi$ is equal to $\varphi_c$, the localized components do not appear. Thus, the localized components are induced only when anomalous tunnelling is effective. This result suggests that the localized components are crucial to understand anomalous tunnelling. It is to be noted that the probability densities of the localized components are negative\(^4\). Due to their negativity, the probability density of scattering components can be comparably large even near the potential barrier without totally raising the probability density of $\psi_l(0)$. Accordingly, the localized components have a role to spread the scattering components across the potential barrier, and the appearance of the localized component is one of the origins of anomalous tunnelling.

4.2. Observation of anomalous tunnelling

In order to describe realistic parameters properly, we add the bars to the dimensionless parameters again as in the equation (9). Throughout the above calculations, we have assumed that the system size $L$ is large enough compared to the wavelength $\lambda$ of the excitations and neglected effects of the edges. However, effects of the edges can be significant under the following situation. Let us consider the experimental setup described in section 2. In this setup, we consider that a Bogoliubov excitation propagating to the potential barrier is created as the initial state by using the

\[^4\text{The localized components always appear in states } \psi_l(x) \text{ and } \psi_r(x) \text{ superposed with the scattering components, and the local probability density of the states always remains positive totally.}\]
Bragg pulse. After the excitation is scattered by the potential barrier, the reflected and transmitted waves propagate to the edges of the system. As long as these waves do not reach the edges, the assumption of neglecting effects of the edges can be valid. On the other hand, once these waves reach the edges, the edges dramatically affect the excitation and the assumption becomes no longer valid. Therefore, the time in which the excitation propagates should be smaller than \( L/c_s \) in order to relate our calculations to real experiments, where \( c_s \) is the Bogoliubov sound speed.

Since the width of the potential barrier is finite in real systems, we consider a rectangular potential barrier,

\[
V(x) = V_0 \left( \frac{d}{2} - |x| \right),
\]

where \( V \) and \( d \) are the height and the width of the potential barrier, respectively. The anomalous tunnelling is predicted also for this rectangular potential barrier [13]. When the barrier is so high that \( 1/\kappa_0 \xi, e^{-\kappa_0 d} \ll 1 \) is satisfied, the peak width of the transmission coefficient can be expressed as [28]

\[
\frac{\Delta_{\text{rec}}}{g n_0} \sim \frac{2\sqrt{2} e^{-\kappa_0 d}}{\kappa_0 \xi}, \quad (70)
\]

where

\[
\kappa_0 = \sqrt{\frac{2m}{\hbar^2} (V - \mu)}.
\]

One needs to create excitations with energy comparable to \( \Delta_{\text{rec}} \) for the observation of anomalous tunnelling in experiments. For that reason, the values of two parameters with length dimension are restricted. One is the barrier width \( d \), which corresponds to thickness of the laser beam. The other is the system size, which is the axial length \( L \) of the system. The healing length \( \xi \) and the barrier intensity \( \kappa_0 \xi \) determine the restrictions.

We consider the experimental setup of [18]. The healing length is \( \xi \sim 1 \mu m \) in that experiment, because the total number of \(^{87}\)Rb atoms, the system size and the radial harmonic trap frequency are \( N \sim 3000 \), \( L \sim 80 \mu m \) and \( 2\pi \omega_\perp \sim 40 \text{ kHz} \). We see from equation (70) that the peak width decreases as the barrier width increases. This means that the potential barrier should be as narrow as possible. Since one can narrow down the laser beam waist, which corresponds to the barrier width, to the value comparable to the wavelength, the laser width can be \( d \sim 1 \mu m \) experimentally with barriers created by a blue-detuned laser.

We shall give the restriction on the system size. The anomalous tunnelling occurs in a low-energy region where the Bogoliubov spectrum is phononlike. Phonons with energy comparable to or smaller than \( \Delta_{\text{rec}} \) are available if the axial size of the system \( L \) is larger than the wavelength \( \lambda_\Delta \), where \( \lambda_\Delta \) is easily calculated from equation (70) as \( \lambda_\Delta /\xi \sim \kappa_0 \xi e^{\kappa_0 d} \). We set \( d = 1 \mu m \) and \( \xi = 1 \mu m \). With a barrier height of \( \kappa_0 \xi = 3 \), \( L \) should be larger than \( 60 \mu m \). Since the system size is \( L \sim 70 \mu m \) in the experimental setup of [18], they barely overcomes this restriction. Moreover, one can enlarge the system size by changing the position of the optical end caps. Thus, anomalous tunnelling is expected to occur in the uniform BEC if one can use a sufficiently narrow laser beam to create the potential barrier.

A procedure to observe anomalous tunnelling is as follows. We consider two BECs separated by a potential barrier. One can control the phase difference by moving the potential barrier at
the velocity \(-\hbar q/m\), where \(q\) is given by equation (A.16) (see appendix). The potential barrier should be moved around the centre of the trap so that the edges do not affect the experiment. At first, one stimulates the condensate on the left-hand side into phonon excitations with energy comparable to or smaller than \(\Delta_{\text{rec}}\) by using the Bragg pulse. The phonon with wavelength \(\lambda\) takes \(\lambda/c_s\) to pass through the barrier. After longer time than \(\lambda_{\Delta}/c_s\), we detect the number of transmitted excitations \(n_t\) and that of reflected excitations \(n_r\) by the time of flight absorption images, and the transmission coefficient is given by \(T = n_t/(n_t + n_r)\). Thus, the observation of the anomalous tunnelling may be possible in an optimized setup.

The dramatic \(\varphi\)-dependence of the transmission coefficient discussed in section 3 enables us to determine or estimate the phase difference between two condensates by measuring the transmission coefficient. The phase difference is near \(n\pi\) if most excitations at low energy transmit across the potential barrier, while it is near \((n + 1/2)\pi\) if most excitations at low energy are reflected by the potential barrier, where \(n\) is an arbitrary integer.

5. Conclusion

In summary, we have investigated the \(\varphi\)-dependence of the tunnelling of elementary excitations in a bosonic S-I-S junction by solving the Bogoliubov equations, and we have found a significant \(\varphi\)-dependence of the transmission coefficient.

Calculating the probability density of the localized components, we have discussed mechanisms of anomalous tunnelling. We have shown that the localized component arises only when anomalous tunnelling appears; this means that the prominence of the localized components is one of the origins of anomalous tunnelling.

We have discussed the feasibility of observing anomalous tunnelling in experiments. We expect that anomalous tunnelling may be observed in BECs trapped in a box-shaped potential.

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Appendix. Josephson relation for a rectangular potential barrier

We have already obtained the relation between the supercurrent of the condensate and the phase difference for a strong \(\delta\)-function potential barrier given by equation (28), and we have shown that the relation satisfies the Josephson relation. In this appendix, we shall calculate the relation between the supercurrent of the condensate and the phase difference for a rectangular potential barrier, and show that the Josephson relation is satisfied also in this case.

We consider a BEC in the box-shaped trap mentioned in section 2. Instead of a \(\delta\)-function potential barrier, we adopt a rectangular potential barrier with width \(d\) expressed in
the dimensionless form as
\[ V(x) = V\theta \left( \frac{d}{2} - |x| \right). \]  
(A.1)

We shall solve equations (12) and (13) for this potential barrier and obtain the relation between \( q \) and \( \varphi \).

Outside the barrier, one obtains the solution of the same form as equations (18) and (20),
\[ A^2 = \tilde{\gamma}(x)^2 + q^2, \]  
(A.2)
\[ S(x) - S\left( \pm \frac{d}{2} \right) = \int_{\pm d/2}^{x} dx \frac{q}{A^2} = q \left( x \mp \frac{d}{2} \right) + \text{sgn}(x) \left[ \tan^{-1}\left( \frac{\tilde{\gamma}(x)}{q} \right) - \tan^{-1}\left( \frac{\tilde{\gamma}(\pm \frac{d}{2})}{q} \right) \right]. \]  
(A.3)
where
\[ \tilde{\gamma}(x) = \sqrt{1 - q^2 \tanh \left( \sqrt{1 - q^2} \left( |x| - \frac{d}{2} + x_0 \right) \right)} \]  
(A.4)

Under the barrier, the general solution takes the form [26]
\[ A^2 = a^2 + \frac{\text{sn}^2(\sqrt{\beta_+} x, \sigma)}{\text{cn}^2(\sqrt{\beta_+} x, \sigma)} \beta_- \]  
(A.5)
where
\[ a = A(0), \]  
(A.6)
\[ \sigma^2 = \frac{\beta_+ - \beta_-}{\beta_+}, \]  
(A.7)
\[ \beta_{\pm} = \frac{3a^2 + \kappa_0^2 \pm \sqrt{(\kappa_0^2 + a^2)^2 - (4q^2/a^2)}}{2}, \quad \kappa_0 = \sqrt{2(V - \mu)}. \]  
(A.8)

Here, \( \text{cn}(u, q) \) and \( \text{sn}(u, q) \) are the Jacobi elliptic functions. Assuming \( \kappa_0, e^{\kappa_0d} \gg 1 \), one can neglect the last term of equation (12). In this case, one finds the solution
\[ A^2 = \frac{a^2 + (q/\kappa_0a)^2}{2} \cosh 2\kappa_0x + \frac{a^2 - (q/\kappa_0a)^2}{2}, \]  
(A.9)
\[ S\left( \frac{d}{2} \right) - S\left( -\frac{d}{2} \right) = \int_{-d/2}^{d/2} dx \frac{q}{A^2} \approx \pi - 2 \tan^{-1}\left( \frac{\kappa_0a^2}{q} \right). \]  
(A.10)

The constants \( x_0 \) and \( a \) can be determined by the boundary conditions at \( x = d/2 \),
\[ \Psi_0\left( \frac{d}{2} + 0 \right) = \Psi_0\left( \frac{d}{2} - 0 \right), \]  
(A.11)
\[
\frac{d\Psi_0}{dx}\bigg|_{\xi+0} = \frac{d\Psi_0}{dx}\bigg|_{\xi-0}.
\]  
(A.12)

Substituting equations (A.3) and (A.10) into equation (26), we obtain

\[
\varphi = \pi - 2 \tan^{-1}\left(\frac{\kappa_0 d^2}{q}\right) + qd + 2 \left[ \tan^{-1}\left(\frac{\tilde{\gamma}(x)}{q}\right) - \tan^{-1}\left(\frac{\tilde{\gamma}(\pm \frac{d}{2})}{q}\right) \right].
\]  
(A.13)

Expanding equations (A.11), (A.12) and (A.13) into power series of \(\kappa_0^{-1}\) and \(e^{-\kappa_0 d}\), one obtains

\[
\tilde{\gamma}\left(\frac{d}{2}\right) \simeq \frac{1}{\kappa_0},
\]  
(A.14)

\[
a \simeq \sqrt{2(1 + \cos \varphi)} \frac{e^{-\kappa_0 d}}{\kappa_0},
\]  
(A.15)

\[
q \simeq \frac{2e^{-\kappa_0 d}}{\kappa_0} \sin \varphi.
\]  
(A.16)

We can clearly see from equation (A.16) that the relation between the supercurrent and the phase difference for the rectangular potential barrier takes the form of the Josephson relation as well as the case of the \(\delta\)-function potential barrier.

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