Stringy Hodge numbers and Virasoro algebra

Victor V. Batyrev
Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
e-mail: batyrev@bastau.mathematik.uni-tuebingen.de

Abstract

Let $X$ be an arbitrary smooth $n$-dimensional projective variety. It was discovered by Libgober and Wood that the product of the Chern classes $c_1(X)c_{n-1}(X)$ depends only on the Hodge numbers of $X$. This result has been used by Eguchi, Jinzenji and Xiong in their approach to the quantum cohomology of $X$ via a representation of the Virasoro algebra with the central charge $c_n(X)$.

In this paper we define for singular varieties $X$ a rational number $c_{1,n-1}^{st}(X)$ which is a stringy version of the number $c_1c_{n-1}$ for smooth $n$-folds. We show that the number $c_{1,n-1}^{st}(X)$ can be expressed in the same way using the stringy Hodge numbers of $X$. Our results provides an evidence for the existence of an approach to quantum cohomology of singular varieties $X$ via a representation of the Virasoro algebra whose central charge is the rational number $e_{st}(X)$ which equals the stringy Euler number of $X$. 
1 Introduction

Let $X$ be an arbitrary smooth projective variety of dimension $n$. The $E$-polynomial of $X$ is defined as

$$E(X; u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

where $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$ are Hodge numbers of $X$. Using the Hirzebruch-Riemann-Roch theorem, Libgober and Wood [10] has proved the following equality (see also the papers of Borisov [6] and Salamon [12]):

**Theorem 1.1**

$$\frac{d^2}{du^2} E_{st}(X; u, 1) |_{u=1} = \frac{3n^2 - 5n}{12} c_n(X) + \frac{c_1(X) c_{n-1}(X)}{6}.$$

By Poincaré duality for $X$, one immediately obtains [6, 12]:

**Corollary 1.2** Let $X$ be an arbitrary smooth $n$-dimensional projective variety. Then $c_1(X) c_{n-1}(X)$ can be expressed via the Hodge numbers of $X$ using the following equality

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(X) \left( p - \frac{n}{2} \right)^2 = \frac{n}{12} c_n(X) + \frac{1}{6} c_1(X) c_{n-1}(X),$$

where

$$c_n(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)$$

is the Euler number of $X$.

**Corollary 1.3** Let $X$ be an arbitrary smooth $n$-dimensional projective variety with $c_1(X) = 0$. Then the Hodge numbers of $X$ satisfy the following equation

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(X) \left( p - \frac{n}{2} \right)^2 = \frac{n}{12} \sum_{p,q} (-1)^{p+q} h^{p,q}(X),$$

In particular, for hyper-Kähler manifolds $X$ this equation reduces to

$$2 \sum_{j=1}^{2n} (-1)^j (3j^2 - n) b_{2n-j}(X) = nb_{2n}(X),$$

where

$$b_i(X) = \sum_{p+q=i} h^{p,q}(X)$$

is $i$-th Betti number of $X$. 
Remark 1.4 We that if $X$ is a $K3$-surface, then the relation (1.3) is equivalent to the equality $c_2(X) = 24$. For smooth Calabi-Yau 4-folds $X$ the relation (1.3) has been observed by Sethi, Vafa, and Witten [11] (it is equivalent to the equality $c_4(X) = \frac{1}{2}(8 - h^{1,1}(X) + h^{2,1}(X) - h^{3,1}(X))$).

if $h^{1,0}(X) = h^{2,0}(X) = h^{3,0}(X) = 0$).

There are a lot of examples of Calabi-Yau varieties $X$ having at worst Gorenstein canonical singularities which are hypersurfaces and complete intersections in Gorenstein toric Fano varieties [1, 3]. It has been shown in [2] that for all these examples of singular Calabi-Yau varieties $X$ one can define so called stringy Hodge numbers $h^{p,q}_{st}(X)$. Moreover, the stringy Hodge numbers of Calabi-Yau complete intersections in Gorenstein toric varieties agree with the topological mirror duality test [4]. It was a natural question posed in [5], whether one has the same identity for stringy Hodge numbers of singular Calabi-Yau varieties as for usual Hodge numbers of smooth Calabi-Yau manifolds, i.e.

$$
\sum_{p,q} (-1)^{p+q} h^{p,q}_{st}(X) \left( p - \frac{n}{2} \right)^2 = \frac{n}{12} \sum_{p,q} (-1)^{p+q} h^{p,q}_{st}(X) = \frac{n}{12} e_{st}(X). \tag{1}
$$

The purpose of this paper is to show that the formula (1) holds true. Moreover, one can define a rational number $e^{1,-1}_{st}(X)$ which is a stringy version $c_1(X)c_{n-1}(X)$ such that the stringy analog of (1) holds true provided the stringy Hodge numbers of $X$ exist.

2 Stringy Hodge numbers

Recall our general approach to the notion of stringy Hodge numbers $h^{p,q}_{st}(X)$ for projective algebraic varieties $X$ with canonical singularities (see [3]). Our main definition in [3] can be reformulated as follows:

**Definition 2.1** Let $X$ be an arbitrary $n$-dimensional projective variety with at worst log-terminal singularities, $\rho : Y \rightarrow X$ a resolution of singularities whose exceptional locus $D$ is a divisors with normally crossing components $D_1, \ldots, D_r$. We set $I := \{1, \ldots, r\}$ and $D_J := \bigcap_{j \in J} D_j$ for all $J \subset I$. Define the stringy $E$-function of $X$ to be

$$
E_{st}(X; u, v) := \sum_{J \subset I} E(D_J; u, v) \prod_{j \in J} \left( \frac{uv - 1}{(uv)^{a_j+1} - 1} - 1 \right),
$$

3
where the rational numbers \(a_1, \ldots, a_r\) are determined by the equality
\[
K_Y = \rho^* K_X + \sum_{i=1}^r a_i D_i.
\]

Then the **stringy Euler number** of \(X\) is defined as
\[
e_{\text{st}}(X) := \lim_{u,v \to 1} E_{\text{st}}(X; u, v) = \sum_{J \subset I} c_{n-|J|}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right),
\]
where \(c_{n-|J|}(D_J)\) is the Euler number of \(D_J\) (we set \(c_{n-|J|}(D_J) = 0\) if \(D_J\) is empty).

**Definition 2.2** Let \(X\) be an arbitrary \(n\)-dimensional projective variety with at worst Gorenstein canonical singularities. We say that **stringy Hodge numbers of \(X\) exist**, if \(E_{\text{st}}(X; u, v)\) is a polynomial, i.e.,
\[
E_{\text{st}}(X; u, v) = \sum_{p,q} a_{p,q}(X) u^p v^q.
\]
Under the assumption that \(E_{\text{st}}(X; u, v)\) is a polynomial, we define the **stringy Hodge numbers** \(h_{\text{st}}^{p,q}(X)\) to be \((-1)^{p+q} a_{p,q}\).

**Remark 2.3** In the above definitions, the condition that \(X\) has at worst log-terminal singularities means that \(a_i > -1\) for all \(i \in I\); the condition that \(X\) has at worst Gorenstein canonical singularities is equivalent for \(a_i\) to be nonnegative integers for all \(i \in I\) (see [9]).

The following statement has been proved in [3]:

**Theorem 2.4** Let \(X\) be an arbitrary \(n\)-dimensional projective variety with at worst Gorenstein canonical singularities. Assume that stringy Hodge numbers of \(X\) exist. Then they have the following properties:
(i) \(h_{\text{st}}^{0,0}(X) = h_{\text{st}}^{n,n}(X) = 1\);
(ii) \(h_{\text{st}}^{p,q}(X) = h_{\text{st}}^{n-p,n-q}(X)\) and \(h_{\text{st}}^{p,q}(X) = h_{\text{st}}^{q,p}(X)\) \(\forall p, q\);
(iii) \(h_{\text{st}}^{p,q}(X) = 0\) \(\forall p, q > n\).

**3 The number \(c_{\text{st}}^{1,n-1}(X)\)**

**Definition 3.1** Let \(X\) be an arbitrary \(n\)-dimensional projective variety \(X\) having at worst log-terminal singularities and \(\rho : Y \to X\) is a desingularization with normally crossing irreducible components \(D_1, \ldots, D_r\) of the exceptional locus. We define the number
\[
c_{\text{st}}^{1,n-1}(X) := \sum_{J \subset I} \rho^* c_1(X)c_{n-|J|-1}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right),
\]
where $\rho^* c_1(X)c_{n-\lvert J \rvert -1}(D_J)$ is considered as the intersection number of the 1-cycle $c_{n-\lvert J \rvert -1}(D_J) \in A_1(D_J)$ with the $\rho$-pullback of the class of the anticanonical $\mathbb{Q}$-divisor of $X$.

**Remark 3.2** It is not clear a priori that the number $c_{\text{st}}^{1,n-1}(X)$ in the above the definition does not depend on the choice of a desingularization $\rho$. Later we shall see that it is the case.

**Proposition 3.3** For any smooth $n$-dimensional projective variety $V$, one has

$$\frac{d}{du} E(V; u, 1)_{u=1} = \frac{n}{2} c_n(V).$$

**Proof.** By definition of $E$-polynomials, we have

$$\frac{d}{du} E(V; u, 1)_{u=1} = \sum_{p,q} p(-1)^{p+q} h^{p,q}(V).$$

The Poincaré duality $h^{p,q}(V) = h^{n-p,n-q}(V) \forall p, q$ implies that

$$\sum_{p,q} \left( p - \frac{n}{2} \right) (-1)^{p+q} h^{p,q}(V) = 0.$$

Hence,

$$\sum_{p,q} p(-1)^{p+q} h^{p,q}(V) = \frac{n}{2} \sum_{p,q} (-1)^{p+q} h^{p,q}(V) = \frac{n}{2} c_n(V).$$

$\square$

**Proposition 3.4** For any $n$-dimensional projective variety $X$ having at worst log-terminal singularities, one has

$$\frac{d}{du} E_{\text{st}}(X; u, 1)_{u=1} = \frac{n}{2} e_{\text{st}}(X).$$

**Proof.** By definition 2.1, we have

$$E_{\text{st}}(X; u, 1) = \sum_{J \subseteq I} E(D_J; u, 1) \prod_{j \in J} \left( \frac{u - 1}{u^{a_j+1} - 1} - 1 \right).$$

Applying 3.3 to every smooth submanifold $D_J \subset Y$, we obtain

$$\frac{d}{du} E_{\text{st}}(X; u, 1)_{u=1} = \sum_{J \subseteq I} \frac{n - \lvert J \rvert}{2} c_{n-\lvert J \rvert}(D_J) \prod_{j \in J} \left( -\frac{a_j}{a_j+1} \right) +$$

$$+ \sum_{J \subseteq I} \frac{\lvert J \rvert}{2} c_{n-\lvert J \rvert}(D_J) \prod_{j \in J} \left( -\frac{a_j}{a_j+1} \right) = \frac{n}{2} E_{\text{st}}(X).$$

$\square$
Proposition 3.5 Let $V$ be a smooth projective algebraic variety of dimension $n$ and $W \subset V$ a smooth irreducible divisor on $V$ or empty divisor (the latter means that $\mathcal{O}_V(W) \cong \mathcal{O}_V$). Then
\[ c_1(\mathcal{O}_V(W))c_{n-1}(V) = c_{n-1}(W) + c_1(\mathcal{O}_W(W))c_{n-2}(W), \]
where $c_{n-1}(W)$ is considered to be zero if $W = \emptyset$.

Proof. Consider the short exact sequence
\[ 0 \to T_W \to T_V|_W \to \mathcal{O}_W(W) \to 0, \]
where $T_W$ and $T_V$ are tangent sheaves on $W$ and $V$. It gives the following the relation between Chern polynomials
\[ (1 + c_1(\mathcal{O}_W(W)t)(1 + c_1(D)t + c_2(D)t^2 + \cdots + c_{n-1}(D)t^{n-1}) = \]
\[ = 1 + c_1(T_V|_W)t + c_2(T_V|_W)t^2 + c_{n-1}(T_V|_W)t^{n-1}). \]
Comparing the coefficients by $t^{n-1}$ and using $c_{n-1}(T_V|_W) = c_1(\mathcal{O}_V(W))c_{n-1}(V)$, we come to the required equality. \qed

Corollary 3.6 Let $Y$ be a smooth projective variety, $D_1, \ldots, D_r$ smooth irreducible divisors with normal crossings, $I := \{1, \ldots, r\}$. Then for all $J \subset I$ and for all $j \in J$ one has
\[ c_1(\mathcal{O}_{D_{J\setminus\{j\}}}(D_j))c_{n-|J|}(D_{J\setminus\{j\}}) - c_{n-|J|}(D_j) = c_1(\mathcal{O}_{D_J}(D_j))c_{n-|J|-1}(D_J), \]
where $D_J$ is the complete intersection $\bigcap_{j \in J} D_j$.

Proof. One sets in $X \setminus V := D_{J\setminus\{j\}}$ and $W := D_J$. \qed

Proposition 3.7 Let $\rho : Y \to X$ be a desingularization as in $\text{[3.1]}$. Then
\[ \sum_{J \subset I} c_1(D_J)c_{n-|J|-1}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) = c_{1,n-1}^{\text{st}}(X) + \]
\[ + \sum_{J \subset I} \left( \sum_{j \in J} (a_j + 1)c_{n-|J|}(D_J) \right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right). \]

Proof. Using the formula
\[ c_1(Y) = \rho^*c_1(X) + \sum_{i \in I} -a_i c_1(\mathcal{O}_Y(D_i)) \]
and the adjunction formula for every complete intersection $D_J (J \subset I)$, we obtain
\[ c_1(D_J) = \rho^*c_1(X)|_{D_J} + \sum_{j \in J} (-a_j - 1)c_1(\mathcal{O}_{D_J}(D_j)) + \sum_{j \in I \setminus J} (-a_j)c_1(\mathcal{O}_{D_J}(D_j)). \]
Therefore
\[
\sum_{J \subset I} c_1(D_J) c_{n-|J|-1}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) = c_{st}^{1,n-1}(X) +
\]
\[
+ \left( \sum_{j \in J} (-a_j - 1) c_1(O_{D_J}(D_j)) c_{n-|J|-1}(D_J) \right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) +
\]
\[
+ \left( \sum_{j \in I \setminus J} (-a_j) c_1(O_{D_J}(D_j)) c_{n-|J|-1}(D_J) \right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right).
\]

Using (3.6) we obtain
\[
\sum_{j \in J} (-a_j - 1) c_1(O_{D_J}(D_j)) c_{n-|J|-1}(D_J) =
\]
\[
= \sum_{j \in J} (-a_j - 1) \left( c_1(O_{D_{J \setminus \{j\}}}(D_j)) c_{n-|J|(D_{J \setminus \{j\}})} - c_{n-|J|(D_J)} \right).
\]

By substitution (3.6) to (3), we come to the required equality. \(\square\)

**Theorem 3.8** Let \(X\) be an arbitrary \(n\)-dimensional projective variety with at worst log-terminal singularities. Then
\[
\frac{d^2}{du^2} E_{st}(X; u, 1)_{|u=1} = \frac{3n^2 - 5n}{12} e_{st}(X) + \frac{1}{6} c_{st}^{1,n}(X).
\]

**Proof.** Using the equalities
\[
\frac{d}{du} \left( \frac{u - 1}{u^{a+1} - 1} - 1 \right)_{|u=1} = \frac{-a}{2(a + 1)}, \quad \frac{d^2}{du^2} \left( \frac{u - 1}{u^{a+1} - 1} - 1 \right)_{|u=1} = \frac{a(a + 2)}{6(a + 1)}
\]

Together with the identities in (1.1) and (3.3) for every submanifold \(D_J \subset Y\), we obtain
\[
\frac{d^2}{du^2} E_{st}(X; u, 1)_{|u=1} = \sum_{J \subset I} c_1(D_J) c_{n-|J|-1}(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) +
\]
\[
+c_1(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) +
\]
\[
+ \sum_{J \subset I} \frac{(n - |J|)|J| c_{n-|J|}(D_J)}{2} \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) +
\]
\[
+ \sum_{J \subset I} \frac{c_{n-|J|}(D_J)(|J| - 1)|J|}{4} \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right) +
\]
\[
+ \sum_{J \subset I} \frac{c_{n-|J|}(D_J)(|J| - 1)|J|}{4} \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right).
\]
\[ + \sum_{J \subset I} \frac{c_{n-|J|}(D_J)(-\sum_{j \in J}(a_j + 2))}{6} \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right). \]

By 3.7, the first term of the above equals
\[ \frac{1}{6} c_{1,n-1} \left( X \right) + \frac{1}{6} \sum_{J \subset I} \left( \sum_{j \in J}(a_j + 1)c_{n-|J|}(D_J) \right) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right). \]

Now the required statement follows from the equality
\[ \sum_{j \in J}(a_j + 1) \frac{6}{6} + \frac{3(n - |J|)^2 - 5(n - |J|)}{12} + \frac{(n - |J||J|}{2} + \]
\[ + \frac{|J| - 1)|J|}{4} + \frac{- \sum_{j \in J}(a_j + 2)}{6} = \frac{3n^2 - 5n}{12}. \]

\[ \square \]

**Corollary 3.9** The number \( c_{1,n} \left( X \right) \) does not depend on the choice of the desingularization \( \rho : Y \to X \).

**Proof.** By 3.4 and 3.8, \( c_{1,n} \left( X \right) \) can be computed in terms of derivatives of the stringy E-function of \( X \). But the stringy E-function does not depend on the choice of a desingularization \( \boxed{\rho} \).

**Corollary 3.10** Let \( X \) be a projective variety with at worst Gorenstein canonical singularities. Assume that the stringy Hodge numbers of \( X \) exist. Then
\[ \sum_{p,q} (-1)^{p+q} h_{p,q} \left( X \right) \left( p - \frac{n}{2} \right)^2 = \frac{n}{12} \epsilon_{st} \left( X \right) + \frac{1}{6} c_{1,n-1} \left( X \right). \]

**Proof.** The equality follows immediately from 3.8 using the properties of the stringy Hodge numbers \( \boxed{2.4} \).

**Corollary 3.11** If the canonical class of \( X \) is numerically trivial, then \( c_{1,n} \left( X \right) = 0 \). In particular, for Calabi-Yau varieties with at worst Gorenstein canonical singularities we have
\[ \frac{d^2}{du^2} E_{st} \left( X; u, 1 \right)\bigg|_{u=1} = \frac{3n^2 - 5n}{12} \epsilon_{st} \left( X \right), \]

and therefore stringy Hodge numbers of \( X \) satisfy the identity (\( \boxed{7} \)) provided these stringy numbers exist.
4 Virasoro Algebra

Recall that the Virasoro algebra with the central charge $c$ consists of operators $L_n$ ($m \in \mathbb{Z}$) satisfying the relations

$$[L_n, L_m] = (n - m)L_{n+m} + c\frac{n^3 - n}{12}\delta_{n+m,0} \quad n, m \in \mathbb{Z}.$$ 

For arbitrary compact Kähler manifold $X$, Eguchi et. al have proposed in [7, 8] a new approach to its quantum cohomology and to its Gromov-Witten invariants for all genera $g$ using so called the Virasoro condition:

$$L_nZ = 0, \forall n \geq -1,$$

where

$$Z = \exp F = \exp \left( \sum_{g \geq 0} \lambda^{2g-2} F_g \right)$$

is the partition function of the topological $\sigma$-model with the target space $X$ and $F_g$ the free energy function corresponding to the genus $g$. In this approach, the central charge $c$ acts as the multiplication by $c_n(X)$. Moreover, all Virasoro operators $L_n$ can be explicitly written in terms of elements of a basis of the cohomology of $X$, their gravitational descendants and the action of $c_1(X)$ on the cohomology by the multiplication. In particular the commutator relation

$$[L_1, L_{-1}] = 2L_0$$

implies the precisely the identity of Libgober and Wood in the form

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(X) \left( \frac{n+1}{2} - p \right) \left( p - \frac{n-1}{2} \right) = \frac{1}{6} \left( \frac{3-n}{2} c_n(X) - c_1(X)c_{n-1}(X) \right).$$

Now let $X$ be a projective algebraic variety with at worst log-terminal singularities. We conjecture that there exists an analogous approach to the quantum cohomology as well as to the Gromov-Witten invariants of $X$ for all genera using the Virasoro algebra in such a way that for any resolution of singularities $\rho : Y \rightarrow X$ the corresponding Virasoro operators can be explicitely computed via the numbers $a_i$ appearing in the formula

$$K_X = \rho^* K_X + \sum_{i=1}^r a_i D_i$$

and bases in cohomology of all complete intersections $D_J$ together with the multiplicative actions of $c_1(D_J)$ in them. We consider our main result [3,8] as an evidence in favor of this conjecture.
References

[1] V.V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Algebraic Geom., 3 (1994), 493-535.

[2] V.V. Batyrev, D. Dais, Strong McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry, Topology, 35 (1996), 901-929.

[3] V.V. Batyrev, L.A. Borisov, Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds, in Mirror Symmetry II, (eds. S.-T. Yau), pp.65-80 (1995).

[4] V.V. Batyrev, L.A. Borisov, Mirror Duality and String-Theoretic Hodge Numbers, Invent. Math., 126 (1996), 183-203.

[5] V.V. Batyrev, Stringy Hodge numbers of varieties with Gorenstein canonical singularities, Preprint 1997, alg-geom/9711018.

[6] L.A. Borisov, On the Betti numbers and Chern classes of varieties with trivial odd cohomology groups, Preprint 1997, alg-geom/9703023.

[7] T. Eguchi, K. Hori and Ch.-Sh. Xiong, Quantum cohomology and Virasoro algebra, Phys. Lett. B402 (1997), 71-80.

[8] T. Eguchi, M. Jinzenji and Ch.-Sh. Xiong, Quantum Cohomology and Free Field Representation, hep-th/9709152.

[9] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the Minimal Model Program, Adv. Studies in Pure Math. 10 (1987), 283-360.

[10] A.S. Libgober and J.W. Wood, Uniqueness of the complex structure on Kahler manifolds of certain homotopy types, J. Diff. Geom. 32 no. 1, (1990) 139–154.

[11] S. Sethi, C. Vafa, and E. Witten, Constraints on Low-Dimensional String Compactifications, Nucl.Phys. B480 (1996), 213-224.

[12] S.M. Salamon, On the cohomology of Kahler and hyper-Kahler manifolds. Topology 35, no. 1 (1996), 137–155.