MULTILINEAR MULTIPLIER THEOREMS AND APPLICATIONS

LOUKAS GRAFAKOS, DANQING HE, HANH VAN NGUYEN, AND LIXIN YAN

ABSTRACT. We obtain new multilinear multiplier theorems for symbols of restricted smoothness which lie locally in certain Sobolev spaces. We provide applications concerning the boundedness of the commutators of Calderón and Calderón-Coifman-Journé.

1. Introduction

The theory of multilinear multipliers has made significant advances in recent years. An \( n \)-dimensional \( m \)-linear multiplier is a bounded function \( \sigma \) on \((\mathbb{R}^n)^m\) associated with an \( m \)-linear operator \( T_\sigma \) on \( \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) in the following way:

\[
T_\sigma(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) \sigma(\xi_1, \ldots, \xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi_1 \cdots d\xi_m,
\]

where \( f_j, j = 1, \ldots, m, \) are Schwartz functions in \( \mathbb{R}^n, \) and \( \hat{f}_j(\xi_j) = \int_{\mathbb{R}^n} f_j(x) e^{-2\pi i x \cdot \xi_j} dx \) is the Fourier transform of \( f_j. \) A classical result of Coifman and Meyer [9, 10] says that if for all sufficiently large multiindices \( \alpha_1, \ldots, \alpha_m \in (\mathbb{Z}^+ \cup \{0\})^n \) we have

\[
\left| \partial^{\alpha_1}_{\xi_1} \cdots \partial^{\alpha_m}_{\xi_m} \sigma(\xi_1, \ldots, \xi_m) \right| \lesssim (|\xi_1| + \cdots + |\xi_m|)^{-\sum(|\alpha_1| + \cdots + |\alpha_m|)}
\]

for all \( (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^n)^m \setminus \{(0, \ldots, 0)\}, \) then \( T_\sigma \) admits a bounded extension from \( L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) when \( 1 < p_1, \ldots, p_m \leq \infty, \) \( 1/p = 1/p_1 + \cdots + 1/p_m, \) and \( 1 \leq p < \infty. \) The extension of this theorem to indices \( p > 1/m \) was simultaneously obtained by Kenig and Stein [29] (when \( m = 2 \)) and Grafakos and Torres [22]. This theorem provides an \( m \)-linear extension of Mikhlin’s classical linear multiplier result [30]. Hörmander [25] obtained an improvement of Mikhlin’s theorem showing that when \( m = 1, \) \( T_\sigma \) maps \( L^{p_1}(\mathbb{R}^n) \) to \( L^{p_1}(\mathbb{R}^n), \) \( 1 < p_1 < \infty \) under the weaker condition

\[
\sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{s/2}(\sigma(2^j \cdot) \hat{\Psi}) \right\|_{L^2(\mathbb{R}^n)} < \infty,
\]

where \( s > n/2 \) and \( \hat{\Psi} \) is a smooth function supported in an annulus centered at the origin.

Here \( \Delta \) is the Laplacian and \( (I - \Delta)^{s/2} \) is an operator given on the Fourier transform side.
by multiplication with \((1 + 4\pi^2|\xi|^2)^{s/2}\). Hörmander’s theorem was extended to \(L^r\)-based Sobolev spaces and to indices \(p_1 \leq 1\), with \(L^{p_1}\) replaced by the Hardy space \(H^{p_1}\), by Calderón and Torchinsky \([5]\).

The adaptation of Hörmander’s theorem to the multilinear setting was first obtained by Tomita \([10]\). This theorem was later extended by Grafakos and Si \([20]\) to the range \(p < 1\) by replacing \(L^2\)-based Sobolev spaces by \(L^r\)-based Sobolev spaces. The endpoint cases where some \(p_j\) are equal to infinity were treated by Grafakos, Miyachi, and Tomita \([18]\). Fujita and Tomita \([13]\) provided weighted extensions of these results and also noticed that the operator \((I - \Delta)^{s/2}\) in \((\mathbb{R}^n)^m\) can be replaced by \((I - \Delta_{\xi_1})^{s_{1/2}} \cdots (I - \Delta_{\xi_m})^{s_{m/2}}\), where \(\Delta_{\xi_j}\) is the Laplacian in the \(\xi_j\)th variable. The bilinear version of the Calderón and Torchinsky theorem was proved by Miyachi and Tomita \([31]\), while the \(m\)-linear version (for general \(m\)) was proved by Grafakos and Nguyen \([16]\) and Grafakos, Miyachi, Nguyen, and Tomita \([17]\).

To study certain multilinear singular integrals, such as multicommutators, there is a need for a multilinear multiplier theorem that can handle symbols on \((\mathbb{R}^n)^m\) which, for instance, have one derivative in each variable but no two derivatives in a given variable. We notice that in the case where \(s_j\) are positive integers for all \(j\), replacing \((I - \Delta)^{s/2}\) on \((\mathbb{R}^n)^m\) by \((I - \Delta_{\xi_1})^{s_{1/2}} \cdots (I - \Delta_{\xi_m})^{s_{m/2}}\), as in Fujita and Tomita \([13]\), reflects the following decay condition for the derivatives of \(\sigma\)

\[
(1.4) \quad |\partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \cdots \partial_{\xi_m}^{\beta_m} \sigma(\xi_1, \ldots, \xi_m)| \lesssim (|\xi_1| + \cdots + |\xi_m|)^{-\sum_{j=1}^m |\beta_j|},
\]

where each multiindex \(\beta_j\) satisfies \(|\beta_j| \leq s_j\). In this case a given coordinate of \(\xi_j\) could be differentiated as many as \(s_j\) times. In this article we study multipliers that satisfy the following coordinate-wise version of (1.4)

\[
(1.5) \quad |\partial_{\xi_{11}}^{\beta_{11}} \cdots \partial_{\xi_{1n}}^{\beta_{1n}} \partial_{\xi_{21}}^{\beta_{21}} \cdots \partial_{\xi_{2n}}^{\beta_{2n}} \cdots \partial_{\xi_{m1}}^{\beta_{m1}} \cdots \partial_{\xi_{mn}}^{\beta_{mn}} \sigma(\xi_1, \ldots, \xi_m)| \lesssim (|\xi_1| + \cdots + |\xi_m|)^{-\sum_{j=1}^m \sum_{\ell=1}^n |\beta_{j\ell}|},
\]

where \(\xi_j = (\xi_{j1}, \ldots, \xi_{jn})\) and each \(\beta_{j\ell}\) is at most \(s_j/n\). Condition (1.5) weakens the Coifman-Meyer hypothesis (1.2) and also (1.4) in the sense that it does not allow any one-dimensional variable to be differentiated more than an appropriate number of times.

We now state our first main result concerning the operator \(T_\sigma\) in (1.1). Here and throughout the \(i\)th coordinate of the vector \(\xi_j\) in \(\mathbb{R}^n\) is denoted by \(\xi_{ji}\). We denote partial derivatives in the \(\xi_{ji}\) variable by \(\partial_{\xi_{ji}}\). Also the Laplacian \(\Delta_{\xi_j}\) on \(\mathbb{R}^n\) is given by \(\partial_{\xi_{j1}}^2 + \cdots + \partial_{\xi_{jn}}^2\). We have a result that extends condition (1.5) in the Sobolev space.
setting. We define \((I - \partial^2_{\xi_{it}})^{2r_i}f(\xi)\) as the linear operator \(((1 + 4\pi^2|\eta|^{2})^{2r_i}\hat{f}(\eta))^{\gamma}(\xi)\) related to the multiplier \((1 + 4\pi^2|\eta|^{2})^{2r_i}\).

**Theorem 1.1.** Suppose that \(1 \leq r \leq 2\) and \(\gamma_{i\ell} > 1/r\) for all \(1 \leq i \leq m\) and \(1 \leq \ell \leq n\). Let \(\sigma\) be a bounded function on \(\mathbb{R}^{mn}\) such that

\[
\sup_{j \in \mathbb{Z}} \left\| \prod_{1 \leq i \leq m} \prod_{1 \leq \ell \leq n} (I - \partial^2_{\xi_{it}})^{2r_i} \right\|_{L^r(\mathbb{R}^{mn})} = A < \infty,
\]

where \(\hat{\Psi}\) is a smooth function supported in the annulus \(\frac{1}{2} \leq |(\xi_1, \ldots, \xi_m)| \leq 2\) in \(\mathbb{R}^{mn}\) that satisfies

\[
\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}(\xi_1, \ldots, \xi_m)) = 1, \quad \text{for all } (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^n)^m \setminus \{0\}.
\]

If \(1 < p_i < \infty, i = 1, \ldots, m,\) satisfy \(\max_{1 \leq i \leq m} \max_{1 \leq \ell \leq n} \gamma_{i\ell} < \min_{1 \leq i \leq m} p_i\) and \(\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m},\) then we have

\[
\|T_\sigma\|_{L^p(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A.
\]

Taking \(\gamma_{i\ell} = \gamma_i/n\) for all \(\ell \in \{1, \ldots, n\}\) and using simple embeddings between Sobolev spaces we deduce the following consequence of Theorem 1.1.

**Corollary 1.2.** Let \(1 \leq r \leq 2\) and suppose that \(\gamma_i > n/r\) for all \(i = 1, \ldots, m\). Let \(\sigma\) be a bounded function on \(\mathbb{R}^{mn}\) such that

\[
\sup_{j \in \mathbb{Z}} \left\| (I - \Delta_{\xi_1})^{2r_1} \cdots (I - \Delta_{\xi_m})^{2r_m} \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^r(\mathbb{R}^{mn})} = A < \infty,
\]

where \(\Psi\) is as in Theorem 1.1. Then (1.7) holds where \(p_i\) are as in Theorem 1.1.

We also provide an endpoint case of Corollary 1.2 when all \(p_i = 1\). Let \(H^1(\mathbb{R}^n)\) denote the classical Hardy space on \(\mathbb{R}^n\). We note that when \(m = 1\), boundedness for \(T_\sigma\) is known to hold from \(H^1\) to \(L^1\).

**Theorem 1.3.** Suppose that \(1 \leq r \leq 2\) and that \(\gamma_i > n\) for all \(i = 1, \ldots, m\). Let \(\sigma\) be a bounded function on \(\mathbb{R}^{mn}\) which satisfies (1.8). Then we have

\[
\|T_\sigma\|_{H^1(\mathbb{R}^n) \times \cdots \times H^1(\mathbb{R}^n) \rightarrow L^{1/m, \infty}(\mathbb{R}^n)} \lesssim A.
\]

Another extension of the Coifman-Meyer multiplier theorem is in the multiparameter setting. In this case (1.2) is relaxed to

\[
\left| \partial_{\xi_{i_1}}^{\alpha_1} \cdots \partial_{\xi_{i_n}}^{\alpha_n} \partial_{\xi_{i_1}}^{\alpha_2} \cdots \partial_{\xi_{i_n}}^{\alpha_2} \cdots \partial_{\xi_{i_1}}^{\alpha_m} \cdots \partial_{\xi_{i_n}}^{\alpha_m} \sigma(\xi_1, \ldots, \xi_m) \right| \lesssim (|\xi_{i_1}| + \cdots + |\xi_{i_m}|)^{-\alpha_1 + \cdots + \alpha_m} \cdots (|\xi_{i_1}| + \cdots + |\xi_{i_m}|)^{-\alpha_1 + \cdots + \alpha_m}
\]
for sufficiently large indices $\alpha_{i\ell}$. Such a condition was first considered by Muscalu, Pipher, Tao, and Thiele [35, 36], who obtained boundedness for the associated operator in the case $m = 2$, i.e., from $L^{p_1} \times L^{p_2}$ to $L^p$ when $1/p_1 + 1/p_2 = 1/p$ and $1/2 < p < \infty$.

In this article we also prove a multilinear multiplier theorem that extends condition (1.10). Precisely, we study multilinear multipliers that satisfy

$$
|\partial^{\beta_{j1}}_{\xi_{j1}} \cdots \partial^{\beta_{jn}}_{\xi_{jn}} \sigma(\xi_1, \ldots, \xi_m)|
\lesssim (|\xi_{j1}| + \cdots + |\xi_{mn}|)^{-(\beta_{j1} + \cdots + \beta_{mn})}
$$

with $\beta_{ji}$ restricted. To handle the case of fractional derivatives we state our condition in terms of Sobolev spaces. We denote by $(I - \partial^2_{\xi_{j1}})^{\gamma_{j1}}$ the operator given on the Fourier transform side by multiplication by $(1 + 4\pi^2|y_{j1}|^2)^{\gamma_{j1}}$, where $y_{j}$ is the dual variable of $\xi_{j}$.

We now state our multiparameter version of Theorem 1.1 which extends the results in [35, 36] for Hörmander type multipliers with minimal smoothness in a way that avoids time-frequency analysis.

**Theorem 1.4.** Let $1 \leq r \leq 2$ and $\gamma_{ij} > 1/r$ for all $1 \leq i \leq m$ and $1 \leq \ell \leq n$. Suppose that $\sigma$ is a bounded function on $\mathbb{R}^{mn}$ such that

$$
\sup_{k_1, \ldots, k_n \in \mathbb{Z}} \left\| \prod_{j=1}^{m} (I - \partial^2_{\xi_{j1}})^{\gamma_{j1}} \cdots (I - \partial^2_{\xi_{jn}})^{\gamma_{jn}} \left[ \sigma(D_{k_1, \ldots, k_n} \xi) \prod_{\ell=1}^{n} \hat{\Psi}_\ell(\xi_1, \ldots, \xi_m) \right] \right\|_{L^r(\mathbb{R}^{mn})} = A < \infty,
$$

where

$$
D_{k_1, \ldots, k_n} \xi = \begin{bmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{m1} & \xi_{m2} & \cdots & \xi_{mn}
\end{bmatrix},
$$

for some $\hat{\Psi}_\ell$ smooth functions on $\mathbb{R}^m$ supported in the annulus $1/2 \leq |\eta| \leq 2$ satisfying

$$
\sum_{k \in \mathbb{Z}} \hat{\Psi}_\ell(2^{-k} \eta) = 1, \quad \text{for all } \eta \in \mathbb{R}^m \setminus \{0\}.
$$

If $1 < p_i < \infty$, $i = 1, \ldots, m$, satisfy $\max \max_{1 \leq i \leq m} \frac{1}{\gamma_{i\ell}} < \min_{1 \leq i \leq m} p_i$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then we have

$$
\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A.
$$

A version of Theorem 1.4 was proved by Chen and Lu [6] when $r = m = 2$ and when the differential operator $(I - \partial^2_{\xi_{j1}})^{\gamma_{j1}} \cdots (I - \partial^2_{\xi_{jn}})^{\gamma_{jn}}$ is replaced by $(I - \Delta_{\xi_{j}})^{\gamma_{j}}$, where $\gamma_{j} = \gamma_{j1} + \cdots + \gamma_{jn}$; besides allowing $r$ to be less than 2 and $m \geq 2$, Theorem 1.4 improves that of Chen and Lu [6] in the sense that only a restricted number of derivatives falls on each coordinate, while in [6] all derivatives could fall on a single coordinate $\xi_{j}$ of the multiplier. An immediate consequence of Theorem 1.4 is the following:
Corollary 1.5. Let \( \sigma_\ell(\xi_{1\ell}, \ldots, \xi_{m\ell}) \) be bounded functions on \( \mathbb{R}^m \) for \( 1 \leq \ell \leq n \). Let \( \sigma(\xi_1, \ldots, \xi_m) = \prod_{\ell=1}^n \sigma_\ell(\xi_{1\ell}, \ldots, \xi_{m\ell}) \), where \( \xi_i = (\xi_{i1}, \ldots, \xi_{in}) \in \mathbb{R}^n \), \( 1 \leq i \leq m \). Suppose that for some \( \gamma_{i\ell} \) and \( r \) as in Theorem 1.4 we have
\[
\text{sup}_{1 \leq \ell \leq n} \text{sup}_{k \in \mathbb{Z}} \| (I - \partial_{\xi_{1\ell}}^2)^{\frac{\gamma_{1\ell}}{2}} \cdots (I - \partial_{\xi_{m\ell}}^2)^{\frac{\gamma_{m\ell}}{2}} \left[ \sigma_\ell(2k) \hat{\Psi}_\ell \right] \|_{L^r(\mathbb{R}^m)} = B < \infty
\]
where \( \hat{\Psi}_\ell \) is a smooth function supported in an annulus in \( \mathbb{R}^m \) that satisfies (1.12). Then for \( \max_{1 \leq \ell \leq n} (\frac{1}{\gamma_{i\ell}}, 1) < p_i < \infty \) for all \( i = 1, \ldots, m \) and \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \) we have
\[
\| T_\sigma \|_{L^p(\mathbb{R}^n) \times \cdots \times L^{pm}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim B^n.
\]
As an application, we use this corollary to give a short proof of the boundedness of Calderón-Coifman-Journé commutators (Proposition 6.7) where the results in [6, 35, 36] are not applicable.

Finally, we use arrows to denote elements of \( \mathbb{R}^{nm} \), i.e., \( \vec{\xi} = (\xi_1, \ldots, \xi_m) \), where \( \xi_j \in \mathbb{R}^n \).

2. Preliminaries

The following lemma will be useful in the sequel.

Lemma 2.1. Let \( \gamma_{i\ell}, \gamma_{j}, \gamma > 0 \), \( 1 \leq i \leq m, \ 1 \leq j, \ell \leq n \). Let \( D^\Gamma \) be a differential operator on \( \mathbb{R}^{mn} \) of one of the following three types:
\[
\begin{align*}
&\prod_{i=1}^m \prod_{\ell=1}^n (I - \partial_{\xi_{i\ell}}^2)^{\frac{\gamma_{i\ell}}{2}}, \\
&(I - \Delta \xi_1)^{\frac{\gamma}{2}} \cdots (I - \Delta \xi_m)^{\frac{\gamma}{2}}, \\
&(I - \Delta \xi_1 - \cdots - \Delta \xi_m)^{\frac{\gamma}{2}}.
\end{align*}
\]

Let \( 1 < \rho \leq r < \infty \) and let \( \phi \) be a smooth function with compact support. Then there is a constant \( C = C(\rho, r, \phi, n, \gamma_{i\ell}, \gamma_{j}, \gamma) \) such that
\[
\| D^\Gamma (\phi f) \|_{L^\rho(\mathbb{R}^{mn})} \leq C \| D^\Gamma (f) \|_{L^r(\mathbb{R}^{mn})}
\]
is valid for all Schwartz functions \( f \) on \( \mathbb{R}^{mn} \).

Moreover, if \( D_\delta \) is an operator of one of the following three types:
\[
\begin{align*}
&\prod_{i=1}^m \prod_{\ell=1}^n (I - \partial_{\xi_{i\ell}}^2)^{\frac{\delta_{i\ell}}{2}}, \\
&(I - \Delta \xi_1)^{\frac{\delta}{2}} \cdots (I - \Delta \xi_m)^{\frac{\delta}{2}}, \\
&(I - \Delta \xi_1 - \cdots - \Delta \xi_m)^{\frac{\delta}{2}}
\end{align*}
\]
then for $D^\Gamma$ and $D_\delta$ of the same type and $\delta > 0$ we have

\[
(2.2) \quad \|D^\Gamma D_{-\delta}(\phi f)\|_{L^1(\mathbb{R}^{mn})} \leq C'\|D^\Gamma f\|_{L^1(\mathbb{R}^{nm})}
\]

for all Schwartz functions $f$ on $\mathbb{R}^{mn}$. Here $C'$ is a constant depending on $\phi, n, \gamma_i \epsilon, \gamma_j, \gamma, \delta$.

**Proof.** Estimate (2.1) could be derived by versions of the Kato-Ponce inequality adapted to the types of operators in question, such versions are given in [19, Section 5]. In the case where $D^\Gamma = (I - \Delta)^{\frac{\gamma}{2}}$, the proof of (2.1) is also given in [15, Lemma 7.5.7]. The idea in this reference also works in this setting. We provide a sketch: we embed $D^\Gamma$ in the analytic family of differential operators $D^z\Gamma$ (in which all $\gamma$’s are multiplied by $z$) and reduce matters to the inequality

\[
\|D^z\Gamma(\phi D^{-z}\Gamma f)\|_{L^p(\mathbb{R}^{mn})} \lesssim \|f\|_{L^r(\mathbb{R}^{mn})}.
\]

Let us assume that $\gamma_i \epsilon, \gamma_j, \gamma$ are rational numbers; if the case of rational numbers is proved, then by continuity we can deduce the result for all positive numbers as follows: on the right of the inequality we obtain a constant that is polynomial in $\gamma_i \epsilon, \gamma_j$ or $\gamma$. But each function $D^\Gamma(\phi f)$ and $D^\Gamma f$ is continuous in $\gamma_i \epsilon, \gamma_j$ or $\gamma$. Using this continuity we obtain the conclusion for all $\gamma_i \epsilon, \gamma_j, \gamma$ positive reals.

To prove (2.1) we interpolate between the cases where $z = it$ and $z = 2N + it$, where $N$ is a natural number and common multiple of all the denominators of $\gamma_i \epsilon, \gamma_j, \gamma$. At the endpoint cases $z = it$ and $z = 2N + it$, the $D^{it}\Gamma$ and $D^{-it}\Gamma$ are $L^p$ bounded with bounds that grow at most polynomially in $t$ (and in the $\gamma$’s), while $D^{2N}\Gamma$ is expanded via Leibniz’s rule. Applying the Hörmander multiplier theorem and Hölder’s inequality (to estimate the $L^p$ norm over the support of $\phi$ by the $L^r$ norm over the entire space) we obtain the claimed assertion in the cases where $z = it$ and $z = 2N + it$ with bounds that grow at most polynomially in $t$. Interpolation for analytic families of operators yields the claimed conclusion.

We now turn our attention to (2.2) which is equivalent to

\[
(2.3) \quad \|D^\Gamma D_{-\delta}(\phi D^{-\Gamma} D_{-\delta}(f))\|_{L^1(\mathbb{R}^{mn})} \leq C'\|f\|_{L^1(\mathbb{R}^{mn})}
\]

and observe that $D_{-\delta} = (D_\delta)^{-1}$.

We embed the operator $f \mapsto D^\Gamma D_{-\delta}(\phi D^{-\Gamma} D_{-\delta}(f))$ into the analytic family of operators $f \mapsto D^{z\Gamma} D_{-\delta}(\phi D^{-z\Gamma} D_{-\delta}(f))$ and we obtain (2.3) as a consequence of interpolation between the points $z = it$ and $z = 2N + it$, where $N$ is as before and $t$ is real. At the endpoint $z = it$ we have that $D^{\pm it\Gamma} D_{-\delta}$ is a convolution operator with an integrable
kernel and so
\[ \| D^{it\Gamma} D_{-\delta}(\phi D^{-it\Gamma} D_{-\delta}(f)) \|_{L^1} \lesssim \| \phi D^{-it\Gamma} D_{-\delta}(f) \|_{L^1} \lesssim \| D^{-it\Gamma} D_{-\delta}(f) \|_{L^1} \lesssim \| f \|_{L^1}, \]
with constants bounded by polynomial expressions of the \( \gamma \)'s and \( |t| \). When \( z = 2N + it \) we have
\[ (2.4) \quad \| D^{it\Gamma} D_{-\delta} D^{2\Gamma N} (\phi D^{-it\Gamma} D_{-\delta}(f)) \|_{L^1} \lesssim \| D^{2\Gamma N} (\phi D^{-2\Gamma N - it\Gamma} D_{-\delta}(f)) \|_{L^1} \]
and we expand the \( D^{2\Gamma N} \) derivative via Leibniz's rule. Then \( \| D^{2\Gamma N} (\phi G) \|_{L^1} \) is bounded by a constant multiple of a sum of terms like \( \| D^k (G) \|_{L^1} \) where \( 0 \leq k \leq 2\Gamma N \) and \( D^k \) has the same type as \( D^\Gamma \). Each operator of the form \( D^k D^{-2\Gamma N} D^{-it\Gamma} D_{-\delta} \) is given by convolution with an integrable kernel. At the end we control the right hand side of (2.4) by a constant multiple of \( \| f \|_{L^1} \), with a constant bounded by polynomial expressions of the \( \gamma \)'s and \( |t| \). This concludes the sketch of proof. \( \square \)

We will also need a reverse square function inequality associated with Littlewood-Paley operators acting on each variable separately. We denote variables in \( \mathbb{R}^n \) by \( (z_1, \ldots, z_n) \), where each \( z_j \) lies in \( \mathbb{R}^l \). Fix a smooth function \( \widehat{\Psi} \) supported in an annulus in \( \mathbb{R}^l \) satisfying \( \sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}z) = 1 \) for all \( z \neq 0 \). For \( j \in \mathbb{Z} \), define a Littlewood-Paley operator
\[ \Delta_j^{(k)}(f) = (\widehat{f}(z_1, z_2, \ldots, z_n) \widehat{\Psi}(2^{-j}z_k))^\vee \]
acting on functions \( f \) on \( \mathbb{R}^n \). We need the following result.

**Lemma 2.2.** For \( f \in L^p(\mathbb{R}^n) \) with \( 1 < p < \infty \) we have
\[ (2.5) \quad \left\| \left( \sum_{j_1} \cdots \sum_{j_n} |\Delta_j^{(1)}(f) | \cdots |\Delta_j^{(n)}(f) | \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)}. \]

Conversely, for \( 0 < p < \infty \) there exists a constant \( C \) such that for any \( f \) in \( L^2(\mathbb{R}^n) \) satisfying
\[ (2.6) \quad \| \left( \sum_{j_1} \cdots \sum_{j_n} |\Delta_j^{(1)}(f) | \cdots |\Delta_j^{(n)}(f) | \right)^{1/2} \|_{L^p} < \infty \]
we have
\[ (2.7) \quad \| f \|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{j_1} \cdots \sum_{j_n} |\Delta_j^{(1)}(f) | \cdots |\Delta_j^{(n)}(f) | \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}. \]

**Proof of Lemma 2.2.** The proof of (2.5) is well known and is omitted; see for instance [13, Theorem 6.1.6] when \( l = 1 \) but the same idea works for all \( l \). So we now focus on (2.7) which we prove inductively. The case \( n = 1 \) is the reverse of the Littlewood-Paley...
inequality when \( p > 1 \). When \( n = 1 \) and \( p \leq 1 \), then by \cite{[15]} Theorem 2.2.9 there is a polynomial \( Q \) on \( \mathbb{R}^l \) such that

\[
\|f - Q\|_{H^p(\mathbb{R}^l)} \lesssim \left\| \left( \sum_{j_1} |\Delta_{j_1}^{(1)}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^l)} < \infty.
\]

Since \( f \) lies in \( L^2(\mathbb{R}^l) \), it follows that \( f - Q \) is a locally integrable function which lies in \( H^p(\mathbb{R}^l) \) and thus \( \|f - Q\|_{L^p} \lesssim \|f - Q\|_{H^p(\mathbb{R}^l)} < \infty \). Therefore \( Q = 0 \) and (2.7) follows.

Assume that the assertion is valid for \( n \). We will prove the case \( n + 1 \). Let \( r_k \) be the Rademacher functions reindexed by \( k \in \mathbb{Z} \). Applying (2.7) to \( g = \sum_k f_k r_k \) we obtain

\[
\int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \left( \sum_k |f_k(x_1, \ldots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n
\]

\[
\lesssim \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \int_0^1 \left| \sum_k f_k(x_1, \ldots, x_n) r_k(t_{n+1}) \right|^p dt_{n+1} dx_1 \cdots dx_n
\]

\[
= C \int_0^1 \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} |g(x_1, \ldots, x_n)|^p dx_1 \cdots dx_n dt_{n+1},
\]

where we used the property of Rademacher functions; see for instance \cite{[14]} Appendix C. By the induction hypothesis, the preceding expression is bounded by a multiple of

\[
\int_0^1 \int_{\mathbb{R}^l} \left( \sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g(x_1, \ldots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n dt_{n+1}
\]

\[
\lesssim \int_0^1 \int_{\mathbb{R}^l} \int_{[0,1]^n} \left| \sum_{j_1} \cdots \sum_{j_n} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g(x_1, \ldots, x_n) \prod_{i=1}^n r_{j_i}(t_i) \right|^p dt_1 \cdots dt_n d\vec{x} dt_{n+1}
\]

\[
\approx \int_{\mathbb{R}^l} \int_{[0,1]^{n+1}} \left| \sum_{j_1, \ldots, j_n, k} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1, \ldots, x_n) r_k(t_{n+1}) \prod_{i=1}^n r_{j_i}(t_i) \right|^p dt_1 \cdots dt_{n+1} d\vec{x}
\]

\[
\lesssim \int_{\mathbb{R}^l} \left( \sum_{j_1} \cdots \sum_{j_n} \sum_k |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1, \ldots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n,
\]

once again the properties of Rademacher functions were used and \( d\vec{x} = dx_1 \cdots dx_n \).

It follows that

\[
\int_{\mathbb{R}^{n+1}} |f(x_1, \ldots, x_{n+1})|^p dx_1 \cdots dx_{n+1}
\]

\[
\lesssim \int_{\mathbb{R}^{n+1}} \sup_{t > 0} |(\varphi_t \ast f(x_1, \ldots, x_n))(x_{n+1})|^p dx_{n+1} dx_1 \cdots dx_n
\]

\[
\lesssim \int_{\mathbb{R}^{n+1}} \left( \sum_{j_{n+1}} |\Delta_{j_{n+1}}^{(n+1)} f(x_1, \ldots, x_{n+1})|^2 \right)^{p/2} dx_{n+1} dx_1 \cdots dx_n
\]
\[
\approx \int_{\mathbb{R}^{n+1}} \left( \sum_{j_n+1} |\Delta_{j_n+1}^{(n+1)} f(x_1, \ldots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_{n+1}
\]

\[
\lesssim \int_{\mathbb{R}^{n+1}} \left( \sum_{j_1} \cdots \sum_{j_n} \sum_{j_n+1} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} \Delta_{j_n+1}^{(n+1)} f(x_1, \ldots, x_n, x_{n+1})|^2 \right)^{p/2} dx_1 \cdots dx_n dx_{n+1},
\]

where in the last step we use the inequality in the preceding alignment. To make this argument precise, we work with finitely many terms and then then pass to limit using Fatou’s lemma.

\[\square\]

**Remark 2.3.** In both (2.5) and (2.7) we do not need the full set of variables. For example, we have

\[
\left\| \left( \sum_{j_1 \in \mathbb{Z}} \cdots \sum_{j_q \in \mathbb{Z}} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_q}^{(q)} f(x)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \approx \|f\|_{L^p(\mathbb{R}^n)}
\]

for any \(1 \leq q \leq n\) by applying Lemma 2.2 to \(f\) as a function of \((x_1, \ldots, x_q)\).

**Remark 2.4.** As a consequence of (2.7) one can derive the following inequality:

\[
\|f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{H^p(\mathbb{R}^l \times \cdots \times \mathbb{R}^l)} \quad \text{for } f \in L^2(\mathbb{R}^n), \ 0 < p \leq 1
\]

where \(H^p(\mathbb{R}^l \times \cdots \times \mathbb{R}^l)\) denotes the multiparameter Hardy space; on this see [23].

## 3. The proof of Theorem 1.1

**Proof.** For \(1 \leq k \neq l \leq m\), we introduce sets

\[
U_{k,l} = \left\{ (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^n)^m : \max_{j \neq k,l} |\xi_j| \leq \frac{11}{10} |\xi_k| \leq \frac{11}{50m} |\xi_l| \right\}
\]

and

\[
W_{k,l} = \left\{ (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^n)^m : \max_{j \neq k,l} |\xi_j| \leq \frac{11}{10} |\xi_k|, \frac{1}{10m} |\xi_l| \leq |\xi_k| \leq 2|\xi_l| \right\}.
\]

We now construct smooth homogeneous of degree zero functions \(\Phi_{k,l}\) and \(\Psi_{k,l}\) supported in \(U_{k,l}\) and \(W_{k,l}\), respectively, and such that

\[
\sum_{1 \leq k \neq l \leq m} \left( \Phi_{k,l}(\xi_1, \ldots, \xi_m) + \Psi_{k,l}(\xi_1, \ldots, \xi_m) \right) = 1
\]

for every \((\xi_1, \ldots, \xi_m)\) in \((\mathbb{R}^n)^m \setminus \{0\}\); such functions can be constructed following the hint of Exercise 7.5.4 in [15]. In the support of \(\Phi_{k,l}\) the vector with the largest magnitude is \(\xi_l\), while in the support of \(\Psi_{k,l}\) the vector with the largest magnitude is \(\xi_l\) and the one
with the second largest magnitude is $\xi_k$. This partition of unity induces the following decomposition of $\sigma$:

$$\sigma = \sum_{j=1}^{m} \sum_{k=1 \atop k \neq j}^{m} (\sigma \Phi_{j,k} + \sigma \Psi_{j,k}) .$$

We will prove the required assertion for each piece of this decomposition, i.e., for the multipliers $\sigma \Phi_{j,k}$ and $\sigma \Psi_{j,k}$ for each pair $(j,k)$ in the previous sum. In view of the symmetry of the decomposition, it suffices to consider the case of a fixed pair $(j,k)$ in the sum in (3.2). To simplify the notation, we fix the pair $(m,m - 1)$; thus, for the rest of the proof we fix $j = m$ and $k = m - 1$, and we prove boundedness for the $m$-linear operators whose symbols are $\sigma_1 = \sigma \Phi_{m,m-1}$ and $\sigma_2 = \sigma \Psi_{m,m-1}$. These correspond to the $m$-linear operators $T_{\sigma_1}$ and $T_{\sigma_2}$, respectively. Note that $\sigma_1$ is supported in the set where

$$\max(|\xi_1|, \ldots, |\xi_{m-2}|) \leq \frac{1}{10} |\xi_{m-1}| \quad \text{and} \quad |\xi_{m-1}| \leq \frac{1}{5m} |\xi_m| .$$

Also $\sigma_2$ is supported in the set where

$$\max(|\xi_1|, \ldots, |\xi_{m-2}|) \leq \frac{1}{10} |\xi_{m-1}| \quad \text{and} \quad \frac{1}{10m} \leq \frac{|\xi_{m-1}|}{|\xi_m|} \leq 2 .$$

Fix a Schwartz function $\theta$ whose Fourier transform is supported in the annulus $\frac{1}{2} \leq |\xi| \leq 2$ and $\sum_{j \in \mathbb{Z}} \widehat{\theta}(2^{-j} \xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Associated with $\theta$ we define the Littlewood–Paley operator $\Delta_\theta^j(g) = g \ast \theta_{2^{-j}}$, where $\theta_t(x) = t^{-n}\theta(t^{-1}x)$ for $t > 0$. The function $\theta$ can be extended to the function $\Theta$ defined on $\mathbb{R}^{nm}$ by setting $\widehat{\Theta}(\xi) = \widehat{\Theta}(\xi_1, \ldots, \xi_m) = \widehat{\Theta}(\xi_1 + \cdots + \xi_m)$. For given Schwartz functions $f_j$ we have

$$\Delta_\theta^j(T_{\sigma_1}(f_1, \ldots, f_m))(x)$$

$$= \int_{\mathbb{R}^{mn}} \widehat{\theta}(2^{-j}(\xi_1 + \cdots + \xi_m)) \sigma_1(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi$$

$$= \int_{\mathbb{R}^{mn}} \widehat{\Theta}(2^{-j} \xi) \sigma_1(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi .$$

Note that for all $\xi = (\xi_1, \ldots, \xi_m)$ in the support of the function $\widehat{\Theta}(2^{-j} \xi) \sigma_1(\xi)$, we always have $2^{j-2} \leq |\xi_m| \leq 2^{j+2}$. Therefore we can take a Schwartz function $\eta$ whose Fourier transform is supported in $\frac{1}{8} \leq |\xi_m| \leq 8$ and identical to 1 on $\frac{1}{4} \leq |\xi_m| \leq 4$ and insert the factor $\widehat{\eta}(2^{-j} \xi_m)$ into the above integral without changing the outcome. More specifically

$$\Delta_\theta^j(T_{\sigma_1}(f_1, \ldots, f_m))(x)$$

$$= \int_{\mathbb{R}^{mn}} \widehat{\Theta}(2^{-j} \xi) \sigma_1(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_{m-1}(\xi_{m-1}) \hat{f}_m(\xi_m) \widehat{\eta}(2^{-j} \xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi .$$
Now define \( \hat{\Psi}_s(\xi) = \sum_{|k| \leq 4} \hat{\Psi}(2^{-k}\xi) \) and note that \( \hat{\Psi}_s(2^{-j}\xi) \) is equal to 1 on the annulus \( \{ \xi \in \mathbb{R}^m : 2^{-j-4} \leq |\xi| \leq 2^{-j+4} \} \) which contains the support of \( \sigma_1 \hat{\Theta}(2^{-j} \cdot) \). Then we write

\[
\Delta_j^\rho(T_{\sigma_1}(f_1, \ldots, f_m))(x) = \int_{\mathbb{R}^m} \hat{\Psi}_s(2^{-j}\xi) \hat{\Theta}(2^{-j}\xi) \sigma_1(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_{m-1}(\xi_{m-1}) \hat{f}_m(\xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} d\xi.
\]

Taking the inverse Fourier transform, we obtain that \( \Delta_j^\rho(T_{\sigma_1}(f_1, \ldots, f_m))(x) \) is equal to

\[
(3.3) \quad \int_{(\mathbb{R}^n)^m} 2^{mnj} (\sigma_1^j \hat{\Psi}_s \hat{\Theta})^\vee 2^j (x - y_1), \ldots, 2^j (x - y_m)) \prod_{i=1}^{m-1} f_i(y_i) (\Delta_j^\eta f_m)(y_m) d\vec{y},
\]

where \( d\vec{y} = dy_1 \cdots dy_m \), and \( \sigma_1^j(\xi_1, \xi_2, \ldots, \xi_m) = \sigma_1(2^j \xi_1, 2^j \xi_2, \ldots, 2^j \xi_m) \).

Recall our assumptions that \( \max \max_{1 \leq i \leq m, 1 \leq \ell \leq n} \frac{1}{\gamma_i} < r \) and \( \max \max_{1 \leq i \leq m, 1 \leq \ell \leq n} \frac{1}{\gamma_i} < \min(p_1, \ldots, p_m) \).

If \( r > 1 \) we pick \( \rho \) such that \( 1 < \rho < 2 \) and \( \max \max_{1 \leq i \leq m, 1 \leq \ell \leq n} \frac{1}{\rho} < \rho < \min(p_1, \ldots, p_m, r) \).

If \( r = 1 \), we set \( \rho = 1 \).

Define a weight for \((y_1, \ldots, y_m) \in (\mathbb{R}^n)^m\) by setting

\[
w_{\vec{y}}(y_1, \ldots, y_m) = \prod_{1 \leq i \leq m} \prod_{1 \leq \ell \leq n} (1 + 4\pi^2 |y_{i\ell}|^2)^{\frac{\rho \ell}{2}}.
\]

Let us first suppose that \( \rho > 1 \). We have

\[
|\Delta_j^\rho(T_{\sigma_1}(f_1, \ldots, f_{m-1}, f_m))(x)| \leq \int_{(\mathbb{R}^n)^m} w_{\vec{y}}(2^j(x - y_1), \ldots, 2^j(x - y_m)) |(\sigma_1^j \hat{\Psi}_s \hat{\Theta})^\vee 2^j (x - y_1), \ldots, 2^j (x - y_m))|\]

\[
\times \frac{2^{mnj} |f_1(y_1) \cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|}{w_{\vec{y}}(2^j(x - y_1), \ldots, 2^j(x - y_m))} d\vec{y}
\]

\[
\leq \left[ \int_{(\mathbb{R}^n)^m} \left| (w_{\vec{y}}(\sigma_1^j \hat{\Psi}_s \hat{\Theta})^\vee)(2^j(x - y_1), \ldots, 2^j(x - y_m)) \right|^{\rho \ell} d\vec{y} \right]^\frac{1}{\rho \ell}
\]

\[
\times 2^{mnj} \left( \int_{(\mathbb{R}^n)^m} \frac{|f_1(y_1) \cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|}{w_{\vec{y}}(2^j(x - y_1), \ldots, 2^j(x - y_m))} d\vec{y} \right)^\frac{1}{\rho}
\]

\[
\leq C \left( \int_{(\mathbb{R}^n)^m} \left| w_{\vec{y}}(y_1, \ldots, y_m)(\sigma_1^j \hat{\Psi}_s \hat{\Theta})^\vee(y_1, \ldots, y_m) \right|^{\rho \ell} d\vec{y} \right)^\frac{1}{\rho \ell}
\]

\[
\times \left( \int_{(\mathbb{R}^n)^m} \left( \frac{2^{mnj} |f_1(y_1) \cdots f_{m-1}(y_{m-1})(\Delta_j^\eta f_m)(y_m)|}{\prod_{\ell=1}^{n}(1 + 2^j |x_\ell - y_{i\ell}|)^{\rho \gamma_i} \cdots (\prod_{\ell=1}^{n}(1 + 2^j |x_\ell - y_{m\ell}|)^{\rho \gamma_i}} \right) d\vec{y} \right)^\frac{1}{\rho}
\]

\[
\leq C \left\| \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{x_{i\ell}}^2)^{\frac{\rho \ell}{2}} (\sigma_1^j \hat{\Psi}_s \hat{\Theta}) \right\|_{L^\rho} \prod_{i=1}^{m-1} \left( \int_{\mathbb{R}^n} \frac{2^{jn} |f_i(y_i)|^\rho}{\prod_{\ell=1}^{n}(1 + 2^j |x_\ell - y_{i\ell}|)^{\rho \gamma_i}} d\vec{y} \right)^\frac{1}{\rho \ell}
\]
\[ \leq C' \parallel \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{\xi_{\ell i}}^2)^{\gamma_i^{\ell}} \parallel_{L^p} \left[ \prod_{i=1}^{m-1} \mathcal{M}(|f_i|^p(x)) \right] \mathcal{M}(|f_m|^p(x))^{\frac{1}{p}} \]

where \( \mathcal{M} \) is the strong maximal function given as \( \mathcal{M} = M^{(1)} \circ \cdots \circ M^{(n)} \), where \( M^{(j)} \) is the Hardy-Littlewood maximal operator acting in the \( j \)th variable. Here we made use of the hypothesis that \( \gamma_{i\ell} \rho > 1 \) and we used the Hausdorff-Young inequality, which is possible since \( 1 \leq \rho < 2 \). Now using (2.1) we obtain

\[ \left\| \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{\xi_{\ell i}}^2)^{\gamma_i^{\ell}} \right\|_{L^p} \leq A. \]

We now turn to the case where \( r = 1 \) in which case \( \rho = 1 \). We choose \( \gamma'_{i\ell} < \gamma_{i\ell} \) and \( \delta > 0 \) such that

\[ \frac{1}{\gamma_{i\ell}} = \frac{1}{\gamma'_{i\ell} + \delta} < \frac{1}{\gamma'_{i\ell} - \delta} < \frac{1}{r} = 1 \]

for all \( i, \ell \). The preceding argument with \( \gamma'_{i\ell} - \delta \) in place of \( \gamma_{i\ell} \) yields that is bounded by

\[ |\Delta^\theta\{T_{\sigma_i}(f_1, \ldots, f_m)\}| \leq C' \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{\xi_{\ell i}}^2)^{\gamma_i^{\ell} - \delta} (\sigma_i^j \widehat{\psi}_i \widehat{\theta}) \left\| \prod_{i=1}^{m-1} \mathcal{M}(|f_i|) \mathcal{M}(|f_m|^p) \right\|_{L^1} \leq A. \]

In view of (2.2) we obtain

\[ \leq \prod_{i=1}^{m} \prod_{\ell=1}^{n} (I - \partial_{\xi_{\ell i}}^2)^{\gamma_i^{\ell} + \delta} (\sigma_i^j \widehat{\psi}_i \widehat{\theta}) \left\| \prod_{i=1}^{m-1} \mathcal{M}(|f_i|) \mathcal{M}(|f_m|^p) \right\|_{L^1} \leq A. \]

Thus, we have obtained the estimate

\[ |\Delta^\theta\{T_{\sigma_i}(f_1, \ldots, f_m)\}(x)| \leq A \left[ \prod_{i=1}^{m-1} \mathcal{M}(|f_i|^p(x))^{\frac{1}{p}} \mathcal{M}(|f_m|^p(x))^{\frac{1}{p}} \right] \]

from which it follows that

\[ \left( \sum_{j \in \mathbb{Z}} |\Delta^\theta\{T_{\sigma_j}(f_1, \ldots, f_m)\}|^2 \right)^{\frac{1}{2}} \leq A \left[ \prod_{i=1}^{m-1} \mathcal{M}(|f_i|^p(x))^{\frac{1}{p}} \mathcal{M}(|f_m|^p(x))^{\frac{1}{p}} \right] \left( \sum_{j \in \mathbb{Z}} \mathcal{M}(|\Delta^\theta f_m|^p)^{\frac{1}{p}} \right)^{\frac{1}{2}}. \]

The claimed bound follows by applying Hölder’s inequality with exponents \( p_1, \ldots, p_m \) and using the boundedness of \( \mathcal{M} \) on \( L^{p_i/p} \), \( i = 1, \ldots, m \), and the Fefferman-Stein [12] vector-valued Hardy-Littlewood maximal function inequality on \( L^{p_m/p} \). (Note \( 1 < 2/p \leq 2 \).)

Next we deal with \( \sigma_2 \). Using the notation introduced earlier, we write

\[ T_{\sigma_2}(f_1, \ldots, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_2}(f_1, \ldots, f_m, \Delta^\theta f_m). \]
We introduce another Littlewood–Paley operator $\Delta^\theta_d$, which is given on the Fourier transform by multiplying with a bump $\widehat{\zeta}(2^{-j}\xi)$, where $\zeta$ is equal to one on the annulus $\{\xi \in \mathbb{R}^n : \frac{1}{2^k} \leq |\xi| \leq 4\}$ with $\frac{1}{2^k} \leq \frac{1}{2^{m+1}}$, vanishes off the annulus $\frac{1}{2^{m+1}} \leq |\xi| \leq 8$, and $\sum_j \zeta(2^{-j}\xi) = k + 3$. The key observation in this case is that

(3.4) $T_{\sigma_2}(f_1, \ldots, f_{m-1}, \Delta^\theta_d f_m) = T_{\sigma_2}(f_1, \ldots, f_{m-2}, \Delta^\theta_d f_{m-1}, \Delta^\theta_d f_m)$.

As in the previous case, we have

$$T_{\sigma_2}(f_1, \ldots, f_{m-2}, \Delta^\theta_d f_{m-1}, \Delta^\theta_d f_m)(x) = \int_{(\mathbb{R}^n)^m} \sigma_2(\xi) \prod_{i=1}^{m-2} \hat{f}_i(\xi_i) \Delta^\theta_d f_{m-1}(\xi_{m-1}) \Delta^\theta_d f_m(\xi_m) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} \, d\xi.$$  

The integrand in the right-hand side of (3.5) is supported in $\frac{1}{2}2^j \leq |\xi_1| + \cdots + |\xi_m| \leq \frac{11m+2j}{5}$. Thus one may insert the factor

$$\Psi_*(2^{-j}\xi_1, \ldots, 2^{-j}\xi_m) = \sum_{|\xi| \leq m+1} \Psi(2^{-j-k}\xi_1, \ldots, 2^{-j-k}\xi_m)$$

in the integrand.

A similar calculation as in the case for $\sigma_1$ yields the estimate

$$|T_{\sigma_2}(f_1, \ldots, f_{m-2}, \Delta^\theta_d f_{m-1}, \Delta^\theta_d f_m)| \lesssim A \left( \prod_{i=1}^{m-2} M(|f_i|^\rho)^{\frac{j}{p}} \right) M(|\Delta^\theta_d f_{m-1}|^\rho)^{\frac{j}{p}} M(|\Delta^\theta_d f_m|^\rho)^{\frac{j}{p}}.$$ 

Summing over $j$ and taking $L^p$ norms yields

$$\left\| T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m) \right\|_{L^p(\mathbb{R}^n)} \leq C A \left( \prod_{i=1}^{m-2} M(|f_i|^\rho)^{\frac{j}{p}} \right) \left( \sum_{j \in \mathbb{Z}} M(|\Delta^\theta_d f_{m-1}|^\rho)^{\frac{j}{p}} M(|\Delta^\theta_d f_m|^\rho)^{\frac{j}{p}} \right)_{L^p} \leq C A \left( \prod_{i=1}^{m-2} M(|f_i|^\rho)^{\frac{j}{p}} \right) \left( \sum_{j \in \mathbb{Z}} M(|\Delta^\theta_d f_{m-1}|^\rho)^{\frac{j}{p}} \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}} M(|\Delta^\theta_d f_m|^\rho)^{\frac{j}{p}} \right)^{\frac{1}{2}}.$$ 

Applying Hölder’s inequality, the boundedness of $M$ on $L^{p_i/p_i}, i = 1, \ldots, m - 1$, and the Fefferman-Stein [12] vector-valued Hardy-Littlewood maximal function inequality on $L^{p_{m-1}/p}$ or on $L^{p_m/p}$ (noting that $1 < 2/p \leq 2$) concludes the proof of the theorem.  

**Remark 3.1.** In case I we obtained the estimate

$$|\Delta^\theta_d (T_{\sigma_1}(f_1, \ldots, f_{m-1}, f_m))| \lesssim A \left( \prod_{i=1}^{m-1} M(|f_i|^\rho)^{\frac{j}{p}} \right) M(|\Delta^\theta_d f_m|^\rho)^{\frac{j}{p}}.$$
In case II we obtained the estimate
\[|T_{\sigma_j}(f_1, \ldots, f_{m-1}, \Delta_j^0 f_m)| \lesssim A \left( \prod_{i=1}^{m-2} \mathcal{M}(|f_i|^{\rho}) \right)^{1/2} \mathcal{M}(|\Delta_j^0 f_{m-1}|^{\rho})^{1/2} \mathcal{M}(|\Delta_j^0 f_m|^{\rho})^{1/2}.\]

By symmetry for any \(k_0 \neq j_0\) in \(\{1, \ldots, m\}\) we have for \(\sigma_{j_0,k_0}\)
\[|\Delta_j^0(T_{\sigma_{j_0,k_0}}(f_1, \ldots, f_m))| \lesssim A \left( \prod_{1 \leq i \leq m, i \neq j_0} \mathcal{M}(|f_i|^{\rho}) \right)^{1/2} \mathcal{M}(|\Delta_j^0 f_{j_0}|^{\rho})^{1/2},\]

and for \(\sigma_{j_0,k_0}\)
\[|T_{\sigma_{j_0,k_0}}(f_1, \ldots, \Delta_j^0 f_{j_0}, \ldots, f_m)| \lesssim A \left( \prod_{1 \leq i \leq m, i \neq j_0, i \neq k_0} \mathcal{M}(|f_i|^{\rho}) \right)^{1/2} \mathcal{M}(|\Delta_j^0 f_{j_0}|^{\rho})^{1/2} \mathcal{M}(|\Delta_j^0 f_{k_0}|^{\rho})^{1/2}.\]

4. The proof of Theorem 1.3

Proof. For \(1 \leq k \neq l \leq m\), recall the sets \(U_{k,l}\) and \(W_{k,l}\) and the functions \(\Phi_{k,l}\) and \(\Psi_{k,l}\) in the proof of Theorem 1.1. Letting \(\sigma_{k,l} = \sigma_{\Phi_{k,l}}\) and \(\sigma_{k,l}^2 = \sigma_{\Psi_{k,l}}\), we write
\[
\sigma = \sum_{1 \leq k \neq l \leq n} (\sigma_{k,l}^1 + \sigma_{k,l}^2).
\]

By the symmetry, it suffices to consider the case where \(k = m - 1\) and \(l = m\). We establish the claimed estimate for \(T_{\sigma_1}\) and \(T_{\sigma_2}\) with \(\sigma_1 = \sigma_{m-1,m}^1\) and \(\sigma_2 = \sigma_{m-1,m}^2\).

We first consider \(T_{\sigma_1}(f_1, \ldots, f_m)\), where \(f_j\) are fixed Schwartz functions. We will prove
\[
|\Delta_j^0(T_{\sigma_1}(f_1, \ldots, f_m))|^{1/2} \lesssim A \|f_1\|_{H^1(\mathbb{R}^n)} \cdots \|f_m\|_{H^1(\mathbb{R}^n)}.
\]

Let \(H^{1/m, \infty}\) denote the weak Hardy space of all bounded tempered distributions whose smooth maximal function lies in weak \(L^{1/m}\). Given \(0 < p < \infty\), for \(F\) in \(L^2(\mathbb{R}^n)\) there is a polynomial \(Q\) on \(\mathbb{R}^n\) such that
\[
\|F - Q\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C_{p,n} \|F - Q\|_{H^{0, \infty}(\mathbb{R}^n)} \approx \left( \sum_j |\Delta_j[F]|^2 \right)^{1/2} \|F - Q\|_{L^{p, \infty}(\mathbb{R}^n)},
\]

by a result of He [24]. But the fact that \(F\) lies in \(L^2\) implies that \(Q = 0\). Applying (4.2) with \(F = T_{\sigma_1}(f_1, \ldots, f_m)\), for which we observe that \(\|T_{\sigma_1}(f_1, \ldots, f_m)\|_{L^2(\mathbb{R}^n)} < \infty\) for Schwartz functions \(f_j\), we conclude from (4.1) that (1.9) holds for \(\sigma_1\).

To verify (1.1), we recall (3.3) and set \(\omega_{\gamma_i}(y) = (1 + 4\pi^2|y|^2)^{\gamma_i/2}\) for \(y \in \mathbb{R}^n\). Choose \(\gamma'_i\) and \(\delta > 0\) such that \(n < \gamma'_i - \delta < \gamma'_i < \gamma'_i + \delta = \gamma_i\) for all \(1 \leq i \leq m\).

Now we rewrite
\[
|\Delta_j^0(T_{\sigma_1}(f_1, \ldots, f_m))(x)|
\]
\[
\leq \int_{(\mathbb{R}^n)^m} \left\{ \prod_{i=1}^m \omega_{\gamma_i' - \delta}(2^i(x - y_i)) \right\} \left| (\sigma_1^j \hat{\Psi}, \hat{\Theta})^\vee (2^i(x - y_1), \ldots, 2^i(x - y_m)) \right| \\
\times \frac{2^{mnj} |f_1(y_1)| \cdots |f_{m-1}(y_{m-1})| |(\Delta_j f_m)(y_m)|}{\prod_{i=1}^m \omega_{\gamma_i' - \delta}(2^i(x - y_i))} d\vec{y}
\]

(4.3) \[ \lesssim \left\| \left( \prod_{i=1}^m \omega_{\gamma_i' - \delta}(\sigma_1^j \hat{\Psi}, \hat{\Theta})^\vee \right) \left( \prod_{i=1}^{m-1} M(f_i)(x) \right) M(\Delta_j^\eta f_m)(x) \right\|_{L^\infty} \lesssim A
\]
as a consequence of the fact that \( \gamma_i' - \delta > n \) for all \( 1 \leq i \leq m \). Here \( M \) is the uncentered Hardy-Littlewood maximal operator. In view of the Hausdorff-Young inequality, the first factor in (4.3) is bounded by

\[
\left\| \prod_{i=1}^m (I - \Delta_{\xi_i})^{\gamma_i' - \delta}(\sigma_1^j \hat{\Psi}, \hat{\Theta}) \right\|_{L^1} \lesssim \left\| \prod_{i=1}^m (I - \Delta_{\xi_i})^{\gamma_i' - \delta}(\sigma(2^j \cdot) \hat{\Psi}) \right\|_{L^1} \lesssim A
\]

where the penultimate inequality is a consequence of (2.2) and that \( \gamma_i' + \delta = \gamma_i \).

Thus, we proved

\[
|\Delta_j^\theta(T_{\sigma_1}(f_1, \ldots, f_m))| \lesssim A \left( \prod_{i=1}^{m-1} M(f_i) \right) M(\Delta_j^\eta f_m).
\]

Using the preceding inequality we obtain

\[
\|T_{\sigma_1}(f_1, \ldots, f_{m-1}, f_m)\|_{H^{1/m, \infty}(\mathbb{R}^n)} \\
\lesssim \left\| \left\{ \sum_j |\Delta_j^\theta(T_{\sigma_1}(f_1, \ldots, f_m))|^2 \right\} \right\|_{L^{1/m, \infty}(\mathbb{R}^n)} \\
\lesssim A \left\| \left\{ \sum_j M(\Delta_j^\eta f_m)^2 \right\} \right\|_{L^{1, \infty}(\mathbb{R}^n)} \prod_{i=1}^{m-1} \|M(f_i)\|_{L^{1, \infty}(\mathbb{R}^n)} \\
\lesssim A \left\| \left\{ \sum_j |\Delta_j^\eta f_m|^2 \right\} \right\|_{L^1(\mathbb{R}^n)} \prod_{i=1}^{m-1} \|f_i\|_{L^1(\mathbb{R}^n)} \\
\lesssim A \prod_{i=1}^m \|f_i\|_{H^1(\mathbb{R}^n)}.
\]

This proves estimate (1.9) for \( \sigma_1 \).

Next we deal with \( \sigma_2 \). From (3.4), we have

\[
T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_2}(f_1, \ldots, f_{m-1}, \Delta_j^\theta f_m),
\]
where $T_{\sigma_2}(f_1, \ldots, f_{m-2}, \Delta_j^\xi f_{m-1}, \Delta_j^\theta f_m)$ is defined in (3.5). A similar calculation as in the case for $\sigma_1$ yields the estimate

$$|T_{\sigma_2}(f_1, \ldots, f_{m-2}, \Delta_j^\xi f_{m-1}, \Delta_j^\theta f_m)| \lesssim A\left(\prod_{i=1}^{m-2} M(f_i)\right) M(\Delta_j^\xi f_{m-1}) M(\Delta_j^\theta f_m).$$

Summing over $j$, taking $L^{1/m, \infty}$ quasinorms and applying the Littlewood-Paley characterization of $H^1$ we deduce

$$\|T_{\sigma_2}(f_1, \ldots, f_{m-1}, f_m)\|_{L^{1/m, \infty}(\mathbb{R}^n)} \lesssim A \left( \prod_{i=1}^{m-2} M(f_i) \sum_{j \in \mathbb{Z}} M(\Delta_j^\xi f_{m-1}) M(\Delta_j^\theta f_m) \right)_{L^{1/m, \infty}(\mathbb{R}^n)}$$

$$\lesssim A \left( \prod_{i=1}^{m-2} M(f_i) \right) \left( \sum_{j \in \mathbb{Z}} M(\Delta_j^\xi f_{m-1})^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} M(\Delta_j^\theta f_m)^2 \right)^{1/2} \|L^{1/m, \infty}(\mathbb{R}^n)}$$

$$\lesssim A \left( \prod_{i=1}^{m-2} \|f_i\|_{L^{1}(\mathbb{R}^n)} \right) \left( \sum_{j \in \mathbb{Z}} |\Delta_j^\xi f_{m-1}|^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} |\Delta_j^\theta f_m|^2 \right)^{1/2} \|L^{1}(\mathbb{R}^n)}$$

$$\lesssim A \prod_{i=1}^{m} \|f_i\|_{H^1(\mathbb{R}^n)}.$$

This concludes the proof of Theorem 1.3.

5. THE PROOF OF THEOREM 1.4

We provide the proof of Theorem 1.4 next, which is similar to the proof of Theorem 1.3 but could be read independently.

Since the detailed proof of Theorem 1.3 is notationally cumbersome, we first present a proof in the case where $m = 4$ and $n = 3$, i.e., the case of 4 variables and 3 coordinates. This case captures all the ideas of the general case. Then we discuss the general case at the end.

Consider the following matrix of the coordinates of all variables:

$$\begin{bmatrix}
\xi_{11} & \xi_{12} & \xi_{13} \\
\xi_{21} & \xi_{22} & \xi_{23} \\
\xi_{31} & \xi_{32} & \xi_{33} \\
\xi_{41} & \xi_{42} & \xi_{43}
\end{bmatrix} = \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix}.$$
Along each column we encounter two cases: the case where the largest coordinate is larger than all the other ones (case I) and the other case where the largest coordinate is comparable to the second largest (case II). Such a splitting along all columns produces 8 cases. We only study a representative of these 8 cases, and in each one of those we make an arbitrary assumption about the largest variable. The case below illustrates the general one. Assume that:

- along column 1: case I (largest in modulus variable is $\xi_{41}$);
- along column 2: case II (largest in modulus variable is $\xi_{42}$ and second largest is $\xi_{12}$);
- along column 3: case I (largest in modulus variable is $\xi_{23}$).

We denote the symbol associated with this case by

$$
\tau = \sigma_{I,I,I}^{(41),(42,12),23}.
$$

This symbol is obtained by multiplying $\sigma$ by a function of the form

$$
\Phi\left(\frac{\xi_{11}}{\xi_{41}}, \frac{\xi_{21}}{\xi_{41}}, \frac{\xi_{31}}{\xi_{41}}\right)\Phi\left(\frac{\xi_{12}}{\xi_{42}}, \frac{\xi_{22}}{\xi_{42}}, \frac{\xi_{32}}{\xi_{42}}\right)\Psi\left(\frac{\xi_{13}}{\xi_{23}}, \frac{\xi_{33}}{\xi_{23}}, \frac{\xi_{43}}{\xi_{23}}\right)
$$

where $\Phi(u_1, u_2, u_3)$ is supported in $[0, \frac{1}{206}] \times [0, \frac{1}{206}] \times [0, \frac{1}{206}]$ while $\Psi(u)$ is supported in $[\frac{1}{100}, 2]$; see the proof of Theorem 1.1 or [15] (pages 570-571 or Exercise 7.5.4).

Fix a Schwartz function $\theta$ whose Fourier transform is supported in $[\frac{1}{2}, 2] \cup [-2, -\frac{1}{2}]$ and satisfies $\sum_{j \in \mathbb{Z}} \hat{\theta}(2^{-j}v) = 1$ for $v \in \mathbb{R} \setminus \{0\}$. Associated with $\theta$ we define the Littlewood–Paley operator $\Delta_j^{(i)}(f) = f \ast \theta_{2^{-j}}$, where $\theta_t(u) = t^{-n}\theta(t^{-1}u)$ for $t > 0$ and $\ast$ denotes the convolution in the $i$th variable. In a Littlewood-Paley operator $\Delta_j^{(k)}$ the upper letter inside the parenthesis indicates the coordinate on which it acts, so $1 \leq k \leq 3$. We write

$$
T_\tau(f_1, f_2, f_3, f_4) = \sum_{j_1} \sum_{j_2} \sum_{j_3} T_\tau(f_1, \Delta_j^{(3)}f_2, f_3, \Delta_j^{(2)}\Delta_j^{(1)}f_4)
$$

and we have

$$
T_\tau(f_1, \Delta_j^{(3)}f_2, f_3, \Delta_j^{(2)}\Delta_j^{(1)}f_4)(x) = \int_{\mathbb{R}^{12}} \tau(\xi) \hat{f}_1(\xi_1) \hat{\theta}(2^{-j_1}\xi_{23}) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) \hat{\theta}(2^{-j_2}\xi_{42}) \hat{\theta}(2^{-j_3}\xi_{41}) \hat{f}_4(\xi_4) e^{2\pi ix(\xi_1 + \xi_2 + \xi_3 + \xi_4)} d\xi.
$$

Since $\xi_{41}$ is the largest variable among $\xi_{11}, \xi_{21}, \xi_{31}, \xi_{41}$, we have that

$$
|\xi_{41}| \leq |\xi_{11}| + |\xi_{21}| + |\xi_{31}| + |\xi_{41}| \leq \frac{232}{200} |\xi_{41}|
$$

and since $\xi_{42}$ is the largest variable among $\xi_{12}, \xi_{22}, \xi_{32}, \xi_{42}$, we have that

$$
|\xi_{42}| \leq |\xi_{12}| + |\xi_{22}| + |\xi_{32}| + |\xi_{42}| \leq \frac{232}{200} |\xi_{42}|.
$$
Likewise
\[ |\xi_{23}| \leq |\xi_{13}| + |\xi_{23}| + |\xi_{33}| + |\xi_{43}| \leq \frac{232}{200} |\xi_{23}|. \]

We may therefore insert in the preceding integral the function
\[ \hat{\Omega}(D_{-j_1,-j_2,-j_3}(\xi_1, \xi_2, \xi_3, \xi_4)) = \hat{\Theta}(2^{-j_1}(\xi_{11} + \xi_{21} + \xi_{31} + \xi_{41})) \hat{\Theta}(2^{-j_3}(\xi_{13} + \xi_{23} + \xi_{33} + \xi_{43})), \]
where \( \hat{\Theta}(u) = \hat{\theta}(u/2) + \hat{\theta}(u) + \hat{\theta}(2u) \); notice that \( \hat{\Theta} \) equals 1 on the support of \( \hat{\theta} \). We denote by \( \tilde{\Delta}_j \) the Littlewood-Paley operators associated to \( \Theta \). For the same reason we may also insert the function
\[
\hat{\Psi}_j^*(D_{-j_1,-j_2,-j_3}(\xi_1, \xi_2, \xi_3, \xi_4)) = \hat{\Psi}_j^*(2^{-j_1}(\xi_{11}, \xi_{21}, \xi_{31}, \xi_{41})) \hat{\Psi}_j^*(2^{-j_2}(\xi_{12}, \xi_{22}, \xi_{32}, \xi_{42})) \hat{\Psi}_j^*(2^{-j_3}(\xi_{13}, \xi_{23}, \xi_{33}, \xi_{43}))
\]
where
\[
\hat{\Psi}_j^*(u_1, u_2, u_3, u_4) = \sum_{|k| \leq 1} \hat{\Psi}(2^{-k}(u_1, u_2, u_3, u_4)),
\]
and \( \hat{\Psi}_j^* \) is as in the hypotheses of the theorem. Let
\[
D_{j_1,j_2,j_3}(\xi_1, \xi_2, \xi_3, \xi_4) = \begin{pmatrix}
\xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\
\xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\
\xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\
\xi_{41} & \xi_{42} & \xi_{43} & \xi_{44}
\end{pmatrix}
\begin{pmatrix}
2^{j_1} \\
2^{j_2} \\
2^{j_3}
\end{pmatrix}
\]
and
\[ \tau^{j_1,j_2,j_3}(\xi_1, \xi_2, \xi_3, \xi_4) = \tau(D_{j_1,j_2,j_3}(\xi_1, \xi_2, \xi_3, \xi_4)). \]

Additionally, in case II there is the second largest variable which is comparable to the largest one. Therefore we can take a Schwartz function \( \eta \) whose Fourier transform is supported in \([\frac{1}{256}, 8] \cup [-8, -\frac{1}{256}] \) and identical to 1 on \([\frac{1}{128}, 4] \cup [-4, -\frac{1}{128}] \) and insert the factor \( \tilde{\eta}(2^{-j_2}\xi_{12}) \) into the above integral without changing the outcome. Let us denote the Littlewood-Paley operator associated with \( \eta \) by \( \Delta_j \).

We may therefore rewrite
\[
T_r(f_1, \Delta_{j_1}^2 f_2, f_3, \Delta_{j_2}^2 \Delta_{j_1}^1 f_4) = T_r(\Delta_{j_2}^2 f_1, \Delta_{j_1}^3 f_2, f_3, \Delta_{j_2}^2 \Delta_{j_1}^1 f_4)
= \Delta_{j_1}^1 \Delta_{j_3}^3 T_r(\Delta_{j_2}^2 f_1, \Delta_{j_1}^3 f_2, f_3, \Delta_{j_2}^2 \Delta_{j_1}^1 f_4).
\]
Manipulations with the Fourier transform give that the above can be expressed as
\[
\int_{\mathbb{R}^4} 2^{4(j_1+j_2+j_3)} \left( \tau^{j_1,j_2,j_3} \hat{\Psi}_j^* \hat{\Omega} \right)^{\vee} \left( D_{j_1,j_2,j_3}(x-y_1, x-y_2, x-y_3, x-y_4) \right) (\Delta_{j_2}^2 f_1)(y_1)(\Delta_{j_1}^3 f_2)(y_2)(f_3)(y_3)(\Delta_{j_2}^2 \Delta_{j_1}^1 f_4)(y_4) dy_1 dy_2 dy_3 dy_4.
\]
If \( r = 1 \), set \( \rho = 1 \). If \( r > 1 \) pick \( \rho \) such that \( 1 < \rho < 2 \) and that
\[
\frac{1}{\sum_{1 \leq i \leq \rho} \gamma_{1i}} \leq \frac{1}{\min(p_1, \ldots, p_m, r)}.
\]

Setting \( \omega(y) = (1 + 4\pi^2|y|^2)^\frac{\rho}{2} \) for \( y \in \mathbb{R} \), we write
\[
|T_\tau(f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4)(x_1, x_2, x_3)|
\leq \int_{\mathbb{R}^{12}} 2^{\frac{2\ell_1 + \ell_2 + \ell_3}{\rho}} \prod_{i=1}^{3} \omega_{\gamma_{1i}}(2^{\ell_i}(x_{\ell_i} - y_{\ell_i})) \left( \tau^{j_1,j_2,j_3} \hat{\Psi} \hat{\Omega} \right)^{\frac{\rho}{2}}(D_{j_1,j_2,j_3}(x, y)) dy_2 dy_3 dy_4.
\]

We now apply Hölder’s inequality with exponents \( \rho \) and \( \rho' \) to obtain the estimate
\[
(5.1)
\]
where we used that \( \rho \gamma_{1i} > 1 \) for all \( i, \ell \) and also that
\[
\left\| \prod_{i=1}^{4} \prod_{\ell=1}^{3} (I - \partial_{\ell i}^2) \frac{2^{\tau}}{2^{\frac{2\ell_1 + \ell_2 + \ell_3}{\rho}}} (\tau^{j_1,j_2,j_3} \hat{\Psi} \hat{\Omega}) \right\|_{L^\rho} \lesssim \left\| \prod_{i=1}^{4} \prod_{\ell=1}^{3} (I - \partial_{\ell i}^2) \frac{2^{\tau}}{2^{\frac{2\ell_1 + \ell_2 + \ell_3}{\rho}}} (\tau^{j_1,j_2,j_3} \hat{\Psi} \hat{\Omega}) \right\|_{L^\rho} \lesssim A
\]
which is a consequence of Lemma 2.1 and of the fact that \( \hat{\Psi} \hat{\Omega} \) is a finite sum of \( \hat{\psi}_i \)'s.

We now use (5.1) to estimate our operator. We write
\[
T_\tau(f_1, f_2, f_3, f_4) = \sum_{j_1} \sum_{j_2} \sum_{j_3} \Delta^{(1)}_{j_1} \Delta^{(3)}_{j_3} T_\tau(\Delta^{(2)}_{j_2}, f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4).
\]

Let \( \mathcal{M} \) denote the strong maximal function. For each \( j_1 \) and \( j_3 \) we have the pointwise estimate
\[
\left| \Delta^{(1)}_{j_1} \Delta^{(3)}_{j_3} \sum_{j_2} T_\tau(\Delta^{(2)}_{j_2}, f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4) \right|
\leq CA \sum_{j_2} \mathcal{M}(\Delta^{(2)}_{j_2} f_1)^{\frac{1}{2}} \mathcal{M}(\Delta^{(3)}_{j_3} f_2)^{\frac{1}{2}} \mathcal{M}(f_3)^{\frac{1}{2}} \mathcal{M}(\Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4)^{\frac{1}{2}}
\leq CA \left( \sum_{j_2} \mathcal{M}(\Delta^{(2)}_{j_2} f_1)^{\frac{1}{2}} \mathcal{M}(\Delta^{(3)}_{j_3} f_2)^{\frac{1}{2}} \mathcal{M}(f_3)^{\frac{1}{2}} \mathcal{M}(\Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]
We now apply Lemma 2.2 (hypothesis (2.6) is easy to check), more precisely by Remark 2.3 to write
\[ \|T_\tau(f_1, f_2, f_3, f_4)\|_{L^p} \lesssim \left( \sum_{j_1} \sum_{j_3} \left| M_1^{(1)} M_3^{(3)} \sum_{j_2} T_\tau(\overline{\Delta}^{(2)}_{j_2} f_1, \Delta^{(3)}_{j_3} f_2, f_3, \Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4) \right|^2 \right)^{1/2} \]
and using the preceding estimate we control this expression by
\[ A \left( \sum_{j_2} \mathcal{M}(\overline{\Delta}^{(2)}_{j_2} f_1)^{\frac{2}{p}} \right)^{1/2} \left( \sum_{j_3} \mathcal{M}(\Delta^{(3)}_{j_3} f_2)^{\frac{2}{p}} \right)^{1/2} \mathcal{M}(\Delta f_3^{\rho})^{\frac{2}{p}} \left( \sum_{j_2} \sum_{j_1} \mathcal{M}(\Delta^{(2)}_{j_2} \Delta^{(1)}_{j_1} f_4)^{\frac{2}{p}} \right)^{1/2} \].
The required conclusion follows by applying Hölder’s inequality, the Fefferman-Stein inequality \cite{12}, and Lemma 2.2 using the facts that \( 1 < \rho < 2 \) and \( \rho < p_i \) for all \( i \).

We show now how to modify the above proof to obtain the general case. To do so, we introduce some notation. We consider the set \( \{1, 2, \ldots, n\} \) that indexes the columns of the \( m \times n \) matrix \( (\xi_{kl})_{1 \leq k \leq m, 1 \leq l \leq n} \). We split the set \( \{1, 2, \ldots, n\} \) into two pieces I and II, by placing \( l \in I \) if the \( l \)th column follows in the first case (where there the largest variable dominates all the other ones) and placing \( l \in II \) if the \( l \)th column follows in the second case (where there the largest variable and the second largest are comparable). To make the notation a bit simpler, without loss of generality we suppose that \( I = \{1, \ldots, q\} \) and \( II = \{q + 1, \ldots, n\} \) for some \( q \). Notice that one of these sets could be empty.

Recall the notation for the Littlewood-Paley operators \( \Delta^{(l)}_j \) as in the case \( m = 4, n = 3 \). For the purposes of this theorem we introduce a slightly more refined notation using two upper indices in \( \Delta^{(k,l)}_j \). The first index shows the function \( f_k \) on which \( \Delta^{(k,l)}_j \) acts and the second one the coordinate \( \xi_{kl} \) of the variable \( \xi_k \) on which \( \Delta^{(k,l)}_j \) acts.

Define a map
\[ u : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\} \]
such that for each \( l \), \( u(l) \) denotes the index such that \( \xi_{u(l)l} \) is largest among \( \xi_{kl} \). Also define a map
\[ \bar{u} : \{q + 1, \ldots, n\} \to \{1, 2, \ldots, m\} \]
such that \( \xi_{\bar{u}(l)l} \) is second largest among \( \xi_{kl} \). We always have \( \bar{u}(l) \neq u(l) \) for all \( l \) in \( \{q + 1, \ldots, n\} \). We also define
\[ \Delta^{(u(r),r)}_j f = \Delta^{(u(r),r)}_j (f_1, \ldots, f_m) = (f_1, \ldots, \Delta^{(r)}_j f_{u(r)}, \ldots, f_m) \]
and we extend this definition to the case where \( \Delta^{(u(i_1),i_1)}_{j_{i_1}} \cdots \Delta^{(u(i_r),i_r)}_{j_{i_r}} \) acts on \( (f_1, \ldots, f_m) \). Additionally, we use the definitions of \( \tilde{\Delta}_j \) and \( \overline{\Delta}_j \) as introduced in the special case \( m = 4, n = 3 \).
Let $\tau$ be the multilinear multiplier associated with a given fixed mapping $u$. We write

$$T_\tau(f_1, \ldots, f_m) = \sum_{j_1, \ldots, j_n \in \mathbb{Z}} T_\tau[\Delta_j^{(u(1),1)} \cdots \Delta_j^{(u(n),n)}(f_1, \ldots, f_m)]$$

$$= \sum_{j_1, \ldots, j_n \in \mathbb{Z}} \Delta_j^{(u(1),1)} \cdots \Delta_j^{(u(q),q)} \sum_{j_{q+1}, \ldots, j_m \in \mathbb{Z}} T_\tau[q \Delta_j^{(u(\kappa),\kappa)} \prod_{\lambda=q+1}^n \Delta_j^{(u(\lambda),\lambda)} \Delta_j^{(\mu(\lambda),\lambda)} f].$$

The estimates in the case $m = 4$ and $n = 3$ show that the term in the interior sum satisfies

$$\left| \Delta_j^{(u(1),1)} \cdots \Delta_j^{(u(q),q)} \sum_{j_{q+1}, \ldots, j_m \in \mathbb{Z}} T_\tau[q \Delta_j^{(u(\kappa),\kappa)} \prod_{\lambda=q+1}^n \Delta_j^{(u(\lambda),\lambda)} \Delta_j^{(\mu(\lambda),\lambda)}(f_1, \ldots, f_m)] \right| \lesssim A \sum_{j_{q+1}, \ldots, j_m \in \mathbb{Z}} \prod_{i=1}^m \mathcal{M}\left( \prod_{1 \leq \kappa \leq q} \Delta_j^{(i,\kappa)} \prod_{q+1 \leq \lambda \leq n} \Delta_j^{(i,\lambda)} \prod_{q+1 \leq \mu \leq n} \Delta_j^{(i,\mu)} f_i \right)^{\frac{1}{p}} \prod_{i=1}^m \mathcal{M}\left( \prod_{1 \leq \kappa \leq q} \Delta_j^{(i,\kappa)} \prod_{q+1 \leq \lambda \leq n} \Delta_j^{(i,\lambda)} \prod_{q+1 \leq \mu \leq n} \Delta_j^{(i,\mu)} f_i \right)^{\frac{1}{p}}$$

where $u^{-1}[i] = \{k \in \{1, \ldots, n\} : u(k) = i\}$ and with the understanding that if any of the index sets is empty, then the corresponding Littlewood-Paley operators do not appear. Applying the Cauchy-Schwarz inequality $m - q$ times successively for the indices $j_{q+1}, j_{q+1}, \ldots, j_m$ we estimate the last displayed expression by

$$A \prod_{i=1}^m \left[ \sum_{j_\lambda \in \mathbb{Z}} \sum_{j_\mu \in \mathbb{Z}} \mathcal{M}\left( \prod_{1 \leq \kappa \leq q} \Delta_j^{(i,\kappa)} \prod_{q+1 \leq \lambda \leq n} \Delta_j^{(i,\lambda)} \prod_{q+1 \leq \mu \leq n} \Delta_j^{(i,\mu)} f_i \right)^{\frac{1}{p}} \right]^{\frac{1}{2}}$$

When $I \neq \emptyset$, we use Lemma 2.2 and (5.2) to obtain

$$\left\| T_\tau(f_1, \ldots, f_m) \right\|_{L^p} \lesssim \left\| \sum_{j_1, \ldots, j_q \in \mathbb{Z}} \Delta_j^{(u(1),1)} \cdots \Delta_j^{(u(q),q)} \sum_{j_{q+1}, \ldots, j_m \in \mathbb{Z}} T_\tau[q \Delta_j^{(u(\kappa),\kappa)} \prod_{\lambda=q+1}^n \Delta_j^{(u(\lambda),\lambda)} \Delta_j^{(\mu(\lambda),\lambda)} f] \right\|_{L^p}$$

$$\lesssim A \left\| \sum_{j_1, \ldots, j_q \in \mathbb{Z}} \prod_{i=1}^m \left[ \sum_{j_\lambda \in \mathbb{Z}} \sum_{j_\mu \in \mathbb{Z}} \mathcal{M}\left( \prod_{1 \leq \kappa \leq q} \Delta_j^{(i,\kappa)} \prod_{q+1 \leq \lambda \leq n} \Delta_j^{(i,\lambda)} \prod_{q+1 \leq \mu \leq n} \Delta_j^{(i,\mu)} f_i \right)^{\frac{1}{p}} \right]^{\frac{1}{2}} \right\|_{L^p}.$$
\[ \zeta A \left( \prod_{i=1}^{m} \left( \sum_{j_{\kappa} \in \mathbb{Z}} \sum_{j_{\lambda} \in \mathbb{Z}} \sum_{j_{\mu} \in \mathbb{Z}} \sum_{1 \leq \kappa \leq q} \sum_{1 \leq \lambda \leq n} \sum_{1 \leq \mu \leq n} \mathcal{M} \left( \prod_{j_{\kappa} \in \mathbb{Z}} \Delta_{j_{\kappa}}^{(i, \kappa)} \prod_{j_{\lambda} \in \mathbb{Z}} \Delta_{j_{\lambda}}^{(i, \lambda)} \prod_{j_{\mu} \in \mathbb{Z}} \Delta_{j_{\mu}}^{(i, \mu)} f_{i} \right) \right)^{\frac{2}{p}} \right) \right)_{L^{p}}. \]

Otherwise, when \( I = \emptyset \), from (5.2) we can see that \( T_{r}(f_{1}, \ldots, f_{m}) \) is controlled by

\[ \left( \sum_{j_{\kappa} \in \mathbb{Z}} \sum_{j_{\lambda} \in \mathbb{Z}} \sum_{j_{\mu} \in \mathbb{Z}} \sum_{1 \leq \kappa \leq q} \sum_{1 \leq \lambda \leq n} \sum_{1 \leq \mu \leq n} \mathcal{M} \left( \prod_{j_{\kappa} \in \mathbb{Z}} \Delta_{j_{\kappa}}^{(i, \kappa)} \prod_{j_{\lambda} \in \mathbb{Z}} \Delta_{j_{\lambda}}^{(i, \lambda)} \prod_{j_{\mu} \in \mathbb{Z}} \Delta_{j_{\mu}}^{(i, \mu)} f_{i} \right) \right)_{L^{p}} \]

At this point we apply Hölder’s inequality and the Fefferman-Stein inequality \([12]\) using the facts that \( 1 < p < 2 \) and \( \rho < p_{i} \) for all \( i \). Then we control \( \|T_{r}(f_{1}, \ldots, f_{m})\|_{L^{p}} \) by a constant multiple of

\[ A \prod_{i=1}^{m} \left( \sum_{j_{\kappa} \in \mathbb{Z}} \sum_{j_{\lambda} \in \mathbb{Z}} \sum_{j_{\mu} \in \mathbb{Z}} \sum_{1 \leq \kappa \leq q} \sum_{1 \leq \lambda \leq n} \sum_{1 \leq \mu \leq n} \mathcal{M} \left( \prod_{j_{\kappa} \in \mathbb{Z}} \Delta_{j_{\kappa}}^{(i, \kappa)} \prod_{j_{\lambda} \in \mathbb{Z}} \Delta_{j_{\lambda}}^{(i, \lambda)} \prod_{j_{\mu} \in \mathbb{Z}} \Delta_{j_{\mu}}^{(i, \mu)} f_{i} \right) \right)_{L^{p_{i}}} \]

or

\[ A \prod_{i=1}^{m} \left( \sum_{j_{\kappa} \in \mathbb{Z}} \sum_{j_{\lambda} \in \mathbb{Z}} \sum_{j_{\mu} \in \mathbb{Z}} \sum_{1 \leq \kappa \leq q} \sum_{1 \leq \lambda \leq n} \sum_{1 \leq \mu \leq n} \mathcal{M} \left( \prod_{j_{\kappa} \in \mathbb{Z}} \Delta_{j_{\kappa}}^{(i, \kappa)} \prod_{j_{\lambda} \in \mathbb{Z}} \Delta_{j_{\lambda}}^{(i, \lambda)} \prod_{j_{\mu} \in \mathbb{Z}} \Delta_{j_{\mu}}^{(i, \mu)} f_{i} \right) \right)_{L^{p_{i}}} \]

and by the Littlewood-Paley theorem the last expression is bounded by \( A \) times the product of the \( L^{p_{i}} \) norms of the \( f_{i} \).

**Remark 5.1.** We see from the proof that we do not use the property that \( \xi_{k} \in \mathbb{R} \), so the same argument generalizes our result to the case when each \( f_{k} \) is defined on \( \mathbb{R}^{d} \) with \( \xi_{k} \in \mathbb{R}^{d} \). This covers [6, Theorem 1.10], as we claimed in the introduction.

### 6. Applications: Calderón-Coifman-Journé commutators

#### 6.1. Calderón commutator

In 1965 Calderón [2] introduced the (first-order) commutator

\[ C_{1}(f; a)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x - y)^{2}} f(y) dy, \]

where \( a \) is the derivative of a Lipschitz function \( A \) and \( f \) is a test function on the real line. It is known that \( C_{1} \) is a bounded operator in \( L^{p}(\mathbb{R}) \), \( 1 < p < \infty \), if \( A \) is a Lipschitz function on \( \mathbb{R} \) and

\[ \|C_{1}(f; a)\|_{L^{p}(\mathbb{R})} \leq C_{p} \|a\|_{L^{\infty}(\mathbb{R})} \|f\|_{L^{p}(\mathbb{R})}, \quad 1 < p < \infty. \]
See Calderón [2, 3] and Coifman-Meyer [8] for its history.

Viewed as a bilinear operator acting on the pair \((f, a)\), then the operator \(C_1\) can be written as a bilinear multiplier operator

\[
C_1(f; a)(x) = -i\pi \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \hat{a}(\eta) \left( \text{sgn}(\eta)\Phi(\xi/\eta) \right) e^{2\pi i x (\xi + \eta)} d\xi d\eta,
\]

where \(\Phi\) is the following Lipschitz function on the real line:

\[
\Phi(s) = \begin{cases} 
-1, & s \leq -1; \\
1 + 2s, & -1 < s \leq 0; \\
1, & s > 0.
\end{cases}
\]

The operator \(C_1\) is too singular to fall under the scope of multilinear Calderón-Zygmund theory [22]. However it was shown to be bounded from \(L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})\) to \(L^{p}(\mathbb{R})\) when \(1 < p_1, p_2 < \infty\) and \((1/p_1 + 1/p_2)^{-1} = p > 1/2\); see C. Calderón [4]. See also Coifman-Meyer [8] and Duong-Grafakos-Yan [11]. The boundedness of \(C_1\) on \(L^{p}\) for \(p \geq 1\) was also studied by Muscalu [33] via time-frequency analysis.

In this work we will apply Theorem 1.1 to obtain a direct proof of the boundedness of \(C_1\) from \(L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})\) to \(L^{p}(\mathbb{R})\) in the full range of \(p > 1/2\). Our proof is based on exploiting the (limited) smoothness of the function \(\Phi\), measured in terms of a Sobolev space norm. A partial result using a similar idea in this direction with the restriction \(p > 2/3\) has been obtained by [32].

For \(r \geq 1\) and \(\gamma > 0\), we recall the Sobolev space \(L^r_\gamma(\mathbb{R}^n)\), \(\gamma > 0\) of all functions \(g\) with \(\|(I - \Delta)^{\gamma/2}g\|_{L^p} < \infty\). For \(\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)\), we denote by \(L^r_{\vec{\gamma}}(\mathbb{R}^n)\) the class of distributions \(f\) such that

\[
\left\| \prod_{1 \leq \ell \leq n} (I - \partial_\ell^2)^{\gamma_\ell/2} f \right\|_{L^r(\mathbb{R}^n)} < \infty.
\]

It is easy to verify using multiplier theorems that \(L^r_{\vec{\gamma}}(\mathbb{R}^n) \subset L^r_\gamma(\mathbb{R}^n)\), where \(\gamma = |\vec{\gamma}| = \gamma_1 + \cdots + \gamma_n\). The spaces \(L^r_\gamma(\mathbb{R}^n)\) are sometimes referred to as Sobolev spaces with dominating mixed smoothness in the literature, see [37] for more details and references.

To begin, we need the following characterizations of Sobolev norms, given by Stein [38, 39, Lemma 3, p. 136].

**Lemma 6.1** (Stein). (i) Let \(0 < \alpha < 1\) and \(2n/(n + 2\alpha) < p < \infty\). Then \(f \in L^p_\alpha(\mathbb{R}^n)\) if and only if \(\|f\|_{L^p_\alpha(\mathbb{R}^n)} \simeq \|f\|_{L^p(\mathbb{R}^n)} + \|I_\alpha(f)\|_{L^p(\mathbb{R}^n)}\) where

\[
I_\alpha(f)(x) = \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n + 2\alpha}} dy \right)^{1/2}.
\]
(ii) Let $1 \leq \alpha < \infty$ and $1 < p < \infty$. Then $f \in L^p_\alpha(\mathbb{R}^n)$ if and only if $f \in L^p_{\alpha-1}(\mathbb{R}^n)$ and for $1 \leq j \leq n$, $\frac{\partial f}{\partial x_j} \in L^{p}_{\alpha-1}(\mathbb{R}^n)$. Furthermore, we have

$$\|f\|_{L^p_\alpha(\mathbb{R}^n)} \lesssim \|f\|_{L^p_{\alpha-1}(\mathbb{R}^n)} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{L^p_{\alpha-1}(\mathbb{R}^n)}.$$ 

Throughout this section fix a nondecreasing smooth function $h$ on $\mathbb{R}$ such that

$$h(t) = \begin{cases} 3, & \text{if } t \in [4, +\infty); \\ \text{smooth}, & \text{if } t \in [2, 4); \\ t, & \text{if } t \in [1/8, 2); \\ \text{smooth}, & \text{if } t \in [1/32, 1/8); \\ 1/16, & \text{otherwise.} \end{cases}$$

(6.4)

Lemma 6.2. Let $u$ be a function supported in the rectangle

$$(6.5) \quad \{(y_1, y_2) : |y_1| \leq 101/100, 1/4 \leq y_2 \leq 7/4\}$$

in $\mathbb{R}^2$ such that $\nabla u \in L^\infty(\mathbb{R}^2)$, and $u(x) \in L^r_\gamma(\mathbb{R}^2)$ with $1 < \gamma < 2$, $2/\gamma < r < 1/(\gamma - 1)$. Define $U(y_1, y_2) = u(y_1/h(y_2), y_2)$. Then $U \in L^r_\gamma(\mathbb{R}^2)$ and

$$\|U\|_{L^r_\gamma(\mathbb{R}^2)} \leq C (\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^r_\gamma(\mathbb{R}^2)}).$$

Proof. Because of Lemma 6.1 it suffices to show for $\alpha = \gamma - 1$ and $2/(1 + \alpha) < r < 1/\alpha$ that $U \in L^r_1(\mathbb{R}^2)$, $I_\alpha(U) \in L^r(\mathbb{R}^2)$ and $I_\alpha(\partial_j U) \in L^r(\mathbb{R}^2)$ with $j = 1, 2$. The first assertion follows trivially by checking the derivatives directly while the second one is verified in a way similar to the third one, where we adapt an argument found in Triebel [111] Section 4.3] with a suitable change of variables.

Next, we show that $I_\alpha(\partial_1 U) \in L^r(\mathbb{R}^2)$. We will estimate the following expression

$$\|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{\|y - y'|^{2+2\alpha}} dy \right)^{r/2} dy'.$$

Denote by $B$ a finite ball centered at 0 containing the support of $\partial_1 U$. Then it is easy to check that, since $\partial_1 U \in L^\infty$, $r(1 + \alpha) = r\gamma > 2$,

$$\|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)} \leq C \left( \|\nabla u\|_{L^\infty} + \int_{3B} \left( \int_{3B} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{\|y - y'|^{2+2\alpha}} dy \right)^{r/2} dy' \right),$$

where $C$ is a constant depending on $B$.

Denote $x = (x_1, x_2)$, $y = (y_1, y_2)$. One writes $y = \varphi(x)$ and $x = \psi(y)$ in the form

$$\begin{cases} y_1 = \varphi_1(x_1, x_2) = x_1 h(x_2), \\
y_2 = \varphi_2(x_1, x_2) = x_2 \end{cases}$$
and
\[
\begin{align*}
\begin{cases}
  x_1 = \psi_1(y_1, y_2) = y_1/h(y_2), \\
  x_2 = \psi_2(y_1, y_2) = y_2,
\end{cases}
\end{align*}
\]
where \( h \) is a function defined in (6.4). By the change of variables \( y = \varphi(x) \) with \( |\det \varphi'(x)| < C < \infty \), direct computations give
\[
\begin{align*}
\partial_1 U(y) &= \frac{\partial}{\partial y_1} u(\psi(y)) \cdot \frac{1}{h(y_2)} =: \partial_1 u(\psi(y)) \cdot \frac{1}{h(y_2)}, \\
\partial_2 U(y) &= -\frac{\partial}{\partial y_1} u(\psi(y)) \cdot \frac{y_1 h'(y_2)}{h(y_2)} + \frac{\partial}{\partial y_2} u(\psi(y)) = -\partial_1 u(\psi(y)) \cdot \frac{y_1 h'(y_2)}{h(y_2)} + \partial_2 u(\psi(y)),
\end{align*}
\]
and the fact that \( |\psi(y) - \psi(y')| \leq \max\{\|\nabla \psi_1\|_\infty, \|\nabla \psi_2\|_\infty\} |y - y'| \), we have
\[
\begin{align*}
\|I_\alpha(\partial_1 U)\|_{L^r(R^2)} &\leq C\|\nabla u\|_{L^\infty(R^2)} + C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|\psi(y) - \psi(y')|^{2+2\alpha}} dy' \right)^{r/2} dy' \\
&\leq C\|\nabla u\|_{L^\infty(R^2)} + C \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \frac{|\partial_1 u(\psi(y)) - \partial_1 u(\psi(y'))|^2}{|h(y_2)|^2} dy' \right]^{r/2} dy' \\
&\leq C\|\nabla u\|_{L^\infty(R^2)} + C \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \frac{|h(x_2)|^2}{|x - x'|^{2+2\alpha}} |\det \varphi'(x)| dx \right]^{r/2} \|\varphi'(x')\|_1 dx' \\
&\leq C\|\nabla u\|_{L^\infty(R^2)} + C \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \frac{|\partial_1 u(x) - \partial_1 u(x')|^2}{|h(x_2)|^2} dx \right]^{r/2} dx'.
\end{align*}
\]
Now take \( \eta(x_1, x_2) \in C_0^\infty(\mathbb{R}^2) \) assuming value 1 on the support of \( \partial_1 u \) so that the support of \( \eta \) is just a bit larger than that of \( \partial_1 u \), and \( h(x_2) = x_2 \) on the support of \( \eta \). Define \( \tilde{h}(x_1, x_2) = \eta(x_1, x_2)/h(x_2) \) and then write
\[
\frac{\partial_1 u(x)}{h(x_2)} = \frac{\partial_1 u(x')}{h(x_2')}
= \partial_1 u(x) \tilde{h}(x) - \partial_1 u(x') \tilde{h}(x')
= \left[ \partial_1 u(x) - \partial_1 u(x') \right] \tilde{h}(x') + \partial_1 u(x) [\tilde{h}(x) - \tilde{h}(x')],
\]
which yields
\[
\begin{align*}
\|I_\alpha(\partial_1 U)\|_{L^r(R^2)} &\leq C\|\nabla u\|_{L^\infty(R^2)} + C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|\partial_1 u(x) - \partial_1 u(x')|^2}{|x - x'|^{2+2\alpha}} dx \right)^{r/2} dx' \\
&\quad + C\|\nabla u\|_{L^\infty(R^2)} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|\tilde{h}(x) - \tilde{h}(x')|^2}{|x - x'|^{2+2\alpha}} dx_1 dx_2 \right)^{r/2} dx_1 dx_2' \\
&\leq C\|\nabla u\|_{L^\infty(R^2)} + C\|\partial_1 u\|_{L^r_\alpha(R^2)} + C\|\nabla u\|_{L^\infty} \|\tilde{h}\|_{L^r_\alpha(R^2)}
\end{align*}
\]
\[
\leq C \left( \| \nabla u \|_{L^\infty(\mathbb{R}^2)} + \| u \|_{L^r_\alpha(\mathbb{R}^2)} \right)^r.
\]

A similar argument as the one above shows that
\[
\| I_\alpha(\partial_2 U) \|_{L^r_\beta(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|\partial_2 U(y) - \partial_2 U(y')|^r}{|y - y'|^{2 + 2\alpha}} \, dy' \right)^{r/2} \, dy \\
\leq C \| \nabla u \|_{L^\infty(\mathbb{R}^2)} + C \| \partial_1 u \|_{L^r_\alpha(\mathbb{R}^2)} + C \| \partial_2 u \|_{L^r_\alpha(\mathbb{R}^2)} \\
\leq C \left( \| \nabla u \|_{L^\infty(\mathbb{R}^2)} + \| u \|_{L^r_\alpha(\mathbb{R}^2)} \right)^r.
\]

Also, by repeating the preceding argument we obtain,
\[
\| I_\alpha(U) \|_{L^r_\beta(\mathbb{R}^2)} \leq C \left( \| u \|_{L^\infty(\mathbb{R}^2)} + \| u \|_{L^r_\alpha(\mathbb{R}^2)} \right) \leq C \| u \|_{L^r_\alpha(\mathbb{R}^2)},
\]
where we used the Sobolev embedding theorem in the last inequality with \(\gamma r > 2\). The proof of Lemma 6.2 is now complete. \(\square\)

For \(g, h\) on \(\mathbb{R}\) define a the tensor \(g \otimes h\) as the following function on \(\mathbb{R}^2\) by setting
\[(g \otimes h)(\xi, \eta) = g(\xi)h(\eta) .\]

**Lemma 6.3.** Let \(f \in L^r_\alpha(\mathbb{R})\) supported in \([-1, 1]\), and \(\hat{\varTheta}\) is a smooth function supported in an annulus centered at 0 with size comparable to 1, then we have
\[
\| f \otimes \hat{\varTheta} \|_{L^r_\alpha(\mathbb{R}^2)} \leq C \| f \|_{L^r_\alpha(\mathbb{R})}.
\]

**Proof.** We use the same idea as in the proof of Lemma 6.2. It suffices to prove that \(f \otimes \hat{\varTheta} \in L^r_\alpha(\mathbb{R}^2)\) and that \(I_\alpha(\partial^\beta(f \otimes \hat{\varTheta})) \in L^r(\mathbb{R}^2)\) with \(|\beta| = 1\). It is easy to check that \(\| f \otimes \hat{\varTheta} \|_{L^r_\alpha} \leq C \| f \|_{L^r_\alpha}\), so we only prove that \(I_\alpha(\partial^\xi(f \otimes \hat{\varTheta})) \in L^r(\mathbb{R}^2)\).

Note that \(f \otimes \hat{\varTheta}\) is compactly supported and we can choose a function \(\varphi(\xi, \eta) \in C^\infty(\mathbb{R}^2)\) assuming 1 on the support of \(f \otimes \hat{\varTheta}\) and therefore \(f \otimes \hat{\varTheta} = f(\xi)\varphi(\xi, \eta)\hat{\varTheta}(\eta)\varphi(\xi, \eta)\). Then
\[
\int_{\mathbb{R}^2} | I_\alpha(\partial^\xi(f \otimes \hat{\varTheta})) |^r d\xi d\eta \text{ is split into the parts}
\]
\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|(f(\xi)\varphi(\xi, \eta) - f'(\xi')\varphi(\xi', \eta'))\hat{\varTheta}(\eta')\varphi(\xi', \eta')|^2}{(|(\xi, \eta) - (\xi', \eta')|^{2 + 2\alpha})} \, d\xi' d\eta' \right)^{r/2} \, d\xi d\eta,
\]
and
\[
\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{|(f(\xi)\varphi(\xi, \eta)\hat{\varTheta}(\eta)\varphi(\xi, \eta) - \hat{\varTheta}(\eta')\varphi(\xi', \eta')|^2}{(|(\xi, \eta) - (\xi', \eta')|^{2 + 2\alpha})} \, d\xi' d\eta' \right)^{r/2} \, d\xi d\eta.
\]
We prove only that the first one is finite since the latter can be proved similarly.

To prove the boundedness of the first one, we split it further via the identity
\[
f'(\xi)\varphi(\xi, \eta) - f'(\xi')\varphi(\xi', \eta') = (f'(\xi) - f'(\xi'))\varphi(\xi, \eta) + f'(\xi')(\varphi(\xi, \eta) - \varphi(\xi', \eta')).
\]
which, by Lemma 6.1, is bounded by $\|f\|_{L^r(\mathbb{R}^2)}$. For the other part, a simple change of variable $\eta' \rightarrow (\eta - \eta')/(\xi - \xi')$ shows that it is equal to

$$C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} \frac{|f'(\xi) - f'(\xi')|^2}{|\xi - \xi'|^{1+2\alpha}} d\xi' \right)^{r/2} |\varphi(\xi, \eta)| d\xi d\eta,$$

which, by Lemma [6.1], is bounded by $\|f\|_{L^r(\mathbb{R}^2)}$ since $\varphi \in C_0^\infty(\mathbb{R}^2)$. \hfill $\Box$

**Lemma 6.4.** Let $\gamma \in (1, 2)$ and $1 < r < \frac{1}{\gamma-1}$. Then $\|\Phi \varphi\|_{L^r(\mathbb{R})} < \infty$, where $\varphi$ is a smooth function with compact support, and $\Phi$ is the function in (6.3).

**Proof.** To obtain the claim, we need to show that $D^\gamma(\varphi \Phi) = \left((1 + |\xi|^2)^{\gamma/2} \hat{\varphi} \Phi\right)^\vee \in L^r(\mathbb{R})$. Since

$$\|D^\gamma(\varphi \Phi)\|_{L^r(\mathbb{R})} \approx \|\varphi \Phi\|_{L^r(\mathbb{R})} + \left\|\left(|\xi|^\gamma \hat{\varphi} \Phi\right)^\vee\right\|_{L^r(\mathbb{R})},$$

and trivially $\varphi \Phi \in L^r(\mathbb{R})$, we reduce the proof to establishing $\left\|\left(|\xi|^\gamma \hat{\varphi} \Phi\right)^\vee\right\|_{L^r(\mathbb{R})} < \infty$. By the Kato-Ponce inequality for homogeneous type [7], [35], [19], it suffices to show that $(|\xi|^\gamma \hat{\Phi})^\vee$ lies in $L^r(\mathbb{R})$. Indeed, for $\gamma \in (1, 2)$ we write

$$\hat{\Phi}(\xi)|\xi|^\gamma = \frac{1}{\xi} \hat{\Phi}(\xi)|\xi|^\gamma = \frac{1}{2\pi i} \frac{1}{\xi} \hat{\Phi}(\xi)|\xi|^\gamma$$

$$= -\frac{1}{\pi} \frac{1}{\xi} \chi_{[-1,0]}(\xi)|\xi|^\gamma = -\frac{1}{\pi} \frac{1}{\xi} \frac{e^{2\pi i \xi} - 1}{2\pi i \xi} |\xi|^\gamma$$

$$= -\frac{1}{\pi} \frac{1}{2\pi i} |\xi|^\gamma - 1 |\xi|^{\gamma-2} = -\frac{1}{2\pi^2} (e^{2\pi i \xi} - 1) |\xi|^{\gamma-2}.$$

Taking inverse Fourier transforms we obtain that

$$(\hat{\Phi}(\xi)|\xi|^\gamma)^\vee(x) = c_\gamma(|x+1|^{1-\gamma} - |x|^{1-\gamma})$$

and this function lies in $L^r(\mathbb{R})$ when $1 < r < \frac{1}{\gamma-1}$ and $\gamma$ is very close to 2. \hfill $\Box$

The preceding result can be lifted to $\mathbb{R}^2$ as follows.

**Lemma 6.5.** Let $\gamma \in (1, 2)$ and $1 < r < \frac{1}{\gamma-1}$, and let $\theta$ be a function supported in $\frac{1}{2} \leq |\xi| \leq 2$ on the real line. Define $U(\xi, \eta) = \Phi(\xi/\eta)\theta(\xi/\eta)\hat{\psi}(\xi, \eta)$, where $\hat{\psi}$ is a smooth function supported in an annulus centered at zero. Then $\|U\|_{L^r(\mathbb{R}^2)} < \infty$.

**Proof.** Set

$$u(\xi, \eta) = \Phi(\xi)\theta(\xi)\hat{\psi}(\xi, \eta)$$

and

$$U(\xi, \eta) = \Phi(\xi/\eta)\theta(\xi/\eta)\hat{\psi}(\xi, \eta).$$
Since \( h(\eta) = \eta \) on the support of the function \( U \). We now apply Lemma 6.2 to obtain
\[
\|U\|_{L^r_\gamma(\mathbb{R}^2)} \leq C \left( \|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^\gamma_\sigma(\mathbb{R}^2)} \right).
\]
Thus, it is enough to show that \( \|u\|_{L^\gamma_\sigma(\mathbb{R}^2)} < \infty \). We introduce a compactly supported smooth function \( \hat{\Theta}(\eta) \) which is equal to 1 on the support of \( \eta \mapsto \theta(\xi) \hat{\Psi}(\xi, \eta) \) for any \( \xi \). the Kato-Ponce inequality ([28] [19]) allows us to estimate the Sobolev norm of \( u \) as follows:
\[
\|u\|_{L^\gamma_\sigma(\mathbb{R}^2)} = \|\Phi(\xi) \theta(\xi) \hat{\Theta}(\eta) \hat{\Psi}(\xi, \eta)\|_{L^\gamma_\sigma(\mathbb{R}^2)} \\
\leq \|\Phi(\xi) \theta(\xi) \hat{\Theta}(\eta)\|_{L^\gamma_\sigma(\mathbb{R}^2)} \|\hat{\Psi}(\xi, \eta)\|_{L^\infty(\mathbb{R}^2)} \\
+ \|\hat{\Theta}(\xi, \eta)\|_{L^\gamma_\sigma(\mathbb{R}^2)} \|\Phi(\xi) \theta(\xi) \hat{\Theta}(\eta)\|_{L^\infty(\mathbb{R}^2)}.
\]
We are left with establishing \( \|\Phi(\xi) \theta(\xi) \hat{\Theta}(\eta)\|_{L^\gamma_\sigma(\mathbb{R}^2)} < \infty \), since all other terms on the right of the above inequality are finite. This is achieved via Lemmas 6.4 and 6.3. Thus the proof of Lemma 6.5 is complete.

Using these ideas we are able to deduce the following result concerning \( C_1 \).

**Proposition 6.6.** The Calderón commutator \( C_1 \) maps \( L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \) to \( L^p(\mathbb{R}) \) when \( 1/p_1 + 1/p_2 = 1/p \), \( 1 < p_1, p_2 < \infty \), and \( 1/2 < p < \infty \).

**Proof.** Note that \( \sigma(\xi, \eta) = \text{sgn}(\eta) \Phi(\xi/\eta) \) has an obvious modification which is continuous on \( \mathbb{R}^2 \setminus \{0\} \). We denote the latter by \( \text{sgn}(\eta) \Phi(\xi/\eta) \) as well since there is no chance to introduce any confusion.

We introduce a smooth function with compact support \( \theta \) on the real line which is supported in two small intervals, say, of length \( 1/100 \) centered at the points \(-1\) and \(0\). Then we write
\[
1 = \theta(\xi/\eta) + 1 - \theta(\xi/\eta)
\]
and we decompose the function \( \text{sgn}(\eta) \Phi(\xi/\eta) = \sigma_1(\xi, \eta) + \sigma_2(\xi, \eta) \), where \( \sigma_1(\xi, \eta) = \text{sgn}(\eta) \Phi(\xi/\eta) \theta(\xi/\eta) \) and \( \sigma_2(\xi, \eta) = \text{sgn}(\eta) \Phi(\xi/\eta)(1 - \theta(\xi/\eta)) \). Let \( \hat{\Psi} \) be a smooth bump supported in the annulus \( 1/2 < |(\xi, \eta)| < 3/2 \) in \( \mathbb{R}^2 \). The function \( \sigma_2 \) is smooth away from zero and \( \sigma_2 \hat{\Psi} \) lies in \( L^r_\gamma(\mathbb{R}^2) \) for any \( r, \gamma > 1 \) Also, \( \sigma_1 \hat{\Psi} \) lies in \( L^r_\gamma(\mathbb{R}^2) \) with \( r\gamma > 1 \). In view of Lemma 6.5 Then Corollary 1.2 implies the required conclusion. \( \square \)
6.2. Commutators of Calderón-Coifman-Journé. Now we focus on the boundedness properties of the following \(n\)-dimensional version of \(C_1\):

\[
C_1^{(n)}(f, a)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y) \left( \prod_{i=1}^{n} \frac{1}{(y_i - x_i)^2} \right) \int_{x_1}^{y_1} \cdots \int_{x_n}^{y_n} a(u_1, \ldots, u_n) \, du_1 \cdots du_n \, dy,
\]

where \(f\) is a function on \(\mathbb{R}^n\), and \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\). The operator \(C_1^{(n)}\) was introduced by a suggestion of Coifman when \(n = 2\). The \(L^2 \times L^\infty \to L^2\) bound for \(C_1^{(2)}\) was studied by Aguirre \[11\] and Journé \[26, 27\], namely,

\[
\|C_1^{(2)}(f, a)\|_{L^2(\mathbb{R}^2)} \leq C\|a\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^2(\mathbb{R}^2)}.
\]

For general \(n \geq 2\), boundedness for \(C_1^{(n)}\) from \(L^{p_1} \times L^{p_2}\) to \(L^p\) for \(p > 1/2\), can be derived by Muscalu’s work on Calderón commutators on polydiscs \[34, \text{Theorem 6.1}\] via time-frequency analysis.

In this section we will apply Corollary \[1.5\] to obtain a direct proof of the boundedness of \(C_1^{(n)}\) from \(L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)\) to \(L^p(\mathbb{R}^n)\) in the full range of \(p > 1/2\).

**Proposition 6.7.** Let \(1 < p_1, p_2 < \infty\), \(1/2 < p < \infty\) and \(1/p = 1/p_1 + 1/p_2\). Then the operator \(C_1^{(n)}(f, a)\) is bounded from \(L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)\) into \(L^p(\mathbb{R}^n)\), i.e.,

\[
\|C_1^{(n)}(f, a)\|_{L^p(\mathbb{R}^n)} \leq C_p \|a\|_{L^{p_1}(\mathbb{R}^n)}\|f\|_{L^{p_2}(\mathbb{R}^n)}.
\]

**Proof.** The operator \(C_1^{(n)}(f, a)\) is a bilinear operator which can also be expressed in bilinear Fourier multiplier form as

\[
C_1^{(n)}(f, a)(x) = (-i\pi)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{f}(\xi_1, \ldots, \xi_n) \hat{a}(\eta_1, \ldots, \eta_n) e^{2\pi i x \cdot (\xi + \eta)} m(\xi; \eta) \, d\xi \, d\eta,
\]

where the symbol \(m\) is given by

\[
m(\xi; \eta) = \prod_{i=1}^{n} \left[ \text{sgn}(\eta_i) \Phi\left( \frac{\xi_i}{\eta_i} \right) \right],
\]

and \(\xi = (\xi_1, \ldots, \xi_n)\) and \(\eta = (\eta_1, \ldots, \eta_n)\). Since \(m(\xi, \eta) = \prod_{i=1}^{n} \sigma(\xi_i, \eta_i)\) is a product of \(n\) equal pieces, by Corollary \[1.3\] it suffices to verify that \(\sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\hat{\psi}\|_{L^r_{1/2, \gamma/2}(\mathbb{R}^2)} = B < \infty\). Note that \(\sigma(2^k \cdot)\hat{\psi} \in L^r_{1/2, \gamma/2}(\mathbb{R}^2)\) uniformly in \(k\) by Proposition \[6.6\] so they are also in \(L^r_{\gamma/2, \gamma/2}(\mathbb{R}^2)\) uniformly due to that \(L^r_{\gamma}(\mathbb{R}^2) \subset L^r_{\gamma/2, \gamma/2}(\mathbb{R}^2)\). We complete the proof of Proposition \[6.7\] \(\square\)
References

[1] J. Aguirre, *Multilinear Pseudo-differential operators and paraproducts*, Thesis (Ph.D.)–Washington University in St. Louis, 1981, 155 pp.

[2] A. P. Calderón, *Commutators of singular integrals*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1092–1099.

[3] A. P. Calderón, *Commutators, singular integrals on Lipschitz curves and applications*, Proc. Intern. Congress Math. Helsinki (1978), 85–96.

[4] C. P. Calderón, *On commutators of singular integrals*, Studia Math. **53** (1975), 139–174.

[5] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, Adv. Math. **24** (1977), 101–171.

[6] J. Chen and G. Lu, *Hörmander type theorems for multi-linear and multi-parameter Fourier multipliers operators with limited smoothness*, Nonlinear Anal. **101** (2014) 98–112.

[7] M. Christ and M. Weinstein, *Dispersion of small-amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. (1991) **100**, 87–109.

[8] R. R. Coifman and Y. Meyer, *On commutators of singular integral and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[9] R. R. Coifman and Y. Meyer, *Commutateurs d’intégrales singulières et opérateurs multilineaires*, Ann. Inst. Fourier, Grenoble **28** (1978), 177–202.

[10] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudodifférentiels*, Astérisque **57** (1978).

[11] X. T. Duong, L. Grafakos and L. X. Yan, *Multilinear operators with non-smooth kernels and commutators of singular integrals*, Trans. Amer. Math. Soc. **362** (2010), 2089–2113.

[12] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. **93** (1971), 107–115.

[13] M. Fujita and N. Tomita, *Weighted norm inequalities for multilinear Fourier multipliers*, Trans. Amer. Math. Soc. **364** (2012), 6335–6353.

[14] L. Grafakos, *Classical Fourier Analysis, 3rd Edition*, GTM 249, Springer, New York, 2014.

[15] L. Grafakos, *Modern Fourier Analysis, 3rd Edition*, GTM 250, Springer, New York, 2014.

[16] L. Grafakos and H. V. Nguyen, *Multilinear Fourier multipliers with minimal Sobolev regularity, I*, Colloq. Math., to appear.

[17] L. Grafakos, A. Miyachi, H. V. Nguyen and N. Tomita, *Multilinear Fourier multipliers with minimal Sobolev regularity, II*, submitted.

[18] L. Grafakos, A. Miyachi and N. Tomita, *On multilinear Fourier multipliers of limited smoothness*, Can. J. Math. **65** (2013), 299–330.

[19] L. Grafakos and S. Oh, *The Kato-Ponce inequality*, Comm. Partial Differential Equations **39** (2014), 1128–1157.

[20] L. Grafakos and Z. Si, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angew. Math. **668** (2012), 133–147.

[21] L. Grafakos and R. H. Torres, On multilinear singular integrals of Calderón-Zygmund type. Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000). *Publ. Mat.* (2002), Extra, 57–91.

[22] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), 124-164.

[23] Y. Han, J. Li, G. Lu, and P. Wang, $H^p \rightarrow H^p$ boundedness implies $H^p \rightarrow L^p$ boundedness, Forum Math. **23** (2011), 729–756.

[24] D. He, *Square function characterization of weak Hardy spaces*, J. Fourier Anal. Appl. **20** (2014), 1083–1110.

[25] L. Hörmander, *Estimates for translation invariant operators in $L^p$ spaces*, Acta Math. **104** (1960), 93–140.

[26] J.-L. Journé, *Calderón–Zygmund operators on product spaces*, Rev. Mat. Iberoamericana, **1** (1985), 55–91.

[27] J.-L. Journé, *Two problems of Calderón–Zygmund operators on product spaces*, Ann. Inst. Fourier, Grenoble, **38** (1988), 111–132.
[28] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math. XLI (1988), 891–907.
[29] C. E. Kenig and E. M. Stein, Multilinear estimates and fractional integration, Math. Res. Letters, 6 (1999), 1–15.
[30] S. G. Mikhlin, On the multipliers of Fourier integrals, (Russian) Dokl. Akad. Nauk SSSR (N.S.) 109 (1956), 701–703.
[31] A. Miyachi and N. Tomita, Minimal smoothness conditions for bilinear Fourier multipliers, Rev. Mat. Iberoamericana 29 (2013), 495–530.
[32] A. Miyachi and N. Tomita, Boundedness criterion for bilinear Fourier multiplier operators, Tohoku Math. J. (2) 66 (2014), 55–76.
[33] C. Muscalu, Calderón commutators and the Cauchy integral on Lipschitz curves revisited I. First commutator and generalizations, Rev. Mat. Iberoam. 30 (2014), 727–750.
[34] C. Muscalu, Calderón commutators and the Cauchy integral on Lipschitz curves revisited III. Poly-disc extensions, Rev. Mat. Iberoam. 30 (2014), 1413–1437.
[35] C. Muscalu, J. Pipher, T. Tao and C. Thiele, Bi-parameter paraproducts, Acta Math. 193 (2004), 269–296.
[36] C. Muscalu, J. Pipher, T. Tao and C. Thiele, Multi-parameter paraproducts, Rev. Mat. Iberoamericana 22 (2006), 963–976.
[37] H. H. Schmeisser, H. Triebel, Topics in Fourier analysis and function spaces, Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology], 42. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987. 300 pp.
[38] E. M. Stein, The characterization of functions arising as potentials, Bull. Amer. Math. Soc. 67 (1961), 102–104.
[39] E. M. Stein, Singular integral and differentiability properties of functions, Princeton Univ. Press, Princeton, NJ, 1970.
[40] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, J. Funct. Anal. 259 (2010), 2028–2044.
[41] H. Triebel, Theory of function spaces II, Monographs in Mathematics, 84 Birkhäuser Verlag, Basel, 1992. viii+370 pp.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA MO 65211, USA
E-mail address: grafakosl@missouri.edu

DEPARTMENT OF MATHEMATICS, SUN YAT-sen UNIVERSITY, GUANGZHOU, 510275, P. R. CHINA
E-mail address: hedanqing@mail.sysu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487, USA
E-mail address: hvnguyen@ua.edu

DEPARTMENT OF MATHEMATICS, SUN YAT-sen UNIVERSITY, GUANGZHOU, 510275, P. R. CHINA
E-mail address: mcsylx@mail.sysu.edu.cn