Abstract

In this paper, we describe the randomized QLP (RQLP) algorithm and its enhanced version (ERQLP) for computing the low rank approximation to $A$ of size $m \times n$ efficiently such that $A \approx QLP$, where $L$ is the rank-$k$ lower-triangular matrix, $Q$ and $P$ are column orthogonal matrices. The theoretical cost of the implementation of RQLP and ERQLP only needs $O(mnk)$. Moreover, we derive the upper bounds of the expected approximation error $E[|\sigma_j(A) - \sigma_j(L)|/\sigma_j(A)]$ for $j = 1, \cdots, k$, and prove that the $L$-values of the proposed methods can track the singular values of $A$ accurately. These claims are supported by extensive numerical experiments.

Keywords: singular value decomposition, column pivoted QR decomposition, QLP decomposition, randomized QLP decomposition

AMS subject classifications: 15A18, 15A23, 65F99.

1. Introduction

Given an $m \times n$ matrix $A$, consider a randomized algorithm for computing rank-$k$ ($k \leq \text{rank}(A) \leq n \leq m$) approximate decomposition efficiently

$$A \approx QLP^T,$$

where $Q$ and $P$ are column orthogonal matrices and $L$ is lower-triangular matrix. The diagonal elements of $L$ are called $L$-values.

One way for gaining SVD-type information is the column pivoted QR (CPQR) decomposition, which admits the form $A \Pi = QR$, where $\Pi$ is a permutation matrix, factor $R$ is a upper-triangular matrix and the diagonal elements of $R$ are called the $R$-values, and $Q$ is orthogonal matrix. The partial CPQR can compute a rank-$k$ approximate column subspace of $A$ spanned by the leading $k$ columns in $A \Pi$. To improve the low-rank approximation, specialized pivoting strategies will result in some rank-revealing QR factorizations, see [1, 4, 5, 6] and the $R$-values can roughly approximate the singular values of $A$ [11, 12].

As another candidate of the SVD, G. W. Stewart [16] proposed a so called pivoted QLP decomposition. The algorithm consists of two CPQR decompositions. To be precise, the matrix $A \Pi_0$ is first factored as $A \Pi_0 = \hat{Q}R$, then the matrix $R^T \Pi_1$ is factored as $R^T \Pi_1 = \hat{P}L^T$, resulting in

$$A \approx QLP^T = \hat{Q}\Pi_1L\hat{P}^T\Pi_0^T$$

with $Q = \hat{Q}\Pi_1$ and $P = \Pi_0\hat{P}$, where $Q$ and $P$ are orthogonal, $\Pi_0$ and $\Pi_1$ are permutation matrices, and $L$ is lower triangular. It was shown that the $L$-values can track the singular values

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of $A$ far better than the $R$-values, see \[12\]. Similarly, the truncated pivoted QLP decomposition can also be used as a low rank approximation of $A$.

We see that the pivoted QLP decomposition applies twice pivoted QR decomposition, once to $A$ and once to $R^T$. The Householder triangularization with column pivoting requires about $(2mn^2 - 2n^3/3)$ floats when matrix $A$ is of full-rank. The additional work in the reduction to lower triangular form is $4n^3/3$ floats, see \[3\]. Thus when $m = n$ the pivoted QLP decomposition requires twice as much work as the pivoted QR decomposition. Therefore, it is a motive of this paper to design a fast algorithm to reduce the computation cost of the pivoted QLP decomposition without losing too much accuracy.

Our paper is organized as follows. In Section 2, we propose the randomized QLP decomposition algorithm and its enhanced version. Next, the asymptotic convergence rates of $L$-values are discussed in Section 3. Then, we give the block version of randomized QLP decomposition algorithm in Section 4. Finally, we provide the numerical experiments to confirm the effectiveness of proposed algorithms in Section 5 and some conclusion remarks in last section.

Throughout the paper $\sigma_i(M)$ will denote the $i$-th singular value of $M$ in descending order, and $\sigma_{\min}(M)$ is the smallest singular value of $M$. $\|M\|_F$ denotes the Frobenius norm, $\|M\|_2 = \sigma_1(M)$ is the spectral norm.

2. Randomized QLP decomposition

Let $V$ be a given $m \times k$ ($k \leq n \leq m$) matrix with independent columns, the orthogonal projector of the $R(V)$ is defined by

$$P_V = V(V^TV)^{-1}V^T,$$

where $R(X)$ denotes the range of a matrix $X$. Especially, when $V$ is orthonormal, $P_V = VV^T$. It is easy to verify \[8\] that

$$(A - P_V A)^T (A - P_V A) = A^T A - B^T B,$$

where $B = V^T A$. Based on the fact that for a matrix $M$, $\|M\|_F^2 = tr(M^TM)$, where $tr(X)$ means the trace of a matrix $X$, thus

$$\|A - P_V A\|_F^2 = \|A\|_F^2 - \|B\|_F^2. \quad (2.1)$$

Suppose the orthogonal basis vectors of $k$-dimensional dominant subspace of $A$ form the orthonormal matrix $V$, then we have $A \approx P_V A$. It also can be written as

$$\|A - P_V A\|_F \approx \min_{\text{rank}(X) \leq k} \|A - X\|_F. \quad (2.2)$$

There are similar results in the 2-norm.

Randomized range finder \[9\] is usually a crucial step in the randomized matrix approximation framework, which can be used to approximate the matrix $V$ above. The basic idea is to use random sampling to identify the subspace capturing the dominant actions of a matrix. To illustrate, we present a basic randomized range finder scheme as follows, see \[2, 8, 13, 14, 18\].

1. Draw a Gaussian random matrix $\Omega$ of size $n \times \ell$.
2. Compute a sampling matrix $Y = A\Omega$ of size $m \times \ell$.
3. Orthonormalize the columns of $Y$ to form the $m \times \ell$ matrix $V$, e.g., using the QR, SVD, etc.

The number of columns $\ell$ is usually slightly larger than the target rank $k$ because we can obtain more accurate approximations of this form. We refer to this discrepancy $p = \ell - k$ as the over-sampling parameter. Usually, $p = 5$ or $p = 10$ is often sufficient, see \[9\].
2.1. The randomized QLP decomposition

The randomized QLP (RQLP) decomposition can be split into two computational stages. The first is to construct a low-dimensional subspace and build an orthonormal matrix $V$ of size $m \times \ell$ that captures the action of $A$ via a randomized range finder. The second is to restrict the matrix to the subspace and then compute the QLP decomposition of the reduced matrix $B = V^T A$, as described below.

1. Set $B = V^T A$ so that $P_V A = V B$.
2. Compute the pivoted QLP decomposition so that $B = Q_B L_P^T B$.
3. Set $Q = V Q_B$, and $P = P_B$.

The pseudo-codes of rank-$k$ RQLP algorithm is given in Algorithm 2.1.

**Algorithm 2.1 RANDOMIZED QLP DECOMPOSITION ALGORITHM**

**Input:** $A \in \mathbb{R}^{m \times n}$, a target rank $k \geq 2$, an oversampling parameter $p \geq 2$.

**Output:** $[Q, L, P] = \text{RQLP}(A, k, p)$.

1. $\Omega = \text{rand}(n, \ell)$.
2. $Y = A \Omega$.
3. $[V, \sim] = \text{qr}(Y)$.
4. $B = V^T A$.
5. $[Q_0, R_0, \Pi_0] = \text{cpqr}(B)$.
6. $[Q_1, L^T, \Pi_1] = \text{cpqr}(R_0^T)$.
7. $Q = V Q_0 \Pi_1$, $P = \Pi_0 Q_1$.

Assume that the error matrix is $E$ in the randomized range finder, i.e., $E = A - V V^T A$.

Then we can have

$$A = Q L P^T + E. \quad (2.3)$$

The bounds on the probability of a large deviation given in [3, Theorem 10.5] for the randomized range finder can directly apply to the output of the RQLP algorithm too.

**Corollary 2.1.** Under the hypotheses of Algorithm 2.1 construct the orthonormal matrix $V$ according to the randomized range finder and perform the pivoted QLP decomposition to the reduced matrix $B = V^T A$ such that $B = Q_B L_P^T B$. If $p > 2$, then the expected approximation error

$$\mathbb{E} \left[ \| A - Q L P^T \|_F \right] \leq \left( 1 + \frac{k}{p - 1} \right)^{1/2} \left( \sum_{j=k+1}^{n} \sigma_j^2 \right)^{1/2}, \quad (2.4)$$

where $\mathbb{E}$ denotes expectation, $Q = V Q_B$ and $P = P_B$.

When the error matrix $E$ is measured in the spectral norm, as opposed to the Frobenius norm, the randomized scheme is slightly further removed from optimality. For details, see [3, Theorem 10.6].
2.2 The enhanced RQLP decomposition

In the case that the given matrix $A$ is singular, the CPQR decomposition in the first step is reasonable, because it orders the initial singular values of $A$ and moves the zeros to the bottom of the matrix, i.e.,

$$A\Pi = Q_0 \begin{bmatrix} R_0 \\ 0 \end{bmatrix},$$

where $R_0$ is upper-triangular matrix. Like the QR algorithm for computing the SVD of a real upper triangular matrix [2], Huckaby and Chan [11] showed that if we perform the QLP iteration, the $L$-values produced by QLP iteration can track the singular values of $A$ far better than the ones produced by pivoted QLP decomposition.

That is to say, we continue to take the QR decomposition with no pivoting of the transpose of the $R$ factor produced by the last step, i.e.,

$$R_{i-1}^T = Q_i R_i, \quad i = 1, 2, \cdots, (2.5)$$

the error bounds

$$\left| \sigma_j(R_{11}^{(i)})^{-1} - \sigma_j(A)^{-1} \right| \sigma_j(A)^{-1} \quad \text{will be improved by a quadratic factor in each step for } j = 1, 2, \cdots, k,$$

where the $k \times k$ upper triangular matrix $R_{11}^{(i)}$ is the first diagonal block of $R_i$, see [11].

Inspired by this, we use the unpivoted inner QR decomposition to $R_0$ in Algorithm 2.1 and we can get the enhanced randomized QLP algorithm, which is denoted by ERQLP. For $d$ times inner QR decomposition, the pseudo-codes of ERQLP algorithm is given in Algorithm 2.2 where $d$ is an even number. A similar argument can also be used when $d$ is odd.

**Algorithm 2.2 Enhanced randomized QLP decomposition algorithm**

**Input:** $A \in \mathbb{R}^{m \times n}$, a target rank $k \geq 2$, an oversampling parameter $p \geq 2$ and an even number of inner QR iterations $d$.

**Output:** $[Q, L, P] = \text{ERQLP}(A, k, p, d)$.

1. $\Omega = \text{rand}(n, \ell)$. % Gaussian random matrix $\Omega$ is of size $n \times \ell$
2. $Y = A\Omega$.
3. $[V, \sim] = \text{qr}(Y)$. % Range finder
4. $B = V^T A$.
5. $[Q^{(0)}, R^{(0)}, \Pi] = \text{cpqr}(B)$. % The first CPQR decomposition
6. for $i = 1, \cdots, d$ % $d$ times inner QR decomposition
   7. $[Q^{(i)}, R^{(i)}] = \text{qr}([R^{(i-1)}]^T)$. % Inner QR decomposition
8. end
9. $Q = V Q^{(0)} Q^{(2)} \cdots Q^{(d)}$, $L = [R^{(d)}]^T$, $P = \Pi Q^{(1)} Q^{(3)} \cdots Q^{(d-1)}$.

2.3 Computational complexity

In this subsection, we analyze the theoretical cost of the implementation of Algorithm 2.1–2.2 and we compare it to those of the truncated pivoted QLP decomposition.

Let $C_{mm}$, $C_{qr}$ and $C_{cpqr}$ denote the scaling constants for the cost of executing the matrix-matrix multiplication, the full QR decomposition and the CPQR decomposition, respectively. More specifically, we assume that

1. multiplying two matrices of size $m \times n$ and $n \times r$ costs $C_{mm}mnr$. 


2. performing a QR decomposition with no pivoting of a matrix of size $m \times n$, with $m \geq n$, costs $C_{qr}mn^2$.

3. performing a CPQR decomposition of a matrix of size $m \times n$, with $m \geq n$, costs $C_{cpqr}mn^2$.

The execution time for Algorithm 2.1 is easily seen to be

$$T_{RQLP} \sim 2C_{mm}mn\ell + (C_{mm} + C_{qr})m\ell^2 + C_{cpqr}n\ell^2 + C_{cpqr}\ell^3.$$ 

Similarly, the execution time for Algorithm 2.2 is

$$T_{ERQLP} \sim 2C_{mm}mn\ell + (C_{mm} + C_{qr})m\ell^2 + C_{cpqr}n\ell^2 + d(C_{mm} + C_{qr})\ell^3.$$ 

To compute the rank-$\ell$ pivoted QLP decomposition, we require to apply partial CPQR decomposition on the $m$-by-$n$ matrix $A$, i.e., $A\Pi(:,1:\ell) = QR$, and compute the full CPQR decomposition of the $n$-by-$\ell$ matrix $[R_{11} \ R_{12}]^T$. This gives a total execution time of

$$T_{QLP} \sim C_{cpqr}mn\ell + C_{cpqr}n\ell^2.$$ 

Compared with $T_{QLP}$ by omitting the lower order terms, we see that $T_{RQLP}$ and $T_{ERQLP}$ have the same order of magnitude in cost flops. But the truncated QLP is very time-consuming to permute the data required in CPQR for the $m$-by-$n$ matrix $A$, while the most flops for RQLP and ERQLP are spent on the matrix-matrix multiplications which is the so-called nice BLAS-3 operations, and the original large-scale matrix $A$ is visited only twice.

3. Error analysis

In this section, we assess the ability of the approximate QLP decomposition, i.e., RQLP and ERQLP, to capture the singular values. We start with the following lemmas.

**Lemma 3.1.** [7, Theorem 3.1] Let $A$ be an $m$-by-$n$ matrix and $V$ be a matrix with orthonormal columns. Then $\sigma_j(A) \geq \sigma_j(V^TA)$ for $1 \leq j \leq \min(m,n)$.

**Lemma 3.2.** [7, Theorem 5.6] Under the hypotheses of Algorithm 2.1 construct the column orthogonal matrix $V$ and the reduced matrix $B = V^TA$ according to the randomized range finder, and let $B_k$ be the rank-$k$ truncated SVD of $B$. Then for any $p \geq 2$,

$$\mathbb{E}[\sigma_j(V B_k)] \geq \frac{\sigma_j(A)}{\sqrt{1 + C^2\tau_j^2}},$$

where $C = 4e\sqrt{\ell(\sqrt{n-\ell} + p + \sqrt{\ell} + 7)}$, $\tau_j = \sigma_{k+1}(A)/\sigma_j(A)$, $j = 1, \ldots, k$.

For all $1 \leq j \leq k$, Lemma 3.2 implies that

$$\mathbb{E} \left[ \frac{\sigma_j(A) - \sigma_j(V B_k)}{\sigma_j(A)} \right] \leq 1 - \frac{1}{\sqrt{1 + C^2\tau_j^2}}.$$  \hspace{1cm} (3.1)

Then we have the following theorem to illustrate that the $L$-values of RQLP can track the singular values of $A$ precisely.

Theorem 3.1. Under the hypotheses of Algorithm \[2.1\] compute the pivoted QLP decomposition so that \( B = Q_B L P_B^T \) and partition the lower triangular matrix \( L \) into diagonal blocks \( L_{11} \) and \( L_{22} \) and off-diagonal block \( L_{21} \), where \( L_{11} \) is of size \( k \)-by-\( k \). Assume that the bounds
\[
\gamma \geq \sigma_k(B)/\sqrt{k(n-k+1)}, \quad \|L_{22}\|_2 \leq \sigma_{k+1}(B)\sqrt{(k+1)(n-k)} \tag{3.2}
\]
hold, where \( \gamma = \sigma_{\min}(L_{11}) \) and \( \rho = \|L_{22}\|_2/\gamma < 1 \). Then for \( 1 \leq j \leq k \),
\[
E \left[ \frac{\sigma_j(A) - \sigma_j(L_{11})}{\sigma_j(A)} \right] \leq 1 - \frac{1}{\sqrt{1 + \gamma^2 j^2}} + O \left( \frac{\|L_{21}\|_2^2}{(1 - \rho^2)\gamma^2} \right), \tag{3.3}
\]
where \( \tau_j = \sigma_{k+1}(A)/\sigma_j(A) \).

Proof. The rank revealing QR decomposition provides bounds on the singular values of \( B \) in terms of the norms of the blocks. By Theorem 3.1 of Reference \[15\], when the bounds (3.2) hold, we can have
\[
\frac{\sigma_j(L_{11})}{\sigma_j(B)} \geq \left[ 1 - \frac{\|L_{21}\|_2^2}{(1 - \rho^2)\gamma^2} \right]^{1/2}
\]
for \( 1 \leq j \leq k \), which is equivalent to
\[
\frac{\sigma_j(L_{11})}{\sigma_j(B)} \geq 1 - O \left( \frac{\|L_{21}\|_2^2}{(1 - \rho^2)\gamma^2} \right).
\]
The above relation can be further rewritten as
\[
\sigma_j(B) - \sigma_j(L_{11}) \leq \sigma_j(B)O \left( \frac{\|L_{21}\|_2^2}{(1 - \rho^2)\gamma^2} \right). \tag{3.4}
\]
Let \( V \) be the column orthogonal matrix constructed by Algorithm \[2.1\]. Using Lemma \[3.1\] we have
\[
\sigma_j(B) = \sigma_j(V^T A) \leq \sigma_j(A).
\]
From this, the formula (3.4) is reduced to
\[
\frac{\sigma_j(B) - \sigma_j(L_{11})}{\sigma_j(A)} \leq O \left( \frac{\|L_{21}\|_2^2}{(1 - \rho^2)\gamma^2} \right). \tag{3.5}
\]
On the other hand,
\[
E [\sigma_j(A) - \sigma_j(L_{11})] = E [\sigma_j(A) - \sigma_j(V B_k)] + E [\sigma_j(V B_k) - \sigma_j(L_{11})]
\]
\[
= E [\sigma_j(A) - \sigma_j(V B_k)] + E [\sigma_j(B) - \sigma_j(L_{11})].
\]
Thus,
\[
E \left[ \frac{\sigma_j(A) - \sigma_j(V B_k)}{\sigma_j(A)} \right] = E \left[ \frac{\sigma_j(A) - \sigma_j(V B_k)}{\sigma_j(A)} \right] + E \left[ \frac{\sigma_j(B) - \sigma_j(L_{11})}{\sigma_j(A)} \right].
\]
Combing (3.1) and (3.5), we arrive at (3.3).

Next, we can obtain the error upper bounds for the \( L \)-values of ERQLP similarly.
Lemma 3.3. [11, Theorem 4.1] Let \( B \) be an \( \ell \)-by-\( n \) matrix, \( n \geq \ell \geq k \) and the singular values of \( B \) satisfy \( \sigma_k(B) > \sigma_{k+1}(B) \). The upper triangular matrix \( R^{(0)} \) is the \( R \)-factor in the first pivoted QR decomposition of \( B \), i.e., \( B \Pi = Q^{(0)} R^{(0)} \) and then apply \( i \) times inner unpivoted QR decomposition to \( [R^{(i-1)}]^T \), i.e., \( [R^{(i-1)}]^T = Q^{(i)} R^{(i)} \) (\( i = 1, 2, \cdots \)). Partition the upper triangular matrix \( R^{(i)} \) into diagonal block \( R^{(i)}_{11} \) and \( R^{(i)}_{22} \) and off-diagonal block \( R^{(i)}_{12} \), where \( R^{(i)}_{11} \) is of size \( k \)-by-\( k \). Assume the bounds in (3.6) hold. Then for \( j = 1, \cdots , k \),

\[
\frac{\sigma_j(R^{(i)}_{11}) - \sigma_j(B)}{\sigma_j(B)^{-1}} \leq \left( \frac{\sigma_{k+1}(B)}{\sigma_k(B)} \right)^{2i} \cdot O \left( \frac{n^{(4i+1)/2} R^{(i)}_{22}}{[1 - (\rho(i))^2] (\gamma(i))^2} \right). \tag{3.7}
\]

From the interlacing property of singular values, \( \sigma_j(B) \geq \sigma_j(R^{(i)}_{11}) \). The formula (3.7) implies that

\[
\frac{\sigma_j(B) - \sigma_j(R^{(i)}_{11})}{\sigma_j(B)} \leq \left( \frac{\sigma_{k+1}(B)}{\sigma_k(B)} \right)^{2i} \cdot O \left( \frac{n^{(4i+1)/2} R^{(i)}_{22}}{[1 - (\rho(i))^2] (\gamma(i))^2} \right). \tag{3.8}
\]

It means that if the relative error bound \([\sigma_j(B) - \sigma_j(R^{(i)}_{11})]/\sigma_j(B)\) is improved with a quadratic factor by one inner QR decomposition.

Theorem 3.2. Under the hypotheses of Algorithm 2.2, \( B = Q_B L P_B^T \), where \( Q_B = Q^{(0)} Q^{(2)} \cdots Q^{(d)} \), \( L = [R^{(d)}]^T \), \( P_B = \Pi Q^{(1)} Q^{(3)} \cdots Q^{(d-1)} \), and the upper triangular matrix \( R^{(0)} \) and \( R^{(i)} \) are the \( R \)-factor in the first pivoted QR decomposition and the \( i \)-th inner QR decomposition, respectively, i.e.,

\[
B \Pi = Q^{(0)} R^{(0)} , \quad [R^{(i-1)}]^T = Q^{(i)} R^{(i)},
\]

where \( i = 1, \cdots , d \). Let \( L \) and \( R^{(d)} \) be partitioned as

\[
L = \begin{bmatrix} L_{11} & 0 \\ L_{12} & L_{22} \end{bmatrix}, \quad R^{(d)} = \begin{bmatrix} R^{(d)}_{11} & R^{(d)}_{12} \\ 0 & R^{(d)}_{22} \end{bmatrix}.
\]

Assume that the bounds in (3.6) hold. Then for \( 1 \leq j \leq k \),

\[
\mathbb{E} \left[ \frac{\sigma_j(A) - \sigma_j(L_{11})}{\sigma_j(A)} \right] \leq 1 - \frac{1}{\sqrt{1 + C^2 \tau_j}} + \left( \frac{\sigma_{k+1}(B)}{\sigma_k(B)} \right)^{2d} \cdot O \left( \frac{n^{(4d+1)/2} R^{(i)}_{12} \| R^{(0)}_{12} \|_2}{[1 - (\rho(d))^2] (\gamma(d))^2} \right). \tag{3.9}
\]

Proof. The proof is similar to those given in Theorem 3.1. Based on (3.1) and (3.8), we can get (3.9) directly.

From Theorem 3.2, one can observe that for \( j = 1, 2, \cdots , k \), the larger the iteration number \( d \) in Algorithm 2.2 is, the smaller the expected relative error bound \([\sigma_j(A) - \sigma_j(L_{11})]/\sigma_j(A)\) is. The \( L \)-values from ERQLP will be the better approximations to the singular values of \( A \) than those of RQLP algorithm. But at the same time, the number of operations at each iteration will increase. A lot of numerical tests show that \( d = 2 \) or 4 is an appropriate choice.
4. THE BLOCK VERSION OF RQLP

It is straightforward to convert the RQLP algorithm to a block scheme, denoted by the block RQLP (BRQLP) algorithm. Suppose the target rank \( n \) and the block size \( b \) satisfy \( \ell = \frac{h b}{\ell} \) for some integer \( h \). Partition the Gaussian random matrix \( \Omega \) into slices \( \{\Omega_j\}_{j=1}^h \), each of size \( n \times b \), so that \( \Omega = [\Omega_1, \Omega_2, \ldots, \Omega_h] \), \( Y = A\Omega = [Y_1, Y_2, \ldots, Y_h] \), where \( Y_j = A\Omega_j \) for \( j = 1, \ldots, h \). Analogously, partition the orthonormal matrix \( V \) and the reduced matrix \( B \) in groups of \( b \) columns and \( b \) rows, respectively,

\[
V = [V_1, V_2, \ldots, V_h], \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_h \end{bmatrix}.
\]

Inspired by the block version of the randomized range finder in \[14\], we first initiate the algorithm by setting \( A^{(0)} = A \) and construct the matrices \( \{V_j\}_{j=1}^h \) and \( \{B_j\}_{j=1}^h \) one at a time.

We further compute the unpivoted QR decomposition of \( Y_j \) because the R-factor is never used, i.e.,

\[
[V_j, \sim] = qr(Y_j).
\]

Just like the classical (non-block) Gram-Schmidt orthogonalization procedure \[3\], the round-off errors will cause loss of orthonormality among the columns in \([V_1, V_2, \ldots, V_j]\). The remedy is to adopt the reorthogonalized strategy,

\[
[V_j, \sim] = qr(Y_j - \Sigma_{i=1}^{j-1} V_i V_i^T V_j).
\]

Next, calculate the block reduced matrix \( B_j \) and \( A^{(j)} \) by

\[
B_j = V_j^T A^{(j-1)}, \quad A^{(j)} = A^{(j-1)} - V_j B_j.
\]

Then apply the standard pivoted QLP decomposition to \( B_j \) such that

\[
B_j = Q_B^{(j)} L_j [P_B^{(j)}]^T, \tag{4.4}
\]

and update \( Q_j = V_j Q_B^{(j)} \), \( P_j = P_B^{(j)} \). To be precise, the pseudo-codes of BRQLP algorithm is shown in Algorithm 4.1.

From (4.3), we can have \( B_j = V_j^T A \). It follows that the error resulting from Algorithm 4.1 is

\[
A - QLP^T = A - [Q_1, \ldots, Q_h] \begin{bmatrix} L_1 \\ \vdots \\ L_s \end{bmatrix} [P_1, P_2, \ldots, P_h]^T
\]

\[
= A - [V_1, \ldots, V_h] \begin{bmatrix} B_1 \\ \vdots \\ B_h \end{bmatrix} = A - VB = A - VV^T A. \tag{4.5}
\]

As shown in \[14\], for a fixed Gaussian matrix \( \Omega \), the projectors from the block and unblock version of the randomized range finder are identical. Thus, the average-case analysis of Algorithm \[2.1\] can be similarly applied to Algorithm 4.1.

For the block randomized QLP algorithm, we assume that it stops after \( s \) steps. Then, the runtime of BRQLP decomposition is

\[
T_{BRQLP} \sim \sum_{j=1}^h \frac{3C_{mm}mnb + 2C_{qr}mb^2 + 2(j-1)C_{mm}mb^2 + C_{cpqr}nb^2 + C_{mm}mb^2}
\sim \sum_{j=1}^h \frac{3C_{mm}mnl + \frac{2}{h}C_{qr}ml^2 + C_{mm}ml^2 + \frac{1}{h}C_{cpqr}nl^2 + \frac{1}{h}C_{mm}ml^2}.
\]
Algorithm 4.1 BLOCK RANDOMIZED QLP DECOMPOSITION ALGORITHM

Input: \( A \in \mathbb{R}^{m \times n} \), a target rank \( k \geq 2 \), an oversampling parameter \( p \geq 2 \), a block size \( b \).
Output: \( [Q, L, P] = \text{BRQLP}(A, k, p, b) \).

1. \( \Omega = \text{rand}(n, \ell) \). % Gaussian random matrix \( \Omega \) is of size \( n \times \ell \)
2. for \( j = 1, \ldots, h \)
3. \( \Omega_{j} = \Omega(:, (j - 1)b + 1 : jb) \), \( Y_{j} = A\Omega_{j} \).
4. \( [V_{j}, \sim] = \text{qr}(Y_{j}) \). % Range finder
5. \( [V_{j}, \sim] = \text{qr}(V_{j} - \Sigma_{t=1}^{j-1}V_{t}V_{t}^{T}V_{j}) \). % Reorthogonalized process
6. \( B_{j} = V_{j}^{T}A^{(j-1)} \).
7. \( A^{(j)} = A^{(j-1)} - V_{j}B_{j} \).
8. \( [Q_{B}^{(j)}, L_{j}, P_{B}^{(j)}] = \text{qlp}(B_{j}) \). % Standard QLP decomposition
9. \( Q_{j} = V_{j}Q_{B}^{(j)} \), \( P_{j} = P_{B}^{(j)} \).
10. end
11. \( Q = [Q_{1}, Q_{2}, \cdots, Q_{h}] \), \( L = \text{blkdiag}(L_{1}, L_{2}, \cdots, L_{h}) \), \( P = [P_{1}, P_{2}, \cdots, P_{h}] \).

Comparing with RQLP, we see that the BRQLP involves one additional term of \( C_{mmn}\ell \), but needs less time to execute full QR decomposition.

5. Numerical experiments

In this section, we give several examples to illustrate that the randomized algorithms are as accurate as the classical methods. All experiments are carried out on a Founder desktop PC with Intel(R) Core(TM) i5-7500 CPU 3.40 GHz by MATLAB R2016(a) with a machine precision of \( 10^{-16} \).

Example 5.1. Specifically, like Tropp in [17], two synthetic matrices are given by its SVD forms:
\[
A = U^{(A)}\Sigma^{(A)}(V^{(A)})^{T} \in \mathbb{R}^{n \times n},
\]
where \( U^{(A)} \in \mathbb{R}^{n \times n} \) and \( V^{(A)} \in \mathbb{R}^{n \times n} \) are random orthogonal matrices, the \( \Sigma^{(A)} \) is diagonal with entries given by the following rules.

i). Polynomially decaying spectrum (pds): \( \Sigma^{(A)} = \text{diag}(1, \cdots, 1, 2^{-s}, 3^{-s}, \cdots, (n - t + 1)^{-s}) \),

ii). Exponentially decaying spectrum (eds): \( \Sigma^{(A)} = \text{diag}(1, \cdots, 1, 2^{-s}, 2^{-2s}, \cdots, 2^{-(n-t)s}) \),

where the nonnegative constants \( t \) and \( s \) control the rank of the significant part of the matrix and the rate of decay, respectively.

Example 5.2. The remaining testing problems are from P. C. Hansen’s Regularization Tools (version 4.1) [10]. The ill-conditioned matrix \( A \) are generated by the discretization of the Fredholm integral equation with the first kind square integrable kernel
\[
\int_{a}^{b} K(y, z) f(z) \, dz = g(y), \quad c \leq y \leq d.
\]

When the Galerkin discretization method is used, we choose the examples heat and phillips.
In the following test, we compare the proposed algorithms against several existing algorithms in terms of execution time and accuracy. For more details, we fix the matrix size \( m = n \) and set the target rank \( k = 120 \). The over-sampling parameter is \( p = 5 \). The approximate errors for the singular values of \( A \) are defined by

\[
\text{QLP/RQLP/ERQLP : } \text{err} = \max \{|\sigma_j(A) - \sigma_j(L)| : j = 1, \ldots, k\}.
\]

In Figure 1–Figure 2, we display the ability of \( R \)-values computed by CPQR and \( L \)-values computed by QLP, RQLP and ERQLP, to capture the singular values of the input matrix as described in Section 3. It shows that the \( L \)-values can track the singular values of \( A \) far better than the \( R \)-values.

Table 1–Table 4 show the measured total CPU times \( T \) (in seconds) and approximate error \( \text{err} \), which lead us to make several observations:

1. Comparing the accuracy of QLP with its randomized variants, QLP and RQLP have almost the same approximation error. However, when two or four times inner QR iterations are taken, ERQLP performs a higher approximation accuracy than RQLP and QLP.

2. Comparing the speed of QLP with its randomized variants, randomized QLP algorithms (RQLP and ERQLP) are decisively faster than QLP in all cases. In more detail, we see that RQLP and ERQLP have the similar speed, with RQLP being slightly faster.

| \( n \) | QLP \( T(s) \) | QLP \( \text{err} \) | RQLP \( T(s) \) | RQLP \( \text{err} \) | ERQLP \( d = 2 \) \( T(s) \) | ERQLP \( d = 2 \) \( \text{err} \) | ERQLP \( d = 4 \) \( T(s) \) | ERQLP \( d = 4 \) \( \text{err} \) |
|---|---|---|---|---|---|---|---|---|
| 2000 | 2.757 | 9.55e-02 | 0.042 | 9.32e-02 | 0.078 | 3.58e-02 | 0.082 | 2.50e-02 |
| 4000 | 24.414 | 5.34e-02 | 0.145 | 5.02e-02 | 0.282 | 5.20e-02 | 0.267 | 2.97e-02 |
| 6000 | 74.175 | 6.36e-02 | 0.312 | 6.20e-02 | 0.551 | 2.80e-02 | 0.554 | 2.09e-02 |

| \( n \) | QLP \( T(s) \) | QLP \( \text{err} \) | RQLP \( T(s) \) | RQLP \( \text{err} \) | ERQLP \( d = 2 \) \( T(s) \) | ERQLP \( d = 2 \) \( \text{err} \) | ERQLP \( d = 4 \) \( T(s) \) | ERQLP \( d = 4 \) \( \text{err} \) |
|---|---|---|---|---|---|---|---|---|
| 2000 | 2.741 | 1.65e-01 | 0.041 | 1.68e-01 | 0.079 | 1.22e-01 | 0.083 | 1.07e-02 |
| 4000 | 11.901 | 1.76e-01 | 0.145 | 1.75e-01 | 0.274 | 1.45e-01 | 0.255 | 9.46e-02 |
| 6000 | 75.822 | 1.69e-01 | 0.277 | 1.65e-01 | 0.544 | 1.09e-01 | 0.527 | 7.95e-02 |

| \( n \) | QLP \( T(s) \) | QLP \( \text{err} \) | RQLP \( T(s) \) | RQLP \( \text{err} \) | ERQLP \( d = 2 \) \( T(s) \) | ERQLP \( d = 2 \) \( \text{err} \) | ERQLP \( d = 4 \) \( T(s) \) | ERQLP \( d = 4 \) \( \text{err} \) |
|---|---|---|---|---|---|---|---|---|
| 2000 | 4.039 | 8.62e-02 | 0.086 | 8.62e-02 | 0.136 | 2.16e-02 | 0.137 | 7.96e-03 |
| 4000 | 27.572 | 8.62e-02 | 0.235 | 8.62e-02 | 0.411 | 2.16e-02 | 0.427 | 7.96e-03 |
| 6000 | 95.160 | 8.62e-02 | 0.567 | 8.62e-02 | 0.965 | 2.16e-02 | 0.958 | 7.96e-03 |
Table 4: Computational time and error of different factorizations for the example phillips.

| n    | QLP  | RQLP | ERQLP (d = 2) | ERQLP (d = 4) |
|------|------|------|---------------|---------------|
|      | $T(s)$ | err  | $T(s)$ | err  | $T(s)$ | err  | $T(s)$ | err  |
| 2000 | 3.194 | 7.12e-01 | 0.086 | 7.10e-01 | 0.157 | 3.88e-01 | 0.156 | 2.62e-01 |
| 4000 | 25.026 | 7.12e-01 | 0.264 | 7.06e-01 | 0.436 | 3.86e-01 | 0.463 | 2.72e-01 |
| 6000 | 91.232 | 7.12e-01 | 0.563 | 7.08e-01 | 0.944 | 4.15e-01 | 0.974 | 2.26e-01 |

Figure 1: The approximations to the singular values of $A$ for the examples pds (left) and eds (right). Rank control parameter $t = 30$. Rate of decay control parameters are set by $s = 2$ and $s = 1/20$ for pds and eds, respectively. The matrix size is $4000 \times 4000$.

Figure 2: The approximations to the singular values of $A$ for the example heat (left) and phillips (right). The matrix size is $4000 \times 4000$.

6. Concluding remarks

Based on the randomized range finder algorithm, two randomized QLP decomposition are proposed: RQLP and ERQLP, which can be used for producing standard low rank factorization, like a partial QR decomposition or a partial singular value decomposition and have several attractive properties.

1. The theoretical cost of the implementation of RQLP and ERQLP only need $O(mnk)$.
2. It is easy to convert the RQLP algorithm to a block scheme.
3. The $L$-values from ERQLP can track the singular values of $A$ better than QLP, in particular, much better than the $R$-values given by CPQR. The computational time of randomized QLP
variants is much less than the original deterministic one. And we provide mathematical justification and numerical experiments for these phenomena.

Moreover, the randomized QLP decomposition are suitable for the fixed-rank determination problems in practice. Different from (2.1), we can seek $V$ and $B$ with a suitable rank such that

$$
\|E\|_F^2 = \|A\|_F^2 - \|B\|_F^2 \leq \varepsilon,
$$

where $\varepsilon$ is the accuracy tolerance, and set $\|E\|_F$ as the error indicator which leads to a auto-rank randomized QLP algorithm. That is to say, the rank of factor matrices needs to be automatically determined by some accuracy conditions. We will continue to further in-depth study from the viewpoint of both theory and computations for this problem in future.

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