Abstract

The study of the two-shell system started in “Pair of null gravitating shells I and II” is continued. The pull back of the Liouville form to the constraint surface, which contains complete information about the Poisson brackets of Dirac observables, is computed in the singular double-null Eddington-Finkelstein (DNEF) gauge. The resulting formula shows that the variables conjugate to the Schwarzschild masses of the intershell spacetimes are simple combinations of the values of the DNEF coordinates on these spacetimes at the shells. The formula is valid for any number of in- and out-going shells. After applying it to the two-shell system, the symplectic form is calculated for each component of the physical phase space; regular coordinates are found, defining it as a symplectic manifold. The symplectic transformation between the initial and final values of observables for the shell-crossing case is written down.
1 Introduction

The present paper is the third in a series devoted to the two-shell system. The first paper, Ref. [1], and the second, Ref. [2], will be referred to as I and II. In I, all classical solutions that have a regular center on the left are described, and the space of solutions is parametrized by three discrete and four continuous parameters. The space of solutions is a candidate for the physical phase space and the parameters are candidates for Dirac observables. In II, the action functional for a single shell due to Louko, Whiting and Friedman [3] is generalized to any number of in- and out-going shells. The pull back $\Theta_\Gamma$ of the corresponding Liouville form $\Theta$ to the constraint surface $\Gamma$ is transformed into coordinates consisting of embeddings, embedding momenta, and Dirac observables. Some general properties of the pull back, such as gauge invariance, are shown. They enable us to accomplish the transformation explicitly, and this is the main task of the present paper.

The central idea of this paper is to calculate $\Theta_\Gamma$ in double-null Eddington-Finkelstein (DNEF) coordinates. This is a kind of gauge, but a singular one: Both the metric and the embeddings corresponding to this gauge are discontinuous at the shells and diverging at any Schwarzschild horizon. We shall show that this singularity does not influence the calculated value of $\Theta_\Gamma$. That follows from the way $\Theta_\Gamma$ transforms under ordinary gauge transformations. What is the motivation for using the singular gauge?

In fact, we first calculated it in regular gauges. The results revealed that the DNEF coordinates had a special role: their values at the shells appeared in the final formula for $\Theta_\Gamma$. Moreover, the calculations were very long while the final formula was very simple, indicating that working with DNEF coordinates from the start could lead to simplifications. This turns out to be right, although the calculation is still far from being trivial. One can perhaps say that the Liouville form chooses itself the spacetime coordinates in which it likes to be expressed.

Originally, $\Theta_\Gamma$ has a form of an integral over a Cauchy surface. Each Cauchy surface has a boundary consisting of the regular center and the infinity; each such surface intersects the shells at intersection points. Due to the discontinuity of the DNEF gauge at the shells, the contribution of each intersection point to $\Theta_\Gamma$ is non-zero, whereas the contributions from the regular center and the infinity in this gauge both vanish (cf. [4], where the only non-zero contribution comes from the infinity). Thus, $\Theta_\Gamma$ can be cast as a sum over all intersection points in which the summands have a standard form. In this way, a general formula can be shown to be valid for any number of in- or out-going shells.

The sum over the intersection points is of course associated with a particular Cauchy surface. Let us consider two such surfaces. If the shells do not cross between these two Cauchy surfaces, then the corresponding summands have the same value on
each surface. However, if the shells do cross between the surfaces, the corresponding
summands for the Cauchy surface below and above the crossing point are related
by a highly non-trivial canonical transformation.

The plan of the paper is as follows. In Sec. 2, the calculation of $\Theta_{\Gamma}$ in the DNEF
gauge is justified and accomplished. The result is a general formula that expresses
$\Theta_{\Gamma}$ in terms of some Dirac observables. These are the Schwarzschild masses of the
intershell spacetime pieces and some combinations of the values of DNEF coordinates
on these pieces at the shells. The canonical transformation between the observables
below and above the crossing point is calculated.

In Sec. 3, the formula is specialized to the system of two shells. Some complete
sets of Dirac observables as coordinates on the various components and regions of
the physical phase space are considered. The physical phase space is then given
the structure of a symplectic manifold. On each component of the physical phase space,
we find a global chart with respect to which the components of the symplectic form
are $C^\infty$ and regular, in the sense that the matrix of the components has a nowhere
vanishing determinant. In particular, in the case of shell-crossing, the “singular”
case in which both shells lie on horizons (denoted by $C_{00}$ in I) turns out to be a
smooth surface in the phase space. Finally, Sec. 4 contains some conclusions and
outlook, speculating about the prospective quantum theory.

2 Calculation of the Liouville form in a singular
gauge

Let $\Sigma$ be a Cauchy surface defined by an embedding $(U(\rho), V(\rho))$. In II, the pull
back $\Theta_{\Gamma}$ to the constraint surface $\Gamma$ of the Liouville form $\Theta$ has been written as a sum
of various contributions from different parts of $\Sigma$: First, there are contributions from
each spacetime point (denoted by $p$) where a shell intersects $\Sigma$. If the intersection is
with a single (in-going or out-going) shell, each point $p$ contributes by a single term
(cf. II, Eq. (46))

$$pdr \, .$$

If the intersection is a point $p$ where an in-going and an out-going shells cross each
other, then the contribution is

$$p_{\text{out}}dr + p_{\text{in}}dr \, .$$

Second, there are contributions from each connected volume cut out from $\Sigma$ by the
shell intersections. Each such volume contributes by the boundary terms

$$(fdU + gdV + hi do^j - \varphi db)_{\rho=b} - (fdU + gdV + hi do^j - \varphi da)_{\rho=a} \, ,$$
where \( \rho = a \) and \( \rho = b \), \( a < b \), are the boundary points of the volume (cf. Eq. (60) of II), \( \phi^i \) are Dirac observables, \( f, g, h_i \) and \( \varphi \) are functions of \( \phi^i, U(\rho) \) and \( V(\rho) \) defined by Eqs. (51)–(55) of II:

\[
\begin{align*}
  f &= \frac{RR_U}{2} \ln \left( \frac{U'}{V'} \right) + F(U, V, \phi^i), \\
  g &= \frac{RR_V}{2} \ln \left( \frac{U'}{V'} \right) + G(U, V, \phi^i), \\
  h_i &= \frac{RR_i}{2} \ln \left( \frac{U'}{V'} \right) + H_i(U, V, \phi^i), \\
  \varphi &= RR_U U' - RR_V V' - \frac{R}{2} (R_U U' + R_V V') \ln \left( \frac{U'}{V'} \right) \\
  &\quad - FU' - GV' + \phi(U, V, \phi^i),
\end{align*}
\]

The boundaries \( a \) and \( b \) can correspond either to the regular center \( \rho = 0 \) of \( \Sigma \), or to a shell intersection \( \rho = r \) with \( \Sigma \), or to the infinity \( \rho = \infty \) of \( \Sigma \). Finally, there is a contribution due to the infinity,

\[-N_\infty E_\infty dt\]

(cf. Eq. (46) of II).

In this paper, we shall collect all contributions at each particular boundary and transform it in several steps to a general simple form. We shall denote the contribution from the center by \( \Theta_0 \), from any shell intersection point \( r \) by \( \Theta_r \) and from the infinity by \( \Theta_\infty \). Thus, we have

\[\Theta_f = \Theta_0 + \sum_r \Theta_r + \Theta_\infty,\]

where

\[\Theta_0 = -(f du + gdV + h_i d\phi^i)_{\rho=0}\]

and

\[\Theta_\infty = \lim_{\rho=\infty} (f du + gdV + h_i d\phi^i) - N_\infty E_\infty dt.\]

The shortest way to calculate \( \Theta_f \) found as yet uses the DNEF gauge; this is, however, a singular gauge (cf. Sec. 3 of I), and some justification is in order.

In Sec 3.2.2 of II, gauge transformation of the functions \( F, G \) and \( H_i \) in the formulæ (1)–(3) have been calculated starting from the requirement that the Liouville form remains invariant. The invariance is in fact pointwise, that is, the integrand of the Liouville form in any volume part is invariant at any point \( \rho \). Calculating the
form in a singular gauge such as DNEF coordinates then makes sense. At the points where the gauge is regular the integrand has the same value as for a $C^1$ gauge. The DNEF gauge is singular where the embedding intersects a horizon; the value of the integrand at such a point can be defined as the limit from the left or from the right because the integrand is continuous in a regular gauge.

This argument shows that the difference of the boundary terms obtained from the integration over any volume part is gauge invariant and can be calculated in the DNEF gauge. The only point at which caution is necessary is the expression of the shell variables $r$ and $p$ in terms of embedding variables and Dirac observables because the embeddings are not continuous at the shell. Let us, therefore, generalize formulae (30) and (31) of II,

$$p_{\text{out}} = -R(r)\Delta_r(R,U)U'(r),$$

$$p_{\text{in}} = R(r)\Delta_r(R,V)V'(r).$$

for the momentum and calculate the corresponding formulae for $r$. Again, the idea is that the value of $p$ and $r$ is gauge invariant and we just have to express it in the singular gauge.

Let us start with $p_{\text{out}}$. Let us label the volume parts of $\Sigma$ adjacent to the shell by the index $K = l, r$, $l$ meaning left and $r$ meaning right from the shell. The corresponding pieces of Schwarzschild spacetimes are denoted by $M_K$, the Schwarzschild masses by $M_K$ and the maximal extensions of $M_K$ by $\overline{M}_K$. We denote the singular gauge within $\overline{M}_K$ by $U_K$ and $V_K$. The coordinates $U$ and $V$ that occur in Eqs. (8) and (9) represent a $C^1$ gauge (see Sec. 2.2 of II). Hence, we have

$$p_{\text{out}} = -R(r)\left[ \left( \frac{\partial R}{\partial U} \right)_r - \left( \frac{\partial R}{\partial U} \right)_l \right] U'(r)$$

$$= -R(r) \left[ \left( \frac{\partial R}{\partial U} \right)_r \left( \frac{\partial U}{\partial \rho} \right)_r - \left( \frac{\partial R}{\partial U} \right)_l \left( \frac{\partial U}{\partial \rho} \right)_l \right]$$

$$= -R(r) \left[ \left( \frac{\partial R}{\partial U} \right)_r \left( \frac{\partial U}{\partial \rho} \right)_{\rho=r} - \left( \frac{\partial R}{\partial U} \right)_l \left( \frac{\partial U}{\partial \rho} \right)_{\rho=r} \right],$$

and the generalized equation reads

$$p_{\text{out}} = -R(r)\Delta_r(R,U)U' .$$

Similarly,

$$p_{\text{in}} = R(r)\Delta_r(R,V)V' .$$

As for $r$, we have to express it in terms of the coordinates of the shell with respect to the singular gauge. Thus, we have immediately

$$U_K(r) = u_K ,$$

$$V_K(r) = v_K ,$$

and

$$4$$
where \( u_K \) and \( v_K \), \( K = r, l \) are the new variables to describe the position of the shell. As the functions \( U_K(\rho) \) and \( V_K(\rho) \) are monotonous, the relation between \( r \) and any of \( u_K \) and \( v_K \) is well defined for each \( K = l, r \). Of course, the four parameters \( u_l, v_l, u_r, v_r \) are redundant for the description of the shell position that is determined by just one parameter \( r \), but the Liouville form which results at the end of the calculation contains only one combination of the four parameters.

If the shell is marginally bound, that is, if it lies at a Schwarzschild horizon of both spacetimes left and right, then either \( u_K \) or \( v_K \) diverges. However, since the Liouville form is gauge invariant, we obtain the “proper” value for it at these points of phase space as follows. First, we calculate the Liouville form in a singular gauge for all cases but for the marginally bound ones. Second, we transform it into some regular coordinates on the phase space. Third, we take the limits to the marginally bound cases. This will be done in Sec. 3.2.

2.1 Contribution from a single shell

Let us calculate \( \Theta_r \) in the case that \( p, \rho = r \), is an intersection point of \( \Sigma \) with a single shell. We assume that the shell does not lie at the Schwarzschild horizons of \( M_K \) for \( K = l, r \). Let the embedding defining \( \Sigma \) be described by the pair of functions \((U_K(\rho), V_K(\rho))\) in \( M_K \) for each \( K \). Then all contributions to \( \Theta_r \) are (cf. Eqs. (51)–(55) of II):

\[
\Theta_r = -\Delta_r \left[ fdU + gdV + h_i do^i + \varphi dr \right] + p dr = \\
-\Delta_r \left[ \left( \frac{1}{2} \ln \left( \frac{U'}{V'} \right) RR_U + F \right) (dU + U' dr) \right] \\
-\Delta_r \left[ \left( \frac{1}{2} \ln \left( \frac{U'}{V'} \right) RR_V + G \right) (dV + V' dr) \right] \\
-\Delta_r \left[ \left( \frac{1}{2} \ln \left( \frac{U'}{V'} \right) RR_i + H_i \right) do^i \right] \\
+\Delta_r \left[ RR_U U' dr - RR_V V' dr \right] + p dr .
\]  

(14)

The symbol \( \Delta_r \) (that is defined in I, below Eq. (19)) contains the indices \( l \) and \( r \) implicitly.

Now, we can start simplifying and transforming the right-hand side of Eq. (14). The first step is based on the properties of the function \( R \) along the shell: it is continuous, and it is defined, from both sides, by

\[
R(\mathbf{r}) := R_K \left( U_K(\mathbf{r}), V_K(\mathbf{r}), o^i \right) .
\]  

(15)
Differentiating this equation, we obtain the relation
\[
d(R(r)) = R_{K,U}(r)\left(dU_K(r) + U'_K(r)dr\right) \\
+ R_{K,V}(r)\left(dV_K(r) + V'_K(r)dr\right) + R_{K,i}(r)do^i . \tag{16}
\]
Moreover, taking differentials of Eqs. (12) and (13) yields
\[
d\left(U_K(r)\right) = dU_K(r) + U'_K(r)dr = du_K \tag{17}
\]
and
\[
d\left(V_K(r)\right) = dV_K(r) + V'_K(r)dr = dv_K . \tag{18}
\]
Hence, Eq. (14) simplifies to
\[
\Theta_r = -\Delta_r \left[ \frac{1}{4} \ln \left( -\frac{U'}{V'} \right) d\left(R^2\right) + Fdu + Gdv + H_ido^i \right] \\
+ \Delta_r \left[ RR,U'U'dr - RR,V'V'dr \right] + pdr . \tag{19}
\]
Further transformations depend on whether the shell at \( p \) is out- or in-going. Let us assume that it is out-going; the procedure for the in-going shells is analogous.

In the next step, we calculate the value of the jumps that occur in Eq. (19). The idea is that the embedding \( \Sigma \) is \( C^1 \); if the transformation between our singular gauge and a \( C^1 \) gauge were known, one could compute the jumps. However, such a transformation is outlined by Lemma 2 of I. At a point of an out-going shell that does not lie at a Schwarzschild horizon (either a crossing of a shell and a horizon or a shell lying at a horizon), we can apply the Lemma with the result: The transformation of a regular gauge \( U \) and \( V \) to \( U_l \) and \( V_l \) is
\[
U = U_l , \quad V = V_l \tag{20}
\]
left from the shell, and to \( U_r \) and \( V_r \), it is determined by
\[
U = U_r - u_r + u_l \tag{21}
\]
and
\[
R_r(u_r, V_r(M_r)) = R_l(u_l, V, M_l) , \tag{22}
\]
where \( R_K(U_K, V_K, M_K) \) expresses the radius coordinate \( R \) in the spacetime \( M_K \) as a function of the DNEF coordinates \( U_K \) and \( V_K \) and the mass \( M_K, K = l, r \).
Let us further recall that we admit only $C^1$ embeddings or else the variational principle in II does not lead to the desired equations of motion (cf. Sec. 2 of II). Hence, the derivatives $U'$ and $V'$ of the embedding functions $U(\rho)$ and $V(\rho)$ must be continuous across the shell. By differentiating Eq. (22) with respect to $V$ right from the shell we find that

$$\frac{dV_r(V)}{dV} = \frac{R_{l,V_r}(u_l, V, M_l)}{R_{r,V_r}(u_r, V_r(V), M_r)}.$$  \hfill (23)

Eq. (23) implies that for all points $V(\rho)$ right from the shell we have

$$R_{r,V_r}(\rho)V'_r(\rho) = R_{l,V}(u_l, V(\rho), M_l)V'(\rho).$$  \hfill (24)

Eq. (21) implies for all points $V(\rho)$ left from the shell:

$$R_{l,V}(\rho)V'_l(\rho) = R_{l,V}(\rho)V'(\rho).$$  \hfill (25)

We learn from Eqs. (24) and (23) that

$$R_{r,V_r}(\rho)V'_r(\rho) = R_{l,V}(\rho)V'_l(\rho)$$  \hfill (26)

and, therefore, since $R(\rho)$ is continuous at $\rho = r$, that

$$\Delta_r \left[ RR_{r,V}V' \right] = 0.$$  \hfill (27)

Moreover, Eq. (14) gives

$$\Delta_r \left[ RR_{r,U}U' \right] + p = 0.$$  \hfill (28)

Next, let us consider the discontinuity in the logarithm in Eq. (19) which can be expressed as

$$\Delta_r \left[ \ln \left( \frac{-U'}{V'} \right) \right] = \ln \left( \frac{U'_r V'_l}{V'_r U'_l} \right).$$  \hfill (29)

Since $U(\rho)$ is continuous at $\rho = r$, we find from Eqs. (20) and (21) that

$$U'_r(\rho) = U'_l(\rho).$$  \hfill (30)

Eqs. (26) and (30) can be used to simplify (24) and express it in terms of the derivatives $R_{l,V}$ at $\rho = r$,

$$\Delta_r \left[ \ln \left( \frac{-U'}{V'} \right) \right] = \ln \left( \frac{R_{r,V_r}(\rho)}{R_{l,V_l}(\rho)} \right).$$  \hfill (31)
The derivative $R_{K,V_K}$ as well as its sign can be determined in terms of the radius coordinate $R_K$ and the mass $M_K$ if the position of the point $p$ is further specified. It can lie in different quadrants of the extended Schwarzschild spacetime $\mathcal{M}_K$. The quadrants have been labeled by sign pairs $(\alpha, \beta)$ in Sec. 2.1 of I. The values of $\alpha$ and $\beta$ also determine which DNEF coordinates are well defined in the quadrant $(\alpha, \beta)$: they are $U^\alpha$ and $V^\beta$.

Suppose that the intersection $p$ lies in the quadrant $(\alpha_K, \beta_K)$ of the Schwarzschild spacetime $\mathcal{M}_K$. We must have $\alpha_r = \alpha_l$ because of the argument of matching divergences of Sec. 2.2 in I. Then the dependence of the function $R_K$ on the coordinates $U_K$ and $V_K$ is given by Eq. (5) of I:

$$R = 2M_K\kappa \left[ \alpha_K \beta_K \exp \left( \frac{-\alpha_K U^\alpha_K + \beta_K V^\beta_K}{4M_K} \right) \right], \quad (32)$$

where $\kappa$ is the Kruskal function (cf. Eq. (6) of I). From the definition of $\kappa$, the following equations result

$$(\kappa(x) - 1)e^{\kappa(x)} = x$$

and

$$\kappa'(x) = \frac{1}{\kappa(x)e^{\kappa(x)}}.$$  

Using these equations to calculate $R_{K,V_K}$, we obtain

$$\frac{\partial R_K}{\partial V^\beta_K} = \frac{\beta_K}{2} \left( 1 - \frac{2M_K}{R_K} \right). \quad (33)$$

We know that the expression

$$\alpha \beta \left( 1 - \frac{2M}{R} \right)$$

is always positive. Since $\alpha$ is the same from both sides, the expression on the right-hand side of Eq. (33) has the same sign for both values of $K$ and

$$\ln \left( \frac{\partial R}{\partial V^\beta_K} \frac{\partial V^\beta_K}{\partial R} \right) = \ln \left| \frac{1 - 2M_r/R}{1 - 2M_l/R} \right| = \Delta_r \left( \ln \left| 1 - \frac{2M}{R} \right| \right) = \Delta_r (\ln |R - 2M|).$$

Thus, using Eqs. (27), (28) and (31), we obtain

$$\Theta_r = -\frac{1}{4} d\left( R^2(r) \right) \Delta_r (\ln |R - 2M|) - \Delta_r (Fdu + Gdv + H_id\omega) \quad (34)$$

Formula (34) holds at each intersection of an out-going shell with $\Sigma$ that is not a point of Schwarzschild horizon, $R = 2M_K$. Indeed, the first term diverges at such an intersection.
In the next step, we use the solutions for the functions \( F, G \) and \( H_i \) that have been found in Sec. 3.3 of II: in the DNEF gauge,

\[
\begin{align*}
F_K &= 0, \quad (35) \\
G_K &= 0, \quad (36) \\
H_{Ki} &= -\left(\alpha_K U_K^\alpha + \beta_K V_K^\beta\right) \frac{M_{K,i}}{2}, \quad (37)
\end{align*}
\]

\( K = l, r \). Using the formulae (12) and (13), we find that

\[
\Theta_r = -\frac{1}{4} d\left(R^2(r)\right) \Delta_r (\ln |R - 2M|) + \frac{1}{2} \Delta_r \left[(\alpha u + \beta v)dM \right]. \quad (38)
\]

Eq. (12) and the definition of \( \kappa \) (Eq. (6) of I) imply

\[
\frac{-\alpha_K u_K + \beta_K v_K}{2} = R + 2M_K \ln \left| \frac{R}{2M_K} - 1 \right|. \quad (39)
\]

Substituting this for \( \beta_K v_K \), we can cast Eq. (38) as follows:

\[
\Theta_r = \Delta_r \left[ \alpha u dM - \frac{1}{4} d\left(R^2\right) \ln |R - 2M| + RdM + 2M \ln \left| \frac{R - 2M}{2M} \right| dM \right]. \quad (40)
\]

For the last two terms in the bracket, however, the following identity can be easily shown to hold:

\[
-\frac{1}{4} d\left(R^2\right) \ln |R - 2M| + RdM + 2M \ln \left| \frac{R - 2M}{2M} \right| dM
\]

\[
= d \left[ -\frac{1}{4} (R^2 - 4M^2) \ln |R - 2M| - M^2 \ln(2M) + \frac{1}{8} R^2 + \frac{1}{2} M^2 \right]. \quad (41)
\]

Subtracting the corresponding exact form from \( \Theta_r \) leaves us with

\[
\Theta_r = \Delta_r \left[ \alpha u dM \right] = \alpha_r u_r \, dM_r - \alpha_l u_l \, dM_l. \quad (42)
\]

For an ingoing shell we find in an analogous way that

\[
\Theta_r = \Delta_r \left[ \beta v dM \right] = \beta_r v_r \, dM_r - \beta_l v_l \, dM_l. \quad (43)
\]

These two formulae summarize our treatment of single shells. Observe that the singularity in \( \Theta_r \) due to shell crossing a horizon at \( p \) has disappeared: Eq. (43) is well defined everywhere at any shell that does not lie at a horizon.
2.2 Contribution from a crossing point of two shells

Let us next consider the case in which the embedding passes through the crossing point $p$ of an in-going with an out-going shell. The form $\Theta_r$ at the crossing point $\rho = r$ is given by modified Eq. (14), where we have to write $p_{\text{out}} dr + p_{\text{in}} dr$ instead of $pd\rho$, and where the operator $\Delta_r$ has a different meaning: it does not denote jumps across one but across two shells. This ought to be explained in more details. The intersection of the two shells defines four spacetime regions around the crossing point. Let us denote them by $\mathcal{M}_K$, $K = l, r, u, d$, where the subscripts stand for left, right, up, down. The embedding passes from the spacetime region on the left, $\mathcal{M}_l$, to the one on the right, $\mathcal{M}_r$, without entering in the upper or lower intermediate regions. The operator $\Delta_r$ in (45) thus refers to the jump from $\mathcal{M}_l$ to $\mathcal{M}_r$. Let us choose the coordinates in each region $\mathcal{M}_K$ to be the corresponding DNEF coordinates $U_K$ and $V_K$. The shells are described by Eqs. (12) and (13) but now $K = r, l, u, d$. Hence, the coordinates $u_K, v_K$ in $\mathcal{M}_K$ satisfy Eqs. (17)–(18) for all $K$.

Since $R$ as a function on spacetime is continuous even at shell crossings, Eq. (15) still holds, now for all four values of $K = l, r, d, u$. Hence, Eq. (16) also holds, in particular for $K = l$ and $r = r$, so we can again collect all derivatives of $R$ as it has been done in Sec. 2.1. The result is analogous to Eq. (19):

$$\Theta_r = -\Delta_r \left[ \frac{1}{4} \ln \left( -\frac{U'(r)}{V'(r)} \right) d \left( R^2 \right) + F(r)du + G(r)dv + H_i do^i \right]$$

$$+ \Delta_r \left[ RR_{,U} U' dr - RR_{,V} V' dr \right] + p_{\text{out}} dr + p_{\text{in}} dr. \quad (44)$$

We can even use Eqs. (10) and (11) for the momenta with $\Delta_r$ meaning the jump across the two shells. Indeed, let us start with a regular gauge so that Lemma 1 of II is applicable. On the Lemma, the jump in the $U$-derivative across an out-going shell is continuous along the shell and it is zero across the in-going one. As the jump of any quantity across two shells is the sum of jumps across each, we have the required property, and we can proceed with the transformation to the singular gauge as in the case on a single shell. Hence, we obtain the formula:

$$\Theta_r = -\Delta_r \left[ \frac{1}{4} \ln \left( -\frac{U'(r)}{V'(r)} \right) d \left( R^2(r) \right) + F(r)du + G(r)dv + H_i(r)do^i \right]. \quad (45)$$

Let us foliate a neighborhood of the crossing point by a $C^1$ family of embeddings. The foliation is described in $\mathcal{M}_K$ by $U_K(\tau, \rho)$, $V_K(\tau, \rho)$ where the crossing point corresponds to $\tau = t$, $\rho = r$; i.e., $U_K(t, r) = u_K$ and $V_K(t, r) = v_K$. Let the foliation intersect the ingoing shell at $\rho = r_{\text{in}}(t)$ and the outgoing shell at $\rho = r_{\text{out}}(t)$ so that $r_{\text{in}}(t) = r = r_{\text{out}}(t)$. In particular, this means that if $\tau \leq t$ then $U_K(\tau, r_{\text{out}}(t)) = u_K$.
for $K = l, d$ and $V_K(\tau, r_{in}(\tau)) = v_K$ for $K = d, r$, while if $\tau \geq t$ then $V_K(\tau, r_{in}(\tau)) = v_K$ for $K = l, u$ and $U_K(\tau, r_{out}(\tau)) = u_K$ for $K = u, r$.

To calculate the jump in the logarithm term on the right-hand side of Eq. (15), we again apply Lemma 2 of I. At each point of the four “legs” of the crossing (trajectories of the shells outside $p$), we can find $C^1$ coordinates and obtain equations for the jumps in $V'$ and $U'$ analogous to Eqs. (26) and (30). Therefore, taking the limit $\tau = t$, we obtain equations like (26) and (30) for all four spacetime regions around the crossing point:

$$R_{r,V}(r) V'_r(r) = R_{u,V}(r) V'_u(r) , \quad U'_r(r) = U'_u(r) , \quad (46)$$
$$R_{l,V}(r) V'_l(r) = R_{d,V}(r) V'_d(r) , \quad U'_l(r) = U'_d(r) , \quad (47)$$
$$R_{u,U}(r) U'_u(r) = R_{l,U}(r) U'_l(r) , \quad V'_u(r) = V'_l(r) , \quad (48)$$
$$R_{r,U}(r) U'_r(r) = R_{d,U}(r) U'_d(r) , \quad V'_r(r) = V'_d(r) . \quad (49)$$

Computing $R_{K,V}$ and $R_{K,U}$ from Eq. (32) as for the single shell, we can cast Eqs. (46)–(49) in the following convenient form:

$$\frac{V'_r(r)}{V'_u(r)} = \frac{\beta_u (R(r) - 2M_u)}{\beta_r (R(r) - 2M_r)} , \quad \frac{U'_r(r)}{U'_u(r)} = 1 , \quad (50)$$
$$\frac{V'_l(r)}{V'_d(r)} = \frac{\beta_d (R(r) - 2M_d)}{\beta_l (R(r) - 2M_l)} , \quad \frac{U'_l(r)}{U'_d(r)} = 1 , \quad (51)$$
$$\frac{U'_u(r)}{U'_l(r)} = \frac{\alpha_l (R(r) - 2M_l)}{\alpha_u (R(r) - 2M_u)} , \quad \frac{V'_u(r)}{V'_l(r)} = 1 , \quad (52)$$
$$\frac{U'_d(r)}{U'_r(r)} = \frac{\alpha_r (R(r) - 2M_r)}{\alpha_d (R(r) - 2M_d)} , \quad \frac{V'_d(r)}{V'_r(r)} = 1 . \quad (53)$$

Let us view them as algebraic equations determining the derivatives $U'$ and $V'$ in terms of $R(r)$, the masses $M_l$, $M_r$, $M_u$, $M_d$ and the various indices $\alpha$ and $\beta$. Then it is easy to see that not all of them are independent. Multiplying the equations for $U'$ in Eqs. (52) and (53) and using the $U'$-equations from (50) and (51) lead to the condition

$$\frac{\alpha_l (R(r) - 2M_l)}{\alpha_u (R(r) - 2M_u)} = \frac{\alpha_d (R(r) - 2M_d)}{\alpha_r (R(r) - 2M_r)} , \quad (54)$$

while an analogous procedure for $V'$-equations yields

$$\frac{\beta_r (R(r) - 2M_r)}{\beta_u (R(r) - 2M_u)} = \frac{\beta_d (R(r) - 2M_d)}{\beta_l (R(r) - 2M_l)} . \quad (55)$$
Let us now recall the argument of matching divergence of Sec. 2.2 in I which implies that not all of the indices \( \alpha \) and \( \beta \) are independent. In general, two regions on opposite sides of an outgoing shell should have the same \( \alpha \), and two regions on opposite sides of an ingoing shell should have the same \( \beta \). In our case, this means that \( \alpha_u = \alpha_r \), \( \alpha_l = \alpha_d \), \( \beta_u = \beta_l \) and \( \beta_r = \beta_d \). Therefore, Eqs. (54) and (55) both reduce to the Dray-'t Hooft-Redmount condition (5) and (6) (cf. Eq. (7) of I),

\[
\frac{R(r) - 2M_l}{R(r) - 2M_u} = \frac{R(r) - 2M_d}{R(r) - 2M_r} ,
\]

determining the Schwarzschild radius \( R(r) \) of the crossing point in terms of the masses of the four regions. These variables are therefore not all independent.

Let us now return to the logarithm term in Eq. (45) and use Eqs. (50)–(53) to calculate its discontinuity from \( M_l \) to \( M_r \). The discontinuity

\[
\Delta_r \left[ \ln \left( - \frac{U'}{V'} \right) \right] = \ln \left( \frac{U'(r)V'(r)}{V'(r)U'(r)} \right)
\]

can be expressed in various ways by using different pairs of equations from the set (54)–(53). Of course, because of the Dray-'t Hooft-Redmount condition (56) all these alternative expressions are equivalent.

The simplest and most symmetric of these is

\[
\Delta_r \left[ \ln \left( - \frac{U'}{V'} \right) \right] = \ln \left| \frac{R(r) - 2M_d}{R(r) - 2M_u} \right| ,
\]

which brings Eq. (43) to the form

\[
\Theta_r = - \frac{1}{4} d(R^2(r)) \ln |R(r) - 2M_d| + \frac{1}{4} d(R^2(r)) \ln |R(r) - 2M_u|
\]

\[
+ \frac{1}{2} (\alpha_r u_r + \beta_r v_r) \, dM_r - \frac{1}{2} (\alpha_l u_l + \beta_l v_l) \, dM_l .
\]

Notice that we have again chosen the particular solutions (35)–(37) for the functions \( F, G \) and \( H_i \).

The last two terms in Eq. (59) can be written with the help of Eq. (33). Then, applying the identity (41), we obtain our final expression for the contribution to the Liouville form from a crossing point, involving the discontinuities \( \Delta_r \) from the left to the right region and \( \Delta_t \) from the lower to the upper region:

\[
\Theta_r = \frac{1}{2} \Delta_t \left[ (\alpha u + \beta v) \, dM \right] + \frac{1}{2} \Delta_r \left[ (\alpha u + \beta v) \, dM \right]
\]

\[
= \frac{1}{2} (\alpha u u_u + \beta u v_u) \, dM_u - \frac{1}{2} (\alpha d u_d + \beta d v_d) \, dM_d
\]

\[
+ \frac{1}{2} (\alpha_r u_r + \beta_r v_r) \, dM_r - \frac{1}{2} (\alpha_l u_l + \beta_l v_l) \, dM_l .
\]

Again, this formula holds only if no shell at the crossing is marginally bound.
2.3 The general formula for all contributions

Before we conclude this section let us verify that the above formula for a crossing point agrees with the previous formulae (42) and (43) for ingoing and outgoing shells. Indeed, one can consider an embedding starting from the left and then passing either above or below or through a crossing point. If it passes above, it crosses first the ingoing shell and then the outgoing shell. The total contribution from both shells is the sum of the contributions (42) and (43):

$$
\Theta_{\text{above}} = \beta_u v_u dM_u - \beta_l v_l dM_l + \alpha_r u_r dM_r - \alpha_u u_u dM_u .
$$

(61)

If it passes below, it crosses the outgoing shell first and the ingoing one second. By summing the contributions (42) and (43) we now get

$$
\Theta_{\text{below}} = \alpha_d u_d dM_d - \alpha_l u_l dM_l + \beta_r v_r dM_r - \beta_d v_d dM_d .
$$

(62)

Expressions (61) and (62) are equivalent, and their symmetric sum indeed recovers the result (60) obtained by letting the embedding pass through the crossing point. We can prove the equivalence as follows.

The difference $\Theta_{\text{above}} - \Theta_{\text{below}}$ can be written as

$$
\Theta_{\text{above}} - \Theta_{\text{below}} = -\chi_l dM_l + \chi_d dM_d + \chi_u dM_u - \chi_r dM_r ,
$$

where

$$
\chi_K := -\alpha_K u_K + \beta_K v_K .
$$

Eq. (39) allows us to express $\chi$’s in terms of the four masses $M_K$,

$$
\chi_K = 2R(r) + 4M_K \ln |R(r) - 2M_K| - 4M_K \ln (2M_K) ,
$$

(63)

if we use formula (56) to calculate $R(r)$:

$$
R(r) = 2\frac{M_l M_r - M_d M_u}{M_l + M_r - M_d - M_u} .
$$

(64)

In this way $\Theta_{\text{above}} - \Theta_{\text{below}}$ is a well defined form on the four-dimensional space spanned by $M_l, M_r, M_d$ and $M_u$. Observe that, from the point of view of canonical transformations, our “old momenta” are $M_l, M_r$ and $M_d$, while the “new momenta” are $M_l, M_r$ and $M_u$ and the form $\Theta_{\text{above}} - \Theta_{\text{below}}$ is the differential of a generating function $F$ that depends on the old and new coordinates, if we can show that it is an exact form. Let us show that.

First, we observe that $\Theta_{\text{above}} - \Theta_{\text{below}}$ is invariant with respect to the swaps

$$
M_r \leftrightarrow M_l ,\ M_d \leftrightarrow M_u .
$$

(65)
Hence, it is sufficient to show that

\[-\frac{\partial \chi_l}{\partial M_d} = \frac{\partial \chi_d}{\partial M_l}, \quad \frac{\partial \chi_l}{\partial M_r} = \frac{\partial \chi_r}{\partial M_l}, \quad \frac{\partial \chi_d}{\partial M_u} = \frac{\partial \chi_u}{\partial M_d}.\]

Second, we observe that \(\chi_K\) depends on the masses with index different from \(K\) only through \(R(r)\). Hence the above equations can be written as

\[-\frac{\partial \chi_l}{\partial R(r)} \frac{\partial R(r)}{\partial M_d} = \frac{\partial \chi_d}{\partial R(r)} \frac{\partial R(r)}{\partial M_l}, \quad \frac{\partial \chi_l}{\partial R(r)} \frac{\partial R(r)}{\partial M_r} = \frac{\partial \chi_r}{\partial R(r)} \frac{\partial R(r)}{\partial M_l}, \quad \frac{\partial \chi_d}{\partial R(r)} \frac{\partial R(r)}{\partial M_u} = \frac{\partial \chi_u}{\partial R(r)} \frac{\partial R(r)}{\partial M_d}.\]  

(66) 

(67) 

(68)

Eq. (63) yields

\[\frac{\partial \chi_K}{\partial R(r)} = \frac{2R(r)}{R(r) - 2M_K},\]  

(69)

and Eq. (64) implies

\[\frac{\partial R(r)}{\partial M_l} = 2 \frac{(M_r - M_d)(M_r - M_u)}{(M_t + M_r - M_d - M_u)^2}\]

and

\[\frac{\partial R(r)}{\partial M_d} = 2 \frac{(M_u - M_t)(M_u - M_r)}{(M_t + M_r - M_d - M_u)^2};\]

the other formulae can be obtained through the swaps (63). We also find that

\[R(r) - 2M_t = -2 \frac{(M_t - M_d)(M_t - M_u)}{M_t + M_r - M_d - M_u}\]

and

\[R(r) - 2M_d = 2 \frac{(M_d - M_t)(M_d - M_r)}{M_t + M_r - M_d - M_u};\]

the other two differences resulting by the swaps. It follows that

\[\frac{\partial R(r)}{\partial M_t} = -\frac{R(r) - 2M_r}{M_t + M_r - M_d - M_u}\]

(70)

and

\[\frac{\partial R(r)}{\partial M_d} = \frac{R(r) - 2M_u}{M_t + M_r - M_d - M_u}.\]

(71)

Substituting Eqs. (63), (64) and (71) or their suitable swaps into Eqs. (66)–(68), we easily show the equality if we use Eq. (56) once more.
Since the form $\Theta_{\text{above}} - \Theta_{\text{below}}$ is exact, its integral along a curve depends only on the end points of the curve. Let us calculate such an integral in the $M$-space. Let the curve start at the origin and consist of four straight segments: $(0,0,0,0) \rightarrow (M_d,0,0,0)$, $(M_d,0,0,0) \rightarrow (M_d,M_u,0,0)$, $(M_d,M_u,0,0) \rightarrow (M_d,M_u,M_r,0)$ and $(M_d,M_u,M_r,0) \rightarrow (M_d,M_u,M_r,M_l)$. An elementary integration yields

$$
\mathcal{F} = -2M_tM_r + 2M_dM_u
- 2M_t^2 \ln \frac{|M_t - M_d||M_t - M_u|}{M_t(M_t + M_r - M_d - M_u)}
- 2M_r^2 \ln \frac{|M_r - M_d||M_r - M_u|}{M_r(M_t + M_r - M_d - M_u)}
+ 2M_d^2 \ln \frac{|M_d - M_l||M_d - M_r|}{M_d(M_t + M_r - M_d - M_u)}
+ 2M_u^2 \ln \frac{|M_u - M_l||M_u - M_r|}{M_u(M_t + M_r - M_d - M_u)}.
$$ (72)

Strictly speaking, this formula holds only for that part of the $M$-space where the masses $M_d$ and $M_u$ are larger than $M_t$ and $M_r$ because the integration curve would otherwise cross the corresponding singularities of the integrand. But one guesses that this formula holds in all generic subcases as it stands. We can check this as follows.

The function $\mathcal{F}$ is to be a generating function of the canonical transformation that brings the Liouville form $\Theta_{\text{below}}$ into $\Theta_{\text{above}}$. It depends on the old $(M_t, M_d, M_r)$ and the new $(M_t, M_u, M_r)$ momenta and the usual generating formulae must hold (as some new momenta coincide with some old ones, differences of coordinates appear instead of the coordinates themselves):

$$
\frac{\partial \mathcal{F}}{\partial M_d} = -(\beta_d v_d - \alpha_d u_d),
$$ (73)

$$
\frac{\partial \mathcal{F}}{\partial M_u} = -(\beta_u v_u - \alpha_u u_u),
$$ (74)

$$
\frac{\partial \mathcal{F}}{\partial M_t} = (\beta_t v_t - \alpha_t u_t),
$$ (75)

$$
\frac{\partial \mathcal{F}}{\partial M_r} = (\beta_r v_r - \alpha_r u_r).
$$ (76)

A simple calculation shows that Eqs. (73)–(75) are valid in all subcases with $a \neq 0$ and $b \neq 0$ for our guessed $\mathcal{F}$ as given by Eq. (72).

The transformation itself can be calculated as follows. Let us introduce the following notation

$$
q_1 = -\alpha_t u_t, \quad q_2 = \beta_r v_r, \quad q_3 = \alpha_d u_d - \beta_d v_d,
$$

$$
p_1 = -M_t, \quad p_2 = -M_r, \quad p_3 = -M_d,
$$

$$
Q_1 = -\beta_t v_t, \quad Q_2 = \alpha_r u_r, \quad Q_3 = -\alpha_u u_u + \beta_u v_u,
$$

$$
P_1 = -M_t, \quad P_2 = -M_r, \quad P_3 = -M_u.
$$
Then $\Theta_{\text{below}} = \sum -q_n dp_n$ and $\Theta_{\text{above}} = \sum -Q_n dP_n$. Eq. (39) for $K = d$ gives

$$R(r) = -2p_3\kappa \left[ \alpha_d \beta_d \exp \left( \frac{q_3}{4p_3} \right) \right]$$  \hspace{1cm} (77)$$
and Eq. (64) yields

$$P_3 = \frac{p_1p_2 - (p_1 + p_2 - p_3)p_3\kappa \left[ \alpha_d \beta_d \exp \left( \frac{q_3}{4p_3} \right) \right]}{p_3 - p_3\kappa \left[ \alpha_d \beta_d \exp \left( \frac{q_3}{4p_3} \right) \right]}.$$  \hspace{1cm} (78)$$
Using Eq. (39) again, we obtain

$$Q_3 = 2R(r) - 4P_3 \ln \left| \frac{R(r) + 2P_3}{-2P_3} \right|,$$  \hspace{1cm} (79)$$
where $R(r)$ and $P_3$ are given by Eqs. (77) and (78). Then,

$$P_1 = p_1, \quad P_2 = p_2$$  \hspace{1cm} (80)$$
and Eq. (39) implies

$$Q_1 = q_1 - 2R(r) + 4p_1 \ln \left| \frac{R(r) + 2p_1}{-2p_1} \right|,$$  \hspace{1cm} (81)$$
$$Q_2 = q_2 - 2R(r) + 4p_2 \ln \left| \frac{R(r) + 2p_2}{-2p_2} \right|,$$  \hspace{1cm} (82)$$
where again $R(r)$ is to be substituted from Eq. (77). The desired transformation is described by Eqs. (77)–(82); it is a rather involved one.

Finally, let us make the following observation. Let us view the operator $\Delta_r$ as the jump in the positive spacelike direction and the operator $\Delta_t$ as the jump in the positive timelike direction. This orientation of spacetime is fixed by our particular definition of DNDF coordinates. Then, we see that along an outgoing shell $\Delta_r = -\Delta_t$, and along an ingoing shell $\Delta_r = \Delta_t$. By substituting these relations into expression (60) we recover our previous expressions (42) and (43) as special cases. Let us therefore summarize all our results of this section by rewriting expressions (42), (43) and (60) concisely as

$$\Theta = \Delta_t \left[ R^* dM \right] + \Delta_r \left[ T dM \right],$$  \hspace{1cm} (83)$$
where

$$R^*_K := \frac{-\alpha_K u_K + \beta_K v_K}{2},$$
and

$$T_K := \frac{\alpha_K u_K + \beta_K v_K}{2}.$$  

This holds in general, with $\rho = r$ corresponding to the point of intersection of the shell(s) with the chosen embedding if none of the shells is marginally bounded.
2.4 Contributions from center and from infinity

For the center and infinity, we shall again utilize the freedom in the gauge choice and compute in the DNEF coordinates in the spacetime part surrounding the origin, which we denote by $\mathcal{M}_0$ and the part near infinity, which will be denoted by $\mathcal{M}_\infty$. For simplicity we shall drop the subscripts 0 and $\infty$ from $U$ and $V$ in this section.

2.4.1 At the center

The contributions to the Liouville from $\rho = 0$ are given by Eq. (6), where the functions $f$, $g$ and $h_i$ are defined by Eqs. (1)–(3). The solution (35), (36) and (37) for the functions $F$, $G$, $H_i$ is trivial in the Minkowski part of spacetime,

$$F = 0 \quad G = 0 \quad H_i = 0,$$

and therefore Eq. (3) reduces to an expression involving only the logarithm terms in $f$, $g$ and $h_i$.

At the regular center $R = 0$ (where $\rho = 0$) all embeddings have to satisfy the boundary condition

$$U'(0) = -V'(0),$$

which guarantees that the embedded hypersurfaces avoid conical singularities. Eq. (84) reduces all logarithm terms in Eqs. (1)–(3) to zero. The functions $f$, $g$ and $h_i$ and consequently the Liouville form at $\rho = 0$ is therefore trivial:

$$\Theta_0 = 0.$$

2.4.2 At infinity

The contributions to the Liouville from $\rho \to \infty$ are given by Eq. (6). At infinity, we restrict the foliation to be parallel to the $T = \text{const}$, where $T$ is the Schwarzschild time coordinate in $\mathcal{M}_\infty$; more precisely, we assume that

$$R(\rho) \to \rho + O(\rho^{-1}), \quad T(\rho) \to T_\infty + O(\rho^{-1})$$

for the Schwarzschild coordinates $T$ and $R$ along any embedding. We also assume that the foliation parameter $t$ satisfies

$$t \to T_\infty$$

asymptotically so that $N_\infty = 1$. Finally, we replace $E_\infty$ by $M_\infty$, $M_\infty$ being the total mass of the system. Then we can write

$$\Theta_\infty = \lim_{\rho \to \infty} (f dU + g dV + h_i d\phi^i) - M_\infty dT_\infty.$$
The functions $f$, $g$ and $h_i$ have always the same functional form, given by Eqs. (1)–(3) and $F$, $G$ and $H_i$ are given by Eqs. (35)–(37), where $M_K$ is replaced by $M_\infty$.

Let us start from the only non-trivial contribution, Eq. (37), and use the fact that $V + U = 2T$. As $\rho$ approaches infinity the time coordinate $T$ approaches the asymptotic time $T_\infty$, which means that the total contribution from $H_i$ is just

$$H_i do^i = -T_\infty dM_\infty.$$  

(89)

This term together with the term $-M_\infty dT_\infty$ in Eq. (88) yields an exact form. The remaining contributions to the Liouville form in Eq. (88) are proportional to the logarithm terms and can be summarized as follows:

$$\Theta_\infty = \frac{R}{2} \ln \left( -\frac{U''}{V} \right) \left( R_{,U}dU + R_{,V}dV + R_{,i}do^i \right).$$  

(90)

Recall that in a Schwarzschild region of mass $M$ the Eddington-Finkelstein coordinates are related to the coordinates $T$ and $R$ by

$$U = T - R^*, \quad V = T + R^*, \quad R^* = R + 2M\ln \left| \frac{R}{2M} - 1 \right|. \quad (91)$$

(92)

As $\rho \to \infty$, it follows from Eq. (92) that $R^*$ has the asymptotic expansion

$$R^*(\rho) \to \rho + 2M\ln \left( \frac{\rho}{2M} \right) + O(\rho^{-1}). \quad (93)$$

Eq. (91) determines how the Eddington-Finkelstein coordinates approach infinity,

$$U(\rho) \to -\rho - 2M\ln \left( \frac{\rho}{2M} \right) + T_\infty + O(\rho^{-1}), \quad (94)$$

$$V(\rho) \to +\rho + 2M\ln \left( \frac{\rho}{2M} \right) + T_\infty + O(\rho^{-1}), \quad (95)$$

and hence how their derivatives $U'$ and $V'$ approach it:

$$U'(\rho) \to -1 + O(\rho^{-1}), \quad (96)$$

$$V'(\rho) \to 1 + O(\rho^{-1}). \quad (97)$$

The way in which the differentials $dU$ and $dV$ increase as $\rho \to \infty$ follows directly from Eqs. (94) and (95),

$$dU(\rho) \to -2dM\ln \left( \frac{\rho}{2M} \right) + dT_\infty + 2dM + O(\rho^{-1}), \quad (98)$$

$$dV(\rho) \to +2dM\ln \left( \frac{\rho}{2M} \right) + dT_\infty - 2dM + O(\rho^{-1}), \quad (99)$$

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while the way in which the logarithm in Eq. (90) behaves is determined from Eqs. (94) and (97):

\[
\ln \left( -\frac{U'}{V'} \right) (\rho) \to O(\rho^{-1}).
\] (100)

The final terms whose behavior we need to determine are the terms multiplying the logarithm in Eq. (90). Eqs. (32) and the fact that we are in the (++) quadrant imply that the derivatives of \( R \) are related to \( R \) and \( M \) by

\[
R_{U} = -\frac{1}{2} \left( 1 - \frac{2M}{R} \right) = -R_{V},
\]

\[
R_{i} = \frac{R}{M} M_{i} - \frac{1}{2} \left( 1 - \frac{2M}{R} \right) \frac{V - U}{M} M_{i}.
\] (101)

After some simple calculations it follows that

\[
R_{U}(\rho) \to -\frac{1}{2} + \frac{M}{\rho} + O(\rho^{-3}), \quad R_{V}(\rho) \to \frac{1}{2} - \frac{M}{\rho} + O(\rho^{-3}),
\]

\[
R_{i}(\rho) \to 2M_{i} - 2M_{i} \ln \left( \frac{\rho}{2M} \right) + O(\rho^{-1}\ln(\rho)).
\] (102)

Putting together Eqs. (98), (99) and (102) it is not difficult to show that

\[
\left( R_{U}dU + R_{V}dV + R_{i}d\phi \right)(\rho) \to O\left( \rho^{-1}\ln(\rho) \right).\] (103)

When Eqs. (100) and (103) are used in Eq. (90) they yield the final result; the total contribution to the Liouville form from infinity vanishes:

\[
\Theta_{\infty} = 0.
\] (104)

3 Algebra of Dirac’s observables

All calculations done and all results obtained as yet in this paper have been entirely general in the sense that they hold for a system containing any number of out- and in-going null spherical shells. In the present section, we return to our original system of two shells. Our final aim is to express the Liouville form in terms of a chosen set of Dirac observables for all cases studied in I and so to find the symplectic structure of the physical phase space. In the previous section, the Liouville form has been reduced to the sum over contributions from shells that intersect a Cauchy surface. Each intersection of the surface either with single shells or with crossing points of shells contributes according to the general formula (83).
3.1 The parallel shells: Cases A and B

The subspacetimes between the shells are denoted by $\mathcal{M}_K$, where the index $K = l, m, r$ (see Fig. 1 in I). The spacetime region $\mathcal{M}_l$ is flat and lies entirely in the $(+, +)$ quadrant. The masses of the other two regions are $M_m$ and $M_r$. The DNEF coordinates of each region $\mathcal{M}_K$ are denoted by $U_K, V_K$. The regions $\mathcal{M}_m$ and $\mathcal{M}_r$ lie in the $(+, +) \cup (-, +)$ part of their corresponding Schwarzschild extensions. The two shells are characterized by the index $s = 1, 2$. The position of the $s$-shell with respect to the chart $U_K, V_K$ covering the region $\mathcal{M}_K$ is defined by $V_K^+ = v_{Ks}^+$. By applying Eq. (83) to case A we find the following total Liouville form:

$$\Theta_{\Gamma} = v_{m1} \, dM_m - v_{m2} \, dM_m + v_{r2} \, dM_r .$$

(105)

It is clear from (105) that the momentum conjugate to the coordinate $v_{r2}$ is $-M_r$ and the momentum conjugate to the difference $v_{m1} - v_{m2}$ is $-M_m$. These four coordinates describe the physical phase space of the system, which is the subset of $\mathbb{R}^4$ defined by the inequalities

$$M_m > 0 , \quad M_r > M_m .$$

(106)

The coordinate $v_{r2}$ can take any values between $-\infty$ and $+\infty$, while $v_{m1} - v_{m2}$ is negative in case A and positive in case A’.

Different coordinates can be defined by the canonical transformation

$$M_{(1)} = M_m , \quad M_{(2)} = M_r - M_m ,$$

$$v_{(1)} = v_{m1} - v_{m2} + v_{r2} , \quad v_{(2)} = v_{r2} .$$

These coordinates are associated more closely to the individual shells: $M_{(1)}$ is the mass of the first shell and $M_{(2)}$ is that of the second. Moreover, the coordinate $v_{(1)}$ can be considered as the advanced time of the first shell with respect to the continuous time coordinate at the past null infinity. That is, we can choose a gauge so that $v_{m2} = v_{r2}$ and the coordinate $V$ is continuous across the shell.

In terms of these coordinates the physical phase space is the subset of $\mathbb{R}^4$ defined by

$$M_{(1)} > 0 , \quad M_{(2)} > 0 .$$

(107)

The variables $v_{(2)}$ and $v_{(1)}$ are free to take any values between $-\infty$ and $+\infty$ but $v_{(2)} > v_{(1)}$ in case A while $v_{(2)} < v_{(1)}$ in case A’. In case B and B’, the subspacetimes are again denoted by $\mathcal{M}_K$, $K = l, m, r$, and the masses of the two non-flat regions are $M_m$ and $M_r$. The left region lies in the $(+, +)$ quadrant of the flat spacetime while the other two in the $(+, +) \cup (+, -)$ part of their Schwarzschild extension.
The position of the s-shell is defined by $U^+_K = u_Ks$. Using (83) we find that the total Liouville form is

$$\Theta = u_{m1} \, dM_m - u_{m2} \, dM_m + u_{r2} \, dM_r.$$  

(108)

The description of the phase space is similar to the description above, and analogous choices of natural physical coordinates can be made.

### 3.2 The crossing shells: Cases C or C’, and subcases

Cases C and C’ are obtained from each other by interchanging the values 1, 2 of the label s. So we can consider only one of these cases. Let us also drop s since the positions of the shells are already distinguished by the letters u and v. The four spacetime regions are denoted by $\mathcal{M}_K$ where $K = l, r, u, d$. The various subcases are pictured in Figs. 4–9 in I. The spacetime region $\mathcal{M}_l$ is flat and the masses of the other regions are $M_r, M_u$ and $M_d$. The left region lies in the $(+, +)$ quadrant. The other regions may lie in a union of two or four quadrants of their Schwarzschild extension depending on whether each shell lies above, exactly on, or below its corresponding horizon. The various possibilities are captured by simple relationships between the masses $M_r, M_u$ and $M_d$. For this reason, the indices a and b have been introduced in I, defined by

$$a := \text{sgn}(M_r - M_u), \quad b := \text{sgn}(M_r - M_d).$$  

(109)

Their meaning is the following: Because of the flatness of the left region the only shell boundaries that can lie on horizons are the boundaries between $\mathcal{M}_d$ and $\mathcal{M}_r$ and between $\mathcal{M}_u$ and $\mathcal{M}_r$. If a shell lies above the horizon (corresponding to an unbound state) the difference between the Schwarzschild mass on its right and that on its left is positive, if it lies below the horizon (bound state) this difference is negative, and if it lies exactly on the horizon (marginally bound state) it is zero. The indices a and b hence provide all the necessary information and characterize the subcases $C_{ab}$ by the nine combinations of their values $(+, 0, -)$.

The values of the signs $\alpha_K$ and $\beta_K$ for each case $C_{ab}$ have been found in I: the relations

$$\alpha_l = +1, \quad \beta_l = +1, \quad \alpha_d = +1, \quad \beta_u = +1$$

hold for all cases; if $a \neq 0$,

$$\alpha_u = \alpha_r = a$$

and if $b \neq 0$,

$$\beta_d = \beta_r = b.$$  

If $a = 0$ the coordinates $u_u$ and $u_r$ diverge, and if $b = 0$ $v_d$ and $v_r$ diverge.
Let us consider the cases with $a \neq 0$: $C_{++}$, $C_{+0}$, $C_{+-}$, $C_{-+}$, $C_{-0}$ and $C_{--}$. The coordinates $v_l$, $v_u$, $u_u$ and $u_r$ are regular, and we can write the form $\Theta_{\text{above}}$ as follows:

$$\Theta_{\text{above}} = (-au_u + v_u) dM_u + au_r dM_r .$$

This region of the phase space can, therefore, be considered as a Darboux chart with the coordinate pairs

$$- au_u + v_u, -M_u ; \quad au_r, -M_r .$$

Next, consider the cases with $b \neq 0$: $C_{++}$, $C_{0+}$, $C_{-+}$, $C_{0-}$ and $C_{--}$. Here, $u_t$, $u_d$, $v_d$ and $v_r$ are regular, and $\Theta_{\text{below}}$ is

$$\Theta_{\text{below}} = (u_d - bv_d) dM_d + bv_r dM_r .$$

Hence, this region of the phase space is covered by the Darboux chart with coordinate pairs

$$- u_d - bv_d, -M_d ; \quad bv_r, -M_r .$$

The two charts overlap in $C_{++} \cup C_{+-} \cup C_{-+} \cup C_{--}$. The (canonical) transformation between the two charts in the overlapping region is given by the formulae (77), (78), (79), (80) and (82), where we have to set:

$$q_2 = bv_r , \quad q_3 = u_d - bv_d , \quad p_2 = -M_r , \quad p_3 = -M_d$$

and

$$Q_2 = au_r , \quad Q_3 = -au_u + v_u , \quad P_2 = -M_r , \quad P_3 = -M_u ,$$

and $p_1 = P_1 = 0$. Let us write down the transformation explicitly:

$$Q_2 = q_2 + 4p_3\bar{\kappa} + 4p_2 \ln \left| \frac{p_2 - p_3\bar{\kappa}}{p_2} \right| ,$$

$$Q_3 = -4p_3\bar{\kappa} - \frac{4\bar{\kappa}}{\bar{\kappa} - 1} (p_2 - p_3) \ln \left| \frac{p_2 - p_3\bar{\kappa}}{p_2 - p_3} \right| ,$$

$$P_2 = p_2 ,$$

$$P_3 = \frac{\bar{\kappa}}{\bar{\kappa} - 1} (p_2 - p_3) ,$$

where $\bar{\kappa}$ is a shorthand for

$$\bar{\kappa} = \kappa \left[ b \exp \left( \frac{q_3}{4p_3} \right) \right] .$$
and the coordinates \( q_1 \) and \( Q_1 \) do not occur in the formulae. The generating function \( F' \) for the transformation is given by Eq. (72) if we set \( M_l = 0 \) in it:

\[
F' = 2M_uM_d - 2M^2 \ln \left[ \frac{M_r - M_d|M_r - M_u}{M_r(M_r - M_d - M_u)} \right] + 2M^2 \ln \left[ \frac{M_r - M_d}{M_r - M_d - M_u} \right] + 2M^2 \ln \left[ \frac{M_r - M_u}{M_r - M_d - M_u} \right].
\]

The charts cover the whole phase space with the exception of case \( C_{00} \).

To find coordinates that are regular at \( C_{00} \), we first use Eq. (82) rewritten in terms of the three masses,

\[
Q_2 = q_2 - \frac{4M_dM_u}{M_d + M_u - M_r} - 4M_r \ln \left| \frac{(M_r - M_d)(M_r - M_u)}{M_r(M_d + M_u - M_r)} \right|
\]

and consider the fact that \( Q_2 \) diverges at \( M_u = M_r \) while \( q_2 \) does when \( M_d = M_r \).

It follows that the combination

\[
Q_2 + 4M_r \ln |M_r - M_u|
\]

is always regular.

Let us introduce a new coordinate \( q \) by

\[
Q_2 = q - \frac{4M_dM_u}{M_d + M_u - M_r} - 4M_r \ln \left| \frac{M_r - M_u}{M_d + M_u - M_r} \right|. \tag{121}
\]

This should be regular; the motivation for addition of further terms is that

\[
\bar{v}_u = q - 4(M_r - M_u) \ln \left| \frac{M_r - M_u}{M_d + M_u - M_r} \right|
\]

hence, \( q \to \bar{v}_u \) for \( M_u \to M_r \). Here, the coordinate \( \bar{v}_u \) is defined by \( \bar{v}_u := Q_2 + Q_3 \); we would have \( \bar{v}_u = v_u \) if we shift \( v_u \) and \( u_u \) so that \( u_u = u_r \). If we express \( Q_3 \) in Eq. (79) in terms of masses,

\[
Q_3 = \frac{4M_dM_u}{M_d + M_u - M_r} + 4M_u \ln \left| \frac{M_r - M_u}{M_d + M_u - M_r} \right|. \tag{123}
\]

we can see that Eq. (122) holds. The choice of \( \bar{v}_u \) instead of \( q \) does not, however, lead to differentiable components of the Liouville form.

Let us view Eqs. (121) and (123), together with \( P_2 = -M_r \) and \( P_3 = -M_u \) as transformation to new variables \( q, M_d, M_u \) and \( M_r \). Then

\[
\Theta_F = -Q_2dP_2 - O_3dP_3
= qdM_r + \frac{2M_d(M_r - M_u)}{M_d + M_u - M_r}d(M_r - M_u) - \frac{2(M^2_r - M^2_u)}{M_d + M_u - M_r}dM_d
+ d\left[-2(M^2_r - M^2_u) \ln \left| \frac{M_r - M_u}{M_d + M_u - M_r} \right| \right].
\]
If we subtract the singular exact form, the rest seems to be regular. But we have also to calculate the determinant of the symplectic tensor in order to show the regularity. Some simplification can be achieved in the coordinates $q, x, y$ and $z$ defined by

$$x = M_r - M_u, \quad y = M_r + M_u, \quad z = M_d.$$ 

The corresponding components of the symplectic form $\Omega = d\Theta_\Gamma$ are

$$\Omega_{qx} = 1/2, \quad \Omega_{qy} = 1/2, \quad \Omega_{qz} = 0,$$

$$\Omega_{xy} = 0, \quad \Omega_{xz} = 2 \frac{x^2 - yz}{(z - x)^2}, \quad \Omega_{yz} = 2 \frac{x^2 - xz}{(z - x)^2}$$

and the determinant is

$$\det \Omega = \frac{z^2}{(z - x)^4} (x - y)^2 = \frac{4M_d^2M_u^2}{(M_d + M_u - M_r)^4}.$$ 

The C-component $\mathcal{P}_C$ of the physical phase space is determined by the boundaries (cf. I, Eqs. (9) and (10))

$$M_d > 0, \quad M_u > 0,$$

and

$$0 < M_r < M_d + M_u.$$ 

Hence, $\Omega$ is $C^\infty$ and non-degenerate everywhere in $\mathcal{P}_C$. In particular, case $C_{00}$, which is defined by $M_d = M_u = M_r$, is a smooth surface in $\mathcal{P}_C$.

4 Conclusions and outlook

In the three papers I, II and the present one, complete sets of Dirac observables for the system of two null-matter shells have been found and their Poisson bracket have been determined. The result is exceedingly simple and analogous to what was found earlier for the single shell [4], but the calculation itself, in spite of many improvements, has still been tedious. We don’t give up hopes that methods exist to make the calculation truly simpler.

Let us turn to the question of quantization. It seems that the similarity of the present results to those of [4] suggests that at least some quantization methods can be imported from [4]. More specifically, there does not seem to be any reason why each shell of the pair could not bounce at the regular center exactly as the single shell did in [4]. Some singularities that afflict the classical solutions could so be again avoided by the quantum theory.

On the other hand, the pair of shells is crucially different from the single shell in one aspect: the shells can intersect and they have a non-trivial interaction at the
intersection. The transformation (116)–(120) describes the change of the (initial) Dirac observables before into the (final) ones after the crossing. It can so represent the dynamical evolution through the crossing and it ought to be derivable from the Hamiltonian of the system. It may be difficult to find such a Hamiltonian on account of the transformation being rather involved. Perhaps there is another choice of Dirac observables that simplifies the transformation.

This is a technical problem. However, there is another very interesting aspect of the crossing that may lead to a great change in results. This is the fact that the external shell can become bound after the crossing. For the values of the initial observables $q_3$ and $p_3$ such that

$$\kappa \left[ \exp \left( \frac{q_3}{4p_3} \right) \right] \in \left( 1, \frac{p_2}{p_3} \right),$$

the Eqs. (118) and (119) yield

$$M_r \leq M_u$$

and a black hole forms, while for

$$\kappa \left[ \exp \left( \frac{q_3}{4p_3} \right) \right] \in \left( \frac{p_2}{p_3}, \infty \right)$$

the external shell can reach infinity. Here, we assume that $p_2/p_3 > 1$ so $M_r > M_d$ and the external shell before the crossing is not bound. This indicates that the pair of shells might only sometimes re-expand to infinity and the other times form a “quantum black hole”. Recall that the single shell always re-expands and reach the infinity (cf. [7]). The possibility that a black hole forms would, therefore, constitute a qualitatively new result. We hope to be able to construct the quantum theory soon.

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