TORSION OF THE KHOVANOV HOMOLOGY

ALEXANDER N. SHUMAKOVITCH

Abstract. Khovanov homology is a recently introduced invariant of oriented links in $\mathbb{R}^3$. It categorifies the Jones polynomial in the sense that the (graded) Euler characteristic of the Khovanov homology is a version of the Jones polynomial for links. In this paper we study torsion of the Khovanov homology. Based on our calculations, we formulate several conjectures about the torsion and prove weaker versions of the first two of them. In particular, we prove that all non-split alternating links have their integer Khovanov homology almost determined by the Jones polynomial and signature. The only remaining indeterminacy is that one cannot distinguish between $\mathbb{Z}_2$ factors in the canonical decomposition of the Khovanov homology groups for different values of $k$.

1. Introduction

Let $L$ be an oriented link in the Euclidean space $\mathbb{R}^3$ represented by a planar diagram $D$. In a seminal paper, Mikhail Khovanov [Kh1] assigned to $D$ a family of abelian groups $H^{i,j}(L)$, whose isomorphism classes depend on the isotopy class of $L$ only. These groups are defined as homology groups of an appropriate (graded) chain complex $C(D)$ with integer coefficients. The main property of the Khovanov homology is that it categorifies the Jones polynomial. More specifically, let $J_L(q)$ be a version of the Jones polynomial of $L$ that satisfies the following identities (called the Jones skein relation and normalization):

$$-q^{-2}J_{\begin{array}{c}
\circ
\end{array}}(q) + q^2J_{\begin{array}{c}
\circ
\end{array}}(q) = (q - 1/q)J_{\begin{array}{c}
\circ
\end{array}}(q); \quad J_{\begin{array}{c}
\circ
\end{array}}(q) = q + 1/q. \quad (1.1)$$

The skein relation should be understood as relating the Jones polynomials of three links whose planar diagrams are identical everywhere except in a small disk, where they are different as depicted in (1.1). The normalization fixes the value of the Jones polynomial on the trivial knot. $J_L(q)$ is a Laurent polynomial in $q$ for every link $L$ and is completely determined by its skein relation and normalization.

The gist of the categorification is that the (graded) Euler characteristic of the Khovanov chain complex equals $J_L(q)$:

$$J_L(q) = \sum_{i,j} (-1)^i q^j h^{i,j}(L), \quad (1.2)$$

where $h^{i,j}(L) = \text{rk}(H^{i,j}(L))$, the Betti numbers of $H(L)$. The reader is referred to [BN1, Kh1] for detailed treatment.

There are several numerical conjectures about the Khovanov homology. We recall them briefly below.

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Given a link $L$, the ranks $h^{i,j}(L)$ of its Khovanov homology can be arranged into a table with columns and rows numbered by $i$ and $j$, respectively (see Figure 1). A pair of entries in such a table is said to be a “knight move” pair, if these entries have the same (positive) value and their $i$- and $j$-positions in the table differ by 1 and 4, respectively. This “knight move” rule is depicted in Figure 1, where $a$ can be an arbitrary positive integer.

1.A. Conjecture (Bar-Natan, Garoufalidis, Khovanov [BN1]). Let $L$ be a knot. Consider the table of Khovanov ranks $h^{i,j}(L)$ for $L$. If one subtracts 1 from two adjacent entries in the column $i = 0$, then the remaining entries are arranged in “knight move” pairs.

1.B. Example. Figure 1 illustrates Conjecture 1.A for the knot 77. The two 1’s to subtract are shown inside circles with a gray background and the rest of circles joined by lines depict the “knight move” pairs.

Remark. In fact, different “knight move” pairs are allowed to overlap. The common entry in this case is simply the sum of the overlapping entries from both pairs. For example, the knot 133663, whose homology are presented in section A.4 from the Appendix, has two overlapping pairs $(h^{1,-1}, h^{2,3})$ and $(h^{2,-3}, h^{3,7})$. This confusion will be cleared up after we give a more rigorous statement of this Conjecture in section 2.3.

Remark. Conjecture 1.A was proved by Eun Soo Lee [L1, L2] for the special case of H-thin knots (see below), in particular for all alternating knots.

Let $R$ be a commutative ring with unity. In this paper, we are mainly interested in the cases when $R = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$.

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**Figure 1.** Pattern of the “knight move” rule. Ranks of the Khovanov homology of the knot 77 that illustrates Conjecture 1.A.
1.C. Definitions (cf. Khovanov [Kh2]). A link $L$ is said to be homologically thin over a ring $R$ or simply $RH$-thin if its Khovanov homology groups with coefficients in $R$ are supported on two adjacent diagonals $2i - j = const$. A link $L$ is said to be homologically slim or simply $H$-slim if it is $ZH$-thin and all its homology groups supported on the upper diagonal have no torsion. A link $L$ that is not $RH$-thin is said to be $RH$-thick.

1.D. If a link is $ZH$-thin, then it is $QH$-thin as well. Conversely, a $QH$-thick link is necessarily $ZH$-thin. If a link is $H$-slim, then it is $\mathbb{Z}_p H$-thin for every prime $p$.

1.E. Examples. The knot $7_7$ is $QH$-thin since the free part of its homology is supported on the diagonals $2i - j = \pm 1$ (see Figure 1). In fact, it is $H$-slim as well (see Theorem 1.F below). The first $QH$-thick knot is $8_{19}$ (see Figure 2).

Remark. Most of the $ZH$-thin knots are $H$-slim. The first prime $ZH$-thin knot that is not $H$-slim is the mirror image of $16_{n197566}$. It is also $\mathbb{Z}_2H$-thick.

1.F. Theorem (Lee [L1, L2]). Every oriented non-split alternating link $L$ is $H$-slim and the Khovanov homology of $L$ is supported on the diagonals $2i - j = \sigma(L)\pm 1$, where $\sigma(L)$ is the signature of $L$.

Remark. This theorem was originally conjectured by Bar-Natan, Garoufalidis, and Khovanov [BN1, G] in a somewhat weaker form. Their conjecture stated that every non-split alternating link is $QH$-thin and not $H$-slim. Lee proved a stronger version (see [L1], Corollary 4.3).

Khovanov homology is very difficult to compute by hand. Conjecture 1.A above was formulated based on extensive computations by Dror Bar-Natan using a Mathematica software package that he developed [BN1]. He computed ranks of the Khovanov homology for all prime knots with up to 11 crossings. In 2002, the author developed KhoHo [Sh] that used reductions of the Khovanov chain complex to compute its homology faster and over $\mathbb{Z}$. All the conjectures below about torsion of the Khovanov homology were formulated based on computations done with KhoHo.

As of this writing, the integer Khovanov homology is known for all prime knots with at most 16 crossings, all prime links with at most 14 crossings, as well as many thousands other knots and links.

Remark. For several years, KhoHo was the fastest program for computing Khovanov homology. It works efficiently for knots and links with up to 17–19 crossings. Only in Summer 2005 a significantly faster program by Bar-Natan and Green [BN2, BNG] was written.

As it turns out, torsion of the Khovanov homology has very special properties and is at least as interesting and important as the free part of the homology. First of all, every knot and link considered, except the unknot, the Hopf link, their connected sums, and disjoint unions, has torsion of order 2. If proved, this could lead to a (relatively) easy way to detect the unknot.

Conjecture 1. Khovanov homology of every non-split link except the trivial knot, the Hopf link, and their connected sums has 2-torsion, that is, torsion elements of order 2.

Torsion of orders other than 2 appears very seldom in the Khovanov homology. Among all 1,701,936 prime knots with at most 16 crossings, all alternating ones
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Figure 2. Torsion version of the “knight move” rule. Khovanov homology of the knots $8_{19}$ and $9_{42}$ that are both $\mathbb{Q}H$- and $\mathbb{Z}H$-thick but are T-fancy and T-simple, respectively.

have 2-torsion only. 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion. One of the first such knots is the $(4, 5)$-torus knot. Its Khovanov homology is presented in section A.5 from the Appendix.

Remark. The original version of this paper contained a conjecture that no link has torsion of odd order in its Khovanov homology. This turned out to be false. The first known counter-example is the $(5, 6)$-torus knot [BN2] with 3-torsion. It has 24 crossings. Nonetheless, a large class of links is proved to have 2-torsion only, see Theorem 1 below.

Let $t_{p^k}^{i,j}(L)$, where $p$ is a prime number and $k \geq 1$, be the $p^k$-rank of $H^{i,j}(L)$, that is, the multiplicity of $\mathbb{Z}_{p^k}$ in the canonical decomposition of $H^{i,j}(L)$. Let also $T_{p^k}^{i,j}(L) = \sum_{k=1}^{\infty} t_{p^k}^{i,j}(L)$. The complete information about the canonical decomposition of all the groups $H^{i,j}(L)$ can be combined into a table, where an $(i, j)$-entry contains the corresponding rank $h^{i,j}(L)$ and a comma separated list of $t_{p^k}^{i,j}(L)$ for all relevant $p$ and $k$ with the subscript indicating the torsion order. For example, a hypothetical entry of $5, 3, 2, 1, 4$ means that the corresponding group is $\mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ (see Figure 2 as well as the Appendix).

Similar to the case of ranks, torsion for a majority of knots and links fits into very regular patterns that are explained below (see Figures 2 and 3).

1.G. DEFINITIONS. A link $L$ is said to be weakly torsion simple or just WT-simple if 1) it satisfies Conjecture 1.A; 2) it has no torsion of odd order; 3) for every “knight move” pair of value $a$ that comprises entries $(i, j)$ and $(i + 1, j + 4)$, one has that $T_{p^k}^{i+1,j+2}(L) = a$; and 4) all $2^k$-ranks of $L$ fit into such patterns (see Figure 2, where the torsion corresponding to a “knight move” pair is depicted in a gray square). $L$ is said to be torsion simple or T-simple if it is WT-simple and has torsion of order 2 only. It follows that $t_{2^k}^{i+1,j+2}(L) = a$ in this case. A link $L$ that is not WT-simple is said to be T-fancy.

Remark. Strictly speaking, Conjecture 1.A is stated for knots only. Nonetheless, it was generalized (and proved) by Lee [L2] to the case of QH-thin links (see Theorem 2.3.C). Throughout this paper we are going to refer to a link $L$ as being
Torsion of the Khovanov homology

| Number of crossings | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Number of prime non-alternating knots | 3   | 8   | 42  | 185 | 888 | 5110| 27436| 168030| 1008906 |
| Number of T-fancy knots | 1   | 0   | 6   | 11  | 71  | 322 | 1736 | 10889 | 64341  |
| Number of T-rich knots | 0   | 0   | 0   | 0   | 4   | 14  | 177  | 1413  |        |

Figure 3. Number of prime T-fancy and T-rich knots

T-simple or T-fancy with understanding that this notion is assumed to be applicable, that is, \( L \) is either a knot or a \( \mathbb{Q}\mathbb{H} \)-thin link.

1.H. Examples. The knot 9_42 is T-simple (see Figure 2) and the knot 8_19 is the first T-fancy one. Both of them are \( \mathbb{Q}\mathbb{H} \)- and \( \mathbb{Z}\mathbb{H} \)-thick. Theorems 1.F and 3 (see below) imply that the only T-fancy knots are non-alternating ones. Figure 3 lists the number of prime T-fancy knots with at most 16 crossings.

Remark. The torsion of a T-simple link is completely determined by the ranks of the Khovanov homology. In particular, for non-split alternating links it is completely determined by the Jones polynomial and signature \([L_2]\).

Conjecture 2. Every \( H \)-slim link is T-simple. In particular, every non-split alternating link is T-simple.

Most of the T-fancy links have their torsion ranks never greater than the value of the corresponding “knight move” pair.

1.I. Definition. A T-fancy link \( L \) is said to be torsion rich or just T-rich if it satisfies Conjecture 1.A, has no torsion of odd order (that is, all torsion elements have order \( 2^k \) for some \( k \)) and there is at least one value of \((i, j)\) such that \( T_2^{i+1,j+2}(L) \) is greater than the value of the corresponding “knight move” pair \((h^{i,j}, h^{i+1,j+4})\).

1.J. Example. The first T-rich knot has 13 crossings. It is the knot 13_93663 mentioned above. This knot is also the first one whose homology is supported on 4 adjacent diagonals. Figure 3 lists the number of prime T-rich knots with at most 16 crossings.

In \([Kh2]\) Khovanov defined a reduced version of his homology. The graded Euler characteristic of this reduced homology is again a version of the Jones polynomial. More specifically, it satisfies the same skein relation from (1.1) but is normalized to be 1, as opposite to \( q + 1/q \), on the unknot. The reduced Khovanov homology is always supported on exactly one diagonal less than the standard one. Very few knots have torsion in the reduced homology and all known examples of those that have are T-rich.

Conjecture 3. A knot is T-rich if and only if its reduced Khovanov homology has torsion.

The following conjecture is a bit optimistic, but its confirmation would be very exciting, since the torsion seems to be easier to read from the diagram than the rest of the homology.
Conjecture 4. If two knots have the same torsion in their Khovanov homology, then they have the same ranks as well. In other words, the Khovanov homology of a knot is completely determined by its torsion.

Remark. There are many examples of knots which have the same ranks in the Khovanov homology but different torsion. Some of the first ones are 14_{9933} and the mirror image of 15_{129763}. The former one is T-simple, while the latter one is T-fancy.

In this paper we prove the following results.

Theorem 1. Khovanov homology of every H-slim link has no torsion of order \( p^k \) for any odd prime \( p \) and \( k \geq 1 \).

Corollary 2. Khovanov homology of every non-split alternating link has no torsion of order \( p \) for any \( p \) other than a power of 2.

We prove Theorem 1 by showing that Conjecture 1.A holds true for every H-slim link not only for rational homology, but also for homology over \( \mathbb{Z}_p \) for all odd prime \( p \). This is done using a slight modification of Lee’s methods from [L2].

Theorem 3 (cf. Conjecture 2). Every H-slim link is WT-simple.

Corollary 4. Every non-split alternating link is WT-simple. In particular, the integer Khovanov homology of non-split alternating links is all but determined by the Jones polynomial and signature except that one cannot distinguish between \( \mathbb{Z}_2 \) factors in the canonical decomposition of the Khovanov homology groups for different values of \( k \).

Remark. Lee proved that the rational Khovanov homology of non-split alternating links is completely determined by the Jones polynomial and signature, see [L2].

Corollary 5 (cf. Conjecture 1). Every alternating link except the trivial knot, the Hopf link, their connected sums and disjoint unions has torsion of order 2.

Remark. Marta Asaeda and Jozef Przytycki gave [AP] an independent proof of Corollary 5. Moreover, they proved that an adequate link that satisfies some additional conditions has torsion of order 2 as well. Contrary to our approach, their proof is constructive. They explicitly find a generator in the Khovanov chain complex that gives rise to an appropriate torsion element. More recently, Milena Pabiniak, Jozef Przytycki, and Radmila Sazdanović provided similar treatments for semi-adequate links [PPS, PS].

Theorem 3 is a corollary of Theorem 3.2.A that establishes a structure of an exact sequence on Khovanov homology over \( \mathbb{Z}_2 \). To complete the proof of Conjecture 2, one only has to show that every H-slim link has no torsion elements of order \( 2^k \) for \( k \geq 2 \).

This paper is organized as follows. Section 2 contains main definitions and facts about the Khovanov homology that are going to be used in the paper. Theorems 1 and 3 and Corollary 5 are proved in sections 4, 3, and 3.3, respectively. The Appendix contains information about standard and reduced Khovanov homology of knots whose torsion has some remarkable properties.
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2. Khovanov chain complex and its properties

In this section we briefly recall the main ingredients of the Khovanov homology theory. Our exposition follows the one by Viro, whose paper [V] is recommended for a full treatment.

2.1. Generators and the differential of the Khovanov chain complex. Let $D$ be a planar diagram representing an oriented link $L$. Assign a number $\pm 1$, called a sign, to every crossing of $D$ according to the rule depicted in Figure 4. The sum of such signs over all the crossings is called the writhe number of $D$ and is denoted by $w(D)$.

![Positive crossing and positive marker](image1)

![Negative crossing and negative marker](image2)

Figure 4. Positive and negative crossings

Figure 5. Positive and negative markers and the corresponding smoothings of a diagram.

At every crossing of $D$, the diagram locally divides the plane into 4 quadrants. A choice of a pair of antipodal quadrants at a crossing can be depicted on the diagram with the help of a marker, which can be either positive or negative (see Figure 5). A collection of markers chosen at every crossing of a diagram $D$ is called a (Kauffman) state of $D$. There are, clearly, $2^n$ different states, where $n$ is the number of crossings of $D$. Denote by $\sigma(s)$ the difference between the numbers of positive and negative markers in a given state $s$.

Given a state $s$ of a diagram $D$, one can smooth $D$ at every crossing with respect to the corresponding marker from $s$ (see Figure 5). The result is a family $D_s$ of disjointly embedded circles. Denote the number of these circles by $|s|$.

Let $s$ be a state of a diagram $D$. Equip each circle from $D_s$ with either a plus or minus sign. We call the result an enhanced (Kauffman) state of $D$ that belongs to $s$. There are exactly $2^{|s|}$ different enhanced states that belong to a given state $s$. Denote by $\tau(S)$ the difference between the numbers of positively and negatively signed circles in a given enhanced state $S$. 
With every enhanced state \( S \) belonging to a state \( s \) of a diagram \( D \) one can associate two numbers:

\[
i(S) = \frac{w(D) - \sigma(s)}{2}, \quad j(S) = -\frac{\sigma(s) + 2\sigma(S) - 3w(D)}{2}.
\]

Since both \( w(D) \) and \( \sigma(s) \) are congruent modulo 2 to the number of crossings, \( i(S) \) and \( j(S) \) are always integer.

Fix \( i, j \in \mathbb{Z} \). It was shown by Viro [V] that the Khovanov chain group \( C^{i,j}(D) \) is generated by all the enhanced states of \( D \) with \( i(S) = i \) and \( j(S) = j \). With the basis of the chain groups chosen, the Khovanov differential \( d^{i,j} : C^{i,j}(D) \to C^{i+1,j}(D) \) can be described by its matrix, called the **incidence matrix** in this context. The elements of the incidence matrix are called **incidence numbers** and are denoted by \( (S_1 : S_2) \), where \( S_1 \) and \( S_2 \) are enhanced states (that is, generators) from \( C^{i,j}(D) \) and \( C^{i+1,j}(D) \), respectively.

The incidence number \( (S_1 : S_2) \) is zero unless all of the following 3 conditions are met:

I. the markers from \( S_1 \) and \( S_2 \) differ at one crossing of \( D \) only and at this crossing the marker from \( S_1 \) is positive, while the marker from \( S_2 \) is negative;

Remark. If this condition is satisfied, \( D_{S_2} \) is obtained from \( D_{S_1} \) by either joining two circles into one or splitting one circle into two and, hence, \( |S_2| = |S_1| \pm 1 \).

II. the common circles of \( D_{S_1} \) and \( D_{S_2} \) have the same signs;

III. one of the following 4 conditions are met:

1) \( |S_2| = |S_1| - 1 \), both joining circles from \( D_{S_1} \) are negative and the resulting circle from \( D_{S_2} \) is negative as well;
2) \( |S_2| = |S_1| - 1 \), the joining circles from \( D_{S_1} \) have different signs and the resulting circle from \( D_{S_2} \) is positive;
3) \( |S_2| = |S_1| + 1 \), the splitting circle from \( D_{S_1} \) is positive and both the resulting circles from \( D_{S_2} \) are positive as well;
4) \( |S_2| = |S_1| + 1 \), the splitting circle from \( D_{S_1} \) is negative and the resulting circles from \( D_{S_2} \) have different signs.

If all three conditions above are satisfied, then the incidence number \( (S_1 : S_2) \) is defined to be equal to \((-1)^t\), where \( t \) is defined as follows. Choose some order on the crossings of \( D \). Let the crossing, where one changes the marker to get from \( S_1 \) to \( S_2 \), have number \( k \) in this order. Then \( t \) is the number of negative markers in \( S_1 \) whose order number is greater than \( k \). As it turns out, the resulting homology does not depend on the choice of the crossing order. More details can be found in [BN1, V].

2.2. **Reduced Khovanov homology**. Let \( D \) be a diagram of a link \( L \). Pick a base point on \( D \) that is not a crossing. Let \( \bar{C}(D) \) be a subcomplex of \( C(D) \) generated by all the enhanced states of \( D \) that have a positive sign on the circle that the base point belongs to. The homology \( \bar{H}(L) \) of this subcomplex is called the **reduced Khovanov homology** of \( L \). It can be shown that if \( L \) is a knot, then its reduced homology does not depend on the choice of the base point. In general, the reduced homology of a link might depend on the component that the base point is chosen on. See [Kh2] for more details.
2.2.A. (Khovanov [Kh2], cf. (1.2)). The graded Euler characteristic of $\bar{C}(D)$ is a version of the Jones polynomial of $L$:

$$\bar{J}_L(q) = J_L(q)/(q + 1/q) = \sum_{i,j} (-1)^i q^j h^{i,j}(L). \tag{2.1}$$

where $h^{i,j}(L)$ are the Betti numbers of $\bar{H}^{i,j}(L)$.

$\bar{J}_L(q)$ is completely determined by the following identities (cf. (1.1)):

$$-q^{-2}\bar{J}_t(q) + q^2 \bar{J}_t(q) = (q - 1/q)\bar{J}_t(q); \quad \bar{J}_{\bigwedge}(q) = 1. \tag{2.2}$$

2.2.B. (Khovanov [Kh2]). For any link $L$, its reduced Khovanov homology $\tilde{H}(L)$ over $\mathbb{Q}$ is supported on exactly one diagonal less than the standard one.

2.2.C. Corollary. If $L$ is a $\mathbb{Q}$H-thin link (in particular, a non-split alternating link), then $\tilde{H}(L)$ is supported on exactly one diagonal. If follows that $\bar{J}_L(q)$ is alternating, that is, its coefficients have alternating signs. More precisely, if $\bar{J}_L(q) = \sum_{i\in\mathbb{Z}} c_i q^{2i+\gamma}$, where $\gamma$ is the number of components of $L$ modulo 2, then $\bar{J}_L(q)$ is alternating if and only if $(-1)^{i+j} c_i c_j \geq 0$ for all $i$ and $j$.

2.3. Khovanov polynomial and its torsion version. Let $L$ be a link and $\text{Kh}(L)(t,q) = \sum_{i,j} t^i q^j \text{Kh}^{i,j}(L)$ be the Poincaré polynomial in variables $t$ and $q$ of its Khovanov homology. This polynomial is called the Khovanov polynomial of $L$. Now Conjecture 1.A can be reformulated in the following way.

2.3.A. (Rigorous statement of Conjecture 1.A). Let $L$ be a knot. Then there exists a polynomial $\text{Kh}'(L)$ in $t^{\pm 1}$ and $q^{\pm 1}$ with non-negative coefficients only and an even integer $s = s(L)$ such that

$$\text{Kh}'(L) = q^{s-1}(1 + q^2 + (1 + t q^4)\text{Kh}'(L)). \tag{2.3}$$

In other words, there exist non-negative integers $g^{i,j}(L)$ such that

$$h^{i,j}(L) = g^{i,j}(L) + g^{i-1,j-4}(L) + \varepsilon^{i,j}, \tag{2.4}$$

where $\varepsilon^{0,s=\pm 1} = 1$ and $\varepsilon^{i,j} = 0$ if $i \neq 0$ or $j \neq s \pm 1$.

Remark. It is clear from the construction that $g^{i,j}(L)$ is the coefficient of the term $t^i q^{j-s+1}$ in $\text{Kh}'(L)$. It must be non-zero for finitely many values of the pair $(i,j)$ only.

2.3.B. If $L$ is a $\mathbb{Q}$H-thin knot, then the polynomial $\text{Kh}'(L)$ contains powers of $t^\pm 1$ only. Let $\text{Kh}'(L) = \sum_{i\in\mathbb{Z}} a_i t^i q^{2i}$. In this case $g^{i,2i+s-1}(L) = a_i$ and, hence,

$$h^{i,2i+s-1}(L) = g^{i,2i+s-1}(L) + g^{i-1,2i+s-5}(L) + \varepsilon^{i,2i+s-1} = a_i + \delta_{i0}; \tag{2.5}$$

$$h^{i,2i+s+1}(L) = g^{i,2i+s+1}(L) + g^{i-1,2i+s-3}(L) + \varepsilon^{i,2i+s+1} = a_{i-1} + \delta_{i0}, \tag{2.6}$$

where $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ is the Kronecker delta. All other $h^{i,j}(L)$ are zero.

Theorem 1.F implies that $s(L) = -\sigma(L)$ for all alternating knots $L$, where $\sigma(L)$ is the signature of $L$.

The following theorem is a counterpart of Conjecture 1.A for the case of $\mathbb{Q}$H-thin links.
2.3.C. Theorem (Lee [L2]). Let $L$ be an $m$-component oriented $\mathbb{Q}$H-thin link (for example, a non-split alternating link). Let $\ell_{k,l}$ be the linking number of the $k$-th and $l$-th components of $L$ and let $\sigma(L)$ be the signature of $L$. Then

$$K\!h(L) = q^{-\sigma(L)-1} \left( 1 + q^2 \sum_{E \subset \{2, \ldots, m\}} (tq^2)^{2\sum_{i \in E, l \in E} \ell_{k,l}} + (1 + tq^4)K'\!h(L)(tq^2) \right)$$

for some polynomial $K'\!h(L)$ with non-negative coefficients.

2.3.D. Definition. For a given link $L$, its torsion Khovanov polynomial $K\!h_T$ in variables $t^{\pm 1}$ and $Q_{p^k}^{\pm 1}$ is defined as $K\!h_T(L)(t, Q_2, Q_3, Q_4, \ldots) = \sum_{i,j,p,k} t^i Q_j^k t^{1,2}_{p^k}(L)$, where $i$ and $j$ are arbitrary, $k \geq 1$, and $p$ runs through all prime numbers. Recall that $t^{i,j}_{p^k}(L)$ is the $p^k$-rank of $\mathcal{H}^{i,j}(L)$ and $T^{i,j}_{p^k}(L) = \sum_{k=1}^{\infty} t^{i,j}_{p^k}(L)$.

The following proposition provides a straightforward reformulation of Definitions 1.G and 1.I in terms of $K\!h_T$.

2.3.E. 1. A link $L$ is $T$-simple if and only if $K\!h_T(L)$ depends on the variables $t^{\pm 1}$ and $Q_{2^k}^{\pm 1}$ only and $K\!h_T(L)(t, q) = t q^{s+1} K'\!h(L)$. The latter equality holds true if and only if $t^{1,2}_{2^k}(L) = g^{r-1,2^{-2}}(L)$ for all $i$ and $j$ (see 2.3.B).

2. A link $L$ is $WT$-simple if and only if $K\!h_T(L)$ depends on the variables $t^{\pm 1}$ and $Q_{2^k}^{\pm 1}$ only and $K\!h_T(L)(t, q, q, \ldots) = t q^{s+1} K'\!h(L)$. The latter equality holds true if and only if $T^{i,j}_{2^k}(L) = g^{r-1,2^{-2}}(L)$ for all $i$ and $j$ (see 2.3.B).

3. A link $L$ is $T$-rich if and only if $K\!h_T(L)$ depends on the variables $t^{\pm 1}$ and $Q_{2^k}^{\pm 1}$ only and $t q^{s+1} K'\!h(L) - K\!h_T(L)(t, q, q, \ldots)$ contains some terms with negative coefficients.

3. Khovanov homology with $\mathbb{Z}_2$ coefficients

Denote by $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ the Khovanov homology over $\mathbb{Z}_2$ (instead of $\mathbb{Z}$) of an oriented link $L$ and by $h^{i,j}_{\mathbb{Z}_2}(L)$ the corresponding Betti numbers. In this section we construct an acyclic differential $\mathbf{v}$ of bidegree $(0, 2)$ on $\mathcal{H}^{i,j}_{\mathbb{Z}_2}(L)$ and prove Theorem 3.

3.1. Construction of the $(0, 2)$-differential. Let $D$ be a planar diagram of a link $L$ and $S$ be some enhanced state of $D$. Denote by $\mathcal{N}(S)$ the set of all enhanced states of $D$ that have the same markers and signs on all the circles as $S$ except one circle where '$+$' is replaced with '$-'$. It is a straightforward verification that for every enhanced state $S' \in \mathcal{N}(S)$, one has $\sigma(S') = \sigma(S)$ and $\tau(S') = \tau(S) - 2$. Hence, $i(S') = i(S)$ and $j(S') = j(S) + 2$.

3.1.A. Definition. A differential $\nu^{i,j} : C^{i,j}(D; \mathbb{Z}_2) \to C^{i,j+2}(D; \mathbb{Z}_2)$ of bidegree $(0, 2)$ on the Khovanov chain groups with coefficients in $\mathbb{Z}_2$ is defined on the generators of $C^{i,j}(D; \mathbb{Z}_2)$ as $\nu^{i,j}(S) = \sum_{S' \in \mathcal{N}(S)} S'$ (see Figure 6).

![Figure 6. Action of $\nu$ on generators of the chain groups.](image-url)
To show that \( \nu \) is indeed a differential, i.e. \( \nu^2 = 0 \), we observe that for every enhanced states \( S \) and \( S'' \) that have the same markers and signs on all the circles except two circles where \( S \) has \( '+' \), while \( S'' \) has \( '-' \), \( S'' \) appears exactly twice in \( \nu(\nu(S)) \).

3.1.C. **Lemma.** \( \nu \) is isomorphic to \( \nu \) that to the \((0,\cdot)\) base case of \( \nu \).

**Proof.** Let \( Z_n \) be the set of all \( n \) sequences of signs \( '+' \) and \( '-' \) and let \( Z'_n \subset Z_n \) consists of all sequences with the difference between the numbers of minuses and pluses being exactly \( k \) (with \( k \equiv n \mod 2 \)). \( Z_n \) and \( Z'_n \) have \( 2^n \) and \( b_n = \binom{n}{n+k} \) elements, respectively. Denote by \( G_n^k \) the group \( Z'_2 \) whose factors are enumerated by the elements from \( Z'_n \).

One can construct a differential \( \mu_n^k : Z_n^k \to Z_n^{k+2} \) similarly to \( \nu \): a generator of \( G_n^k \) corresponding to a sequence \( \rho \) from \( Z_n^k \) is mapped into the sum of the generators of \( G_n^{k+2} \) corresponding to all the sequences obtained from \( \rho \) by changing exactly one \( '+' \) into a \( '-' \).

Let \( G_n \) be the complex

\[
0 \to G_{n-2} \overset{\mu_{n-2}^2}{\to} G_{n-4} \to \cdots \overset{\mu_{n-2}^2}{\to} G_n \to 0
\]  

(3.1)

It follows from the definition of the Khovanov chain complex that the complex

\[
\cdots \overset{\nu^{i,j-2}}{\longrightarrow} C_{i,j-2}(D,\mathbb{Z}_2) \overset{\nu^{i,j-2}}{\longrightarrow} C_{i,j}(D,\mathbb{Z}_2) \overset{\nu^{i,j+2}}{\longrightarrow} C_{i,j+2}(D,\mathbb{Z}_2) \overset{\nu^{i,j+2}}{\longrightarrow} \cdots
\]  

(3.2)

is isomorphic to a direct sum of \( G_{[s]} \) with various shifts, where \( s \) runs over all the Kauffman states of \( D \) such that \( i(s) = i \).

We claim that the complex \( G_n \) is acyclic. Let us prove this by induction on \( n \). The base case of \( n = 1 \) is trivial. Denote by \( G_n^- \) the subcomplex of \( G_n \) that is obtained by choosing only those sequences that have \( '-' \) in the first position. Then \( G_n^- \) is isomorphic to \( G_{n-1} \) and, hence, is acyclic by the induction hypothesis. It follows that \( G_n \) has the same homology as \( G_n/\partial G_n^- \). But the later is again isomorphic to \( G_{n-1} \). Hence, \( G_n \) is acyclic as well.

This completes the proof of the lemma.

3.1.B. **Lemma.** \( \nu \) commutes with the Kauffman differential \( d \) (over \( \mathbb{Z}_2 \)).

The proof is elementary and is left to the reader as an exercise.

3.2. **Patterns in \( \mathbb{Z}_2 \) homology.** Lemma 3.1.C implies that \( \nu \) can be extended to the \((0,2)\) differential \( \nu \) on the Khovanov homology over \( \mathbb{Z}_2 \). This differential is also acyclic, although this does not follow from Lemma 3.1.B directly.

3.2.A. **Theorem.** \( \nu \) is acyclic. In particular, for every fixed \( i \) the following sequence is exact:

\[
\cdots \overset{\nu^{i,j-2}}{\longrightarrow} \mathcal{H}^{i,j-2}_{\mathbb{Z}_2}(L) \overset{\nu^{i,j-2}}{\longrightarrow} \mathcal{H}^{i,j}_{\mathbb{Z}_2}(L) \overset{\nu^{i,j}}{\longrightarrow} \mathcal{H}^{i,j+2}_{\mathbb{Z}_2}(L) \overset{\nu^{i,j+2}}{\longrightarrow} \cdots
\]  

(3.3)

Consequently, \( \sum_{j \in \mathbb{Z}} (-1)^{j} \mathcal{H}^{i,2j+\gamma}_{\mathbb{Z}_2}(L) = 0 \) for every \( i \), where \( \gamma \) is the number of components of \( L \) modulo 2.

The following proof is due to Khovanov. It replaced the original one that was too technical and unnecessarily complicated.
PROOF. Choose a base point $b$ somewhere on the diagram $D$ away from the crossings. In [Kh2] Khovanov introduced another differential $X^{i,j} : C^{i,j}(D) \to C^{i,j-2}(D)$ of bidegree $(0,-2)$ on the chain groups $C^{i,j}(D)$ with $\mathbb{Z}$ coefficients. It is defined as follows. Let $S \in C^{i,j}(D)$ be some enhanced state. If the circle of $S$ that contains $b$ has a positive sign, then $X^{i,j}(S) = 0$. Otherwise $X^{i,j}(S) = S' \in C^{i,j-2}(D)$, where $S'$ is obtained from $S$ by changing the sign of the circle that contains $b$ from $'-$ to $'+$.

It follows immediately from the definition that $X \circ X = 0$, that is, $X$ is indeed a differential.

It is easy to check that $X$ commutes with the Khovanov differential $d$ and, hence, can be extended to the $(0,-2)$-differential $\nu$ on the Khovanov homology. We will abuse the notation slightly and denote reductions of $X$ and $\nu$ modulo 2 by the same symbols.

We first claim that $\nu \circ X + X \circ \nu = \text{id}$. Indeed, let $S$ be some enhanced state of $D$. If the circle of $S$ that contains $b$ has a positive sign, then $\nu(X(S)) = 0$. Moreover, $\nu(S)$ is a sum of enhanced states such that all but one of them have positive signs on their circles that contain $b$. Hence, $X(\nu(S)) = S$. On the other hand, if the circle of $S$ containing $b$ has a negative sign, then all the enhanced states from $\nu(S)$ have negative signs on their circles containing $b$ and $X(\nu(S))$ is the sum of all the states that are obtained from $S$ by changing the sign of the circle that contains $b$ from $'-$ to $'+$ and changing the sign of some other circle from $'+$ to $'-$.

Moreover, $X(S)$ has one more positive sign than $S$ and $\nu(X(S))$ is the sum of all the same states as $X(\nu(S))$ plus $S$ itself. The claim follows.

Since $\nu$ and $X$ both commute with the differential $d$, one has that $\nu \circ X + X \circ \nu = \text{id}$ on the homology level as well. It follows that $\nu$ is acyclic. Indeed, if $\alpha \in H^{i,j}(D)$ such that $\nu(\alpha) = 0$, then $\nu(\nu(\alpha)) = \alpha$, that is, $\alpha$ lies in the image of $\nu$.

3.2.B. COROLLARY. $\nu$ is acyclic on $H_{Z_2}(L)$ as well.

3.2.C. COROLLARY. Since the reduced Khovanov homology over $\mathbb{Z}_2$ is isomorphic to the kernel of $\nu$, it follows that $H_{Z_2}^{i,j}(L) \cong H_{Z_2}^{i+1,j+1}(L)$ for every $i, j$.

3.2.D. COROLLARY. Let $L$ be an H-slim link. Then it is $\mathbb{Z}_2$-thin by 1.D, that is, its $\mathbb{Z}_2$-homology are supported on the diagonals $2i - j = -s \pm 1$. Theorem 3.2.A implies that $h_{Z_2}^{i,2i+s-1}(L) = h_{Z_2}^{i,2i+s+1}(L)$ for every $i$.

PROOF OF THEOREM 3. Let $L$ be an H-slim knot. It follows from Theorem 1 that $L$ has torsion of order $2^k$ only. It remains to show that $T^{i,j}_{Z_2}(L) = g^{i-1,j-2}(L)$ for all $i$ and $j$ (see 2.3.E). Since $T^{i,j}_{Z_2}(L) = g^{i-1,j-2}(L) = 0$ for $j \neq 2i + s - 1$ and $g^{i-1,2i+s-3}(L) = a_{i-1}$ in notation of 2.3.B, we only need to prove that $T^{i,2i+s-3}_{Z_2}(L) = a_{i-1}$.

Observe now that $h^{i,j}_{Z_2}(L) = h^{i,j}(L) + T^{i,j}_{Z_2}(L) + T^{i+1,j}_{Z_2}(L)$. It follows from 2.3.B that $h^{i,2i+s-1}_{Z_2}(L) = a_i + \delta_0 + T^{i,2i+s-1}_{Z_2}(L)$ and $h^{i,2i+s+1}_{Z_2}(L) = a_{i+1} + \delta_0 + T^{i+1,2i+s+1}_{Z_2}(L)$. Corollary 3.2.D implies that

$$T^{i,2i+s-1}_{Z_2}(L) - a_{i-1} = T^{i+1,2i+s+1}_{Z_2}(L) - a_i$$

for all $i$. Hence, $T^{i,2i+s-1}_{Z_2}(L) - a_{i-1} = \text{const}$, for some constant independent of $i$. Since the support of the Khovanov homology is finite, there exists $i$ such that $T^{i,2i+s-1}_{Z_2}(L) - a_{i-1} = 0$. It follows that the constant must be zero.

The case of $L$ being a link can be considered similarly.
3.3. Proof of Corollary 5. Let $L$ be an alternating link with $m$ components. Then its Jones polynomial has the form $J_L(q) = \sum c_i q^{2i+\gamma}$, where $\gamma = m \mod 2$ (cf. Corollary 2.2.C). Define $d(L) = |\bar{J}_L(\sqrt{-1})| = \sum |c_i|$. In fact, $d(L) = |\det(L)|$, where $\det(L)$ is the determinant of $L$, hence the notation.

3.3.A. Theorem (Thistlethwaite [Th], Theorem 1). Let $L$ be a prime non-split alternating link that admits an irreducible alternating diagram with $n$ crossings, and let $\bar{J}_L(q) = \sum c_i q^{2i+\gamma}$ with $c_u \neq 0$ and $c_v \neq 0$ be its Jones polynomial. Then $v - u = n$ and $c_i c_{i+1} \leq 0$ for every $u \leq i < v$. If, moreover, $L$ is not a $(2,k)$-torus link, then $c_i \neq 0$ for every $u \leq i \leq v$.

3.3.B. Lemma. Let $L$ be an alternating link with $m$ components. Then $d(L) \geq 2^{m-1}$. Moreover, if $L$ is not the trivial knot, the Hopf link, their connected sums or disjoint unions, then $d(L) > 2^{m-1}$.

Proof. Assume first that $L$ is non-split and prime. It is easy to deduce from (2.2) (see also [J]) that $\bar{J}_L(1) = \sum c_i = (-2)^{m-1}$. Hence, $d(L) \geq |\bar{J}_L(1)| = 2^{m-1}$. It only remains to show that if $L$ is neither the trivial knot, nor the Hopf link, then the polynomial $\bar{J}_L(q)$ has both strictly positive and strictly negative coefficients. Theorem 3.3.A implies this for all $L$ except the trivial knot and $(2,k)$-torus links.

Denote a $(2,k)$-torus link by $TL_k$. It easily follows from (2.2) that $d(TL_k) = k$. Indeed, if one changes a positive crossing of $TL_k$ into a negative one, one gets $TL_{k-2}$, and if one smooths such a crossing, one obtains $TL_{k-1}$. Recall now that a $(2,k)$-torus link has at most 2 components and that the Hopf link is exactly the $(2,2)$-torus link.

Now the Lemma follows from the fact that the Jones polynomial of the connected sum and disjoint union of links $L_1$ and $L_2$ is equal to $\bar{J}_{L_1}(q)\bar{J}_{L_2}(q)$ and $(q + 1/q)J_{L_1}(q)\bar{J}_{L_2}(q)$, respectively. □

3.3.C. Lemma. Let $L$ be an alternating link that is not the trivial knot, the Hopf link, their connected sums and disjoint unions. Then rank $\mathcal{H}(L) > 2^m$, where $m$ is the number of components of $L$ and rank $\mathcal{H}(L) = \sum_{i,j} h^{i,j}(L)$ is the total rank of the Khovanov homology.

Proof. Consider first the case when $L$ is non-split. Theorem 2.3.C implies that rank $\mathcal{H}(L) \geq 2^m$. Assume that this rank is $2^m$. In this case $Kh'(L)$ must be 0 and

$$J_L(q) = Kh(L)(-1,q) = q^{-\sigma(L)}((q + 1/q) \left( \sum_{E \subseteq \{2,\ldots,m\}} (-q^2)^{2 \sum_{i \in E} \ell_{k,i}} \right)).$$

(3.5)

Since $\bar{J}_L(q) = J_L(q)/(q + 1/q)$, one has

$$d(L) = |\bar{J}_L(\sqrt{-1})| = \sum_{E \subseteq \{2,\ldots,m\}} 2^{\sum_{i \in E} \ell_{k,i}} = 2^{m-1}. \quad (3.6)$$

This contradicts Lemma 3.3.B. Hence, rank $\mathcal{H}(L) > 2^m$.

The general case follows from the fact that rank $\mathcal{H}(L)$ is multiplicative under disjoint union (see [Kh1], Corollary 12). □

Let us now finish the proof of Corollary 5. Lemma 3.3.C states that rank $\mathcal{H}(L) > 2^m$ and, hence, $Kh'(L) \neq 0$ (in notation of Theorem 2.3.C). Since $L$ is WT-simple by Theorem 3, it follows from 2.3.E that $Kh_T(L) \neq 0$ as well. Hence, $L$ has non-trivial torsion. Since $L$ is WT-simple, some torsion elements must be of order 2. □
4. Torsion of order $p$ of the Khovanov homology

This section is devoted to proving Theorem 1. We start by showing that the Khovanov homology over $\mathbb{Z}_p$ of an H-slim link satisfy Conjecture 1.A as well.

4.1. Khovanov homology with $\mathbb{Z}_p$ coefficients. Let $L$ be an oriented link and $p$ be an odd prime number. Denote by $H^{i,j}_{\mathbb{Z}_p}(L)$ the Khovanov homology of $L$ over $\mathbb{Z}_p$ and by $h^{i,j}_{\mathbb{Z}_p}(L)$ their Betti numbers. Let $Kh_{\mathbb{Z}_p}(L)(t,q) = \sum_{i,j} t^i q^j h^{i,j}_{\mathbb{Z}_p}(L)$ be the corresponding Poincaré polynomial.

4.1.A. Theorem (cf. Theorem 2.3.C and [L2], Theorems 1.2 and 1.4). Let $L$ be an $m$-component oriented H-slim link, for example, a non-split alternating link. Then $Kh_{\mathbb{Z}_p}(L)$ satisfies identity (2.7) for the original Khovanov polynomial with some other polynomial $Kh'_p(L)$ instead of $Kh(L)$. If $L$ is an H-slim knot, then this identity becomes

$$Kh_{\mathbb{Z}_p}(L) = q^{-\sigma(L)-1}(1 + q^2 + (1 + tq^4)Kh'_p(L)(tq^2)), \quad (4.1)$$

where $\sigma(L)$ is the signature of $L$.

Proof. We will show that the methods used by Lee to prove Theorem 2.3.C for the Khovanov homology with $\mathbb{Q}$ coefficients work in our $\mathbb{Z}_p$ case as well if $p$ is odd prime. Only the main steps are to be outlined and the reader is assumed to be familiar with [L2].

First of all, we define the Lee differential $\Phi$ of bidegree $(1,4)$ on the Khovanov chain complex $C$. The corresponding incidence numbers $(S_1 : S_2)_\Phi$ of two enhanced states $S_1 \in C^{i,j}_{\mathbb{Z}_p}(D)$ and $S_2 \in C^{i,j+1}_{\mathbb{Z}_p}(D)$ are defined in a similar way to the original ones from page 8 with the only difference being in condition III:

The incidence number $(S_1 : S_2)_\Phi$ is zero unless all of the following 3 conditions are met, in which case $(S_1 : S_2)_\Phi = \pm 1$ with the sign defined as before:

I$\Phi$. the markers from $S_1$ and $S_2$ differ at one crossing of $D$ only and at this crossing the marker from $S_1$ is positive, while the marker from $S_2$ is negative;

II$\Phi$. the common circles of $DS_1$ and $DS_2$ have the same signs;

III$\Phi$. one of the following two conditions are met:

1) $|S_2| = |S_1| - 1$, both joining circles from $DS_1$ are positive and the resulting circle from $DS_2$ is negative;

2) $|S_2| = |S_1| + 1$, the splitting circle from $DS_1$ is positive and both the resulting circles from $DS_2$ are negative;

It is easy to see that $\Phi$ is indeed a differential and (anti)commutes with the Khovanov differential $d$. Lee’s proofs from [L2] can be applied to our version of $\Phi$ to show that it also commutes with the isomorphisms induced on $H_{\mathbb{Z}_p}(L)$ by the Reidemeister moves. Hence, $\Phi$ gives rise to a well defined differential on $H_{\mathbb{Z}_p}(L)$.

Consider now yet another differential $\Phi + d$ on $C_{\mathbb{Z}_p}(D)$. It can be best described by changing the labels on the circles comprising enhanced states from ‘$+$’ and ‘$-$’ to $a = (\cdot') + (\cdot')$ and $b = (\cdot') - (\cdot')$. In this notation one has a new third condition on the incidence numbers $(S_1 : S_2)_{\Phi+d}$:

III$\Phi+d$. one of the following 4 conditions are met:

1) $|S_2| = |S_1| - 1$, both joining circles from $DS_1$ and the resulting circle from $DS_2$ are marked with $a$, then $(S_1 : S_2)_{\Phi+d} = \pm 2$;
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2) \(|S_2| = |S_1| - 1\), both joining circles from \(D_{S_1}\), and the resulting circle from \(D_{S_2}\) are marked with \(b\), then \((S_1 : S_2)_{\Phi d} = \mp 2\);
3) \(|S_2| = |S_1| + 1\), the splitting circle from \(D_{S_1}\), and both the resulting circles from \(D_{S_2}\) are marked with \(a\), then \((S_1 : S_2)_{\Phi d} = \pm 1\);
4) \(|S_2| = |S_1| + 1\), the splitting circle from \(D_{S_1}\), and both the resulting circles from \(D_{S_2}\) are marked with \(b\), then \((S_1 : S_2)_{\Phi d} = \pm 1\).

Denote by \(H(D)\) the homology with respect to \(\Phi + d\). It can be shown that \(H(D)\) is invariant under the Reidemeister moves, so that we can safely write \(H(L)\) instead. The fact that \(p \neq 2\) is crucial here, as the proof involves division by 2 (see [L2]).

Theorem 4.4 from [L2] that states

\[
H(L) \cong \frac{\ker(\Phi : \mathcal{H}_{Z_p}(L) \to \mathcal{H}_{Z_p}(L))}{\text{im}(\Phi : \mathcal{H}_{Z_p}(L) \to \mathcal{H}_{Z_p}(L))} \quad (4.2)
\]

still holds true without changes. One needs to use the fact that \(L\) is \(Z_p\)-H-slim, since it is H-slim, here.

The only non-trivial generalization is proving an analogue of Theorem 4.2 of [L2] that \(\dim_{Z_p} H(L) = 2^m\). Lee’s proof uses Hodge theory arguments which are not applicable to the \(Z_p\) case. Fortunately for us, Hodge theory is only used to provide a lower bound on \(\dim_{\mathbb{Q}} H(L; \mathbb{Q})\). It follows that \(2^m \leq \dim_{\mathbb{Q}} H(L; \mathbb{Q}) \leq \dim_{Z_p} H(L),\) where the former inequality is provided by Theorem 4.2 from [L2], and the latter one by the Universal Coefficient Theorem. In particular, all the enhanced states of \(D\), such that at every crossing the two touching circles have different labels, are linearly independent in \(H(L)\). Such states are in one-to-one correspondence with all the orientations of \(L\) (see [L2]). Lee’s proof of the fact that \(\dim_{\mathbb{Q}} H(L; \mathbb{Q}) \leq 2^m\) still works without changes for \(Z_p\). Hence \(2^m \leq \dim_{Z_p} H(L) \leq 2^m\) and \(\dim_{Z_p} H(L) = 2^m\).

Filling the remaining technical gaps is left to the reader. \(\square\)

4.2. Proof of Theorem 1. Let \(L\) be an H-slim link. Since \(h^{i,j}_{Z_p}(L) = h^{i,j}(L) + T^{i,j}_{p}(L) + T^{i+1,j}_{p}(L)\), it follows from Theorems 2.3.C and 4.1.A that \(h_{Z_p}(L) = h(L)\) are arranged in “knight move” pairs everywhere without having to subtract anything (cf., for example, Conjecture 1.A). Hence,

\[
T^{i,2i-\sigma(L)-3}_{p}(L) = T^{i+1,2i-\sigma(L)+3}_{p}(L) + T^{i+2,2i-\sigma(L)+3}_{p}(L) \quad (4.3)
\]

Since the support of the Khovanov homology is finite, all \(T^{i,j}_{p}(L)\) must be zero. \(\square\)

APPENDIX

This section contains information about standard and reduced Khovanov homology of knots whose torsion has some remarkable properties. The knot pictures below were generated using Robert Scharein’s program KnotPlot [S].

A.1. How to read the tables. Columns and rows of the tables below are marked with \(i\)- and \(j\)-grading of the Khovanov homology, respectively. For the standard homology, the \(j\)-grading is always odd and the corresponding table entries are printed in boldface. The reduced homology have their \(j\)-grading even. They occupy places between the main rows.

Only entries representing non-trivial groups are shown. An entry of the form \(a, b_2, c_4\) means that the corresponding group is \(\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c\).
A.2. The knot $8_{19}$. This is the first QH-thick knot. It is T-fancy as well.

|    | 0  | 1  | 2  | 3  | 4  | 5  |
|----|----|----|----|----|----|----|
| 17 |    |    |    | 1  |    |    |
| 15 |    |    |    | 1  |    |    |
| 13 |    | 1  |    | 1  | 1  |    |
| 11 |    | 1  | 1  | 1  |    | 1  |
|  9 | 1  |    |    |    |    |    |
|  7 | 1  |    |    |    |    |    |
|  5 | 1  |    |    |    |    |    |

Table 1. The knot $8_{19}$ and its Khovanov homology (standard and reduced).

A.3. The knot $9_{42}$. This knot is QH-thick but T-simple. Conjecture 2 states that every T-fancy knot should be H-thick as well. This example shows that the converse is not true in general.

|    | -4 | -3 | -2 | -1 |  0 |  1 |  2 |
|----|----|----|----|----|----|----|----|
|  7 |    |    |  1 |    |    |    |    |
|  5 |    |    |    |    |  1 |  1 |  1 |
|  3 |    |    |    |  1 |  1 |    |    |
|  1 |    |    |  1 |  1 |    |  1 |  1 |
| -1 |    |    |    |    |  1 |  1 |  1 |
| -3 |    | -1 |  1 |  1 |    |    |    |
| -5 | -1 |    | -1 |    |    |    |    |
| -7 |    |    |  1 |  1 |    |    |    |

Table 2. The knot $9_{42}$ and its Khovanov homology (standard and reduced).
A.4. The knot $13_{3663}^n$. This is the first T-rich knot (the groups with excessive torsion are $\mathcal{H}^{-3,-7}$, $\mathcal{H}^{-3,-5}$, $\mathcal{H}^{-2,-5}$, $\mathcal{H}^{-2,-3}$, $\mathcal{H}^{0,-1}$, $\mathcal{H}^{0,1}$, $\mathcal{H}^{1,1}$, and $\mathcal{H}^{1,3}$). This knot has 2-torsion in the reduced homology as well. This supports the claim of Conjecture 3 that a knot is T-rich if and only if its reduced Khovanov homology has torsion. This knot is also the first one whose homology is supported on 4 diagonals. The only other knots with 13 crossings or less that share the same properties are $13_{4587}^n$, $13_{4639}^n$, and $13_{5016}^n$.

|   | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|----|----|----|----|----|----|---|---|---|---|---|---|---|---|
| 13 |    |    |    |    |    |    |   |   | 1 |    |    |    |    |    |
| 11 |    |    |    |    |    |    |   | 1 |    |    |    |    |    |    |
| 9  |    |    |    |    |    |    |   | 1 | 1 |    |    |    |    |    |
| 7  |    |    |    |    |    |    |   | 1 | 1 | 1 |    |    |    |    |
| 5  |    |    |    |    |    |    |   | 1 | 1 | 2 |    |    |    |    |
| 3  |    |    |    |    |    |    |   | 1 | 1 |    | 2, 1 | 1 |    |    |
| 1  |    |    |    |    |    |    |   | 1 | 1 | 1 |    | 1, 1 | 1 |    |
| -1 |    |    |    |    |    |    |   | 1 | 1 | 1 | 1 |    |    |    |
| -3 |    |    |    |    | 2 |    |   | 1 | 1 | 1 | 1 | 1 | 1 |    |
| -5 |    |    | 1 | 1 | 1 |    |   | 1 | 1 | 1 | 1 |    |    |    |
| -7 | 1  | 1  | 1  | 1  | 1  | 1  |   | 1  |    |    |    |    |    |    |
| -9 | 1  | 1  |    |    |    |    |   | 1  |    |    |    |    |    |    |
| -11| 1  |    |    |    |    |    |   | 1  |    |    |    |    |    |    |

Table 3. Standard and reduced Khovanov homology of the knot $13_{3663}^n$. 
A.5. The (4,5)-torus knot. This is one of the first knots whose Khovanov homology has torsion of order 4. Its minimal diagram has 15 crossings. There are no knots with 13 crossings or less that have torsion of order other than 2. This knot is also T-rich and has 2-torsion in reduced homology (cf. Conjecture 3).

|    | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| 29 | 1  | 1  | 1  |    |    |    |    |    |    |    |    |
| 27 | 1  | 1  |    |    |    |    |    |    |    |    |    |
| 25 | 1  |    |    |    | 1  | 1  |    |    |    |    |    |
| 23 | 1  | 1  | 1  | 1  |    | 1  |    |    |    |    |    |
| 21 | 1  | 1  | 1  |    | 1  |    |    |    |    |    |    |
| 19 | 1  | 1  |    | 1  |    |    | 1  |    |    |    |    |
| 17 | 1  |    | 1  |    |    | 1  |    |    |    |    |    |
| 15 | 1  | 1  |    |    |    |    |    | 1  |    |    |    |
| 13 |    |    | 1  |    |    |    |    |    |    |    |    |
| 11 |    |    |    | 1  |    |    |    |    |    |    |    |

Table 4. Standard and reduced Khovanov homology of the (4,5)-torus knot.

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DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, PHILLIPS HALL, 801 22nd ST. NW, SUITE #739, WASHINGTON, DC 20052, U.S.A.
E-mail address: Shurik@gwu.edu