Passivity and Microlocal Spectrum Condition

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Abstract. In the setting of vector-valued quantum fields obeying a linear wave-equation in a globally hyperbolic, stationary spacetime, it is shown that the two-point functions of passive quantum states (mixtures of ground- or KMS-states) fulfill the microlocal spectrum condition (which in the case of the canonically quantized scalar field is equivalent to saying that the two-point function is of Hadamard form). The fields can be of bosonic or fermionic character. We also give an abstract version of this result by showing that passive states of a topological ∗-dynamical system have an asymptotic pair correlation spectrum of a specific type.

1 Introduction

A recurrent theme in quantum field theory in curved spacetime is the selection of suitable states which may be viewed as generalizations of the vacuum state familiar from quantum field theory in flat spacetime. The selection criterion for such states should, in particular, reflect the idea of dynamical stability under temporal evolution of the system. If a spacetime possesses a time-symmetry group (generated by a timelike Killing vector field), then a ground state with respect to the corresponding time-evolution appears as a good candidate for a vacuum-like state. More generally, any thermal equilibrium state for that time-evolution should certainly also be viewed as a dynamically stable state. Ground- and thermal equilibrium states, and mixtures thereof, fall into the class of the so-called “passive” states, defined in [34]. An important result by Pusz and Woronowicz [33] asserts that a dynamical system is in a passive state exactly if it is impossible to extract energy from the system by means of cyclic processes. Since the latter form of passivity, i.e. the validity of the second law of thermodynamics, expresses a thermodynamical stability which is to be expected to hold generally for physical dynamical systems, one would expect that passive states are natural candidates for physical (dynamically stable) states in quantum field theory in curved spacetime, at least when the spacetime, or parts of it, posses time-symmetry groups. This point of view has been expressed in [3].
In this work we study the relationship between passivity of a quantum field state and the microlocal spectrum condition for free quantum fields on a stationary, globally hyperbolic spacetime. The microlocal spectrum condition (abbreviated, \( \mu SC \)) is a condition restricting the form of the wavefront sets, \( WF(\omega_n) \), of the \( n \)-point distributions \( \omega_n \) of a quantum field state [4, 35]. For quasifree states, it suffices to restrict the form of \( WF(\omega_2) \); see relation (1.1) near the end of this Introduction for a definition of \( \mu SC \) in this case.

There are several reasons why the \( \mu SC \) may rightfully be viewed as an appropriate generalization of the spectrum condition (i.e. positivity of the energy in any Lorentz frame), required for quantum fields in flat spacetime, to quantum field theory in curved spacetime. Among the most important is the proof by Radzikowski [35] (based on mathematical work by Duistermaat and Hörmander [12]) that, for the free scalar Klein-Gordon field on any globally hyperbolic spacetime, demanding that the two-point function \( \omega_2 \) obeys the \( \mu SC \) is equivalent to \( \omega_2 \) being of Hadamard form. This is significant since it appears nowadays well-established to take the condition that \( \omega_2 \) be of Hadamard form as criterion for physically (dynamically stable) quasifree states for linear quantum fields on curved spacetime in view of a multitude of results, cf. e.g. [15, 14, 31, 42, 43, 47] and references given therein. Moreover, \( \mu SC \) has several interesting structural properties which are quite similar to those of the usual spectrum condition, and allow to some extent similar conclusions [1, 3, 14]. It is particularly worth mentioning that one may, in quasifree states of linear quantum fields fulfilling \( \mu SC \), covariantly define Wick-products and develop the perturbation theory for \( P(\phi) \)-type interactions along an Epstein-Glaser approach generalized to curved spacetime [3, 7]. Also worth mentioning is the fact that \( \mu SC \) has proved useful in the analysis of other types of problems in quantum field theory in curved spacetime [36, 30, 13].

In view of what we said initially about the significance of the concept of passivity for quantum field states on stationary spacetimes one would be inclined to expect that, on a stationary, globally hyperbolic spacetime, a passive state fulfills the \( \mu SC \), at least for quasifree states of linear fields. And this is what we are going to establish in the present work.

We should like to point out that more special variants of such a statement have been established earlier. For the scalar field obeying the Klein-Gordon equation on a globally hyperbolic, static spacetime, Fulling, Narcowich and Wald [16] proved that the quasifree ground state with respect to the static Killing vector field has a two-point function of Hadamard form, and thus fulfills \( \mu SC \), as long as the norm of the Killing vector field is globally bounded away from zero. Junker [26] has extended this result by showing that, if the spacetime has additionally compact spatial sections, then the quasifree KMS-states (thermal equilibrium states) at any finite temperature fulfill \( \mu SC \). But the requirement of having compact Cauchy-surfaces, or the constraint that the static Killing vector field have a norm bounded globally away from zero, exclude several interesting situations from applying the just mentioned results. A prominent example is Schwarzschild spacetime, which possesses a static timelike Killing flow, but the norm of the Killing vector field tends to zero as one approaches the horizon along any Cauchy-surface belonging to the static foliation. In [28] (cf. also [17]), quasifree ground- and KMS-states with respect to the Killing flow on Schwarzschild spacetime have been constructed for the scalar Klein-Gordon field, and it has long been conjectured that the two-point functions of these states...
are of Hadamard form. However, when trying to prove this along the patterns of [16] or [26], who use the formulation of quasifree ground- and KMS-states in terms of the Klein-Gordon field’s Cauchy-data, one is faced with severe infra-red problems even for massive fields upon giving up the constraint that the norm of the static Killing vector field be globally bounded away from zero. This has called for trying to develop a new approach to proving $\mu$SC for passive states, the result of which is our Theorem 5.1; see further below in this Introduction for a brief description. As a corollary, our Thm. 5.1 shows that the quasifree ground- and KMS-states of the scalar Klein-Gordon field on Schwarzschild spacetime satisfy $\mu$SC (thus their two-point functions are of Hadamard form). [We caution the reader that this does not show that these states or rather, their “doublings” defined in [28], were extendible to Hadamard states on the whole of the Schwarzschild-Kruskal spacetime. There can be at most one single quasifree, isometry-invariant Hadamard state on Schwarzschild-Kruskal spacetime and this state necessarily restricts to a KMS-state at Hawking temperature on the (“outer, right”–) Schwarzschild-part of Schwarzschild-Kruskal spacetime, cf. [31, 29].]

We turn to summarizing the contents of the present work. In Chapter 2, we will introduce the notion of “asymptotic pair correlation spectrum” of a state $\omega$ of a topological $\ast$-dynamical system. This object is to be viewed as a generalization of the wavefront set of the two-point function $\omega_2$ in the said general setting, see [14] for further discussion. We then show that for (strictly) passive states $\omega$ the asymptotic pair correlation spectrum must be of a certain, asymmetric form. This asymmetry can be interpreted as the microlocal remnant of the asymmetric form of the spectrum that one would obtain for a ground state.

Chapter 3 will be concerned with some aspects of wavefront sets of distributions on test-sections of general vector bundles. Sec. 3.1 contains a reformulation of the wavefront set for vector-bundle distributions along the lines of Prop. 2.2 in [44]. We briefly recapitulate some notions of spacetime geometry, as far as needed, in Sec. 3.2. In Sec. 3.3 we quote the propagation of singularities theorem (PST) for wave-operators acting on vector bundles, in the form used later in Chap. 5, from [8, 12].

In Sec. 4.1 we introduce, following [32], the Borchers algebra of smooth test-sections with compact support in a vector bundle over a Lorentzian spacetime, and briefly summarize the connection between states on the Borchers algebra, their GNS-representations, the induced quantum fields, and the Wightman $n$-point functions. We require that the quantum fields associated with the states are, in a weak sense, bosonic or fermionic, i.e. they fulfill a weak form of (twisted) locality. A quite general formulation of (bosonic or fermionic) quasifree states will be given in Sec. 4.3.

Chapter 5 contains our main result, saying that for a state $\omega$ on the Borchers algebra associated with a given vector bundle, over a globally hyperbolic, stationary spacetime $(M, g)$ as base manifold, the properties

(i) $\omega$ is (strictly) passive,

(ii) $\omega$ fulfills a weak form of (twisted) locality, and

(iii) $\omega_2$ is a bi-solution up to $C^\infty$ for a wave operator,
imply

$$\text{WF}(\omega_2) \subset \mathcal{R},$$  \hspace{1cm} (1.1)

where $\mathcal{R}$ is the set of pairs of non-zero covectors $(q, \xi; q', \xi') \in T^* M \times T^* M$ so that $g^{\mu\nu}\xi_{\nu}$ is past-directed and lightlike, the base points $q$ and $q'$ are connected by an affinely parametrized, lightlike geodesic $\gamma$, and both $\xi$ and $-\xi'$ are co-tangent to $\gamma$, or $\xi = -\xi'$ if $q = q'$.

Following [7], we say that the quasifree state with two-point function $\omega_2$ fulfills the $\mu\text{SC}$ if the inclusion (1.1) holds. If one had imposed the additional requirement that $\omega$ (resp., the associated quantum fields) fulfill appropriate vector-bundle versions of the CCR or CAR, one would conclude that

$$\text{WF}(\omega_2) = \mathcal{R},$$

as is e.g. the case for the free scalar Klein-Gordon field (cf. [35]). Moreover, for a quasifree state $\omega$ on the Borchers algebra of a vector bundle over any globally hyperbolic spacetime one can show that imposing CCR or CAR implies that $\omega_2$ is of Hadamard form (appropriately generalized) if and only if $\text{WF}(\omega_2) = \mathcal{R}$. The discussion of these matters will be contained in a separate article [38].

2 Passivity and Asymptotic Pair Correlation Spectrum

Let $\mathcal{A}$ be a $C^*$-algebra with unit and $\{\alpha_t\}_{t \in \mathbb{R}}$ a one-parametric group of automorphisms of $\mathcal{A}$, supposed to be strongly continuous, that is, $||\alpha_t(A) - A|| \to 0$ as $t \to 0$ for each $A \in \mathcal{A}$. Moreover, let $D(\delta)$ denote the set of all $A \in \mathcal{A}$ such that the limit

$$\delta(A) := \lim_{t \to 0} \frac{1}{t}(\alpha_t(A) - A)$$

exists. One can show that $D(\delta)$ is a dense $*$-subalgebra of $\mathcal{A}$, and $\delta$ is a derivation with domain $D(\delta)$.

Following [34], one calls a state $\omega$ on $\mathcal{A}$ passive if for all unitary elements $U \in D(\delta)$ which are continuously connected to the unit element,\footnote{i.e. there exists a continuous curve $[0, 1] \ni t \mapsto U(t) \in D(\delta)$ with each $U(t)$ unitary and $U(0) = 1_A$, $U(1) = U$.} the estimate

$$\omega(U^* \frac{1}{t}\delta(U)) \geq 0$$  \hspace{1cm} (2.1)

is fulfilled. As a consequence, $\omega$ is invariant under $\{\alpha_t\}_{t \in \mathbb{R}}$: $\omega \circ \alpha_t = \omega$ for all $t \in \mathbb{R}$. Furthermore, it can be shown (cf. [34]) that ground states or KMS-states at inverse temperature $\beta \geq 0$ for $\alpha_t$ are passive, as are convex sums of such states. (In Appendix A we will summarize some basic properties of ground states and KMS-states. Standard references include [3, 39].)
However, the significance of passive states is based on two remarkable results in [34]. First, a converse of the previous statement is proven there: If a state is completely passive, then it is a ground state or a KMS-state at some inverse temperature \( \beta \geq 0 \). Here a state is called **completely passive** if, for each \( n \in \mathbb{N} \), the product state \( \otimes^n \omega \) is a passive state on \( \otimes^n \mathcal{A} \) with respect to the dynamics \( \{ \otimes^n \alpha_t \}_{t \in \mathbb{R}} \).

Secondly, the following is established in [34]: the dynamical system modelled by \( \mathcal{A} \) and \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is in a passive state precisely if it is impossible to extract energy from the system by means of cyclic processes. In that sense, passive states may be viewed as good candidates for physically realistic states of any dynamical system since for these states the second law of thermodynamics is warranted.

In the present section we are interested in studying the asymptotic high frequency behaviour of passive states along similar lines as developed recently in [13]. We shall, however, generalize the setup since this will prove useful for developments later in this work. Thus, we assume now that \( \mathcal{A} \) is a topological *-algebra with a locally convex topology and with a unit element (cf. e.g. [40]). We denote by \( S \) the set of continuous semi-norms on \( \mathcal{A} \).

Moreover, we say that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is a continuous one-parametric group of *-automorphisms of \( \mathcal{A} \) if for each \( t \), \( \alpha_t \) is a topological *-automorphism of \( \mathcal{A} \), and if the group action is locally bounded and continuous in the sense that for each \( \sigma \in S \) there is \( \sigma' \in S \), \( r > 0 \) with \( \sigma(\alpha_t(A)) \leq \sigma'(A) \) for all \( |t| < r \), \( A \in \mathcal{A} \), and \( \sigma(\alpha_t(A) - A) \to 0 \) as \( t \to 0 \) for each \( A \in \mathcal{A} \). Then we refer to the pair \( (\mathcal{A}, \{ \alpha_t \}_{t \in \mathbb{R}}) \) as a **topological *-dynamical system**. Using the fact that for all \( A,B \in \mathcal{A} \) and \( \sigma \in S \), the maps \( C \mapsto \sigma(AC) \) and \( C \mapsto \sigma(CB) \) are again continuous semi-norms on \( \mathcal{A} \), one deduces by a standard argument that also \( \sigma(\alpha_s(A)\alpha_t(B) - AB) \to 0 \) as \( s,t \to 0 \).

A continuous linear functional \( \omega \) on \( \mathcal{A} \) will be called a state if \( \omega(A^*A) \geq 0 \) for all \( A \in \mathcal{A} \) and if \( \omega(1_A) = 1 \). Furthermore, we say that \( \omega \) is a ground state, or a KMS-state at inverse temperature \( \beta > 0 \), for \( \{ \alpha_t \}_{t \in \mathbb{R}} \), if the functions \( t \mapsto \omega(A\alpha_t(B)) \) are bounded for all \( A,B \in \mathcal{A} \), and if \( \omega \) satisfies the ground state condition (A.1) or the KMS-condition (A.2) given in Appendix A, respectively.

Now we call a family \( (A_\lambda)_{\lambda \geq 0} \) with \( A_\lambda \in \mathcal{A} \) a **global testing family** in \( \mathcal{A} \) provided there is for each \( \sigma \in S \) an \( s \geq 0 \) (depending on \( \sigma \) and on the family) such that

\[
\sup_\lambda \lambda^s \sigma(A_\lambda^*A_\lambda) < \infty.
\]

The set of all global testing families will be denoted by \( \mathbf{A} \).

Let \( \omega \) be a state on \( \mathcal{A} \), and \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} \). Then we say that \( \xi \) is a regular direction for \( \omega \), with respect to the continuous one-parametric group \( \{ \alpha_t \}_{t \in \mathbb{R}} \), if there exists some \( h \in C^\infty_0(\mathbb{R}^2) \) and an open neighbourhood \( V \) of \( \xi \) in \( \mathbb{R}^2 \setminus \{0\} \) such that\[^2\]

\[
\sup_{k \in V} \left| \int e^{-i\lambda t - k \cdot h(t)} \omega\left(\alpha_{t_1}(A_\lambda)\alpha_{t_2}(B_\lambda)\right) dt \right| = O^\infty(\lambda) \quad \text{as } \lambda \to 0
\]

holds for all global testing families \( (A_\lambda)_{\lambda \geq 0}, (B_\lambda)_{\lambda \geq 0} \in \mathbf{A} \).

\[^2\]We shall write \( \varphi(\lambda) = O^\infty(\lambda) \) as \( \lambda \to 0 \) iff for each \( s \in \mathbb{N} \) there are \( C_s, \lambda_s > 0 \) so that \( |\varphi(\lambda)| \leq C_s \lambda^s \) for all \( 0 < \lambda < \lambda_s \).
Then we define the set \( \text{ACS}_A^2(\omega) \) as the complement in \( \mathbb{R}^2 \setminus \{0\} \) of all \( k \) which are regular directions for \( \omega \). We call \( \text{ACS}_A^2(\omega) \) the \textit{global asymptotic pair correlation spectrum} of \( \omega \). The asymptotic pair correlation spectrum, and more generally, asymptotic \( n \)-point correlation spectra of a state, may be regarded as generalizations of the notion of wavefront set of a distribution in the setting of states on a dynamical system. We refer to [44] for considerable further discussion and motivation. The properties of \( \text{ACS}_A^2(\omega) \) are analogous to those of \( \text{ACS}^2(\omega) \) described in [44, Prop. 3.2]. In particular, \( \text{ACS}_A^2(\omega) \) is a closed conic set in \( \mathbb{R}^2 \setminus \{0\} \). It is evident that, if \( \omega \) is a finite convex sum of states \( \omega_i \), then \( \text{ACS}_A^2(\omega) \) is contained in \( \bigcup \text{ACS}_A^2(\omega_i) \).

Now we are going to establish an upper bound for \( \text{ACS}_A^2(\omega) \), distinguished by a certain asymmetry, for all \( \omega \) in a subset \( \mathcal{P} \) of the set of all passive states, to be defined next:

We define \( \mathcal{P} \) as the set of all states on \( \mathcal{A} \) which are of the form

\[
\omega(A) = \sum_{i=1}^{m} \rho_i \omega_i(A), \quad A \in \mathcal{A},
\]

where \( m \in \mathbb{N} \), \( \rho_i > 0 \), \( \sum_{i=1}^{m} \rho_i = 1 \), and each \( \omega_i \) is a ground state or a KMS-state at some inverse temperature \( \beta_i > 0 \) (note that \( \beta_i = 0 \) is not admitted) on \( \mathcal{A} \) with respect to \( \{\alpha_t\}_{t \in \mathbb{R}} \). The states in \( \mathcal{P} \) will be called \textit{strictly passive}.

We should like to remark that in the present general setting where \( \mathcal{A} \) is not necessarily a \( C^* \)-algebra, the criterion for passivity given at the beginning in (2.1) may be inappropriate since it could happen that \( D(\delta) \), even if dense in \( \mathcal{A} \), doesn’t contain sufficiently many unitary elements. In the \( C^* \)-algebraic situation, (2.1) entails the slightly weaker variant

\[
\omega(A - \delta(A)) \geq 0
\]

for all \( A = A^* \in D(\delta) \), and one may take this as substitute for the condition of passivity of a state in the present more general framework (supposing that \( D(\delta) \) is dense). In fact, each \( \omega \in \mathcal{P} \) is \( \{\alpha_t\}_{t \in \mathbb{R}} \)-invariant and satisfies (2.7) (see Appendix A), and in the \( C^* \)-algebraic situation, every \( \omega \in \mathcal{P} \) also satisfies (2.1).

**Proposition 2.1.** Let \( (\mathcal{A}, \{\alpha_t\}_{t \in \mathbb{R}}) \) be a topological \(*\)-dynamical system as described above.

1. Let \( \omega \in \mathcal{P} \). Then
   - either \( \text{ACS}_A^2(\omega) = \emptyset \),
   - or \( \text{ACS}_A^2(\omega) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_1 + \xi_2 = 0, \xi_2 \geq 0\} \).

2. Let \( \omega \) be an \( \{\alpha_t\}_{t \in \mathbb{R}} \)-invariant KMS-state at inverse temperature \( \beta = 0 \). Then
   - either \( \text{ACS}_A^2(\omega) = \emptyset \),
   - or \( \text{ACS}_A^2(\omega) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_1 + \xi_2 = 0\} \).

**Proof.** 1.) By assumption \( \omega \) is continuous, hence we can find a seminorm \( \sigma \in S \) so that \( |\omega(A)| \leq \sigma(A) \) for all \( A \in \mathcal{A} \). Thus there are positive constants \( c \) and \( s \) so that

\[
\omega(\alpha_t(A^*_A A)) = \omega(A^*_A A) \leq c \cdot (1 + \lambda^{-1})^s
\]

(2.6)
holds for all \( t \in \mathbb{R} \). In the first equality, the invariance of \( \omega \) was used, and in the second, condition (2.2) was applied. Thus, for any Schwartz-function \( \hat{h} \in \mathcal{S}(\mathbb{R}^2) \), and any \((A_\lambda)_{\lambda>0}\), \((B_\lambda)_{\lambda>0}\) in \( A \), one obtains that the following function holds for all \( \lambda > 0 \) and \( k \in \mathbb{R}^2 \),

\[
w_\lambda(k) := \int e^{-ik\cdot\hat{h}(t)}\omega(\alpha_{t_1}(A_\lambda)\alpha_{t_2}(B_\lambda)) \, dt
\]

depends smoothly on \( k \) and satisfies the estimate

\[
|w_\lambda(\lambda^{-1}k)| \leq c'(|k| + \lambda^{-1} + 1)^r
\]

with suitable constants \( c' > 0, r \in \mathbb{R} \). Hence, this function satisfies the assumptions of Lemma 2.2 in [44]. Application of the said Lemma entails the following: Suppose that for any open neighbourhood \( V \) of \( \xi \in \mathbb{R}^2 \setminus \{0\} \) we can find some \( \hat{h} \in \mathcal{S}(\mathbb{R}^2) \) with \( \hat{h}(0) = 1 \) and

\[
\sup_{k \in V} \left| \int e^{-i\lambda^{-1}k\cdot\hat{h}(t)}\omega(\alpha_{t_1}(A_\lambda)\alpha_{t_2}(B_\lambda)) \, dt \right| = O^\infty(\lambda) \quad \text{as } \lambda \to 0 \tag{2.7}
\]

for all \((A_\lambda)_{\lambda>0},(B_\lambda)_{\lambda>0} \in A \). Then this implies that the analogous relation holds with \( \hat{h} \) replaced by \( \phi \cdot \hat{h} \) for any \( \phi \in C_0^\infty(\mathbb{R}^2) \) when simultaneously \( V \) is replaced by some slightly smaller neighbourhood \( V' \) of \( \xi \). Consequently, relation (2.7) — with \( \hat{h} \in \mathcal{S}(\mathbb{R}^2), \hat{h}(0) = 1 \) — entails that \( \xi \) is absent from \( ACS^2_A(\omega) \).

2.) Some notation needs to be introduced before we can proceed. For \( f \in \mathcal{S}(\mathbb{R}) \), we define

\[
(\tau_s f)(s') := f(s' - s) \quad \text{and} \quad f'(s') := f(-s'), \quad s, s' \in \mathbb{R}.
\]

Then we will next establish

\[
\omega \circ \alpha_t = \omega \quad \Rightarrow \quad ACS^2_A(\omega) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}: \xi_1 + \xi_2 = 0\}. \tag{2.8}
\]

To this end, let \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} \) be such that \( \xi_1 + \xi_2 \neq 0 \), and pick some \( \delta > 0 \) and an open neighbourhood \( V_\xi \) of \( \xi \) so that \( |k_1 + k_2| > \delta \) for all \( k \in V_\xi \).

Now pick two functions \( h_j \in C_0^\infty(\mathbb{R}) \) \((j = 1, 2)\) such that their Fourier-transforms \( \hat{h}_j(t_j) = \frac{1}{\sqrt{2\pi}} \int e^{-i\cdot\hat{h}_j(p)} \, dp \) have the property \( \hat{h}_j(0) = 1 \). Define \( \hat{h} \in \mathcal{S}(\mathbb{R}^2) \) by \( \hat{h}(t) := \hat{h}_1(t_1)\hat{h}_2(t_2) \).

Then observe that one can find \( \lambda_0 > 0 \) such that the functions

\[
g_{\lambda,k}(p) := ((\tau_{-\lambda^{-1}(k_1+k_2)}\hat{h}_1) \cdot \hat{h}_2)(p), \quad p \in \mathbb{R}, \tag{2.9}
\]

vanish for all \( k = (k_1, k_2) \in V_\xi \) and all \( 0 < \lambda < \lambda_0 \). Consequently, also the functions

\[
f_{\lambda,k}(p) := (\tau_{-\lambda^{-1}k_2}g_{\lambda,k})(p), \quad p \in \mathbb{R}, \tag{2.10}
\]

vanish for all \( k \in V_\xi \) and all \( 0 < \lambda < \lambda_0 \). Denoting the Fourier-transform of \( f_{\lambda,k} \) by \( \hat{f}_{\lambda,k} \), one obtains for all \( k \in V_\xi, 0 < \lambda < \lambda_0 \):

\[
0 = \int \hat{f}_{\lambda,k}(s) \omega(A_\lambda\alpha_s(B_\lambda)) \, ds = \int e^{-i\lambda^{-1}(k_1+k_2)s'} e^{-i\lambda^{-1}k_2s'} \hat{h}_1(s') \hat{h}_2(s' + s) \omega(A_\lambda\alpha_s(B_\lambda)) \, ds' \, ds = \int e^{-i\lambda^{-1}k\cdot\hat{h}(t)} \omega(\alpha_{t_1}(A_\lambda)\alpha_{t_2}(B_\lambda)) \, dt
\]
for all testing-families \((A_\lambda)_{\lambda > 0}, (B_\lambda)_{\lambda > 0}\). Invariance of \(\omega\) under \(\{\alpha_t\}_{t \in \mathbb{R}}\) was used in passing from the second equality to the last. In view of step 1.) above, this shows (2.8).

3.) In a further step we will argue that

\[
\omega \text{ ground state } \Rightarrow ACS^2_A(\omega) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_2 \geq 0\}. \quad (2.11)
\]

So let again \(h_j\) and \(\hat{h}_j\) as above, and \(f_{\lambda, k}\) as in (2.10) with Fourier-transform \(\hat{f}_{\lambda, k}\). Let \(\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}\) have \(\xi_2 < 0\). Then there is an open neighbourhood \(V_\xi\) of \(\xi\) and an \(\epsilon > 0\) so that \(k_2 < -\epsilon\) for all \(k \in V_\xi\). The support of \(f_{\lambda, k}\) is contained in the support of \(\tau_{\lambda^{-1}k_2}h_2\), and there is clearly some \(\lambda_0 > 0\) such that \(\text{supp } \tau_{\lambda^{-1}k_2}h_2 \subset (-\infty, 0)\) for all \(k = (k_1, k_2) \in V_\xi\) as soon as \(0 < \lambda < \lambda_0\). By the characterization of a ground state given in (A.1), and using also the \(\{\alpha_t\}_{t \in \mathbb{R}}\)-invariance of a ground state, one therefore obtains

\[
\sup_{k \in V_\xi} \left| \int e^{-i \lambda^{-1}k \cdot t} \hat{h}_j(t) \omega(A_\lambda \alpha_t(B_\lambda)) \, dt \right| = \sup_{k \in V_\xi} \left| \int \hat{f}_{\lambda, k}(s) \omega(A_\lambda \alpha_s(B_\lambda)) \, ds \right| = 0 \quad \text{if } 0 < \lambda < \lambda_0
\]

for all \((A_\lambda)_{\lambda > 0}, (B_\lambda)_{\lambda > 0} \in A\). Relation (2.11) is thereby proved.

4.) Now we turn to the case

\[
\omega \text{ KMS at } \beta > 0 \Rightarrow ACS^2_A(\omega) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_2 \geq 0\}. \quad (2.12)
\]

Consider a \(\xi \in \mathbb{R}^2 \setminus \{0\}\) with \(\xi_2 < 0\) and pick some \(\epsilon > 0\) and an open neighbourhood \(V_\xi\) of \(\xi\) so that \(k_2 < -\epsilon\) for all \(k = (k_1, k_2) \in V_\xi\). Choose again \(h_j\) and \(\hat{h}_j\) as above and define correspondingly \(g_{\lambda, k}\) and \(f_{\lambda, k}\) as in (2.9) and (2.10), respectively. Denote again their Fourier-transforms by \(\hat{g}_{\lambda, k}\) and \(\hat{f}_{\lambda, k}\). Note that \(g_{\lambda, k}\) and \(f_{\lambda, k}\) are in \(C^\infty_0(\mathbb{R})\) for all \(\lambda > 0\) and all \(k \in \mathbb{R}^2\), so their Fourier-transforms are entire analytic. Moreover, a standard estimate shows that

\[
\sup_{\lambda > 0, \mathbb{R}^2} \int |\hat{g}_{\lambda, k}(s + i \beta)| \, ds \leq c' < \infty. \quad (2.13)
\]

One calculates

\[
\hat{f}_{\lambda, k}(s + i \beta) = e^{\lambda^{-1}k_2\beta} e^{-i \lambda^{-1}k_2s} \hat{g}_{\lambda, k}(s + i \beta), \quad s \in \mathbb{R},
\]

and now the KMS-condition (A.2) yields for all \((A_\lambda)_{\lambda > 0}, (B_\lambda)_{\lambda > 0} \in A,\)

\[
\left| \int \hat{f}_{\lambda, k}(s) \omega(A_\lambda \alpha_s(B_\lambda)) \, ds \right| = \left| e^{\lambda^{-1}k_2\beta} \int e^{-i \lambda^{-1}k_2s} \hat{g}_{\lambda, k}(s + i \beta) \omega(\alpha_s(B_\lambda)A_\lambda) \, ds \right| \\
\leq e^{\lambda^{-1}k_2\beta} c' \cdot c'' (1 + \lambda^{-1}) s', \quad \lambda > 0, \ k \in \mathbb{R}^2,
\]

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for suitable \(c'', s' > 0\), where (2.13) and (2.14) have been used. Making also use of the \(\{\alpha_t\}_{t \in \mathbb{R}}\)-invariance of \(\omega\) one finds, with suitable \(\gamma > 0\),

\[
\sup_{k \in V_T} \left| \int e^{-i\lambda^{-1}k \cdot t} \hat{h}(t) \omega(\alpha_{t_1}(A_\lambda)\alpha_{t_2}(B_\lambda)) \, dt \right| \\
= \sup_{k \in V_T} \left| \int \hat{f}_{\lambda,k}(s) \omega(A_\lambda\alpha_s(B_\lambda)) \, ds \right| \\
\leq \gamma e^{-\lambda^{-1}e^\beta(1 + \lambda^{-1}) s'} = O^\infty(\lambda) \quad \text{as } \lambda \to 0
\]

for all \((A_\lambda)_{\lambda > 0}, (B_\lambda)_{\lambda > 0} \in \mathbf{A}\). This establishes statement (2.12).

5.) Combining now the assertions (2.8), (2.11) and (2.12), one can see that for each \(\omega \in \mathcal{P}\) there holds

\[
ACS^2_{\mathbf{A}}(\omega) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_1 + \xi_2 = 0, \xi_2 \geq 0\}.
\]

Since the set on the right-hand side obviously has no proper conic subset in \(\mathbb{R}^2 \setminus \{0\}\), one concludes that statement (1) of the Proposition holds true.

6.) As \(\omega\) is KMS at \(\beta = 0\), this means that it is a trace: \(\omega(AB) = \omega(BA)\). Since \(\omega\) is also \(\{\alpha_t\}_{t \in \mathbb{R}}\)-invariant, we have

\[
ACS^2_{\mathbf{A}}(\omega) \subset \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_1 + \xi_2 = 0\}.
\]

The set on the right hand side has precisely two proper closed conic subsets

\[
W_\pm := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\} : \xi_1 + \xi_2 = 0, \pm \xi_2 \geq 0\}.
\]

These two sets are disjoint, \(W_+ \cap W_- = \emptyset\), and we have \(W_+ = -W_-\). Hence, since \(\omega\) is a trace, one can argue exactly as in [44, Prop. 4.2] to conclude that either \(ACS^2_{\mathbf{A}}(\omega) \subset W_+\) or \(ACS^2_{\mathbf{A}}(\omega) \subset W_-\) imply \(ACS^2_{\mathbf{A}}(\omega) = \emptyset\). This establishes statement (2) of the Proposition. \(\square\)

Hence we see that strict passivity of \(\omega\) results in its \(ACS^2_{\mathbf{A}}(\omega)\) being asymmetric. This is due to the fact that, roughly speaking, the negative part of the spectrum of the unitary group implementing \(\{\alpha_t\}_{t \in \mathbb{R}}\) in such a state is suppressed by an exponential weight factor. It is worth noting that this asymmetry is not present for KMS-states at \(\beta = 0\). Such states at infinite temperature would hardly be regarded as candidates for physical states, and they can be ruled out by the requirement that \(ACS^2_{\mathbf{A}}(\omega)\) be asymmetric.

**Remark 2.2.** One can modify or, effectively, enlarge the set of testing families by allowing a testing family to depend on additional parameters: Define \(\mathbf{A}^s\) as the set of all families \((A_{y,\lambda})_{\lambda > 0, y \in \mathbb{R}^m}\) where \(m \in \mathbb{N}\) is arbitrary (and depends on the family) having the property that for each semi-norm \(\sigma \in S\) there is an \(s \geq 0\) (depending on \(\sigma\) and on the family) such that

\[
\sup_{\lambda, y} \lambda^s \sigma(A_{y,\lambda}^* A_{y,\lambda}) < \infty. \quad (2.14)
\]
Then the definition of a regular direction $k \in \mathbb{R}^2 \setminus \{0\}$ for a state $\omega$ of the dynamical system $(A, \{\alpha_t\}_{t \in \mathbb{R}})$ may be altered through declaring $\xi$ a regular direction iff there are an open neighbourhood $V$ of $\xi$ and a function $h \in C_0^\infty(\mathbb{R}^2)$, $h(0) = 1$, so that

$$\sup_{k \in V} \sup_{y,z} \left| \int e^{-i\lambda^{-1}k \cdot t} h(t) \omega(\alpha_{t_1}(A_{y,\lambda})\alpha_{t_2}(B_{z,\lambda})) \, dt \right| = O^\infty(\lambda) \quad \text{as } \lambda \to 0$$

holds for any pair of elements $(A_{y,\lambda})_{\lambda > 0, y \in \mathbb{R}^m}, (B_{z,\lambda})_{\lambda > 0, z \in \mathbb{R}^n}$ in $A^2$. This makes the set of regular directions a priori smaller, and if we define $\text{ACS}^2_{A^2}(\omega)$ as the complement of all $\xi \in \mathbb{R}^2 \setminus \{0\}$ that are regular directions for $\omega$ according to the just given, altered definition then clearly we have, in general, $\text{ACS}^2_{A^2}(\omega) \supset \text{ACS}^2_{A}(\omega)$. However, essentially by repeating — with somewhat more laborious notation — the proof of Prop. [2.1], one can see that the statements of Prop. [2.1] remain valid upon replacing $\text{ACS}^2_{A}(\omega)$ by $\text{ACS}^2_{A^2}(\omega)$. We shall make use of that observation later.

### 3 Wavefront Sets and Propagation of Singularities

#### 3.1 Wavefront Sets of Vectorbundle-Distributions

Let $\mathfrak{X}$ be a $C^\infty$ vector bundle over a base manifold $N$ ($n = \dim N \in \mathbb{N}$) with typical fibre isomorphic to $\mathbb{C}^r$ or to $\mathbb{R}^r$; the bundle projection will be denoted by $\pi_N$. (We note that here and throughout the text, we take manifolds to be $C^\infty$, Hausdorff, 2nd countable, finite dimensional and without boundary.) We shall write $C^\infty(\mathfrak{X})$ for the space of smooth sections of $\mathfrak{X}$ and $C_0^\infty(\mathfrak{X})$ for the subspace of smooth sections with compact support. These spaces can be endowed with locally convex topologies in a like manner as for the corresponding test-function spaces $\mathcal{E}(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$, cf. [4] [10] for details. By $(C^\infty(\mathfrak{X}))'$ and $(C_0^\infty(\mathfrak{X}))'$ we denote the respective spaces of continuous linear functionals, and by $C^\infty(\mathfrak{X}_U)$ the space of all smooth sections in $\mathfrak{X}$ having compact support in the open subset $U$ of $N$.

For later use, we introduce the following terminology. We say that $\rho$ is a local diffeomorphism of some manifold $X$ if $\rho$ is defined on some open subset $U_1 = \text{dom} \rho$ of $X$ and maps it diffeomorphically onto another open subset $U_2 = \text{Ran} \rho$ of $X$. If $U_1 = U_2 = X$, then $\rho$ is a diffeomorphism as usual. Let $\rho$ be a (local) diffeomorphism of the base manifold $N$. Then we say that $R$ is a (local) bundle map of $\mathfrak{X}$ covering $\rho$ if $R$ is a smooth map from $\pi_N^{-1}(\text{dom} \rho)$ to $\pi_N^{-1}(\text{Ran} \rho)$ so that, for each $q$ in $\text{dom} \rho$, $R$ maps the fibre over $q$ linearly into the fibre over $\rho(q)$. If this map is also one-to-one and if $R$ is also a local diffeomorphism, then $R$ will be called a (local) morphism of $\mathfrak{X}$ covering $\rho$. Moreover, let $(\rho_x)_{x \in B}$ be a family of (local) diffeomorphisms of $N$ depending smoothly on $x \in B$ where $B$ is an open neighbourhood of $0 \in \mathbb{R}^s$ for some $s \in \mathbb{N}$. Then we call $(R_x)_{x \in B}$ a family of (local) morphisms of $\mathfrak{X}$ covering $(\rho_x)_{x \in B}$ if each $R_x$ is a morphism of $\mathfrak{X}$ covering $\rho_x$, depending smoothly on $x \in B$.

Note that each bundle map $R$ of $\mathfrak{X}$ covering a (local) diffeomorphism $\rho$ of $N$ induces a (local) action on $C_0^\infty(\mathfrak{X})$ in form of a continuous linear map $R^* : C_0^\infty(\mathfrak{X}_\text{dom} \rho) \to C_0^\infty(\mathfrak{X}_\text{Ran} \rho)$ given by

$$R^* f := R \circ f \circ \rho^{-1}, \quad f \in C_0^\infty(\mathfrak{X}_\text{dom} \rho). \quad (3.1)$$
Given a local trivialization of $\mathfrak{X}$ over some open $U \subset N$, this induces a one-to-one correspondence between $C_0^\infty(\mathfrak{X}_U)$ and $\oplus^r D(U)$, inducing in turn a one-to-one correspondence between $(C_0^\infty(\mathfrak{X}_U))'$ and $\oplus^r D'(U)$. Now let $u \in (C_0^\infty(\mathfrak{X}_U))'$ and let $(u_1, \ldots, u_r) \in \oplus^r D'(U)$ be the corresponding $r$-tuple of scalar distributions on $U$ induced by the local trivialization of $\mathfrak{X}$ over $U$. The wavefront set $WF(u)$ of $u \in (C_0^\infty(\mathfrak{X}_U))'$ may then be defined as the union of the wavefront sets of the components $u_a$, i.e.

$$WF(u) := \bigcup_{a=1}^r WF(u_a), \quad (3.2)$$

cf. $\S$ 3. It is not difficult to check that this definition is, in fact, independent of the choice of local trivialization of $\mathfrak{X}$ over $U$, and thus yields a definition of $WF(u)$ for all $u \in (C_0^\infty(\mathfrak{X}))'$ having the properties familiar of the wavefront set of scalar distributions on the base manifold $N$, so that $WF(u)$ is a conical subset of $T^*N \setminus \{0\}$.

Another characterization of $WF(u)$ may be given in the following way. Let $q \in U$ and $\xi \in T_q^*N \setminus \{0\}$. Choose any chart for $U$ around $q$, thus identifying $q$ with $0 \in \mathbb{R}^n$ and $\xi$ with $\xi \in T_0^*\mathbb{R}^n \equiv \mathbb{R}^n$ via the dual tangent map of the chart. With respect to the chosen coordinates, we introduce

- translations: $\tilde{\rho}_x(y) := y + x$, and
- dilations: $\tilde{\delta}_\lambda(y) := \lambda y$

on a sufficiently small coordinate ball around $y = 0$ and taking $\lambda > 0$ and the norm of $x \in \mathbb{R}^n$ small enough so that the coordinate range isn’t left. Via pulling these actions back with help of the chart they induce families of local diffeomorphisms $(\rho_x)_x \in B$ and $(\delta_\lambda)_{0 < \lambda < \lambda_0}$ of $U$ for sufficiently small index ranges.

Now let $F_q(\mathfrak{X})$ be the set of all families $(f_\lambda)_{\lambda > 0}$ of sections in $\mathfrak{X}$ with

(i) $f_\lambda \in C_0^\infty(\mathfrak{X}_{\delta_\lambda K})$ for some open neighbourhood $K$ of $q$ when $\lambda$ is sufficiently small

(ii) For each continuous seminorm $\sigma$ on $C_0^\infty(\mathfrak{X})$ there is $s \geq 0$ so that $\sup_{\lambda} \lambda^s \sigma(f_\lambda) < \infty$.

With these conventions, we can formulate:

**Lemma 3.1.** $(q, \xi)$ is not contained in $WF(u)$ if and only if the following holds:

For any family $(R_x)_{x \in B}$ of local morphisms of $\mathfrak{X}$ covering $(\rho_x)_{x \in B}$ there is some $h \in D(\mathbb{R}^n)$ with $h(0) = 1$, and an open neighbourhood $V$ of $\xi$ (in $\mathbb{R}^n \equiv T^*_q N$), such that for all $(f_\lambda)_{\lambda > 0} \in F_q(\mathfrak{X})$ one has

$$\sup_{k \in V} \left| \int e^{-i\lambda^{-1} k \cdot x} h(x) u(R^*_x f_\lambda) \, dx \right| = O^\infty(\lambda) \quad \text{as} \quad \lambda \to 0. \quad (3.3)$$

**Proof.** Select a local trivialization of $\mathfrak{X}$ over $U$. With respect to it, there are smooth $GL(r)$-valued functions $(R^*_b(x))_{a,b=1}^r$ of $x$ such that

$$u(R^*_x f_\lambda) = R^*_b(x) u_a(f^*_x \circ \rho^{-1}_x).$$

$^3$We assume that the reader is familiar with the concept of the wavefront set of a scalar distribution, which is presented e.g. in the textbooks $[23]$, $[4]$, $[37]$.

$^4$Summation over repeated indices will be assumed from now on. See also footnote $5$. 

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Now suppose that \((q, \xi)\) is not in WF\((u)\), so that \((q, \xi)\) isn’t contained in any of the WF\((u_a)\). Then, making use of the fact that the wavefront set of a scalar distribution may be characterized in terms of the decay properties of its localized Fourier-transforms in any coordinate chart (cf. [25]) in combination with Prop. 2.1 and Lemma 2.2 in [44], one obtains immediately the relation (3.3). Conversely, assume that (3.3) holds. Since \((R^a_b(x))_{a,b=1}^r\) is in GL\(r\) for each \(x\) and depends smoothly on \(x\), we can find a smooth family \((S_{c}^{b}(x))_{b,c=1}^r\) of functions of \(x\) so that \(S_{c}^{b}(x)R_{b}^a(x) = \delta_c^a\), \(x \in B\). Since (3.3) holds, one may apply Lemma 2.2 of [44] to the effect that for some open neighbourhood \(V'\) of \(\xi\) and for all \((0, \ldots, \varphi_{\lambda}, \ldots, 0)_{\lambda > 0} \in \mathbf{F}_q(\mathcal{X})\) where only the \(c\)-th entry is non-vanishing, one has

\[
\sup_{k \in V'} \left| \int e^{-i\lambda^{-1}k \cdot x} h(x) u_c(\varphi_{\lambda} \circ \rho_x^{-1}) \, dx \right| = \sup_{k \in V'} \left| \int e^{-i\lambda^{-1}k \cdot x} S_{c}^{b}(x) R_{b}^a(x) u_a(\varphi_{\lambda} \circ \rho_x^{-1}) \, dx \right| = O^{\infty}(\lambda) \quad \text{as } \lambda \to 0.
\]

Then one concludes from Prop. 2.1 in [44] that \((q, \xi)\) isn’t contained in WF\((u_c)\) for each \(c = 1, \ldots, r\).

A very useful property is the behaviour of the wavefront set under (local) morphisms of \(\mathcal{X}\). We put on record here the following Lemma without proof, which may be obtained by extending the proof for the scalar case in [25] together with some of the arguments appearing in the proof of Lemma 3.1.

**Lemma 3.2.** Let \(U_1\) and \(U_2\) be open subsets of \(N\), and let \(R : \mathcal{X}_{U_1} \to \mathcal{X}_{U_2}\) be a vector bundle map covering a diffeomorphism \(\rho : U_1 \to U_2\). Let \(u \in \left(C^\infty_0(\mathcal{X}_{U_1})\right)'\). Then it holds that

\[
\WF(R^* u) \subset \mathcal{T}D\rho^{-1}\WF(u),
\]

where \(\mathcal{T}D\rho^{-1}\) denotes the transpose (or dual) of the tangent map of \(\rho^{-1}\). If \(R\) is even a bundle morphism, then the inclusion (3.4) specializes to an equality.

### 3.2 Briefing on Spacetime Geometry

Since several concepts of spacetime geometry are going to play some role lateron, we take the opportunity to introduce them here and establish the corresponding notation. We refer to the standard references [46, 23] for a more thorough discussion and also for definition of some well-established terminology that is not always introduced explicitly in the following.

Let us assume that \((M, g)\) is a spacetime, so that \(M\) is a smooth manifold of dimension \(m \geq 2\), and \(g\) is a Lorentzian metric having signature \((+, -, \ldots, -)\). It will also be assumed that the spacetime is time-orientable, and that a time-orientation has been chosen. Then one introduces, for any subset \(G\) of \(M\), the corresponding future/past sets \(J^\pm(G)\), consisting of all points lying on piecewise smooth, continuous future/past-directed causal curves emanating from \(G\). A subset \(G' \subset M\) is, by definition, *causally separated* from \(G\)
if it has void intersection with $\overline{J^+(G)} \cup \overline{J^-(G)}$. Thus a pair of points $(q, p) \in M \times M$ is called causally separated if $q$ is causally separated from $p$ or vice versa, since this relation is symmetric.

A smooth hypersurface $\Sigma$ in $M$ is called a Cauchy-surface if each inextendible causal curve in $(M, g)$ intersects $\Sigma$ exactly once. Spacetimes $(M, g)$ possessing Cauchy-surfaces are called globally hyperbolic. It can be shown that a globally hyperbolic spacetime admits smooth one-parametric foliations into Cauchy-surfaces.

Globally hyperbolic spacetimes have a very well-behaved causal structure. A certain property of globally hyperbolic spacetimes will be important for applying the propagation of singularities theorem in Section 5, so we mention it here: Let $v$ be a non-zero lightlike vector in $T_q M$ for some $q \in M$. It defines a maximal smooth, affinely parametrized geodesic $\gamma: I \to M$ with the properties $\gamma(0) = q$ and $\frac{d}{dt}\gamma(t)|_{t=0} = v$ where ‘maximal’ here refers to choosing $I$ as the largest real interval ($I$ is taken as a neighbourhood of 0, and may coincide e.g. with $\mathbb{R}$) where $\gamma$ is a smooth solution of the geodesic equation compatible with the specified data at $q$. Then $\gamma$ is both future- and past-inextendible (see e.g. the argument in [35, Prop. 4.3]), and consequently, given an arbitrary Cauchy-surface $\Sigma \subset M$, there is exactly one parameter value $t \in I$ so that $\gamma(t) \in \Sigma$.

3.3 Wave-Operators and Propagation of Singularities

Suppose that we are given a time-oriented spacetime $(M, g)$. Then let $\mathfrak{U}$ be a vector bundle with base manifold $M$, typical fibre isomorphic to $\mathbb{C}^r$, and bundle projection $\pi_M$. Moreover, we assume that there exists a morphism $\Gamma$ of $\mathfrak{U}$ covering the identity map of $M$ which is involutive ($\Gamma \circ \Gamma = \text{id}_\mathfrak{U}$) and acts anti-isomorphically on the fibres; in other words, $\Gamma$ acts like a complex conjugation in each fibre space. Therefore, the $\Gamma$-invariant part $\mathfrak{U}^\Gamma$ of $\mathfrak{U}$ is a vector bundle over the base $M$ with typical fibre $\mathbb{R}^r$.

A linear partial differential operator $P: C^\infty_0(\mathfrak{U}) \to C^\infty_0(\mathfrak{U})$ will be said to have metric principal part if, upon choosing a local trivialization of $\mathfrak{U}$ over $U \subset M$ in which sections $f \in C^\infty_0(\mathfrak{U}_U)$ take the component representation $(f^1, \ldots, f^r)$, and a chart $(x^\mu)^m_{\mu=1}$, one has the following coordinate representation for $P$:

$$(P f)^a(x) = g^{\mu \nu}(x)\partial_\mu \partial_\nu f^a(x) + A^{\nu a}_\mu(x)\partial_\nu f^b(x) + B^a_b(x) f^b(x).$$

Here, $\partial_\mu$ denotes the coordinate derivative $\frac{\partial}{\partial x^\mu}$, and $A^{\nu a}_\mu$ and $B^a_b$ are suitable collections of smooth, complex-valued functions. Observe that thus the principal part of $P$ diagonalizes in all local trivializations (it is “scalar”).

If $P$ has metric principal part and is in addition $\Gamma$-invariant, i.e.

$$\Gamma^* P \circ \Gamma^* = P,$$

then we call $P$ a wave operator. In this case, $P$ leaves the space $C^\infty_0(\mathfrak{U}^\Gamma)$ of $\Gamma^*$-invariant sections invariant. As an aside we note that then there is a covariant derivative (linear

\footnote{Greek indices are raised and lowered with $g^{\mu \nu}(x)$, latin indices with $\delta^\mu_\nu$.}
connection) $\nabla^{(P)}$ on $\mathfrak{V}$ together with a bundle map $v$ of $\mathfrak{V}$ covering $\text{id}_M$ such that

$$Pf = g^{\mu\nu}\nabla^{(P)}_\mu\nabla^{(P)}_\nu f + v^* f$$

for all $f \in C_0^\infty(\mathfrak{V})$; this covariant derivative is given by

$$2 \cdot \nabla^{(P)}_{\text{grad}_\varphi} f = P(\varphi f) - \varphi P(f) - (\Box_g \varphi) f$$

for all $\varphi \in C_0^\infty(M, \mathbb{R})$ and $f \in C_0^\infty(\mathfrak{V})$, where $\Box_g$ denotes the d’Alembert-operator induced by the metric $g$ on scalar functions [20].

Before we can state the version of the propagation of singularities theorem that will be relevant for our considerations later, we need to introduce further notation. By $\mathfrak{V} \boxtimes \mathfrak{V}$ we denote the outer product bundle of $\mathfrak{V}$. This is the $C^\infty$-vector bundle over $M \times M$ whose fibres over $(q_1, q_2) \in M \times M$ are $\mathfrak{V}_{q_1} \otimes \mathfrak{V}_{q_2}$ where $\mathfrak{V}_{q_j}$ denotes the fibre over $q_j$ ($j = 1, 2$), and with base projection defined by

$$v_{q_1} \otimes v'_{q_2} \mapsto (q_1, q_2) \quad \text{for} \quad v_{q_1} \otimes v'_{q_2} \in \mathfrak{V}_{q_1} \otimes \mathfrak{V}_{q_2}.$$ 

Note also that the conjugation $\Gamma$ on $\mathfrak{V}$ induces a conjugation $\otimes^2 \Gamma$ on $\mathfrak{V} \boxtimes \mathfrak{V}$ by anti-linear extension of the assignment

$$\otimes^2 \Gamma( v_{q_1} \otimes v'_{q_2} ) := \Gamma v_{q_1} \otimes \Gamma v'_{q_2}, \quad q_j \in M.$$ 

The definition of $\otimes^n \mathfrak{V}$, the $n$-fold outer tensor product of $\mathfrak{V}$, should then be obvious, and likewise the definition of $\otimes^n \Gamma$.

Going to local trivializations and using partition of unity arguments, it is not difficult to see that the canonical embedding $C_0^\infty(\mathfrak{V}) \otimes C_0^\infty(\mathfrak{V}) \subset C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V})$ is dense ([10]). Moreover, if we take some $L \in (C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V}))'$, then it induces a bilinear form $\Lambda$ over $C_0^\infty(\mathfrak{V})$ by setting

$$\Lambda(f, f') = L(f \otimes f'), \quad f, f' \in C_0^\infty(\mathfrak{V}). \quad (3.6)$$

Clearly $\Lambda$ is then jointly continuous in both entries. On the other hand, if $\Lambda$ is a bilinear form over $C_0^\infty(\mathfrak{V})$ which is separately continuous in both entries ( $f \mapsto \Lambda(f, f')$ and $f \mapsto \Lambda(f', f)$ are continuous maps for each fixed $f'$), then the nuclear theorem implies that there is an $L \in (C_0^\infty(\mathfrak{V} \boxtimes \mathfrak{V}))'$ inducing $\Lambda$ according to (3.3) [10]. These statements generalize to the case of $n$-fold tensor products in the obvious manner.

Now define $\mathcal{N} := \{(q, \xi) \in T^*M \setminus \{0\} : g^{\mu\nu}(q)\xi_\mu\xi_\nu = 0\}.$

Moreover, define for each pair $(q, \xi; q', \xi') \in \mathcal{N} \times \mathcal{N}$:

$$(q, \xi) \sim (q', \xi') \quad \text{iff there exists an affine parametrized lightlike geodesic } \gamma \text{ in } (M, g) \text{ connecting } q \text{ and } q' \text{ and such that } \xi \text{ and } \xi' \text{ are co-tangent to } \gamma \text{ at } q \text{ and } q', \text{ respectively.}$$

Here, we say that $\xi$ is co-tangent to $\gamma$ at $q = \gamma(s)$ if $\left.\frac{d}{dt}\right|_{t=s} \gamma(t)^\mu = g^{\mu\nu}(q)\xi_\nu$, where $t$ is the affine parameter. Therefore, $(q, \xi) \sim (q', \xi')$ means $\xi$ and $\xi'$ are parallel transports of $\xi$.

---

6The notation $(q, \xi) \in T^*M$ means that $\xi \in T^*_q M$, i.e. $q$ denotes the base point of the cotangent vector $\xi$. 
each other along the lightlike geodesic $\gamma$ connecting $q$ and $q'$. Note that the possibility $q = q'$ is included, in which case $(q, \xi) \sim (q', \xi')$ means $\xi = \xi'$. One can introduce the following two disjoint future/past-oriented parts (with respect to the time-orientation of $(M, g)$) of $N$,

$$N_\pm := \{(q, \xi) \in N \mid \pm \xi \succ 0\} ,$$

(3.7)

where $\xi \succ 0$ means that the vector $\xi^\mu = g^{\mu\nu}\xi_\nu$ is future-pointing.

The relation "~" is obviously an equivalence relation between elements in $N$. For $(q, \xi) \in N$, the corresponding equivalence class is denoted by $B(q, \xi)$; it is a bi-characteristic strip of any wave operator $P$ on $\mathfrak{W}$ since such an operator has metric principal part and therefore its bi-characteristics are lightlike geodesics (see, e.g. [30]).

Now we are ready to state a specialized version of the propagation of singularities theorem (PST) which is tailored for two-point distributions that are solutions (up to $C^\infty$-terms) of wave operators, and which derives as a special case of the PST in [8]. We should like to point out that the formulation of the PST in [8] (extending arguments developed in [12] for the scalar case) is considerably more general in two respects: First, it applies, with suitable modifications, not only to linear second order differential operators with metric principal part, but to pseudo-differential operators on $C^\infty_0(\mathfrak{W})$ that have a so-called ‘real principal part’ (of which ‘metric principal part’ is a special case, note also that a metric principal part is homogeneous). Secondly, the general formulation of the PST gives not only information about the wavefront set of a $u \in (C^\infty_0(\mathfrak{W}))'$ which is a solution up to $C^\infty$-terms of a pseudo-differential operator $A$ having real principal part (i.e. $WF(Au) = \emptyset$), but even describes properties of the polarization set of such a $u$. The polarization set $WF_{pol}(u)$ of $u \in (C^\infty_0(\mathfrak{W}))'$ is a subset of the direct product bundle $T^*M \oplus \mathfrak{W}$ over $M$ and specifies which components of $u$ (in a local trivialization of $\mathfrak{W}$) have the worst decay properties in Fourier-space near any given base point in $M$; the projection of $WF_{pol}(u)$ onto its $T^*M$-part coincides with the wavefront set $WF(u)$. The reader is referred to [8] for details and further discussion, and also to [33, 24] for a discussion of the polarization set for Dirac fields on curved spacetimes. As a corollary to the PST formulated in [8] together with Lemma 6.5.5. in [12] (see also [30] for an elementary account), one obtains the following:

**Proposition 3.3.** Let $P$ be a wave operator on $C^\infty_0(\mathfrak{W})$ and define for $w \in (C^\infty_0(\mathfrak{W} \boxtimes \mathfrak{W}))'$ the distributions $w_{(P)}, w^{(P)} \in (C^\infty_0(\mathfrak{W} \boxtimes \mathfrak{W}))'$ by

$$w_{(P)}(f \otimes f') := w Pf \otimes f',$$

$$w^{(P)}(f \otimes f') := w(f \otimes Pf'),$$

(3.8)

for all $f, f' \in C^\infty_0(\mathfrak{W})$.

Suppose that $WF(w_{(P)}) = \emptyset = WF(w^{(P)})$. Then it holds that

$$WF(w) \subset N \times N$$

and

$$(q, \xi; q', \xi') \in WF(w) \text{ with } \xi \neq 0 \text{ and } \xi' \neq 0 \Rightarrow B(q, \xi) \times B(q', \xi') \subset WF(w).$$
4 Quantum Fields

4.1 The Borchers Algebra

We begin our discussion of linear quantum fields obeying a wave equation by recalling the definition and basic properties of the Borchers-algebra \[ \mathcal{B} \].

Let \( \mathfrak{U} \) denote a vector bundle over the base-manifold \( M \) as in the previous section. Then consider the set

\[
\mathcal{B} := \{ f \equiv (f_n)_{n=0}^{\infty} : f_0 \in \mathbb{C}, f_n \in C_0^\infty(\bigotimes^n \mathfrak{U}), \text{ only finitely many } f_n \neq 0 \}
\]

where \( \bigotimes^n \mathfrak{U} \) denotes the \( n \)-fold outer product bundle of \( \mathfrak{U} \), cf. Sec. 3. The set \( \mathcal{B} \) is a priori a vector space, but one may also introduce a \( * \)-algebraic structure on it: A product \( f \cdot g \) for elements \( f, g \in \mathcal{B} \) is given by defining the \( n \)-th component \( (f \cdot g)_n \) to be

\[
(f \cdot g)_n := \sum_{i+j=n} f_i \otimes g_j.
\]

Here, \( f_i \otimes g_j \) is understood as the element in \( C_0^\infty(\bigotimes^n \mathfrak{U}) \) induced by the canonical embedding \( C_0^\infty(\bigotimes^n \mathfrak{U}) \otimes C_0^\infty(\bigotimes^n \mathfrak{U}) \subset C_0^\infty(\bigotimes^n \mathfrak{U}) \). Observe that \( \mathcal{B} \) possesses a unit element \( 1_\mathcal{B} \), given by the sequence \( ((1_\mathcal{B})_n)_{n=0}^{\infty} \) having the number 1 in the 0-th component while all other components vanish. Moreover, for \( f \in \mathcal{B} \) one can define \( f^* \) by setting

\[
f^*_n(q_1, \ldots, q_n) := \bigotimes^n \Gamma f_n(q_n, \ldots, q_1), \quad q_j \in M,
\]

for the \( n \)-th component of \( f^* \) where \( \Gamma \) denotes the complex conjugation assumed to be given on \( \mathfrak{U} \). This yields an anti-linear involution on \( \mathcal{B} \). With these definitions of product and \( * \)-operation, \( \mathcal{B} \) is a \( * \)-algebra.

Furthermore, \( \mathcal{B} \) has a natural ‘local net structure’ in the sense that one obtains an inclusion-preserving map \( M \supset \emptyset \mapsto \mathcal{B}(\emptyset) \subset \mathcal{B} \) taking subsets \( \emptyset \) of \( M \) to unital \( * \)-subalgebras \( \mathcal{B}(\emptyset) \) of \( \mathcal{B} \) upon defining \( \mathcal{B}(\emptyset) \) to consist of all \( (f_n)_{n=0}^{\infty} \) for which \( \text{supp} f_n \subset \emptyset, \quad n \in \mathbb{N} \).

Another simple fact is that (local) morphisms of \( \mathfrak{U} \) commuting with \( \Gamma \) can be lifted to (local) automorphisms of \( \mathcal{B} \). To this end, let \( (R_x)_{x \in \mathcal{B}} \) be a family of (local) morphisms of \( \mathfrak{U} \) covering \( (\rho_x)_{x \in \mathcal{B}} \), and assume that \( \Gamma R_x = R_x \Gamma \) for all \( x \). Suppose that \( \emptyset \subset M \) is in the domain of \( \rho_x \); then define a map \( \alpha_x \) on \( \mathcal{B}(\emptyset) \) by setting for \( f \in \mathcal{B}(\emptyset) \) the \( n \)-th component,

\[
(\alpha_x f)_n := \bigotimes^n R^*_x f_n,
\]

for \( f \in \mathcal{B}(\emptyset) \). This yields a \( * \)-isomorphism \( \alpha_x : \mathcal{B}(\emptyset) \to \mathcal{B}(\rho_x(\emptyset)) \).

We will now turn \( \mathcal{B} \) into a locally convex space by giving it the topology of the strict inductive limit of the topological vector spaces

\[
\mathcal{B}_n := \mathbb{C} \oplus \bigoplus_{k=1}^n C_0^\infty(\bigotimes^k \mathfrak{U}), \quad n \in \mathbb{N}.
\]
This topology is known as the \textit{locally convex direct sum topology} (ex. \cite[Chap. II, \S 4 n° 5]{4}). Some important properties of $\mathcal{B}$, equipped with this topology are given in the following lemma, the proof of which will be deferred to Appendix B.

\textbf{Lemma 4.1.} With the topology given above, $\mathcal{B}$ is complete and a topological $\ast$-algebra. Moreover, a linear functional $u : \mathcal{B} \to \mathbb{C}$ is continuous if and only if there is a sequence $(u_n)_{n=0}^{\infty}$ with $u_0 \in \mathbb{C}$ and $u_j \in (C_0^\infty(\mathbb{R}^n\mathfrak{V}))'$ for $j \in \mathbb{N}$ so that

$$u(f) = u_0 f_0 + \sum_{j \in \mathbb{N}} u_j(f_j), \quad f \in \mathcal{B}.$$  

(4.3)

If $\alpha$ is a $\ast$-automorphism lifting a morphism $R$ of $\mathfrak{V}$ to $\mathcal{B}$ as in (4.2), then $\alpha$ is continuous. Moreover, let $(R_x)_{x \in \mathcal{B}}$ be a family of morphisms of $\mathfrak{V}$ depending smoothly on $x$ with $\Gamma R_x = R_x \Gamma$ and $R_0 = \text{id}_{\mathfrak{V}}$, and let $(\alpha_x)_{x \in \mathcal{B}}$ be the family of $\ast$-automorphisms of $\mathcal{B}$ induced according to (4.2). Then for each $f \in \mathcal{B}$ it holds that

$$\alpha_x(f) \to f \quad \text{for} \quad x \to 0,$$  

(4.4)

and there is a constant $r > 0$ such that to each continuous semi-norm $\sigma$ of $\mathcal{B}$ one can find another semi-norm $\sigma'$ with the property

$$\sigma(\alpha_x(f)) \leq \sigma'(f), \quad |x| \leq r, \quad f \in \mathcal{B}.$$  

(4.5)

\section*{4.2 States and Quantum Fields}

A state $\omega$ on $\mathcal{B}$ is a continuous linear form on $\mathcal{B}$ which fulfills the positivity requirement $\omega(f^*f) \geq 0$ for all $f \in \mathcal{B}$. By Lemma 4.1 such a state $\omega$ is completely characterized by a set $\{\omega_n|n \in \mathbb{N}_0\}$ of linear functionals $\omega_n \in (C_0^\infty(\mathbb{R}^n\mathfrak{V}))'$, the so-called $n$-point functions.

The positivity requirement allows it to associate with any state $\omega$ a Hilbertspace $\ast$-representation by the well-known Gelfand-Naimark-Segal (GNS) construction (or the Wightman reconstruction theorem \cite{41}). More precisely, given a state on $\mathcal{B}$, there exists a triple $(\varphi, D \subset \mathcal{H}, \Omega)$, called GNS-representation of $\omega$, possessing the following properties:

(a) $\mathcal{H}$ is a Hilbertspace, and $D$ is a dense linear subspace of $\mathcal{H}$.

(b) $\varphi$ is a $\ast$-representation of $\mathcal{B}$ on $\mathcal{H}$ by closable operators with common domain $D$.

(c) $\Omega$ is a unit vector contained in $D$ which is cyclic, i.e. $D = \varphi(\mathcal{B})\Omega$, and has the property that

$$\omega(f) = \langle \Omega, \varphi(f)\Omega \rangle, \quad f \in \mathcal{B}.$$  

Furthermore, the GNS-representation is unique up to unitary equivalence. We refer to \cite[Part II]{40} for further details on $\ast$-representations of $\ast$-algebras as well as for a proof of these statements and references to the relevant original literature.

Therefore, a state $\omega$ on $\mathcal{B}$ induces a quantum field — that is to say, an operator-valued distribution

$$C_0^\infty(\mathfrak{V}) \ni f \mapsto \Phi(f) := \varphi(f), \quad f = (0, f, 0, 0, \ldots),$$  

(4.6)
where the \( \Phi(f) \) are, for each \( f \in C_0^\infty(\mathfrak{W}) \), closable operators on the dense and invariant domain \( \mathcal{D} \) and one has \( \Phi(\Gamma^* f) \subset \Phi(f)^* \) where \( \Phi(f)^* \) denotes the adjoint operator of \( \Phi(f) \). Conversely, such a quantum field induces states on \( \mathcal{B} \): Given some unit vector \( \psi \in \mathcal{D} \), the assignment

\[
\omega^{(\psi)}(c \cdot 1_\mathcal{B}) := c, \quad c \in \mathbb{C},
\]

\[
\omega^{(\psi)}(f^{(1)} \otimes \cdots \otimes f^{(m)}) := \langle \psi, \Phi(f^{(1)}) \cdots \Phi(f^{(m)}) \psi \rangle, \quad f^{(j)} \in C_0^\infty(\mathfrak{W}),
\]

defines, by linear extension, a state \( \omega^{(\psi)} \) on \( \mathcal{B} \). (Obviously this generalizes from vector states to mixed states.)

If the quantum field \( \Phi \) is an observable field, then one would require commutativity at causal separation, and this means

\[
\Phi(f)\Phi(f') = \Phi(f')\Phi(f)
\]

whenever the supports of \( f \) and \( f' \) are causally separated. Such commutative behaviour (locality) of \( \Phi \) at causal separation is characteristic of bosonic fields. On the other hand, a field \( \Phi \) is fermionic if it anti-commutes at causal separation (twisted locality), i.e.

\[
\Phi(f)\Phi(f') = -\Phi(f')\Phi(f)
\]

for causally separated supports of \( f \) and \( f' \). The general analysis of quantum field theory so far has shown that the alternative of having quantum fields of bosonic or fermionic character may largely be viewed as generic at least for spacetime dimensions greater than 2 \([24, 11, 19]\).

If \( \omega \) is a state on \( \mathcal{B} \) inducing via its GNS-representation a bosonic field, then it follows that the commutator \( \omega_2^{(-)} \) of its two-point function, defined by

\[
\omega_2^{(-)}(f \otimes f') := \frac{1}{2}(\omega_2(f \otimes f') - \omega_2(f' \otimes f)), \quad f, f' \in C_0^\infty(\mathfrak{W}),
\]

vanishes as soon as the supports of \( f \) and \( f' \) are causally separated. If, on the other hand, \( \omega \) induces a fermionic field, then the anti-commutator,

\[
\omega_2^{(+)}(f \otimes f') := \frac{1}{2}(\omega_2(f \otimes f') + \omega_2(f' \otimes f)), \quad f, f' \in C_0^\infty(\mathfrak{W}),
\]

of its two-point function vanishes when the supports of \( f \) and \( f' \) are causally separated.

For our purposes in Sec. \( \[\] \) we may assume a weaker version of bosonic or fermionic behaviour of quantum fields: We shall later suppose that \( \omega_2^{(+)} \) or \( \omega_2^{(-)} \) is smooth \((C^\infty)\) at causal separation. The definition relevant for that terminology is as follows:

**Definition 4.2.** Let \( w \in (C_0^\infty(\mathfrak{W} \otimes \mathfrak{W}))' \). We say that \( w \) is **smooth at causal separation** if

\[
WF(w_\Omega) = \emptyset
\]

where \( \Omega \) is the set of all pairs of points \((q, q') \in M \times M \) which are causally separated in \((M, g)\) \([\] \) and \( w_\Omega \) denotes the restriction of \( w \) to \( C_0^\infty((\mathfrak{W} \otimes \mathfrak{W})_\Omega) \).

---

\(^7\Omega\) is an open subset in \( M \times M \) due to global hyperbolicity.
4.3 Quasifree States

Of particular interest are quasifree states associated with quantum fields obeying canonical commutation relations (CCR) or canonical anti-commutation relations (CAR). A simple way of introducing them is via the characterization of such states given in [29] which we will basically follow here. Note, however, that in this reference the map $K$ in (4.7) is defined on certain quotients of $C^\infty_0(\mathfrak{V}^0)$ while we define $K$ on $C^\infty_0(\mathfrak{V}^0)$ itself (recall that $C^\infty_0(\mathfrak{V}^0)$ is the space of $\Gamma^*$-invariant sections). This is due to the fact that we haven’t imposed CCR or CAR for states on the Borchers algebra, so the notion of quasifree states given here is, in this respect, more general.

Let $\mathfrak{h}$ be a complex Hilbertspace (the so called ‘one-particle Hilbertspace’) and $F_{\pm}(\mathfrak{h})$ the bosonic/fermionic Fock-space over $\mathfrak{h}$. By $a_{\pm}(.)$ and $a^\dagger_{\pm}(.)$ we denote the corresponding annihilation and creation operators, respectively. The Fock-vacuum vector will be denoted by $\Omega_{\pm}$. Then we say that a state $\omega$ on $\mathcal{B}$ is a (bosonic/fermionic) quasifree state if there exists a real-linear map $K : C^\infty_0(\mathfrak{V}^0) \rightarrow \mathfrak{h}$ (4.7)

whose complexified range is dense in $\mathfrak{h}$, such that the GNS-representation $(\varphi, \mathcal{D} \subset \mathcal{H}, \Omega)$ of $\omega$ takes the following form: $\mathcal{H} = F_{\pm}(\mathfrak{h})$, $\Omega = \Omega_{\pm}$, and

$$\Phi(f) = \frac{1}{\sqrt{2}} \left( a_{\pm}(K(f)) + a^\dagger_{\pm}(K(f)) \right), \quad f \in C^\infty_0(\mathfrak{V}^0),$$

where $\Phi(.)$ relates to $\varphi(.)$ as in (4.6).

Quasifree states are in a sense the most simple states. It is, however, justified to consider prominently those states since for quantum fields obeying a linear wave-equation, ground- and KMS-states turn out to be quasifree in examples. Any quasifree state $\omega$ is entirely determined by its two-point function, i.e. by the map

$$C^\infty_0(\mathfrak{V}) \times C^\infty_0(\mathfrak{V}) \ni (f^{(1)}, f^{(2)}) \mapsto \omega(f^{(1)} \otimes f^{(2)}) = \langle \Omega, \Phi(f^{(1)})\Phi(f^{(2)})\Omega \rangle,$$

in the sense that the $n$-point functions

$$\omega_n(f^{(1)} \otimes \cdots \otimes f^{(n)}) = \langle \Omega, \Phi(f^{(1)}) \cdots \Phi(f^{(n)})\Omega \rangle, \quad f^{(j)} \in C^\infty_0(\mathfrak{V})$$

vanish for all odd $n$, while the $n$-point functions for even $n$ can be expressed as polynomials in the variables $\omega_2(f^{(i)} \otimes f^{(j)}), i, j = 1, \ldots, n$. This attaches particular significance to the two-point functions for quantum fields obeying linear wave equations. We refer to [3, 29, 1] for further discussion of quasifree states and their basic properties.

5 Passivity and Microlocal Spectrum Condition

In the present section we will state and prove our main result connecting passivity and microlocal spectrum condition for linear quantum fields obeying a hyperbolic wave equation on a globally hyperbolic, stationary spacetime.

First, we need to collect the assumptions. It will be assumed that $\mathfrak{V}$ is a vector bundle, equipped with a conjugation $\Gamma$, over a base manifold $M$ carrying a time-orientable
Lorentzian metric \( g \), and that \((M, g)\) is globally hyperbolic. Moreover, we assume that the spacetime \((M, g)\) is stationary, so that there is a one-parametric \( C^\infty \)-group \( \{\tau_t\}_{t \in \mathbb{R}} \) of isometries whose generating vector field, denoted by \( \partial^r \), is everywhere timelike and future-pointing (with respect to a fixed time-orientation). We recall that the notation \( N_\pm \) for the future/past-oriented parts of the set of null-covectors \( N \) has been introduced in [B.7], and note that \((q, \xi) \in N_\pm \) iff \( \pm \xi(\partial^r) > 0 \). It is furthermore supposed that there is a smooth one-parametric group \( \{T_t\}_{t \in \mathbb{R}} \) of morphisms of \( \mathcal{W} \) covering \( \{\tau_t\}_{t \in \mathbb{R}} \), and a wave operator \( P \) on \( C_0^\infty(\mathcal{W}) \), having the following properties:

\[
\Gamma \circ T_t = T_t \circ \Gamma, \quad T_t^* P = P T_t^*, \quad t \in \mathbb{R}.
\]

Now let \( \mathcal{B} \) again denote the Borchers algebra as in Sec. [4]. The automorphism group induced by lifting \( \{T_t\}_{t \in \mathbb{R}} \) on \( \mathcal{B} \) according to (3.8) will be denoted by \( \{\alpha_t\}_{t \in \mathbb{R}} \). Whence, by Lemma 4.1, \( (\mathcal{B}, \{\alpha_t\}_{t \in \mathbb{R}}) \) is a topological \( * \)-dynamical system.

Recall that a state \( \omega \) on \( \mathcal{B} \) is, by definition, contained in \( \mathcal{P} \) if it is a convex combination of ground- or KMS-states at strictly positive inverse temperature for \( \{\alpha_t\}_{t \in \mathbb{R}} \).

**Theorem 5.1.** Let \( \omega \in \mathcal{P} \) and let \( \omega_2 \) be the two-point distribution of \( \omega \) (see Sec. [4]). Suppose that \( \text{WF}(\omega_2^{(P)}) = \emptyset = \text{WF}(\omega_2^{(P)}) \) where \( \omega_2^{(P)} \) and \( \omega_2^{(P)} \) are defined as in (3.8), and suppose also that the symmetric part \( \omega_2^{(+)} \) or the anti-symmetric part \( \omega_2^{(-)} \) of the two-point distribution is smooth at causal separation (Definition [3.3]).

Then it holds that \( \text{WF}(\omega_2) \subset \mathcal{R} \) where \( \mathcal{R} \) is the set

\[
\mathcal{R} := \{(q, \xi; q', \xi') \in N_- \times N_+: (q, \xi) \sim (q', -\xi')\}. \tag{5.1}
\]

**Proof. 1.)** Let \( q \) be any point in \( M \). Then there is a coordinate chart \( \kappa = (y^0, y) = (y^0, y^1, \ldots, y^{m-1}) \) around \( q \) so that, for small \( |t| \),

\[
\kappa \circ \tau_t = \tau_t \circ \kappa
\]

holds on a neighbourhood of \( q \), where

\[
\tau_t(y^0, y) := (y^0 + t, y).
\]

In such a coordinate system, we can also define “spatial” translations

\[
\rho_\underline{x}(y^0, y) := (y^0, y + \underline{x})
\]

for \( \underline{x} = (x^1, \ldots, x^{m-1}) \) in a sufficiently small neighbourhood \( \underline{B} \) of the origin in \( \mathbb{R}^{m-1} \). Let \( (R_\underline{x})_{\underline{x} \in \underline{B}} \) be any smooth family of local morphisms around \( q \) covering \( (\rho_\underline{x})_{\underline{x} \in \underline{B}} \), where

\[
\rho_\underline{x} := \kappa^{-1} \circ \rho_\underline{x} \circ \kappa
\]

(\( \rho_\underline{x} \) is defined on a sufficiently small neighbourhood of \( q \)). Now let \( q' \) be another point, and choose in an analogous manner as for \( q \) a coordinate system \( \kappa' \), and \( (\rho'_\underline{x})_{\underline{x} \in \underline{B}'} \) and \( (R'_\underline{x})_{\underline{x} \in \underline{B}'} \).

**2.)** In a further step we shall now establish the relation

\[
\text{WF}(\omega_2) \subset \{(q, \xi; q', \xi') \in (T^* M \times T^* M) \setminus \{0\} : \xi(\partial^r) + \xi'(\partial^r) = 0, \ \xi'(\partial^r) \geq 0\}. \tag{5.2}
\]
Since we have $WF(\omega_2) \subset N \times N$ by Prop. 3.2, this then allows us to conclude that

$$WF(\omega_2) \subset \{(q, \xi; q', \xi') \in N_- \times N_+ : \xi (\partial^T) + \xi' (\partial^T) = 0\},$$

and we observe that thereby the possibility $(q, \xi; q', \xi') \in WF(\omega_2)$ with $\xi = 0$ or $\xi' = 0$ is excluded, because that would entail both $\xi = 0$ and $\xi' = 0$.

For proving (5.2) it is in view of Lemma 3.1 and according to our choice of the co-ordinate systems $\kappa$, $\kappa'$ and corresponding actions $(R_\xi)_{\xi \in B}$ and $(R'_\xi)_{\xi' \in B'}$ sufficient to demonstrate that the following holds:

There is a function $h \in C^\infty_0(\mathbb{R}^m \times \mathbb{R}^m)$ with $h(0) = 1$, and for each $(\xi; \xi') = (\xi_0, \xi; \xi_0', \xi') \in (\mathbb{R}^m \times \mathbb{R}^m) \setminus \{0\}$ with $\xi_0 + \xi_0' \neq 0$ or $\xi_0' < 0$ there is an open neighbourhood $V \subset (\mathbb{R}^m \times \mathbb{R}^m) \setminus \{0\}$ such that

$$\sup_{(k'; l') \in V} \left| \int e^{-i\lambda^{-1}(tk_0 + zk')} e^{-i\lambda^{-1}(t'k_0' + z'k')} h(t, z; t', z') \omega_2((T_t^* R^*_\xi \otimes T_{t'}^* R^*_{\xi'}) f_\lambda) \, dt \, dt' \, dz \, dz' \right| = O^\infty(\lambda)$$

as $\lambda \to 0$ holds for all $(f_\lambda)_{\lambda > 0} \in F_q(\mathfrak{V})$ and all $(f_{\lambda}^*)_{\lambda > 0} \in F_{q'}(\mathfrak{U})$. (The notation $k = (k_0, k')$ should be obvious.) However, making use of part (c) of the statement of Prop. 2.1 in [44], for proving (5.4) it is actually enough to show that there are $h$ and $V$ as above so that

$$\sup_{(k'; l') \in V} \left| \int e^{-i\lambda^{-1}(tk_0 + zk')} e^{-i\lambda^{-1}(t'k_0' + z'k')} h(t, z; t', z') \omega_2(T_t^* R^*_\xi f_\lambda \otimes T_{t'}^* R^*_{\xi'}) \, dt \, dt' \, dz \, dz' \right| = O^\infty(\lambda)$$

as $\lambda \to 0$ holds for all $(f_\lambda)_{\lambda > 0} \in F_q(\mathfrak{V})$ and all $(f_{\lambda}^*)_{\lambda > 0} \in F_{q'}(\mathfrak{U})$.

In order now to exploit the strict passivity of $\omega$ via Prop. 2.1, we define the set $B^s$ of testing families with respect to the Borchers algebra $\mathcal{B}$ in the same manner as we have defined the set $A^s$ of testing families for the algebra $\mathcal{A}$ in Remark 2.2. In other words, a $\mathcal{B}$-valued family $(f_{\lambda})_{\lambda > 0, z \in \mathbb{R}^n}$ is a member of $B^s$, for arbitrary $n \in \mathbb{N}$, whenever for each continuous seminorm $\sigma$ on $\mathcal{B}$ there is some $s \geq 0$ so that

$$\sup_{\lambda > 0} \lambda^s \sigma(f_{\lambda}^*) f_{\lambda} < \infty.$$}

Now if $(f_\lambda)_{\lambda > 0}$ is in $F_q(\mathfrak{V})$, then $(f_{\lambda}^*)_{\lambda > 0, z \in B}$ defined by

$$f_{\lambda}^* := (0, R^*_\xi f_\lambda, 0, 0, \ldots)$$

is easily seen to be a testing family in $B^s$. The same of course holds when taking any $(f_{\lambda}^*)_{\lambda > 0} \in F_{q'}(\mathfrak{U})$ and defining $(f_{\lambda}^*)_{\lambda > 0, z' \in B'}$ accordingly.

Since $\omega \in \mathcal{P}$, it follows from Prop. 2.1 and Remark 2.2 that, with respect to the time-translation group $\{\alpha_t\}_{t \in \mathbb{R}}$,

$$ACS^2_B(\omega) \subset \{(\xi_0, \xi_0') \in \mathbb{R}^2 \setminus \{0\} : \xi_0 + \xi_0' = 0, \xi_0' \geq 0\}.$$}

And this means that there is some $h_0 \in C^\infty_0(\mathbb{R}^2)$ with $h_0(0) = 1$, and for each $(\xi_0, \xi_0') \in \mathbb{R}^2 \setminus \{0\}$ with $\xi_0 + \xi_0' \neq 0$ or $\xi_0' < 0$ an open neighbourhood $V_0$ in $\mathbb{R}^2 \setminus \{0\}$ so that

$$\sup_{(k_0, k_0') \in V_0, z, z'} \left| \int e^{-i\lambda^{-1}(tk_0 + t'k_0')} h_0(t, t') \omega(\alpha_t f_{\lambda}^* \alpha_{t'}^* f_{\lambda}^*) \, dt \, dt' \right| = O^\infty(\lambda)$$

as $\lambda \to 0$ holds for all $(f_\lambda)_{\lambda > 0} \in F_q(\mathfrak{V})$.
that also \( \omega \) can be drawn assuming instead that
\[
\text{for all } f, \quad h^*(f) = f \quad \text{for small enough } \lambda.
\]

Hence, upon taking
\[
V = \{(k_0, k_0'; k', k') : (k_0, k_0') \in V_0, \quad k, k' \in \mathbb{R}^{m-1}\}
\]
and \( h(t, x; t', x') = h_0(t, t') h(x, x') \), where \( h \) is in \( C_0^\infty(\mathbb{R}^{m-1} \times \mathbb{R}^{m-1}) \) with \( h(0) = 1 \), and with \( h_0 \) and \( h \) having sufficiently small supports, it is now easy to see that (5.7) entails the required relation (5.5), proving (5.2), whence (5.3) is also established.

3.) Now we shall show the assumption that \( \omega_2^+ \) is smooth at causal separation to imply
\[
\text{that also } \omega_2^- \quad \text{and hence, } \omega_2 \text{ itself is smooth at causal separation. The same conclusion can be drawn assuming instead that } \omega_2^-(\cdot) \text{ is smooth at causal separation. We will present the proof only for the first mentioned case, the argument for the second being completely analogous.}
\]

We define \( \Omega \) as the set of pairs of causally separated points \( (q, q') \in M \times M \). The restriction of \( \omega_2 \) to \( C_0^\infty((\mathfrak{M} \boxtimes \mathfrak{M})_\Omega) \) will be denoted by \( \omega_{20} \). By assumption, \( \omega_{20}^+ \) has empty wavefront set and therefore \( \text{WF}(\omega_{20}) = \text{WF}(\omega_{20}^-) \). Since \( (q, q') \in \Omega \) iff \( (q', q) \in \Omega \), the ‘flip’ map \( \rho : (q, q') \mapsto (q', q) \) is a diffeomorphism of \( \Omega \). Then
\[
R : \mathfrak{M}_q \boxtimes \mathfrak{M}_{q'} \ni v_q \otimes v_{q'} \mapsto v_{q'} \otimes v_q \in \mathfrak{M}_{q'} \boxtimes \mathfrak{M}_q \tag{5.8}
\]
is a morphism of \( (\mathfrak{M} \boxtimes \mathfrak{M})_\Omega \) covering \( \rho \). Thus one finds
\[
[R^*(f \otimes f')](q, q') = f'(q) \otimes f(q'),
\]
implying
\[
\omega_{20}^-(R^*(f \otimes f')) = \omega_{20}^-(f' \otimes f) = -\omega_{20}^-(f \otimes f')
\]
for all \( f \otimes f' \in C_0^\infty((\mathfrak{M} \boxtimes \mathfrak{M})_\Omega) \). Noting that multiplication by constants different from zero doesn’t change the wavefront set of a distribution, this entails, with Lemma 3.2
\[
\text{WF}(\omega_{20}^-) = \text{WF}(\omega_{20}^- \circ R^*) = \imath D \rho^{-1} \text{WF}(\omega_{20}^-). \tag{5.9}
\]

Now it is easy to check that
\[
\imath D \rho^{-1}(q, \xi; q', \xi') = (q', \xi'; q, \xi)
\]
for all \( (q, \xi; q', \xi') \in T^* M \times T^* M \), and this implies
\[
\imath D \rho^{-1}(N_- \times N_+) = N_+ \times N_. \tag{5.10}
\]

However, since we already know from (5.8) that \( \text{WF}(\omega_{20}) \subset N_- \times N_+ \) and \( \text{WF}(\omega_{20}^+) = \emptyset \), we see that \( \text{WF}(\omega_{20}^-) \subset N_- \times N_+ \). Combining this with (5.9) and (5.10) yields
\[
\text{WF}(\omega_{20}^-) \subset (N_- \times N_+) \cap (N_+ \times N_) = \emptyset.
\]
4.) Now we will demonstrate that the wavefront set has the form (5.1) for points \((q,q)\) on the diagonal in \(M \times M\), by demonstrating that otherwise singularities for causally separated points would occur according to the propagation of singularities (Prop. 3.3). To this end, let \((q,\xi; q,\xi')\) be in \(WF(\omega_2)\) with \(\xi\) not parallel to \(\xi'\). In view of the observation made below (5.3) that we must have \(\xi \neq 0\) and \(\xi' \neq 0\), we obtain from Prop. 3.3 \(B(q,\xi) \times B(q,\xi') \subset WF(\omega_2)\). For any Cauchy surface of \(M\), one can find \((p,\eta; p',\eta')\) in \(B(q,\xi) \times B(q,\xi')\) with \(p\) and \(p'\) lying on that Cauchy surface because of the inextendibility of the bi-characteristics. Since \(\xi\) is not parallel to \(\xi'\), one can even choose that Cauchy surface so that \(p \neq p'\) (if such a choice were not possible, the bi-characteristics through \(q\) with cotangent \(\xi\) and \(\xi'\) would coincide). But this is in contradiction to the result of 3.) since \(p\) and \(p'\) are causally separated. Hence, only \((q,\xi; q,\xi')\) with \(\xi = \lambda\xi', \lambda \in \mathbb{R}\) can be in \(WF(\omega_2)\). Applying the constraint \(\xi(\partial^\tau) + \xi'(\partial^\tau) = 0\) found in (5.3) gives \(\lambda = -1\). Together with the other constraint \(WF(\omega_2) \subset N_- \times N_+\) of (5.3), we now see that if \((q,\xi; q,\xi')\) is in \(WF(\omega_2)\) it must be in \(\mathbb{R}\).

5.) It will be shown next that \(\omega_2\) is smooth at points \((q,q')\) in \(M \times M\) which are causally related but not connected by any lightlike geodesic: Suppose \((q,\xi; q',\xi')\) were in \(WF(\omega_2)\) with \(q, q'\) as described. Using global hyperbolicity and the inextendibility of the bi-characteristics, we can then find \((p,\eta)\) in \(B(q,\xi)\) with \(p\) lying on the same Cauchy surface as \(q'\). As \(p\) cannot be equal to \(q'\) by assumption, it must be causally separated from \(q'\), and so we have by Prop. 3.3 a contradiction to 3.). Thus, \(\omega_2\) must indeed be smooth at \((q, q')\).

6.) Finally, we consider the case of points \((q,q')\) connected by at least one lightlike geodesic: Let \((q,\xi; q',\xi')\) be in \(WF(\omega_2)\). To begin with, we assume additionally that \(\xi\) is not co-tangential to any of the lightlike geodesics connecting \(q\) and \(q'\). As in 4.) we then find \((p,\eta; p',\eta')\) in \(B(q,\xi) \times B(q',\xi')\) with \(p\) and \(p'\) lying on the same Cauchy surface and \(p \neq p'\), thus establishing a contradiction to 3.).

To cover the remaining case, let \(\xi\) be co-tangential to one of the lightlike geodesics connecting \(q\) and \(q'\). As a consequence, we find \(\eta\) with \((q',\eta) \in B(q,\xi)\). By 4.), we have \(\eta = -\xi', \xi' > 0\), showing \((q,\xi; q',\xi')\) to be in \(\mathbb{R}\).

We conclude this article with a few remarks. First we mention that for the canonically quantized scalar Klein-Gordon field, \(WF(\omega_2) \subset \mathbb{R}\) implies \(WF(\omega_2) = \mathbb{R}\) and thus the two-point function of every strictly passive state is of Hadamard form, see [34]. Results allowing similar conclusions for vector-valued fields subject to CCR or CAR will appear in [38].

In [27], quasifree ground states have been constructed for the scalar Klein-Gordon field on stationary, globally hyperbolic spacetimes where the norm of the Killing vector field is globally bounded away from zero. Our result shows that they all have two-point functions of Hadamard form. As mentioned in the introduction, quasifree ground- and KMS-states have also been constructed for the scalar Klein-Gordon field on Schwarzschild spacetime [28], and again we conclude that their two-point functions are of Hadamard form.
In [18], massive vector fields are quantized on globally hyperbolic, ultrastatic space-times using (apparently) a ground state representation, and our methods apply also in this case.

Appendix

A Ground- and KMS-States, Passivity

Let \((A, \{\alpha_t\}_{t \in \mathbb{R}})\) be a topological \(\ast\)-dynamical system as described in Section 2. We recall that a continuous linear functional \(\omega : A \rightarrow \mathbb{C}\) is called a state if \(\omega(A^\ast A) \geq 0\) for all \(A \in A\) and \(\omega(1_A) = 1\). Now let \(\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int e^{-ipt} f(p) dp, f \in C_0^\infty(\mathbb{R})\), denote the Fourier-transform. Note that \(\hat{f}\) extends to an entire analytic function of \(t \in \mathbb{C}\). Then a convenient way of defining ground- and KMS-states is the following:

The state \(\omega\) is a ground state for \((A, \{\alpha_t\}_{t \in \mathbb{R}})\) if \(\mathbb{R} \ni t \mapsto \omega(A\alpha_t(B))\) is, for each \(A,B \in A\), a bounded function and if moreover,

\[
\int_{-\infty}^{\infty} \hat{f}(t)\omega(A\alpha_t(B)) dt = 0, \quad A,B \in A, \quad (A.1)
\]

holds for all \(f \in C_0^\infty((-\infty,0))\).

The state \(\omega\) is a KMS state at inverse temperature \(\beta > 0\) for \((A, \{\alpha_t\}_{t \in \mathbb{R}})\) if \(\mathbb{R} \ni t \mapsto \omega(A\alpha_t(B))\) is, for each \(A,B \in A\), a bounded function and if moreover,

\[
\int_{-\infty}^{\infty} \hat{f}(t)\omega(A\alpha_t(B)) dt = \int_{-\infty}^{\infty} \hat{f}(t+i\beta)\omega(\alpha_t(B)A) dt, \quad A,B \in A, \quad (A.2)
\]

holds for all \(f \in C_0^\infty(\mathbb{R})\).

The state \(\omega\) is a KMS state at inverse temperature \(\beta = 0\) if \(\omega\) is \(\{\alpha_t\}_{t \in \mathbb{R}}\)-invariant and a trace, i.e.

\[
\omega(AB) = \omega(BA), \quad A,B \in A. \quad (A.3)
\]

(Note that we have here additionally imposed \(\{\alpha_t\}_{t \in \mathbb{R}}\)-invariance in the definition of KMS state at \(\beta = 0\). Other references define a KMS state at \(\beta = 0\) just by requiring it to be a trace. The invariance doesn’t follow from that, cf. [4].)

We note that various other, equivalent definitions of ground- and KMS-states are known (mostly formulated for the case that \((A, \{\alpha_t\}_{t \in \mathbb{R}})\) is a \(C^\ast\)-dynamical system), see e.g. [3] and [33] as well as references cited there.

The term ‘KMS’ stands for Kubo, Martin and Schwinger who introduced and used the first versions of condition (A.2). The significance of KMS-states as thermal equilibrium states, particularly for infinite systems in quantum statistical mechanics, has been established in [22].

The following properties of any ground- or KMS-state at inverse temperature \(\beta > 0, \omega\), are standard in the setting of \(C^\ast\)-dynamical systems, and the proofs known for this case carry over to topological \(\ast\)-dynamical systems:
(i) $\omega$ is $\{\alpha_t\}_{t \in \mathbb{R}}$-invariant

(ii) $\omega(A^\dagger \delta(A)) \geq 0$ for all $A = A^* \in D(\delta)$

(where $\delta$ and $D(\delta)$ are as introduced at the beginning of Section 2).

Let us indicate how one proceeds in proving these statements. We first consider the case where $\omega$ is a ground state. Since $\mathcal{A}$ contains a unit element, the ground state condition (A.1) says that for any $A \in \mathcal{A}$ the Fourier-transform of the function $t \mapsto \omega(\alpha_t(A))$ vanishes on $(-\infty, 0)$. For $A = A^*$, that Fourier-transform is symmetric and hence is supported at the origin. As $t \mapsto \omega(\alpha_t(A))$ is bounded, its Fourier-transform can thus only be a multiple of the Dirac-distribution. This entails that $t \mapsto \omega(\alpha_t(A))$ is constant. By linearity, this carries over to arbitrary $A \in \mathcal{A}$, and thus $\omega$ is $\{\alpha_t\}_{t \in \mathbb{R}}$-invariant.

Now we may pass to the GNS-representation $(\varphi, D \subset \mathcal{H}, \Omega)$ of $\omega$ (cf. Sec. 4 where this object was introduced for the Borchers-algebra, but the construction can be carried out for topological $\ast$-algebras, see [40]) and we observe that, if $\omega$ is invariant, then $\{\alpha_t\}_{t \in \mathbb{R}}$ is in the GNS-representation implemented by a strongly continuous unitary group $\{U_t\}_{t \in \mathbb{R}}$ leaving $\Omega$ as well as the domain $D = \varphi(\mathcal{A})\Omega$ invariant. This unitary group is defined by

$$U_t \varphi(A)\Omega := \varphi(\alpha_t A)\Omega, \quad A \in \mathcal{A}, \ t \in \mathbb{R}.$$

Since it is continuous, it possesses a selfadjoint generator $H$, i.e. $U_t = e^{itH}$, and the ground state condition implies that the spectrum of $H$ is contained in $[0, \infty)$. Therefore, one has for all $A \in D(\delta)$,

$$\frac{1}{i} \omega(A^* \delta(A)) = \langle \varphi(A)\Omega, H \varphi(A)\Omega \rangle \geq 0$$

and this entails property (ii).

Now let $\omega$ be a KMS-state at inverse temperature $\beta > 0$. For the proof of its $\{\alpha_t\}_{t \in \mathbb{R}}$-invariance, see Prop. 4.3.2 in [39]. Property (ii) is then a consequence of the so-called ‘auto-correlation lower bounds’, see [39, Thm. 4.3.16] or [5, Thm. 5.3.15]. (Note that the proofs of the cited theorems generalize to the case where $\omega$ is a state on a topological $\ast$-algebra.)

**B  Proof of Lemma 4.1**

We now want to give the proof of Lemma 4.1. First, we state some properties of the strict inductive limit of a sequence of locally convex spaces, the topology given to $\mathcal{B}$ being a specific example. See for example [II, §4] for proofs as well as for further details.

Let $(E_n)_{n=1}^\infty$ be a sequence of locally convex linear spaces such that $E_n \subset E_{n+1}$ and the relative topology of $E_n$ in $E_{n+1}$ coincides with the genuine topology of $E_n$ for all $n \in \mathbb{N}$. Let $E$ be the inductive limit of the $E_n$, denote by $\pi_n : E_n \hookrightarrow E$ the canonical imbeddings of the $E_n$ into $E$ and let $F$ be some locally convex space. In this situation, we have:

(a) $E$ is a locally convex space.

(b) A map $f : E \rightarrow F$ is continuous iff $f \circ \pi_n$ is continuous for each $n$ in $\mathbb{N}$. 

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(c) A family of maps \((f_i)_i, f_i : E \to F\) is equicontinuous iff the family \((f_i \circ \pi_n)_i\) is equicontinuous for each \(n\) in \(\mathbb{N}\).

(d) The relative topology of \(E_n\) in \(E\) coincides with the genuine topology of \(E_n\).

(e) If the \(E_n\) are complete, so is \(E\).

Now we can prove the Lemma:

Because of (e), \(B\) is complete. The characterization (1.3) of the continuous linear forms on \(B\) is a special case of (b).

We want to check now that \(B\) is a topological \(*\)-algebra, i.e. that its \(*\)-operation is continuous and its multiplication \(m : B \times B \to B\) separately continuous in both entries:

For \(f \in B\) let \(m_f : B \to B\) be the right multiplication with \(f\) and denote by \([f]\) the smallest integer such that \(f_k = 0\) for all \(k > [f]\). By (b), showing continuity of \(m_f\) amounts to showing the continuity of the maps \(m_f \circ \pi_n : B_n \to B_{n+[f]}\) where by (d), we can take the topologies involved to be the genuine topologies of the respective spaces. As those topologies are direct sum topologies with finitely many summands, the question of continuity can be further reduced, finding that \(m_f\) is continuous iff the maps

\[
C_0^\infty(\mathbb{R}^k\mathfrak{M}) \ni g \mapsto f \otimes g \in C_0^\infty(\mathbb{R}^{n+k}\mathfrak{M})
\]

are continuous for all \(f \in C_0^\infty(\mathbb{R}^k\mathfrak{M})\), \(k \in \mathbb{N}\). That this is indeed the case can be checked by taking recourse to the topologies of the \(C_0^\infty(\mathbb{R}^k\mathfrak{M})\). Therefore, the maps \(m_f \circ \pi_n\) are continuous for all \(n\), which in turn shows the continuity of right multiplication on \(B\).

In the same way, the proof of continuity of the \(*\)-operation reduces to showing continuity of the \(*\)-operation (1.1) on \(C_0^\infty(\mathbb{R}^n\mathfrak{M})\), which in turn shows easily that the continuity of the \(*\)-operation can be proven completely analogous to that of right-multiplication or inferred from it, using the continuity of the \(*\)-operation. Therefore, \(B\) equipped with the locally convex direct sum topology is indeed a topological \(*\)-algebra.

For the proof of the last statements of the lemma, let \(\alpha_x\) be a \(*\)-homomorphism of \(B\) which is the lift of a morphism \(R_x\) of \(\mathfrak{M}\) covering \(p_x\) as stated in Lemma 4.1. Because of (b) and (d) above, \(\alpha_x\) is continuous if its restrictions \(\alpha_x \circ \pi_n : B_n \to B_n\) are continuous which in turn is the case, iff the maps \(\mathbb{R}^nR_x^* : C_0^\infty(\mathbb{R}^n\mathfrak{M}) \to C_0^\infty(\mathbb{R}^n\mathfrak{M})\) are continuous.

That the \(\mathbb{R}^nR_x^*\) are indeed continuous follows from density of \(\mathbb{R}^nC_0^\infty(\mathfrak{M})\) in \(C_0^\infty(\mathbb{R}^n\mathfrak{M})\) together with the continuity of \(R_x^*\) on \(C_0^\infty(\mathfrak{M})\), the latter of which can again be checked by inspection of the topology on \(C_0^\infty(\mathfrak{M})\).

For the proof of the continuity property (1.3), note that \([\alpha_x(f)] = [f]\) for all \(x\), thus it suffices to prove the convergence of \(\alpha_x(f)\) for \(x \to 0\) in the topology induced on \(B_{[f]}\). But this convergence is implied by the assumed smoothness of \(R_x\) (hence of \(R_x^*\)) in \(x\) together with (d). The proof of (1.3) amounts to showing that \([\alpha_x]|_{|x| \leq r}\) is an equicontinuous set of maps. By (c) and (d), the proof can in the by now familiar way be reduced to proving equicontinuity of \((R_x^*)_{|x| \leq r}\) for some \(r > 0\). For the proof of the latter, note that because of the assumed smoothness of \(\rho_x\) in \(x\) we find \(r > 0\) such that for each compact set \(K \subset M\) the set \(\bigcup_{|x| < r} \rho_x(K)\) is contained in some other compact set. Inspection of the topology of \(C_0^\infty(\mathfrak{M})\) shows that this enables one to find to each given seminorm \(\eta\) on \(C_0^\infty(\mathfrak{M})\) another seminorm \(\eta'\) such that for all \(f\) in \(C_0^\infty(\mathfrak{M})\), \(\eta(R_x^*f) \leq \eta'(f)\) holds for \(|x| \leq r\), thus proving the desired equicontinuity.
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