Henon’s Isochrone Model

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ABSTRACT

Hénon sought the most general spherical potential in which the radial periods of orbits depended only on energy. He named this potential the isochrone, and discovered that it provided a good representation of data for globular clusters. He sought an explanation in terms of resonant relaxation. The role that resonant relaxation might play in globular clusters is still an open question, but the isochrone potential is guaranteed a role in dynamical astronomy because it is the most general potential in which closed-form expressions for angle-action coordinates are available. I explain how this property makes the isochrone invaluable for the powerful technique of torus mapping. I also describe flattened isochrone models, which enable us to explore a powerful general method of generating self-consistent stellar systems.

Key words: solar neighbourhood – Galaxy: kinematics and dynamics – methods: data analysis

1 INTRODUCTION

I have been asked to speak about Hénon’s isochrone model. This was covered in three papers, submitted to Ann Ap in French between May 1958 and November 1959 (Hénon 1959a,b, 1960). The first of these papers is by far the most non-trivial. In it Hénon derives the isochrone potential as the most general spherical potential in which the radial period is independent of angular momentum at a given energy. Since this paper appeared in French and has a subtle line of argumentation I have decided to allocate a good deal of space to reproducing, and I hope clarifying, its argument. The other two papers are much more straightforward and I will not cover them in detail. Instead I will describe the role the isochrone potential plays in torus mapping, and a family of flattened isochrone models which I have recently published. I believe these applications will ensure that Hénon’s isochrone plays a significant role in astrophysics throughout the coming decade.

2 THE FIRST ISOCHRONE PAPER

The paper of May 1958 (Hénon 1959a) points out that the gravitational potentials of both a homogeneous mass distribution and a point mass have the property that the radial period $T_r$ of an orbit depends on the orbit’s energy $E$ alone: all orbits of a given value of $E$ have the same value of $T_r$, irrespective of their angular momentum $L$. Hénon asks “what is the most general gravitational potential for which this property holds?”

He writes down the integral $\int dr/v_r$ that determines $T_r$ and makes a change of variable to $x = 2r^2$. Then the integral becomes

$$T_r = \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 x - L^2 - f(x)}}$$

where $f(x) \equiv 2\Phi[r(x)]$. He plots the curve $y = f(x)$ defined by the potential and points out that the argument of the radical on the bottom of (1) is the vertical distance between this curve and the straight line $y = Ex - L^2$. Increasing $L$ has the effect of moving the straight line down, and at a critical value $L_c(E)$ the line touches the convex curve $y = f(x)$ at $x_0$ (Fig. 1). Hénon introduces a new independent variable $u$ through

$$u^2 \equiv f(x) - (Ex - L_c^2)$$

along with the convention that $u < 0$ when $x < x_0$ and $u > 0$ when $x > x_0$. Geometrically, $u^2$ is the vertical distance between the tangent $y = Ex - L_c^2$ and the the curve $y = f(x)$. For $L < L_c$ the line $y = Ex - L^2$ intersects the curve $y = f(x)$ twice: at $x_1 < x_0$ and at $x_2 > x_0$. These points of intersection are implicitly defined by the equation

$$Ex_i - L^2 - f(x_i) = 0 \quad (i = 1, 2).$$

The quantity

$$\lambda^2 = L_c^2 - L^2$$
The points $x_1$, $x_2$ are functions of both $E$ (which determines the slopes of the lines in Fig. 1 and $L^2$ (which determines the $y$ intercepts of these lines). We next differentiate our expressions (3) and (14) with respect to first $E$ and then $L^2$:

$$x_1 + E \frac{\partial x_1}{\partial E} \frac{df}{dx} \frac{dx_i}{dE} = 0$$

$$2(x_2 - x_1) \left( \frac{\partial x_2}{\partial E} - \frac{\partial x_1}{\partial E} \right) = (L_c^2 - L^2) \frac{dt}{dE} + t \frac{dL^2}{dE}$$

$$E \frac{\partial x_1}{\partial L^2} - 1 - \frac{df}{dx} \frac{dx_i}{dL^2} = 0$$

$$2(x_2 - x_1) \left( \frac{\partial x_2}{\partial L^2} - \frac{\partial x_1}{\partial L^2} \right) = -t.$$  

We use the first of these equations to eliminate $\partial x_i/\partial E$ from the second equation. This operation yields

$$2(x_2 - x_1) \left( \frac{x_2}{f_2 - E} - \frac{x_1}{f_1 - E} \right) = (L_c^2 - L^2) \frac{dt}{dE} + t \frac{dL^2}{dE}$$

where $f_2 \equiv df/dx|_{x_2}$, etc. But from the third of equations (16) we have

$$\frac{\partial x_1}{\partial L^2} = \frac{1}{E - f_i^2}.$$  

so equation (17) can be written

$$-2(x_2 - x_1) \left( x_2 \frac{\partial x_2}{\partial L^2} - x_1 \frac{\partial x_1}{\partial L^2} \right) = (L_c^2 - L^2) \frac{dt}{dE} + \frac{dL^2}{dE}.$$  

Between this equation and the last of equations (16) we can solve for the individual derivatives of $x_1$ and $x_2$:

$$2(x_2 - x_1)^2 \frac{\partial x_2}{\partial L^2} = (L_c^2 - L^2) \frac{dt}{dE} + \frac{dL^2}{dE} + \frac{1}{2} x_1 t$$

$$2(x_2 - x_1)^2 \frac{\partial x_1}{\partial L^2} = (L_c^2 - L^2) \frac{dt}{dE} + \frac{dL^2}{dE} + \frac{1}{2} x_2 t.$$  

Adding these equations, we obtain

$$(x_2 - x_1)^2 \frac{\partial(x_1 + x_2)}{\partial L^2} = (L_c^2 - L^2) \frac{dt}{dE} + \frac{dL^2}{dE} + \frac{1}{2} t(x_1 + x_2).$$

We use equation (14) to eliminate $(x_2 - x_1)^2$ from this equation and rearrange the result:

$$\frac{\partial(x_1 + x_2)}{\partial L^2} - \frac{x_1 + x_2}{2(L_c^2 - L^2)} = - \frac{1}{t} \frac{dt}{dE} - \frac{1}{L_c^2 - L^2} \frac{dL^2}{dE}. $$

This is a first-order, linear differential equation. Its integrating factor is $\sqrt{L_c^2 - L^2}$, so its general solution follows from

$$(x_1 + x_2) \sqrt{L_c^2 - L^2} = - \frac{1}{t} \frac{dt}{dE} \int \frac{dL^2}{L_c^2 - L^2} - \frac{L_c^2}{L_c^2 - L^2} \frac{dL^2}{dE} \int \frac{dL^2}{\sqrt{L_c^2 - L^2}}$$

$$= \frac{1}{t} \frac{dt}{dE} \int \left(L_c^2 - L^2\right)^{3/2} + \frac{dL^2}{dE} \int 2 \sqrt{L_c^2 - L^2} + \cos(\bar{x})$$

Since a non-zero value of the constant of integration would cause $x_1 + x_2$ to diverge as $L \to L_c$, it must be set to zero, and we have finally

$$x_1 + x_2 = \frac{1}{t} \frac{dt}{dE} \left(L_c^2 - L^2\right) + 2 \frac{dL^2}{dE}.$$  

This equation specifies as a function of the angular momentum $L$ the distance in Fig. 1 between the points of intersection of the potential’s curve and the straight line. Hence

$$t \equiv 4L_c^4 \pi^2.$$
it implicitly specifies the curve and thus the potential. To tease out that specification we first write down the quadratic equation that has roots at $x_i$:

$$[2x - (x_1 + x_2)]^2 = (x_1 - x_2)^2$$  \hspace{1cm} (25)

and use equations (24) and (14) to eliminate $x_1 + x_2$ and $x_2 - x_1$:

$$2x - \frac{2}{3} \frac{d \ln t}{d E} (L_\gamma^2 - L_\delta^2) - 2 \frac{dL_\gamma^2}{dE} = t(L_\gamma^2 - L_\delta^2).$$  \hspace{1cm} (26)

Now we eliminate $L^2 = Ex_i - f(x_i)$ (eq. 3)

$$2x - \frac{2}{3} \frac{d \ln t}{d E} (L_\gamma^2 - Ex_i + f(x_i)) - 2 \frac{dL_\gamma^2}{dE} = t(L_\gamma^2 - Ex_i + f(x_i)).$$

This equation holds only when $x = x_i$, so we set it to that value and rearrange to

$$\left(\frac{2}{3} \frac{d \ln t}{d E} f_i - \left(2 + \frac{2}{3} \frac{d \ln t}{d E} \right) x_i + \left(2 \frac{2}{3} \frac{d \ln t}{d E} + \frac{2}{3} \frac{dL_\gamma^2}{dE} \right) \right)^2 = t(f_i + L_\gamma^2 - Ex_i).$$  \hspace{1cm} (28)

Since $(x_i, f_i)$ is the locus of a point on the potential’s curve, we now have the explicit equation of that curve. The coefficients $d \ln t/dE$ etc that form this relation between $x_i$ and $f_i$ are functions of $E$, but the curve specified by the relation does not depend on $E$; $E$ specifies the slope of the straight lines along which individual points $(x_i, f_i)$ lie, not the curve itself.

The standard equation of a parabola is

$$y^2 = 4ax.$$  \hspace{1cm} (29)

When we make a general linear transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$  \hspace{1cm} (30)

this becomes

$$(\delta y' + \gamma x')^2 = 4a(\alpha x' + \beta y' + x_0)$$  \hspace{1cm} (31)

This equation has the same form as our equation (28) for the curve of the potential, so requiring $T_r$ to depend only on $E$ implies that the curve of $r^2\Phi$ versus $r^2$ is a parabola. $\Phi$ may diverge at the centre, but it will not do so more strongly than $1/r$, so $f = 2r^2\Phi \to 0$ as $r \to 0$, and the origin always lies on the potential’s parabola. Moreover

$$\frac{df}{dx} = \frac{d(x \Phi)}{dx} = \Phi + x \frac{d\Phi}{dx}$$  \hspace{1cm} (32)

so the gradient of the parabola at the origin is the central value of $\Phi$. Also, the gradient of the chord from the origin to an arbitrary point $(x, f)$ on the parabola is $f/x = \Phi(r)$; in particular, the value of the potential at infinity is given by the parabola’s asymptotic gradient.

To avoid the complex expressions in (28), we reformulate the expression in the form (31)

$$(\delta y + \gamma x + y_0)^2 = 4a(\alpha x + \beta y) + y_0^2,$$  \hspace{1cm} (33)

where the constant on the right has been chosen to ensure that the origin lies on the parabola. As $x \to \infty$, $f = x \Phi$ will grow too, and the left side of the equation will grow faster than the right side unless $\delta f$ tends to $\gamma x$. Hence

$$\delta \Phi_{\infty} = -\gamma$$  \hspace{1cm} (34)

and we may rewrite (33) in the form

$$[\delta (\Phi - \Phi_{\infty})]x + y_0]^2 = 4a(\alpha x + \beta \Phi + y_0^2$$  \hspace{1cm} (35)

If we choose to set the zero point of $\Phi$ at $x = \infty$, this quadratic equation for $\Phi(x)$ becomes

$$\delta^2 x \Phi^2 + (2\delta y_0 - 4\alpha \beta)\Phi - 4\alpha \alpha = 0,$$  \hspace{1cm} (36)

so

$$\Phi(x) = \frac{(4\alpha \beta - 2\delta y_0) \pm \sqrt{(2\delta y_0 - 4\alpha \beta)^2 + 16\delta^2 x \alpha \alpha}}{2\delta^2 x}$$  \hspace{1cm} (37)

We cast this into a more familiar form by multiplying top and bottom by the top with the optional sign reversed

$$\Phi(x) = -\frac{8\alpha \alpha}{(4\alpha \beta - 2\delta y_0) \pm \sqrt{(2\delta y_0 - 4\alpha \beta)^2 + 16\delta^2 x \alpha \alpha}}$$  \hspace{1cm} (38)

The plus sign must be chosen to ensure that the central potential is finite. Then setting

$$GM \equiv \frac{\sqrt{2\alpha \alpha}}{\delta}$$  \hspace{1cm} and  \hspace{1cm} $$b = \frac{\alpha \beta}{\sqrt{2\alpha \alpha}}$$  \hspace{1cm} (39)

we obtain the isochrone potential in its classic form

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}.$$  \hspace{1cm} (40)

Hénon plots the potential and its radial force out to $5b$. He gives the formula for the mass contained within radius $r$ and, taking the derivative of this formula, gives the generating density $\rho(r)$. He remarks that the density is everywhere positive and that $\rho \sim r^{-3}$ at large $r$. He plots the projected density out to $r = 5b$ together with data from five globular clusters, and the model is seen to fit the data to within the variation between clusters.

Then he asks “why do clusters resemble the isochrone?” He remarks that perturbations in the cluster potential due to individual stars will cause the cluster’s density profile to evolve until some stable configuration is reached. Normally the endpoint of this evolution is taken to be when the velocity distribution has become Maxwellian, but this distribution is only possible in the isothermal sphere, which necessarily has infinite mass.

Hénon argues that stars that share the same radial period are resonantly coupled, so they will share energy on a shorter timescale than stars that have different radial periods. In any other model than an isochrone, the set of resonantly coupled stars will contain stars that differ in energy. Consequently these stars will exchange energy. Hénon hypothesises that in these exchanges stars with more energy will gain energy from those with more energy. He further hypothesises that these energy changes will cause the mass distribution to evolve towards the isochrone model. Once this distribution has been reached, net energy exchanges between resonantly coupled stars will cease, and thus the cluster will cease to evolve.

Hénon recognises that a satisfactory exploration of this idea is very hard and proposes a preliminary test of concept. To develop such a test he considers “hyperbolic models” in which the curve $f(x)$ is a hyperbola rather than a parabola. For these models at fixed $E, T_r$ decreases with increasing $L$, with the consequence that at given $E$, circular orbits have shorter periods than eccentric orbits. Conversely, in the set of resonantly coupled stars, circular orbits have more energy.
than eccentric orbits. So in a hyperbolic model eccentric orbits will gain energy from from circular orbits. It follows that the circular orbits will shrink while eccentric orbits grow in size. But in any cluster the outer regions are dominated by eccentric orbits, so resonant interactions in a hyperbolic model will enhance the density in the outer regions, reducing the central concentration of the model. A plot of the density profiles of the isochrone and a hyperbolic model shows that the latter is more centrally concentrated than the former. Therefore resonant interactions in the latter will drive the hyperbolic model in the direction of the isochrone.

In the concluding section Hénon remarks that whatever the validity of the hypothesis of resonant interactions, the isochrone constitutes a realistic cluster model for which all quantities are analytically available.

3 ISOCRONE PAPERS TWO AND THREE

The second paper (Hénon 1959b) computes orbits in the isochrone. Hénon uses \( U = 1 - \Phi/\Phi(0) \) as the radial coordinate, explicitly integrating \( dU/dt = (dU/dr)r \) so obtain both \( t(U) \) and the angular coordinate \( \psi(U) \). He simplifies these formulae by introducing the potentials \( U_1 \) and \( U_2 \) at peri- and apo-centre. He tabulates \( U_1, r_i, T_i \) and the mean density along the orbit for several values of \( E \) and \( L \). He plots a few of these orbits.

The third paper (Hénon 1960) uses Eddington’s inversion formula to compute the isochrone’s distribution function (df) \( f(E) \). Hénon tabulates both \( f(E) \) and \( dN/dE \), the number of stars with a given energy.

4 WHY IS THE ISOCRONE IMPORTANT?

From the current perspective, the isochrone is one member of a family of useful spherical systems: the other members are the Jaffe (1983) model and the Hernquist (1990) model. These models are all derived from simple functional forms of \( \Phi(r) \):

\[
\begin{align*}
\Phi_{\text{Jaffe}}(r) &= -4\pi G \rho b^2 \ln(1 + b/r) \\
\Phi_{\text{Hern}}(r) &= -2\pi G \rho b^2 / (1 + r/b),
\end{align*}
\]

and for each one has analytic forms of both the density \( \rho(r) \) and the ergodic df \( f(E) \). What uniquely distinguishes the isochrone is the availability of analytic formulae for the actions and angles of its orbits. This availability is implicit in Hénon’s second paper but he does not mention this. The relevant formulae appear to have been first given by Saha (1991).

4.1 Role of the isochrone in torus mapping

Regular orbits are those whose time series \( x(t), z(t), \) etc., are quasiperiodic: when these time series are Fourier transformed, all the frequencies that occur can be expressed as integer linear combinations of three fundamental frequencies. Arnold (1978) shows that a quasiperiodic orbit admits three isolating integrals, which confine the orbit to a three-torus in six-dimensional phase space. These tori are null in the sense that the Poincaré invariant \( \sum_i \int dx_i \cdot dv_i \) of every two surface in the torus vanishes. The natural labels of the tori are the actions

\[
J_i = \frac{1}{2\pi} \int_{\gamma_i} \mathbf{v} \cdot d\mathbf{x}
\]

where for \( i \neq j \) and \( \gamma_i \) are two closed paths around the torus that cannot be distorted into one another while staying within the torus. These are the only labels of individual tori that can be complemented by canonically conjugate variables \( \theta_i \), the “angle” variables. The \( \theta_i \) specify positions with a torus.

If a canonical transformation is used to map a null torus, it remains a null torus, and if the image torus can be restricted to a constant-energy surface, it becomes an orbital torus: a surface to which an orbiting particle is confined by motion under the Hamiltonian. The idea behind torus mapping is to arrange for the Hamiltonian to be constant on a null torus of pre-determined actions by adjusting the parameters of a canonical transformation such that \( H(x, \mathbf{v}) = \) constant on the image torus (McGill & Binney 1991). If phase space – or a non-negligible part of it – can be foliated with orbital tori in this way, the images of the angle-action coordinates of the analytic tori become angle-action coordinates of the Hamiltonian \( H \). Any analytic tori can be used for this purpose, but in practice one tries to use tori that are as similar as possible to the orbital tori of \( H \). In this regard the isochrone potential is well suited to mapping tori into orbital tori for many galactic potentials, and it was the choice of McGill & Binney (1991). The harmonic-oscillator potential (which is really a limiting case of the isochrone potential) has also been used for torus mapping (Kaasalainen & Binney 1994).

If the Hamiltonian \( H \) is not integrable (i.e., many of its orbits are not quasiperiodic) it will prove impossible to arrange for \( H \) to be constant on the image torus for some or all action values. Nevertheless, one can foliate phase space with the image tori that result from minimising the variance of \( H \) over the image torus. Then if we define \( \overline{H(J)} \) to be the angle-averaged value of \( H \) on the image torus with actions \( J \), \( \overline{H} \) becomes an integrable Hamiltonian with explicitly known angle-action coordinates, and the difference \( \Delta(x, \mathbf{v}) = H(x, \mathbf{v}) - \overline{H} \) becomes a small perturbation of this integrable Hamiltonian that yields the original Hamiltonian. Thus torus mapping allows us to study motion in a general Hamiltonian as perturbation of a very close integrable Hamiltonian (Kaasalainen & Binney 1994).

For over a decade after its introduction, torus mapping found little application. In the last several years it has proved a valuable tool for modelling our Galaxy (McMillan & Binney 2008; McMillan 2011; McMillan & Binney 2013) and I believe it will play a significant role in the scientific exploitation of the billion-Euro Gaia survey.

4.2 Flattened isochrones

Since Hénon gave is the isochrone’s df \( f(H) \) and we know how to write \( H \) as a function of the actions \( J \), it is trivial to express the isochrone’s \( f \) as a function of \( J \).

Since angle-action coordinates are canonical, the Jacobian between these coordinates and ordinary \( (x, \mathbf{v}) \) coordinates is unity, and the density of stars in angle-action space is \( f(J) \). Moreover, the range of each angle coordinate is al-
ways (0, 2π), so the phase-space volume occupied by orbits with actions in d^3J is (2π)^3 d^3J and thus the density of stars in three-dimensional action space is (2π)^3 f(J). That is, to within an uninteresting constant factor, the function f(J) that completely characterises the dynamics of a stellar population is the density of stars in a readily imagined three-dimensional space.

In an ergodic model such as Hénon’s isochrone, the action-space density of stars is constant on surfaces H(J) = constant, which are roughly triangular surfaces k · J = constant, where k is a vector whose direction is determined by H. In this ergodic model all three velocity dispersions are equal.

If we shift stars over surfaces of constant H, so the action-space density of stars becomes non-uniform on these surfaces, we generate a model that has velocity anisotropy: if we shift the stars towards the J_3 axis, we generate radial anisotropy. If we shift the stars away from the J_3 axis, we flatten the model. So long as we leave unchanged the number of stars on each constant-E surface, the radial density profile is essentially unchanged: the model may become elongated, or flattened, and it may develop velocity anisotropy, but it retains essentially the same radial density profile.

A significant advantage of considering the DF to be a function of the actions rather than a function of energy and other isolating integrals is that it is then very easy to find the gravitational potential of the self-consistent model that has the given DF. One simply guesses a potential Φ_0 and on a spatial grid evaluates the density μ_{1/2}(x) = ∫ d^3v f(J) implied by the DF in that potential. One then solves Poisson’s equation for the corresponding potential Φ_{1/2}(x) at the grid points. Then one takes as a new guess of the self-consistent potential

\[ Φ_1(x) = (1 + γ)Φ_{1/2}(x) - γΦ_0(x) \]  

with γ ≃ 0.5, and repeats the process, which converges after 3–5 iterations – see Figure 2.

A prerequisite for this program is the ability to evaluate the action integrals given an arbitrary point (x, v). A very convenient technique, which is remarkably accurate for modified isochrone models, is the “Stäckel Fudge” of Binney (2012), and this is the technique used by Binney (2014) to explore flattened isochrone models.

There are several ways in which one can change an ergodic DF into a flattened model and it remains unclear what the best approach is. Binney (2014) replaced each action J_i in the ergodic DF by α_i J_i. If, for example, α_i > 1, the new DF decreases with increasing J_i more rapidly than in the ergodic DF and the model becomes tangentially biased, and conversely if α_i < 1. If α_z > 1, motion perpendicular to the equatorial plane is being discouraged, and the model becomes flattened.

5 CONCLUSIONS

Hénon’s isochrone is a fine example of the kind of curiosity-driven research that is so much discouraged by the current funding environment. There is no way that Hénon could have justified to a grants committee his effort to find the most general potential that made T_v independent of angular momentum. I suppose he was first just curious, and then became fascinated by the puzzle posed by the search for the isochrone. As Section 1 shows, solving this puzzle required a tour de force, and was surely not done in some spare afternoon.

Once Hénon had found the isochrone, he was impressed by how well it represented globular clusters. And rightly so, because it is really very surprising that a potential chosen to have orbits with a particular property should even have a non-negative generating density; that it also have a non-negative DF is improbable; that it should also provide an excellent fit to globular clusters is astounding.

Hénon’s attempt at a physical explanation of the closeness of globular clusters to the isochrone is highly ingenious and shows deep physical insight. Within the last decade resonant relaxation has become fashionable in studies of galactic nuclei and of planetary systems, but in 1958 Hénon’s use of the idea was ground-breaking. That said, I don’t think it was convincing. Yes stars with common values of T_v can more readily exchange energy than stars with unrelated frequencies, but it is far from clear that the sense of that exchange will be to establish equipartition of energy: the essence of the gravitational catastrophe is that in self-gravitating systems, stars with less energy lose energy to those with more, so in self-gravitating systems wealth inequalities grow as fast as they do just now in the United States. Moreover, why the focus on stars that have equal T_v, rather than T_φ?

Hénon was recognised that he was only scratching the surface of how a globular cluster might evolve through resonant relaxation. I think this problem remains an intriguing issue in dynamics, and one that might now be addressed with the help of techniques, torus mapping and taking actions as the arguments of DFs, that make extensive use of Hénon’s delightful isochrone.
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REFERENCES

Arnold V.I., 1978, “Mathematical methods of classical mechanics”, New York, Springer
Binney, J., 2012, MNRAS, 426, 1324
Binney, J., 2014, MNRAS, 440, 787
Eddington, A.S., 1915, MNRAS, 76, 37
Hénon M., 1959a, Ann Ap, 22, 126
Hénon M., 1959b, Ann Ap, 22, 491
Hénon M., 1960, Ann Ap, 23, 474
Hernquist L., 1990, ApJ, 356, 359
Jaffe W., 1983, MNRAS, 202, 995
Kaasalainen, M. & Binney, J., 1994a, MNRAS, 268, 1033
Kaasalainen, M. & Binney, J., 1994b, PhRvL, 73, 2377
McGill C., Binney J., 1990, MNRAS, 244, 634
McMillan P.J., 2011, MNRAS, 418, 1565
McMillan P.J., Binney J.J., 2008, MNRAS, 390, 429
McMillan P.J., Binney, J., 2013, MNRAS, 433, 1411
Saha P., 1991, MNRAS, 248, 494