Fourier–Mukai partners of Enriques and bielliptic surfaces in positive characteristic

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1 Introduction

In this note we prove the following result.

Main Theorem. If $X$ is an Enriques (resp. bielliptic) surface over an algebraically closed field $k$ of characteristic at least 3 (resp. 5), then it does not have any nontrivial Fourier–Mukai partners.

This was proven over the complex numbers by Bridgeland and Maciocia [2]. Their argument uses Torelli theorems to reduce the result to lattice-theoretic assertions, and thus relies heavily on the complex numbers. Standard spreading out methods allow one to deduce the general characteristic 0 case from the complex case, but these do nothing helpful in positive characteristic.

Our approach to proving this in positive characteristic is to lift derived equivalences to characteristic 0 and specialize isomorphisms using the Matsusaka–Mumford theorem. The strategy is similar to that taken in [8]. In the present situation, the central issue is the question of whether the fibers of the kernel, viewed as complexes on $Y$, are smooth points in the moduli space of complexes, something that is automatic for K3 surfaces since obstruction spaces themselves vanish. While the obstruction spaces for simple complexes on Enriques and bielliptic surfaces are nonvanishing, we can nevertheless show that the obstruction classes vanish by using equivariant techniques similar to those originally used in [2].

In Section 2 we briefly discuss the positive characteristic version of the equivariance results of [3]. In Section 3 we study the deformation theory of complexes on Enriques and bielliptic surfaces and produce an isomorphism $\text{Def}_X \rightarrow \text{Def}_Y$ of deformation spaces that gives us a way to lift kernels. In Section 4 we complete the proof of the main theorem.

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2 Equivariant lifts of derived equivalences

Fix a base scheme $S = \text{Spec}(A)$ with $A$ a Henselian local ring with algebraically closed residue field, and fix proper smooth morphisms $X \to S$ and $Y \to S$. Suppose the relative dualizing sheaves $\omega_{X/S}$ and $\omega_{Y/S}$ are torsion of order $n$, with $n$ invertible in $S$. Given trivializations $\omega_{X/S}^\otimes n \to \mathcal{O}_X$ and $\omega_{Y/S}^\otimes n \to \mathcal{O}_Y$, let $X \to X$ and $Y \to Y$ be the finite étale covers corresponding to the associated classes in $H^1(X, \mu_n)$ and $H^1(Y, \mu_n)$, respectively.

Bridgeland and Maciocia [3] studied the problem of lifting derived equivalences $D(X) \to D(Y)$ to $D(X) \to D(Y)$ when $S = \text{Spec} C$. Their methods can be adapted to prove a relative form of this result. With the assumption that $n$ is invertible in $A$, the proofs written in [3] translate word-for-word to our setting. We have chosen not to reproduce the proofs here, but rather briefly indicate how things work.

**Proposition 2.1.** Given a Fourier–Mukai equivalence $\Phi_P : D^b(X/S) \to D^b(Y/S)$ with kernel $P \in D^b(X \times_S Y/S)$, there exists $\bar{P} \in D^b(X \times_S Y/S)$ so that $\Phi_{\bar{P}} : D^b(X) \to D^b(Y)$ is an equivalence and the following diagrams commute:

$$
\begin{array}{ccc}
D^b(X/S) & \xrightarrow{\Phi_P} & D^b(Y/S) \\
\pi_X \downarrow & & \downarrow \pi_Y \\
D^b(X/S) & \xrightarrow{\Phi_{\bar{P}}} & D^b(Y/S)
\end{array}
\quad
\begin{array}{ccc}
D^b(X/S) & \xrightarrow{\Phi_{\bar{P}}^{-1}} & D^b(Y/S) \\
\pi_X \downarrow & & \downarrow \pi_Y \\
D^b(X/S) & \xrightarrow{\Phi_P^{-1}} & D^b(Y/S)
\end{array}
\quad (2.1.1)
$$

An equivalence $\Phi_{\bar{P}}$ making the diagrams (2.1.1) commute is called lift of $\Phi_P$.

**Sketch of proof.** The arguments of [3, Lemma 4.4 and Proposition 2.5(b)] work verbatim in the relative case to produce $\bar{P}$ so that $\Phi_{\bar{P}}$ is a lift of $\Phi_P$. The methods rely on a simple theorem about equivariant objects for cyclic coverings that we state here. Suppose $Z$ is a proper $A$-scheme, $L$ is an invertible sheaf on $Z$, and $\sigma : L^\otimes n \to \mathcal{O}_Z$ is an isomorphism. Write

$$
\bar{Z} = \text{Spec}_Z \bigoplus_{i=1}^{n-1} L^\otimes i \to Z
$$

for the $\mu_n$-cover associated to the pair $(L, \sigma)$. Given an object $E$ of $D_{\text{Qcoh}}(Z)$ (the bounded derived category with quasi-coherent cohomology), there is some $\tilde{E} \in D_{\text{Qcoh}}(\bar{Z})$ such that $p_* \tilde{E} \cong E$ if and only if there is an isomorphism $E \to E \otimes L$. By properties of kernels of Fourier–Mukai equivalences, the backwards direction of this statement implies that lifts of kernels of Fourier–Mukai kernels in our setting exist. The proof roughly goes like this: proceed by induction and use exactness of tensoring with $L$ to reduce to the case that $\tilde{E}$ is a quasi-coherent sheaf. The isomorphism gives an action of $\bigoplus_{i=0}^{n-1} L^\otimes i$ on $E$. Sheaves with such an action that is furthermore compatible with $\sigma$ are naturally equivalent to pushforwards $\mathcal{O}_Z$-modules. There is a subtle point here: one needs to know that the action of $\bigoplus L^i$ on $E$ can be made compatible via $\sigma$ with the $\mathcal{O}_Z$-module action, and for that one might need to take $n$th roots of elements of $\mathcal{O}_Z(Z)$. Since $Z$ is proper over $A$ and $A$ is Henselian with algebraically closed residue field, $\mathcal{O}_Z(Z)$ is a finite product of Henselian local rings with algebraically closed residue fields. Thus, since $n$ is invertible in $A$, we can take $n$th roots in $\mathcal{O}_Z(Z)$.

Once a lift $\bar{P}$ of $P$ exists, one can prove that $\Phi_{\bar{P}}$ is an equivalence by appealing to a derived form of the Nakayama lemma. That is: $\Phi_{\bar{P}}$ has left and right adjoints, with kernels $\bar{L}$ and $\bar{R}$. The adjunctions induce maps on kernels $\bar{L} \circ \bar{P} \to \Delta_* \mathcal{O}_X$ and $\Delta_* \mathcal{O}_Y \to \bar{P} \circ \bar{R}$. The restrictions of these morphisms to
the residue field of $A$ are isomorphisms by \cite[Lemma 4.3(a)]{3}, whose proof works over any algebraically closed field in which $n$ is invertible. Hence, by Nakayama’s Lemma, $\Phi_P$ is an equivalence. 

\section{Lifting equivalences}

In this section we study the problem of deforming equivalences between Enriques and bielliptic surfaces from positive characteristic to characteristic 0. The key to lifting equivalences lies in showing that the fibers of the kernel over $X$ are smooth points in the stack of simple perfect complexes on $Y$.

Throughout this section, given a (bielliptic or Enriques) surface $X$, we will let $\pi : \tilde{X} \to X$ denote the canonical cover.

\subsection{Lemmas on the deformation theory of complexes}

Let $A \to A_0$ a square-zero extension of rings with kernel $I$.

\begin{lemma}
Let $\gamma : Z \to X$ be a finite étale morphism of flat schemes over $A$. Given a relatively perfect complex $P \in D(Z_{A_0})$, the natural map
\[ \text{Ext}^2_{Z_0}(P, P \otimes I) \to \text{Ext}^2_{Z_0}(L\gamma^*P, L\gamma^*P \otimes I) \]

sends the obstruction to deforming $P$ over $A$ to the obstruction to deforming $L\gamma^*P$ over $A$.
\end{lemma}

\begin{proof}
The proof of this result we offer here is undoubtedly far from ideal. Unfortunately, there does not appear to be an argument in the literature that is general enough to work without resorting to the techniques used in \cite[Section 3]{7}. The idea there is as follows: given the complex $P$, one replaces it by a “good resolution” $P' \to P$, which is a quasi-isomorphism in which $P'$ has terms of the form $\bigoplus j!O_U$ for étale morphisms $j : U \to X$ with $U$ affine. The complex $P'$ is itself $K$-flat (in the sense of \cite{13}), so it can be used to compute derived tensor products. The obstruction class arises by computing good resolutions of $P$ over $A_0$ and over $A$ and then making an explicit map

\[ Q \to P \otimes_{A_0} I[1] \]

in $D^b(X_0)$, where $Q$ is the homotopy limit of the derived adjunction map $P \otimes_A A_0 \to P$. (Details may be found in \cite[Construction 3.2.8]{7}.)

Consider the Cartesian diagram

\[ \begin{array}{ccc}
U' & \rightarrow & Z \\
\downarrow{j'} & & \downarrow{\gamma} \\
U & \rightarrow & X,
\end{array} \]

To establish the appropriate functoriality of the obstruction class, it suffices to show that $\gamma^*j_!O_U = j'_!(\gamma')^*$. By flat base change we have a canonical isomorphism of functors $j^*\gamma_* = \gamma'_*(j')^*$, giving rise to a canonical isomorphism $\gamma^*j_! = j'_!(\gamma')^*$. This gives the desired result.
\end{proof}

\begin{lemma}
Suppose $Z$ is a smooth projective surface over $A$ such that $\omega_{Z/A} \cong O_Z$. Given a relatively perfect simple complex $P$ on $Z_0$, the trace map
\[ \text{Ext}^2_{Z_0}(P, P) \to \text{Ext}^2_{Z_0}(O, O) \]

3
sends the obstruction to deforming \( P \) to \( Z \) to the obstruction to deforming the determinant \( \det P \in \text{Pic}(Z_0) \) to \( Z \).

**Proof.** This is a bit of a folk theorem that lacks a general write-up. The closest thing to a proof that applies in the present case is the proof (but not the statement!) of [14, Theorem 3.23]. There is also a derived version of the statement in [12, Section 3], but [ibid.] works over \( \mathbb{C} \) rather than a general base. Finally, there is a similar statement at the end of [6], which works over general Noetherian bases, but there are flatness assumptions that do not hold in the present context (or, more generally, in the context of infinitesimal deformation problems where the base is not a field). \( \square \)

**Lemma 3.1.3.** Suppose \( \gamma : Z \to X \) is a finite flat morphism of smooth proper relative surfaces over \( A \) such that \( \omega_{Z/A} \cong \mathcal{O}_Z \). Suppose \( Q \) is a relatively perfect complex on \( X \) such that

1. the invertible sheaf \( \det Q \) is unobstructed with respect to the extension \( Z_0 \subset Z \);
2. we have
   \[
   L \gamma^* Q \cong \bigoplus_{i=1}^m Q_i
   \]
   for some \( m \) invertible in \( A_0 \) with each \( Q_i \) a simple relatively perfect complex on \( Z \);
3. for each \( i = 2, \ldots, m \) we have
   \[
   \det Q_i \cong \det Q_1 \otimes \Lambda_i
   \]
   for some \( \Lambda_i \in \text{Pic}(Z) \) that is unobstructed with respect to the extension \( Z_0 \subset Z \).

Then the complex \( Q \) is unobstructed.

**Proof.** By Lemma 3.1.1, it suffices to show that \( L \gamma^* Q \) is unobstructed, and further by Lemma 3.1.2 it suffices to show that \( \det(Q_1) \) is unobstructed. By assumption,

\[
L \gamma^* Q \cong \det(Q_1)^{\otimes m} \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_n.
\]

Since the \( \Lambda_i \) and \( \det(Q) \), and hence \( \gamma^* \det(Q) \) are liftable, we have \( m \text{ob}(\det Q_1) = 0 \). Since \( m \) is invertible in \( A_0 \), we conclude that \( \text{ob}(\det Q_1) = 0 \). \( \square \)

**Lemma 3.1.4.** Let \( X \) be an Enriques (resp. bielliptic) surface over an Artinian local ring \( A \) with algebraically closed residue field \( k \) of characteristic at least 3 (resp. 5); let \( n \) be the order of \( \omega_X \) in \( \text{Pic}(X) \). Given a section \( x \in X(A) \), there is a decomposition

\[
L \pi_Y^* \Phi_P(\mathcal{O}_x) \cong \bigoplus_{i=1}^n Q_i
\]

in \( D(\tilde{Y}) \) that satisfies the conditions of Lemma 3.1.3.

**Proof.** Suppose

\[
\Phi_P : D(\tilde{X}) \to D(\tilde{Y})
\]

is the canonical covering equivalence induced by \( \Phi_P \) as described in Section 2. By the commutativity of diagrams (2.1.1), there is an isomorphism

\[
L \pi_Y^* \Phi_P(\mathcal{O}_x) \cong \Phi_P(L \pi_X^*(\mathcal{O}_x)).
\]
On the other hand, \( L \pi_X^*(\mathcal{O}_x) = \mathcal{O}_{x_1} \oplus \cdots \oplus \mathcal{O}_{x_n} \), where \( \{x_1, \ldots, x_n\} = \pi_X^{-1}(x) \). Note that \( \mathcal{O}_{x_1}, \ldots, \mathcal{O}_{x_n} \) all have the same Mukai vector. It follows that

\[
\Phi_{x_1}, \ldots, \Phi_{x_n} \in \mathcal{D}(\tilde{Y})
\]

have the same Mukai vectors. Thus, the determinants of \( \Phi_{x_1}, \ldots, \Phi_{x_n} \) differ by elements of \( \text{Pic}^0(\tilde{Y}) \). Since \( Y \) is either Enriques or bielliptic (so that \( \tilde{Y} \) is either K3 or abelian), the elements of \( \text{Pic}^0(\tilde{Y}) \) are unobstructed. The hypothesis of Lemma 3.1.3 is thus satisfied. \( \square \)

### 3.2 An isomorphism of deformation functors

Suppose \( k \) is algebraically closed of characteristic at least 3 (resp. 5) and \( \mathcal{D}(X) \to \mathcal{D}(Y) \) is a Fourier–Mukai equivalence with \( X \) an Enriques (resp. bielliptic) surface.

**Lemma 3.2.1.** If \( X \) is Enriques (resp. bielliptic) then \( Y \) is Enriques (resp. bielliptic).

**Proof.** Derived equivalence of smooth \( k \)-schemes preserves the dimension and the order of the canonical sheaf. Thus, \( Y \) is a surface with torsion canonical sheaf. In addition, given an \( \text{H}^* \) and a Fourier–Mukai equivalence 

\[
\Phi_P : \mathcal{D}(X) \to \mathcal{D}(Y),
\]

we have induced isomorphisms

\[
\Phi_P^{\text{odd}} : H^{\text{odd}}(X) \to H^{\text{odd}}(Y) \quad \text{and} \quad \Phi_P^{\text{even}} : H^{\text{even}}(X) \to H^{\text{even}}(Y)
\]

([4, Lemma 3.1]). This, together with Poincaré duality, implies that derived equivalent surfaces have the same \( \ell \)-adic Betti numbers. We conclude that \( Y \) is Enriques (resp. bielliptic) using Bombieri-Mumford classification of surfaces in positive characteristic [1]. \( \square \)

Given a morphism \( W \to B \), let \( \mathcal{P} \text{erf}_{W/B} \) denote the stack of geometrically simple universally glueable relatively perfect complexes (as described in [7]). Since the objects are simple, \( \mathcal{P} \text{erf}_{W/B} \) is a \( \mathbb{G}_m \)-gerbe over an algebraic space that we will denote \( \text{Perf}_{W/B} \).

**Proposition 3.2.2.** Let \( A_0 \) be an Artinian augmented \( W(k) \)-algebra and let \( \Phi_{P_0} : \mathcal{D}(X_0) \to \mathcal{D}(Y_0) \) be a relative Fourier–Mukai equivalence for a pair \( X_0, Y_0 \) of Enriques or bielliptic surfaces over \( A_0 \), which has an associated morphism 

\[
X_{A_0} \to \text{Perf}_{Y_0/A_0}.
\]

For any \( x \in X(A_0) \), the associated complex \( \Phi(k(x)) \in \mathcal{D}(Y_{A_0}) \) lies in the smooth locus of \( \text{Perf}_{Y_0/A_0} \).

**Proof.** This follows immediately from Lemma 3.1.4 \( \square \)

**Theorem 3.2.3.** Under the same hypotheses as Proposition 3.2.2, there is an isomorphism of formal deformation functors

\[
\rho : \text{Def}_{Y_0} \to \text{Def}_{X_0}
\]

such that for any deformation \( \iota : Y_0 \hookrightarrow Y_A \in \text{Def}_{Y_0}(A) \) there is a complex \( P_A \in \mathcal{D}(Y_A \times \rho(Y_A)) \) with \( L \iota^* P_A \cong P_0 \).
Proof. Since the smooth locus of $\text{Perf}_{Y_0/A_0}$ is open, Proposition 3.2.2 implies that $X$ itself lands in the smooth locus. By [8, Lemma 5.2], the image of $X$ is an open subscheme of $\text{Perf}_{Y_0/A_0}$. Since open subschemes of the smooth locus canonically lift, we get an induced deformation $X_A$. Consider the preimage
\[ X_A := \text{Perf}_{Y_0/A_0} \times_{\text{Perf}_{Y_0/A_0}} X_A. \]
This is a $\mathbb{G}_m$-gerbe over $X_A$ [7, Corollary 4.3.3] that is trivializable over $A_0$, and there is a complex $\mathcal{P}_A$ on $X_A \times_{\text{Spec} A} Y_A$ and an invertible twisted sheaf $M_0$ on $X_A$ such that $\mathcal{P}_k \cong P|_{X_A} \otimes M_0$. Since $H^2(X_0, \mathcal{O}) = 0$, the sheaf $M_0$ deforms to some invertible $X_A$-twisted sheaf $M$ over all of $X_A$. Taking the pushforward of $\mathcal{P}_A \otimes M$ to $X_A \times Y_A$ gives a deformation of $P_0$, as desired.

Theorem 3.2.4. Given a complete local Noetherian ring $R$ with residue field $k$ and a lift $Y_R \to \text{Spec} R$ of $Y/k$, there is a lift $X_R \to \text{Spec} R$ and a relative Fourier–Mukai equivalence $\Phi_{P_R} : D(X_R) \to D(Y_R)$.

Proof. Given a deformation $Y_R$ of $Y$, from Theorem 3.2.3 we get induced formal deformations of $X$ and $P$ to $X_R \in \text{Def}_X(R)$ and $\Psi_R \in D(X_R \times_{\text{Spec} R} Y_R)$.

Since $H^2(X, \mathcal{O}) = 0$, any ample invertible sheaf on $X$ lifts to $X_R$, so we can algebraize $X_R$ to the completion of a relative Enriques surface $X_R$. The Grothendieck Existence Theorem for perfect complexes [7, Proposition 3.6.1] algebraizes $\Psi_R$ to a complex
\[ P_R \in D(X_R \times_{\text{Spec} R} Y_R). \]

As in [8, Proof of Theorem 6.1], Nakayama’s lemma then implies that $P_R$ is a relative Fourier–Mukai equivalence, as desired.

4 Proof of the Main Theorem

We can now complete the proof of the main theorem. Given an equivalence $F : D(X) \to D(Y)$, by [10, Theorem 3.2.1], there is some kernel $P \in D(X \times Y)$ so that $F$ is naturally isomorphic to the Fourier–Mukai equivalence $\Phi_P$. Enriques or bielliptic surfaces of characteristic at least 3 or 5 can be lifted to (a possibly finite extension of) the Witt vectors; see [5, Proposition 6.1] and Partsch [1]. By Theorem 3.2.4, given any lift $Y_R$ of $Y$ over a finite flat $W(k)$-algebra $R$ there are induced deformations $X_R$ and $P_R \in D(X_R \times_{\text{Spec} R} Y_R)$ giving a relative Fourier–Mukai equivalence. By [7, Proposition 3.6.1], the generic fiber of $P_R$ induces an equivalence on the generic fibers of $X_R$ and $Y_R$, which are defined over the field $K := \text{Frac}(R)$ of characteristic 0: $\Phi_K : D(X_K) \simeq D(Y_K)$. Hence, for any field extension $L/K$, the base change $P_L \in D(X_L \times_Y Y_L)$ induces a Fourier–Mukai equivalence $D(X_L) \simeq D(Y_L)$. In particular we may choose an embedding $K \to \mathbb{C}$, yielding a Fourier–Mukai equivalence $\Phi_{PC} : D(X_C) \simeq D(Y_C)$.

By [2, Propositions 6.1, 6.2], $X_C$ and $Y_C$ are isomorphic. Spreading out, we find a finite extension $K'$ of $K$ such that there is an isomorphism $X_{K'} \simeq Y_{K'}$ over $K'$. The normalization $R'$ of $R$ in $K'$ is a DVR with residue field $k$ and fraction field $K'$. Since invertible sheaves are unobstructed, this isomorphism preserves relative polarizations over $R'$, and we may use [9, Theorem 2] to conclude that the special fibers of $X_{R'}$ and $Y_{R'}$ are isomorphic, as desired.
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