Discrete Nonlinear Schrödinger Equations Free of the Peierls-Nabarro Potential

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We derive a class of discrete nonlinear Schrödinger (DNLS) equations for general polynomial nonlinearity whose stationary solutions can be found from a reduced two-point algebraic problem. It is demonstrated that the derived class of discretizations contains subclasses conserving classical norm or a modified norm and classical momentum. These equations are interesting from the physical standpoint since they support stationary discrete solitons free of the Peierls-Nabarro potential. As a consequence, even in highly-discrete regimes, solitons are not trapped by the lattice and they can be accelerated by even weak external fields. Focusing on the cubic nonlinearity we then consider a small perturbation around stationary soliton solutions and, solving corresponding eigenvalue problem, we (i) demonstrate that solitons are stable; (ii) show that they have two additional zero-frequency modes responsible for their effective translational invariance; (iii) derive semi-analytical solutions for discrete solitons moving at slow speed. To highlight the unusual properties of solitons in the new discrete models we compare them with that of the classical DNLS equation giving several numerical examples.

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I. INTRODUCTION

Recent theoretical and experimental results have demonstrated that the fundamental properties of waves can be precisely engineered by introducing a periodic modulation of the medium characteristics. In particular, there appear unique possibilities to control the propagation of light in photonic structures with a periodic modulation of the optical refractive index [1], and manage the flow of atomic Bose-Einstein condensates (BEC) in periodic potentials [2]. Additional flexibility in tailoring wave dynamics can be realized through nonlinear self-action, that may appear due to various physical mechanisms such as light-matter interactions for optical beams or atom-atom scattering in BEC. One of the key effects of nonlinearity is the suppression of the natural tendencies of localized wave packets to broaden due to dispersion or diffraction, supporting the formation of optical lattice solitons [3, 4] and self-localized atomic states [5, 6].

The excitation of lattice solitons and their dynamics can be described by the discrete nonlinear Schrödinger (DNLS) equation when the wave propagation is primarily defined by tunneling between neighboring potential wells [3, 7]. This special regime of energy flow results in special properties of discrete solitons. It was predicted theoretically [8–12] and observed experimentally [13, 14] that the discrete solitons can freely propagate through the lattice below a certain energy threshold, whereas at higher energies the solitons become trapped at a particular lattice site. This phenomenon is related to the presence of the self-induced Peierls-Nabarro potential (PNp) barrier, defining the difference of energies between the solitons whose centers are positioned at a lattice site or in-between the neighboring lattice sites.

The ability to enhance or suppress PNp may allow for precise control over transmission of higher-energy wave packets through the lattice. The PNp vanishes for discrete solitons in the framework of the Ablowitz-Ladik (AL) equations [15, 16] which are integrable. One of the features of the AL model is nonlocal nonlinearity, as the medium response is defined by the wave amplitudes at the neighboring lattice sites, in contrast to the local on-site nonlinearity that leads to strong PNp [8]. However, the AL model has not been directly connected to a specific physical application. On the other hand, more recently, it was demonstrated that nonlocal nonlinearity of a more general type representing real physical systems can indeed lead to strong reduction of PNp [17, 18] for lattice solitons. It was also shown that PNp can be reduced (and even reversed) in the case of a saturable nonlinear response [19].

In this paper, we consider a class of DNLS equations featuring a general type of nonlinearity characterized by a nonlocal response as well as an arbitrary polynomial dependence on the intensity, that includes the case of nonlocal saturable nonlinearity. We show that, under certain conditions, the PNp can be made exactly zero. Quite remarkably, the corresponding equations are generally not integrable, leading to nontrivial soliton dynamics. We present examples of equations that conserve the total en-
ergy (or norm), suggesting that the predicted effects can be observed in specially engineered periodic structures.

Our presentation is structured as follows. In section II, we present the general setup of the continuum and discrete models of the present work. In section III, we derive the class of PNP-free DNLS equations and extract two subclasses conserving different physical quantities. The case of cubic (Kerr) nonlinearity is considered in detail in section IV. Finally, in section V, we briefly summarize our findings and present our conclusions.

II. SETUP

The PNP potential is a feature of discrete models and does not appear for continuous and translationally invariant nonlinear Schrödinger (NLS) equation. We demonstrate below that, by performing appropriate discretization of the continuous NLS equation, it is possible to obtain a broad class of DNLS equations where PNP is also absent. We note that a conceptually similar approach has been developed for Klein-Gordon type lattices [20–27] and is now starting to emerge in NLS settings as well [28]. We also note in passing the interesting connection of the continuous NLS equation, it is possible to demonstrate below that, by performing appropriate discretization of the class of PNp-free DNLS equations and extract two analogues of Eq. (1) of the form:

\[ \psi(x,t) = f(x)e^{i\omega t}, \]

we reduce it to an ordinary differential equation for the real function \( f(x) \),

\[ D(x) \equiv f'' - 2\omega f + 2fG'(f^2) = 0, \]

having the first integral

\[ u(x) \equiv (f')^2 - 2\omega f^2 + 2G(f^2) + C = 0, \]

where \( C \) is the integration constant.

We then identify discretizations of Eq. (10) of the form

\[ D(f_-, f_n, f_+) = 0, \]

such that solutions to the three-point discrete Eq. (12) can be found from a reduced two-point problem

\[ u(f_-, f_n) \equiv \frac{1}{2\hbar^2} (f_n - f_-)^2 - 2\omega f^- f_n + 2G(f_-, f_n) + C = 0, \]

which is a discrete version of Eq. (11), assuming that \( G(f_-, f_n) \) reduces to \( G(f^2) \) in the continuum limit \( (\hbar \rightarrow 0) \). In the present study, we will set \( C = 0 \), which is
sufficient for obtaining the single-humped stationary solutions. However, the case of arbitrary $C$ enables one to construct all possible stationary solutions to the corresponding discrete model [32].

Taking into account that Eq. (10) is the static Klein-Gordon equation with the potential

$$V(f) = \omega f^2 - G(f^2),$$

(14)
a wide class of discretizations solving the auxiliary problem has been offered in the very recent work of [23, 24].

For example, discretizing the left-hand side of the identity (1/2)du/dt = D(x), we obtain the discrete version of Eq. (10),

$$D_t(f_-, f_n, f_+) = \frac{u(f_n, f_+) - u(f_-, f_n)}{f_+ - f_-} = 0. \quad (15)$$

Formally, $D_t(f_-, f_n, f_+) = 0$ is a three-point problem but, clearly, its solutions can be found from the two-point problem $u(f_-, f_n) = 0$ and thus, the auxiliary problem is solved. We note, in passing, that this type of argument was first proposed in [22].

Before we come back to our main problem of finding PNP-free discretizations for Eq. (1), we should remark that among the solutions to the auxiliary problem we should select the ones which can be rewritten in terms of $\psi_n$ and $\psi_n^*$ in the desired form of Eq. (3). This can be done easily, e.g., when $D_1$ given by Eq. (15) is written in a non-singular form (i.e., when the denominator cancels with an appropriate factoring of the numerator). This always occurs if $G(\xi)$ is polynomial and if $u(f_-, f_n)$ possesses the symmetry

$$u(f_-, f_n) = u(f_n, f_-). \quad (16)$$

We thus focus on $G(\xi)$ in the form of Taylor expansion,

$$G(|\psi|^2) = \sum_{k=1}^{\infty} a_k |\psi|^{2k}, \quad (17)$$

with real coefficients $a_k$.

The most general polynomial, two-point discrete version of Eq. (17), possessing the symmetry $G(f_-, f_n) = G(f_n, f_-)$ is

$$G(\psi_-) = \sum_{k=1}^{\infty} a_k Q_{2k}(f_-, f_n), \quad (18)$$

where $Q_{2k}(f_-, f_n)$ involves only the terms of order $2k$:

$$Q_{2k}(f_-, f_n) = \frac{\sum_{i=0}^{k} q_{i,k} (f_i f_n^{2k-i} + f_n^{2k-i} f^{-i})}{2 \sum_{i=0}^{k} q_{i,k}}, \quad (19)$$

with $q_{i,k}$ being free parameters such that $\sum_{i=0}^{k} q_{i,k} \neq 0$.

Then we substitute Eq. (13) with $G(f_-, f_n)$ given by Eq. (18) into Eq. (15) to obtain

$$-\omega f_n + \frac{1}{2h^2} (f_+ - f) - 2f + f_+ + \sum_{k=1}^{\infty} a_k S_{2k-1}(f_-, f_n, f_+) \sum_{i=0}^{k} q_{i,k} = 0, \quad (20)$$

where the function

$$S_{2k-1}(f_-, f_n, f_+) = q_{0,k} \sum_{m=0}^{2k-1} f^m f_+^{2k-1-m}$$

$$+ \sum_{i=1}^{k} q_{i,k} \left( f_i f_n^{2k-i-1} + f_n^{2k-i-1} f_i \right), \quad (21)$$

involves only the terms of order $2k - 1$.

B. Main problem

PNP-free discretization of NLS equation Eq. (1) has the form:

$$i \dot{\psi}_n + \frac{1}{2h^2} (\psi_- - 2\psi_n + \psi_+ + \frac{1}{2} \sum_{k=1}^{\infty} a_k S_{2k-1}(\psi_-, \psi_n, \psi_+) = 0, \quad (22)$$

where $S_{2k-1}(\psi_-, \psi_n, \psi_+)$ is any function that, upon substituting Eq. (4), reduces to $S_{2k-1}(f_-, f_n, f_+) e^{i\omega t}$, with $S_{2k-1}(f_-, f_n, f_+)$ given by Eq. (21). Indeed, the stationary solutions to Eq. (22), satisfying the three-point problem, Eqs. (20) and (21), can be found from the reduced two-point problem, Eqs. (13), (18), and (19).

To complete the construction of the PNP-free DNLS equation we need to obtain a suitable function $S_{2k-1}(\psi_-, \psi_n, \psi_+)$ reducible by the ansatz Eq. (4) to $S_{2k-1}(f_-, f_n, f_+) e^{i\omega t}$. Clearly, $S_{2k-1}(f_-, f_n, f_+)$ can have several counterparts $S_{2k-1}(\psi_-, \psi_n, \psi_+).$ For example, both $|\psi_n|^2 \psi_-$ and $\psi_n^2 \psi_-$ give $(f_n f_+) e^{i\omega t}$ upon substituting Eq. (4).

As it can be seen from Eq. (21), the typical term of $S_{2k-1}(f_-, f_n, f_+)$ is $f^l f_+^m$ with $k + l + m = 2k - 1$, and $k, l, m \geq 0$. This term can be transformed, e.g., to

$$f_{-}^{l} f_{n}^{m} f_{+}^{m} \rightarrow \psi_- |\psi_{k-1}^{n}| \psi_{k+m}^{m},$$

$$\psi_+ |\psi_{l-1}^{n}| \psi_{l+m}^{m},$$

$$\psi_+ |\psi_{l-1}^{n}| \psi_{l+m}^{m},$$

(23)

for $k > 0$, $l > 0$, and $m > 0$, correspondingly.

It can also be transformed, e.g., to

$$f_{-}^{l} f_{n}^{m} f_{+}^{m} \rightarrow \psi_- |\psi_{k-1}^{n}| \psi_{k+m}^{m},$$

(24)

for $k > 0$ and $l > 1$, or to

$$f_{-}^{l} f_{n}^{m} f_{+}^{m} \rightarrow \left( \psi_{n}^{2} \right)^* |\psi_{-}^{k} \psi_{l-1}^{m} \psi_{l+m}^{m}|,$$

(25)

for $k > 1$ and $l > 2$, and so on.

One can see that the number of possibilities rapidly increases with increase in the order of the term, $2k - 1$. 


C. PNp-free DNLS equation conserving $N$

Let us consider the following PNp-free DNLS equation

$$
i\psi_n + \frac{1}{2\hbar^2} (\psi_n - 2\psi_n + \psi_+),$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} a_k S_{2k-1}(\psi_-, \psi_n, \psi_+) = 0,$$  \hspace{1cm} \text{(26)}

where

$$S_{2k-1}(\psi_-, \psi_n, \psi_+) = \psi_n \sum_{i=1}^{k} q_{i,k} \left| \psi_n^{i-1} \right| \sum_{m=0}^{2k-i-1} \left| \psi_m^{2k-i-1-m} \right|$$

$$+ \left| \psi_n^{2k-i-1} \right| \sum_{m=0}^{i-1} \left| \psi_m^{i-1-1-m} \right|. \hspace{1cm} \text{(27)}$$

For this model, the classical norm of Eq. (6), is conserved.

To construct this equation we had to set $q_{0,k} = 0$ to drop the first sum in the right-hand side of Eq. (21), because it does not contain $f_n$.

It is possible to construct some other DNLS equations conserving $N$ and one example will be given later for the Kerr nonlinearity. However, the above equation is interesting because it involves only the on-site nonlinearities modified through inter-site coupling.

Stationary solutions to Eq. (26) can be found from the two-point problem Eq. (13) with $G(f_-, f_n)$ given by Eq. (18), where

$$Q_{2k}(f_-, f_n) = \sum_{i=1}^{k} q_{i,k} \left( f_i f_n^{2k-i} + f_+ f_n^{2k-i} \right), \hspace{1cm} \text{(28)}$$

i.e., in comparison with Eq. (19), we simply drop the term with $i = 0$.

D. PNp-free DNLS equation conserving $\tilde{N}$ and $P$

In Eq. (19), let us drop all the terms corresponding to odd $i$. We can write the result in the form

$$Q_{2k}(f_-, f_n) = \sum_{i=0}^{[k/2]} q_{i,k} \left( f_2 f_n^{2k-2i} + f_+ f_n^{2k-2i} \right), \hspace{1cm} \text{(29)}$$

where $[k/2]$ is the largest integer not greater than $k/2$.

Substituting Eqs. (13), (18), and (29) into Eq. (15) we obtain

$$-\omega f_n + \frac{1}{2\hbar^2} (f_- - 2f_n + f_+)$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} a_k S_{2k-1}(f_-, f_n, f_+) = 0,$$  \hspace{1cm} \text{(30)}

where

$$S_{2k-1}(f_-, f_n, f_+) = (f_- + f_+) \sum_{m=0}^{k-1} f_-^{2m} f_+^{2(k-m-1)}$$

$$+ \sum_{i=1}^{[k/2]} q_{i,k} \left( f_2 f_n^{2k-i} + f_+ f_n^{2k-i} \right), \hspace{1cm} \text{(31)}$$

The resulting dynamical model (corresponding to (30) and Eq. (31)) of the form

$$i\psi_n + \frac{1}{2\hbar^2} (\psi_n - 2\psi_n + \psi_+)$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} a_k S_{2k-1}(\psi_-, \psi_n, \psi_+) = 0,$$  \hspace{1cm} \text{(32)}

with

$$S_{2k-1}(\psi_-, \psi_n, \psi_+) = \psi_n \sum_{i=0}^{[k/2]} q_{i,k} \left( f_2 f_n^{2k-2i} + f_+ f_n^{2k-2i} \right)$$

$$+ \sum_{i=0}^{[k/2]} q_{i,k} \left( f_2 f_n^{2k-i} + f_+ f_n^{2k-i} \right), \hspace{1cm} \text{(33)}$$

respects the conservation laws of both the modified norm of Eq. (7) and the momentum of Eq. (8). Stationary solutions to Eq. (32) can be found from the two-point problem Eqs. (13), (18), and (29).

IV. CUBIC (KERR) NONLINEARITY

A. Examples of PNp-free models

Let us study in detail Eq. (1) with Kerr nonlinearity, i.e., with $G(|\psi|^2) = |\psi|^2$,

$$i\psi_t + \frac{1}{2} \delta_{xx} |\psi|^2 \psi = 0.$$  \hspace{1cm} \text{(34)}$$

We then have $G(|\psi|^2) = |\psi|^4/2$, i.e., in Eq. (17), $a_2 = 1/2$ and $a_k = 0$ for $k \neq 2$.

To construct PNp-free DNLS equations we write Eq. (20) and Eq. (21) for the case of Kerr nonlinearity as

$$\omega f_n + \frac{1}{2\hbar^2} (f_- - 2f_n + f_+) + \frac{S_3(f_-, f_n, f_+)}{4(\alpha + \beta + \gamma)} = 0,$$  \hspace{1cm} \text{(35)}
and
\[ S_3(f_-, f_n, f_+) = \alpha (f_n^2 + f_- f_n^2 + f_+ f_n^2 + f_+^3) \\
+ \beta (f_+^3 + f_- f_+ + f_+^2 + f_n^2) f_n \\
+ 2\gamma (f_+ + f_-) f_n^2, \] (36)

where we introduced the following shorter notations for the free parameters: \( \alpha = \eta_{0,2}, \beta = \eta_{1,2}, \) and \( \gamma = \eta_{2,2}. \)

Solutions to the three-point problem of Eq. (35) can be found from the following two-point problem [Eqs. (13), (18), and (19)]
\[ \frac{1}{h^2} (f_n - f_-)^2 - 2\omega f f_n + \alpha (f_n^4 + f_n^4) + \beta (f_n^4 - f_n^3 + f_n + f_n^2) \]
\[ + \frac{2\gamma f_n^2}{2(\alpha + \beta + \gamma)} = 0. \] (37)

The ensuing PNp-free DNLS equation with Kerr non-linearity is
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_+ - 2\psi_n + \psi_+) + S_3(\psi_-, \psi_n, \psi_+) \]
\[ = 0. \] (38)

where \( S_3(\psi_-, \psi_n, \psi_+) \) is any function that, upon substituting Eq. (4), reduces to \( e^{i\omega t} S_3(f_-, f_n, f_+) \) [i.e., respecting the phase invariance of the equation], with \( S_3(f_-, f_n, f_+) \) given by Eq. (36). Some guidelines on how to construct \( S_3(\psi_-, \psi_n, \psi_+) \) can be found in Sec. III B.

From Eq. (26) and Eq. (27), at \( a_2 = 1/2 \) and \( a_k = 0 \) for \( k \neq 2 \), we obtain the Kerr-type DNLS equation conserving the classical norm \( N \)
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_+ - 2\psi_n + \psi_+) \]
\[ + \frac{\psi_n}{4(\alpha + \gamma)} \left[ \beta (|\psi_-|^2 + |\psi_-|^2 + |\psi_+|^2 + |\psi_+|^2) \right. \]
\[ + 2\gamma (|\psi_-|^2 + |\psi_+|^2) \] \[ = 0. \] (39)

The two-point equation for finding the amplitudes of stationary solutions to Eq. (39) is Eq. (37) with \( \alpha = \beta = 0 \).

Similarly, from Eq. (32) and Eq. (33), we obtain the Kerr-type DNLS equation conserving modified norm \( \tilde{N} \) and momentum \( \tilde{P} \)
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_+ - 2\psi_n + \psi_+) \]
\[ + \frac{\psi_- + \psi_+}{4(\alpha + \gamma)} \left[ \alpha (|\psi_-|^2 + |\psi_+|^2) + 2\gamma |\psi_n|^2 \right] = 0. \] (40)

DNLS equations of the form of (39) and (40) do not, of course, exhaust the list of possible PNp-free models with Kerr nonlinearity. To give one more example, we note that the last term of Eq. (36) can be used to produce the following DNLS equation conserving classical norm
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_- - 2\psi_n + \psi_+) + \frac{1}{4} (\psi_- + \psi_+) |\psi_n|^2 \]
\[ + \frac{1}{4} (\psi_-^2 + \psi_+^2) \psi_n^2 = 0. \] (41)

Amplitudes of stationary solutions to Eq. (41) can be found from Eq. (37) at \( \alpha = \beta = 0 \).

B. Soliton solutions

We now compare some properties of the classical DNLS equation,

model I :
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_- - 2\psi_n + \psi_+) + |\psi_n|^2 \psi_n = 0, \] (42)

with those of the \( N \)-conserving model of Eq. (39) with \( \beta = 0 \),

model II :
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_- - 2\psi_n + \psi_+) \]
\[ + \frac{\psi_n}{2} (|\psi_-|^2 + |\psi_n|^2) = 0; \] (43)

and those of the \( \tilde{N} \)- and \( P \)-conserving model of Eq. (40) with \( \gamma = 0 \),

model III :
\[ i\dot{\psi}_n + \frac{1}{2h^2} (\psi_- - 2\psi_n + \psi_+) \]
\[ + \frac{\psi_- + \psi_+}{4} (|\psi_-|^2 + |\psi_+|^2) = 0. \] (44)

All three models share the same continuum limit, the integrable NLS equation Eq. (34), and thus, in the regime of weak discreteness (small lattice spacing \( h \)), their soliton solutions of the form of Eq. (4) can be expressed approximately as
\[ \psi_n(t) = \frac{A}{\cosh[Ah(n - x_0)]} \exp[-i(A^2/2)t], \] (45)

where \( A \) and \( \omega = A^2/2 \) are the soliton amplitude and frequency, respectively.

The approximate solution of Eq. (45) contains the free parameter \( x_0 \) defining the soliton position. However, in contrast to the NLS equation of Eq. (34), where \( x_0 \) can be chosen arbitrarily due to translational invariance, the DNLS models usually have stationary soliton solutions only for a discrete set of values of \( x_0 \) (e.g. on-site, \( x_0 = 0 \), and inter-site, \( x_0 = 1/2 \)). This is true, for example, for
the classical DNLS of model I and for the Salerno model [33], among others. The models II and III, by construction, are among the members of a wider class of DNLS equations proposed in this paper, where stationary soliton solutions exist for any \( x_0 \), or, in other words, they can be placed anywhere with respect to the lattice; otherwise put, the Peierls-Nabarro potential is absent for stationary solutions of these models.

Let us now describe the exact soliton solutions to the models I, II, and III.

An explicit formula does not exist for the stationary soliton solutions of model I. Such solutions can be obtained using the fixed point algorithms [34], but, as mentioned above, only for \( x_0 = 0 \) and \( x_0 = 1/2 \).

Model II, model of Eq. (41), and also model of Eq. (40) at \( \alpha = 0 \), have the same equations for the amplitudes of stationary solutions. However, the latter model, as mentioned above, is the integrable DNLS equation [15, 16] and thus, an exact stationary solutions for these models can be obtained explicitly in the form

\[
\psi_n(t) = \frac{1}{h} \frac{\sinh \mu}{\cosh \mu[n-x_0]} \exp i \omega t,
\]

where \( x_0 \) is the parameter defining the soliton position and it can obtain any value from \([0,1)\). The soliton frequency \( \omega = h^{-2}(1 - \cosh \mu) \) and amplitude \( A = h^{-2} \sinh \mu \) are expressed in terms of the free parameter \( \mu > 0 \).

Model III has the solutions of the form of Eq. (4) with \( f_n \) derivable from the two-point problem

\[
\frac{1}{h^2} (f_n - f_\pm)^2 - 2 \omega f_- f_n + \frac{1}{2} (f_\pm^2 + f_n^2) = 0.
\]

The soliton can be constructed by setting an arbitrary value for \( f_- \) (or \( f_+ \)) in the range \([A_m, A_s]\) and finding \( f_n \) (or \( f_- \)) from the quartic Eq. (47). Quantities \( A_m \) and \( A_s \) are the amplitudes of solitons centered between two lattice sites and on a lattice site, respectively. We have \( A_m = \sqrt{2} \omega \), and \( A_s \) can be found from the condition that two distinct real roots of Eq. (47) merge into a multiple root. The arbitrariness in the choice of initial value of \( f_- \) (or \( f_+ \)) implies the absence of the Peierls-Nabarro potential and the possibility to place the soliton anywhere with respect to the lattice.

### C. Soliton’s internal modes

Let us study the stability of stationary soliton solutions for the models described in Sec. IV B. In this study we calculate the soliton’s internal modes and frequencies of these modes.

Following the methodology of the paper [35], to study the stability of the solution Eq. (4), we consider the complex perturbation \( \epsilon_n(t) \) in the frame rotating with the periodic solution:

\[
\psi_n(t) = [f_n + \epsilon_n(t)] e^{i \omega t}.
\]

Substituting Eq. (48) into the classical DNLS equation Eq. (42) (model I) we find that the linearized equation satisfied by \( \epsilon_n(t) \) is

\[
i \epsilon_n - \omega \epsilon_n + \frac{1}{2 h^2} (\epsilon_- - 2 \epsilon_n + \epsilon_+) + 2 f_n^2 \epsilon_n + f_n^2 \epsilon_n^* = 0.
\]

Similarly we obtain the linearized equations for \( \epsilon_n(t) \) for the \( N \)-conserving model II of Eq. (43),

\[
i \epsilon_n - \omega \epsilon_n + \frac{1}{2 h^2} (\epsilon_- - 2 \epsilon_n + \epsilon_+) + \frac{1}{2} f_n (f_- + f_+) [\epsilon_+ + \Re(\epsilon_n)] + \frac{1}{2} f_n^2 [\Re(\epsilon_-) + \Re(\epsilon_+)] = 0,
\]

and for the \( \bar{N} \)- and \( P \)-conserving model III of Eq. (44),

\[
i \epsilon_n - \omega \epsilon_n + \frac{1}{2 h^2} (\epsilon_- - 2 \epsilon_n + \epsilon_+) + \frac{1}{2} (f_- + f_+) [\epsilon_- + \epsilon_+] + \frac{1}{4} (f_-^2 + f_+^2) (\epsilon_- + \epsilon_+ + \epsilon_+^*) = 0.
\]

Defining \( \epsilon_n(t) = a_n(t) + i b_n(t) \), the linearized equations can be written as

\[
\begin{pmatrix}
\dot{a}_n \\
\dot{b}_n
\end{pmatrix} =
\begin{pmatrix}
0 & \Omega \\
J & 0
\end{pmatrix}
\begin{pmatrix}
a_n \\
b_n
\end{pmatrix},
\]

where vectors \( a \) and \( b \) contain \( a_n \) and \( b_n \), respectively. Stationary soliton solutions are linearly stable if and only if the eigenvalue problem, \( \det(J\Omega - \lambda^2 I) \), has only real and nonpositive solutions for \( \lambda^2 \) [35].

In our stability analysis, a soliton, having a frequency \( \omega = 1 \), is placed at the middle of chain of 400 sites with periodic boundary conditions. Different magnitudes of the discreteness parameter \( h \) and different positions of solitons with respect to the lattice, \( x_0 \), are considered.

Most of eigenfrequencies \( -\lambda^2 \) appear within the band between \( \omega \) and \( \omega + 2/\hbar^2 \), where \( \omega \) is the frequency of soliton and \( 2/\hbar^2 \) is the maximum frequency of the linear spectrum of trivial solution, \( f_n = 0 \). We do not show these eigenvalues in the figures. Eigenvalues appearing outside of this band are related to soliton’s internal modes. The spectra of solitons in the models I, II, and III always contain two zeroes. However, spectra of solitons in the PNP-free models (including models II and III), due to their effective translational invariance, always contain two additional zeroes.

Results for the classical DNLS equation (model I) are presented in Fig. 1 for (a) \( x_0 = 0 \) and (b) \( x_0 = 1/2 \). In (a), the on-site soliton is stable because all \( \lambda^2 \) are real and nonpositive, while in (b), the inter-site soliton is unstable because we have positive \( \lambda^2 \) (shown by open circles). The solid line shows the bottom edge of the linear band and dotted line shows the always existing eigenvalues \( \lambda^2 = 0 \).

In Fig. 2, the results for the \( N \)-conserving, PNP-free DNLS equation (model II) are presented. The always existing eigenvalues \( \lambda^2 = 0 \) are shown by dotted line and
FIG. 1: Model I, spectrum of the soliton with frequency $\omega = 1$ (bottom edge of the continuous spectrum) for (a) stable on-site and (b) unstable inter-site configurations. Spectrum always contains a pair of zero-frequency modes shown by dashed lines. For $h < 0.4$ discreteness is weak and the soliton in (a) has a mode with nearly zero frequency corresponding to the translational mode of the continuum NLS equation. The frequency of this mode grows rapidly with increase in $h$ for $h > 0.4$ and it enters the continuous band at $h = 0.75$ where soliton loses its mobility.

the additional two zeroes are shown by dots; the latter reflect the translational invariance of the soliton. Note that panel (b) corresponds to the soliton having $x_0 = 1/4$, i.e., placed non-symmetrically with respect to the lattice points. For $h > 1.25$, below the linear spectrum, there exists a soliton internal mode. This mode was not observed for the soliton placed at $x_0 = 1/2$.

Similar results for the $\hat{N}$- and $P$-conserving, PNP-free DNLS equation (model III) are shown in Fig. 3. In contrast to models I and II, soliton in model III has the internal modes lying not only below, but also above the linear band. These modes are shown in Fig. 4 by dots and the upper edge of the linear spectrum is shown by the solid line.

FIG. 2: Model II, spectrum of the soliton with frequency $\omega = 1$ for (a) on-site and (b) asymmetric configurations. Both configurations are stable. A soliton internal mode bifurcates from the bottom edge of the continuous frequency band at rather large $h$. In addition to an always existing pair of zero-frequency modes (shown by dashed lines), for any $x_0$, there exist another pair of zero-frequency modes (shown by dots), reflecting the translational invariance of the soliton and the absence of the Peierls-Nabarro potential.

FIG. 3: Model III, bottom part of the spectrum of the soliton with frequency $\omega = 1$ for (a) on-site and (b) inter-site configurations. Both configurations are stable. Soliton internal mode bifurcates from the bottom edge of the continuous frequency band. For any $x_0$, the spectrum contains two pairs of zero-frequency modes, which is a common feature for the PNP-free models.

D. Mobility of solitons

Solving the eigenvalue problem formulated in Sec. IV C, we find solutions of corresponding DNLS equation of the form of Eq. (48). Solutions are accurate for small amplitudes of the eigenvectors, $\epsilon_n(t)$. Here we examine the solutions corresponding to the translational eigenmode (since “kicks” along this eigendirection may be responsible for/related to motion in these lattices).

We define the position of discrete soliton as its center of mass:

$$S(t) = \frac{\sum n N_n(t)}{\sum N_n(t)},$$

(52)

where $N_n = |\psi_n|^2$; then the soliton velocity is $v = dS/dt$. This definition is used for $N$-conserving models I and II, and for $\hat{N}$-conserving model III, in Eq. (52), we naturally use $\hat{N}_n = (1/4)(\psi_n^{+} \psi_n^{+} + \psi_n^{+} \psi_n^{-} + \psi_n^{-} \psi_n^{+} + \psi_n^{-} \psi_n^{-})$ instead of $N_n$.

Setting initial conditions according to Eq. (48) with
The soliton frequency is

\[ \omega(x, t) \]

for different initial velocities. Soliton frequency is \( \omega = 1 \).

For all three models we boost the solitons initially placed at \( x_0 = 0 \).

As Fig. 1(a) suggests, the soliton of model I has a translational mode with nearly zero frequency only for \( h < 0.4 \). For such a small \( h \), discreteness is weak and model I can be regarded as weakly perturbed continuum NLS equation, which supports moving solitons. However, the frequency of the translational mode increases rapidly for \( h > 0.4 \) and it enters the continuum frequency band at \( h = 0.75 \) when the soliton completely loses its mobility.
We have described a general and systematic method of constructing spatial discretizations on NLS-type models, whose stationary solutions can be obtained from a two-point difference problem. In this setting, finding stationary solutions becomes tantamount to solving the NLS equation with the modified norm \( N \). While model \( I \) conserves the classical norm \( N \), model \( II \) conserves modified norm \( N \) and momentum \( p \).

V. CONCLUSIONS

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Acknowledgments

The soliton kinematics in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation.

In Fig. 5 we show the time evolution of soliton's velocity in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation.

Solutions of the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation.

For the case of cubic nonlinearity we demonstrate that the solitons do not propagate in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation. The soliton velocity is conserved in the PNP-free DNLS equations in the classical NLS equation while the soliton propagates along the classical NLS equation.

We believe that our results may suggest novel possibilities for engineering nonlinear lattices and optimization of discrete solitons in the various physical contexts.

The soliton kinematics in the classical NLS equation and the PNP-free DNLS equations in the classical NLS equation.

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