Asymptotic iteration method for the eigenfunctions and eigenvalue analysis in Schrodinger equation with modified anisotropic nonquadratic potential

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Abstract. Analysis eigenfunctions and eigenvalue for the modified anisotropic nonquadratic potential in Schrodinger equation is solved using asymptotic iteration method (AIM). Cylindrical coordinates is used to separate three dimensional Schrodinger equation into three one dimensional Schrodinger type equation and then it is solved using AIM to obtain the eigenfunction and eigenvalue. The nonrelativistic energy equation are calculated, while the wavefunctions are discussed.

1. Introduction
The wavefunctions and energy spectra are important fundamental in quantum mechanics to describe behaviour of microscopic particles. Schrodinger equation for the central and non-central potentials have been studied in many research. Numerous studies have analyzed the bound state of charged particle moving in vector potential and non-central scalar potential, such as an electron moving in a Coulomb field with simultaneously presence of Aharonov-Bohm field,[1-2] Hulthen potential,[3] Hylleraas potential,[4-6] or ring-shaped-oscillator potential.[7-9] Very recently, some methods are used to derive the energy spectrum and the wavefunction of a charge particle moving in non-central potential, such as supersymmetric quantum mechanics with the idea of shape invariance, [9-16] factorization method, [17-18] Nikiforov-Uvarov method [19-20] and asymptotic iteration method (AIM).[21-24]. In this paper, will be investigated Schrodinger equation with a modified anisotropic nonquadratic potential using asymptotic iteration method (AIM) [21-24]. In previous research, the anisotropic nonquadratic potential has been investigated using Path integral method, [25] algebraic solution method, [26] and semi-classical treatment [27]. AIM is a powerful tool to determine eigenvalue and eigenfunction of second order differential equation. The modified anisotropic non-quadratic potential is given as [25-27]

$$V(r, \theta, z) = \frac{1}{\alpha^2} \left( e^{\frac{-x^2+y^2}{\alpha}} \right)^{1/2} \left( \frac{\beta}{\gamma \sqrt{x^2+y^2}} + \frac{\gamma x}{\gamma \sqrt{x^2+y^2}} \right) \frac{\alpha^2 + \beta^2 \sinh z}{\cosh z}$$  

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$$\alpha, \beta, \gamma, \alpha', \beta'$$ are positive constant. By applying the cylindrical coordinate, the potential which is expressed in equation (1) is reduced into a simple and separable variable potential. For cylindrical coordinate we have

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}; x = r \cos \theta; y = r \sin \theta; z$$

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with \(0 < r \leq \infty, 0 \leq \theta \leq 2\pi, 0 < z < \infty\) and therefore the potential energy in equation (1) becomes

$$0 < r \leq \infty, 0 \leq \theta \leq 2\pi, 0 < z < \infty$$

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\[ V(r, \theta, z) = \frac{1}{\alpha^2} \left[ e^{-\imath \mu r} \right] + \frac{\beta + \gamma \cos \theta}{r^2 \sin^2 \theta} + \frac{\alpha' + \beta' \sinh z}{\cosh^2 z} \]  

(3)

In cylindrical coordinate by using equation (2), the three dimensional Schrodinger equation for an anisotropic non-quadratic in equation (3) is given by

\[-\frac{\hbar^2}{2m} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] \psi(\theta, r, z) + \frac{1}{\alpha^2} \left[ e^{-\imath \mu r} \right] + \frac{\beta + \gamma \cos \theta}{r^2 \sin^2 \theta} + \frac{\alpha' + \beta' \sinh z}{\cosh^2 z} \psi(\theta, r, z) = E \psi(\theta, r, z) \]

(4)

2. Asymptotic Iteration Method

Asymptotic Iteration Method (AIM) is used to solve second order differential equation in terms:

\[ y''(x) = 2 \left( \frac{x^{n+1}}{1-x^{-N+2}} \right) y'(x) - \frac{W x^{N+2}}{1-\beta x^{N+2}} \]

(5)

where \( \lambda_0(k) \neq 0 \) and \( s_o(k) \) are coefficients of the differential equation and are well defined functions as well as sufficiently differentiable. On the other hand, equation (5) can be written in term:

\[ y''(x) = 2 \left( \frac{x^{n+1}}{1-x^{-N+2}} \right) y'(x) \]

The one-dimensional Schrodinger equation can be reduced into hypergeometric or confluent hypergeometric type differential equation by suitable changes of variables, and then changes it into the differential equation which has the form in equation (5). The solution of equation (5) can be obtained by using iteration of \( \lambda_i \) and \( s_i \):

\[ \lambda_i(k) = \lambda_{i-1}' + \lambda_{i-1} s_0(k) - s_{i-1}' + s_{i-1} \lambda_{i-1}, i = 1, 2, 3, ... \]

(6)

Eigenvalues can be obtained using equation (8): \[23]\[
\lambda_i(k) - \lambda_{i-1}(k) s_0(k) = 0 \]

Equation (8) is AIM-type differential equation which eigenfunctions can be obtained by using equation (9) below [24,28]

\[ y_n(k) = (-1)^n C (N + 2)^n \frac{\Gamma}{2} F_1 (-n, p + n, \sigma, bk^{N+2} ) \]

(9)

where, \[ \Gamma = \frac{\Gamma(n+\sigma)}{\Gamma(\sigma)} \]

Here \( C \) is normalization constant and \( F_1 \) is hypergeometric function. Equations (9-10) is used to obtain wavefunctions of the Schrodinger equation. The non-relativistic energy equation can be formulated by equating eigenvalue by using equations (7-8).

3. Schrodinger Equation for Modified Anisotropic Non-quadratic Potential

The three dimensional Schrodinger equation with an anisotropic non-quadratic potential which is expressed in equation (4) is solved by using variable separation method by setting the total wave function as \( \psi(r, \theta, z) = R(r) P(\theta) Z(z) \), so we have the axial part, angular part and radial part, given as

\[ \frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2} = \frac{\alpha' + \beta' \sinh z}{\cosh^2 z} = \lambda_1 \]

(10)

\[ \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{2m}{\hbar^2} \left( \frac{\beta + \gamma \cos \theta}{r^2 \sin^2 \theta} \right) = \lambda_2 \]

(11)
\[
\begin{align*}
&\frac{r}{R} \frac{\partial}{\partial r} \left( \frac{\partial R}{\partial r} \right) - \frac{2m}{\hbar^2} \left( \frac{r^2}{\alpha^2} e^{-\alpha r} \right) + \frac{2m}{\hbar^2} Er^2 + \lambda_r r^2 = -\lambda_r \tag{13}
\end{align*}
\]

with \(\lambda_1, \lambda_2\) is variable separation constant.

3.1. Solution of Z part Schrodinger equation

The Schrodinger equation of axial (\(z\)) part in equation (11) is rewritten as

\[
\frac{\partial^2 Z(z)}{\partial z^2} + \frac{2m}{\hbar^2} \left( \frac{\alpha + \beta \sinh z}{\cosh^2 z} + \lambda_z \frac{\hbar^2}{2m} \right) Z = 0 \tag{14}
\]

Equation (14) is solved by AIM with change of variable which, \(\sinh z = i(1 - 2u)\)

so,

\[
\cosh^2 z = 4u(1 - u); \quad \frac{d^2}{du^2} = -u(1 - u) \frac{d^2}{du^2} - \frac{1 - 2u}{du} \tag{15}
\]

By using equations (15-16) in equation (14), we get

\[
\left(1 - u\right)f = \left[2 \chi + \frac{1}{2}\right] - \left(2 \chi + 2 \varepsilon + 1\right) u \right] f + \left[\lambda_1 - \left(\chi + \varepsilon\right)^2\right] f = 0 \tag{19}
\]

where

\[
\chi = \frac{1}{2} \sqrt{\frac{2m}{\hbar^2} (\alpha' + \beta') + \frac{1}{4}} + \frac{1}{4}; \quad \varepsilon = \frac{1}{2} \sqrt{\frac{2m}{\hbar^2} (\alpha' - \beta') + \frac{1}{4}} + \frac{1}{4} \tag{20}
\]

Equation (19) must be transformed to AIM type, so we obtain eigenvalues and eigenfunction by using AIM. By divide equation (19) with \(u(1 - u)\), yields the second order differential equation,

\[
f = \left[2 \chi + \frac{1}{2}\right] - \left(2 \chi + 2 \varepsilon + 1\right) u \right] f + \left[\lambda_1 - \left(\chi + \varepsilon\right)^2\right] f = 0 \tag{21}
\]

Equation (21) is AIM type differential equation. By using equation (5) and from equation (21), we get

\[
\lambda_0 = \frac{2 \chi + 2 \varepsilon + 1}{u(1 - u)} \left(2 \chi + \frac{1}{2}\right) - \frac{2 \chi + \frac{1}{2}}{u}; \quad s_n = \frac{\left(\chi + \varepsilon\right)^2 - \lambda_1}{u(1 - u)} \tag{22}
\]

Eigenvalue is obtained from equation (7-8) and equation (21), is given as

\[
\lambda_n = s_n, \downarrow \lambda_n, s_n \rightarrow \lambda_n = \left(\chi + \varepsilon + n\right)^2 \tag{23}
\]

Equation (23) is eigenvalue from axial part of Schrodinger equation, which is generalized by using Matlab software. Equation (23), where \(\chi\) and \(\varepsilon\) in equation (20), \(n\) is axial (\(z\)) quantum number, will be used to find energy spectra. Then, we can obtain eigenfunction or wavefunction using equation (6, 9-10). By compare equation (6) and equation (21) and then using equation (10), we get:

\[
f = (-1)^n C_n \left(2 \chi + \frac{1}{2}\right) \left\{\begin{array}{cc}
-1 & 0, 2 \chi + 2 \varepsilon + n, 2 \alpha + \frac{1}{2}, u
\end{array}\right\} \tag{24}
\]

By substitute equation (24) in equation (18), and by using: \(\sinh z = i(1 - 2u)\rightarrow u = \frac{i\sinh z + 1}{2}\), we get wavefunction of the axial part for groundstate and the first excitation condition,

\[
n_0 = 0 \rightarrow Z_0(z) = \left\{\begin{array}{cc}
\left(\frac{i\sinh z + 1}{2}\right) (1 - i\sinh z)^\chi&\chi; n_1 = 1 \rightarrow Z_1(z) = \left\{\begin{array}{cc}
\left(\frac{i\sinh z + 1}{2}\right) (1 - i\sinh z)^\chi&\chi
\end{array}\right\}
\right\} \tag{25}
\]
The solution of Schrödinger equation for axial part provides the value of variable separation constant, \( \lambda_1 \), and the wave functions of axial part.

### 3.2. The solution of angular Schrödinger equation

The Schrödinger equation for angular part which is expressed in equation (12) can be rewritten as

\[
\frac{d^2 P(\theta)}{d\theta^2} - \frac{2m}{\hbar^2} \left[ \beta + \gamma \cos \theta - \frac{\hbar^2}{2m} \lambda_2 \right] P = 0
\]

Equation (26) can be transformed to hypergeometry type differential equation by using

\[
\cos \theta = 1 - 2x \rightarrow \sin^2 \theta = 4x(1-x); \quad \frac{dx}{d\theta} = x(1-x) \frac{d^2}{dx^2} + \left( \frac{1}{2} - x \right) \frac{d}{dx}
\]

so equation (27) become,

\[
x(1-x) \frac{d^2 P}{dx^2} + \left( \frac{1}{2} - x \right) \frac{dP}{dx} - \frac{2m}{\hbar^2} \left[ \beta + \gamma (1-2x) - \frac{\hbar^2}{2m} \lambda_2 \right] P = 0
\]

and then by parameter

\[
P = x^\eta (1-x)^\kappa g(x)
\]

we get hypergeometry type differential equation for angular part of Schrödinger equation:

\[
x(1-x) g^* \left[ \left( \frac{2\eta + 1}{2} \right) - \left( 2\eta + 2\kappa + 1 \right)x \right] g + \left[ -\lambda_2 - (\eta + \kappa)^2 \right] g = 0
\]

Equation (30) is divided with \( x(1-x) \), yields

\[
g^* \left[ \left( \frac{2\eta + 1}{2} \right) - \left( 2\eta + 2\kappa + 1 \right)x \right] g + \left[ -\lambda_2 - (\eta + \kappa)^2 \right] g = 0
\]

where,

\[
\eta = \frac{1}{2} \sqrt{\frac{2m}{\hbar^2} \left( \beta + \gamma \right) + \frac{1}{4} \kappa} + \frac{1}{4} \kappa
\]

Equation (31) is AIM type differential equation. By compare equation (5) and equation (31), we obtain \( \lambda_{ng} \) and \( s_0 \) and by using equation (7-8), we obtain the generalized eigenvalue,

\[
\Delta_n = s_n \lambda_{ng} - s_0 \lambda_0 \rightarrow -\lambda_{2ng} = (\eta + \kappa + n_\theta)^2
\]

where \( \lambda_{2ng} \) is separation variable constant in angular part, \( n_\theta \) is angular quantum number, \( \eta \) and \( \kappa \) are constants which is expressed in equation (32). Then for angular wavefunction, we compare equation (31) with equation (6), and by using equation (9-10), we have

\[
g(x) = (-1)^n C_1 \left( \frac{1}{2} \right)^{n_\theta} (2\eta + 1/2)_{n_\theta} F_{n_\theta} \left( -n_\theta, 2\eta + 2\kappa + n_\theta, 2\eta + 1/2, x \right)
\]

From equation (31) and equation (34), and by using:

\[
\cos \theta = 1 - 2x \rightarrow x = \frac{1 - \cos \theta}{2}
\]

we obtain the angular wavefunction, for groundstate and the first excited,

\[
n_\theta = 0 \rightarrow P_0(\theta) = \left( \frac{1 - \cos \theta}{2} \right)^{1/2} \left( \frac{1 + \cos \theta}{2} \right)^{1/2} C_1 ; n_\theta = 1 \rightarrow P_1(\theta) = \left( \frac{1 - \cos \theta}{2} \right)^{1/2} \left( \frac{1 + \cos \theta}{2} \right)^{1/2} C_1 \left( 2\eta + 1/2 \right)^{1/2} \left( 2\eta + 2\kappa + 1/2 \right)^{1/2}
\]

### 3.3 Solution of radial Schrodinger equation

From radial part, equation (13) is multiplied with \( R(r)/r^2 \) so we have

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) - \frac{2m}{\hbar^2} \left[ \frac{1}{\alpha^2} \left( 1 - e^{-\alpha r} \right) \right] - \frac{2m}{\hbar^2} \frac{E - \lambda_2 - \lambda_i}{r^2} R = 0
\]

By setting \( R(r) = (U(r)/\sqrt{r}) \) in equation (36) we obtain

\[
\frac{d^2 U}{dr^2} - \frac{2m}{\hbar^2} \left[ \frac{1}{\alpha^2} \left( 1 - e^{-\alpha r} \right) \right] - \frac{E - \lambda_2 - \lambda_i}{r^2} U = 0
\]

Equation (37) is solved by using approximation of the centrifugal term [29-30] for \( \frac{r}{\alpha} \ll<<1 \) is given as
\[
\left( \frac{1}{r^2} \right) \equiv \left( \frac{1}{4\alpha^2} \right) \left( d_0 + \frac{1}{\sinh^2 \left( r/2\alpha \right)} \right)
\]  
(38)

with \( d_0 = 1/12 \). In term of hyperbolic functions,
\[
\left( \frac{1}{\alpha^2} \right) \left( \frac{1}{1 - e^{-\alpha r}} \right) = \left( \frac{1}{\alpha^2} \right) \left( \frac{1}{4 \sinh^2 \left( r/2\alpha \right)} \right)
\]  
(39)

and then by using equation (38-39), equation (37) is written as
\[
U = - \left( \frac{2m/h^2}{4\alpha^2 \sinh^2 \left( r/2\alpha \right)} \right) \left( \frac{\lambda_2 + 1}{4\alpha} \right) \left( \frac{\lambda_2 + \alpha^2}{4\alpha} \right) = 0
\]  
(40)

and by setting change of variable:
\[
\coth \left( r/2\alpha \right) = 1 - 2w, \quad \frac{d^2}{dw^2} + \frac{1}{\alpha^2} \left( w(w-1)^2 \right) \frac{d^2}{dw^2} + \frac{1}{\alpha^2} w(w-1)(2w-1) \frac{d}{dw}
\]  
(41)

in equation (40), we get
\[
w(1-w) \frac{dU}{dw^2} + (1-2w) \frac{dU}{dw} - \left( \frac{\lambda_2 + 1}{4\alpha} \right) \frac{2m}{h^2} \frac{\lambda_2}{w(1-w)} \frac{d}{dw} \left( \frac{\lambda_2 + \alpha^2}{4\alpha} \right) = 0
\]  
(42)

To reduce equation (42) in terms of hypergeometry type, we set
\[
U = w^2 (1-w)^2 h(w)
\]  
(43)

By inserting equation (43) into equation (42), we obtain
\[
w(1-w) h^* + \left[ (2\zeta + 1) - (2\zeta + 2\tau + 2) w \right] h^* \left[ -\left( \frac{\lambda_2 + 1}{4\alpha} \right) - \left( \frac{2m}{h^2} \right) \right] (\zeta + \tau)^2 - \zeta - \tau = 0
\]  
(44)

where,
\[
\zeta = \sqrt{\frac{2m}{h^2} \alpha^2 - \lambda_2 \alpha^2 + \frac{1}{4} d_0 \left( \lambda_2 + \frac{1}{4} \right)} \quad \tau = \sqrt{\frac{2m}{h^2} \alpha^2 E - \lambda_2 \alpha^2 + \frac{1}{4} d_0 \left( \lambda_2 + \frac{1}{4} \right)}
\]  
(45)

Equation (45) is reduced to AIM type by divide it with \( w(1-w) \), so equation (45) become
\[
h^* + \left[ \left( (2\zeta + 1) - (2\zeta + 2\tau + 2) w \right) / w(1-w) \right] h^* \left[ -\left( \frac{\lambda_2 + 1/4}{4\alpha} \right) - \left( \frac{2m}{h^2} \right) \right] (\zeta + \tau)^2 - \zeta - \tau = 0
\]  
(46)

and then we compare equation (46) with equation (5) to obtain \( \lambda_0, s_0 \) and by using equation (7-8), can be obtained the generalized eigenvalue in following relation:
\[
\Delta_n = s_n \lambda_{z_{\omega}} - s_{n-1} \lambda_n \rightarrow \left( 2m/h^2 \right) - \left( \frac{\lambda_2 + 1}{4\alpha} \right) = (\zeta + \tau + n_{\omega}) (\zeta + \tau + n) + 1
\]  
(47)

From generalized eigenvalue in equation (47), we obtain the non-relativistic energy equation, by inserting \( \zeta \) and \( \tau \) which is expressed in equation (45) to equation (47). The non-relativistic energy equation of Schrodinger equation for modified anisotropic nonquadratic potential is,
\[
E = \frac{\hbar^2}{8m \lambda_{z_{\omega}}^2} \left[ \left( \frac{2m}{h^2} - \frac{\lambda_{z_{\omega}}}{2} - n_{\omega} \right)^2 + 4\alpha^2 \lambda_{z_{\omega}} + d_0 \left( \lambda_{z_{\omega}} + \frac{1}{4} \right) \right]
\]  
(48)

where \( \lambda_{z_{\omega}} \) is separation variable constant of angular part in equation (33), \( \lambda_{n_{\omega}} \) is separation variable constant of axial part in equation (23), \( n_{\omega} \) is radial quantum number. Table 1 are non-relativistic energy with variation potential constants \( \alpha, \beta, \gamma, \alpha', \beta' \) in \( n_{\omega} = 1, n_\alpha = 0, n_e = 1 \) In Table 1, \( E(\alpha) \) is energy as a function of \( \alpha \), \( E(\alpha') \) is energy as a function of \( \alpha' \) and so on.
Table 1. Non-relativistic energy corresponding to $n_z=1, n_y=0, n_x=1, d_0=1/12$ of particle under the influence of an anisotropic non-quadratic potential

| Value of $r$ | $E(\alpha)$ | $E(\alpha')$ | $E(\beta)$ | $E(\beta')$ | $E(\gamma)$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 1           | 0.3425      | 0.3425      | 0.3425      | 0.3425      | 0.3425      |
| 2           | 0.2410      | 0.2535      | 0.0295      | 0.7535      | 0.3959      |
| 3           | 0.2222      | 0.2166      | 0.0295      | 1.2166      | 0.5606      |
| 4           | 0.2156      | 0.1970      | 0.2107      | 1.6970      | 0.5606      |
| 5           | 0.2125      | 0.1849      | 0.6772      | 2.1849      | 0.6576      |

Table 1 show that the increasing radial constant $\alpha$, and axial constant $\alpha'$, cause the value of energies is smaller, while the energies value increase due to the value of angular constants $\gamma', \beta$ is larger.

Then for radial wavefunction, we compare equation (6) with equation (46) and by using equation (9-10), we get

$$ h(w) = (-1)^n \mathcal{C}_n (1) \left( 2\zeta + 1 \right) r \mathcal{P}_n \left( -n_y, 2\zeta + 2r - 1 + n_y, 2\zeta + 1, w \right) $$

(49)

By inserting equation (49) into equation (43), and by change of variable:

$$ \coth(r/2\alpha) = 1 - 2w \to w = \left( 1 - \coth(r/2\alpha) \right)/2 $$

(50)

we obtain radial wavefunction, for groundstate and the first excited, are given as

$$ n_y = 0 \to \left( 1 - \coth(r/2\alpha) \right) \mathcal{C}_n, n_y = 1 \to \mathcal{P}(\theta) = \left( 1 - \coth(r/2\alpha) \right) \mathcal{C}_n (2\zeta + 1) \left( 1 - \coth(r/2\alpha) \right) $$

(51)

4. Conclusion

Schrodinger equation for the modified anisotropic non-quadratic potential is investigated by asymptotic iteration method. By using cylindrical coordinate, three dimensional Schrodinger equation is reduced to one radial Schrodinger equation, one angular Schrodinger equation, and one axial Schrodinger equation. By using suitable variable substitution and asymptotic iteration method, the eigenvalue and eigenfunction are obtained. The non-relativistic energy are calculated in variation value of potentials constants and the wavefunctions are expressed in hypergeometry term.

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