DOMINANCE COMPLEX AND VERTEX COVER NUMBER

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Abstract. The dominance complex \( D(G) \) of a simple graph \( G = (V, E) \) is the simplicial complex consisting of subsets of \( V \) whose complements are dominating. We show that the connectivity of \( D(G) \) plus 2 is a lower bound for the vertex cover number \( \tau(G) \).

1. Introduction

For a simple graph \( G = (V, E) \), a subset \( S \) of \( V \) is a dominating set in \( G \) if every vertex \( v \) in \( G \) belongs to \( S \) or is adjacent to an element in \( S \). The dominance complex \( D(G) \) is the simplicial complex consisting of the subsets of \( V \) whose complements are dominating.

Dominance complex was considered in Ehrenborg and Hetyei [1], and there are few classes of graphs such that the homotopy types of their dominance complexes are determined. However, the known examples seem to suggest that there is a certain relationship between the topology of \( D(G) \) and the vertex cover number \( \tau(G) \). In fact, Ehrenborg and Hetyei showed that the dominance complex of a forest is homotopy equivalent to a sphere, and Marietti and Testa [4] in fact showed \( S^{\tau(G)-1} \simeq D(G) \) when \( G \) is a forest. Taylan [7] generalize this result to chordal graphs. Taylan [7] also determined the homotopy types of the \( P_3 \)-devoid complexes of cycles, which coincide with the dominance complexes of cycles. Namely, she showed

\[
D(C_4 t) \simeq S^{2t-1} \vee S^{2t-1} \vee S^{2t-1}, \quad D(C_{4t+i}) \simeq S^{2t+i-2} \quad (i = 1, 2, 3).
\]

Note that the vertex cover number of \( C_n \) is \( \lceil n/2 \rceil \).

These results seem to suggest that there is a relationship between the connectivity of \( D(G) \) and \( \tau(G) \), and the purpose in this note is to show that a certain homotopy invariant of \( D(G) \) gives a lower bound for the vertex cover number \( \tau(G) \).

For a topological space \( X \), let \( \text{conn}_{\mathbb{Z}_2}(X) \) be the largest number \( n \) such that \( i \leq n \) implies \( \tilde{H}_i(X; \mathbb{Z}_2) = 0 \), and call it the \( \mathbb{Z}_2 \)-homological connectivity of \( X \). Note that Hurewicz theorem implies \( \text{conn}(X) \leq \text{conn}_{\mathbb{Z}_2}(X) \), where \( \text{conn}(X) \) is the connectivity of \( X \). Then our main result is formulated as follows:

Theorem 1. For every simple graph \( G \), the following inequality holds:

\[
\text{conn}_{\mathbb{Z}_2}(D(G)) + 2 \leq \tau(G)
\]
Corollary 2. For every simple graph $G$, the dominance complex $D(G)$ is not contractible.

Note that in the cases mentioned above, the equality $\text{conn}_{\mathbb{Z}_2}(D(G)) + 2 = \tau(G)$ holds except for the case $G = C_{4t+1}$. In this exceptional case, these numbers differ by 1.

In the proof, we show that the suspension of the Alexander dual of $D(G)$ has a free $\mathbb{Z}_2$-action, and it contains $S^{\alpha(G)-1}$ as a $\mathbb{Z}_2$-subspace. Here $\alpha(G) = |V| - \tau(G)$ is the size of a maximum independent set.

Acknowledgement. The author is supported by JSPS KAKENHI 19K14536.

2. Proofs

We start with describing the dominance complex as an independence complex of a hypergraph. A hypergraph $\mathcal{H} = (X, H)$ is a pair consisting of a set $X$ with a multi-set $H$ on $X$. We consider that every hypergraph is finite, i.e. $X$ and $H$ are finite. A subset $\sigma$ of $X$ is independent if there is no element in $H$ containing $\sigma$. Then the independent sets of $\mathcal{H}$ form an (abstract) simplicial complex $I(\mathcal{H})$, and we call it the independence complex of $\mathcal{H}$.

Next we recall the Alexander dual of independence complexes of hypergraphs. Let $K$ be an (abstract) simplicial complex with underlying set $X$. Then the combinatorial Alexander dual $K^\vee$ is the simplicial complex consisting of the subsets of $K$ whose complement is a non-face of $K$. Then a simplex of the Alexander dual $I^\vee(\mathcal{H})$ of the independence complex $I(\mathcal{H})$ is a subset $\sigma$ of $X$ such that $\sigma \cap \tau = \emptyset$ for some $\sigma \in H$. Recall that a subset of $X$ which intersects every hyperedge of $\mathcal{H}$ is said to be transversal. Thus $I^\vee(\mathcal{H})$ is the simplicial complex consisting of non-transversal sets.

Let $B_{\mathcal{H}}$ be the associated bipartite graph of the hypergraph $\mathcal{H}$. Namely, the vertex set of $B_{\mathcal{H}}$ is the disjoint union $V \sqcup H$ of $V$ and $H$, and $v \in V$ and $h \in H$ are adjacent if and only if $v \in h$. Then Nagel and Reiner [8] actually showed the following homotopy equivalence:

Theorem 3 (Proposition 6.2 of [8]). For every hypergraph $\mathcal{H}$, there is a following homotopy equivalence:

$$\Sigma(I^\vee(\mathcal{H})) \simeq I(B_{\mathcal{H}})$$

Here $\Sigma$ denotes the suspension.

We consider the case of dominance complex. As Ehrenborg and Hetyei noted in [11], the dominance complex $D(G)$ of a simple graph $G$ is simply described as the independence complex of some hypergraph $\mathcal{D}_G$ defined as follows: The underlying set of $\mathcal{D}_G$ is the vertex set $V(G)$ of $G$, and the set of hyperedges of $\mathcal{D}_G$ is the multi-set $\{N[v] \mid v \in V(G)\}$. Here $N[v]$ denotes the set $\{v\} \cup \{w \in V \mid \{v, w\} \in E(G)\}$. Then it is easy to see $D(G) = I(\mathcal{D}_G)$. 


Next we describe the associated hypergraph of $D_G$. Define the graph $G^{\infty}$ as follows: The vertex set of $G^{\infty}$ is $\{+, -\} \times V(G)$, and the set of edges of $G^{\infty}$ is

$$E(G^{\infty}) = \{(+(v), (-, w)) \mid v \in N[w]\}.$$ 

Clearly, $G^{\infty}$ is the associated bipartite graph of $D_G$, and Theorem 3 implies the following:

**Corollary 4.** There is a following homotopy equivalence:

$$\Sigma(D^\gamma(G)) \simeq I(G^{\infty})$$

Note that $G^{\infty}$ has a natural involution exchanging $(+, v)$ and $(-, v)$, and we write $\gamma$ to indicate it. Then $\gamma$ induces a $\mathbb{Z}_2$-action of $I(G^{\infty})$.

**Lemma 5.** The $\mathbb{Z}_2$-action of $I(G^{\infty})$ is free.

**Proof.** Let $\sigma$ be a simplex of $I(G^{\infty})$. It suffices to show $\sigma \cap \gamma \sigma = \emptyset$. Suppose $\sigma \cap \gamma \sigma \neq \emptyset$ and let $(\varepsilon, v) \in \sigma \cap \gamma \sigma$. Thus $(\varepsilon, v) \in \gamma \sigma$ implies $(-\varepsilon, v) \in \sigma$. This means $(+, v), (-, v) \in \sigma$. Since $\sigma$ is an independent set in $G^{\infty}$, this is a contradiction. \hfill $\square$

For a free $\mathbb{Z}_2$-space $X$, the coindex $\coind(X)$ of $X$ is the largest integer $n$ such that there is a continuous $\mathbb{Z}_2$-map from $S^n$ to $X$. Here we consider the involution of $S^n$ as the antipodal map. Recall that $\alpha(G)$ denotes the size of a maximum independent set of a simple graph $G$.

**Lemma 6.** The complex $I(G^{\infty})$ has a $\mathbb{Z}_2$-subcomplex which is $\mathbb{Z}_2$-homeomorphic to $S^{\alpha(G) - 1}$. In particular, we have $\alpha(G) - 1 \leq \coind(I(G^{\infty}))$.

**Proof.** Let $A_n$ be the boundary of $(n + 1)$-dimensional cross polytope. Namely, the vertex set of $A_n$ is $\{\pm 1, \ldots, \pm (n + 1)\}$ and a subset $\sigma$ of it is a simplex if and only if there is no $i$ with $\{\pm i\} \subset \sigma$. Then $|A_n|$ is homeomorphic to $S^n$.

Let $\sigma = \{v_1, \ldots, v_{\alpha(G)}\}$ be a maximum independent set of $G$. Define the simplicial map $f : A_{\alpha(G) - 1} \to I(G^{\infty})$ by sending $+i$ to $(+, v_i)$ and $-i$ to $(-, v_i)$. This is clearly an inclusion from $A_{\alpha(G) - 1}$ to $I(G^{\infty})$ which is a $\mathbb{Z}_2$-equivariant. \hfill $\square$

Next we observe that the coindex of a free $\mathbb{Z}_2$-space $X$ gives a restriction of the homology groups of $X$. Let $h\text{-dim}_{\mathbb{Z}_2}(X)$ be the maximum integer $n$ such that $\tilde{H}_n(X; \mathbb{Z}_2) \neq 0$. Then we have the following:

**Lemma 7.** For a finite free $\mathbb{Z}_2$-simplicial complex $X$, the following inequality holds:

$$\coind(X) \leq h\text{-dim}_{\mathbb{Z}_2}(X)$$

**Proof.** Suppose $n = \coind(X) > h\text{-dim}_{\mathbb{Z}_2}(X)$. Let $\overline{X}$ denote the orbit space of $X$ and $w_1(X)$ the 1st Stiefel-Whitney class of the double cover $X \xrightarrow{p} \overline{X}$ (see [2] or [3]). Since there is a
\(\mathbb{Z}_2\)-map \(S^n \to X\) and \(w_1(S^n)^n \neq 0\), the naturality of \(w_1\) implies \(0 \neq w_1(X)^n \in H^n(X; \mathbb{Z}_2)\). By the Gysin sequence of the double cover (see [6]), we have the following exact sequence:

\[
H^k(X; \mathbb{Z}_2) \xrightarrow{\delta} H^k(X; \mathbb{Z}_2) \xrightarrow{\cup w_1(X)} H^{k+1}(X; \mathbb{Z}_2)
\]

Since \(h:\text{dim}_{\mathbb{Z}_2}(X) < n\), we have that \(H^n(Z; \mathbb{Z}_2) = 0\) and hence the map \(H^n(X; \mathbb{Z}_2) \xrightarrow{\cup w_1(X)} H^{n+1}(X; \mathbb{Z}_2)\) is injective. Thus we have \(w_1(X)^{n+1} \neq 0\). By induction, we have that \(0 \neq w_1(X)^k \in H^k(X; \mathbb{Z}_2)\) for every \(k > n\). This is a contradiction since \(X\) is a finite complex. □

We are now ready to prove Theorem 1. Set \(k = \text{conn}_{\mathbb{Z}_2}(D(G))\). The combinatorial Alexander duality theorem (see [5]) implies \(h:\text{dim}_{\mathbb{Z}_2}(D^\vee(G)) = |V| - k - 4\). Thus we have

\[
\alpha(G) - 1 \leq \text{coind}(I(G^\infty)) \leq h:\text{dim}_{\mathbb{Z}_2}(I(G^\infty)) = h:\text{dim}_{\mathbb{Z}_2}(\Sigma D^\vee(G)) = |V| - k - 3.
\]

Here the first and second inequalities follow from Lemma 7 and Lemma 6, respectively. Thus we have

\[
\text{conn}_{\mathbb{Z}_2}(D(G)) = k + 2 \leq |V| - \alpha(G) = \tau(G)
\]

This completes the proof.

References

[1] R. Ehrenborg, G. Hetyei; The topology of independence complex, Eur. J. Comb. 27 (2006) 906-923.
[2] D.N. Kozlov; Combinatorial algebraic topology, Springer, Berlin, Algorithms and Computation in Mathematics, Vol. 21, 2008.
[3] M. Marietti, D. Testa; A uniform approach to complexes arising from forests, Electron. J. Comb. 15 (2008).
[4] M. Marietti, D. Testa; Cores of simplicial complexes, Discrete Comput. Geom. 40 (2008), 444-468.
[5] E. Miller, B. Sturmfels; Combinatorial Commutative Algebra. Graduate Texts in Mathematics. 227. New York, NY: Springer-Verlag, 2005.
[6] J.W. Milnor, J.D. Stasheff; Characteristic Classes, Princeton University Press, 1974.
[7] D. Taylan; Matching trees for simplicial complexes and homotopy type of devoid complexes of graphs, Order, 33 (2016), 459-476.
[8] U. Nagel, V. Reiner; Betti numbers of monomial ideals and shifted skew shapes, Electron. J. Comb. 16 (2009).

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