PROJECTIVE FLATNESS OVER KLT SPACES AND UNIFORMISATION OF VARIETIES WITH NEF ANTI-CANONICAL DIVISOR

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Abstract. We give a criterion for the projectivisation of a reflexive sheaf on a klt space to be induced by a projective representation of the fundamental group of the smooth locus. This criterion is then applied to give a characterisation of finite quotients of projective spaces and Abelian varieties by $\mathbb{Q}$-Chern class (in)equalities and a suitable stability condition. This stability condition is formulated in terms of a naturally defined extension of the tangent sheaf by the structure sheaf. We further examine cases in which this stability condition is satisfied, comparing it to K-semistability and related notions.

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1. Introduction

1.1. Stability, the Miyaoka-Yau Inequality and quasi-étale uniformisation. Let $X$ be a $\mathbb{Q}$-Fano $n$-fold; that is, let $X$ be a normal, projective, $n$-dimensional variety with at worst klt singularities such that $-K_X$ is $\mathbb{Q}$-ample. Generalising a classical result, it has been shown in [GKPT19, Thm. 6.1] that if the tangent sheaf $\mathcal{T}_X$ is stable with respect to the anticanonical polarisation $-K_X$, then its first two $\mathbb{Q}$-Chern classes, which are well-defined for all spaces with klt singularities, satisfy the $\mathbb{Q}$-Bogomolov-Gieseker Inequality,

\[
\frac{n-1}{2n} \cdot \hat{c}_1 \left( \Omega_X^{[1]} \right)^2 \cdot [-K_X]^{n-2} \leq \hat{c}_2 \left( \Omega_X^{[1]} \right) \cdot [-K_X]^{n-2}.
\]

(1.0.1)
As part of the present investigation, we will generalise (1.0.1) to the case where $\mathcal{T}_X$ is semistable with respect to the anticanonical polarisation $-K_X$, see Section 5.2 below. In analogy to the case of manifolds with ample canonical bundle and as a generalisation of a classical result on Kähler-Einstein Fano manifolds, [CO75], one expects more, namely a $\mathbb{Q}$-Miyaoka-Yau Inequality of the form
\begin{equation}
\frac{n}{2(n+1)} \cdot \tilde{c}_1 \left( \Omega_X^{[1]} \right)^2 \cdot \left[-K_X\right]^{n-2} \leq \tilde{c}_2 \left( \Omega_X^{[1]} \right) \cdot \left[H\right]^{n-2}.
\end{equation}

The canonical extension. Section 7.1 shows by way of classical examples that without an additional stability assumption (1.0.2) will not hold. This paper discusses an algebro-geometric (semi)stability notion that is stronger than (semi)stability of $\mathcal{T}_X$ and guarantees that the $\mathbb{Q}$-Miyaoka-Yau Inequality (1.0.2) holds: (semi)stability of the canonical extension. The canonical extension is a reflexive sheaf $\mathcal{E}_X$ on $X$ that appears in the middle of the short exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{E}_X \to \mathcal{T}_X \to 0 \]
whose extension class is given by the first Chern class of the $\mathbb{Q}$-Cartier divisor $-K_X$. For the first Chern classes of line bundles over manifolds the construction is classical, see [Ati57], and generalisations of it have appeared in many other problems of Kähler geometry, see for instance [Tia92, Sem92, Don02, GW20]. Section 4 discusses the construction of the canonical extension in the singular case. Note that the $\mathbb{Q}$-Miyaoka-Yau Inequality (1.0.2) is in fact nothing but the $\mathbb{Q}$-Bogomolov-Gieseker inequality for the canonical extension sheaf $\mathcal{E}_X$.

1.2. Main results. Using the canonical extension, we may now formulate the main results. Our point of departure is the following.

**Proposition 1.1** (Miyaoka-Yau Inequality and Semistable Canonical Extension). Let $X$ be an $n$-dimensional, projective klt space. If there exists an ample Cartier divisor $H$ on $X$ such that the canonical extension $\mathcal{E}_X$ is semistable with respect to $H$, then
\begin{equation}
\frac{n}{2(n+1)} \cdot \tilde{c}_1 \left( \Omega_X^{[1]} \right)^2 \cdot \left[H\right]^{n-2} \leq \tilde{c}_2 \left( \Omega_X^{[1]} \right) \cdot \left[H\right]^{n-2}.
\end{equation}

Definition 21 on page 4 recalls the notion of a klt space. We prove Proposition 1.1 in Section 5.2. Section 6 discusses criteria to guarantee stability of the canonical extension $\mathcal{E}_X$.

**Remark 1.2** (Kähler-Einstein manifolds). If $(X, \omega)$ is a Kähler-Einstein Fano manifold, Tian has shown in [Tia92, Thm. 2.1] that the bundle $\mathcal{E}_X$ admits a Hermitian-Yang-Mills metric with respect to $\omega$, and is therefore $\omega$-semistable. The Miyaoka-Yau Inequality (1.0.2) then holds by [Kob87, Thm. IV.4.7], with equality if and only if the metric has constant holomorphic sectional curvature and $X \cong \mathbb{P}^n$, see [Kob87, Thm. IV.4.16] or [Tia96, Thm. 2.3]. The next result generalises this to klt varieties with nef anticanonical classes.

**Theorem 1.3** (Quasi-étale uniformisation if $-K_X$ is nef). Let $X$ be a projective variety. Assume that $X$ has at most klt singularities\footnote{equivalently: Assume that the pair $(X, 0)$ is klt.} and that its anti-canonical class $-K_X$ is nef. Then, the following statements are equivalent.
\begin{enumerate}
\item There exists an ample Cartier divisor $H$ on $X$ such that the canonical extension $\mathcal{E}_X$ is semistable with respect to $H$ and such that equality holds in (1.1.1):
\begin{equation}
\frac{n}{2(n+1)} \cdot \tilde{c}_1 \left( \Omega_X^{[1]} \right)^2 \cdot \left[H\right]^{n-2} = \tilde{c}_2 \left( \Omega_X^{[1]} \right) \cdot \left[H\right]^{n-2}.
\end{equation}
\item The variety $X$ is a quotient of the projective space or of an Abelian variety by the action of a finite group of automorphisms that acts without fixed points in codimension one.
\end{enumerate}
A proof of Theorem 1.3 is given in Section 5.3 below. Section 7.2 discusses many examples where equality in (1.0.2) holds.

Remark 1.4 (Torus quotients). If a variety $X$ satisfies Assumption (1.3.1) and if additionally $K_X \equiv 0$, then $\hat{c}_2(\Omega^{(1)}_X) \cdot [H]^{n-2} = 0$, and we recover the main results of [GKP16b, LT18]. More intricate characterisations of torus quotients can be found in [GKP21].

1.3. Application to Fano varieties. Note that Proposition 1.1 and Theorem 1.3 apply in particular to $Q$-Fano varieties, and give a characterisation of finite quotients of projective spaces among $Q$-Fano varieties $X$ with semistable canonical extension $\mathcal{E}_X$.

In fact, in the last few decades multiple stability notions for Fano manifolds have been introduced, with a view both towards the construction of moduli spaces and the existence question for Kähler-Einstein metrics, see for instance the survey [Xu20]. The klt condition on the singularities appears naturally in this context, see [Oda13]. In Section 6, we prove that stability of the canonical extension can be guaranteed in natural classes of examples, which include all smooth Fano threefolds of Picard number one. There, we also discuss the relation to K-(semi)stability.

Finally, together with a result of Druel-Guenancia-Păun [DGP20, Thm. B] on the semistability of the canonical extension on $Q$-Fano varieties, Proposition 1.1 and Theorem 1.3 immediately imply the following statement. This generalises the classical result discussed in Remark 1.2 above.

Theorem 1.5 (Characterisation of finite quotients of projective spaces). Let $X$ be a $Q$-Fano variety admitting a singular Kähler-Einstein metric. Then, the $Q$-Miyaoka-Yau inequality (1.0.2) holds, with equality if and only if $X \cong \mathbb{P}^n/G$, where $G$ is a finite group of automorphisms acting without fixed points in codimension one. □

1.4. Projectively flat sheaves on singular spaces. The key objects used in our arguments are reflexive sheaves $\mathcal{F}$ such that $\mathcal{F}|_{X_{\text{reg}}}$ is locally free and holomorphically projectively flat; i.e., the projectivisation of $\mathcal{F}|_{X_{\text{reg}}}$ is induced by a projective representation of the fundamental group of $X_{\text{reg}}$. Details are discussed in Section 3.1. The proofs of our main results rely on the following technical criterion for projective flatness, which generalises classic results from the smooth case to the setting of klt spaces. Its proof crucially relies on [LT18].

Proposition 1.6 (Criterion for projective flatness). Let $X$ be an $n$-dimensional, projective klt space and let $H \in \text{Div}(X)$ be ample. Further, let $\mathcal{F}$ be a reflexive sheaf of rank $r$ on $X$. Assume that $\mathcal{F}$ is semistable with respect to $H$ and that its $Q$-Chern classes satisfy the equation

$$\frac{r - 1}{2r} \cdot \hat{c}_2(\mathcal{F})^2 \cdot [H]^{n-2} = \hat{c}_2(\mathcal{F}) \cdot [H]^{n-2}.$$  \hspace{1cm} (1.6.1)

Then, $\mathcal{F}|_{X_{\text{reg}}}$ is locally free and projectively flat.

A proof of Proposition 1.6 is given in Section 5.1 below. Section 3.3 describes the structure of the sheaf $\mathcal{F}$ in more detail. Further applications of Proposition 1.6 are discussed in [GKP21].

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2. Conventions and notation

2.1. Global conventions. Throughout this paper, all schemes, varieties and morphisms will be defined over the complex number field. We follow the notation and conventions
of Hartshorne’s book [Har77]. In particular, varieties are always assumed to be irreducible. We refer the reader to [KM98] for notation around higher-dimensional birational geometry.

**Definition 2.1 (Klt spaces).** A normal, quasi-projective variety $X$ is called a klt space if there exists an effective Weil $\mathbb{Q}$-divisor $\Delta$ such that the pair $(X, \Delta)$ is klt.

2.2. Reflexive sheaves. Given a normal, quasi-projective variety (or normal, irreducible complex space) $X$, we write $\Omega^p_X := (\Omega^p_X)^{\times}$ and refer to this sheaf as the sheaf of reflexive differentials. More generally, given any coherent sheaf $\mathcal{E}$ on $X$, write $\mathcal{E}([m]) := (\mathcal{E}^\vee)^{\times}$ and $\mathrm{det} \mathcal{E} := (\Lambda^{\mathrm{rank} \mathcal{E}} \mathcal{E})^{\times}$. Given any morphism $f : Y \to X$ of normal, quasi-projective varieties (or normal, irreducible, complex spaces), we write $f^*[\mathcal{E}] := (f^* \mathcal{E})^{\times}$.

2.3. Varieties and complex spaces. In order to keep notation simple, do not distinguish between algebraic varieties and their underlying complex spaces, unless there is specific danger of confusion. Along these lines, if $X$ is a quasi-projective complex variety, we write $\pi_1(X)$ for the fundamental group of the associated complex space.

2.4. Covering maps and quasi-étale morphisms. A cover or covering map is a finite, surjective morphism $\gamma : X \to Y$ of normal, quasi-projective varieties (or normal, irreducible complex spaces). The covering map $\gamma$ is called Galois if there exists a finite group $G \subset \mathrm{Aut}(X)$ such that $\gamma$ is isomorphic to the quotient map.

A morphism $f : X \to Y$ between normal varieties (or normal, irreducible complex spaces) is called quasi-étale if $f$ is of relative dimension zero and étale in codimension one. In other words, $f$ is quasi-étale if $\dim X = \dim Y$ and if there exists a closed, subset $Z \subseteq X$ of codimension $\geq 2$ such that $f|_{X\setminus Z} : X \setminus Z \to Y$ is étale.

2.5. Maximally quasi-étale spaces. Let $X$ be a normal, quasi-projective variety (or a normal, irreducible complex space). We say that $X$ is maximally quasi-étale if the natural push-forward map of fundamental groups,

$$\pi_1(X_{\mathrm{reg}}) \xrightarrow{\text{(incl).}} \pi_1(X)$$

induces an isomorphism between the profinite completions, $\mathring{\pi}_1(X_{\mathrm{reg}}) \cong \mathring{\pi}_1(X)$.

**Remark 2.2.** Recall from [FL81, 0.7.B on p. 33] that the natural push-forward map (incl), is always surjective. If $X$ is any klt space, then $X$ admits a quasi-étale cover that is maximally quasi-étale and again a klt space, [GKP16b, Thm. 1.14].

3. Projective flatness

3.1. Projectively flat bundles and sheaves. As pointed out in the introduction, projective flatness is the core concept of this paper. Projectively flat bundles over differentiable manifolds are thoroughly discussed in the literature, for instance in the classic textbook [Kob87]. We are, however, not aware of references that cover the singular case. We have therefore chosen to introduce the relevant notions in some detail here. In a nutshell, we call a projective space bundle projectively flat if it comes from a $\mathbb{P}\mathrm{GL}$-representation of the fundamental group. The following construction and the subsequent definitions make this precise.

**Construction 3.1.** Let $X$ be a normal and irreducible complex space. Given a number $r \in \mathbb{N}$ and a representation of the fundamental group, $\rho : \pi_1(X) \to \mathbb{P}\mathrm{GL}(r + 1, \mathbb{C})$, consider the universal cover $\tilde{X} \to X$ and the diagonal action of $\pi_1(X)$ on $\tilde{X} \times \mathbb{P}^r$. The quotient

$$\mathbb{P}_\rho := \tilde{X} \times \mathbb{P}^r / \pi_1(X)$$

is a complex space that carries the natural structure of a locally trivial $\mathbb{P}^r$-bundle over the original space $X$. 
Definition 3.2 (Projectively flat bundles and sheaves on complex spaces). Let $X$ be a normal and irreducible complex space, let $r \in \mathbb{N}$ be any number and let $\mathbb{P} \to X$ be a locally trivial $\mathbb{P}^r$-bundle. We call the bundle $\mathbb{P} \to X$ (holomorphically) projectively flat if there exists a representation of the fundamental group, $\rho : \pi_1(X) \to \mathbb{P}GL(r+1, \mathbb{C})$, and an isomorphism $\mathbb{P}^r \cong X_{\rho}$ of complex spaces over $X$, where $\mathbb{P}^r$ is the bundle constructed in 3.1 above. A locally free coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called (holomorphically) projectively flat if the associated bundle $\mathbb{P}(\mathcal{F})$ is projectively flat.

Definition 3.3 (Projectively flat bundles and sheaves on complex varieties). Let $X$ be a connected, complex, quasi-projective variety and let $r \in \mathbb{N}$ be any number. An étale locally trivial $\mathbb{P}^r$-bundle $\mathbb{P} \to X$ is called projectively flat if the associated analytic bundle $\mathbb{P}^{(an)} \to X^{(an)}$ is projectively flat. Ditto for coherent sheaves.

On complex manifolds, projective flatness is of course equivalent to the existence of certain connections. We briefly recall the following standard fact.

Fact 3.4 (Projective flatness and connections). Let $X$ be a connected complex manifold and let $\mathcal{F}$ be a locally free coherent sheaf on $X$. Then, the following are equivalent.

(3.4.1) The locally free sheaf $\mathcal{F}$ is projectively flat in the sense of Definition 3.2.

(3.4.2) The locally free sheaf $\mathcal{F}$ admits a holomorphic connection whose curvature tensor is of the form

$$ R = \alpha \cdot \text{Id}_{\mathcal{F}} \in H^0(X, \Omega^2_X \otimes \text{End}(\mathcal{F})) $$

for some holomorphic 2-form $\alpha$ on $X$.

Proof. In the $C^\infty$-setting, this is [Kob87, I.Cor. 2.7 and I.Prop. 2.8]. The proofs carry over to the holomorphic setting mutatis mutandis. □

3.2. Projective flatness and flatness. In our earlier paper [GKP16b] we discussed locally free sheaves $\mathcal{E}$ on singular spaces $X$ that were flat in the sense that $\mathcal{E}$ was defined by a representation $\pi_1(X) \to \text{GL}(r, \mathbb{C})$, in a manner analogous to Construction 3.1. The two notions are of course related.

Proposition 3.5 (Projective flatness and flatness of derived sheaves). Let $X$ be a normal and irreducible complex space and $\mathcal{E}$ a rank-$r$ locally free sheaf on $X$. If $\mathcal{E}$ is projectively flat, then the locally free sheaves $\text{End}(\mathcal{E})$ and $\text{Sym}^r \mathcal{E} \otimes \det \mathcal{E}^*$ are flat in the sense of [GKPT19, Def. 2.13].

Proof. The group morphisms

$$ \text{GL}(r, \mathbb{C}) \to \text{GL}(r^2, \mathbb{C}) \quad \text{and} \quad \text{GL}(r, \mathbb{C}) \to \text{GL}(\mathbb{C}, \mathbb{C}) $$

factor via $\mathbb{P}\text{GL}(r, \mathbb{C})$. □

In case where $X$ is maximally quasi-étale, Proposition 3.5 has the following consequence, which we find remarkable because it can be used to guarantee local freeness of certain sheaves at the singular points of $X$. We will later use it in a setting where $\mathcal{E}$ is constructed so that $\text{End}(\mathcal{E})$ contains the tangent sheaf $\mathcal{T}_X$ as a direct summand and where Chern class equalities guarantee projective flatness of $\mathcal{E}|_{X_{\text{reg}}}$.

Corollary 3.6 (Projective flatness and local freeness of derived sheaves I). Let $X$ be a normal, irreducible complex space that is maximally quasi-étale. Let $\mathcal{E}$ be a reflexive sheaf on $X$ such that $\mathcal{E}|_{X_{\text{reg}}}$ is locally free and projectively flat. Then, the sheaves $(\text{Sym}^r \mathcal{E} \otimes \det \mathcal{E}^*)^*$ and $\text{End}(\mathcal{E})$ are both locally free and flat in the sense of [GKPT19, Def. 2.13].

Proof. The assumption that $X$ is maximally quasi-étale says that the natural morphism of étale fundamental groups,

$$ \pi_1(X_{\text{reg}}) \to \pi_1(X), $$

...
is isomorphic. The arguments from [GKP16b, proof of Thm. 1.14 on p. 26] now apply verbatim to yield the desired extension. Alternatively, use the fact that any finite-dimensional complex representation of \( \pi_1(X_{\text{reg}}) \) extends to a representation of \( \pi_1(X) \) by [Gro70, Thm. 1.2b].

Once we are working over a complex manifold rather than over an arbitrary complex space, the converse statements are also true.

**Proposition 3.7** (Projective flatness and local freeness of derived sheaves II). Let \( X \) be a connected complex manifold and \( \mathcal{F} \) be a locally free coherent sheaf on \( X \) of rank \( r \). Then, the following are equivalent.

1. \( \mathcal{F} \) is projectively flat in the sense of Definition 3.2.
2. \( \mathcal{F} \) is locally free and flat on \( \mathbb{C}^r \).
3. The locally free sheaf \( \mathcal{E} \) is flat in the sense that it admits a flat holomorphic connection.

**Proof.** The implication “(3.7.1) \( \Rightarrow \) (3.7.2)” follows from Fact 3.4. We refer the reader to [Kob87, I.Cor. 2.7 and I.Prop. 2.8] for a proof of the analogous statement in the \( C^\omega \)-setting; again, the arguments carry over to the holomorphic setting mutatis mutandis. The implication “(3.7.2) \( \Rightarrow \) (3.7.1)” is proven in [Bis09, proof of Prop. 2.1], crucially using the fact that \( \text{End}(V) \) is a faithful representation of the complex reductive group \( \text{PGL}(V) \), where \( V \) is a complex vector space of dimension \( r \). The equivalence “(3.7.1) \( \Leftrightarrow \) (3.7.3)” is proven in an entirely similar fashion, by looking at the representation \( \text{Sym}^r(V) \otimes \det(V) \) instead of \( \text{End}(V) \). We leave it to the reader to spell out the details. \( \square \)

**Remark 3.8.** For more information on locally free sheaves with a flat holomorphic connection, the reader is referred to [Ati57, Sect. 7].

The following example shows that under the assumptions of Corollary 3.6 the sheaf \( \mathcal{E} \) itself is not necessarily locally free.

**Example 3.9.** Let \( q : \widetilde{X} \rightarrow \mathbb{P}^1 \) be the second Hirzebruch surface with its natural projection to \( \mathbb{P}^1 \), that is, \( \widetilde{X} := \mathbb{P}_2(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}) \). Let \( E \subset \widetilde{X} \) be the section at infinity, and note that the complement \( \widetilde{X} \setminus E \) is simply connected, since it is the total space of a line bundle over a simply connected space. Next, consider the blow-down \( \pi : \widetilde{X} \rightarrow X \). The variety \( X \) is then isomorphic to the subvariety of \( \mathbb{P}^3 \) given as the cone over a quadric normal curve in \( \mathbb{P}^2 \). Finally, let \( D := \pi(F) \) be the image of a \( q \)-fibre \( F \). The variety \( D \) is then a line through the unique singular point \( P \in X \). Since \( \pi \) is an isomorphism away from \( E \), the smooth locus \( X_{\text{reg}} \) of \( X \) is simply connected; in particular, \( X \) is maximally quasi-étale. The Weil divisorial sheaf \( \mathcal{E} := \mathcal{O}_X(D) \) is not locally free at \( P \); however, \( \mathcal{O}(\mathcal{E}) \cong (\mathcal{E}_X(D) \otimes \mathcal{O}_X(-D))^{\oplus \kappa} \cong \mathcal{O}_X \) is locally free and flat on \( X \).

### 3.3. Projective flatness on maximally quasi-étale spaces.

The proofs of our main results require that we consider a setting where \( X \) is normal and maximally quasi-étale, and \( \mathcal{F} \) is a coherent, reflexive sheaf on \( X \) whose restriction \( \mathcal{F}|_{X_{\text{reg}}} \) is locally free and projectively flat in the sense of Definition 3.2.

#### 3.3.1. Extension and local structure.

In this section we show that the structure of \( \mathcal{F} \) at the singular points of \( X \) is eventually rather simple.

**Proposition 3.10** (Extension of projectively flat \( \mathbb{P}^r \)-bundles). Let \( X \) be a normal, irreducible complex space that is maximally quasi-étale. Then, any representation \( \rho^* : \pi_1(X_{\text{reg}}) \rightarrow \text{PGL}(r, \mathbb{C}) \) factors via a representation of \( \pi_1(X) \) as follows,

\[
\begin{array}{ccc}
\pi_1(X_{\text{reg}}) & \overset{(\text{incl})}{\longrightarrow} & \pi_1(X) \\
\rho^* & \longrightarrow & \text{PGL}(r, \mathbb{C})
\end{array}
\]
In particular, any projectively flat \( \mathbb{P}^{r-1} \)-bundle \( \mathbb{P}_{\text{reg}} \to X_{\text{reg}} \) extends to a projectively flat projective bundle \( \mathcal{F}_X \to X \).

Proof. Write \( G := \text{Image}(\rho^r) \). Since \( \mathbb{P}\text{GL}(r, \mathbb{C}) \) is a linear group, so is \( G \). In particular, it follows from Malcev’s theorem, [Weh73, Thm. 4.2], that \( G \) is residually finite. As in the proof of Corollary 3.6, using the arguments from [GKP16b, proof of Thm. 1.14 on p. 26] we find that the representation \( \rho^s \) is induced by a representation of \( \pi_1(X) \), which yields the desired extension. \( \square \)

Proposition 3.10 has further consequences. While we cannot expect that the sheaf \( \mathcal{F} \) applies to settings where the preimage is not locally free near the singular points of \( X \), it turns out that \( \mathcal{F} \) locally (in the analytic topology) always looks like a direct sum of rank \( \mathcal{F} \) copies of the same Weil divisorial sheaf.

Proposition 3.11 (Local description of projectively flat sheaves). Let \( X \) be a normal, irreducible complex space that is maximally quasi-étale. Let \( \mathcal{F}_{\text{reg}} \) be a locally free and projectively flat coherent sheaf on \( X_{\text{reg}} \). Then, the following holds.

(3.11.1) If \( U \subseteq X \) is any simply connected open subset, then there exists an invertible sheaf \( \mathcal{L}U_{\text{reg}} \in \text{Pic}(U_{\text{reg}}) \), unique up to isomorphism, such that
\[
\mathcal{F}_{\text{reg}}|_{U_{\text{reg}}} \cong (\mathcal{L}U_{\text{reg}})^{\oplus \text{rank } \mathcal{F}_{\text{reg}}}.
\]

(3.11.2) The sheaf \( \mathcal{F}_{\text{reg}} \) extends from \( X_{\text{reg}} \) to a reflexive sheaf \( \mathcal{F}_{X} \) on \( X \) if and only if for every simply connected open \( U \subseteq X \), the sheaf \( \mathcal{L}U_{\text{reg}} \) extends from \( U_{\text{reg}} \) to a reflexive coherent sheaf \( \mathcal{L}U \) of rank one on \( U \). In this case,
\[
\mathcal{F}_{X}|U \cong (\mathcal{L}U)^{\oplus \text{rank } \mathcal{F}_{\text{reg}}}.
\]

Proof. Write \( r := \text{rank } \mathcal{F} \). We have seen in Proposition 3.10 that the projectively flat \( \mathbb{P}^{r-1} \)-bundle \( \mathbb{P}_{\text{reg}} := \mathcal{P}(\mathcal{F}_{\text{reg}}) \) extends to a projectively flat \( \mathbb{P}^{r-1} \)-bundle \( \mathcal{P}_X \) on \( X \). By construction, the restriction of \( \mathcal{P}_X \) to any simply connected open subset \( U \subseteq X \) is trivial and so is its restriction to \( U_{\text{reg}} \). In particular, we have an isomorphism
\[
\mathcal{P}(\mathcal{F}_{\text{reg}})|_{U_{\text{reg}}} \cong \mathcal{F}_{\text{reg}}|_{U_{\text{reg}}} \cong U_{\text{reg}} \oplus \mathcal{F}_U(\mathcal{O}^{\oplus r+1}_{U_{\text{reg}}}).
\]

It follows that \( \mathcal{F}_{\text{reg}}|_{U_{\text{reg}}} \) and \( \mathcal{O}^{\oplus r+1}_{U_{\text{reg}}} \) differ only by a twist with an invertible \( \mathcal{L}U_{\text{reg}} \in \text{Pic}(U_{\text{reg}}) \). This shows (3.11.1).

To prove (3.11.2), consider one of the open sets \( U \) and write \( \iota : U_{\text{reg}} \to U \) for the obvious inclusion. Since push-forward respects direct summands, we find that
\[
\iota_*(\mathcal{F}_{\text{reg}}|_{U_{\text{reg}}}) \cong (\mathcal{L}U_{\text{reg}})^{\oplus r},
\]
and the left side is coherent if and only if every summand of the right side is. \( \square \)

3.3.2. Pull-back. As one consequence of the local description of projectively flat sheaves, we find that they satisfy a surprising pull-back property. We believe that the following corollary is of independent interest and include it here for later reference, even though to do not use it in the sequel.

Corollary 3.12 (Pull-back of projectively flat sheaves). Let \( X \) be a normal, irreducible complex space that is maximally quasi-étale. Let \( \mathcal{F} \) be a reflexive sheaf on \( X \) such that \( \mathcal{F}|_{X_{\text{reg}}} \) is locally free and projectively flat. If \( \varphi : Y \to X \) is any morphism where \( Y \) is irreducible, smooth and where \( \text{img}(\varphi) \not\subset X_{\text{sing}} \), then the reflexive pull-back \( \varphi|^{-1} \mathcal{F} \) is locally free and projectively flat.

Remark 3.13. Note that Corollary 3.12 applies to settings where the preimage \( \varphi^{-1}(X_{\text{sing}}) \) is a divisor in \( Y \). One relevant case is where \( \varphi \) is the inclusion map of a smooth curve \( Y \subset X \) that passes through the singular locus.
Proof of Corollary 3.12. For convenience of notation, write \( \mathcal{F}_Y := \varphi^* \mathcal{F} \). We begin by showing that \( \mathcal{F}_Y \) is locally free. As this question is local over \( X \), we may assume without loss of generality that \( X \) is simply connected. Proposition 3.11 will then allow to find a Weil divisorial sheaf \( \mathcal{L} \) on \( X \) such that \( \mathcal{F} = \mathcal{L}^{\oplus \text{rank } \mathcal{F}} \), which implies \( \mathcal{F}_Y = (\varphi^* \mathcal{L})^{\oplus \text{rank } \mathcal{F}} \). Local freeness of \( \mathcal{F}_Y \) follows since \( \varphi^* \mathcal{L} \) is Weil divisorial on the smooth space \( X \) and hence invertible.

It remains to show that \( \mathcal{F}_Y \) is projectively flat. By Proposition 3.7 it suffices to show that \( \text{End}(\mathcal{F}_Y) \) is flat. This will be established by showing that the natural morphism

\[
\varphi^* \text{End}(\mathcal{F}) \to \text{End}(\mathcal{F}_Y)
\]

is isomorphic. But we have already seen in Corollary 3.6 that \( \text{End}(\mathcal{F}) \) is locally free and flat, and then so is its pull-back \( \varphi^* \text{End}(\mathcal{F}) \). To be more precise, recall that the Morphism (3.13.1) comes to be as a composition

\[
\varphi^* \text{End}(\mathcal{F}) \longrightarrow \text{End}(\varphi^* \mathcal{F}) \longrightarrow \text{End}(\varphi^* \mathcal{F}).
\]

The existence of the first morphism in (3.13.2) follows from the observation that any endomorphism of \( \mathcal{F} \) induces an endomorphism of the pull-back \( \varphi^* \mathcal{F} \). The existence of the second morphism in (3.13.2) follows from the observation that any endomorphism of any sheaf induces an endomorphism morphism of its reflexive hull.

The question "Is (3.13.1) isomorphic?" is again local over \( X \), so that we may again assume without loss of generality that \( \mathcal{F} = \mathcal{L}^{\oplus \text{rank } \mathcal{F}} \). As before, we consider the invertible sheaf \( \mathcal{L}_Y := (\varphi^* \mathcal{L})^\oplus \in \text{Pic}(Y) \). As both arrows in (3.13.2) respect the direct sum decomposition, it suffices to show that the natural morphism

\[
\mathcal{O}_Y \cong \varphi^* \mathcal{O}_X \cong \varphi^* \text{End}(\mathcal{L}) \to \text{End}(\mathcal{L}_Y) \cong \mathcal{O}_Y
\]

is isomorphic. The assumption \( \text{Im}(\varphi) \not\subset X_{\text{sing}} \) implies that (3.13.3) is generically injective, hence injective. But (3.13.3) is also surjective, as the pull-back of the identity section \( \text{Id}_{\mathcal{L}} \in \text{End}(\mathcal{L})(X) \), which generates \( \text{End}(\mathcal{L}) \), maps to the identity \( \text{Id}_{\mathcal{L}_Y} \in \text{End}(\mathcal{L}_Y)(Y) \), which generates \( \text{End}(\mathcal{L}_Y) \).

\[\square\]

4. The canonical extension

The following construction is classical and well-known in the smooth setup, see [Ati57], and has been studied for various questions in Kähler geometry, for example in [Tia92, Don02, GW20].

Construction 4.1 (Extensions induced by \( \mathbb{Q} \)-Cartier divisors). Let \( X \) be a normal variety and \( D \) be a \( \mathbb{Q} \)-Cartier Weil divisor on \( X \). Choose any integer \( m \in \mathbb{N}_{>0} \) such that \( m \cdot D \) is Cartier, and let \( c_1(\mathcal{O}_X(m \cdot D)) \in H^1(X, \Omega^1_X) \) be the first Chern class of the locally free sheaf \( \mathcal{O}_X(m \cdot D) \). Spelled out: if \( \mathcal{U} = (U_a) \) is a trivialising covering for \( \mathcal{O}_X(m \cdot D) \) with transition functions \( g_{a,b} \in \mathcal{O}^*_X(U_{a,b}) \), then the first Chern class \( c_1(\mathcal{O}_X(m \cdot D)) \) is the image of the Čech cohomology class

\[
\left[ \frac{dg_{a,b}}{g_{a,b}} \right]_{a,b \in \mathcal{I}} \in \check{H}^1(\mathcal{U}, \Omega^1_X)
\]

in \( H^1(X, \Omega^1_X) \). We can then assign to \( D \) the cohomology class

\[
c_1(D) := \frac{1}{m} c_1(\mathcal{O}_X(m \cdot D)) \in H^1(X, \Omega^1_X).
\]

Observe that the class \( c_1(D) \) is independent of the choice of \( m \). Using the canonical identification \( H^1(X, \Omega^1_X) \cong \text{Ext}^1(\mathcal{O}_X, \Omega^1_X) \) and the standard interpretation of first Ext-groups, [Har77, III.Prop. 6.3] and [Eis95, Ex. A.3.26], in this way we obtain the isomorphism class of an extension

\[
0 \to \Omega^1_X \to \mathcal{W}_D \to \mathcal{O}_X \to 0.
\]
We refer to (4.1.1) as the extension of $\Omega_X^1$ by $\mathcal{O}_X$ induced by $D$. Functoriality of Ext-groups with respect to restriction to open subsets implies that (4.1.1) splits on any affine open subset of $X$. In particular, we find that the extension defined by $D$ is locally splitable. This allows to dualise (4.1.1) in order to obtain a likewise locally splitable extension of $\mathcal{T}_X$ by $\mathcal{O}_X$, 

(4.1.2) \quad 0 \to \mathcal{O}_X \to E \to \mathcal{T}_X \to 0.

**Remark 4.2** (Reflexivity of $E_D$). In the setting of Construction 4.1, recall that $\mathcal{T}_X$ is reflexive. This implies in particular that the middle term $E_D$ of the locally splitable Extension (4.1.2) is likewise reflexive. Alternatively, use that $E_D \cong \mathcal{H}om(W_D, \mathcal{O}_X)$ to reach the same conclusion.

**Remark 4.3** (Functoriality in morphisms). In the setting of Construction 4.1, if $\gamma : Y \to X$ is any morphism of normal varieties, then $c_1(\gamma^*D)$ is the image of $c_1(D)$ under the obvious map

$$H^1(\mathcal{O}_Y) : H^1(X, \Omega_X^1) \to H^1(Y, \Omega_Y^1).$$

If $\gamma$ is étale, this implies that $\mathcal{W}_{\gamma^*D} \cong \gamma^*\mathcal{W}_D$ and $E_{\gamma^*D} \cong \gamma^*E_D$.

**Remark 4.4** (Geometric realisation). Over $X_{\text{reg}}$, where $\mathcal{O}_X(D)$ is locally free with associated line bundle $L \to X_{\text{reg}}$, the sequence (4.1.2) is just the Atiyah extension [Ati57, Thm. 1]. It coincides with the natural exact sequence

$$0 \to \mathcal{O}_{X_{\text{reg}}} \xrightarrow{e} \mathcal{T}_L(-\log X_{\text{reg}})|_{X_{\text{reg}}} \to \mathcal{T}_{X_{\text{reg}}} \to 0,$$

where we identified $X_{\text{reg}}$ with the zero section of $L$, and $e$ is given by the vector field coming from the natural $C^*$-action; see also [Tia92, p. 407]. Similarly, if $D$ is Cartier with associated line bundle $L$ over $X$, then (4.1.2) is

$$0 \to \mathcal{O}_X \xrightarrow{e} \mathcal{T}_L(-\log X)|_X \to \mathcal{T}_X \to 0.$$

To see this, use the fact that $L \to X$ is locally trivial to see that $L$ is normal, so that $\mathcal{T}_L(-\log X)$ is reflexive, and then again to obtain that $\mathcal{T}_L(-\log X)|_X$ stays reflexive.

**Definition 4.5** (The canonical extension). If $X$ is a normal projective variety such that $K_X$ is $\mathbb{Q}$-Cartier, we apply Construction 4.1 to the divisor $D = -K_X$, in order to obtain a locally splittable extension with reflexive middle term,

(4.5.1) \quad 0 \to \mathcal{O}_X \to E \to \mathcal{T}_X \to 0.

We refer to (4.5.1) as the canonical extension of $\mathcal{T}_X$ by $\mathcal{O}_X$. Abusing language somewhat, we will also refer to the sheaf $E_X$ as the canonical extension.

**Remark 4.6** (Functoriality in morphisms). In the setting of Definition 4.5, given any quasi-étale morphism $\gamma : Y \to X$ of normal varieties, then $\gamma_*E \cong \gamma^!E_X$ restricted to $X_{\text{reg}}$, use Remarks 4.2 and 4.3, as well as the fact that two reflexive sheaves agree if they agree on a big open set.

**Remark 4.7** (The $\mathbb{Q}$-Fano case). If $X$ is a $\mathbb{Q}$-Fano variety, then by definition $-K_X$ is $\mathbb{Q}$-ample and as a consequence, the canonical extension is highly non-trivial. In particular, if $C \subset X_{\text{reg}}$ is any smooth projective curve $C$ inside the smooth locus of $X$, then the restricted sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_X|_C \to \mathcal{T}_X|_C \to 0$$

is exact and not globally splitable.
5. Proofs of the main results

5.1. Proof of Proposition 1.6 ("Criterion for projective flatness"). We consider the endomorphism sheaf $\mathcal{E} := \mathcal{E}nd(\mathcal{F}, \mathcal{F})$. The sheaf $\mathcal{E}$ is then likewise reflexive, semistable with respect to $H$ by [GKP16a, Prop. 4.4], and its first $\mathbb{Q}$-Chern class vanishes. Equation (1.6.1) will then guarantee that

$$\hat{c}_1(\mathcal{E}) \cdot [H]^{n-1} = 0 \quad \text{and} \quad \hat{c}_2(\mathcal{E}) \cdot [H]^{n-2} = 0.$$ 

Indeed, vanishing of the first term has already been mentioned above. Regarding the second term we therefore observe that by $[\mathcal{GKPT19}, \text{Thm. 3.13.2}]$, it suffices to show that $\hat{c}_2(\mathcal{E}|_S) = 0$, where $S$ is a general, and hence klt, complete intersection surface for a suitable multiple of $H$. We notice that for such a surface $S$ we have $\mathcal{E}|_S \cong \mathcal{E}nd(\mathcal{F}|_S, \mathcal{F}|_S)$ and that, by $[\mathcal{GKPT19}, \text{Thm. 3.13.2}]$ again, equation (1.6.1) implies that $\hat{\Lambda}(\mathcal{F}|_S) = 0$, where $\hat{\Lambda}$ is the $\mathbb{Q}$-Bogomolov discriminant, see $[\mathcal{GKPT19}, \text{Def. 3.15}]$. The standard calculus for Chern classes of vector bundles together with $[\mathcal{GKPT19}, \text{Thm. 3.13.1}]$ then implies that $\hat{c}_2(\mathcal{E}|_S) = \hat{\Lambda}(\mathcal{F}|_S) = 0$, as desired. We are hence in the position to apply $[\mathcal{LT18}, \text{Thm. 1.4}]$ to see that $\mathcal{E}|_{X_{\kappa}}$ is locally free and carries a flat holomorphic connection; Proposition 3.7 then gives the claim.

5.2. Proof of Proposition 1.1 ("Miyaoka-Yau Inequality and Semi-stable Canonical Extension"). We have already remarked that Inequality (1.1.1) is equivalent to the $\mathbb{Q}$-Bogomolov-Gieseker Inequality for the reflexive sheaf $\mathcal{E}_X$,

$$(5.0.1) \quad \hat{\Lambda}(\mathcal{E}_X) \cdot [H]^{n-2} \geq 0$$

where $\hat{\Lambda}$ is the $\mathbb{Q}$-Bogomolov discriminant, as introduced in $[\mathcal{GKPT19}, \text{Def. 3.15}]$. To begin the proof in earnest, let $m \in \mathbb{N}$ be sufficiently large such that $m \cdot H$ is very ample, let

$$(H_1, \ldots, H_{n-2}) \in (m \cdot H)^{\times(n-2)}$$

be a general tuple of hypersurfaces, and let $S := H_1 \cap \cdots \cap H_{n-2}$ be the associated complete intersection surface in $X$. Recall that...

- the surface $S$ is a klt space, $[\mathcal{KM98}, \text{Lem. 5.17}]$.
- the restricted sheaf $\mathcal{E}_S := \mathcal{E}_X|_S$ is reflexive, $[\mathcal{Gro66}, \text{Thm. 12.2.1}]$.
- the sheaf $\mathcal{E}_S$ is semistable with respect to $H_S := H|_S$, $[\mathcal{Fle84}, \text{Thm. 1.2}]$, and
- Inequality (5.0.1) is equivalent to $\hat{\Lambda}(\mathcal{E}_S) \geq 0$, $[\mathcal{GKPT19}, \text{Thm. 3.13.2}]$.

Next, recall from $[\mathcal{GKPT19}, \text{Thm. 3.13.1}]$ that there exists a normal surface $\hat{S}$ and a finite cover $\gamma : \hat{S} \to S$ such that...

- the reflexive pull-back $\mathcal{E}_{\hat{S}} := \gamma^*\mathcal{E}_S$ is locally free, and
- the desired Inequality $\hat{\Lambda}(\mathcal{E}_{\hat{S}}) \geq 0$ is equivalent to $\Delta(\mathcal{E}_{\hat{S}}) \geq 0$, where $\Delta$ is the standard Bogomolov discriminant.

Also, recall from $[\mathcal{HL10}, \text{Lem. 3.2.2}]$ that $\mathcal{E}_{\hat{S}}$ is semistable with respect to the ample divisor $H_{\hat{S}} := \gamma^*H$. Finally, let $\pi : \tilde{S} \to \hat{S}$ be a resolution of singularities. Then, $\mathcal{E}_{\tilde{S}} := \pi^*\mathcal{E}_{\hat{S}}$ is semistable with respect to the nef divisor $H_{\tilde{S}} := \pi^*H_{\hat{S}}$, $[\mathcal{GKP16a}, \text{Prop. 2.7 and Rem. 2.8}]$, and $\Delta(\mathcal{E}_{\tilde{S}}) = \Delta(\mathcal{E}_{\hat{S}})$. But $\Delta(\mathcal{E}_{\tilde{S}}) \geq 0$ by $[\mathcal{GKP16a}, \text{Thm. 5.1}]$, which ends the proof of Proposition 1.1.

5.3. Proof of Theorem 1.3 ("Quasi-étale uniformisation if $-K_X$ nef"). To prepare for the proof, observe that if $\gamma : Y \to X$ is any quasi-étale cover, then the following holds.

(5.1.1) Recall from $[\mathcal{KM98}, \text{Prop. 5.20}]$ that $Y$ is again klt and notice that $-K_Y = \gamma^*(-K_X)$ is also $\mathbb{Q}$-nef.

\footnote{Especially, see the preprint version of \textit{loc. cit.}, where the construction of $\mathbb{Q}$-Chern classes is carried out in detail.}
(1.3.2) The calculus of \( \mathbb{Q} \)-Chern classes, [GKPT19, Lem. 3.16], shows that equality holds in the \( \mathbb{Q} \)-Miyaoka-Yau inequality for \( Y \) (with respect to the ample divisor \( \gamma' H \)) if and only if it holds for \( X \) (with respect to \( H \)).

(1.3.3) We have seen in Remark 4.6 that \( \delta_Y \equiv \gamma'^{-1} \delta_X \). Recalling that the reflexive sheaf \( \gamma'^{-1} \delta_X \) is semistable with respect to \( \gamma' H \) if and only if the sheaf \( \gamma' \delta_X / \text{tor} \) is semistable with respect to \( \gamma' H \), it follows from [HL10, Lem. 3.2.2] that \( \delta_Y \) is semistable with respect to \( \gamma' H \) if and only if \( \delta_X \) is semistable with respect to \( H \).

**Direction (1.3.2) \( \Rightarrow \) (1.3.3).** Assume that \( X \) is of the form \( X = Y/G \), where \( Y \) is the projective space or an Abelian variety, and where \( G < \text{Aut}(Y) \) is a finite group acting fixed-point free in codimension one. Denote the quotient map by \( \gamma : Y \rightarrow X \), observe that \( \gamma \) is quasi-étale and let \( H \in \text{Div}(X) \) be any ample Cartier divisor. No matter whether \( Y = \mathbb{P}^n \) or \( Y \) is Abelian, it is well-known in either case that the quotient space \( X \) has klt singularities, that equality holds in the Miyaoka-Yau Inequality for \( \gamma' H \) and that the canonical extension \( \delta_Y \) is semistable with respect to \( \gamma' H \). Since \( \gamma \) is quasi-étale, Items (5.1.1)–(5.1.3) will then imply the same for \( X \) and for the divisor \( H \).

**Direction (1.3.1) \( \Rightarrow \) (1.3.2).** Assume that \( X \) satisfies the assumptions in (1.3.1) and choose a Galois, maximally quasi-étale cover \( \gamma : Y \rightarrow X \) as provided by [GK16b, Thm. 1.5]. Items (5.1.1)–(5.1.3) guarantee that \( Y \) reproduces the Assumptions (1.3.1). Replacing \( X \) by \( Y \), we may (and will) therefore assume from now on that \( X \) itself is maximally quasi-étale. With these additional assumptions, we aim to show that \( X \) is isomorphic to \( \mathbb{P}^n \) or to an Abelian variety, from which the main claim follows.

As \( \delta_X \) is \( H \)-semistable and equality holds in (1.1.1), Proposition 1.6 ("Criterion for projective flatness") implies that \( \delta_X|_{\mathbb{P}^n} \) is projectively flat. As a consequence, we may apply Corollary 3.6 and find that \( \delta_{\text{nd}}(\delta_X) \) is locally free and flat. By construction of \( \delta_X \) as a locally splittable extension, we can then locally write

\[
\delta_X = \mathcal{O}_X \oplus \mathcal{T}_X \quad \text{and} \quad \delta_{\text{nd}}(\delta_X) = \mathcal{O}_X \oplus \mathcal{T}_X \oplus \Omega_X^{[1]} \oplus \delta_{\text{nd}}(\mathcal{T}_X).
\]

In particular, we find that the locally free sheaf \( \delta_{\text{nd}}(\delta_X) \) locally contains \( \mathcal{T}_X \) as a direct summand, which is therefore also locally free. The positive solution of the Lipman-Zariski conjecture for klt spaces, [GKKP11, Thm. 6.1] or [Dru14, Thm. 3.8], then already implies that \( X \) is smooth.

There is more that we can say. Assumptions (1.3.1) also imply that the \( \mathbb{Q} \)-twisted bundle\(^3\)

\[
\delta_X \left( \frac{1}{\pi^2} \cdot [\text{det} \delta_X^\ast] \right) = \delta_X \left( \frac{1}{\pi^2} \cdot [K_X] \right)
\]

is nef, [Nak04, Chapt. IV, Thm. 4.1 and Chap. II, Def. 6.4], and then so is the quotient \( \mathcal{T}_X \left( \frac{1}{\pi^2} \cdot [K_X] \right) \), see [Laz04, Thm. 6.2.12]. Given that \( K_X \) is anti-nef, we find that \( \mathcal{T}_X \) is already nef. This has consequences: replacing \( X \) by a suitable étale cover, we may assume without loss of generality that the Albanese map \( \text{alb}(X) : X \rightarrow \text{Alb}(X) \) is a submersion and that its fibres are connected Fano manifolds whose tangent bundles are likewise nef, [DPS94, Prop. 3.9 and Thm. 3.14].

If \( \text{alb}(X) \) has zero-dimensional fibres, then \( \text{alb}(X) \) is étale, the variety \( X \) is therefore Abelian, and the proof ends here. We will therefore assume of the remainder of the proof that \( \text{alb}(X) \) has positive-dimensional fibres, and let \( F \subseteq X \) be such a fibre. Choose any curve inside \( F \) and consider its normalisation, say \( \eta : C \rightarrow F \). Knowing that \( F \) is Fano, we find that \( \deg \eta'(-K_X) = \deg \eta'(-K_F) > 0 \), and nefness of \( \mathcal{T}_X \left( \frac{1}{\pi^2} \cdot [K_X] \right) \) implies that \( \eta^\ast \mathcal{T}_X \) is already ample, [Laz04, Prop. 6.2.11]. As one consequence, we see that \( F = X \), for otherwise there would be a non-trivial surjection

\[
\eta^\ast \mathcal{T}_X \longrightarrow \eta^\ast \mathcal{T}_{\text{alb}/X} \cong \mathcal{O}_C^{\oplus \text{codim}_F X},
\]

\(^3\)see [Laz04, Sect. 6.2] for the notation used here
which cannot exist if \( \eta^* \mathcal{F} \) is ample. But then \( X \) is a Fano manifold and hence simply connected, see [Tak00, Thm. 1.1] and references there. Together with Proposition 3.11, we conclude that there exists a (Cartier) divisor \( D \in \text{Div}(X) \) such that \( \mathcal{E}_X \cong \mathcal{O}_X(D)^{\oplus n+1} \).

Taken together with the exact sequence (4.5.1), this implies that

\[
\mathcal{O}_X ((n+1) \cdot D) \cong \det \mathcal{E}_X \cong \mathcal{O}_X (-K_X).
\]

We find that \( c_1(X) = (n+1) \cdot c_1(D) \) and that \( D \) is ample. The classic Kobayashi-Ochiai Theorem, [KO73, Cor. on p. 32], then shows that \( X \cong \mathbb{P}^n \). This concludes the proof of Theorem 1.3. \( \square \)

6. Stability criteria for the canonical extension

In order to apply Proposition 1.1 we need to find criteria guaranteeing that the canonical extension \( \mathcal{E}_X \) of a given variety \( X \) is semistable.

6.1. Destabilising subsheaves in \( \mathcal{E}_X \) when \( \mathcal{F}_X \) is semistable. It is known that the tangent sheaves of many Fanos are semistable with respect to the anti-canonical polarisation\(^4\), and it seems natural to ask about semistability of \( \mathcal{E}_X \) for these varieties. While even in these cases one cannot hope that \( \mathcal{E}_X \) will always be semistable\(^5\), we describe potential destabilising subsheaves in \( \mathcal{E}_X \).

**Notation 6.1.** Throughout the present Section 6 we will mostly be concerned with Fano manifolds. To keep notation short, semistability is (unless otherwise stated) always understood with respect to the canonical class, and we denote the slope with respect to the anti-canonical class by \( \mu(\bullet) \) rather than the more correct \( \mu_{-K_X}(\bullet) \).

**Proposition 6.2.** Let \( X \) be a \( \mathbb{Q} \)-Fano variety of dimension \( n \). Suppose that \( \mathcal{F}_X \) is semistable with respect to \( -K_X \) and that \( \mathcal{I} \subset \mathcal{E}_X \) is a proper, saturated subsheaf with \( \mu(\mathcal{I}) \geq \mu(\mathcal{E}_X) \). Then, the composed map \( \mathcal{I} \hookrightarrow \mathcal{E}_X \twoheadrightarrow \mathcal{F}_X \) is injective, and there are inequalities

\[
(6.2.1) \quad \mu(\mathcal{E}_X) \leq \mu(\mathcal{I}) \leq \mu(\mathcal{F}_X).
\]

**Proof.** If the composed map \( \psi : \mathcal{I} \rightarrow \mathcal{F}_X \) is not injective, then \( \ker \psi \) is a subsheaf of \( \mathcal{E}_X \) of rank one and the inequality \( \mu(\mathcal{I}) \geq \mu(\mathcal{E}_X) \) yields \( \mu(\text{Image} \psi) > \mu(\mathcal{F}_X) \), thus contradicting the semistability of \( \mathcal{F}_X \). It follows that the composed map \( \psi \) is injective and Inequalities (6.2.1) are simply the assumption and the semistability of \( \mathcal{F}_X \). \( \square \)

We now restrict our attention to Fano manifolds with \( \rho(X) = 1 \), where the first Chern classes can and will be considered as numbers; for instance, \( c_1(-K_X) = r \) is the index of \( X \).

**Proposition 6.3.** In the setting of Proposition 6.2 assume additionally that \( X \) is smooth with \( \rho(X) = 1 \), and let \( r \) be the index of \( X \). Let \( m \) be the rank of \( \mathcal{I} \). Then, \( m < n \) and

\[
(6.3.1) \quad \frac{r \cdot m}{n+1} \leq c_1(\mathcal{I}) \leq \frac{r \cdot m}{n}.
\]

**Proof.** The Inequalities (6.3.1) are simply reformulations of (6.2.1). To show the bound for \( m \), we argue by contradiction and assume that \( m = n \). By Proposition 6.2, \( \mathcal{I} \) is then a reflexive subsheaf of \( \mathcal{F}_X \). If \( \mathcal{I} = \mathcal{F}_X \), then the sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_X \rightarrow \mathcal{F}_X \rightarrow 0
\]

\(^4\)It was in fact conjectured until recently that the tangent bundle of every Fano manifold \( X \) with Picard number \( \rho(X) = 1 \) is stable. While this conjecture was shown in an impressive number of cases, it was recently disproved by Kanemitsu. We refer the reader to Kanemitsu’s paper [Kan20] for details.

\(^5\)See [Tia92, Thm. 3.2] for an elementary example where \( \mathcal{F}_X \) is semistable while \( \mathcal{E}_X \) is not.
is globally split, contradicting Remark 4.7. It follows that $\mathcal{F} \subseteq \mathcal{F}_X$ and that $\text{det} \mathcal{F} \subseteq \mathcal{O}_X(-K_X)$. In other words, $c_1(\mathcal{F}) < c_1(-K_X) = r$. The left inequality in (6.3.1) will then read
\[
\frac{r \cdot n}{n+1} \leq c_1(\mathcal{F}) \leq r - 1,
\]
which implies that $r \geq n + 1$. It follows, for instance by [Keb02, Thm. 1.1], that $X \cong \mathbb{P}^n$, where $\mathcal{O}_X$ is known to be stable. This contradicts the defining property of $\mathcal{F}$.

**Corollary 6.4.** Let $X$ be a Fano manifold with $\rho(X) = 1$. If $\mathcal{F} \subset \mathcal{O}_X$ is destabilising, then $\text{rank}(\mathcal{F}) \geq 2$.

**Proof.** If $\mathcal{F}$ had rank one, then $\mathcal{F}$ would be an ample line bundle which at the same time is a subsheaf of $\mathcal{F}_X$ by Proposition 6.2. But then $X \cong \mathbb{P}^n$ by Wahl’s theorem, [Wah83, Thm. 1], so that again a contradiction has been reached.

**Corollary 6.5.** Let $X$ be a Fano manifold of index one with $\rho(X) = 1$. Then $\mathcal{O}_X$ is stable.

**Proof.** Suppose that $\mathcal{F} \subset \mathcal{O}_X$ is destabilising. Recall from [PW95, Prop. 2.2] that $\mathcal{F}_X$ is stable. Proposition 6.3 therefore applies to bound the integer $c_1(\mathcal{F})$ as follows
\[
0 < \frac{1 \cdot m}{n+1} \leq c_1(\mathcal{F}) \leq \frac{1 \cdot m}{n} < 1.
\]
This is absurd.

In spite of the elementary results above, the precise relation between semistability of $\mathcal{F}_X$ and $\mathcal{O}_X$ is not clear to us. We would like to pose the following question.

**Problem 6.6.** Let $X$ be a Fano manifold where $\mathcal{F}_X$ is (semi)stable with respect to $-K_X$. Suppose that $\rho(X) = 1$. Does this guarantee that the canonical extension $\mathcal{O}_X$ is (semi)stable as well?

6.2. $\lambda$-stability. In his work [Tia92] on Fano manifolds carrying a Kähler-Einstein metric, Tian introduced the following notion.

**Definition 6.7 ($\lambda$-stability, [Tia92, Def. 1.3]).** Let $X$ be a normal projective variety and $H$ an ample divisor. Let $\lambda > 0$ be a real number. We say that a torsion free coherent sheaf $\mathcal{E}$ is $\lambda$-stable with respect to $H$ if the slope inequality $\mu_H(\mathcal{E}) < \lambda \cdot \mu_H(\mathcal{E})$ holds for all saturated subsheaves $\mathcal{F} \subset \mathcal{E}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$. Analogously for $\lambda$-semistable.

**Example 6.8.** Tian [Tia92, Thm. 2.1] has shown that the tangent bundles of Fano Kähler-Einstein manifolds of dimension $n$ and with $b_2 = 1$ are $\frac{\lambda}{n+1}$-semistable.

An elementary calculation following [Tia92, Prop. 1.4] relates (semi)stability of the canonical extension sheaf to $\lambda$-(semi)stability of $\mathcal{F}_X$.

**Lemma 6.9 ($\lambda$-stability criterion).** Let $X$ be a $\mathbb{Q}$-Fano variety of dimension $n$. Suppose that the tangent sheaf $\mathcal{F}_X$ is $\frac{\lambda}{n+1}$-(semi)stable. Then, the canonical extension sheaf $\mathcal{O}_X$ is (semi)stable.

**Remark 6.10.** We do not expect the sufficient criterion of Observation 6.9 to be necessary.

6.3. Weighted complete intersections. We will now spell out a few situations where $\frac{n}{n+1}$-stability of $\mathcal{F}_X$ can be guaranteed. Given a Fano manifold $X$ with $b_2(X) = 1$, or equivalently Picard number $\rho(X) = 1$, we denote by $\mathcal{O}_X(1)$ the ample generator of $\text{Pic}(X) \cong \mathbb{Z}$. Furthermore, as usual for any coherent sheaf $\mathcal{F}$ on $X$ and $r \in \mathbb{Z}$ we write $\mathcal{F}(r) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes r}$, and as above we write $\omega_X \cong \mathcal{O}_X(-r)$ with $r$ the index of $X$.

Many of the Fano manifolds of interest to us are weighted complete intersections in weighted projective space, in the following (restrictive) sense, e.g. see [Fuj80].
Definition 6.11 (Weighted complete intersection). A polarised variety \((X, L)\) of dimension \(n\) — i.e., a pair consisting of a variety \(X\) of dimension \(n\) and an ample line bundle \(L\) on \(X\) — is called a weighted complete intersection of type \((a_1, \ldots, a_r)\) in \(\mathbb{P}(d_0, d_1, \ldots, d_{n+r})\) if the graded algebra

\[
R(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k)
\]

has a system of generators consisting of homogeneous elements \(\xi_0, \xi_1, \ldots, \xi_{n+r}\) with \(\deg \xi_j = d_j\) for all \(j = 0, 1, \ldots, n + s\) such that the relation ideal among the \(\{\xi_j\}\) is generated by \(s\) homogeneous polynomials \(f_1, \ldots, f_s\) with \(\deg f_i = a_i\) for all \(i = 1, \ldots, s\).

If \(s = 1\), we say that \((X, L)\) is a weighted hypersurface of degree \(a_1\) in \(\mathbb{P}(d_0, d_1, \ldots, d_{n+r})\).

Remark 6.12. It is noted in [Fuj80, (3.4)] that the above definition is equivalent to the one adopted by Mori in [Mor75, Defn. 3.1]. In particular, if \(X\) is smooth, and \(X \hookrightarrow \mathbb{P}(d_0, d_1, \ldots, d_{n+r})\) is the natural embedding, then \(\mathbb{P}(d_0, d_1, \ldots, d_{n+r})\) is smooth along \(X\), see [Mor75, Prop. 1.1].

Proposition 6.13. Let \(X\) be a Fano manifold of dimension \(n \geq 3\) with \(p(X) = 1\) and index \(r \leq n\). Write \(\delta := n - r\). Assume that \((X, \mathcal{O}_X(1))\) is a weighted complete intersection with

\[
h^0(X, \Omega^q_X(p)) = 0, \quad \text{for all} \quad 1 \leq p < n - \frac{\delta}{\delta + 1}(n + 1).
\]

Then, \(\mathcal{F}_X\) is \(\mathbb{Q}\)-semistable.

Proof. We argue as in [PW95, Cor. 0.3]. Let \(\mathcal{F} \subset \mathcal{F}_X\) be a saturated and hence reflexive subsheaf of rank \(1 \leq q < n\) and write \(\mathcal{F} \cong \mathcal{O}_X(k)\). Then, the inequality \(\mu(\mathcal{F}) < \mu(\mathcal{F}_X)\) to be proven reads

\[
\frac{k}{q} = \mu(\mathcal{F}) \leq \frac{n}{n + 1} \cdot \mu(\mathcal{F}_X) = \frac{n}{n + 1} \cdot \frac{r}{n} = \frac{n - \delta}{n + 1}.
\]

We estimate the slope \(\mu(\mathcal{F})\) by relating it to the existence of twisted forms. Taking determinants, we obtain an inclusion \(\det \mathcal{F} \subset \wedge^q \mathcal{F}_X\), and hence a non-vanishing result

\[
0 \neq h^0(X, \wedge^q \mathcal{F}_X \otimes \mathcal{O}_X(-k)) = h^0(X, \Omega^{\delta - q}_X(r - k)).
\]

Using the argument in the proof of Corollary 6.4 and the \(\mathbb{Q}\)-semistability of \(\mathcal{F}_X\), we see that we may assume \(q \geq 2\). Since \(X \cong \text{Proj}(R(X, \mathcal{O}_X(1)))\) is smooth, its affine cone \(\text{Spec}(R(X, \mathcal{O}_X(1)))\) has an isolated singularity at the origin. This observation together with Remark 6.12 allows us to apply a theorem of Flenner on the non-existence of twisted forms, [Fle81, Satz 8.11], and infer that \(r - k \geq n - q\). Equivalently, we found a bound

\[
\mu(\mathcal{F}) \geq \frac{k}{q} \leq \frac{q - \delta}{q}.
\]

To make use of (6.13.4), consider the inequality

\[
(\delta + 1) \cdot q \leq \delta \cdot (n + 1).
\]

There are two cases. If (6.13.5) holds, then

\[
\mu(\mathcal{F}) \leq \frac{q - \delta}{q} \leq \frac{n - \delta}{n + 1},
\]

and Inequality (6.13.2) follows. So, suppose for the remainder of this proof that (6.13.5) does not hold. But then Assumption (6.13.1) implies that \(h^0(X, \Omega^{n-q}_X(n - q)) = \{0\}\). In view of the non-vanishing result (6.13.3), this implies that \(r - k \geq n - q + 1\). Writing this inequality in terms of slopes as in (6.13.4), and then using that \(q < n + 1\) by definition, we obtain

\[
\frac{k}{q} \leq \frac{q - (\delta + 1)}{q} \leq \frac{n - \delta}{n + 1}.
\]

\[\text{See also [PW95, Thm. 0.2(d)]}\]
establishing (6.13.2) and completing the proof. □

Remark 6.14. Assumption (6.13.1) is empty in case $X$ has index one; hence the tangent bundle of Fano weighted complete intersections of index one in weighted projective spaces is always $\frac{n}{n+1}$-semistable. At the other end of possible indices, if $r = n+1$, then $X \cong \mathbb{P}^n$, whose tangent bundle is $\frac{n}{n+1}$-semistable, but not $\frac{n}{n+1}$-stable, as any inclusion $O_{\mathbb{P}^n}(1) \rightarrow \mathcal{R}_X$ shows; if $r = n$, then $X$ is a (hyper)quadric, $[KO73]$, and $T_X$ is $\frac{n}{n+1}$-stable.

6.4. Smooth Fano n-folds with Picard number one. In concrete situations, Assumption (6.13.1) is often easily verified, as we will see now.

Proposition 6.15. Let $X$ be a Fano manifold of dimension $n \geq 3$ with $\rho(X) = 1$. If $X$ has index $r = n-1$, then $\mathcal{R}_X$ is $\frac{n}{n+1}$-semistable with respect to $-K_X$.

Proof. By assumption, $X$ has index $r = n-1$, so $X$ is a del Pezzo manifold, which are classified, see [IP99, Thm. 3.3.1 and Rem. 3.3.2] or [Fuj90, (8.11)]. In particular, all such manifolds with $\rho = 1$ are complete intersections in a weighted projective space or linear sections of the Grassmannian $G(2,5)$ embedded into $\mathbb{P}^9$ by the Plücker embedding.

We first treat the case of complete intersections, for which we will verify Condition (6.13.1). The classification exhibits four cases:

- $X$ is a hypersurface of degree 6 in $\mathbb{P}(1,1,\ldots,2,3)$,
- $X$ is a hypersurface of degree 4 in $\mathbb{P}(1,1,\ldots,1,2)$,
- $X$ is a cubic in $\mathbb{P}^{n+1}$,
- $X$ is the intersection of two quadrics in $\mathbb{P}^{n+2}$.

We treat the first three cases simultaneously and leave the fourth to the reader. Let $\mathbb{P}$ denote the relevant (weighted) projective space and let $d$ be the degree of $X \subset \mathbb{P}$. Recall from Remark 6.12 that $\mathbb{P}$ is smooth along $X$, so that we have an exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow \Omega_{\mathbb{P}}^k \rightarrow \Omega_X^k \rightarrow 0.$$ 

Tensoring with $\mathcal{R}_X(k)$ and taking cohomology, things come down to showing that

$$(6.15.1) \quad H^0(\mathbb{P}, \Omega_{\mathbb{P}}^k(k)) = \{0\} \quad \text{and} \quad H^1(\mathbb{P}, \Omega_{\mathbb{P}}^k(k-d)) = \{0\} \quad \text{for all } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$ 

The right-hand side of (6.15.1) is settled by [Fle81, Satz 8.11(1.b)], whereas the left-hand side follows from

$$H^0(\mathbb{P}, \Omega_{\mathbb{P}}^k(k)) = \{0\} \quad \text{and} \quad H^1(\mathbb{P}, \Omega_{\mathbb{P}}^k(k-d)) = \{0\} \quad \text{for all } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$ 

Both are of course classical, see, e.g., [Dol82, Thm. 2.3.2 and Cor. 2.3.4; notation on p. 47].

Thus, it remains to treat linear sections of the Grassmannian $G := G(2,5)$ embedded by Plücker in $\mathbb{P}^9$ (in dimensions 3 up to 6). If dim $X$ = 6, then $X = G(2,5)$ is Kähler-Einstein, so we are done by Example 6.8. Therefore, we are reduced to $3 \leq n = \dim X \leq 5$. Suppose $\mathcal{R}_X$ is not $\frac{n}{n+1}$-semistable and consider a $\frac{n}{n+1}$-destabilising subsheaf $\mathcal{F} \subset \mathcal{R}_X$. Since $\mathcal{R}_X$ is stable, [PW95, Thm. 2.3], and has slope $\frac{1}{n}$, elementary slope considerations give dim $X \geq 4$, rank($\mathcal{F}$) = $n-1$, and $c_1(\mathcal{F}) = n-2$. Similar to (6.13.3), taking determinants hence yields

$$(6.15.2) \quad H^0(X, \Omega_X^{1}(1)) \neq \{0\}.$$ 

We will see that this is impossible. First, suppose that $n = 5$ and look at the structure sheaf sequence for the hyperplane section $X \subset G \subset \mathbb{P}^9$, twisted with $\Omega_X^{1}(1)$. Since

$$h^0(\mathbb{G}, \Omega_{\mathbb{G}}^{1}(1)) = h^1(\mathbb{G}, \Omega_{\mathbb{G}}^{1}(1)) = h^2(\mathbb{G}, \Omega_{\mathbb{G}}^{1}) = 0$$

by [PW95, Lem. 0.1], and since $h^1(\mathbb{G}, \Omega_{\mathbb{G}}^{1}) = b_2(\mathbb{G}) = 1$, we see that $h^0(X, \Omega_X^{1}(1)|_X) = 1$ and $h^1(X, \Omega_X^{1}(1)|_X) = 0$. Therefore, taking cohomology of the twisted conormal sequence

$$0 \rightarrow \mathcal{R}_X \rightarrow \Omega_X^{1}|_X \rightarrow \Omega_X^{1} \rightarrow 0$$

yields
we obtain

\[ H^0(X, \Omega_X^1(1)) = H^1(X, \Omega_X^1(1)) = \{0\}, \]

contradicting (6.15.2) and proving the assertion for \( n = 5 \).

Second, look at a further hyperplane section \( Y \subset X \), i.e., the case \( n = 4 \), and repeat the above argument: use the non-vanishing (6.15.2), but this time for \( Y \), together with

\[ H^0(X, \Omega_X^1(1)) = H^1(Y, \Omega_Y^1(1)) = \{0\}, \]

see (6.15.3), and \( h^1(X, \Omega_X^1) = b_2(X) = 1 \) to get

\[ h^0(Y, \Omega_Y^1(1)|_Y) = 1, \]

which then contradicts the vanishing obtained from the conormal sequence for \( Y \) in \( X \).

\[ \square \]

Corollary 6.16. Let \( X \) be a smooth Fano threefold. If \( \rho(X) = 1 \), then \( K_X \) is \( \frac{2}{n+1} \)-semistable, and the canonical extension \( \mathcal{O}_X \) is semistable.

Proof. Fano threefolds of Picard number one and index \( r = 1, 3, 4 \) are taken care of by Remark 6.14, whereas those of index \( r = 2 \) are covered by Proposition 6.15. Lemma 6.9 implies the statement about the canonical extension.

\[ \square \]

Remark 6.17. Using the classification of Fano manifolds of coindex 3 and the same methods, Theorem 6.15 certainly holds in case \( r = \dim X = 2 \) as well.

6.5. Relation to K-stability. On the one hand, many of the examples considered in the preceding section can be shown to be K-polystable (at least in each single case) technically much more advanced methods; see the overview provided in [Xu20, Sect. 6.12]. Then, the positive solution of the YTD-conjecture shows that these admit a Kähler-Einstein metric, which in turn implies that \( K_X \) is \( \frac{2}{n+1} \)-semistable by Tian’s original result, see Example 6.8. Note that according to loc. cit., it is an open question whether all smooth Fano threefolds of Picard number one are K-semistable.

On the other hand, if \( X \) is a linear section of \( \mathbb{G}(2, 5) \) of dimension 4 or 5, then the pair \( (X, -K_X) \) is not K-semistable, see [Fuj17]. Together with Proposition 6.15 and Lemma 6.9, this shows that \( K_X \) might be semistable (making Proposition 1.1 and Theorem 1.3 applicable) even for K-unstable Fano manifolds \( X \), which in particular do not carry Kähler-Einstein metrics.

Finally, we note that [Li17, Thm. 1.5] together with the degree estimate [Tia92, (4.11)] implies that for any Fano manifold \( X \) such that \((X, -K_X)\) is K-semistable the canonical extension \( \mathcal{O}_X \) is semistable with respect to \( -K_X \), as seems to be well-known to experts.\(^7\)

7 Examples

7.1. Necessity of stability assumptions. The following elementary examples showing that certain stability conditions are necessary for the Q-Miyaoka-Yau inequality and the characterisation of the extreme case to hold are well-known to experts. We recall them for the convenience of the reader.

Example 7.1 (Projectivised bundle). Consider the 4-dimensional Fano manifold \( X := \mathbb{P}_2(\mathcal{O}(3)) \). Then \( -K_X = 800 \), whereas \( -K_X^2 = 296 \). It follows that \( X \) violates the Bogomolov-Gieseker Inequality (1.0.1) as well as the Miyaoka-Yau Inequality (1.0.2). In particular, it follows from [HL10, Thm. 3.4.1] that \( K_X \) is not semistable with respect to \( -K_X \).

Example 7.2 (Weighted projective space). The three-dimensional weighted projective space \( X := \mathbb{P}(1, 1, 1, 3) \) has one isolated canonical Gorenstein singularity of type \( \frac{4}{3}(1, 1, 1) \), which admits a crepant resolution. Moreover, \( X \) is Fano and its anticanonical volume

\[^7\text{We thank Henri Guenancia for confirming our corresponding educated guess.}\]
is \([-K_X]^3 = 72\); see [Sta17] and [Pro05, Ex. 1.4]. Clearly, \(\chi(\mathcal{O}_X) = 1\), so that [Rei87, Cor. 10.3] implies that
\[
[-K_X] \cdot \frac{c_2}{2}\left(\Omega_X^{[1]}\right) = [-K_X] \cdot c_2(X) = 24.
\]
Hence, \(X\) realises equality in the \(\mathbb{Q}\)-Bogomolov-Gieseker Inequality (1.0.1) and violates the \(\mathbb{Q}\)-Miyaoka-Yau Inequality (1.0.2).

Remark 7.3 (Explanation of Example 7.2). Prokhorov [Pro05, Thm 1.5] proved that any Fano threefold \(X\) with only canonical Gorenstein singularities satisfies the bound \((-K_X)^3 \leq 72\), and in case of equality either \(X \cong \mathbb{P}(1,1,1,3)\) or \(X \cong \mathbb{P}(1,1,1,3)\). With this in mind, consider \(Y := \mathbb{P}_{23}(\mathcal{O}_{23} \oplus \mathcal{O}_{23}(3))\) and \(\sigma : Y \to Z\) be the blow down of the exceptional section; then \(Z\) is a three-dimensional Fano Gorenstein variety of index 2 with a canonical singularity and \([-K_Y]^3 = 72\); consequently, \(Z \cong \mathbb{P}(1,1,1,3)\). Using this realisation, it is easy to check that \(\mathcal{F}_{T(1,1,1,3)}\) is not semistable. In fact, \(\mathcal{F}_{T(1,1,1,3)}\) is a torsion free destabilising subsheaf.

If \(X\) is a smooth Fano threefold, the above phenomena do not occur, even without assuming \(\mathcal{F}_X\) to be semistable. In fact, classification shows that
\[
[-K_X]^3 \leq 4^3 = 64,
\]
for any smooth Fano threefold, [MM82, MM83, IP99, MM03]. Since
\[
-K_X \cdot c_2(X) = 24 \cdot \chi(X, \mathcal{O}_X) = 24,
\]
Inequality (1.0.2) holds. Moreover, equality in (1.0.2) occurs if and only if \(X = \mathbb{P}^3\). This remains true for threefolds with at worst Gorenstein terminal singularities, since these admit smoothings by [Nam97]. We refer the reader to [Pro05] for a further discussion.

7.2. Equality in the Miyaoka-Yau Inequality (1.0.2). Examples of Fano varieties realising the bound (1.0.2) are produced by the following result.

**Proposition 7.4.** Let \(V\) be an \(n\)-dimensional complex vector space, \(n \geq 2\). Let \(G < \text{GL}(V)\) be a finite subgroup having the following two properties.

1. \(\{e\} \) is a homothety.
2. No element of \(G\) is a quasi-reflection.\(^9\)

Let \(W = \text{direct sum of } V\) with a 1-dimensional trivial \(G\)-representation, \(W := \mathbb{C} \oplus V\). Then, the quotient map \(\gamma : \mathbb{P}(W) \to \mathbb{P}(W)/G\) for the induced \(G\)-action on \(\mathbb{P}(W)\) is quasi-\(\acute{e}tale\) with Galois group \(G\).

**Proof.** Introduce linear coordinates \(z_1, \ldots, z_n\) on \(V\), \(z_0\) on \(\mathbb{C}\), and the corresponding homogeneous coordinates \([z_0 : z_1 : \ldots : z_n]\) on \(\mathbb{P}(W)\). By construction, the point \([1 : 0 : \ldots : 0]\) in \(\mathbb{P}(W)\) is fixed by \(G\). Moreover, the \(G\)-invariant (and hence \(\gamma\)-saturated) neighbourhood \(U := \{z_0 \neq 0\}\) is \(G\)-equivariantly isomorphic to \(V\) in such a way that \([1 : 0 : \ldots : 0]\) is mapped to \(0 \in V\). In particular, \(G\) acts effectively on \(\mathbb{P}(W)\). As \(G\) does not have any quasi-reflections, the restriction of \(\gamma\) to \(U\) is quasi-\(\acute{e}tale\).

It therefore remains to exclude ramification of \(\gamma\) along \(Z := \mathbb{P}(W) \setminus U\), which is \(G\)-equivariantly isomorphic to \(\mathbb{P}(V)\), where \(G\) acts on the latter space via its inclusion into \(\text{GL}(V)\). Assume that there is an element \(g \in G\) that fixes \(Z\) pointwise. Then, \(g\) acts on \(V\) via a homothety, so that \(g = \{e\}\) by assumption. Therefore, \(\gamma\) is unramified at the generic point of \(Z\) and therefore quasi-\(\acute{e}tale\), as claimed. \(\square\)

\(^8\)Note that the "basket" is empty due to the existence of a crepant resolution.

\(^9\)An element \(g \in \text{GL}(V)\) is called quasi-reflection if \(1\) is an eigenvalue of \(g\) with geometric multiplicity equal to \(\dim V - 1\).
Example 7.5. Let \( \xi \) be a non-trivial third root of unity, and let \( G = \mathbb{Z}/3\mathbb{Z} \) act on \( V = \mathbb{C}^2 \) by
\[
[m] \cdot (z_1, z_2) = (\xi^m \cdot z_1, \xi^{2m} \cdot z_2).
\]
The action is faithful, so that we may consider \( G \) as a subgroup of \( \text{GL}(2, \mathbb{C}) \). Moreover, the assumptions of Proposition 7.4 are fulfilled, so that \( \mathbb{P}(\mathbb{C} \oplus V)/G \) is a \( \mathbb{Q} \)-Fano surface realising equality in the \( \mathbb{Q} \)-Miyaoka-Yau Inequality (1.0.2).

Example 7.6. Let \( G \) be a non-Abelian finite simple group, and let \( \rho : G \to \text{GL}(V) \) be any non-trivial finite-dimensional representation; \( \dim V = n \) is necessarily greater than two. As ker\((\rho)\) is a normal subgroup of \( G \), the representation is faithful, so that \( G < \text{GL}(V) \). Similarly, as \( G \) is non-Abelian, \( Z(G) \) has to be trivial; in particular, \( G \) does not contain homotheties. Moreover, the subgroup of \( G \) generated by quasi-reflections is normal, [Pri67, proof of Prop. 6], so likewise trivial, since the Shephard-Todd classification of complex reflection groups, [LT09, Chap. 8], does not contain non-Abelian simple groups. Consequently, \( G \) fulfills the assumptions of Proposition 7.4 above, and \( \mathbb{P}(\mathbb{C} \oplus V)/G \) is a \( \mathbb{Q} \)-Fano \( n \)-fold realising equality in the \( \mathbb{Q} \)-Miyaoka-Yau Inequality (1.0.2).

As subexamples, we may take \( G = A_5 \) and \( \rho \) the three-dimensional (real) representation realising \( A_5 \) as the rotational symmetry group of the icosahedron, or \( G \) the monster group with its smallest non-trivial representation, which has dimension 196,883.

We find that every finite group appears as the Galois group for one of our examples.

Corollary 7.7. Let \( G \) be any finite group. Then, there exists a finite-dimensional, faithful complex representation \( \rho : G \to \text{GL}(W) \) such that the quotient map \( \gamma : \mathbb{P}(W) \to \mathbb{P}(W)/G \) for the induced \( G \)-action on \( \mathbb{P}(W) \) is quasi-étaile with Galois group \( G \).

Proof. Let \( V_0 \) be a faithful complex representation of \( G \). Then, the direct sum \( V := V_0 \oplus \mathbb{C} \) of \( V_0 \) with the trivial representation is also faithful, and no element of \( G \setminus \{e\} \) acts on \( V \) by a homothety.

We now follow the argument given in [Bra20, proof of Cor. 5]. If there is no element of \( G \) acting via a quasi-reflection on \( V \), then by Proposition 7.4 above we may take \( W = V \oplus \mathbb{C} \), the direct sum of \( V \) with a further trivial representation, on which \( G \) also acts without non-trivial homotheties. Otherwise, we consider
\[
V' := V \oplus V.
\]
Note that no element of \( G \setminus \{e\} \) acts on \( V' \) by a homothety. Suppose that there is a \( g \in G \) acting as a quasi-reflection on \( V' \). Consider \( V' \) as a representation of the subgroup \( \langle g \rangle \subset G \) generated by \( g \); by assumption, this contains a pointwise fixed hyperplane \( H \). As the representation \( V' \) is obviously reducible, a non-trivial decomposition into subrepresentations being given by (7.7.1), it follows from the first paragraph of the proof of [1J12, Theorem] that one of the copies of \( V \) has to be contained in \( H \). In other words, \( \langle g \rangle \), and in particular \( g \), acts trivially on \( V \), contradicting faithfulness of \( V \). We may then apply Proposition 7.4 again in order to conclude.

Corollary 7.7 is particularly interesting because of the following topological observation, which shows that every finite group appears as the fundamental group of the smooth locus in a \( \mathbb{Q} \)-Fano variety that realises equality in the \( \mathbb{Q} \)-Miyaoka-Yau Inequality (1.0.2).

Lemma 7.8 (Topological properties of \( \mathbb{P}^n/G \)). Let \( X \cong \mathbb{P}^n/G \) for some finite subgroup \( G < \text{Aut}_\mathbb{C}(\mathbb{P}^n) \) such that the quotient map \( \gamma : \mathbb{P}^n \to X \) is quasi-étaile. Then, \( X \) is simply connected and \( \pi_1(X_{\text{reg}}) = G \).

Proof. Simple connectedness of \( X \) follows from the fact that every \( g \in G \) has a fixed point in the simply connected space \( \mathbb{P}^n \) together with [Arm65, main result, first sentences of intro]. To show that \( \pi_1(X_{\text{reg}}) = G \), note that \( \gamma^{-1}(X_{\text{reg}}) \subset \mathbb{P}^n \) has a complement of codimension at least two by assumption, and is therefore simply connected. Moreover, \( G \) acts freely on it with quotient \( X_{\text{reg}} \), so that \( \pi_1(X_{\text{reg}}) = G \), as claimed.
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