A systematic review of Bezier-like Triangular in surface reconstruction

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Abstract. The application of spline triangular surface has recently been critically discussed in Computer Aided Geometry Design (CAGD). This type of surface has significantly contributed to many important areas especially in reconstructing medical images such as bone and organs. The reconstructed surface is used in further study for example behavioural analysis, safety analysis and surgical planning. Many successful methods have been developed for example Bezier, Ball and Timmer triangular surfaces. In this paper, a comprehensive review of the literature on the triangular surface is presented. The advantages and disadvantages of existing methods on triangular surface are discussed and highlighted in this paper. The methods will be compared in terms of basis function development and the results of produced surface are shown visually. The main objective of this paper is to assist researchers in deciding the best method to be used in surface reconstruction.

1. Introduction

Computer Aided Geometry Design (CAGD) is a discipline related to the computational aspect of geometric design. CAGD has long been widely used for the effectiveness of mathematical applications especially in the construction of curves and surfaces. CAGD is an area that was initially developed to expand the advantage of computer to the industry especially in the fields of automotive, aerospace and shipbuilding [1]. The emergence of CAGD in recent decades has motivated various studies of curve and surfaces smoothing techniques. The objective of the study is to confront real life problems by using CAGD application. Most researchers are in competition to produce the best smoothing techniques of curve and surfaces using CAGD. The most popular CAGD techniques are Bezier and B-spline curves and surfaces.

Spline function is one type of CAGD smoothing technique for curve and surface reconstruction which have been used in every possible industry in the past decade. In fact, the spline function has been evolved before the emergence of computer modelling [2]. The reason for the great interest in spline function is there are many desirable features that make it a suitable method for various areas of application where it is required to compute unknown functions. One desirable feature of spline function is that it smoothen the curve and enables faster surface reconstruction. Spline function has many advantages compared to existing methods. For example, it is easy to work computationally because of the stability, simplicity, piecewise polynomial form, and algorithmic efficiency for evaluation, derivatives, and integrations [3]. The application of spline function have been used extensively in the area of Approximation Theory, Numerical Analysis, Engineering, business, medical, architecture, manufacturing, animations, graphics modelling and scientific areas [4–6].

The curve and surface reconstruction of spline function can be represented into two forms, which are rectangular and triangular [5]. Both are used for describing bivariate polynomial surface by using tensor product or barycentric coordinates. In bivariate polynomial surface, the tensor product is referring
to rectangular and barycentric coordinates applied in triangular form [7]. Generally, the curve and surface reconstruction are displayed in rectangular form due to its requirement that the control point must be organized as a regular rectangular structure [3]. Wang et al. [8] developed a rapid manufacturing method for internal and external rectangular spline shafts by using a combination of laser cutting welding which is used in agricultural machines.

Xu [9] generalized bi-cubic natural spline interpolation on a rectangular domain from the cubic natural spline interpolation on an interval based on its properties. Also, rectangular meshes to define bivariate rational cubic ball function in the specific area of image zooming have been used in [10] in order to deal with image processing. Rectangular spline is also employed by [11] to present a complex topological structure which is motivated by the magneto hydrodynamic (MHD) simulation. The simulation is applied to the fusion devices edge plasma similar to Tokamaks with Isogeometric analysis.

Unfortunately, some data points are not set in regular rectangular structure which leads to ambiguous normal curve and surface reconstruction. This situation makes surface modelling difficult, especially for multi slice images because the data points are required to be the same for the connected slices [12]. Therefore, points reparameterization is used to fit with suitable data points in order to get exact connection. In addition, [11] explained an extraordinary vertex is required to ensure the rectangular meshes are not associated only with simpler topological structures which is needed more than that for the target application. According to [5], an alternative way to solve the problem of spline rectangular function is by partitioning the model into triangles. In this paper, the spline triangular function will be the main technique to be discussed.

The spline triangular function has greatly contributed to many important areas especially in reconstructing medical images such as bone and organs. The medical image reconstruction is used to assist medical personnel in behavioural analysis, safety analysis, and surgical planning. The superiority of spline triangular function over spline rectangular function is that it is well represented in closed surface with arbitrary topology. In the surface fitting, mesh reconstruction is mostly triangular in nature and the interpolation is often easily solved with spline triangular function [5]. Furthermore, the spline triangular function has three desirable properties which are generality, locality, and lower degree [5]. The spline triangular function has many successful development methods which are Bezier triangular, B-spline triangular, Timmer triangular, Ball triangular, and so on.

The rest of the paper is organized as follows. The formulation of Bezier triangular is explained in Section 2. Section 3 discusses the formulation of Bezier-like triangular which are Ball and Timmer triangular. The discussion on construction of Bezier triangular as well as Ball and Timmer triangular are presented in Section 4. Section 5 concludes the paper.

2. Bezier Triangular

A French car company, Citroen, hired a young mathematician named Paul de Faget de Casteljau in 1959 to solve theoretical problem posed by blueprint-to-computer challenge [7,13–15]. At that time, he began to develop systems for curves and surfaces design by introducing control polygon which had never been applied before in the reconstruction design. The system was adopted from Bernstein polynomials and is still known today as the de Casteljau algorithm. The control polygon was created to utilize points near the curve and surface design. The changing of control polygon influenced curve and surface to change. Citroen kept Paul de Casteljau’s work secret for a long time. Bezier triangular was introduced by Paul de Casteljau in the late fifties after he realized there is a need to extend the curve and surface for Bezier with a more natural generalization rather than tensor product [7,13,16]. Paul de Casteljau found the Bezier curve has a symmetric formulation in barycentric coordinates terms. He immediately generalized the triangular from patch to surface.

Paul de Casteljau’s work remained unknown and his work for triangular Bezier patch was not discovered until 1975 by Boehm [16]. Boehm revealed de Casteljau’s work and improved the triangular development. Another researcher named Malcom Arthur Sabin generated triangular Bezier patch in 1977 by using Bernstein polynomials without knowing the existing work of Paul de Casteljau. The development of Bezier triangular was extended by other early researchers such as Frederickson in 1970, Gerald Farin in 1979, and Sablonniere in 1982 after they realized that piecewise surfaces can be defined over regular triangulations [7,13,16].
Bezier triangular is one of the Bezier curve and surface techniques that are often used for producing smoothness result [17]. Basically, the smoothness of curve and surface is influenced by the level of continuity. There are two types of continuity which are parametric and geometric. The first order parametric continuity will be denoted as C¹ while the first order geometric continuity is G¹. Since the continuity is important in smoothness result, [18] constructed an approximate continuity of C¹ and G¹ for Bezier triangular in order to get better shape quality with lower costs as well as lower degree patch. Approximation continuities are used because curve and surface does not fulfil the actual continuities requirement even though the resulting surface appears to be smooth visually. Therefore, the approximation continuity of C¹ and G¹ are the best way to implement actual continuity in the curve and surface reconstruction.

Furthermore, G¹ continuity has been applied by [19] to interpolate data in R³ by using degree four of Bezier triangular with bivariate interpolant. The application was visualized in three-dimensions (3D) which has the same level of visual smoothness with the real G¹ continuity. Moreover, the continuity of C¹ and G¹ were used in the construction of new Timmer triangular with cubic degree as in the study by [20]. As Bezier triangular was extended from univariate Bezier basis function, Timmer triangular also was constructed by Timmer basis function with univariate extension.

Bezier triangular is also used to interpolate scattered data. The research of [21] construct new cubic Bezier-like triangular patches with application in scattered data interpolation. The basis function of new cubic Bezier-like triangular patches is controlled by three shape parameters. The research used C¹ continuity condition and 10 control points implemented by using cubic precision method on each adjacent triangle. In addition, the scattered data was also interpolated by using Bezier triangular with quartic degree as had been studied by [22].

The Bezier triangular quartic patch was used for shape-preserving interpolation with C¹ continuity and 15 control points. Draman et al. [23] employed rational Bezier triangular with degree four which is quartic. The Bezier triangular is used to construct a surface by comprising two composite triangles with C¹ continuity. This study also used three shape parameters similar to the study by [21]. The same researchers [24] extended the development of rational Bezier triangular to interpolate scattered data points of Rainfall with two different approaches which are global and local methods. The data points of rainfall distribution are collected from several areas in Malaysia.

A numerical analysis approach of Isogeometric Analysis (IGA) integrates the concepts of Computer Aided Design (CAD) and Computer Aided Engineering (CAE) [25]. The IGA has been used for Bezier triangular in many approaches. Silva et al. [25] discussed rational Bezier triangular of two-dimensional (2D) geometric modelling for IGA. Another study of Bezier triangular in IGA was done by [26] using rational function with C¹ continuity. The Bezier triangular was used to analyse Kirchhoff plate with a feature-preserving automatic meshing algorithm that admits localized geometric features. There is another study of Kirchhoff plate development with shell elements which also used rational Bezier triangular in the context of IGA [27]. The formulation of Kirchhoff-Love shell required higher continuity between the elements due to higher order PDEs in the problem description. Moreover, [28] used rational Bernstein-Bezier triangular in order to overcome the limitation of IGA. IGA has its limitation on parameterization from surface to volume whenever it deals with real world problems in engineering analysis.

Lewanowicz et al. [29] proved a polynomial approximation problems of rational triangular Bezier surfaces with prescribed boundary control points. The surface problems are convenient in practice and need to be imported into another system. The idea of using dual bivariate constraints of Bernstein polynomials plays a significant role in the research. The high efficiency of the technique was contributed by Bernstein polynomial dual bivariate constraints and numerical computation adaptive scheme of a double integrals which involved rational function of the surfaces. Progressive Iterative Approximation (PIA) is generally used for interpolating and approximating a given set of points with supremacy on clear geometric meaning and convergence stability [30]. Since PIA is useful in CAGD, a preconditioned PIA for rational Bezier triangular was employed by [30] in order to get good result with time-consuming reduction in approximating and interpolating data points. This is because the rational Bezier triangular is stated as more efficient and reliable in data interpolation performance.
Bezier triangular emphasize its important for medical imaging. Basically, Computed Tomography (CT) scan image is a non-invasive medical imaging device which obtained a set of data points of 2D and 3D shape that can reconstructed by using Bezier triangular [31]. Solid representation is very rarely used because it is difficult to reconstruct and requires more time to compute. In that case, medical image of human bone was reconstructed by using Bezier triangular in the study by [31]. The human bone image before and after smoothing by using Bezier triangular are shown in Figure 1. The human bone image was obtained from CT scan images with 3D data points. This study proved Bezier triangular can be used in implant design, biomechanical analysis, and other applications that require solid models.

Tolani et al. [32] studied kinematic modelling of human arm such as elbow workspace by using Bezier triangular. The technique was certainly suitable for reconstructing algorithm for the model since the surface was subdivided and prunes data searching efficiently by an oct-tree data structure. Furthermore, the surface was obtained from intersection of sphere and surface from the model. Beside CT scan image, MRI is the other medical imaging device that can obtain 2D and 3D data points. As stated by [33], data points of tumour were extracted from MRI device used for reconstruction by using Delaunay triangulation in order to estimate volume to ratio of the tumour. The technique was also used for calculating the distance between points on head and the points on tumour.

![Figure 1. Resulting Image of Human Bone](image)

The application of Bezier triangular is not only for medical imaging but also for spray painting of robot. In industrial robotics, it has become complex to control the spraying robot due to its complex surface and many parameters. The first step of spraying robot path is to construct a surface modelling of workpiece. Chen et al. [34] generated a new path planning scheme for spray painting robot with Bezier triangular named B-B triangle. A workpiece has been chosen to reconstruct by using Bezier triangle as shown in Figure 2. They aim to have a smooth path and satisfactory performance for all the joints. This scheme is extended by [35] since the spraying robot makes the real operation become difficult. The same application Bezier triangular on spray painting of robot is studied by [36]. The researchers found two types of spray trajectory optimization which are spray space path and end-effector moving speed. They proved that a large number of spraying experiments were influential in finding the best initial spray trajectory. In the paper, Bezier triangular is used to model a basin in U-direction and V-direction spatial path with 10 control vertices (see Figure 3).

![Figure 2.](image)
The use of Delaunay triangular is well-known for generating triangular mesh. The Delaunay triangular is basically used for maximising the minimal value of angles in the triangle as well as avoiding the skinny triangle. In fact, the Delaunay triangular could also preserve the original surface topology and does not involve the deletion of sample data points [37]. Based on [38], Delaunay triangular generates triangle with poor quality. Thus, the researchers improved the triangular by using simple sweep line algorithm. Table 1 summarized several works extended from Bezier triangular to polygonise triangular such as Delaunay triangular, Timmer triangular, Bernstein-Bezier triangular, and Ball triangular.

### Table 1. Findings of Bezier Triangular.

| Proposed Method | Types of Bezier Triangular | Findings |
|-----------------|-----------------------------|----------|
| Awang and Rahmat [37], 2017 | Delaunay Triangular | • Cubic degree.  
• Visualized in 2D surface.  
• Scattered and non-uniform data points distribution.  
• Represented by using MATLAB Graphical User Interface (GUI) which is easy to analyse, interpret, and draw conclusions from the result obtained. |
| Karim and Saaban [39], 2018 | Ball Triangular and Delaunay Triangular | • Cubic degree.  
• Scattered data is used to visualize the geometrical images of the surface data.  
• $C^1$ continuity.  
• The scheme is tested to visualize the terrain data collected at central region of Malaysia. |
| Ali et al. [40], 2019 | Timmer Triangular and Delaunay Triangular | • Cubic degree.  
• Scattered and non-uniform data points distribution.  
• $C^1$ and $G^1$ continuity.  
• Presented using MATLAB and Mathematica software.  
• Aim for producing high accuracy for image and surface reconstruction. |
| Ludwig et al. [41], 2019 | Bernstein-Bezier Triangular | • $C^1$ and $G^1$ inter-element continuity.  
• Rotation-free Kirchhoff-Love formulation is used.  
• Formulation is free from transverse-shear locking.  
• Relies on high polynomial degree to mitigate membrane locking.  
• New finite element procedure for thin plate and shells. |
| Liu and Jeffers [42], 2019 | Delaunay Triangular | • Quartic and sextic degree.  
• Involved IGA.  
• Rational function is used for domain triangulation of complex geometries.  
• Feature-preserving Delaunay discretization coupled with a local refinement technique. |
3. Bezier-like Triangular: Timmer and Ball

3.1. Bezier-like Basis Function

Bezier-like curves can be expressed in terms of Bernstein polynomials, defined explicitly by [43]:

$$B_i^n (u) = \binom{n}{i} u^i (1-u)^{n-i},$$

(1)

where the binomial coefficients are given by

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \text{if} \quad 0 \leq i \leq n$$

(2)

Equation (1) and (2) can be extended to degree three which is cubic and has $n=3$. Therefore, the equation for cubic degree can be defined as follows:

$$
\begin{align*}
B_0^3 (u) &= (1-u)^3 \\
B_1^3 (u) &= 3u(1-u)^2 \\
B_2^3 (u) &= 3u^2(1-u) \\
B_3^3 (u) &= u^3
\end{align*}
$$

(3)

Equation (3) is called Bezier basis function for degree three. The generalization of cubic Bezier-like curve basis function for degree three has two shape parameters which are $\alpha$ and $\beta$. The shape parameters can be adjusted as desired in order to change the shape of a curve. The basis function of Bezier-like curve defined for $t \in [0,1]$ are as below [44]:

$$
\begin{align*}
F_0^3 (u) &= (1-u)^2 \left( 1 + (2-\alpha)u \right) \\
F_1^3 (u) &= \alpha (1-u)^2 u \\
F_2^3 (u) &= \beta u^2 (1-u) \\
F_3^3 (u) &= u^3 \left( 1 + (2-\beta)(1-u) \right)
\end{align*}
$$

(4)

The speciality of the basis function can be defined as cubic Bezier curve when $\alpha = \beta = 3$, cubic Ball curve when $\alpha = \beta = 2$, cubic Timmer curve when $\alpha = \beta = 4$, and straight line when $\alpha = \beta = 0$.

Figure 4 shows a control polygon with three different curves which are cubic Bezier, cubic Timmer, and cubic Ball based on their basis function.

Figure 4. A Control Polygon with Three Different Curves.
3.2. Bezier-like Triangular Basis Function

Bezier triangular is represented by a smooth parametric surface with a set of control points and basis function. Based on [43], three vertices \( V_1, V_2, \) and \( V_3 \) correspond to the barycentric coordinates \((1,0,0),(0,1,0),\) and \((0,0,1)\) which are denoted as \( u, v, \) and \( w \) respectively. Any point of the triangle can be written as
\[
V = uV_1 + vV_2 + wV_3, \tag{5}
\]
with \( u + v + w = 1. \)

Bezier triangular of order \( n \) can be defined in terms of a set of \( 0.5(n+1)(n+2) \) control points \( P_{ijk} \) for indices, \( i \geq 0, j \geq 0, k \geq 0, \) and \( i + j + k = n. \) The Bezier triangular was constructed by using de Casteljau algorithm.

\[
P(u,v,w) = \sum_{i,j,k \leq n} P_{ijk} B_{ijk}^n (u,v,w) \tag{6}
\]
for \( 0 \leq u + v + w \leq 1, u + v + w \leq 1 \) where \( B_{ijk}^n (u,v,w) \) are the basis functions of polynomial described in the following way:
\[
B_{ijk}^n (u,v,w) = \frac{n!}{i!j!k!} u^i v^j w^k. \tag{7}
\]

Equation (6) is called bivariate because it has one variable that is dependent on the other two variables which is \( w = 1 - u - v. \)

Bezier-like triangular is defined with three shape parameters which are \( \alpha, \beta, \) and \( \gamma \) and \( i + j + k = 3 \) based on equation (6), which can be written as [45]:
\[
P(u,v,w) = w^3 \left( 1 + \gamma (w-1) \right) P_{003} + w^2 \gamma v (\gamma + 2) P_{012} + \gamma v^2 w (\beta + 2) P_{021} + v^2 (1 + \beta (v-1)) P_{030} + w^2 u (\gamma + 2) P_{002} + 6uvw P_{111} + v^2 u (\beta + 2) P_{120} + u^2 w (\alpha + 2) P_{201} + u^3 v (\alpha + 2) P_{210} \tag{8}
\]

Based on equation (8), cubic Bezier triangular can be defined when \( \alpha = \beta = \gamma = 1 \) which can be written in the form:
\[
P(u,v,w) = w^3 P_{003} + 3vw^2 P_{012} + 3v^2 w P_{021} + v^3 P_{030} + 3uw^2 P_{102} + 6uvw P_{111} + 3uv^2 P_{120} + 3u^2 w P_{201} + 3u^3 v P_{210} + u^3 P_{300} \tag{9}
\]

Equation (8) can be expanded to defined Ball triangular when \( \alpha = \beta = \gamma = 0. \) The cubic Ball triangular is defined as follows:
\[
P(u,v,w) = w^3 P_{003} + 2vw^2 P_{012} + 2v^2 w P_{021} + v^3 P_{030} + 2uw^2 P_{102} + 6uvw P_{111} + 2uv^2 P_{120} + 2u^2 w P_{201} + 2u^3 v P_{210} + u^3 P_{300} \tag{10}
\]

The cubic Bezier-like triangular defined by (8) can be reduced to cubic Timmer triangular when \( \alpha = \beta = \gamma = 2 \) as below:
\[
P(u,v,w) = w^2 (2w - 1)P_{003} + 4vw^2P_{012} + 4v^2wP_{021} + v^2(2v - 1)P_{000} + 4uv^2P_{102} + 6uvwP_{111} + 4uv^2P_{120} + 4u^2wP_{201} + 4u^2vP_{210} + u^2(2u - 1)P_{300}
\]

Based on equation (9), a control net of cubic Bezier triangular, which consists of a few vertices and 10 control points with Bezier basis function, are shown in Figure 5.

![Figure 5. Control Net of Cubic Bezier Triangular with (a) Control Points; (b) Bezier Basis Function.](image)

The control net is shown for degree cubic which led to the sum of all subscripts in each of the vertices to also be three. The de Casteljau algorithm can be represented by [20]:

Given:
A triangular array of points \( P_{i,j,k} \in E^3; i+j+k = n \) and a point in \( E^2 \) with barycentric coordinate \( u \).

Set:
\[
P_i^r(u) = uP_{i,j,k+1}^r(u) + vP_{i,j,k+2}^{r-1}(u) + wP_{i,j,k+3}^{r-1}(u)
\]

where \( r = 1,...,n \) and \( i+j+k = n-r \).

Bezier triangular properties are based on de Casteljau algorithm which is listed as follows [20]:

a) Affine invariance:
De Casteljau algorithm uses linear interpolation since the linear interpolation is an affine map.

b) Invariance under affine parameter transformations:
After an affine transformation, a point \( u \) in the de Casteljau algorithm will have the same barycentric coordinates \( u \).

c) Convex hull:
The property is satisfied since for \( 0 \leq u,v,w \leq 1 \), each of the \( P_i^r \) is a convex combination of the previous \( P_{i,j,k+1}^{r-1} \).

4. Discussion
A set of data points is used for constructing Bezier triangular and other existing triangular such as Ball triangular and Timmer triangular. The set of data points consisting of 10 control points used for constructing Bezier triangular as well as Ball and Timmer triangular are listed in Table 3 as shown in Table 2. The constructions of 2D cubic patch are shown in Figure 6.
Table 2. 2D Control Points.

| x  | 0 | 4 | 8 | 12 | 2 | 6 | 10 | 4 | 8 | 6 |
|----|---|---|---|----|---|---|----|---|---|---|
| y  | 0 | 0 | 0 | 0 | 4 | 6 | 4 | 8 | 8 | 12 |

![Bezier Triangular](image1)

![Ball Triangular](image2)

![Timmer Triangular](image3)

(a) (b) (c)

**Figure 6.** 2D Data Points Cubic Patch Construction of (a) Bezier Triangular; (b) Ball Triangular; (c) Timmer Triangular.

The construction is then applied to 3D data points as shown in Figure 7 for Bezier triangular, Figure 8 for Ball triangular, and Figure 9 for Timmer triangular. The 3D data points are listed in Table 3 below.

Table 3. 3D Control Points.

| x  | 70 | 90 | 140 | 180 | 50 | 100 | 180 | 100 | 130 | 120 |
|----|----|----|-----|-----|----|-----|-----|-----|-----|-----|
| y  | 30 | 50 | 70   | 65   | 90 | 100 | 160 | 190 | 220 | 400 |
| z  | 10 | 20 | 20   | 10   | 20 | 30  | 20  | 30  | 30  | 10  |
Figure 7. 3D Data Points Cubic Patch Construction of Bezier Triangular, (a) Control Polygon; (b) without Control Polygon; (c) with Control Polygon.

Figure 8. 3D Data Points Cubic Patch Construction of Ball Triangular, (a) without Control Polygon; (b) with Control Polygon.
5. Conclusion

Computer Aided Geometry Design (CAGD) is widely applied to this day especially on the surface of the spline triangle. This is because the triangular surface has contributed a lot to various key sectors such as medicine in the reconstruction of medical images such as bones and organs. With this reconstruction, further studies can be continued such as behavioural analysis, safety analysis, and surgical planning. Various successful methods have been discussed such as the Bezier, Ball, and Timmer triangular. This paper has discussed a comprehensive review of the literature on triangular surfaces. The literature explained the existing methods with advantages and disadvantages. The basis function of Bezier, Ball, and Timmer are discussed based on the Bezier-like basis function. Comparisons of such surfaces have been shown visually. Therefore, this paper can help researchers in determining the best method to be used in surface reconstruction.

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References

[1] Ali F A M, Karim S A A, Dass S C, Skala V, Saaban A, Hasan M K and Hashim I 2020 Visualizing the Energy of Scattered Data by Using Cubic Timmer Triangular Patches J. Phys. Conf. Ser. 1366

[2] Majeed A, Abbas M, Qayyum F, Miura K T, Misro M Y and Nazir T 2020 Geometric Modeling
using New Cubic Trigonometric B-spline Functions with Shape Parameter. *Mathematics* 8 1–25

[3] Loop C 1994 Smooth Spline Surfaces over Irregular Meshes *Proc. 21st Annu. Conf. Comput. Graph. Interact. Tech.* 303–310

[4] Schumaker L L *Spline Functions* Spline Functions: Computational Methods

[5] Xu D 2002 *Incremental Algorithms for the Design of Triangular-based Spline Surfaces* (University of Pennsylvania)

[6] Hadi N A and Alias N 2019 3-dimensional human head reconstruction using cubic spline surface on CPU-GPU platform *ACM Int. Conf. Proceeding Ser. Part F1479* 16–20

[7] Farin G 1984 *A History of Curves and Surfaces in CAGD* (Elsevier) pp 1–21

[8] Wang S, Hu Z, Gu F, Peng B, Chen Y, Wu F and Wang Y 2021 A Rapid Manufacturing Method for Rectangular Spline Based on Laser Cutting and Welding *Trans. ASABE* 64 117–126

[9] Xu W 2020 Elements of Bi-cubic Polynomial Natural Spline Interpolation for Scattered Data: Boundary Conditions Meet Partition of Unity Technique *Stat. Optim. Inf. Comput.* 8 994–1010

[10] Hadi N A 2015 *G2 Parametric Curve and Surface Fitting using Beta-spline* (Universiti Teknologi Mara)

[11] Wu M, Mourrain B, Galligo A and Nkonga B 2017 *Hermite Type Spline Spaces over Rectangular Meshes with Complex Topological Structures* *Commun. Comput. Phys.* 21 835–866

[12] Farin G, Hoschek J and Kim M-S 2002 *A History of Curves and Surfaces in CAGD* Handbook of Computer Aided Geometric Design (Elsevier) pp 1–21

[13] Hoyle N 2006 Automated Multi-Stage Geometry Parameterization of Internal Fluid Flow Applications

[14] Brennan A M 2020 Measure, modulation and metadesign: NC fabrication in industrial design and architecture *J. Des. Hist.* 33 66–82

[15] Farin G 1986 *Triangular Bernstein-Bézier Patches* *Comput. Aided Geom. Des.* 3 83–127

[16] Abdul Karim S A, Saaban A and Muthuvalu M S 2014 Bézier triangular patches for closed surface *Appl. Math. Sci.* 8 355–366

[17] Liu Y 2008 *Triangular Bézier Surfaces with Approximate Continuity* (University of Waterloo)

[18] Vlachkova K and Radev K 2021 Interpolation of Data in R3 using Quartic Triangular Bézier Surfaces

[19] Ali F A M, Karim S A A, Dass S C, Skala V, Saaban A, Hasan M K and Hashim I 2019 New Cubic Timmer Triangular Patches with Cl and G1 Continuity *J. Teknol.* 81 1–11

[20] Karim S A A, Saaban A, Skala V, Ghaffar A, Nisar K S and Baleanu D 2020 Construction of New Cubic Bézier-like Triangular Patches with Application in Scattered Data Interpolation *Adv. Differ. Equations 2020*

[21] Karim S A A, Saaban A and Nguyen V T 2020 Scattered Data Interpolation using Quartic Triangular Patch for Shape-preserving Interpolation and Comparison with Mesh-free Methods *Symmetry (Basel).* 12

[22] Abdul Karim S A, Saad N and Kannan R 2021 C1 Surface Interpolation using Quartic Rational Triangular Patches *Advanced Methods for Processing and Visualizing the Renewable Energy* ed S A A Karim, N Saad and R Kannan (Springer) pp 89–99

[23] Draman N N C, Karim S A A, Hashim I and Ping Y W Rainfall Scattered Data Interpolation using Rational Quartic Triangular Patches *Theoretical, Modelling and Numerical Simulations Toward* 855–873

[24] Engvall L and Evans J A 2016 Isogeometric Triangular Bernstein-Bézier Discretizations:
Automatic Mesh Generation and Geometrically Exact Finite Element Analysis *Comput. Methods Appl. Mech. Eng.* **304** 378–407

[29] Lewanowicz S, Keller P and Woźny P 2017 Constrained Approximation of Rational Triangular Bézier Surfaces by Polynomial Triangular Bézier Surfaces *Numer. Algorithms* **75** 93–111

[30] Liu C, Han X and Li J 2020 Preconditioned Progressive Iterative Approximation for Triangular Bezier Patches and Its Application *J. Comput. Appl. Math.* **366**

[31] Mun D and Kim B C 2017 Three-dimensional Solid Reconstruction of a Human Bone from CT Images using Interpolation with Triangular Bezier Patches *J. Mech. Sci. Technol.* **31** 3875–3886

[32] Tolani D, Badler N and Gallier J 2000 A Kinematic Model of the Human Arm Using Triangular Bezier Spline Surfaces 1–36

[33] Sakthi Bharathi A and Manimegalai D 2015 3D Digital Reconstruction of Brain Tumor from MRI Scans using Delaunay Triangulation and Patches *ARPN J. Eng. Appl. Sci.* **10** 9227–9232

[34] Chen W, Sun C, Liu H, Liu J and Tang Y 2017 Path Planning Scheme for Spray Painting Robot with Bézier Curves on Complex Curved Surfaces *Proc. - 2017 32nd Youth Acad. Annu. Conf. Chinese Assoc. Autom. YAC* 2017 698–703

[35] Chen W, Wang X, Ge H and Wen Y 2019 Trajectory Optimization for Spray Painting Robot on Bezier-Bernstein Algorithm *Proc. 2018 Chinese Autom. Congr. CAC* 2018 3389–3394

[36] Chen W, Liu J, Tang Y and Ge H 2019 Automatic Spray Trajectory Optimization on Bézier Surface *Electron.* **8**

[37] Awang N and Rahmat R W 2017 Reconstruction of Smooth Surface by using Cubic Bezier Triangular Patch in GUI **2** 61–69

[38] Hadi N A, Farhani A and Dahalan W M 2021 An Improved Simple Sweep Line Algorithm for Delaunay Refinement Triangulation *Adv. Eng. Process. Technol. II* 263–270

[39] Jaafar W N W 2018 *Visualization of Curve and Surface Data using Rational Cubic Ball Functions* (University Sains Malaysia)

[40] Ali F A M, Karim S A A, Dass S C, Skala V, Hasan M K and Hashim I 2019 Efficient Visualization of Scattered Energy Distribution Data by Using Cubic Timmer Triangular Patches *Energy Efficiency in Mobility Systems* pp 181–197

[41] Ludwig T, Hühne C and De Lorenzis L 2019 Rotation-free Bernstein–Bézier Elements for Thin Plates and Shells - Development and Validation *Comput. Methods Appl. Mech. Eng.* **348** 500–534

[42] Liu N and Jeffers A E 2019 Feature-preserving Rational Bézier Triangles for Isogeometric Analysis of Higher-order Gradient Damage Models *Comput. Methods Appl. Mech. Eng.* **357**

[43] Farin G and Kim M 2002 Handbook of Computer Aided Geometric Design 2002

[44] Ahmad A 2014 A Generalization of Bezier-like Curve *J. Sci. Math. Technol.* **1** 56–68

[45] Karim S A A, Saaban A, Skala V, Ghaffar A, Nisar K S and Baleanu D 2020 Construction of New Cubic Bézier-like Triangular Patches with Application in Scattered Data Interpolation *Adv. Differ. Equations* **2020** 1–22