No scalar hair theorem for neutral Neumann stars: static massive scalar fields nonminimally coupled to gravity

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Abstract

In a recent paper, Hod proved that spherically symmetric Dirichlet reflecting compact stars cannot support static scalar fields nonminimally coupled to gravity. In the present paper, we study the validity of no hair theorems for compact stars with Neumann surface boundary conditions. We find that Neumann compact stars cannot support static massive scalar field hairs with a generic dimensionless field-curvature coupling parameter.

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I. INTRODUCTION

The classical no hair theorem [1]-[8] plays an important role in the development of black hole theories. It states an intriguing property that asymptotically flat black holes cannot support minimally coupled static scalar fields, for recent progress see references [9]-[23] and reviews [24, 25]. It was usually believed that this no hair property is due to the existence of black hole absorbing horizons.

However, no scalar hair behavior also appears in the horizonless spacetime. It was firstly proved that asymptotically flat neutral Dirichlet reflecting horizonless compact stars cannot support massive scalar field hairs [26]. In the asymptotically dS gravity, massive scalar, vector and tensor fields also cannot exist outside neutral horizonless Dirichlet reflecting compact stars [27]. Then whether no hair theorem exists in the charged horizonless gravity is still an question to be answered. In fact, it was shown that static scalar fields cannot condense outside reflecting shells of large radii [28–30]. Moreover, large charged reflecting stars cannot support static scalar field hairs [31–35]. It was also showed that scalar fields cannot exist outside compact stars with Neumann surface boundary conditions [36].

All front no scalar hair theorems only consider minimally coupled scalar field hairs. Interestingly, black hole no scalar hair theorem also holds with nonminimal field-curvature couplings [37–40]. Considering nonminimal coupling parameter $\xi$, compact stars with generic boundary conditions can rule out the existence of scalar hairs in ranges $\xi < 0$ and $\xi > \frac{1}{4}$ and compact stars with Dirichlet reflecting boundary conditions can also rule out exterior scalar hair for $0 \leq \xi \leq \frac{1}{4}$ [41-43]. So it is interesting to study no scalar hair properties with other boundary conditions in the range $0 \leq \xi \leq \frac{1}{4}$. In this work, we plan to investigate the no nonminimal scalar hair behavior in the background of horizonless compact stars with Neumann boundary conditions.

The rest of this work is as follows. We construct the gravity model of static scalar fields nonminimally coupled to gravity in the background of Neumann compact stars. We find that no scalar hair theorem holds for generic coupling parameters. At last, we summarize the main results.
II. NO HAIR THEOREM FOR SCALAR FIELDS OUTSIDE NEUMANN STARS

We study the gravity model of massive scalar fields nonminimally coupled to the compact star gravity. And the asymptotically flat spherically symmetric spacetime reads 37–41

\[ ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \] (1)

The functions \( \nu \) and \( \lambda \) only depend on the radial coordinate \( r \). We define the radial coordinate \( r = r_s \) as the star radius. Asymptotic flatness of the spacetime requires the behaviors 37

\[ \nu(r \to \infty) \sim O\left(\frac{1}{r}\right), \quad \lambda(r \to \infty) \sim O\left(\frac{1}{r}\right). \] (2)

The Lagrange density with scalar fields nonminimally coupled to gravity is 40, 41

\[ \mathcal{L} = R - \xi R\psi^2 - |\nabla_\alpha \psi|^2 - \mu^2 \psi^2, \] (3)

where \( \psi(r) \) is the scalar field with mass \( \mu \). We label \( R \) as scalar Ricci curvature of the spacetime. Since we are interested in the asymptotically flat background, there is

\[ R(r \to \infty) \to 0. \] (4)

The dimensionless physical parameter \( \xi \) describes the nonminimal coupling strength between scalar fields and curvature. Hod proved that compact stars with generic boundary conditions can rule out the existence of scalar hairs in ranges \( \xi < 0 \) and \( \xi > \frac{1}{4} \) 41. It remains to study no hair theorem in the background of compact regular star for parameters satisfying

\[ 0 \leq \xi \leq \frac{1}{4}. \] (5)

The scalar field equation is 41

\[ \psi'' + \left(\frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2}\right)\psi' - (\mu^2 + \xi R)e^\lambda \psi = 0. \] (6)

At the star surface, we take the Neumann boundary condition. Around the infinity, the scalar field asymptotically behaves in the form

\[ \psi \sim A \cdot \frac{1}{r}e^{-\mu r} + B \cdot \frac{1}{r}e^{\mu r}, \] (7)

where \( A \) and \( B \) are integral constants. The physical solution requires \( B = 0 \) 40. It yields boundary conditions

\[ \psi'(r_s) = 0, \quad \psi(\infty) = 0. \] (8)
The Ricci scalar curvature is
\[ R = -\frac{8\pi}{1 - 8\pi \xi^2 \psi^2} \{ e^{-\lambda} \{ \xi \left( \frac{12}{r} + 3\nu' - 3\lambda' \} \psi \psi' + 6\xi \psi'' + (6\xi - 1)(\psi')^2 \} - 2\mu^2 \psi^2 \}. \] (9)

Substituting (9) into the scalar field equation (6), we arrive at
\[ F \cdot \psi'' + \left[ F \cdot \left( \frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2} \right) + 8\pi \xi (6\xi - 1) \psi \psi' \right] \psi' - \mu^2 e^\lambda (1 + 8\pi \xi \psi^2) \psi = 0, \] (10)
where we have defined \( F(r, \xi) = 1 + 8\pi \xi (6\xi - 1) \psi^2. \)

We divide the proof of no hair theorem into three cases
\[ \psi(r_s) = 0, \ \psi(r_s) > 0 \ \text{and} \ \psi(r_s) < 0. \] (11)

It is known that the nontrivial scalar field cannot exist for \( \psi(r_s) = 0 \) [41]. Considering the symmetry \( \psi \rightarrow -\psi \) of equation (10), it remains to prove no hair theorem for
\[ \psi(r_s) > 0. \] (12)

In the range \( 0 \leq \xi \leq \frac{1}{4} \), there is \( F(r, \xi) \geq 0 \), which is important in the following analysis [41]. The proof of \( F(r, \xi) \geq 0 \) is as follows. In the range \( \frac{1}{6} \leq \xi \leq \frac{1}{4} \), there is \( F(r, \xi) \geq 0 \). For \( 0 \leq \xi < \frac{1}{6} \), \( F(r, \xi) \) cannot switch signs. Otherwise, \( F(r, \xi) \) vanishes at some point \( r_0 \). And at this point \( r = r_0 \), there is the relation
\[ 8\pi \xi (6\xi - 1) \psi'^2 = \mu^2 e^\lambda (1 + 8\pi \xi \psi^2). \] (13)
In the regime \( 0 \leq \xi < \frac{1}{6} \), the functional expression on the left side of (13) is non-positive whereas the functional expression on the right side of (13) is positive definite. One therefore concludes that the radial function \( F(r, \xi) \) cannot switch signs. At the infinity, \( F(r, \xi) \) behaves as
\[ F(r \rightarrow \infty, \xi) \rightarrow 1 + 8\pi \xi (6\xi - 1) \psi(\infty)^2 = 1 > 0. \] (14)
So we find \( F(r, \xi) > 0 \) in the case of \( 0 \leq \xi \leq \frac{1}{6} \). As a summary, for \( 0 \leq \xi \leq \frac{1}{4} \), there is the relation
\[ F(r, \xi) \geq 0. \] (15)

We divide the analysis into two cases
\[ \psi''(r_s) \leq 0 \ \text{and} \ \psi''(r_s) > 0. \] (16)

In the case of \( \psi''(r_s) \leq 0 \), we obtain following relations at the star surface \( r = r_s \) as
\[ \{ \psi^2 > 0, \ \psi' = 0 \ \text{and} \ \psi'' \leq 0 \} \ \text{for} \ \ r = r_s. \] (17)
At the star radius \( r_s \), relations (15), (17) and \( 0 \leq \xi \leq \frac{1}{4} \) give the characteristic inequality

\[
F \cdot \psi'' + \left[ F \cdot \left( \frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2} \right) + 8\pi\xi(6\xi - 1)\psi\psi' \right] \psi\psi' - \mu^2 e^{-\lambda}(1 + 8\pi\xi\psi^2)\psi^2 < 0,
\]

which is in contradiction with equation (10).

In another case of \( \psi''(r_s) > 0 \), also considering the condition \( \psi'(r_s) = 0 \), we will have \( \psi'(r) > 0 \) around \( r_s \). With increase of the radial coordinate, the scalar field firstly becomes more positive and finally approaches zero at the infinity. In this case, there is at least one positive maximum extremum point \( r = r_{peak} \) between the star surface \( r_s \) and the infinity boundary. At this extremum point, the scalar field is characterized by following relations

\[
\{ \psi^2 > 0, \quad \psi' = 0 \quad \text{and} \quad \psi\psi'' \leq 0 \} \quad \text{for} \quad r = r_{peak}.
\]

At this extremum point \( r = r_{peak} \), relations (15), (19) and \( 0 \leq \xi \leq \frac{1}{4} \) lead to the inequality

\[
F \cdot \psi'' + \left[ F \cdot \left( \frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2} \right) + 8\pi\xi(6\xi - 1)\psi\psi' \right] \psi\psi' - \mu^2 e^{-\lambda}(1 + 8\pi\xi\psi^2)\psi^2 < 0.
\]

It can be easily seen that relation (20) is in contradiction with equation (10). So nontrivial scalar field solution of equation (10) cannot exist. Here we prove no nonminimally coupled scalar hair theorem for \( 0 \leq \xi \leq \frac{1}{4} \). Also considering known results that compact stars with generic boundary conditions cannot support scalar hairs in ranges \( \xi < 0 \) and \( \xi > \frac{1}{4} \) [41], we conclude that scalar hair cannot form outside regular neutral Neumann stars for any coupling parameter \( \xi \).

III. CONCLUSIONS

In the background of spherically symmetric regular Neumann stars, we studied no hair theorem for static massive scalar fields nonminimally coupled to the asymptotically flat gravity. We considered the field-curvature coupling and included scalar fields’ backreaction on the background. We obtained the characteristic inequalities (18) at the star surface and (20) at extremum points, which are in contradiction with the scalar field equation (10). It means that there is no nontrivial scalar field solution. At last, we concluded that asymptotically flat spherically symmetric regular Neumann stars cannot support the existence of exterior massive scalar field hairs for generic nonminimal coupling parameters.
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