Advanced calculation of the deflection of worm shafts with FEM

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Abstract. Worm gears are gear units with a shaft angle of mostly 90°. They are used in a variety of industrial applications due to large gear ratios and high transmittable torques. The high sliding ratio in the tooth contact requires a hard/soft material combination that is insensitive to scuffing. Worm shafts usually are made of case-hardened steel and worm wheels of a copper-tin-bronze alloy. Since damage predominantly occurs at the wheel (pitting, wear, tooth fracture, etc.), a wide range of calculation methods for load capacity and service life were developed. Regarding the worm shaft, only the deflection is examined, because high displacements of the shaft shifts the contact pattern, which leads to transmission errors and may lead to increased wear. Common calculation methods, e.g. according to DIN 3996 and ISO/TS 14521 provide a good approximation of the deflection. However, simplifications are made in favour of the manageability of the calculation method. Geometric modifications on the worm such as reduced tooth thickness or different lead angles cannot be taken into account with common methods. In order to close this gap, a new approach was presented by Norgauer in 2021, however, significantly overestimating the stiffening effect of the gearing in some cases by neglecting the helical winding of the worm teeth. For efficiency-optimal design of minimally thin worm shafts, approaches that go beyond the deflection are also missing. Investigations by the authors on worm gears using the finite element method (FEM) show a deviation of the bending line of worm shafts from the common calculation methods. The FEM calculations simulate the tooth meshing under load by a driving torque on the shaft, whereas the standard calculations simplify the load distribution on the worm shaft as a radial force introduced at a point. In addition to the magnitude of the radial forces, the axial tooth force component was identified as an influence factor on the bending line since it causes a displacement of the maximum bending location. Especially for increasing diameter factors the displacement of the bearings under load must be taken into account since they may be in the same range as the deflection. Furthermore, the helical winding of the teeth around the shaft leads to an average stiffening effect of the gearing. The cross-section depends on the gear meshing position and causes a periodic fluctuation of the area moment of inertia and a wobbling motion of the shaft. The changing load distribution on several teeth causes a periodic change of the lever arms and an additional dependence of the deflection from the mesh position. An analytical calculation method for the deflection of worm shafts is proposed. It aims at the precise calculation of the spatial expression of the bending line, extends the commonly used simplified model and takes the newly identified factors into account. The bending lines in both radial directions are considered separately. The values of the new introduced factors are derived from the FEM results. Finally, the calculated bending lines are compared with the FEM results and verified.
1. Introduction
Worm gear units enable high torque and speed ratios in a small installation space with an axis cross angle of usually $\Sigma = 90^\circ$. They are helical gears and, in addition to the rolling component in the tooth contact, also have a high design-related sliding component. Due to the sliding, they are preferably designed in a hard-soft material pairing that is insensitive to scuffing, with the worm usually made of case-hardened steel and the wheel of copper-tin bronze as shown in Figure 1. Due to the pairing of the hard steel with the relatively soft bronze material, worm gears usually fail in operation due to excessive damage to the worm wheel.

![Figure 1. Worm shaft and worm wheel.](image)

Therefore, previous research on the wear of worm gears in Germany has concentrated almost exclusively on the investigation of the wheel in terms of geometry and materials. Worm shafts, on the other hand, are designed exclusively against deflection, since this may cause a displacement of the contact pattern and consequently to increased local wear due to unfavorable load transmission. The exact course of the bending line remains unknown, since only the maximum value is calculated by the simplifications made in common calculation methods. In some cases, the geometry of worm shafts is modified in favor of other advantages (e.g. duplex worms, reduction of the tooth thickness to increase the durability of the worm wheel teeth). This cannot be considered in common calculation methods.

2. State of the art
Calculation methods for determining the deflection of the worm shaft are proposed in DIN 3996 [1], ISO/TS 14521 [2], AGMA 6022 [3] as well as the British Standards BS 721 [4]. Norgauer [5] compared the proposed calculation methods and showed the differences between them. These methods are all based on the simplification of the worm geometry as a round profile which is radially loaded by a force and calculated according to Bernoulli’s principle. The radial load is calculated in [1] and [2] according to Niemann/Winter [6] and considers the friction in the tooth contact. The deflection of the worm shaft is approximated according to [1] for symmetrical arranged bearings by equation (1):

$$\delta_m = 2 \cdot 10^{-6} \cdot l_1^3 \cdot F_{tm2} \cdot \frac{\sqrt{\tan^2(y_m + \arctan(\mu_{zm})) + \tan^2\alpha_0/\cos^2\gamma_m}}{(1 \cdot d_f^4)^3}$$

Both DIN 3996 and the ISO/TS 1451 use the root diameter extended by 10% as substitute diameter to consider the stiffening effect of the worm teeth. The factor is based on experiments carried out by Lutz [7], who loaded various worm shafts on a pulser with a radial force, measured the deflection and derived the factor from the results. AGMA 6022 calculates the deflection with the root diameter and therefore always calculates higher values than [1] and [2]. The calculation is on the safe side.

Norgauer [5] developed an extended analytical approach for calculating the moment of inertia for standard and geometric modified worm teeth. The tooth form of the worm is re-calculated from the machining tool simplified as trapezoid in the face section and added to the moment of inertia of the root diameter. Both the rotation of the trapezoids due to the position as well as the distance of the centroids of the tooth areas to the common center according to Steiner’s theorem are considered. Based
on the calculations, he derives an equivalent diameter to consider the stiffening effect of the toothing. While this approach allows for the consideration of geometric modifications, it overestimates the stiffening effect of the gearing in some cases because of the negligence of the helical winding of the worm teeth over the length of the gearing.

The finite element method offers the possibility to analyze the bending of the worm shaft with increased accuracy, but also increased computational effort. Norgauer [5] proved the suitability of the finite element method for calculating the deflection of worm shafts but did not simulate the complex load transmission in the tooth contact; therefore those results highly agree with the common standard calculation methods [1], [2], [3] and the experiments according to Lutz [7]. However, the full potential of the finite element method is not exploited due to the simplified load on the worm. Hermes [8] successfully calculate the contact behavior and load transfer between the flanks with FEM, Reißmann [9] investigated the tooth root stress on worm wheels by simulating the loaded tooth contact. However, the subject of their investigations was not the deflection of the worm shafts.

3. Advanced calculation of the worm shaft deflection with FEM
The FEM modelling aims at the realistic calculation of the worm shafts bending line by simulating the load distribution in the tooth contact as well as the deformation of the worm teeth, the wheel and the displacement of the bearings. The load transmission in the contact is evaluated for reference.

3.1. Scope of investigation, FEM- modelling and calculation method
The investigated geometries shown in Figure 2 according to DIN 3975 [10] where modelled with CAD. The axis distance is a = 100mm, gear ratio i = 20,5, axial modulus m₀ = 4mm, normal pressure angle α₀ = 20° and number of threads z₁ = 2. One geometry of each diameter factor is modelled with normal and one with reduced tooth thickness on the worm. The main gear data is listed in Table 1:

| Table 1. Main gear data |
|------------------------|
|                         | S0700 | S0725 | S0900 | S0925 |
| q [°]                   | 7,00  | 9,00  |       |       |
| γₙ [°]                  | 15,95 | 12,53 |       |       |
| xₙ [-]                  | 0,00  | -0,25 | 0,00  | -0,25 |
| sₘₙ [mm]               | 6,28  | 4,28  | 6,28  | 4,28  |
| λₙₙ [mm²]              | 629,70| 508,00| 1,028,80| 871,00|
| b₁ [mm]                | 75,00 |       | 64,90 |       |
| d₁₁ [mm]               | 18,00 |       | 26,00 |       |
| d₁₃ [mm]               | 28,00 |       | 36,00 |       |
| d₁₅ [mm]               | 36,00 |       | 44,00 |       |
| b₂ [mm]                | 26,70 |       | 29,40 |       |
| d₂₁ [mm]               | 162,00|       | 154,00|       |
| d₂₃ [mm]               | 172,00|       | 164,00|       |
| d₂₅ [mm]               | 180,00|       | 172,00|       |
| d₂₇ [mm]               | 184,00|       | 176,00|       |

The load is introduced into the FEM model by applying the drive torque T₁ on the worm shaft. The different values of the drive torque cause a normal force Fₙ on the tooth flank of 10,000 N for each geometry. By converting the equations for normal and tangential force according to DIN 3996 [1] and Niemann/Winter [6], T₁ is calculated as follows:

$$T₁ = \frac{d₃}{2000} \cdot Fₙ \cdot \cos αₙ \cdot \left( \sin γₘ + μ₂ \cdot \cos γₘ \right)$$ (2)
The friction coefficient $\mu_z$ between the tooth surfaces is calculated according to DIN 3996 [1] and assumed to be constant in the flank contact of the FEM model. For realistic modeling, a fixed/loose bearing arrangement with a double-row angular contact ball bearing (A) at $z = l_1$ as fixed and a single-row deep groove ball bearing (B) at $z = 0$ as loose support is used with symmetrical arrangement of the bearings and a spacing of $l_1 = 135\text{mm}$. To determine the stiffness of each roller bearing, the reaction forces of the bearings were calculated in a first step using FEM and assuming undeformable bearings. In a second step, the bearing stiffness of each bearing were calculated according to ISO/TS 16281 [11] using the reaction forces from the first step. The gear housing stiffness is neglected in the calculation. The schematic of the model is shown in Figure 3, the introduced load, friction coefficient and bearing stiffnesses are summarized in Table 2:

### Table 2. Load and boundary conditions

| Größe | unit | S07   | S09   |
|-------|------|-------|-------|
| $T_1$ | [Nm] | 49,4  | 54,1  |
| $\mu_z$ | [-]  | 0,1048| 0,1057|
| $cAx$ | [Nmm$^{-1}$] | 174.467| 178.546|
| $cAy$ | [Nmm$^{-1}$] | 174.466| 178.551|
| $cAz$ | [Nmm$^{-1}$] | 272.811| 272.981|
| $cBx$ | [Nmm$^{-1}$] | 69.912 | 73.504|
| $cBy$ | [Nmm$^{-1}$] | 69.385 | 73.024|

The components are simulated as deformable bodies made of the commonly used reference materials 16MnCr5 (worm) and 12SnCu-C-GZ (wheel) according to DIN 3996. At the worm wheel, a certain amount of plasticization of the material is expected under load due to the small yielding strength, Young’s modulus and Poisson Ratio of bronze. To consider this effect, a bilinear isotropic hardening model as shown in Figure 4 is used. It allows to approximate the plastic deformation of the material above the yield strength by expanding the stress-strain curve above the yield point with the tangential modulus $E_T$. The material data used in the simulations is summarised in Table 3:

### Table 3. Material properties according to [12] and [13]

| unit | 16MnCr5 | 12CuSn-C-GZ |
|------|---------|-------------|
| $E$  | [Nmm$^{-2}$] | 206.000,00 | 88.300,00 |
| $E_T$ | [Nmm$^{-2}$] | 1.000,00 | 500,00 |
| $\nu$ | [-]  | 0,30 | 0,35 |
| $\sigma_B$ | [Nmm$^{-2}$] | 1.000,00 | 280,00 |
| $\sigma_S$ | [Nmm$^{-2}$] | 695,00 | 150,00 |

#### Figure 3. Schematic of the FEM model

#### Figure 4. Bilinear isotropic hardening

### 3.2. Results of the FEM-calculations

The result data from the FEM calculations for all analysed engagement positions are evaluated and further processed in post-processing tools. The comparison with established calculation programs such as SNETRA [7] provides very good agreements. The contact lines between the flanks are determined for each contact position by calculating a regression line from the contact elements with pressure $p > 0$ MPa and projected into the xy-plane. Secondly, the load distribution over the contact lines is calculated from the pressure transmitted by the surface contact elements and the associated element...
area, depending on the mesh size. In the model, modifications of the tooth thickness leads to a changed load transmission in the contact for both worm geometries as shown in Table 4:

**Table 4. Load transfer between worm and wheel flanks on the contact lines in [N/mm]**

| q = 7 | xs = 0.00 | xs = -0.25 |
|-------|-----------|------------|
| ![Graph](image1.png) | ![Graph](image2.png) | ![Graph](image3.png) |

The tooth modification causes locally increased line loads in the run-out area and a shift in direction of the tip diameter which, according to the model, would have a negative effect on the wear of the flanks and thus the service life of the gearing. The changes may also have a minor effect on the deflection of the worm due to the shifted load transmission. The smaller gear geometries of the worms with diameter factor q = 7 show increased line loads and stronger gradients than the ones with q = 9.

The bending lines of the worm shafts are evaluated as the directional displacement in x and y of the mesh nodes on the worm axis. The displacement in axial direction z is not considered. In Figure 5, the bending lines of the worm shafts for different engagement positions in the various planes and in three dimensions are summarized. In direction of the tangential force (xz-plane), the bending line forms symmetrically with the maximum almost exactly between the bearing points while in the yz-plane, it is significantly shifted in direction of the axial force in z-direction caused by the lever arm to the x-axis. The displacement of the bearings is more relevant for higher diameter factors as the maximum bending value decreases. The xy-plane shows the spatially curved course of the bending line as a result due to different bearing displacements which is very similar for all simulated geometries. Due to the changing load transmission, the change of the force reactions on the bearings and the variable stiffness of the helically-shaped toothing, all simulations show a dependency of the worm angle φ₁. To quantify those influences on the bending lines, the maximum deflection values and their corresponding locations on the z-axis as well as the amplitudes between the highest and the lowest maximum value are calculated in the xz- and the yz-plane and summarised in Table 5. The maximum three-dimensional value is defined as the maximum of all values δₓᵧ calculated from δₓ and δᵧ for every z coordinate by δₓᵧ = \sqrt{δₓ(z)^2 + δᵧ(z)^2}. The location of the maximum in z is not necessarily the same as for the deflection values in x- and y-direction. All results show a good correlation to the deflection results calculated with DIN 3996. For higher diameter factors, the deviation increases due to the consideration of the bearing stiffness in FEM.
Figure 5. Bending lines in the xz-, yz- and the xy-plane and in 3 dimensions as calculated with FEM

All simulations show a shifting of the maximum value of the bending lines in direction of the axial force as well as a dependence of the course on the angle $\phi_1$. For smaller diameter factors, considering the modification of the tooth thickness is more important while for higher diameter factors the displacement of the bearings is more relevant. The maximum deflection values calculated according to DIN 3996 [1] are the same for standard and modified geometries which is highly inaccurate.

Table 5. FEM-Results compared to DIN3996

|                  | S0700 | S0725 | S0900 | S0925 |
|------------------|-------|-------|-------|-------|
| $\delta_y,_{\text{max}}$ | [mm] | 0,0941 | 0,1190 | 0,0375 | 0,0394 | 0,1190 | 0,1131 | 0,0366 | 0,0391 | 0,1290 | 0,1249 | 0,0378 | 0,0398 |
| $\delta_z,_{\text{max}}$ | [mm] | 85,0   | 69,0   | 64,0   | 65,0   | 85,0   | 69,0   | 64,0   | 65,0   | 77,0   | 77,0   | 77,0   | 77,0   |
| $\delta_x,_{\text{max}}$ | [mm] | 0,1379 | 0,1609 | 0,0507 | 0,0538 | 0,1379 | 0,1609 | 0,0507 | 0,0538 | 0,1609 | 0,1609 | 0,0507 | 0,0507 |
| $\delta_{xy,\text{max}}$ | [mm] | 0,1479 | 0,1479 | 0,0331 | 0,0331 | 0,1479 | 0,1479 | 0,0331 | 0,0331 | 0,1479 | 0,1479 | 0,0331 | 0,0331 |
| $\delta_{y,\text{max}} / \delta_{m}$ | [-] | 0,9324 | 1,09455 | 1,5317 | 1,6253 | 0,9324 | 1,09455 | 1,5317 | 1,6253 | 0,9324 | 1,09455 | 1,5317 | 1,6253 |
| $\delta_{x,\text{max}}$ | [mm] | 0,0077 | 0,00107 | 0,0014 | 0,0020 | 0,0077 | 0,00107 | 0,0014 | 0,0020 | 0,0077 | 0,00107 | 0,0014 | 0,0020 |
| $\delta_{y,\text{max}}$ | [mm] | 0,0047 | 0,0036 | 0,0011 | 0,0008 | 0,0047 | 0,0036 | 0,0011 | 0,0008 | 0,0047 | 0,0036 | 0,0011 | 0,0008 |
| $\delta_{x,\text{max}}$ | [mm] | 0,0077 | 0,0082 | 0,0013 | 0,0016 | 0,0077 | 0,0082 | 0,0013 | 0,0016 | 0,0077 | 0,0082 | 0,0013 | 0,0016 |

4. Extended analytic approach for calculating the deflection of the worm shaft

In order to calculate the bending line for different worm geometries with higher accuracy and avoiding the extensive effort of FEM, an analytical approach is presented in the following. The influencing factors identified from the FEM results as shown in 3.2. as well as the relevant dimensions are shown in Figure 6 for two rotation angles $\phi_1$. The bending line is defined in the xz- and the yz-plane and is calculated by equation (3):

$$w_{\text{total}} = \sqrt{w_{xz}(z, \phi_1)^2 + w_{yz}(z, \phi_1)^2}$$ (3)
4.1. Influences on the course of the bending line

Due to the changing load transmission in the contact between the flanks, the resulting *focus of the force transmission* changes periodically depending on the engagement position. This may cause a change of the lever arms of the toothing forces, the bending moments as well as the bending lines in different engagement positions. The coordinates of the load transmission centre as shown in Figure 7 were derived from the FEM-results. The offset of the graphs on the left side compared to the ones on the right is due to the different starting angle of the worm in the simulation. For \( \phi_1 = 112.5^\circ \) the FEM-calculation for model S0725 did not converge, hence this value is missing in the graphs.

**Figure 6.** Dimensions and influences on the course of the bending line

**Figure 7.** Coordinates of the force introduction points as a function of the rotation angle, derived from FEM

The *axial tooth force* causes a tilting of the worm shaft and a displacement of the maximum deflection in working direction of the axial force in the yz-plane as shown in 3.2. due to the eccentric load transmission relative to the worm axis with an average of \( d_{el1}/2 \). Although it is not significant in the xz-plane, the influence of the axial force is taken into account for both directions in the presented approach. Roller bearings exhibit compliance under load due to their geometric design and the elasticity of the materials. Depending on the *stiffnesses of the bearings*, the displacement may be in a similar range like the deflection of the shaft. Since the displacement of the bearings of the worm shaft additionally reduces the actual bending, it should be considered. For the calculation, the bearings supporting the worm wheel are considered as non-deformable due to their relatively larger size.
compared to the bearings on the worm shaft. The displacements $u$ of the bearings in $x$- and $y$-direction due to the reaction forces are summarised in matrix $U_L$ and calculated by equation 4:

$$
U_L = F_L \cdot c_L = \begin{bmatrix}
F_{Ax} \cdot c_{L,Ax} & F_{Bx} \cdot c_{L,Bx} \\
F_{Ay} \cdot c_{L,Ay} & F_{By} \cdot c_{L,By}
\end{bmatrix}
$$

According to Norgauer [5], the surface moment of inertia of the worm teeth in the face section can be approximated by a calculation from the tool shape. All dimensions of the tooth form in the face section, which is approximated as a trapezoid, are calculated from the geometric diameters of the worm, the tooth thickness, the axial modulus, the number of teeth as well as the pressure angle. The dimensions of the trapezoid as shown in Figure 8 (*) are calculated according to Norgauer [5]. The moment of inertia of the cross-section of the worm is calculated by summing the moment of inertia of the root diameter, the rotated tooth areas depending on the number of teeth and the Steiner’s theorem. With an extreme value analysis, Norgauer calculates the worst case of the rotated cross-section of the worm with the smallest moment of inertia representing the whole worm shaft. This consideration still overestimates the stiffening effect since the teeth are wound around the base body of the shaft in a helical shape. The resulting rotation of the helical toofing along the worm axis corresponds to a rotation as a function of the axis coordinate $z$ according to Figure 8:

$$
I = \frac{\pi}{64} \cdot d_{ers}(\varphi_1)^4
$$

The used substitutional diameter $d_{ers}$ is described by a sinus function according to equation (6):

$$
d_{ers} = d_{f1} \cdot k_{stiff} = d_{f1} \cdot \left[ a_i \cdot \sin(b_i \cdot (\varphi_1 - c_i)) + d_i \right]
$$
The amplitude \( a_i \) of the oscillation of the moment of inertia of the worm shaft is calculated by equation (7) for the investigated geometries. It is based on the empiric data calculated with FEM and considers the gearing length compared to the worm shaft length:

\[
a_i = \frac{\min(I)}{\max(I)} \frac{b_1}{l_1} \tag{7}
\]

The period length \( b_1 = 2 \) is constant because the moments of inertia are equal in size every half rotation. The phase shift factor \( c_i \) describes the offset between the moments of inertia around the \( x- \) and \( y- \) axis. Derived from the FEM-results, in \( y \)-direction it is \( c_i = 0 \) and in \( x \)-direction \( c_i = \pi/4 \). The factor \( d_i \) corresponds to the mean moment of inertia of the shaft geometry related to the toothing length \( b_1 \). For the investigated geometries, \( d_i \) is calculated by equation (8):

\[
d_i = 4 \left( d_{f1}^4 \cdot \frac{\pi}{64} + \text{mean}(l_m) \cdot \frac{b_1}{l_1} \cdot \frac{d_{f1}}{d_{a1}} \cdot \frac{r_m}{a_0} \right) \cdot \frac{64}{\pi} \cdot d_{f1}^{-1} \tag{8}
\]

Equations (7) and (8) due meet the characteristics of the investigated geometries evaluated from the FEM analysis but need to be further validated by a more comprehensive study. The equations are likely to be modified in the future to accurately calculate a larger variety of different geometries. The moment of inertia is calculated by summing up every value for every cross-section of the worm shaft divided by the amount \( n_z \) of calculated cross-sections:

\[
l(z) = \sum_{z=0}^{b_1} \left( \sum_{l=1}^{z_1} \left( \frac{I_{x0} + I_{y0}}{2} + \frac{I_{x0} - I_{y0}}{2} \cdot \cos(2 \cdot \varphi_{\text{geom}}) + \left[ \sin(\varphi_{\text{geom}}) \cdot r_{SP} \right] \cdot A_{TP} \right) \right) \cdot n_z^{-1} \tag{9}
\]

The geometric rotation angle is described as a function of the \( z \)-coordinate:

\[
\varphi_{\text{geom}} = \frac{i - 1}{z_1} \cdot 2\pi + \frac{z}{m_x \cdot z_1 \cdot \pi} \cdot 2\pi \tag{10}
\]

To calculate the mean stiffening effect of the toothing, the mean moment of inertia of the worm shaft is further calculated for every rotation angle \( \varphi_i \) divided by the amount \( n_{\varphi_1} \) of calculated rotation angles:

\[
l_m = \frac{\sum_{\varphi_1=0}^{2\pi} l(z)}{n_{\varphi_1}}, \quad \text{with} \quad \varphi_{\text{geom}} = \frac{i - 1}{z_1} \cdot 2\pi + \frac{z}{m_x \cdot z_1 \cdot \pi} \cdot 2\pi + \varphi_1 \tag{11}
\]

In Table 6 the values of the factors \( a \) and \( d \) calculated from the FEM-results are compared to the ones calculated with the analytic approach. Overall, they show a good agreement for the investigated gear geometries.

| Table 6. Accuracy of the analytical approach vs FEM-results |
|-----------------------------------------------------------|
|               | S0700   | S0725   | S0900   | S0925   |
| \( d_{\text{FEM}} \) [-] | 1,2639  | 1,2174  | 1,0954  | 1,0795  |
| \( a_{\text{FEM}} \) [-]  | 0,0207  | 0,0193  | 0,0092  | 0,0099  |
| \( d_{\text{analyt}} \) [-] | 1,2618  | 1,2149  | 1,1329  | 1,1074  |
| \( a_{\text{analyt}} \) [-] | 0,0181  | 0,0190  | 0,0091  | 0,0094  |
| \( d_{\text{dev}} \) [%]   | -0,17   | -0,20   | 3,42    | 2,58    |
| \( a_{\text{dev}} \) [%]    | -12,53  | -1,37   | -1,50   | -5,16   |
4.2. Extended mechanical calculation model
The identified factors are considered in the mechanical model shown in Figure 9 with the tooth force components of the worm calculated according to DIN 3996 [1]. Assuming small deformations, the calculation of the bending line in the xz- and the yz-plane is done by applying the superposition principle. The position of the force introduction is variable in all three directions and described by the unknown functions \( x(\varphi_1) \), \( y(\varphi_1) \) and \( l(\varphi_1) \). The dependence on the angle \( \varphi_1 \) is caused by the changing load transmission. To calculate the internal reactions, the beam is divided into two sections 1 and 2 at the load introduction point:

\[
F_L = \begin{bmatrix} F_{Ax} & F_{Bx} \\ F_{Ay} & F_{By} \\ F_{Az} & F_{Bz} \end{bmatrix} = \begin{bmatrix} (F_{tm1} \cdot l(\varphi_1) - F_{xm1} \cdot x_{SP}(\varphi_1)) - F_{tm1} \cdot l(\varphi_1) & (F_{xm1} \cdot x_{SP}(\varphi_1) - F_{tm1} \cdot l(\varphi_1)) \\ (F_{rm1} \cdot l(\varphi_1) - F_{xm1} \cdot y_{SP}(\varphi_1)) - F_{rm1} & (F_{xm1} \cdot y_{SP}(\varphi_1) - F_{rm1} \cdot l(\varphi_1)) \\ -F_{xm1} & 0 \end{bmatrix}
\]

(12)

For calculating the moment load inside the material, the beam is divided into two sections at the force initiation point. The moments \( M \) in the beam are calculated by:

\[
M = \begin{bmatrix} M_{xz,1} & M_{xz,2} \\ M_{yz,1} & M_{yz,2} \end{bmatrix} = \begin{bmatrix} F_{Ax} \cdot z & F_{Ax} \cdot z + F_{tm1} \cdot (z - l(\varphi_1)) + F_{xm1} \cdot x_{SP}(\varphi_1) \\ F_{Ay} \cdot z & F_{Ay} \cdot z + F_{rm1} \cdot (z - l(\varphi_1)) + F_{xm1} \cdot y_{SP}(\varphi_1) \end{bmatrix}
\]

(13)

Considering the dependence of the moment of inertia \( I \) from the rotational angle \( \varphi_1 \) of the worm shaft as shown in 4.1. the differential equation of the second derivation of the bending line is solved according to [14]:
\[ w'' = -\frac{M}{E \cdot I(\varphi_1)} \]  

By double integrating (14) for each section and plane, the bending line is described by four equations as summarised in the matrix (15):

\[
W_{ges} = \begin{bmatrix} w_{xz,1} & w_{xz,2} \\ w_{yz,1} & w_{yz,2} \end{bmatrix} \cdot E^{-1} \cdot I(\varphi_1)^{-1}
\]  

With the equations for each section in each plane as shown in equations (16) to (19):

\[
w_{xz,1} = \left( -\frac{1}{6} \cdot F_{Ax} \cdot z^3 + c_{1,xz,1} \cdot z + c_{1,xz,2} \right) 
\]

\[
w_{yz,1} = \left( -\frac{1}{6} \cdot F_{Ay} \cdot z^3 + c_{1,yz,1} \cdot z + c_{1,yz,2} \right) 
\]

\[
w_{xz,2} = \left( -\frac{1}{6} \cdot (F_{Ax} + F_{tm1}) \cdot z^3 + \frac{1}{2} \left( F_{tm1} \cdot l(\varphi_1) - F_{xm1} \cdot \frac{x(\varphi_1)}{2} \right) \cdot z^2 + c_{1,xz,3} \cdot z + c_{1,xz,4} \right) 
\]

\[
w_{yz,2} = \left( -\frac{1}{6} \cdot (F_{Ay} + F_{rm1}) \cdot z^3 + \frac{1}{2} \left( F_{rm1} \cdot l(\varphi_1) - F_{xm1} \cdot \frac{y(\varphi_1)}{2} \right) \cdot z^2 + c_{1,yz,3} \cdot z + c_{1,yz,4} \right) 
\]

The equations for the bending lines contain eight unknown integration constants \( c_1 \). They can be determined by formulating a linear equation system from four boundary conditions for each plane. The boundary conditions are derived from the displacement of the bearing points and the continuity in the transition between the sections of the bending line. They are similar in both planes, for the xz-plane those are formulated as follows: \( w_{xz,1}(z=0) = u_{l,Ax}, \ w_{xz,2}(z=l_1) = u_{l,Bx}, \ w_{xz,1}(z=l(\varphi_1)) = w_{xz,2}(z=l(\varphi_1)) \) and \( w_{xz,1}'(z=l(\varphi_1)) = w_{xz,2}'(z=l(\varphi_1)) \). Finally, the resulting bending line is calculated by equation (3).

5. Comparison between the analytic and the FEM results and discussion

The presented analytic approach is used to recalculate the geometries and loads calculated with FEM. In the analytic approach, all identified influencing factors on the bending line are considered. The tooth forces are calculated according to Niemann/Winter the bearing stiffnesses correspond to those from the FEM models. Due to a missing analytical approach for the force application point, it is taken into account on the basis of the evaluation of the FEM results with regard to the center of the load transmission as shown in 4.1. A general approximation as the reference diameter for the force application point may be appropriate, but the intention is to show the potential of the approach.

Table 7 shows the results of the bending lines calculated with FEM (black) and with the analytic approach (red). The calculated results show a good agreement with the ones calculated with FEM but with a slight deviation thus to the complex load transmission between the flanks which cannot be considered analytically. For the diameter factor of 7, the results are on the safe side. The bending lines for the diameter factor of 9 show a good agreement. For all calculations, the displacement of the bearings is smaller in the FEM-calculation than the analytical calculated displacements.
Table 7. Comparison of the bending lines calculated with FEM and analytically

| q = 7 | xs = 0.00 | | xs = -0.25 |
|------|-----------|------|-----------|

6. Parameters, indices and abbreviations
All parameters, indices and abbreviations used in this article are listed in Chyba! Nenalezen zdroj odkazů..

Table 8. Parameters, indices and abbreviations

| Symbol | Unit | Parameter               | Symbol | Unit | Parameter               |
|--------|------|-------------------------|--------|------|-------------------------|
| a      | mm   | Center distance         | α      | °    | Pressure angle          |
| b₁     | mm   | Tooth width of the worm | γₘ     | °    | Lead angle              |
| b₂     | mm   | Tooth width of the wheel| δ(z)   | mm  | Deflection at coordinate z |
| c₁     | -    | Integration constants  | δₐ     | mm  | Amplitude of the deflection |
| dᵟ₁    | mm   | Root diameter           | δₗₘ    | mm  | Maximum deflection      |
| dᵟ₂    | mm   | Reference diameter      | δᵣₘ    | mm  | According to DIN3996   |
| dᵟ₃    | mm   | Tip diameter            | δₓₘ    | mm  | Deflection in x-direction |
| dᵟ₄    | mm   | Outer diameter          | δᵧₘ    | mm  | Deflection in y-direction |
| i      | -    | Gear ratio              | δₓᵧₘ   | mm  | Deflection in three dimensions |
| l₁     | mm   | Distance between bearings A and B | ε      | -   | Elastic strain          |
| l₁₁    | mm   | Distance from bearing A to tooth force | η      | -   | Efficiency factor       |
| mₛ     | mm   | Axial modulus           | μₛ     | -   | Mean tooth coefficient of friction |
| p      | N/mm²| Contact pressure between surfaces | ν      | -   | Poisson ratio           |
| q      | -    | Diameter factor         | σₙ     | N/mm²| Tensile strength        |
| rₛₚ    | mm   | Radius of center of gravity of tooth area | σₛₖ   | N/mm²| Yield strength          |
| Sₘₚₓ   | mm   | Tooth thickness in axial section | φ₁     | °   | Rotation angle of the worm |
| uₛₚₖ   | mm   | Displacement of the bearings | Σ      | °   | Axis cross angle        |
| w      | mm   | Equation of the bending line | w''    | -   | Second derivation of w  |
7. Conclusion
The new developed analytical approach allows a more accurate calculation of the bending line considering the axial force, the load transmission, the bearing displacements as well as the periodically changing moment of inertia without using FEM. By neglecting different input parameters, the level of detail can be chosen by the user. For a general and fast design approach, the method according to DIN 3996 is recommended for gear units with high geometric similarity to the standard reference.

The new calculation method may be used for increased accuracy in the calculation of the load distribution in the tooth contact, for the calculation of geometries deviating from the standard - in particular for efficiency-optimized thin worm shafts, optimized design for bend-critical geometries as well as an approach for the calculation of bending stresses in the worm shaft. Due to the small scope of the FEM-investigation, the results need to be further verified for different geometries and loads. Also, further experimental verification is reasonable.

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