ALEKSIANDROV’S ESTIMATES FOR ELLIPTIC EQUATIONS WITH DRIFT IN A MORREY SPACES CONTAINING $L_d$

HONGJIE DONG AND N.V. KRYLOV

Abstract. In this note, we obtain a version of Aleksandrov’s maximum principle when the drift coefficients are in Morrey spaces, which contains $L_d$, and when the free term is in $L_p$ for some $p < d$.

1. Introduction and main result

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, ..., x^d)$. Define

$$D_i = \partial/\partial x^i, \quad D = (D_i), \quad D_{ij} = D_i D_j, \quad D^2 = (D_{ij}),$$

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad B_r = B_r(0).$$

Let $S$ be the set of symmetric $d \times d$ matrices and, for $\delta \in (0, 1]$, let

$$S_\delta = \{a \in S : \delta |\xi|^2 \leq a^{ij} \xi^i \xi^j \leq \delta^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d\}.$$

We take a measurable $S_\delta$-valued function $a$ on $B_1$ and a measurable $\mathbb{R}^d$-valued function $b$ on $B_1$ and set

$$L = a^{ij} D_{ij} + b^i D_i.$$

The goal of this article is to prove the following result in which $d_0 = d_0(d, \delta) \in (d/2, d)$ is specified later.

**Theorem 1.1.** Let $R \in (0, \infty)$, $d_0 < q < d$, $p \in (d_0 d/q, d)$, $v \in W^2_p(B_R)$ and assume that for any $x \in B_R$ and $\rho \leq 2R$

$$\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} I_{B_R}(y) |b|^q(y) dy \leq \bar{b} \rho^{-q}, \quad (1.1)$$

where $\bar{b} > 0$ is sufficiently small depending only on $d, \delta, p, q$ (independent of $R$). Then in $B_R$ we have

$$v \leq \sup_{\partial B_R} v + N R^{2 - d/p} \|L v\|_{L_p(B_R)}, \quad (1.2)$$

where $N$ depends only on $d, \delta, p$, and $q$.

2010 Mathematics Subject Classification. 35B50, 35B45, 35J15.

Key words and phrases. Aleksandrov’s maximum principle, Morrey spaces.

H. Dong is partially supported by the Simons Foundation, grant # 709545.
The first results about estimates of the maximum of solutions of elliptic equations even not uniformly elliptic in terms of the $L^d$-norms of the free terms belong to A.D. Aleksandrov in 1960, see Theorem 8 in [2], where he already considered $b \in L^d$. The proofs are given in 1963 in [3].

There was considerable interest in reducing $L^d$-norm of the free term to $L^{d_0}$ norm with $d_0 < d$. This was achieved by Cabré [5] for bounded $b$ and by Fok in [6] for $b \in L^{d_0}$. In [12] the second named author allowed $b \in L^d$ and the free term in $L^{d_0}$ with $d_0 < d$. This made it possible to develop in [10] a $W^{2,p}_d$-solvability theory for linear equations with $b \in L^d$ and $p < d$ (without assuming that the $L^d$-norm of $b$ is small or the domain is small).

Applied to fully nonlinear equations we can now treat $W^{2,d_0}_d$-solvability with “the coefficients” of the first order terms in $L^d$ (see [9]). In this paper we are doing the next step by reducing the integrability property of $b$ further down.

**Remark 1.1.** Observe that (1.1) may be satisfied but $b \notin L^d$. There is a hope that once condition (1.1) is satisfied in all balls of fixed radius $R$ (perhaps small) and arbitrary centers and $\rho \leq 2R$, estimate (1.2) will hold for all $R$ with $N$ that also depends on $R$. For $R$ small and, say $b$ with compact support, condition (1.1) is indeed satisfied if $b \in L^d$ because by Hölder’s inequality the left-hand side of (1.1) is less than $N(d)\|b\|_{L^d(B_R)}^d$. It is then tempting to claim that our Theorem 1.1 contains Aleksandrov’s result specified for uniformly nondegenerate equations with bounded $a$ or contains the corresponding result of [11] (see Corollary 3.1 there) However in Aleksandrov’s result and in [11] the constants depend only on the $L^d$-norm of $b$ and not on the rate with which $\|b\|_{L^d(B_R(x))} \to 0$ as $R \to 0$.

On the other hand, one cannot drop the smallness assumption of $b$. For instance, if $R = 1$, $a^{ij} = \delta^{ij}$, $b(x) = -cx/|x|^2$ and $v(x) = 1 - |x|^2$, then $Lu = -2d + 2c$ and if $c \geq d$, $Lu \geq 0$ and (1.2) is false.

It turns out that (1.2) is true for any $u \in W^{2,p}_d(B_1)$ if $c$ is small enough. Indeed, observe that if $|x| \geq 2\rho$ the integral

$$\int_{B_\rho(x)} I_{B_1}(y)|y|^{-q} dy$$

is less than $\rho^{-q}|B_\rho(x)|$ and if $|x| \leq 2\rho$, the said integral is dominated by

$$\int_{B_{3\rho}} |y|^{-q} dy = N(d)\rho^{n-q}.$$

This example has a deep connection with the so-called form-boundedness condition from [7]. We also stress one more time that in this example $b \notin L^d$.

2. Proof of Theorem 1.1

By using change of scale we see that it suffices to concentrate on $R = 1$. However since we also need $B_2$, we keep $R$ free for a while.
For $p \in [1, \infty)$ and $\mu \in (0, d/p)$ introduce Morrey’s space $E_{p,\mu}(B_R)$ as the set of $g \in L_p(B_R)$ such that
\[ \|g\|_{E_{p,\mu}(B_R)} := \sup_{\rho \leq 2R, x \in B_R} \rho^\mu \|g\|_{L_p(B_R, \rho)} < \infty, \]
where $B_{R,\rho}(x) = B_R \cap B_\rho(x)$ and
\[ \|g\|_{L_p(\Gamma)} = \left( \frac{1}{|\Gamma|} \int_\Gamma |g|^p \, dx \right)^{1/p}. \]

Let
\[ E_{p,\mu}^2(B_R) = \{ u : u, Du, D^2 u \in E_{p,\mu}(B_R) \}. \]
and provide $E_{p,\mu}^2(B_R)$ with an obvious norm. The case $R = \infty$ is not excluded and we drop $B_\infty = \mathbb{R}^d$ from our notation.

**Theorem 2.1.** Let $1 < \mu \leq d/p$, $p > 1$. Define $q$ ($> p$) from
\[ \frac{1}{q} = \frac{1}{p} - \frac{1}{\mu p}. \]
Then for any $u \in W^1_p(\mathbb{R}^d)$
\[ \|u\|_{E_{q,\mu p/q}} \leq N(d, p, \mu) \|Du\|_{E_{p,\mu}}. \tag{2.1} \]

Proof. As it follows from Secs. 1, 2, Ch. V of [13], for almost any $x$ we have
\[ |u(x)| \leq N(d) \int_{\mathbb{R}^d} |Du(y)||x - y|^{-d + 1} \, dy. \]
After that (2.1) follows from Theorem 3.1 of [1]. The theorem is proved.

This theorem is quite remarkable because it allows us to estimate higher powers of $u$ compared with the usual Sobolev embedding theorem at the expense of requiring $Du$ be slightly better. It is the first crucial point in the proof of Theorem 1.1. Another one, well expected, is Theorem 2.3.

**Corollary 2.2.** Let $1 < p < q < d$ and $b \in E_{q,1}(B_1)$. Set $\mu = q/p$ Then for any $u \in E_{p,\mu}^2(B_2)$ we have
\[ \|b|Du|\|_{E_{p,\mu}(B_1)} \leq N\|b\|_{E_{q,1}(B_1)}\|u\|_{E_{p,\mu}^2(B_2)}, \]
where the constants $N$ depend only on $d, p, q$.

Proof. Take $x \in B_1$, $\rho \leq 2$, and take $\zeta \in C^\infty_0(\mathbb{R}^d)$ such that $\zeta = 1$ on $B_1$, $\zeta = 0$ outside $B_{2\rho}$, and $|\zeta| + |D\zeta| + |D^2\zeta| \leq N = N(d)$. By using Hölder’s inequality we see that
\[ \rho^\mu \|b|Du||_{L_p(B_{1,\rho}(x))} \leq N \rho \|b\|_{L_q(B_{1,\rho}(x))} \rho^{\mu - 1} \|D(\zeta u)||_{L_{q'}(B_{\rho}(x))}, \]
where $q' = pq/(q - p)$ and the constant $N$ arose because $|B_{1,\rho}(x)|$ is not quite $|B_\rho(x)|$. Furthermore, since $\mu - 1 = \mu p/q'$ and $1/q' = 1/p - 1/(\mu p)$ by Theorem 2.1
\[ \rho^{\mu - 1} \|D(\zeta u)||_{L_{q'}(B_{\rho}(x))} \leq N\|D^2(\zeta u)||_{E_{p,\mu}} \leq N\|u||_{E_{p,\mu}^2(B_2)}. \]
This obviously leads to the desired result.
For \( u'' \in S \) introduce a Pucci function

\[
P(u'') = \sup_{a \in \mathcal{S}_3} \text{tr}(a u'').
\]

In the following by \( d_0 \) we denote the constant called \( n_0 \) in [4] corresponding to the domain \( B_2 \) and the operator \( P(D^2 u) \). Note that \( d_0 = d_0(d, \delta) \in (d/2, d) \).

**Theorem 2.3.** Let \( d_0 < p < q < d \) and set \( \mu = q/p \). Assume that a nonnegative bounded \( b \in E_{q,1}(B_1) \) and \( b = 0 \) outside \( B_1 \). Then there is a \( \tilde{b} = \tilde{b}(d, \delta, q, p) > 0 \) such that if \( \|b\|_{E_{q,1}(B_1)} \leq \tilde{b} \), then for any \( f \in E_{p,\mu}(B_2) \) there exists a unique \( u \in E^2_{p,\mu}(B_2) \cap C(B_2) \) satisfying

\[
P(D^2 u) + b|Du| + f = 0
\]

in \( B_2 \) and equal zero on \( \partial B_2 \). Moreover, we have

\[
\|u\|_{E^2_{p,\mu}(B_2)} \leq N\|f\|_{E_{p,\mu}(B_2)},
\]

where \( N \) depends only on \( d, \delta, q, \) and \( p \).

Proof. The existence and uniqueness of solution follows directly from Theorem 4.1 of [4] due to the assumption that \( b \) is bounded. To prove (2.3) it suffices to observe that by the same Theorem 4.1 of [4]

\[
\|u\|_{E^2_{p,\mu}(B_2)} \leq N\|b\|_{E_{p,\mu}(B_2)} \leq N\|b\|_{E_{p,\mu}(B_2)} + N\|f\|_{E_{p,\mu}(B_2)},
\]

where, thanks to Corollary 2.2, the first term can be absorbed into the left-hand side if \( \|b\|_{E_{q,1}(B_1)} \) is sufficiently small. The theorem is proved.

Now we prove Theorem 1.1 in case \( R = 1 \) when it takes the following form.

**Theorem 2.4.** Let \( d_0 < q < d \), \( p \in (d_0 d/q, d) \), \( v \in W^2_p(B_1) \). Assume that \( b \in E_{q,1}(B_1) \) and \( \|b\|_{E_{q,1}(B_1)} \leq \tilde{b} \), where \( \tilde{b} \) is taken from Theorem 2.3. Then in \( B_1 \) we have

\[
v \leq \sup_{\partial B_1} v + N\|L(\nu - \nu)\|_{L_p(B_1)},
\]

where \( N \) depends only on \( d, \delta, q, \) and \( p \).

Proof. If we replace \( L \) in (2.4) with \( I_{|b|>n} + I_{|b|\leq n} L \) and assume that our assertion is true, then by passing to the limit as \( n \to \infty \) and using the dominated and monotone convergence theorems we obtain (2.4) as is. It follows that we may assume that \( b \) is bounded and then that \( v, b, \) and \( a \) are smooth. After that by subtracting from \( v \) the solution \( w \) of \( Lw = 0 \) in \( B_1 \) with boundary value \( w = v \), we reduce the situation to the one in which \( v = 0 \) on \( \partial B_1 \). Then set \( f = (Lv) - I_{B_1} \), extend \( b \) as zero outside \( B_1 \), and define \( u \) as a solution of (2.2) in \( B_2 \) with zero boundary data. According to [14] or [8] there is such solution \( u \) which belongs to \( W^2_p(B_2) \) for any \( r > 1 \). By the maximum principle, \( u \geq 0 \) in \( B_2 \) and, since \( L(u - v) \leq Lu + f \leq 0 \) in \( B_1 \), again by the maximum principle \( v \leq u \) in \( B_1 \).
Furthermore, in light of Hölder’s inequality, \( u \in E^2_{r,\nu}(B_2) \) for any \( r \in (1, \infty) \) and \( \nu \in (0, d/r) \). Now, by embedding theorems, (2.3), and again Hölder’s inequality, for \( d_0 < r < q \) and \( \nu = q/r \),

\[
    u \leq N \|u\|_{W^2_{r}(B_2)} \leq N \|u\|_{E^2_{r,\nu}(B_2)} \leq N \|f\|_{E^r_{r,\nu}(B_2)} \leq N \|f\|_{L^r_{d/q}(B_2)}.
\]

After that it only remains to note that \( rd/q = p \) for \( r = pq/d \) (\( r \in (d_0, q) \) with \( \nu = q/r = d/p > 1 \)). The theorem is proved.

**Remark 2.1.** As an intermediate result we have proved that if \( v, a, b \) are smooth and \( v = 0 \) on \( \partial B_1 \), then

\[
    v \leq N \|(Lv) - \|_{E^r_{q/p,d/p}(B_1)}.
\]

**References**

[1] D. Adams, *A note on Riesz potentials*, Duke Math. J., Vol. 42 (1975), No. 4, 765-778.

[2] A. D. Aleksandrov, *Certain estimates for the Dirichlet problem*, Dokl. Akad. Nauk SSSR, Vol. 134 (1960), 1001–1004 (Russian); translated as Soviet Math. Dokl., Vol. 1 (1961) 1151–1154.

[3] A. D. Aleksandrov, *Uniqueness conditions and estimates for the solution of the Dirichlet problem*, Vestnik Leningrad. Univ., Vol. 18 (1963), No. 3, 5-29 in Russian; English translation in Amer. Mat. Soc. Transl., Vol. 68 (1968), No. 2, 89-119.

[4] S.S. Byun, M. Lee, and D.K. Palagachev, *Hessian estimates in weighted Lebesgue spaces for fully nonlinear elliptic equations*, J. Differential Equations, Vol. 260 (2016), No. 5, 4550-4571.

[5] X. Cabré, *On the Aleksandrov-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., Vol. 48 (1995), 539–570.

[6] K. Fok, *A nonlinear Fabes-Stroock result*, Comm. PDEs, Vol 23 (1998), No. 5-6, 967–983.

[7] D. Kinzebulatov and Yu.A. Semenov, *Brownian motion with general drift*, Stoch. Proc. Appl., Vol. 130 (2020), No. 5, 2737-2750.

[8] N.V. Krylov, *“Sobolev and viscosity solutions for fully nonlinear elliptic and parabolic equations”*, Mathematical Surveys and Monographs, 233, Amer. Math. Soc., Providence, RI, 2018.

[9] N.V. Krylov, *Linear and fully nonlinear elliptic equations with \( L_d \)-drift*, Comm. PDE, Vol. 45 (2020), No. 12, 1778–1798.

[10] N.V. Krylov, *Elliptic equations with VMO a, b \( \in \ L_d \), and c \( \in \ L_{d/2} \)*, Trans. Amer. Math. Sci., Vol. 374 (2021), No. 4, 2805-2822.

[11] N.V. Krylov, *On stochastic equations with drift in \( L_d \)*, arXiv:2001.04008.

[12] N.V. Krylov, *On stochastic Itô processes with drift in \( L_d \)*, arXiv:2001.03660.

[13] E. Stein, *“Singular integrals and differentiability properties of functions”*, Princeton University Press, Princeton, NJ, 1970.

[14] N. Winter, *\( W^{2,p} \) and \( W^{1,p} \)-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, Z. Anal. Anwend., Vol. 28 (2009), No. 2, 129–164.

(H. Dong) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

Email address: Hongjie_Dong@brown.edu

(N. V. Krylov) 127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455

Email address: nkrylov@umn.edu