Emptiness formation probability in the domain-wall six-vertex model

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Abstract. The emptiness formation probability in the six-vertex model with domain wall boundary conditions is considered. This correlation function allows one to address the problem of limit shapes in the model. We apply the quantum inverse scattering method to calculate the emptiness formation probability for the inhomogeneous model. For the homogeneous model, the result is given both in terms of certain determinant and as a multiple integral representation.

1. Introduction

A special kind of fixed boundary conditions, the so-called domain wall boundary conditions, was first considered for the six-vertex model by Korepin in seminal paper [1]. In [2], Izergin showed that the partition function of the model on the finite lattice can be found exactly in terms of certain determinant; see also paper [3] for details. It was later shown in [4, 5], by studying the thermodynamic limit, that the free energy per site is different with respect to the case of periodic boundary conditions. This fact hints at spatial separation of phases (e.g., ferroelectric order and disorder), which is confirmed both numerically [6–8] and analytically [9, 10].

To get some details about the phase separation phenomena, e.g., to find the shape of the spatial curve separating the phases, or limit shape, one has to know some appropriate correlation function. The problem of computing the correlation functions in the ‘domain-wall’ six-vertex model has been addressed in papers [11–14] where some correlation functions near the boundary were found.

In this paper, we continue the study of correlation functions of the six-vertex model with domain wall boundary conditions. Specifically, we consider here a particular non-local correlation function, the emptiness formation probability (EFP). This function describes the probability of having a set of consecutive horizontal edges along a given column, all in a given state; we consider here the case when the set starts from the top boundary and extends inside the lattice. This correlation function allows one to address the problem of limit shapes in the model [15].

To compute EFP, we follow the lines of papers [12, 14, 16] where the quantum inverse scattering method (QISM) [17, 18] and some facts from the theory of orthogonal polynomials were used. Mostly following ideas (as well as notations) of [14], we represent EFP in certain determinantal form, which is shown here to be also equivalent to some multiple integral. This last representation recalls analogous
multiple integral representations for correlation functions of quantum spin chains [19–22].

The paper is organized as follows. In the next Section we start with giving some definitions and fixing some notations. The quantum inverse scattering method in application to the model is considered in Section 3. The core calculation of EFP for the inhomogeneous model is contained in Section 4. The homogeneous limit is performed in Section 5, where the main result is given both in terms of certain determinant and as a multiple integral representation. Section 6 is devoted to discussion of equivalent multiple integral representations for EFP.

2. Some definitions and notations

2.1. The model. The six-vertex model is a statistical mechanics model in which the local states are associated with edges of a square lattice, and the Boltzmann weights are assigned to its vertices. The states can take two values, which are often denoted by arrows pointing along the edge. Among the sixteen possible arrow configurations around a vertex only six are allowed (having nonzero Boltzmann weights), with equal number of incoming and outgoing arrows. In this paper we consider the model on a lattice having both \( N \times N \) lattice with the boundary states fixed in a special way: all arrows on the left and right boundaries are outgoing while on the top and bottom boundaries all arrows are incoming. Such a model is called the six-vertex model with domain wall boundary conditions.

In the six-vertex model with invariance under reversal of all arrows there are three possible values for Boltzmann weights at each vertex, usually denoted as \( a, b, \) and \( c \). To use the quantum inverse scattering method (QISM) for calculations we will consider the inhomogeneous version of the model, in which the weights of the vertex being at the intersection of \( k \)-th horizontal line and \( \alpha \)-th vertical line are

\[
a_{\alpha k} = a(\lambda_{\alpha}, \nu_k), \quad b_{\alpha k} = b(\lambda_{\alpha}, \nu_k), \quad c_{\alpha k} = c,
\]

where

\[
a(\lambda, \nu) = \sin(\lambda - \nu + \eta), \quad b(\lambda, \nu) = \sin(\lambda - \nu - \eta), \quad c = \sin(2\eta)
\]

and we enumerate vertical lines (labelled by Greek indices) from right to left, and horizontal lines (labelled by Latin indices) from top to bottom. The parameters \( \lambda_1, \ldots, \lambda_N \) are assumed to be all different; the same is assumed about \( \nu \)'s. After applying QISM we set these parameters equal within each set, \( \lambda_{\alpha} = \lambda \) and \( \nu_k = \nu \) and, without losing generality, we can assume that \( \nu = 0 \). In this way we obtain the homogenous model; the indicated procedure will be referred to as homogeneous limit.

The partition function of the inhomogeneous model is defined in a standard way as the sum over all possible configurations, each configuration being assigned its Boltzmann weight, which is the product of all vertex weights over the lattice,

\[
Z_N = \sum_C \prod_{\alpha=1}^{N} \prod_{k=1}^{N} w_{\alpha k}(C).
\]

Here \( w_{\alpha k}(C) \) takes values \( w_{\alpha k}(C) = a_{\alpha k}, b_{\alpha k}, c_{\alpha k} \), depending on the configuration \( C \). Because of (2.1), \( Z_N = Z_N(\lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N) \) where \( \lambda \)'s and \( \nu \)'s are
regarded as ‘variables’; \( \eta \) is regarded as a parameter (having the meaning of a ‘coupling constant’) and it is often omitted in notations. In QISM the dependence on \( \lambda \)'s and \( \nu \)'s play an important role (in particular, \( Z_N \) is invariant under permutations within each set of variables).

### 2.2. QISM formulation

We now define the main objects of QISM in relation to the model. First, let us consider vector space \( \mathbb{C}^2 \) and denote its basis vectors as the spin-up and spin-down states

\[
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

To each lattice row and column we associate vector space \( \mathbb{C}^2 \). We also use the convention that upward and right arrows correspond to the ‘spin up’ state while downward and left arrows correspond to the ‘spin down’ state.

Next, let us introduce the quantum \( L \)-operator, which can be defined as a matrix of the Boltzmann weights. Namely, to each vertex being intersection of the \( \alpha \)-th vertical line (column) and the \( k \)-th horizontal line (row) we associate the operator \( L_{\alpha k}(\lambda_\alpha, \nu_k) \) which acts in the direct product of two vector spaces \( \mathbb{C}^2 \): in the ‘horizontal’ space \( \mathcal{H}_k = \mathbb{C}^2 \) (associated with the \( k \)-th row) and in the ‘vertical’ space \( \mathcal{V}_\alpha = \mathbb{C}^2 \) (associated with the \( \alpha \)-th column). We regard arrow states on the top and right edges of the vertex as ‘in’ indices of the \( L \)-operator while those on the bottom and left edges as ‘out’ ones. Explicitly, the \( L \)-operator reads

\[
L_{\alpha k}(\lambda_\alpha, \nu_k) = \sin(\lambda_\alpha - \nu_k + \eta \sigma^z_k) \sin(2\eta)(\tau^-_\alpha \sigma^z_k + \tau^+_\alpha \sigma^-_k).
\]

Here \( \tau \)'s (\( \sigma \)'s) are Pauli matrices of the corresponding vertical (horizontal) vector spaces.

Further, we introduce the monodromy matrix, which is an ordered product of \( L \)-operators. We define monodromy matrix here as a product of \( L \)-operators along a column, regarding the corresponding vertical space \( \mathcal{V}_\alpha \) as an ‘auxiliary’ space. The horizontal spaces \( \mathcal{H}_k \) will be regarded as ‘quantum’ spaces; the space \( \mathcal{H} = \bigotimes_{k=1}^N \mathcal{H}_k \) is therefore the total quantum space. In defining the monodromy matrix it is convenient to think of \( L \)-operator as acting in \( \mathcal{V}_\alpha \otimes \mathcal{H} \) and, moreover, writing it as 2-by-2 matrix in \( \mathcal{V}_\alpha \), with the entries being quantum operators (acting in \( \mathcal{H} \)),

\[
L_{\alpha k}(\lambda_\alpha, \nu_k) = \begin{pmatrix} \sin(\lambda_\alpha - \nu_k + \eta \sigma^z_k) & \sin(2\eta)(\tau^-_\alpha \sigma^z_k + \tau^+_\alpha \sigma^-_k) \\ \sin(2\eta)\sigma^-_k & \sin(\lambda_\alpha - \nu_k - \eta \sigma^z_k) \end{pmatrix}_{[\mathcal{V}_\alpha]}.
\]  \hspace{1cm} (2.3)

Here the subscript indicates that this is a matrix in \( \mathcal{V}_\alpha \) and \( \sigma^l_k \) (\( l = +, -, z \)) denote quantum operators in \( \mathcal{H} \) acting as Pauli matrices in \( \mathcal{H}_k \) and identically elsewhere. The monodromy matrix is defined as

\[
T_{\alpha}(\lambda_\alpha) = L_{\alpha N}(\lambda_\alpha, \nu_N) \cdots L_{\alpha 2}(\lambda_\alpha, \nu_2) L_{\alpha 1}(\lambda_\alpha, \nu_1) = \begin{pmatrix} A(\lambda_\alpha) & B(\lambda_\alpha) \\ C(\lambda_\alpha) & D(\lambda_\alpha) \end{pmatrix}_{[\mathcal{V}_\alpha]}.
\]

The operators \( A(\lambda) = A(\lambda; \nu_1, \ldots, \nu_N) \), etc, act in \( \mathcal{H} \). They play an important role in QISM.

Operators \( A(\lambda), B(\lambda), C(\lambda), \) and \( D(\lambda) \), admit simple graphical interpretation as columns of the lattice, with top and bottom arrows fixed. Let us introduce ‘all
spins down’ and ‘all spins up’ states
\[ |\uparrow\rangle = \bigotimes_{k=1}^{N} |\uparrow\rangle_k, \quad |\downarrow\rangle = \bigotimes_{k=1}^{N} |\downarrow\rangle_k, \] (2.4)
where \(|\uparrow\rangle_k\) and \(|\downarrow\rangle_k\) are basis vectors of \(\mathcal{H}_k\). In the case of domain wall boundary conditions each column corresponds to an operator \(B(\lambda_\alpha)\) (where \(\alpha\) is the number of the column) while vectors (2.4) describe states on the right and left boundaries; the partition function reads:
\[ Z_N = \langle \downarrow | B(\lambda_N) \cdots B(\lambda_2) B(\lambda_1) | \uparrow \rangle. \]

2.3. Izergin-Korepin formula. In [1] Korepin established recursion relations for the partition function, and in [2] Izergin showed that these relations are satisfied by the following explicit expression (see also [3] for details)
\[ Z_N = \prod_{\alpha=1}^{N} \prod_{k=1}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k) \det \mathcal{M}, \] (2.5)
where \(\mathcal{M}\) is \(N\)-by-\(N\) matrix with entries
\[ \mathcal{M}_{\alpha k} = \varphi(\lambda_\alpha, \nu_k), \quad \varphi(\lambda, \nu) = \frac{c}{a(\lambda, \nu)b(\lambda, \nu)}, \] (2.6)
while \(a(\lambda, \nu)\), \(b(\lambda, \nu)\) and \(c\) are defined in (2.2), and function \(d(\lambda, \lambda')\), standing in the pre-factor of (2.5), is
\[ d(\lambda, \lambda') := \sin(\lambda - \lambda'). \]

In the next Section we sketch a proof of (2.5), originally given in [12], which uses exclusively the Yang-Baxter algebra. The method is useful since it can be generalized to the case of correlation functions (in contrast to the original approach of Korepin and Izergin, [3]).

In the homogeneous limit, i.e., when \(\lambda_\alpha = \lambda\) and \(\nu_k = 0\), expression (2.5) simplifies to
\[ Z_N = \frac{[\sin(\lambda - \eta) \sin(\lambda + \eta)]^{N^2}}{\prod_{n=1}^{N-1} (n!)^2} \det \mathcal{N} \] (2.7)
where \(N\)-by-\(N\) matrix \(\mathcal{N}\) has entries
\[ \mathcal{N}_{\alpha k} = \partial^2_{\lambda} \varphi(\lambda), \quad \varphi(\lambda) := \varphi(\lambda, 0) = \frac{\sin(2\eta)}{\sin(\lambda - \eta) \sin(\lambda + \eta)}. \]
Expression (2.7) was given for the first time in [2]; the derivation of (2.7) from (2.5) was explained in detail in [3].

2.4. Emptiness formation probability. Let us denote by \(F_N^{(r,s)}\) the probability of having all arrows on \(s\) first horizontal edges (counted, as usual, from the top of the lattice) between \(r\)-th and \((r+1)\)-th columns, to be all pointing left. Using operator formalism we can define this probability as
\[ F_N^{(r,s)} = Z^{-1} \langle \downarrow | B(\lambda_N) \cdots B(\lambda_{r+1}) \pi_1 \cdots \pi_s B(\lambda_r) \cdots B(\lambda_1) | \uparrow \rangle. \] (2.8)
Here \(\pi_j\) denotes the projector on the spin-down state (which correspondingly fixes the arrow to be pointing left),
\[ \pi_j = \frac{1}{2} (1 - \sigma^z_j). \]
We shall call correlation function (2.8) as emptiness formation probability (EFP), adopting the name of similar object from the quantum spin chain context [18].

In this respect we comment that the name ‘emptiness formation probability’ for the quantity defined by formula (2.8) has to understood with some care, since it is in fact corresponds to a different object. Indeed, if one regards, following the common practice in QISM, the vector $|⇑⟩$ as an ‘empty’ state and $B$’s as ‘creation’ operators over this state, then our definition actually corresponds rather to some ‘fullness formation probability’. Nevertheless, we shall follow the tradition and call this quantity EFP.

One can consider, instead of EFP defined by (2.8), the true emptiness formation probability, defining it by replacing all $π$’s in (2.8) by $\bar{π}_j = 1 - π_j$. However, contrarily to the standard situation in quantum spin chains in absence of external field, such quantity cannot be related to EFP defined by (2.8). This is due to the fact that in the domain-wall six-vertex model the local polarization is non-vanishing almost everywhere over the lattice, or, in other words, the spin-reversal symmetry is broken by the boundary conditions.

Our choice of correlation function (2.8) is motivated by its further application to study limit shapes of the model. Indeed, because of peculiarity of both the domain-wall boundary conditions and the six-vertex model rule of conservation of incoming and outgoing arrows through each lattice vertex, EFP (2.8) actually measures the probability that all vertices in the top-left $(N - r) \times s$ sublattice have the same configuration of arrows, namely, all arrows point to the left or downwards. Hence EFP measures ferroelectric order or ‘freezing’ of states. The limit shape arises in some appropriate scaling limit and corresponds to some curve where EFP jumps from one to zero as the size of this $(N - r) \times s$ sublattice increases (see [15] for further details).

3. QISM and recurrence relations

3.1. Yang-Baxter algebra. One of the most basic relations of QISM is the so-called “RLL” relation [18, 23, 24], which reads

$$ R_{αα′}(λ, λ′)[L_{αk}(λ, ν) \otimes L_{α′k}(λ′, ν)] = [L_{αk}(λ′, ν) \otimes L_{α′k}(λ, ν)] R_{αα′}(λ, λ′). $$

Here $R_{αα′}(λ, λ′)$, called the $R$-matrix, is a matrix acting in the direct product of two auxiliary vector spaces, $V_α \otimes V_{α′}$, and it can be conveniently represented as a 4-by-4 matrix (we assume that the first space refers to the 2-by-2 blocks, while the second one to the entries in the blocks):

$$ R_{αα′}(λ, λ′) = \begin{pmatrix} f(λ′, λ) & 0 & 0 & 0 \\ 0 & g(λ′, λ) & 1 & 0 \\ 0 & 1 & g(λ′, λ) & 0 \\ 0 & 0 & 0 & f(λ′, λ) \end{pmatrix}_{[V_α \otimes V_{α′}]}.$$

Here the functions $f(λ′, λ)$ and $g(λ′, λ)$ are

$$ f(λ′, λ) = \frac{\sin(λ - λ′ + 2η)}{\sin(λ - λ′)}, \quad g(λ′, λ) = \frac{\sin(2η)}{\sin(λ - λ′)}.$$

This $R$-matrix is also sometimes referred to as XXZ chain $R$-matrix, due to relation of the six-vertex model with Heisenberg XXZ quantum spin chain. It is to be
mentioned that here and below we are mainly following notations and conventions of book [18].

The importance of the RLL relation above is that it implies the following relation, which, in turn, can be called RTT relation,

\[ R_{\alpha\alpha'}(\lambda, \lambda') [T_{\alpha}(\lambda) \otimes T_{\alpha'}(\lambda')] = [T_{\alpha}(\lambda') \otimes T_{\alpha'}(\lambda)] R_{\alpha\alpha'}(\lambda, \lambda'). \] (3.1)

This relation contains all commutation relations between the operators \( A(\lambda) \), \( B(\lambda) \), \( C(\lambda) \), and \( D(\lambda) \). The algebra of these operators is called the Yang-Baxter algebra, or the quantum algebra of monodromy matrix. Among the sixteen relations contained in (3.1) the following two of them will be used explicitly below, namely,

\[ B(\lambda) B(\lambda') = B(\lambda') B(\lambda). \] (3.2)

and

\[ A(\lambda) B(\lambda') = f(\lambda, \lambda') B(\lambda') A(\lambda) + g(\lambda', \lambda) B(\lambda) A(\lambda'). \] (3.3)

3.2. The ‘two-site model’. Let us consider the following decomposition of the monodromy matrix

\[ T(\lambda) = T_2(\lambda) T_1(\lambda), \]

where \( T_1(\lambda) \) is defined as a product of the first several \( L \)-operators while \( T_2(\lambda) \) is the product of the remaining ones. Such a decomposition is sometimes called ‘two-site model’ [18]. We shall consider here the case when \( T_1(\lambda) \) consists of just one \( L \)-operator,

\[ T_2(\lambda) = L_{\alpha_1 N}(\lambda, \nu_N) \cdots L_{\alpha_2}(\lambda, \nu_2), \quad T_1(\lambda) = L_{\alpha_1}(\lambda, \nu_1). \] (3.4)

Introducing the operators \( A_1(\lambda) \), \( A_2(\lambda) \), etc, as operator-valued entries of the corresponding monodromy matrices \( T_1(\lambda) \), \( T_2(\lambda) \), respectively, taking into account that

\[ B(\lambda) = A_2(\lambda) B_1(\lambda) + B_2(\lambda) D_1(\lambda), \]

and using (2.3), we have

\[ B(\lambda) = A_2(\lambda) c \sigma_1^{-1} + B_2(\lambda) [a(\lambda, \nu_1) \pi_1 + b(\lambda, \nu_1) \bar{\pi}_1] \]

\[ = \begin{pmatrix} b(\lambda, \nu_1) B_2(\lambda) & 0 \\ c A_2(\lambda) & a(\lambda, \nu_1) B_2(\lambda) \end{pmatrix} \] \[ \in \mathcal{H}_1 \].

The lower-triangle structure of operator \( B(\lambda) \) as a matrix in \( \mathcal{H}_1 \) leads to the property that the product of several \( B \)'s has the form

\[ B(\lambda_n) \cdots B(\lambda_1) = \begin{pmatrix} E_{11}(\lambda_1, \ldots, \lambda_n) & 0 \\ E_{21}(\lambda_1, \ldots, \lambda_n) & E_{22}(\lambda_1, \ldots, \lambda_n) \end{pmatrix} \] \[ \in \mathcal{H}_1 \]. (3.5)

The diagonal entries \( E_{11}(\lambda_1, \ldots, \lambda_n) \), \( E_{22}(\lambda_1, \ldots, \lambda_n) \) are simply proportional to \( B_2(\lambda_n) \cdots B_2(\lambda_1) \), while the non-diagonal entry, \( E_{21}(\lambda_1, \ldots, \lambda_n) \), reads

\[ E_{21}(\lambda_1, \ldots, \lambda_n) = \sum_{\alpha=1}^n \prod_{\beta=\alpha+1}^n a(\lambda_\beta, \nu_1) \cdot c \cdot \prod_{\beta=1}^{n-1} b(\lambda_\beta, \nu_1) \]

\[ \times B_2(\lambda_n) \cdots B_2(\lambda_{\alpha+1}) A_2(\lambda_\alpha) B_2(\lambda_{\alpha-1}) \cdots B_2(\lambda_1). \] (3.6)

It is to be stressed that due to (3.2) entries in (3.5) are totally symmetric under permutations of \( \lambda \)'s. While this is completely evident for the diagonal entries, such a property is rather non-trivial for expression (3.6), and is a consequence of commutation relation (3.3) for operators \( B_2(\lambda) \) and \( A_2(\lambda) \).
3.3. The key relation. In dealing with the ‘two-site model’ it is useful to consider the corresponding decomposition of the vectors ‘all spins up’ and ‘all spins down’, e.g., \(|\uparrow\rangle = |\uparrow_1\rangle \otimes |\uparrow_2\rangle\). To fit (3.4) we set \(|\uparrow_1\rangle = |\uparrow\rangle_1\) and \(|\uparrow_2\rangle = \otimes_{k=2}^N |\uparrow\rangle_k\). Obviously, we have

\[
\langle \downarrow_1 | B(\lambda_n) \cdots B(\lambda_1) | \uparrow \rangle = E_{21}(\lambda_1, \ldots, \lambda_n) | \uparrow_2 \rangle. \tag{3.7}
\]

Taking into account that

\[
A_2(\lambda) | \uparrow_2 \rangle = \prod_{k=2}^N a(\lambda, \nu_k) | \uparrow_2 \rangle
\]

we can use (3.3) to reduce RHS of (3.7) in terms of \(B_2\)'s only (applied to vector \(|\uparrow_2\rangle\)). The result reads

\[
E_{21}(\lambda_1, \ldots, \lambda_n) | \uparrow_2 \rangle = c n \prod_{\alpha=1}^n b(\lambda_\beta, \nu_1) \prod_{\beta=1}^n f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^N a(\lambda_\alpha, \nu_k) \\
\times B_2(\lambda_\alpha) \cdots B_2(\lambda_{\alpha+1}) B_2(\lambda_{\alpha-1}) \cdots B_2(\lambda_1) | \uparrow_2 \rangle. \tag{3.8}
\]

To get a hint about how this formula can be derived, it is sufficient to look at the term \(\alpha = n\) in (3.6) which is proportional to \(A_2(\lambda_n) B_2(\lambda_{n-1}) \cdots B(\lambda_1)\). This is the only term which contributes to the \(\alpha = n\) term of (3.8) (containing \(B(\lambda_{n-1}) \cdots B(\lambda_1)\)) after applying commutation relation (3.3); moreover, only the first term in RHS of (3.3) contributes to the \(\alpha = n\) term in (3.8). The remaining terms in (3.8) are just due to the total symmetry with respect to permutations of \(\lambda\)’s.

Formulæ (3.7) and (3.8) express a vector containing \(B\)'s in terms of vectors containing \(B_2\)'s, so they can be seen as a recurrence relation with respect to \(N\), the number of lattice sites. Choosing a specific value of \(n\) (the number of \(B\)'s) and specifying also the component of this vector, one can obtain recurrence relations for some important ‘scalar’ quantities, such as partition function and EFP of the domain-wall six-vertex model. In turn, these recurrence relations can be explicitly solved.

3.4. Recurrence relation for \(Z_N\). As an illustration of the approach let us consider here how the recurrence relation for \(Z_N\) can be derived and solved. The recurrence relation emerges when \(n\) is specified to the value \(n = N\), and taking the scalar product of (3.7) with the vector \(|\downarrow_2\rangle\). In this case relation (3.8) gives

\[
Z_N = c N \prod_{\alpha=1}^N b(\lambda_\beta, \nu_1) \prod_{\beta=1}^N f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^N a(\lambda_\alpha, \nu_k) Z_{N-1}[\lambda_\alpha; \nu_1]. \tag{3.9}
\]

Here \(Z_{N-1}[\lambda_\alpha; \nu_1]\) denotes the partition function of the domain-wall six-vertex model on \((N-1) \times (N-1)\) lattice, with the sets \(\lambda\)'s and \(\nu\)'s such that they do not contain \(\lambda_\alpha\) and \(\nu_1\), namely, they are \(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_{\alpha+1}, \ldots, \lambda_N,\) and \(\nu_2, \ldots, \nu_N\), respectively (in other words, the square brackets indicate independence of these variables, in comparison with the ‘original’ sets \(\lambda_1, \ldots, \lambda_N\) and \(\nu_1, \ldots, \nu_N\)).

Relation (3.9) obviously represents a recurrence relation for the partition function with respect to the size of the lattice. The initial condition to the recurrence
is $Z_1 = c$. It is to be emphasized that in (3.9) values of $\lambda$’s and $\nu$’s are completely arbitrary.

In order to prove that Izergin-Korepin formula indeed solves (3.9), one can just substitute (2.5) into both sides of this relation, and verify whether this is an identity or not. After substituting (2.5) into (3.9), and cancelling many factors, one is left with

$$\det \mathcal{M} = \frac{\prod_{k=1}^{N} d(\nu_k, \nu_k)}{\prod_{\alpha=1}^{N} a(\lambda_\alpha, \nu_1)} \sum_{\alpha=1}^{N} (-1)\alpha-1 g(\lambda_\alpha) \det \mathcal{M}_{[\alpha;1]}.$$  

(3.10)

Here $\mathcal{M}_{[\alpha;1]}$ denotes the $(N-1) \times (N-1)$ matrix obtained from matrix $\mathcal{M}$ (2.6) by removing $\alpha$-th row and the first column. The function $g(\lambda)$ is defined by

$$g(\lambda) := \frac{\prod_{\alpha=1}^{N} e(\lambda_\alpha, \lambda)}{\prod_{k=1}^{N} b(\lambda, \nu_k)},$$  

(3.11)

where

$$e(\lambda, \lambda') := \sin(\lambda - \lambda' + 2\eta).$$  

(3.12)

It is useful to note that the sum in (3.10) can be written as the determinant

$$\begin{vmatrix} g(\lambda_1) & \varphi(\lambda_1, \nu_2) & \ldots & \varphi(\lambda_1, \nu_N) \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ g(\lambda_N) & \varphi(\lambda_N, \nu_2) & \ldots & \varphi(\lambda_N, \nu_N) \end{vmatrix}.$$  

The main point is that for each value $\lambda = \lambda_\alpha$ ($\alpha = 1, \ldots, N$) the function $g(\lambda) = g(\lambda; \lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N)$, where $\lambda_1, \ldots, \lambda_N$ and $\nu$’s are to be regarded as parameters, can be represented as follows

$$g(\lambda_\alpha) = \sum_{k=1}^{N} \varphi(\lambda_\alpha, \nu_k) \frac{\prod_{\alpha=1}^{N} a(\lambda_\alpha, \nu_k)}{\prod_{j \neq k}^{N} d(\nu_k, \nu_j)}.$$  

(3.13)

Noting that the ratio of the two products here does not depend on $\alpha = 1, \ldots, N$, it is easy to see that relation (3.10) is indeed fulfilled.

The only nontrivial point in this derivation is identity (3.13) which can be proven by various methods (e.g., by induction). Let us mention here a hint (even if not completely rigorous) on how such kind of relations can be deduced: at values $\lambda = \nu_k + \eta$ ($k = 1, \ldots, N$) function $g(\lambda)$ has simple poles; the sum in RHS of (3.13) is nothing but the sum over these poles, in analogy with meromorphic function expansion.

4. Calculation of EFP

4.1. Recurrence relation. Let us denote

$$\bar{F}^{(r,s)}_N := Z_N F^{(r,s)}_N,$$

that is, we will consider temporarily the sole matrix element of (2.8), or the ‘numerator’ of the correlation function. Choosing in (3.7) $n = r$ and taking the scalar product of this vector with $\langle \psi_2 | B_2(\lambda_N) \cdots B_2(\lambda_r+1) \rangle$, one can easily see that (3.8)
implies the following recurrence relation

\[ F^{(r,s)}_N = \prod_{\beta=r+1}^{N} a(\lambda_\beta, \nu_1) c \sum_{\alpha=1}^{r} \prod_{\beta=1, \beta \neq \alpha}^{r} b(\lambda_\beta, \nu_1) \prod_{\beta=1, \beta \neq \alpha}^{r} f(\lambda_\alpha, \lambda_\beta) \prod_{k=2}^{N} a(\lambda_\alpha, \nu_k) \]

\[ \times F^{(r-1,s-1)}_{N-1} [\lambda_\alpha; \nu_1]. \] (4.1)

As above, the ‘dependence’ on some \( \lambda_\alpha \) and \( \nu_k \) enclosed in brackets indicates that the sets of \( \lambda \)'s and \( \nu \)'s on which the quantity depends, has \( N-1 \) elements each, with \( \lambda_\alpha \) and \( \nu_k \) missing in the corresponding set (i.e., brackets indicate no dependence on these variables).

In the case \( s = 1 \) EFP describes the ‘boundary’ polarization, which was considered in [12]. For all \( r = 1, \ldots, N \), we have

\[ \tilde{F}^{(r-1,0)}_{N-1} [\lambda_\alpha; \nu_1] = Z_{N-1} [\lambda_\alpha; \nu_1] \] (4.2)

and therefore we can just plug Izergin-Korepin formula into RHS of (4.1) in order to obtain EFP at \( s = 1 \). Indeed, taking into account that

\[ Z_{N-1} [\lambda_\alpha, \nu_1] = \frac{(-1)^{\alpha-1} N! \prod_{j=1}^{N} d(\lambda_\alpha, \lambda_j)}{a(\lambda_\alpha, \nu_1) \prod_{\beta=1, \beta \neq \alpha}^{N} a(\lambda_\beta, \nu_1) b(\lambda_\beta, \nu_1) \prod_{k=2}^{N} a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)} \times \frac{\det M_{[\alpha;1]}}{\det M} \] (4.3)

we arrive at the expression

\[ F^{(r,1)}_N = \frac{1}{\det M} \cdot \frac{\prod_{k=2}^{N} d(\nu_1, \nu_k)}{\prod_{j=1}^{N} a(\lambda_\alpha, \nu_1) \prod_{\alpha=r+1}^{N} b(\lambda_\alpha, \nu_1)} \sum_{\alpha=1}^{r} (-1)^{\alpha-1} g_r(\lambda_\alpha) \det M_{[\alpha;1]}. \] (4.4)

Here the function \( g_r(\lambda) := g_r(\lambda; \lambda_1, \ldots, \lambda_N; \nu_1, \ldots, \nu_N) \), is given by

\[ g_r(\lambda) := \prod_{\alpha=r+1}^{N} d(\lambda_\alpha, \lambda) \prod_{\alpha=1}^{r} e(\lambda_\alpha, \lambda) \prod_{k=1}^{N} b(\lambda, \nu_k), \]

where function \( e(\lambda, \lambda') \) is defined in (3.12). For \( r = N \) function \( g_r(\lambda) \) is just function \( g(\lambda) \) defined in (3.11).

4.2. The cases \( s = 2 \) and \( s = 3 \). Using (4.1) and (4.4) we can derive EFP in the case of \( s = 2 \). Indeed, for \( \alpha = 1, \ldots, r \), using (4.4), we can write

\[ F^{(r-1,2)}_{N-1} [\lambda_\alpha; \nu_1] = \frac{1}{\det M_{[\alpha;1]}} \prod_{k=3}^{N} d(\nu_2, \nu_k) \prod_{\beta=1, \beta \neq \alpha}^{r} \frac{1}{a(\lambda_\beta, \nu_2)} \prod_{\beta=r+1}^{N} b(\lambda_\beta, \nu_2) \]

\[ \times \sum_{\beta=1}^{r} (-1)^{\beta-1} + \chi(\beta, \alpha) \frac{b(\lambda_\beta, \nu_1)}{e(\lambda_\alpha, \lambda_\beta)} g_2(\lambda_\beta) \det M_{[\alpha;2;1]}. \] (4.5)

Here \( \chi(\beta, \alpha) = 1 \) if \( \beta > \alpha \), and \( \chi(\beta, \alpha) = 0 \) otherwise. Substituting in (4.5) the expression for \( \det M_{[\alpha;1]} \) which follows from relation (4.3), and switching to the
non-normalized quantity, we get
\[
\overline{F}_{N-1}^{(r-1,1)}[\lambda;\nu] = \frac{Z_N}{\det \mathcal{M}} \prod_{\beta=1}^{N} \frac{d(\lambda_\beta, \lambda_\alpha)}{a(\lambda_\beta, \nu_1) a(\lambda_\beta, \nu_r)} \prod_{k=2}^{N} \frac{d(\nu_1, \nu_k)}{a(\lambda_\alpha, \nu_k) b(\lambda_\alpha, \nu_k)} \\
\times \prod_{k=3}^{N} \frac{1}{a(\lambda_\beta, \nu_k)} \prod_{\beta=1}^{r} \frac{1}{b(\lambda_\beta, \nu_2)} \prod_{\beta=r+1}^{N} \frac{1}{b(\lambda_\beta, \nu_2)} \\
\times \sum_{\beta=1}^{r} (-1)^{\beta-1+\chi(\beta, \alpha)} \frac{b(\lambda_\beta, \nu_1)}{e(\lambda_\alpha, \lambda_\beta)} g_r(\lambda_\beta) \det \mathcal{M}_{[\alpha, \beta; 1, 2]}.
\]
Plugging this expression into the recurrence relation, after all cancellations, we finally find
\[
F_N^{(r,2)} = \frac{1}{\det \mathcal{M}} \prod_{j=1}^{3} \prod_{\alpha=1}^{N} \frac{d(\nu_1, \nu_k)}{a(\lambda_\alpha, \nu_j) a(\lambda_\alpha, \nu_j)} \prod_{\alpha=r+1}^{N} b(\lambda_\alpha, \nu_1) b(\lambda_\alpha, \nu_2) \\
\times \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} (-1)^{\alpha+\beta+\chi(\beta, \alpha)} \frac{d(\lambda_\alpha, \nu_2) b(\lambda_\beta, \nu_1)}{e(\lambda_\alpha, \lambda_\beta)} g_r(\lambda_\alpha) g_r(\lambda_\beta) \det \mathcal{M}_{[\alpha, \beta; 1, 2]}.
\]
Expression (4.6) can be further used to find EFP for \( s = 3 \), by repeating the procedure just explained. We quote here only the result,
\[
F_N^{(r,3)} = \frac{1}{\det \mathcal{M}} \prod_{j=1}^{3} \prod_{\alpha=1}^{N} \frac{d(\nu_1, \nu_k)}{a(\lambda_\alpha, \nu_j) a(\lambda_\alpha, \nu_j)} \prod_{\alpha=R+1}^{N} b(\lambda_\alpha, \nu_1) b(\lambda_\alpha, \nu_2) \\
\times \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{\gamma=1}^{r} (-1)^{\alpha+\beta+\gamma+\chi(\gamma, \alpha)+\chi(\gamma, \beta)+\chi(\beta, \alpha)} g_r(\lambda_\alpha) g_r(\lambda_\beta) g_r(\lambda_\gamma) \\
\times \frac{d(\lambda_\alpha, \nu_2) a(\lambda_\beta, \nu_3) a(\lambda_\gamma, \nu_1) b(\lambda_\beta, \nu_1) b(\lambda_\gamma, \nu_1) b(\lambda_\gamma, \nu_1)}{e(\lambda_\alpha, \lambda_\beta) e(\lambda_\alpha, \lambda_\gamma) e(\lambda_\beta, \lambda_\gamma)} \times \det \mathcal{M}_{[\alpha, \beta, \gamma; 1, 2, 3]}.
\]

4.3. Result for generic \( s \). Inspecting formulae (4.4), (4.6) and (4.7) it is rather straightforward to guess the result for generic values of \( s \). The following expression for EFP is valid:
\[
F_N^{(r,s)} = \frac{1}{\det \mathcal{M}} \prod_{j=1}^{s} \prod_{\alpha=1}^{N} \frac{d(\nu_1, \nu_k)}{a(\lambda_\alpha, \nu_j) a(\lambda_\alpha, \nu_j)} \prod_{\beta=r+1}^{N} b(\lambda_\beta, \nu_1) b(\lambda_\beta, \nu_2) \\
\times \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \cdots \sum_{\alpha_s=1}^{r} \sum_{\alpha_s \neq \alpha_{s-1}} \cdots \sum_{\alpha_1=1}^{r} \sum_{\alpha_2 \neq \alpha_1} \cdots (-1)^{s + \sum_{k=1}^{s} \alpha_k \sum_{j \leq k \leq s} \chi(\alpha_k, \alpha_j) \sum_{j=1}^{s} g_r(\lambda_\alpha) \\
\times \prod_{1 \leq j < k \leq s} \frac{a(\lambda_\alpha, \nu_1) b(\lambda_\alpha, \nu_1)}{e(\lambda_\alpha, \lambda_\alpha)} \det \mathcal{M}_{[\alpha_1, \ldots, \alpha_s; 1, \ldots, s]}.
\]
It can be shown directly that formula (4.8) solves recurrence relation (4.1). The calculation in fact repeats the one above for the case of \( s = 2 \).
Let us focus on writing EFP for the case of \((N - 1) \times (N - 1)\) lattice, namely, we are interested in \(F_{N-1}^{(r-1,s-1)}[\lambda_1, \nu_1]\). To be more precise, in adapting expression (4.8) to this case, apart from using the fact that the sum is now \((s - 1)\)-fold, it is important also to take into account that the function \(g_r(\lambda)\) implicitly depends on \(N\), so that switching from \(N\) to \((N - 1)\) implies some extra factor. In all, after all these preparations, which are mostly devoted to fit the notations, the expression which has to be substituted into RHS of (4.1) reads:

\[
F_{N-1}^{(r-1,s-1)}[\lambda_1, \nu_1] = \frac{1}{\det \mathcal{M}[\alpha_1;1]} \prod_{j=2}^{s} \prod_{\substack{j \neq 1 \neq \alpha \neq \alpha_j, \ldots, \alpha_{s-1}}} d(\nu_j, \nu_k) \prod_{j=1}^{N} a(\lambda_{j}, \nu_j) \prod_{j=r+1}^{s} b(\lambda_j, \nu_j) \\
\times \sum_{\substack{\alpha_2 \neq 1 \neq \alpha \neq \alpha_j, \ldots, \alpha_{s-1} \neq \alpha_1}} (-1)^{s-r+1+\sum_{\alpha_2 \neq 1 \neq \alpha \neq \alpha_j, \ldots, \alpha_{s-1}} \chi(\alpha_2, \alpha_1)} \\
\times \prod_{j=1}^{s} \prod_{2 \leq k \leq \nu} a(\lambda_{\alpha_j}, \nu_k) b(\lambda_{\alpha_k}, \nu_j) \times \frac{e(\lambda_{\alpha_j}, \lambda_{\alpha_k})}{e(\lambda_{\alpha_1}, \lambda_{\alpha_s})} \\
\times \det \mathcal{M}[\alpha_1, \ldots, \alpha_s;1, \ldots, s].
\]  

(4.9)

Again, as in the case of \(s = 2\) considered above, one can use relation (4.3) to eliminate \(\det \mathcal{M}[\alpha_1;1]\) in favour of \(\det \mathcal{M}\) in (4.9). With this point taken into account, cancellation of various factors shows that recurrence relation (4.1) is indeed fulfilled.

Apart from showing that (4.8) is a solution to (4.1), it is also useful to mention that the corresponding 'initial conditions' are also satisfied. Indeed, for \(s \leq r\), such a condition is just the generalization of condition (4.2), namely

\[
\tilde{F}_{N-s}^{(r-s,0)}[\lambda_1, \ldots, \lambda_r; \nu_1, \ldots, \nu_s] = Z_{N-s}[\lambda_1, \ldots, \lambda_r; \nu_1, \ldots, \nu_s].
\]

For \(s > r\), the corresponding condition is

\[
F_{N-r}^{(0,s-r)}[\lambda_1, \ldots, \lambda_r; \nu_1, \ldots, \nu_r] = 0.
\]

Note, that this relation implies the relation \(F_{N}^{(r,s)} = 0\) which must hold whenever \(s > r\), and which follows directly from the definition of EFP (2.8).

5. EFP in the homogeneous limit

5.1. The procedure. The homogeneous limit can be performed along the lines of papers [3, 14]. We start with writing

\[
\lambda_\alpha = \lambda + \xi_\alpha,
\]

where \(\xi\)'s will be set equal to zero in the limit (as well as \(\nu\)'s). Keeping \(\xi\)'s nonzero (and different from each other), and using the fact that for a function \(f(x)\), regular near \(x = \lambda\), one has \(\exp(\xi \partial_\xi) f(\lambda + \varepsilon)|_{\varepsilon = 0} = f(\lambda + \xi)\), we can bring (4.8) into a form
which involves some determinant:

$$F^{(r,s)}_N = \frac{1}{\det \mathcal{M}} \prod_{j=1}^{s} \prod_{\lambda = j+1}^{N} d(\nu_j, \nu_k) \prod_{\eta = j+1}^{N} b(\lambda, \nu_j)$$

$$\times \left| \begin{array}{c}
\exp(\xi_1 \partial_{\lambda_1}) \ldots \exp(\xi_1 \partial_{\lambda_{s}}) \varphi(\lambda_1, \nu_{s+1}) \ldots \varphi(\lambda_1, \nu_N) \\
\ldots \\
\exp(\xi_N \partial_{\lambda_1}) \ldots \exp(\xi_N \partial_{\lambda_{s}}) \varphi(\lambda_N, \nu_{s+1}) \ldots \varphi(\lambda_N, \nu_N)
\end{array} \right| \times \prod_{j=1}^{s} g_1(\lambda + \varepsilon_j) \prod_{1 \leq j < k \leq s} a(\lambda + \varepsilon_j, \nu_k) b(\lambda + \varepsilon_k, \nu_j) e(\lambda + \varepsilon_j, \lambda + \varepsilon_k) \bigg|_{\varepsilon_1 = \ldots = \varepsilon_s = 0}. \tag{5.1}
$$

It is to be emphasized that this expression is valid for the inhomogeneous model (no homogeneous limit yet); it represents an equivalent way of writing the multiple sum in (4.8).

Let us now perform the limit \(\nu_1 \to 0, \nu_2 \to 0, \ldots, \nu_N \to 0\), in this order, at each stage keeping the contribution of leading order in the corresponding variable; next we do the same with \(\xi_j\)'s, in the order \(\xi_1 \to 0, \xi_2 \to 0, \ldots, \xi_N \to 0\), again keeping only the contributions of leading order. The first line of (5.1) gives

$$\frac{(-1)^{(N-s-1)/(N-s)}}{a^{s} b^{(N-r)s} \det \mathcal{N}} \prod_{j=1}^{s} \frac{1}{(N-j)!} \frac{1}{\nu_{j+2}^{2!} \nu_{j+3}^{3!} \ldots \nu_{N-s-1}^{(N-s-1)!}} \frac{1}{\xi_1^{N-1} \xi_2^{N-2} \ldots \xi_{N-1}^{N-1}},$$

where \(a := a(\lambda, 0)\), \(b := b(\lambda, 0)\). The second line of (5.1) gives

$$\frac{(-1)^{(N-s-1)/(N-s)}}{a^{s} b^{(N-r)s} \det \mathcal{N}} \left| \begin{array}{c}
1 \ldots 1 \\
\partial_{\lambda_1} \ldots \partial_{\lambda_{s}} \\
\ldots \\
\partial_{\lambda_1}^{N-1} \ldots \partial_{\lambda_{s}}^{N-1} \\
\partial_{\varepsilon_1} \ldots \partial_{\varepsilon_{s}} \\
\ldots \\
\partial_{\varepsilon_1}^{N-1} \ldots \partial_{\varepsilon_{s}}^{N-1}
\end{array} \right| \times \left( \frac{\nu_{s+2}^{2!} \nu_{s+3}^{3!} \ldots \nu_{N-s-1}^{(N-s-1)!}}{(N-s)!} \right) \left( \frac{\xi_1^{N-1} \xi_2^{N-2} \ldots \xi_{N-1}^{N-1}}{(N-1)!} \right). \tag{5.2}
$$

The terms in the third line of (5.1) give

$$\prod_{j=1}^{s} \left[ \frac{-\sin(\varepsilon_j - 2\eta)\varepsilon_j^{N-r}}{\sin(\varepsilon_j + \lambda - \eta)^N} \right] \prod_{1 \leq j < k \leq s} \frac{\sin(\varepsilon_j + \lambda + \eta) \sin(\varepsilon_k + \lambda - \eta)}{\sin(\varepsilon_j - \varepsilon_k + 2\eta)}. \tag{5.3}
$$

In all, after all cancellations, EFP in the homogeneous model reads

$$F^{(r,s)}_N = \frac{(-1)^s \prod_{j=1}^{s} \prod_{\lambda = j+1}^{N} \prod_{\eta = j+1}^{N} [\sin(\varepsilon_j + \lambda - \eta)]^{N}}{a^{s} b^{(N-r)s} \det \mathcal{N}} \left| \begin{array}{c}
\varphi(\lambda) \ldots \partial_{\lambda}^{N-s-1} \varphi(\lambda) \\
\varphi(\lambda) \ldots \partial_{\lambda}^{N-s} \varphi(\lambda) \\
\ldots \\
\varphi(\lambda) \ldots \partial_{\lambda}^{2N-s-2} \varphi(\lambda) \\
\varphi(\lambda) \ldots \partial_{\varepsilon_1}^{N-1} \varphi(\lambda) \\
\ldots \\
\varphi(\lambda) \ldots \partial_{\varepsilon_1}^{N-1} \varphi(\lambda)
\end{array} \right| \times \prod_{j=1}^{s} \left( \frac{\sin(\varepsilon_j)^{N-r} \sin(\varepsilon_j - 2\eta)^r}{\sin(\varepsilon_j + \lambda - \eta)^N} \right) \left( \frac{\sin(\varepsilon_j + \lambda + \eta) \sin(\varepsilon_k + \lambda - \eta)}{\sin(\varepsilon_j - \varepsilon_k + 2\eta)} \right) \bigg|_{\varepsilon_1 = \ldots = \varepsilon_s = 0}. \tag{5.3}
$$

Note, that in the determinant here we have changed the order of columns, in comparison with formulae (5.1) or (5.2).
5.2. Orthogonal polynomials representation. Formula (5.3) for EFP of the homogenous model, involving \(N \times N\) determinant, can be also represented in terms of some \(s \times s\) determinant. Such an equivalent representation can be further used to obtain a multiple integral representation (having \(s\) integrations), similar to those for correlation functions of quantum spin chains [19–22].

The derivation of the \(s \times s\) determinant representation from (5.3) is based on the following facts. Let \(\{P_n(x)\}_{n=0}^{\infty}\) be a set of orthogonal polynomials,

\[
\int P_n(x)P_m(x)\mu(x)\,dx = h_n\delta_{nm},
\]

where the integration domain is assumed over the real axis, the weight \(\mu(x)\) is real nonnegative, and we choose \(h_n\)'s such that \(P_n(x)\)'s are monic (i.e., the leading coefficient is equal to one), \(P_n(x) = x^n + \ldots\). Let \(c_n\) denote \(n\)-th moment of the weight \(\mu(x)\),

\[
c_n = \int x^n\mu(x)\,dx \quad (n = 0, 1, \ldots).
\]

The orthogonality condition (5.4) and standard properties of determinants allow one to prove that

\[
\begin{vmatrix}
  c_0 & c_1 & \ldots & c_{n-1} \\
  c_1 & c_2 & \ldots & c_n \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n-1} & c_n & \ldots & c_{2n-2}
\end{vmatrix} = h_0h_1\cdots h_{n-1}.
\]

More generally, the following formula is valid

\[
\begin{vmatrix}
  c_0 & c_1 & \ldots & c_{n-k-1} & 1 & 1 & \ldots & 1 \\
  c_1 & c_2 & \ldots & c_{n-k} & x_1 & x_2 & \ldots & x_k \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n-1} & c_n & \ldots & c_{2n-k-1} & x_1^{n-1} & x_2^{n-1} & \ldots & x_k^{n-1}
\end{vmatrix}
= h_0\cdots h_{n-k-1} \begin{vmatrix}
  P_{n-k}(x_1) & \ldots & P_{n-k}(x_k) \\
  P_{n-1}(x_1) & \ldots & P_{n-1}(x_k)
\end{vmatrix}.
\]

The proof of this formula can be given by the methods explained in [25]. We shall use (5.5) with \(n = N\) and \(k = s\).

To apply (5.5) to expression (5.3) we need first to identify the orthogonality weight. Since in our case, by definition, \(c_n = \partial_n^r\varphi(\lambda)\), the weight can be obtained by representing the function \(\varphi(\lambda)\) via Laplace transform:

\[
\varphi(\lambda) = \int e^{\lambda x}\Phi(x)\,dx.
\]

Here function \(\Phi(x) = \Phi(x; \eta)\) is independent of \(\lambda\), but depends on \(\eta\) as a parameter (indeed, in our case \(\mu(x) = \mu(x; \lambda, \eta)\) where \(\lambda\) and \(\eta\) are to be considered as parameters, and \(\mu(x) = e^{\lambda x}\Phi(x)\)).

The explicit form of function \(\Phi(x)\) depends on the particular choice of the physical regime in the six-vertex model. For example, let us consider the so-called disordered regime \(|\Delta| < 1\). In this case \(\lambda\) and \(\eta\) in our standard parametrization of weights \(a = \sin(\lambda + \eta), b = \sin(\lambda - \eta)\), and \(c = \sin(2\eta)\) are both real and satisfy
The other two regimes, the ferroelectric ($\Delta > 1$) and anti-ferroelectric ($\Delta < -1$) can be approached when $\lambda$ and $\eta$ are complex, satisfying $\text{Re} \lambda = \text{Re} \eta = 0$ or $\text{Re} \lambda = \text{Re} \eta = \pi/2$, respectively. In fact, for these regimes the weight $\mu(x)$ can be found from (5.6) using the proper analytical continuation in the parameters $\lambda$ and $\eta$. The integral in RHS of (5.6) in these cases is replaced by a sum coming from the simple poles of the integrand, or, in other words, the corresponding measure $\mu(x)\,dx$ turns out to be discrete (see [5] for explicit formulae). For our construction below the actual choice of the regime, and hence the explicit expression for the orthogonality weight $\mu(x)$, is irrelevant since only the corresponding orthogonal polynomials will enter the formulae explicitly (for instance, one can think of the case of the disordered regime; the resulting expressions in terms of $P_n(x)$’s for other regimes turn out to be essentially the same).

To make the use of (5.5) convenient in the framework of representation (5.3), we define, following [14], the functions

$$
\omega(\varepsilon) = \frac{\sin(\lambda + \eta)}{\sin(\lambda - \eta)} \frac{\sin \varepsilon}{\sin(\varepsilon - 2\eta)}, \quad \omega(\varepsilon) = \frac{\sin(\lambda - \eta)}{\sin(\lambda + \eta)} \frac{\sin \varepsilon}{\sin(\varepsilon + 2\eta)}.
$$

They are functions of $\varepsilon$, with $\lambda$ and $\eta$ regarded as parameters, as indicated in the notations (below we omit the dependence on $\lambda$ and $\eta$ whenever possible). We also define

$$
\rho(\varepsilon) = \frac{\sin(\lambda - \eta)}{\sin(2\eta)} \frac{\sin(\varepsilon - 2\eta)}{\sin(\varepsilon + \lambda - \eta)}, \quad \rho(\varepsilon) = \frac{\sin(\lambda + \eta)}{\sin(2\eta)} \frac{\sin(\varepsilon + 2\eta)}{\sin(\varepsilon + \lambda + \eta)}.
$$

There are useful relations

$$
\rho(\varepsilon) = \frac{1}{\omega(\varepsilon) - 1}, \quad \rho(\varepsilon) = \frac{1}{1 - \omega(\varepsilon)}.
$$

Noting that

$$
\frac{\sin(\varepsilon + \lambda + \eta)}{\sin(\varepsilon - 2\eta)} \frac{\sin(\varepsilon + \lambda - \eta)}{\sin(\varepsilon + 2\eta)} = \frac{1}{\varphi^2(\varepsilon_1)\varphi(\varepsilon_2)} \frac{1}{\omega(\varepsilon_1)\omega(\varepsilon_2) - 1},
$$

and defining

$$
K_n(x) = \frac{n! \varphi^{n+1}}{h_n} P_n(x)
$$

we have, in virtue (5.5), the following orthogonal polynomials representation:

$$
F^{(r,s)}_N(x) = (-1)^r \left[ \begin{array}{c} K_{N-s}(\partial_{x_1}) \cdots K_{N-s}(\partial_{x_s}) \\ K_{N-1}(\partial_{x_1}) \cdots K_{N-1}(\partial_{x_s}) \end{array} \right] \int \left\{ \prod_{j=1}^s [\omega(\varepsilon_j)]^{N-r} [\rho(\varepsilon_j)]^N \right\} \prod_{1 \leq j < k \leq s} \rho(\varepsilon_j)\rho(\varepsilon_k) \frac{1}{\omega(\varepsilon_j)\omega(\varepsilon_k) - 1} |_{\varepsilon_1 = \ldots = \varepsilon_s = 0}.
$$
5.3. Multiple integral representation. Let $H_N^{(r)}$ denotes the probability of having the $c$-weight vertex of the first row (from the top of the lattice) at $r$-th position (from the right). This quantity can be related to EFP at $s = 1$, which describes in this case essentially the one-point correlation function (polarization) at the boundary, via $H_N^{(r)} = F_N^{(r,1)} - F_N^{(r-1,1)}$ (we refer for details to [12] where $F_N^{(r,1)}$ was denoted as $G_N^{(r)}$). Using (5.10) and taking into account the first relation in (5.8), we have (see also [14])

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon) \frac{[\omega(\varepsilon)]^{N-r}}{[\omega(\varepsilon) - 1]^{N-1}} \bigg|_{\varepsilon = 0}. \tag{5.11}$$

Let us define the generating function:

$$h_N(z) = \sum_{r=1}^{N} H_N^{(r)} z^{r-1}. \tag{5.12}$$

The key identity, for the derivation of a multiple integral representation for EFP, is

$$K_{N-1}(\partial_\varepsilon) f(\omega(\varepsilon)) \bigg|_{\varepsilon = 0} = \frac{1}{2\pi i} \oint_{C_0} \frac{(z - 1)^{N-1}}{z^N} h_N(z) f(z) \, dz. \tag{5.13}$$

Here $f(z)$ is some function, regular near $z = 0$, and $C_0$ is a simple closed counterclockwise contour around the origin.

To prove this identity, we note first that since $f(z)$ is assumed regular at $z = 0$, one can further think of it as a polynomial of degree $(N - 1)$, since higher powers in $z$ do not contribute to either side of (5.13) (note that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$). It therefore suffices to prove (5.13) for $f(z)$ being a monomial. Next, let us define auxiliary quantities

$$V_N^{(p)} = K_{N-1}(\partial_\varepsilon) [\omega(\varepsilon)]^{N-p} \bigg|_{\varepsilon = 0}, \quad (p = 1, \ldots, N). \tag{5.14}$$

Evaluating the integral in RHS of (5.13), one finds that (5.13) is nothing but the relation

$$\vec{v} = (-1)^{N-1} \vec{A} \vec{h}, \tag{5.15}$$

where the vectors $\vec{v}$ and $\vec{h}$ have components $v_p = V_N^{(p)}$ and $h_r = H_N^{(r)}$, and matrix $\vec{A}$ has entries $A_{pr} = (-1)^{p-r} (N-1)_{p-r}$. Obviously, matrix $\vec{A}$ is lower-triangular, and moreover it can be represented as

$$\vec{A} = (I - \mathcal{E})^{N-1}, \tag{5.16}$$

where $\mathcal{E}$ denotes the lower triangular matrix, with entries standing under the main diagonal equal to one and all other entries being zeroes, i.e., $\mathcal{E}_{pr} = \delta_{p,r+1}$. Inverting matrix $\vec{A}$ in (5.15) with the help of (5.16), one arrives immediately to the expression for $H_N^{(r)}$ in terms of $V_N^{(p)}$’s, which follows from formula (5.11) and definition (5.14). This proves identity (5.13).

As a result, applying identity (5.13) to (5.10) and taking into account relations (5.8) and definitions (5.7), we obtain the following multiple integral representation
for EFP:

\[
F_{N}^{(r,s)} = \left( -\frac{1}{2\pi i} \right)^{s} \oint_{C_{0}} \cdots \oint_{C_{0}} \frac{h_{N}(z_{1})}{z_{1}^{r}(z_{1} - 1)} \cdots \frac{h_{N}(z_{s})}{z_{s}^{r}(z_{s} - 1)} \left| \begin{array}{cccc}
\frac{h_{N-1}(z_{1})}{z_{1}^{r}(z_{1} - 1)} & \cdots & \frac{h_{N-1}(z_{s})}{z_{s}^{r}(z_{s} - 1)} \\
\frac{1}{z_{1}^{r-1}(z_{1} - 1)^{2}} & \cdots & \frac{1}{z_{s}^{r-1}(z_{s} - 1)^{2}} \\
\frac{h_{N-s+1}(z_{1})}{z_{1}^{r-s+1}(z_{1} - 1)^{s}} & \cdots & \frac{h_{N-s+1}(z_{s})}{z_{s}^{r-s+1}(z_{s} - 1)^{s}} \\
\end{array} \right|
\]

Here \(\tilde{z}_{j}\)'s are functions of the corresponding \(z_{j}\)'s,

\[
\tilde{z}_{j} = \frac{b^{2}z_{j}}{(a^{2} + b^{2} - c^{2})z_{j} - a^{2}},
\]

where \(a\), \(b\), and \(c\) are the homogeneous six-vertex model weights (formula (5.18) follows from (5.7) by expressing function \(\tilde{\omega}(\varepsilon)\) in terms of \(\omega(\varepsilon)\) and using \(a = \sin(\lambda + \eta)\), \(b = \sin(\lambda - \eta)\), and \(c = \sin(2\eta)\)).

6. More on the multiple integral representation

6.1. Preliminaries. Formula (5.17) gives EFP in terms of a multiple integral. This integral representation can be transformed into some other forms which can be useful for further study of EFP (e.g., for finding the limit shape). Our aim in this section is to derive some of these representations.

The first point to be mentioned in this respect is that the integrand of (5.17) involves some determinant, which is an antisymmetric function with respect to permutations of the integration variables \(z_{1}, \ldots, z_{s}\). Because of this antisymmetry, only the antisymmetric part of the double product, with respect to permutations of these variables, actually contributes to the multiple integral in (5.17).

Let us introduce the parametrization

\[
\tilde{z}_{j} = \omega(\xi_{j}), \quad \tilde{\xi}_{j} = \tilde{\omega}(\xi_{j})
\]

where the functions \(\omega(\xi) = \omega(\xi; \lambda, \eta)\) and \(\tilde{\omega}(\xi) = \tilde{\omega}(\xi; \lambda, \eta)\) are as in (5.7) (note that upon this parametrization relation (5.18) is satisfied automatically). With this parametrization the double product in (5.17) coincides with the expression whose antisymmetric part has been found in paper [20] (see appendix C of that paper). Namely, using a mixed set of notations, the antisymmetric part of the double product, with respect to permutations of the variables \(z_{1}, \ldots, z_{s}\), can be written as

\[
\text{Asym} \prod_{z_{1}, \ldots, z_{s}} \frac{(\tilde{z}_{j} - 1)(z_{k} - 1)}{\tilde{z}_{j}z_{k} - 1}
\]

\[
= \frac{1}{s!} \prod_{1 \leq j < k \leq s} \left( z_{k} - z_{j} \right) \prod_{j,k=1 \atop j \neq k}^{s} \frac{1}{b^{2}z_{j}z_{k} - (a^{2} + b^{2} - c^{2})z_{j} + a^{2}}
\]

\[
\times \frac{a^{s(s-1)}c^{s(s-2)}}{\prod_{j=1}^{s} |\sin(\xi_{j} - 2\eta)|^{s-1}} Z_{s}(\lambda + \xi_{1}, \ldots, \lambda + \xi_{s}).
\]
denoted as $Z$ with $\nu \lambda$ ent. Choosing some particular of weight $b$ account one has therefore the expression

\[ Z = \text{heterogeneous six-vertex model on } N \times \lambda \text{ via } \gamma \text{ limit, one can introduce variables } \xi \text{ or later use we mention here that function } \gamma(2.1) \text{ and } (2.2), \text{ but with all } \nu \text{'s equal zero.}

The expression in the last line of (6.2) is some symmetric polynomial in variables $z_1, \ldots, z_s$. Our aim below will be to show that this expression admits a representation in terms of some determinant. Furthermore, due to this representation it turns out possible to clarify the meaning of the determinant in (5.17) which appears to be related to the partition function of the partially inhomogeneous model on the $N \times N$ lattice with $s$ inhomogeneities (out of $N$ possible).

### 6.2. Back to the partition function.

Let us consider the ‘partially’ inhomogeneous six-vertex model on $N \times N$ lattice, whose partition function will be denoted as $Z_N(\lambda_1, \ldots, \lambda_N)$. The weights of the model are given by (2.1) and (2.2), with $\nu_1 = \nu_2 = \cdots = \nu_N = 0$, while $\lambda_1, \lambda_2, \ldots, \lambda_N$ are kept in general to be different. Choosing some particular $\lambda$ as a value for $\lambda_1, \lambda_2, \ldots, \lambda_N$ in the homogenous limit, one can introduce variables $\xi_1, \xi_2, \ldots, \xi_N$ by

\[ \lambda_\alpha = \lambda + \xi_\alpha, \quad (6.3) \]

i.e., such that the homogenous limit will correspond to making all $\xi$’s vanish. To simplify notations we shall write $a(\lambda_\alpha) := a(\lambda_\alpha, 0)$ and $b(\lambda_\alpha) := b(\lambda_\alpha, 0)$. Since both the homogeneous and inhomogeneous models will be considered here simultaneously, we shall write $a(\lambda)$ and $b(\lambda)$ for the weights of the homogenous model (where $\lambda$ is the same as in (6.3)), rather than simply $a$ and $b$. We come back however to these simplified notations in the last subsection.

An important role below will be played by the variables $u_1, \ldots, u_N$, defined by

\[ u_j = \gamma(\xi_j), \quad \gamma(\xi) = \frac{a(\lambda) b(\lambda + \xi)}{b(\lambda) a(\lambda + \xi)}. \quad (6.4) \]

For later use we mention here that function $\gamma(\xi)$ is related to function $\omega(\xi)$ (5.7) via $\gamma(\xi) = \omega(-\lambda + \eta - \xi)$.

Our aim here will be to study the quantity $\tilde{Z}_N(\xi_1, \ldots, \xi_N)$, which depends also on $\eta$ and $\lambda$ as parameters, and is defined by

\[ Z_N(\lambda_1, \ldots, \lambda_N) = Z_N(\lambda, \ldots, \lambda) \tilde{Z}_N(\xi_1, \ldots, \xi_N). \]

As we shall see, there is a nice representation for this “bare” partition function in terms of variables (6.4).

To illustrate the idea, it is useful to consider first the case when only one inhomogeneity is present. Without lack of generality we can assume that this is the inhomogeneity of the first row (i.e., $\xi_2 = \cdots = \xi_N = 0$). The domain wall boundary conditions admit only one vertex of weight $c$ in the first row; if this vertex is at $r$-th position (counted, as usual, from the right) then the first $(r - 1)$ vertices are of weight $b$ while the remaining $(N - r)$ vertices are of weight $a$. Taking this into account one has therefore the expression

\[ Z_N(\lambda_1, \lambda, \ldots, \lambda) = \sum_{r=1}^{N} \left[ \frac{a(\lambda_1)}{a(\lambda)} \right]^{N-r} \left[ \frac{b(\lambda_1)}{b(\lambda)} \right]^{r-1} H_N^{(r)}(\lambda, \ldots, \lambda) Z_N(\lambda, \ldots, \lambda). \]
Note that here $H_N^{(c)}(\lambda, \ldots, \lambda)$ denotes the correlation function of the homogeneous model (recall that this function describes the probability of having the $c$-weight vertex at $r$-th position on the first row). Recalling definition (5.12), this last expression implies that

$$\tilde{Z}_N(\xi_1, 0, \ldots, 0) = \left[ \frac{a(\lambda_1)}{a(\lambda)} \right]^{N-1} h_N(u_1). \quad (6.5)$$

It is to be emphasized that the variables $\xi_1, \lambda_1$, and $u_1$ are related to each other by (6.3) and (6.4); $\lambda$ is to be regarded as a parameter, entering also function $\gamma(\xi)$ in (6.4).

To consider the general case, let us introduce functions $h_{N,s}(u_1, \ldots, u_s)$, where the second subscript in the notation refers to the number of arguments ($s = 1, \ldots, N$)

$$h_{N,s}(u_1, \ldots, u_s) = \prod_{1 \leq j < k \leq s} (u_k - u_j)^{-1} \left| \begin{array}{cccc} u_1^{-s} h_{N,s+1}(u_1) & \cdots & u_{s-1} h_{N,s+1}(u_s) \\ u_1^{-s+1} h_{N,s+2}(u_1) & \cdots & u_{s-2} h_{N,s+2}(u_s) \\ \vdots & & \vdots & \vdots \\ (u_1 - 1)^{s-1} h_N(u_1) & \cdots & (u_s - 1)^{s-1} h_N(u_s) \end{array} \right|. \quad (6.6)$$

These functions are symmetric polynomials of degree $(N-1)$ in each of their variables. They satisfy the relation (recall that $h_N(1) = 1$)

$$h_{N,s+1}(u_1, \ldots, u_s, 1) = h_{N,s}(u_1, \ldots, u_s). \quad (6.7)$$

This relation says that these functions can be in fact constructed iteratively, starting with $h_{N,N}(u_1, \ldots, u_N)$; one has then, in particular, $h_{N,1}(u_1) = h_N(u_1)$. The "bare" partition function, in the most general case of $N$ inhomogeneities, reads

$$\tilde{Z}_N(\xi_1, \ldots, \xi_N) = \prod_{j=1}^{N} \left[ \frac{a(\lambda_j)}{a(\lambda)} \right]^{N-1} h_{N,N}(u_1, \ldots, u_N). \quad (6.8)$$

Using relation (6.7) one can easily obtain the corresponding formula in the case of $s$ (out of $N$) inhomogeneities.

It is worth to mention that $h_{N,s}(u_1, \ldots, u_s)$ introduced in (6.6), besides describing the partition function with $s$ inhomogeneities, turns out to coincide, modulo a trivial factor and the substitution $u_j \rightarrow z_j$, with the determinant in (5.17). We will exploit this observation below in discussing equivalent multiple integral representations for EFP.

### 6.3. Proof of the formula for $\tilde{Z}_N(\xi_1, \ldots, \xi_N)$

Representation (6.8) for $\tilde{Z}_N(\xi_1, \ldots, \xi_N)$ (for the special case of $\lambda = \pi/2$) appeared for the first time in paper [26]; since the proof in that paper was lacking, we sketch it here for completeness.

Let us begin with Izergin-Korepin formula, specialized to the case of $\nu_1 = \cdots = \nu_N = 0$, which reads

$$Z_N(\lambda_1, \ldots, \lambda_N) = \frac{\prod_{j=1}^{N} a(\lambda_j) b(\lambda_j)^{N}}{\prod_{1 \leq j < k \leq N} d(\lambda_k, \lambda_j) \prod_{n=0}^{N-1} n!} \left| \begin{array}{cccc} \varphi(\lambda_1) & \cdots & \varphi(\lambda_N) \\ \partial_{\lambda_1} \varphi(\lambda_1) & \cdots & \partial_{\lambda_N} \varphi(\lambda_N) \\ \vdots & \vdots & \vdots \\ \partial_{\lambda_1}^{N-1} \varphi(\lambda_1) & \cdots & \partial_{\lambda_N}^{N-1} \varphi(\lambda_N) \end{array} \right|. $$
Taking into account that \( \varphi(\lambda) = c/a(\lambda)b(\lambda) \), and passing to the orthogonal polynomial representation (see formulae (5.5) and (5.9)), we get

\[
\tilde{Z}_N(\xi_1, \ldots, \xi_N) = \frac{[\varphi(\lambda)]^{N(N-1)/2}}{\prod_{j=1}^N [\varphi(\lambda_j)]^{N} \prod_{1 \leq j < k \leq N} d(\lambda_k, \lambda_j)} \times \begin{bmatrix} K_0(\partial_{\lambda_1}) & \ldots & K_0(\partial_{\lambda_N}) \\ K_1(\partial_{\lambda_1}) & \ldots & K_1(\partial_{\lambda_N}) \\ \vdots & \ddots & \vdots \\ K_{N-1}(\partial_{\lambda_1}) & \ldots & K_{N-1}(\partial_{\lambda_N}) \end{bmatrix} \prod_{j=1}^N \varphi(\lambda_j).
\]

Now, taking into account that

\[
u_k - u_j = \frac{[a(\lambda)]^2}{a(\lambda_j)a(\lambda_k)} \varphi(\lambda)d(\lambda_k, \lambda_j)
\]

and

\[
\frac{\varphi(\lambda)a(\lambda)}{\varphi(\lambda_j)a(\lambda_j)} = \frac{a(\lambda_j)}{a(\lambda)} u_j,
\]

we arrive at

\[
\tilde{Z}_N(\xi_1, \ldots, \xi_N) = \prod_{j=1}^N \left[ \frac{a(\lambda_j)}{a(\lambda)} \right]^{N-1} \frac{\prod_{j=1}^N u_j^{N-1}}{\prod_{1 \leq j < k \leq N} (u_k - u_j)} 
\times \frac{1}{\prod_{j=1}^N \varphi(\lambda_j)} \begin{bmatrix} K_0(\partial_{\lambda_1}) & \ldots & K_0(\partial_{\lambda_N}) \\ K_1(\partial_{\lambda_1}) & \ldots & K_1(\partial_{\lambda_N}) \\ \vdots & \ddots & \vdots \\ K_{N-1}(\partial_{\lambda_1}) & \ldots & K_{N-1}(\partial_{\lambda_N}) \end{bmatrix} \prod_{j=1}^N \varphi(\lambda_j).
\]

Comparing with (6.8), we see that to complete the proof we only need to express the second line in terms of \( u \)’s.

For this purpose let us again turn to the special case when only one inhomogeneity is present. In this case Izergin-Korepin formula boils down to

\[
Z_N(\lambda_1, \ldots, \lambda) = \frac{[a(\lambda_1)b(\lambda)]^N [a(\lambda)b(\lambda)]^{N(N-1)}}{(N-1)! \prod_{n=0}^{N-2} (n!)^2 [d(\lambda_1, \lambda)]^{N-1}} \begin{bmatrix} \varphi(\lambda) & \ldots & \partial_{\lambda}^{N-2} \varphi(\lambda) & \varphi(\lambda_1) \\ \partial_{\lambda} \varphi(\lambda) & \ldots & \partial_{\lambda}^{N-3} \varphi(\lambda) & \partial_{\lambda} \varphi(\lambda_1) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{\lambda}^{N-1} \varphi(\lambda) & \ldots & \partial_{\lambda}^{2N-3} \varphi(\lambda) & \partial_{\lambda}^{N-1} \varphi(\lambda_1) \end{bmatrix}.
\]

We have

\[
\tilde{Z}_N(\xi_1, 0, \ldots, 0) = \frac{1}{[\varphi(\lambda_1)]^N [d(\lambda_1, \lambda)]^{N-1}} K_{N-1}(\partial_{\lambda_1}) \varphi(\lambda_1).
\]

Taking into account that (6.9) implies that \( d(\lambda_1, \lambda) = (u_1 - 1)a(\lambda_1)[a(\lambda)\varphi(\lambda)]^{-1} \) and also using (6.10), we have

\[
\tilde{Z}_N(\xi_1, 0, \ldots, 0) = \left[ \frac{a(\lambda_1)}{a(\lambda)} \right]^{N-1} \left( \frac{u_1}{u_1 - 1} \right)^{N-1} \frac{1}{\varphi(\lambda_1)} K_{N-1}(\partial_{\lambda_1}) \varphi(\lambda_1).
\]
Now, comparing this expression with (6.5), we find that
\[
\frac{1}{\varphi(\lambda_j)} K_n(\partial_{\lambda_j})\varphi(\lambda_j) = \left(\frac{u_j - 1}{u_j}\right)^n h_{n+1}(u_j).
\]
Finally, plugging this formula into (6.11), one arrives at (6.8), which is thus proven.

### 6.4. Equivalent representations.
Coming back to the expression in the last line of formula (6.2), one immediately finds, using (6.8) (and also taking into account (6.1), (6.4), and (5.8)) that
\[
\frac{Z_s(\lambda + \xi_1, \ldots, \lambda + \xi_s)}{\prod_{j=1}^s \sin(\xi_j - 2\eta) \sin(\eta)} = \frac{Z_s}{\varphi(a)^{s-1} \varphi(b)^{s-1}} \prod_{j=1}^s \left(\frac{z_j - 1}{u_j}\right)^{s-1} h_{s,s}(u_1, \ldots, u_s).
\] (6.12)
Here \(Z_s\) stands for \(Z_s(\lambda, \ldots, \lambda)\) and below we also come back to our simplified notations \((a = a(\lambda, 0), b = b(\lambda, 0))\). Furthermore, we shall use the following notations
\[
\Delta := \frac{a^2 + b^2 - c^2}{2ab} = \cos(2\eta), \quad t := \frac{b}{a} = \frac{\sin(\lambda - \eta)}{\sin(\lambda + \eta)}.
\]
Exploiting formulae (5.7) and (6.4) it is straightforward to find that
\[
u_j = -\frac{z_j - 1}{(t^2 - 2t\Delta)z_j + 1}.
\]
This formula, by the way, shows that (6.12) is indeed a polynomial in \(z_1, \ldots, z_s\) (recall that in general the functions \(h_{N,s}(u_1, \ldots, u_s)\) are symmetric polynomials of degree \((N - 1)\) in each variable). In all, for the antisymmetrical part of the double product in (5.17), we obtain
\[
\text{Asym}_{z_1, \ldots, z_s} \prod_{1 \leq j < k \leq s} \frac{(z_j - 1)(z_k - 1)}{z_jz_k - 1} = \frac{Z_s}{s!a^{s(s-1)/2}c^s} \prod_{1 \leq j < k \leq s} (z_k - z_j) \times \prod_{j=1}^s \frac{[(t^2 - 2t\Delta)z_j + 1]^{s-1} h_{s,s}(u_1, \ldots, u_s)}{(t^2 z_j z_k - 2t\Delta z_j z_k + 1)}.\]
Taking account that the determinant in (5.17) can be represented in terms of the function \(h_{N,s}(z_1, \ldots, z_s)\), we arrive therefore at the following equivalent multiple integral representation for EFP
\[
F_N^{(r,s)} = \frac{(-1)^{s(s+1)/2}Z_s}{s!(2\pi i)^{s(s-1)/2}c^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{1 \leq j < k \leq s} (z_k - z_j)^2 \times \prod_{j,k=1}^s \frac{1}{(t^2 z_j z_k - 2t\Delta z_j z_k + 1)} \prod_{j=1}^s \frac{[(t^2 - 2t\Delta)z_j + 1]^{s-1}}{z_j(z_j - 1)^s} \times h_{N,s}(z_1, \ldots, z_s) h_{s,s}(u_1, \ldots, u_s) \, dz_1 \cdots dz_s.\] (6.13)
This formula is the main result of the present paper.

As a comment to this result, let us also mention that due to (6.8) one can obtain from (6.13) a multiple integral representation, in which the integrand is expressed in terms of the partition functions. Indeed, since \(z_j\)'s and \(u_j\)'s are given by (5.7) and (6.4) where functions \(\omega(\xi)\) and \(\gamma(\xi)\) appear to be related to each other.
as \( \gamma(\xi) = \omega(-\lambda + \eta - \xi) \), or \( \omega(\xi) = \gamma(-\lambda + \eta - \xi) \), one may prefer to use \( \xi_1, \ldots, \xi_s \) as the integration variables, and furthermore (6.8) implies that

\[
h_{N,s}(\omega(\xi_1), \ldots, \omega(\xi_s)) = \frac{Z_N(\eta - \xi_1, \ldots, \eta - \xi_s, \lambda, \ldots, \lambda)}{Z_N(\lambda, \ldots, \lambda)} \prod_{j=1}^s \left[ \frac{a(\lambda)}{a(\eta - \xi_j)} \right]^{N-1}. \]

As a result, we obtain the following alternative multiple integral representation

\[
F_N^{(r,s)} = \frac{(-1)^{Ns+s(s+1)/2}a^{(N-r)s}b^{rs}}{s!(2\pi i)^s} \int_{C_0} \cdots \int_{C_0} \prod_{1 \leq j < k \leq s} [\sin(\xi_k - \xi_j)]^2 \times \prod_{j,k=1 \atop j \neq k}^s \frac{1}{\sin(\xi_j - \xi_k + 2\eta)} \prod_{j=1}^s \frac{1}{\sin(\xi_j - 2\eta)}^{N-r} \times Z_N(\eta - \xi_1, \ldots, \eta - \xi_s, \lambda, \ldots, \lambda) \prod_{j=1}^s d\xi_1 \cdots d\xi_s. \tag{6.14}
\]

This formula evidently recalls multiple integral representations which appear for correlation functions in quantum spin chains (see, e.g., [19–22]).

The important and essentially nontrivial object in the integrand of (6.14) is the partition function \( Z_N \), with \( s \) shifted arguments. While the presence of the second one, \( Z_s \), is a standard consequence of the Yang-Baxter algebra, the first partition function, \( Z_N \), together with other factors in the integrand, is due to the specificities of both the boundary conditions (domain-wall ones) and the particular correlation function under consideration (EFP).

In conclusion, let us briefly discuss formula (6.13). This representation may seem not completely explicit since it is heavily based on the use of the generating function \( h_N(z) \). We expect however representation (6.13) to appear most useful when addressing the problem of limit shapes in the model. In the case of vertex weights obeying the free-fermion condition \( (\eta = \pi/4) \), function \( h_N(z) \) is known explicitly and particularly simple. This case has been considered in [15]. Fortunately, function \( h_N(z) \) is also known at \( \eta = \pi/6 \) and \( \eta = \pi/3 \), for \( \lambda = \pi/2 \). The first case corresponds to enumeration of alternating sign matrices and the second one to a particular example of their weighted enumeration (the so-called 3-enumeration). For this reason we expect that formula (6.13) may appear also very useful out of the free-fermion case, and, at least, allows one to solve the long-standing problem of the limit shape of large alternating sign matrices.

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