Contents

Preface
1. Introduction 4
2. Compactly generated spaces 11
3. Weakly principal and principal bundles 16
4. McCord classifying spaces and principal bundles 22
5. Spaces with amenable fundamental group 25
6. Weak equivalences 38
7. Elements of homological algebra 42
8. Bounded cohomology and the fundamental group 54
9. The covering theorem 64
10. An algebraic analogue of the mapping theorem 74

Appendices
A.1. The differential of the standard resolution 84
A.2. The complex $B(G^*, U)^G$ 86
A.3. The second bounded cohomology group 87
A.4. Functoriality with coefficients 89
A.5. Straight and Borel straight cochains 90
A.6. Double complexes 92

References 92
Preface

The sources of the bounded cohomology theory. The bounded cohomology theory was introduced by Gromov in his fundamental paper [Gro1]. It was born in a rich and diverse geometric context. This context includes the works of Milnor, Hirsch–Thurston, Sullivan, Smillie, and others on flat bundles and their characteristic classes, especially the Euler class. It includes also Mostow’s rigidity for hyperbolic manifolds and Thurston’s ideas [T] about the geometry of 3-manifolds. But the main motivation came from riemannian geometry and, in particular, from Cheeger’s finiteness theorem. To quote Gromov [Gro1],

The main purpose of this paper is to provide new estimates from below for the minimal volume in terms of the simplicial volume defined in section (0.2).

Here the minimal volume of a riemannian manifold $M$ is the infimum of the total volume of complete riemannian metrics on $M$ subject to the condition that the absolute value of all sectional curvatures is $\leq 1$. The simplicial volume of $M$ is a topological invariant of $M$ defined in terms of the singular homology theory. The bounded cohomology theory emerged as the most efficient tool to deal with the simplicial volume. For further details of this story we refer the reader to [Gro1], Introduction, and [Gro2], Sections 5.34–5.43.

The origins of the present paper. These are much less glamorous. Gromov’s paper [Gro1] consists of two types of results: the geometric and topological results motivating and applying the bounded cohomology theory; and the results dealing with the bounded cohomology theory proper. Back in the early 1980–is I found the results of the first type fairly accessible, but failed in my attempts to understand the proofs of the general results about the bounded cohomology, such as the vanishing of the bounded cohomology of simply-connected spaces.

Still, I was fascinated by this theorem and wanted to know a proof. The only option available was to prove it myself. I found a proof based on a modification of Cartan–Serre killing homotopy group process [CS] and Dold–Thom construction [DT]. Using the Dold–Thom results apparently required to limit the theory by spaces homotopy equivalent to countable CW-complexes, but this was sufficient for Gromov’s applications. Emboldened by this success, I found proofs of most of Gromov’s basic theorems. These proofs were based on a strengthening of the vanishing theorem, the language of the homological algebra, as suggested by R. Brooks [Br], and the sheaf theory. I did not attempt to deal with the last part of [Gro1], which is devoted to relative bounded cohomology and applications to complete riemannian manifolds.

This work was reported in my 1985 paper [I1]. The present paper grew out of a modest project to correct the fairly inadequate English translation of [I1] and to typeset it in LaTeX. This project was started in early 1990–is and was abandoned two times due to the lack of convenient tools to code moderately complicated commutative diagrams. The third attempt quickly got out of control and lead to a major revision of the theory.
A stimulus for the revision was provided by a remark of Th. Bühler in the Introduction to his monograph [Bü]. Referring to the theorem about vanishing of the bounded cohomology of a simply connected space $X$ homotopy equivalent to a countable CW-complex, he wrote

The proof of this result is quite difficult and not very well understood as is indicated by the strange hypothesis on $X$ (Gromov does not make this assumption explicit, his proof is however rather sketchy to say at least). The reason for this is the fact that the complete proof given by Ivanov uses the Dold–Thom construction which necessitates the countability assumption.

In the present paper all results are proved without the countability assumption. While keeping the core ideas intact, this requires reworking the theory from the very beginning. An argument going back to Eilenberg [E] allows to extend most of the results to arbitrary topological spaces.

In a separate development, I realized that the power of the relative homological algebra is rather limited. The bounded cohomology groups are real vector spaces equipped with a canonical semi-norm, which is the *raison d'être* of Gromov’s theory. While relative homological algebra provides a convenient framework for discussing the bounded cohomology groups as topological vector spaces, it does not allow to efficiently recover the canonical semi-norms. By this reason the relative homological algebra is moved in the present paper from the forefront to the background. In particular, the bounded cohomology of groups are defined in terms of the standard resolutions, in contrast with [I$_1$], where they were defined in terms of strong relatively injective resolutions, and a trick was used to recover the canonical semi-norms.

**Spaces vs. groups.** In an agreement with the spirit of Gromov’s paper [Gro$_1$], the focus of the present paper is on the bounded cohomology of spaces, and the bounded cohomology of discrete groups are treated as a tool to study the bounded cohomology of spaces.

By now the bounded cohomology theory of groups is a well established subject in its own right, dealing with locally compact groups and especially with the Lie groups and lattices in them. Although some of the ideas of [I$_1$] found another home in this theory, the present paper is intended to provide the foundations for the bounded cohomology theory of spaces but not of groups. The monographs by N. Monod [Mo] and Th. Bühler [Bü] discuss the foundation of the bounded cohomology theory of groups from two rather different points of view.

The focus of the cohomology theory of groups is on the cohomology with non-trivial coefficients. By this reason in this paper the bounded cohomology theory of groups is discussed in the case of general coefficients, in contrast with [I$_1$], where the case of non-trivial coefficients was left to the reader. Given the results of the paper, it is a routine matter to generalize the bounded cohomology theory of spaces to cohomology with local coefficients. But in order not to obscure the main ideas, the exposition is limited by the case of trivial coefficients.

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* Some remarks of M. Blank [Bl$_1$], [Bl$_2$] apparently imply that later Th. Bühler realized that the countability assumption is not crucial for the methods of [I$_1$], but did not wrote down any details.
1. Introduction

**Singular cohomology.** Let us recall the definition of the singular cohomology of topological spaces. Anticipating the theory of bounded cohomology, we will restrict our attention by cohomology with coefficients in \( \mathbb{R} \). Let \( X \) be a topological space. For each integer \( n \geq 0 \) let \( \Delta_n \) be the standard \( n \)-dimensional simplex and let \( S_n(X) \) be the set of continuous maps \( \Delta_n \to X \), i.e. the set of \( n \)-dimensional singular simplices in \( X \). The real \( n \)-dimensional singular cochains are defined as arbitrary functions \( S_n(X) \to \mathbb{R} \). They form a group and even a vector space over \( \mathbb{R} \), which is denoted by \( C^n(X) \). The formula

\[
\partial f(\sigma) = \sum_{i=0}^{n+1} (-1)^i f(\partial_i \sigma),
\]

where \( \partial_i \sigma \) is the \( i \)-th face of the singular simplex \( \sigma \), defines a map

\[
\partial : C^n(X) \to C^{n+1}(X).
\]

As is well known, \( \partial \circ \partial = 0 \). The real singular cohomology groups \( H^*(X) \) of the space \( X \) are defined as the cohomology groups of the complex

\[
0 \to C^0(X) \xrightarrow{\partial} C^1(X) \xrightarrow{\partial} C^2(X) \xrightarrow{\partial} \cdots.
\]

**Bounded cohomology.** Gromov [Gro] modified the above definition in a minor, at the first sight, way. He replaced the space \( C^n(X) \) of all functions \( S_n(X) \to \mathbb{R} \) by the space \( B^n(X) \) of bounded functions \( S_n(X) \to \mathbb{R} \). Such functions are called *bounded \( n \)-cochains*. Clearly,

\[
\partial(B^n(X)) \subset B^{n+1}(X)
\]

and hence

\[
0 \to B^0(X) \xrightarrow{\partial} B^1(X) \xrightarrow{\partial} B^2(X) \xrightarrow{\partial} \cdots,
\]

is a complex of real vector spaces. The *bounded cohomology spaces* of \( X \) are defined as the cohomology spaces of this complex and are denoted by \( \hat{\mathbb{H}}^*(X) \). In more details,

\[
\hat{\mathbb{H}}^n(X) = \text{Ker} (\partial | B^n(X)) / \text{Im} (\partial | B^{n-1}(X)).
\]

The bounded cohomology spaces \( \hat{\mathbb{H}}^n(X) \) are real vector spaces carrying a *canonical semi-norm*, their *raison d’être*. In order to define this semi-norm, let us recall that \( B^n(X) \) is a
Banach space with the norm $\| \cdot \|$ defined as

$$\| f \| = \sup_{\sigma \in S_n(X)} |f(\sigma)|.$$ 

The cohomology $\widehat{H}^n(X)$ inherits a semi-norm $\| \cdot \|$ from the norm on $B^n(X)$ in an obvious manner. Namely, if $c \in \widehat{H}^n(X)$, then

$$\| c \| = \inf \| f \|,$$

where the infimum is taken over all cochains $f \in B^n(X)$ representing the cohomology class $c$. It may happen that $c \neq 0$ but $\| c \| = 0$. This is possible if and only if the image of $\partial : B^{n-1}(X) \to B^n(X)$ is not closed.

The bounded cohomology have the same functorial properties as the usual ones. Thus each continuous map $X \to Y$ induces a map $\widehat{H}^*(Y) \to \widehat{H}^*(X)$, and the maps induced by homotopic maps $X \to Y$ are equal. In particular, $\widehat{H}^*(X) = 0$ if $X$ is contractible. The proofs are the same as in the singular cohomology theory, since the chain maps and the chain homotopies used in these proofs map bounded cochains to bounded cochains.

**Bounded cohomology of groups.** The *bounded cohomology* $\widehat{H}^n(G)$ of a discrete group $G$ are defined by Gromov (see [Gro1], Section 2.3) as the bounded cohomology of an Eilenberg–MacLane space $K(G, 1)$. But this definition is hardly used by Gromov directly.

There is also an algebraic definition in the same spirit as the definition of the bounded cohomology of spaces. One starts with the definition of the classical cohomology groups $H^n(G, \mathbb{R})$ based on the standard resolution of the trivial $G$-module $\mathbb{R}$ and replaces arbitrary real-valued functions by the bounded ones. In more details, let $B(G^n)$ be the vector space of bounded real-valued functions $G^n \to \mathbb{R}$. The usual supremum norm turns $B(G^n)$ into a Banach space, and the action of $G$ defined by

$$(h \cdot f)(g_1, \ldots, g_n) = f(g_1, \ldots, g_{n-1}, g_nh)$$

turns $B(G^n)$ into a $G$-module. Let us consider the sequence

$$0 \to B(G) \xrightarrow{d_0} B(G^2) \xrightarrow{d_1} B(G^3) \xrightarrow{d_2} \cdots,$$

where the *differentials* $d_n$ are defined by the formula

$$d_n(f)(g_0, g_1, \ldots, g_{n+1}) = (-1)^{n+1} f(g_1, g_2, \ldots, g_{n+1})$$

$$+ \sum_{i=0}^{n} (-1)^{n-i} \left( f(g_0, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) \right).$$
A standard calculation shows that \( d_{n+1} \circ d_n = 0 \) for all \( n \geq 0 \), i.e. that \( B(G^{•+1}) \) together with the differentials \( d_• \) is a complex. A motivation for the formula defining \( d_n \), which also leads to a non-calculational proof of the identity \( d_{n+1} \circ d_n = 0 \), is discussed in Appendix 1.

The differentials \( d_n \) commute with the action of \( G \), and hence the subspaces of \( G \)-invariant vectors form a subcomplex. The bounded cohomology spaces \( \hat{H}^n(G) \) of \( G \) are defined as the cohomology spaces of this subcomplex. Such a definition is also contained in [Gro1], albeit hidden inside of a proof. See [Gro1], the last page of Section 3.3.

**Amenable groups.** The notion of an amenable group was introduced by von Neumann [N]. Morally speaking, a group \( G \) is amenable if one can assign to each bounded function \( G \to \mathbb{R} \) a real number deserving to be called its mean or the average value in a way invariant under the translations by the elements of \( G \). More formally, \( G \) is called amenable if there exists a bounded linear functional \( B(G) \to \mathbb{R} \) invariant under the natural right action of \( G \) on the space of such functionals and having the norm \( \leq 1 \). Amenable groups play a central role in the bounded cohomology theory. The following theorem is the first illustration of this role.

**Theorem.** Suppose that \( X \) is a path-connected space homotopy equivalent to a CW-complex and having amenable fundamental group. Then \( \hat{H}(X) = 0 \) for all \( i \geq 1 \).

See Theorem 5.6. In particular, \( \hat{H}(X) = 0 \) for all \( i \geq 1 \) if \( X \) is simply-connected and homotopy equivalent to a CW-complex. While this theorem is remarkable by itself, the following stronger form of it is crucial for the methods of the present paper. Recall that contracting chain homotopy for a complex is a chain homotopy between its identity map and the zero map.

**Theorem.** Suppose that \( X \) is a path-connected space homotopy equivalent to a CW-complex. If \( \pi_1(X) \) is amenable, then the complex \( B^•(X) \) admits a contracting chain homotopy consisting of bounded maps with the norm \( \leq 1 \).

See Theorem 5.10. These two theorems are the main results of Section 5. Sections 2 – 4 are devoted to the tools used in their proofs. Section 4 is a review of McCord’s theory of classifying spaces [McC], which replaced the Dold–Thom construction used in [I1]. McCord [McC] works in the category of compactly generated spaces, which are discussed in Section 2. Section 3 is devoted to a modification of the theory of principal bundles used in Section 5.

**Theorem.** Suppose that \( X, A \) are path-connected spaces and \( \varphi: A \to X \) is a weak equivalence, i.e. a map inducing isomorphisms of all homotopy groups. Then the induced map

\[
\varphi^*: \hat{H}^n(X) \to \hat{H}^n(A)
\]

of the bounded cohomology groups is an isometric isomorphism for all \( n \).
See Theorem 6.4. By well known theorems of J.H.C. Whitehead, this theorem allows to extend the two previous theorems to arbitrary topological spaces and reduce most of the questions about bounded cohomology to the case of spaces homotopy equivalent to CW-complexes.

**Homological algebra.** Section 7 begins with a fragment of homological algebra needed to work with the bounded cohomology of discrete groups. While most of the results of this sort appear to be direct adaptations of the classical results, we included detailed proofs. The reason was very well spelled out by N. Monod in the Introduction to his monograph [Mo].

Well, first of all, such “obvious transliterations” often just fail to hold true. Furthermore, it happens also that usual proofs do not yield the most accurate statement in our setting. This is illustrated e.g. by the fundamental lemma on comparison of the resolutions.

The difficulties with the fundamental lemma encountered by N. Monod [Mo] are exactly the same as in [I1] and the present paper. The main result of Section 7 is Theorem 7.6, one of the two main tools to deal with these difficulties. It is too technical to state it in the introduction, but the point is that it allows sometimes to replace the existence statement of the fundamental lemma by an explicit construction with an adequate control of the norms. Theorem 7.6 extends Theorem 3.6 of [I1] to the case of twisted coefficients. In his situation, N. Monod deals with this difficulty by proving an analogue of Theorem 3.6 of [I1]. See [Mo], Theorem 7.3.1.

**The bounded cohomology of spaces and of groups.** This is the topic of Section 8. Suppose that $G$ is a discrete group and $p: \mathcal{X} \to X$ is a locally trivial principal $G$-bundle. Then $G$ acts freely on $\mathcal{X}$, the quotient space $\mathcal{X}/G$ is equal to $X$, and $p$ is a covering space projection. One can construct a $G$-equivariant morphism of complexes

$$r_\ast : B(G^{*+1}) \to B^*(\mathcal{X}).$$

consisting of maps with the norm $\leq 1$. This construction is the second main tool for overcoming difficulties with the fundamental lemma. It is fairly flexible and works also in other situations. See, for example, Appendix 5. The morphism $r_\ast$ induces a map

$$\hat{H}^*(G) \to \hat{H}^*(X)$$

with the norm $\leq 1$. If $\pi_1(\mathcal{X})$ is amenable, then $B^*(\mathcal{X})$ admits a contracting chain homotopy consisting of bounded maps with the norm $\leq 1$. This allows to apply Theorem 7.6 and conclude that there exists a $G$-equivariant morphism of complexes

$$u_\ast : B^*(\mathcal{X}) \to B(G^{*+1})$$

consisting of maps with the norm $\leq 1$. 

7
The morphism $u_*$ induces a map $\hat{H}^*(X) \to \hat{H}^*(G)$ with the norm $\leq 1$. The two maps between $\hat{H}^*(X)$ and $\hat{H}^*(G)$ turn out to be mutually inverse. This leads to the following fundamental theorem.

**Theorem.** If $\pi_1(\mathcal{X})$ is amenable, then the maps

$$u_* : \hat{H}^*(X) \xrightarrow{\sim} \hat{H}^*(G) : r_*,$$

induced by $u_*$ and $r_*$ respectively, are mutually inverse isometric isomorphisms.

See Theorem 8.2 for a formally stronger result. This theorem, in particular, immediately implies that there exists an isometric isomorphism $\hat{H}^*(X) \to \hat{H}^*(\pi_1(X))$. With these tools at hand, it is an easy matter to prove the functoriality of this isomorphism and other expected properties. See Section 8. As the first application of these tools to the bounded cohomology of spaces, we prove the first of two main Gromov’s theorems about bounded cohomology, called by him the **Mapping theorem**.

**The Mapping theorem.** Let $X, Y$ be two path-connected spaces and let $\varphi : Y \to X$ be a continuous map. If the induced homomorphism of the fundamental groups

$$\varphi_* : \pi_1(Y) \to \pi_1(X)$$

is surjective with amenable kernel, then $\varphi^* : \hat{H}^*(X) \to \hat{H}^*(Y)$ is an isometric isomorphism.

See Theorem 8.4. This is a far reaching generalization of the vanishing theorem for spaces with amenable fundamental group. Another generalization deals with coverings of spaces. In order to state it, let us call a subset $Y$ of a topological space $X$ amenable if for every path-connected component $Z$ of $Y$ the image of the inclusion homomorphism $\pi_1(Z) \to \pi_1(X)$ is an amenable subgroup of $\pi_1(X)$. Let us call a covering of a space $X$ by its subsets nice if it is either an open covering, or is a locally finite closed covering, and the space $X$ satisfies some mild technical assumptions (see the assumption (C) at the beginning of Section 9).

**The covering theorem.** Let $\mathcal{U}$ is a covering of a space $X$ by amenable subsets. Suppose that $\mathcal{U}$ is either a nice covering, or a finite open covering. Let $N$ be the nerve of the covering $\mathcal{U}$ and $|N|$ be the geometric realization of $N$. Then the canonical homomorphism $\hat{H}^*(X) \to H^*(X)$ can be factored through the canonical homomorphism $H^*(|N|) \to H^*(X)$.

See Theorem 9.1 and Theorem 9.3 for the cases of the nice coverings and of finite open coverings respectively. The covering theorem is a generalization and at the same time a much more precise version of the second of two main Gromov’s theorems about bounded cohomology, called by him the **Vanishing theorem**.
The **Vanishing theorem.** Suppose that a manifold $X$ can be covered by open amenable subsets in such a way that every point of $X$ is contained in no more than $m$ elements of this covering. Then the canonical homomorphism $\hat{H}^i(X) \rightarrow H^i(X)$ vanishes for $i \geq m$.

Here we implicitly assume that manifolds are required to be paracompact (this is needed to insure the niceness of the covering if it is infinite). Since Gromov developed the bounded cohomology theory for the sake of applications to riemannian manifolds, and all riemannian manifolds are paracompact, this seems to be a reasonable assumption.

The proofs of the last two theorems are independent of the theory of bounded cohomology of groups, i.e. of Sections 7 and 8. Using the sheaf theory, these theorems are deduced from the vanishing of the bounded cohomology of spaces with amenable fundamental groups.

The **mapping theorem for groups.** Suppose that $A$ is a normal amenable subgroup of a group $\Gamma$. Then the quotient map $\Gamma \rightarrow \Gamma/A$ induces an isometric isomorphism

$$\alpha^* : \hat{H}^*(\Gamma/A) \rightarrow \hat{H}^*(\Gamma).$$

By using Eilenberg–MacLane spaces and isometric isomorphisms $\hat{H}^*(X) \rightarrow \hat{H}^*(\pi_1(X))$ one can easily deduce this theorem from the Mapping theorem. Nevertheless, a proof based only on the results of Section 7 is given in Section 10. The main reason is that the algebraic proof is a drastically simplified version of the main part of the proof of the vanishing theorem of the bounded cohomology of spaces with amenable fundamental groups. As such, it is quite instructive. The algebraically-minded readers may prefer to start with Sections 7 and 10.

**Twisted coefficients.** Another reason for presenting an algebraic proof of the mapping theorem for groups is that this is the best place to go beyond the definition of the bounded cohomology with twisted coefficients and to prove something non-trivial about them. The main theorem of Section 10, namely Theorem 10.7, is a version of the mapping theorem for groups for the bounded cohomology with twisted coefficients, although not with arbitrary ones. In order for the proof to work, the module of coefficients $U$ should admit an invariant averaging procedure for $U$-valued functions on amenable groups. In a related context of the continuous cohomology of Banach algebras, B. Johnson [J] noticed that this is the case when $U$ is the Banach dual of some other Banach module, and in Section 10 we work only with such modules of coefficients. This class of coefficients plays a central role in the work of N. Monod, and one may think that this is “the right” class of coefficients for the bounded cohomology theory.

In the context of the bounded cohomology of discrete groups this class of coefficients was used for the first time by G.A. Noskov [No], who proved a weaker version of Theorem 10.7. Contrary to his claim, his proof cannot be adjusted to prove that the induced map is an isometry. Cf. the above discussion of the difficulties with the fundamental lemma.
Means of vector-valued functions. They play a central role in the bounded cohomology theory with twisted coefficients. One of the tools used may be of independent interest.

Let $U = V^*$ be a Banach space dual to some Banach space $V$. Let $A$ be a set and let $B(A, U)$ be the Banach space of maps $f: A \to U$ such that the real-valued function $\|f\|$ is bounded. Then every mean $M: B(A) \to \mathbb{R}$ leads to a mean $m: B(A, U) \to U$. The mean $m$ commutes with all operators $L^*: U \to U$ adjoint to bounded operators $L: V \to V$.

If $A$ is a group and $M$ is an invariant mean, then $m$ is also invariant. See Lemma 10.1. The construction $M \mapsto m$ commutes with push-forwards by surjective homomorphisms (see Sections 5 and 10 for the definition). This is an obvious, but crucial observation.

Bounded cohomology of spaces with twisted coefficients. The theory of bounded cohomology of spaces can be fairly routinely extended to the case of twisted (better known as local) coefficients in $\pi_1(X)$-modules dual to Banach $\pi_1(X)$-modules. The properties of the above construction of means of vector-valued functions ensure that all arguments in Section 5 still work. Apparently, there is no averaging procedure for more general coefficients, except in hardly interesting special cases, such as for spaces with finite fundamental group.

Appendices. There are six appendices, complementing the main part of the text.

A.1. This appendix is devoted to a conceptual motivation of the definition of the cohomology of groups, both the bounded and the usual ones. There is no reason to overload the main part of the text by this categorical approach.

A.2. This appendix goes in the opposite direction and gives an explicit description of the subcomplex of invariant subspaces of $B(G^*)$.

A.3. This appendix is devoted to a proof of a theorem to the effect that $\hat{H}^2(G)$ is always a Hausdorff (and hence a Banach) space. See Theorem A.3.1. This result is due independently to Sh. Matsumoto and Sh. Morita [MM] and the author [I2].

A.4. In this appendix the functorial properties of the bounded cohomology of groups with coefficients are presented in full generality, complementing Sections 8 and 10.

A.5. In this appendix the methods of Section 8 are applied to the complex of straight Borel cochains, playing an important role in Gromov's paper [Gro1].

A.6. This appendix is just a statement of a theorem about double complexes used in Section 9 and an old-fashioned reference for its proof.
2. Compactly generated spaces

**Test spaces and test maps.** Following [tD] (see [tD], Section 7.9), a test space is defined as a compact Hausdorff space, and a test map is defined as a continuous map \( f : C \to X \) from a test space to a topological space \( X \).

**Compactly generated spaces.** A subset \( A \) of a topological space \( X \) is called **compactly closed** if for every test map \( f : C \to X \) the preimage \( f^{-1}(A) \) is closed. If \( A \) is closed in \( X \), then \( A \) is compactly closed. For a topological space \( X \) the space \( k(X) \) is defined as the topological space having the same set of points as \( X \) and compactly closed subsets of \( X \) as the closed sets. The space \( X \) is called **compactly generated** if \( k(X) = X \) as topological spaces, i.e. if every compactly closed subset of \( X \) is closed.

**2.1. Lemma.** Suppose that \( C \) is a test space. A set-theoretic map \( f : C \to X \) is continuous as a map into \( X \) if and only if it is continuous as map into \( k(X) \).

**Proof.** If \( f \) is continuous as a map into \( X \), then \( f \) is a test map. If \( A \) is closed in \( k(X) \), then \( A \) is compactly closed in \( X \) and hence \( f^{-1}(A) \) is closed. Hence \( f \) is continuous as a map into \( k(X) \). If \( f \) is continuous as a map into \( k(X) \) and \( A \) is closed in \( X \), then \( A \) is compactly closed and hence \( f^{-1}(A) \) is closed. It follows that \( f \) is continuous as a map into \( X \).  

**2.2. Lemma.** Suppose that \( X, Y \) are topological spaces.

(i) The identity map \( k(X) \to X \) is continuous.

(ii) \( k(X) \) is a compactly generated space, i.e. \( k(k(X)) = k(X) \).

(iii) If \( F : X \to Y \) is continuous, then \( F \) is continuous as a map \( k(X) \to k(Y) \).

**Proof.** The part (i) is obvious. Lemma 2.1 implies that the properties of being compactly closed in \( X \) and in \( k(X) \) are equivalent. The part (ii) follows. If \( f : C \to X \) is a test map, then \( F \circ f \) is a test map \( C \to Y \). Therefore, if \( A \) is compactly closed in \( Y \), then

\[
    f^{-1}(F^{-1}(A)) = (F \circ f)^{-1}(A)
\]

is closed in \( C \). It follows that \( F^{-1}(A) \) is compactly closed in \( X \). The part (iii) follows.

**2.3. Lemma.** Suppose that \( p : X \to Y \) is a quotient map (i.e. \( p \) is a continuous surjective map and \( Y \) has the quotient topology with respect to \( p \)). If \( X \) is compactly generated, then \( Y \) is compactly generated.
Proof. Let \( B \subset Y \) be compactly closed subset. Since \( p \) is a quotient map, \( B \) is closed if \( p^{-1}(B) \) is closed. If \( f : C \to X \) is a test map, then \( p \circ f : C \to Y \) is a test map and hence

\[
f^{-1}(p^{-1}(B)) = (p \circ f)^{-1}(B)
\]

is closed. It follows that \( p^{-1}(B) \) is compactly closed, and hence is closed in \( X \). It follows that \( B \) is closed in \( Y \). The lemma follows. ■

2.4. Lemma. Suppose that \( Y \subset X \) is a subspace of a compactly generated space \( X \). If \( Y \) is either closed or open, then \( Y \) is compactly generated.

Proof. Suppose that \( Y \) is closed. Let \( B \) be a compactly closed subset of \( Y \). We need to prove that \( B \) is closed in \( Y \), or, equivalently, that \( B \) is closed in \( X \). If \( f : C \to X \) is a test map, then \( D = f^{-1}(Y) \) is closed in \( C \). Since \( C \) is compact, \( D \) is also compact and hence the map \( g : D \to Y \) is a test map. Since \( B \) is compactly closed in \( Y \), it follows that \( g^{-1}(B) \) is closed in \( D \) and hence in \( C \). But \( B \subset Y \) implies that \( f^{-1}(B) = g^{-1}(B) \) and hence \( f^{-1}(B) \) is closed in \( C \). It follows that \( B \) is compactly closed in \( X \) and hence is closed in \( X \). Therefore, \( B \) is closed in \( Y \). Since \( B \) was an arbitrary compactly closed subset of \( Y \), it follows that the space \( Y \) is compactly generated.

Suppose now that \( Y \) is open. Let \( B \) be a compactly closed subset of \( Y \). In order to prove that \( B \) is closed in \( Y \), it is sufficient to prove that \( Y \smallsetminus B \) is open in \( Y \), or, equivalently, that \( Y \smallsetminus B \) is open in \( X \). Since \( X \) is compactly generated, it is sufficient to prove that \( f^{-1}(Y \smallsetminus B) \) is open for every test map \( f : C \to X \). Suppose that \( y \in f^{-1}(Y \smallsetminus B) \). Then \( f^{-1}(Y) \) is an open neighborhood of \( y \) in \( C \). Since \( C \) is compact Hausdorff and hence is regular, there exists an open set \( U \subset C \) such that \( y \in U \) and the closure \( \overline{U} \) is contained in \( f^{-1}(Y) \). Then \( \overline{U} \) is a compact Hausdorff space and hence \( f \) induces a test map \( g : \overline{U} \to Y \). Since \( B \) is compactly closed, the preimage \( g^{-1}(B) \) is closed in \( \overline{U} \) and hence

\[
f^{-1}(Y \smallsetminus B) \cap \overline{U} = \overline{U} \smallsetminus g^{-1}(B)
\]

is open in \( \overline{U} \). It follows that \( f^{-1}(Y \smallsetminus B) \cap U \) is open in \( U \) and hence is open in \( C \). Therefore \( f^{-1}(Y \smallsetminus B) \) is a neighborhood of \( y \) in \( C \). It follows that \( f^{-1}(Y \smallsetminus B) \) is open in \( C \), and hence \( Y \smallsetminus B \) is open in \( X \) and \( B \) is closed in \( Y \). ■

Products. The usual cartesian product of topological spaces \( X, Y \) will be denoted by \( X \times_c Y \). Even if \( X, Y \) are compactly generated, then \( X \times_c Y \) does not need to be compactly generated. In order to deal with this issue, a new product \( X \times_k Y = k(X \times_c Y) \) is introduced.

2.5. Lemma. If \( X, Y \) are compactly generated, then \( X \times_k Y \) is the product in the category of all compactly generated spaces and continuous maps.
Proof. By Lemma 2.2 (i) the map $X \times_k Y \to X \times_c Y$ is continuous. Since the projections $X \times_c Y \to X, Y$ are continuous, the same projections considered as maps $X \times_k Y \to X, Y$ are also continuous. Suppose that $Z$ is a compactly generated space and that $f$ and $g$ are continuous maps $Z \to X$ and $Z \to Y$ respectively. Since $X \times_c Y$ is the product in the category of all topological spaces, there is a unique map $(f, g) : Z \to X \times_c Y$ with the components $f, g$. By Lemma 2.2 (iii) the map $(f, g)$ leads to a continuous map $k(f, g) : k(Z) \to k(X \times_c Y)$ with the components $f, g$. Since $Z = k(Z)$, this completes the proof. ■

2.6. Lemma. If $X$ is compactly generated and $Y$ is locally compact Hausdorff, then $X \times_c Y$ is compactly generated and hence $X \times_k Y = X \times_c Y$.

Proof. The following proof largely follows the proof of Theorem 4.3 in [St]. Let $A$ be a compactly closed subset of $X \times_c Y$ and $(x, y)$ be a point in the complement of $A$. Since $Y$ is locally compact Hausdorff, there is a compact (and Hausdorff) neighborhood $U$ of $y$ in $Y$. The inclusion $i : x \times_c U \to X \times_c Y$ is a test map and hence $A \cap (x \times_c U) = i^{-1}(A)$ is closed. Therefore, replacing $U$ by a smaller compact neighborhood of $y$ if necessary, we can assume that $A \cap (x \times_c U) = \emptyset$. Let $B$ be the projection of $A \cap (X \times U)$ to $X$. Then $x \notin B$ by the choice of $U$. If $f : C \to X$ is a test map, then

$$f \times_c \text{id}_U : C \times_c U \to X \times_c Y$$

is also a test map. Since $A$ is compactly closed, $(f \times_c \text{id}_U)^{-1}(A)$ is closed in $C \times_c U$. Since $U$ is compact, the projection of this preimage to $C$ is closed in $C$. But this projection is equal to $f^{-1}(B)$. It follows that $f^{-1}(B)$ is closed for any test map $f$ and hence $B$ is closed. Since $x \notin B$ and $y \in U$, the set $(X \setminus B) \times_c U$ is a neighborhood of $(x, y)$ disjoint from $A$. Since $(x, y)$ is an arbitrary point in the complement of $A$, it follows that $A$ is closed in $X \times_c Y$. ■

2.7. Lemma. Suppose that $X, Y, Z$ are compactly generated spaces and $p : X \to Y$ is a quotient map (i.e. $p$ is a continuous surjective map and $Y$ has the quotient topology with respect to $p$). Then $p \times_k \text{id}_Z : X \times_k Y \to Y \times_k Z$ is a quotient map.

Proof. If spaces $X, Y, Z$ are Hausdorff, there is a direct proof in the same spirit as the proof of Lemma 2.6. See [St], Theorem 4.4. In the general case it is more convenient to reduce this lemma to a theorem about function spaces. See [tD], Theorem 7.9.19. ■

Weakly Hausdorff spaces. A topological space $X$ is called weakly Hausdorff if $f(C)$ is closed for every test map $f : C \to X$. In the category of compactly generated spaces weakly Hausdorff spaces play the same role as Hausdorff spaces in the category of all topological spaces. By Lemma 2.10 below weakly Hausdorff compactly generated spaces can be defined in the same way as separated schemes in the algebraic geometry. The impatient readers may skip the next three lemmas (which will not be used directly) and go to the last subsection of this section.
2.8. Lemma.

(i) A Hausdorff space is weakly Hausdorff.

(ii) A weakly Hausdorff space is a $T_1$-space.

(iii) A subspace of a weakly Hausdorff space is weakly Hausdorff.

(iv) If $X$ is weakly Hausdorff, then $k(X)$ is also weakly Hausdorff.

(v) If $X$ is a weakly Hausdorff space, then $f(C)$ is Hausdorff for every test map $f: C \to X$.

(vi) If $X, Y$ are weakly Hausdorff spaces, then $X \times kY$ is weakly Hausdorff.

Proof. The properties (i)–(iii) are trivial. Let us prove (iv). Suppose that $X$ is weakly Hausdorff. If $f: C \to k(X)$ is a test map, then by Lemma 2.1 $f$ is also a test map $C \to X$. Since $X$ is weakly Hausdorff, the image $f(C)$ is closed in $X$. It follows that $f(C)$ is closed in $k(X)$. This proves that $k(X)$ is weakly Hausdorff. This proves (iv).

Let us prove (v). Suppose that $x, y \in f(C)$ and $x \neq y$. By Lemma 2.8(ii) the sets $\{x\}$ and $\{y\}$ are closed and hence the preimages $f^{-1}(x)$ and $f^{-1}(y)$ are closed. Since these preimages are disjoint and the space $C$, being compact and Hausdorff, is normal, there exist disjoint open sets $U, V \subset C$ containing $f^{-1}(x), f^{-1}(y)$ respectively. Then $C \sim U$ and $C \sim V$ are closed in $C$ and hence are compact and Hausdorff. Since $X$ is weakly Hausdorff, the images $f(C \sim U)$ and $f(C \sim V)$ of these complements are closed and hence the complements $f(C) \sim f(C \sim U)$ and $f(C) \sim f(C \sim V)$ of these images in $f(C)$ are open in $f(C)$. Since these complements are disjoint and contain $x$ and $y$ respectively, this proves that there are two disjoint open sets in $f(C)$ containing the points $x, y$ respectively. This proves (v).

The claim (vi) follows from (v) and the standard properties of the usual product $\times c$, and (vi) immediately follows from (iv) and (vi). 

2.9. Lemma. Suppose that $X$ is weakly Hausdorff. Then $A$ is compactly closed in $X$ if and only if for every compact Hausdorff subspace $D$ of $X$ the intersection $A \cap D$ is closed in $D$.

Proof. Suppose that $A$ is compactly closed. If $D$ is a compact Hausdorff subspace of $X$, then the inclusion map $D \to X$ is a test maps. Since $A \cap D$ is the preimage of $A$ under this inclusion map, $A \cap D$ is closed in $D$.

Conversely, suppose that $A$ satisfies the condition from the lemma. If $f: C \to X$ is a test map, then $f(C)$ is Hausdorff by Lemma 2.8(v) and hence is a compact Hausdorff subspace of $X$. Therefore $f(C) \cap A$ is closed in $f(C)$ and hence $f^{-1}(A) = f^{-1}(f(C) \cap A)$ is closed in $C$. It follows that $A$ is compactly closed.
2.10. Lemma. If $X$ is compactly generated, then $X$ is weakly Hausdorff if and only if the diagonal $D_X = \{(x, x) \mid x \in X\}$ is closed in $X \times_k X$.

Proof. Suppose that $X$ is weakly Hausdorff. Let $f : C \to X \times_k X$ be an arbitrary test map. Let $f_1, f_2$ be the components of $f$. By Lemma 2.8(v) the images $f_1(C)$ and $f_2(C)$ are compact Hausdorff subspaces, and hence $K = f_1(C) \cup f_2(C)$ is also compact Hausdorff. It follows that $D_K$ is closed in $K \times_c K$ and hence in $K \times_k K$. It follows that $f^{-1}(D_K)$ is closed. But $f^{-1}(D_X) = f^{-1}(D_K)$ and therefore $f^{-1}(D_X)$ is closed. Since $X \times_k X$ is compactly generated, it follows that $D_X$ is closed in $X \times_k X$.

Suppose now that $D_X$ is closed in $X \times_k X$. Let $f : C \to X$ be a test map. We need to show that the image $f(C)$ is closed in $X$. Let $g : D \to X$ be some other test map. Then

$$f \times g : C \times_c D = C \times_k D \to X \times_k X$$

is a test map into $X \times_k X$. Therefore $(f \times g)^{-1}(D_X)$ is closed in $C \times_c D$. Since

$$g^{-1}(f(C)) = \text{pr}_2\left((f \times g)^{-1}(D_X)\right)$$

and the spaces $C, D$ are compact Hausdorff space, this implies that $g^{-1}(f(C))$ is closed in $C$. Since $X$ is compactly generated, it follows that $f(C)$ is closed in $X$. Since $f$ is an arbitrary test map, this proves that $X$ is weakly Hausdorff. ■

CW-complexes. At the end of the day, all spaces used in this paper turn out to be Hausdorff, at least if the bounded cohomology theory is limited to Hausdorff spaces. But the proofs depend on the theory of topological groups in the category of compactly generated spaces and their classifying spaces developed by McCord [McC] in the context of weakly Hausdorff compactly generated spaces. See Section 4 for an outline of this theory.

The topological groups needed are, in general, topological groups in the category of compactly generated spaces but not the topological groups in the usual sense. The reason is that they are CW-complexes and the product $\times_c$ of two CW-complexes is not necessarily a CW-complex.

We will need the classifying spaces of these topological groups only when they happen to be CW-complexes. While usually CW-complexes are assumed to be Hausdorff from the very beginning, this assumption is superfluous. If CW-complexes are defined in terms of the consecutive attaching of cells, then one can prove that they are normal, and in particular, Hausdorff spaces. See [H], Proposition A.3. If CW-complexes are defined in terms of the characteristic maps of cells, it is sufficient to assume that the space is only weak Hausdorff. Lemma 2.8(v) assures that the proof of the equivalence of this definition with the other one applies to weakly Hausdorff spaces. See [H], the proofs of Propositions A.1 and A.2.
3. Weakly principal and principal bundles

**Weakly principal bundles.** Suppose that \( G \) is simultaneously a group and a topological space. Initially no conditions relating the group structure with the topological structure are imposed. The product in \( G \) is denoted by the dot: \( (a, b) \mapsto a \cdot b \). A map \( p : E \rightarrow B \) is said to be a *weakly principal bundle* with the fiber \( G \) if the following conditions hold.

(i) \( p : E \rightarrow B \) is a Serre fibration, i.e. the homotopy covering property holds for cubes, or, what is the same, for standard simplices \( \Delta_n, \ n \geq 0 \).

(ii) A free action of \( G \) on \( E \) is given. No continuity requirements are imposed at this stage. The fibers \( p^{-1}(y), \ y \in E, \) are orbits of this action and hence \( B = E/G \) as a set.

(iii) If \( \tau : \Delta_n \rightarrow E \) is a singular simplex in \( E \), then every singular simplex \( \sigma : \Delta_n \rightarrow E \) such that \( p \circ \sigma = p \circ \tau \) has the form

\[
\sigma : x \mapsto g(x) \cdot \tau(x)
\]

for some *continuous* map \( g : \Delta_n \rightarrow G \). We will abbreviate (3.1) as \( \sigma = g \cdot \tau \).

By the condition (ii) for any two set-theoretic maps \( \sigma, \tau : \Delta_n \rightarrow E \) such that \( p \circ \sigma = p \circ \tau \) there is a unique map \( g : \Delta_n \rightarrow G \) such that \( \sigma = g \cdot \tau \). The point of the condition (iii) is in the requiring \( g \) to be continuous if \( \tau \) and \( \sigma \) are continuous.

For \( n = 0 \) the continuity requirement in (iii) is vacuous, and this special case of (iii) follows from (ii). In fact, this special case means that every non-empty fiber is an orbit of \( G \).

The property (ii) implies that all fibers are non-empty. In view of this the property (i) implies that for every singular simplex \( \rho : \Delta_n \rightarrow B \) there exists a *lift* of \( \rho \) to \( E \), i.e. a singular simplex \( \tau : \Delta_n \rightarrow E \) such that \( \rho = p \circ \tau \).

**What is really needed.** The definition of weakly principal bundles presents an intermediate ground between the classical definition of principal \( G \)-bundles for topological groups \( G \) and the properties needed for the applications we have in mind. The continuity requirement in (iii) isn’t crucial. What is really needed is the following.

(a) There are sets \( L_n \) of maps \( \Delta_n \rightarrow G \) with the following property. If \( \tau : \Delta_n \rightarrow E \) is a singular simplex, then every lift of \( p \circ \tau \) has the form \( g \cdot \tau \) with \( g \in L_n \).

(b) Each set \( L_n \) is invariant under the action of the group \( \Sigma_{n+1} \) of symmetries of \( \Delta_n \).

(c) The sets \( L_n \) form a *coherent system* in the following sense. For every simplicial embedding \( i : \Delta_{n-1} \rightarrow \Delta_n \) the set \( L_{n-1} \) is equal to the set \( \{ g \circ i \mid g \in L_n \} \).
Locally trivial principal bundles. Suppose that $G$ is a topological group in the sense of the category of compactly generated spaces. This means $G$ is a group and a compactly generated space, the inverse $a \mapsto a^{-1}$ is a continuous map $G \to G$, and the multiplication is continuous as a map $G \times_k G \to G$, but may be not continuous as a map $G \times_c G \to G$.

For the rest of this section we assume that $p : E \to B$ is a continuous map of compactly generated spaces, and that an action of $G$ on $E$, continuous as a map $E \times_k G \to E$, is given. The map $p$ together with the action of $G$ on $E$ is said to be a locally trivial principal bundle in the category of compactly generated spaces if $B$ can be covered by open subsets $U$ such that $p$ is trivial over $U$, i.e. there exists a homeomorphism $h$ such that the diagram

$$
p^{-1}(U) \xrightarrow{h} U \times_k G \xrightarrow{p_U} U
$$

(where $p_U$ is the projection to the factor $U$) is commutative both as the diagram of topological spaces and as the diagram of $G$-sets, where $G$ acts on the set $U \times G$ in the obvious manner.

3.1. Lemma. If $p$ is a locally trivial principal bundle, then $p$ is a weakly principal bundle.

Proof. Suppose that $p$ is a locally trivial principal bundle.

Since $U \times_k G$ is the product in the category of compactly generated spaces, the standard proof of the fact that every locally trivial bundle is a Serre fibration applies without any changes to this situation. One can also use the fact that for a compact Hausdorff space $X$ (in particular, for a cube) the map $X \to U \times_k G$ is continuous if and only if it is continuous as a map $X \to U \times_c G$. This allows to deduce Serre's property in the present framework from the Serre's property for the usual locally trivial bundles. Therefore $p$ satisfies the condition (i).

Since $G$ acts freely on the product $U \times G$ and $U = (U \times G)/G$ as a set, $p$ satisfies (ii).

It remains to prove (iii). As we pointed out above, the map $g$ satisfying (3.1) exists and is unique. We need only to check that it is continuous. Since the continuity is a local property, we may assume that the image $p \circ \tau(\Delta_n)$ is contained in an open set $U$ such that $p$ is trivial over $U$. By composing everything with the map $h$ from the above diagram, we see that it is sufficient to prove that if $\tau, \sigma : \Delta_n \to U \times_k G$ are continuous maps such that

$$p_U \circ \tau = p_U \circ \sigma,$$

then the unique map $g : \Delta_n \to G$ such that $\sigma = g \cdot \tau$ is continuous.
Let $f = p_U \circ \tau = p_U \circ \sigma$. Since $\Delta_n$ is a compact Hausdorff space and hence is a compactly generated space, there exist continuous maps $t, s : \Delta_n \to G$ such that

$$\tau(x) = (f(x), t(x)) \quad \text{and} \quad \sigma(x) = (f(x), s(x))$$

for all $x \in \Delta_n$. Obviously, $g(x) = s(x) \cdot t(x)^{-1}$ for all $x \in \Delta_n$. Since $G \times_k G$ is a product in the category of compactly generated spaces, the map $x \mapsto (s(x), t(x))$ is continuous as a map $\Delta_n \to G \times_k G$. Since the inverse map $G \to G$ and the product map $G \times_k G \to G$ are continuous, it follows that $g$ is continuous.

**Numerable bundles.** Recall that a covering of a space is called *numerable* if there exists a partition of unity subordinated to this covering. The bundle $p : E \to B$ is said to be numerable if there exists an open numerable covering $\mathcal{U}$ of $B$ such that $p$ is trivial over every element $U \in \mathcal{U}$. As is well known, $p$ is numerable if $B$ is paracompact.

The remaining part of this section is devoted to the proof of Theorem 3.6 below. The author hoped to find it in textbooks, but failed even with research papers.

**3.2. Lemma.** Suppose that locally trivial principal bundle $p : E \to B$ is numerable and that $f : B' \to B$ is a continuous map. Let $p' : E' \to B'$ be the bundle induced from $p$ by $f$. If $f$ is a homotopy equivalence, then the canonical map $f^\sim : E' \to E$ is a homotopy equivalence.

**Proof.** We refer to [tD] for the standard results of the bundle theory. Their proofs in [tD] apply without any changes to compactly generated spaces and locally trivial bundles in the category of compactly generated spaces, if one takes into the account the fact that by Lemma 2.6 $X \times_c [0, 1]$ is compactly generated when $X$ is compactly generated.

Let $g : B \to B'$ be a homotopy equivalence inverse to $f$ and let $p'' : E'' \to B$ be the bundle induced from $p'$ by $g$. Let $g^\sim : E'' \to E'$ be the canonical map. The bundle $p''$ is induced from the bundle $p$ by $f \circ g$. Since $p$ is numerable and $f \circ g$ is homotopic to the identity $id_B$, the bundle $p''$ is isomorphic to $p$ over $B$. See [tD], Theorem 14.3.2. By composing $g^\sim$ with an isomorphism $E \to E''$ from $p$ to $p''$ we get a bundle map

$$E \xrightarrow{h} E' \xrightarrow{p'} E'' \xrightarrow{g^\sim} E' \xrightarrow{p} E$$

The composition $f^\sim \circ h$ is a bundle map covering $f \circ g$. Since $f \circ g$ is homotopic to the identity $id_B$, the homotopy lifting theorem (see [tD], 14.3.4) implies that $f^\sim \circ h$ is homo-
topic to a bundle map \( j : E \to E \) covering the identity map \( \text{id}_B \). Since the map \( j \) covers a homeomorphism, it is itself a homeomorphism (see [tD], remarks after Proposition 14.1.6). It follows that \( f^\sim \circ h : E \to E \) is a homotopy equivalence.

Since the bundle \( p' \) is induced from a numerable bundle, it is itself numerable. Therefore the homotopy lifting theorem applies to the bundle map \( h \circ f^\sim : E' \to E' \) covering \( g \circ f \) and a homotopy connecting \( g \circ f \) with \( \text{id}_B' \). It follows that \( h \circ f^\sim \) is homotopic a bundle map \( j' : E \to E \) covering the identity map \( \text{id}_B \). As above, the map \( j' \) is a homeomorphism, and hence \( h \circ f^\sim \) is homotopy equivalent. Since both maps \( f^\sim \circ h \) and \( h \circ f^\sim \) are homotopy equivalences, the map \( f^\sim \) is a homotopy equivalence, as claimed. ■

**CW-complexes.** Suppose now that \( B \) is a CW-complex, and let \( B_n \) be the \( n \)-th skeleton of \( B \). Then \( B_n \) is obtained from \( B_{n-1} \) by glueing a collection of \( n \)-dimensional discs. In more details, let us denote by \( D^n \) the disjoint union of these discs, and by \( S^{n-1} \) the disjoint union of their boundary spheres, \( S^{n-1} \subset D^n \). Then \( B_n \) is obtained by glueing to \( B_{n-1} \) the space \( D^n \) by a continuous map \( \alpha_n : S^{n-1} \to B_{n-1} \). Let \( B_{n-1} \sqcup D^n \) be the disjoint union of \( B_{n-1} \) and \( D^n \). There is a continuous map

\[
\varphi_n : B_{n-1} \sqcup D^n \to B_n
\]

equal to the inclusion on \( B_{n-1} \), to \( \alpha_n \) on \( S^{n-1} \subset D^n \), and inducing a homeomorphism

\[
D^n \cup S^{n-1} \to B_n \cup B_{n-1}.
\]

Moreover, the topology of \( B_n \) is the quotient topology of \( B_{n-1} \sqcup D^n \) induced by \( \varphi_n \).

**Locally trivial principal bundles over CW-complexes.** We continue to assume that \( B \) is a CW-complex and keep the above notations related to \( B \). Let \( E_n = p^{-1}(B_n) \) and let \( \pi_n : E_n \to B_{n-1} \sqcup D^n \) be the bundle induced from \( p \) by \( \varphi_n \). Then \( E_n = E_{n-1} \sqcup E_n \), where \( E_n = \pi^{-1}_n(D^n) \). Let \( \Phi_n : E_{n-1} \sqcup E_n = E_n \to E_n \) be the canonical map. Then the diagram

\[
\begin{array}{ccc}
E_{n-1} \sqcup E_n & \xrightarrow{\Phi_n} & E_n \\
\downarrow{\pi_n} & & \downarrow{p} \\
B_{n-1} \sqcup D^n & \xrightarrow{\varphi_n} & B_n
\end{array}
\]

is commutative.
3.3. **Lemma.** \( \Phi_n \) is a quotient map and hence \( E_n \) can be obtained by glueing \( E_n \) to \( E_{n-1} \) along a continuous map \( \pi_n^{-1}(S^{n-1}) \to E_{n-1} \).

**Proof.** It is sufficient to prove that \( B_n \) can be covered by open sets \( U \) such that \( \Phi_n \) is a quotient map over \( U \), i.e. such that the map

\[
\chi : \pi_n^{-1}(\varphi_n^{-1}(U)) \to p^{-1}(U)
\]

induced by \( \Phi_n \) is a quotient map. Since \( p \) is locally trivial, it is sufficient to show that \( \chi \) is a quotient map if the bundle \( E_n \to B_n \) is trivial over \( U \). Suppose that this is the case. Then \( \pi_n \) is trivial over \( \varphi_n^{-1}(U) \) and there is a commutative diagram

\[
\begin{array}{ccc}
\pi_n^{-1}(\varphi_n^{-1}(U)) & \xrightarrow{\chi} & p^{-1}(U) \\
\downarrow h' & & \downarrow h \\
\varphi_n^{-1}(U) \times_k G & \longrightarrow & U \times_k G,
\end{array}
\]

where the vertical arrows are homeomorphisms and the lower horizontal arrow is the map \( \varphi_n^{-1}(U) \to U \) multiplied by \( \text{id}_G \). Since \( B \) is a CW-complex, the map \( \varphi_n^{-1}(U) \to U \) is a quotient map. By Lemma 2.7 this implies that the lower horizontal arrow is a quotient map. Hence the top horizontal arrow \( \chi \) is also a quotient map. \( \blacksquare \)

3.4. **Lemma.** The topology of \( E \) is the weak topology defined by the subspaces \( E_n \).

**Proof.** The proof is similar to the proof of Lemma 3.3. It is sufficient to prove this claim locally, over open sets \( U \subset B \) such that \( p \) is trivial over \( U \). Therefore, it is sufficient to prove that \( U \times_k G \) has the weak topology defined by subspaces \( (U \cap B_n) \times_k G \). Since the space \( B \) has the weak topology defined by the subspaces \( B_n \), the subspace \( U \) has the weak topology defined by the subspaces \( U \cap B_n \). It remains to use the fact that this property survives multiplication \( \times_k Z \) by any compactly generated space \( Z \). See [St], Theorem 10.1. \( \blacksquare \)

3.5. **Lemma.** If the fiber \( G \) of the bundle \( p : E \to B \) is a CW-complex, then \( E \) is homotopy equivalent to a CW-complex.

**Proof.** The space \( E_0 \) is a CW-complex, being a disjoint union of fibers which are assumed to be CW-complexes. Let \( X_0 = E_0 \). Suppose that CW-complexes \( X_m \) and homotopy equivalences \( f_m : E_m \to X_m \) are already defined for \( m \leq n-1 \) and that \( X_l \) is a subcomplex of \( X_m \) and \( f_m \) extends \( f_l \) if \( l \leq m \).
By Lemma 3.3 $E_n$ results from glueing $E_n$ to $E_{n-1}$ along a continuous map

$$g_n : \pi_n^{-1}(S^{n-1}) \longrightarrow E_{n-1}.$$ 

Since all components of $D^n$ are contractible, the induced bundle $\pi_n$ is trivial over $D^n$ and hence there exists a homeomorphism

$$E_n \longrightarrow D^n \times_k G$$

taking $\pi_n^{-1}(S^{n-1})$ to $S^{n-1} \times_k G$. We may treat this homeomorphism as an identification. Since $G$ is assumed to be a CW-complex, the product $D^n \times_k G$ admits a structure of a CW-complex such that $S^{n-1} \times_k G$ is a subcomplex. Therefore $E_n$ is the result of glueing of the CW-complex $D^n \times_k G$ to $E_{n-1}$ by the map $g_n$ defined on the subcomplex $S^{n-1} \times_k G$.

This implies, in particular, that $E_{n-1}$ is a neighborhood deformation retract of $E_n$ in the sense of [St], and hence satisfies the homotopy extension property (see [P], Lecture 2, Proposition 3, for example). Hence the homotopy equivalence $f_{n-1} : E_{n-1} \longrightarrow X_{n-1}$ extends to a homotopy equivalence between $E_n$ and the result of glueing of $D^n \times_k G$ to $X_{n-1}$ by

$$f_{n-1} \circ g_n : S^{n-1} \times_k G \longrightarrow X_{n-1}.$$ 

See [tD], Proposition 5.1.10, or [P], Lecture 2, p. 92, or [Bro], Theorem 7.5.7.

Let $h_n$ be a cellular map homotopic to $f_{n-1} \circ g_n$, and let $X_n$ be the result of glueing of the CW-complex $D^n \times_k G$ to $X_{n-1}$ by $h_n$. Then $X_n$ is a CW-complex containing $X_{n-1}$ as a subcomplex and $E_n$ is homotopy equivalent to $X_n$ also. Moreover, $f_{n-1}$ extends to a homotopy equivalence $f_n : E_n \longrightarrow X_n$ (see [Bro], Corollary 7.5.5, for example).

Let $X$ be the union of the CW-complexes $X_n$, and let $f : E \longrightarrow X$ be the map equal to $f_n$ on $X_n$. Then $X$ has a unique structure of a CW-complex such that every $X_n$ is a subcomplex. The topology of $X$ is the weak topology defined by the subspaces $X_n$. Lemma 3.4 implies that $f$ is continuous. For every $n$ the subspace $X_{n-1}$, being a subcomplex of $X_n$, is a neighborhood deformation retract of $X_n$. Also, as we proved above, for every $n$ the subspace $E_{n-1}$ is a neighborhood deformation retract of $E_n$. Therefore, the fact that each $f_n$ is a homotopy equivalence implies that $f$ is a homotopy equivalence. See, for example, [tD], Proposition 5.2.9, or [P], Lecture 10, Additional Material, Theorem 1 and Proposition 2. ■

3.6. **Theorem.** Suppose that $p : E \longrightarrow B$ is a numerable locally trivial principal bundle with the fiber $G$. If $G$ is a CW-complex and $B$ is homotopy equivalent to a CW-complex, then $E$ is homotopy equivalent to a CW-complex.

**Proof.** This follows from Lemmas 3.5 and 3.2. ■
4. McCord classifying spaces and principal bundles

**The spaces** \( B(G, X) \). Let \( X \) be a weakly Hausdorff compactly generated space and let \( G \) be a topological group in the category of weakly Hausdorff compactly generated spaces. Suppose that a base point \( * \) of \( X \) is chosen. Let \( B(G, X) \) be the set of functions \( u : X \to G \) such that \( u(*) = e \), where \( e \) is the unit of \( G \), and \( u(x) = e \) for all but finitely many points \( x \in X \). Then \( B(G, X) \) is itself a group with respect to the point-wise multiplication \( \oplus \) of maps \( X \to G \), defined by the formula \( (u \oplus v)(x) = u(x) \cdot v(x) \). The unit of this group is the map \( e \) taking the value \( e \) at all points of \( X \). Let \( g \in G \). For \( x \in X \), \( x \neq * \), let \( gx \) be the map \( X \to G \) taking the value \( g \) at \( x \) and the value \( e \) at all other points of \( X \). Let \( g * = e \).

The set \( B(G, X) \) is equipped with a topology in the following manner. For each \( n \geq 0 \), let be \( B_n(G, X) \) the subset of functions \( u \) taking the value of \( e \) at all but \( \leq n \) points of \( X \). Clearly, \( B_0(G, X) = \{e\} \) and for \( n \geq 1 \) the set \( B_n(G, X) \) consists of elements of the form

\[
g_1x_1 \oplus g_2x_2 \oplus \ldots \oplus g_nx_n.
\]

The space \( B_n(G, X) \) is equipped with the topology of a quotient space induced by the map

\[
\mu_n : (G \times X)^n \to B_n(G, X),
\]

defined by the formula

\[
\mu_n((g_1, x_1), (g_2, x_2), \ldots, (g_n, x_n)) = g_1x_1 \oplus g_2x_2 \oplus \ldots \oplus g_nx_n.
\]

Each \( B_n(G, X) \) is a closed subspace of \( B_{n+1}(G, X) \) by Lemma 6.2 of [McC]. Therefore

\[
B_0(G, X) \subset B_1(G, X) \subset B_2(G, X) \subset \ldots
\]

is a sequence of spaces, whose union is equal to \( B(G, X) \). The set \( B(G, X) \) is equipped with the weak topology defined by the spaces \( B_n(G, X) \). If \( X \) is homeomorphic to \( S^0 \), i.e. if \( X \) consists of two points, then \( B(G, X) = B_1(G, X) = G \).

**4.1. Theorem.** The space \( B(G, X) \) is a weakly Hausdorff compactly generated space. If \( G \) is a topological abelian group, then \( B(G, X) \) is also a topological abelian group.

**Proof.** See [McC], Lemma 6.5 and Proposition 6.6. ■

**4.2. Theorem.** Suppose that \( G \) is a discrete abelian group and that \( X \) is a simplicial complex equipped with the weak topology. Then \( B(G, X) \) admits structure of a CW-complex.
**Proof.** See [McC], Section 7. ■

**Induced maps.** Let $G$ be a topological group and $\varphi : X \to Y$ be a continuous map of based weakly Hausdorff compactly generated spaces. If either $G$ is abelian or the map $\varphi$ is injective on $X \setminus \{\ast\}$, then $\varphi$ induces a continuous map

$$\varphi_* : B(G, X) \to B(G, Y)$$

acting by the formula

$$\varphi_*\left(g_1x_1 \oplus g_2x_2 \oplus \cdots \oplus g_nx_n\right) = g_1\varphi(x_1) \oplus g_2\varphi(x_2) \oplus \cdots \oplus g_n\varphi(x_n).$$

The assumptions about $G$ and $\varphi$ are needed for $\varphi_*$ to be correctly defined. In order to see this, it is better to describe $\varphi_*$ in terms of functions $X, Y \to G$. In these terms

$$\varphi_*(u)(y) = \sum_{x \in \varphi^{-1}(y)} u(x).$$

There is no natural way to order the points in the preimage $\varphi^{-1}(y)$. In order for the sum to be correctly defined, it is sufficient to assume either that $G$ is abelian, or that every sum consists of 0 or 1 terms, except, perhaps, the sum corresponding to $y = \ast$. In other terms, it is sufficient to assume that either $G$ is abelian, or $\varphi$ is injective on $X \setminus \{\ast\}$.

**4.3. Theorem.** Let $G$ be an abelian topological group and $\varphi : X \to Y$ be a continuous map of based weakly Hausdorff compactly generated spaces. If $\varphi$ is a quotient map, then $\varphi_*$ is also a quotient map. If $\varphi$ is a closed injective map, then $\varphi_*$ also has this property.

**Proof.** See [McC], Proposition 6.7. ■

**4.4. Theorem.** Let $G$ be an abelian topological group and $\varphi_t : X \to Y$, $t \in [0, 1]$ be a homotopy of continuous maps of based weakly Hausdorff compactly generated spaces. Then

$$(\varphi_t)_* : B(G, X) \to B(G, Y)$$

is a homotopy. If $X, Y$ are homotopy equivalent, then $B(G, X)$ and $B(G, Y)$ are homotopy equivalent. In particular, if $X$ is contractible, then so is $B(G, X)$.

**Proof.** See [McC], Proposition 6.10. ■
**Principal bundles.** Let $G$ be a discrete group and $(X, A)$ be a pair of based simplicial complexes equipped with the weak topology. Suppose that either the group $G$ is abelian, or $(X, A)$ is homeomorphic to $(I, \partial I)$, where $I$ is an interval and $\partial I$ is its boundary. Let $i : A \to X$ be the inclusion map and $p : X \to X/A$ be the quotient map. The maps $i$ and $p$ induce maps

$$B(G, A) \xrightarrow{i_*} B(G, X) \xrightarrow{p_*} B(G, X/A)$$

If $G$ is abelian, then Theorem 4.3 implies that $i_*$ is a closed embedding and $p_*$ is a quotient map. The map $i_*$ together with the group operation $\oplus$ on $B(G, X)$ define an action of $B(G, A)$ on $B(G, X)$. By [McC], Lemma 8.3, $p_*$ induces a canonical homeomorphism

$$B(G, X)/B(G, A) \to B(G, X/A).$$

If $(X, A)$ is homeomorphic to $(I, \partial I)$, this is true if the topology of $B(G, I)$ and $B(G, \partial I)$ is defined slightly differently. See [McC], Section 9. With this topology $B(G, I)$ is contractible and since $B(G, \partial I)$ is equal to $G$, the map $p_*$ is the universal cover of a $K(G, 1)$-space.

**4.5. Theorem.** Under the above assumptions, the map $p_* : B(G, X) \to B(G, X/A)$ is a numerable locally trivial principal bundle with the fiber $B(G, A)$.

**Proof.** See [McC], Theorems 8.8 and 9.17.

**Eilenberg-MacLane spaces and universal bundles.** Let $G$ be a discrete group and $n$ be a natural number. Suppose that either $G$ is abelian or $n = 1$. Let $D^n$ be the standard $n$-dimensional disc and $S^{n-1}$ be its boundary sphere. Let us identify $D_n/S^{n-1}$ with the standard $n$-sphere $S^n$. By Theorem 4.5 there is a principal numerable bundle

$$p^n_G : B(G, D^n) \to B(G, S^n)$$

with the fiber $B(G, S^{n-1})$. If $n = 1$, then the fiber is $G$ and the bundle is a covering space. By Theorem 4.4 the space $B(G, D^n)$ is contractible. By Section 3 the bundle $p^n_G$ is a Serre fibration. An induction by $n$, starting with $B(G, S^0) = G$, and using the homotopy sequence of the Serre fibration $p^n_G$ to go from $n - 1$ to $n$, shows that $B(G, S^n)$ is an Eilenberg-MacLane space of the type $K(G, n)$. Cf. [McC], Corollary 10.6. Since $B(G, D^n)$ is contractible, the bundle $p^n_G$ is a universal bundle in the category of compactly generated spaces. Its base $B(G, S^n)$ is a $K(G, n)$-space and its fiber $B(G, S^{n-1})$ is a $K(G, n - 1)$-space. By Section 3 $p^n_G$ is a weakly principal bundle with the fiber $B(G, S^{n-1})$.

If $n \geq 2$, then the spaces $B(G, D^n)$, $B(G, S^n)$, $B(G, S^{n-1})$ are CW-complexes by Theorem 4.2. For the further use, let us replace the above $p^n_G$ by the universal cover of a CW-complex of the type $K(G, 1)$. Obviously, it is a weakly principal bundle with the fiber $G$. 

24
5. Spaces with amenable fundamental group

**Means.** For a set $S$ we will denote by $B(S)$ the vector space of all bounded real functions on $S$. As is well known, $B(S)$ is a Banach space with the norm

$$\|f\| = \sup_{s \in S} |f(s)|.$$  

A linear functional $m : B(S) \to \mathbb{R}$ is called a mean on $B(S)$ if $|m(f)| \leq \|f\|$ for all $f \in B(S)$ and $m(1) = 1$, where $1(s) = 1$ for all $s \in S$. In other words, $m$ is a bounded functional such that $\|m\| \leq 1$ and $m(1) = 1$. Usually means are defined by requiring that

$$\text{(5.1)} \quad \inf f \leq m(f) \leq \sup f$$

for all $f \in B(S)$, where the infimum and the supremum are taken over the set $S$. This definition motivates the term mean. The two definitions are equivalent by the following lemma.

5.1. **Lemma.** A linear functional $m$ is a mean if and only if (5.1) holds for all $f \in B(S)$.

**Proof.** Let $f \in B(S)$, and $f' = f - \inf f$. Then $\|f'\| = \sup f - \inf f$ and hence

$$m(f) = m(f') + \inf f \leq (\sup f - \inf f) + \inf f = \sup f.$$  

By applying this inequality to $-f$ in the role of $f$, we see that $\inf f \leq m(f)$. Conversely, suppose that (5.1) holds. This immediately implies that $m(1) = 1$. Since $\|f\|$ is equal to the largest of the numbers $\sup f$ and $-\inf f = \sup(-f)$, the inequality (5.1) applied to $f$ and $-f$ implies that $\|m\| \leq 1$. ■

**Invariant means and amenable groups.** Let $G$ be a group acting on a set $S$ from the right. Then $G$ acts on $B(S)$ from the left by the formula $g \cdot f(s) = f(s \cdot g)$, where $g \in G$, $f \in B(S)$, $s \in S$. A mean $m$ on $B(S)$ is called right invariant if $m(g \cdot f) = m(f)$ for all $g \in G$, $f \in B(S)$. Usually the right invariant means will be called simply invariant means. If there exists an invariant mean on $B(G)$ with respect to the action of the group $G$ on itself by the right translations, then $G$ is said to be amenable.

One can define in an obvious way the notion of a left invariant mean. It turns out that if $G$ is amenable, then there exists a mean on $B(G)$ which is simultaneously right and left invariant (see [Gr], Lemmas 1.1.1 and 1.1.3), but we will not need this result.

5.2. **Theorem.** If a group is abelian, then it is amenable.

**Proof.** See [Gr], Theorem 1.2.1. ■
Free transitive actions. Suppose that a group $G$ acts on a set $S$ on the left and that this action is free and transitive. Then every $s \in S$ defines a bijection $r_s : G \rightarrow S$ by the rule $g \mapsto g.s$, where $g \in G$. If $s, t \in S$, then the bijections $r_s$ and $r_t$ differ by a right translation of $G$. Indeed, since the action is transitive, there exists $h \in G$ such that $t = h \cdot s$ and hence

$$r_t(g) = g \cdot t = g \cdot (h \cdot s) = (g \cdot h) \cdot s = r_s(g \cdot h).$$

It follows that if $f \in B(S)$, then

$$f \circ r_t(g) = f \circ r_s(g \cdot h) = h \cdot (f \circ r_s)(g)$$

for every $g \in G$. In other terms, $f \circ r_t = h \cdot (f \circ r_s)$. If $m$ is a mean on $B(G)$ and $s \in S$, then $f \mapsto m(f \circ r_s)$ is a mean on $B(S)$. If the mean $m$ is invariant and $s, t \in S$, then

$$m(f \circ r_t) = m(h \cdot (f \circ r_s)) = m(f \circ r_s),$$

and hence the mean $f \mapsto m(f \circ r_s)$ on $B(S)$ does not depend on the choice of $s$. The mean $f \mapsto m(f \circ r_s)$ on $B(S)$ is said to be induced by $m$ and the action of $G$ on $S$ (which should be free and transitive).

Push-forwards. Let $\pi : G \rightarrow H$ be a surjective homomorphism. The map $f \mapsto f \circ \pi$ is a bounded operator $B(H) \rightarrow B(G)$. In fact, its norm is obviously equal to 1. If $m$ is an invariant mean on $B(G)$, then the map $\pi_* m$ defined by the formula

$$\pi_* m : f \mapsto m(f \circ \pi)$$

is an invariant mean on $B(H)$ called the push-forward of $m$ by $\pi$.

Coherent sequences of invariant means. Let $G$ be a topological group. Suppose that $G$ is either abelian, or discrete and amenable. For a topological space $X$ let $G^X$ be the group of continuous maps $X \rightarrow G$ considered as a discrete group. Recall that $\Delta_n$ is the standard $n$-dimensional simplex. If $G$ is abelian, then the group $G^{\Delta_n}$ is abelian, and hence is amenable by Theorem 5.2. If $G$ is discrete and amenable, then $G^{\Delta_n}$ is equal to $G$ and hence is amenable. Let $M_n$ be the set of invariant means on $B(G^{\Delta_n})$, so $M_n \subset B(G^{\Delta_n})^\ast$. Suppose that $m_0, m_1, \ldots, m_n, \ldots$ is an either finite or infinite sequence of invariant means $m_n \in M_n$. If it is a finite sequence, then $n$ ranges over $0, 1, 2, \ldots, N$ for some $N \geqslant 0$, if it is an infinite sequence, then $n$ ranges over all non-negative integers $0, 1, 2, \ldots$. If $i : \Delta_{n-1} \rightarrow \Delta_n$ is a simplicial embedding, then the map $\sigma \mapsto \sigma \circ i$ is a homomorphism

$$\pi^i : G^{\Delta_n} \rightarrow G^{\Delta_{n-1}},$$

26
which is obviously surjective. The sequence is called coherent if

\( \pi_i^* m_n = m_{n-1} \)

for every \( n \) such that \( m_n \) is defined and every \( i \) as above. Equivalently,

\( m_n (f \circ \pi^i) = m_{n-1}(f) \)

for every simplicial embedding \( i : \Delta_{n-1} \rightarrow \Delta_n \) and \( f \in B(G^\Delta_{n-1}) \) if \( m_n \) is defined.

\( \Sigma_{n+1} \)-invariant means on \( B(G^\Delta_n) \). The natural action of the symmetric group \( \Sigma_{n+1} \) on \( \Delta_n \) induces an action of the group \( \Sigma_{n+1} \) on the group \( G^\Delta_n \). In turn, this action induces an action of \( \Sigma_{n+1} \) on the dual space \( B(G^\Delta_n)^* \). Obviously, this action leaves the space of means on \( B(G^\Delta_n) \) invariant. Moreover, since the group \( \Sigma_{n+1} \) acts on \( G^\Delta_n \) by automorphisms, this action leaves the set \( M_n \) of invariant means invariant. We will say that an invariant mean \( m \) on \( B(G^\Delta_n) \) is \( \Sigma_{n+1} \)-invariant if \( m \) is fixed by the action of \( \Sigma_{n+1} \) on the dual space \( B(G^\Delta_n)^* \). If \( m \) is an arbitrary invariant mean on \( B(G^\Delta_n) \), then

\[
\frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} m^\sigma
\]

is a \( \Sigma_{n+1} \)-invariant mean, where the action of \( \Sigma_{n+1} \) is written as \( (m, \sigma) \mapsto m^\sigma \).

Suppose that \( m_n \) is a \( \Sigma_{n+1} \)-invariant mean on \( B(G^\Delta_n) \) and that \( i : \Delta_{n-1} \rightarrow \Delta_n \) is a simplicial embedding. Let us consider the push-forward

\[
m_{n-1} = \pi_i^* m_n.
\]

Since \( m_n \) is \( \Sigma_{n+1} \)-invariant, the push-forward \( m_{n-1} \) is independent of the choice of the simplicial embedding \( i \) and is \( \Sigma_n \)-invariant. Since \( m_{n-1} \) is independent of the choice of \( i \), the coherence condition (5.2) holds for every simplicial embedding \( i : \Delta_{n-1} \rightarrow \Delta_n \).

5.3. Theorem. Let \( G \) be a topological group and \( N \geq 0 \). If \( G \) is either abelian, or discrete and amenable, then there exists a coherent sequence of invariant means \( m_0, m_1, m_2, \ldots, m_N \).

Proof. As we saw above, under assumptions of the theorem the groups \( G^\Delta_n \) are amenable. In particular, there exist an invariant mean \( m_N \) on \( B(G^\Delta_N) \). Moreover, \( m_N \) can be chosen to be \( \Sigma_{n+1} \)-invariant. Let \( i : \Delta_{N-1} \rightarrow \Delta_N \) be an arbitrary simplicial embedding and let

\[
m_{N-1} = \pi_i^* m_N.
\]

By repeating this construction we get a sequence \( m_N, m_{N-1}, m_{N-2}, \ldots, m_0 \) of invariant means satisfying (5.2) for all \( n \leq N \). ■
Coboundary maps. Let us review the basic notations related to the coboundary maps. Let $Z$ be a topological space. For every $n = 1, 2, \ldots$ and every $k = 0, 1, \ldots, n$, let

$$\partial_k : S_n(Z) \rightarrow S_{n-1}(Z)$$

be the map taking a singular simplex $\tau : \Delta_n \rightarrow Z$ into its $k$-th face $\partial_k \tau : \Delta_{n-1} \rightarrow Z$. By the definition, $\partial_k \tau = \tau \circ i_k$ for some simplicial embedding $i_k : \Delta_{n-1} \rightarrow \Delta_n$. The coboundary of a cochain $f \in C^{n-1}(Z)$ is the cochain $\partial f \in C^n(Z)$ defined by the formula

$$\partial f(\tau) = \sum_{k=0}^{n} (-1)^k f(\partial_k \tau) = \sum_{k=0}^{n} (-1)^k f \circ \partial_k(\tau),$$

where $\tau \in S_n(Z)$.

5.4. Theorem. Let $G$ be a topological group. Suppose that $G$ is either abelian, or discrete and amenable. Let $p : X \rightarrow Y$ be a weakly principal bundle with the fiber $G$. Then every coherent sequence $\{m_n\}$ leads to a sequence of homomorphisms

$$p_* : B^n(X) \rightarrow B^n(Y)$$

defined for the same $n$ as $\{m_n\}$ and such that $p_* \circ p^* = id$, $\|p_*\| = 1$, and $p_*$ commutes with the coboundaries of the chain complexes $B^*(X)$, $B^*(Y)$ whenever this makes sense.

Proof. In what follows the number $n$ is assumed to be such that $m_n$ is defined.

For each singular simplex $\sigma : \Delta_n \rightarrow Y$ let us denote by $C_\sigma$ the set of singular simplices $\tau : \Delta_n \rightarrow X$ such that $p \circ \tau = \sigma$. Since $p$ is a weakly principal bundle with the fiber $G$, the group $G^{\Delta_n}$ acts on $C_\sigma$ from the left and this action is free and transitive. Therefore the mean $m_n$ together with this action induces a mean on $B(C_\sigma)$, which we denote by $m_\sigma$.

Let us define a linear map $p_* : B^n(X) \rightarrow B^n(Y)$ as follows. For a cochain $f \in B^n(X)$ let $p_*(f) \in B^n(Y)$ be the cochain taking the value

$$p_*(f)(\sigma) = m_\sigma(f \mid C_\sigma)$$

on each singular simplex $\sigma : \Delta_n \rightarrow Y$, where $f \mid C_\sigma$ is the restriction of $f$ to the set $C_\sigma$. Since $m_\sigma : B(C_\sigma) \rightarrow \mathbb{R}$ is a mean, $\|p_*(f)\| \leq \|f\|$ for all $f \in B^n(Y)$. It follows that $\|p_*\| \leq 1$. At the same time $m_\sigma(1) = 1$ implies that $\|p_*\| \geq 1$ and hence $\|p_*\| = 1$. In addition, $m_\sigma(1) = 1$ implies that $m_\sigma(a1) = a$ for every $a \in \mathbb{R}$ and hence $p_* \circ p^* = id$.

It remains to show that the maps $p_*$ commute with the coboundaries of the chain complexes $B^*(X)$, $B^*(Y)$. Let $\sigma : \Delta_n \rightarrow Y$ be a singular $n$-simplex, $i : \Delta_{n-1} \rightarrow \Delta_n$ be a simplicial
embedding, and \( \tau = \sigma \circ i \). Let \( \delta : C_\sigma \rightarrow C_\tau \) be the map \( \rho \mapsto \rho \circ i \). We claim that

\[
(5.4) \quad m_\tau(f) = m_\sigma(f \circ \delta)
\]

for all \( f \in B(C_\tau) \). In order to prove this, let us choose an arbitrary simplex \( \overline{\sigma} \in C_\sigma \). Then the simplex \( \overline{\tau} = \overline{\sigma} \circ i \) belongs to \( C_\tau \). The diagram

\[
\begin{array}{ccc}
G^\Delta_n & \xrightarrow{\pi^i} & G^\Delta_{n-1} \\
\downarrow r_{\overline{\sigma}} & & \downarrow r_{\overline{\tau}} \\
C_\sigma & \xrightarrow{\delta} & C_\tau
\end{array}
\]

is obviously commutative. The maps \( r_{\overline{\sigma}}, r_{\overline{\tau}} \) are bijections and can be used to identify the top row of the diagram with the bottom row. By the definition of \( m_\sigma \) and \( m_\tau \) this identification turns \( m_n \) and \( m_{n-1} \) into \( m_\sigma \) and \( m_\tau \) respectively. Hence (5.4) follows from (5.3).

Now we are ready to prove that the maps \( p_* \) commute with the differentials. Let \( f \in B^{n-1}(X) \) and let \( \sigma : \Delta_n \rightarrow Y \) be a singular \( n \)-simplex in \( Y \). Then

\[
p_*(\partial f)(\sigma) = m_n(\partial f \mid C_\sigma) = m_n \left( \sum_{k=0}^{n} (-1)^k f \circ \partial_k \mid C_\sigma \right) = \sum_{k=0}^{n} (-1)^k m_n \left( f \circ \partial_k \mid C_\sigma \right).
\]

For every \( k \) the map \( \partial_k \) takes \( C_\sigma \) into \( C_\tau \), where \( \tau = \partial_k \sigma \). The map \( C_\sigma \rightarrow C_\tau \) induced by \( \partial_k \) is nothing else but the above map \( \delta \) corresponding to \( i = i_k \). Hence (5.4) implies that

\[
m_n \left( f \circ \partial_k \mid C_\sigma \right) = m_{n-1} \left( f \mid C_{\partial_k \sigma} \right).
\]

It follows that

\[
p_*(\partial f)(\sigma) = \sum_{k=0}^{n} (-1)^k m_n \left( f \circ \partial_k \mid C_\sigma \right) = \sum_{k=0}^{n} (-1)^k m_{n-1} \left( f \mid C_{\partial_k \sigma} \right) = \sum_{k=0}^{n} (-1)^k p_*(f)(\partial_k \sigma) = \partial p_*(f)(\sigma)
\]

Therefore \( p_* \circ \partial = \partial \circ p_* \). This completes the proof. ■
5.5. Corollary. Under assumptions of Theorem 5.4 the induced map

\[ p^* : \hat{H}^i(Y) \to \hat{H}^i(X) \]

of the bounded cohomology groups is injective for every \( i \).

**Proof.** Let \( N \) be a natural number \( \geq i + 1 \). By Theorem 5.3 there exists a coherent sequence \( m_0, m_1, m_2, \ldots, m_N \) for \( G \). By Theorem 5.4 for each \( i \leq N \) there exists a homomorphism

\[ p_* : B^i(X) \to B^i(Y) \]

such that \( p_* \circ p^* = \text{id} \) and these homomorphisms commute with the coboundary maps. Hence for each \( i = 0, 1, \ldots, N - 1 \) these homomorphisms induce a homomorphism

\[ p_* : \hat{H}^i(X) \to \hat{H}^i(Y) \]

which is a left inverse to \( p^* : \hat{H}^i(Y) \to \hat{H}^i(X) \). It follows that \( p^* \) is injective. ■

**Killing homotopy groups.** Let \( X \) be a path-connected space homotopy equivalent to a CW-complex. Suppose that \( \pi_i(X) = 0 \) for all \( i < n \) and let \( \pi = \pi_n(X) \). For \( n \geq 2 \), let

\[
\begin{align*}
B_\pi &= B(\pi, S^n), \\
E_\pi &= B(\pi, D^n), \\
G_\pi &= B(\pi, S^{n-1}),
\end{align*}
\]

be the spaces from Section 4. As we saw in Section 4, the spaces \( B_\pi, E_\pi, \) and \( G_\pi \) are CW-complexes, the spaces \( B_\pi \) and \( G_\pi \) are Eilenberg–MacLane spaces of the types \( K(\pi, n) \) and \( K(\pi, n-1) \) respectively and the space \( E_\pi \) is contractible. By Theorem 4.1 \( G_\pi \) is a topological group in the category of compactly generated spaces. With these notations, the principal numerable bundle (4.1) takes the form

\[(5.5) \quad p^*_n : E_\pi \to B_\pi \]

and has \( G_\pi \) as the fiber. If \( n = 1 \), then we take as (5.5) the universal cover of any CW-complex which is a \( K(\pi, 1) \)-space. By Lemma 3.1 \( p^*_n \) is a weakly principal bundle. In particular, \( p^*_n \) is a Serre fibration. The homotopy sequence of \( p^*_n \) has 0’s at all places except

\[
\pi_n(B_\pi) \to \pi_{n-1}(G_\pi).
\]

This boundary map is nothing else but \( \text{id}_\pi : \pi \to \pi \). Since \( X \) is homotopy equivalent to a CW-complex and \( B_\pi \) is a CW-complex of the type \( K(\pi, n) \), there exists a map

\[ j : X \to B_\pi \]
such the the induced map of homotopy groups $\pi_n(j)$ is an isomorphism. Let

$$p : Y \rightarrow X$$

be the bundle induced from the bundle $p^n_\pi$ by the map $j$. It has the same fiber $G_\pi$ as the bundle $p^n_\pi$. Since $p^n_\pi$ is a Serre fibration, the bundle $p$ is a Serre fibration also. By comparing the homotopy sequences of the bundles $p$ and $p^n_\pi$, we see that the boundary map

$$\pi_n(X) = \pi \rightarrow \pi_{n-1}(G_\pi)$$

is an isomorphism. Since $\pi_m(G_\pi) = 0$ for $m \neq n - 1$, it follows that

$$\pi_n(Y) = 0 \quad \text{and} \quad p_* : \pi_m(Y) \rightarrow \pi_m(X)$$

is an isomorphism for $m \neq n$. One may say that the space $Y$ resulted from killing the $n$-th homotopy group of $X$. The classical version of this construction is due to Cartan and Serre [CS] and uses instead of the principal bundle $p^n_\pi$ the path space Serre fibration

$$PK(\pi, n) \rightarrow K(\pi, n),$$

where $PK(\pi, n)$ is the space of paths in a $K(\pi, n)$-space starting at a fixed point.

Since the bundle $p$ is induced from the bundle $p^n_\pi$ and $p^n_\pi$ is a numerable locally trivial principal bundle, $p$ is also a numerable locally trivial principal bundle. By Theorem 3.6 this implies that $Y$ is homotopy equivalent to a CW-complex. This allows to apply the same construction to $Y$ and $n + 1$ in the roles of $X$ and $n$ respectively, and continue in this way.

**Iterated killing.** Let $X$ be a path-connected space homotopy equivalent to a CW-complex. One can start killing homotopy groups with the first non-zero group and then iterate the construction. This procedure is also due to Cartan and Serre [CS]. It leads to a sequence of maps

$$(5.6) \quad \ldots \quad p_n \rightarrow X_n \quad p_{n-1} \rightarrow X_{n-1} \quad p_{n-2} \rightarrow \ldots \quad p_2 \rightarrow X_2 \quad p_1 \rightarrow X_1$$

such that $X_1 = X$, $\pi_i(X_n) = 0$ if $i < n$, $\pi_i(X_n) = \pi_i(X)$ if $i \geq n$, and each map

$$p_n : X_{n+1} \rightarrow X_n$$

is a weakly principal bundle having a topological group $G_n$ as a fiber. The group $G_n$ is abelian for $n \geq 2$ and is discrete and isomorphic to $\pi_1(X)$ for $n = 1$. Each space $X_n$ is homotopy equivalent to a CW-complex, and each group $G_n$ is a CW-complex.
Partial contracting homotopies. Let X be a topological space, and let n be a natural number. A partial contracting homotopy up to dimension n is a sequence of homomorphisms

\[ \mathbf{R} \xleftarrow{K^0} B^0(X) \xleftarrow{K^1} B^1(X) \xleftarrow{K^2} \cdots \xleftarrow{K^n} B^n(X) \]

such that \( \partial \circ K^i + K^{i+1} \circ \partial = \text{id} \) for all \( 0 \leq i \leq n - 1 \), where \( \partial \) is, as usual, the coboundary operator. It is called bounded if all \( K^i \) are bounded operators, and strictly bounded if \( \| K^i \| \leq 1 \) for all \( 0 \leq i \leq n \).

Suppose that X is path-connected and \( \pi_i(X) = 0 \) for all \( i < n \). Then there exists a strictly bounded partial contracting homotopy up to dimension n. A construction of such partial homotopy is implicitly contained in the most proofs of the vanishing of the singular cohomology groups of contractible spaces. Since we need to deal with non-contractible spaces and need to control the norm of the involved maps, we present such a construction from the scratch.

Let us begin with constructing a sequence of maps

\[ \{1\} \xrightarrow{L^0} S_0(X) \xrightarrow{L^1} S_1(X) \xrightarrow{L^2} \cdots \xrightarrow{L^n} S_n(X) \]

such that for every \( \sigma \in S_i(X), \ i \leq n - 1 \), the equality

\[ (5.7) \quad \partial L^{i+1}(\sigma) + \sum_{k=0}^{i} (-1)^k L^i(\partial_k \sigma) = \sigma. \]

holds in the chain group \( C_i(X) \). Let us choose a base point \( b \in X \) and set \( L^0(1) = \beta \), where \( \beta \) is the 0-simplex with the image \{b\}. Given a singular 0-simplex \( \sigma : \Delta_0 \rightarrow X \), let us connect its only vertex with the base point \( b \) by a path in \( X \). Such a path leads to a singular 1-simplex \( L^1(\sigma) : \Delta_1 \rightarrow X \) such that \( \partial L^1(\sigma) = \sigma - L^0(1) \). Next, if \( \sigma : \Delta_1 \rightarrow X \) is a singular 1-simplex, then the three 1-simplices \( L^1(\partial_0 \sigma), \sigma, L^1(\partial_1 \sigma) \) form a loop. If \( n \geq 2 \), then \( X \) is simply-connected and hence this loop is contractible in \( X \). In this case there exists singular 2-simplex \( L^2(\sigma) : \Delta_2 \rightarrow X \) such that \( \partial L^2(\sigma) = \sigma - L^1(\partial_0 \sigma) + L^1(\partial_1 \sigma) \). Since \( X \) is \((n-1)\)-connected, we can continue in this way until we get all the maps \( L_i \) needed.

Now we can define \( K^{i+1} : B^{i+1}(X) \rightarrow B^i(X) \) as the map induced by

\[ L^{i+1} : S_i(X) \rightarrow S_{i+1}(X). \]

Since \( K^i \) is induced by a map between the sets of singular simplices, its norm is \( \| K^i \| \leq 1 \). It follows from (5.7) that \( K^i \) with \( i \leq n \) form a strictly bounded partial contracting homotopy.
5.6. Theorem. Suppose that $X$ is a path-connected space homotopy equivalent to a CW-complex and having amenable fundamental group. Then $\hat{H}^i(X) = 0$ for all $i \geq 1$.

Proof. Iterated killing of homotopy groups leads to a sequence of maps

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1$$

such that $X_1 = X$, $\pi_i(X_n) = 0$ if $i < n$, $\pi_i(X_n) = \pi_i(X)$ if $i \geq n$, and each map

$$p_n : X_{n+1} \to X_n$$

is a principal $G_n$-bundle for a topological group $G_n$ which is either abelian, or discrete and amenable. Let $i \geq 1$, and let us choose some $n \geq i$. By Corollary 5.5 the maps

$$p_m^* : \hat{H}^i(X_m) \to \hat{H}^i(X_{m+1})$$

are injective for all $m$. This implies that the induced map

$$(5.8) \quad (p_1 \circ p_2 \circ \cdots \circ p_{n+1})^* : \hat{H}^i(X) \to \hat{H}^i(X_{n+1})$$

is injective. Since $X_{n+1}$ is path-connected and $\pi_i(X_{n+1}) = 0$ for all $i < n + 1$, there is a bounded partial contracting homotopy up to dimension $n + 1$ for $X_{n+1}$. Since $n \geq i$, it follows that $\hat{H}^i(X_{n+1}) = 0$, and the injectivity of (5.8) implies that $\hat{H}^i(X) = 0$. ■

Countable abelian groups. If $\pi$ is a countable abelian group, then for every $n \geq 1$ there is an Eilenberg–MacLane space $G_\pi$ of the type $K(\pi, n-1)$ which is a topological group in usual sense. This was proved by Milnor [M2], Corollary to Theorem 3. The space $G_\pi$ is the geometric realization of a semi-simplicial complex and hence is a CW-complex. Alternatively, one can construct such a topological group $G_\pi$ by applying the Dold–Thom [DT] construction $AG(\bullet)$ to a countable simplicial complex which is a Moore space of the type $(\pi, n-1)$.

Milnor’s universal bundle [M2] is a map of the form (5.5) which is a principal $G_\pi$-bundle in the classical sense, and hence is a weakly principal bundle with the fiber $G_\pi$. If the base of the Milnor’s universal bundle is not a CW-complex, then one can replace this universal bundle by another one induced from it by a weak homotopy equivalence. Even better, one can start with a countable CW-complex $B_\pi$ which is an Eilenberg–MacLane space of the type $K(\pi, n)$. By [M1], Theorem 5.2(3), there exists a principal bundle with all the desired properties.

It follows that if all homotopy groups $\pi_n(X)$ are countable, for example if $X$ is homotopy equivalent to countable CW-complex, then one can prove Theorem 5.6 without using compactly generated spaces and the McCord classifying spaces from Section 4.
Contracting chain homotopies. The rest of this section is devoted to the proof of the existence of strictly bounded contracting chain homotopies $K^i$ defined for all $i \geq 0$ at once. This result will not be needed till Section 8.

5.7. Theorem. Suppose that $G$ is either an abelian topological group, or an amenable discrete group. Then there exists a coherent sequence of invariant means \( \{m_n\}, \ n = 0, 1, 2, \ldots \).

Proof. Let $\mathcal{M}_n$ be the set of all $\Sigma_{n+1}$-invariant means on $B(G^{\Delta_n})$. By remarks preceding Theorem 5.3 $\mathcal{M}_n \neq \emptyset$ and if $i : \Delta_{n-1} \to \Delta_n$ is a simplicial embedding, then the map $m \mapsto \pi^i_* m$ is independent on $i$ and takes $\mathcal{M}_n$ to $\mathcal{M}_{n-1}$. Thus we have a projective system

\[
(5.9) \quad \mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \mathcal{M}_2 \leftarrow \ldots .
\]

The set $\mathcal{M}_n$ is a subset of the set $\mathcal{M}_n$ of all invariant means and hence is contained in the unit ball of the dual Banach space $B(G^{\Delta_n})^*$. The set $\mathcal{M}_n$ is obviously closed in the weak* topology. By the Banach-Alaoglu theorem (see [R], Theorem 3.15) the unit ball is compact in this topology, and hence $\mathcal{M}_n$ is also compact. The map $f \mapsto f \circ \pi^i$ is a bounded operator

\[
B(G^{\Delta_{n-1}}) \longrightarrow B(G^{\Delta_n})
\]

It follows that the dual map $B(G^{\Delta_n})^* \longrightarrow B(G^{\Delta_{n-1}})^*$ is also bounded, and hence is continuous in the weak* topology. The push-forward map $m \mapsto \pi^i_* m$ is the restriction of this dual map to the set of invariant means and hence is continuous in the weak* topology. It follows that (5.9) is a projective system of continuous in the weak* topology maps. Since the sets $\mathcal{M}_n$ are compact in this topology, the limit of this projective system is nonempty (cf. [Bo], Chap. 1, Sec. 9, n° 6, Proposition 8). Any point of this limit is a coherent sequence. $\blacksquare$

Compatible partial contracting homotopies. Suppose that in the sequence

\[
(5.10) \quad \ldots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \ldots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1
\]

the groups $\pi_i(X_n) = 0$ for $i < n$, and each $p_n$ is a principal $G_n$-bundle for a topological group $G_n$ which is either abelian or discrete and amenable. Suppose that for each $n \geq 1$

\[
\mathbb{R} \xleftarrow{K_n^0} B^0(X_n) \xleftarrow{K_n^1} B^1(X_n) \xleftarrow{K_n^2} \ldots \xleftarrow{K_n^{n-1}} B^{n-1}(X_n)
\]

is a partial contracting homotopy. These partial homotopies are said to be compatible if

\[
(5.11) \quad p_{n*} \circ K_{n+1}^i \circ p_n^* = K_n^i
\]

for every $i \leq n - 1$, where $p_{n*} : B^i(X_{n+1}) \to B^i(X_n)$ are the maps from Theorem 5.4. By
Theorem 5.3 one can assume that they are defined for \( i \leq n - 1 \). Suppose that for each \( n \)

\[
\begin{array}{ccccccc}
1 & \xrightarrow{L_0^n} & S_0(X_n) & \xrightarrow{L_1^n} & S_1(X_n) & \xrightarrow{L_2^n} & \cdots & \xrightarrow{L_{n-1}^n} & S_{n-1}(X_n)
\end{array}
\]

is a sequence of maps such that for every \( \sigma \in S_i(X_n), \ i \leq n - 2 \), the equality

\[
(5.12) \quad \partial L_{i+1}^n(\sigma) + \sum_{k=0}^{i} (-1)^k L_i^n(\partial_k \sigma) = \sigma
\]

holds in the chain group \( C_{i+1}(X_n) \). Sequences \( L^n \) are said to be compatible if

\[
(5.13) \quad p_n \circ L_i^{n+1} = L_i^n \circ p_n
\]

for every \( i \leq n - 2 \), where we denote by the same symbol \( p_n \) the map \( S_i(X_{n+1}) \rightarrow S_i(X_n) \) induced by \( p_n : X_{n+1} \rightarrow X_n \).

5.8. Lemma. If there exist compatible sequences \( L^1, L^2, L^3, \ldots \), then there exist compatible strictly bounded partial homotopies \( K_1, K_2, K_3, \ldots \).

Proof. Suppose that \( L^n \) are compatible sequences. For \( i \leq n \) let

\[
K_i^n : B^i(X_n) \rightarrow B^{i-1}(X_n)
\]

be the map induced by \( L_i^n : S_{i-1}(X_n) \rightarrow S_i(X_n) \).

The identity (5.7) implies that the maps \( K_i^n \) with \( i \leq n \) form a partial contracting homotopy. Since \( K_i^n \) is induced by a map between the sets of singular simplices, its norm is \( \|K_i^n\| \leq 1 \). Therefore this partial contracting homotopy is strictly bounded.

It remains to check the compatibility condition (5.11). If \( f \in B^i(X_n) \), then

\[
p_n^* \circ K_{i+1}^n \circ p_n^* (f) = p_n^* \circ K_{i+1}^n (f \circ p_n) = p_n^* (f \circ p_n \circ L_i^{n+1}) = p_n^* (f \circ L_i^n \circ p_n) = p_n^* \circ p_n^* (f \circ L_i^n) = f \circ L_i^n = K_i^n (f),
\]

where on the second line we used the compatibility condition (5.13). This proves (5.11).
5.9. Lemma. If for every \( n \) the topological group \( G_n \) is \((n - 2)\)-connected, then there exist compatible sequences \( L^1, L^2, L^3, \ldots \).

**Proof.** Let us choose the base points \( b_n \in X_n \) in such a way that \( p_n(b_{n+1}) = b_n \) for all \( n \). Let \( m \geq 1 \). Suppose that we already constructed the sequences \( L^1, L^2, L^3, \ldots, L^m \) such that \( L^m_0(1) = b_n \) for all \( n \leq m \) and (5.13) holds for every \( n \leq m - 1 \) and \( i \leq n - 2 \).

Let us construct \( L^{m+1}_i \) in such a way that \( L^{m+1}_i(1) = b_{m+1} \) and (5.13) holds for \( n = m \) and every \( i \leq n - 2 \). Suppose that \( 1 \leq k \leq m \) and the maps

\[
L^{m+1}_i : S^{i-1}(X_{m+1}) \to S^i(X_{m+1})
\]

are already constructed for \( i \leq k - 1 \). Arguing by induction, we may assume that (5.13) holds if \( n = m \) and \( i \leq k - 1 \). Let us construct the map \( L^{m+1}_k \). If \( k = m \), then (5.13) with \( n = m \) imposes no restrictions on \( L^{m+1}_k \) and we can construct \( L^{m+1}_m \) as in Section 5.

Hence we may assume that \( k \leq m - 1 \). In the rest of the proof we will omit the subscripts and superscripts of \( L \). Let \( \sigma : \Delta_{k-1} \to X_m \) be an arbitrary singular \((k - 1)\)-simplex, and let

\[
\sigma = p_m(\overline{\sigma}) \quad \text{and} \quad \tau = L(\sigma).
\]

We need to define \( L(\overline{\sigma}) \) in such a way that \( p_m \circ L(\overline{\sigma}) = \tau \). The boundary \( \partial L(\overline{\sigma}) \) is already defined and we can take as \( L(\overline{\sigma}) \) any singular \( k \)-simplex \( \overline{\tau} : \Delta_k \to X_{m+1} \) such that

\[
\partial \overline{\tau} = \partial L(\overline{\sigma}) \quad \text{and} \quad p_m(\overline{\tau}) = \tau.
\]

The boundary \( \partial L(\overline{\sigma}) \) defines a continuous map \( \rho : b\Delta_k \to X_{m+1} \), where \( b\Delta_k \) is the geometric boundary of the simplex \( \Delta_k \). The inductive assumption implies that

\[
p_m \circ \rho = \tau|b\Delta_k,
\]

where \( \tau|b\Delta_k \) is the restriction of \( \tau \) to \( b\Delta_k \). Let \( \overline{\tau} : \Delta_k \to X_{m+1} \) be an arbitrary singular \( k \)-simplex such that \( p_m \circ \overline{\tau} = \tau \). Since \( p_m \) is a principal \( G_m \)-bundle, the map \( \rho \) differs from the restriction \( \overline{\tau}|b\Delta_j \) by a continuous map

\[
d : b\Delta_k \to G_m.
\]

Since \( b\Delta_k \) is homeomorphic to the sphere of dimension \( k - 1 \leq (m - 1) - 1 = m - 2 \) and \( G_m \) is \((m - 2)\)-connected, one can extend \( d \) to a continuous map

\[
D : \Delta_k \to G_m
\]

and take as \( \overline{\tau} \) the map differing from \( \overline{\tau} \) by \( D \). Since the singular simplex \( \overline{\sigma} : \Delta_{k-1} \to X_m \) was arbitrary, this completes the induction step and hence the proof of the lemma. ■
5.10. Theorem. Let \( X \) be a path-connected space homotopy equivalent to a CW-complex. If \( \pi_1(X) \) is amenable, then there exists a strictly bounded contracting homotopy

\[
\begin{array}{cccc}
\mathbb{R} & \leftarrow & K^0 & \leftarrow \ B^0(X) & \leftarrow & K^1 & \leftarrow & B^1(X) & \leftarrow & K^2 & \leftarrow & \cdots,
\end{array}
\]

i.e. a chain homotopy between \( \text{id} \) and \( 0 \), such that \( \|K_n\| \leq 1 \) for all \( n \).

Proof. The iterated killing of homotopy groups leads to a sequence of maps and spaces of the form (5.10) such that \( X_1 = X \), \( \pi_i(X_n) = 0 \) if \( i < n \), and each map \( p_n \) is a principal \( G_n \)-bundle for a topological group \( G_n \) which is either abelian or is discrete and amenable. Each \( G_n \) is an Eilenberg–MacLane space of type \( K(\pi_n(X), n-1) \), and hence is \((n-2)\)-connected for every \( n \). By Lemma 5.9 there exist compatible sequences \( L^1, L^2, L^3, \ldots \), and by Lemma 5.8 there exist compatible strictly bounded partial homotopies \( K^1, K^2, K^3, \ldots \).

In order to speak about compatible partial homotopies, it was sufficient to know that the maps

\[
p_n^*: B^i(X_{n+1}) \longrightarrow B^i(X_n)
\]

from Theorem 5.4 are defined for all \( i \leq n - 1 \). But in view of Theorem 5.7 one may assume that they are defined for all \( i \geq 0 \). Let \( K^i: B^i(X) \longrightarrow B^{i-1}(X) \) be the map

\[
K^i = p_1^* \circ \ldots \circ p_m^* \circ K_m^i \circ p_m^* \circ \ldots \circ p_1^*,
\]

where \( m \) is any integer \( \geq i + 1 \). Condition (5.11) implies that this definition does not depend on the choice of \( m \). The sequence \( K^* \) turns out to be a contracting homotopy. Indeed,

\[
\partial \circ K^i + K^{i+1} \circ \partial = \partial \circ p_1^* \circ \ldots \circ p_1^* + \ldots + \partial \circ p_1^* = \partial \circ p_1^* \circ p_m^* \circ \ldots \circ p_1^*
\]

\[
= p_1^* \circ \ldots \circ p_m^* \circ \partial \circ K_m^i \circ p_m^* \circ \ldots \circ p_1^*
\]

\[
= p_1^* \circ \ldots \circ p_m^* \circ K_m^{i+1} \circ p_m^* \circ \ldots \circ p_1^*
\]

\[
= p_1^* \circ \ldots \circ p_m^* \circ K_m^i \circ p_1^* + \partial \circ p_1^* \circ \ldots \circ p_1^* = \text{id}
\]

for every \( m \geq i + 1 \), where we used the facts that \( \partial \circ K_m^i + K_{m+1}^i \circ \partial = \text{id} \) and that \( p_n^* \), \( p_n^* \) commute with the differentials. Moreover, \( \|K^i\| \leq 1 \) because all the norms \( \|p_n^*\|, \|K_n^i\|, \|p_n^*\| \) are \( \leq 1 \). Hence \( K^* \) is the promised chain homotopy. \( \blacksquare \)
6. Weak equivalences

The \( l_1 \)-norm of chains. The \( l_1 \)-norm \( \| c \|_1 \) of a singular chain

\[
c = \sum_{\sigma} c_{\sigma} \sigma \in C_m(X),
\]

where \( c_{\sigma} \in \mathbb{R} \) and the sum is taken over all singular simplices \( \sigma \in S_m(X) \), is defined as

\[
\| c \|_1 = \sum_{\sigma} |c_{\sigma}|
\]

By the definition of singular chains, the sums above involve only finite number of non-zero coefficients \( c_{\sigma} \), and hence \( \| c \|_1 \) is a well defined real number (i.e. is \( < \infty \)). The \( l_1 \)-norm is one of the main notions of Gromov’s theory, but in this section it plays only a technical role.

Natural chain homotopies. Let \( \iota_n : \Delta_n \longrightarrow \Delta_n \) be the identity map of \( \Delta_n \) considered as a singular simplex \( \iota_n \in S_n(\Delta_n) \). There exist singular chains \( h_n \in C_{n+1}(\Delta_n \times I) \) such that

\[
(6.1) \quad \partial h_n = \iota_n \times 1 - \iota_n \times 0 - \sum_{i=0}^{n} (-1)^i (\partial_i \iota_n \times 1)_\ast (h_{n-1}),
\]

for all \( n \geq 0 \), where \( \partial_i \iota_n : \Delta_{n-1} \longrightarrow \Delta_n \) is the \( i \)-th face of the singular simplex \( \iota_n \). Such chains are provided by natural chain homotopy between the morphisms

\[
C_\ast(\Delta_n) \longrightarrow C_\ast(\Delta_n \times I)
\]

induced by the maps \( \iota_n \times 0, \iota_n \times 1 : \Delta_n \longrightarrow \Delta_n \times I \), or, better, such chains are the main ingredient in the construction of natural chain homotopy between the morphisms

\[
C_\ast(Y) \longrightarrow C_\ast(Y \times I)
\]

induced by the maps \( \text{id}_Y \times 0, \text{id}_Y \times 1 : Y \longrightarrow Y \times I \) for an arbitrary space \( Y \).

\( k \)-connected pairs of spaces. Let \( X \) be a topological space and \( A \) be a subspace of \( X \). As usual, let \( D^m \) be the standard \( m \)-dimensional disc and \( S^{m-1} \) be its boundary sphere. The pair \( (X, A) \) is said to be \( k \)-connected if for \( m \leq k \) every map

\[
f : (D^m, S^{m-1}) \longrightarrow (X, A)
\]

is homotopic relatively to \( A \) to a map having image in \( A \).
**Bounded Eilenberg complexes.** We need a bounded cohomology version of a classical construction from the singular homology theory going back to Eilenberg [E]. For natural number $k$ let $(\Delta_n)_k$ be the union of all $\leq k$-dimensional faces of the standard $n$-simplex $\Delta_n$. Let $S_n(X, A)_k$ be the set of all singular simplices $\sigma: \Delta_n \rightarrow X$ such that

$$\sigma\left((\Delta_n)_k\right) \subset A.$$ 

If $\sigma \in S_n(X, A)_k$, then $\partial_i \sigma \in S_{n-1}(X, A)_k$ for every face $\partial_i \sigma$ of $\sigma$. Let $B^n(X, A)_k$ be the space of all bounded functions $S_n(X, A)_k \rightarrow \mathbb{R}$. The space $B^n(X, A)_k$ is a Banach space with respect to the supremum norm. The coboundary maps

$$d_n: B^n(X, A)_k \rightarrow B^{n+1}(X, A)_k$$

are defined by the same formula as for $B^\ast(X)$. The spaces $B^n(X, A)_k$ together with the coboundary maps $d_n$ form a cochain complex denoted by $B^\ast(X, A)_k$. Let $\widehat{H}^n(X, A)_k$ be the cohomology groups of this complex. These cohomology groups inherit a semi-norm from the Banach norm of the spaces $B^n(X, A)_k$. The obvious restriction map

$$\rho: B^\ast(X) \rightarrow B^\ast(X, A)_k$$

commutes with the boundary operators and hence induces homomorphisms

$$\rho_\ast: \widehat{H}^n(X) \rightarrow \widehat{H}^n(X, A)_k$$

6.1. **Lemma.** If $(X, A)$ is $k$-connected, then $\rho_\ast$ is an isometric isomorphism for all $n$.

**Proof.** If $(X, A)$ is $k$-connected, then one can assign to each $\sigma \in S_n(X)$ a homotopy $P(\sigma): \Delta_n \times I \rightarrow X$, where $I = [0, 1]$, in such a way that the following conditions hold.

(i) $P(\sigma)_0 = \sigma$.

(ii) $P(\sigma)_1 \in S_n(X, A)_k$.

(iii) If $\sigma \in S_n(X, A)_k$, then $P(\sigma)_t$ is a constant homotopy.

(iv) $P(\sigma) \circ (\partial_i \sigma \times \text{id}_I) = P(\partial_i \sigma)$ for each face $\partial_i \sigma$ of $\sigma$.

The construction is similar to the one of the maps $L^n$ in Section 5. See [Sp], Section 7.4, Lemma 7 and Theorem 8, or [tD], Section 9.5.
Let us define \( \alpha : B^n(X, A)_k \rightarrow B^n(X) \) by
\[
\alpha(f)(\sigma) = f(P(\sigma)_1).
\]

By the property (ii) the map \( \alpha \) is well defined, by (iv) the map \( \alpha \) is a chain map, and (iii) implies that \( \rho \circ \alpha \) is the identity map.

Let us now define \( k_{n+1} : B^{n+1}(X) \rightarrow B^n(X) \) by
\[
k_{n+1}(f)(\sigma) = f\left(P(\sigma)_*(h_n)\right),
\]
where in the right hand side the cochain \( f \in B^{n+1}(X) \) is interpreted as a linear functional \( C_{n+1}(X) \rightarrow \mathbb{R} \). A priori \( k_{n+1}(f) \) is only a singular cochain, and we need to check that this cochain is bounded. But, obviously,
\[
|k_{n+1}(f)(\sigma)| \leq \|f\| \cdot \|P(\sigma)_*(h_n)\|_1 \leq \|f\| \cdot \|h_n\|_1,
\]
and hence the cochain \( k_{n+1}(f) \) is indeed bounded. Moreover, the above inequalities imply that \( k_{n+1} \) is a bounded operator and \( \|k_{n+1}\| \leq \|h_n\|_1 \), but this is not needed for the proof.

The properties (i) and (iv) together with (6.1) imply that \( k_* \) is a chain homotopy between \( \alpha \circ \rho \) and the identity morphism of \( B^*(X) \). Cf. [Sp], Section 7.4, the proof of Lemma 7, or [tD], the end of the proof of Theorem 9.5.1. It follows that \( \alpha \circ \rho \) induces the identity map of the cohomology of \( B^*(X) \), i.e. of the bounded cohomology groups \( \hat{H}^n(X) \). Since \( \rho \circ \alpha \) is the identity map, it follows that the chain maps \( \alpha \) and \( \rho \) induce the mutually inverse isomorphisms between \( \hat{H}^n(X) \) and \( \hat{H}^n(X, A)_k \).

It remains to prove that these isomorphisms are isometric. It is sufficient to prove that both induced isomorphisms in cohomology are bounded with the norm \( \leq 1 \). In order to prove this, it is sufficient to prove that \( \alpha \) and \( \rho \) are bounded operators with the norm \( \leq 1 \). The operator \( \rho \) is defined as a restriction and hence its norm is \( \leq 1 \). The operator \( \alpha \) is dual to a map between sets of simplices, and hence \( \|\alpha\| \leq 1 \). In more details,
\[
|\alpha(f)(\sigma)| = |f(P(\sigma)_1)| \leq \|f\|
\]
because \( P(\sigma)_1 \) is a singular simplex (compare with the inequalities for the norm \( \|k_{n+1}\| \)). It follows that \( \|\alpha\| \leq 1 \). This completes the proof. \( \blacksquare \)

6.2. Theorem. If the pair \( (X, A) \) is \( k \)-connected, then the homomorphism
\[
i^* : \hat{H}^n(X) \rightarrow \hat{H}^n(A)
\]
induced by the inclusion \( i : A \rightarrow X \) is an isometric isomorphism for \( n \leq k - 1 \).
Proof. If \( n \leq k \), then, obviously, \( S_n(X, A)_k = S_n(A) \) and hence \( B^n(X, A)_k = B^n(A) \). Moreover, the restriction map \( \rho \) is nothing else but the restriction map \( B^n(X) \to B^n(A) \). In view of this, the theorem follows from Lemma 6.1. (In order to ensure the isomorphism property for \( n = k \), an isomorphism of cochains in dimension \( k + 1 \) is needed.) ■

6.3. Theorem. Suppose that the spaces \( X, A \) are path-connected and the map \( \varphi : A \to X \) is a \( k \)-equivalence, i.e. the induced map of homotopy groups

\[
\varphi_* : \pi_n(A, a) \to \pi_n(X, \varphi(a))
\]

is an isomorphisms for \( n \leq k - 1 \) and an epimorphisms for \( n = k \). Then

\[
\varphi^* : \hat{H}^n(X) \to \hat{H}^n(A)
\]

is an isometric isomorphism for \( n \leq k - 1 \).

Proof. The mapping cylinder \( Z \) of \( \varphi \) contains \( X \) as a deformation retract and \( A \) as a subspace. Moreover, the inclusion \( A \to Z \) is homotopic in \( Z \) to the composition of \( \varphi \) with the inclusion \( X \to Z \). As in the usual cohomology theory, this reduces the theorem to the case when the map \( A \to X \) is the inclusion of a subspace. But if \( A \) is a subspace of \( X \), then the map \( A \to X \) is a \( k \)-equivalence if and only if the pair \( (X, A) \) is \( k \)-connected. Therefore, the theorem follows from Theorem 6.2. ■

6.4. Corollary. Suppose that the spaces \( X, A \) are path-connected and the map \( \varphi : A \to X \) is a weak equivalence, i.e. the induced map of homotopy groups

\[
\varphi_* : \pi_n(A, a) \to \pi_n(X, \varphi(a))
\]

is an isomorphisms for all \( n \). Then

\[
\varphi^* : \hat{H}^n(X) \to \hat{H}^n(A)
\]

is an isometric isomorphism for all \( n \). ■

Extending the results to arbitrary topological spaces. As is well known, every path-connected space is weakly homotopy equivalent to a CW-complex. In view of this, Corollary 6.4 allows to extend the main results of the bounded cohomology theory from spaces homotopy equivalent to CW-complexes to arbitrary spaces. For example, the conclusions of Theorems 5.6 and 5.10 hold for arbitrary spaces.
Elements of homological algebra

**Bounded G-modules.** Let $G$ be a discrete group. A *bounded left $G$-module* is defined as a real semi-normed space $V$ together with a left action of $G$ on $V$ such that $\|g \cdot v\| \leq \|v\|$ for all $g \in G$ and $v \in V$. We will call the bounded left $G$-modules simply $G$-modules. If $V$ and $W$ are two $G$-modules, then a *$G$-morphism* from $V$ to $W$ is defined as a bounded linear operator $V \to W$ commuting with the action of $G$.

For every semi-normed space $V$ there is an action of $G$ on $V$ defined by $g \cdot v = v$ for all $g \in G$, $v \in V$. This action and the corresponding structure of a $G$-module on $V$ are called *trivial*. The simplest semi-normed space is $\mathbb{R}$ with the absolute value function being the semi-norm. The corresponding trivial $G$-module $\mathbb{R}$ is the simplest $G$-module.

For a bounded left $G$-module $V$ let $B(G, V)$ be the space of functions $f : G \to V$ such that

$$\|f\| = \sup \{ \|f(g)\| \mid g \in G \} \leq \infty.$$  

The space $B(G, V)$ is a Banach space with the norm $\|f\|$, and the action of $G$ defined by

$$(h \cdot f)(g) = h \cdot (f(gh))$$

turns it into a bounded $G$-module. If $V$ is a trivial $G$-module, then this action takes the form

$$(h \cdot f)(g) = f(gh).$$

**Relatively injective $G$-modules.** A $G$-morphism of $G$-modules $i : V_1 \to V_2$ is said to be *strongly injective* if there exists a bounded linear map $\sigma : V_2 \to V_1$ such that $\sigma \circ i = \text{id}$ and $\|\sigma\| \leq 1$. Here the map $\sigma$ is not assumed to be a morphism of $G$-modules. Obviously, a strongly injective $G$-morphism is injective.

A $G$-module $U$ is said to be *relatively injective* if for every strongly injective $G$-morphism of $G$-modules $i : V_1 \to V_2$ and any $G$-morphism of $G$-modules $\alpha : V_1 \to U$ there exists a $G$-morphism $\beta : V_2 \to U$ such that $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\alpha\|$. See the following diagram.

\[
\begin{array}{c}
V_1 & \xrightarrow{i} & V_2 \\
\alpha \downarrow & & \downarrow \beta \\
U & \xleftarrow{\sigma} & \\
\end{array}
\]

The following lemma provides us with all relatively injective modules we will need.
7.1. Lemma. For every $G$-module $V$ the $G$-module $B(G, V)$ is relatively injective.

Proof. Suppose that we are in the situation of the above diagram with $U = B(G, V)$. Given $i$ and $\alpha$, we need to construct $\beta$. Let us define $\beta$ by the formula

$$\beta(w)(g) = g^{-1} \cdot (\alpha \circ \sigma(g \cdot w)(1)),$$

where $w \in V_2$ and $g \in G$. A calculation, which we, contrary to the tradition, do not omit, shows that $\beta$ commutes with $G$ and $\beta \circ i = \alpha$. Namely, if $w \in V_2$ and $g, h \in G$, then

$$\beta(h \cdot w)(g) = g^{-1} \cdot (\alpha \circ \sigma((gh) \cdot w)(1))$$

$$= g^{-1} \cdot (\alpha \circ ((gh) \cdot w)(1))$$

$$= (h(gh)^{-1}) \cdot (\alpha \circ ((gh) \cdot w)(1))$$

$$= h \cdot (\alpha \circ ((gh) \cdot w)(1))$$

and hence $\beta$ is a $G$-morphism. Also, if $v \in V_1$, then

$$\beta(i(v))(g) = g^{-1} \cdot (\alpha \circ (g \cdot i(v))(1))$$

$$= g^{-1} \cdot (\alpha \circ i(g \cdot v)(1))$$

$$= g^{-1} \cdot (\alpha(g \cdot v)(1)) = g^{-1} \cdot ((g \cdot \alpha(v))(1))$$

$$= g^{-1} \cdot (g \cdot (\alpha(v)(g)))$$

$$= g^{-1} \cdot g \cdot (\alpha(v)(g)) = \alpha(v)(g)$$

and hence $\beta \circ i = \alpha$. Finally, $\|\beta\| \leq \|\alpha\|$ because for every $w \in V_2$

$$\|\beta(w)(g)\| = \|g^{-1} \cdot (\alpha \circ (g \cdot w)(1))\| \leq \|\alpha \circ (g \cdot w)(1)\|$$

$$\leq \|\alpha\| \cdot \|\sigma\| \cdot \|g \cdot w\| \leq \|\alpha\| \cdot \|w\|$$

$\blacksquare$
Resolutions. A resolution or, more precisely, a G-resolution of a G-module $U$ is defined as an exact sequence of G-modules and G-morphisms of the form

\[(7.1) \quad 0 \rightarrow U \xrightarrow{d_{-1}} U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} U_2 \xrightarrow{d_2} \cdots .\]

A contracting homotopy for the resolution (7.1) is a sequence of bounded operators

\[
\begin{array}{cccccccc}
U & K_0 & U_0 & K_1 & U_1 & K_2 & U_2 & \ldots \\
\end{array}
\]

such that

\[(7.2) \quad d_{n-1} \circ K_n + K_{n+1} \circ d_n = \text{id}\]

for $n \geq 1$, $K_0 \circ d_{-1} = \text{id}_U$, and $\|K_n\| \leq 1$ for all $n$.

The resolution (7.1) is said to be relatively injective if all G-modules $U_n$ are relatively injective, and strong if it admits a contracting homotopy. A split resolution

\[(7.3) \quad 0 \rightarrow U \xleftarrow{d_{-1}} U_0 \xleftarrow{K_0} U_1 \xleftarrow{K_1} U_2 \xleftarrow{K_2} \cdots ,\]

is defined as a strong resolution together with a contracting homotopy.

7.2. Lemma. If (7.3) is a split resolution, then $d_{-1}$ is strongly injective, as also the morphisms

\[d'_n : U_n/\text{Ker} \ d_n \rightarrow U_{n+1} .\]

induced by morphisms $d_n$ for all $n = 0, 1, 2, \ldots$.

Proof. Since $K_0 \circ d_{-1} = \text{id}$ and $\|K_n\| \leq 1$, the morphism $d_{-1}$ is strongly injective. Let

\[q_n : U_n \rightarrow U_n/\text{Ker} \ d_n\]

be the canonical projection and let $\sigma_n = q_n \circ K_{n+1} : U_{n+1} \rightarrow U_n/\text{Ker} \ d_n$. The homotopy identity (7.2) implies that

\[q_n \circ d_{n-1} \circ K_n + q_n \circ K_{n+1} \circ d_n = q_n .\]

The exactness of (7.1) implies that $\text{Im} \ d_{n-1} = \text{Ker} \ d_n$ and hence $q_n \circ d_{n-1} = 0$. Therefore
\[ \sigma_n \circ d_n = q_n \circ K_{n+1} \circ d_n = q_n. \] It follows that \( \sigma_n \circ d'_n = \text{id}. \)

Obviously, \( \|q_n\| \leq 1 \) and therefore

\[ \|\sigma_n\| = \|q_n \circ K_{n+1}\| \leq \|q_n\| \cdot \|K_{n+1}\| \leq 1 \cdot 1 = 1. \]

These properties of \( \sigma_n \) imply that \( d'_n \) is strongly injective. ■

7.3. Lemma. Suppose that (7.3) is a split resolution of \( U \) and

\[ 0 \rightarrow V \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots, \tag{7.4} \]

is a complex of \( G \)-modules, i.e. \( d_{n+1} \circ d_n = 0 \) for \( n \geq -1. \) If all \( G \)-modules \( V_n, \ n \geq 0, \) are relatively injective, then any \( G \)-morphism \( u : U \rightarrow V \) can be extended to a \( G \)-morphism from the resolution (7.3) to the complex (7.4), i.e. to a commutative diagram

\[ 0 \rightarrow U \xrightarrow{d_{-1}} U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} U_2 \xrightarrow{d_2} \cdots \]

\[ \xrightarrow{u} \]

\[ 0 \rightarrow V \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots, \tag{7.5} \]

in which all maps \( u_i, \ i \geq 0, \) are \( G \)-morphisms.

**Proof.** Since \( d_{-1} \) is strongly injective by Lemma 7.2 and the \( G \)-module \( V_0 \) is relatively injective, there exists a \( G \)-morphism \( u_0 : U_0 \rightarrow V_0 \) such that

\[ u_0 \circ d_{-1} = d_{-1} \circ u, \]

i.e. the leftmost square of (7.5) is commutative. Suppose that \( G \)-morphisms \( u_0, u_1, \ldots, u_n \) are already constructed and that involving them squares of (7.5) are commutative. Then

\[ d_n \circ u_n \circ d_{n-1} = d_n \circ d_{n-1} \circ u_{n-1} = 0 \]

and hence \( d_n \circ u_n \) is equal to zero on the image \( \text{Im} d_{n-1}. \) Since \( \text{Im} d_{n-1} = \text{Ker} d_n, \) the \( G \)-morphism \( d_n \circ u_n \) induces a \( G \)-morphism

\[ u'_n : U_n / \text{Ker} d_n \rightarrow V_{n+1}. \]
Since the morphism $d'_n$ from Lemma 7.2 is strongly injective and $V_{n+1}$ is relatively injective, there exists a $G$-morphism $u_{n+1} : U_{n+1} \to V_{n+1}$ such that $u'_n = u_{n+1} \circ d'_n$ and hence

$$d_n \circ u_n = u_{n+1} \circ d_n.$$  

The induction completes the proof of the existence of $G$-morphisms $u_0, u_1, u_2, \ldots$.

7.4. Lemma. Under the assumptions of Lemma 7.3, every two extensions $u \cdot$ of $u$ are chain homotopic by a chain homotopy consisting of $G$-morphisms.

Proof. It is sufficient to prove that if in the diagram (7.5) $u = 0$, then the chain map $u \cdot$ is chain homotopic to zero. Suppose that $u = 0$. Then $u_0 \circ d_{-1} = d_{-1} \circ u = 0$. Since $\text{Im} d_{-1} = \text{Ker} d_0$, it follows that $u_0$ defines a morphism

$$u'_0 : U_0/\text{Ker} d_0 \to V_0.$$  

Since $d'_0$ is a strongly injective $G$-morphism and $V_0$ is a relatively injective $G$-module, there exists a $G$-morphism $k_1 : U_1 \to V_0$ such that $k_1 \circ d'_0 = u'_0$ and hence $k_1 \circ d_0 = u_0$. Suppose that for $m = 1, 2, \ldots, n$ morphisms

$$k_m : U_m \to V_{m-1}$$

are already constructed in such a way that

$$d_{m-1} \circ k_m + k_{m+1} \circ d_m = u_m$$

if $1 \leq m \leq n - 1$. Then $k_n \circ d_{n-1} = u_{n-1} - d_{n-2} \circ k_{n-1}$ and therefore

$$\left( u_n - d_{n-1} \circ k_n \right) \circ d_{n-1} = u_n \circ d_{n-1} - d_{n-1} \circ k_n \circ d_{n-1}$$

$$= u_n \circ d_{n-1} - d_{n-1} \circ (u_{n-1} - d_{n-2} \circ k_{n-1})$$

$$= (u_n \circ d_{n-1} - d_{n-1} \circ u_{n-1}) + d_{n-1} \circ d_{n-2} \circ k_{n-1}$$

$$= 0 + 0 = 0.$$  

It follows that the $G$-morphism $u_n - d_{n-1} \circ k_n$ defines a $G$-morphism

$$u'_n : U_n/\text{Ker} d_n \to V_n.$$  

46
Since $d'_n$ is strongly injective and $V_n$ is a relatively injective, there exists a G-morphism

$$k_{n+1} : U_{n+1} \rightarrow V_n$$

such that $k_{n+1} \circ d'_n = u'_n$. Then

$$k_{n+1} \circ d_n = u_n - d_{n-1} \circ k_n$$

and therefore $d_{n-1} \circ k_n + k_{n+1} \circ d_n = u_n$. An induction completes the proof of the existence of a chain homotopy between $u_\bullet$ and zero. ■

**The norm of morphisms in an extension.** It would be nice if the extension $u_\bullet$ in Lemma 7.3 could be chosen in such a way that $\|u_i\| \leq \|u\|$ for all $i \geq 0$. It is instructive to see what estimate of the norms is implicit in the proof of Lemma 7.3.

For every vector $x \in U_n/\ker d_n$ and every $\varepsilon > 0$ there exists a vector $y \in U_n$ such that $q_n(y) = x$ and $\|y\| \leq \|x\| + \varepsilon$. It follows that $u'_n(x) = d_n \circ u_n(y)$ and hence

$$\|u'_n(x)\| = \|d_n \circ u_n(y)\| \leq \|d_n \circ u_n\| \cdot \|y\|$$

$$\leq \|d_n \circ u_n\| \cdot \|x\| + \|d_n \circ u_n\| \cdot \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\|u'_n(x)\| \leq \|d_n \circ u_n\| \cdot \|x\|$. Therefore $\|u'_n\| \leq \|d_n \circ u_n\|$. Since the morphism $d'_n$ is strongly injective and the G-module $V_{n+1}$ is relatively injective, there exists a G-morphism $u_{n+1} : U_{n+1} \rightarrow V_{n+1}$ such that

$$\|u_{n+1}\| \leq \|u'_n\| \leq \|d_n \circ u_n\| \leq \|d_n\| \cdot \|u_n\|$$

and $u'_n = u_{n+1} \circ d'_n$. It follows that

$$\|u_{n+1}\| \leq \|u\| \cdot \prod_{i=-1}^{n} \|d_i\|.$$

Suppose that $\|d_n\| \leq n + 2$, as is the case for the so-called standard resolution of $V$ and for resolutions defined by topological spaces. Then the above proof leads to the estimate

$$\|u_n\| \leq (n + 1)! \cdot \|u\|,$$

implicitly contained in the work of R. Brooks [Br]. But by Theorem 7.6 below, for the standard resolution there is indeed an extension $u_\bullet$ such that $\|u_i\| \leq \|u\|$ for all $i \geq 0$. 47
The $G$-modules $B(G^{n+1}, U)$. For a bounded left $G$-module $U$ and $n \geq 0$ let $B(G^{n+1}, U)$ be the space of functions $f : G^{n+1} \to U$ such that

$$
\| f \| = \sup \left\{ \| f(g_0, g_1, \ldots, g_n) \| \mid (g_0, g_1, \ldots, g_n) \in G^{n+1} \right\} \leq \infty.
$$

$B(G^{n+1}, U)$ is a Banach space with the norm $\| f \|$, and the action $\cdot$ of $G$ defined by

$$(h \cdot f)(g_0, g_1, \ldots, g_n) = h \cdot f(g_0, g_1, \ldots, g_nh)$$

turns it into a bounded $G$-module. Let the $G$-module $b(G^{n+1}, U)$ be equal to $B(G^{n+1}, U)$ as a Banach space, but with the action $\ast$ of $G$ defined by

$$(h \ast f)(g_0, g_1, \ldots, g_n) = h \cdot f(g_0, g_1, \ldots, g_n).$$

Obviously, $B(G^{n+1}, U) = B(G, B(G^n, U))$ and hence $b(G^{n+1}, U)$ is a relatively injective $G$-module by Lemma 7.1. If $U$ is a trivial $G$-module, then $b(G^{n+1}, U)$ is also trivial.

The standard resolution. For a bounded left $G$-module $U$ let us consider the sequence

$$(7.6) \quad 0 \to U \xrightarrow{d_{-1}} B(G, U) \xrightarrow{d_0} B(G^2, U) \xrightarrow{d_1} B(G^3, U) \xrightarrow{d_2} \cdots,$$

where $d_{-1}(v)(g) = v$ for all $v \in U$, $g \in G$, and

$$d_n(f)(g_0, g_1, \ldots, g_{n+1}) = (-1)^{n+1} f(g_1, g_2, \ldots, g_{n+1})$$

$$+ \sum_{i=0}^{n} (-1)^{n-i} f(g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}),$$

for all $n \geq 0$ and $g_0, g_1, \ldots, g_{n+1} \in G$. Obviously, the maps $d_n$ are $G$-morphisms, and a standard calculation shows that $d_{n+1} \circ d_n = 0$ for all $n \geq -1$, i.e. that (7.6) is a complex. See Appendix 1 for a non-calculational proof. Let us consider also the sequence

$$(7.7) \quad U \xleftarrow{K_0} B(G, U) \xleftarrow{K_1} B(G^2, U) \xleftarrow{K_2} B(G^3, U) \xleftarrow{\cdots},$$

where $K_n(f)(g_0, \ldots, g_{n-1}) = f(g_0, \ldots, g_{n-1}, 1)$ for all $g_0, g_1, \ldots, g_{n-1} \in G$. Obviously, $\| K_n \| \leq 1$ for all $n \geq 0$. A much easier standard calculation shows that (7.7) is a contracting homotopy for (7.6). It follows that the sequence (7.6) is exact. Therefore the chain complex (7.6) together with the contracting homotopy (7.7) is a strong resolution of $U$. This resolution is called the standard resolution. It is relatively injective by Lemma 7.1.
Comparing a split resolution with the standard one. Suppose that

\[ 0 \xrightarrow{d_{-1}} U \xleftarrow{K_0} U_0 \xrightarrow{d_0} U_1 \xleftarrow{K_1} U_2 \xrightarrow{d_1} U_3 \xleftarrow{K_2} \ldots, \]

is a split resolution of \( U \). Let us define the maps

\[ k_n, u_n : U_n \rightarrow B(G^{n+1}, U), \]

where \( n \geq 0 \), by the formulas

\[
k_n(f)(g_0, \ldots, g_n) = K_0(g_0 \cdot K_1(\ldots K_{n-1}(g_{n-1} \cdot K_n(g_n \cdot f))\ldots)),
\]

\[
u_n(f)(g_0, \ldots, g_n) = (g_0 g_1 \ldots g_n)^{-1} \cdot \left( k_n(f)(g_0, \ldots, g_n) \right).
\]

An equivalent way to define \( u_n \) is to set \( u_{-1} = \text{id}_U \) and use the recursive relation

\[ u_m(f)(g_0, \ldots, g_m) = g_m^{-1} \cdot \left( u_{m-1}(K_m(g_m \cdot f))(g_0, \ldots, g_{m-1}) \right). \]

in order to define \( u_m \) for \( m \geq 0 \).

7.5. Lemma. The maps \( u_n \) are \( G \)-morphisms.

Proof. Obviously, \( k_n(h \cdot f)(g_0, \ldots, g_n) = k_n(f)(g_0, \ldots, g_n h) \). Therefore

\[
u_n(h \cdot f)(g_0, \ldots, g_n)
\]

\[
= (g_0 g_1 \ldots g_n)^{-1} \cdot k_n(h \cdot f)(g_0, \ldots, g_n)
\]

\[
= (g_0 g_1 \ldots g_n)^{-1} \cdot k_n(f)(g_0, \ldots, g_n h)
\]

\[
= \left( h \cdot (g_0 g_1 \ldots g_n h)^{-1} \right) \cdot k_n(f)(g_0, \ldots, g_n h)
\]

\[
= h \cdot \left( (g_0 g_1 \ldots g_n h)^{-1} \cdot k_n(f)(g_0, \ldots, g_n h) \right)
\]

\[
= h \cdot \left( u_n(f)(g_0, \ldots, g_n h) \right) = (h \cdot u_n(f))(g_0, \ldots, g_n).
\]

It follows that \( u_n(h \cdot f) = h \cdot u_n(f) \) and hence \( u_n \) is a \( G \)-morphism. □
7.6. Theorem. The sequence of maps $u_\bullet = \{ u_n \}$ is a morphism of resolutions

\[
\begin{array}{ccccccc}
0 & \longrightarrow & U & \longrightarrow & U_0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & \cdots \\
\downarrow u_{-1} & & \downarrow u_0 & & \downarrow u_1 & & \downarrow u_2 & & \\
0 & \longrightarrow & U & \longrightarrow & B(G, U) & \longrightarrow & B(G^2, U) & \longrightarrow & B(G^3, U) & \longrightarrow & \cdots 
\end{array}
\]

extending $u_{-1} = id_U$ and such that $\| u_n \| \leq 1$ for all $n \geq 0$.

Proof. Since $\| K_m \| \leq 1$ and $\| g \cdot f \| \leq \| f \|$ for all $g \in G$, $f \in U_m$, $m \geq 0$,

\[
u_n(f)(g_0, \ldots, g_n) \leq \| f \|
\]

for every $g_0, \ldots, g_n \in G$ and $f \in U_n$, and hence $\| u_n(f) \| \leq \| f \|$ for every $f \in U_n$.

It follows that $\| u_n \| \leq 1$ for every $n$.

It remains to check that $d_n \circ u_n = u_{n+1} \circ d_n$ for all $n \geq -1$. We will prove this using induction by $n$. Since $d_{-1}$ is a $G$-morphism and $K_0 \circ d_{-1} = id$,

\[
u_0(d_{-1}(v))(g) = g^{-1} \cdot K_0(g \cdot d_{-1}(v)) = g^{-1} \cdot g \cdot v = v.
\]

At the same time $u_{-1} = id_U$ and hence

\[
d_{-1}(u_{-1}(v))(g) = d_{-1}(v)(g) = v
\]

It follows that $d_{-1} \circ u_{-1} = u_0 \circ d_{-1}$. Suppose now that $d_{n-1} \circ u_{n-1} = u_n \circ d_{n-1}$ and prove that $d_n \circ u_n = u_{n+1} \circ d_n$. By the relation (7.8) with $m = n + 1$ we have

\[
u_{n+1}(f)(g_0, \ldots, g_{n+1})
\]

\[
= g_{n+1}^{-1} \cdot \left( u_n(K_{n+1}(g_{n+1} \cdot d_n(f)))(g_0, \ldots, g_n) \right)
\]

\[
= g_{n+1}^{-1} \cdot \left( u_n(K_{n+1}(d_n(g_{n+1} \cdot f)))(g_0, \ldots, g_n) \right)
\]

\[
= g_{n+1}^{-1} \cdot \left( u_n(K_{n+1} \circ d_n(g_{n+1} \cdot f))(g_0, \ldots, g_n) \right).
\]
The homotopy identity (7.2) implies that

\[ u_n(K_{n+1} \circ d_n(g_{n+1} \cdot f))(g_0, \ldots, g_n). \]

\[ = u_n(g_{n+1} \cdot f - d_{n-1} \circ K_n(g_{n+1} \cdot f))(g_0, \ldots, g_n) \]

\[ = u_n(g_{n+1} \cdot f)(g_0, \ldots, g_n) - u_n(d_{n-1} \circ K_n(g_{n+1} \cdot f))(g_0, \ldots, g_n) \]

\[ = u_n(g_{n+1} \cdot f)(g_0, \ldots, g_n) - u_n \circ d_{n-1}(K_n(g_{n+1} \cdot f))(g_0, \ldots, g_n) \]

\[ = u_n(g_{n+1} \cdot f)(g_0, \ldots, g_n) - d_{n-1}(u_{n-1}(K_n(g_{n+1} \cdot f)))(g_0, \ldots, g_n), \]

where at the last step we used the inductive assumption \( u_n \circ d_{n-1} = d_{n-1} \circ u_{n-1} \). The definition of \( d_{n-1} \) together with the recursive relation (7.8) for \( m = n \) imply that the last displayed expression is equal to

\[ u_n(g_{n+1} \cdot f)(g_0, \ldots, g_n) = (-1)^n u_{n-1}(K_n(g_{n+1} \cdot f))(g_0, \ldots, g_{n-1}) \]

\[ - \sum_{i=0}^{n-1} (-1)^{n-1-i} u_{n-1}(K_n(g_{n+1} \cdot f))(g_0, \ldots, g_i g_{i+1}, \ldots, g_n) \]

\[ = u_n(g_{n+1} \cdot f)(g_0, \ldots, g_n) - (-1)^n g_{n+1} \cdot \left( u_n(f)(g_1, \ldots, g_{n+1}) \right) \]

\[ - \sum_{i=0}^{n-1} (-1)^{n-1-i} g_{n+1} \cdot \left( u_n(f)(g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \right) \]

\[ = g_{n+1} \cdot \left( u_n(f)(g_1, \ldots, g_{n+1}) \right) + (-1)^{n+1} g_{n+1} \cdot \left( u_n(f)(g_1, \ldots, g_{n+1}) \right) \]

\[ + \sum_{i=0}^{n-1} (-1)^{n-i} g_{n+1} \cdot \left( u_n(f)(g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \right) \]

\[ = (-1)^{n+1} g_{n+1} \cdot \left( u_n(f)(g_1, \ldots, g_{n+1}) \right) \]

\[ + \sum_{i=0}^{n} (-1)^{n-i} g_{n+1} \cdot \left( u_n(f)(g_0, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \right) \]

\[ = g_{n+1} \cdot \left( d_n(u_n(f))(g_0, \ldots, g_{n+1}) \right). \]

51
By combining the above calculations, we conclude that

\[ u_{n+1}(d_n(f))(g_0, \ldots, g_{n+1}) = d_n(u_n(f))(g_0, \ldots, g_{n+1}) \]

for all \( f \in U_n \) and all \( g_0, g_1, \ldots, g_{n+1} \in G \). Therefore \( u_{n+1} \circ d_n = d_n \circ u_n \).

**Bounded cohomology of discrete groups.** For any G-module \( W \) let

\[ W^G = \{ v \in W \mid g v = v \text{ for all } g \in G \} \]

be the subspace of \( G \)-invariant elements in \( W \). Given a \( G \)-module \( U \) and a \( G \)-resolution (7.1) of \( U \), we can form the complex of subspaces of invariant elements

(7.9) \hspace{1cm}

\[ 0 \hspace{1cm} U_0^G \hspace{1cm} d_0 \hspace{1cm} U_1^G \hspace{1cm} d_1 \hspace{1cm} U_2^G \hspace{1cm} d_2 \hspace{1cm} \cdots . \]

From now on, we will often shorten the notations for resolutions and complexes by omitting explicit references to the maps \( d_n \), where \( n \geq -1 \) or \( n \geq 0 \). So, the resolution (7.1) can be denoted simply by \( U_* \), and the complex (7.9) by \( U_*^G \). Let

\[ H^n(G, U) = \frac{\ker(d_n \mid U_n^G)}{\operatorname{im}(d_{n-1} \mid U_{n-1}^G)} \]

be the \( n \)-th cohomology space of the complex (7.9). Being a sub-quotient of the semi-normed space \( U_n \), it inherits from \( U_n \) a semi-norm. This semi-norm is not a norm if the image \( d_{n-1}(U_{n-1}^G) \) is not closed. But if \( U_n \) is a Banach space, this semi-norm is complete in the sense that the quotient by the subspace of elements with the norm 0 is a Banach space.

There is a preferred resolution of any \( G \)-module \( U \), namely, the standard resolution (7.6). The **bounded cohomology of** \( G \) **with coefficients in** \( U \) are defined as the cohomology spaces

\[ \hat{H}^n(G, U) = \mathcal{H}^n(G, B(G^{**+1}, U)^G) = H^n(B(G^{**+1}, U)^G) \]

of the complex of invariant subspaces of the standard resolution. See Appendix 2 for a more explicit description of this subcomplex. The spaces \( \hat{H}^n(G, U) \) are semi-normed real vector spaces. The bounded cohomology with coefficients in \( \mathbb{R} \) considered as a trivial \( G \)-module will be denoted simply by \( \hat{H}^n(G) \). The space \( \hat{H}^2(G) \) is always Hausdorff. See Appendix 3. If \( n \geq 3 \), then \( \hat{H}^n(G) \) may happen to be non-Hausdorff, as was shown by T. Soma [So1], [So2].

**Bounded cohomology and other resolutions.** Suppose that \( U, V \) are two \( G \)-modules and \( u : U \rightarrow V \) be a \( G \)-morphism. Suppose that \( U_* \) is a strong relatively injective resolution of \( U \) and \( V_* \) is a strong relatively injective resolution of \( V \).
By Lemma 7.3 there exists a $G$-morphism $u_\ast: U_\ast \to V_\ast$ extending $u$, and by Lemma 7.4 such an extension $u_\ast$ is unique up to chain homotopies consisting of $G$-morphisms. Being a $G$-morphism, $u_\ast$ defines a homomorphism

$$u_\ast: \mathcal{H}^n(G, U_\ast) \to \mathcal{H}^n(G, V_\ast)$$

for every $n \geq 0$. Since $u_\ast$ is unique up to chain homotopies consisting of $G$-morphisms, $u_\ast$ depends only on $u$, and since $u_\ast$ consists of bounded maps, all maps $u_\ast$ are bounded.

Suppose now that $U_\ast, U'_\ast$ are two resolutions of the same $G$-module $U$, which are both strong and relatively injective. Then the identity map $\text{id}_U$ extends to $G$-morphisms

$$i_\ast: U_\ast \leftrightarrow U'_\ast : i'_\ast$$

which, in turn, leads to canonical homomorphisms

$$i_\ast: \mathcal{H}^n(G, U_\ast) \leftrightarrow \mathcal{H}^n(G, U'_\ast) : i'_\ast$$

for every $n \geq 0$. The composition $i'_\ast \circ i_\ast: U_\ast \to U_\ast$ extends the identity $G$-morphism $\text{id}_U$. But the identity morphism of $U_\ast$ also extends $\text{id}_U$. By Lemma 7.4 this implies that $i'_\ast \circ i_\ast$ is chain homotopic to the identity morphism of $U_\ast$ and hence the map

$$\mathcal{H}^n(G, U_\ast) \to \mathcal{H}^n(G, U_\ast)$$

induced by $i'_\ast \circ i_\ast$ is equal to the identity for every $n \geq 0$. It follows that the composition $i'_\ast \circ i_\ast$ is equal to the identity. By the same argument $i_\ast \circ i'_\ast$ is equal to the identity. It follows that $i_\ast, i'_\ast$ are mutually inverse isomorphism of vector spaces. Since $i_\ast, i'_\ast$ are bounded, they are isomorphisms of topological vector spaces.

It follows that up to a canonical isomorphism $\mathcal{H}^n(G, U_\ast)$ does not depend on the choice of the resolution $U_\ast$ as a topological vector space. In particular, it is isomorphic to $\hat{\mathcal{H}}^n(G, U)$ as a topological vector space. But its semi-norm depends on $U_\ast$.

7.7. Theorem. Suppose that $U_\ast$ is a strong relatively injective resolution of a $G$-module $U$. Then for every $n \geq 0$ there exists a canonical isomorphism of topological vector spaces

$$\mathcal{H}^n(G, U_\ast) \to \hat{\mathcal{H}}^n(G, U)$$

which is a bounded operator of norm $\leq 1$.

Proof. The existence of such an isomorphism follows from Theorem 7.6, and its independence on any choices follows from Lemma 7.4.  ■
8. Bounded cohomology and the fundamental group

**Discrete principal bundles.** Let $G$ be a discrete group. Suppose that $p : \mathcal{X} \longrightarrow X$ is a locally trivial principal right $G$-bundle. Then the group $G$ acts freely on $\mathcal{X}$ from the right, the quotients space $\mathcal{X}/G$ is equal to $X$, and $p$ is a covering space projection. The action of $G$ on $\mathcal{X}$ induces a left action of $G$ on the vector spaces $B^n(\mathcal{X})$ and thus turns them into left $G$-modules. The projection of the bundle $p$ induces isometric isomorphisms

$$p^* : B^n(X) \longrightarrow B^n(\mathcal{X})^G$$

commuting with the differentials, and hence an isometric isomorphism of the complexes

$$p^* : B^*(X) \longrightarrow B^*(\mathcal{X})^G$$

Of course, the cohomology of $B^*(X)$ is the bounded cohomology $\hat{H}^*(X)$.

**Strictly acyclic bundles.** The bundle $p : \mathcal{X} \longrightarrow X$ is said to be strictly acyclic if $\mathcal{X}$ admits strictly bounded contracting homotopy in the sense of Theorem 5.10. By Theorem 5.10 this is the case if the fundamental group $\pi_1(\mathcal{X})$ is amenable. If $p$ is strictly acyclic, then $B^*(\mathcal{X})$ is a strong resolution of the trivial $G$-module $\mathbb{R}$ and the cohomology of the complex $B^*(\mathcal{X})^G$ are the cohomology denoted in Section 7 by $\mathcal{H}^*(G, B^*(\mathcal{X}))$. It follows that in this case

$$\hat{H}^*(X) = \mathcal{H}^*(G, B^*(\mathcal{X})).$$

On the other hand, if $p$ is strictly acyclic, then by Theorem 7.6 there exists a $G$-morphism

$$u_* : B^*(\mathcal{X}) \longrightarrow B(G^{*+1})$$

from $B^*(\mathcal{X})$ to the standard $G$-resolution of $R$ extending $id_R$ and consisting of maps $u_n$ of the norm $\leqslant 1$. Since $B(G^{*+1})$ is relatively injective, by Lemma 7.4 $u_*$ is unique up to chain homotopies. By passing to $G$-invariants and then to the cohomology, $u_*$ leads to a map

$$u(p)_* : \mathcal{H}^*(G, B^*(\mathcal{X})) \longrightarrow \mathcal{H}^*(G, B(G^{*+1})) = \hat{H}^*(G)$$

depending only on the action of $G$ on $\mathcal{X}$, i.e. only on the principal $G$-bundle $p$. In view of (8.1) and the definition of $\hat{H}^*(G)$, the map $u(p)_*$ can be interpreted as a map

$$u(p)_* : \hat{H}^*(X) \longrightarrow \hat{H}^*(G).$$

It depends only on the principal $G$-bundle $p$. Since $u_*$ consists on the maps of the norm $\leqslant 1$, the norm of $u(p)_*$ is also $\leqslant 1$. 

54
A morphism \( B(G^{*+1}) \to B^*(\mathcal{X}) \). Let us construct a morphism of resolutions

\[
  r_* : B(G^{*+1}) \to B^*(\mathcal{X})
\]

extending \( \text{id}_R \) and consisting of maps \( r_n \) of the norm \( \leq 1 \).

Let \( F \) be a fundamental set for the action of \( G \) on \( \mathcal{X} \). i.e. a subset \( F \subset \mathcal{X} \) such that \( F \) intersects each \( G \)-orbit in exactly one point. Let \( n \geq 0 \) and let \( v_i \) be the \( i \)-th vertex of \( \Delta_n \). Let

\[
  s_n : S_n(\mathcal{X}) \to G^{n+1}
\]

be the map defined as follows. For a singular simplex \( \sigma: \Delta_n \to \mathcal{X} \) let

\[
  s_n(\sigma) = (g_0, g_1, \ldots, g_n),
\]

where \( g_0, g_1, \ldots, g_n \) are the unique elements of \( G \) such that

\[
  \sigma(v_0) \in F g_n,
\]

\[
  \sigma(v_1) \in F g_{n-1} g_n,
\]

\[
  \sigma(v_2) \in F g_{n-2} g_{n-1} g_n,
\]

\[
  \ldots \ldots \ldots,
\]

\[
  \sigma(v_{n-1}) \in F g_1 \ldots g_{n-1} g_n,
\]

\[
  \sigma(v_n) \in F g_0 g_1 \ldots g_{n-1} g_n.
\]

Let \( r_n : B(G^{n+1}) \to B^n(\mathcal{X}) \) be the map defined by the formula

\[
  r_n(f)(\sigma) = f(s_n(\sigma))
\]

A direct verification shows that \( r_* \) commutes with the differentials and hence is a morphism of resolutions. Obviously, \( r_* \) extends \( \text{id}_R \) and consists of maps \( r_n \) with the norm \( \leq 1 \).

A better way to see that \( r_* \) commutes with the differentials is to note that \( s_* \) commutes with the face operators of the singular complex \( S_*(\mathcal{X}) \) and of the nerve \( NG \) of the category \( G \) associated with the group \( G \) (see Appendix 1). This immediately implies that \( r_* \) commutes with the differentials.
8.1. Lemma. The \( G \)-modules \( B^n(\mathcal{X}) \) are relatively injective for all \( n \).

**Proof.** Let \( F \) be a fundamental set for the action of \( G \) on \( \mathcal{X} \). Let \( S_n(\mathcal{X}, F) \) be the set of singular simplices \( \Delta_n \rightarrow \mathcal{X} \) taking the first vertex \( v_0 \) of \( \Delta_n \) into \( F \). If \( \sigma : \Delta_n \rightarrow \mathcal{X} \) is a singular simplex, then \( \sigma = \tau g \) for unique \( \tau \in S_n(\mathcal{X}, F) \) and \( g \in G \). Hence the map

\[
S_n(\mathcal{X}, F) \times G \rightarrow S_n(\mathcal{X})
\]

defined by \( (\tau, g) \mapsto \tau g \) is a bijection. This bijection is equivariant with respect to the obvious right action of \( G \) on \( S_n(\mathcal{X}, F) \times G \) and the action of \( G \) on \( S_n(\mathcal{X}) \) induced by the action of \( G \) on \( \mathcal{X} \). Therefore this bijection leads to an isometric isomorphism of \( G \)-modules

\[
B(G, B(S_n(\mathcal{X}, F))) \rightarrow B^n(\mathcal{X})
\]

where the Banach space \( B(S_n(\mathcal{X}, F)) \) is considered as a trivial \( G \)-module. In view of this isomorphism Lemma 7.1 implies that the \( G \)-module \( B^n(\mathcal{X}) \) is relatively injective. \( \blacksquare \)

8.2. Theorem. If \( p : \mathcal{X} \rightarrow X \) is strictly acyclic, then \( u(p)_* \) is an isometric isomorphism. The inverse map is induced by \( r : B(G^{*+1}) \rightarrow B^*(\mathcal{X}) \).

**Proof.** The morphisms of resolutions \( u_* \) and \( r_* \) induce maps

\[
u_* : H^n(G, B^*(\mathcal{X})) \xrightarrow{\nu} H^n(G, B(G^{*+1})),\]

which have the norm \( \leq 1 \) together with \( u_n \) and \( r_n \). If the compositions \( r_* \circ u_* \) and \( u_* \circ r_* \) are equal to the identity, then \( r_* \) and \( u_* \) are mutually inverse isomorphisms, and since they both have the norm \( \leq 1 \), even isometric isomorphisms. In order to prove that \( r_* \circ u_* \) and \( u_* \circ r_* \) are equal to the identity, it is sufficient to prove that the compositions

\[
u_* \circ r_* : B(G^{*+1}) \rightarrow B(G^{*+1}) \quad \text{and} \quad r_* \circ u_* : B^*(\mathcal{X}) \rightarrow B^*(\mathcal{X})
\]

are chain homotopic to the identity by chain homotopies consisting of \( G \)-morphisms. Both \( u_* \circ r_* \) and \( r_* \circ u_* \) extend \( \text{id}_R \), as also do the identity morphisms of \( B(G^{*+1}) \) and \( B^*(\mathcal{X}) \). But \( B(G^{*+1}) \) is relatively injective by Lemma 7.1 and \( B^*(\mathcal{X}) \) is relatively injective by Lemma 8.1. Hence \( u_* \circ r_* \) and \( r_* \circ u_* \) are chain homotopic to the identity by Lemma 7.4. \( \blacksquare \)

8.3. Theorem. Let \( X \) be a path-connected space and let \( \Gamma = \pi_1(X) \). Suppose that \( A \) is a normal amenable subgroup of \( \Gamma \). Then there exists a canonical isometric isomorphism

\[
\hat{H}^*(X) \rightarrow \hat{H}^*(\Gamma/A).
\]

In particular, there exists a canonical isometric isomorphism \( \hat{H}^*(X) \rightarrow \hat{H}^*(\pi_1(X)) \).
Proof. In view of Section 6 we can assume that X is a CW-complex. Let $p: \mathcal{X} \to X$ be the covering space of $X$ corresponding to the subgroup $A$ of $\Gamma = \pi_1(X)$, and let $G = \Gamma/A$. Then $G$ acts freely on $\mathcal{X}$ and $\mathcal{X}/G = X$. Hence $p: \mathcal{X} \to X$ is a locally trivial principal $G$-bundle. Since $\pi_1(\mathcal{X})$ is isomorphic to $A$ and hence is amenable, Theorem 5.10 implies that the bundle $p$ is strictly acyclic. Therefore the theorem follows from Theorem 8.2. ■

8.4. Theorem (Mapping theorem). Let $X, Y$ be two path-connected spaces and $\varphi: Y \to X$ be a continuous map. If the induced homomorphism of the fundamental groups

$$\varphi_*: \pi_1(Y) \to \pi_1(X)$$

is surjective with amenable kernel, then $\varphi^*: \hat{\mathbb{H}}^*(X) \to \hat{\mathbb{H}}^*(Y)$ is an isometric isomorphism.

Proof. In view of Section 6 we can assume that $X, Y$ are CW-complexes. Let $\Gamma = \pi_1(Y)$, $G = \pi_1(X)$, and $A = \ker \varphi_*$. Then $G = \Gamma/A$. Let $p: \mathcal{X} \to X$ be the universal covering space of $X$ and $q: \mathcal{Y} \to Y$ be the covering space of $Y$ induced from $p$ by $\varphi$. Then $q$ corresponds to the subgroup $A \subset \Gamma = \pi_1(Y)$. The group $G$ acts on both $\mathcal{X}$ and $\mathcal{Y}$ and the canonical map $\Phi: \mathcal{Y} \to \mathcal{X}$ is $G$-equivariant, i.e. $\Phi$ is a morphism of $G$-bundles. By Theorem 5.10 both $p$ and $q$ are strictly acyclic. Hence there exist morphisms

$$u_*: B^*(\mathcal{X}) \to B(G^{*+1}) \quad \text{and} \quad v_*: B^*(\mathcal{Y}) \to B(G^{*+1})$$

extending $\text{id}_R$. The triangle

$$
\begin{array}{ccc}
B^*(\mathcal{X}) & \xrightarrow{\Phi^*} & B^*(\mathcal{Y}) \\
\downarrow{u_*} & & \downarrow{v_*} \\
B(G^{*+1}) & & 
\end{array}
$$

does not need to be commutative, but it is commutative up to a chain homotopy consisting of $G$-morphisms by Lemma 7.4. It follows that the triangle

$$
\begin{array}{ccc}
\hat{\mathbb{H}}^*(X) & \xrightarrow{\varphi^*} & \hat{\mathbb{H}}^*(Y) \\
\downarrow{u(p)_*} & & \downarrow{u(q)_*} \\
\hat{\mathbb{H}}^*(G), & & 
\end{array}
$$

which one gets after passing to the subspaces of $G$-invariants and then to cohomology, is commutative. The slanted arrows of this triangle are isometric isomorphisms by Theorem 8.2. Hence the theorem follows from the commutativity of this triangle. ■
Functoriality. Let $\alpha : \Gamma \to G$ be a homomorphism of discrete groups. If $U$ is a $G$-module, then $\alpha$ induces an action of $\Gamma$ on $U$ by the rule $h \cdot f = \alpha(h) \cdot f$ for $h \in \Gamma, f \in U$. The resulting structure of a $\Gamma$-module on $U$ is said to be induced by $\alpha$. If $U$ is a $G$-module, $V$ is a $\Gamma$-module, and $u : U \to V$ is a $\Gamma$-morphism with respect to the induced by $\alpha$ structure of a $\Gamma$-module on $U$ then $U^G \subset U^\Gamma$ and hence $u$ induces a map

$$u(G, \Gamma) : U^G \to V^\Gamma.$$ 

The obvious maps $\alpha^* : B(G^{n+1}) \to B(\Gamma^{n+1})$ are $\Gamma$-morphisms and commute with the differentials of the standard resolutions. Together all these maps define an $\Gamma$-morphism of resolutions $B(G^{*+1}) \to B(\Gamma^{*+1})$ and hence the maps $\alpha^*(G, \Gamma)$ induce a map

$$\alpha^* : \hat{H}^*(G) \to \hat{H}^*(\Gamma).$$

in the bounded cohomology. See Appendix 4 for a more general version of functoriality.

8.5. Theorem. Let $X, Y$ be path-connected spaces and $\varphi : Y \to X$ is a continuous map. Let $\varphi_* : \pi_1(Y) \to \pi_1(X)$ be the induced map of the fundamental groups. Then the square

$$\begin{array}{ccc}
\hat{H}^*(X) & \xrightarrow{\varphi^*} & \hat{H}^*(Y) \\
\downarrow & & \downarrow \\
\hat{H}^*(\pi_1(X)) & \xrightarrow{(\varphi_*)^*} & \hat{H}^*(\pi_1(Y)),
\end{array}$$

in which the vertical arrows are canonical isomorphisms from Theorem 8.3, is commutative.

Proof. In view of Section 6 we can assume that $X, Y$ are CW-complexes. Let $\Gamma = \pi_1(Y)$, $G = \pi_1(X)$, and $\alpha = \varphi_*$. Let $p : \mathcal{X} \to X$ and $q : \mathcal{Y} \to Y$ be the universal coverings of $X$ and $Y$ respectively. There exists a map $\Phi : \mathcal{Y} \to \mathcal{X}$ such that the square

$$\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\Phi} & \mathcal{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\varphi} & X
\end{array}$$

is commutative and $\Phi$ is $\alpha$-equivariant, i.e. if $h \in \Gamma$ and $y \in \mathcal{Y}$, then

$$\Phi(h \cdot y) = \alpha(h) \cdot \Phi(y).$$
The coverings $p$ and $q$ are locally trivial right $G$-bundle and $\Gamma$-bundle respectively. By Theorem 5.10 both $p$ and $q$ are strictly acyclic and hence there exist morphisms

$$u_* : B^\bullet(\mathcal{X}) \to B(G^{*+1}) \quad \text{and} \quad v_* : B^\bullet(\mathcal{Y}) \to B(\Gamma^{*+1})$$

extending $\text{id}_R$. The square

$$\begin{array}{ccc}
B^\bullet(\mathcal{X}) & \xrightarrow{\Phi^*} & B^\bullet(\mathcal{Y}) \\
\downarrow u_* & & \downarrow v_* \\
B(G^{*+1}) & \xrightarrow{\alpha^*} & B(\Gamma^{*+1})
\end{array}$$

does not need to be commutative, but it is commutative up to a chain homotopy consisting of $G$-morphisms by Lemma 7.4. It follows that the square

$$\begin{array}{ccc}
\hat{H}^\bullet(X) & \xrightarrow{\Phi^*} & \hat{H}^\bullet(Y) \\
\downarrow u(p)_* & & \downarrow u(q)_* \\
\hat{H}^\bullet(G) & \xrightarrow{\alpha^*} & \hat{H}^\bullet(\Gamma)
\end{array}$$

which one gets after passing to the subspaces of $G$-invariants and then to cohomology, is commutative. The vertical arrows of this square are the canonical isomorphisms from Theorem 8.3. Hence the theorem follows from the commutativity of this square. ■

**Bounded vs. classical cohomology.** For any space $X$ the inclusions $B^n(X) \to C^n(X)$ commute with the differentials and hence induce a canonical map

$$h_X : \hat{H}^\bullet(X) \to H^\bullet(X)$$

from bounded cohomology to the usual singular cohomology with real coefficients. In order to define an analogue of this map for groups, one needs to define first the classical cohomology of a group $G$. Let $C(G^n)$ be the vector space of all functions $G^n \to R$ and let the sequence

$$0 \to R \xrightarrow{d_1} C(G) \xrightarrow{d_0} C(G^2) \xrightarrow{d_1} C(G^3) \xrightarrow{d_2} \cdots ,$$

be defined by the same formulas as for $B(G^{*+1})$. This sequence is a complex of the classical $G$-modules (without any boundedness conditions). One may consider the subcomplex of its $G$-invariant elements and defined the classical cohomology $H^\bullet(G)$ of the group $G$ with co-
coefficients in the trivial G-module R as the cohomology of this subcomplex. The inclusions $B(G^{n+1}) \to C(G^{n+1})$ commute with the differentials and hence induce a canonical map

$$h_G : \hat{H}^*(G) \to H^*(G)$$

from the bounded cohomology of G to the classical cohomology of G with real coefficients.

**The classical cohomology and the fundamental group.** Let G be a discrete group and let $p : \mathcal{X} \to X$ be a principal right G-bundle. As before, $p$ induces an isomorphism

$$p^* : C^*(X) \to C^*(\mathcal{X})^G.$$  

The spaces $C^*(\mathcal{X})$ are relatively injective G-modules in the classical sense. The proof is similar to the proof of Lemma 8.1. The classical version of Lemma 7.3 leads to a G-morphism

$$(8.3) \quad C(G^{n+1}) \to C^*(\mathcal{X})$$

unique up to a chain homotopy by Lemma 7.4. By taking the spaces of G-invariant elements and passing to cohomology one gets a canonical map

$$H^*(G) \to H^*(X).$$

Suppose now that X is a path-connected space and let $G = \pi_1(X)$. If X admits a universal covering space, then taking $p$ to be the universal covering leads to a canonical map

$$(8.4) \quad H^*(G) \to H^*(X).$$

In general, one needs to begin by replacing X by a weak equivalent CW-complex.

**8.6. Theorem.** Let X be a path-connected space and let $G = \pi_1(X)$. Then the square

$$\begin{array}{ccc}
\hat{H}^*(G) & \xrightarrow{h_G} & H^*(G) \\
\downarrow & & \downarrow \\
\hat{H}^*(X) & \xrightarrow{h_X} & H^*(X),
\end{array}$$

where the left vertical arrow is the inverse of the canonical isomorphism from Theorem 8.3 and the right vertical arrow is the map (8.4), is commutative.
Proof. As usual, we may assume that X is a CW-complex. Let \( p : \tilde{X} \to X \) be the universal covering of X. The left vertical arrow is an instance of the inverse of the canonical map from Theorem 8.2 and hence is induced by a morphism \( r_* : B(G^{*+1}) \to B^*(\tilde{X}) \). The square

\[
\begin{array}{ccc}
B(G^{*+1}) & \xrightarrow{r_*} & C(G^{*+1}) \\
\downarrow & & \downarrow \\
B^*(\tilde{X}) & \xrightarrow{r^*} & C^*(\tilde{X})
\end{array}
\]

where the horizontal arrows are inclusions and the right vertical arrow is the map (8.3), does not need to be commutative, but it is commutative up to a chain homotopy consisting of classical G-morphisms by the classical analogue of Lemma 7.4. After passing to G-invariants and then to cohomology, this implies that the square from the theorem is commutative.

The Eilenberg–MacLane spaces. Perhaps, the easiest way to explain to a topologist what is the cohomology of groups is to define the cohomology of a group G as the cohomology of an Eilenberg–MacLane space \( K(G, 1) \). For the bounded cohomology such a definition is equivalent to the definition from Section 7 by Theorem 8.3. Moreover, the two definitions are equivalent in a functorial manner, as follows from Theorem 8.5. For the classical cohomology the two definitions are equivalent by the classical analogue of Theorem 8.3, and this equivalence is functorial by the classical analogue of Theorem 8.5. These classical analogues are well known and are easier than their bounded cohomology counterparts, but apply only to Eilenberg–MacLane spaces.

Let X be a path-connected space, \( G = \pi_1(X) \), and let \( \varphi : X \to K(G, 1) \) be a map inducing the identity map of the fundamental groups \( G = \pi_1(X) \to \pi_1(K(G, 1)) \). Such a map \( \varphi \) always exists if X is homotopy equivalent to a CW-complex. Then the diagram

\[
\begin{array}{ccc}
\hat{H}^*(K(G, 1)) & \xrightarrow{h_{K(G, 1)}} & H^*(K(G, 1)) \\
\downarrow_{\varphi^*} & & \downarrow_{\varphi^*} \\
\hat{H}^*(X) & \xrightarrow{h_X} & H^*(X)
\end{array}
\]

is commutative and, moreover, is canonically isomorphic to the diagram of Theorem 8.6. The proof is left to the reader as an exercise.
The Hirsch–Thurston class of groups. Hirsch and Thurston [HT] defined a class of groups \( \mathcal{C} \) as the smallest class containing all amenable groups and such that if \( G, H \in \mathcal{C} \), then the free product \( G \ast H \in \mathcal{C} \), and if \( G \in \mathcal{C} \) and \( G \) is a subgroup of finite index in \( K \), then \( K \in \mathcal{C} \).

The class \( \mathcal{C} \) is closed under passing to subgroups of finite index. One can prove this by induction using Kurosh’s theorem about subgroups of free products of groups (see, for example, [Se], Chapter I, Section 5.5) and the fact that every subgroup of an amenable group is amenable (see [Gr], Theorem 1.2.5). We leave details to the reader.

8.7. Theorem. If \( G \in \mathcal{C} \), then \( h_G : \hat{H}^*(G) \to H^*(G) \) is the zero homomorphism.

Proof. To begin with, Theorem 5.6 together with Theorem 8.3 implies that \( \hat{H}^*(G) = 0 \) if \( G \) is an amenable group. Therefore in this case \( \hat{H}^*(G) \to H^*(G) \) is a zero homomorphism. Suppose that \( \hat{H}^*(G_i) \to H^*(G_i) \) are zero homomorphisms for \( i = 1, 2 \). The canonical embeddings \( G_i \to G_1 \ast G_2 \), where \( i = 1, 2 \), induce the horizontal arrows of the diagram

\[
\begin{array}{ccc}
\hat{H}^*(G_1 \ast G_2) & \to & \hat{H}^*(G_1) \oplus \hat{H}^*(G_2) \\
\downarrow & & \downarrow \\
H^*(G_1 \ast G_2) & \to & H^*(G_1) \oplus H^*(G_2).
\end{array}
\]

Obviously, this diagram is commutative. It is well known that the lower horizontal arrow is an isomorphism. Therefore the commutativity of the diagram implies that the homomorphism \( \hat{H}^*(G_1 \ast G_2) \to H^*(G_1 \ast G_2) \) is equal to zero. Finally, suppose that \( G \) is a subgroup of finite index of a group \( K \) and \( \hat{H}^*(G) \to H^*(G) \) is a zero homomorphism. The inclusion \( G \to K \) induces the horizontal arrows of the commutative diagram

\[
\begin{array}{ccc}
\hat{H}^*(K) & \to & \hat{H}^*(G) \\
\downarrow & & \downarrow \\
H^*(K) & \to & H^*(G).
\end{array}
\]

Since the module of coefficients is the trivial module \( \mathbb{R} \), the lower horizontal arrow is injective. It follows that \( \hat{H}^*(K) \to H^*(K) \) is a zero homomorphism. The theorem follows. ■

Remark. It may happen that \( G \in \mathcal{C} \), but \( \hat{H}^*(G) \neq 0 \). For example, this is the case if \( G \) is a free group with \( \geq 2 \) free generators. See [Br] or [Gro1].
Closed manifolds of negative curvature. Hirsch and Thurston [HT] suggested that if \( M \) is a closed riemannian manifold of negative curvature, then \( \pi_1(M) \notin \mathcal{C} \). See [HT], remarks after Corollary 1.3. Theorem 8.7 leads to a natural proof of this conjecture.

If \( M \) is orientable, then \( \tilde{H}^*(M) \rightarrow H^*(M) \) is non-zero by a theorem of Thurston. In fact, the image of this homomorphism contains the fundamental class of \( M \). See [Gro_1], [T], or [IY] for a proof. It follows that in this case \( \tilde{H}^*(\pi_1(M)) \rightarrow H^*(\pi_1(M)) \) is non-zero and therefore \( \pi_1(M) \notin \mathcal{C} \) by Theorem 8.7. If \( M \) is not orientable, one can pass to the orientation covering of \( M \) and conclude that \( \pi_1(M) \) contains a subgroup of index 2 not belonging to \( \mathcal{C} \). Since \( \mathcal{C} \) is closed under passing to subgroups of finite index, this completes the proof.

The first proof of this conjecture is due to N. Gusevskii [Gus] and based on completely different ideas. See [I_1], Section (5.4) for some remarks about his proof. I refer to them mostly in order to correct a blatant translation mistake in the English translation of [I_1]. The last phrase of the footnote in Section (5.4) should be “It is amazing that this argument went unnoticed”.

The role of the homological algebra. The proofs in this section rely in an essential manner on Lemma 7.4, i.e. on the homotopy uniqueness of morphisms between the resolutions. At the same time they do not rely on Lemma 7.3 about the existence of such morphisms, despite the fact that the involved resolutions such as \( B(G^{*+1}) \) and \( B^*(\mathcal{X}) \) are strong and relatively injective and hence Lemma 7.3 applies. Instead of using Lemma 7.3 all the needed morphisms are constructed explicitly. The reason is that Lemma 7.3 does not provide the estimate \( \leq 1 \) for the norm, which is needed to prove that induced maps in cohomology are isometries.

In contrast with the bounded case, the singular cochain complex \( C^*(\mathcal{X}) \) of the universal covering space \( \mathcal{X} \) of a space \( X \) is only rarely acyclic. By this reason in the classical situation one cannot expect to have an analogue of the map \( u(p)_* \) and one has to resort to the classical analogue of Lemma 7.3 which leads to a map going in the opposite direction. The use of this classical analogue causes no problem since there is no norm to take care of anyhow.

The definitions of strongly injective morphisms and relatively injective modules. Somewhat surprisingly, the details of these definitions do not play any essential role. The requirements \( \| \sigma \| \leq 1 \) in the definition of strongly injective morphisms, and \( \| \beta \| \leq \| \alpha \| \) in the definition of relatively injective modules could be relaxed to the boundedness of the operators \( \sigma \) and \( \beta \) without affecting any of the above results except the parenthetical discussion of the norm of morphisms of an extension after Lemma 7.4. These requirements are imposed because they are met every time one needs these notions and potentially they are useful. In contrast, the assumption \( \| K_n \| \leq 1 \) in the definition of strong resolutions plays a key role in Theorem 7.6 and hence in most of the results about isomorphisms being isometries.

Other situations. The technique developed in this section is quite flexible. In Appendix 5 this is illustrated by applying it to straight Borel cochains, playing an important role in [Gro_1].

63
9. The covering theorem

Coverings. Let $\mathcal{U}$ be a covering of a space $X$ by subsets of $X$. We will assume that

(C) either the covering $\mathcal{U}$ is open and $X$ is hereditary paracompact, i.e. every open subset of $X$ is paracompact, or $\mathcal{U}$ is closed and locally finite and $X$ is paracompact.

The condition (C) is needed to apply the sheaf theory. It holds for every covering of a paracompact manifold by open subsets and for every locally finite covering of a CW-complex by its CW-subcomplexes. After the main results are proved, one can extend them at least to all finite open coverings by replacing $X$ by the geometric realization of its singular complex and using the invariance of bounded cohomology under weak equivalences. See Theorem 9.3.

A path-connected subset $Y$ of a space $X$ is said to be amenable if the image of the inclusion homomorphism $\pi_1(Y) \to \pi_1(X)$ is an amenable group. For example, if the fundamental group $\pi_1(Y)$ is amenable, the $Y$ is amenable. This follows from the fact that any quotient group of an amenable group is amenable. An arbitrary subset of a space $X$ is said to be amenable if all its path-components are amenable. A covering of a space $X$ is said to be amenable if it satisfies the condition (C) and all its elements are amenable.

The nerve of a covering. Recall that the nerve $N$ of a covering $\mathcal{U}$ is a simplicial complex in the sense, for example, of [Sp], Section 3.1. The vertices of $N$ are in a one-to-one correspondence with the set $\mathcal{U}$ and the simplices of $N$ are finite sets of vertices such that the intersection of the corresponding elements of $\mathcal{U}$ is non-empty. For each simplex $\sigma$ of $N$ we denote by $|\sigma|$ the intersection of the elements of the covering $\mathcal{U}$ corresponding to the vertices of $\sigma$. We will assume that the set of the vertices of $N$ is ordered by a linear order $<.$

9.1. Theorem (Covering theorem). Suppose that $\mathcal{U}$ is an amenable covering of a space $X$. Let $N$ be the nerve of the covering $\mathcal{U}$ and let $|N|$ be the geometric realization of the nerve $N$. Then the canonical homomorphism $\hat{H}^*(X) \to H^*(X)$ can be factored through the canonical homomorphism $H^*(|N|) \to H^*(X)$. In other terms, the diagram of the solid arrows

\[
\begin{array}{ccc}
\hat{H}^*(X) & \to & H^*(X) \\
& \searrow & \\
& H^*(|N|) & \\
\end{array}
\]

(9.1)

can be completed to a commutative diagram by a dashed arrow.
**Proof.** We start by reducing the theorem to its special case when the fundamental groups of all components of elements of $\mathcal{U}$ and of finite intersections of elements of $\mathcal{U}$ are amenable. Let $U$ be such a component, and let $\alpha$ be a loop in $U$ contractible in $X$. Let $X'$ be the result of attaching a two-dimensional disc along the loop $\alpha$ to $X$. Since $\alpha$ is contractible in $X$, the fundamental groups of $X'$ and $X$ are the same, and hence (by Theorem 8.4) the bounded cohomology of $X'$ and $X$ are the same. If we include the attached disc into all elements of the covering $\mathcal{U}$ containing $U$, we will get a covering $\mathcal{U}'$ of the $X'$. Clearly, $\mathcal{U}'$ has the same nerve as $\mathcal{U}$. Moreover, there is a canonical homomorphism $H^*(X') \longrightarrow H^*(X)$.

The glueing operation preserves the paracompactness assumptions. See [FP], Proposition A.5.1(v) and Exercise 5 in the Appendix (pp. 273 and 305). If $\mathcal{U}$ is closed and locally finite, then the same is true for $\mathcal{U}'$. If $\mathcal{U}$ is open, then $\mathcal{U}'$ is not open in general. In this case one needs to replace every $V \in \mathcal{U}'$ by an appropiate open neighborhood of $V$ in $X'$ having $V$ as a deformation retract. It follows that we can assume that $\mathcal{U}'$ satisfies the condition (C).

Therefore, if the theorem is true for $X'$ and $\mathcal{U}'$, then it is true for $X$ and $\mathcal{U}$. By attaching discs in this way one can kill the kernel of $\pi_1(U) \longrightarrow \pi_1(X)$ and hence turn $U$ into a subspace with amenable fundamental group (since the image of $\pi_1(U) \longrightarrow \pi_1(X)$ is amenable by the assumption). Attaching such discs to all components $U$ as above reduces the theorem to the case where all $\pi_1(U)$ are amenable. In the rest of the proof we will consider only this case.

For $p \geq 0$ let $N_p$ be the set of $p$-dimensional simplices of the nerve $N$. For $p, q \geq 0$ let

$$
C^p(N, B^q) = \bigoplus_{\sigma \in N_p} B^q(|\sigma|).
$$

Let us consider for every $\sigma \in N_p$ the complex

$$
(9.2) \quad 0 \longrightarrow \mathbb{R} \longrightarrow B^0(|\sigma|) \xrightarrow{d_0} B^1(|\sigma|) \xrightarrow{d_1} \cdots.
$$

For each $p \geq 0$ the direct sum of these complexes over all $\sigma \in N_p$ is the complex

$$
(9.3) \quad 0 \longrightarrow C^p(N) \longrightarrow C^p(N, B^0) \xrightarrow{d_0} C^p(N, B^1) \xrightarrow{d_1} \cdots,
$$

where $C^p(N)$ is the space of real simplicial $p$-cochains of $N$.

Let $p \geq 0$ and let $\sigma$ be a $p$-simplex $\sigma$ of $N$. Let

$$
v_0 < v_1 < \ldots < v_p
$$

be the vertices of $\sigma$ listed in the increasing order. For $i = 0, 1, \ldots, p$ let $\partial_i \sigma = \sigma \setminus \{v_i\}$. 65
In other terms, $\partial_i \sigma$ is the simplex with the vertices $v_0, \ldots, \tilde{v}_i, \ldots, v_p$. Let

$$\Delta_{\sigma, i} : B^q(|\partial_i \sigma|) \to B^q(|\sigma|)$$

be the map induced by the inclusion $|\sigma| \to |\partial_i \sigma|$, and

$$i_\sigma : B^q(|\sigma|) \to C^p(N, B^q),$$

$$p_\sigma : C^p(N, B^q) \to B^q(|\sigma|)$$

be the inclusion of and the projection to a direct summand respectively. Let us define maps

$$\delta_i, \delta : C^{p-1}(N, B^q) \to C^p(N, B^q)$$

by the formulas

$$\delta_i = \sum_{\sigma \in N_p} i_\sigma \circ \Delta_{\sigma, i} \circ p_{\partial_i \sigma} \quad \text{and} \quad \delta = \sum_{i=0}^p (-1)^i \delta_i.$$

Each of the maps $\delta_i$, as also the map $\delta$, defines a chain map

$$0 \to C^{p-1}(N) \to C^{p-1}(N, B^0) \to C^{p-1}(N, B^1) \to \ldots$$

$$0 \to C^p(N) \to C^p(N, B^0) \to C^p(N, B^1) \to \ldots,$$

Let $\Delta : B^q(X) \to B^q(|\sigma|)$ be the map induced by the inclusion $|\sigma| \to X$, and let

$$\Delta = \bigoplus_{\sigma \in N_0} \Delta_{\sigma}$$

The map $\Delta$ defines a chain map

$$0 \to R \to B^0(X) \to B^1(X) \to \ldots$$

$$0 \to C^0(N) \to C^0(N, B^0) \to C^0(N, B^1) \to \ldots,$$
All these chain maps together form a commutative diagram

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & R & \longrightarrow & B^0(X) & \longrightarrow & B^1(X) & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(N) & \longrightarrow & C^0(N, B^0) & \longrightarrow & C^0(N, B^1) & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(N) & \longrightarrow & C^1(N, B^0) & \longrightarrow & C^1(N, B^1) & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^2(N) & \longrightarrow & C^2(N, B^0) & \longrightarrow & C^2(N, B^1) & \longrightarrow & \ldots \\
& & \ldots & & \ldots & & \ldots & & \\
\end{array}
\]

(9.4)

A standard computation shows that the columns of this diagram are complexes. In particular, the family \( C^p(N, B^q) \) \( p, q \geq 0 \) together with the differentials \( d_*, \delta \) is a double complex.

Let us construct the homomorphism \( \hat{H}^*(X) \longrightarrow H^*(|N|) \). Let \( T \) be the total complex of the above double complex. By the definition,

\[
T^n = \bigoplus_{p + q = n} C^p(N, B^q)
\]

and the differential \( d : T^n \longrightarrow T^{n+1} \) is defined by the formula

\[
d|C^p(N, B^q) = d_q + (-1)^p \delta.
\]

The diagram (9.4) leads to the maps

\[
B^*(X) \longrightarrow T^* \quad \text{and} \quad C^*(N) \longrightarrow T^*,
\]

where \( B^*(X) \) and \( C^*(N) \) are the top row and the left column of the diagram (9.4) without \( R \). The second of these two maps commutes with the differentials only up to a sign, but still induces a map in cohomology. The cohomology of the complex \( B^*(X) \) are nothing else but the bounded cohomology \( \hat{H}^*(X) \). The cohomology of the complex \( C^*(N) \) are, by definition, the cohomology of the simplicial complex \( N \). The latter are known to be equal to \( H^*(|N|) \).
Therefore, there are two canonical maps

\[ \hat{H}^*(X) \rightarrow H^*(T) \quad \text{and} \quad H^*(|N|) \rightarrow H^*(T). \]

As we will see, the second of these two maps is an isomorphism. Assuming this for a moment, one can get the promised homomorphism \( \hat{H}^*(X) \rightarrow H^*(|N|) \) as the composition of the first map with the inverse of the second. By a well known theorem about double complexes (which is stated in Appendix 6 for the convenience of the reader), in order to prove that the homomorphism \( H^*(|N|) \rightarrow H^*(T) \) is an isomorphism, it is sufficient to prove that all rows of (9.4), starting with the second one, are exact. Each of these rows is the complex (9.3) for some \( p \geq 0 \) and hence is a direct sum of several complexes of the form (9.2). For every simplex \( \sigma \) the fundamental group \( \pi_1(|\sigma|) \) is amenable and hence \( \hat{H}^n(|\sigma|) = 0 \) for \( n > 0 \) by Theorem 5.6. It follows that the complexes (9.2) are exact, and hence the rows of (9.4) are also exact.

This completes the construction of the homomorphism \( \hat{H}^*(X) \rightarrow H^*(|N|) \). It remains to prove the commutativity of the diagram (9.1). As the first step towards the proof of the commutativity, let us replace in the construction of the diagram (9.4) the spaces \( B^q(|\sigma|) \) by the spaces \( \gamma^q(|\sigma|) \), and the spaces \( B^q(X) \) by the spaces \( \gamma^q(X) \). The new diagram has the same left column as the old one, its top row is the usual complex of the singular cochains of \( X \), and every term \( C^p(N, B^q) \) is replaced by the term \( C^p(N, \gamma^q) \). There is an obvious canonical map from the diagram (9.4) to the new diagram. In order to save some space, let us represent these two diagrams in the following self-explanatory manner:

\[
\begin{array}{ccc}
B^*(X) & \rightarrow & C^*(X) \\
\downarrow & & \downarrow \\
C^*(N) & \rightarrow & C^*(N, B^*) \\
& & \\
& & \\
& & \\
\end{array}
\]

Of course, the cohomology of the complex \( \gamma^*(X) \), i.e. of the top row of the second diagram, is nothing else but the usual singular cohomology \( H^*(X) \). If the columns of the second diagram were exact, we could easily complete the proof by referring to the same theorem about double complexes as above. But these columns are not exact in general, and our next goal is to find a similar diagram with exact columns. We will use the sheaf theory to this end.

For every \( q \geq 0 \) let \( \mathcal{C}^q \) be the presheaf on \( X \) assigning to every open subset \( U \subset X \) the real vector space of the real-valued cochains \( C^q(U) \) and to every inclusion \( U \subset V \) the restriction homomorphism \( C^q(V) \rightarrow C^q(U) \). Let \( \gamma^q \) be the sheaf associated with the presheaf \( \mathcal{C}^q \). The maps \( d_q : C^q(U) \rightarrow C^{q+1}(V) \) define a morphism of presheaves \( d_q : \mathcal{C}^q \rightarrow \mathcal{C}^{q+1} \). Let \( d_q : \gamma^q \rightarrow \gamma^{q+1} \) be the associated morphism of sheaves. Let us replace in the construction of the diagram (9.4) the spaces \( B^q(|\sigma|) \) by the spaces \( \gamma^q(|\sigma|) \), and the spaces \( B^q(X) \) by the spaces \( \gamma^q(X) \). Of course, we need also to use the maps \( d_q : \gamma^q(|\sigma|) \rightarrow \gamma^{q+1}(|\sigma|) \)
instead of the maps \( d_q : B^q(|\sigma|) \to B^{q+1}(|\sigma|) \) and the sheaf restriction maps

\[
\gamma^q(|\delta_i \sigma|) \to \gamma^q(|\sigma|)
\]

instead of the maps \( \Delta_{\sigma, i} \). This leads to a new analogue

\[
\begin{array}{c}
\gamma^*(X) \\
\downarrow \\
C^*(N) \to C^*(N, \gamma^*)
\end{array}
\]

of the diagram (9.4). Let \( C^*(N, \gamma^q) \) be the \( q \)-th column of the this diagram without the term \( \gamma^q(X) \). By definition, the cohomology of the complex \( C^*(N, \gamma^q) \) is nothing else but the cohomology of the covering \( \mathcal{U} \) with coefficients in the sheaf \( \gamma^q \). It follows that the 0-th cohomology group of this complex is equal to \( \gamma^q(X) \). See, for example, [Go], Chapter II, Theorem 5.2.2, which applies in view of the assumption (C).

We claim that the higher cohomology groups of this complex are 0. The proof consists of references to the sheaf theory; the classical book [Go] by R. Godement still seems to be the best source. Note first that \( \gamma^q \) is a soft sheaf if \( X \) is paracompact and is a flasque sheaf if \( X \) is hereditary paracompact. See [Go], Chapter II, Example 3.9.1. It remains to apply Theorem 5.2.3 from [Go], Chapter II, assuming that the family of supports \( \Phi \) from this theorem is the family of all closed subsets of \( X \). This is a paracompactifying family by [Go], Chapter II, Section 3.2. If \( \mathcal{U} \) is open and \( X \) is hereditary paracompact, then \( \gamma^q \) is a flasque sheaf and the higher cohomology vanish by the part (a) of this theorem. If \( \mathcal{U} \) is closed and locally finite and \( X \) is paracompact, then \( \gamma^q \) is a soft sheaf and the higher cohomology vanish by the part (c) of this theorem. In view of the assumption (C) this proves our claim.

Let us combine our three diagrams into one as follows.

\[
\begin{array}{c}
\gamma^*(X) \\
\downarrow \\
C^*(N, \gamma^*)
\end{array}
\]

The map from the second diagram to the third one is induced by the canonical maps of the
presheaves \( C^q \) to the associated sheaves \( \gamma^q \). Like \( C^p(N, B^q)_{p,q \geq 0} \), the families

\[ C^p(N, C^q)_{p,q \geq 0} \quad \text{and} \quad C^p(N, \gamma^q)_{p,q \geq 0} \]

are double complexes. Let us denote their total complexes by \( T_C^* \) and \( T_\gamma^* \) respectively. The previous diagram leads to the diagram

\[
\begin{array}{cccccc}
B^*(X) & \longrightarrow & C^*(X) & \longrightarrow & \gamma^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
T^* & \longrightarrow & T_C^* & \longrightarrow & T_\gamma^* \\
\downarrow & & \downarrow & & \downarrow \\
C^*(N) & \longrightarrow & C^*(N) & \longrightarrow & C^*(N)
\end{array}
\]

of complexes and chain maps, which, in turn, leads to the diagram

\[
\begin{array}{cccccc}
\hat{H}^*(X) & \longrightarrow & H^*(X) & \longrightarrow & H^*(\gamma^*(X)) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(T^*) & \longrightarrow & H^*(T_C^*) & \longrightarrow & H^*(T_\gamma^*) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(|N|) & \longrightarrow & H^*(|N|) & \longrightarrow & H^*(|N|)
\end{array}
\]

of cohomology groups, which is, like all the previous ones, obviously commutative. As we saw above, the homomorphism \( H^*(|N|) \longrightarrow H^*(T^*) \) is an isomorphism. Similarly, since the columns of (9.5) are exact, the theorem about double complexes (see Appendix 6) implies that \( j_\gamma : H^*(\gamma^*(X)) \longrightarrow H^*(T_\gamma^*) \) is an isomorphism. Moreover,

\[ k : H^*(X) \longrightarrow H^*(\gamma^*(X)) \]

is also an isomorphism. See [Sp], Example 6.7.9. Recall that the map

\[ \hat{H}^*(X) \longrightarrow H^*(|N|) \]

was defined as \( i^{-1} \circ j \). Now we see that the map \( i^{-1} \circ j : \hat{H}^*(X) \longrightarrow H^*(|N|) \) can be in-
cluded into the commutative diagram

\[
\begin{array}{ccc}
\hat{H}^*(X) & \to & H^*(X) \\
 i^{-1} \circ j & \downarrow & k^{-1} \circ j^{-1} \circ i_Y \\
 & H^*(|N|) & \\
\end{array}
\]

having the canonical map \( \hat{H}^*(X) \to H^*(X) \) as the horizontal arrow. This diagram has the promised form (9.1).

It remains only to settle the question if the map \( k^{-1} \circ j^{-1} \circ i_Y \) is the same as the canonical map \( H^*(|N|) \to H^*(X) \). From the point of view of the sheaf theory, the most natural way to define the canonical map \( H^*(|N|) \to H^*(X) \) is to define it as the map \( k^{-1} \circ j^{-1} \circ i_Y \). If this approach is adopted, the question disappears. The readers who nevertheless prefer some other definition of this map may compare their preferred definition with this one. In any case, this issue does not belong to the bounded cohomology theory.

9.2. Theorem (Vanishing theorem). If \( X \) admits an amenable covering such that every point of \( X \) is contained in no more that \( m \) elements of this covering, then the canonical homomorphism \( \hat{H}^i(X) \to H^i(X) \) vanishes for \( i \geq m \).

Proof. This immediately follows from Theorem 9.1.

The singular simplicial sets. For a topological space \( Y \) let \( s_*(Y) \) be the singular simplicial set of \( Y \). It is a simplicial set having as its \( n \)-simplices the singular simplices \( \Delta_n \to Y \) in \( Y \). As usual, we denote by \( |S| \) the geometric realization of a simplicial set \( S \). As is well known, there is a canonical map \( |s_*(Y)| \to Y \) and this map is a weak homotopy equivalence. If \( Y \subset X \), then \( s_*(Y) \) is a simplicial subset of \( s_*(X) \) and \( |s_*(Y)| \subset |s_*(X)| \). Moreover, the subset \( |s_*(Y)| \) is a closed subset of \( |s_*(X)| \). Indeed, if \( s_*(Y) \) contains a simplex, then \( s_*(Y) \) contains all its faces, and hence \( |s_*(Y)| \) is a union of (geometric realizations of) closed simplices. Since \( |s_*(X)| \) is equipped with the weak topology determined by the closed simplices (it is actually a CW-complex), this implies that \( |s_*(Y)| \) is a closed subset.

9.3. Theorem. Suppose that \( \mathcal{U} \) is a finite open covering of a space \( X \) by amenable path-connected subsets. Then the conclusion of Theorem 9.1 holds.

Proof. Let \( s_*(\mathcal{U}) \) be the union of \( s_*(U) \) over all \( U \in \mathcal{U} \). Clearly, \( s_*(\mathcal{U}) \) is closed under the face and degeneracy operators and therefore is a simplicial subset of \( s_*(X) \). The sets \( |s_*(U)| \) with \( U \in \mathcal{U} \) form a closed finite covering of \( |s_*(\mathcal{U})| \). Obviously, the nerve of this covering is the same as the nerve \( N \) of \( \mathcal{U} \). By a well known theorem of singular homology
theory the inclusion $s_{\ast}(\mathcal{U}) \to s_{\ast}(X)$ induces an isomorphism of homology and cohomology groups. See [Sp], Theorem 4.4.14. It follows that the inclusion $|s_{\ast}(\mathcal{U})| \to |s_{\ast}(X)|$ also induces an isomorphism of homology and cohomology groups. Let us consider the diagram

\[ \begin{array}{ccc}
\hat{H}^\ast(X) & \to & H^\ast(X) \\
\downarrow & & \downarrow \\
\hat{H}^\ast(|s_{\ast}(X)|) & \to & H^\ast(|s_{\ast}(X)|) \\
\downarrow & & \downarrow \\
\hat{H}^\ast(|s_{\ast}(\mathcal{U})|) & \to & H^\ast(|s_{\ast}(\mathcal{U})|) \\
& & \hat{H}^\ast(|N|) 
\end{array} \]

where the horizontal arrows are the canonical maps from the bounded to the usual cohomology, the vertical arrows are induced by the canonical map $|s_{\ast}(X)| \to X$ and the inclusion $|s_{\ast}(\mathcal{U})| \to |s_{\ast}(X)|$, and the lower triangle comes from Theorem 9.1 applied to the finite closed covering of $|s_{\ast}(\mathcal{U})|$ by the sets $|s_{\ast}(U)|$ with $U \in \mathcal{U}$. This diagram is obviously commutative. Since the right vertical arrows are isomorphisms, the theorem follows. \[\blacksquare\]

**Remark.** The fact that the bounded cohomology depend only on the fundamental groups was used only at the very first step of the proof of Theorem 9.1. The assumption that the covering consists of amenable subsets can be replaced by the assumption that the bounded cohomology of elements of the covering and of all their finite intersections vanish. This turns the Covering Theorem into a bounded cohomology version of Leray’s theorem about coverings. The author believes that this analogy with Leray’s theorem helps to put in the proper context and to understand the Vanishing Theorem, one of the main results of the “abstract” part of [Gro1].

**Remark.** In [I1] the author suggested that it may be interesting to look for a pure group-theoretic version of the Theorems 9.1 and 9.2, without realizing that a part of proof of Theorem 8.7 is such a version of a special case of Theorem 9.2 (corresponding to an amenable covering by two subsets). This idea still looks attractive, but in view of the work of R. Brown [Bro] it seems that an algebraic analogue of these theorems should deal with groupoids. Cf. [Bl1], [Bl2].

**Remark.** It may happen that the assumptions of Vanishing Theorem hold but $\hat{H}^i(X)$ for some $i \geq m$ is non-zero. For example, if $X$ is a wedge of two circles, then $\hat{H}^2(X) \neq 0$. See [Br] or [Gro1]. At the same time the obvious covering of $X$ by two circles is amenable.
Small simplices. At the end of the proof of Theorem 9.1 we used the fact that the homomorphism $k : H^*(X) \to H^*(\gamma^*(X))$ is an isomorphism. This is a corollary of the fact that the singular cohomology can be computed by taking only small simplices into account. See [Sp], Theorem 4.4.14 for a precise form of the latter claim. This is not the case for the bounded cohomology, as one can prove by using the ideas of the proof of Theorem 9.1.

Let us outline such a proof. Let $\mathcal{U}$ be an amenable open covering of $X$ and let $N$ be its nerve. As in the proof of Theorem 9.1, we may assume that all finite intersections of the elements of $\mathcal{U}$ have amenable fundamental groups. Let $B^q$ be the analogue for the bounded cochains of the presheaf $C^q$ and let $\beta^q$ be the sheaf associated with the presheaf $B^q$. Let

$$\beta^*(X)$$

be the bounded analogue of (9.4) and (9.5). The family $C^p(N, \beta^q)_{p,q \geq 0}$ is a double complex. Let $T_\beta$ be its total complex. The sheaf $\beta^q$ is soft if $X$ is paracompact and is flasque if $X$ is hereditary paracompact by the same reason as $\gamma^q$. It follows that the canonical map

$$j_\beta : H^*(\beta^*(X)) \to H^*(T_\beta^*)$$

is an isomorphism. Suppose now that the bounded cohomology can be computed by using only the small simplices. Then for every open $Y \subset X$ the canonical map

$$\hat{H}^*(Y) \to H^*(\beta^*(Y))$$

is an isomorphism. Therefore

$$H^i(\beta^*(|\sigma|)) = \hat{H}^i(|\sigma|) = 0$$

for every $i \geq 1$ and every simplex $\sigma$. It follows that all rows of the diagram (9.6), starting with the second one, are exact and hence the canonical map $H^*(|N|) \to H^*(T_\beta^*)$ is an isomorphism. Since the maps

$$H^i(\beta^*(X)) \to H^*(T_\beta^*) \quad \text{and} \quad \hat{H}^*(X) \to H^*(\beta^*(X))$$

are also isomorphisms, it follows that $H^*(|N|)$ is canonically isomorphic to $\hat{H}^*(X)$. Since the condition (C) is very weak, it is easy to see that this cannot be true. It is also not difficult to give an example of a covering $\mathcal{U}$ for which $H^*(|N|)$ is not isomorphic to $\hat{H}^*(X)$. In fact, the covering of the wedge of two circles by these two circles is such an example.
10. An algebraic analogue of the mapping theorem

The $G$-modules $F(G^{n+1}, U)$. Let $U$ be a $G$-module and $n \geq 0$. Let $F(G^{n+1}, U)$ be equal to $B(G^{n+1}, U)$ as a Banach space, but with the action $\cdot$ of $G$ defined by

$$(h \cdot c)(g_0, g_1, \ldots, g_n) = h \cdot (c(g_0 h, g_1 h, \ldots, g_n h)).$$

The map $s : G^{n+1} \longrightarrow G^{n+1}$ defined by

$$s(g_0, g_1, \ldots, g_n) = (g_n, g_{n-1} g_n, \ldots, g_0 g_1 \ldots g_n)$$

is an equivariant bijection from $G^{n+1}$ with the right action

$$(g_0, g_1, \ldots, g_n) \cdot h = (g_0, g_1, \ldots, g_n h)$$

of $G$ to $G^{n+1}$ with the right action

$$(g_0, g_1, \ldots, g_n) \cdot h = (g_0 h, g_1 h, \ldots, g_n h)$$

of $G$. It follows that the map

$$s^* : F(G^{n+1}, U) \longrightarrow B(G^{n+1}, U)$$

defined by $s^*(c) = c \circ s$ is an isomorphism of $G$-modules. Obviously, $s^*(c) = f$, where

$$f(g_0, g_1, \ldots, g_n) = c(g_n, g_{n-1} g_n, \ldots, g_0 g_1 \ldots g_n),$$

The homogeneous form of the standard resolution. Let us consider the sequence

$$0 \longrightarrow U \xrightarrow{d_{-1}} F(G, U) \xrightarrow{d_0} F(G^2, U) \xrightarrow{d_1} F(G^3, U) \xrightarrow{d_2} \cdots,$$

where $d_{-1}(v)(g) = v$ for all $v \in U$, $g \in G$, and

$$d_n(c)(g_0, g_1, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(g_0, \ldots, \hat{g}_i, \ldots, g_{n+1}),$$

for all $n \geq 0$ and $g_0, g_1, \ldots, g_{n+1} \in G$. A direct verification shows that the maps $s^*$ define an isomorphism of the above sequence with the standard resolution (7.6) of $U$. 

74
Given \( n \geq 0 \) and \( i = 1, 2, \ldots, n \), let

\[ \text{pr}_{i,G} : G^n \rightarrow G^{n-1} \]

be the projection of \( G^n \) onto the product of all factors of \( G^n = G \times G \times \ldots \times G \) except the \( i \)-th one. In terms of these projections the differential \( d_n \) takes the form

\[ d_n(c)(g) = \sum_{i=0}^{n+1} (-1)^i c \circ \text{pr}_{i,G}(g), \]

where \( n \geq 0 \) and \( g \in G^{n+2} \).

**Means of vector-valued functions.** Let \( U \) be a Banach space. Given a set \( S \), we denote by \( B(S,U) \) the vector space of all maps \( f : S \rightarrow U \) such that the function \( s \mapsto \| f(s) \| \) from \( S \) to \( \mathbb{R} \) is bounded. The vector space \( B(S,U) \) is a Banach space with the norm

\[ \| f \| = \sup_{x \in S} \| f(x) \|. \]

A *mean* on \( B(S,U) \) is defined as a linear map

\[ m : B(S,U) \rightarrow U \]

such that \( \| m \| \leq 1 \) and \( m(\text{const}_u) = u \) for all \( u \in U \), where \( \text{const}_u \) is the constant function with the value \( u \), i.e. \( \text{const}_u(s) = u \) for all \( s \in S \). For the trivial \( G \)-module \( U = \mathbb{R} \) this definition reduces to the definition of a mean on \( B(S) \).

Suppose that a group \( G \) acts on a set \( S \) from the right. Then \( G \) acts on the left on the space \( B(S,U) \) by the formula \( g \cdot f(s) = f(s \cdot g) \), where \( g \in G \), \( f \in B(S,U) \), \( s \in S \). The mean \( m \) on \( B(S,U) \) is called *right invariant* if \( m(g \cdot f) = m(f) \) for all \( g \in G \), \( f \in B(S,U) \). As usual, we call the right invariant means simply *invariant means*, and \( G \) is always considered together with its action on itself by the right translations.

Suppose now \( G \) acts on a set \( S \) on the left and this action is free and transitive. Then every \( s \in S \) defines a bijection \( r_s : G \rightarrow G \) by the rule \( g \mapsto g \cdot s \), where \( g \in G \). If \( s, t \in S \), then the bijections \( r_s \) and \( r_t \) differ by a right translation of \( G \). Cf. Section 5. If \( m \) is a mean on \( B(G,U) \) and \( s \in S \), then \( f \mapsto m(f \circ r_s) \) is a mean on \( B(S,U) \). By the same reason as in Section 5 the mean \( f \mapsto m(f \circ r_s) \) on \( B(S,U) \) does not depend on the choice of \( s \). This mean is said to be *induced by \( m \) and the action of \( G \) on \( S \)*. The *push-forwards* of invariant means by surjective homomorphisms are defined in exactly the same way as in Section 5.

The following construction goes back at least to B.E. Johnson [J]. See [J], Theorem 2.5 and its proof. The key idea is to work with the *dual* Banach spaces.
A construction of means of vector-valued functions. Let $A$ be a set and let $U = V^*$ be a Banach space dual to some other Banach space $V$. A function $f \in B(A, U)$ can be considered as a function $f : A \times V \to \mathbb{R}$ linear by the second argument and such that

$$|f(a, v)| \leq c \|v\|$$

for some $c \in \mathbb{R}$ and all $a \in A$. Then $\|f\|$ is equal to the infimum of $c$ such that this inequality holds. In particular, if $a \in A$ and $v \in V$, then

$$|f(a, v)| \leq \|f\| \cdot \|v\| \quad (10.1)$$

Suppose that $M : B(A) \to \mathbb{R}$ is a mean. Given a vector $v \in V$, let us define a function $f_v : A \to \mathbb{R}$ by the rule $f_v(a) = f(a, v)$. By (10.1) the function $f_v$ is bounded and, moreover, $\|f_v\| \leq \|f\| \cdot \|v\|$. Let us define a map $m(f) : V \to \mathbb{R}$ by

$$m(f)(v) = M(f_v).$$

Since $f(a, \bullet)$ is a linear functional, $f_v$ linearly depends on $v$. Since $M$ is a linear functional, this implies that $m(f)$ is a linear functional on $V$. It is bounded of the norm $\leq \|f\|$ because

$$\|m(f)(v)\| \leq \|f_v\| \leq \|f\| \cdot \|v\|.$$

It follows that $m(f) \in V^* = U$ and $\|m(f)\| \leq \|f\|$. 

10.1. Lemma. The map $m : f \to m(f)$ is a mean $B(A, U) \to U$. If $L : V \to V$ is a bounded operator and $L^* : V^* = U \to U$ is its adjoint operator, then

$$m(L^* \circ f) = L^*(m(f)).$$

If $A$ is a group and $M$ is an invariant mean, then $m$ is an invariant mean also.

Proof. Since $f_v$ linearly depends on $f$ and $M$ is linear, $m$ is linear. By the inequality $\|m(f)\| \leq \|f\|$ its norm $\leq 1$. Suppose that $f(a) = u \in U = V^*$ for all $a \in A$. Then $f_v(a) = u(v)$ for all $a \in A$ and $v \in V$, i.e. all $f_v$ are constant functions. It follows that $m(f)(v) = u(v)$ for all $v \in V$ and hence $m(f) = u$. Therefore $m$ is indeed a mean. If $a \in A$ and $v \in V$, then $L^* \circ f(a, v) = f(a, L(v))$. Therefore $(L^* \circ f)(v) = f_{L(v)}$ and

$$m(L^* \circ f)(v) = M((L^* \circ f)(v)) = M(f_{L(v)})$$

$$= m(f)(L(v)) = L^*(m(f))(v)$$

It follows that $m(L^* \circ f) = L^*(m(f))$. The last statement of the lemma is obvious. ■
Dual modules. Suppose that \( A \) is a group and \( V \) is a Banach space and a right \( A \)-module. Then \( U = V^* \) has a canonical structure of a left \( A \)-module. Namely, if \( u : V \to \mathbb{R} \) is a bounded linear functional and \( a \in A \), then \( a \cdot u \) is the linear functional \( v \mapsto u(v \cdot a) \).

For every \( a \in A \) let \( L(a) : V \to V \) be the operator \( v \mapsto v \cdot a \) of the right multiplication by \( a \) in \( V \). Then the adjoint operator \( L(a)^* : U \to U \) is the operator \( u \mapsto a \cdot u \) of the left multiplication by \( a \) in \( U \).

10.2. Lemma. In the situation of Lemma 10.1, suppose that \( U \) is a right \( A \)-module. Then \( m : B(A, U) \to U \) is a morphism of \( A \)-modules.

Proof. If \( a \in A \) and \( f \in B(A, U) \), then by definition \( a \cdot f = L(a)^* \circ f \circ r_a \), where \( r_a : A \to A \) is the right shift \( b \mapsto ba \) by \( a \) in \( A \). By Lemma 10.1 \( m \) is an invariant mean and hence \( m(f \circ r_a) = m(f) \). Moreover, Lemma 10.1 implies that

\[
m(a \cdot f) = m(L(a)^* \circ f \circ r_a) = L(a)^*(m(f \circ r_a)) = L(a)^*(m(f)) = a \cdot m(f).
\]

It follows that \( m \) is indeed a morphism of \( A \)-modules. \( \square \)

Quotients by normal subgroups. Let \( \Gamma \) be a discrete group, \( A \) be a normal subgroup of \( \Gamma \), and \( U \) be a \( \Gamma \)-module. Let \( G = \Gamma/A \) and \( \alpha : \Gamma \to G \) be the quotient map. Then \( \alpha \) turns \( B(G, U) \) into a \( \Gamma \)-module, which can be described as follows. Let us consider \( G \) as the group of the right cosets. Then \( \Gamma \) acts on \( G \) from the right by the rule \( g \cdot h = gh \), where \( h \in \Gamma \) and \( g = A \gamma \) for some \( \gamma \in \Gamma \) is a right coset, and acts on \( B(G, U) \) from the left by the rule

\[
h \cdot f(A \gamma) = h \cdot (f(A \gamma h))
\]

The obvious map \( \alpha^* : B(G, U) \to B(\Gamma, U) \) induced by \( \alpha \) is a \( \Gamma \)-morphism.

Quotients by normal amenable subgroups. Let \( \Gamma \) be a group and \( V \) be a Banach space and a right \( \Gamma \)-module. Then \( U = V^* \) is a left \( \Gamma \)-module. Suppose that \( A \) is a normal amenable subgroup of \( \Gamma \). Then \( U \) is also an \( A \)-module in a canonical way. Let \( M : B(A) \to \mathbb{R} \) be an invariant mean and let \( m : B(A, U) \to \mathbb{R} \) be the defined by \( M \) invariant mean on \( B(A, U) \).

As before, let \( G = \Gamma/A \) and \( \alpha : \Gamma \to G \) be the quotient map. The group \( A \) acts freely and transitively from the left on each right coset \( A \gamma \), where \( \gamma \in \Gamma \). Given \( \gamma \in \Gamma \), let the mean

\[
m_\gamma : B(A \gamma, U) \to U
\]

be defined by \( m_\gamma(f) = m(f \circ r_\gamma) \), where \( r_\gamma : A \to A \gamma \) is the map \( a \mapsto a \gamma \). The mean
$m_\gamma$ is nothing else but the mean on $B(A\gamma, U)$ induced by $m$ and the left action of $A$ on $A\gamma$, which is free and transitive. In particular, $m_\gamma$ depends only on the coset $A\gamma$, but not on the choice of a representative $\gamma$ of this coset. Cf. the beginning of Section 5. Let

$$m_* : B(\Gamma, U) \rightarrow B(G, U)$$

be the map defined by the formula

$$m_*(f)(A\gamma) = m_\gamma(f|A\gamma),$$

where $f|A\gamma$ is the restriction of $f$ to $A\gamma$. It is well defined because $m_\gamma$ depends only on $A\gamma$.

**10.3. Lemma.** The map $m_*$ is a $\Gamma$-morphism, $m_* \circ \alpha^* = id$, and $\|m_*\| \leq 1$.

**Proof.** Let us prove first that $m_*$ commutes with the $\Gamma$-actions. Let $h \in \Gamma$ and $f \in B(\Gamma, U)$. Then by the definition of the $\Gamma$-action

$$h \cdot f = L(h)^* \circ f \circ r_h,$$

where $r_h : \Gamma \rightarrow \Gamma$ is the right shift $\delta \mapsto \delta h$ by $h$ in $\Gamma$.

Let us consider an arbitrary $\gamma \in \Gamma$ and prove that the values of $m_*(h \cdot f)$ and $h \cdot m_*(f)$ on the coset $A\gamma$ are equal. The value of $m_*(h \cdot f)$ is

$$m_*(h \cdot f)(A\gamma) = m_*(L(h)^* \circ f \circ r_h)(A\gamma)$$

$$= m_\gamma(L(h)^* \circ f \circ r_h|A\gamma)$$

$$= m(L(h)^* \circ \varphi),$$

where $\varphi : A \rightarrow U$ is the map $a \mapsto f \circ r_h(a\gamma) = f(a\gamma h)$, and therefore

$$m_*(h \cdot f)(A\gamma) = L(h)^*(m(\varphi))$$

in view of Lemma 10.1. On the other hand

$$h \cdot m_*(f)(A\gamma) = L(h)^*(m_*(f)(A\gamma h))$$

$$= L(h)^*(m_\gamma h(f|A\gamma h))$$

$$= L(h)^*(m(\psi)),$$
where $\psi : A \rightarrow U$ is the map $a \mapsto f \circ r_{A'}(a) = f(a \gamma h)$, and therefore

$$h \cdot m_*(f)(A \gamma) = L(h)^* (m(\psi)).$$

Since, obviously, $\varphi = \psi$, it follows that

$$m_*(h \cdot f)(A \gamma) = h \cdot m_*(f)(A \gamma)$$

for all $\gamma \in \Gamma$ and hence $m_*(h \cdot f) = h \cdot m_*(f)$. This proves that $m_*$ is a $\Gamma$-morphism. The last two claims of the lemma immediately hold because all $m_\gamma$ are means. \[\square\]

**Invariant means on $B(A^n)$**. We need an analogue of the coherent systems of invariant means from Section 5. Let $A$ be an amenable group. Then $A^n$ are amenable for all $n$ by [Gr], Theorem 1.2.6. Recall that

$$\mathbf{pr}_{i,A} : A^n \rightarrow A^{n-1}$$

is the projection onto the product of all factors of $A^n$ except the $i$-th one. A sequence

$$M_0, M_1, \ldots, M_n, \ldots$$

of invariant means $M_n \in B(A^n)^*$, where $n = 0, 1, 2, \ldots$, is called **coherent** if

$$\left(\mathbf{pr}_{i,A}\right)_* M_n = M_{n-1}$$

for every $n, i$ as above. Equivalently, $M_n(f \circ \mathbf{pr}_{i,A}) = M_{n-1}(f)$ for every $n, i$ as above.

**$\Sigma_n$-invariant means on $B(A^n)$**. The symmetric group $\Sigma_n$ acts on $A^n$ by permuting the factors. This action induces actions of $\Sigma_n$ on $B(A^n)^*$ and on the set of invariant means on $B(A^n)$. We will say that an invariant mean $M \in B(A^n)^*$ is a $\Sigma_n$-invariant mean if $M$ is fixed by the action of $\Sigma_n$. If $M \in B(A^n)^*$ is an invariant mean, then

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} M^\sigma$$

is a $\Sigma_n$-invariant mean, where the action of $\Sigma_n$ is written as $(M, \sigma) \mapsto M^\sigma$. Suppose that $M_n$ is a $\Sigma_n$-invariant mean on $B(A^n)$ and that $1 \leq i \leq n$. Let

$$M_{n-1} = \left(\mathbf{pr}_{i,A}\right)_* M_n.$$

Since $M_n$ is $\Sigma_n$-invariant, $M_{n-1}$ is independent of the choice of $i$ and is $\Sigma_{n-1}$-invariant. Since $M_{n-1}$ is independent of the choice of $i$, the condition (10.2) holds for every $i$. 

79
10.4. **Theorem.** There exists a coherent sequence of invariant means $M_0, M_1, M_2, \ldots$.

**Proof.** The proof is completely similar to the proof of Theorem 5.7. ■

**Invariant means on** $B(A^n, U)$. Let $U$ be a Banach space. A sequence

$$m_0, m_1, \ldots, m_n, \ldots$$

of invariant means $m_n : B(A^n, U) \to U$, where $n = 0, 1, 2, \ldots$, is called coherent if

$$\left( \text{pr}_{i, A} \right)_* m_n = m_{n-1}$$

for every $n, i$ as above. Equivalently, $m_n (f \circ \text{pr}_{i, A}) = m_{n-1} (f)$ for every $n, i$ as above.

10.5. **Theorem.** If $U = V^*$ is the dual to a Banach space $V$, then there exists an infinite coherent sequence $m_0, m_1, m_2, \ldots$, where $m_n$ is an invariant mean $B(A^n, U) \to U$.

**Proof.** The construction of an invariant mean $m$ on $B(A^n, U)$ by an invariant mean $M$ on $B(A^n)$ commutes with push-forwards. Hence the theorem follows from Theorem 10.4. ■

10.6. **Theorem.** Suppose that $A$ is a normal amenable subgroup of a group $\Gamma$ and that $U$ is a left $\Gamma$-module dual to a right $\Gamma$-module. Let $G = \Gamma/A$ and let $\alpha : \Gamma \to G$ be the quotient map. Then there exists a $\Gamma$-morphism

$$\alpha_* : F(\Gamma^{n+1}, U) \to F(G^{n+1}, U)$$

of $\Gamma$-resolutions of the $\Gamma$-module $U$ such that $\alpha_* \circ \alpha^* = \text{id}$ and $\| \alpha_* \| \leq 1$.

**Proof.** The proof follows the lines of the proof of Theorem 5.4.

Let $\{m_n\}$ be a coherent sequence of invariant means $B(A^n, U) \to U$. For every $n$ the group $A^n$ is a normal subgroup of $\Gamma^n$ and $\alpha$ induces an isomorphism

$$\alpha^n : \Gamma^n/A^n \to G^n.$$ 

By applying Lemma 10.3 to $\Gamma^n, A^n, m_n$ in the roles of $\Gamma, A, m$ respectively, we get maps

$$m_{n*} : B(\Gamma^n, U) \to B(G^n, U)$$

such that $m_{n*} \circ \alpha^{n*} = \text{id}$, $\| m_{n*} \| \leq 1$, and each $m_{n*}$ is a $\Gamma^n$-morphism.
The map $m_n^*$ can be also considered as a map

$$m_n^* : F(\Gamma^n, U) \to F(G^n, U)$$

with the same properties. The structure of $\Gamma$-modules on vector spaces $F(\Gamma^n, U), F(G^n, U)$ is induced from the structure of $\Gamma^n$-modules by the diagonal homomorphism

$$h \mapsto (h, h, \ldots, h) \in \Gamma^n,$$

where $h \in \Gamma$. It follows that the maps $m_n^*$ are morphisms of $\Gamma$-modules. Let us prove that the maps $m_n^*$ commute with the differentials of the homogeneous standard resolutions.

As the first step, let us prove that

$$m_{n+1}^*(c \circ pr_{i, \Gamma}) = m_n^*(c) \circ pr_{i, G}. \tag{10.4}$$

for every $c \in F(\Gamma^n, U)$ and $0 \leq i \leq n$. Let us evaluate both sides of (10.4) on an arbitrary $g \in G^{n+1} = \Gamma^{n+1}/\Lambda^{n+1}$.

Let $\gamma \in \Gamma^{n+1}$ be some representative of the coset $g$, and let

$$g(i) = pr_{i, G}(g) \quad \text{and} \quad \gamma(i) = pr_{i, \Gamma}(\gamma).$$

Then $\gamma(i)$ is a representative of the coset $g(i)$. By the definition of push-forwards the left hand side of (10.4) is equal to $m_{n+1}(\varphi)$, where $\varphi : \Lambda^{n+1} \to U$ is the map

$$\varphi(a) = c \circ pr_{i, \Gamma}(a\gamma) = c(pr_{i, \Lambda}(a)\gamma(i)). \tag{10.5}$$

The right hand side is equal to $m_n(\psi)$, where $\psi : \Lambda^n \to U$ is the map

$$\psi(b) = c(b\gamma(i)). \tag{10.6}$$

By comparing (10.5) and (10.6) we see that $\varphi = \psi \circ pr_{i, \Lambda}$ and hence

$$m_{n+1}(\varphi) = \left((pr_{i, \Lambda})_* m_n\right)(\psi).$$

In view of the coherence condition (10.3) this implies that the values of both sides of (10.4) on $g$ are equal. Since $g$ is arbitrary, this proves (10.4).
Let us prove that \( m_{n+1} \circ d_{n-1} = d_{n-1} \circ m_n \). Let \( c \in F(\Gamma^n, U) \) and \( g \in G^{n+1} \). Then

\[
m_{n+1}(d_{n-1}(c))(g) = m_{n+1} \left( \sum_{i=0}^{n} (-1)^i c \circ \text{pr}_{i, G} \right)(g)
\]

\[
= \sum_{i=0}^{n} (-1)^i m_{n+1}(c \circ \text{pr}_{i, G})(g)
\]

\[
= \sum_{i=0}^{n} (-1)^i m_n(c)(g(i))
\]

\[
= \sum_{i=0}^{n} (-1)^i m_n(c) \circ \text{pr}_{i, G}(g) = d_n(m_n(c))(g).
\]

Therefore \( m_{n+1} \circ d_{n-1} = d_{n-1} \circ m_n \). Let \( \alpha_* \) be equal to \( m_n \) on \( F(\Gamma^n, U) \). By the above results, \( \alpha_* \) is a morphism of resolutions with the required properties. \( \blacksquare \)

### 10.7. Theorem

In the situation of Theorem 10.6 there is a canonical isometric isomorphism

\[
\hat{H}^*(G, U^A) \longrightarrow \hat{H}^*(\Gamma, U).
\]

**Proof.** Let \( \alpha^* : F(G^{*+1}, U) \longrightarrow F(\Gamma^{*+1}, U) \) be the morphism induced by \( \alpha \) and let \( \alpha_* \) be the morphism from Theorem 10.6. Both of them are \( \Gamma \)-morphisms and hence induce maps

\[
(10.7) \quad \alpha^* : F(G^{*+1}, U)^\Gamma \rightleftharpoons F(\Gamma^{*+1}, U)^\Gamma : \alpha_*.
\]

Together with morphisms of resolutions, the latter maps have the norm \( \leq 1 \). By the definition, the cohomology of the complex \( F(\Gamma^{*+1}, U)^\Gamma \) are the bounded cohomology \( \hat{H}^*(\Gamma, U) \).

In order to identify the cohomology of the complex \( F(G^{*+1}, U)^\Gamma \), let us consider the complex \( F(G^{*+1}, U)^A \) of \( A \)-invariants of the complex \( F(G^{*+1}, U) \). Since the subgroup \( A \subset \Gamma \) acts trivially on \( G = \Gamma/A \), for every \( n \geq 0 \) the subspace \( F(G^n, U)^A \) is equal to \( F(G^n, U^A) \). Obviously, \( U^A \) is a \( G \)-module and hence \( F(G^n, U^A) \) is also a \( G \)-module. Moreover,

\[
F(G^{*+1}, U)^\Gamma = \left( F(G^{*+1}, U)^A \right)^G = F(G^{*+1}, U^A)^G.
\]

Therefore \( F(G^{*+1}, U)^\Gamma \) is equal to the complex of the \( G \)-invariants of \( F(G^{*+1}, U^A) \) of the \( G \)-module \( U^A \) and the morphisms (10.7) can be interpreted as morphisms

\[
(10.8) \quad \alpha^* : F(G^{*+1}, U^A)^G \rightleftharpoons F(\Gamma^{*+1}, U)^\Gamma : \alpha_*.
\]

82
Theorem 10.6 implies that the induced map in cohomology
\[
α_* \circ α^* : \hat{H}^*(G, U^A) \rightarrow \hat{H}^*(G, U^A)
\]
is equal to the identity. In order to prove that the induced map in cohomology
\[
α^* \circ α_* : \hat{H}^*(\Gamma, U) \rightarrow \hat{H}^*(\Gamma, U)
\]
is equal to the identity, it is sufficient to prove that the \(\Gamma\)-morphism
\[
α^* \circ α_* : F(\Gamma^{•+1}, U) \rightarrow F(\Gamma^{•+1}, U)
\]
is chain homotopic to the identity. But the \(\Gamma\)-resolution \(F(\Gamma^{•+1}, U)\) of \(U\) is strong and relatively injective and \(α^* \circ α_*\) extends \(\text{id}_U\). Therefore Lemma 7.4 implies that \(α^* \circ α_*\) is chain homotopic to the identity morphism of \(F(\Gamma^{•+1}, U)\). It follows that the maps (10.8) are mutually inverse. Since both of them have the norm \(\leq 1\), they are isometric isomorphisms. ■

**Relative injectivity of \(\Gamma\)-modules** \(B(G^n, U)\). For the proof of Theorem 10.7 there is no need to know if \(B(G^n, U)\) is relatively injective as a \(\Gamma\)-module, in contrast with the \(\Gamma\)-modules \(B(\Gamma^n, U)\). The relative injectivity of latter is needed for applying Lemma 7.4. Still, \(B(G^n, U)\) is relatively injective as a \(\Gamma\)-module. This follows from Theorem 10.6 and the next lemma.

**10.8. Lemma.** Let \(V\) be a relatively injective \(\Gamma\)-module and \(W\) be a retract of \(V\) in the sense that there exist \(\Gamma\)-morphisms \(i: W \rightarrow V\) and \(p: V \rightarrow W\) such that \(p \circ i = \text{id}_W\) and \(\|i\|, \|p\| \leq 1\). Then \(W\) is also relatively injective.

**Proof.** Let us consider the diagram

\[
\begin{array}{c}
V_1 \\
\alpha \downarrow \beta \downarrow \gamma \\
W \\
\quad i \\
\quad p \quad V,
\end{array}
\]

where \(k: V_1 \rightarrow V_2\) is a given strongly injective \(\Gamma\)-morphism, \(\alpha\) is a given \(\Gamma\)-morphism, and we need to find a \(\Gamma\)-morphism \(\beta\) such that \(\beta \circ k = \alpha\) and \(\|\beta\| \leq \|\alpha\|\). Since \(V\) is relatively injective, there exists a \(\Gamma\)-morphism \(\gamma\) such that \(\gamma \circ k = i \circ \alpha\) and

\[
\|\gamma\| \leq \|i \circ \alpha\| \leq \|i\| \|\alpha\| \leq \|\alpha\|.
\]

Let \(\beta = p \circ \gamma\). Then \(\beta \circ k = p \circ i \circ \alpha = \alpha\) and \(\|\beta\| \leq \|p\| \|\gamma\| \leq \|\alpha\|\). ■
A.1. The differential of the standard resolution

**Semi-simplicial sets.** A *semi-simplicial set* consists of sets $S_0, S_1, S_2, \ldots$ and the maps

$$\partial_i : S_n \to S_{n-1} \quad \text{and} \quad \delta_i : S_n \to S_{n+1}$$

defined for $0 \leq i \leq n$ and called the *face* and *degeneracy operators* respectively. They should satisfy some well known relations which we will not use explicitly. Neither will we use the degeneracy operators. The elements of $S_n$ are called *$n$-simplices*.

**Nerve of a category.** The *nerve* $\mathcal{N} \mathcal{C}$ of a category $\mathcal{C}$ is a semi-simplicial set having as its $n$-simplices the sequences of objects and morphisms of $\mathcal{C}$ of the form

$$O_0 \leftarrow a_0 \ O_1 \leftarrow a_1 \ O_2 \leftarrow a_2 \ \ldots \leftarrow a_{n-2} \ O_{n-1} \leftarrow a_{n-1} \ O_n .$$

The $i$-th face of such a simplex are obtained by removing the object $O_{n-i}$. The objects $O_0$ and $O_n$ are removed together with the arrow respectively ending or starting at the removed object. If an object $O_i$ with $0 < i < n$ is removed, then two morphisms

$$O_{i-1} \leftarrow a_{i-1} \ O_i \leftarrow a_i \ O_{i+1} \quad \text{are replaced by} \quad O_{i-1} \leftarrow a_{i-1} \circ a_i \ O_{i+1} .$$

The operator $\delta_i$ acts by replacing the object $O_{n-i}$ by the identity morphism $O_{n-i} \to O_{n-i}$.

**A category associated to a group.** A group $G$ gives rise to a category $\mathcal{G}$ having the elements of $G$ as objects and exactly one morphism between any two objects. For every $k, g \in G$ there is a morphism $kg \leftarrow g$ denoted also by $k$ by an abuse of notations. The composition $g_1 \circ g_2$, when defined, is the morphism $g_1 g_2$. The $n$-simplices of $\mathcal{N} \mathcal{G}$ are diagrams of the form

$$\bullet \leftarrow g_0 \ \bullet \leftarrow g_1 \ \bullet \leftarrow g_2 \ \ldots \leftarrow \bullet \leftarrow g_{n-2} \ g_{n-1}g_n \leftarrow g_{n-1} .$$

The bullets $\bullet$ on this diagram stand for the objects $O_i = g_ig_{i+1} \ldots g_n$. The $n$-simplices of $\mathcal{N} \mathcal{G}$ are in one-to-one correspondence with the sequences $(g_0, g_1, g_2, \ldots, g_n) \in G^{n+1}$, and we will identify them with such sequences. Then the face operators take the form

$$\partial_0(g_0, g_1, g_2, \ldots, g_n) = (g_1, g_2, \ldots, g_{n-2}, g_{n-1}g_n),$$

$$\partial_i(g_0, g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_{n-i-1}g_{n-i}, \ldots, g_n) \quad \text{for} \quad 0 < i < n,$$

$$\partial_n(g_0, g_1, g_2, \ldots, g_n) = (g_1, g_2, \ldots, g_n).$$
The group $G$ acts on $G$ from the right. The action of $h \in G$ takes an object $g$ to the object $gh$ and a morphism $kg \leftarrow g$ to the morphism $kgh \leftarrow gh$. i.e. it takes a morphism

$$\bullet \leftarrow k \bullet \quad \text{to a morphism of the same form} \quad \bullet \leftarrow k \bullet.$$ 

This action induces the right action of $G$ on the nerve $N\mathcal{G}$. The action of $h \in G$ on the $n$-simplex $(g_0, g_1, g_2, \ldots, g_n)$ takes it to $(g_0, g_1, g_2, \ldots, g_n h)$.

**The nerve $N\mathcal{G}$ and the standard resolution.** Let $B(N\mathcal{G}_n, \mathbb{R})$ be the space of bounded functions on the set $N\mathcal{G}_n$. The right action of $G$ on $N\mathcal{G}$ induces a right action of $G$ on $N\mathcal{G}$ and hence a left action on $B(N\mathcal{G}_n, \mathbb{R})$, turning $B(N\mathcal{G}_n, \mathbb{R})$ into a $G$-module. Let

$$\partial^n : B(N\mathcal{G}_n, \mathbb{R}) \longrightarrow B(N\mathcal{G}_{n+1}, \mathbb{R})$$

be the restriction to the bounded cochains of the standard coboundary map, i.e. let

$$\partial^n(f)(\sigma) = \sum_{i=0}^{n+1} (-1)^i f(\partial_i \sigma).$$

for every $f \in B(N\mathcal{G}_n, \mathbb{R})$. The identification of $N\mathcal{G}_n$ with $G_{n+1}$ leads to an identification of $B(N\mathcal{G}_n, \mathbb{R})$ and $B(G_{n+1}, \mathbb{R})$ as left $G$-modules. Comparing the formulas for $\partial^n$ and $d_n$ shows that this identification turns $\partial^n$ into the differential $d_n$. This interpretation of the standard resolution can be routinely extended to the case of non-trivial coefficients.

**A proof that the standard resolution $B(G^{n+1}, \mathbb{R})$ is a complex.** It is sufficient to show that

$$0 \longrightarrow \mathbb{R} \xrightarrow{\partial_{-1}} B(N\mathcal{G}_0, \mathbb{R}) \xrightarrow{\partial_0} B(N\mathcal{G}_1, \mathbb{R}) \xrightarrow{\partial_1} \cdots,$$

where $\partial_{-1}(a)(\sigma) = a$ for every $a \in \mathbb{R}, \sigma \in N\mathcal{G}_0$, is a complex. Removing two objects from an $(n+2)$-simplex $\sigma \in N\mathcal{G}_{n+2}$ leads to a simplex $\tau \in N\mathcal{G}_n$ which does not depend on the order in which these two objects were removed. Given $f \in B(N\mathcal{G}_n, \mathbb{R})$, the value $f(\tau)$ enters the tautological double sum expressing

$$\partial^{n+1}(\partial^n(f)(\sigma))$$

twice, corresponding to the two orders of removing two objects. But removing the first object changes the number (the subscript) of the second one by 1 if the first is to the right of the second on the above diagram, and does not change the number if on the left. It follows that $f(\tau)$ enters the double sum twice, but with different signs. Therefore all terms of this double sum cancel, and hence $\partial^{n+1} \circ \partial^n = 0$ for all $n \geq 0$. The equality $\partial_0 \circ \partial_{-1} = 0$ is immediate. This proof can be routinely generalized to the case of non-trivial coefficients.
A.2. The complex $B(G^{n+1}, U)^G$

The invariant subspaces. The subspaces $B(G^{n+1}, U)^G$ admit an explicit description. Let $f \in B(G^{n+1}, U)$. Then $f$ is invariant with respect to the action of $G$ if and only if

$$h \cdot (f(g_1, \ldots, g_n, g_{n+1} h)) = f(g_1, \ldots, g_n, g_{n+1})$$

for all $g_1, \ldots, g_n, g_{n+1}, h \in G$. By taking $h = g_{n+1}^{-1}$, we see that

$$f(g_1, \ldots, g_n, g_{n+1}) = g_{n+1}^{-1} \cdot (f(g_1, \ldots, g_n, 1))$$

if $f$ is $G$-invariant. It follows that the map

$$i^n : B(G^{n+1}, U)^G \rightarrow B(G^n, U)$$

defined by the formula $i^n(f)(g_1, \ldots, g_n) = f(g_1, \ldots, g_n, 1)$ is a canonical isomorphism.

The complex of invariant subspaces. By using isomorphisms $i^n$ we see that the bounded cohomology $\hat{H}^\ast(G, U)$ is equal to the cohomology of the complex

$$0 \rightarrow U \xrightarrow{\delta_0} B(G, U) \xrightarrow{\delta_1} B(G^2, U) \xrightarrow{\delta_2} \cdots,$$

where $\delta_0(v)(g) = v$ for all $v \in U$, $g \in G$, and

$$\delta_n(f)(g_0, g_1, \ldots, g_n) = (-1)^{n+1} f(g_1, g_2, \ldots, g_n)$$

$$+ \sum_{i=0}^{n-1} (-1)^{n-i} f(g_0, \ldots, g_i g_{i+1}, \ldots, g_n)$$

$$+ g_{n+1}^{-1} \cdot (f(g_0, \ldots, g_{n-1}))$$

for $n \geq 1$ and $g_0, g_1, \ldots, g_n \in G$. For the first differential $\delta_1$ this formula takes the form

$$\delta_1(f)(g_0, g_1) = f(g_1) - f(g_0 g_1) + g_1^{-1} \cdot (f(g_0)).$$

If $U$ is a trivial $G$-module, then the formula for $\delta_1$ takes even simpler form

$$\delta_1(f)(g_0, g_1) = f(g_0) + f(g_1) - f(g_0 g_1).$$
A.3. The second bounded cohomology group

**Pseudo-limits.** Let $N$ the set of all positive integers. The space $B(N)$ can be considered as the space of bounded sequences of real numbers. Let $L(N)$ be the subspace of sequences $f$ such that there exists limit of $f(n)$ for $n \to \infty$. A *pseudo-limit* is defined as a bounded linear functional $l: B(N) \to \mathbb{R}$ such that

$$l(f) = \lim_{n \to \infty} f(n)$$

for all $f \in L(N)$ and $\|l\| \leq 1$. Let us prove that pseudo-limits exist.

Let us define $l(f)$ for $f \in L(N)$ as the limit of $f$. Then $l$ is a linear functional $L(N) \to \mathbb{R}$ such that $|l(f)| \leq \|f\|$ for all $f \in L(N)$. By applying Hahn–Banach theorem to $l$ and the norm $\| \cdot \|$ on $B(N)$ we see that $l$ can be extended to a linear functional on $B(N)$ in such a way that $|l(f)| \leq \|f\|$ for all $f \in B(N)$. Obviously, such an extension is a pseudo-limit.

Let $s: N \to N$ be the map $n \mapsto n + 1$. One can prove that there exist pseudo-limits $l$ such that $l(f) = l(f \circ s)$ for all $f \in B(N)$. Such a pseudo-limit is called a *Banach limit*. See, for example, [R], Exercise 4 to Chapter 3. But for our purposes pseudo-limits are sufficient.

A.3.1. **Theorem (see [MM], [I2]).** For every group $G$ the semi-norm on $\hat{H}^2(G)$ is a norm.

**Proof.** We will use the description of the complex $B(G^{*+1}, \mathbb{R})^G$ from Appendix 2. It is sufficient to prove that $\text{Im} \, \delta_1$ is closed. In order to do this it is sufficient, in turn, to find a bounded left inverse to $\delta_1$, i.e. a bounded operator $P$ such that $P \circ \delta_1 = \text{id}$.

Indeed, suppose that $P \circ \delta_1 = \text{id}$ and let $Q = \text{id} - \delta_1 \circ P$. Then

$$Q \circ \delta_1 = (\text{id} - \delta_1 \circ P) \circ \delta_1 = \delta_1 - \delta_1 \circ (P \circ \delta_1) = \delta_1 - \delta_1 = 0$$

and hence $\text{Im} \, \delta_1 \subset \text{Ker} \, Q$. On the other hand, if $x \in \text{Ker} \, Q$, then $x - \delta_1 \circ P(x) = 0$ and hence $x = \delta_1(P(x))$ belongs to the image of $\delta_1$. Therefore $\text{Ker} \, Q \subset \text{Im} \, \delta_1$. By combining this with the already proved opposite inclusion, we see that $\text{Im} \, \delta_1 = \text{Ker} \, Q$. Since $Q$ is a bounded operator together with $\delta_1$ and $P$, its kernel is closed and hence $\text{Im} \, \delta_1$ is closed.

It remains to construct a bounded left inverse $P$. Such an inverse should, in particular, recover $f \in B(G)$ if $\delta_1(f)$ is known. The kernel of $\delta_1$ consists of the bounded homomorphisms $G \to \mathbb{R}$, as it follows from the explicit formula for $\delta_1$. Since every bounded homomorphism $G \to \mathbb{R}$ is obviously equal to zero, $\delta_1$ is injective. Therefore $f \in B(G)$ is indeed determined by $\delta_1(f)$. Let us try to find an explicit way to recover $f$ from $\delta_1(f)$. 87
Let $f \in B(G)$ and let $A = \delta_1(f)$. Then

$$A(g, g^{i-1}) = f(g) + f(g^{i-1}) - f(g^i)$$

for every $g \in G$ and $i \geq 0$. Let us consider an arbitrary element $g \in G$. If $m \geq 1$, then

$$f(g^m) = f(g) + f(g^{m-1}) - A(g, g^{m-1})$$

$$= f(g) + f(g) + f(g^{m-2}) - A(g, g^{m-1}) - A(g, g^{m-2})$$

$$\ldots$$

$$= mf(g) - \sum_{i=1}^{m-1} A(g, g^i)$$

and hence

$$f(g) = \frac{1}{m} f(g^m) + \frac{1}{m} \sum_{i=1}^{m-1} A(g, g^i).$$

Since $f$ is bounded,

$$\lim_{m \to \infty} \frac{1}{m} f(g^m) = 0.$$

It follows that

$$f(g) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m-1} A(g, g^i).$$

The last formula recovers $f$ by $A = \delta_1(f)$. In order to construct $P$, it is sufficient to replace the limit in this formula by a pseudo-limit $l$. Indeed, if $A \in B(G^2)$, then

$$\left| \frac{1}{m} \sum_{i=1}^{m-1} A(g, g^i) \right| \leq \|A\|$$

and hence we can define $P(A)$ by the formula

$$P(A)(g) = l\left( \sum_{i=1}^{m-1} \frac{1}{m} A(g, g^i) \right).$$

Obviously, $\|P(A)\| \leq \|A\|$ and hence $P$ is a bounded operator. In fact, $\|P\| \leq 1$. Since $l$ is equal to the limit on $L(N)$, the above calculations show that $P$ is a left inverse to $\delta_1$. ■
A.4. Functoriality with coefficients

**Change of groups.** Let $\alpha : \Gamma \rightarrow G$ be a homomorphism. Then any $G$-module $V$ can be turned into a $\Gamma$-module by defining the action of $\Gamma$ by the rule $(\gamma, v) \mapsto \alpha(\gamma) \cdot v$. This $\Gamma$-module is denoted by $\alpha V$ and said to be the result of the *change of groups* by $\alpha$. Obviously,

$$V^G \subset \alpha V^\Gamma.$$ 

Suppose now that $U$ is a $\Gamma$-module and $V$ is a $G$-module. A linear map $u : V \rightarrow U$ is said to be an $\alpha$-*morphism* if it is a $\Gamma$-morphism of $\Gamma$-modules $\alpha V \rightarrow U$.

**The induced maps.** Let $u : V \rightarrow U$ be an $\alpha$-morphism. It leads to the canonical maps

$$(\alpha, u)^* : B(G^{n+1}, V) \rightarrow B(\Gamma^{n+1}, U)$$

defined by the formula

$$(\alpha, u)^*(f)(\gamma_0, \gamma_1, \ldots, \gamma_n) = u\left(f(\alpha(\gamma_0), \alpha(\gamma_1), \ldots, \alpha(\gamma_n))\right).$$

A trivial check shows that the maps $(\alpha, u)^*$ are $\alpha$-morphisms and commute with the differentials of the standard resolutions. Therefore the maps $(\alpha, u)^*$ define a $\Gamma$-morphism

$$(\alpha, u)^* : \alpha B(G^{*+1}, V) \rightarrow B(\Gamma^{*+1}, U)$$

of $\Gamma$-resolutions. Since

$$B(G^n, V)^G \subset _\alpha B(G^n, V)^\Gamma,$$

for every $n \geq 0$, this $\Gamma$-morphism leads to a morphism of complexes

$$B(G^{*+1}, V)^G \rightarrow _\alpha B(\Gamma^{*+1}, V)^\Gamma$$

and then to homomorphisms

$$(\alpha, u)^* : \hat{H}^n(G, V) \rightarrow \hat{H}^n(\Gamma, U)$$

of the bounded cohomology spaces. If $U = V = R$ with the trivial action of $\Gamma$, $G$ respectively and if $u = \text{id}_R$, then the homomorphism $(\alpha, u)^*$ is equal to $\alpha^*$ from Section 8.

The norm of the maps $(\alpha, u)^*$ is obviously $\leq \|u\|$. 

89
A.5. Straight and Borel straight cochains

**Borel straight cochains.** Let $G$ be a discrete group and let $p : \mathcal{X} \to X$ be a locally trivial principal right $G$-bundle. A cochain $f \in B^n(\mathcal{X})$ is called **straight** if $f(\sigma)$ depends only on the vertices of $\sigma$. We will identify straight $n$-cochains with functions $\mathcal{X}^{n+1} \to \mathbb{R}$, and the space of straight $n$-cochains with the space $B(\mathcal{X}^{n+1})$. A straight cochain $f \in B(\mathcal{X}^{n+1})$ is said to be a **Borel straight cochain** if $f : \mathcal{X}^{n+1} \to \mathbb{R}$ is a Borel function, i.e. if the preimage of every Borel subset of $\mathbb{R}$ is a Borel subset of $\mathcal{X}^{n+1}$. The space of Borel straight $n$-cochains is denoted by $B(\mathcal{X}^{n+1})$. The image of $d_{-1} : \mathbb{R} \to B^0(\mathcal{X})$ consists of constant functions and hence is contained in $B(\mathcal{X})$. Each space $B(\mathcal{X}^{n+1})$ is in a natural way a $G$-module, and they form a subcomplex $B(\mathcal{X}^\cdot + 1)$ of the complex $B^\cdot(\mathcal{X})$.

**A.5.1. Lemma.** The $G$-modules $B(\mathcal{X}^{n+1})$ are relatively injective for all $n$.

**Proof.** Let $F$ be a Borel fundamental set for the action of $G$ on $\mathcal{X}$ and let $V^n$ be the space of bounded Borel functions $F \times \mathcal{X}^n \to \mathbb{R}$. Let

$$I^n : B(\mathcal{X}^{n+1}) \to B(G, V^n)$$

be the map given by the formula

$$I^n(f)(y_0, y_1, \ldots, y_n) = f(y_0g, y_1g, \ldots, y_ng),$$

where $g \in G$, $y_0 \in F$, and $y_1, \ldots, y_n \in \mathcal{X}$. A routine check shows that $I^n$ is an isometric isomorphism of $G$-modules. Therefore $B(\mathcal{X}^{n+1})$ is relatively injective by Lemma 7.1.

**A.5.2. Lemma.** The complex $B(\mathcal{X}^\cdot + 1)$ together with the map $d_{-1} : \mathbb{R} \to B(\mathcal{X})$ is a strong resolution of the trivial $G$-module $\mathbb{R}$.

**Proof.** Let $b \in \mathcal{X}$ and let

$$K_n : B(\mathcal{X}^{n+1}) \to B(\mathcal{X}^n) \quad \text{and} \quad K_0 : B(\mathcal{X}) \to \mathbb{R}$$

be the maps defined by the formulas

$$K_n(f)(y_1, \ldots, y_n) = f(b, y_1, \ldots, y_n) \quad \text{and} \quad K_0(f) = f(b).$$

Obviously, $K_n(f)$ is a Borel function if $f$ is, and hence this definition is correct. A standard check shows that $K_\cdot$ is a contracting homotopy. It remains to point out that $\|K_n\| \leq 1$. ■
A.5.3. Theorem. If the fundamental group $\pi_1(\mathcal{X})$ is amenable, then the map

$$\mathcal{J} : H^*\left(G, \mathcal{B}(\mathcal{X}^{*+1})\right) \rightarrow H^*\left(G, B^*(\mathcal{X})\right) = \hat{H}^*(X)$$

induced by the inclusion $\mathcal{B}(\mathcal{X}^{*+1}) \rightarrow B^*(\mathcal{X})$ is an isometric isomorphism.

Proof. The resolutions $B^*(\mathcal{X})$ and $\mathcal{B}(\mathcal{X}^{*+1})$ are strong resolutions by Theorem 5.10 and Lemma A.5.2 respectively. Hence by Theorem 7.6 there exist $G$-morphisms of resolutions

$$u_\cdot : B^*(\mathcal{X}) \rightarrow B(G^{*+1}) \quad \text{and} \quad v_\cdot : \mathcal{B}(\mathcal{X}^{*+1}) \rightarrow B(G^{*+1})$$

unique up to chain homotopy. By Theorem 8.2 the map

$$u(p)_\cdot : H^*\left(G, B^*(\mathcal{X})\right) \rightarrow \hat{H}^*(G)$$

induced by $u_\cdot$ is an isometric isomorphism. We claim that the map

$$v(p)_\cdot : H^*\left(G, \mathcal{B}(\mathcal{X}^{*+1})\right) \rightarrow \hat{H}^*(G)$$

induced by $v_\cdot$ is also an isometric isomorphism. The proof is almost the same as the proof of Theorem 8.2. The key part of the latter is the construction of a morphism of resolutions $r_\cdot$. If the fundamental set $F$ used in the construction of $r_\cdot$ is a Borel set, then the image of $r_\cdot$ consists of Borel straight cochains. It follows that $r_\cdot$ defines a morphism of resolutions

$$s_\cdot : B(G^{*+1}) \rightarrow \mathcal{B}(\mathcal{X}^{*+1})$$

extending $\text{id}_R$ and consisting of maps with the norm $\leq 1$. The rest of the proof requires only replacing $B^*(\mathcal{X})$ by $\mathcal{B}(\mathcal{X}^{*+1})$. Hence $v(p)_\cdot$ is an isometric isomorphism. The maps $u(p)_\cdot$, $v(p)_\cdot$, and $\mathcal{J}$ form a triangle diagram

![Triangle Diagram](image)

This triangle is commutative because all maps are induced by morphisms of resolutions and the resolution $B(G^{*+1})$ is relatively injective. Cf. the proofs of commutativity in Section 8. Since $u(p)_\cdot$, $v(p)_\cdot$ are isometric isomorphisms, $\mathcal{J}$ is an isometric isomorphism also. ■
A.6. Double complexes

A.6.1. Theorem. Let $(K^p,q)_{p,q \geq 0}$ be a double complex with the differentials

$$d : K^p,q \rightarrow K^{p,q+1} \quad \text{and} \quad \delta : K^p,q \rightarrow K^{p+1,q}$$

and let $(T^n)_{n \geq 0}$ be its total complex. Let $L^p$ be the kernel of the differential

$$d : K^p,0 \rightarrow K^p,1.$$ 

Then $L^*$ together with the differential $\delta$ is a subcomplex of the total complex $T^*$. If the complexes $(K^p,*,d)$ are exact, then the homomorphism $H^*(L^*) \rightarrow H^*(T^*)$ induced by the inclusion $L^* \rightarrow T^*$ is an isomorphism.

Proof. This is a special case of Theorem 4.8.1 from the Chapter I of the book [Go]. Its standard proof is based on the properties of the spectral sequences associated with a double complex. But it can be also proved directly, by using the diagram chase. ■

References

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94