Transition from Quantum Chaos to Localization in Spin Chains

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Recent years have seen an increasing interest in quantum chaos and related aspects of spatially extended systems, such as spin chains. However, the results are seemingly contradictory as generic approaches suggest the presence of many-body localization while analytical calculations for certain system classes, here referred to as the “self-dual case”, prove adherence to universal (chaotic) spectral behavior. We address these issues studying the level statistics in the vicinity of the latter case, thereby revealing transitions to many-body localization as well as the appearance of several non-standard random-matrix universality classes.

*Introduction* — An early root of quantum chaos \cite{1–4} was the study of the level statistics in typical many-body systems like nuclei. Attempting to relate quantum to classical chaos, the complexity of the many-body dynamics brought single-particle systems into the focus. In recent years, the interest turned back to many-body systems \cite{5,11}. Ironically, although the dynamics in such systems is rather complicated, it is not always chaotic, examples are integrable dynamics, various types of regular collective motion \cite{12,13} and many-body localization (MBL) \cite{14,19}.

We consider spin chains that presently attract considerable interest as they can be realized experimentally with cold atoms \cite{20,25} or as chains on surfaces \cite{26,27}. On the theoretical side various different directions in many-body chaos converge in spin chains, such as thermalization in finite systems \cite{28,32}, localization and entanglement effects \cite{33–40}, spectral properties \cite{41–50}, characterized e.g. by the spectral form factor, and the classical-quantum transition \cite{9,51–55}.

We are interested in chains of $N$ spin-1/2 particles for arbitrary $N$. We employ duality: the unitary propagator in time corresponds to a nonunitary “propagation” in particle number governed by an operator dual to the time evolution operator \cite{46,56}. By this we calculate the spectral form factor that characterizes correlations between the eigenvalues of a system, as a function of time in the disorder free case. The Prosen group \cite{47} extended this including disorder and confirmed the predictions made by Random Matrix Theory (RMT) \cite{57,58} for long times. The dual operator can be unitary \cite{46}, this situation was coined “self-dual case” \cite{48}. *In the self-dual case* the exact RMT result for the spectral form factor applies at all fixed times in the thermodynamic limit of infinitely long disordered chains \cite{48}.

Here we establish the symmetry property of the kicked Ising chain similar to the charge conjugation \cite{1}; as a result at the point of self-duality its spectrum is described by one of the “new” ensembles of RMT rather than the expected circular orthogonal ensemble (COE). Then we study the stability of the RMT results as we go away from self-duality addressing the questions: How does the transition occur between RMT and the localized regime? Specifically, how does the spectral form factor change for short times if we move away from the self-dual case? How does it change in dependence of time revealing ergodic behavior or localization in the long-time limit? Such an investigation is all the more important since a variety of seemingly close models of the disordered spin-1/2 chains exhibit qualitatively different spectral properties obeying RMT \cite{17,48} or indicating MBL \cite{41}.

We derive an exact analytical expression for the form factor for long chain near the self-dual case. Larger violation of self-duality brings about an explosive growth of the spectral form that becomes sharp in the limit $N \to \infty$ similar to a thermodynamic phase transition leading finally to MBL. These results are much more general as they hold for arbitrary self-dual models.

*Kicked Spin Chain* — We study a periodically kicked disordered chain of $N$ spins with the Pauli matrices $\hat{\sigma}_n = (\hat{\sigma}^x_n, \hat{\sigma}^y_n, \hat{\sigma}^z_n)$, the Ising coupling $J$, the magnetic field $b_z$ in $z$ direction and the site dependent field $h_n$ in $x$ direction modeling disorder \cite{38,46,48,51,54}. The time evolution operator per period is $U = \hat{U}_I\hat{U}_b$, where

$$
\hat{U}_I = \exp\left(-iJ \sum_{n=1}^{N} \hat{\sigma}^z_n \hat{\sigma}^z_{n+1}\right) \\
\hat{U}_b = \exp\left(-i \sum_{n=1}^{N} h_n \hat{\sigma}^z_n\right) \exp\left(-ib_z \sum_{n=1}^{N} \hat{\sigma}^x_n\right)
$$

(1)

are the Ising and the magnetic field part, respectively. In the $\hat{\sigma}^z_n$ product eigenbasis $|\hat{\sigma}^N\rangle = |\sigma_1,\ldots,\sigma_N\rangle$, its dimension is $2^N \times 2^N$. The in general nonunitary dual operator $\hat{W}$ of dimension $2^T \times 2^T$ in the basis $|\hat{\sigma}^T\rangle = |\sigma_1,\ldots,\sigma_T\rangle$ is especially suited for the small $T$ regime.

It fulfills $\text{Tr} \, \hat{U}^T = \text{Tr} \, \hat{W}$ with $\hat{W} = \prod_{n=1}^{N} \hat{W}_n$. Here $\hat{W}_n = \hat{W}_I \hat{W}_n b$, with

$$
\langle \hat{\sigma}^T | \hat{W}_I | \hat{\sigma}^T \rangle = \exp\left(-iJ \sum_{t=1}^{T} \sigma_t \sigma_i\right), \\
\langle \hat{\sigma}^T | \hat{W}_n b | \hat{\sigma}^T \rangle = \delta_{\hat{\sigma}^T} \exp\left(-ih_n \sum_{t=1}^{T} \sigma_t\right) \prod_{t=1}^{T} R_{\sigma_t \sigma_{t+1}},
$$

(2)

where $R_{\sigma_t \sigma_{t+1}}$ are random variables distributed uniformly on $[0,1]$.
in the basis $|\bar{\sigma}^T\rangle$, and $R_{\sigma_1\sigma_2\cdots\hat{\sigma}_i\cdots\hat{\sigma}\cdots} = R_{11} = R_{1-1} = \cos b_x$, $R_{-11} = R_{1-1} = -i\sin b_x$. The condition for self-duality, i.e. for the unitarity of $W$, is $J = b_x = \pi/4$. This is so because $\prod_{t=1}^T R_{\sigma_t\sigma_{t+1}}$ then transforms into $(-1)^T2^{-T/2}$ with $2\nu$ being the number of domain walls in the dual ring $\sigma_1, \ldots, \sigma_T$ leading, up to the factor $2^{-T/2}$, to a unitary operator $W_n,b$. Including this factor into $\hat{W}_I$ transforms this operator into the $T$ dimensional (unitary) discrete Fourier transform.

Spectral Form Factor — The quantity of interest is the spectral form factor

$$K_N(T) = \left\langle \left| \text{Tr} \hat{U}^{T^2} \right|^2 \right\rangle = \left\langle \left| \text{Tr} \prod_{n=1}^N \hat{W}_n \right|^2 \right\rangle, \quad (3)$$

where $\langle \ldots \rangle$ denotes the disorder average over $h_n$. The operators $\hat{W}_n$ possess symmetries with respect to cyclic permutations $\hat{P}_C$ and reflection $\hat{P}_R$ with $\hat{P}_C |\sigma_1, \ldots, \sigma_T\rangle = |\sigma_2, \ldots, \sigma_T, \sigma_1\rangle$ and $\hat{P}_R |\sigma_1, \ldots, \sigma_T\rangle = |\sigma_T, \ldots, \sigma_2, \sigma_1\rangle$. For odd $T$ the symmetrized basis

$$|\eta^{(k)}_n\rangle = A \sum_{t=1}^T \hat{P}_C e^{\frac{2\pi i k t}{T}} |\bar{\sigma}^T\rangle, \quad k = 1, \ldots, T-1$$

$$|\eta^{(0,\pm)}_n\rangle = A'(1 \pm \hat{P}_R) \sum_{t=1}^{T-1} \hat{P}_C |\bar{\sigma}^T\rangle, \quad k = 0 \quad (4)$$

allows to decompose $\hat{W}_n$ into irreducible blocks $\hat{W}_n^{(k)}$; if $T$ is even the block with $k = T/2$ needs additional desymmetrization.

Following [48] we introduce the squared space with the product basis $|\bar{\sigma}\bar{\sigma}'\rangle = |\bar{\sigma}\rangle |\bar{\sigma}'\rangle$, operators in that space will be denoted by calligraphic letters. The advantage is that averaging of the form factor over the disorder can be performed analytically. We consider the limit of strong disorder [59] when the result simplifies to

$$K_N(T) = \text{Tr} \hat{A}^N, \quad \hat{A} = \hat{P}(\hat{W} \otimes \hat{W}^* \hat{P}) \quad (5)$$

with $\hat{W}$ obtained from $\hat{W}_n$ by replacing the $h_n$ by their average. The projector $\hat{P}$ leaves unchanged the states $|\bar{\sigma}\bar{\sigma}'\rangle$ with $\sum_{t=1}^T \sigma_t = \sum_{t=1}^T \sigma'_t$ and annihilates all others. We then express the form factor as

$$K_N(T) = \sum_{kk'} \sum_j \left[ \lambda^{(kk')}_{jj} \right]^N, \quad (6)$$

where we used the block diagonal structure of $\hat{W} = \oplus_k \hat{W}^{(k)}$ and $\hat{A} = \oplus_{kk'} \hat{A}^{(kk')}$ in the basis $|\eta^{(k)}_n\rangle |\eta^{(k')}_{n'}\rangle$. The eigenvalues of $\hat{A}^{(kk')}$ are denoted by $\lambda^{(kk')}_{jj}$.

The basis truncation by $\hat{P}$ shifts the vast majority of eigenvalues of $\hat{W} \otimes \hat{W}^*$ inside the unit circle.

However, each unitary block $\hat{A}^{(kk)}$ has an eigenvalue $\lambda^{(kk)}_{jj}$ associated with the trivial eigenvector $\sum_{n} |\eta^{(k)}_n\rangle |\eta^{(k')}_{n'}\rangle$. Counting the number of blocks in the symmetry-reduced representation of the dual operator and considering $\hat{W}^{(k)} = W^{(T-k)}$ we obtain that in the self-dual case, $T > 5$ and odd (respectively even), the operator $\hat{A}$ has $2T$ (respectively $2T+1$) eigenvalues one associated with the trivial eigenvectors (there can be more in the exceptional cases when an additional symmetry is present[48]); in the limit $N \to \infty$ they give rise to the form factor close to the RMT predictions for COE. This result is based on unitarity and symmetry of the dual problem, and may be extended to other self-dual systems.

Symmetry Relations — The operators $\hat{U}$ and $\hat{W}_n$ fulfill the symmetry relations

$$\hat{U}' = (-i)^N \hat{\Sigma}_y^N \hat{U}^* \hat{\Sigma}_y^N, \quad \hat{W}'_n = (-i)^T \hat{\Sigma}_y^T \hat{W}_n^* \hat{\Sigma}_y^T, \quad (7)$$

where $\hat{\Sigma}_y^N = \oplus_{t=1}^N \hat{\sigma}_t^y$; the unprimed operators are evaluated at $J = \pi/4 - \Delta J$ and the primed ones at $J = \pi/4 + \Delta J$, $h_n$ and $b_x$ are arbitrary. The proof of the first relation is given in [61]; a more cumbersome proof for the dual operators will be published elsewhere.

The symmetries (7) have important consequences for the spectral properties of $\hat{U}$ and $\hat{W}$. They imply for $\Delta J = 0$ that the eigenphases of $\hat{W}_M = \exp(iTN\pi/4) \hat{W}$ and $\hat{U}_M = \exp(iN\pi/4) \hat{U}$ come in complex conjugated pairs which single out the phases 0 and $\pm \pi$ in the spectrum. After an appropriate transformation of the basis set these matrices become symplectic for $T$ respectively $N$ odd and orthogonal for $T$ respectively $N$ even [61].

In Fig. 1 we show the numerically computed eigenphase densities $\rho(\varphi)$ of the block 0+ of $\hat{W}_M$ for different $T$. For sufficiently large $N$ we find perfect agreement with the predictions by the following “new” ensembles of RMT: the circular quaternion ensemble (CQE) for $T$ odd and the circular real ensemble (CRE) for $T$ even. Both are characterized by quadratic level repulsion, a quadratic

![FIG. 1: Disorder averaged eigenphase density $\rho(\varphi)$ of the block 0+ of $\hat{W}_M$ for $N = 13, T = 5$ (left) and $T = 6$ (right). The delta like peak at zero at the right plot indicates the zero mode. This corresponds to Eq. (5) for $n = 9$ and $n = 12$, respectively, after proper normalization.](image-url)
behaviour around zero and $\pm \pi$ and the density \[ \rho(\phi) = 1 \pm \frac{\sin n\phi}{n \sin \phi}. \] (8)

For odd $T$ only the minus sign is realized with odd $n = \dim \hat{W}^{(k)} + 1$. For even $T$ both signs occur and $n = \dim \hat{W}^{(k)} - 1$ can be even and odd such that $\rho(\phi)$ can have a minimum or a maximum in the vicinity of 0 and $\pm \pi$; at the points of minima disorder protected eigenphases similar to Majorana zero modes \[64\] appear in the spectrum, see Fig. 1, bottom. The operator $\hat{U}$ possesses additionally to Eq. (7) time reversal symmetry \[1\] leading to a distribution of the eigenphases of $\hat{U}_M$ according to $T_+ \text{CQE}$ for $N$ odd and $T_+ \text{CRE}$ for $N$ even characterized by a linear (quadratic) behavior close to zero, $\pm \pi$ for $N$ odd (even) and a linear eigenphase repulsion \[64\], see Fig. 2. In the vicinity of 0, $\pm \pi$ the spectra of $\hat{W}_M$ and $\hat{U}_M$ reduce to those of Hamiltonians of “new” symmetry classes, see Ref. \[65, 66\].

Dependence on $\Delta J$ and $T$ — The numerically calculated dependence of $K_N(T)$ on $\Delta J$ is shown for different $N$ in Fig. 3. The right relation in Eq. (7) implies the symmetry of the plot around $J = \pi/4$. When we move away from $J = \pi/4$ the form factor first slightly decreases, then forms a plateau and finally increases exponentially. In the limit $N \to \infty$ the plateau shrinks while the increase becomes ever sharper reminiscent of a phase transition. According to Eq. (9) the dominant contributions to the form factor of long chains result only from the largest eigenvalues that smoothly transform into the unit eigenvalues in the self-dual case; see Fig. 4. In case of the exponential growth of $K_N(T)$ the situation further simplifies, as the only significant contribution results from the largest eigenvalue of the block $\hat{A}^{(0+,0+)}$. Due to Eq. (9), $\lambda_k = \max_j \lambda_j^{(kk)}$ is an even function of $\Delta J$, thus

$$\lambda_k = 1 + B_k \Delta J^2 + O(\Delta J^4).$$

We obtain for the coefficients $B_k$ \[11\],

$$B_k = -\frac{2T(T-1)}{2^r-1} - B_{0\pm} = \pm \frac{2T(T-1)}{2^{(r-1)/2} \pm 1}$$

with $B_{0-} = 0$ for $T \leq 5$. Although these expressions are exact for prime $T$ they provide a good interpolation formula also for other $T$, see Fig. 1 of Ref. \[11\]. In the limit $N \to \infty$, $\Delta J \to 0$, $N\Delta J^2 \equiv x = \text{const.}$, the relation $\lambda_k^N \approx \exp(N\Delta J^2 B_k)$ becomes exact yielding the spectral form factor

$$K_N(T) = e^{xB_0+} + e^{xB_0-} + 2(T-1)e^{xB_k}. $$

We find a good agreement that improves with increasing $N$ between this expression and the exact numerical result.
Issue of Many-Body Localization — We plot in Fig. 5 $K_N(T)$ as function of $N$. After an erratic behavior for small $N$ and an RMT transition region we find an exponential growth with $N$. This hints at a localization effect. To clarify this, we consider a toy model of a set of $\mu$ chaotic non-interacting non-identical quantum systems each belonging to a RMT universality class, e.g. the COE with the form factor $K_{COE}(T)$. The energy spectrum of the whole set will be a direct sum of the spectra of separate systems and the form factor will be the $\mu$-th power of $K_{COE}(T)$.

Now consider a disordered chain of $N$ spins suspected to undergo localization, with localization length of some $N_c$. For $N < N_c$ the chain length is too small for the localization to occur and the form factor observed is $K_{COE}(T)$. For larger $N$ the eigenphase spectrum will be a direct sum of $N/N_c = \mu$ spectra with the form factor $(K_{COE}(T))^\mu$. The plot of $\ln K_{N}(T)$ as function of the chain length $N$ would thus consists of a horizontal stretch at $N < N_c$ and a tilted line with inclination $\tan \phi = \ln K_{COE}(T)/N_c$ for $N > N_c$.

Returning to the system under discussion, we recall that in the limit of long chains the form factor reduces to the $N$-th power of the largest eigenvalue $\lambda_{0+}$ of $\hat{A}$. Consequently the time dependent localization length is found as $N_c(T) = \ln K_{COE}/\ln \lambda_{0+} \approx \ln 2T/(B_{0+}\Delta J^2)$.

Whereas the latter relation is restricted to the regime of small $\Delta J$, our dual operator approach allows to decide upon localization or ergodic behavior for general $\Delta J$. We show $\lambda_{0+}$ in Fig. 6 in dependence of odd $T$ (see Fig. 2 in Ref. [61] for $T$ even) for the full range of $\Delta J$ where the system undergoes a transition from the behavior around $\Delta J = 0$, where the eigenvalue $\lambda_{0+}$ is almost flat, to the integrable case of $\Delta J = \pi/4$ marked by strong oscillations. In-between we find a qualitative change in the behavior of the eigenvalue as its decay is first drastically diminished, then oscillations occur with the amplitude decreasing with time. Remarkably, at any second data point, $\lambda_{0+}$ grows with $\Delta J$ up to $\Delta J \approx 0.6$ and then decreases leading to counteroscillating curves for $\Delta J$ smaller and larger than 0.6.

We thus expect that for large times and $\Delta J$ above a certain threshold, the decay and oscillations stop such that $N_c(T)$ tends to a limit larger than 1 indicating localization. For an independent check we consider the average spacing ratio $\langle r \rangle = \langle \max(\varphi_{n+1} - \varphi_n, \varphi_{n-1} - \varphi_n) \rangle / \min(\varphi_{n+1} - \varphi_n, \varphi_{n-1} - \varphi_n)$ where $\varphi_{n-1}$, $\varphi_n$, $\varphi_{n+1}$ are adjacent eigenphases [67]. It is shown in Fig. 7 and displays a clear transition between the ergodic and localized behavior at a $\Delta J$ consistent with the value estimated above, which seems to become sharp in the thermodynamic limit $N \to \infty$.

Conclusions — The disordered spin chain with parameters satisfying the self-duality condition is a rare example of a system in which absence of many-body localization and adherence to the RMT predictions in the thermodynamic limit have been analytically proven. We investigated the break-up of these properties in the case of deviations from the self-dual situation. A basic ingredient was the symmetry relation for the operators $\hat{U}$ and $\hat{W}$, with respect to the self-dual case. We explored its important consequences on the eigenphase density which are in accordance with non-standard RMT ensembles.

FIG. 5: Logarithm of the form factor for $T = 7$ time steps versus the chain length $N$. The erratic small $N$ behavior is shown in the inset.

FIG. 6: Coefficient $\lambda_{0+}$ versus $T$ for $T$ odd with color coded values of $\Delta J$. 
Furthermore, we studied $K_N(T)$ in dependence of $\Delta J$, $T$ and $N$ numerically and also explained our findings analytically. Finally, we can identify a transition to MBL in the system. This allows, first, to establish a connection between the Refs. [47] and [48] by providing the time dependence of the spectral form factor. Second, in view of Eq. (11), the relationship between Refs. [48] and [11] becomes obvious, the RMT behavior shows up only in a narrow region in the vicinity of the self-dual situation turning to a localized behavior of an interacting many-body system for increased $\Delta J$.

The methods developed here can be used to study many other quantities, as e.g. correlation functions and the entanglement entropy for spin chains which are in the focus of experimental and theoretical research.

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We find good agreement with numerical results already for $\xi \approx 1$ [60].

In preparation.

Supplemental Material

https://en.wikipedia.org/wiki/Circular_ensemble

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Proof of the Symmetry Relations with respect to the Self-Dual Line — Here we start with the proof of the left relation in Eq. (7) in the main text. First, $\hat{U}_I$ remains invariant under the simultaneous conjugation with $\hat{\Sigma}^N_y$ and complex conjugation. For the matrix elements of $\hat{U}_I$ consider the operator $\sum_{n=1}^{N} \hat{\sigma}^n \sigma_{n+1}$. We get in the $\sigma^N$ basis $(N-2\nu) + 2\nu = N-4\nu$ with $2\nu$ being the number of domain walls yielding for $\hat{U}_I$ in Eq. (7) $(-1)^{\nu \exp [i(N-4\nu)\Delta J]} e^{iN\pi/4}$ and for $\hat{U}_I^*$ $(-1)^{\nu \exp [i(N-4\nu)\Delta J]} e^{-iN\pi/4}$. This proves relation (7) for $\hat{U}_I$. The relation for $\hat{W}_I$ can be obtained in a similar way [1].

Symmetry of $\hat{U}_M$ and $\hat{W}_M$ on the Self-Dual Line — Consider the first of the relations (7) in the self-dual case, $J = b_x = \pi/4$ rewritten for $\hat{U}_M$, $\hat{U}_M = \hat{\Sigma}^N_y \hat{U}_M^* \hat{\Sigma}^N_y$. (12)

The operator $\hat{\Sigma}^N_y$ has non-zero matrix elements only between the basis states $|\sigma\rangle = |\sigma_1, \ldots, \sigma_N\rangle$ and $| - \sigma\rangle$, i.e., with all spins flipped. Let $\eta(\sigma) = 1$ if the number of spins-up in $|\sigma\rangle$ is even and $-1$ otherwise; obviously $\eta(-\sigma) = (-1)^n \eta(\sigma)$. Then we have,

$$
\langle \sigma | \hat{\Sigma}^N_y | \sigma' \rangle = (-1)^n \langle \sigma | \hat{\Sigma}^N_y | -\sigma \rangle = i^n \eta(\sigma) \delta_{\sigma,-\sigma'}.
$$

(13)

Let us group the basis states into pairs $|\sigma\rangle, | - \sigma\rangle$. First consider $N$ odd and choose $|\sigma\rangle$ in each pair such that $\eta(\sigma) = +1$. In that basis set $\hat{\Sigma}^N_y = iN\hat{\Omega}$ where $\hat{\Omega}$ is the fundamental symplectic matrix in one of its canonical representation; namely $\hat{\Omega}$ is composed of $2^{N-1}$ blocks

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

on its diagonal. Rewriting (12) as

$$
\hat{U}_M^{-1} = \hat{\Omega} \hat{U}_M^* \hat{\Omega},
$$

(14)

where $T$ stands for transposition, we see that $\hat{U}_M$ is symplectic unitary.

If $N$ is even both members of the pair $|\sigma\rangle, | - \sigma\rangle$ have the same $\eta$. The matrix $\Sigma^N_y$ is again block-diagonal, however its $2 \times 2$ blocks are now $\eta(\sigma)\delta^2$. After the transformation within each pair

$$
|u(\sigma)\rangle = \frac{1}{\sqrt{2}} (|\sigma\rangle + \eta | - \sigma\rangle),
$$

$$
|v(\sigma)\rangle = \frac{1}{i\sqrt{2}} (|\sigma\rangle - \eta | - \sigma\rangle).
$$

(15)

the unitary matrix $\hat{U}_M$ becomes real, i.e., orthogonal; it which follows from (12), after some algebra.

The proof for the dual operator $\hat{W}_M$ is analogous. Note that $\hat{U}_M$ obeys, apart from (12), time reverse invariance whereas $\hat{W}_M$ does not.

Derivation of the Formula for $B_k$ — Here we sketch the derivation of the coefficients $B_k$ given in Eq. (10) of the main text. As one important input we first determine the dimension of the subspaces labeled by $k$. Defining projection operators onto that subspaces

$$
\hat{Q}^{(0)} = \frac{1}{2T} (1 + \hat{P}_T) \sum_{t=1}^{T} \left( \hat{P}_C \right)^t,
$$

$$
\hat{Q}^{(k)} = \frac{1}{T} \sum_{t=1}^{T} \exp^{i2\pi t/T} \left( \hat{P}_C \right)^t,
$$

(16)

we get for $T$ prime the dimensions of the subspaces by calculating the traces of the latter operators [1]

$$
M_{0,\pm} = \left( \frac{2^T + 2(T-1) \pm T^2(T+1)/2}{2T} \right),
$$

$$
M_k = \left( \frac{2^T - 2}{T} \right).
$$

(17)

Thereby we extend the analysis of Ref. [2].

Furthermore, we need the relation

$$
B_k = \frac{1}{2M_k} \frac{\partial^2 H(k,j)}{\partial j^2} \bigg|_{j=0}
$$

with

$$
H(k,j) = \text{Tr} \hat{W}^{(k)} (\hat{W}^{(k)})^\dagger
$$

(18)

To show this we start from Eq. (5) in the main text in the limit $\xi \to \infty$ allowing for the replacement of $\hat{O}_I$ by $\hat{P}$. We insert into the expression for $K_N(T)$ in Eq. (5) in the main text a power series of $\hat{W}$

$$
\hat{W} = \hat{W}_0 + \Delta J \hat{W}_1 + \Delta J^2 \hat{W}_2/2 + O(\Delta J^3).
$$

(19)

The zeroth order term yields the spectral form factor in the self-dual case $K^{SD}_N(T)$, the first order vanishes due to the symmetry relation (8) in the main text and the second order reduces up to terms of order $\Delta J^3$ to

$$
K_N(T) - K^{SD}_N(T)
$$

$$
= 2N\Delta J^2 \text{Tr} \left\{ \hat{P} \left( \hat{W}_0 \otimes \hat{W}_0^* \right) \hat{P} \right\}^{N-1}
$$

$$
\hat{P} \left( \hat{W}_2 \otimes \hat{W}_0^* + \hat{W}_0 \otimes \hat{W}_2^* + 2\hat{W}_1 \otimes \hat{W}_1^* \right) \hat{P} \right\}.
$$

(20)

Using that $N$ is large, the $N - 1$st power of $\hat{P} \left( \hat{W}_0 \otimes \hat{W}_0^* \right) \hat{P}$ in the last equation acts as a projector onto its eigenvector

$$
\eta^{(k)}_0 = \frac{1}{\sqrt{M_k}} \sum_{i=1}^{M_k} |\eta_i^{(k)}\rangle \langle \eta_i^{(k)}|^\dagger
$$

(21)
to the eigenvalues of largest magnitude. The vectors \( |\eta(k)\rangle \) are defined in Eq. (4) of the main text. Rewriting now first Eq. (20) and second Eq. (18) in terms of the matrix elements of the dual operator given in Eq. (2) of the main text, relation (15) can be shown [1].

This relation can be used finally to obtain Eq. (10) of the main text. Therefore \( H(k,J) \) can be rewritten as

\[
H(k,J) = \frac{1}{T} \sum_{t=1}^{T} e^{2\pi i k t / T} Z^{(t)}(J)
\]

(22)

with

\[
Z^{(t)}(J) = \frac{1}{2T} \sum_{\sigma,\sigma' = \pm 1} \exp \left[ -iJ \sum_{\tau=1}^{T} \sigma_{\tau} (\sigma'_{T-\tau} - \sigma'_{T+\tau}) \right].
\]

(23)

The resulting sums in the last equation can be calculated for \( T \) prime [1] yielding

\[
\left. \frac{\partial^2 Z^{(t)}}{\partial J^2} \right|_{\Delta J = 0} = 8T(T - 1)
\]

(24)

and thus the first relation in Eq. (10) in the main text. For the sectors \( 0\pm \) the function \( H(k,J) \) defined in Eq. (18) can be written in the form

\[
H(0\pm, J) = H_1(J) \pm H_2(J)
\]

(25)

with

\[
H_1(J) = \frac{1}{2T} \sum_{t=1}^{T} Z^{(t)}(J), \quad H_2(J) = \frac{1}{2T} \sum_{t=1}^{T} S^{(t)}(J)
\]

(26)

with

\[
S^{(t)}(J) = \frac{1}{2T} \sum_{\sigma,\sigma' = \pm 1} \exp \left[ -iJ \sum_{\tau=1}^{T} \sigma_{\tau} (\sigma'_{T-\tau} - \sigma'_{T+\tau}) \right].
\]

(27)

Computing the sums in the last equation for \( T \) prime [1], we obtain with

\[
\left. \frac{\partial^2 S^{(t)}}{\partial J^2} \right|_{\Delta J = 0} = 4T(T - 1)2^{(T+1)/2}
\]

(28)

the second part of Eq. (10) in the main text as well.

**Dominant coefficient \( \lambda_{0+} \) —** Here we show first how good our analytical expressions for \( B_k \) that we derive for \( T \) prime above are for other values of \( T \). We consider here the coefficient \( \lambda_{0+} \) that we find in the main text dominant in the long chain limit \( N \to \infty \). Therefore we plot in Fig. 8 at first as full line the expression in Eq. (10) of the main text for the coefficient \( B_{0+} \) and second the results from full numerical calculations as dots. We observe that the expression (10) provides a good interpolation also for nonprime \( T \)-values as well, especially for the odd ones.

Furthermore, for reasons of clarity we left out in Fig. 6 in the main text the values of \( \lambda_{0+} \) for \( T \) even that we provide in Fig. 9. They show a similar structure as in Fig. 6 indicating that the implications for localization properties we drew in the main text from the behavior of \( \lambda_{0+} \) for \( T \) odd remain valid for \( T \) even as well. The curves at the bottom in Fig. 9 correspond to the ones in the perturbative regime, e.g. the one shown in Fig. 8.

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[1] in preparation
[2] C. Pineda, T. Prosen, Phys. Rev. E 76, 061127 (2007).