The ring of cyclic quasi-symmetric functions and its non-Escher subring are introduced in this paper. A natural basis consists of fundamental cyclic quasi-symmetric functions; for the non-Escher subring they arise as toric $P$-partition enumerators, for toric posets $P$ with a total cyclic order. The associated structure constants are determined by cyclic shuffles of permutations. We then prove the following positivity phenomenon: for every non-hook shape $\lambda$, the coefficients in the expansion of the Schur function $s_\lambda$ in terms of fundamental cyclic quasi-symmetric functions are nonnegative. The proof relies on the existence of a cyclic descent map on the standard Young tableaux (SYT) of shape $\lambda$. The theory has applications to the enumeration of cyclic shuffles and SYT by cyclic descents. We conclude by providing a cyclic analogue of Solomon’s descent algebra.

1. Introduction

The graded rings Sym (sometimes denoted $\Lambda$) and QSym, of symmetric and quasi-symmetric functions, respectively, have many applications to enumerative combinatorics, as well as to other branches of mathematics; see, e.g., [31, Ch. 7]. This paper introduces two intermediate objects: the graded ring $cQSym$ of cyclic quasi-symmetric functions, and its non-Escher subring $cQSym^-$.

The rings Sym, QSym and $cQSym$ may be defined via invariance properties. A formal power series $f \in \mathbb{Z}[[x_1, x_2, \ldots]]$ of bounded degree is symmetric if for any $t \geq 1$, any two sequences $i_1, \ldots, i_t$ and $j_1, \ldots, j_t$
of distinct positive integers (indices), and any sequence \(m_1, \ldots, m_t\) of positive integers (exponents), the coefficients of \(x_i^{m_1} \cdots x_u^{m_t}\) and \(x_j^{m_1} \cdots x_{j_1}^{m_t}\) in \(f\) are equal. We call \(f\) quasi-symmetric if for any \(t \geq 1\), any two increasing sequences \(i_1 < \cdots < i_t\) and \(j_1 < \cdots < j_t\) of positive integers, and any sequence \(m_1, \ldots, m_t\) of positive integers, the coefficients of \(x_i^{m_1} \cdots x_i^{m_t}\) and \(x_j^{m_1} \cdots x_{j_1}^{m_t}\) in \(f\) are equal. We define \(f\) to be cyclic quasi-symmetric if for any \(t \geq 1\), any two increasing sequences \(i_1 < \cdots < i_t\) and \(j_1 < \cdots < j_t\) of positive integers, any sequence \(m = (m_1, \ldots, m_t)\) of positive integers, and any cyclic shift \(m' = (m'_1, \ldots, m'_t)\) of \(m\), the coefficients of \(x_i^{m_1} \cdots x_i^{m_t}\) and \(x_j^{m'_1} \cdots x_{j_1}^{m'_t}\) in \(f\) are equal. In fact, the set \(cQSym\) of cyclic quasi-symmetric functions is a graded ring satisfying

\[
\text{Sym} \subseteq cQSym \subseteq QSym.
\]

We remark that all algebras (and coalgebras) in this paper are defined over \(\mathbb{Z}\), so that all structure constants are (mostly nonnegative) integers. Consequently, they may also be defined over an arbitrary field.

Toric posets were recently introduced by Develin, Macauley and Reiner \[8\]. A toric analogue of \(P\)-partitions is presented in Section 3.2. Toric \(P\)-partition enumerators, in the special case of total cyclic orders, form a convenient \(\mathbb{Q}\)-basis for a ring \(cQSym^-\), which is a subring of \(cQSym\). A slightly extended set actually forms a \(\mathbb{Q}\)-basis for \(cQSym\) itself. The elements of this basis are called fundamental cyclic quasi-symmetric functions, are indexed by cyclic compositions of a positive integer \(n\) (equivalently, by cyclic equivalence classes of nonempty subsets \(J \subseteq [n] := \{1, 2, \ldots, n\}\), and are denoted \(F_{n,J}^\text{cyc}\). Normalized versions of them actually form \(\mathbb{Z}\)-bases for \(cQSym\) and \(cQSym^-\); see Lemma \[2.21\] and Observation \[2.25\] below. The full ring \(cQSym\) contains, in particular, all symmetric functions. Its subring \(cQSym^-\) may be described as non-Escher, or combinatorial, forming a natural framework for the study of cyclic descent sets of combinatorial objects, possessing by definition the non-Escher property.

A toric analogue of Stanley’s fundamental decomposition lemma for \(P\)-partitions \[30\] Lemma 3.15.3], given in Lemma \[3.15\] below, is applied to provide a combinatorial interpretation of the resulting structure constants in terms of shuffles of cyclic permutations (more accurately: cyclic words), as follows.

For a finite set \(A\) of size \(a\), let \(\mathcal{S}_A\) be the set of all bijections \(u : [a] \to A\), viewed as words \(u = (u_1, \ldots, u_a)\). Elements of \(\mathcal{S}_A\) will be called bijective words, or simply words. Call \(A\) the support of \(u\). If \(A = [a]\) then \(\mathcal{S}_A\) is the symmetric group \(S_a\), whose elements are genuine permutations.

If \(A\) is a finite set of integers, or any finite totally ordered set, define the cyclic descent set of \(u \in \mathcal{S}_A\) by

\[
\text{cDes}(u) := \{1 \leq i < a : u_i > u_{i+1}\} \subseteq [a],
\]

with the convention \(u_{a+1} := u_1\). A cyclic word \([u] \in \mathcal{S}_A/\mathbb{Z}_a\) is an equivalence class of elements of \(\mathcal{S}_A\) under the cyclic equivalence relation \((u_1, \ldots, u_a) \sim (u_{i+1}, \ldots, u_a, u_1, \ldots, u_i)\) for all \(i\). A cyclic shuffle of two cyclic words \([u]\) and \([v]\) with disjoint supports is any cyclic equivalence class \([w]\) represented by a shuffle \(w\) of a representative of \([u]\) and a representative of \([v]\). The set of all cyclic shuffles of \([u]\) and \([v]\) is denoted \([u] \shuffle_{\text{cyc}} [v]\), and is a collection of cyclic equivalence classes.

The following cyclic analogue of the product formula for (ordinary) fundamental quasi-symmetric functions \[31\] Ex. 7.93] provides a combinatorial interpretation for the structure constants of \(cQSym^-\).

**Theorem 1.1.** Let \(C = A \cup B\) be a disjoint union of finite sets of integers. For each \(u \in \mathcal{S}_A\) and \(v \in \mathcal{S}_B\), one has the following expansion:

\[
F_{[C], \text{cDes}(u)}^\text{cyc} F_{[B], \text{cDes}(v)}^\text{cyc} = \sum_{[w] \in [u] \shuffle_{\text{cyc}} [v]} F_{[C], \text{cDes}(w)}^\text{cyc}.
\]

See Theorem \[3.22\] below.

Recall that a skew shape is called a ribbon if it does not contain a 2 \(\times\) 2 square. The following positivity phenomenon is proved in Section 4.

**Theorem 1.2.** For every skew shape \(\lambda/\mu\) which is not a connected ribbon, all the coefficients in the expansion of the skew Schur function \(s_{\lambda/\mu}\) in terms of normalized fundamental cyclic quasi-symmetric functions are nonnegative integers.

A more precise statement, which provides a combinatorial interpretation for the coefficients, is given in Corollary \[4.6\] below. The proof relies on the existence of a cyclic extension of the descent map on standard Young tableaux (SYT) of shape \(\lambda/\mu\), which was proved in \[2\]. Using Postnikov’s result regarding toric
Schur functions, one deduces that the coefficients in the expansion of a non-hook Schur function $s_\lambda$ in terms of fundamental cyclic quasi-symmetric functions are equal to certain Gromov-Witten invariants, see Remark 4.11 below.

Applications to the enumeration of SYT and cyclic shuffles of permutations with prescribed cyclic descent set or prescribed number of cyclic descents follow from this theory. Using a ring homomorphism from $cQSym$ to the ring of formal power series $\mathbb{Z}[[q]]_\circ$, with product defined by $q^i \odot q^j := q^{\max(i,j)}$, Theorem 1.1 implies the following result.

**Theorem 1.3.** Let $A$ and $B$ be two disjoint sets of integers, with $|A| = m$ and $|B| = n$. For each $u \in \mathfrak{S}_A$ and $v \in \mathfrak{S}_B$, if $\text{cdes}(u) = i$ and $\text{cdes}(v) = j$ then the number of cyclic shuffles $[w]$ in $[u] \shuffle_{\text{cyc}} [v]$ with $\text{cdes}(w) = k$ is equal to

$$\frac{k(m - i)(n - j) + (m + n - k)ij}{(m + j - i)(n + i - j)} \binom{m + j - i}{k - i} \binom{n + i - j}{k - j}.$$ 

See Corollary 5.11 below. More enumerative applications are given in Section 5.

The group ring $\mathbb{Z}[\mathfrak{S}_n]$ has a distinguished subring, Solomon’s descent algebra $\mathfrak{D}_n$, with basis elements

$$D_I := \sum_{\pi \in \mathfrak{S}_n, \text{Des(}\pi) = I} \pi \quad (I \subseteq [n - 1]).$$

Cellini [6] and others looked for an appropriate cyclic analogue. We provide a partial answer, using an operation dual to the product in $\mathfrak{D}_n$ — the internal coproduct $\Delta_n$ on $\mathbb{Q}Sym_n$; see Section 6 for its definition. The analogue turns out to be a module, rather than an algebra.

**Theorem 1.4.** $c\mathbb{Q}Sym_n$ and $c\mathbb{Q}Sym_n^-$ are right coideals of $\mathbb{Q}Sym_n$ with respect to the internal coproduct:

$$\Delta_n(c\mathbb{Q}Sym_n) \subseteq c\mathbb{Q}Sym_n \otimes \mathbb{Q}Sym_n$$

and

$$\Delta_n(c\mathbb{Q}Sym_n^-) \subseteq c\mathbb{Q}Sym_n^- \otimes \mathbb{Q}Sym_n.$$ 

The structure constants for $c\mathbb{Q}Sym_n^-$ are nonnegative integers.

For a more detailed description, including a combinatorial interpretation of the coefficients, see Theorem 6.1.

**Corollary 1.5.** Let $c_2^{[n]}_{0,n}$ be the set of equivalence classes, under cyclic rotations, of subsets $\emptyset \subseteq J \subseteq [n]$. Defining

$$cD_A := \sum_{\pi \in \mathfrak{S}_n, \text{cDes(}\pi) \in A} \pi \quad (A \in c_2^{[n]}_{0,n}),$$

the additive free abelian group

$$c\mathfrak{D}_n := \text{span}_\mathbb{Z}\{cD_A : A \in c_2^{[n]}_{0,n}\}$$

is a left module for Solomon’s descent algebra $\mathfrak{D}_n$.

In general, $c\mathfrak{D}_n$ is is not an algebra, namely, is not closed under the multiplication in $\mathbb{Z}[\mathfrak{S}_n]$.

We conclude the paper with final remarks and open problems.

2. **The ring of cyclic quasi-symmetric functions**

Recall from [14] the following basic definitions:

A quasi-symmetric function is a formal power series $f \in \mathbb{Z}[x_1, x_2, \ldots]$ of bounded degree such that, for any $t \geq 1$, any two increasing sequences $i_1 < \cdots < i_t$ and $i'_1 < \cdots < i'_t$ of positive integers, and any sequence $(m_1, \ldots, m_t)$ of positive integers, the coefficients of $x_{i_1}^{m_1} \cdots x_{i_t}^{m_t}$ and $x_{i'_1}^{m_1} \cdots x_{i'_t}^{m_t}$ in $f$ are equal. Denote by $\mathbb{Q}Sym$ the set of all quasi-symmetric functions, and by $\mathbb{Q}Sym_n$ the set of all quasi-symmetric functions which are homogeneous of degree $n$. 

**Remark 1.** Let $\alpha 

Corollary 1.5. Let $c_2^{[n]}_{0,n}$ be the set of equivalence classes, under cyclic rotations, of subsets $\emptyset \subseteq J \subseteq [n]$. Defining

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**Remark 1.** Let $\alpha 

Corollary 1.5. Let $c_2^{[n]}_{0,n}$ be the set of equivalence classes, under cyclic rotations, of subsets $\emptyset \subseteq J \subseteq [n]$. Defining

$$cD_A := \sum_{\pi \in \mathfrak{S}_n, \text{cDes(}\pi) \in A} \pi \quad (A \in c_2^{[n]}_{0,n}),$$

the additive free abelian group

$$c\mathfrak{D}_n := \text{span}_\mathbb{Z}\{cD_A : A \in c_2^{[n]}_{0,n}\}$$

is a left module for Solomon’s descent algebra $\mathfrak{D}_n$.

In general, $c\mathfrak{D}_n$ is is not an algebra, namely, is not closed under the multiplication in $\mathbb{Z}[\mathfrak{S}_n]$.
For any positive integer \( n \), there is a natural bijection \( c_{\text{comp}} : 2^{[n-1]} \to \text{Comp}_n \), from the power set of \([n-1] = \{1, \ldots, n-1\}\) to the set of all compositions of \( n \), defined by

\[
J = \{j_1 < \cdots < j_t\} \subseteq [n-1] \Rightarrow c_{\text{comp}}(J) := (j_1, j_2 - j_1, \ldots, j_t - j_{t-1}, n - j_t).
\]

In particular, \( c_{\text{comp}}(\emptyset) = (n) \).

The monomial quasi-symmetric function corresponding to a subset \( J \subseteq [n-1] \) is defined by

\[
M_{n,J} := \sum x_{i_1} \cdots x_{i_n},
\]

where the sum extends over all sequences \((i_1, \ldots, i_n)\) of positive integers such that \( j \in J \Rightarrow i_j < i_{j+1} \) and \( j \not\in J \Rightarrow i_j = i_{j+1} \). Equivalently, letting \( c_{\text{comp}}(J) = (m_0, \ldots, m_t) \) where \( t = \#J \),

\[
M_{n,J} = \sum_{i_0 < \cdots < i_t} x_{i_0}^{m_0} \cdots x_{i_t}^{m_t}.
\]

The set \( \{M_{n,J} : J \subseteq [n-1]\} \) forms a basis for the additive abelian group \( \text{QSym}_n \).

Similarly, the fundamental quasi-symmetric function corresponding to a subset \( J \subseteq [n-1] \) is defined by

\[
F_{n,J} := \sum x_{i_1} \cdots x_{i_n},
\]

where the sum extends over all sequences \((i_1, \ldots, i_n)\) of positive integers such that \( j \in J \Rightarrow i_j < i_{j+1} \) and \( j \not\in J \Rightarrow i_j \leq i_{j+1} \). For all \( J \subseteq [n-1] \) we have

\[
F_{n,J} = \sum_{K \supseteq J} M_{n,K}
\]

and consequently, by inclusion-exclusion,

\[
M_{n,J} = \sum_{K \supseteq J} (-1)^{\#(K \setminus J)} F_{n,K}.
\]

Thus, the set \( \{F_{n,J} : J \subseteq [n-1]\} \) forms a basis for the additive abelian group \( \text{QSym}_n \).

We shall now define cyclic analogues of these concepts.

**Definition 2.1.** A cyclic quasi-symmetric function is a formal power series \( f \in \mathbb{Z}[[x_1, x_2, \ldots]] \) of bounded degree such that, for any \( t \geq 1 \), any two increasing sequences \( i_1 < \cdots < i_t \) and \( i'_1 < \cdots < i'_t \) of positive integers, any sequence \( m = (m_1, \ldots, m_t) \) of positive integers, and any cyclic shift \( m' = (m'_1, \ldots, m'_t) \) of \( m \), the coefficients of \( x_{i_1}^{m_1} \cdots x_{i_t}^{m_t} \) and \( x_{i'_1}^{m'_1} \cdots x_{i'_t}^{m'_t} \) in \( f \) are equal.

Denote by \( c\text{QSym} \) the set of all cyclic quasi-symmetric functions, and by \( c\text{QSym}_n \) the set of all cyclic quasi-symmetric functions which are homogeneous of degree \( n \).

**Observation 2.2.** \( \text{QSym}, c\text{QSym} \) and the set \( \text{Sym} \) of symmetric functions (sometimes denoted \( \Lambda \)) are graded abelian groups satisfying

\[
\text{Sym} \subseteq c\text{QSym} \subseteq \text{QSym}. \tag{2.1}
\]

It is not too difficult to check that they are also rings, that is, closed under multiplication of formal power series. For \( \text{Sym} \) and \( \text{QSym} \) this is well-known. The proof for \( c\text{QSym} \), with a combinatorial interpretation of the structure constants, is deferred to the end of Section 3; see Proposition 3.25.

### 2.1. Monomial cyclic quasi-symmetric functions.

For any positive integer \( n \), let \( 2^{[n]} \) be the set of all subsets of \([n]\), and let \( c_0^{[n]} \) be the set of all nonempty subsets of \([n]\). There is a natural map \( c_{\text{comp}} : c_0^{[n]} \to \text{Comp}_n \), defined by

\[
J = \{j_1 < \cdots < j_t\} \subseteq [n] \mapsto c_{\text{comp}}(J) := (j_2 - j_1, \ldots, j_t - j_{t-1}, j_1 - j_t + n).
\]

In particular, \( c_{\text{comp}}(\emptyset) = (n) \) for any \( j \in [n] \), while \( c_{\text{comp}}(\emptyset) \) is undefined.

Clearly, if \( J' \) is a cyclic shift of \( J \) then \( c_{\text{comp}}(J') \) is a cyclic shift of \( c_{\text{comp}}(J) \). The converse is also true—in fact, if \( c_0^{[m]} \) (respectively, \( c\text{Comp}_n \)) denotes the set of equivalence classes of elements of \( 2^{[n]} \) (respectively, \( \text{Comp}_n \)) under cyclic shifts, then the induced map \( c_{\text{comp}} : c_0^{[m]} \to c\text{Comp}_n \) is a bijection.

Note that \( c_0^{[m]} \) is the set of orbits of \( 2^{[m]} \) under the natural action of the cyclic group \( \mathbb{Z}/m\mathbb{Z} \). On the other hand, \( \text{Comp}_n \) contains compositions of varying lengths \( 1 \leq t \leq n \), so that \( c\text{Comp}_n \) consists of orbits of the corresponding groups \( \mathbb{Z}/t\mathbb{Z} \).
Remark 2.3. Burnside’s Lemma, applied to the natural action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on $2^{[n]}$, implies that

$$#c_{0}^{[n]} = \frac{1}{n} \sum_{d|n} \varphi(d)(2^{n/d} - 1) = \left( \frac{1}{n} \sum_{d|n} \varphi(d)2^{n/d} \right) - 1,$$

where $\varphi$ is Euler’s totient function.

Definition 2.4. For a nonempty subset $J = \{j_1 < \cdots < j_t\} \in 2^{[n]}_0$, let $cc_n(J) = (m_1, \ldots, m_t)$. The monomial cyclic quasi-symmetric function corresponding to $J$ is defined by

$$M_{n,J}^{\text{cy}} := \sum_{i_1 < \cdots < i_t} \sum_{j=1}^t x_{i_1}^{m_1} \cdots x_{i_t}^{m_{t+i-1}},$$

where indices are added cyclically in $[t]$, meaning $m_{j+i} = m_j$. Define also $M_{n,\emptyset}^{\text{cy}} := 0$.

Lemma 2.5. (From monomial to cyclic monomial) For every subset $J \subseteq [n]$

$$M_{n,J}^{\text{cy}} = \sum_{J' \subseteq [n]} M_{n,(J-j)\cap [n-1]},$$

where $J - j := \{k - j : k \in J \} \subseteq [n]$ with subtraction interpreted cyclically modulo $n$.

Proof. For $J = \emptyset$ both sides are zero. For $J = \{j_1 < \cdots < j_t\}$ nonempty with $cc_n(J) = (m_1, \ldots, m_t)$, the claim follows directly from Definition 2.4 since, for each $k \in [t]$,

$$M_{n,(J-j)\cap [n-1]} = M_{n,\{j_{k+1-\cdots-j_{k-1}}\} + j_k + n} = \sum_{i_1 < \cdots < i_t} x_{i_1}^{m_1} \cdots x_{i_t}^{m_{t+i-1}}. \qed$$

Example 2.6.

| $J \subseteq [2]$ | $M_{2,J}^{\text{cy}}$ |
|-----------------|-----------------|
| $\emptyset$     | 0               |
| \{1\} or \{2\}  | $M_{2,\emptyset}$ |
| \{1, 2\}        | $2M_{2,\{1\}}$  |

| $J \subseteq [3]$ | $M_{3,J}^{\text{cy}}$ |
|-----------------|-----------------|
| $\emptyset$     | 0               |
| \{1\} or \{2\}  | $M_{3,\emptyset}$ |
| \{1, 2\} or \{2, 3\} or \{1, 3\} | $M_{3,\{1\}} + M_{3,\{2\}}$ |
| \{1, 2, 3\}     | $3M_{3,\{1, 2\}}$ |

| $J \subseteq [4]$ | $M_{4,J}^{\text{cy}}$ |
|-----------------|-----------------|
| $\emptyset$     | 0               |
| \{1\} or \{2\}  | $M_{4,\emptyset}$ |
| \{1, 2\} or \{2, 3\} or \{3, 4\} or \{1, 4\} | $M_{4,\{1\}} + M_{4,\{3\}}$ |
| \{1, 3\} or \{2, 4\} | $2M_{4,\{2\}}$ |
| \{1, 2, 3\} or \{1, 2, 4\} or \{1, 3, 4\} or \{2, 3, 4\} | $M_{4,\{1, 2\}} + M_{4,\{1, 3\}} + M_{4,\{2, 3\}}$ |
| \{1, 2, 3, 4\} | $4M_{4,\{1, 2, 3\}}$ |

Observation 2.7. (Cyclic invariance) If $J' \in 2^{[n]}$ is a cyclic shift of $J \in 2^{[n]}$ then $M_{n,J'}^{\text{cy}} = M_{n,J}^{\text{cy}}$.

As we shall see, a suitable set of functions $M_{n,J}^{\text{cy}}$ will form a basis for the vector space $cQSym_n \otimes \mathbb{Q}$. In order to get a $\mathbb{Z}$-basis for $cQSym_n$, they need to be normalized; see Subsection 2.3.
2.2. Fundamental cyclic quasi-symmetric functions.

**Definition 2.8.** Let \( P := \{1, 2, 3, \ldots\} \). For each subset \( J \subseteq [n] \) denote by \( P_{n,J}^{\text{cyc}} \) the set of all pairs \( (w, k) \) consisting of a word \( w = (w_1, \ldots, w_n) \in P^n \) and an index \( k \in [n] \) satisfying

(i) The word \( w \) is “cyclically weakly increasing” from index \( k \), namely \( w_k \leq w_{k+1} \leq \ldots \leq w_1 \leq \ldots \leq w_{k-1} \).

(ii) If \( j \in J \setminus \{k-1\} \) then \( w_j < w_{j+1} \), where indices are computed modulo \( n \). (This condition is vacuous if \( J = \emptyset \) or \( J = \{k-1\} \).)

**Remark 2.9.** The index \( k \) is uniquely determined by the word \( w \), unless all the letters of \( w \) are equal; in which case, any index \( k \in [n] \) will do by (i), but (ii) implies that either \( J = \{k-1\} \) or \( J = \emptyset \). Each of these “constant words” is therefore counted in \( P_{n,J}^{\text{cyc}} \) just once when \#\( J = 1 \), but \( n \) times when \( J = \emptyset \).

**Example 2.10.** Let \( n = 5 \) and \( J = \{1, 3\} \). The pairs \((12345, 1)\), \((23312, 4)\) and \((23122, 3)\) are in \( P_{5,\{1,3\}}^{\text{cyc}} \) (see Figure 1), but the pairs \((12354, 1)\), \((22312, 4)\) and \((23112, 3)\) are not.

![Figure 1](image.png)

**Figure 1.** The pairs \((12345, 1)\), \((23312, 4)\) and \((23122, 3)\) in \( P_{5,\{1,3\}}^{\text{cyc}} \).

**Definition 2.11.** The **fundamental cyclic quasi-symmetric function** corresponding to a subset \( J \subseteq [n] \) is defined by

\[
F_{n,J}^{\text{cyc}} := \sum_{(w,k) \in P_{n,J}^{\text{cyc}}} x_{w_1} x_{w_2} \cdots x_{w_n}.
\]

**Example 2.12.**

\[
\begin{align*}
F_{5,\{1,3,5\}}^{\text{cyc}} &= 5 \sum_{i_1 < i_2 < i_3 < i_4 < i_5} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} \\
&+ 2 \sum_{i_1 < i_2 < i_3 < i_4} \left( x_{i_1}^2 x_{i_2} x_{i_3} x_{i_4} + x_{i_1} x_{i_2}^2 x_{i_3} x_{i_4} + x_{i_1} x_{i_2} x_{i_3}^2 x_{i_4} + x_{i_1} x_{i_2} x_{i_3} x_{i_4}^2 \right) \\
&+ \sum_{i_1 < i_2 < i_3} \left( x_{i_1}^2 x_{i_2}^2 x_{i_3} + x_{i_1} x_{i_2}^2 x_{i_3}^2 + x_{i_1} x_{i_2} x_{i_3}^2 + x_{i_1} x_{i_2} x_{i_3} x_{i_4}^2 \right)
\end{align*}
\]

and

\[
\begin{align*}
F_{6,\{2,4,6\}}^{\text{cyc}} &= 6 \sum_{i_1 < i_2 < i_3 < i_4 < i_5 < i_6} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} x_{i_6} \\
&+ 3 \sum_{i_1 < i_2 < i_3 < i_4 < i_5} x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5} \left( x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} + x_{i_5} \right) \\
&+ 3 \sum_{i_1 < i_2 < i_3 < i_4} \left( x_{i_1}^2 x_{i_2} x_{i_3} x_{i_4} + x_{i_1} x_{i_2}^2 x_{i_3} x_{i_4} + x_{i_1} x_{i_2} x_{i_3}^2 x_{i_4} + x_{i_1} x_{i_2} x_{i_3} x_{i_4}^2 \right) \\
&+ 3 \sum_{i_1 < i_2 < i_3} x_{i_1}^2 x_{i_2} x_{i_3}^2.
\end{align*}
\]

**Lemma 2.13.** (Fundamental vs. monomial) For any subset \( J \subseteq [n] \)

\[
F_{n,J}^{\text{cyc}} = \sum_{K \supseteq J} M_{n,K}^{\text{cyc}}
\]
and
\[ M_{n,J}^{\text{cyc}} = \sum_{K \supset J} (-1)^{\#(K \setminus J)} F_{n,K}^{\text{cyc}}. \]

**Proof.** For each subset \( J \subseteq \{1, \ldots, n\} \) denote by \( Q_{n,J}^{\text{cyc}} \) the set of all pairs \((w, k)\) consisting of a word \( w = (w_1, \ldots, w_n) \in \mathbb{P}^n \) and an index \( k \in \{1, \ldots, n\} \) which satisfy

(i) The word \( w \) is “cyclically weakly increasing” from index \( k \), namely \( w_k \leq w_{k+1} \leq \cdots \leq w_n \leq w_1 \leq \cdots \leq w_{k-1} \).

(ii) \( k-1 \in J \) and, for \( j \in \{1, \ldots, n\} \setminus \{k-1\} \): \( j \in J \setminus \{k-1\} \iff w_j < w_{j+1} \), where indices are computed modulo \( n \). (Thus \( Q_{n,J}^{\text{cyc}} = \emptyset \) for \( J = \emptyset \).

Definition \ref{def:n_j} can now be written in the form
\[ M_{n,J}^{\text{cyc}} = \sum_{(w,k) \in Q_{n,J}^{\text{cyc}}} x_{w_1}x_{w_2} \cdots x_{w_n} \quad (\forall J \subseteq \{1, \ldots, n\}). \]

The sets \( P_{n,J}^{\text{cyc}} \) from Definition \ref{def:p_n_j} are clearly disjoint unions
\[ P_{n,J}^{\text{cyc}} = \bigcup_{K \supset J} Q_{n,K}^{\text{cyc}} \quad (\forall J \subseteq \{1, \ldots, n\}) \]
and thus, by Definition \ref{def:m_n_k}
\[ F_{n,J}^{\text{cyc}} = \sum_{K \supset J} M_{n,K}^{\text{cyc}} \quad (\forall J \subseteq \{1, \ldots, n\}). \]

The other claim follows by the principle of inclusion-exclusion. \( \square \)

**Proposition 2.14.** (From fundamental to cyclic fundamental) For any subset \( J \subseteq \{1, \ldots, n\} \)
\[ F_{n,J}^{\text{cyc}} = \sum_{i \in \{1, \ldots, n\}} F_{n, (J-i) \cap \{1, \ldots, n\}}, \]
where \( J - i := \{k - i : k \in J\} \subseteq \{1, \ldots, n\} \) with subtraction interpreted cyclically modulo \( n \).

**Proof.** For any \( J \subseteq \{1, \ldots, n\} \), by Lemma \ref{lem:f_n_j} and Lemma \ref{lem:m_n_k}
\[ F_{n,J}^{\text{cyc}} = \sum_{J \subseteq K \subseteq \{1, \ldots, n\}} M_{n,K}^{\text{cyc}} = \sum_{J \subseteq K \subseteq \{1, \ldots, n\}} \sum_{i \in K} M_{n,(K-i) \cap \{1, \ldots, n\}} = \sum_{i \in \{1, \ldots, n\}} \sum_{J \cup \{i\} \subseteq K \subseteq \{1, \ldots, n\}} M_{n,(K-i) \cap \{1, \ldots, n\}}. \]

Denoting \( K' := K - i \) and \( K'' := K' \cap \{1, \ldots, n\} \), it follows that
\[ F_{n,J}^{\text{cyc}} = \sum_{i \in \{1, \ldots, n\}} \sum_{(J-i) \cap \{1, \ldots, n\} \subseteq K' \subseteq \{1, \ldots, n\}} M_{n,K''} = \sum_{i \in \{1, \ldots, n\}} \sum_{(J-i) \cap \{1, \ldots, n\} \subseteq K'' \subseteq \{1, \ldots, n\}} F_{n,(J-i) \cap \{1, \ldots, n\}}. \]

**Example 2.15.** For the functions in Example \ref{ex:example}
\[ F_{5,\{1,3,5\}}^{\text{cyc}} = F_{5,\{1,3\}} + F_{5,\{2,4\}} + F_{5,\{1,3,4\}} + F_{5,\{2,3\}} + F_{5,\{1,2,4\}} \]
and
\[ F_{6,\{2,4,6\}}^{\text{cyc}} = 3F_{6,\{2,4\}} + 3F_{6,\{1,3,5\}}. \]

**Lemma 2.16.**

(1) \( F_{n,\emptyset}^{\text{cyc}} = nF_{n,\emptyset} = nh_n \) and \( F_{n,\{1, \ldots, n\}}^{\text{cyc}} = nF_{n,\{1, \ldots, n\}} = ne_n \) are symmetric functions.

(2) **Cyclic invariance:** If \( J' \in \mathcal{P} \) is a cyclic shift of \( J \in \mathcal{P} \) then \( F_{n,J'}^{\text{cyc}} = F_{n,J}^{\text{cyc}}. \)

(3) **Linear dependence:** \( \sum_{J \subseteq \{1, \ldots, n\}} (-1)^{\#J} F_{n,J}^{\text{cyc}} = 0. \)

**Proof.** (1) and (2) are immediate from Proposition \ref{prop:proposition} (3) follows from the second formula in Lemma \ref{lem:m_n_k} for \( J = \emptyset \), since \( M_{n,\emptyset}^{\text{cyc}} = 0 \). \( \square \)
2.3. Normalization. In order to get a basis for the vector space \( cQSym_n \otimes \mathbb{Q} \), we need to take a suitable set of representatives of the functions defined in the previous subsections. In order to get an actual \( \mathbb{Z} \)-basis for \( cQSym_n \), we also need to normalize them. We recall and augment some of our previously used notations as follows: \( 2^{[n]} \) denotes the subsets of \( [n] = \{1, 2, \ldots, n\} \), and \( 2^{[n]}_0 \) denotes the nonempty subsets of \( [n] \), while \( c2^{[n]}_0 \) and \( c2^{[n]}_0 \) respectively, denote their collections of equivalence classes under cyclic shifting.

Definition 2.17. For any \( J \subseteq [n] \) let \( D_J := \{i \in \mathbb{Z}/n\mathbb{Z} : J+i \equiv J \pmod{n}\} \) be the stabilizer of \( J \) under the action of \( \mathbb{Z}/n\mathbb{Z} \) by cyclic shifts, and let \( d_J := |D_J| \). Define the normalized monomial cyclic quasi-symmetric function \( \hat{M}_{n,J} \) by

\[
\hat{M}_{n,J} := \frac{1}{d_J} M_{n,J}^{\text{cyc}} \quad (\forall J \subseteq [n]).
\]

For any orbit \( A \in c2^{[n]}_0 \) define

\[
M_{n,A}^{\text{cyc}} := M_{n,A}^{\text{cyc}}, \quad D_A := D_J, \quad d_A := d_J \quad \text{and} \quad \hat{M}_{n,A}^{\text{cyc}} := \hat{M}_{n,J}^{\text{cyc}},
\]

where \( J \) is any element of the orbit \( A \). By Observation 2.7 these are all well-defined (i.e., independent of the choice of \( J \) in \( A \)). Note also that, for each \( A \in c2^{[n]}_0 \), \( d_A \cdot |A| = n \), and therefore

\[
\hat{M}_{n,A}^{\text{cyc}} = \frac{1}{n} \sum_{J \in A} M_{n,J}^{\text{cyc}} \quad (\forall A \in c2^{[n]}_0).
\]

Example 2.18. For \( n = 6 \) and \( A = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\} \in c2^{[6]}_0 \), \( cc_n(A) = (3, 3) \) and

\[
M_{n,A}^{\text{cyc}} = 2 \sum_{i_1 < i_2} x_{i_1}^3 x_{i_2}^3.
\]

Since \( d_A = 2 \),

\[
\hat{M}_{n,A}^{\text{cyc}} = \sum_{i_1 < i_2} x_{i_1}^3 x_{i_2}^3.
\]

Lemma 2.19. \( \{ \hat{M}_{n,A}^{\text{cyc}} : A \in c2^{[n]}_0 \} \) is a \( \mathbb{Z} \)-basis for \( cQSym_n \).

Proof. It follows from Definition 2.4 that, for any \( A \in c2^{[n]}_0 \), the coefficient of each monomial in \( M_{n,A}^{\text{cyc}} \) is \( d_A \).

Therefore \( \hat{M}_{n,A}^{\text{cyc}} \in cQSym_n \) is a sum of monomials with all coefficients equal to 1. The set \( \{ \hat{M}_{n,A}^{\text{cyc}} : A \in c2^{[n]}_0 \} \) clearly \( \mathbb{Z} \)-spans \( cQSym_n \) and is linearly independent, since each monomial of degree \( n \) appears in \( \hat{M}_{n,A}^{\text{cyc}} \) for exactly one \( A \in c2^{[n]}_0 \). Note also that \( \hat{M}_{n,\{\emptyset\}}^{\text{cyc}} = M_{n,\{\emptyset\}}^{\text{cyc}} = 0 \).

Definition 2.20. For \( J \subseteq [n] \) let \( D_J \) and \( d_J \) be as in Definition 2.17. Define the normalized fundamental cyclic quasi-symmetric function \( \hat{F}_{n,J}^{\text{cyc}} \) by

\[
\hat{F}_{n,J}^{\text{cyc}} := \frac{1}{d_J} F_{n,J}^{\text{cyc}} \quad (\forall J \subseteq [n]).
\]

For any orbit \( A \in c2^{[n]}_0 \) define

\[
F_{n,A}^{\text{cyc}} := F_{n,A}^{\text{cyc}} \quad \text{and} \quad \hat{F}_{n,A}^{\text{cyc}} := \hat{F}_{n,J}^{\text{cyc}},
\]

where \( J \) is any element of the orbit \( A \). By Lemma 2.16(2), these are all well-defined (i.e., independent of the choice of \( J \in A \)). Also, as for the monomial functions,

\[
\hat{F}_{n,A}^{\text{cyc}} = \frac{1}{n} \sum_{J \in A} F_{n,J}^{\text{cyc}} \quad (\forall A \in c2^{[n]}_0).
\]

Lemma 2.21.

(1) \( \hat{F}_{n,\emptyset}^{\text{cyc}} = F_{n,\emptyset} = h_n \) and \( \hat{F}_{n,\{n\}}^{\text{cyc}} = F_{n,\{n-1\}} = e_n \) are symmetric functions.
(2) **Linear dependence:**

\[
\sum_{A \in c2^{[n]}} (-1)^{r(A)} \hat{F}^{\text{cyc}}_{n,A} = 0,
\]

where the rank of an orbit \( A \in c2^{[n]} \) is \( r(A) := |J| \), for any element \( J \in A \); the notation is meant to distinguish \( r(A) \) from the size \#A of the orbit \( A \).

(3) For any \( C \in c2^{[n]} \), the set \( B_C := \{ \hat{F}^{\text{cyc}}_{n,A} : A \in c2^{[n]} \setminus \{C\} \} \) is a \( \mathbb{Z} \)-basis for \( c\text{QSym}_n \). In particular, for \( C = [\emptyset] \), \( B_0 := \{ \hat{F}^{\text{cyc}}_{n,A} : A \in c2^{[n]}_0 \} \) is a \( \mathbb{Z} \)-basis for \( c\text{QSym}_n \).

**Proof.** (1) follows immediately from Observation 2.16(1) and Definition 2.20 since \( d_{\emptyset} = d_[n] = n \).

(2) Write Lemma 2.16(3) in the form

\[
\sum_{J \subseteq [n]} (-1)^{|J|} d_J \hat{F}^{\text{cyc}}_{n,J} = 0,
\]

or equivalently

\[
\sum_{A \in c2^{[n]}} (-1)^{r(A)} \#A \cdot d_A \hat{F}^{\text{cyc}}_{n,A} = 0.
\]

Clearly

\[
\#A \cdot d_A = n
\]

for any orbit \( A \in c2^{[n]} \), yielding the claimed formula.

(3) Note that, at this point, we know that \( \hat{F}^{\text{cyc}}_{n,A} \in c\text{QSym}_n \oplus \mathbb{Q} (\forall A \in c2^{[n]} \) but not necessarily \( \hat{F}^{\text{cyc}}_{n,A} \in c\text{QSym}_n \). This is part of the claim, and will be proved below.

For \( A, B \in c2^{[n]} \) write \( A \leq B \) if there exist \( J \in A \) and \( K \in B \) such that \( J \subseteq K \) (equivalently, for any \( J \in A \) there exists \( K \in B \) such that \( J \subseteq K \); equivalently, for any \( K \in B \) there exists \( J \in A \) such that \( J \subseteq K \)). It is easy to see that \( \leq \) is a partial order on \( c2^{[n]} \). For \( J, K \subseteq [n] \), let \( d_{J,K} := \#D_{J,K} \) where:

\[
D_{J,K} := \{ i \in \mathbb{Z}/n\mathbb{Z} : J \subseteq K + i \} = \{ i \in \mathbb{Z}/n\mathbb{Z} : J - i \subseteq K \}.
\]

The number \( d_{J,K} \) is invariant under cyclic shifts of either \( J \) or \( K \), and can therefore be used to define \( d_{A,B} \) for any \( A, B \in c2^{[n]} \). Clearly, \( d_{A,B} > 0 \iff A \leq B \). We claim that \( d_{A,B} \) is divisible by both \( d_A \) and \( d_B \); if one fixes representatives \( J \) and \( K \) for \( A \) and \( B \), respectively, then one has

(i) \( \frac{d_{A,B}}{d_A} = \#\{ K' \in B : J \subseteq K' \} \), as \( i \mapsto K + i \) gives a \( d_B \)-to-1 surjection \( D_{J,K} \to \{ K' \in B : J \subseteq K' \} \),

(ii) \( \frac{d_{A,B}}{d_B} = \#\{ J' \in A : J' \subseteq K \} \), as \( i \mapsto J - i \) gives a \( d_A \)-to-1 surjection \( D_{J,K} \to \{ J' \in A : J' \subseteq K \} \).

Then the first formula in Lemma 2.13 provides the second equality in the following rewriting:

\[
F^{\text{cyc}}_{n,A} = F^{\text{cyc}}_{n,J} = \sum_{K' \subseteq [n], J \subseteq K} M^{\text{cyc}}_{n,K'} = \sum_{B \geq A} M^{\text{cyc}}_{n,B} \cdot \#\{ K' \in B : J \subseteq K' \} = \sum_{B \geq A} \frac{d_{A,B}}{d_B} M^{\text{cyc}}_{n,B}
\]

where the last equality uses (i) above. Dividing the far left and right sides of (2.3) by \( d_A \) gives:

\[
\hat{F}^{\text{cyc}}_{n,A} = \sum_{B \geq A} \frac{d_{A,B}}{d_A} \hat{M}^{\text{cyc}}_{n,B}.
\]

An analogous rewriting using the second formula in Lemma 2.13 leads to the following:

\[
\hat{M}^{\text{cyc}}_{n,A} = \sum_{B \geq A} (-1)^{r(B) - r(A)} \frac{d_{A,B}}{d_A} \hat{F}^{\text{cyc}}_{n,B}.
\]

It now follows from Equation (2.4) together with Lemma 2.19 that \( \hat{F}^{\text{cyc}}_{n,A} \in c\text{QSym}_n \) for each \( A \in c2^{[n]} \). Furthermore, Equation (2.5) and Lemma 2.19 imply that \( \{ \hat{F}^{\text{cyc}}_{n,A} : A \in c2^{[n]} \} \) spans \( c\text{QSym}_n \) over \( \mathbb{Z} \). Since \( c\text{QSym}_n \) is a torsion-free abelian group of rank \( \#c2^{[n]} \), by Lemma 2.19 the linear dependence (2) above completes the proof. \( \square \)
Example 2.22. The matrix whose columns expand \( \{ F_{n,J}^{cyc} \} \) in terms of \( \{ M_{n,K}^{cyc} \} \) is given below for \( n = 2, 3 \) and 4, indexing rows and columns by representatives of the equivalence classes in \( c_{20}^{[n]} \).

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

Here are the same matrices for \( \{ d_J^{-1} F_{n,J}^{cyc} \} \) and \( \{ d_K^{-1} M_{n,K}^{cyc} \} \):

\[
\begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 3 & 3 \\
0 & 1 & 2 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 4 & 2 & 4 & 4 \\
0 & 1 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Remark 2.23. The degree 0 homogeneous component \( cQSym_0 \) is, of course, isomorphic to \( \mathbb{Z} \). We formally define \( F_0^{cyc} = M_0^{cyc} = 1 \) to get monomial and fundamental bases for it.

2.4. The non-Escher subring \( cQSym^- \). For many combinatorial applications, in which the non-Escher property of cyclic descent sets (see Definition 4.2) is manifest, it is natural to consider a certain subgroup \( cQSym^- \) of \( cQSym_n \).

Definition 2.24. For any positive integer \( n \), let \( 2_{n,0}^{[n]} \) be the set of all subsets of \( [n] \) other than the empty set and \( [n] \) itself. Let \( c_{20}^{[n]} \) be the set of equivalence classes of elements of \( 2_{n,0}^{[n]} \) under cyclic shifts. Define,
for each \( n \geq 1 \),
\[
c\text{QSym}_n^- := \text{span}_\mathbb{Z} \left\{ \widehat{F}_{n,A}^{\text{cyc}} : A \in \mathbb{Z}_0^{[n]} \right\} = \text{span}_\mathbb{Z} \left\{ \widehat{F}_{n,J}^{\text{cyc}} : \emptyset \subseteq J \subseteq [n] \right\}.
\]

By Lemma 2.21 \( c\text{QSym}_n^- \) is a free abelian subgroup of corank one in \( c\text{QSym}_n \). Define
\[
c\text{QSym}^- := \bigoplus_{n \geq 0} c\text{QSym}_n^-,
\]
where \( c\text{QSym}_0^- := \mathbb{Z} \).

**Observation 2.25.** For each \( n \geq 1 \), the set \( \{ \widehat{F}_{n,A}^{\text{cyc}} : A \in \mathbb{Z}_0^{[n]} \} \) is a \( \mathbb{Z} \)-basis for \( c\text{QSym}_n^- \).

In fact, \( c\text{QSym} \) is a ring and \( c\text{QSym}^- \) a subring; see Proposition 3.25. We call \( c\text{QSym}^- \) the *non-Escher subring* of \( c\text{QSym} \).

### 3. Toric posets and cyclic \( P \)-partitions

In this section we take a little excursion into cyclic analogues of posets that were called *toric posets* [8], and develop a theory of cyclic \( P \)-partitions for them; in particular, a cyclic analogue of Stanley’s fundamental decomposition lemma for \( P \)-partitions is provided. Just as the fundamental quasi-symmetric functions \( F_{n,j} \) are \( P \)-partition enumerators for certain (labeled) total orders, the fundamental cyclic quasi-symmetric functions \( F_{n,j}^{\text{cyc}} \) will be shown to be cyclic \( P \)-partition enumerators for certain (labeled) total cyclic orders. This will be used to prove that \( c\text{QSym}^- \) is a ring and to study its structure constants.

#### 3.1. Ordinary \( P \)-partitions for directed acyclic graphs

We first review here \( P \)-partitions for a labeled poset, rephrased very slightly in terms of a directed acyclic graph. Although this rephrasing is trivial, it is better for thinking later about the cyclic/toric version.

**Definition 3.1.** A *directed acyclic graph* (DAG) on \( \{1,2,\ldots,n\} \) is a subset \( \overrightarrow{D} \) of the cartesian product \( \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} \), or a binary relation (written either as \( i \rightarrow j \) in \( \overrightarrow{D} \), or \( i \overset{\overrightarrow{D}}{\to} j \)) containing no directed cycles \( i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k-1 \rightarrow i_k = i_1 \) with \( k > 1 \). In particular, it is

- *antisymmetric*, that is, one cannot have both \( i \rightarrow j \) and \( j \rightarrow i \) in \( \overrightarrow{D} \), and
- *irreflexive*, that is, one cannot have \( i \rightarrow i \) in \( \overrightarrow{D} \).

When one has two DAGs \( \overrightarrow{D}_1, \overrightarrow{D}_2 \) (on the same set of vertices) with \( \overrightarrow{D}_1 \subseteq \overrightarrow{D}_2 \), then one says that \( \overrightarrow{D}_2 \) *extends* \( \overrightarrow{D}_1 \).

A DAG \( \overrightarrow{P} \) is *transitive* if \( i \rightarrow j \) and \( j \rightarrow k \) in \( \overrightarrow{P} \) implies \( i \rightarrow k \) in \( \overrightarrow{P} \). Transitive DAGs are called *posets*.

The *transitive closure* \( \overrightarrow{P} \) of a DAG \( \overrightarrow{D} \) is the poset extending \( \overrightarrow{D} \) obtained from \( \overrightarrow{D} \) by adding in \( i_1 \rightarrow i_k \) whenever one has a chain \( i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{k-1} \rightarrow i_k \) in \( \overrightarrow{D} \).

A poset \( \overrightarrow{P} \) is called a *total* (or *linear*) *order* if for every pair \( (i,j) \) with \( 1 \leq i < j \leq n \), it contains either \( i \rightarrow j \) or \( j \rightarrow i \). In other words, \( \overrightarrow{P} \) is a *transitive tournament*. It is easily seen that this means that \( \overrightarrow{P} = \overrightarrow{w} \) is the transitive closure of the DAG having \( w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_n \) for some (unique) permutation \( w = (w_1,w_2,\ldots,w_n) \) in the symmetric group \( \mathfrak{S}_n \).

Denote by \( \mathcal{L}(\overrightarrow{D}) \) the set of all permutations \( w \) in \( \mathfrak{S}_n \) for which \( \overrightarrow{w} \) extends \( \overrightarrow{D} \).

**Definition 3.2.** A *\( \overrightarrow{D} \)-partition* is a function \( f : \{1,2,\ldots,n\} \to \{0,1,2,\ldots\} \) for which

- \( f(i) \leq f(j) \) whenever \( i \rightarrow j \) in \( \overrightarrow{D} \), and
- \( f(i) < f(j) \) whenever \( i \rightarrow j \) in \( \overrightarrow{D} \), but \( i >_\mathbb{Z} j \).

Denote by \( \mathcal{A}(\overrightarrow{D}) \) the set of all \( \overrightarrow{D} \)-partitions \( f \).

**Lemma 3.3.** (*Fundamental lemma of \( \overrightarrow{D} \)-partitions [30] Lemma 3.15.3*) For any DAG \( \overrightarrow{D} \), one has a decomposition of \( \mathcal{A}(\overrightarrow{D}) \) as the following disjoint union:
\[
\mathcal{A}(\overrightarrow{D}) = \bigsqcup_{w \in \mathcal{L}(\overrightarrow{D})} \mathcal{A}(\overrightarrow{w}).
\]

The following proposition explains why theory of DAGs is the same as that of posets.
Proposition 3.4. If $\tilde{D}_2$ extends $\tilde{D}_1$ then one has inclusions

\[ \mathcal{L}(\tilde{D}_2) \subseteq \mathcal{L}(\tilde{D}_1), \]
\[ \mathcal{A}(\tilde{D}_2) \subseteq \mathcal{A}(\tilde{D}_1), \]

with equality in both cases if $\tilde{D}_2$ is the transitive closure of $\tilde{D}_1$.

3.2. Toric DAGs, toric posets, and toric P-partitions. The toric case is mildly trickier, because one must consider the following equivalence relation on DAGs.

Definition 3.5. Say that a DAG $D$ has $i_0$ in $\{1, 2, \ldots, n\}$ as a source (respectively, sink) if $D$ contains no arrows/relations of the form $j \rightarrow i_0$ (respectively, of the form $i_0 \rightarrow j$).

Say that $D'$ is obtained from $D$ by an elementary equivalence or flip at $i_0$ if $i_0$ is either a source or sink of $D$ and one obtains $D'$ by reversing all the arrows in $D$ incident with $i_0$. Define the equivalence relation $\equiv$ on DAGs to be the reflexive-transitive closure of the elementary equivalences, that is, $D \equiv D'$ if and only if there exists a (possibly empty) sequence of flips one can apply starting with $D$ to obtain $D'$.

Definition 3.6. A toric DAG is the $\equiv$-equivalence class $[\tilde{D}]$ of a DAG $\tilde{D}$.

Example 3.7. Here is an example of a toric DAG $[\tilde{D}_1]$:

Here is another toric DAG $[\tilde{D}_2]$:

Definition 3.8. Say that $[\tilde{D}_2]$ torically extends $[\tilde{D}_1]$ if there exist $\tilde{D}' \in [\tilde{D}_1]$ for $i = 1, 2$ with $\tilde{D}' \subseteq \tilde{D}_2$.

A certain toric extension, called the toric transitive closure, will be particularly important.

Definition 3.9. Say that $i \rightarrow j$ is implied from toric transitivity in a DAG $\tilde{D}$ if there exist in $\tilde{D}$ both a chain $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$ and a direct arrow $i_1 \rightarrow i_k$ such that $i = i_a, j = i_b$ for some $1 \leq a < b \leq k$.

The toric transitive closure of $\tilde{D}$ is the DAG $\tilde{P}$ obtained by adding in all arrows $i \rightarrow j$ implied from toric transitivity in $\tilde{D}$; see Figure 2 where the existence of all the solid arcs in $\tilde{D}$ implies the existence of all the dotted arcs in its toric transitive closure. Note that the toric transitive closure of $\tilde{D}$ is a subset of its usual transitive closure.

A DAG $\tilde{D}$ is toric transitively closed if it equals its toric transitive closure.

Proposition 3.10. If $\tilde{D}_1 \equiv \tilde{D}_2$, then $\tilde{D}_1$ is toric transitively closed if and only if the same is true for $\tilde{D}_2$.

Proof. It suffices to check this when $\tilde{D}_1, \tilde{D}_2$ differ by a flip at some node $j$. Note that for every set of vertices $\{i_1, \ldots, i_k\}$ for which $\tilde{D}_1$ contains all the solid arcs shown in Figure 2 it will contain all of the dotted arcs before the flip at node $j$ if and only if $\tilde{D}_2$ contains the corresponding arcs after the flip; this is obvious if $j \notin \{i_1, \ldots, i_k\}$, and straightforward to check when $j \in \{i_1, \ldots, i_k\}$. □

1It is not hard to see, e.g., from [5, Prop. 4.2], that the toric transitive closure of any DAG will already be toric transitively closed. That is, one does not need to iterate the closure procedure to reach a DAG which is toric transitively closed.
**Definition 3.11.** A toric DAG \([\vec{D}]\) is a toric pose\(^2\) if \([\vec{D}]\) is toric transitively closed for one of its \(\equiv\)-class representatives \(\vec{D}\), or equivalently, by Proposition 3.10, for all such representatives \(\vec{D}\).

**Definition 3.12.** A total cyclic order is a toric poset \([\vec{w}]\) for some \(w = (w_1, \ldots, w_n)\) in \(\mathfrak{S}_n\), that is,
\[
[\vec{w}] = \{(w_1, w_2, \ldots, w_{n-1}, w_n),
(w_2, w_3, \ldots, w_n, w_1),
\vdots
(w_n, w_1, \ldots, w_{n-2}, w_{n-1})\}.
\]
In other words, a total cyclic order is a toric poset with at least one (equivalently, all) of its \(\equiv\)-class representatives being a total (linear) order.

Denote by \(L^{\text{tor}}([\vec{D}])\) the set of all total cyclic orders \([\vec{w}]\) which torically extend \([\vec{D}]\).

**Remark 3.13.**
1. A total cyclic order \([\vec{w}]\) may be viewed as a coset \(w\mathbb{Z}_n \in \mathfrak{S}_n / \mathbb{Z}_n\), where \(\mathbb{Z}_n\) is the subgroup of \(\mathfrak{S}_n\) generated by the \(n\)-cycle \((2,3,\ldots,n,1)\).
2. Total cyclic orders may be identified with \(n\)-cycles in \(\mathfrak{S}_n\).
3. Total cyclic orders may be geometrically visualized as \(n\) dots in a directed cycle labeled by \(1,\ldots,n\) with no repeats. These configurations are called cyclic permutations, and will be used in the study of cyclic shuffles, see Figure 3.

**Definition 3.14.** A toric \([\vec{D}]\)-partition is a function \(f : \{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots\}\) which is a \(\vec{D}'\)-partition for at least one DAG \(\vec{D}'\) in \([\vec{D}]\). Let \(A^{\text{tor}}([\vec{D}])\) denote the set of all toric \([\vec{D}]\)-partitions.

**Lemma 3.15.** For any DAG \(\vec{D}\), one has a decomposition of \(A^{\text{tor}}([\vec{D}])\) as the following disjoint union:
\[
A^{\text{tor}}([\vec{D}]) = \bigcup_{[\vec{w}] \in L^{\text{tor}}([\vec{D}])} A^{\text{tor}}([\vec{w}]).
\]

*Proof.* The assertion about disjointness already follows, since by definition, one has
\[
A^{\text{tor}}([\vec{w}]) = \bigcup_{w' \in [\vec{w}]} \mathcal{A}(\vec{w}'),
\]
\[\text{This isn’t quite the way that it was defined in } [8], \text{ but essentially equivalent, via } [8] \text{ Theorem 1.4].\]
and this union is disjoint by Lemma 3.3. For the union assertion, one has these equalities, justified below:

$$\mathcal{A}^\mathrm{tor}(\lceil \vec{D} \rceil) = (i) \bigcup_{\vec{D}' \in \lceil \vec{D} \rceil} \mathcal{A}(\vec{D}') \subseteq (ii) \bigcup_{\vec{D}' \in \lceil \vec{D} \rceil} \bigcup_{w' \in \mathcal{L}(\vec{D}')} \mathcal{A}(\vec{w}')$$

Equality (i) is the definition of $$\mathcal{A}^\mathrm{tor}(\lceil \vec{D} \rceil)$$, equality (ii) is Lemma 3.3, and equality (iv) uses 3.1 again.

For equality (iii), one needs to show that $$w' \in \mathcal{L}(\vec{D}')$$ for some $$\vec{D}' \in \lceil \vec{D} \rceil$$ if and only if one has $$\vec{w}' \in \lceil \vec{w} \rceil$$ for some $$\lceil \vec{w} \rceil \in \mathcal{L}^\mathrm{tor}(\lceil \vec{D} \rceil)$$. The forward implication is straightforward: if $$w' \in \mathcal{L}(\vec{D}')$$ for some $$\vec{D}' \in \lceil \vec{D} \rceil$$, then $$\vec{w}' \supseteq \vec{D}'$$, so that $$\lceil \vec{w}' \rceil$$ torically extends $$\lceil \vec{D}' \rceil = \lceil \vec{D} \rceil$$.

For the reverse implication, assuming that $$\vec{w}' \in \lceil \vec{w} \rceil \in \mathcal{L}^\mathrm{tor}(\lceil \vec{D} \rceil)$$, then there must exist $$\vec{D}'' \in \lceil \vec{D} \rceil$$ with $$\vec{D}'' \subseteq \vec{w}$$, and the same sequence of flips will take $$\vec{D}''$$ to some $$\vec{D}' \subseteq \vec{w}'$$. One then has $$\vec{D}' \in \lceil \vec{D}'' \rceil = \lceil \vec{D} \rceil$$ and $$w' \in \mathcal{L}(\vec{D}')$$, as desired. \(\square\)

### 3.3. Cyclic \(P\)-partition enumerators.

**Definition 3.16.** Given a toric poset $$\lceil \vec{D} \rceil$$ on \(\{1, 2, \ldots, n\}\), define its cyclic \(P\)-partition enumerator

$$F^\mathrm{cyc}_{\lceil \vec{D} \rceil} := \sum_{f \in \mathcal{A}^\mathrm{tor}(\lceil \vec{D} \rceil)} x_{f(1)}x_{f(2)} \cdots x_{f(n)}.$$ 

A special case of these enumerators are the fundamental cyclic quasi-symmetric functions, defined in Subsection 2.2. Recall the notations $$P^\mathrm{cyc}_{n,J}$$ and $$F^\mathrm{cyc}_{n,J}$$. 

**Proposition 3.17.** If $$w \in \mathcal{S}_n$$ has $$c\text{Des}(w) = J$$, then

$$\mathcal{A}^\mathrm{tor}(\lceil \vec{w} \rceil) = \{f \in \mathbb{P}^n : \exists i \ (f, i) \in P^\mathrm{cyc}_{n,J}\}.$$ 

Consequently,

$$F^\mathrm{cyc}_{\lceil \vec{w} \rceil} = F^\mathrm{cyc}_{n,J}.$$ 

In particular, this shows that $$F^\mathrm{cyc}_{\lceil \vec{w} \rceil}$$ lies in $$\mathcal{CQSym}^-$$.

**Proof.** By definition, $$\mathcal{A}^\mathrm{tor}(\lceil \vec{w} \rceil) = \bigcup_{w' \in \lceil \vec{w} \rceil} \mathcal{A}(\vec{w}')$$. It is not hard to see that if $$w = (w_1, \ldots, w_n)$$ and $$w' = (w_1, w_{i+1}, \ldots, w_n, w_1, w_2, \ldots, w_{i-1})$$, then

$$\mathcal{A}(\vec{w}') = \{f \in \mathbb{P}^n : (f, i) \in P^\mathrm{cyc}_{n,J}\}.$$ 

\(\square\)

An immediate consequence of Lemma 3.15 is then the following.

**Proposition 3.18.** For any toric poset $$\lceil \vec{D} \rceil$$, one has the following expansion

$$F^\mathrm{cyc}_{\lceil \vec{D} \rceil} = \sum_{\lceil \vec{w} \rceil \in \mathcal{L}^\mathrm{tor}(\lceil \vec{D} \rceil)} F^\mathrm{cyc}_{n, c\text{Des}(w)}.$$ 

In particular, this shows that $$F^\mathrm{cyc}_{\lceil \vec{D} \rceil}$$ lies in $$\mathcal{CQSym}^-$$.

We now use this fact to expand products of of basis elements $$\{F^\mathrm{cyc}_{n,J}\}$$ back in the same basis. The key notion is that of a cyclic shuffle of two total cyclic orders.

First recall the notion of shuffles of permutations. For a finite set $$A$$ of size $$a$$, let $$\mathcal{S}_A$$ be the set of all bijections $$w : [a] \to A$$, viewed as words $$w = (w_1, \ldots, w_a)$$. If $$A = [a]$$ then $$\mathcal{S}_A$$ is of course the symmetric
Theorem 3.22. F is a cyclic quasi-symmetric function.

Proof. Consider the toric poset $\mathcal{D}$ where $\mathcal{D}$ is the disjoint union of the posets $\mathcal{U}$ and $\mathcal{V}$, and its associated cyclic quasi-symmetric function $F_{[\mathcal{D}]}$. E.g., if $u, v$ are as in Example 3.21, then $[\mathcal{D}]$ is represented by this $\mathcal{D}$.

\[ F_{[A],\text{cDes}(u)} \cdot F_{[B],\text{cDes}(v)} = \sum_{[w] \in [\mathcal{U}] \cup_{\text{cyc}} [\mathcal{V}]} F_{[C],\text{cDes}(w)}^\text{cyc} \]
On one hand, the definition of $F_{[D]}^{\text{cyc}}$ lets one express it as a product as follows:

\[(3.2) \quad F_{[D]}^{\text{cyc}} = F_{[u]}^{\text{cyc}} \cdot F_{[v]}^{\text{cyc}} = F_{|A|,\text{cDes}(u)}^{\text{cyc}} \cdot F_{|B|,\text{cDes}(v)}^{\text{cyc}}.\]

On the other hand, one can see from the definition that

\[\mathcal{L}^{|\text{tor}}(([D])) = [u] \sqcup \text{cyc } [v],\]

and hence Proposition 3.18 shows that

\[(3.3) \quad F_{[D]}^{\text{cyc}} = \sum_{[w] \in \mathcal{L}^{|\text{tor}}(([D]))} F_{[w]}^{\text{cyc}} = \sum_{[w] \in [u] \sqcup \text{cyc } [v]} F_{[C],\text{cDes}(w)}^{\text{cyc}}.\]

Equating the right hand sides of (3.2) and (3.3) gives the proposition. \qed

**Proposition 3.23.** Let $A$ and $B$ be disjoint sets of integers, of cardinalities $a$ and $b$ respectively. For each $u = (u_1, u_2, \ldots, u_a) \in \mathcal{S}_A$ and $v = (v_1, v_2, \ldots, v_b) \in \mathcal{S}_B$ there are $(a+b-1)!$ cyclic shuffles in $[u] \sqcup \text{cyc } [v]$.

**Proof.** The number of such cyclic shuffles $[w]$ is $b \times (a+b-1)!$.

This number is $b \cdot (a+b-1)! = \frac{(a+b-1)!}{(a-1)!b!}$. \qed

**Example 3.24.** Let’s compute $F_{3,\{1\}}^{\text{cyc}} \cdot F_{2,\{1\}}^{\text{cyc}}$ using Theorem 3.22. First pick $u \in \mathcal{S}_3$ and $v \in \mathcal{S}_{5}\setminus\{9\}$ such that $\text{cDes}(u) = \{1\}$ and $\text{cDes}(v) = \{1\}$, say $u = (3, 1, 2)$ and $v = (5, 4)$. Since $|A| = 3$ and $|B| = 2$, Proposition 3.23 implies that there are $\frac{5!}{3!2!} = 12$ cyclic shuffles $[w]$ in $[u] \sqcup \text{cyc } [v]$. These cyclic shuffles are listed below:

| $w \in [w]$ with $w_1 = 3$ | $\text{cDes}(w)$ |
|-----------------------------|------------------|
| $\{1, 4, 5\}$              | (1, 4, 5)        |
| $\{1, 5\}$                 | (1, 5)           |
| $\{1, 3, 5\}$              | (1, 3, 5)        |
| $\{2, 4\}$                 | (2, 4)           |
| $\{2, 3\}$                 | (2, 3)           |
| $\{3\}$                    | (3)              |
| $\{3, 5\}$                 | (3, 5)           |
| $\{3, 4\}$                 | (3, 4)           |
| $\{3, 4, 5\}$              | (3, 4, 5)        |

Consequently,

\[F_{3,\{1\}}^{\text{cyc}} \cdot F_{2,\{1\}}^{\text{cyc}} = F_{5,\{1,4,5\}}^{\text{cyc}} + 2F_{5,\{1,3,5\}}^{\text{cyc}} + F_{5,\{1,3,4\}}^{\text{cyc}} + 2F_{5,\{1,5\}}^{\text{cyc}} + 2F_{5,\{2,5\}}^{\text{cyc}} + 2F_{5,\{2,4\}}^{\text{cyc}} + F_{5,\{2,3\}}^{\text{cyc}} + F_{5,\{3\}}^{\text{cyc}} = F_{5,\{1,2,3\}}^{\text{cyc}} + 3F_{5,\{1,2,4\}}^{\text{cyc}} + 2F_{5,\{1,2,5\}}^{\text{cyc}} + 5F_{5,\{1,3\}}^{\text{cyc}} + F_{5,\{1\}}^{\text{cyc}}.\]

**Proposition 3.25.** $c\text{QSym}$ and $c\text{QSym}^-$ are graded rings.

**Proof.** By Lemma 2.21 in order to show that $c\text{QSym}$ is closed under multiplication it suffices to show that, for any two subsets $J \subseteq [a]$ and $K \subseteq [b]$, the product $F_{a,J}^{\text{cyc}} \cdot F_{b,K}^{\text{cyc}}$ lies in $c\text{QSym}_{a+b}$. Similarly, by Observation 2.25 in order to show that $c\text{QSym}^-$ is closed under multiplication it suffices to show that this product lies in $c\text{QSym}^-_{a+b}$ whenever both $J$ and $K$ are nonempty and proper, that is, $\emptyset \subsetneq J \subsetneq [a]$ and $\emptyset \subsetneq K \subsetneq [b]$.

Let us start from the latter claim. Indeed, any $\emptyset \subsetneq J \subsetneq [a]$ is the cyclic descent set of some $u \in \mathcal{S}_a$, and similarly for $\emptyset \subsetneq K \subsetneq [b]$ and $v \in \mathcal{S}_b$. By Theorem 3.22 $F_{a,J}^{\text{cyc}} \cdot F_{b,K}^{\text{cyc}}$ is a sum of terms of the form $F_{a+b,L}^{\text{cyc}}$ with $\emptyset \subsetneq L \subsetneq [a+b]$, and therefore belongs to $c\text{QSym}^-_{a+b}$. Normalizing, it follows that $F_{a,J}^{\text{cyc}} \cdot F_{b,K}^{\text{cyc}} \in c\text{QSym}^-_{a+b} \otimes \mathbb{Q}$. \qed
Since each of the two factors is a sum of monomials with integer coefficients, the same holds for the product, which is therefore genuinely in $c\text{QSym}_m$.

As for the somewhat larger $c\text{QSym}$, it remains to show that $\hat{F}_{a,[a]}^{\text{cyc}} \cdot f \in c\text{QSym}$ for every $a \geq 1$ and $f \in c\text{QSym}$; note that, because of the linear dependence in Lemma 2.21(2), there is no need to check $\hat{F}_{a,[0]}^{\text{cyc}} \cdot f$. Indeed, $\hat{F}_{a,[a]}^{\text{cyc}} = e_a$ and, for any sequence $i_1 < \cdots < i_t$ of indices and any sequence $m = (m_1, \ldots, m_t)$ of positive exponents,

$$
\left( \text{coefficient of } x_{i_1}^{m_1} \cdots x_{i_t}^{m_t} \text{ in } e_a \cdot f \right) = \sum_{A \subseteq \{i_1, \ldots, i_t\}} \left( \text{coefficient of } \prod_{j \in A} x_j \text{ in } f \right).
$$

Since $f$ is cyclic quasi-symmetric, this sum will be unchanged when we replace $m = (m_1, \ldots, m_t)$ by any cyclic shift $m' = (m'_1, \ldots, m'_t)$ and replace $i_1 < \cdots < i_t$ by any $i'_1 < \cdots < i'_t$.

**Corollary 3.26.** The structure constants of $c\text{QSym}$, with respect to the normalized fundamental basis, are nonnegative integers.

**Proof.** Nonnegativity follows from Theorem 3.22 and integrality follows from Proposition 3.25. □

An analogous statement holds for $c\text{QSym}$; see Proposition 4.8 below.

### 3.4. The involution $\omega$

Let $\pi_0$ be the longest permutation in the symmetric group $\mathfrak{S}_n$, defined by $\pi_0(i) := n + 1 - i$ ($1 \leq i \leq n$). Recall that the involution $\omega$ on $\mathfrak{S}_n$ defined by $\omega(\pi) := \pi_0 \pi$ ($\forall \pi \in \mathfrak{S}_n$) induces, via the mapping $\pi \mapsto \text{Des}(\pi)$, the involutions (also denoted $\omega$) $J \mapsto [n-1] \setminus J$ on $2^{[n-1]}$, $F_{n,J} \mapsto F_{n,[n-1]\setminus J}$ on $c\text{QSym}$, and, consequently, $s_\lambda \mapsto s_{\lambda'}$ on $c\text{QSym}$ (see §7.6 and Theorem 7.14.5). In fact, $\omega$ is a ring automorphism of $\text{QSym}$; see [22 Corollary 2.4] and [31 Ex. 7.94a].

**Observation 3.27.** $c\text{QSym}$ and $c\text{QSym}^-$ are invariant under the involution $\omega$. Explicitly,

$$
\omega \left( \hat{F}_{n,J}^{\text{cyc}} \right) = \hat{F}_{n,[n]\setminus J}^{\text{cyc}} \quad (\forall J \subseteq [n]).
$$

The corresponding restrictions of $\omega$ are ring automorphisms of both $c\text{QSym}$ and $c\text{QSym}^-$.  

**Proof.** By Proposition 2.14 for any $J \subseteq [n]$

$$
\omega \left( F_{n,J}^{\text{cyc}} \right) = \omega \left( \sum_{i \in [n]} F_{n,(J-i) \cap [n-1]} \right) = \sum_{i \in [n]} F_{n,[n-1]\setminus(J-i)} = \sum_{i \in [n]} F_{n,([n]\setminus(J-i)) \cap [n-1]} = F_{n,[n]\setminus J}^{\text{cyc}}.
$$

Since $d_J = d_{[n]\setminus J}$, the same claim holds also for the normalized functions $\hat{F}_{n,J}^{\text{cyc}}$.

Finally, since $\omega$ is an automorphism of $\text{QSym}$, it is also an automorphism of each of its invariant subrings $c\text{QSym}$ and $c\text{QSym}^-$. □

### 4. Expansion of Schur functions in terms of fundamental cyclic quasi-symmetric functions

Recall that $\text{QSym}$, $c\text{QSym}$ and the set $\text{Sym}$ of symmetric functions (sometimes denoted $\Lambda$) are graded rings satisfying

$$
\text{Sym} \subseteq c\text{QSym} \subseteq \text{QSym}.
$$

In this section we prove the following theorem.

**Theorem 4.1.** For every skew shape $\lambda/\mu$ which is not a connected ribbon, all the coefficients in the expansion of the skew Schur function $s_{\lambda/\mu}$ in terms of normalized fundamental cyclic quasi-symmetric functions are nonnegative integers.

Theorem 4.1 follows from Corollary 4.6 below. The cyclic descent map on SYT of a given shape plays a key role in the proof; let us recall the relevant definition and main result from [2].
Definition 4.2 ([2 Definition 2.1]). Let \( \mathcal{T} \) be a finite set, equipped with an arbitrary map (called descent map) \( \text{Des} : \mathcal{T} \rightarrow 2^{[n-1]} \). A cyclic extension of \( \text{Des} \) is a pair \((c\text{Des}, p)\), where \( c\text{Des} : \mathcal{T} \rightarrow [n] \) is a map and \( p : \mathcal{T} \rightarrow \mathcal{T} \) is a bijection, satisfying the following axioms: for all \( T \in \mathcal{T} \):

- (extension) \( c\text{Des}(T) \cap [n-1] = \text{Des}(T) \),
- (equivariance) \( c\text{Des}(p(T)) = 1 + c\text{Des}(T) \),
- (non-Escher) \( \emptyset \subseteq c\text{Des}(T) \subseteq [n] \).

Example 4.3. Let \( \mathcal{T} \) be \( \mathfrak{S}_n \), the symmetric group on \( n \) letters. The classical descent set of a permutation \( \pi = (\pi_1, \ldots, \pi_n) \in \mathfrak{S}_n \) is

\[
\text{Des}(\pi) := \{1 \leq i \leq n - 1 : \pi_i > \pi_{i+1}\} \subseteq [n-1],
\]

where \( [m] := \{1, 2, \ldots, m\} \). Its cyclic descent set, as defined by Cellini [6], is

\[
(4.1) \quad c\text{Des}(\pi) := \{1 \leq i \leq n : \pi_i > \pi_{i+1}\} \subseteq [n],
\]

with the convention \( \pi_{n+1} := \pi_1 \). The pair \((c\text{Des}, p)\) satisfies the axioms of Definition 4.2 for \( p : \mathfrak{S}_n \rightarrow \mathfrak{S}_n \) defined by \( p(\pi) := \pi \circ c \), where \( c \) is the cyclic shift mapping \( i \) to \( i - 1 \) (mod \( n \)).

There is also an established notion (see, e.g., [31 p. 361]) of a descent set for a standard (Young) tableau \( T \) of skew shape \( \lambda/\mu \) (and size \( n \)):

\[
\text{Des}(T) := \{1 \leq i \leq n - 1 : i + 1 \text{ appears in a lower row of } T \text{ than } i\} \subseteq [n-1].
\]

For example, the following standard Young tableau \( T \) of shape \( \lambda/\mu = (4, 3, 2)/(1, 1) \) has \( \text{Des}(T) = \{2, 3, 5\} \):

\[
\begin{array}{ccc}
1 & 2 & 7 \\
3 & 5 \\
4 & 6
\end{array}
\]

For the special case of rectangular shapes, Rhoades [27] constructed a cyclic extension \((c\text{Des}, p)\) of \( \text{Des} \) satisfying the axioms of Definition 4.2 where \( p \) acts on \( \text{SYT} \) via Schützenberger’s jeu-de-taquin promotion operator. For almost all skew shapes there is a general existence result, as follows.

Theorem 4.4 ([2 Theorem 1.1]). Let \( \lambda/\mu \) be a skew shape with \( n \) cells. The descent map \( \text{Des} \) on \( \text{SYT}(\lambda/\mu) \) has a cyclic extension \((c\text{Des}, p)\) if and only if \( \lambda/\mu \) is not a connected ribbon. Furthermore, for all \( J \subseteq [n] \), all such cyclic extensions share the same cardinalities \( |c\text{Des}^{-1}(J)| \).

We shall now provide, in Theorem 4.5 and Corollary 4.6, cyclic analogues of the classical result [31 Theorem 7.19.7] (first proved in [14, Theorem 7]).

Theorem 4.5. For every skew shape \( \lambda/\mu \) of size \( n \), which is not a connected ribbon, and for any cyclic extension \((c\text{Des}, p)\) of \( \text{Des} \) on \( \text{SYT}(\lambda/\mu) \),

\[
n s_{\lambda/\mu} = \sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^\text{cyc}
\]

where \( s_{\lambda/\mu} \) is the Schur function indexed by \( \lambda/\mu \).

Proof. Denoting

\[
m^\text{cyc}(J) := \#\{T \in \text{SYT}(\lambda/\mu) : c\text{Des}(T) = J\} \quad (\forall J \subseteq [n]),
\]

it follows from Lemma 2.13 that

\[
\sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^\text{cyc} = \sum_{J \subseteq [n]} m^\text{cyc}(J) F_{n,j}^\text{cyc} = \sum_{J \subseteq [n]} m^\text{cyc}(J) \sum_{i \in [n]} F_{n,(J-i) \cap [n-1]}.
\]

The equivariance axiom from Definition 4.2 translates to cyclic invariance of the multiplicities \( m^\text{cyc} \):

\[
m^\text{cyc}(J - i) = m^\text{cyc}(J) \quad (\forall J \subseteq [n], i \in [n]).
\]

This, in turn, implies that

\[
\sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^\text{cyc} = \sum_{J \subseteq [n]} \sum_{i \in [n]} m^\text{cyc}(J - i) F_{n,(J-i) \cap [n-1]} = n \sum_{J \subseteq [n]} m^\text{cyc}(J) F_{n,J \cap [n-1]}.
\]
Distinguishing the cases \( n \not\in J \) and \( n \in J \), and changing notation accordingly, we can write
\[
\sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^{\text{cyc}} = n \sum_{J \subseteq [n-1]} \left( m_{\text{cyc}}^{\text{cyc}}(J) + m_{\text{cyc}}^{\text{cyc}}(J \cup \{n\}) \right) F_{n,J}.
\]
Denoting
\[
m(J) := \#\{ T \in \text{SYT}(\lambda/\mu) : \text{Des}(T) = J \} \quad (\forall J \subseteq [n-1]),
\]
the extension axiom from Definition 4.2 is equivalent to
\[
m(J) = m_{\text{cyc}}^{\text{cyc}}(J) + m_{\text{cyc}}^{\text{cyc}}(J \cup \{n\}) \quad (\forall J \subseteq [n-1]),
\]
which now gives
\[
\sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^{\text{cyc}} = n \sum_{J \subseteq [n-1]} m(J) F_{n,J}.
\]
The classical formula of Gessel [14, Theorem 7]
\[
s_{\lambda/\mu} = \sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)} = \sum_{J \subseteq [n-1]} m(J) F_{n,J}
\]
now completes the proof. \(\square\)

The following corollary shows that the coefficients in the expression of a skew Schur function in terms of the basis of normalized fundamental cyclic quasi-symmetric functions are equal to the corresponding fiber sizes of (any) cyclic descent map.

**Corollary 4.6.** Under the assumptions of Theorem 4.3,
\[
s_{\lambda/\mu} = \sum_{A \in c2^{[n]}_0} m_{\text{cyc}}(A) \hat{F}_{n,A}^{\text{cyc}}
\]
where
\[
m_{\text{cyc}}(A) := m_{\text{cyc}}^{\text{cyc}}(J) = \#\{ T \in \text{SYT}(\lambda/\mu) : c\text{Des}(T) = J \} \quad \left( \forall J \in A \in c2^{[n]}_0 \right).
\]
This number is independent of the choice of \( J \) in \( A \). In fact,
\[
m_{\text{cyc}}(A) = \frac{1}{\# A} \sum_{J \subseteq A} m_{\text{cyc}}^{\text{cyc}}(J) = \frac{1}{\# A} \#\{ T \in \text{SYT}(\lambda/\mu) : c\text{Des}(T) \in A \}.
\]

**Proof.** By Theorem 4.3 and the above notations, using the non-Escher axiom from Definition 4.2
\[
s_{\lambda/\mu} = \frac{1}{n} \sum_{T \in \text{SYT}(\lambda/\mu)} F_{n,c\text{Des}(T)}^{\text{cyc}} = \frac{1}{n} \sum_{\emptyset \subseteq J \subseteq [n]} m_{\text{cyc}}^{\text{cyc}}(J) F_{n,J}^{\text{cyc}} = \frac{1}{n} \sum_{A \in c2^{[n]}_0} \#A \cdot m_{\text{cyc}}^{\text{cyc}}(A) F_{n,A}^{\text{cyc}}.
\]
By Definition 2.20 and the equality \( \#A \cdot d_A = n \),
\[
s_{\lambda/\mu} = \frac{1}{n} \sum_{A \in c2^{[n]}_0} \#A \cdot d_A \cdot m_{\text{cyc}}^{\text{cyc}}(A) \hat{F}_{n,A}^{\text{cyc}} = \sum_{A \in c2^{[n]}_0} m_{\text{cyc}}^{\text{cyc}}(A) \hat{F}_{n,A}^{\text{cyc}},
\]
as claimed. \(\square\)

If \( \lambda/\mu \) is a connected ribbon then a corresponding cyclic descent map \( c\text{Des} \) does not exist, by Theorem 4.4. Nevertheless, one can combine the dual Jacobi-Trudi identity [31, Corollary 7.16.2] with Proposition 3.29 to get an expansion of \( s_{\lambda/\mu} \) as a linear combination of \( \hat{F}_{n,A}^{\text{cyc}} \ (A \in c2^{[n]}_0) \) with (not necessarily positive) integer coefficients. In the special case that \( \lambda/\mu \) is a hook, this result can be stated very explicitly using a simpler approach.

**Proposition 4.7.** For every \( 0 \leq k \leq n-1 \),
\[
n s_{(n-k,1^k)} = \sum_{J \subseteq [n]} (-1)^{|J|-k-1} F_{n,J}^{\text{cyc}}
\]
and
\[ s_{(n-k,1^k)} = \sum_{A \in \mathfrak{S}_n^{(n)}} (-1)^{r(A)}k-1 \tilde{F}_{n,A}^{\text{cyc}}. \]

Note that one of the summands is always \( \tilde{F}_{n,\{\emptyset\}}^{\text{cyc}} \notin \text{cQSym} \).

Proof. First recall from [1, Theorem 3.11(a)] that for every \( 1 \leq k \leq n - 1 \), the multisets of cyclic descent sets of SYT of shape \((n-k+1,1^k)/(1) = (1^k) \oplus (n-k)\) is the set of all subsets of \([n]\) of size \(k\), each of them appearing exactly once; namely,
\[(4.2) \quad \{c\text{Des}(T) : T \in \text{SYT}((1^k) \oplus (n-k))\} = \{J \subseteq [n] : |J| = k\}.
\]
Equivalently, in the notation of the proof of Theorem 4.5 for this shape:
\[
m^{\text{cyc}}(J) = \begin{cases} 1, & \text{if } |J| = k; \\ 0, & \text{otherwise} \end{cases} \quad (\forall J \subseteq [n]).
\]
Thus, by Theorem 4.5
\[(4.3) \quad n s_{(1^k) \oplus (n-k)} = \sum_{J \subseteq [n]} F_{n,J}^{\text{cyc}}.
\]
The argumentation above is valid for \(1 \leq k \leq n - 1\), but Formula (4.3) actually holds also for \(k = 0\) and \(k = n\), by Lemma 2.16(1). (Theorem 4.5 cannot be used in these cases, since the corresponding shapes \((n)\) and \((1^n)\) are connected ribbons.)

On the other hand, by the definition of (skew) Schur functions in terms of semistandard tableaux,
\[ s_{(1^k) \oplus (n-k)} = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})} \quad (1 \leq k \leq n - 1).
\]
Thus
\[ s_{(n-k,1^k)} = \sum_{j=k+1}^n (-1)^{j-k-1} s_{(1^j) \oplus (n-j)} \quad (0 \leq k \leq n - 1).
\]
Applying Equation (4.3) to the RHS completes the proof. \(\square\)

We are now in a position to extend Corollary 3.26 to \text{cQSym}.

**Proposition 4.8.** The structure constants of \text{cQSym}, with respect to the normalized fundamental basis, are nonnegative integers.

Proof. We already know that the structure constants are integers. Corollary 3.26 shows that \( \hat{F}_{a,a}^{\text{cyc}} \cdot \hat{F}_{b,b}^{\text{cyc}} \) is a nonnegative linear combination of basis elements \( \hat{F}_{a+b,C}^{\text{cyc}} \) whenever \( A \in \mathfrak{S}_0^{[a+b]} \) and \( B \in \mathfrak{S}_0^{[b]} \). If \( A = \{[a]\} \) and \( B = \{[b]\} \) then, by Corollary 4.6,
\[
\hat{F}_{a,[a]}^{\text{cyc}} \cdot \hat{F}_{b,[b]}^{\text{cyc}} = e_a \cdot e_b = s_{(1^a) \oplus (1^b)} = \sum_{C \in \mathfrak{S}_0^{(a+b)}} m_{\text{cyc}}(C) \hat{F}_{a+b,C}^{\text{cyc}}
\]
where the skew shape \((1^a) \oplus (1^b)\), consisting of two disconnected columns \((1^a)\) and \((1^b)\), is not a connected ribbon and thus the coefficients
\[ m_{\text{cyc}}(C) = \frac{1}{\# C} \cdot \# \{T \in \text{SYT}((1^a) \oplus (1^b)) : c\text{Des}(T) \in C\}
\]
are clearly nonnegative. It remains to show nonnegativity when \( A = \{[a]\} \) but \( B \in \mathfrak{S}_0^{[b]} \). In that case, choose a bijective word \( w_0 \in \mathfrak{S}_{a+1,\ldots,a+b} \) such that \( c\text{Des}(w_0) \in B \). Then, by Proposition 2.14
\[
\hat{F}_{a,[a]}^{\text{cyc}} \cdot \hat{F}_{b,B}^{\text{cyc}} = F_{a,[a]} \cdot \frac{1}{d_B} \sum_{i \in [b]} F_{b,\text{Des}(w_0)^c}
\]
where \( c \in \mathfrak{S}_b \) maps each \( i \) to \( i + 1 \) \((\text{mod } b)\). According to the classical Theorem 3.19
\[
\hat{F}_{a,[a]}^{\text{cyc}} \cdot \hat{F}_{b,B}^{\text{cyc}} = \frac{1}{d_B} \sum_{i \in [b]} \sum_{w \in \mathfrak{S}_{a,a-1,\ldots,1}} F_{a+b,\text{Des}(w)^c}.
\]
Denoting
\[ W := \bigsqcup_{i \in [b]} (a, a-1, \ldots, 1) \sqcup w_0c^i, \]

it suffices to show that the set of bijective words \( W \), with the usual descent map \( \text{Des} : W \to [a + b - 1] \), has a cyclic extension in the sense of Definition 4.2.

Indeed, define a map \( \text{cDes}^* : W \to [a + b] \) as a slight deformation of Cellini’s \( \text{cDes} \), defined in (4.1):
\[
\text{cDes}^*(w) := \begin{cases} 
\text{Des}(w) \sqcup \{a + b\}, & \text{if } w(a + b) > w(1) \text{ or } w(1) = a; \\
\text{Des}(w), & \text{otherwise } \quad (\forall w \in W)
\end{cases}
\]

and define \( p(w) \) as the word obtained from \( w \) by cyclically shifting the letters which are bigger than \( a \) one position to the right modulo \( a + b \), and inserting the letters \( a, a-1, \ldots, 1 \) in the remaining positions in decreasing order. One can verify that \( p : W \to W \) is a bijection which shifts \( \text{cDes}^* \) cyclically, namely
\[
\text{cDes}^*(p(w)) = 1 + \text{cDes}^*(w) \pmod{a + b} \quad (\forall w \in W).
\]

For example, if \( a = 4, b = 3, \) and \( w = 4375261 \) then \( p(w) = 4327516 \) and \( p^2(w) = 6432751 \), yielding \( \text{cDes}^*(w) = \{1, 3, 4, 6, 7\}, \text{cDes}^*(p(w)) = \{1, 2, 4, 5, 7\}, \) and \( \text{cDes}^*(p^2(w)) = \{1, 2, 3, 5, 6\} \).

One concludes that
\[
\sum_{w \in W, \text{Des}(w)} F_{a+b, \text{Des}(w)} = \frac{1}{a+b} \sum_{i \in [a+b]} \sum_{w \in W, \text{Des}(p^i(w))} F_{a+b, \text{Des}(p^i(w))} = \frac{1}{a+b} \sum_{w \in W} F_{a+b, \text{cDes}^*(w)}
\]
completing the proof. \( \square \)

Recall the cyclic ribbon Schur function, defined as
\[
\tilde{s}_{cc_n}(J) := \sum_{\emptyset \neq I \subseteq J} (-1)^{(J \setminus I)} h_{cc_n}(I) \in \text{Sym}_n \quad (\forall \emptyset \neq J \subseteq [n]).
\]
The following cyclic analogue of [14, Theorem 3] holds.

**Lemma 4.9.** For every (homogeneous of degree \( n \)) symmetric function \( f \in \text{Sym}_n \), the unique expansion
\[
f = \sum_{A \in c_2^{[n]}_0} c_A \hat{F}_{n,A}^{\text{cyc}}
\]
satisfies, for each representative \( J \) of each orbit \( A \in c_2^{[n]}_0 \):
\[
c_A = \langle f, \tilde{s}_{cc_n}(J) \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is the usual inner product of symmetric functions.

**Proof.** Write \( f \) in the monomial basis of \( \text{Sym}_n \):
\[
f = \sum_{\lambda \vdash n} \langle f, h_{\lambda} \rangle m_{\lambda}.
\]
Clearly,
\[
m_{\lambda} = \sum_{A : \text{par}(A) = \lambda} \hat{M}_{n,A}^{\text{cyc}} = n^{-1} \sum_{J : \text{par}(J) = \lambda} M_{n,J}^{\text{cyc}}
\]
where the second sum is over all the subsets \( \emptyset \neq J \subseteq [n] \) such that the corresponding cyclic composition \( cc_n(J) \) can be reordered to the partition \( \lambda \) (and the first sum is over all cyclic orbits \( A \) of such subsets).
Thus

\[ f = n^{-1} \sum_{\lambda \vdash n} \langle f, h_\lambda \rangle \sum_{J : \text{par}(J) = \lambda} M^{\text{cyc}}_{n,J} \]

\[ = n^{-1} \sum_{\emptyset \neq J \subset [n]} \langle f, h_{cc_n(J)} \rangle M^{\text{cyc}}_{n,J} \]

\[ = n^{-1} \sum_{\emptyset \neq J \subset [n]} \langle f, h_{cc_n(J)} \rangle \sum_{K \supseteq J} (-1)^{\#(K \setminus J)} F^{\text{cyc}}_{n,K} \]

\[ = n^{-1} \sum_{\emptyset \neq K \subset [n]} F^{\text{cyc}}_{n,K} \sum_{\emptyset \neq J \subset K} (-1)^{\#(K \setminus J)} \langle f, h_{cc_n(J)} \rangle \]

\[ = n^{-1} \sum_{\emptyset \neq K \subset [n]} F^{\text{cyc}}_{n,K} \langle f, s_{cc_n(K)} \rangle \]

\[ = \sum_{A \in c_2^{[n]}} \tilde{F}^{\text{cyc}}_{n,A} \langle f, s_{cc_n(A)} \rangle. \]

In the last expression, $cc_n(A)$ is shorthand for $cc_n(K)$ for an arbitrary $K \in A$. \qed

**Corollary 4.10.** Let $f$ be a symmetric function, homogeneous of degree $n$. The coefficients in the expansion of $f$ as a linear combination of \( \{ \tilde{F}^{\text{cyc}}_{n,A} : A \in c_2^{[n]} \} \) are integral if and only if the coefficients in its expansion as a linear combination of Schur functions are integral.

**Proof.** Let $c_A(f)$ denote the coefficient of $\tilde{F}^{\text{cyc}}_{n,A}$ in the expansion of $f$ in the above normalized fundamental basis. Clearly,

\[ c_A(f) = \sum_{\lambda \vdash n} \langle f, s_\lambda \rangle c_A(s_\lambda) \quad (\forall A \in c_2^{[n]}). \]

By Corollary 4.6 and Proposition 4.7 for every partition $\lambda \vdash n$ and $A \in c_2^{[n]}$

\[ c_A(s_\lambda) \in \mathbb{Z}. \]

We conclude that if the coefficients in the expansion of $f$ as a linear combination of Schur functions are integral then so are the coefficients in its expansion in the normalized fundamental basis.

For the opposite direction, assume that all the coefficients $c_A(f)$ are integral. Then, by Lemma 4.9

\[ \langle f, s_{cc_n(J)} \rangle \in \mathbb{Z} \quad (\forall \emptyset \neq J \subset [n]). \]

The definition of $s_{cc_n(J)}$ as an alternating sum of complete homogeneous symmetric functions $h_{cc_n(I)}$ is equivalent, by the Principle of Inclusion and Exclusion, to

\[ h_{cc_n(J)} = \sum_{\emptyset \neq I \subseteq J} s_{cc_n(I)} \quad (\forall \emptyset \neq J \subseteq [n]), \]

and therefore

\[ \langle f, h_{cc_n(J)} \rangle \in \mathbb{Z} \quad (\forall \emptyset \neq J \subseteq [n]). \]

By the Jacobi-Trudi identity [31, Theorem 7.16.1], any Schur function $s_\lambda$ is an alternating sum of complete homogeneous symmetric functions, so that

\[ \langle f, s_\lambda \rangle \in \mathbb{Z} \quad (\forall \lambda \vdash n), \]

as claimed. \qed

**Remark 4.11.** By Lemma 4.9 together with [24, Proposition 3.14], for every non-hook shape $\lambda$, the coefficient of $\tilde{F}^{\text{cyc}}_{n,[J]}$ in $s_\lambda$ is equal to the coefficient of $s_\lambda$ in the Schur expansion of Postnikov’s toric Schur function $s_{\mu(J)} / \mu(J)$. It follows from Postnikov’s result [20, Theorem 5.3] that that these coefficients are equal to certain Gromov–Witten invariants.
5. Enumerative applications

5.1. Distribution of cyclic descents over cyclic shuffles. The following classical result is due to Stanley.

Proposition 5.1 (Ex. 3.161; see also [17] section 2.4 for a combinatorial proof). Let $A$ and $B$ be two disjoint sets of integers. For each $u \in \mathcal{S}_A$ and $v \in \mathcal{S}_B$, the distribution of the descent set over all shuffles of $u$ and $v$ depends only on $\text{Des}(u)$ and $\text{Des}(v)$.

We have the following cyclic analogue. Recall that for every $u \in \mathcal{S}_A$, $\text{cDes}([u])$ is defined up to cyclic rotation.

Proposition 5.2. Let $A$ and $B$ be two disjoint sets of integers. For each $u \in \mathcal{S}_A$ and $v \in \mathcal{S}_B$, the distribution of the cyclic descent set over all cyclic shuffles of $[u]$ and $[v]$ depends only on $\text{cDes}([u])$ and $\text{cDes}([v])$.

Proof. This is an immediate consequence of Theorem 3.22.

Consider now $\mathbb{Z}[[q]]$, the ring of formal power series in $q$, as a (free abelian) additive group with generators $(q^n)_{n=0}^\infty$, and define a new product by

$$q^i \odot q^j := q^{\max(i,j)},$$

extended linearly. We obtain a (commutative and associative) ring, to be denoted $\mathbb{Z}[[q]]_\odot$.

Consider also the ring $\mathbb{Z}[[x]] = \mathbb{Z}[x_1, x_2, \ldots ]$, with the usual product. Define a map $\Psi : \mathbb{Z}[[x]] \to \mathbb{Z}[[q]]_\odot$ by

$$\Psi(x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}) := q^{r_{i_k}} \quad (k > 0, i_1 < \ldots < i_k, m_1, \ldots , m_k > 0)$$

and

$$\Psi(1) := 1,$$

extended linearly.

Observation 5.3. $\Psi$ is a ring (Z-algebra) homomorphism.

We now compute the images, under $\Psi$, of various quasi-symmetric functions. All explicit formulas will be stated in $\mathbb{Z}[[q]]$, with the standard multiplication.

Lemma 5.4. For any positive integer $n$ and subset $J \subseteq [n-1]$,

$$\Psi(M_{n,J}) = \left(\frac{q}{1-q}\right)^{|J|+1} = (1-q)\sum_r \binom{r}{|J|+1}q^r$$

and

$$\Psi(F_{n,J}) = \frac{q^{|J|+1}}{(1-q)^n} = (1-q)\sum_r \binom{n-r-|J|-1}{n-1}q^r.$$

Proof. Let $(m_0, \ldots , m_j) = \text{co}_n(J)$ be the composition of $n$ corresponding to $J \subseteq [n-1]$, with $j = |J|$. Then

$$\Psi(M_{n,J}) = \sum_{i_0 < \ldots < i_j} \Psi(x_{i_0}^{m_0} \cdots x_{i_j}^{m_j}) = \sum_{i_0 < \ldots < i_j} q^{r_{i_j}} = \sum_{r \geq j+1} \binom{r-j-1}{j}q^r = \left(\frac{q}{1-q}\right)^{j+1}$$

and, consequently,

$$\Psi(F_{n,J}) = \sum_{J \subseteq K \subseteq [n-1]} \Psi(M_{n,K}) = \sum_{k=j}^{n-1} \binom{n-1-j-k}{-j} \left(\frac{q}{1-q}\right)^{k+1} \frac{q^{j+1}}{(1-q)^n} = \frac{q^{j+1}}{(1-q)^n}.$$ 

This proves the two explicit formulas. The corresponding expansions into formal power series, after division by an extra factor $1-q$ (for reasons that will become apparent later), are standard.

Lemma 5.5. For any positive integers $m$, $n$ and subsets $J \subseteq [m-1]$ and $K \subseteq [n-1]$,

$$\Psi(M_{m,J} \cdot M_{n,K}) = (1-q)\sum_r \binom{r}{|J|+1}\binom{r}{|K|+1}q^r$$

and

$$\Psi(F_{m,J} \cdot F_{n,K}) = (1-q)\sum_r \binom{r+m-|J|-1}{m}\binom{r+n-|K|-1}{n}q^r.$$
Proof. Let \( (m_0, \ldots, m_j) = \text{co}_m(J) \) and \( (n_0, \ldots, n_k) = \text{co}_n(K) \), where \( j = |J| \) and \( k = |K| \). Then
\[
\Psi(M_{m,j} \cdot M_{n,k}) = \sum_{i_0 < \cdots < i_j \atop i'_0 < \cdots < i'_k} \Psi(x_{i_0}^{m_0} \cdots x_{i_j}^{m_j} \cdot x_{i'_0}^{n_0} \cdots x_{i'_k}^{n_k}).
\]
Since
\[
\# \{(i_0, \ldots, i_j, i'_0, \ldots, i'_k) : i_0 < \cdots < i_j, i'_0 < \cdots < i'_k, \max(i_j, i'_k) = r\} = \binom{r}{j+1} \binom{r}{k+1} - \binom{r-1}{j+1} \binom{r-1}{k+1},
\]
it follows that
\[
\Psi(M_{m,j} \cdot M_{n,k}) = \sum_r \left[ \binom{r}{j+1} \binom{r}{k+1} - \binom{r-1}{j+1} \binom{r-1}{k+1} \right] q^r = (1-q) \sum_r \binom{r}{j+1} \binom{r}{k+1} q^r.
\]
Consequently,
\[
\Psi(F_{m,j} \cdot F_{n,k}) = \sum_{J \subseteq J'} \sum_{K \subseteq K'} \Psi(M_{m,j'} \cdot M_{n,k'})
\]
\[
= (1-q) \sum_r \sum_{j'=j}^{m-1} \sum_{k'=k}^{n-1} \binom{m-1-j}{j'} \binom{n-1-k}{k'} \sum_r \binom{r}{j'+1} \binom{r}{k'+1} q^r
\]
\[
= (1-q) \sum_r \sum_{j'=j}^{m-1} \binom{m-1-j}{j'} \binom{n}{n-1-k} \sum_r \binom{r}{j'+1} \binom{r}{k'+1} q^r
\]
\[
= (1-q) \sum_r \binom{r+m-1-j}{m} \binom{r+n-1-k}{n} q^r.
\]
\[
\square
\]

**Theorem 5.6.** Let \( A \) and \( B \) be two disjoint sets of integers, with \( |A| = m \) and \( |B| = n \). For each \( u \in \mathfrak{S}_A \) and \( v \in \mathfrak{S}_B \), the distribution of descent number over all shuffles of \( u \) and \( v \) is given by
\[
\sum_{w \in u \shuffle v} q^{\text{des}(w)} = (1-q)^{m+n+1} \sum_r \binom{r+m-\text{des}(u)}{m} \binom{r+n-\text{des}(v)}{n} q^r.
\]

**Proof.** By Theorem 3.19,
\[
F_{m,\text{Des}(u)} \cdot F_{n,\text{Des}(v)} = \sum_{w \in u \shuffle v} F_{m+n,\text{Des}(w)}.
\]
Apply the mapping \( \Psi \) to both sides: on one hand, by Lemma 5.5,
\[
\Psi(F_{m,\text{Des}(u)} \cdot F_{n,\text{Des}(v)}) = (1-q) \sum_r \binom{r+m-\text{des}(u)-1}{m} \binom{r+n-\text{des}(v)-1}{n} q^r.
\]
On the other hand, by Lemma 5.4,
\[
\Psi\left( \sum_{w \in u \shuffle v} F_{m+n,\text{Des}(w)} \right) = \sum_{w \in u \shuffle v} q^{\text{des}(w)+1} (1-q)^{m+n}.
\]
It follows that
\[
\sum_{w \in u \shuffle v} q^{\text{des}(w)} = (1-q)^{m+n+1} \sum_r \binom{r+m-\text{des}(u)-1}{m} \binom{r+n-\text{des}(v)-1}{n} q^{r-1}
\]
\[
= (1-q)^{m+n+1} \sum_r \binom{r+m-\text{des}(u)}{m} \binom{r+n-\text{des}(v)}{n} q^r.
\]
\[
\square
\]

**Corollary 5.7.** Let \( A \) and \( B \) be two disjoint sets of integers, with \( |A| = m \) and \( |B| = n \). For each \( u \in \mathfrak{S}_A \) and \( v \in \mathfrak{S}_B \), if \( \text{des}(u) = i \) and \( \text{des}(v) = j \) then the number of shuffles \( w \) in \( u \shuffle v \) with \( \text{des}(w) = k \) is equal to
\[
\binom{m+j-i}{k-i} \binom{n+i-j}{k-j}.
\]
By Lemma 2.14 and Lemma 5.4, we have:

\[
(1 - q)^{m+n+1} \sum_r \binom{r+m-i}{m} \binom{r+n-j}{n} q^r.
\]

We therefore want to show that:

\[
(1 - q)^{m+n+1} \sum_r \binom{r+m-i}{m} \binom{r+n-j}{n} q^r = \sum_t \binom{m+j-i}{t-i} \binom{n+i-j}{t-j} q^t.
\]

Proof. By Theorem 5.6, the number of shuffles of \(q\) is equal to the coefficient of \(q^k\) in the product:

\[
(1 - q)^{m+n+1} \sum_r \binom{r+m-i}{m} \binom{r+n-j}{n} q^r.
\]

We shall use the triple-binomial identity [18, (5.28)]

\[
\sum_{\ell} \binom{m-x+y}{\ell} \binom{n+x-y}{n-\ell} \binom{x+\ell}{m+n} = \binom{x}{m} \binom{y}{n},
\]

valid for nonnegative integers \(m\) and \(n\) and arbitrary commuting indeterminates \(x\) and \(y\); summation is over the admissible integer values of \(\ell\), say \(0 \leq \ell \leq n\). It is equivalent to the Pfaff-Saalschütz hypergeometric identity [18, (5.97)]; we have renamed some of the variables to avoid confusion with our current notation.

In (5.1) substitute \(x := m + r - i\), \(y := n + r - j\) and \(\ell := n + i - t\) to get:

\[
(1 - q)^{m+n+1} \sum_r \binom{r+m-i}{m} \binom{r+n-j}{n} q^r = \sum_t \binom{m+j-i}{t-i} \binom{n+i-j}{t-j} q^t.
\]

It follows that:

\[
\sum_r \sum_{t \leq r} \binom{m+j-i}{t-i} \binom{n+i-j}{t-j} q^t = \sum_r \binom{m+r-i}{t-i} \binom{n+r-j}{t-j} q^t.
\]

Since:

\[
(1 - q)^{-(m+n+1)} = \sum_{s \geq 0} \binom{m+n+s}{m+n} q^s,
\]

the LHS of (5.3) is equal to:

\[
(1 - q)^{-(m+n+1)} \sum_t \binom{m+j-i}{t-i} \binom{n+i-j}{t-j} q^t,
\]

completing the proof. \(\square\)

Now we turn to the cyclic analogues.

**Lemma 5.8.** For any positive integer \(n\) and subset \(J \subseteq [n]\),

\[
\Psi(M_{n,J}^{\text{cy}}) = \frac{|J||q^{|J|}}{(1-q)^{|J|}} = (1-q) \sum_r |J| \binom{r}{|J|} q^r
\]

and

\[
\Psi(F_{n,J}^{\text{cy}}) = \frac{|J||q^{|J|} + (n-|J|)|q^{|J|+1}}{(1-q)^n} = (1-q) \sum_r \left( r + n - |J| - 1 \right) q^r.
\]

Proof. By Lemma 2.3 and Lemma 5.4

\[
\Psi(M_{n,J}^{\text{cy}}) = \sum_{J \subseteq J} \Psi(M_{n,(J-J) \cap [n-1]}) = \frac{|J||q^{|J|}}{(1-q)^{|J|}} = (1-q) \sum_r |J| \binom{r}{|J|} q^r.
\]

By Lemma 2.14 and Lemma 5.4

\[
\Psi(F_{n,J}^{\text{cy}}) = \sum_{i \in [n]} \Psi(F_{n,(i-i) \cap [n-1]}) = \frac{|J||q^{|J|} + (n-|J|)|q^{|J|+1}}{(1-q)^n} = (1-q) \sum_r c_r(n, |J|) q^r,
\]

where \(c_r(n, |J|)\) is the \(r\)-th coefficient of \(q^{|J|}\) in the product:

\[
(1 - q)^{m+n+1} \sum_r \binom{r+m-i}{m} \binom{r+n-j}{n} q^r.
\]
where

\[ c_r(n, j) = j \binom{r + n - j}{n} + (n - j) \binom{r + n - j - 1}{n} = \binom{r + n - j - 1}{n - 1} \cdot j(r + n - j) + (n - j)(r - j) = \binom{r + n - j - 1}{n - 1} \cdot r. \]

\[
\Psi(M_{m,J}^{\text{cyc}} \cdot M_{n,K}^{\text{cyc}}) = (1 - q) \sum_r |J| \binom{r}{|J|} |K| \binom{r}{|K|} q^r
\]

and

\[
\Psi(F_{m,J}^{\text{cyc}} \cdot F_{n,K}^{\text{cyc}}) = (1 - q) \sum_r \left( \frac{r + m - |J| - 1}{m - 1} \right) \left( \frac{r + n - |K| - 1}{n - 1} \right) r^2 q^r.
\]

**Lemma 5.9.** For any positive integers \( m, n \) and subsets \( J \subseteq [m] \) and \( K \subseteq [n] \),

\[
\Psi(M_{m,J}^{\text{cyc}} \cdot M_{n,K}^{\text{cyc}}) = (1 - q) \sum_r |J| \binom{r}{|J|} |K| \binom{r}{|K|} q^r
\]

and

\[
\Psi(F_{m,J}^{\text{cyc}} \cdot F_{n,K}^{\text{cyc}}) = (1 - q) \sum_r \left( \frac{r + m - |J| - 1}{m - 1} \right) \left( \frac{r + n - |K| - 1}{n - 1} \right) r^2 q^r.
\]

**Proof.** By Lemma 2.5 and Lemma 5.5

\[
\Psi(M_{m,J}^{\text{cyc}} \cdot M_{n,K}^{\text{cyc}}) = \sum_{j \in [m]} \sum_{k \in [n]} \Psi(M_{m,(j \cap [m-1]} \cdot M_{n,(k \cap [n-1]} \)

\[
= |J| \cdot |K| \cdot (1 - q) \sum_r \left( \frac{r}{|J|} \right) \binom{r}{|K|} q^r.
\]

By Lemma 2.14 and Lemma 5.5

\[
\Psi(F_{m,J}^{\text{cyc}} \cdot F_{n,K}^{\text{cyc}}) = \sum_{r \in [m]} \sum_{k \in [n]} \Psi(F_{m,(j \cap [m-1]} \cdot F_{n,(k \cap [n-1]} \)

\[
= (1 - q) \sum_r c_r(m, |J|) \cdot c_r(n, |K|) \cdot q^r
\]

where, as in the proof of Lemma 5.8

\[
c_r(m, j) = j \binom{r + m - j}{m} + (m - j) \binom{r + m - j - 1}{m} = \binom{r + m - j - 1}{m - 1} \cdot j(r + m - j) + (m - j)(r - j) = \binom{r + m - j - 1}{m - 1} \cdot r.
\]

\[
\square
\]

**Theorem 5.10.** Let \( A \) and \( B \) be two disjoint sets of integers, with \( |A| = m \) and \( |B| = n \). For each \( u \in \mathcal{S}_A \) and \( v \in \mathcal{S}_B \), the distribution of cyclic descent number over all cyclic shuffles of \([u]\) and \([v]\) is given by

\[
\sum_{[w] \in [u] \cup [v]} q^{\text{cdes}(w)} = (1 - q)^{m+n} \sum_r \left( \frac{r + m - \text{cdes}(u) - 1}{m - 1} \right) \left( \frac{r + n - \text{cdes}(v) - 1}{n - 1} \right)rq^r.
\]

**Proof.** By Theorem 3.22

\[
F_{m,c\text{Des}(u)}^{\text{cyc}} \cdot F_{n,c\text{Des}(v)}^{\text{cyc}} = \sum_{[w] \in [u] \cup [v]} F_{m+n,c\text{Des}(w)}^{\text{cyc}}.
\]

Apply the mapping \( \Psi \) to both sides: on one hand, by Lemma 5.9

\[
\Psi(F_{m,c\text{Des}(u)}^{\text{cyc}} \cdot F_{n,c\text{Des}(v)}^{\text{cyc}}) = (1 - q) \sum_r \left( \frac{r + m - \text{cdes}(u) - 1}{m - 1} \right) \left( \frac{r + n - \text{cdes}(v) - 1}{n - 1} \right) r^2 q^r.
\]

\[
\square
\]
On the other hand, by Lemma 5.8

\[ \Psi\left( \sum_{[w] \in [u]^{\pi Cyc}} P_{m+n,cDes(w)}^{\pi Cyc} \right) = (1 - q) \sum_{[w] \in [u]^{\pi Cyc}} \sum_{r} \left( \frac{r + m + n - cdes(w) - 1}{m + n - 1} \right) rq^r. \]

It follows that

\[ \sum_{[w] \in [u]^{\pi Cyc}} \left( \frac{r + m + n - cdes(w) - 1}{m + n - 1} \right) = \left( \frac{r + m - cdes(u) - 1}{m - 1} \right) \left( \frac{r + n - cdes(v) - 1}{n - 1} \right) \quad (\forall r > 0). \]

This also holds for \( r = 0 \), since the cyclic descent number of a permutation is strictly positive, so that both sides are equal to zero. Using

\[ \frac{q^r}{(1 - q)^{m+n}} = \sum_{r} \left( \frac{r + m + n - c - 1}{m + n - 1} \right) q^r \quad (\forall c > 0), \]

this can written, equivalently, as

\[ \sum_{[w] \in [u]^{\pi Cyc}} q^{cdes(w)} = (1 - q)^{m+n} \sum_{r} \left( \frac{r + m - cdes(u) - 1}{m - 1} \right) \left( \frac{r + n - cdes(v) - 1}{n - 1} \right) rq^r. \]

\[ \square \]

Corollary 5.11. Let \( A \) and \( B \) be two disjoint sets of integers, with \(|A| = m \) and \(|B| = n \). For each \( u \in \mathfrak{S}_A \) and \( v \in \mathfrak{S}_B \), if \( cdes(u) = i \) and \( cdes(v) = j \) then the number of cyclic shuffles \([u]w\) in \([u]^{\pi Cyc}v\) with \( cdes(w) = k \) is equal to

\[ a(m, n, i, j, k) := k \left( \frac{m + j - i - 1}{k - i} \right) \left( \frac{n + i - j - 1}{k - j} \right) + (m + n - k) \left( \frac{m + j - i - 1}{k - i} \right) \left( \frac{n + i - j - 1}{k - j} \right) \]

\[ = \frac{k(m - i)(n - j) + (m + n - k)ij}{(m + j - i)(n + i - j)} \left( \frac{m + j - i}{k - i} \right) \left( \frac{n + i - j}{k - j} \right). \]

Proof. Similar to the proof of Corollary 5.7 above. By Theorem 5.10 the number of cyclic shuffles of \([u]w\) and \([v]w\) with cyclic descent number \( k \) is equal to the coefficient of \( q^k \) in the product

\[ (1 - q)^{m+n} \sum_{r} \left( \frac{r + m - i - 1}{m - 1} \right) \left( \frac{r + n - j - 1}{n - 1} \right) rq^r. \]

We therefore want to show that

\[ (1 - q)^{m+n} \sum_{r} \left( \frac{r + m - i - 1}{m - 1} \right) \left( \frac{r + n - j - 1}{n - 1} \right) rq^r = \sum_{t} a(m, n, i, j, t)q^t. \]

Using identity 5.2 with \( m - 1 \) and \( n - 1 \) instead of \( m \) and \( n \), it follows that

\[ \sum_{r} \left( \frac{r + m - i - 1}{m - 1} \right) \left( \frac{r + n - j - 1}{n - 1} \right) rq^r = \sum_{r \geq t} \left( \frac{m + j - 1}{t - i} \right) \left( \frac{n + i - j - 1}{t - j} \right) \left( \frac{r - t + m + n - 2}{m + n - 2} \right) rq^r. \]

Since

\[ (1 - q)^{-(m+n)} = \sum_{s \geq 0} \left( \frac{s + m + n - 1}{m + n - 1} \right) q^s \]

and

\[ \binom{r - t + m + n - 2}{m + n - 2} = \binom{r - t + m + n - 1}{m + n - 1} + (m + n - t - 1) \binom{r - t + m + n - 1}{m + n - 1}. \]
it follows that the RHS of (5.4) is equal to \((1 - q)^{-(m + n)}\) times
\[
\sum_t \left( \frac{m + j - i - 1}{t - i} \right) \left( \frac{n + i - j - 1}{t - j} \right) t q^t + \sum_t \left( \frac{m + j - i - 1}{t - i} \right) \left( \frac{n + i - j - 1}{t - j} \right) (m + n - t - 1) q^{t+1}
\]
\[
= \sum_t \left( \frac{m + j - i - 1}{t - i} \right) \left( \frac{n + i - j - 1}{t - j} \right) t + \left( \frac{m + j - i - 1}{t - i} \right) \left( \frac{n + i - j - 1}{t - j} \right) (m + n - t) \right] q^t
\]
so that
\[
(1 - q)^{m+n} \sum_r \left( \frac{r + m - i - 1}{m - 1} \right) \left( \frac{r + n - j - 1}{n - 1} \right) r q^r = \sum_t a(m, n, i, j, t) q^t,
\]
as claimed. \(\square\)

5.2. Distribution of cyclic descents over SYT. Let us first restate Corollary 4.9 in an equivalent form.

Lemma 5.12. For any skew shape \(\lambda/\mu\) of size \(n\), which is not a connected ribbon, for any cyclic extension \((\text{cDes}, p)\) of \(\text{Des}\) on \(\text{SYT}(\lambda/\mu)\), and for any subset \(\emptyset \subseteq J \subseteq [n]\), the fiber size
\[
\#\{T \in \text{SYT}(\lambda/\mu) : \text{cDes}(T) = J\}
\]
is equal to the coefficient of \(F_{n, [J]}^{\text{cyc}}\) in the expansion of \(s_{\lambda/\mu}\) in terms of normalized fundamental cyclic quasi-symmetric functions.

It follows that identities involving Schur functions may be interpreted as statements about the distribution of \(\text{cDes}\) over standard Young tableaux. Several examples will be given below.

Write \(\lambda/\mu = \nu^1 \oplus \cdots \oplus \nu^t\) to indicate that \(\nu^1, \ldots, \nu^t\) are the connected components of the skew shape \(\lambda/\mu\), ordered from southwest to northeast. It is well known that
\[
(5.5) \quad s_{\nu^1 \oplus \cdots \oplus \nu^t} = s_{\nu^1 \cdots \nu^t}.
\]

Proposition 5.13. Let \(\lambda \vdash m\) and \(\mu \vdash n\) be non-hook partitions. If \(A_\lambda \subseteq \Sigma_m\) and \(A_\mu \subseteq \Sigma_{m+1, m+n}\) are sets of permutations satisfying
\[
\sum_{T \in \text{SYT}(\lambda)} x^{\text{cDes}(T)} = \sum_{\pi \in A_\lambda} x^{\text{cDes}(\pi)} \quad \text{and} \quad \sum_{T \in \text{SYT}(\mu)} x^{\text{cDes}(T)} = \sum_{\pi \in A_\mu} x^{\text{cDes}(\pi)}
\]
then
\[
\sum_{T \in \text{SYT}(\lambda \oplus \mu)} x^{\text{cDes}(T)} = \frac{1}{mn} \sum_{\sigma \in A_\lambda} \sum_{\tau \in A_\mu} x^{\text{cDes}(\sigma \circ \tau)}
\]

Proof. First, by Theorem 4.5
\[
s_{\lambda \oplus \mu} = \frac{1}{m+n} \sum_{T \in \text{SYT}(\lambda \oplus \mu)} F_{m+n, c\text{Des}(T)}^{\text{cyc}}.
\]
On the other hand, by Equation (5.5) combined with Theorem 4.5 Theorem 3.22 and our assumptions regarding the equidistribution of \(\text{cDes}\) on corresponding sets of permutations and tableaux,
\[
s_{\lambda \oplus \mu} = s_{\lambda} s_{\mu} = \frac{1}{m} \sum_{T \in \text{SYT}(\lambda)} F_{m, c\text{Des}(T)}^{\text{cyc}} \frac{1}{n} \sum_{T \in \text{SYT}(\mu)} F_{n, c\text{Des}(T)}^{\text{cyc}} = \frac{1}{mn} \sum_{\sigma \in A_\lambda} \sum_{\tau \in A_\mu} F_{m+n, c\text{Des}(\sigma \circ \tau)}^{\text{cyc}} = \frac{1}{mn(m+n)} \sum_{\sigma \in A_\lambda} \sum_{\tau \in A_\mu} \sum_{\nu \in \Sigma_{m+n}} F_{m+n, c\text{Des}(\nu)}^{\text{cyc}}.
\]
One concludes that
\[
\sum_{T \in \text{SYT}(\lambda \oplus \mu)} F_{m+n, \text{cDes}}^{\text{cyc}}(T) = \frac{1}{mn} \sum_{\sigma \in A_{\lambda}} \sum_{w \in \mathcal{G}_{m+n} : [\bar{w}] \in [\bar{\sigma}][\mathcal{W}_{\text{cyc}}^{\text{cyc}}(\bar{\tau})]} F_{m+n, \text{cDes}}^{\text{cyc}}(w).
\]

Now, fiber sizes of the cDes map on both SYT(\lambda \oplus \mu) and \{ w \in \mathcal{G}_{n+m} : [\bar{w}] \in [\bar{\sigma}][\mathcal{W}_{\text{cyc}}^{\text{cyc}}(\bar{\tau})] \} are invariant under cyclic rotation. Comparing coefficients on both sides of the above equation thus completes the proof. \(\square\)

The next application of Lemma 5.12 relates the distribution of cDes on permutations to its distribution on tableaux of various straight shapes \(\lambda\).

For every positive integer \(n\) define the corresponding multivariate cyclic Eulerian polynomial by

\[
\mathcal{G}_{n}^{\text{cDes}}(t) := \mathcal{G}_{n}^{\text{cDes}}(t_1, \ldots, t_n) := \sum_{\pi \in \mathcal{S}_n} \prod_{i \in \text{cDes}(\pi)} t_i.
\]

For every skew shape \(\lambda/\mu\) which is not a connected ribbon denote

\[
\mathcal{G}_{\lambda/\mu}^{\text{cDes}}(t) := \mathcal{G}_{\lambda/\mu}^{\text{cDes}}(t_1, \ldots, t_n) := \sum_{T \in \text{SYT}(\lambda/\mu)} \prod_{i \in \text{cDes}(T)} t_i.
\]

Let \(f^n := \#\text{SYT}(\lambda)\).

**Theorem 5.14.** ([2] Theorem 1.2) For every positive integer \(n\)

\[
\mathcal{G}_{n}^{\text{cDes}}(t) = \sum_{\text{non-hook } \lambda^n} f^n \mathcal{G}_{\lambda}^{\text{cDes}}(t) + \sum_{k=1}^{n-1} \binom{n-2}{k-1} \mathcal{G}_{(1^k) \oplus (n-k)}^{\text{cDes}}(t),
\]

where the last summation is over skew shapes \((1^k) \oplus (n-k), 1 \leq k \leq n-1, \text{ consisting of one column of size } k \text{ and one row of size } n-k\).

**Proof.** First, by Pieri’s rule

\[
s_{(1^k) \oplus (n-k)} = s_{(1^k)} s_{(n-k)} = s_{(n-k, 1^k)} + s_{(n-k+1, 1^{k-1})}.
\]

Hence

\[
\sum_{k=1}^{n-1} \binom{n-2}{k-1} s_{(1^k) \oplus (n-k)} = \sum_{k=0}^{n-1} \binom{n-1}{k} s_{(n-k, 1^k)} = \sum_{k=0}^{n-1} f_{(n-k, 1^k)} s_{(n-k, 1^k)}.
\]

Combining this with Equation (5.5) and iterations of Pieri’s rule we get

\[
s_{(1^n)} = s_{1^n} = \sum_{\text{non-hook } \lambda^n} f^n \mathcal{G}_{\lambda} = \sum_{\text{non-hook } \lambda^n} f^n \mathcal{G}_{\lambda} + \sum_{k=1}^{n-1} \binom{n-2}{k-1} s_{(1^k) \oplus (n-k)}.
\]

The bijection \(f : \mathcal{S}_n \rightarrow \text{SYT}(1^n)\), which sends a permutation \(w\) to the standard Young tableau whose entries are \(w^{-1}(1), \ldots, w^{-1}(n)\) (read from southwest to northeast), preserves descent sets and therefore also the distribution of cyclic descent sets. Lemma 5.12 completes the proof. \(\square\)

In a similar fashion one can simplify the proofs of Theorem 5.1 and Corollary 7.1 in [2].

Here is another, more specific, application. For \(2 \leq k \leq n-2\) consider two shapes obtained by adding a cell to the hook \((n-k, 1^{k-1}) \vdash n-1\):

1. Add a cell at the inner corner, to get \((n-k, 2, 1^{k-2})\).
2. Add a disconnected cell at the northeast corner, to get \((n-k, 1^{k-1}) \oplus (1)\).

**Example 5.15.** For \(n = 6\) and \(k = 3\), the two shapes obtained from the hook \((3,1,1)\) are

\[(3,2,1)\quad \text{and} \quad (3,1,1) \oplus (1)\]
The following proposition shows that the fiber sizes of cyclic descent maps on these two shapes differ by at most one. This explains the slight difference between Theorem 3.11(b) and Theorem 3.11(c) (see also Proposition 3.17 and Proposition 3.18) in [1].

**Proposition 5.16.** For every \(2 \leq k \leq n - 2\) and \(\emptyset \subset J \subset [n]\), if \(|J| = k\) then
\[
\# \{T \in SYT(\{(n - k,1^{k-1}) \oplus (1)\}) : \text{cDes}(T) = J\} = \# \{T \in SYT(\{n-k,2,1^{k-2}\}) : \text{cDes}(T) = J\} = 1.
\]
If \(|J| \neq k\) then this difference is 0.

**Proof.** By Pieri’s rule,
\[
s_{(n-k,1^{k-1})\oplus(1)} = s_{(n-k,1^{k-1})} s_{(1)} = s_{(n-k+1,1^{k-1})} + s_{(n-k,2,1^{k-2})} + s_{(n-k,1^k)}.
\]
Combining this with Equation (5.6) one obtains
\[
\text{Proposition 3.17 and Proposition 3.18}) in [1].
\]

It turns out [14, Theorem 11] that fundamental quasi-symmetric functions in the variables ordered accordingly. Let \(s_{(n-k,1^{k-1})\oplus(1)} = s_{(n-k,2,1^{k-2})} + s_{(1^k)\oplus(n-k)}\).

By Lemma 5.12 this identity is equivalent to
\[
\sum_{T \in SYT(\{(n-k,1^{k-1}) \oplus (1)\})} x^{\text{cDes}(T)} = \sum_{T \in SYT(\{n-k,2,1^{k-2}\})} x^{\text{cDes}(T)} + \sum_{T \in SYT((1^k)\oplus(n-k))} x^{\text{cDes}(T)}.
\]
Equation (4.2), describing the second summand on the RHS, completes the proof. \(\square\)

6. The Internal Coproduct and the Cyclic Descent Module

The ring of quasi-symmetric functions admits two natural coproducts: the inner (or internal) coproduct, whose dual is the product in Solomon’s descent algebra [14], and the outer (or graded) coproduct, whose dual is the product in the Hopf algebra of non-commutative symmetric functions [13 22]. The current section studies a cyclic analogue of the inner coproduct; for an analogue of the outer coproduct see Problem 7.2 below.

Let us start with a short review of the internal coproduct on QSym, following [14]. Consider two countable sets of variables, \(X = \{x_1 < x_2 < \ldots\}\) and \(Y = \{y_1 < y_2 < \ldots\}\), labeled by positive integers and (totally) ordered accordingly. Let \(XY := \{x_iy_j : i, j \geq 1\}\), ordered lexicographically:
\[
x_i,y_i < x_j,y_j \iff i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_1 < j_2).
\]
It turns out [14, Theorem 11] that fundamental quasi-symmetric functions in the variables \(XY\) can be expressed in terms of corresponding functions of \(X\) and \(Y\) separately:
\[
F_{n,\text{Des}(\pi)}(XY) = \sum_{\sigma_1,\sigma_2 \in S_n \atop \sigma_2 \sigma_1 = \pi} F_{n,\text{Des}(\sigma_1)}(X) \cdot F_{n,\text{Des}(\sigma_2)}(Y) \quad (\forall \pi \in S_n).
\]

It follows that, for any three subsets \(I, J, K \subseteq [n-1]\), the nonnegative integer
\[
a_{K}^{I,J} := \# \{(\sigma_1, \sigma_2) \in S_n \times S_n : \text{Des}(\sigma_1) = I, \text{Des}(\sigma_2) = J, \sigma_2 \sigma_1 = \pi\}
\]
indeed depends only on \((I, J)\) and \(K\) but not on the choice of \(\pi\), as long as \(\text{Des}(\pi) = K\). We can therefore write
\[
F_{n,K}(XY) = \sum_{I,J \subseteq [n-1]} a_{K}^{I,J} F_{n,I}(X) \cdot F_{n,J}(Y),
\]
and define a coproduct \(\Delta_n : \text{QSym}_n \to \text{QSym}_n \otimes \text{QSym}_n\) by
\[
\Delta_n(F_{n,K}) := \sum_{I,J \subseteq [n-1]} a_{K}^{I,J} F_{n,I} \otimes F_{n,J} \quad (\forall K \subseteq [n-1]).
\]
This turns \(\text{QSym}_n\) into a coalgebra [14].

There is also a dual structure. Consider the following elements of the group ring \(\mathbb{Z}[S_n]\):
\[
D_I := \sum_{\pi \in S_n \atop \text{Des}(\pi) = I} \pi \quad (I \subseteq [n-1]).
\]
The discussion above implies that
\[
D_J \cdot D_I = \sum_{K \subseteq [n-1]} a_{K}^{I,J} D_K \quad (\forall I, J \subseteq [n-1]),
\]
where multiplication is that of the group ring. It follows that the additive free abelian group
\[
\mathcal{D}_n := \text{span}_Z \{ D_I : I \subseteq [n-1] \}
\]
is actually a subring of \( \mathbb{Z}[\mathfrak{S}_n] \), known as Solomon’s descent algebra \[23\].

The ring \( \text{cQSym} \) of cyclic quasi-symmetric functions is a subring of \( \text{QSym} \). What is its status with respect to the internal coproduct? Is there a dual structure?

**Theorem 6.1.** \( \text{cQSym}_n \) and \( \text{cQSym}^-_n \) are right coideals of \( \text{QSym}_n \) with respect to the internal coproduct:

\[
\Delta_n(\text{cQSym}_n) \subseteq \text{cQSym}_n \otimes \text{QSym}_n \quad (\forall n \geq 0)
\]

and

\[
\Delta_n(\text{cQSym}^-_n) \subseteq \text{cQSym}^-_n \otimes \text{QSym}_n \quad (\forall n \geq 0).
\]

Explicitly, in \( \text{cQSym}^-_n \):

\[
\Delta_n(\hat{F}^{\text{cyc}}_{n,B}) = \sum_{\substack{A \in \mathbb{C}_{0,n}^\circ J \subseteq [n-1]}} \left( \frac{dA}{dB} \right)_{\tilde{\alpha}} \hat{F}^{\text{cyc}}_{cQSym,J} \otimes F_{n,J} \quad (\forall B \in \mathbb{C}_{0,n}^\circ).
\]

where the number
\[
\tilde{\alpha} := \# \{(\sigma_1, \sigma_2) \in \mathfrak{S}_n \times \mathfrak{S}_n : c\text{Des}(\sigma_1) \in A, \text{Des}(\sigma_2) = J, \sigma_2 \sigma_1 = \pi \}
\]
depends on \( A, B \in \mathbb{C}_{0,n}^\circ \) and \( J \subseteq [n-1] \) but not on \( \pi \), as long as \( c\text{Des}(\pi) \in B \). The structure constants \( \frac{dA}{dB} \) are nonnegative integers. In addition,

\[
\Delta_n(\hat{F}^{\text{cyc}}_{n,B}) = \Delta_n(h_n) = \sum_{\lambda \vdash n} s_{\lambda} \otimes s_{\lambda}
\]

and

\[
\Delta_n(\hat{F}^{\text{cyc}}_{n,[n]}) = \Delta_n(e_n) = \sum_{\lambda \vdash n} s_{\lambda} \otimes s_{\lambda'}
\]

belong to \( \text{Sym}_n \otimes \text{Sym}_n \subseteq \text{cQSym}_n \otimes \text{QSym}_n \).

**Proof.** First, consider a permutation \( \pi \in \mathfrak{S}_n \). By Proposition \[2.14\]

\[
F^{\text{cyc}}_{n,c\text{Des}(\pi)} = \sum_{i \in [n]} F_{n,(\pi \cdot c^i) \cap [n-1]}. \]

Denoting by \( c \in \mathfrak{S}_n \) the permutation mapping \( i \mapsto i + 1 \) \( (1 \leq i \leq n - 1) \) and \( n \mapsto 1 \), clearly

\[
c\text{Des}(\pi) - i = c\text{Des}(\pi \cdot c^i) \quad (\forall i)
\]

and therefore

\[
F^{\text{cyc}}_{n,c\text{Des}(\pi)} = \sum_{i \in [n]} F_{n,c\text{Des}(\pi \cdot c^i) \cap [n-1]} = \sum_{i \in [n]} F_{n,\text{Des}(\pi \cdot c^i)}.
\]

Using \[6.1\], it follows that

\[
F^{\text{cyc}}_{n,c\text{Des}(\pi)}(XY) = \sum_{i \in [n]} F_{n,\text{Des}(\pi \cdot c^i)}(X) \cdot F_{n,\text{Des}(\sigma_2)}(Y)
\]

\[
= \sum_{i \in [n]} \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}_n \atop \sigma_2 \sigma_1 = \pi} F_{n,\text{Des}(\sigma_1 \cdot c^i)}(X) \cdot F_{n,\text{Des}(\sigma_2)}(Y)
\]

and therefore

\[
\Delta_n(F^{\text{cyc}}_{n,c\text{Des}(\pi)}) = \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}_n \atop \sigma_2 \sigma_1 = \pi} F^{\text{cyc}}_{n,c\text{Des}(\sigma_1)} \otimes F_{n,\text{Des}(\sigma_2)} \in \text{cQSym}_n \otimes \text{QSym}_n \quad (\forall \pi \in \mathfrak{S}_n).
\]
Denoting \( A := [c\operatorname{Des}(\sigma_1)] \), \( J := \operatorname{Des}(\sigma_2) \) and \( B := [c\operatorname{Des}(\pi)] \), this can be written as

\[
\Delta_n(F_{n,B}^{\text{cyc}}) = \sum_{A \in cZ_n} \tilde{a}_{B}^{A} J_{n,A} \otimes F_{n,J} \quad (\forall B \in cZ_n)
\]

and the normalized version follows. The structure constants \( \tilde{a}_{B}^{A} J_{n,A} \) are clearly nonnegative rational numbers.

We can prove their integrality by a general indirect argument, as follows: \( \hat{F}_{n,B}^{\text{cyc}} \in c\text{QSym}_n \subseteq \text{QSym}_n \), and therefore \( \Delta_n(\hat{F}_{n,B}^{\text{cyc}}) \in \text{QSym}_n \otimes \text{QSym}_n \) is a linear combination, with integer coefficients, of the basis elements \( F_{n,I} \otimes F_{n,J} \). Expanded in \( \mathbb{Z}[X]_n \otimes \mathbb{Z}[Y]_n \), it is a linear combination of tensor products of monomials, again with integer coefficients. On the other hand, the above computation shows that it belongs to \( \mathbb{Q} \otimes c\text{QSym}_n \otimes \text{QSym}_n \), and thus is a linear combination of basis elements \( \hat{F}_{n,A}^{\text{cyc}} \otimes F_{n,J} \), or equivalently basis elements \( \hat{M}_{n,A}^{\text{cyc}} \otimes M_{n,J} \), with rational coefficients. Each monomial in \( \mathbb{Z}[X]_n \otimes \mathbb{Z}[Y]_n \) appears in exactly one of the latter basis elements, with coefficient 1, and thus the coefficients in this basis (as well as in the other basis) are actually integers.

Any subset of \([n]\), other than \( \emptyset \) and \([n]\), has the form \( c\operatorname{Des}(\pi) \) for some \( \pi \in \mathfrak{S}_n \); it remains to check \( \Delta_n(\hat{F}_{n,\emptyset}^{\text{cyc}}) \) and \( \Delta_n(\hat{F}_{n,[n]}^{\text{cyc}}) \).

Indeed,

\[
\Delta_n(\hat{F}_{n,\emptyset}^{\text{cyc}}) = \Delta_n(F_{n,\emptyset}) = \Delta_n(F_{n,\operatorname{Des}(id)}) = \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma_1)} \otimes F_{n,\operatorname{Des}(\sigma_2)}
\]

\[
= \sum_{\sigma \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma)} \otimes F_{n,\operatorname{Des}(\sigma^{-1})}
\]

and, denoting by \( w_0 \) the involution (longest element) in \( \mathfrak{S}_n \) mapping \( i \mapsto n+1-i \) (\( \forall i \in [n] \)),

\[
\Delta_n(\hat{F}_{n,[n]}^{\text{cyc}}) = \Delta_n(F_{n,[n-1]}) = \Delta_n(F_{n,\operatorname{Des}(w_0)}) = \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma_1)} \otimes F_{n,\operatorname{Des}(\sigma_2)}
\]

\[
= \sum_{\sigma \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma w_0)} \otimes F_{n,\operatorname{Des}(\sigma^{-1})}.
\]

Recall now [31, Theorem 7.23.2], which can be stated as the pair of identities

\[
\sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda = \sum_{\sigma \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma)} \otimes F_{n,\operatorname{Des}(\sigma^{-1})}
\]

and

\[
\sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda' = \sum_{\sigma \in \mathfrak{S}_n} F_{n,\operatorname{Des}(\sigma w_0)} \otimes F_{n,\operatorname{Des}(\sigma^{-1})}.
\]

Using them leads to the claimed expressions

\[
\Delta_n(\hat{F}_{n,\emptyset}^{\text{cyc}}) = \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda \in \text{Sym}_n \otimes \text{Sym}_n
\]

and

\[
\Delta_n(\hat{F}_{n,[n]}^{\text{cyc}}) = \sum_{\lambda \vdash n} s_\lambda \otimes s_\lambda' \in \text{Sym}_n \otimes \text{Sym}_n.
\]

\[\square\]

**Remark 6.2.** \( c\text{QSym}_n \) and \( c\text{QSym}_n^- \) are not left coideals of \( \text{QSym}_n \) with respect to the internal coproduct, already for \( n = 3 \); and they are not coalgebras, already for \( n = 4 \).

The invariance of the coefficients \( \tilde{a}_{B}^{A} J_{n,A} \) gives the following dual structure: In addition to

\[
D_J := \sum_{\pi \in \mathfrak{S}_n} \pi \quad (J \subseteq \{n-1\}),
\]
define also
\[ cD_A := \sum_{\pi \in \mathfrak{S}_n, \text{cDes} (\pi) \subseteq A} \pi \quad (A \in c2_{0,n}^{[n]}). \]

The discussion above implies that
\[ D_J \cdot cD_A = \sum_{B \in c2_{0,n}^{[n]}} a_{B}^{A} J cD_B \quad (\forall I \subseteq [n-1], A \in c2_{0,n}^{[n]}), \]

where multiplication is in the group ring \( \mathbb{Z}[\mathfrak{S}_n] \).

**Corollary 6.3.** The additive free abelian group
\[ c\mathfrak{D}_n := \text{span}_\mathbb{Z}\{cD_A : A \in c2_{0,n}^{[n]}\} \]

is a left module for Solomon’s descent algebra \( \mathfrak{D}_n \).

In general, \( c\mathfrak{D}_n \) is not a right module of \( \mathfrak{D}_n \) and not an algebra.

### 7. Open Problems and Final Remarks

We close with several remarks and questions.

Recall that one has a tower of ring extensions
\[ \text{(7.1)} \quad \text{Sym}_Q[x_1, \ldots, x_n] \subseteq cQ\text{Sym}_Q[x_1, \ldots, x_n] \subseteq \text{QSym}_Q[x_1, \ldots, x_n] \subseteq \mathbb{Q}[x_1, \ldots, x_n]. \]

For each ring extension \( R \subset S \) within this tower, it is natural to ask about the structure of \( S \) as an \( R \)-module. For example, one might ask whether \( S \) is a free \( R \)-module, or one might ask for a minimal generating set of \( S \) as an \( R \)-module. In this setting of graded \( Q \)-algebras, the latter question is equivalent to asking for elements of \( S \) whose images in the quotient ring \( S/(R+) \) form a \( Q \)-vector space basis, where \( R+ \) denotes the elements of strictly positive degree within \( R \).

Garsia and Wallach [12] proved that the ring \( \text{QSym}_Q[x_1, \ldots, x_n] \) of quasi-symmetric functions in \( n \) variables (over \( Q \)) is a free module over the ring \( \text{Sym}_Q[x_1, \ldots, x_n] \) of symmetric functions in these variables. This suggests asking the same question for the leftmost inclusion in (7.1). However, one can check that \( c\text{QSym}_Q[x_1, \ldots, x_n] \) is not free over \( \text{Sym}_Q[x_1, \ldots, x_n] \) starting at \( n = 4 \), via a Hilbert series calculation: the quotient of their Hilbert series in \( q \) has negative coefficients as a power series, beginning \( 1 + 2q^6 + q^7 + 2q^8 + q^9 − q^{10} + \cdots \).

Aval, Bergeron and Bergeron [3] studied \( \mathbb{Q}[x_1, \ldots, x_n] \) as a module over \( \text{QSym}_Q[x_1, \ldots, x_n] \), which is known to not be a free module. One might still ask whether \( \mathbb{Q}[x_1, \ldots, x_n] \) happens to be free as a module over \( c\text{QSym}_Q[x_1, \ldots, x_n] \), but not surprisingly, this fails starting at \( n = 3 \); the relevant quotient of two Hilbert series expansion that begins \( 1 + 2q + 4q^4 + q^3 − q^6 − 2q^7 \cdots \). On the other hand, in [3] the authors provide a minimal generating set for \( \mathbb{Q}[x_1, \ldots, x_n] \) as a \( \text{QSym}_Q[x_1, \ldots, x_n] \)-module, with cardinality given by a Catalan number, and compute a simple expression for the Hilbert series the quotient \( \mathbb{Q}[x_1, \ldots, x_n]/(\text{QSym}_Q[x_1, \ldots, x_n]) \). One might ask analogous questions about the quotient \( \mathbb{Q}[x_1, \ldots, x_n]/(c\text{QSym}_Q[x_1, \ldots, x_n]) \), which we have not explored.

Regarding the middle inclusion in (7.1), one can check that \( \text{QSym}[x_1, \ldots, x_n] \) is not a free module over \( c\text{QSym}_Q[x_1, \ldots, x_n] \) starting at \( n = 3 \); the relevant quotient of two Hilbert series expansion that begins \( 1 + q^3 + 2q^4 + 2q^5 + q^6 − q^9 \cdots \). We have not explored the dimension and Hilbert series of the quotient \( \text{QSym}[x_1, \ldots, x_n]/(c\text{QSym}_Q[x_1, \ldots, x_n]) \).

A natural goal is to study specific specializations of cyclic quasi-symmetric functions, e.g., the principal specialization defined by \( x_i := q^{i-1} \) for all \( i \). There are possible applications to various statistics of cyclic permutations. For example, we are looking for a notion of cyclic major index, which will provide cyclic analogues to [30] Proposition 7.19.12 and related identities.

A Schur-positivity phenomenon, involving cyclic quasi-symmetric functions, was presented in Section [4]. It is desired to find more results of this type. Here is a conjectured one. It was proved in [10] Cor. 7.7 that,
for every positive integer $0 < k < n$, the cyclic quasi-symmetric function

$$\sum_{\pi \in S_n} F_{n, \text{cdes}(\pi)}$$

is symmetric and Schur-positive. Computational experiments suggest the following (stronger) cyclic version.

**Conjecture 7.1.** For every $\emptyset \subsetneq J \subsetneq [n]$ the cyclic quasi-symmetric function

$$\sum_{\pi \in S_n} F_{n, \text{cDes}(\pi)}^{\text{cyc}} = \sum_{\pi \in S_n} F_{n, \text{cDes}(\pi)}$$

is symmetric and Schur-positive.

A related problem is the following. The proof of Theorem 4.4 in [2] is indirect and involves Postnikov’s toric Schur functions. A constructive proof, providing an explicit combinatorial definition of the cyclic descent map, was found most recently by Brice Huang [19]. Finding a bijective proof for Theorem 5.14 is now desired. Such a bijection would provide an effective way to construct subsets $A_{\lambda}$ and $A_{\mu}$ as in Proposition 5.13. A generalization of Proposition 5.13 to hook components is further desired. A solution would provide a far-reaching generalization of [11, Theorem 1].

The inner coproduct on QSym was studied in Section 6.

**Problem 7.2.** Define a cyclic analogue of the outer (graded) coproduct on QSym, and study its duality with a suitable version of the product on non-commutative symmetric functions.

Finally, cyclic descents were introduced by Cellini [6] in the search for subalgebras of Solomon’s descent algebra. An important subalgebra of the descent algebra is the peak algebra.

**Problem 7.3.** Define and study cyclic peaks and a cyclic peak algebra.

For cyclic peaks in Dyck paths see [1, §5.1].

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