DICRITICAL DIVISORS AFTER S.S. ABHYANKAR AND
I. LUENGO.

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Abstract. In [AL11], S.S Abhyankar and I. Luengo introduce a new theory of
dicritical divisors in the most general framework. Here we simplify and generalize
their results (see Theorems 3.1 and 3.2).

1. Introduction

The notion of dicritical divisor appeared at the beginning of the 20th century, in
the study of isolated singularities of complex planar differential equations [Dul06].
Given a germ $\omega$ of a holomorphic differential 1-form singular at the origin $0 \in \mathbb{C}^2$, the
singularity is called dicritical if there exists an infinite number of irreducible pairwise
distinct (germs of) invariant curves passing through 0. In this case, the resolution
of singularities [Sei68] leads to the following notion (see e.g. [MM80]): a dicritical
divisor - if it exists - is an irreducible component of the exceptional divisor which
is transverse to the foliation defined by $\omega$. An important example is given by the
case where $\omega$ has a first integral which is a meromorphic function $f/g$, considered in a
neighborhood of one of its poles. There the foliation is given by the pencil of curves
$\{\lambda f + \mu g = 0, \lambda, \mu \in \mathbb{C}\}$. This case is related to the Jacobian problem in dimension
2. Indeed, any polynomial map of $\mathbb{C}^2$ extends to a rational map of $\mathbb{P}_2(\mathbb{C})$ over certain
points at infinity (see [TW94] and Section 4 below).

In connection to the Jacobian problem, S.S. Abhyankar and I. Luengo introduce
in [AL11] an algebraic version of the dicritical divisors, in the most general context.
Set a point $p \in \mathbb{P}_2(\mathbb{C})$, and consider the corresponding local ring $R = \mathcal{O}_{\mathbb{P}_2(\mathbb{C}),p}$.
Pick $\frac{f}{g} \in QF(R)$ the quotient field of $R$, $\frac{f}{g}$ irreducible. It is well-known that, by a finite
sequence of blow-ups of points, one can monomialize the ideal $(f,g)$:

$$\text{Spec } R \leftarrow X_1 \leftarrow \cdots \leftarrow X_\nu.$$ 

Let $E = \bigcup_i E_i$ be the exceptional divisor in $X_\nu$. For any $i$, we define $\tilde{f}, \tilde{g}$ by $f = h\tilde{f}$,
$g = h\tilde{g}$ where $h = \text{GCD}(f,g)$ locally at $x \in E_i$. The couple $(f,g)$ defines a morphism:

$$\phi_{(f,g,i)} : E_i \to \mathbb{P}_1(\mathbb{C})$$

$$x \mapsto (\tilde{f}(x), \tilde{g}(x)).$$

With this notations, a dicritical divisor is therefore a divisor $E_i$ for which $\phi_{(f,g,i)}$ is
surjective. In other words, through any point of $E_i$ passes the strict transform of a
curve of the pencil. At the origin of their more general definition (2.2), S.S. Abhyankar
and I. Luengo make the following key observation: all which precedes is equivalent to
supposing that the resudu of $\frac{f}{g}$ is transcendental over the residue field of $\mathcal{O}_{X_\nu,\eta_i}$, the
latter being a discrete valuation ring ($\eta_i$ is the generic point $E_i$); the only hypothesis
being henceforth that $R$ is a 2 dimensional regular local ring.

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2. Definitions and preliminary results.

**Notation 2.1.** From now on, we will use the following notations. Let $R$ be a noetherian regular local ring of dimension 2. We denote by $QF(R)$ its quotient field, $\mathfrak{m}$ its maximal ideal and $K := R/\mathfrak{m}$ its residue field. We consider also a discrete valuation ring $R_v$ which dominates $R$ and such that $QF(R_v) = QF(R)$. In the other words, $R_v$ is a prime divisor of $R$ in the sense of [Abh50]. We denote by $v : QF(R) \to \mathbb{Z} \cup \{\infty\}$ the corresponding valuation and $\mathfrak{m}_v$ the maximal ideal. The residue map is denoted by $\text{Res}_v : R_v \to K_v$, where $K_v := R_v/\mathfrak{m}_v$ and $K$ is identified with $R/(R \cap \mathfrak{m}_v)$. Note that $\text{trdeg}_K K_v = 1$. Given a regular system of parameters $(x, y)$ of $\mathfrak{m}$, such a valuation $v$ is said to be (algebraically) monomial with respect to $(x, y)$ if for any polynomial expansion $P(x, y) = \sum_{a,b} \lambda_{a,b} x^ay^b$ with $\lambda_{a,b} \in R \setminus \mathfrak{m}$, one has $v(P) = \min\{av(x) + bv(y) \mid \lambda_{a,b} \neq 0\}$ [Tei03, Definition 3.22].

**Definition 2.2.** Let $z \in QF(R)$, $z \neq 0$. We call dicritical divisor of $z$ any prime divisor $R_v$ of $R$ such that $z \in R_v$ and $\text{Res}_v(z)$ is transcendental over $K$.

We will use the following results and notations [Abh56, Definition 3, Proposition 3] adapted to our context.

**Definition 2.3.** Let $X \to \text{Spec}(R)$ be the blow-up of $\text{Spec}(R)$ along $\mathfrak{m}$ [Har77, Definition p. 163]. Let $x$ be the center of $v$ over $X$ [Har77, Theorem 4.7 p. 101]. $\hat{R} = \mathcal{O}_{X,x}$ is called blow-up (or quadratic transform) of $R$ along $v$. More simply, let $(x, y)$ be a regular system of parameters of $\mathfrak{m}$. Suppose for instance that $v(x) \leq v(y)$. We denote $S := \hat{R}^\mathfrak{m} \cap \mathfrak{m}_v$. We have $\hat{R} := \hat{R}^\mathfrak{m} |_S$. Then the valuation $v$ has center $S, \hat{R}$.

By induction, we call sequence of blow-ups of $R$ along $v$ the sequence $R = R_0 \subsetneq \cdots \subsetneq R_i \subsetneq R_{i+1} \subsetneq \cdots$ where for any $i \in \mathbb{N}$, $R_{i+1}$ is the blow-up of $R_i$ along $v$.

**Proposition 2.4** (Abhyankar). Let $(R_j)_{j \in \mathbb{N}}$ be the sequence of blow-ups of $R$ along $v$. There is a unique $v \in \mathbb{N}$ such that for any $j, j' \in \mathbb{N}$ with $j \leq v < j'$, we have $R_j \neq R_v = R_j'$ with $\dim(R_j) = 2 > 1 = \dim(R_j')$. Moreover, $R_v$ is a prime divisor of $R_\nu$ with $K_v$ pure transcendental extension of $K_\nu := R_\nu/\mathfrak{m}_v$ of degree 1.

**Remark 2.5.** Let $(x_\nu, y_\nu)$ be a regular system of parameters of $\mathfrak{m}_\nu$. Then the valuation $v$ is the $\mathfrak{m}_\nu$-adic valuation, which is of course monomial with respect to $(x_\nu, y_\nu)$, and $R_v = R_\nu |_{(x_\nu)}(x_\nu)$. Besides, if we denote $K_v = K'(t)$ where $K' = R_\nu/\mathfrak{m}_\nu$ is the relative algebraic closure of $K$ in $K_v$ and $t \in K_v$ is transcendental over $K$, then $[K' : K] < \infty$.

3. The main theorems.

The following result is the main theorem of [AL11]:

**Theorem 3.1.** Let $z \in QF(R)$, $z \neq 0$. Let $R_v$ be a dicritical divisor of $z$. Suppose that there is $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $m \in \mathbb{N}$ such that $zx^m \in R$. Then the element $t$ of $(2.5)$ can be chosen so that $\text{Res}_v(z) \in K'[t]$.

**Proof.** We proceed by induction on $\nu$ which is finite by (2.4).

**Case $\nu = 0$.** In this case, the valuation $v$ is the $\mathfrak{m}$-adic valuation. Let $(x, y)$ be a regular system of parameters of $\mathfrak{m}$. By hypothesis, $z = \frac{f}{x^m}$ and $v(f) = m$. Therefore we write $f$ as:

$$ f = \sum_{a+b=m,a,b \in \mathbb{N}} \lambda_{a,b} x^a y^b, \quad \lambda_{a,b} \in R, \text{ with } \lambda_{a,b} \in R \setminus \mathfrak{m} \text{ for some } a,b. \quad (1) $$
After blowing-up, in $R[n_{\mathfrak{m}} x]$, $v$ is the $x$-adic valuation with valuation ring $R[n_{\mathfrak{m}} x(x)]$. So we have:

$$z = \sum_{a+b=m, a,b \in \mathbb{N}} \lambda_{a,b} x^{a+b-m} \left( \frac{y}{x} \right)^b = \sum_{a+b=m, a,b \in \mathbb{N}} \lambda_{a,b} \left( \frac{y}{x} \right)^b, \quad \lambda_{a,b} \in R.$$  

Since $\text{Res}(z)$ is transcendental over $K$, there is at least one $\lambda_{a,b} \in R \setminus \mathfrak{m}$ with $\text{Res}(\lambda_{a,b}) \neq 0$ and $b > 0$. So we obtain:

$$\text{Res}(z) = \sum_{a+b=m, a,b \in \mathbb{N}} \text{Res}(\lambda_{a,b}) t^b, \quad t := \text{Res} \left( \frac{y}{x} \right).$$

Case $\nu \geq 1$. In $R = R_0$, we consider the following dichotomy: either $v(y) \geq v(x)$ or $v(y) < v(x)$.

Suppose that $v(y) \geq v(x)$. Then $R_1$ is the localisation of $R[n_{\mathfrak{m}} x]$ at the center of $v$. In $R_1$, we have $z = \frac{\nu}{\nu_{\mathfrak{m}}}$, where $\nu(f)$ is the $\mathfrak{m}$-adic order of $f$ and $f_1 \in R_1 \subset R_0$ is the strict transform of $f$. Since $v(z) = 0$ and $v(f_1) \geq 0$, $v(x) \geq 0$, we have $m \geq \nu(f)$. The hypotheses of the theorem hold for $R_1$: by induction on $\nu$, we obtain the desired result.

Suppose now that $v(y) < v(x)$. There are two subcases. Either there exists $i$, $1 \leq i \leq \nu$, such that, in the sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_i \subset \cdots \subset R_{\nu},$$

the inverse image of $x^m$ in $R_i$ has only one component. Then the hypotheses of the theorem hold for $R_i$: by induction on $\nu$, we obtain the desired result.

Or there is no such $i$. Then the center of $v$ is always at the origin of one of the two usual affine charts of the blow-ups. The valuation $v$ is monomial defined by

$$v(x) = \alpha, \; v(y) = \beta, \quad \alpha, \beta \in \mathbb{N} \text{ with } \alpha > \beta. \quad (2)$$

Since $\nu \geq 1$ and since we are at the origin of a chart in $R_1$, we have $R_1 = R[n_{\mathfrak{m}} x y]_{(x,y)}$.

With the notations of (1), we have

$$f = \sum_{a+b \geq m, a,b \in \mathbb{N}} \lambda_{a,b} x^a y^b, \quad \lambda_{a,b} \in R,$$

with $\lambda_{a,b} \in R \setminus \mathfrak{m}$ for at least one couple $(a,b)$ such that $\alpha a + b \beta - m \alpha = 0$.

By (2), since $\alpha > \beta$, if we have $\alpha a + b \beta - m \alpha = (a + b - m) \alpha + b(\beta - \alpha) = 0$, then $(a + b - m) \alpha \geq 0$. So we always have $a + b \geq m$ in the preceding sum. So we have:

$$z = \sum_{a+b \geq m, a,b \in \mathbb{N}} \lambda_{a,b} \left( \frac{x}{y} \right)^{a+b-m}, \quad \lambda_{a,b} \in R,$$

with $\lambda_{a,b} \in R \setminus \mathfrak{m}$ for at least one couple $(a,b)$ such that $\alpha a + b \beta - m \alpha$.

The hypotheses of the theorem hold in $R_1$ for such a $z$: by induction on $\nu$, we obtain the desired result. \hfill \Box

The following result generalises Theorem 3.1 and [AL11] Remark (7.4) (II).

**Theorem 3.2.** Let $z \in QF(R)$, $z \neq 0$. Let $R_0$ be a dicritical divisor of $z$. Suppose that there exist a regular system of parameters $(x,y)$ of $\mathfrak{m}$ and $a_0,b_0 \in \mathbb{N}$ such that $f = z x^{a_0} y^{b_0} \in R$.

(1) If $v$ is not monomial with respect to $(x,y)$, then the element $t$ of $[2.4]$ can be chosen so that $\text{Res}_v(z) \in K'[t]$. 

(2) If $v$ is monomial with respect to $(x,y)$, then $z = x^{a_0} y^{b_0} \cdot g$, with $g \in R_0 \setminus \mathfrak{m}$.
(2) If $v$ is monomial with respect to $(x, y)$, we denote $v(x) = \alpha$, $v(y) = \beta$ ($\alpha, \beta \in \mathbb{N}^*$), $\gamma := a\alpha + b\beta = v(f)$ and:

$$f = \sum_{a\alpha + b\beta \geq \gamma} \lambda_{a,b} x^a y^b, \quad \text{with } \lambda_{a,b} \in R$$

and $B_f := \{ b \in \mathbb{N} \mid \lambda_{a,b} \in R \setminus \mathfrak{m}, a\alpha + b\beta = \gamma \} \neq \emptyset$.

(a) If $\text{Card}(B_f) \geq 2$, then the element $t$ of (2.7) can be chosen so that $\text{Res}_v(z) \in K'[t]$ if and only if $b_0 \leq \min(B_f)$ or $b_0 \geq \max(B_f)$.

(b) If $B_f = \{ b_1 \}$, then we have $b_0 \neq b_1$ and the element $t$ of (2.7) can be chosen so that $\text{Res}_v(z) \in K'[t]$.

Remark 3.3. In the case (2), one can translate the condition on $b_0$ in terms of the Newton polygon associated to $f$ relatively to $(x, y)$. Denote by $D$ the line of equation $a\alpha + b\beta = \gamma$ in the plane $(a, b) \in \mathbb{R}^2$ and by $s_1$, respectively $s_2$, its point with coordinates $(a_1, b_1)$ where $b_1 = \min(B_f)$, respectively $(a_2, b_2)$ where $b_2 = \max(B_f)$. So $s_0 := (a_0, b_0) \in D$ and the segment $[s_1, s_2]$ is an edge (possibly reduced to a vertex) of the Newton polygon of $f$. The condition $b_0 \leq \min(B_f)$ or $b_0 \geq \max(B_f)$ is equivalent to supposing that $s_0 \notin [s_1, s_2]$.

Remark 3.4. Note that

$$\tilde{f} := \sum_{(a,b) \in B_f} \text{Res}(\lambda_{a,b}) U^a V^b \in K[U, V] \quad \text{(E)}$$

may be seen as $\text{In}_v(f) \in \text{gr}_v R = K[U, V]$ where $\text{In}_v(x) = U$ and $\text{In}_v(y) = V$. By definition in [Sp90, p.108], $\text{gr}_v R := \oplus_{\rho \in \mathbb{N} I_p/I_{p^+}}$ where $I_p := \{ w \in R \mid v(w) \geq \rho \}$ and $I_{p^+} := \{ w \in R \mid v(w) > \rho \}$. The relation (E) is proven in [Thi03, Remark 3.23 (2)]. See also [Hir67, Definition 2.9 and Remark 2.10] with $\Delta = \{(a,b) \mid a\alpha + b\beta \geq 1\}$, which inspired M. Spivakovsky, B. Teissier and many others.

If $a_0, b_0 \neq 0$, since $z$ and $f$ are fixed, then $x, y$ are fixed up to multiplication by invertibles. So $U, V$ are fixed up to multiplication by a scalar. Therefore $B_f$ is fixed.

If $a_0 = 0$ or $b_0 = 0$, by symmetry between $x$ and $y$ (see Remark 3.3), we may assume that $b_0 = 0$. In this case, our condition (2)(a) is trivially verified, even if $B_f$ may not be uniquely defined anymore (if $\alpha \beta$, we can replace $y$ by $y + \lambda_{a,b} x^{\beta/\alpha}$, i.e. $V$ by $V + \text{Res}(\lambda_{a,b}) U^{1/\alpha}$ which modifies $B_f$).

Proof. (1) Since the valuation is not monomial with respect to $x, y$, there is an index $i \in \{1, \ldots, v\}$ such that, in $R_v$, we write $x^i y^j = x^m u$ with $x_i$ parameter of $\mathfrak{m}_i$ and $u \in R_i$ invertible. Then we are reduced to the hypotheses of Theorem 3.1.

(2) Suppose now that the valuation is monomial with respect to $x, y$. We consider the ring $R_v$, with parameters $x_v, y_v$, for which the valuation $v$ is $m$-adic. We denote $x = x_v^{k_1} y_v^{k_2}$ and $y = x_v^{l_1} y_v^{l_2}$, where $k_1 l_2 - k_2 l_1 = 1$. The exponents $(c, d)$ of the monomials $x^{c} y^{d}$ in $R_v$ are obtained from the exponents $(a, b)$ of the corresponding monomials $x^a y^b$ by application of a special linear matrix (with the notations of Remark 3.3) it is the planar linear transformation changing the line $D$ into $\tilde{D} : c + d = \gamma$:

$$\begin{pmatrix} c \\ d \end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{where } A = \begin{pmatrix} k_1 & l_1 \\ k_2 & l_2 \end{pmatrix} \quad \text{with } \det(A) = k_1 l_2 - k_2 l_1 = 1.$$ 

So we obtain $\text{ord}_{x_v, y_v}(f) = c_0 + d_0 = \gamma$ and:

$$z = \sum_{c+d > c_0+d_0} \lambda_{a,b} x^{c} y^{d} \quad ; \quad \begin{pmatrix} c \\ d \end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

(2)(a) A linear map preserves barycenters, so $b_0 \leq \min(B_f)$, respectively $b_0 \geq \max(B_f)$, if and only if $d_0 \leq \min(\tilde{B}_f)$, respectively $d_0 \geq \max(\tilde{B}_f)$, where $\tilde{B}_f = \text{(a) We thank the referee of the J. of Algebra who pointed out the problem of the unicity of $B_f$.}
\[ \{d = k_2a + l_2b \mid b \in B_f \}. \] In the first case, we denote the change of coordinates of the last blow-up by \((x_\nu, y_\nu) \mapsto (x_\nu, \frac{z_\nu}{y_\nu})\) with \(\frac{z_\nu}{y_\nu} \in R \setminus m\). Setting \(t := \text{Res} \left( \frac{z_\nu}{y_\nu} \right)\), we compute as desired:

\[
\text{Res}(z) = \sum_{c+d=\gamma} \text{Res}(\lambda_{a,b})t^d, \quad \text{with } (c,d) = (k_1a + l_1b, k_2a + l_2b);
\]

\[
= \sum_{c+d=\gamma} \frac{t^{d_0}}{d_0} \text{Res}(\lambda_{a,b})t^{d-d_0} \in K[t] \setminus K, \quad d \geq d_0.
\]

In the second case, we make the other change of coordinates and we obtain also \(\text{Res}(z) \in K[t] \setminus K\) with \(t := \text{Res} \left( \frac{w_\nu}{x_\nu} \right)\). On the other hand, if min\((B_f) < b_0 < \max(B_f)\) (which implies that \(\max(B_f) - \min(B_f) \geq 2\)), for with instance \(t := \text{Res} \left( \frac{w_\nu}{x_\nu} \right)\), we obtain:

\[
\text{Res}(z) = \text{Res}(\lambda_{a_1,b_1})t^{d_1-d_0} + \cdots + \text{Res}(\lambda_{a_2,b_2})t^{d_2-d_0}, \quad \text{with } d_1 - d_0 < 0 < d_2 - d_0.
\]

We note [AL11 p.1] that \(t'\) is another generator of \(K_v = K(t)\) over \(K\) if and only if there exists \(\rho_1, \rho_2, \theta_1, \theta_2 \in K\) such that \(t := \frac{\rho_1t' + \rho_2t + \theta_1}{\theta_1t + \theta_2}\), and \(\rho_1\theta_2 - \rho_2\theta_1 \neq 0\). Thus we can write \(\text{Res}(z)\) as follows:

\[
\text{Res}(z) = \frac{\text{Res}(\lambda_{a_1,b_1})(\theta_1t' + \theta_2)^{d_2-d_0} + \cdots + \text{Res}(\lambda_{a_2,b_2})(\rho_1t' + \rho_2)^{d_2-d_0}}{\theta_1t' + \theta_2} = \frac{\text{Res}(\lambda_{a_1,b_1})(\theta_1t' + \theta_2)^{d_2-d_0}}{\theta_1t' + \theta_2} \cdot \frac{1 + \cdots + \text{Res}(\lambda_{a_2,b_2})(\rho_1t' + \rho_2)^{d_2-d_0}}{\theta_1t' + \theta_2}.
\]

In the case where \(\theta_1\rho_1 = 0\), \(\text{Res}(z)\) cannot be a polynomial in \(t'\). If \(\theta_1\rho_1 \neq 0\), to have \(\text{Res}(z)\) polynomial in \(t'\), we need \(\frac{\theta_1}{\rho_1}\), respectively \(\frac{\rho_2}{\theta_2}\), is a root of order \(d_0 - d_1\), respectively \(d_2 - d_0\), of the numerator. So we would have \(\frac{\theta_1}{\rho_1}\) root of \(\rho_1t' + \rho_2\), and \(\frac{\rho_2}{\theta_2}\) root of \(\theta_1t' + \theta_2\), contradicting the fact that \(\rho_1\theta_2 - \rho_2\theta_1 \neq 0\).

(b) With the preceding notations, when \(\min(B_f) = \max(B_f) = b_1\) (i.e. when the Newton polygon of \(f\) has only one vertex \(s_1 = s_2\)), necessarily \(b_0 \neq b_1\). Indeed, if not, we would, we would have \(\text{Res}(z) = \text{Res}(\lambda_{a_1,b_1}) \in K\), which would contradict the fact that \(\text{Res}(z)\) is transcendental over \(K\), and consequently that \(K_v\) is a dicritical divisor of \(f\).

\[ \square \]

4. The polynomial case.

In this section, we resume the notion of dicritical divisor introduced in [AL11 Section (6.2)] in the case of the ring \(k[x,y]\) of bivariate polynomials over a field \(k\). This notion is adapted to the Jacobian problem in dimension 2.

**Definition 4.1.** Let \(f \in k[x,y] \setminus k\). We call **dicritical divisor** of \(f\) any discrete valuation ring \(R_v\) of \((k[x,y])\) such that \(k[x,y] \not\subseteq R_v\) and \(k(f) \subset R_v\) with \(\text{Res}(f)\) transcendental over \(k\).

A polynomial map \(f \in k[x,y]\) is defined everywhere but at infinity, where it becomes a rational function. Let \(F(X : Y : Z) = Z^mf \left( \frac{x}{y}, \frac{y}{z} \right), m = \text{deg}(f)\), be the homogenized of \(f\) on \(\mathbb{P}_2(k)\). This function has points of indetermination \(\{F(X : Y : Z) = 0\}\), which are the points at infinity of the curve defined by \(f\). The center of \(v\) is in \(\text{Spec}(k[\phi, \psi])\) with \((\phi, \psi) = (\frac{1}{b}, \frac{1}{y})\) if \(x \not\in R_v\) (open set \(X \neq 0\) of \(\mathbb{P}_2(k)\)) or \((\phi, \psi) = (\frac{1}{b}, \frac{1}{y})\) if \(x \in R_v\) (open set \(Y \neq 0\) of \(\mathbb{P}_2(k)\)). For instance, in the first case, we obtain:

\[
f(x,y) = f(\phi, \psi) = \sum_{a,b} \lambda_{a,b}x^ay^b = x^m \sum_{a,b} \lambda_{a,b} \left( \frac{x}{y} \right)^{m-(a+b)} (\frac{y}{z})^b = \sum_{a,b} \lambda_{a,b} \phi^{m-(a+b)} \psi^b.
\]
Thus $f$ defines a rational function $\tilde{f}(\phi, \psi) \in QF(R) = k(x, y)$ for which $R_v$ is a dicritical divisor and such that $\phi^m \tilde{f} \in R$. In other words, a dicritical divisor of $f$ in the sense of (4.1) corresponds to a dicritical divisor of $\tilde{f}$ in the sense of (2.2) where $R$ is the local ring at a point at infinity of $f(x, y) = 0$. Moreover, at these points the hypotheses of (3.1) hold for $\tilde{f}$. As in the preceding section, we denote $K_v = k'[t]$ with $k'$ relative algebraic closure of $k$ in $K_v$ and $t$ transcendental over $k$. We deduce that:

**Corollary 4.2.** Let $R_v$ a dicritical divisor of $f \in k[x, y] \setminus k$ in the sense of (4.1). Then the element $t$ of (2.5) can be chosen so that $\text{Res}(f) \in k'[t]$.

This result can be seen as a complement to the study of the dicritical divisors at infinity for polynomials in two complex variables [Fou96]. The existence of these divisors in the general case (4.1) is not obvious. We leave to the reader the pleasure of reading the masterful argument (I) of [AL11 Section (6.2)].

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