SOLVING THE STRONGLY COUPLED 2D GRAVITY

III: STRING SUSCEPTIBILITY AND
TOPOLOGICAL N-POINT FUNCTIONS

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Abstract

We spell out the derivation of novel features, put forward earlier in a letter, of two dimensional gravity in the strong coupling regime, at $C_L = 7, 13, 19$. Within the operator approach previously developed, they neatly follow from the appearance of a new cosmological term/marginal operator, different from the standard weak-coupling one, that determines the world sheet interaction. The corresponding string susceptibility is obtained and found real contrary to the continuation of the KPZ formula. Strongly coupled (topological like) models—only involving zero-mode degrees of freedom—are solved up to sixth order, using the ward identities which follow from the dependence upon the new cosmological constant. They are technically similar to the weakly coupled ones, which reproduce the matrix model results, but gravity and matter quantum numbers are entangled differently.
1 Introduction

The truncation of the chiral operator algebra at $C_L = 7, 13, 19$ seems to be the key to the strong coupling regime of two dimensional gravity\footnote{Evidence that a similar mechanism is at work for $W_3$ gravity are given in ref.\cite{jones}.}. After the first hints\cite{first_hints}, the truncations theorems were proven at the level of primaries (for half integer spins) in ref.\cite{primary_truncations}, the final complete proof being given in a recent paper of ours\cite{our_paper}. In ref.\cite{5}, and in the present article, we analyse further the physics of these theories and begin solving the topological models proposed in ref.\cite{4}, where gravity is in the strong coupling regime.

In the strong coupling regime, the screening charges are complex, so that the dimensions of primaries are not real in general. These truncation theorems express the fact that, nevertheless, at the special values of $C_L$ the operator algebra of the chiral intertwinors—that appear as chiral components of the powers of the Liouville exponentials—has a consistent restriction to Verma modules with real Virasoro highest weights. Thus one is led to consider that strongly coupled two dimensional gravity should be expressible entirely in terms of the corresponding chiral components. This brings in drastic changes with respect to the weakly coupled regime. It rules out the Liouville exponentials, which are the well known local fields of the present description of 2D gravity in the conformal gauge, both classically and at the weakly coupled quantum level\cite{3}. However, a new set of local fields may be constructed, which is consistent with the truncation theorems. Then the corresponding physics follows quite naturally, using methods very similar to the weak coupling ones.

In ref.\cite{4} we emphasized that the transition from weak to strong coupling may be characterized by a deconfinement of chirality. Indeed, in the weak coupling regime, the Liouville exponentials may be consistently restricted to the sector where left and right chiral components have the same Virasoro weight, so that we may say that chirality (of gravity) is confined. In this regime, the quantum numbers associated with each screening charge may take independent values. In the strong coupling regime, on the contrary, the reality condition forces us to link the quantum numbers associated with the two screening charges, for each chirality. This is however incompatible with the equality between left and right weights, so that the (gravity) chirality fluctuates: it is deconfined. The main motivation to study the strongly coupled regime is of course the building of the (so called) non critical strings. Indeed, the balance of central charges prevents us from doing so in the weak coupling regime. The actual construction is still too complicated beyond the three point functions, but we have set up\cite{4,5} models—which we call topological since their only degrees of freedom are zero modes—where gravity is strongly coupled. We shall present details about their solution which were left out from ref.\cite{5}.

The article is organised as follows. In section 2 we derive the string susceptibility, drawing a parallel with the weak coupling calculation. In section 3, we study the three point functions of our topological models. It was derived earlier\cite{4} only with the same screening choice for the three legs, but we need to treat the mixed case. Section 4 is devoted to the derivation of the N point function from the Ward identities generated by taking derivatives with respect to the strong-coupling cosmological constant. Using Mathematica, we derive irreducible irreducible vertices, up to six
legs, which give a consistent perturbation theory up to sixth order. This is similar to earlier studies in the weak coupling regime \[10\] but we let the screening number fluctuate, contrary to this reference. Consistency of the perturbative expansion of the topological models leads us to counting rules for its relation with the Feynman path integrals, which could not be guessed a priori. However, at the end we verify that the same rules also hold in the weak coupling regime, if the screening numbers are allowed to fluctate. As a matter of fact, the two types of topological models (i.e. the continuous version of matrix models, and our new ones) are found to be very similar technically, although gravity and matter quantum numbers are entangled completely differently.

2 The string susceptibility

2.1 The weak coupling case revisited

To begin with, we review the derivation in the weak coupling regime, following essentially ref.\[7\]. We shall provide details left out previously which will shed light on the strong coupling case. In particular, it will be useful to draw a close parallel with the DDK argument. With our notations, the latter follows from the following effective action, with cosmological constant \(\mu_c\), background metric \(\hat{g}\):

\[
S_{\mu_c}^{(w)} = -\frac{1}{8\pi} \int dz d\bar{z} \sqrt{-\hat{g}} \{ \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + Q_L R^{(2)} \Phi + \mu_c e^{\alpha_- \Phi} \},
\]

(2.1)

The two screening charges \(\alpha_\pm\) and \(Q_L\) are related as usual:

\[
\alpha_- \alpha_+ = 2, \quad \alpha_- + \alpha_+ = Q_L,
\]

(2.2)

and the Liouville central charge is such that

\[
C_L = 1 + 3Q_L^2.
\]

(2.3)

For \(C_L > 25\), \(\alpha_\pm\) are real; this is the weak coupling regime. For \(25 > C_L > 1\), \(\alpha_\pm\) are complex; this is the strong coupling situation which we want to study further in the present article. The Virasoro weight of the operator \(\exp(-\beta \Phi)\) is \(\Delta_\nu = -\frac{1}{2} \beta (\beta + Q_L)\).

Recall the basic point of DDK. Consider the correlator

\[
\left< \prod_{\ell} e^{-\beta_{\ell} \Phi(z_{\ell}, \bar{z}_{\ell})} \right>_{\nu_h}^{(\mu_c)} = \left< \prod_{\ell} e^{-\beta_{\ell} \phi(z_{\ell}, \bar{z}_{\ell})} \right>_{\nu_h}^{\mu_c},
\]

(2.4)

\(\nu_h\) is the number of handles. The notation is that \(\Phi\) is the Liouville operator, while \(\phi\) is the \(c\)-number field that appears in the functional integral. The latter is unchanged if we change variable by letting \(\phi = \tilde{\phi} - \ln(\mu_c)/\alpha_-\). This gives

\[
\left< \prod_{\ell} e^{-\beta_{\ell} \Phi(z_{\ell}, \bar{z}_{\ell})} \right>_{\nu_h}^{(\mu_c)} = \left< \prod_{\ell} e^{-\beta_{\ell} \phi(z_{\ell}, \bar{z}_{\ell})} \right>_{1}^{\nu_h \mu_c \beta_{\ell} + Q_L (1 - \nu_h) / \alpha_-}.
\]

(2.5)

\[3\] The upper index “\(w\)” is to recall that this effective action is relevant to the weak coupling regime.
the last term comes from the linear term of the action and the Gauss-Bonnet theorem
\[ \frac{1}{8\pi} \int \int dz \bar{z} R^{(2)} = (1 - \nu_h). \] (2.6)

Now, we compare this procedure with the corresponding discussion in the operator
method. There, the operator algebra of each chiral component has a quantum group
symmetry of the type \( U_q(sl(2)) \odot \hat{U}_q(sl(2)) \), with
\[ q = e^{ih}, \quad \hat{q} = e^{\hat{h}}, \quad h = \pi \frac{\alpha^2}{2}, \quad \hat{h} = \pi \frac{\alpha^2}{2}. \] (2.7)

The two quantum group parameters are related by
\[ h \hat{h} = \pi^2, \quad h + \hat{h} = \frac{C_L - 13}{6}. \] (2.8)

The first relation shows that they are, in a sense, dual pairs. The above symbol \( \odot \) has a special meaning which was much discussed before[11, 4, 8]. The Hilbert
space of states is of course a direct sum of products of a left and a right Virasoro
Verma modules. These are characterized by the eigenvalue of the rescaled zero-
mode momenta \( \varpi, \bar{\varpi} \) of the Bäcklund free field with chiral components \( \vartheta_R(x) \), \( \vartheta_L(x) \) defined by writing on the cylinder,
\[ \vartheta_R(x) = q_0 + i p_0 x + i \sum_{n \neq 0} e^{-inx} p_n \frac{n}{n}, \quad \varpi = \frac{2i}{\alpha_-} p_0, \]
\[ \vartheta_L(x) = \bar{q}_0 + i \bar{p}_0 \bar{x} + i \sum_{n \neq 0} e^{-inx} \bar{p}_n \frac{n}{n}, \quad \varpi = \frac{2i}{\alpha_-} \bar{p}_0. \] (2.9)

A right Verma module is characterized by the highest-weight eigenvalue
\[ \Delta(\varpi) = \frac{h}{4\pi} (\varpi^2 - \bar{\varpi}^2), \quad \varpi_0 = 1 + \frac{\pi}{h} \] (2.10)
of \( L_0 \), with similar formulae for the left Verma modules. For \( \varpi = \pm \varpi_0 \), \( \Delta \) vanishes. This describes the two \( SL(2,C) \) invariant states. In ref.[7], it was shown that
the transformation law Eq.2.5 corresponds to the following definition for arbitrary
cosmological constant:
\[ e^{-\beta \Phi(\mu_c)} = \mu_c^{\beta/\alpha_-} \mu_c^{-\varpi/2} e^{-\beta \Phi} \mu_c^{-\varpi/2}. \] (2.11)

From this it is easy to recover the standard results for \( \nu_h = 0, \) and 1. First, consider
the sphere (\( \nu_h = 0 \)). In the operator method, the left hand side of Eq.2.5 is given by
\[ \left\langle \prod_{\ell} e^{-\beta_\ell \Phi(z_\ell, \bar{z}_\ell)} \right\rangle_{\mu_c}^{(0)} = \left\langle -\varpi_0, -\bar{\varpi}_0 \right| \prod_{\ell} e^{-\beta_\ell \Phi} \mu_c^{-\varpi_0, \bar{\varpi}_0} \right. . \] (2.12)
where the notation \( \left| \varpi_0, \bar{\varpi}_0 > \right. \) represents the state with left and right weights \( \Delta = \bar{\Delta} = 0 \). Making use of Eq.2.11, we get
\[ \left\langle \prod_{\ell} e^{-\beta_\ell \Phi(z_\ell, \bar{z}_\ell)} \right\rangle_{\mu_c}^{(0)} = \mu_c^{\sum_\ell \beta_\ell} \left\langle -\varpi_0, -\bar{\varpi}_0 \right| \mu_c^{-\varpi/2} \prod_{\ell} e^{-\beta_\ell \Phi} \mu_c^{-\varpi/2} \left| \varpi_0, \bar{\varpi}_0 > \right. . \]

\[ 4 \] For recent developments in this connection see refs.[8, 9].
The contribution of the term \( Q_L R^{(2)} \Phi \) of the effective action is recovered when \( \mu_{\pm}^\pm \) hit the left and right vacua, since, according to Eq. 2.2, \( Q_L/\alpha_- = \varpi_{0} \), and the result follows for \( \nu_{h} = 0 \). Next, for the torus \( \nu_{h} = 1 \), one should take the trace in the Hilbert space:

\[
\left\langle \prod_{\ell} e^{-\beta_{\ell} \Phi(z_{\ell}, \bar{z}_{\ell})} \right\rangle_{(\mu_{c})}^{(1)} = \text{Tr} \left[ \prod_{\ell} e^{-\beta_{\ell} \Phi} \right].
\]  

(2.13)

Making again use of Eq. 2.11 we now get

\[
\left\langle \prod_{\ell} e^{-\beta_{\ell} \Phi(z_{\ell}, \bar{z}_{\ell})} \right\rangle_{(0)}^{(\mu_{c})} = \mu_{c} \sum_{\ell} \beta_{\ell} \text{Tr} \left[ \mu_{c}^{-\varpi/2} \prod_{\ell} e^{-\beta_{\ell} \mu_{c} \varpi/2} \right] = \mu_{c} \sum_{\ell} \beta_{\ell} \left\langle \prod_{\ell} e^{-\beta_{\ell} \Phi(z_{\ell}, \bar{z}_{\ell})} \right\rangle_{(0)}^{(1)}.
\]

Thus, we get the same result without the term \( Q_L/\alpha_- \). This agrees with Eq. 2.5 with \( \nu_{h} = 1 \).

In order to compute the string susceptibility, we next consider

\[
Z^{(\nu_{h})}(A) \equiv \left\langle \delta \left[ \int dzd\bar{z} e^{\alpha_- \Phi - A} \right] \right\rangle_{(\nu_{h})}^{(\mu_{c})}.
\]  

(2.14)

It is clear, from the form of the \( \mu_{c} \) dependence of \( \exp(-\beta \Phi) \) that, without entering into the detailed definition of the \( \delta \) function, we should have

\[
\delta \left[ \int dzd\bar{z} e^{\alpha_- \Phi - A} \right]_{\mu_{c}} = \mu_{c}^{-\varpi/2} \delta \left[ \int dzd\bar{z} \mu_{c}^{-1} e^{\alpha_- \Phi - A} \right]_{1} \mu_{c}^{\varpi/2}.
\]  

(2.15)

Consider the sphere, one gets

\[
Z^{(0)}_{(\mu_{c})}(A) = \left\langle -\varpi_{0}, -\varpi_{0} | \mu_{c}^{-\varpi/2} \delta \left[ \int dzd\bar{z} \mu_{c}^{-1} e^{\alpha_- \Phi - A} \right]_{1} \mu_{c}^{\varpi/2} | \varpi_{0}, \varpi_{0} \right>. \tag{2.16}
\]

Assuming that for large area \( Z^{(0)}_{(\mu_{c})}(A) \sim A^{\gamma_{\text{str}} - 3} \), this gives the well known result

\[
\gamma_{\text{str}} = 2 - Q/\alpha_- = 1 - \frac{\pi}{h}.
\]  

(2.16)

A similar discussion also gives back the standard result for \( \nu_{h} = 1 \).

Next we describe the cosmological dependence of the chiral components which is such that Eq. 2.11 holds. Our basic assumption will be that it remains the same for the strong coupling regime. Contact with the quantum group classification is made by letting

\[
\beta = \alpha_- J + \alpha_+ \tilde{J}.
\]  

(2.17)

Then the corresponding local Liouville exponential operator is given by[3, 5]

\[
e^{-\beta \Phi(z, \bar{z})} = \sum_{m, \tilde{m}} \tilde{V}_{m, \tilde{m}}(z) \nu_{m, \tilde{m}}(\bar{z}).
\]  

(2.18)

This form is dictated by locality and closure under fusion. The notation for the chiral operators refers to a particular normalization whose precise expression will
be needed later on. In ref.\[1\], it was remarked that Eq.(2.11) is derivable from the following generalised Weyl transformation law

\[
\begin{align*}
\tilde{V}^{(J \tilde{J})}_{m \tilde{m}}(\mu) &= \mu^{J+\tilde{J}\pi/\hbar} \mu^{-\omega/2} \tilde{V}^{(J \tilde{J})}_{m \tilde{m}}(\mu) \mu^{\omega/2}, \\
\tilde{V}^{(\tilde{J} J)}_{m \tilde{m}}(\bar{\mu}) &= \tilde{\mu}^{\tilde{J}+J\pi/\hbar} \tilde{\mu}^{-\bar{\omega}/2} \tilde{V}^{(\tilde{J} J)}_{m \tilde{m}}(\bar{\mu}) \tilde{\mu}^{\bar{\omega}/2}.
\end{align*}
\] (2.19)

Taking two different parameters $\mu$ and $\tilde{\mu}$ is only relevent for the strong coupling case. For the weak coupling one, the left and right quantum numbers are all equal (chirality is confined) so that in deriving Eq.(2.11), one only deals with the product $\mu\bar{\mu}$. The previous discussion is immediately recovered with $\mu_c = \mu\bar{\mu}$.

### 2.2 The strong coupling regime

At this point we turn to the strong coupling regime. Now $1 < C_L < 25$. The screening charges $\alpha_\pm$ are complex and related by complex conjugation. Thus complex weights appear in general. The weight of a chiral operator $\tilde{V}^{(J \tilde{J})}_{m \tilde{m}}$ is equal to $\Delta(\omega_{J\tilde{J}})$, with $\omega_{J\tilde{J}} = \omega_0 + 2J + 2\tilde{J}\pi/\hbar$. We shall denote by $|J, \tilde{J}> >$ the corresponding highest weight state. There are two types of exceptional cases such that the weights are real. These are the states $|J, \tilde{J}> >$ (resp. $|J-1, J> >$), with negative (resp. positive) weights. One could try to work with the corresponding Liouville exponentials $\exp[-J(\alpha_- - \alpha_+)|\Phi]$ (resp. $\exp[(J+1)\alpha_- - \alpha_+)|\Phi]$), but this would be inconsistent, since these operators do not form a closed set under fusing and braiding. Moreover, the chiral vertex operators are such that

\[
< J_2, \tilde{J}_2|\tilde{V}^{(J \tilde{J})}_{m \tilde{m}}|J_1, \tilde{J}_1 > \propto \delta_{J_1, J_2-m} \delta_{\tilde{J}_1, \tilde{J}_2-\tilde{m}}.
\] (2.20)

As a result, it follows from Eqs.(2.18) that Liouville exponential operators do not preserve the reality condition for highest weights just recalled. The basic problem is that Eq.(2.18) involves the $\tilde{V}$ operators with arbitrary $m \tilde{m}$, while the reality condition forces us to only use $\tilde{V}$ operators of the type

\[
V^{(J)}_{m,+} \equiv \tilde{V}^{(-J-1)}_{-m,m}, \quad V^{(J)}_{m,-} \equiv \tilde{V}^{(J)}_{m,m}.
\] (2.21)

Next recall that the truncation theorems hold for

\[ C = 1 + 6(s + 2), \quad s = 0, \pm 1. \] (2.22)

For these values, there exists a closed chiral operator algebra restricted to the physical Hilbert space

\[
\mathcal{H}_{\text{phys}}^\pm \equiv \bigoplus_{r=0}^{1+\pm 1} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}^\pm_{r/2(2\pm s)+n/2}.
\] (2.23)

where $\mathcal{H}^\pm_J$ denotes the Verma modules with highest weights $|J+1/2 - 1/2, J> >$. The physical operators $\chi^{(J)}_{\pm}$ are defined for arbitrary $2J \in \mathbb{Z}/(2 \mp s)$, and $2J_1 \in \mathbb{Z}/(2 \mp s)$, to be such that

\[
\chi^{(J)}_{\pm} \mathcal{P}_{\mathcal{H}_{J_1}^\pm} = \sum_{\nu = \pm + m \in \mathbb{Z}_+} (-1)^{2(\pm s)(2J_1 + \nu + 1)/2} V^{(J)}_{m, \pm} \mathcal{P}_{\mathcal{H}_{J_1}^\pm}. \] (2.24)

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5 By the symbol $\mathbb{Z}/(2 \pm s)$, we mean the set of numbers $r/(2 \pm s) + n$, with $r = 0, \cdots, 1 \pm s$, $n$ integer; $\mathbb{Z}$ denotes the set of all positive or negative integers, including zero.

6 $\mathbb{Z}_+$ denotes the set of non negative integers.
where $\mathcal{P}_{\mathcal{H}_{J_1}^+}$ is the projector on $\mathcal{H}_{J_1}^\pm$. According to the formulae previously recalled, their weights are, respectively,

$$\Delta^-(J, C_L) = -\frac{C_L - 1}{6} J(J + 1), \quad \Delta^+(J, C_L) = 1 + \frac{25 - C_L}{6} J(J + 1). \quad (2.25)$$

Note that the cases of positive and negative weights are completely separated operatorially, even though we discuss them simultaneously to avoid repetitions. Moreover, the definition of $V_{m_+}^{(J)}$ is not symmetric between the two screening charges. For the left components, it is appropriate to make the other choice

$$V_{m_+}^{(J)} \equiv \tilde{V}_{m_+}^{(-J,J)} - m_+,$$

$$V_{m_-}^{(J)} \equiv \tilde{V}_{m_-}^{(J,J)} - m_-.$$  

(2.26)

With this, one defines the left physical fields by a formula similar to Eq. 2.24:

$$\chi^{(J)} \pm P_{\mathcal{H}_{J_1}^\pm} = \sum_{\nu \equiv J + m \in \mathbb{Z}} (-1)^{(J\pm)(\nu+1)/2} \frac{25 - C_L}{6} \chi^{(J)} \pm \nu(J + 1). \quad (2.27)$$

Since Eqs. 2.25 are invariant under $J \rightarrow -J - 1$, we get the same formulae for the weights of the left physical operators just defined:

$$\Delta^-(J, C_L) = -\frac{C_L - 1}{6} J(J + 1), \quad \Delta^+(J, C_L) = 1 + \frac{25 - C_L}{6} J(J + 1), \quad (2.28)$$

The particular way to define the left and right components just summarised has another motivation, which we stressed already before. Complex conjugation exchanges the two screening charges, so that the operators $\chi_{+}^{(J)}$, and $\chi_{+}^{(J)}$ are not hermitian. The present choice ensures that the amplitudes are nevertheless invariant under complex conjugation if we exchange left and right quantum numbers.

Now we have enough recollection to turn back to the string susceptibility. As is well known, the basic tool of the weak coupling derivation, was the existence of the cosmological term $\exp(\alpha_{+} - \Phi)$, of weight $(1,1)$, whose integral over $z \bar{z}$ defines the invariant area and may be added to the free field action (see Eq. 2.1) without breaking conformal invariance. In the strong coupling regime, this cosmological term is not acceptable since it does not preserve the reality conditions. On the other hand, Eqs. 2.25 2.28 are clearly such that $\Delta^+(0, C_L) = \Delta^+(0, C_L) = 1$. As we pointed out earlier, the corresponding operator

$$\mathcal{Y}^{0,0} = \chi^{(0)}_{+}(z) \bar{\chi}^{(0)}_{+}(\bar{z}). \quad (2.29)$$

defines a new cosmological term and is local. Thus the area element of the strong coupling regime is $\chi^{(0)}_{+}(z) \bar{\chi}^{(0)}_{+}(\bar{z}) dz d\bar{z}$. It is factorized into a simple product of a single $z$ component by a $\bar{z}$ component. The effective action Eq. 2.1 should be modified accordingly. However, it is reasonable to assume that the behaviour under global rescaling (global Weyl transformation) of the chiral components will remain essentially the same. Moreover, the reality condition recalled above led us [5] to take $\bar{\mu} = \mu^{h/\pi}$, with $\mu$ real. According to Eq. 2.19, this gives

$$\chi_{+}^{(J)}(\mu) = \mu^{-J-1+J\pi/h} \chi_{+}^{(J)}(\mu^{-w/2}) \quad (\mu^{w/2})$$

$$\bar{\chi}_{+}^{(J)}(\mu) = \mu^{-J-h/\pi} \bar{\chi}_{+}^{(J)}(\mu^{-w/2}) \quad (\mu^{-w/2}) \quad (2.30)$$

7 This is actually a notational convenience, since clearly the two choices are exchanged by letting $J \rightarrow -J - 1$.
For the cosmological term this gives
\[ \chi_{(\mu_0)}^{0,0} = \mu_c^{-1} \mu_c^{-(\omega + \omega h/\pi)/4} \nu_{(1)}^{0,0} \mu_c^{(\omega + \omega h/\pi)/4}, \] (2.31)
where we have let \( \mu_c = \mu^2 \), so that the overall factor becomes \( \mu_c^{-1} \) as it should. From here, the computation of the string susceptibility, already summarised in ref.\[\text{[3]}\] goes as follows. Consider, now the expectations values similar to Eq.\[\text{2.14}\]
\[ \mathcal{Z}^{(\nu h)}_{\mu c}(A) \equiv \left\langle \delta \left[ \int dzd\bar{z} \chi_{+}^{(0)}(\mu_c) - A \right]_{\mu_c}^{(\nu h)} \right\rangle. \] (2.32)

At this time, we only have the operator method at our disposal, so that we may define the right hand side only for \( \nu_h = 0 \), and \( \nu_h = 1 \) as, respectively,
\[ \mathcal{Z}^{(0)}_{\mu c}(A) \equiv < -\omega_0, -\omega_0 | \delta \left[ \int dzd\bar{z} \nu^{0,0} - A \right]_{\mu_c} | \omega_0, \omega_0 >, \]
\[ \mathcal{Z}^{(1)}_{\mu c}(A) \equiv \mathrm{Tr} \left\{ \delta \left[ \int dzd\bar{z} \nu^{0,0} - A \right]_{\mu_c} \right\}. \] (2.33)

Again we shall not need the detailed definition of the delta function. Only the Weyl transformation, analogous to Eq.\[\text{2.11}\] will matter, that is
\[ \delta \left[ \int dzd\bar{z} \nu^{0,0} - A \right]_{\mu_c} = \mu_c^{-1} (\omega + \omega h/\pi)/4 \delta \left[ \int dzd\bar{z} \nu^{0,0} - A \right]_{\mu_c}^{(\omega + \omega h/\pi)/4}. \] (2.34)

From there on the calculation proceeds exactly as in the weak coupling regime, and we shall not repeat it. The basic point is that the factors \( \mu_c^{1/2} (\omega + \omega h/\pi)/4 \) give real eigenvalues when they hit the left or right vacua, since \( \omega_0 (1 + h/\pi) = (C_L - 1)/6. \) This is contrast with the weak coupling factors \( \mu_c^{1/2} \) that would give complex result here. It came out from our choice of \( \mu, \tilde{\mu} \), which therefore ensures the reality of the string susceptibility. For \( \nu_h = 0 \) one finds \( \gamma_{\mathrm{str}} = (2 - s)/2 \). Comments about this result were given in ref.\[\text{[3]}\].

### 3 The three-point functions

#### 3.1 The gravity coupling constants revisited

The Verma modules of gravity are conveniently characterized by the eigenvalue of the rescaled momentum \( \omega \). The spectrum of eigenvalue is of the form \( \omega_{J, \tilde{J}} = \omega_0 + 2J + 2\tilde{J} \pi/h. \) The starting point to derive the three-point functions is the expression\[\text{[1]}\] of the matrix element of the \( \tilde{V} \) fields recalled above between such highest weight states. It is convenient to rewrite it under the form
\[ < -\omega_3 | \hat{V}^{(J, \tilde{J})}_{m_0} | \omega_2 > = \delta_{J_2 - J_3 - m_0} \delta_{\tilde{J}_2 - \tilde{J}_3 - m_0} g_{-\omega_4} \]
\[ g_{-\omega_4} = (-1)^{\nu_0} (i/2)^{\nu_0 + \tilde{\nu}} H_{\nu \tilde{\nu}}(\omega_1) H_{\nu \tilde{\nu}}(\omega_2) H_{\nu \tilde{\nu}}(\omega_3) \]
\[ \frac{H_{\nu \tilde{\nu}}(\omega_1 + 2\tilde{\nu}/h, \omega_2 - \omega_3)}{H_{\nu \tilde{\nu}}(\omega_1, \omega_2 - \omega_3)} \] (3.1)

with
\[ \omega_i = \omega_0 + 2J_i + 2\tilde{J}_i, \quad i = 1, 2, 3; \]
\[ \nu + \tilde{\nu} \frac{\pi}{h} = \frac{1}{2} (\omega_1 + \omega_2 + \omega_3 - \omega_0) \equiv \nu^c \] (3.2)
The $g$‘s are called coupling constants. Each factor $H_{\nu\nu'}(\varpi)$ was shown to be expressible as a sum over path. It is convenient to write it under the form (with $\varpi = \varpi_0 + 2J + 2J \frac{\pi}{h}$)

$$H_{\nu\nu'}(\varpi) = \left(\frac{-\pi}{h}\right)^{(1/4 + \hat{J})\nu} \left(\frac{-h}{\pi}\right)^{(1/4 + J)\nu} \left\| \frac{\varpi}{\varpi - \nu} \right\|. \quad (3.3)$$

In general we define the symbol $\left\| \frac{B^e}{A^e} \right\|$ for $B^e - A^e \equiv \nu^e = \nu + \hat{\nu} \frac{\pi}{h}$, with $\nu$ and $\hat{\nu}$ positive integers, as the following product of factors along a general path

$$\left\| \frac{B^e}{A^e} \right\| = \prod_{n=0}^{N-1} \left\{ \left(\frac{-h}{\pi}\right)^{R_{e-1/2}} \hat{F} \left[ \frac{R_{e+1} + \hat{R}_{e+1}}{2} - \frac{\pi}{2h} \right] \right\}^\epsilon_{e}/2, \quad (3.4)$$

where the product is taken along any path going from $A^e$ to $B^e$ with intermediate points $R_{e,\ell}$, such that

$$R_{e+1}^e - R_{e,\ell}^e = \epsilon_{\ell} + \frac{\pi}{h} \hat{\epsilon}_{\ell}, \quad \epsilon_{\ell} = 0, \pm 1, \hat{\epsilon}_{\ell} = 0, \pm 1, \quad \epsilon_{\ell} \hat{\epsilon}_{\ell} = 0; \quad (3.5)$$

Note that we shall always be in the case where $R_{\ell}$ and $\hat{R}_{\ell}$ are rational, while $\frac{h}{\pi}$ is not. Thus the last equation is unambiguous. We shall not bother about the precise choice of sheet for square root of gamma functions. It should be specified according to ref.[4]. It is not important since it will only appear in the final leg factors. This product does not depend upon the choice of path[4]. Up to the factor we have put in front of Eq.(3.3) it coincides with the one introduced in refs.[4], [11].

One may visualise the path as made up with $N$ segments of coordinates $\epsilon_{\ell}$, $\hat{\epsilon}_{\ell}$, starting from the point specified by $R_0^e$. Using the fact that $\epsilon_{\ell} \hat{\epsilon}_{\ell} = 0$, one may rewrite the factors $(-\pi/h)R_{e-1/2}$, and $(-h/\pi)R_{e-1/2}$ may be rewritten, respectively as $(-\pi/h)(R_{\ell} + \hat{R}_{\ell+1}-1)/2$, and $(-h/\pi)(R_{\ell} + \hat{R}_{\ell+1}-1)/2$, so that our definition only involves the midpoints $(R_{e,\ell} + \hat{R}_{e,\ell+1})/2$ of each segment. It is easy to verify that we have the composition law

$$\left\| \frac{B^e}{A^e} \right\| \left\| \frac{C^e}{B^e} \right\| = \left\| \frac{C^e}{A^e} \right\| \quad (3.6)$$

So far, we assumed that $\nu$ and $\hat{\nu}$ are positive integers. This last relation allows us to extend the definition to arbitrary signs. Next, it is useful to use the fact that $F(z)$ satisfies the relation $F(1 - z) = (F(z))^{-1}$ to derive the identity

$$\left\| \frac{B^e}{A^e} \right\| = \left\| \frac{1 + \frac{\pi}{h} - B^e}{1 + \frac{\pi}{h} - A^e} \right\|. \quad (3.7)$$

---

8 This may be seen, in particular, by making use of ref.[4]
Returning to Eq.\((3.1)\), one finds that it may be written as

\[
g_{\omega_1,\omega_2} = (-1)^{\nu \tilde{\nu}} \left(\frac{i}{2}\right)^{\nu + \tilde{\nu}} \left(-\frac{\pi}{h}\right)^{(\nu - \tilde{\nu})/2} \prod_{i=1}^{4} \frac{\omega_i - \nu^c}{\omega_i - \nu^c} = 0. \tag{3.8}
\]

The last term has been retransformed using Eq.\((3.7)\) so that it takes the same form as the other three with \(\omega = 0\). Letting \(\omega_4 = 0\), we may thus write compactly

\[
g_{\omega_1,\omega_2} = (-1)^{\nu \tilde{\nu}} \left(\frac{i}{2}\right)^{\nu + \tilde{\nu}} \left(-\frac{\pi}{h}\right)^{(\nu - \tilde{\nu})/2} \prod_{i=1}^{4} \frac{\omega_i - \nu^c}{\omega_i - \nu^c} \tag{3.9}
\]

### 3.2 The dressings

#### 3.2.1 The weak coupling case

In the weak coupling regime, one represents matter by another copy of the Liouville theory with a different central charge. Following our previous works the symbols pertaining to matter are noted with a prime (or, if convenient, with an index \(M\)). The relations between matter and gravity parameters are

\[
C_M = 26 - C_L, \quad h/\pi = -\pi/h', \quad \alpha'_\pm = \mp \alpha_\pm. \tag{3.10}
\]

One constructs local fields in analogy with Eq.\(2.18\):

\[
e^{-\left(J'\alpha'_- + \tilde{J}'\alpha'_+\right)} \Phi'(z, \bar{z}) = \sum_{m, \bar{m}} \tilde{V}^r_{m, \bar{m}}(\nu) \tilde{V}^r_{m, \bar{m}}(\nu) (z). \tag{3.11}
\]

\(\Phi'(z, \bar{z})\) is the matter field (it commutes with \(\Phi(z, \bar{z})\)). There are two possible dressing of these operators by gravity such that the total weights are \(\Delta = \Delta = 1\). The first is achieved by considering the vertex operators

\[
\mathcal{W}^{J', \tilde{J}'} \equiv e^{-\left(-\tilde{J}' - 1\right)\alpha_- + J'\alpha_+}\Phi - \left(J'\alpha'_- + (J')\alpha'_+\right)\Phi'. \tag{3.12}
\]

In particular for \(J' = \tilde{J}' = 0\), we get the cosmological term \(\exp(\alpha_-\Phi)\). For the associated Liouville zero modes, this means that \(\omega_{J, \tilde{J}} = -\omega_{J, \tilde{J}}\). The other choice of dressing leads to the operators

\[
\mathcal{W}_{\text{conjugate}}^{J', \tilde{J}'} = e^{-\left(J'\alpha_- - (J' + 1)\alpha_+\right)}\Phi - \left(J'\alpha'_- + (J')\alpha'_+\right)\Phi'. \tag{3.13}
\]

#### 3.2.2 The strong coupling case

We consider another copy of the strongly coupled theory, with central charge \(C_M = 26 - C_L\). Since this gives \(C_M = 1 + 6(-s + 2)\), we are also at the special values, and the truncation theorems applies to matter as well. This “string theory” has no transverse degree of freedom, and is thus topological. The complete dressed vertex operator are now

\[
\mathcal{V}^{J, \tilde{J}} = \chi^r(J) \chi^\dagger(J) \chi^{-r}(-J), \tag{3.14}
\]

\[
\mathcal{V}_{\text{conjugate}}^{J, \tilde{J}} = \chi^r(J) \chi^\dagger(-J) \chi^{-r}(J). \tag{3.15}
\]

As in the weak coupling formula, operators relative to matter are distinguished by a prime. The gravity part is taken to have positive weights so that it includes the cosmological term.
3.3 The matter coupling constants revisited

It is convenient to define symbols $\left\langle B^e_{\alpha e} \right\rangle'$ related to matter, similar to $\left\langle B^e \right\rangle$. Let

$$\left\langle B^e_{\alpha e} \right\rangle' = \prod_{\ell=0}^{N-1} \left\{ \left( \frac{\pi}{h'} \right)^{R^e_{\ell - 1/2}} F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{h'}{2\pi} \right] \epsilon_{\ell}/2 \right\} \times$$

$$\prod_{\ell=0}^{N-1} \left\{ \left( \frac{\pi}{h} \right)^{R^e_{\ell - 1/2}} F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{\pi}{2h'} \right] \right\}^{\gamma_{\ell}/2}. \quad (3.16)$$

It is the direct analogue of Eq.3.3, apart from a simple modification, namely, we put the factors $(\pi/h')^{R^e_{\ell - 1/2}}$ and $(h'/\pi)^{R^e_{\ell - 1/2}}$, instead of $(-\pi/h)^{R^e_{\ell - 1/2}}$, and $(-h'/\pi)^{R^e_{\ell - 1/2}}$. As we already know[7, 4, 5], tremendous simplifications occur when gravity and matter are put together. Our definitions are such that this appears neatly the level of the product symbols. Indeed, let us derive the following basic relations

$$\left\langle B^e_{\alpha e} \right\rangle' = \left\langle 1 + \frac{\pi}{h} A^e \right\rangle = \left\langle \frac{\pi}{h} (1 - A^e) \right\rangle. \quad (3.17)$$

The equality between the last two expressions is a direct consequence of Eq.3.7. In order to derive the first equality, one transforms the first line of Eq.3.16 by writing

$$F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{h'}{2\pi} \right] = \left\{ F \left[ 1 - \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{\pi}{2h} \right] \right\}^{-1}$$

Let us define the associated path in Liouville theory by $R^e_{\ell} = 1 - R^e_{\ell}$, so that the second term becomes the inverse of the Liouville factor $F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{\pi}{2h} \right]$. On the other hand, an easy calculation shows that for the other $F$ we have

$$F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{\pi}{2h} \right] = F \left[ \frac{R^e_{\ell} + R^e_{\ell+1}}{2} - \frac{h}{2\pi} \right].$$

It follows from the relations given above that $R^e_{\ell} = 1 - R^e_{\ell}$, and $\hat{R}^e_{\ell} = R^e_{\ell}$. Therefore the associated path of gravity is such that $\epsilon_{\ell} = \epsilon'_{\ell}$, $\hat{\epsilon}_{\ell} = -\epsilon'_{\ell}$. This allows to verify that all the other factors also match, and Eq.3.17 follows.

Finally, the matter coupling is given by

$$\hat{g}^{-\omega'_{\ell}}_{\omega_{\ell}', \omega_{\ell}'} = \left( \frac{i}{2} \right)^{\nu' + \nu'} \left( \frac{\pi}{h'} \right)^{(\nu' - \nu')/2} \prod_{i=1}^{4} \left\langle \frac{\pi}{h} \right\rangle \left\langle \omega_{\ell}' + \nu_{e} \right\rangle \left\langle \omega_{\ell}' - \nu_{e} \right\rangle. \quad (3.18)$$

3.4 The leg factors

In this part, we re-discuss the simplifications that occur when gravity and matter are put together at the most basic level, that is, when the coupling constants are multiplied. As originally discussed in [7], and as we just saw, the dressing conditions have a simple expression in terms of the $\omega'$'s, namely, one must have $\omega' = \pm \hat{\omega}$. The two choices of sign corresponds to the two possible screening charges, and are treated much in the same way. There are in fact only two really different situations, the first when one takes the same dressing condition on all three legs—this is the case we have considered in our previous works—and the second when one of the screening charges is of the opposite type. Let us discuss them in turn.
3.4.1 The same choice for every legs

This situation was discussed at length before. We return to it briefly as a preparation for the mixed screening choice. First consider the case where $\varpi'_i = \varpi_i \equiv \frac{h}{\pi} \varpi_i$. Note that, since by definition $\varpi_4 = 0$, the same screening condition holds (trivially) for $i = 4$. Applying the formulae given above one finds that $\nu^{\epsilon} = \frac{h}{\pi}(\nu^{\epsilon} + 1)$, and

$$\begin{pmatrix} \varpi'_i \\ \varpi'_i - \nu^{\epsilon} \end{pmatrix} = \begin{pmatrix} \varpi_i \\ \varpi_i - \nu \end{pmatrix}.$$

Making use of Eq.3.6 this leads to

$$\begin{pmatrix} \varpi'_i \\ \varpi'_i - \nu^{\epsilon} \end{pmatrix} = \begin{pmatrix} \varpi_i \\ \varpi_i - \nu \end{pmatrix} = \left\{ \left( -\frac{\nu}{\hbar} \right)^{2\nu + 1} F[\varpi_i] \right\}^{-1/2}.$$

Altogether, we get a product of leg factors

$$g_{\varpi_1 \varpi_2} g_{\varpi_3 \varpi_4} = (-1)^{\nu_3} \left( \frac{i}{2} \right)^{2\nu - 1} \left( -\frac{\hbar}{\pi} \right)^{2\nu} \prod_{i=1}^{4} \frac{1}{\sqrt{F[\varpi_i]}}. \quad (3.19)$$

The other screening condition $\varpi'_i = -\frac{h}{\pi} \varpi_i$ is treated similarly, obtaining

$$g_{\varpi_1 \varpi_2} g_{\varpi_3 \varpi_4} = (-1)^{\nu_3} \left( \frac{i}{2} \right)^{2\nu - 1} \left( -\frac{\hbar}{\pi} \right)^{2\nu} \prod_{i=1}^{4} \frac{1}{\sqrt{F[\varpi_i]}}. \quad (3.20)$$

3.4.2 The mixed case

The coupling constant $g_{\varpi_1 \varpi_2} g_{\varpi_3 \varpi_4}$ is completely symmetric and, by relabelling, we may always choose the third leg to be the one which differs from the others. Again there are two cases. First choose $\varpi'_i = \frac{h}{\pi} \varpi_i$ for $i = 1, 2$, and $\varpi'_3 = -\frac{h}{\pi} \varpi_3$. The relation between $\nu^{\epsilon}$ and $\nu^{\epsilon'}$ may be written as $\nu^{\epsilon'} = \frac{h}{\pi}(\nu^{\epsilon} - \varpi_3 + 1)$. A simple modification of a calculation performed above then gives, for $i = 1, 2$,

$$\begin{pmatrix} \varpi'_i \\ \varpi'_i - \nu^{\epsilon} \end{pmatrix} = \begin{pmatrix} \varpi_i \\ \varpi_i - \nu \end{pmatrix} = \begin{pmatrix} \varpi_i - \nu^{\epsilon} + \varpi_3 \\ \varpi_i + 1 \end{pmatrix} \begin{pmatrix} \varpi_i - \nu \end{pmatrix}$$

$$= \begin{pmatrix} \varpi_i \\ \varpi_i + 1 \end{pmatrix} \begin{pmatrix} \varpi_i - \nu^{\epsilon} + \varpi_3 \\ \varpi_i - \nu \end{pmatrix}. \quad (3.21)$$

where we have used the multiplication law Eq.3.6. For $i = 3$, we write instead $\nu^{\epsilon'} = -\frac{2\nu}{\pi} - 1 + \frac{2}{\pi}(\varpi_1 + \varpi_2)$. One now gets

$$\begin{pmatrix} \varpi'_3 \\ \varpi'_3 - \nu^{\epsilon} \end{pmatrix} = \begin{pmatrix} \varpi_3 \\ \varpi_3 - \nu \end{pmatrix} = \begin{pmatrix} \varpi_3 - \nu^{\epsilon} + \varpi_1 + \varpi_2 \\ \varpi_3 - \nu \end{pmatrix}. \quad (3.22)$$

The last term in Eq.3.6 may be treated in the same way as the third, and the result is given by the last equation with $\varpi_3 \to 0$. Altogether, one gets

$$g_{\varpi_1 \varpi_2} g_{\varpi_3 \varpi_4} = \left( \frac{i}{2} \right)^{2(\varpi_3 - \varpi_1 - \varpi_2)} \left( -\frac{\nu}{\hbar} \right)^{\nu + 1} \frac{1}{\sqrt{F[\varpi_1]}} \frac{1}{\sqrt{F[\varpi_2]}} \frac{1}{\sqrt{F[\varpi_3]}}. \quad (3.23)$$
When one performs the computation, one first arrives at the right hand side multiplied by the factor
\[
\left| \frac{\omega_1 + \omega_3 - \nu^e}{\omega_1 - \nu^e} \right| \left| \frac{\omega_2 + \omega_3 - \nu^e}{\omega_2 - \nu^e} \right| \left| \frac{\omega_1 + \omega_2 + \omega_3 - \nu^e}{\omega_3 - \nu^e} \right| - \nu^e.
\]
Making use of Eq. 3.17, one may verify that it is equal to one, thereby completing the derivation. Thus in this case also the result is a product of leg factors. A similar calculation allows us to deal with the other choice, obtaining
\[
g_{\omega_1 \omega_2}^{\omega_3} g_{\omega_1}^{\omega_2} g_{\omega_3}^{\omega_1} = \left( \frac{i}{2} \right)^{2(J_3 - J_1 - J_2) + 1} \left( -\frac{h}{\pi} \right)^{\nu - \nu} \left( \frac{1}{F[0]} \right)^2 \left( \frac{1}{F[\omega_1]} \right)^2 \left( \frac{1}{F[\omega_2]} \right)^2 \left( \frac{1}{F[\omega_3]} \right)^2.
\]

\section{3.5 The three point functions.}

The computation makes use of the relations just derived for the coupling constants.

Let us first consider the weak coupling case for comparison. One gets with the same screening on every leg\footnote{\textsuperscript{9}Up to separate multiplicative factors on each leg which we drop. See refs. \cite{4}, \cite{11} for detailed computations of these factors, in some cases.}:
\[
\left\langle \prod_{\ell} \mathcal{W}_{\nu_\ell, t_\ell} \right\rangle = \left( g_{\omega_3}^{\omega_1, \omega_2} g_{\omega_1}^{\omega_2, \omega_3} g_{\omega_2}^{\omega_1, \omega_3} \right)^2 = \frac{1}{F[0]} \prod_{i=1}^{3} \frac{1}{F[\omega_i]}. \tag{3.25}
\]
\[
\left\langle \prod_{\ell=1}^{3} \mathcal{W}_{\nu_\ell, t_\ell}^{\text{conj}} \right\rangle = \left( g_{\omega_3}^{\omega_1, \omega_2} g_{\omega_1}^{\omega_2, \omega_3} g_{\omega_2}^{\omega_1, \omega_3} \right)^2 = \frac{1}{F[0]} \prod_{i=1}^{3} \frac{1}{F[\omega_i]}. \tag{3.26}
\]
In the weak coupling regime with one different screening, one gets
\[
\left\langle \mathcal{W}_{\nu_\ell, t_\ell}^{\text{conj}} \prod_{\ell=1}^{3} \mathcal{W}_{\nu_\ell, t_\ell} \right\rangle = \left( g_{\omega_3}^{\omega_1, \omega_2} g_{\omega_1}^{\omega_2, \omega_3} g_{\omega_2}^{\omega_1, \omega_3} \right)^2 = \frac{1}{F[0]} \prod_{i=1}^{3} \frac{1}{F[\omega_i]}. \tag{3.27}
\]

Next consider the strong coupling case. First, in order to make contact with our previous work\footnote{\textsuperscript{4}} let us consider the completely symmetric case
\[
\left\langle \prod_{\ell} \mathcal{V}_{\nu_\ell, t_\ell} \right\rangle = g_{\omega_3}^{\omega_1, \omega_2} g_{\omega_1}^{\omega_2, \omega_3} g_{\omega_2}^{\omega_1, \omega_3} g_{\omega_3}^{\omega_1, \omega_2} = \frac{1}{F[0]} \prod_{i=1}^{3} \frac{1}{F[\omega_i]}. \tag{3.28}
\]
Next we shall rather make use of the non symmetric three-point function
\[
\left\langle \prod_{\ell=1}^{3} \mathcal{V}_{\nu_\ell, t_\ell} \right\rangle = g_{\omega_3}^{\omega_1, \omega_2} g_{\omega_1}^{\omega_2, \omega_3} g_{\omega_2}^{\omega_1, \omega_3} g_{\omega_3}^{\omega_1, \omega_2} = \frac{1}{F[0]} \prod_{i=1}^{3} \frac{1}{F[\omega_i]}. \tag{3.29}
\]
All the expressions of coupling constants we have obtained involve a factor $1/F(0)$ which is divergent. We drop it from now on, since it may be reabsorbed by an overall change of the normalization of the three point functions which will not matter for the forthcoming discussion.
4 N-point functions

4.1 Strong coupling regime

As shown in ref.[10], and as we will recall below, the key tools in deriving the higher point functions of the weak coupling regime are the Ward identities which may be derived from the path integral formulation Eq.(4.1) by taking derivatives with respect to the cosmological constant $\mu_c$. In so doing, the details of the effective action are not important. One only uses the fact that the dependence upon this parameter in Eq.(4.1) is entirely contained—by definition—in the last term of $S^{(w)}_{\mu_c}$. In the strong coupling regime another cosmological term ($V^{0,0}$) appears, and it is reasonable to assume that the higher point functions will be determined from the associated ward identities. In this case, however, the vertex operators create and destroy gravity chirality, so that the effective action cannot be constructed out of an ordinary world sheet boson. It should rather involve two independent chiral bosons with opposite chiralities. We leave its derivation for further studies, since its explicit form does not matter so much at present. We only assume that this theory can be described by an effective action including the new cosmological term:

$$S_{\mu_c} = S_0 + \mu_c \int V^{0,0}$$  \hspace{1cm} (4.1)

where the action $S_0$ does not depend on the cosmological constant $\mu_c$. So, we consider N-point correlators

$$\int e^{-S_{\mu_c} \frac{\partial}{\partial \mu_c}} \frac{\partial}{\partial \mu_c} \left( \frac{\partial}{\partial \mu_c} \right) \cdots \frac{\partial}{\partial \mu_c} \left( \frac{\partial}{\partial \mu_c} \right) \left( e^{S_{\mu_c}} \right) \quad \text{(4.2)}$$

where the integral sign stands for the functional integration and the integration over the position of the $N$ insertion points. For notational simplicity we drop the primes of the spins appearing in the vertex operators. As before, making use of Eq.(4.3) one derives the scaling law of these correlators:

$$\left( \frac{\partial}{\partial \mu_c} \right) \left( \frac{\partial}{\partial \mu_c} \right) \cdots \frac{\partial}{\partial \mu_c} \left( \frac{\partial}{\partial \mu_c} \right) \left( e^{S_{\mu_c}} \right) \quad \text{(4.3)}$$

with

$$P_i = P(J_i, \overline{J}_i) = \frac{Q M}{4} \left[ \alpha_+ (J_i + \frac{1}{2}) + \alpha_- (J_i + \frac{1}{2}) \right]. \quad \text{(4.4)}$$

Derivation with respect to the cosmological constant of these $N$ point functions gives $N+1$ point functions with one external momentum put to zero:

$$\left( \frac{\partial}{\partial \mu_c} \right) \left( \frac{\partial}{\partial \mu_c} \right) \cdots \frac{\partial}{\partial \mu_c} \left( \frac{\partial}{\partial \mu_c} \right) \left( e^{S_{\mu_c}} \right) \quad \text{(4.5)}$$

If this model can indeed be described by such an effective action, its perturbative expansion must yield Feynman rules. We already computed the three point vertices previously and we represent it graphically with an outgoing momentum for the leg $(J_1, \overline{J}_1)$:

$$\left( \frac{\partial}{\partial \mu_c} \right) \left( \frac{\partial}{\partial \mu_c} \right) \cdots \frac{\partial}{\partial \mu_c} \left( \frac{\partial}{\partial \mu_c} \right) \left( e^{S_{\mu_c}} \right) \quad \text{(4.6)}$$
with

\[ L_{J,J} = \frac{1}{\sqrt{F \left[ (1 - \frac{4}{N})(1 + 2J) \right] F\left(\left(1 - \frac{4}{N}\right)(1 + 2\overline{J})\right)}} \] (4.7)

\[ L_{J,J}^{\text{conj}} = \frac{1}{\sqrt{F \left[ -(1 - \frac{4}{N})(1 + 2J) \right] F\left(\left(1 - \frac{4}{N}\right)(1 + 2\overline{J})\right)}} \] (4.8)

from previous sections. The particular three-point function

\[ \left\langle \mathcal{V}^{\text{conj}}_{J_i,1,-J_i,-1,J_i} \mathcal{V}_{0,0}^{J_i} \mathcal{V}_{J_i} \right\rangle_{\mu_c} = \mu_c^{2P_i-1} L_{J_i,J_i}^{\text{conj}} L_{0,0} L_{J_i,J_i} \] (4.9)

is the derivative of the two point function. This allows us to compute it by taking the primitive and get

\[ \left\langle \mathcal{V}^{\text{conj}}_{J_i,1,-J_i,-1,J_i} \mathcal{V}_{J_i} \right\rangle_{\mu_c} = \frac{1}{2F_i} \mu_c^{2P_i} L_{J_i,J_i}^{\text{conj}} L_{J_i,J_i} \] (4.10)

by using that \( L_{0,0} = 1 \), which can be checked for every special value\(^{[10]} \) \( C_L = 7, 13, 19 \). For amplitudes with truncated external legs, such a two point function is the inverse of the propagator, and its value is consequently given by\(^{[11]} \)

\[ p_{\mu c}(J_i, \overline{J}_i) = P_i \frac{\mu_c^{-2P_i}}{(L_{J_i,J_i}^{\text{conj}} L_{J_i,J_i})} = \frac{J_i \overline{J}_i}{2} \] (4.11)

On the other hand, by taking derivatives, one gets higher order correlators. The derivatives of the three point functions yields four point functions with one external moment put to zero. However, the \( N \)-point functions should be symmetric in their \( N \) legs when expressed in terms of the variables we are using. (this is true for \( N = 3 \)) and we get by symmetrisation the generic four-point function

\[ \left\langle \mathcal{V}^{\text{conj}}_{J_1,1,-J_1,-1,J_2,J_2} \mathcal{V}_{J_3,J_4} \mathcal{V}_{J_3,J_4} \right\rangle_{\mu_c} = \mu_c^{P_1+P_2+P_3+P_4-\frac{\overline{z}+2}{2}} \]

\[ \left( \frac{Q^M}{4} \left[ \alpha'_-(J_1 + J_2 + J_3 + J_4 + 1) + \alpha'_+(J_1 + J_2 + J_3 + J_4 + 1) \right] \right) - 1 \] (4.12)

If we believe in this effective action description, it must yield Feynman rules by perturbative expansion. This total four point function\(^{[4,12]} \) can therefore be obtained from the following diagrams

\[ \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \] (4.13)

the last one being the one particule irreducible amplitude. It is clear on this example that, by choosing one incoming and \( N - 1 \) outgoing momenta, we have consistently

\(^{10}\) This property is again very specific to these central charges.

\(^{11}\) We rescaled by an unexplained factor \((1/2)\) which will turn out necessary so that the Feynman rules match with the derivation ones.
restricted ourselves to Feynman rules which only involve one particle irreducible vertices of the same type. Our purpose here is to get those one particle irreducible vertices, which are the basic ingredients of the theory. To do this, we shall subtract from the total four point function \ref{4.12} the first three diagrams of \ref{4.13} which we shall compute from the previous Feynman rules (the propagator \ref{4.11} and the vertices \ref{4.6} with their selection rules\footnote{Of the type \(J_1 + J_2 - J_{12}\) integer...}).

Let us figure out what shall give such a computation. In a diagram of the type
\[ J_2J_2 \quad J_3J_3 \quad J_4J_4, \]
and in any diagram, it can easily be verified that the leg factors coming from the propagators and the ones coming from the vertices cancel out. There only remains the leg factors attached to the external legs and we will consequently stop writing them down\footnote{We shall moreover only write formulae for \(\mu_c = 1\) as the general dependency in \(\mu_c = 1\) is now well known.}. So, the value of such a diagram is essentially given by the momentum \(P(J,J)\) given in \ref{4.4} coming from the propagator \ref{4.11} for \(J_i = J,\bar{J} = J\). We can then easily imagine that the sum of three such diagrams (see \ref{4.13}) can give something not very different from the total four point function \ref{4.12}, and hence by difference a quite simple four point one particle irreducible function.

However, it does not work this way. Due to the screening charges, or equivalently to the selection rules of the three point functions, the momentum is not conserved. In terms of spins, it means that in the previous diagram one can have \(J = J_3 + J_4 - \nu_2\) and \(\bar{J} = \bar{J}_3 + \bar{J}_4 - \bar{\nu}_2\) where \(\nu_2\) and \(\bar{\nu}_2\) are the number of screening charges attached to the second vertex (right hand side). These numbers can be any positive integers provided they do not exceed the total number of screening charges \(\nu = J_2 + J_3 + J_4 - J_1, \bar{\nu} = \bar{J}_2 + \bar{J}_3 + \bar{J}_4 - \bar{J}_1\) (this stems from the selection rules for the first vertex). And so, the only consistent way to proceed is to sum over all the possible internal momenta or spins \((J,\bar{J})\). So, the value of the diagram \ref{4.14} will be
\[
\sum_{\nu_2=0}^{\nu} \sum_{\bar{\nu}_2=0}^{\bar{\nu}} \frac{Q_M}{4}[\alpha'(J_3 + J_4 - \nu + \frac{1}{2}) + \alpha'_+(\bar{J}_3 + \bar{J}_4 - \bar{\nu} + \frac{1}{2})].
\]

We notice that these selection rules give arithmetic progressions for the spins \(J\) and \(\bar{J}\) and that the propagator is linear in these spins. It is hence clear that such a sum shall yield an overall (combinatorial) factor equal to the number of terms in it \(((\nu + 1)(\bar{\nu} + 1)\) in this case). It can be verified on each case. As a consequence the summation of the first three diagrams of \ref{4.13} over their possible internal momenta will not give something close to the value of the total four point function \ref{4.12}, as they include this extra combinatorial factor. We then conclude that the right value of the total four point function should include this factor. We consequently define the physical \(N\) point amplitudes by
\[
A^{(N)}_{\mu,\nu}((J_1,\bar{J}_1), (J_2,\bar{J}_2)\ldots (J_N,\bar{J}_N)) = \left(\begin{array}{c} \nu+N-3 \\ N-3 \end{array}\right) \left(\begin{array}{c} \bar{\nu}+N-3 \\ N-3 \end{array}\right) \langle \nu_{J_1,\bar{J}_1}^\text{conj} \nu_{J_2,\bar{J}_2} \nu_{J_3,\bar{J}_3} \ldots \nu_{J_N,\bar{J}_N} \rangle.
\]
The combinatorial factors of the type \( \binom{\nu + N - 3}{N - 3} \) are the number of ways of placing \( \nu \) screening charges on the \( N - 2 \) vertices of a \( N \) point function.\(^{14}\)

Of course, this modifies the derivation rule \( 4.15 \) in the following way:

\[
\begin{align*}
\text{Phys}^N_{A_{(\nu, \overline{\nu})}} ((J_1, \overline{J}_1), (J_2, \overline{J}_2), \ldots, (J_N, \overline{J}_N), (0, 0)) &= \left(\frac{\nu + N - 2}{N - 2}\right) (\frac{\overline{\nu} + N - 2}{N - 2}) \cdot \frac{\partial}{\partial \mu_c} \text{Phys}^N_{A_{(\nu, \overline{\nu})}} ((J_1, \overline{J}_1), (J_2, \overline{J}_2), \ldots, (J_N, \overline{J}_N)),
\end{align*}
\]

which gives now for the four point function

\[
\begin{align*}
\text{Phys}^{(N+1)}_{A_{(\nu, \overline{\nu})}} ((J_1, \overline{J}_1), (J_2, \overline{J}_2), (J_3, \overline{J}_3), (J_4, \overline{J}_4)) &= (\nu + 1)(\overline{\nu} + 1) \cdot \left(1 + \frac{Q_M}{4} \left[ \alpha'_-(J_1 + J_2 + J_3 + J_4 + 1) + \alpha'_+(J_1 + J_2 + J_3 + J_4 + 1) \right] \right)
\end{align*}
\]

which differs from \( 4.12 \) by the simple factor \( (\nu + 1)(\overline{\nu} + 1) \).

Substracting from this total four point function the first three reducible diagrams of \( 4.13 \), which are given by sums of the type \( 4.15 \), one gets the one particle irreducible correlator

\[
\begin{align*}
\text{Phys}^{A_{(\nu, \overline{\nu})}} ((J_1, \overline{J}_1), (J_2, \overline{J}_2), (J_3, \overline{J}_3), (J_4, \overline{J}_4)) &= (\nu + 1)(\overline{\nu} + 1) \cdot \frac{1}{4} \left( -2 + s \right) + \frac{Q_M}{2} (\alpha'_- \nu + \alpha'_+ \overline{\nu})
\end{align*}
\]

which does not depend on the momenta.

It works similarly for the higher order correlators, with a very quick growing complexity however. First of all, every diagram gets more complicated: instead of the double sum \( 4.17 \) the \( N \) point diagrams may include \( 2(N - 3) \) nested sums. Then, the number of diagrams go from 4 diagrams with 2 different skeletons for the four point function (cf \( 4.13 \)) to 26 diagrams of 5 different (oriented) skeletons\(^{15}\) for the five point function, that we represent here\(^{16}\)

\[
\begin{align*}
\text{Diagram} &= \text{Diagram} \, + \, \text{Diagram} \, + \, \text{Diagram} \, + \, \text{Diagram} \, + \, \text{Diagram} \, + \, \text{Diagram},
\end{align*}
\]

\(^{14}\) This can be obtained by developping in \( x \) the expression \( (1 + x + x^2 \ldots)^{N-2} = 1/(1 - x)^{N-2} \). Its coefficient of order \( \nu \) is \( (N - 2)(N - 1) \ldots (N + \nu - 3)/\nu! \), which is equal to the number of choices \( \binom{\nu + N - 3}{N - 3} \). It is remarquable that it may be expressed as a number of choices of \( N - 3 \) (or \( \nu \)) objects among \( \nu + N - 3 \), however we have no interpretation of it by now.

\(^{15}\) It is easy to see that the new derivation rule \( 4.17 \) does not change anything to our previous computation of the two point function from the three point one, as the number of screenings is zero in that case.

\(^{16}\) We say here that two Feynman diagrams have the same oriented skeleton if they can be “superposed” so that the arrows coincide. So, the diagrams with the same oriented skeleton are obtained by exchanging the \( N - 1 \) incoming legs \( (J_2, \ldots, J_N) \). The exchange with the outgoing leg \( J_1 \) gives a different oriented skeleton and leads to different computations because of the different selection rules and directions of internal arrows (first and second diagrams of \( 4.20 \) for instance).

\(^{17}\) We only draw the five oriented skeletons.
to 236 diagrams with 12 different oriented skeletons at six points, 2752 diagrams
with 34 skeletons at seven points... The five point function seems already almost
out of reach of human computational capabilities. This is why we used a computer
and the “Mathematica” program for symbolic calculation. We defined an oriented
diagram as a tree of which root is the outgoing momentum. We generated all of
them recursively, keeping only one diagram with each different skeleton in order
to speed up computation. Each of them was then computed recursively, the result
being given by multiple sums. They could be computed thanks to appropriate rules
for simplification and computation of sums. And we eventually obtained all the
five point function:

\[
\Pi^{(5)}_{(\nu, \bar{\nu})}((J_1, \bar{J}_1), (J_2, \bar{J}_2), (J_3, \bar{J}_3), (J_4, \bar{J}_4), (J_5, \bar{J}_5)) = \left(\frac{\nu^2}{2}\right) \left(\frac{\bar{\nu}^2}{2}\right) \left(B^{(5)}(\nu, \bar{\nu}) - \sum_{i=1}^{5} (P_i)^2\right) \tag{4.21}
\]

where \(B^{(5)}(\nu, \bar{\nu})\) is the following polynomial of degree 2 in \(\nu, \bar{\nu}\)

\[
B^{(5)}(\nu, \bar{\nu}) = 2 + 2a_+ + 2a_- + a_+^2 + 2a_-a_+ + cn^2
\]

\[
= \frac{5a_-\nu}{3} - \frac{5a_+\bar{\nu}}{3} - \frac{11a_+^2\nu}{12} - \frac{11a_-^2\bar{\nu}}{12} + \frac{a_-^2\nu^2}{4} + \frac{a_+^2\bar{\nu}^2}{4} + \frac{10a_-\nu - a_+a_-\bar{\nu}}{9} \tag{4.22}
\]

with \(a_\pm = Q_M\alpha_\pm^\prime/4\),

and for the six point function

\[
\Pi^{(6)}_{(\nu, \bar{\nu})}((J_1, \bar{J}_1), (J_2, \bar{J}_2), (J_3, \bar{J}_3), (J_4, \bar{J}_4), (J_5, \bar{J}_5), (J_6, \bar{J}_6)) = \left(\frac{\nu^2}{3}\right) \left(\frac{\bar{\nu}^2}{3}\right) \left[B^{(6)}(\nu, \bar{\nu}) + \frac{3}{2}(2+s) - a_-\nu - a_+\bar{\nu}\right] \left[\sum_{i=1}^{6} (P_i)^2\right] \tag{4.23}
\]

where \(2+s\) comes from \(4 + 2a_- + 2a_+\), with

\[
B^{(6)}(\nu, \bar{\nu}) = \frac{1}{16} \left(-96 - 144a_+ - 140a_+^2 - 46a_+^3 - 144a_- - 280a_-a_- - 138a_+a_- - 140a_-^2 - 138a_+a_-^2 - 46a_+a_- + 104a_-a_+ - a_+a_-\nu + a_+^2a_-\nu + 114a_+^2\nu \right.
\]

\[
+42a_+a_-^2\nu + 49a_+^3\nu - 28a_+^2\nu^2 + 6a_+a_-^2\nu^2 - 16a_-^2\nu^3 + 2a_+a_-^2\nu^3 + 104a_-a_-\nu + 114a_+a_-^2\bar{\nu} \right.
\]

\[
+49a_+^3\bar{\nu} + 18a_+a_-\bar{\nu} + 42a_+a_-^2\bar{\nu} + a_+a_-\bar{\nu} - 104a_+a_-\nu - a_+a_-\bar{\nu} - 36a_+a_-\nu - 36a_+a_-^2\nu \bar{\nu}
\]

\[
+17a_+a_-^2\nu^2 - 28a_+a_-^2\nu^2 - 16a_+a_-^2\nu^2 + 6a_+a_-^2\nu^2 + 17a_+a_-^2\nu^2 + 2a_+a_-^2\nu^3\right). \tag{4.24}
\]

The polynomials \(B^{(5)}\) and \(B^{(6)}\) are rather complicated and by now we have no
interpretation of them. On the other hand, the way these irreducible correlation
functions depend on the momenta is particularly interesting. First, they are symmetric
in all of their legs, including the outgoing one, which is not the case of the
reducible diagram. Second, they are symmetric under the exchange of $P_i$ into $-P_i$ (or equivalently $J_i, \tilde{J}_i$ into $-J_i - 1, -\tilde{J}_i - 1$), as we only have even powers of the momenta ($0$ or $2$). This is necessary to have a really symmetric correlator as one leg is outgoing (this was not true for the total correlators, see [4.18 e.g.]). This feature was already noticed in the weak coupling regime [10], where higher order correlators where computed at $c = 1$ in the case without screening charges (for matter). It was obtained there that the irreducible $N$ point functions only depend on momenta at even power $2 \text{Int}((N - 3)/2)$, which is similar to the present results in the strong coupling regime in the case with screenings. This symmetry of the irreducible correlation functions seems to be a good check of consistency of this description by an effective action.

4.2 Weak coupling regime

We draw a parallel with the case of weak coupling in order to show that many things in the previous derivation are not specific to the strong coupling regime. Similarly, we use two copies of the construction of the weak coupling regime, one for matter and the other for gravity, and get the dressed operators already introduced in Eqs. 3.12, 3.13. One knows from the so-called Seiberg bound that this second type of dressing is to be chosen only for negative momenta of matter, and this is why we reverse the spins in the “conj” operators. This was discussed in details in ref. [10], and it was shown there that the integral representations give non vanishing results only for one negative momentum, the others being positive. We shall simply follow this guide here and not go through the whole discussion as our main purpose is simply to show that many aspects of the previous derivations were not specific of the strong coupling regime. So we only consider (at least in a first step) correlators of the type

$$\langle \mathcal{W}_{\text{conj}}^{-J_1 - 1, -\hat{J}_1 - 1} \mathcal{W}^{J_2, \hat{J}_2} \cdots \mathcal{W}^{J_N, \hat{J}_N} \rangle_{\mu_c}. \quad (4.25)$$

Their scaling law was computed in Eq. 2.3 and may be rewritten

$$\langle \mathcal{W}_{\text{conj}}^{-J_1 - 1, -\hat{J}_1 - 1} \mathcal{W}^{J_2, \hat{J}_2} \cdots \mathcal{W}^{J_N, \hat{J}_N} \rangle_{\mu_c} \sim \mu_c \sum_i Q_i - \left(\frac{N}{2} - 1\right)(1 + \pi h) \quad (4.26)$$

with

$$Q_i = - \left(J_i + \frac{1}{2}\right) + \left(\hat{J}_i + \frac{1}{2}\right) \frac{\pi h}{\alpha_-} = \frac{\beta_i}{\alpha_-} + \frac{1}{2} \left(1 + \frac{\pi h}{\alpha_-}\right) = \frac{\beta_i}{\alpha_-} + \frac{Q}{2\alpha_-}. \quad (4.27)$$

The three point functions are obtained from previous computations and are equal to one, up to (different) leg factors:

$$\langle \mathcal{W}_{\text{conj}}^{-J_1 - 1, -\hat{J}_1 - 1} \mathcal{W}^{J_2, \hat{J}_2} \mathcal{W}^{J_3, \hat{J}_3} \rangle_{\mu_c} = M_{J_1, \hat{J}_1}^\text{conj} M_{J_2, \hat{J}_2} M_{J_3, \hat{J}_3} \mu_c^{Q_1 + Q_2 + Q_3 - \frac{1}{2}(1 + \pi h)} \quad (4.28)$$
with
\[ M_{\mathcal{J}, \mathcal{J}} = \frac{1}{F(1 + 2 \mathcal{J} + \frac{\pi}{4} \mathcal{J})}, \quad M_{\mathcal{J}, \mathcal{J}}^\text{conj} = \frac{1}{F(1 + 2 \mathcal{J} + \frac{\pi}{4} \mathcal{J})}. \] (4.29)

From the particular three point function with one momentum put to zero, one deduces the two point function by integration. The propagator is given by its inverse and is again given by the moment:
\[ P_{\mu c}(\mathcal{J}_i, \hat{\mathcal{J}}_i) = Q_i = \frac{\mathcal{J}_i \hat{\mathcal{J}}_i}{P} \] (4.30)
where from now on we omit the leg factors and the \( \mu_c \) dependence.

All these derivations seem really similar to the ones performed in the case of strong coupling. And it is actually possible to show that, up to some correspondence, all the diagrammatic computations are equivalent. Although the underlying physics is completely different the following correspondence
\[ V_{\mathcal{J}_i, \mathcal{J}_i} \rightarrow W_{\mathcal{J}_i, \hat{\mathcal{J}}_i}, \quad \mathcal{J}_i \rightarrow \hat{\mathcal{J}}_i, \quad P_i \rightarrow Q_i, \quad a_- \equiv Q_M \alpha'/4 \rightarrow -1, \quad a_+ \equiv Q_M \alpha'/4 \rightarrow \pi/h = \alpha^2/2 \equiv \rho \] (4.31)
relates both diagramatics. This can easily be checked for the dependence in \( \mu_c \) (Eqs. 4.3 and 4.26 respectively\(^1\)), for the momenta (Eqs. 4.4 and 4.27), for the three point function (Eqs. 4.6 and 4.28 respectively), and for the propagators (Eqs. 4.11 and 4.30). Consequently the higher order correlators and diagrams obtained either by derivation or from these Feynman rules obey the same correspondence. So we get immediately the one particle irreducible correlators of the weak coupling regime by replacing \( \mathcal{J}_i, a_-, a_+ \) by \( \hat{\mathcal{J}}_i, -1, \rho \) respectively. The four point one particle irreducible amplitude is now given by
\[ 1^{PI}_A^{(4)}((\mathcal{J}_1, \hat{\mathcal{J}}_1), (\mathcal{J}_2, \hat{\mathcal{J}}_2), (\mathcal{J}_3, \hat{\mathcal{J}}_3), (\mathcal{J}_4, \hat{\mathcal{J}}_4)) = (\nu + 1)(\bar{\nu} + 1) \frac{1}{2} (-1 - \rho - \nu + \rho \bar{\nu}) \] (4.32)
where it was used that the first term in 4.19 comes from \(-(2+s)/4 = -1+(2-s)/4 = -1-(a_-+a_+)/2\) which gives via the correspondence \(-1 - ((-1) + (\pi/h))/2 = -(1 + \rho)/2 = -(1 + \rho)/2. \) The five point function is
\[ 1^{PI}_A^{(5)}((\mathcal{J}_1, \hat{\mathcal{J}}_1), (\mathcal{J}_2, \hat{\mathcal{J}}_2), (\mathcal{J}_3, \hat{\mathcal{J}}_3), (\mathcal{J}_4, \hat{\mathcal{J}}_4), (\mathcal{J}_5, \hat{\mathcal{J}}_5)) \]

\(^1\)Up to a factor 1/2 here again
\(^19\) Using \((2+s)/2 = 2 - (2-s)/2 = 2 + a_- + a_+ \rightarrow 2 + (-1) + (\pi/h) = 1 + \pi/h.\)
The restriction of the previous formulae to this case yields numbers fluctuate, contrary to ref. [10]. We may expect progress in the future.

Our Feynman perturbation involves novel features as compared with the previous discussion of ref. [10], but we showed that they also appear in the weak coupling regime, once we let the screening elements. A basic point is the deconfinement of chirality whose world sheet meaning is still mysterious. One way to gain insight would be to study our topological models, and their descriptions of the DDK type which should include a pair of chiral bosons of opposite chiralities instead of the Liouville field. Our Feynman perturbation physics just described is needed. In particular one would like to reach a geometrical understanding of our new cosmological term and of its associated area element. A basic point is the deconfinement of chirality whose world sheet meaning is still mysterious. One way to gain insight would be to study our topological models, and their descriptions of the DDK type which should include a pair of chiral bosons of opposite chiralities instead of the Liouville field. Our Feynman perturbation involves novel features as compared with the previous discussion of ref. [10], but we showed that they also appear in the weak coupling regime, once we let the screening numbers fluctuate, contrary to ref. [10]. We may expect progress in the future.

5 Outlook

The general conclusion of the present study is that the strong coupling physics is governed by the new cosmological term, very much the way the weak coupling arises from the standard Liouville theory. Of course a better understanding of the strong coupling physics just described is needed. In particular one would like to reach a geometrical understanding of our new cosmological term and of its associated area element. A basic point is the deconfinement of chirality whose world sheet meaning is still mysterious. One way to gain insight would be to study our topological models, and their descriptions of the DDK type which should include a pair of chiral bosons of opposite chiralities instead of the Liouville field. Our Feynman perturbation involves novel features as compared with the previous discussion of ref. [10], but we showed that they also appear in the weak coupling regime, once we let the screening numbers fluctuate, contrary to ref. [10]. We may expect progress in the future.
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