Molecular characterizations of variable anisotropic Hardy spaces with applications to boundedness of Calderón–Zygmund operators

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Abstract

Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) be a variable exponent function satisfying the globally log-Hölder continuous condition and \( A \) a general expansive matrix on \( \mathbb{R}^n \). Let \( H_A^{p(\cdot)}(\mathbb{R}^n) \) be the variable anisotropic Hardy space associated with \( A \) defined via the non-tangential grand maximal function. In this article, via the known atomic characterization of \( H_A^{p(\cdot)}(\mathbb{R}^n) \), the author establishes its molecular characterization with the known best possible decay of molecules. As an application, the author obtains a criterion on the boundedness of linear operators on \( H_A^{p(\cdot)}(\mathbb{R}^n) \), which is used to prove the boundedness of anisotropic Calderón–Zygmund operators on \( H_A^{p(\cdot)}(\mathbb{R}^n) \). In addition, the boundedness of anisotropic Calderón–Zygmund operators from \( H_A^{p(\cdot)}(\mathbb{R}^n) \) to the variable Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) is also presented. All these results are new even in the classical isotropic setting.

Keywords

Expansive matrix · (variable)Hardy space · Molecule · Calderón–Zygmund operator

Mathematics Subject Classification

42B35 · 42B30 · 42B20 · 46E30

1 Introduction

The main purpose of this article is to establish a molecular characterization of the variable anisotropic Hardy space \( H_A^{p(\cdot)}(\mathbb{R}^n) \) from [28], where \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) is a variable exponent function satisfying the so-called globally log-Hölder continuous condition [see (2.4) and (2.5)] and \( A \) a general expansive matrix on \( \mathbb{R}^n \) (see

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Definition 2.1). It is well known that a molecule is a natural generalization of a atom with the support condition replaced by some decay condition. Note that, to obtain the boundedness of linear operators, which are bounded on $L^2(\mathbb{R}^n)$, from the classic Hardy space $H^p(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, it suffices to show that the $L^p(\mathbb{R}^n)$ norm of the image of these operators acting on $(p, 2, s)$-atoms can be uniformly controlled by a harmless positive constant (see [19, Corollary 1.3]). However, it is complicated to investigate the boundedness of these operators on $H^p(\mathbb{R}^n)$ due to the fact that the image of these operators acting on atoms may no longer be atoms. Fortunately, instead of atoms, one can use molecules because many of these linear operators (for instance, one of the most basic operators in harmonic analysis, Calderón–Zygmund operators) usually map an atom into a harmless positive constant multiple of a related molecule, which further implies the desired boundedness. Thus, the molecular characterization plays a key role in studying the boundedness of many important operators on Hardy-type spaces; see, for instance, [10, 11, 24, 27, 31, 32, 36]. For more progress about molecules, we refer the reader to [9, 17, 18].

Recall that, as a generalization of the classical Hardy space $H^p(\mathbb{R}^n)$, the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, with the constant exponent $p$ replaced by a variable exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$, was first introduced by Nakai and Sawano [33] and, independently, by Cruz-Uribe and Wang [13] with some weaker assumptions on $p(\cdot)$ than those used in [33]. Later, Sawano [34], Yang et al. [39] and Zhuo et al. [41] further completed the real-variable theory of these variable Hardy spaces. For more progress about function spaces with variable exponents, we refer the reader to [1, 2, 12, 15, 16, 22, 23, 37, 38, 40]. On the other hand, motivated by the important role of discrete groups of dilations in wavelet theory, Bownik [4] originally introduced the anisotropic Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$, with $p \in (0, \infty)$, which also gave a unified framework of the real-variable theory of both the classical Hardy space and the parabolic Hardy space of Calderón and Torchinsky [8]. Later on, Bownik et al. [5] further extended the anisotropic Hardy space to the weighted setting. Nowadays, the anisotropic setting has proved useful not only in developing function spaces arising in harmonic analysis, but also in many other branches such as the wavelet theory (see, for instance, [3, 4, 14]) and partial differential equations (see, for instance, [7, 21]).

The year before last, Liu et al. [28] introduced the variable anisotropic Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ associated with $A$, via the non-tangential grand maximal function and established its various real-variable characterizations, respectively, by means of the radial or the non-tangential maximal functions, atoms, finite atoms, the Lusin area function, the Littlewood–Paley $g$-function or $g_A^*$-function. As applications, the boundedness of the maximal operators of the Bochner–Riesz and the Weierstrass means from $H^{p(\cdot)}_A(\mathbb{R}^n)$ to the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ was also obtained in [28]. Moreover, these real-variable characterizations of the space $H^{p(\cdot)}_A(\mathbb{R}^n)$ have proved very useful in the study on the real interpolation between $H^{p(\cdot)}_A(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ (see [30]).

Nevertheless, the molecular characterization of $H^{p(\cdot)}_A(\mathbb{R}^n)$, which can be conveniently used to prove the boundedness of many important operators (for instance, Calderón–Zygmund operators) on the space $H^{p(\cdot)}_A(\mathbb{R}^n)$, is still missing. Therefore, to further complete the real-variable theory of variable anisotropic Hardy spaces
$H^p_{\infty} (\mathbb{R}^n)$, in this article, we establish a molecular characterization of the space $H^p_A (\mathbb{R}^n)$, where the range of the used decay index $\varepsilon$ is the known best possible in some sense [see Remark 3.1(iv)]. As an application, we then obtain a criterion on the boundedness of linear operators on $H^p_A (\mathbb{R}^n)$ (see Theorem 4.1), which further implies the boundedness of anisotropic Calderón–Zygmund operators on $H^p_A (\mathbb{R}^n)$. Finally, the boundedness of anisotropic Calderón–Zygmund operators from $H^p_A (\mathbb{R}^n)$ to the variable Lebesgue space $L^p (\mathbb{R}^n)$ is also presented.

The organization of the remainder of this article is as follows.

In Sect. 2, we first give some notation used throughout this article and then recall some notions on expansive matrices and homogeneous quasi-norms as well as variable Lebesgue spaces and variable anisotropic Hardy spaces.

Section 3 is devoted to establishing a molecular characterization of $H^p_A (\mathbb{R}^n)$ (see Theorem 3.1). To this end, we first introduce the variable anisotropic molecular Hardy space $H^p_{\infty} (\mathbb{R}^n)$ (see Definition 3.2). Then, by the known atomic characterization of $H^p_A (\mathbb{R}^n)$ obtained in [28, Theorem 4.8] (see also Lemma 3.3), we easily find that $H^p_A (\mathbb{R}^n) \subset H^p_{\infty} (\mathbb{R}^n)$ and the inclusion is continuous. Thus, to prove Theorem 3.1, it suffices to show that $H^p_{\infty} (\mathbb{R}^n)$ is continuously embedded into $H^p_A (\mathbb{R}^n)$. Note that, to show the embedding of this type, the well-known strategy is to decompose a molecule into an infinite linear combination of the related atoms (see, for instance, [32, (7.4)] or [26, (3.23)]), which does not work in the present situation because the uniformly upper bound estimate of the dual bases of the natural projection of each molecule on the infinite annuli of a dilated ball (see [32, (7.2)] or [26, (3.18)]) is still unknown due to its anisotropic structure. To overcome this difficulty, by borrowing some ideas from the proofs of [27, Theorem 3.12] and [33, Theorem 5.2] as well as fully using the integral size condition of a molecule [see Definition 3.1(i)], we directly estimate the non-tangential maximal function of a molecule on the infinite annuli of a dilated ball [see (3.6)] and then obtain that $H^p_{\infty} (\mathbb{R}^n) \subset H^p_A (\mathbb{R}^n)$ with continuous inclusion, which completes the proof of Theorem 3.1. Here, we should point out that, in the proof of [27, Theorem 3.12], the useful properties of the growth function $\phi$ play a key role (see [27, (3.5)]); however, that approach is obviously invalid in the present situation due to its variable exponent setting. Instead, we use a technical lemma, which can reduce some estimates related to $L^p (\mathbb{R}^n)$ norms for some series of functions into dealing with the $L^q (\mathbb{R}^n)$ norms of the corresponding functions [see Lemma 3.2 and (3.5)]. In addition, in [33, Theorem 5.2], Nakai and Sawano established a molecular characterization of the variable isotropic Hardy space $H^p (\mathbb{R}^n)$; however, the molecule used in [33] is associated with a particular pointwise size condition, which is much stronger than the integral size condition of a molecule used in Theorem 3.1. In this sense, the conclusion of Theorem 3.1 also improves the corresponding one of [33, Theorem 5.2].

As applications, in Sect. 4, we investigate the boundedness of anisotropic Calderón–Zygmund operators from $H^p_A (\mathbb{R}^n)$ to itself (see Theorem 4.2) or to the variable Lebesgue space $L^p (\mathbb{R}^n)$ (see Theorem 4.3). To this end, by the known finite atomic characterization of $H^p_A (\mathbb{R}^n)$ and the molecular characterization of $H^p_A (\mathbb{R}^n)$ presented in Sect. 3, we first establish a useful criterion on the boundedness of linear operators on $H^p_A (\mathbb{R}^n)$ (see Theorem 4.1), which shows that, for any given
linear operator $T$, if it maps each atom into a related molecule, then $T$ has a unique bounded linear extension on $H^p_A(\mathbb{R}^n)$. Applying this criterion and an auxiliary lemma from [31] (see also Lemma 4.4), we then prove Theorem 4.2. Moreover, a procedure similar to that used in the proof of Theorem 4.2 with some technical modifications shows that Theorem 4.3 also holds true.

Finally, we make some conventions on notation. We always let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $0_n$ be the origin of $\mathbb{R}^n$. For any given multi-index $\gamma := (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\gamma| := \gamma_1 + \cdots + \gamma_n$ and $\partial^\gamma := (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n}$.

We denote by $C$ a positive constant which is independent of the main parameters, but may vary from line to line. The symbol $f \lesssim g$ means $f \leq Cg$, and if $f \lesssim g \lesssim f$, then we write $f \sim g$. If $f \leq Cg$ and $g = h$ or $g \leq h$, we then write $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leq h$. In addition, for any set $E \subset \mathbb{R}^n$, we denote by $\mathbb{I}_E$ its characteristic function, by $E^c$ the set $\mathbb{R}^n \setminus E$ and by $|E|$ its $n$-dimensional Lebesgue measure. For any $r \in [1, \infty]$, we denote by $r'$ its conjugate index, namely $1/r + 1/r' = 1$, and by $\lfloor t \rfloor$ the largest integer not greater than $t$ for any $t \in \mathbb{R}$.

## 2 Preliminaries

In this section, we recall the notions of expansive matrices and variable anisotropic Hardy spaces (see, for instance, [4, 28]).

The following definition of expansive matrices is from [4].

**Definition 2.1** A real $n \times n$ matrix $A$ is called an expansive matrix (shortly, a dilation) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the set of all eigenvalues of $A$.

Let $b := |\det A|$. Then, by [4, p. 6, (2.7)], we know that $b \in (1, \infty)$. From the fact that there exist an open ellipsoid $\Delta$, with $|\Delta| = 1$, and $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$ (see [4, p. 5, Lemma 2.2]), it follows that, for any $i \in \mathbb{Z}$, $B_i := A^i \Delta$ is open, $B_i \subset rB_i \subset B_{i+1}$ and $|B_i| = b^i$. For any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, an ellipsoid $x + B_i$ is called a dilated ball. Throughout this article, we always use $\mathcal{B}$ to denote the set of all such dilated balls, namely

$$\mathcal{B} := \{x + B_i : x \in \mathbb{R}^n, i \in \mathbb{Z}\},$$

and let

$$\tau := \inf \{k \in \mathbb{Z} : r^k \geq 2\}.$$
Definition 2.2 Let $A$ be a given dilation. A measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ is called a \textit{homogeneous quasi-norm}, associated with $A$, if

(i) $x \neq 0_n$ implies $\rho(x) \in (0, \infty)$;
(ii) for each $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$;
(iii) there exists a constant $C \in [1, \infty)$ such that, for any $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq C[\rho(x) + \rho(y)]$.

For any given dilation $A$, by [4, p. 6, Lemma 2.4], we can use the \textit{step homogeneous quasi-norm} $\rho$ defined by setting, for any $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{i \in \mathbb{Z}} b^i 1_{B_{i+1} \setminus B_i}(x) \quad \text{when } x \neq 0_n,$$

for convenience.

For any measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$, let

$$p_- := \essinf_{x \in \mathbb{R}^n} p(x), \quad p_+ := \esups_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p := \min\{p_-, 1\}. \quad (2.3)$$

Denote by $P(\mathbb{R}^n)$ the \textit{set of all measurable functions} $p(\cdot)$ satisfying $0 < p_- \leq p_+ < \infty$.

Let $f$ be a measurable function on $\mathbb{R}^n$ and $p(\cdot) \in P(\mathbb{R}^n)$. Then, the \textit{modular functional} (or, for simplicity, the \textit{modular}) $\omega_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\omega_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx,$$

and the \textit{Luxemburg} (also called \textit{Luxemburg–Nakano}) \textit{quasi-norm} $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ by setting

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \omega_{p(\cdot)}(f / \lambda) \leq 1 \right\}.$$

Moreover, the \textit{variable Lebesgue space} $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ satisfying that $\omega_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

A function $p(\cdot) \in P(\mathbb{R}^n)$ is said to satisfy the \textit{globally log-Hölder continuous condition}, denoted by $p(\cdot) \in C_{\log}^g(\mathbb{R}^n)$, if there exist two positive constants $C_{\log}(p)$ and $C_{\infty}$, and a constant $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1 / \rho(x - y))}, \quad (2.4)$$

and

$$|p(x) - p_\infty| \leq \frac{C_{\infty}}{\log(e + \rho(x))}. \quad (2.5)$$

Recall also that a \textit{Schwartz function} is an infinitely differentiable function $\phi$ satisfying, for any $\ell \in \mathbb{Z}_+$ and multi-index $\alpha \in \mathbb{Z}_+^n$. 
\[ \| \phi \|_{a, \alpha} := \sup_{x \in \mathbb{R}^n} |\rho(x)|^\alpha |\partial^\alpha \phi(x)| < \infty. \]

Let \( S(\mathbb{R}^n) \) be the set of all Schwartz functions as above, equipped with the topology determined by \( \{ \| \cdot \|_{a, \alpha} \}_{a \in \mathbb{Z}_+^n, \alpha \in \mathbb{Z}_+} \), and \( S'(\mathbb{R}^n) \) its dual space, equipped with the weak-* topology. For any \( N \in \mathbb{Z}_+ \), denote by \( S_N(\mathbb{R}^n) \) the following set:

\[
\left\{ \phi \in S(\mathbb{R}^n) : \| \phi \|_{S_N(\mathbb{R}^n)} := \sup_{a \in \mathbb{Z}_+^n, |a| \leq N} \sup_{x \in \mathbb{R}^n} |\partial^a \phi(x)| \max \{ 1, |\rho(x)|^N \} \leq 1 \right\}.
\]

Throughout this article, for any \( \phi \in S(\mathbb{R}^n) \) and \( i \in \mathbb{Z} \), let \( \phi_i(\cdot) := b^{-i} \phi(A^{-i} \cdot) \).

Let \( \lambda_-, \lambda_+ \in (1, \infty) \) be two numbers such that

\[ \lambda_- \leq \min \{|\lambda| : \lambda \in \sigma(A)\} \leq \max \{|\lambda| : \lambda \in \sigma(A)\} \leq \lambda_+. \]

In particular, when \( A \) is diagonalizable over \( \mathbb{C} \), we can let

\[ \lambda_- := \min \{|\lambda| : \lambda \in \sigma(A)\} \quad \text{and} \quad \lambda_+ := \max \{|\lambda| : \lambda \in \sigma(A)\}. \]

Otherwise, we can choose them sufficiently close to these equalities in accordance with what we need in our arguments.

**Definition 2.3** Let \( \phi \in S(\mathbb{R}^n) \) and \( f \in S'(\mathbb{R}^n) \). The non-tangential maximal function \( M_\phi(f) \), associated with \( \phi \), is defined by setting, for any \( x \in \mathbb{R}^n \),

\[ M_\phi(f)(x) := \sup_{y} |f * \phi_i(y)|. \]

Moreover, for any given \( N \in \mathbb{N} \), the non-tangential grand maximal function \( M_N(f) \) of \( f \in S'(\mathbb{R}^n) \) is defined by setting, for any \( x \in \mathbb{R}^n \),

\[ M_N(f)(x) := \sup_{\phi \in S_N(\mathbb{R}^n)} M_\phi(f)(x). \]

The following variable anisotropic Hardy spaces were first introduced by Liu et al. in [28, Definition 2.4].

**Definition 2.4** Let \( p(\cdot) \in C^\log(\mathbb{R}^n) \) and \( N \in \mathbb{N} \cap \{(1/p - 1) \log b/\log \lambda_- + 2, \infty \} \), where \( p \) is as in (2.3). The variable anisotropic Hardy space \( H^{p(\cdot)}_A(\mathbb{R}^n) \) is defined by setting

\[ H^{p(\cdot)}_A(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : M_N(f) \in L^{p(\cdot)}(\mathbb{R}^n) \}, \]

and for any \( f \in H^{p(\cdot)}_A(\mathbb{R}^n) \), let \( \| f \|_{H^{p(\cdot)}_A(\mathbb{R}^n)} := \| M_N(f) \|_{L^{p(\cdot)}(\mathbb{R}^n)}. \)

Observe that, in [28, Theorem 3.10], it was proved that the space \( H^{p(\cdot)}_A(\mathbb{R}^n) \) is independent of the choice of \( N \) as in Definition 2.4.
3 Molecular characterization of $H^p_A(\mathbb{R}^n)$

In this section, we characterize $H^p_A(\mathbb{R}^n)$ by means of molecules. Recall that, for any $q \in (0, \infty]$ and measurable set $E \subset \mathbb{R}^n$, the Lebesgue space $L^q(E)$ is defined to be the set of all measurable functions $f$ on $E$ such that, when $q \in (0, \infty)$,

$$
\|f\|_{L^q(E)} := \left[ \int_E |f(x)|^q \, dx \right]^{1/q} < \infty,
$$

and

$$
\|f\|_{L^\infty(E)} := \text{ess sup}_{x \in E} |f(x)| < \infty.
$$

The following definition of anisotropic $(p(\cdot), r, s, \varepsilon)$-molecules is from [31].

**Definition 3.1** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $r \in (1, \infty]$, $s \in \left[\left( \frac{1}{p_-} - 1\right) \frac{\ln b}{\ln \lambda_-}, \infty \right) \cap \mathbb{Z}_+$, and $\varepsilon \in (0, \infty)$, where $p_-$ is as in (2.3). A measurable function $m$ is called an anisotropic $(p(\cdot), r, s, \varepsilon)$-molecule, associated with some dilated ball $B := x_0 + B_{i_0} \in \mathcal{B}$ with $x_0 \in \mathbb{R}^n$, $i_0 \in \mathbb{Z}$ and $\mathcal{B}$ as in (2.1), if

(i) for each $j \in \mathbb{Z}_+$, $\|m\|_{L^r(B_j(0))} \leq b^{-j\varepsilon} |B|^1 r \|m\|_{L^r(B)}^{-1}$, where $B_j(0) := B$ and, for any $j \in \mathbb{N}$, $B_j = B_j(x_0 + B_{i_0}) := x_0 + (A^jB_{i_0}) \setminus (A^{j-1}B_{i_0})$;

(ii) for any multi-index $a \in \mathbb{Z}^n_+$ with $|a| \leq s$, $\int_{\mathbb{R}^n} m(x)x^a \, dx = 0$.

In what follows, for convenience, we always call an anisotropic $(p(\cdot), r, s, \varepsilon)$-molecule simply by a $(p(\cdot), r, s, \varepsilon)$-molecule. Now, using $(p(\cdot), r, s, \varepsilon)$-molecules, we introduce the variable anisotropic molecular Hardy space $H^p_A(r, s, \varepsilon)(\mathbb{R}^n)$ as follows.

**Definition 3.2** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$, $r$, $s$ and $\varepsilon$ be as in Definition 3.1. The variable anisotropic molecular Hardy space $H^p_A(r, s, \varepsilon)(\mathbb{R}^n)$ is defined to be the set of all $f \in S'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), r, s, \varepsilon)$-molecules, $\{m_i\}_{i \in \mathbb{N}_0}$ associated, respectively, with $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B}$ such that

$$
f = \sum_{i \in \mathbb{N}} \lambda_i m_i \quad \text{in} \quad S'(\mathbb{R}^n).
$$

Moreover, for any $f \in H^p_A(r, s, \varepsilon)(\mathbb{R}^n)$, let

$$
\|f\|_{H^p_A(r, s, \varepsilon)(\mathbb{R}^n)} := \inf \left\{ \left[ \sum_{i \in \mathbb{N}} \left| \frac{\lambda_i}{\|1_{B^{(i)}}\|_{L^p(\mathbb{R}^n)}} \right|^p \right]^{1/p} \right\},
$$

where $1_{B^{(i)}}$ denotes the characteristic function of the ball $B^{(i)}$. 
where the infimum is taken over all decompositions of $f$ as above and $p$ as in $(2.3)$.

To establish the molecular characterization of $H_A^{p(r)}(\mathbb{R}^n)$, we need several technical lemmas. First, Lemma 3.1 is just [30, Remark 4.4(i)].

**Lemma 3.1** Let $p(\cdot) \in C^0(\mathbb{R}^n)$, $r \in (0, p)$ and $i \in \mathbb{Z}_+$. Then, there exists a positive constant $C$ such that, for any sequence $\{\overline{B}^{(k)}\}_{k \in \mathbb{N}} \subset \mathcal{B}$,

$$\left\| \sum_{k \in \mathbb{N}} 1_{A^i \overline{B}^{(k)}} \right\|_{L^p(\mathbb{R}^n)} \leq C b^{i/r} \left\| \sum_{k \in \mathbb{N}} 1_{B^{(k)}} \right\|_{L^p(\mathbb{R}^n)} .$$

By Lemma 3.1 and an argument similar to that used in the proof of [20, Lemma 3.15] with some slight modifications, we obtain the following useful conclusion; the details are omitted.

**Lemma 3.2** Let $r(\cdot) \in C^0(\mathbb{R}^n)$ and $q \in [1, \infty) \cap (r_+, \infty)$ with $r_+$ as in $(2.3)$. Assume that $\{\lambda_i\}_{i \in \mathbb{N}} \subset C$, $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B}$ and $\{a_i\}_{i \in \mathbb{N}} \subset L^q(\mathbb{R}^n)$ satisfy, for any $i \in \mathbb{N}$, supp $a_i \subset A^i B^{(i)}$ with some fixed $i_0 \in \mathbb{Z}$,

$$\|a_i\|_{L^q(\mathbb{R}^n)} \leq \frac{|B^{(i)}|^{1/q}}{\|1_{B^{(i)}}\|_{L^{r}(\mathbb{R}^n)}},$$

and

$$\left\| \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i| 1_{B^{(i)}}}{\|1_{B^{(i)}}\|_{L^{r}(\mathbb{R}^n)}} \right]^\frac{r}{q} \right\|_{L^{q}(\mathbb{R}^n)}^{1/r} < \infty,$$

where $r$ is as in $(2.3)$. Then,

$$\left\| \left[ \sum_{i \in \mathbb{N}} | \lambda_i a_i |^r \right]^{1/r} \right\|_{L^{q}(\mathbb{R}^n)} \leq C \left\| \left\{ \sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i| 1_{B^{(i)}}}{\|1_{B^{(i)}}\|_{L^{r}(\mathbb{R}^n)}} \right]^\frac{r}{q} \right\}^{1/r} \right\|_{L^{q}(\mathbb{R}^n)},$$

where $C$ is a positive constant independent of $\lambda_i$, $B^{(i)}$ and $a_i$.

The succeeding definitions of both anisotropic $(p(\cdot), r, s)$-atoms and variable anisotropic atomic Hardy spaces $H_A^{p(r),s}(\mathbb{R}^n)$ are from [28].

**Definition 3.3** (i) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $r \in (1, \infty]$ and $s$ be as in $(3.1)$. An anisotropic $(p(\cdot), r, s)$-atom (shortly, a $(p(\cdot), r, s)$-atom) is a measurable function $a$ on $\mathbb{R}^n$ satisfying

\begin{align*}
(i_1) & \quad \text{supp } a \subset B, \text{ where } B \in \mathcal{B} \text{ with } \mathcal{B} \text{ as in (2.1)}; \\
(i_2) & \quad \|a\|_{L^r(\mathbb{R}^n)} \leq \frac{|B|^{1/r}}{\|1_{B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}};
\end{align*}
(i) \[ \int_{\mathbb{R}^n} a(x) x^\gamma \, dx = 0 \text{ for any } \gamma \in \mathbb{Z}_+^n \text{ with } |\gamma| \leq s. \]

(ii) Let \( p(\cdot) \in C^\log(\mathbb{R}^n) \), \( r \in (1, \infty) \) and \( s \) be as in (3.1). The variable anisotropic atomic Hardy space \( H_A^{p(\cdot),r,s}(\mathbb{R}^n) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) satisfying that there exist a sequence \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and a sequence of \( (p(\cdot),r,s) \)-atoms, \( \{a_i\}_{i \in \mathbb{N}} \), supported, respectively, in \( \{B(i)\}_{i \in \mathbb{N}} \subset \mathcal{B} \) such that

\[ f = \sum_{i \in \mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \]

Moreover, for any \( f \in H_A^{p(\cdot),r,s}(\mathbb{R}^n) \), let

\[ \|f\|_{H_A^{p(\cdot),r,s}(\mathbb{R}^n)} := \inf \left\{ \sqrt[p]{\sum_{i \in \mathbb{N}} \left[ \frac{|\lambda_i|^p}{\|1_{B(i)}\|_{L^p(\mathbb{R}^n)}} \right]} \right\}^{1/p} \bigg\| L^{1/p}(\mathbb{R}^n) \bigg\}, \]

where the infimum is taken over all decompositions of \( f \) as above.

We also need the following atomic characterizations of \( H_A^{p(\cdot)}(\mathbb{R}^n) \) established in [28, Theorem 4.8].

**Lemma 3.3** Let \( p(\cdot) \in C^\log(\mathbb{R}^n) \), \( r \in (\max\{p_+, 1\}, \infty) \) with \( p_+ \) as in (2.3), \( s \) be as in (3.1) and \( N \in \mathbb{N} \cap \{\lceil \frac{1}{p} (1 - 1) \frac{\ln b}{\ln \lambda} \rceil + 2, \infty\} \) with \( p_+ \) as in (2.3). Then, \( H_A^{p(\cdot)}(\mathbb{R}^n) = H_A^{p(\cdot),r,s}(\mathbb{R}^n) \) with equivalent quasi-norms.

The following two lemmas are, respectively, from [38, Remark 2.1(i)] and [4, p. 8, (2.11), p. 5, (2.1) and (2.2) and p. 17, Proposition 3.10].

**Lemma 3.4** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). Then, for any \( s \in (0, \infty) \) and \( f \in L^{p(\cdot)}(\mathbb{R}^n) \),

\[ \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^s. \]

In addition, for any \( \lambda \in \mathbb{C} \) and \( f, g \in L^{p(\cdot)}(\mathbb{R}^n) \), \( \|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \) and

\[ \|f + g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p, \]

where \( p_+ \) is as in (2.3).

**Lemma 3.5** Let \( A \) be some fixed dilation. Then,

(i) for any \( i \in \mathbb{Z} \), we have

\[ B_i + B_i \subset B_{i+\tau} \quad \text{and} \quad B_i + (B_{i+\tau})^c \subset (B_i)^c, \]

where \( \tau \) is as in (2.2).
(ii) there exists a positive constant $C$ such that, for any $x \in \mathbb{R}^n$, when $k \in \mathbb{Z}_+$,
\[ C^{-1}(\lambda_+)^k|x| \leq |A^kx| \leq C(\lambda_+)^k|x|, \]
and, when $k \in \mathbb{Z} \setminus \mathbb{Z}_+$,
\[ C^{-1}(\lambda_+)^k|x| \leq |A^kx| \leq C(\lambda_+)^k|x|; \]

(iii) for any given $N \in \mathbb{N}$, there exists a positive constant $C_{(N)}$, depending on $N$, such that, for any $f \in S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,
\[ M_N^0(f)(x) \leq M_N(f)(x) \leq C_{(N)}M_N^0(f)(x), \]
where $M_N^0$ denotes the radial grand maximal function of $f \in S'(\mathbb{R}^n)$ defined by setting, for any $x \in \mathbb{R}^n$,
\[ M_N^0(f)(x) := \sup_{\phi \in \mathfrak{D}_x(\mathbb{R}^n)} \sup_{i \in \mathbb{Z}} |f * \phi_i(x)|. \]

Let $L^1_{\text{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Recall that, for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the anisotropic Hardy–Littlewood maximal function $M_{\text{HL}}(f)$ is defined by setting, for any $x \in \mathbb{R}^n$,
\[ M_{\text{HL}}(f)(x) := \sup_{k \in \mathbb{Z}} \sup_{y \in x+B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)| \, dz = \sup_{x \in B \in \mathfrak{B}} \frac{1}{|B|} \int_B |f(z)| \, dz, \quad (3.2) \]

where $\mathfrak{B}$ is as in (2.1).

The following Fefferman–Stein vector-valued inequality of the maximal operator $M_{\text{HL}}$ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is just [30, Lemma 4.3].

**Lemma 3.6** Let $\nu \in (1, \infty]$. Assume that $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfies $1 < p_- \leq p_+ < \infty$. Then, there exists a positive constant $C$ such that, for any sequence $\{f_k\}_{k \in \mathbb{N}}$ of measurable functions,
\[ \left\| \left\{ \sum_{k \in \mathbb{N}} [M_{\text{HL}}(f_k)]^\nu \right\}^{1/\nu} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{N}} |f_k|^\nu \right)^{1/\nu} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \]

with the usual modification made when $\nu = \infty$, where $M_{\text{HL}}$ denotes the Hardy–Littlewood maximal operator as in (3.2).

Now, we state the main result of this section as follows.

**Theorem 3.1** Let $p(\cdot)$, $r$, $s$ and $N$ be as in Lemma 3.3 and $\varepsilon \in ((s + 1) \log p(\lambda_+ / \lambda_-), \infty)$.

Then, $H^{p(\cdot)}_A(\mathbb{R}^n) = H^{p(\cdot), r, s, \varepsilon}_A(\mathbb{R}^n)$ with equivalent quasi-norms.
\textbf{Proof} Let \( p(\cdot), r \) and \( s \) be as in Lemma 3.3. Then, by Lemma 3.3, we know that 
\( H_A^{p(\cdot)}(\mathbb{R}^n) = H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \) with equivalent quasi-norms. Moreover, by the definitions of both \( H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \) and \( H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \), we find that \( H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \subset H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \) and this inclusion is continuous. Thus, \( H_A^{p(\cdot)}(\mathbb{R}^n) \subset H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \) with continuous inclusion.

Conversely, for any \( f \in H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \), without loss of generality, we can assume that \( f \) is not the zero element of \( H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n) \). Then, from Definition 3.2, it follows that there exist a sequence \( \{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C} \) and a sequence of \( (p(\cdot), r, s, \varepsilon) \)-molecules, \( \{m_i\}_{i \in \mathbb{N}} \), associated, respectively, with \( \{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathcal{B} \) such that

\[
f = \sum_{i \in \mathbb{N}} \lambda_im_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),
\] 

(3.3)

and

\[
\|f\|_{H_A^{p(\cdot), r, s, \varepsilon}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i \in \mathbb{N}} \left\| \frac{\lambda_i}{\lambda_i - \lambda} \right\|_B^{p(\cdot)} \right\|_{L^{r(\cdot)}(\mathbb{R}^n)} \right\|^{1/p}_{L^s(\mathbb{R}^n)},
\]

where \( p \) is as in (2.3). Obviously, there exist two sequences \( \{x_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n \) and \( \{\ell_i\}_{i \in \mathbb{N}} \subset \mathbb{Z} \) such that, for any \( i \in \mathbb{N} \), \( x_i + B_{\ell_i} = B^{(i)} \). By (3.3), it is easy to see that, for any \( N \in \mathbb{N} \cap \{ \left( \frac{1}{p} - 1 \right) \frac{\ln b}{\ln \lambda} + 2, \infty \} \) and \( x \in \mathbb{R}^n \),

\[
M_N(f)(x) \leq \sum_{i \in \mathbb{N}} |\lambda_i|M_N(m_i)(x)1_{x_i + A^*B_{\ell_i}}(x)
\]

\[
+ \sum_{i \in \mathbb{N}} |\lambda_i|M_N(m_i)(x)1_{(x_i + A^*B_{\ell_i})\setminus A^{(i)}B_{\ell_i}}(x) =: I_1 + I_2.
\] 

(3.4)

To deal with \( I_1 \), for any \( r \in (\max\{p_+, 1\}, \infty) \), \( \varepsilon \) as in Theorem 3.1 and \( i \in \mathbb{N} \), from the boundedness of \( M_N \) on \( L^r(\mathbb{R}^n) \) (see [29, Remark 2.10]) and Definition 3.1(i), we deduce that

\[
\|M_N(m_i)\|_{L^r(\mathbb{R}^n)} \lesssim \|m_i\|_{L^r(\mathbb{R}^n)} \lesssim \sum_{j \in \mathbb{Z}_+} \|m_i\|_{L^r(U_j(B^{(i)}))}
\]

\[
\lesssim \sum_{j \in \mathbb{Z}_+} b^{-je} \frac{|B^{(i)}|^{1/r}}{\|1_{B^{(i)}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B^{(i)}|^{1/r}}{\|1_{B^{(i)}}\|_{L^{s(\cdot)}(\mathbb{R}^n)}},
\]

where \( U_0(B^{(i)}) := B^{(i)} \) and, for any \( j \in \mathbb{N} \),

\[
U_j(B^{(i)}) = U_j(x_i + B_{\ell_i}) := x_i + (A^jB_{\ell_i}) \setminus (A^{j-1}B_{\ell_i}).
\]

This, combined with Lemma 3.2, implies that
\[ \| I_1 \|_{L^p(\mathbb{R}^n)} \lesssim \left\| \sum_{i \in \mathbb{N}} \left[ \lambda_j^i |M_N(m_j)1_{A^\rho_j}] \right]^2 \right\|_{L^p(\mathbb{R}^n)}^{1/p} \]

\[ \lesssim \left\| \sum_{i \in \mathbb{N}} \left[ \left| \lambda_j^i \right|1_{B^{0,i}} \right]^2 \right\|_{L^p(\mathbb{R}^n)}^{1/p} \]

\[ \sim \| f \|_{L^p_{\lambda,\rho,j}(\mathbb{R}^n)}. \] (3.5)

For the term \( I_2 \), suppose that \( P \) is a polynomial of degree not greater than \( s \). Then, by Definition 3.1 and the Hölder inequality, we find that, for any \( N \in \mathbb{N} \), \( \phi \in \mathcal{S}_N(\mathbb{R}^n) \), \( t \in \mathbb{Z} \) and \( x \in (x_i + B_{r_i+t,k+1}) \) with \( i \in \mathbb{N} \),

\[ (m_i \ast \phi_j)(x) \]

\[ = b^{-t} \left| \int_{\mathbb{R}^n} m_i(y) \phi(A^{-t}(x-y)) \, dy \right| \]

\[ \leq b^{-t} \sum_{j \in \mathbb{Z}_+} \left| \int_{U(x_i + B_{r_i})} m_i(y) \left[ \phi(A^{-t}(x-y)) - P(A^{-t}(x-y)) \right] \, dy \right| \]

\[ \leq b^{-t} \sum_{j \in \mathbb{Z}_+} \sup_{y \in A^{-t}(x_i + B_{r_i, t})} |\phi(y) - P(y)| \left| \int_{U(x_i + B_{r_i})} m_i(y) \, dy \right| \]

\[ \lesssim b^{c \cdot r - t} \sum_{j \in \mathbb{Z}_+} b^{j/r} \sup_{y \in A^{-t}(x_i + B_{r_i, t})} |\phi(y) - P(y)| \, dy \]

\[ \lesssim b^{c \cdot r - t} \left\| 1_{x_i + B_{r_i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} \sum_{j \in \mathbb{Z}_+} b^{(1/r - \varepsilon) j} \sup_{y \in A^{-t}(x_i + B_{r_i, t})} |\phi(y) - P(y)|. \]

Assume that \( x \in [x_i + (B_{r_i+t+k+1} \setminus B_{r_i+t+k})] \) for some \( k \in \mathbb{Z}_+ \). Without loss of generality, we can assume that \( s = \left\lfloor (1/p - 1) \ln b / \ln \lambda_- \right\rfloor \) and \( N = s + 2 \), which implies that \( b^{\lambda_-^{k+1}} \leq b^{N} \). Then, by Lemma 3.5, an estimation similar to that used in the proof of [27, (3.9)] and the fact that \( \varepsilon \in ((s+1) \log_b(\lambda_+ / \lambda_-), \infty) \), we conclude that, for any \( i \in \mathbb{N} \),

\[ M_N(m_j)(x) \lesssim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} \sum_{j \in \mathbb{Z}_+} \left\{ b^{-N_k} + b^{-(s+1) \log_b(\lambda_+ / \lambda_-)} \right\} \]

\[ \times \max \left\{ b^{-N_k}, (b^{\lambda_-^{k+1}})^{-k} \right\} \]

\[ \lesssim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} \left( b^{\lambda_-^{k+1}} \right)^{-k} \]

\[ \sim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} b^{-k} b^{-(s+1) \ln \lambda_+ / \ln b} \]

\[ \lesssim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} b^{c \cdot r (s+1) \ln \lambda_+ / \ln b + 1} b^{-(\varepsilon r + k)(s+1) \ln \lambda_+ / \ln b + 1} \]

\[ \lesssim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} \left[ \rho(x - x_i) \right]^\beta \]

\[ \lesssim \left\| 1_{B^{0,i}} \right\|_{L^\infty(\mathbb{R}^n)}^{-1} \left[ M_{\text{HL}}(1_{B^{0,i}})(x) \right]^\beta, \]
where

$$\beta := \left( \frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{p}.$$ 

From this and Lemmas 3.4 and 3.6, it follows that

$$\|I_2\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \sum_{i \in \mathbb{N}} \frac{\|\lambda_i\|_{\mathbb{R}^n}}{\|1_{B^{(i)}}\|_{L^{p_0}(\mathbb{R}^n)}} \left[ M_{\mathcal{H}L}(1_{B^{(i)}}) \right]^\beta \left\| \lambda_i \right\|_{L^{p_0}(\mathbb{R}^n)} \leq \left\{ \sum_{i \in \mathbb{N}} \frac{\|\lambda_i\|_{\mathbb{R}^n}}{\|1_{B^{(i)}}\|_{L^{p_0}(\mathbb{R}^n)}} \right\}^{1/p} \left\| f \right\|_{L^{p_0}(\mathbb{R}^n)}.$$

This, together with (3.4), (3.5) and Lemma 3.4 again, implies that

$$\|f\|_{H_{A}^{p_0}(\mathbb{R}^n)} = \|M_N(f)\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \|f\|_{H_{A}^{\lambda,p,\alpha,s,e}(\mathbb{R}^n)}^\beta,$$

which completes the proof of Theorem 3.1. \(\square\)

**Remark 3.1**

(i) When \(A := d I_{n \times n}\) for some \(d \in \mathbb{R}\) with \(|d| \in (1, \infty)\), here and thereafter, \(I_{n \times n}\) denotes the \(n \times n\) unit matrix, \(H_{A}^{p_0}(\mathbb{R}^n)\) and \(H_{A}^{\lambda,p,\alpha,s,e}(\mathbb{R}^n)\) become, respectively, the classical isotropic variable Hardy space (see [13, 33]) and variable molecular Hardy space. In this case, Theorem 3.1 includes [33, Theorem 5.2] as a special case.

(ii) Recall that, in [27, Theorem 3.12], the authors established the molecular characterizations of the anisotropic Musielak–Orlicz Hardy space \(H_{A}^{\phi}(\mathbb{R}^n)\) with \(\phi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)\) being an anisotropic growth function (see [25, Definition 3]). By [28, Remark 2.5(iii)], we know that the anisotropic Musielak–Orlicz Hardy space \(H_{A}^{\phi}(\mathbb{R}^n)\) and the variable anisotropic Hardy space \(H_{A}^{p_0}(\mathbb{R}^n)\) in this article cannot cover each other and hence neither do [27, Theorem 3.12] and Theorem 3.1.

(iii) Very recently, in [31, Theorem 2.8], Liu et al. obtained the molecular characterizations of variable anisotropic Hardy–Lorentz spaces \(H_{A}^{p(\cdot),q}(\mathbb{R}^n)\) with \(p(\cdot) \in C^{log}(\mathbb{R}^n)\) and \(q \in (0, \infty)\). It is easy to see that the space \(H_{A}^{p(\cdot)}(\mathbb{R}^n)\), in this article, is not covered by the space \(H_{A}^{p(\cdot),q}(\mathbb{R}^n)\) since the exponent \(q \in (0, \infty)\).
in $H^p_A(\mathbb{R}^n)$ is only a constant. Thus, Theorem 3.1 is neither covered by [31, Theorem 2.8].

(iv) We should also point out that [27, Theorem 3.12] and [31, Theorem 2.8], respectively, require the decay index $\varepsilon$ to belong to

$$\left(\max\left\{1, (s+1) \log_b (\lambda_+/\lambda_-)\right\}, \infty\right)$$

and

$$\left(\max\left\{1, (s+1) \log_b (\lambda_+)\right\}, \infty\right),$$

either of which is just a proper subset of

$$((s+1) \log_b (\lambda_+/\lambda_-), \infty),$$

from Theorem 3.1. In this sense, we improve the range of $\varepsilon$ in the molecular characterizations. In particular, when $A$ is as in (i) of this remark and $p(\cdot) \equiv p \in (0, \infty)$, the space $H^p_A(\mathbb{R}^n)$ becomes the classical isotropic Hardy space $H^p(\mathbb{R}^n)$ and $\log_b (\lambda_+/\lambda_-) = 0$. In this case, Theorem 3.1 gives a molecular characterization of $H^p(\mathbb{R}^n)$ with the known best possible decay of molecules, namely $\varepsilon \in (0, \infty)$.

4 Some applications

In this section, as applications, we first establish a criterion on the boundedness of linear operators on $H^p_A(\mathbb{R}^n)$. Then, applying this criterion, we obtain the boundedness of anisotropic Calderón–Zygmund operators on $H^p_A(\mathbb{R}^n)$. In addition, the boundedness of these operators from $H^p_A(\mathbb{R}^n)$ to the variable Lebesgue space $L^p(\mathbb{R}^n)$ is also presented.

First, we recall the notion of variable anisotropic finite atomic Hardy spaces $H^{p(\cdot),r,s}_A, \text{fin}(\mathbb{R}^n)$ from [28, Definition 5.1].

Definition 4.1 Let $p(\cdot) \in C^\log(\mathbb{R}^n)$, $r \in (1, \infty]$, $s$ be as in (3.1) and $A$ a dilation. The variable anisotropic finite atomic Hardy space $H^{p(\cdot),r,s}_{A, \text{fin}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist an $I \in \mathbb{N}$, a finite sequence $\{\lambda_i\}_{i \in [1,I]} \subset \mathbb{C}$ and a finite sequence of $(p(\cdot), r, s)$-atoms, $\{a_i\}_{i \in [1,I]}$, supported, respectively, in $\{B^{(i)}\}_{i \in [1,I]} \subset \mathcal{B}$ such that

$$f = \sum_{i=1}^I \lambda_i a_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H^{p(\cdot),r,s}_{A, \text{fin}}(\mathbb{R}^n)$, let

$$\|f\|_{H^{p(\cdot),r,s}_{A, \text{fin}}(\mathbb{R}^n)} := \inf\left\{\left\|\sum_{i=1}^I \left|\frac{\lambda_i}{\|1_B^{(i)}\|_{L^p(\mathbb{R}^n)}}\right|^p\right\|^{1/p}_{L^p(\mathbb{R}^n)}\right\},$$

with $p$ as in (2.3), where the infimum is taken over all decompositions of $f$ as above.
Denote by $C(\mathbb{R}^n)$ the set of all continuous functions and by $C_c^\infty(\mathbb{R}^n)$ the set of all infinite differentiable functions with compact support. The succeeding three lemmas are just, respectively, [28, Theorem 5.4 and Lemma 7.3] and [30, Lemma 5.4].

**Lemma 4.1** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (\max\{p_+ \cdot 1\}, \infty]$ and $s$ be as in (3.1), where $p_+$ is as in (2.3).

(i) If $r \in (\max\{p_+ \cdot 1\}, \infty)$, then $\| \cdot \|_{H^p_{A, fin}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^p_{A, fin}(\mathbb{R}^n)$;

(ii) $\| \cdot \|_{H^p_{A, \alpha, s}(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^p_{A, \alpha, s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

**Lemma 4.2** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then, $H^p_{A, fin}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^p_{A, fin}(\mathbb{R}^n)$.

**Lemma 4.3** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_- \in (1, \infty)$. Then, there exists a positive constant $C$ such that, for arbitrary two subsets $E_1, E_2$ of $\mathbb{R}^n$ with $E_1 \subset E_2$, $C^{-1} \left( \frac{|E_1|}{|E_2|} \right)^{\frac{1}{p_-}} \leq \frac{\|1_{E_1}\|_{L^{p}(\mathbb{R}^n)}}{\|1_{E_2}\|_{L^{p}(\mathbb{R}^n)}} \leq C \left( \frac{|E_1|}{|E_2|} \right)^{\frac{1}{p_+}}$.

Applying the above three lemmas and Theorem 3.1, we establish a criterion on the boundedness of linear operators on $H^p_{A, fin}(\mathbb{R}^n)$ as follows.

**Theorem 4.1** Assume that $T$ is a linear operator defined on the set of all measurable functions. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (\max\{p_+ \cdot 1\}, \infty]$ with $p_+$ as in (2.3) and $\tilde{s}$ be as in (3.1) with $s$ replaced by $\tilde{s}$. If there exist some $k_0 \in \mathbb{Z}$ and a positive constant $C$ such that, for any $(p(\cdot), r, \tilde{s})$-atom $\tilde{a}$ supported in some dilated ball $x_0 + B_{i_0} \in \mathfrak{B}$ with $x_0 \in \mathbb{R}^n$, $i_0 \in \mathbb{Z}$ and $\mathfrak{B}$ as in (2.1), $\frac{1}{C} T(\tilde{a})$ is a $(p(\cdot), r, s, \epsilon)$-molecule associated with $x_0 + B_{i_0 + k_0}$, where $s$ and $\epsilon$ are as in Theorem 3.1, then $T$ has a unique bounded linear extension on $H^p_{A, fin}(\mathbb{R}^n)$.

**Proof** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (\max\{p_+ \cdot 1\}, \infty]$ and $\tilde{s} \in [(1/p_--1)\ln b/\ln \lambda_-], \infty) \cap \mathbb{Z}_+$, with $p_-$ as in (2.3). We next prove Theorem 4.1 by considering two cases.

**Case 1.** $r \in (\max\{p_+ \cdot 1\}, \infty)$. In this case, for any $f \in H^p_{A, fin}(\mathbb{R}^n)$, by Definition 4.1, we know that there exist some $f \in \mathbb{N}$, three finite sequences $\{\lambda_i\}_{i \in [1, J]} \subset \mathbb{C}$, $\{x_i\}_{i \in [1, J]} \subset \mathbb{R}^n$ and $\{\epsilon_i\}_{i \in [1, J]} \subset \mathbb{Z}$, and a finite sequence of $(p(\cdot), r, \tilde{s})$-atoms, $\{a_{\lambda_i, x_i, \epsilon_i}\}_{i \in [1, J]}$, supported, respectively, in $\{x_i + B_{\epsilon_i}\}_{i \in [1, J]} \subset \mathfrak{B}$ such that $f = \sum_{i=1}^{J} \lambda_i q_i$ in $S'(\mathbb{R}^n)$ and
\[ \|f\|_{H^p_{A,\text{fin}}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i=1}^I \left[ \frac{1}{\|1_{x_i+B_{\ell_i}}\|_{L^p(\mathbb{R}^n)}} \|\lambda_i 1_{x_i+B_{\ell_i}}\|_{L^\infty(\mathbb{R}^n)} \right]^2 \right\}^{1/2} \right\|_{L^p(\mathbb{R}^n)} \quad (4.1) \]

From this and the linearity of \( T \), it is easy to see that \( T(f) = \sum_{i=1}^I \lambda_i T(a_i) \) in \( \mathcal{S}'(\mathbb{R}^n) \), where, for any \( i \in [1, I] \cap \mathbb{N} \), \( \lambda_i T(a_i) \) with \( C \) being a positive constant independent of \( i \) is a \((p(\cdot), r, s, \varepsilon)\)-molecule associated with \( x_i + B_{\ell_i+k_0} \) with \( s, \varepsilon \) and \( k_0 \) as in Theorem 4.1. By this, Theorem 3.1, Definition 3.2, Lemmas 4.3, 3.6 and 3.4, (4.1) and Lemma 4.1, we further conclude that, for any \( f \in H^p_{A,\text{fin}}(\mathbb{R}^n) \),

\[
\|T(f)\|_{H^p_{A}(\mathbb{R}^n)} \sim \|T(f)\|_{H^p_{A,\text{fin}}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i=1}^I \left[ \frac{\lambda_i}{\|1_{x_i+B_{\ell_i}}\|_{L^{\infty}(\mathbb{R}^n)}} \right]^{1/\omega} \right\}^{1/\omega} \right\|_{L^{p/\omega}(\mathbb{R}^n)} ^{\omega/p} \quad (4.2)
\]

where \( \omega \in (0, p) \) is a constant.

Now, let \( f \in H^p_{A}(\mathbb{R}^n) \). Then, by the obvious density of \( H^p_{A,\text{fin}}(\mathbb{R}^n) \) in \( H^p_{A}(\mathbb{R}^n) \) with respect to the quasi-norm \( \| \cdot \|_{H^p_{A}(\mathbb{R}^n)} \), we find that there exists a Cauchy sequence \( \{f_j\}_{j \in \mathbb{N}} \subset H^p_{A,\text{fin}}(\mathbb{R}^n) \) such that

\[ \lim_{j \to \infty} \left\| f_j - f \right\|_{H^p_{A,\text{fin}}(\mathbb{R}^n)} = 0. \]

This, combined with the linearity of \( T \) and (4.2), implies that, as \( j, m \to \infty \),

\[ \left\| T(f_j) - T(f_m) \right\|_{H^p_{A}(\mathbb{R}^n)} = \left\| T(f_j - f_m) \right\|_{H^p_{A}(\mathbb{R}^n)} \lesssim \left\| f_j - f_m \right\|_{H^p_{A}(\mathbb{R}^n)} \to 0. \]

Thus, \( \{T(f_j)\}_{j \in \mathbb{N}} \) is also a Cauchy sequence in \( H^p_{A}(\mathbb{R}^n) \). From this and the completeness of \( H^p_{A}(\mathbb{R}^n) \), it follows that there exists some \( g \in H^p_{A}(\mathbb{R}^n) \) such that \( g = \lim_{j \to \infty} T(f_j) \) in \( H^p_{A}(\mathbb{R}^n) \). Then, let \( T(f) := g \). By this and (4.2), we know that \( T(f) \) is well defined, and moreover, for any \( f \in H^p_{A}(\mathbb{R}^n) \),
In this case, by Lemmas 4.2 and 3.3, it is easy to see that $C_y p_b$. This finishes the proof of Theorem 4.1.

\[ \|T(f)\|_{H^p_A(\mathbb{R}^n)} \lesssim \limsup_{j \to \infty} \left[ \|T(f) - T(f_j)\|_{H^p_A(\mathbb{R}^n)} + \|T(f_j)\|_{H^p_A(\mathbb{R}^n)} \right] \]
\[ \sim \limsup_{j \to \infty} \|T(f_j)\|_{H^p_A(\mathbb{R}^n)} \]
\[ \lesssim \lim_{j \to \infty} \|f\|_{H^p_A(\mathbb{R}^n)} \sim \|f\|_{H^p_A(\mathbb{R}^n)^*} \]

which completes the proof of Theorem 4.1 in Case 1).

Case 2. $r = \infty$. In this case, by Lemmas 4.2 and 3.3, it is easy to see that $H^{p(\cdot),\infty}_A(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}_A(\mathbb{R}^n)$. By this, repeating the proof of Case 1 with some slight modifications, we conclude that Theorem 4.1 also holds true when $r = \infty$. This finishes the proof of Theorem 4.1.

Next, we consider the boundedness of anisotropic Calderón–Zygmund operators from $H^{p(\cdot)}_A(\mathbb{R}^n)$ to itself [or to $L^{p(\cdot)}(\mathbb{R}^n)$]. To this end, we first recall the notion of anisotropic Calderón–Zygmund operators from [4, p. 60, Definition 9.1] as follows.

**Definition 4.2** A locally integrable function $K$ on

\[ \Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}, \]

is called an **anisotropic Calderón–Zygmund standard kernel** if there exist two positive constants $C$ and $\delta$ such that, for any $x, y, \tilde{x}, \tilde{y} \in \Omega$,

\[ |K(x, y)| \leq \frac{C}{\rho(x - y)} \quad \text{when} \quad x \neq y, \]

\[ |K(x, y) - K(x, \tilde{y})| \leq C \frac{[\rho(y - \tilde{y})]^{\delta}}{[\rho(x - y)]^{1+\delta}} \quad \text{when} \quad \rho(x - y) \geq b^{2^r} \rho(y - \tilde{y}), \]

and

\[ |K(x, y) - K(\tilde{x}, y)| \leq C \frac{[\rho(x - \tilde{x})]^{\delta}}{[\rho(x - y)]^{1+\delta}} \quad \text{when} \quad \rho(x - y) \geq b^{2^r} \rho(x - \tilde{x}), \]

with $\tau$ as in (2.2). Moreover, a linear operator $T$ is called an **anisotropic Calderón–Zygmund operator** if it is bounded on $L^2(\mathbb{R}^n)$ and there exists an anisotropic Calderón–Zygmund standard kernel $K$ such that, for any $f \in L^2(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$,

\[ T(f)(x) = \int_{\text{supp} f} K(x, y)f(y) \, dy. \]

In what follows, for any $\nu \in \mathbb{N}$, denote by $C^\nu(\mathbb{R}^n)$ the set of all functions on $\mathbb{R}^n$ whose derivatives with order not greater than $\nu$ are continuous. Since we are interested in the boundedness of anisotropic Calderón–Zygmund operators on $H^{p(\cdot)}_A(\mathbb{R}^n)$...
with \( p(\cdot) \in C^{\log}(\mathbb{R}^n) \), we need to increase the smooth hypothesis on the corresponding kernel \( K \) as follows, which originates from [4, p. 61, Definition 9.2].

**Definition 4.3** Let \( v \in \mathbb{N} \). An anisotropic Calderón–Zygmund operator \( T \) is called an
anisotropic Calderón–Zygmund operator of order \( v \) if its kernel \( K \) is a \( C^v(\mathbb{R}^n) \) function with respect to the second variable \( y \) and there exists a positive constant \( C \) such that, for any \( \alpha \in \mathbb{Z}^n_+ \) with \( 1 \leq |\alpha| \leq v, m \in \mathbb{Z} \) and \( x, y \in \Omega \) with \( \rho(x - y) = b^m \),

\[
\left| \partial_y^n K(x, A^{-m} y) \right| \leq C[\rho(x - y)]^{-1} = Cb^{-m},
\]

(4.4)

where, for any \( x, y \in \mathbb{R}^n \) satisfying \( x \neq A^m y \), \( \tilde{K}(x, y) := K(x, A^m y) \).

**Remark 4.1** By [31, Remark 4.3], we know that, for any \( v \in \mathbb{N} \), the classical isotropic Calderón–Zygmund operator of order \( v \) (see [35, p. 289]) is an operator as in Definition 4.3 in the case when \( A := d1_{n \times n} \) for some \( d \in \mathbb{R} \) with \( |d| \in (1, \infty) \). For more details related to the kernels satisfying (4.4), we refer the reader to [4, p. 61, Example].

Motivated by [4, p. 64, Definition 9.4], we introduce the following vanishing moment condition.

**Definition 4.4** Let \( p(\cdot) \in C^{\log}(\mathbb{R}^n), v \in \mathbb{N}, s_0 := [(1/p_- - 1)\ln b/\ln \lambda_] \) with \( p_- \) as in (2.3) and

\[
\frac{1}{p_-} - 1 < \frac{(\ln \lambda_-)^2}{\ln b \ln \lambda_+} v.
\]

An anisotropic Calderón–Zygmund operator \( T \) of order \( v \) is said to satisfy that \( T^*(x^\alpha) = 0 \) for any \( \alpha \in \mathbb{Z}^n_+ \) with \( |\alpha| \leq s_0 \) if, for any \( f \in L^2(\mathbb{R}^n) \) with compact support and satisfying that, for any \( \gamma \in \mathbb{Z}^n_+ \) with \( |\gamma| \leq v, \int_{\mathbb{R}^n} f(x)^\gamma dx = 0 \), it holds true that, for any \( \alpha \in \mathbb{Z}^n_+ \) with \( |\alpha| \leq s_0, \int_{\mathbb{R}^n} T(f(x))^\alpha dx = 0 \).

The following useful conclusion is just [31, Lemma 4.10].

**Lemma 4.4** Let \( p(\cdot), v, s_0 \) be as in Definition 4.4. Assume that \( r \in (1, \infty] \) and \( T \) is an anisotropic Calderón–Zygmund operator of order \( v \) satisfying \( T^*(x^\alpha) = 0 \) for any \( \alpha \in \mathbb{Z}^n_+ \) with \( |\alpha| \leq s_0 \). Then, there exists a positive constant \( C \) such that, for any \( (p(\cdot), r, v - 1) \)-atom \( \tilde{a} \) supported in some dilated ball \( x_0 + B_{i_0} \in \mathfrak{B} \) with \( x_0 \in \mathbb{R}^n, i_0 \in \mathbb{Z} \) and \( \mathfrak{B} \) as in (2.1), \( \frac{1}{C} T(\tilde{a}) \) is a \( (p(\cdot), r, s_0, \varepsilon) \)-molecule associated with \( x_0 + B_{i_0 + \tau + 1} \), where

\[
\varepsilon := v \log_b(\lambda_-) + 1/r'
\]

and \( \tau \) is as in (2.2).
Then, we have the following boundedness of anisotropic Calderón–Zygmund operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to itself (see Theorem 4.2) or to $L^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 4.3).

**Theorem 4.2** Let $p(\cdot), \nu, s_0$ be as in Definition 4.4. Assume $T$ is an anisotropic Calderón–Zygmund operator of order $\nu$ and satisfies $T^*(x) = 0$ for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s_0$. Then, there exists a positive constant $C$ such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,

$$\|T(f)\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

**Proof** Indeed, Theorem 4.2 is an immediate corollary of Theorem 4.1 and Lemma 4.4. This finishes the proof of Theorem 4.2. \qed

**Theorem 4.3** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$. Assume that $T$ is an anisotropic Calderón–Zygmund operator of order $\nu$ with $\nu \in [s_0 + 1, \infty)$, where $s_0$ is as in Definition 4.4. Then, there exists a positive constant $C$ such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,

$$\|T(f)\|_{L^{\nu(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$  \hspace{1cm} (4.5)

**Proof** Let $p(\cdot) \in C^\log(\mathbb{R}^n)$, $r \in (\max\{1, p_+\}, \infty)$ and 

$$s \in \left\{(1/p_- - 1)\ln b/\ln \lambda_- , \infty\right\} \cap \mathbb{Z}_+,$$

where $p_+$ and $p_-$ are as in (2.3). We next prove this theorem by two steps.

**Step 1.** In this step, we show that, for any $f \in H_{A, \text{fin}}^{p(\cdot), r, s}(\mathbb{R}^n)$, (4.5) holds true. To this end, for any $f \in H_{A, \text{fin}}^{p(\cdot), r, s}(\mathbb{R}^n)$, from Definition 4.1, it follows that there exist some $I \in \mathbb{N}$, three finite sequences $\{\lambda_i\}_{i \in [1, I]} \subset \mathbb{C}$, $\{\ell_i\}_{i \in [1, I]} \subset \mathbb{N}$ and $\{s_i\}_{i \in [1, I]} \subset \mathbb{Z}$, and a finite sequence of $(p(\cdot), r, s)$-atoms, $\{t_i\}_{i \in [1, I]}$, supported, respectively, in $\{x_i + B_{\ell_i}\}_{i \in [1, I]} \subset \mathcal{B}$ such that $f = \sum_{i=1}^I \lambda_i t_i$ in $S'(\mathbb{R}^n)$ and

$$\|f\|_{H_{A, \text{fin}}^{p(\cdot), r, s}(\mathbb{R}^n)} \sim \left\{ \sum_{i=1}^I \left| \frac{\lambda_i}{1 + (x_i + B_{\ell_i})^s} \right| f_{\nu(\cdot)} \right\}^{1/2}.$$

By the linearity of $T$ and Lemma 3.4, it is easy to see that

$$\|T(f)\|_{L^{\nu(\cdot)}(\mathbb{R}^n)} \leq \sum_{i=1}^I |\lambda_i| T(a_i) 1_{x_i + B_{\ell_i} + s}$$

$$+ \sum_{i=1}^I |\lambda_i| T(a_i) 1_{(x_i + B_{\ell_i})^s} \hspace{1cm} (4.7)$$

$$=: I_1 + I_2.$$  

For $I_1$, choose $h \in L^{p(\cdot)/p'(\cdot)}(\mathbb{R}^n)$ satisfying that $\|h\|_{L^{p(\cdot)/p'(\cdot)}(\mathbb{R}^n)} \leq 1$ and
Then, by Lemma 3.4 and the Hölder inequality, we find that, for any $t \in (1, \infty)$ with $p_+ < tp < r$,

\[
(1)_1^p \lesssim \left\| \sum_{i=1}^{I} |\lambda_i| \| T(a_i) \|_{L^{p}(\mathbb{R}^n)}^p \right\|_{L^{p_+}(\mathbb{R}^n)}^p \\
\sim \int_{\mathbb{R}^n} \sum_{i=1}^{I} |\lambda_i| \left[ T(a_i)(x) \right] \left\| 1_{x+B_{\ell_i+t}}(x) \right\|_{L^{p}(\mathbb{R}^n)} dx \\
\leq \sum_{i=1}^{I} \left\| T(a_i) \left\|_{L^{p}(\mathbb{R}^n)}^p \right\| 1_{x+B_{\ell_i+t}}(x) \right\|_{L^{p}(\mathbb{R}^n)}^{1/t} \left\| 1_{x+B_{\ell_i+t}}(x) \right\|_{L^{p}(\mathbb{R}^n)}^{1/t} \left\| h \right\|_{L^{p}(\mathbb{R}^n)} dx.
\]

By this, the boundedness of $T$ on $L^u(\mathbb{R}^n)$ for any $u \in (1, \infty)$ (see [4, p. 60]), Definition 3.3 and the Hölder inequality again, we conclude that

\[
(1)_1^p \lesssim \sum_{i=1}^{I} |\lambda_i| \left\| 1_{x+B_{\ell_i+t}} \right\|_{L^{p}(\mathbb{R}^n)} \lesssim \sum_{i=1}^{I} \frac{1}{\left| B_{\ell_i+t} \right|} \int_{B_{\ell_i+t}} [h(x)]^{1/p'} dx \right\|_{L^{p}(\mathbb{R}^n)}^{1/p'} dx \\
\lesssim \sum_{i=1}^{I} \left\| T(a_i) \right\|_{L^{p}(\mathbb{R}^n)} \left\| 1_{x+B_{\ell_i+t}}(x) \right\|_{L^{p}(\mathbb{R}^n)} \left\| M_{\mathbb{H}L}(h^{1/p'}) \right\|_{L^{p}(\mathbb{R}^n)}^{1/p'} dx \\
\lesssim \left\| M_{\mathbb{H}L}(h^{1/p'}) \right\|_{L^{p}(\mathbb{R}^n)}^{1/p'} \left\| M_{\mathbb{H}L}(h^{1/p'}) \right\|_{L^{p}(\mathbb{R}^n)}^{1/p'} dx.
\]

On the other hand, by the fact that $p_+/p \in (0, t)$, we know that $(p\cdot)/p' \in (t', \infty]$. From this, Lemma 3.1, [30, Lemma 3.3(ii)], Lemma 3.4, the fact that $\left\| h \right\|_{L^{p_+}(\mathbb{R}^n)} \lesssim 1$ and (4.6), we further deduce that
Remark 4.2

(i) Let $\nu \in \mathbb{N}$ and $p \in (0, 1]$ satisfy
\[
\frac{1}{p} - 1 \leq \frac{(\ln \lambda)\varepsilon^2}{\ln b \ln \lambda} \nu.
\] (4.8)

If \(p(\cdot) \equiv p\), then the spaces \(H^p_A(\mathbb{R}^n)\) and \(L^p(\mathbb{R}^n)\) become, respectively, the anisotropic Hardy space \(H^p_A(\mathbb{R}^n)\) of Bownik [4] and the Lebesgue space \(L^p(\mathbb{R}^n)\). In this case, by Theorems 4.2 and 4.3, we know that, for any \(\nu \in \mathbb{N}\) and \(p \in (0, 1]\) as in (4.8), the anisotropic Calderón–Zygmund operator of order \(\nu\) (see Definition 4.3) is bounded from \(H^p_A(\mathbb{R}^n)\) to itself [or to \(L^p(\mathbb{R}^n)\)], which are just, respectively, [4, p. 68, Theorem 9.8 and p. 69, Theorem 9.9].

Moreover, let \(A := dI_{n \times n}\) for some \(d \in \mathbb{R}\) with \(|d| \in (1, \infty)\), \(\nu = 1\). Then, \(\frac{\ln \lambda \varepsilon^2}{\ln b \ln \lambda} \nu = \frac{1}{n}\) and \(H^p_A(\mathbb{R}^n)\) and \(L^p(\mathbb{R}^n)\) become the classical isotropic Hardy space \(H^p(\mathbb{R}^n)\) and the Lebesgue space \(L^p(\mathbb{R}^n)\), respectively. In this case, by Theorems 4.2 and 4.3 and Remark 4.1, we further conclude that, for any \(p \in (\frac{n}{n+1}, 1]\), the classical Calderón–Zygmund operator is bounded from \(H^p(\mathbb{R}^n)\) to itself [or to \(L^p(\mathbb{R}^n)\)], which is a well-known result (see, for instance, [35]).

(ii) When \(A := dI_{n \times n}\) for some \(d \in \mathbb{R}\) with \(|d| \in (1, \infty)\), the space \(H^p_A(\mathbb{R}^n)\) becomes the variable Hardy space \(H^{p(\cdot)}(\mathbb{R}^n)\) (see [13, 33]). In this case, Theorems 4.2 and 4.3 are new.

(iii) Very recently, Bownik et al. [6] obtained the boundedness of a kind of more general anisotropic Calderón–Zygmund operators (see [6, Definition 5.4]) from the anisotropic Hardy space \(H^p(\Theta)\) to itself or to the Lebesgue space \(L^p(\mathbb{R}^n)\) (see, respectively, [6, Theorems 5.11 and 5.12]), where \(\Theta\) is a continuous multi-level ellipsoid cover of \(\mathbb{R}^n\) (see [6, Definition 2.1]). Obviously, the space \(H^p_A(\mathbb{R}^n)\), in this article, is not covered by the space \(H^p(\Theta)\) since the exponent \(p\) in \(H^p(\Theta)\) is only a constant. Therefore, Theorems 4.2 and 4.3 are neither covered by [6, Theorems 5.11 and 5.12].

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**References**

1. Almeida, V., Betancor, J.J., Rodríguez-Mesa, L.: Anisotropic Hardy-Lorentz spaces with variable exponents, Can. J. Math. 69(6), 1219–1273 (2017)
2. Almeida, A., Hästö, P.: Besov spaces with variable smoothness and integrability. J. Funct. Anal. 258(5), 1628–1655 (2010)
3. Barrios, B., Betancor, J.J.: Anisotropic weak Hardy spaces and wavelets. J. Funct. Spaces Appl. Art. ID 809121, 17 pp (2012)
4. Bownik, M., Li, B., Li, J.: Variable anisotropic singular integral operators. Preprint (2020); arXiv:2004.09707 [math.FA]
5. Bownik, M., Wang, L.-A.D.: A PDE characterization of anisotropic Hardy spaces. Preprint
6. Bownik, M.: Anisotropic Hardy spaces and wavelets. Mem. Am. Math. Soc. 164(781), vi+122pp (2003)
7. Bownik, M., Li, B., Yang, D., Zhou, Y.: Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators. Indiana Univ. Math. J. 57(7), 3065–3100 (2008)
8. Calderón, A.-P., Torchinsky, A.: Parabolic maximal functions associated with a distribution. Adv. Math. 16, 1–64 (1975)
9. Cleanthous, G., Georgiadis, A.G., Nielsen, M.: Molecular decomposition of anisotropic homogeneous mixed-norm spaces with applications to the boundedness of operators. Appl. Comput. Harmon. Anal. 47(2), 447–480 (2019)
10. Cruz-Uribe, D.V., Fiorenza, A.: Variable Lebesgue spaces. Foundations and harmonic analysis, applied and numerical harmonic analysis, Birkhäuser. Springer, Heidelberg (2013)
11. Diening, L., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Math. 2017, Springer, Heidelberg (2011)
12. Grafakos, L., Torres, R.H.: Pseudodifferential operators with homogeneous symbols. Michigan Math. J. 46(2), 261–269 (1999)
13. Han, Y., Lee, M.-Y., Lin, C.-C.: Atomic decomposition and boundedness of operators on weighted Hardy spaces. Can. Math. Bull. 55(2), 303–314 (2012)
14. Kempka, H., Vybíral, J.: Lorentz spaces with variable exponents. Math. Nachr. 287(8–9), 938–954 (2014)
15. Lee, M.-Y., Lin, C.-C.: The molecular characterization of weighted Hardy spaces. J. Funct. Anal. 188(2), 442–460 (2002)
16. Liu, J., Weisz, F., Yang, D., Yuan, W.: Anisotropic Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. 266(5), 2111–2160 (2019)
17. Liu, J., Yang, D., Yuan, W.: Anisotropic Hardy spaces with variable exponents and generalized Campanato spaces. J. Math. Anal. Appl. 456(1), 356–393 (2017)
18. Liu, J., Yang, D., Yuan, W.: Anisotropic Hardy spaces with variable exponents and their real interpolation. J. Math. Anal. Appl. 464(2), 1232–1259 (2018)
19. Liu, J., Zheng, Z.: Weighted anisotropic Hardy spaces and their applications. Sci. China Math. 59(9), 1669–1720 (2016)
35. Stein, E.M.: Harmonic analysis: Real-Variable Methods, orthogonality, and oscillatory integrals. Princeton mathematical series 43, monographs in harmonic analysis III. Princeton University Press, Princeton (1993)

36. Taibleson, M.H., Weiss, G.: The molecular characterization of certain Hardy spaces. In: Representation theorems for Hardy spaces, pp. 67–149, Astérisque, 77, Soc. Math. France, Paris (1980)

37. Xu, J.: Variable Besov and Triebel-Lizorkin spaces. Ann. Acad. Sci. Fenn. Math. 33(2), 511–522 (2008)

38. Yang, D., Zhuo, C., Nakai, E.: Characterizations of variable exponent Hardy spaces via Riesz transforms. Rev. Mat. Complut. 29(2), 245–270 (2016)

39. Yan, X., Yang, D., Yuan, W., Zhuo, C.: Variable weak Hardy spaces and their applications. J. Funct. Anal. 271(10), 2822–2887 (2016)

40. Zhuo, C., Sawano, Y., Yang, D.: Hardy spaces with variable exponents on RD-spaces and applications. Dissertationes Math. (Rozprawy Mat.) 520, 1–74 (2016)

41. Zhuo, C., Yang, D., Yuan, W.: Interpolation between $H^{p(x)}(\mathbb{R}^n)$ and $L^{\infty}(\mathbb{R}^n)$: real method. J. Geom. Anal. 28(3), 2288–2311 (2018)