Fading-Resilient Super-Orthogonal Space-Time Signal Sets: Can Good Constellations Survive in Fading?

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Abstract

In this correspondence, first-tier indirect (direct) discernible constellation expansions are defined for generalized orthogonal designs. The expanded signal constellation, leading to so-called super-orthogonal codes, allows the achievement of coding gains in addition to diversity gains enabled by orthogonal designs. Conditions that allow the shape of an expanded multidimensional constellation to be preserved at the channel output, on an instantaneous basis, are derived. It is further shown that, for such constellations, the channel alters neither the relative distances nor the angles between signal points in the expanded signal constellation.

Index Terms

Euclidean distance, fading channels, geometrical uniformity, space-time codes, constellation space invariance, fading resilience.

I. INTRODUCTION

Encoding jointly along spatial and temporal dimensions has received considerable attention over the recent years, and the concerted research effort has led to improved understanding of both block and trellis designs of space-time codes (see, for example, [1], [2], [3], [4], [8], [9], [10], [11], [12], [13], [14], [15], [16]); by comparison, geometric considerations have been sporadic (see, e.g., [5], [7])—perhaps due to the perception that multiplicative distortions incurred as a result of fading can destroy symmetries. Contrary to any such perception, Schulze proved in [5] that flat fading channels leave invariant the shape of (generalized) orthogonal space-time constellations (or codematrixes)—although he viewed his results mainly as a geometrical interpretation for the optimal detection of all orthogonal and generalized orthogonal space-time constellations, which are linearly decodable; note that generalized orthogonal designs are alternatively called space-time block codes. Recently, Gharavi-Alkhansari and Gershman [6] examined the same invariance property for an orthogonal space-time constellation, and used it to explain why optimal decoding reduces to symbol-by-symbol decoding.

There exists an alternative motivation for examining the conditions that allow the shape of a constellation to be preserved in fading; it pertains to code designs that rely on certain geometrical properties of the (multidimensional) constellation, such as the spectrum of relative
Euclidean distances. When performance is viewed on an instantaneous basis, rather than on average, the observed relative distance between two valid points (codewords) depends on the effect of multiplicative distortion (fading) on the two points; if an instantaneous realization of the channel distorts valid candidate points differently, a less likely point may appear more likely at the channel output, with respect to receiver observations\(^1\). When instantaneous performance—as a function of Euclidean distances—is relevant, it becomes crucial to be able to preserve the shape of the signal constellation for any realization of the channel (multiplicative distortion). This is the motivation for considering the resilience of the constellation shape to fading; other implications of fading resilience are discussed below.

In [5], a necessary and sufficient condition [8] for orthogonality of a space-time constellation—such as arising, e.g., from Radon-Hurwitz constructions [2]—shows that when a detector operates to detect individual coordinates\(^2\), the detection equation at any receive antenna is such that the equivalent channel leaves invariant—up to a scaling factor—the distances between the (potentially transmitted) multidimensional space-time constellation points, as well as their respective angles; one can recognize this invariance to be a form of resilience to fading of the (generalized) orthogonal space-time constellation, whose shape is, in effect, preserved (up to a scaling factor) in spite of the multiplicative distortions due to flat fading. Note that the above assumption about coordinate-wise detection implies that a multidimensional space-time constellation point from \(\mathbb{C}^{n_0}, n_0 \in \mathbb{N}\), is viewed (by the detector) as a point from \(\mathbb{R}^{2n_0}\), via a well-known isometric transformation (see, e.g., [5, eq. (1)]). As demonstrated in [5], the invariance property applies directly to space-time codematrices from (generalized) orthogonal designs [4] or from unitary designs [8]—mainly because such designs allow any space-time constellation point to be expressed as a linear combination of basis matrices (see proof in [5]). It is known that even an orthogonal space-time block code that has full-rate is, in essence, a space-time modulator; i.e., it can provide diversity gain in flat fading channels, but no coding gain (as redundancy is inserted in the spatial dimension, and the inherent repetition in the time dimension provides as good a coding redundancy as repetition codes do)\(^3\). Linearly decodable, real, generalized orthogonal designs (respectively complex unitary designs) for \(N\) transmit antennas can be viewed as \(T \times N\) real matrices (respectively \(N \times \check{N}\) complex matrices); they are (non-surjective) mappings from \(\mathbb{R}^{2K}\) to \(\mathbb{R}^{TN}\) (respectively to \(\mathbb{R}^{2N^2}\)), where \(T\) is the number of channel uses (symbol epochs) covered by a codematrix [8, Definition 4]. In light of the need to add coding gain, the natural question is whether this fading resilience can be preserved when coding redundancy is added—preferably, without modifying the spectral efficiency or expanding the bandwidth; this, in turn, requires that the space-time constellation be extended beyond the orthogonal set. Such constructions have been reported by Ionescu et al. [9], and later generalized by Siwamogsatham and Fitz [12], [13] and by Seshadri and Jafarkhani [10], [11], who dubbed such codes ‘super-orthogonal’. It is shown in this correspondence that Schulze’s result for (generalized) orthogonal designs can

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\(^1\)When performance, e.g. pairwise error probability, is averaged over channel realizations the result only depends on constellation properties, in isolation of fading; this makes the effect of instantaneous channel distortions transparent to performance on average.

\(^2\)That is, it provides information on real coordinates of complex symbols that make up the space-time constellation, rather than on the complex entities themselves; this is always the case.

\(^3\)E.g., when simulating in AWGN channels an Alamouti code [2] with two transmit and one receive antennas, whereby the complex elements mapped to \(2 \times 2\) space-time complex matrices are drawn from a 4PSK constellation, one obtains the familiar uncoded performance of 4PSK in AWGN—provided that both schemes use the same total energy per channel use, or equivalently the (average) received bit energy values per receive antenna are the same.
be extended to a larger family of space-time constellations, and to space-time codes that are not linearly decodable. If, in addition, the space-time constellation and the redundancy scheme itself have additional symmetries related to the shape of the constellation and the codebook—e.g., geometrical uniformity [7]—then the important implication would be that symmetries such as geometrical uniformity can be preserved after passing through the fading channel. This, in turn, should motivate efforts to embed symmetry enabling structures into codes designed for fading channels. In particular, fading resilient symmetries are enablers for extending the concept of geometrical uniform codes to multiple-inputmultiple-output (MIMO) fading channels—e.g., by using such constellations along with a more powerful redundancy scheme, such as a turbo, multilevel, or low-density parity-check (LDPC) code.

II. FADED RESILIENCE VIA GEOMETRICALLY INVARIANT PROPERTIES

Let \( i = \sqrt{-1} \) and consider a linearly decodable, complex, linear, generalized orthogonal design for rate \( K/T \) for \( N \) transmit antennas, which maps a vector \( s = [z_1, \ldots, z_K]^T \in \mathbb{C}^K \) of \( K \) complex symbols \( z_k = x_k + iy_k, k = 0, \ldots, K - 1 \), to semiunitary complex \( T \times N \) matrices \( S \in \mathcal{M}_{T,N}(\mathbb{C}) \); semiunitarity means that \( S^H S = \|s\|^2 I_N \) (even when \( T \neq N \)), and the linearly decodable assumption leads, in one aspect, to the constraint \( T \geq N \). The constraint \( T \geq N \) can be dropped if one considers codes that are not linearly decodable. In another aspect, pursuant to the isometry \( \mathcal{I} : \mathbb{C}^K \mapsto \mathbb{R}^{2K} \) that maps \( s \) to the \( 2K \)-dimensional real vector \( \chi \equiv \{\Re\{z_1\}, \Im\{z_1\}, \ldots, \Re\{z_K\}, \Im\{z_K\}\}^T = \mathcal{I}(s) \), linearity in the arbitrary symbols \( z_k, k = 0, \ldots, K - 1 \), means that there exist \( 2K \) basis matrices of size \( T \times N \), with complex elements, such that

\[
S = \sum_{l=0}^{2K-1} \chi_l \beta_l \in \mathcal{O}, \quad \forall \chi \in \mathbb{R}^{2K}
\]

(1)

\[
= \sum_{l=1}^{K} (x_l \beta_{2l-2} + y_l \beta_{2l-1}) = \sum_{l=1}^{K} (z_l \beta_l^+ + z_l^* \beta_l^-),
\]

(2)

where the asterisk represents complex conjugation, and [8]

\[
\beta_l^\pm = \frac{1}{2} (\beta_{2l-2} \pm i \beta_{2l-1});
\]

(3)

a necessary and sufficient condition for \( S^H S = \|s\|^2 I_N \) is

\[
\beta_l^H \beta_p + \beta_p^H \beta_l = 2 \delta_{lp} I_N, \ l, p = 0, \ldots, 2K - 1,
\]

(4)

where \( I_N \) is the \( N \times N \) identity matrix.

Remark 1: The rate \( K/T \) mentioned above represents only a symbol rate, which does not indicate in any way a (finite) spectral efficiency—unless the complex symbols are restricted to a common finite constellation \( \mathcal{Q} \) such as \( m \)-PSK, with \( m \) some integer power of 2; in other words, the complex symbols \( z_k \)'s (or the real \( 2K \)-tuple \( \chi \)) can assume arbitrary complex (real) values (\( \mathcal{O} \) is non-countable).

As long as \( \chi \in \mathbb{R}^{2K} \), the set \( \mathcal{O} \) spanned by the basis \( \{\beta_l\}_{l=0}^{2K-1} \) over \( \mathbb{R} \) is a vector space. Specifying a (finite) spectral efficiency means, e.g., restricting the complex symbols \( z_k, k = 0, \ldots, K - 1 \), to a common finite constellation \( \mathcal{Q} \), e.g. \( m \)-PSK; this will produce a multidimensional space-time constellation with a finite cardinality, denoted \( \mathcal{G} \subset \mathcal{O} \) in the sequel;
nevertheless, eqs. (1) and (4) still hold because \( Q \subset \mathbb{C} \) and, respectively, because restricting \( z_k, k = 0, \ldots, K - 1 \), to \( Q \) does not modify the basis expansion in \( O \). Note that (1), (4) directly lead to

\[
(S - S')^H (S - S') = \|\chi - \chi'\|^2 I_N, \quad \forall S, S' \in O.
\]  

(5)

Since the complex Radon-Hurwitz eqs. (4) are invariant to multiplication of all matrices in a generator set by \( \zeta \in \mathbb{C}, |\zeta| = 1 \), it follows that \( \{\beta_l\}_{l=0}^{2K-1} \) is a basis in \( O \) if and only if \( \{\beta_l \zeta\}_{l=0}^{2K-1} \) is.

An expansion (see below) of the finite space-time constellation \( G \)—as practiced, e.g., in [9], [12], [13], [10], [11]—does not necessarily remain within the limits of the generalized orthogonal design \( O \), and orthogonality of pairwise differences [see (5)] is not necessarily preserved in the expanded constellation.

A. Constellation Expansions and Their Properties

As mentioned above, adding coding redundancy without modifying the spectral efficiency requires that the finite space-time constellation be extended beyond the set \( G \) of orthogonal matrices. Consider a multidimensional space-time constellation \( G \) from a generalized complex orthogonal design \( O \), and an expansion of \( G \) via a symmetry or by multiplication with some unitary \( N \times N \) matrix \( U \). A first-tier expanded constellation is

\[
G_e \overset{\text{def}}{=} G \cup GU.
\]  

(6)

and has been introduced in [9]. Specifically, with a 4PSK constellation on each of \( N = 2 \) transmit antennas, [9] used a symmetry operation (characterized further in [7, Section II.B]) to expand an orthogonal set of sixteen matrices obtained by mapping all \( K \)-tuples of 4PSK elements to \( T \times N \) matrices, where \( K = T = 2 \); after expansion, pairwise differences are in general non-orthogonal (no longer verify (5)), and the symmetry operation used in [9] corresponds to right multiplication by the unitary matrix \([1 0 \quad 0 -1]\)—recognized to be a particular case of the ‘super-orthogonal’ construction from [10], [11]. Note that any symmetry can be described as multiplication by a unitary matrix of appropriate size.

Remark 2: It should be stressed here that, whenever the intention is to guarantee some geometrical invariance property of the expanded constellation \( G_e \), the preferred method for expanding \( G \) should be some symmetry operation, rather than an arbitrary unitary transformation—which, in turn, should arise simply as a consequence of the symmetry itself; the reason is, of course, the very nature of the expected result, which is some form of geometrical invariance.

As already noted, \( GU \not\subset O \), in general, because \( GU \) is not necessarily in the span of \( \{\beta_l\}_{l=0}^{2K-1} \); thereby, orthogonality of pairwise differences after a constellation expansion that does not alter the spectral efficiency will be lost. Nevertheless, if \( S \in G \), then \( (SU)^H SU = \|s\|^2 I_N \) and

\[
SU = \sum_{l=0}^{2K-1} \chi_l \beta_l', \quad \forall \chi \in \mathbb{R}^{2K}
\]  

(7)

\[
\beta_l' = \beta_l U, \quad \forall l = 0, \ldots, 2K - 1
\]  

(8)

As discussed above [8], \( \{\beta_l\}_{l=0}^{2K-1} \) verify the complex Radon-Hurwitz eqs. (4), while \( \{\beta_l\}_{l=0}^{2K-1} \) verify

\[
\beta_l^H \beta_p' + \beta_p^H \beta_l' = 2\delta_{lp} I_N, \quad l, p = 0, \ldots, 2K - 1;
\]  

(9)
however, a similar property does not necessarily hold for two basis matrices from the different sets \( \{ \beta_i \}_{i=0}^{2K-1} \), \( \{ \beta_i' \}_{i=0}^{2K-1} \).

Since \( U \) is unitary if and only if \( U \zeta \) is unitary—provided that \( \zeta \in \mathbb{C}, |\zeta| = 1 \)—expansions via \( U \zeta \) and \( U \) should be simultaneously characterizable as applying \( U \) to either \( G_\zeta \) or \( \tilde{G} \).

**Lemma 3:** Let \( Q \subset \mathbb{C} \) be a (finite) complex constellation, and \( \{ \beta_i \}_{i=0}^{2K-1} \) a generator set for \( G_\zeta \) over \( \mathcal{I}(\mathbb{Q}^K) \), such that any \( S \in G \) verifies (1), (2) with \( z_k \in Q \). Let \( \zeta \in \mathbb{C}, |\zeta| = 1 \). Then

\[
\tilde{G} \text{ def } G_\zeta = \{ \tilde{S} \text{ def } S_\zeta | S \in G \} \subset O,
\]

\[
\tilde{S} = \sum_{l=1}^{K} [\Re \{ z_l \zeta \eta_{2l-2} + \Im \{ z_l \zeta \eta_{2l-1} \} ] , \quad \tilde{S} = \sum_{l=1}^{K} [\tilde{x}_l \eta_{2l-2} + \tilde{y}_l \eta_{2l-1} ] , \quad \tilde{x}_l + i \tilde{y}_l \text{ def } \tilde{z}_l \in \mathbb{Q}^2 \zeta ,
\]

\[
\eta_{2l-2} \text{ def } \zeta ( \Re \{ \zeta \} \beta_{2l-2} - \Im \{ \zeta \} \beta_{2l-1} ) , \quad l = 1, \ldots, K ,
\]

\[
\eta_{2l-1} \text{ def } \zeta ( \Re \{ \zeta \} \beta_{2l-1} + \Im \{ \zeta \} \beta_{2l-2} ) , \quad l = 1, \ldots, K .
\]

Moreover, \( \{ \eta \}_{i=0}^{2K-1} \subset O \)

\[
\eta_l^H \eta_p + \eta_p H \eta_l = 2 \delta_{lp} I_N , \quad l, p = 0, \ldots, 2K - 1 . \tag{15}
\]

**Proof:** A sketch of proof is as follows. The fact that \( \{ \eta \}_{i=0}^{2K-1} \subset O \) is obvious; simple manipulations of (13), (14), (4) prove (15) directly. To prove (11) it suffices to re-write the terms in the second summation of (2) as \( z_2 \beta^\gamma + z_2 \beta^\gamma = z_2 (\zeta^\gamma \beta^\gamma + z_2 \zeta^\gamma \beta^\gamma) = (z_2 \zeta^\gamma + (z_2) \zeta^\gamma)^\eta^\gamma, \)

where \( \eta_2^\gamma = \beta^\gamma \zeta \) and \( \eta_2^\gamma = \beta^\gamma \zeta \), followed by straightforward manipulations and by finally multiplying (2) by \( \zeta \).

Lemma 3 shows that an expansion of \( G \) by \( G_\zeta = \tilde{G}(\zeta I_N) \) simply changes the generator set and the alphabet (from \( Q \) to \( Q_\zeta \)), and is indiscernible (from \( O \)) in the sense that \( G_\zeta \subset O \). Therefore expansions of the form \( G_\zeta = G \cup \tilde{G} \cup G \zeta \) differ from those of the form \( G_\zeta = G \cup \tilde{G} U \) only in that \( U \) operates on a different subset of \( O \) (\( G_\zeta \) vs. \( \tilde{G} \)). Clearly, \( \zeta \in \mathbb{C}, |\zeta| = 1 \) preserves the constellation energy.

**Definition 4:** If \( \zeta \in \mathbb{C}, |\zeta| = 1, \zeta \neq 1 \), and \( U \neq I_N \) is an \( N \times N \) unitary matrix, then a first-tier, indirect (direct), discernible constellation expansion of \( G \) is \( G_e = G \cup \tilde{G} \cup G \zeta \) \((G_e = G \cup \tilde{G} U)\), where \( \tilde{G} \cup \tilde{G} \neq G (\tilde{G} U \neq G) \) and \( U \) has either more than two distinct eigenvalues, or all real eigenvalues.\(^4\)

Consider a direct discernible constellation expansion of \( G \) to \( G \cup \tilde{G} \), where matrices \( S, SU \) verify (1), (7) \( \forall S \in G \).

**Theorem 5:** If \( G_e \) of (6) is a first-tier, direct, discernible expansion by \( U \neq \pm I_N \) of a multidimensional space-time constellation \( G \) from a generalized complex orthogonal design, having a generator set \( \{ \beta_i \}_{i=0}^{2K-1} \), and if \( \{ \beta_i \}_{i=0}^{2K-1} \) is the generator set for \( G' \defequiv \tilde{G} U \) that verifies (8), then no element of the set \( \{ \beta_i' \}_{i=0}^{2K-1} \) is a linear combination, over \( \mathbb{R} \), of the matrices \( \beta_i', l = 0, \ldots, 2K - 1 \).

**Proof:** Assume to the contrary that \( \beta_{q_0}' = \beta_{q_0} U = \sum_{q=0}^{2K-1} t_q \beta_q' \), where \( t \defequiv [t_0, \ldots, t_{2K-1}]^T \in \mathbb{R}^{2K} \). It can be easily verified, using (9), that \( \sum_{q=0}^{2K-1} t_q^2 = 1 \). First, assume that at least two components of \( t \) are nonzero. Then, for some nonzero \( t_{q_1}, q_1 \neq q_0, \beta_{q_1} = t_{q_1} \beta_{q_0} U - t_{q_1}^{-1} \sum_{q \neq q_1, q_0} t_q \beta_q - \)

\(^4\)Via Lemma 3, this accommodates constellation expansions by a unitary (not necessarily Hermitian) matrix that has complex eigenvalues, but only arising as a rotation of a set of real eigenvalues.
the code matrix selected for transmission verifies either
is selected for transmission from the
element of
and the dimensionality condition implicit in (17) is well-defined.

or, after using (4), \( t_{q_0}^{-1}U^H - 2t_{q_0}t_{q_1}^{-1}I + t_{q_1}U = 0 \). Then \( U^H = 2t_{q_0}I - U \), and unitarity of \( U \) translates into \( U \) verifying the equation

\[
U^2 - 2t_{q_0}U + I = 0. \tag{16}
\]

Assume that \( U \) verifies a (monic) polynomial equation of degree smaller than two, namely \( U + m_0I = 0; \) then, \( U = -m_0I \), and unitarity together with the assumption that \( U \) has real eigenvalues imply that \( U = \pm I_N \), which contradicts the hypothesis. Then, necessarily, (16) is the minimum equation of \( U \). But \( t^2 - 2t_{q_0}t + 1 = 0 \) has roots \( t^{(1,2)} = t_{q_0} \pm \sqrt{t_{q_0}^2 - 1} \), with \( t_{q_0} < 1 \); thereby, since the irreducible (in \( \mathbb{C} \), in this case) factors of the minimum polynomial divide the characteristic polynomial, it follows that the distinct eigenvalues of \( U \) are the distinct roots among \( \{t^{(1)}, t^{(2)}\} \), which do have, indeed, unit magnitude, but nonzero imaginary parts—again contradicting the hypothesis. Finally, assume that only one component of \( t \) is nonzero, say \( \beta_{q_0} = \beta_{q_1}U = \beta_{q_1}, \) \( q_1 \neq q_0 \). Then (4) is equivalent to \( U^H + U = 0 \Leftrightarrow U^2 + I = 0 \), and the minimal polynomial \( t^2 + 1 = 0 \) has non-real roots \( \pm i \)—again contradicting the hypothesis. This completes the proof.

Since \( G_\zeta \subset O \), as discussed above, a similar contradiction as the one used above can be employed to infer directly

**Corollary 6:** If \( G_\zeta = \hat{G} \cup G \cup U_\zeta \) is a discernible expansion then \( (G_\zeta \setminus G) \cap O = \{0\} \).

Thereby, Theorem 5 leads directly to a direct sum structure via

**Corollary 7:** Any discernible expanded constellation \( G_\zeta \) is naturally embedded in a direct sum of two \( 2K \)-dimensional vector sub-spaces of \( M_{T,N}(\mathbb{C}) \), and

\[
S = \sum_{i=0}^{2K-1} \chi_i \beta_i + \sum_{i=0}^{2K-1} \chi'_i \beta'_i, \quad \forall S \in G_\zeta. \tag{17}
\]

**B. Implications of Discernible Constellation Expansions**

In all cases where the Euclidean distance between points from the multidimensional constellation \( G_\zeta \) is relevant [15], [14], [16], [7], the Euclidean, or Frobenius, norm of \( S \in G_\zeta \) is important; then, \( S \) can be identified via an isometry with a vector from \( \mathbb{R}^{2TN} \), where \( 2TN \) is the total number of real coordinates in \( S \) when using the expanded constellation \( G_\zeta \). Therefore, since \( S \in G_\zeta \) is completely described by the \( 2 \cdot 2 \cdot K \) real coordinates of the embedding space (see (17)), it follows that the first tier expansion uses \( 4K \) of the available \( 2TN \) diversity degrees of freedom. Note that, since when \( N \geq 2 \) the maximum rate for square matrix embeddable space-time block codes (unitary designs) is at most one [8, Theorem 1], it follows that \( K \leq T \) and the dimensionality condition implicit in (17) is well-defined.

**C. Fading Resilience**

In order to show that \( G_\zeta = \hat{G} \cup G \cup U \) is resilient to flat fading, assume that a code matrix \( c \in G_\zeta \), is selected for transmission from the \( N \) transmit antennas during \( T \) time epochs; an arbitrary element of \( G_\zeta \) (denoted \( S \) in above paragraphs) verifies (17), and either the \( \chi_k \) coefficients or the \( \chi'_k \) coefficients vanish. Without loss of generality, assume there is one receive antenna. Clearly, the code matrix selected for transmission verifies either \( c \in \hat{G} \) or \( c \in G_\zeta \setminus \hat{G} \); assume first the
former, i.e. all $\chi'_k$ coefficients vanish in (17). The observation vector during the $T$ time epochs is given by

$$r = c h + n_c,$$

where $h = [h_1 \ h_2 \ \cdots \ h_N]^T$ is the vector of complex multiplicative fading coefficients and $n_c$ is complex AWGN with variance $\sigma^2 = N_0/2$ in each real dimension. Given $h$ and $n_c$, when $\chi'_k$’s are all zeros, the received vector is simply

$$r = \sum_{k=0}^{2K-1} \chi_k \eta_k + n_c,$$

where $\eta_k = \beta_k h$ for $k = 0, 1, \cdots, 2K - 1$. By eq. (4), it can be shown that $\Re\{\langle \eta_k, \eta_l \rangle\} = \|h\|^2 \delta_{kl}$. Define $g_k$ as the real vector corresponding to $\eta_k$ as follows:

$$\eta_k \leftrightarrow \|h\| g_k$$

for $k = 0, 1, \cdots, 2K - 1,$

where $\leftrightarrow$ denotes the correspondence between complex and real vectors. Clearly, $g_k$’s are real orthonormal vectors. Also define the real vectors corresponding to $r$ and $n_c$ respectively as follows: $r \leftrightarrow y$ and $n_c \leftrightarrow n$. Then, the received real vector

$$y = \|h\| \sum_{k=0}^{2K-1} \chi_k g_k + n.$$

Define $G = [g_0 \ g_1 \ \cdots \ g_{2K-1}], \quad \chi = [\chi_0 \ \cdots \ \chi_{2K-1}]^T,$

then

$$y = \|h\| G \chi + n.$$

Similarly, when $c \in G_c \setminus \mathcal{G}$, i.e. all $\chi_k$’s in (17) vanish, the following equation holds:

$$y' = \|h\| \sum_{k=0}^{2K-1} \chi'_k g'_k + n',$$

where $r \leftrightarrow y'$ and $\beta_k h \leftrightarrow \|h\| g'_k$. That is,

$$y' = \|h\| G' \chi' + n',$$

where $G' = [g'_0 \ g'_1 \ \cdots \ g'_{2K-1}], \quad \chi' = [\chi'_0 \ \chi'_1 \ \cdots \ \chi'_{2K-1}]^T$. Conditioned on whether the transmitted signal point is selected from $\mathcal{G}$ or from $G_c \setminus \mathcal{G}$, one can first define $\chi_\oplus = [\chi^T \ \chi'^T]^T$, $\eta_\oplus = [n^T \ n'^T]^T$, and $y_\oplus = [y^T \ y'^T]^T$, where either half of the real coefficients vanish, then express the received signal in both cases as

$$y_\oplus = \|h\| G_\oplus \chi_\oplus + n_\oplus$$

(18)

where $G_\oplus$ is the $2 \cdot 2 \cdot T \times 2 \cdot 2 \cdot K$ matrix

$$\begin{bmatrix} G & 0 \\ 0 & G' \end{bmatrix}.$$

It is easy to verify that $G_\oplus^T G_\oplus = I_{2 \cdot 2 \cdot K}$. Hence, $y_\oplus$ preserves the distances and angles of $\chi_\oplus$—up to the scaling factor $\|h\|$ and noise.

A final discussion pertains to the side information on whether the transmitted signal point belongs to $\mathcal{G}$ or $G_c \setminus \mathcal{G}$:

1) Representing the multidimensional points in $G_c$—and their respective Euclidean distances—in terms of vectors coordinates $(\chi_k, \chi'_k)$ rather than matrix entries, was preferred above only because it simplified the analysis;

2) The side information mentioned above is naturally available at the receiver during hypothesis testing—since any tested point in $G_c$ belongs to an unique sub constellation, thereby allowing one to form $\chi_\oplus$ by appropriate zero-padding; then, for each hypothesis, the nonzero received (i.e., observed) coordinates can be easily padded with leading or trailing
zeroes, in order to form $y_{\oplus}$ and match the standing hypothesis about the transmitted point. Thereby, when testing various $\chi_{\oplus}$ vectors—from a constellation $G_e$ with a given shape—performance is determined precisely by the distances and angles between $y_{\oplus}$ vectors; if the latter match the distances and angles between points in $G_e$ (up to noise, and a scaling factor due to fading), then the shape of $G_e$ is preserved, and other symmetry properties of $G_e$ become relevant when they exist.

3) Equivalently, rather than calculating the Euclidean distances between multidimensional points from $G_e$ in terms of vector coordinates $\chi_k$, $\chi'_k$, the decoder may (and usually does) compute them as Frobenius norms of (respective difference) matrices. (Euclidean distances between $\chi_{\oplus}$ vectors and Frobenius norms of their corresponding difference matrices are the same—with proper normalization.)

4) For example, in the space-time trellis codes from [9], the branches departing from, and converging to, any state use signal points from one subconstellation; when a maximum likelihood receiver tests any branch, the originating state of the branch together with the associated information bits determine a point from a precise subconstellation.

Hence, the decoder on the receiver side does, naturally, have access to the side information during hypothesis testing, and thereby benefits from shape invariance.

In summary, the fading channel, up to scaling and noise, leaves invariant the shape in the expanded signal constellation $G_e$. Although the maximum likelihood decoding for the expanded signal constellation is no longer linear, the decoding process benefits from this property nonetheless.

III. Example

In this section, we illustrate the above results with the expanded signal constellation in [9].

The expanded signal constellation in [9] over QPSK is shown in Table I. The entries in the codematrixes in Table I are the indices of the signal points in Table II. It is clear that the first 16 matrices, $C_i$ ($0 \leq i \leq 15$), are of the form $\begin{bmatrix} A & B^* \\ B & -A^* \end{bmatrix}$, and hence can be expressed as linear combinations of the following four base matrices:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Denote these four base matrices as $\beta_k$, $k = 0, 1, 2, 3$, and the first 16 code matrices can be represented by the linear combinations $\sum_{k=0}^{3} \chi_k \beta_k$. Similarly the other 16 code matrices, $C_i$ ($16 \leq i \leq 31$), are of the form $\begin{bmatrix} A & -B^* \\ B & A^* \end{bmatrix}$, and can be represented with linear combinations of four different base matrices $\beta'_k$, $k = 0, 1, 2, 3$:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

It can be verified that $\beta_k$'s satisfy Eq. (4) and so do the $\beta'_k$'s. However, it can be shown that the property does not necessarily hold when two matrices are from two different groups. The latter generator set is obtained from the former via $\beta'_k = \beta_k U$, $k = 0, \ldots, 3$, where $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let

\(^5\)Same appears to be true of codes (not extended constellation!) from [11].
\[ \{ C_i \}_{i=0}^{31} = \{ C_i \}_{i=0}^{15} \cup \{ C_i \}_{i=16}^{31} \\
= \left\{ \sum_{k=0}^{3} (\chi_k \beta_k + \chi_k' \beta_k') : \chi_k \in \{ -1, 1 \} \right. \] \\
\left. \text{and } \chi_k' = 0 \right\} \\
\cup \left\{ \sum_{k=0}^{3} (\chi_k \beta_k + \chi_k' \beta_k') : \chi_k' \in \{ -1, 1 \} \right. \] \\
\left. \text{and } \chi_k = 0 \right\}.

We also remark that the space-time trellis codes in [9] are such that the branches departing from, and converging to, any state are all labelled by codemats from either \( G \) or \( GU \). As such, the
side information mentioned above is accessible to the decoder.

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