ANALYSIS OF A DELAYED FREE BOUNDARY PROBLEM FOR TUMOR GROWTH

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(Communicated by Bei Hu)

ABSTRACT. In this paper we study a delayed free boundary problem for the growth of tumors. The establishment of the model is based on the diffusion of nutrient and mass conservation for the two process proliferation and apoptosis (cell death due to aging). It is assumed the process of proliferation is delayed compared to apoptosis. By $L^p$ theory of parabolic equations and the Banach fixed point theorem, we prove the existence and uniqueness of a local solutions and apply the continuation method to get the existence and uniqueness of a global solution. We also study the asymptotic behavior of the solution, and prove that in the case $c$ is sufficiently small, the volume of the tumor cannot expand unlimitedly. It will either disappear or evolve to a dormant state as $t \to \infty$.

1. Introduction. Over the last thirty years, a variety of partial differential equation models for tumor growth or therapy have been developed, cf.[2-4, 9, 13-14, 17-18] and references therein. Most of those models are based on the reaction diffusion equations and mass conservation law. Analysis of such free boundary problems has drawn great interest, and many interesting results have been established, cf.[1, 5-8, 10-12, 15-17, 19-23] and references therein.

In this paper we study the following problem:

$$
\frac{c}{\partial t} \sigma(r,t) = \Delta_r \sigma(r,t) - \lambda \sigma(r,t), \quad 0 \leq r < R(t), \quad t > 0,
$$

$$
\frac{\partial \sigma}{\partial r}(0,t) = 0, \quad \sigma(R(t),t) = \sigma_\infty, \quad t > 0,
$$

$$
\frac{d}{dt} \left( \frac{4\pi R^3(t)}{3} \right) = 4\pi \int_0^{R(t-\tau)} \mu \sigma(r,t-\tau)r^2 dr - 4\pi \int_0^{R(t)} \mu \tilde{\sigma} r^2 dr, \quad t > 0,
$$

$$
\sigma(r,t) = \psi(r,t), \quad 0 \leq r \leq R(t), \quad -\tau \leq t \leq 0,
$$

$$
R(t) = \varphi(t), \quad -\tau \leq t \leq 0
$$

2000 Mathematics Subject Classification. Primary: 35K57, 35Q92; Secondary: 39B12.

Key words and phrases. Tumors, Parabolic equations, Global solution, Asymptotic behavior.
where $r$ is the radial variable scaled by the tumor-cell radius, $t$ is the time variable scaled by the tumor-cell doubling time, the variable $\sigma(r, t)$ represents the scaled nutrient concentration at radius $r$ and time $t$ and the variable $R(t)$ represents the scaled radius of the tumor at time $t$, $\Delta r = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r})$. The term $\lambda \sigma$ in (1.1) is the scaled consumption rate of nutrient in a unit volume within a unite time interval; $\mu \sigma$ is the scaled proliferation rate of tumor cells in a unit volume (=the number new-born cells in a unite volume within a unite time interval); $\sigma_{\infty}$ reflects scaled constant supply of nutrient that the tumor receives from its surface, $\psi$ and $\varphi$ are given initial date of $\sigma$ and $R$. $c$ is a constant and $c \ll 1$(cf[12]).

The study of the effects of time delays in the growth of tumors by using the methods of mathematical models was initiated by[2]. Experiments suggest that changes in the proliferation rate can trigger changes in apoptotic cell loss and that these changes do not occur instantaneously: they are mediated by growth factors expressed by the tumor cells (see[2]). In [2], the author considered two ways of modifying the standard model of avascular tumor growth by incorporating into the net proliferation rate a time-delayed factor. In the first case, the delay represents the time taken for cells to undergo mitosis. In the second case, the delay represents the time for changes in the proliferation rate to stimulate compensatory changes in apoptotic cell loss. The models presented in [2] are quasi-stationary version (i.e. $c=0$) of free boundary problem with a necrotic core inside the tumor and the nutrient is consumed by tumor cells with constant rate. Numerical and asymptotic techniques were used to show how a tumor’s growth dynamics are affected by including such delay terms. In particular for the first case, the author showed that the size of the delay does not affect the limiting behavior of the tumor. Recently, Foryś and Bodnar [10] made a rigorous analysis of the model with time delay presented in Byrne [2] of the first case in the framework of delay differential equations. In [10], they only considered a simpler model without a necrotic core inside the tumor. They mainly considered a tumor’s growth dynamics and showed that how the time delay affect the tumor growth. They proved existence of periodic solutions for some parameter and global stability of steady-state solutions for other ones.

The idea of the model studied in this paper come from the paper Byrne [2], Foryś and Bodnar [10] and Cui and Xu [8]. The model is established by modifying the model A. Friedman [12] by considering the time delay effect as the first case in Byrne [2](i.e. the time delay represents the time taken for cells to undergo mitosis). Obviously, this is a fully non-stationary version a free boundary problem for tumor growth with a time delay. The limiting case where $c = 0$ (i.e. the quasi-stationary version) is studied by [8]. In [8] rigorous analysis of the limiting case of the model is given. Final mathematical formulations to the limiting case is a retarded differential equation of the form

$$R'(t) = f(R(t), R(t - \tau)), t > 0$$
$$R(t) = \varphi(t), -\tau \leq t \leq 0.$$  

By using a comparison method, the authors discussed the dynamical behavior of solutions to the model. They proved that the dynamical behavior of solutions of the problem is similar to that of solutions for corresponding problem without time delay. In the limiting case where $c = 0$ Eq.(1), (2) can be solved exactly, and the exact expression of the evolution equation for $R$ can be obtained. This is clearly not the case for present model and the method used in [8] can not be used to present model. Using Banach fixed point theorem, a compare method and
some mathematical techniques, we mainly prove the existence and uniqueness of the global solution to the problem and asymptotic behavior of the solutions to the problem. The results show that in the case \( c \) is sufficiently small and \( \sigma_\infty > \bar{c} \), the volume of the tumor cannot expand unlimitedly. It will evolve to a dormant state as \( t \to \infty \) which is similar to that of the corresponding problem without time delay (see [12]). We also show that in the case \( c \) is sufficiently small and \( \sigma_\infty < \bar{c} \), the volume of the tumor also cannot expand unlimitedly. It will disappear.

The paper is arranged as follows: In Section 2 we prove the existence and uniqueness of the global solution to the system (1)-(5). Section 3 is devoted to the asymptotic behavior of the solutions to the system (1)-(5). In the last Section we give some conclusions.

2. Global existence and uniqueness. We shall prove a global existence and uniqueness theorem for the problem (1)-(5) under the following assumptions:

\( (A_1) \) \( \varphi \in C[-\tau,0], \varphi(t) > 0 \) for \( -\tau \leq t \leq 0 \), and there exists a constant \( \delta > 0 \) such that \( \min_{-\tau \leq t \leq 0} \varphi \geq \delta \).

\( (A_2) \) \( \psi \in C([0, \infty) \times [-\tau,0]), \psi(r,t) = \sigma_\infty, r \geq R(t); 0 \leq \sigma(r,t) \leq \sigma_\infty, r \leq R(t) \).

\( (A_3) \) \( \psi(r,0) = \psi_0(r) \) is twice weakly differentiable on \([0, R(0)], \psi'' \in L^\infty[0, R(0)], \psi''(0) = 0, \psi_0(R(0)) = \sigma_\infty \).

For a given number \( T > 0 \) and a given positive function \( R \in C[0,T] \) we introduce the following notations:

\[ Q_T^R = \{(x,t) \in R^3 \times R^1 : |x| < R(t), 0 < t \leq T\}, \]

\[ Q_T^R = \text{the closure of } Q_T^R, \]

\[ W^{2,1}_p(Q_T^R) = \{\sigma(x,t) \in L^p(Q_T^R) : \partial_x^k \partial_t^k \sigma(x,t) \in L^p(Q_T^R) \text{ for } |\alpha|+2k \leq 2, 1 \leq p < \infty\}, \]

\[ C^{2+\lambda,1+\lambda}(Q_T^R) = \text{the Hölder space on the parabolic domain } Q_T^R (0 < \lambda < 1). \]

\[ D_{p,\sigma}(B_{R_0}) = \text{the trace space of } B_{p,\sigma}(B_{R_0}), \]

where \( B_{p,\sigma}(B_{R_0}) = W^{2,1}_p(Q_T^R) \cap \{\sigma \in C(Q_T^R) : \sigma(R_0,t) = \sigma_\infty, 0 \leq t \leq T\} \) and \( B_{R_0} \) denotes the ball in \( R^3 \) centered at origin with radius \( R_0 \).

**Lemma 2.1.** (see [20] Lemma 1 or [7] Lemma 2.1) Let \( c, T \) be given positive numbers. Let \( R(t) \in C^1[0,T], R(t) > 0 \) for all \( 0 \leq t \leq T \), and \( R(0) = R_0 \). Let \( \psi(0,x) \in D_{p,\sigma_\infty} \) for some \( \frac{5}{2} < p < \infty \), and \( F \in C(Q_T^R) \). Then the following initial value problem:

\[ c\sigma_t = \Delta \sigma + F(x, t), \quad (x,t) \in C(Q_T^R), \]

\[ \sigma(R(t),t) = \sigma_\infty, \quad 0 < t \leq T, \]

\[ \sigma(x,0) = \psi(x,0), \quad |x| \leq R_0 \]

has a unique solution \( \sigma \) in the sense that satisfies the following three conditions: \( (1) \) \( \sigma \in W^{2,1}_p(Q_T^R) \subset C(Q_T^R); \) \( (2) \) \( \sigma \) satisfies the equation (7) a.e. in \( Q_T^R; \) \( (3) \) \( \sigma \) satisfies the conditions (8) and (9). Moreover, the following assertions hold:

\( (i) \) If \( \psi(x,0) \) and \( F(x,t) \) are spherically symmetric in \( x \) then \( \sigma \) is also spherically symmetric in \( x \).

\( (ii) \) There exists a positive constant \( C \) depending on \( \sigma_\infty, c \) and \( \|R(t)\|_{L^\infty[0,T]}, \|\frac{1}{R(t)}\|_{L^\infty[0,T]}, \|R'(t)\|_{L^\infty[0,T]} \) such that

\[ \|\sigma\|_{W^{2,1}_p(Q_T^R)} \leq C(\sigma_\infty + \|\psi(x,0)\|_{D_{p,\sigma_\infty}} + \|F\|_{L^p(Q_T^R)}). \]
Our main results of this section are as follows.

**Theorem 2.2.** Assume that the conditions \((A_1) - (A_3)\) are satisfied. Then the system \((1) - (2)\) has a unique solution \(\sigma(r, t), R(t)\) for all \(t \geq -\tau\). Moreover, the following estimates hold:

(i) \(0 \leq \sigma(r, t) \leq \sigma_\infty, 0 \leq r \leq R(t), t \geq 0,\)

(ii) \(R(0)e^{-\frac{\bar{\sigma}}{2}t} \leq R(t) \leq ae^{\frac{\bar{\sigma}}{2}t}, \) for \(t \geq 0; \) \(\delta \leq R(t) \leq C\) for \(\tau \leq t \leq 0,\) where \(a = \sqrt{1 + \mu \sigma_\infty \tau} |\varphi|, |\varphi| = \max_{-\tau \leq \varphi(t), b = \mu(\sigma_\infty + \bar{\sigma}).}\)

(iii) For any \(T > 0, -\frac{1}{3} \mu \bar{\sigma}\) \(\leq \frac{R(t)}{R(t)} \leq \frac{M}{\bar{\sigma}}\) for \(0 < t \leq T,\) where \(M = \mu(\sigma_\infty e^{3\sqrt{\mu_0 + k_0}T} - \bar{\sigma}).\)

**Proof.** Clearly, \(\bar{\sigma} = 0\) and \(\sigma_\infty\) are respectively lower and upper solutions of the system \((1), (2)\) and \((4)\). We have \(0 \leq \sigma(r, t) \leq \sigma_\infty, 0 \leq r \leq R(t), t \geq 0.\) Then

\[
\frac{1}{R^2(t)} \int_0^{R(t)} \mu \bar{\sigma} r^2 dr \leq \frac{dR(t)}{dt} \leq \frac{\mu}{3R^2(t)} [\sigma_\infty (R^3(t) - \tau R^3(t)], t > 0,
\]

which implies that \(R(t) \geq R(0)e^{-\frac{\bar{\sigma}}{2}t}\) and \(\frac{dR(t)}{dt} \leq \mu \sigma_\infty \omega(t - \tau) - \mu \bar{\sigma} \omega(t),\) here \(\omega(t) = R^3(t)\). By Lemma 3.1 in \([8]\) and Theorem 3.1 in \([15]\), we have \(\omega(t) \leq a^3 e^{bt}\), which implies \(R(t) \leq ae^{\frac{b}{3}t}\). From (11) and (ii) we can get (iii).

For arbitrary \(T > 0,\) we introduce a metric space \((S_T, d)\) as follows: The set \(S_T\) consists of vector functions \((\sigma(r, t), R(t))\), where \(\sigma(r, t)\) is defined on \([0, \infty) \times [-\tau, T], R(t)\) is defined on \([-\tau, T]\), and they satisfy the following conditions:

(i) \(R \in C[-\tau, T] \cap C^1[0, T], R(t) = \varphi(t), -\tau \leq t \leq 0,\) and

\[
R(0)e^{-\frac{\bar{\sigma}}{2}t} \leq R(t) \leq ae^{\frac{\bar{\sigma}}{2}t}, \text{ for } 0 < t \leq T,
\]

here \(M = \mu(\sigma_\infty e^{3\sqrt{\mu_0 + k_0}T} - \bar{\sigma}).\)

(ii) \(\sigma \in C([0, \infty) \times [-\tau, T]),\) and

\[
\sigma(r, t) \leq \sigma_\infty, \text{ for } 0 \leq r \leq R(t), 0 < t \leq T,
\]

\[
\sigma(r, t) = \sigma_\infty, \text{ for } r \geq R(t), 0 < t \leq T,
\]

\[
\sigma(r, t) = \psi(r, t), \text{ for } -\tau < t \leq 0.
\]

The metric \(d\) is defined by

\[
d((\sigma_1, R_1), (\sigma_2, R_2)) = \max_{r \geq 0, -\tau \leq t \leq T} |\sigma_1(r, t) - \sigma_2(r, t)| + \max_{-\tau \leq t \leq T} |R_1(t) - R_2(t)|.
\]

It is clear that \((S_T, d)\) is a complete metric space.

We define a mapping \(F : (\sigma(r, t), R(t)) \rightarrow (\bar{\sigma}(r, t), \bar{R}(t))\) in the following way:

\[
\frac{c \partial \bar{\sigma}}{\partial r} = D_r \bar{\sigma}(r, t) - \lambda \sigma(r, t), 0 < r < R(t), t > 0,
\]

\[
\frac{\partial \bar{\sigma}}{\partial r}(0, t) = 0, \bar{\sigma}(\bar{R}(t), t) = \sigma_\infty, t > 0,
\]

\[
\frac{d\bar{R}(t)}{dt} = \frac{\mu \bar{R}(t)}{R^3(t)} \int_0^{R(t-\tau)} \sigma(r, t-\tau) r^2 dr - \int_0^{R(t)} \bar{\sigma} r^2 dr, t > 0,
\]
\[ \bar{\sigma}(r, t) = \psi(r, t), \quad 0 < r < R(t), \quad -\tau \leq t \leq 0, \]  
\[ \bar{R}(t) = \varphi(t), \quad -\tau \leq t \leq 0 \]  
In the following, we prove \( F \) is a mapping from \( S_T \) to \( S_T \).

If \( (\sigma(r, t), R(t)) \in S_T \), set \( G(t) = \frac{1}{\bar{R}(t)} \int_0^{R(t)-r} \sigma(r, t - \tau) r^2 dr - \int_0^{\bar{R}(t)} \bar{\sigma} r^2 dr \).

Then we have 
\[ -\frac{\mu}{R} \leq G(t) \leq \frac{1}{\bar{R}^2} \left( \frac{R(t) - \tau}{R(t)} \right)^3 - \bar{\sigma} \leq \frac{1}{R}, \quad 0 \leq t \leq T. \]

Since \( \bar{R}(t) = \varphi(0)e^{\int_0^t G(s) ds} \), noticing \( \varphi \in C[-\tau, 0] \) we have \( \bar{R}(t) \in C[-\tau, T] \cap C^1[0, T] \) and
\[ R(0)e^{-\frac{\mu}{R}t} \leq \bar{R}(t) \leq R(0)e^{\frac{1}{R}t} \leq ae^{\frac{1}{R}t}, \quad \text{for } 0 < t \leq T, \]
here \( M \) as before. Then \( \bar{R}(t) \) satisfies the condition (i).

Since \( \bar{R}(t) \in C^1[0, T] \), noticing assumption (A4) by Lemma 2.1 (cf. Lemma 2.1 in [7]) we see that the problem
\[ c \frac{\partial \bar{\sigma}}{\partial t} = \Delta \bar{\sigma}(r, t) - \lambda \bar{\sigma}(r, t), \quad 0 < r < R(t), t > 0, \]
\[ \frac{\partial \bar{\sigma}}{\partial r}(0, t) = 0, \quad \bar{\sigma}(\bar{R}(t), t) = \sigma_{\infty}, t > 0, \]
\[ \bar{\sigma}(r, t) = \psi(r, t), \quad 0 < r < R(t), -\tau \leq t \leq 0, \]
has a unique solution \( \bar{\sigma} = \bar{\sigma}(r, t) \in W_{p,1}^1(Q_{\bar{R}}^T) \) for any \( \frac{\lambda}{\tau} < p < \infty \). By the embedding theory \( W_{p,1}^1(Q_{\bar{R}}^T) \subset C^\lambda(Q_{\bar{R}}^T), \lambda = 2 - \frac{\lambda}{p} \). Noticing \( \psi \in C((0, \infty) \times [-\tau, 0] \), we have \( \bar{\sigma} \in C((0, \infty) \times [-\tau, T] \). Since \( \sigma(r, t) \) is nonnegative, by comparison, we have \( \bar{\sigma} \leq \sigma_{\infty} \) for \( 0 \leq r \leq \bar{R}(t), 0 \leq t \leq T. \)

Extending \( \bar{\sigma} \) to \( [0, \infty) \times [0, T] \) such that \( \bar{\sigma}(r, t) = \sigma_{\infty} \) for \( r \geq \bar{R}(t), 0 \leq t \leq T, \) we have \( \bar{\sigma}(r, t) \) satisfies the condition (ii).

Therefore, \( (\bar{\sigma}(r, t), \bar{R}(t)) \) \( S_T \). Then \( F \) is a mapping from \( S_T \) to \( S_T \) follows.

Next, we prove the mapping \( F \) is a contraction mapping for small \( T \). Let \( (\sigma_i, R_i) \in S_T, (i = 1, 2) \) and denote \( F(\sigma_i, R_i) = (\bar{\sigma}_i, \bar{R}_i), (i = 1, 2) \). From (15), we have for any \( 0 \leq t \leq T, \)
\[ |\bar{R}_1 - \bar{R}_2| = R_0 |e^{\int_0^t G_1(s) ds} - e^{\int_0^t G_2(s) ds}| \leq TR_0 e^\frac{\lambda M T}{R_0} \max_{0 \leq t \leq T} |G_1(t) - G_2(t)|, \]
where
\[ G_i(t) = \frac{\mu}{R_i^4(t)} \int_0^{R_i(t)-r} \sigma(r, t - \tau) r^2 dr - \int_0^{R_i(t)} \bar{\sigma} r^2 dr \quad (i = 1, 2). \]

By (12) we have
\[ \max_{0 \leq t \leq T} |G_1(t) - G_2(t)| \leq C(T)d(\sigma_1, R_1), (\sigma_2, R_2)). \]

Substituting this estimate to (22) we have
\[ \max_{0 \leq t \leq T} |\bar{R}_1(t) - \bar{R}_2(t)| \leq TC(T)d(\sigma_1, R_1), (\sigma_2, R_2)). \]

Noticing that
\[ \max_{-\tau \leq t \leq 0} |\bar{R}_1(t) - \bar{R}_2(t)| = \max_{-\tau \leq t \leq 0} |\varphi(t) - \varphi(t)| = 0 \leq TC_1(T)d(\sigma_1, R_1), (\sigma_2, R_2)), \]
we have
\[ \max_{-\tau \leq t \leq T} |\bar{R}_1(t) - \bar{R}_2(t)| \leq TC(T)d(\sigma_1, R_1), (\sigma_2, R_2)). \]
Denote
\[ m(t) = \min_{0 \leq t \leq T} \{ \bar{R}_1(t), \bar{R}_2(t) \}, \quad M(t) = \max_{0 \leq t \leq T} \{ \bar{R}_1(t), \bar{R}_2(t) \}, \]
\[ h(r, t) = \bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t) \quad (0 \leq r \leq m(t), 0 \leq t \leq T). \]

Then \( h \) satisfies the problem
\[
\begin{cases}
ch_t = \Delta h - \mu(\sigma_1 - \sigma_2), (r, t) \in Q^m_T \\
\frac{\partial h}{\partial r}(0, t) = 0, 0 \leq t \leq T \\
h(m(t), t) = \bar{\sigma}_1(m(t), t) - \bar{\sigma}_2(m(t), t), 0 < t \leq T \\
h(r, t) = 0, 0 \leq r \leq R(t), -\tau \leq t \leq 0.
\end{cases}
\]

Then by maximum principle we have
\[
\begin{align*}
\max_{(r, t) \in Q^m_T} |h(t)| &= \max_{0 \leq t \leq T} |\bar{\sigma}_1(m(t), t) - \bar{\sigma}_2(m(t), t) - h_1(m(t), t)| \\
&\leq \max_{0 \leq t \leq T} |\bar{\sigma}_1(m(t), t) - \bar{\sigma}_2(m(t), t)| + \sup_{[0, \infty) \times [0, T]} |h_1(r, t)|.
\end{align*}
\]

It follows that
\[
\max_{Q^m_T} |\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)| = \max_{Q^m_T} |h| \leq \max_{Q^m_T} |h_1| + \max_{Q^m_T} |h_2| \\
\leq 2 \sup_{[0, \infty) \times [0, T]} |h_1(r, t)| + \max_{0 \leq t \leq T} |\bar{\sigma}_1(m(t), t) - \bar{\sigma}_2(m(t), t)|. \tag{29}
\]

Then
\[
\begin{align*}
\max_{r \geq 0, 0 \leq t \leq T} |\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)| &\leq 2 \sup_{[0, \infty) \times [0, T]} |h_1(r, t)| \\
&\quad + \max_{m(t) \leq r \leq M(t), 0 \leq t \leq T} |\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)|. \tag{30}
\end{align*}
\]

From (27), we have
\[
\sup_{[0, \infty) \times [0, T]} |h_1(r, t)| \leq \max_{r \geq 0, 0 \leq t \leq T} |\sigma_1(r, t) - \sigma_2(r, t)|. \tag{31}
\]

For \( m(t) \leq r \leq M(t) \) and \( 0 \leq t \leq T \) and \( p > 5 \), by Lemma 2.1 (ii) and the embedding theorem, we have...
\[
|\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)| \leq |\bar{\sigma}_1(r, t) - \sigma_\infty| + |\bar{\sigma}_2(r, t) - \sigma_\infty|
\]
\[
= |\bar{\sigma}_1(r, t) - \sigma_1(\tilde{R}_1(t), t)| + |\bar{\sigma}_2(r, t) - \sigma_2(\tilde{R}_2(t), t)|
\]
\[
\leq \sup_{0 \leq \xi \leq \tilde{R}(t)} (|\frac{\partial \bar{\sigma}_1}{\partial r}(\xi, t)| + |\frac{\partial \bar{\sigma}_2}{\partial r}(\xi, t)|)|\tilde{R}_1(t) - \tilde{R}_2(t)|
\]
\[
\leq C(T) \max_{0 \leq t \leq T} |\tilde{R}_1(t) - \tilde{R}_2(t)|.
\]

Noticing that
\[
\max_{r \geq 0, -\tau \leq t \leq 0} |\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)| = \max_{r \geq 0, -\tau \leq t \leq 0} |\psi(r, t) - \psi(r, t)| = 0
\]
\[
\leq TC(T)d(\sigma_1, (R_1), (\sigma_2, R_2)),
\]
we have
\[
\max_{r \geq 0, -\tau \leq t \leq T} |\bar{\sigma}_1(r, t) - \bar{\sigma}_2(r, t)| \leq TC(T)d(\sigma_1, (R_1), (\sigma_2, R_2)).
\]

By (25) and (34), we have
\[
d((\bar{\sigma}_1, \tilde{R}_1), (\bar{\sigma}_2, \tilde{R}_2)) \leq TC(T)d(\sigma_1, (R_1), (\sigma_2, R_2)).
\]

Therefore, \( F \) is a contraction mapping for small \( T \).

In the following, we prove the solution exists for all \( t > 0 \). If not, the maximal existence time interval \([-\tau, T^*]\) where \( T^* > 0 \) is finite. By Theorem 2.1(ii) and (iii), we know \( \|R(t)\|_{L^\infty[-\tau, T^*]}, \|\frac{1}{R(t)}\|_{L^\infty[-\tau, T^*]}, \|R'(t)\|_{L^\infty[0, T^*]} \) are bounded. Noticing above proof, we have that there exists \( T_1 \) such that for any \( t_0 \in (0, T^*) \) satisfying \([t_0 - \tau, t_0] \subset [-\tau, T^*)\), a solution to the problem (1.1)-(1.5) exists on the time interval \([t_0 - \tau, t_0 + T_1]\). By the uniqueness it follows that all solutions obtained in this way are equal in their common existence interval, so the solution can be extended to the time interval \([-\tau, T_1 + T^*)\) which contrary to the assumption on \( T^* \). This completes the proof.

3. Asymptotic behavior of the solutions to the system (1)-(5). In this section, we study asymptotic behavior of the solutions to (1)-(5). First we consider the case \( \sigma_\infty < \bar{\sigma} \).

**Theorem 3.1.** Assume that the conditions \((A_1) - (A_3)\) are satisfied. If \( \sigma_\infty < \bar{\sigma} \), then for any \( c > 0 \) and the initial function \( \varphi \), there holds
\[
\lim_{t \to \infty} R(t) = 0.
\]

**Proof.** By Theorem 2.2 (i) and the Eq.(3), we have
\[
-\frac{\mu}{R^2(t)} \int_0^{R(t)} \bar{\sigma} r^2 dr \leq \frac{dR(t)}{dt} \leq \frac{\mu}{3R^2(t)} |\sigma_\infty(R^3(t) - \bar{\sigma} R^3(t))|, \quad t > 0.
\]

From the left inequality above we can get
\[
R(t) \geq R(0)e^{-\frac{\mu t}{R^2(t)}}.
\]

Set \( \omega(t) = R^3(t) \), by the right inequality of (35) we have
\[
\frac{d\omega(t)}{dt} \leq \mu \sigma_\infty \omega(t - \tau) - \mu \bar{\sigma} \omega(t).
\]
Set $c$ is the unique real value root of the equation $z = -\mu \tilde{\sigma} + \mu \sigma_\infty e^{-r z}$. By $\sigma_\infty < \tilde{\sigma}$ we readily have $c < 0$. Consider the following initial value problem:

$$\frac{dx(t)}{dt} = \mu \sigma_\infty x(t - \tau) - \mu \tilde{\sigma} x(t), \quad t > 0; \quad x(t) = C^3 e^{ct}, \quad -\tau \leq t \leq 0.$$ 

The solution to the above problem is $x(t) = C^3 e^{ct}$. Since when $-\tau \leq t \leq 0$, $\omega(t) \leq x(t)$, by Lemma 3.1 in [8], we have for $t \geq -\tau$, $\omega(t) \leq x(t)$ i.e., $R(t) \leq Ce^{\lambda t} \to 0, t \to \infty$.

Next, we consider the case $\sigma_\infty > \tilde{\sigma}$. By Theorem 2.1 [12], the system (1)-(5) has a unique stationary solution $(\sigma_s(r), R_s), R_s > 0$ in this case. In the following, we prove that $(\sigma_s(r), R_s)$ is asymptotically stable provided $c$ is small.

Consider the initial problem

$$\Delta_r v(r, t) = \lambda v, \quad 0 < r < R(t), \quad t > 0, \quad (36)$$

$$\frac{\partial v}{\partial r}(0, t) = 0, \quad v(R(t), t) = \sigma_\infty, \quad t > 0, \quad (37)$$

$$\frac{d}{dt} \frac{4\pi R^3(t)}{3} = 4\pi \int_0^{R(t-\tau)} \mu v(r, t-\tau) r^2 dr - 4\pi \int_0^{R(t)} \mu \tilde{\sigma} r^2 dr, \quad t > 0, \quad (38)$$

$$R(t) = \varphi(t), -\tau \leq t \leq 0. \quad (39)$$

The solution to (36), (37) is

$$v(r, t) = \frac{\sigma_\infty R(t)}{\sinh \sqrt{\lambda} r} \frac{\sinh \sqrt{\lambda} r}{r} \quad (40)$$

**Lemma 3.2.** Let $(\sigma(r, t), R(t))$ is the solution to (1)-(5). Assume that the conditions (A1) - (A3) are satisfied and for some $0 < T \leq \infty$ and $\varepsilon > 0$

$$|R'(t)| \leq L \leq L_0, \varepsilon \leq R(t) \leq \frac{1}{\varepsilon}. \quad (41)$$

Assume further that $0 \leq r \leq R_0$, 

$$|\sigma(r, 0) - v(r, 0)| \leq M \leq M_0. \quad (42)$$

Then there exist positive constants $c_0 \quad C$ independent of $c, T, L, M$ and $R_0$ but depend on $\varepsilon, L_0, M_0$ such that

$$|\sigma(r, t) - v(r, t)| \leq C(c + e^{-\Delta r}) \quad (43)$$

for arbitrary $0 \leq r \leq R(t), 0 \leq t < T$ and $0 < c \leq c_0$.

**Proof.** By direct computation, we have

$$\frac{\partial v}{\partial t} = \sigma_\infty \frac{R'(t) \sinh \sqrt{\lambda} r}{r \sinh \sqrt{\lambda} R(t)} + \frac{\sqrt{\lambda} r \cosh \sqrt{\lambda} R(t) R'(t) \sinh \sqrt{\lambda} r}{(r \sinh \sqrt{\lambda} R(t))^2}$$

Hypothesis (41) implies that

$$|\frac{\partial v}{\partial t}| \leq CL,$$

for $0 < r < R(t), t \geq 0$, where $C$ depends only on $\sigma_\infty, \lambda$ and $\varepsilon$. Let $\sigma_+(r, t) = v \pm C e^{-\Delta r}$. Then

$$c \frac{\partial \sigma_+}{\partial t} - \Delta_r \sigma_+ + \lambda \sigma_+ \geq -CL c + CLc = 0$$
By (37) and (42), we have
\[
\frac{\partial \sigma_+}{\partial t}(0, t) = 0, \sigma_+(R(t), t) > \sigma_\infty, \text{ for } t > 0, \sigma_+(r, 0) \geq \sigma_0(r) \text{ for } 0 \leq r \leq R(0).
\]
Then by comparison principle we obtain
\[
\sigma_+(r, t) \geq \sigma(r, t) \text{ for } 0 \leq r \leq R(t), 0 \leq t < T.
\]
Similarly arguments can prove that
\[
\sigma_-(r, t) \leq \sigma(r, t) \text{ for } 0 \leq r \leq R(t), 0 \leq t < T.
\]
Hence (42) holds. This completes the proof.

Lemma 3.3. Let \( p(x) = \frac{\cosh x - 1}{x^2} \). Then the following assertions hold:
1. \( p'(x) < 0 \) for all \( x > 0 \), and \( \lim_{x \to +0} p(x) = \frac{1}{\hat{\sigma}} \), \( \lim_{x \to -\infty} p(x) = 0 \).
2. \( \frac{xp''(x)}{p'(x)} \) is strictly decreasing for any \( x > 0 \), and \(-2 < \frac{xp''(x)}{p'(x)} < 1\).
3. \( x^2p(x) \) is strictly monotone increasing for \( x > 0 \).
4. \( \lim_{x \to +0} xp'(x) = 0 \).

Proof. The proof of (1) can be found in [12] and the proof of (3),(4) can be found in [8]. Next we prove (2), from Lemma 3.3 [6] we know that
\[
\frac{xp''(x)}{p'(x)} = \frac{2(sinh^3 x - x^3 \cosh x)}{(x^2 + x \cosh x sinh x - 2 sinh^2 x) \sinh x} - 2,
\]
and \( \frac{xp''(x)}{p'(x)} \) is strictly monotone decreasing for all \( x > 0 \). By simple computation, it follows that
\[
\lim_{x \to -\infty} \frac{xp''(x)}{p'(x)} = -2.
\]
From [23] we know that \( \lim_{x \to 0} \frac{xp''(x)}{p'(x)} = 1 \), then we have \(-2 < \frac{xp''(x)}{p'(x)} < 1\). This completes the proof.

Lemma 3.4. Let \((\sigma(r, t), R(t))\) is the solution to (1)-(5). Assume that the conditions \((A_1)-(A_3)\) are satisfied. If \( \sigma_\infty > \hat{\sigma} \), assume for some \( \varepsilon > 0, \varepsilon \leq R(t) \leq \frac{1}{\varepsilon} \) for \(-\tau \leq t \leq 0 \). Then there exists a positive constant \( c_0 \) independent of \( c, R(t) \) for \(-\tau \leq t \leq 0 \) such that
\[
\frac{1}{2} \min(R_s, \varepsilon e^{-\frac{4\varepsilon^2}{\varepsilon^2 - 1}}) < R(t) < 2 \max(R_s, \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon} |\mu\sigma - \frac{1}{\varepsilon} - \mu|\tau})
\]
for arbitrary \( t \geq 0 \) \( 0 < c \leq c_0 \).

Proof. From (35) and the assumption \( \varepsilon \leq R(t) \leq \frac{1}{\varepsilon} \) for \(-\tau \leq t \leq 0 \), we have
\[
\varepsilon e^{-\frac{4\varepsilon^2}{\varepsilon^2 - 1}} < R(t) < \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon} |\mu\sigma - \frac{1}{\varepsilon} - \mu|\tau} \text{ for } 0 \leq t \leq \tau.
\]
Assume that (44) is not valid for some \( t \). It follows that there exists \( T > \tau \) such that for \( 0 \leq t < T \),
\[
\frac{1}{2} \min(R_s, \varepsilon e^{-\frac{4\varepsilon^2}{\varepsilon^2 - 1}}) < R(t) < 2 \max(R_s, \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon} |\mu\sigma - \frac{1}{\varepsilon} - \mu|\tau})
\]
and either \( R(T) = 2 \max(R_s, \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon} |\mu\sigma - \frac{1}{\varepsilon} - \mu|\tau}) \) or \( R(T) = \min(R_s, \varepsilon e^{-\frac{4\varepsilon^2}{\varepsilon^2 - 1}}) \).
If \( R(T) = 2 \max(R_s, \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon} |\mu\sigma - \frac{1}{\varepsilon} - \mu|\tau}) \), then
\[
R'(T) \geq 0.
\]
By (35) and the fact that for $0 \leq t < T$,
\[ \frac{1}{2} \min(R_s, \varepsilon e^{-\frac{\mu\sigma}{R_s}}) < R(t) < 2 \max(R_s, \frac{1}{\varepsilon} e^{\frac{1}{2}\mu\sigma \sqrt{t} - \mu \sigma |\tau|}), R_s), \]
we have $R'(t) \leq L_0$, $L_0$ is a positive constant independent of $c$ and $T$. Obviously $|\sigma(r, 0) - v(r, 0)| \leq \sigma_\infty$. By Lemma 3.2, it follows
\[ |\sigma(r, t) - v(r, t)| \leq C(c + e^{-\frac{\lambda r}{\tau}}) \] (46)
for arbitrary $0 \leq r \leq R(t), 0 \leq t < T$ and $0 < c \leq c_0$. Then we have for $t > \tau$

\[
R'(t) = \frac{1}{R^2(t)} \int_0^{R(t-\tau)} \mu \sigma(r, t-\tau)r^2 dr - \int_0^{R(t)} \mu \sigma r^2 dr
\]
\[
\leq \frac{1}{R^2(t)} \int_0^{R(t-\tau)} \mu v(r, t-\tau)r^2 dr + \frac{C}{3} R \left( c + e^{-\frac{\lambda(t-\tau)}{\tau}} \right) R^3 (t-\tau) - \frac{1}{3} \mu \sigma R(t)
\]
\[
= \frac{1}{3} R(t) \left( 3\mu \sigma \infty p(\sqrt{\lambda} R(t-\tau)) \left( \frac{R(t-\tau)}{R(t)} \right)^3 - \lambda \tilde{\sigma} + \mu C(c + e^{-\frac{\lambda(t-\tau)}{\tau}}) \left( \frac{R(t-\tau)}{R(t)} \right)^3 \right)
\]
It follows that for $T > \tau$
\[ R'(T) \leq \frac{1}{3} R(T) \left( 3\mu \sigma \infty p(\sqrt{\lambda} R(T)) - \lambda \tilde{\sigma} + \mu C(c + e^{-\frac{\lambda(T-\tau)}{\tau}}) \right). \]
where we have used the fact $x^2 p(x)$ for $x > 0$ is monotone increasing (Lemma 3.3 (3)). From Lemma 3.3 (1) we known function $p(x)$ is monotone decreasing for any $x > 0$, noticing $R(T) > R_s$ we have $3\mu \sigma \infty p(\sqrt{\lambda} R(T)) - \lambda \tilde{\sigma} < 0$, then if $c_0$ is sufficiently small and $0 < c \leq c_0$ it follows that $R'(T) < 0$ which contracts to the fact $R'(T) \geq 0$.

If $R(T) = \frac{1}{2} \min(\varepsilon, R_s)$ similar arguments can prove the desired assertion. This completes the proof. □

**Lemma 3.5.** Let $G(x, y) = \mu \sigma \infty p(\sqrt{\lambda} y)(x^2 y)^3 - \frac{1}{3} \mu \tilde{\sigma}$. Consider initial value problems
\[ R^+(t) = \int_0^{R^+(t)} G(R^+(t), R^+(t-\tau)) + \text{Cac} \left( \frac{R^+(t-\tau)}{R^+(t)} \right)^3], t > 0, \] (47)
\[ R^+(t) = \varphi(t), -\tau \leq t \leq 0. \] (48)
and
\[ R^-(t) = \int_0^{R^-(t)} G(R^-(t), R^-(t-\tau)) - \text{Cac} \left( \frac{R^-(t-\tau)}{R^-(t)} \right)^3], t > 0, \] (49)
\[ R^-(t) = \varphi(t), -\tau \leq t \leq 0. \] (50)
where $C$ is a positive constant independent $\alpha$ and $c$. Then for $\sigma_\infty > \tilde{\sigma}$, there exist positive constants $\alpha_0, c_0$ such that for $\alpha \in (0, \alpha_0], c \in (0, c_0]$ the equations $G(x, x) \pm \text{Cac} = 0$ has respectively unique solutions $R^+_x$. For any positive initial value function $\varphi(t), -\tau \leq \tau \leq 0$ and $\alpha \in (0, \alpha_0), c \in (0, c_0]$ the solutions to problems (47),(48) and (49),(50) which we respectively denote as $R^+_x$ and $R^-_x$ converge respectively to $R^+_x$ as $t \rightarrow \infty$ for $\alpha \in (0, \alpha_1], c \in (0, c_1].$

**Proof.** Noticing that the function $p(y)$ is monotone decreasing for $y > 0$ and $0 < p(y) < \frac{1}{3}$ and $\sigma_\infty > \tilde{\sigma}$, we see that there exist positive constants $\alpha_1, c_1$ such that for $\alpha \in (0, \alpha_1], c \in (0, c_1]$ the equations $G(x, x) \pm \text{Cac} = 0$ has respectively unique solutions $R^+_x$.\[\sqrt{\lambda}\]}
In the following we prove that for any positive initial function \( \varphi(t) \in C[-\tau, 0] \), there exists positive constants \( \alpha_2, c_2 \) such that for \( \alpha \in (0, \alpha_2], c \in (0, c_2] \) the corresponding solutions of the equations to the initial problems (3.13),(3.14) and (3.15),(3.16) which we respectively denote as \( R^k(t) \), converge respectively to \( R^k_\ast \) as \( t \to \infty \). First, we consider the problem (47),(48). By Lemma 3.3(3), we see that the function \( G(x, y) \) is monotone increasing for \( y > 0 \). Let \( H(x, y) = G(x, y) + C\alpha c(\frac{y}{x})^3 \).

Then we readily have \( H(x, y) \) is monotone increasing for \( y > 0 \). By the fact that \( p(y) \) is monotone decreasing for \( y > 0 \) and \( H(x, x) = \lambda \sigma \in p(\sqrt{x}) - \lambda \sigma + C\alpha c \), noticing \( H(R^+_x, R^+_x) = 0 \), we have \( H(x, x) > 0 \) for \( x < R^+_x \) and \( H(x, x) < 0 \) for \( x > R^+_x \). By Lemma 3.4 [8] we have \( \lim_{t \to \infty} R^+_x(t) = R^+_x \). Next we consider the problem (49),(50).

Let
\[
K(x, y) = G(x, y) - C\alpha c(\frac{y}{x})^3 = [\mu \sigma \in p(\sqrt{x}) - C\alpha c](\frac{y}{x})^3 - \frac{1}{3} \mu \sigma. 
\]

Let \( \xi = \sqrt{x} \eta, \eta = \sqrt{y}, h(\eta) = \eta^3 p(\eta) - (\lambda \sigma \in c)^{-1} \alpha c(\eta)^3 \). Then
\[
K(x, y) = K(\xi, \eta) = [\mu \sigma \in p(\eta) - C\alpha c(\eta^3 - \frac{1}{3} \mu \sigma = \lambda \sigma \in c^{-1} \eta^3 h(\eta) - \frac{1}{3} \mu \sigma. 
\]

To prove \( \frac{\partial K}{\partial \eta} > 0 \) for all \( x > 0, 0 < y < M_1 \), where \( M_1 \) is a positive constant. We only need to prove \( h'(\eta) > 0 \) for all \( 0 < \eta < \sqrt{M_1} =: M \). By direct computation, we obtain
\[
h'(\eta) = \eta^2 [3p(\eta) + \eta p'(\eta)] - 3(\lambda \sigma \in c)^{-1} \alpha c. 
\]

and
\[
(3p(\eta) + \eta p'(\eta))' = 4p'(\eta) + \eta p''(\eta). 
\]

By Lemma 3.3 (1) and (4) we have
\[
\lim_{\eta \to 0^+} 3p(\eta) + \eta p'(\eta) = 1, \lim_{\eta \to \infty} 3p(\eta) + \eta p'(\eta) = 0, 
\]

and by Lemma 3.3 (1), (2) we get for all \( x > 0 \)
\[
4p'(\eta) + \eta p''(\eta) < 2p'(\eta) + \eta p''(\eta) < 0. 
\]

Then we have for any given positive initial function \( \varphi, -\tau \leq t \leq 0 \), there exists \( M > |\varphi|, |\varphi| = \max_{-\tau \leq t \leq 0} \varphi \) (actually \( M \) can be chosen more larger) such that \( 3p(M) + Mp'(M) > 0 \), and for \( 0 < \eta \leq M \),
\[
3p(\eta) + \eta p'(\eta) \geq 3p(M) + Mp'(M) > 0. 
\]

Then there exists \( \alpha_2, c_2 \) and \( c_2 \) is sufficiently small satisfying
\[
3p(M) + Mp'(M) - (\lambda \sigma \in c)^{-1} \alpha c_2 > 0. 
\]

Let \( c_0 = \min\{c_1, c_2\}, a_0 = \min\{\alpha_1, \alpha_2\}. \) For \( c \in (0, a_0], \alpha \in (0, a_0], G(x, x) \pm C\alpha c = 0 \) has respectively unique solutions \( R^+_x \). From \( R^+_x \) satisfies Eq. \( G(x, x) - C\alpha c = 0 \) we can get \( R^+_x < \sqrt{y} p^{-1}(\frac{\sigma}{M_0}) \). So we can choose \( M > R^+_x. \) Similarly arguments as Lemma 3.3 and 3.4 in [8] can lead to the desired assertion for problem (49),(50), we omit the details here.

\[\Box\]

**Lemma 3.6.** Assume that the conditions \((A_1) - (A_3)\) are satisfied. Assume further that for all \( t > -\tau \),
\[
C_* \leq R(t) \leq C^*. 
\]

\[\Box\]
where \( C_* \) and \( C^* \) are two constants independent of \( c \) and \( \alpha \). If \( \sigma_\infty > \bar{\sigma} \), Then there exist positive constants \( c_0, 0, T_0 \) and \( C \) independent of \( c \) such that the following assertions holds: If \( 0 < c \leq c_0 \), for any \( \alpha \in (0, \alpha_0] \), if the inequalities

\[
|R(t) - R_s| \leq \alpha, |\sigma(r, t) - \sigma_s(r)| \leq \alpha.
\]

(52)

hold for all \( 0 \leq r \leq R(t), t \geq -\tau \) and \( |R'(t)| \leq \alpha \) holds for all \( 0 \leq r \leq R(t), t \geq 0 \), then also the inequalities

\[
|R(t) - R_s| \leq C\alpha(c + e^{-\bar{\sigma}t}), |R'(t)| \leq C\alpha(c + e^{-\bar{\sigma}t}), |\sigma(r, t) - \sigma_s(r)| \leq C\alpha(c + e^{-\bar{\sigma}t})
\]

(53)

hold for all \( 0 \leq r \leq R(t), t \geq T_0 + \tau \).

Proof. Since

\[
\frac{1}{R^2(t)}[\int_0^{R(t-\tau)} \mu_\omega(r, t-\tau)r^2dr - \int_0^{R(t)} \mu_\omega r^2dr] = R(t)G(R(t), R(t-\tau)),
\]

(54)

where \( G(R(t), R(t-\tau)) = \mu_\omega \sigma_\infty p(\sqrt{\lambda}R(t-\tau))(\frac{R(t-\tau)}{R(t)})^3 - \frac{1}{4}\mu \bar{\sigma} \). By Lemma 3.2, (3), (52) and \( |R'(t)| \leq \alpha \) holds for all \( 0 \leq r \leq R(t), t \geq 0 \), we can get for \( t > \tau \)

\[
|R'(t) - R(t)G(R(t), R(t-\tau))|
\]

\[
= |\frac{1}{R^2(t)}[\int_0^{R(t-\tau)} \mu_\omega(r, t-\tau)r^2dr - \int_0^{R(t)} \mu_\omega r^2dr]|
\]

\[
\leq \frac{1}{3} \mu R(t)\alpha(c + e^{\frac{-\lambda(t-\tau)}{\sqrt{\lambda}}})(\frac{R(t-\tau)}{R(t)})^3.
\]

Noticing for \( t \geq 2\tau \), \( e^{\frac{-\lambda(t-\tau)}{\sqrt{\lambda}}} \leq e^{-\frac{\lambda t}{\sqrt{\lambda}}} \), it follows that for \( t > 2\tau \)

\[
R(t)[G(R(t), R(t-\tau)) - C_2\alpha(c + e^{\frac{-\lambda(t-\tau)}{\sqrt{\lambda}}})(\frac{R(t-\tau)}{R(t)})^3] \leq R'(t)
\]

\[
\leq R(t)[G(R(t), R(t-\tau)) + C_2\alpha(c + e^{\frac{-\lambda(t-\tau)}{\sqrt{\lambda}}})(\frac{R(t-\tau)}{R(t)})^3]
\]

(55)

where \( C_2 \) is a positive constant independent of \( \alpha \) and \( c \). Here and hereafter for easy of notation we use the same notation to denote various different positive constants independent of \( c \) and \( \alpha \). Consider initial value problems

\[
R'(\pm t) = R(\pm t)[G(R(\pm t), R(\pm (t-\tau))) \pm \alpha c(\frac{R(\pm (t-\tau))^3}{R(\pm t)})^3], t > 0,
\]

\[
R(\pm t) = \varphi(\pm t), -\tau \leq \pm t \leq 0.
\]

By Lemma 3.5 we know that there exist positive constants \( \alpha_0, c_0 \) such that for \( \alpha \in (0, \alpha_0], c \in (0, c_0] \) the equations \( G(x, x) \pm \alpha c(\frac{R(\pm (t-\tau))^3}{R(\pm t)})^3 \) has respectively unique solutions \( R^\pm(\pm t) \), and the corresponding solutions of the equations to the initial problem above which we respectively denote as \( R^\pm(\pm t) \), converge respectively to \( R^\pm_s \) as \( t \to \infty \).

By the fact \( p(x) \) is monotone decreasing for all \( x > 0 \), we can get

\[
|R^\pm_s - R_s| \leq \alpha C c.
\]

(56)

Actually, since \( R^\pm_s \) respectively satisfies the equations \( p(\sqrt{\lambda}R^\pm_s) - \frac{\lambda}{2\sqrt{\lambda}} = \mp \alpha c \) and \( \frac{\lambda}{2\sqrt{\lambda}} \) satisfies the equation \( p(\sqrt{\lambda}R_s) - \frac{\lambda}{2\sqrt{\lambda}} = 0 \), by the fact (51) and \( p(x) \) is monotone decreasing for all \( x > 0 \), we readily have \( |R^\pm_s - R_s| \leq \alpha C c \). Then there exists
$M^* > 0$ and $T > 0$ such that for all $t > T$, $R^\pm(t) \leq M^*$. If we choose $M > M^*$, comparison principle (cf. [8] Lemma 3.1) implies for all $t \geq T$

$$R^-(t) \leq R(t) \leq R^+(t).$$

By linearizing Eq. (47) at the stationary point $R^+_s$, we have

$$R^+(t) = -aR^+(t) + bR^+(t - \tau),$$

where $a = 3[\mu c^2 p(\sqrt{\lambda}R^+_s) + Cac]$, $b = 3[\mu c^2 p(\sqrt{\lambda}R^+_s) + Cac] + \mu c^2 p'(\sqrt{\lambda}R^+_s)\sqrt{\lambda}R^+_s$.

The characteristic equations of Eq. (58) is

$$z = -az + be^{-\tau z},$$

Similarly by linearizing Eq. (49) at the stationary point $R^-_s$, we have

$$R^-(t) = -AR^-(t) + BR^-(t - \tau),$$

where $A = 3[\mu c^2 p(\sqrt{\lambda}R^-_s) - Cac]$, $B = 3[\mu c^2 p(\sqrt{\lambda}R^-_s) - Cac] + \mu c^2 p'(\sqrt{\lambda}R^-_s)\sqrt{\lambda}R^-_s$.

The characteristic equations of Eq. (60) is

$$z = -Az + Be^{-\tau z},$$

Since $p'(\sqrt{\lambda}R^\pm_s) < 0$ and $3p(\sqrt{\lambda}R^\pm_s) + p'(\sqrt{\lambda}R^\pm_s)\sqrt{\lambda}R^\pm_s = x^{-2}(x^3 p(x))_{x=\sqrt{\lambda}R^\pm_s} > 0$ we have $a > b > 0$ and $A > B > 0$ provided $c \in (0, c_0]$ and $\alpha \in (0, \alpha_0]$. This implies that all complex roots of Eq. (59) and (61) are located in the left-half plane. Then by Theorem 2.4 in [15] we have there exist positive constants $K$, $\theta$ and $T_1(\geq 2\tau)$ such that for any $t \geq T_1$

$$|R^\pm(t) - R^\pm_s| \leq Ke^{-\theta t}|\varphi(t) - R^\pm_s|.$$

Then noticing (56) for any $t \geq T_0 := \max\{T_1, T\}$

$$|R(t) - R_s| \leq \max |R^\pm(t) - R_s|$$

$$\leq |R^\pm(t) - R^\pm_s| + |R^\pm_s - R_s|$$

$$\leq Ke^{-\theta t}|\varphi(t) - R^\pm_s| + Cac$$

$$\leq Ke^{-\theta t}(|\varphi(t) - R_s| + |R_s - R^\pm_s|) + Cac$$

$$\leq C\alpha(e + e^{-\theta t}).$$

By the mean value theorem and the fact that $v_s(r) = \sigma_s(r)$, noticing (51) we have

$$|v(r, t) - \sigma_s(r)| = |v(r, t) - v_s(r)| \leq C |R(t) - R_s| \leq C\alpha$$

for $0 \leq r \leq R(t), t \geq 0$. It follows that

$$|\sigma(r, t) - v(r, t)| \leq |\sigma(r, t) - \sigma_s(r)| + |v(r, t) - \sigma_s(r)| \leq C\alpha$$

for $0 \leq r \leq R(t), t \geq 0$. In particular, $|\sigma_0(r) - v(r, 0)| \leq C\alpha$ for $0 \leq r \leq R(0)$. Since $|R(t)| \leq \alpha$ for all $t \geq 0$, by Lemma 3.2 we have there exists a positive constant $c_0$ independent $c$ and $\alpha$ such that

$$|\sigma(r, t) - v(r, t)| \leq C(c + e^{-\frac{q}{3\alpha}}) \leq C(c + e^{-\frac{q}{3\alpha}})$$

(62)

for arbitrary $0 \leq r \leq R(t), t \geq 0$ and $0 < c \leq c_0$. 
Set \( f(t) = \frac{1}{R(t)} \left[ \int_0^{R(t)} \mu \sigma (r, t - \tau) r^2 dr - \int_0^{R(t)} \mu \sigma r^2 dr \right] \). Then we have for \( t \geq \tau \)
\[
|R(t) f(t) - R(t) G(R(t), R(t - \tau))| = \left| \frac{1}{R^2(t)} \int_0^{R(t)} \mu |\sigma(r, t - \tau) - \sigma(r, t)| r^2 dr \right| 
\leq \frac{1}{3 \mu R(t)} c \left( e^{e^{-\frac{1}{\epsilon^2} \mu \sigma}} + \frac{\mu c \sigma}{R(t)} \right)^3.
\]
Noticing (51) we have for \( t \geq 2\tau \)
\[
|R(t) f(t) - R(t) G(R(t), R(t - \tau))| \leq C \alpha \left( e^{e^{-\frac{1}{\epsilon^2} \mu \sigma}} \right). \tag{63}
\]
Since
\[
|G(R(t), R(t - \tau)) - G(R_s, R_s)| = |\mu \sigma \infty p(\sqrt{\lambda} R(t - \tau)) \left( \frac{R(t - \tau)}{R(t)} \right)^3 - \mu \sigma \infty p(\sqrt{\lambda} R_s) \left( \frac{R_s}{R_s} \right)^3|,
\]
by mean value theorem and (51) we have for \( t \geq T_0 + \tau \)
\[
|G(R(t), R(t - \tau)) - G(R_s, R_s)| \leq C \left( |R(t) - R_s| + |R(t - \tau) - R_s| \right) \leq C \alpha \left( e^{e^{-\frac{1}{\epsilon^2} \mu \sigma}} \right).
\]
Then by equations \( R'(t) = R(t) f(t), (3.29) \) and inequality(51), it follows that
\[
|R'(t)| \leq C \alpha \left( e^{e^{-\frac{1}{\epsilon^2} \mu \sigma}} \right). \tag{63}
\]
The proof of Lemma 3.6 is complete. \( \square \)

**Theorem 3.7.** Assume that the conditions \((A_1) - (A_3)\) are satisfied. Let \( \sigma(r, t), R(t) \) be the solution of the system (1)-(5). If \( \sigma_\infty > \sigma \), then for any \( \varepsilon > 0 \) there exist corresponding positive constants \( c_0, \gamma \) and \( C \) such that if \( \varepsilon \leq \varphi(t) \leq \frac{1}{\epsilon^2} \) and \( 0 < c \leq c_0 \) hold, then \( |R(t) - R_s| \leq C e^{-\gamma t}, |R'(t)| \leq C e^{-\gamma t}, |\sigma(r, t) - \sigma_s(r)| \leq C e^{-\gamma t} \) for all \( t \geq T_0 + \tau, 0 \leq r \leq R(t) \).

**Proof.** By Lemma 3.4 we see that there exists a positive constant \( c_0 \) such that if \( 0 < c \leq c_0 \) then for all \( t \geq 0 \)
\[
\frac{1}{2} \min(R_s, e^{-\frac{\mu c \sigma}{R}}) < R(t) < 2 \max(R_s, e^{-\frac{\mu c \sigma}{R}}),
\]
without loss of generality we may assume that (51) holds for all \( t \geq 0 \). It follows that \( |R(t) - R_s| \leq C^* + R_s =: \alpha_1 \) for all \( t \geq 0 \). By (35) we have for all \( t \geq 0 \), \( |R'(t)| \leq \frac{\mu c \sigma 64C^*}{4c_0^2} =: \alpha_2 \). Obviously \( |\sigma(r, t) - \sigma_s(r)| \leq 2\sigma_\infty \) holds for all \( 0 \leq r \leq R(t), t \geq -\tau \). We see the conditions of Lemma 3.6 hold for \( \alpha = \alpha_0 =: \max\{\alpha_1, \alpha_2, 2\sigma_\infty\} \). Let \( C \) and \( \theta \) be as in (53) and take \( c_0 \) smaller and \( T_0 \) larger (if necessary) such that \( C(e + e^{-\theta T}) < \frac{1}{2} \). Then by successively applying Lemma 3.6 over the time interval \( [nT_0 + \tau, \infty)(n = 1, 2, \cdots) \) as in [12] one can get the desired assertion. \( \square \)

**4. Conclusion.** In this paper we have studied a delayed mathematical model for the tumor growth. The model is established by modifying the model A. Friedman [12] by considering the time delay effect as in Byrne [2] and the time delay represents the time taken for cells to undergo mitosis. The existence and the uniqueness of a solution for all \( t \geq 0 \) and analysis of the asymptotic behavior of stationary solutions were shown. From the results of rigorous analysis of this paper we see that the time delay in cell proliferation does not change the tendency of the tumor to a dormant state under a realistic assumption that \( c \) (i.e., the tumor doubling time is
more larger compared to the time scale of diffusion of nutrient within the tumor) is small.

Acknowledgments. This work is partially supported by the Natural National Science Foundation of China (No. 10771223, 10926128) and Natural Science Foundation of Guangdong Province (No. 9251064101000015). The author would like to express his great thanks to Professor Shangbin Cui for his continuous encouragements. The author also expresses his thanks to the references for their valuable suggestions on modification of the original manuscript.

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Received December 2009; revised February 2010.

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