On a Navier-Stokes-Allen-Cahn model with inertial effects

Gianluca Favre
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
E-mail: gianluca.favre@univie.ac.at

Giulio Schimperna
Dipartimento di Matematica, Università di Pavia,
Via Ferrata 5, I-27100 Pavia, Italy
E-mail: giusch04@unipv.it

May 24, 2018

Abstract
A mathematical model describing the flow of two-phase fluids in a bounded container $\Omega$ is considered under the assumption that the phase transition process is influenced by inertial effects. The model couples a variant of the Navier-Stokes system for the velocity $u$ with an Allen-Cahn-type equation for the order parameter $\varphi$ relaxed in time in order to introduce inertia. The resulting model is characterized by second-order material derivatives which constitute the main difficulty in the mathematical analysis. Actually, in order to obtain a tractable problem, a viscous relaxation term is included in the phase equation. The mathematical results consist in existence of weak solutions in 3D and, under additional assumptions, existence and uniqueness of strong solutions in 2D. A partial characterization of the long-time behavior of solutions is also given and in particular some issues related to dissipation of energy are discussed.

Key words: two-phase fluid, Allen-Cahn, Navier-Stokes, inertial effects, viscous relaxation, existence and uniqueness.

AMS (MOS) subject classification: 35Q35, 35K10, 35L82, 76D05, 80A22.

1 Introduction

In this paper we are concerned with the mathematical analysis of the following PDE system describing the evolution of a phase-changing fluid through the variables $u$ (macroscopic velocity of the flow) and $\varphi$ (order parameter, normalized in such a way that the values $\varphi = \pm 1$ represent the pure states):

$$u_t + u \cdot \nabla u + \nabla p - \Delta u = -\text{div}(\nabla \varphi \otimes \nabla \varphi), \quad (1.1)$$
$$\text{div} u = 0, \quad (1.2)$$
$$\pi = \varphi_t + u \cdot \nabla \varphi - \epsilon \Delta \varphi + \sigma \varphi, \quad (1.3)$$
$$\pi_t + \pi + \delta u \cdot \nabla \pi - \Delta \varphi + f(\varphi) = 0, \quad (1.4)$$

with parameters $\delta \geq 0$, $\sigma \geq 0$ and $\epsilon > 0$ whose role will be clarified in the sequel. The evolution is assumed to take place in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, over a bounded but otherwise arbitrary time reference interval $(0, T)$. More precisely, in order to avoid complications related with the interaction with the boundary, we will assume that $\Omega$ is the unitary flat torus $\Omega \sim [0,1]^d$. Correspondingly, we will take periodic boundary conditions for all variables. This may be seen as the simplest situation where one can attempt a mathematical analysis of the system; we however remark that at least a part of our results could be extended to other types of conditions at the price of some additional technicality in the proofs.
Relation (1.1), with the incompressibility constraint (1.2), is the standard Navier-Stokes system where the right hand side of (1.1) accounts for the extra-stress due to the effects of phase transition. On the other hand, the change of phase is governed by a proper version of the Allen-Cahn equation, where inertial effects are taken into account. Indeed, from a mechanical point of view, the phase transition phenomenon may be thought to be driven by the microscopic motion of molecules (we refer to the monograph [15] for a general discussion), which motivates the occurrence of an inertial term leading to a finite propagation speed of the phase transition front. The function $f$ in (1.4) represents the derivative of a configuration potential $F$ of double-well type whose minima are attained in correspondence of the pure phases of the material. Mathematically speaking, $F$ will be assumed to have a polynomial growth at infinity (of arbitrary order) and to be $\lambda$-convex (cf. (2.5) below). It is worth remarking that, even in dimension $d = 2$, the polynomial behavior at infinity seems to be necessary in order for our arguments to work; in particular, our results do not extend to more general potentials.

In the above formulation, the Allen-Cahn equation for the phase parameter has been split into the coupled relations (1.3), (1.4). Indeed, this choice seems more natural because it gives a better description of the transport effects in terms of the auxiliary variable $\pi$, which will play an important role from the mathematical viewpoint. Actually, on the one hand, it is easy to rewrite relations (1.3)-(1.4) eliminating $\pi$:

$$
\varphi_{tt} + (1 + \sigma)\varphi_t + \sigma \varphi + \mathbf{u}_t \cdot \nabla \varphi + \mathbf{u} \cdot \nabla \left( (1 + \delta)\varphi_t + (1 + \delta \sigma)\varphi + \delta \mathbf{u} \cdot \nabla \varphi - \delta \lambda \Delta \varphi \right) - \epsilon \Delta \varphi_t - (1 + \epsilon) \Delta \varphi + f(\varphi) = 0,
$$

but, on the other hand, the equivalence between (1.3)-(1.4) and (1.5) holds just at a formal level. Indeed, at least in the framework of weak solutions (here and below this terminology is borrowed from the standard one for Navier-Stokes), one can effectively provide a control of the variable $\pi$, whereas in (1.5) there appear a number of “bad” quantities, like the second order “inertial” term $\varphi_{tt}$, the cubic terms coming from the combination of the transport terms in (1.3)-(1.4), and the term $\mathbf{u}_t \cdot \nabla \varphi$ (depending on the macroscopic acceleration $\mathbf{u}_t$), for which a very poor regularity is available in the setting of weak solutions. Instead, in the decoupled formulation (1.3)-(1.4) most of the “bad” contributions are absorbed into the terms $\pi_t$ and $\delta \mathbf{u} \cdot \nabla \pi$ which can be controlled much more easily.

It is also worth commenting a bit the presence of the term $-\epsilon \Delta \varphi$ in (1.3), which reflects into the corresponding summand $-\epsilon \Delta \varphi_t$ in the combined equation (1.5) and in fact acts as a viscous regularization of the phase variable. This type of effect may be experimentally observed at least as far as $\epsilon > 0$ is very small, and corresponds to a strong damping in the phase transition. In the present setting, however, that contribution is included mainly for mathematical reasons; indeed, omitting it (i.e., taking $\epsilon = 0$), the resulting system seems to present, even in 2D, insurmountable mathematical difficulties, especially related to the control of the right hand side of (1.1) (see also Remark 4.3 below for additional comments).

Second order in time relaxations of the Allen-Cahn and Cahn-Hilliard equations have been considered in several different contexts and under various mathematical assumptions. We may mention, without any claim of completeness, the papers [3] [17] [15] (see also the references therein). In these contributions, thermal effects have also been considered, leading to the so-called phase-field models with inertia. Inertial effects at the microscopic level have been observed also in (nematic) liquid crystals (see, e.g., [11] [16]). On the other hand, at least up to our knowledge, phase transition models with inertia have not been studied so far in the case of two-phase fluids, i.e., when the process is influenced by a macroscopic velocity satisfying some version of the Navier-Stokes system. Generally, the mathematical description of two-phase (or two-component) fluid flows is a very important and actual research topic and the related literature is extremely vast. Just to give an idea, we may mention the works [1] [2] [3] [7] [12] [13] [21] and the references quoted there. In spite of the fact that a large number of physically significant situations are addressed in the quoted works, the problem considered here does not seem to have been previously studied elsewhere.

Coming to a description of our results, we start observing that the only a priori information that is always guaranteed for solutions to our system is the energy estimate, which is a direct consequence of the variational structure of the model (cf. Subsec. B.1 below for more details). On the other hand, this bound is sufficient only for proving existence of weak solutions. Note that it is thanks to the viscosity term (the last summand in (1.3)) that we can obtain some information on the second space
derivatives of \( \varphi \). Actually, for \( \epsilon = 0 \) taking the limit of the right-hand side of (1.1) would be probably hopeless due to lack of compactness. Indeed, here it does not seem possible to obtain the strong \( L^2 \)-convergence of \( \nabla \varphi \) via contractive arguments (as is usually done for the decoupled semilinear wave equation with subcritical nonlinearity), due to the presence of the transport terms.

Once we have obtained existence of weak solutions (which we can do both in 2D and in 3D) looking for additional properties is more difficult, even in the 2D case, and we can only prove a number of partial results holding under additional regularity assumptions on coefficients and data. Entering details, in 2D we can try to construct strong solutions (such a classification is mutated by the standard terminology used for Navier-Stokes) and, indeed, we can prove their existence, but only in the case when the second order transport term is neglected, i.e., one has \( \delta = 0 \) in (1.4). In the class of strong solutions one can also prove uniqueness. This result holds, with a conditional nature, also in 3D (where existence of strong solutions is not known, of course), or for \( \delta > 0 \).

A further question we address is related to energy dissipation. Actually, due to the absence of external sources and to the choice of periodic boundary conditions, one expects that at least a part of the kinetic and chemical energy of the body is gradually converted into heat. On the other hand, we can explicitly prove this fact only in two cases, i.e. when either \( \sigma > 0 \) in (1.3) or \( \delta = 1 \) in (1.4). As will be explicit from the proofs, when \( \sigma > 0 \) (and apparently only in that case) one can control, uniformly in time, the contribution coming from the combination of the additional viscosity term \( -\sigma \Delta \varphi \) with the non-monotone semilinear term \( f(\varphi) \). It may also be worth observing that, in place of \( \sigma \varphi \), one could consider a more general semilinear contribution \( \sigma \varphi \) in (1.3), which (under suitable conditions on the function \( \sigma \)) would provide a nonlinear damping effect (see, e.g., [3, 5, 8, 10, 11, 19, 20, 24] for related models); the resulting mathematical problem will be possibly addressed in a forthcoming paper. On the other hand, in the case \( \delta = 1 \), corresponding to the situation when transport effects with the same transport speed occur both for \( \varphi \) and for \( \pi \), we can prove dissipation of energy even for \( \sigma = 0 \) thanks to a cancellation property. It is worth noting that, in the language of dynamical systems, this leads to existence of an absorbing set in the “energy space”, a fact that may serve as a starting point for studying the long-time behavior of solution trajectories (this may be also the object of future work). On the other hand, in the case when \( \sigma = 0 \) and \( \delta = 1 \) the dissipativity of the physical energy remains as an open question.

We conclude here with the plan of the paper. The next section is devoted to presenting our precise mathematical assumptions on coefficients and data and a number of preliminary considerations. Then, our main results will be stated in the subsequent Section 3. The core of the proofs, including the basic a priori estimates and the arguments used for passing to the limit in the approximation, will be then presented in Section 4. Finally, a possible construction of regularized solutions by means of a Faedo-Galerkin scheme will be given in Section 5.

2 Preliminaries

2.1 Notation and functional setup

We will note as \( \Omega \) the unit flat torus \([0, 1]^d\), with \( d = 2 \) or \( d = 3 \). As is customary, all functions defined on \( \Omega \) will be implicitly assumed to satisfy periodic boundary conditions in a suitable sense. This will not be emphasized in the notation, for the sake of simplicity. For instance, we will set \( H := L^2(\Omega) \) and \( V := H^1(\Omega) \) implicitly assuming \( \Omega \)-periodicity. These spaces will be used as function spaces for the phase variable; the same symbols \( H \) and \( V \) will be used also for denoting vector- or tensor-valued functions (we may write, for instance, \( \nabla \varphi \in H \)). The standard scalar product in \( H \) will be noted as \( \langle \cdot, \cdot \rangle \). Since the immersion \( V \subset H \) is continuous and dense, identifying \( H \) with \( H' \) through the above scalar product we obtain the Hilbert triplet \((V, H, V')\) for the phase variable.

Concerning the velocity function \( \mathbf{u} \), we set

\[
C^\infty_{\text{div}}(\Omega) = \{ \mathbf{u} \in [C^\infty(\Omega)]^d : \text{div} \mathbf{u} = 0 \}, \quad V_{\text{div}} := C^\infty_{\text{div}}(\Omega)^{H^1(\Omega)}, \quad H_{\text{div}} := C^\infty_{\text{div}}(\Omega)^{L^2(\Omega)},
\]

where, again, \( \Omega \)-periodicity is still implicitly subsumed everywhere. The spaces \( H_{\text{div}} \) and \( V_{\text{div}} \) are seen as (closed) subspaces of \( H \) and \( V \) (more precisely, of \( H^d \) and \( V^d \)), respectively, and in particular they are endowed with the corresponding norms. Then, the embedding \( V_{\text{div}} \subset H_{\text{div}} \) is continuous and
dense, which permits us to identify \( V' \) with its topological dual \( V'' \) by means of the scalar product of \( H_{\text{div}} \), still denoted by \( (\cdot, \cdot) \), and to construct the “velocity Hilbert triplet” \((V_{\text{div}}, H_{\text{div}}, V''_{\text{div}})\). We recall that \( V_{\text{div}} \) is endowed with the bilinear form \((u, v) := (\nabla u, \nabla v)\) for all \( u, v \in V_{\text{div}} \). Moreover, given a generic Banach spaces \( X \) (and particularly in the cases \( X = V, X = V_{\text{div}} \)), we will denote by \((\cdot, \cdot)\) the duality between \( X' \) and \( X \). As is customary, we define the trilinear \( V_{\text{div}} \)-continuous form

\[
 b(u, v, w) = \int_{\Omega} (u \cdot \nabla v) \cdot w \, dx \quad \forall u, v, w \in V_{\text{div}} \tag{2.2}
\]

and the bilinear form \( B : V_{\text{div}} \times V_{\text{div}} \to V''_{\text{div}} \) defined by

\[
 (B(u, v), w) = b(u, v, w) \quad \forall u, v, w \in V_{\text{div}}.
\]

Using Ladyzhenskaya’s inequality (see, e.g., [23] Chap. 5)), it turns out that the following properties hold for every \( u, v \) and \( w \in V_{\text{div}} \):

\[
|b(u, v, w)| \leq c\|u\|_{H_{\text{div}}}^{1/2}\|u\|_{V_{\text{div}}}^{1/2}\|v\|_{V_{\text{div}}}\|w\|_{V_{\text{div}}} \quad \text{if} \quad d = 3, \tag{2.3}
\]

\[
|b(u, v, w)| \leq c\|u\|_{H_{\text{div}}}^{1/2}\|u\|_{V_{\text{div}}}^{1/2}\|v\|_{V_{\text{div}}}\|w\|_{V_{\text{div}}}^{1/2} \quad \text{if} \quad d = 2. \tag{2.4}
\]

### 2.2 Assumptions on the potential

We let \( F \in C^2(\mathbb{R}; \mathbb{R}) \) and setting \( f := F' \) we assume that

\[
 K(|s|^p + 1) \geq f'(s) \geq k|s|^p - \lambda \tag{2.5}
\]

for some \( K, k, p > 0, \lambda \geq 0 \) and every \( s \in \mathbb{R} \). Namely, \( f \) grows at infinity as a power-like function of exponent \( p + 1 > 1 \) (i.e., it is strictly superlinear). Note that the above implies in particular that \( F \) is (at least) \( \lambda \)-convex, i.e., \( F'' = f' \) is everywhere greater than \(-\lambda \), with true convexity holding in the case \( \lambda = 0 \).

A practical and common example of a phase potential satisfying the above is the standard **double well potential** having the expression \( F(s) = (s^2 - 1)^2 \) for which (2.5) holds of course for \( p = 2 \).

The above relation has several useful consequences, some of which are listed below. First, by integration one can easily deduce

\[
 K'(|s|^{p+1} + 1) \geq f(s) \text{ sign } s \geq k'|s|^{p+1} - \lambda', \tag{2.6}
\]

and, integrating again,

\[
 K''(|s|^{p+2} + 1) \geq F(s) \geq k''|s|^{p+2} - \lambda'', \tag{2.7}
\]

where \( K', k', K'', k'' > 0 \) and \( \lambda', \lambda'' \geq 0 \) are computable constants depending only on \( K, k, p, \lambda \). Moreover, combining (2.6) and (2.7), it is not difficult to deduce that

\[
 \kappa_p(|s|^{p+2} + F(s)) - c_p \leq f(s) \leq C_p(|s|^{p+2} + 1) \quad \forall s \in \mathbb{R}, \tag{2.8}
\]

where the constants \( \kappa_p, C_p > 0 \) and \( c_p \geq 0 \) depend only on the given values of the parameters in (2.5).

Moreover, thanks to \( p + 2 > 2 \), we also observe that for any (large) \( M > 0 \) and (small) \( \varepsilon > 0 \) there exists \( c(M, \varepsilon) > 0 \) such that

\[
 M|s|^2 \leq \varepsilon F(s) + c(M, \varepsilon) \quad \forall s \in \mathbb{R}. \tag{2.9}
\]

### 2.3 Initial data and conservation properties

In the sequel we will note the spatial mean of a generic function \( v \) defined over \( \Omega \) as

\[
 v_\Omega := \frac{1}{|\Omega|} \int_\Omega v \, dx = \int_\Omega v \, dx, \tag{2.10}
\]

the second equality holding because \( \Omega \) is the unit torus. We also recall the Poincaré-Wirtinger inequality

\[
 \|v - v_\Omega\|_H \leq c\|
abla v\|_H, \tag{2.11}
\]
holding for any \( v \in V \) and for some \( c > 0 \). In particular, thanks to periodic boundary conditions, the above holds with \( v_\Omega = 0 \) when \( v = D_x z \) for some \( z \in H^2(\Omega) \) and \( i \in \{1, \ldots, d\} \).

We observe that, due to the choice of periodic boundary conditions, any hypothetic solution to our problem satisfies some conservation properties. First of all, integrating (1.1) over \( \Omega \), we actually have
\[
\frac{d}{dt} u_\Omega = 0,
\]
(2.12)
i.e., the mean value of the velocity is conserved in time. Such a property corresponds to the conservation of (total) momentum in absence of external forces.

Then, in order to reduce technical complications we shall always assume that
\[
\begin{align*}
\mathbf{u}_0 &\in H^{\text{div}}, \\
(\mathbf{u}_0)_\Omega &= 0.
\end{align*}
\]
Indeed the case when \((\mathbf{u}_0)_\Omega \neq 0\) could be reduced to the present one by rewriting the system in terms of the translated variable \( \mathbf{u} - u_\Omega \).

Next, integrating (1.3) and (1.4), we deduce, respectively,
\[
\begin{align*}
\frac{d}{dt} \phi_\Omega + \sigma \phi_\Omega &= \pi_\Omega, \\
\frac{d}{dt} \pi_\Omega + \pi_\Omega + \int_\Omega f(\phi) \, dx &= 0,
\end{align*}
\]
(2.14)
(2.15)
whence, combining the above relations, we obtain (here primes denote derivation in time)
\[
(\phi_\Omega)^{\prime\prime} + (1 + \sigma)(\phi_\Omega)^{\prime} + \sigma \phi_\Omega + \int_\Omega f(\varphi) \, dx = 0.
\]
(2.16)
This relation rules the evolution of the total mass of either component of the binary fluid. Note, however, that in itself it does not suffice in order to prove that \( \phi_\Omega \) remains bounded uniformly in time.

### 3 Main Results

Basically we can distinguish our results into two classes. The first one refers to the regularity setting of \textit{weak solutions} (this essentially means that the initial data have the sole regularity corresponding to the finiteness of the physical energy). In such a framework, we can prove existence both in 2D and in 3D (Theorem 1) for \( \delta \geq 0 \). Moreover, in all cases except when \( \sigma = 0 \) and \( \delta \neq 1 \), we can also prove an energy dissipation principle (Prop. 2), i.e. the fact that, whatever is the magnitude of the energy at the initial time, solution trajectories (or, at least those solution trajectories that arise as cluster points of approximating families; indeed, uniqueness is not known at this level) tend to have an energy configuration that is below some computable threshold depending only on the parameters of the system (and not on the magnitude of the initial data). In the language of dynamical systems this corresponds to the existence of a \textit{uniformly absorbing set}. Though this property is very natural in our setting (periodic boundary conditions and no external source), in the sense that one expects that energy is progressively converted into heat as the system evolves, surprisingly we cannot prove it under the most general assumptions.

The second family of results refer to the class of \textit{strong solutions}, hence requiring some more smoothness of initial data. Unfortunately, even in 2D, we can prove existence of such solutions (Theorem 3) only in the case when \( \delta = 0 \), i.e. the second-order transport effect is neglected. In this case, we can also prove uniqueness of strong solutions (Theorem 4). The result holds both in 2D and in 3D, having of course a conditional nature in the latter case.

\textbf{Remark.} When \( \sigma = 0 \), the lack of a dissipative estimate in the general case is tied to the occurrence of the viscosity term \(-\epsilon \Delta \varphi\) in (1.3). Indeed, at least in the physical case when \( F \) is nonconvex, the energy inequality (cf. (4.7) below) may contain a nonpositive term on the left hand side and also the information (2.16) on the spatial means seems not sufficient to control it uniformly in time. Hence
an estimate of the energy can be obtained only using Gronwall’s inequality, which may lead to an exponential growth in time. On the other hand, in order to avoid this (unexpected) behavior, we may try to adapt the argument standardly used to prove dissipativity of solutions to the semilinear wave equation. However, we then face the occurrence of the transport terms, which apparently cannot be controlled in the general case, whereas they vanish thanks to incompressibility if \( \delta = 1 \) (and only in that case).

We are now ready to state our first result devoted to existence of weak solutions:

**Theorem 1** (Existence of weak solutions, \( d = 2, 3 \)). Let \( d \in \{2, 3\} \), let \( \epsilon > 0 \), \( \sigma \geq 0 \) and \( \delta \geq 0 \). Let also Assumption \((2.5)\) hold. Let \( T > 0 \) and let the initial data satisfy

\[
\begin{align*}
    u_0 &\in H_{\text{div}}, \quad (u_0)_\Omega = 0, \quad \varphi_0 \in V \cap L^{p+2}(\Omega), \quad \pi_0 \in H.
\end{align*}
\]

Then there exists at least one triple \((u, \varphi, \pi)\) belonging to the regularity class

\[
\begin{align*}
    u &\in W^{1,q}(0, T; V_{\text{div}}') \cap L^\infty(0, T; H_{\text{div}}) \cap L^2(0, T; V_{\text{div}}), \\
    \varphi &\in L^\infty(0, T; V) \cap L^\infty(0, T; L^{p+2}(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
    \varphi_t &\in L^2(0, T; L^s(\Omega)), \\
    \pi &\in L^\infty(0, T; H), \\
    \pi_t &\in L^2(0, T; (W^{1,3})'(\Omega)),
\end{align*}
\]

where

\[
q = s = 2 \quad \text{if} \quad d = 2, \quad q = 4/3, \quad s = 3/2 \quad \text{if} \quad d = 3,
\]

satisfying system \((1.1)-(1.3)\) in the following weak sense

\[
\begin{align*}
    \langle u_t, v \rangle - \int_\Omega (u \otimes u) : \nabla v \, dx + (u \cdot \nabla u, \nabla v) &= \int_\Omega (\nabla \varphi \otimes \nabla \varphi) : \nabla v \, dx, \\
    \pi &= \varphi_t + u \cdot \nabla \varphi - \epsilon \Delta \varphi + \sigma \varphi, \quad \text{a.e. in} \ \Omega, \\
    \langle \pi_t, v \rangle + (\pi, v) - \delta \int_\Omega \pi u \cdot \nabla v \, dx + (\nabla \varphi, \nabla v) + \int_\Omega f(\varphi)v \, dx &= 0,
\end{align*}
\]

for almost every \( t \in (0, T) \), every \( v \in V_{\text{div}} \) and every \( v \in W^{1,3}(\Omega) \), and complying with the initial conditions

\[
\begin{align*}
    u|_{t=0} &= u_0, \quad \varphi|_{t=0} = \varphi_0, \quad \pi|_{t=0} = \pi_0
\end{align*}
\]

almost everywhere in \( \Omega \). The triple \((u, \varphi, \pi)\) will be noted as a weak solution in the sequel.

**Remark.** It is worth observing that condition \((3.1)\) corresponds exactly to the finiteness of the physical energy (cf. \((1.7)\) below) at the initial time. Indeed, assumption \( \varphi_0 \in L^{p+2}(\Omega) \) is equivalent to asking \( F(\varphi_0) \in L^1(\Omega) \) due to \((2.5)\).

Since \( T > 0 \) is arbitrary, we can assume that the weak solutions provided by Theorem 1 could be extended to be defined for any time \( t \in [0, \infty) \). In this perspective we can prove the following

**Proposition 2** (Dissipativity). Let the hypotheses of Theorem 1 hold and let us additionally assume that, either \( \sigma > 0 \), or \( \sigma = 0 \) and \( \delta = 1 \). Then there exists a constant \( C_0 \) independent of the initial data (but depending on the other parameters of the system) and a time \( T_0 \) depending only on the “energy” of the initial datum, i.e. on the quantity

\[
\|u_0\|_{H_{\text{div}}} + \|\varphi_0\|_V + \|\varphi_0\|_{L^{p+2}(\Omega)} + \|\pi_0\|_H.
\]

such that any weak solution provided by Theorem 1 emanating from this initial datum satisfies

\[
\|u(t)\|_{H_{\text{div}}} + \|\varphi(t)\|_V + \|\varphi(t)\|_{L^{p+2}(\Omega)} + \|\pi(t)\|_H \leq C_0 \quad \forall t \geq T_0,
\]

**Remark.** In the terminology of dynamical systems, the constant \( C_0 \) in estimate \((3.13)\) may be interpreted as the “radius” of an absorbing set with respect to the norms specified there (which in turn somehow quantify the magnitude of the energy).
Remark. Since the uniqueness of weak solutions is not known at this level, the dissipativity property in (3.13) has to be carefully interpreted. Indeed, its proof is obtained by passing to the limit in an analogue relation holding at the approximate level (i.e., for some regularized solution that has better properties). Hence, any weak solution that is a limit point of a sequence of approximate solutions turns out to satisfy it. On the other hand, we cannot exclude that there might exist “bad” weak solutions, unrelated to the approximation scheme, that do not satisfy (3.13).

Theorem 3 (Existence of strong solutions). Let \( d = 2, \epsilon > 0, \delta = 0, \sigma \geq 0 \) and let Assumption (2.5) hold. Let also \( T > 0 \) and let the initial data satisfy
\[
\begin{align*}
  &u_0 \in V_{\text{div}}, \quad \varphi_0 \in H^2(\Omega), \quad \pi_0 \in V. 
\end{align*}
\]
Then there exists almost one strong solution, namely a triple \((u, \varphi, \pi)\) with
\[
\begin{align*}
u &\in H^1(0,T; V_{\text{div}}) \cap L^\infty(0,T; V_{\text{div}}) \cap L^2(0,T; H^2_{\text{div}}(\Omega)), \\
\varphi &\in H^1(0,T; V) \cap L^\infty(0,T; H^2(\Omega)) \cap L^2(0,T; H^3(\Omega)), \\
\pi &\in H^1(0,T; H) \cap L^\infty(0,T; V),
\end{align*}
\]
validating system (1.1)-(1.4) in the following sense:
\[
\begin{align*}
u_t + \text{div}(\nu \otimes \nu) + \nabla p + \text{div}(\nabla \varphi \otimes \nabla \varphi) - \Delta \nu &= 0 \quad \text{a.e. in } (0,T) \times \Omega, \\
\text{div} \nu &= 0 \quad \text{a.e. in } (0,T) \times \Omega, \\
\pi &= \varphi_t + \nu \cdot \nabla \varphi - \epsilon \Delta \varphi + \sigma \varphi \quad \text{a.e. in } (0,T) \times \Omega, \\
\pi_t + \pi - \Delta \varphi + f(\varphi) &= 0 \quad \text{a.e. in } (0,T) \times \Omega
\end{align*}
\]
an the initial conditions as in (3.11).

Theorem 4 (Uniqueness of strong solutions). Let \( d \in \{2,3\}, \sigma \geq 0, \epsilon > 0 \) and let Assumption (2.5) hold. Let also \( T > 0 \) and let \((u_i, \varphi_i, \pi_i), i = 1,2\) be a couple of strong solutions in the sense of the previous theorem, and satisfying in particular the regularity conditions (3.15)-(3.17) emanating from the same initial datum \((u_0, \varphi_0, \pi_0)\) satisfying (3.11). Moreover, let either \( \delta = 0 \) or \( \delta > 0 \) together with the additional assumption
\[
\pi_i \in L^2(0,T; W^{1,3}(\Omega)) \quad \text{for some } i \in \{1,2\}.
\]
Then \((u_1, \varphi_1, \pi_1) = (u_2, \varphi_2, \pi_2)\) almost everywhere in \((0,T) \times \Omega\).

Remark. The uniqueness result is conditional in two aspects. First of all, existence of strong solutions clearly cannot be obtained for \( d = 3 \). Moreover, for \( \delta > 0 \) (a case which is also not covered by Theorem 3), we can prove that two hypothetical strong solutions emanating from the same initial datum coincide only in the case when either of the two satisfies the additional regularity (3.22).

4 Proofs

In this section we will prove Theorems 1, 3, 4 as well as Proposition 2. We start presenting a number of basic a-priori estimates by working directly, though formally, on system (1.1)-(1.4) without referring explicitly to any regularization or approximation. Actually, the variational structure underlying system (1.1)-(1.4) is rather simple and for this reason it seems to be worth starting with the computations leading to the energy inequality. This actually constitutes the main a priori information that any (reasonably defined) solution should satisfy. Then, a possible approach via a Faedo-Galerkin regularization compatible with the a priori estimates will be outlined in Section 5 below.

In what follows we will note as \( c, \kappa \) some generic positive constants depending only on the given data of the problem but independent of the final time \( T \), with \( \kappa \) being used in estimates from below. The values of \( c, \kappa \) will be allowed to vary on occurrence, and will be assumed to be independent of any hypothetical regularization or approximation parameter. Specific values of \( \kappa, c \) will be noted as \( \kappa_i, c_i, i \geq 1 \).
4.1 Energy estimate

We start testing \((1.1)\) with \(u\) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^{2}}^{2} + \|\nabla u\|_{H^{2}}^{2} - \int_{\Omega} (\nabla \varphi \otimes \nabla \varphi) : \nabla u \, dx = 0, \tag{4.1}
\]

where we also used the incompressibility constraint \((1.2)\) and the periodic boundary conditions. Next, we test \((4.4)\) with \(\pi\). Using again periodicity and incompressibility we deduce

\[
\frac{1}{2} \frac{d}{dt} \|\pi\|_{H^{2}}^{2} + \|\pi\|_{H^{2}}^{2} + \int_{\Omega} \nabla \pi \cdot \nabla \varphi \, dx + \int_{\Omega} f(\varphi) \pi \, dx = 0. \tag{4.2}
\]

Now we exploit the expression of \(\pi\) \((1.3)\) in order to compute the last two integrals in \((4.2)\). Firstly we have

\[
\int_{\Omega} \nabla \pi \cdot \nabla \varphi \, dx = \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_{H^{2}}^{2} + |\nabla \Delta \varphi + \epsilon \|\nabla \varphi\|_{H^{2}}^{2} + \int_{\Omega} \nabla \pi \cdot \nabla (u \cdot \nabla \varphi) \, dx, \tag{4.3}
\]

whereas the second term, thanks also to incompressibility, gives

\[
\int_{\Omega} f(\varphi) \pi \, dx = \frac{d}{dt} \int_{\Omega} F(\varphi) \, dx + \epsilon \int_{\Omega} f'(\varphi) |\nabla \varphi|^{2} \, dx + \sigma \int_{\Omega} f(\varphi) \pi \, dx. \tag{4.4}
\]

Replacing \((4.3)-(4.4)\) into \((4.2)\), we deduce

\[
\frac{d}{dt} \left( \frac{1}{2} \|\pi\|_{H^{2}}^{2} + \frac{1}{2} |\nabla \varphi|_{H^{2}}^{2} + \int_{\Omega} F(\varphi) \, dx \right) + \|\pi\|_{H^{2}}^{2} + \|\nabla \varphi\|_{H^{2}}^{2} + |\nabla \Delta \varphi + \epsilon \|\nabla \varphi\|_{H^{2}}^{2}
+ \sigma \int_{\Omega} f(\varphi) \pi \, dx + \int_{\Omega} \nabla \pi \cdot \nabla (u \cdot \nabla \varphi) \, dx + \epsilon \int_{\Omega} f'(\varphi) |\nabla \varphi|^{2} \, dx = 0. \tag{4.5}
\]

We now notice that, by standard integrations by parts,

\[
\int_{\Omega} \nabla \varphi \cdot \nabla (u \cdot \nabla \varphi) \, dx = \int_{\Omega} (\nabla \varphi \otimes \nabla \varphi) : \nabla u \, dx + \frac{1}{2} \int_{\Omega} u \cdot \nabla |\nabla \varphi|^{2} \, dx, \tag{4.6}
\]

and the second summand on the right hand side vanishes by incompressibility. Then, summing \((4.1)\) together with \((4.6)\) and taking \((4.5)\) into account, we arrive at the energy estimate

\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|_{H^{2}}^{2} + \frac{1}{2} |\nabla \varphi|_{H^{2}}^{2} + \int_{\Omega} F(\varphi) \, dx \right) + \|\nabla u\|_{H^{2}}^{2} + \|\pi\|_{H^{2}}^{2}
+ \sigma |\nabla \varphi|_{H^{2}}^{2} + |\nabla \Delta \varphi + \epsilon \|\nabla \varphi\|_{H^{2}}^{2} + \sigma \int_{\Omega} f(\varphi) \pi \, dx + \epsilon \int_{\Omega} f'(\varphi) |\nabla \varphi|^{2} \, dx = 0. \tag{4.7}
\]

Now, since \(F\) is just assumed to be \(\lambda\)-convex, in general the last term on the left hand side may be nonpositive. In order to control it, we need to exploit the Gronwall inequality, which may lead to an exponential growth of the energy with time. Indeed, recalling \((2.8)\) and Assumption \((2.9)\) (and using in particular the \(\lambda\)-convexity of \(F\)), taking a generic \(C > 0\) we deduce

\[
\frac{d}{dt} \left[ \|u\|_{H^{2}}^{2} + \|\pi\|_{H^{2}}^{2} + \|\nabla \varphi\|_{H^{2}}^{2} + \int_{\Omega} 2F(\varphi) \, dx + C \right] + 2\|\nabla u\|_{H^{2}}^{2} + 2\|\pi\|_{H^{2}}^{2}
+ 2(\sigma - \lambda \epsilon) |\nabla \varphi|_{H^{2}}^{2} + 2(\sigma - \lambda \epsilon) |\nabla \Delta \varphi + \epsilon \|\nabla \varphi\|_{H^{2}}^{2} + 2\sigma \kappa (\|\varphi\|_{L^{2}}^{p-2} + \int_{\Omega} F(\varphi) \, dx) \leq 2\sigma \kappa \tag{4.8}
\]

and the coefficient \(2(\sigma - \lambda \epsilon)\) may be actually nonpositive (and it is always nonpositive when \(\sigma = 0\) and \(\lambda > 0\)).

Let us then note as \(G\) the sum of the terms in square brackets in \((1.8)\). Now, recalling that \(u\) has zero spatial mean by \((3.1)\) and \((2.12)\), the Poincaré-Wirtinger inequality implies

\[
\|\nabla u\|_{H^{2}}^{2} \geq \kappa \|u\|_{H^{2}}^{2}. \tag{4.9}
\]
for some $\kappa > 0$. Hence, rearranging terms, (4.8) gives
\[
\frac{d}{dt} G + \|\nabla u\|^2_{H_{div}} + \kappa \|u\|^2_{H_{div}} + 2\|\pi\|_H^2 + \sigma \kappa p \|\nabla \varphi\|^2_H + 2 \sigma \kappa p \int_\Omega F(\varphi) \, dx + 2 \kappa \|\Delta \varphi\|_H^2 + 2 \sigma \kappa p \|\varphi\|^{p+2}_{L^{p+2}(\Omega)} \leq 2 \sigma c_p + (\sigma \kappa p - 2 \sigma + 2 \kappa \lambda) \|\nabla \varphi\|^2_H. \tag{4.10}
\]

Now, using (2.7) and (2.9), it is clear that
\[
\kappa, c > 0 \text{ depending on the various parameters of the problem. Now, when } \sigma = 0 \text{ the right hand side needs to be estimated by using Gronwall. Indeed, in that case (4.12), using (4.11), may be rewritten as}
\]
\[
\frac{d}{dt} G + \kappa G + \|\nabla u\|^2_{H_{div}} + 2 \epsilon \|\Delta \varphi\|_H^2 + 2 \sigma \kappa p \|\varphi\|^{p+2}_{L^{p+2}(\Omega)} \leq c(1 + \|\nabla \varphi\|^2_H). \tag{4.12}
\]
\[
\text{for some computable constants } \kappa, c > 0 \text{ depending on the various parameters of the problem. Now, when } \sigma = 0 \text{ the right hand side needs to be estimated by using Gronwall. Indeed, in that case (4.12), using (4.11), may be rewritten as}
\]
\[
\frac{d}{dt} G + \kappa G + \|\nabla u\|^2_{H_{div}} + 2 \epsilon \|\Delta \varphi\|_H^2 \leq c_1 G + c_2, \tag{4.13}
\]
\[
\text{for some constants } c_1, c_2 > 0, \text{ whence Gronwall’s inequality entails}
\]
\[
G(t) \leq G(0)e^{c_1 t} + \frac{c_2}{c_1} \tag{4.14}
\]
\[
\text{as well as}
\]
\[
\int_0^t (\|\nabla u\|^2_{H_{div}} + 2 \epsilon \|\Delta \varphi\|_H^2) \, ds \leq c(G(0)e^{c_1 t} + t). \tag{4.15}
\]
\[
\text{On the other hand, when } \sigma > 0 \text{ it is clear that we can avoid exponential growth of the energy. Indeed, in that case the right hand side of (4.12) can be controlled in the following way:}
\]
\[
c \|\nabla \varphi\|^2_{H_{div}} = -c \int_\Omega \varphi \Delta \varphi \, dx \leq c \|\Delta \varphi\|^2_H + \frac{c}{\epsilon} \|\varphi\|^2_H \leq c \|\Delta \varphi\|^2_H + \sigma \kappa p \|\varphi\|^{p+2}_{L^{p+2}(\Omega)} + c(\sigma, \epsilon, p), \tag{4.16}
\]
\[
\text{where in the last computation we have used (2.9). As a consequence, in place of (4.13) we get from (4.12) the better relation}
\]
\[
\frac{d}{dt} G + \kappa G + \|\nabla u\|^2_{H_{div}} + c \|\Delta \varphi\|^2_H + \sigma \kappa p \|\varphi\|^{p+2}_{L^{p+2}(\Omega)} \leq c_3, \tag{4.17}
\]
\[
\text{whence we immediately deduce}
\]
\[
G(t) \leq G(0)e^{-\kappa_1 t} + \frac{c_3}{\kappa_1} \tag{4.18}
\]
\[
\text{and, integrating (4.17) over the generic interval } (t, t + 1), t \geq 0,
\]
\[
\int_t^{t+1} (\|\nabla u\|^2_{H_{div}} + c \|\Delta \varphi\|^2_H + \sigma \kappa p \|\varphi\|^{p+2}_{L^{p+2}(\Omega)}) \, dt \leq G(0)e^{-\kappa_1 t} + c_3 + \frac{c_3}{\kappa_1}. \tag{4.19}
\]

### 4.2 Dissipative estimate for $\sigma = 0$ and $\delta = 1$

We start with testing (1.3) by $\pi$ to obtain
\[
(\pi, \varphi_t) = \|\pi\|_H^2 - (u \cdot \nabla \varphi, \pi) + c(\Delta \varphi, \pi). \tag{4.20}
\]
Next, assuming $\delta = 1$, we test (1.4) by $\varphi$. Integrating by parts in time, we obtain
\[
(\pi, \varphi_t) = \frac{d}{dt}(\pi, \varphi) + (\pi, \varphi) + (u \cdot \nabla \varphi, \varphi) + \|\nabla \varphi\|^2_H + (f(\varphi), \varphi). \tag{4.21}
\]
Combining the above relations, we deduce
\[
\frac{d}{dt} (\pi, \varphi) + (\pi, \varphi) + \|\nabla \varphi\|_{H}^2 + (f(\varphi), \varphi) = \|\pi\|_{H}^2 + (\mathbf{u} \cdot \nabla \varphi, \pi) - (\mathbf{u} \cdot \nabla \pi, \varphi) + \epsilon(\Delta \varphi, \pi).
\] (4.22)

Now, we observe that, due to \(\delta = 1\) and incompressibility, the transport terms actually vanish:
\[
(\mathbf{u} \cdot \nabla \varphi, \pi) + (\mathbf{u} \cdot \nabla \pi, \varphi) = \int_{\Omega} \mathbf{u} \cdot \nabla (\varphi\pi) \, dx = -\int_{\Omega} \varphi\pi \, div \mathbf{u} \, dx = 0.
\]
Hence relation (4.22) reduces to
\[
\frac{d}{dt} (\pi, \varphi) + (\pi, \varphi) + \|\nabla \varphi\|_{H}^2 + (f(\varphi), \varphi) = \|\pi\|_{H}^2 + \epsilon(\Delta \varphi, \pi).
\] (4.23)

Let us now multiply the above relation by \(\eta > 0\) and sum the result to (4.8). We get
\[
\frac{d}{dt} \left[ \|\mathbf{u}\|_{H_{av}}^2 + \|\pi\|_{H}^2 + \|\nabla \varphi\|_{H}^2 + \int_{\Omega} 2F(\varphi) \, dx + C + \eta(\pi, \varphi) \right]
+ 2\|\nabla \mathbf{u}\|_{H_{av}}^2 + (2 - \eta)\|\pi\|_{H}^2 + 2\|\Delta \varphi\|_{H}^2 + \eta\|\nabla \varphi\|_{H}^2 + \eta(f(\varphi), \varphi)
\leq 2\lambda \epsilon \|\nabla \varphi\|_{H}^2 + \eta(\Delta \varphi, \pi) - \eta(\pi, \varphi)
\] (4.24)
and we need to control the terms on the right hand side. First of all, we observe that
\[
\eta(\Delta \varphi, \pi) - \eta(\pi, \varphi) \leq \frac{\eta\epsilon^2}{2} \|\Delta \varphi\|_{H}^2 + \frac{\eta}{2} \|\varphi\|_{H}^2 + \eta\|\pi\|_{H}^2.
\] (4.25)
On the other hand,
\[
2\lambda \epsilon \|\nabla \varphi\|_{H}^2 = 2\lambda \epsilon \int_{\Omega} (-\Delta \varphi) \varphi \, dx \leq \frac{\epsilon}{2} \|\Delta \varphi\|_{H}^2 + 2\lambda^2 \epsilon \|\varphi\|_{H}^2.
\] (4.26)
Replacing (4.25) into (4.24), we then deduce
\[
\frac{d}{dt} \left[ \|\mathbf{u}\|_{H_{av}}^2 + \|\pi\|_{H}^2 + \|\nabla \varphi\|_{H}^2 + \int_{\Omega} 2F(\varphi) \, dx + C + \eta(\pi, \varphi) \right]
+ 2\|\nabla \mathbf{u}\|_{H_{av}}^2 + (2 - \eta)\|\pi\|_{H}^2 + \frac{3\epsilon}{2} \|\Delta \varphi\|_{H}^2 + \eta\|\nabla \varphi\|_{H}^2 + \eta(f(\varphi), \varphi) \leq \left(\frac{\eta}{2} + 2\lambda^2 \epsilon\right) \|\varphi\|_{H}^2.
\] (4.27)
Then, let us first fix \(\eta\) such that \(0 < \eta \leq \min\{\epsilon^{-1}, 1/2\}\). As a consequence, using also (2.9), we have
\[
\eta(\pi, \varphi) \geq -\frac{1}{4} \|\pi\|_{H}^2 - \frac{1}{4} \|\varphi\|_{H}^2 \geq -\frac{1}{4} \|\pi\|_{H}^2 - \int_{\Omega} F(\varphi) \, dx - c,
\] (4.28)
\[
\eta(\pi, \varphi) \leq \frac{1}{4} \|\pi\|_{H}^2 + \frac{1}{4} \|\varphi\|_{H}^2,
\] (4.29)
for some \(c > 0\). Hence, for such choice of \(\eta\), noting as \(\mathcal{D}\) the sum of the terms in square brackets in (4.27), we may choose \(C > 0\) so that
\[
\frac{1}{2} \|\varphi\|_{H}^2 + \frac{3}{4} \|\pi\|_{H}^2 + \|\mathbf{u}\|_{H_{av}}^2 + \int_{\Omega} F(\varphi) \, dx \leq \mathcal{D}
\leq \|\varphi\|_{H}^2 + \frac{5}{4} \|\pi\|_{H}^2 + \|\mathbf{u}\|_{H_{av}}^2 + \int_{\Omega} 2F(\varphi) \, dx + C.
\] (4.30)
We also observe that, thanks to (2.8) and (2.9), the right hand side of (4.27) can be estimated as follows
\[
\left(\frac{\eta}{2} + 2\lambda^2 \epsilon\right) \|\varphi\|_{H}^2 \leq \frac{\eta}{2} (f(\varphi), \varphi) + c(\eta, \epsilon, \lambda),
\] (4.31)
and the nonconstant term can be absorbed by the corresponding one on the left hand side. Moreover, the term depending on \( f \) on the left hand side can be estimated from below by means of (4.38). Hence, using once more (4.27), we see that the differential inequality
\[
\frac{d}{dt}D + \kappa_2 D + \kappa_2 \|\nabla u\|_{H_{\text{div}}}^2 + \epsilon \|\Delta \varphi\|_H^2 \leq c_4,
\]  
with the constants \( \kappa_2, c_4 > 0 \) depending on \( \epsilon, \lambda \) also through the choice of \( \eta \). As a consequence, we deduce
\[
D(t) + \int_t^{t+1} \left( \kappa_2 \|\nabla u(s)\|_{H_{\text{div}}}^2 + \epsilon \|\Delta \varphi(s)\|_H^2 \right) \, ds \leq D(0) e^{-\kappa_2 t} + \frac{c_4}{\kappa_2} + c_4, \quad \forall t \geq 0. \]  

4.3 Proof of Theorem 1: existence of weak solutions

In the sequel we consider a sequence \( \{ (u_n, \varphi_n, \pi_n) \} \), \( n \in \mathbb{N} \), of triplets satisfying system (4.14)-(4.15) in a suitable sense and complying, uniformly with respect to \( \kappa, c \), we assume that relation (4.12) is satisfied with constants \( \kappa, c \) independent of \( n \). Hence, either (4.14)-(4.15) or (4.18)-(4.19) also hold with constants independent of \( n \). As a consequence, we can prove that any limit point (in a suitable sense) of a generic (nonrelabelled) subsequence of \( \{ (u_n, \varphi_n, \pi_n) \} \) is a weak solution in the sense of Theorem 1. In Section 5 we will see how the present argument can be adapted to a Galerkin approximation scheme.

That said, we first observe that, from (4.14)-(4.15) (or from (4.18)-(4.19)), using the definition of \( G \) (cf. (4.8)) and the structure property (4.11), we may deduce the following relations:

\[
\begin{align*}
|u_n| & \to u \quad \text{weakly star in } L^\infty(0, T; H_{\text{div}}) \cap L^2(0, T; V_{\text{div}}), \quad (4.34) \\
\varphi_n & \to \varphi \quad \text{weakly star in } L^\infty(0, T; V) \cap L^\infty(0, T; L^{p+2}(\Omega)) \cap L^2(0, T; H^{2}(\Omega)), \quad (4.35) \\
\pi_n & \to \pi \quad \text{weakly star in } L^\infty(0, T; H). \quad (4.36)
\end{align*}
\]

Here and below all the convergence properties will be intended to hold up to the extraction of (non-relabelled) subsequences of \( n \to \infty \).

We now deduce some further consequence of the energy estimate that will be needed in order to take the limit of the product terms in the system. We start dealing with the velocity. We then observe that for any \( v \in V_{\text{div}}, \)
\[
\int \Omega (u_n \cdot \nabla u_n) \cdot v \, dx = - \int \Omega (u_n \otimes u_n) : \nabla v \, dx \leq \|u_n\|_{L^4(\Omega)}^2 \|\nabla v\|_H. \]  

Now, for \( d = 2 \), thanks to Ladyzhenskaya’s inequality, (4.31) implies
\[
\|u_n\|_{L^4(0, T; L^4(\Omega))} \leq c, \]  
whence from (4.37) we obtain
\[
\|u_n \cdot \nabla u_n\|_{L^2(0, T; V_{\text{div}})} \leq c. \]  

On the other hand, for \( d = 3 \), Sobolev’s embeddings and interpolation only give
\[
\|u_n\|_{L^{5/2}(0, T; L^4(\Omega))} \leq c, \]  
whence from (4.37) we deduce
\[
\|u_n \cdot \nabla u_n\|_{L^{5/2}(0, T; V_{\text{div}})} \leq c. \]  

Analogously, from (4.35) we have
\[
\begin{align*}
\|\nabla \varphi_n\|_{L^4(0, T; L^4(\Omega))} & \leq c \quad \text{for } d = 2, \quad (4.42) \\
\|\nabla \varphi_n\|_{L^{5/3}(0, T; L^4(\Omega))} & \leq c \quad \text{for } d = 3, \quad (4.43)
\end{align*}
\]
whence
\[
\begin{align*}
\|\nabla \varphi_n \otimes \nabla \varphi_n\|_{L^2(0, T; H)} & \leq c \quad \text{for } d = 2, \quad (4.44) \\
\|\nabla \varphi_n \otimes \nabla \varphi_n\|_{L^{5/3}(0, T; H)} & \leq c \quad \text{for } d = 3. \quad (4.45)
\end{align*}
\]
Now, testing (1.1) (written for $u_n, \varphi_n$) by $\mathbf{v} \in V_{\text{div}}$ and rearranging, we have

$$\langle u_{n,t}, \mathbf{v} \rangle = -\langle \nabla u_n, \nabla \mathbf{v} \rangle + \langle \nabla \varphi_n \otimes \nabla \varphi_n, \nabla \mathbf{v} \rangle + \langle u_n \otimes u_n, \nabla \mathbf{v} \rangle,$$

whence, using (4.39), (4.41) and (4.44)-(4.46), we easily infer

$$u_{n,t} \to u_t \quad \text{weakly in } L^q(0,T; V_{\text{div}}'),$$

where $q = 2$ if $d = 2$ and $q = 4/3$ if $d = 3$. Combining the above with (4.31) and applying the Aubin-Lions lemma, we then obtain

$$u_n \to u \quad \text{strongly in } C^0_0([0,T]; V_{\text{div}}') \cap L^2(0,T; H_{\text{div}}).$$

We now move to considering the behavior of $\varphi_n$. To this aim, we first notice that, using (4.38) and (4.42) in 2D, and using the first (4.34), the last (4.35) and Sobolev’s embeddings in 3D, there follows

$$\|u_n \cdot \nabla \varphi_n\|_{L^2(0,T;L^s(\Omega))} \leq c,$$

where $s = 3/2$ if $d = 3$ and $s = 2$ if $d = 2$. Hence, comparing terms in (1.3) and using again (4.34)-(4.36) it is not difficult to deduce

$$\varphi_n, t \to \varphi_t \quad \text{weakly in } L^2(0,T; L^s(\Omega)),$$

with $s$ as above. Hence, applying once more the Aubin-Lions lemma, we infer

$$\varphi_n \to \varphi \quad \text{strongly in } C^0_0([0,T]; H) \cap L^2(0,T; V).$$

As a consequence, we obtain that

$$\nabla \varphi_n \otimes \nabla \varphi_n \to \nabla \varphi \otimes \nabla \varphi \quad \text{strongly in } L^1(0,T; L^1(\Omega)).$$

This allows us to take the limit $n \to \infty$ in relation (4.46) to get back (4.38). Moreover, it is clear that, from (4.38) and (4.51), we also obtain

$$\nabla \varphi_n \to \nabla \varphi \quad \text{strongly in } L^1(0,T; L^1(\Omega)).$$

Actually, relations (4.52) and (4.53) could be improved, anyway they suffice for the sequel. Indeed, we can now write (1.3) at the $n$-level, take the limit $n \to \infty$, and get back (3.9).

Finally, we need to pass to the limit in (1.4). Then, let us notice that (4.35) and (2.6) entail

$$\|f(\varphi_n)\|_{L^\infty(0,T; L^{\frac{4q}{3}}(\Omega))} \leq c.$$

Combining this fact with the pointwise convergence resulting from (4.51) we then deduce

$$f(\varphi_n) \to f(\varphi) \quad \text{weakly star in } L^\infty(0,T; L^{\frac{4q}{3}}(\Omega)) \quad \text{and strongly in } L^1(0,T; L^1(\Omega)).$$

Finally, if $\delta > 0$, we need to take care of the convection term in (1.4). Combining (4.34) with (4.36) and using interpolation with the 2D embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < \infty$, we actually deduce that

$$\|u_n \cdot \nabla \varphi_n\|_{L^2(0,T; L^r(\Omega))} \leq c_r,$$

where $r = 3/2$ for $d = 3$ and $r \in [1,2]$ for $d = 2$. Then, using the second of (4.38), we infer

$$u_n \pi_n \to u \pi \quad \text{weakly in } L^2(0,T; L^r(\Omega)).$$

Hence, for any $v \in W^{1,3}(\Omega)$, using incompressibility and periodic boundary conditions, we deduce

$$\int_\Omega (u_n \cdot \nabla \pi_n) v \, dx = - \int_\Omega \pi_n (u_n \cdot \nabla v) \, dx \to - \int_\Omega (u \cdot \nabla v) \, dx \quad \text{weakly in } L^2(0,T).$$
and in particular the distributional divergence $\text{div}(\pi u)$ lies in the dual space $L^2(0,T;(W^{1,3})'(\Omega))$.  

Using (4.31)-(4.36) together with (4.31) and (4.36), comparing terms in the $n$-version of (4.4), and using continuity of the embedding $L^\infty(0,T;L^{\frac{p-2}{p}}(\Omega)) \subset L^2(0,T;(W^{1,3})'(\Omega))$, we then obtain

$$
\pi_{n,t} \rightharpoonup \pi_t \quad \text{weakly in } L^2(0,T;(W^{1,3})'(\Omega)).
$$

This fact, by the Aubin-Lions lemma and (4.36), also yields the strong convergence

$$
\pi_n \rightarrow \pi \quad \text{strongly in } C^0([0,T];V').
$$

The above relations permit us to pass to the limit in (4.4), which may be reinterpreted as an equality in $L^2(0,T;(W^{1,3})'(\Omega))$ (or also as an equality in $(W^{1,3})'(\Omega)$, almost everywhere in time). Namely, in the limit we get back exactly (3.10). Note also that the regularity conditions (3.2)-(3.4) in the statement of Theorem 1 are a direct consequence of (4.31)-(4.36), (4.47)-(4.50) and (4.59).

Finally, we observe that the strong convergence relations (4.48), (4.51), (4.60) imply

$$
(u_n,\varphi_n,\pi_n)|_{t=0} \rightarrow (u,\varphi,\pi)|_{t=0} \quad \text{strongly in } V'_\text{div} \times H \times V'.
$$

Hence, assuming that the triplet $(u_n,\varphi_n,\pi_n)$ satisfies an initial condition of the form

$$
(u_n,\varphi_n,\pi_n)|_{t=0} = (u_0,n,\varphi_0,n,\pi_0,n),
$$

where $(u_0,n,\varphi_0,n,\pi_0,n)$ tends to $(u_0,\varphi_0,\pi_0)$ in a suitable way, letting $n \nearrow \infty$, we obtain (3.11) in the limit, which concludes the proof of Theorem 1.

Remark. It is worth discussing a bit more the occurrence of the viscous regularization term in (1.3) at the light of the a priori estimates. Actually, if that term is omitted (i.e., if $\epsilon = 0$), the latter convergence in (4.36) would be lacking and we would not have any $L^p$-information on second space derivatives of $\varphi$. Hence, the $L^2(0,T;V)$-convergence in (4.51) would also be missing and we could not take the limit of the right hand side of (1.4) as specified in (1.52).

4.4 Proof of Proposition 2: dissipativity

In the case $\sigma > 0$ we can take advantage of estimate (4.18), whereas for $\sigma = 0$ and $\delta = 1$ we have relation (4.33). Of course, these bounds hold a priori for the approximating solutions $(u_n,\varphi_n,\pi_n)$. On the other hand, they pass to the limit $n \nearrow \infty$ because the quantities on the right hand sides are independent of $n$ and we can use semicontinuity of norms with respect to weak or weak star convergence when we take the limit. Hence we obtain that (4.18), or (4.33), also holds for limit solution(s) $(u,\varphi,\pi)$. Noting that the functionals $G$ (cf. (4.8)) and $D$ (cf. (4.27)) control both from above and from below the norms specified in the statement, it is then apparent that (4.18) (or 4.33) implies the desired bound (4.13) for every $t \geq T_0$, with $T_0$ depending only on the magnitude of the initial energy, i.e., on the norms in (4.12).

Remark. One may wonder if the dissipative estimate (4.13) (possibly combined with the regularity estimates obtained in the proof of existence of strong solutions) could be used to prove existence of the global attractor at least in the 2D case. We do not address this interesting issue here, but we limit ourselves to observe that the problem seems nontrivial and its resolution may require the use of some careful decomposition method. Indeed, while the variables $u$ and $\varphi$ enjoy some regularization property (if they are, respectively, in $H^1_\text{div}$ and in $V$ at the initial time, then they are in $V'_\text{div}$ and in $H^2(\Omega)$ for some small $t > 0$), i.e., they have a parabolic behavior, on the other hand the variable $\pi$ has a hyperbolic behavior, i.e. it does not seem to regularize in time (compare (3.5) for $\pi$ with (3.2) and (3.3) for $u$ and $\varphi$).

4.5 Proof of Theorem 3: existence of strong solutions

The proof is based on some additional regularity estimates. As before, we work directly on system (1.1)-(1.4). We postpone to the next section the justification of this argument at the light of the approximation scheme.
First of all, we multiply (1.1) with $-\Delta u$ and integrate over $\Omega$ to obtain
\[
0 = \int_\Omega \left( -u_1 \Delta u + |\Delta u|^2 - \text{div}(\nabla \varphi \otimes \nabla \varphi) \Delta u - \nabla p \Delta u - \text{div}(u \otimes u) \Delta u \right) \, dx \tag{4.63}
\]
\[
= \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 + \int_\Omega \left( - (D^2 \varphi \nabla \varphi) \cdot \Delta u - \Delta \varphi (\nabla \varphi \cdot \Delta u) \right) \, dx.
\]
Indeed, the convection term on the first row vanishes due to $d = 2$ (see, e.g., [23, p. 242]).

Now using Ladyzhenskaya's and Young's inequalities with 2D Sobolev embeddings, we estimate the last two terms as follows:
\[
\left| \int_\Omega \left( - (D^2 \varphi \nabla \varphi) \cdot \Delta u - \Delta \varphi (\nabla \varphi \cdot \Delta u) \right) \, dx \right|
\leq \|\nabla \varphi\|_{L^4(\Omega)} \|D^2 \varphi\|_{L^2(\Omega)} \|\Delta u\|_{H_{\text{av}}} (4.64)
\leq \alpha \|\Delta u\|_{H_{\text{av}}}^2 + c_a \|\nabla \varphi\|_V \|D^2 \varphi\|_H (4.65)
\leq \alpha \|\Delta u\|_{H_{\text{av}}}^2 + c_a \|\Delta \varphi\|_H^2 + c_a \|\Delta \varphi\|_H^2,
\]
where $\alpha > 0$ denotes small constants to be chosen later and the constants $c_a > 0$ are correspondingly large. In the above computation we used the uniform in time $V$-bound for $\varphi$ resulting from the energy bound (cf. (4.35)) and elliptic regularity.

Next, we deal with the Allen-Cahn system. Actually, testing (1.3) with $\Delta \pi$, summing the results, and performing standard manipulations (note in particular that a couple of terms cancel out), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \pi\|_{H^2}^2 + \|\Delta \varphi\|_{H^2}^2 + \|\nabla \varphi\|_{H^2}^2 + \sigma \|\Delta \varphi\|_{H^2}^2 + \epsilon \|\nabla \Delta \varphi\|_{H^2}^2
\leq - \int_\Omega f'(\varphi) \nabla \varphi \nabla \pi \, dx + \int_\Omega (\nabla u \nabla \varphi) \cdot \nabla \Delta \varphi \, dx + \int_\Omega (D^2 \varphi u) \cdot \nabla \Delta \varphi \, dx =: \sum_{j=1}^3 I_j (4.66)
\]
and we have to control the terms $I_j$ on the right hand side. We actually have
\[
|I_1| \leq c \int_\Omega (1 + |\varphi|^p) \|\nabla \varphi\| \|\nabla \pi\| \, dx
\leq c \|\varphi\|_{L^1(\Omega)} \|\nabla \varphi\| \|\varphi\|_H \|\nabla \varphi\|_H \leq c \|\nabla \varphi\|_V \|\nabla \varphi\|_V \|\varphi\|_V
\leq c (1 + \|\varphi\|_{H^2(\Omega)}) \|\nabla \varphi\|_H
\leq \alpha \|\nabla \varphi\|_H^2 + c_a \|\Delta \varphi\|_H^2 (4.67)
\]
\[
|I_2| \leq \|\nabla u\|_{H_{\text{av}}} \|\nabla \varphi\|_H \|\nabla \varphi\|_H \|\nabla \varphi\|_V \|\Delta \varphi\|_H
\leq \alpha \|\nabla u\|_{H_{\text{av}}}^2 + \alpha \|\Delta \varphi\|_H^2 + c_a \|\nabla \varphi\|_H^2 \|\varphi\|_V \|\nabla \varphi\|_V
\leq c (1 + \|\varphi\|_{H^2(\Omega)}) \|\varphi\|_V
\leq \alpha \|\nabla \varphi\|_V^2 + c_a \|\Delta \varphi\|_H^2 (4.68)
\]
\[
|I_3| \leq \|u\|_{L^4(\Omega)} \|D^2 \varphi\|_{L^2(\Omega)} \|\nabla \varphi\|_H
\leq c \|\nabla \varphi\|_{H_{\text{av}}} \|\nabla \varphi\|_H \|\varphi\|_H \|\Delta \varphi\|_H
\leq c \|\nabla \varphi\|_H \|\Delta \varphi\|_H + c_a \|\nabla \varphi\|_{H_{\text{av}}} \|\Delta \varphi\|_H^2.
\]
where $\alpha$ and $c_{\alpha}$ are as above (of course their value can vary on occurrence) and we have repeatedly used assumption (2.5), 2D-Sobolev embeddings, Young’s and Ladyzhenskaya’s inequalities, and the information already obtained with the energy bound (cf. (4.34), (4.35)).

Now we sum (4.63) to (4.65). Using (4.64), (4.66), (4.67) and (4.68), we then deduce

$$\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{H^{1}_{\text{div}}}^2 + \| \nabla \pi \|_{H^{1}_{\text{div}}}^2 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \| \nabla \pi \|_{H^{1}_{\text{div}}}^2 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \sigma \| \alpha \|_{H^{1}_{\text{div}}}^2 + \epsilon \| \nabla \Delta \varphi \|_{H^{1}_{\text{div}}}^2 \leq \frac{d}{dt} \| \nabla u \|_{H^{1}_{\text{div}}}^2 + \| \nabla \pi \|_{H^{1}_{\text{div}}}^2 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \sigma \| \alpha \|_{H^{1}_{\text{div}}}^2 + \epsilon \| \nabla \Delta \varphi \|_{H^{1}_{\text{div}}}^2,$$  \hspace{1cm} (4.69)

Now, we can take the various $\alpha$’s small enough (in a way that also depends on $\epsilon$) so that the above reduces to

$$\frac{d}{dt} \| \nabla u \|_{H^{1}_{\text{div}}}^2 + \| \nabla \pi \|_{H^{1}_{\text{div}}}^2 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \| \nabla \pi \|_{H^{1}_{\text{div}}}^2 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \sigma \| \alpha \|_{H^{1}_{\text{div}}}^2 + \epsilon \| \nabla \Delta \varphi \|_{H^{1}_{\text{div}}}^2 \leq c(1 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2 + \| \nabla u \|_{H^{1}_{\text{div}}}^2),$$ \hspace{1cm} (4.70)

This inequality has the structure

$$\frac{d}{dt} \mathcal{E}_1(t) + \mathcal{D}_1(t) \leq m(t)(1 + \mathcal{E}_1(t))$$ \hspace{1cm} (4.71)

where $\mathcal{E}_1$ denotes the quantity in square brackets on the left hand side, $\mathcal{D}_1$ is the sum of the remaining terms, and the function

$$t \mapsto m(t) := c(1 + \| \Delta \varphi \|_{H^{1}_{\text{div}}}^2) \hspace{1cm} (4.72)$$

belongs to $L^1(0, T)$ thanks to the $L^2(0, T; H^2(\Omega))$-bound for $\varphi$ following from the energy estimate (cf. (4.35)).

Applying Gronwall’s Lemma to the function $t \mapsto \mathcal{E}_1(t)$ we then conclude that

$$u \in L^\infty(0, T; V_{\text{div}}),$$

$$\pi \in L^\infty(0, T; V),$$

$$\varphi \in L^\infty(0, T; H^2(\Omega)).$$ \hspace{1cm} (4.73)

Note that the improved regularity assumptions on the initial data (3.14) have also been used here.

Then, integrating once more (4.71) in time we also deduce

$$u \in L^2(0, T; H^3(\Omega)),$$

$$\nabla \Delta \varphi \in L^2(0, T; H),$$ \hspace{1cm} (4.74)

whence, using once more elliptic regularity we infer

$$\varphi \in L^2(0, T; H^3(\Omega)).$$ \hspace{1cm} (4.75)

Finally, comparing terms in the equations of the system and using the above bounds, standard manipulations permit us to deduce also

$$u_t \in L^2(0, T; H_{\text{div}}),$$

$$\varphi_t \in L^2(0, T; V),$$

$$\pi_t \in L^2(0, T; H),$$ \hspace{1cm} (4.76)

which is the last bound we need.

Now, in order to complete the proof, we interpret the information obtained above in the framework of an approximation scheme. In this respect, (4.73), (4.74), (4.75), and (4.76) can be seen as a-priori estimates uniform with respect to the approximation parameter $n$. Consequently, the convergence relations (4.34) - (4.36) may also be improved in the corresponding way. Finally, we may observe that, in view of the improved regularity, all terms in equations (1.1) and (1.2) lie in some $L^p$-space and may be consequently interpreted in the pointwise sense, as noted in the statement of the theorem. This completes the proof of Theorem 3.
4.6 Proof of Theorem 4: uniqueness

Let us given a pair of strong solutions \((u_1, \phi_1, \pi_1)\) and \((u_2, \phi_2, \pi_2)\) of equations (4.1)-(4.4) emanating from the same initial datum \((u_0, \phi_0, \pi_0)\) satisfying (4.3). We will show that the two solutions do coincide. We perform the proof in the more difficult case \(d = 3\) (as said, in this situation the result is conditional because strong solutions are not known to exist); of course the argument extends to \(d = 2\) where we actually have better embeddings. That said, we put \(u := u_1 - u_2\), \(\phi := \phi_1 - \phi_2\), \(\pi := \pi_1 - \pi_2\) and \(p := p_1 - p_2\). Then, writing (4.1) for the two solutions, and taking the difference, we get

\[
u - \Delta u + \text{div}(u \otimes u_1) + \text{div}(u_2 \otimes u) + \nabla p + \text{div}(\nabla \phi \otimes \nabla \phi_1) + \text{div}(\nabla \phi_2 \otimes \nabla \phi) = 0.
\]

(4.77)

Proceeding in the same way for (1.3) and (4.4) we obtain

\[
\pi = \phi_1 + u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2 - \epsilon \Delta \phi + \sigma \phi,
\]

(4.78)

\[
\pi + \sigma \cdot u_1 \cdot \nabla \phi + \frac{3}{6} \nabla \phi_2 \cdot \nabla \phi - f(\phi_1) - f(\phi_2) = 0.
\]

(4.79)

We now work on system (4.77)-(4.79) with the aim of getting a contraction estimate. We start testing (4.77) with \(u\) to get

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^2}^2 + \|\nabla u\|_{H^1}^2 = \int \left( (\nabla \phi \otimes \nabla \phi_1) + (\nabla \phi_2 \otimes \nabla \phi) \right) \cdot \nabla u \, dx
\]

(4.80)

Let us now provide a bound of the terms \(K_i\) on the right hand side. First, we have

\[
|K_1| \leq \|\nabla u\|_{H^2} \|\nabla \phi\|_{L^2} (\|\nabla \phi_1\|_{L^2} + \|\nabla \phi_2\|_{L^2})
\]

(4.81)

\[
\leq \epsilon \|\nabla u\|_{H^2} \|\nabla \phi\|_{L^2}^1 \|\nabla \phi_1\|_{L^2} \leq \frac{1}{2} \|\nabla u\|_{H^2}^2 + \frac{c}{6} \|\Delta \phi\|_{H^1} + c \|\nabla \phi\|_{L^2}^2,
\]

where we have used Sobolev’s embeddings, Young’s inequality, regularity (4.10) both for \(\phi_1\) and for \(\phi_2\), and elliptic regularity results (in order to control the \(L^2\)-norm of \(\nabla \phi\) with the \(H\)-norm of \(\Delta \phi\)). Secondly, we have

\[
|K_2| \leq \|\nabla u\|_{H^2} \|u\|_{L^2}^1 (\|u_1\|_{L^2} + \|u_2\|_{L^2})
\]

(4.82)

\[
\leq \epsilon \|\nabla u\|_{H^2} \|u\|_{L^2}^2 \leq \frac{1}{8} \|\nabla u\|_{H^2}^2 + c \|u\|_{H^2}^2,
\]

having used regularity (4.10) both for \(u_1\) and for \(u_2\), Sobolev’s embeddings, and the Poincaré-Wirtinger inequality (recall that both \(u_1\) and \(u_2\) have zero spatial mean).

Next, we multiply (4.78) by \(-\Delta \phi\) to deduce

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_{H^2} + \epsilon \|\Delta \phi\|_{H^1}^2 + \sigma \|\nabla \phi\|_{H^1}^2 = -(\pi, \Delta \phi) + (u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2, \Delta \phi).
\]

(4.83)

Correspondingly, we multiply (4.79) by \(\pi\). Standard manipulations give

\[
\frac{1}{2} \frac{d}{dt} \|\pi\|_{H^3} + \|\pi\|_{H^3}^2 = -\delta (u_1 \cdot \nabla \pi + u \cdot \nabla \pi_2, \pi) + (\pi, \Delta \phi) - (f(\phi_1) - f(\phi_2), \pi).
\]

(4.84)

Combining the previous two relations we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \phi\|_{H^2} + \frac{1}{2} \frac{d}{dt} \|\pi\|_{H^3} + \epsilon \|\Delta \phi\|_{H^1}^2 + \sigma \|\nabla \phi\|_{H^1}^2 + \|\pi\|_{H^3}^2
\]

(4.85)

\[
= (u_1 \cdot \nabla \phi, \Delta \phi) + (u \cdot \nabla \phi_2, \Delta \phi) - \delta (u_1 \cdot \nabla \pi, \pi) - \delta (u \cdot \nabla \pi_2, \pi) - (f(\phi_1) - f(\phi_2), \pi)
\]

\[
=: J_1 + J_2 + J_3 + J_4 + J_5,
\]

and we need to control the quantities \(J_i\) on the right hand side. First of all,

\[
|J_1| \leq \|u_1\|_{L^3} \|\nabla \phi\|_{L^1} \|\Delta \phi\|_{H^1}
\]

(4.86)

\[
\leq \|u_1\|_{L^3} \|\nabla \phi\|_{H^1}^{1/2} \|\Delta \phi\|_{H^1}^{3/2} \leq \frac{c}{6} \|\Delta \phi\|_{H^1} + c \|\nabla \phi\|_{H^1}.
\]
where we used interpolation, regularity \((3.15)\) for \(u_1\), and elliptic regularity results for \(\varphi\). Next, using the fact that \(u\) has zero spatial mean and the regularity \((5.16)\) for \(\varphi_2\), we obtain

\[
|J_2| \leq \|u\|_{L^3(\Omega)} \|\nabla \varphi_2\|_{L^5(\Omega)} \|\Delta \varphi\|_H
\]

\[
\leq \|u\|_{H_{av}}^{2/3} \|\nabla u\|_{H_{av}}^{1/2} \|\varphi_2\|_{H^2(\Omega)} \|\Delta \varphi\|_H \leq \frac{1}{8} \|\nabla u\|_{H_{av}}^2 + \frac{\epsilon}{6} \|\Delta \varphi\|_H^2 + c_\epsilon \|u\|_{H_{av}}^2.
\]

On the other hand, by incompressibility,

\[
J_3 = -\frac{\delta}{2} \int_{\Omega} u_1 \cdot \nabla \pi^2 \, dx = 0,
\]

whereas the subsequent term, for \(\delta > 0\), can be controlled only in case we have the additional regularity \((3.22)\) (here for \(i = 2\)):

\[
|J_4| \leq \frac{\delta}{2} \|u\|_{L^\infty(\Omega)} \|\nabla \varphi_2\|_{L^5(\Omega)} \|\pi\|_H
\]

\[
\leq \frac{1}{8} \|\nabla u\|_{H_{av}} + c\delta^2 \|\nabla \varphi_2\|_{L^3(\Omega)}^2 \|\pi\|^2_H.
\]

Finally, using \((2.5)\), we have

\[
|J_5| \leq c \int_{\Omega} (1 + |\varphi_2|^p + |\varphi_1|^p)|\pi| |\varphi| \, dx \leq c\|\varphi\|^2_H + c\|\pi\|^2_H.
\]

Now, we can take the sum of \((4.80)\) and \((4.85)\). Using \((4.81)-(4.82)\) and \((4.80)-(4.90)\), we then deduce

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H_{av}}^2 + \frac{1}{2} \|\nabla u\|_{H_{av}}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_H + \frac{1}{2} \frac{d}{dt} \|\pi\|_H + \frac{\epsilon}{2} \|\Delta \varphi\|_H^2 + \sigma \|\varphi\|^2_H
\]

\[
\leq c\|\nabla \varphi\|^2_H + c\|u\|^2_{H_{av}} + c(1 + \delta^2 \|\nabla \varphi_2\|_{L^3(\Omega)}^2) \|\pi\|_H^2 + c\|\pi\|^2_H.
\]

To control the last term on the right hand side, we need to test \((4.78)\) with \(\varphi\). This procedure yields

\[
\frac{1}{2} \frac{d}{dt} \|\varphi\|^2_H + c\|\nabla \varphi\|^2_H + \sigma \|\varphi\|^2_H = \int_{\Omega} \pi \varphi \, dx - \int_{\Omega} (u_1 \cdot \nabla \varphi) \varphi \, dx - \int_{\Omega} (u \cdot \nabla \varphi_2) \varphi \, dx =: H_1 + H_2 + H_3.
\]

Now, \(H_2\) is readily seen to be 0 thanks to incompressibility. On the other hand, it is easy to see that

\[
|H_1| \leq c\|\varphi\|^2_H + c\|\pi\|^2_H,
\]

\[
|H_3| \leq \|u\|_{L^\infty(\Omega)} \|\nabla \varphi_2\|_{L^5(\Omega)} \|\varphi\|_H \leq \frac{1}{2} \|\nabla u\|_{H_{av}}^2 + c\|\pi\|^2_H.
\]

Hence, summing \((4.92)\) to \((4.91)\), neglecting some positive quantities on the left hand side, and taking \((4.93)-(4.94)\) into account, we finally arrive at

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H_{av}}^2 + \frac{1}{2} \|\nabla u\|_{H_{av}}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|^2_H + \frac{1}{2} \frac{d}{dt} \|\pi\|^2_H + \frac{\epsilon}{2} \|\Delta \varphi\|^2_H + \sigma \|\varphi\|^2_H + \frac{1}{2} \frac{d}{dt} \|\varphi\|^2_H
\]

\[
\leq c\|\nabla \varphi\|^2_H + c\|u\|^2_{H_{av}} + c(1 + \delta^2 \|\nabla \varphi_2\|_{L^3(\Omega)}^2) \|\pi\|_H^2 + c\|\pi\|^2_H.
\]

Hence, exploiting in the case \(\delta > 0\) the additional regularity assumption \((3.22)\), we can use Gronwall’s lemma in the above relation to deduce that \((u_1, \varphi_1, \pi_1)\) coincides with \((u_2, \varphi_2, \pi_2)\) over the whole of \((0, T)\), which actually concludes the proof of the theorem.

**Remark.** It is clear that, in the case when the initial data for \((u_1, \varphi_1, \pi_1)\) and \((u_2, \varphi_2, \pi_2)\) do not coincide with each other, then one can obtain from \((4.95)\) a continuous dependence estimates in the norms specified in \((3.14)\). Namely, one has

\[
\|u_1(t) - u_2(t)\|_{H_{av}}^2 + ||\varphi_1(t) - \varphi_2(t)||_H^2 + ||\pi_1(t) - \pi_2(t)||_H^2
\]

\[
\leq C(T) \left( \|u_1(0) - u_2(0)\|_{H_{av}}^2 + ||\varphi_1(0) - \varphi_2(0)||_H^2 + ||\pi_1(0) - \pi_2(0)||_H^2 \right)
\]

for any \(t \in [0, T]\), with the constant \(C(T)\) on the right hand side depending on the “strong” norms of the two solutions specified in \((3.15)-(3.17)\) (and in \((3.22)\) in the case \(\delta > 0\)).
5 Galerkin approximation

In this section we present a possible construction of an approximate solution \((u_n, \varphi_n, \pi_n)\) by means of a Faedo-Galerkin scheme. Since the procedure is rather standard and follows an approach already used for similar models (see, e.g., [22, p. 284] or [9]), we will only give some highlights leaving the remaining details to the reader.

That said, we let \(H_{\text{div}, 0}\) and \(V_{\text{div}, 0}\) denote the subspaces, respectively, of \(H_{\text{div}}\) and of \(V_{\text{div}}\), consisting of the functions having zero spatial mean. Then we can define the Stokes operator as an unbounded linear operator on \(H_{\text{div}, 0}\) by setting \(A = -P\Delta, D(A) = H^2(\Omega) \cap V_{\text{div}, 0}\), where \(P : L^2(\Omega) \to H_{\text{div}}\) is the Leray projector [25]. Notice that

\[
(Au, v) = (u, v) = (\nabla u, \nabla v) \quad \forall u \in D(A), \quad \forall v \in V_{\text{div}, 0}.
\]

Moreover, the operator \(A\) is positive and self-adjoint on \(H_{\text{div}, 0}\). Hence, we can take as a Galerkin base of \(H_{\text{div}, 0}\) the family \(\{w_i\}_{i \in \mathbb{N}}\) of the (properly normalized and linearly independent) eigenfunctions of \(A\). Next, noting as \(I\) the identity operator of \(H\), we consider the unbounded linear operator \(B\) of \(H\) defined as \(B := I - \Delta\) with domain \(D(B) = H^2(\Omega)\) (note that \(\Omega\)-periodicity is still implicitly assumed also at this level). Hence, \(B\) is also positive and self-adjoint and we can take as a Galerkin base of \(H\) the family \(\{\tau_i\}_{i \in \mathbb{N}}\) of the (normalized and linearly independent) eigenfunctions of \(B\). For any \(m \in \mathbb{N}\), we can then define the \(m\)-dimensional subspaces \(W_m := \text{span}\{w_1, \ldots, w_m\} \subset H_{\text{div}, 0}\) and \(T_m := \text{span}\{\tau_1, \ldots, \tau_m\} \subset H\). We also denote as \(P_m : H_{\text{div}, 0} \to W_m\) and \(\Pi_m : H \to T_m\) the orthogonal projectors onto \(W_m\) and \(T_m\), respectively.

Finally, it is convenient to replace \(f\) with a suitable regularization \(f_n\) depending on a further approximation parameter \(n\). More precisely, we assume that \(f_n \in C^1(\mathbb{R})\) with

\[
|f_n(r)| + |f'_n(r)| \leq C_n \quad \forall n \in \mathbb{N}, \quad r \in \mathbb{R}, \tag{5.1}
\]

with the constants \(C_n > 0\) of course going to infinity as \(n \to \infty\). An explicit expression of \(f_n\) can be easily constructed by suitably truncating \(f\) outside some bounded interval \(I_n\) increasing with respect to \(n\), for example \(I_n = [-n, n]\). Then we obtain as a byproduct that \(f_n\) converges to \(f\) uniformly on compact sets of \(\mathbb{R}\) as \(n \to \infty\).

With this machinery at hand, we can look for a Faedo-Galerkin solution of the form

\[
u_m = \sum_{j=1}^{m} u_j^{m}(t)w_j, \quad \varphi_m = \sum_{j=1}^{m} \varphi_j^{m}(t)\tau_j, \quad \pi_m = \sum_{j=1}^{m} \pi_j^{m}(t)\tau_j,
\]

satisfying the following discretized system:

\[
(\partial_t \nu_m, w) + (\nu_m \cdot \nabla \nu_m, w) + (A \nu_m, w) = (\nabla \varphi_m \otimes \nabla \varphi_m, \nabla w), \quad \forall w \in V_m, \tag{5.2}
\]

\[
\pi_m = \partial_t \varphi_m + \Pi_m(\nu_m \cdot \nabla \varphi_m) + \epsilon B \varphi_m + (\sigma - \epsilon) \varphi_m, \tag{5.3}
\]

\[
\partial_t \pi_m + \pi_m + \delta \Pi_m(\nu_m \cdot \nabla \pi_m) + B \varphi_m - \varphi_m + \Pi_m(f_n(\varphi_m)) = 0, \tag{5.4}
\]

complemented with the initial conditions

\[
u_m|_{t=0} = P_m u_0, \quad \varphi_m|_{t=0} = \Pi_m \varphi_0, \quad \pi_m|_{t=0} = \Pi_m \pi_0. \tag{5.5}
\]

Since all nonlinear terms in the above system have at least a locally Lipschitz dependence on their arguments, it then turns out that existence of a local in time solution to (5.2)-(5.4) with the initial conditions (5.5) can be obtained by means of the classical Cauchy theorem for ODE’s. It is also worth stressing that the above constructed approximate solution, despite being identified by the sole subscript \(m\), depends in fact on both approximation parameters \(n\) and \(m\).

For any fixed \(n \in \mathbb{N}\), we will now let \(m\) tend to infinity so to obtain a solution \((u_n, \varphi_n, \pi_n)\) depending only on \(n\). To this aim, we just reproduce the a priori bounds obtained in the previous section working now on the Faedo-Galerkin scheme. Indeed, all the computations can be repeated just with some small technical difference mainly related to the presence of a regularized nonlinear function \(f_n\) in place of the original \(f\). The only point we would like to remark refers to the use of (5.3) to compute the product term now taking the form \((\Pi_m(f_n(\varphi_m)), \pi_m)\) (compare with (4.4)). Indeed,
in the present setting this gives rise to the quantity \( \Pi_m(f_n(\varphi_m)) \Pi_m(\mathbf{u}_m \cdot \nabla \varphi_m) \) which does not necessarily vanish because both factors are projected onto the finite dimensional subspace \( T_m \). On the other hand, we can easily see that

\[
(\Pi_m(f_n(\varphi_m)), \Pi_m(\mathbf{u}_m \cdot \nabla \varphi_m)) = (\Pi_m(f_n(\varphi_m)) - f_n(\varphi_m), \mathbf{u}_m \cdot \nabla \varphi_m)
\]

\[
\leq c_n \|\mathbf{u}_m\|_{H^1} \|\nabla \varphi_m\|_{H^1},
\]

where we also used incompressibility and condition (5.1). Note that the additional contribution on the right hand side, at fixed \( n \in \mathbb{N} \), can be controlled using Gronwall (giving rise to an additional quantity in the approximate energy inequality). On the other hand, as we take \( m \not\to \infty \), the projection operators disappear and we do no longer face any additional term when repeating the estimates on \((\mathbf{u}_n, \varphi_n, \pi_n)\).

It is also worth observing that, as we take \( m \not\to \infty \), we can rely on a set of a priori estimates that are uniform with respect to the time variable. Hence, as a consequence of standard extension arguments it will turn out that, despite Galerkin solutions \((\mathbf{u}_n, \varphi_n, \pi_n)\) might be defined only on small time intervals, the solutions \((\mathbf{u}_n, \varphi_n, \pi_n)\) obtained in the limit can be thought to be defined for every \( t \in (0, \infty) \). As a consequence, we can take \((\mathbf{u}_n, \varphi_n, \pi_n)\) as a regularized solution to our system depending on the sole parameter \( n \). Moreover, the a-priori estimates obtained before are also uniform with respect to \( n \in \mathbb{N} \). Hence, to let \( n \not\to \infty \), we can proceed as described in Subsection 1.3 with the sole differences related to the occurrence of the regularized nonlinearity \( f_n \). Nevertheless, even here, the needed modifications are almost straightforward. Indeed, using the strong (hence a.e. pointwise) convergence of \( \varphi_n \) (cf. (4.51)) and the fact that \( f_n \to f \) uniformly on compact subsets of \( \mathbb{R} \), it is easy to realize that, in place of (4.55), there holds

\[
f_n(\varphi_n) \to f(\varphi) \text{ weakly star in } L^\infty(0,T;L^\infty(\Omega)) \text{ and strongly in } L^1(0,T;L^1(\Omega)).
\]

Then, the rest of the argument works up to minor adaptations.

Acknowledgments. The present paper benefits from the support of the Italian MIUR-PRIN Grant 2015PA5MP7 “Calculus of Variations” and of the GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica) for GS. GF has been supported by the Austrian Science Fund (FWF) grant W1245 and by Vienna Doctoral School of Mathematics.

References

[1] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal., 194 (2009), 463–506.
[2] D.M. Anderson, G.B. McFadden and A.A. Wheeler, Diffuse-interface methods in fluid mechanics, Annu. Rev. Fluid Mech., Palo Alto, CA, Vol. 30, 1998, 139–165.
[3] V. Barbu, I. Lasiecka and M.A. Rammaha, On nonlinear wave equations with degenerate damping and source terms, Trans. Amer. Math. Soc., 357 (2005), 2571–2611.
[4] T. Blesgen, A generalization of the Navier-Stokes equation to two-phase flows, J. Physics D (Applied Physics), 32 (1999), 1119–1123.
[5] L. Bociu and I. Lasiecka, Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping, Discrete Contin. Dyn. Syst., 22 (2008), 835–860.
[6] G. Bonfanti, M. Frémond and F. Luterotti, Existence and uniqueness results to a phase transition model based on microscopic accelerations and movements, Nonlinear Anal. Real World Appl., 5 (2004), 123–140.
[7] F. Boyer, *Nonhomogeneous Cahn-Hilliard fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), 225–259.

[8] I. Chueshov and I. Lasiecka, *Attractors for second-order evolution equations with a nonlinear damping*, J. Dynam. Differential Equations, 16 (2004), 469–512.

[9] P. Colli, S. Frigeri and M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, J. Math. Anal. Appl., 386 (2012), 428–444.

[10] F. Dell’Oro and V. Pata, *Long-term analysis of strongly damped nonlinear wave equations*, Nonlinearity, 24 (2011), 3413–3435.

[11] F. Dell’Oro and V. Pata, *Strongly damped wave equations with critical nonlinearities*, Nonlinear Anal., 75 (2012), 5723–5735.

[12] M. Eleuteri, E. Rocca and G. Schimperna, *On a non-isothermal diffuse interface model for two-phase flows of incompressible fluids*, Discrete Contin. Dyn. Syst., 35 (2015), 119–138.

[13] M. Eleuteri, E. Rocca and G. Schimperna, *Existence of solutions to a two-dimensional model for nonisothermal two-phase flows of incompressible fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), 1431–1454.

[14] E. Feireisl, E. Rocca, G. Schimperna and A. Zarnescu, *On a hyperbolic system arising in liquid crystals modeling*, J. Hyperbolic Differ. Equ., 15 (2018), 15–35.

[15] M. Frémond, *Non-smooth Thermomechanics*, Springer-Verlag, Berlin, 2002.

[16] F. Gay-Balmaz and C. Tronci, *The helicity and vorticity of liquid-crystal flows*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467 (2011), 1197–1213.

[17] M. Grasselli and V. Pata, *Existence of a universal attractor for a parabolic-hyperbolic phase-field system*, Adv. Math. Sci. Appl., 13 (2003), 443–459.

[18] M. Grasselli, A. Miranville, V. Pata and S. Zelik, *Well-posedness and long time behavior of a parabolic-hyperbolic phase-field system with singular potentials*, Math. Nachr., 280 (2007), 1475–1509.

[19] A. Kh. Khanmamedov, *Global attractors for strongly damped wave equations with displacement dependent damping and nonlinear source term of critical exponent*, Discrete Contin. Dyn. Syst., 31 (2011), 2497–2522.

[20] I. Lasiecka and D. Toundykov, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Anal., 64 (2006), 1757–1797.

[21] C. Liu and J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Phys. D, 179 (2003), 211–228.

[22] A. Miranville, *Some mathematical models in phase transition*, Discrete Contin. Dyn. Syst. Ser. S, 7 (2014), 271–306.

[23] J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge University Press, 2001.

[24] L. Roder and T. Tbou, *Stabilization of the wave equation with localized nonlinear damping*, J. Differential Equations, 145 (1998), 502–524.

[25] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam, 1984.