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Constructing two-qubit gates with minimal couplings

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Couplings between quantum systems are frequently less robust and harder to implement than controls on individual systems; thus, constructing quantum gates with minimal interactions is an important problem in quantum computation. In this paper we study the optimal synthesis of two-qubit quantum gates for the tunable coupling scheme of coupled superconducting qubits and compute the minimal interaction time of generating any two-qubit quantum gate.

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I. INTRODUCTION

Superconducting systems are among the leading candidates for the implementation of quantum information processing applications [1]. Due to the ubiquitous bath degrees of freedom in the solid-state environment, the time over which quantum coherence can be maintained remains limited, although significant progress in lengthening that time has been achieved. A key challenge is how to produce accurate quantum gates and how to minimize their duration such that the number of operations within $T_2$ meets the error correction threshold. Concomitantly, progress has been made in applying optimal control techniques to steer quantum systems in a robust, relaxation-minimizing [2,3], or time optimal way [4–6].

In order to perform multiqubit operations, one needs a reliable method to perform switchable coupling between the qubits, i.e., a coupling mechanism that can be easily turned on and off. Over the past few years, there has been considerable interest in this question, both theoretically [7–11] and experimentally [12–22]. As a practically relevant and illustrative example, we consider the tunable coupling scheme for flux qubits and considering building two-qubit gates using minimal coupling time. This is an extension of previous studies of time optimal control [4], which considers constructing quantum gates using one coupling Hamiltonian, to the problem of constructing quantum gates using two or more coupling Hamiltonians.

II. SYNTHESISIZATION OF TWO-QUBIT GATES

In this paper, we will take the tunable coupling scheme proposed in [23] as our main model for superconducting quantum computing, but the methods can be easily generalized to the other schemes. The tunable coupling scheme in [23] uses an extra high-frequency qubit to obtain the parametric ac-modulated coupling, as shown in Fig. 1. As discussed in [23], this coupling scheme has some useful features including an optimal point for the effective coupling energy. At such a point the effective coupling energy is insensitive to low-frequency flux noise, so two-qubit oscillations can be expected to be long lasting. This coupling scheme has been recently realized experimentally [24].

The effective Hamiltonian of this system is [23]

$$H = \frac{1}{2} \sum_{j=1}^{2} \left[ J_{12}^{(j)} \sigma_x^{(j)} - u_j(t) \sigma_z^{(j)} \right] - J_{12}(t) \sigma_x^{(1)} \sigma_x^{(2)}. \tag{1}$$

In this Hamiltonian, the coupling $J_{12}(t)$ is modulated sinusoidally at the angular frequency $\omega_{\pm}=(\Delta_2 \pm \Delta_1)$. The essence of the coupling scheme is seen by considering a general modulation of the form

$$J_{12}(t) = g_0 + g_+ (t) \cos(\omega_+ t) + g_- (t) \cos(\omega_- t). \tag{2}$$

To perform single-qubit operations we use Rabi oscillations driven by a resonant microwave control field,

$$u_j(t) = 2\Omega_j \cos(\Delta_j t + \phi_j(t)). \tag{3}$$

In this setup all the temporal dependence of the Hamiltonian is assumed to arise from the time-dependent flux of the applied fields. The rotating wave approximation, which is also valid if cross couplings are taken into account, results in a rotating frame Hamiltonian of the form

![FIG. 1. By modulating external magnetic field on qubit 3, we can turn on and off the effective coupling between qubit 1 and qubit 2.](Image)

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**Remark 1.** \(x <_y\) means that \(x\) lies in the convex hull of \(\{(\varepsilon_1 y_1, \varepsilon_2 y_2, \varepsilon_3 y_3) | \varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1, \Pi_i \varepsilon_i = 1, \pi \text{ is permutation on } \{1, 2, 3\}\} \). From now on, when we say constructing a gate in time \(T\), we mean constructing the gate with coupling time less or equal to \(T\), aided by local operations. **Theorem 1** [5]. A unitary gate \(U \in \text{SU}(4)\) can be generated in coupling time \(T\) if and only if we can simulate a Hamiltonian \(\tilde{H}\) in coupling time \(T\), such that \(\tilde{U} \prec \tilde{U}^\dagger \) or \(\tilde{U}^\dagger \prec \tilde{U}\). For a given Hamiltonian \(H(t)\), the Hamiltonians that can be simulated within time \(T\) using \(H(t)\) and local control are the Hamiltonians with the canonical forms of the following:

\[
\begin{array}{l}
\left\{ \theta_1 \sigma_x \otimes \sigma_y + \theta_2 \sigma_y \otimes \sigma_x + \theta_3 \sigma_z \otimes \sigma_z \right\} \\
(\theta_1, \theta_2, \theta_3) < \int_0^T \tilde{U}(t) dt, \quad \theta_1 \equiv \theta_2 \equiv |\theta_3| \quad \end{array}
\]

This theorem reduces our problem to the integration of the canonical form of

\[
H(t) = g_s(t)(\sigma_1^x \sigma_2^x - \sigma_1^y \sigma_2^y) + g_l(t)(\sigma_1^x \sigma_2^y + \sigma_1^y \sigma_2^x)
\]

(9)

while we absorbed the constant \(\frac{1}{2}\) into \(g_s(t)\) and \(g_l(t)\) and here \(g_s(t)\) and \(g_l(t)\) are both non-negative. We need to find out the boundary of \(\int_0^T \tilde{U}(t) dt\), as all the Hamiltonians that can be simulated within time \(T\) is \(s\) majorized by one of the points on the boundary. We study the problem in two cases:

**Case I.** The magnetic fields of frequencies \(\omega_s\) and \(\omega_l\) can be generated power independently, i.e., \(g_s, g_l\) are constrained independently. Let us say they take values independently on \(\{0, \omega\}\).

Using Proposition 1, we get the canonical parameter of the Hamiltonian (9), which is

\[
g_s(t) + g_l(t), |g_s(t) - g_l(t)|, 0). \]

We will see that it is actually \(s\) majorized by

\(A + B, |A - B|, 0)\)

since

\[
g_s(t) + g_l(t) \leq A + B,
\]

\[
g_s(t) + g_l(t) + |g_s(t) - g_l(t)| = \max\{2g_s(t), 2g_l(t)\}
\]

\[
\leq \max\{2A, 2B\}
\]

\[
= A + B + |A - B|,
\]

so \((g_s(t) + g_l(t), |g_s(t) - g_l(t)|, 0) < (A + B, |A - B|, 0)\) for all \(t\). Then given a unitary matrix \(U \in \text{SU}(4)\), the minimal coupling time needed to generate \(U\) is the minimal \(T\) such that

\[
\tilde{U} \prec ((A + B)T, |A - B|T, 0)
\]

or

\[
y \in \text{R}^3 \text{ (denoted } x <_y \text{) if}
\]

\[
x_1^y \leq y_1^y,
\]

\[
x_1^y + x_2^y + x_3^y \leq y_1^y + y_2^y + y_3^y,
\]

\[
x_1^y + x_2^y - x_3^y \leq y_1^y + y_2^y - y_3^y.
\]

(8)
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\[ \hat{\theta}' + \frac{\pi}{2}(-1,0,0)\nabla,(A + B)T,|A - B|T,0 \]

holds.

For example, suppose we want to generate a controlled-
NOT (CNOT) gate,

\[ \text{CNOT} = \exp \left[ -i \frac{\pi}{4} \left( \sigma_z \otimes \sigma_z - \sigma_x \otimes I - I \otimes \sigma_x + \frac{1}{2} I \right) \right]. \]

The canonical parameter of CNOT is \((\frac{\pi}{4},0,0)\). The two in-
equalities coincide in this case, so the minimal coupling time
needed is the minimal T such that

\[ \left( \frac{\pi}{4},0,0 \right) \nabla,(A + B)T,|A - B|T,0, \]

which is \(\frac{\pi}{2}M(A + B)\).

Case 2. The magnetic fields of frequencies \(\omega_{\pm}\) are gen-
erated power correlated, i.e., \(g_+\) and \(g_-\) are jointly constrained,

\[ g_+^2 + g_-^2 \leq M^2. \]

In this case the situation is more subtle as there is no single point, like \((A + B,|A - B|,0)\), that \(s\) majorizes all the points in the region.

Since \(x \nabla,(D\hat{\theta})\) when \(D \geq 1\), to be at the boundary, \(g_+(t)\) and \(g_-(t)\) should take maximal values. Let \(g_+(t) = M \sin \alpha(t)\) and \(g_-(t) = M \cos \alpha(t)\), where \(\alpha(t) \in [0, \frac{\pi}{2}]\) (as \(\alpha\) and \(\frac{\pi}{2} - \alpha\) give the same canonical parameter, we will just consider \(\alpha \in [0, \frac{\pi}{2}]\)), then integration of these canonical parameters give a set of boundary points,

\[ \left( \int_0^T g_+(t) + g_-(t) dt, \int_0^T g_-(t) - g_+(t) dt, 0 \right). \]

In the case that \(\alpha\) is constant, we obtain

\[ \left( \int_0^T g_+(t) + g_-(t) dt, \int_0^T g_-(t) - g_+(t) dt, 0 \right) = TM(\sin \alpha + \cos \alpha \cos \alpha - \sin \alpha, 0) \]

\[ = TM(\sqrt{2} \sin (\alpha + \frac{\pi}{4}), \sqrt{2} \cos (\alpha + \frac{\pi}{4}), 0) \]

(10)

In the general case where \(\alpha\) varies in time, we get

\[ \left( \int_0^T g_+(t) + g_-(t) dt, \int_0^T g_-(t) - g_+(t) dt, 0 \right) \]

\[ = \left( \int_0^T M \sin(\alpha(t)) + M \cos(\alpha(t)) dt, \int_0^T M \cos(\alpha(t)) - M \sin(\alpha(t)) dt, 0 \right) \]

\[ = \left( \sum_{i=1}^{T/\delta} (\sin(\alpha(t_i)) + \cos(\alpha(t_i))) \delta t, \sum_{i=1}^{T/\delta} (\cos(\alpha(t_i)) - \sin(\alpha(t_i))) \delta t, 0 \right) \]

(12)

This is just the convex combination of the points we ob-
tained by constant \(\alpha\), and lies in the interior of those points,
so all the boundary points are achieved by constant \(\alpha\).

So given a unitary matrix \(U \in SU(4)\), the minimal time
needed to generate \(U\) is the minimal \(T\) such that \(\exists \alpha \in [0, \frac{\pi}{2}]\),

\[ \hat{\theta}' \nabla,(T M(\sqrt{2} \sin (\alpha + \frac{\pi}{4}), \sqrt{2} \cos (\alpha + \frac{\pi}{4}), 0) \]

or

\[ \hat{\theta}' + \frac{\pi}{2}(-1,0,0)\nabla,(T M(\sqrt{2} \sin (\alpha + \frac{\pi}{4}), \sqrt{2} \cos (\alpha + \frac{\pi}{4}), 0) \]

Again taking CNOT, for example, the minimal time needed to
generate it is the minimal \(T\) such that

\[ \left( \frac{\pi}{4},0,0 \right) \nabla,(T M(\sqrt{2} \sin (\alpha + \frac{\pi}{4}), \sqrt{2} \cos (\alpha + \frac{\pi}{4}), 0) \]

holds for one \(\alpha \in [0, \frac{\pi}{2}]\). It is obvious that \(\alpha = \frac{\pi}{4}\) gives the
minimal time \(T = \frac{\pi M}{4} + 2M\). So to generate CNOT, we should take
\(g_+(t) = g_-(t) = \frac{\pi M}{4} + 2M\), turning on the coupling for a time \(\frac{\pi M}{4} + 2M\).

This control sequence generates the term \(\exp[-i \frac{\pi}{4} \sigma_z \otimes \sigma_z]\), which is locally equivalent to the CNOT gate.

III. CONCLUSION

To perform large-scale quantum information processing, it
is necessary to control the interactions between individual
qubits while retaining quantum coherence. To this end, su-
perconducting circuits allow for a high degree of flexibilitiy.
In this paper, we found the minimal coupling time required
to generate arbitrary two-qubit quantum gate in the super-
conducting quantum computing scheme. We reduced this
problem to the problem of simulating a desired Hamiltonian
using two Hamiltonians and single-qubit operations and de-
duced explicit forms to compute the minimal time needed
to control the system. The results of this work might be useful
for guiding the design of pulses in superconducting quantum
computing experiments. Minimizing the time taken to per-
form an operation is certainly helpful in the presence of finite
decoherence times. If one were able to take into account the
specific form and time dependence of the environmental cou-
pling, one might be able to devise more robust schemes.
While it lies outside the scope of the current paper, it might
be interesting to combine the time optimal control techniques
with methods that take advantage of correlations in the noise,
e.g., implementing refocusing sequences to counteract the
effects of 1/f noise [31,32]. Such optimization in the face of
the combination of different types of constraints is a chal-
enging problem in both classical and quantum control
theories.
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