Symmetric Groups and Expander Graphs

Martin Kassabov

March 29, 2022

Abstract

We construct explicit generating sets $S_n$ and $\tilde{S}_n$ of the alternating and the symmetric groups, which turn the Cayley graphs $C(\text{Alt}(n), S_n)$ and $C(\text{Sym}(n), \tilde{S}_n)$ into a family of bounded degree expanders for all $n$. This answers affirmatively an old question which has been asked many times in the literature. These expanders have many applications in the theory of random walks on groups, card shuffling and other areas.

Introduction

A finite graph is called an expander if for any (not too big) set of vertices there are many edges leaving this set. This implies that expander graphs are highly connected and have a small diameter. Such graphs have many practical applications, for example in construction of computer networks.

Using simple counting arguments it can be shown that the random $k$-regular graphs are expanders for $k \geq 5$. However these expanders are not sufficient for many applications where one needs explicit families of expander graphs. Constructing such examples is a difficult problem.

A natural candidate for a family of expanders are the Cayley graphs $C(G_i, S_i)$ of a sequence of finite groups $G_i$ with respect to some (suitably chosen) generating sets $S_i$. It is known that if there is a uniform bound for the size of the generating sets $S_i$ then the expanding properties of the Cayley graphs are related to the representation theory of the groups $G_i$, more specifically to their Kazhdan constants.

Using this connection G. Margulis in [29] gave the first explicit construction of a family of expanders, using the Kazhdan property $T$ of $\text{SL}_3(\mathbb{Z})$. Currently there are several different constructions of expanders using the representation theory of infinite groups — typically one finds a finitely generated infinite group $G$ with a ‘nice’ representation theory (usually the group has some variant of property $T$, property $\tau$, Selberg property etc.). In this case the Cayley graphs

\textit{2000 Mathematics Subject Classification}: Primary 20B30; Secondary 05C25, 05E15, 20C30, 20F69, 60C05, 68R05, 68R10.

\textit{Key words and phrases}: expanders, symmetric groups, alternating groups, random permutations, property T, Kazhdan constants.
of (some) finite quotients $G_i$ of $G$ with respect to the images of a generating set $S$ of the big group form an expander family. It is very interesting to ask when can one do the opposite — which leads to the following difficult problem (see [26]):

**Problem 1** Let $G_i$ be an infinite family of finite groups. Is it possible to make their Cayley graphs expanders using suitably chosen generating sets?

Currently there is no theory which can give a satisfactory answer to this question. The answer is known only in a few special cases: If the family of finite groups comes from a finitely generated infinite group with property $T$ (or its weaker versions) then the answer is YES.\(^1\) Also if all groups in the family are “almost” abelian then the answer is NO (see [23]) and this is essentially the only case where a negative answer of Problem 1 is known.

A natural family of groups which are sufficiently far from abelian is the family of all symmetric groups. The special case of Problem 1 for the symmetric groups, i.e., the existence of a generating sets which make their Cayley graphs expanders, is an old open question which has been asked several times in the literature, see [3, 25, 26, 28].

The asymptotic as $n \to \infty$ of Kazhdan constant of the symmetric group $\text{Sym}(n)$ with respect to some natural generating sets are known, see [5]. Unfortunately in all known examples the Kazhdan constant goes to zero as the size of the symmetric groups increase (even though in many cases the sizes of the generating sets are not bounded), which suggest that Problem 1 has a negative answer for the family of all symmetric groups.

On the other hand the symmetric group $\text{Sym}(n)$ can be viewed as a general linear group over “the field” with one element, see [12]. In [17], it is shown that the Cayley graphs of $\text{SL}_n(\mathbb{F}_p)$ for any prime $p$ and infinitely many $n$ can be made expanders simultaneously by choosing a suitable generating sets. Using the previous remark this presents a strong supporting evidence that Problem 1 has a positive answer.

The main result of this paper\(^2\) answers affirmatively Problem 1 in the case of alternating/symmetric groups.

**Theorem 2** For all $n$ there exists an explicit generating set $S_n$ (of size at most $L$) of the alternating group $\text{Alt}(n)$, such that the Cayley graphs $\mathcal{C}(\text{Alt}(n), S_n)$ form a family of $\epsilon$-expanders. Here, $L$ and $\epsilon > 0$ are some universal constants.

The proof uses the equivalence between family of expanders and groups with uniformly bounded Kazhdan constants. Using bounded generation and relative Kazhdan constant of some small groups, we can obtain lower bounds for the

---

\(^1\)The opposite is also true: For any infinite family of finite groups $G_i$ the existence of generating sets $S_i$ such that the Cayley graphs $\mathcal{C}(G_i, S_i)$ are a family of expanders is equivalent to the existence of a finitely generated subgroup of $\prod G_i$ which has a variant of property $T$ (more precisely property $\tau$ with respect to the induced topology from the product topology on $\prod G_i$).

\(^2\)This result was announced in [16].
Kazhdan constants of the symmetric groups Sym(n) with respect to several different generating sets. All these estimates rely on Theorem 1.6 whose proof uses upper bounds for the characters of the symmetric group and estimates of the mixing time of random walks.

Theorem 2 has many interesting applications: First, it provides one of the few constructions of an expander family of Cayley graphs C(Gi, Si) such that the groups Gi are not obtained as quotients of some infinite group having a variant of Kazhdan property T. The other constructions which do not use a variant of property T are based on an entirely different idea — the so called ‘zig-zag’ product of graphs, for details see [2, 30, 32, 35].

Second, the automorphism groups of the free group Aut(Fn) can be mapped onto infinitely many alternating groups, see [10]. This, together with Theorem 2, provides a strong supporting evidence for the conjecture that Aut(Fn) and Out(Fn) have property T. This conjecture if correct, will imply that the product replacement algorithm has a logarithmic mixing time, see [27] for details.

Third, Theorem 2 implies that for a fixed C, the expanding constant of Alt(n) with respect to the set SCn is large enough. The size of the set SCn is independent on n, and if n is sufficiently large then |SCn| < 10−30n1/30. The last inequality allows us to use the expander C(Alt(n); SCn) as a ‘seed’ graph for recursive construction of expanders suggested by E. Rozenman, A. Shalev and A. Widgerson in [35]. This will be one of the few constructions of an infinite family of expander graphs which are purely combinatorial, i.e., it does not use any representation theory. This construction produces a family of expander graphs from the automorphism groups of n-regular rooted tree of depth k. A slight modification of this construction gives another recursive expander family based on Alt(nk) for fixed large n and different k-s.

Theorem 2 implies the analogous result for the symmetric groups:

**Theorem 3** For all n there exists an explicit generating set ˜Sn (of size at most L) of the alternating group Sym(n), such that the Cayley graphs C(Sym(n), ˜Sn) form a family of ε-expanders. Here L and ε > 0 are some universal constants.

The rest of the paper is organized as follows: Section 1 contains definitions and a sketch of the proof of Theorem 1.6 which is a weaker version of Theorem 2. The detailed proof of this theorem is contained in Sections 2 and 3. Section 4 explains how Theorems 2 and 3 can be derived from Theorem 1.6. Section 5 concludes the paper with some comments about possible modifications and applications of Theorem 2.

---

3 As mentioned before if C(Gi, Si) are expanders, then there exists an infinite group with a variant of property T. The main point here is that we prove that the Cayley graphs are expanders without using the representation of this infinite group.

4 More precisely we have that the spectral gap of the normalized Laplacian of the Cayley graph is very close to 1.
Acknowledgements: I wish to thank Alex Lubotzky and Nikolay Nikolov for their encouragement and useful discussions during the work on this project. I am also very grateful to Yehuda Shalom and Efim Zelmanov for introducing me to the subject.

1 Outline

Let us start with one of the equivalent definitions of expander graphs:

Definition 1.1 A finite graph $\Gamma$ is called an $\epsilon$-expander for some $\epsilon \in (0, 1)$ if for any subset $A \subseteq \Gamma$ of size at most $|\Gamma|/2$ we have $|\partial(A)| > \epsilon|A|$. Here $\partial(A)$ is the set of vertices of $\Gamma \setminus A$ of edge distance 1 to $A$. The largest such $\epsilon$ is called the expanding constant of $\Gamma$.

Constructing families of $\epsilon$-expanders with a large expanding constant $\epsilon$ and bounded valency is an important practical problem in computer science, because the expanders can be used to construct concentrators, super concentrators, contractors and etc. For an excellent introduction to the subject we refer the reader to the book [25] by A. Lubotzky and to [22].

If we consider only graphs $\Gamma_i$ were the degree of each vertex is at most $k$, then all graphs $\Gamma_i$ are $\epsilon$-expanders for some $\epsilon$, if any of the following equivalent conditions holds:

5. The Cheeger constants of $\Gamma_i$ are uniformly bounded away from zero;
6. The Laplacian of $\Gamma_i$ have a uniformly bounded spectral gap;

In this case we also have that the lazy random walk on the graph $\Gamma_i$ mixes in $O(\log |\Gamma_i|)$ steps.

The graphs which appear in many applications arise from finite groups — they are Cayley with respect to some generating set or their quotients. In that case we have an additional equivalent condition — the Cayley graphs $C(G_i; S_i)$ of $G_i$, with respect to generating sets $S_i$ are $\epsilon$-expanders for some $\epsilon$ if and only if the Kazhdan constants $K(G_i; S_i)$ are uniformly bounded away from zero.

The original definition of Kazhdan property $T$ uses the Fell topology of the unitary dual of a group, see [21]. We are interested not only in property $T$, which automatically holds for any finite group, but also in the related notion of Kazhdan constants. The following definitions, which addresses the notion of the Kazhdan constants, are equivalent to the usual definitions of relative property $T$ and property $T$.

\footnote{If the degree of the graphs is not bounded, there are two different notions of expander graphs – one corresponding to Definition 1.1 and a bound of the Cheeger constant; and a second one, coming from a bound on the spectral gap of the Laplacian. In the rest of the paper we will use the second definition, which is more restrictive.}
Definition 1.2 Let $G$ be a discrete group generated by a finite set $S$ and let $H$ be a subset of $G$. Then the pair $(G, H)$ has relative property $T$ if there exists $\epsilon > 0$ such that for every unitary representation $\rho : G \rightarrow U(H)$ on a Hilbert space $H$ without $H$ invariant vectors and every vector $v \neq 0$ there is some $s \in S$ such that $||\rho(s)v - v|| > \epsilon||v||$. The largest $\epsilon$ with this property is called the relative Kazhdan constant for $(G, H)$ with respect to the set $S$ and is denoted by $K(G, H; S)$.

The group $G$ has Kazhdan property $T$ if the pair $(G, G)$ has relative $T$ and the Kazhdan constant for the group $G$ is $K(G; S) := K(G, G; S)$.

The property $T$ depends only on the group $G$ and does not depend on the choice of the generating set $S$, however the Kazhdan constant depends also on the generating set.

It is clear that any finite group $G$ has property $T$, because it has only finitely representations generated by a single vector. If the generating set $S$ contains all elements of the group $G$, then we have the inequalities

$$2 \geq K(G; G) \geq \sqrt{2}.$$ 

This follows from the following observation: if a unit vector $v \in H$ is moved by at most $\sqrt{2}$ by any element of the group $G$, then the whole orbit $Gv$ is contained in some half space and its center of mass is not zero. The $G$ invariance of the orbit gives that the center of mass is a non-zero $G$-invariant vector in $H$. However the resulting Cayley graphs are complete graphs with $|G|$ vertices and very ‘expensive’ expanders.

As mention before to prove Theorem 2, it is enough to prove that there exist generating sets, the Kazhdan constants of which are uniformly bounded away from 0. We will start with a similar result for $\text{SL}_n(F_p)$ for a fixed $n \geq 3$, which also proves that $\text{SL}_n(Z)$ has property $T$ and gives an estimate for the Kazhdan constant with respect to the generating set consisting of the elementary matrices.

The standard proofs of property $T$ for arithmetic groups, like $\text{SL}_n(Z)$, use the representation theory of Lie groups and are not quantitative, i.e., they do not lead to any estimate of the Kazhdan constants. Our approach is based on ideas from [36]: we start with the trivial estimate

$$K(\text{SL}_n(F_p); \text{SL}_n(F_p)) \geq \sqrt{2}$$

and will make several changes of the initial generating set $\text{SL}_n(F_p)$, which will decrease its size and keep some estimate of the Kazhdan constant. Finally we will end with a generating $S_{n,p}$ of size independent on $p$ such that $K(\text{SL}_n(F_p); S_{n,p})$ is bounded away from 0.

There are two propositions which allow us to estimate the Kazhdan constant if we change the generating set. The first one is a quantitative version of the fact that property $T$ for a discrete group does not depend on the generating set.

Proposition 1.3 Let $S$ and $S'$ be two finite generating sets of a group $G$, such that $S' \subset S^k$, i.e., all the elements of $S'$ can be written as short products of
elements form $S$. Then we have
\[ \mathcal{K}(G; S) \geq \frac{1}{k} \mathcal{K}(G; S'). \]

**Proof.** If $v$ is an $\epsilon$ almost invariant vector for the set $S$ then
\[
\|\rho(s_1 \ldots s_k)v - v\| = \left\| \sum_{j=1}^{k} \rho(s_1 \ldots s_{j-1}) (\rho(s_j)v - v) \right\| \leq \sum_{j=1}^{k} \|\rho(s_j)v - v\| \leq k\epsilon.
\]
This shows that $v$ is $k\epsilon$ almost invariant vector for the set $S^k$ and in particular for $S'$. If we start with $\epsilon = \frac{1}{k} \mathcal{K}(G; S')$ then $v$ is $\mathcal{K}(G; S')$ almost invariant vector for $S'$, which gives the existence of nonzero invariant vectors in $\mathcal{H}$. □

The second proposition uses relative property $T$ to enlarge the generating set by adding a subgroup to it:

**Proposition 1.4** If $H$ is a normal subgroup of a group $G$ generated by a set $S$ then:
\[ \mathcal{K}(G; S) \geq \frac{1}{2} \mathcal{K}(G, H; S) \mathcal{K}(G; S \cup H). \]

**Proof.** Let $\rho : G \to U(\mathcal{H})$ is a unitary representation and let $v$ be $\epsilon$ almost invariant vector for the set $S$. We can write $\mathcal{H} = \mathcal{H}^\parallel \oplus \mathcal{H}^\perp$, where $\mathcal{H}^\parallel$ is the space of all $H$ invariant vectors in $\mathcal{H}$ and $\mathcal{H}^\perp$ is its orthogonal complement. These spaces are $G$-invariant because $H$ is a normal subgroup. This decomposition gives $v = v^\parallel + v^\perp$, where both components are $\epsilon$-almost invariant vectors. However there are no $H$ invariant vectors in $\mathcal{H}^\perp$ and relative property $T$ of $(G, H)$ implies
\[
||\rho(s)v^\perp - v^\perp|| \geq \mathcal{K}(G, H; S)||v^\perp||
\]
for some $s \in S$, which implies that $||v^\perp|| \leq \epsilon \mathcal{K}(G, H; S)^{-1}$. Thus, for any $h \in H$ we have
\[
||\rho(h)v - v|| = ||\rho(h)v^\perp - v^\perp|| \leq 2||v^\perp|| \leq 2\epsilon \mathcal{K}(G, H; S)^{-1}.
\]
If we start with $\epsilon = \frac{1}{k} \mathcal{K}(G, H; S) \mathcal{K}(G; S \cup H)$ then the above inequality gives that $v$ is $\mathcal{K}(G; S \cup H)$-almost invariant for both $H$ and $S$ therefore there exists an invariant vector in $\mathcal{H}$. □

In practice one often uses a variant of this proposition for several subgroup $H_i$ simultaneously.

**Proposition 1.5** a) Let $H_i$ and $N_i$ be subgroups of a group $G$ such that $H_i \triangleleft N_i < G$ and $N_i$ is generated by $S_i$. If $\mathcal{K}(N_i, H_i; S_i) \geq \alpha$ and $S_i \subset S$ for all $i$ then
\[ \mathcal{K}(G; S) \geq \frac{1}{2} \alpha \mathcal{K}(G; \cup H_i). \]
Let $N$ be a group generated by a set $S$ and let $H \triangleleft N$. If $\pi_i$ are homomorphisms from $N$ in a group $G$, then
\[
\mathcal{K}(G; \cup \pi_i(S)) \geq \frac{1}{2} \mathcal{K}(N, H; S) \mathcal{K}(G; \cup \pi_i(H)).
\]

Using these simple propositions Y. Shalom was able to estimate the Kazhdan constant of $\text{SL}_n(\mathbb{Z})$ with respect to the set $S$ of all elementary matrices with $\pm 1$ off the diagonal. Let $EM_{i,j}$ denote the subgroup of elementary matrices of the form $\text{Id} + ne_{i,j}$ and let $EM = \bigcup_{i\neq j} EM_{i,j}$.

The relative Kazhdan constant of the pair $(\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ was estimated by Burger, see also [9], to be:
\[
\mathcal{K}(\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; S) \geq 1/10,
\]
where $S$ is the set consisting of the 4 elementary matrices in $\text{SL}_2(\mathbb{Z})$ together with the standard basis vectors of $\mathbb{Z}^2$ and their inverses. The proof is a quantitative version of the fact that there are only a few $\text{SL}_2(\mathbb{Z})$ invariant measures on the torus $\mathbb{R}/\mathbb{Z}^2$.

Using embeddings $\pi_{i,j}$ of $\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ in $\text{SL}_n(\mathbb{Z})$ such that the image of $\mathbb{Z}^2$ under $\pi_{i,j}$ contains $E_{i,j}$, by Proposition 1.5 we have
\[
\mathcal{K}(\text{SL}_n(\mathbb{Z}), S) \geq \frac{1}{2} \mathcal{K}(\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; S) \mathcal{K}(\text{SL}_n(\mathbb{Z}); \cup EM_{i,j}) \geq \frac{1}{20} \mathcal{K}(\text{SL}_n(\mathbb{Z}), EM).
\]

The next step in the proof uses bounded generation of $\text{SL}_n(\mathbb{Z})$ with respect to the set of the elementary matrices. This implies that
\[
\mathcal{K}(\text{SL}_n(\mathbb{Z}); EM) \geq \frac{1}{N} \mathcal{K}(\text{SL}_n(\mathbb{Z}); EM^N) = \frac{1}{N} \mathcal{K}(\text{SL}_n(\mathbb{Z}); SL_n(\mathbb{Z})) \geq \frac{\sqrt{2}}{N},
\]
where $N \geq \frac{4}{3} n^2 + 60$.

Combining these two inequalities one obtains a lower bound for the Kazhdan constant
\[
\mathcal{K}(\text{SL}_n(\mathbb{Z}), S) > \frac{1}{30n^2 + 1200}.
\]
In particular, this implies that there is a bound for the Kazhdan constants $\mathcal{K}(\text{SL}_n(F_p); S)$ which is independent on $p$, implying that the Cayley graphs $\mathcal{C}(\text{SL}_n(F_p); S)$ of the finite groups $\text{SL}_n(F_p)$ are expanders.\footnote{The bounded generation of $\text{SL}_n(\mathbb{Z})$ is a deep result due to Carter and Keller, see \cite{4} and \cite{3}. For our purposes we need only the Kazhdan constant of the finite groups $\text{SL}_n(\mathbb{Z}/s\mathbb{Z})$ and we need bounded generation of these groups, which follows immediately from the standard row reduction algorithm. Similar result is also valid for the finite quotients of the groups $\text{SL}_n(\mathbb{Z}[x_1, \ldots, x_k])$, although the bounded generation for these groups is not known, see \cite{13}.}

\footnote{In \cite{13}, it is shown that $(\mathcal{K}(\text{SL}_n(\mathbb{Z}), S))^{-1} = O(\sqrt{n})$. This yields an asymptotically exact estimate for the expanding constant of the Cayley graphs of $\text{SL}_n(F_p)$ with respect to the set of all elementary matrices.}
Using relative property $T$ of the pair $\text{SL}_2(R) \rtimes R^2, R^2$ for finitely generated noncommutative rings $R$, the Cayley graphs of $\text{SL}_n(\mathbb{F}_q)$ for any prime power $q$ and infinitely many $n$ can be made expanders simultaneously by choosing a suitable generating set, see [17]. An important building block in this construction is that the group $\text{SL}_n(\mathbb{F}_q)$ can be written as a product of 20 abelian subgroups and this number is independent on $n$ and $q$.

These methods can not be applied to the symmetric/alternating groups. Most of the estimates for the relative Kazhdan constant $\mathcal{K}(G, H; S)$ use that there are no invariant measures on the dual $\hat{H}$ of the group $H$ under the action of the normalizer of $H$ in $G$. The quantitative versions of this fact are known only if $\hat{H}$ is well understood, which is the case only if the subgroup $H$ is (or is very close to) an abelian subgroup.

If we start with finite generating sets $S_n$ of bounded size of the alternating groups $\text{Alt}(n)$, then we can find only finitely many abelian groups $H_n$ (the number depending only on the size of the generating set $S_n$) such that the relative Kazhdan constants can be estimated easily. Thus would allow us to bound $\mathcal{K}(\text{Alt}(n); S_n)$ with $\mathcal{K}(\text{Alt}(n); E_n)$, where $E_n$ is a union of a bounded number of abelian subgroups. We can use Proposition 1.3 to estimate $\mathcal{K}(\text{Alt}(n); E_n)$ if $E_n = \text{Alt}(n)$ for some $k$, i.e., if each alternating group $\text{Alt}(n)$ is a product of a fixed number of abelian subgroups.

However, the finite symmetric/alternating groups do not have this property — the size of $\text{Sym}(n)$ or $\text{Alt}(n)$ is approximately $n^n$ and every abelian subgroup has no more than $2^n$ elements, thus one needs at least $\ln n$ subgroups. This suggests that $\text{Alt}(n)$ are “further from the abelian groups” than all other finite simple groups, and therefore they should have more expanding properties. Unfortunately, this also significantly complicates the construction of expanders based on the alternating groups, because the above method can not be applied without significant modifications.

The main idea of the proof of Theorem 2 is a modification of the above method — the difference is that we are looking for a set $E_n$, which is a union of finitely many abelian subgroups in $\text{Sym}(n)$, such that the Kazhdan constants $\mathcal{K}(\text{Sym}(n); E_n)$ are uniformly bounded, even though there is no $k$ such that $E_n = \text{Sym}(n)$ for all $n$. There is a natural construction of the set $E_n$ if $n$ has a specific form. If $n = (2^{3s} - 1)^6$ for some $s$, we shall construct a group $\Delta$ generated by a set $S$ and an abelian subgroup $\Gamma$ in it, such that the relative Kazhdan constants $\mathcal{K}(\Delta, \Gamma; S)$ are bounded away from 0. There are several natural embeddings $\pi_i : \Delta \to \text{Alt}(n)$ and the Kazhdan constant $\mathcal{K}(\text{Alt}(n); \cup \pi_i(\Gamma))$ can be estimated. These two bounds give us a bound for $\mathcal{K}(\text{Alt}(n); \cup \pi_i(S))$, which will prove the following:

**Theorem 1.6** If $N = (2^{3s} - 1)^6$ for some $s \geq 6$ there exists a generating set $S_N$ (of size at most 200) of the alternating group $\text{Alt}(N)$, such that the Cayley graphs $C(\text{Alt}(N), S_N)$ form a family of $\epsilon$-expanders. Here $\epsilon$ is a universal constant.

---

It is interesting that the full symmetric group on an infinite set can be written as a product of 250 abelian subgroups, see [1].
Sketch of the proof of Theorem 1.6. This sketch explains the main idea of the proof — the complete proof is in Sections 2 and 3. We will think that the alternating group \( \text{Alt}(N) \) acts on a set of \( N \) points which are arranged into \( d = 6 \) dimensional cube of size \( K = 2^{3s} - 1 \). The group \( \Gamma \) is a direct product of \( K^{d-1} = K^5 \) cyclic groups of order \( K \). This group can be embedded in \( \text{Alt}(N) \) in \( d \) different ways as follows: the image of each cyclic group in \( \Gamma \) under \( \pi_i \) permutes the points on a line, parallel to the \( i \)-th coordinate axis. An other way to think about \( E_i = \pi_i(\Gamma) \) is as part of the subgroup of \( \text{Alt}(N) \) which preserves all coordinates but the \( i \)-th one.

The group \( \Gamma \) can be embedded in a group \( \Delta \) generated by a set \( S \), such that the embeddings \( \pi_i \) can be extended to \( \Delta \). The group \( \Delta \) is a product of many copies of \( \text{SL}_3(F_2) = \text{EL}_3(\text{Mat}_2(F_2)) \) and can be viewed as \( \text{EL}_3(R) \), where the ring \( R \) is a product of many copies of the matrix ring \( \text{Mat}_2(F_2) \). An important observation is that the ring \( R \) has a generating set whose size is independent on \( s \). Using results from [17], mainly the relative Kazhdan constant

\[
\mathcal{K}(\text{EL}_2(\mathbb{Z}(x_1, \ldots, x_k)) \ltimes \mathbb{Z}(x_1, \ldots, x_k)^2, \mathbb{Z}(x_1, \ldots, x_k)^2; F)
\]

for some set \( F \). This allows us to obtain an estimate for \( \mathcal{K}(\Delta; \overline{S}) \)

\[
\mathcal{K}(\Delta, \Gamma; \overline{S}) \geq \mathcal{K}(\Delta; \overline{S}) \geq \frac{1}{550}
\]

(1)

and to compare \( \mathcal{K}(\text{Alt}(N); \cup \pi_i(\overline{S})) \) and \( \mathcal{K}(\text{Alt}(N); \cup \pi_i(\Gamma)) \).

We will finish the proof of Theorem 1.6 using Theorem 3.1 from Section 3 which gives us that

\[
\mathcal{K}(\text{Alt}(N); \cup E_i) \geq \frac{1}{70},
\]

(2)

where \( E_i = \pi_i(\Gamma) \), provided that \( s > 6 \).

9 The proof of this inequality uses directly the representation theory of the alternating group. Let \( \rho \) be a unitary representation of \( \text{Alt}(N) \) in a Hilbert space \( \mathcal{H} \) and let \( v \in \mathcal{H} \) be \( \epsilon \)-almost invariant vector with respect to the set \( E \). We want to prove that if \( \epsilon < 1/70 \) then \( \mathcal{H} \) contains an invariant vector.

First, we will split the representation \( \rho \) into two components — one corresponding to partitions \( \lambda \) with \( \lambda_1 < N - h \) and a second one, containing all other partitions (here \( h \) depends on \( N \) and will be determined later). This decomposition of the representation \( \rho \) into two components \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), using the first part of the partition \( \lambda \) borrows ideas from [34]. There, Y. Roichman uses a similar argument to show that the Cayley graphs of the alternating group with respect to a conjugacy class with a large number of non-fixed points have certain expanding properties.

First we shall prove that the projection \( v_1 \) of the vector \( v \) in the first component \( \mathcal{H}_1 \) is small provided that \( h \gg K \ln K \). It can be shown that for a fixed \( k \)

\[
\text{If } s \leq 6, \text{ using the same methods we can also obtain a bound for the Kazhdan constant like } \mathcal{K}(\text{Alt}(N); \cup \pi_i(\Gamma)) \geq 1/1000, \text{ but this requires more careful analysis.}
\]
the ball $E^k$ almost contains an entire conjugancy class $C$ containing all cycles of length approximately $K^{d-1/3} \ln K$. Using estimates for the values of characters of the symmetric group we can show that if $h \gg K \ln K$ then

$$\left\| \frac{1}{|C|} \sum_{g \in C} gv_1 \right\| \ll ||v_1||.$$  

The vectors $v$ and $v_1$ are almost invariant with respect to the set $\cup \pi_i(\Gamma)$ and therefore with respect to $C$. This implies that

$$\left\| \frac{1}{|C|} \sum_{g \in C} gv_1 - v_1 \right\| \leq 47 \epsilon. \quad (3)$$

The two inequalities gives that $||v_1|| \leq 47 \epsilon + 0.07$.

That projection $v_2$ of $v$ in the second component $H_2$ is close to an invariant vector if $h < K^{d/4}$. Here the idea is that the space $H_2$ can be embedded into a vector space with a basis $B$ consisting all ordered tuples of size $h$. It can be shown that the mixing time of the random walk on the set $B$, generated by $E$ has a small mixing time — this is possible in part because $|E| \gg |B|$, but the proof uses the specific structure of the set $E$. This implies that the vector

$$U^Kv_2, \quad \text{where} \quad U = \frac{1}{|E|} \sum_{g \in E} g$$

is very close to some invariant vector $v_0$. On the other hand this vector is close to $v_2$, therefore

$$||v_2 - v_0|| \leq 16 \epsilon. \quad (4)$$

Equations (3) and (4) were obtained using the assumptions $h \gg K \ln K$ and $h < K^{d/4}$ — these restrictions can be satisfied only if $K^{d/4} \gg K \ln K$, i.e., $d > 4$. In order to simplify the argument we require that $d$ is even, which justifies our choice of $d = 6$ and $N = K^6$. In this case, $h$ have to satisfy the inequalities $K \ln K \ll h < K^{3/2}$, therefore we chose $h = \frac{1}{2}K^{3/2}$ and define $H_1$ to be the sub-representation of $H$ corresponding to all partitions with $\lambda_1 < N - \frac{1}{2}K^{3/2}$.

Equations (3) and (7) give us that

$$||v - v_0|| \leq 63 \epsilon + 0.07.$$  

In particular this implies that $v_0 \neq 0$ if $\epsilon$ is small enough and the representation $H$ contains an invariant vector and finishes the proof of Theorem 1.6.1.

Finally equations (4) and (2) together with proposition 1.5 imply that

$$\mathcal{K}(\text{Alt}(N) \cup \pi_i) \geq \frac{1}{2} \mathcal{K}(\text{Alt}(N); \cup \pi_i) \mathcal{K}(\Delta, \Gamma; \bar{S}) > 10^{-5},$$

which completes the proof of Theorem 1.6 modulo the results in Sections 2 and 3. In section 4 we derive Theorems 2 and 3 from Theorem 1.6.
2 The groups $\Gamma$ and $\Delta$

As mentioned before, we will think that the alternating group $\text{Alt}(N)$ acts on a set of $N$ points which are arranged into a 6-dimensional cube of size $K = 2^{3s} - 1$ and we will identify these points with ordered 6-tuples of nonzero elements from the field $\mathbb{F}_{2^{s}}$. For the rest of this section we will assume that $s$, $K$ and $N$ are fixed and that $s > 6$. For any associative ring $R$, $\text{EL}_3(R)$ denotes the subgroup of the $3 \times 3$ invertible matrices with entries in $R$, generated by all elementary matrices.

Let $H$ denote the group $\text{SL}_3(\mathbb{F}_2) = \text{EL}_3(\text{Mat}_s(\mathbb{F}_2))$. The group $H$ has a natural action on the set $V \setminus \{0\}$ of $K$ nonzero elements of a vector space $V$ of dimension $3s$ over $\mathbb{F}_2$. The elements of $H$ act by even permutations on $V \setminus \{0\}$, because $H$ is a finite simple group and does not have $\mathbb{Z}/2\mathbb{Z}$ as a factor.

We can identify $V$ with the field $\mathbb{F}_{2^{3s}}$ — the existence of a generator for the multiplicative group of $\mathbb{F}_{2^{3s}}$ implies that some element of $H = \text{GL}_3(\mathbb{F}_2)$ acts as a $K$-cycle on $V \setminus \{0\}$, because we work over a field characteristic 2. This requirement is not necessary and can be avoided if we modify the proof of Theorem 3.1 for other possible choices of the group $H$, which also give bounded degree expanders in infinitely many alternating groups see Section 5.

Let $\Delta$ be the direct product of $K^{d-1}$ copies of the group $H$. The group $\Delta$ can be embedded into $\text{Alt}(N)$ in 6 different ways which we denote by $\pi_i$, $i = 1, \ldots, d$. The image of each copy of $H$ under $\pi_i$ acts as $\text{SL}_3(\mathbb{F}_2)$ on a set of $K = 2^{3s} - 1$ points where all coordinates but the $i$th one are fixed. The existence of an element of order $K$ in $H$ implies that $\Delta$ contains an abelian subgroup $\Gamma$ isomorphic to $(\mathbb{Z}/K\mathbb{Z}) \times K^{d-1}$.

Another way to think of this group is as follows — $\Delta$ is a product of copies of $\text{SL}_3(\mathbb{F}_2)$, i.e.,

$$
\Delta \simeq (\text{SL}_3(\mathbb{F}_2))^d \simeq \text{EL}_3(\text{Mat}_s(\mathbb{F}_2))^d \simeq \text{EL}_3(\text{Mat}_s(\mathbb{F}_2)^d),
$$

i.e., $\Delta \simeq \text{EL}_3(R)$ where $R$ denotes the product of $K^{d-1}$ copies of the matrix ring $\text{Mat}_s(\mathbb{F}_2)$.

**Lemma 2.1** For any $s$ the ring $R$ is generated by $2 + \lceil 3(d-1)/s \rceil \leq 5$ elements.

**Proof.** The matrix algebra $\text{Mat}_s(\mathbb{F}_2)$ can be generated as a ring by 1 and two elements $\bar{\alpha}$ and $\bar{\beta}$, for example we can take $\bar{\alpha} = e_{2,1}$ and $\bar{\beta} = \sum e_{i, i+1}$. By construction, the ring $R$ is

$$
R = \text{Mat}_s(\mathbb{F}_2)^d
$$

We can think that the copies of $\text{Mat}_s(\mathbb{F}_2)$ are indexed by tuples of length

$$
t = \lceil \log_{\text{Mat}_s(\mathbb{F}_2)} K \rceil = \left\lceil \frac{(d-1) \log_2 K}{s^2} \right\rceil \leq \left\lceil \frac{3(d-1)}{s} \right\rceil.
$$
of elements in Mat$_s$($F_2$). Let us define the elements $\alpha$, $\beta$ and $\gamma_i$, $i = 1, \ldots, t$ in Mat$_s$($F_2$)$^{\times K_5}$ as follows: All components of $\alpha$ and $\beta$ are equal to $\bar{\alpha}$ and $\bar{\beta}$, the components of $\gamma_i$ in the copy of Mat$_s$($F_2$) indexed by $(p_1, \ldots, p_t)$ is equal to $p_i$. It is no difficult to show that the elements $\alpha$, $\beta$ and $\gamma_i$ generate Mat$_s$($F_2$)$^{\times K_{d-1}}$ as associative ring. □

Any generating set of a ring $R$ gives a generating set of the group EL$_3(R)$:

Corollary 2.2 For any $s$ the group $\Delta$ can be generated by a set $\bar{S}$ consisting of $18 + 6\lceil 3(d-1)/s \rceil \leq 36$ involutions (elementary matrices in EL$_3(R)$).

Proof. Using the definition of EL$_3(R)$ we can see that if the ring $R$ is generated by $\alpha_k$ then the group EL$_3(R)$ is generated by the set

$$\bar{S} = \{\text{Id} + e_{i,j} \mid i \neq j\} \cup \{\text{Id} + \alpha_k e_{i,j} \mid i \neq j\}.$$ 

The set $\bar{S}$ consists of involutions because the ring $R$ has characteristic 2. □

The main result in this section is:

Theorem 2.3 The Kazhdan constant of the group $\Delta$ with respect to the set $\bar{S}$ is

$$\mathcal{K}(\Delta; \bar{S}) \geq \frac{1}{550}.$$ 

Proof. The proof uses bounded generation of EL$_3(R)$ and the following Theorem 2.4 from [17] (see also Theorem 3.4 in [36]):

Theorem 2.4 Let $R$ be an associative ring generated by $1, \alpha_1, \ldots, \alpha_t$. Let $F_1$ be set of $4(t+1)$ elementary matrices in EL$_2(R)$ with $\pm 1$ and $\pm \alpha_i$ off the diagonal and $F_2$ be the set of standard basis vectors in $R^2$ and their inverses. Then the relative Kazhdan constant of the pair (EL$_2(R) \ltimes R^2, R^2$) is

$$\mathcal{K}(\text{EL}_2(R) \ltimes R^2, R^2; F_1 \cup F_2) \geq \frac{1}{\sqrt{18(\sqrt{t} + 3)}}.$$ 

Proof. This is only a short sketch of proof, for details the reader is referred to [17] or [36] in the case of commutative ring. We may assume that the ring $R$ is the free associative ring generated by $\alpha_i$. For any unitary representation $\rho : \text{EL}_2(R) \ltimes R^2 \rightarrow U(\mathcal{H})$ and a unit vector $v \in \mathcal{H}$ we can construct a measure $\mu_v$ on the dual $\widehat{R^2}$.

If the vector $v$ is an almost invariant under the set $F_1$ then we have

$$|\mu_v(B) - \mu_v(gB)| \ll 1,$$

for any Borel set $B \subset \widehat{R^2}$ and any $g \in F_1$. This show that $\mu_v$ is almost invariant measure on $\widehat{R^2}$. The vector $v$ is almost invariant for $F_2$ therefore $\mu_v(B_i) \ll 1$ for some specific Borel sets $B_i$. It is known that there are only few EL$_2(R)$
invariant measures on $\hat{R}^2$ and almost all of them give large measures to the sets $B_i$. It can be shown that the above inequalities imply that $\mu(\{0\}) > 0$. By definition $\mu(\{0\})$ is square of the length of the projection of $v$ onto the subspace of $H$ invariant vectors. Thus $\mu(\{0\}) > 0$ implies the existence of $H$ invariant vectors in $\mathcal{H}$.

Using 6 different embeddings of $\text{EL}_2(R) \ltimes R^2$ into $\text{EL}_3(R)$ we can obtain the following implication of the above theorem:

$$\mathcal{K}(\Delta; \bar{S}) \geq \frac{1}{2} \mathcal{K}(\text{EL}_2(R) \ltimes R^2; R^2; F_1 \cup F_2) \mathcal{K}(\Delta; \text{GEM}) \geq \frac{1}{6\sqrt{2}(3 + \sqrt{5})} \mathcal{K}(\Delta; \text{GEM})$$

(5)

where $\text{GEM}$ is the set of all generalized elementary matrices in $\text{EL}_3(R)$, i.e., the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

up to permuting the rows and the columns.

**Lemma 2.5** Any element $g$ from the group $\Delta$ can be written as a product of 17 elements from the $\text{GEM}$, i.e., we have $\Delta = \text{GEM}^{17}$.

**Proof.** Let $g \in \text{EL}_3(R)$. With three additional left multiplications by GEMs we can transform $g$ to a $3 \times 3$ block matrix where the last column is trivial, with an extra 3 left multiplications by GEMs we can make the second column trivial. Finally with one GEM we can transform $g$ to a matrix which differs form the identity only in the top-left corner, i.e., we have:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \xrightarrow{3} \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} * & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ \[\] The entry in the top-left corner is an element in $\text{SL}_3(\mathbb{F}_2)^{\times^{K^{d-1}}}$ and thus is a group commutator of two invertible elements in $R$. As such it can be written as a product of 10 generalized elementary matrices in $\text{EL}_3(R)$. This shows that every matrix in $\text{SL}_3(R)$ can be written as a product of 17 generalized elementary matrices which are in the set $\text{GEM}$. \[\]

**Remark 2.6** This proof works only for finite rings $R$, which have the property that any element in $[R^*, R^*]$ is a commutator of two elements in the multiplicative group $R^*$.

This together with proposition [1.3] implies
Corollary 2.7 We have
\[ K(\Delta; GEM) \geq \sqrt{\frac{2}{17}}. \]

The above corollary together with equation 5 imply
\[ K(\Delta; \bar{S}) \geq \frac{1}{6\sqrt{2(3 + \sqrt{5})}} K(\Delta; GEM) \geq \frac{1}{6(3 + \sqrt{5})} \times 17 > \frac{1}{550}, \]
which finishes the proof of Theorem 2.3. □

Using the embedding \( \pi_i \) and proposition 1.5 we obtain:
\[ K(\text{Alt}(N); \cup \pi_i(\bar{S})) > \frac{1}{1100} K(\text{Alt}(N); \cup \pi_i(\Gamma)). \]

3 Representations of Alt(\( N \))

In this section we will use the same notation as in Section 2. Let \( N = K^d \) for some odd \( K \) and \( d = 6 \). We will think that Alt(\( N \)) acts on the points in a \( d \) dimensional cube of size \( K \). Let \( \Gamma \) denote the group \((\mathbb{Z}/K\mathbb{Z})^d \). The group \( \Gamma \) has \( d \) embeddings \( \pi_i \) in Alt(\( N \)). The image of each cyclic subgroup under \( \pi_i \) shifts the points on some line parallel to the \( i \)-th coordinate axis. Let \( E_i = \pi_i(\Gamma) \) denote the images of \( \Gamma \) and let \( E \) be the union
\[ E = \bigcup \pi_i(\Gamma). \]

The main result in this section is the following:

**Theorem 3.1** The Kazhdan constant of Alt(\( N \)) with respect to the set \( E \) satisfies
\[ K(\text{Alt}(N); E) \geq \frac{1}{70}, \]
provided that \( K \) is odd and \( K > 10^6 \).

**Remark 3.2** The proof of Theorem 3.1 gives that
\[ \lim \inf K(\text{Alt}(N); E) \geq \frac{1}{60} \quad \text{as} \quad K \to \infty \]
and the proof of this statement is slightly easier because we can ignore many terms which tend to 0 as \( K \to \infty \). If \( K \ll 10^6 \) this method gives very weak bounds for the Kazhdan constant, however we believe that Theorem 3.1 also holds for small \( K \).

**Proof.** Let \( \rho : \text{Alt}(N) \to U(\mathcal{H}) \) be a unitary representation of the alternating group, and let \( v \in \mathcal{H} \) be \( \epsilon \)-almost invariant unit vector for the set \( E \). We will
show that $H$ contains an invariant vector if $\epsilon$ is small enough. Without loss of
generality we may assume that $H$ is generated by $v$ as an $\text{Alt}(N)$ module.

The irreducible representations of the symmetric group $\text{Sym}(N)$ are parameter-
ized by the partitions $\lambda$ of $N$. Almost the same is true the representations
of the alternating group, but the correspondence in this case is not 1-to-1,
see [11, 13]. We are going to avoid this problem by inducing the representation
$\rho$ to the symmetric group and working with the induced representation

$$\rho^s = \text{Ind}^\text{Sym}(N)_{\text{Alt}(N)} \rho : \text{Sym}(N) \to U(\mathcal{H}^s).$$

Without loss of generality we can assume that $H \subset H^s$, then $H^s$ is also gener-
ated, as $\text{Sym}(N)$-module, by the vector $v$. We can decompose $H^s$ as

$$H^s = \bigoplus \mathcal{H}_\lambda,$$

where $\mathcal{H}_\lambda$ is the sum of all irreducible components in $H$ which correspond to
the partition $\lambda$. We can group these terms in three parts and break $\rho^s$ as a sum
of three representations

$$H^s = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where

$$\mathcal{H}_1 = \bigoplus_{\lambda_1 < N - h}^{\lambda_1' < N - h} \mathcal{H}_{\lambda}, \quad \mathcal{H}_2 = \bigoplus_{\lambda_1 \geq N - h} \mathcal{H}_{\lambda}, \quad \mathcal{H}_3 = \bigoplus_{\lambda_1' \geq N - h} \mathcal{H}_{\lambda},$$

where $\lambda'$ denotes the dual partition and $h = \frac{1}{2}K^{3/2}$. The action of $\text{Alt}(N)$ on
$\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ is the same for any partition $\lambda$, thus $\mathcal{H}_2$ and $\mathcal{H}_3$ are isomorphic
as representations of the alternating group. Without loss of generality we may
assume that the unit vector $v = v_1 + v_2$ has components only in $\mathcal{H}_1$ and $\mathcal{H}_2$.

In this section we will use several probability arguments, which are based on
the following lemma:

**Lemma 3.3** Suppose there we have $lk$ urns grouped in $l$ boxes of size $k$. If we
put $p$ balls in these urns such that different balls go to different urns, then the
probability of having at least $q$ balls in the first box is less than

$$P(l, k, p, q) = \binom{p}{q} \left( \frac{k}{lk - p} \right)^q \leq \left( \frac{cp}{ql} \right)^q \exp \left( \frac{qp}{kl} - p \right).$$

If we drop the restriction that different balls go to different urns then we can
omit the exponential factor from the above estimate.

**Proof.** Since different balls have to go to different urns, the relative probability
that some ball ends in the first box is not exactly $1/l$, but it is clear that this relative probability is less than $\frac{1}{lk - p}$. This allows us to bound the number of
balls in the first box using the binomial distribution with $\lambda = \frac{k}{lk - p}$. Let

$$B_{p, \lambda}(x) = (1 - \lambda + \lambda x)^p = \sum b_i x^i$$
denote the generating function of the binomial distribution with parameters $\lambda$ and $p$. It is clear that the probability of having at least $q$ balls in the first box is less than

$$P = \sum_{i \geq q} b_i.$$ 

We have that

$$\frac{1}{q!} \frac{d^q B_{p, \lambda}(x)}{dx^q} = \sum_{i \geq q} \left( \frac{p}{q} \right) b_i x^{i-q} \geq \sum_{i \geq q} b_i x^{i-q},$$

i.e., $P \leq \frac{1}{q!} B_{p, \lambda}(1)^{(q)}$. The explicit formula for $B_{p, \lambda}(x)$ gives us

$$P \leq \frac{1}{q!} \left. \frac{d^q (1 - \lambda + \lambda x)^p}{dx^q} \right|_{x=1} = \left( \frac{p}{q} \right) \lambda^q,$$

which implies that

$$P(l, k, p, q) = \left( \frac{p}{q} \right) \left( \frac{k}{lk-p} \right)^q.$$ 

The second inequality follows from the estimates

$$\left( \frac{p}{q} \right) \leq \frac{p^q}{q!} \leq \frac{p^q}{(q/e)^q} = \left( \frac{ep}{q} \right)^q$$

and

$$\frac{lk}{lk-p} = 1 + \frac{p}{lk-p} < \exp \left( \frac{p}{lk-p} \right).$$

□

Next we prove the following theorem, which together with estimates for the values of the characters of the symmetric group from [33], implies that the component $v_1$ of $v$ is short:

**Theorem 3.4** The ball $E^{47}$ of radius 47 generated by the set $E$ contains almost all cycles in $\text{Alt}(N)$ of length approximately $K^{d-1}/3 \ln K$, i.e.,

$$\left| C_L \setminus E^{47} \right| \leq \frac{1}{100} |C_L|,$$

where $C_L$ is the conjugacy class in $\text{Alt}(N)$ of all cycles of length $L$, which is the largest integer of the form $1 + a(K - 1)$ less than $K^{d-1}/3 \ln K$.

**Proof.** Let $M$ denote the set of points in the $d$ dimensional cube with first coordinate equal to 1.

**Lemma 3.5** Let us choose $L$ distinct points in the $d$ dimensional cube of size $K$. With high probability (more than .99) we can move these points to the set $M$ using only 2 elements from $E$, provided that $K \geq 10^6$. 

16
**Proof.** The probability of having at least \(K/\ln K\) points, which differ only in the first or second coordinate is less than

\[
P_1 = K^{d-2} \times P\left(K^{d-2}, K^2, \frac{K^{d-1}}{3\ln K}, \ln K\right) \leq K^{d-2} \left(\frac{e^{K^{d-1}}}{K^{d-2} \times \ln K}\right) \exp\left(\frac{K^{d-1}}{3\ln K} \times \ln K\right) \leq K^{d-2} \left(\frac{e}{3}\right)^{\ln K} \exp\left(\frac{1}{10}\right) \leq \exp(-70),
\]

where the final inequality holds only if \(K\) is large enough. Also the probability of having at least \(\ln K\) points which differ only in the second coordinate is less than

\[
P_2 = K^{d-1} \times P\left(K^{d-1}, K, \frac{K^{d-1}}{3\ln K}, \ln K\right) \leq K^{d-1} \left(\frac{e^{K^{d-1}}}{K^{d-1} \times \ln K}\right) \exp\left(\frac{K^{d-1}}{3\ln K} \times \ln K\right) \leq K^{d-1} \left(\frac{e}{3(\ln K)^2}\right)^{\ln K} \exp\left(\frac{1}{2K}\right) \leq K^{-0.35}
\]

The above inequalities imply that almost surely (with probability more than \(1 - P_1 - P_2 > .99\)) there are no more than \(\ln K\) points in each line parallel to the second coordinate axis and no more than \(K/\ln K\) points in each square parallel to the first and second coordinate axis.

We claim that if we have a set \(b\) of \(L\) points in such good position then there exists \(g \in E_2\) such that there are no two points in \(gb\) which differ only in the first coordinate.

Such an element \(g\) can be constructed as follows: Let us enumerate the lines parallel to the second coordinate axis and let \(b_i\) be the subset of \(b\) consisting of all points on the first \(i\) lines. We will prove by induction that there exists \(g_i\) such that there are no two points in \(g_ib_i\) which differ only by the first coordinate. The base case is trivial and \(i = K^5\) will prove the claim. The induction step is the following: The action of \(g_{i+1}\) on all the lines but the \(i\)th will be the same as the action of \(g_i\). There are \(K\) possibilities for the action on the \(i\)th line. The number of bad choices (the ones where there are two points in \(g_{i+1}b_{i+1}\) which differ only in the first coordinate) is at most \((s_i - t_i) \times t_i\), where \(t_i\) is the number of points from \(b\) on the \(i + 1\)st line and \(s_i\) is the number of points from \(b\) in the square parallel to the first and second coordinate axis containing the \(i + 1\) line. By assumption the points in \(B\) are in a good position, therefore \(t_i \leq \ln K\) and \(s_i \leq K/\ln K\), thus

\[
(s_i - t_i) \times t_i \leq (s_i - 1)t_i \leq (\ln K - 1) \times K/\ln K < K.
\]

This shows there is some good choice for the action of \(g_{i+1}\) on the \(i + 1\)st line. Thus we can define the action of \(g_{i+1}\) on the \(i + 1\)st line such that no two points
in $g_{i+1}b_{i+1}$ differ only in their first coordinate, which proves the induction step and the claim.

If no two points from $gb$ differ only in the first coordinate, we can find $h \in E_1$ such that $hgb$ is a subset of $M$. □

**Lemma 3.6** For any permutation $\sigma \in \text{Sym}(K^{d-1})$ acting on the points in $M$ there exist an element $t \in E^{4d-5}$ such that the restriction of $t$ on this square is the same as $\sigma$.

**Proof.** Notice that inside $E_1E_2E_3 \ldots E_{d-1}E_dE_1E_2E_3 \ldots E_{d-1}E_dE_1 \subset E^{4d-5}$ which preserves $M$ and acts on it as $\sigma$. □

**Lemma 3.7** For any integer $a$ less than $\frac{K^{d-1} - 1}{K - 1}$ there is an element in $c_0 \in E_2E_3E_4E_5E_6$ which is a cycle of length $1 + a(K - 1)$ in the face $M$.

**Proof.** Chose a lines in $M$ such that each line is parallel to some coordinate axis and their union is a tree. Then the products of the shifts on these lines in any order is a cycle on their union which contains exactly $1 + a(K - 1)$ points. By definition each shift is in some $E_i$, which proves that there is an element in $E_2 \ldots E_6$ which acts as a cycle of length $1 + a(K - 1)$. □

Now we can finish the proof of Theorem 3.4. Let $c$ be any cycle of length $L$. By lemma 3.5 almost surely the support of $c$ can be moved to $M$ by some element in $E_2$. Lemma 3.6 gives us the $c$ can be conjugated to $c_0$ by an element in $E_2 \times E_3 = E_21$, i.e., that $c \in E^{47}$. □

Having that $E^{47}$ contains almost a whole conjugancy class we can use the Roichman [33] character estimates: Let $\lambda$ be a partition of $N$ and $C_L$ be the conjugancy class in $\text{Sym}(N)$ consisting of all $L$-cycles. Then the character $\chi_\lambda$ of the irreducible representation corresponding to the partition $\lambda$ satisfies the inequality:

$$|\chi_\lambda(C_L)| \leq \chi_\lambda(id) \max \left\{ \frac{\lambda_1}{N}, \frac{\lambda'_1}{N'}, \frac{3}{4} \right\}^{L-5}.$$  \hspace{1cm} (6)

Let

$$C = \frac{1}{|C_L|} \sum_{g \in C_L} \rho^*(g).$$
By definition we have $Cv_\lambda = \frac{\chi_\lambda(C_L)}{\chi_\lambda(id)}v_\lambda$ for any $v_\lambda \in \mathcal{H}_\lambda$. By equation (6) we have

$$||Cv_1|| \leq ||v_1|| \max_{\lambda} \max_{\lambda_1 < \lambda} \left\{ \frac{\lambda_1}{N}, \frac{\lambda_1^4}{N^4} \right\} \leq \frac{L_5}{5} \leq ||v_1|| \exp\left(\frac{-h(L-5)}{4N} \right) \leq \frac{1}{2K^{3/2}} \left( \frac{K^{d-1} - K - 4}{3\ln K} \right) \leq \frac{K^{1/2}}{24\ln K} \left( 1 - 3(K + 4)K^{1-d} \ln K \right) \leq e^{-3||v_1||}.$$  

On the other hand we have

$$||Cv_1 - v_1|| \leq ||Cv - v|| \leq \frac{1}{|C_L|} \sum_{g \in C_L} ||\rho^s(g)v - v|| \leq \frac{1}{|C_L|} \sum_{g \in C_L \cap E^{47}} ||\rho^s(g)v - v|| + \frac{1}{|C_L|} \sum_{g \in C_L \setminus E^{47}} ||\rho^s(g)v - v|| \leq \frac{1}{|C_L|} \sum_{g \in C_L \cap E^{47}} 47\epsilon + \frac{1}{|C_L|} \sum_{g \in C_L \setminus E^{47}} 2 \leq 47\epsilon + 2 |C_L \setminus E^{47}| \leq 47\epsilon + 0.02,$$

because the elements in $E^{47}$ move the vector $v$ by at most $47\epsilon$ and the other elements move $v$ by less than $2$. Combining the above two equations gives us

$$||v_1|| \leq ||Cv_1 - v_1|| + ||Cv_1|| \leq 47\epsilon + 0.02 + e^{-3||v_1||} \leq 47\epsilon + 0.02 + e^{-3} < 47\epsilon + 0.07. \quad (7)$$

In order to show that $v_2$ is close to an invariant vector we use entirely different argument.

Let $\mathcal{B}$ be the set of all ordered $h$-tuples of points in the six dimensional cube with side $K$. Let $\mathcal{L}$ is the Hilbert space with basis $\mathcal{B}$ with the natural action of $\text{Sym}(N)$ on it.

First we will prove the next lemma which will allow us to work with the vector space $\mathcal{L}$. 

19
**Lemma 3.8** There is a injective homomorphism of $\text{Sym}(N)$-modules

$$i : \mathcal{H}_2 \rightarrow \mathcal{L}.$$  

**Proof.** Let decompose $\mathcal{H}_2$ into sum of irreducible representaions

$$\mathcal{H}_2 \simeq \bigoplus_{\lambda \geq N-h} \mathcal{H}_\lambda \simeq \bigoplus_{\lambda \geq N-h} m_\lambda V_\lambda,$$

where $V_\lambda$ irreducible representing of $\text{Sym}(N)$ corresponding to $\lambda$ and $m_\lambda$ is its multiplicity in $\mathcal{H}$. By assumption the space $\mathcal{H}$ is generated by the vector $v$, therefore the multiplicities $m_\lambda$ are 0 or 1. The representation theory of $\text{Sym}(N)$ gives us that $V_\lambda$ can be embedded in $\mathcal{L}$, which completes the proof. □

We will prove that the projection $v_0$ of $v_2$ onto the space of invariant vectors in $\mathcal{L}$ is close to $v_2$ by show that the random walk generated by $E$ mixes rapidly (in a fixed number of steps) on the basis $\mathcal{B}$ of $\mathcal{L}$. We will reduce the problem to a random walk on the $h$-tuples of points in the square of size $K^{d/2}$.10

Define the operators $U_i$, for $i = 1, \ldots, 6$ on the space $\mathcal{L}$ with basis $\mathcal{B}$ by

$$U_i b = \frac{1}{|E_i|} \sum_{g \in E_i} gb.$$  

Notice that $U_i U_i = U_i$, because $E_i$ is a subgroup of $\text{Alt}(N)$. Let $Q_1$ and $Q_2$ denote the operators $Q_1 = U_1 U_2 U_3$ and $Q_2 = U_4 U_5 U_6$.

The next theorem is a quantitative version of the observation that the random walk on $\mathcal{B}$ generated by $E = \bigcup E_i$ mixes in few steps, provided that $h < K^{3/2}$.

**Theorem 3.9** The entries of the matrix $\{a_{b,b'}\}$ of the operator $Q_2 Q_1 Q_2 Q_1$ defined by

$$Q_2 Q_1 Q_2 Q_1 b = \sum_{b' \in \mathcal{B}} a_{b,b'} b',$$

satisfy the inequality

$$a_{b,b'} \geq \frac{1}{|\mathcal{B}|} \left( 1 - \frac{h^2}{K^3} \right).$$

This inequalities imply that the operator norm of $Q_2 Q_1 Q_2 Q_1 - P_0$ is less than $1 - \frac{h^2}{K^3}$, where $P_0$ is the projection onto the space of $\text{Sym}(N)$ invariant vectors in $\mathcal{L}$.

**Proof.** Let $\mathcal{B}_1 (\mathcal{B}_2)$ denote the sets of all $h$-tuples from $\mathcal{B}$ such that there are no two points with the same first (last) three coordinates.

10Here we use the assumption that $d$ is even.
Lemma 3.10 The entries of the matrices \( \{ q_{1; b, b'} \} \) and \( \{ q_{2; b, b'} \} \) of the operators \( Q_1 \) and \( Q_2 \) satisfy

\[
\sum_{b' \in B_1} q_{1; b, b'} \geq 1 - \frac{h^2}{2K^3} \quad \text{and} \quad \sum_{b' \in B_2} q_{2; b, b'} \geq 1 - \frac{h^2}{2K^3}.
\]

**Proof.** Let \( p_{i, j, k} \) be the number of points from \( B \) which have 4th, 5th and 6th coordinate equal to \( i, j, \) and \( k \) respectively. Let's choose some ordering of the triples \( \{ i, j, k \} \) and let \( p_s \) be the numbers \( p_{i, j, k} \) in this order.

Then the number of elements in \( E_1E_2E_3 \) which sends \( b \) to an element in \( B_1 \) is at least

\[
Q_{1; b} \geq \prod_s \left( K^{3K^2} - \left( \frac{p_s^2}{2} + p_s \sum_{t < s} p_t \right) \times K^{3K^2-3} \right) = K^{3K^5} \prod_s \left( 1 - \frac{p_s^2/2 + \sum_{t < s} p_t}{K^3} \right) \geq
\]

\[
K^{3K^5} \left( 1 - \frac{\sum_s p_s^2/2 + \sum_s \sum_{t < s} p_t}{K^3} \right) = K^{3K^5} \left( 1 - \frac{\sum_s p_s^2}{2K^3} \right) = K^{3K^5} \left( 1 - \frac{h^2}{2K^3} \right) = |E_1| \times |E_2| \times |E_3| \times \left( 1 - \frac{h^2}{2K^3} \right),
\]

because there are \( K^{3K^2} \) possibilities for the action of \( E_1E_2E_3 \) on the cube corresponding to \( s \), and the number of bad ones (such that \( gb \not\in B_1 \), because some point from this cube have the same first 3 coordinates as one of the previous points) is at most

\[
\sum_{i=1}^{p_s} (i - 1 + \sum_{t < s} p_t) \times K^{3K^2-3} \leq \left( \frac{p_s^2}{2} + p_s \sum_{t < s} p_t \right) \times K^{3K^2-3}.
\]

Using the definition of \( Q_1 \) and \( q_{1; b, b'} \) we can see that

\[
\sum_{b' \in B_1} q_{1; b, b'} = \frac{Q_{1; b}}{|E_1| \times |E_2| \times |E_3|} \geq 1 - \frac{h^2}{2K^3}.
\]

\( \square \)

Lemma 3.11 If \( b \in B_1 \) and \( b' \in B_2 \) then the number

\[
P(b, b') = \frac{|\{ g_1 \in E_1, | g_1g_2g_3g_4g_5g_6b = b' \}|}{|E_1| \times |E_2| \times |E_3| \times |E_4| \times |E_5| \times |E_6|} = K^{-6h}.
\]

does not depend on \( b_1 \) and \( b_2 \).
Proof. Let \( g_i \in E_i \) satisfy \( g_1 \ldots g_6 = b' \). If \( g_1 \ldots g_6 \) sends the tuple \( (a_1, \ldots, a_6) \) to the tuple \( (b_1, \ldots, b_6) \), then \( g_6 \) acts on the line \( (a_1, \ldots, a_5, *) \) as shift by \( b_6 - a_6 \); \( g_5 \) acts on the line \( (a_1, \ldots, a_4, *, b_6) \) as shift by \( b_5 - a_5 \) and so on. Thus \( g_1 \ldots g_6 = b' \) determines the action of each \( g_i \) on exactly \( h \) lines (the conditions \( b \in B_1 \) and \( b' \in B_2 \) imply that all these lines are different). This shows that
\[
P(h, b') = \left( \frac{K^{h_6} - h}{K^{h_6}} \right)^6 = K^{-6h}.
\]
\[\blacksquare\]

Now we can finish the proof of Theorem 3.9. We can write \( Q_2Q_1Q_2Q_1b \) as \( Q_2(Q_1Q_2)(Q_1b) \), therefore
\[
a_{b,b'} = \sum_{c,c' \in B} q_{1; c} P(c, c') q_{2; c', b'} \geq \sum_{c \in B_1, c' \in B_2} q_{1; c} P(c, c') q_{2; c', b'} =\]
\[
= K^{-6h} \sum_{c \in B_1, c' \in B_2} q_{1; c} q_{2; c', b'} = K^{-6h} \sum_{c \in B_1} q_{1; c} \sum_{c' \in B_1} q_{1; c} \geq \]
\[
\geq K^{-6h} \left( 1 - \frac{h^2}{2K^6} \right)^2.
\]
The size of the basis \( B \) is
\[
|B| = \prod_{i=1}^{h} (K^6 - i + 1) = K^{6h} \prod_{i=1}^{h} \left( 1 - \frac{i + 1}{K^6} \right) \geq \]
\[
\geq K^{6h} \left( 1 - \sum_{i=1}^{h} \frac{i + 1}{K^6} \right) \geq K^{6h} \left( 1 - \frac{h^2}{2K^6} \right).
\]
Theorem 3.9 follows from the above estimates and the inequality
\[
\left( 1 - \frac{h^2}{2K^6} \right)^2 \left( 1 - \frac{h^2}{2K^6} \right) \geq \left( 1 - \frac{h^2}{K^3} \right) .
\]
\[\blacksquare\]

Let \( v_0 \) be the projection of \( v_1 \) on the space of invariant vectors in \( L \). Using Theorem 3.9 we can see
\[
||Q_2Q_1Q_2Q_1v_1 - v_0|| = ||Q_2Q_1Q_2Q_1v_1 - v_0|| \leq \frac{h^2}{K^3} ||v_1 - v_0||,
\]
but we also have
\[
||Q_2Q_1Q_2Q_1v_1 - v_1|| \leq ||Q_2Q_1Q_2Q_1v - v|| \leq 12\epsilon,
\]
which implies that
\[
||v_1 - v_0|| \leq 12\epsilon + \frac{h^2}{K^3} ||v_1 - v_0||.
\]
Substituting $h = \frac{1}{2}K^{3/2}$ in the above inequality yields
\[ ||v_1 - v_0|| \leq \frac{1}{1 - \frac{1}{4}} 12\epsilon = 16\epsilon. \tag{8} \]

This finishes the proof of Theorem 3.1, because equations (7) and (8) imply that
\[ ||v - v_0|| \leq ||v_2|| + ||v_1 - v_0|| \leq 47\epsilon + 0.07 + 16\epsilon = 63\epsilon + 0.07. \]
The last expression is less then 1 if $\epsilon = 1/70$, therefore $v_0$ is not zero and it is an invariant vector in $\mathcal{H} \subset \mathcal{H}$.

As mentioned in the introduction Theorem 1.6 follows immediately from Proposition 1.5 and Theorems 2.4 and 3.1.

4 Proof of Theorems 2 and 3

Theorems 2 and 3 follow easily from Theorem 1.6.

**Proof of Theorem 2** By Theorem 1.6 the alternating groups $\text{Alt}(n_s)$ are expanders with respect to some generating set $F_{n_s}$ for $n_s = (2^{3s} - 1)^6$. The sequence $\{n_s\}_s$ grows exponentially. Thus, for any sufficiently large $n > 10^{38}$ there exists $s$ such that
\[ 1 < \frac{n}{n_s} < \max_s \left( \frac{(2^{3s+3} - 1)^6}{(2^{3s} - 1)^6} \right) \leq \max_s \left( 8 + \frac{7}{2^{3s} - 1} \right)^6 < 3 \times 10^5. \]

Using the butterfly lemma it can be shown that the group $\text{Alt}(n)$ can be written as a product of several copies of $\text{Alt}(n_s)$ embedded in $\text{Alt}(n)$. The number of copies $P$ is at most
\[ P \leq 3\lceil n/n_s \rceil + 3 < 10^6. \]

Let $\pi_i$ denote the $P$ embeddings of $\text{Alt}(n_s)$ in $\text{Alt}(n)$ Taking the union of $\pi_i(F_{n_s})$ one obtains a generating set $F_n$ of $\text{Alt}(n)$. By Propositions 1.3 and 1.4 we have
\[ \mathcal{K}(\text{Alt}(n); F_n) \geq \frac{1}{2} \mathcal{K}(\text{Alt}(n); \cup \pi_i(\text{Alt}(n_s))) \mathcal{K}(\text{Alt}(n_s); F_{n_s}) \geq \frac{\sqrt{2}}{P} \mathcal{K}(\text{Alt}(n_s); F_{n_s}) \geq 10^{-12}, \]
because every element in $\text{Alt}(n)$ can be written as a product of no more than $P$ elements from $\cup \pi_i(\text{Alt}(n_s))$. By the construction it is clear that the size of $F_n$ is bounded above by $10^9$.

If $n$ is small then the above argument does not work, because we cannot find $n_s < n$ with the desired properties. But there are only finitely many $n$ less...
than $10^{38}$. It is clear for each such $n$ that there exist $\epsilon_n > 0$ and a generating set $F_n$ with less than $10^9$ elements such that $\mathcal{K}(\text{Alt}(n); F_n) \geq \epsilon_n$.

Thus for each $n$ we have constructed a generating set $F_n$ of the alternating group $\text{Alt}(n)$ of size at most $L = 10^9$ such that $\mathcal{K}(\text{Alt}(n); F_n) \geq \epsilon$, where $\epsilon = \min\{10^{-12}, \inf_n \epsilon_n\} > 0$, which completes the proof of Theorem 2. □

**Proof. of Theorem 3** The alternating group $\text{Alt}(n)$ is a subgroup of index 2 inside $\text{Sym}(n)$. Let $t \in \text{Sym}(n) \setminus \text{Alt}(n)$ be an odd permutation. If $F_n$ is an expanding generating set of $\text{Alt}(n)$ then $\tilde{F}_n = F_n \cup \{t\}$ is an expanding generating set of $\text{Sym}(n)$ and the Kazhdan constants are almost the same.

Using the trivial inequality

$$\mathcal{K}(\text{Sym}(n); \text{Alt}(n); F_n \cup \{t\}) \geq \mathcal{K}(\text{Alt}(n); F_n)$$

and proposition 1.4 we can see that

$$\mathcal{K}(\text{Sym}(n); F_n \cup \{t\}) \geq \frac{1}{2} \mathcal{K}(\text{Alt}(n); F_n) \mathcal{K}(\text{Sym}(n); \text{Alt}(n) \cup \{t\}).$$

However it is clear that any element in $\text{Sym}(n)$ can be written as a product of two elements from $\text{Alt}(n) \cup \{t\}$, thus

$$\mathcal{K}(\text{Sym}(n); \text{Alt}(n) \cup \{t\}) \geq \frac{\sqrt{2}}{2}.$$

These two inequalities give us

$$\mathcal{K}(\text{Sym}(n); F_n \cup \{t\}) \geq \frac{\sqrt{2}}{4} \mathcal{K}(\text{Alt}(n); F_n) \geq \frac{1}{3} \mathcal{K}(\text{Alt}(n); F_n),$$

which finishes the proof of Theorem 3. □

**Remark 4.1** In the above proofs of Theorems 2 and 3 we have obtained huge generating sets $F_n$ and $\tilde{F}_n$ with very weak bounds for the Kazhdan constants. Using a more careful analysis it is possible to decrease the size of $F_n$ and to improve the bounds for the Kazhdan constants. We can prove that for all sufficiently large $n$ there exist generating sets $F_n$ and $\tilde{F}_n$ of the alternating and the symmetric groups such that

- $|F_n| \leq 20$ and $|\tilde{F}_n| \leq 20$,
- both $F_n$ and $\tilde{F}_n$ consist only of involutions,
- $\mathcal{K}(\text{Alt}(n); F_n) \geq 10^{-7}$ and $\mathcal{K}(\text{Sym}(n); \tilde{F}_n) \geq 10^{-7}$.

**Remark 4.2** The bounds for the Kazhdan constants in Theorems 2 and 3 are explicit in the case $n > 10^{40}$. This restriction comes from the condition $K \geq 10^6$ in Theorem 3.4. We suspect that some modification in the proof will allow us to weaken requirement.
5 Comments

In this section we will briefly discuss some variations of the construction.

As in sections 1.5 and 1.6, we will denote \( N = K^d \), where \( d \) is fixed. Let \( H \) be any group which acts transitively on the set of \( K \) points, we will also assume that all elements of \( H \) act as even permutations, i.e., we have \( H \hookrightarrow \text{Alt}(K) \).\(^{11}\)

We will define the group

\[
\Delta(H) = H \times K^{d-1}.
\]

This group can be embedded in \( \text{Alt}(N) \) in \( d \) different ways using \( \pi_i \). Let \( E = \cup \pi_i(\Delta(H)) \) be the union of the images of these embeddings.

The main result in section 1.6, Theorem 3.1, says that

\[
\mathcal{K}(\text{Alt}(N); \cup \pi_i(\Delta(Z/KZ))) \geq \frac{1}{60},
\]

provided that \( d = 6 \) and \( K \) is large enough. This result can be generalized to any transitive group:

**Theorem 5.1** Let \( H \) be a transitive group acting on \( K \) points then

\[
\mathcal{K}(\text{Alt}(N); \cup \pi_i(\Delta(H))) \geq \frac{1}{60},
\]

provided that \( d = 6 \) and \( K \) is large enough.

**Proof.** The proof is almost the same as the one of Theorem 3.1. The only difference is that Lemma 3.7 does not hold. However there is a weaker analog of this lemma:

**Lemma 5.2** For any integer \( a \) less than \( K^{d-2} \) there is an element \( c_0 \) in \( E_2 \), which acts on \( M \) as a permutation with \( Ka \) non-fixed points.

**Proof.** Let \( g \in H \) be an element without fixed points — such an element exists in any transitive group because the average number of fixed points is 1. The \( c_0 \) is the image of an element in \( \Delta(H) \) which is equal to \( g \) in \( a \) copies of \( H \) and is identity in the other copies. \( \square \)

The same argument as in the proof of Theorem 3.1 gives that \( E^{47} \) (even \( E^{43} \)) contains almost all elements in the conjugancy class \( C \), which contains \( c_0 \). By construction the permutations in this conjugancy class contain \( L \) non-fixed points, where \( L \) is approximately \( K^{d-1}/3 \ln K \).

For such conjugancy classes there are character estimates, see \(^{33}\), similar to \(^{34}\):

\[
|\chi_{\lambda}(C)| \leq \chi_{\lambda}(id) \max \left\{ \frac{\lambda_1}{N}, \frac{\lambda_1'}{N}, q \right\}^{cL}, \tag{9}
\]

\(^{11}\)If the last requirement is not satisfied then it is impossible to make the Cayley graphs of groups \( \Delta(H) \) expanders with respect to any generating set of bounded size.
where $c$ and $q$ are some universal constants. The precise values of these constants are not in the literature but using the proof of Theorem 1 from [33] one can obtain estimates for these constants. Our computations show that we can use $q = 1 - 10^{-3}$ and $c = 10^{-3}$.

The estimate in (7) continues to hold if $K$ is sufficiently large. The bound depends on the values of the constants $c$ and $q$ above. A careful computation of the constants $c$ and $q$ shows that we can use $q = 1 - 10^{-3}$ and $c = 10^{-3}$, which gives that if $K > 10^{10}$ then $K$ is sufficiently large and the estimate (7) holds.

The rest of the proof is the same as in Theorem 3.1. □

Actually we have shown that

$$\liminf_{K \to \infty} K(\text{Alt}(N); \bigcup \pi_i(\Delta(\mathbb{Z}/K\mathbb{Z}))) \geq \frac{1}{C(d)},$$

if $d = 6$ and $C(6) = 60$. This result is also valid for larger values of $d$ with $C(d) = 10d$ and the proof is essentially the same.

It is interesting to see if a similar result holds for small values of $d$: In the case $d = 1$ the question does not make sense because the group $\text{Alt}(N)$ is not generated by the set $E$. If $d = 2$, we think that it is not possible to obtain a bound for the Kazhdan constant which is independent on $K$, because the group $\Delta(H)$ is not large enough.

Using more careful analysis of the random walk generated by $E$ on the set of $h$ tuples, it is possible to show that there is a uniform bound for the Kazhdan constant if $d = 4$ or $d = 5$. We do not know whether such result is also valid in the case $d = 3$.

In order to use the groups $\Delta(H)$ to construct bounded degree expanders we need to find a family of groups $H_s$ which acts transitively on $K_s$ points such that there is a uniform lower bound for the Kazhdan constants:

$$\mathcal{K}(H_s^{K_s^{d-1}}; S_s)$$

for some generating sets $S_s$. In our proof we used the groups $H_s = \text{SL}_{3s} (\mathbb{F}_2)$ acting on $F_2^{3s} \setminus \{0\}$. A more natural family of groups with these properties is $\text{SL}_n (\mathbb{F}_q)$ acting on $\mathbb{F}_q^n \setminus \{0\}$ or on the projective space $P^{n-1} \mathbb{F}_q$, where $n \geq 3$ is fixed and $q$ is a power of a prime number. In the case $n = 4$ and $q = 2^s$ we can even find an element in $\text{SL}_4(\mathbb{F}_{2^s})$ which acts as a long cycle on $P^3 \mathbb{F}_{2^s}$. In this case we will obtain slightly better Kazhdan constants with respect to some generating set of $\text{Alt}(N)$ where $N = (2^{3s} + 2^{2s} + 2^s + 1)^6$.

Using the groups $\text{SL}_{3s} (\mathbb{F}_2)$ has some advantages: it is possible to generate the product of $2^{s^2}$ copies of the matrix algebra $\text{Mat}_s (\mathbb{F}_2)$ by just three elements. This allows us to construct generating sets $S_s$ of $\Delta(\text{SL}_{3s} (\mathbb{F}_2))^{\times s}$ such that

$$\mathcal{K}(\Delta(\text{SL}_{3s} (\mathbb{F}_2))^{\times s}; S_s) \geq \frac{1}{1000}.$$
Using these generating sets we can make the Cayley graphs of the product of \( s \) copies of \( \text{Alt}(N) \) expanders.

Another advantage is the following: Let \( \tilde{\Delta} \) be the infinite product of the groups \( \Delta(\text{SL}_{3s}(\mathbb{F}_2)) \). Inside \( \tilde{\Delta} \) there is a dense subgroup \( \Delta \) generated by the set \( S \), which projects to \( S_s \) in every factor. Using the methods from \cite{17}, it can be shown that the the pro-finite completion of the group \( \tilde{\Delta} \) is slightly larger than \( \Delta \) and it has property \( \tau \). We can use this group to obtain a dense subgroup \( G \) inside

\[
\prod_s \text{Alt} ((2^{3^s} - 1)^6)
\]

which also has property \( \tau \). \( G \) is the first example of a dense subgroup in the product of infinitely many alternating groups which does not map onto \( \mathbb{Z} \), for more details see \cite{15} and for other examples of dense subgroups in such products see \cite{31}.

Theorem 2 can be viewed as a major stop towards proving the conjecture suggested by Alex Lubotzky \cite{24,12}

**Conjecture 5.3** Let \( G_i \) be the family of all non-abelian finite simple groups. There exists a generating sets \( S_i \) (with uniformly bounded size) such that the Cayley graphs \( C(G_i, S_i) \) form a family of \( \epsilon \)-expanders for some fixed \( \epsilon > 0 \).

A strong supporting evidence for this conjecture is the following well known fact (see \cite{23,26,36}): The groups of a fixed Lie type over different finite fields form an expander family, provided that the rank of the Lie group is at least 2. However both the size of the generating set and the expanding constant depend on the rank. This shows that almost all non-abelian finite simple groups can be put into infinitely many families such the groups in each family can be made expanders. Another supporting evidence for this conjecture is that for any non-abelian finite simple group there exist 4 generators such that the diameter of the corresponding Cayley graph is logarithmic in the size of the group (see \cite{4} and \cite{20}). However in these examples it is known that the Cayley graphs are not expanders.

As with many similar results, one expects that the proof of Conjecture 5.3 will use the classification of the finite simple groups.

The first major step towards proving Conjecture 5.3 was made in \cite{17} — there it is shown that the Cayley graphs of \( \text{SL}_n(\mathbb{F}_q) \) for any prime power \( q \) and infinitely many \( n \) can be made expanders simultaneously by choosing a suitable generating sets. This can be generalized to all families of finite simple groups of Lie type of rank at least 2.

The results in this paper prove the Lubotzky conjecture in case of the alternating groups, which was believed that this is the most difficult case. These results, together with some new ones in the rank one case, are combined in \cite{18}, which almost proves Conjecture 5.3.

\footnote{This conjecture was around for a long time, however it had never appeared in writing because the affirmative answer was not known even in many simple cases.}
References

[1] Miklós Abért, *Symmetric groups as products of abelian subgroups*, Bull. London Math. Soc. **34** (2002), no. 4, 451–456. MR1897424 (2002m:20006)

[2] Noga Alon, Alexander Lubotzky, and Avi Wigderson, *Semi-direct product in groups and zig-zag product in graphs: connections and applications (extended abstract)*, 42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001), IEEE Computer Soc., Los Alamitos, CA, 2001, pp. 630–637. MR1948752

[3] L. Babai, G. Hetyei, W. M. Kantor, A. Lubotzky, and Á. Seress, *On the diameter of finite groups*, 31st Annual Symposium on Foundations of Computer Science, Vol. I, II (St. Louis, MO, 1990), IEEE Comput. Soc. Press, Los Alamitos, CA, 1990, pp. 857–865. MR1150735

[4] L. Babai, W. M. Kantor, and A. Lubotzky, *Small-diameter Cayley graphs for finite simple groups*, European J. Combin. **10** (1989), no. 6, 507–522. MR1022771 (91a:20038)

[5] Roland Bacher and Pierre de la Harpe, *Exact values of kazhdan constants for some finite groups*, Journal of Algebra (1994), no. 163, 495–515. MR1262716 (95b:20018)

[6] M. Burger, *Kazhdan constants for SL(3, Z)*, J. Reine Angew. Math. **413** (1991), 36–67. MR1089795 (92c:22013)

[7] David Carter and Gordon Keller, *Bounded elementary generation of SL_n(O)*, Amer. J. Math. **105** (1983), no. 3, 673–687. MR704220 (85f:11083)

[8] ______, *Elementary expressions for unimodular matrices*, Comm. Algebra **12** (1984), no. 3-4, 379–389. MR737253 (86a:11023)

[9] Ofer Gabber and Zvi Galil, *Explicit constructions of linear size superconcentrators*, 20th Annual Symposium on Foundations of Computer Science (San Juan, Puerto Rico, 1979), IEEE, New York, 1979, pp. 364–370. MR598118 (83g:68097)

[10] Robert Gilman, *Finite quotients of the automorphism group of a free group*, Canad. J. Math. **29** (1977), no. 3, 541–551. MR0435226 (55 #8186)

[11] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978. MR513828 (80g:20019)

[12] ______, *Representations of general linear groups*, London Mathematical Society Lecture Note Series, vol. 94, Cambridge University Press, Cambridge, 1984. MR776229 (86j:20036)
[13] Gordon James and Adalbert Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981. MR644144 (83k:20003)

[14] Martin Kassabov, *Kazhdan Constants for $\text{SL}_n(\mathbb{Z})$*, arXiv:math.GR/0311487.

[15] ———, *Property Tau and Subgroup Growth*, in preparation.

[16] ———, *Symmetric Groups and Expanders*, arXiv:math.GR/0503204.

[17] ———, *Universal lattices and unbounded rank expanders*, arXiv:math.GR/0502237.

[18] Martin Kassabov, Alex Lubotzky, and Nikolay Nikolov, *Finite simple groups and expanders*, in preparation.

[19] Martin Kassabov and Nikolay Nikolov, *Universal lattices and Property Tau*, arXiv:math.GR/0502112.

[20] Martin Kassabov and Tim R. Riley, *Diameters of Cayley graphs of $\text{SL}_n(\mathbb{Z}/k\mathbb{Z})$*, arXiv:math.GR/0502221.

[21] D. A. Každan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. 1 (1967), 71–74. MR0209390 (35 #288)

[22] Maria Klawe, *Limitations on explicit constructions of expanding graphs*, SIAM J. Comput. 13 (1984), no. 1, 156–166. MR731033 (85k:68077)

[23] A. Lubotzky and B. Weiss, *Groups and expanders*, Expanding graphs (Princeton, NJ, 1992), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 10, Amer. Math. Soc., Providence, RI, 1993, pp. 95–109. MR1235570 (95b:05097)

[24] Alexander Lubotzky, *private communication*.

[25] ———, *Discrete groups, expanding graphs and invariant measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994. MR1308046 (96g:22018)

[26] ———, *Cayley graphs: eigenvalues, expanders and random walks*, Surveys in combinatorics, 1995 (Stirling), London Math. Soc. Lecture Note Ser., vol. 218, Cambridge Univ. Press, Cambridge, 1995, pp. 155–189. MR1358635 (96k:05081)

[27] Alexander Lubotzky and Igor Pak, *The product replacement algorithm and Kazhdan’s property (T)*, J. Amer. Math. Soc. 14 (2001), no. 2, 347–363 (electronic). MR1815215 (2003d:60012)
[28] Alexander Lubotzky and A. Žuk, *On property \( \tau \).

[29] G. A. Margulis, *Explicit constructions of expanders*, Problemy Peredači Informacii 9 (1973), no. 4, 71–80. MR0484767 (58 #4643)

[30] Roy Meshulam and Avi Wigderson, *Expanders in group algebras*, Combinatorica 24 (2004), no. 4, 659–680. MR2096820

[31] László Pyber, *Groups of intermediate subgroup growth and a problem of Grothendieck*, Duke Math. J. 121 (2004), no. 1, 169–188. MR2031168 (2004k:20056)

[32] Omer Reingold, Salil Vadhan, and Avi Wigderson, *Entropy waves, the zig-zag graph product, and new constant-degree expanders and extractors (extended abstract)*, 41st Annual Symposium on Foundations of Computer Science (Redondo Beach, CA, 2000), IEEE Comput. Soc. Press, Los Alamitos, CA, 2000, pp. 3–13. MR1931799

[33] Yuval Roichman, *Upper bound on the characters of the symmetric groups*, Invent. Math. 125 (1996), no. 3, 451–485. MR1400314 (97e:20014)

[34] ______, *Expansion properties of Cayley graphs of the alternating groups*, J. Combin. Theory Ser. A 79 (1997), no. 2, 281–297. MR1462559 (98g:05070)

[35] Eyal Rozenman, Aner Shalev, and Avi Wigderson, *A new family of cayley expanders*, (2004).

[36] Yehuda Shalom, *Bounded generation and Kazhdan’s property (T)*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 145–168 (2001). MR1813225 (2001m:22030)

Martin Kassabov,
Cornell University, Ithaca, NY 14853-4201, USA.
*e-mail: kassabov@math.cornell.edu*