Universal corrections to scaling for block entanglement in spin-1/2 $XX$ chains

Pasquale Calabrese$^1$ and Fabian H L Essler$^2$

$^1$ Dipartimento di Fisica, dell'Università di Pisa and INFN, Pisa, Italy
$^2$ The Rudolf Peierls Centre for Theoretical Physics, Oxford University, Oxford OX1 3NP, UK
E-mail: calabres@df.unipi.it and f.essler1@physics.ox.ac.uk

Received 23 June 2010
Accepted 9 August 2010
Published 31 August 2010

Abstract. We consider the Rényi entropies $S_n(\ell)$ in the one-dimensional spin-1/2 Heisenberg $XX$ chain in a magnetic field. The case $n = 1$ corresponds to the von Neumann ‘entanglement’ entropy. Using a combination of methods based on the generalized Fisher–Hartwig conjecture and a recurrence relation connected to the Painlevé VI differential equation we obtain the asymptotic behaviour, accurate to order $O(\ell^{-3})$, of the Rényi entropies $S_n(\ell)$ for large block lengths $\ell$. For $n = 1, 2, 3, 10$ this constitutes the 3, 6, 10, 48 leading terms respectively. The $o(1)$ contributions are found to exhibit a rich structure of oscillatory behaviour, which we analyse in some detail both for finite $n$ and in the limit $n \to \infty$.

Keywords: integrable spin chains (vertex models), Painlevé equations, entanglement in extended quantum systems (theory)

ArXiv ePrint: 1006.3420
1. Introduction

Let $|\Psi\rangle$ be the ground state of an extended quantum mechanical system and $\rho = |\Psi\rangle\langle\Psi|$ its density matrix. In order to quantify the bipartite entanglement in the ground state one divides the Hilbert space into a part $\mathcal{A}$ and its complement $\mathcal{B}$ and considers the reduced
density matrix $\rho_A = \text{Tr}_B \rho$ of subsystem $A$. A measure of the quantum entanglement in the ground state is provided by the Rényi entropies [1]

$$S_n = \frac{1}{1-n} \ln \text{Tr} \rho_A^n. \quad (1)$$

The particular case $n = 1$ of (1) is known as the von Neumann entropy $S_1$ and it is usually called simply entanglement entropy. However, the knowledge of $S_n$ for different $n$ characterizes the full spectrum of non-zero eigenvalues of $\rho_A$ (see e.g. [2]) and provides significantly more information on the entanglement than the more widely studied von Neumann entropy.

Of particular interest is the universal scaling behaviour exhibited by $S_n$ at quantum critical points. For a one-dimensional critical system whose scaling limit is described by a conformal field theory (CFT) of central charge $c$ and $A$ being an interval of length $\ell$ embedded in an infinite system, the asymptotic large-$\ell$ behaviour of the Rényi entropies is given by [3]–[5]

$$S_n(\ell) \sim \frac{c}{6} \left( 1 + \frac{1}{n} \right) \ln \ell + c'_n. \quad (2)$$

Here $c'_n$ is a non-universal constant. The scaling behaviour (2) has been verified both analytically and numerically for a variety of quantum spin chains whose scaling limits are described by CFTs, see e.g. [6]–[17] as well as in direct field theory calculations [18]. In one-dimensional systems these entanglement entropies provide a very useful way for determining the central charge $c$ that characterizes the behaviour at conformally invariant critical points. While other methods for determining $c$ such as the finite-size scaling of the ground state energy [19, 20] require the knowledge of certain non-universal properties such as the velocity of sound, the large-$\ell$ behaviour of the entanglement provides a direct measure of $c$ as is apparent from equation (2). For this reason a scaling analysis of $S_n$ is increasingly used in numerical studies of quantum phase transitions in one-dimensional systems [21]–[33].

In such applications $S_n(\ell)$ is computed numerically and the large-$\ell$ behaviour is then fitted to the form (2). It has been observed that the asymptotic result is sometimes obscured by large, and often oscillatory, corrections to scaling [34, 35, 24, 25]. In [35], on the basis of both exact and numerical results, it has been argued that these corrections are in fact universal and encode information about the underlying CFT beyond what is captured by the central charge alone. More precisely, they give access to the scaling dimensions of some of the most relevant operators in the underlying CFT. This conjecture of [35] has been recently confirmed by using perturbed CFT arguments [36].

A precise characterization of the subleading terms in $S_n(\ell)$ is then desirable for two reasons. First, the knowledge of their structure will be helpful when using (2) to extract the central charge from numerical computations of $S_n(\ell)$. Second, the subleading terms can be used to infer the scaling dimensions of certain operators in the CFT characterizing the quantum critical point. This motivates the present study, in which we significantly extend our recent calculation [35] of the subleading corrections to the Rényi entropies in the $XX$ chain.
1.1. Spin-1/2 XX chain

The Hamiltonian of the XX model on an infinite one-dimensional chain is

$$H = -\sum_{l=-\infty}^{\infty} \frac{1}{2}[\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y] - h\sigma_l^z,$$

(3)

where $\sigma_l^x, \sigma_l^y, \sigma_l^z$ are the Pauli matrices at site $l$. The Jordan–Wigner transformation

$$c_l = \left(\prod_{m<l} \sigma_m^z\right) \frac{\sigma_l^x + i\sigma_l^y}{2}$$

(4)

maps this model to a quadratic Hamiltonian of spinless fermions

$$H = -\sum_{l=-\infty}^{\infty} c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l + 2h(c_l^\dagger c_l - \frac{1}{2}).$$

(5)

Here $h$ represents the chemical potential for the spinless fermions $c_l$, which satisfy canonical anti-commutation relations $\{c_l, c_m^\dagger\} = \delta_{l,m}$. The Hamiltonian (5) is diagonal in momentum space and for $|h| < 1$ the ground state is a partially filled Fermi sea with Fermi momentum

$$k_F = \arccos |h|.$$  

(6)

In the following we will always assume that $|h| < 1$ so that we are dealing with a gapless theory.

1.2. Entanglement entropy of the XX chain: Jin–Korepin result

A key result regarding entanglement measures in the XX chain is due to Jin and Korepin [8], who obtained the leading large-$\ell$ behaviour of $S_n$. Their result takes the form

$$S_n^{JK}(\ell) = \frac{1}{6} \left(1 + \frac{1}{n}\right) \ln(2\ell \sin k_F) + E_n,$$

(7)

where the constant $E_n$ has the integral representation

$$E_n = \left(1 + \frac{1}{n}\right) \int_0^\infty \frac{dt}{t} \left[ \frac{1}{1 - n^{-2}} \left(\frac{1}{n \sinh t/n} - \frac{1}{\sinh t}\right) \frac{1}{\sinh t} - \frac{e^{-2t}}{6}\right].$$

(8)

The objective of our work is to determine the subleading corrections to $S_n^{JK}(\ell)$ for large, finite block lengths $\ell$. It is therefore convenient to introduce quantities $d_n(\ell)$

$$d_n(\ell) \equiv S_n(\ell) - S_n^{JK}(\ell),$$

(9)

to which we will refer throughout.

The remainder of this paper is organized as follows. For the sake of clarity we first present a summary of our results in section 2. We then turn to the details of our derivations. In section 3 we briefly review one of our key tools, the generalized Fisher–Hartwig conjecture. The latter is used in section 4 to determine all ‘harmonic’ corrections to the Rényi entanglement entropies. In order to go beyond the generalized Fisher–Hartwig conjecture we utilize recent developments related to random matrix theory. These are introduced in section 5 and used to determine ‘non-harmonic’ terms in the asymptotic expansion for the von Neumann and Rényi entropies in sections 6 and 7 respectively. Comparisons between our analytic expansion and numerical results are presented in section 8.

doi:10.1088/1742-5468/2010/08/P08029
2. Summary of results

This section contains a summary of our results.

2.1. Rényi entropies of the $XX$ chain: general result

Our full result for $d_n(\ell)$ can be cast in the form

$$d_n(\ell) = \frac{2}{n-1} \sum_{p,q=1}^{\infty} (-1)^p L_k^{(2p(2q-1))/n} (Q_{n,q})^p \left[ \frac{\cos(2k_F\ell p)}{p} + \frac{A_q \sin(2k_Fp\ell)}{L_k} \right]$$

$$+ \left[ B_{p,q}^{(n)} e^{2ip\ell} + \text{h.c.} \right] + \frac{1}{L_k^2} \frac{n+1}{285n^3} \left( 15(3n^2 - 7) + (49 - n^2) \sin^2 k_F \right)$$

$$+ \mathcal{O}(L_k^{-3}),$$

(10)

where

$$L_k = 2\ell |\sin k_F|,$$

(11)

$$A_q = \left[ 1 + 3 \left( \frac{2q - 1}{n} \right)^2 \right] \cos k_F,$$

(12)

$$Q_{n,q} = \left[ \frac{\Gamma(1/2 + (2q - 1)/2n)}{\Gamma(1/2 - (2q - 1)/2n)} \right]^2,$$

(13)

$$B_{p,q}^{(n)} = \frac{2q - 1}{6n} \left[ \left( 5 + \frac{7(2q - 1)^2}{n^2} \right) \sin^2(k_F) - 15 \left( \frac{2q - 1}{n^2} + 1 \right) \right]$$

$$- \frac{p}{4} \left[ \left( 1 + 3 \left( \frac{2q - 1}{n} \right)^2 \right) \cos(k_F) \right]^2.$$

(14)

The leading contribution to $d_n(\ell)$ has already been announced in [35] and is given by

$$d_n(\ell) = \frac{2 \cos(2k_F\ell)}{1 - n} (2\ell |\sin k_F|)^{-2/n} Q_{n,1} + \mathcal{O}(\ell^{-\min[4/n,2]}).$$

(15)

While equation (10) provides an infinite number of contributions, for a given fixed value of $n$ only a finite number of them will be smaller than the leading neglected term, which is always of order $\mathcal{O}(L_k^{-3})$. To be specific, in the cases $n = 1, 2, 3, 10$ equation (10) gives the leading 1, 4, 8, 46 terms in the asymptotic expansion of $d_n(\ell)$ and hence the leading 3, 6, 10, 48 terms in the expansion of $S_n(\ell)$.

2.2. Rényi entropies of the $XX$ chain: explicit results for $S_2(\ell)$ and $S_3(\ell)$

In the special cases $n = 2$ and $3$ our results read

$$d_2(\ell) = -\frac{2Q_{2,1} \cos(2k_F\ell)}{L_k} + \frac{1}{L_k^2} \left[ Q_{2,1}^2 \cos(4k_F\ell) \right.$$

$$- \frac{7Q_{2,1} \cos k_F}{2} \sin(2k_F\ell) + \frac{5 + 3 \sin^2 k_F}{64} \left] + \mathcal{O}(L_k^{-3}),$$

(16)

doi:10.1088/1742-5468/2010/08/P08029
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

and

\[ d_3(\ell) = -\frac{Q_{3,1} \cos(2k_F \ell)}{L_{\lambda}^{2/3}} + \frac{Q_{3,1}^2 \cos(4k_F \ell)}{2L_{\lambda}^{4/3}} - \frac{4Q_{3,1} \cos k_F}{3L_{\lambda}^{5/3}} \sin(2k_F \ell) \]

\[ - \frac{Q_{3,1}^3 \cos(6k_F \ell)}{3L_{\lambda}^2} + \frac{15 + 2 \sin^2 k_F}{243L_{\lambda}^2} + \frac{4 \cos(k_F)Q_{3,1}^2 \sin(4k_F \ell)}{3L_{\lambda}^{7/3}} \]

\[ + \frac{Q_{3,1}^3 \sin(8k_F \ell)}{4L_{\lambda}^{5/3}} + \frac{2Q_{3,1}(111 - 62 \sin^2(k_F)) \cos(2k_F \ell)}{81L_{\lambda}^8} + O(L_{\lambda}^{-3}). \quad (17) \]

2.3. Rényi entropies of the XX chain: limit \( n \to \infty \)

In the limit \( n \to \infty \) infinitely many terms in (10) combine to generate a logarithmic contribution, whose general expression is given in equation (70). It assumes a particularly simple form at half-filling \( k_F = \pi/2 \)

\[ d_{\infty}(\ell) \simeq \frac{\pi^2}{24 \ln(2\ell \ell)} \left\{ \begin{array}{ll} 2 & \ell \text{ odd,} \\ -1 & \ell \text{ even,} \end{array} \right. \quad (18) \]

where \( b = \exp(-\Psi(1/2)) \approx 7.12429 \).

2.4. von Neumann entropy of the XX chain

In the special case \( n = 1 \) corresponding to the von Neumann entropy all oscillating contributions to (10) vanish. This explains why it is easier to determine the central charge from \( S_1 \) than from Rényi entropies with \( n \geq 2 \) (this is no longer true in the presence of boundaries [34], where it is found that oscillations persist in the limit \( n \to 1 \)). Specializing equation (10) to \( n = 1 \) we obtain

\[ S_1 \simeq \frac{1}{3} \ln \ell + c'_1 - \frac{1}{12\ell^2} \left( \frac{1}{5} + \cot k_F^2 \right). \quad (19) \]

In this expression, the \( \ell^{-2} \) power-law behaviour is universal [35,13].

3. Entanglement entropy in the XX model

Let us return to the spin-1/2 XX model on an infinitely long chain (3). The reduced density matrix of a block of \( \ell \) contiguous sites can be expressed as

\[ \rho_A = \det C \exp \left( \sum_{j,l \in A} [\ln(C^{-1} - 1)]_{jl} c_j^\dagger c_l \right), \quad (20) \]

where the correlation matrix \( C \) has matrix elements

\[ C_{nm} = \langle c_m^\dagger c_n \rangle = \frac{\sin k_F (m - n)}{\pi(m - n)}. \quad (21) \]

As a real symmetric matrix \( C \) can be diagonalized by a unitary transformation

\[ U C U^\dagger \equiv \delta_{lm}(1 + \nu_m)/2. \quad (22) \]
This implies that the reduced density matrix $\rho_\ell$ is uncorrelated in the transformed basis. The Rényi entropies can be expressed in terms of the eigenvalues $\nu_l$ as

$$S_n(\ell) = \sum_{l=1}^\ell e_n(\nu_l), \quad \text{with} \quad e_n(x) = \frac{1}{1-n} \ln \left( \left( \frac{1+x}{2} \right)^n + \left( \frac{1-x}{2} \right)^n \right).$$

(23)

More details about this procedure can be found in, e.g., [6,16,37]. We note that the above construction refers to the block entanglement of Fermionic degrees of freedom. However, in the case considered here, the non-locality induced by the Jordan–Wigner transformation does not affect the reduced density matrix. In fact, it can be seen to mix only spins inside the block. This ceases to be the case when two or more disjoint intervals are considered [38,39] and other techniques need to be employed [40] in order to recover CFT predictions [41,42,38].

The representation (23) is particularly convenient for numerical computations: the eigenvalues $\nu_m$ of the $\ell \times \ell$ correlation matrix $C$ are determined by standard linear algebra methods and $S_n(\ell)$ is then computed using equation (23). In order to obtain the universal behaviour in the limit of large block lengths $\ell \to \infty$ we follow [8]. We introduce the determinant

$$D_\ell(\lambda) = \det((\lambda + 1)I - 2C) \equiv \det(G).$$

(24)

In the eigenbasis of $C$ the determinant is simply a polynomial of degree $\ell$ in $\lambda$ with zeroes $\{\nu_j|j = 1, \ldots, \ell\}$, i.e.

$$D_\ell(\lambda) = \prod_{j=1}^\ell (\lambda - \nu_j).$$

(25)

This implies that the Rényi entropies have the integral representation

$$S_n(\ell) = \frac{1}{2\pi i} \oint d\lambda \ e_n(\lambda) \frac{d \ln D_\ell(\lambda)}{d\lambda},$$

(26)

where the contour of integration encircles the segment $[-1, 1]$. The matrix $G$ is an $\ell \times \ell$ Toeplitz matrix, i.e. its matrix elements depend only on the difference between row and column indices

$$G_{jk} = g_{j-k}. \quad (27)$$

In the theory of Toeplitz matrices an important role is played by the Fourier transform $g(\theta)$ of $g_l$

$$g_l = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} g(\theta).$$

(28)

The function $g(\theta)$ is called the symbol and in our case takes the form

$$g(\theta) = \begin{cases} 
\lambda + 1 & \theta \in [k_F, 2\pi - k_F] \\
\lambda - 1 & \theta \in [0, k_F] \cup [2\pi - k_F, 2\pi].
\end{cases} \quad (29)$$

On the interval $[0, 2\pi]$ the function $g(\theta)$ has two discontinuities at $\theta_1 = k_F$ and $\theta_2 = 2\pi - k_F$. 

doi:10.1088/1742-5468/2010/08/P08029

7
3.1. The generalized Fisher–Hartwig conjecture

The Fisher–Hartwig conjecture [43] gives the asymptotic behaviour of the determinant of a Toeplitz matrix in the limit where the dimension \( \ell \) of the matrix becomes large. This has been used by Jin and Korepin [8] to derive the leading large \( \ell \) asymptotic behaviour of the Rényi entanglement entropies. As stressed in [8], for the Toeplitz matrices defined by the symbol (29) the Fisher–Hartwig conjecture has been proven by Basor [44].

In order to employ the Fisher–Hartwig conjecture one needs to express the symbol \( g(\theta) \) of a Toeplitz matrix in the form

\[
g(\theta) = f(\theta) \prod_{r=1}^{R} e^{ib_{r}[\theta - \theta_{r} - \pi \text{sgn}(\theta - \theta_{r})] (2 - 2 \cos(\theta - \theta_{r}))^{a_{r}}},
\]

where \( R \) is an integer, \( a_{r}, b_{r} \) and \( \theta_{r} \) are constants and \( f(\theta) \) is a smooth function with winding number zero. The Fisher–Hartwig conjecture then states that the large-\( \ell \) asymptotic behaviour of the corresponding Toeplitz determinant is given by

\[
D_{\ell} \sim F[f(\theta)]^{\ell} \left( \prod_{j=1}^{R} (\ell^{a_{j}^{2}} - b_{j}^{2}) \right) E,
\]

where \( F[f(\theta)] = \exp(1/(2\pi) \int_{0}^{2\pi} \text{d} \theta \ln f(\theta)) \) and \( E \) is a known function of \( f(\theta), a_{r}, b_{r}, \) and \( \theta_{r} \). In our case it is straightforward to express the symbol in the canonical form (30). As \( g(\theta) \) has two discontinuities in \([0, 2\pi]\) we have \( R = 2 \). It is useful to define a function

\[
\beta_{\lambda} = \frac{1}{2\pi i} \ln \left[ \frac{\lambda + 1}{\lambda - 1} \right],
\]

where the branch cut of the logarithm is chosen such that

\[
-\pi \leq \arg \left[ \frac{\lambda + 1}{\lambda - 1} \right] < \pi.
\]

Inserting the ansatz

\[
b_{1} = -b_{2}, \quad a_{1,2} = 0, \quad f(\theta) = f_{0} = \text{const}
\]

into (30) gives

\[
g(\theta) = f_{0} e^{2ib_{2}k_{F}} \begin{cases} 1 & \theta \in [k_{F}, 2\pi - k_{F}] \\ e^{-2\pi i b_{2}} & \theta \in [0, k_{F}] \cup [2\pi - k_{F}, 2\pi] \end{cases}.
\]

Comparing (35) to (29) we conclude that we require

\[
b_{2} = \beta_{\lambda} + m,
\]

where \( m \) is an arbitrary integer number. We further identify

\[
f_{0} = (\lambda + 1)e^{-2ib_{2}k_{F}} = (\lambda + 1)e^{-2ik_{F}m}e^{-2ik_{F}\beta_{\lambda}}.
\]

The integer \( m \) labels the different inequivalent representations of the symbol \( g(\theta) \), see [44]. In their work Jin and Korepin employed the Fisher–Hartwig conjecture for the \( m = 0 \).
representation and obtained the following result for the large-\( \ell \) asymptotics of \( D_\ell(\lambda) \) [8]

\[
D_\ell^{JK}(\lambda) \sim \left[ (\lambda + 1) \left( \frac{\lambda + 1}{\lambda - 1} \right)^{-(k_F/\pi)} \right]^{\ell} L_k^{-2\beta_0^2(\lambda)} G^2(1 + \beta_\lambda) G^2(1 - \beta_\lambda),
\]

(38)

where \( L_k = 2|\sin k_F| \) has been introduced in (11). Inserting (38) into (26) and carrying out the integral leads to the result for the asymptotic behaviour of the Rényi entropy reported in equation (7). Expression (7) provides the leading behaviour of \( S_n(\ell) \) for large block lengths \( \ell \). It is the purpose of our work to determine (universal) subleading contributions to (7). This is achieved by noting that for the case when the symbol \( g(\theta) \) has several inequivalent representations labelled by an integer \( m \) the asymptotics of the corresponding Toeplitz determinant is given by the so-called \emph{generalized Fisher–Hartwig conjecture} (gFHC) [44], which reads

\[
D_\ell(\lambda) \sim \sum_m e^{l_0^{(m)}(\ell)} e^{-2\sum_r (b_r^{(m)})^2} E^{(m)}.
\]

(39)

In our case, the various parameters in (39) are given by

\[
l_0^{(m)} = \ln(j_0^{(m)}) = \ln(\lambda + 1) - 2ik_F/\beta_\lambda - 2ik_F m,
\]

(40)

\[
b_2^{(m)} = -b_1^{(m)} = \beta_\lambda + m,
\]

(41)

where \( m \) are integers and

\[
E^{(m)} = [2 - 2\cos(2k_F)]^{-2(\beta_\lambda + m)^2} [G(1 + \beta_\lambda + m)G(1 - \beta_\lambda - m)]^2.
\]

(42)

Here \( G(z) \) is the Barnes \( G \)-function [45]. We note that the gFHC has been used to determine the large-distance asymptotics of various two-point correlation functions in [46, 47]. Important properties of the gFHC in our case are as follows.

(i) The exponential increase is representation independent and governed by the exponent

\[
\text{Re}(l_0^{(m)}) = \text{Re} \left[ \ln(\lambda + 1) \right] - \frac{k_F}{\pi} \text{Re} \left[ \ln \left( \frac{\lambda + 1}{\lambda - 1} \right) \right].
\]

(43)

(ii) The leading oscillatory behaviour depends on the representation and is given by

\[
\text{Im}(l_0^{(m)}) = \text{Im} \left[ \ln(\lambda + 1) \right] - \frac{k_F}{\pi} \text{Im} \left[ \ln \left( \frac{\lambda + 1}{\lambda - 1} \right) \right] - 2k_F m.
\]

(44)

(iii) The power-law correction depends on the representation and is characterized by the exponents

\[
\alpha_m = (b_1^{(m)})^2 + (b_2^{(m)})^2 = -2(\beta_\lambda + m)^2.
\]

(45)

The real parts of these exponents are

\[
\text{Re}(\alpha_m) = -2\text{Re}(\beta_\lambda^2) - 2m(2\text{Re}(\beta_\lambda)).
\]

(46)

In conjunction with the inequality \(-1 \leq 2\text{Re}(\beta_\lambda) < 1 \) this establishes that

\[
\text{Re}(\alpha_m) \leq \text{Re}(\alpha_0).
\]

(47)

Equality in (47) holds only for \( m = 1 \) and \( \text{Re}(\beta_\lambda) = -\frac{1}{2} \), which corresponds to the case \(-1 < \lambda < 1 \).
We note that point (iii) is crucial for equation (38) to give the correct asymptotic behaviour of \( S_n(\ell) \). Along the integration contour in (26) we always have \( \text{Im}(\lambda) \neq 0 \). Representations with \( m \neq 0 \) therefore give subleading corrections, which we are going to analyse in section 4.

The full result of the generalized Fisher–Hartwig conjecture for the Toeplitz determinant takes the form

\[
D_\ell \sim (\lambda + 1)^{\ell} \left( \frac{\lambda + 1}{\lambda - 1} \right)^{-(k_F \ell / \pi)} \sum_{m \in \mathbb{Z}} L_k^{-2(m+\beta_\lambda)^2} e^{-2ik_F m \ell} [G(m+1+\beta_\lambda)G(1-m-\beta_\lambda)]^2.
\]

(48)

4. Corrections to the scaling for entanglement

The leading corrections to scaling for the Rényi entropies are obtained from the ‘harmonic’ terms given by the generalized Fisher–Hartwig conjecture. It follows from (46) that the most important corrections arise from the first two contributions with \( m = \pm 1 \). Keeping only the three terms corresponding to \( m = -1, 0, 1 \) in (48) we obtain the following expression for the asymptotics of the determinant \( D_\ell(\lambda) \)

\[
D_\ell \sim D_\ell^{JK}(1 + \Psi_\ell(\lambda)),
\]

(49)

Here \( D_\ell^{JK}(\lambda) \) is given in equation (38). Using \( G(1+x)/G(x) = \Gamma(x) \) we can rewrite the last formula as

\[
D_\ell(\lambda) \sim D_\ell^{JK}(1 + \Psi_\ell(\lambda)),
\]

(50)

It follows from the factorized form of (50) that the contributions of the correction terms to the entropies are easier to calculate than the contribution of the leading term \( D_\ell^{JK} \) itself. This will enable us to obtain a full analytic answer. For large \( L_k \) we have (we recall that \( d_n(\ell) = S_n(\ell) - S_n^{JK}(\ell) \))

\[
d_n(\ell) \sim \frac{1}{2\pi i} \oint d\lambda \frac{\ln [1 + \Psi_\ell(\lambda)]}{\lambda} = \frac{1}{2\pi i} \oint d\lambda \frac{\ln e_n(\lambda) \frac{d\Psi_\ell(\lambda)}{d\lambda}}{\lambda} + \cdots.
\]

(51)

The contour integral can be written as the sum of two contributions infinitesimally above and below the interval \([-1, 1]\) respectively, i.e.

\[
d_n(\ell) \sim \frac{1}{2\pi i} \left[ \int_{-1+ie}^{1+ie} - \int_{-1-ie}^{1-ie} \right] d\lambda \frac{\ln e_n(\lambda) \frac{d\Psi_\ell(\lambda)}{d\lambda}}{\lambda}.
\]

(52)

This shows that we only require the discontinuity across the branch cut. The only discontinuous function is \( \beta_\lambda \), which for \(-1 < x < 1\) behaves as

\[
\beta_{x \pm ie} = -iw(x) \mp \frac{1}{2}, \quad \text{with} \quad w(x) = \frac{1}{2\pi} \ln \frac{1+x}{1-x}.
\]

(53)
We now change variables from $\lambda$ to $w$

$$\lambda = \tanh(\pi w), \quad -\infty < w < \infty.$$  

(54)

We have

$$\left[ L_k^{-2+\beta} \frac{\Gamma^2(1+\beta)}{\Gamma^2(\beta)} \right]_{\beta=-iw} \sim L_k^{4iw} \gamma^2(w),$$

$$\left[ L_k^{-2-\beta} \frac{\Gamma^2(1-\beta)}{\Gamma^2(\beta)} \right]_{\beta=-iw} \sim -L_k^{-4iw} \gamma^2(-w),$$

where we have dropped terms of order $O(L_k^{-4})$ compared to the leading ones and we have defined

$$\gamma(w) = \frac{\Gamma(1/2-\beta)}{\Gamma(1/2+\beta)}.$$  

(55)

Integrating by parts and using

$$\frac{d}{dw} e_n(\tanh(\pi w)) = \frac{\pi n}{1-n} (\tanh(n\pi w) - \tanh(\pi w)),$$

we arrive at

$$d_n(\ell) \sim \frac{\ln}{2(1-n)} \int_{-\infty}^{\infty} dw (\tanh(\pi w) - \tanh(n\pi w))$$

$$\times \left[ e^{-2ik\ell} L_k^{4iw} \gamma^2(w) - e^{2ik\ell} L_k^{-4iw} \gamma^2(-w) \right] + \cdots.$$  

(57)

For large $\ell$ the leading contribution to the integral arises from the poles closest to the real axis. These are located at $w_0 = i/2n$ ($w_0 = -i/2n$) for the first (second) term in (57). Evaluating their contributions to the integral gives

$$d_n(\ell) = \frac{2\cos(2k\ell)}{1-n} \left( 2\ell |\sin k\ell| \right)^{-2/n} \gamma^2 \left[ \frac{\Gamma(1/2 + (1/2n))}{\Gamma(1/2 - (1/2n))} \right]^2 + o(\ell^{-2/n}).$$  

(58)

This result implies that at half-filling ($k_F = \frac{\pi}{2}$) and $n > 1$ the corrections are positive (negative) for odd (even) $\ell$.

4.1. Subleading corrections

Equation (58) describes the asymptotic behaviour in the limit $L_k \to \infty$, $n$ fixed. It provides a good approximation for large, finite $\ell$ as long as $\ln(L_k) \gg n$. This is a strong restriction already for moderate values of $n$. For example, $L_k$ is required to be larger than $10^4$ for $n = 10$. For practical purposes it is useful to know the corrections to $S_n(\ell)$ for large $\ell$ but $\ln(L_k)$ not necessarily much larger than $n$. In this regime there are two main sources of corrections to (58).

(i) The integral (56) is no longer dominated by the poles closest to the real axis and contributions from further poles need to be included. These give rise to corrections proportional to $L_k^{-2q/n}$, with $q$ integer.
(ii) Further terms in the expansion of the logarithm in equation (51) need to be taken into account. The corresponding contributions are proportional to $e^{\pm i2pk_F\ell}$ with $p = 2, 3, \ldots$.

At half-filling (zero magnetic field) the situation is different in that terms with odd $p$ all give rise to an overall factor $(-1)^\ell$ and hence modify the staggered contribution to $S_n(\ell)$, while terms with even $p$ add to the smooth (non-oscillatory) part already present in $S_n^{JK}(\ell)$.

We now take both types of corrections into account. We first consider the series expansion of the logarithm in equation (51)

$$\ln[1 + \Psi_\ell(\lambda)] = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}(\Psi_\ell(\lambda))^p}{p}. \quad (59)$$

Recalling the explicit expression (50) for $\Psi_\ell(\lambda)$ leads to a binomial sum

$$(\Psi_\ell(\lambda))^p = \left(e^{-2ik_F\ell}L_k^{-2(1+2\beta)}c_{\beta}\lambda + e^{2ik_F\ell}L_k^{-2(1-2\beta)}c_{-\beta}\lambda\right)^p$$

$$= \sum_{q=0}^{p} \binom{p}{q} e^{2ik_F\ell(2q-p)}L_k^{-2p}L_k^{-4(p-2q)\beta}c_{\beta}^p q\text{ }c_{-\beta}^q, \quad (60)$$

where we have introduced the shorthand notation $c_{\beta} = (\Gamma(1 + \beta)/\Gamma(-\beta))^2$. When calculating the discontinuity across the branch cut running from $\lambda = -1$ to 1 all terms other than $q = 0$ and $q = p$ give rise to terms that are subleading in $L_k$. Hence we may approximate

$$(\Psi_\ell(\tanh(\pi w) + i\epsilon))^p - (\Psi_\ell(\tanh(\pi w) - i\epsilon))^p \approx e^{-2ik_F\ell p}L_k^{4iw\gamma}L_k^{-iw-1/2}$$

$$\quad + e^{2ik_F\ell p}L_k^{-4iw\gamma}L_k^{-iw+1/2}. \quad (61)$$

The analogue of (57) then reads

$$d_n(\ell) \sim \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \ln \int_{-\infty}^{\infty} dw(\tanh(\pi w) - \tanh(n\pi w))$$

$$\quad \times \left[e^{-2ik_F\ell p}L_k^{4iw\gamma}2p(w) - e^{2ik_F\ell p}L_k^{-4iw\gamma}2p(-w)\right]. \quad (62)$$

The integral is carried out by contour integration, taking the two terms in square brackets into account separately. The first (second) contribution has simple poles in the upper (lower) half plane at $w_q = i((2q-1)/2n)$ ($w_q = -i((2q-1)/2n)$), where $q$ is a positive integer such that $2q - 1 \neq n, 3n, 5n, \ldots$. Contour integration then gives

$$d_n(\ell) = \frac{2}{1 - n} \sum_{p,q=1}^{\infty} \frac{(-1)^{p+1}}{p} \cos(2k_F\ell p) L_k^{-(2p(2q-1)/n)}(Q_n,q)^p + \mathcal{O}(L_k^{-1-2/n}), \quad (63)$$

where the constants $Q_n,q$ have been defined in (13). In the sum over $q$ the special values $2q - 1 \neq n, 3n, 5n, \ldots$ are to be omitted. In particular, this means that for $n = 1$ all these corrections are absent. Equation (63) is one of the main results of our work. It shows
that there are contributions to the Rényi entropies with oscillation frequencies that are arbitrary multiples of $2k_F$.

At half-filling ($k_F = \pi/2$) certain simplifications occur. For even $\ell$ we find

$$d_n(\ell) \sim \frac{2}{1 - n} \left[ (2\ell)^{-2/n}Q_{n,1} - (2\ell)^{-4/n}Q_{n,1}^2 \frac{Q_{n,1}}{2} + (2\ell)^{-6/n}Q_{n,1}^3 \frac{3}{2} + Q_{n,3} \right] + \cdots,$$

while for odd $\ell$ we obtain

$$d_n(\ell) \sim \frac{-2}{1 - n} \left[ (2\ell)^{-2/n}Q_{n,1} + (2\ell)^{-4/n}Q_{n,1}^2 \frac{Q_{n,1}}{2} + (2\ell)^{-6/n}Q_{n,1}^3 \frac{3}{2} + Q_{n,3} \right] + \cdots.$$  

In all of the above analysis we have ignored contributions to the generalized Fisher–Hartwig conjecture with $|m| > 1$. While these lead to oscillatory contributions with frequencies that are integer multiples of $2k_F$ they are suppressed by additional powers of $\ell^{-1}$ and hence are subleading, even in the case where $n$ is not small.

It is apparent from (63) that the limit $n \to \infty$ deserves special attention. $S_\infty(\ell)$ is known in the literature as single copy entanglement [48]. Here it is necessary to sum up an infinite number of contributions in order to extract the large-$\ell$ asymptotics. We note that the large-$n$ limit is not only of academic interest, but will provide information on the behaviour of $S_n(\ell)$ in the regime $n \gg \ln L_k$, $L_k \gg 1$.

4.2. Large $n$ limit of $S_n(\ell)$

In order to investigate the limit $n \to \infty$ we consider equation (62), but now first take the parameter $n$ to infinity and then carry out the resulting integrals. This gives

$$d_\infty(\ell) \sim \frac{i}{2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \int_{-\infty}^{\infty} dw \left( \text{sgn}(w) - \tanh(\pi w) \right)$$

$$\times \left[ e^{-2ik_FpL_k^{4w^p} \left[ \gamma(w) \right]^{2p}} e^{2ik_FpL_k^{-4w^p} \left[ \gamma(-w) \right]^{2p}} \right].$$

$$= -\sum_{p=1}^{\infty} \frac{(-1)^p}{p} \left[ e^{-2ik_Fp\text{Im} \int_0^{\infty} dw \left[ 1 - \tanh(\pi w) \right] L_k^{4w^p} \left[ \gamma(w) \right]^{2p}} \right]$$

$$- e^{2ik_Fp\text{Im} \int_0^{\infty} dw \left[ 1 - \tanh(\pi w) \right] L_k^{-4w^p} \left[ \gamma(-w) \right]^{2p}}.$$  

Using that the first singularity in the upper (lower) half plane occurs at $w = i/2$ ($w = -i/2$) we deform the contours to run parallel to the real axis with imaginary parts $i/4$ and $-i/4$ respectively, i.e. for the first term we use

$$\int_0^{\infty} dw f(w) = \int_0^{i/4} dw f(w) + \int_{i/4}^{\infty+i/4} dw f(w).$$

It is straightforward to show that the second integral contributes only to order $O(1/L_k)$ and does not give rise to logarithmic corrections. Hence the leading contribution is of the
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

\[ d_\infty(\ell) \sim \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \left[ e^{-2i k_F \ell p} \Re \int_0^{1/4} dz (1 - i \tan(\pi z)) L_k^{-4zp} (\gamma(iz))^{2p} \right. \]
\[ \left. + e^{2i k_F \ell p} \Re \int_0^{1/4} dz (1 + i \tan(\pi z)) L_k^{-4zp} (\gamma(iz))^{2p} \right] \]
\[ = 2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \cos(2k_F \ell p) \int_0^{1/4} dz \ e^{-4zp \ln L_k} (\gamma(iz))^{2p}. \]  
(67)

For large \( L \) the dominant contribution to the integral is obtained by expanding \( (\gamma(iz))^{2p} \) in a power series around \( z = 0 \)

\[ d_\infty(\ell) \sim 2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \cos(2k_F \ell p) \int_0^{1/4} dz e^{-4zp \ln L_k} (1 + 4pz \Psi(1/2) + \cdots) \]
\[ = 2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} \cos(2k_F \ell p) \left[ \frac{1}{4 \ln L_k} + \frac{\Psi(1/2)}{4 \ln^2 L_k} + \cdots \right] + O(L^{-1}), \]  
(68)

where \( \Psi(x) \) is the digamma function. The leading contribution can be expressed in terms of the dilogarithm function \( \text{Li}_2(x) \) using

\[ 2 \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} \cos(2k_F \ell p) = \text{Li}_2(-e^{i2k_F \ell}) + \text{Li}_2(-e^{-i2k_F \ell}). \]  
(69)

In the half-filled case (\( k_F = \pi/2 \)) our result takes a particularly simple form

\[ d_\infty(\ell) \sim \frac{1}{2 \ln L_k} \sum_{p=1}^{\infty} \frac{(-1)^p(\ell+1)}{p^2} = \begin{cases} \frac{1}{2 \ln L_k} \frac{\pi^2}{6} & \ell \text{ odd}, \\ -\frac{1}{2 \ln L_k} \frac{\pi^2}{12} & \ell \text{ even}. \end{cases} \]  
(70)

Summing some of the subleading terms in (68) to all orders in \( \ln(L_k)^{-1} \) leads to an expression of the form

\[ d_\infty(\ell) \sim \frac{\pi^2}{24 \ln(bL_k)} \left\{ \begin{array}{ll} 2 & \ell \text{ odd}, \\ -1 & \ell \text{ even}, \end{array} \right. \]  
(71)

where \( b = \exp(-\Psi(1/2)) \approx 7.12429 \). This is found to be in good agreement with numerical computations.

5. Connection with random matrix theory

Keating and Mezzadri [49] have shown that the Toeplitz determinant \( D_\ell(\lambda) \) is related to an important quantity in random matrix theory, namely the gap probability for the circular unitary ensemble (CUE). The generating function \( E_{\ell}^{\text{CUE}}[(0, \phi); \xi] \) (in the following we will
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

drop all arguments to ease notations) for the probability of finding exactly \( k \) eigenvalues \( e^{i\theta} \) within the segment \( \theta \in (\pi - \phi, \pi] \) of the unit circle is given by [50]

\[
E_{\ell}^{\text{CUE}} \equiv \frac{1}{(2\pi)^{\ell!}} \left( \int_{-\pi}^{\pi} -\xi \int_{-\phi}^{\pi} \right) d\theta_1 \ldots \left( \int_{-\pi}^{\pi} -\xi \int_{-\phi}^{\pi} \right) d\theta_{\ell} \prod_{1 \leq j < k \leq \ell} |e^{i\theta_j} - e^{i\theta_k}|^2.
\]

This is equal to the determinant of the Toeplitz matrix [50]

\[
W_{ij} = w_{i-j}, \quad \text{with} \quad w_n = \delta_{n,0} + \frac{\xi}{2\pi^2}(-1)^{n+1}\frac{e^{i\phi} - 1}{n}. \tag{72}
\]

It then follows that for \( \xi = 2/(\lambda + 1) \) and \( \phi = 2k_F \) we have [49]

\[
D_{\ell}(\lambda) = (\lambda + 1)^{\ell} E_{\ell}^{\text{CUE}}. \tag{73}
\]

For any value of \( \ell \) the generating function \( E_{\ell}^{\text{CUE}} \) can be determined from a recurrence relation connected to the Painlevé VI transcendent [50]. The recurrence relation reads

\[
x_{\ell}x_{\ell-1} - c = \frac{1 - x_{\ell}^2}{2x_{\ell}} [(\ell + 1)x_{\ell+1} + (\ell - 1)x_{\ell-1}] - \frac{1 - x_{\ell-1}^2}{2x_{\ell-1}} [\ell x_{\ell} + (\ell - 2)x_{\ell-2}], \tag{74}
\]

where \( c = \cos k_F \) and the initial values are

\[
x_{-1} = 0, \quad x_0 = 1, \quad x_1 = -\frac{\xi}{\pi} \frac{\sin k_F}{1 - (\xi/\pi)k_F}. \tag{75}
\]

The generating function is related to \( x_\ell \) by

\[
\frac{E_{\ell+1}^{\text{CUE}} E_{\ell}^{\text{CUE}}}{(E_{\ell}^{\text{CUE}})^2} = 1 - x_{\ell}^2 = \frac{D_{\ell+1} D_{\ell-1}}{D_{\ell}^2}. \tag{76}
\]

For the sake of completeness we quote the values of the generating function for \( \ell = 0 \) and 1

\[
E_0^{\text{CUE}} = 1, \quad E_1^{\text{CUE}} = 1 - \frac{\xi}{2\pi} \phi. \tag{77}
\]

5.1. Leading large-\( \ell \) asymptotics of \( x_\ell \)

In [49] it was suggested to combine the asymptotic results (38) following from the Fisher–Hartwig conjecture with the recurrence relation (74) in order to obtain further corrections to the large-\( \ell \) behaviour of the Rényi entropies. Inserting (38) into (76) suggests that [49]

\[
x_{\ell} = \frac{\sqrt{2} |\beta_\lambda|}{\ell} + O(\ell^{-2}). \tag{78}
\]

However, a numerical solution of the recurrence relation shows that (78) does not generally provide the correct large-\( \ell \) asymptotics of \( x_\ell \). The reason for this is as follows. When we substitute the ‘full’ result (50) of the generalized Fisher–Hartwig conjecture into (74) we find that the contributions due to the representations with \( m = \pm 1 \) behave as \( \ell^{-1\pm2j_\lambda} \) for large \( \ell \). For any \( \Re(\beta_\lambda) \neq 0 \) one of these will dominate over the contribution arising from the \( m = 0 \) term that gives rise to equation (78). In other words subleading contributions to \( D_\ell(\lambda) \) give rise to the leading large-\( \ell \) behaviour of \( x_\ell \)!

doi:10.1088/1742-5468/2010/08/P08029
We now show in more detail how to extract the large-$\ell$ behaviour of $x_\ell$ from that of $D_\ell(\lambda)$. In order to keep things simple, we focus on the case $\text{Re}(\beta_\lambda) > 0$. Here we may neglect the terms with $|m| > 1$ and $m = -1$ in (48), which leads to

$$\frac{D_{\ell+1}D_{\ell-1}}{D_\ell^2} \sim \left[ 1 + \frac{2\beta_\lambda^2}{\ell^2} \right] (1 - 4a_0^2 e^{2ik_F\ell} - 2 + 4\beta_\lambda \sin^2 k_F) + \cdots,$$

where we have introduced

$$a_0 = (2 \sin k_F)^{-1+2\beta_\lambda} \frac{\Gamma(1 - \beta_\lambda)}{\Gamma(\beta_\lambda)}.
\tag{80}$$

The contribution in square brackets arises from the $m = 0$ Fisher–Hartwig term and is the result quoted in [49]. For $\text{Re}(\beta_\lambda) > 0$ this term is subleading and we obtain instead

$$x_\ell \sim (-1)^\ell e^{ik_F\ell} \ell^{-1+2\beta_\lambda} \frac{\Gamma(1 - \beta_\lambda)}{\Gamma(\beta_\lambda)} \quad \text{Re}(\beta_\lambda) > 0.
\tag{81}$$

Here we have fixed the sign of $x_\ell$ by requiring that the expression (81) asymptotically satisfies the recurrence relation (74). The analogous analysis in the case $\text{Re}(\beta_\lambda) < 0$ gives

$$x_\ell \sim (-1)^\ell e^{-ik_F\ell} \ell^{-1-2\beta_\lambda} \frac{\Gamma(1 + \beta_\lambda)}{\Gamma(-\beta_\lambda)} \quad \text{Re}(\beta_\lambda) < 0.
\tag{82}$$

We may combine equations (81) and (82) into a single equation

$$x_\ell \sim \frac{(-1)^\ell}{\ell} \left[ e^{ik_F\ell}(2\ell \sin k_F)^{2\beta_\lambda} \frac{\Gamma(1 - \beta_\lambda)}{\Gamma(\beta_\lambda)} + e^{-ik_F\ell}(2\ell \sin k_F)^{-2\beta_\lambda} \frac{\Gamma(1 + \beta_\lambda)}{\Gamma(-\beta_\lambda)} \right] + \cdots.
\tag{83}$$

We emphasize that (83) must not be understood as giving the two leading terms in the large-$\ell$ asymptotic expansion of $x_\ell$ because, e.g., for $\text{Re}(\beta_\lambda) > \frac{1}{6}$ there are other contributions to $x_\ell$ that decay more slowly than $\ell^{-1-2\beta_\lambda}$. In order to check the result (83) we have solved the recurrence relation (74) numerically for a number of different values of $\lambda$ and $k_F$. In Figure 1 we compare the asymptotic expression (83) against the numerically computed values for $x_\ell$. The agreement is seen to be excellent in all cases.

5.2. Asymptotic expansion for $x_\ell$ and analytic corrections to the gFHC expression for $D_\ell(\lambda)$

We now turn to the derivation of contributions to the large-$\ell$ asymptotic expansion for $D_\ell(\lambda)$ that are not contained in the gFHC. This will be achieved by utilizing the recurrence relation (74).

We expect the asymptotics of $D_\ell(\lambda)$ to be such that each harmonic term predicted by gFHC is multiplied by a function analytic in $1/\ell$. Restricting our attention to the first three harmonic terms (i.e. $m = -1, 0, 1$), this leads to an expansion of the form

$$\frac{D_\ell}{D_\ell^{\text{DK}}} \sim \left[ 1 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots \right] + a_0^2 \ell e^{2ik_F\ell} \left[ 1 + \frac{a_1}{\ell} + \frac{a_2}{\ell^2} + \cdots \right]
\tag{84}$$

$$+ b_0^2 \ell e^{-2ik_F\ell} \left[ 1 + \frac{b_1}{\ell} + \frac{b_2}{\ell^2} + \cdots \right],$$

doi:10.1088/1742-5468/2010/08/P08029 16
Universal corrections to scaling for block entanglement in spin-\(1/2\) \(XX\) chains

\[ \text{Figure 1.} \text{ Real part of } x_\ell \text{ as a function of } \ell \text{ for several values of } \lambda \text{ and } k_F. \text{ The points are obtained from a numerical solution of the recurrence relation (74). The continuous lines are obtained from the asymptotic prediction (83) by the replacement } (-1)^\ell \to e^{i\pi \ell}. \text{ The first two panels correspond to the same value } \beta_\lambda \approx 0.323792 - 0.128075i \text{ but two different values of } k_F. \text{ The last panel corresponds to } \beta_\lambda \approx 0.015892 - 0.0003966i \text{ and hence the contributions of both terms in equation (83) are important. In all cases we observe good agreement of the theoretical prediction (83) with the numerical data.} \]

\[ \text{where } a_0 \text{ is given by (80) and} \]

\[ b_0 = (2 \sin k_F)^{-1-2\beta_\lambda} \Gamma(1 + \beta_\lambda) \Gamma(-\beta_\lambda). \]  

\[ \text{(85)} \]

\[ \text{We note that by definition we have} \]

\[ (2 \sin k_F)^2 a_0 b_0 = -\beta_\lambda^2. \]  

\[ \text{(86)} \]

\[ \text{We now proceed in a straightforward albeit extremely tedious way.} \]

(i) We first insert (84) into (76) in order to obtain an expression for the asymptotic expansion for \(x_\ell\).

(ii) We input the resulting expression into the recurrence relation (74) for \(x_\ell\) and determine the parameters characterizing the asymptotic expansion of \(x_\ell\) order by order in \(\ell^{-1}\).

The result of step (i) for \(\text{Re}(\beta_\lambda) > 0\) is

\[ x_\ell \text{asy} = \sum_{j=0}^{3} (-1)^j \left[ 1 + \frac{\alpha_{j1}}{\ell} + \frac{\alpha_{j2}}{\ell^2} + \frac{\alpha_{j3}}{\ell^3} + \cdots \right] a_0^{2j} e^{2ik_F j\ell} \ell^j (-2 + 4\beta_\lambda) \]

\[ + \sum_{j=1}^{2} \ell^{-4j\beta_\lambda} e^{-2ik_F j\ell} \left[ q_{j0} + \frac{q_{j1}}{\ell} + \frac{q_{j2}}{\ell^2} + \cdots \right] + \cdots, \]

\[ \text{(87)} \]
where we have written only the terms required for our purposes and where have introduced the quantity
\[ x^\text{asy}_\ell = (-1)^\ell e^{ik_F \ell} \ell^{-1+2\beta_\lambda} 2a_0 \sin k_F. \] (88)

The explicit expressions for the coefficients \( o_{jl} \) and \( q_{jl} \) in terms of the expansion coefficients \( a_j, b_j, \) and \( c_j \) characterizing the large-\( \ell \) asymptotics of \( D_\ell \) are reported in appendix A.

Step (ii) consists of substituting (87) in the recurrence relation (74) and determining the coefficients \( o_{jl} \) and \( q_{jl} \). The nonlinearity of (74) renders this a very difficult task, because terms at different orders in \( \ell^{-1} \) in \( x_\ell \) contribute to the same order in the recurrence relation. For this reason it is crucial to retain sufficiently many terms in (87). The results of this procedure are reported in appendix B.

Combining the results reported in appendices A and B then yields the desired expressions for the expansion coefficients \( a_j, b_j, \) and \( c_j \)

\[ c_1(\beta_\lambda) = 2\beta_\lambda^3 \cot k_F, \]
\[ c_2(\beta_\lambda) = \frac{\beta_\lambda^2}{6} (-1 + 7\beta_\lambda^2 + 12\beta_\lambda^4 - 3\beta_\lambda^6 (5 + 4\beta_\lambda^2) \csc^2 k_F), \]
\[ b_j(\beta_\lambda) = c_j(\beta_\lambda + 1), \quad j = 1, 2, \]
\[ a_j(\beta_\lambda) = c_j(\beta_\lambda - 1), \quad j = 1, 2. \] (89)

6. Corrections to the von Neumann entropy

Having determined the asymptotic expansion for \( D_\ell(\lambda) \) we may now use (26) to calculate additional subleading contributions to the Rényi entropies. We first consider the von Neumann entropy (the case \( n = 1 \)), in which as we have seen above all harmonic contributions vanish. Taking the limit \( n \to 1 \) in (26) and following through the same steps as in section 4 we find

\[ d_1(\ell) \sim \frac{1}{2} \int_{-\infty}^{\infty} dw \frac{\pi w}{\cosh^2 \pi w} \left[ c_1^+ - c_1^- \right] + \frac{2c_2^+ - (c_1^+)^2 - 2c_2^- + (c_1^-)^2}{2\ell^2}, \] (90)

where we have defined
\[ c_j^\pm = c_j(-iw \mp \frac{1}{2}), \quad j = 1, 2. \] (91)

Here \( c_{1,2} \) are given by (89) and we have used
\[ \lim_{n \to 1} \frac{\tanh(\pi w) - \tanh(n\pi w)}{1 - n} = \frac{\pi w}{\cosh^2 \pi w}. \] (92)

As \( c_1^+ - c_1^- \) is an even function of \( w \) the \( \mathcal{O}(\ell^{-1}) \) contribution in (90) vanishes. The \( \mathcal{O}(\ell^{-2}) \) contribution can be calculated analytically using the integrals
\[ \int_{-\infty}^{\infty} dw \frac{\pi w^2}{\cosh^2 \pi w} = \frac{1}{6}, \quad \int_{-\infty}^{\infty} dw \frac{\pi w^4}{\cosh^2 \pi w} = \frac{7}{120}, \] (93)

which gives the final result
\[ d_1(\ell) \sim -\frac{1}{12\ell^2} \left[ \frac{1}{5} + \cot^2 k_F \right]. \] (94)

The simplicity of this answer suggests the existence of a much more straightforward derivation than ours.
7. Corrections to the Rényi entropies

The case of the Rényi entropies is more complicated because the contribution of the harmonic terms does not vanish. Our starting point is the expansion (84) for the Toeplitz determinant, which we express in the form

\[
\frac{D_\ell}{D^{(K)}_\ell} \sim 1 + \Psi_\ell(\beta_\lambda) + \frac{\delta \Psi_\ell^{(1)}(\beta_\lambda)}{\ell} + \frac{\delta \Psi_\ell^{(2)}(\beta_\lambda)}{\ell^2} + \cdots. \tag{95}
\]

Here \(\Psi_\ell(\beta_\lambda)\) are the contributions we have taken into account previously in section 4. The logarithm of the Toeplitz determinant is expanded as

\[
\ln \left[ \frac{D_\ell}{D^{(K)}_\ell} \right] \sim \ln \left[ 1 + \Psi_\ell(\beta_\lambda) + \frac{\delta \Psi_\ell^{(1)}(\beta_\lambda)}{\ell} + \frac{\delta \Psi_\ell^{(2)}(\beta_\lambda)}{\ell^2} \right]
\]
\[
\approx \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \left\{ [\Psi_\ell(\beta_\lambda)]^p + \frac{\delta \Psi_\ell^{(1)}(\beta_\lambda)}{\ell} \frac{\delta \Psi_\ell^{(2)}(\beta_\lambda)}{\ell^2} \right. \\
\left. + \frac{1}{\ell^2} \frac{[\delta \Psi_\ell^{(1)}(\beta_\lambda)]^2 [\Psi_\ell(\beta_\lambda)]^{p-2}}{\ell^2} \right\}
\]
\[
\equiv \chi^{(0)}(\beta_\lambda) + \frac{\chi^{(1)}(\beta_\lambda)}{\ell} + \frac{\chi^{(2)}(\beta_\lambda)}{\ell^2}. \tag{96}
\]

Following through the same steps as in section 4 we then arrive at the following expansion for the Rényi entropies

\[
d_n(\ell) \sim \frac{\ln}{2(1-n)} \int_{-\infty}^{\infty} dw (\tanh(\pi w) - \tanh(n\pi w)) \\
\times \left[ \chi^{(0)}(\beta_\lambda) + \frac{\chi^{(1)}(\beta_\lambda)}{\ell} + \frac{\chi^{(2)}(\beta_\lambda)}{\ell^2} \right]_{\beta_\lambda = -iw - \frac{\ell}{2}}^{\beta_\lambda = -iw + \frac{\ell}{2}}
\]
\[
= d_n^{(0)}(\ell) + d_n^{(1)}(\ell) + d_n^{(2)}(\ell). \tag{97}
\]

The contribution \(d_n^{(0)}(\ell)\) has been determined in section 4. The other two contributions are calculated by the same method as in section 4 and we find

\[
d_n^{(1)}(\ell) \sim \frac{2 \cos(k_F)}{1-n} \sum_{p,q=1}^{\infty} (-1)^{p+1} \sin(2k_F p \ell) L_k^{-1-2p(2q-1)/n} \\
\times \left[ 1 + 3 \left( \frac{2q-1}{n} \right)^2 \right]^{2p} \frac{\Gamma((1/2) + ((2q-1)/2n))^{2p}}{\Gamma((1/2) - ((2q-1)/2n))}, \tag{98}
\]
\[
d_n^{(2)}(\ell) \sim \frac{2}{n-1} \sum_{p,q=1}^{\infty} (-1)^{p} L_k^{-2-2p(2q-1)/n} \left[ \frac{\Gamma((1/2) + ((2q-1)/2n))^{2p}}{\Gamma((1/2) - ((2q-1)/2n))} \right]^{2p} B_p^{(n)} e^{2i p k_F \ell} + \text{h.c.}
\]
\[
+ \frac{1}{\ell^2} \frac{1}{1440n^3} \left( 49 - n^2 + \frac{15(3n^2 - 7)}{\sin^2(k_F)} \right). \tag{99}
\]

Explicit expressions for the coefficients \(B_p^{(n)}\) are given in appendix C.

doi:10.1088/1742-5468/2010/08/P08029 19
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

Figure 2. Top left: $|d_2(\ell)|$ at half-filling ($k_F = \pi/2$) compared to the asymptotic expression. Top right: $|1/d_\infty(\ell)|$ at half-filling (straight lines for even/odd $\ell$ respectively). The agreement is seen to be excellent even for moderate values of $\ell$. Bottom: $|d_n(\ell)|^{-1}$ as a function of $\ell$ for several values of $n$ and $k_F = \pi/2$. The straight lines show the asymptotic results (71) in the limit $n \to \infty$ for even (upper curve) and odd $\ell$ (lower curve). We see that for large $n$ the correction $d_n(\ell)$ exhibits a logarithmic increase up to a block size $\ln \ell \sim n$, when the power-law asymptotic behaviour starts to be seen (as we are plotting $|d_n(\ell)|^{-1}$ the asymptotic behaviour corresponds to an $\ell^{2/n}$ power-law increase with $\ell$).

8. Numerical results

Given our asymptotic expansion a natural question to ask is how well it approximates the Rényi entropies for large but finite block lengths $\ell$. In order to address this question we will now present a number of comparisons between our asymptotic result and numerically exact expressions for $S_n(\ell)$. The latter are obtained by determining the eigenvalues of the Toeplitz matrix $C_{nm}$ in equation (21) and computing $S_n$ from equation (23).

8.1. Leading contributions to $d_n(\ell)$

In the top two panels of figure 2 we plot the absolute value of $d_n(\ell)$ for $n = 2$ and $\infty$ at $k_F = \pi/2$ and compare it to the leading asymptotic expressions (15) and (71) respectively. We see that the asymptotic expressions give good agreement with the numerically exact results even for moderate values of $\ell$.

The next issue we turn to is the behaviour of $S_n(\ell)$ for large, finite values of $n$. In this case the asymptotic power-law $d_n(\ell) \propto \ell^{-2/n}$ only emerges for very large block lengths.
8.2. Corrections to the von Neumann entropy $S_1(\ell)$

For the von Neumann entropy $S_1$ all the oscillating terms vanish and the predicted large-$\ell$ asymptotic behaviour is given in equation (19). In figure 3 we plot $-\ell^2d_1(\ell)$ as a function of $\ell$ and compare it with the prediction (19). We see that the agreement is excellent, which indicates that further corrections are very small. We note that for vanishing magnetic field ($k_F = \pi/2$) the amplitude of the $O(\ell^{-2})$ correction term is numerically small ($1/60$) so that the corresponding contribution to $S_1(\ell)$ becomes negligible already for relatively small $\ell$. At least in the particular case of the $XX$ model in zero field this shows that the central charge is most conveniently extracted from finite-size scaling studies of $S_1(\ell)$ rather than higher Rényi entropies.

8.3. Subleading contributions to $d_n(\ell)$

For $n > 1$ the structure of subleading corrections to scaling for $S_n(\ell)$ is significantly richer. Explicit expressions, accurate to order $O(\ell^{-3})$, for the cases $n = 2$ and 3 are
Figure 4. Corrections to scaling for the Rényi entropy $S_2(\ell)$ at $k_F = \pi/4$. Upper panel: $d_2(\ell)$ as a function of $\ell$. Dots are the numerical results while the continuous line corresponds to the asymptotic expression (16). Lower panel: rescaled subleading corrections $D_2(\ell)$ defined in (100) as a function of $\ell$. The agreement between the asymptotic expression (continuous line) and numerical data (dots) confirms that (16) is correct to order $o(\ell^{-2})$.

given in equations (16) and (17) respectively. A comparison of these results for $n = 2$ with numerical computations is presented in figure 4. We have chosen $k_F = \pi/4$ so as to be able to separate the oscillation frequencies of the various contributions. The top panel in figure 4 presents a comparison of the asymptotic expression for $d_2(\ell)$ (continuous lines) with numerical computations (dots) and shows good agreement even for small $\ell$. In order to better assess the accuracy of the asymptotic results we introduce the rescaled, subtracted quantity

$$D_2(\ell) = [d_2(\ell) - d_2^{asy}(\ell)]\ell^2,$$

(100)

where $d_2^{asy}(\ell)$ represents the leading correction given in equation (15). By construction $D_2(\ell)$ should tend to a sum of oscillatory terms with fixed amplitudes for large $\ell$. The four-sublattice oscillatory behaviour of $D_2(\ell)$ predicted by (16) is clearly visible and as expected we observe excellent agreement between the numerical and asymptotic results. For larger values of $n$, the corrections arising from the ‘analytic’ part of $D_\ell(\lambda)$ are less important than the harmonic contributions. Thus the most relevant terms in the asymptotic expansion are those given in equation (63). In fact, the first analytic correction has an exponent $-1 - 2/n$ and at a given $n$ appears only after $[n/2]$ harmonic contributions with exponents $-2p/n$ with $p = 1, 2, \ldots$. It has already been observed in [35] that for higher values of $n$ the first subleading order does not suffice to give an accurate description of the Rényi entropies.

In figure 5 we show a comparison of the corrections $d_n(\ell)$ for $n = 10, 20$ and $k_F = \pi/4$ with the asymptotic result, equation (63). Step by step we take into account further terms
Figure 5. Corrections $d_n(\ell)$ for $n = 10$ (top) and $n = 20$ (bottom) for $k_F = \pi/4$. The numerical data are well described by equation (63), but more terms are needed to obtain the same degree of accuracy when $n$ is increased. In both plots the various continuous curves correspond to equation (63) with one (red curve), two (green curve), three (blue curve), etc terms in (63) retained.

in the asymptotic expression (63) until we obtain good agreement with the numerical data. We observe that for $n = 10$ three terms in equation (63) are enough to reproduce the data, while for $n = 20$ we need five terms to have the same accuracy.

9. Conclusions

In this work we have determined the asymptotic behaviour of the Rényi entropies $S_n(\ell)$ in the spin-1/2 XX model for large block lengths $\ell$. A summary of our results has been presented in section 2. While we have considered the specific case of the spin-1/2 XX chain in a magnetic field, some features we find are in fact universal. In particular, the scaling of the leading oscillatory term (15) has been observed for the XXZ model in zero magnetic field in [35]. The corresponding exponent is modified to $\ell^{-2K/n}$, where $K$ is the Luttinger liquid parameter. This is in full agreement with recent perturbed CFT calculations [36]. As we have emphasized repeatedly, a precise knowledge of the structure of the oscillating terms in $S_n(\ell)$ is useful for extracting properties such as the central charge and scaling dimensions of certain operators at quantum critical points. They furthermore can be used for analysing numerical studies of more complicated quantities such as the entanglement of two disjoint intervals [38, 40].

Oscillating behaviour has also been observed in numerical studies of other entanglement estimators [51] such as the valence-bond entanglement. A natural question is whether these can be determined for certain models using the free fermion techniques we employed for the XX case as well.
Finally we would like to remark that our results carry over directly to the critical Ising chain. According to [12], the R`enyi entropies in the critical Ising model (with $c = 1/2$) are related to those of the spin-1/2 $XX$ chain in zero magnetic field ($k_F = \pi/2$) by

$$S_{1s}^\ell(\ell) = \frac{1}{2}S^X_{2\ell}(2\ell, k_F = \pi/2).$$

**Acknowledgments**

This work was supported in part by the EPSRC under grant EP/D050952/1 (FHLE) and by the ESF network INSTANS. We thank John Cardy and Bernard Nienhuis for helpful discussions.

**Appendix A. Relation between the expansion coefficients for $D_\ell$ and $x_\ell$**

In this appendix we report the relations between the coefficients $a_i$, $b_i$ and $c_i$ in the asymptotic expansion (84) of the Toeplitz determinant $D_\ell(\lambda)$ and the expansion coefficients $o_{j\ell}$, $q_{j\ell}$ characterizing the large-$\ell$ behaviour (87) of the auxiliary quantities $x_\ell$. The following relations hold:

$$o_{j1} = i(1 - 2\beta_\lambda)\cot k_F + \frac{a_1 - c_1}{2}(2j + 1),$$

$$o_{j2} = (j + \frac{1}{2})(a_2 - c_2) + (1 - 3\beta_\lambda + 3\beta_\lambda^2) + i\cot k_F(2j + 1 - (j + 1)\beta)(a_1 - c_1) + \frac{2j + 1}{8}(a_1 - c_1)((2j - 1)a_1 - (2j + 3)c_1) - \delta_{j0}\left[\frac{1 - \beta_\lambda}{2\sin k_F}\right]^2,$$

$$q_{j0} = -\beta_\lambda^2,$$

$$q_{j1} = \beta_\lambda^2\left[\frac{a_1 - c_1}{2} + i(1 - 2\beta_\lambda)\cot k_F\right] - c_1,$$

$$q_{j2} = \frac{(a_2 - c_2)\beta_\lambda^2}{2} - 3c_2 + \frac{a_1c_1(\beta_\lambda^2 + 2)}{4} - \frac{3a_1^2\beta_\lambda^2}{8} + \frac{c_1^2(\beta_\lambda^2 + 8)}{8} - \frac{\beta_\lambda^2(1 - 2\beta_\lambda + 5\beta_\lambda^2)}{2} + ((a_1 - c_1)\beta_\lambda^2 + (1 - 2\beta_\lambda)c_1)i\cot k_F + \frac{\beta_\lambda^2(3 - 14\beta_\lambda + 15\beta_\lambda^2)}{4\sin^2 k_F},$$

$$q_{20} = 0,$$

$$q_{21} = \beta_\lambda^2\left(\frac{a_1 + b_1}{2} - 4\beta_\lambda i\cot k_F\right) - \beta_\lambda^2(\beta_\lambda^2 + 1)c_1.$$  

(A.1)

We have derived relations for some other coefficients such as $o_{03}$ and $o_{13}$, but they are not needed for our purposes and we therefore refrain from reporting them here.

**Appendix B. Expansion coefficients for $x_\ell$**

Substituting the expansion (87) into the recurrence relation (74) gives a set of consistency relations for the coefficients $q_{j\ell}$ and $o_{j\ell}$ in (87). These read

$$q_{20} = q_{30} = q_{40} = q_{21} = q_{31} = q_{41} = 0,$$  

(B.1)

$$o_{11} = 3o_{01} - 2(1 - 2\beta_\lambda)i\cot k_F,$$  

(B.2)

doi:10.1088/1742-5468/2010/08/P08029
In terms of the constants $b$, $B$, $B_{pq}$

\begin{align}
q_{11} &= q_{10}(-a_{10} + 2b_1 \cot k_F), \\
o_{j1} &= a_{01} + 2j(a_{01} - (1 - 2\beta_1) \cot k_F), \\
o_{02} &= \frac{\beta_1(-1 + 12\beta_1 - 32\beta_1^2 + 27\beta_1^4)}{6} + \frac{\beta_1(2 - 13\beta_1 + 32\beta_1^2 - 18\beta_1^4)}{4 \sin^2 k_F}, \\
q_{12} &= \beta_1^2 a_{02} - \beta_1^4(4 + 9\beta_1^2) + \frac{\beta_1^4}{2 \sin^2 k_F}(13 + 18\beta_1^2).
\end{align}

Appendix C. Expressions for the coefficients $B_{pq}$

We recall the expressions for the coefficients $c_{1,2}$ and $b_{1,2}$ (89)

\begin{align}
c_1(\beta_1) &= 2\beta_1^3 \cot k_F, \\
c_2(\beta_1) &= \frac{\beta_1^2}{6}(-1 + 7\beta_1^2 + 12\beta_1^4 - 3\beta_1^4(5 + 4\beta_1^2) \csc^2 k_F), \\
b_j(\beta_1) &= c_j(1 + \beta_1), \quad j = 1, 2.
\end{align}

In terms of the constants $b_{j,q}^+, c_{j,q}^+$

\begin{align}
b_{j,q}^+ &= b_j(\frac{2q - 1}{2n} - \frac{1}{2}), \\
c_{j,q}^+ &= c_j(\frac{2q - 1}{2n} - \frac{1}{2}), \quad j = 1, 2,
\end{align}

the coefficients $B_{pq}$ are given by

\begin{align}
B_{pq}^{(n)} &= 2 \sin^2(k_F) \left[ b_{2,q}^+ - \frac{(b_{1,q}^+)^2}{2} - c_{2,q}^+ + \frac{(c_{1,q}^+)^2}{2} + \frac{p}{2}(b_{1,q}^+ - c_{1,q}^+)^2 \right].
\end{align}

If $2q - 1 = n, 3n, 5n, \ldots$ we instead have $B_{pq}^{(n)} = 0$. An explicit expression is

\begin{align}
B_{pq}^{(n)} &= \frac{x}{3} \left[ (5 + 28x^2) \sin^2(k_F) - 15(4x^2 + 1) \right] - \frac{p}{4} \left[ (1 + 12x^2) \cos(k_F) \right]^2 \bigg|_{x=(2q-1)/2n}.
\end{align}

References

[1] Amico L, Fazio R, Osterloh A and Vedral V, Entanglement in many-body systems, 2008 Rev. Mod. Phys. 80 517

[2] Calabrese P and Cardy J (ed), Entanglement entropy in extended systems, 2009 J. Phys. A: Math. Theor. 42 500301

[3] Calabrese P and Lelevre A, Entanglement spectrum in one-dimensional systems, 2008 Phys. Rev. A 78 032329

[4] Holzhey C, Larsen P and Wilczek F, Geometric and renormalized entropy in conformal field theory, 1994 Nucl. Phys. B 424 443

[5] Calabrese P and Cardy J, Entanglement entropy and quantum field theory, 2004 J. Stat. Mech. P06002

[6] Vidal G, Latorre J I, Rico E and Kitaev A, Entanglement in quantum critical phenomena, 2003 Phys. Rev. Lett. 90 227902

doi:10.1088/1742-5468/2010/08/P08029
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

Latorre J I, Rico E and Vidal G, *Ground state entanglement in quantum spin chains*, 2004 Quantum Inf. Comput. 4 048

[7] Peschel I, *On the entanglement entropy for a XY spin chain*, 2004 J. Stat. Mech. P12005

[8] Jin B-Q and Korepin V E, *Quantum spin chain, Toeplitz determinants and Fisher–Hartwig conjecture*, 2004 J. Stat. Phys. 116 79

[9] Its A R, Jin B-Q and Korepin V E, *Entanglement in XY spin chain*, 2005 J. Phys. A: Math. Gen. 38 2975

[10] Zhou H-Q, Barthel T, Fjaerestad J O and Schollwoeck U, *Entanglement entropy and twist fields*, 2008 J. High Energy Phys. JHEP11(2008)076

[11] De Chiara G, Montangero S, Calabrese P and Fazio R, *Entanglement entropy dynamics in Heisenberg chains*, 2006 J. Stat. Mech. P03001

[12] Igloi F and Juhasz R, *Exact relationship between the entanglement entropies of XY and quantum Ising chains*, 2008 Europhys. Lett. 81 57003

[13] Nienhuis B, Campostrini M and Calabrese P, *Entanglement, combinatorics and finite-size effects in spin-chains*, 2009 J. Stat. Mech. P02063

[14] Caraglio M and Gliozzi F, *Entanglement entropy and twist fields*, 2008 J. High Energy Phys. JHEP11(2008)076

[15] Alba V, Fagotti M and Calabrese P, *Entanglement entropy of excited states*, 2009 J. Stat. Mech. P10020

[16] Latorre J I and Riera A, *A short review on entanglement in quantum spin systems*, 2009 J. Phys. A: Math. Theor. 42 504002

[17] Gliozzi F and Tagliacozzo L, *Entanglement entropy and the complex plane of replicas*, 2010 J. Stat. Mech. P01002

[18] Casini H and Huerta M, *A finite entanglement entropy and the c-theorem*, 2004 Phys. Lett. B 600 142

[19] Casini H and Huerta M, *Entanglement and alpha entropies for a massive scalar field in two dimensions*, 2005 J. Stat. Mech. P12012

[20] Casini H, Fosco C D and Huerta M, *Entanglement and alpha entropies for a massive Dirac field in two dimensions*, 2005 J. Stat. Mech. P07007

[21] Cardy J L, Castro-Alvaredo O A and Doyon B, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, 2008 J. Stat. Phys. 130 129

[22] Castro-Alvaredo O A and Doyon B, *Bi-partite entanglement entropy in massive 1 + 1-dimensional quantum field theories*, 2009 J. Phys. A: Math. Theor. 42 504006

[23] Casini H and Huerta M, *Entanglement entropy in free quantum field theory*, 2009 J. Phys. A: Math. Theor. 42 504007

[24] Nishioka T, Ryu S and Takayanagi T, *Holographic entanglement entropy: an overview*, 2009 J. Phys. A: Math. Theor. 42 504008

[25] Headrick M, *Entanglement Rényi entropies in holographic theories*, 2010 arXiv:1006.0047

[26] Affleck I, *Universal term in the free energy at a critical point and the conformal anomaly*, 1986 Phys. Rev. Lett. 56 742

[27] Laeuchli A and Kollath C, *Spreading of correlations and entanglement after a quench in the one-dimensional Bose–Hubbard model*, 2008 J. Stat. Mech. P05018

[28] Xavier J C, *Entanglement entropy, conformal invariance and the critical behavior of the anisotropic spin-S Heisenberg chains: a DMRG study*, 2010 Phys. Rev. B 81 224404

[29] Feiguin A, Trebst S, Ludwig A W W, Troyer M, Kitaev A, Wang Z and Freedman M H, *Interacting anyons in topological quantum liquids: the golden chain*, 2007 Phys. Rev. Lett. 98 160409

[30] Song H F, Rachel S and Le Hur K, *General relation between entanglement and fluctuations in one dimension*, 2010 Phys. Rev. B 82 012405

[31] Campostrini M and Vicari E, *Scaling of bipartite entanglement in one-dimensional lattice systems, with a trapping potential*, 2010 J. Stat. Mech. P08020

[32] Campostrini M and Vicari E, *Quantum critical behavior and trap-size scaling of trapped bosons in a one-dimensional optical lattice*, 2010 Phys. Rev. A 81 063614

[33] Legeza O, Solyom J, Tincani L and Noack R M, *Entropic analysis of quantum phase transitions from uniform to spatially inhomogeneous phases*, 2007 Phys. Rev. Lett. 99 087203

[34] Tagliacozzo L, de Oliveira T R, Iblisdir S and Latorre J I, *Scaling of entanglement support for matrix product states*, 2008 Phys. Rev. B 78 024410

doi:10.1088/1742-5468/2010/08/P08029

26
Universal corrections to scaling for block entanglement in spin-1/2 XX chains

Pollmann F, Mukerjee S, Turner A M and Moore J E, Theory of finite-entanglement scaling at one-dimensional quantum critical points, 2009 Phys. Rev. Lett. 102 255701

Schuch N, Wolf M M, Verstraete F and Cirac J I, Entropy scaling and simulability by matrix product states, 2008 Phys. Rev. Lett. 100 030504

Perez-Garcia D, Verstraete F, Wolf M M and Cirac J I, Matrix product state representations, 2007 Quantum Inf. Comput. 7 401

Forrester P J and Witte N S, Discrete Painlevé equations, orthogonal polynomials on the unit circle and
Laflorencie N, Sorensen E S, Chang M-S and Affleck I, Boundary effects in the critical scaling of
Cardy J and Calabrese P, Unusual corrections to scaling in entanglement entropy
Eisler V and Garmon S S, Fano resonances and entanglement entropy
Calabrese P, Campostrini M, Essler F and Nienhuis B, Parity effects in the scaling of block entanglement in
Peschel I and Eisler V, Reduced density matrices and entanglement entropy in free lattice models
On reduced density matrices for disjoint subsystems
Igloi F and Peschel I, Entanglement entropy of two disjoint blocks in critical Ising models, 2010 Phys. Rev. B 81 060411

Calabrese P, Campostrini M, Essler F and Nienhuis B, Parity effects in the scaling of block entanglement in gapless spin chains, 2010 Phys. Rev. Lett. 104 095701

Cardy J and Calabrese P, Unusual corrections to scaling in entanglement entropy, 2010 J. Stat. Mech. P04023

Peschel I and Eisler V, Reduced density matrices and entanglement entropy in free lattice models, 2009 J. Phys. A: Math. Theor. 42 504003

Alba V, Tagliacozzo L and Calabrese P, Entanglement entropy of two disjoint blocks in critical Ising models, 2010 Phys. Rev. B 81 060411

Igloi F and Peschel I, On reduced density matrices for disjoint subsystems, 2010 Europhys. Lett. 89 40001

Fagotti M and Calabrese P, Entanglement entropy of two disjoint blocks in XY chains, 2010 J. Stat. Mech. P04016

Furukawa S, Pasquier V and Shiraiishi J, Mutual information and compactification radius in a c = 1 critical phase in one dimension, 2009 Phys. Rev. Lett. 102 170602

Calabrese P, Cardy J and Tonni E, Entanglement entropy of two disjoint intervals in conformal field theory, 2009 J. Stat. Mech. P11001

Fisher M E and Hartwig R E, Toeplitz determinants: some applications, theorems, and conjectures, 1968 Adv. Chem. Phys. 15 333

Basor E L and Tracy C A, The Fisher–Hartwig conjecture and generalizations, 1991 Physica A 177 167
Basor E L and Morrison K E, The Fisher–Hartwig conjecture and Toeplitz eigenvalues, 1994 Linear Algebr. Appl. 202 129

See e.g. http://mathworld.wolfram.com/BarnesG-Function.html

Ovchinnikov A A, Fisher–Hartwig conjecture and the correlators in XY spin chain, 2007 Phys. Lett. A 366 357

Franchini F and Abanov A G, Asymptotics of Toeplitz determinants and the emptiness formation probability for the XY spin chain, 2005 J. Phys. A: Math. Gen. 38 5069

Eisert J and Cramer M, Single-copy entanglement in critical spin chains, 2005 Phys. Rev. A 72 42112

Peschel I and Zhao J, On single-copy entanglement, 2005 J. Stat. Mech. P11002

Orus R, Latorre J I, Eisert J and Cramer M, Half the entanglement in critical systems is distillable from a single specimen, 2006 Phys. Rev. A 73 060303

Keating J P and Mezzadri F, Random matrix theory and entanglement in quantum spin chains, 2004 Commun. Math. Phys. 252 543

Keating J P and Mezzadri F, Entanglement in quantum spin chains, symmetry classes of random matrices, and conformal field theory, 2005 Phys. Rev. Lett. 94 050501

Forrester P J and Witte N S, Discrete Painlevé equations, orthogonal polynomials on the unit circle and N-recurrences for averages over U(N)–PVI τ -functions, 2003 arXiv:math-ph/0308036

doi:10.1088/1742-5468/2010/08/P08029

27
Universal corrections to scaling for block entanglement in spin-1/2 \(XX\) chains

Forrester P J and Witte N S, *Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems*, 2006 Constr. Approx. 24 201

[51] Alcaraz F C and Rittenberg V, *Shared information in stationary states at criticality*, 2010 J. Stat. Mech. P03024

Kallin A B, González I, Hastings M B and Melko R, *Valence bond and von Neumann entanglement entropy in Heisenberg ladders*, 2009 Phys. Rev. Lett. 103 117203

Alet F, Capponi S, Laflorencie N and Mambrini M, *Valence bond entanglement entropy*, 2007 Phys. Rev. Lett. 99 117204