Unbiased Multilevel Monte Carlo: Stochastic Optimization, Steady-state Simulation, Quantiles, and Other Applications

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Abstract

We present general principles for the design and analysis of unbiased Monte Carlo estimators in a wide range of settings. Our estimators possess finite work-normalized variance under mild regularity conditions. We apply our estimators to various settings of interest, including unbiased optimization in Sample Average Approximations, unbiased steady-state simulation of regenerative processes, quantile estimation and nested simulation problems.

1 Introduction

In this paper, we propose simple and yet powerful techniques that can be used to delete bias that often arise in the implementation of Monte Carlo computations in a wide range of decision making and performance analysis settings, for instance, stochastic optimization and quantile estimation, among others.

There are two key advantages of the estimators that we will present. Firstly, they can be easily implemented in the presence of parallel computing processors, yielding estimates whose accuracy improves as the size of available parallel computing cores increase while keeping the work-per-processor bounded in expectation. Secondly, the confidence intervals can be easily produced in settings in which variance estimators might be difficult to obtain (e.g. in stochastic optimization problems whose asymptotic variances depend on Hessian information).

To appreciate the advantage of parallel computing with bounded cost per parallel processor, let us consider a typical problem in machine learning applications, which usually involves a sheer amount of data. Because of the technical issues that arise in using the whole data set for training, one needs to resort techniques such as stochastic gradient descent, which is not easy to fully run in parallel, or sample-average approximations, which can be parallelized easily but it carries a systematic bias. For both the optimal value function and the optimal policies, we provide estimators that are unbiased, possess finite variance, and can be implemented in finite expected termination time. Thus, our estimators can be directly implemented in parallel with each parallel processor being assigned an amount of work which is bounded in expectation.

A second example that we shall consider as an application of our techniques arises in steady-state analysis of stochastic systems. A typical setting of interest is to compute a long-term average of expected cost or reward for running a stochastic system. This problem is classical in the literature of stochastic simulation and it has been studied from multiple angles. Our simple approach provides another way such that the steady-state analysis of regenerative processes can be done without any bias. A key characteristic is that the approach we study involves the same principle underlying the stochastic optimization setting mentioned in the previous paragraph.

Other types of problems that we are able to directly address using our methodology include computing unbiased estimators of quantiles and unbiased estimators of nested expectations. In addition to being unbiased, all of the estimators have finite work-normalized variance and can be simulated in finite expected termination time, which makes their implementation in parallel computation straightforward.

Applications such as stochastic optimization and quantile estimation allow us to highlight the fact that the variance estimates of our Monte Carlo estimators are straightforward to produce. These variance estimates are important for us to generate asymptotically accurate confidence intervals. In contrast, even though asymptotically unbiased estimators may be available, sometimes these estimators require information about

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Hessians (as in the optimization setting) or even density information (as in quantile estimation applications) to produce accurate confidence intervals, while our estimators do not require this type of information.

Our estimator relates to the multilevel-Monte Carlo method developed in [13]. We apply the debiasing techniques introduced in [19] and [18]. Since the introduction of these techniques, several improvements and applications have been studied, mostly in the context of stochastic differential equations and partial differential equations with random input, see for example [15], [1], [16] and [9].

In [21], a stratified sampling technique is introduced in order to show that the debiasing in multi-level Monte Carlo can be achieved virtually at no cost in either asymptotic efficiency or sample complexity relative to the standard (biased) MLMC estimator. The results of [21] can be applied directly to our estimators in order to improve the variance, but the qualitative rate of convergence (i.e. $O(1/\varepsilon^2)$ remains the same).

A recent and independent paper [14] also studies the nested simulation problem. While the techniques that we propose here are very related to theirs, our framework is postulated in greater generality, which involves not only nested simulation problems but other applications as described earlier as well.

Another recent paper [11] studies multi-level Monte Carlo in the context of stochastic optimization, but their setting is different from what we consider here and they give a completely different class of algorithms which are not unbiased. Another recent paper [8] also develops optimal $O(1/\varepsilon^2)$ sample complexity algorithms for nested simulation problems, but they focus on specific settings and do not develop a framework as general as what we consider here.

The rest of the paper is organized as follows. In Section 2 we discuss the general principle which drives the construction of our unbiased estimators. Then, we apply these principles to the different settings of interest, namely, unbiased estimators for non-linear functions of expectations, stochastic convex optimization, quantile estimation and nested simulation problems, in later sections.

## 2 The General Principles

The general principles are based on the work of [19]. Suppose that one is interested in estimating a quantity of the form $\theta(\mu) \in \mathbb{R}$, where $\mu$ is a generic probability distribution, say with support in a subset of $\mathbb{R}^d$, and $\theta(\cdot)$ is a non-linear map.

A useful example to ground the discussion in the mind of the reader is $\theta(\mu) = g(E_{\mu} [X])$, where $g : \mathbb{R}^d \to \mathbb{R}$ is a given function (with regularity properties which will be discussed in the sequel). We use the notation $E_{\mu} (\cdot)$ to denote the expectation operator under the probability distribution $\mu$. For the sake of simplicity, we will later omit the subindex $\mu$ when the context is clear.

We consider the empirical measure $\mu_n$ of iid samples $\{X_i \in \mathbb{R}^d : 1 \leq i \leq n\}$, i.e.,

$$\mu_n (dx) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} (dx),$$

where $\delta_{X_i} (\cdot)$ is the point mass at $X_i$ for $i = 1, \ldots, n$. The sample complexity of producing $\mu_n (\cdot)$ is equal to $n$. By the strong law of large numbers for empirical measures (Varadarajan’s Theorem), $\mu_n \to \mu$ almost surely in the Prohorov space.

Under mild continuity assumptions, we have that

$$\theta (\mu_n) \to \theta (\mu) \quad (1)$$

as $n \to \infty$ and often one might expect that $\theta (\mu_n)$ is easy to compute. Then, $\theta (\mu_n)$ becomes a natural and reasonable estimator for $\theta (\mu)$. However, there are several reasons that make it desirable to construct an unbiased estimator with finite variance, say $Z$, for $\theta (\mu)$; even if $(\theta (\mu_n))_{n \geq 1}$ is asymptotically normal in the sense that $n^{1/2} (\theta (\mu_n) - \theta (\mu)) \Rightarrow \mathcal{N} (0, \sigma^2_\theta)$ with some $\sigma^2_\theta > 0$. First, as we mentioned in the introduction, if one copy of $Z$ can be produced in finite expected time, averaging the parallel replications of $Z$ immediately yields an estimate of $\theta (\mu)$, whose accuracy then can be increased by the Central Limit Theorem as the number of parallel replications of $Z$ increases. Second, the variance of $Z$ can be estimated with the natural variance estimator of iid replications of $Z$, when $\sigma^2_\theta$ may be difficult to evaluate from the samples (e.g. if $\theta (\mu)$ represents some quantile of $\mu$).

Our goal is to construct a random variable $Z$ such that

$$E [Z] = \theta (\mu), \text{ Var} (Z) < \infty,$$
and the expected sample complexity to produce $Z$ is bounded. To serve this goal, we first construct a sequence of random variables $\{\Delta_m : m \geq 1\}$ satisfying the following properties:

Assumption 1. General assumptions

(i) There exists some $\alpha, c \in (0, \infty)$ such that $E\left[|\Delta_m|^2\right] \leq c \cdot 2^{-\alpha m}$,

(ii) $\sum_{m=0}^{\infty} E[\Delta_m] = \theta(\mu)$,

(iii) If $C_m$ is the computational cost of producing one copy of $\Delta_m$ (measured in terms of sampling complexity), then $E[C_m] \leq c' \cdot 2^m$ for some $c' \in (0, \infty)$.

If we are able to construct the sequence $\{\Delta_m : m \geq 1\}$ satisfying Assumption 1, then we can construct an unbiased estimator for $Z$. First, sample $N$ from geometric distribution with success parameter $r$, so that $p(k) = P(N = k) = r(1 - r)^k$ for $k \geq 0$. The parameter $r \in (0, 1)$ will be optimized shortly. At this point it suffices to assume that $r \in \left(\frac{1}{2}, 1 - \frac{1}{\sqrt{1 + \alpha}}\right)$.

Once the distribution of $N$ has been specified, the estimator that we consider takes the form

$$Z = \frac{\Delta_N}{p(N)} \tag{2}$$

where $N$ is independent of the iid sequence $\{\Delta_m\}_{m=1}^{\infty}$. Note that the estimator possesses finite variance because $r < 1 - \frac{1}{\sqrt{1 + \alpha}}$.

$$E\left[Z^2\right] = \sum_{k=0}^{\infty} E\left[Z^2 \mid N = k\right] p(k) = \sum_{k=0}^{\infty} E\left[\frac{\Delta_k^2}{p(k)^2} \mid N = k\right] p(k) \tag{3}$$

$$= \sum_{k=0}^{\infty} E\left[\frac{\Delta_k^2}{p(k)}\right] \leq c \sum_{k=0}^{\infty} \frac{2^r \cdot 2^{-(1 + \alpha)k}}{p(k)} = \frac{c}{r} \sum_{k=0}^{\infty} \left(\frac{2r}{(1 + \alpha)(1 - r)}\right)^k < \infty.$$ 

Moreover, the unbiasedness of the estimator is ensured by Assumption 1(iii)

$$E[Z] = \sum_{k=0}^{\infty} E[Z \mid N = k] p(k) = \sum_{k=0}^{\infty} E\left[\frac{\Delta_k}{p(k)}\right] p(k) = \theta(\mu).$$

Finally, because $r > 1/2$, the expected sampling complexity of producing $Z$, denoted by $C$, is finite precisely by Assumption 1(iii)

$$E[C] = E[C_N] \leq c' \sum_{k=0}^{\infty} 2^{rk} p(k) = r c' \sum_{k=0}^{\infty} (2(1 - r))^k < \infty. \tag{4}$$

In [19] the choice of $N$ is optimized in terms of the $E[\Delta_n^2]$ and $E[C_n]$. In [6] a bound on the work-normalized variance, namely the product

$$\sum_{k=0}^{\infty} \frac{2^{r(1 + \alpha)k}}{p(k)} \times \sum_{k=0}^{\infty} 2^{rk} p(k),$$

corresponding to the bounds in the right-hand side of [3] and [4] is optimized, and the result shows optimal choice of $p(k)$'s corresponds to choosing $N$ geometrically distributed with $r = 1 - 1/2$ when $\alpha = 1$. Following the same logic, the optimal choice of $N$ should be geometrically distributed with $r = 1 - 2^{-1/(1 + \alpha)}$ for the general $\alpha > 0$ case and we advocate this choice for the construction of $Z$.

The contribution of our work is to study the construction of the $\Delta_m$'s based on the sequence $\{\mu_n : n \geq 1\}$ satisfying Assumption 1 as we now explain. Now our focus is on explaining the high-level ideas at an informal level and provide formal assumptions later for different settings.

Suppose that there exists a function $T^\mu_\mu : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\frac{d}{dt} \theta(\mu + t(\mu_n - \mu)) \bigg|_{t=0} = \int T^\mu_\mu(x) d(\mu_n - \mu) = E_{\mu_n}[T^\mu_\mu(X)] - E_{\mu}[T^\mu_\mu(X)].$$

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Typically, $T^\theta_\mu (\cdot)$ corresponds to the Riesz representation (if it exists) of the derivative of $\theta (\cdot)$ at $\mu$. Going back to the case in which $\theta (\mu) = g (E\mu [X])$, assuming that $g (\cdot)$ is differentiable with derivative $Dg (\cdot)$, we have

$$\frac{d}{dt} g (E\mu [X] + t (E\mu_n [X] - E\mu [X])) \bigg|_{t=0} = Dg (E\mu [X]) \cdot (E\mu_n [X] - E\mu [X]),$$

so in this setting $T^\theta_\mu (x) = Dg (\int z\mu (dz)) \cdot x = Dg (E\mu [X]) \cdot x$. Now, suppose that $\theta (\cdot)$ is smooth in the sense that

$$\theta (\mu) = \theta (\mu_n) + (E\mu_n [T^\theta_\mu (X)] - E\mu_n [T^\theta_\mu (X)]) + \epsilon (n, \mu_n),$$

where

$$|\epsilon (n, \mu_n)| = O_p \left( \left( \left( E\mu_n [T^\theta_\mu (X)] - E\mu_n [T^\theta_\mu (X)] \right)^2 \right) \right).$$

This control of the error of the first order Taylor expansion of the map $t \mapsto \theta (\mu + t (\mu_n - \mu))$ around $t = 0$. So, in the context in which $\theta (\mu) = g (E\mu [X])$, if $g (\cdot)$ is twice continuously differentiable, then we have

$$\epsilon (n, \mu_n) = \frac{1}{2} (E\mu_n [X] - E\mu_n [X])^T \cdot (D^2 g) (E\mu [X]) \cdot (E\mu_n [X] - E\mu [X]) + o (1),$$

as $n \rightarrow \infty$.

The key ingredient in the construction of the sequence $\{\Delta_m : m \geq 1\}$ is an assumption of the form

$$\sup_{n \geq 1} n^2 E \left[ \left| E\mu_n [T^\theta_\mu (X)] - E\mu_n [T^\theta_\mu (X)] \right|^4 \right] < \infty,$$

this assumption will typically be followed as an strengthening of a Central Limit Theorem expansion to the limit $\mu_n \rightarrow \mu$ as $n \rightarrow \infty$, which would typically yield

$$n^{1/2} \{ E\mu [T^\theta_\mu (X)] - E\mu_n [T^\theta_\mu (X)] \} \rightarrow W,$$

as $n \rightarrow \infty$ for some $W$. Under (6) the construction of $\Delta_n$, satisfying Assumption 1(iii), proceeds as follows. Let

$$\mu^{E}_n (dx) = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{(X_{2i})} (dx), \quad \mu^{O}_n (dx) = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{(X_{2i-1})} (dx)$$

and set for $n \geq 1$,

$$\Delta_n = \theta (\mu^{2n+1}) - \frac{1}{2} \left( \theta (\mu^{E}_n) + \theta (\mu^{O}_n) \right).$$

The key property behind the construction for $\Delta_n$ in (7) is that

$$\mu^{2n+1} = \frac{1}{2} \left( \mu^{E}_n + \mu^{O}_n \right),$$

so a linearization of $\theta (\mu)$ will cancel the first order effects implied in approximating $\mu$ by $\mu^{2n+1}, \mu^{O}_n$ and $\mu^{E}_n$. In particular, using (5) directly we have that

$$|\Delta_n| \leq |\epsilon (2^{2n+1}, \mu^{2n+1})| + \left| \epsilon (2^n, \mu^{O}_n) \right| + \left| \epsilon (2^n, \mu^{E}_n) \right|,$$

consequently due to (6) we have that

$$E \left[ |\Delta_n|^2 \right] = O (2^{-2n}).$$

Once (8) is in place, verification of Assumption 1(iii) is straightforward because

$$E [\theta (\mu^{2n+1})] = E [\theta (\mu^{2n+1})] - E [\theta (\mu^{2n+1})],$$

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so if we define
\[ \Delta_0 = \theta(\mu_2), \]
then
\[ \sum_{n=0}^{\infty} E(\Delta_n) = \theta(\mu). \]
Assumption 1(iii) follows directly because the sampling complexity required to produce \( \Delta_m \) is \( C_m = 2^{m+1} \) (assuming each \( X_i \) required a unit of sample complexity).

The rest of the paper is dedicated to the analysis of (7). The abstract approach described here, in terms of the derivative of \( \theta(\mu) \), sometimes is cumbersome to implement under the assumptions that are natural in the applications of interest (for example stochastic optimization). So, we may study the error in (7) directly in later applications, but we believe that keeping the high-level intuition described here is useful to convey the generality of the main ideas.

3 Non-linear functions of expectations and applications

We first apply the general principle to the canonical example considered in our previous discussion, namely
\[ \theta(\mu) = g \left( \int y d\mu(y) \right) = g(E_\mu[X]). \]
Let \( \nu = E_\mu[X] \). We will impose natural conditions on \( g(\cdot) \) to make sure that the principles discussed in Section 2 can be directly applied.

We use \((X_k : k \geq 1)\) to denote an iid sequence of copies of the random variable \( X \in \mathbb{R}^d \) from distribution \( \mu \). For \( k \geq 1 \), we define
\[ X^O_k = X_{2k-1} \quad \text{and} \quad X^E_k = X_{2k}. \]
Note that the \( X^O \)'s correspond to \( X_k \)'s indexed by odd values and the \( X^E \)'s correspond to the \( X_k \)'s indexed by even values. For \( k \in \mathbb{N}_+ \), let
\[ S_k = X_1 + \ldots + X_k \]
and similarly let
\[ S^O_k = X^O_1 + \ldots + X^O_k, \]
\[ S^E_k = X^E_1 + \ldots + X^E_k. \]
In this setting, we may define
\[ \Delta_n = g \left( \frac{S_{2n+1}}{2^{n+1}} \right) - \frac{1}{2} g \left( \frac{S^O_{2n}}{2^n} \right) + g \left( \frac{S^E_{2n}}{2^n} \right) \]
for \( n \geq 0 \) and let the estimator to be
\[ Z = \frac{\Delta_N}{p(N)} + g(X_1), \quad (9) \]
where \( N \) was defined in Section 2.

We now impose precise assumptions on \( g(\cdot) \), so that Assumption 1 can be verified for \( \Delta_n \). We summarize our discussion in Theorem 1 next.

**Theorem 1.** Suppose that the following assumptions are forced:

1. Suppose that \( g : \mathbb{R}^d \to \mathbb{R} \) has linear growth of the form \( |g(x)| \leq c_1 (1 + \|x\|_2) \) for some \( c_1 > 0 \), where \( \|\cdot\|_2 \) denotes the \( l_2 \) norm in Euclidian space,

2. Suppose \( g \) is continuously differentiable in a neighborhood of \( \nu = E[X] \), and \( Dg(\cdot) \) is locally Holder continuous with exponent \( \alpha > 0 \), i.e.,
\[ \|Dg(x) - Dg(y)\|_2 \leq \kappa(x) \|x - y\|_2^\alpha, \]
where \( \kappa(\cdot) \) is bounded on compact sets.
3. $X$ has finite $3(1 + \alpha)$ moments, i.e. $E \left[ |X|^{3(1+\alpha)} \right] < \infty$.

Then, $E[Z] = g(E[X]), \text{Var}(Z) < \infty$ and the sampling complexity required to produce $Z$ is bounded in expectation.

Proof. We first show the unbiasedness of the estimator $Z$. From 4 we have that

$$|g(S_n/n)|^2 \leq c_1 (1 + \|S_n/n\|_2^2).$$

And because of 3 we have $E\left[ |g(S_n/n)|^2 \right] < \infty$, which implies that $g(S_n/n)$ is uniformly integrable. For each $n \geq 0$,

$$E[\Delta_n] = E\left[ g\left( S_{2n+1}/2^{n+1} \right) \right] - E\left[ g\left( S_{2n}/2^n \right) \right].$$

With the condition in 2 that $g$ is continuous in a neighborhood of $\nu$, we derive

$$E[Z] = E\left[ \frac{\Delta_n}{p(N)} \right] + E[g(X_1)] = \sum_{n=1}^{\infty} E[\Delta_n] + E[g(X_1)] = \lim_{n \to \infty} E\left[ g\left( S_{2n}/2^n \right) \right] = E(g(X)).$$

Next we show $E[\Delta_n^2] = O \left( 2^{-(1+\alpha)n} \right)$ for all $n \geq 0$. We pick $\delta > 0$ small enough so that $g(\cdot)$ is continuously differentiable in a neighborhood of $\nu$ and the locally Holder continuous condition holds as well.

$$|\Delta_n| = |\Delta_n| I\left( \max \left( \|S_{2n}^O/2^n - \nu\|_2, \|S_{2n}^E/2^n - \nu\|_2 \right) > \delta/2 \right) + |\Delta_n| I\left( \|S_{2n}^O/2^n - \nu\|_2 \leq \delta/2, \|S_{2n}^E/2^n - \nu\|_2 > \delta/2 \right) \leq |\Delta_n| I\left( \|S_{2n}^O/2^n - \nu\|_2 > \delta/2 \right) + |\Delta_n| I\left( \|S_{2n}^E/2^n - \nu\|_2 > \delta/2 \right)

$$

When $\|S_{2n}^O/2^n - \nu\|_2 \leq \delta/2$ and $\|S_{2n}^E/2^n - \nu\|_2 \leq \delta/2$, we have $\|S_{2n+1}/2^{n+1} - \nu\|_2 \leq \delta$ and $\|S_{2n}^O/2^n - S_{2n}^E/2^n\|_2 \leq \delta$, thus

$$\Delta_n = \frac{1}{2} \left( g\left( S_{2n+1}/2^{n+1} \right) - g\left( S_{2n}/2^n \right) \right) + \frac{1}{2} \left( g\left( S_{2n}/2^n \right) - g\left( S_{2n+1}/2^{n+1} \right) \right)$$

$$= \frac{1}{4} Dg\left( \xi_n^O \right)^T \frac{S_{2n}^E}{2^n} \frac{S_{2n}^O}{2^n} - \frac{S_{2n}^O}{2^n} + \frac{1}{4} Dg\left( \xi_n^E \right)^T \frac{S_{2n}^O}{2^n} - \frac{S_{2n}^E}{2^n}$$

$$= \frac{1}{4} \left( Dg\left( \xi_n^O \right) - Dg\left( \xi_n^E \right) \right)^T \frac{S_{2n}^E - S_{2n}^O}{2^n}.$$

where $\xi_n^O$ is some value between $S_{2n}^O/2^n$ and $S_{2n+1}/2^{n+1}$, and $\xi_n^E$ is some value between $S_{2n}^E/2^n$ and $S_{2n+1}/2^{n+1}$. It is not hard to see that

$$\|\xi_n^O - \xi_n^E\|_2 \leq \left\| \frac{U_n^O + U_n^E}{2} \right\|_2 \left( \frac{S_{2n}^O}{2^n} - \frac{S_{2n}^E}{2^n} \right) \leq \frac{S_{2n}^E}{2^n} - \frac{S_{2n}^O}{2^n} \leq 2^{-(1+\alpha)n}.$$

Hence, using the fact that $\kappa(\cdot)$ is bounded on compact sets, we have that there exists a deterministic constant $c \in (0, \infty)$ (depending on $\delta$) such that

$$E\left[ |\Delta_n|^2 I\left( \|S_{2n}/2^n - \nu\|_2 \leq \delta/2, \|S_{2n}^E/2^n - \nu\|_2 \leq \delta/2 \right) \right] \leq cE\left( \frac{S_{2n}^O - S_{2n}^E}{2^n} \right)^{2(1+\alpha)} = O \left( 2^{-(1+\alpha)n} \right),$$

where the last estimate follows from 5.

On the other hand, in order to analyze,

$$E \left[ |\Delta_n|^2 I\left( \|S_{2n}/2^n - \nu\|_2 > \delta/2 \right) \right].$$

(10)
If we could assume that the $X_i$’s have a finite moment generating function in a neighborhood of the origin it would be easy to see that $\{10\}$ decays at a speed which is $o \left( 2^{-n(1+\alpha)} \right)$ (actually the rate would be superexponentially fast in $n$). However, we are not assuming the existence of a finite moment generating function, but we are assuming the existence of finite second moments. The intuition that we will exploit is that the large deviations event that is being introduced in $\{10\}$ would be driven (in the worst case) by a large jump (that is, we operate based on intuition borrowed from large deviations theory for heavy-tailed superexponentially fast in $n$).

We then obtain that by selecting $\delta' > 0$ small to be determined in the sequel, the set

$$A_n = \{1 \leq i \leq 2^n : \|X_i - v\|_2 \geq 2^{n(1-\delta')}\}$$

and $N_n = |A_n|$. In simple words, $N_n$ is the number of increments defining $S_{2^n}^0$ which are large. Note that

$$E \left[ |\Delta_n|^2 I \left( \|S_{2^n}^0/2^n - \nu\|_2 > \delta/2 \right) \right] = E \left[ |\Delta_n|^2 I \left( \|S_{2^n}^0/2^n - \nu\|_2 > \delta/2 \right) I (N_n = 0) \right] + E \left[ |\Delta_n|^2 I \left( \|S_{2^n}^0/2^n - \nu\|_2 > \delta/2 \right) I (N_n \geq 1) \right].$$

We can easily verify using Chernoff’s bound that for any $\gamma > 0$, we have that

$$P \left( \|S_{2^n}^o/2^n - \nu\|_2 > \delta/2 | N_n = 0 \right) = o \left( 2^{-n\gamma} \right),$$

this implies that

$$E \left[ |\Delta_n|^2 I \left( \|S_{2^n}^0/2^n - \nu\|_2 > \delta/2 \right) I (N_n = 0) \right] \leq E \left[ |\Delta_n|^{2(1+\alpha)} \right]^{1/(1+\alpha)} P \left( \|S_{2^n}^o/2^n - \nu\|_2 > \delta/2, N_n = 0 \right)^{\alpha/(1+\alpha)} = o \left( 2^{-n(1+\alpha)} \right).$$

On the other hand, note that

$$2^{-2n} E \left[ \|X_i - v\|_2^2 I \left( \|X_i - v\|_2 > 2^{n(1-\delta')(1+\alpha)} \right) \right] = 2^{-2n+1} \int_{2^{n(1-\delta')(1+\alpha)}}^\infty \frac{t P \left( \|X_i - v\|_2 > t \right) dt}{t^{2+3\alpha}} \leq 2^{-2n+1} \int_{2^{n(1-\delta')(1+\alpha)}}^\infty \frac{E \left( \|X_i - v\|_2^{3(1+\alpha)} \right) dt}{t^{2+3\alpha}} = O \left( 2^{-2n-n(1+3\alpha)(1-\delta')(1+\alpha)} \right).$$

Using the previous estimate, it follows easily that

$$E \left[ |\Delta_n|^2 I (N_n = 1) \right] = O \left( 2^{n \cdot 2^{-2n-n(1+3\alpha)(1-\delta')(1+\alpha)}} \right).$$

The previous expression is $O \left( 2^{-2n} \right)$ if $\delta' > 0$ is chosen sufficiently small. Similarly, for any fixed $k$,

$$E \left[ |\Delta_n|^2 I (N_n = k) \right] = O \left( 2^{(k-2)n \cdot 2^{-n \cdot k(1+3\alpha)(1-\delta')(1+\alpha)}} \right) = O \left( 2^{-2n} \right).$$

On the other hand,

$$P (N_n \geq m) = O \left( 2^{nm \cdot 2^{-nm \cdot 3(1-\delta')(1+\alpha)}} \right).$$

We then obtain that by selecting $\delta' > 0$ sufficiently small and $m$ large so that

$$m \cdot \alpha \left( 3(1-\delta') - \frac{1}{1+\alpha} \right) \geq 2,$$

we conclude

$$E \left[ |\Delta_n|^2 I (N_n \geq m) \right] \leq E \left[ |\Delta_n|^{2(1+\alpha)} \right]^{1/(1+\alpha)} P (N_n \geq m)^{\alpha/(1+\alpha)} = O \left( 2^{-2n} \right).$$
Consequently, we have that
\[ E \left[ \| \Delta_n \|^2 I \left( \frac{\| S_{2n}^n \|}{2^n} > \delta/2 \right) I (N_n \geq 1) \right] \]
\[ \leq \sum_{k=1}^{m-1} E \left[ \| \Delta_n \|^2 I (N_n = k) \right] + E \left[ \| \Delta_n \|^2 I (N_n \geq m) \right] = O (2^{-2n}). \]

A similar analysis yields that
\[ E \left[ \| \Delta_n \|^2 I \left( \max \left( \frac{\| S_{2n}^n \|}{2^n} - \nu \| \cdot \|_2 \right), \frac{\| S_{2n}^n \|}{2^n} - \nu \| \cdot \|_2 \right) > \delta/2 \right) \right] = O (2^{-2n}), \]
therefore the estimator \( Z \) has finite variance.

Finally, the sampling complexity of producing one copy of \( \Delta_n \) is
\[ C_n = 2^{n+1} + c = O (2^n) \]
with some constant \( c > 0 \).

### 3.1 Application to steady-state regenerative simulation

The context of steady-state simulation provides an important instance in which developing unbiased estimators is desirable. Recall that if \( (W(n) : n \geq 0) \) is a positive recurrent regenerative process taking values on some space \( \mathcal{Y} \), then for all measurable set \( A \), we have the following limit holds with probability one

\[ \pi(A) := \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} I (W(n) \in A) = \frac{E_0 \left[ \sum_{n=0}^{\tau-1} I (W(n) \in A) \right]}{E_0 [\tau]}, \]

where the notation \( E_0 \) indicates that \( W(\cdot) \) is zero-delayed under the associated probability measure \( P_0 (\cdot) \). The limiting measure \( \pi(\cdot) \) is the unique stationary distribution of the process \( W(\cdot) \); for additional discussion on regenerative processes see the appendix on regenerative process in [3], and also [2]. Most ergodic Markov chain that arise in practice are regenerative; certainly all irreducible and positive recurrent countable state-space Markov chains are regenerative.

A canonical example which is useful to keep in mind to conceptualize a regenerative process is the waiting time sequence of the single server queue. In which case, it is well known that the waiting time of the \( n \)-th customer, \( W(n) \), satisfies the recursion \( W(n+1) = \max (W(n) + Y(n+1), 0) \), where the \( Y(n) \)'s form an iid sequence of random variables with negative mean. The waiting time sequence regenerates at zero, so if \( W(0) = 0 \), the waiting time sequence forms a zero-delayed regenerative process. Let \( f(\cdot) \) be a bounded measurable function and write
\[ X_1 = \sum_{n=1}^{\tau-1} f (W(n)) \quad \text{and} \quad X_2 = \tau, \]
then we can estimate the stationary expectation \( E_\pi f (W) \) via the ratio
\[ E_\pi [f (W)] = \frac{E_0 [X_1]}{E_0 [X_2]} . \] (11)

Since \( \tau \geq 1 \), it follows that for \( g(x_1, x_2) = x_1/x_2 \), assumptions can be easily verified and therefore Theorem 1 applies, we need to assume that \( E \left( |X_1|^{3+\varepsilon} \right) < \infty \) and \( E \left( \tau^{3+\varepsilon} \right) < \infty \) for some \( \varepsilon > 0 \).

### 3.2 Additional applications

In addition to steady-state simulation, ratio estimators such (11) arise in the context of particle filters and state-dependent importance sampling for Bayesian computations, see [10] and [17].

In the context of Bayesian inference, one is interested in estimating expectations from some density \( (\pi(y) : y \in \mathcal{Y}) \) for the form \( \pi(y) = h(y)/\gamma \), where \( h(\cdot) \) is a non-negative function with a given (computable) functional form and \( \gamma > 0 \) is a normalizing constant which is not computable, but is well defined (i.e. finite) and ensures that \( \pi(\cdot) \) is indeed a well defined density on \( \mathcal{Y} \). Since \( \gamma > 0 \) is unknown one must resort to
techniques such as Markov chain Monte Carlo or sequential importance sampling to estimate \( E_q [f(Y)] \) (for any integrable function \( f(\cdot) \)), see for instance \cite{17}.

Ultimately, the use of sequential importance samplers or particle filters relies on the identity

\[
E_q [f(Y)] = E_q \left[ \frac{h(Y)}{q(Y)} f(Y) \right] / E_q \left[ \frac{h(Y)}{q(Y)} \right],
\]

where \((q(y) : y \in Y)\) is a density on \( Y \) and \( E_q [\cdot] \) denotes the expectation operator associated to \( q(\cdot) \) (and we use \( P_q(\cdot) \) for the associated probability). Of course, we must have that the likelihood ratio \( \pi(Y)/q(Y) \) well defined almost surely with respect to \( P_q(\cdot) \) and

\[
E_q \left[ \frac{\pi(Y)}{q(Y)} \right] = 1.
\]

Thus, by using sequential importance sampling or particle filters one produces a ratio estimator \cite{12} and therefore the application of our result in this setting is very similar to the one described in the previous subsection. The verification of Theorem \( 1 \) requires additional assumption on the selection of \( q(\cdot) \), which should have heavier tails than \( \pi(\cdot) \) in order to satisfy Assumption 3.

4 Stochastic Convex Optimization

In this section we study a wide range of stochastic optimization problems and we show that the general principle applies. This section studies situations in which, going back to Section \( 2 \), the derivative \( T^\theta \) may be difficult to characterize and analyze, but the general principle is still applicable. So, in this section we study its applications directly.

Consider the following constrained stochastic convex optimization problem

\[
\min_{\beta} \quad f(\beta) = E_\mu [F(\beta, X)] \\
\text{s.t.} \quad G(\beta) \leq 0,
\]

where \( D = \{\beta \in \mathbb{R}^d : F(\beta) \leq 0\} \) is a nonempty closed subset of \( \mathbb{R}^d \). \( f \) is a convex map from \( \mathbb{R}^d \) to \( \mathbb{R} \). \( G(\beta) = (g_1(\beta), \ldots, g_m(\beta))^T \) is a vector-valued convex function for some \( m \in \mathbb{N} \). \( X \) is a random vector whose probability distribution \( \mu \) is supported on a set \( \Omega \subset \mathbb{R}^k \), and \( F : D \times \Omega \rightarrow \mathbb{R} \).

Let \( \beta_* \) denote the optimal solution and \( f_* = f(\beta_*) \) denote the optimal objective value. Lagrangian of problem \( 13 \) is

\[
L(\beta, \lambda) = f(\beta) + \lambda^T G(\beta).
\]

If \( f(\cdot) \) and \( g_i(\cdot) \)'s are continuously differentiable for \( i = 1, \ldots, m \), the following Karush-Kuhn-Tucker (KKT) conditions are sufficient and necessary for optimality:

\[
\nabla f(\beta_*) + \nabla G(\beta_*) \lambda_* = 0,
\]

\[
G(\beta_*) \leq 0,
\]

\[
\lambda_+ G(\beta_*) = 0,
\]

\[
\lambda_* \geq 0,
\]

where \( \lambda_* \in \mathbb{R}^m \) is the Lagrangian multiplier corresponding to \( \beta_* \).

One of the standard tools in such settings is the method of Sample Average Approximation (SAA), which consists in replacing the expectations by the empirical means. Suppose we have \( n \) iid copies of the random vector \( X \), denoted as \( \{X_1, \ldots, X_n\} \), we solve the following optimization problem

\[
\min_{\beta} \quad f_n(\beta) = \frac{1}{n} \sum_{i=1}^n F(\beta, X_i) \\
\text{s.t.} \quad G(\beta) \leq 0
\]

as an approximation to the original problem \( 13 \). Let \( \beta_n \) denote the optimal solution and let \( \hat{f}_n = f_n(\beta_n) \) denote the optimal value of the SAA problem \( 19 \). The traditional SAA approach is to use them as estimators to the true optimal solution \( \beta_* \) and optimal target value \( f_* \) of the problem \( 13 \). Although the SAA estimators are easy to construct and consistent, they are biased. Proposition 5.6 in \cite{20} shows \( E[\hat{f}_n] \leq f_* \) for any \( n \in \mathbb{N} \).
We construct unbiased estimators for the optimal solution and optimal value of problem (13) by utilizing the SAA estimators. Let $\beta_{2n+1}$, $\beta_{2n}^O$, $\beta_{2n}^E$ denote the SAA optimal solutions as

$$
\beta_{2n+1} = \arg\min_{G(\beta) \leq 0} f_{2n+1}(\beta) = \arg\min_{G(\beta) \leq 0} \frac{1}{2n+1} \sum_{i=1}^{2n+1} F(\beta, X_i),
$$

$$
\beta_{2n}^O = \arg\min_{G(\beta) \leq 0} f_{2n}^O(\beta) = \arg\min_{G(\beta) \leq 0} \frac{1}{2n} \sum_{i=1}^{2n} F(\beta, X_i^O),
$$

$$
\beta_{2n}^E = \arg\min_{G(\beta) \leq 0} f_{2n}^E(\beta) = \arg\min_{G(\beta) \leq 0} \frac{1}{2n} \sum_{i=1}^{2n} F(\beta, X_i^E).
$$

Let $\hat{f}_{2n+1} = f_{2n+1}(\beta_{2n+1})$, $\hat{f}_{2n}^O = f_{2n}^O(\beta_{2n}^O)$ and $\hat{f}_{2n}^E = f_{2n}^E(\beta_{2n}^E)$ denote the SAA optimal values. Similarly we let $\lambda_{2n+1}$, $\lambda_{2n}^O$ and $\lambda_{2n}^E$ denote the corresponding Lagrange multipliers.

We define

$$
\Delta_n = \hat{f}_{2n+1} - \frac{1}{2} \left( \hat{f}_{2n}^O + \hat{f}_{2n}^E \right) \quad \text{and} \quad \tilde{\Delta}_n = \beta_{2n+1} - \frac{1}{2} (\beta_{2n}^O + \beta_{2n}^E)
$$

for all $n \geq 0$, then the estimator of the optimal value $f_*$ is

$$
Z = \frac{\Delta_n}{p(N)} + \hat{f}_1
$$

(20) and the estimator of the optimal solution $\beta_*$ is

$$
Z = \frac{\tilde{\Delta}_n}{p(N)} + \beta_1,
$$

(21)

where $N$ was defined in Section 2. We now impose assumptions in this setting so that Assumption 1 for the general principles can be verified for both $\Delta_n$ and $\tilde{\Delta}_n$.

**Assumption 2.** Stochastic convex optimization assumptions:

(i) The feasible region $D \subset \mathbb{R}^d$ is compact.

(ii) $f$ has a unique optimal solution $\beta_* \in D$.

(iii) $F(\cdot, X)$ is finite, convex and twice continuously differentiable on $D$ a.s.

(iv) There exists a locally bounded measurable function $\kappa: \Omega \to \mathbb{R}_+$, $\gamma > 0$ and $\delta > 0$ such that

$$
|F(\beta', X) - F(\beta, X)| \leq \kappa(X) \|\beta' - \beta\|^\gamma
$$

for all $\beta, \beta' \in D$ with $\|\beta' - \beta\| \leq \delta$ and $X \in \Omega$; and $\kappa(X)$ has finite moment generating function in a neighborhood of the origin.

(v) Define,

$$
M_\beta(t) = E[\exp(t(F(\beta, X_t) - f(\beta)))]
$$

(22) and assume that there exists $\delta_0 > 0$ and $\sigma^2 > 0$ such that for $|t| \leq \delta_0$,

$$
\sup_{\beta \in D} M_\beta(t) \leq \exp(\sigma^2 t^2/2).
$$

(vi) There is $\delta_0^* > 0$ and $t > 0$ such that

$$
\sup_{\|\beta - \beta_*\| \leq \delta_0^*} E[\exp(t \|\nabla_{\beta} F(\beta, X)\|)] < \infty.
$$

(vii) $E \left[ \left\| \nabla_{\beta, \beta}^2 F(\beta, X) \right\|^p \right] < \infty$ with some $p > 2$.

(viii) $G(\beta) = (g_1(\beta), \ldots, g_m(\beta))^T$ and $g_i(\cdot)$ is twice continuously differentiable convex function for all $1 \leq i \leq m$.

(ix) $\beta \in D$ such that $G(\beta) < 0$ (Slater conditions ensures strong duality).

(x) LICQ holds at $\beta_*$, i.e., the gradient vectors $\{\nabla g_i(\beta_*) : g_i(\beta_*) = 0\}$ are linearly independent (LICQ is the weakest condition to ensure the uniqueness of Lagrangian multiplier; see [22] for instance).

(xi) Strict complementarity condition holds, i.e., $\lambda_\alpha(i) > 0$ when $g_i(\beta_*) = 0$ for all $i = 1, \ldots, m$.

We summarize the discussion of unbiased estimator for the optimal solution $\beta_*$ as Theorem 3 in Section 4.1 and unbiased estimator for the optimal objective value $f_*$ as Theorem 3 in Section 4.2.
4.1 Unbiased estimator of optimal solution

In this section, we will utilize the large deviation principles for the SAA optimal solutions develop in [23]. We first provide the following lemma to summarize the LDP.

**Lemma 2.** If Assumptions 2(i), 2(ii), 2(iv), 2(v) hold, then for every $\epsilon > 0$, there exist positive constants $c(\epsilon)$ and $\alpha(\epsilon)$, independent of $n$, such that for $n$ sufficiently large

$$P (\|\beta_n - \beta_*\| > \epsilon) \leq c_\epsilon e^{-2^n \alpha(\epsilon)},$$

where $\alpha(\epsilon)$ is locally quadratic at the origin, i.e., $\alpha(\epsilon) = \alpha_0 \epsilon^2$ as $\epsilon \to 0$ with $\alpha_0 > 0$.

**Proof.** If Assumptions 2 is in force, then for every $\epsilon > 0$, there exist positive constants $c(\epsilon)$ and $\alpha(\epsilon)$, independent of $n$, such that for $n$ sufficiently large

$$P (\|\beta_n - \beta_*\| > \epsilon) \leq c_\epsilon e^{-2^n \alpha(\epsilon)},$$

where $\alpha(\epsilon)$ is locally quadratic at the origin, i.e., $\alpha(\epsilon) = \alpha_0 \epsilon^2$ as $\epsilon \to 0$ with $\alpha_0 > 0$.

**Theorem 3.** If Assumptions 2 is in force, then $E [\bar{Z}] = \beta_*$, $\text{Var}(\bar{Z}) < \infty$ and the computation complexity required to produce $\bar{Z}$ is bounded in expectation.

**Proof.** If Assumptions 2(i), 2(ii), 2(iv), 2(v), 2(x) and 2(xi) hold, the following result is given on page 171 of [20]

$$2^{n/2} \left[ \frac{\beta_n - \beta_*}{\lambda_n - \lambda_*} \right] \Rightarrow \mathcal{N} \left( 0, \Sigma \right),$$

where

$$J = \begin{bmatrix} H & A \\ A^T & 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$H = \nabla^2_{\beta \beta} L (\beta_*, \lambda_*) \in \mathbb{R}_{d \times d}, \ A \text{ is the matrix whose columns are formed by vectors } \nabla g_i (\beta) \text{ when } g_i (\beta) = 0 \text{ for } i = 1, \ldots, m, \ \text{and } \Sigma = E \left[ (\nabla F (\beta_*, X) - \nabla f (\beta_*)) (\nabla F (\beta_*, X) - \nabla f (\beta_*))^T \right].$$

Non-singularity of $J$ is guaranteed by Assumptions 2(x) and 2(xi).

We first show $\bar{Z}$ is unbiased. For $n \geq 0$,

$$E \left[ \Delta_n \right] = E [\beta_{n+1}] - E [\beta_n].$$

Since the feasible region $\mathcal{D} \subset \mathbb{R}_d$ is closed and bounded by Assumption 2(i) $\{\beta_n : n \geq 0\}$ is uniformly integrable. With $\beta_n \to \beta_*$ in [23], we have

$$E \left[ \bar{Z} \right] = \sum_{n=1}^{\infty} E \left[ \Delta_n \right] + E [\beta_1] = \lim_{n \to \infty} E [\beta_n] = \beta_*.$$

We next prove $\text{Var}(\bar{Z}) < \infty$ by showing $E \left[ \Delta_n \Delta_n^T \right] = O \left( 2^{-(1+\alpha)n} \right)$ with some $\alpha > 0$. Let $m (\beta_*) = E \left[ \nabla^2_{\beta \beta} F (\beta_*, X) \right]$. The key ingredients are - firstly we use the large deviation principle (LDP) of $\{\beta_n : n \geq 0\}$ to get moderate deviation estimates for $\{\beta_n : n \geq 0\}$, secondly to use extended contraction principle with
modified optimization problems to translate the LDP to the sequence of Lagrange multipliers \( \{\lambda_{2^n} : n \geq 0\} \).

For the first part, we have by Lemma 2 that
\[
P(\|\beta_{2^n} - \beta_\ast\| \geq \epsilon) = \exp(-2^n \alpha(\epsilon) + o(2^n))
\]
for all \( \epsilon > 0 \) sufficiently small and \( \alpha(\epsilon) = \alpha_0\epsilon^2(1 + o(1)) \) as \( \epsilon \to 0 \) for \( \alpha_0 > 0 \). This yields moderate deviation estimates for \( \{\beta_{2^n} : n \geq 0\} \). In particular, we let \( \epsilon \to 0 \) at a speed of the the form \( \epsilon = 2^{-m} \) for \( 1/4 < \rho < 1/2 \) and the limit above still provide the correct rate of convergence, i.e.,
\[
P(\|\beta_{2^n} - \beta_\ast\| \geq 2^{-m^n}) = \exp\left(-\alpha_0 2^{(4-2\rho)n} + o\left(2^{(4-2\rho)n}\right)\right).
\]

Then to translate this LDP to \( \{\lambda_{2^n} : n \geq 0\} \), we consider a family of modified optimization problems (indexed by \( \eta \))
\[
\min f_\eta(\beta) = E_\mu [F(\beta, X)] + \eta^T \beta, \quad G(\beta) \leq 0,
\]
and its associated optimal solution, \( \beta(\eta) \), with the associated Lagrange multiplier, \( \lambda(\eta) \), for the modified problem. It follows that \( \lambda(\cdot) \) is continuously differentiable as a function of \( \eta \) in a neighborhood of the origin, this is a consequence of Assumptions 2(2) and 2(12). Both \( \beta(\eta) \) and \( \lambda(\eta) \) are characterized by the following KKT conditions
\[
\nabla f(\beta(\eta)) + \nabla G(\beta(\eta)) \lambda(\eta) = -\eta, \quad G(\beta(\eta)) \leq 0, \quad \lambda(\eta)^T G(\beta(\eta)) = 0, \quad \lambda(\eta) \geq 0.
\]

By one of the KKT optimality conditions specified in 15 for the SAA problem we have that
\[
0 = \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla_\beta F(\beta_{2^n}, X_i) + \nabla G(\beta_{2^n}) \lambda_{2^n}.
\]

The previous equality implies that
\[
\nabla_\beta f(\beta_{2^n}) + \nabla_\beta G(\beta_{2^n}) \cdot \lambda_{2^n} = -\bar{\eta}_{2^n},
\]
where
\[
\bar{\eta}_{2^n} = \frac{1}{2^n} \sum_{i=1}^{2^n} (\nabla_\beta F(\beta_{2^n}, X_i) - \nabla_\beta f(\beta_{2^n})).
\]

Written in this form, we can identify that \( \beta_{2^n} = \beta(\bar{\eta}_{2^n}) \) and \( \lambda_{2^n} = \lambda(\bar{\eta}_{2^n}) \). We already know that \( \{\beta_{2^n} : n \geq 0\} \) has a large deviations principle, so the LDP can be derived for \( \{\bar{\eta}_{2^n} : n \geq 0\} \) by Theorem 2.1 of 12 with Assumption 2(12). Furthermore, the LDP can then be derived for the Lagrange multipliers \( \{\lambda_{2^n} = \lambda(\bar{\eta}_{2^n}) : n \geq 0\} \) because \( \lambda(\cdot) \) is continuously differentiable as a function of \( \eta \) in a neighborhood of the origin, as we mentioned earlier.

Then, from (29) it follows by Taylor expansion that
\[
0 = \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla_\beta F(\beta_{2^n}, X_i) + \nabla G(\beta_{2^n}) \lambda_{2^n} = \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla_\beta F(\beta_\ast, X_i) + \frac{1}{2^n} \sum_{i=1}^{2^n} (\nabla_\beta F(\beta_{2^n}, X_i) - \nabla_\beta F(\beta_\ast, X_i)) + \nabla G(\beta_{2^n}) \lambda_{2^n}
\]
\[
= \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla_\beta F(\beta_\ast, X_i) + \nabla G(\beta_\ast) \lambda_\ast + \left(\frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2_\beta F(\beta_\ast, X_i) - m(\beta_\ast)\right) (\beta_{2^n} - \beta_\ast)
\]
\[
+ m(\beta_\ast) + \lambda^T_\ast \nabla^2 G(\beta_\ast) \cdot (\beta_{2^n} - \beta_\ast) + \nabla G(\beta_\ast) (\lambda_{2^n} - \lambda_\ast)
\]
\[
+ \hat{R}_{\eta, (\beta)} + \hat{R}_{\eta, (\lambda)} + \hat{R}_{\eta, (\beta, \lambda)},
\]
where
\[ R_{n,(β,β)} = O(\|β_{2n} - β_*\|^2), \]
\[ R_{n,(λ,λ)} = O(\|λ_{2n} - λ_*\|^2), \]
\[ R_{n,(β,λ)} = O(\|β_{2n} - β_*\|\|λ_{2n} - λ_*\|). \]

Let
\[ \bar{R}_n = \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2 β F (β_*, X_i) - m(β_*), (β_{2n} - β_*) \right). \]

and let \( \Lambda_1 = m(β_*) + λ_* T \nabla^2 G(β_*) \in \mathbb{R}_{d \times d}, \Lambda_2 = \nabla G(β_*) \in \mathbb{R}_{d \times m}. \) Then we can rewrite (30) as
\[ \Lambda_1 (β_{2n} - β_*) + \Lambda_2 (λ_{2n} - λ_*) = - \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla β F (β_*, X_i) + \nabla G(β_*) λ_* + \bar{R}_n + \bar{R}_{n,(β,β)} + \bar{R}_{n,(λ,λ)} + \bar{R}_{n,(β,λ)} \right). \]

Note that by Holder’s inequality,
\[ E \left[ \bar{R}_n R_n^T \right] \leq E \left[ \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2 β F (β_*, X_i) - m(β_*), (β_{2n+1} - β_*) \right) \right] \leq E \left[ \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2 β F (β_*, X_i) - m(β_*), (β_{2n+1} - β_*) \right) \right]^{2/p} E \left[ \left( β_{2n+1} - β_* \right)^{2p/(p-2)} \right]^{(p-2)/p}, \]

where
\[ E \left[ \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2 β F (β_*, X_i) - m(β_*), (β_{2n+1} - β_*) \right) \right] = O \left( 2^{-np/2} \right) \]

by \([5]\), and by using the moderate large deviation estimate
\[ E \left[ \left( \frac{1}{2^n} \sum_{i=1}^{2^n} \nabla^2 β F (β_*, X_i) - m(β_*), (β_{2n+1} - β_*) \right) \right]^{2/p} E \left[ \left( β_{2n+1} - β_* \right)^{2p/(p-2)} \right]^{(p-2)/p} = O \left( 2^{-2pn/p} \right). \]

Combining them together we get \( E \left[ \bar{R}_n R_n^T \right] = O (2^{-(1+2\rho)n}) \). Similarly we can get
\[ E \left[ \bar{R}_{n,(β,β)} R_{n,(β,β)}^T \right] = O (2^{-4m}), \]
\[ E \left[ \bar{R}_{n,(λ,λ)} R_{n,(λ,λ)}^T \right] = O (2^{-4m}), \]
\[ E \left[ \bar{R}_{n,(β,λ)} R_{n,(β,λ)}^T \right] = O (2^{-4m}). \]

Because
\[ \Lambda_1 \left( β_{2n+1} - \frac{1}{2} (β_{2n}^O + β_{2n}^E) \right) + \Lambda_2 \left( λ_{2n+1} - \frac{1}{2} (λ_{2n}^O + λ_{2n}^E) \right) \]
\[ = \bar{R}_{n+1} - \frac{1}{2} \left( \bar{R}_{n}^O + \bar{R}_{n}^E \right) - R_{n+1,(β,β)} - \frac{1}{2} \left( R_{n,(β,β)}^O + R_{n,(β,β)}^E \right) \]
\[ + R_{n+1,(λ,λ)} - \frac{1}{2} \left( R_{n,(λ,λ)}^O + R_{n,(λ,λ)}^E \right) - R_{n+1,(β,λ)} - \frac{1}{2} \left( R_{n,(β,λ)}^O + R_{n,(β,λ)}^E \right), \]

we have that \( E \left[ \tilde{A}_n \tilde{A}_n^T \right] = O (2^{-4\rho n}). \) Note that \( \rho \in (1/4, 1/2) \), it satisfies Assumption \([1(1)]\) of the general principles of unbiased estimators in Section \([2]\).

The computational cost for producing \( \tilde{A}_n \), denoted by \( C_n \), is of order \( O(2^n) \). After generating \( 2^{n+1} \) iid copies of \( X's \), we can use Newton’s method or other root-finding algorithms to solve the KKT condition for optimal solution, or use other classic tools such as subgradient method or interior point method. \( \square \)
4.2 Unbiased estimator of optimal value

**Theorem 4.** If Assumption 2 is in force, then \( E[Z] = f_* \), \( \text{Var}(Z) < \infty \) and the computation complexity required to produce \( Z \) is bounded in expectation.

**Proof.** Finite expected computation complexity of producing \( \Delta_n \) has been discussed in the proof to Theorem 3. We now show the unbiasedness of estimator \( Z \). Since \( f_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} F(\beta, X_i) \rightarrow f(\beta) \), uniformly on \( D \), with the result of Proposition 5.2 in [20] we have \( \hat{f}_n \rightarrow f_* \) w.p.1 as \( n \rightarrow \infty \). If Assumption 2(v) is in force, \( \{\hat{f}_n : n \geq 0\} \) is uniformly integrable, hence

\[
E[Z] = \lim_{n \rightarrow \infty} E[\hat{f}_n] = f_*.
\]

We next prove the estimator \( Z \) has finite variance by showing \( E[\Delta_n^2] = O(2^{-4\rho n}) \) with \( \rho > 1/4 \). By Taylor expansion around the unique true optimal solution \( \beta_* \) and the KKT condition,

\[
\Delta_n = f_{2n+1}(\beta_{2n+1}) - \frac{1}{2} \left(f_{2n}^O \left(\beta_{2n}^O\right) + f_{2n}^E \left(\beta_{2n}^E\right)\right)
= \frac{1}{2n+1} \sum_{i=1}^{2n+1} F(\beta_{2n+1}, X_i) - \frac{1}{2} \left(\frac{1}{2n} \sum_{i=1}^{2n} F(\beta_{2n}^O, X_i^O) + \frac{1}{2n} \sum_{i=1}^{2n} F(\beta_{2n}^E, X_i^E)\right)
= \frac{1}{2n+1} \sum_{i=1}^{2n+1} \nabla_{\beta} F(\beta_*, X_i) + \nabla G(\beta_*) \lambda_* \left(\beta_{2n+1} - \beta_*\right) + R_{n+1}
- \frac{1}{2} \left(\frac{1}{2n} \sum_{i=1}^{2n} \nabla_{\beta} F(\beta_*, X_i^O) + \nabla G(\beta_*) \lambda_* \left(\beta_{2n}^O - \beta_*\right) + R_n^O\right)
- \frac{1}{2} \left(\frac{1}{2n} \sum_{i=1}^{2n} \nabla_{\beta} F(\beta_*, X_i^E) + \nabla G(\beta_*) \lambda_* \left(\beta_{2n}^E - \beta_*\right) + R_n^E\right)
- \lambda_*^T \nabla G(\beta_*)^T \left(\beta_{2n+1} - \frac{1}{2} \left(\beta_{2n}^O + \beta_{2n}^E\right)\right),
\]

where \( R_{n+1} = O\left(\|\beta_{2n+1} - \beta_*\|^2\right) \), \( R_n^O = O\left(\|\beta_{2n}^O - \beta_*\|^2\right) \) and \( R_n^E = O\left(\|\beta_{2n}^E - \beta_*\|^2\right) \). By using the moderate LDP explained in the proof of Theorem 3 with \( \rho \in (1/4, 1/2) \), we have

\[
E\left[R_n^2\right] \leq c_1 E\left[\|\beta_{2n} - \beta_*\|^4 I\left(\|\beta_{2n} - \beta_*\| \geq 2^{-\rho n}\right)\right] + c_2 E\left[\|\beta_{2n} - \beta_*\|^4 I\left(\|\beta_{2n} - \beta_*\| < 2^{-\rho n}\right)\right] = O\left(2^{-4\rho n}\right).
\]

Also similar analysis as (31), (32) and (33) in the proof of Theorem 3 yields

\[
E\left[\left(\frac{1}{2n+1} \sum_{i=1}^{2n+1} \nabla_{\beta} F(\beta_*, X_i) + \nabla G(\beta_*) \lambda_* \left(\beta_{2n+1} - \beta_*\right)\right)^2\right] = O\left(2^{-(1+2\rho)n}\right).
\]

Combining the fact that \( E\left[\Delta_n^2\right] = O\left(2^{-4\rho n}\right) \), we finally get \( E\left[\Delta_n^2\right] = O\left(2^{-4\rho n}\right) \).

4.3 Applications and numerical examples

4.3.1 Linear Regression

Linear regression is to solve the following optimization problem

\[
\min_{\beta \in \mathbb{R}^{p+1}} \text{MSE} = E_\mu \left[F(\beta, (X, y))\right] = E_\mu \left[\left(y - X^T \beta\right)^2\right],
\]

where \( X \in \mathbb{R}^{p+1} \) is called independent variables, whose first coordinate is 1, and \( y \) is real valued response called dependent variable. The pair \((X, y)\) is from distribution \( \mu \). The goal is to find the optimal \( \beta_* \) that minimizes the mean-squared-error (MSE).
In many of the real-world problems, we normally have the distribution \( \mu \) being the empirical measure of all the data available \( \{(X_i, y_i) : 1 \leq i \leq n_0\} \), where \( n_0 \) denote the total number of data points we have. When \( n_0 \) is enormous, it would be difficult and slow to load all the data and do computation at once. Like we mentioned in previous sections, we can take a subsample of the whole dataset to solve the corresponding SAA linear regression, but it results significant estimation bias. With the unbiased estimators (20) and (21), we can take relatively small subsamples and solve them on multiple processors in parallel, without any bias.

We have \( F(\beta, (X, y)) = (y - X^T \beta)^2 \) strictly convex and twice continuously differentiable in \( \beta \), so the optimizer is unique. To have all the required conditions listed in Assumption 2 satisfied, we can let \( G(\beta) = (g_1(\beta), g_1(\beta), \ldots, g_{2(p+1)}(\beta))^T \) with \( g_{2i-1}(\beta) = e_i^T \beta - M \) and \( g_{2i}(\beta) = -e_i^T \beta - M \) for \( 1 \leq i \leq p + 1 \), with \( M > 0 \) sufficiently large, so that the unique optimizer \( \beta_* \) is in the interior of \( D = \{ \beta \in \mathbb{R}^{p+1} : G(\beta) \leq 0 \} \). Then all the conditions follow naturally.

The numerical experiment is to test how the unbiased estimators perform on some real-world dataset. We use Beijing air pollution data (downloaded from the website of UCI machine learning repository), which has 43824 data points, real-valued PM2.5 concentration and 11 real-valued independent variables including time of a day, temperature, pressure, wind direction and speed, etc. We first use the entire dataset to get the true optimal solution \( \beta_* \) and optimal value \( f_* \) as baselines of the experiment. Then we repeat the SAA approach and our unbiased method for 10000 times; for the SAA problem, each time we randomly sample a subset of fixed size, while for the unbiased method we randomly sample a subset of size \( 2N + 1 \) with \( N \) geometrically distributed in \( \{B, B+1, B+2, \ldots \} \). We call such integer value \( B \) “burning size”. In Chapter 2 we have \( B = 0 \), which leads to the smallest possible dataset we can get is of size 1. To better control the variance, our experiment uses \( B = 10 \).

The left plot of Figure 1 has two curves. The red curve shows how \( \|\beta_{SAA} - \beta_*\|_\infty \) changes as we increase the number of replications, whereas the blue curve shows the same \( l_\infty \) distance between the mean of the unbiased estimators and \( \beta_* \). At the beginning, both estimators are volatile, though the SAA estimator has relatively smaller variance than the unbiased estimator, but they stabilize when the number of replications is around 2500 and finally are both close enough to the true optimizer \( \beta_* \). The right plot of Figure 1 shows how the optimal value estimators from SAA and unbiased method perform as we increase the number of replications. The blue dashed horizontal line indicates the level of true optimal value \( f_* \), i.e. the MSE computed by using the entire dataset), the green curve corresponds to the average MSE of SAA problems and the red curve corresponds to the average MSE of unbiased method. Clearly the unbiased estimator outperform the other as it gets close to \( f_* \) after some initial fluctuation, however the SAA estimator gives consistent negative bias, which verifies the theoretic results given in the SAA literature as we mentioned earlier.
4.3.2 Regularized Regressions

Two classic regularized regression techniques are Ridge and LASSO, with $l_2$ and $l_1$ penalty added to the target value function respectively. Ridge regression is to solve the following optimization problem

$$
\min_{\beta \in \mathbb{R}^{p+1}} E_{\mu} \left[ (y - X^T \beta)^2 \right] + \lambda \| \beta \|_2^2, \tag{35}
$$

where $\lambda \geq 0$ is the shrinkage parameter. An equivalent way to express the Ridge regression is

$$
\min E_{\mu} \left[ (y - X^T \beta)^2 \right] \quad \text{s.t.} \quad \beta^T \beta \leq t,
$$

where $t \propto 1/\lambda$ has one-to-one correspondence with each shrinkage parameter $\lambda$ in (35).

Similarly we can express LASSO as

$$
\min E_{\mu} \left[ (y - X^T \beta)^2 \right] \quad \text{s.t.} \quad (-1)^{r_1} \beta_1 + \ldots + (-1)^{r_p} \beta_p \leq t, \quad r_i \in \{0,1\} \quad \text{for all} \quad i = 1, \ldots, p, \tag{36}
$$

with $t \geq 0$.

For both problem we can verify the conditions in Assumptions 2 and use the method proposed to construct unbiased estimators.

4.3.3 Logistic Regression

Logistic regression is to solve the following optimization problem

$$
\min_{\beta \in \mathbb{R}^n} f(\beta) = E_{\mu} [F(\beta, (X,y))] = E_{\mu} \left[ -\log \left(1 + \exp \left(-y \beta^T X\right)\right)\right], \tag{37}
$$

where $X \in \mathbb{R}^{p+1}$ has its first coordinate being 1, and $y \in \{-1, 1\}$ is the label of the class that the data point $(X, y)$ falls in. The pair $(X, y)$ is from some distribution $\mu$. The classic logistic regression is to find the optimal coefficient $\beta_\ast$ to maximize the log-likelihood, and we give in (37) an equivalent problem to minimize the negative of the log-likelihood, i.e., $\min f(\beta)$ with $f$ being strict convex.

We run an numerical experiment to check how our unbiased estimators perform, compared to the SAA estimators of both the optimal solution and the optimal objective value. The dataset we use is some online advertising campaign data from Yahoo research, which has 2801523 data points, each has 22 real-valued features and one response $y \in \{-1, 1\}$ indicating whether it is a click or not. We first use the entire dataset to get the true optimal solution $\beta_\ast$ and optimal value $f_\ast$ as baselines. Then, for the SAA method and our unbiased estimating method, we run 10000 replications each to see whether they are able to produce a good estimation to $\beta_\ast$ and $f_\ast$. Again we use the burning size $B$ equal to 10 here.

In Figure 2 the left plot shows how SAA estimator (in red) and the unbiased estimator (in blue) approach the true optimal solution $\beta_\ast$ as we increase the size of replications, and similarly the right plot shows the performance of both estimators for the optimal value $f_\ast$, which is represented by the level of the blue dashed line. In both cases, our unbiased estimators beat the SAA estimators in terms of unbiasedness.

5 Quantile Estimation

Suppose $(X_k : k \geq 1)$ are iid with cumulative distribution function $F(x) = \mu([(-\infty,x]) = P(X \leq x)$ for $x \in \mathbb{R}$. We define $x_p = x_p(\mu) = \inf\{x \geq 0 : F(x) \geq p\}$ to be the $p$-quantile of distribution $\mu$ for any given $0 < p < 1$. If $F(\cdot)$ is continuous we have that

$$
F(x_p) = p.
$$

Connecting to the general framework from Section 2 here we have $\theta(\mu) := x_p(\mu)$.

We first impose some assumptions.
Assumption 3. Distributional quantile assumptions:

(i) $F$ is at least twice differentiable in some neighborhood of $x_p$,
(ii) $F''(x)$ is bounded in the neighborhood,
(iii) $F''(x_p) = f(x_p) > 0$,
(iv) $E[X^2] < \infty$.

Note that Assumptions 3(i), 3(ii) and 3(iii) ensure $x_p$ is the unique $p$-quantile of distribution $\mu$. By Bahadur representation of sample quantiles in [4], we have

\[
Y_n = x_p + \frac{np - Z_n}{n f_{\mu}(x_p)} + R_n,
\]

where

\[
Y_n = (1 - w_n)X_{[np]} + w_nX_{[np]+1}, \quad w_n = np - [np] \in [0, 1),
\]

i.e., the sample $p$-quantile of sample $(X_1, \ldots, X_n)$, $Z_n = \sum_{i=1}^n I(X_i \leq x_p)$ and $R_n = O(n^{-3/4} \log n)$ as $n \to \infty$ almost surely.

Lemma 5. If Assumption 3(iv) is in force, $\sup_{n \geq 1/p} E[Y_n^2] < \infty$.

Proof. Just follow Bahadur’s proof. Let

\[
G_n(x, \omega) = (F_n(x, \omega) - F_n(x_p, \omega)) - (F(x) - F(x_p)),
\]

and let $I_n$ be an open interval $(x_p - a_n, x_p + a_n)$ with the constant $a_n \sim \log n / \sqrt{n}$ as $n \to \infty$. Define

\[
H_n(\omega) = \sup \{|G_n(x, \omega)| : x \in I_n\}.
\]

By Lemma 1 in [4], $H_n(\omega) \leq K_n(\omega) + \beta_n$ with $\beta_n = O(n^{-3/4} \log n)$, $\sum_n p(K_n \geq \gamma_n) < \infty$ and $\gamma_n = cn^{-3/4} \log n$. By Lemma 2 in [4] we have $Y_n \in I_n$ for sufficiently large $n$ w.p.1. Let $n_\ast = \sup_n \{K_n \geq \gamma_n \text{ or } Y_n \notin I_n\} < \infty$, then for all $n \geq 1/p$

\[
E[Y_n^2] = E[Y_n^2 I(n \leq n_\ast)] + E[Y_n^2 I(n > n_\ast)],
\]

where

\[
E[Y_n^2 I(n \leq n_\ast)] \leq E\left[\sum_{i=1}^n X_i^2 I(n \leq n_\ast)\right] \leq n_\ast E[X^2] < \infty,
\]

Figure 2: Logistic regression test on AOL’s campaign data
and

\[
E \left[ Y_n^2 I (n > n_*) \right] = E \left[ \left( x_p + \frac{np - Z_n}{nf(x_p)} + R_n \right)^2 I (n > n_*) \right] \\
\leq 3x_p^2 + \frac{3pq}{nf(x_p)^2} + 3E \left[ R_n^2 I (n > n_*) \right] \\
\leq 3x_p^2 + \frac{3pq}{nf(x_p)^2} + 3E \left[ H_n^2 I (n > n_*) \right] \\
\leq 3x_p^2 + \frac{3pq}{nf(x_p)^2} + 6\gamma^2_n + 6\beta_n^2 < \infty.
\]

Combining these two parts together we can conclude \( \sup_n E \left[ Y_n^2 \right] < \infty \).

We let \( Y_{2n+1} \) denote the sample \( p \)-quantile of \((X_1, \cdots, X_{2n+1})\), let \( Y_{2n}^O \) denote the sample \( p \)-quantile of the odd indexed sub-sample \((X_1^O, \cdots, X_{2n}^O)\) and let \( Y_{2n}^E \) denote the sample \( p \)-quantile of the even indexed sub-sample \((X_1^E, \cdots, X_{2n}^E)\). Then, define

\[
\Delta_n = Y_{2n+1} - \frac{1}{2} (Y_{2n}^O + Y_{2n}^E).
\]

Let \( n_b = \min \{ n \in \mathbb{N} : n \geq 1/p \} \). We let the geometrically distributed random variable \( N \) to take values on \( \{n_b, n_b + 1, \ldots \} \) with \( p(n) = P(N = n) > 0 \) for all \( n \geq n_b \). Define the estimator to be

\[
Z = \frac{\Delta_N}{p(N)} + Y_{2n_q}.
\]

**Theorem 6.** If Assumption \( 3 \) are in force, then \( E[Z] = x_p, Var(Z) < \infty \) and the computation complexity required to produce \( Z \) is bounded in expectation.

**Proof.** We first show the unbiasedness of \( Z \). Uniform integrability of \( \{Y_{2n} : n \geq n_b\} \) is established in Lemma 5 with Assumption \( 3(iv) \) holds true, so we have

\[
E[Z] = \sum_{n=n_b}^{\infty} E[\Delta_n] + E[Y_{2n_b}] = \lim_{n \to \infty} E[Y_{2n}] = E[\lim_{n \to \infty} Y_{2n}] = x_p.
\]

We next show \( Var(Z) < \infty \). With \( 3(E) \) we have

\[
\Delta_n = \left( x_p + \frac{2n+1 - Z_{2n+1}}{2n+1f(x_p)} + R_{2n+1} \right) - \frac{1}{2} \left[ \left( x_p + \frac{2n - Z_{2n}^O}{2n f(x_p)} + R_{2n}^O \right) + \left( x_p + \frac{2n - Z_{2n}^E}{2n f(x_p)} + R_{2n}^E \right) \right] \\
= R_{2n+1} - \frac{1}{2} (R_{2n}^O + R_{2n}^E) \\
= O \left( n \cdot 2^{-3n/4} \right) \quad w.p.1,
\]

thus

\[
\Delta_n^2 = O \left( n^2 \cdot 2^{-3n/2} \right).
\]

If we choose \( p(n) = r(1 - r)^{n-n_b} \) with \( r < 1 - \frac{1}{2\sqrt{2}} \) for \( n \geq n_b \), then

\[
E \left[ \frac{\Delta_n}{p(N)} \right] = \sum_{n=n_b}^{\infty} E[\Delta_n] \frac{p(n)}{p(n)} < \infty,
\]

hence \( Var(Z) < \infty \).

Finally we show the computation cost of generating \( \Delta_n \) is finite in expectation. Each replication of \( Z \) involves simulating \( 2^{N+1} \) independent copies of \( X \). If we adopt the selection method based on random partition introduced in \( 7 \), then it will cost us \( O(2^{N+1}) \) time to identify the sample \( p \)-quantiles \( Y_{2n+1}, Y_{2n}^O \) and \( Y_{2n}^E \). Therefore by letting \( N \) be an independent geometrically distributed random variable with success parameter \( r \in (1/2, 1 - 2^{-3/2}) \), \( Z \) is an unbiased estimator of the true unique \( p \)-quantile \( x_p \) and it has finite work-normalized variance.
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References

[1] A. Agarwal and E. Gobet. Finite variance unbiased estimation of stochastic differential equations. In 2017 Winter Simulation Conference (WSC), pages 1950–1961, Dec 2017.

[2] S. Asmussen. Applied Probabilities and Queues. Springer-Verlag, 2 edition, 2000.

[3] S. Asmussen and P. Glynn. Stochastic Simulation: Algorithms and Analysis. Springer-Verlag, 2008.

[4] R. R. Bahadur. A note on quantiles in large samples. The Annals of Mathematical Statistics, 37(3):577–580, 06 1966.

[5] Bengt Von Bahr. On the convergence of moments in the Central Limit Theorem. The Annals of Mathematical Statistics, 36(3):808–818, 1965.

[6] Jose H. Blanchet and Peter W. Glynn. Unbiased Monte Carlo for optimization and functions of expectations via multi-level randomization. In Proceedings of the 2015 Winter Simulation Conference, WSC’15, pages 3656–3667. IEEE Press, 2015.

[7] Manuel Blum, Robert W. Floyd, Vaughan Pratt, Ronald L. Rivest, and Robert E. Tarjan. Time bounds for selection. Journal of Computer and System Sciences, 7(4):448 – 461, 1973.

[8] K. Bujok, B. M. Hambly, and C. Reisinger. Multilevel simulation of functionals of Bernoulli random variables with application to basket credit derivatives. Methodology and Computing in Applied Probability, 17(3):579–604, Sep 2015.

[9] Dan Crisan, Pierre Del Moral, Jeremie Houssineau, and Ajay Jasra. Unbiased multi-index Monte Carlo. Stochastic Analysis and Applications, 36(2):257–273, 2018.

[10] P. Del Moral. Feynman-Kac Formulae Genealogical and Interacting Particle Systems with Applications. Springer-Verlag, 2004.

[11] Steffen Dereich and Thomas Mueller-Gronbach. General multilevel adaptions for stochastic approximation algorithms. arXiv:1506.05482, 2017.

[12] F. Gao and X. Zhao. Delta method in large deviations and moderate deviations for estimators. The Annals of Statistics, 39(2):1211–1240, 2011.

[13] Michael B. Giles. Multilevel Monte Carlo path simulation. Operations Research, 56(3):607–617, 2008.

[14] Michael B. Giles and Abdul-Lateef Haji-Ali. Multilevel nested simulation for efficient risk estimation. arXiv:1802.05016, Feb 2018.

[15] Michael B. Giles and Lukasz Szpruch. Antithetic multilevel Monte Carlo estimation for multi-dimensional SDEs without Lévy area simulation. The Annals of Applied Probability, 24(4):1585–1620, 08 2014.

[16] Amirreza Khodadadian, Leila Taghizadeh, and Clemens Heitzinger. Optimal multilevel randomized quasi-monte-carlo method for the stochastic drift–diffusion–poisson system. Computer Methods in Applied Mechanics and Engineering, 329:480 – 497, 2018.

[17] J. S. Liu. Monte Carlo Strategies in Scientific Computing. Springer, 2008.

[18] Don McLeish. A general method for debiasing a Monte Carlo estimator. Monte Carlo Methods and Applications, 17(4):301–315, 2012.

[19] Chang-Han Rhee and Peter W. Glynn. Unbiased estimation with square root convergence for SDE models. Operations Research, 63(5):1026–1043, 2015.
[20] A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming*. Society for Industrial and Applied Mathematics, 2009.

[21] Matti Vihola. Unbiased estimators and multilevel monte carlo. *Operations Research*, 66(2):448–462, 2018.

[22] Gerd Wachsmuth. On LICQ and the uniqueness of lagrange multipliers. *Operations Research Letters*, 41(1):78–80, 2013.

[23] H. Xu. Uniform exponential convergence of sample average random functions under general sampling with applications in stochastic programming. *Journal of Mathematical Analysis and Applications*, 368, 2010.