SPECIALIZATION OF CANONICAL HEIGHTS ON ABELIAN VARIETIES

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Abstract. Given a family of abelian varieties over a quasiprojective smooth curve $T^0$ over a global field and a point $P$ on the generic fiber, we show that the Néron-Tate canonical height $h_{A_t}(P_t)$ of $P_t$ along each fiber is exactly equal to a Weil height $h_{\overline{T}(t)}$ given by an adelic metrized line bundle $\overline{M}$ on the unique smooth projective curve $T$ containing $T^0$. As a consequence, we show that a conjecture of Zhang on the finiteness of small-height specializations of $P$ is equivalent to $\overline{M}$ being big.

1. Introduction

Let $K$ be a number field or a transcendence degree one function field of any characteristic, and let $T^0$ be a smooth, quasiprojective curve over $K$. Suppose we have an abelian scheme $A \to T^0$, so that every fiber $A_t$ is an abelian variety. By fixing a line bundle on $A$ which restricts to a symmetric and ample line bundle on each fiber, we may specify the Néron-Tate height $h_{A_t}$ on each fiber in a consistent way.

Fix a section $P : T^0 \to A$ with specializations $P_t$. If we let $F = K(T^0)$ be the function field of $T$, this corresponds to an $F$ point on the generic fiber $A_\eta$. Following a line of study begun by Silverman and Tate, we ask how the canonical height $h_{A_t}(P_t)$ varies as we vary the parameter $t$ along $T^0(K)$. Let $T$ be the unique smooth projective curve containing $T^0$, and let $h_T(t)$ be any height on $T$ corresponding to a degree one line bundle on $T$. The general formulation of several theorems and conjectures is that

$$h_{A_t}(P_t) = h_A(P)h_T(t) + \text{(error term)},$$

Where $h_A$ is the Néron-Tate canonical height on $A_\eta$.

Silverman conjectured [Sil82], and Tate proved [Tat83 Main Theorem] that if $A$ is a family of elliptic curves, a divisor $D_P$ can be found on $T$ so that replacing $h_T$ with $h_{D_P}$ reduces the error term to $O(1)$. Call [Cal89] and [Gre89] prove similar results in higher dimension. A series of subsequent results of Silverman culminating in [Sil91] further classify the nature of the bounded error function for families of elliptic curves, and work of Biesel, Holmes, and de Jong [BHdJ17] makes similar refinements in higher dimension. More recently, DeMarco and Mavraki [DM20, Theorem 1.1] showed how to eliminate the error term entirely for families of elliptic curves.

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curves by replacing $h_T$ with a specific height constructed using adelic metrized line bundles. Here, we generalize their result to any dimension, resolving this question in full for abelian varieties.

**Theorem 1.1.** Let $P \in A(F)$ such that $h_A(P) \neq 0$. The function

$$t \mapsto h_A(P_t),$$

defined for all but finitely many $t \in T$, can be extended to a height function $h_{\overline{M}}$ on all of $T$ which is given by a nef adelic line bundle $\overline{M}$ on $T$ whose underlying line bundle $M$ is ample.

This is proved by a completely different method from that of DeMarco and Mavraki. We use Yuan-Zhang’s theory of adelic line bundles and vector heights \cite{YZ21} to construct a canonical adelic line bundle, essentially extending Tate’s limiting method from $\mathbb{R}$-valued height functions to the underlying geometry. Once this theory is established, the existence of a Néron model for $A$ provides the needed global geometric setting on which to complete the argument.

**Remark 1.2.** By the Lang-Néron Theorem (see \cite{Con06}), if $A_\eta$ has trivial $F/K$-trace, the condition that $h_A(P) \neq 0$ is equivalent to $P$ being non-torsion. In general, it’s equivalent to $P$ being outside a torsion coset of $\text{Tr}_{F/K}(A_\eta)$.

With a metrized line bundle giving $h_A(P_t)$ exactly, one could use equidistribution arguments to show, for example, that if $P, Q \in A(F)$ are two points whose heights are both non-zero and there exists a sequence $\{t_n\} \subset T(\overline{Q})$ such that both $h_{A_t}(P_{t_n})$ and $h_{A_{t_n}}(Q_{t_n})$ tend to zero as $n \to \infty$, then $P$ and $Q$ specialize to torsion points at the exact same places. This is likely vacuous, however, due to the following conjecture of Zhang.

**Conjecture 1.3.** (\cite{Zha98}) Suppose $A : T^0 \to A$ is a non-isotrivial family of abelian varieties as above, with $K$ a number field, and that $A_\eta$ is simple of dimension at least two. For each non-torsion section $P : T^0 \to A$ defined over $\overline{Q}$, there exists an $\epsilon > 0$ such that

$$\{t \in T^0(\overline{Q}) : h_A(P_t) \leq \epsilon\}$$

is finite.

Instead, using Theorem 1.1 we can restate Zhang’s conjecture as a positivity statement for self-intersections.

**Corollary 1.4.** Let $\overline{M}$ be the metrized line bundle defined in Theorem 1.1 so that $h_A(P_t) = h_{\overline{M}}(t)$. Then the conclusion of Zhang’s conjecture is equivalent to

$$\overline{M}^2 > 0.$$
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2. Vector-valued heights

We begin with an overview of adelic line bundles and vector-valued height functions, in so much as is needed here. For a more complete treatment, see [YZ21, Ch. 2] and [Car20, Sec. 2] for more development of the function field setting.

We will work in two parallel settings. The first is for varieties over a field, in which all constructions are purely geometric, and which we will call the geometric case. The second is for varieties over Spec ℤ, which we will call the arithmetic case. Note that when K is a number field, we will require both settings, as the height on A_q is geometric, while that on each fiber is arithmetic.

Geometric case: Fix any field k, and let U be a quasi-projective k variety. A projective model for U is a projective variety X along with an open immersion U → X defined over k. We write Div(X) for the group of Cartier divisors on X and Pr(X) for the subgroup of principal Cartier divisors.

Arithmetic case: Let U → Spec ℤ be a quasi-projective arithmetic variety. In this setting, a projective model for U is a projective arithmetic variety X → Spec ℤ with an open embedding U → X over Spec ℤ. Write Div(X) for the group of arithmetic divisors on X, and Pr(X) for the subgroup of principal arithmetic divisors.

In both cases the projective models for U form an inverse system via dominating morphisms. Using pullbacks on that system, we define the following, first in the geometric case. If U → Spec k is a quasi-projective variety,

\[ \hat{\text{Div}}(U/k)_{\text{mod}} := \lim_{U \hookrightarrow X} \text{Div}(X)_{\mathbb{Q}} \quad \text{and} \quad \hat{\text{Pr}}(U/k)_{\text{mod}} := \lim_{U \hookrightarrow X} \text{Pr}(X)_{\mathbb{Q}}. \]

An element of \( \hat{\text{Div}}(U/k)_{\text{mod}} \) is called effective if it comes from an effective divisor in some \( \hat{\text{Div}}(X)_{\mathbb{Q}} \). If U → Spec ℤ is a quasi-projective arithmetic variety, we define

\[ \hat{\text{Div}}(U/\mathbb{Z})_{\text{mod}} := \lim_{U \hookrightarrow X} \hat{\text{Div}}(X)_{\mathbb{Q}} \quad \text{and} \quad \hat{\text{Pr}}(U/\mathbb{Z})_{\text{mod}} := \lim_{U \hookrightarrow X} \hat{\text{Pr}}(X)_{\mathbb{Q}}. \]

An element of \( \hat{\text{Div}}(U/\mathbb{Z})_{\text{mod}} \) is called effective if it comes from an effective arithmetic divisor in some \( \hat{\text{Div}}(X)_{\mathbb{Q}} \), meaning that the finite part of the divisor is effective, and the Green’s function is non-negative.

Next we define a topology on both of these groups, stemming from effectivity. Since the argument is identical in both the arithmetic and geometric cases we provide it only for the arithmetic case.

Let U be a quasi-projective arithmetic variety. Effectivity provides a partial ordering on \( \hat{\text{Div}}(U/\mathbb{Z})_{\text{mod}} \). Fix some projective model X_0 and a strictly effective arithmetic
divisor $\overline{D}_0$ on $X_0$ such that the support of $\overline{D}_0$ is equal to $X_0 \setminus U$. We call such $\overline{D}_0$ a boundary divisor. For $\epsilon \in \mathbb{Q}_{>0}$, define a basis of epsilon balls around 0 by

$$B(\epsilon, 0) := \left\{ \overline{E} \in \widehat{\text{Div}}(U/\mathbb{Z})_{\text{mod}} : \epsilon \overline{D}_0 \pm \overline{E} \text{ are both effective} \right\}.$$ 

Via translation, this defines a topology on all of $\widehat{\text{Div}}(U)_{\text{mod}}$, and it is easy to check that this topology does not depend on the choice of $\overline{D}_0$.

We now define $\widehat{\text{Div}}(U/\mathbb{Z})$ to be the completion of $\widehat{\text{Div}}(U/\mathbb{Z})_{\text{mod}}$ with respect to this topology. $\widehat{\text{Div}}(U/k)$ is defined identically, except without a Green’s function attached to each divisor. Finally, define the groups of adelic line bundles in each setting:

$$\widehat{\text{Pic}}(U/\mathbb{Z}) := \widehat{\text{Div}}(U/\mathbb{Z})/\widehat{\text{Pr}}(U/\mathbb{Z})_{\text{mod}},$$

$$\widehat{\text{Pic}}(U/k) := \widehat{\text{Div}}(U/k)/\widehat{\text{Pr}}(U/k)_{\text{mod}}.$$

To further justify notating these as Picard groups, elements of $\widehat{\text{Pic}}(U/k)$ (resp. $\widehat{\text{Pic}}(U/\mathbb{Z})$) can be represented by sequences $\{X_i, \psi_i, L_i, \ell_i\}_{i \geq 1}$ (resp. $\{X_i, \psi_i, \overline{L}_i, \ell_i\}_{i \geq 1}$) where $X_i$ is a projective model for $U$ with a morphism $\psi_i : X_i \to X_1$, where $L_i$ (resp. $\overline{L}_i$) is a $\mathbb{Q}$-line bundle (resp. Hermitian $\mathbb{Q}$-line bundle) on $X_i$ and $\ell_i$ is a rational section of $L_i \otimes \psi^* L_1^{-1}$ with support contained in $X_i \setminus U$. The equality of this representation with the definitions above is shown in [YZ21, Lemma 2.5.1].

We will typically just write $\overline{L} = \{X_i, L_i\} \in \widehat{\text{Pic}}(U/k)$ or $\overline{L} = \{X_i, \overline{L}_i\} \in \widehat{\text{Pic}}(U/\mathbb{Z})$. From the conditions on $\ell_i$, we get a well-defined restriction map $\overline{L} \mapsto L \in \text{Pic}(U)_{\mathbb{Q}}$.

Such a sequence is Cauchy provided that $\{\text{div}(\ell_i)\}$ (resp. $\{\widehat{\text{div}}(\ell_i)\}$) is Cauchy under the topology defined above. A sequence converges to zero if there exists a sequence of rational sections $s_i$ of $L_i$ such that $\ell_i = s_i \otimes \psi^*_i s_1^{-1}$, and such that $\{\text{div}(s_i)\}$ (resp. $\{\widehat{\text{div}}(s_i)\}$) is itself Cauchy.

We call an adelic line bundle nef if it is isomorphic to a sequence where every (Hermitian) line bundle is nef, and we call an adelic line bundle integrable if it can be written as the difference of two nef ones. Denote the cones of nef elements

$$\widehat{\text{Pic}}(U/k)_{\text{nef}} \quad \text{and} \quad \widehat{\text{Pic}}(U/\mathbb{Z})_{\text{nef}}$$

and the subgroups of integrable elements

$$\widehat{\text{Pic}}(U/k)_{\text{int}} \quad \text{and} \quad \widehat{\text{Pic}}(U/\mathbb{Z})_{\text{int}}.$$

2.1. The relative setting. From here on out, we unify the notation for the base $k$ and base $\mathbb{Z}$ settings. Let $b$ be either Spec $\mathbb{Z}$ or Spec $k$. Let $K$ be a field which is finitely generated over $\mathbb{Q}$, if $b = \text{Spec} \mathbb{Z}$, or finitely generated over $k$, when $b = \text{Spec} k$. An open model for $K$ is a quasi-projective $b$-variety $V$ with function field $K$. 

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Next, let $X$ be a quasi-projective variety over $K$. An open model for $X/K$ consists of an open model $V$ for $K$, together with a quasi-projective and flat morphism $U \to V$ whose generic fiber is $X \to \text{Spec } K$.

The open models for $X$ form an inverse system via inclusion. Taking the limit over this system, we define

$$\hat{\text{Pic}}(X/b) := \lim_{U \to V} \hat{\text{Pic}}(U/b).$$

When $X = \text{Spec } K$, we will simply write $\hat{\text{Pic}}(K/b)$. Restricting these limits to the nef and integrable elements, we define the nef cone and integrable subgroup

$$\hat{\text{Pic}}(X/b)_{\text{nef}} \subset \hat{\text{Pic}}(X/b)_{\text{int}} \subset \hat{\text{Pic}}(X/b).$$

We will write elements as $\mathcal{T} \in \hat{\text{Pic}}(X/b)$. Given such $\mathcal{T} = \{X_i, \mathcal{L}_i\}_{i \geq 0} \in \hat{\text{Pic}}(X/b)$, we can restrict via base change to

$$\mathcal{T}_K := \{X_{i,K}, \mathcal{L}_{i,K}\} \in \hat{\text{Pic}}(X/K),$$

as each $X_{i,K}$ is a projective model for $X$. Since, by construction, all $\mathcal{L}_{i,K}$ agree on $X \subset X_{i,K}$, we have a well defined restriction to $X$ which we denote simply $L \in \text{Pic}(X)$. If $X^0$ is a smooth, quasi-projective $K$ curve, since $X^0$ has a unique projective model $X$ over $K$,

$$(2.1) \quad \hat{\text{Pic}}(X^0/K) = \text{Pic}(X)_{\mathbb{R}} := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$  

When $K$ is a number field or a function field of transcendence degree one over $k$ and $X$ is a projective $K$-variety, $\hat{\text{Pic}}(X/b)$ corresponds to the group of adelic metrized line bundles, as defined by Zhang [Zha95b]. There also exists an analytic theory of adelic line bundles over larger fields, for example [YZ21, Sec. 3], [Car20, Sec. 2], but we will not require it here.

2.2. Pullbacks. Given a map $f : Y \to X$, since a priori the models defining $\hat{\text{Pic}}(Y/b)$ and $\hat{\text{Pic}}(X/b)$ may not be compatible, it is not immediately apparent that we can define pullbacks of adelic line bundles. We rectify that.

**Lemma 2.2.** (C.f. [YZ21] Sec. 2.5.5) Let $f : Y \to X$ a morphism of quasi-projective varieties which is flat over $b$. Then there exists a well-defined pullback morphism

$$f^* : \hat{\text{Pic}}(X/b) \to \hat{\text{Pic}}(Y/b).$$

**Proof.** Assume for simplicity of notation that $b = \text{Spec } \mathbb{Z}$; the only difference when $b = \text{Spec } k$ is the lack of Hermitian metrics. Let $\mathcal{T} \in \hat{\text{Pic}}(X/b)$ be represented by a Cauchy sequence $\{X_i, \mathcal{L}_i\}$. By moving to smaller open subvarieties as needed, we may extend $f$ to a morphism $f : \mathcal{W} \to \mathcal{U}$ of open models for $X$ and $Y$, and it suffices to prove that there exists a well defined pullback $\hat{\text{Pic}}(\mathcal{U}/b) \to \hat{\text{Pic}}(\mathcal{W}/b)$.  

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Let $\mathcal{Y}$ be a projective model for $\mathcal{W}$. By Raynaud’s flattening theorem [RG71], by blowing up along $\mathcal{Y}\setminus\mathcal{W}$, we may assume that $f : \mathcal{W} \to \mathcal{X}$ extends to a flat morphism $f_i : \mathcal{Y}_i \to \mathcal{X}_i$. Replacing $\mathcal{Y}_i$ with a morphism which dominates it, we may further assume that there is a morphism $ψ_i : \mathcal{Y}_i \to \mathcal{Y}_0$ for every $i$, which is the identity on $\mathcal{W}$.

We now have a compatible system of models on which to pull back $L$. Define $f^*L := \{\mathcal{Y}_i, ψ_i, f^*_i\mathcal{L}_i, f^*_iℓ_i\}_{i \geq 0}$.

Since effective Cartier divisors pull back to effective Cartier divisors under both flat and dominant maps, and the topology on $\hat{\text{Pic}}(\mathcal{W}/b)$ does not depend on the choice of boundary divisor, the topology on $\hat{\text{Pic}}(\mathcal{U}/b)$ induces that on $\hat{\text{Pic}}(\mathcal{W}/b)$ via pullback of any boundary divisor on $\mathcal{X}_0$, and the sequence defining $f^*L$ is Cauchy with respect to this topology.

□

3. Canonical heights

Definition 3.1. Let $X$ be a quasi-projective $K$ variety, with $K$ and $b$ as in the previous section. Let $L ∈ \hat{\text{Pic}}(X/b)_{\text{int}}$, and suppose that $L ∈ \text{Pic}(X)$ is ample. Let $P ∈ X(K)$. We define the vector valued $L$-height of $P$ to be $h_L(P) := P^*L ∈ \hat{\text{Pic}}(K/b)_{\text{int}}$.

If $P$ is defined over some finite extension $K'$, we compose with the map $\hat{\text{Pic}}(K'/b)_{\text{nef}} → \hat{\text{Pic}}(K/b)_{\text{nef}}$ induced by the norm functor $N_{K'/K}$ and scale the result by $1/[K' : K]$, so that this is a relative height which does not depend on the choice of $K'$. For simplicity, we will usually omit this formalism and assume without loss of generality that $P$ is defined over $K$.

When $K$ is a number field or a transcendence degree one function field over $k$, taking the limit of the arithmetic degrees produces a Weil height [Zha95b], which we notate $\hat{h}_L(P) := \hat{\text{deg}}(h_L(P))$, and thus vector-valued heights extend the theory of $\mathbb{R}$-valued heights.

Lemma 3.2. (C.f. [YZ21] Sec. 6.1.1, [Car20] Sec. 4.1) Let $f : X → X$ be an endomorphism of projective varieties over $\overline{K}$ with a polarization, i.e. an ample $\mathbb{Q}$-line bundle $L ∈ \text{Pic}(X)\mathbb{Q}$ and a rational number $q > 1$ such that $f^*L = qL$. Then there exists a nef adelic line bundle $\mathcal{L}_f ∈ \hat{\text{Pic}}(X/b)_{\text{nef}}$ extending $L$ such that $f^*\mathcal{L}_f = q\mathcal{L}_f$.

In particular, there exists a canonical height $h_f := h_{\mathcal{L}_f}$ such that for all $P ∈ X(\overline{K})$, $h_f(f(P)) = qh_f(P)$.
Proof. We again prove this only in the base $\Z$ case, as the base $k$ case is identical besides the lack of Hermitian metrics. To start, we can find an open model $U \to V$ over $b$ for $X \to \text{Spec}K$ on which $f$ extends to a $V$-morphism $f : U \to U$, and over which there exists an extension $L \in \text{Pic}(U)_Q$ of $L$ such that $f^*L = qL$. Fix a projective model $X_0 \to B$ for $U \to V$, and a nef extension $\mathcal{L}_0 \in \text{Pic}(X_0)_Q$ of $\mathcal{L}$. By shrinking $V$ if needed, we may assume $B \setminus V$ is an effective Cartier divisor.

Write $f^m$ to mean the $m$-th iterate of $f$. For each $m \geq 1$, define $f_m : X_m \to X_0$ to be the normalization of the composition $X \xrightarrow{f^m} X \hookrightarrow X_0$, and define

$$\mathcal{L}_m := \frac{1}{q_m}f_m^*\mathcal{L}_0.$$  

For each $m \geq 0$, let $\mathcal{X}_m$ be a projective model for $U$ which dominates both $X_m$ and $X_{m+1}$ via $\phi_m : \mathcal{X}_m' \to \mathcal{X}_m$ and $\rho_m : \mathcal{X}_m' \to \mathcal{X}_{m+1}$, both of which restrict to the identity on $U$. Further assume that for $m \geq 1$, there exists a morphism $\tau_m : \mathcal{X}_m' \to \mathcal{X}_0'$ which commutes with $\phi_m$, $\rho_m$ and $f_m$. By construction $\tau_m$ extends $f^m$ on $U$. Since $\phi_m^*\mathcal{L}_0 - \rho_m^*\mathcal{L}_1$ is trivial on $U$, we can find a boundary divisor $D_0$ supported on $B \setminus V$ such that for the topology defined by $\pi^*D_0$, we have $\phi^*_m\mathcal{L}_0 - \rho^*_m\mathcal{L}_1 \in B(1,0)$. Now

$$\phi_m^*\mathcal{L}_m - \rho_m^*\mathcal{L}_{m+1} = q^{-m}\tau_m^*(\phi_0^*\mathcal{L}_0 - \rho_0^*\mathcal{L}_1),$$

and thus

$$\phi_m^*\mathcal{L}_m - \rho_m^*\mathcal{L}_{m+1} \in B(q^{-m},0).$$

We get a Cauchy sequence

$$\mathcal{L}_f := \{\mathcal{X}_m, \mathcal{L}_m\}_{m \geq 0} \in \widehat{\text{Pic}}(X/b)_{\text{nef}},$$

which is easily seen to satisfy the desired conditions. \hfill \Box

4. Proof of Theorem 1.1

Let $K$ be a number field and $b = \text{Spec} \Z$, or let $K$ be a transcendence degree one function field with constant field $k$, and $b = \text{Spec} k$. Let $T^0$ be a smooth quasiprojective curve over $K$, and let $F = K(T^0)$ be its function field. Let $A$ be a family of abelian varieties over $T^0$, and let $T$ be the unique smooth projective curve with function field $F$.

Write $\mathfrak{h}_A = \mathfrak{h}[2]$ for the $\widehat{\text{Pic}}(F/K)_{\text{int}}$-valued canonical height on the generic fiber $A_{\eta}$, and on each fiber $A_t$ over $t \in T^0(\overline{K})$, write $\mathfrak{h}_{A_t}$ for the $\widehat{\text{Pic}}(K/b)_{\text{int}}$-valued canonical height on $A_t$. As defined in the previous section, the usual $\R$-valued Néron-Tate canonical heights are

$$h_A(P) = \widehat{\text{deg}}(\mathfrak{h}_A(P)), \quad h_{A_t}(P_t) = \widehat{\text{deg}}(\mathfrak{h}_{A_t}(P_t))$$

We show that if $h_A(P) \neq 0$, there exists a nef adelic line bundle $\mathcal{M} \in \widehat{\text{Pic}}(K/b)_{\text{nef}}$ such that on the $\overline{K}$-points of $T$ over which $A$ has good reduction, the function

$$t \mapsto \mathfrak{h}_{A_t}(P_t)$$

is quasipositive.
is equal to a vector-valued height function \( h_{\overline{M}} \), defined on all of \( T \).

Proof. Let \( A \to T \) be the Néron model for \( A_\eta \). This is a quasiprojective \( K \)-variety. Replacing \( F \), and thus \( T \), with a finite extension will not alter the result, as the heights involved are invariant under finite extensions. Thus, we may assume that \( A \to T \) has semiabelian reduction [BLR90 Ch. 7.4, Thm 1]. We also write \( P : T \to A \) for the unique section extending \( P \in A(F) \), by the Néron mapping property.

Since \( A \) has abelian reduction at all but finitely many fibers, and the bad fibers have finitely many components, by replacing the point \( P \) with \( nP \) if needed, we may assume that the section \( P \) is contained in the connected component of the identity, which we denote \( A^\circ \). This replacement scales the canonical height on both \( A \) and on each abelian fiber \( A_t = A_t \) by \( n^2 \), and thus does not affect the result.

Let \( X \to b \) be a projective model for \( A^\circ \) over \( b \). Fix a symmetric and ample line bundle \( L \) on \( A_\eta \), and extend this to a nef line bundle \( \mathcal{L} \) on \( X \). By applying the normalization construction from the proof of Theorem 3.2, we can find a second projective model \( X' \to b \) with a morphism \([-1] : X' \to X \)

extending the automorphism \([-1] \) on \( A^\circ \). Fix a third model \( X_0 \) dominating both \( X \) and \( X' \). We can then define a nef line bundle

\[
\mathcal{L}_0 = \frac{1}{2} (\mathcal{L} + [-1]^* \mathcal{L}) \in \text{Pic}(X_0)_\mathbb{Q},
\]

where we use the implied pullbacks from \( X \) and \( X' \) to \( X_0 \).

We adapt an argument of Green [Gre89]. Consider the difference \([n]^* \mathcal{L}_0|_{A^\circ} - n^2 \mathcal{L}_0|_{A^\circ} \). By the theorem of the cube, this becomes trivial when restricted to \( A_\eta = A_\eta \). Then, since \( A^\circ \) is smooth and has integral fibers over \( T \), this difference is equal to the pullback under \( A^\circ \to T \) of a line bundle on \( T \), by [Gro67 Cor. 21.4.13 of Ch. 4, Errata et Addenda]. Finally, pulling back by the identity section \( E : T \to A^\circ \), we see that \([n]^* \mathcal{L}_0|_{A^\circ} - n^2 \mathcal{L}_0|_{A^\circ} \) must be trivial. Thus,

\([n]^* \mathcal{L}_0|_{A^\circ} = n^2 \mathcal{L}_0|_{A^\circ} \).

Remark 4.1. For a different version of the above argument extending a polarization to \( A^\circ \), one can use Moret-Bailly’s extension of cubical structure, [MB85 Thm. 3.5], as in [K98 Sec. 4].

Next, apply Lemma 3.2 to extend \( L \) to a nef adelic line bundle \( \overline{L} = \{ \mathcal{L}_m, \overline{\mathcal{L}}_m \} \in \hat{\text{Pic}}(A/b)_{\text{nef}} \) such that

\([2]^* \overline{L} = 4 \overline{L} \).

In general, the model adelic line bundles \( \overline{L}_m \) may all differ along a boundary divisor. Using the above construction extending the polarization to \( A^\circ \), however, the line
bundles $L_{m,K}$ can be made to all agree on $\mathcal{A}$. Since the section $P$ lands fully within $\mathcal{A}$, we have a well-defined pullback

$$\overline{M} = P^*L \in \widehat{\text{Pic}}(T/b)_{\text{int}},$$

since each $P^*(L_{m,K})$ is the same line bundle on $T$.

By construction, $h_M(t) = t^*\overline{M} = P^*_tT_t = h_{A_t}(P_t)$ for all $t \in T^0(K)$. Since $T$ is a curve over $K$, the $\mathbb{R}$-valued Néron-Tate height of $P$ is simply $\deg(M) = \deg((P^*L)_K) \in \mathbb{Q}$. Since $P$ does not have height zero, we conclude that $M$ is ample, and that $\overline{M}$ is nef, as the pullback of a nef adelic line bundle.

**Remark 4.2.** Without the Néron model, which exists for abelian varieties, but not in general for dynamical systems (see for example [Hsi96]), the above construction would instead produce only $\overline{M} \in \widehat{\text{Pic}}(T^0/b)$. Since $T^0$ has a unique projective closure, this gives an $\mathbb{R}$-line bundle $M \in \text{Pic}(T)_{\mathbb{R}}$, but it remains an open question whether the adelic structure extends over all of $T$, i.e. whether the metrics defined by each $L_m$ converge to an adelic metric on $T$.

4.1. **Proof of Corollary 1.4.** Since $\overline{M}$ is nef, we know that $\overline{M}^2 \geq 0$. Define

$$e_1 := \sup_{\text{open } U \subset T} \inf_{t \in U(K)} h_{\overline{M}}(t).$$

It can easily be seen that $e_1 \geq 0$, and is positive if and only if Zhang’s conjecture holds.

By Zhang’s essential inequalities [Zha95a, Thm 5.2],

$$e_1 \geq \frac{\overline{M}^2}{\deg \overline{M}} \geq \frac{e_1}{2}.$$ 

From this, the equivalence of the restatement of the conjecture in terms of the bigness of $\overline{M}$ is immediate.

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