A static axisymmetric exact solution of $f(R)$-gravity

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Abstract

We present an exact, axially symmetric, static, vacuum solution for $f(R)$ gravity in Weyl’s canonical coordinates. We obtain a general explicit expression for the dependence of $df(R)/dR$ upon the $r$ and $z$ coordinates and then the corresponding explicit form of $f(R)$, which must be consistent with the field equations. We analyze in detail the modified Schwarzschild solution in prolate spheroidal coordinates. Finally, we study the curvature invariants and show that, in the case of $f(R) \neq R$, this solution corresponds to a naked singularity.

Keywords: $f(R)$-gravity, static axisymmetric, exact solution

1. Introduction

In recent years, $f(R)$ theories of gravity have gained much attention as promising candidates to overcome the issues posed by dark energy in the standard cosmological model (c.f. reference [1, 2] for a recent review). There has been a stimulating debate in their study, leading us to a number of interesting results, particularly in the context of exact solutions.

Due to the highly non-linear nature of the $f(R)$-field equations, finding exact solutions is indeed a difficult task. Following astrophysical motivations, solutions with spherical symmetry have been the most widely studied [3]. However, there has also been a growing interest in finding exact solutions with cylindrical symmetry [4]. To the best of the authors’ knowledge (except for [5, 6]), there is no fully integrated, explicit and exact axially symmetric solution of $f(R)$-gravity.

In this paper, we consider static vacuum solutions of $f(R)$ theories in Weyl coordinates. In particular, we obtain the explicit dependence of $df(R)/dR$ upon the coordinates $\rho$ and $z$. This in turn, allows us to get the corresponding explicit form of $f(R)$. Finally, we analyze in detail the solutions of the modified field equations corresponding to the Schwarzschild solution in cylindrical coordinates.

2. $f(R)$ Field equations for a static axially symmetric space-time

The $f(R)$ action is given by

$$S = \int \left( \frac{1}{16\pi G} f(R) + \mathcal{L}_m \right) \sqrt{-g} \, d^4x,$$  \hspace{1cm} (1)

where $G$ is the gravitational constant, $R$ is the curvature scalar and $\mathcal{L}_m$ is the matter Lagrangian. The field equation resulting from this action are

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G (T^g_{ab} + T^m_{ab}), \hspace{1cm} (2)$$

where the ‘gravitational’ stress-energy tensor is

$$8\pi G T^g_{ab} = T^g_{ab}, \hspace{1cm} (3)$$

and

$$T^g_{ab} = \frac{1}{F_R} \left[ \frac{g_{ab}}{2} (f(R) - RF_R) + \nabla_a \nabla_b F_R \left( \delta^c_{[a} \delta^d_{b]} - g_{ab} T^c_{cd} \right) \right].$$

Here $F_R \equiv df(R)/dR$ and $T^m_{ab} \equiv T^m_{ab}/F_R$, where $T^m_{ab}$ is the stress-energy tensor obtained from the matter Lagrangian $\mathcal{L}_m$ in the action (1).

Equivalently, we can write (2) in the form

$$F_R R_{ab} - \frac{1}{2} f(R) g_{ab} = \nabla_a \nabla_b F_R + g_{ab} \Box F_R = 8\pi G T^m_{ab}. \hspace{1cm} (4)$$

Taking the trace of this expression we obtain the relation between $f(R)$ and its derivative $F_R$

$$F_R R - 2f(R) + 3\Box F_R = 8\pi G T^m. \hspace{1cm} (5)$$

We are interested in the static axially symmetric solutions of (4). To this end, let us consider the Weyl- Lewis-Papapetrou metric in cylindrical coordinates is [7]

$$ds^2 = -e^{2\phi} \, dt^2 + e^{-2\phi} [d\rho^2 + d\lambda^2 + e^{2\lambda} (d\rho^2 + dz^2)], \hspace{1cm} (6)$$

where $\phi$ and $\lambda$ are continuous functions of $\rho$ and $z$. Using the trace equation (5), the modified Einstein field equations (4) become

$$F_R R_{ab} - \nabla_a \nabla_b F_R - 8\pi G T^m_{ab} = g_{ab} B, \hspace{1cm} (7)$$

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where \( B = \frac{1}{4} (F_R R - \Box F_R - 8\pi G T_m^m) \). The non-zero components of the Ricci tensor are

\[
\begin{align*}
R_{00} &= e^{4\phi - 2\lambda} \nabla^2 \phi, \\
R_{11} &= \rho^2 e^{4\phi - 2\lambda} \nabla^2 \phi, \\
R_{22} &= -\nabla^2 \lambda + \nabla^2 \phi + \frac{2}{\rho} \phi, \\
R_{33} &= -\nabla^2 \lambda + \nabla^2 \phi - 2 \phi_z^2, \\
R_{23} &= \frac{1}{\rho} \phi_z - 2 \phi \phi_z.
\end{align*}
\] (8a)-(8e)

while a straightforward computation of the curvature scalar yields

\[
R = 2 e^{2\phi - 2\lambda} (-\nabla^2 \lambda + \nabla^2 \phi + \frac{1}{\rho} \lambda + \phi_z^2 + \phi_{z}^2),
\] (9)

where \( \nabla^2 \) is the usual Laplace operator in cylindrical coordinates.

From (7) and (8) we obtain the following system of equations:

\[
\begin{align*}
\frac{1}{8\pi G} [F_R R_{00} - 8\pi G T_{00}^0] &= B, \\
\frac{1}{8\pi G} [F_R R_{11} - 8\pi G T_{11}^1] &= B, \\
\frac{1}{8\pi G} [F_R R_{22} - 8\pi G T_{22}^2] &= B, \\
\frac{1}{8\pi G} [F_R R_{33} - 8\pi G T_{33}^3] &= B, \\
F_R R_{23} - 8\pi G T_{23}^m - F_{R,23} &= 0.
\end{align*}
\] (10a)-(10e)

This allows us to write down the independent field equations, i.e.

\[
\begin{align*}
\nabla^2 \phi &= -\frac{4\pi G}{F_R} e^{2(\lambda - \phi)} [T_{00}^m - T_{11}^1], \\
\lambda_{\rho} &= \rho (\phi_{\rho}^2 - \phi_z^2) + \frac{4\pi G \rho}{F_R} e^{2(\lambda - \phi)} [T_{22}^2 - T_{33}^3] \\
&+ \frac{\rho}{2 F_R} [F_{R,22} - F_{R,33}], \\
\lambda_z &= 2 \rho \phi \phi_z + \frac{8\pi G \rho}{F_R} T_{23}^m + \rho F_{R,23}.
\end{align*}
\] (11a)-(11c)

To find a general solution to the above equations is indeed a difficult task. Nevertheless, in the following sections we will discuss some particular solutions to (11a).

3. \( f(R) \) vacuum solutions for a static axially symmetric space-time

For simplicity, we restrict ourselves to the vacuum case, i.e. we make

\[
\begin{align*}
\nabla^2 \phi &= 0, \\
\lambda_{\rho} &= \rho (\phi_{\rho}^2 - \phi_z^2) + \frac{\rho}{2 F_R} (F_{R,\rho \rho} - F_{R,\rho \rho}), \\
\lambda_z &= 2 \rho \phi \phi_z + \frac{\rho F_{R,\rho z}}{F_R}, \\
\rho F_{R,\rho \rho} (F_{R,\rho \rho} - F_{R,\rho \rho}) + \rho F_R \nabla^2 (F_{R,\rho \rho}) &+ F_{R,\rho \rho} (F_R - 2 \rho F_{R,\rho}) = 0, \\
f(R) &= \frac{1}{2} F_R R + \frac{3}{2} \nabla^2 F_R.
\end{align*}
\] (12a)-(12f)

One can see that equation (12d) is the integrability condition for \( \lambda \). Note that for an arbitrary \( f(R) \), the system (12) may become inconsistent. Therefore, we look for the class of functions \( f(R) \) compatible with (12). It is an easy exercise to prove that

\[
\Box F_R = e^{2(\lambda - \phi)} (F_{R,\rho \rho} + F_{R,\rho \rho}).
\] (13)

Thus, substituting (13) in (12f) and using (12e) we obtain

\[
f(R) = \frac{1}{2} F_R R \left[ 1 - \frac{3 W(\rho)}{2 F_R} \right],
\] (14)

where

\[
W(\rho) = \frac{F_{R,\rho \rho} + F_{R,\rho \rho}}{\lambda_{\rho \rho} + \lambda_{\rho \rho} + \phi_{\rho}^2 + \phi_{\rho}^2}.
\] (15)

In order to obtain some analytical solutions to the system (12), we need to make some further simplifying assumptions. First, suppose that it is possible to write

\[
F_R (\rho, z) = U(\rho) V(z).
\] (16)

Then, substituting back into (12d) we have

\[
\rho^{-1} U^{-2} \left[ 2 dU \left( \frac{dU}{d\rho} - U \right) - 2 \rho U \frac{d^2 U}{d\rho^2} \right] = \left( V \frac{dV}{dz} \right)^{-1} \left( V \frac{d^2 V}{dz^2} - \frac{d^2 V}{dz^2} \frac{dV}{dz} \right).
\] (17)

Equating each side to a separation constant, \( \ell^2 \), we obtain the third order pair of ordinary differential equations

\[
\rho^{-1} U^{-2} \left[ 2 dU \left( \frac{dU}{d\rho} - U \right) - 2 \rho U \frac{d^2 U}{d\rho^2} \right] = \ell^2 \] (18a)
\[
\left( V \frac{dV}{dz} \right)^{-1} \left( V \frac{d^2 V}{dz^2} - \frac{d^2 V}{dz^2} \frac{dV}{dz} \right) = \ell^2. \] (18b)

Let us re-write (18a) as

\[
\frac{dM(\rho)}{d\rho} + \frac{M(\rho)}{\rho} = \frac{\ell^2}{2}.
\] (19)
where \( M(\rho) = U^{-1} dU/d\rho \). One can solve this immediately to obtain
\[
M(\rho) = U^{-1} dU/d\rho = -n - 2\rho \frac{\dot{\rho}}{4},
\]
which has the solution
\[
U(\rho) = c\rho^n e^{-2\rho^2/4},
\]
where \( c \) and \( n \) are integration constants. Now, one can easily show that
\[
V(\rho) = e^{bc},
\]
where \( b \) is an arbitrary constant, is solution of (18b) if both, \( b \) and \( l \), satisfy the condition
\[
b^2 = 0.
\]
Thus, we have that some possible solutions for \( F_R \) are [c.f. equation (16)]

(i) \( b = 0 \) and \( l = 0 \). In this case
\[
F_R = c\rho^n.
\]

(ii) \( b \neq 0 \) and \( l = 0 \). In this case
\[
F_R = c\rho^n e^{blc}.
\]

(iii) \( b = 0 \) and \( l \neq 0 \). In this case
\[
F_R = c\rho^ne^{-\rho^2l/4}.
\]
Consequently, by substituting (24) [or (25)] in (14) and using (12) we obtain that \( f(R) \) must satisfy the consistency condition
\[
f(R) = 2R \frac{df}{dR},
\]
whose solution is simply
\[
f(R) = k R^{1/2},
\]
where \( k \) is an arbitrary constant.

Substituting (25) in both, equations (12b) and (12c), we have
\[
\lambda_\rho = \rho (\phi_\rho^2 - \phi_z^2) + \frac{1}{2\rho} [n(n - 1) - b^2 \rho^2],
\]
\[
\lambda_z = 2\rho \phi_\rho \phi_z + bn,
\]
respectively. Whereas, by taking \( F_R = c\rho^n e^{-\rho^2l/4} \) we obtain from (14)
\[
f(R) = \frac{1}{2} F_R R \left[ 1 - 3L(\rho) \right],
\]
with
\[
L(\rho) = \frac{(4\rho^2 - 4n)^2 - 4(\rho^2 + 4n)}{(3\rho^2 + 4n - 4)(\rho^2 - 4n)}.
\]
Finally, substituting (26) in (12b) and (12c), we have
\[
\lambda_\rho = \rho (\phi_\rho^2 - \phi_z^2) + \frac{1}{32\rho} [(4\rho^2 - 4n)^2 - 4(4\rho^2 + 4n)],
\]
\[
\lambda_z = 2\rho \phi_\rho \phi_z,
\]
respectively.

As we can see from equations (29) and (32), the function \( \lambda \) can be calculated by means of a line integral. Although \( \nabla^2 \phi = 0 \) is a linear differential equation, the equations for \( \lambda \) manifest the non-linearity of the “modified” Einstein field equations.

The usual Einstein vacuum equations for the static axisymmetric spacetime we have
\[
\nabla^2 \phi = 0,
\]
\[
\lambda_\rho = \rho (\phi_\rho^2 - \phi_z^2),
\]
\[
\lambda_z = 2\rho \phi_\rho \phi_z.
\]

The Laplace equation may be solved by using various coordinates in the Euclidean 3-space and then the function \( \lambda \) can be calculated [c.f. chapter 20 in [7]]. We observe the following points:

1. If \( F_R = c\rho^n e^{blc} \), then we can obtain a vacuum static axially symmetric solution \((\phi, \lambda)\) of the vacuum ‘modified’ Einstein field equations from the vacuum Einstein field equations \((\bar{\phi}, \bar{\lambda})\) using the transformation
\[
\phi = \bar{\phi},
\]
\[
\lambda = \bar{\lambda} + \ln \left[ k \rho^{(n-1)/2} \right] - \frac{b^2}{4} \rho^2 + bn.
\]

2. Similarly, if \( F_R = c\rho^n e^{-\rho^2l/4} \) we make
\[
\phi = \bar{\phi},
\]
\[
\lambda = \bar{\lambda} + \frac{1}{32} \left\{ \frac{\bar{\phi}}{\rho} - 2(2n + 1) \bar{\phi} \rho^2 + \ln \left[ k \rho^{(n(n-1)/2)} \right] \right\}.
\]

In both cases, \( k \) is an appropriate constant.

Thus, one can say that equations (34) and (35) represent a Weyl class of solutions in \( f(R) \)-gravity. Moreover, one can expect that some properties of the curvature of the seed solution will be inherited to the modified ones.

4. A particular solution for a vacuum static axially symmetric space-time

Here we present an application of the results obtained in the previous section. First, we assume a given metric potential, \( \phi \) say [c.f. equation (6)], and then we find the other one by means of the two equations \( F_R = c\rho^n e^{blc} \), and \( F_R = c\rho^n e^{-\rho^2l/4} \). Let us work in prolate spheroidal coordinates \((x, y)\), with \( x \in [1, \infty) \) and \( y \in [-1, 1] \). These are related to the cylindrical coordinates \((\rho, z)\) through the relations
\[
\rho^2 = m^2(x^2 - 1)(1 - y^2) \quad \text{and} \quad z = mxy.
\]

The line element (6) becomes [c.f. equation (4.5.18) in [8]]
\[
dx^2 = -e^{2\phi} dx^2 + m^2 e^{\phi} \left( x^2 - y^2 \right) \left[ \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right] + m^2 e^{-2\phi} (x^2 - 1)(1 - y^2) d\phi^2.
\]
The Einstein vacuum equations in $f(R)$ gravity for a static axially symmetric space-time can be cast into the form

$$w^2\phi = 0$$

$$\lambda_x = \lambda + \beta \rho_x + \Omega z_x,$$

$$\lambda_y = \lambda + \beta \rho_y + \Omega z_y,$$

where

$$\phi = \frac{1}{2} \ln \left[ \frac{x - 1}{x + 1} \right]$$

and

$$\lambda = \frac{1}{2} \ln \left[ \frac{x^2 - 1}{x^2 + y^2} \right]$$

are the metric potentials corresponding to the usual Schwarzschild solution in standard general relativity. Note that we can relate the prolate coordinates $(x, y)$ to the usual Schwarzschild coordinates $(r, \theta)$ through the relations (c.f. equation (4.5.19) in [8])

$$r = m(x + 1) \quad \text{and} \quad \theta = \arccos(y).$$

This in turn allows us to relate them to (36) by

$$\rho^2 = [(r - m)^2 - m^2] \sin^2(\theta) \quad \text{and} \quad z = (r - m) \cos(\theta).$$

It becomes clear that $\rho = 0$ corresponds to $x = 1$ and $r = 2m$. Thus, in the forthcoming discussion, the reader should be aware that the domain of the prolate coordinates is defined from the horizon up to infinity. Notice as well that

$$\beta = \frac{\rho}{2F_R}(F_R_{\rho\rho} - F_R_{\rho z}) \quad \text{and} \quad \Omega = \frac{\rho F_R_{\rho z}}{F_R}.$$

Thus, in the same way as the last case, we will obtain a explicit form of $\lambda$ by considering the different values of $F_R$.

4.1. $F_R = c\rho^2 e^{\eta \rho}$

Here we have

$$\lambda_x = \frac{x(1 - y^2)}{(x^2 - 1)(x^2 - y^2)} + \frac{n(n - 1)x}{2(x^2 - 1)}$$

$$- \frac{b^2 m^2 y}{2}(1 - y^2) + bmnx,$$

$$\lambda_y = \frac{y}{x^2 - y^2} - \frac{n(n - 1)y}{2(1 - y^2)}$$

$$+ \frac{b^2 m^2 y}{2}(x^2 - 1) + bmnx,$$

whose solution is

$$\lambda = \lambda + \frac{n(n - 1)}{4} \ln [(x^2 - 1)(1 - y^2)]$$

$$+ \frac{b^2 m^2 Q}{4} + bmnx,$$

$$Q = x^2 y^2 - x^2 - y^2.$$
Figure 1: Kretchman invariant (top plots) and Ricci curvature scalar (bottom plots) as functions of $x$ for $F_R = c \rho^{n} e^{bn}$ with $n = 1, 2$ and $n = 3$ for the values $b = 0$ (left plots) and $b = 1$ (right plots).
Figure 2: Kretchman invariant (top plots) and Ricci curvature scalar (bottom plots) as functions of $x$ for $F_R = c^p e^{-\rho^2/\bar{k}}$ with $n = 1, 2$ and $n = 3$ for the values $l = 0$ (left plots) and $l = 1$ (right plots).
4.2. $F_R = c \rho^n e^{-\bar{\rho}^2/\rho^2}$

Now, in this case

$$\lambda_x = \tilde{\lambda}_x + \frac{x}{32(x^2 - 1)} P, \quad (54a)$$

$$\lambda_y = \tilde{\lambda}_y - \frac{y}{32(1 - y^2)} P, \quad (54b)$$

$$P = 16n(n - 1) + \bar{\rho}^2 - 4\bar{\rho}^2(2n + 1)^2],$$

and the solution is written as

$$\lambda = \lambda + \frac{n(n - 1)}{4} \ln [(x^2 - 1)(1 - y^2)]$$

$$+ \frac{\bar{\rho}^2 m^2}{128}[\bar{\rho}^2 m^2(\bar{Q} + 2) + 8(2n + 1)\bar{Q}]. \quad (55)$$

Just as in the previous case, we analyse the curvature invariants to look for singularities. Again, we split our study into two cases

1. $l = 0$. The curvature invariants are

$$R = -\frac{h_3(x, y; n)}{(x + 1)^2} \quad (56)$$

and

$$K = \frac{h_4(x, y; n)}{2(x - 1)^2 m^4(x + 1)^6(-1 + y^2)^2}. \quad (57)$$

Same as before, the equatorial plane $n = 1$ solution reduces to Schwarzschild

$$R_{n=1} = 0 \quad (58)$$

and

$$K_{n=1} = 32 \frac{1}{(x + 1)^6 m^4}. \quad (59)$$

2. $l \neq 0$. This case shares the same singular structure as the $l = 0$ case. As can be seen in figure 2 for $n = 1, 2, 3$. Note that for these solutions, $n = 1$ is always regular at the horizon. However, taking $n > 1$ always produces a naked singularity at $x = 1$.

Finally, let us note that, in every case, the curvature invariants converge to zero in large $x$ limit.

5. Closing remarks

The issue of static and axially symmetric solutions in $f(R)$-gravity is a timely topic in the context of the exact solutions. In this paper, we have presented an axially symmetric static vacuum solution in Weyl coordinates for $f(R)$ gravity. In particular, from the integrability condition of one of the metric potentials of the Weyl-Lewis-Papapetrou line element and using the method of separation of variables we have obtained a general explicit expression for the dependence of $df(R)/dR$ on the $\rho$ and $z$ coordinates and, therefore, the corresponding general explicit form of $f(R)$. Working in prolate spheroidal coordinates, we have analysed in detail the ‘Schwarzschild’ solution to the modified field equations. We have shown that these particular static and axially symmetric vacuum solutions of $f(R) \neq R$ correspond to naked singularities, as can be seen in the right hand column in figures 1 and 2. In particular, one observes that the singularity structure of the case $F_R = c \rho^n e^{-\bar{\rho}^2/\rho^2}$ is very sensitive to the value of $b$ as can be seen in figure 1 where the solution ceases to be regular at the horizon and becomes a naked singularity, even in the $n = 1$ case.

Finally, it is worth noting that the potentials (44) and (55) were found by integrating the corresponding system of differential equations [equations (38)]. However, these solutions can be obtained directly by using the transformations (34) and (35). This result allows us to generate axially symmetric solutions for $f(R)$ from known seeds of the Weyl and Schwarzschild solutions to the Einstein field equations.

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References

[1] T. Sotiriou, V. Faraoni, $f(r)$ theories of gravity, Reviews of Modern Physics 82 (1) (2010) 451.
[2] S. Nojiri, S. Odintsov, Unified cosmic history in modified gravity: from $f(r)$-theory to lorentz non-invariant models, Physics Reports 505 (2) (2011) 59–144.
[3] T. Multamäki, I. Vilja, Spherically symmetric solutions of modified field equations in $f(r)$ theories of gravity, Physical Review D 74 (6) (2006) 064022.
[4] A. Azadi, D. Momeni, M. Nouri-Zonoz, Cylindrical solutions in metric $f(r)$ gravity, Physics Letters B 670 (3) (2008) 210–214.
[5] S. Capozziello, M. De Laurentis, A. Stabile, Axially symmetric solutions in $f(r)$-gravity, Classical and Quantum Gravity 27 (16) (2010) 165008.
[6] J. Cembranos, A. de la Cruz-Dombriz, P. Romero, Kerr-newman black holes in $f(r)$ theories, arXiv preprint arXiv:1109.4519.
[7] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact solutions of Einstein’s field equations, Cambridge University Press, 2003.
[8] M. Carmeli, Group theory and general relativity: representations of the Lorentz group and their applications to the gravitational field, Imperial College Press, 1977.