Abstract. In this paper we construct the \(q\)-analogue of Barnes’s Bernoulli numbers and polynomials of degree 2, which is an answer to a part of Schlosser’s question. Finally, we will treat the \(q\)-analogue of the sums of powers of consecutive integers.

1. Introduction

In 1713, J. Bernoulli first discovered the method which one can produce those formulae for the sum \(\sum_{j=1}^{n} j^k\), for any natural numbers \(k\) (cf. [1],[3],[6],[7],[15],[22]). The Bernoulli numbers are among the most interesting and important number sequences in mathematics. These numbers first appeared in the posthumous work “Ars Conjectandi” (1713) by Jakob Bernoulli(1654-1705) in connection with sums of powers of consecutive integers (Bernoulli(1713) or D. E. Smith(1959) see [15]).

Let \(q\) be an indeterminate which can be considered in complex number field, and for any integer \(k\) define the \(q\)-integer as

\[
[k]_q = \frac{q^k - 1}{q - 1}, \quad \text{(cf. [11],[12],[13],[16],[17])}.
\]

Note that \(\lim_{q \to 1} [k]_q = k\). Recently, many authors studied \(q\)-analogue of the sums of powers of consecutive integers.

In [6], Garrett and Hummel gave a combinatorial proof of a \(q\)-analogue of \(\sum_{k=1}^{n} k^3 = (\binom{n+1}{2})^2\) as follows:

\[
\sum_{k=1}^{n} q^{k-1} \left( \frac{1-q^k}{1-q} \right)^2 \left( \frac{1-q^{k-1}}{1-q^2} + \frac{1-q^{k+1}}{1-q^2} \right) = \left[ \frac{n+1}{2} \right]_q^2,
\]

where

\[
\left[ \frac{n}{k} \right]_q = \prod_{j=1}^{k} \frac{1-q^{n+1-j}}{1-q^j}
\]
denotes the $q$-binomial coefficient. Garrett and Hummel, in their paper, asked for a simpler $q$-analogue of the sum of cubes. As a response to Garrett and Hummel’s question, Warnaar gave a simple $q$-analogue of the sum of cubes as follows:

$$\sum_{k=1}^{n} q^{2n-2k} \frac{(1 - q^k)^2}{1 - q^2} = \left[ \frac{n + 1}{2} \right]_q^2.$$  

(1.1)

In [21], Schlosser took up on Garrett and Hummel’s second question. Especially, he studied the $q$-analogues of the sums of consecutive integers, squares, cubes, quarts and quints. He obtained his results by employing specific identities for very-well-poised basic hypergeometric series. However, Schlosser could not find the $q$-analogue of the sums of powers of consecutive integers of higher order, and left it as question. In [15], T. Kim evaluated sums of powers of consecutive $q$-integers as follows:

For any positive integers $n, k(> 1), h \in \mathbb{Z}$, let

$$S_{n,q}^h(k) = \sum_{j=0}^{n} q^h[k]^n [k]_q.$$  

Then, he obtained the interesting formula for $S_{n,q}(k)$ below:

$$S_{n,q}(k) = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \beta_q[k]^j [k]_q^{n+1-j} - \frac{(1 - q^{n+1}) \beta_{n+1,q}}{n+1},$$

where $\beta_q$ are the modified Carlitz’s $q$-Bernoulli numbers. Indeed, this formula is exactly a $q$-analogue of the sums of powers of consecutive integers due to Bernoulli.

Recently, the problem of $q$-analogues of the sums of powers have attracted the attention of several authors([8],[9],[15],[19],[21],[23]). Let

$$S_{m,n}(q) = \sum_{k=1}^{n} [k]_q^2 [k]_q^{m-1} q^{(n-k)^{m+1}}.$$  

(1.2)

Then Warnaar[23] (for $m = 3$) and Schlosser[21] gave formulae for $m = 1, 2, 3, 4, 5$ as the meaning of the $q$-analogue of the sums of consecutive integers, squares, cubes, quarts and quints. By two families of polynomials and Vandermonde determinant, Guo and Zeng[9] found the formulae for the $q$-analogue of the sums of consecutive integers (for $m = 1, 2, ..., 5$). They recovered the formulae of Warnaar and Schlosser, for $m = 6, 7, ..., 11$. In [21], Schlosser speculated on the existence of a general formula for $S_{m,n}(q)$, which is defined in (1.2), and left it as an open problem.

By using T. Kim technical method to construct $q$-Bernoulli numbers and polynomials in [10],[11],[12],[13],[14],[15],[16],[19], we construct the $q$-analogue of Barnes’ Bernoulli numbers and polynomials of degree 2, which is an answer to a part of Schlosser’s question. Finally, we give some formulae for $S_{m,n}(q)$. 

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We define generating function $F_{k,q}^*(t)$ of the $q$-Bernoulli numbers $\beta_{n,k,q}^* (n \geq 0)$ as follows:

$$F_{k,q}^*(t) = -t \sum_{j=0}^{\infty} q^{k-j}[j]_q^2 \exp(t[j]_q q^{\frac{k-j}{2}}) = \sum_{n=0}^{\infty} \frac{-\beta_{n,k,q}^*}{n!} (\text{cf. [10],[11],[12],[13],[14],[15],[16]}).$$

**Theorem 1.** Let $n, k$ be positive integers. Then

$$\beta_{n,k,q}^* = \left(\frac{1}{1-q}\right)^{n-1} \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m q^{\frac{(n-1)(k-1)}{2} + k + m - 2}}{(1-q^{m-n-2})(1-q^{m-n-1})}.$$ 

We define generating function $F_{k,q}^*(t;k)$ of the $q$-Bernoulli polynomials $\beta_{n,k,q}^*(k) (n \geq 0)$ as follows:

$$F_{k,q}^*(t;k) = -t \sum_{j=0}^{\infty} q^{-j}[j+k]_q^2 \exp(t[j+k]_q q^{\frac{j+k}{2}}) = \sum_{m=0}^{\infty} \frac{\beta_{n,k,q}^*(k)t^n}{n!}.$$ 

**Theorem 2.** Let $n, k$ be positive integers. Then

$$\beta_{n,k,q}^*(k) = \frac{q^{\frac{k-1}{2}}}{[2]_q(1-q)^{n-2}} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left(\frac{mq^{k(m-1)}}{1-q^{m-n-2}} - \frac{mq^{k(m+1)}}{1-q^{m-n-1}}\right).$$

**Theorem 3.** (General formula for $S_{m,n}(q)$) Let $n, k$ be positive integers. Then

$$S_{n,k}(q) = \sum_{j=0}^{k-1} [j]_q^2 [j]_q^{n-1} q^{\frac{(n+1)(k-j)}{2}} = \frac{\beta_{n,k,q}^*(k) - \beta_{n,k,q}^*}{n}.$$ 

**2. Preliminary**

Let $n, k$ be positive integers ($k > 1$), and let

$$S_n(k) = \sum_{j=1}^{k-1} j^n = 1^n + 2^n + \ldots + (k-1)^n.$$ 

It is well-known that

$$S_1(k) = \frac{1}{2}k^2 - \frac{1}{2}k,$$

$$S_2(k) = \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k,$$

$$S_3(k) = \frac{1}{4}k^4 - \frac{1}{2}k^3 + \frac{1}{4}k^2,$$

$$\ldots.$$
Thus we have the following three conjectures:
(I) $S_n(k)$ is a polynomial in $k$ of degree $n + 1$ with leading coefficient $\frac{1}{n+1}$,
(II) The constant term of $S_n(k)$ is zero, i.e., $S_n(0) = 0$,
(III) The coefficient of $k^n$ in $S_n(k)$ is $-\frac{1}{2}$.
Therefore, $S_n(k)$ is a polynomial in $k$ of the form
$$S_n(k) = \frac{1}{n+1} k^{n+1} - \frac{1}{2} k^n + a_{n-1} k^{n-1} + \cdots + a_1 k.$$

We note that
$$\frac{d}{dk} S_n(k) = k^n - \frac{n}{2} k^{n-1} + \cdots.$$

To make life easier, we put the first two conjectures together and we reach the following conjecture, which is what Jacques Bernoulli (1654-1705) claimed more than three hundred years ago.

**Bernoulli:** There exists a unique monic polynomial of degree $n$, say $B_n(x)$, such that
$$S_n(k) = \sum_{j=1}^{k-1} j^n = 1^n + 2^n + \cdots + (k-1)^n = \int_0^k B_n(x) dx.$$

As the $q$-analogue of $S_n(k)$, Schlosser[21] considered the existence of general formula on $S_{m,n}(q)$, and he gave the below values:
$$\sum_{k=1}^{n} [k]_{q^2} [k]_q^{-1} q^{(n-k)\frac{m+1}{2}},$$
where $m = 1, 2, ..., 5$.

Indeed,
$$\sum_{k=1}^{n} [k]_{q^2} [k]_q q^{\frac{1}{2} (n-k)} = \frac{[n]_q [n+1]_q [n+\frac{1}{2}]_q}{[1]_q [2]_q [\frac{3}{2}]_q}, \quad m = 2,$$
$$\sum_{k=1}^{n} [k]_{q^2} [k]_q^2 q^{2 (n-k)} = \left[\frac{n+1}{2}\right]_q^2, \quad m = 3,$$
$$\sum_{k=1}^{n} [k]_{q^2} [k]_q^3 q^{\frac{3}{2} (n-k)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{n+\frac{1}{2}})}{(1-q)(1-q^2)(1-q^{\frac{3}{2}})} \times \left(\frac{(1-q^n)(1-q^{n+1})}{(1-q)^2} - \frac{q^n (1-q^{\frac{1}{2}})}{1-q^{\frac{3}{2}}}\right), \quad m = 4,$$
$$\sum_{k=1}^{n} [k]_{q^2} [k]_q^4 q^{3 (n-k)} = \frac{(1-q^n)^2 (1-q^{n+1})^2}{(1-q)^2 (1-q^2)(1-q^3)} \times \left(\frac{(1-q^n)(1-q^{n+1})}{4 (1-q)^2} - \frac{q^n (1-q)}{1-q^2}\right), \quad m = 5.$$
T. Kim[15] proved the smart formula for the $q$-analogue of $S_n(k)$ as follows:

$$
\sum_{k=0}^{n-1} q^k [k]_q = \frac{1}{2} \left( [n]_q^2 - \frac{[2n]_q}{[2]_q} \right)
$$

and

$$
\sum_{k=0}^{n-1} q^{k+1} [k]^2 = \frac{1}{3} [n]_q^3 - \frac{1}{2} \left( [n]_q^2 - \frac{[2n]_q}{[2]_q} \right) - \frac{1}{3} [3n]_q.
$$

3. Proof of Main Theorems

We define generating function $F^*_k(t)$ of the $q$-Bernoulli numbers $\beta^*_{n,k,q}$ ($n \geq 0$) as follows:

$$
F^*_k(t) = -t \sum_{j=0}^{\infty} q^{k-j} [j]_q^2 \exp(t[j]_q^{k-j}) = \sum_{n=0}^{\infty} \frac{\beta^*_{n,k,q}}{n!} t^n.
$$

Proof of Theorem 1. Let

$$
\sum_{n=0}^{\infty} \frac{\beta^*_{n,k,q}}{n!} t^n = -t \sum_{j=0}^{\infty} q^{k-j} [j]_q^2 \exp(t[j]_q^{k-j}).
$$

By using Taylor series in the above then we have

$$
\sum_{n=0}^{\infty} \frac{\beta^*_{n,k,q}}{n!} t^n = -t \sum_{j=0}^{\infty} q^{k-j} [j]_q^2 \sum_{n=0}^{\infty} \frac{[j]_q^n}{n!} t^n.
$$

By using some elementary calculations in the above, we have

$$
\sum_{n=0}^{\infty} \frac{\beta^*_{n,k,q}}{n!} t^n = -t \sum_{j=0}^{\infty} q^{k-j} [j]_q^2 \sum_{n=0}^{\infty} \frac{[j]_q^n}{n!} t^n
\times \sum_{n=0}^{\infty} \left\{ \left( \frac{1}{1-q} \right)^n q^{n(k-j)} \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{jm} \right\} \frac{t^n}{n!}
\times \sum_{j=0}^{\infty} \left( q^{mj-j} - \frac{m_j}{2} (1-q^{2j}) \right) \frac{t^n}{n!}.
$$
By using geometric power series in the above and after some calculations, we obtain

\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^* t^n}{n!} = -t \sum_{n=0}^{\infty} \left( \frac{1}{1-q} \right)^n \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m q^{m+k+\frac{1}{2}(k-1)-1}(1-q^2)}{(1-q^{-1+m-\frac{1}{2}})(1-q^{1+m-\frac{1}{2}})} n!
\]

For \(n = 0\), then \(\beta_{0,k,q}^* = 0\). Thus, we have

\[
\sum_{n=1}^{\infty} \frac{\beta_{n,k,q}^* t^n}{n!} = -t \sum_{n=1}^{\infty} \left( \frac{1}{1-q} \right)^{n-1} \sum_{m=1}^{n-1} \frac{(-1)^{m-1} q^{m+k+\frac{1}{2}(n-1)(k-1)-2}}{(1-q^{-2+m-\frac{n-1}{2}})(1-q^{m-\frac{n-1}{2}})} (n-1)!
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{1-q} \right)^{n-1} \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m q^{m+k+\frac{1}{2}(n-1)(k-1)-2}}{(1-q^{m-\frac{n-1}{2}})(1-q^{m-\frac{n-1}{2}})} n!
\]

By comparing the coefficients of \(\frac{t^n}{n!}\) on both sides of the above equation, we easily arrive at the desired result. \(\Box\)

We define generating function \(F_{k,q}^*(t; k)\) of the \(q\)-Bernoulli polynomials \(\beta_{n,k,q}^*(k) (n \geq 0)\) as follows:

\[
F_{k,q}^*(t; k) = -t \sum_{j=0}^{\infty} q^{-j} [j + k]_{q^2} \exp(t[j + k]_q^{-1}) = \sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k) t^n}{n!}.
\]

**Proof of Theorem 2.** Let

\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k) t^n}{n!} = -t \sum_{j=0}^{\infty} q^{-j} [j + k]_{q^2} \exp(t[j + k]_q^{-1}).
\]

By using Taylor expansion of \(e^x\) in the above, we have

\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k) t^n}{n!} = -t \sum_{j=0}^{\infty} q^{-j} [j + k]_{q^2} \sum_{n=0}^{\infty} [j + k]_q^{n} \frac{t^n}{n!}.
\]
By using some elementary calculations in the above, we have
\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k)t^n}{n!} = -t \sum_{j=0}^{\infty} q^{-j}[j+k]_q^2
\]
\[
\times \sum_{n=0}^{\infty} \left\{ \left( \frac{1}{1-q} \right)^n \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{m(j+k)- \frac{n}{2}} \right\} \frac{t^n}{n!}
\]
\[
= \frac{t}{q^2-1} \sum_{n=0}^{\infty} \left( \frac{1}{1-q} \right)^n \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{mk}
\]
\[
\times \sum_{j=0}^{\infty} \left( 1 - q^{2(j+k)} \right) q^{-j+m-\frac{in}{2}} \frac{t^n}{n!}.
\]

By using geometric power series in the above and after some calculations, we obtain
\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k)t^n}{n!} = -t \sum_{n=0}^{\infty} \frac{1}{[2]_q} \left( \frac{1}{1-q} \right)^{n-1}
\]
\[
\times \sum_{m=0}^{n-1} \binom{n}{m} (-1)^{m+1} q^{\frac{n-1}{2}(k-1)+k+m-1} \left( 1 - q^{1+m-\frac{n+1}{2}} \right) \left( n-1 \right)! \frac{t^n}{n!}
\]

Thus, we get
\[
\sum_{n=0}^{\infty} \frac{\beta_{n,k,q}^*(k)t^n}{n!} = -t \sum_{n=0}^{\infty} \frac{1}{[2]_q} \left( \frac{1}{1-q} \right)^{n-1}
\]
\[
\times \sum_{m=0}^{n} \binom{n}{m} (-1)^{m} \left( \frac{q^{mk}}{1-q^{m-1-\frac{n}{2}}} - \frac{q^{(m+2)k}}{1-q^{m+1-\frac{n}{2}}} \right) \frac{t^n}{n!}
\]

and
\[
\sum_{n=1}^{\infty} \frac{\beta_{n,k,q}^*(k)t^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{[2]_q} \left( \frac{1}{1-q} \right)^{n-2}
\]
\[
\times \sum_{m=0}^{n} \binom{n}{m} (-1)^{m} \left( \frac{mq^{(m-1)k}}{1-q^{m-\frac{n-1}{2}}} - \frac{mq^{(m+1)k}}{1-q^{m-\frac{n-1}{2}}} \right) \frac{t^n}{n!}
\]

By comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equations, we easily arrive at the desired result. \( \square \)

Proof of Theorem 3. Let
\[
- \sum_{j=0}^{\infty} q^{-j}[j+k]_q^2 \exp(t[j+k]_q q^\frac{k}{2}) + \sum_{j=0}^{\infty} q^{k-j}[j]_q^2 \exp(t[j]q q^{-j}) \quad (3.3)
\]
\[
= \sum_{j=0}^{\infty} q^{k-j}[j]_q^2 \exp(t[j]q q^{-j}).
\]
By using Theorem 1 and Theorem 2 and (3.3), we easily arrive at the desired result. Our proof of Theorem 3 runs parallel that of Theorem 1 and Theorem 2 above, so we choose to omit the details involved.

**Remark 1.** The Barnes double zeta function is defined by

$$\zeta_2(s, w \mid w_1, w_2) = \sum_{m,n=0}^{\infty} (w + mw_1 + nw_2)^{-s}, \quad \text{Re}(s) > 2, \text{ cf. [2]},$$

for complex number $w \neq 0, w_1, w_2$ with positive real parts.

The Barnes’ polynomial: Barnes[2] introduced $r$-tuple Bernoulli polynomials $rS_m(u; \tilde{w})$ by,

$$F_r(t; u; \tilde{w}) = \sum_{k=1}^{r} \frac{rS_1^{(k+1)}(u; \tilde{w})(-1)^k}{t^{k-1}}\left(\prod_{j=1}^{r} (1 - e^{-w_j t})\right) + \sum_{m=1}^{\infty} \frac{rS_m'(u; \tilde{w})}{m!} (-1)^{m-1} t^m,$$

for $|t| < \min \left\{ \frac{2\pi}{|w_1|}, \ldots, \frac{2\pi}{|w_r|} \right\}$. Here $w_1, w_2, \ldots, w_r$ are complex number with positive real parts, $\tilde{w} = (w_1, w_2, \ldots, w_r)$ and $rS_1^{(k)}(u; \tilde{w})$ the $k$-th derivative of $rS_1(u; \tilde{w})$ with respect to $u$.

Substituting $u = -1$, $w_1 = w_2 = -1$ and $r = 2$ into (3.4), we have

$$F_2(t; -1; -1) = \frac{te^t}{(1 - e^t)^2}.$$ 

In (3.1),

$$\lim_{q \to 1} F_{k,q}^*(t) = \lim_{q \to 1} -t \sum_{j=0}^{\infty} q^{-j} [j]_q^2 \exp(t[j]_q q^{\frac{k-j}{2}}) = -F_2(t; -1; -1; -1).$$

4. **Further Remarks and Observations on a class of $q$-Zeta Functions**

In this section, by using generating functions of $F_{k,q}^*(t)$ and $F_{k,q}^*(t; k)$, we produce new definitions of $q$-polynomials and numbers. These generating functions are very important in the case of defining $q$-zeta function. Therefore, by using these generating functions and Mellin transformation, we will define the $q$-zeta function.

By applying Mellin transformation in (3.2), we obtain

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_{k,q}^*(-t; k) dt = \sum_{n=0}^{\infty} \frac{[n+k]_q^2 q^{-n+\frac{k+2}{2}}}{[n+k]_q^s},$$
where $\Gamma(s)$ denotes the Euler gamma function.

For $s \in \mathbb{C}$, we define

$$\zeta^*_k(q)(s) = \sum_{n=0}^{\infty} \frac{[n+k]_q^{(k-n)/(2-s)}}{[n+k]_q^s}, \quad \Re(s) > 2.$$ 

By Mellin transformation in (3.1), we obtain

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} F^*_k(-t) dt = \sum_{n=0}^{\infty} \frac{[n]_q^{(k-n)/(2-s)}}{[n]_q^s} = \zeta^*_k(q)(s), \quad \Re(s) > 2.$$

For any positive integer $n$, Cauchy Residue Theorem in the above equation, we have

$$\zeta^*_k(q)(1-n) = -\frac{\beta^*_n}{n}.$$ 

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