A NEW PROOF OF THE PYTHAGOREAN THEOREM
INSPIRED BY NOVEL CHARACTERIZATIONS OF UNITAL
ALGEBRAS

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Abstract. A new proof of the Pythagorean Theorem is presented, utilizing
George Birkhoff’s version of the postulates of Euclidean geometry as incorpo-
rating \( \mathbb{R} \). Generalization of the mechanism of proof forwards novel characteri-
zations of real finite-dimensional unital associative algebras which, along with
the accompanying functor, are also presented and applied.

Phrased in modern terms, Decartes’ introduction of the Cartesian plane as
emerging from the classical Euclidean plane entails construction of an isometric
isomorphism between the intrinsic vector space resulting from the Euclidean plane
with a distinguished origin and the direct sum \( \mathbb{R} \oplus \mathbb{R} \) augmented by the inherited
Euclidean notion of length (the four postulate formulation of Euclidean geometry
due to Birkhoff [1], which fundamentally incorporates \( \mathbb{R} \), provides a rigorous
environment for effecting that program). The isomorphism relies on the parallel
postulate, and an important feature is that such an isomorphism can be constructed
from any pair of perpendicular lines through a chosen origin in the Euclidean plane.
We will present a proof of the Pythagorean Theorem based on that feature. The
proof is believed to be new - at least the idea of this proof is not found in the
large compendium [2], the more recent survey [3], or the present version of the
Wikipedia article “Pythagorean theorem”. More interesting than the proof itself
is generalization of the basic rationale to the setting of real finite-dimensional uni-
tal associative algebras, which leads to novel characterizations of these algebras in
relation to a powerful functor. In fact, the proof was originally inspired by the
algebraic application rather than the other way around [4].

Section 1 presents a synopsis of the proof, and Section 2 generalizes the proof’s
argument to real finite-dimensional unital associative algebras. Full details of the
proof, which are straightforward given the synopsis, are relegated to Appendix A.

1. The new proof of the Pythagorean Theorem

The original motivation for the Cartesian plane was that it can provide a conve-
nient arena for computations relating to features of the classical Euclidean plane.
An example relevant to our current endeavor concerns the “Euclidean length func-
tion” \( \ell \) which specifies the length of a line segment from a distinguished origin \( O \)
to any other point \( P \) of the Euclidean plane. Define a “direction” to be any point
\( C \) of the unit circle around the origin. The directional derivative of \( \ell \) at \( P \) in ref-

ence to a given direction \( C \) can be defined in terms of the labeling of \( O, P, C \)
by \( (0, 0), (x, y), (u, v) \in \mathbb{R} \oplus \mathbb{R} \) as determined by the above isomorphism resulting
from a selected pair of perpendicular lines intersecting at \( O \) (“a pair of orthogonal
coordinate axes through the origin”), via the function \( D\ell(x, y; u, v) \) supplying the
Synopsis of the argument. From the Mean Value Theorem of one-dimensional Calculus, and a demonstration of the continuity of \( D\ell(x, y; u, v) \) with respect to a fixed selected direction \((u, v)\) with \((x, y)\) varying in a domain excluding the origin, it is easy to show that \( D\ell(x, y; u, v) = \frac{\partial(x, y)}{\partial u} u + \frac{\partial(x, y)}{\partial v} v \), where \( \frac{\partial(x, y)}{\partial x} \) and \( \frac{\partial(x, y)}{\partial y} \) are the directional derivatives of \( \ell \) at the point labeled by \((x, y)\) with respect to the directions associated with the first and second coordinate axes (these directions being determined by the direction of increasing values of \( \mathbb{R} \) on the respective coordinate axes). Thus, for fixed \((x, y) \neq (0, 0)\), \( D\ell(x, y; u, v) \) is a linear functional for \((u, v)\) now varying over all of \( \mathbb{R} \). Evidently, this linear functional is the dual vector of \( \left( \frac{\partial(x, y)}{\partial u}, \frac{\partial(x, y)}{\partial v} \right) \). It follows that \( D\ell(P; C) \) is a linear functional for fixed \( P \) with \( C \) now varying throughout the Euclidean plane vector space. In view of the isomorphism, the dual of this linear functional is the point \( G \) of the Euclidean plane vector space labeled by \( \left( \frac{\partial(x, y)}{\partial x}, \frac{\partial(x, y)}{\partial y} \right) \) (i.e., the isomorphism maps vectors to vectors and dual vectors to dual vectors). The point \( G \) as obtained from \( D\ell(x, y; u, v) \) via the isomorphism is thereby invariant under change to an alternative pair of orthogonal coordinate axes through the origin since, as observed above, the linear functional \( D\ell(P; C) \) as determined by \( D\ell(x, y; u, v) \) is invariant under this change. Thus, we can choose an alternative pair of orthogonal coordinate axes through the origin (with coordinate axes and components of points distinguished by the symbol \( \ell \)) such that \( P \) is on the positive portion of the first coordinate axis of the new coordinate system, so that the Euclidean plane vector space point \( P \) originally labeled by \((x, y)\) is now labeled by \((x', 0)\), and \( G \) is now labeled by \( \left( \frac{\partial(x', 0)}{\partial x'}, \frac{\partial(x', 0)}{\partial y'} \right) = (1, 0) \), where the right-hand-side is an easy calculation since determination of the two directional derivatives on the left-hand-side only involves evaluation of \( \ell \) on the coordinate axes. Thus, according to the (isometric) isomorphism, we have \( \ell(G) = 1 \) (i.e., \( G \) is a point on the unit circle) and \( G \) is labeled by a positive multiple of \((x', 0)\) (i.e, \( G \) is labeled by a point on the ray from the origin through \((x', 0)\)). From the Euclidean geometry postulates, \( \ell \) is a degree-1 positive homogeneous function. This means that with respect to the original pair of orthogonal coordinate axes through the origin, we have \( \left( \frac{\partial(x, y)}{\partial x}, \frac{\partial(x, y)}{\partial y} \right) = \left( \frac{x}{\ell(x, y)}, \frac{y}{\ell(x, y)} \right) \), since the right-hand-side (uniquely) is both a point on the unit circle while also being a positive multiple of \((x, y)\) - as are now seen to be required for a labeling of \( G \) via the isomorphism. With respect to this last equation, one-dimensional indefinite integration (anti-differentiation) can be applied separately to the equality of the first coordinates, and then to the equality of the second coordinates. Reconciliation of the two resulting expressions for \( \ell \) in the context of the nonnegativity of \( \ell \) and its degree-1 positive homogeneity, quickly leads to \( \ell(x, y) = \sqrt{x^2 + y^2} \), from which the Pythagorean Theorem easily
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2. Novel characterizations of real finite-dimensional unital associative algebras arising from the proof’s structure

2.1. The characterizations. Surprisingly, the form of the argument presented in the proof synopsis is quite robust as regards its potential for generalization. The climax of the synopsis centers on two equations that can be recast as,

\begin{align}
\text{(2.1)} \quad s &= \ell(s)\nabla \ell(s), \\
\text{(2.2)} \quad \ell(\nabla \ell(s)) &= 1,
\end{align}

for \( s \in \mathbb{R}^n \) not at the origin (with \( n = 2 \), though we will not use that restriction going forward), and “\( \nabla \ell(s) \)” being the ordered \( n \)-tuple of coordinate-wise one-dimensional derivatives as in the proof synopsis. Indeed, solution of the above two displayed equations by themselves yields the Euclidean norm. But (2.2) is itself just a consequence of (2.1) and degree-1 positive homogeneity of \( \ell(s) \) (simply apply \( \ell \) to both sides of (2.1) and invoke the homogeneity condition). And (2.1) is appealing on its own, since given the degree-1 positive homogeneity constraint it seems to be a “simplest” expression for a relationship between \( s \) and \( \ell(s) \). That is, the Euler Homogeneous Function Theorem states that \( s \cdot \nabla \ell(s) = \ell(s) \). For \( \alpha > 0 \), replacing \( s \) by \( \alpha s \) then leads to \( \alpha s \cdot \nabla \ell(\alpha s) = \ell(\alpha s) = \alpha \ell(s) = \alpha s \cdot \nabla \ell(s) \), implying \( \nabla \ell(\alpha s) = \nabla \ell(s) \) (where \( \nabla \ell(\alpha s) \) means that \( \nabla \ell \) is evaluated at \( \alpha s \)). So, although \( s \) and \( \nabla \ell(s) \) are both members of \( \mathbb{R}^n \), they behave very differently when \( s \) is replaced by \( \alpha s \). On the other hand, for degree-1 positive homogeneous \( \ell \), the expressions \( s \) and \( \ell(s) \nabla \ell(s) \) do behave the same way when \( s \) is replaced by \( \alpha s \) (each expression simply being multiplied by \( \alpha \)). So what could be more natural than linking them via (2.1) (and thus forwarding the Euclidean norm)?

But without unduly compromising the simplicity of the rationale, we could alternatively propose that (2.1) be replaced by

\begin{align}
\text{(2.3)} \quad Ls &= \ell(s)\nabla \ell(s),
\end{align}

for a linear transformation \( L : \mathbb{R}^n \to \mathbb{R}^n \), since we would still have \( Ls \) and \( \ell(s)\nabla \ell(s) \) behaving the same when \( s \) is replaced by \( \alpha s \), \( \alpha > 0 \). Applying the dot product with \( s \) to both sides of (2.3), the Euler Homogeneous Function Theorem then implies

\begin{align}
\text{(2.4)} \quad s' Ls &= s \cdot Ls = \ell^2(s),
\end{align}

where \( L \) in this equation is understood to be the matrix associated with the above linear transformation \( L \) with respect to the standard basis of \( \mathbb{R}^n \). According to (2.3), \( L \) must be a real symmetric matrix since \( Ls \) is a gradient (i.e., the right-hand-side of (2.3) is equal to \( \frac{1}{2} \nabla \ell^2(s) \)). According to (2.4), \( L \) must be positive semi-definite. From the Polarization Identity, it is seen that the above leads to arbitrary inner product spaces on \( \mathbb{R}^n \) - a nice, if simple, generalization.

However, the above argument can easily be further generalized to the task of attaching a “norm” to \( \mathbb{R}^n \) when the latter is the vector space of elements of a unital associative algebra. That is, the basic rationale of the argument is easily modified to allow the new feature of the inverse of an element to influence selection of a norm -
something not explicitly addressed by the well-recognized “usual norm” of a finite-dimensional associative algebra as the determinant of an element’s image under the left regular representation (e.g., the algebra’s norm as defined by Bourbaki [5], and whose norm format is also fundamental more generally, such as in Algebraic Number Theory).

Thus, we wish to exploit the existence of an element’s inverse to aid in construction of a “norm-like” real function $U(s)$ on the units of the algebra. By analogy with the vector space argument in the first two paragraphs of this section, one recognizes that $s^{-1}$ and $\nabla U(s)$ are members of $\mathbb{R}^n$ if $s$ is a unit of the algebra, and $\nabla U(\alpha s) = \nabla U(s)$ for $\alpha > 0$ if degree-1 positive homogeneity is again mandated. While the expressions $s^{-1}$ and $\nabla U(s)$ thereby behave differently when $\alpha s$ replaces $s$, this time it is $s^{-1}$ and $\nabla U(s)$ that behave the same under that replacement, in that both expressions are simply multiplied by $\frac{1}{\alpha}$ - mirroring the situation that pertains to the first paragraph of this section. So now, based on that latter treatment (and its “minimalist” approach for attaching a norm to a vector space), we might be tempted to equate $s^{-1}$ and $\nabla U(s)$ - except that the latter is the gradient of $\log U(s)$, while $s^{-1}$ is in general not a gradient. That problem resolves if we can find a linear transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Ls^{-1}$ is a gradient, i.e., satisfies the exterior derivative condition,

$$d \left( \left[ Ls^{-1} \right] \cdot ds \right) = d \left( (ds)'Ls^{-1} \right) = 0.$$ 

Here and in the sequel the standard basis on $\mathbb{R}^n$ is assumed, so that $L$ in (2.5) is the matrix associated with the above linear transformation with respect to the standard basis.

But there is also the Euler Homogeneous Function Theorem to deal with, wherein 

$$\frac{s \cdot \nabla U(s)}{U(s)} = 1.$$ 

Denoting the multiplicative identity of the algebra as $1$, we can define $\|1\|^2$ to be the number of independent entries on the main diagonal of the matrices comprising the left regular representation of the algebra. We are free to constrain $L$ by the requirement that $1 \cdot (L1) = 1'L1 = \|1\|^2$. It follows that we can now propose,

$$Ls^{-1} = \|1\|^2 \frac{\nabla U(s)}{U(s)} = \left( \frac{\|1\|^2}{U(s)} \right) \nabla U(s),$$

if $L$ satisfies both (2.5) and

$$s'Ls^{-1} = s \cdot Ls^{-1} = \|1\|^2,$$

since constraint (2.7) is a necessary condition for a solution of (2.6) to be degree-1 positive homogeneous (e.g., take the dot product of both sides of (2.6) with $s$ and apply the Euler Homogeneous Function Theorem). Note that (2.7) must hold if $s$ is replaced by $s^{-1}$ as both are units, so we expect $L$ to be a real symmetric matrix. Indeed, $L$ is akin to an inner product matrix, and an important feature is that $L$ need not be positive semi-definite (exploitable in a relativistic physics context [6]). Thus, (2.7) can be interpreted to mean that multiplicative inverses $s, s^{-1}$ also behave inversely as vector space members with respect to inner product $L$. If $L$ satisfying (2.5) and (2.7) can be found, then one can provide $U(s)$ by integrating (2.6), at least in some simply connected neighborhood of $1$.

Equation (2.6) is analogous to (2.3) (which we emphasize with the format of the second equality in (2.3)), and (2.7) is analogous to (2.4). But the marked difference
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is that for the “naked” vector space application (where (2.3) and (2.4) pertain), \(L\) can be any real symmetric \((n \times n)\)-matrix (allowing for possibly complex-valued \(\ell(s)\)), while in the unital algebra application (where (2.6) and (2.7) pertain) the additionally required (2.5) very significantly constrains real symmetric \(L\). It is this constraint (which also allows for the further constraint (2.7)) that makes things interesting, because different algebras have very different collections of admissible choices for \(L\), and these differences distinguish the algebras in novel ways.

So let us now formalize our terminology.

Definition 2.1. For a unital associative algebra whose vector space of elements is \(\mathbb{R}^n\),

- A unital neighborhood \(\mathcal{N}\) is an open simply connected neighborhood of 1 consisting only of units.
- With respect to the standard basis, a Proto-norm is a real symmetric matrix \(L\) such that for \(s \in \mathcal{N}\),

\[d\left([Ls^{-1}] \cdot ds\right) = d\left((s^{-1})'L(ds)\right) = 0.\]

- \(L\) is a normalized Proto-norm if for \(s \in \mathcal{N}\) it satisfies both (2.5) and

\[s' Ls^{-1} = \|1\|^2,\]

where \(\|1\|^2\) is the number of independent entries on the main diagonal of the matrices comprising the left regular representation of the algebra.
- An incomplete Unital Norm is a function, \(\mathcal{U} : \mathcal{N} \to \mathbb{R}\), with \(\mathcal{U}(1) = 1\), and for which there is a normalized Proto-norm \(L\) such that

\[Ls^{-1} = \|1\|^2 \frac{\nabla \mathcal{U}(s)}{\mathcal{U}(s)}.\]

- An incomplete Unital Norm is singular if \(L\) is singular.
- \(\mathcal{U}(s)\) is a Unital Norm if (2.6) holds when \(\mathcal{N}\) is replaced by the entire space of units of the algebra.
- For \(s\) in the space of units, if \(\mathcal{U}(\hat{s}) \equiv \lim_{\hat{s} \to s} \mathcal{U}(s)\) exists for each member \(\hat{s}\) of the closure of the space of units, then \(\mathcal{U}\) is a closed Unital Norm.
- The algebra’s Unital Norm family is the set of all incomplete Unital Norms implied by the algebra.
- The algebra’s Proto-norm family is the set of all Proto-norms implied by the algebra, and similarly for its normalized Proto-norm family.

Note that Definition 2.1 would still make sense if instead of requiring associativity of the algebra we only require that the units have unique inverses. The latter “extended” version of the definition will be exploited later on.

We now have the easy,

Theorem 2.1. A unital neighborhood \(\mathcal{N}\) exists for any unital associative algebra whose vector space of elements is \(\mathbb{R}^n\). A normalized Proto-norm \(L\) implies an incomplete Unital Norm on \(\mathcal{N}\) as the nonnegative function \(\mathcal{U} : \mathcal{N} \to \mathbb{R}\), with

\[\mathcal{U}(s) = e^{\frac{1}{\|1\|^2} \int_1^s [Lt^{-1}] dt}.\]

For \(\alpha > 0\), if \(s\) and \(\alpha s\) are both in \(\mathcal{N}\), and the line segment with endpoints 1 and \(\frac{1}{\alpha} 1\) is also in \(\mathcal{N}\), then

\[\mathcal{U}(\alpha s) = \alpha \mathcal{U}(s).\]
Suppose there is a smooth path \( P \) from \( 1 \) to \( s \) which is inside \( N \) such that \( P^{-1} \) is also inside \( N \), where \( P^{-1} \) is the set consisting of the inverses of the members of \( P \). Then

\[
U(s)U(s^{-1}) = 1.
\]

**Proof.** The usual topology on any \( \mathbb{R}^m \) is assumed. To see that a unital neighborhood exists, it is sufficient to observe that such a neighborhood clearly exists for the left regular representation of the algebra (where one works with members of the matrix algebra \( \mathbb{R}^{n \times n} \), and the inverse of a unit is simply the reciprocal of the determinant multiplied by the adjugate matrix).

Existence of normalized Proto-norm \( L \) implies (2.5) and (2.7). Satisfaction of the first of those equations along with the required \( U(1) = 1 \), results in the path-independent integral expression,

\[
\log U(s) = \frac{1}{\|1\|^2} \int_1^s [Lt^{-1}] \cdot dt,
\]

which yields (2.8).

For \( s \) and \( \alpha s \) in \( N \), and the line segment from \( 1 \) to \( \frac{1}{\alpha} 1 \) also in \( N \), with \( \alpha > 0 \), we have

\[
\log U(\alpha s) = \frac{1}{\|1\|^2} \int_1^{\alpha s} [Lt^{-1}] \cdot dt = \frac{1}{\|1\|^2} \int_{1\alpha}^s [L(\alpha u)^{-1}] \cdot d(\alpha u)
\]

\[
= \frac{1}{\|1\|^2} \int_1^s [Lu^{-1}] \cdot du + \frac{1}{\|1\|^2} \int_{1\alpha}^1 [Lu^{-1}] \cdot du
\]

\[
= \log U(s) + \log \alpha,
\]

where the integral from \( \frac{1}{\alpha} 1 \) to \( 1 \) can be easily evaluated along the line segment with those endpoints, and we have used (2.7) in evaluation of that integral to obtain the term \( \log \alpha \) on the right-hand-side of the final equation. Thus, we have \( U(\alpha s) = \alpha U(s) \).

Now suppose there is a smooth path \( P \) from \( 1 \) to \( s \) which is inside \( N \) such that \( P^{-1} \) is also inside \( N \). We then have,

\[
\log U(s^{-1}) = \frac{1}{\|1\|^2} \int_1^{s^{-1}} [Lt^{-1}] \cdot dt
\]

\[
= \frac{1}{\|1\|^2} \int_1^s [Ly] \cdot d(y^{-1})
\]

\[
= \frac{1}{\|1\|^2} \left( (y^{-1} : [Ly]) \bigg|_1^s - \int_1^s y^{-1} \cdot d(Ly) \right)
\]

\[
= -\frac{1}{\|1\|^2} \int_1^s y^{-1} \cdot L(dy) = -\frac{1}{\|1\|^2} \int_1^s [Ly^{-1}] \cdot dy
\]

\[
= -\log U(s),
\]

where we have used a change of variable (contemplating the left regular representation, it is appreciated that \( y = t^{-1} \) is a diffeomorphism on \( N \), commutativity of a linear transformation and a differential, the property that \( L \) is a real symmetric matrix, and (2.7). Equation (2.9) then follows. \( \square \)

Before getting into further particulars, we point out that the above leads to novel characterizations of real finite dimensional unital associative algebras. First, for any
unital neighborhood, Theorem 2.1 demonstrates that each normalized Proto-norm uniquely implies an incomplete Unital Norm given by (2.8). Conversely,

**Corollary 2.2.** An incomplete Unital Norm uniquely implies a normalized Proto-norm.

**Proof.** For a unital neighborhood $\hat{\mathcal{N}}$, let $(\hat{\mathcal{N}})^{-1}$ be the set of inverses of the members of $\hat{\mathcal{N}}$. The algebra’s left regular representation, and the matrix inverse formula from Laplace expansion, indicate that $(\hat{\mathcal{N}})^{-1}$ is homeomorphic to $\hat{\mathcal{N}}$. Thus, $(\hat{\mathcal{N}})^{-1}$ is an open set containing 1. Based on the usual topology of $\mathbb{R}^n$, we can take $\hat{\mathcal{N}}$ to be an open ball around 1. Invoking the left regular representation, it is easy to see that by making $\hat{\mathcal{N}}$ smaller, the open set $(\hat{\mathcal{N}})^{-1}$ can be contained in an open ball around 1 of arbitrarily small radius, and where we can take this latter open ball to be unital neighborhood $\mathcal{N}$. It then follows from (2.6) that a Proto-norm associated with an incomplete Unital Norm $\mathcal{U}(s)$ on $\mathcal{N}$ must be unique. This is because (2.6) implies that two such Proto-norms $L_1, L_2$ would be such that $(L_1 - L_2)s^{-1}$ is zero, where $s^{-1}$ can be an arbitrary member of $(\hat{\mathcal{N}})^{-1} \subset \mathcal{N}$, and (as previously noted) $(\hat{\mathcal{N}})^{-1}$ is an open set containing 1. □

It is shown in [7] that a unital associative algebra whose vector space of elements is $\mathbb{R}^n$ always has a non-empty Unital Norm family. Thus, from (2.6) and Corollary 2.2, any member of an algebra’s Unital Norm family characterizes the algebra’s inverse operation on its units, i.e., characterizes the equivalence class of algebras sharing the algebra’s mapping $s \to s^{-1}$ on the orthogonal complement of the kernel of the Proto-norm associated with the Unital Norm family member.

### 2.2. Tabulated examples.

For the remainder of Section 2, “algebra” will be taken to mean a unital associative algebra whose vector space of elements is $\mathbb{R}^n$, unless indicated otherwise.

To compute a Proto-norm family of some algebra, whose family members are thereby real symmetric $(n \times n)$-matrices with respect to the standard basis, we start with a generic symmetric $(n \times n)$-matrix each of whose entries is a real parameter (e.g., $\alpha, \beta, \gamma, \ldots$). The $\frac{n(n+1)}{2}$ entries in the upper triangular portion of the matrix are initially each a different parameter, and the other entries of the matrix are determined by symmetry. The uncurling constraint (2.5) is then applied, which leads to some matrix entries possibly being given the value zero, and others being some other linear combination of the original $\frac{n(n+1)}{2}$ parameters. Thus, the Proto-norm family is described by a parametrized matrix, whose entries are linear combinations of components of a $m$-dimensional parameter vector, with $1 \leq m \leq \frac{n(n+1)}{2}$, where $m$ is the number of distinct parameters ultimately present in the parametrized matrix. We consider that the parametrized matrix is equivalent to the set of real matrices given by all possible realizations of the parameter vector as it ranges over $\mathbb{R}^m$ (where a “realization” is a choice of a real value for each component of the parameter vector), and in the sequel the latter interpretation of any parametrized matrix is always understood. Thus, the Proto-norm family is a $m$-dimensional space represented by the above parametrized matrix.

Without any difficulty, we can now generate the following table of Proto-norm and Unital Norm families for various algebras.
| algebra          | Algebraic Norm | Unital Norm family | Proto-norm family                |
|------------------|----------------|--------------------|----------------------------------|
| $\mathbb{R}^{n \times n}$ | $(\det(s))^n$ | $(\det(s))^\frac{1}{n}$ | $\alpha \ (n^2 \times n^2)$-“transpose matrix” |
| $\mathbb{C}$     | $x^2 + y^2 = r^2$ | $\sqrt{x^2 + y^2} e^{\beta \arctan(\frac{y}{x})} = re^{\beta \theta}$ | $\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$ |
| $\mathbb{L}$     | $x^2 - y^2$ | $\sqrt{x^2 - y^2} e^{\beta \arctanh(\frac{y}{x})} = (x^2 - y^2)^{\frac{1}{2}} \left(\frac{x+y}{x-y}\right)^{\frac{1}{2}}$ | $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ |
| $\mathbb{R} \oplus \mathbb{R}$ | $xy$ | $(xy)^{\frac{1}{2}} \left(\frac{x}{y}\right)^{\frac{1}{2}}$ | $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ |
| $\mathbb{D}$     | $x^2$ | $xe^{\beta \frac{1}{2}}$ | $\begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}$ |
| $\mathbb{H}$     | $(x^2 + y^2 + z^2 + w^2)^2$ | $\sqrt{x^2 + y^2 + z^2 + w^2}$ | diag{\(\alpha, -\alpha, -\alpha, -\alpha\)} |
| Cayley-Dixon seq., after $\mathbb{H}$ | (Euclidean norm)$^2$ | Euclidean Norm | diag{\(\alpha, -\alpha, -\alpha, \ldots, -\alpha\)} |
| Spin Factor Jordan | (Minkowski norm)$^2$ | Minkowski norm | diag{\(\alpha, \alpha, \ldots, \alpha\)} |
| $\bigoplus_{i=1}^{n} \mathbb{R}$ | $\prod x_i$ | $(\prod x_i^\sigma)^{\frac{1}{\sigma}}$, $\sum \sigma_i = n$ | diag{\(\sigma_1, \ldots, \sigma_n\)} |

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \leftrightarrow (x, y, z) \in \mathbb{R}^3$$
$$x^2y$$

$$\begin{bmatrix} x & z \\ 0 & x \end{bmatrix} \leftrightarrow (x, z, w) \in \mathbb{R}^3$$
$$x^3$$

$$\begin{bmatrix} x & z \\ 0 & v \end{bmatrix} \leftrightarrow (x, v, z, w) \in \mathbb{R}^4$$
$$x^4$$

$$\begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \leftrightarrow (x, y, v, z, w) \in \mathbb{R}^5$$

$$x^4y$$

$$\mathbb{u}T_n$$
$$x^n$$

$$x_1 e^{\beta_1 \gamma_1 \ldots \beta_n \gamma_n} \left(\frac{x}{y}\right)^{\frac{1}{2}} e^{\beta \frac{1}{2}}$$

$$\{s I : s \in \mathbb{u}T_n, s_1, n \neq 0\}$$

Table I
We have included the Cayley-Dixon algebras and the Spin Factor Jordan Algebras in the table, even though most of these are not associative. However, these algebras can be formulated as \(*\)-algebras whose Hermitian elements are real, so that both the “usual norm program”, as well as the Unital-norm program envisioned by Theorem 2.1 can be carried out (as in [7]). Thus, the table heading “Algebraic Norm” means the usual norm or its extension to not associative \(*\)-algebras whose Hermitian elements are real, and the headings “Unital Norm family” and “Proto-norm family” include the extended version of Definition 2.1 as indicated immediately following the definition’s statement. \(\mathbb{C}\) is the algebra of split complex numbers, and \(\mathbb{D}\) is the algebra of dual numbers. \(u\mathbb{T}_n\) is the algebra of \((n \times n)\) upper triangular Toeplitz matrices, and in its row of the table, \(P_{\gamma_2, \ldots, \gamma_n} \left( \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1} \right)\) is a multivariate polynomial in the indicated components of an element \((x_1, \ldots, x_n)\) with the subscripted coefficients as parameters. \(\mathbb{I}\) is the exchange matrix (the \((n \times n)\)-matrix whose entries on the anti-diagonal are 1, and whose other entries are 0). Computational details resulting in the table entries are found in [7].

The usual norm is defined from the solitary mapping given by the left regular representation of the algebra. But an incomplete Unital Norm is defined by any mapping in the normalized Proto-norm family associated with the algebra. The Proto-norm family is thereby a potentially much richer reflection of an algebra’s structure than the usual norm - and Table I provides examples of that.

To begin with, from a glance at Table I it would appear that one member of the Proto-norm family frequently supplies a Unital Norm that gives results that are essentially the same (up to an exponent) as the usual norm. This observation is made precise in [7], where it is also shown that an algebra’s Proto-norm family always exists, it is at least one-dimensional, and its Unital Norm family is always nonempty. The following Table I observations then relate to the case where multiple Unital Norms are associated with an algebra.

- The Dual numbers (row 5), and the algebra of upper triangular Toeplitz matrices in general (row 14), provide instances where the Unital Norms are especially impressive due to their “dominance” as regards sensitivity to all algebra element components compared to the usual norm, which in these cases is sensitive to only a single element component. Other algebras with this greater element component sensitivity of the Unital Norm can be easily constructed, as upper triangular matrix algebras for which almost all of their elements are defective matrices.

- It may be possible to reconstruct a unit from its incomplete Unital Norm values. For example, a unit in any of the two-dimensional real unital algebras can be reconstructed from the two norm values determined by two of its incomplete Unital Norms, if the unit is in a unital neighborhood of \(\mathbb{I}\).

- The non-units are apparently associated with an algebra-dependent type of Unital Norm family singularity reflecting the topology of the space of units.

- Some algebras are “essentially the same” as their Proto-norm families in a certain sense. For example, when the members of the left regular representation of the algebra of complex numbers, or of the algebra of dual numbers, or of the algebra of real upper triangular Toeplitz matrices, are each multiplied by the exchange matrix, the result in each case is the Proto-norm family of the algebra (and, in fact, the left regular representation of the split-complex numbers is \textit{identical} to its Proto-norm family). In cases
like these, every member of the algebra has dual roles in that it implies a first linear transformation as its image under the left regular representation, along with a second linear transformation (again acting on the algebra itself) as a Proto-norm.

There is another suggestive feature of Table I, to which we now turn.

2.3. The functor. A superficial perusal of the Table I’s last column seems to indicate that various of the algebras have relationships through their Proto-norm families. That notion is confirmed in the construction of a functor, as follows.

We define $\mathcal{A}$ to be the category whose objects are the unital associative algebras whose vector space of elements is $\mathbb{R}^n$ where $n$ is any particular positive integer, and whose morphisms are algebra epimorphisms. Composition of morphisms is simply composition of the epimorphisms, and thereby associative. The identity morphisms obviously exist. We always assume the standard basis on $\mathbb{R}^n$, and with this understanding in the sequel we will use the same symbol, e.g., $K$, to denote an epimorphism as well as its associated matrix under the standard basis, depending on the context.

We define category $\mathcal{P}$ as follows. An object in this category is a symmetric parametrized $(n \times n)$-matrix where $n$ is any particular positive integer, and where each matrix entry is a linear combination of the components of a $m$-parameter vector, $m \geq 1$, such that the set of real matrices implied by all realizations of the $m$-vector of parameters is an $m$-dimensional space. Consider two objects $P_1, P_2$ of $\mathcal{P}$, where $P_1$ is a $(j \times j)$-matrix parametrized by the components of a $k$-parameter vector and $P_2$ is a $(q \times q)$-matrix parametrized by the components of a $r$-parameter vector. There is a morphism $P_1 \xrightarrow{p_1} P_2$ if and only if there exists a real $(q \times j)$-matrix $V$ such that $VV'$ is nonsingular and $V'P_2V \subset P_1$, where the latter subset expression is interpreted to mean that the set of real matrices implied by $V'P_2V$ through all realizations of the $r$-parameter vector associated with $P_2$, is a subset of the set of real matrices implied by $P_1$ through all realizations of the associated $k$-parameter vector. If it exists, the morphism is clearly unique. Note the requirement that $VV'$ be nonsingular assures that the object $V'P_2V$ will have the same number of parameters as $P_2$, since nonsingularity of $VV'$ assures that $V'V'P_2V'V'$ has as many parameters as $P_2$, while $V'P_2V$ cannot have a greater number of parameters than $P_2$. The identity morphisms obviously exist.

Now consider the expression $P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} P_3$. It implies that there exists a real matrix $V_2$ such that $V_2P_1V_2 \subset P_2$ with $V_2V_2'$ nonsingular, and there exists a real matrix $V_1$ such that $V_1V_2P_2V_1 \subset P_1$ with $V_1V_1'$ nonsingular. This implies $V_1'V_2'P_3V_2V_1 \subset P_1$, i.e., $(V_2V_1)'P_3(V_2V_1) \subset P_1$. For $P_1 \xrightarrow{f} P_3$ to exist, it is then sufficient that $(V_2V_1)'V_2V_1' = V_2(V_1V_1')V_2'$ be nonsingular. To see that the latter is indeed nonsingular, first recall that $V_1V_1'$ is nonsingular. This is the matrix of some linear transformation, so there is a basis such that $V_1V_1'$ can be replaced by the identity matrix. With respect to this basis, $V_2$ is replaced by the matrix $\bar{V}_2$, and the matrix $\bar{V}_2\bar{V}_2'$ is replaced by the matrix $V_2V_2'$, which is then evidently nonsingular since $V_2V_2'$ is nonsingular. But with respect to the new basis, $V_2(V_1V_1')V_2'$ becomes $\bar{V}_2\bar{V}_2'$, and is thus nonsingular, as was required to be shown. Thus, $P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} P_3$ implies the existence of $P_1 \xrightarrow{f} P_3$, which is unique since all category $\mathcal{P}$ morphisms are unique. We define $p_1 \circ p_2 \equiv f$. Associativity of morphism composition is obvious.
A mapping $\mathbf{F}$ from category $\mathcal{A}$ to category $\mathcal{P}$ is defined as follows.

$\mathbf{F}[A]$ is the Proto-norm family of an object $A$ in category $\mathcal{A}$ as defined in Definition 2.1. Based on comments at the beginning of Section 2.2, $\mathbf{F}[A]$ is obviously an object of category $\mathcal{P}$. For an algebra epimorphism, implying the category $\mathcal{A}$ morphism $A_1 \xrightarrow{a} A_2$, we define $\mathbf{F}(a)$ to be the category $\mathcal{P}$ morphism $\mathbf{F}[A_1] \xrightarrow{\mathbf{F}(a)} \mathbf{F}[A_2]$. If this morphism exists, it is unique (a feature of category $\mathcal{P}$ morphisms in general, as noted above). So, in order to demonstrate that $\mathbf{F}$ is a functor, we only need to show that this morphism does indeed exist, and that $\mathbf{F}$ satisfies the requisite commutative diagram.

**Theorem 2.3.** $\mathbf{F} : \mathcal{A} \rightarrow \mathcal{P}$, is a covariant functor.

**Proof.** We will first show that the category $\mathcal{A}$ morphism $A_1 \xrightarrow{a} A_2$ implies the existence of category $\mathcal{P}$ morphism $\mathbf{F}[A_1] \xrightarrow{\mathbf{F}(a)} \mathbf{F}[A_2]$.

Morphism $a$ results from an algebra epimorphism $K : A_1 \rightarrow A_2$. Let $1_{A_2}$ be the multiplicative identity on $A_2$. Then $1_{A_2} = K(s^{-1} s) = K(s)s(s^{-1})$, so that $(Ks)^{-1} = K(s^{-1})$. By definition, given the Proto-norm family $\mathbf{F}[A_2]$ associated with $A_2$ and an arbitrary real matrix $\tilde{F}[A_2]$ in the set of real matrices implied by $\mathbf{F}[A_2]$, we have $d \left( (s_2^{-1})' \tilde{F}[A_2](s_2) \right) = 0$ in a unital neighborhood of $1_{A_2}$, where $s_2$ is notation for a variable in $A_2$. Taking $s_1$ to be notation for a variable in $A_1$, we then have

$$0 = d \left( (s_2^{-1})' \tilde{F}[A_2](s_2) \right) = d \left( ((Ks_1)^{-1})' \tilde{F}[A_2](s_1) \right) = d \left( (s_1^{-1})' [K' \tilde{F}[A_2]K] (s_1) \right),$$

using the equation at the end of the second sentence of this paragraph, and the fact that a differential commutes with a linear transformation. Equation (2.11) is simply an instance of (2.5), so it follows that symmetric matrix $K' \mathbf{F}[A_2]K$ is a Proto-norm associated with $A_1$. Consequently, $K' \mathbf{F}[A_2]K \subset \mathbf{F}[A_1]$. Furthermore, $KK'$ is nonsingular since $K$ is an epimorphism. Thus, criteria for the existence of morphism $\mathbf{F}(a)$ are fulfilled (with $V = K$).

Commutativity of the following diagram is then evident from the uniqueness of category $\mathcal{P}$ morphisms, and the way composition of category $\mathcal{P}$ morphisms is defined,

$$\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{F}[A_1] & \xrightarrow{\mathbf{F}(a_1)} & \mathbf{F}[A_2] & \xrightarrow{\mathbf{F}(a_2)} & \mathbf{F}[A_3]
\end{array}$$

Thus, $\mathbf{F}$ is a covariant functor.

We then have the following useful results. The first is immediate.

**Corollary 2.4.** An epimorphism from one algebra to another algebra can exist only if there is a category $\mathcal{P}$ morphism associated with their respective Proto-norm families.

**Corollary 2.5.** An epimorphism from a first algebra to a second algebra can exist only if the dimension of the Proto-norm family of the first algebra is greater than or equal to the dimension of the Proto-norm family of the second algebra.
Proof. Given algebra epimorphism $K : A_1 \to A_2$, suppose $A_1$ is a $j$-dimensional algebra with a Proto-norm family utilizing a $k$-parameter vector (and thereby the Proto-norm family is a $k$-dimensional space), and $A_2$ is a $q$-dimensional algebra with a Proto-norm family utilizing a $r$-parameter vector (and thereby the Proto-norm family is a $r$-dimensional space). Evidently, $K$ is a $(q \times j)$-matrix with $q \leq j$ and $KK'$ is nonsingular (since $K$ is an epimorphism). The proof of Theorem 2.3 has shown that $K'F[A_2]K \subset F[A_1]$. Thus, $K'F[A_2]K$ cannot have more parameters than $F[A_1]$ (i.e., the dimension of $K'F[A_2]K$ cannot exceed the dimension of $F[A_1]$). Furthermore, $K'F[A_2]K$ has at least as many parameters as $KK'F[A_2]KK'$, but it has less than or equal to the number of parameters of $F[A_2]$. However, $KK'F[A_2]KK'$ has the same number of parameters as $F[A_2]$ since $KK'$ is nonsingular. Hence, the dimension of $F[A_2]$ cannot exceed the dimension of $F[A_1].$ □

Corollary 2.6. If the Proto-norm family of an algebra is one-dimensional and contains a nonsingular member, then the algebra is simple.

Proof. Suppose algebra $A_1$ has a non-trivial ideal and its Proto-norm family is one-dimensional and has a nonsingular member. It follows that $F[A_1]$ is represented by $\alpha L_1$, $\alpha \in \mathbb{R}$, where $L_1$ is a real nonsingular $(j \times j)$-matrix. Since $A_1$ has a non-trivial ideal, there exists an algebra epimorphism $K : A_1 \to A_2$ where $A_2$ has smaller dimension than $A_1$. Corollary 2.5 indicates that the Proto-norm family of $A_2$ cannot have dimension greater than one, and thus $F[A_2]$ is represented by $\alpha L_2$ where $L_2$ is a real $(q \times q)$-matrix with $q < j$. From the proof of Theorem 2.3, we must then have that the nonzero matrix $KK'L_2K$ is a member of the Proto-norm family $F[A_1]$. But the determinant of the nonzero matrix $KK'L_2K$ is zero since $K$ has a non-trivial kernel, while the only member of the Proto-norm family $F[A_1]$ having zero determinant is the zero matrix (corresponding to $\alpha = 0$), since $L_1$ is necessarily nonsingular. It follows that $K$ cannot exist, and so $A_1$ is simple. □

A consequence of Corollary 2.6 is an alternative demonstration that the algebra of real $(n \times n)$-matrices is simple, since it can be shown that the Proto-norm family of this algebra is one-dimensional [7].

Note that the converse of Corollary 2.6 is not true. For example, $\mathbb{C}$ is simple but has a two-dimensional Proto-norm family (as depicted in Table I).

In the proof of Theorem 2.3, the transformation $F[A_2] \to K'F[A_2]K$ plays a key role. The transformation is invertible since $K$ is an algebra epimorphism from $A_1$ to $A_2$. That is, for $P \equiv F[A_2]$ and $P \equiv K'F[A_2]K = K'PK'$, we have $P = (K'K')^{-1}K'PK'(KK')^{-1}$. Despite fundamental differences, the format of the transformation of Proto-norm families in the first sentence of this paragraph is reminiscent of a similarity transformation of matrices, with $K'$ substituting for $K^{-1}$. In the case where $K$ is invertible (i.e., the epimorphism is an isomorphism), the analogy can be made explicit as follows.

Definition 2.2. $M_1$ and $M_2$ as members of matrix rings are similar if they are matrices representing the same linear transformation with respect to different bases.

Definition 2.3. $P_1$ and $P_2$ as objects in category $P$ are similar if they are Proto-norm families of isomorphic algebras.

Theorem 2.7. If $M_1$ and $M_2$ are similar matrices as in Definition 2.2, then $M_1 = K^{-1}M_2K$ for some nonsingular matrix $K$. 
Theorem 2.8. If $P_1$ and $P_2$ are similar Proto-norm families as in Definition 2.3, then $P_1 = K'P_2K$ for some nonsingular matrix $K$.

Proof. If $P_1$ and $P_2$ are similar objects of category $P$, then by definition there exists an algebra isomorphism $K : A_1 \to A_2$ where $A_1, A_2$ are the algebras whose Proto-norm families are $P_1, P_2$, respectively. Using the same argument as in the second paragraph of the proof of Theorem 2.3 we have both $K'P_2K \subset P_1$ and $(K^{-1})^{-1}P_1K^{-1} \subset P$. It follows that $P_1 = K'P_2K$.

The above proof immediately implies,

Corollary 2.9. The dimension of a Proto-norm family is invariant under an isomorphism of algebras.

Despite Theorem 2.8 important distinctions between the Proto-norm family transformation and the matrix similarity transformation are that $K$ in the theorem statement is not unique (there is an equivalence class of such matrices) and there is no analogue of group notions related to conjugacy classes.

The following are used in Section 2.1.2.

Definition 2.4. An object of $P$ is nonnegative if the determinants of all of its implied real matrices are nonnegative, it is nonpositive if none of the determinants of its implied real matrices are positive, and it is otherwise indefinite.

Corollary 2.10. If there is an isomorphism between two algebras then their Proto-norm families are either both nonnegative, both nonpositive, or both indefinite. Furthermore, the determinants of the respective Proto-norm families are dependent on the same number of parameters.

Proof. If $K : A_1 \to A_2$ is the algebra isomorphism, then it follows from Theorem 2.8 that $\det(F[A_1]) = \det(K'F[A_2]K) = \det(K'K)\det(F[A_2])$. Corollary 2.10 immediately follows.

2.4. Table I examples of the functor in action. Table I readily provides examples of application of the functor. We denote the algebras in rows 1 through 14 (the first column of Table I) as $A_1, A_2, \ldots, A_{14}$.

2.4.1. Row 1. We have already noted that since $\mathbb{R}^{n \times n}$ is associated with a 1-parameter Proto-norm family, the functor indicates that this algebra is simple (Corollary 2.6).

2.4.2. Rows 3 and 4. The algebras pertaining to these rows, $\mathbb{C}$ and $\mathbb{R} \oplus \mathbb{R}$, are isomorphic. Their Proto-norm family members are given by the parametrized matrices as the final entry in row 3 and the final entry in row 4 of Table I. If we set $\sigma_1 = \alpha - \beta$ and $\sigma_2 = \alpha + \beta$, defining a new set of two independent parameters, then the parametrized matrices $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ and $\begin{bmatrix} \alpha - \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix}$ represent the same set of real matrices, i.e., they are the same Proto-norm family. For $V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = V' \begin{bmatrix} \alpha - \beta & 0 \\ 0 & \alpha + \beta \end{bmatrix} V$. That is, the morphism of Proto-norm families exists, as required. As indicated in remarks following the statement of Corollary 2.9 other satisfactory choices for $V$ exist, and these include the matrix defining the algebra isomorphism between $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{C}$ (as expected from the proof of Theorem 2.8).
While \( \mathbb{C} \) and \( \mathbb{R} \oplus \mathbb{R} \) are isomorphic, an isomorphism between any two of \( \mathbb{C}, \mathbb{C}, \) and \( \mathbb{D} \) is excluded by Corollary 2.10.

2.4.3. **Rows 1, 2, 3, 4, and 5.** The \( n = 1 \) case of row 1 refers to the algebra \( \mathbb{R} \), and its Proto-norm family, i.e., the category \( P \) object \( F[\mathbb{R}] \), is represented by the parametrized matrix \( [\sigma] \). The row 2 algebra is \( \mathbb{C} \), and its Proto-norm family \( F[\mathbb{C}] \) is represented by the category \( P \) object \( \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \). There is no morphism pointing from \( F(\mathbb{C}) \) to \( F(\mathbb{R}) \). That is, there is clearly no nonzero real matrix \( V \) such that \( V'[\sigma]V \subset \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \), since all the real matrix members of \( V'[\sigma]V \) must have determinant zero - while \( \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \) has determinant zero if and only if \( \alpha, \beta = 0 \) and is thereby the zero matrix.

On the other hand, consider the algebra in row 4, \( \mathbb{R} \oplus \mathbb{R} \). There is clearly a morphism from \( F[\mathbb{R} \oplus \mathbb{R}] \) to \( F[\mathbb{R}] \) - i.e., take \( V = [1, 0] \), so that \( V'[\sigma]V = \begin{bmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset F[\mathbb{R} \oplus \mathbb{R}] = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \). Thus, unlike the case of \( \mathbb{C} \), an epimorphism from \( \mathbb{R} \oplus \mathbb{R} \) to \( \mathbb{R} \) is not precluded. And indeed, the epimorphism does exist, resulting from the ideal \( \{0\} \oplus \mathbb{R} \). A similar argument applies to existence of an epimorphism from \( \mathbb{C} \) (row 3) to \( \mathbb{R} \), and existence of an epimorphism from \( \mathbb{D} \) (row 5) to \( \mathbb{R} \).

2.4.4. **Rows 10 and 4.** There is a morphism between \( A_{10} \) and \( \mathbb{R} \oplus \mathbb{R} \) associated with the ideal \( \{0, z \} : z \in \mathbb{R} \}. Corollary 2.4 then implies that there must exist a morphism from \( F(A_{10}) \) and \( F(A_4) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 10 and 4 of Table I.

2.4.5. **Rows 13, 12, and 4.** There is an epimorphism from \( A_{13} \) to \( \mathbb{R} \oplus \mathbb{R} \) associated with the ideal \( \{z, w, v \} : z, w, v \in \mathbb{R} \}. Corollary 2.4 then implies that there must exist a morphism from \( F(A_{13}) \) and \( F(A_4) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 13 and 4 of Table I.

On the other hand, as regards \( A_{13} \) and its subalgebra \( A_{12} \), applying Corollary 2.5 to the relevant Proto-norm families in Table I indicates that there can be no morphism from \( F(A_{13}) \) to \( F(A_{12}) \) - precluding an epimorphism of the respective algebras.

2.4.6. **Rows 12, and 5.** There is an epimorphism from \( A_{12} \) to \( \mathbb{D} \) associated with the ideal \( \{0, w \} : w, v \in \mathbb{R} \}. Corollary 2.4 then implies that there must exist a morphism from \( F(A_{12}) \) to \( F(A_5) \). Indeed, existence of the latter morphism is obvious from the last entries in rows 12 and 5 of Table I.

2.4.7. **Rows 11 and 5.** There is an epimorphism from \( A_{11} \) to \( A_{5} \) associated with the ideal \( \{w \} : w \in \mathbb{R} \}. Accordingly, the morphism \( F(A_{11}) \rightarrow F(A_{5}) \).
must exist - and it clearly does by inspection of the depicted Proto-norm families at the end of rows 11 and 5.

2.4.8. **Row 6, the Quaternion algebra** $\mathbb{H}$. A simple calculation shows that $F(\mathbb{H})$ is given by the parametrized matrix $\text{diag}\{\alpha, -\alpha, -\alpha, -\alpha\}$. Corollary 2.6 then indicates that $\mathbb{H}$ is simple. This is an alternative to the ideal-based argument that $\mathbb{H}$ is simple because it can be shown to be a division algebra.

Independent of the above, Theorem 2.3 indicates that there cannot be an epimorphism from $\mathbb{H}$ to its subalgebra $\mathbb{C}$ since their Proto-norm families depicted in row 6 and row 2 preclude a morphism pointing from $F(\mathbb{H})$ to $F(\mathbb{C})$, due to Corollary 2.5.

The presentation in this section by no means exhausts things we would like to do with the formalism. More results can be found in [7].

**Appendix A. Details of proof of the Pythagorean Theorem**

To flesh out the proof synopsis in Section 1, in this appendix we will

- Present the four postulates of Birkhoff’s formulation of Euclidean geometry, along with the implied vector space and degree-1 positive homogeneous length function $\ell$,
- Derive the resulting isometric isomorphism with $\mathbb{R} \oplus \mathbb{R}$ based on a pair of orthogonal coordinate axes through a distinguished origin in Euclidean space,
- Prove that the length function $\ell$ is continuously differentiable on domains excluding the origin,
- Show that, for fixed $(x, y) \neq (0, 0)$ with $(u, v)$ varying over $\mathbb{R} \oplus \mathbb{R}$, the expression $D\ell(x, y; u, v)$ is a linear functional having the form prescribed in the synopsis.

A.1. **Birkhoff’s postulates for plane Euclidean geometry, and the resulting “Euclidean plane with distinguished origin” as a vector space, with a homogeneous length function.** Slightly reworded, Birkhoff’s postulates entail the following [1]:

1. “Postulate of Line Measure”: There is a bijection between $\mathbb{R}$ and the set of points on any line, so that the length of any line segment is the absolute value of the difference of the real numbers corresponding to the segment’s endpoints.
2. “Point-Line Postulate”: There is one and only one line that contains any two given distinct points of the plane.
3. “Postulate of Angle Measure”: There is a bijection between the set of rays from a point $O$ and the real numbers (mod $2\pi$) such that, for a line not containing $O$, continuously varying a point on the line implies continuous variation of the bijection-assigned real number value of the ray it lies on. The angle formed by a first ray from $O$ assigned real value $\alpha_1$ and a second ray from $O$ assigned real value $\alpha_2$, is $\alpha_1 - \alpha_2$ (mod $2\pi$).
4. “Postulate of Similarity”: If a triangle has two sides having lengths related to the lengths of two sides of another triangle by the same positive proportionality constant, and the absolute values of the angles formed by these two sides of each triangle are equal, then these are similar triangles, i.e.,
they have the same three angles up to a sign, and the lengths of the third side of the two triangles are related by the same proportionality constant.

Regarding the identification of points of the plane with points of \( \mathbb{R} \times \mathbb{R} \), we’ll let \([1]\) speak for itself: “We are now prepared to define a rectangular coordinate system \((x, y)\) with axes the directed perpendicular lines \(Ox, Oy\) intersecting in the origin \(O\). Choose a system of numeration along \(Ox, Oy\) so that \(O\) is marked \(0\) on both lines, and the numbers increase algebraically in the positive direction. Drop the unique perpendiculars from \(P\) to \(Ox\) and \(Oy\) respectively meeting these lines at \(Px\) and \(Py\). The rectangular coordinates of \(P\) are then the numbers \(x_P\) and \(y_P\) attached to \(Px\) and \(Py\) respectively. These are evidently uniquely determined numbers.” This bijection, explicitly derived in \([1]\), is a straightforward consequence of Playfair’s “parallel axiom” (Theorem IX in \([1]\)) - which follows from the Postulate of Similarity without use of the Pythagorean Theorem.

Birkhoff’s formulation of Euclidean geometry makes intrinsic use of \(\mathbb{R}\). Accordingly, we take \(\mathbb{R}\) to be as understood in Real Analysis, and we thus assume the accompanying notions of limits and continuity, as well as Differential Calculus in one dimension.

We can now fashion a vector space with a length function out of the Euclidean plane with origin \(O\).

**Definition A.1.** The Euclidean plane vector space is constituted as follows:

1. There is a chosen point \(O\) of the Euclidean plane termed “the origin”.
2. There is a length function \(\ell\) such that for \(P\) in the Euclidean plane we have \(\ell(P)\) as the length of the line segment from the origin to \(P\).
3. For \(P, P'\) in the Euclidean plane vector space that are not collinear with the origin \(O\), we use the parallel postulate (a consequence of the Postulate of Similarity) to define \(P + P'\) to be a third point \(P''\) in the Euclidean plane vector space as the point of intersection of two unique lines, the first line being the line through \(P\) parallel to the line segment from the origin to \(P'\), and the second line being the line through \(P'\) parallel to the line segment from the origin to \(P\). The case of points \(O, P, P'\) being collinear is addressed using the Postulate of Line Measure: If \(P\) and \(P'\) do not lie on the same ray from \(O\), then \(P''\) has length given by the absolute value of the difference of the lengths of the two points, and resides on the ray from the origin through the point \(P\) or \(P'\) as determined by which point has the greater length. If \(P\) and \(P'\) do lie on the same ray from \(O\), then \(P''\) has length equal to the sum of the lengths of \(P\) and \(P'\), and lies on the ray from the origin on which \(P\) and \(P'\) lie.
4. Multiplication of a point \(P\) by \(\alpha \in \mathbb{R}\) is defined so that, for \(\alpha \neq 0\) and \(P\) not at the origin, \(\alpha P\) is one of two points on the line containing the origin and \(P\) that have length \(|\alpha|\ell(P)\). If \(\alpha > 0\), the point \(\alpha P\) is such that \(P\) and \(\alpha P\) lie on the same ray from the origin (and the new point is termed “parallel” to \(P\)). For \(\alpha < 0\), the point \(\alpha P\) is such that \(P\) and \(\alpha P\) do not lie on the same ray from the origin (and the new point is termed “antiparallel” to \(P\)). For \(\alpha = 0\), or if \(P\) is the origin, the point \(\alpha P\) is the origin.

It follows that \(\ell\) is a degree-1 positive homogeneous function.
A.2. The isometric isomorphism between the Euclidean plane vector space and \( \mathbb{R} \oplus \mathbb{R} \).

**Lemma A.1.** Consider the bijection, \( B : \) Euclidean plane vector space \( \rightarrow \mathbb{R} \oplus \mathbb{R} \), that results from a pair of orthogonal coordinate axes through the distinguished origin in the Euclidean plane. Suppose \( \mathbb{R} \oplus \mathbb{R} \) is endowed with a length function similarly notated to that of the Euclidean plane vector space, i.e., \( \ell : \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \), such that \( \ell(x, y) = \ell(P_{x, y}) \) where \( P_{x, y} \) is the member of the Euclidean plane vector space corresponding to \( (x, y) \) under the bijection. Then the bijection is an isometric isomorphism.

**Proof.** Using the bijection \( B \), we can denote points of the Euclidean plane vector space as \( P_{x,y} \), with \( (x, y) \in \mathbb{R} \oplus \mathbb{R} \). From the structure of the Euclidean plane vector space (Definition A.1), we can write \( P_{x,y} = P_{x,0} + P_{0,y} \). Consequently, \( P_{x1,y1} + P_{x2,y2} = P_{x1,0} + P_{0,y1} + P_{x2,0} + P_{0,y2} = P_{x1+x2,0} + P_{0,y1+y2} = P_{x1+x2,y1+y2} \).

We then have,

\[
B(P_{x1,y1} + P_{x2,y2}) = B(P_{x1+x2,y1+y2}) = (x1 + x2, y1 + y2) = (x1, y1) + (x2, y2) \tag{A.1}
\]

This, along with (A.1), indicates that \( B \) is a linear isomorphism from the Euclidean plane vector space to \( \mathbb{R} \oplus \mathbb{R} \). By definition, \( \ell(B(P_{x,y})) = \ell(x, y) = \ell(P_{x,y}) \). Consequently, \( B \) is an isometric isomorphism. \( \square \)

A.3. The length function \( \ell(s) \) is continuously differentiable on domains excluding \( s = 0 \).

**Definition A.2.** The value of the directional derivative of the length function \( \ell : \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \) at a point \( (x, y) \neq (0, 0) \) with respect to direction \( (u, v) \) is given by

\[
D\ell(x,y;u,v) \equiv \lim_{\epsilon \to 0} \frac{\ell((x, y) + \epsilon(u,v)) - \ell(x, y)}{\epsilon}. \tag{A.3}
\]

**Lemma A.2.** Directional derivatives of \( \ell \) in all directions exist at each point exclusive of the origin.

**Proof.** From Definition A.1, \( \ell(\alpha P) = |\alpha| \ell(P) \), i.e., \( \ell \) is a degree-1 positive homogenous function. Also note that, using a parallelogram construction, the length of any line segment in Euclidean plane vector space can be expressed using \( \ell \) and vector subtraction. That is, for \( P, P' \in \) Euclidean plane vector space, the length of the line segment with endpoints \( P, P' \) is \( \ell(P - P') \). Using the triangle inequality, the continuity of \( \ell \) immediately follows (i.e., \( |\ell(P') - \ell(P)| \leq \ell(P' - P) \), the implied “reverse triangle inequality”).

The Postulate of Line Measure assures that every non-origin point is a positive multiple of a direction. Thereby, evaluation of (A.3) immediately indicates that the directional derivative exists in the directions parallel and antiparallel to \( (x, y) \neq (0, 0) \), and these values are respectively +1 and −1. Thus, consider the directional derivative in some other direction. We assert that the expression inside the limit in (A.3) is monotonic as \( \epsilon \to 0^+ \) and is monotonic as \( \epsilon \to 0^- \). If that were not the case, since \( \ell \) is continuous, there would have to be two choices of \( \epsilon \) having the same
sign such that the expressions inside that limit are equal. We can call these choices $\epsilon'$ and $\frac{1}{k}\epsilon'$, with $0 < k < 1$. Using the degree-1 positive homogeneity of $\ell$, simple manipulation of the equality of the two expressions leads to

$$\ell((x, y) + \epsilon'(u, v)) - \ell(k(x, y) + \epsilon'(u, v)) = \ell(x, y) - \ell(k(x, y)) = (1 - k)\ell(x, y).$$

But according to the (reverse) triangle inequality, the absolute value of the left-hand-side above must be strictly less than the absolute value of the right-hand-side, since $k \neq 1$, and $(u, v)$ and $(x, y)$ are not collinear. Thus, (A.4) is not possible, so the above monotonicity assertion is verified. Furthermore, the triangle inequality indicates that the expression inside the limit in (A.3) is bounded by the values of the the directional derivatives of $\ell$ in the directions parallel and antiparallel to $(x, y)$. A monotone convergence argument then establishes the existence of the directional derivative.  

**Lemma A.3.** The directional derivative of $\ell$ in a given fixed direction as a function over the points of the plane is continuous at each point exclusive of the origin.

**Proof.** From the existence of directional derivatives in all directions at $(x, y) \neq (0, 0)$ (Lemma A.2), and the degree-1 positive homogeneity of $\ell$, it easily follows from the directional derivative expression (A.3) that a fixed-direction directional derivative of $\ell$ at any point distinct from the origin depends only on the ray on which the point lies (i.e., it is constant on a ray). This means that to demonstrate continuity of the directional derivative at $(x, y) \neq (0, 0)$, the argument need only evaluate the limit of the difference of the directional derivatives at $(x, y)$ and $(x', y')$ as the ray on which the latter point lies approaches the ray on which the former point lies - so it is sufficient to examine the case where $(x', y')$ approaches $(x, y)$ along the circle with center $(0, 0)$ containing $(x, y)$. In fact, the derivative is continuous at $(x, y)$, if and only if it is continuous at the point of the unit circle $(\frac{x}{\|x\|}, \frac{y}{\|y\|})$.

Thus, without loss we can assume that $(x, y)$ and $(x', y')$ are both points on the unit circle centered at the origin, so that it is then sufficient to show that the difference of their directional derivatives in a fixed direction $(u, v)$ tends to zero as $(x', y')$ approaches $(x, y)$ along the unit circle. That is, we only need to show that the directional derivative in the fixed $(u, v)$-direction is a continuous function over the points of the unit circle. But, the directional derivative in a direction $(u, v)$ for a point $(x, y)$ on the unit circle depends only on the angle formed by $(x, y)$ and $(u, v)$, since the directional derivative is coordinate system-independent, and the latter vectors both have unit length. Hence, the following symmetry pertains: demonstrating continuity of the directional derivative in a fixed direction as a function over all points of the unit circle, is equivalent to demonstrating continuity of the directional derivative taken at a fixed point of the unit circle as a function over all directions.

To demonstrate continuity of the latter function, we invoke Proposition 24 of book 1 of Euclid’s Elements, which states that if two sides of one triangle are congruent to two sides of a second triangle, and the [absolute value of the] included angle of the first triangle is larger than the [absolute value of the] included angle of the second triangle, then the third side of the first triangle is longer than the third side of the second triangle. Now label the rays from the origin via $\theta \in (-\pi, \pi]$, with $\theta = 0$ labeling the ray containing $(x, y)$. From the Postulate of Angle Measure, it follows that $\theta$ continuously labels the set of directions. Applying the above
Proposition 24 to the directional derivative expression \([A.3]\), it is easily seen that the directional derivative at \((x, y)\) as a function of the direction labeled by \(\theta\) is monotonic for \(\theta \in (-\pi, 0]\) and also monotonic for \(\theta \in [0, \pi]\) - and is thus a continuous function from a well known result that a derivative which exists at each point of a closed interval, and is monotonic on that interval, is continuous on the interval. In view of the statement at the end of the last paragraph, this establishes the continuity of the directional derivative taken in a fixed direction over the non-origin points of the plane.

\[ \Box \]

A.4. **The directional derivative** \(D\ell(x, y; u, v)\) **has the prescribed form.** Suppose non-origin point \(P\) in the Euclidean plane vector space is labeled by \((x, y)\) via the isomorphism. Lemma \(A.2\) states that all directional derivatives exist at \((x, y)\). Let \(\frac{\partial \ell(x, y)}{\partial x}\) and \(\frac{\partial \ell(x, y)}{\partial y}\) be the respective directional derivative values of \(\ell\) at \((x, y)\) in the \((1, 0)\) and \((0, 1)\) directions. From the mean value theorem of one-dimensional Calculus,

\[
\ell\left((x, y) + \epsilon(u, v)\right) - \ell(x, y) = \left[\ell\left((x, y) + \epsilon(u, 0)\right) - \ell(x, y)\right] + \left[\ell\left((x, y) + \epsilon(u, v)\right) - \ell\left((x, y) + \epsilon(u, 0)\right)\right] \leq \frac{\partial \ell(p_1)}{\partial x} u\epsilon + \frac{\partial \ell(p_2)}{\partial y} v\epsilon,
\]

where \(p_1\) is in the line segment with endpoints \((x, y)\) and \((x, y) + \epsilon(u, 0)\), and \(p_2\) is in the line segment with endpoints \((x, y) + \epsilon(u, 0)\) and \((x, y) + \epsilon(u, v)\), and by constructing an appropriate parallelogram we already know that the respective lengths of these line segments are just \(u\epsilon\) and \(v\epsilon\) since the line segments are parallel to the \(x\)-axis and \(y\)-axis and the values of \(\ell\) are known on these axes (i.e., \(\ell(a, 0) = |a|\) and \(\ell(0, b) = |b|\) from the Postulate of Line Measure).

We know from Lemma \(A.3\) that the directional derivative of \(\ell\) in a fixed direction is continuous at \((x, y) \neq (0, 0)\). Thus, dividing both sides of \((A.5)\) by \(\epsilon\) and taking the limit as \(\epsilon \to 0\), we find

\[
D\ell(x, y; u, v) = \lim_{\epsilon \to 0} \frac{\ell\left((x, y) + \epsilon(u, v)\right) - \ell(x, y)}{\epsilon} = \frac{\partial \ell(x, y)}{\partial x} u + \frac{\partial \ell(x, y)}{\partial y} v.
\]

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