THE NEWTON POLYGON OF A PLANAR SINGULAR CURVE

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Abstract. Suppose that a planar algebraic curve \( C \) defined over a valuation field by an equation \( F(x, y) = 0 \) has a point \( p \) of multiplicity \( m \). Valuations of the coefficients of \( F \) define a subdivision of the Newton polygon \( \Delta \) of the curve \( C \).

This point \( p \) of multiplicity \( m \) imposes certain linear conditions on the coefficients of \( F \). Properties of the matroid associated with these conditions can be visualized on the subdivision of \( \Delta \) by means of tropical geometry. Roughly speaking, there are \( \frac{3}{8}m^2 \) distinguished points in \( \Delta \) which are responsible for the singularity.

0. Introduction

Consider a family of planar algebraic complex curves \( C_{t_k} = \{(x, y)|F_{t_k}(x, y) = 0\} \), where \( t_k \to 0 \). Suppose we know only the asymptotics of the coefficients of the polynomials \( F_{t_k}(x, y) \). Is it possible to extract any meaningful information from this knowledge? Unexpectedly, many properties of a generic curve in this family are visible from such a viewpoint.

We may treat a family of complex curves as a single curve \( C \) over the field \( K \) of series of the form \( \sum_{\alpha \in I} c_\alpha t^\alpha \). Specializing \( t = t_k, t_k \in \mathbb{C} \), we obtain a family of complex curves \( C_{t_k} \). The field \( K \) is equipped with a valuation map \( \text{val}(\sum_{\alpha \in I} c_\alpha t^\alpha) := -\min_{\alpha \in I} \alpha \); the valuation of an element is a measure of its asymptotic behavior as \( t \) tends to 0.

Fix a non-empty finite subset \( \mathcal{A} \subset \mathbb{Z}^2 \). Throughout this article, we consider a curve \( C \) given by an equation \( F(x, y) = 0 \), where

\[
F(x, y) = \sum_{(i,j) \in \mathcal{A}} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{K}^*.
\]

Assuming that \( a_{ij} \) converge near 0 as functions \( a_{ij}(t) \) of \( t \), one can substitute \( t \) by \( t_k \to 0, t_k \in \mathbb{C} \) and end up with a family of curves defined by equations \( \sum a_{ij}(t_k)x^i y^j = 0 \). While \( t_k \) tends to 0, the coefficient \( a_{ij}(t_k) \) is approximately \( t_k^{-\text{val}(a_{ij})} \).

The Newton polygon \( \Delta(\mathcal{A}) \) of the curve \( C \) is the convex hull of \( \mathcal{A} \) in \( \mathbb{R}^2 \). The extended Newton polygon \( \tilde{\Delta} \) of the curve \( C \) is the convex hull of the set \( \{(i, j), s \in \mathbb{R}^2 \times \mathbb{R}|(i, j) \in \mathcal{A}, s \leq \text{val}(a_{ij})\} \). Projecting down all the faces of \( \tilde{\Delta} \), we get a subdivision of \( \Delta(\mathcal{A}) \) by the images of the faces.

Tropical geometry, from the point of view of this article, forgets everything about the curve \( C \) except the extended Newton polygon \( \tilde{\Delta} \) and asks which information remains. Note, that only the valuations of the coefficients of \( F \) are required to construct \( \tilde{\Delta} \).

A point \( p \) is of multiplicity \( m \) for \( C \) if all the partial derivatives of \( F \) up to order \( m - 1 \) are zero at \( p \); for positive characteristic, one should slightly change the definition. A natural question arises: is it visible tropically, i.e. on the subdivision of \( \Delta(\mathcal{A}) \), that \( C \) has a point of multiplicity \( m \)?

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A point \( p \) of multiplicity \( m \) imposes \( \frac{m(m+1)}{2} \) linear conditions on \( a_{ij} \) and it happens that asymptotically the influence appears almost as expected: with a few exceptional situations, on the subdivision we will see a special collection of faces with the sum, denoted by \( \text{Inf}(p) \), of their areas at least \( \frac{1}{2}m^2 \) in some cases and at least \( \frac{3}{8}m^2 \) in other cases (Figure 1). These faces, or rather integer points there, are “responsible” for the singularity at \( p \). So, Figure 1 reveals an example of the geometric footprints of the matroid associated with the above linear conditions.

To dig deeper and distinguish the two above cases we shall introduce the tropical curve. By definition, non-archimedian amoeba of \( C \) is \( \text{Val}(C) = \{(\text{val}(x), \text{val}(y))|(x, y) \in C\} \), it is a particular example of a tropical curve. It is known that \( \text{Val}(C) \) is a graph with vertices and straight edges, which is combinatorially dual to the subdivision of \( \Delta(A) \), i.e. its vertices correspond to the faces of the subdivision and its edges correspond to the edges of the subdivision.

For a point \( p = (p_1, p_2) \) of multiplicity \( m \) for \( C \) one can ask about properties of \( P = (\text{val}(p_1), \text{val}(p_2)) \), in terms of combinatoric of \( \text{Val}(C) \). If \( P \) is on an edge of \( \text{Val}(C) \), then the above estimate is \( \frac{1}{2}m^2 \) (Theorem 2); if \( P \) is a vertex of \( \text{Val}(C) \), then the estimate for the sum of areas turns out to be \( \frac{3}{8}m^2 \) (Theorem 3), which is a bit mystical and unexpected.

These theorems give new easy visible purely tropical properties which testify to the presence of a singular point; also we get an extension of the classical notion of tropical multiplicity.

The previous research in this direction has been done for \( m = 2 \) in [22, 23] (H. Markwig, T. Markwig, E. Shustin), for inflection points in [4] (E. Brugué, L. López de Medrano) and for cusps in [13] (Y. Ganor); general theorems about tropicalization of \( \Delta \)-discriminants are obtained in [5, 7, 6] (A. Dickenstein, E. Feichtner, B. Sturmfels, L. Tabera). Realization of tropical singular points, i.e. lifting them to the complex world, via patchworking is discussed in [27] (E. Shustin, I. Tyomkin).

Lattice width is the most frequent notion in our arguments, it already proved to be useful elsewhere, e.g., the article [9] uses it to estimate the gonality of a general curve with a given Newton polygon. The minimal genera of surfaces dual to a given 1-dimensional cohomology class in a three-manifold are related to the lattice width of its Alexander polynomial ([19], [12]). A good survey about lattice geometry and related combinatorial problems can be found in [5].

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![Figure 1](image-url)
1. AFFINE AND TORIC GEOMETRY

Let $\mathbb{T}$ denote $\mathbb{R} \cup \{-\infty\}$, $\mathbb{T}$ is usually called the tropical semi-ring. Let $\mathbb{F}$ be an arbitrary (possibly finite) field. By $\mathbb{K}$ we denote the field $\mathbb{F}\{\{t\}\}$ of generalized Puiseux series, namely the field
\[ \mathbb{K} = \mathbb{F}\{\{t\}\} = \left\{ \sum_{\alpha \in I} c_{\alpha} t^{\alpha} | c_{\alpha} \in \mathbb{F}, I \subset \mathbb{R} \right\}, \]
where $t$ is a formal variable and $I$ is a well-ordered set, i.e. each of its nonempty subsets has a least element. Define a valuation map $\text{val} : \mathbb{K} \to \mathbb{T}$ by the rule $\text{val}(\sum_{\alpha \in I} c_{\alpha} t^{\alpha}) := -\min_{\alpha} \alpha$ and $\text{val}(0) := -\infty$.

The map $\text{val}$ possesses the following properties:

- $\text{val}(ab) = \text{val}(a) + \text{val}(b)$,
- $\text{val}(a + b) = \max(\text{val}(a), \text{val}(b))$ if $\text{val}(a) \neq \text{val}(b)$,
- $\text{val}(a + b) \leq \max(\text{val}(a), \text{val}(b))$ if $\text{val}(a) = \text{val}(b)$.

Different constructions of Puiseux series and their properties are listed in the article [24]. Throughout this paper, the field $\mathbb{K}$ is used as the main field; but in fact, we can consider any other valuation field $\mathbb{K}$ – this will not change the results.

**Definition 1.1.** The tropicalization $\text{Trop}(V) \subset \mathbb{T}^n$ of an algebraic variety $V \subset \mathbb{K}^n$ is the image of $V$ under the map $\text{val}$ applied coordinate-wise. (See the examples below.)

The set $\text{Val}(V)$ is the non-Archimedean amoeba of $V$ in the terminology of Kapranov ([16]).

Fix a finite subset $\mathcal{A} \subset \mathbb{Z}^2$ once and for all. Throughout this article we consider a curve $C$ given by an equation $F(x, y) = \sum a_{ij} x^i y^j$, $(i, j) \in \mathcal{A}, a_{ij} \in \mathbb{K}^\times$.

**Remark 1.2.** We do not suppose that $C$ is irreducible or reduced.

Now we recall some basic notions of tropical geometry. To a curve $C$ given by an equation $F(x, y) = 0$ we associate a subdivision of its Newton polygon $\Delta = \text{ConvexHull}(\mathcal{A})$ by the following procedure. Consider the extended Newton polyhedron
\[ \tilde{\Delta} = \text{ConvexHull}\left( \bigcup \{(i, j, x)(i, j) \in \mathcal{A}, x \leq \text{val}(a_{ij})\} \right) \subset \mathbb{R}^3. \]

The projection of the edges of $\tilde{\Delta}$ to the first two coordinates gives us the subdivision of the Newton polygon. Hence a curve $C$ produces the tropical curve $\text{Val}(C)$ and a subdivision of $\Delta(A)$. This subdivision is dual to $\text{Val}(C)$ in the following sense: each vertex $Q$ of $\text{Val}(C)$ corresponds to some face $d(Q)$ of the subdivision; each edge $E$ of $\text{Val}(C)$ corresponds to some edge $d(E)$ in the subdivision, the direction of the edge $d(E)$ is perpendicular to the direction of $E$; and if $Q$ is an end of $E$, then $d(Q)$ contains $d(E)$ (see Example 2.1).

The tropical curve $\text{Val}(C)$ is equal to the set of non-smooth points of the piecewise linear function $\text{Val}(F) = \max (ix + jy + \text{val}(a_{ij}))$ ([16], Theorem 2.1.1), i.e. to the set of points $(x, y) \in \mathbb{T}^2$ where this maximum is attained at least twice. From this description, it is easily seen that each vertex of $\tilde{\Delta}$ corresponds to a connected component of $\mathbb{T}^2 \setminus \text{Val}(C)$. By definition, the multiplicity of an edge $E$ of $\text{Val}(C)$ is the changing of slope of $\text{Val}(F)$ in the direction orthogonal to $E$ (see Figure 2, page 10).

**Example 1.3.** Suppose $\text{Val}(F)$ is equal to $i_1 x + j_1 y + \text{val}(a_{i_1 j_1})$ on one side of an edge $E$ and $i_2 x + j_2 y + \text{val}(a_{i_2 j_2})$ on the other side of $E$. Therefore $E$ is locally defined by the equation $(i_1 - i_2)x + (j_1 - j_2)y + (\text{val}(a_{i_1 j_1}) - \text{val}(a_{i_2 j_2})) = 0$, the endpoints of $d(E)$ are $(i_1, j_1), (i_2, j_2)$, and the multiplicity of $E$ is equal to the integer length of $d(E)$, which is $gcd(i_1 - i_2, j_1 - j_2)$ by definition.
The reader should be familiar with the notions mentioned above, or is kindly requested to refer to [25] \[11] [10] [18].

**Definition 1.4.** If the leading term in the Taylor expansion of $F$ at a point $p$ has degree $m$, then $m = \mu_p(C)$ is called the *multiplicity* of $p$.

Another way to say the same thing is to define $\mu_p(C)$ for $p = (p_1, p_2)$ as the maximal $m$ such that the polynomial $F$ belongs to the $m$-th power of the ideal of the point $p$, i.e. $F \in \langle x - p_1, y - p_2 \rangle^m$ in the local ring of the point $p$.

**Example 1.5.** The condition for a point $p$ to be of multiplicity one for $C$ means that $p \in C$. Multiplicity greater than one implies that $p$ is a singular point of $C$.

**Example 1.6.** Consider a generic curve $C'$ of degree $d$ given by an equation $F'(x, y) = \sum a_{ij}x^iy^j$. The point $(0, 0)$ is of multiplicity $m$ for the curve $C'$ if and only if for all $i, j$ with $i + j < m$, one has $a_{ij} = 0$. As a consequence, for each point $p \in \mathbb{K}^2$ the condition that $\mu_p(C') = m$ can be rewritten as a system of $\frac{m(m+1)}{2}$ linear equations in the coefficients $\{a_{ij}\}$ of $F'$.

**Lemma 1.7.** Suppose $ad - bc = 1$ where $a, b, c, d \in \mathbb{Z}$. The transformation $\Psi : (x, y) \mapsto (x^ay^b, x^cy^d)$ preserves multiplicity at the point $p = (1, 1)$, i.e. $\mu_{(1,1)}(C) = \mu_{(1,1)}(\Psi(C))$.

**Proof.** We only need to verify that $\langle x - 1, y - 1 \rangle = \langle x^ay^b - 1, x^cy^d - 1 \rangle$ in the local ring of $(1, 1)$. If $a, b \geq 0$, then $x^ay^b - 1 = (x - 1 + 1)^a(y - 1 + 1)^b - 1 = (x - 1)H_1 + (y - 1)H_2$; if $a \geq 0, b < 0$, then we remember that we can multiply by $G, G(1,1) \neq 0$, therefore $x^{-b}(x^ay^b - 1) = (x - 1 + 1)^a - (y - 1 + 1)^b = (x - 1)H_1 + (y - 1)H_2$; etc. The map $\Psi^{-1}$ is also given by an integer matrix, hence we repeat the above arguments and finally get $\langle x - 1, y - 1 \rangle = \langle x^ay^b - 1, x^cy^d - 1 \rangle$. \hfill $\square$

**Definition 1.8.** One says that a map $f$ *tropicalizes* to a map $\text{Val}(f)$ if the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{K}^k & \xrightarrow{f} & \mathbb{K}^k \\
\downarrow\text{Val} & & \downarrow\text{Val} \\
T^k & \xrightarrow{\text{Val}(f)} & T^k 
\end{array}
$$

**Proposition 1.9.** A map $\Psi : (x, y) \mapsto (x^ay^b, x^cy^d)$ tropicalizes to the integer affine map $Val(\Psi) : (x, y) \mapsto (ax + by, cx + dy)$. \hfill $\square$

So, $\text{Val}(\Psi)$ acts on $\mathbb{Z}^2$, the Newton polygon, and the extended Newton polygon. We will consider only $SL(2, \mathbb{Z})$-invariant notions, therefore we can always assume that a given edge of a tropical curve is horizontal.

**Proposition 1.10.** A map $\Psi : (x, y) \mapsto (rx, qy)$ with $r, q \in \mathbb{K}^*$ tropicalises to the map $\text{Trop}(\Psi) : (x, y) \mapsto (x + \text{val}(r), y + \text{val}(q))$. \hfill $\square$

This map $\Psi$ can be viewed as coordinate-changing, it affects coefficients of an equation $F = \sum a_{ij}x^iy^j$ by $\text{val}(a'_{ij}) = \text{val}(a_{ij}) - l(i, j)$, with $l(i, j) = i \cdot \text{val}(r) + j \cdot \text{val}(q)$. In terms of the extended Newton polygon $\mathcal{A}$, this means subtracting a linear function $l(i, j)$, therefore the subdivision of the Newton polygon of $F$ does not change. This is not surprising because $\text{Val}(\Psi)$ is a translation. Thus, properties of a curve $C'$ with a singularity at a given point $(p_1, p_2) \in (\mathbb{K}^*)^2$ do not depend on the point $(p_1, p_2)$ since we would like to study its tropicalization $\text{Val}(C')$.

Thus we always consider a curve $C$ such that $\mu_{(1,1)}(C) = m$. Note that $(1, 1)$ tropicalizes to the point $(0, 0) \in T^2$.

Let us consider a non-empty finite set $B \subset \mathbb{Z}^2$. 
Definition 1.11. The lattice width of $B$ in a direction $u \in \mathbb{Z}^2$ is the number $\omega_u(B) = \max_{x,y \in B} u \cdot (x - y)$. The minimal lattice width $\omega(B)$ is defined to be $\min_{u \in \mathbb{Z}^2} \omega_u(B)$.

Recall that integer length of an interval is the number of integer points on it minus one; the set $\text{ConvexHull}(B)$ is the intersection with $\mathbb{Z}^2$ of ConvexHull$(B)$ taken inside $\mathbb{R}^2$. We say that $B$ is a convex polygon if $B = \text{ConvexHull}(B)$.

Definition 1.12. A finite set $B \subset \mathbb{Z}^2$ is called m-thick in the following cases:

- $B$ is empty or ConvexHull$(B)$ is 1-dimensional and its integer length is at least $m$,
- ConvexHull$(B)$ is 2-dimensional and the minimal lattice width $\omega(B)$ is at least $m$,
- ConvexHull$(B)$ is 2-dimensional and for each $u \in \mathbb{Z}^2$, if ConvexHull$(B)$ has lattice width $\omega_u = m - a$ with $a > 0$, then ConvexHull$(B)$ has two sides of integer length at least $a$, and these sides are perpendicular to $u$.

Proposition 1.13. If a convex polygon $B \subset \mathbb{Z}^2$ is m-thick and has at most one vertical side, then $\omega_{(1,0)}(B) \geq m$. If $B$ is m-thick and has no parallel sides, then $\omega(B) \geq m$. □

Lemma 1.14. (The lemma about reducibility.) If $\mu_{(1,1)}(C) = m$ and for some $a > 0$, $u = (u_1, u_2)$ we have $\omega_u(A) = m - a$, then $C$ contains a rational component $(s^{u_1}, s^{u_2})$. Furthermore, $\Delta(A)$ contains two sides of integer length at least $a$, and these sides are perpendicular to $u$.

Proof. By Lemma L14 and Proposition L13 it is enough to prove the lemma only for $u = (1, 0)$. Let us restrict the equation $F$ to the line $y = 1$. It has a root of multiplicity $m$, but the degree of the restriction is $m - a$, therefore $F$ is identically 0 on $y = 1$, hence $F$ is divisible by $y - 1$. Divide $F$ by the maximal power of $y - 1$, say $b$. Clearly $b \geq a$, otherwise we can repeat the above argument. Therefore $F$ is divisible by $(y - 1)^a$, additionally this means that both vertical sides of $A$ have length at least $a$. □

Remark 1.15. We just proved that if $\mu_{(1,1)}(C) = m$, then $A$ is m-thick. In particular, if $\omega_u(A) = 0$ (hence $a = m$), then $A$ is an interval of integer length at least $m$.

Definition 1.16. For $\mu \in \mathbb{R}$, denote by $A_\mu$ the set $\{(i, j) \in A | \text{val}(a_{ij}) \geq \mu\}$.

The following theorem describes the set of valuations of the coefficients $a_{ij}$ of $F(x, y)$. Other theorems make extensive use of it.

Theorem 1. (The M-thickness Theorem.) For each real number $\mu$, the set $A_\mu$ is m-thick.

Proof. This theorem is proved by reducing to Lemma L15. Suppose $F \in \langle x - 1, y - 1 \rangle^m$. Therefore $F = f_0(x - 1)^m + f_1(x - 1)^{m-1}(y - 1) + \cdots + f_m(y - 1)^m$, where $f_k$ are Laurent polynomials. Taking on the both sides of the equality only those monomials $c_i x^{e_i} y^j, c_{ij} \in F$ with $\text{val}(c_{ij} x^{e_i} y^j) \geq \mu$, we get a new Laurent polynomial $\hat{F} \in \langle x - 1, y - 1 \rangle^m$ such that $\hat{F}$ contains monomials only from $A_\mu$. It is clear that if $A_\mu$ is non-empty, then the equation $\hat{F}(x, y) = 0$ defines a curve $\hat{C}$ with $\mu_{(1,1)}(\hat{C}) = m$. Then we apply Lemma L14 for $\hat{F}$.

To conclude this section we prove a simple lemma which will be crucial latter.

Lemma 1.17. Suppose $h : [a, b] \to \mathbb{R}$ is smooth and concave on the interval $[a, b]$. Define $\hat{h}_{[a,b]}(x)$ as the length of the subinterval of $[a, b]$ where $h$ is larger than $h(x)$; $\hat{h}_{[a,b]}(x) = \max\{y \in [a, b] | h(y) \geq h(x)\}$. Therefore $\int_a^b \hat{h}_{[a,b]}(x)dx = (b - a)^2/2$.

Proof of the lemma. Without loss of generality $h(a) \geq h(b) = 0$. Let $q$ be the point where the maximum of $h$ is attained. On the intervals $[c, q]$ and $[q, b]$ the function $h$ is invertible, say $h_1(h(x)) = x$ for $x \in [a, q]$ and $h_2(h(x)) = x$ for $x \in [q, b]$. For $y \in [0, h(a)]$, define $h_1(y) = a$. Let us
integrate $\hat{h}_{[a,b]}(x)$ along the $y$-axis. We have to change the measure that we use in integration. The integral becomes

$$\int_{h(q)}^{0} (-h_1(y) + h_2(y))d(-h_1(y) + h_2(y)).$$

Here $h_1(h(q)) = h_2(h(q)) = q, h_1(0) = a, h_2(0) = b$. Define $H(y)$ as $h_2(y) - h_1(y)$. Therefore

$$\int_{a}^{b} \hat{h}_{[a,b]}(x)dx = \int_{h(q)}^{0} H(y)d(H(y)) = H^2(0)/2 = (b-a)^2/2.$$

**Remark 1.18.** We proved the lemma for differentiable functions, but the proof also works for piecewise linear functions.

2. Formulation of main theorems

In this section, we state the main results of this article. We start with the following example:

**Example 2.1.** Consider a curve $C'$ defined by the equation $F(x, y) = 0$, where

$$F(x, y) = t^{-3}xy^3 - (3t^{-3} + t^{-2})xy^2 + (3t^{-3} + 2t^{-2} - 2t^{-1} - 3)x + t^{-2}x^2y^2 - (2t^{-2} - t^{-1})x^2y + (t^{-2} - t^{-1} - 3)x^2 + t^{-1}y - (t^{-1} + 1) + x^3$$

The point $p = (1, 1)$ is a point of multiplicity 3 for the curve $C'$, so let us look at properties of the point $P = (\text{val}(1), \text{val}(1)) = (0, 0)$ on the curve $\text{Val}(C')$. The curve $\text{Val}(C')$ is equal to the set of non-smooth points of the function

$$\text{Val}(F(x, y)) = \max(3 + x + 3y, 3 + x + 2y, 3 + x + y, 3 + x, 2 + 2x + 2y, 2 + 2x + y, 2 + 2x, y + 1, 1, 3x)$$

The curve $\text{Val}(C')$ divides the plane into regions. In Figure 2 we have written the value of $\text{Val}(F(x, y))$ on each region.

The point $P = (0, 0)$ lies on an edge $E$ of multiplicity $m = 3$ and divides it in the ratio 1:2. This example also reveals $\mathcal{D}_p, \mathcal{N}_f$ which we will now introduce. Here $\mathcal{D}_p$ is the set of edges which are dual to the horizontal edges of $\text{Val}(C')$. Thus, in this example, $\mathcal{D}_p$ is the set of vertical edges in $\Delta(A)$. The number $\mathcal{N}_f(p)$ is the sum of the areas of the faces which are incident to at least one edge in $\mathcal{D}_p$. Here $\mathcal{N}_f(p)$ is $2 + 5/2 + 1 = 11/2$ which is greater than $m^2/2 = 3^2/2$. These facts are incarnations of a general phenomenon.
Definition 2.2. For a point $q \in C \subset (\mathbb{K}^*)^2$ we pick $\text{Val}(q) = Q \in \text{Val}(C)$, then we take the set $\{E_i\}_{i=1..k}$ of all the edges of $\text{Val}(C)$ such that the usual (not tropical) line spanned by $E_i$ passes through $Q$. We denote by $\mathfrak{Dep}(q)$ the set $\{d(E_i)\}_{i=1..k}$ of edges of the subdivision. The name $\mathfrak{Dep}(q)$ is supposed to mean that it is the set of edges dependent on the point $q$.

Note that vertices $A_j$ lying on an edge $E_i$ correspond to faces $d(A_j)$ containing $d(E_i)$.

Definition 2.3. If $Q$ lies on an edge of $\text{Val}(C)$, then the number $\mathfrak{Inf}(q)$ is the sum of the areas of the faces which are incident to at least one edge in $\mathfrak{Dep}(q)$. If $Q$ is a vertex of $\text{Val}(C)$, then by $\mathfrak{Inf}(q)$ we denote the same sum as above and by $\overline{\mathfrak{Inf}}(q)$ we denote the same sum but with the area of the face $d(A_j)$ taken with coefficient 2.

Proposition 2.4. The set $\mathfrak{Dep}(q)$ depends only on $\text{Val}(q) = Q$, therefore we can write $\mathfrak{Dep}(q)$ and $\mathfrak{Inf}(q)$ instead of $\mathfrak{Dep}(Q)$, $\mathfrak{Inf}(Q)$.

Remark 2.5. It is easy to see that if $Q = (0,0)$, then $\mathfrak{Dep}(Q)$ consists of exactly those edges which are projections of horizontal edges of $A$ (Figure 3). For a point $Q = (q_1, q_2)$, the set $\mathfrak{Dep}(Q)$ consists of projections of edges parallel to the plane $z + q_1i + q_2j = 0$.

Denote by $P = (\text{val}(p_1), \text{val}(p_2))$ the tropicalization of the point $(p_1, p_2)$ of multiplicity $m$ on $C$. Recall that without loss of generality $P = (0,0)$ by Propositions 1.9, 1.10.

Now we can see the exceptional situations we mentioned in the introduction.

Definition 2.6. A curve $C$ is good if it does not contain a rational component with parametrization $(p_1s^{u_1}, p_2s^{u_2})$ where $s$ is a local parameter, $u = (u_1, u_2) \in \mathbb{Z}^2$, and $(p_1, p_2)$ is the point of multiplicity $m$ on the curve $C$.

In this paper the important facts have names which we use to refer to them. The following theorems describe a governorship, i.e. the area of influence on the subdivision (this explains the notation $\mathfrak{Inf}$), of the singular point $p$:

Theorem 2. (The Main Technical Theorem.) If $C$ is good and the point $P$ is on an edge $E$ of $\text{Val}(C)$ and $P$ is not a vertex of $\text{Val}(C)$, then $\mathfrak{Inf}(P) \geq \frac{1}{2}m^2$ and the integer length of $d(E)$ is at least $m$.

In this case, in the subdivision $\Delta(A)$, we see a collection of faces which have parallel sides (see Figure 1a).

Theorem 3. (The Exertion Theorem.) If $C$ is good and the point $P$ is a vertex of $\text{Val}(C)$, then $\overline{\mathfrak{Inf}}(P) \geq \frac{3}{2}m^2$. Furthermore, if we count the area of $d(P)$ with multiplicity 2, then the obtained sum $\mathfrak{Inf}(P)$ is at least $\frac{1}{2}m^2$.

In this case, we will see a collections of faces with parallel sides which are attached to the face dual to the point $P$ (see Figure 1b).

Our theorems work only for good curves. The following example illustrates the problem with curves which are not good.

Example 2.7. Consider a curve $C'$ defined by a polynomial $F_k(x, y) = (x - 1)^k(y - 1)^{m-k}$. The Newton polygon of $F_k$ is the rectangle with vertices $(0,0), (k,0), (0,m-k), (k,m-k)$, it is $m$-thick and its area is $k(m-k)$ which is frequently less than $\frac{3}{2}m^2$. $C'$ consists of the line $y = 1$ with multiplicity $k$ and the line $x = 1$ with multiplicity $m-k$, hence it is not good, but $\mu_{(1,1)}(C') = m$.

Remark 2.8. In fact, one can completely describe the matroid associated with the linear conditions imposed by such a singular point $(\mathbb{K})$ if $\text{char} \mathbb{K} = 0$, namely, the sets of the type $A \setminus \{(i,j) | G(i,j) = 0\}$, where $G$ is a polynomial of degree no more than $m-1$, are all the dependent sets, minimal by
inclusion. If we take $G$ to be a product of $m - 1$ lines, then it easily implies the $m$-thickness property for $A_s$, and the latter carries more or less all geometric information about the matroid and can also be used in positive characteristic.

3. A proof of the Main Technical Theorem

Here we prove the **Main Technical Theorem**. Firstly, we introduce some notation and face the key ideas.

**Definition 3.1.** Suppose that the edges $E_1, E_2, \ldots, E_n$ of the curve $\text{Val}(C)$ lie on a line and $\bigcup_{i=1}^{n} E_i$ is connected. In such a case, the union $\bigcup_{i=1}^{n} E_i$ is called a *long edge*. In particular, each edge $E_i$ is a long edge also. A long edge is called *maximal* if it is not contained in a bigger long edge.

So, a long edge is just a union of consecutive edges on a line. In Example 2.1 three horizontal edges give us a maximal long edge.

It follows from Lemma 1.7 and Proposition 1.9 that without loss of generality $P = (0, 0)$ and the edge $E$ containing $P$ is horizontal, hence $d(E)$ is vertical.

Let us take the maximal long edge $\bigcup_{i=1}^{m} E_i$ containing the edge $E$, and let $E_i = E$. It is clear that $d(E_i) \in \text{Dep}(P)$. We denote by $A_1$ the leftmost vertex of $\text{Val}(C)$ on the long edge and by $A_2, A_3, \ldots$ all the consequent vertices, if they exist, numbered from left to right. We denote by $E_i$ the edge of $\text{Val}(C)$ such that $E_i \subset E$ and the left end of $E_i$ is the point $A_i$.

Recall that for each edge $E'$ of $\text{Val}(C)$, there is a dual edge $d(E')$ in the subdivision of the Newton polygon of the curve $C$. We denote by $L(d(E'))$ the lifting of an edge $d(E')$ in the boundary of $\tilde{A}$.

![Figure 3. A part of the extended Newton polyhedron above a long edge. The set $\text{Dep}(0, 0)$ consists of the projection of the edges $L(d(E_i))$, which are depicted as thick black horizontal intervals, while a section of the extended Newton polygon by a horizontal plane is marked in gray. The projection onto the $xz$-plane is also depicted.](image)

In Example 2.1 $F$ can be written as $t^{-3}x(y-1)^3+t^{-2}x(x-1)(y-1)^2+t^{-1}(x-1)^2(y-1)+(x-1)^3$. This illustrates that the extended Newton polygon is made of layers of $m$-thick sets, namely $\text{supp}(x(y-1)^3), \text{supp}(x(x-1)(y-1)^2), \text{supp}(x-1)^2(y-1)), \text{supp}(x-1)^3$ in this example.

**The M-thickness Theorem** gives us information about $\text{val}(a_{ij})$. This permits to recover some properties of the subdivision. We will be interested in the sum of the areas of the faces $d(A_i)$ where $A_i$ is on a long edge through the point $(0, 0)$. A face $d(A_i)$ is the projection of a face of $A$ with at least one edge $L(d(E_i))$. These edges are colored in red in Figure 3. Let us meditate on the extended Newton polyhedron $A$ in Figure 3. Its horizontal sections are $m$-thick, therefore it follows from Lemma 1.14 that the width of a section passing through a red edge $L(d(E_i))$ plus the length of $L(d(E_i))$ is at least $m$. Now we project $A$ onto the $xz$-plane (Figure 3) and denote obtained function by $z = g(x)$. Now the lengths of the edges $d(E_i)$ can be interpreted in terms of $\hat{g}$ and by
Lemma 1.17 we get an estimate for $\int \hat{g}$ which is more or less the same as $\sum \text{area}(d(A_i))$. This is the key observation. Squirming through a number of lemmas we get the desired estimate.

We study two distinct cases: when the point $(0,0)$ (which is the tropicalization of the point $(1,1)$) lies on an edge of $\text{Val}(C)$ or is a vertex of $\text{Val}(C)$. The latter will be discussed in the following sections of the paper.

I emphasize that in this section the situation near only one long edge is considered; there may possibly be other edges in $\mathcal{D\Phi}(P)$ besides $d(E_i)_{i=1..n}$.

The following lemmas prove that Figure 3 reveals all the important for us features of $\tilde{A}$.

**Lemma 3.2.** The edge $L(d(E))$ has direction $(0,1,0)$ and it is higher than all other points of $\tilde{A}$.

Proof. Refer to Figure 2. Recall that $(0,0)$ is on the edge $E$. Therefore, the top end $(k,j_1)$ of $d(E)$ represents the tropical monomial $\text{val}(a_{kj_1}) + kx + j_1y$ of $\text{Val}(F)$, it dominates other monomials in the region higher than the edge $E$, and the bottom end $(k,j_2)$ of $d(A)$ represents the monomial $\text{val}(a_{kj_2}) + kx + j_1y$ which dominates other monomials in the region lower than the edge $E$. Therefore $\text{val}(a_{kj_1}) + kx + j_1y$ and $\text{val}(a_{kj_2}) + kx + j_1y$ are equal on the edge $E$, in particular at the point $(0,0)$; therefore $\text{val}(a_{kj_1}) = \text{val}(a_{kj_2}) = s$, hence $L(d(E))$ is horizontal. Furthermore, $\max_{x,y}(\text{val}(a_{kj_1} + ix + jy) = \text{val}(a_{kj_1}) = s$ at the point $(0,0)$, and if $\text{val}(a_{kj}) = s = \text{val}(a_{kj_1})$, then $i = k$, otherwise $P$ is a vertex of $\text{Val}(C)$. Then $j \leq j_1$ because of the maximality of $\text{val}(a_{kj_1}) + kx + j_1y$ in the region higher than $E$; and $j_2 \leq j$ by symmetric reasoning. □

By Theorem 1 the set $d(E) = A_{s}$ is $m$-thick, hence the edge $d(E)$ has integer length at least $m$ and we have proved the first part of The Main Technical Theorem. □

Now we will prove that $\mathfrak{If}(P) \geq \frac{1}{4}m^2$.

Let $x_i$ be the $x$-coordinate of the edge $d(E_i)$. We denote the $y$-coordinates of its endpoints by $y_i < y^i$, and the integer length of $d(E_i)$ by $m_i = y^i - y_i$.

**Remark 3.3.** As we see in Figure 3 the area of a face $d(A_{i+1})$, which is a summand of $\mathfrak{If}(P)$, is no less than $\frac{1}{2}(m_{i+1} - m_i)(x_{i+1} - x_i)$. For the unity of notation we denote by $x_0$ the $x$-coordinate of the leftmost vertex on the picture, and set by definition $m_0 = 0$, therefore the area of the leftmost face is no less than $\frac{1}{2}(m_1 - m_0)(x_1 - x_0)$.

Consider the face $d(A_1)$. It has vertical side $d(E_1)$ on the right and we suppose for now that there exists the leftmost vertex of the face $d(A_1)$. (See Figure 4) It is convenient to treat the leftmost vertex of the face $d(A_1)$ as a vertical edge of zero length. Let us abuse notation and name this vertex $d(E_0)$, although there is no edge $E_0$. Proceeding further, define its length $m_0 = 0$, and coordinates $(x_0, y_0)$ will be the coordinates of $d(E_0)$. One can do the same procedure and define $d(E_{n+1}), m_{n+1}$ if the rightmost vertex of $d(A_n)$ does exist, i.e., the maximal long edge, containing $E$, is not infinite on the right.

Now the sum $\mathfrak{If}(P)$ which we seek to estimate may be written as follows:

$$\mathfrak{If}(P) \geq \frac{1}{2} \sum_{i=0}^{n+1} (m_i + m_{i+1})(x_{i+1} - x_i)$$

where $i$ runs from 0 or 1 up to $n$ or $n+1$ depending on the finiteness of the maximal long edge. The next lemma is obvious in Figure 4 the height of a red edge which is on the left side of the picture is greater then the heights $\text{val}(a_{ij})$ of the points $(i,j)$ which lie to its left. Refer also to Figure 2.

**Lemma 3.4.** Consider an edge $E_q$ where $q < l$. Therefore, for each $(i,j) \in A$ with the property $i < x_q$ or $i = x_q, j < y_q$ or $i = x_q, j > y^q$, the valuation $\text{val}(a_{ij})$ is less than $\text{val}(a_{x_qy^q}) = \text{val}(a_{x_qy^q})$. 

Proof. For each two consecutive edges \( d(E_i), d(E_{i+1}) \), there is a face of the polyhedron \( \tilde{A} \), this face is spanned on the edges \( L(d(E_i)), L(d(E_{i+1})) \). The edges \( d(E_i) \) are all parallel to \( d(E_i) \), therefore all the edges \( L(d(E_i)) \) are parallel to each other as well. Denote \( \text{val}_i := \text{val}(a_{x,qy}) = \text{val}(a_{x,y_q}) \).

Provided that \( A \) is convex, all points \( (i,j) \text{ val}(a_{ij}) \) lie under each plane passing through a face of \( \tilde{A} \). This finishes the proof of the lemma and implies \( \text{val}_1 < \text{val}_2 < \cdots < \text{val}_l = s > \text{val}_{l+1} > \cdots > \text{val}_n \).

\[ \square \]

Remark 3.5. At the same time we proved the symmetric statement for \( q > l \). This lemma should convince you that the actual picture is as drawn in Figure 3. The above Lemma 3.4 works even for \( q = 0 \) and also for \( q = n + 1 \) if \( d(E_0), d(E_{n+1}) \) are defined.

\[ \square \]

Remark 3.6. Hereafter we will not write and also for “\( q = 0, n + 1 \)”, the only meaning of the number \( n \) is a counter for the edges. So we will use 1 and \( n \) everywhere, without mentioning that this could be 0 or \( n + 1 \) correspondingly, depending on the finiteness of the maximal long edge.

![Figure 4](image_url)

Figure 4. The number \( x_i \) is the \( x \)-coordinate of the edge \( d(E_i) \) in (A). By definition \( g(x_i) = \text{val}_i \) in (B). Also \( \hat{g}(a) \) and \( \hat{g}(b) \) are presented. The key observation is that \( \hat{g}(x_i) + m_i \geq m \). Furthermore, \( \hat{g}(at + b(1-t)) \geq t\hat{g}(a) + (1-t)\hat{g}(b) \), this proves the concavity of \( \hat{g} \) on \([x_0, x_3]\) and on \([x_3, x_6]\).

By Theorem 1 each set \( A_{\text{val}_q} \) is \( m \)-thick for each \( q = 0 \ldots n \). Recall that \( d(E_q) = \{(x_q, y)|y_q \leq t \geq y', y_q - y = m_q \} \) and \( \text{val}_q = \text{val}(a_{x_q, y_q}) \) is the height (i.e. \( z \)-coordinate) of \( L(d(E_q)) \).

Lemma 3.7. If \( C \) is good and \( \mu_{(1,1)}(F) = m \), then \( x_n - x_0 \geq m \).

Proof. Let us suppose that \( x_n - x_0 < m \). If there exists neither \( d(E_0) \) nor \( d(E_{n+1}) \), then we are in the situation of Lemma [1,14] and by Definition [2,6] the curve \( C \) is not good, but we prohibit such curves. If \( d(E_0) \) exists and \( d(E_{n+1}) \) does not exist, then \( A_{\text{val}_0} \) is on the left side of the line \( x = x_n \) and if \( x_n - x_0 < m \) then the set \( A_{\text{val}_0} \) has \( \omega_{(1,0)} < m \) and has no two vertical sides, which contradicts the fact that \( A_{\mu} \) is \( m \)-thick for each \( \mu \) (Proposition [1,13]). If both \( d(E_0) \) and \( d(E_{n+1}) \) exist, then we apply the above argument for \( A_{\text{max}(\text{val}_0, \text{val}_{n+1})} \).

The next lemma describes a section on Figure 3. We draw a horizontal section through the edge \( L(d(E_q)) \) at height \( \text{val}_q \). This edge lies on the right-hand side of the picture and the section passes between edges with heights \( \text{val}_r, \text{val}_{r+1} \) on the left hand side of the picture. The claim is that the sum of the width of this section and the length of the red edge is at least \( m \).

Lemma 3.8. Suppose \( q > l \) and \( \text{val}_{l+1} \geq \text{val}_q > \text{val}_r \) where \( r < l \). Then \( m_q + x_q - x_r \geq m \).

Proof. Take \( A_{\text{val}_q} \) and refer to Figure 3. There \( q \) is equal to \( l + 1 \), and \( r \) is equal to \( l - 1 \). By definition \( A_{\text{val}_q} \) is contained in the projection to \( \mathbb{R}^2 \) of the horizontal section \( z = \text{val}_q \) of the extended
Newton polyhedron $\tilde{A}$. Lemma 3.13 implies that $A_{val_q}$ is inside the strip $\{(i,j)|x_i \leq i < x_q\}$ and it is $m$-thick. Therefore the length $m_q$ is at least $m - (x_{r+1} - x_q)$. □

The advantage of this lemma is that it is given in terms of the Newton polygon (see Definition 6.4). Still, it is not enough to prove The Main Technical Theorem. To reinforce it we have to meditate a little more on the extended Newton polyhedron.

Consider the set of faces of $\tilde{A}$ with sides $L(d(E_i)), i = 0 \ldots n$. Now project the boundary of $\tilde{A}$ to the $x$-$z$-plane. Each edge $L(d(E_i))$ projects to the point $(x_i, val_i)$ (Figure 3 shows the source and Figure 4 shows the result). Let $z(a)$ be the maximal $z$-coordinate of points with $x = a$ in the image of the projection.

Define a function $g(x) = z(x)$. The projection of the face stretched on the edges $L(d(E_i)), L(d(E_{i+1}))$ coincides with the graph of $g$ on the interval $[x_i, x_{i+1}]$; the graph of $g$ in turn appears there as the line between $(x_i, val_i), (x_{i+1}, val_{i+1})$ (see Figure 3).

Define a function $\hat{g}(x)$ to be the length of the interval excised out of the line $z = g(x)$ by the graph of $g$ (see Figure 4).

Lemma 3.9. The length $m_i$ of the edge $d(E_i)$ is no less than $m - \hat{g}(x_i)$.

Proof. Suppose that the projection of the excised interval onto the $x$-axis is $[x_i, x']$. In fact, $x' - x_i \geq \omega(1,0)(\{z = val_i\} \cap \tilde{A})$. Therefore $A_{val_i}$ is inside the strip $\{(x,y)|x_i \leq x \leq x'\}$. Together with Lemma 1.14 this implies the present lemma. □

Remark 3.10. In fact, $A_{val_i}$ is contained in $\{z = val_i\} \cap \tilde{A}$ but does not necessary coincide with it.

I want to estimate $\text{Infl}(P)$, which is greater or equal then the sum of the areas of the faces $d(A_i), i = 1 \ldots n$. Consider the following piecewise linear function $f$ on the interval $[x_0, x_n]$; define $f(x) = m_i, i = 1 \ldots n$, $f$ is linear on $[x_i, x_{i+1}]$. Clearly $\text{Infl}(P) \geq \int_{x_0}^{x_n} f(x)dx$.

Lemma 3.11. On the interval $[x_0, x_n]$, we have $f(x) + \hat{g}(x) \geq m$.

Proof. The inequality is satisfied at $x_i$ for each $i$, by Lemma 3.9. Consider an interval $[x_i, x_{i+1}]$. If $\hat{g}$ is linear on it, then Lemma 3.9 gives us $f(x) + \hat{g}(x) \geq m$. If $\hat{g}$ is not linear, then it is concave on $[x_i, x_{i+1}]$. For an illustration, see Figure 4. For each two points $a, b \in [x_1, x_i]$ and $t \in [0,1]$ we have $\hat{g}(at + b(1-t)) \geq t\hat{g}(a) + (1-t)\hat{g}(b)$. But $f$ is linear on $[x_i, x_{i+1}]$, and the inequality is satisfied at the endpoints, therefore it is satisfied at each point. □

Proposition 3.12. By Lemma 3.7 there are points $b, c \in [x_0, x_n]$ such that $c - b = m$ and $\hat{g}_{[x_0,x_n]} = \hat{g}_{[b,c]}$.

If we can chose $b, c$ in such a way that $g(b) = g(c)$, then we trivially have $\hat{g}_{[x_0, x_n]} = \hat{g}_{[b,c]}$. If we cannot chose such $c, b$, then $b = x_0$ or $c = x_n$ and we also have $\hat{g}_{[x_0,x_n]} = \hat{g}_{[b,c]}$. □

Final step of the proof of the Theorem. The following computation completes the proof, applying Lemma 1.17

\[ S \geq \int_{x_0}^{x_{n+1}} f(x)dx \geq \int_a^b f(x)dx \geq \int_a^b (m - \hat{g}_{[b,c]}(x))dx \geq m(b - a) - (b - a)^2/2 = m^2/2. \]

Remark 3.13. Lemma 1.17 may be easily generalized to higher dimensions.

4. Singular point in a vertex

Here for $C$ with a point (1,1) of multiplicity $m$ we prove

Theorem 3 (The Exertion Theorem). If $C$ is good and the point $P = \text{Val}(1,1)$ is a vertex of $\text{Val}(C)$, then $\text{Infl}(P) \geq 3m^2/2$. Furthermore, if we count the area of $d(P)$ with multiplicity 2, then the obtained sum $\text{Infl}(P)$ is at least $1/2m^2$. 


The numbers \( \text{Infl}(P), \overline{\text{Infl}}(P) \) are given by Definition 2.3, and, using the combinatorial description below, we reduce this theorem to pure plane geometry problems about \( m \)-thick sets. Firstly, we need some definitions and a preparation lemma.

For a vector \( u \in \mathbb{Z}^2 \), we take the line \( l_u(P) \) passing through \( P \) with slope \( u \).

**Definition 4.1.** We denote by \( \text{dep}_u(P) \) the set of all edges in the connected component, containing \( P \), of the intersection of \( \text{Val}(C) \) with the line \( l_u(P) \), by \( \text{infl}_u(P) \) the set of all vertices of \( \text{Val}(P) \) lying on edges in \( \text{dep}_u(P) \), and we define \( \text{dep}(P) = \bigcup_{u \in \mathbb{Z}^2} \text{dep}_u(P) \), \( \text{infl}(P) = \bigcup_{u \in \mathbb{Z}^2} \text{infl}_u(P) \).

**Remark 4.2.** It is easy to see that \( \text{Dep}(P) = \bigcup_{E \in \text{dep}(P)} d(E) \) and \( \text{Infl}(P) = \sum_{Q \in \text{infl}(P)} \text{area}(d(Q)) + \text{area}(d(P)) \).

Recall that a good \( C \) does not contain rational components of type \((x_0 s^{u_1}, y_0 s^{u_2})\), and by the [lemma about reducibility][1] the Newton polygon of \( C \) has lattice width at least \( m \) in all directions.

**Lemma 4.3.** For each direction \( u \in \mathbb{Z}^2 \) such that \( d(P) \) has at most one side perpendicular to \( u \), the width \( \omega_u(d(P)) \) is at least \( m \).

**Proof.** This follows from Lemma 1.14 because \( d(P) \) is \( A_{\mu'} \) where \( \mu' \) is maximal among those \( \mu \) for which \( A_{\mu} \) is not empty. \( \square \)

**Definition 4.4.** For an \( m \)-thick set \( B \) and a direction \( u \in \mathbb{Z}^2 \) we define \( \text{def}_u(B) = \max(m - \omega_u(B), 0) \).

**Remark 4.5.** For each \( m \)-thick set \( B \) we have \( \omega_u(B) + \text{def}_u(B) \geq m \) for all \( u \in \mathbb{Z}^2 \).

Consider the lattice width \( a = \omega_u(d(P)) \) of \( d(P) \) in the direction \( u \) of an edge \( E \), containing \( P \).

**Lemma 4.6.** (The preparation lemma.) If \( a < m \) then a) the face \( d(P) \) has two sides of length at least \( \text{def}_u(d(P)) \), and these sides are perpendicular to the vector \( u \); and b) \( \sum_{Q \in \text{infl}(P)} \text{area}(d(Q)) - \text{area}(d(P)) \geq \text{def}_u(d(P))^2 \).

**Proof of a).** This follows from Lemma 1.14. \( \square \)

**Proof of b).** Refer to Figure 5, the faces, which contribute to \( \sum_{Q \in \text{infl}(P)} \text{area}(d(Q)) - \text{area}(d(P)) \), are enclosed by magenta lines. Look at the set \( \{(i, j) \in \mathbb{Z}^2 \} \) where \( \text{val}(a_{ij}) \) is maximal. It contains

![Figure 5](image-url)
vertices of $d(P)$ and maybe some integer points inside $d(P)$. We denote by $S_1$ the shorter of the two perpendicular to $u$ sides of $d(P)$. It follows from Lemma 4.14 that the length of $S_1$ is at least $def_u(d(P))$.

As in the proof of The Main Technical Theorem, we consider the sets $A_\mu$ for different $\mu$. One can cover $d(P) \setminus S_1$ by a vertical lines. Therefore, one can imagine cutting out these a vertical lines from all the $A_\mu$ and shrinking the rest. Furthermore, the situation looks like a singular point of multiplicity $m - a$ on an edge. Proceeding similarly to the proof of The Main Technical Theorem, one gets an estimate that the sum of areas of the faces is at least $(m - a)^2/2$.

Now we are ready to prove The Exertion Theorem. Unfortunately, there is no conceptual proof, only combinatorial computations. All the area calculations are quite straightforward and we only indicate the main ideas behind them.

Proof. We would like to estimate two sums

$$\overline{3m}(P) = \sum_{Q \in \text{int}(P)} \text{area}(d(Q)), \overline{3m}(P) = \text{area}(d(P)) + \sum_{Q \in \text{ext}(P)} \text{area}(d(Q))$$

Exertion Theorem follows from The preparation lemma and two following lemmas.

Lemma 4.7. For a convex $m$-thick $B$ we have $\text{area}(B) + \frac{1}{2} \sum_{u \in \mathbb{Z}^2} \text{def}_u(B)^2 \geq \frac{3}{8}m^2$

Proof. See the next section.

Lemma 4.8. For a convex $m$-thick $B$ we have $2 \cdot \text{area}(B) + \frac{1}{2} \sum_{u \in \mathbb{Z}^2} \text{def}_u(B)^2 \geq \frac{1}{2}m^2$

Proof. See the next section.

Remark 4.9. The name Exertion Theorem comes from the fact that asymptotically, the singular point $P$ mostly has influence on $\overline{3m}(P)$ coefficients of a curve. Moreover, if a tropical curve has a number of singular points in general position, then each coefficient is under a governorship of at most two singular points and the coefficients in $d(P)$ is under governorship of $P$ only. For this reason and for purposes of the article, we prove estimates for $\overline{3m}(P)$ and for $\overline{3m}(P)$.

5. Two combinatorial lemmas

Consider a convex $m$-thick polygon $B$. Suppose that the minimal lattice width of $B$ is attained in the horizontal direction and is equal to $a$. Pick two points $M, L$ on the left vertical side of $B$ and points $N, K$ on the right one in such a way that the distances $ML$ and $NK$ equal $m - a$ and so $MNLK$ is a parallelogram. Let us call it initial parallelogram, see Figure 6 (A).

Using $SL(2, \mathbb{Z})$ change of coordinates we can suppose that the $y$-coordinate of $M$ minus $y$-coordinate of $N$ is less than $a$ and non-negative. Denote this difference by $b$.

Proposition 5.1. The width $\omega_{(0,1)}(MNLK)$ of the parallelogram $MNLK$ in the direction $(0, 1)$ is equal to $m - a + b$. The width $\omega_{(1,1)}(MNLK)$ is equal to $m - b$, and if $b > 0$, then $B$ is bigger than $MNLK$ because $B$ is $m$-thick.

Lemma 5.2. $2 \cdot \text{area}(B \setminus (MNLK)) + \frac{1}{2} \sum_{u \in \mathbb{Z}^2, u \neq (1,0)} \text{def}_u(B)^2 \geq a^2/2$

Proof. Suppose $\omega_{(0,1)}(B) = m - x$. Therefore, if $x < a - b$, then $B$ must have two horizontal sides $M_1 M_2, K_1 K_2$ of lengths at least $x$. Therefore $B$ contains a polygon like the one depicted in Figure 6 (B). With the purpose of estimating the area from below, we force the arrangement of sides to be the worst, like at the bottom in Figure 6 (B). At the top, we should move the horizontal side to the left as much as possible, while preserving the convexity of $B$. We denote by $x_1, x_2$ the increments of $\omega_{(0,1)}$ at the top of the picture and at the bottom. All notation is presented in Figure 6 and the picture serves as the main illustration tool for the following computations.
We have \( x \geq m - (m - a + b + x_1 + x_2) \) and we can suppose that inequality holds. So, \( x_1 + x_2 = a - b - x \).

The minimal area of \( MM_1M_2NKK_1K_2L \setminus MNKL \) is \( a(x_1 + x_2)/2 + x(b + x_1 + b + x_2)/2 \) and it is attained when the bottom horizontal edge is in the extremal right position, and the top edge is in the extremal left position.

Therefore \( \frac{1}{2} \det_{(0,1)}(B)^2 + \text{area}(B \setminus (MNKL)) \geq x^2/2 + 2(a(x_1 + x_2)/2 + x(b + x_1 + b + x_2)/2) = (x^2/2 + a(a - b - x) + x(a + b - x)) = (a(a - b) + xb - x^2/2), \) therefore \( \min(a(a - b) + xb - x^2/2) = \min(a(a - b), (a - b)a + (a - b)(b - \frac{a^2}{2}).) \)

The minimum, which we denote \( S_{0,1} \), is equal to \( a(a - b) \) if \( b \geq a/3 \) and \( (a(a - b) + \frac{b}{2}a - 3b)(a - b)/2 \) if \( b \geq a/3 \).

Now for \( \omega_{(1,1)} \) we define \( y_1, y_2 \) and \( N_1N_2, L_1L_2 \) as above.

With the notation \((y, y_1, y_2)\) instead of \((x, x_1, x_2)\), for the direction \((1,1)\) we get \( y_1 + y_2 + y = b \).

The additional area is at least \( \frac{1}{2} \det_{(1,1)}(B)^2 + \text{area}(B \setminus (MNKL)) \geq y^2 + a(y_1 + y_2) + (a - b + y_1 + a - b + y_2) = (y^2 + a(b - y) + y(2a + b - y)). \)

The minimum, which we denote \( S_{1,1} \), is equal to \( ab \) if \( b \leq 2a/3 \) and \( ab + b(2a - 3b)/2 \) if \( b \geq 2a/3 \).

Let us find \( S^w = \min_b \max(S_{0,1}, S_{1,1}) \). If \( a/3 \leq b \leq 2a/3 \), then \( S_{0,1} + S_{1,1} = a^2 \) and we are done.

If \( b \leq a/3 \) then \( b = 0 \) and \( S_{0,1} = a^2/2 \) or \( b = a/3 \), and we get \( S_{0,1} = 2a^2/3 \). If \( b \geq 2a/3 \), then \( b = a \) and we get \( S_{1,1} = a^2/2 \), or \( b = 2a/3 \) and we get \( S_{1,1} = 2a^2/3 \).

\[ \square \]

Lemma 5.3. For a convex \( m \)-thick \( B \) we have \( 2 \cdot \text{area}(B) + \frac{1}{2} \sum_{u \in \mathbb{Z}_2} \det_u(B)^2 \geq \frac{1}{2} m^2 \)

Proof follows from the previous Lemma. Indeed, \((m - a)^2/2 + 2(a(m - a)) + a^2/2 \geq m^2/4 \) because \( a(m - a) \geq 0 \).

Now we will exploit widths of \( B \) in the directions \((0,1), (1,1), (1,0) \). Consider the both directions \((0,1), (1,1) \), therefore we have all \( x, y, x_1, y_1, x_2, y_2 \) and \( B \) contains \( MM_1M_2NN_1N_2KK_1K_2LL_1L_2 \). These points are allowed to coincide. Refer to Figure 7.

Figure 6. Dual picture to a singular point on an edge.

Lemma 5.4. The thick intervals \( M_1M_2, N_1N_2, K_1K_2, L_1L_2 \) either share common vertex (like \( K_1K_2, L_1L_2 \) in the bottom of Figure 7), or are maximally far from each other (like \( M_1M_2, N_1N_2 \) at the top of the picture).

Proof. Moving the intervals, preserving \( x, y, x_1, y_1, x_2, y_2 \), and simple arguments by linearity prove the lemma.

Let \( A_1 \) denote the minimal area of the top augmented piece when \( N_1 \neq M_2 \), i.e. \( N_1N_2 \) and \( M_1M_2 \) are maximally far from each other. Let \( A_2 \) denote the minimal area of the bottom augmented piece when \( L_1 \neq K_2 \), i.e. \( L_1L_2 \) and \( K_1K_2 \) are maximally far from each other. Let \( A_3 \) denote the minimal
area of the top augmented piece when \( N_1 = M_2 \). Let \( A_4 \) denote the minimal area of the bottom augmented piece when \( L_1 = K_2 \).

**Lemma 5.5.** \( A_1 + A_3 = A_2 + A_4 \)

Proof. Computing \( \omega_{0,1}(B), \omega_{1,1}(B) \), we get \( x_1 + x_2 + m - a + b = m - x, y_1 + y_2 + m - b = m - y \). Therefore as before, there are relations \( x_1 + x_2 = a - b - x, y_1 + y_2 = b - y \). If the thick intervals are far from each other, then the additional area in the top is \( A_1 = (b + x_1)a - ab/2 - (a - x - y)(x_1 + b - y_1 - y)/2 - y(x_1 + b - y_1 - y + x_1 + b - y_1)/2 = 1/2(ax_1 - yx_1 - yb + yy_1 + ay_1 + ay + xx_1 + xb - xy_1 - xy) \).

The opposite \( A_2 \) can be obtained by changing \( x_1 \) by \( x_2 \) and \( y_1 \) by \( y_2 \):

\[
A_2 = 1/2(ax_2 - yx_2 - yb + yy_2 + ay_2 + ay + xx_2 + xb - xy_2 - xy) = 1/2(a^2 - ax_1 + yx_1 + by - yy_1 - y^2 - ay_1 - ay - xx_1 - x^2 + xb + xy_1 + xy)
\]

If the thick intervals have a common vertex, we get

\[
A_3 = 1/2(yy_1 + (b + x_1)x + ax_1 + (b + x_1)(a - b - x_1 - y) - x_1(a - b - x_1 + y_1 - x)) = 1/2(yy_1 + xx_1 + ax_1 + ab - b^2 - bx_1 + by_1)
\]

\[
A_4 = 1/2(yy_2 + xx_2 + ax_2 + ab - b^2 - bx_2 + by_2) = 1/2(-yy_1 - y^2 - xx_1 - x^2 + a^2 - ab - ax_1 + bx_1 - by_1 + b^2)
\]

By direct computation we get \( A_1 - A_3 = A_4 - A_2 \). If \( A_1 < A_3 \), then \( A_4 < A_2 \), so the minimum sum of the areas of the augmented pieces is \( A_1 + A_4 \) or \( A_2 + A_3 \). Suppose that the minimum is attained in the case \( A_1 + A_4 \).

**Lemma 5.6.** \( \text{area}(MM_1M_2NN_1N_2KK_1K_2LL_1L_2 \setminus M NK L) + \frac{1}{2}(def_{(0,1)}(B)^2 + def_{(1,1)}(B)^2) \geq 3a/8b^2 \)

Proof. \( A_1 + A_4 + x^2/2 + y^2/2 = 1/2(a^2 - ab + b^2 + xb + y(a - b - x) + x_1(b - y) + y_1(a - b - x)). \) Minimizing, we get \( x_1 = y_1 = 0 \). Next, \( y = 0, x = 0 \). Next, minimizing by \( b \) we get \( 3/8a^2 \).

One can say that in a more elegant way, that area \( \text{area}(MM_1M_2NN_1N_2KK_1K_2LL_1L_2 \setminus M NK L) + \frac{1}{2}(def_{(0,1)}(B)^2 + def_{(1,1)}(B)^2) \geq 1/2(a^2 - xy_2 + y_1x_1) \). Due to our conditions, the minimum is attained when \( x_1 = x_2 = 0 \) and \( x_2 = a - b, y_2 = b \). \( \square \)
Lemma 5.7. For a convex $m$-thick $B$ we have $\text{area}(B) + \frac{1}{2} \sum_{a \in \mathbb{Z}} \text{def}_a(B)^2 \geq \frac{3}{8} m^2$

Using previous Lemma $(m-a)^2/2 + a(m-a) + \frac{3}{8}a^2 \geq \frac{3}{8}m^2$ and equality is attained if $a = m$. □

Remark 5.8. As a side effect, for a special case $a = m$ we also proved

Theorem (11), based on [2], p.716, formula $II_3$, p.715 formula $I)$. Let $B \subset \mathbb{Z}^2$ be a convex polygon and $\omega(B) = a$. Therefore area$(B) \geq \frac{3}{8}a^2$.

Furthermore, we found extremal cases and exact bounds $\text{area}(B) \geq \frac{3}{2}k^2$ if $\omega(B) = 2k$ and $\text{area}(B) \geq \frac{1}{2}(3k^2 + 3k + 1)$ if $\omega(B) = 2k + 1$.

6. Tropical points of multiplicity $m$

“The forces of our minds are clumsy forces, and crush the truth a little in taking hold of it.”

H. G. Wells

Let us furnish the last section with an example that shows that the question “what are the conditions that several singular points impose on the equation of a curve?” is not simple at all.

Example 6.1. For an even $m = 2k$, examine the polygon $T$ of minimal area with minimal lattice width $\omega(T) = m$. This is the triangle with vertices $(0,2k), (k,0), (2k,k)$. The triangle $T$ comes as the support set of the polynomial $(y^2 + x - 3xy + x^2y)^k = 0$ which defines a curve $C$ passing through $(1,1)$ with multiplicity $m$. The area of $T$ is $\frac{8}{3}m^2$ which shows that the estimate in Exertion Theorem is sharp.

Suppose we consider curves over $\mathbb{C}$, then the equations $\frac{\partial y + r}{\partial x} F(x,y) = 0$, $q + r < k$ on the coefficients of a polynomial $F = \sum_{(i,j) \in T} a_{ij} x^i y^j$ give us the set of necessary and sufficient conditions such that $\mu_{(1,1)}(C) = m$. Note that among them there are at least

$$\frac{2k(2k+1) - 3k^2 - 3k - 2}{2} = \frac{k^2 - k - 2}{2}$$

linearly dependent ones. Here $\frac{2k(2k+1)}{2}$ is the number of equations and $\frac{3k^2+3k+2}{2}$ is the number of variables, i.e. the number of integer points inside $T$. To see one more phenomenon we consider the set

$$\mathcal{A} = \text{ConvHull}((0,0), (3,1), (6,3), (6,4), (3,6), (1,3))$$

which is above-mentioned triangle (with $k = 3$) with three additional points $(1,3), (3,1), (6,4)$. Nevertheless, the only curve $C$ with support in $\mathcal{A}$ and $\mu_{(1,1)}(C) = 6$ is given by the equation $(y^2 + x - 3xy + x^2y)^3 = 0$. Hence adding three new monomials $a_{13} x^3 y^3 + a_{31} x^3 y + a_{64} x^6 y^4$ does not add new degrees of freedom and $a_{13} = a_{31} = a_{64}$ are always 0! Still, one can argue that this curve is reducible.

Example 6.2. Consider the curve $C'$ given by the equation $(x^2 y + xy^2 - 3xy + x^2 y)^8 + xy^4(x-1)^8 = 0$. It is irreducible, $\mu_{(1,1)}(C') = 8$ and the number of integer points in the Newton polygon of $C'$ is 35 which is lesser than the number of linear condition, namely, 36.

Let us consider the following definition, given in the introduction:

Definition 6.3. A point $p$ on a tropical hypersurface $H$ is called of multiplicity $m$ for $H$ if there exists a hypersurface $H' \subset (\mathbb{K}^*)^n$ and a point $p' \in H'$ of multiplicity $m$ such that $\text{Val}(H') = H, \text{Val}(p') = p$.

One of the most beautiful advantages of tropical geometry is that if one wants to solve a problem in classic geometry, for example, counting curves [26, 1], estimating the dimension of secant varieties [8].
or constructing an example [4] [15], one can *tropicalize* the problem and then, in pure tropical-combinatorial terms, solve it. Thus, we are looking for an *intrinsic* definition of a tropical point of multiplicity $m$ on a tropical curve, which means: for a given tropical curve and a point on it, determine whether this is point of multiplicity $m$ for this curve, in the sense of the above definition.

So, we present new witnesses of the existence of a tropical singular point of high multiplicity, which extends the definition, given in [25](Mikhalkin).

We say that edges $E_1, E_2 \in \mathcal{Ep}(P)$ are *complimentary* if they have the same direction, therefore $d(E_1)$ and $d(E_2)$ lie on a line through $P$, and we demand that $d(E_1) \cup d(E_2)$ contains points on the line lying on the both sides on $P$.

**Definition 6.4.** For a point $P \in \text{Val}(C)$, we say that $\mu_A(C) = m$ if a) The *Witness Condition* is satisfied: for any two complimentary edges $E_1, E_2$ with lengths $m_1, m_2$, the integer distance between them is at least $m - \max(m_1, m_2)$; and b) if $P$ is not a vertex of $\text{Val}(C)$ then The *Main Technical Theorem* is satisfied and if $P$ is a vertex, then The *Exertion Theorem* is satisfied.

It follows from Lemma 3.8 that Definition 6.3 implies Definition 6.4.

**Example 6.5.** The edge $E$ with the tropicalization of singular point is complimentary to itself, therefore $0 \geq m - \text{length}(E)$, therefore the length of $E$ is at least $m$.

So one can take Definition 6.4 as a new definition of a tropical point of multiplicity $m$, extending the old one:

**Definition 6.6.** ([25]) One can say that a point $p$ on a tropical curve $C$ is of multiplicity $m$ if the local stable intersection $L_{\text{stable}} C|_p$ of any tropical line $L$ through $p$ with the curve $C$ is at least $m$.

If there is no long edge $\bigcup_{i=1}^m E_i$, where $n > 1$ through a point $p$ of multiplicity $m$, then the old definition and the new one coincide: if $n > 1$ then Definition 6.6 is equivalent to Lemma 3.7 and Definition 6.4 provides new information about the sum of the areas of some faces.

**Remark 6.7.** Nevertheless, the reader can add more necessary conditions. For example, for a long edge $A_1 A_2 \ldots A_n$ passing through $A$, there should exist numbers $\text{val}_i$ for each edge $A_i A_{i+1}$ such that Lemma 3.9 is also satisfied. This can be done in pure tropical terms, but the resulting formula is too far from to be nice.

We proved that if a point $P$ is the tropicalization of a singular point of multiplicity $m$, then $P$ satisfies the definition above, so, we wrote down only necessary conditions. Nevertheless an ambiguity remains: for a singular point on an edge $E$, all points on $E$ are singular by the definition above. So, it would be more appropriate to say that an edge containing a point $A$ may contain a singular point if $\text{Inf}(P) \geq \frac{1}{2} m^2$. Nevertheless, it is possible to determine via tropical modifications where it is a singular point if both $x_0, x_{n+1}$ exist and the distance $x_{n+1} - x_0$ is equal to $m$.

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