EXPONENTIAL FIELDS AND CONWAY’S OMEGA-MAP

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Abstract. Inspired by Conway’s surreal numbers, we study real closed fields whose value group is isomorphic to the additive reduct of the field. We call such fields omega-fields and we prove that any omega-field of bounded Hahn series with real coefficients admits an exponential function making it into a model of the theory of the real exponential field. We also consider relative versions with more general coefficient fields.

1. Introduction

We study real closed valued fields $K$, with a convex valuation ring $O(1) \subseteq K$ satisfying the property that the value group $v(K^\times)$ is isomorphic to the additive reduct $(K,+,<)$ of the field, where $v$ is the valuation induced by $O(1)$. We call omega-field a field with this property. The name is motivated by the fact that any omega-field admits a map akin to Conway’s omega-map $x \mapsto \omega^x$ on the field of surreal numbers $\textbf{No}$ [3] or its fragments $\textbf{No}(\lambda)$ studied in [6], where $\lambda$ is an $\epsilon$-number. We need to recall that any real closed field $K$ admits a section of the valuation, hence it has a multiplicative subgroup $G \subseteq K^>0$, called a group of monomials, given by the image of the section. Since $G$ is a multiplicative copy of $v(K^\times)$, we have that $K$ is an omega-field if and only if it admits an isomorphism

$$\omega : (K,+,0,<) \cong (G,\cdot,1,<),$$

and we shall call omega-map any such isomorphism. The prototypical example is Conway’s omega-map on the surreal numbers, and in analogy with the surreal case, we use the exponential notation $\omega^x$ to denote the image of $x$ under $\omega$.

Here we explore the relation between omega-fields and exponential fields, where an exponential field is a real closed field $K$ admitting an exponential map, that is an isomorphism $\exp : (K,+,0,<) \cong (K^>0,\cdot,1,<)$. We shall freely write $e^x$ rather than $\exp(x)$ when convenient. Note that $\omega^x$ should not be read as $e^{\omega \log(x)}$ (the easiest way to see why is that the map $x \mapsto \omega^x$, if there is such an omega-map, is not continuous in the order topology of $K$). While in general there are no containments between the class of fields admitting an omega-map and that of fields admitting an exponential map, a non-trivial inclusion of the former in the latter can be obtained by restricting the analysis to $\kappa$-bounded Hahn fields, as discussed below.
In general, any real closed valued field $K$ with monomials $G$ is isomorphic to a truncation closed subfield (see Definition 2.8 (1)) of the Hahn field $k((G))$ [13], where $k \cong O(1)/o(1)$ is the residue field and we write $o(1)$ for the maximal ideal of $O(1)$. For the sake of simplicity in this introduction we focus on the typical case $k = \mathbb{R}$, but our results hold more generally assuming that the residue field $k$ is a model of $T_{an,exp}$, the theory of the real exponential field $\mathbb{R}_{exp}$ with all restricted analytic functions [5]. The full Hahn field $\mathbb{R}((G))$ is always naturally a model of the theory of restricted analytic functions $T_{an}$ [5], but it never admits an exponential function if $G \neq 1$ [10]. However, for every regular uncountable cardinal $\kappa$, there is a group $G$ such that the $\kappa$-bounded Hahn field $\mathbb{R}((G))_\kappa$ does admit an exponential function [12]. We thus restrict our analysis to fields of the form $K = \mathbb{R}((G))_\kappa$ (without assuming a priori that they admit an exponential map). Our first result is the following. The case $G = \text{No}(\kappa)$ with $\kappa$ regular uncountable is in [6].

**Theorem (3.8).** Every omega-field of the form $\mathbb{R}((G))_\kappa$ admits an exponential function making it into a model of $T_{an,exp}$.

Our work was partly motivated by the desire to understand the connections between the surreal numbers, with its various subfields studied in [1, 2], and the exponential fields of the form $\mathbb{R}((G))_\kappa$ constructed by S. Kuhlmann and S. Shelah in [12]. We shall prove that the latter are not always omega-fields (Theorem 4.5), but they are omega-fields if and only if $G$ is order-isomorphic to $G^{>1}$ (Theorem 4.1); in this case, given a chain isomorphism $\psi : G \cong G^{>1}$, there is an omega-map satisfying $\omega^g = e^{\psi(g)}$ for all $g \in G$.

Let us now discuss Theorem 3.8 in more detail. We show that given $K = \mathbb{R}((G))_\kappa$ and an omega-map $\omega : K \cong G$, we can construct an exponential function directly starting from $\omega$ and an auxiliary chain isomorphism

$$h : (K, <) \cong (K^{>0}, <),$$

where by chain we mean linearly ordered set. Any choice of $h$ yields an exponential field (Theorem 3.4) and at least one choice of $h$ will yield a model of $T_{an,exp}$ (Theorem 3.8). Varying $h$ we can thus produce a variety of exponential fields; some of them are models of $T_{exp}$, while all the others are not even $o$-minimal (Theorem 3.10), depending on the growth properties of $h$ (Definition 2.11).

To define the exponential function, it is more convenient to first define a logarithm $\log : K^{>0} \to K$ and let $\exp$ be the compositional inverse $\log$. To this aim we start by putting

$$\log(\omega^{\varepsilon}) = \omega^{h(x)}$$

for $x \in K$ and $\log(1 + \varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n!}$ for $\varepsilon \in o(1)$. Note that such infinite sums make sense in the $\kappa$-bounded Hahn field $\mathbb{R}((G))_\kappa$.

The extension of $\log$ to the whole of $K^{>0}$ is then carried out guided by the principle that $log$ takes products into sums and $\omega$ takes sums into products. We simply extend this to infinite sums. More precisely, $\log$ is determined by $\log(\sum_{i<\alpha} \omega^{\gamma_i} r_i) = \sum_{i<\alpha} \omega^{h(\gamma_i)} r_i$, and $\log(\sum_{i<\alpha} \omega^{\gamma_i} r_i + \varepsilon) = \log(\omega^{\gamma_i}) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n!}$, where $\varepsilon \in o(1)$ and $r \in \mathbb{R}$. Another way to express the spirit of the construction is that we first define $\log$ on the representatives of the multiplicative archimedean classes $\omega^x$, then we extend it to the representatives of the additive archimedean classes $\omega^x$, and finally to the whole of $K$. It is not difficult to show that this construction always yields an exponential field. We now need to show that there is at least one function $h$ such that the exponential field $K$ arising from $\omega$ and $h$ as above is a
model of $T_{\exp}$. A necessary condition is that the exponential map grows faster than any polynomial, or equivalently, that its inverse log grows slower than $x^{1/n}$ for all positive $n \in \mathbb{N}$. This translates into the condition $h(x) < r \cdot \omega^x$ for every $x \in K$ and $r \in \mathbb{R}_{>0}$. We shall abbreviate the above with $h(x) \sim \omega^x$.

Since $\omega^x$ is discontinuous (its values are the representatives of the archimedean classes), and $h$ is continuous in the order topology of $K$ (being a chain isomorphism from $K$ to $\mathbb{R}_{>0}$), the existence of such an $h$ is not immediate. In the case of Gonshor’s $h$ on the surreal numbers [7], the condition $h(x) \sim \omega^x$ is forced by the inductive definition of $h$. However, this cannot be generalized to our more general setting where similar inductive definitions make no sense, and we use instead a bootstrapping procedure (Lemma 3.6). Granted this, the final exp on $K$ is easily seen to yield a model of $T_{\exp}$ using [15, 5] (Theorem 3.8).

All the logarithms considered in this paper are **analytic** (Definition 2.10): for $\varepsilon \in o(1)$, the function $x \mapsto \log(1 + x)$ is given by the familiar Taylor expansion $\log(1 + \varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$, whereas for $g \in G$, $\log(g)$ is a purely infinite element of $\mathbb{R}((G))$, and for $r \in \mathbb{R}$, $\log(r)$ is the usual real logarithm.

Theorems 3.4 and 3.8 produce analytic logarithms satisfying two additional restrictions: $\log(\omega^x) \in G$ for all $x \in K$, and $\log$ brings “infinite products” to “infinite sums”. It turns out, however, that all analytic logarithms arise in this way, up to changing the omega-map $\omega : K \cong G$. More precisely, we have the following classification result.

**Theorem** (Corollary 4.2). Every analytic logarithm on an omega-field of the form $K = \mathbb{R}((G))$, arising from some omega-map $x \mapsto \omega^x$ and some chain isomorphism $h : K \cong \mathbb{R}_{>0}$.

The surreal numbers fit into the above picture if we allow $\kappa$ to be the proper class of all ordinals and $G$ to be the image of Conway’s omega-map $x \mapsto \omega^x$. Gonshor’s exponentiation is induced by the omega-map and Gonshor’s function $h$ [7]; by the above results, any other analytic logarithm on $\mathbb{N}$ arises in this way, possibly after replacing Conway’s omega-map with another isomorphism from $\mathbb{N}$ to its group of monomials and Gonshor’s $h$ with another chain isomorphism.

## 2. Preliminaries

### 2.1. Valuations.

Let $K$ be an ordered field (possibly with additional structure) and let $O(1) \subseteq K$ be a convex subring. Then $O(1)$ is the valuation ring of a valuation $v$ and we denote by $o(1)$ the unique maximal ideal of $O(1)$. If $K$ is real closed, it has a subfield $k \subseteq K$ isomorphic to the residue field $O(1)/o(1)$ of the valuation, namely we can write $O(1) = k + o(1)$. We shall always assume in the sequel that $K$ is real closed and $O(1), o(1), k$ are as above.

**Definition 2.1.** For $x, y \in K$ we define:
- $x \leq y$ if $|x| \leq |y|$ for some $c \in O(1)$ (domination);
- $x \asymp y$ if $x \leq y$ and $y \leq x$ (comparability);
- $x \prec y$ if $x \leq y$ and $x \neq y$ (strict domination);
- $x \sim y$ if $x - y \prec x$ (is asymptotic to $y$).

With these notations we have $O(1) = \{ x : x \leq 1 \}$ and $o(1) = \{ x : x < 1 \}$.

**Definition 2.2.** A multiplicative subgroup $G$ of $K^{<0}$ is a group of **monomials** if it consists in a family of representatives for each $\asymp$-class. In other words a group
of monomials is an embedded copy of the value group. It is well known that any real closed field admits a group of monomials.

**Remark 2.3.** For $x, y \in K$ we have:

- $x \prec y$ if and only if $c|x| < |y|$ for all $c \in O(1)$ (or equivalently for all $c \in k$);
- $x \asymp y$ if and only if $x = cy(1 + \varepsilon)$ for some $c \in k^\times$ and $\varepsilon \in o(1)$;
- $x \sim y$ if and only if $x = y(1 + \varepsilon)$ for some $\varepsilon \in o(1)$.

- If $x \neq 0$ there are unique $r \in k^\times$, $g \in G$, $\varepsilon \in o(1)$ such that $x = gr(1 + \varepsilon)$.

2.2. **Hahn groups.** By a chain we mean a linearly ordered set. We describe a well known procedure to build an ordered group starting from a chain.

**Definition 2.4.** Given a chain $\Gamma$ and an ordered abelian group $(C, +, \prec)$, the $\Gamma$-product of $C$ is the abelian group of all functions $f : \Gamma \to C$ with reverse well-ordered support $\{ \gamma \in \Gamma : f(\gamma) \neq 0 \}$ and pointwise addition, ordered by declaring $f > 0$ if $f(\gamma) > 0$, where $\gamma$ is the biggest element in the support.

If we write $G$ in additive notation, a typical element of $G$ can be written in the form $\sum_{\gamma \in \Gamma} \gamma r_\gamma$, representing the function sending $\gamma \in \Gamma$ to $r_\gamma \in C$, while $G$ itself is denoted $\sum_{\gamma \in \Gamma} C$. We prefer however to use a multiplicative notation and write $G$ as $\prod_{\gamma \in \Gamma} C$ and a typical element of $G$ as $\prod_{\gamma \in \Gamma} \gamma r_\gamma$. In this notation the multiplication is given by

$$\left( \prod_{\gamma \in \Gamma} t^{\gamma r_\gamma} \right) \left( \prod_{\gamma \in \Gamma} t^{\gamma' r'_\gamma} \right) = \prod_{\gamma \in \Gamma} t^{\gamma(r_\gamma + r'_\gamma)}$$

Since the supports are reverse well-ordered, we can fix a decreasing enumeration $(\gamma_i : i < \alpha)$ of the support, where $\alpha$ is an ordinal, and write an element of $\prod t^{FC}$ in the form

$$f = \prod_{i < \alpha} t^{\gamma_i r_i} \in \prod t^{FC}.$$ 

According to our conventions, $f > 1 \iff r_0 > 0$ and $t^\gamma > t^\beta \iff \gamma > \beta$.

If $\Gamma$ has only one element, we may identify $\prod t^{FC}$ with a multiplicative copy $t^C$ of $(C, +, \prec)$.

When $C = (\mathbb{R}, +, \prec)$, we obtain the **Hahn group** over $\Gamma$, which can be characterized as a maximal ordered group with a set of archimedean classes of the same order type as $\Gamma$ [8]. Recall that two positive elements are in the same archimedean class if each of them is bounded, in absolute value, by an integer multiple of the other.

**Notation 2.5.** Let $\kappa$ be a regular cardinal. If in the definition of the $\Gamma$-product we only allow supports of reverse order type $< \kappa$, we obtain the $\kappa$-bounded version

$$\left( \prod t^{FC} \right)_\kappa \subseteq \prod t^{FC}.$$

We shall also consider the case when $\Gamma$ is a proper class and $\kappa = \text{On}$, in which case $(\prod t^{FC})_{\text{On}}$ is the ordered group of all functions $f : \Gamma \to C$ whose support is a reverse well ordered set (rather than a reverse well ordered class).

\[1\] Other authors prefer to use well-ordered supports, but one can pass from one version to the other reversing the order of $\Gamma$. 
2.3. **Hahn fields.** Given a field $k$ and a multiplicative ordered abelian group $G$, let $k((G))$ denote the Hahn field with coefficients in $k$ and monomials in $G$. The underlying additive group of $k((G))$ coincides with the $G$-product of $k$: its elements are functions $f : G \to k$ with reverse well-ordered supports, which we write either in the form $f = \sum_{g \in G} g r_g$, where $r_g = f(g)$, or in in the form

$$f = \sum_{i<\alpha} g_i r_i$$

where $\alpha$ is an ordinal, $(g_i)_{i<\alpha}$ is a decreasing enumeration of the support, and $r_i = f(g_i) \in k^*$. Addition is defined componentwise and multiplication is given by the usual Cauchy product. We order $k((G))$ according to the sign of the leading coefficient, namely $f > 0 \iff r_0 > 0$.

**Remark 2.6.** It can be proved that if $k$ is real closed and $G$ is divisible, then $k((G))$ is real closed [9]. Moreover, $k((G))$ is **spherically complete**: any decreasing intersection of valuation balls has a non-empty intersection.

**Notation 2.7.** Inside $k((G))$, we let $O(1)$ be the valuation ring of all the elements $x$ with $|x| \leq r$ for some $r \in k$, and $o(1)$ be the corresponding maximal ideal. We then have $O(1) = k + o(1)$. With respect to this valuation ring, $k$ is a copy of the residue field and $G$ is a group of monomials. We shall use similar notations for any subfield $\mathbb{K} \subseteq k((G))$ containing $k$ and $G$.

2.4. **Restricted analytic functions.** A family $(f_i)_{i \in I}$ of elements of $k((G))$ is **summable** if the union of the supports of the elements $f_i$ is reverse well-ordered and, for each $g \in G$, there are only finitely many $i \in I$ such that $g$ is in the support of $f_i$. In this case $\sum_{i \in I} f_i$ is defined as the element $f = \sum g r_g$ of $k((G))$ whose coefficients are given by $r_g = \sum_{i \in I} r_{g, i} \in k$ where $r_{g, i}$ is the coefficient of $g$ in $f_i$. This makes sense since, given $g \in G$, only finitely many $r_{g, i}$ are non-zero.

By Neumann’s lemma [14] for any power series $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ with coefficients in $k$ and for any $\varepsilon \prec 1$ in $k((G))$, the family $(a_n \varepsilon^n)_{n \in \mathbb{N}}$ is summable, so we can evaluate $P(x)$ at $\varepsilon$ obtaining an element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n$ of $k((G))$. Similar considerations apply to power series in several variables.

**Definition 2.8.** Let $\mathbb{K} \subseteq k((G))$ be a subfield. We say that $\mathbb{K}$ is an **analytic subfield** if

1. $\mathbb{K}$ is truncation closed: if $\sum_{i<\alpha} g_i r_i$ belongs to $\mathbb{K}$, then $\sum_{i<\beta} g_i r_i$ belongs to $\mathbb{K}$ for every $\beta \leq \alpha$;
2. $\mathbb{K}$ contains $k$ and $G$;
3. If $P(x) = \sum_{n \in \mathbb{N}} a_n x^n$ is a power series with coefficients in $k$ and $\varepsilon \prec 1$ is in $\mathbb{K}$, then the element $P(\varepsilon) = \sum_{n \in \mathbb{N}} a_n \varepsilon^n \in k((G))$ lies in the subfield $\mathbb{K}$.

Similarly for power series in several variables.

We recall that $T_{an}$ is the theory of the real field with all analytic functions restricted to the poly-intervals $[-1, 1]^n \subseteq \mathbb{K}^n$ [5]. (By rescaling, we can equivalently use any other closed poly-interval.)

**Fact 2.9.** We have:

1. The field $\mathbb{R}((G))$ admits a natural interpretation of the analytic functions restricted to the poly-interval $[-1, 1]^n \subseteq \mathbb{K}$, making $\mathbb{K}$ into a model of $T_{an}$.
2. The same holds for any analytic subfield of $\mathbb{R}((G))$, and in particular for $\mathbb{R}((G))_{\kappa}$ for every regular uncountable $\kappa$. 

(3) More generally, if \( k \) is a model of \( T_{an} \), then any analytic subfield \( K \) of \( k((G)) \) is naturally a model of \( T_{an} \).

The proof of (1) is in [5] and is based on a quantifier elimination result in the language of \( T_{an} \). The other points follow by the same argument. We interpret the restricted analytic functions in the analytic subfield \( K \subseteq k((G)) \) as follows. Given a real analytic function \( f \) converging on a neighbourhood of \([-1, 1]^n \cap \mathbb{R}^n \), we need to define \( f(x + \varepsilon) \) where \( x \in [-1, 1]^n \cap \mathbb{R}^n \) and \( \varepsilon \in o(1)^n \subseteq K^n \). We do this by using the Taylor expansion \( f(x + \varepsilon) = \sum_i \frac{D^i f(x)}{i!} \varepsilon^i \) where \( i \) is a multi-index in \( \mathbb{N}^n \). Here \( D^i f(x) \in k \) (using the fact that \( k \) is a model of \( T_{an} \)) and the infinite sum is in the sense of the Hahn field \( k((G)) \).

### 2.5. Exponential fields.

A prelogarithm on a real closed field \( K \) is a morphism from \( (K^{>0}, \cdot, 1, <) \) to \( (K, +, 0, <) \) and a logarithm is a surjective prelogarithm. An exponential map is the compositional inverse of a logarithm, that is an isomorphism from \( (K, +, 0, <) \) to \( (K^{>0}, \cdot, 1, <) \). We say that \( K \) is an exponential field if it has an exponential map. Given a logarithm \( \log \), we write \( \exp \) for the composition \( \exp = \log^{-1} \). The multiplicative group \( K \) is a model of \( \prelog \) precisely when \( \exp \) is an exponential map. Namely any element \( x \) can be uniquely written in the form \( x = \sum_{i \in \alpha} g_i r_i \) with \( g_i \in G^{>1} \) for all \( i \).

**Condition (1)** is the compositional inverse of a logarithm, that is an isomorphism from \( (K, +, 0, <) \) to \( (K^{>0}, \cdot, 1, <) \). We say that \( K \) is an exponential field if it has an exponential map. Given a logarithm \( \log \), we write \( \exp \) for the corresponding exponential map and we write \( e^x \) instead of \( \exp(x) \) when convenient. Now assume \( k \) has a logarithm and consider the Hahn field \( k((G)) \). It turns out that if \( G \neq 1 \), \( k((G)) \) never admits a logarithm extending that on \( k \) [10]. On the other hand if \( \kappa \) is a regular uncountable cardinal, then for suitable choices of \( G \), the logarithm on \( k \) can be extended to \( k((G))^{\kappa} \), and when \( k = \mathbb{R} \) this can be done in such a way that \( k((G))^{\kappa} \) is a model of \( T_{exp} [12] \).

**Definition 2.10.** Let \( k \) be an exponential field and let \( K \) be an analytic subfield of \( k((G)) \), for instance \( K = k((G))^{\kappa} \) with \( \kappa \) regular uncountable. An analytic logarithm on \( K \) is a logarithm \( \log : K^{>0} \to K \) with the following properties:

1. \( \log : K^{>0} \to K \) extends the given logarithm on \( k \).
2. \( \log(1 + \varepsilon) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\varepsilon^i}{i} \) for all \( \varepsilon < 1 \) in \( K \) (the assumption \( \varepsilon < 1 \) ensures the summability).
3. \( \log(G) = K^\ast \), where \( K^\ast := k((G^{>1})) \cap K \) is the group of purely infinite elements, namely the series of the form \( \sum_{i \in \alpha} g_i r_i \) with \( g_i \in G^{>1} \) for all \( i \).

Conditions (1) and (2) are rather natural, and ensure that the restrictions of \( \log(1 + x) \) to small finite intervals agree with the natural \( T_{an} \)-interpretations of Fact 2.9. A motivation for (3) is the following. The multiplicative group \( K^{>0} \) admits a direct sum decomposition

\[ K^{>0} = G k^{>0} (1 + o(1)), \]

namely any element \( x \) of \( K^{>0} \) can be uniquely written in the form \( x = gr(1 + \varepsilon) \) where \( r \in k^{>0} \), \( g \in G \) and \( \varepsilon \in o(1) \). Applying \( \log \) to both sides of the above equation, we get (by (1) and (2)) a direct sum decomposition

\[ K = \log(G) \oplus k \oplus o(1) \]

of the additive group \( (K, +) \). Indeed by (1) we have \( \log(k^{>0}) = k \) and \( \log(K^{>0}) = K \), while (2) implies that the logarithm maps \( 1 + o(1) \) bijectively to \( o(1) \) with inverse given by \( \exp(\varepsilon) = \sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{n!} \). We have thus proved that \( \log(G) \) is a direct complement of \( O(1) = k + o(1) \). Although there are several choices for such a complement, the most natural one is \( \log(G) = K^\ast \), as required in point (3), since it is the unique one closed under truncations.
2.6. Growth axiom and models of $T_{\exp}$. Ressayre proved in [15] that an exponential field is a model of $T_{\exp}$ if and only if it satisfies the elementary properties of the real exponential restricted to $[0, 1]$ and satisfies the growth axiom scheme $x \geq n^2 \rightarrow \exp(x) > x^n$ for all $n \in \mathbb{N}$.

**Definition 2.11.** Given an analytic subfield $K \subseteq \mathbb{k}((G))$, we say that an analytic logarithm $\log : \mathbb{K}^{>0} \rightarrow \mathbb{K}$ satisfies the **growth axiom at infinity** if $\log(x) < x^{1/n}$ for all $x > k$ and all positive integers $n$.

**Proposition 2.12.** If $k$ is a model of $T_{\mathit{an}, \exp}$ (for instance $k = \mathbb{R}$) and $k \subseteq \mathbb{k}((G))$ is an analytic subfield with an analytic logarithm satisfying the growth axiom at infinity, then $k$ (with the natural interpretation of the symbols) is a model of $T_{\mathit{an}, \exp}$.

**Proof.** This follows from [15, 5] but we include some details. The inverse exp of an analytic logarithm is easily seen to satisfy $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in o(1)$. Since moreover exp extends the given exponential on $k$, it follows that the restriction of exp to $[-1, 1]$ agrees with the natural $T_{\mathit{an}}$-interpretation of Fact 2.9. This shows that $k$ is a model of $T_{\exp}[[-1, 1]]$, as it is in fact the restriction of a model of $T_{\mathit{an}}$ to a sublanguage. Since the interpretation of exp grows faster than any polynomial (by the growth axiom at infinity plus the fact that $k$ is a model of $T_{\exp}$), we can conclude by the axiomatisations of [15, 5]. □

The above result rests on the quantifier elimination result for $T_{\mathit{an}, \exp}$. We do not know whether it suffices that $k$ is a model of $T_{\exp}$ to obtain that $\mathbb{k}((G))_{\kappa}$ is a model of $T_{\exp}$ (or even $T_{\exp}[0, 1]$).

3. Omega fields

**Definition 3.1.** A real closed field $(\mathbb{K}, \cdot, <)$ with a convex valuation ring $O(1)$ and corresponding group of monomials $G \subseteq \mathbb{K}^{>0}$ is an **omega-field** if $(\mathbb{K}, \cdot, <)$ is isomorphic to $(G, \cdot, <)$ as an ordered group. Given an omega-field $\mathbb{K}$ we shall call **omega-map** any isomorphism of ordered groups

$$\omega : (\mathbb{K}, \cdot, <, \cdot) \cong (G, \cdot, <).$$

Since the group $G$ of monomials is isomorphic to the value group of $\mathbb{K}$, we have that $\mathbb{K}$ is an omega-field if and only if $(\mathbb{K}, \cdot, <)$ is isomorphic to its value group. The definition of omega-map is inspired by Conway’s omega map $\omega^\mathbb{K}$ on the surreal numbers. We recall that the surreals can be presented in the form $\mathbb{No} = \mathbb{R}((\omega^{\mathbb{No}}))_{\mathbb{On}}$, with the image of the omega-map being the group $\omega^{\mathbb{No}}$ of monomials. The subscript $\mathbb{On}$ indicates that we only consider series whose support is a set, rather than a proper class. The surreals should thus be considered as a bounded Hahn field rather than a full Hahn field.

3.1. Construction of omega-fields. In the sequel let $\kappa$ be a regular uncountable cardinal.

**Theorem 3.2.** Given an exponential field $k$, there is a group $G$ such that the field $\mathbb{K} = k((G))_{\kappa}$ admits an omega-map $\omega : \mathbb{K} \rightarrow G$.

When $k = \mathbb{R}$ one can take $G = \mathbb{No}(\kappa)$ as in [6]. In the general case the proof is a variant of a similar construction in [12]. Given a chain $\Gamma$ and an additive ordered group $C$ (in our application $C = (\mathbb{k}, \cdot, <)$), let $H(\Gamma)$ denote, in the following Lemma, the ordered group $(\prod t^{FC})_{\kappa}$.
Lemma 3.3. Fix a chain $\Gamma_0$ and a chain embedding $\eta_0 : \Gamma_0 \to H(\Gamma_0)$ (for instance $\eta_0(\gamma) = \gamma$). Then there is a chain $\Gamma \supseteq \Gamma_0$ and a chain isomorphism $\eta : \Gamma \cong H(\Gamma)$ extending $\eta_0$.

Proof. We consider $H$ as a functor from chains to ordered abelian groups: if $j : \Gamma' \to \Gamma''$ is a chain embedding, we define $H(j) : H(\Gamma') \to H(\Gamma'')$ as the group embedding which sends $\prod_i t^{\gamma_i} r_i$ to $\prod_i t^{j(\gamma_i)} r_i$. We do an inductive construction in $\kappa$-many steps. At a certain stage $\beta < \kappa$ we are given

$$G_\beta = H(\Gamma_\beta)$$

and a chain embedding $\eta_\beta : \Gamma_\beta \to G_\beta$ together with embeddings $j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta$ for $\alpha < \beta$. Let $\Gamma_{\beta+1}$ be a chain isomorphic to $(G_\beta, <)$ (for instance $G_\beta$ itself considered as a chain) and fix a chain isomorphism $f_\beta : G_\beta \to \Gamma_{\beta+1}$. Now let $j_\beta : \Gamma_\beta \to \Gamma_{\beta+1}$ be the composition $f_\beta \circ \eta_\beta$ and let $G_{\beta+1} = H(\Gamma_{\beta+1})$. We can then find a commutative diagram of embeddings

$$
\begin{array}{ccc}
\Gamma_\beta & \xrightarrow{\eta_\beta} & H(\Gamma_\beta) \\
\downarrow{j_\beta} & & \downarrow{f_\beta} \\
\Gamma_{\beta+1} & \xrightarrow{\eta_{\beta+1}} & H(\Gamma_{\beta+1}),
\end{array}
$$

by letting $\eta_{\beta+1} = H(j_\beta) \circ f_\beta^{-1}$. We can now define $j_{\beta,\beta+1} = j_\beta$ and $j_{\alpha,\beta+1} = j_{\beta,\beta+1} \circ j_{\alpha,\beta}$ for $\alpha < \beta$.

We iterate the construction along the ordinals. At a limit stage $\lambda$, let $\Gamma_\lambda = \lim_{\beta < \lambda} \Gamma_\beta$ and let $j_{\beta,\lambda} : \Gamma_\beta \to \Gamma_\lambda$ be the natural embedding for $\beta < \lambda$.

We then define $\eta_\lambda : \Gamma_\lambda \to H(\Gamma_\lambda)$ as the composition

$$\Gamma_\lambda = \lim_{\beta < \lambda} \Gamma_\beta \to \lim_{\beta < \lambda} H(\Gamma_\beta) \to H(\lim_{\beta < \lambda} \Gamma_\beta) = H(\Gamma_\lambda).$$

More explicitly, for each $\gamma \in \Gamma_\lambda$, pick some $\beta < \lambda$ and $\theta \in \Gamma_\beta$ such that $\gamma = j_{\beta,\lambda}(\theta)$, and define $\eta_\lambda(\gamma) \in G_\lambda$ as the image under $H(j_{\beta,\lambda}) : G_\beta \to G_\lambda$ of $\eta_\beta(\theta) \in G_\beta$. Since $\kappa$ is regular, when we arrive at stage $\kappa$ we have an isomorphism

$$\eta_\kappa : \Gamma_\kappa \cong G_\kappa$$

and we can define $\Gamma = \Gamma_\kappa$ and $\eta = \eta_\kappa$. \(\square\)

Proof of Theorem 3.2. By Lemma 3.3, there is a chain $\Gamma$ and a chain isomorphism

$$\eta : \Gamma \cong G = H(\Gamma)$$

(2)

Now let $K = k((G))_\kappa$ and define an omega-map $\omega : K \to G$ by setting

$$\omega^\Sigma_{i < \alpha} g_i r_i = \prod_{i < \alpha} t^{\gamma_i r_i}.$$ 

where $g_i = \eta(\gamma_i)$. In particular $\omega^\eta(\gamma) = t^{\gamma}$ for every $\gamma \in \Gamma$. \(\square\)

3.2. The logarithm. In the sequel let $\kappa$ be a regular uncountable cardinal. Our next goal is to prove the following theorem.

Theorem 3.4. Every omega-field of the form $K = \mathbb{R}((G))_\kappa$ admits an analytic logarithm. More generally, if $k$ is an exponential field, then every omega-field of the form $K = k((G))_\kappa$ admits an analytic logarithm.
Proof. We construct a logarithm depending both on the omega-map and on an auxiliary function $h$. Let $h : K \rightarrow K^{>0}$ be a chain isomorphism (any ordered field admits such a function, for instance $h(x) = (-x + 1)^{-1}$ for $x \leq 0$ and $h(x) = x + 1$ for $x \geq 0$). For $x \in K$, we let

$$\log(\omega^x) = \omega^{h(x)}.$$  

This defines log on the subclass $\omega^x$ of $G$, which we call the class of fundamental monomials. They can be seen as the representatives of the multiplicative archimedean classes.

Next we define $\log(g)$ for an arbitrary $g$ in $G$. Since $G = \omega^K$, we can write $g = \omega^x$ for some $x \in K$. We then write $x = \sum_{i<\alpha} g_ir_i = \sum_{i<\alpha} \omega^{x_i}r_i$ and set $\log(g) = \sum_{i<\alpha} \omega^{h(x_i)}r_i$. Summing up, the definition of $\log|G$ takes the form

$$\log \left( \omega \sum_{i<\alpha} \omega^{x_i}r_i \right) = \sum_{i<\alpha} \omega^{h(x_i)}r_i. \tag{4}$$

The idea is that $\omega \sum_{i<\alpha} g_ir_i$ should be thought as an infinite product $\prod_{i<\alpha} \omega^{g_ir_i}$, and we stipulate that log maps infinite products into infinite sums.

We can now extend log to the whole of $K^{>0}$ as follows. For $x \in K^{>0}$ we write $x = gr(1+\varepsilon)$ with $g \in G$, $r \in K^{>0}$ and $\varepsilon < 1$, and we define

$$\log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}. \tag{5}$$

The infinite sum makes sense because the terms under the summation sign are summable and the sum belongs to $k((G))_\kappa$ (because $\kappa$ is regular and uncountable).

We must verify that with these definitions log is an analytic logarithm (Definition 2.10). It is not difficult to see that log is an increasing morphism from $(K^{>0},\cdot,1,\varepsilon)$ to $(K_+,\cdot,0,\varepsilon)$. To prove the surjectivity let us first observe that $k = \log(k^{>0}) \subseteq \log(K^{>0})$. Moreover, for $\varepsilon < 1$ we have $\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$ with inverse given by $e^\varepsilon = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n}$, and therefore $\log(1+o(1)) = o(1)$. Since $K = K^\uparrow + k + o(1)$, to finish the proof of the surjectivity it suffices to show that $\log(G) = K^\uparrow$. So let $x = \sum_{i<\alpha} g_ir_i \in K^\uparrow$, namely $g_i \in G^{>1}$ for all $i$. We must show that $x$ is in the image of log. Since $h : K \rightarrow K^{>0}$ is surjective and $G = \omega^K$, we have $G^{>1} = \omega^{(K^{>0})} = \omega^{h(K)}$, so we can choose $x_i \in K$ so that $g_i = \omega^{h(x_i)}$ for all $i$. Now by definition $\log(\omega \sum_{i<\alpha} \omega^{x_i}r_i) = \sum_{i<\alpha} \omega^{h(x_i)}r_i = x$ concluding the proof of surjectivity.

In the above theorem we have considered $k((G))_\kappa$, rather than an arbitrary analytic subfield $K$ of $k((G))$, because for the proof to work we need to know that whenever $\sum_{i<\alpha} \omega^{x_i}r_i \in K$, we also have $\sum_{i<\alpha} \omega^{h(x_i)}r_i \in K$.

Definition 3.5. We call $\log_{\omega,h} : K^{>0} \rightarrow K$ the analytic logarithm induced by the omega-map $\omega : K \rightarrow G$ and the chain isomorphism $h : K \rightarrow K^{>0}$ as given by (4)-(5) in the proof of Theorem 3.4, and we call $\exp_{\omega,h}$ its compositional inverse.

3.3. Getting a logarithm satisfying the growth axiom. The structures constructed so far are exponential fields, but not necessarily models of $T_{exp}$. In this section we show how to get models of $T_{exp}$. We need the following lemma to take care of the growth axiom at infinity.
Lemma 3.6. Let $K = k((G))_{\alpha}$ be equipped with an omega-map $\omega : K \cong G$. Then there exists a chain isomorphism $h : K \to K^{>0}$ such that $h(x) \prec \omega^x$ for all $x \in K$.

Proof. The idea is a bootstrapping procedure. Given an $h$ we produce a log and an exp, and given the exp we produce a new $h$. We then glue together a couple of $h$'s obtained in this way to produce the final $h$.

To begin with, consider the following chain isomorphism $K \to K^{>0}$, definable in any ordered field:

$$
h_0(x) = \begin{cases} x + 1 & \text{for } x \geq 0 \\
\frac{1}{1-x} & \text{for } x < 0,
\end{cases} \quad h_1(x) = \begin{cases} \frac{1}{2}x + 1 & \text{for } x \geq 0 \\
\frac{1}{1-x} & \text{for } x < 0.
\end{cases}
$$

Definition 3.5 yields two logarithmic functions $\log_0 = \log_{\omega,h_0}$ and $\log_1 = \log_{\omega,h_1}$ on $K((G))_{\alpha}$ associated with $h_0$ and $h_1$ (and the given omega-map). Since $h_1(x) \leq h_0(x)$, we have $\log_1(x) \leq \log_0(x)$ for all $x \in K^{>1}$. The corresponding exponential functions $\exp_0, \exp_1$ satisfy the opposite inequality: $\exp_0(x) \leq \exp_1(x)$ for $x > 0$.

We claim that $\exp_0(x) \prec \omega^x$ for $x > k$ and $\exp_1(x) \prec \omega^x$ for $x \preceq -\omega^3$.

Indeed, note that $h_0(x) > x$ for all $x \in K$ and $h_1(x) < x$ for $x > 2$. Taking the compositional inverse we obtain $x > h_0^{-1}(x)$ for all $x \in K$ and $x < h_1^{-1}(x)$ for $x > 2$.

Now let $y \in K^{>k}$, and let $r \omega^x$ be the leading term of $y$ (where $r \in K^{>0}$, $x \in K^{>0}$). Then

$$
\exp_0(y) \prec \exp_0(2r\omega^x) = \omega^{2r\omega_0^{-1}(x)} \prec \omega^{\frac{1}{2}\omega^x} \prec \omega^y,
$$
since $2r\omega^y - y > k$, $y - \frac{r}{2}\omega^y > k$, and $\omega^{h_0^{-1}(x)} \prec \omega^y$.

Similarly, $h_1(x) < x$ for all $x \in K^{>2}$. Let $y \in K^{>\omega^3}$, and let $r \omega^x$ be the leading term of $y$. Then $r \in K^{>0}$, $x \in K^{>2}$ and

$$
\exp_1(y) \succ \exp_1\left(\frac{r}{2}\omega^x\right) = \omega^{\frac{r}{2}\omega_1^{-1}(x)} \succ \omega^{\frac{1}{2}\omega^x} \succ \omega^y.
$$

Letting $z = -y \preceq -\omega^3$, we obtain $\exp_1(z) = \frac{1}{\exp_1(y)} \preceq \frac{1}{\omega^y} = \omega^z$, and the claim is proved.

We can now build the final chain isomorphism $h : K \to K^{>0}$ by taking the functions $\exp_0, \exp_1$ restricted to suitable convex subsets of $K$, and defining $h$ on the complement as an increasing function in such a way that globally $h$ is increasing and bijective. A concrete choice can be the following. Let $c = \exp_1(-\omega^3) > 0$. Define

$$
h(x) = \begin{cases} \exp_0(x) & \text{for } x > k \\
2c + x & \text{for } 0 < x \leq 1 \\
2c + \frac{c}{x} & \text{for } -\omega^3 \leq x \leq 0 \\
\exp_1(x) & \text{for } x < -\omega^3.
\end{cases}
$$

By construction, $h$ is a chain isomorphism $h : K \to K^{>0}$: it is order preserving because $\exp_0, \exp_1$ are themselves chain isomorphisms, and it is surjective since $\exp_0(K^{>k}) = K^{>k}$, $\exp_1(({-\infty},-\omega^3)) = (0,c)$. Moreover, $h(x) \prec \omega^x$ for all $x \in K$, as desired:

- if $x > k$, then $h(x) = \exp_0(x) \prec \omega^x$;
- if $0 < x \leq 1$, then $h(x) = 2c + x \preceq 1 \prec \omega^x$;
- if $-\omega^3 \leq x \leq 0$, then $h(x) = c = \exp_1(-\omega^3) \prec \omega^{-\omega^3} \preceq \omega^x$;
- if $x < -\omega^3$, then $h(x) = \exp_1(x) \prec \omega^x$. 

We next show that an \( h \) as constructed above is sufficient to guarantee the growth axiom at infinity.

**Lemma 3.7.** Let \( \log = \log_{1, h} : \mathbb{K}^0 \to \mathbb{K} \) be as in Definition 3.5. If \( h \) satisfies \( h(x) < \omega^x \) for every \( x \in \mathbb{K} \), then \( \log(y) < y^r \) for all positive \( r \in \mathbb{k} \) and all \( y > k \) (where \( y^r \) is defined as \( e^{r \log(y)} \)).

**Proof.** Assume \( h(x) < \omega^x \) for all \( x \in \mathbb{K} \). This means that \( h(x) < \omega^x r \) for all \( r \in \mathbb{K}^0 \). Let \( y = \omega^{x^r} \). Then \( \log(y) = \log(\omega^{x^r}) = \omega^h(x) < \omega^{x^r} r = y^r \). We have thus proved that \( \log(y) < y^r \) for \( y \) of the form \( \omega^{x^r} \) and \( r \in \mathbb{K}^0 \).

We now prove the inequality for \( y \) of the form \( \omega^x \), where \( x \in \mathbb{K}^0 \). To this aim we write the exponent \( x \) in the form \( \sum_{i<\alpha} \omega^x r_i \) and observe that \( r_0 > 0 \) and \( \log(\omega^x) = \log(\omega^{\sum_{i<\alpha} \omega^x r_i}) = \sum_{i<\alpha} \log(\omega^{\omega^x r_i}) r_i \). By the special case we have \( \log(\omega^{\omega^x r_i}) < \omega^{\omega^x a} \leq \omega^{\omega^r a} \) for every \( i < \alpha \) and \( a \in \mathbb{K}^0 \). Letting \( a = r r_0/2 \) it follows that \( \log(\omega^x) < \omega^{\omega^r a} = (\omega^{\omega^r r_0})^{\frac{r}{r_0}} < (\omega^{ \frac{r}{r_0} })^{\frac{r}{r_0}} = \omega^{xr} \).

For a general \( y > k \), write \( y \) in the form \( \omega^x s(1 + \varepsilon) \) with \( s \in \mathbb{K}^0 \), \( x > 0 \) and \( \varepsilon < 1 \), and observe that \( \log(y) < \log(2s) + \log(\omega^x) < (\omega^x)^{\frac{r}{r_0}} < y^r \) for any \( r \in \mathbb{K}^0 \). \( \square \)

In the case when the residue field \( \mathbb{k} \) is archimedian, the statement in the conclusion of Lemma 3.7 is equivalent to the growth axiom at infinity (Definition 2.11). We are now ready for the main result of this section.

**Theorem 3.8.** Every omega-field of the form \( \mathbb{K} = \mathbb{K}((G))_\kappa \) admits an analytic logarithm making it into a model of \( T_{\text{an,exp}} \). More generally, if \( \mathbb{k} \) is a model of \( T_{\text{an,exp}} \), then every omega-field of the form \( \mathbb{K} = \mathbb{k}((G))_\kappa \) admits an analytic logarithm making it into a model of \( T_{\text{an,exp}} \).

**Proof.** By Proposition 2.12 and Lemma 3.7. \( \square \)

### 3.4. Growth axiom and o-minimality.

We now discuss the connections between the growth axiom and o-minimality (see [4] for the development of the theory of o-minimal structures).

**Lemma 3.9.** Let \( \mathbb{K} \) be an o-minimal exponential field. Note that \( \exp \) must be differentiable and by a linear change of variable, we can assume that \( \exp'(0) = 1 \). Then \( \exp(x) > x^n \) for all positive \( n \in \mathbb{N} \) and all \( x > 0 \).

**Proof.** Given a definable differentiable unary function \( f : \mathbb{K} \to \mathbb{K} \) in an o-minimal expansion of a field, its derivative \( f' \) is definable, and if \( f' \) is always positive, then \( f \) is increasing. It follows that if \( f, g \) are definable differentiable functions satisfying \( f(a) \leq g(a) \) and \( f'(x) < g'(x) \) for all \( x \geq a \), then \( f(x) < g(x) \) for every \( x > a \). Starting with \( 0 < \exp(x) \) and integrating we then inductively obtain that for each positive \( k, n \in \mathbb{N} \) there is a positive \( c \in \mathbb{N} \) such that \( k x^n \leq e^x \) for all \( x > c \). \( \square \)

By the above observation and Ressayre’s axiomatization [15], an exponential field is a model of \( T_{\text{exp}} \) if and only if it satisfies the complete theory of restricted exponentiation and it is o-minimal.

**Theorem 3.10.** Assume \( \mathbb{K} = \mathbb{K}((G))_\kappa \) has an omega-map \( \omega : \mathbb{K} \cong G \). Fix a chain isomorphism \( h : \mathbb{K} \cong \mathbb{K}^0 \) and put on \( \mathbb{K} \) the logarithm induced by \( \omega \) and \( h \) as in Definition 3.5. Then \( \mathbb{K} \) is either a model of \( T_{\text{exp}} \) or it is not even o-minimal.
Proof. We have already seen that if \( h(x) < \omega^x \) for all \( x \in \mathbb{K} \), then \( \mathbb{K} \) is a model of \( T_{\exp} \) (Theorem 3.8). Now suppose that \( h(x) \not< \omega^x \) for some \( x \). Then there is some \( n \in \mathbb{N}^{>0} \) such that \( h(x) \geq \frac{1}{n} \omega^x \). Letting \( y = \omega^\frac{k}{n} \omega^x \), we have \( \log(y) = \frac{1}{n} \log(\omega^x) = \frac{1}{n} \frac{1}{n} \omega^h(x) \geq \frac{1}{n} \omega^h(x) = \frac{1}{n} y \), hence \( y^n \geq e^k \), contradicting o-minimality by Lemma 3.9 (since \( \exp \) extends the real exponential function, we have \( \exp'(0) = 1 \), so the hypothesis of the lemma are satisfied).

4. Other exponential fields of series

4.1. Criterion for the existence of an omega-map. In this section we try to classify all possible analytic logarithms on \( k((G))_\kappa \). We show that in the case of omega-fields every analytic logarithm arises from an omega-map and some \( h \).

**Theorem 4.1.** Assume that \( \mathbb{K} = k((G))_\kappa \) has an analytic logarithm \( \log \). Then:

1. \( \mathbb{K} \) has an omega-map \( \omega : \mathbb{K} \cong G \) if and only if \( G \) is isomorphic to \( G^{>1} \) as a chain;
2. moreover, if \( G \cong G^{>1} \), there is an omega-map and a chain isomorphism \( h : \mathbb{K} \cong \mathbb{K}^{>0} \) such that the logarithm induced by \( \omega \) and \( h \) coincides with the original logarithm.

**Proof.** First note that \( \mathbb{K} \), being an ordered field, is always isomorphic to \( \mathbb{K}^{>0} \) as a chain. If there is an omega-map \( \omega : \mathbb{K} \cong G \), we obtain an induced isomorphism from \( G = \omega^\mathbb{K} \) to \( G^{>1} = \omega^\mathbb{K}^{>0} \).

For the opposite direction, assume that \( G \) is isomorphic to \( G^{>1} \) as a chain and let \( \psi : G \cong G^{>1} \) be a chain isomorphism. Define \( \omega : \mathbb{K} \rightarrow G \) by:

\[
\omega \sum_{i<\alpha} g_i r_i = \sum_{i<\alpha} \psi(g_i) r_i.
\]

In particular we have \( \omega^g = e^{\psi(g)} \). Clearly \( \omega \) is a morphism from \( (\mathbb{K}, +, 0, <) \) to \( (G, \cdot, 1, <) \) and to prove that it is an omega-map it only remains to verify that it is surjective. To this aim recall that \( \log(G) = \mathbb{K}^\uparrow \) (by definition of analytic logarithm), so for the corresponding \( \exp \) we have \( G = \exp(\mathbb{K}^\uparrow) \). Since \( e^{\sum_{i<\alpha} \psi(g_i) r_i} \) is an arbitrary element of \( \exp(\mathbb{K}^\uparrow) \), the surjectivity of \( \omega \) follows. Now since \( \psi : G \cong G^{>1} \) and \( G = \omega^\mathbb{K} \), there is a chain isomorphism \( h : \mathbb{K} \rightarrow \mathbb{K}^{>0} \) such that:

\[
\psi(\omega^x) = \omega^h(x).
\]

Since \( e^{\psi(\omega^x)} = \omega^x \), we obtain \( \omega^x = e^{\omega^h(x)} \) and therefore \( \log(\omega^x) = \omega^h(x) \). It then follows that \( \log \) coincides with the analytic logarithm induced by \( \omega \) and \( h \). \qed

**Corollary 4.2.** Every analytic logarithm on an omega-field of the form \( \mathbb{K} = k((G))_\kappa \) arises from some omega-map and some chain isomorphism \( h : \mathbb{K} \cong \mathbb{K}^{>0} \).

4.2. The iota-map. Our next goal is to show that \( k((G))_\kappa \) may have an analytic logarithm without being an omega-field. This will be proved in the next subsection. Here we recall the following two results from [12] with a sketch of the proofs for the reader’s convenience (considering that the notations are different). We use the same notation \( H(\Gamma) = (\prod t^C)_\kappa \) employed in Lemma 3.3, with \( C = (k, +, <) \).

**Fact 4.3 ([12]).** Let \( k \) be an exponential field. Let \( \Gamma \) be a chain and suppose there is an isomorphism of chains \( \iota : \Gamma \cong H(\Gamma)^{>1} \). Let \( G = H(\Gamma) \) and let \( \mathbb{K} = k((G))_\kappa \). Then:

1. there is an analytic logarithm \( \log : \mathbb{K}^{>0} \rightarrow \mathbb{K} \) such that \( \log(t^\gamma) = \iota(\gamma) \in G \).
(2) if \( k \) is a model of \( T_{an,\exp} \) and \( \iota(\gamma) < t^{\gamma r} \) for each \( r \in k^{>0} \), then \( \log \) satisfies the growth axiom at infinity, thus making \( K \) into a model of \( T_{an,\exp} \).²

Proof. Define \( \log = \log_c \) on \( G \) by

\[
\log((\prod_{i<\alpha} t^{\gamma_i r_i})) = \sum_{i<\alpha} \iota(\gamma_i) r_i \in k((G^{>1}))_\kappa.
\]

Given \( x \in K^{>0} \), write \( x = gr(1+\varepsilon) \) for some \( r \in k^{>0} \), \( g \in G \) and \( \varepsilon \in o(1) \); now define \( \log(x) = \log(g) + \log(r) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n} \), where \( \log(r) \) refers to the given logarithm on \( k \), and observe that since \( \varepsilon < 1 \) and \( \kappa > \omega \) the infinite sum belongs to \( K = k((G))_\kappa \). Clearly \( \log \) is an analytic logarithm and (1) is proved.

The verification of point (2) is as in Theorem 3.4. □

Fact 4.4 ([12]). Fix a chain \( \Gamma_0 \) and a chain embedding \( \iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1} \) (for instance \( \iota_0(\gamma) = t^\gamma \)). Then:

1. there is a chain \( \Gamma \supseteq \Gamma_0 \) and a chain isomorphism \( \iota : \Gamma \cong H(\Gamma)^{>1} \) extending \( \iota_0 \);
2. if \( \iota(\gamma) < t^{\gamma r} \) for every \( \gamma \in \Gamma_0 \) and \( r \in C^{>0} \), then \( \iota(\gamma) < t^{\gamma r} \) for every \( \gamma \in \Gamma \) and \( r \in C^{>0} \).

Proof. The proof of (1) is similar to the proof of Lemma 3.3, the only difference is that we use \( H(\Gamma)^{>1} \) instead of \( H(\Gamma) \). Starting with the initial chain embedding \( \iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1} \) we inductively produce chain embeddings \( \iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1} \) and \( j_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\beta \) for \( \alpha < \beta \). The step from \( \beta \) to \( \beta + 1 \) is based on the following diagram

\[
\begin{array}{c}
\Gamma_\beta \\
\downarrow \iota_\beta \\
\Gamma_{\beta+1} \\
\downarrow j_{\beta+1} \\
H(\Gamma_{\beta+1})^{>1}
\end{array}
\quad
\begin{array}{c}
H(\Gamma_\beta)^{>1} \\
\downarrow f_\beta \\
H(\Gamma_{\beta+1})^{>1}
\end{array}
\quad
\begin{array}{c}
\Gamma_\beta \\
\downarrow j_{\beta} \\
H(\Gamma_{\beta+1})^{>1}
\end{array}
\]

where \( \Gamma_{\beta+1} \) is a chain isomorphic to \( H(\Gamma_{\beta+1})^{>1} \), \( f_\beta \) is a chain isomorphism, and the embeddings \( j_\beta \) and \( \iota_{\beta+1} \) are defined so that the diagram commutes. Limit stages are handled as in Lemma 3.3. Finally we set \( \Gamma = \Gamma_\kappa = \lim_{\beta<\kappa} \Gamma_\beta \) and \( \iota = \iota_\kappa \) and observe that \( \iota : \Gamma \to H(\Gamma)^{>1} \) is a chain isomorphism.

To prove (2), we show by induction on \( \beta < \kappa \) that \( \iota_\beta(\gamma) < t^{\gamma r} \) for every \( \gamma \in \Gamma_\beta \) and \( r \in C^{>0} \), provided this holds for \( \beta = 0 \). Since limit stages are easy, it suffices to prove the induction step from \( \beta \) to \( \beta + 1 \). So let \( \eta \in \Gamma_{\beta+1} \). Then \( \eta = f_\beta(x) \) for some \( x = \prod_i t^{\gamma_i r_i} = (\prod_i t^{\Gamma_\beta C})^{>1} \). The embedding \( \iota_\beta \) sends \( \eta \) to \( \prod_i t^{\iota_\beta(\gamma_i) r_i} \) where \( j_\beta = f_\beta \circ \iota_\beta \) is the embedding of \( \Gamma_\beta \) into \( \Gamma_{\beta+1} \). We must prove that \( \prod_i t^{\iota_\beta(\gamma_i) r_i} < t^{\eta r} \) for every \( r \in C^{>0} \). This is equivalent to saying \( \iota_\beta(\gamma_0) < \eta \), which in turn is equivalent to \( \iota_\beta(\gamma_0) < \prod_i t^{\gamma_i r_i} \). The latter inequality follows from the inductive hypothesis and the proof is complete. □

4.3. A model without an omega-map. We can now show that there are fields of the form \( \mathbb{R}(\langle G \rangle)_\kappa \) which admit an analytic logarithm but not an omega-map.

Theorem 4.5. Given a regular uncountable cardinal \( \kappa \), there is \( G \) such that the field \( K = \mathbb{R}(\langle G \rangle)_\kappa \) has an analytic logarithm making it into a model of \( T_{\exp} \) but \( G \) is not isomorphic to \( G^{>1} \) as a chain (so \( K \) is not an omega-field).

²In the cited paper the authors consider \( k = \mathbb{R} \), but the general case is the same.
Proof. Start with the chain $\Gamma_0 = \omega_1 \times \mathbb{Z}$ ordered lexicographically and the initial embedding $\iota_0 : \Gamma_0 \to \left( \prod_{k} t^{\mathbb{G}_k} \right)^{>1}$ given by $\iota_0((\alpha,n)) = t^{(\alpha,n-1)}$. Define $\Gamma = \lim_{\beta<\kappa} \Gamma_\beta$ and $\iota : \Gamma \cong H(\Gamma)^{>1}$ as in Fact 4.4 and note that $\iota(\gamma) < t^{\gamma r}$ for every $\gamma \in \Gamma$ and $r \in \mathbb{G}^{>1}$ (since this holds for $\iota_0$ and is preserved at later stages). Now take $G = H(\Gamma)$ and put on the field $\mathbb{K} = \mathbb{k}(G)_\kappa$ the log induced by $\iota$ as in Fact 4.3. By the above inequalities the logarithm satisfies the growth axiom at infinity, so $\mathbb{K}$ is a model of $T_{\text{exp}}$. It remains to show that $G \not\cong G^{>1}$ as a chain. Note that the image of $\iota_0 : \Gamma_0 \to H(\Gamma_0)^{>1} = \Gamma_1$ is cofinal and cofinal in $H(\Gamma_0)^{>1}$. It follows that for each $\beta \leq \kappa$, the image of $\iota_\beta : \Gamma_\beta \to H(\Gamma_\beta)^{>1}$ is cofinal and cofinal in $H(\Gamma_\beta)^{>1} = \Gamma_{\beta+1}$. Likewise, by an easy induction, for each $\beta \geq 0$ the image of $\Gamma_0$ in $\Gamma_\beta$ is initial and cofinal. In particular the image of $\Gamma_0$ in the final chain $\Gamma_\kappa = \Gamma \cong H(\Gamma)^{>1}$ is cofinal and coinitial. Since $\Gamma_0$ has cofinality $\omega_1$ and coinitiality $\omega$, it follows that $\Gamma$ and $H(\Gamma)^{>1}$ have cofinality $\omega_1$ and coinitiality $\omega$. Now observe that $1/x$ is an order-reversing bijection from $H(\Gamma)^{<1}$ to $H(\Gamma)^{>1}$, and therefore $H(\Gamma) = H(\Gamma)^{<1} \cup 1 \cup H(\Gamma)^{>1}$ has cofinality and coinitiality both equal to $\omega_1$. We conclude that $G = H(\Gamma)$ cannot be chain isomorphic to $G^{>1}$, because they have different coinitiality. \qed

5. Omega-groups

A group isomorphic to the value group of an omega-field will be called omega-group. It would be interesting to give a characterization of the omega-groups. As a partial result, we characterise those groups $G$ such that $\mathbb{k}((G))_\kappa$ is an omega-field. We also clarify the relation between having an omega-map and having an analytic logarithm.

Proposition 5.1. Let $\mathbb{K}$ be a field of the form $\mathbb{k}((G))_\kappa$. Then:

1. If $\mathbb{K}$ is an omega-field, then $G$ is isomorphic to $\left( \prod_{i} t^{G_i} \right)^{>1}$, where the chain $\Gamma$ is order-isomorphic to (the underlying chain of) $G$ itself;
2. If $\mathbb{K}$ has an analytic logarithm, then $G$ is isomorphic to $\left( \prod_{i} t^{G_i} \right)^{>1}$, where $\Gamma$ is order-isomorphic to $G^{>1}$.

Proof. (1) The elements of $\mathbb{K}$ can be written in the form $\sum_{i<\alpha} g_i r_i$. So the elements of $G$ are of the form $\omega \sum_{i<\alpha} g_i r_i$. This corresponds to the element $\prod_{i<\alpha} t^{g_i r_i} \in \left( \prod_{i=1}^{\mathbb{G}_k} \right)^{>1}$ via an isomorphism.

(2) Since $\log(G) = \mathbb{K}^1$, we have $G = \exp(\mathbb{K}^1)$, and therefore an element $g$ of $G$ can be written in the form $\exp(\sum_{i<\alpha} g_i r_i)$ with $g_i \in G^{>1}$ and $r_i \in \mathbb{k}$. This corresponds to $\prod_{i<\alpha} t^{g_r r_i} \in \left( \prod_{i=1}^{\mathbb{G}_k} \right)^{>1}$ via an isomorphism. \qed

In the following corollary we abstract some of the properties of the groups considered above. We refer to [11] for the definition of the value-set.

Corollary 5.2. Let $\mathbb{K}$ be a field of the form $\mathbb{k}((G))_\kappa$.

1. If $\mathbb{K}$ has an analytic logarithm, then $G$ is a $\mathbb{k}$-module, the value set $\Gamma$ of $G$ is order isomorphic to $G^{>1}$, and all the $\mathbb{k}$-archimedean components of $G$ are isomorphic to the additive group of $\mathbb{k}$.
2. If $\mathbb{K}$ is an omega-field, the same properties hold (as in particular $\mathbb{K}$ has an analytic logarithm) and in addition $G$ is isomorphic to $G^{>1}$ as a chain.
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