Effective field theories for QCD with rooted staggered fermions

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ABSTRACT

Even highly improved variants of lattice QCD with staggered fermions show significant violations of taste symmetry at currently accessible lattice spacings. In addition, the “rooting trick” is used in order to simulate with the correct number of light sea quarks, and this makes the lattice theory nonlocal, even though there is good reason to believe that the continuum limit is in the correct universality class. In order to understand scaling violations, it is thus necessary to extend the construction of the Symanzik effective theory to include rooted staggered fermions. We show how this can be done, starting from a generalization of the renormalization-group approach to rooted staggered fermions recently developed by one of us. We then explain how the chiral effective theory follows from the Symanzik action, and show that it leads to “rooted” staggered chiral perturbation theory as the correct chiral theory for QCD with rooted staggered fermions. We thus establish a direct link between the renormalization-group based arguments for the correctness of the continuum limit and the success of rooted staggered chiral perturbation theory in fitting numerical results obtained with the rooting trick. In order to develop our argument, we need to assume the existence of a standard partially-quenched chiral effective theory for any local partially-quenched theory. Other technical, but standard, assumptions are also required.
I. INTRODUCTION

On a hypercubic lattice in four dimensions, the continuum limit of lattice QCD with staggered fermions \[1\] contains four “tastes” of mass-degenerate quarks per staggered fermion field \[2, 3, 4, 5\]. Hence, if we introduce a separate staggered fermion field for each physical light-quark flavor (up, down, and strange), the continuum limit consists of QCD containing four up, four down, and four strange quarks.

A simple solution to this problem is to adjust for the excessive multiplicity by taking the fourth root of the fermion determinant for each staggered fermion field \[6\]. Heuristically, if the staggered determinant factorizes into four identical determinants in the continuum limit, one for each taste, taking the fourth root corrects for the taste multiplicity. The desired theory, QCD with one up, down and strange quark each is then obtained in the continuum limit. Since the staggered determinant is positive for any real, nonzero bare quark mass \(m\), and the continuum determinant is (formally) positive for positive quark mass, the positive fourth root should be chosen.\(^2\) The continuum quark mass is proportional to \(|m|\), which undergoes only a multiplicative renormalization, because staggered fermions have one exact chiral symmetry.

This procedure, the “fourth-root trick,” raises a number of questions \[9, 10, 11\]. The fourth root of a determinant cannot in general be written as a Grassmann integral with a local action. Therefore, the first question is whether the theory defined by the fourth-root trick is local and unitary.

In Ref. \[12\] we showed that, as might be expected, the fourth-root staggered theory is not local at nonzero lattice spacing \(a\). Continuing correlation functions defined in the Euclidean theory to Minkowski space will lead to violations of unitarity at \(a \neq 0\), on a distance scale set by the lightest particles in the theory, the Goldstone bosons. For examples of this, see Ref. \[13\], as well as Sec. 6 of Ref. \[14\], which we will revisit later in this paper.

The origin of these diseases can be traced back to the taste symmetry-breaking part of the staggered Dirac operator. This taste-breaking part corresponds to a dimension-five irrelevant operator. Thus, in the local, unrooted staggered theory, all taste symmetry-breaking effects are expected to vanish in the continuum limit, where exact \(U(4)\) taste symmetry will be restored for each of the four up, four down, and four strange quarks present in that theory.

The leading power-law scaling of irrelevant operators is characteristic of any local and renormalizable theory, such as in particular the unrooted staggered theory. This brings us to the second question: Does the same scaling persist in the fourth-root theory? Two related considerations make it natural to address this question via a Renormalization-Group (RG) approach. To begin with, the RG framework allows us to define what we mean by the continuum limit. This is done by performing \(n+1\) blocking steps\(^3\) on the original lattice theory, with its fine spacing \(a_f = a\), each time increasing the lattice spacing by a factor of two, to arrive at an RG-blocked theory formulated on a lattice with a coarse spacing \(a_c = 2^{n+1}a_f\). Keeping \(a_c\) fixed and small in physical units, \(a_c \ll \Lambda_{QCD}^{-1}\), while sending \(n \to \infty\) (and thus \(a_f \to 0\)), one obtains a coarse-lattice theory describing the continuum physics. An RG framework is also natural because the restoration of taste symmetry is only expected to occur on distance scales much larger than the original lattice cutoff \(a_f\).

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1 We assume the usual choice of only a single-site bare mass term.

2 For the case of an odd number of quarks with negative quark mass, see Refs. \[4, 5\].

3 See Sec. \[III\] for an explanation of the convention \(a_c/a_f = 2^{n+1}\) \[15\].
blocking removes the short-distance fluctuations while modifying the action of the remaining degrees of freedom by local terms only. When we increase the number of blocking steps \( n \), the blocked theory becomes more taste symmetric, and we eventually recover exact taste symmetry in the continuum limit \( n \to \infty \).

Using this RG framework, it was argued in Ref. [15] that the continuum limit of QCD with rooted staggered fermions is a local theory that belongs to the correct universality class. There are strong arguments that the fourth-root theory, while nonlocal, is nevertheless renormalizable [10, 16, 17], and this is the fundamental reason behind the validity of its continuum limit. The detailed reasoning is based on a number of technical assumptions, all of which are very similar to the assumptions needed to establish the nature of the continuum limit for the unrooted staggered theory. Further analytic and numerical work aimed at confirming the technical assumptions of Ref. [15] would add direct and strong evidence for the validity of the fourth-root trick. For full details, we refer to Ref. [15]; for shorter, more intuitive accounts, we refer to Refs. [10, 11]. We stress that one key element—the anticipated scaling of the taste-breaking effects—has been corroborated by extensive numerical studies [18, 19, 20, 21].

Assuming that the rooted staggered theory has the correct continuum limit, this leaves us with a third question. While the anticipated scaling of taste-breaking effects is observed, these effects are clearly not negligible at present [13, 18, 19, 20, 21]. It is therefore imperative to take lattice artifacts into account in the effective continuum field theories (EFTs) such as the Symanzik effective theory (SET) or chiral perturbation theory (ChPT). The latter provides a central tool for analyzing the numerical data and performing the chiral and continuum extrapolations in the light-quark sector. In the case of rooted staggered fermions, we thus need to construct EFTs that take the discretization effects into account, including those that correspond to the nonlocal behavior of the theory at \( a \neq 0 \). The construction of such EFTs is the subject of this paper.

For the pseudo-scalar Goldstone-boson physics, a candidate EFT already exists; it is provided by staggered ChPT [22] with the replica rule (rSChPT), or “rooted staggered ChPT” [23]. (Extensions to higher order [24], and to heavy-light meson [25] and baryon [26] rSChPT were recently given.) An argument for the validity of rSChPT was presented in Ref. [14], and reviewed in Refs. [10, 11]. The key feature of Ref. [14] is that the argument takes place completely within the context of chiral effective theories, and the replica rule is justified only in that context. Here we will need to introduce a somewhat different version of the replica rule, which will be justified in addition at the level of the fundamental lattice theory, but which will ultimately give the same results in the chiral theory. A detailed comparison of the two approaches will be made in Sec. V C.

The overall goal in the current paper is to extend the standard procedure for the construction of ChPT for a local lattice theory to QCD with rooted staggered fermions. The standard procedure consists of two steps. The SET [27] is constructed first. This can be done order by order in perturbation theory, but it is generally assumed that the SET is valid nonperturbatively as well. We will assume throughout that this includes partially quenched theories [16]. In particular, we will assume that locality suffices, and that unitarity (which may be lost in partially quenched theories) is not necessary. Once the correct form of the SET has been established, its symmetries can be used to construct ChPT. Since the SET organizes the low-energy effective theory as a systematic expansion in the lattice spacing, one automatically obtains the chiral theory as an expansion in the lattice spacing as well.

Establishing that EFTs can be constructed following the usual rules for QCD with rooted
staggered fermions thus constitutes a fundamental step in understanding the effects of rooting at nonvanishing lattice spacing. The main thrust of this paper is the construction of the SET for the rooted theory; obtaining the corresponding ChPT is then straightforward, and we show that it is indeed given by rSChPT. We emphasize that our construction applies to all commonly used versions of staggered fermions: standard (unimproved) staggered [1], Asqtad [28], HYP [29], Fat7bar [30], HISQ [31], etc. The only requirement is that the action have the usual staggered symmetries. The size of the discretization effects is of course different with different versions of staggered fermions, but their form (and appearance at each order in \( a_f \)) is the same.

It is also important to note that the effective theories we ultimately construct are those for the relevant rooted staggered theory on the original (fine) lattice. The RG framework is used only as a tool in the derivation of these effective theories. Nevertheless, it is an indispensable tool: the conclusions of Ref. [15] have to be valid in order for our construction of the EFTs to make any sense. We will assume this to be the case.

The difficulty in constructing EFTs for the rooted theory is the following. Consider for simplicity a staggered theory with a common power, denoted \( n_r \), of the fermion determinant for each staggered flavor in the theory. As long as \( n_r \) is a positive integer the lattice theory is local, and the construction of EFTs proceeds as usual. In order to arrive at the fourth-root theory, however, we must set \( n_r = 1/4 \). Our task is to ensure that a replica continuation may be performed: a well-defined procedure must be devised to reach the value \( n_r = 1/4 \) at the level of an EFT, and the procedure must be consistent with the \( n_r \)-dependence of the underlying lattice theory.

In a diagrammatic EFT calculation, the dependence on the number of (sea) quarks arises in two ways. First, there is explicit dependence arising through loop diagrams. In addition, the coupling constants of the EFT (the Symanzik coefficients in the case of the SET, and the low-energy constants in the case of ChPT) depend in an unknown way on the underlying lattice theory, including in particular on the number of replicas \( n_r \). It is the latter dependence that makes our task nontrivial. In principle, one may envisage two basic obstructions to the replica continuation of the coupling constants in the EFT. Mathematically, a unique analytic continuation off the positive integers (which in the case at hand is where the theory is local) does not exist. Also, it could be that the replica continuation we have in mind will encounter a singularity precisely at the desired point \( n_r = 1/4 \).

The dependence of the underlying lattice theory on the number of replicas \( n_r \) is both perturbative and nonperturbative; this means that proving that no obstacle to the replica continuation is present would be tantamount to solving the theory nonperturbatively. The key observation that makes our task nevertheless tractable is that, after a large number \( n \) of RG blocking steps, the taste-symmetry breaking effects are very small: the unrooted staggered theory with integer \( n_r \) is very close to a \( U(4) \) taste-invariant theory. The rooted theory, with \( n_r = 1/4 \), is then also very close to a local lattice theory, for which the standard construction of EFTs is valid. Indeed, the “re-weighted” taste-invariant theories introduced in Ref. [15] are local whenever \( n_r \) is a multiple of 1/4. The proximity of these local theories makes it possible to construct the SET and, later, ChPT, for the rooted theory.

We will reach the SET for the rooted theory starting from the SET for the corresponding re-weighted, taste-invariant theory. The flavors of the taste-invariant theory will always

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4 The discussion generalizes easily to the isospin limit \( m_u = m_d = m_\ell \), where one takes the square root of a single staggered flavor with (bare) mass \( m_\ell \).
be kept in one-to-one correspondence with those of the continuum-limit theory. In the taste-invariant theory the dependence of the Symanzik coefficients on the physical quarks is nonperturbative, and unknown, as usual. This does not pose any difficulty, because the number of physical flavors is never varied.

During the intermediate steps of the derivation the parameter \( n_r \) will take on a related, but different technical meaning. The precise definitions will be given and explained in Sec. III below. As already mentioned above, we first approximate the staggered theory by a local, taste-invariant theory that belongs to the correct universality class. The (rooted) staggered theory will then be reached from the taste-invariant theory by “turning on” smoothly the taste-breaking effects. The dependence on \( n_r \) of the lattice theory will come only from the taste-breaking effects, which are nonlocal (for noninteger \( n_r \)) but small. The difference between corresponding taste-invariant and staggered theories is of order the fine lattice spacing \( a_f \) of the original (unblocked) lattice. This will allow us to show that all the lattice correlation functions are polynomials in \( n_r \) to any fixed order in the expansion in \( a_f \). The degree of the \( n_r \)-polynomial is less than the order of the \( a_f \)-expansion. The \( n_r \)-dependence of the Symanzik coefficients can then be determined unambiguously. It follows that the replica continuation is nowhere singular in the complex \( n_r \) plane, to the given order in \( a_f \). Finally, after performing the replica continuation, the parameter \( n_r \) resumes its original role as the power of the staggered determinant in the lattice theory. The further transition to ChPT is essentially a repeat of the same reasoning. As will become clear later on, we do have to assume that a chiral effective theory can be constructed for any local, but partially-quenched, theory. This was already emphasized in Refs. [10, 14].

The outline of this paper is as follows. In Sec. II we consider the symmetries of staggered fermions in some detail. We derive the form in which shift symmetry [4] is realized in the SET, and thus in any other EFT derived from the SET. A quick overview of the most important observations of that section is given at its beginning, and any reader not interested in the details can skip the remainder of the section.

In Sec. III we come to the main part of this paper, the construction of the SET for QCD with rooted staggered fermions. We generalize the staggered theory to a class of partially-quenched theories in which it is possible to implement the program outlined above. In Sec. IV we discuss the SET to quadratic order in the lattice spacing in more detail, in order to illustrate the general construction. In Sec. V we make the transition to the chiral effective theory, and demonstrate that it is indeed given by rSChPT. As an example, we work out in rSChPT the leading-order contribution to the connected scalar two-point function, following the calculation in Ref. [14]. We then compare the present derivation of rSChPT to that of Ref. [14], using the respective discussions of the scalar two-point function to make the comparison concrete. The final section contains our conclusions. A brief account of this work was presented at Lattice 2007 [32].

II. SYMMETRIES OF THE SYMANZIK EFFECTIVE ACTION FOR STAGGERED FERMIONS

Here we discuss the symmetries of unrooted staggered fermions that are most relevant for this paper, and the way they appear at the level of the SET. We begin with an overview of the main results of this section. In the following subsections we will then give a more detailed discussion.
1. The staggered fermion action is invariant under shift symmetry, which, in the continuum limit, enlarges to the product of $SU(4)$ taste symmetry and translation symmetry. At the level of the SET, the taste part of shift symmetry takes the form of the $32$-element group $\Gamma_4$ generated by a set of four-dimensional Dirac gamma matrices $\xi_\mu$, with

\[ \{ \xi_\mu, \xi_\nu \} = 2\delta_{\mu\nu}, \quad \mu, \nu \in \{1, 2, 3, 4\} . \]  

This result was derived to order $a^2$ in Ref. [22]. Here we give a general argument that makes it clear that the result is true to all orders in $a$. On the continuum quark fields $q$ used in the SET, the generating elements of $\Gamma_4$ can be chosen to act according to

\[ q \rightarrow \xi_\mu q, \quad \bar{q} \rightarrow \bar{q} \xi_\mu . \]  

Here the field $q_{\beta b}$ has a Dirac spin index $\beta$ and an $SU(4)$ taste index $b$, with the matrices $\xi_\mu$ acting on the latter.

2. On the lattice, a taste-basis field $\psi$ carrying the same indices as the continuum quark field $q$ is related to the one-component field $\chi$ by a unitary transformation [5, 33]

\[ \psi = Q\chi, \quad \overline{\psi} = \overline{\chi} Q^\dagger . \]  

The field $\psi$ lives on a coarse lattice whose spacing is twice that of the original staggered action. The ultra-local, unitary matrix $Q$ maps the one-component variables $\chi$ on the sixteen sites of each even hypercube to the sixteen components of $\psi$ on the single corresponding coarse-lattice site. The transformation $Q$ is required to be gauge covariant, and its choice is not unique. As a result hypercubic rotational symmetry is somewhat complicated in the taste basis. Of course, since the one-component and taste bases are related by a unitarity transformation, the physical consequences of all staggered symmetries are preserved.

A somewhat different taste-basis operator, that we will refer to as the “RG taste–basis” Dirac operator [15, 34], is defined by a Gaussian smearing of the unitary transformation (2.3). The resulting inverse Dirac operator satisfies

\[ D^{-1}_{\text{taste}} = \frac{1}{\alpha} + Q D^{-1}_{\text{stag}} Q^\dagger , \]  

where $D_{\text{stag}}$ is the Dirac operator in the one-component formulation, and $\alpha$ is a parameter of order $1/a$. Even though the theories described by $D_{\text{stag}}$ and $D_{\text{taste}}$ are no longer related by a simple, unitary basis transformation, they are physically completely equivalent, because the propagators differ only by a contact term. The advantage of the Gaussian-smeared transformation is that discarding the taste-breaking part of $D_{\text{taste}}$ does not introduce any fermion doublers [12, 34]. Because the staggered theory and the taste-invariant theory have a similar fermion content, one can interpolate smoothly between them. This will prove useful for the derivation of the SET.

In the one-component formulation, shift symmetry is a unitary transformation on the fields $\chi$ and $\overline{\chi}$ (cf. Eq. (2.11) in the next subsection). Since $Q$ is unitary, the same is also true for the fields $\psi$ and $\overline{\psi}$, and from this it follows that the theory in the RG taste basis is also invariant under shift symmetry.

\[ ^5 \text{ For a detailed discussion of rotational symmetry in this framework, see Ref. [15].} \]
3. Because of staggered symmetries, discretization errors for theories with staggered fermions start at order $a^2$ \[35\]. This is not obvious if one considers staggered fermions in the taste basis of Refs. \[5, 33\], or in the modified form used in the RG analysis of Refs. \[15, 34\], where taste-breaking terms occur in the action starting at order $a$. In this case, shift symmetry connects the leading, taste-invariant term in the lattice action with the order $a$ taste-breaking term, \textit{i.e.}, their relative strength is fixed. There exists a local field redefinition that brings the taste-basis lattice action into a form where the taste violations are explicitly of order $a^2$, and shift symmetry is realized as in Eq. (2.2), again up to order $a^2$ terms \[36\]. More generally, the momentum-space basis used in the derivation of Eq. (2.2) can be related to the taste basis by a non-local field redefinition. Because the construction of the SET proceeds order by order in $a$, the field redefinition in effect becomes local. Therefore, the SETs constructed in the taste basis and in the staggered (or momentum-space) basis are always related by a local field redefinition.

4. Staggered fermions have an exact chiral symmetry when $m = 0$, often referred to as $U(1)_\epsilon$ symmetry, taking the form \[2\]

$$\chi(x) \to e^{i\theta_\epsilon(x)}\chi(x), \quad \overline{\chi}(x) \to e^{i\theta_\epsilon(x)}\overline{\chi}(x), \quad \epsilon(x) = (-1)^{x_1+x_2+x_3+x_4}. \quad (2.5)$$

For $m = 0$, this implies that

$$\{D_{\text{taste}}, \gamma_5 \otimes \xi_5\} = \frac{2}{\alpha} D_{\text{taste}}(\gamma_5 \otimes \xi_5) D_{\text{taste}}, \quad (2.6)$$

where $\gamma_5$ acts on the spin index, and $\xi_5 = \xi_1\xi_2\xi_3\xi_4$ acts on the taste index \[12, 34\]. In other words, $D_{\text{taste}}$ is a Ginsparg–Wilson operator \[37\] with respect to $U(1)_\epsilon$ symmetry.

Before we proceed, we return to the relation of our analysis and that of Ref. \[22\]. The SET at order $a^2$ was determined in Ref. \[22\] by enumerating the allowed dimension-6 lattice operators consistent with the lattice symmetries, including shift symmetry. It was then shown that shift symmetry is represented on the corresponding continuum operators as a $\Gamma_4$ symmetry. A more direct method of determining the SET, which we follow here, is to enumerate continuum operators. This leads to the result of point 1, that shift symmetry always implies a taste $\Gamma_4$ symmetry of the SET.

In the subsections following below, we will discuss some of these observations in more detail. These subsections are not needed for the construction of the SET for rooted staggered fermions, which can be found in Sec. \[III\].

\textbf{A. Diagrammatic argument}

Our first argument for claim 1 above is essentially perturbative, and assumes that we are working in the momentum-space representation of the one-component basis. This result may be considered a corollary of Ref. \[4\]. To keep it self-contained, a summary of relevant facts from Ref. \[4\] has been included in the discussion below.

We will consider diagrams with $n$ external fermion and $r$ external gauge-field lines, corresponding to an operator which appears at a certain order in the SET. On the lattice, because of the phase factors which appear in the staggered action, momentum is conserved modulo
\( \pi \) (in this section we work in lattice units), and any such diagram will have an overall delta function for momentum conservation of the form

\[
\delta(p_1 + \cdots + p_n + k_1 + \cdots + k_r + \Pi) ,
\]  

(2.7)

where \( \Pi \) is a vector with components 0 or \( \pi \). The delta function is the periodic delta function with period \( 2\pi \). The (lattice) quark and anti-quark momenta are \( p_i, i = 1, \ldots, n \) and the gluon momenta \( k_j, j = 1, \ldots, r \).

Because we are interested in an operator in the SET, we may take all physical external momenta small. Fermion doubling then implies that on every quark line we need to split the momenta as

\[
p_i = q_i + \pi A_i ,
\]

(2.8)

in which \( q_i \) lives in the reduced Brillouin zone \((-\pi/2 < q_{i\mu} \leq \pi/2)\), and \( \pi A_i = \pi A_i \), with

\[
A_i \in \{(0, 0, 0, 0), (1, 0, 0, 0), \ldots, (1, 1, 1, 1)\} .
\]

(2.9)

We now take all physical momenta \( q_i \) and \( k_j \) small — so small that their sum has no components as large as \( \pm \pi \). The delta function in Eq. (2.7) thus factorizes into

\[
\delta(q_1 + \cdots + q_n + k_1 + \cdots + k_r) \delta(\pi A_1 + \cdots + \pi A_n + \Pi) .
\]

Now consider what happens to this diagram under a shift

\[
\chi(x) \rightarrow \zeta_\mu(x)\chi(x + \hat{\mu}) ,
\]

(2.11)

\[\overline{\chi}(x) \rightarrow \overline{\chi}(x + \hat{\mu})\zeta_\mu(x) ,\]

\[U_\nu(x) \rightarrow U_\nu(x + \hat{\mu}) ,\]

\[\zeta_\mu(x) = (-1)^{x_\mu+1+\cdots+x_4} = e^{i\pi \zeta_\mu \cdot x} ,\]

where the last equality defines \( \pi \zeta_\mu \). In momentum space (with \( \chi(x) = \int_p e^{i p \cdot x} \chi(p) \)), this takes the form

\[
\chi(p_i) = \chi(q_i + \pi A_i) \rightarrow e^{i(q_i + \pi A_i)_\mu} \chi(q_i + \pi A_i + \pi \zeta_\mu) .
\]

(2.12)

Applying a shift in the \( \mu \) direction to all external legs of our diagram, and noting that the \( j \)th external gluon line is multiplied by a factor \( e^{i(k_j)_\mu} \) under a shift, we obtain the total factor

\[
e^{i(q_1 + \cdots + q_n + k_1 + \cdots + k_r)_\mu} ,
\]

(2.13)

which, by virtue of the first delta function in Eq. (2.10), is equal to one. Therefore, we may omit these (small-momentum) phase factors in the shift (2.12). We conclude that the diagram is invariant under the modified symmetry

\[
\chi(q_i + \pi A_i) \rightarrow e^{i(\pi A)_\mu} \chi(q_i + \pi A_i + \pi \zeta_\mu) , \quad i = 1, \ldots, n ,
\]

(2.14)

which does not act on the gluon fields. The transformation (2.14) generates a representation of the group \( \Gamma_4 \) acting on the quark fields. Indeed, applying the transformation first in the \( \mu \) direction, and then in the \( \nu \) direction, one obtains (dropping the index \( i \))

\[
\chi(q + \pi A) \rightarrow e^{i(\pi A + \pi \zeta_\mu)_\nu} e^{i(\pi A)_\mu} \chi(q + \pi A + \pi \zeta_\mu + \pi \zeta_\nu) .
\]

(2.15)
For $\mu = \nu$, we have $(\pi_{\nu})_{\mu} = 0$ (cf. Eq. (2.11)), and Eq. (2.15) thus reduces to the identity. For $\mu \neq \nu$,

$$
\zeta_\mu(x + \nu) = \zeta_\mu(x) \Rightarrow e^{i(\pi_{\nu})_\nu} = +1 , \quad \mu > \nu ,
$$

$$
\zeta_\mu(x + \nu) = -\zeta_\mu(x) \Rightarrow e^{i(\pi_{\nu})_\nu} = -1 , \quad \mu < \nu ,
$$

implying that shifts anti-commute, just like the generators of $\Gamma_4$. We may make contact with Eq. (2.12) by introducing

$$
\phi_A(q) \equiv \chi(q + \pi_A) .
$$

The transformation Eq. (2.14) can now be written as

$$
\phi_A(q) \rightarrow \sum_B (\Xi_\mu)_{AB} \phi_B(q) ,
$$

for some $16 \times 16$ matrices $\Xi_\mu$ satisfying the Dirac algebra

$$
\{ \Xi_\mu, \Xi_\nu \} = 2\delta_{\mu\nu} .
$$

Finally, we can perform a basis transformation such that $\Xi_\mu = 1 \otimes \xi_\mu$, and transform back to position space to obtain Eq. (2.2).

Our argument shows that the diagram is invariant under the symmetry (2.2) if it is invariant under shift symmetry (2.12). The group $\Gamma_4$ may thus be used to restrict the form of the SET in accordance with the shift symmetry of the underlying lattice theory. This is a considerable simplification, because the group $\Gamma_4$ does not mix operators of different dimensions, i.e., of different orders in the Symanzik expansion.

The same reasoning goes through in a theory in which the staggered fermion fields carry a flavor index $\ell = 1, \ldots, n_f$: one simply labels the fields $\chi_\ell$ and $\overline{\chi}_\ell$ in Eq. (2.11) with the extra index $\ell$. Since the gauge fields also transform under shift symmetry, the same shift symmetry acts on all staggered fields simultaneously. It thus follows that the discrete symmetry $\Gamma_4$ acts in the same way on all staggered fields $\chi_\ell$, and does not enlarge to the group $(\Gamma_4)^{n_f}$.

As an aside, we note that the invariance of the diagram under shift symmetry has implications for the second delta function in Eq. (2.10). Naively, it would seem to follow that $\Pi$ just has to be equal to the sum over all $\pi_{A_i}$, but in general this is not sufficient. The reason is that the vertex can contain explicit periodic functions of the external momenta, which leads to additional sign factors under a shift. This is best illustrated with an example. Consider a lattice vertex of the form

$$
\sum_{A,B} \delta(q_1 + q_2 + k) \delta(\pi_A + \pi_B + \Pi) \cos (q_1 + k + \pi_A)_\nu \overline{\chi}(q_2 + \pi_B) \chi(q_1 + \pi_A) A_\nu(k) ,
$$

in which we split $p_1 = q_1 + \pi_A$, $p_2 = q_2 + \pi_B$, and take $q_{1,2}$ and $k$ to be small. Performing a shift on the $\chi$ and $\overline{\chi}$ fields results in (dropping a factor $\delta(q_1 + q_2 + k)$)

$$
\sum_{A,B} \delta(\pi_A + \pi_B + \Pi) \cos (q_1 + k + \pi_A)_\nu e^{i(\pi_{\nu} + \pi_B)_\mu} \overline{\chi}(q_2 + \pi_B + \pi_{\nu}) \chi(q_1 + \pi_A + \pi_{\nu}) A_\nu(k)
$$

$$
\times \chi(q_2 + \pi_B + \pi_\nu) \chi(q_1 + \pi_A + \pi_\nu) A_\nu(k)
$$

$$
= \sum_{A,B} \delta(\pi_A + \pi_B + \Pi) \cos (q_1 + k + \pi_A)_\nu e^{i(\pi_{\nu} + \pi_B)_\mu} \overline{\chi}(q_2 + \pi_B) \chi(q_1 + \pi_A) A_\nu(k) ,
$$
where we used that $(\pi_\omega)_\mu = 0$ and that $2\pi_\omega = 0 \mod 2\pi$. The vertex is thus invariant if $\Pi_\mu + (\pi_\omega)_\nu = 0 \mod 2\pi$. An example of such a $\Pi$ is $\pi_\eta$, which is defined by the phase factors which appear in the staggered action:

$$\eta_\mu(x) \equiv e^{i\pi_\eta x} \equiv (-1)^{x_1 + \cdots + x_{\nu-1}}. \quad (2.22)$$

**B. Group-theoretical argument**

There is a very simple group-theoretical way to derive the same result. Let $S_\mu$ be the shift in the $\mu$ direction. All elements of the shift-symmetry group can be generated from the basic four shifts, and it is thus sufficient to consider only the $S_\mu$. In any irreducible representation of the group, $S_\mu$ looks like

$$S_\mu \rightarrow e^{iq_\mu \Xi_\mu}, \quad (2.23)$$

with $-\pi/2 < q_\mu \leq \pi/2$ the physical momentum in lattice units, and the matrices $\Xi_\mu$ generate a representation of $\Gamma_4$ [38]. All irreducible representations are either “bosonic,” if each $\Xi_\mu$ is mapped onto $\pm 1$ (sixteen choices), or “fermionic,” if the $\Xi_\mu$ are chosen to satisfy the Dirac algebra (2.19). Any field appearing in an EFT for the staggered theory (such as the SET or ChPT) transforms in some representation of $S_\mu$ under a shift (i.e., with some choice of $q_\mu$ and $\Xi_\mu$).

Now we use that any continuum EFT is also invariant under continuum translations, which, on a continuum field $\Phi$ with momentum $q$, act as

$$\Phi(q) \rightarrow e^{iq\cdot r}\Phi(q), \quad (2.24)$$

for a translation over a displacement $r$. We may thus choose $r$ such that $q \cdot r = -q_\mu$, follow $S_\mu$ by this translation, and again obtain a symmetry of the EFT. This symmetry is precisely the one generated by the $\Xi_\mu$, i.e., a representation of $\Gamma_4$.

**C. Taste basis**

The arguments in the previous subsections made use of the momentum basis of the one-component formalism. The Feynman rules for the staggered theory in the one-component basis [4] were (assumed to have been) used in the derivation of the SET. Also, the group-theoretical argument works naturally on the momentum basis, since that is where irreducible representations of the staggered symmetry group live [38]. Alternatively, one could have started from the taste basis. The SET derived from the taste basis will not look the same as that derived from the one-component formalism; but the two SETs should be physically equivalent. Since the one-component and taste bases are related by a (nonlocal) unitary transformation in momentum space [5], one expects that the SETs derived from them, too, will be related by a field redefinition. Moreover, to any finite order in $a$, the SET-level field redefinition should be local, because the same is true for the unitary transformation between the two bases, when expanded to the corresponding finite order in $a$.

We illustrate this in the free massless theory, working to order $p^2$ in the Symanzik expansion. On the taste basis, shift symmetry takes on the form [5, 36]

$$\psi(y) \rightarrow \frac{1}{2}((\xi_\mu + \gamma_5\gamma_\mu\xi_5)\psi(y) + (\xi_\mu - \gamma_5\gamma_\mu\xi_5)\psi(y + \hat{\mu})). \quad (2.25)$$
The field $\psi$, introduced in Eq. (2.3), is in this case given explicitly by

$$\psi_{\beta b}(y) = \frac{1}{2^{3/2}} \sum_A (\gamma_A)_{\beta b} \xi(2y + A), \quad (2.26)$$

where $\gamma_A = \gamma_1^A \gamma_2^A \gamma_3^A \gamma_4^A$, and $A$ runs over the set (2.9). The normalization factor in Eq. (2.26) differs from that in Ref. [5] because we take $\psi$ to be in lattice units of the coarser lattice; whereas Ref. [5] works in physical units. In momentum space, Eq. (2.25) looks like

$$\psi(p) \to e^{ip_{\mu}/2} (\xi_\mu \cos (p_\mu/2) - i\gamma_5 \gamma_\mu \xi_5 \sin (p_\mu/2)) \psi(p) \quad (2.27)$$

$$= e^{ip_{\mu}/2} \left( \xi_\mu - i\frac{1}{2} \gamma_5 \gamma_\mu \xi_5 p_\mu + \mathcal{O}(p^3) \right) \psi(p).$$

The factor $e^{ip_{\mu}/2}$ corresponds to the factor $e^{iq_\mu}$ in Eq. (2.12), because the lattice spacings differ by a factor two. Dropping the factor $e^{ip_{\mu}/2}$ on the same grounds as in Sec. II A, it is easily verified that the transformation (2.27) becomes a generating element of $\Gamma_4$, and that it is a symmetry of the order-$a$ SET in the taste representation,

$$S_{\text{free}} = \sum_\mu \int_{-\pi}^{\pi} \frac{d^4 p}{(2\pi)^4} \overline{\psi}(p) \left( i\gamma_\mu p_\mu + \frac{1}{2} \gamma_5 \xi_5 p_\mu^2 + \mathcal{O}(p^3) \right) \psi(p). \quad (2.28)$$

This may also be written as

$$S_{\text{free}} = \sum_\mu \int_{-\pi}^{\pi} \frac{d^4 p}{(2\pi)^4} \left( \overline{\psi}(p)i\gamma_\mu p_\mu \psi(p) + \frac{1}{2} \overline{\psi}(p)\gamma_5 \xi_5 p_\mu^2 \psi(p) + \mathcal{O}(p^3) \right), \quad (2.29)$$

where we consider the field $\psi_{\beta b}$ as a $4 \times 4$ matrix.

In momentum space, the transformation relating the one-component and taste representations is [5, 36, 39]

$$\psi(p) = \frac{1}{2^{11/2}} \sum_{A,B} (-1)^{A-B} \gamma_A \phi_B(q) e^{iq \cdot A}, \quad (2.30)$$

$$\overline{\psi}(p) = \frac{1}{2^{11/2}} \sum_{A,B} (-1)^{A-B} \overline{\gamma}_A \overline{\phi}_B(q) e^{-iq \cdot A},$$

where again $A$ and $B$ take values in the set (2.9), and where $q = p/2$. The field $\phi(q)$ was defined in Eq. (2.17). The transformation (2.30) is indeed nonlocal, but its expansion to any finite order in $a$ is local. For instance, upon expanding $e^{\pm i q \cdot A} = 1 \pm i q \cdot A + \mathcal{O}(q^2)$, and starting from Eq. (2.29), this field redefinition brings the action (2.28) into the form

$$S_{\text{free}} = \sum_{\mu} \sum_{AB} \int_{-\pi/2}^{\pi/2} \frac{d^4 q}{(2\pi)^4} \overline{\phi}_A(q) \left( i(\Gamma_\mu)_{AB} q_\mu + \mathcal{O}(q^3) \right) \phi_B(q), \quad (2.31)$$

where the $\Gamma_\mu$ matrices form a 16-dimensional representation of the Dirac algebra and commute with the taste matrices $\Xi_\nu$ defined in Eq. (2.18). Note that Eq. (2.31) is expressed in units of the fine lattice spacing.
Let us also briefly consider the RG taste representation defined by Eq. (2.4) in the massless free theory. To order $p^2$ the action is given by [12]

$$
\sum_\mu \int_{-\pi}^{\pi} \frac{d^4 p}{(2\pi)^4} \overline{\psi}(p) \left( i \gamma_\mu p_\mu + \frac{1}{\alpha} p_\mu^2 + \frac{1}{2} \gamma_5 \xi_5 \xi_\mu p_\mu^2 + O(p^3) \right) \psi(p) .
$$

(2.32)

This action is invariant under $U(1)_\epsilon$ symmetry in “Ginsparg–Wilson–Lüscher” (GWL) [37, 40] form. In the free theory, this symmetry looks like (again to order $a$)

$$
\delta \psi(p) = \gamma_5 \xi_5 \left( 1 - \frac{2}{\alpha} \sum_\mu i \gamma_\mu p_\mu + O(p^2) \right) \psi(p) ,
$$

(2.33)

$$
\delta \overline{\psi}(p) = \overline{\psi}(p) \gamma_5 \xi_5 .
$$

In this case, we may first carry out a field redefinition

$$
\psi(p) \rightarrow \left( 1 + \frac{1}{\alpha} \sum_\mu i \gamma_\mu p_\mu \right) \psi(p) ,
$$

(2.34)

$$
\overline{\psi}(p) \rightarrow \overline{\psi}(p) ,
$$

followed by (2.30), to bring the action into a form without terms of order $a$. Note that Eq. (2.34) is nothing but the free-field, order-$a$ form of the field redefinition

$$
\psi \rightarrow (1 - D/\alpha)^{-1} \psi
$$

(2.35)

with here $D = D_{\text{taste}}$, which transforms the GWL form of $U(1)_\epsilon$ symmetry into a standard $\gamma_5 \xi_5$ symmetry [41].

As a final note, we observe that to this order in $a$, field redefinitions can be carried out such that the resulting action is invariant under the full $U(4)$ taste symmetry. This turns out to be true to all orders in $a$ in the free theory [4], but not in the interacting theory.

### III. DERIVATION OF THE SYMANZIK EFFECTIVE ACTION

We begin by considering a theory with $n_r$ replicas of one staggered fermion with bare mass $m$, in the RG taste basis. For now, $n_r$ will be a positive integer. We perform $n + 1$ RG blocking steps, labeled $k = 0, 1, \ldots, n$, following the blocking procedure of Ref. [15]. The special $k = 0$ step is used to carry out the transition from the one-component to the taste basis, cf. Eq. (2.4). In this step the number of fermion degrees of freedom is not thinned out; in each subsequent step they are thinned out by a factor $2^4 = 16$. The partition function for this theory can be written as

$$
Z(n_r) = \int DU \prod_{k=0}^{n} D\mathcal{V}^{(k)} B_n \left( n_r; U, \{ \mathcal{V}^{(k)} \} \right) \text{Det}^{n_r} (D_{\text{taste}, n}) .
$$

(3.1)

The notation here is as follows: The gauge field on the original lattice, with spacing $a_f$, is denoted by $U$. The spacing of the $k$-th blocked lattice is $a_k = 2^{k+1} a_f$, and the gauge field on that lattice is $\mathcal{V}^{(k)}$. The spacing of the final, coarse lattice is $a_c = 2^{n+1} a_f$. The Boltzmann
weight for the collection of gauge fields, original and blocked, is \( B_n \left( n_r; U, \{ \mathcal{V}^{(k)} \} \right) \). It is composed of three parts: the original gauge action, the gauge-field blocking kernels,\(^6\) and a short-distance contribution to the effective gauge-field action, \( n_r \delta S_{\text{eff}} \), coming from integrating out the fermions on all lattices except the last one, where

\[
e^{-\delta S_{\text{eff}}} = \prod_{k=0}^{n} \det \left( G_k^{-1} \right). \tag{3.2}
\]

The operators \( D_{\text{taste},k} \) and \( G_k^{-1} \) are recursively given by

\[
\begin{align*}
D_{\text{taste},k}^{-1} &= \alpha_k^{-1} + Q^{(k)} D_{\text{taste},k-1}^{-1} Q^{(k)}, \quad k = 1, \ldots, n, \\
G_k^{-1} &= D_{\text{taste},k-1} + \alpha_k Q^{(k)} Q^{(k)\dagger}, \quad k = 1, \ldots, n.
\end{align*}
\]

The blocking parameter \( \alpha_k \) is of order \( 1/a_k \). The blocking kernel at the \( k \)-th step, \( Q^{(k)} = Q^{(k)}(\mathcal{V}^{(k-1)}) \), gauge-covariantly averages the fermion fields over \( 2^4 \) hypercubes on the \( (k-1) \)-st lattice. For the \( k = 0 \) step, \( D_{\text{taste},0} = D_{\text{taste}} \) is defined in Eq. (2.4), and \( G_0^{-1} = D_{\text{stag}} + \alpha_0 Q^{(0)\dagger} Q^{(0)} \), where \( \alpha_0 = \alpha \) and \( Q^{(0)} = Q \) are those introduced in Eq. (2.4). Recall that the special \( k = 0 \) blocking kernel is unitary; all other blocking kernels are not.

For small momenta, \( Q^{(k)\dagger} Q^{(k)} \approx 1 \), and with \( \alpha_k \sim 1/a_k \) it thus follows that the eigenvalues of \( G_k^{-1} \) are at least of order \( 1/a_k \), making \( \delta S_{\text{eff}} \) a short-distance contribution to the effective gauge action.\(^7\) While this can be proved in the free case \([34]\), in the interacting case this is an assumption that is already necessary for the conventional RG picture to work in local, renormalizable theories. The nature of this assumption is discussed in detail in Ref. \([15]\); here we will assume it to be correct. It follows that \( \delta S_{\text{eff}} \) remains local when we take \( n_r \) to be any real number.\(^8\)

The fermionic contribution to long-distance physics then resides entirely in the \( n_r \)-th power of the determinant of \( D_{\text{taste},n} \) in Eq. (3.1). The problems with locality of the rooted theory originate with taking \( n_r \to 1/4 \) in this power. Our task will be to perform a faithful replica continuation at the level of the SET. As explained in the introduction, this is not straightforward. Calculations in the effective theories, the SET or ChPT, lead to explicit dependence on \( n_r \) (for instance, through loops). But there is also implicit dependence through the couplings that appear in the effective theory, which is in general nonperturbative, and not known.

Our strategy will be to first approximate the fourth-root theory by a local (“re-weighted”) theory. The fermions of this theory do not carry a taste degree of freedom; they are taste singlets. The multiplicity of taste-singlet fermions, \( n_s \), will always be chosen to match the fermion spectrum of the target continuum theory. Therefore we will never have to perform any “replica continuation” in \( n_s \); rather, \( n_s \) will always be kept a positive integer. In our construction, the unknown dependence of the couplings in the effective theory on the fermions will be due to the taste-singlet fermions only.

---

\(^6\) We do not integrate over any of the gauge fields; this can be postponed to the end. The explicit expression for \( B_n \left( n_r; U, \{ \mathcal{V}^{(k)} \} \right) \) is given in Ref. \([15]\).

\(^7\) Much smaller eigenvalues are allowed, as long as the corresponding eigenmodes are localized on a distance of at most order \( a_k \). Such modes would not affect the long-distance physics.

\(^8\) We keep \( n_r \) in the range where the gauge coupling is asymptotically free.
The fourth-root theory will be reached from the taste-singlet theory by “turning on” the taste-breaking effects that introduce the nonlocal behavior. This is where a replica continuation away from the integers will be needed. Because of the smallness of the taste-breaking effects, the replica continuation will be under control. Indeed, we will show that to any order in $a_f$, the dependence of the taste-breaking effects on $n_r$ is polynomial, with a degree less than the order in the $a_f$-expansion.

We start by splitting $D_{\text{taste},n}$ into a taste-singlet part and a taste-breaking part with vanishing trace in taste space,

$$D_{\text{taste},n} = D_{\text{inv},n} + \Delta_n,$$

$$D_{\text{inv},n} = \tilde{D}_{\text{inv},n} \otimes 1,$$

$$\tilde{D}_{\text{inv},n} = \frac{1}{4} \text{tr}_{ts}(D_{\text{taste},n}),$$

where $\text{tr}_{ts}$ denotes the trace in taste space, and $1$ is the taste identity matrix. Following Ref. [15] we assume that, in the coarse-lattice theory, $\Delta_n$ scales like

$$\|a_c \Delta_n\| \lesssim \frac{a_f}{a_c}. \quad (3.5)$$

This estimate is valid modulo logarithmic corrections to the leading power-law scaling. For extensive discussions of this scaling assumption, we refer to Ref. [15] (see also Refs. [10, 11]). Here we only observe that, in any theory with integer $n_r$, this assumption is needed to establish that unrooted staggered fermions have the usually assumed continuum limit. However, by exploiting the proximity of the local re-weighted theory after a large number $n$ of blocking steps, it was argued that the scaling (3.5) is also valid in theories with fractional $n_r$. In this paper, we will assume this to be the case.

Using this split, we generalize the determinant in Eq. (3.1) to

$$\text{Det}^{n_r} (D_{\text{taste},n}) \rightarrow \text{Det}^{n_s} \left( \tilde{D}_{\text{inv},n} \right) \frac{\text{Det}^{n_r} (D_{\text{inv},n} + t \Delta_n)}{\text{Det}^{n_r} (D_{\text{inv},n})}, \quad (3.6a)$$

while also replacing

$$B_n \left( n_r; U, \{V^{(k)}\} \right) \rightarrow B_n \left( n_s/4; U, \{V^{(k)}\} \right). \quad (3.6b)$$

The generalized theory reduces to Eq. (3.1) if we set $n_s = 4n_r$ and $t = 1$. This generalization has two important properties. First, if $n_s = 4n_r$ and $n_r$ assumes physically interesting values, i.e., multiples of $1/4$, then $n_s$ is an integer. Second, when $n$ is large enough, $D^{-1}_{\text{inv},n} \Delta_n$ is small enough (in an ensemble-average sense) that we may expand

$$\frac{\text{Det}^{n_r} (D_{\text{inv},n} + t \Delta_n)}{\text{Det}^{n_r} (D_{\text{inv},n})} = \exp \left[ n_r \text{Tr} \log \left( 1 + t D^{-1}_{\text{inv},n} \Delta_n \right) \right]$$

$$= \exp \left[ -n_r \text{Tr} \left( \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} t^\ell (D^{-1}_{\text{inv},n} \Delta_n)^\ell \right) \right]. \quad (3.7)$$

The parameter $t$ interpolates between the taste-invariant operator $D_{\text{inv},n}$ at $t = 0$ and the (blocked) staggered operator at $t = 1$. In addition, $t$ is a book-keeping device. The power
of $t$ is, evidently, the same as the power of $D^{-1}_{\text{inv},n}\Delta_n$. As we explain in detail in Sec. III B below, for the construction of the effective theories we may use the bound

$$\|D^{-1}_{\text{inv},n}\Delta_n\| \lesssim \frac{a_f}{a_c} = \frac{1}{2^{n+1}} \equiv \epsilon_n . \quad (3.8)$$

(The $\sim$ sign has a meaning similar to that in Eq. (3.5).) We conclude that the $t$-expansion is an expansion in powers of $a_f$ for the taste-breaking effects.

For $t = 0$, the determinant ratio (3.7) collapses to one. The taste-invariant theory at $t = 0$ is thus local for any integer $n_s$, and independent of $n_r$. The staggered theory is reached by expanding as in Eq. (3.7), eventually setting $t = 1$. The rooted staggered theory is obtained by setting $n_r$ to a quarter-integer value. When we construct the SET to any finite order in $a_f$, the maximal power of $t$ will be limited by that order.\footnote{Note that $a_f$-dependence which does not involve taste-symmetry breaking may result from other sources besides the determinant ratio (3.7).} By Eq. (3.7), the maximal power of $n_r$ is bounded by the power of $t$. (Because of taste-tracelessness of $\Delta_n$, the maximal power of $n_r$ is in fact strictly less than the power of $t$.) The maximal power of $n_r$ is thus (strictly) less than the order in $a_f$. Therefore, at fixed $n_s$ and to any finite order in $a_f$, the dependence of any correlation function on $n_r$, and thus of the SET that reproduces it, will be polynomial. This implies that, at the level of the SET, the replica continuation in $n_r$ to quarter-integer values will be well-defined, resulting in the “staggered SET with the replica rule.” What this means is the following: We start with integer $n_r$. The effective action is then given in terms of a set of Symanzik coefficients which are unknown functions of $n_s$, but depend polynomially on $n_r$ (we may already set $t = 1$). With this action, one calculates correlation functions which again depend polynomially on $n_r$ (to any finite order in $a_f$), with $n_r$ dependence coming from the Symanzik coefficients and from loops. Finally, one sets $n_r = n_s/4$, and the resulting correlation function is precisely that of the rooted staggered theory. The following subsections contain a more detailed argument on how this works.

We comment in passing that, for $t = 1$, we may also interpolate between the taste-singlet local theory at $n_r = 0$, and the (rooted) staggered theory at $n_r = n_s/4$ by varying $n_r$ instead of $t$. While the two ways of moving from the taste-singlet to the staggered theory are mathematically equivalent, we find the argument more transparent if the transition is done by varying $t$.

\section{A. The generalized theory}

In order to define the SET we first need a complete definition of the generalized staggered theory, coupled to sources in order to generate all correlation functions. Returning to integer $n_r$, the theory defined by Eq. (3.6) contains $n_s$ taste-singlet fermions with Dirac operator $\tilde{D}_{\text{inv},n}$, $n_r$ generalized staggered fermions with Dirac operator $D_{\text{inv},n} + t\Delta_n$, and $4n_r$ ghosts with Dirac operator $\tilde{D}_{\text{inv},n}$. Introducing sources $H = (\bar{\eta}, \eta, \tilde{\eta})$ and $\overline{H} = (\bar{\eta}, \eta, \tilde{\eta})$ for the taste-singlet, generalized staggered, and ghost fields respectively, we define the partition function
of the generalized theory as

$$Z_n(t, n_r, n_s; H, \overline{H}) = \int \mathcal{D}U \prod_{k=0}^{n} \mathcal{D}\psi^{(k)} B_n \left( \frac{n_s}{4}; U, \{\psi^{(k)}\} \right)$$

$$\times \Det^{n_s} (\tilde{D}_{\text{inv},n}) \frac{\Det^{n_r} (D_{\text{inv},n} + t\Delta_n)}{\Det^{n_r} (D_{\text{inv},n})} \exp \left[ \overline{\eta} (\tilde{D}_{\text{inv},n} \times I_{n_s}) \eta \right]$$

$$\times \exp \left[ \overline{\eta} (D_{\text{inv},n} + t\Delta_n)^{-1} \otimes I_{n_s} \eta + \overline{\eta} (D_{\text{inv},n}^{-1} \otimes I_{n_r}) \eta \right]$$

$$= \int \mathcal{D}U \prod_{k=0}^{n} \mathcal{D}\psi^{(k)} B_n \left( \frac{n_s}{4}; U, \{\psi^{(k)}\} \right)$$

$$\times \Det^{n_s} (\tilde{D}_{\text{inv},n}) \exp \left[ -n_r \Tr \left( \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} (D_{\text{inv},n}^{-1} \Delta_n)^{\ell} \right) \right]$$

$$\times \exp \left[ \overline{\eta} (\tilde{D}_{\text{inv},n} \times I_{n_s}) \eta + \overline{\eta} (\tilde{D}_{\text{inv},n}^{-1} \otimes I_{4n_r}) \eta \right]$$

$$\times \exp \left[ \overline{\eta} \left( \sum_{\ell=1}^{\infty} (-1)^{\ell} \ell \left( D_{\text{inv},n}^{-1} \Delta_n \right)^{\ell} D_{\text{inv},n}^{-1} \otimes I_{n_r} \right) \eta \right].$$

Here $I$ stands for the identity matrix, with dimensions as indicated by the subscript. This is a theory with two lattice parameters, $a_c$ and $a_f$. Alternatively, we may trade $a_f$ for the small parameter $\epsilon_n$ of Eq. (3.8). In the second expression we give the explicit expansions in the book-keeping parameter $t$. As explained above, for fixed $n_s$ correlation functions expanded to some finite power in $a_f$ are polynomial in $n_r$. For $t = 1$, $n_r = n_s/4$, and $\tilde{\eta} = \eta = \bar{\eta} = \bar{\eta} = 0$, Eq. (3.9) is precisely the theory of $n_s$ degenerate, fourth-rooted staggered fermions.

The generalized theory has a vector-like $U(n_s|4n_r) \times U(n_r)$ graded symmetry; $U(n_s|4n_r)$ acts on the taste-singlet and ghost fields, and $U(n_r)$ on the generalized staggered field. For $t = 0$ the symmetry enlarges to $U(n_s + 4n_r|4n_r)$. The discrete symmetries include hypercubic rotations and axis reversal [4]. In the staggered sector, for $t = 1$ this is augmented by shift symmetry, and (softly broken) $U(1)$, symmetry in GWL form for each flavor. The vector and axial staggered symmetries expand to a $U(n_s) \times U(n_r)$ chiral symmetry group [23]. There is no chiral symmetry in the taste-singlet and ghost sectors, because the GWL version of $U(1)$ symmetry mixes the taste-invariant and noninvariant parts of the blocked staggered Dirac operator [12].

We are now ready to discuss the SET for the generalized theory. As long as $n_r$ is a positive integer, the lattice theory is partially quenched but local, and we will assume that an SET for this theory exists in Euclidean space. The effective theory can be written in terms of continuum fields $\Psi = (\bar{q}, q, \bar{q}, q)$ and $\overline{\Psi} = (\bar{q}, \bar{q}, \bar{q}, q)$ for the taste-singlet, generalized staggered and ghost fields, respectively, as well as a continuum gluon field $A_\mu$. As explained above, its parameters (the couplings multiplying each operator in the Symanzik expansion) are polynomials in $n_r$ if we work to a finite order in $a_f$; while their dependence on $n_s$ is

---

10 We remind the reader that $D_{\text{inv},n}$ and $\Delta_n$ in Eq. (3.4) are defined in the RG taste basis, cf. Eq. (2.4), and not in the standard taste basis of Refs. [4,33].

11 It is sufficient to consider the SET in Euclidean space, since we will postpone the continuation to Minkowski space until after the continuum limit has been taken [11].
unknown. Only the \( n_s \) dependence survives in the continuum limit, where the determinant ratio \((3.7)\) collapses to one.\(^\text{12}\)

For general \( t, n_s \) and \( n_r \), the fundamental cutoff is the lattice spacing of the generalized theory, \(a_c\). The SET is the effective theory for quarks and gluons with momenta much smaller than \(1/a_c\). However, the lattice theory contains an additional small parameter, \(\epsilon_n = a_f/a_c\), cf. Eq. \((3.8)\). It will be useful for our purposes to think of the Symanzik expansion as an expansion in \(a_f = \epsilon_n a_c\), with Symanzik coefficients that depend on \(a_c\).\(^\text{13}\) The effective theory can be divided into three different sectors, corresponding to three different types of operators that can occur. The (generalized) staggered sector consists of operators made out of staggered fields \(q\) and \(\bar{q}\) only. Likewise, the taste-singlet–ghost sector consists of operators made out of the “auxiliary” fields \(\hat{\psi} = (\hat{q}, \hat{\bar{q}})\) and \(\hat{\bar{\psi}} = (\hat{\bar{q}}, \hat{\bar{\bar{q}}})\) only. Finally there is the mixed sector, where each operator is made out of both staggered and auxiliary fields. (Of course, all operators may contain gluon fields.)

In order to establish the validity of rSChPT in Sec. \(\text{V}\) we will not need to know the explicit form of the SET in full generality. In fact, we need only consider the staggered sector of the SET. Disregarding the auxiliary and mixed sectors, the resulting SET, defined in terms of the quark fields \(q\) and \(\bar{q}\) and the gluon fields, is invariant under all symmetries of the generalized staggered operator \(D_{\text{inv},n} + t \Delta_n\). For \(t = 0\) this includes taste-replica symmetry \(U(4n_r)\), while for \(t = 1\) this includes the smaller group \(\Gamma_4\), as well as softly broken \(U(1)\) symmetry.

For the remainder of this subsection we set \(t = 1\), and thus \(D_{\text{inv},n} + \Delta_n = D_{\text{taste},n}\) reduces to the RG-blocked operator of Eq. \((3.1)\). Symmetries that act on the space-time coordinates often take a complicated form under RG blocking. In particular, shift symmetry is realized in a complicated way. First, the RG blocking leading to Eqs. \((3.1)\) and \((3.9)\) was started in the RG taste basis defined in Eq. \((2.4)\), and shift symmetry is thus realized as a gauge-covariant form of Eq. \((2.25)\). Second, the transition to the RG taste basis was followed by \(n\) additional RG blocking steps.

The physical consequences of any exact lattice symmetry of the underlying staggered theory, nevertheless, cannot be lost by RG blocking. The reason is the existence of a pull-back mapping of every coarse-lattice operator to a fine-lattice operator \([15]\). For \(n_r = n_s/4\), where the taste-singlet and ghost determinants drop out, this mapping gives rise to exact equality of corresponding observables. In other words, the coarse-lattice observables are a subset of the original fine-lattice staggered observables. The pull-back mapping extends to \(n_r \neq n_s/4\). Consider the expectation value of a product of coarse-lattice staggered fermion (and gauge) fields. By undoing the RG-blocking gaussian transformations of the fermions, this can be rewritten as an expectation value of a corresponding product of fine-lattice staggered fields (that depends in addition on the original and blocked gauge fields). Because the Boltzmann weight of the generalized theory contains the taste-singlet and ghost determinants, expectation values will not be the same as in the original staggered theory. But since the fine-lattice symmetries are unchanged, pulled-back coarse-lattice observables will still transform under all the staggered symmetries. Together with other observables constructed from the fine-lattice staggered fields, they must fall into representations of all these symmetries. This implies that the physical consequences of the full set of staggered symmetries remain intact.

\(^{12}\) We observe that at nonzero \(a_c\) but \(a_f \to 0\), \(i.e.,\) in the limit \(n \to \infty\), the lattice action is a perfect action.

\(^{13}\) In the following subsection, we will argue that no negative powers of \(a_f\) can appear.
The \( t = 1 \) staggered-sector SET must therefore be invariant under all the symmetries listed in Sec. \[\] If we derive the SET using the taste basis some of these symmetries will take a complicated form. In particular, shift symmetry will mix different orders in \( a = a_f \). But other continuum fields can always be chosen by suitable field redefinitions such that shift symmetry resumes the simple form of Eq. \( (2.2) \) at the level of the SET. Moreover, a SET-level field redefinition will also eliminate any \( a_c \)-dependence of the SET that originates from the matching to the coarse-lattice interpolating fields.\[\] The only remaining dependence of the staggered-sector SET on \( a_c \) originates at this stage from the presence of the taste-singlet and ghost determinants in the underlying theory \( (3.9) \).

Recall now that the group generated by the four elementary shifts \( S_\mu \) contains translations by \( 2a_f \). At the level of the SET shift symmetry enlarges to the direct product of the group \( \Gamma_4 \) and the continuous translation group. In the continuum limit \( a_f \to 0 \) the discrete group \( \Gamma_4 \) enlarges to the full taste/replica symmetry group \( SU(4n_r) \) (with \( \Gamma_4 \) embedded such that it acts identically on all \( n_r \) replicas).

The conclusion of the above arguments is that, for \( t = 1 \) and for any positive integer values of \( n_s \) and \( n_r \), the generalized staggered sector of the SET assumes exactly the same structure, as an expansion in the fine lattice spacing \( a_f \), as the standard staggered SET for \( n_r \) staggered fields. To order \( a_f^2 \), this SET is derived in Ref. \[22\] (for \( n_r = 1 \)) and Ref. \[23\] (for arbitrary \( n_r \)), and is written down explicitly in Ref. \[24\]. However, the Symanzik coefficients of the staggered-sector SET of the generalized theory are not the same functions of the parameters of the underlying theory as in the ordinary staggered SET. In the generalized theory, the Symanzik coefficients depend on \( n_s \) and \( a_c \), parameters not present in the ordinary staggered theory. Dependence on \( n_s \) arises because of contributions from taste-singlet loops. In addition, the \( n_r \) dependence (at fixed \( n_s \)) of the Symanzik coefficients is different from that of the ordinary staggered SET, because of contributions from ghost loops. Indeed, the reason why the auxiliary sector was introduced in the first place, is that—unlike the original staggered theory—the SET of the generalized theory depends polynomially on \( n_r \) to any order in \( a_f \), as long as \( n_s \) is held fixed.

We are now ready to make contact with the rooted theory. In order to reach the SET of the rooted theory we hold \( n_s \) fixed and choose \( t = 1 \). For any \( t \), we may perform the replica continuation \( n_r \to n_s/4 \) in any correlation function at any given order in the loop expansion.\[\] Indeed, because the Symanzik coefficients are polynomials in \( n_r \) to any desired order in \( a_f \), this continuation from integer values of \( n_r \) is well-defined. Now, recall that the taste-singlet and ghost sectors of the generalized theory \( (3.9) \) cancel (for vanishing sources) when we set \( n_r = n_s/4 \). As explained above, this finally eliminates all the remaining dependence of the staggered-sector SET on the coarse spacing \( a_c \), leaving only the dependence on the fine spacing \( a_f \). We have thus succeeded in constructing the replica-continued SET for the original blocked theory, Eq. \( (3.1) \), for any quarter-integer value of \( n_r \), and to the desired order in \( a_f \).

Putting everything together, we have shown that the familiar staggered SET for integer \( n_r \), derived to order \( a_f^2 \) in Ref. \[22, 23\], and written down explicitly and extended to order

\[\] Via the pull-back, the coarse-lattice operators may be regarded as a particular set of interpolating fields on the fine lattice as well. The freedom in making field redefinitions at the level of the SET thus parallels the freedom, discussed in Appendix B of Ref. \[11\], to choose different sets of interpolating fields on the fine lattice.

\[\] For further discussion of the replica continuation, see Sec. \[\]
can be used to compute any correlation function of interest to the desired order in \( a_f \). The result should then be replica-continued to quarter-integer values of \( n_r \). This continuation provides the correct prescription for calculating any correlation function in the rooted theory from the staggered SET. Of course, in practice we will not know the precise coefficients of powers of \( n_r \) in the Symanzik coefficients; indeed in practical situations the Symanzik coefficients must be treated as unknown numbers, to be fitted from numerical data. However, it suffices for our argument to know that the dependence is polynomial. When we continue in \( n_r \), we then need only continue the explicit \( n_r \) dependence coming from loops, giving a result as usual in terms of unknown Symanzik coefficients.

### B. Power counting

A cornerstone in the argument of the previous section is the expansion in Eq. (3.9), which is convergent if the norm of \( D^{-1}_{\text{inv},n} \Delta_n \) is small enough. In this subsection, we consider this condition in more detail. There are two issues to be considered: the effect of insertions of \( \Delta_n \), as well as the size of the full object in which we expand, \( D^{-1}_{\text{inv},n} \Delta_n \).

In general, the SET for a lattice theory with lattice spacing \( a \) is constructed by matching correlation functions in an expansion in \( a p \), with \( p \ll 1/a \) a generic momentum, to the underlying lattice theory. To make the matching possible in perturbation theory, one should also take \( p \gg \Lambda_{\text{QCD}} \). The Symanzik coefficients are extracted by computing suitable one-particle irreducible correlation functions in the lattice theory, taking all the (nonexceptional) external momenta to be of order \( p \) [27]. For the part coming from the fermions, this amounts to expanding \( D^{-1}_{\text{latt}} \) around \( D^{-1}_{\text{cont}} \), namely to an expansion in \( D^{-1}_{\text{cont}}(D_{\text{latt}} - D_{\text{cont}}) \), where \( D_{\text{cont}} \) is the Dirac operator for the continuum-limit theory, and \( D_{\text{latt}} \) is the Dirac operator of the lattice theory. Because \( D_{\text{latt}} - D_{\text{cont}} \) is an irrelevant operator, we expect \( \|D_{\text{latt}} - D_{\text{cont}}\| \ll a p^2 \). Also, on dimensional grounds, \( \|D^{-1}_{\text{cont}}\| \sim 1/p \). Putting it together we conclude that \( \|D^{-1}_{\text{cont}}(D_{\text{latt}} - D_{\text{cont}})\| \sim a p \) is the relevant estimate for the construction of the SET. Observe that this argument is insensitive to the long-distance physics, because the effective infrared cutoff on the loop momenta is \( p \), and by assumption \( p \gg \Lambda_{\text{QCD}} \). In particular, the estimates are independent of the quark masses.

In the above argument we have implicitly assumed that the momentum flowing through a particular (sub-)diagram is of order \( p \). This need not be true for sub-diagrams with a non-negative degree of divergence, where all ultraviolet momenta may contribute significantly to the loop integrals. In general, counter terms will need to be added in order to absorb contributions from such diagrams; in a renormalizable theory there are only a finite number of counter terms that need to be adjusted. Symmetries may exclude (some of) these counter terms.

Let us now study how these general considerations enter the construction of the SET for the generalized theory (3.9). Our starting point will be the \( t = 0 \) taste-singlet theory. This theory is local, because \( n_s \) is integer. In order to reach the generalized staggered theory from the taste-singlet theory, we have to expand the propagator \( (D_{\text{inv},n} + t \Delta_n)^{-1} \) around \( D_{\text{inv},n}^{-1} \), and eventually set \( t = 1 \). The object in which we are expanding is thus \( D_{\text{inv},n}^{-1} \Delta_n \). Since \( \Delta_n \) is an irrelevant operator (cf. Eq. (3.5)), repeating the above general arguments leads to the estimate \( \|D_{\text{inv},n}^{-1} \Delta_n\| \sim a_f p \), if the momentum flowing through the diagram is order \( p \).

As noted above, we must separately consider sub-diagrams with a non-negative degree of divergence. The contributions of such sub-diagrams depend crucially on the number of blocking steps \( n \), as we now explain.
Consider first what happens for \( k = n = 0 \), namely, when we have performed only the first special RG step that takes the fermions from the one-component to the taste basis. We then have \( a_c = 2a_f \). When we extract the Symanzik coefficients from a lattice calculation, the loop momenta live on the coarse lattice. But since the coarse and fine lattice spacings differ only by a factor of two, the loop momentum can go as high as \( p \sim 1/a_f \). In the divergent sub-diagrams we thus have \( \| D_{inv,n}^{-1} \Delta_n \| \sim 1 \). Indeed, for \( a_c = 2a_f \), the generalized staggered theory will develop \( O(1/a_c) = O(1/a_f) \) mass terms, since shift symmetry and \( U(1) \) symmetry (for any \( t \neq 1 \)) are broken at the (common) lattice scale.\(^{16}\)

The situation is qualitatively different after a large number \( n \) of RG steps has been performed. Because the lattice calculation is performed on the coarse lattice,\(^ {17} \) the maximal momentum that can flow through any sub-diagram is now of order \( 1/a_c \), and one arrives at the estimate (3.5) for the magnitude of insertions of \( \Delta_n \). The estimate \( \| D_{inv,n}^{-1} \Delta_n \| \sim a_f p \) still holds, but, what has changed is that now the maximal value that \( p \) can reach is \( 1/a_c \ll 1/a_f \). The conclusion is that, for extracting the Symanzik coefficients, the appropriate estimate is just that of Eq. (3.8):

\[
\| D_{inv,n}^{-1} \Delta_n \| \lesssim a_f/a_c .
\]

This estimate is valid in the taste-singlet, \( t = 0 \) theory, on the same grounds as for any other local theory, and we will thus assume that it is valid nonperturbatively as well. This is all we need, because the staggered theory is constructed as an expansion in \( D_{inv,n}^{-1} \Delta_n \) around the taste-singlet theory.

We end this subsections with three comments. First, it should be noted that, in Ref. \[15\], the bound

\[
\| D_{inv,n}^{-1} \Delta_n \| \lesssim a_f/(ma_c^2)
\]

was used, with \( m \) the renormalized quark mass after \( n \) RG steps. Clearly, the bound (3.11) is far weaker than (3.10), and it implies that the chiral \( (m \to 0) \) limit can be taken only after the continuum \( (a_f \to 0) \) limit. In Ref. \[15\], this was necessary in order to place a uniform bound on the difference between any taste-singlet correlation function and the corresponding rooted correlation function on any (including the most infrared) scale, thereby establishing the existence of the (correct) continuum limit for the rooted theory. In contrast, assuming that the scaling (3.5) holds, the bound (3.11) is much too generous for the derivation of the SET for the generalized theory (3.9), as we have seen above. In particular, it follows that this SET is well-defined in the chiral limit, as is the chiral effective theory that can be derived from the SET. The requirement that the chiral limit for staggered fermions be taken after the continuum limit \[8, 43, 44, 45, 46\] is then reproduced by calculations within staggered ChPT \[45\]. Note that, while Ref. \[45\] finds many standard quantities for which the limits commute in SChPT, other quantities for which the limits do not commute are also discussed.

Our second comment is that the original staggered theory has no power divergences, because of shift and \( U(1) \), symmetry. This is therefore also true for the \( n \)-times blocked staggered theory (3.1), and for the corresponding SET. Moreover, for large \( n \), the SET for the generalized theory (3.9) at arbitrary values for \( t \in [0, 1) \) is related to the SET at \( t = 1 \) by a convergent expansion in \( t \), equivalently in \( \epsilon_n = a_f/a_c \). The implication is that, for all

\(^{16}\) The breaking of shift symmetry is qualitatively the same as in the theory studied in Ref. \[42\].

\(^{17}\) See Ref. \[15\] for a detailed discussion on how the coarse-lattice diagrammatic calculation is related to a calculation in the underlying fine-lattice staggered theory.
t, the SET for the generalized theory (3.9) has no power divergences in $1/a_f$, but only in $1/a_c$. Examples of this are given in Sec. IV below.

Finally, we remark that the framework introduced here resolves a concern, discussed in Ref. [10], about the renormalizability of the rooted staggered theory. The concern is the following: the complete notion of renormalizability requires not only that (infinite) counterterms can be chosen to make amplitudes finite, but also that the finite parts of counterterms can be chosen to bring the theory into a given scheme. While we know that the staggered theory is renormalizable for integer $n_r$, for non-integer $n_r$ this notion of renormalizability requires that the finite parts of counterterms, as well as the infinite parts, are polynomial in $n_r$ to any finite order in perturbation theory. In Ref. [10], the condition on the finite parts was introduced as an additional assumption, albeit a plausible one. Here, such a separate assumption is unnecessary. Under the assumptions of the RG approach [15], the taste-singlet (re-weighted) theory, defined by setting $t = 0$ in Eq. (3.9), is a local theory of $n_s$ fermions, that moreover becomes a perfect-action lattice theory in the limit $a_f \to 0$, for any fixed $a_c$. Thus one expects its renormalizability to follow straightforwardly by standard arguments. The rooted staggered theory is then reached by expanding in $t$, and setting $t = 1$ and $n_r = n_s/4$. Because of the bound (3.10), the expansion in $t$ just brings in positive powers of $a_f$, and all finite (and infinite) parts of the counterterms are unaffected for any $n_r$. Thus the rooted staggered theory is renormalizable if the taste-singlet theory is. In addition, the two theories have the same counterterms.

C. Partial quenching

Unlike other lattice discretizations of QCD, the continuum limit of the rooted staggered theory is, inherently, a partially-quenched theory [10, 14, 16, 46]. This remains true when we consider the staggered sector of our generalized lattice theory (3.9) all by itself. Let us work out the example of a target theory with $n_s$ degenerate quarks. Our starting point is the generalized lattice theory with the same $n_s$, and with $t = 1$. In order to obtain the set of all correlation functions of the physical $n_s$-flavor theory in the continuum limit, we need to let the combination of replica and taste indices of the external lines assume precisely $n_s$ distinct values. This can, for example, be accomplished by fixing the taste index of the external legs to a single value (for example, 1), and letting the replica indices take on $n_s$ values (for example 1, 2, ..., $n_s$). Alternatively, we could use all four taste indices and only $[n_s/4]$ replica indices, where the square brackets denote rounding up to the next integer. (In this case, unless $n_s/4$ is already an integer, not all taste indices will be used in conjunction with each replica index.) Many other similar choices, as well as other types of embeddings for certain classes of physical correlation functions [8, 14], are also possible. Prior to the replica continuation, the lattice theory is local. The source term in Eq. (3.9) must accommodate all the degrees of freedom, as specified above, that will be used in physical correlation functions. Therefore, we must consider only theories where $n_r$, the (still integer!) number of staggered replicas, is not smaller than $[n_s/4]$.

When we perform the replica continuation we set the power of the staggered and ghost determinants in Eq. (3.9) to $n_r = n_s/4$. Since we have already set $t = 1$, if we turn off all sources, the partition function of the generalized theory reduces to the rooted partition function, in its RG-blocked dress (3.1). During the replica continuation of any correlation function, by definition we hold fixed all indices of the external legs, including in particular the replica (and taste) indices. This means that the number of replicas in the source term of
Eq. (3.9) must stay equal to or larger than $[n_s/4]$. The mismatch created between the power of the staggered (or ghost) determinant and the multiplicity of the corresponding external sources means that the staggered sector has in itself been partially-quenched unless $n_s$ is a multiple of 4.

After the replica continuation, the correlation functions of the EFT reproduce those of the rooted lattice theory to the same order in $a_f$. We stress again that the replica continuation at the level of the EFT is well defined because, as we have shown, to any order in $a_f$ the $n_r$-dependence in the underlying lattice theory (3.9) assumes the form of a finite-degree polynomial.

In our above example, be it before or after the replica continuation, the $\text{replica} \times \text{taste}$ multiplicity of the staggered fields used to generated physical correlation functions is equal to or larger than $4[n_s/4]$, which is to be compared with the $n_s$ physical flavors of the target theory. As a result, the total number of available valence degrees of freedom will in general exceed the physical number, and, when we finally take the continuum limit, the physical correlation functions will form a proper subset of the set of all (partially-quenched) correlation functions. This conclusion is in fact valid for any target theory. The only exception is a target theory in which the multiplicity of every mass-degenerate quark species is divisible by four, in which case the theory may be obtained in the continuum limit of an unrooted staggered theory.

Another conclusion is that the partially-quenched representation obtained in the continuum limit is not unique. The only restriction is that the set of all partially-quenched correlation functions must be large enough to accommodate all the physical correlation functions of the target continuum theory. With the minimal choice of replicas on the external lines, $[n_s/4]$, the vector $\text{replica} \times \text{taste}$ symmetries are represented as a $U(4[n_s/4] \mid 4[n_s/4] - n_s)$ graded group on the continuum-limit correlation functions. Had we initially allowed for $n' \geq [n_s/4]$ values of the replica index on the external legs, all the physical correlation functions of the target theory would still be reproduced once we performed the replica continuation (followed by the continuum limit). But there would be more ways of embedding a given physical correlation function in the space of all correlation functions. Correspondingly, the $\text{replica} \times \text{taste}$ symmetries would be represented as an $U(4n' \mid 4n' - n_s)$ graded group. The arbitrariness in picking a range $n' \geq [n_s/4]$ for the external-legs replica index thus entails the existence of infinitely many partially-quenched representations in the continuum limit, all of which share the same physical subspace.

In the rooted theory, closed (“sea-quark”) fermion loops as well as (“valence-quark”) fermion lines attached to external legs both originate from the same staggered fields. Therefore the sea and valence masses are equal, and there is no clear-cut distinction between the sea and valence sectors. This is a necessary condition for the emergence of a unitary, physical subspace in the continuum limit.

In practice, it is often useful to explore unitarity-violating correlation functions in which the valence-quark mass is allowed to vary away from the sea-quark mass. This situation is what is usually referred to as partial quenching. As we have just explained, the continuum limit of the rooted theory is automatically a partially-quenched theory, albeit with equal sea and valence masses. If it is desired to study different sea and valence masses, it is straightforward to add a (generalized-)staggered valence sector to the generating functional (3.9),

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18 Correlation functions lying outside of the physical subset may exhibit various types of pathological behavior [16, 46].
by simply inserting a factor

$$\exp \left[ \overline{\eta}_v \left( D_{\text{inv},n}^v + t_v \Delta_n \right)^{-1} \otimes I_{n_v} \right] \eta_v$$

(3.12)

into the integrand. The superscript \( v \) on \( D_{\text{inv},n}^v \) indicates that a different quark mass may have been chosen in the valence sector. For \( t_v = 1 \) the valence sector has all staggered symmetries. Again, for \( a_f \) small enough, an expansion can be set up in \( t_v \), just as before.

In Eq. (3.12), \( \eta_v \) and \( \overline{\eta}_v \) are sources for any desired number \( n_v \) of valence (generalized) staggered fields. To avoid confusion we stress that, even if the valence-sector source term (3.12) has been added to the generating functional (3.9), we cannot dispose of the original source terms. The reason is that, if we want to consider the SET for both sea and valence quarks, we need sources for both in order to match the complete set of partially-quenched correlation functions between the lattice and the effective theory. With the valence sector (3.12) in place, the replica \( \times \) taste symmetries form an \( U(4n' + 4n_v | 4n' + 4n_v - n_s) \) graded group in the continuum limit. (Of course, these symmetries will be softly broken by unequal sea and valence masses.) As before, \( n' \) is the number of distinct values of the replica index that we have allowed for the staggered fields with sea-quark mass on the external legs.

In summary, we have seen that partial quenching occurs at three distinct levels. The generalized theory (3.9) is partially quenched to begin with, because, to keep the taste-breaking effects under control, we had to introduce a taste-singlet sector and a taste-invariant ghost sector. During the replica continuation, the staggered sector undergoes a second-stage partial quenching, created by the mismatch between the power of the determinant and the multiplicity of the sources. Last, if we are interested in different valence and sea masses, we need to introduce a “conventional” valence sector, cf. Eq. (3.12).

IV. EXAMPLES

It is instructive to consider some aspects of the SET to second order in \( a_f \) in more detail.\(^{19} \)

The SET can be written as an expansion in \( a_f, t \) and \( n_r \), and thus takes the general form

$$S(\Psi, \bar{\Psi}, A; a_f, t, n_r) = \sum_{i=0}^{\infty} \sum_{j=0}^{i-1} \sum_{k=0}^{j} (a_f)^i t^j (n_r)^k S_{i,j,k}(\Psi, \bar{\Psi}, A).$$

(4.1)

Here we already took into account that each power of \( t \) has to come with at least one power of \( a_f \), and that each power of \( n_r \) has to be lower than the power of \( t \) (it cannot be equal because \( \text{tr}_{ts}(\Delta_n) = 0 \)). Equation (4.1) is manifestly polynomial in \( n_r \) to any fixed, finite order in \( a_f \). Here we allow all types of quarks (taste-singlet, generalized staggered and ghost) to appear on the external legs. The staggered sector is obtained by setting \( \hat{q} = \bar{q} = \hat{q} = 0 \). The coefficients in \( S_{i,j,k} \) depend on both \( a_c \) and \( n_s \) in all sectors. Because of this, one cannot in general conclude that terms linear in \( a_f \) have to be multiplied by dimension-five operators, etc. As already explained in Sec. IIIA for \( t = 1 \) we may assume that a correlation function calculated in the SET does not depend on \( a_c \) if we set \( n_r = n_s/4 \) after the calculation. In this section, we will consider \( n_r \) integer.

\(^{19} \) In this section we return to the theory defined by Eq. (3.9). The inclusion of valence quarks with a mass unequal to that of the sea quarks, as described in Sec. IIIIC is straightforward.
Because of the way the RG-blocked theory is constructed, for general $t$ the preferred basis for (the generalized staggered sector of) the SET is the RG-taste basis. Using this basis while restricting ourselves to the (generalized) staggered sector, and to $i \leq 2$, the expansion (4.1) takes the explicit form

$$S^{\text{quad}}(q, \bar{q}, A; a_f, t, n_r) = S_{0,0,0}(q, \bar{q}, A)$$

$$+ a_f [S_{1,0,0}(q, \bar{q}, A) + tS_{1,1,0}(q, \bar{q}, A)]$$

$$+ a_f^2 [S_{2,0,0}(q, \bar{q}, A) + tS_{2,1,0}(q, \bar{q}, A) + t^2S_{2,2,0}(q, \bar{q}, A) + n_r t^2 S_{2,2,1}(q, \bar{q}, A)] .$$

The $n_r$-dependent term (the last term) is at this order the only one coming from the expansion of the determinant ratio in Eq. (3.9). The other $t$-dependent terms come from the expansion of the staggered source term in that equation. We note that $S_{1,0,0}$ is taste invariant, because of taste invariance of the $t = 0$ theory. Furthermore, $S_{2,2,1}$ is taste invariant too, because the factor of $n_r t^2$ originates from the determinant ratio in Eq. (3.9), which does not affect the symmetry structure of the SET. The taste structure of the SET is determined by the external legs, which correspond to the source terms in Eq. (3.9). Since the two allowed insertions of $\Delta_n$ have been “used up” by the determinant ratio, only the taste-invariant part of the source term contributes to $S_{2,2,1}$.

If we set $t = 1$ then, as discussed in Sec. III A there exist a field redefinition that brings $S^{\text{quad}}$ to the familiar form of Ref. [22] for $n_r = 1$, or to the form of Refs. [23, 24] for $n_r > 1$. In particular, the redefinition removes the terms linear in $a_f$. The Symanzik coefficients are equal to those of Refs. [22, 23, 24] if one also chooses $n_s = 4n_r$, a multiple of four. For general $n_s$ and $n_r$, the staggered SET is that of Refs. [23, 24], but the coefficients are different functions of $n_r$. This form of $S^{\text{quad}}$ is the one needed for the construction of rSChPT [23], which we will discuss in Sec. V.

The taste-invariant operator $D_{\text{inv}, n}$ has no chiral symmetry, even when the chiral limit is taken in the underlying staggered theory, and we would thus naively expect a linearly divergent mass term of the form $\bar{q}q/a_c$. However, for large $n$, the taste-invariant theory is close to the theory with $t = 1$ in the sense explained in Sec. III B. In order to deviate from the $t = 1$ staggered theory, at least one power of $a_f$, coming from an insertion of $\Delta_n$, is needed. Equivalently, the $1/a_c$ linear divergence has to be multiplied by at least one factor of $\epsilon_n = a_f/a_c$. In fact, even a mass term with magnitude $\sim \epsilon_n/a_c = a_f/a_c^2$ cannot occur. To see this, note that we may write

$$D^{-1}_{\text{taste}, n} = D^{-1}_{\text{inv}, n} - D^{-1}_{\text{inv}, n} \Delta_n D^{-1}_{\text{inv}, n} + \ldots .$$

This shows that the order $a_f$ difference between the $t = 0$ and $t = 1$ theories has to break taste, and therefore a taste-singlet difference has to be of order $a_f^2$. Singlet mass terms can thus only occur in $S_{2,0,0}$ and $S_{2,2,0}$, with opposite coefficients such that they cancel at $t = 1$.

Next, let us consider nonsinglet mass terms, i.e., terms of the form $\overline{7}Kq/a_c$ with some (momentum-independent) kernel $K$ for which $\text{tr}_\text{ts}(K) = 0$. At order $a_f$ a nonsinglet mass term can only be part of $S_{1,1,0}$, because $S_{1,0,0}$ is taste invariant. However, staggered symmetries at $t = 1$ forbid such terms in $S_{1,1,0}$, thus excluding this possibility. At order $a_f^2$, a

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20 In particular, the Symanzik coefficients of all taste-breaking four-fermion operators in the SET are independent of $n_r$ and depend only on $n_s$.
nonsinglet mass term can only appear in \( tS_{2,1,0} + t^2S_{2,2,0} \) because \( S_{2,0,0} \) and \( S_{2,2,2} \) are taste invariant. Let us assume that a bilinear \( \mathbf{7}Kq \) appears with coefficient \( c_1 \) in \( S_{2,1,0} \), and with coefficient \( c_2 \) in \( S_{2,2,0} \). Staggered symmetries then imply that \( tc_1 + t^2c_2 = 0 \) at \( t = 1 \), and thus \( c_1 + c_2 = 0 \). Any nonsinglet mass term at order \( a_f^2 \) is therefore proportional to \( t(t-1) \). Simply put, there has to be a factor \( t \) in order to break taste symmetry, and a factor \( t - 1 \) to break staggered symmetries, which include \( \Gamma_4 \) and \( U(1)_c \).

In order to exclude various contributions to the nonsinglet mass terms in the above argument, we used the fact that mass terms cannot be introduced or removed by field redefinitions. As we now explain, the same is not true for operators of dimension five or higher: they cannot be excluded by arguments based on field redefinitions. With the taste basis of Eq. (4.2), we know from Eq. (2.28) that taste nonsinglet Wilson-like dimension-five operators will already appear in \( S_{1,1,0} \). Of course, being nonsinglet, such terms will have to vanish at \( t = 0 \). In addition, because of staggered symmetries, a local field redefinition can be found removing such terms at \( t = 1 \). However, this same field redefinition applied to the SET at \( t \neq 1 \) will, in general, introduce taste-breaking terms at \( t = 0 \). So, all we can conclude is that \emph{before} the field redefinition such terms are proportional to \( t \), while \emph{after} the field redefinition they are proportional to \( t - 1 \). We cannot conclude that they are proportional to \( t(t-1) \). In the case of the mass terms discussed above, stronger conclusions are possible, because dimension-three terms cannot be removed by a field redefinition.

V. STAGGERED CHIRAL PERTURBATION THEORY

In this section, we will discuss the transition from the SET to staggered ChPT, or SChPT. For integer \( n_r \) and \( n_s = 4n_r \) the derivation was first given in Ref. [22] (for \( n_r = 1 \)) and Ref. [23] (for \( n_r > 1 \)), and we refer to those papers for details on the explicit construction of the SChPT chiral Lagrangian. Here we will focus on the continuation to \( n_r = n_s/4 \), with \( n_s \) as always a positive integer.

A. The transition to staggered chiral perturbation theory

In the previous section we explained how the appropriate SET for a rooted staggered theory can be constructed. Holding \( n_s \) fixed, the Symanzik coefficients are polynomials in \( n_r \), and thus have no singularities at quarter-integer values of \( n_r \). The rooted staggered SET is obtained as a replica rule: calculate correlation functions to a given order in \( a_f \), then set \( n_r = n_s/4 \). For the next step—the transition to ChPT—we must again retain both \( n_s \) and \( n_r \) as independent variables. In ChPT, as for the SET, the replica continuation in \( n_r \) will be well-defined at fixed \( n_s \), and SChPT with the replica rule, namely rSChPT, will be recovered after the continuation to \( n_r = n_s/4 \).

When we calculate correlation functions using the SET for the generalized theory (3.9), dependence on \( n_r \) occurs in two ways: through the polynomial dependence of the Symanzik coefficients, and through fermion loops. Once we have calculated a certain correlation function to some order in \( a_f \) and to a given order in the loop expansion, the dependence on \( n_r \) is thus explicitly known. Technically, this dependence will not be a polynomial, because only the inverse quark propagators, and not the quark propagators themselves, depend polynomially on \( n_r \). However, each quark propagator can be re-expanded around that of the \( t = 0 \) theory in terms of \( a_f \), and thus \( n_r \), just as in the underlying lattice theory (Eq. (3.9)).
This sets the stage for the derivation of the appropriate chiral theory for QCD with rooted staggered fermions. The continuum chiral theory is an effective theory for low-energy scales where only Goldstone bosons can appear on the external lines. It can be organized as an expansion in \( p/\Lambda_{\chi} \), where \( \Lambda_{\chi} \sim 1 \text{ GeV} \) is the chiral scale separating other hadrons from the Goldstone bosons \[47\]. The chiral effective theory can be generalized to include discretization errors, in an expansion in \( a = a_f \). The chiral effective theory is to be constructed by matching its correlation functions to those of the underlying theory in a double expansion in \( p/\Lambda_{\chi} \) and \( a_f p \). In practice, the low-energy constants (LECs) of the chiral theory cannot be calculated by analytic methods, and are determined by fitting experimental or numerical data.

For positive integer \( n_s \) and \( n_r \), the underlying lattice theory is local, as is the SET, and the transition to the chiral theory is more or less standard \[22, 23, 48, 49\]. In addition, the estimate \( (3.10) \) is still expected to hold, even though it cannot be checked in perturbation theory, because in this case the correct degrees of freedom for \( p \lesssim \Lambda_{\chi} \) are no longer quarks and gluons. Using the expansion \( (3.9) \) just as in Sec. III, this implies that the LECs of the chiral theory again have to be polynomials in \( n_r \). Finally, setting \( t = 1 \) and performing the continuation to \( n_r = n_s/4 \) we recover the replica-continued SChPT, or rSChPT, of Refs. \[14, 23\].

We assume here that the contributions of ghosts and taste-singlet quarks in the sea will cancel to all orders in the partially quenched ChPT once we put \( n_r = n_s/4 \). All differences between the current rSChPT and the standard rSChPT \[14, 23\] (which does not have the taste-singlet and ghost sectors) will then disappear in the limit \( n_r = n_s/4 \), as long as we choose not to put ghosts and taste-singlet quarks on the external lines. Since the ghost and taste-singlet Dirac operators and masses are identical, this cancellation is trivial at the QCD level, but not completely trivial beyond one loop at the chiral level.\[22\] We believe, though, that the cancellation is almost certainly true order by order in SChPT, and that it will probably be possible to construct a “quark flow” proof of this. This completes our argument that rSChPT is the correct chiral theory for QCD with rooted staggered fermions.

### B. An example

It is instructive to see how our approach works in a concrete example. We will re-consider the leading-order contribution in rSChPT to the connected scalar two-point function, previously described in detail in Sec. 6 of Ref. \[14\]. Adding a scalar source \( s(x) \) to the generating functional, this two-point function is defined as the connected part of the second derivative with respect to this source (setting \( s = 0 \) after taking the derivatives). Adapting it to our generalized theory, Eq. (27) of Ref. \[14\] takes the form\[23\]

\[
Z(s) = \frac{\int \mathcal{D}U \prod_{k=1}^{n} \mathcal{D}V^{(k)} \; B_n \left( \frac{s}{4} \right) \; \text{Det}^{n_r} \left( D_{\text{taste,n}} + s \otimes 1 \right) \; \text{Det}^{(n_s - 4n_r)} \left( \tilde{D}_{\text{inv,n}} + s \right)}{\int \mathcal{D}U \prod_{k=1}^{n} \mathcal{D}V^{(k)} \; B_n \left( \frac{s}{4} \right) \; \text{Det}^{n_r} \left( D_{\text{taste,n}} \right) \; \text{Det}^{(n_s - 4n_r)} \left( \tilde{D}_{\text{inv,n}} \right)}, \quad (5.1)
\]

\[21\] Again, the only element of this transition that is not absolutely standard is the assumption that all steps can be carried out for partially-quenched theories, since the generalized theory \( (3.9) \) is partially quenched.

\[22\] We thank S. Sharpe for emphasizing this point to us.

\[23\] The connection with the method and notation of Ref. \[14\] is explained in Sec. V C.
where we only indicated the \( n_s \) dependence of \( B_n \) explicitly, cf. Eq. (3.9). Here we have chosen \( t = 1 \), but have not yet set \( n_r = n_s/4 \). It is important to keep \( n_r \) integral at this stage in order to develop the chiral theory; keeping \( n_r \neq n_s/4 \) also allows us to highlight the different ways in which \( n_s \) and \( n_r \) appear.

In Eq. (5.1), we are starting from the fact that correlation functions generated in the rooted staggered theory by the taste-singlet meson source \( s(x) \otimes 1 \) are identical, in the continuum limit, to the desired correlations generated by \( s(x) \) in the target QCD theory. (See Eq. (12) of Ref. [8].) Note, however, that we have coupled \( s(x) \) not only to the staggered quarks but also to the ghost and taste-singlet quarks. This keeps the expansion in \( n_r \) under control because the staggered and ghost contributions differ only by the small taste-violating term \( \Delta_n \). Requiring that the taste-singlet and ghost quarks cancel at \( n_r = n_s/4 \) then implies that \( s(x) \) also couples to the taste-singlet quarks.

Even without a replica continuation, the lattice theory defined by Eq. (5.1) is, as we discussed already above, a partially quenched theory with \( n_r \) staggered fermions, \( n_s \) taste-singlet fermions, and \( 4n_r \) taste-singlet ghosts. It differs from Eq. (3.9) in the way it is coupled to sources. Of course, the correlation functions that are generated by taking derivatives with respect to \( s(x) \) can also be generated by taking joint derivatives with respect to \( H(x) \) and \( \bar{H}(x) \) (with one each for each space-time point). Regardless of which type of source is used, the dynamics is that of the sea-quark loops, and is controlled by the determinants in Eq. (3.9). Since in this subsection we are only interested in the scalar two-point function, the formulation with the source \( s(x) \) is simpler. Note that here we need the complete effective theory, including taste-singlet and mixed sectors, because the source \( s(x) \) couples to all quarks.

At leading order in ChPT, the scalar two-point function consists of a sum over one-loop diagrams, with pseudo-scalar mesons on the loop (cf. Fig. 2 of Ref. [14]). Since \( s(x) \) couples to all bilinears, staggered, taste-singlet, and ghost, all types of pseudo-scalar mesons contribute to these diagrams, including fermionic mesons made out of quarks and ghosts, and mesons made only out of ghosts. Because the taste-singlet quarks and ghost have the same Dirac operator \( \tilde{D}_{\text{inv},n} \), the result for the scalar two-point function that we will give below is that of a theory with \( n_s - 4n_r \) taste-singlet quarks, irrespective of the value (and in particular, sign) of \( n_s - 4n_r \). In the interest of brevity, therefore, the discussion below will simply assume that we are dealing with a theory with a positive number \( n_s - 4n_r \) of taste-singlet quarks (as well as \( n_r \) staggered quarks).

In Ref. [14] it was shown that, as expected, in the one-flavor theory (for which \( n_s = 4n_r = 1 \) only the non-Goldstone, heavy pseudo-scalar taste-singlet state (the “\( \eta' \)”)) contributes to this two-point function in the continuum limit, despite the presence of fifteen additional light pions in the underlying staggered theory. That this has to happen follows from the general discussion given in Ref. [8]. Here we will not repeat the details of the calculation given in Ref. [14], but only keep track of how the results change in the generalized setup of the present paper, and see how \( n_r \) and \( n_s \) appear in the final result. With Ref. [14], we keep the singlet pseudo-scalar state in the calculation for pedagogical reasons.

There are now three kinds of pions, those made out of staggered quarks, those made out of taste-singlet quarks, and “mixed pions,” made out of staggered and taste-singlet quarks. The leading-order masses of the pseudo-scalars in the staggered sector are given by

\[
M_\Xi^2 = 2\mu m + a_f^2 \Delta_\Xi ,
\]

where \( \Xi \in \{I, \xi_\mu, i\xi_\mu\xi_\nu(\mu > \nu), i\xi_\mu\xi_5, \xi_5 \} \) labels the taste of each of the sixteen staggered
pseudo-scalars (for each replica), and the $\Delta_\Xi$ are four LECs\(^{24}\) representing the taste splittings; \(m\) is the quark mass. Then there are pions made out of only taste-singlet quarks, with mass\(^{25}\)

\[
M_{ts}^2 = 2\mu m + a_f^2 \Delta_{ts}.
\]

(5.3)

Finally, there are mixed pseudo-scalars made out of one taste-singlet and one staggered quark. The mass of the latter can be parametrized, to leading order, as\(^{50}\)

\[
M_{mix}^2 = 2\mu m + 2a_f^2 \Delta_{mix},
\]

(5.4)

with, in general, $\Delta_{mix} \neq \Delta_{ts}$. The fact that the mass of the mixed mesons does not depend on their staggered taste follows, as in Ref.\(^{50}\), from shift symmetry, which forbids taste-violating staggered bilinears, and therefore forbids taste-violating four-quark operators with one staggered and one taste-singlet bilinear. Note that all the above masses (in particular, \(M_{I}\)) are the pseudo-scalar masses before including the effect of the anomaly.

The LECs $\mu$, $\Delta_\Xi$, $\Delta_{ts}$ and $\Delta_{mix}$ have unknown dependence on \(n_s\), but do not depend on \(n_r\). For $\mu$ this is obvious, because it is a continuum LEC, and the continuum theory does not depend on \(n_r\) at all, but only on \(n_s\). (Recall that, in the continuum limit, the determinants ratio (3.7) goes to one.) Because $\Delta_\Xi$ represents an order $a_f^2$ effect, it can, according to our general arguments, be at most linear in \(n_r\). In practice, it is independent of \(n_r\), because symmetry-breaking terms of order $a_f^2$ in the SET do not originate from the determinant ratio but only from the source term in Eq. (3.9) (cf. the discussion below Eq. (4.2)); similar arguments apply for $\Delta_{mix}$ and $\Delta_{ts}$. At higher order there will be \(n_r\)-dependent corrections to Eqs. (5.2) through (5.4) coming from insertions of the operator \(S_{2,2,1}\) in Eq. (4.2). The taste-singlet and mixed mesons also contribute to our scalar two-point function as long as \(n_r \neq n_s/4\).

Of course, the singlet pseudo-scalar (the "\(\eta'\)") will not be a Goldstone boson. It will pick up a mass that does not vanish in the chiral and continuum limits. In the continuum limit, the $\eta'$ mass is given by

\[
M_{\eta'}^2 = 2\mu m + n_s n_0^2 / 3,
\]

(5.5)

where \(n_0^2\) is the double-hairpin parameter (cf. Ref.\(^{23}\)).

Again, since the continuum limit does not depend on \(n_r\), the parameter \(m_0^2\) does not depend on \(n_r\).\(^{26}\) Away from the continuum limit, mixing takes place in the neutral meson sector because of different scaling violations in \(M_I^2\) and \(M_{ts}^2\). This mixing leads to the

\(^{24}\) $\Delta_{\xi_s} = 0$ because this taste corresponds to the exact Goldstone bosons.

\(^{25}\) The operator $\tilde{D}_{inv,n}$ has no chiral symmetry, and the taste-singlet quark mass is additively renormalized by an amount of order $a_f^2$ (see Sec. IV). The quantity $\Delta_{ts}$ represents the effect of this renormalization on the meson mass. In the case of the mixed pseudo-scalar mass, Eq. (5.3), such renormalization is absorbed in $\Delta_{mix}$, which must be present in any case.

\(^{26}\) There are in general corrections of order $a_f^2$, as well as momentum-dependent contributions, to this parameter, but they do not invalidate our conclusions. Following Ref.\(^{14}\), other hairpin contributions of order $a_f^2$ will be ignored as well.
appearance of pseudo-scalar mesons with masses $M_{\pm}$ given by

$$M_{\pm}^2 = \frac{1}{2} \left( n_s \frac{m_0^2}{3} + M_I^2 + M_{\pm}^2 \pm \sqrt{\left( n_s \frac{m_0^2}{3} \right)^2 - 2(n_s - 8n_r)\frac{m_0^2}{3}a_f^2 \Delta + a_f^4 \Delta^2} \right),$$

$$a_f^2 \Delta \equiv M_I^2 - M_{ts}^2 = a_f^2(\Delta_I - \Delta_{ts}).$$

(5.6)

In the continuum limit, $\Delta = 0$ and $M_I^2 = M_{ts}^2 \equiv 2\mu m$, so the expression for $M_{\pm}^2$ simplifies to Eq. [5.5].

In order to give the expression for the scalar two-point function, we define single-particle propagators

$$D_A(p) = \frac{1}{p^2 + M_A^2}, \quad A = \Xi, \, ts, \, mix,$$

(5.7)

and hairpin “double poles”

$$X_{I,ts}(p) = X_{ts,I}(p) = \frac{1}{(p^2 + M_{ts}^2)(p^2 + M_I^2)}.$$

(5.8)

For $\Delta = 0$, all hairpin double poles become equal, and $D_I(p) = D_{ts}(p)$.

The result for the Fourier transform $\tilde{G}(p)$ of the scalar two-point function is

$$\tilde{G}(q) = \mu^2 \int \frac{d^4 p}{(2\pi)^4} \left\{ 2n_s^2 \sum_{\Xi} D_{\Xi}(p)D_{\Xi}(p + q) + 16n_r(n_s - 4n_r)D_{\text{mix}}(p)D_{\text{mix}}(p + q) + 2(n_s - 4n_r)^2D_{ts}(p)D_{ts}(p + q) - 8n_r\frac{m_0^2}{3}(D_I(p)X_{I,I}(p + q) + D_I(p + q)X_{I,I}(p)) - 2(n_s - 4n_r)\frac{m_0^2}{3}(D_{ts}(p)X_{ts,ts}(p + q) + D_{ts}(p + q)X_{ts,ts}(p)) + \left( \frac{m_0^2}{3} \right)^2 \left[ 32n_s^2X_{I,I}(p)X_{I,I}(p + q) + 2(n_s - 4n_r)^2X_{ts,ts}(p)X_{ts,ts}(p + q) + 16n_r(n_s - 4n_r)X_{I,ts}(p)X_{I,ts}(p + q) \right] \right\}. \quad (5.9)$$

The explicit factors $m_0^2/3$ can be eliminated from this expression by using the relation

$$\frac{m_0^2}{3} = \frac{1}{n_s} (M_+^2 + M_-^2 - M_I^2 - M_{ts}^2).$$

(5.10)

As discussed above, if we expand out the masses $M_{\pm}^2$ in powers of $a_f$, the $n_r$ dependence of Eq. [5.9] is polynomial. The $n_s$ dependence is not polynomial because the LECs $\mu$, $\Delta_{\Xi}$, $\Delta_{ts}$ and $\Delta_{\text{mix}}$ depend on $n_s$ implicitly in an unknown way.
Let us compare the result (5.9) to a similar calculation, done in the taste-singlet theory obtained by replacing $D_{taste,n}$ with $D_{inv,n}$ in Eq. (5.1). To order $a_f^2$, this corresponds to setting $M_I^2 = M_{mix}^2 = M_{ts}^2$. The expression for $M_+^2$ (cf. (5.6)) again simplifies to (5.5), except that $2\mu m$ is replaced with $M_{ts}^2$, because $M_{ts}^2$ may still include discretization errors. Instead of Eq. (5.9) we now arrive at

$$\tilde{G}(q) \rightarrow 2\mu^2 \int \frac{d^4p}{(2\pi)^4} \left\{ (n_s^2 - 1) \frac{1}{p^2 + M_{ts}^2} \frac{1}{(p + q)^2 + M_{ts}^2} \right. \right.

$$

$$+ \left. \frac{1}{p^2 + M_{t',ts}^2} \frac{1}{(p + q)^2 + M_{t',ts}^2} \right\},$$

$$M_{t',ts}^2 = M_{ts}^2 + n_s \frac{m_0^2}{3}.$$ 

As expected, this result is $n_r$-independent. The first term on the right-hand side is recognized as the anticipated contribution of the $n_s^2 - 1$ degenerate Goldstone pions of a theory with $n_s$ (mass-degenerate) flavors.

Replacing $D_{taste,n}$ with $D_{inv,n}$ means that the product of determinants in the denominator of Eq. (5.1) collapses to $\text{Det}^{n_s}(\tilde{D}_{inv,n})$, with a similar simplification in the numerator. Our calculation thus explicitly demonstrates how we may consider the rooted staggered theory as a local taste-singlet theory with small, nonlocal corrections of order $a_f^2$, which, to any fixed order in $a_f$, are polynomial in $n_r$. Our example also illustrates how the nonlocality of the rooted staggered theory manifests itself in the low-energy EFT: while Eq. (5.11) satisfies unitarity, Eq. (5.9), at $a_f \neq 0$, does not. This is most easily seen by noting the presence of the minus signs multiplying various terms in Eq. (5.9), in what should be (in a unitary theory) a positive definite correlation function.

### C. Comparison with Reference [14]

The present work may be compared with the complementary argument for the validity of rSChPT given in Ref. [14]. That argument starts from ChPT for a rooted theory with four degenerate flavors of staggered fermions, which thus describes four mass-degenerate quark species. The underlying lattice theory is local, trivially, because it contains the fourth power of the fourth-rooted staggered determinant. Staying entirely within the ChPT framework, Ref. [14] then treats the nondegenerate case by perturbing in the quark masses. An assumption of the analyticity of the expansion around positive quark mass is required at this point. In addition, the replica rule (called the “replica trick” in Ref. [14]) needs to be introduced because the theory becomes nonlocal as one moves away from the degenerate limit. Finally, one of the four masses can be made so large that that quark decouples from the chiral effective theory (at which point it can be thought of as the charm quark). Using an assumption about the details of decoupling, one arrives at rSChPT for three light quarks. The decoupling assumption leaves a small potential loophole in the argument of Ref. [14].

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27 By making use of the general sources in Eq. (3.9) this conclusion applies to any physical correlation function of interest. A by-product is that the generalized theory (3.9) provides an alternative framework to that discussed in Appendix B of Ref. [10] for solving the “valence rooting” problem.
While the three-flavor chiral theory goes over, in the continuum limit, to the standard three-flavor chiral theory of QCD, it is not guaranteed that the LECs have the same numerical values as in QCD. (In the initial four-flavor case, the correctness of the LECs is guaranteed, however.)

Here, we have started instead from the fundamental lattice theory (in RG-blocked form) and have shown how rSChPT may be derived from it, via the SET. The replica rule is given definite meaning in the fundamental theory, so its appearance in the EFTs is completely natural. In contrast, the replica rule in Ref. [14] has, by construction, meaning only at the chiral level. It is for that reason that a distinction was made in Ref. [14] between the power of the staggered determinant at the QCD level, which was called $R$, and the number of replicas introduced at the chiral level, $n_r$. Here, because the replica rule is justified at the QCD level, we need make no such distinction. We do however need to introduce the number of flavors of the taste-singlet quarks, $n_s$, which affects LECs in a nonperturbative (and hence unknown) way in order that the $n_r$ dependence be completely controlled (indeed, polynomial). Thus Ref. [14] and the current work represent two different generalizations of the staggered theory. In the limit $R = n_r = n_s/4$, the two generalizations agree. Since this is the limit we need to take at the end of any rSChPT calculation, it is clear that the two versions of rSChPT give the same results.

Another advantage of the present approach is that it allows us to dispense with the assumptions about decoupling and about the analyticity of the mass expansion. This means that the current argument closes the loophole mentioned above. The continuum low-energy constants are automatically those of QCD with the correct number of flavors.

On the other hand, the current argument, based as it is on Ref. [15], inherits the assumptions of that work. The key assumptions have already been mentioned in the Introduction and explained in Sec. III. They are that:

- The effective action $\delta S_{\text{eff}}$, generated by integrating out fermions on finer lattices, is local.

- The perturbative scaling laws apply, implying that the dimension-five taste-breaking operator $\Delta_n$ goes to zero like $a_f$ (times logarithms) in the continuum limit. This in turn is based on the highly plausible assumption that the theory is renormalizable to all orders in perturbation theory for any $n_r$.

The assumption of taste-symmetry restoration is needed in Ref. [14] too, but only for integer $n_r$, where the scaling argument is completely standard. The argument of Ref. [14] works entirely within the chiral theory, and the resulting rSChPT then implies the symmetry restoration (in the chiral sector) for the rooted case. We also note that, in the RG framework, there is an alternative route to establish the validity of the continuum limit while relying only on the scaling of $\Delta_n$ in the taste-singlet (re-weighted) theory [11]. Since the latter theory is local by the first assumption, the validity of the scaling assumption needed for the RG treatment is very plausible. We remind the reader that there is considerable numerical evidence for the continuum restoration of taste symmetry in the rooted case [13, 18, 19, 20, 21].

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28 We again are assuming that the contributions of ghosts and taste-singlet quarks in the sea cancel to all orders in partially quenched ChPT once there are the same number of ghosts and taste-singlet quarks, \textit{i.e.}, once $n_r = n_s/4$. 

31
Both the present arguments and those of Ref. [14] rely heavily on the validity of the standard partially quenched chiral theory [16] for describing partially quenched fundamental theories that are local. We also need to assume here that the SET exists for partially-quenched theories, as long as the lattice theory is local.

The calculation of the scalar two-point function, presented in Sec. V B, may now be compared to the corresponding calculation in Sec. 6 of Ref. [14]. Note that Ref. [14] considers only the one-flavor case as an example, so to make the connection, we must put \( n_s = 1 \). The result here, Eq. (5.9), then corresponds directly to Eq. (41) of Ref. [14]. We can in fact make the connection at the quark flow level: The first two lines of Eq. (5.9) correspond to Figs. 3(a) and (d) of Ref. [14], the next two lines correspond to Figs. 3(b) and (c), and the last two lines correspond to Fig. 3(e). It is straightforward to check that, if we set \( n_r = n_s/4 = 1/4 \) in Eq. (5.9), and \( R = n_r = 1/4 \) in Eq. (41) of Ref. [14], the results are identical.

VI. CONCLUSION

In this paper we presented a theoretical argument that rSChPT [23] is the correct chiral theory for QCD with rooted staggered fermions. Much evidence in favor of this claim already existed, both on the theoretical side [14], as well as on the numerical side [13, 18, 19, 20]. Here we showed that it is possible to extend the usual construction of the Symanzik effective theory and chiral perturbation theory, to the rooted staggered case. Our arguments apply equally well to any staggered quark action that has the usual staggered symmetries, for example standard (unimproved) staggered [1], Asqtad [28], HYP [29], Fat7bar [30], or HISQ [31] quarks. The version of staggered quarks used will not effect the form of the discretization effects summarized by the effective theory, but does effect the size of these effects, which is reflected in the size of the LECs.

The effective theories are first constructed for a taste-singlet local theory with \( n_s \) physical fermion flavors (the \( t = 0 \) theory of Eq. (3.9)). The rooted, nonlocal staggered theory is then reconstructed as an expansion in the lattice spacing of the underlying staggered theory (i.e., \( a_f \)), by moving smoothly from \( t = 0 \) to \( t = 1 \). In this framework, the dependence on \( n_r \) is polynomial to any finite order in \( a_f \) and to any finite order in the loop expansion. The effective theories, however, are in the first instance only known at integer values of \( n_r \), where they are fairly standard. The polynomial dependence on \( n_r \) allows us to to make the replica continuation of any correlation function, computed order-by-order in the effective theory for integer \( n_r \), to \( n_r = n_s/4 \). Once the value \( n_r = n_s/4 \) is reached, the correct correlation functions of the underlying rooted lattice theory are recovered.

The ability to extend standard techniques for the derivation of the SET and ChPT to rooted staggered fermions does not preclude various sicknesses in the rooted theory at nonzero \( a_f \). Indeed, in Ref. [12] we argued that the rooted theory is nonlocal at nonzero \( a_f \), due to the taste-breaking induced splittings in hadron taste multiplets. It is essential that the replica-continued SET and SChPT reproduce the nonlocal behavior. This happens because loop corrections calculated in these theories have to be continued to noninteger number of staggered replicas as well, and the replica-continued amplitudes cannot be reproduced from

\[ 29 \] For the SET, the relevant loop expansion is the one in fermion loops; for ChPT it is the chiral loop expansion.
any local Lagrangian. An explicit example of this was worked out in Sec. 6 of Ref. [14]; we revisited this example in Sec. V B in our generalized framework.

It is important to list the assumptions that underlie our arguments. The most important assumption is that QCD with rooted staggered fermions has the desired continuum limit. This conclusion, in turn, is based on a number of technical and testable assumptions, as explained in detail in Ref. [15] (see also Refs. [10, 11]). If this conclusion were to turn out to be incorrect, that would also invalidate the analysis presented here. Turning this around, we consider the success of fitting high-precision numerical results with rSChPT as direct evidence that the conclusion of Ref. [15] is, in fact, valid.

In order to keep the replica continuation under control, in Eq. (3.9) we temporarily treated the number of dynamical quarks in the theory \( n_s \) and the power of the staggered determinant \( n_r \) as independent. Because \( 4n_r \) ghosts are needed, we also have to assume that the construction of the SET and ChPT goes through in the standard way for partially-quenched (but local) theories. This second assumption is very common in applications of EFTs to lattice QCD. However, one should keep in mind that, while partially quenched ChPT [16] is by now standard, its foundations are not as firm as for ordinary, unquenched, ChPT. See Ref. [51] for a discussion of this point.

A third assumption is the technical observation that \( D^{-1}_{inv,n} \Delta_n \) has to scale as \( a_f p \), with \( p \) the momentum scale at which a correlation function in the effective theory is matched to the underlying theory. An exception are short-distance contributions coming from sub-diagrams with non-negative degree of divergence in which \( D^{-1}_{inv,n} \Delta_n \) can become as large as \( a_f/a_c \) at most. The end result, the estimate (3.10), is crucial for establishing that the \( n_r \)-dependence of the generalized theory (3.9) is polynomial, to any finite order in \( a_f \). Again, we consider this assumption as noncontroversial, because it underlies the standard derivation of EFTs for local lattice theories, and because it is used only in the \( t = 0 \) theory, which is local by our first assumption. The weaker, quark-mass dependent bound on \( D^{-1}_{inv,n} \Delta_n \) used in Ref. [15] is not needed for the derivation of the effective theories, and both the SET and the chiral theory are valid in the chiral limit. We emphasize here that the physically sensible approach for any staggered theory (rooted or not) is to avoid the region \( m \ll a_f^2 \Lambda_{QCD}^3 \), where lattice artifacts may dominate [8, 44, 45].

In the actual construction of a SET or a chiral theory, use is made of the symmetries of the underlying theory. Particularly important symmetries for staggered fermions are \( U(1)_c \), chiral symmetry and shift symmetry, and we discussed in detail how these are realized at the level of the SET. Generalizing a result previously derived to order \( a_f^2 \) in Ref. [22], we showed that for the SET, shift symmetry enlarges to the direct product of the continuum translation group and the finite discrete group \( \Gamma_4 \). Since this observation holds for the SET, it also holds for any EFT derived from the SET. Finally, we note that our arguments also apply to the cases of rSChPT with baryons or heavy-light mesons.

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30 We note that the same assumption, coupled with the framework introduced in this paper, can be used to make more plausible the argument for perturbative renormalizability of the rooted theory. See the discussion at the end of Sec. III B.
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