Magic angles and current-induced topology in twisted nodal superconductors

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Motivated by the recent achievements in the realization of strongly correlated and topological phases in twisted van der Waals heterostructures, we study the low-energy properties of a twisted bilayer of nodal superconductors. It is demonstrated that the spectrum of the superconducting Dirac quasiparticles close to the gap nodes is strongly renormalized by twisting and can be controlled with magnetic fields, current, or interlayer voltage. In particular, the application of an interlayer current transforms the system into a topological superconductor, opening a topological gap and resulting in a quantized thermal Hall effect with gapless, neutral edge modes. Close to the “magic angle,” where the Dirac velocity of the quasiparticles is found to vanish, a correlated superconducting state breaking time-reversal symmetry is shown to emerge. Estimates for a number of superconducting materials, such as cuprate, heavy fermion, and organic nodal superconductors, show that twisted bilayers of nodal superconductors can be readily realized with current experimental techniques.

Introduction: Controlling the properties and phases of the neutral Bogoliubov-de Gennes (BdG) quasiparticles in superconductors [1, 2] is an outstanding challenge in condensed matter physics. In particular, topologically nontrivial BdG bands [3, 4] hold the promise of hosting the exotic Majorana fermion excitations [5, 6] that can be used to perform topological quantum computation [7]. Moreover, the impact of interactions between the BdG quasiparticles on experimental observations has remained poorly understood even though they are expected to play an important role in nodal [8–13], topological [14], and strongly correlated [15–18] superconductors. However, despite many considered materials [19–23] and nanostructure setups [5, 24–27], the controlled realization of topological and correlated phases of the BdG quasiparticles remains an open problem. Fundamentally, low-energy BdG quasiparticles are composed of a particle and a hole and are therefore charge neutral [1, 2], limiting the utility of electric-field based control commonly used in various semiconductor applications.

Recently, a new paradigm in the engineering of correlated and topological phases has emerged [28], known as “twistronics” [29] or moiré materials [30], that utilizes stacking of two-dimensional materials with an interlayer rotation (i.e. twist as in Fig. 1) to achieve novel properties [31]. Motivated by the discoveries of correlated insulators and superconductivity in twisted bilayer graphene (TBG) [32–35], twisting has been applied to construct multilayer devices based on graphene [36–39] and transition metal dichalcogenides [40–44]; extensions to topological surface states [45–46] and ultra-cold gases [47–49] have been also proposed. In addition to correlation-driven phases, twistronics has been instrumental in realizing topological phases, such as the Chern insulator [37–39, 50].

Here, we demonstrate that twisted bilayers of two-dimensional nodal superconductors (TBSC) can realize topological and interacting superconducting phases that are tunable by experimentally accessible parameters such as applied current, magnetic field, or voltage. The Dirac velocity of the BdG quasiparticles near the zero of the excitation gap (i.e. nodes) is strongly renormalized by the interlayer tunneling and vanishes at a “magic” value of twist angle where a quadratic band touching-type dispersion is found. Displacement field between the layers, Zeeman splitting and an in-plane current can be all used to tune the dispersion, bringing the Dirac dispersion back or creating a BdG Fermi surface. Applying an interlayer current results in a fully gapped topological superconducting state at any nonzero twist angle (even away from the magic value) with a quantized thermal Hall effect, similar to that of chiral superconductors [51, 52], while an in-plane magnetic field creates a network of topological domains with alternating Chern numbers and chiral edge modes between them. Finally, close to the magic angle even weak interactions between the BdG quasiparticles are shown to result in a (secondary) instability to a time-reversal symmetry breaking superconducting state. The recently demonstrated superconductivity in monolayers of high-$T_c$ cuprates [53] shows that this platform is accessible to current experimental investigation, and we discuss a number of other candidate materials.

Magic angle in twisted nodal superconductor bilayers: We first construct a continuum low-energy model of the TBSC (illustrated in Fig. 1 (a) for the case of a d-wave superconductor with a cuprate-like Fermi surface). For a single layer, the quasiparticles in the vicinity of the gap node momentum $K_N$ on the Fermi surface are described by the Dirac Hamiltonian in the Gor’kov-Nambu space:

$$H_N(k) = v_F \cdot k s_3 + v_\Delta \cdot k \Delta ,$$

where $v_\Delta = \partial_\Delta \Delta (k = K_N)$; $\Delta = \tau_1$ for a singlet SC and $\Delta = (d(k) \cdot s)\tau_1$, $d^2(k) = 1$ for a triplet SC.

For the twisted bilayer, we follow the approach of Refs. [31, 54] to include the effect of the interlayer electron tunneling $t(r-r')c_1^\dagger_{1,s}(r)c_{2,s}(r')+\text{H.c.}$, where 1 and 2
of the node Hamiltonian (1) with a momentum shift $k \parallel Q_N$. The full Hamiltonian can then be written as two copies the scale of\( Q \parallel K_N \), tunneling with $v_F \parallel K_N$ while $v_\Delta \parallel Q_N \perp K_N$. Using the notation $\sigma_i$ for Pauli matrices acting in the layer space we obtain for this case:

$$H(k) = v_F k_{\|} \tau_3 + v_\Delta k_{\perp} \hat{\Delta} - \frac{1}{2} v_\Delta \theta K_N \Delta \sigma_3 + t \tau_3 \sigma_1, \quad (2)$$

where $k_{\|}$ is along $K_N$ and $k_{\perp}$ is along $Q_N$. The first three terms represent the Hamiltonian of the shifted nodes in each layer while the last term describes the tunneling between them. Note that Eq. (S8) does not result from a truncation of the Hamiltonian in momentum space \[31\] and thus its applicability is not limited to $t \ll v_\Delta \theta K_N$.

Additionally, for the case of nodes not being in a reflection plane, relevant for, e.g., nodal s-wave states in Fe-based superconductors \[59, 60\], $v_F$ and $v_\Delta$ in (S8) are replaced with their projections along $K_N$ and $Q_N$, respectively, and an additional term $\frac{1}{2} (v_F \cdot Q_N) \sigma_3 \tau_3$ appears. The effect of this term on the dispersion can be important close to the magic angle and will be analyzed below, but first we focus on (S8) itself.

We now discuss the evolution of the quasiparticle spectrum of TBSC as a function of the twist angle [Fig. 1(c)]. The eigenenergies of Eq. (S8) are given by

$$E_{k_{\|}}^2(k) = (v_F k_{\|})^2 + (v_\Delta k_{\perp})^2 + t^2 (1 + \alpha^2)$$

\[\pm 2t \sqrt{(v_F k_{\|})^2 + (v_\Delta k_{\perp})^2} \alpha^2 + t^2 \alpha^2, \quad (3)\]

where $\alpha = \frac{\nu s_\theta K_N}{2t}$ is the dimensionless parameter characterizing the twist angle. In the absence of a twist, the Fermi surfaces of the two layers overlap and are expected to hybridize forming bonding and antibonding Fermi surfaces, the nodes being where the high-symmetry line crosses each [Fig. 1(e), left panel]. Indeed, for $\alpha = 0$ (i.e. $\theta = 0$) one recovers from (3) two anisotropic Dirac cones at $k_{\|} = \pm t/v_F, k_{\perp} = 0$ with the energies given by

$$E_{k_{\|,\perp}}^2(k) = v_F^2 (k_{\|} \pm t/v_F)^2 + v_\Delta^2 k_{\perp}^2, \quad (4)$$

where one observes that the Dirac velocities are the same as in the case of a single layer (1). On increasing the twist angle, one finds two Dirac cones at $k_{\|} = \pm \sqrt{1 - \alpha^2} t/v_F, k_{\perp} = 0$ for $\alpha < 1$, and two Dirac cones at $k_{\|} = 0, k_{\perp} = \pm \sqrt{1 - \alpha^2} t/v_\Delta$ for $\alpha > 1$. Near each of the nodes, the spectrum has a linear dispersion similar to Eq. (4) with the velocities $v_F, v_\Delta$ replaced by the renormalized velocities

$$\tilde{v}_{F,\Delta} = \sqrt{1 - \min\{\alpha^2, \alpha^{-2}\}} v_{F,\Delta}. \quad (5)$$

Thus, on increasing the twist angle, the nodes initially move toward one another along $k_{\|}$ and become more hybridized [Fig. 1(c)] while at high twist angles $\alpha > 1$ stand for two layers and $s$ for the spin index. At low twist angles $\theta \ll 1$, the tunneling occurs between the states with momenta $k$ and $\tilde{k}$ related by $k - \tilde{k} = -\theta (\hat{x} \times (k + G))$ with an amplitude $t(k + G)$, where $t(q)$ is the Fourier transform of $t(r - r')$ and $G$ is an inverse lattice vector. Unlike graphene, the position of the Dirac node in a nodal SC is not fixed to a high-symmetry point and thus $|K_{N} + G| > |K_{N}|$. Assuming further $t(q)$ to decay on the scale of $|G|$ as a function of the in-plane momentum $q$ \[31, 54\] we arrive at the conclusion, that for a generic node position within the Brillouin zone, tunneling with $G \neq 0$ near $K_N$ can be neglected.\[55\].

In momentum space, the tunneling then takes the form

$$t_{c_{1,s}(k - Q_N)} c_{2,s}(k) H.c., \text{ where } Q_N = -\theta [\hat{x} \times K_N] \text{ and } t = t(K_N)/\Omega, \Omega \text{ being the unit cell area} \[31\]. The full Hamiltonian can then be written as two copies of the node Hamiltonian (1) with a momentum shift $H_N(k \pm Q_N/2) [\text{Fig. 1(b)] coupled by the tunneling } t.$ Furthermore, in most of the known cases of nodal superconductors \[56–58\], the nodes are required by symmetry to be in a reflection plane, such that $v_F \parallel K_N$ while $v_\Delta \parallel Q_N \perp K_N$. Using the notation $\sigma_i$ for Pauli matrices acting in the layer space we obtain for this case: $H(k) = v_F k_{\|} \tau_3 + v_\Delta k_{\perp} \hat{\Delta} - \frac{1}{2} v_\Delta \theta K_N \Delta \sigma_3 + t \tau_3 \sigma_1, \quad (2)$

FIG. 1. Magic-angles in twisted nodal superconductors. (a) Momentum-space schematic of a twisted nodal superconductor (by angle $\theta$) exemplified by a $d$-wave superconductor with a sign changing gap (from blue to red). (b) Near the nodes ($K_N$ and $\tilde{K}_N$) the BdG quasiparticles of the two layers have a Dirac cone dispersion shifted by a vector $Q_N = \theta K_N$ with respect to one another. (c) Evolution of the low-energy BdG band structure near a node in presence of interlayer hybridization: on increasing the twist angle $\theta$, the Dirac points move towards one another, merging into a quadratic band touching (QBT) at $\theta = \theta_{MA}$; for $\theta > \theta_{MA}$, they separate in the orthogonal direction in the Brillouin zone. Dashed (solid) lines mark the unhybridized (hybridized at $\theta = 0$) Fermi surfaces of the two layers, $k_{\|}$ is along $v_F$ at $\theta = 0$ and $k_{\perp}$ is along $Q_N$. $\hat{\Delta}$ is an inverse lattice vector, relevant for, e.g., nodal s-wave states in Fe-based superconductors \[59, 60\], $v_F$ and $v_\Delta$ in (S8) are replaced with their projections along $K_N$ and $Q_N$, respectively, and an additional term $\frac{1}{2} (v_F \cdot Q_N) \sigma_3 \tau_3$ appears. The effect of this term on the dispersion can be important close to the magic angle and will be analyzed below, but first we focus on (S8) itself.
the nodes move apart along $\mathbf{v}_\Delta$ and become progressively decoupled. Most importantly, the BdG Dirac velocity vanishes at $\alpha = 1$ corresponding to a magic angle $\theta_{MA} = 2t/(v_\Delta K_N)$.

At the magic angle itself, an effective Hamiltonian for energies much lower than $t$ can be obtained by projecting Eq. (S8) to the subspace spanned by the two zero-energy eigenstates at $k_\parallel, k_\perp = 0$ and using second-order perturbation theory [55]. For the singlet case ($\Delta = \tau_3$),

$$H_{MA}(k) = \frac{-(v_F k_\parallel)^2 + (v_\Delta k_\perp)^2}{2t} \eta_3 - \frac{v_F v_\Delta k_\parallel k_\perp}{t} \eta_1,$$

where the $\eta$ matrices act in the subspace spanned by the zero-energy solutions of Eq. (S8). For the triplet case, orienting the spin quantization axis along $d(K_N)$ results in two copies of Eq. (S9), related by $v_\Delta \rightarrow -v_\Delta$.

This Hamiltonian describes a quadratic band touching (QBT) [Fig. 1(c), middle], that also occurs at the magic angle of TBG [32, 61]. The spectrum $E_{MA}(k) \approx \pm (v_F k_\parallel)^2 + (v_\Delta k_\perp)^2$ is characterized by an anisotropic effective mass $m_\parallel = \frac{1}{v_F}$ and $m_\perp = \frac{1}{v_\Delta}$. For a two-dimensional system, this spectrum possesses a finite density of states (DOS) at zero energy: $\nu = \frac{2t}{v_F v_\Delta}$. To obtain an order-of-magnitude estimate, we approximate $v_\Delta \sim \frac{k_\Delta}{V_F}$, where $\Delta_0$ is the estimate for the superconducting gap maximum value and the size of the Fermi surface being of the order $k_F$. This results in $\nu \sim \frac{2}{\Delta_0} \pi v_0$, where $v_0 = \frac{m}{\tau}$ is the density of states in the normal state. Interestingly, $\nu$ can represent a rather large fraction of the normal state DOS, especially if the superconducting gap is not too large.

A question may be raised of whether the enhanced DOS at $\theta_{MA}$ in the superconducting state affects the self-consistency equation for the superconducting gap; in the Supplementary material [55] we show that the corrections due to the presence of the QBT are small by a parameter $\sim (t/\Delta_0)^3 \log^{-1} \Lambda/\Delta_0$ (where $\Lambda$ is the high-energy cutoff for the pairing kernel) at low temperatures and can be neglected. The physical reason for this suppression is that the most pronounced effects of tunneling are confined to the nodal region, where the order parameter itself vanishes.

### Tuning the BdG quasiparticle dispersion with external fields

We now show that the dispersion of TBSGs near the magic angle can be dramatically modified by a number of external parameters that are currently accessible in experiment. For each external perturbation, we first identify a corresponding additional term in terms of Eq. (S8), which can then be projected to the $\eta$ basis of Eq. (S9) to determine the resulting spectrum.

The resulting effects of external fields on the spectrum are summarized in Table I. The interlayer displacement field (applied in an experimental setup with a back-gate) corresponds to the chemical potentials ($\tau_3$) in the two layers being different ($\sigma_1$); when projected to the basis of Eq. (S9) this term turns into a $\text{const} \cdot \eta_1$ contribution.

| Tuning parameter                          | Term added to Eq. (S8) | Spectrum         |
|--------------------------------------------|------------------------|------------------|
| Interlayer displacement field              | $\tau_3 \sigma_3$      | Dirac point      |
| Zeeman field $h \parallel d$              | $s \parallel d$        | Fermi surface    |
| Zeeman field $h \perp d$                   | $s \perp d$            | QBT (at MA)      |
| In-plane supercurrent                      | $\tau_0 \sigma_0$      | Fermi surface    |
| Interplane supercurrent                    | $i\tau_3 \Delta$       | Gapped           |

It results in a splitting of the QBT into two Dirac points, such that the QBT is avoided for all twist angles. In terms of Fig. 1, on increasing the twist angle, two Dirac points move towards one another, but avoid collapsing into a QBT, by preemptively turning in the direction of $\pm \mathbf{v}_\Delta$. As has been mentioned after Eq. (S8), if the nodes are not symmetry protected ones, a $\tau_3 \sigma_3$ term would also appear, with its magnitude given by $\frac{1}{2}(v_F \cdot Q_N)$. For the magic-angle effects to be observable, this term has to be much smaller than $t$ at the MA, i.e. $v_{F,\perp} \ll v_\Delta$, where $v_{F,\perp} = (v_F \cdot Q_N)/(Q_N)$. A Zeeman magnetic field $h \cdot \mathbf{d}$, for a singlet SC or triplet SC with $d \parallel \mathbf{d}$, commutes with Eq. (S8), resulting in a spectrum that splits into two sectors with energies $E_{0,\pm}(k) \pm h$. This results in the formation of compensated quasiparticle pockets of opposite spin, as has been predicted in d-wave superconductors [62, 63]. However, the size of the resulting pockets would be affected by the renormalization of the Dirac velocity in TBSC, Eq. (5). In particular, the field-induced DOS is $\nu(h) \approx \frac{2}{\Delta_0} \pi v_0 h$. More intriguingly, this effect can be used to shift the quasiparticle occupation to within the miniband, formed by the reconstruction of the Brillouin zone by the moiré pattern. This represents an analogue of electrostatic gating for the neutral BdG quasiparticles. In TBG, gating to commensurate moiré filling fractions has lead to the observation of correlated states near the magic angle [30]; thus Zeeman field (and an in-plane current, see below) may appear useful in controlling the correlations in TBSC too. For a triplet TBSC with $d \perp \mathbf{h}$, the Zeeman term has the same commutation properties with respect to Eq. (S8) as $\tau_3$ [55] and can be absorbed into a shift of $k_\parallel$, preserving the QBT at the magic angle.

Finally, we consider the effect of a supercurrent flow in TBSC, that can be induced by applying an external current bias. For a single layer, the in-plane supercurrent corresponds to a finite Cooper pair momentum $Q_P$, such that $v_F \cdot k_{\tau_3} \rightarrow v_F \cdot k_{\tau_3} + v_F \cdot Q_P$ in Eq. (1). The effect of the new term is to produce quasiparticle pockets, similar to the Zeeman field, albeit without spin polariza-
tion [64], which has been also observed experimentally [65, 66]. As with the Zeeman field, in-plane supercurrent effects in TBSC should be boosted by proximity to the MA in TBSC and can be used to efficiently "gate" the BdG quasiparticles.

The effect of an interlayer supercurrent [Fig. 2(a)] is dramatically different. Microscopically, it corresponds to a non-zero phase difference between the order parameters in the two layers $\Delta_1 \rightarrow \Delta_1 e^{i\varphi/2}, \Delta_2 \rightarrow \Delta_2 e^{-i\varphi/2}$, related via the current-phase relation $I(\varphi) = I_c \sin \varphi$ can be shown to hold down to exponentially small temperatures $T \sim 2te^{-\Delta_0/\lambda_t}$ [68] [55]. It follows then, that $\varphi$ is monotonically increasing as a function of the applied current up to a maximal value of $\varphi = \pi/2$, corresponding to the critical interlayer current $I_c$. The low-energy Hamiltonian of TBSC accounting for $\varphi$ takes the form

$$H(k, \varphi) = v_F k_\parallel \tau_3 + v_\Delta k_\perp \cos(\varphi/2) \Delta - \alpha \cos(\varphi/2) \Delta \alpha \tau_3 + \tau_3 \beta_1 + v_\Delta k_\perp \sin(\varphi/2) i \tau_3 \Delta \beta_3 - \alpha \sin(\varphi/2) i \tau_3 \Delta, \quad (7)$$

where a renormalization of the values of $v_\Delta$ and $\alpha$ with respect to Eq. (S8) is observed, in addition to two new terms not present in Eq. (S9). Their presence results in the opening of a finite spectral gap $\Delta_f$ for any nonzero $\alpha$:

$$\Delta_f(\varphi) = \begin{cases} 2t |\alpha \sin \varphi/2| & |\alpha| < \cos \varphi/2, \\ \frac{t_\parallel \sin \varphi}{\sqrt{\alpha^2 + \sin^2 \varphi}} & |\alpha| > \cos \varphi/2. \end{cases} \quad (8)$$

This result is especially transparent for $\alpha \leq \cos \varphi/2$, where in the absence of the current the Dirac points are at $k_\perp = 0$. For $\varphi \neq 0$, the only new term introduced in Eq. (7) at $k_\perp = 0$ is $-\alpha t_\parallel \tau_3 \Delta \sin(\varphi/2)$ which anticommutes with all the remaining ones, leading to a Dirac gap equal to $2t |\alpha \sin(\varphi/2)|$. This term also corresponds to the appearance of an imaginary part of the SC order parameter that does not vanish at the node, similar to the situation in SC breaking time-reversal symmetry (e.g., $s + id$ or $p_x + ip_y$ ones [52, 69]).

Indeed, this analogy can be traced down to the simultaneous breaking of mirror and time reversal symmetries in the twisted bilayer (in our case, the time-reversal symmetry breaking is induced by the current). For simplicity, we follow the case of $d_{x^2-y^2}$ SC here, but the argument can be straightforwardly generalized. The twisted second layer breaks the mirror symmetries of the first layer, leading to a mixing of $d_{x^2-y^2}$ and $d_{xy}$ order parameters being possible, while breaking of time-reversal symmetry introduces an imaginary part to the order parameters $d_{x^2-y^2} \rightarrow e^{i\Phi} d_{x^2-y^2}$ and $d_{xy} \rightarrow e^{i\Phi} d_{xy}$. Generically, $\Phi_{x^2-y^2} \neq \Phi_{xy}$ as it is not required by any symmetry. In this case, the corresponding superconducting gap $\Delta(k) = e^{i\Phi_{x^2-y^2}} \Delta_{x^2-y^2}(k) + e^{i\Phi_{xy}} \Delta_{xy}(k)$ can be seen to never vanish, as the real and imaginary parts of $\Delta(k)$ do not vanish simultaneously for any $k$.

In Fig. 2(b), we present the maximal value of the current-induced gap $\Delta_f(\varphi = \varphi_{Max})$ (for $\varphi$ between 0 and $\pi/2$ corresponding to the stable supercurrent branch) as a function of the twist angle. One observes that the maximal gap value equal to $t$ is reached at $\theta = \theta_{MA}/\sqrt{2}$.

Current-induced topological states: We now show that the gapped state induced by the interlayer Josephson current is topological, belonging to class C for the singlet case [3] and characterized by a $Z$ topological invariant, i.e. the Chern number $C$. To simplify the discussion, we consider only the situation at the magic angle where the system is described by Eq. (S9). Since the gap does not close for all nonzero twist angles, the quantized topological properties at any twist angle are identical to those at the magic angle. Projecting the last term in Eq. (7) to the basis of Eq. (S9), we obtain the term $-\alpha t \eta_2$; the second-to-last term only leads to corrections of order $\sim \sin^2(\varphi/2) (v_\Delta k_\perp)^2/t$ to the QBT dispersion, not affecting its topological properties.

Indeed, adding the term $-\alpha t \eta_2$ to Eq. (S9) results in a finite Chern number

$$C = \frac{1}{2\pi} \int dk_\perp dk_\parallel F_{xy}(k_\perp, k_\parallel) = \text{sgn}[\alpha], \quad (9)$$

where $F_{xy}(k_\perp, k_\parallel)$ is the Berry connection [70], as is expected from a merger of two gapped Dirac points.

Having demonstrated the finite Chern number for a single “valley” formed by two nodes, we need to check whether the Chern numbers generated by the different valleys sum up to a non-zero number. For this purpose, we consider the correspondence between the Hamiltonians Eq. (S9) for different pairs of nodes. Let us first study two adjacent nodes on a single Fermi surface [Fig. 2(e)]: while the direction of the Fermi velocity changes smoothly between the two, the direction of $v_\Delta$ has to change orientation, leading to $v_\Delta \rightarrow -v_\Delta$ and $\alpha \rightarrow -\alpha$ in Eq. (7). As a result, at the magic angle, Eq. (S9), the second ($\eta_2$) term as well as the current-induced $-\alpha t \eta_2$ term change sign. Consequently, the Chern number does not change between two adjacent valleys in TBSC. This argument is general and applies to any type of TBSC (d-wave, p-wave, etc.). Note that a similar argument shows that in a triplet TBSC, the Chern numbers of two spin sectors for a single valley are the same. Thus, the total Chern number of this state is equal to the number of nodes in a single layer. A similar situation occurs in chiral superconductors [52], e.g. for a $d_{x^2-y^2} + id_{xy}$ superconductor on a square lattice, $v_\Delta$ and $d_{xy}$ change sign between the adjacent Dirac nodes, such that the Chern numbers of individual gapped Dirac points (which are 1/2) sum up. Indeed, both states are characterized by the Chern number and belong to the same $C$ class [3], which implies the existence of neutral gapless chiral edge modes that result in
a quantized thermal (and spin for the singlet case) Hall conductance $[51, 52] \kappa_{xy}/T = C(\pi/6)k_B^2/h$.

**Topological domains induced by an in-plane field.** The opening of a topological gap by the interplane current suggests another possibility for its experimental observation - in a parallel magnetic field. As we have shown above, the conventional current-phase relation holds in TBSC at low twist angles, allowing for the TBSC to be viewed as a Josephson junction. The external parallel magnetic field penetrates the TBSC in the form of Josephson vortices $[71]$, where the interlayer phase difference is position-dependent and goes from 0 to $2\pi$ [Fig. 2(d)]. This implies, that the current-induced gap changes sign across the vortex core $\Delta J(\phi(x))$. Thus, the vortex core represents a domain wall between two gapped states with Chern number equal to $\pm C$. For fields much larger then the lower critical field, the phase difference in the junction is set to $\phi(x) = \frac{2\pi H_d}{\Phi_0} x$, where $d$ is the effective junction thickness and $H$ is the external field $[72]$, resulting in a domain size of $\Delta x = \frac{\Phi_0}{2\pi d}$.

Let us now discuss the experimental signatures of this domain state. One notices that the contribution of the edge states to the total thermal Hall conductance should vanish due to the presence of domains with opposite Chern numbers. However, each individual Josephson vortex core represents a chiral thermal transport channel (Fig. 2(d)) with $\kappa_{xx}/T = \frac{4C\pi k_B^2}{6h}$, allowing their observation in a thermal transport experiment similar to the case of charge transport through domain walls in double-layer graphene $[73]$. Unlike that case, the modes stemming from different valleys in TBSC have the same chirality, suggesting that their presence should not be affected by the field orientation within the plane. Moreover, with increasing field, more domain walls are created, connecting the thermal contacts. We predict that $\kappa_{xx}/T$ will oscillate then, as a function of field, between $\frac{4N\pi k_B^2}{6h}$ and 0 depending on the number of vortex lines connecting the contacts, being odd or even, respectively.

**Symmetry broken correlated phases near the magic-angle:** We now explore the role of interactions between the BdG quasiparticles close to the magic angle. Above, we have shown that the density of states at the magic-angle is finite due to the presence of a QBT. In this case, correlations may manifest themselves as instabilities already at weak coupling $[74]$. To analyze the likely correlated states to emerge at the magic-angle in TBSC, we study the order parameter susceptibilities defined as $\chi_A(T) = -\frac{\partial^2}{\partial T^2} T \sum_{\tau_{\sigma_n}, k} \log(\xi_n - H(k) - W\hat{A})$, where $\hat{A}$ is a basis matrix of the form $\tau_\alpha \otimes \sigma\otimes s_c$, representing the order parameter. The critical temperature is determined by $\chi_A(T) = \frac{2}{\chi_A}$, $\lambda_\Lambda$ is the coupling constant in the respective channel. We assume the interlayer interactions to be much weaker than the intralayer ones and thus we only consider orderings that do not involve layer degrees of freedom, i.e. $\hat{A} = \tau_\alpha \otimes \sigma_\otimes s_c$.

To simplify the discussion, we first address the singlet $\Delta = \tau_1$ case. Of all the possible order parameters, only the $\tau_2$ (or its spinful version $\tau_2\sigma_{1,2,3}$) order has a weak-coupling instability, with susceptibility $\chi_{\tau_2}(T \rightarrow 0) \approx \frac{1}{4} \log \frac{2\pi C}{\gamma \tau}$ at $T \rightarrow 0$, where $\gamma$ is the Euler-Mascheroni constant. The susceptibilities for the other orders remain finite at $T = 0$. The $\tau_2$ order parameter corresponds to a secondary superconducting instability, while the purely imaginary character of the order parameter indicates a broken time-reversal symmetry state, such as a $d + is$ state $[69]$. Indeed, a number of competing SC states may be expected in systems with non-phononic pairing mechanisms $[75–78]$. Depending on the type of the subleading SC instability, the sign of the order parameter may change between the nodes, affecting
the topology of the state. For example, for an s-wave secondary instability, the order parameter sign will remain the same, resulting in a total zero Chern number, similar to the quantum valley Hall state in TBG [79]. On the other hand, for a $d_{xy}$ instability in a $d_{x^2-y^2}$ TBSC, the resulting state will have Chern number equal to the number of nodes, similar to the supercurrent-induced state discussed above.

The results above for the $\hat{A} = \tau_2$ instability apply also to the triplet TBSC case. Unlike the singlet case, $\hat{A} = \tau_1 (h \cdot \mathbf{s})$ has a weak-coupling instability only for $h \perp \mathbf{d}$, which has the same susceptibility as $\chi_{\tau_2}$. Above we considered the order parameters that do not break translational symmetry; in principle, order parameters such as spin-, charge-, or pair-density waves can couple different nodes, opening a gap. However, their properties would likely depend on the particular Fermi surface geometry and hence we leave the consideration of these order parameters for future studies focused on specific materials.

Away from the magic angle, the spectrum has Dirac nodes instead of a QBT; while no weak-coupling instability is expected, one expects the secondary instability temperature $T^*$ to remain finite close to the magic-angle. The equation for $T^*(\theta)$ for the $\tau_2$ order parameter is given by

$$
\log \frac{T^*(\theta)}{T_0} = \int_0^\infty \frac{d\varepsilon}{2\pi} \int_0^{2\pi} d\eta \left[ -\frac{\tanh \frac{\varepsilon}{2T}}{\varepsilon} + \frac{\tanh \frac{\sqrt{\varepsilon^2 - 2t^2(\alpha - 1)^2} + t^2(\alpha - 1)^2}{2T}}{\varepsilon} \right].
$$

Equation (10) is consistent with Eq. (S15).

The results of a numerical solution of Eq. (10) are presented in Fig. 3: the critical temperature is found to vanish for the deviation from magic-angle larger than a critical one: $|\theta - \theta_{MA}| = 2\pi e^{-\frac{\gamma}{\theta_{MA} T_0}}$. 

**Discussion:** Let us briefly recall our findings, focusing on the predictions for experiments. The reduction of the quasiparticle Dirac velocity Eq. (5) and the gap opening induced by the current, magnetic field [Fig. 2(a,d)] or interaction effects [Fig. 3] near $\theta_{MA} = 2t/(v_{DA}K_N)$ affect the quasiparticle density of states and can be directly revealed with scanning tunneling microscopy (STM) and thermal transport experiments. Alternatively, angle-resolved photoemission spectroscopy can detect the change in the position of nodes with the twist angle [Fig. 1(c)]. STM or superconducting spectroscopy [80] can also reveal the gapless chiral edge modes in the topological SC state [Fig. 2(a,d)]. For the current-induced state, the most striking signature of the edge modes is the quantized thermal (and spin, for the singlet case) Hall conductances [51].

The value of $\theta_{MA}$ being of the order of $t/\Delta_0$, where $\Delta_0$ is the maximal SC gap value, implies that for the observation of strong twist effects in TBSC the interlayer hybridization much weaker than $\Delta_0$ is preferred. Reducing $t$ can be practically achieved by introducing an insulating barrier between the two layers, similar to conventional Josephson junctions. In our study we found the effects of hybridization to be most pronounced at $\theta_{MA}$, and suppressed if the twist angle is further increased (see, e.g., Eqs. 5,S15). On the other hand, increasing the twist angle between nodal superconductors is known to suppress the critical superconducting current at small $t$, eventually suppressing it to zero at special angles dictated by symmetry (e.g., 45° in a $d$-wave superconductor). This dramatically alters the current-phase relation $I(\varphi)$ allowing the subdominant effect to become important; in particular, a spontaneous phase transition into the chiral topological SC state breaking time-reversal symmetry is predicted [81–84]. However, the spontaneously generated topological gap should be smaller than the one induced by an interlayer current at the magic angle since it is an effect of higher order expansion in $t$. On dimensional grounds, one expects the gap to be of the order $t^2/\Delta_0$, consistent with Eq. (S15).

Another important question is that of disorder, as the nodal superconductors are usually strongly affected by it [85] due to the presence of gapless excitations close to the gap nodes. Consequently, we expect the gapped states of TBSC to be robust against weak disorder. In particular, the current-induced topological state [Fig. 2(a)] is expected to be protected against weak to moderate disorder as the Chern number can not change continuously [3, 86]. Similar reasoning applies to the case of temperature; for $T \ll \Delta_j$ thermal effects can be ignored. On the other hand, temperature provides an additional control parameter, as the value of $v_{DA}$ should decrease with temperature, vanishing at $T_c$. An increasing temperature consequently leads to an enhanced $\theta_{MA}$ value, which can be used to achieve magic-angle conditions if the device is initially at $\theta > \theta_{MA}(T = 0)$. It will...
be interesting in future work to consider the role of disorder in the twist angle [87–89] in TBSCs, which is known to play a role in current twistoric experiments [35, 90].

Let us now discuss the possible materials that can be used to realize TBSC. Of all the unconventional superconductors [91], the most readily available platform seems to be provided by the high-Tc d-wave cuprate superconductors, that have been recently become available in extremely thin bi- or single-layer form [53, 92]. A further advantage of this system is that v_F and v_Δ can be controlled by gating the system, as they strongly depend on carrier concentration [93]. Using the values of v_Δ for optimal doping [93], one can estimate the magic angle value; one obtains θ_{MA} ≈ 14° [55] due to strongly suppressed tunneling along the nodal direction [94]. Organic superconductors are also known for their quasi-2D or 1D structures [91]. Importantly, d-wave SC has been achieved in a monolayer of the quasi-2D (BETS)₂GaCl₄ [95]. Using the interlayer hopping deduced from magnetotransport [96] and assuming a ~ cos 2θ d-wave gap, one obtains θ_{MA} ≈ Δ_0/t = 1° [55].

An alternative to these materials are iron-based superconductors, where a sign-changing s-wave superconductivity occurs [59, 60] that may possess accidental or symmetry-enforced nodes. These are generally characterized by a more three-dimensional character then cuprates; however, superconducting FeSe monolayers have been grown on SrTiO₃ substrates [97]. While in that case the gap is nodeless, it would be interesting to explore whether other iron-based SCs can remain nodal in monolayer form. Also, important advances have been achieved in the fabrication of monolayer SCs based on transition metal dichalcogenides [98]. While no experimental evidence of nodal behavior has been reported so far, the likely existence of the nodes in the superconducting order parameter symmetry is still under debate [58], large twist angles through a qualitatively different physical mechanism than what we have identified here.

Moving to materials not yet available in single-layer form, one expects materials with quasi-2D structure to be most promising. Out of the heavy-fermion superconductors, CeCoIn₅ is characterized by the largest 2D anisotropy [102]. The estimate based on the STM experiments [103] yields θ_{MA} ≈ 14° [55] due to the heavy effective masses reflected in a low value of hopping parameters. Another material of interest is Sr₂RuO₄, where the order parameter symmetry is still under debate [58], with both singlet and triplet candidate states proposed. What is important for our discussion is the high degree of two-dimensionality observed in ARPES [104] and the likely existence of the nodes in the superconducting gap [105, 106] that appear to be symmetry-enforced [106]. However, the extremely small SC gap, with a maximum of about 350 μeV [106] implies that to observe the magic angle in Sr₂RuO₄-based TBSC the interlayer tunneling has to be reduced first, by, e.g. an insulating layer introduced between the monolayers, as discussed above.

To conclude, we have shown that twisted bilayers of nodal superconductors can realize topological superconductivity and interacting phases of the neutral BdG quasiparticles “on demand.” The quasiparticle dispersion undergoes a dramatic reconstruction near the “magic” value of the twist angle where their Dirac velocity vanishes, and even weak interactions lead to a time-reversal symmetry breaking transition. We have shown that while electrostatic gating reduces the renormalization effects, while keeping the Dirac points at zero energy, Zeeman field effects and in-plane current act as an effective ”gate” for the quasiparticles, allowing control of their occupation. Most importantly, we have demonstrated that applying an interlayer current bias opens a topological gap in the system that manifests itself in quantized spin and thermal Hall responses while the orbital effect of an in-plane magnetic field creates a network of chiral domains separated by Josephson vortex cores. We have discussed a number of experiments that can be used to verify these predictions and identified several materials that host nodal superconductivity in monolayers. We have thus shown that TBSC can be readily realized with present-day experimental techniques and systems, creating a novel, highly tunable, platform for topological and strongly correlated superconductivity.

Note Added: While we were completing this manuscript Refs. [83, 84] appeared, that find topological superconductivity in twisted nodal superconductors at large twist angles through a qualitatively different physical mechanism than what we have identified here.

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Supplementary Material: Magic angles and current-induced topology in twisted nodal superconductors

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DETAILS OF THE LOW-ENERGY MODEL DERIVATION FOR TBSC

We consider a twisted bilayer of nodal superconductors in the low energy regime. To describe the resulting BdG Hamiltonian in a generic way, we use the Balian-Werthammer spinors in layers \( l = 1, 2 \) with

\[
\Phi_{l}^{\dagger}(k) = [\epsilon_{l,\uparrow}^{\dagger}(k), c_{l,\downarrow}(-k), c_{l,\downarrow}^{\dagger}(k), -c_{l,\uparrow}(-k)]
\]

and denote matrices acting in Gor’kov-Nambu and spin space by \( \tau_{i} \) and \( s_{i} \), respectively.

The tunneling between layers can be generally written as

\[
H_{\text{tun}} = \sum_{\mathbf{R}, \mathbf{R}' \tau} \Phi_{l}^{\dagger}(\mathbf{R}) T(\mathbf{R}, \mathbf{R}') \Phi_{2}(\mathbf{R}) + \text{h.c.}
\]  

(S2)

For simplicity, we work under the following assumptions: (1) tunneling is spin-independent; (2) only charge tunneling \((T \sim \tau_{3} \text{ in Gor’kov-Nambu space})\) is considered; (3) two-center approximation \((T(\mathbf{R}, \mathbf{R}') = T(\mathbf{R} - \mathbf{R}')\) is employed [31]. (2) can be understood as neglecting the off-diagonal hopping in Gor’kov-Nambu space correspond to interlayer pairing \( c_{1,\downarrow}(k)c_{2,\uparrow}(\mathbf{k}') \); the latter can arise in the mean-field BdG Hamiltonian only from the decoupling of interlayer interactions which we neglect with respect to the intralayer ones. Note that the corresponding correlations \( \langle c_{1,\uparrow}(k)c_{2,\uparrow}(\mathbf{k}') \rangle \) can still be non-zero. Taking the above into account, the tunneling term takes the form

\[
H_{\text{tun}} = \sum_{\mathbf{R}, \mathbf{R}' \tau} t(\mathbf{R} - \mathbf{R}') \Phi_{l}^{\dagger}(\mathbf{R}') \tau_{3} \Phi_{2}(\mathbf{R}) + \text{h.c.}
\]  

(S3)

Now, following the approach of [31], we approximate the tunneling \((S3)\) for the fermions close to the nodal points \( K_{N} \) and small twist angles. In Fig. S1 the schematic of the Brillouin zone reconstruction in a twisted bilayer is presented for a square Brillouin zone and a Fermi surface appropriate for the cuprates. The tunneling matrix element \( t_{k,\mathbf{k}'} \) between states with momentum \( k \) and \( \mathbf{k}' \) takes the form

\[
t_{k,\mathbf{k}'} = \sum_{\mathbf{G}} \frac{t_{\mathbf{k} + \mathbf{G}}}{\Omega} \delta_{\mathbf{k} + \mathbf{G}, \mathbf{k}'}
\]

where \( \Omega \) is the unit cell area, \( t_{\mathbf{q}} \) is the continuous Fourier transform of \( t(r) \) and \( \mathbf{G} \) and \( \hat{\mathbf{G}} \) are the reciprocal lattice vector of the original and twisted BZ, respectively. We assumed a one-atom unit cell and the shift between twisted layers to be zero (the results do not depend on it). From the momentum-space picture (Fig. S1) one sees, that as \( k_{N} \) is at a generic point of the Brillouin zone, \( |\mathbf{K}_{N} + \hat{\mathbf{G}}| \neq |\mathbf{K}_{N}| \). Moreover, if the node is sufficiently close to the \( \Gamma \) point, i.e. \( |\mathbf{K}_{N}| \ll |\mathbf{G}| \), it follows also that \( |\mathbf{K}_{N}| \ll |\mathbf{K}_{N} + \hat{\mathbf{G}}| \). Assuming that \( t_{\mathbf{q}} \) decays on the scale of inverse BZ size, all terms except the one with \( \mathbf{G}, \hat{\mathbf{G}} = 0 \) can be neglected. At small twist angles we can further approximate \( \mathbf{k} \approx \mathbf{k} - [\hat{\mathbf{x}} \times \mathbf{k}] \theta \equiv \mathbf{k} + Q_{N} \). We can then approximate the tunneling term as

\[
H_{\text{tun}} \approx t \sum_{k} \Phi_{l}^{\dagger}(k) \tau_{3} \Phi_{2}(k + Q_{N}) + \text{h.c.},
\]  

(S4)

where \( t \) is a constant and \( k \) is now measured from \( K_{N} \) - the node momentum.

Note that the tunneling occurs with a momentum shift \(-Q_{N} \), when tunneling \( 2 \rightarrow 1 \) and \( Q_{N} \) for \( 1 \rightarrow 2 \), which implies that tunneling with other momenta (e.g. \( 2Q_{N} \) and so on) is not obtained from multiple hopping. As the tunneling acts between the layers, \( Q_{N} \) shift can only be followed by \(-Q_{N} \), i.e. restoring to the initial point. It is then evident, that no tunneling between different nodes may occur in \((S4)\) as \( Q_{N} \ll K_{N} \). At low twist angles, the coupling between nodes may then only arise if \( \mathbf{K}_{N} + \hat{\mathbf{G}} \) coincides in momentum space with the other node. Such a \( \hat{\mathbf{G}} \) satisfies \( \hat{\mathbf{G}} - \mathbf{G} = \mathbf{K}_{N}' - \mathbf{K}_{N} \). Generically the nodes are not expected to be very closely spaced, i.e. \( |\mathbf{K}_{N}' - \mathbf{K}_{N}| \sim K_{N} \). Thus \( \hat{\mathbf{G}} \) does not satisfy this condition as \( |\mathbf{K}_{N} - \mathbf{K}_{N}'| \sim \theta K_{N} \approx K_{N} \). For finite \( \theta \) the condition can be satisfied in principle as \( |\hat{\mathbf{G}} - \mathbf{G}| \sim \theta \mathbf{G} \); however, as has been noted above, we neglect tunneling with \( \hat{\mathbf{G}} \neq 0 \).

The full Hamiltonian can then be written as (shifting the momenta by \(-Q_{N}/2\))

\[
\hat{H}(k) = \sum_{k} \Phi_{1,k-Q_{N}/2}^{\dagger}(k-Q_{N}/2) \tau_{3} \Phi_{1,k-Q_{N}/2} + \Phi_{2,k-Q_{N}/2}^{\dagger}(k+Q_{N}/2) \tau_{3} \Phi_{2,k-Q_{N}/2} + \Phi_{1,k-Q_{N}/2}^{\dagger}(k-Q_{N}/2) \Delta \Phi_{1,k-Q_{N}/2} + \Phi_{2,k-Q_{N}/2}^{\dagger}(k+Q_{N}/2) \Delta \Phi_{2,k-Q_{N}/2} + t \Phi_{1,k}^{\dagger}(k) \tau_{3} \Phi_{2}(k+Q_{N}) + t \Phi_{2}(k+Q_{N}) \tau_{3} \Phi_{1}(k),
\]  

(S5)
where $\varepsilon(k)$ is the quasiparticle dispersion and $\Delta(k)$ - the superconducting gap and $k$ is measured from $K_N$. At low twist angles one can expand $\varepsilon(k \pm Q_N/2) \approx v_F \cdot (k \pm Q_N/2)$, $\Delta(k \pm Q_N/2) \approx v_\Delta \cdot (k \pm Q_N/2)$.

Introducing the spinors $\Phi^\dagger(k) = [\Phi^\dagger_1(k - Q_N/2), \Phi^\dagger_2(k + Q_N/2)]$ and denoting the Pauli matrices acting in the layer space by $\sigma_i$ the Hamiltonian can be rewritten in a compact form:

$$\hat{H}_0(k) = \sum_k \Phi^\dagger_k \left( v_F \cdot k \sigma_3 - \frac{v_F \cdot Q_N}{2} \tau_3 \sigma_3 + v_\Delta \cdot k \Delta \sigma_3 - \frac{v_\Delta \cdot Q_N}{2} \Delta \sigma_3 + t \tau_3 \sigma_1 \right) \Phi^\dagger_k $$

Equation (2) of the main text is obtained then for the case $v_F \parallel K_N$, $v_F \perp v_\Delta$, that corresponds to the nodes being at a high-symmetry line, such as in the case of symmetry-protected gap nodes in a non s-wave superconductor. In
what follows we assume this condition to hold (see the discussion in the main text for the effects of the violation of this condition).

To simplify the derivations, in what follows below we will also use notations:

\[ \xi \equiv v_F \cdot k, \ \delta \equiv \nu_\Delta \cdot k, \ \delta_0 \equiv \frac{v_\Delta \cdot Q}{2}. \]  

(S7)

Note that using the notation in the main text we have \( \xi = v_F k_\parallel \) and \( \delta = v_\Delta k_\perp \).

**Effective theory at the MA**

Here we derive the projected \( 2 \times 2 \) Hamiltonian at the magic angle. The calculations below are for the singlet case; the generalization to triplet case is straightforward: a rotation in spin space brings \( d(K_N) \rightarrow (0, 0, |d(K_N)|) \), such that \( \Delta = \tau_1 s_3 \), and the full \( 8 \times 8 \) Hamiltonian can be rewritten in a block-diagonal form, with the spin-up and spin-down blocks related by \( v_\Delta \rightarrow -v_\Delta \). At the magic angle, the Hamiltonian Eq.(2) of the main text takes the form:

\[ H(k) = \xi \tau_3 + \delta \tau_1 + t \tau_3 \sigma_1 - t \tau_1 \sigma_3. \]  

(S8)

The zero-energy eigenvectors at \( \xi = \delta = 0 \) are \( |a\rangle = (1, 1, 1, -1)/2 \) and \( |b\rangle = (-1, 1, 1, 1)/2 \). These states are equal superpositions of particles and holes and thus have zero charge, but the spin is well-defined. If we project the Hamiltonian (S8) to the subspace spanned by \( |a\rangle, |b\rangle \) we obtain exactly zero. One can note that \( |a\rangle, |b\rangle \) are eigenvectors of the last two terms in (S8); however the first two \( H' = \xi \tau_3 + \delta \tau_1 \) can lead to virtual transitions out of the subspace. Computing the second-order corrections due to these terms in second-order perturbation theory \( \delta H_{\alpha,\beta=a,b}(k) = -\sum_{\gamma=c,d} \frac{(a|H'|\gamma)(\gamma|H'|\beta)}{2t} \), where \( |c\rangle = (1, -1, 1, 1)/2 \), \( |d\rangle = (1, 1, -1, 1)/2 \) are the excited states with energy \( 2t \), we get:

\[ H_{MA}(k) = -\frac{\xi^2 - \delta^2}{2t} \eta_3 - \frac{\xi \delta}{t} \eta_1, \]  

(S9)

where \( \eta \) matrices act in the \( |a\rangle, |b\rangle \) space. The eigenvalues of this Hamiltonian \( \pm \sqrt{\xi^2 + \delta^2} \) indeed correspond to the eigenvalues of (S8) expanded in \( \xi \) and \( \delta \) to the second order.

**External perturbations**

Using (S9) we first discuss what spectrum types can emerge in the presence of perturbations \( H_{MA}(k) \rightarrow H_{MA}(k) + \delta H \) represented by Pauli matrices in \( \eta \) space.

- We first consider

\[ \delta H = a \eta_2, \]

which results in a fully gapped spectrum with a gap \( 2|a| \). As is shown below the full gap remains intact even for large \( \xi, \delta \), where the full Hamiltonian beyond the projected one needs to be considered.

- Next we study

\[ \delta H = -a \eta_3 - b \eta_1 \]

results in the following equations for the positions of the nodes:

\[ \xi^2 - \delta^2 = 2ta; \ \xi \delta = tb. \]

The solutions are given by:

\[ \xi_0 = \pm \sqrt{ta + t\sqrt{a^2 + b^2}}; \ \delta_0 = \pm \frac{tb}{\sqrt{ta + t\sqrt{a^2 + b^2}}}, \]

where the second root of the resulting biquadratic equation for \( \xi \) is always negative and is thus omitted. Note that expanding (S9) around one of the two node positions \((\xi_0, \delta_0)\) generically results in a Dirac point spectrum.
Finally, for
\[ \delta H = \alpha \eta_0, \]
a constant term simply shifting the energy of the QBT. It results in the appearance of a finite Fermi surface.

We now consider the perturbation terms of the full Hamiltonian (S8) and discuss their possible physical origins. We consider first the singlet case and perturbation terms of the general form \( W_0 \hat{A} = \tau_i \sigma_j \). Below we consider all the possible \( i \) and \( j \) and for each case indicate the corresponding term in the projected hamiltonian (S9).

- \( \hat{A} = \tau_1, \tau_3, \tau_1 \sigma_3, \tau_3 \sigma_1 \rightarrow (\delta \rightarrow \delta + W_0), (\xi \rightarrow \xi + W_0), - \eta_3, \eta_3 \); these terms are already contained in (S8) and only lead to a renormalization of the initial model parameters (SC gap, chemical potential, twist angle or interlayer hopping).
- \( \hat{A} = 1 \equiv \tau_0 \sigma_0 \rightarrow \eta_0 \) Note that this term is *not* equivalent to a chemical potential shift represented by \( \tau_3 \) in the Nambu notation. It leads to a creation of a single Fermi surface and can be realized in two ways.

First, a nonzero in-plane supercurrent results in \( \Delta \rightarrow \Delta e^{i \varphi} \) leading to \( H \rightarrow H + F \cdot q \). Second possibility is a Zeeman term \( s_i \). It commutes with the Hamiltonian, resulting in two independent sectors with different signs of the term. Note that the two mechanisms above result in different parity properties: the supercurrent-generated term is odd under parity and thus creates a doubly degenerate electron or hole pocket at each node, while the Zeeman term would create a non-degenerate coinciding electron *and* hole pocket (nodal line) at each node.

- \( \hat{A} = \tau_2 \rightarrow \eta_2 \): The spectrum is fully gapped and the lowest eigenvalues at the magic angle are given by:
\[
E = \pm \sqrt{[\sqrt{\xi^2 + \delta^2 + t^2} - t]^2 + W_0^2}.
\]

Such a perturbation can be implemented by applying an interlayer bias current (see below); a formation of the superconducting order parameter with respect to intralayer one. A nonzero phase to the interalyer order parameter with respect to intralayer one. Normal current will be nonzero only above the critical current value, where the value of the expression for the normal interlayer current. Application of a bias current in the SC state would result only in a subleading superconducting order with a phase of \( \pi/2 \). Thus, an extension of (S8) to include additional spaces (such as spin or node number ones) would not lead to any new type of spectrum, since the commutation relations of any new term would fall onto one of the cases considered in Tab. II.
External perturbations for triplet SCs

We now consider the case of a single-component triplet superconductor. In this case, triplet SC order parameter near a node takes the form $\delta \tau_1 (s \cdot d)$, where $s_i$ are Pauli matrices in spin space. Consequently, the analog of (S8) is:

$$\tilde{H} = \sum_{k_x > 0} \Phi^\dagger(k)H(k)\Phi(k)$$

(S10)

$$H(k) = \delta \tau_1 (s \cdot d) + \xi \tau_3 - t \tau_1 (s \cdot d)\sigma_3 + t \tau_3 \sigma_1,$$

where the $k$ summation is restricted to half the Brillouin zone to avoid $k \rightarrow -k$ redundancy. Let us now discuss perturbations. For perturbations without spin matrices it is convenient to perform an SU($k$) where the $\delta \tau_1 \rightarrow \tau_1$ spin rotation that brings $\delta \tau_1 \rightarrow \pm \tau_1$ and the spectrum is determined as for (S8).

For perturbations involving spin in the form of the matrix $(h \cdot s)$ there are two cases:

- **$h \parallel d$** As above, the problem may be reduced to two copies of (S8) with $(h \cdot s), (d \cdot s) \rightarrow \pm h, d$.
- **$h \perp d$**: Choosing the quantization axis along $d$, we apply a unitary transformation $U = U^\dagger = \frac{1 - \eta_1}{2} \tau_3 + \frac{1 + \eta_1}{2}$ (i.e. spin-down component is multiplied by $\tau_3$). The Hamiltonian (S10) is transformed to:

$$UH(k)U^\dagger = d\delta \tau_1 + \xi \tau_3 - dt \tau_1 \sigma_3 + t \tau_3 \sigma_1,$$

whereas the perturbation Hamiltonian is given by

$$W_0 U\sigma_a \tau_b (h \cdot s) U^\dagger = \begin{cases} W_0 \sigma_a \tau_3 \tau_b (h \cdot s) & (b = 0, 3) \\ -W_0 \sigma_a \tau_3 \tau_b s_3 (h \cdot s) & (b = 1, 2) \end{cases}.$$

In both cases spin part of the perturbation is trivially diagonalized and the overall eigenvalues correspond to two copies of (S8) with a perturbation $\pm W_0 h\sigma_a \tau_3 \tau_b$ for $(b = 0, 3)$ and $\pm iW_0 h\sigma_a \tau_3 \tau_2$ for $(b = 0, 2)$. Thus the spectrum in the presence of perturbation can be determined from Tab. II by identifying the commutation relations of the perturbing operator with $(h \cdot s) \rightarrow \pm h$ multiplied with $\tau_3$ (which has $-++$ signature) with the terms in (S8).

Physically, for $h \parallel d$ all perturbations have similar physical effects as the ones without spin matrices. For $h \perp d$, on the other hand, there are new effects. First, $\tau_3 (h \cdot s)$ results in a full gap with the example of $p + ip$ state ($d \parallel x, h \parallel y$ or vice versa). Another way to create a full gap ($\sim \eta_2$ term in the reduced Hamiltonian (S9)) is with $\sigma_2 \tau_3 (h \cdot s)$, which is more complicated physically. The Zeeman field perpendicular to $d$ results in a spectrum same as for $\pm \tau_3$ perturbation, i.e. shifts the QBT in momentum space rather then creating a nodal line as for $h \parallel d$.

Self-consistency equation in TBSC

The presence of a finite DOS at the magic angle raises the question of whether the magnitude of the superconducting gap may be affected by its presence. Here we show that the resulting corrections to the gap equation at low temperatures are perturbatively small. Here we present calculations for the singlet SC case for simplicity. Assuming the interaction between two layers to be much weaker then the intralayer one, the self-consistency equations are given...
by:
\[
\Delta_1(T, \mathbf{k}) = T \sum_{\mathbf{k}', \mathbf{k}''} V_{SC}^{(1)}(\mathbf{k}, \mathbf{k}'') F_1(i\varepsilon', \mathbf{k}');
\]
\[
\Delta_2(T, \mathbf{k}) = T \sum_{\mathbf{k}', \mathbf{k}''} V_{SC}^{(2)}(\mathbf{k}, \mathbf{k}'') F_2(i\varepsilon', \mathbf{k}'),
\]
where \( F_1/2(i\varepsilon', \mathbf{k}') \) is the anomalous Green’s function in one of the layers. The anomalous Green’s function is:
\[
F_1(i\varepsilon, \mathbf{k}) = \frac{\Delta_1(\varepsilon^2 + \xi^2 + \Delta_2^2) + t^2\Delta_2}{(\varepsilon^2 + \xi^2 + \Delta_2^2 + (\varepsilon^2 + \xi^2)(\Delta_1^2 + \Delta_2^2) + 2t^2(\varepsilon^2 - \xi^2) + 2t^2\Delta_1\Delta_2 + \Delta_1^2\Delta_2 + \Delta_2^2\Delta_2 + t^4};
\]
(recall that \( \xi \) and \( \delta \) are defined in Eq. (S7)); \( F_2(i\varepsilon, \mathbf{k}) \) is obtained from the above by exchanging \( 1 \leftrightarrow 2 \).

Taking for simplicity a separable form of the interaction \( V_{SC}(\mathbf{k}, \mathbf{k}') = V_{SC}(f(\mathbf{k})f(\mathbf{k}') \) we find that the solution takes the form \( \Delta_n(T, \mathbf{k}) = \Delta_0(T)f(\mathbf{k}) \); \( \Delta_b(T, \mathbf{k}) = \Delta_0(T)f(\mathbf{k}) \), which is a consequence of the \( \varphi_0 \rightarrow -\varphi_0 \) symmetry of the bilayer. Using the expansion \( \Delta_1(2) = \delta - (\pm)\delta_0 \) near the nodes the equation for the amplitude of the order parameter \( \Delta_0(T) \) takes the form (using \( f(\mathbf{k}) \approx (\delta - \delta_0)/\Delta_0 \)):
\[
\Delta_0 = -V_{SC} \frac{T}{\Delta_0} \sum_{\varepsilon_n, \mathbf{k}} I(\delta, \xi, \varepsilon_n);
\]
\[
I(\delta, \xi, \varepsilon_n) = \frac{(\delta - \delta_0)^2(\varepsilon^2 + \xi^2 + (\delta + \delta_0)^2) + t^2(\delta^2 - \delta^2_0)}{(\varepsilon^2 + \xi^2 + \delta^2 + t^2)^2 - 4t^2\xi^2 + 2\delta_0^2(\varepsilon^2 + \xi^2 - \delta^2 - \delta^2_0)}.
\]
For \( \varepsilon_n, \xi \gg t \) the integrand is approximately:
\[
I(\delta, \xi, \varepsilon_n)|_{\varepsilon_n, \xi \gg t} \approx \frac{(\delta - \delta_0)^2}{\varepsilon^2_n + \xi^2 + (\delta - \delta_0)^2};
\]
that can be shown to be independent of \( \delta_0 \) with a variable shift \( \delta \rightarrow \delta + \delta_0 \). Indeed, the expression above corresponds to the case \( t = 0 \) when the layers are simply decoupled. The integral can be estimated as follows:
\[
T \sum_{\varepsilon_n, \mathbf{k}} I(\delta, \xi, \varepsilon_n)|_{\varepsilon_n, \xi \gg t} \approx \frac{1}{(2\pi)^3 v_F v_\delta} \int_{\Delta_0}^{\infty} \frac{d\delta}{\Delta_0} \int d\xi d\varepsilon \frac{\delta^2}{\varepsilon_n^2 + \xi^2 + \delta^2} \approx \frac{2\Delta_0^3}{3(2\pi)^2 v_F v_\delta} \left( \frac{1}{3} + \log \frac{\Lambda_0}{\Delta_0} \right),
\]
where \( \Lambda_0 \) is the cutoff for the \( \xi \) integral.

Thus, for \( \varepsilon, \xi \gg t, \delta_0 \), on the other hand, the most important question is whether there is a divergence near the nodes. As it is expected to be strongest (if present) at the magic angle we study the case \( \delta_0 = t \). The integrand can be written as:
\[
\delta I(\delta, \xi, \varepsilon_n)|_{\varepsilon_n, \xi \ll t, \delta_0 = t} = \frac{(\delta - t)^2(\varepsilon^2 + \xi^2) + \delta^2(\delta^2 - t^2)}{(\varepsilon^2 + \xi^2 + \delta^2)^2 + 4\varepsilon^2 t^2};
\]
and its contribution to the integral can be estimated assuming an upper cutoff \( \Lambda_0 \) and a lower one \( \Delta_0 \). The result is
\[
\sim \frac{t^2 \delta^2_0}{(2\pi)^2 v_F v_\delta} \log^{-1}(\Lambda_0/\Delta_0).
\]
At low values of \( \varepsilon, \xi \ll \delta_0 \), on the other hand, the most important question is whether there is a divergence near the nodes. As it is expected to be strongest (if present) at the magic angle we study the case \( \delta_0 = t \). The integrand can be written as:
\[
I(\delta, \xi, \varepsilon_n)|_{\varepsilon, \xi \ll t, \delta_0 = t} = \frac{(\delta - t)^2(\varepsilon^2 + \xi^2) + \delta^2(\delta^2 - t^2)}{(\varepsilon^2 + \xi^2 + \delta^2)^2 + 4\varepsilon^2 t^2}.
\]
Close to the QBT at \( \xi, \delta = 0 \) the integrand is approximately
\[
I(\delta, \xi, \varepsilon_n)|_{\varepsilon, \xi \ll t, \delta_0 = t} \approx \frac{1}{4} \frac{\xi^2 - \delta^2 + (\xi^2 + \delta^2)\delta^2/t^2}{(\xi^2 + \delta^2)^2/4t^2 + \varepsilon^2};
\]
where a linear in $\delta$ term in the numerator is omitted as it vanishes after integration. The contribution of $I(\delta, \xi, \varepsilon_n)|_{\xi, \varepsilon_n t = \delta_0}$ to the sum in the gap equation can be evaluated assuming a cutoff $\sim t$ yielding

$$T \sum_{\varepsilon_n, k} I(\delta, \xi, \varepsilon_n)|_{\xi, \varepsilon_n t = \delta_0} \sim \frac{\pi t^4}{16(2\pi)^2 v_F v_b},$$

which is smaller by a factor $(t^3/\Delta_0^2) \log^{-1}(\Lambda_0/\Delta_0)$ then the leading term (S13) that is independent of twist angle.

**Spectrum in the presence of interlayer phase difference**

In the presence of an interlayer phase difference $\varphi$ the order parameters in the two layers are $\Delta_1(\mathbf{k}) = \Delta(\mathbf{k}) e^{i \varphi/2} = \Delta(\mathbf{k})(\cos(\varphi/2) + i \sin(\varphi/2))$, $\Delta_2(\mathbf{k}) = \Delta(\mathbf{k})(\cos(\varphi/2) + i \sin(\varphi/2))$. In Gor’kov-Nambu space the imaginary part of the gap is written with $\tau_2$ instead of $\tau_1$; as $\tau_2 = i \tau_3 \tau_1$ Eq. 7 of the main text is obtained. The spectrum is

$$E^2(\mathbf{k}) = \epsilon^2 + t^2 + \delta^2 + \delta_0^2 \pm \sqrt{4t^2 \epsilon^2 + 4\delta^2 \delta_0^2 + 2t^2 (\delta^2 + \delta_0^2)} - 2t^2 (\delta^2 - \delta_0^2) \cos \varphi.$$  \hspace{1cm} (S14)

Now we show that for $\varphi \neq 0$ the spectrum (S14) is always gapped. First, we can rewrite (S14) in the form:

$$E^2(\mathbf{k}) = \left[ t \pm \sqrt{\epsilon^2 + \delta^2 \left( \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} \right) + \delta_0^2 \cos^2 \frac{\varphi}{2}} \right]^2 + \delta^2 \left( \cos^2 \frac{\varphi}{2} - \frac{\delta_0^2}{t^2} \right) + \delta_0^2 \sin^2 \frac{\varphi}{2},$$

using which one can show that for $\delta_0^2 / t^2 < \cos^2 \frac{\varphi}{2}$ the dispersion minimum is at $\delta = 0, \xi = \pm \sqrt{t^2 - \delta_0^2 \cos^2 \frac{\varphi}{2}}$ and the lowest eigenvalues are $\pm |\delta_0 \sin \frac{\varphi}{2}|$.

More generally, the conditions for the minimum of (S14) (with $- \text{sign taken}$) are:

$$2\delta - \frac{4t\delta}{2} \left( \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} \right) = 0;$$
$$2\xi - \frac{4t\xi}{2} \left( \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} \right) + \delta_0^2 \cos^2 \frac{\varphi}{2} = 0.$$

There are three cases: $\delta = 0, \xi \neq 0$ leads to the result above. For $\delta \neq 0, \xi = 0$, we get $\delta = \left( \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} \right)^2 t^2 - \delta_0^2 \cos^2 \frac{\varphi}{2}$ and $E^2 = t^2 + \delta_0^2 - \frac{\delta_0^2 \cos^2 \frac{\varphi}{2}}{\delta_0^2 + \sin^2 \frac{\varphi}{2}} \left( \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} \right)^2 t^2$. If both $\xi$ and $\delta$ are assumed to be non-zero, it is necessary that $\delta_0^2 / t^2 = \cos^2 \frac{\varphi}{2}$; in that case there is a ring of minima with $\sqrt{\epsilon^2 + \delta^2} = t^2 - \delta_0^2 \cos^2 \frac{\varphi}{2} = t^2 (1 - \cos^4 \frac{\varphi}{2})$ and $E^2 = \delta_0^2 \sin^2 \frac{\varphi}{2}$. Thus, in each case the minimum of the gap is always non-zero. Summarizing the above, the spectral gap is given by (cf. Eq. (8) of the main text):

$$\Delta_f(\varphi) = \begin{cases} 2 |\delta_0 \sin \frac{\varphi}{2}| & \frac{\delta_0^2}{t^2} \leq \cos^2 \frac{\varphi}{2}; \\ \frac{\delta_0^2}{t^2} + \sin^2 \frac{\varphi}{2} & \frac{\delta_0^2}{t^2} \geq \cos^2 \frac{\varphi}{2}. \end{cases} \hspace{1cm} (S15)$$

We stress that the above expression is always nonzero.

**Projected Hamiltonian near MA in the presence of Josephson current**

We can also consider the projection of the Hamiltonian to the low-energy eigenstates at MA. The new terms appearing in the Hamiltonian are (we specify here the singlet SC case)

$$\delta H = v_{\Delta} k_{\perp} \sin(\varphi/2) \tau_2 \sigma_3 - t \sin(\varphi/2) \tau_2.$$  \hspace{1cm} (S16)
Apart from the $\tau_2$ term leading to the gap, an additional term $v_{\Delta k_{\perp}} \sigma_3 \tau_2$ that is however: (1) small close to QBT $\delta \ll t$ (2) does not result in the gap closing (see (S14)). Furthermore, if projected to the basis of (S9), $\tau_2 \sigma_3$ yields zero, while second-order perturbation theory results in a contribution $-\frac{\sin^2(\phi/2)(v_{\Delta k_{\perp}})^2}{2t}$ that can be neglected for $\phi \ll 1$. As the topological properties do not depend on the gap value as long as it’s nonzero (and it doesn’t close on increasing $\phi$), this term can be dropped when discussing topological properties.

**Current-phase relation in TBSC**

The Josephson current-phase relation can be obtained from the derivative of the free energy of the bilayer with respect to the phase difference $I(\phi) = \frac{2e}{h} \frac{dF(T, \phi)}{d\phi}$ [67]. The free energy is given by

$$F(T, \phi) = -2T \sum_{\varepsilon_n} \int \frac{dk}{(2\pi)^2} \log \left\{ \varepsilon_n^2 + (\xi + t)^2 + \delta^2 \right\}$$

$$-4 \sin^2 \frac{\phi}{2} \delta^2 \varepsilon_n^2 - 2 \delta^2 \varepsilon_n^2 \cos \phi + 2 \delta^2 (\varepsilon_n^2 + \xi^2 - \delta^2) + \delta_0^2 \right\},$$

where the $2$ in front is due to spin. Calculating the current yields

$$I(\phi) = \frac{2e}{h} \frac{dF(T, \phi)}{d\phi} = \frac{4e}{h} T \sum_{\varepsilon_n} \frac{1}{(2\pi)^2 v_F \varepsilon_n} \int d\xi \delta$$

$$2t^2 (\delta^2 - \delta_0^2) \sin \phi$$

$$\left( \varepsilon_n^2 + \xi^2 + \delta^2 \right) + 2 \varepsilon_n^2 (t^2 + \delta_0^2) + 2 \varepsilon_n^2 (t^2 - \delta_0^2) - 4 \delta^2 t^2 \sin^2 \frac{\phi}{2} + (t^2 - \delta_0^2)^2 + 4 \delta^2 t^2 \sin^2 \frac{\phi}{2}.$$  \hspace{1cm} (S17)

Where the upper cutoff for the $\delta$ integral is $\Delta_0$. We can divide the sum into high- and low-energy parts. The former one, assuming $\xi, \delta \gg t, \delta_0, T$ can be approximated by

$$I(\phi)_{|\xi, \delta \gg t, \delta_0} \approx \frac{4e}{h} \frac{1}{(2\pi)^2 v_F \varepsilon_n} \int_{\sim t, \delta_0}^{\Delta_0} \int d\xi d\delta \sin \phi$$

$$\frac{2t^2 \delta^2 \sin \phi}{\left( \varepsilon_n^2 + \xi^2 + \delta^2 \right)^2} \approx \frac{8et^2 \sin \phi}{(2\pi)^2 v_F \varepsilon_n} \Delta_0.$$  \hspace{1cm} (S18)

The low-energy part $\xi, \delta \ll t, \delta_0$ can be estimated as follows. The effects of this part are expected to be most pronounced near the magic angle, since the density of states near zero energy is the largest in this case. As increasing $\phi$ enhances the spectral gap, we may furthermore focus on the case of small phase $\phi \ll 1$. The characteristic values of $\xi$ and $\delta$ can be deduced from the dispersion at the magic angle being $\frac{\xi^2 + t^2}{2t}$ and the current-induced gap $\Delta_0 \sim t \sin(\phi/2)$ implying $\xi^2, \delta^2 \sim t^2 \sin(\phi/2)$, which is also evident from (S18). Moreover, for $t^2 |\sin(\phi/2)| \gg |\delta_0^2 - t^2| \sim 2t v\Delta K_N |\theta - \theta_{MA}|$ and thus $|\sin(\phi/2)| \gg \Delta_0 |\theta - \theta_{MA}|/t$ one can neglect the quadratic terms in $\xi$ and $\delta$ with respect to the quartic ones (using $\sin^2(\phi/2) \ll 1$). One also observes that characteristic $\varepsilon_n^2$ values are of the order $t^2 \sin^2(\phi/2)$ which can be neglected with respect to $\xi^2, \delta^2$, leading to the estimate:

$$\delta I(\phi) = -\frac{4e}{h} \frac{1}{(2\pi)^2 v_F \varepsilon_n} T \sum_{\varepsilon_n} \int_{\xi}^{\delta \leq \delta_0} d\xi d\delta \sin \phi$$

$$= T = 0 - \frac{et^3}{2\pi h v_F \Delta} \sin \phi \log \left\{ \frac{1}{\sin \pi t} \right\}$$

$$= T = t \sin(\phi/2) - \frac{et^3}{2\pi h v_F \Delta} \sin \phi \log \frac{2et \gamma}{\pi T}.$$  \hspace{1cm} (S19)

There is a logarithmic singularity at low values of $\phi$, however, its effect is important only for $\phi < \frac{1}{\Delta_0} e^{-4\Delta_0/(\pi t)}$ and $T < te^{-4\Delta_0/(\pi t)}$ where both limits expected to be extremely small for $t \ll \Delta_0$. As the gap maximum is attained at $\phi \approx \pi/2$ we neglect this contribution, resulting in the conventional current-phase relation $I(\phi) \sim \sin(\phi)$.

**SPONTANEOUS SYMMETRY BREAKING AT MA**

**Order parameter susceptibilities**

The self-consistency equation for an order parameter are in channel $\hat{A}$ is given by:
where $\nu = \frac{1}{\pi\epsilon_F^cD}$. Here we compare the susceptibilities $\hat{\chi}_A$ for intralayer order parameters (i.e. that do not include $\sigma_{1,2,3}$), as the interlayer interaction is likely to be much smaller then the intralayer one. In particular, to estimate the susceptibilities we will break them down into contributions from low and high energies.

Without spin, matrices $\tau_3, \tau_1, \tau_1\sigma_3, \tau_3\sigma_1$ are already contained in the Hamiltonian (S8) and do not constitute an order parameter, however, their spinful versions (i.e. multiplied by $s_i$) do; for singlet superconductor or $s \parallel d$ the susceptibilities are equal to those calculated if the spin matrix was replaced with an identity matrix. For $s \perp d$, as was discussed above (see discussion after (S10)), the resulting spectrum (and thus the susceptibility) is identical to the one for the order parameter with $s \rightarrow \tau_3$ (or $i\tau_3$ for orders containing $\tau_{1,2}$ such that the result is a Hermitian matrix).

- $\xi, \delta \gg t, T$: In this case the expressions for the susceptibilities are simplified. Furthermore, as two terms in the Hamiltonian (S8) can be neglected, with the remaining ones not containing $\sigma$ matrices, the susceptibility depends only on the $\tau$ component of the order parameter:

$$\chi_{\tau_0}(i\epsilon_n, \xi, \delta) \approx \frac{2}{\epsilon_n^2 + \xi^2 + \delta^2}; \xi, \delta \gg t, T$$

Integrating over $\epsilon$ and $\Lambda_0 > |\xi| > t$ and $\Delta_0 > |\delta| > t$ one obtains:

$$\chi_{\tau_0}(i\epsilon_n, \xi, \delta) \approx O \left( t \log \frac{\Delta_0}{\Lambda_0} \right)$$

One observes that there is a large $4\Delta_0 \log \frac{\Delta_0}{\Lambda_0}$ contribution to the superconducting instabilities $\tau_{1,2}$.

- $\xi, \delta \ll t$: Here we expand the $\chi_A(\{i\epsilon_n, \xi, \delta\})$ near the QBT; in particular we look for the ones that have the strongest singularity at the origin $\epsilon_n, \xi, \delta \rightarrow 0$; the regular ones would then give a contribution of the order $t$. Of all the order parameters considered, only the $\tau_2$ channel results in a logarithmic singularity with the respective susceptibilities being equal to:

$$\chi_{\tau_2}(i\epsilon_n, \xi, \delta) \approx t \frac{\log 2}{{\pi T}}$$

where $\gamma$ is the Euler-Mascheroni constant.

*Estimating the critical temperature of the quasiparticle interaction-induced instability within the superconducting phase*

If we assume that at zero twist angle the system is stable, the order parameters that have most propensity to develop at the magic angle are those that show a singular $\chi(T)$. These are $\tau_2$ order parameter for the singlet case and additionally $\tau_1s_\perp$ for the triplet case, where the direction of $s_\perp$ is orthogonal to $d$. The critical temperatures are obtained approximately by combining (S21) and (S22) (here the notations are for the singlet case):

$$T^{*\tau_2/\tau_1s_\perp} = \frac{2t\gamma}{\pi} \left( \frac{2\Lambda_0}{\Delta_0} \right)^{\frac{\Delta_0}{\Lambda_0}} e^{-\frac{2\gamma}{\pi T} + \frac{\Delta_0}{\Lambda_0}}$$

Assumption of no secondary instability at zero twist angle suggests that $\frac{1}{\lambda_T} > \frac{2\Lambda_0}{\Delta_0} \left( \log \frac{2\Lambda_0}{\Delta_0} + 1 \right)$. Thus, $T_c$ is expected to be less then or of the order $t$. To find how $T_c$ is suppressed away from the magic angle we write the susceptibility $\chi_{\tau_2}(i\epsilon_n, \xi, \delta)$ in the following form:

$$\chi_{\tau_2}(i\epsilon_n, \xi, \delta) \approx -\frac{2}{\epsilon_n^2 + \xi^2 + \delta^2 + t^2 + \delta_0^2}$$

(S24)
For $T, |\delta_0 - t| \ll t$ one has for $\varepsilon_n \sim T$: $\xi^2 + \delta^2 \sim Tt \gg \varepsilon_n^2$, and thus the expression can be simplified to capture the most divergent part:

$$\chi_{\tau_2}(i\varepsilon_n, \xi, \delta) \approx \frac{-4t^2}{4\varepsilon_n^2 t^2 + (\xi^2 + \delta^2)^2 + 4T^2(\delta_0 - t) + 4t(\delta_0 - t)(\xi^2 - \delta^2)}.$$  \hfill (S25)

Then one can write:

$$\chi_{\tau_2}(T = T^*) - \chi_{\tau_2}^{=\delta_0}(T = T^*) \rightarrow$$

$$\log \frac{T^*}{T_0} = \int_0^\infty d\varepsilon \int_0^{2\pi} \frac{d\eta}{2\pi} \frac{\tanh \sqrt{\varepsilon^2 - 2|\delta_0 - t|\varepsilon \cos(2\eta) + (\delta_0 - t)^2}}{\sqrt{\varepsilon^2 - 2|\delta_0 - t|\varepsilon \cos(2\eta) + (\delta_0 - t)^2}}$$ \hfill (S26)

where cylindrical coordinates $\varepsilon = \frac{\xi^2 + \delta^2}{2t}, \eta$ for $\xi, \delta$ integration have been used.

**DETAILS OF $\theta_{MA}$ ESTIMATES FOR MATERIAL CANDIDATES**

- **Cuprates**: $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+y}$ The dominant interplane hopping in this case has the form $t \sim (\cos k_x - \cos k_y)^2$ that vanishes near the gap nodes [107]; more recent estimates [94] suggest that there is a nonzero tunneling along the nodal direction. Using the values of $v_\Delta$ for optimal doping [93] and the interlayer tunneling parameters from [94], one obtains $\theta_{MA} \approx 13.8^\circ$.

- **(BETS)$_2\text{GaCl}_4$** The interlayer hopping is of the order 0.21 meV [96]. Assuming a cos2$\theta$ d-wave gap with a maximum of 12 meV [95], one obtains $\theta_{MA} \approx t/\Delta_0 = 1^\circ$.

- **Heavy fermions**: $\text{CeCoIn}_5$ is characterized by the anisotropy $m_c/m_a = 5.6$ [102]. Due to the heavy effective mass, we assume the hopping to be mostly due to $f$ electrons, estimating the c-axis hopping from the in-plane one [103] and the mass anisotropy as $t_c \sim m_c/m_a * t_a \approx 0.15$ meV. The gap maximum is known to be around 0.6 meV [103], which yields $\theta_{MA} \approx t/\Delta_0 \approx 14^\circ$.

- **Sr$_2$RuO$_4$** has been initially seen as a candidate for chiral triplet p-wave superconductivity [58], with recent experiments [108–110] having put this idea into question; instead, a diverse range of unconventional superconducting states, with both singlet [78, 111–114] and triplet [78, 111, 115] order parameters are currently discussed. Experimentally, the superconducting gap has been shown to have extremely shallow minima, ultimately indistinguishable from nodes [105, 106], suggesting Sr$_2$RuO$_4$ to be a nodal superconductor.

ARPES experiments reveals that this materials is highly two-dimensional [104, 116], with the observed effects of the out-of-plane dispersion suggesting an interplane hopping being of the order of a few meV (e.g., 2.5 meV in [113]). Sr$_2$RuO$_4$ has an extremely small SC gap, with a maximum of about 350µeV. This implies, that to observe the magic angle in Sr$_2$RuO$_4$-based TBSC, the interlayer tunneling has to be reduced first, by, e.g. an insulating layer introduced between the monolayers, as discussed in the main text.