Multi-normed spaces based on non-discrete measures and their tensor products

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Abstract. Lambert discovered a new type of structures situated, in a sense, between normed spaces and abstract operator spaces. His definition was based on the notion of amplifying a normed space by means of the spaces $\ell^n_2$. Later, several mathematicians studied more general structures (‘$p$-multi-normed spaces’) introduced by means of the spaces $\ell^n_p$, $1 \leq p \leq \infty$. We pass from $\ell^n_p$ to $L_p(X, \mu)$ with an arbitrary measure. This becomes possible in the framework of the non-coordinate approach to the notion of amplification. In the case of a discrete counting measure, this approach is equivalent to the approach in the papers mentioned.

Two categories arise. One consists of amplifications by means of an arbitrary normed space, and the other consists of $p$-convex amplifications by means of $L_p(X, \mu)$. Each of them has its own tensor product of objects (the existence of each product is proved by a separate explicit construction). As a final result, we show that the ‘$p$-convex’ tensor product has an especially transparent form for the minimal $L_p$-amplifications of $L_q$-spaces, where $q$ is conjugate to $p$. Namely, tensoring $L_q(Y, \nu)$ and $L_q(Z, \lambda)$, we obtain $L_q(Y \times Z, \nu \times \lambda)$.

Keywords: $L$-space, $L$-boundedness, general $L$-tensor product, $p$-convex tensor product.

§ 1. Introduction

The subject of this paper is a rather far-reaching generalization (in several steps) of a construction introduced by Lambert in his PhD thesis [1] (his supervisor Wittstock was one of the founding fathers of operator space theory); see also [2]. Informally speaking, this construction is intermediate between the classical structure of a normed space and the structure of an abstract operator space (also known as a quantum space). The latter is studied in the widely known textbooks [3]–[6]; see also [7].

Lambert suggested endowing a given linear space $E$ with a sequence of norms in the spaces of columns of any size consisting of vectors in $E$. We stress that he used not matrices (as in the theory of operator spaces), but their first columns. These norms must satisfy two axioms, ‘contractiveness’ and ‘convexity’, stated in terms of

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the spaces $\ell^n_2$, $n \in \mathbb{N}$. Lambert called the resulting objects ‘Operatorfolgenräume’
and defined two tensor products, ‘maximal’ (see [1], §3.1.1) and ‘minimal’ (see [1], §3.1.3), in the resulting category. The former may be regarded as a predecessor of
the tensor product introduced in §5 below.

Lambert’s theory had various connections with the classical theory of normed
spaces and the theory of operator spaces. Later, a team of mathematicians, starting
from essentially different problems related to Banach lattices, arrived at more gen-
eral structures, but still in the framework of the ‘coordinate approach’ based on the
consideration of columns of arbitrary size. First, there were Dales and Polyakov [8],
later joined by Daws, Pham, Ramsden, Laustsen, Oikhberg and Troitsky [9]–[11].
These authors created a rich and ramified theory, from which we are most inter-
ested in the so-called $p$-multi-normed spaces [11] satisfying an analogue of the con-
tractiveness axiom, but now in terms of the spaces $\ell^n_p$ with an arbitrary fixed $p$,
$1 \leq p \leq \infty$. The ‘best’ of these structures also satisfy an analogue of the convexity
axiom, so-called $p$-convexity (Lambert had $p = 2$).

This paper pursues two aims. First, we extend the class of the structures in ques-
tion by passing from $p$-multi-normed spaces to their ‘continuous’ versions. Namely,
we replace the base space $\ell^p$ by $L_p(X, \mu)$ with an arbitrary measure. To do this, we
replace the coordinate approach by the so-called non-coordinate approach to what
we call an amplification.

In the context of operator spaces, the latter approach was known to specialists
and systematically studied in [7] (see also [12]). Its essence is as follows. Instead of
a sequence of norms on all spaces of columns of vectors in $E$, we consider a norm on
a single space $L \otimes E$, where $L$ is a chosen and fixed ‘base’ space, for example,
$L_p(X, \mu)$. In the case of a discrete counting measure, this approach is equivalent
to the coordinate approach in the papers cited above. However, it gives a better
perspective in the general case. In this approach, the axiom of contractiveness
is transformed into the condition that the normed space $L \otimes E$ is a contractive
module over $B(L)$. Concerning the axiom of $p$-convexity, it now seems that its non-
coordinate version can be defined only under certain rather restrictive assumptions
on $L$ that make this space ‘similar’ to $L_p(X, \mu)$.

We are mainly interested in tensor products of the objects introduced. (In this
paper we call these objects $L$-spaces.) We introduce two essentially different tensor
products. The first, $\otimes_L$, is defined in the case of general base spaces endowed
with the additional structure of a bilinear operator $\diamondsuit : L \times L \to L$ with certain
natural properties. The other tensor product, denoted by $\otimes_{pL}$ and called $p$-convex,
is constructed only in the class of $p$-convex $L$-spaces with base space $L_p(X, \mu)$.
Each of these tensor products is defined in terms of the universal property for its
own class of bilinear operators, and its existence is proved by giving an explicit
construction.

We give several examples. In particular, we show that the tensor product $\otimes_L$
takes a transparent form for $L$-spaces that have the biggest of all possible norms.
It is perhaps more interesting that the second tensor product takes a transparent
concrete form for the $L_p(X, \mu)$-spaces $L_q(\cdot)$, where $1 < p < \infty$ and $q := p/(p−1)$,
this time endowed with the minimal norm. Namely, up to a complete isometric
isomorphism of $L_p(X, \mu)$-spaces (an analogue of complete isometric isomorphisms of operator spaces), we have

$$L_q(Y, \nu) \hat{\otimes}_{pL} L_q(Z, \lambda) = L_q(Y \times Z, \nu \times \lambda).$$

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§ 2. L-spaces and L-quantization

As usual, we denote by $B(E, F)$ the space of all bounded operators between normed spaces $E$ and $F$, and endow it with the operator norm. We write $B(E)$ instead of $B(E, E)$. The identity operator on $E$ will be denoted by $1_E$. Two projections $P$ and $Q$ on $E$ are said to be orthogonal if $PQ = QP = 0$.

The symbol $\otimes$ is used for the algebraic tensor product of linear spaces and for elementary tensors. The symbols $\otimes_{pr}$ and $\otimes_{in}$ denote the non-completed projective and injective tensor products of normed spaces.

We choose and fix a normed space $L$ (for now arbitrary) and call it the base space. We write $B$ instead of $B(L)$.

In what follows we need the threefold notion of amplification mentioned in the introduction. We first amplify linear spaces, then linear operators, and finally bilinear operators. Note that these amplifications differ from those used in the theory of quantum spaces (see [7], for example). The amplification of a linear space $E$ is the tensor product $L \otimes E$. We usually abbreviate this notation to $LE$ and denote elementary tensors (such as $\xi \otimes x$, where $\xi \in L$, $x \in E$) by $\xi x$. Note that $LE$ is a left module over the algebra $B$ with the outer multiplication $\cdot$, which is well defined by the formula $a \cdot (\xi x) := a(\xi)x$.

Definition 2.1. A seminorm on $LE$ is called an $L$-seminorm on $E$ if the left $B$-module $LE$ is contractive, that is, $\|a \cdot u\| \leq \|a\|\|u\|$ for all $a \in B$ and $u \in LE$.

A space $E$ endowed with an $L$-seminorm is called a seminormed $L$-space. If the seminorm in question is actually a norm, then we naturally speak of a normed $L$-space and in this case we usually omit the word ‘normed’.

Example 2.2 (principal). Let $(X, \mu)$ be a measure space. Our principal example of a base space is $L_p(X, \mu)$, where $1 \leq p \leq \infty$. We use the textbook [13] as the main reference in measure theory. For simplicity we always assume that all our measures have a countable basis (thus the set of atoms is at most countable). When there is no danger of misunderstanding, we speak of a measure space $X$ and the normed space $L_p(X)$. In particular, $X \times Y$ denotes the Cartesian product of measure spaces.

Remark 2.3. After translation into ‘index-free’ language, the papers cited above are devoted to the case when $X = \mathbb{N}$ with the counting measure. In particular, Lambert’s spaces are those with $L = \ell_2$, and in the Dales–Polyakov spaces we have either $L = \ell_\infty$ or $L = \ell_1$. Finally, when $L = \ell_p$, the notion of $L$-space is equivalent to that of $p$-multi-normed space of Dales–Laustsen–Oikhberg–Troitsky [11], § 2.2.

A seminormed $L$-space $E$ becomes a seminormed space in the ordinary sense if we put $\|x\| := \|\xi x\|$ for all $x \in E$, where $\xi \in L$ is an arbitrary vector of norm 1.
Clearly, the result is independent of the choice of $\xi$. The resulting seminormed space is called the underlying space of the given $L$-space, and the latter is called an $L$-quantization of the former. We use this terminology by analogy with quantization in operator space theory (see, for example, [14], [3] or [7]). Clearly, $\|\xi x\| = \|\xi\|\|x\|$ for all $\xi \in L$ and $x \in E$.

It is easy to verify that the space $C$ of scalars has a unique $L$-quantization, which is obtained by identifying $LC$ with $L$.

**Proposition 2.4.** Let $E$ be a seminormed $L$-space with a normed underlying space. Then the $L$-seminorm on $LE$ is a norm.

*Proof.* Take any $u \in LE$, $u \neq 0$, and represent it as $\sum_{k=1}^{n} \xi_k x_k$, where the $\xi_k$ are linearly independent, $\|\xi_1\| = 1$ and $x_1 \neq 0$. Clearly, there is a $T \in B$ with $\|T\xi_1\| = 1$ and $T\xi_k = 0$ for $k > 1$. Then, by Definition 2.1,

$$\|T\|\|u\| \geq \|T \cdot u\| = \|x_1\| > 0. \quad \square$$

**Example 2.5.** Generally speaking, every normed space has many $L$-quantizations. We distinguish two of them. The $L$-space denoted by $E_{\max}$ (resp. $E_{\min}$) has the $L$-norm obtained by endowing $LE$ with the norm of $L \otimes_{pr} E$ (resp. $L \otimes_{in} E$). We denote the norms of these spaces by $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ respectively. The corresponding quantizations of $E$ are said to be maximal and minimal respectively. Clearly, the $L$-norm of $E_{\max}$ is the greatest of all $L$-norms of $L$-quantizations of $E$. The adjective ‘minimal’ will be justified later.

**Example 2.6.** We want to introduce an $L$-quantization of the projective tensor product $E \otimes_{pr} F$ of two normed spaces provided that one of the tensor factors (say, $F$) is itself an $L$-space.

Consider the linear isomorphism

$$\beta: L(E \otimes F) \rightarrow E \otimes_{pr} (LF): \xi(x \otimes y) \mapsto x \otimes \xi y$$

and introduce a norm in $L(E \otimes F)$ by setting $\|U\| := \|\beta(U)\|$. The space $E \otimes_{pr} (LF)$, being the projective tensor product of a normed space and a contractive $B$-module, also has the standard structure of a contractive $B$-module. Since $\beta$ is a $B$-module morphism, the same holds for $L(E \otimes F)$. Thus $E \otimes F$ becomes an $L$-space. We denote the norm of the corresponding underlying space simply by $\|\cdot\|$, and the norm of $E \otimes_{pr} F$ by $\|\cdot\|_{pr}$. We must show that $\|\cdot\| = \|\cdot\|_{pr}$.

Take an arbitrary $u \in E \otimes F$. Since $\|\cdot\|$ is clearly a cross-norm, we have $\|u\| \leq \|u\|_{pr}$. It remains to show that $\|\xi u\|' \geq \|u\|_{pr}$ for every $\xi \in L$ with $\|\xi\| = 1$.

Identifying the $B$-modules $L(E \otimes F)$ and $E \otimes_{pr} (LF)$ by means of $\beta$, we represent $\xi u$ as

$$\sum_{k=1}^{n} x_k \otimes w_k, \quad x_k \in E, \quad w_k \in LF.$$  

Take a functional $f: L \rightarrow C$ such that $f(\xi) = 1$ and $\|f\| = 1$, and define an operator (of norm 1) $T: L \rightarrow L: \eta \mapsto f(\eta)\xi$. Clearly, $T \cdot w_k = \xi y_k$ for some $y_k \in F$, ...
\(k = 1, \ldots, n\). Therefore,
\[
\sum_{k=1}^{n} \|x_k\|\|w_k\| \geq \sum_{k=1}^{n} \|x_k\|\|T \cdot w_k\| = \sum_{k=1}^{n} \|x_k\|\|y_k\|.
\]

But
\[
\xi u = T \cdot (\xi u) = \sum_{k=1}^{n} x_k \otimes T \cdot w_k = \xi \left( \sum_{k=1}^{n} x_k \otimes y_k \right).
\]

Hence \(u = \sum_{k=1}^{n} x_k \otimes y_k\). It follows that
\[
\sum_{k=1}^{n} \|x_k\|\|w_k\| \geq \|u\|_{pr}.
\]

By the definition of the projective norm, this yields the desired bound \(\|\xi u\|' \geq \|u\|_{pr}\).

Remark 2.7. In [7], the amplification of \(E\) is defined as \(F \otimes E\), where \(F\) is the space of bounded finite-rank operators on some Hilbert space \(L\). However, it is worth mentioning that the norm on \(F \otimes E\) endowing \(E\) with the structure of an (abstract) operator space is not always an \(L\)-norm in the sense of Definition 2.1. The simplest counterexample is \(E := \mathcal{B}(L)\) with the standard quantum norm. The following assertion can be proved using estimates obtained by Tomiyama [15]. Let \(\tau: \mathcal{F} \to \mathcal{F}\) be the operator sending every matrix to its transpose. Then for every \(C > 0\) there is a \(u \in FE\) such that, although \(\|\tau\| = 1\), we have \(\|\tau \cdot u\| > C\|u\|\).

\[\text{§ 3. L-bounded linear and bilinear operators}\]

Suppose that we are given an operator \(\varphi: E \to F\) between linear spaces. For brevity we write \(\varphi_{\infty}\) for the operator \(1_L \otimes \varphi: LE \to LF\) (sending \(\xi x\) to \(\xi \varphi(x)\)) and call it the amplification of \(\varphi\). Clearly, \(\varphi_{\infty}\) is a morphism of left \(B\)-modules.

Definition 3.1. An operator \(\varphi: E \to F\) between seminormed \(L\)-spaces is said to be \(L\)-bounded if the operator \(\varphi_{\infty}\) is bounded. Then we set \(\|\varphi\|_{Lb} := \|\varphi_{\infty}\|\). Similarly, in terms of \(\varphi_{\infty}\), we define \(L\)-contractive and \(L\)-isometric operators as well as \(L\)-isometric isomorphisms.

If \(\varphi\) is bounded as an operator between the corresponding underlying seminormed spaces, then we (just) say that is bounded and, as usual, denote its operator seminorm by \(\|\varphi\|\). Clearly, every \(L\)-bounded operator \(\varphi: E \to F\) is bounded, and \(\|\varphi\| \leq \|\varphi\|_{Lb}\).

Some operators between \(L\)-spaces are ‘automatically’ \(L\)-bounded provided that they are bounded as operators between the underlying spaces. Here is the first phenomenon of that kind.

Proposition 3.2. Let \(E\) be an \(L\)-space. Then every bounded functional \(f: E \to \mathbb{C}\) is \(L\)-bounded and \(\|f\|_{Lb} := \|f\|\).
Proof. Recall that
\[ \|f_\infty(u)\| = \max\{\langle f_\infty(u), \alpha \rangle, \alpha \in L^*, \|\alpha\| = 1\}. \]
For every \( \alpha \) we set \( x_\alpha := (\alpha \otimes 1_E)(u) \in E \). Then, representing \( u \) as a sum of elementary tensors, we obtain that \( \langle f_\infty(u), \alpha \rangle = f(x_\alpha) \). We now fix an arbitrary \( \eta \in L \) with \( \|\eta\| = 1 \) and consider the operator \( S: L \to L: \xi \mapsto \alpha(\xi)\eta \). Clearly, \( \|S\| = \|\alpha\| = 1 \). But the same representation of \( u \) implies that \( S \cdot u = \eta x_\alpha \).
Therefore we have \( \|x_\alpha\| = \|\eta x_\alpha\| \leqslant \|S\| \|u\| = \|u\| \). It follows that \( |\langle f_\infty(u), \alpha \rangle| \leqslant \|f\| \|x_\alpha\| = \|f\| \|u\| \) and \( \|f_\infty\| = \|f\| \).

As a corollary, for every \( L \)-space \( E \) and every \( u \in LE \) we have \( \|u\| \geqslant \sup\{\|f_\infty(u)\|\} \), where the supremum is taken over all \( f \in E^* \) with \( \|f\| \leqslant 1 \). But this supremum is exactly \( \|u\|_{\text{min}} \). This justifies the term ‘minimal’ in Example 2.5.

We now want to define amplifications of bilinear operators. To do this, we need a certain additional structure connected with our base space, namely, a fixed bilinear operator, which is denoted in what follows by \( \diamond: L \times L \to L \) and called the \( \diamond \)-operation. Let us write \( \xi \diamond \eta \) instead of \( \diamond(\xi, \eta) \). A \( \diamond \)-operation is said to be metric if we always have \( \|\xi \diamond \eta\| = \|\xi\| \|\eta\| \).

Suppose that \( E \) is a linear space, \( \xi \in L \) and \( u \in LE \). Then we set \( \xi \diamond u := T_\xi \cdot u \), where \( T_\xi \in B \) takes \( \eta \) to \( \xi \diamond \eta \). Thus, this version of the \( \diamond \)-operation is well defined on elementary tensors by the equality \( \xi \diamond \eta x := (\xi \diamond \eta)x \). We similarly define \( u \diamond \eta \in LE \) by the equality \( \xi x \diamond \eta := (\xi \diamond \eta)x \). When the \( \diamond \)-operation is metric, we have \( T_\xi = \|\xi\| S \), where \( S \) is an isometry. Therefore, if \( E \) is an \( L \)-space, then
\[ \|\xi \diamond u\| \leqslant \|\xi\| \|u\| \quad \text{and, similarly,} \quad \|u \diamond \eta\| \leqslant \|\eta\| \|u\|. \]  

(3.1)

**Example 3.3** (principal). Suppose that our base space is \( L_p(X) \) in Example 2.2. We say that two measure spaces are of the same type if
a) they simultaneously have or do not have a continuous (= non-atomic) part,
b) their sets of atoms have the same cardinality.

It is well known (see, for example, [13], Corollary 9.12.18, and also [16], §14, or [17], III.A) that the spaces \( L_p(X) \) and \( L_p(Y) \) with \( p \neq 2 \) are isometrically equivalent if and only if the corresponding measure spaces are of the same type. It follows immediately that the spaces \( L_p(X) \) and \( L_p(X \times X) \) with \( p \neq 2 \) are isometrically isomorphic if and only if \( X \) has no atoms, exactly one atom or an infinite (necessarily countable) set of atoms. Thus in these three cases we can choose an isometric isomorphism \( i: L_p(X \times X) \to L_p(X) \) and fix it throughout the whole paper. Having done this, we define \( \diamond: L_p(X) \times L_p(X) \to L_p(X) \) as the composite \( i \vartheta_X \), where the bilinear operator \( \vartheta_X: L_p(X) \times L_p(X) \to L_p(X \times X) \) takes a pair \( (x, y) \) to the function \( x(s)y(t) \) of two variables \( s, t \in X \). Clearly, this \( \diamond \)-operation is metric.

**Remark 3.4.** Of course, when \( p = 2 \) (that is, when we are dealing with Hilbert spaces), a metric \( \diamond \)-operation exists for all measure spaces.

For the rest of the paper we assume that our base space is endowed with a metric \( \diamond \)-operation.
Let $\mathcal{R} : E \times F \to G$ be a bilinear operator between linear spaces. Its *amplification* is the bilinear operator $\mathcal{R}_\infty : LE \times LF \to LG$ associated with the 4-linear operator $L \times E \times L \times F \to LG : (\xi, x, \eta, y) \mapsto (\xi \diamond \eta)\mathcal{R}(x, y)$.

Thus $\mathcal{R}_\infty$ is well defined on elementary tensors by the equality $\mathcal{R}_\infty(\xi x, \eta y) = (\xi \diamond \eta)\mathcal{R}(x, y)$.

**Definition 3.5.** A bilinear operator $\mathcal{R}$ between $L$-spaces is said to be $L$-*bounded* (resp. $L$-*contractive*) if its amplification is bounded (resp. contractive). We put $\|\mathcal{R}\|_{L_b} := \|\mathcal{R}_\infty\|$.

It is easy to show that the property of a bilinear operator in our principal example of being $L$-bounded does not depend on the concrete choice of $\diamond$ or, equivalently, on the choice of an isometric isomorphism between $L_p(X \times X)$ and $L_p(x)$. Let $\mathcal{R}$ be an $L$-bounded bilinear operator. Then the equality $\mathcal{R}_\infty(\xi x, \eta y) = (\xi \diamond \eta)\mathcal{R}(x, y)$ implies that $\mathcal{R}$ is bounded as an operator between the corresponding underlying spaces and $\|\mathcal{R}\| \leq \|\mathcal{R}\|_{L_b}$. At the same time, as with linear operators, $L$-boundedness sometimes follows automatically from ‘classical’ boundedness.

**Proposition 3.6.** Let $E$ and $F$ be $L$-spaces, $f : E \to \mathbb{C}$ and $g : F \to \mathbb{C}$ bounded functionals. Then the bilinear functional $f \times g : E \times F \to \mathbb{C} : (x, y) \mapsto f(x)g(y)$ is $L$-bounded and $\|f \times g\|_{L_b} = \|f\|\|g\|$.

**Proof.** Since $\|f \times g\| = \|f\|\|g\|$, it suffices to show that $\|f \times g\|_{L_b} \leq \|f\|\|g\|$. Indeed, combining the obvious formula $(f \times g)_\infty(u, v) = f_\infty(u) \diamond g_\infty(v)$ with Proposition 3.2, we see that

$$\|(f \times g)_\infty(u, v)\| \leq \|f\|\|g\|\|u\|\|v\|.$$

□

In the following proposition $E$ is a normed space, $F$ is an $L$-space. We denote the $L$-quantization (considered in Example 2.6) of $E \otimes_{pr} F$ again by $E \otimes_{pr} F$. This leads to no ambiguity.

**Proposition 3.7.** The canonical bilinear operator $\vartheta : E_{max} \times F \to E \otimes_{pr} F : (x, y) \mapsto x \otimes y$ between the corresponding $L$-spaces is $L$-contractive.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
LE_{max} \times LF & \xrightarrow{\vartheta_{\infty}} & LE(E \otimes_{pr} F) \\
\beta_0 \times 1_{LF} & & \downarrow \beta \\
(E \otimes_{pr} L) \times LF & \xrightarrow{S} & E \otimes_{pr} LF,
\end{array}
$$
where the bilinear operator $S$ acts by the rule $(x \otimes \xi, v) \mapsto x \otimes (\xi \diamond v)$, $\beta$ is the isometric operator in Example 2.6, and the ‘flip’ $\beta_0$ is its special case when $F := \mathbb{C}$. The diagram is obviously commutative and $S$ is contractive. As a corollary, when $w \in LE_{\text{max}}$ and $v \in LF$ we have $\|\vartheta_\infty(w, v)\| \leq \|w\|\|u\|$. □

§ 4. The general $L$-tensor product

Throughout this section we fix two arbitrarily chosen $L$-spaces $E$ and $F$. Moreover, let $\mathcal{U}$ be a subclass of the class of all normed $L$-spaces.

**Definition 4.1.** A pair $(\Theta, \theta)$, consisting of an element $\Theta \in \mathcal{U}$ and an $L$-contractive bilinear operator $\theta: E \times F \to \Theta$, is called a tensor product of $E$ and $F$ relative to $\mathcal{U}$ if, for every $G \in \mathcal{U}$ and every $L$-bounded bilinear operator $R: E \times F \to G$ there is a unique $L$-bounded operator $R: \Theta \to G$ such that the diagram

$$
\begin{array}{ccc}
E \times F & \overset{\theta}{\longrightarrow} & \Theta \\
\downarrow & & \downarrow R \\
G & \overset{R}{\longrightarrow} & G
\end{array}
$$

is commutative and, moreover, $\|R\|_{Lb} = \|R\|_{Lb}$.

Such a pair is unique in the following sense. If two pairs $(\Theta_k, \theta_k)$, $k = 1, 2$, satisfy the given definition for a certain $\mathcal{U}$, then there is an $L$-isometric isomorphism $I: \Theta_1 \to \Theta_2$ such that $I\theta_1 = \theta_2$. This is a particular case of a general categorical observation concerning the uniqueness of an initial object in a category (see, for example, [18], [19], Theorem 2.73). However, the question of the existence of such a pair depends on our luck with the choice of $\mathcal{U}$.

**Definition 4.2.** The tensor product of $E$ and $F$ relative to the class of all normed $L$-spaces is called the non-completed general $L$-tensor product of our spaces.

We shall prove the existence of such a pair by an explicit construction.

We first need an ‘extended’ version of the diamond multiplication, this time between the elements of amplifications of linear spaces. Given any $u \in LE$ and $v \in LF$, we consider the element $u \diamond v := \vartheta_\infty(u, v) \in L(E \otimes F)$, where $\vartheta: E \times F \to E \otimes F$ is the canonical bilinear operator. In other words, this ‘diamond operation’ is well defined by the equality $\xi x \diamond \eta y := (\xi \diamond \eta)(x \otimes y)$.

**Proposition 4.3.** Every $U \in L(E \otimes F)$ can be represented as

$$
U = \sum_{k=1}^{n} a_k \cdot (u_k \diamond v_k)
$$

for some positive integer $n$, $a_k \in B$, $u_k \in LE$, $v_k \in LF$, $k = 1, \ldots, n$.

**Proof.** Since every element of $L(E \otimes F)$ is a sum of elements of the form $\xi(x \otimes y)$, $\xi \in L$, $x \in E$, $y \in F$, it suffices to verify the assertion for elements of this form. Take an arbitrary vector $\eta \in L$, $\eta \neq 0$, and an arbitrary operator $a \in B$ such that $a(\eta \diamond \eta) = \xi$. Then, clearly, $\xi(x \otimes y) = a \cdot (u \diamond v)$, where $u := \eta x$, $v := \eta y$. □
As a corollary, the operator
\[ B \otimes_{pr} LE \otimes_{pr} LF \to L(E \otimes F): (a \otimes u \otimes v) \mapsto a \cdot (u \otimes v) \]
is surjective. Therefore the space \( L(E \otimes F) \) can be endowed with the seminorm of the corresponding quotient space. We denote this seminorm by \( \| \cdot \|_L \). In other words,
\[
\|U\|_L := \inf \left\{ \sum_{k=1}^n \|a_k\| \|u_k\| \|v_k\| \right\}, \tag{4.2}
\]
where the infimum is taken over all possible representations of \( U \) in the form (4.1).

Being a quotient module of the module \( B \otimes_{pr} [LE \otimes_{pr} LF] \), which is certainly contractive, the seminormed \( B \)-module \( (L(E \otimes F), \| \cdot \|_L) \) is itself contractive. Therefore \( \| \cdot \|_L \) is an \( L \)-seminorm on \( E \otimes F \). We denote the resulting \( L \)-space by \( E \otimes_L F \).

Observe the obvious estimate
\[
\|u \otimes v\|_L \leq \|u\|_L \|v\|_L, \quad u \in LE, \quad v \in LF. \tag{4.3}
\]

Since \( u \otimes v = \vartheta_{\infty}(u, v) \), we see that \( \vartheta \) with values in \( E \otimes_L F \) is \( L \)-contractive.

Since \( \|x \otimes y\| = \| (\xi \otimes \xi) x \otimes y \| \) for all \( \xi \in L \) with \( \|\xi\| = 1 \), we obtain the following estimate in the underlying space of \( E \otimes_L F \):
\[
\|x \otimes y\| \leq \|x\| \|y\|, \quad x \in E, \quad y \in F. \tag{4.4}
\]

**Proposition 4.4.** Let \( G \) be an \( L \)-space, \( \mathcal{R}: E \times F \to G \) an \( L \)-bounded bilinear operator, and \( R: E \otimes_L F \to G \) the associated linear operator. Then \( R \) is \( L \)-bounded and \( \| \mathcal{R} \|_{L^b} = \| R \|_{L^b} \).

**Proof.** Take any \( U \in L(E \otimes_L F) \) and represent it as in (4.1). Since \( R_{\infty} \) is a \( B \)-module morphism, we see from the obvious equality \( R_{\infty}(u \otimes v) = \mathcal{R}_{\infty}(u, v) \) that
\[
R_{\infty}(U) = \sum_{k=1}^n a_k \cdot \mathcal{R}_{\infty}(u_k, v_k).
\]
Therefore,
\[
\|R_{\infty}(U)\| \leq \sum_{k=1}^n \|a_k\| \| \mathcal{R}_{\infty}(u_k, v_k)\| \leq \| \mathcal{R} \|_{L^b} \sum_{k=1}^n \|a_k\| \|u_k\| \|v_k\|.
\]
Combining this with (4.2), we obtain that \( \| R \|_{L^b} \leq \| \mathcal{R} \|_{L^b} \). The reverse inequality follows from (4.3). \( \square \)

**Proposition 4.5.** (As a matter of fact), \( \| \cdot \|_L \) is a norm.

**Proof.** By Proposition 2.4 it suffices to show that \( \|\xi w\|_L \neq 0 \) for every non-zero elementary tensor \( \xi w \), where \( w \in E \otimes_L F \). Since \( E \) and \( F \) are normed spaces, there are bounded functionals \( f: E \to \mathbb{C}, \ g: F \to \mathbb{C} \) such that \( (f \otimes g)w \neq 0 \). In Proposition 4.4, we put
\[
\mathcal{R} := f \times g: E \times F \to \mathbb{C}.
\]
Since \( \mathcal{R} \) is \( L \)-bounded (by Proposition 3.6), the operator \( (f \otimes g)_\infty: L(E \otimes_L F) \to L \) is bounded. At the same time, \( (f \otimes g)_\infty(\xi w) = [(f \otimes g)(w)] \xi \neq 0 \). \( \square \)
Combining the facts accumulated, we immediately obtain the desired existence theorem.

**Theorem 4.6.** The pair \((E \otimes_L F, \vartheta)\) is a non-completed general \(L\)-tensor product of \(E\) and \(F\).

For some concrete tensor factors, the tensor product introduced also becomes more concrete.

**Theorem 4.7.** Let \(E\) and \(F\) be the spaces in Example 2.6. Suppose that \(L := L_p(X)\), where \(X\) satisfies the conditions in Example 3.3 and the \(\diamond\)-operation is taken from the same example. Then there is an \(L\)-isometric isomorphism \(I: E_{\text{max}} \otimes_L F \rightarrow E \otimes_{\text{pr}} F\) acting as the identity operator on the common underlying linear space of our \(L\)-spaces.

**Proof.** Consider the operator \(I\) associated with \(\vartheta\) (see Proposition 3.7). It acts as indicated in the statement and it is \(L\)-contractive by Theorem 4.6. We need only show that \(I_\infty\) does not decrease the norms of vectors.

Take any \(U \in L(E \otimes F)\). Identifying \(L(E \otimes F)\) with \(E \otimes L F\), we can represent \(U\) as

\[
\sum_{k=1}^n x_k \otimes v_k, \quad x_k \in E, \quad v_k \in L F.
\]

Let \(q\) be conjugate to \(p\). If \(p < \infty\), then we choose an arbitrary \(e \in L\) with \(\|e\| = 1\), write \(e^* \in L_q(X)\) for a function of norm 1 such that

\[
\int_X e(s)e^*(s)\, ds = 1
\]

and consider the operator \(j: L_p(X \times X) \rightarrow L_p(X)\) taking every function \(f(s, t)\) to

\[
g(t) := \int_X f(s, t)e^*(s)\, ds.
\]

But if \(p = \infty\), then we put \(e(s) \equiv 1\) and consider the operator \(j: L_p(X \times X) \rightarrow L_p(X)\) taking every function \(f(s, t)\) to \(g(t) := \text{ess sup}|f^t|\), where \(f^t(s) := f(s, t)\). In both cases we put \(T := ji^{-1} \in B\) and see that \(\|T\| = 1\). Moreover, representing each \(v_k\) as a sum of elementary tensors in \(L F\), we easily obtain that

\[
U = T \cdot \left[ \sum_{k=1}^n e x_k \otimes v_k \right].
\]

By (4.2), this yields the bound \(\|U\|_L \leq \sum_{k=1}^n \|x_k\|\|v_k\|\) and, as a corollary,

\[
\|U\|_L \leq \inf \left\{ \sum_{k=1}^n \|x_k\|\|v_k\| \right\},
\]

where the infimum is taken over all representations of \(U\) in the form indicated.

Now look at \(I_\infty(U)\). This is the same sum \(\sum_{k=1}^n x_k \otimes v_k\) regarded as an element of the normed space \(E \otimes_{\text{pr}} L F\). Hence \(\|I_\infty(U)\|\) is exactly the infimum in (4.5). This yields the desired estimate \(\|I_\infty(U)\| \geq \|U\|_L\). □
Remark 4.8. One can easily show that $E_{\text{max}} \otimes_L F_{\text{max}} = [E \otimes_{pr} F]_{\text{max}}$ up to an $L$-isometric isomorphism for every base space $L$ and all normed spaces $E$ and $F$. In particular, for a Hilbert space $H$ we have $H_{\text{max}} \otimes_L H_{\text{max}} = \mathcal{N}_0(H)_{\text{max}}$, where $\mathcal{N}_0(H)$ is the space of finite-rank operators on $H$, equipped with the trace class norm.

§ 5. $p$-convexity and the $p$-convex tensor product

To introduce the next idea, we need one more structure on the base space apart from the $\triangle$-operation. Namely, we say that $L$ is a stratified space if $L$ is endowed with a family $P$ of projections of norm 1 (or 0) such that $P, Q \in P$ implies $PQ = QP \in P$, and if $P, Q \in P$ are orthogonal, then $P + Q \in P$. Projections in $P$ are said to be proper.

Example 5.1 (principal). Let $L := L_p(X)$. Given a measurable subset $X'$ of $X$, we write $P_{X'} \in B(L_p(X))$ for the projection acting by the rule $f \mapsto f\chi$, where $\chi$ is the characteristic function of $X'$. Of course, if the measure of $X'$ is positive, then $\|P_{X'}\| = 1$. We shall identify the image of this projection with $L_p(X')$. Clearly, such projections are orthogonal if and only if the intersection of the corresponding sets has measure zero. It is also obvious that the family of projections of the form indicated satisfies the conditions stated above. When speaking of $L_p(X)$ as a stratified space, we shall always mean this particular family.

In what follows, given any $n$-tuple of numbers $\lambda_k \geq 0$, $k = 1, \ldots, n$, we understand the expression $(\sum_{k=1}^n \lambda_k^p)^{1/p}$ with $p = \infty$ as $\max\{\lambda_1, \ldots, \lambda_n\}$.

We shall need the following obvious assertion for future reference.

Proposition 5.2. Suppose that $X$ and $Y$ are measure spaces, $S_k \in B(L_p(X), L_p(Y))$, $k = 1, \ldots, n$, and $X'_k$, $Y'_k$ are two families of pairwise-disjoint measurable subsets of $X$, $Y$ respectively. Then

$$\left\| \sum_{k=1}^n P_{Y'_k}S_kP_{X'_k} \right\| \leq \max\{\|S_k\|, k = 1, \ldots, n\}.$$

Let $u$ be an element of a seminormed $L$-space. A projection $P \in B$ is called a support of $u$ if $P \cdot u = u$.

Definition 5.3. Let $L$ be a stratified space. For $1 \leq p \leq \infty$, a seminormed $L$-space $E$ is said to be $p$-convex if, for every $u, v \in LE$ with orthogonal proper supports, we have

$$\|u + v\| \leq (\|u\|^p + \|v\|^p)^{1/p}.$$

As an immediate corollary, the following inequality holds for all $u_1, \ldots, u_n \in LE$ with pairwise-orthogonal proper supports in $P$:

$$\left\| \sum_{k=1}^n u_k \right\| \leq \left(\sum_{k=1}^n \|u_k\|^p\right)^{1/p}.$$
When $L := \ell_p$, this definition is equivalent to that of a $p$-convex $p$-multi-normed space in [11]. For completeness we also mention the theory of $p$-operator spaces of Daws [20]; see also the earlier papers by Pisier [21] and Le Merdi [22].

The base space $L$ is said to be $p$-convex if it becomes $p$-convex after identification with $LC$. Needless to say, $L^p(X)$ is $p$-convex as a base space.

Clearly, for every stratified space $L$, all seminormed $L$-spaces are $1$-convex. It is also clear that every $p$-convex space is $r$-convex for all $r$ with $1 \leq r < p$.

**Proposition 5.4.** When $L$ is $p$-convex (in particular, when $L := L^p(X)$), every $L$-space $E$ with minimal quantization is $p$-convex.

**Proof.** If $u, v$ have orthogonal supports in $LE$, then for every $f \in E^*$ the elements $f_\infty(u), f_\infty(v)$ have orthogonal supports in $L$. Therefore,

$$\|f_\infty(u + v)\|^p \leq \|f_\infty(u)\|^p + \|f_\infty(v)\|^p.$$  

It remains to take the supremum on the right-hand side over all $f \in E^*$ with $\|f\| = 1$, and then do the same on the left-hand side. □

At the same time, the maximal quantization of a normed space $E$ need not be $p$-convex when $p > 1$ and $L = L^p(X)$. The simplest counterexample is $E := \ell_1$.

One can construct other $p$-convex spaces from given $p$-convex spaces. For example, it is easy to show that if $E_k$ ($k = 1, 2$) are $q_k$-convex $L$-spaces and $p \leq \min\{q_1, q_2\}$, then the $L$-space $E_1 \oplus E_2$ endowed with the $L$-norm of the $\ell_p$-sum of the normed spaces $LE_1$ and $LE_2$ is $p$-convex.

We now suppose that the base space $L$ is stratified and has a $\diamondsuit$-operation. Fix $L$-spaces $E$ and $F$.

**Definition 5.5.** The tensor product of $E$ and $F$ relative to the class of all $p$-convex $L$-spaces is called the non-completed $p$-convex tensor product of $E$ and $F$.

Unfortunately, at the moment we can prove the existence of such a tensor product only under rather restrictive additional conditions on $\diamondsuit$ and especially on $\mathcal{P}$, which make our triple $(L, \mathcal{P}, \diamondsuit)$ too close to the triples arising in the case of base spaces $L^p(X)$. Therefore we have decided not to bother the reader with the list of these conditions. Instead, we assume for the rest of the paper that the base is $L^p(X)$ with a measure space $X$ having either no atoms or an infinite set of atoms.

We call a measure space of this sort convenient. The family of proper projections on $L^p(X)$ and the $\diamondsuit$-operation are defined as in Examples 5.1 and 3.3 respectively. Note that the case of a single atom (permitted in Example 3.3) is now forbidden; otherwise Proposition 5.6 below fails to be true.

An isometric operator on $L^p(X)$ is said to be proper if its image coincides with that of some proper projection or, equivalently, with $L^p(X')$ for some measurable $X' \subset X$. Two proper isometries are said to be disjoint if the intersection of their images consists only of zero or, equivalently, if the corresponding projections are orthogonal. Since $X$ is convenient, it contains an infinite family of pairwise-disjoint measurable subsets of the same type as $X$ (see Example 3.3). Here is an immediate corollary.
Proposition 5.6. There is an infinite family of pairwise-disjoint proper isometries acting on $L_p(X)$.

Given a proper isometry $I \in \mathcal{B}$ with image $L_p(X')$, we write $I^* \in \mathcal{B}$ for the operator $I^{-1}P_{X'}$, that is, the co-isometric operator (= quotient map) acting as $I^{-1}$ on $L_p(X')$ and vanishing on the complementary subspace $L_p(X \setminus X')$. Clearly, if $I_k$, $k = 1, \ldots, n$, are pairwise-disjoint proper isometries, then

$$I_k^* I_l = \delta_{kl} 1_L.$$ \hspace{1cm} (5.1)

We proceed to the explicit construction of the $p$-convex tensor product.

Proposition 5.7. Let $E$, $F$ be linear spaces. Then every element $U \in L(E \otimes F)$ can be represented as

$$a \cdot \sum_{k=1}^{n} I_k \cdot (u_k \diamond v_k),$$ \hspace{1cm} (5.2)

where $a \in \mathcal{B}$ and $I_k$ are pairwise-disjoint proper isometries.

Proof. Write $U$ in the form (4.1). By Proposition 5.6 there are $n$ proper pairwise-disjoint isometries $I_k$. Consider the following element of $L(E \otimes F)$:

$$\left(\sum_{k=1}^{n} a_k I_k^*\right) \cdot \left(\sum_{l=1}^{n} I_l \cdot (u_l \diamond v_l)\right).$$

By (5.1), it is equal to $U$. \Box

From now on we assume that we are given two arbitrary (not necessarily $p$-convex!) $L$-spaces $E$ and $F$. To every $U \in L(E \otimes F)$ we assign the number

$$\|U\|_{pL} := \inf \left\{ \|a\| \left(\sum_{k=1}^{n} \|u_k\|^p \|v_k\|^p\right)^{1/p} \right\},$$ \hspace{1cm} (5.3)

where the infimum is taken over all possible representations of $U$ in the form (5.2).

We mention an obvious assertion.

Proposition 5.8. For all $U \in L(E \otimes F)$ and $a \in \mathcal{B}$ we have $\|a \cdot U\|_{pL} \leq \|a\| \|U\|_{pL}$.

The following proposition is less obvious.

Proposition 5.9. The function $U \mapsto \|U\|_{pL}$ is a seminorm on $L(E \otimes F)$.

Proof. Suppose that

$$U = a \cdot \sum_{k=1}^{n} I_k' \cdot (u_k' \diamond v_k'), \quad V = b \cdot \sum_{l=1}^{m} I_l'' \cdot (u_l'' \diamond v_l''),$$

where the $I_k'$ are proper pairwise-disjoint isometries and the same is true for the $I_l''$.

We make an arbitrary choice of another pair $I_U, I_V \in \mathcal{B}$ of disjoint isometries and observe that

$$U + V = (aI_U^* + bI_V^*) \cdot \left(\sum_{k=1}^{n} I_k I_k' \cdot (u_k' \diamond v_k') + \sum_{l=1}^{m} I_l I_l'' \cdot (u_l'' \diamond v_l'')\right).$$
Clearly, the composites $I_U I'_k$ and $I_V I''_l$ are proper isometries and, taken all together, are pairwise disjoint. Therefore, by (5.3) and Proposition 5.8,

$$
\|U + V\|_{pL} \leq \|aI^*_U + bI^*_V\| \left( \sum_{k=1}^{n} \|u^1_k\|^p v^1_k|^p + \sum_{l=1}^{m} \|u^2_l\|^p v^2_l|^p \right)^{1/p}.
$$

(5.4)

Our next aim is to obtain the estimate $\|aI^*_U + bI^*_V\| \leq (\|a\|^q + \|b\|^q)^{1/q}$, where $q$ is conjugate to $p$. (As usual, we assume that 1 and $\infty$ are conjugate to each other.)

Take any $\xi \in L$ with $\|\xi\| \leq 1$ and denote the proper projections corresponding to our isometries by $P_U = I_U I^*_U$ and $P_V = I_V I^*_V$. Using Hölder’s inequality, we obtain that

$$
\|(aI^*_U + bI^*_V)(\xi)\| \leq \|(aI^*_U P_U)(\xi)\| + \|(bI^*_V P_V)(\xi)\| \leq \|a\| \|P_U(\xi)\| + \|b\| \|P_V(\xi)\|
$$

\leq (\|a\|^q + \|b\|^q)^{1/q}(\|P_U(\xi)\|^p + \|P_V(\xi)\|^p)^{1/p}.

Since our projections are orthogonal, the second factor is equal to $(\|P_U(\xi) + P_V(\xi)\|^p)^{1/p}$ and, therefore, does not exceed $\|P_U + P_V\| = 1$. This proves the desired estimate.

Clearly, we can obtain representations of $U$ by multiplying $a$ by a constant and dividing all the $u^1_k$ by that constant. This can also be done for $V$. Hence we can assume that

$$
\|a\|^q = \sum_{k=1}^{n} \|u^1_k\|^p v^1_k|^p \quad \text{and} \quad \|b\|^q = \sum_{l=1}^{m} \|u^2_l\|^p v^2_l|^p
$$

when $1 < p < \infty$. We can also assume that $\|a\| = \|b\| = 1$ when $p = 1$, and $\max\{\|u^1_k\|, k = 1, \ldots, n\} = \max\{\|u^2_l\|, l = 1, \ldots, m\}$ when $p = \infty$.

In the first case,

$$
\|U + V\|_{pL} \leq (\|a\|^q + \|b\|^q)^{1/q}(\|a\|^q + \|b\|^q)^{1/p} = \|a\|^q + \|b\|^q.
$$

Since $q - 1 = q/p$, we have $\|a\|^q = \|a\|((\sum_{k=1}^{n} \|u^1_k\|^p v^1_k|^p)^{1/p}$. Together with the corresponding inequality for $\|b\|^q$, this yields that

$$
\|U + V\|_{pL} \leq \|a\| \left( \sum_{k=1}^{n} \|u^1_k\|^p v^1_k|^p \right)^{1/p} + \|b\| \left( \sum_{l=1}^{m} \|u^2_l\|^p v^2_l|^p \right)^{1/p}.
$$

(5.5)

We easily verify that the same estimate holds in the remaining cases.

Thus, in all cases, taking the infimum as in (5.3), we obtain the triangle inequality $\|U + V\|_{pL} \leq \|U\|_{pL} + \|V\|_{pL}$.

The property of seminorms concerning multiplication by scalars is obvious. \(\square\)

We denote the resulting seminormed $L$-space by $E \otimes_{pL} F$.

**Proposition 5.10.** This $L$-space is $p$-convex.
Proof. Suppose that $U, V \in \mathbf{L}(E \otimes F)$ have orthogonal supports $P_1$ and $P_2$. Choose arbitrary representations of $U$ and $V$ as in Proposition 5.9 and take the corresponding $I_U$ and $I_V$. The estimate (5.4) holds. Clearly, we may assume that $a = P_1 a$, $b = P_2 b$ and also $\|a\| = \|b\| = 1$. Putting $P_U := I_U I_U^*$ and $P_V := I_V I_V^*$, we thus have $aI_U^* = P_1 aI_U^* P_U$ and $bI_V^* = P_2 bI_V^* P_V$. It follows by Proposition 5.2 that

$$\|aI_U^* + bI_V^*\| \leq \max\{\|aI_U^*\|, \|bI_V^*\|\} \leq \max\{\|a\|, \|b\|\} = 1.$$  

Hence

$$\|U + V\|_{pL} \leq \left( \sum_{k=1}^{n} \|a_k\|^p \|v_k^1\|^p + \sum_{l=1}^{m} \|a_l^2\|^p \|v_l^2\|^p \right)^{1/p}.$$  

It remains to take the corresponding infima. □

As in the case of the tensor product $\otimes^L$, we have

$$\|u \otimes v\|_{pL} \leq \|u\| \|v\|, \quad u \in LE, \quad v \in LF. \quad (5.6)$$

It follows that the canonical bilinear operator $\vartheta: E \times F \rightarrow E \otimes_{pL} F$ (compare with $\vartheta: E \times F \rightarrow E \otimes^L F$ in §4) is $L$-contractive.

The argument in the proof of (4.4) gives the following estimate in the underlying seminormed space of $E \otimes_{pL} F$:

$$\|x \otimes y\| \leq \|x\| \|y\|, \quad x \in E, \quad y \in F. \quad (5.7)$$

Proposition 5.11. Let $G$ be a $p$-convex $L$-space, $\mathcal{R}: E \times F \rightarrow G$ an $L$-bounded bilinear operator, and $R: E \otimes_{pL} F \rightarrow G$ the associated linear operator. Then $R$ is $L$-bounded and $\|\mathcal{R}\|_{bL} = \|R\|_{bL}$.

Proof. Take any $U \in \mathbf{L}(E \otimes F)$ and write it in the form (5.2). Since $R_\infty$ is a $\mathcal{B}$-module morphism, we see that

$$R_\infty(U) = a \cdot \left( \sum_{k=1}^{n} I_k \cdot \mathcal{R}_\infty(u_k, v_k) \right).$$

Look at the elements $I_k \cdot \mathcal{R}_\infty(u_k, v_k) \in LG$. They have pairwise-orthogonal supports (namely, $I_k I_k^*$) and $G$ is $p$-convex. Therefore we obtain that

$$\|R_\infty(U)\| \leq \|a\| \left( \sum_{k=1}^{n} \|I_k \cdot \mathcal{R}_\infty(u_k, v_k)\|^p \right)^{1/p} \leq \|a\| \left( \sum_{k=1}^{n} \|\mathcal{R}_\infty(u_k, v_k)\|^p \right)^{1/p}$$

$$\leq \|a\| \left( \sum_{k=1}^{n} \|\mathcal{R}\|_{bL} \|u_k\|^p \|v_k\|^p \right)^{1/p} = \|\mathcal{R}\|_{bL} \|a\| \left( \sum_{k=1}^{n} \|u_k\|^p \|v_k\|^p \right)^{1/p}.$$  

Hence $\|R\|_{bL} \leq \|\mathcal{R}\|_{bL}$. The reverse inequality follows from (5.6). □

Proposition 5.12. (As a matter of fact), $\| \cdot \|_{pL}$ is a norm.

Proof. Needless to say that $\mathcal{C}$ is a $p$-convex $L$-space. Hence Proposition 5.11 applies and the proof of Proposition 4.5 can be repeated with obvious modifications. □
Combining the relevant propositions, we obtain the following existence theorem.

**Theorem 5.13.** The pair \((E \otimes_p L, \vartheta)\) is a non-completed \(p\)-convex tensor product of the \(L\)-spaces \(E\) and \(F\).

This theorem may be regarded as a far-reaching generalization of Lambert’s most important results on the maximal tensor product of his ‘Operatorfolgenräume’ (see [1], pp. 73–78). We recall that all such constructions emanate from the papers of Blecher–Paulsen [23] and Effros–Ryan [24] about projective tensor products of operator spaces.

**Remark 5.14.** We do not discuss here the ‘non-discrete’ version of another (the so-called minimal) tensor product introduced by Lambert (in the framework of the ‘coordinate’ approach) for 2-convex \(l_2\)-spaces in [1], §3.1.3.

§ 6. \(p\)-convex tensor products of the spaces \(L_q(\cdot)\)

In conclusion we describe a situation when the \(p\)-convex tensor product for \(L := L_p(X)\) with some convenient \(X\) takes an especially transparent form. It turns out that the ‘best’ tensor factors for \(1 < p < \infty\) are the spaces \(L_q(\cdot), q = p/(p - 1)\), with the minimal quantization as in Example 2.5.

Throughout this section we assume that all the \(L\)-spaces are endowed with the minimal quantization.

Let \(Y\) and \(Z\) be measure spaces. Consider a linear operator

\[
J : L_p(Y) \otimes L_p(Z) \to L_p(Y \times Z)
\]

which is well defined by the rule \(x \otimes y \mapsto x(s)y(t), s \in Y, t \in Z\). We easily see that it is injective and its image consists of degenerate functions of two variables, that is, functions of the form

\[
\sum_{k=1}^{n} f_k(s)g_k(t), \quad f_k \in L_p(Y), \quad g_k \in L_p(Z).
\]

This image is a normed subspace of \(L_p(Y \times Z)\), which is dense when \(p < \infty\). We denote it by \(L_p(Y) \otimes^p L_p(Z)\). Clearly, this space can be identified with the tensor product \(L_p(Y) \otimes L_p(Z)\) endowed with the corresponding induced norm.

**Proposition 6.1.** Let \(A : L_p(Y_1) \to L_p(Y_2)\) and \(B : L_p(Z_1) \to L_p(Z_2)\) be bounded linear operators. Then the operator

\[
A \otimes B : L_p(Y_1) \otimes^p L_p(Z_1) \to L_p(Y_2) \otimes^p L_p(Z_2)
\]

is also bounded and \(\|A \otimes B\| \leq \|A\|\|B\|\).

**Proof.** Every \(u \in L_p(Y_1) \otimes^p L_p(Z_1)\) is a function of the form

\[
\sum_{k=1}^{n} f_k(s)g_k(t), \quad f_k \in L_p(Y_1), \quad g_k \in L_p(Z_1).
\]
If \( p < \infty \), then, by Fubini’s theorem,
\[
\|A \otimes B(u)\| = \left( \int_{Z_2} \left[ \int_{Y_2} A \left( \sum_{k=1}^{n} ((Bg_k(t)f_k)(s)) \right)^p ds \right] dt \right)^{1/p}
\]
\[
\leq \|A\| \left( \int_{Z_2} \left[ \int_{Y_2} \sum_{k=1}^{n} (Bg_k(t)f_k)_{L_p(Y_2)}^p dt \right] \right)^{1/p}
\]
\[
= \|A\| \left( \int_{Y_2} \left[ \int_{Z_2} \sum_{k=1}^{n} (Bg_k(s)f_k)_{L_p(Z_2)}^p ds \right] dt \right)^{1/p}
\]
\[
= \|A\| \|B\| \left( \int_{Y_2} \left[ \sum_{k=1}^{n} (f_k(s)g_k)_{L_p(Z_2)}^p \right] ds \right)^{1/p}
\]
\[
= \|A\| \|B\| \left( \int_{Y_2} \left[ \sum_{k=1}^{n} (f_k(s)g_k)_{L_p(Z_2)}^p d(s, t) \right] \right)^{1/p}
\]
\[
= \|A\| \|B\| \left( \int_{Y_2 \times Z_2} \left[ \sum_{k=1}^{n} (f_k(s)g_k(t))_{L_p(Z_2)}^p \right] d(s, t) \right)^{1/p}
\]
\[
= \|A\| \|B\| \left( \int_{\mathbb{R}^n} \left[ \sum_{k=1}^{n} (f_k(s)g_k(t))_{L_p(Z_2)}^p \right] d(s, t) \right)^{1/p}
\]
\[
= \|A\| \|B\| \left( \int_{\mathbb{R}^n} \left[ \sum_{k=1}^{n} (f_k(s)g_k(t))_{L_p(Z_2)}^p \right] \right)^{1/p}
\]
When \( p = \infty \), the desired fact is proved in a similar way, but replacing Fubini’s theorem for the functions \( h \in L_\infty(Y, Z) \), \( h_s \in L_\infty(Z) : h_s(t) := h(s, t) \) and \( h^t \in L_\infty(Y) : h^t(s) := h(s, t) \) by the equality
\[
\text{ess sup} |h| = \text{ess sup} \left[ \text{ess sup} |h_s| \right] = \text{ess sup} \left[ \text{ess sup} |h^t| \right].
\]

We recall that the norm on the injective tensor product \( E \otimes_{\text{in}} F \) of two normed spaces can be expressed in terms of an isometric operator from \( E \otimes_{\text{in}} F \) to \( B(E', F) \), where \( E' \) is an arbitrary subspace of \( E^* \) such that \( \|x\| = \sup \{ \|f(x)\|, f \in E', \|f\| = 1 \} \) for all \( x \in E \) (for example, if \( E' \) is a dense subspace of \( E^* \), or a predual of \( E \) if such a predual exists); see, for example, [25], pp. 62, 63 or [26], § 4, [27], pp. 45, 46 (and also, of course, [28]). In the particular case of the spaces \( L_p(\cdot), \) the relevant assertion takes the following form. Write \( \langle f, g \rangle \) for the classical duality \( (f, g) \mapsto \int_Y f(t)g(t) dt \) between \( L_p(Y) \) and \( L_q(Y) \), and write \( \langle \langle u, v \rangle \rangle \) for the duality between \( L_p(Y) \otimes^p L_q(Z) \) and \( L_q(Y) \otimes^q L_p(Z) \) which is well defined on elementary tensors by the equality \( \langle \langle y_1 \otimes z_1, y_2 \otimes z_2 \rangle \rangle = \langle y_1, y_2 \rangle \langle z_1, z_2 \rangle \). In what follows, \( Y, Z, Y_1, \ldots \) are arbitrary measure spaces.

**Proposition 6.2.** i) **There is an isometric operator**
\[
\mathcal{I} : L_p(Y) \otimes_{\text{in}} L_q(Z) \to B(L_p(Z), L_p(Y))
\]
(in particular, \( \mathcal{I} : LL_q(Z) \to B(L_p(Z), L_p) \)) well defined by saying that it takes \( y \otimes z \) to the operator acting by the rule \( z' \mapsto \langle z', z \rangle y \).

ii) **When \( 1 < p < \infty \) (or, equivalently, \( 1 < q < \infty \)), there is an isometric operator**
\[
\mathcal{J} : [L_p(Y_1) \otimes^p L_p(Z_1)] \otimes_{\text{in}} [L_q(Y_2) \otimes^q L_q(Z_2)] \to B(L_p(Y_2) \otimes^p L_p(Z_2), L_p(Y_1) \otimes^p L_p(Z_1))
\]
well defined by saying that it takes \( u \otimes v \) to the operator acting by the rule \( v' \mapsto \langle \langle v', v \rangle \rangle u \).
Proof. Part i) follows since $L_p(Z)$ is the dual (or the predual if $p=1$) of $L_q(Z)$.

Part ii) follows since $(L_q(Y_2) \otimes^q L_q(Z_2))^*$ is equal to $L_p(Y_2 \times Z_2)$ when $1 < p < \infty$, and the latter is dense in $L_p(Y_2) \otimes^p L_p(Z_2)$. □

**Proposition 6.3.** Let $Y$ and $Z$ be measure spaces. Then the bilinear operator

$(R): L_q(Y) \times L_q(Z) \to L_q(Y) \otimes^q L_q(Z)$

sending the pair $(y, z)$ to $y \otimes z = y(s)z(t)$, $s \in Y$, $t \in Z$, is $\mathbf{L}$-contractive.

*Proof.* Consider the bilinear operator

$$S: LL_q(Y) \times LL_q(Z) = (L_p(X) \otimes_{in} L_q(Y)) \times (L_p(X) \otimes_{in} L_q(Z))$$

$$\to [L_p(X) \otimes^p L_p(X)] \otimes_{in} [L_q(Y) \otimes^q L_q(Z)]$$

which is well defined on elementary tensors by the rule

$$(\xi \otimes y, \eta \otimes z) \mapsto (\xi \otimes \eta) \otimes (y \otimes z).$$

Clearly, $R_\infty = (i_0 \otimes \mathbf{1})S$, where $i_0: [L_p(X) \otimes^p L_p(X)] \to L_p(X)$ is the restriction of the isometric isomorphism $i$ introduced in Example 3.3, and $\mathbf{1}$ is the identity operator on $L_q(Y) \otimes^q L_q(Z)$. Since we are dealing with the injective tensor product, $i_0 \otimes \mathbf{1}$ is an isometry along with $i_0$ and $\mathbf{1}$ (‘the injective property’; see, for example, [26], § 4). Therefore it suffices to prove that $S$ is contractive.

Consider the diagram

$$\begin{array}{c}
(L_p(X) \otimes_{in} L_q(Y)) \times (L_p(X) \otimes_{in} L_q(Z))
\quad \xrightarrow{S} \quad [L_p(X) \otimes^p L_p(X)] \otimes_{in} [L_q(Y) \otimes^q L_q(Z)]
\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad pandas...
Of course, every space \( L_q(\cdot) \), \( q < \infty \), contains the dense subspace \( L_q^0(\cdot) \) formed by the linear combinations of the characteristic functions of subsets of finite measure (we can assume that these subsets are disjoint). Therefore, in view of (5.7), it suffices to show that \( R_\infty \) does not decrease the norms of (finite) sums of elementary tensors of the form \( \xi(y \otimes z) \), where \( \xi \in L_p(X) \), \( y \in L_q^0(Y) \), \( z \in L_q^0(Z) \).

It is easy to see that every such sum can be represented in the form \( \sum_{k,l} \xi_{k,l} y_k \otimes z_l \), where \( y_k \) and \( z_l \) are functions of norm 1 that are proportional to the characteristic functions of pairwise-disjoint subsets \( Y_k \subseteq Y \) and \( Z_l \subseteq Z \) respectively.

We look at \( R_\infty(U) \). Of course, it is a certain sum \( \sum_{k,l} \xi_{k,l} e_{k,l} \), where the \( e_{k,l} \in L_q(Y \times Z) \) are functions of norm 1 that are proportional to the characteristic functions of the subsets \( Y_k \times Z_l \subseteq Y \times Z \). In other words,

\[
e_{k,l} = y_k \otimes z_l \in L_q(Y) \otimes^q L_q(Z) \subseteq L_q(Y \times Z).
\]

Therefore, by Proposition 6.2(i), \( \|R_\infty(U)\| \) is the norm of the operator

\[
S : L_p(Y \times Z) \to L_p(X)
\]

taking \( h(s,t) \) to \( \sum_{k,l} \langle h, e_{k,l} \rangle \xi_{k,l} \).

We now return to our original \( U \). Clearly, there are pairwise-disjoint subsets \( X^1_k \), \( k = 1, \ldots, n \), and \( X^2_l \), \( l = 1, \ldots, m \), of finite measure in \( X \).

We write \( \eta_k \) for the function of norm 1 in \( L = L_p(X) \) which is proportional to the characteristic function of \( X^1_k \) and put

\[
u := \sum_{k=1}^n \eta_k y_k \in LL_q(Y).
\]

By Proposition 6.2, \( \|u\| \) is the norm of the operator

\[
S_u : L_p(Y) \to L_p(X)
\]

taking \( g \) to \( \sum_{k=1}^n \langle g, y_k \rangle \eta_k \). Since we obviously have

\[
S_u = \sum_{k=1}^n P_{X^1_k} S_{u,k} P_{X^1_k},
\]

where \( S_{u,k} : g \mapsto \langle g, y_k \rangle \eta_k \), we obtain that \( \|S_u\| \leq \max\{\|S_{u,k}\| ; k = 1, \ldots, n\} \) (see Proposition 5.2). But we have \( \|S_{u,k}(g)\| \leq \|g\| \|y_k\| \|\eta_k\| \), whence \( \|S_{u,k}\| \leq 1 \) and, therefore, \( \|S_u\| \leq 1 \). Thus we have \( \|u\| \leq 1 \). Using the same argument, we set \( v := \sum_{l=1}^m \zeta_l z_l \in LL_q(Z) \) for similarly chosen \( \zeta_l \) and establish that \( \|v\| \leq 1 \). As a matter of fact, we have \( \|u\| = \|v\| = 1 \), but we do not need this here.

Clearly, \( u \odot v = \sum_{k,l} (\eta_k \odot \zeta_l) y_k \otimes z_l \). Therefore it follows from the definition of the \( L \)-norm on \( L_q(Y) \otimes_p L_q(Z) \) (see (5.3)) that the theorem will be proved if we can find an operator \( T \in B = B(L_p(X)) \) such that \( T(\eta_k \odot \zeta_l) = \xi_{k,l} \) (whence \( U = T \cdot (u \odot v) \)) and \( \|T\| \leq \|S\| \).

Choose a contractive operator \( i_{k,l} : L_p(X^1_k \times X^2_l) \to L_p(Y^1_k \times Z_l) \) of norm 1 that sends the constant function \( \eta_k(s) \zeta_l(t) \), \( s \in X^1_k \), \( t \in X^2_l \), to some constant
function $e_{k,t}^*$ of the same norm. For example, one can choose $i_{k,l} := gf$, where $f$ is a functional of norm 1 sending the first of the constants mentioned to 1 and $g$ is the operator taking 1 to $e_{k,t}^*$. After this, identifying each $L_p(Y_k \times Z_l)$ and $L_p(X_1^k \times X_2^l)$ with the corresponding subspaces of $L_p(Y \times Z)$ and $L_p(X \times X)$ respectively, we introduce the operator

$$D := S \left( \sum_{k,l} Q_{k,l} i_{k,l} P_{k,l} \right) : L_p(X \times X) \to L_p(X),$$

where $P_{k,l}$ is the natural projection of $L_p(X \times X)$ onto $L_p(X_1^k \times X_2^l)$, and $Q_{k,l}$ is the natural projection of $L_p(Y \times Z)$ onto $L_p(Y_k \times Z_l)$. We see that the operators $\sum_{k,l} Q_{k,l} i_{k,l} P_{k,l}$ and $i_{k,l}$ take $\eta_k(s)\zeta_l(t)$ to the same constant $e_{k,t}^*$, and the obvious equality $\langle e_{k,t}^*, e_{k,l} \rangle = 1$ implies that $S(e_{k,l}^*) = \xi_{k,l}$. Therefore $D$ takes $\eta_k(s)\zeta_l(t)$ to $\xi_{k,l}$.

Furthermore, we obtain by Proposition 5.2 that

$$\left\| \sum_{k,l} Q_{k,l} i_{k,l} P_{k,l} \right\| \leq \max \{\|Q_{k,l} i_{k,l} P_{k,l}\|\} = 1.$$

It follows that $\|D\| \leq \|S\|$. Finally, we recall the isometric isomorphism $i: L_p(X \times X) \to L_p(X)$ (see Example 3.3). In particular, this map takes every function of the form $\eta(s)\zeta(t)$ (identified with $\eta \otimes \zeta \in L_p(X) \otimes^p L_p(X)$) to $\eta \Join \zeta$. We see from this that the operator $T := Di^{-1}: L_p(X) \to L_p(X)$ is exactly what we need. □

Remark 6.5. Nowhere in the paper have we assumed that our normed spaces are complete. However, as a rule, the principal notions and facts have ‘complete’ versions. An $L$-space is said to be complete (or Banach) if its underlying normed space is complete. As in the ‘classical’ context, every $L$-space has a completion. Its definition and properties (as well as the existence theorem) can be obtained by repeating (with obvious modifications) what was said in Ch. 4 of [7] in the case of operator spaces. Moreover, both tensor products introduced above have their ‘Banach’ versions. One need only take the class of complete $L$-spaces (resp. complete $p$-convex $L$-spaces) for $\Omega$ in Definition 4.2 (resp. Definition 5.5). Then, in particular, we obtain the following version of Theorem 6.4.

If $L := L_p(X)$ and $1 < p < \infty$, then the completed $p$-convex tensor product of the $L$-spaces $L_q(Y)$ and $L_q(Z)$ is $L_q(Y \times Z)$.

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