Modeling the Aerodynamic Lift Produced by Oscillating Airfoils at Low Reynolds Number

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Abstract

For present study, setting Strouhal Number (St) as control parameter, numerical simulations for flow past oscillating NACA-0012 airfoil at $10^3$ Reynolds Numbers (Re) are performed. Temporal profiles of unsteady forces; lift and thrust, and their spectral analysis clearly indicate the solution to be a period-1 attractor for low Strouhal numbers. This study reveals that aerodynamic forces produced by plunging airfoil are independent of initial kinematic conditions of airfoil that proves the existence of limit cycle. Frequencies present in the oscillating lift force are composed of fundamental ($f_s$), even and odd harmonics ($3f_s$) at higher Strouhal numbers. Using numerical simulations, shedding frequencies ($f_s$) were observed to be nearly equal to the excitation frequencies in all the cases. Unsteady lift force generated due to the plunging airfoil is modeled by modified van der Pol oscillator. Using method of multiple scales and spectral analysis of steady-state CFD solutions, frequencies and damping terms in the van der Pol oscillator model are estimated. We prove the applicability of this model to all planar motions of airfoil; plunging, pitching and flapping. An important aspect of currently-proposed model is capturing the time-averaged value of aerodynamic lift coefficient.

Keywords

Reduced-Order Modeling, Nonlinear Dynamics, Low Reynolds Number Aerodynamics, Oscillating Airfoils, Limit Cycle

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1 Introduction

To propose efficient and better designs for small swimming and flying unmanned vehicles, understanding of the unsteady mechanisms to generate lift and thrust forces at very low Reynolds numbers (Re) is of key importance. Being a highly nonlinear system, fluid flowing over these vehicles carries great complexities. Since Knoller [10] and Betz [4] revealed the mechanisms for production of thrust due to vortex pattern behind oscillating and pitching airfoils, respectively, this field has attracted the attention of many researchers around the globe. Recently, due to the advent of biomimicking flying (micro air vehicles) and swimming robots (underwater vehicles), many research efforts in this direction are being presented. Ho et al. [7], Triantafyllou et al. [27], Wang [30], Lehmann [11] and Shyy et al. [20] have provided detailed reviews regarding progress in the field of unsteady aerodynamics for flapping flight of insects, birds and robots. Study of mechanism for generation of unsteady forces is still to be understood completely due to a wide spectrum of parameters that are involved. Both the experimental and currently available numerical techniques require costly resources involving a large amount in terms of time and money. Considering this fact, researchers have also focussed towards development of the reduced order models. These models are based upon the reduced number of states for a dynamical system. These models can be built by either phenomenological or direct model-reduction approaches. For phenomenological modeling technique, the behavior of a parameter from the response of a physical system is modeled by a set of ordinary differential equations. Self-excited oscillators are great examples of this type covering the range from electric circuits to the aero-elastic phenomena. On the contrary, a direct approach requires computation of coherent structures in the flow to develop a reduced-order model [22]. Proper-orthogonal decomposition (POD) based reduced-order models are classical examples in this category. It forms the reduced-order basis functions from the flow snapshots capturing optimal energy of a dynamical system. The governing equations, such as the Navier-Stokes equations, are projected onto these reduced basis to form a reduced-order model thereby reducing the system from millions of degrees of freedom to order of ten. These models have been successfully employed for flow control, design, optimization, and uncertainty quantification [1]. In the phenomena-based approach for the development of reduced order models, work by Skop et al. [23] is of vital importance. They introduced the Skop-Griffin parameter in order to propose a modified van der Pol oscillator model to predict the lift of a cylinder coupled with a linear model for mathematical representation of structure’s motion. This model is a center of focus in the study of vortex-induced vibrations. In Ref. [24], they extended their model from rigid to elastic cylinders. Identifying the cubic nonlinearity in lift force and quadratic relation between lift and drag forces for a circular cylinder, Nayfeh et al. [16,18] used method of multiple scales [14,15,17] and proposed the first-order accurate reduced order self-excited oscillator models for steady and transient parts of the aerodynamic forces. Following the higher order spectral analysis technique [5,6] for the identification of nonlinear parameters, Qin [19] developed forced van der Pol oscillator models for the rotational, inline and transversal oscillations of circular cylinders considering primary resonance with soft and hard excitation cases at Re = 10^4. Janajreh et al. [8] extended this study for rotational oscillations of circular cylinders at Re = 10^2. Marzouk et al. [12] presented a second order accurate model for the steady-state lift and drag forces for stationary circular cylinders at different Re values. Keeping eccentricity as the control parameter and using method of harmonic balance, Akhtar et al. [2] performed the numerical simulations for flow over the elliptical structures and approximated the transient and steady state behavior of the lift and drag forces using a combined van der Pol-Duffing oscillator model. These
afore-mentioned studies help enhance our understanding regarding the nonlinear mechanism for production of the unsteady aerodynamic forces by the bluff bodies. To extend this technique for the aerodynamic bodies, Ellenrieder [28] adopted the methodology proposed by Skop et al. [25] to present following oscillator model for time-dependent lift produced by the airfoils undergoing forced plunging motion;

\[
\ddot{C}_L - \omega_s G (C_{L_o}^2 - 4Q^2) \dot{C}_L + \omega_s^2 C_L = \omega_s F \dot{h}
\]

where \(\omega_s\) is the fundamental vortex shedding frequency, \(C_{L_o}\) is the maximum amplitude of the time-dependent lift force coefficient, \(h\) is an instantaneous position of airfoil while plunging and, \(G\) and \(F\) are the constants determined from the empirical formulation proposed by Skop et al. [25]. In the current study, we first show the existence of limit-cycle behavior in the response of an airfoil performing forced oscillatory motion. Here, the response of this nonlinear system is determined in terms of lift and thrust forces. We consider different sets of initial kinematic conditions of the oscillating airfoil. Identifying the nonlinearities in the response, we present a modified forced van der Pol oscillator model for lift-force coefficient. The analytical solution of this nonlinear mathematical model is derived using method of multiple scales; a powerful perturbation technique. The strength of this technique to solve the perturbation problems lies in the fact that both the small and large values of system’s states can be handled by the involvement of slow and fast time-scales. This reduced-order model not only captures the temporal details of the the lift force but also it predicts its spectral composition accurately. The purpose of this study is to provide a computational tool for estimating the response of this physical system quickly. To determine the parameters of the presented reduced-order model, we employ numerical data obtained from CFD (computational fluid dynamics) simulations using ANSYS Fluent [3]. We perform these CFD simulations for a range of St values at \(Re = 10^3\). These simulations are thoroughly validated and used here as the first step towards development of the reduced-order models. The manuscript is organized as follows. Section 2 provides necessary details on our CFD simulation methodology for flow past the oscillating airfoils along with the validation studies. Using different sets of the kinematic initial conditions and states of this nonlinear system in section 3, we show the limit-cycle existence in the response of the oscillation airfoils. Mathematical formulation and derivation of a reduced-order model based on a modified forced van der Pol oscillator equation is presented in section 4. Applicability of this model for lift produced by plunging, pitching and flapping airfoils is shown in section 5. This model is capable of accurately predicting the nonlinear behavior of unsteady \(C_L\) of oscillating airfoils. To model its time-averaged value \(\bar{C}_L\), we introduce a different type of quadratic nonlinearity to the self-excited van der Pol oscillator model and describe its details in section 6. Section 7 explains the overall behavior of the linear and nonlinear damping terms with respect to increasing control parameter; the Strouhal number (defined in section 2). We also show the results of predictive settings for the present model to compute unsteady \(C_L\) and its spectra for the intermediate values of Strouhal number where the model parameters were calculated through interpolation of the data-set from CFD solutions. In section 8, we conclude and summarize our present work.

## 2 Numerical Methodology

For the present study, we simulate flow past an oscillating NACA-0012 airfoils by solving two-dimensional incompressible Navier-Stokes equations using ANSYS-Fluent; a finite-volume based
commercial software. The Navier-Stokes equations in their integral form can be written as;

\[ \int_V \frac{\partial \rho u}{\partial t} dV + \oint \rho \phi \bar{v} \cdot d\bar{A} - \oint \Gamma \phi \nabla \phi \cdot d\bar{A} - \int_V S_\phi dV = 0 \] (2)

where \( \rho \) is the density of fluid, \( \bar{v} \) represents the velocity vector, \( \bar{A} \) shows the surface-area vector, \( \Gamma \phi \) is the diffusion term, \( \nabla \phi \) denotes the gradient term and \( S_\phi \) shows the source term. Since the present case employs moving mesh technique [3], the source term is zero. The temporal term is approximated using the first-order implicit method. Second-order upwind scheme is employed for numerically approximating the convective term. Assuming incompressible flow presently, velocity and pressure terms are coupled through the pressure implicit with splitting of operator (PISO) algorithm. To avoid effect of disturbances on boundaries, radius of the O-type domain is kept at 25c, where c is the chord-length of airfoil, as shown in Fig. 1. Flow domain is meshed using unstructured triangular cells. We use high grid resolution near the airfoil surface to resolve the boundary layer and to capture the wake characteristics downstream of the airfoil. Airfoil surface is resolved using 400 nodes. Dynamic meshing techniques; spring analogy and remeshing, are employed in the vicinity of moving airfoil for grid transition that allows the adjustment of the grid in accordance with the instantaneous position of airfoil. Numerical solutions of flow-fields are

![Figure 1: Schematic of Geometry and Fluid-Domain](image)

highly dependent on the suitability and accuracy of boundary conditions. In Fluent, motion of an object may be defined by a user-defined function (UDF) which is a computer code written in the C-language environment coupled with the Fluent-Macros. Forced motion of an airfoil can be of three types; plunging, pitching and flapping (combination of plunging and pitching). The schematic of these motion are shown in Fig. 2. Pithing and plunging motion can be modeled as;

\[
\text{Plunging}: h(t) = h_0 \cos(2\pi ft + \phi_h) \\
\text{Pitching}: \alpha(t) = \alpha_0 \sin(2\pi ft)
\] (3)

where \( \alpha_0 \) is the maximum pitching amplitude and \( \phi_h \) denotes the phase-angle. Dirichlet conditions
are employed on the inlet boundary and the pressure outlet condition is used for the outflow boundary. At this boundary, static pressure is specified. For incompressible flows, pressure on the boundary is determined by taking the average of specified values on the cell faces and computed values of the static pressure on the corresponding cell-centers. All other flow variables are computed by extrapolation of the computed values from the inner domain. The Reynolds number, defined as $U_\infty c/\nu$, is calculated using appropriate values of the kinematic viscosity $\nu$, keeping all other parameters equal to unity. Strouhal number (St) is defined as $St = 2f h_o/U_\infty$ where $h_o$ is the maximum amplitude, $f$ is the excitation frequency for oscillating airfoil in Hertz, and $U_\infty$ is the free-stream fluid velocity. It is considered as the primary governing parameter for investigating unsteady behavior of oscillating airfoils. $h_o$ depicts the wake-width behind a plunging airfoil. For pitching motion, total vertical excursion traversed by the trailing-edge of airfoil approximates the wake-width. St is varied by changing plunging and pitching amplitudes while incoming reference flow velocity and oscillation frequency are kept fixed. In the present study, numerical simulations are initialized by the uniform inlet velocity boundary conditions. To perform these simulations, we employ $1.18440 \times 10^5$ number of cells in the whole domain and 2000 time-steps per oscillation cycle of the airfoil. Details for the grid-convergence, time-step refinement, and validation studies are available in Ref. [9]. Lift and thrust forces are computed by integrating pressure and shear stresses over the surface of airfoil. Their corresponding coefficients, $C_L$ and $C_T$ respectively are computed.
Using time-period of an oscillation cycle ($\tau = 1/f$), corresponding time-averaged coefficients are calculated through following relation:

$$\bar{C} = \frac{1}{\tau} \int_{t}^{t+\tau} C(t) dt$$

3 Existence of Limit-Cycle

Limit cycles can be used to model various nonlinear oscillatory systems from the real world. Limit-cycle exists if energy input to the dynamical system is balanced by an energy dissipation mechanism and the system’s response limits itself onto either a periodic, a quasi-periodic, or a chaotic attractor [13]. Hence this phenomenon represents a closed trajectory in the phase space. To characterize the behavior of a nonlinear system as a limit cycle, it needs to be tested for various initial conditions. Based upon a reduced-order model for $C_L$, von Ellenrieder [28] and von Ellenrieder et al. [29] presented the aerodynamic response of plunging airfoil as a limit cycle. To obtain numerical solution for governing ordinary differential equation, all the initial conditions were described in the form of $C_L$ and its time-derivatives. Use of the actual initial kinematic conditions seems more appropriate to investigate the limit-cycle behavior of any dynamical system. For current research-work, we consider a broader spectrum of the initial conditions through inclusion of initial position where the airfoil starts its motion from. There can be infinite starting positions between positive and negative amplitudes of an oscillating airfoil. In this study, initial conditions refer to the initial kinematic states of plunging motion for the numerical simulations. These include following scenarios;

1. Does plunging airfoil undergo upstroke or downstroke first?
2. Which position does it start its oscillation from?

We choose three positions to start the airfoil’s oscillatory motion. These are the top-most, mean (mid) and the bottom-most positions. Airfoil can perform upstroke initially starting from the bottom-most or mean position and it may undergo downstroke starting from mean or the top-most position. $\phi_h = 0^\circ$ gives initial downstroke starting from the maximum positive amplitude of oscillating airfoil. Using these conditions of initial stroke and position, four cases are required to be studied for one set of values of oscillation frequency and amplitude. To analyze this phenomena here, we consider four initial conditions described earlier in terms of the starting position and stroke of a plunging airfoil. We compute the response of this dynamical system in terms of the aerodynamic lift and thrust coefficients denoted as $C_L$ and $C_T$, respectively. Using the temporal histories and phase maps of these dynamical states, we observe similarity in their nonlinear character for different initial conditions. In Fig. 3, $C_L$ and $C_T$ for four initial conditions of plunging airfoil are presented. We observe the same phase difference in their temporal histories as given in the forced motion of plunging airfoil. Another notable feature is the same amplitude of $C_L$ for all the initial conditions. Phase of $C_L$ is exactly equal to that of heave motion. But for $C_T$, phase depends
Figure 3: Comparison of $C_L$ and $C_T$ for different initial positions and strokes of plunging airfoil for $St = 0.10$ in (a) and (c); and for $St = 0.30$ in (b) and (d).

upon magnitude of $\phi$ only and not its sign (where we can say $270^\circ = -90^\circ$). Here, we support the existence of limit-cycle using aerodynamics forces and their time-derivatives as the states of flow past a plunging airfoil through phase maps shown in Fig. 4 for $St = 0.10$ and $0.30$. Periodic steady-state solutions for the aerodynamic force coefficients are presented for this purpose. Like bluff-body aerodynamics [2,12,18], the frequency of unsteady $C_T$ is twice of that for the unsteady $C_L$. All the initial conditions lead us to the same period-$n$ attractor thus proving independence of this system from initial kinematic states of oscillating airfoil for the range of parameters considered here.

4 Reduced Order Modeling

Young et al. [31] explained the existence of vortex-lock in phenomena for plunging airfoil for a range of flow and kinematic parameters. In such cases, natural vortex-shedding frequency comes out to be equal to forcing frequency of oscillating airfoil. Presence of higher-order harmonics in spectra of unsteady aerodynamic force coefficients for flow over oscillating airfoils exhibits this phenomena as nonlinear. This is a well-known fact in vibrations of bluff-bodies like cylinders and cables as well [21, 26]. In current study, spectra of the unsteady lift force are analyzed to identify the prominent frequency components and their corresponding harmonics present in the signal. For given range of $St$, (from 0.05 to 0.5), fundamental vortex shedding $f_s$ frequency appears to be equal
Figure 4: Phase maps using $C_L$ and $C_T$ their time-derivatives for different initial positions and strokes of plunging airfoil at $St = 0.10$ in (a) and (c); and for $St = 0.30$ in (b) and (d) to the forced plunging frequency ($\Omega/2\pi$), equal to 0.5Hz of the airfoil. It indicates that these all are the cases related to the primary resonance. First even and odd harmonics show prominent peaks. As we increase $St$, even harmonic starts getting larger amplitude. Figure 5 shows three shedding cycles represented by unsteady $C_L$ and its relevant amplitude-spectra. The amplitudes of energetic fundamental, first even and odd harmonics are also indicated being employed here for model-reduction. An asymmetric forced van der Pol oscillator model is used here to analytically represent the lift force produced by plunging NACA-0012 airfoil. Due to the presence of even harmonics in $C_L$-spectra, we introduce a quadratic nonlinear term containing a multiple of $C_L\dot{C}_L$.

Choice of this term to model the quadratic nonlinearity is made due to a phase-difference of nearly $\pi/2$ (or its odd integral multiples) between the fundamental and first even harmonics. Assuming this system as weekly damped and having a soft excitation;

$$\ddot{C}_L + \omega^2_s C_L = \epsilon[\mu\dot{C}_L - \alpha C_L\dot{C}_L - \gamma C_L^2\dot{C}_L + F_0\cos(\Omega t + \Gamma)]$$

$\omega_s$ is the vortex-shedding frequency, $\epsilon$ is a book-keeping parameter while $\mu$, $\alpha$ and $\gamma$ represent the linear, quadratic and cubic damping coefficients, respectively. All these parameters are positive real numbers. Although no vortex-shedding from NACA-0012 is observed at zero angle-of-attack at this Re, we consider this case related to primary resonance ($\Omega \approx \omega_s$) in a mathematical sense;

$$\Omega = \omega + \epsilon \sigma$$
Using Eq. 7 and $F_0$ as the amplitude, excitation may be expressed as:

$$E(t) = F_0 \cos[(\omega_s + \epsilon \sigma)t + \Gamma]$$  

(8)

Assuming time scales are $T_0 = t$, $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$, we have;

$$E(t) = F_0 \cos[(\omega_s T_0 + \sigma T_1 + \Gamma]$$  

(9)

Using Chain rule for differentiation, we can write;

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + ...$$  

(10)

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0 D_1) + ...$$  

(11)

Let second-order approximate solution for Eq. 6 be;

$$C_L(t) = C_{L0}(T_0, T_1, T_2) + \epsilon C_{L1}(T_0, T_1, T_2) + \epsilon^2 C_{L2}(T_0, T_1, T_2)$$  

(12)

We expand all the terms in Eq. 6 as follows;

$$\ddot{C}_L = (D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + 2\epsilon^2 D_0 D_2)(C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2})$$

$$= D_0^2 C_{L0} + \epsilon D_0^2 C_{L1} + \epsilon^2 D_0^2 C_{L2} + 2\epsilon D_0 D_1 C_{L0} + 2\epsilon^2 D_0 D_1 C_{L1} + \epsilon^2 D_1^2 C_{L0} + 2\epsilon^2 D_0 D_2 C_{L0}$$  

(13)

$$\omega_s^2 C_L = \omega_s^2 (C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2})$$

$$= \omega_s^2 C_{L0} + \epsilon \omega_s^2 C_{L1} + \epsilon^2 \omega_s^2 C_{L2}$$  

(14)

$$\epsilon \mu \ddot{C}_L = \epsilon \mu (D_0 + \epsilon D_1 + \epsilon^2 D_2 + ...)(C_{L0} + \epsilon C_{L1} + \epsilon C_{L2})$$

$$= \epsilon \mu D_0 C_{L0} + \epsilon^2 C_{L1} + \epsilon^2 \mu D_1 C_{L0}$$  

(15)
To solve Eq. 19, all the terms on right hand side need expansion.

\[-\epsilon \alpha C_L \dot{C}_L = -\epsilon \alpha (C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2})(D_0 + \epsilon D_1 + \epsilon^2 D_1)(C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2})
= -\epsilon \alpha C_{L0}(D_0 C_{L0}) - \epsilon^2 \alpha C_{L0}(D_0 C_{L1})
- \epsilon^2 \alpha C_{L0}(D_1 C_{L0}) - \epsilon^2 \alpha C_{L1}(D_0 C_{L0})\]  
(16)

\[-\epsilon \gamma C_L^2 C_L^2 = -\epsilon \gamma [C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2}]^2(D_0 + \epsilon D_1 + \epsilon^2 D_1)(C_{L0} + \epsilon C_{L1} + \epsilon^2 C_{L2})
= -\epsilon \gamma C_{L0}^2(D_0 C_{L0}) - \epsilon^2 \gamma C_{L0}^2(D_0 C_{L1})
- \epsilon^2 \gamma C_{L0}^2(D_1 C_{L0}) - 2 \epsilon^2 \gamma C_{L0} C_{L1}(D_0 C_{L0})\]  
(17)

\(\epsilon F_0 \cos(\omega_s T_0 + \sigma T_1 + \Gamma)\) remains in its present form. Equating like powers of \(\epsilon\) on both sides of Eq. 6, we get;

\(\mathcal{O}(\epsilon^0) : \quad D^2_o C_{L0} + \omega_s C_{L0} = 0\)  
(18)

\(\mathcal{O}(\epsilon^1) : \quad D^2_o C_{L1} + \omega_s C_{L1} = -2D_o D_1 C_{L0} + \mu D_o C_{L0} - \alpha C_{L0}(D_0 C_{L0}) - \gamma C_{L0}^2(D_0 C_{L0})
+ F_0 \cos(\omega_s T_0 + \sigma T_1 + \Gamma)\)  
(19)

\(\mathcal{O}(\epsilon^2) : \quad D^2_o C_{L2} + \omega_s C_{L2} = -2D_o D_1 C_{L1} - D^2_1 C_{L0} - 2D_o D_2 C_{L0} + \mu(D_0 C_{L1}) - \alpha C_{L0}(D_0 C_{L1})
- \alpha C_{L0}(D_1 C_{L0}) - \alpha C_{L1}(D_0 C_{L0})
- \gamma C_{L0}^2(D_0 C_{L1}) - \gamma C_{L0}^2(D_1 C_{L0}) - 2 \gamma C_{L0} C_{L1}(D_0 C_{L0})\)  
(20)

Solution of Eq. 18 is;

\(C_{L0}(t) = A_o(T_1, T_2) \cos[\omega_s T_0 + \theta_o(T_1, T_2)]\)  
(21)

To solve Eq. 19, all the terms on right hand side need expansion.

\[-2 \frac{\partial^2 C_{L0}}{\partial T_o \partial T_1} = 2 \omega_s \frac{\partial A_o}{\partial T_1} \sin(\omega_s T_0 + \theta_o) + 2 \omega_s A_o \frac{\partial \theta_o}{\partial T_1} \cos(\omega_s T_0 + \theta_o)\]  
(22)

\[\mu \frac{\partial C_{L0}}{\partial T_o} = -\mu \omega_s A_o \sin(\omega_s T_0 + \theta_o)\]  
(23)

\[-\alpha C_{L0} \frac{\partial C_{L0}}{\partial T_o} = \frac{1}{2} \alpha \omega_s A_o^2 \sin(2 \omega_s T_0 + 2 \theta_o)\]  
(24)

\[-\gamma C_{L0}^2 \frac{\partial C_{L0}}{\partial T_o} = \frac{1}{4} \gamma \omega_s A_o^3 \sin(3 \omega_s T_0 + 3 \theta_o)\]  
(25)

\(F_0 \cos(\omega_s T_0 + \sigma T_1 + \Gamma) = F_0 \cos(\omega_s T_0 + \theta_s + \sigma T_1 + \Gamma - \theta_o)\)  
(26)

\(= F_0 \cos(\omega_s T_0 + \theta_s) \cos(\sigma T_1 + \Gamma - \theta_o) - F_0 \sin(\omega_s T_0 + \theta_s) \sin(\sigma T_1 + \Gamma - \theta_o)\)  
(27)
Modulation equation may be obtained by combining multiples of \( \sin(\omega_s T + \theta_o) \);

\[
2\omega_s \frac{\partial A_o}{\partial T_1} - \mu \omega_s A_o + \frac{\gamma \omega_s A_o^3}{4} - \frac{F_o}{2\omega_s} \sin(\sigma T_1 + \Gamma + \theta_o) = 0
\]  
(28)

\[
\frac{\partial A_o}{\partial T_1} = \frac{\mu A_o}{2} - \frac{\gamma A_o^3}{8} - \frac{F_o}{2\omega_s} \sin(\sigma T_1 + \Gamma + \theta_o)
\]  
(29)

By combining multiples of \( \cos(\omega_s T + \theta_o) \), second modulation equation may be constructed.

\[
2\omega_s A_o \frac{\partial \theta_o}{\partial T_1} = \frac{F_o}{2\omega_s} \cos(\sigma T_1 + \Gamma + \theta_o)
\]  
(30)

To make the system autonomous, we assume;

\[
\eta = \sigma T_1 + \Gamma - \theta_o
\]

\[
\theta_o = \sigma T_1 + \Gamma - \eta
\]

\[
\frac{\partial \theta_o}{\partial T_1} = \sigma - \frac{\partial \eta}{\partial T_1}
\]

So, Eq. 30 may be written as;

\[
\frac{\partial \eta}{\partial T_1} = \sigma + \frac{F_o}{2\omega_s A_o} \cos(\eta)
\]  
(31)

Thus amplitude and phase are governed by;

\[
\dot{A}_o = \frac{\mu A_o}{2} - \frac{\gamma A_o^3}{8} - \frac{F_o}{2\omega_s} \sin(\sigma T_1 + \Gamma + \theta_o)
\]  
(32)

\[
\dot{\eta} = \sigma + \frac{F_o}{2\omega_s A_o} \cos(\eta)
\]  
(33)

To get the steady-state motion, time-derivative of both amplitude and phase needs to be zero.

\[
\frac{\mu A_o}{2} - \frac{\gamma A_o^3}{8} = \frac{F_o}{2\omega_s} \sin(\sigma T_1 + \Gamma + \theta_o)
\]  
(34)

\[
\sigma A_o = \frac{F_o}{2\omega_s} \cos(\eta)
\]  
(35)

Squaring and adding both Eq. 34 and 35, we get;

\[
\frac{F_o^2}{4\omega_s^2} = \left( \frac{\mu A_o}{2} + \frac{\alpha A_o^3}{8} \right)^2 + A_o^2 \sigma^2
\]  
(36)

Identifying the modulating terms, we get following governing expression for \( C_{L1}(t) \);

\[
\frac{\partial^2 C_{L1}}{\partial T_o^2} + \omega_s^2 C_{L1} = \alpha \omega_s \frac{A_o^2}{2} \sin(2\omega_s T_0 + 2\theta_o) + \gamma \omega_s \frac{A_o^3}{4} \sin(3\omega_s T_0 + 3\theta_o)
\]  
(37)
To get particular solution, we can write;

$$C_{L1p} = K \sin(2\omega_s T_o + 2\theta_o) + M \sin(3\omega_s T_o + 3\theta_o)$$  \hspace{1cm} (38)$$

$$\frac{\partial C_{L1p}}{\partial T_o} = 2\omega_s K \cos(2\omega_s T_o + 2\theta_o) + 3\omega_s M \cos(3\omega_s T_o + 3\theta_o)$$  \hspace{1cm} (39)$$

$$\frac{\partial^2 C_{L1p}}{\partial T_o^2} = -4\omega_s^2 K \sin(2\omega_s T_o + 2\theta_o) - 9\omega_s^2 M \sin(3\omega_s T_o + 3\theta_o)$$  \hspace{1cm} (40)$$

Putting these values of \(C_{L1p}\), \(\frac{\partial C_{L1p}}{\partial T_o}\) and \(\frac{\partial^2 C_{L1p}}{\partial T_o^2}\) into Eq. 37 and equating like terms, we get \(K = -\frac{\alpha A_o^2}{6\omega_s}\) and \(M = -\frac{\gamma A_o^3}{32\omega_s}\). Hence, \(C_{L1}(t)\) comes out to be;

$$C_{L1p}(t) = -\frac{\alpha A_o^2}{6\omega_s} \sin(2\omega_s t + 2\theta_o) - \frac{\gamma A_o^3}{32\omega_s} \sin(3\omega_s t + 3\theta_o)$$  \hspace{1cm} (41)$$

Thus, complete solution for \(C_L(t)\) is;

$$C_L(t) = A_o \cos(\omega_s t + \theta_o) - \epsilon \frac{\alpha A_o^2}{6\omega_s} \cos(2\omega_s t + 2\theta_o + \frac{\pi}{2}) - \epsilon^2 \frac{\gamma A_o^3}{32\omega_s} \cos(3\omega_s t + 3\theta_o + \frac{\pi}{2}) + ...$$  \hspace{1cm} (42)$$

For autonomous dynamical system in complex notation, we have;

$$C_L(t) = \frac{A_o}{2} [e^{i(\Omega t + \Gamma - \eta)} + e^{-i(\Omega t + \Gamma - \eta)}] - \frac{\alpha A_o^2}{12\omega_s} [e^{i(2\Omega t + 2\Gamma - 2\eta + \frac{\pi}{2})} + e^{-i(2\Omega t + 2\Gamma - 2\eta + \frac{\pi}{2})}] - \frac{\gamma A_o^3}{64\omega_s} [e^{i(3\Omega t + 3\Gamma - 3\eta + \frac{\pi}{2})} + e^{-i(3\Omega t + 3\Gamma - 3\eta + \frac{\pi}{2})}]$$  \hspace{1cm} (43)$$

Now, we describe the procedure for identification of linear and nonlinear parameters. Performing the spectral analysis of unsteady lift profiles, values for fundamental vortex shedding frequency \(\omega\), its corresponding amplitude \(A_o\) and amplitudes \(A_2\) and \(A_3\) of first even and odd harmonics, respectively can be measured. Comparing the assumed approximate solution of van der Pol oscillator model in Eq. 12 with that from multiple scales method in Eq. 42, nonlinear damping parameters; \(\alpha\) and \(\gamma\), can be calculated as;

$$\alpha = \frac{6\omega_s A_2}{A_o^2}$$  \hspace{1cm} (44)$$

$$\gamma = \frac{32\omega_s A_3}{A_o^3}$$  \hspace{1cm} (45)$$

External detuning parameter is;

$$\sigma = \frac{\Omega - \omega}{\epsilon}$$  \hspace{1cm} (46)$$
Setting phase $\Gamma$ of the excitation, $\eta$ can be measured from the phase $\phi$ of $C_L(\Omega)$ component in Fourier Transform of lift signal:

$$\eta = \Gamma - \phi[C_L(\Omega)] \quad (47)$$

Forcing amplitude $F_0$ is:

$$F_0 = \frac{2\omega_s A_0 \sigma}{\cos(\eta)} \quad (48)$$

Linear damping $\mu$ can be calculated as:

$$\mu = \frac{\gamma A_0^2}{4} - \frac{F_0 \sin(\eta)}{A_0 \omega_s} \quad (49)$$

5 RESULTS & DISCUSSION

When airfoil starts oscillation in a fluid, a reverse von Karman vortex street is observed in the wake which leads to generation of time-varying aerodynamic forces. At low St, these aerodynamic forces are periodic in nature which may be composed of several harmonics of fundamental vortex shedding frequency. We compare the solution of van der Pol oscillator model with those from CFD, both in temporal and spectral domains. The proposed model not only captures temporal profiles accurately but also strong harmonics in their spectra. We present the validity of the proposed reduced-order model for three types of forced motions of airfoil; plunging, pitching and flapping.

5.1 Plunging Airfoil

Plunging airfoil performs oscillatory motion along vertical direction. It portraits a single-degree of freedom system. Figure 6a shows comparison of aerodynamic lift force from CFD simulations and the proposed model in time domain. Solution for the ordinary differential equation (ODE) of proposed reduced-order model is obtained by numerical integration using Runge-Kutta method of order-4. It is clear from these figures that current model fulfills the requisites effectively. To capture the inherent nonlinear characteristics of lift force, the results of the proposed model should carry the same spectral components of the signal. It can be analyzed by comparing the spectra of signals from CFD and reduced-order model. $C_L$-spectra of CFD and van der Pol oscillator model are shown in Fig. 6b. It shows that the proposed model not only captures the fundamental frequency with accurate amplitude but also the other harmonic components are well predicted by this model. To develop a database for linear and nonlinear damping parameters, force amplitude, forced and vortex shedding frequencies, Table 1 presents sample dynamic parameters for a range of St.

5.2 Pitching Airfoil

Although pitching motion of airfoil is a different degree-of-freedom but its response in terms of aerodynamic forces resembles those of plunging airfoil. Table 2 presents sample values for spectral and ROM parameters for different values of St. In Fig. 7, we show comparison of CFD solution with that of van der Pol oscillator model.
Figure 6: Comparison of CFD solution and the proposed model for Plunging Airfoil at St = 0.45
(a) Time Histories (Solid Lines: CFD and Circles: ROM) (b) Amplitude-Spectrum (Solid Lines: CFD and Dashed Lines: ROM)

Table 1: Spectral and ROM Parameters for Lift of Plunging Airfoil

| Parameters/St | 0.15     | 0.30     | 0.45     |
|---------------|----------|----------|----------|
| $f_s$         | 0.4982   | 0.5003   | 0.4998   |
| $A_0$         | 1.908    | 6.051    | 10.3     |
| $A_2$         | 0.001039 | 0.01653  | 0.06951  |
| $A_3$         | 0.08816  | 0.3482   | 0.2933   |
| $\eta$        | 37.0033° | 48.2008° | 53.0429° |
| $\mu$         | 1.1382   | 1.4396   | 0.7144   |
| $\alpha$      | 0.0054   | 0.0085   | 0.0123   |
| $\gamma$      | 1.2714   | 0.1580   | 0.0270   |
| $F_o$         | 0.1760   | 0.1513   | 0.0870   |

5.3 Flapping Airfoil

As described earlier, flapping motion is a combination of plunging and pitching motions. Practically, swimming and flying species or robots employ synchronized pitching and plunging while flapping their wings. Looking at the similar nature of response, we model lift force of flapping wings using same oscillator model. Table 3 shows sample values of spectral and ROM parameters for flapping airfoil while Fig. 8 shows comparison of CFD and ROM solutions. CFD simulations presented here were carried out for $0.05 \leq h_o \leq 0.50$, $\alpha_o = 10^\circ$, $f = 0.5$Hz and $Re = 10^3$. It also proves the suitability of this model for this complex phenomenon.
Table 2: Spectral and ROM Parameters for Lift of Pitching Airfoil

| Parameters/St | 0.10 | 0.20 | 0.30 |
|---------------|------|------|------|
| $f_s$         | 0.5002 | 0.5001 | 0.5003 |
| $A_0$         | 0.8367 | 2.049 | 3.576 |
| $A_2$         | 0.0005242 | 0.0008243 | 0.002885 |
| $A_3$         | 0.0136 | 0.1203 | 0.3186 |
| $\eta$       | $-0.4177^\circ$ | 6.376$^\circ$ | 10.8025$^\circ$ |
| $\mu$        | 0.4098 | 1.4757 | 2.2212 |
| $\alpha$     | 0.0141 | 0.0037 | 0.0043 |
| $\gamma$     | 2.3351 | 1.4059 | 0.7008 |
| $F_0$        | 0.0072 | 0.0081 | 0.2209 |

Figure 7: Comparison of CFD solution and the proposed model for Pitching Airfoil at St = 0.30  
(a) Time Histories (Solid Lines: CFD and Circles: ROM) (b) Amplitude-Spectrum (Solid Lines: CFD and Dashed Lines: ROM)

6 Improved Model

Like usual unsteady signals, we may decompose $C_L$ into two components; time-averaged value $\bar{C}_L$ and fluctuating component $C_l$.

$$C_L = \bar{C}_L + C_l$$  \hspace{1cm} (50)

Although motion of airfoil during plunging, pitching and flapping is symmetrical about its mean position, yet there exists a non-zero time-averaged value of $C_L$ [32]. Presence of non-zero time-averaged value of a signal along with an even harmonic shows quadratic nonlinearity in the system [17]. In section 4, we attempt to model the quadratic nonlinearity using a term $C_L\dot{C}_L$. It is chosen due to a phase-difference of $\pi/2$ (or its integral multiple) between fundamental and first even
Table 3: Spectral and ROM Parameters for Lift of Flapping Airfoil

| Parameters/St | 0.05  | 0.20  | 0.35  |
|---------------|-------|-------|-------|
| $f_s$         | 0.5017| 0.5003| 0.4999|
| $A_0$         | 0.6721| 2.093 | 5.24  |
| $A_2$         | 0.02461| 0.0288| 0.04316|
| $A_3$         | 0.03737| 0.07439| 0.1265|
| $\eta$       | $-31.6684^\circ$| $60.7861^\circ$| $60.1451^\circ$|
| $\mu$        | 1.4077| 0.8865| 0.6364|
| $\alpha$     | 1.0304| 0.1240| 0.0456|
| $\gamma$     | 12.4164| 0.8161| 0.0924|
| $F_o$         | 0.0467| 0.0542| 0.0450|

Figure 8: Comparison of CFD solution and the proposed model for Pitching Airfoil at St = 0.20
(a) Time Histories (Solid Lines: CFD and Circles: ROM) (b) Amplitude-Spectrum (Solid Lines: CFD and Dashed Lines: ROM)

harmonic in the unsteady $C_L$ signal. This model can capture the unsteady details effectively but it does not predict $\bar{C}_L$ that is an important feature of aerodynamics in case of oscillating airfoils. Considering lesser magnitudes of forcing function amplitude $F_o$, a possible singularity at $\eta = \pi/2$ and to take care of $\bar{C}_L$, we propose another version of the modified van der Pol oscillator's ordinary differential equation to model the lift of oscillating airfoil.

\[
\ddot{C}_L + \omega^2 s C_L = \epsilon [\mu \dot{C}_L - \alpha C_L^2 - \gamma C_L^2 \dot{C}_L] \tag{51}
\]
Solving this system using the method of multiple scales, following second-order solution is obtained:

\[ C_L(t) = A_o \cos(\omega t) + \epsilon \left[ \frac{\alpha A_o^2}{2 \omega^2} - \frac{\alpha A_o^2}{6 \omega^2} \cos(2\omega t + 2\beta_o) \right. \]
\[ \left. - \frac{\alpha A_o^3}{32\omega} \cos(3\omega t + 3\beta_o) \right] \]  \hspace{1cm} (52)

Modulation equations in this case are:

\[ \dot{A}_o = \mu A_o - \frac{\gamma A_o^3}{8} \]  \hspace{1cm} (53)

\[ \dot{\beta}_o = \frac{\mu^2}{8\omega} + \frac{\mu \gamma A_o^2}{8\omega} - \frac{11\gamma^2 A_o^4}{256\omega} + \frac{5\alpha^2 A_o^2}{12\omega^3} \]  \hspace{1cm} (54)

Linear and nonlinear damping parameters are identified as:

\[ \mu = \frac{8\omega A_3}{A_o} \]  \hspace{1cm} (55)

\[ \alpha = \frac{3\omega^2 A_2}{A_o^2} \]  \hspace{1cm} (56)

\[ \gamma = \frac{4\mu}{A_o^2} \]  \hspace{1cm} (57)

From Eq. 52, \( \bar{C}_L \) is:

\[ \bar{C}_L = \frac{2\alpha \mu}{\gamma \omega^2} \]  \hspace{1cm} (58)

To prove the accuracy of this model, we compare its results with those from CFD in Fig 9. CFD results for St=0.20 and 0.40 were used here to extract the ROM parameters. The strength of this model lies in its capability to model time-averaged value of the lift force. This model, too, behaves well in time as well as spectral domains. We also show a database for ROM parameters for different St values in Table 4. Comparing damping parameters with those in Table 1 for lift of plunging airfoil, we observe almost equal magnitudes. It is a manifestation for capability of this improved reduced-order model to predict actual unsteady characteristics in lift force signal.

7 Predictive Settings

To summarize the results of the proposed model and its suitability for responses of various kinematics of airfoil, Fig. 10 shows the variations of damping coefficients as functions of St. Positive values of \( \mu, \alpha \) and \( \gamma \) indicate the presence of limit cycle. Despite a difference in level of their magnitudes in case of three different kinematics of airfoil, similar trend for variation can be observed for quadratic (\( \alpha \)) and cubic (\( \gamma \)) damping-coefficients. This similar nature of nonlinear dynamic parameters proves the effectiveness of the present reduced-order model. Magnitudes of nonlinear
Figure 9: Comparison of solutions from CFD and the improved ROM for Plunging Airfoil (a) Time Histories (Solid Lines: CFD and Circles: ROM) (b) Amplitude-Spectrum (Solid Lines: CFD and Dashed Lines: ROM)
Table 4: Parameters for Lift of Plunging Airfoil in Case of Improved ROM

| Parameters/St | 0.15  | 0.30  | 0.45  |
|--------------|-------|-------|-------|
| µ            | 1.1613| 1.4462| 0.7157|
| α            | 0.0085| 0.0134| 0.0194|
| γ            | 1.2760| 0.1580| 0.0270|
| $\bar{C}_L$(ROM) | 0.0046| 0.0316| 0.1083|
| $\bar{C}_L$(CFD) | 0.0063| 0.0352| 0.1190|

Figure 10: Plots of Dynamic Parameters for Varying St

Damping parameters decrease due to increasing amplitude $A_0$ for fundamental harmonic as can be seen in Eq. 44 and 45. Linear damping $\mu$ for $C_L$ of pitching airfoil comes out to be a linear function of St while it shows quite similar nonlinear behavior for plunging and flapping motions. Now, we examine the performance of our reduced-order models in predictive settings for $St = 0.18$. For this purpose, the selection of this St value is justified from Fig. 10 where we observe higher gradients in the trends of both linear and nonlinear damping parameters. We find out the linear, quadratic and cubic damping values using data points presented in Fig. 10 through cubic interpolation scheme. We compute the solution of the proposed van der Pol oscillator model by numerical integration.
and compare its performance with the CFD results. Figure 11 shows that the reduced-order model well satisfies the requirement not only in time-domain but also in the spectral domain even at the point around highest variation in the available data. To quantitatively test the model, we calculate the percentage errors in the amplitudes of unsteady $C_L$ from CFD and predicted values of reduced order model, and the fundamental frequency in the spectra of both. These values come out to be 7.91% and zero, respectively. While comparing both the spectra, we see small deviation in the levels of higher harmonics. $C_L$ comes out to be 0.009 from CFD and the numerical solution of the model in predictive settings gives 0.006 which are quite close to each other. Errors and deviations from actual CFD results may be removed by using more data points as samples in the interpolation scheme.

Figure 11: Comparison of CFD solution and the proposed model for Pitching Airfoil at St = 0.20 
(a) Time Histories (Triangles: CFD and Solid Line: ROM) (b) Amplitude-Spectrum (Dashed Line: CFD and Solid Lines: ROM)

8 Conclusions

In the present paper, we identify the existence of a limit-cycle behavior through aerodynamic forces produced by oscillating airfoils. It is done through various initial kinematic states that has greater physical significance. Noting the presence of even and odd harmonics in $C_L$-spectra for plunging, pitching and flapping airfoils, we propose a phenomenological reduced-order model by employing an asymmetric forced van der Pol oscillator model. Parameters for this models are calculated through results of CFD simulations. By numerically integrating the governing ROM ODE, we present comparison of its solutions with those of CFD. Results are quite promising for both temporal and spectral domains. Real strength of this proposed model lies in its applicability to aerodynamic lift forces of plunging, pitching and flapping airfoils that are altogether different degrees-of-freedom of the relevant structure. Similarity of nonlinear behavior in all of these phenomenon are also proven by plotting nonlinear damping coefficients versus Strouhal number. This model helps measure the aerodynamic forces of oscillating streamlined body quickly and accurately without involvement of costly experimental equipment or time-consuming complex CFD simula-
tions. This model is limited in a sense that it cannot predict $C_L$. To overcome this deficiency and, considering the lower amplitudes of forcing functions and a probable singularity condition in forced van der Pol oscillator model, we propose another model that carries $C_L^2$ term to model quadratic nonlinearity with no forcing function. This model captures $C_L$ quite effectively. Magnitudes of linear and nonlinear dampings in Table 3 and 4 are quite close to each other.

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