Evolving traversable wormholes satisfying the energy conditions in the presence of pole dark energy

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We consider the evolution of traversable wormhole geometries in the inflationary, radiation- and matter-dominated eras, and dynamic wormholes with a traceless energy-momentum tensor (EMT), within the recently proposed pole dark energy model. We show that the evolving radiation- and matter-dominated wormhole spacetimes satisfy the null energy condition (NEC), but possess negative energy densities at late times, thus violating the weak energy condition (WEC) in this specific domain. However, we demonstrate with a specific example that the traceless EMT evolving wormholes, supported by conformally invariant massless fields, in principle, could satisfy the WEC, and consequently the NEC, at all times and for all values of the radial coordinate. Thus, one may imagine a scenario in which these geometries originate in the Planckian era through quantum gravitational processes. Inflation could then provide a natural mechanism for the enlargement of these Planckian wormholes, where their FLRW background evolution is governed by pole dark energy. For the first time in the literature, specific dynamical 4-dimensional solutions are presented that satisfy the NEC and WEC everywhere and everywhen.

I. INTRODUCTION

General relativistic traversable wormholes as cosmic compact objects, and theoretically engineered as hypothetical short-cuts in spacetime [1, 2], are threaded and sustained by exotic matter, which is a fluid that violates the null energy condition (NEC). While an extensive variety of wormhole structures have been explored in the contexts of general relativity and its alternative theories from different aspects [3–20], evolving wormholes under the effect of cosmic fluids, such as dark energy and radiation fields, are a relatively outstanding interesting topic. One way to study this subject is to embed a wormhole in a Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which permits the geometry to evolve in a cosmological background [21–36]. A further advantage of these evolving wormholes, as compared to their static counterparts, is their ability to satisfy the energy conditions in arbitrary finite intervals of time [37, 38].

The line element of an evolving wormhole, used throughout this work, is given by

$$ds^2 = -e^{2\Phi(r)} dt^2 + a^2(t) \left[ \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega^2 \right],$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the linear element of the unit sphere, and $\Phi(r)$, $b(r)$ and $a(t)$ are the redshift and shape functions and the scale factor, respectively. In order to describe a wormhole, the following conditions need to be satisfied: $b(r_0) = r_0$, $1 - b(r)/r \geq 0$ and $b(r) - b(r)/r < 0$, where $r_0$ is the wormhole throat, which represents a minimum radius in the wormhole spacetime [1]. The last inequality translates the flaring-out condition, and through the Einstein field equations, it imposes the violation of the NEC [1–4]. As the violation of the energy conditions is a somewhat problematic issue, it is important to minimize these violations [30, 39–44].

In this paper, we study the evolution of traversable wormholes in a FLRW background within the recently proposed pole dark energy model [45]. In this model, used to explain dark energy, the Lagrangian is the summation of the potential $V$ and a kinetic term of the form $-k (\nabla \sigma)^2 / 2a^2$ with a pole of order $p$ and residue $k$ at $\sigma = 0$, and thus, the $p = 2$ and $V = 0$ case corresponds to a minimal k-essence model up to the first order of approximation [46]. The kinetic term can be transformed to a canonical scalar field form, where the resultant transformed Lagrangian of the model could give rise to an observationally viable dark energy equation of state evolution, given by $\omega(z) < -0.9$, an outcome which occurs even for transformed potentials $V(\phi)$ with the forms that could not normally produce a reliable behavior for the dark energy equation of state [45]. Due to their quantum stability and attractor features, these models with kinetic terms including a pole have been employed for studying inflation. A multipole dark energy model has also been proposed [47]. Here, we explore the possibility that evolving wormhole geometries may be supported by this model, in a manner analogous to more standard equations of state [48–54]. Furthermore, we explore the energy conditions for matter threading these traversable wormhole geometries.

The paper is outlined in the following manner: In Sec.
II, we present the action and the field equations of the pole dark energy model. In Sec. III, we analyse evolving wormholes in the inflationary, radiation- and matter-dominated eras, as well as wormholes with a traceless EMT, and explore the validity of the null and weak energy conditions for the solutions obtained. Finally, in Sec. IV, we summarize and discuss our results.

II. ACTION AND FIELD EQUATIONS

The action of the pole dark energy model is written as

\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} + \mathcal{L}_\sigma + \mathcal{L}_m \right), \tag{2} \]

where \( R \) is the scalar curvature, \( \kappa \) is related to Newton’s constant, \( \mathcal{L}_m \) is the matter Lagrangian density and the Lagrangian of the scalar field \( \sigma \) is given by \[ \tag{3} \]
\[ \mathcal{L}_\sigma = -\frac{1}{2} \frac{k}{\sigma^p} \nabla_\mu \sigma \nabla^\mu \sigma - V(\sigma), \]

in which the pole resides at \( \sigma = 0 \) and has a residue \( k \) and order \( p \). Varying the action (2) with respect to the metric \( g_{\mu\nu} \) and scalar field \( \sigma \), yields the field equations

\[ G_{\mu\nu} = \kappa \left( T_{\mu\nu}^\sigma + T_{\mu\nu}^m \right), \tag{4} \]

\[ 0 = \frac{1}{\sigma^{p+1}} \nabla_\sigma \nabla^\sigma \sigma - \frac{k}{\sigma^p} \nabla_\sigma \nabla^\rho \sigma + \frac{dV(\sigma)}{d\sigma}, \tag{5} \]

respectively, where

\[ T_{\mu\nu}^\sigma = \frac{k}{\sigma^p} \nabla_\mu \sigma \nabla_\nu \sigma \]

\[ - \frac{1}{2} \frac{k}{\sigma^{p+1}} \nabla_\sigma \nabla^\rho \sigma + \frac{dV(\sigma)}{d\sigma} g_{\mu\nu}, \]

is the scalar field energy-momentum tensor (EMT), \( G_{\mu\nu} \) is the Einstein tensor and \( T_{\mu\nu}^m \) is the matter EMT. For the wormhole solutions considered here, the matter EMT is given by \[ \tag{6} \]
\[ T_{\mu\nu}^m = \rho u^\mu u^\nu - \tau n^\mu n^\nu + p(n^\mu n^\rho + n^\rho n^\nu), \]

where \( \rho \) is the energy density, \( \tau \) is the radial tension (which is equivalent to a negative radial pressure), i.e., \( \tau = -\rho_r \), \( p \) is the tangential pressure and \( u^\mu \) and \( n^\mu \) are the unit timelike and spacelike vectors, respectively \[ \tag{7} \]
\[ T_{\mu\nu}^\sigma = \text{diag}(-\rho, -\tau, p, p). \]

Note that the kinetic term in Eq. (3) can be transformed to a canonical form \( -\left( \nabla \phi \right)^2 / 2 \) via \[ \tag{8} \]
\[ \sigma = \begin{cases} (2-p)^{2/(2-p)} \phi^{2/(2-p)} & \text{for } p \neq 2, \\ e^{\pm \phi / \sqrt{\tilde{k}}} & \text{for } p = 2, \end{cases} \]

With this transformed canonical Lagrangian of the scalar field in hand, the field equations are given by

\[ G_{\mu\nu} = \kappa \left( T_{\mu\nu}^\sigma + T_{\mu\nu}^m \right), \tag{9} \]

\[ 0 = -\nabla_\sigma \nabla^\rho \phi + \frac{dV(\phi)}{d\phi}, \tag{10} \]

where \( T_{\mu\nu}^\sigma = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \phi \nabla^\sigma \phi - V(\phi) g_{\mu\nu}, \) with \( V(\phi) = V(\sigma(\phi)) \), and \( \sigma(\phi) \) can be read from Eq. (6).

Eq. (7) may be expressed as the following effective Einstein field equation, \( G_{\mu\nu} = \kappa T_{\mu\nu}^{\text{eff}} \), where the effective EMT is given by \[ T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^\sigma + T_{\mu\nu}^m. \]

The necessary condition to have a wormhole geometry is the violation of the generalized NEC \[ \tag{11} \]
\[ \text{NEC}, \]

i.e., \( T_{\mu\nu}^{\text{eff}} k^\mu k^\nu < 0 \). Indeed, one may, in principle, impose that the matter EMT satisfies the NEC, i.e., \( T_{\mu\nu}^{\text{eff}} k^\mu k^\nu \geq 0 \) and thus, the pole dark energy plays the role of the exotic matter in order to support the geometry. More specifically, taken into account the above considerations, the condition \( T_{\mu\nu}^{\text{eff}} k^\mu k^\nu < 0 \) implies \( 0 \leq T_{\mu\nu}^m k^\mu k^\nu < -T_{\mu\nu}^{\text{eff}} k^\mu k^\nu \). We show below that this is indeed possible, and consequently we construct specific dynamical 4-dimensional solutions that satisfy the NEC everywhere and everywhere.

It is also interesting to note the role played by the scale factor, given in the line element of an evolving geometry \[ \tag{12} \]
\[ ds^2 = dz^2 + dr^2 + r^2 d\varphi^2, \]

Here, the wormhole slice is given by the following metric

\[ ds^2 = \frac{a^2(t) dr^2}{1 - b(r)/r} + a^2(t) r^2 d\varphi^2, \tag{13} \]

and confronting the coefficients of \( d\varphi^2 \), provides the following relations

\[ \hat{r} = a(t) r \big|_{t=\text{const}}, \tag{14} \]

\[ d\varphi^2 = a^2(t) d\varphi^2 \big|_{t=\text{const}}, \tag{15} \]

We emphasize, in particular, that when considering derivatives, that Eqs. (11) and (12) do not represent a “coordinate transformation”, but rather a “rescaling” of the \( r \) coordinate on each \( t = \text{constant slice} \).

With respect to the \( z, \hat{r}, \varphi \) coordinates, the “wormhole” form of the metric will be preserved if the metric on the embedded slice has the form

\[ ds^2 = \frac{d\hat{r}^2}{1 - b(\hat{r})/\hat{r}} + \hat{r}^2 d\varphi^2, \tag{16} \]

where \( b(\hat{r}) \) has a minimum at some \( \hat{b}(\hat{r}_0) = \hat{r}_0 \). Eq. (10) can be rewritten in the form of Eq. (16) by using Eqs. (11) and (12) and \( b(\hat{r}) = a(t) b(r) \). The evolving wormhole will have the same overall size and shape relative to the \( z, \hat{r}, \varphi \) coordinate system, as the initial wormhole had relative to the initial \( z, r, \varphi \) embedding space coordinate system. This is due to the fact that the embedding space corresponds to \( z \), \( r \) coordinates which “scale” with time (each embedding space corresponds to a particular value of \( t = \text{constant} \)).
Following the embedding procedure outlined in Ref. [1], and using Eqs. (9) and (13), one deduces that
\[
\frac{dz}{dr} = \pm \left( \frac{\bar{r}}{b(\bar{r})} - 1 \right)^{-1/2} \frac{d\bar{z}}{d\bar{r}}. \tag{14}
\]
Eq. (14) implies
\[
\bar{z}(\bar{r}) = \pm \int \left( \frac{d\bar{r}}{(\bar{r}/b(\bar{r}) - 1)^{1/2}} \right) = \pm a(t) \int \frac{\bar{r}}{r - b} - 1 \right)^{-1/2} \, dr \\
= \pm a(t) \bar{z}(r). \tag{15}
\]

Thus, taking into account Eqs. (12) and (15), we verify that the relation between the embedding space at any time \( t \) and the initial embedding space at \( t = 0 \) is given by
\[
ds^2 = d\bar{z}^2 + r^2 d\varphi^2 = a^2(t) \left( d\bar{z}^2 + dr^2 + r^2 d\varphi^2 \right). \tag{16}
\]

Relative to the \( \bar{z}, \bar{r}, \varphi \) coordinate system the wormhole maintains the same size, as the scaling of the embedding space compensates for the evolution of the wormhole. However, the wormhole will change size relative to the initial \( t = 0 \) embedding space.

Writing the analog of the "flaring out condition" [1] for the evolving wormhole we have \( d^2\bar{r}(\bar{z})/d\bar{z}^2 > 0 \) at or near the throat. Thus, taking into account the above expressions, we have
\[
d\frac{d^2\bar{r}(\bar{z})}{d\bar{z}^2} = \frac{1}{a(t)} \frac{b - b' r}{2b^2} = \frac{1}{a(t)} \frac{d^2 r(z)}{dz^2} > 0, \tag{17}
\]
at or near the throat. We also deduce the expressions \( \bar{b}'(\bar{r}) = \bar{db}/d\bar{r} = \bar{b}'(r) = db/dr \), so that one may rewrite the right-hand side of Eq. (17) relative to the barred coordinates as
\[
d\frac{d^2\bar{r}(\bar{z})}{d\bar{z}^2} = \left( \frac{\bar{b} - \bar{b}'(\bar{r})}{2b^2} \right) > 0, \tag{18}
\]
at or near the throat. One verifies that using the barred coordinates, the flaring out condition Eq. (18), has the same form as for the static wormhole.

In this paper, we consider a specific class of wormhole solutions with a constant redshift function, \( \Phi = \text{const} \). Using the metric (1), the gravitational field equations (7) provide
\[
\rho(t, r) = \rho_b(t) - \rho_\psi(t) + \frac{b'}{r^2 a^2}, \tag{19}
\]
\[
\tau(t, r) = \tau_b(t) - \tau_\psi(t) + \frac{b}{r^3 a^2}, \tag{20}
\]
\[
p(t, r) = -\tau_b(t) + \tau_\psi(t) - \frac{b'}{2r^2 a^2} + \frac{b}{2r^3 a^2}, \tag{21}
\]
where \( \rho_b(t) = 3H^2 \), \( \tau_b(t) = H^2 + 2\dot{a}/a \), \( \rho_\psi(t) = \dot{\phi}^2/2 + V(\phi) \), and \( \tau_\psi(t) = -\dot{\phi}^2/2 + V(\phi) \), in which \( H = \dot{a}/a \).

Here, the overdot and prime denote derivatives with respect to \( t \) and \( r \), respectively. For notational simplicity, we consider \( \kappa = 1 \). Note that if one fixes \( a \) to unity and excludes the background evolution and the dark energy contribution, we recover the well-known equations of motion of the Morris-Thorne wormhole [1].

From Eq. (8), the scalar field equation of motion is given by \( \ddot{\phi} + 3H \dot{\phi} + \ddot{V}/\phi = 0 \), with \( \phi = \phi(t) \). In order to solve this equation numerically, we re-write it in terms of dimensionless functions of \( a \). To this effect, we use the following definitions: \( \ddot{\phi} = \dot{a}\dot{\phi}'(a) + a^2 \phi''(a) \), \( \ddot{V} = \dot{a}\phi'(a) \), \( \ddot{a} = Ha + Ha \), \( H = \dot{H}(a) \), and \( \dot{a} = Ha \), where here the prime denotes a derivative with respect to the scale factor. We also define \( U = V/3H^2 \) and \( E = H/\dot{a} \), where \( H_0 \) is the present value of the Hubble parameter. Thus, we obtain the following differential equation:
\[
\phi''(a)a^2 E^2(a) + \phi'(a)aE(a) [4E(a) + aE'(a)] + 3\frac{dU}{d\phi} = 0. \tag{22}
\]

In addition to this, Eqs. (19)-(21) can be re-expressed as
\[
\frac{\rho}{3H_0^2} = \frac{b'(r)}{3a^2 H_0^2 r^2} - \frac{1}{6} a^2 E^2(a) \phi'^2(a) \\
- \frac{\tau}{3H_0^2} = \frac{b(r)}{3a^2 H_0^2 r^2} + \frac{1}{6} a^2 E^2(a) \phi'^2(a) \\
P \frac{3H_0^2} = - \frac{b'(r)}{6a^2 H_0^2 r^2} + \frac{b(r)}{6a^2 H_0^2 r^3} - \frac{1}{6} a^2 E^2(a) \phi'^2(a) \\
+ U(\phi) - \frac{2}{3} aE(a) E'(a) - E^2(a), \tag{23-25}
\]

To solve Eq. (22) for \( \phi(a) \), we have to deduce \( E \). By applying a barotropic equation of state \( \tau_b = -\omega_0 \rho_b \) for the background, we find \( E = a^{-3(\omega_0 + 1)/2} \). For the inflationary, radiation- and matter-dominated eras, the parameter \( \omega_0 \) is equal to \(-1, 1/3 \) and 0, respectively.

Relative to the potential \( V(\phi) \), from Eq. (6), we see that a power law potential \( V \sim \sigma^n \) transforms to another power law potential of the form \( \phi^{2n/(2-n)} \). For \( p < 2 \), the signs of the initial and transformed potential powers are the same while the transformed one is steeper and so is less interesting for inflation or dark energy close to a cosmological constant like behavior. For \( p > 2 \), the signs flip, i.e., a monomial potential is transformed to an inverse power law one and vice versa. This is significant as for canonical scalar fields, a monomial potential causes a thawing dark energy scenario which begins with a cosmological constant like state at high redshift and deviates from this as it evolves at later times. On the other hand, an inverse power law potential exhibits freezing dark energy behavior at early times, i.e., it could possess a dynamical attractor behavior with a constant equation of state parameter \( \omega_0 = -\sigma \phi_\psi \rho_b(\phi) \) and then advances towards a cosmological constant behavior at later times [55]. Therefore, the pole dark energy model can produce
the features of freezing, possibly attractor, fields from monomial potentials and thawing fields from an initial inverse power law potential. Here, we use the power law potential for $\sigma$ with $n > 0$ and $p > 2$ which causes an inverse power law potential for $\phi$ of the form $V \sim \phi^{-\alpha}$ in which $\alpha = 2n/(p - 2)$. Note that any value of $\alpha$ could be obtained by different sets of $n(>0)$ and $p(>2)$.

In order to study the behavior of the EMT components $\rho$, $\tau$ and $p$ given by Eqs. (19)-(21) and the corresponding energy conditions, we have to choose a suitable shape function $b(r)$ for our wormhole structure. For this purpose, we choose $b(r) = r_0 (r_0/r)^q$ where $r_0$ is the wormhole throat and $b(r_0) = r_0$. As mentioned in the Introduction, the shape function satisfies $1 - b(r)/r \geq 0$ and the flaring-out condition $b'(r) - b(r)/r < 0$. These conditions lead to $1 - b(r)/r = 1 - (r_0/r)^{1+q} \geq 0$ and $b'(r) - b(r)/r = -(1 + q) (r_0/r)^{1+q} < 0$, respectively. As one can see, both of them are satisfied provided $q > -1$ (Note that $r_0/r \leq 1$). For specific values of $q$ used in the following analysis, we show the satisfaction of these conditions for the shape function in Fig. 1.

**III. EVOLVING TRAVERSABLE WORMHOLES AND ENERGY CONDITIONS**

In this section, we study the evolution of traversable wormholes in the inflationary, radiation– and matter–dominated eras, as well as evolving wormholes with a traceless EMT, in the presence of pole dark energy. We will also explore the null and weak energy conditions for our solutions. The weak energy condition (WEC) is expressed in terms of the energy density $\rho$, radial tension $\tau$ and tangential pressure $p$ as $\rho \geq 0$, $\rho - \tau \geq 0$ and $\rho + p \geq 0$, respectively. The last two inequalities, i.e., $\rho - \tau \geq 0$ and $\rho + p \geq 0$ correspond to the NEC.

In the following, we consider $U(\phi) = \phi^{-\alpha}$ and $b(r) = r_0 (r_0/r)^q$. Then, with $E = a^{-3(\omega_\phi+1)/2}$ which arises from the background equation of state $\tau_0 = -\omega_\phi \rho_0$ and using the dimensionless definitions, Eqs. (19)-(21) lead to

$$\frac{\rho - \tau}{3H_0^2} = \frac{6 - a^2 \phi^2(a)}{a^{3(\omega_\phi+1)}} - \phi^{-\alpha}(a) - \frac{qr_0 (\frac{r_0}{r})^q}{3a^2H_0^2r^3},$$

which immediately leads to $\rho - \tau < 0$ for $q > -1$, which satisfies the flaring-out condition at the throat. Thus, both the NEC and the WEC are violated, if one seeks for a traversable wormhole in this region. It is, however, remarkable that $\omega_\phi$ is physically viable for this case, i.e., $\omega_\phi(z) < -0.9$, with some $\alpha$ values less than unity, according to our numerical analysis.

**A. Inflation era**

For the inflationary era with $\omega_\phi = -1$, Eq. (27) reduces to

$$\frac{\rho - \tau}{3H_0^2} = \frac{-a^2}{3} \phi^2(a) - \frac{r_0 (\frac{r_0}{r})^q (q + 1)}{3a^2H_0^2r^3},$$

which is depicted in Fig. 2(a). It is physically viable since $\omega_\phi(z) < -0.9$. In Fig. 3, the
FIG. 2. The behavior of $\omega_\phi$ vs $z$ for the radiation-dominated era, the matter-dominated era and the traceless EMT case. Note that the horizontal axis is logarithmic. See the text for more details.

FIG. 3. The behaviors of $\rho$, $\rho - \tau$ and $\rho + p$, respectively, versus $z$ for different values of $r$ in the radiation-dominated era ($\omega_b = 1/3$) with $U(\phi) = \phi^{0.1}$ and $q = 2$. Note that both the horizontal and vertical axes are logarithmic. The $\gamma$-shaped part in subfigure (a) shows the point at which $\rho$ meets zero and changes its sign.

FIG. 4. The behaviors of $\rho$, $\rho - \tau$ and $\rho + p$, respectively, versus $z$ for different values of $r$ in the matter-dominated era ($\omega_b = 0$) with $U(\phi) = \phi^{0.2}$ and $q = 0.5$. Both horizontal and vertical axes are logarithmic. The $\gamma$-shaped part in subfigure (a) shows the point at which $\rho$ attains zero and consequently changes sign.

behaviors of $\rho$, $\rho - \tau$ and $\rho + p$ versus $z$ for different values of $r$ are shown where $q = 2$, i.e., $b(r) = r_0 (r_0/r)^2$, which satisfies all required conditions. As one can see, at earlier times, the wormhole geometry satisfies the WEC. As time passes, $\rho$, $\rho - \tau$ and $\rho + p$ decrease. This occurs for the throat as well as other wormhole radii. Eventually, at late times, the energy density $\rho$ becomes negative, as depicted in Fig. 3(a), whereas $\rho - \tau$ and $\rho + p$ remain positive (see Figs. 3(b) and 3(c)). Thus, the NEC is satisfied at late times, contrary to the WEC. Consequently, the NEC is satisfied by these evolving traversable wormhole solutions at all times and for all values of $r$, including the wormhole throat. Note that the energy density of the throat becomes negative earlier than other radii, as is transparent from Fig. 3(a).

C. Matter-dominated era

The behavior of $\omega_\phi$ versus $z$ in the matter-dominated era where $\omega_b = 0$, $E = a^{-3/2}$ and $a(t) \propto t^{2/3}$ with $U(\phi) = \phi^{0.2}$ is depicted in Fig. 2(b). It is also physically
viable since \( \omega_{\phi}(z) < -0.9 \). In Fig. 4, the behaviors of \( \rho, \rho - \tau \) and \( \rho + p \) with respect to \( z \) for different values of \( r \) are shown for \( q = 0.5 \), i.e., \( b(r) = r_0(r_0/r)^{0.5} \) which satisfies all the required conditions. As exhibited in Fig. 4, the wormhole geometry satisfies the WEC at earlier times, and as it evolves in time, the quantities \( \rho, \rho - \tau \) and \( \rho + p \) decrease. The throat and the other wormhole radii behave in this manner, and at late times, the energy density \( \rho \) becomes negative. However, the quantities \( \rho - \tau \) and \( \rho + p \) remain positive at late times (Figs. 4(b) and 4(c)). Therefore, the NEC is satisfied by these dynamical wormhole solutions at all times for all values of \( r \), including the wormhole throat, as in the previous example. However, the WEC is violated only at late times, as depicted by Fig. 4(a). In addition to this, the energy density of the throat becomes negative earlier than for regions of larger radii.

**D. Wormholes with traceless EMT**

Considering the traceless EMT, i.e., \( -\rho - \tau + 2p = 0 \), we obtain from Eqs. (23)-(25) that

\[
2aE(a)E'(a) + \frac{1}{3}E(a)^2 \left[ 12 + a^2\phi'(a)^2 \right] - 4U(\phi) + \frac{2}{3a^2H_0^2} \frac{b'(r)}{r^2} = 0.
\]

The behavior of \( \omega_{\phi} \) versus \( z \) with \( U(\phi) = \phi^{0.1} \) is depicted in Fig. 2(c). It is physically viable since \( \omega_{\phi}(z) < -0.9 \). In Fig. 5, the behaviors of \( \rho, \rho - \tau \) and \( \rho + p \), with respect to \( z \) for different values of \( r \) are shown. Note that \( \rho \) is independent of the radial coordinate \( r \) for the \( q = 0 \) case, as can be seen from Eq. (26). Figure 5 shows that \( \rho, \rho - \tau \) and \( \rho + p \) decrease as time evolves. However, they remain positive at all times and consequently the NEC and WEC are always satisfied. This occurs for the wormhole throat as well as other wormhole radii. It is interesting to note that at a specified time/redshift, the quantity \( \rho - \tau \) increases for increasing values of the radius, and the minimum value corresponds to the throat, as depicted by Fig. 5(b). On the other hand, \( \rho + p \) decreases for increasing values of the radius, and has a maximum at the throat, as is depicted by Fig. 5(c).

**IV. DISCUSSION AND CONCLUSION**

In this work, using the recently proposed pole dark energy model, we explored the evolution of traversable wormhole geometries in a FLRW background, in particular, in the inflationary, radiation– and matter–dominated eras. In addition to these solutions, we also analysed dynamical wormholes with a traceless EMT. A central theme in this work was the study of the energy conditions, and it was shown explicitly that the evolving radiation– and matter–dominated wormhole spacetimes satisfy the NEC, but possess negative energy densities at late times, thus violating the WEC in this specific domain. Nevertheless, inflating traversable wormhole geometries always violate both the NEC and WEC. On the other hand, it was shown for a specific example that the traceless EMT evolving wormholes satisfies both the NEC and WEC at all times.

These solutions can be thought to be embedded in a scenario where inflation provides a natural mechanism for the enlargement of submicroscopic Planckian wormholes, that originated via quantum gravitational processes, to macroscopic size. Their subsequent evolution is governed by pole dark energy. In fact, it was shown that Lorentzian wormholes in a flat de Sitter back-
ground could serve this purpose [21]. Subsequent work on evolving wormholes, conformally related to static Morris-Thorne wormhole geometries were also found to exist for finite intervals of time, with the EMT satisfying the WEC in specific ranges [37, 38]. The role of extra compact decaying dimensions have also been dealt with in the context of simple models involving an exponential inflation and a Kaluza–Klein type inflationary scenario [38].

Finally, to the best of our knowledge, the evolving traversable wormhole geometries considered in this work, are the first found in the literature, in four-dimensions, to present solutions in a cosmological background constructed by normal matter. More specifically, the NEC and WEC are satisfied everywhere and everywhen. Thus, these novel results motivate further work in this interesting branch of research. Work along these lines is presently underway.

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[1] M. S. Morris and K. S. Thorne, Wormholes in Spacetime and Their Use for Interstellar Travel: A Tool For Teaching General Relativity, Am. J. Phys. 56, 395 (1988).
[2] M. S. Morris, K. S. Thorne and U. Yurtsever, Wormholes, Time Machines, and the Weak Energy Condition, Phys. Rev. Lett. 61, 1446 (1988).
[3] M. Visser, Lorentzian wormholes: From Einstein to Hawking, AIP press, New York (1995).
[4] F. S. N. Lobo, Wormholes, Warp Drives and Energy Conditions, Springer, Switzerland (2017).
[5] R. Korolev, F. S. N. Lobo and S. V. Sushkov, General constraints on Horndeski wormhole throats, Phys. Rev. D 101, 124057 (2020) [arXiv:2004.12382].
[6] G. Antoniou, A. Bakopoulos, P. Kanti, B. Kleihaus and J. Kunz, Novel Einstein-scalar-Gauss-Bonnet wormholes without exotic matter, Phys. Rev. D 101, 024033 (2020) [arXiv:1904.13091].
[7] T. Tangphati, A. Chatrabhuti, D. Samart and P. Channuie, Thin-shell wormholes in de Rham-Gabadadze-Tolley massive gravity, Eur. Phys. J. C 80, 722 (2020) [arXiv:1912.12208].
[8] E. Papantonopoulos and C. Vlachos, Wormhole solutions in modified Brans-Dicke theory, Phys. Rev. D 101, 064025 (2020) [arXiv:1912.04005].
[9] N. Godani and G. C. Samanta, Traversable Wormholes in $R + \alpha R^n$ Gravity, Eur. Phys. J. C 80, 30 (2020) [arXiv:2001.00010].
[10] I. Fayyaz and M. F. Shamir, Wormhole Structures in Logarithmic-Corrected $R^2$ Gravity, Eur. Phys. J. C 80, 430 (2020) [arXiv:2005.10023].
[11] A. Restuccia and F. Tello-Ortiz, A new class of f(R)-gravity model with wormhole solutions and cosmological properties, Eur. Phys. J. C 80, 580 (2020).
[12] A. Banerjee, M. K. Jasim and S. G. Ghosh, Traversable wormholes in $f(R,T)$ gravity satisfying the null energy condition with isotropic pressure, arXiv:2003.01545.
[13] B. Lazov, P. Nedkova and S. Yazadjiev, Uniqueness theorem for static phantom wormholes in Einstein-Maxwell-dilaton theory, Phys. Lett. B 778, 408 (2018) [arXiv:1711.00290].
[14] A. A. Kirillov and E. P. Savelova, Wormhole as a possible accelerator of high-energy cosmic-ray particles, Eur. Phys. J. C 80, 45 (2020) [arXiv:1902.05742].
[15] D. Bak, C. Kim and S. H. Yi, Experimental Probes of Traversable Wormholes, JHEP 12, 005 (2019) [arXiv:1907.13465].
[16] Z. Xu, M. Tang, G. Cao and S. N. Zhang, Possibility of traversable wormhole formation in the dark matter halo with anisotropic pressure, Eur. Phys. J. C 80, 70 (2020).
[17] K. Jusufi, P. Channuie and M. Jamil, Traversable Wormholes Supported by GUP Corrected Casimir Energy, Eur. Phys. J. C 80, 127 (2020) [arXiv:2002.01341].
[18] T. Berry, F. S. N. Lobo, A. Simpson and M. Visser, Thin-shell traversable wormhole crafted from a regular black hole with asymptotically Minkowski core, Phys. Rev. D 102, 064054 (2020) [arXiv:2008.07046].
[19] S. Fallows and S. F. Ross, Making near-extremal wormholes traversable, arXiv:2008.07946.
[20] J. Maldacena and A. Milekhin, Humanly traversable wormholes, arXiv:2008.06618.
[21] T. A. Roman, Inflating Lorentzian wormholes, Phys. Rev. D 47, 1370 (1993) [arXiv:gr-qc/9211012].
[22] L. A. Anchordoqui, D. F. Torres, M. L. Trobo and S. E. Perez Bergliaffa, Evolving wormhole geometries, Phys. Rev. D 57, 829 (1998), [arXiv:gr-qc/9710026].
[23] A. V. B. Arellano and F. S. N. Lobo, Evolving wormhole geometries within nonlinear electrodynamics, Class. Quant. Grav. 23, 5811 (2006), [arXiv:gr-qc/0608003].
[24] E. Ebrahimi and N. Riazi, (n+1)-Dimensional Lorentzian Wormholes in an Expanding Cosmological Background, Astrophys. Space Sci. 321, 217 (2009), [arXiv:0905.3882].
[25] E. Ebrahimi and N. Riazi, Expanding (n + 1)-Dimensional Wormhole Solutions in Brans-Dicke Cosmology, Phys. Rev. D 81, 024036 (2010), [arXiv:0910.4116].
[26] M. R. Bordbar and N. Riazi, Time-dependent wormhole in an inhomogeneous spherically symmetric space-time with a cosmological constant, Astrophys. Space Sci. 331, 315 (2011).
[27] S. N. Sajadi and N. Riazi, Expanding Lorentzian wormholes in $R^2$ gravity, Prog. Theor. Phys. 126, 753 (2011).
[28] M. Cataldo, S. Bahamonde and F. Arostica, (n + 1)-dimensional Lorentzian evolving wormholes supported by polytropic matter, Eur. Phys. J. C 73, 2517 (2013).
[29] M. R. Setare and A. Sepehri, Role of higher-dimensional evolving wormholes in the formation of a big rip singularity, Phys. Rev. D 91, 063523 (2015), [arXiv:1612.05077].

[30] M. Kord Zangeneh, F. S. N. Lobo and N. Riazi, Higher-dimensional evolving wormholes satisfying the null energy condition, Phys. Rev. D 90, 024072 (2014), [arXiv:1406.5703].

[31] M. Cataldo, P. Labrana, S. del Campo, J. Crisostomo and P. Salgado, Evolving Lorentzian wormholes supported by phantom matter with constant state parameters, Phys. Rev. D 78, 104006 (2008) [arXiv:0810.2715].

[32] M. Cataldo, S. del Campo, P. Minning and P. Salgado, Evolving Lorentzian wormholes supported by phantom matter and cosmological constant, Phys. Rev. D 79, 024005 (2009) [arXiv:0812.4436].

[33] M. Cataldo and S. del Campo, Two-fluid evolving Lorentzian wormholes, Phys. Rev. D 85, 104010 (2012) [arXiv:1204.0753].

[34] T. Harada, H. Maeda and B. J. Carr, Self-similar cosmological solutions with dark energy. I. Formulation and asymptotic analysis, Phys. Rev. D 77, 024022 (2008) [arXiv:0707.0528].

[35] H. Maeda, T. Harada and B. J. Carr, Self-similar cosmological solutions with dark energy. II. Black holes, naked singularities and wormholes, Phys. Rev. D 77, 024023 (2008) [arXiv:0707.0530].

[36] H. Maeda, T. Harada and B. J. Carr, Cosmological wormholes, Phys. Rev. D 79, 044034 (2009) [arXiv:0901.1153].

[37] S. Kar, Evolving wormholes and the weak energy condition, Phys. Rev. D 49, 862 (1994).

[38] S. Kar and D. Sahdev, Evolving Lorentzian Wormholes, Phys. Rev. D 53, 722 (1996), [arXiv:gr-qc/9506094].

[39] M. R. Mehdizadeh, M. Kord Zangeneh and F. S. N. Lobo, Einstein-Gauss-Bonnet traversable wormholes satisfying the weak energy condition, Phys. Rev. D 91, 084004 (2015), [arXiv:1501.04773].

[40] M. R. Mehdizadeh, M. Kord Zangeneh and F. S. N. Lobo, Higher-dimensional thin-shell wormholes in third-order Lovelock gravity, Phys. Rev. D 92, 044022 (2015), [arXiv:1506.03427].

[41] M. Kord Zangeneh, F. S. N. Lobo and M. H. Dehghani, Traversable wormholes satisfying the weak energy condition in third-order Lovelock gravity, Phys. Rev. D 92, 124049 (2015), [arXiv:1510.07089].

[42] T. Harko, F. S. N. Lobo, M. K. Mak and S. V. Sushkov, Modified-gravity wormholes without exotic matter, Phys. Rev. D 87, 067504 (2013), [arXiv:1301.6878].

[43] S. Capozziello, F. S. N. Lobo and J. P. Mimoso, Energy conditions in modified gravity, Phys. Lett. B 730, 280 (2014), [arXiv:1312.0784].

[44] S. Capozziello, F. S. N. Lobo and J. P. Mimoso, Generalized energy conditions in Extended Theories of Gravity, Phys. Rev. D 91, 124019 (2015), [arXiv:1407.7293].

[45] E. V. Linder, Pole Dark Energy, Phys. Rev. D 101, 023506 (2020), [arXiv:1911.01060].

[46] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, k-Inflation, Phys. Lett. B 458, 209 (1999), [arXiv:hep-th/9904075].

[47] C.-J. Feng, X.-H. Zhai and X.-Z. Li, Multi-pole Dark Energy, arXiv:1912.10830.

[48] S. V. Sushkov, Wormholes supported by a phantom energy, Phys. Rev. D 71, 043520 (2005), [arXiv:gr-qc/0502084].

[49] F. S. N. Lobo, Phantom energy traversable wormholes, Phys. Rev. D 71, 084011 (2005), [arXiv:gr-qc/0502099].

[50] F. S. N. Lobo, Stability of phantom wormholes, Phys. Rev. D 71, 124022 (2005), [arXiv:gr-qc/0506001].

[51] F. S. N. Lobo, Chaplygin traversable wormholes, Phys. Rev. D 73, 064028 (2006), [arXiv:gr-qc/0511003].

[52] F. S. N. Lobo, Van der Waals quintessence stars, Phys. Rev. D 75, 024023 (2007), [arXiv:gr-qc/0610118].

[53] A. DeBenedictis, R. Garattini and F. S. N. Lobo, Phantom stars and topology change, Phys. Rev. D 78, 104003 (2008), [arXiv:0808.0839].

[54] F. S. N. Lobo, F. Parsaei and N. Riazi, New asymptotically flat phantom wormhole solutions, Phys. Rev. D 87, 084030 (2013), [arXiv:1212.5806].

[55] R. R. Caldwell and E. V. Linder, The Limits of Quintessence, Phys. Rev. Lett. 95, 141301 (2005).