CATEGORIES OF NETS

JOHN C. BAEZ, FABRIZIO GENOVESE, JADE MASTER, AND MICHAEL SHULMAN

ABSTRACT. We present a unified framework for Petri nets and various variants, such as pre-nets and Kock’s whole-grain Petri nets. Our framework is based on a less well-studied notion that we call Σ-nets, which allow finer control over whether tokens are treated using the collective or individual token philosophy. We describe three forms of execution semantics in which pre-nets generate strict monoidal categories, Σ-nets (including whole-grain Petri nets) generate symmetric strict monoidal categories, and Petri nets generate commutative monoidal categories, all by left adjoint functors. We also construct adjunctions relating these categories of nets to each other, in particular showing that all kinds of net can be embedded in the unifying category of Σ-nets, in a way that commutes coherently with their execution semantics.

1. INTRODUCTION

A Petri net is a seemingly simple thing:

\[
\begin{array}{c}
\text{It consists of “places” (drawn as circles) and “transitions” (drawn as boxes), with directed edges called “arcs” from places to transitions and from transitions to places. The idea is that when we use a Petri net, we place dots called “tokens” in the places, and then move them around using the transitions:}
\end{array}
\]

Thanks in part to their simplicity, Petri nets are widely used in computer science, chemistry, biology and other fields to model systems where entities interact and change state [18, 30].

Ever since the work of Meseguer and Montanari [28], parallels have been drawn between Petri nets and symmetric strict monoidal categories (SSMCs). Intuitively, a Petri net can be interpreted as a presentation of such a category, by using its places to generate a monoid of objects, and its transitions to generate the morphisms. An object in the SSMC represents a “marking” of the net—a given placement of tokens in it—while a morphism represents a “firing sequence”: a sequence of transitions that carry markings to other markings. One of the advantages of this “execution
semantics” for a net is that it can be compositionally interfaced to other structures using monoidal functors.

However, the apparent simplicity of Petri nets hides many subtleties. There are various ways to make the definition of Petri net precise. For example: is there a finite set of arcs from a given place to a given transition (and the other way around), or merely a natural number? If there is a finite set, is this set equipped with an ordering? Furthermore, what is a morphism between Petri nets? A wide variety of answers to these questions have been explored in the literature.

Different answers are good for different purposes. In the “individual token philosophy”, we allow a finite set of tokens in each place, and tokens have their own individual identity. In the “collective token philosophy”, we merely allow a natural number of tokens in each place, so it means nothing to switch two tokens in the same place [19].

Moreover, the idea of using SSMCs to represent net semantics, albeit intuitive, presents subtleties of its own. There has been a great deal of work on this subject [2, 8, 12, 17, 16, 27, 31, 32, 33]. Nevertheless, we still lack a general answer describing the relations between nets and SSMCs.

2. Dramatis personæ

Our goal is to bring some order to this menagerie. Our attitude is that though there may be multiple kinds of Petri net, each should freely generate a monoidal category of an appropriate sort, and these processes should be left adjoint functors. Furthermore, in practical applications we want this functor to be as easy to compute as possible [16]. Finally, the different categories of nets should be related to each other by functors that are coherent with known functors between categories of monoidal categories of different sorts.

More specifically, we will consider three kinds of monoidal categories, and three corresponding categories of nets:

\[
\begin{array}{c}
\text{StrMC} & \overset{\cong}{\longrightarrow} & \text{SSMC} & \overset{\cong}{\longrightarrow} & \text{CMC} \\
\downarrow & & \downarrow & & \downarrow \\
\text{PreNet} & \overset{\cong}{\longrightarrow} & \Sigma\text{-net} & \overset{\cong}{\longrightarrow} & \text{Petri}
\end{array}
\]

On the top row we have:

- **StrMC**, with strict monoidal categories as objects and strict monoidal functors as morphisms.
- **SSMC**, with symmetric strict monoidal categories as objects and strict symmetric monoidal functors as their morphisms. A **symmetric strict monoidal category** is a symmetric monoidal category whose monoidal structure is strictly associative and unital; its symmetry may not be the identity.
- **CMC**, with commutative monoidal categories as objects and strict symmetric monoidal functors as morphisms. A **commutative monoidal category** is a symmetric strict monoidal category where the symmetry is the identity.
These three kinds of monoidal categories are freely generated by three kinds of nets, on the bottom row of the diagram:

- **PreNet**, with pre-nets as objects. A pre-net consists of a set \( S \) of **places**, a set \( T \) of **transitions**, and functions \( T \stackrel{s,t}{\rightarrow} S^* \times S^* \), where \( S^* \) is the underlying set of the free monoid on \( S \). We describe the category \( \text{PreNet} \), and its adjunction with \( \text{StrMC} \), in Section 4. These ideas are due to Bruni, Meseguer, Montanari and Sassone [8].

- **Σ-net**, with Σ-nets as objects. A Σ-net consists of a set \( S \) and a discrete opfibration \( T \rightarrow PS \times PS^{\text{op}} \), where \( PS \) is the free symmetric strict monoidal category generated by a set of objects \( S \) and no generating morphisms. We describe the category \( \Sigma\text{-net} \) in Section 5, and its adjunction with \( \text{SSMC} \) in Theorem 7.3.

- **Petri**, with Petri nets as objects. A Petri net, as we will use the term, consists of a set \( S \), a set \( T \), and functions \( T \stackrel{s,t}{\rightarrow} N[S] \times N[S] \), where \( N[S] \) is the free commutative monoid on \( S \). We describe the category \( \text{Petri} \), and its adjunction with \( \text{CMC} \), in Section 3. This material can be found in some of our earlier work [2, 27].

These three notions of net obviously have a similar flavor. Their parallel relationships to the three notions of monoidal category is made even clearer when we note that regarded as discrete categories, \( S^* \) and \( N[S] \) are respectively the free monoidal category and the free commutative monoidal category generated by \( S \), and that every functor between discrete categories is a discrete opfibration.

Besides the three adjunctions between the categories on the top row and those on the bottom row, in which the left adjoints point upward, there are also adjunctions running horizontally across the diagram: adjoint pairs in the top row and bottom right and an adjoint triple in the bottom left, with left adjoints drawn above their right adjoints. In Section 7, we examine these adjunctions in detail.

Of particular importance are the right adjoint mapping Petri nets to Σ-nets, and the left adjoint mapping pre-nets to Σ-nets. We think of these as “embedding” the collective token world (Petri nets) and the individual token world (pre-nets) into the unifying context of Σ-nets. In the case of Petri nets, the functor is literally an embedding (i.e., fully faithful), and since it is a right adjoint it preserves all limits (though not all colimits). In the case of pre-nets, the functor is faithful but not full; but it is an equivalence onto a slice category of Σ-net, and preserves all colimits and all connected limits (such as pullbacks). These embeddings also respect the most common categorical semantics: in Section 7 we will show that the left adjoints \( \text{PreNet} \rightarrow \text{SSMC} \) and \( \text{Petri} \rightarrow \text{CMC} \) both factor through Σ-net.

The images of pre-nets and Petri nets in Σ-nets have a large intersection, consisting of those nets in which no places are ever duplicated in the inputs or outputs of any transition. These are the nets for which there is no difference between the individual and collective token philosophies. As we shall see, general Σ-nets allow more fine-grained control than either pre-nets or Petri nets: for example, some transitions may obey the individual token philosophy while others obey the collective token philosophy.

Our work is closely related to that of Kock [21]. He refers to Σ-nets as “digraphical species”, and sketches a proof, different from ours, that there is an adjunction relating them to \( \text{SSMC} \). But his focus is on a fourth notion of net: “whole-grain Petri nets”. He sketches a proof that these are the image of pre-nets inside Σ-net,
which we detail in Section 8 (so that in particular, whole-grain Petri nets also generate symmetric strict monoidal categories); but he has nothing to say about their relationship to Petri nets as traditionally conceived.

3. Petri Nets

Symmetric monoidal categories are a general algebraic framework to represent processes that can be performed in sequence and in parallel. Because Petri nets represent schematics for such processes, we expect them to freely generate symmetric monoidal categories. In fact they generate a special sort of symmetric monoidal categories: commutative ones.

Definition 3.1. Let Petri be the category where:

- An object is a Petri net: a pair of functions \( T \overset{s,t}{\rightarrow} N[S] \), where \( N[S] \) denotes the underlying set of the free commutative monoid on \( S \).
- A morphism from \( T_1 \overset{s_1,t_1}{\rightarrow} N[S_1] \) to \( T_2 \overset{s_2,t_2}{\rightarrow} N[S_2] \) is a pair of functions \( f: S_1 \rightarrow S_2, g: T_1 \rightarrow T_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
N[S_1] & \overset{N[f]}{\rightarrow} & N[S_1] \\
N[f] \downarrow & & \downarrow N[f] \\
N[S_2] & \overset{N[f]}{\leftarrow} & N[S_2]
\end{array}
\]

where \( N[f] \) denotes the unique monoid homomorphism extending \( f \).

Definition 3.2. A commutative monoidal category is just a commutative monoid object in \( \mathbf{Cat} \). Equivalently, it is a strict monoidal category \( (C, \otimes, I) \) such that for all objects \( a \) and \( b \) and morphisms \( f \) and \( g \) in \( C \)

\[ a \otimes b = b \otimes a \quad \text{and} \quad f \otimes g = g \otimes f. \]

A morphism of commutative monoidal categories is a strict monoidal functor. We write \( \mathbf{CMC} \) for the category of commutative monoidal categories and such morphisms between them.

A commutative monoidal category can be seen as a particularly strict sort of symmetric monoidal category. Ordinarily, symmetric monoidal categories are equipped with “symmetry” isomorphisms

\[ \sigma_{x,y}: x \otimes y \overset{\sim}{\rightarrow} y \otimes x \]

for every pair of objects \( x \) and \( y \). In a commutative monoidal category \( x \otimes y \) is equal to \( y \otimes x \), so we can — and henceforth will — make it symmetric by choosing \( \sigma_{x,y} \) to be the identity for all \( x \) and \( y \). Any morphism of commutative monoidal categories then becomes a strict symmetric monoidal functor.

The following adjunction shows that Petri nets are the right sort of generating data for commutative monoidal categories.

\[ \begin{array}{ccc}
\text{Petri} & \xleftarrow{F_{\text{Petri}}} & \mathbf{CMC} \\
\downarrow & & \downarrow \text{U}_{\text{Petri}} \\
\text{Petri} & \xrightarrow{F_{\text{Petri}}} & \mathbf{CMC}
\end{array} \]

Proposition 3.3. There is an adjunction
whose left adjoint sends a Petri net $P$ to the commutative monoidal category $F_{\text{Petri}}(P)$ where:

- **Objects** are markings of $P$, i.e., elements of the free commutative monoid on its set of places.
- **Morphisms** are generated inductively by the following rules:
  
  - for each place $s$ there is an identity $1_s : s \to s$
  
  - for each transition $\tau$ of $P$, there is a morphism going from its source to its target
  
  - for every pair of morphisms $f : x \to y$ and $f' : x' \to y'$, there is a morphism $f \otimes f' : x \otimes x' \to y \otimes y'$
  
  - for every pair of composable morphisms $f : x \to y$ and $g : y \to z$, there is a morphism $g \circ f : x \to z$

  and quotiented to satisfy the axioms of a commutative monoidal category.

**Proof.** This is a special case of [27, Theorem 5.1]. See also [2, Lemma 9]. $\Box$

As noted in Section 1, we view this construction as associating to each net a monoidal category of its “executions”. The objects of this category are markings that accord with the collective token philosophy: for instance, if $p$ and $q$ are places, the object $2p + 3q$ has two tokens on $p$ and three tokens on $q$, but no way to distinguish between the former two tokens or between the latter three. Similarly, the morphisms in this category are equivalence classes of firing sequences. This interpretation is particularly captivating if we represent morphisms in a monoidal category using string diagrams.

However, the equivalence relation on firing sequences that determines when two define the same morphism is very coarse when we take the commutative monoidal category freely generated by a Petri net. Indeed, if $f, g : x \to x$ are morphisms in a commutative monoidal category, the following sequence of equations holds:

$$
\frac{x \leftarrow f \rightarrow x}{x} = \frac{x \leftarrow f}{x} = \frac{\frac{x \leftarrow x}{x} \rightarrow x}{x} = \frac{x \leftarrow g}{x} \rightarrow f \rightarrow x
$$

These equations imply that given any two firing sequences $f$ and $g$ that start and end at some marking $x$, the commutative monoidal category cannot distinguish whether they act independently or whether $f$ acts on the tokens already processed by $g$. When $x$ is the tensor unit, the equations above hold in any symmetric monoidal category. But in a commutative monoidal category, the above equations hold for any object $x$.

4. **Pre-nets**

The shortcomings of commutative monoidal categories we presented in the last section are overcome by using symmetric monoidal categories where the symmetries are not necessarily identity morphisms. One popular approach first builds monoidal categories and then freely adds symmetries. In 1991 Joyal and Street [20] introduced “tensor schemes”, which can be used to describe free strict monoidal categories. In 2001, essentially the same idea was introduced by Bruni, Meseguer, Montanari and Sassone [8] under the name “pre-nets”. However, for these authors, the use of pre-nets to describe free strict monoidal categories was just the first stage of a procedure to obtain free symmetric strict monoidal categories. We recall this procedure now.
Definition 4.1. Let \( \text{PreNet} \) be the category where:

- An object is a **pre-net**: a pair of functions \( T \xrightarrow{s,t} S^* \), where \( S^* \) is the underlying set of the free monoid on \( S \).

- A morphism from \( T_1 \xrightarrow{s_1,t_1} S_1^* \) to \( T_2 \xrightarrow{s_2,t_2} S_2^* \) is a pair of functions \( f: S_1 \rightarrow S_2, g: T_1 \rightarrow T_2 \) such that the following diagram commutes, where \( f^* \) denotes the unique monoid homomorphism extending \( f \):

\[
\begin{array}{ccc}
S_1^* & \xrightarrow{s_1} & T_1 & \xrightarrow{t_1} & S_1^* \\
\downarrow{f^*} & & \downarrow{g} & & \downarrow{f^*} \\
S_2^* & \xleftarrow{s_2} & T_2 & \xleftarrow{t_2} & S_2^*
\end{array}
\]

Graphically, a pre-net looks very similar to a Petri net, and we follow the convention of [4] by decorating arcs with numbers to indicate their input/output order in a transition, as in:

![Petri net diagram]

This denotes that the place in the top left is used as the first and third input of the transition, while the place in the bottom right is the second input. For the output place, no decorations are needed as the ordering is unambiguous.

Pre-nets give rise to strict monoidal categories as follows.

**Proposition 4.2.** There is an adjunction

\[
\text{PreNet} \xrightleftharpoons{\text{U}_{\text{PreNet}}} \text{StrMC}
\]

whose left adjoint sends a pre-net \( Q \) to the strict monoidal category where:

- **Objects** are elements of the free monoid on the set of places.

- **Morphisms** are generated inductively by the following rules:
  - for each place \( s \) there is an identity \( 1_s: s \rightarrow s \)
  - for each transition \( \tau \) of \( P \), there is a morphism going from its source to its target
  - for every pair of morphisms \( f: x \rightarrow y \) and \( f': x' \rightarrow y' \), there is a morphism \( f \otimes f': x \otimes x' \rightarrow y \otimes y' \)
  - for every pair of composable morphisms \( f: x \rightarrow y \) and \( g: y \rightarrow z \), there is a morphism \( g \circ f: x \rightarrow z \)

and quotiented to satisfy the axioms of a strict monoidal category.

**Proof.** This is [27, Prop. 6.1]. □

The above adjunction can be composed with one defined in Proposition 7.1 to obtain an adjunction between pre-nets and strict symmetric monoidal categories:

\[
\text{PreNet} \xrightleftharpoons{\text{U}_{\text{PreNet}}} \text{StrMC} \xrightleftharpoons{\text{F}_{\text{StrMC}}} \text{SSMC}
\]
The composite adjunction

\[
\begin{array}{c}
\text{PreNet} \\
\downarrow \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
\text{SSMC} \\
F^* \\
U^*
\end{array}
\]

is used to obtain the categorical operational semantics of pre-nets under the individual token philosophy. This adjunction was first presented by Bruni, Meseguer, Montanari, and Sassone \[8\] with codomain a full subcategory of SSMC and was later refined to above form in \[27\].

This composite adjunction has also been used to give a categorical semantics for Petri nets \[7, 8, 31\]. For this, given a Petri net \(P\), one first chooses a pre-net \(Q\) having \(P\) as its underlying Petri net, and the forms the symmetric strict monoidal category \(F^*(Q)\). However this semantics is not functorial, thanks to the arbitrary choice involved.

The category PreNet is better behaved than Petri. The latter is not even cartesian closed, for essentially the same reasons described in \[10, 26\], but the former is cartesian closed, and even a topos:

**Proposition 4.3.** The category PreNet is equivalent to a presheaf category.

**Proof.** It suffices to construct a category \(C\) so that functors from \(C\) to \(\text{Set}\) are the same as pre-nets. Let \(C\) have an object \(p\), and for every pair of natural numbers \((n, m)\), let \(C\) contain an object \(t(m, n)\). Here \(p\) stands for ‘places’, while \(t(m, n)\) stands for ‘transitions with \(m\) inputs and \(n\) outputs’. Besides identity morphisms, \(C\) contains \(m\) morphisms \(s_i: t(m, n) \to p\) representing the source maps and \(n\) morphisms \(t_j: t(m, n) \to p\) representing the target maps. Composition in \(C\) is trivial.

A pre-net \(T \xrightarrow{(s,t)} S^* \times S^*\) can be identified with the functor \(C \to \text{Set}\) that sends \(p\) to the set of places \(S\), sends the object \(t(m, n)\) to the subset of \(T\) consisting of transitions with \(m\) inputs and \(n\) outputs, and sends the morphisms \(s_i, t_j: t(m, n) \to p\) to the functions that map each transition to its \(i\)-th input and \(j\)-th output. A morphism of pre-nets \((f, g)\) can then be identified with a natural transformation between such functors, with the \(p\)-component given by \(g\) and with the \((t(m, n))\)-components given by the restrictions of \(f\) to the set of transitions with \(m\) inputs and \(n\) outputs. Naturality follows from the commutative diagrams in Definition 4.1. \(\Box\)

One downside of pre-nets is that ordering the inputs and outputs of transitions seems artificial in many applications where Petri nets are heavily used \[16, 35\]. This ordering also greatly restricts the available morphisms between pre-nets. For example, there is no morphism between the following pre-nets:

![Diagram of pre-nets](image)

even though there is a morphism between their underlying Petri nets, in which the ordering information has been forgotten.

Furthermore, in the symmetric strict monoidal category \(F_\ast (P)\) coming from a pre-net \(P\), none of the symmetries \(\sigma_{x,y}: x \otimes y \to y \otimes x\) are identities, except when
$x$ or $y$ is the unit object. Perhaps more importantly, if a transition in the pre-net $F_\star(P)$ gives a morphism
\[ t: x_1 \otimes \cdots \otimes x_m \to y_1 \otimes \cdots \otimes y_n, \]
where $x_i, y_j$ are objects coming from places of $P$, the composite of $t$ with symmetries that permute the inputs $x_i$ and outputs $y_j$ is only equal to $f$ if both these permutations are the identity.

Thus, the SSMCs obtained from pre-nets exemplify an extreme version of the individual token philosophy. Not only does each token have its own individual identity, switching two tokens before or after executing the morphism corresponding to a transition always gives a different morphism.

5. $\Sigma$-nets

We have seen that Petri nets generate symmetric monoidal categories naturally suited to the collective token philosophy, where the identity of individual tokens does not matter at all, so switching two never has an effect. On the other hand, we have just seen that pre-nets generate symmetric monoidal categories suited to an extreme version of the individual token philosophy, in which switching tokens always has an effect.

Now we introduce a new kind of nets, called $\Sigma$-nets, which in some sense lie between these two extremes. In a $\Sigma$-net, one has control over which permutations of the input or output of a transition alter the morphism it defines and which do not. This finer ability to control the action of permutations allow $\Sigma$-nets to behave either like Petri nets or pre-nets—or in a mixed way.

**Lemma 5.1.** There is a forgetful functor
\[ Q: \text{SSMC} \to \text{Set} \]
that sends a symmetric strict monoidal category to its set of objects and sends a strict symmetric monoidal functor to its underlying function on objects. $Q$ has a left adjoint
\[ P: \text{Set} \to \text{SSMC} \]
that sends a set $S$ to the symmetric strict monoidal category $PS$ having (possibly empty) words in $S$ as objects, and permutations as morphisms.

**Proof.** See Sassone [31, Sec. 3] or Gambino and Joyal [14, Sec. 3.1].

**Definition 5.2.** A $\Sigma$-net is a set $S$ together with a functor
\[ N: PS \times PS^{op} \to \text{Set}. \]

A morphism between $\Sigma$-nets $PS_1 \times PS_1^{op} \xrightarrow{g} \text{Set}$ and $PS_2 \times PS_2^{op} \xrightarrow{M} \text{Set}$ is a pair $(g, \alpha)$ where $g: S_1 \to S_2$ is a function and $\alpha$ is a natural transformation filling the following diagram:
This defines the category $\Sigma$-net.

The definition of $\Sigma$-net may seem unintuitive, but it is easily explained. Suppose $N: PS \times PSp \to \text{Set}$ is a $\Sigma$-net. We call $S$ its set of places. Objects of $PS$ are words of places. Given $m, m' \in PS$, we call an element of $N(m, m')$ a transition with source $m$ and target $m'$. In Theorem 7.7 we describe how to freely generate a symmetric strict monoidal category $C$ from the $\Sigma$-net $N$. In this construction, the transitions of $N$ give morphisms that generate all the morphisms in $C$.

More precisely: the objects of $C$ are words of places. The tensor product of objects is given by concatenation of words, and the symmetry in $C$ acts by permuting places in a word. Each transition $t \in N(m, m')$ gives a morphism $t: m \to m'$ in $C$, and all other morphisms are generated by composition, tensor product and symmetries.

Given a transition $t \in N(m, m')$, the action of the functor $N: PS \times PSp \to \text{Set}$ on morphisms describes what happens to the corresponding morphism $t: m \to m'$ when we permute the places in its source and target: it gets sent to some other transition (possibly the same one). We say two transitions $t \in N(m, m')$ and $u \in N(n, n')$ are in the same transition class if and only if there exists a morphism $\sigma: (m, m') \to (n, n')$ in $PS \times PSp$ such that

$$N(\sigma)(t) = u.$$

**Example 5.3.** There is a $\Sigma$-net $N$ with just two places, say $p$ and $q$, and just two transitions, $t_1 \in N(pq, \epsilon)$ and $t_2 \in N(qp, \epsilon)$, where $\epsilon$ stands for the empty word. There are two morphisms in $PS \times PSp$ with domain $(pq, \epsilon)$, namely the identity and the swap $(pq, \epsilon) \to (qp, \epsilon)$. Since there is a unique function between any two singleton sets, $N$ of this swap must map $t_1$ to $t_2$. Thus, both $t_1$ and $t_2$ lie in the same transition class, and this $\Sigma$-net has just one transition class.

**Example 5.4.** Now consider a $\Sigma$-net $M$ with just two places $p$ and $q$ and exactly four transitions, with $M(pq, \epsilon) = \{ t_1, u_1 \}$ and $M(qp, \epsilon) = \{ t_2, u_2 \}$. We set $M$ of the swap $(pq, \epsilon) \to (qp, \epsilon)$ to act as the function $\{ t_1, u_1 \} \to \{ t_2, u_2 \}$ sending $t_1$ to $t_2$ and $u_1$ to $u_2$. This $\Sigma$-net has exactly two transition classes: $t_1$ and $t_2$ represent one transition class, and $u_1$ and $u_2$ represent the other. Note that if we instead define the action of $M$ on the swap to send $t_1$ to $u_2$ and $t_2$ to $u_1$, then we would still have two transition classes; in fact this would be an isomorphic $\Sigma$-net.

Examples 5.3 and 5.4 illustrate the situation when no place occurs more than once in the source or target of any transition. In this case, any two nonempty values of the functor $N$ will be related by at most one morphism in $PS \times PSp$, and the functorial action of $N$ on such a morphism provides a way to canonically identify their values.

**Example 5.5.** Next consider a net $O$ with one place $p$ and one transition, namely $t \in O(pp, \epsilon)$. There are still two morphisms in $PS \times PSp$ with domain $(pp, \epsilon)$, the identity and the swap, but now both have $(pp, \epsilon)$ as codomain as well. There is still only one transition class, but now $t$ is mapped to itself by both morphisms $(pp, \epsilon) \to (pp, \epsilon)$, the identity and the swap.

Given a group $G$ acting on a set $X$, the isotropy group of $x \in X$ is the subgroup of $G$ consisting of elements that map $x$ to itself. Thus, in Example 5.5, unlike Example 5.4, we are seeing a transition with a nontrivial isotropy group. In
fact, because permutations act trivially on all transitions, the $\Sigma$-net of Example 5.5 belongs to the image of Petri under the functor $G_{petri} : \text{Petri} \to \Sigma\text{-net}$ described in Proposition 7.4.

**Example 5.6.** Next consider a $\Sigma$-net $Q$ with one place $p$ and precisely two transitions $t_1, t_2$ with $Q(pp, \epsilon) = \{t_1, t_2\}$. Suppose that $Q$ of the identity $(pp, \epsilon) \to (pp, \epsilon)$ acts as the identity function (as it must), while $Q$ of the swap acts by $t_1 \mapsto t_2$ and $t_2 \mapsto t_1$. Then $t_1$ and $t_2$ represent the same transition class, so there is once again only one transition class. The isotropy groups of $t_1$ and $t_2$ are trivial. In fact, because permutations act freely on the transitions in every transition class, this $\Sigma$-net belongs to the image of $\text{PreNet}$ under the functor $F_{pre} : \text{PreNet} \to \Sigma\text{-net}$ described in Theorem 7.3.

**Example 5.7.** Now let us give an example blending features from Examples 5.5 and 5.6. For this, we create a $\Sigma$-net $R$ that has one place $p$ and three transitions $t_1, t_2, u \in R(pp, \epsilon)$, such that $t_1$ and $t_2$ are order-sensitive while $u$ is not. This $\Sigma$-net maps $(ss, \epsilon)$ to $\{t_1, t_2, u\}$ and everything else to the empty set. The action of the swap automorphism of $(ss, \epsilon)$ switches $t_1, t_2$ and fixes $u$. As a result, this $\Sigma$-net has two transition classes: $t_1, t_2$ are both representatives of one transition class, while $u$ represents the other. This $\Sigma$-net is not in the image of $G_{petri} : \text{Petri} \to \Sigma\text{-net}$ or $F_{pre} : \text{PreNet} \to \Sigma\text{-net}$; it mixes the two worlds.

There is a graphical way to represent a $\Sigma$-net: we draw it in three dimensions, where a transition class is depicted like a tank containing all the permutations that act trivially on an arbitrarily chosen transition in this transition class. Below, you can see two examples of $\Sigma$-nets:

The left one is Example 5.5, where the identity and the swap both act trivially. The right one is Example 5.6, where only the identity acts trivially. The $\Sigma$-nets of Examples 5.3 and 5.7 are instead:

Note that although each transition class is a set of transitions, and each tank represents a single transition class, the pictures inside that tank do not represent the transitions in that class. Rather, these pictures represent the isotropy group of a single arbitrarily chosen transition that belongs to the class in question. The number of transitions and the size of the isotropy group are inversely related: their product is the cardinality of the total symmetry group with the given inputs and outputs. In the simple examples drawn above, this total symmetry group is $S_2$: 
thus when a transition class contains one transition this transition has isotropy group \(S_2\), and when it contains two transitions, each has trivial isotropy group.

Albeit intuitive, this graphical formalism quickly becomes disadvantageous for large nets. In this case, we can instead draw our nets as we usually draw Petri nets, and decorate the transitions with the relevant isotropy groups—and when this group is trivial, we can omit it. In this style of drawing, the nets of Examples 5.5 and 5.6 look as follows:

\[ S_2 \]

6. Perspectives on the category \(\Sigma\text{-net}\)

The definition of \(\Sigma\text{-nets}\) in Section 5 gives what may be called a “profunctorial” perspective: a \(\Sigma\text{-net}\) is a functor from \(P S \times P S^{\text{op}}\) to \(\text{Set}\), which is the same as a profunctor from \(P S\) to itself. This perspective will be useful in constructing the adjunction between \(\Sigma\text{-net}\) and \(\text{SSMC}\) in Theorem 7.7. However, there are other perspectives on \(\Sigma\text{-nets}\), leading to two alternative descriptions of \(\Sigma\text{-net}\), useful for other purposes.

6.1. The presheaf perspective.

**Theorem 6.1.** \(\Sigma\text{-net}\) is equivalent to a presheaf category.

**Proof.** We construct a category \(D\) so that functors from \(D\) to \(\text{Set}\) can be identified with \(\Sigma\text{-nets}\). To construct \(D\), we take the category \(C\) from Proposition 4.3 and throw in extra automorphisms of each object \(t(m, n)\), making its automorphism group \(S_m \times S_n\). For a source map \(s_i: t(m, n) \to p\) and an automorphism \((\sigma, \tau) \in S_m \times S_n\), we set the composite \(s_i \circ (\sigma, \tau)\) equal to \(s_{\sigma(i)}\). Similarly, for a target map \(t_j: t(m, n) \to p\), we set the composite \(t_j \circ (\sigma, \tau)\) equal to \(t_{\tau(j)}\). Then, for each \(\Sigma\text{-net} N: P S \times P S^{\text{op}} \to \text{Set}\), there is a corresponding functor \(\nu: D \to \text{Set}\) defined as follows. It sends the object \(p \in D\) to the set of places of \(N\). It sends each object \(t(m, n) \in D\) to the disjoint union of the sets \(N(a, b)\) over all \(a \in P S\) with length \(m\) and \(b \in P S\) with length \(n\). It sends the morphisms \(s_i, t_j: t(m, n) \to p\) to the functions that map any transition to its \(i\)th input and \(j\)th output. Finally, this functor \(\nu\) sends the permutations \((\sigma, \tau)\) to the natural actions of the symmetric group on the transitions of \(N\). For a morphism of \(\Sigma\text{-nets} (g, \alpha): N \to N'\), there is a natural transformation between their functors whose \(p\)-component is given by \(g\) and whose \((t(m, n))\)-components are given by disjoint unions of the components of \(\alpha\).

One can check that the resulting functor from \(\Sigma\text{-net}\) to \(\text{Set}^D\) is an equivalence. □

Theorem 6.1 fills an important gap between Petri nets and graphs. Indeed, whereas the category of graphs and their morphisms is a presheaf category, the category Petri is not. Theorem 6.1 has a lot of nice consequences, such as:

- **\(\Sigma\text{-net}\)** is complete and cocomplete. This is particularly important since many compositional approaches to Petri nets rely on colimits; for example, composition of open Petri nets is done using pushouts (see Section 9), while tensoring them is done using coproducts [2].
- **\(\Sigma\text{-net}\)** is a topos, and thus an adhesive category [24], so it admits a theory of double pushout rewriting [23]. This is relevant as double pushout rewriting
is a widely used technique to transform graph-like structures in the literature [13]. The internal logic of toposes is very rich, and understanding its implications for Σ-nets is an interesting direction for future work.

The category $D$ in Theorem 6.1 is equivalent to Kock’s category of “elementary graphs” [21, 1.8]. Thus, Σ-net is equivalent to his category of “digraphical species” [21, 1.14].

6.2. The groupoidal perspective. The profunctorial and presheaf perspectives highlight the transitions of a Σ-net over its transition classes. Sometimes, however, we want to work directly with the transition classes; we now describe a third perspective that permits this.

Firstly, it is well-known [25, Theorem 2.1.2] that a functor $N: PS \times PS^{op} \to \text{Set}$ is equivalent to a discrete opfibration $T \to PS \times PS^{op}$. In addition to this, a morphism of Σ-nets is equivalently a commutative square

$$
\begin{array}{ccc}
T_1 & \xrightarrow{g} & T_2 \\
\downarrow & & \downarrow \\
PS_1 \times PS_1^{op} & \xrightarrow{Pf \times Pf^{op}} & PS_2 \times PS_2^{op}
\end{array}
$$

Note that this looks much more similar to the definitions of the categories PreNet and Petri. The set of objects of $T$ here is the disjoint union of the sets $N(p, p')$, i.e., the transitions rather than the transition classes. The transition classes are the isomorphism classes of the groupoid $T$. To contract these down to single objects, we can replace it by an equivalent groupoid that is skeletal, i.e., there are no morphisms $x \to y$ for objects $x \neq y$, or equivalently each isomorphism class contains exactly one object. After such a replacement the functor $T \to PS \times PS^{op}$ is no longer a discrete opfibration, but it is still faithful; and we now have to allow the morphisms to commute only up to isomorphism.

**Theorem 6.2.** The category of Σ-nets is equivalent to the following category:

- Its objects are faithful functors $T \to PS \times PS^{op}$, where $S$ is a set and $T$ is a skeletal groupoid.

- Its morphisms are squares that commute up to specified isomorphism

$$
\begin{array}{ccc}
T_1 & \xrightarrow{g} & T_2 \\
\downarrow & & \downarrow \\
PS_1 \times PS_1^{op} & \xrightarrow{Pf \times Pf^{op}} & PS_2 \times PS_2^{op}
\end{array}
$$

modulo the equivalence relation that two such morphisms $(f, g, \theta)$ and $(f', g', \theta')$ are considered equal if $f = f'$ and there is a natural isomorphism $\phi: g \Rightarrow g'$ such that

$$
\begin{array}{ccc}
T_1 & \xrightarrow{g} & T_2 \\
\downarrow_{\phi} & & \downarrow \\
PS_1 \times PS_1^{op} & \xrightarrow{Pf \times Pf^{op}} & PS_2 \times PS_2^{op}
\end{array} =
$$

$$
\begin{array}{ccc}
T_1 & \xrightarrow{g} & T_2 \\
\downarrow & & \downarrow_{\phi'} \\
PS_1 \times PS_1^{op} & \xrightarrow{Pf \times Pf^{op}} & PS_2 \times PS_2^{op}
\end{array}
$$
Note that since \( T_2 \to PS_2 \times PS_2^{op} \) is faithful, such a \( \phi \) is unique if it exists. Thus, the category described in the theorem is in fact equivalent to the evident 2-category having as 2-morphisms natural isomorphisms \( \phi \) as above.

**Proof.** Define a category \( P \) as in the theorem, but where the objects allow \( T \) to be any groupoid. Then there is a functor \( \Sigma\text{-net} \to P \), since discrete opfibrations are faithful and strictly commutative squares also commute up to isomorphism. This functor is faithful, since if \( \theta \) and \( \theta' \) are identities so is \( \phi \), by the faithfulness of \( T_2 \to PS_2 \times PS_2^{op} \). Moreover, any morphism in \( P \) whose target \( T_2 \to PS_2 \times PS_2^{op} \) is a discrete opfibration has a representative that commutes strictly, since we can lift the isomorphism \( \theta \) to an isomorphism \( \phi \) with \( \theta' \) an identity. Thus, the functor \( \Sigma\text{-net} \to P \) is also full.

Let the **pseudo slice 2-category** over a groupoid \( B \) be the 2-category with groupoids over \( B \) as objects, triangles commuting up to natural isomorphism as morphisms, and the evident 2-morphisms \([29, \text{Definition 3.2}]\). Any groupoid over \( B \) is equivalent, in the pseudo slice 2-category of \( B \), to a fibration \([36, \text{Theorem 6.7}]\), which in the groupoid case is the same as an opfibration. If \( f: A \to B \) is faithful then this opfibration will be as well, so \( f \) is equivalent to a discrete opfibration. Since equivalences in the pseudo slice 2-category yield isomorphisms in \( P \), the functor \( \Sigma\text{-net} \to P \) is also essentially surjective, and hence an equivalence.

The category described in the theorem is a full subcategory of \( P \), so it suffices to show that every object of \( P \) is isomorphic to one where \( T \) is skeletal. But any groupoid is equivalent to a skeletal one, and such an equivalence preserves faithfulness and yields an isomorphism in \( P \). \( \square \)

Note that the construction in the final paragraph taking a groupoid to a skeletal one preserves connected components, while in the output each connected component has exactly one object. Thus, in the representation described in Theorem 6.2 the objects of the groupoid \( T \) really are precisely the transition classes. Since the transition classes of a \( \Sigma \)-net correspond to the transitions of a Petri net, we can think of a \( \Sigma \)-net as a Petri net together with, for each transition, (1) a lifting of its source and target multisets to words, and (2) an **isotropy group** that acts faithfully on those words, i.e., maps injectively to the subgroup of \( \Sigma_m \times \Sigma_n \) that fixes both of those words.

**Example 6.3.** If we start from a transition in a Petri net \( t: 3a + 2b \to 4c \), then we could lift it to a transition in a \( \Sigma \)-net by defining \( t: aaabb \to ccccc \) and equipping it with any subgroup of \( \Sigma_3 \times \Sigma_2 \times \Sigma_4 \), which describes the “degree of collectivization” of \( t \). If the isotropy group is trivial, then our \( \Sigma \)-net behaves like a pre-net—tokens are “fully individualized”—whereas if it is as large as possible then it behaves as a Petri net—tokens are “fully collectivized”. This idea is heavily used in the next section to describe the adjunctions between Petri, PreNet and \( \Sigma \)-net.
7. Description of the adjunctions

Now we describe in detail all the adjunctions between the categories in play. We again include the diagram of Section 2, but now with most of the functors labeled.

\[
\begin{array}{cccc}
\text{StrMC} & \xrightarrow{\mathcal{F}_{\text{SSMC}}} & \text{SSMC} & \xrightarrow{\mathcal{F}_{\text{CMC}}} \\
& \mathcal{U}_{\text{StrMC}} & & \mathcal{U}_{\text{SSMC}} \\
\text{PreNet} & \xrightarrow{\mathcal{F}_{\text{SSMC}}} & \Sigma \text{-net} & \xrightarrow{\mathcal{F}_{\text{Petri}}} \\
& \mathcal{H}_{\text{pre}} & & \mathcal{G}_{\text{pre}} \\
\end{array}
\]

The adjunctions in the top row can be constructed using standard tools, such as the adjoint functor theorem or the adjoint lifting theorem.

**Proposition 7.1.** There is an adjunction

\[
\text{StrMC} \dashv \begin{array}{c}
\mathcal{F}_{\text{SSMC}} \\
\mathcal{U}_{\text{StrMC}}
\end{array} \text{SSMC}.
\]

Here, \(\mathcal{U}_{\text{StrMC}}\) freely adds symmetries to a strict monoidal category, while \(\mathcal{U}_{\text{SSMC}}\) sends any symmetric strict monoidal category to its underlying strict monoidal category.

**Proposition 7.2.** There is an adjunction

\[
\text{SSMC} \dashv \begin{array}{c}
\mathcal{F}_{\text{CMC}} \\
\mathcal{U}_{\text{SSMC}}
\end{array} \text{CMC}.
\]

\(\mathcal{F}_{\text{SSMC}}\) takes a symmetric strict monoidal category and imposes a law saying that all symmetries are identity morphism, while \(\mathcal{U}_{\text{SSMC}}\) sends any commutative monoidal category to its underlying symmetric strict monoidal category.

The adjunction between \text{Petri} and \text{CMC} was recalled in Proposition 3.3, while that between \text{PreNet} and \text{StrMC} was recalled in Proposition 4.2. We now cover the middle column and bottom row of the diagram, which are new.

**Theorem 7.3.** There is a triple of adjoint functors

\[
\begin{array}{c}
\text{PreNet} \\
\mathcal{F}_{\text{pre}} \\
\mathcal{G}_{\text{pre}} \\
\mathcal{H}_{\text{pre}}
\end{array} \Sigma \text{-net}.
\]

**Proof.** For this proof it is most convenient to work with the presheaf perspective. In the proof of Proposition 4.3 we described a category \(\mathcal{C}\) such that \(\text{PreNet} \cong [\mathcal{C}, \text{Set}]\) and in the proof of Theorem 6.1 we described a category \(\mathcal{D}\) such that \(\Sigma \text{-net} \cong [\mathcal{D}, \text{Set}]\). Recall that \(\mathcal{D}\) is built by starting with the objects and morphisms of \(\mathcal{C}\) and adding new morphisms and equations. The inclusion gives a functor \(i : \mathcal{C} \to \mathcal{D}\) which induces a functor

\[
\Sigma \text{-net} \cong [\mathcal{D}, \text{Set}] \xrightarrow{\mathcal{(-)}_{\text{pre}}} [\mathcal{C}, \text{Set}] \cong \text{PreNet}
\]

given by precomposition with \(i\). The composite functor above is the forgetful functor \(\mathcal{G}_{\text{pre}}\). Therefore \(\mathcal{G}_{\text{pre}}\) has a left adjoint \(\mathcal{F}_{\text{pre}} : \text{PreNet} \to \Sigma \text{-net}\) given by left Kan
extension along \( i \) and a right adjoint \( H_{\text{pre}} : \text{PreNet} \to \Sigma\text{-net} \) given by right Kan extension along \( i \).

Let us spell out what the functors \( F_{\text{pre}}, G_{\text{pre}}, H_{\text{pre}} \) do in detail, making use of our graphical representation.

\( F_{\text{pre}} \) constructs tanks.: For this functor we work in the groupoid representation of \( \Sigma\text{-nets} \). A pre-net \( T : (s,t) \to S^* \times S^* \) is sent to the \( \Sigma\text{-net} \)
\[ T^{(s,t)} : S^* \times S^* \to P S \times P S^{\text{op}}, \]
using the fact that \( S^* \) is the set of objects of \( P S \). A morphism of pre-nets \((f,g) : (s_1,t_1) \to (s_2,t_2)\) induces a morphism of \( \Sigma\text{-nets} \): \( g : T_1 \to T_2 \) lifts to a morphism between discrete groupoids and \( f : S_1 \to S_2 \) lifts to a functor \( P S_1 \times P S_1^{\text{op}} \to P S_2 \times P S_2^{\text{op}} \). The relevant square as in Theorem 6.2 commutes strictly.

\( F_{\text{pre}} \) takes a pre-net and builds from it a \( \Sigma\text{-net} \) with trivial isotropy groups. Graphically, this amounts to enclosing every transition of the given pre-net in a tank:

In particular, the transition classes of \( F_{\text{pre}}(N) \) are the transitions of \( N \), and each such class contains as many transitions as possible.

\( G_{\text{pre}} \) explodes tanks.: For this functor we work in the profunctor representation. A \( \Sigma\text{-net} \)
\[ N : P S \times P S^{\text{op}} \to \text{Set} \]
is sent to the pre-net having \( S \) as its set of places and the disjoint union of all sets \( N(a,b) \), for any \( a, b \) objects of \( P S \), as its set of transitions. For each transition, input and output places are defined using the inverse image of \( N \). That is, the transitions of \( G_{\text{pre}} N \) are the transitions of \( N \), with their grouping into classes and their isotropy groups forgotten.

We can give a different interpretation of this using the groupoid perspective. Suppose \( T : N \to P S \times P S^{\text{op}} \) is a \( \Sigma\text{-net} \). Then for each object \( t \) of \( T \) such that \( N(t) \) is a pair of strings of length \( m \) and \( n \) there will be \( S_m \times S_n / \text{hom}_T(t,t) \) transitions in \( G_{\text{pre}} N \), where \( S_n \) denotes the group of permutations over a string of \( n \) elements. Graphically, this is represented by “exploding” a tank with \( m \) inputs and \( n \) outputs and introducing \( m!n!/k \) pre-net transitions, where \( k \) is the number of elements in the tank.
In the image above, we see the behavior of $G_{\text{pre}}$ on a $\Sigma$-net having a transition with trivial isotropy group, while in the image below $G_{\text{pre}}$ is used on a $\Sigma$-net having a transition with 2-element isotropy group.

$H_{\text{pre}}$ matches transitions.: While $F_{\text{pre}}$ builds as many tanks as we can get from a pre-net’s transitions, $H_{\text{pre}}$ bundles pre-net transitions sharing the same inputs/outputs modulo permutations, whenever they complete their corresponding symmetry groups. For instance, in the figure below transitions $x$ and $y$ complete the permutation group $S_2 \times S_1$, and hence they give rise to the tank denoted with $\langle x, y \rangle$. The same happens for transitions $x$ and $z$, giving rise to tank $\langle x, z \rangle$.

The following pre-net does not have enough transitions to complete the symmetry group of its inputs/outputs. As such, $H_{\text{pre}}$ cannot match this transition with anything, and does not produce any tank.

In the following case, the pre-net has a repeated input. $H_{\text{pre}}$ is then able to match the transition with itself, producing a maximally commutative tank.

Looking at these examples, we see that in general the transitions of $N$ do not correspond directly to either the transitions of $H_{\text{pre}}(N)$ or the transition classes of $H_{\text{pre}}(N)$.

**Proposition 7.4.** There is an adjunction

$$\Sigma\text{-net} \xleftarrow{\quad F_{\text{pet}}} \xrightarrow{\quad G_{\text{pet}}} \text{Petri}.$$ 

**Proof.** Note first that Petri is by definition precisely the comma category $\left(\text{Set} \downarrow (N[-] \times N[-])\right)$. Similarly, if we identify a $\Sigma$-net with a functor $N: PS \times PS^{op} \to \text{Set}$
and thereby with a discrete opfibration \( N \rightarrow PS \times PS^{\text{op}} \), then \( \Sigma\text{-net} \) becomes identified with the full subcategory of the comma category \((\text{Cat} \downarrow (P(-) \times P(-)^{\text{op}}))\) consisting of the discrete opfibrations.

Now note that \( \text{Set} \) is a reflective full subcategory of \( \text{Cat} \), with reflector \( \pi_0 \) that takes the set of connected components of a category. Moreover, we have \( \pi_0(PS \times PS^{\text{op}}) \cong N[S] \times N[S] \). Thus, Lemma 7.5, proven below (and applied with \( D = \text{Cat} \), \( C = E = \text{Set} \), and \( K = P(-) \times P(-)^{\text{op}} \)), shows that we have an adjunction

\[
(Cat \downarrow (P(-) \times P(-)^{\text{op}})) \xrightarrow{F} (\text{Set} \downarrow (N[-] \times N[-])) = \text{Petri}
\]

in which the left adjoint \( F \) applies \( \pi_0 \) to both domain and codomain, and the right adjoint \( G \) pulls back along the unit \( PS \times PS^{\text{op}} \rightarrow N[S] \times N[S] \). Therefore, it suffices to observe that this right adjoint takes values in discrete opfibrations, hence in \( \Sigma\text{-net} \).

**Lemma 7.5.** Let \( E \) be a reflective subcategory of \( D \), with reflector \( \pi : D \rightarrow E \), and let \( K : C \rightarrow D \) be a functor where \( D \) has pullbacks. Then there is an adjunction

\[
(D \downarrow K) \xrightarrow{F} (E \downarrow (\pi \circ K)).
\]

**Proof.** Let \( \eta_X : X \rightarrow \pi X \) denote the unit of the reflection. Then for any \( f : S_1 \rightarrow S_2 \) in \( C \), we have \( \eta_{KS_1} \circ Kf = \pi Kf \circ \eta_{KS_1} \) by naturality; we denote this common map by \( \eta_f \). Now there is a profunctor between \((D \downarrow K)\) and \((E \downarrow (\pi \circ K))\) defined to take \( T_1 \rightarrow KS_1 \) and \( T_2 \rightarrow \pi KS_2 \) (where \( T_1 \in D \) and \( T_2 \in E \)) to the set of pairs \((f, g)\) where \( f : S_1 \rightarrow S_2 \) in \( C \) and \( g : T_1 \rightarrow T_2 \) in \( D \) make the following square commute:

\[
\begin{array}{ccc}
T_1 & \xrightarrow{g} & T_2 \\
\downarrow & & \downarrow \\
KS_1 & \xrightarrow{\eta_f} & \pi KS_2.
\end{array}
\]

This profunctor is representable on both sides, because any such square factors uniquely in both of the following ways:

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\eta_{T_1}} & \pi T_1 \\
\downarrow & & \downarrow \\
KS_1 & \xrightarrow{\pi KS_1} & \pi KS_2
\end{array}, \quad \begin{array}{ccc}
T_1 & \longrightarrow & T_2 \\
\downarrow & & \downarrow \\
KS_1 & \xrightarrow{\pi KS_1} & \pi KS_2.
\end{array}
\]

On the left, the factorization is by the universal property of \( \eta_{T_1} \), while on the right it is by the universal property of the pullback. Therefore, there is an adjunction \( F \dashv G \) as desired, where \( F \) takes \( T_1 \rightarrow KS_1 \) to \( \pi T_1 \rightarrow \pi KS_1 \), and \( G \) takes \( T_2 \rightarrow \pi KS_2 \) to the pullback of \( T_2 \) to \( KS_2 \).

Note that by construction, this adjunction is a reflection, i.e., the right adjoint \( G_{\text{pet}} \) is fully faithful. We can illustrate the action of \( F_{\text{pet}} \) and \( G_{\text{pet}} \) with examples.

**\( F_{\text{pet}} \) deflates tanks:** In the groupoid perspective, this functor takes a \( \Sigma\text{-net} \) \( T \xrightarrow{N} PS \times PS^{\text{op}} \) and maps it to the Petri net having the underlying set of objects of \( T \) as transitions, \( S \) as places, and input/output functions induced by the mapping on objects of \( N \). The action on a morphism \((g, f)\) is...
obtained by restricting the functor $g$ to its mapping on objects. Graphically, $F_{\text{pet}}$ just deflates tanks, replacing each tank by a single transition:

\[
\begin{array}{c}
\text{F}_{\text{pet}}
\end{array}
\]

In particular, the transitions of $F_{\text{pet}}(N)$ are the transition classes of $N$.

$G_{\text{pet}}$ builds tanks as big as possible.: Petri nets are mapped under $G_{\text{pet}}$ to corresponding $\Sigma$-nets that have the largest isotropy groups possible. Consider a transition $t$ in a Petri net $N$. Its inputs and outputs will be a couple of unordered strings of length $n, m$, respectively. Pick any ordering for these strings, and call them $a, b$, respectively. Finally, let $G_t$ be the subgroup of $S_n \times S_m$ that fixes the pair of strings $(a, b)$.

$N$ is mapped to a $\Sigma$-net $T \xrightarrow{G_{\text{pet}}N} PS \times PS^{\text{op}}$ whose groupoid $T$ has transitions of $N$ as objects and, for each $t$ in $T$, $G_t$ as its group of endomorphisms. $N$ maps $t$ to the ordering $(a, b)$ we have chosen before. It can be seen that picking different orderings of the input/output of each transition gives isomorphic results.

Graphically, out of each Petri net we build a corresponding $\Sigma$-net that has its tanks as full as possible:

\[
\begin{array}{c}
\text{G}_{\text{pet}}
\end{array}
\]

Thus, the transition classes of $G_{\text{pet}}(N)$ are the transitions of $N$.

Remark 7.6. Note that $F_{\text{pre}}$ and $G_{\text{pet}}$ both build a $\Sigma$-net whose transition classes are the transitions of a pre-net or Petri net. On the other hand, $G_{\text{pre}}$ and $F_{\text{pet}}$ are “dual”, in that they build a pre-net or Petri net whose transitions are, respectively, the transitions or the transition classes of a $\Sigma$-net. In particular, the composite $F_{\text{pet}} \circ F_{\text{pre}}$ preserves transitions: it is the functor $\text{PreNet} \to \text{Petri}$ that simply forgets the ordering of inputs and outputs. Its right adjoint $G_{\text{pre}} \circ G_{\text{pet}}$ explodes each transition of a Petri net into as many transitions of a pre-net as possible, giving its inputs and outputs all possible orderings.

The last adjunction to construct is the one in the middle column:
Theorem 7.7. There is an adjunction

\[
\begin{array}{ccc}
\Sigma\text{-net} & \xrightarrow{F_{\Sigma\text{-net}}} & \text{SSMC} \\
\downarrow & & \downarrow \\
\left\uparrow U_{\Sigma\text{-net}} \right. & & \left. \uparrow U_{\Sigma\text{-net}} \right.
\end{array}
\]

Two proofs of Theorem 7.7 were sketched by Kock [21, §§6–7]. Our proof is more similar to the proof of [27, Theorem 5.1], which is a generalization of Propositions 4.2 and 3.3 involving a Lawvere theory \(Q\): these two propositions follow by taking \(Q\) to be the theory of commutative monoids and the theory of monoids, respectively. In that proof, an adjunction between \(Q\)-nets and \(Q\)-categories (i.e., \(Q\)-algebras in \(\text{Cat}\)) was obtained as the composite of two adjunctions where the intermediate category consists of \(Q\)-graphs: graphs internal to the category of \(Q\)-algebras. Note that these \(Q\)-graphs have operations coming from the Lawvere theory \(Q\), which act both on vertices and edges, but they lack the ability to compose edges (i.e., morphisms) that one has in a \(Q\)-category.

Our desired adjunction here is not a special case of [27, Theorem 5.1], since the symmetries in a symmetric monoidal category cannot be represented by a structure on the object set alone. However, we can perform a similar factorization through a category containing only the monoidal operations. We begin by reducing the problem from strict symmetric monoidal categories to (colored) props.

Definition 7.8. A (colored) prop consists of a set \(S\), a strict symmetric monoidal category \(B\), and a strict symmetric monoidal functor \(i: PS \to B\) that is bijective on objects. A morphism of props consists of a function \(S \to S'\) and a strict symmetric monoidal functor \(B \to B'\) making the evident square commute. We denote the category of props by \(\text{PROP}\).

Lemma 7.9. There is an adjunction

\[
\begin{array}{ccc}
\text{PROP} & \xrightarrow{F_2} & \text{SSMC} \\
\downarrow & & \downarrow \\
\left\uparrow U_2 \right. & & \left. \uparrow U_2 \right.
\end{array}
\]

Proof. For a prop \((S, B, i)\) we define \(F_2(S, B, i) = B\). And for a strict symmetric monoidal category \(B\), we let \(S\) be the set of objects of \(B\), so that we have a strict symmetric monoidal functor \(PS \to B\). Now we factor this functor as a bijective-on-objects functor \(i: PS \to B'\) followed by a fully faithful one \(p: B' \to B\). Then \(B'\) can be given a symmetric strict monoidal structure making both \(i\) and \(p\) strict symmetric monoidal functors, and we define \(U_2(B) = (S, B', i)\).

Therefore, it will suffice to construct an adjunction between \(\Sigma\text{-net}\) and \(\text{PROP}\). We work with the profunctor representation of \(\Sigma\)-nets. Let \(U_1: \text{PROP} \to \Sigma\text{-net}\) be the functor sending \((S, B, i)\) to \((S, N)\) where \(N(a_1, a_2) = \hom_B(i(a_2), i(a_1))\). This is the functor we aim to construct a left adjoint of. As in [27], we do this “fiberwise” for a fixed \(A\), then piece the fiberwise adjunctions together.

Lemma 7.10. We have a commutative triangle

\[
\begin{array}{ccc}
\Sigma\text{-net} & \xleftarrow{U_1} & \text{PROP} \\
\downarrow & & \downarrow \\
\text{Set} & & \text{Set}
\end{array}
\]

in which the two diagonal functors are split fibrations and \(U_1\) is cartesian.
Proof. The two diagonal functors send \((S, N)\) to \(S\) and \((S, B, i)\) to \(S\), respectively. To show the left-hand diagonal functor is a split fibration, let \((S, H)\) ∈ \(\Sigma\text{-net}\) and \(g : S' \to S\); then \((S', N \circ (P g \times P g^{op}))\) is the domain of a cartesian lifting. For the right-hand functor, given \((S, B, i)\) ∈ PROP and \(g : S' \to S\), the composite \(i \circ g: PS' \to B\) may no longer be bijective on objects, but we can factor it as a bijective-on-objects functor \(i' : PS' \to B'\) followed by a fully faithful one \(g' : B' \to B\). These are both again strict symmetric monoidal functors, and the induced map \((S', B', i')\) → \((S, B, i)\) is cartesian. Finally, \(U_1\) is cartesian by construction, since \(g'\) is fully faithful.

Let \(U_{1,S}: \Sigma\text{-net}_S \to \text{PROP}_S\) denote the restriction of \(U_1\) to the fibers over a particular set \(S\). We will construct a left adjoint \(F_{1,S}\) of this functor, then piece these together fiberwise.

Following the proof of [27], we need to decompose the structure of a prop with object set \(S\) into the “monoidal piece” and the “composition piece”. This can be accomplished as follows. Batanin and Markl [6] define a duoidal category to be a category \(C\) with two monoidal structures \((\cdot, J)\) and \((\circ, I)\) and additional natural morphisms

\[
I \to J \quad I \to I \cdot I \quad J \circ J \to J
\]

\[
(A \cdot B) \circ (C \cdot D) \to (A \circ C) \cdot (B \circ D)
\]

satisfying axioms that say \((\cdot, J)\) is a pseudomonoid structure on \((C, \circ, I)\) in the 2-category of lax monoidal functors. It is \(\circ\text{-symmetric}\) if \(\circ\) is a symmetric monoidal structure and the above maps commute with the symmetry in an evident way.

In a duoidal category, the monoidal structure \(\cdot\) lifts to a monoidal structure on the category of \(\circ\text{-monoids}\). A \(\cdot\text{-monoid}\) in this monoidal category of \(\circ\text{-monoids}\) is called a duoid. Similarly, if the duoidal category is \(\circ\text{-symmetric}, then \(\cdot\) lifts to the category of commutative \(\circ\text{-monoids}, and a monoid therein is called a \(\circ\text{-commutative duoid}.\)

Lemma 7.11. There is a \(\circ\text{-symmetric duoidal structure on} \Sigma\text{-net}_S \text{ whose category of} \circ\text{-commutative duoids is equivalent to} \text{PROP}_S.\)

Proof. Note that \(\Sigma\text{-net} = \text{Prof}(PS, PS)\) is the hom-category of \(PS\) in the bicategory \(\text{Prof}\) of categories and profunctors. Since this is an endo-hom-category in a bicategory, it has a monoidal structure given by composition in \(\text{Prof}, which we call \(\cdot\) (thus \(J\) is the hom-functor of \(A\)). The monoidal structure \(\circ\) is given by convolution:

\[
(H \circ K)(x, z) = \int^{a,b,c,d} PS(x, ab) \times H(a, c) \times K(b, d) \times PS(cd, z).
\]

with \(I(x, y) = PS(x, \epsilon) \times PS(\epsilon, y)\).

A \(\cdot\text{-monoid}\) is a monad on \(PS\) in the bicategory \(\text{Prof}, which is well-known to be equivalent to a category \(B\) with a bijective-on-objects functor \(PS \to B\. Applying the Yoneda lemma, we find that a \(\circ\text{-monoid structure on such a} B\) consists of morphisms

\[
B(a, c) \times B(b, d) \to B(ab, cd)
\]

that are suitably compatible. This extends the monoidal structure of \(PS\) to the arrows of \(B\) (it is already defined on the objects of \(B\) since they are the same as the objects of \(A\)). Compatibility with the duoidal exchange morphism says that
this action is functorial, while compatibility with the map \( J \circ J \to J \) says that it extends the functorial action of the monoidal structure on \( PS \). The associativity and unitality of a \( \circ \)-monoid says \( B \) has a strict monoidal structure and the functor \( PS \to B \) is strict monoidal. Finally, the symmetry of \( \circ \) switches \( H \) and \( K \) and composes with the symmetry isomorphisms in \( PS \) on either side; thus \( \circ \)-commutativity of a duoid makes \( B \) a symmetric strict monoidal category and \( PS \to B \) a strict symmetric monoidal functor. \( \square \)

In fact, an analogous result holds with \( PS \) replaced by any symmetric monoidal category. A more abstract construction of this duoidal structure was given by Garner and López Franco [15, Proposition 51], while the identification of its duoids follows from their Proposition 49 and the remarks after Proposition 54. Note that the adjective “commutative” in [15] is used with a different meaning than ours; we repeat that for us, “\( \circ \)-commutative” simply means that the monoid structure with respect to \( \circ \) is commutative in the ordinary sense for a monoid object in a symmetric monoidal category.

Note that both monoidal structures \( \star \) and \( \circ \) of \( \Sigma \)-net \( S \) preserve colimits in each variable. We can now work at a higher level of abstraction.

Lemma 7.12. For any cocomplete \( \circ \)-symmetric duoidal category \( C \) such that \( \star \) and \( \circ \) preserve colimits in each variable, the forgetful functor

\[ \circ : \text{CommDuoid}(C) \to C \]

has a left adjoint.

Proof. Recall that free commutative monoids exist in any cocomplete monoid category whose tensor product preserves colimits in each variable, given by

\[ FX = \bigoplus_n X^{\text{con}} / \Sigma_n \]

where \( X^{\text{con}} / \Sigma_n \) denotes the \( n \)th tensor power of \( X \) quotiented by the action of the \( n \)th symmetric group. Indeed, commutative monoids are monadic over such a category. Thus, the category of commutative \( \circ \)-monoids in our \( C \) is monadic over \( C \).

Moreover, since \( \circ \) preserves colimits in each variable, by standard arguments it preserves reflexive coequalizers and sequential colimits in both variables together. Thus \( X \mapsto X^{\text{con}} \) also preserves reflexive coequalizers and sequential colimits, hence so does the functor \( F \) and thus the monad for commutative \( \circ \)-monoids. It follows that reflexive coequalizers and sequential colimits in the category of commutative \( \circ \)-monoids are computed as in \( C \), and therefore are preserved in each variable by the lifted tensor product \( \star \). Therefore, by [22], the free \( \star \)-monoid on a commutative \( \circ \)-monoid exists. Composing these two free constructions, we find that free \( \circ \)-commutative duoids exist. \( \square \)

Lemma 7.13. There is an adjunction

\[ \Sigma \text{-net} \quad \leftarrow \quad \text{PROP.} \]

Proof. By Lemmas 7.11 and 7.12, each fiber functor \( U_{1,S} \) has a left adjoint \( F_{1,S} \); thus it remains to piece these adjoints together. Suppose \((S', N) \in \Sigma \text{-net} \) and \((S, B, i) \in \text{PROP} \). By Lemma 7.10, a morphism \((S', N) \to U_1(S, B, i) \) is equivalently given by
a function \( g: S' \to S \) and a morphism \((S', N) \to U_{1,S'}(S', B', i')\) in \( \Sigma\text{-}\text{net}_{S'} \), where \( P S' \overset{i}{\rightarrow} B' \overset{g}{\rightarrow} B \) is the factorization of \( i \circ Pg \) as a bijective-on-objects functor followed by a fully faithful one. But the latter morphism is equivalently a morphism \( F_{1,S'}(S', N) \to (S', B', i') \) in \( \text{PROP}_{S'} \), hence a morphism \( F_{1,S'}(S', N) \to (S, B, i) \) in \( \text{PROP} \). Thus, defining \( F_1(S', N) = F_{1,S'}(S', N) \) yields a left adjoint to \( U_1 \). (Note that it is unnecessary to ask whether \( F_1 \) is cartesian.)

Proof of Theorem 7.7. Combining Lemmas 7.9 and 7.13 we obtain the composite adjunction:

\[
\Sigma\text{-}\text{net} \xrightarrow{\sim} \text{PROP} \xleftarrow{\sim} \text{SSMC}.
\]

We end this section by considering the commutativity properties of the squares in eq. (1).

Proposition 7.14. There is a natural isomorphism \( G_{\text{pre}} \circ U_{\Sigma,\text{net}} \cong U_{\text{PreNet}} \circ U_{\text{StrMC}} \). Therefore, there is also a natural isomorphism \( F_{\text{StrMC}} \circ F_{\text{PreNet}} \cong F_{\Sigma,\text{net}} \circ F_{\text{pre}} \).

Proof. With our precise definitions, the first isomorphism is actually a strict equality; both functors take a symmetric strict monoidal category \( C \) to the pre-net whose places are the objects of \( C \) and whose transitions from a word \( p \) to a word \( q \) are the morphisms in \( C \) from the tensor product of \( p \) to the tensor product of \( q \). The second isomorphism follows by passage to left adjoints.

Recalling from Section 4 that the composite \( F_{\text{StrMC}} \circ F_{\text{PreNet}} \) has been used to give a categorical semantics for pre-nets, we see that this semantics factors through \( \Sigma\text{-}\text{nets} \).

Proposition 7.15. There is a natural isomorphism \( G_{\text{pet}} \circ U_{\text{Petri}} \cong U_{\Sigma,\text{net}} \circ U_{\text{SSMC}} \). Therefore, there is also a natural isomorphism \( F_{\text{SSMC}} \circ F_{\Sigma,\text{net}} \cong F_{\text{Petri}} \circ F_{\text{pet}} \).

Proof. Again, the first isomorphism is a strict equality: both functors take a commutative monoidal category \( C \) to the \( \Sigma\text{-}\text{net} \) whose places are the objects of \( C \) and whose transitions from \( p \) to \( q \) are the morphisms in \( C \) from the tensor product of \( p \) to the tensor product of \( q \), with symmetries acting trivially. The second isomorphism follows by passage to left adjoints.

Though analogous to Proposition 7.14, Proposition 7.15 does not imply that the categorical semantics of Petri nets factors through \( \Sigma\text{-}\text{nets} \). However, that is also true:

Proposition 7.16. There is a natural isomorphism \( F_{\text{Petri}} \cong F_{\text{SSMC}} \circ F_{\Sigma,\text{net}} \circ G_{\text{pet}} \).

Proof. Let \( N \) be a Petri net and \( C \) a commutative monoidal category; since \( G_{\text{pet}} \) is fully faithful we have natural isomorphisms

\[
\text{Petri}(N, U_{\text{Petri}}(C)) \cong \Sigma\text{-}\text{net}(G_{\text{pet}}(N), G_{\text{pet}}(U_{\text{Petri}}(C)))
\]

\[
\cong \Sigma\text{-}\text{net}(G_{\text{pet}}(N), U_{\Sigma,\text{net}}(U_{\text{SSMC}}(C)))
\]

\[
\cong \text{SSMC}(F_{\Sigma,\text{net}}(G_{\text{pet}}(N)), U_{\text{SSMC}}(C))
\]

\[
\cong \text{CMC}(F_{\text{SSMC}}(F_{\Sigma,\text{net}}(G_{\text{pet}}(N))), C).
\]

Thus \( F_{\text{SSMC}} \circ F_{\Sigma,\text{net}} \circ G_{\text{pet}} \) is left adjoint to \( U_{\text{Petri}} \), hence isomorphic to \( F_{\text{Petri}} \). □
8. Relation to whole-grain Petri nets

We now clarify the relation of our work to Kock’s “whole-grain Petri nets” [21]. We show that a whole-grain Petri net can be thought of as a special sort of Σ-net: one that is free on a pre-net. We first recall Kock’s definition:

**Definition 8.1.** A whole-grain Petri net is a diagram

\[
S \leftarrow I \rightarrow T \leftarrow O \rightarrow S
\]

in which the fibers of the functions \( I \rightarrow T \) and \( O \rightarrow T \) are finite. A morphism of whole-grain Petri nets, sometimes called an etale map, is a diagram

\[
S \leftarrow I \rightarrow T \leftarrow O \rightarrow S
\]

This defines the category \( WGPet \).

**Theorem 8.2.** The category \( WGPet \) is equivalent to the full image of \( F_{pre} : \text{PreNet} \rightarrow \Σ\text{-net} \). In other words, there are functors

\[
\text{PreNet} \xrightarrow{Z_1} WGPet \xrightarrow{Z_2} \Σ\text{-net}
\]

such that \( Z_1 \) is essentially surjective, \( Z_2 \) is fully faithful, and the composite \( Z_2 \circ Z_1 \) is isomorphic to \( F_{pre} \).

**Proof.** Given a pre-net \( s, t : T \rightarrow S^* \times S^* \), let \( I \) be the set of transitions \( u \in T \) equipped with a choice of an element of \( s(u) \), and define \( O \) similarly using \( t(u) \). There are forgetful functions \( I \rightarrow T \) and \( O \rightarrow T \), and maps \( I \rightarrow S \) (resp. \( O \rightarrow S \)) that select the chosen element of \( s(u) \) (resp. \( t(u) \)). This defines a whole-grain Petri net \( Z_1(s,t) \). Note that the fibers of \( I \rightarrow T \) and \( O \rightarrow T \) are not just finite but equipped with a linear ordering, and the morphisms in the image of \( Z_1 \) (which is faithful) are precisely those that preserve these orderings.

To see that \( Z_1 \) is essentially surjective, given a whole-grain Petri net \( N \) we choose linear orderings on each fiber of the maps \( I \rightarrow T \) and \( O \rightarrow T \). These orderings associated each element of \( T \) to two elements of \( S^* \), yielding a pre-net whose image under \( Z_1 \) is isomorphic to \( N \).

Now, given a whole-grain Petri net \( S \leftarrow I \rightarrow T \leftarrow O \rightarrow S \), we define a Σ-net in the presheaf perspective. Its set of places is \( S \), and its \((m,n)\)-transitions are elements \( u \in T \) equipped with a linear ordering on the fibers of \( I \) and \( O \) over \( u \), which we require to have \( m \) and \( n \) elements respectively. These linear orderings enable us to define the source and target maps picking out places, while the permutations act on the linear orderings. This defines the functor \( Z_2 \).

Note that the set \( T \) in a whole-grain Petri net \( N \) is naturally isomorphic to the set of transition classes of \( Z_2(N) \). Thus, to show that \( Z_2 \) is fully faithful it remains to show that a morphism \( \alpha : Z_2(N) \rightarrow Z_2(N') \) uniquely determines the maps \( I \rightarrow I' \) and \( O \rightarrow O' \). Given \( i \in I \) lying over \( t \in T \), choose any ordering on the fibers over \( t \), in which \( i \) appears as the \( k^\text{th} \) element of its fiber. This choice determines a transition \( \hat{\alpha}(t) \) of \( Z_2(N) \), and hence a transition \( \alpha(t) \) of \( Z_2(N') \), which is an element \( \hat{\alpha}(t) \) of \( T' \) with ordered fibers. Then the function \( I \rightarrow I' \) can and must send \( i \) to the \( k^\text{th} \) element of the \( I \)-fiber over \( \hat{\alpha}(t) \). This is independent of...
the choice of ordering because $\alpha$ commutes with the permutation actions, and it is straightforward to check that it indeed defines a morphism $N \to N'$.

Finally, the composite $Z_2 \circ Z_1$ preserves the set of places and replaces each $(m, n)$-transition by $m! n!$ transitions with free permutation action; but this is the same as $F_{\text{pre}}$. □

Another construction of the functor $Z_2$ appears in [21], as a restricted Yoneda embedding or “nerve”. Recall the categories $C$ and $D$ from Proposition 4.3 and Theorem 6.1. In fact $D$ is the full image of the composite of the Yoneda embedding $C \hookrightarrow [C^{\text{op}}, \text{Set}] \simeq \text{PreNet}$ with $Z_1 : \text{PreNet} \to \text{WGPet}$; we can then define $Z_2$ as the composite $\text{WGPet} \to [\text{WGPet}^{\text{op}}, \text{Set}] \to [D^{\text{op}}, \text{Set}] \simeq \Sigma\text{-net}$.

9. Open nets

Various kinds of “open” nets have been proposed, which allow one to build nets by gluing together smaller open nets [3, 5, 9, 34]. In earlier work we introduced a symmetric monoidal double category of open Petri nets, where composing open Petri nets is done by identifying places [2]. For example, here is an open Petri net with one transition:

In addition to a Petri net, it consists of sets $X$ and $Y$ and arbitrary functions from these sets into the set of places. These indicate places at which tokens could flow in or out. We may write this open Petri net as $P : X \to Y$ for short.

Given another open Petri net $Q : Y \to Z$:

we can compose it with $P$ and obtain the following open Petri net $Q \circ P : X \to Z$:

We can also tensor open Petri nets, putting them side by side “in parallel”. This might suggest that open Petri nets should be the morphisms of a symmetric monoidal category. However, composition of open Petri is not strictly associative, and there are very interesting maps between open Petri nets. To get a feeling for these, there is a morphism from this open Petri net:

to this one:
mapping both primed and unprimed entities to the corresponding unprimed ones. This particular morphism maps an open Petri net onto a simpler one. There are also morphisms that include open Petri nets into more complicated ones. For example, the above morphism has two right inverses: two ways to include the bottom open Petri net into the top one.

Given that we can compose open Petri net but also compose maps between them, it is natural to formalize them using a symmetric monoidal double category, which we call \( \mathcal{O}p\text{PenPetri} \). To construct this one can use the following result on ‘structured cospans’ [1, 11], which we state in summary form:

**Lemma 9.1.** ([1, Thm. 3.9]) Let \( X \) be a category with finite colimits. Given a left adjoint \( L : \text{Set} \to X \), there is a symmetric monoidal double category \( \mathcal{O}p\text{pen}(X) \) such that:

- objects are sets,
- vertical 1-morphisms are functions,
- a horizontal 1-cell from \( a \in \text{Set} \) to \( b \in \text{Set} \) is a cospan in \( X \) of this form:

\[
La \longrightarrow x \longleftarrow Lb.
\]

- a 2-morphism is a commutative diagram in \( X \) of this form:

\[
\begin{array}{ccc}
La & \longrightarrow & x \longleftarrow Lb \\
\downarrow{L_f} & & \downarrow{L_g} \\
La' & \longrightarrow & x' \longleftarrow Lb'.
\end{array}
\]

Composition of vertical 1-morphisms is composition of functions. Composition of horizontal 1-cells is composition of cospans in \( X \) via pushout. Horizontal composition of 2-morphisms is also done via pushout, vertical composition of 2-morphisms is done via composition in \( X \), and the tensor product and symmetry are defined using chosen coproducts in \( \text{Set} \) and \( X \).

We obtain the symmetric double category \( \mathcal{O}p\text{en(Petri)} \) by applying this result to the functor \( L : \text{Set} \to \text{Petri} \) that is left adjoint to the functor sending any Petri net to its set of places.

The same sort of construction gives symmetric monoidal double categories of ‘open’ versions of all our favorite kinds of nets and categories. Moreover, the construction in Lemma 9.1 is functorial in the following sense:

**Lemma 9.2.** ([1, Thm. 4.2]) Suppose \( X \) and \( X' \) have finite colimits and this triangle of finitely cocontinuous functors commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\text{Set} & \Downarrow{\phi} & X \\
L & \Downarrow{F} & \Downarrow{L'} \\
\end{array}
\]
Then there a symmetric monoidal double functor $\mathbb{O}pen(F) : \mathbb{O}pen(X) \to \mathbb{O}pen(X')$ that acts as follows on 2-morphisms:

$$
\begin{align*}
La & \xrightarrow{i} x \xleftarrow{o} Lb \\
L^i & \downarrow h \downarrow L^g \\
La' & \xrightarrow{i'} x' \xleftarrow{o'} Lb'
\end{align*}$$

$$
\begin{align*}
L'a & \xrightarrow{Fi \circ \phi_a} Fx \xleftarrow{Fo \circ \phi_b} L'b \\
L'^f & \downarrow Fh \downarrow L'^g \\
L'c & \xrightarrow{Fi' \circ \phi_{a'}} Fy \xleftarrow{Fo' \circ \phi_{b'}} L'd.
\end{align*}
$$

It can be checked that given two composable triangles of the above sort:

$$
\begin{align*}
\xymatrix{ 
& X \\
\text{Set} & F & X' \\
& \text{Set} \\
& F' & X'' \\
& \text{Set} 
}
\end{align*}
$$

we have

$$
\mathbb{O}pen(F' \circ F) = \mathbb{O}pen(F') \circ \mathbb{O}pen(F).
$$

Furthermore, isomorphic triangles give isomorphic symmetric monoidal double functors, so we obtain the following result:

**Theorem 9.3.** There is a diagram of symmetric monoidal double functors

$$
\begin{aligned}
\mathbb{O}pen(\text{StrMC}) & \xrightarrow{\mathbb{O}pen(F_{\text{StrMC}})} \mathbb{O}pen(\text{SSMC}) & \xrightarrow{\mathbb{O}pen(F_{\text{SSMC}})} \mathbb{O}pen(\text{CMC}) \\
\mathbb{O}pen(\text{PreNet}) & \xrightarrow{\mathbb{O}pen(F_{\text{PreNet}})} \mathbb{O}pen(\Sigma\text{-net}) & \xrightarrow{\mathbb{O}pen(F_{\Sigma\text{-net}})} \mathbb{O}pen(\text{Petri})
\end{aligned}
$$

where we get each arrow from one of the left adjoints in eq. (1) using Lemma 9.2, and the squares built from arrows going up or right commute up to 2-isomorphism.

10. **Conclusion and future work**

In this work we have systematized the theory of Petri nets, their variants, and their categorical semantics. To this end, we have shown that the notion of $\Sigma$-net, almost absent from standard Petri net literature, is in fact central. Our framework gives a consistent view of the relations between these interacting notions of net in terms of adjunctions, such that the most important adjunctions present in the literature can be recovered as composites of our fundamental ones.

Our work makes substantial use of tools from homotopy theory and related fields, such as groupoids and fibrations. We believe this will open up exciting new directions of research in the study of distributed systems and network theory in general.
In fact, the relationships between the various notions of net in our work have analogues in topology. On one hand, a manifold can always be given “local coordinates”, but it is too restrictive to ask that such coordinates be preserved strictly by maps between manifolds. Such coordinates can be regarded as analogous to the orderings on sources and targets in a pre-net. On the other hand, when a group acts on a manifold, the quotient topological space may no longer be a manifold, but has singularities at points of non-free action. This “coarse moduli space” can be regarded as analogous to a Petri net, where symmetry information has been lost. Kock’s whole-grain Petri nets are analogous to abstract manifolds themselves: they are free of undesirable “coordinates”, but neither can they have singularities. Finally, our Σ-nets play the role of orbifolds, coordinate-free manifold-like structures that retain the information of “isotropy groups” at singular points, yielding a better-behaved notion of quotient.

ACKNOWLEDGEMENTS

The second author was supported by the project MIUR PRIN 2017FTXR7S “IT-MaTTerS” and by the Lest intervention group.

The fourth author was supported by The United States Air Force Research Laboratory under agreement number FA9550-15-1-0053. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon. The views and conclusions contained herein are those of the author and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the United States Air Force Research Laboratory, the U.S. Government, or Carnegie Mellon University.

REFERENCES

[1] J. C. Baez and K. Courser, Structured cospans. Available as arXiv:1911.04630. (Referred to on page 25.)
[2] J. C. Baez and J. Master, Open Petri nets, Math. Str. Comp. Sci. 30 (2020), 314–341. (Referred to on page 2, 3, 5, 11, 24.)
[3] P. Baldan, F. Bonchi, F. Gadducci and G. V. Monreale, Modular encoding of synchronous and asynchronous interactions using open Petri nets, Sci. Comp. Prog. 109 (2015), 96–124. (Referred to on page 24.)
[4] P. Baldan, R. Bruni, U. Montanari, Pre-Nets, read arcs and unfolding: a functorial presentation, in Recent Trends in Algebraic Development Techniques, Springer, Berlin, 2003, pp. 145–164. (Referred to on page 6.)
[5] P. Baldan, A. Corradini, H. Ehrig, and R. Heckel, Compositional semantics for open Petri nets based on deterministic processes, Math. Str. Comp. Sci. 15 (2005), 1–35. (Referred to on page 24.)
[6] M. Batanin and M. Markl, Centers and homotopy centers in enriched monoidal categories, Adv. Math. 23 (2012), 1811–1858. Available as arXiv:1109.4084. (Referred to on page 20.)
[7] R. Bruni, J. Meseguer, U. Montanari and V. Sassone, Functorial semantics for Petri nets under the individual token philosophy, Electron. Notes Theor. Comput. Sci. 29 (1999). (Referred to on page 7.)
[8] R. Bruni, J. Meseguer, U. Montanari and V. Sassone, Functorial models for Petri nets, Information and Computation 170 (2001), 207–236. (Referred to on page 2, 3, 5, 7.)
[9] R. Bruni, H. C. Melgratti, U. Montanari and P. Sobociński, A connector algebra for C/E and P/T nets’ interactions, Log. Meth. Comp. Sci. 9 (2013), 1–65. Available as arXiv:1307.0204. (Referred to on page 24.)
[10] E. Cheng, A direct proof that the category of 3-computads is not cartesian closed. Available as arXiv:1209.0414. (Referred to on page 7.)
[11] K. Courser, *Open Systems: a Double Categorical Perspective*, Ph.D. thesis, U. C. Riverside, 2020. Available as arXiv:2008.02394. (Referred to on page 25.)

[12] P. Degano, J. Meseguer and U. Montanari, Axiomatizing net computations and processes, in *Logic in Computer Science, 1989*, IEEE, New Jersey, pp. 175–185. Available at https://www.computer.org/csdl/proceedings/lics/. (Referred to on page 2.)

[13] H. Ehrig, M. Pfeader and H.J. Schneider, Graph-grammars: An algebraic approach, in *Switching and Automata Theory, 1973*, SWAT’08, IEEE Conference Record of 14th Annual Symposium, IEEE, 1973, pp. 167–180. (Referred to on page 12.)

[14] N. Gambino and A. Joyal, On operads, bimodules and analytic functors, *Memoirs AMS* 249, 2017. Available as arXiv:1405.7270. (Referred to on page 8.)

[15] R. Garner and I. López Franco, Commutativity, *Jour. Pure Appl. Algebra* 220 (2016), 1707–1751. Available as arXiv:1507.08710. (Referred to on page 21.)

[16] F. Genovese, A. Gryzlov, J. Herold, M. Perone, E. Post and A. Videla, Computational Petri nets: adjunctions considered harmful. Available as arXiv:1904.12974. (Referred to on page 2, 7.)

[17] F. Genovese and J. Herold, Integer Petri nets are compact closed categories, *Electron. Proc. Theor. Comput. Sci.* 287 (2019), 127–144. Available as arXiv:1805.05988. (Referred to on page 2.)

[18] C. Girault and R. Valk, *Petri Nets for Systems Engineering: a Guide to Modeling, Verification, and Applications*, Springer, Berlin, 2013. (Referred to on page 1.)

[19] R. J. van Glabbeek and G. D. Plotkin, Configuration structures, event structures and Petri nets, *Theoretical Computer Science* 410 (2009), 4111–4159. Available as arXiv:0912.4023. (Referred to on page 2.)

[20] A. Joyal and R. Street. The geometry of tensor calculus, I. *Adv. Math.* 88 (1991), 55–112. (Referred to on page 5.)

[21] J. Kock, Elements of Petri nets and processes. Available as arXiv:2005.05108. (Referred to on page 3, 12, 19, 23, 24.)

[22] S. Lack, Note on the construction of free monoids, *Appl. Cat. Str.* 18 (2010), 17–29. Available as arXiv:0802.1946. (Referred to on page 21.)

[23] S. Lack and P. Sobociński, Adhesive categories, in *International Conference on Foundations of Software Science and Computation Structures*, Springer, Berlin, 2004, pp. 273–288. (Referred to on page 11.)

[24] S. Lack and P. Sobociński, Toposes are adhesive, in *International Conference on Graph Transformations*, Lecture Notes in Computer Science 4178, Springer, Berlin, 2006, pp. 184–198. (Referred to on page 11.)

[25] F. Loregian and E. Riehl, Categorical notions of fibration. Available as arXiv:1806.06129. (Referred to on page 12.)

[26] M. Makkai and M. Zawadowski, The category of 3-computads is not cartesian closed, *J. Pure Appl. Alg.* 212 (2008), 2543–2546. (Referred to on page 7.)

[27] J. Master, Petri nets based on Lawvere theories, *Math. Struct. Comp. Sci.* 30 (2020), 833–864. Available as arXiv:1904.09091. (Referred to on page 2, 3, 5, 6, 7, 19, 20.)

[28] J. Meseguer and U. Montanari, Petri nets are monoids, *Information and Computation* 88 (1990), 105–155. (Referred to on page 1.)

[29] E. Palmgren, Groupoids and local cartesian closure, Uppsala University Department of Mathematics Technical Report 21 (2003). Available at http://www2.math.uu.se/~palmgren/gpdlcc.pdf. (Referred to on page 13.)

[30] J. L. Peterson, *Petri Net Theory and the Modeling of Systems*, Prentice–Hall, New Jersey, 1981. (Referred to on page 1.)

[31] V. Sassone, Strong concatenable processes: an approach to the category of Petri net computations, *BRICS Report Series*, Dept. of Computer Science, U. Aarhus, 1994. Available at https://tidsskrift.dk/brics/article/view/21610/19059. (Referred to on page 2, 7, 8.)

[32] V. Sassone, On the category of Petri net computations, in CAAP’92: 17th Colloquium on Trees in Algebra and Programming, Lecture Notes in Computer Science 581, Springer, Berlin, 1992. Available at https://eprints.soton.ac.uk/261951/1/strong-conf.pdf. (Referred to on page 2.)
[33] V. Sassone, An axiomatization of the algebra of Petri net concat-
ena ble processes, Theor. Comput. Sci. 170 (1996), 277–296. Available at 
https://eprints.soton.ac.uk/261820/1/P-of-N-Off.pdf. (Referred to on page 2.)

[34] V. Sassone and P. Sobociński, A congruence for Petri nets, Elec-
tron. Notes Theor. Comput. Sci. 127 (2005), 107–120. Available at
https://eprints.soton.ac.uk/262302/1/petriCongPNGToff.pdf. (Referred to on page 24.)

[35] Statebox Team, The mathematical specification of the Statebox language. Available as
arXiv:1906.07629. (Referred to on page 7.)

[36] N. P. Strickland, $K(n)$-local duality for finite groups and groupoids, Topology 39 (2000),
733–772. (Referred to on page 13.)

Department of Mathematics, U. C. Riverside, Riverside, California 92521
Email address: baez@math.ucr.edu

Department of Informatics, University of Pisa, Pisa, Italia 56127
Email address: fabrizio.romano.genovese@gmail.com

Department of Mathematics, U. C. Riverside, Riverside, California 92521
Email address: jmast003@ucr.edu

Department of Mathematics, University of San Diego, San Diego, California 92110
Email address: shulman@sandiego.edu