Noncommutative Black-Body Radiation:
Implications On Cosmic Microwave Background

Amir H. Fatollahi (1) and Maryam Hajirahimi (2)

1) Mathematical Physics Group, Department of Physics, Alzahra University, P. O. Box 19938, Tehran 91167, Iran
2) Institute for Advanced Studies in Basic Sciences (IASBS), P. O. Box 45195, Zanjan 1159, Iran

fatho@mail.cern.ch
rahimi@iasbs.ac.ir

Abstract

Including loop corrections, black-body radiation in noncommutative space is anisotropic. A direct implication of possible space noncommutativity on the Cosmic Microwave Background map is argued.
There have been arguments supporting the idea that the ordinary picture of space-time breaks down when is probed with sufficiently large momenta and energies. In particular, in an ultra-large momentum transfer experiment a black-hole may be formed, and as long as it lives before its rapid evaporation, an observer experiences limits on information transfer from the volume element comparable in size with the horizon [1]. These kinds of reasoning may lead one to believe in some kinds of space-space and space-time uncertainty relations [1]. As uncertainty relations usually point to noncommutative objects, it is reasonable to consider various versions of noncommutative spacetime theories, among them theories defined on spacetime whose coordinates satisfy the canonical relation

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\lambda^{\mu\nu}, \tag{1} \]

in which \(\lambda^{\mu\nu}\) is an antisymmetric constant tensor. Via recent developments in understanding the dynamics of D-branes of string theory, there has been a renewed interest for studying field theories on spacetimes whose coordinates satisfy the above algebra. In particular, the longitudinal directions of D-branes in the presence of constant B-field background appear to be noncommutative, as are seen by the ends of open strings [2].

The phenomenological implications of possible noncommutative coordinates have been considered in a very large number of works. Among many others, here we can give just a brief list of works, and specially those concerning the phenomenological implications of noncommutative QED. The effect of noncommutativity of space is studied for possible modifications that may appear in high energy scattering amplitudes of particles [3], in energy levels of light atoms [4, 5], and anomalous magnetic moment of electron [6]. The ultra-high energy scattering of massless photons of noncommutative U(1) theory is considered in [7] and the tiny change in the total amplitude is obtained as a function of the total energy. Some other interesting features of noncommutative ED and QED are discussed in [8, 9].

In present work we address the radiation we expect from a black-body in noncommutative space. As we deal with a black-body radiation problem, the natural framework is finite temperature field theory. Noncommutative QED, though renormalizable, shares features suggesting that the present formulation of theory possibly has to be modified to be considered as a true theory. In particular, the IR limits of physical quantities are irregular once they are compared with their counterparts in theories defined in ordinary space. In spite of these mentioned difficulties, one might be hopeful that the results obtained based on the present formulation can still offer a sense for what we
should expect as an indication of noncommutativity, if any after all.

It is understood that field theories on noncommutative spacetime are defined by actions that are essentially the same as in ordinary spacetime, with the exception that the products between fields are replaced by $\star$-product, defined for two functions $f$ and $g$ [10]

$$(f \star g)(x) = \exp \left( \frac{i\lambda^{\mu\nu}}{2} \partial_\mu \partial_\nu \right) f(x)g(y) \mid_{y=x} \quad (2)$$

It can be seen that the $\star$-product is associative, i.e., $f \star g \star h = (f \star g) \star h = f \star (g \star h)$, and so it is not important which two should be multiplied firstly. Though $\star$-product itself is not commutative (i.e., $f \star g \neq g \star f$), we have $\int f \star g = \int g \star f = \int fg$, saying in integrands always one of the stars can be removed.

The pure gauge field sector of noncommutative U(1) theory is defined by the action

$$S_{\text{gauge-field}} = -\frac{1}{4} \int d^4x \, F_{\mu\nu} \star F^{\mu\nu} = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} \quad (3)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$, by definition $[f, g]_\star = f \star g - g \star f$. The action above is invariant under local gauge symmetry transformations

$$A'_\mu = U \star A_\mu \star U^{-1} + \frac{i}{e} U \star \partial_\mu U^{-1} \quad (4)$$

in which $U = U(x)$ is the $\star$-phase, defined by a function $\rho(x)$ via the $\star$-exponential:

$$U(x) = \exp_\star (i\rho) = 1 + i\rho - \frac{1}{2} \rho \star \rho + \cdots , \quad (5)$$

with $U^{-1} = \exp_\star (-i\rho)$, and $U \star U^{-1} = U^{-1} \star U = 1$. Under above transformation, the field strength transforms as $F_{\mu\nu} \to F'_{\mu\nu} = U \star F_{\mu\nu} \star U^{-1}$. We mention that the transformations of gauge field as well as the field strength look like those of non-Abelian gauge theories. Besides we see that the action contains terms which are responsible for interaction between the gauge particles. We see how the noncommutativity of coordinates induces properties on fields and their transformations, as if they were belong to a non-Abelian theory; the subject that how the characters of coordinates and fields may be related to each other is discussed in [11].

The other interesting feature of field theories defined by $\star$-product is that these theories exhibit some aspects very reminiscent of string theory. In particular, in these theories the quanta of fields interact as extended objects, namely electric dipoles [12]. Also in these kinds of field theories one recognizes much more distinct role and behavior than ordinary theories for planar and non-planar Feynman diagrams [13].
As we deal with a black-body radiation problem, the natural framework is finite temperature field theory \[14\]. Finite temperature noncommutative field theory has been the subject of research works \[15\], \[16\], \[17\]. As mentioned, noncommutative U(1) gauge theory is involved by self-interaction of photons, and so beyond the free theory one finds deviations from the expression by ordinary U(1) theory for black-body radiation. The Feynman rules of noncommutative U(1) theory are known \[17\]. Here we consider noncommutativity only for spatial directions, assuming $\lambda^{0i} = 0$. By this one can use the expressions already derived for non-Abelian gauge theory \[18\], except that here the vertex-functions are momentum dependent. The expression for free-energy in unit volume at temperature $T$ at two-loop order is given by ($\hbar = c = 1$, $\beta = T^{-1}$) \[15\]:

$$\mathcal{F}(T) = \mathcal{F}_{\text{isotropic}} + 4e^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\sin^2(\mathbf{k} \times \mathbf{k}')}{\omega'(e^{\beta \omega'} - 1)(e^{\beta \omega} - 1)}$$

in which $\mathcal{F}_{\text{isotropic}}$ represents the part that does not depend on the noncommutativity parameter. As we shall see the other part results in an energy flow that depends on direction. In above $\omega = |\mathbf{k}|$ and $\omega' = |\mathbf{k}'|$, and $\mathbf{k} \times \mathbf{k}' = \frac{1}{2} \lambda^{ij} k_i k_j'$. We mention, as pointed earlier, the expression is convergent both in IR ($\omega, \omega' \to 0$) and UV ($\omega, \omega' \to \infty$) limits. For the more important IR limit, the reason comes back to the fact that noncommutativity scale effectively cuts off interactions at large distances \[15\]. Using the relation $U(T) = \mathcal{F} - T \partial_T \mathcal{F}$, we have for the energy-density $U(T)$

$$U(T) = U_{\text{isotropic}} + 4e^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\sin^2(\mathbf{k} \times \mathbf{k}')}{\omega'(e^{\beta \omega'} - 1)(e^{\beta \omega} - 1)} - 4\beta e^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{\sin^2(\mathbf{k} \times \mathbf{k}')}{(e^{\beta \omega'} - 1)(e^{\beta \omega} - 1)} \left(\frac{e^{\beta \omega}}{\omega'} + \frac{e^{\beta \omega'}}{\omega}\right) \tag{7}$$

One may define the vector $\lambda$ by its components $\lambda_k = \frac{1}{2} \epsilon_{ijk} \lambda^{ij}$. By taking $\lambda = \lambda \mathbf{z}$, one finds $\mathbf{k} \times \mathbf{k}' = \lambda \cdot (\mathbf{k} \times \mathbf{k}') = \lambda (\mathbf{k} \times \mathbf{k}')_z$, and so

$$\mathbf{k} \times \mathbf{k}' = \lambda \omega' \sin \theta \sin \theta' \sin(\phi' - \phi) \tag{8}$$

in which $\mathbf{k}$ and $\mathbf{k}'$ are given as $\mathbf{k} = (\omega, \theta, \phi)$ and $\mathbf{k}' = (\omega', \theta', \phi')$ in spherical coordinates.

By the Taylor expansion of $\sin^2 \alpha \tag{19}$, and taking $\alpha = \mathbf{k} \times \mathbf{k}'$, via (8) one has

$$\sin^2(\mathbf{k} \times \mathbf{k}') = \sum_{m=1}^{\infty} a_m \sin^{2m} \theta \sin^{2m} (\phi' - \phi) \tag{9}$$

in which $a_m = \frac{(-1)^m + 1}{(2m)!} (\lambda \omega' \sin \theta)^{2m}$. So we find

$$\int d\Omega' \sin^2(\mathbf{k} \times \mathbf{k}') = \sum_{m=1}^{\infty} \frac{\pi (-1)^m + 1}{(2m + 1)!} (\lambda \omega' \sin \theta)^{2m} \tag{10}$$
For $U(T, \Omega)$ as the energy-density received from the solid-angle $d\Omega$ we find

$$U(T, \Omega) d\Omega = \left[ \frac{\sigma_0}{4\pi} T^4 + \frac{4e^2}{(2\pi)^6} \sum_{m=1}^{\infty} \frac{\pi (-1)^{m+1} 2^{2m+1}}{(2m+1)!} (\lambda^2 T^{1+4m} J_m \sin^{2m} \theta) \right] d\Omega$$  \hspace{1cm} (11)

in which $\sigma_0 = \frac{\pi^2}{15}$ (Stefan’s constant=$\frac{\pi^2}{60}$), and

$$I_m = \left( \int_0^\infty \frac{s^{2m+1} ds}{(e^s - 1)} \right)^2 - 2 \int_0^\infty \frac{s^{2m+2} e^s ds}{(e^s - 1)^2} \int_0^\infty \frac{s^{2m+1} ds'}{(e^{s'} - 1)}. \hspace{1cm} (12)$$

One finds $I_m = -(4m+3)\zeta^2(2m+2)((2m+1)!)^2$ by the following relations

$$\int_0^\infty \frac{s^{2m+1} ds}{(e^s - 1)} = \zeta(2m+2)(2m+1)!,$$

$$\int_0^\infty \frac{s^{2m+2} e^s ds}{(e^s - 1)^2} = \zeta(2m+2)(2m+2)!,$$

with $\zeta(t)$ as the Riemann zeta-function. Finally we have

$$U(\Omega, T) = \frac{\sigma_0}{4\pi} T^4 - \frac{4\pi e^2}{(2\pi)^6} T^4 \sum_{m=1}^{\infty} (\lambda T^2)^{2m} J_m \sin^{2m} \theta \hspace{1cm} (15)$$

in which $J_m = (-1)^{m+1} 2^{2m+1}(4m+3)\zeta^2(2m+2)(2m+1)!$. In leading order one has

$$U(\Omega, T) = \frac{\sigma_0}{4\pi} T^4 - \frac{7\pi^4}{675} \alpha T^4 (\lambda T^2)^2 \sin^2 \theta + O((\lambda T^2)^4) \hspace{1cm} (16)$$

in which $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$.

We mention as much as one comes out from the noncommutative direction, the radiation is decreasing, giving the minimum for $xy$-plane, $\theta = \frac{\pi}{2}$. This is simply due to the fact that the coupling of photons is related to their momenta. In particular the photons moving in the plane perpendicular to noncommutative direction feel the strongest coupling with respect to others, yielding a decrease in outgoing radiation.

The final comment is about the possible contribution of fermionic degrees of freedom to anisotropy. In fact one can check easily that, as the vertex-function for coupling of fermions to gauge fields depends on $\lambda$ only through a phase factor $[17]$, the expression coming from fermionic degrees of freedom, due to cancellation of two phases, is isotropic.

Although it is hard to imagine that the implications of noncommutativity can be detected in a laboratory black-body radiation, one may look for an indication of noncommutativity in the signals we are getting from the extremely hot seconds of early
universe. In fact, the energy scale that one expects for relevance of noncommutative effects is as much as high and this suggests maybe it has been available for particles only in the early universe. So an excellent way to test the phenomenon related to noncommutativity of spacetime would be the study of what are left for us as early universe’s heir, the most important among them the Cosmic Microwave Background (CMB) radiation. The reason is, CMB map is just a tableau of events which happened at the first epochs of universe, at the decoupling era or much earlier, when the energies were sufficiently high to make relevant possible spacetime noncommutativity. In [20] the consequences of space-time uncertainty relations of the form \( \Delta t \Delta x \geq l_s^2 \) are studied in the context of inflation theory, and possible applications of these relations in better understanding of present CMB data are discussed. Also there have been efforts to formulate and study the noncommutative versions of inflation theory [21]. In [22] by taking the blowing sphere that eventually plays the role of the so-called last-scattering surface as a fuzzy sphere some kinds of explanation is presented for the relatively low angular power spectrum \( C_l \) in small \( l \) region (\( l \simeq 6 \)). Recently in [23] it was studied that how a theory with noncommutative electromagnetic fields - that is considering the fields, rather than the coordinates, noncommutative - may change the pattern we expect to see in polarized CMB data.

As CMB map is in fact nothing more than a black-body radiation pattern which is slightly perturbed by fluctuations, instead of dealing with the implications of noncommutativity on different cosmological models, here we can directly address what one should expect to see in CMB map if in early universe the coordinates had satisfied the algebra (1). According to the expression we obtained, space noncommutativity in early universe modifies the pattern we expect to see in the CMB map sky. Replacing \( \sin^2 \theta \) by a combination of \( P_0(\cos \theta) \) and \( P_2(\cos \theta) \) as zeroth and second Legendre polynomials respectively, in leading order the noncommutative effects modify the monopole and quadrupole moments of angular power spectrum. The temperature \( T_0 \) associated to the black-body, defined by \( \int U(\Omega, T) d\Omega = \sigma_0 T_0^4 \), is modified by the monopole term. The term proportional to \( P_2(\cos \theta) \) results in an anisotropy in the measured power as well as the temperature \( T(\Omega) \) that one may associate to the radiation received from the solid-angle \( d\Omega \). The temperature \( T(\Omega) \) is defined by \( T(\Omega) = \left( \frac{4\pi}{\sigma_0} U(\Omega, T) \right)^{1/4} \). Much effort is currently being devoted to examining the CMB temperature anisotropies measured with the Wilkinson Microwave Anisotropy Probe (WMAP) [24]. It would be extremely important if the present and forthcoming data indicated any significant evidence for canonical noncommutativity in the early universe, a thing which could count
anisotropic radiation among its direct implications.

Acknowledgement: A. H. F. is grateful to M. Khorrami, and specially to A. Hajian for very helpful discussions.

References

[1] S. Doplicher, K. Fredenhagen, and J. E. Roberts, “The Quantum Structure Of Spacetime At The Planck Scale And Quantum Fields,” Commun. Math. Phys. 172 (1995) 187, hep-th/0303037; “Spacetime Quantization Induced By Classical Gravity,” Phys. Lett. B331 (1994) 39.

[2] N. Seiberg and E. Witten, “String Theory And Noncommutative Geometry,” JHEP 9909 (1999) 032, hep-th/9908142; A. Connes, M. R. Douglas, and A. Schwarz, “Noncommutative Geometry And Matrix Theory: Compactification On Tori,” JHEP 9802 (1998) 003, hep-th/9711162; M. R. Douglas and C. Hull, “D-Branes And The Noncommutative Torus,” JHEP 9802 (1998) 008, hep-th/9711165; H. Arfaei and M. M. Sheikh-Jabbari, “Mixed Boundary Conditions And Brane-String Bound States,” Nucl. Phys. B526 (1998) 278, hep-th/9709054.

[3] H. Arfaei and M. H. Yavartanoo, “Phenomenological Consequences Of Noncommutative QED,” hep-th/0010244; J. L. Hewett, F. J. Petriello, and T. G. Rizzo, “Signals For Noncommutative Interactions At Linear Colliders,” Phys. Rev. D64 (2001) 075012, hep-ph/0010354; P. Mathews, “Compton Scattering In Noncommutative Space-Time At The NLC,” Phys. Rev. D63 (2001) 075007, hep-ph/0011332; S.-W. Baek, D. K. Ghosh, X.-G. He, and W. Y. P. Hwang, “Signatures Of Noncommutative QED At Photon Colliders,” Phys. Rev. D64 (2001) 056001, hep-ph/0103068; T. M. Aliev, O. Ozcan, and M. Savci, “The $\gamma\gamma \rightarrow H^0H^0$ Decay In Noncommutative Quantum Electrodynamics,” Eur. Phys. J. C27 (2003) 447, hep-ph/0209205; H. Grosse and Y. Liao, “Pair Production Of Neutral Higgs Bosons Through Noncommutative QED Interactions At Linear Colliders,” Phys. Rev. D64 (2001) 115007, hep-ph/0105090.

[4] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, “Hydrogen Atom Spectrum And The Lamb Shift In Noncommutative QED,” Phys. Rev. Lett. 86 (2001)
“Comments On The Hydrogen Atom Spectrum In The Noncommutative Space,” Eur. Phys. J. C36 (2004) 251, hep-th/0212259

[5] N. Chair and M. M. Sheikh-Jabbari, “Pair Production By A Constant External Field In Noncommutative QED,” Phys. Lett. B504 (2001) 141, hep-th/0009037; M. Haghighat, S. M. Zebarjad, and F. Loran, “Positronium Hyperfine Splitting In Noncommutative Space At The Order $\alpha^6$,” Phys. Rev. D66 (2002) 016005, hep-ph/0109105; M. Haghighat and F. Loran, “Helium Atom Spectrum In Non-Commutative Space,” Phys. Rev. D67 (2003) 096003, hep-th/0206019; M. Caravati, A. Devoto, and W. W. Repko, “Noncommutative QED And The Lifetimes Of Ortho And Para Positronium,” Phys. Lett. B556 (2003) 123, hep-ph/0211463

[6] I. F. Riad and M. M. Sheikh-Jabbari, “Noncommutative QED And Anomalous Dipole Moments,” J. High Energy Phys. 0008 (2000) 045, hep-th/0008132

[7] N. Mahajan, “Noncommutative QED And Gamma Gamma Scattering,” hep-ph/0110148

[8] G. Berrino, S. L. Cacciatori, A. Celi, L. Martucci, and A. Vicini, “Noncommutative Electrodynamics,” Phys. Rev. D67 (2003) 065021, hep-th/0210171; C. A. de S. Pires, “Photon Deflection By A Coulomb Field In Noncommutative QED,” J. Phys. G30 (2004) B41, hep-ph/0410120; S. M. Lietti and C. A. de S. Pires, “Testing Non Commutative QED $\gamma\gamma$ And $\gamma\gamma\gamma$ Couplings At LHC,” Eur. Phys. J. C35 (2004) 137, hep-ph/0402034; T. Mariz, C. A. de S. Pires, and R. F. Ribeiro, “Ward Identity In Noncommutative QED,” Int. J. Mod. Phys. A18 (2003) 5433, hep-ph/0211416; S. I. Kruglov, “Maxwell’s Theory On Non-Commutative Spaces And Quaternions,” Annales Fond. Broglie 27 (2002) 343, hep-th/0110059; “Dirac’s Quantization Of Maxwell’s Theory On Non-Commutative Spaces,” Electromagn. Phenom. 3 (2003) 18, quant-ph/0204137; “Generalized Maxwell Equations And Their Solutions,” Annales Fond. Broglie 26 (2001) 125, math-ph/0110008

[9] Z. Guralnik, R. Jackiw, S. Y. Pi, and A. P. Polychronakos, “Testing Non-Commutative QED, Constructing Non-Commutative MHD,” Phys. Lett. B517 (2001) 450, hep-th/0106044

[10] M. R. Douglas and N. A. Nekrasov, “Noncommutative Field Theory,” Rev. Mod. Phys. 73 (2001) 977, hep-th/0106048
[11] A. H. Fatollahi, “Gauge Symmetry As Symmetry Of Matrix Coordinates,” Eur. Phys. J. C17 (2000) 535, hep-th/0007023; “On Non-Abelian Structure From Matrix Coordinates,” Phys. Lett. B512 (2001) 161, hep-th/0103262; “Electrodynamics On Matrix Space: Non-Abelian By Coordinates,” Eur. Phys. J. C21 (2001) 717, hep-th/0104210.

[12] M. M. Sheikh-Jabbari, “Open Strings In A B-Field Background As Electric Dipoles,” Phys. Lett. B455 (1999) 129, hep-th/9901080; K. Dasgupta and M. M. Sheikh-Jabbari, “Noncommutative Dipole Field Theories,” J. High Energy Phys. 0202 (2002) 002, hep-th/0112064; A. H. Fatollahi and H. Mohammadzadeh, “On The Classical Dynamics Of Charges In Noncommutative QED,” Eur. Phys. J. C36 (2004) 113, hep-th/0404209.

[13] T. Filk, “Divergencies In A Field Theory On Quantum Space,” Phys. Lett. B376 (1996) 53; S. Minwalla, M. Van Raamsdonk, and N. Seiberg, “Noncommutative Perturbative Dynamics,” JHEP 0002 (2000) 020, hep-th/9912072; M. Van Raamsdonk and N. Seiberg, “Comments On Noncommutative Perturbative Dynamics,” JHEP 0003 (2000) 035, hep-th/0002186.

[14] J. I. Kapusta, “Finite Temperature Field Theory,” Cambridge Univ. Press, 1989; M. Le Bellac, “Thermal Field Theory,” Cambridge Univ. Press, 1996; A. Das, “Finite Temperature Field Theory,” World Scientific, 1997.

[15] G. Arcioni and M. A. Vazquez-Mozo, “Thermal Effects In Perturbative Noncommutative Gauge Theories,” JHEP 0001 (2000) 028, hep-th/9912140.

[16] W. Fischler, E. Gorbatov, A. Kashani-Poor, S. Paban, and P. Pouliot, “Evidence For Winding States In Noncommutative Quantum Field Theory,” JHEP 0005 (2000) 024, hep-th/0002067; G. Arcioni, J. L. F. Barbon, J. Gomis, and M. A. Vazques-Mozo, “On The Stringy Nature Of Winding Modes In Noncommutative Thermal Field Theory,” JHEP 0006 (2000) 038, hep-th/0004080; K. Landsteiner, E. Lopez, and M. H. G. Tytgat, “Excitations In Hot Non-Commutative Theories,” JHEP 0009 (2000) 027, hep-th/0006210; Y. Kiem, D. H. Park, and H. Sato, “Open String Derivation Of Winding States In Thermal Noncommutative Field theories,” Phys. Lett. B499 (2001) 321, hep-th/0011119; W. Huang, “High-Temperature Effective Potential Of Noncommutative Scalar Field Theory: Reduction Of Degrees Of Freedom By Noncommutativity,” Phys. Rev. D63 (2001)
A. A. Bytsenko, E. Elizalde, and S. Zerbini, “Effective Finite Temperature Partition Function For Fields On Non-Commutative Flat Manifolds,” Phys. Rev. D64 (2001) 105024, hep-th/0103128; L. C. T. de Brito, M. Gomez, S. Perez, and A. J. da Silva, “Radiative Corrections To The Chern-Simons Term At Finite Temperature In Noncommutative Chern-Simons-Higgs Model,” J. Phys. A37 (2004) 9989, hep-th/0301124; F. T. Brandt, A. Das, J. Frenkel, S. Pereira, and J. C. Taylor, “The Static Effective Action For Non-Commutative QED At High Temperature,” Phys. Rev. D67 (2003) 105010, hep-th/0212090.

[17] F. T. Brandt, A. Das, and J. Frenkel, “Classical Transport Equation In Non-Commutative QED At High Temperature,” Phys. Rev. D66 (2002) 105012, hep-th/0208115.

[18] J. I. Kapusta, “Quantum Chromodynamics At High Temperature,” Nucl. Phys. B148 (1979) 461.

[19] I. S. Gradshteyn and I. M. Ryzhik, “Table Of Integrals, Series, And Products,” Academic Press, 6th Edition (2000), page 42, 1.412 (1).

[20] S. Tsujikawa, R. Maartens, and R. Brandenberger, “Non-Commutative Inflation And The CMB,” Phys. Lett. B574 (2003) 141, astro-ph/0308169; Q. G. Huang and M. Li, “Noncommutative Inflation And The CMB Multipoles,” JCAP 0311 (2003) 001, astro-ph/0308458; G. Calcagni and S. Tsujikawa, “Observational Constrictains On Patch Inflation In Noncommutative Spacetime,” Phys. Rev. D70 (2004) 103514, astro-ph/0407543; G. Calcagni, “Noncommutative Models In Patch Cosmology,” Phys. Rev. D70 (2004) 103525, hep-th/0406006; Y. S. Myung, “Cosmological Parameters In Noncommutative Inflation,” Phys. Lett. B601 (2004) 1, hep-th/0407066; Q. G. Huang and M. Li, “Power Spectra In Spacetime Noncommutative Inflation,” Nucl. Phys. B713 (2005) 219, astro-ph/0311378; H. Kim, G. S. Lee, H. W. Lee, and Y. S. Myung, “Second-Order Corrections To Noncommutative Spacetime Inflation,” Phys. Rev. D70 (2004) 043521, hep-th/0402198; H. Kim, G. S. Lee, and Y. S. Myung, “Noncommutative Spacetime Effect On The Slow-Roll Period Of Inflation,” Mod. Phys. Lett. A20 (2005) 271, hep-th/0402018.

[21] F. Lizzi, G. Mangano, G. Miele, and M. Peloso, “Cosmological Perturbations And Short Distance Physics From Noncommutative Geometry,” JHEP 0206 (2002) 049, hep-th/0203099; F. Nasseri and S. A. Alavi, “Noncommutative Decrumpling Inflation And Running Of The Spectral Index,” Int. J. Mod. Phys. D14
S. A. Alavi and F. Nasseri, “Running Of The Spectral Index In Noncommutative Inflation,” Int. J. Mod. Phys. A20 (2005) 4941, astro-ph/0406477. C. Chu, B. R. Greene, and G. Shiu, “Remarks On Inflation And Noncommutative Geometry,” Mod. Phys. Lett. A16 (2001) 2231-2240, hep-th/0011241. G. Amelino-Camelia and S. Majid, “Waves On Noncommutative Spacetime And Gamma-Ray Bursts,” Int. J. Mod. Phys. A15 (2000) 4301, hep-th/9907110. G. D. Barbosa and N. Pinto-Neto, “Noncommutative Geometry And Cosmology,” Phys. Rev. D70 (2004) 103512, hep-th/0407111. R. H. Brandenberger and J. Martin, “On Signatures Of Short Distance Physics In The Cosmic Microwave Background,” Int. J. Mod. Phys. A17 (2002) 3663, hep-th/0202142. S. Alexander, R. Brandenberger, and J. Magueijo, “Non-Commutative Inflation,” Phys. Rev. D67 (2003) 081301, hep-th/0108190.

M. Fukuma, Y. Kono, and A. Miwa, “Effects Of Space-Time Noncommutativity On The Angular Power Spectrum Of The CMB,” Nucl. Phys. B682 (2004) 377, hep-th/0307029. M. Fukuma, Y. Kono, and A. Miwa, “A Mechanism Of The Large-Scale Damping In The CMB Anisotropy,” Nucl. Phys. B703 (2004) 293, hep-th/0312298. M. Fukuma, Y. Kono, and A. Miwa, “Noncommutative Inflation And The Large-Scale Damping In The CMB Anisotropy,” hep-th/0401153.

J. Gamboa, J. Lopez-Sarrion, and A. P. Polychronakos, “Ultraviolet Modified Photons And Anisotropies In The Cosmic Microwave Background Radiation,” Phys. Lett. B634 (2006) 471, hep-ph/0510113.

C. L. Bennett et al., “First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps And Basic Results,” Astrophys. J. Suppl. 148 (2003) 1, astro-ph/0302207. “First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Foreground Emission,” Astrophys. J. Suppl. 148 (2003) 97, astro-ph/0302208. G. Hinshaw et al., “First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Angular Power Spectrum,” Astrophys. J. Suppl. 148 (2003) 135, astro-ph/0302217. D. N. Spergel et al., “First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination Of Cosmological Parameters,” Astrophys. J. Suppl. 148 (2003) 175, astro-ph/0302209.