The influence of flux balance on the generalized chemical potential in mass transport models

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Abstract. In equilibrium systems, the conservation of the number of particles (or mass) leads to the equalization of the chemical potential throughout the system. Using a non-equilibrium generalization of the notion of a chemical potential, we investigate the influence of the balance of mass fluxes on the generalized chemical potential in the framework of stochastic mass transport models. We focus specifically on the issue of local measurements of the chemical potential. We find that the presence of a branching geometry leads to unequal local measurement results at different points of the system. We interpret these results in terms of mass flux balance, and argue that the conditions for the global definition of the chemical potential still hold, but that local measurements fail to capture the global theoretical value.

Keywords: driven diffusive systems (theory), zero-range processes
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1. Introduction

The question of defining relevant macroscopic control parameters in non-equilibrium systems still remains largely open [1, 2]. At equilibrium, thermodynamics provides us with intensive parameters like temperature, pressure and chemical potential, that are uniform throughout the system even in the presence of heterogeneities, and that can in most cases be easily measured. In non-equilibrium steady states, it is natural to try to find parameters of similar types, and different generalizations of equilibrium concepts have been proposed either through statistical approaches often related to entropy notions [3]–[7], or using generalized fluctuation-dissipation relations in theoretical [8]–[15] and experimental [16]–[19] contexts. A few studies [8], [20]–[22] have more explicitly considered the question of the equalization of such parameters throughout inhomogeneous systems, but this issue has not been settled yet.

Such a question has also been addressed in previous works [23, 24], where the equilibrium definitions of intensive thermodynamic parameters conjugated to conserved quantities have been extended to some classes of non-equilibrium steady-state systems. Provided that a condition called the ‘asymptotic factorization property’ holds [23, 24],

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this approach provides a definition of an out-of-equilibrium chemical potential. The main properties of this generalized notion of a chemical potential have been studied in [24], where it was shown to exhibit some interesting equalization properties in some classes of inhomogeneous non-equilibrium systems. Possible difficulties arising from the non-equilibrium nature of the systems considered and especially from the dynamics at the contact between different systems have also been outlined.

In the present work, we study whether the globally defined chemical potential mentioned above can be evaluated through local measurements, and whether measurements performed on the system at different locations yield the same value, which is a non-trivial issue when the system is inhomogeneous. We investigate in particular how the flux balance—a strong constraint specific to non-equilibrium situations—influences the results of local measurements of the chemical potential. As simple examples of inhomogeneous non-equilibrium systems, we consider different models belonging to the class of mass transport models introduced in [26, 27], in which a globally conserved mass is transferred between neighbouring sites.

The paper is organized as follows. Section 2 presents the generic definition of mass transport models previously introduced in the literature. Section 3 recalls the definition of the generalized notion of a chemical potential, and illustrates on simple mass transport models the local measurement techniques. Section 4 then presents a numerical study of a mass transport model with a three-branch geometry, for which the locally measured values of the chemical potential are found to differ from one branch to the other. In order to elucidate the origin of these discrepancies, a simpler version of the three-branch model, in which masses take only discrete values, is studied in section 5. The generality of the resulting tentative scenario is discussed in section 6.

2. Mass transport models

2.1. The definition

In generic mass transport models, a positive mass $m_i$ resides at each site $i = 1, \ldots, N$ of an arbitrary network. In the most general case, the continuous time stochastic dynamics, which preserves the total mass $M = \sum_{i=1}^{N} m_i$, is defined by rates of transport $\varphi_{ij}(\mu|m_i)$ from site $i$ to site $j$ of the form

$$\varphi_{ij}(\mu|m_i) = v_{ij}(\mu) \frac{f_i(m_i - \mu)}{f_i(m_i)},$$

(1)

where the function $v_{ij}(\mu)$ is identically zero if there is no directed link from $i$ to $j$.

For practical purposes, a simple example consists in the ring geometry. In this case, an amount of mass $\mu$ is transferred from site $i$, containing the mass $m_i$, to site $i+1$ with a probability per unit time $p \varphi_i(\mu|m_i)$, and to site $i-1$ with a probability per unit time $q \varphi_i(\mu|m_i)$, where $q = 1 - p$, as illustrated in figure 1 (note that $N \equiv 0$ and $N + 1 \equiv 1$ due to the periodic boundary conditions). The rate $\varphi_i(\mu|m_i)$ is defined as

$$\varphi_i(\mu|m_i) = v(\mu) \frac{f_i(m_i - \mu)}{f_i(m_i)},$$

(2)

where $v(\mu)$ and $f_i(m)$ are arbitrary positive functions, with $f_i(m)$ possibly site dependent. Thus transport is biased (except if $p = q$), which generates a flux of mass along the ring. To
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Figure 1. Schematic drawing of the dynamics of the mass transport model in a one-dimensional geometry. A fraction $\mu$ of the mass $m_i$ situated at site $i$ is transferred according to the local conditional rates $\varphi_i(\mu|m_i)$, either to site $i+1$ with probability $p$, or to site $i-1$ with probability $q = 1 - p$.

make contact with the generic form given in equation (1), we thus have here $v_{ij}(\mu) = pv(\mu)$ if $j = i + 1$, $v_{ij}(\mu) = qv(\mu)$ if $j = i - 1$, and $v_{ij}(\mu) = 0$ otherwise.

2.2. The factorized steady state

For generic mass transport models, a sufficient condition for the factorization of the probability distribution has been given in [28]. If the condition

$$\sum_{j \neq i} v_{ij}(\mu) = \sum_{j \neq i} v_{ji}(\mu) \quad (3)$$

holds for every site $i$, the steady-state distribution takes the factorized form [26]

$$P\{\{m_i\}\} = \frac{1}{Z(M)} \prod_{i=1}^{N} f_i(m_i) \delta \left( \sum_{i=1}^{N} m_i - M \right), \quad (4)$$

with $\delta(x)$ the Dirac delta function, and where the partition function $Z(M)$ is defined as

$$Z(M) = \int \prod_{i=1}^{N} [dm_i f(m_i)] \delta \left( \sum_{i=1}^{N} m_i - M \right). \quad (5)$$

The condition (3) is automatically satisfied in the above example of the ring geometry. Note that the function $v(\mu)$ does not influence the steady-state distribution, but only the dynamics. When performing numerical simulations, we use throughout the paper $v(\mu) = 1$. 

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3. The notion of a generalized chemical potential

3.1. The definition and the asymptotic factorization condition

A generalized definition of a chemical potential (and of any intensive thermodynamic parameter conjugated to a conserved quantity) in non-equilibrium systems has been proposed in [23,24], in analogy to the equilibrium approach. Considering a system with a globally conserved mass, one can split it into two subsystems $S_1$ and $S_2$, and introduce the conditional probability distribution $\Psi(M_1|M)$ that the mass of $S_1$ is $M_1$, given that the total mass is fixed at a value $M$. The definition of the chemical potential then relies on the so-called ‘asymptotic factorization property’, which basically states that for a large system, the logarithm of $\Psi(M_1|M)$ can be expressed, to leading order, as a sum of a function of $M_1$, a function of $M_2 = M - M_1$, and a function of $M$:

$$\ln \Psi(M_1|M) = \ln Z_1(M_1) + \ln Z_2(M - M_1) - \ln Z(M) + \epsilon(M_1,M), \quad (6)$$

where $\epsilon(M_1,M)$ is required to become negligible with respect to $\ln \Psi(M_1|M)$ in the large system size limit. Qualitatively, the asymptotic factorization condition is satisfied when only short range correlations are present in the system. In particular, if the steady state of the system is factorized as in equation (4), condition (6) holds exactly ($\epsilon = 0$), and $Z$, $Z_1$ and $Z_2$ respectively correspond to the partition functions of the global system and of the subsystems $S_1$ and $S_2$.

Provided that condition (6) is fulfilled [23,24], the approach developed in [23,24] allows one to define an out-of-equilibrium chemical potential $\lambda$ as

$$\lambda = \frac{\partial \ln Z}{\partial M}. \quad (7)$$

In the following, this definition is referred to as the global chemical potential, and we wish to compare it to local measurements, according to the procedures explained below.

3.2. Application to mass transport models with a factorized state

We first apply the above definition of the chemical potential to the simple case of mass transport models with factorized steady state, for which the factorization condition (6) holds without corrections. Let us consider for $f_i(m_i)$ the power-law form $f_i(m_i) = m_i^{\eta_i-1}$ with $\eta_i > 0$ for all $i$. To calculate the generalized chemical potential corresponding to the conserved mass in the system we need to find the dependence of the partition function $Z$ on $M$:

$$Z(M) = \int \prod_{i=1}^{N} [dm_i m_i^{\eta_i-1}] \delta \left( \sum_{i=1}^{N} m_i - M \right), \quad (8)$$

where the integrals are over the positive real axis. A simple rescaling $m_i = x_i M$ reveals the searched for dependence:

$$Z(M) = M^{\sum_{i=1}^{N} \eta_i - 1} \int \prod_{i=1}^{N} [dx_i x_i^{\eta_i-1}] \delta \left( \sum_{i=1}^{N} x_i - 1 \right) = D_N M^{N \eta - 1} \quad (9)$$

Note that this generalized definition of the chemical potential differs by a factor $-1/T$ (where $T$ is the temperature) from the conventional equilibrium definition [25].

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with \( \bar{\eta} = N^{-1} \sum_{i=1}^{N} \eta_i \), and where \( D_N \) is a constant independent of \( M \). The generalized chemical potential is obtained from the derivative of \( \ln Z \)

\[
\lambda = \frac{d \ln Z}{dM} = \frac{N\bar{\eta} - 1}{M} \tag{10}
\]

leading in the thermodynamic limit \( N \to \infty \) to

\[
\lambda = \frac{\bar{\eta}}{\rho} \tag{11}
\]

where \( \rho = M/N \) denotes the average density.

3.3. Local measurement of the chemical potential with a probe

Once the chemical potential has been theoretically defined, an important issue is to know whether it can be measured. One possible way to perform a measurement is to connect to the system considered a probe system, with a much smaller size, in order not to significantly perturb the main system. The probe system is assumed to have a known equation of state, so its chemical potential can be deduced from its average mass. The connection between the two systems is in general local—just like when a thermometer is put into contact with a system to measure its temperature.

As an illustration of the measurement procedure, we consider a mass transport model on a ring with transport rates \( \varphi_i(\mu|m_i) \) defined according to equation (2), where the weight function \( f_i(m) \) is assumed to take the power-law form \( f_i(m) = m^{\eta_i-1} \). This choice is a simple parametrization of the function \( f_i(m_i) \), with a scale invariant form allowing for a straightforward calculation of the chemical potentials; see equations (8)–(11).

For the analytically obtained chemical potential in equation (11) the local exponent \( \eta_i \) is a priori arbitrary. This result only depends on the mean value \( \bar{\eta} \). To test the local measurement technique we perform numerical measurements for inhomogeneous systems with either a slow space dependence of \( \eta_i \), or random uncorrelated values on each site. Only the average value of \( \bar{\eta} \) is chosen to be fixed. For simplicity we set \( p = 1 \) in the numerical simulations. Measuring the chemical potential by connecting a probe system allows us to check whether the local measurement method works well for arbitrary quenched configurations of \( \eta_i \).

A very simple way to obtain a slowly varying value of \( \eta_i \) in a one-dimensional geometry with periodic boundary conditions is to choose a sinusoidal space dependence for \( \eta_i \), of the form \( \eta_i = 2 + \sin(2\pi i/N) \), \( i = 1, \ldots, N \). The space average of \( \eta_i \) is then given by \( \bar{\eta} = 2 \). As for the probe system, we use a homogeneous mass transport model, with site-independent rates \( \varphi_{\text{pr}}(\mu|m_i) \) defined with \( f_{\text{pr}}(m) \equiv 1 \), that we successively attach to sites \( i = N/4 \) and \( 3N/4 \) of the inhomogeneous system (see figure 2). The chemical potential \( \tilde{\lambda} \) of the probe is determined from its known equation of state \( \lambda = 1/\rho_{\text{pr}} \), by measuring the average density \( \rho_{\text{pr}} \) of the probe system. Note that the equation of state is assumed to be valid even in the time-dependent regime, the relaxation of the probe being much faster than that of the main system.

Figure 3 shows that the chemical potential \( \tilde{\lambda}(t) \) of the probe converges to the chemical potential of the main system (computed from its equation of state). The results are seen to be independent of the location where the probe is attached to the system, in spite of the presence of disorder. We further observe that even in a completely disordered system,
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Figure 2. Sketch of the contact of a small probe system (right) with a large system (left). Mass is transferred in the direction of the arrows with the indicated transport rates (see the text).

Figure 3. Chemical potentials $\tilde{\lambda}(t)$ of the probe system of size $N_{pr} = 512$ (upper curves) and $\lambda(t)$ of the main system of size $N = 15870$ (lower curve) during the measurement process, in three different situations: (i) an inhomogeneous system with $\eta_i = 2 + \sin(2\pi i/N)$, and the probe connected to $i = N/4$ (red curve); (ii) the same system, with the probe connected to $i = 3N/4$ (green curve); (iii) an inhomogeneous system with random and uncorrelated values of $\eta_i$ on each site, drawn from a uniform distribution on the interval $1 < \eta_i < 3$ (blue curve). In all cases, the mean value $\overline{\eta} = 2$ is the same. The contact is switched on at time $t = 0$ (vertical dotted line). The initial densities are $\rho = 2$ for the main system and $\rho_{pr} = 0.2$ for the probe. All measurements converge to the theoretically expected value (dashed line).

in which the $\eta_i$ are independent and identically distributed quenched random variables, the measured value of the chemical potential is still the same as long as the mean value $\overline{\eta} = (1/N) \sum_i \eta_i$ remains the same as in the deterministic case.

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3.4. Measurement using local fluctuations

An alternative way to determine a local chemical potential without using an external device is to measure the variance of the fluctuating mass $M_b$ in a small box, as a function of the average density $\rho$ in the box [24]. If correlations are weak enough, the variance is extensive with the box size $N_b$, so

$$\text{Var}(M_b) \equiv \langle M_b^2 \rangle - \langle M_b \rangle^2 = N_b g(\rho).$$

(12)

The function $g(\rho)$ can be determined numerically by plotting the variance as a function of the system size. Then, using the grand canonical ensemble [24], the chemical potential can be related to the function $g(\rho)$ according to

$$g(\rho) = -\frac{d\rho}{d\lambda},$$

(13)

where we denote as $\lambda$ the local value of the chemical potential. Integrating equation (13), one finds

$$\lambda = \int_{\rho}^{\infty} d\rho \frac{d\rho}{g(\rho)},$$

(14)

with the assumption that $\lambda \to 0$ when $\rho \to \infty$. Hence, the knowledge of the function $g(\rho)$ allows for the determination of the local chemical potential $\lambda$. For a detailed description of this procedure see [24].

As we showed in an earlier work [24] this second measurement method, based on fluctuations in subsystems, should always work in this simple one-dimensional geometry as long as the asymptotic factorization condition given in equation (6) is fulfilled. This condition is in particular automatically fulfilled when the steady state is completely factorized. But even in more complicated cases, like for example in systems exhibiting pair-factorized steady states, such a measurement can be meaningful [24]. The first measurement method with an external probe system, described in section 3.3, is somewhat more complicated, because it requires additional conditions on the contact of the probe to the system to ensure flux balance, as explained in [24]. The numerical implementation of the local measurement in section 3.3 does indeed fulfil these requirements.

4. A mass transport model with three branches

4.1. The definition of the model

We now investigate the issue of the local measurement of the chemical potential in a model with a branching geometry, in order to study in more detail the effect of the balance of mass fluxes. We consider a mass transport model with three branches, corresponding to the geometry displayed in figure 4. The three branches are assumed to be oriented. The transfer rate from site $i$ to the neighbouring site is $p\varphi(\mu|m_i)$ along the positive direction (according to the orientation of each branch) and $q\varphi(\mu|m_i)$ along the negative direction. The rate $\varphi(\mu|m)$ is defined according to equation (2), with a site-independent weight function $f(m)$. At the branching points ($A \to B$ or $C$), the probability rates for the transfer to branches $B$ and $C$ are reweighted by factors $\gamma_B$ and $\gamma_C$, as shown in figure 4. For instance, at the lower branching point in figure 4, the transfer from branch $A$ occurs.
Figure 4. Sketch of the model with three branches. Left panel: mass is transported along each oriented branch according to the rate \( p_\varphi(\mu, m) \) in the positive direction, and the rate \( q_\varphi(\mu, m) \) in the negative direction. Right panel: zoom on the branching points, where specific rules are taken into account, some of the transfer rates being reweighted by the factors \( \gamma_B \) and \( \gamma_C \).

with rate \( \gamma_B p_\varphi(\mu|m_i) \) to branch B, and with rate \( \gamma_C p_\varphi(\mu|m_i) \) to branch C. Due to this specific geometry, the probability distribution does not necessarily factorize, even with the choice of transport rates \( \varphi(\mu|m) \) given in equation (2).

In the case of the model with three branches, the branching sites violate the sufficient condition for factorization given as equation (3), which is a strong indication that strict factorization does not hold. Hence the exact solution of this model is not known, and it is not clear a priori whether the present model does or does not satisfy the asymptotic factorization condition (6), which is a key criterion for the existence of a globally defined chemical potential. However, although the strict factorization property is likely to be violated, one might ask whether this violation is ‘localized’ around the branching points and if the local chemical potential in the bulk of the branches remains uniform. It is thus interesting in this situation to perform numerical measurements of the local chemical potential.

4.2. Local measurement of the chemical potential

To test this issue, we measure the chemical potential with a probe system attached to a bulk site of the branch considered, in analogy to the procedure explained in section 3.3. The transport rates are the same in all branches as well as in the probe system, \( \varphi(\mu|m) \equiv 1 \). All three branches have the same number of sites \( N_b \), so the system size is \( N = 3N_b \) (to be specific, the two branching points are included in branch A). Simulations are done with \( p = 1 \) and \( \gamma_B = \gamma_C = \frac{1}{2} \). The results of the numerical implementation of the measurement are shown in figure 5. The locally measured chemical potential is denoted as \( \tilde{\lambda}_\nu \) in branch \( \nu = A, B, \) or C. It turns out that we obtain equal values for the locally measured chemical potential in branches B and C, but a very different value for branch A. In the rest of the paper, we shall try to understand the origin of this discrepancy between the values measured in the different branches.
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Figure 5. Chemical potentials $\tilde{\lambda}_\nu$, $\nu = A, B, C$ measured with a probe system attached to branches $A$, $B$ and $C$, plotted as a function of time $t$ after the contact is switched on at $t = 0$. The transport rates are the same in all branches and in the probe system, $\varphi(\mu|m_i) \equiv 1$. The dynamics at the contact is defined as shown in figure 2. The initial value of the global density is $\rho = 0.5$ in the main system and $\rho_{pr} = 0.2$ in the probe system. All three branches have the same size $N_b = 43348$, and the probe size is $N_{pr} = 1024$. Measurements on each branch have been performed during separate runs.

We first check whether this result, obtained by probing the system, depends on the measurement method. With this aim, we apply the alternative measurement technique presented in section 3.4, that does not require any external device. Let us first define the quantity $g_\nu$ ($\nu = A, B, C$) as the variance of the total mass $M_\nu$ in branch $\nu$ divided by the number of sites $N_\nu$ in this branch:

$$g_\nu = \frac{\langle M_\nu^2 \rangle - \langle M_\nu \rangle^2}{N_\nu}. \tag{15}$$

We have checked numerically that $g_\nu$ does not depend on the size of the subsystem chosen within a given branch. Consequently $g_\nu$ is an intensive quantity depending only on the local density, $g_\nu = g_\nu(\rho_\nu)$.

We choose the same parameters for the dynamics on the three branches as in section 4.2. Each branch is of the same size $N_b$ and the transport rates are given by $\varphi(\mu|m) \equiv 1$ everywhere. The result of a numerical implementation of the measurement of the function $g_\nu(\rho_\nu)$ in the three-branch model is shown in the left panel of figure 6. The functional behaviour of $g_\nu(\rho_\nu)$ is seen to be the same for the three branches, and we denote this function simply as $g(\rho_\nu)$. We further observe in figure 6 that the numerically measured $g(\rho_\nu)$ is very close to the theoretical value $g(\rho_\nu) = \rho^2$ corresponding to a homogeneous system with the same local dynamics $\varphi(\mu|m) = 1$, so we shall use this theoretical expression in the following.

Following the same procedure as the one that led to equation (14), we get

$$\tilde{\lambda}_\nu = \Lambda(\rho_\nu) \equiv \int_{\rho_\nu}^{\infty} \frac{d\rho}{g(\rho)}. \tag{16}$$

This means that the functional behaviour of the local chemical potential $\Lambda$ with density $\rho_\nu$, that is the local equation of state, is the same in the three branches, namely $\Lambda(\rho_\nu) = \rho_{\nu}^{-1}$.
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Figure 6. Left panel: measure of the function $g_\nu(\rho_\nu)$ in each branch A, B and C. The same functional form, very close to the theoretical form $g(\rho) = \rho^2$ corresponding to a homogeneous system, is obtained in each case. Transport rates: $\varphi(\mu|m) \equiv 1$. All branches have the same size $N_\nu = 341$. Right panel: locally measured chemical potential $\tilde{\lambda}_\nu = \Lambda(\rho_\nu)$ in each branch $\nu = A, B, C$, plotted as a function of the total density in the system $\bar{\rho} = M/N$ (from simulations). The expected value of $\tilde{\lambda}_\nu$ for $\varphi(\mu|m) = 1$ is $\tilde{\lambda}_\nu = \rho_\nu^{-1}$ (see the text).

for the specific dynamics chosen here. This result is not surprising since the local dynamics is the same everywhere and although the probability distribution does not factorize, only weak correlations are expected in the bulk of each branch. Taking into account the fact that the local densities $\rho_\nu$ in the three branches are not equal, it follows that the values $\tilde{\lambda}_\nu = \Lambda(\rho_\nu)$ of the local chemical potential of the three branches differ as well: $\tilde{\lambda}_A \neq \tilde{\lambda}_B = \tilde{\lambda}_C$.

4.3. Interpretation in terms of flux balance

This discrepancy can be understood by taking into account the balance of fluxes. To this end, let us compute the flux of mass $\Phi$ crossing a site $i$ in the bulk of a given branch. Considering the rest of the system as a reservoir of mass, one finds for the probability distribution on site $i$

$$p_i(m) = \frac{1}{Q_i} f_i(m) e^{-\tilde{\lambda} m_i},$$

where $Q_i$ is a normalization constant, and $\tilde{\lambda}$ is the local chemical potential. The mass flux $\Phi$ crossing site $i$ can be expressed as

$$\Phi = (p - q) \int_0^\infty dm \, p_i(m) \int_0^m d\mu \varphi_i(\mu|m),$$

which can be rewritten, using equations (17) and (2), as

$$\Phi = (p - q) \int_0^\infty d\mu \mu v(\mu) e^{-\tilde{\lambda} \mu}.$$  

From the expression (19) for the flux $\Phi$, we first note that if the function $v(\mu)$ in the transport rates is identically 1, $v(\mu) \equiv 1$, then the flux is given by $\Phi = 1/\tilde{\lambda}^2$ (note that
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Figure 7. Left panel: rescaled values \( \tilde{\lambda}^{(\nu)} \), defined as \( \tilde{\lambda}^{(\nu)} = \tilde{\lambda}_A \) and \( \tilde{\lambda}^{(\nu)}_{B,C} = \tilde{\lambda}_{B,C}/\sqrt{2} \), plotted as a function of \( \bar{\rho} \), showing a collapse of the data (the same data as in the right panel of figure 6). Right panel: differences in the locally measured chemical potentials of systems B and A (\( \times \)), and systems C and A (+) as a function of \( \bar{\rho} \) compared to the analytical prediction \( \alpha(\bar{\rho}) = 1/(3\bar{\rho}) \) (black line) as discussed in section 6.

Figure 7 shows the rescaled values of the chemical potential for different branches. The left panel displays the rescaled values as a function of \( \bar{\rho} \), and the right panel shows the differences in the locally measured chemical potentials compared to the analytical prediction. These results highlight the impact of flux balance on the generalized chemical potential.

The expression for the flux is independent of the form of \( f(m) \). We use here the local chemical potential \( \tilde{\lambda} \), as the flux is governed by the local dynamics—see equation (19). We shall come back to this point in section 5. In the bulk of the branches, where the correlations are expected to be small, the different fluxes are given by \( \Phi_A = 1/\tilde{\lambda}_A^2 \) and \( \Phi_C = \Phi_B = 1/\tilde{\lambda}_B^2 \) (the fluxes in branches B and C are equal since \( \gamma_B = \gamma_C \)). Moreover we know that the flux in branch A is twice the flux in branch B, which leads to the following relation between the chemical potentials of the different branches:

\[
\tilde{\lambda}_A = \frac{\tilde{\lambda}_B}{\sqrt{2}} = \frac{\tilde{\lambda}_C}{\sqrt{2}}.
\]

This relation has been verified numerically for \( \varphi(\mu|m) \equiv 1 \), where we expect analytically \( \tilde{\lambda}_\nu = \rho_\nu^{-1} \), \( \nu = A, B, C \)—see the right panel of figure 6 and the left panel of figure 7 for the numerical results. We can generalize this flux balance argument to situations where branches B and C do not have the same flux, in which case the relation reads

\[
\tilde{\lambda}_A = \sqrt{\frac{\Phi_B}{\Phi_A}} \tilde{\lambda}_B = \sqrt{\frac{\Phi_C}{\Phi_A}} \tilde{\lambda}_C.
\]

Note that at equilibrium, when the fluxes vanish, this correction is not present and all chemical potentials equalize. But as soon as a little bias is introduced in the dynamics, the value of the chemical potential in branch A differs from that of the two other branches. Let us emphasize that this difference is not perturbative with respect to the bias: \( \tilde{\lambda}_A - \tilde{\lambda}_B \) does not go to zero when \( p - q \to 0 \), but rather remains constant as long as \( p > q \).

From a more theoretical perspective, the discrepancy between \( \tilde{\lambda}_A, \tilde{\lambda}_B \) and \( \tilde{\lambda}_C \) questions the validity of the asymptotic factorization condition (6) required to define a global chemical potential from equation (7). It would thus be interesting to verify explicitly
whether this asymptotic factorization property holds or not. This is a difficult task because (to our knowledge) the exact joint probability distribution is not known for generic rates in the present geometry, as it does not fulfil equation (3). However, the specific case of the zero-range process (ZRP), where masses are discrete, turns out to be solvable and thus deserves to be investigated in more detail.

5. The solvable three-branch model

5.1. The zero-range process and the factorized steady state

The ZRP case, which has been intensively studied in the literature [29], corresponds to the choice \( v(\mu) = \delta(\mu - 1) \), so masses \( m_i \) take integer values, denoted as \( n_i \) in the following. We study here the ZRP in the three-branch geometry illustrated in figure 4. The dynamics is defined as in section 4.1, except that \( \mu \) can only take the value \( \mu = 1 \). To simplify the calculations, we set \( \gamma_B = \gamma_C = 1 \) and choose \( p > q \).

The relation (19) between the flux and the local chemical potential reads in this case \( \Phi_v = e^{-\lambda_v} \). Given that the fluxes are different in the three branches, it is clear that the chemical potentials are not equal. More quantitatively, the balance of fluxes \( \Phi_A = \Phi_B + \Phi_C \) implies \( \lambda_A = -\ln(\Phi_A/\Phi_B) + \lambda_B \). From this expression we see that there is now a shift, given by the logarithm of the ratio of the fluxes, in the value of the chemical potential.

To better understand the origin of this shift, we compute the steady-state distribution of the ZRP with three branches. Interestingly, it has been shown that the steady-state distribution of the ZRP on an arbitrary graph remains factorized [28]. Slightly rephrasing the results of [28], one finds that the distribution \( P(\{n_i\}) \) is given by

\[
P(\{n_i\}) = \frac{1}{Z(M)} \left( \prod_{i=1}^{N} f(n_i) z_i^{n_i} \right) \delta_{\sum_i n_i, M}
\]  

(22)

where \( \delta \) is the Kronecker delta symbol, and where the local ‘fugacities’ \( z_i \) satisfy for all \( i \) the equation

\[
\sum_{j(\neq i)} v_{ij} z_i = \sum_{j(\neq i)} v_{ji} z_j
\]

(23)

with \( v_{ij} \equiv v_{ij}(1) \) defined in equation (1). In the present three-branch model, \( v_{ij} \) is equal to \( p \) if there is a positively oriented link from \( i \) to \( j \), and to \( q \) if the link is negatively oriented; \( v_{ij} = 0 \) in the absence of a link. Note that the fugacities \( z_j \) are defined only up to an overall arbitrary factor. Condition (23) then reads, for all sites \( i \) different from a branching point,

\[
pz_{i-1} - z_i + qz_{i+1} = 0.
\]

(24)

The general solution of this linear equation (valid separately on each branch) is a linear combination of solutions of the form \( z_j = r^j \), with a parameter \( r \) obeying the relation

\[
qr^2 - r + p = 0.
\]

(25)
Equation (25) has two solutions, $r_1 = 1$ and $r_2 = p/q > 1$. As a result, $z_j$ can be expressed on each branch as a linear combination of the two independent solutions $r_1$ and $r_2$,

$$z_j = \tau_\nu \left[ 1 + K_\nu \left( \frac{q}{p} \right)^{N_\nu - j} \right],$$

(26)

where $\tau_\nu$ and $K_\nu$ are constants ($\nu = A, B, \text{or} C$) and $N_\nu$ denotes the number of sites in branch $\nu$. The total number of sites in the system is given by $N = N_A + N_B + N_C + 2$ (the last term accounts for the two nodes). Taking into account equation (23), formulated for the two branching points, one can match the expressions (26) corresponding to different branches and determine the constants $K_\nu$ as well as the ratios $z_A/z_B$ and $z_A/z_C$. In the limit where the sizes $N_A, N_B$ and $N_C$ of the three branches go to infinity, one finds $\tau_A/\tau_B = \tau_A/\tau_C = 2$. As $z_j$ is defined up to an overall prefactor, one can choose for instance $\tau_B = \tau_C = 1$ and $\tau_A = 2$. Given that the terms proportional to $K_\nu$ are exponentially decaying corrections, it turns out that $z_j$ is almost constant, and equal to $\tau_\nu$, on each branch $\nu$.

5.2. Local versus global chemical potentials

As the probability distribution of the ZRP is factorized, the definition of the chemical potential proposed in [23,24] can indeed be applied. Splitting the system into two parts, branch A on one side, and branches B and C on the other side, one obtains

$$\lambda_A = \frac{\partial \ln Z_A}{\partial M_A}, \quad \lambda_{BC} = \frac{\partial \ln Z_{BC}}{\partial M_{BC}},$$

(27)

with $M_A = \sum_{i \in A} n_i$ and $M_{BC} = \sum_{i \in B,C} n_i$ the respective masses of the two subsystems. Since $\tau_A = 2$, we get, neglecting the exponential corrections appearing in equation (26),

$$Z_A = 2^{M_A} Z_A^{(0)}, \quad Z_{BC} = Z_{BC}^{(0)};$$

(28)

where $Z_A^{(0)}$ and $Z_{BC}^{(0)}$ are the standard partition functions of these two subsystems, that would be obtained by taking $z_i = 1$ for all $i$. In the large size limit, one then finds for the chemical potential, using the definition (7),

$$\lambda_A = \ln 2 + \tilde{\lambda}_A, \quad \lambda_{BC} = \tilde{\lambda}_{BC}.$$  

(29)

We thus recover, in this way, the result directly obtained from the balance of fluxes, showing that this result is actually consistent with the asymptotic factorization condition on which the definition of the chemical potential is built (let us recall that the probability distribution of the present ZRP model is factorized). Note that the shift $\ln 2$ is independent of the precise value of the bias $p - q$, as long as this bias is non-zero. If $p = q$, equilibrium is recovered, and the bias vanishes. The presence of this bias is thus a genuine non-equilibrium effect, which appears non-perturbatively, in a discontinuous way.

It is also interesting to note that the shift $\ln 2$ cancels out from the one-site mass distribution. Indeed, one has for a site $i$ in the bulk of branch A, using equation (29),

$$p_i(n) = \frac{1}{Q_A} f(n) 2^n e^{-\lambda_A n} = \frac{1}{Q_A} f(n) e^{-\lambda_A n}$$

(30)
and for a site in branch B or C,

$$p_i(n) = \frac{1}{Q_{BC}} f(n) e^{-\tilde{\lambda}_{BC} n}. \quad (31)$$

Hence both distributions have exactly the same form as a function of the local chemical potential, making the shift undetectable from a local measurement.

6. Discussion: a tentative scenario

We now try to assess more generally, beyond the specific ZRP case, the consistency between the flux balance and the equality of the local chemical potentials. We have seen in the ZRP case that the discrepancy between the local chemical potentials \( \tilde{\lambda}_\nu \) can be explained by the presence of exponential factors that differ from one subsystem to the other. It is tempting to try to generalize this scenario to the case of the mass transport model with continuous masses, although no analytical solution is available in this case.

At a heuristic level, a more general form of equation (28) can be proposed, namely

$$Z_A = e^{\alpha(\bar{\rho}) M_A} Z_A^{(0)}, \quad Z_{BC} = Z_{BC}^{(0)}, \quad (32)$$

since the global average density \( \bar{\rho} \) is fixed, and can thus \textit{a priori} enter as a parameter in the exponential factor. Equation (29) is then replaced by

$$\lambda_A = \alpha(\bar{\rho}) + \tilde{\lambda}_A, \quad \lambda_{BC} = \tilde{\lambda}_{BC}. \quad (33)$$

Assuming that the asymptotic factorization condition holds, one has \( \lambda_A = \lambda_{BC} \), so

$$\alpha(\bar{\rho}) + \tilde{\lambda}_A = \tilde{\lambda}_{BC}. \quad (34)$$

The constant \( \alpha(\bar{\rho}) \) is determined as follows. Given a value of \( \bar{\rho} \), one looks for the densities \( \rho_A(\bar{\rho}) \) and \( \rho_{BC}(\bar{\rho}) \) satisfying the constraints

$$\tilde{\Phi}(\rho_A) = 2\tilde{\Phi}(\rho_{BC}), \quad (35)$$

$$(\bar{\rho}) = \kappa \rho_A + (1 - \kappa) \rho_{BC}, \quad (36)$$

with \( \kappa = N_A/(N_A + N_{BC}) \), and where \( \tilde{\Phi}(\rho) \) is the value of the local flux for a local density \( \rho \). The parameter \( \alpha(\bar{\rho}) \) is then obtained, consistently with equation (34), as

$$\alpha(\bar{\rho}) = \Lambda\left(\rho_{BC}(\bar{\rho})\right) - \Lambda\left(\rho_A(\bar{\rho})\right). \quad (37)$$

In the ZRP case, one recovers \( \alpha = \ln 2 \), independently of the density \( \bar{\rho} \), while with continuous masses, assuming \( v(\mu) = 1 \) and \( f(m) = m^{q-1} \), one finds

$$\alpha(\bar{\rho}) = \eta \frac{1}{\bar{\rho}} \left(1 - 2\kappa + \frac{3\kappa - 1}{\sqrt{2}}\right). \quad (38)$$

This result is in agreement with the numerical findings of section 4.2 where \( \eta = 1 \) and \( \kappa = \frac{1}{3} \), yielding \( \alpha = 1/(3\bar{\rho}) \) (see the right panel of figure 7). Hence, despite the absence of an exact solution of the mass transport model (beyond the specific ZRP case), this tentative scenario provides a consistent explanation of the numerical results obtained for the local chemical potential, showing that the asymptotic factorization condition may still hold while local measurements fail to find an equilibrated chemical potential throughout the system.

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7. Conclusion

In this paper, we have attempted to clarify the influence of the flux balance condition on the generalized chemical potential in non-equilibrium mass transport models. We have seen in particular that even when the asymptotic factorization condition holds, locally measured values of the chemical potential may not equalize, while the theoretically defined ones remain equal. This surprising property has been traced back to the presence of exponential factors appearing in the probability weights of different subsystems. These exponential factors cancel out at the level of the local statistics (for instance the single-site distribution) and are thus locally undetectable. But on the other hand these factors play an essential role in the global statistics, in ensuring the balance of flux.

Quite surprisingly, such factors exhibit a discontinuity between equilibrium and weakly non-equilibrium situations, as exemplified by the exact solution of the ZRP: at equilibrium (zero flux), no exponential factor is present (and local chemical potentials equalize), while the tiniest flux generates (for large systems) a factor $2^M$, leading to a shift $\ln 2$ between the local chemical potentials. We have also shown that a tentative generalization of this scenario, assuming a density-dependent shift $\alpha(\bar{\varphi})$, is consistent with the numerical results.

An open question thus far is that of whether the results presented here extend beyond the specific class of mass transport models. Although the precise range of validity of these results is currently unknown, it is natural to expect the balance of fluxes, which is a generic constraint, to also play an important role in many other non-equilibrium systems, and to generically interfere with local measurements of the chemical potential.

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