AN INFINITE-DIMENSIONAL MANIFOLD STRUCTURE FOR
ANALYTIC LIE PSEUDOGROUPS OF INFINITE TYPE

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ABSTRACT. We construct an infinite-dimensional manifold structure adapted to
analytic Lie pseudogroups of infinite type. More precisely, we prove that any
isotropy subgroup of an analytic Lie pseudogroup of infinite type is a regular
infinite-dimensional Lie group, modelled on a locally convex strict inductive
limit of Banach spaces. This is an infinite-dimensional generalization to the
case of Lie pseudogroups of the classical second fundamental theorem of Lie.

1. Introduction

Our objective in this paper is to prove an integration theorem for the class of
infinite-dimensional Lie algebras of analytic vector fields corresponding to infti-
tesimal actions of Lie pseudogroups of infinite type. More precisely, we shall show
that any isotropy subgroup of an analytic Lie pseudogroup of infinite type is a
regular infinite-dimensional Lie group in the sense of Milnor [Mil 83], modelled
on a locally convex strict inductive limit of Banach spaces. This is an infinite-
dimensional generalization to the case of Lie pseudogroups of infinite type of the
classical second fundamental theorem of Lie for finite-dimensional local Lie al-
gebras of vector fields. The main result of our paper represents the conclusion
of a research program that we have been involved in over the past several years
[KR 97-1, KR 97-2, RK 97, KR 00, KR 01], the aim of which has been to con-
struct a natural infinite-dimensional manifold structure for the parameter space of
analytic Lie pseudogroups of infinite type.

Recall that an analytic Lie pseudogroup $\Gamma^\omega$ of transformations of an analytic
manifold $M$ is a sub-pseudogroup of the pseudogroup of analytic local diffeo-
morphisms of $M$, where the elements of $\Gamma^\omega$ are the analytic solutions of a system $S$
of differential equations which is involutive in the sense of Cartan-Kähler theory.
Lie pseudogroups of infinite type correspond to the case in which the solutions of
$S$ are parametrized by arbitrary functions, while Lie pseudogroups of finite type
correspond to the case where the solutions of $S$ are parametrized by arbitrary con-
stants. A simple example of an analytic Lie pseudogroup of infinite type is given by
the set of conformal local diffeomorphisms of the plane, with $S$ being the Cauchy-
Riemann equations. The standard local action of $PSL(2, \mathbb{R})$ on the real line given
by fractional linear transformations defines a Lie pseudogroup of finite type, with
$S$ being the third-order ordinary differential equation expressing the vanishing of
the Schwarzian derivative. The Lie pseudogroups of finite type give rise to finite-
dimensional local Lie groups [Olv 96]. The modern theory of finite-dimensional Lie
groups can be viewed as a theory of global parameter spaces for Lie pseudogroups
of finite type.

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The classical theory of Lie pseudogroups was founded by Lie and significantly developed by Elie Cartan, who formulated a general structure theory encompassing both Lie pseudogroups of finite and infinite type. Cartan’s approach is based on the construction of $\Gamma^\omega$-invariant differential forms which generalize the Maurer-Cartan forms to the case of Lie pseudogroups of infinite type, and satisfy generalized Maurer-Cartan structure equations \cite{Ca01, Ca37}. These invariant differential forms can be computed explicitly from the differential system $S$ defining the Lie pseudogroup \cite{Ca37}, and lead to many important geometric applications, \cite{OlvPo03-1, OlvPo03-2}. The notion of an involutive G-structure provides a global geometric framework for Cartan’s theory which is well adapted to the study of many classes of Lie pseudogroups, \cite{Ch53}. The Lie pseudogroups arise in this context as local analytic automorphisms of various reductions of the linear frame bundle of the manifold $M$, and the Maurer-Cartan forms are given by the vector-valued canonical one-form to the G-structure. The Cartan-Kähler Theorem and the formulation of the involutivity of $S$ in terms of the characters of the Lie-algebraic tableaux associated to the generalized Maurer-Cartan equations are cornerstones of this theory.

The works of Spencer and of his school \cite{KS72} provide an alternative approach to the study of Lie pseudogroups and their deformations, in which the question of formal integrability is expressed in purely cohomological terms. The fundamental papers by Malgrange \cite{Mal72} on Lie equations give a clear and beautiful account of the main integration theorem for formally integrable differential systems which is central to the work of Spencer, and of the links between this result and the Cartan-Kähler Theorem. Malgrange’s account of the Cartan-Kähler Theorem will play an important role in our paper.

Every analytic Lie pseudogroup gives rise to an infinite-dimensional filtered Lie algebra of vector fields (more precisely a sheaf of filtered Lie algebra germs). These Lie algebras and the geometries for which they act as infinitesimal automorphisms have been studied in depth in a number of important papers, particularly in the transitive case, \cite{GuS64, SiSe65}. It is this local correspondence which is the starting point of our work. In \cite{KR01}, we proved an integration theorem for local Lie algebras of analytic vector fields corresponding to transitive flat Lie pseudogroups of infinite type. More precisely, we proved that any isotropy subgroup of such a Lie pseudogroup is a regular infinite-dimensional Lie group in the sense of Milnor \cite{Mil83}, modelled on a locally convex strict inductive limit of Banach spaces. In the present paper, we dispense with the hypothesis of flatness - a hypothesis which roughly amounts to requiring that the defining equations of the Lie pseudogroup have constant coefficients - and prove an integration theorem for the filtered Lie algebras associated to isotropy subgroups of general analytic Lie pseudogroups. The chart we construct in the non-flat case has a structure similar to the one we had obtained for the flat case in \cite{KR01}, in the sense that it is given by a convergent infinite product of exponentials. However the specific nature of the chart and of the proof of convergence are significantly different in the non-flat case, both at the conceptual and technical levels, and considerably more involved. A key ingredient in the proof is the existence of a bounded filtered basis for the underlying Lie algebra of vector fields. The existence of this basis follows from the basic assumption of involutivity of the underlying differential system and from an important estimate established by Malgrange \cite{Mal72} in the course of his proof of the Cartan-Kähler
Theorem. We then derive the estimates which are necessary to show the convergence of the infinite product of exponentials adapted to this bounded filtered basis. Our differentiable structure is thus similar to the one obtained by Leslie in his fundamental work on the infinite-dimensional manifold structure of the groups of analytic diffeomorphisms of a compact analytic manifold. [L 82].

2. Notation and statement of the main theorem

2.1. **Notation.** We consider \( \mathbb{R}^\nu \) endowed with standard coordinates \( x = (x^1, \ldots, x^\nu) \). For any \( \nu \)-multilplet \( \alpha = (\alpha_1, \ldots, \alpha_\nu) \) of non-negative integers we set

\[
|\alpha| = \alpha_1 + \cdots + \alpha_\nu, \quad |\alpha'| = |\alpha| - 1, \quad \alpha! = \alpha_1! \cdots \alpha_\nu!
\]

and

\[
x^\alpha = (x^1)^{\alpha_1} \cdots (x^\nu)^{\alpha_\nu}.
\]

The \( j \)-th vector of the standard basis is identified with \( \partial_j \) the partial derivative with respect to \( x^j \). It will be convenient to denote by \( \hat{x} \) the sum of the \( \nu \) variables \( x^i \) and by \( \partial_{\hat{x}} \) the sum of the \( \nu \) partial derivatives \( \partial_{x^i} \). Hence

\[
\hat{x} = x^1 + \cdots + x^\nu
\]

and

\[
\partial_{\hat{x}} = \partial_{x^1} + \cdots + \partial_{x^\nu}.
\]

Observe that \( \partial_{\hat{x}} \hat{x} = \nu \).

Let us denote by \( V \) the vector space \( \mathbb{R}^\nu \) and by \( S^k(V^*) \) its space of symmetric covariant tensors of degree \( k \). Any homogeneous polynomial vector field \( V_k \) of degree \( k \) can be identified with an element of \( V \otimes S^k(V^*) \), and can be written as

\[
V_k = \sum_{i=1}^\nu V_i^\alpha \partial_i \text{ where } V_i^\alpha \in S^k(V^*). \quad \text{The space } V \otimes S^k(V^*) \text{ admits}
\]

\[
\left\{ \frac{\alpha!}{\alpha_1! \cdots \alpha_\nu!} x^\alpha \partial_i \right\} \text{ where } i = 1, \ldots, \nu \text{ and } |\alpha| = k
\]

as basis. We denote by \( \| V_k \|_k \) the sup norm associated with this basis, that is,

\[
\| V_k \|_k = \max_{\alpha, i} |V_{k,\alpha}^i|
\]

for any \( V_k = \sum V_{k,\alpha}^i \frac{\alpha!}{\alpha_1! \cdots \alpha_\nu!} x^\alpha \partial_i \).

Let \( \chi(\nu) = \chi_{-1}(\nu) \) denote the Lie algebra of formal vector fields based at the origin 0 of \( \mathbb{R}^\nu \). For every non-negative integer \( q \), let \( \chi_q(\nu) \) denote the Lie subalgebra of formal vector fields tangent up to order \( q \) to the zero vector field. This defines the decreasing filtration naturally associated to \( \chi(\nu) \). In component form any formal vector field \( V \) of \( \chi_q(\nu) \) can be written as \( V = \sum_{i=1}^\nu V_i^\alpha \partial_i \) with \( V_i^\alpha = \sum_{|\alpha| \geq q} v^\alpha_\alpha x^\alpha \).

A formal vector field \( V \) will be said *positive* if its coefficients \( v^\alpha_\alpha \) are positive.

Given a Lie pseudogroup \( \Gamma^\nu \) acting in a neighborhood of 0 in \( \mathbb{R}^\nu \), its formal Lie algebra \( \mathcal{L}(\Gamma) \) is a closed Lie subalgebra of \( \chi(\nu) \) when the latter is endowed with the Tychonov topology. Set \( \mathcal{L}_q(\Gamma) = \mathcal{L}(\Gamma) \cap \chi_q(\nu) \). We have \( \mathcal{L}(\Gamma)/\mathcal{L}_0(\Gamma) \simeq \mathbb{R}^\nu \) whenever \( \Gamma^\nu \) is transitive. We associate to \( \mathcal{L}(\Gamma) \) a flat Lie algebra denoted by \( L(\Gamma) \). By definition, we have

\[
L(\Gamma) = \bigoplus_{q=-\infty}^{\infty} \mathcal{L}_q(\Gamma)/\mathcal{L}_{q+1}(\Gamma).
\]

We note that the quotient space

\[
\gamma_q = \mathcal{L}_{q-1}(\Gamma)/\mathcal{L}_q(\Gamma)
\]

inherits from \( V \otimes S^k(V^*) \) a Banach space structure.
A transitive Lie pseudogroup is said to be flat if its Lie algebra is isomorphic to its associated flat Lie algebra, i.e. \( \mathcal{L}(\Gamma) \simeq L(\Gamma) \). It is well known that the flat Lie algebra arising from a Lie pseudogroup \( \Gamma^\omega \) characterizes the type of geometry associated to \( \Gamma^\omega \). The pseudogroup of local transformations that preserve a volume form and the Hamiltonian pseudogroup of local symplectomorphisms of a symplectic manifold are classical examples of flat Lie pseudogroups.

2.2. Topology. Recall that formal vector field \( V = \sum_{k=0}^{\infty} V_k \) (where, for all \( k \), \( V_k \in V \otimes S^k(V^*) \)) is analytic if its coefficients satisfy
\[
\limsup_k \| V_k \| < \infty.
\]

Let \( \rho \) be a positive real number and let \( \mathcal{L}^\omega_\rho(\Gamma) \) denote the subspace of \( \mathcal{L}^\omega(\Gamma) \) of \( V \)'s such that \( \limsup_k \| V_k \| / \rho^k < +\infty \). We have \( \mathcal{L}^\omega(\Gamma) = \bigcup_{\rho > 0} \mathcal{L}^\omega_\rho(\Gamma) \). Each \( \mathcal{L}^\omega_\rho(\Gamma) \) is naturally endowed with a Banach space structure with the norm
\[
\| V \|_\rho = \sup_k \| V_k \| / \rho^k.
\]

We shall mainly be concerned with isotropy subgroups. In that case we will use preferably the equivalent norm
\[
\| V \|_\rho = \sup_k \| V_k \| / \rho^{k-1}.
\]

For \( \rho < \rho' \) the injection \( \mathcal{L}^\omega_\rho(\Gamma) \hookrightarrow \mathcal{L}^\omega_{\rho'}(\Gamma) \) is continuous and compact. Hence \( \mathcal{L}^\omega(\Gamma) \) is a complete Hausdorff locally convex topological vector space. Its associated topology is the locally convex strict inductive limit topology
\[
\mathcal{L}^\omega(\Gamma) = \lim_{\rho \in \mathbb{N}} \mathcal{L}^\omega_\rho(\Gamma).
\]

This endows \( \mathcal{L}^\omega(\Gamma) \) with a Silva topological Lie algebra structure. We shall denote by \( \mathcal{L}^\omega_{\rho,q}(\Gamma) \) the intersection
\[
\mathcal{L}^\omega_{\rho,q}(\Gamma) = \mathcal{L}^\omega_\rho(\Gamma) \cap \chi_q(\nu),
\]
and by \( \mathcal{L}^\omega_{\rho,q,M}(\Gamma) \) the subset of \( \mathcal{L}^\omega_{\rho,q}(\Gamma) \) consisting of those analytic vector fields \( V \) satisfying
\[
\| V \|_\rho \leq M.
\]

2.3. Statement of the main theorem. Our main objective is to prove an infinite-dimensional version of the classical second fundamental theorem of Lie for isotropy subgroups of analytic Lie pseudogroups. Recall from [KR 01, Theorem 5.3], that the group \( G^\omega_0(\nu) \) of analytic local diffeomorphisms of \( \mathbb{R}^\nu \) fixing the origin has the natural structure of a Gâteaux-analytic Lie group.

**Theorem 1** (Lie II). Any isotropy group of a Lie pseudogroup of analytic transformations in \( \nu \) variables is integrable into a unique connected subgroup \( H \) embedded in \( G^\omega_0(\nu) \). Such a subgroup \( H \) is always a regular analytic Lie group belonging to the class \( \mathcal{CBH}^{\aleph_0} \) of Campbell-Baker-Hausdorff groups of countable order.

Let \( \mathcal{L}^\omega(\Gamma) \subset \chi(\nu) \) be a Lie algebra of local analytic vector fields defined in a neighborhood of the origin \( O \) of \( \mathbb{R}^\nu \). For any integer \( q \), denote by \( \mathcal{L}^\omega_q(\Gamma) \) the subalgebra of vector fields in \( \mathcal{L}^\omega(\Gamma) \) contained in \( \chi_q(\nu) \). Whenever \( q' > q \geq 0 \), the
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subalgebra \( \mathcal{L}_q^\omega (\Gamma) \) is an ideal of \( \mathcal{L}_q^\omega (\Gamma) \). Suppose that for every integer \( q \geq 1 \), we are given a linear section \( \Sigma_q \) of the natural projection

\[
\mathcal{L}_q^{\omega^{-1}}(\Gamma) \to \mathcal{L}_q^{\omega^{-1}}(\Gamma)/\mathcal{L}_q^\omega(\Gamma),
\]
and let

\[
E_q = \Sigma_q(\mathcal{L}_q^{-1}(\Gamma)/\mathcal{L}_q^\omega(\Gamma)).
\]

The isotropy Lie algebra \( \mathcal{L}_0^\omega (\Gamma) \) then decomposes as a direct sum of finite dimensional summands

\[
\mathcal{L}_0^\omega (\Gamma) = \bigoplus_{q=1}^{+\infty} E_q,
\]
where

In the sequel we will denote by \( \Sigma \) the collection of \( \{\Sigma_q\}_{q \in \mathbb{N}} \). We shall prove the following strong boundedness property:

**Lemma 1.** There exist linear sections \( \Sigma_q : \mathcal{L}_q^{\omega^{-1}}(\Gamma)/\mathcal{L}_q^\omega(\Gamma) \to \mathcal{L}_q^{\omega^{-1}}(\Gamma)/\mathcal{L}_q^\omega(\Gamma) \) taking values in the same Banach subspace \( \mathcal{L}_0^\omega (\Gamma) \) of \( \mathcal{L}_0^\omega (\Gamma) \), such that the norms of the continuous operators \( \Sigma_q : \mathcal{L}_q^{\omega^{-1}}(\Gamma)/\mathcal{L}_q^\omega(\Gamma) \to \mathcal{L}_q^{\omega^{-1}}(\Gamma)/\mathcal{L}_q^\omega(\Gamma) \) are uniformly bounded.

The preceding lemma will be used to prove the following theorem:

**Theorem 2** (Lie III). The isotropy Lie algebra \( \mathcal{L}_0^\omega (\Gamma) \) is integrable into a unique connected and simply connected analytic Lie group \( \Gamma_0 \) of the second kind and countable order. In other words \( \Gamma_0 \) is a regular analytic Lie group of class \( C^{BH}_\infty \).

This latter theorem combined with Proposition 4 of our Appendix implies Theorem 1.

3. Estimates

We begin by deriving some basic estimates on the norm of the exponential of a vector field \( X \in \mathcal{L}^\omega_{\rho,n}(\Gamma) \). These estimates will be needed to prove the convergence of the infinite product of exponentials which we will use to define the charts of Theorem 2.

We first consider \( \Gamma^\omega \) the general Lie pseudogroup of \( \nu \) variables acting on \( \mathbb{R}^\nu \). It is clear that \( M \rho^n \frac{\partial^{n+1}}{1-\rho \hat{x}} \partial \hat{x} \) is the maximal positive vector field of \( \mathcal{L}^\omega_{\rho,n}(\Gamma) \) of norm less or equal to \( M \). We call such a vector field a canonical vector field with respect to our bornological decomposition of the Lie algebra.

Let \( X = M \rho^n \frac{\partial^{n+1}}{1-\rho \hat{x}} \partial \hat{x} \) and \( Y = N \rho^r \frac{\partial^{r+1}}{1-\rho \hat{x}} \partial \hat{x} \) be two canonical positive vector fields on \( \mathbb{R}^\nu \).

Then

\[
XY = MN \nu \rho^{n+r} \frac{\partial^{n+r+1}}{1-\rho \hat{x}} \left( \frac{r+1}{1-\rho \hat{x}} + \frac{\rho \hat{x}}{(1-\rho \hat{x})^2} \right) \partial \hat{x}
\]

is also positive. Since in addition

\[
\frac{r+1}{1-\rho \hat{x}} + \frac{\rho \hat{x}}{(1-\rho \hat{x})^2} = \frac{r+1-r\rho \hat{x}}{(1-\rho \hat{x})^2},
\]
we conclude that

\[
XY \ll MN \nu \rho^{n+r} \frac{\partial^{n+r+1}}{1-\rho \hat{x}} \left( \frac{r+1}{1-\rho \hat{x}} \right) \partial \hat{x}.
\]
More generally we obtain with \( Y = N \rho^{x^{-1}} \partial_x \)

\[
XY = MN \nu \rho^{n+r} \frac{\hat{x}^{n+r+1}}{(1 - \rho x)^{\alpha+2}} (r + 1 - (r + 1 - \alpha) \rho \hat{x}) \partial_x
\]

so that whenever \( r + 1 \geq \alpha \)

\[
XY \ll MN \nu \rho^{n+r} \frac{\hat{x}^{n+r+1}}{(1 - \rho x)^{\alpha+2}} \partial_x.
\]

We will deal with sequences of the form \( X^n Y = X(X^{n-1} Y) \) for \( n \) integer. Let \( X = M \rho^{\frac{\hat{x}^{n+1}}{1 - \rho x}} \partial_x \) and \( Y = N \rho^{x^{-1}} \partial_x \). For any integer \( k \) and \( n \geq 2 \)

\[
X^k Y \ll (\nu M)^k N \rho^{kn+r} \frac{\hat{x}^{kn+r+1}}{(1 - \rho x)^{\alpha+2k}} (r + 1)(n + r + 1) \cdots ((k - 1)n + r + 1) \partial_x.
\]

Our next step is the study of \((id - X) \circ e^X\).

Remember that

\[
(id - X) \circ e^X = id - \frac{1}{2!} X^2 + \frac{2}{3!} X^3 + \cdots + \frac{k}{(k+1)!} X^{k+1} + \cdots.
\]

We consider \( X = M \rho^{\frac{\hat{x}^{n+1}}{1 - \rho x}} \partial_x \) with \( n \geq 2 \). Then

\[
X^{k+1} \ll \nu^k M^{k+1} \rho^{(k+1)n} \frac{\hat{x}^{(k+1)n+1}}{(1 - \rho x)^{2k+1}} (n + 1)(2n + 1) \cdots (kn + 1) \partial_x.
\]

But \((n + 1)(2n + 1) \cdots (kn + 1) \leq (n + 1)^2(n + 1) \cdots k(n + 1) = k!(n + 1)^k \) so that

\[
\frac{1}{(k+1)!} X^{k+1} \ll \nu^k M^{k+1} \rho^{(k+1)n} \frac{\hat{x}^{(k+1)n+1}}{(1 - \rho x)^{2k+1}} \frac{(n + 1)^k}{k + 1} \partial_x.
\]

and

\[
\frac{k}{(k+1)!} X^{k+1} \ll \frac{k}{k + 1} (\nu(n + 1))^k M^{k+1} \rho^{(k+1)n+1} \frac{\hat{x}^{(k+1)n+1}}{(1 - \rho x)^{2k+1}} \partial_x.
\]

Notice now that all these vector fields belong to the Banach space \( L^\omega_{\sigma \rho} \) for any real number \( \sigma > 1 \). To evaluate their norm, we note that for any positive integer \( l \), we have

\[
\frac{1}{(1 - \rho y)^{l+1}} = \sum_{m=0}^{+\infty} \binom{m+1}{l} (\rho y)^m.
\]

To obtain an upper bound of the norm of \( \frac{k}{(k+1)!} X^{k+1} \) in \( L^\omega_{\sigma \rho} \), we start from

\[
\frac{k}{(k+1)!} X^{k+1} \ll \frac{k}{k + 1} (\nu(n + 1))^k M^{k+1} \rho^{(k+1)n+1} \frac{\hat{x}^{(k+1)n+1}}{(1 - \rho x)^{2k+1}} \partial_x.
\]
Put $K = \frac{k}{(k+1)!}(\nu(n+1))^kM^{k+1}$ and $\rho_\sigma = \sigma \rho$ so that $\rho = \sigma^{-1}\rho_\sigma$. Therefore

$$\frac{k}{(k+1)!}X^{k+1} \ll K\sigma^{-n(k+1)}\rho_\sigma^n\hat{x}^{n(k+1)+1}\left(\sum_{m=0}^{+\infty} \left(\frac{m+2k}{2k}\right)\sigma^{-m}(\rho_\sigma \hat{x})^m\right) \partial_x $$

It follows that the norm of $\frac{k}{(k+1)!}X^{k+1}$ in $\mathcal{L}_{\sigma,\rho}$ is less than

$$K\sigma^{-n(k+1)}\max_{m\in\mathbb{N}}\left\{\frac{m+2k}{2k}\sigma^{-m}\right\}. $$

Next, we have

**Lemma 3.** For any positive real number $\alpha$, the sequence $\{m^\alpha \sigma^{-m}\}_{m\in\mathbb{N}}$ admits $(\frac{\alpha}{\ln \sigma})^\alpha e^{-\alpha}$ as upper bound.

**Proof.** Put $\phi(m) = m^\alpha \sigma^{-m}$. The derivative $\frac{d\phi}{dm} = \sigma^{-m}m^{\alpha-1}(\alpha - m \ln \sigma)$ vanishes for $m_0 = \frac{\alpha}{\ln \sigma}$. Since $m_0$ corresponds to a maximum, $\phi(m_0)$ provides an upper bound. \qed

Now remark that $\binom{m+2k}{2k} \leq \frac{(m+2k)^{2k}}{(2k)!}$, so that

$$\left(\frac{m+2k}{2k}\right)\sigma^{-m} \leq \frac{(m+2k)^{2k}}{(2k)!}\sigma^{-(m+2k)}\sigma^{2k}. $$

and

$$\max_{m\in\mathbb{N}}\left\{\frac{m+2k}{2k}\sigma^{-m}\right\} \leq \max_{m\in\mathbb{N}}\left\{\frac{m^{2k}(m+2k)^{2k}}{(2k)!}\sigma^{-(m+2k)}\right\}. $$

Using Lemma 3 with $\alpha = 2k$ we get

$$\max_{m\in\mathbb{N}}\left\{\frac{m+2k}{2k}\sigma^{-m}\right\} \leq \frac{\sigma^{2k}}{(2k)!}\left(\frac{2k}{\ln \sigma}\right)^{2k} e^{-2k}. $$

We obtain that way

$$\norm{\frac{k}{(k+1)!}X^{k+1}}_{\mathcal{L}_{\sigma,\rho}} \leq K\sigma^{-n(k+1)}\frac{\sigma^{2k}}{(2k)!}\left(\frac{2k}{\ln \sigma}\right)^{2k} e^{-2k} $$

that is

**Corollary 1.** The norm of $\frac{k}{(k+1)!}X^{k+1}$ in $\mathcal{L}_{\sigma,\rho}$ is bounded as follows

$$\norm{\frac{k}{(k+1)!}X^{k+1}}_{\mathcal{L}_{\sigma,\rho}} \leq M^{k+1}(\nu(n+1))^k\sigma^{-(n-2)k}e^{-2k}\left(\frac{2k}{\ln \sigma}\right)^{2k} \frac{\sigma^{-n}}{(2k)!}. $$

Let us denote this upper bound by $B(n,k,\sigma)$.

$$B(n,k,\sigma) = \frac{M}{\sigma^n}\left(\frac{\sqrt{\nu(n+1)M}}{\sigma^{n-2}\ln \sigma}\right)^{2k}\frac{1}{(2k)!}. $$

From Stirling formula $n! > \frac{1}{2\sqrt{\pi n}}\left(\frac{n}{e}\right)^n$ so that $(2k)! > \left(\frac{2k}{e}\right)^{2k}\sqrt{\pi k}$. Hence

$$B(n,k,\sigma) \leq \frac{M}{\sigma^n}\left(\frac{\sqrt{\nu(n+1)M}}{\sigma^{n-2}\ln \sigma}\right)^{2k}\frac{1}{\sqrt{\pi}}. $$

We conclude that the infinite sum

$$\frac{1}{2!}X^2 + \frac{2}{3!}X^3 + \cdots + \frac{k}{(k+1)!}X^{k+1} + \cdots$$
belongs to \( \mathcal{L}^w_{\sigma, \rho} \) for any \( \sigma \) satisfying \( \sigma \geq \sqrt{\nu (n+1) M} \). Indeed if we put \( q = \frac{\sqrt{\nu (n+1) M}}{\sigma \ln \sigma} \),
\[
\| \sum_{k=1}^{+\infty} \frac{k}{(k+1)!} X^{k+1} \|_{\mathcal{L}_{\sigma, \rho}} \leq \frac{M + \infty}{\sigma^n} \sum_{k=1}^{\infty} \frac{q^{2k}}{\sqrt{k \pi}}
\]
which is less than \( \frac{M \sqrt{\pi^2 \sigma n}}{1 - q^2} \).

We conclude this section on basic estimates by remarking that with \( q = \frac{\sqrt{\nu (n+1) M}}{\sigma \ln \sigma} \), the norm of \( X = M \rho^n \frac{x^{n+1}}{1 - \rho^2} \partial_x \) and \( n \geq 2 \), we have
\[
\frac{1}{k!} X^{k+1} \ll (\nu (n+1))^k (M \rho^n)^{k+1} \frac{\nu^{(k+1)n+1}}{(1 - \rho^2)^{k+1}} \partial_x.
\]
It follows that

**Corollary 2.** For any \( \sigma > 1 \) the norm of \( \frac{1}{k!} X^{k+1} \) in \( \mathcal{L}^w_{\sigma, \rho} \) is bounded as follows
\[
\| \frac{1}{k!} X^{k+1} \|_{\mathcal{L}_{\sigma, \rho}} \leq M^{k+1} (\nu (n+1))^k \sigma^{-n} (2k)^{2k} \left( \frac{2k}{\ln \sigma} \right)^{2k} \frac{q^2}{\pi k!}.
\]

Remark that this upper bound is the same as the one given for \( \frac{1}{k!} X^{k+1} \) since we had replaced \( \frac{k}{(k+1)!} \) by 1. Therefore \( X \circ e^X \) belongs to \( \mathcal{L}_{\sigma, \rho} \) whenever \( q < 1 \) with \( q = \frac{\sqrt{\nu (n+1) M}}{\sigma \ln \sigma} \); we also have
\[
\| \sum_{k=1}^{+\infty} \frac{1}{k!} X^{k+1} \|_{\mathcal{L}_{\sigma, \rho}} \leq \frac{M \sqrt{\pi^2 \sigma n}}{1 - q^2}.
\]

4. **Iterative scheme and proof of Theorem 2**

We will assume in what follows that the uniform boundedness condition of Lemma 4 is satisfied. This assumption will be justified in detail in the Appendix. It will be convenient to represent the sections \( \Sigma_q \) by the introduction of an adapted filtered basis of the Lie algebra \( \mathcal{L}^w (\Gamma) \). A basis \( \mathcal{B} \) of the Lie algebra \( \mathcal{L}^w (\Gamma) \) will be said **filtered** whenever, for all integer \( q \), the subset \( \mathcal{B}_q \) of its elements that belong to \( \mathcal{L}_q^w (\Gamma) \) forms a basis of \( \mathcal{L}_q^w (\Gamma) \). We have

**Lemma 4.** Any Lie algebra \( \mathcal{L}^w (\Gamma) \) satisfying the boundedness condition of Lemma 4 admits a bounded filtered basis \( \mathcal{B} \). Such a basis can be generated by lifting, via the sections \( \Sigma_q \), any bounded homogeneous basis of the associated flat Lie algebra \( \mathcal{L}^w (\Gamma) \). More precisely there exists a constant \( \rho_0 > 0 \) such that for any element \( \varepsilon \) of \( \mathcal{L}_q^{w-1} (\Gamma) / \mathcal{L}_q^w (\Gamma) \), we have \( \| \varepsilon \|_{\mathcal{L}_{\rho_0}} \leq \| \varepsilon \|_{\mathcal{L}_q^w (\Gamma)} \) where \( \varepsilon = \Sigma_q (\varepsilon) \).

The quotient space \( \mathcal{L}_q / \mathcal{L}_p \) for \( p > q \) is isomorphic to the subspace of \( \mathcal{L} \) spanned by the elements in \( \mathcal{B}_q \) that are not in \( \mathcal{B}_p \). We will denote the image of \( \mathcal{L}^q / \mathcal{L}^p \) into \( \mathcal{L}^q \) by the section \( \Sigma \) by \( (\mathcal{L}^q / \mathcal{L}^p)^\Sigma \).

In order to obtain Theorem 2 we need [KR 01] to demonstrate that any local transformation \( \Phi \) of \( \Gamma \) of the form
\[
\Phi = I - \sum_{i=1}^{n} \left( \sum_{|\alpha| \geq 2} \phi^i_{\alpha} x^\alpha \right) \partial_i,
\]
with \( I \) being the identity transformation, can be written uniquely as an infinite product
\[
\cdots \circ \text{Exp} v_n \circ \cdots \circ \text{Exp} v_2 \circ \text{Exp} v_1
\]
where \( v_n \in (\mathcal{L}_n^\omega(\Gamma)/\mathcal{L}_{n+1}^\omega(\Gamma))_\Sigma \) for each integer \( n \) with the sum \( \sum_n v_n \) being element of \( \mathcal{L} \). The existence and uniqueness of a formal solution \( \{v_n\}_{n \in \mathbb{N}} \) is easily established by iteration. The core of the proof consists in checking the analyticity of the series \( \sum_n v_n \).

A basic ingredient in the procedure consists in the decomposition of the “free part” of any vector field with respect to the chosen section \( \Sigma \). We first illustrate the situation in the one-dimensional case.

### 4.1. Decompositions of analytic vector fields.

#### 4.2. The one-dimensional case.

As a warm-up, we begin with the one-dimensional case by considering analytic vector fields in a neighborhood of the origin in \( \mathbb{R} \).

Let first \( p = 2 \) and consider a vector field \( Z = (a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots) \partial_x \) in \( \mathcal{L}^\omega_{p,2,M}(\Gamma) \). This means that its coefficients satisfy the inequality
\[
|a_n| \leq M \rho^n.
\]

Suppose now that the adapted basis contains the vector fields
\[
e_3 = (b_2 x^3 + b_3 x^4 + b_4 x^5 + \cdots) \partial_x
\]
and
\[
e_4 = (c_3 x^4 + c_4 x^5 + \cdots) \partial_x.
\]

Then we decompose \( Z \) as \( Z = X + Y \) where \( X \) represents the “free” part of \( Z \) and is given by
\[
X = \frac{a_2}{b_2} e_3 + (a_3 - \frac{a_2 b_3}{b_2}) \frac{1}{c_3} e_4
\]
and where \( Y \) is given by \( Y = (\alpha_4 x^5 + \alpha_5 x^6 + \cdots + \alpha_n x^{n+1} + \cdots) \partial_x \) with
\[
\alpha_n = a_n - \frac{b_n}{b_2} - a_3 \frac{b_3}{c_3} + a_2 \frac{b_2 c_n}{b_2 c_3}.
\]

Let now \( p = 3 \) and consider a vector field \( Z = (a_3 x^4 + a_4 x^5 + a_5 x^6 + a_6 x^7 \cdots) \partial_x \) in \( \mathcal{L}^\omega_{p,3,M}(\Gamma) \). Suppose that the adapted basis contains the vector fields
\[
e_4 = (c_3 x^4 + c_4 x^5 + c_5 x^6 + c_6 x^7 \cdots) \partial_x,
\]
\[
e_5 = (d_4 x^5 + d_5 x^6 + d_6 x^7 \cdots) \partial_x,
\]
\[
e_6 = (f_4 x^6 + f_5 x^7 \cdots) \partial_x.
\]

We now have a decomposition \( Z = X + Y \) where the “free” part \( X \) of \( Z \) takes the form
\[
X = \frac{a_3}{c_3} e_4 + (a_4 - \frac{a_3 c_4}{c_3}) \frac{1}{d_4} e_5 + \left( a_5 - \frac{a_3 c_5}{c_3} \right) \frac{d_5}{d_4} \frac{1}{f_5} e_6
\]
and where \( Y = (\alpha_6 x^7 + \alpha_7 x^8 + \cdots + \alpha_n x^{n+1} + \cdots) \partial_x \) with
\[
\alpha_n = a_n - \frac{a_3}{c_3} c_n - (a_4 - \frac{a_3 c_4}{c_3}) \frac{d_5}{d_4} - \left( a_5 - \frac{a_3 c_5}{c_3} \right) \frac{d_5}{d_4} \frac{f_n}{f_5}.
\]

By our strong boundedness hypothesis we can find a basis \( \{e_i\} \) for which \( \frac{\rho}{c_3} \leq \rho_0^{n-2}, \frac{c_n}{c_3} \leq \rho_0^{n-3} \) etc. for some positive real number \( \rho_0 > 0 \). It is then always
For each integer $i \in \mathbb{Z}$, we start at $Z_{\rho, \rho, M}$.

Proposition 1. With a basis $X$ of possible to reduce the problem to the case where $\rho_0 = 1$ using the adjoint action $Ad(\rho_0 x)$.

When $p = 2$, we obtain $|\alpha_n| \leq |a_n| + |a_3| + 2 |a_2|$ and when $p = 3$, we have $|\alpha_n| \leq |a_n| + |a_5| + 2 |a_4| + 4 |a_3|$. In the general case we will get

$|\alpha_n| \leq |a_n| + |a_{2p-1}| + 2 |a_{2p-2}| + 2^2 |a_{2p-3}| + \cdots + 2^{p-1} |a_p|$.

Now $|\alpha_n| \leq M \rho^n$ so that

$|\alpha_n| \leq M(\rho^n + \rho^{2p-1}(1 + 2x + 2^2 x^2 + \cdots + 2^{p-1} x^{p-1}))$

with $x = \frac{1}{\rho}$. But $1 + 2x + 2^2 x^2 + \cdots + 2^{p-1} x^{p-1} + \cdots = \frac{1}{1-2x}$ for $2x < 1$. Hence if we start at $\rho \geq 4$ each $|\alpha_n|$ is bounded over by $M(\rho^n + 2\rho^{2p-1})$ so that

$$\|Y\|_{L_x} \leq M \limsup_{k \to 1} (1 + \frac{2}{\rho^k}) = \frac{3}{2} M.$$

Since $X = Y$ we have proved the following

Proposition 1. With a basis $\{e_i\}$ for which $\frac{\rho}{\rho_{\omega}} \leq 1, \frac{\rho_{\omega}}{\rho} \leq 1, \ldots$ for all $n$ the decomposition splits any $Z \in L_{\rho, p, M}(\Gamma)$ into $X + Y$ where $X \in L^{\omega}_{\rho, p, \frac{1}{2} M}(\Gamma)$ and $Y \in L^{\omega}_{\rho, 2p, \frac{1}{2} M}(\Gamma)$.

4.3. The higher dimensional case. Let $Z = Z_{p+1} + Z_{p+2} + \cdots$ be an element of $L^{\omega}_{\rho, p, M}(\Gamma)$. Here each $Z_{p+i}$ represents the homogeneous part of $Z$ of degree $p+i$.

For each integer $i$ denote by $\tilde{Z}_{p+i}$ the image of $Z_{p+i}$ by the chosen section $\Sigma$; i.e. $\tilde{Z}_{p+i} = \Sigma(Z_{p+i}) = \Sigma_{p+i}(Z_{p+i})$. We can rewrite $Z$ as follows

$Z = \tilde{Z}_{p+1} + (Z_{p+2} - \tilde{Z}_{p+1}^{p+2}) + (Z_{p+3} - \tilde{Z}_{p+1}^{p+3}) + \cdots$

where, for each integer $l > i$, $\tilde{Z}_{p+i}^{p+l}$ is the homogeneous part of degree $p+l$ of $\tilde{Z}_{p+i}$.

Since $Z \in L^{\omega}_{\rho, p, \frac{1}{2} M}(\Gamma)$, its homogeneous part of degree $p+1$ satisfies $\|Z_{p+1}\|_{p+1} \leq M \rho^p$. Therefore

$\|\tilde{Z}_{p+1}^{p+l}\|_{p+l} \leq M \rho^p |\rho_0|^{-1} \leq M \rho^p$

for all integers $l \geq 1$ whenever $\rho_0$ has been chosen smaller than 1 with no loss of generality. Denote now by $\tilde{Z}_{p+2}$ the image under $\Sigma$ of the homogeneous part $Z_{p+2} - \tilde{Z}_{p+1}^{p+2}$ of degree $p+2$.

With this notation $Z$ takes the form

$Z = \tilde{Z}_{p+1} + \tilde{Z}_{p+2} + \sum_{l=3}^{+\infty} (Z_{p+l} - \tilde{Z}_{p+1}^{p+l} - \tilde{Z}_{p+2}^{p+l}).$

At that stage we have the following bounds

$\|\tilde{Z}_{p+2}^{p+2}\|_{p+2} = \|Z_{p+2} - \tilde{Z}_{p+1}^{p+2}\|_{p+2} \leq \|Z_{p+2}\|_{p+2} + \|\tilde{Z}_{p+1}^{p+2}\|_{p+2},$

that is

$\|\tilde{Z}_{p+2}^{p+2}\|_{p+2} \leq M \rho^{p+1} + M \rho^p,$

so that

$\|\tilde{Z}_{p+2}^{p+l}\|_{p+l} \leq M \rho^{p+1} + M \rho^p,$

for all integers $l \geq 2$.

Continuing this way we obtain the decomposition of $Z = X + Y$ into a free part $X$ and a remainder $Y$. The free part $X$ takes the form

$X = \tilde{Z}_{p+1} + \tilde{Z}_{p+2} + \cdots + \tilde{Z}_{2p}$
while the remainder $Y$ is
\[ Y = \sum_{i=p+1}^{+\infty} (Z_{p+i} - \bar{Z}_{p+i} - \bar{Z}_{p+2} - \cdots - \bar{Z}_{2p}) \]

Moreover
\[ \|Y_n\| \leq \|Z_n\| + \|\bar{Z}_{p+1}\| + \|\bar{Z}_{p+2}\| + \cdots + \|\bar{Z}_{2p}\| \]
so that
\[ \|Y_{n+1}\| \leq M\rho^n + M\rho^n + (M\rho^{p+1} + M\rho^p) + \cdots + (M\rho^{2p-1} + \cdots + M\rho^p) \]
or
\[ \|Y_{n+1}\| \leq M(\rho^n + \rho^{2p-1}(1 + 2x + 2^2x^2 + \cdots + 2^{p-1}x^{p-1})) \]
with $x = \frac{1}{\rho}$. But $1 + 2x + 2^2x^2 + \cdots + 2^{p-1}x^{p-1} + \cdots = \frac{1}{1-2x}$ for $2x < 1$. Hence if we start at $\rho \geq 4$ each $\|Y_{n+1}\|_{n+1}$ is bounded over by $M(\rho^n + 2\rho^{2p-1})$. The norm of the remainder $Y$ in $L_\rho$ is bounded by
\[ \|Y\|_{L_\rho} \leq M \limsup_{k \geq 1} \{1 + \frac{2}{\rho^k}\} \leq \frac{3}{2} M. \]

Since $X = Z - Y$ we have proved the following

**Proposition 2.** The decomposition splits any $Z \in L_{\rho,p,M}(\Gamma)$ into $X + Y$ where $X \in L_{\rho,p,\frac{\rho}{2} M}(\Gamma)$ and $Y \in L_{\rho,2p,\frac{\rho}{2} M}(\Gamma)$.

4.4. **Proof of Theorem 2 - General iterative scheme.** The iterative step makes us pass from $x - Z$ to $(x - X) \circ e^X - Y \circ e^X = x - W$ where $Z = X + Y$ is the adapted decomposition described above. If $Z$ belongs to $L_{\rho,p,M}(\Gamma)$ then $X$ belongs to $L_{\rho,p,M'}(\Gamma)$ and $Y$ to $L_{\rho,2p,M''=\frac{2}{\rho} M}(\Gamma)$. Therefore we obtain
\[ \|W\|_{L_\rho \rho} \leq \frac{2M'}{\sqrt{\pi} \rho \sigma} \frac{q^2}{1 - q^2} + \|Y\|_{L_\rho \rho}, \]
where $q = \sqrt{\frac{(p+1)M'}{\sigma^2 \ln \sigma}}$ has been chosen to be less than 1. Since $Y$ is of order $2p$, that is $Y \in L_{\rho,2p,M''}(\Gamma)$, we have also $\|Y\|_{L_\rho \rho} \leq \frac{M''}{\sigma^p \rho}$. Finally
\[ \|W\|_{L_\rho \rho} \leq \frac{5M}{\sqrt{\pi} \sigma^p} \frac{q^2}{1 - q^2} + \frac{3M}{2 \sigma^2}, \]

The iteration sends $L_{\rho,p,M}(\Gamma)$ to $L_{\sigma \rho,2p,M}(\Gamma)$ where
\[ q = \sqrt{\frac{(p+1)5M/2}{\sigma^2 \rho \ln \sigma}} < 1 \]
and
\[ \hat{M} = \frac{M}{\sigma^p} \left( \frac{5}{\sqrt{\pi}} \frac{q^2}{1 - q^2} + \frac{3}{2 \sigma^p} \right). \]

With no loss of generality, we can set the value $q$ to $1/2$. We can use the "LambertW" function solve the equation
\[ \sigma^{\frac{p}{2}} \ln \sigma = \sqrt{10(p+1)\hat{M}}. \]
for σ as a function of p and M. This function is defined as the principal branch of the solution of the equation \( z = w \exp(w) \) for \( w \) as a function of \( z \). The Lambert function \( \text{LambertW}(z) \) satisfies the differential equation 
\[
\frac{dw}{dz} = \frac{w}{z(1 + w)}.
\]
More generally, the solution of \( x^a \ln(x) = b \) is given by 
\[
x = e^{\text{LambertW}(ab)}.
\]
and we obtain
\[
\sigma = e^{2\text{LambertW}(\frac{p - 2}{2}) \sqrt{10(p + 1)M}}.
\]
The iteration takes us from \( L_{\rho_n, M_i}(\Gamma) \) to \( L_{\rho_{n+1}, M_{i+1}}(\Gamma) \) where \( n_{i+1} = 2n_i \), \( \rho_{i+1} = \sigma_i \rho_i \), 
\[
M_{i+1} = \frac{M_i}{\sigma_i^n} \left( \frac{5}{3\sqrt{\pi}} + \frac{3}{2\sigma_i^n} \right)
\]
and
\[
\sigma_i = e^{2\text{LambertW}(\alpha_i \cdot (\frac{5}{3\sqrt{\pi}} + \frac{3}{2\sigma_i^n}) - 1) \sqrt{10(\alpha_i + 1)M_i}}.
\]
Obviously \( n_i = 2^n n_0 \). Moreover since \( \frac{5}{3\sqrt{\pi}} \approx 0.94 \) and \( \sigma_i > 1 \) for all integers \( i \) it is clear that \( M_i \leq (\frac{5}{2})^i M_0 \). Therefore 
\[
\rho_{i+1} = \sigma_i \rho_{i-1} \cdots \rho_0 \rho_0
\]
is less than
\[
\rho_{i+1} \leq e^{\sum_{k=0}^{i} 2\text{LambertW}(\frac{5}{3\sqrt{\pi}} + \frac{3}{2\sigma_i^n} - 1) \sqrt{10(\alpha_i + 1)M_i}} \rho_0
\]
But according to the asymptotic properties of \( \text{LambertW} \), which are similar to those of the classical logarithmic function, the infinite sum
\[
\sum_{k=0}^{\infty} \frac{2\text{LambertW}(\alpha_i \cdot (\frac{5}{3\sqrt{\pi}} + \frac{3}{2\sigma_i^n} - 1) \sqrt{10(\alpha_i + 1)M_i})}{2^{2k-2}}
\]
converges. This allows to prove the bornological convergence of the iteration completing the proof of Theorem 2.

5. Appendix: The Malgrange estimate

In this appendix we prove that the Lie algebra of a Lie pseudogroup of analytic local transformations satisfies the boundedness condition of Lemma \( \text{II} \). This property is obtained using Malgrange’s *Appendice sur le théorème de Cartan-Kähler* [Mal 72] [part II p.134] to which the reader is referred.

5.1. General facts.

5.1.1. Notation. Throughout this section we will consider a system of \( r \) equations with \( p \) independent variables and \( q \) dependent ones. The independent variables will be respectively denoted by \( z = (z^1, \ldots, z^p) \), and the dependent ones by \( y = (y^1, \ldots, y^q)^T \). Note that \( y \) will be viewed as a column vector. For each \( i \in \{1, \ldots, p\} \) and \( j \in \{1, \ldots, q\} \) the partial derivative \( \partial_z y^j \) will be denoted by \( y_i^j \). We put \( y_i = (y_i^1, \ldots, y_i^q)^T \). We will use small cap greek letters to represent multi-indices of format \( p \). If \( \alpha = (\alpha_1, \ldots, \alpha_p) \) is such a multi-index, its entries \( \alpha_i \) are nonnegative integers. The greek letter \( \varepsilon \) will be used specifically as follows; we will denote by
\[
\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)
\]
the \(p\)-multiplet with 1 in \(i\)-th position and 0 elsewhere. The null \(p\)-multiplet (0 everywhere) will be represented by 0. As usual \(|\alpha| = \alpha_1 + \cdots + \alpha_p\) (length of \(\alpha\)), \(\alpha! = \alpha_1! \cdots \alpha_p!\) and \(z^\alpha = (z^1)^{\alpha_1} \cdots (z^p)^{\alpha_p}\).

5.1.2. Defining equations. A general linear and homogeneous system of first-order PDEs, that we denote by \((E)\), can be written in matrix form as

\[
A(z)y + \sum_{i=1}^{p} B^i(z)y_i = 0
\]

where the \(r \times q\)-matrices \(A(z), B^i(z)\) are analytic in \(z\). We are working in a neighborhood of the origin for the independent variable \(z\), \(A(z), B^i(z)\) will be represented by their Taylor series centered at the origin i.e.

\[
A(z) = \sum_{|\alpha| \geq 0} A\alpha z^\alpha
\]

and

\[
B^i(z) = \sum_{|\alpha| \geq 0} B^i_\alpha z^\alpha
\]

where \(A\alpha, B^i_\alpha\) are \(r \times q\)-matrices with constant coefficients (real or complex).

From now on we will use the summation convention.

5.2. Recursion.

5.2.1. Formal solution. We will deal with formal solutions \(y\) of the form \(F = f_\gamma z^\gamma\). Such an \(F\) is constructed step-by-step by replacing \(y\) by \(F\) in the equation and solving the corresponding equation at order 0, then 1, then 2 and so on. We focus only on strongly extendible jets (\(jets\) \(fortement\) \(prolongeables\) in Malgrange’s terminology \cite{Mal72} [part II p.138]). The recursive construction is therefore made possible as soon as \(f_\gamma\) for \(|\gamma| = 0, 1\) have been chosen by solving the equation \((E)\) at order 0, i.e. by solving what we will call \((E_0)\).

5.2.2. Equation at order \(l\). From \(y = f_\gamma z^\gamma\) we get \(y_i = (\gamma_i + 1)f_\gamma + \epsilon_i, z^\gamma\) where \(\gamma_i\) is the \(i\)-th entry of \(\gamma = (\gamma_1, \ldots, \gamma_p)\). The equation \((E)\) at order \(l\), that we denote by \((E_l)\), takes the matrix form

\[
\sum_{|\beta| = l} B^i_0(\beta_i + 1) f_{\beta_i + \epsilon_i, z^\beta} = -\left[ \sum_{|\beta| = l} A_\alpha f_{\beta - \alpha, z^\beta} + \sum_{|\beta| = l} B^i_{\beta - \gamma}(\gamma_i + 1)f_{\gamma + \epsilon_i, z^\beta} \right]
\]

where the summation over \(i\) is implicit. Equation \((E_l)\) is precisely, in our case, equation

\[
\sum_{|\alpha| = k} u_\alpha f_{\alpha + \beta, \frac{(\alpha + \beta)!}{\beta!} z^\beta} = \Psi(z, D^k F^{k+l-1})
\]

of Malgrange \cite{Mal72} [part II p.139]. We will denote this equation by \((\Phi_l)\) and, following Malgrange, we will denote by \(G_l = \sum_{|\beta| = l} g_\beta z^\beta\) its right-hand side.

5.3. Norms and the Malgrange estimate.
5.3.1. Norms. In accordance with Malgrange’s notation we will denote by $E_1$ the space $\mathbb{R}^r$ (or $\mathbb{C}^r$) and by $E$ the space $\mathbb{R}^q$ (or $\mathbb{C}^q$). The space of $r \times q$-matrices (over $\mathbb{R}$ or $\mathbb{C}$) will simply be denoted by $\mathcal{M}$. We endow these three vector spaces with the sup norm. Thus for $f_\alpha = (f^{1 \alpha}, \ldots, f^{q \alpha})^T$ in $E$,

$$| f_\alpha | = \max_{i=1, \ldots, q} | f^{i \alpha} |$$

and likewise, for $A_\alpha = [(A^{ij})_{\alpha}]$ in $\mathcal{M}$,

$$| A_\alpha | = \max_{j=1, \ldots, q} | (A^{ij})_\alpha |.$$

In addition we will endow the space $S^l \otimes E$ (resp. $S^l \otimes E_1$ and $S^l \otimes \mathcal{M}$) of degree $l$ homogeneous polynomials (in the $z$ variable) with values in $E$ (resp. $E_1$ and $\mathcal{M}$) with the following norm; if $H = \sum_{|\beta|=l} h_\beta z^\beta$ is such an element then we put

$$\| H \|_l = \max_{|\beta|=l} | h_\beta |$$

where $| h_\beta |$ is the corresponding norm of $h_\beta$ in $E$ (resp. $E_1$ and $\mathcal{M}$).

The following proposition shows to which extent these norms are well adapted.

**Proposition 3.** If $A = A_l(z)$ is a homogeneous polynomial of degree $l$ with values in $\mathcal{M}$ and if $Y = Y_m(z)$ is a homogeneous polynomial of degree $m$ with values in $E$ (column matrix of format $q$) then $AY$ is a homogeneous polynomial of degree $l + m$ with values in $E_1$ (column matrix of format $r$) satisfying

$$\| AY \|_{l+m} \leq q \| A \|_l \| Y \|_m$$

The proof of the preceding proposition is an adaptation of Proposition 3.3 [Mal 72] [part II p.137] taking into account our choice of norms.

5.3.2. The Malgrange estimate. With our choice of norms, the Malgrange estimate [Mal 72] [part II p.139],

$$\max_{|\alpha|=k+l} \alpha! | f_\alpha | r^\alpha \leq C \max_{|\beta|=l} \beta! | g_\beta | r^\beta$$

obtained from equation (Φ1) as a consequence of his Corollary 2.2 [Mal 72] [part II p.135] reads

$$(k + l)! \| F \|_{k+l} \leq \frac{C}{r_0^l} \| G_l \|_l$$

if we consider as a system of $p$ strictly positive numbers$^1$ $r = (r_1, \ldots, r_p)$ the string $r = (r_0, \ldots, r_0)$ (see Section 2 on Polydisques distingués in [Mal 72] [part II p.134]).

In our particular case of first order PDE, this inequality becomes

$$\| F \|_{l+1} \leq \frac{C}{r_0(l+1)} \| G_l \|_l$$

which we denote by (I).$$

$^1$Note that the integer $n$ in Malgrange paper coincides with our $p$.}
Remark 5.1. \( F = f_\alpha z^\alpha \) is analytic if and only if there exist \( M, \rho > 0 \) such that \( \| F \|_l \leq M \rho^l \) for all integer \( l \). This is a consequence of the inequality
\[
\left| \sum_{|\alpha| = l} f_\alpha z^\alpha \right| \leq \sum_{|\alpha| = l} |f_\alpha| \| z \|^{|\alpha|} \leq \left( \sum_{|\alpha| = l} \frac{|f_\alpha|}{|\alpha|!} \right)^{|\alpha|} \| z \| \leq \| F \|_l \leq \| F \|_l (| z | + \cdots + | z^p |)^l.
\]

5.4. Prolongation - Strong boundedness.

5.4.1. Growth of coefficients. From Proposition 3 the norm of the right-hand side \( G_l = \sum_{|\beta| = l} |g_\beta| z^\beta \) of \((E_l)\) satisfies the inequality
\[
\| G_l \|_l \leq q \sum_{s+t=l} \| A \|_s \| F \|_t + \sum_{s>0} \| B^s \|_s \| F_i \|_t
\]
in which intervene the homogeneous parts \((F_i)_t = \sum_{|\gamma|=t} (\gamma_i + 1) f_{\gamma+\varepsilon_i} z^\gamma \) of \( F_i \).

Since the length \( |\gamma + \varepsilon_i| = t + 1 \) for \( |\gamma| = t \) and
\[
(\gamma_i + 1) f_{\gamma+\varepsilon_i} \leq (t + 1) \| F \|_t
\]
we obtain, for \( |\alpha| = t + 1 \),
\[
| f_\alpha | \leq \frac{(t + 1)!}{\alpha!} \| F \|_t.
\]

Therefore
\[
| f_\gamma \varepsilon_i | \leq \frac{(t + 1)!}{(\gamma_i + 1) \gamma!} \| F \|_t
\]
so that
\[
\frac{\gamma!}{\gamma_i!} (\gamma_i + 1) f_{\gamma+\varepsilon_i} \leq (t + 1) \| F \|_t
\]
for all multi-indices \( \gamma \) of length \( t \). As a consequence
\[
\| F_i \|_t \leq (t + 1) \| F \|_t
\]
and finally, introducing two constants \( M, \rho_0 > 0 \) such that \( \| A \|_s, \| B^i \|_s \leq M \rho_0^s \) for all integer \( s \), we get
\[
\| G_l \|_l \leq q \sum_{s+t=l} M \rho_0^s \| F \|_t + \sum_{s+t=l} M \rho_0^s (t + 1) \| F \|_t + \cdots + \| F \|_l.
\]

Lemma 5. The prolongation scheme described by Malgrange leads, in the case of first order linear and homogeneous PDEs, to the inequality
\[
\| F \|_{l+1} \leq C M q \left( \rho_0^l \| F \|_0 + \rho_0^{l-1} \| F \|_1 + \cdots + \| F \|_l \right) + (\rho_0^l \| F \|_l + \rho_0^{l-1} \| F \|_{l-1} + \cdots + \rho_0^0 \| F \|_0),
\]
5.4.2. Strong boundedness.

**Proposition 4** (Strong boundedness). Let $\Lambda$ be a linear and homogeneous analytic system of partial differential equations. Then its vector space of analytic local solutions admits as basis a bounded family $\mathcal{B}$ satisfying the following growth condition: for all $F$ in $\mathcal{B}$ and all positive integers $k$

$$\| F \|_{d+k} \leq K^k \| F \|_d$$

where $K > 0$ is a constant depending only on $\Lambda$, and $d$ represents the smallest degree appearing in $F$.

**Proof.** Such a basis will be constructed using Malgrange prolongation scheme. For instance the “affine part” will be obtained by prolongation of a basis of the vector space solution to $(E_0)$. The “homogeneous part” of degree 2 will be obtained by prolongation of a basis of the vector space solution to $(E_1)$ without right-hand side (no affine part) and so on.

In order to control the growth of the coefficients we associate to each such prolongation $F$ (element of the basis $\mathcal{B}$) a sequence $(\phi_n)$ as follows. For a prolongation representing the affine part we put $\phi_0 = \| F \|_0$, $\phi_1 = \| F \|_1$ and for a prolongation we put $F = F_d + F_{d+1} + \cdots$ representing the homogeneous part of degree $d > 1$ put $\phi_l = 0$ for $l < d$ and $\phi_d = \| F \|_d$. Finally for all integers $l \geq 1$ set

$$\phi_{l+1} = \frac{CMq}{r_0(l+1)}[\rho_0^l \phi_0 + \rho_0^{l-1} \phi_1 + \cdots + \phi_l] + \rho_0 l \phi_l + \rho_0 (l-1) \phi_{l-1} \cdots \rho_0 \phi_0]$$

The sequence $(\phi_n)_{n \in \mathbb{N}}$ bounds the sequence $(\| F \|_n)_{n \in \mathbb{N}}$ from above, i.e. $\| F \|_n \leq \phi_n$ for all integer $n$. Moreover

$$\phi_{l+1} = \frac{CMq}{r_0(l+1)}[\rho_0^l \phi_0 + \cdots + \phi_l] + \rho_0 l \phi_l + \rho_0 (l-1) \phi_{l-1} \cdots \rho_0 \phi_0]$$

that is to say

$$\phi_{l+1} = \frac{CMq}{r_0(l+1)}[\rho_0(l \phi_l) + \phi_l + \rho_0 l \phi_l]$$

or equivalently

$$\phi_{l+1} = \frac{l}{l+1} \rho_0 + \frac{CMq}{r_0(l+1)}(1 + \rho_0 l)\phi_l$$

If we put $K = \rho_0 + \frac{CMq}{r_0} + \frac{CMq}{r_0} \rho_0$ this gives rise to

$$\phi_{l+1} \leq K \phi_l$$

for all integer $l \geq 1$.

Therefore

$$\| F \|_{d+k} \leq \phi_{d+k} \leq K^k \phi_d \leq K^k \| F \|_d$$

as claimed. This inequality holds at least for $d \geq 2$. We include the cases $d = 0$ and $d = 1$ by replacing if necessary $K$ by a larger constant. \hfill \Box

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