GLOBAL ANALYSIS OF NEW GRAVITATIONAL SINGULARITIES IN STRING AND PARTICLE THEORIES  

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ABSTRACT

We present a global analysis of the geometries that arise in non-compact current algebra (or gauged WZW) coset models of strings and particles propagating in curved space-time. The simplest case is the 2d black hole. In higher dimensions these geometries describe new and much more complex singularities. For string and particle theories (defined in the text) we introduce general methods for identifying global coordinates and give the general exact solution for the geodesics for any gauged WZW model for any number of dimensions. We then specialize to the 3d geometries associated with $SO(2,2)/SO(2,1)$ (and also $SO(3,1)/SO(2,1)$) and discuss in detail the global space, geodesics, curvature singularities and duality properties of this space. The large-small (or mirror) type duality property is reformulated as an inversion in group parameter space. The 3d global space has two topologically distinct sectors, with patches of different sectors related by duality. The first sector has a singularity surface with the topology of “pinched double trousers”. It can be pictured as the world sheet of two closed strings that join into a single closed string and then split into two closed strings, but with a pinch in each leg of the trousers. The second sector has a singularity surface with the topology of “double saddle”, pictured as the world sheets of two infinite open strings that come close but do not touch. We discuss the geodesicaly complete spaces on each side of these surfaces and interpret the motion of particles in physical terms. A cosmological interpretation is suggested and comments are made on possible physical applications.

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1. Motivation

In an attempt to formulate solvable models of strings propagating in curved spacetime it was discovered that certain classes of non-compact current algebra models \cite{1} can shed light on gravitational singularities such as black holes in string theory \cite{2} as well as more interesting singularities in higher dimensions \cite{3} \cite{4} \cite{5} \cite{6} \cite{7} \cite{8}. All single time coordinate models based on simple non-compact groups are characterized by the $G/H$ cosets \cite{1} \cite{9} : \( \text{SO}(d-1,2)/\text{SO}(d-1,1) \), \( \text{SO}(d,1)/\text{SO}(d-1,1) \), \( \text{SU}(n,m)/\text{SU}(n) \times \text{SU}(m) \), \( \text{SO}(n,2)/\text{SO}(n) \), \( \text{SO}(2n)^*/\text{SU}(n) \), \( \text{Sp}(2n)^*/\text{SU}(n) \), \( \text{E}_6^*/\text{SO}(10) \), \( \text{E}_7^*/\text{E}_6 \). Naturally, by taking non-simple (direct product) groups, including non-Abelian, \( U(1) \) or \( \mathbb{R} \) factors one can construct extensions of these models \cite{1} provided one demands a total Virasoro central charge \( c = 26 \) (or \( c = 15 \) with supersymmetry). In addition, when there are \( U(1) \) or \( \mathbb{R} \) factors there is a non-trivial way of gauging a total \( U(1) \) or \( \mathbb{R} \) that leads to further models. For example, by combining Witten’s 2d black hole with additional \( U(1) \)’s, black holes, black strings and black p-branes can be constructed in higher dimensions \cite{10} \cite{11} \cite{12} \cite{13}. In all of these models there are interesting duality properties \cite{14} that generalize the kind of duality found in tori and mirror manifolds. The existence of a \textit{discrete} duality and its origins in the general model \cite{1} is given in \cite{1}, while further duality properties based on Killing vectors is given in \cite{15} \cite{16}.

By virtue of being exact conformal models, all of these cosets yield new explicit solutions of Einstein’s equations coupled to matter in \( d = \text{dim}(G/H) \) dimensions by automatically solving the perturbative conformal invariance conditions of \cite{17}. Recall the traditional Lagrangian method for constructing the metric that solves Einstein’s equations: start with a gauged WZW model characterized by \( G/H \) \cite{18}, parametrize the group element, choose a unitary gauge that fixes the gauges of \( H \) completely by setting \( \text{dim} H \) group parameters equal to zero and integrate out the non-propagating \( \text{dim} H \) gauge fields.

\footnote{It must be emphasized that the cosets \( G/H \) are not the left or right cosets in which the subgroup \( H \) acts on one side of the group element. Rather, \( H \) is the “vector” subgroup that acts on both sides \( g \rightarrow \Lambda g \Lambda^{-1} \), where \( \Lambda \in H \). Actually, \( H \) may be generalized \cite{4} to a deformed subgroup \( g \rightarrow \Lambda g \tilde{\Lambda}^{-1} \) where \( \tilde{\Lambda} = g_0 \Lambda g_0^{-1} \) or \( \tilde{\Lambda} = g_0 (\Lambda^T)^{-1} g_0^{-1} \) and \( g_0 \) is a discrete element in complexified \( G \). This deformation is the origin of duality transformations. Its effects can be reproduced as discrete leaps in group space \( g \rightarrow \hat{g} \) by the switching of some signs. In this latter sense the undeformed theory already describes all the dual sectors. More on this point will be said below.}
The remaining group parameters $q^a(\tau, \sigma)$, $a = 1, 2, \cdots, \text{dim}(G/H)$ are interpreted as target space string coordinates and the Lagrangian looks like a non-linear sigma model that describes a string propagating in $\text{dim}(G/H)$ curved spacetime. The gravitational metric $G_{ab}(q)$ is identified and interpreted by examining this effective Lagrangian. Similarly, one can read off a dilaton field $\Phi(q)$ directly from the WZW lagrangian \cite{4} \cite{5}. Remarkably, this metric and dilaton, together with an antisymmetric field $B_{ab}(q)$ which also emerges in a similar way (sometimes zero), automatically solve the coupled Einstein’s equations for $G_{ab}, B_{ab}, \Phi$ in dimension $d = \text{dim}(G/H)$. Therefore, in addition to string theory, this approach is tempting for investigations in General Relativity since it may be viewed as a new method for generating fascinating classical solutions to Einstein’s equations. Even more enticing is the fact that the quantum spectrum is solved exactly by labelling the states group theoretically and computing eigenvalues of Casimir operators.

In string theory, this metric is perturbative from the point of view of conformal invariance and is valid at large $k$, where $k$ is the central extension of the current algebra. In the same large $k$ limit, the algebraic properties of the cosets above indicate \cite{1} that one should expect a $\text{dim}(G/H)$ geometry with a single time coordinate; indeed this is satisfied in intricate ways as will be explicitly seen below. As we will show elsewhere \cite{19} the conformally exact metric and dilaton are computed explicitly by algebraic Hamiltonian methods for any $k$. For the bosonic and heterotic string there are major corrections. However, for the supersymmetric type-II string based on any $G/H$, the exact metric and dilaton are the same as the perturbative metric and dilaton, except for an overall renormalization ($k$ gets replaced by $k - g$, where $g$ is the Coxeter number of $G$). Furthermore, for a particle theory (as opposed to a string theory, see below) the perturbative and exact expressions are identical for any $k$ with or without supersymmetry. Therefore, it is meaningful to study the so called “perturbative” metric for a variety of cases.

Another problem with the perturbative Lagrangian method is that it generates the metric in a patch of the manifold. By choosing a somewhat different unitary gauge one arrives at a metric, in a different coordinate patch, which may bear no resemblance to the previous one (e.g. compare \cite{3} to \cite{4} or \cite{6} in 3d). What is the global space? What are the global coordinates? What is the behaviour of light rays (or slower moving particles) in the geodesically complete space? One needs this information in order to interpret the

\footnote{This procedure which was briefly used in \cite{1} to count the number of time coordinates, was fully explored for $SL(2, \mathbb{R})$ in \cite{2} to construct the 2d black hole metric.}
geometry of spacetime. In two dimensions this problem was solved by rewriting the metric in terms of the globally defined Kruskal coordinates [2]. However, in higher dimensions, in the absence of Killing vectors, we need new methods.

For the reasons mentioned above the new singular geometries that arise in higher dimensions have not been easy to interpret (except for $U(1)$’s times 2d black hole). It is the purpose of the present paper to do so. Our treatment will introduce methods that are completely general and apply to all of the above models, and in fact to any effective metric derived from a gauged Wess-Zumino-Witten model (any number of time coordinates 0, 1, 2, etc.).

2. Global space and geodesics in the general model

Our first observation is that the global coordinates must be $H$-invariant. This will avoid the problem with gauge fixing. In fact, although not immediately obvious, there are precisely $\text{dim}(G/H)$ independent H-invariants $Q^a$ that can be constructed from $\text{dim}G$ group parameters. To illustrate this point, consider $SO(d-1,2)/SO(d-1,1)$. Under the Lorentz subgroup $H = SO(d-1,1)$ the group parameters are classified as a vector $x_\mu$ and an anti-symmetric tensor $a_{\mu\nu}$ in $d$-dimensions. There are precisely $d$ Lorentz invariants that can be constructed from these. For example for $d = 3$ we have $Q^a = (x^2, a^2, \epsilon^{\mu\nu\lambda} x_\mu a_{\lambda\sigma})$, for $d = 4$ we have $Q^a = (x^2, a^2, \epsilon^{\mu\nu\lambda\sigma} a_{\mu\nu} a_{\lambda\sigma}, (x^\mu a_{\mu\nu})^2)$, etc.. The invariants $Q^a$ are related to the gauge fixed group parameters $q^a$ by a change of coordinates $q^a(Q)$, as will be shown below. This example also illustrates why one may get only a patch if the metric is written in terms of gauge fixed group parameters: when $x^2$ is time-like it can be Lorentz transformed (or gauge fixed) to $x_\mu = (x_0, 0, 0, \cdots)$ but if it is space-like or light-like it cannot be put into this form. Thus, the metric will look very different and cover different patches for these two possibilities. However, as an invariant, $x^2$ can take zero, positive and negative values, thus “globally” covering all possibilities. Similar comments apply to all other invariants.

How does one rewrite the metric in terms of H-invariants? We have found an algebraic and systematic approach which also leads to many other results, such as the exact conformal metric, dilaton, etc.. Because it involves a host of other ideas and techniques it will be published as a separate article [19]. Here we will use a more pedestrian approach which agrees with our systematic results. We start with the metric which is computed in any patch and rewrite the group parameters in terms of dot products of $gauge\ fixed\ H$
representations. We make a change of variables from gauge fixed group parameters to these dot products \((\dim(G/H))\) of them and then allow the new coordinates thus identified to take all possible values that invariants can take. This procedure will provide the needed analytic continuation from the original patch to the global space. Then the location and nature of the singularities in the geometry are revealed.

Having global coordinates is not sufficient to get a feeling for the geometry; one also needs to know the behavior of the geodesics. However, the geodesic equation seems completely unmanageable in the complicated metrics that emerge. On the other hand, we have been able to find the general geodesic solution by the following procedure. We first define a gauged point particle theory, which is essentially the dimensional reduction of the familiar WZW model (i.e. all our fields are functions of only \(\tau\), rather than \(\tau, \sigma\)).

\[
S(g) = \frac{k}{4\pi} \int d\tau \ Tr \left( \frac{1}{2} g^{-1} \dot{g} - A_- g^{-1} - A_+ g^{-1} + A_- g A_+ g^{-1} - A_- A_+ \right)
\]

(2.1)

where \(g(\tau) \in G\) is a group element and \(A_{\pm}(\tau)\) are two gauge potentials in the Lie algebra of \(H\). Two gauge potentials are needed for our purposes. The model is gauge invariant under the transformations \(g' = \Lambda g \Lambda^{-1}, A'_{\pm} = \Lambda(A_{\pm} - \partial_\tau) \Lambda^{-1}\), where \(\Lambda(\tau) \in H\). Consider the equations of motion

\[
(g^{-1} D_- g)_H = 0 = (D_+ g g^{-1} - A_+ g^{-1} - A_- g A_+ g^{-1} - A_- A_+ - A_- A_+).
\]

(2.2)

where \(D_\pm = \dot{g} - [A_\pm, g]\), and the subscript \(H\) means a projection to the Lie algebra of \(H\).

These equations may be considered as geodesics in an enlarged space \((\dim(G) + 2\dim(H))\). One avenue for solving these equations is to choose a unitary gauge \((\dim(H)\) conditions), solve for the two gauge potentials and substitute in the remaining equations. The remaining unsolved \(\dim(G/H)\) equations are in fact the geodesic equations. That is, these are the equations of motion that follow from the \(\dim(G/H)\) non-linear sigma model (equivalent to (2.1)) that defines the line element for the metric, \(S = (k/\pi) \int d\tau \ G_{ab}(q) \dot{q}^a \dot{q}^b + \cdots\), and they coincide with the usual geodesic equation for that metric, \(\ddot{q}^a + \Gamma^a_{bc} \dot{q}^b \dot{q}^c = 0\). We may rewrite these geodesic equations in terms of the global coordinates \(Q^a(q)\) described above.

\[\text{The third and fourth equations are derived from a more complicated equation after using the first one and projecting along } H \text{ or } G/H \text{ in the Lie algebra.}\]
It seems hopeless to find solutions for them (see e.g. (6.1)). However, another avenue for solving (2.2) is to choose an axial gauge $A_+ = 0$. In this gauge there is a leftover global $H$-invariance giving all expressions a $H$-covariant form. The last equation yields $A_- = \alpha$ where $\alpha$ is a constant matrix in the Lie algebra of $H$. The first and third equations yield a first integral of the form $g^{-1} D_- g = p$, where $p$ is a constant matrix in the Lie algebra of $G/H$. This equation can be rearranged to the form $\dot{g} = g(p - \alpha) + \alpha g$, and then solved by

$$g = e^{\alpha \tau} g_0 e^{(p-\alpha)\tau}, \quad (2.3)$$

where $g_0$ is a constant group element that characterizes the initial conditions. Finally, replacing this form into the remaining second equation in (2.2) yields a constraint among the constants of integration

$$(g_0(p - \alpha)g_0^{-1})_H + \alpha = 0. \quad (2.4)$$

Let us count the constants of integration. We start with $\dim(G) + \dim H + \dim(G/H)$ parameters in $g_0, \alpha, p$. The constraints and the leftover global $H$-invariance remove $2\dim H$ of them. Therefore, the truly independent and physical ones are $2\dim(G/H)$ in number, which is precisely the number of initial positions $Q^a(0)$ and velocities $\dot{Q}^a(0)$ needed for the general geodesic in $\dim(G/H)$ dimensions. Therefore (2.3)(2.4) contain the general geodesic solution. What remains is the purely group theoretical exercise of projecting from this solution in group space $G$ to the space of $H$-invariants and relating them to the coordinates $Q^a(\tau)$. These will then give the general geodesic solution in the global space!

It was very important that we reformulated the manifold in terms of $H$-invariants because, by virtue of being gauge independent, the solution obtained for the invariants in the axial gauge is indeed the solution in any gauge, and in particular in all the patches of the unitary gauge where the question was first asked.

As already mentioned, in a unitary gauge the Lagrangian (2.1) is rewritten in terms of the line element $ds^2/d\tau^2 = \dot{q}^a \dot{q}^b G_{ab}(q)$. Therefore, if we substitute the covariant solution (2.3)(2.4) in the gauge invariant Lagrangian (2.1) we find the value of $ds^2/d\tau^2$ for the geodesic solution. This gives

$$\frac{ds^2}{d\tau^2} = \frac{1}{2} \text{Tr}(p^2). \quad (2.5)$$
Now by choosing the constant matrix $p$ we have control on whether the geodesic is light-like, time-like or space-like. This feature will allow us to examine below the behavior of light rays in the curved geometries that emerge by taking $Tr(p^2) = 0$.

The above solution was for the self-contained particle theory of (2.1). The string theory has a similar fully general solution as given in [4]. Therefore, we are also equipped to study the string geodesics in these geometries. In fact, by applying techniques similar to those displayed below we can find out the general string motion in curved spaces containing singularities (such as black holes) provided they are generated through a WZW model. The solution exhibited in (2.3)(2.4) simply corresponds to the motion of a string collapsed to a point. In this paper we will not elaborate on the more general string geodesics and invite the interested reader to synthesize the solutions of [4] with the methods of this paper. In this way one can study, for example, the free fall of a string into a black hole.

3. Global space for 3d $SO(2, 2)/SO(2, 1)$ and $SO(3, 1)/SO(2, 1)$

In this section we will apply the general ideas to the specific cases $SO(2, 2)/SO(2, 1)$ and $SO(3, 1)/SO(2, 1)$ to find the global 3d geometry. To begin we will adopt the geometrical analysis of the $SO(2, 2)/SO(2, 1)$ string theory given in [4] which applies unchanged to the point particle theory defined by (2.1). The $SO(3, 1)/SO(2, 1)$ case will be discussed by analytic continuation after the global space for $SO(2, 2)/SO(2, 1)$ is understood. The $SO(2, 2)$ group element was written in terms of 6 parameters that are classified as two Lorentz 3d-vectors $(X_\mu, Y_\mu)$ under the subgroup $SO(2, 1)$. The group element takes the form $g = h(Y)t(X)$, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & h\nu^\nu \end{pmatrix}, \quad t = \begin{pmatrix} b & -bX^\nu \\ bX_\mu & (\eta_\mu^\nu + abX_\mu X^\nu) \end{pmatrix},$$

$$h_{\mu\nu} = \epsilon' \sqrt{1 - Y^2} \eta_{\mu\nu} + \frac{Y_\mu Y_\nu}{1 + \epsilon' \sqrt{1 - Y^2}} + \epsilon_{\mu\nu\lambda} Y^\lambda,$$

$$b = \frac{\epsilon}{\sqrt{1 + X^2}}, \quad a = (1 - b^{-1})/X^2, \quad \epsilon = \pm, \quad \epsilon' = \pm, \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1).$$

Indices are raised or lowered with the Minkowski metric $\eta_{\mu\nu}$. The gauge transformations are Lorentz transformations. The unitary gauge that describes a patch was fixed by taking $X_\mu = \text{tanh}(2r)$ $(0, 0, 1)$ and $Y_\mu = \text{sinh}(2t)$ $(0, \cos \theta, -\sin \theta)$, in which both invariants $(X^2, Y^2)$ are negative (space-like). After solving for $A_\pm$ and substituting in the action,
and taking care of the correct measure in the path integral, the effective Lagrangian
takes the form \( L = \frac{k}{\pi} (ds/d\tau)^2 - \Phi \) where the line element \( ds^2 \) and the
dilaton \( \Phi \) are given in terms of the coordinates \( q^a = (t, r, \theta) \)

\[
ds^2 = dr^2 + \lambda^2(r, \epsilon) [d\theta + \kappa(t, \epsilon') \tan \theta \, dt]^2 - \lambda^{-2}(r, \epsilon) \cos^{-2} \theta \, dt^2
\]

\[
\lambda^2(r, \epsilon) = \frac{\cosh(2r) - \epsilon}{\cosh(2r) + \epsilon}, \quad \kappa(t, \epsilon') = \frac{\sinh(2t)}{\cosh(2t) - \epsilon'}, \quad \epsilon = \pm, \, \epsilon' = \pm
\]

(3.2)

\[
\Phi(r, t, \theta) = \ln \left[ \sinh^2(2r) \cos^2(\theta) (\cosh(2t) - \epsilon') \right] + \text{constant}.
\]

For the particle theory this gives the exact metric and dilaton for any \( k \). Also, as shown in [19], this is the exact metric and dilaton for the supersymmetric \( SO(2, 2)/SO(2, 1) \) type-II superstring for any \( k \), except for an overall quantum renormalization that replaces \( k \) by \( (k - 2) \) in the effective Lagrangian. However, for the purely bosonic or the heterotic string theory this is the large \( k \) conformally perturbative metric and dilaton, while the conformally exact expressions for any \( k \) given in [19] differ from the above.

The various signs \( \epsilon, \epsilon' \) correspond to patches related by duality transformations that will be discussed further below. For every \( \epsilon, \epsilon' \), there are additional sets of analytic continuations into other patches that correspond to other configurations of unitary gauges in different regions of \( X^2, Y^2 \). The 16 patches so obtained form the global space as shown below.

The goal now is to rewrite the metric in terms of an appropriate combination of dot
products \( (X^2, Y^2, X \cdot Y) \) in such a way that a single expression for the metric is valid in all 16 patches. This can be done easily in one patch, and then the expression so obtained is allowed to take values for the entire range of the invariants. This gives the global metric in the global space. We find it convenient to define the following invariants suggested by the form of the group element in (3.1)

\[
b = \epsilon (1 + X^2)^{-1/2}, \quad v = \epsilon' (1 - Y^2)^{1/2} + 1, \quad u = (v - 2)(X \cdot Y)^2/X^2Y^2.
\]

(3.3)

In the unitary gauge for the patch above these reduce to

\[
b = \epsilon \cosh(2r), \quad v = \epsilon' \cosh(2t) + 1, \quad u = \sin^2 \theta (\epsilon' \cosh(2t) - 1).
\]

(3.4)

Using this last equation as a change of coordinates \( Q^a(q) \) we rewrite the metric and dilaton (3.2) in terms of \( Q^a = (v, u, b) \)
\[ ds^2 = \frac{db^2}{4(b^2 - 1)} + \frac{b - 1}{b + 1} \frac{du^2}{4u(v - u - 2)} - \frac{b + 1}{b - 1} \frac{dv^2}{4v(v - u - 2)}, \]

\[ \Phi = \ln[(b^2 - 1)(v - u - 2)] + \Phi_0. \]

The non-zero components of the Ricci tensor are given by

\[ R_{bb} = -\frac{2}{(b^2 - 1)^2}, \quad R_{vv} = \frac{2bv - (b + 1)^2}{v(b - 1)^2(v - u - 2)^2}, \quad R_{uu} = \frac{-2bu + (b - 1)^2}{u(b + 1)^2(v - u - 2)^2}, \]

while the scalar curvature for this metric is

\[ R = \frac{8}{(b^2 - 1)(v - u - 2)} \left( 3 + b^2 - v(b + 1) - u(b - 1) \right), \]

revealing the location of gravitational singularities in the global space. The discussion of the singularity will be postponed to section 4.

In the regions of the global space for which \(|b| \gg 1\) and \(|v - u| \gg 2\) the curvature vanishes, indicating that the metric is flat. For such asymptotic regions, we may exhibit the flat metric by the reparametrization

\[ b \sim \epsilon e^{2z_1}, \quad v \sim \epsilon' e^{2z_0} \sinh^2 z_2, \quad u \sim \epsilon' e^{2z_0} \cosh^2 z_2, \]

\[ ds^2 \sim dz_1^2 + dz_2^2 - dz_0^2 \]

which is valid for large values of \(z_0, z_1\). Since the time coordinate \(z_0\) must get large to reach the asymptotically flat region, this metric must have a cosmological interpretation. This will be discussed in the last section of the paper.

We now determine the globally allowed ranges of \(Q^a = (v, u, b)\) by analytic continuation away from the patch (3.4). First, the correct parametrization of the \(SO(2, 2)\) group element requires that \((X^2 > -1, Y^2 < 1)\) be within the ranges which insure that the square roots in (3.1)(3.3) are real. Furthermore, the existence of the dual patches allows \(Q^a = (v, u, b)\) to take both positive and negative values. This translates to \(b, v\) taking values on the entire real line. The only remaining task is the determination of all allowed values for the Lorentz dot products \((X \cdot Y)^2/X^2Y^2\) when \((X^2, Y^2)\) are in their allowed ranges. It is easy to see that this provides restrictions on the combined ranges of \((v, u, b)\) as follows:
\[(b^2 > 1) \text{ and } (uv > 0), \quad \text{or} \quad [(b^2 < 1) \text{ and } (uv < 0), \text{ excluding } 0 < v < u + 2 < 2.\]

This is then the global space for the $SO(2, 2)/SO(2, 1)$ theory! In Figs.1a,b,c the 16 patches of this space are enumerated as (I), (IIa-IIg), (IIIb), (II'), (II'b,II'c,II'f,II'g), (III'a,III'b). The patch we started from, with $\epsilon = +, \epsilon' = +$, is denoted by (I) in Fig.1a, as can be verified from the values of $Q^a = (v, u, b)$ generated by (3.4). The planes that slice-up the space correspond to the values of $Q^a = (v, u, b)$ at which there is a change of sign for the factors $[(b^2 - 1), (v - u - 2), v, u]$ which appear in the metric (3.3). These crucial sign switches determine the signature of the metric in the various regions of the global space.

The signatures in each patch is given in Fig.1 in the form $(-++)$, $(++-)$, $(+-+)$, which shows the signs of the factors in front of $(dv^2, du^2, db^2)$ in the line element (3.7). It is seen that for any of the $SO(2, 2)/SO(2, 1)$ patches there is always only one time coordinate and two space coordinates, although the role of time switches between $(v, u, b)$ in the various regions. The patches are grouped together in six regions I,II,III,II',III',III' as in the parentheses above. As explained in section 5, each one of the six groups of patches in parentheses is geodesically complete and geodesically disconnected from the others.

For completeness, it is instructive to see how the 16 patches that make up the global $SO(2, 2)/SO(2, 1)$ space (3.9) are parametrized in various unitary gauges. The following table provides this information.

| (patch)$_{\epsilon,\epsilon'}$ | $X, Y, (u,v,b)$ |
|-------------------------------|----------------|
| (I)$_{++}, (II')_{+-}$        | $X_\mu = tanh(2r)(0,0,1), \quad Y_\mu = sinh(2t)(0,\cos\theta,-\sin\theta)$ |
| (II)$g_{-+}, (II)_{+-}$       | $u = (v-2)\sin^2\theta, \quad v = \epsilon\cosh(2t)+1, \quad b = \epsilon\cosh(2r)$ |
| (IIIb)$_{++}, (III'b)_{+-}$   | $X_\mu = tanh(2r)(0,0,1), \quad Y_\mu = sinh(2t)(\sinh\psi,0,\cosh\psi)$ |
| (III'a)$_{+-}, (III'b)_{+-}$  | $u = (v-2)\cosh^2\psi, \quad v = \epsilon\cosh(2t)+1, \quad b = \epsilon\cosh(2r)$ |
| (IIa)$_{+-}, (III'a)_{+-}$    | $X_\mu = tanh(2r)(0,0,1), \quad Y_\mu = \sin(2\phi)(\cosh(t'),0,\sin(t'))$ |
| (IIc)$_{+-}, (II'd)_{+-}$     | $u = (v-2)\sinh^2\phi, \quad v = \epsilon\cosh(2\phi)+1, \quad b = \epsilon\cosh(2r)$ |
| (IIde)$_{+-}$                 | $X_\mu = tanh(2t')(1,0,0), \quad Y_\mu = \sinh(2r')(\sinh\psi,0,\cosh\psi)$ |
|                               | $u = (v-2)\sinh^2\psi, \quad v = \epsilon\cosh(2r')+1, \quad b = \epsilon\cosh(2t')$ |
Fig. 1 shows additional regions with signatures that correspond to zero time coordinates (+ + +), two time coordinates (− − +), (− + −), (+ − −) and three time coordinates (− − −). These cannot be reached from an SO(2, 2) group element. However, by making an analytic continuation to the spaces SO(3, 1)/SO(3) or SO(3, 1)/SO(2, 1) or SO(4)/SO(3) these geometries can be described by the same global metric in (3.7). The corresponding patches are indicated on Fig.1a,b,c. Note that the two purely (space,space,space) = SO(3, 1)/SO(3) regions are non-compact since \( b > 1 \) or \( b < -1 \). This is the region that would be reached from SO(2, 2)/SO(2, 1) by the usual Minkowski-Euclidean continuation of Lorentz vectors and tensors which imply \( X_\mu = (X_0, X_1, X_2) \rightarrow (iX_0, X_1, X_2) \) and \( Y_\mu = (Y_0, Y_1, Y_2) \rightarrow (Y_0, iY_1, iY_2) \) (recall \( A^{\mu\nu} = \epsilon^{\mu\nu\lambda} Y_\lambda \)). This fits well with the current algebra approach to string theory: the conformal field theory that describes strings have the same Virasoro central charge for both of these cases \( c = 3k^2/(k - 1)(k - 2) \), and for \( c = 26 \) a positive value of \( k = (39 \pm \sqrt{325})/23 \) is needed. The positive sign of \( k \) is, of course, crucial in determining the signature of the metric \([4]\) since \( k \) is an overall factor that multiplies the metric in the Lagrangian \([4]\).

The SO(4)/SO(3) and SO(3, 1)/SO(2, 1) regions with signatures (time, time, time) and (time, time, space), etc., do not seem to make physical sense because of the appearance of more than one time coordinate. However, by changing the sign of \( k \) one can convert these to spaces with signatures (space, space, space) = SO(4)/SO(3) or (time, space, space) = SO(3, 1)/SO(2, 1) which do make sense physically, and which are the Euclidean-Minkowski continuations of each other. This is again in accordance with the counting of time coordinates in the current algebra approach to string theory \([1, 3]\). The Virasoro central charge now takes the form \( c = 3K^2/(K + 1)(K + 2) \) for a positive \( K \) (i.e. \( k = -K \)), and has an upper bound of 3. To construct a critical conformal field theory these curved spaces have to be combined with additional spaces in order to reach the critical value of \( c = 26 \). So, the global geometry for the critical string theory depends also on the additional spaces. Thus, the SO(3, 1)/SO(2, 1) or SO(4)/SO(3) based models cannot correspond to a string theory in purely 3 dimensions under any circumstances and have no relation to

\[4\] The computation of the conformally exact metric \([19]\) introduces a renormalization in the overall factor \( k \rightarrow (k - 2) \). Therefore, to maintain the correct signature we must require \( k - 2 > 0 \) which, in turn, demands that we take the positive square root \( k = (39 + \sqrt{325})/23 \). In the supersymmetric theory the central charge is \( c = 9k/2(k - 2) \) which produces \( c = 15 \) for \( k = 20/7 \).
the $c = 26$ 3d-string theory based just on $SO(2,2)/SO(2,1)$ or its Euclidean continuation $SO(3,1)/SO(3)$.

For comparison of these results to the 2 dimensional case based on $SO(2,1)/SO(1,1)$ or the Euclidean continuation $SO(1,2)/SO(2)$, as well as to $SO(3)/SO(2)$, we have written an Appendix.

4. Duality in 3d

In this section we comment on the duality properties of the $SO(2,2)/SO(2,1)$ manifold. The general group theoretical origins of duality transformations was explained in [4]. For the 3d model the duality transformations are generated by switching the signs $\epsilon, \epsilon'$. This is equivalent to considering related dual models where the gauge group is a deformed subgroup (see footnote 1), as in the vector versus axial $U(1)$ in 2d. In any case, the global space of any of these models already contains all the dual regions, and provided it is fully identified as in the previous section, it is sufficient to consider only the undeformed vector subgroup. For comparison, the duality properties in 2d are discussed in Appendix A, including its reformulation as an inversion in group parameter space.

The duality transformations in the 3d manifold which are generated by $\epsilon$ or $\epsilon'$ may be rewritten in the form

\[(v', u', b') = (v, u, -b), \quad (v'', u'', b'') = (2 - v, uv/(2 - v), b).\] (4.1)

The first duality transformation in (4.1) generated by $\epsilon$ flips the following pairs of primed and unprimed patches into each other: $(I, I'g)$, $(IIa, III'c)$, $(IIb, III'b)$, $(IIg, I')$, $(IIIb, II'b)$ while also flipping the pairs of unprimed patches $(IIcd, IIfe)$ and primed patches $(II'c, II'f)$. Similarly, the second duality transformation in (4.1) transforms the following pairs of patches into each other $(I, I')$, $(IIb, II'b)$, $(IIc, II'c)$, $(IIg, II'g)$, $(IIf, II'f)$, $(IIIb, III'b)$ while sending the following patches into themselves $(IIa)$, $(IIde)$, $(III'c)$, as indicated on Fig.1. These generalize the duality properties of the 2d $SO(2,1)/SO(1,1)$ black hole space. As seen from the parametrization of the $SO(2,2)$ group element in (3.1) each duality transformation generated by $(\epsilon, \epsilon')$ makes a discrete leap in $SO(2,2)$ group space. It is interesting to elaborate on these properties by making a change of group parameters

\[X_\mu = \frac{2x_\mu}{x^2 - 1}, \quad Y_\mu = \frac{2y_\mu}{y^2 + 1}.\] (4.2)
As will be seen in the next section the new parameters \((x_{\mu}, y_{\mu})\) are natural for expressing the group element in the spinor representation of \(SO(2, 2)\) just as the old variables were natural for expressing the group element in the vector representation of \(SO(2, 2)\) as in (3.1). The allowed regions \((X^2 > -1, Y^2 < 1)\) are reproduced by letting \(x^2, y^2\) take values anywhere on the real line, that is, \(-\infty < (x^2, y^2) < \infty\). The global variables \(Q^a = (v, u, b)\) become

\[
\begin{align*}
  b &= \frac{1 - x^2}{1 + x^2}, \\
  v &= \frac{2}{1 + y^2}, \\
  u &= \frac{-2(x \cdot y)^2}{x^2(1 + y^2)}. 
\end{align*}
\] (4.3)

From these expressions we figure out that the duality transformation generated by (4.1) simply corresponds to inversions in \((x_{\mu}, y_{\mu})\) space

\[
(x'_{\mu}, y'_{\mu}) = (-\frac{x_{\mu}}{x^2}, y_{\mu}), \quad (x''_{\mu}, y''_{\mu}) = (x_{\mu}, \frac{y_{\mu}}{y^2}).
\] (4.4)

Note that the \((X_{\mu}, Y_{\mu})\) given in (4.2) remain \textit{invariant} under the duality transformations in (4.4) while the group element in (3.1) makes the duality leap just as required by the sign switches of \(\epsilon, \epsilon'\). Again, it is striking how much, these inversion or reflection forms of duality that we have exhibited, resemble the \(R \rightarrow 1/R\) duality of tori or the duality of mirror manifolds (in this connection see also footnote 1).

5. Pinched Double Trousers and Double Saddle Singularities in 3d.

Let us now analyze the properties of the curvature singularity. From (3.3) and (3.7) it is seen that the curvature scalar, the dilaton and metric blow up when

\[
S \equiv (b^2 - 1)(v - u - 2) = 0.
\] (5.1)

Evidently, the singularity resides on the planes \(b = 1, b = -1, v = u + 2\) that can be imagined easily from Fig.1a,b,c when one thinks in three dimensions (equivalently, one has \(x^2 = 0, x^2 = \infty, (x.y)^2 = x^2y^2\) respectively). However, we need to do a little more to unravel some multiple sheeted regions caused by the coordinate singularities at \(u = 0 = v\). It is beneficial to eliminate the coordinate singularities in order to open up regions that are folded into “double sheets” as in the 2d case. The required reparametrization involves taking square roots \(\sqrt{|u|}, \sqrt{|v|}\), but since \(u, v\) can have all signs this needs to be done carefully in various regions so that the global property of the coordinates are preserved. With this in mind let us examine the global variables in the form (4.3) and construct the
combinations \((b+1)v = 4(1+x^2)^{-1}(1+y^2)^{-1}\) and \((b-1)u = 4(x \cdot y)^2(1+x^2)^{-1}(1+y^2)^{-1}\).

From this one concludes that \((b+1)v, (b-1)u\) have the same sign. Examining Fig.1a,b,c one sees that in the unprimed regions I,II,III they are positive and in the primed regions \(\Gamma,\Pi',\Pi'\) they are negative. We will call them the positive and negative regions respectively. Recall that every primed region is dual to some patch of the unprimed one. As we shall further see in section 5 the primed and unprimed regions are geodesically isolated from each other. These observations allow us to define a new set of global coordinates \((\lambda_+, \sigma_+), (\lambda_-, \sigma_-)\) separately in the positive or negative regions respectively

\[
\lambda_+^2 = \pm v(b+1) = 4|(1+x^2)(1+y^2)|^{-1}, \quad \sigma_+^2 = \pm u(b-1) = (x \cdot y)^2\lambda_+^2,
\]

\(-\infty < \lambda_+, \sigma_+ < \infty. \quad (5.2)\)

In terms of the new coordinates the metric and curvature scalar take the following forms in the positive and negative regions

\[
ds^2 = \frac{1}{2S}[\pm db^2 + 2(1 + b) d\lambda_+^2 - 2(b - 1) d\sigma_+^2 + 2 db(\sigma_+ d\lambda_+ - \lambda_+ d\sigma_+)]
\]

\[
R = -\frac{8}{S}[\lambda_+^2 + \sigma_+^2 \pm (b^2 + 3)]
\]

\[
S = \pm 2(b^2 - 1) - (b - 1)\lambda_+^2 + (b + 1)\sigma_+^2
\]

where the singularity factor \(S\) is the same as in (5.1) up to a \(\mp\) sign. The disadvantage of this coordinate system is that the metric is not diagonal, but it has other advantages from the point of view of the geodesics and the overall intuitive view of the singularity. It is also possible to define another set of coordinates \((\rho_\pm, \omega_\pm)\) that have a diagonal metric in the positive and negative regions

\[
\rho_+^2 = \pm v \text{ sign}(b+1), \quad \omega_+^2 = \pm u \text{ sign}(b-1). \quad (5.4)
\]

The metric and curvature now take the form

\[
ds^2 = \frac{db^2}{4(b^2 - 1)} + \frac{1}{S}[\pm (b+1)|b+1| d\rho_+^2 - (b-1)|b-1| d\omega_+^2]
\]

\[
R = -\frac{8}{S}[(b+1)|\rho_+^2| + |b-1|\omega_+^2 \pm (b^2 + 3)]
\]

\[
S = \pm 2(b^2 - 1) - (b - 1)|b+1|\rho_+^2 + (b + 1)|b-1|\omega_+^2.
\]
In Figs.3a,b and Figs.4a,b we show the surface formed by the singularity \( S = 0 \) in the coordinate systems defined above. These pictures were generated using Mathematica 2.0. The clearest interpretation is obtained in the \((\lambda_\pm, \sigma_\pm)\) parametrization.

Let us first discuss the positive region with the \((\lambda_+, \sigma_+, b)\) coordinates. The topology of the surface in Fig.3a is, with some imagination, that of two propagating closed strings, joining into one closed string, and then splitting again into two closed strings. The initial and final closed strings shrink down to a single point just before joining and just after splitting. We call the singularity surface formed by this system of strings the pinched double trousers singularity. In Fig.4a the surface, which is parametrized by \((\rho_+, \omega_+, b)\), is deformed into more regular shapes but retains the same topology of the pinched double trousers. There is also the three dimensional picture that can be imagined with the aid of Fig.1a,b,c in which the positive region is “folded up” and deformed into the regular 3 dimensional blocks labelled by the various unprimed regions I,II,III. From Fig.3a one can intuitively see that the many patches of Fig.1 or Fig.4a have formed some apparently connected and disconnected regions which was not easy to deduce from Figs.1,4. Indeed, the overall feeling conveyed by Fig.3a about the division of the 3d positive space into a connected region (outside of the trousers) and disconnected regions (inside of each pinched leg) is the correct feeling and it will be confirmed by examining the geodesics. The inside of the body of the trousers is not reachable by any geodesic. In fact, this was the \((- - -)\) region of Fig.1b.

At \( b = \pm 1 \) the mapping between Fig.3a and Fig.4a is tricky. Since \( \lambda_\pm = \rho_\pm |b + 1|^{\frac{5}{2}} \) and \( \sigma_\pm = \omega_\pm |b - 1|^{\frac{5}{2}} \), we see that the singularity which consists of a single line along either the \( \lambda_+ \) or the \( \sigma_+ \) axis in Fig.3a has expanded into the caps of the cylinder and the bottoms or caps of the hyperboloids in Fig.4a. The rest of the finite \((\lambda_+, \sigma_+)\) planes at \( b = \pm 1 \) are mapped to either \( \rho_+ = \pm \infty \) or \( \omega_+ = \pm \infty \). The remainder of the finite \((\rho_+, \omega_+)\) planes at \( b = \pm 1 \) in Fig.4a, although they are not part of the surface of the trousers, are also squeezed on top of the singularity lines at \( \lambda_+ = 0 \) or \( \sigma_+ = 0 \) in Fig.4a. Therefore one wonders about the properties of these parts of the \( b = \pm 1 \) planes and in particular about the singularity. For example what happens if a particle attempts to cross from \( |b| < 1 \) to \( |b| > 1 \) in these regions of Fig.4a? Our analysis of geodesics in the next section will address these and other issues. The answer is that geodesics cannot penetrate from \( |b| > 1 \) to \( |b| < 1 \) when both \((\rho_\pm, \omega_\pm)\) are finite, but they can do it by moving through \((\rho_\pm, \pm \infty, 1)\) or \((\pm \infty, \omega_\pm, -1)\). This means that the rest of the \((\rho_\pm, \omega_\pm)\) planes at \( b = \pm 1 \) (which are not shaded in Fig.4a,b) are in fact part of the singularity except at infinity. On the other
hand, in the \((\lambda_{\pm}, \sigma_{\pm}, b)\) coordinates (Fig.3a,b) the only singularities at \(b = \pm 1\) are only along the axes at \(\sigma_{\pm} = 0\) (for \(b = 1\)) or \(\lambda_{\pm} = 0\) (for \(b = -1\)).

The singularity surface in the negative region of Fig.3b or 4b may be called the \textit{double saddle} singularity. It has the topology of the world surface of two infinitely long propagating open strings which come close but do not interact. The upper side of the saddle at \(b > 1\) corresponds to region \(\Gamma'\) of Fig.1a while the lower side of the inverted saddle at \(b < -1\) corresponds to region \(\Pi'\) of Fig.1c. The space in between the saddles are the various parts of region \(\Pi'\) of Fig.1a,b,c : \(\Pi'_{ab}\) for \(b > 1\), \(\Pi'_{g}\) for \(b < -1\) and \(\Pi'_{ef}\) for \(-1 < b < 1\).

As one may deduce intuitively from these pictures \(\Gamma',\Pi,\Pi'\) are geodesically disconnected, while the various parts within each region are geodesically connected. This is confirmed by the geodesic solution.

As explained in section 3, the global space for \(SO(3,1)/SO(3)\) is obtained from the \(SO(2,2)\) notation by the Euclidean continuation of vectors and tensors. This amounts to \(x^2 \to -\vec{x}^2, \ y^2 \to +\vec{y}^2, \ x \cdot y \to i\vec{x} \cdot \vec{y}\). Where \(\vec{x}, \vec{y}\) are 3d Euclidean vectors. Then, \(b = \frac{(1 + \vec{x}^2)}{(1 - \vec{x}^2)}, \ v = \frac{2}{(1 + \vec{y}^2)}\) and \(u = \frac{(v - 2)(\vec{x} \cdot \vec{y})^2}{\vec{x}^2 \vec{y}^2}\). The regions spanned by these invariants (with \(\vec{x}^2, \vec{y}^2 \geq 0\)) are the triangular regions with \(b > 1\) and \(b < -1\), as indicated in Fig.1a,c. the \(\epsilon\) duality flips these two regions while the \(\epsilon'\) duality sends them to themselves. In the \((\lambda_{\pm}, \sigma_{\pm}, b)\) notation the Euclidean continuation amounts to \(\sigma_{\pm} \to i\sigma_{\pm}\). Similarly in the \((\rho_{\pm}, \omega_{\pm}, b)\) notation Euclidean continuation is equivalent to \(\omega_{\pm} \to i\omega_{\pm}\). The \(\pm\) regions now correspond to \(sign(b) = \pm\). Therefore, the expressions for the metric, dilaton, etc. are obtained from the foregoing \(SO(2,2)/SO(2,1)\) expressions \((\ref{eq:5.3})\) \((\ref{eq:5.5})\) by performing this substitution. The singularity surface has its simplest shape in the \((\rho_{\pm}, \omega_{\pm}, b)\) coordinates. It consists of two half-infinite cylinders of radius \(\sqrt{2}\), axis \(b\) and caps at \(b = \pm 1\). The \(SO(3,1)/SO(3)\) global space is the inside of these half cylinders.

Through a similar analysis it is straightforward to discuss the \(SO(3,1)/SO(2,1)\) space. Actually we can obtain all the relevant expressions by making an analytic continuation from \(SO(2,2)/SO(2,1)\) through the substitutions \(x_\mu = (x_0, x_1, x_2) \to (x'_2, i x'_1, x'_0)\) and \(y_\mu = (y_0, y_1, y_2) \to (iy'_2, y'_1, iy'_0)\). The net effect on the invariants is to replace them by \((x^2, y^2, x \cdot y) \to (-x'^2, y'^2, -ix' \cdot y')\). Therefore, although the metric, dilaton, etc. continue to have the same form as \((\ref{eq:3.3})\), the range covered by the coordinates \(Q^a = (v, u, b)\) is different as indicated on Fig.1a,b,c. There are again positive and negative regions which are disconnected and related by duality. As seen from \((\ref{eq:5.2})\) and \((\ref{eq:5.4})\) the metric, dilaton, the singularity surface, etc. are obtained from \((\ref{eq:5.3})\) or \((\ref{eq:5.5})\) by replacing \(\sigma_{\pm} \to -i\sigma_{\pm}\) or \(\omega_{\pm} \to -i\omega_{\pm}\). However, as already pointed out in the previous section, to construct a
critical string theory one needs to combine this space with additional spaces in order to
obtain \( c = 26 \) (for point particles this requirement can be relaxed), and therefore it may
be necessary to take into account the global properties of the total space. For this reason
we will refrain from giving further details about this case in this paper. The \( SO(4)/SO(3) \)
space is similarly discussed by the Euclidean continuation of \( SO(3,1)/SO(2,1) \).

6. Geodesics and global properties.

The geodesic equations that follow from the diagonal line element (3.5) are

\[
\begin{align*}
\ddot{b} - \frac{b\dot{b}^2}{b^2 - 1} - \frac{b - 1}{b + 1} \frac{\dot{u}^2}{u(v-u-2)} - \frac{b + 1}{b - 1} \frac{\dot{v}^2}{v(v-u-2)} &= 0, \\
\ddot{u} + \left( \frac{1}{v-u-2} - \frac{1}{u} \right) \frac{\dot{u}^2}{2} - \frac{\dot{u}\dot{v}}{v-u-2} + \frac{2\dot{u}b}{b^2 - 1} + \frac{(b + 1)^2}{(b - 1)^2} \frac{uv^2}{2v(v-u-2)} &= 0, \\
\ddot{v} - \left( \frac{1}{v-u-2} + \frac{1}{v} \right) \frac{\dot{v}^2}{2} + \frac{\dot{u}\dot{v}}{v-u-2} - \frac{2\dot{v}b}{b^2 - 1} - \frac{(b - 1)^2}{(b + 1)^2} \frac{vu^2}{2u(v-u-2)} &= 0,
\end{align*}
\]

(6.1)

It seems impossible to find a general solution. We might look for special solutions in
which one of the variables is held fixed (e.g. \( b = \text{constant} \)). Note that the correct equation
for this case is to be obtained from the geodesic equations and not by first specializing
the line element (e.g. \( db = 0 \)). There is a difference between first varying the action and
then setting a variable to a constant (correct procedure) versus first setting a variable
to a constant and then varying the action in the remaining variables (wrong procedure).

With the correct procedure, we see that there are no solutions in which \( b = \text{constant} \).
We can also try \( u \) or \( v = \text{constant} \). We find that the only solution of this type is \( u = u_0 \), \( v = v_0 \) and \( b(\tau) = \pm \cosh[\gamma_\pm(\tau - \tau_\pm)] \) for \( |b| > 1 \), and \( b = \cos[\gamma_0(\tau - \tau_0)] \) for \( |b| < 1 \).

Here \( u_0, v_0, \gamma_\pm, \gamma_0, \tau_\pm, \tau_0 \) are integration constants related to initial conditions or boundary
conditions at \( b = \pm 1 \). One can picture these geodesics in three dimensions with the help
of Fig.1a,b,c and Fig.4a,b. They are vertical lines parallel to the \( b \) axis that may end or
bounce at a singularity at either \( b = -1 \) or \( b = 1 \) in other regions. Their fate is determined
by the constants (\(|v_0|, |u_0|\)) or equivalently (\( \rho_\pm, \omega_\pm \)). In region I they lie in \( 1 < b < \infty \), in

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5 The only exception is \( b = \pm 1 \). In this case it is easier to see the solution in the \((\lambda_\pm, \sigma_\pm, b)\)
variables, and is given by: (i) \( b = 1 \), \( \lambda_\pm = \lambda_\pm(0), \sigma_\pm(\tau) = \sigma_\pm(0) + \tau \dot{\sigma}_\pm(0) \) or (ii) \( b = -1 \), \( \lambda_\pm = \lambda_\pm(0) + \tau \dot{\lambda}_\pm(0) \), \( \sigma_\pm(\tau) = \sigma_\pm(0) \). This solution is light-like since it satisfies \( ds^2/d\tau^2 = 0 \)
region IIIb they lie in $-\infty < b < -1$. In region II they may end on the caps of the cylinder or the caps of the hyperboloids, but if they can avoid those they seem to extend from $-\infty$ to $+\infty$. Actually they bounce at $b = \pm 1$ because $ds^2$ has different signs for $b^2 > 1$ and $b^2 < 1$. In region I' they end at $b = 1$, in region III'a,b they end at $b = -1$. In region II' they again bounce at $b = \pm 1$ even when they are not trapped between the two saddles. By evaluating the line element $ds^2/d\tau^2 = \gamma_\pm^2$ or $-\gamma_0^2$, we learn that the portion of the geodesic that lies in $-1 < b < 1$ is time-like and for $|b| > 1$ it is space-like. None of it is light-like.

We see that the direct approach of solving these equations can provide some information about the space but it is limited. However, as discussed in section 2, we can find the exact general geodesic solution through the trick of enlarging the space to the entire $SO(2, 2)$ group space $(X_\mu, Y_\mu)$ plus gauge potentials, finding the solution to the differential equations in the enlarged space and then projecting down to the Lorentz invariants $Q^a = (v, u, b)$, or equivalently $(\lambda_\pm, \sigma_\pm, b)$, etc. Therefore, the first task is to rewrite the solution for the group element in (2.3) in the form (3.1) and then read off the solution $X_\mu(\tau)$ and $Y_\mu(\tau)$. This requires evaluating the exponentials in (2.3) in the form of $4 \times 4$ matrices in the vector representation of $SO(2, 2)$, which requires a lot of algebra. This task is a lot easier in the spinor representation in which it is possible to choose a basis that reduces the $SO(2, 2)$ group element into $2 \times 2$ blocks that correspond to the decomposition $SO(2, 2) \rightarrow SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. In this basis the exponentials are easy to compute since the generators are represented by blocks of just $2 \times 2$ Pauli matrices. Then by using the group theoretical correspondence between the spinor and vector representations given below one can construct the desired solution.

The spinor representation has the added advantage of shedding light on the global properties of the manifold, including the separation of the manifold to positive and negative geometrical regions (see below) and the representation of duality transformations in terms of inversions (see duality section).

Let us first establish a parametrization of the spinor representation and its correspondence to the vector representation in (3.1). We start with the $4 \times 4$ Dirac gamma matrices for three dimensional Minkowski space $\gamma_\mu = \tau_3 \sigma_\mu$, where $\tau_3, \sigma_\mu$ are $2 \times 2$ Pauli matrices acting on a direct product space. We choose the basis $\sigma_\mu = (\sigma_2, i\sigma_1, i\sigma_3)$ which yields $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. From the Dirac matrices one constructs the 6 generators of $SO(2, 2)$ in the $4 \times 4$ spinor representation as follows: $J_\mu = \frac{\gamma_\mu}{2}$, $J_{\mu\nu} = \frac{\sigma_{\mu\nu}}{2} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] = -\epsilon_{\mu\nu\lambda} \tau_0 \sigma_\lambda$, where $\tau_0$ is the identity Pauli matrix. Thus, the Lorentz subgroup is generated by $\tau_0 \sigma_\mu/2$
and the coset is generated by $\tau_3 \sigma_\mu / 2$. By exponentiating these one can construct group elements $h_s, t_s$ in the spinor representation and put them in the form

$$ h_s = \frac{1 - i\tau_0 y}{\pm \sqrt{|1 + y^2|}}, \quad t_s = \frac{1 - i\tau_3 x}{\pm \sqrt{|1 + x^2|}}, \quad y = \sigma_\mu y^\mu, \quad x = \sigma_\mu x^\mu. \quad (6.2) $$

In this equation we have allowed for the possibility that the Lorentz invariants $x^2 = x^\mu x_\mu$, $y^2 = y^\mu y_\mu$ can take any value on the real line. As required, the determinants of the $4 \times 4$ $h, t$ are unity $\det(h_s) = 1 = \det(t_s)$.

To establish a connection to the $4 \times 4$ vector representation given in (3.1) we first define 4 matrices in Dirac space $V_M = (V_0, V_\mu) = (\tau_1 \sigma_0, \tau_2 \sigma_\mu)$. They are orthonormal under the trace, $\text{Tr}(V_M V_N) = 4 \eta_{MN}$, where $\eta_{MN} = \text{diag}(1, 1, -1, -1)$ is the $SO(2, 2)$ metric in the vector representation and can be used to raise or lower the indices $M = (0', 0, 1, 2)$. Under commutation with the 6 generators $(\tau_0 \sigma_\mu, \tau_2 \sigma_\mu)$ the $V_M$ form the 4 dimensional vector representation of $SO(2, 2)$. Therefore, under the action of any $SO(2, 2)$ group element in the spinor representation $g_s$ one finds that these matrices rotate into each other and form a linear space: $g_s V_M g_s^{-1} = (g_v)_M^N V_N$. This means that the coefficients $(g_v)_M^N$ correspond to the $4 \times 4$ vector representation of the $SO(2, 2)$ group element $g_s$ and can be written as

$$ (g_v)_M^N = \frac{1}{4} \text{Tr}(g_s V_M g_s^{-1} V_N). \quad (6.3) $$

This is the construction of the vector representation from the product of two spinor representations. Applying this map to $h_s, t_s$ given in (3.2) we derive their vector representatives $h_v, t_v$ and compare them to the expressions $h, t$ in (3.1). The inverse matrices needed in this computation are $h_s^{-1} = h_s(-y)\text{sign}(1 + y^2)$ and similarly for $t_s^{-1}$. From this simple algebra we derive the relationship between $(X_\mu, Y_\mu)$ and $(x_\mu, y_\mu)$ given in (1.2).

Any group element may be written in the form $g = ht$ in any representation. For the spinor representation in our basis this gives a block diagonal form

$$ g_s = \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix}, \quad g_+ = \frac{(1 - iy)(1 - ix)}{\pm \sqrt{|(1 + y^2)(1 + x^2)|}}, \quad g_- = \frac{(1 - iy)(1 + ix)}{\pm \sqrt{|(1 + y^2)(1 + x^2)|}}, \quad (6.4) $$

where we see that the determinants of each block $\det(g_\pm) = \text{sign}[(1 + x^2)(1 + y^2)]$ could be ±1, while the overall determinant remains at $\det(g) = 1$. Therefore, $SO(2, 2)$ in the spinor representation goes globally beyond $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ by allowing the determinants of the $2 \times 2$ blocks to have the value of $-1$ simultaneously. As we already saw in the
previous sections these signs are closely tied to the globally separate positive and negative geometrical regions! The spinor representation now explains the group theoretical origin of this fact and tells us that we are going beyond $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ when we are in the negative region (double saddle).

We are now in a position to write the spinor representation of the general geodesic solution for the group element given in (2.3)(2.4). The constants of integration are $x_0^\mu, y_0^\mu, p^\mu, \alpha^\mu$ and they are required to obey the constraint $(g_0(\tau_3 p - \tau_0 \alpha)g_0^{-1})_H + \tau_0 \alpha = 0$, where $p = i\sigma_\mu p^\mu$, $\alpha = i\sigma_\mu \alpha^\mu$, and $g_0$ has the form (6.4) with $x_0, y_0$ inserted. The $H$ projection is implemented by dropping all terms that contain $\tau_3$ after multiplying the factors. Then the constraint reduces to two $2 \times 2$ identical blocks of the form

$$
(1 + x_0^2)^{-1}(i[p, x_0] - \alpha - x_0 \alpha x_0) + (1 + y_0^2)^{-1}(i[y_0, \alpha] + \alpha + y_0 \alpha y_0) = 0. \tag{6.5}
$$

In component notation this becomes

$$
M^{\mu \nu} \alpha_\nu = -r \epsilon^{\mu \nu \lambda} x_0 \nu p_\lambda, \quad M^{\mu \nu} = (1 - r) \eta^{\mu \nu} + y_0^\mu y_0^\nu - r x_0^\mu x_0^\nu - \epsilon^{\mu \lambda \nu} y_0 \lambda, \quad r = \frac{1 + y_0^2}{1 + x_0^2}. \tag{6.6}
$$

Using the notation $(a \times b)^\mu = \epsilon^{\mu \nu \lambda} a_\nu b_\lambda$ one can write the generic solution for $\alpha^\mu$ in the form

$$
\alpha^\mu = (x_0 \times p)^\mu - [(x_0 \times p) \cdot y_0 \ (y_0^\mu + (x_0 \times (y_0 \times x_0))^\mu) - (x_0 \times y_0) \cdot (x_0 \times p) \ x_0^\mu]/(x_0 \times y_0)^2. \tag{6.7}
$$

This expression is valid as long as $(x_0 \times y_0)^2 \neq 0$. As seen from (1.2)(3.3)(3.7) this means that at $\tau = 0$ the particle is not right on the singularity $v(0) \neq u(0) + 2.6$

Next we compute the exponentials in (2.3) in the spinor representation. Since every factor splits into $2 \times 2$ blocks as in (6.4) we evaluate each block separately $g_\pm = e^{\alpha^\tau} g_\pm(0)e^{(\pm p - \alpha)^\tau}$. Since $det(e^{\alpha^\tau}) = det(e^{(\pm p - \alpha)^\tau} = 1$, an immediate result is

$$
\alpha^\mu = \beta x_0^\mu - \gamma(1 + x_0^2)/(\gamma^2 + x_0^2)(x_0 \times (x_0 \times p))^\mu + \gamma^2 - 1 \ x_0^2/(\gamma^2 + x_0^2)(x_0 \times p)^\mu, \tag{6.8}
$$

where $\beta, \gamma$ are arbitrary constants.
\[ \text{det}(g_\pm(\tau)) = \text{det}(g_\pm(0)) = \text{sign}((1 + x_0^2)(1 + y_0^2)) \equiv \epsilon_0, \] (6.9)

which shows that geodesics never cross from the positive to the negative region since the sign \( \epsilon_0 \) is time independent. Now we compute the exponentials

\[
e^{\alpha \tau} = c_0(\tau) + i\sigma \cdot \alpha s_0(\tau), \quad e^{(\pm p - \alpha)\tau} = c_\mp(\tau) - i\sigma \cdot (\alpha \mp p) s_\mp(\tau),
\]

\[
c_0(\tau) = \cos(\sqrt{\alpha^2\tau}), \quad s_0(\tau) = \frac{\sin(\sqrt{\alpha^2\tau})}{\sqrt{\alpha^2}},
\]

\[
c_\mp(\tau) = \cos(\sqrt{(\alpha \mp p)^2\tau}), \quad s_\mp(\tau) = \frac{\sin(\sqrt{(\alpha \mp p)^2\tau})}{\sqrt{(\alpha \mp p)^2}}.
\] (6.10)

The resulting 2 \( \times \) 2 matrices \( g_\pm(\tau) \) can be rewritten in the form (6.4) in order to read off the solution for \( x_\mu(\tau) \) and \( y_\mu(\tau) \). While this can certainly be done, we only need the Lorentz invariants which can be extracted as follows

\[
\text{det}(g_+(\tau) + g_-(\tau)) = \frac{4\epsilon_0}{1 + x^2(\tau)} = 2\epsilon_0(b(\tau) + 1),
\]

\[
\text{Tr}(g_+(\tau) + g_-(\tau)) = \pm 4[(1 + x^2(\tau))(1 + y^2(\tau))]^{-\frac{1}{2}} = 2\lambda_{\epsilon_0}(\tau),
\]

\[
\text{Tr}(g_-(\tau) - g_+(\tau)) = \pm 4x(\tau) \cdot y(\tau) [(1 + x^2(\tau))(1 + y^2(\tau))]^{-\frac{1}{2}} = 2\sigma_{\epsilon_0}(\tau).
\] (6.11)

where \( \lambda_{\epsilon_0}, \sigma_{\epsilon_0} \) are the \( \lambda_\pm, \sigma_\pm \) global coordinates defined in the previous section, and the value of \( \epsilon_0 = \pm \) is determined by the initial conditions as in (3.9). Performing these computations gives the result

\[
\frac{\lambda_{\epsilon_0}(\tau)}{\lambda_{\epsilon_0}(0)} = c_0(\tau)[A_1^+ c_+(\tau) + A_1^- c_-(\tau) - A_2^+ s_+ (\tau) - A_2^- s_-(\tau)]
\]

\[
+ s_0(\tau)[A_3^+ c_+(\tau) + A_3^- c_-(\tau) + A_4^+ s_+ (\tau) + A_4^- s_-(\tau)]
\]

\[
\frac{\sigma_{\epsilon_0}(\tau)}{\pm \lambda_{\epsilon_0}(0)} = c_0(\tau)[A_1^+ c_+(\tau) - A_1^- c_-(\tau) - A_2^+ s_+ (\tau) + A_2^- s_-(\tau)]
\]

\[
+ s_0(\tau)[A_3^+ c_+(\tau) - A_3^- c_-(\tau) + A_4^+ s_+ (\tau) - A_4^- s_-(\tau)]
\] (6.12)

\[
\frac{b(\tau)}{b(0)} = c_+(\tau)c_-(\tau) + A_5 s_+(\tau)s_-(\tau) + A_6^+ s_+(\tau)c_-(\tau) + A_6^- s_-(\tau)c_+(\tau).
\]

where the various constants are determined by the initial parameters as follows
\[ \lambda_{\epsilon_0}(0) = \frac{\pm 2}{\sqrt{|(1 + x_0^2)(1 + y_0^2)|}}, \quad b(0) = \frac{1 - x_0^2}{1 + x_0^2}, \quad A_1^\pm = \frac{1}{2}(1 \pm y_0 \cdot x_0) \]

\[ A_2^\pm = \frac{1}{2}(\alpha \pm p) \cdot (y_0 \mp x_0 \mp y_0 \times x_0), \quad A_3^\pm = \frac{1}{2}\alpha \cdot (y_0 \mp x_0 \mp y_0 \times x_0), \]

\[ A_4^\pm = \frac{1}{2}|(1 \pm y_0 \cdot x_0)(\alpha^2 \pm \alpha \cdot p) \mp (\alpha \times p) \cdot (y_0 \mp x_0 \mp y_0 \times x_0)|, \]

\[ A_5 = \alpha^2 - p^2 - \frac{4\alpha \cdot (x_0 \times p)}{1 - x_0^2}, \quad A_6^\pm = \frac{2x_0 \cdot (p \pm \alpha)}{1 - x_0^2}. \]

Using (5.2) we have explicitly checked that the geodesic equations (6.1) are indeed satisfied by the above general solution.

Depending on initial positions and velocities, the arguments of the functions \( c_0, c_\pm, s_0, s_\pm \) may turn out to be real or imaginary, as determined by the \textit{signs}(\alpha^2, (\alpha + p)^2, (\alpha - p)^2) = (\pm, \pm, \pm). \) Accordingly, the solutions may contain oscillating trigonometric functions or their hyperbolic counterparts. The far past and far future position of particles crucially depend on these signs. In the purely trigonometric case, \textit{signs} = (+, +, +), the particle oscillates in the vicinity of the singularity surface and cannot escape from its gravitational pull. The curvature scalar (3.7) is not zero and the particle never reaches the asymptotically flat region. It turns out that this kind of initial condition is possible only in region II, and not in the others. In this purely oscillating solution the particle wobbles around the central blob of Fig.3a and nearby regions of the trousers. When one or two of the signs are negative the particle can get far away from the singularity surface temporarily but periodically returns to parts of it. By computing the \( \tau \to \pm \infty \) asymptotic behavior of the trajectories one finds that the scalar curvature does not vanish, and therefore the particle does not reach the asymptotically flat region. Finally, when all signs are negative, \textit{signs} = (−, −, −), only hyperbolic functions occur in the solution and the particle is found only in the flat regions in the far past and far future of its lifetime. This result is obtained by computing the asymptotic behavior of (6.12) for \( \tau \to \pm \infty \)

\[ \lambda_{\epsilon_0}(\tau) \to A_1^+ e^{(\sqrt{|\alpha^2|} + \sqrt{|(\alpha+p)^2|})|\tau|} + A_1^- e^{(-\sqrt{|\alpha^2|} + \sqrt{|(\alpha+p)^2|})|\tau|} \]

\[ \sigma_{\epsilon_0}(\tau) \to \pm A_4^+ e^{(\sqrt{|\alpha^2|} + \sqrt{|(\alpha-p)^2|})|\tau|} \mp A_4^- e^{(-\sqrt{|\alpha^2|} + \sqrt{|(\alpha-p)^2|})|\tau|} \]

\[ b(\tau) \to B e^{\sqrt{|(\alpha+p)^2|} + \sqrt{|(\alpha-p)^2|})|\tau|} \]

where
\[
A^\pm(\text{sign}(\tau)) = \frac{\lambda e_0(0)}{4} \left[ A_1^\pm + \frac{A_4^\pm}{\sqrt{\alpha^2(\alpha \pm p)^2}} + \text{sign}(\tau)\left(\frac{A_3^\pm}{\sqrt{\alpha^2}} - \frac{A_2^\pm}{\sqrt{|(\alpha \pm p)^2|}}\right)\right]
\]
\[
B(\text{sign}(\tau)) = \frac{b(0)}{4} \left[ 1 + \frac{A_5^\pm}{\sqrt{\alpha^2(\alpha \pm p)^2}} + \text{sign}(\tau)\left(\frac{A_6^+}{\sqrt{|(\alpha + p)^2|}} + \frac{A_6^-}{\sqrt{|(\alpha - p)^2|}}\right)\right].
\]

(6.15)

Comparing to the discussion of the asymptotically flat region in (3.8) we see that indeed, in the purely hyperbolic case, the particle must escape the gravitational pull of the singularity at large past and future times. This type of behavior is possible in all regions I,II,III,I',II',III' through a choice of the initial values at \(\tau = 0\).

From this analysis we arrive at the following important conclusion: A particle which is found in the flat region must have hyperbolic initial conditions \(\text{signs} = (-,-,-)\) since otherwise it could not be there. Thus, if a particle starts out in the flat region, and travels toward the singularity, it must return to another part of the flat region after some time. We may now ask where does such a particle go during its journey? For this we need to discuss the initial conditions as follows.

Substituting \(i\tau_3\sigma_\mu p^\mu\) for the matrix in (2.5) one finds a fixed value for the line element (or Lagrangian) associated with the solution given above

\[
\frac{ds^2}{d\tau^2} = -2p^\mu p_\mu.
\]

(6.16)

This allows one to easily control the signature of the geodesic by choosing time-like, space-like or light-like momenta \(p^\mu\) as an initial condition. The remaining initial conditions \((x_0^\mu, y_0^\mu)\) can also be chosen according to the region one wishes to explore. Table.1 is useful for this purpose. Once one picks one of the regions I,II,III,I',II',III' one can position oneself in it by first choosing \((v(0), u(0), b(0))\), which is equivalent to a choice of parameters and \(\epsilon, \epsilon'\) from Table.1. This determines the vectors \(X^\mu(0), Y^\mu(0)\) from which we deduce \(x_{0\mu} = -X^\mu(0)b(0)/(1 + b(0))\) and \(y_0^\mu = Y^\mu(0)/v(0)\) in a particular Lorentz frame. We can then compute all of the Lorentz invariant constants in (6.13) that determine the geodesics (6.12).

We have written Mathematica and Lotus programs with the above inputs and plotted the geodesics by taking various initial conditions at \(\tau = 0\). We have then examined the location of the particle at both negative and positive values of \(\tau\). These numerical plots reveal very interesting behavior in the vicinity of the singularity. In the purely hyperbolic
case, at large negative proper times (far past) the particle is far away from the singularity at large values of $(\lambda_{e_0}, \sigma_{e_0}, b)$. In a finite amount of proper time the particle approaches the singularity *tangentially* and bounces off from it. Depending on the initial conditions chosen, this may happen several times at various parts of the singularity (i.e. at the legs or body of the pants in the positive region, or at the saddles in the negative region). After a finite amount of proper time, the particle leaves the singularity region and returns to large values of $(\lambda_{e_0}, \sigma_{e_0}, b)$ at large positive proper times (far future).

Such a hyperbolic trajectory is quite interesting, especially when contrasted to the trajectory of a particle that falls into a black hole. In the case of a black hole singularity a particle that falls in never comes back and also cannot send any signals once it passes the horizon. However, in the present case, a particle can start out far away from the singularity in the flat region, fall in, gather information from the neighborhood of the singularity, and come back to another part of the flat region after a finite amount of proper time. Therefore, the notion of a “horizon”, if any, is quite different than the case of a black hole singularity.

If the initial conditions are not purely hyperbolic the numerical plots confirm that the particle is either partially or completely trapped by the singularity as described above. The particle trajectory bounces off various parts of the singularity, and never sticks to it, unlike a black hole. An intuitive physical reason for not sticking to a special point of the singularity is the gravitational attraction of all the other points that form the singularity. Namely, the singularity at any given instant of time is a string, not a point. Therefore, the rest of the string exerts a gravitational force on the particle, thereby not allowing the particle to come to rest at any special point on the string.

An interesting question is whether there might exist closed time-like curves in any of the regions of our global space? To search for these one must allow for the possibility that an external force (such as the firing of a rocket on the spaceship) might change the course of the trajectory at some value of the proper time $\tau$. This is expressed as a change in $p_\mu \rightarrow p'_\mu$ at some value $\tau = \tau_1$. The evolution of the particle trajectory can be computed according to (2.3)(2.4) during $\tau_0 < \tau < \tau_1$, while for $\tau_1 < \tau$ one again uses the same rules, but with initial conditions $g(\tau_1), p'$. The external forces may act more than once. The question of closed time-like trajectories boils down to whether, for positive $p^2, p'^2, \cdots$, there exists a $\tau_2$ such that $g(\tau_2) = \Lambda g_0 \Lambda^{-1}$, where $\Lambda$ is a gauge transformation? We have been unable to answer this question conclusively. However, after staring at many numerical plots, we conjecture that in regions I, III, I', III' closed time-like curves do not seem possible. We are less sure about regions II, II'. However, if closed time-like curves are at all possible in one
of these regions, it could happen only to an observer that is travelling in the vicinity of
the singularity, not while he is in the flat region. Such a closed time-like curve would not
allow this observer to “kill” his mother just before he is born unless his mother was also in
the vicinity of the singularity at the time of the birth as well as at the time of the murder.
Then this, if it occurs, seems to have no consequence on observers that are located in the
flat region. Therefore, it does not seem possible that causality in the flat regions can be
violated. This is probably sufficient not to get into trouble with causality as we know it
in flat regions.

As noted earlier in the text, the metric and dilaton discussed in this paper are conformally
exact only for the type-II superstring (and any type of point particle theory). For
the purely bosonic and heterotic strings the conformally exact metric and dilaton have also
been computed for the 3d and 4d models based on \( SO(2, 2) \) and \( SO(3, 2) \) \[13\]. The exact
versions have singularity surfaces and regions whose properties differ in interesting ways
from those discussed in this paper. These results will be given in a separate publication.
We emphasize that the geodesic analysis discussed in this section is valid in the purely
classical limit in which the dilaton is neglected.

7. Comments on possible physical applications

We have shown that a global analysis can readily be given for all geometries that arise
from gauged WZW models. These are interesting for both particle and string theories and
in either case the small-large duality property is a novel feature worth of further study.
We emphasize that, contrary to common belief, duality is not only a string property, since
the particle action (2.1) shares this feature. In string theory with one time coordinate the
coset models listed in the introduction are of special interest since they are the only known
curved space-time cases for which there is a chance, at least in principle, of solving the
conformal field theory through current algebra methods. In particle theories these models
are also very special since the quantum spectrum can be computed exactly through group
theoretical representation theory for non-compact groups. This presents a rare opportunity
in General Relativity investigations.

The specific 3d model which we have investigated in more detail has a cosmological
interpretation. This was already apparent from the remarks following eq.(3.8). If we
imagine that the “Big Bang” was not a point singularity, but a string singularity, and that
cosmological time begins near the central blob of Fig.3a, then matter and energy that was
created initially will have a future that is determined by the initial location and initial velocity. Let us first consider region I or III, assuming that the “Big Bang” corresponds to the pinch at one of the trouser’s legs. Inside a trouser’s leg all particle trajectories are purely hyperbolic and therefore, all the matter and energy created by the “Big Bang” will eventually travel to the asymptotically flat region. Thus, this singularity may be considered a cousin of a “white hole”. The picture is somewhat different for region II. We have seen that in this region all signs are possible for signs = (±, ±, ±). Therefore, depending on initial conditions, some of the matter and energy will remain trapped (dark matter?) while some other part will escape to the asymptotically flat region. It would be interesting to pursue such models for cosmological applications.

In our discussion of $SO(2,2)/SO(2,1)$ we assumed that we were dealing with a 3d theory. However, we can adjoin a factor of $U(1)$ or $\mathbb{R}$ as a fourth flat dimension, and our entire discussion would then apply to the 3d subspace of a four dimensional theory. Since the $U(1)$ or $\mathbb{R}$ factor absorbs one unit of the Virasoro central charge, $c = 1$, we must choose $k = 5/2$ for the $SO(2,2)_{-5/2}$ current algebra to produce the balance $c = 25$. For a similar supersymmetric theory in 4 dimensions the flat dimension produces $c = 3/2$, therefore $k = 3$ so that with $SO(2,2)_{-3}$ the total central charge is $c = 15$. The fact that $k = 3$ is an integer in this last case may be significant from the point of view of global anomalies.

It was shown in [5] that a heterotic string model can be constructed directly in 4 curved space-time dimensions. The super coset in this case is $SO(3,2)/SO(3,1)$ at level $k = 5$. The perturbative metric and dilaton were given in one of the patches. Clearly, it would be desirable to work out the global analysis of this geometry. One of the interesting features of that model was that it admitted $SU(3) \times SU(2) \times U(1)$ as the flavor symmetry group at level 1. Therefore, one expects a certain number of families of quarks and leptons to emerge as $SU(3)$ triplets and $SU(2)$ doublets.

It is a general hope that a vacuum configuration of the heterotic string correctly describes the low energy spectrum of quark and lepton families. We think that, this notion is more attractive and more believable when the vacuum configuration of the string describes a time dependent cosmological curved space-time. Then one can imagine that the quarks and leptons were produced at the initial “Big Bang”, in the presence of strong gravitational fields (more realistic than flat space), and then travelled along geodesics to flat regions of space where they are observed today. Therefore, 4d curved space-time models of the type described above are very interesting to study these notions. As it was made clear
in the introduction, there are a small number of 4 dimensional conformally exact heterotic string models based on non-compact cosets. Taken, with a cosmological interpretation as above, such models have the potential to describe quantum mechanically the fundamental matter at the earliest times. We are in the process of studying the quantum theory of these models and hope to report on these issues in future publications.

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Appendix

The reader who is familiar with the 2-dimensional $SO(2,1)/SO(1,1)$ manifold that describes a black hole, may find it useful to see it written in our notation and then compare it to the 3-dimensional spaces described above. The 2d theory is reached from the group element in (3.1) by specializing $X_\mu$ to a 2d vector and taking $h_{\mu\nu} = \epsilon' \cosh(\sqrt{u}) + \epsilon_{\mu\nu} \sinh(\sqrt{\pm u})$. As our Lorentz invariant global coordinates we take $Q^a = (u,b)$ where $b$ was defined in (3.1). The global metric and dilaton are then given by

$$ds^2 = \frac{db^2}{4(b^2 - 1)} - \frac{b - 1}{b + 1} \frac{du^2}{4u}, \quad \Phi = \ln(b^2 - 1) + \Phi_0.$$  (8.1)

The global space for $SO(2,1)/SO(1,1)$ is $(u > 0, -\infty < b < \infty)$, as shown in Fig.2. For this case we may rewrite $t = \pm \sqrt{u}$, $du^2/4u = dt^2$ where $-\infty < t < \infty$ is a non-compact coordinate which takes the role of time in the regions marked $I(-+), IIa(+-)$ and the role of space in the region marked $IIIb(+-)$. The regions IIa and IIb are geodesically connected, with IIa outside the horizon and IIb within the horizon, with the black hole at $b = -1$. The region marked $I(-+)$ is geodesically isolated from IIab and contains a naked singularity at $b = -1$. All this is easy to see by transforming to the Kruskal coordinates that are also global coordinates, $b = 1 - 2v_+v_-, \quad v_+^2 = \frac{1}{2}|b - 1|e^{\pm t}$, and for which the metric takes the standard form $ds^2 = dv_+dv_-/(v_+v_- - 1)$ [2]. Note that in the $(u,b)$ parametrization the various regions of the $(v_+,v_-)$ manifold correspond to 2 semi-infinite sheets that extend toward positive $u$ (since $t$ has both signs), sewn together at $u = 0$ along the $b$ axis, and cut at $b = -1$ along the singularity.

Fig.2 also shows the region $(u < 0, -\infty < b < \infty)$ which corresponds to the analytic continuation of the coset $SO(2,1)/SO(1,1)$ to $SO(2,1)/SO(2)$ or $SO(3)/SO(2)$. In these regions we may write $h_{\mu\nu} = \cos\theta + \epsilon_{\mu\nu} \sin\theta$ where $\theta$ is a compact coordinate $0 < \theta < 2\pi$ (note that Minkowski-Euclidean analytic continuation also requires $\epsilon_{01} \rightarrow i\epsilon_{21}$). The metric then describes a cigar a cymbal and a trumpet in the indicated regions [2][20]. In the cigar region $b = \cosh(2r), \quad r > 0$, and $ds^2 = dr^2 + \tanh^2 r \; d\theta^2$. In the trumpet region $b = -\cosh(2r'), \quad r' > 0$ and $ds^2 = (dr')^2 + \coth^2 r' \; d\theta^2$. In the cymbal region $b = \cos(2r''), \quad 0 < r'' < \pi/2$ and $ds^2 = -(dr'')^2 - \tan^2 r'' \; d\theta^2$ (in this region we need to change $k \rightarrow -k$ to get $(++)$ signature). The tip of the cigar touches the zenith of the cymbal at $b = 1$, while the trumpet and cymbal touch their (infinite) skirts at $b = -1$. In the $(u,b)$ parametrization all of these shapes have been deformed to double sheeted strips that lie parallel to the
$b$ axis in the $u$-range $-(2\pi)^2 < u < 0$ and sewn together at $u = 0, -(2\pi)^2$ (periodic in $0 < \theta < 2\pi$). Furthermore the strips are cut at $b = -1$ while sewn at $b = 1$. Further out regions toward negative $u$ repeat periodically the same (cigar, cymbal, trumpet) “music”. The Minkowski-Euclidean analytic continuation is $SO(1,2)/SO(1,1) \rightarrow SO(2,1)/SO(2)$ or $SO(2,1)/SO(1,1) \rightarrow SO(3)/SO(2)$ which corresponds to analytic continuation from positive to negative $u$ in Fig.2.

Let us briefly review the duality properties of the 2d manifold in our notation and point out a new feature. Switching the sign $\epsilon$ is equivalent to $(u', b') = (u, -b)$ for the 2d global coordinates. Fig.2 then shows that duality flips (I,IIa) and (cigar, trumpet), while the regions (IIb) and (cymbal) are self dual. In this process the singularity and the horizon also get interchanged. The group theoretical meaning of this transformation is understood by examining the group element in (3.1) (specialized to 2d): under duality the group element takes a leap in group space. Evidently, this leap does not change the theory, it only reranges the regions. To better understand what is going on it is useful to make a change of coordinates $X_\mu = 2x_\mu/(x^2 - 1)$ which allows one to write $b = \epsilon(1 + X^2)^{-\frac{1}{2}} = (1 - x^2)/(1 + x^2)$. The invariant $x^2$ is anywhere on the real line $-\infty < x^2 < \infty$. The duality transformation $b' = -b$ is now generated by $x'_\mu = -x_\mu/x^2$ which corresponds to an inversion in $x_\mu$ space. Under this inversion $X_\mu$ remains invariant but the group element makes just the required leap. This new version of duality is remarkably similar to the one encountered in tori ($R \rightarrow 1/R$) or mirror manifolds (see also footnote 1).
Figure Captions

Fig.1– Global spaces for $SO(2,2)/SO(2,1)$, $SO(3,1)/SO(2,1)$, $SO(3,1)/SO(3)$ and $SO(4)/SO(3)$, and dual patches.

Fig.2– Global spaces for $SO(2,1)/SO(1,1)$, $SO(2,1)/SO(2)$ and $SO(3)/SO(2)$, and dual patches.

Fig.3a– Pinched double trousers singularity in the positive sector, with $(\lambda_+, \sigma_+, b)$ coordinates.

Fig.3b– Double saddle singularity in the negative sector, with $(\lambda_-, \sigma_-, b)$ coordinates.

Fig.4a– Pinched double trousers singularity in the positive sector, with $(\rho_+, \omega_+, b)$ coordinates.

Fig.4b– Double saddle singularity in the negative sector, with $(\rho_-, \omega_-, b)$ coordinates.
References

[1] I. Bars and D. Nemeschansky, Nucl. Phys. B348 (1991) 89.
[2] E. Witten, Phys. Rev. D44 (1991) 314.
[3] M. Crescimanno, Mod. Phys. Lett. A7 (1992) 489.
[4] I. Bars and K. Sfetsos, Mod. Phys. Lett. A7 (1992) 1091.
[5] I. Bars and K. Sfetsos, Phys. Lett. 277B (1992) 269.
[6] E. S. Fradkin and V. Ya. Linetsky, Phys. Lett. 277B (1992) 73.
[7] P. Horava, Phys. Lett. 278B (1992) 101.
[8] D. Gershon, “Exact Solutions of Four-Dimensional Black Holes in String Theory”, TAUP-1937-91.
[9] I. Bars, “Curved Space-Time Strings and Black Holes”, in Proc. of XXth Int. Conf. on Diff. Geometrical Methods in Physics, Eds. S. Catto and A. Rocha, Vol.2, p.695, (World Scientific, 1992).
[10] J. H. Horn and G. T. Horowitz, Nucl. Phys. B368 (1992) 444.
[11] Giddings and Strominger, Phys. Rev. Lett. 67 (1991) 2930.
[12] E. Raiten, “Perturbation of a Stringy Black Hole”, Fermilab-Pub 91-338-T.
[13] P. Ginsparg and F. Quevedo, “Strings on Curved Space-Times: Black Holes, Torsion, and Duality”, LA-UR-92-640.
[14] A. Giveon, Mod. Phys. Lett. A6 (1991) 2843. ; I. Bars, “String Propagation on Black Holes”, USC-91/HEP-B3.; R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B371 (1992) 269. ; E. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871.
[15] M. Rocek and E. Verlinde, “Duality, Quotients, and Currents”, ITP-SB-91-53.
[16] A. Giveon and M. Rocek, “Generalized Duality in Curved String Backgrounds”, IASSNS-HEP-91-84/.
[17] C. G. Callan, D. Friedan, E. J. Martinec and M. Perry, Nucl. Phys. B262 (1985) 593.
[18] E. Witten, Nucl. Phys. B223 (1983) 422. ; K. Bardakci, E. Rabinovici and B. Saering, Nucl. Phys. B301 (1988) 151. ; K. Gawedzki and A. Kupiainen, Nucl. Phys. B320 (1989) 625. ; H.J. Schnitzer, Nucl. Phys. B324 (1989) 412. ; D. Karabali, Q-Han Park, H.J. Schnitzer and Z.Yang, Phys.Lett. 216B (1989) 307. ; D. Karabali and H.J. Schnitzer, Nucl. Phys. B329 (1990) 649.
[19] I. Bars and K. Sfetsos, in preparation
[20] S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581.