9. Artin’s Reciprocity Law

This section is devoted to a brief presentation of Artin’s reciprocity law in the classical ideal theoretic language. The central part of this article is devoted to studying connections between Artin’s reciprocity law and the proofs of the quadratic reciprocity law using Gauss’s Lemma.

Let \( m \) be a modulus (a formal product of an ideal \( m_0 \) and some infinite primes in some number field), and \( D_m \) the group of fractional ideals coprime to \( m \). An ideal group \( H \) is a satisfying \( P^{(1)}_m \subseteq H \subseteq D_m \). To each finite extension \( K/F \) and each modulus \( m \) we can attach the corresponding Takagi group

\[
T_{K/F}(m) = \{ a \in D_m : a = N_{K/F} \mathfrak{a} \cdot P^{(1)}_m \},
\]

where \( \mathfrak{a} \) runs through the ideals coprime to \( m \) in \( K \), and \( P^{(1)}_m \) the group of principal ideals \( (\alpha) \) with \( \alpha \equiv 1 \mod m \).

The first steps in any approach to class field theory are the two basic inequalities:

1. First Inequality For any finite extension \( K/F \) and any modulus \( m \) we have
   \[
   (D_m : T_{K/F}(m)) \leq (K : F).
   \]

2. Second Inequality. If \( K/F \) is a cyclic extension, then
   \[
   (D_m : T_{K/F}(m)) \geq (K : F)
   \]
   for any sufficiently large modulus \( m \).

Once class field theory is established it turns out that the first inequality can be improved to \( (D_m : T_{K/F}(m)) \mid (K : F) \), and that the second inequality holds for arbitrary abelian extensions. Moreover, the smallest modulus \( m \) for which the second inequality holds is called the conductor of the extension \( K/F \), and the second inequality holds if and only if the modulus \( m \) is a multiple of the conductor.

Cyclotomic Fields. Cyclotomic fields \( K = \mathbb{Q}(\zeta_m) \) are abelian extensions of \( \mathbb{Q} \), hence class fields. In fact, the Takagi group of \( K/\mathbb{Q} \) for the modulus \( m \infty \) is the group \( T_K \) of ideals \( (N_{K/\mathbb{Q}} \alpha) \mathbb{Q}^1_{m \infty} \). Since norms of ideals coprime to \( m \) are congruent to 1 mod \( m \), we have \( T_K = \mathbb{Q}^1_{m \infty} \).

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In Chevalley’s idelic approach the order of these inequalities is reversed.
Quadratic Fields. For quadratic number fields $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d$ we take $m = (d)$ if $d > 0$ and $m = (d)_\infty$ if $d < 0$. Then $T_{K/\mathbb{Q}}\{m\}$ consists of all fractional ideals in $\mathbb{Z}$ that can be written as a product of a norm of an ideal from $K$ and a principal ideal $(a)$ with $a \equiv 1 \mod m$.

Now we claim

Proposition 9.1. Let $K$ be a quadratic number fields with discriminant $d$, and set

$$ m = \begin{cases} (d) & \text{if } d > 0, \\ (d)_\infty & \text{if } d < 0. \end{cases} $$

Then

$$ T_{K/\mathbb{Q}}\{m\} = \{(a) : (\frac{d}{a}) = +1\}. $$

Proof. By definition we have $T_{K/\mathbb{Q}}\{m\} = \{N_{K/\mathbb{Q}}a\} \cdot P_m^{(1)}$, where $a$ runs through the ideals coprime to $m$ in $K$. Since a prime number $p \nmid d$ is the norm of an ideal from $K$ if and only if $(\frac{d}{p}) = +1$, the prime factors of $N_a$ all satisfy $(\frac{d}{p}) = +1$. This shows that $T_{K/\mathbb{Q}}\{m\} \subseteq H_m = \{(a) : (\frac{d}{a}) = +1\} \subseteq D_m$. Since $(D_m : H_m) = 2$ we find that $(D_m : T_{K/\mathbb{Q}}\{m\}) \geq 2$, and the first inequality now implies $D_m = T_{K/\mathbb{Q}}\{m\}$. □

For proving Prop. 9.1 directly one would have to show that every positive integer $a$ coprime to $d$ and satisfying $(\frac{d}{a}) = +1$ can be written as a product $a = rs$, where $r$ (if $d < 0$ then we demand $r > 0$) is a product of primes $p$ with $(\frac{d}{p}) = +1$, and $s \equiv 1 \mod d$.

Artin’s Reciprocity Law. Now let $H$ be an ideal group defined mod $m$: by class field theory there is a class field $K/F$ with Galois group $\text{Gal}(K/F) \cong D_m/H$. The Artin reciprocity map gives an explicit and canonical isomorphism: for a prime ideal $p$ in $F$ unramified in $K$ let $(\frac{K/F}{p})$ denote the Frobenius automorphism of a prime ideal $P$ in $K$ above $p$.

Theorem 9.2 (Artin’s Reciprocity Law). The symbol $(\frac{K/F}{a})$ only depends on the ideal class of $a$ in $D_m/I$, and the Artin map induces an exact sequence:

$$ 1 \longrightarrow H_m \longrightarrow D_m \longrightarrow \text{Gal}(K/F) \longrightarrow 1. $$

In particular, there is an isomorphism

$$ D_m/H \longrightarrow \text{Gal}(K/F) $$

between the generalized ideal class group $D_m/H$ and the Galois group $\text{Gal}(K/F)$ as abelian groups.

We remark in passing that if $F/k$ is a normal extension, then the Artin isomorphism is an isomorphism of $\text{Gal}(k/F)$-modules.

Quadratic Reciprocity I. Already the fact that the Artin map in quadratic number fields is constant on the cosets of $D_m/H$ implies the quadratic reciprocity law. In fact, we may identify the Artin symbol $(\frac{K/F}{p})$ with the Kronecker symbol $(\frac{d}{p})$, and the claim that this only depends on the class generated by $(p)$ in $D_m/T_{K/\mathbb{Q}}\{(d)_\infty\}$ (compare Prop. 9.1) means that $(\frac{d}{p}) = (\frac{d}{q})$ for all primes $p \equiv q \mod d_\infty$. This is Euler’s formulation of the quadratic reciprocity law, which in turn is easily seen to be equivalent to Legendre’s version (see [7]).
Quadratic Reciprocity II. Let \( p \) be a positive prime and put \( p^* = (-1)^{(p-1)/2} \); then \( p^* \equiv 1 \mod 4 \). The quadratic number field \( K = \mathbb{Q}(\sqrt{p^*}) \) is a class field modulo \( p \infty \) of \( \mathbb{Q} \); in fact its associated ideal group is the group \( I^2 P_{p \infty} \), where \( I \) is the group of all fractional ideals in \( \mathbb{Q} \) coprime to \( p \), and \( P_{p \infty} = \{ (a) \in I : a \equiv 1 \mod p \infty \} \).

Now observe that an odd prime \( q \neq p \) splits

- in the Kummer extension \( K/\mathbb{Q} \) if and only if \( (p^*/q) = +1 \);
- in the class field \( K/\mathbb{Q} \) if and only if \( q \equiv a^2 \mod p \infty \), that is, if and only if \( q > 0 \) and \( (q/p) = +1 \).

Comparing these decomposition laws implies the quadratic reciprocity law \( (p^*/q) = (q/p) \) for positive primes \( p, q \). Observe that the Legendre symbol \( (q/p) \) giving the decomposition in the class field can be identified with the Artin symbol for the quadratic extension, which in turn is defined as the Frobenius automorphism of \( K/\mathbb{Q} \).

10. The Transfer Map

In this section we will show that the transfer map, which was first defined by Schur, shows up naturally in class field theory. In the next section we will show that Gauss’s Lemma in the theory of quadratic residues is an example of a transfer, and give a class field theoretical interpretation of its content. As an application, we will prove the quadratic reciprocity law using the transfer in the incarnation of Gauss’s Lemma.

Consider the following situation: let \( F \) be a number field, \( L/F \) an abelian extension, and \( K/F \) some subextension. Let \( G = \text{Gal}(L/F) \) and set \( U = \text{Gal}(L/K) \). Then \( L \) is a class field of \( K \) with respect to some ideal group \( T_{L/K} \), and this ideal group is contained in \( T_{L/F} \), which describes \( L \) as a class field over \( K \). Ideals in the principal class in \( F \) are also contained in the principal class in \( K \).

It is therefore a natural question to ask how the Artin symbols \( (L/F) \) and \( (L/K) \) are related.\(^2\) We know that the Artin map induces isomorphisms

\[
\left( \frac{L/F}{a} \right) : \text{Cl}(F) \longrightarrow G = \text{Gal}(L/F) \quad \text{and} \quad \left( \frac{L/K}{a} \right) : \text{Cl}(K) \longrightarrow U = \text{Gal}(L/K).
\]

\[\begin{array}{c}
\text{F} \\
\downarrow \\
\text{K} \\
\downarrow \\
\text{F}
\end{array} \quad \begin{array}{c}
\text{L} \\
\downarrow \text{Gal}(L/\cdot) \\
\text{K} \\
\downarrow \text{Gal}(L/\cdot) \\
\text{F}
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \\
\text{H} \\
\downarrow \\
\text{G}
\end{array} \quad \begin{array}{c}
\text{L} \\
\downarrow \text{Gal}(L/\cdot) \\
\text{K} \\
\downarrow \text{Gal}(L/\cdot) \\
\text{F}
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \\
\text{H} \\
\downarrow \\
\text{G}
\end{array}
\]

\[\text{Figure 1.}\]

\(^2\)Observe that the ideal \( a \) in \( (L/E) \) should more exactly be written as \( aO_E \). In particular, one has to recall that in the symbol \( (L/F) \), the prime ideal \( p \) in \( K \) need not be prime anymore in \( F \).
It follows from the above that the map sending \((L/F) \mapsto (L/K)\) induces a group homomorphism \(V_{G \to U} : G \to U\). The homomorphism \(V_{G \to U}\) is called the transfer (Verlagerung), and can be computed explicitly as follows: let \(G = \bigcup r_j U\) be a decomposition of \(G\) into disjoint cosets \(r_j U\). For \(g \in G\) write \(g r_i = r_j u_j\) for \(u_j \in U_j\), and set

\[
V_{G \to U}(g) = \prod_j u_j \cdot U',
\]

where \(U'\) is the commutator subgroup of \(U\).

**Transfer in Abelian Groups.** Although the transfer map is usually connected with nonabelian groups since the transfer \(G/G' \to G''/G''\), where \(G = \text{Gal}(K^\infty/K)\) and \(K^\infty\) is the Hilbert class field of \(K\), it is used for studying the capitulation of ideal classes from \(K\) in subfields of its Hilbert class field \(K^1\), the transfer can also be used for abelian groups:

**Lemma 10.1.** Let \(U\) be a subgroup of an abelian group \(G\), and assume that \(G/U\) is a cyclic group of order \(f\). Then \(V_{G \to U}(x) = x^f\) for all \(x \in G\).

**Proof.** Let \(z U\) be a generator of \(G/U\), and write \(G = U \cup z U \cup \ldots \cup z^{f-1} U\). Given any \(x \in G\), we can write \(x = z^i u\) for some \(u \in U\) and find \(V(x) = V(z^i u) = V(z)^i V(u)\); thus we only need to compute \(V(u)\) and \(V(z)\).

1. Since \(u \cdot x^f = x^f h_j(u)\) for \(h_j(u) = u\), we find \(V(u) = \prod h_j(u) = u^f\).

2. Here \(z \cdot z^j = z^{j+1}\), hence \(h_j(z) = 1\) for \(1 \leq j \leq f - 2\) and \(h_{f-1}(z) = z^f \in U\). Thus \(V(z) = z^f\).

Now \(V(x) = V(z^i u) = z^{if} u^f = x^f\) as claimed. \(\square\)

If \(U\) is a subgroup of Klein’s four group \(G\) with order 2, then \(V_{G \to U}\) is the trivial map. There are also examples where \(V\) is surjective:

**Corollary 10.2.** If \(U\) is a subgroup of a finite cyclic group \(G\), then the transfer \(V_{G \to U}\) is surjective.

**Proof.** Let \(G = \langle g \rangle\); then \(V(g) = g(G/U)\) generates a subgroup of order \#U, and since \(U\) is the unique subgroup of \(G\) with this order, the claim follows. \(\square\)

Now assume that \(L/F\) is a cyclic extension, and let \(K/F\) be the subextension fixed by some subgroup \(U\) of \(G = \text{Gal}(L/F)\) (see Fig. 1). The transfer map \(V_{G \to U}\) is surjective by Cor. [10, 2] and its kernel \(H\) fixes a subextension \(F'/F\). Observe that \(\text{Gal}(F'/F) \simeq G/H \simeq U \simeq \text{Gal}(L/K)\). Thus the transfer map gives us an isomorphism between these two Galois groups:

**Proposition 10.3.** Let \(L/F\) be a cyclic extension as above, and let \(p\) be a prime ideal in \(F\) unramified in \(L\). Then \(V_{G \to U}\) induces an isomorphism

\[
G/H \simeq \text{Gal}(F'/F) \to \text{Gal}(L/K) \simeq U.
\]

In particular, the order of \(\sigma_p \cdot H\) is equal to the order of \(V(\sigma_p)\); this implies in particular that \(p \in \text{Spl}(F'/F)\) if and only if \(\sigma_p \in \ker V\).
11. Gauss’s Lemma and the Quadratic Reciprocity Law

Gauss gave eight proofs of the quadratic reciprocity law: his first proof was by induction and used the existence of auxiliary primes discussed in [5]. Gauss’s second proof was based on the genus theory of binary quadratic forms and became the role model for Kummer’s proof of the $p$-th power reciprocity law in $\mathbb{Q}(\zeta_p)$ for regular primes; the inequality between the number of genera and the number of ambiguous ideal classes, which was the central point in Gauss’s second proof, became an important tool in Takagi’s version of class field theory.

Gauss’s third proof used what became known as Gauss’s Lemma in the theory of quadratic residues. In this section, we will show that it has a natural class field theoretic interpretation.

Let $p = 2m + 1$ be an odd prime and $a$ an integer coprime to $p$. A half system modulo $p$ is a set $A = \{a_1, \ldots, a_m\}$ of integers $a_j$ with the property that every coprime residue class modulo $p$ has a unique representative in $A$ or in $-A$. Thus for every $1 \leq j \leq m$ we can write $a \cdot a_j \equiv s_j a_{\pi(j)} \mod p$, where $s_j \in \{\pm 1\}$ and where $\pi$ is a permutation of $A$. Gauss’s Lemma says that the Legendre symbol \((\frac{a}{p})\) is just the product $\prod s_j$ of signs in these $m$ congruences.

Now consider the ideal group $H$ consisting of all ideals $(a)$ with $a > 0$ and $a \equiv 1 \mod p$. The corresponding class field $\mathbb{Q}(m)$ with $m = p\infty$ is $K = \mathbb{Q}(\zeta)$, the field of $p$-th roots of unity, and the Artin map sends a residue class $a \mod p$ to the automorphism $\sigma_a : \zeta \mapsto \zeta^a$. We can identify the ideal class group $G = D_m/H$ with the group $(\mathbb{Z}/p\mathbb{Z})^\times$ by choosing positive generators of ideals.

**Lemma 11.1.** The subgroup $U$ of $G = (\mathbb{Z}/p\mathbb{Z})^\times$ generated by the residue class $-1 \mod p$ fixes the maximal real subfield $K^+ = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ of $K = \mathbb{Q}(\zeta_p)$, and the transfer $V_{G\to U}$ satisfies $V(a) \equiv (\frac{a}{p}) \mod p$.

**Proof.** Since $G$ is abelian, the transfer map $V_{G\to U}$ is a homomorphism $V : G \to U$. By Lemma 10.1 we have $V(a) \equiv a^{(p-1)/2} \mod p$. The claim now follows from Euler’s criterion. \(\square\)

The formula for $V(a)$ in Lemma 11.1 is Gauss’s Lemma. It was observed by various mathematicians that Gauss’s Lemma is an example of the group theoretic transfer map: see e.g. Cartier [2], Delsarte [3], Leutbecher [8] and Waterhouse [9]. The full class field theoretic interpretation given above seems to be new.

Let us now exploit the actual content of the transfer map, namely the connection with the decomposition of prime ideals given in Prop. 10.3 since $G$ is cyclic, the map $V : G \to U$ is onto by Cor 10.2 and $H = \ker V$ is an ideal group whose class field is the quadratic subfield $k = \mathbb{Q}(\sqrt{\eta})$ of $K$. By Prop. 10.3 we have $q \in \text{Spl}(k/\mathbb{Q})$ if and only if $V(\sigma_q) = (\frac{q}{\eta}) = 1$; since $q$ splits in $k/\mathbb{Q}$ if and only if $(\frac{\eta}{q}) = 1$, we deduce

**Theorem 11.2.** The Quadratic Reciprocity Law: for two distinct odd primes $p$ and $q$, we have

\[
\left( \frac{p}{q} \right) = \left( \frac{q}{p} \right) .
\]

Observe that this proof does not require the full power of class field theory; it only uses Artin’s reciprocity law for cyclotomic extensions of $\mathbb{Q}$ (which, as we have seen, was proved by Dedekind), standard properties of the Frobenius automorphism, and
a calculation of the transfer map (Gauss’s Lemma). Actually the transfer map can be eliminated from our arguments, giving the well known proof of the reciprocity law by comparing the splitting of primes in quadratic and cyclotomic extensions. The proof above showing that Gauss’s lemma comes up more or less naturally in the class field theoretical context has a certain charm, however, and perhaps helps to explain why Gauss’s Lemma can be used for proving the quadratic reciprocity law.

In fact, most of the elementary proofs of the quadratic reciprocity law used some form of Gauss’s Lemma (see [7]), and backed with a lot of slicker proofs (see e.g. [4]) it is quite easy to look down upon them; maybe the exposition above can help to rehabilitate Gauss’s Lemma somewhat.

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