Low-Pass Filters: Commentary on a Remark by Feynman

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Abstract

In Feynman’s lectures there is a remark about the limiting value of the impedance of an n-section ladder consisting of purely reactive elements (capacitances and inductances). The remark is that this limiting impedance $z = \lim_{n \to \infty} z_n$ has a positive real part. He notes that this is surprising since the real part of each $z_n$ is zero, therefore it is impossible for the limit to have a positive real part. A recent article in this journal offered an explanation of this paradox based on the fact that realistic impedances have a non-negative real part, but the authors noted that their argument was incomplete. We use the same physical idea, but give a simple argument which shows that the sequence $z_n$ converges like a geometric series. We also calculate the finite speed at which energy is propagated out into the infinite ladder.

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I. INTRODUCTION

In Feynman’s lectures [1] there is a remark about the limiting value of the impedance of an n-section ladder consisting of purely reactive elements (capacitances and inductances). He shows that the impedance obeys the recurrence relation

\[ z_{n+1} = Z_1 + 1/(1/Z_2 + 1/z_n) \]  (1)

If \( \lim_{n \to \infty} z_n := z \) exists then \( z \) is the solution of

\[ z = Z_1 + 1/(1/Z_2 + 1/z) \]  (2)

and this limiting impedance has a positive real part. Feynman points out that this is surprising since the real part of each \( z_n \) is zero, therefore it is impossible for the limit to have a positive real part. A recent article [2] in this journal discussed this paradox as an example of the need for care when taking limits.

A comment [3] on [2] tells us that there is no need to worry about taking limits because there are ways to define infinite series other than as the limit of a sequence of finite series. The author asserts that we should just define the impedance of the infinite ladder to be one of the solutions to equation (2) above. This may be considered as a definition of the impedance of the infinite ladder, but does nothing to remove the surprise we feel that the properties of the so-defined infinite ladder are so different from the properties of \( z_n \) for large \( n \). The feeling of surprise is even more acute in the example given in [3]. This example is the geometric series \( Z_n = \sum_{j=0}^{n} p^j R \) which arises in an infinite ladder of increasing resistances. When \( n \) tends to infinity and \( |p| < 1 \) the sum of this series is \( Z = R/(1 - p) \), and in [3] we are told to use this form even if \( p > 1 \) on the grounds that this is the Borel sum of the series. But this does not remove our astonishment that, according to [3], \( 1 + 2 + 4 + 8 + ... = -1 \), taking \( p = 2 \) for example. Furthermore, contrary to [3], when \( Re p > 1 \) the Borel sum of \( \sum_{j=0}^{\infty} p^j R \) does not exist [4].

The comment [3] gives another argument that is closer to Feynman’s. This is that the impedance \( Z(R) \) of the infinite set of resistances should obey \( Z - R = pZ \), on the grounds that removing the first element of the series leaves still an infinite series. But if we remove the first element of the ladder we get an infinite ladder whose first element is \( pR \), not \( R \). Thus the recursion relation should be \( Z(R) - R = Z(pR) \), and the only rationale we see
for $Z(pR) = pZ(R)$ is that what we mean by $Z(R)$ is $Z(R) = \lim_{n \to \infty} Z_n(R)$. Thus again we need to consider the usual limit, which [3] rejects. Finally the author of [3] points out that we can consider the infinite series as a finite series, appropriately terminated. If we terminate the $n$-section ladder with an active element having resistance $-p^{n+1}R/(p-1)$ then we get $Z = R/(1-p)$ as he wants. But why this value? Why not $-p^{n+1}R/(p-1) + \lambda R$? Then the infinite sum is different. (Another possible rationale, though not one offered in [3], is this: The $Z_n$ obey the recursion relation $Z_{n+1} = R + pZ_n$. This gives $Z_{n+1} = R/(1-p) + p^n(Z_1 - R/(1-p))$. This converges to $R/(1-p)$ for $p < 1$ and for $p > 1$ it converges to the same value if $Z_1 = R/(1-p)$. But then every $Z_n$ equals $Z_1$. This does not seem to be the physical situation envisioned in [3].)

For these reasons we disagree with [3] and think the authors of [2] are right to wonder how to resolve the puzzle noted by Feynman in [1].

So we return to the recursion relation, equation (1), and investigate its convergence. We use the notation of [2]. The authors of [2] suggested that we consider each $Z_1, Z_2$ to have a small imaginary part $r$ and defined the impedance of the infinite ladder to be the iterated limit $\lim_{r \to 0} \lim_{n \to \infty} z_n$, noting that the order in which the limits are taken is crucial. They then gave an argument that this limit exists, based on the contraction mapping principle. Unfortunately their proof is incomplete, as they note, and the reader may be misled about the proper use of the principle. We give here a rigorous proof based on elementary reasoning suitable for an undergraduate classroom and then make some remarks on the contraction mapping principle. We also show that energy is propagated out into the infinite network with a finite velocity which we calculate.

II. EXISTENCE OF THE LIMITING IMPEDANCE

The notation is simplified by defining

$$p_n = z_n/Z_1, \quad p_\pm = z_\pm/Z_1, \quad \text{and} \quad t = Z_2/Z_1,$$

(3)

where $z_\pm$ are the solutions of equation (2). Then equation (1) becomes

$$p_{n+1} = 1 + tp_n/(t + p_n),$$

(4)
the limiting value \( p = p_+ \) or \( p_- \) obeys \( p^2 = p + t \), and \( p_+ = (1/2)(1 \pm \sqrt{1 + 4t}) \). Define the branch of the square root by \( \sqrt{1 + 4t} = a + ib \) with \( a > 0 \). This is always possible when \( t \) is not on the negative real axis and less than \(-1/4\). ( \( t \not\in (-\infty, -1/4) \).) That is, we have cut the complex \( t \)-plane from \(-1/4 \) to \(-\infty \) and all our work is done on one sheet of the Riemann surface. Our first goal is to show that \( \lim_{n \to \infty} p_n = p_+ \). We will see that \( p_- \) is an unstable fixed point. So consider now \( p_1 \neq p_- \).

**Theorem:** For \( p_1 \neq p_- \) and \( \text{Re} \sqrt{1 + 4t} > 0 \) the sequence defined by

\[
p_{n+1} = 1 + tp_n / (t + p_n)
\]

converges to \( p_+ \) at the rate of a geometric series.

**Proof:** It is easily checked that

\[
\frac{p_{n+1} - p_+}{p_{n+1} - p_-} = \left( \frac{p_n t}{p_n + t} - \frac{p_+ t}{p_+ + t} \right) / \left( \frac{p_n t}{p_n + t} - \frac{p_- t}{p_- + t} \right) = \frac{p_n - p_+}{p_n - p_-} \cdot \frac{p_- + t}{p_- + t} = \frac{p_n - p_+}{p_n - p_-} \cdot \frac{p_+^2}{p_-^2}.
\]

Let \( c_n := \frac{p_n - p_+}{p_n - p_-} \) and \( \gamma := \frac{p_-}{p_+} \). Then equation (6) implies

\[
|c_{n+1}| = |c_n|^2 |\gamma|^2,
\]

and \( |\gamma|^2 = ((a - 1)^2 + b^2) / ((a + 1)^2 + b^2) \) is less than 1 if and only if \( a > 0 \). Thus \( |c_n| = |\gamma|^{2n} |c_1| \to 0 \) as \( n \to \infty \) and so \( p_n \to p_+ \). This shows that the limit \( \lim_{n \to \infty} p_n \) does exist for all \( p_1 \neq p_- \), it has positive real part, and the convergence is that of a geometric series.

**Remark 1:** If \( p_1 = p_- \), then \( p_n = p_- \) for all \( n \), so the limit \( \lim_{n \to \infty} p_n = p_- \) does exist in this case also. However, an arbitrarily small perturbation of the initial data \( p_1 = p_- \) results in the limit \( \lim_{n \to \infty} p_n = p_+ \). This means that \( p_+ \) is the stable fixed point for the iterative process (5), while \( p_- \) is an unstable fixed point for this process.

**Remark 2:** There is a geometrical interpretation of the above proof which explains the reasons for formula (6) to hold. Namely, the mapping \( p \to c(p) := (p - p_+)/ (p - p_-) \) maps the complex plane onto itself, taking \( p_+ \) to the origin and \( p_- \) to the point at infinity. In the complex \( c \)-plane the origin is a global attractor.

Apply this now to the low-pass filter discussed by Feynman. In this case \( Z_1 = i\omega L + r \), \( Z_2 = 1 / (i\omega C) + r' \) with \( r, r' \) small and positive, so
\[ 1 + 4t = +A - i\epsilon, A > 0, \text{ for } \omega > \omega_c \]
\[ = -A - i\epsilon, A > 0, \text{ for } \omega < \omega_c. \]  
\[(8)\]

In these formulas \(\epsilon = O(r + r')\) as \(r, r' \to 0\), so \(\epsilon\) is a small positive number, and \(A > 0\) is \(A = |1 - \omega_c^2/\omega^2|\). The cut-off frequency \(\omega_c = 2/\sqrt{LC}\). Choosing the branch of the square root with positive real part gives
\[
\sqrt{1 + 4t} = +\sqrt{A} - (i\epsilon/2)\sqrt{A} + O(\epsilon^2) \text{ for } \omega > \omega_c
\]
\[
= -i\sqrt{A} + (\epsilon/2)\sqrt{A} + O(\epsilon^2) \text{ for } \omega < \omega_c.
\]
\[(9)\]

Here and below \(\sqrt{A} > 0\). Rewrite this now in terms of the limiting impedance \(z_+ = Z_1 p_+ = (Z_1/2)(1 + \sqrt{1 + 4t})\). For \(\omega < \omega_c\) we find
\[
z_+ = [(i\omega L + r)/2][1 - i\sqrt{A} + (\epsilon/2)\sqrt{A} + O(\epsilon^2)] = (\omega L/2)\sqrt{A} + i\omega L/2 + O(\epsilon).
\]
\[(10)\]

The real part of \(z_+\) is positive, consistent with the principle that the real part of the impedance of a passive network cannot be negative. We can now take the limit \(\epsilon \to 0\) and get \(\lim_{\epsilon \to 0} z_+ = (\omega L/2)(\sqrt{A} + i)\). For \(\omega > \omega_c\) we find
\[
z_+ = [(i\omega L + r)/2][1 + \sqrt{A} - (i\epsilon/2)\sqrt{A} + O(\epsilon^2)] = (i\omega L/2)(1 + \sqrt{A}) + O(\epsilon).
\]
\[(11)\]

Again, the real part is positive (so it is physically reasonable), and we can take the limit \(r, r' \to 0\) and find \(\lim_{r \to 0} z_+ = (i\omega L/2)(\sqrt{A} + 1)\). Taking the limit \(\epsilon \to 0\) is equivalent to taking \(r, r' \to 0\).

We have shown that \(z_+ = \lim_{n \to \infty} z_n\) does exist when the impedances have a positive real part. The requirement needed to make the mathematics rigorous is exactly the requirement that makes physical sense: the real part of a physically realistic passive impedance must be positive. Once that is recognized one can describe the physical idealization of perfect impedances by taking the limit as these real parts go to zero. This gives exactly the result obtained by Feynman. We have treated a low-pass filter here, but the same analysis applies to a high-pass filter in which the inductance and capacitance are interchanged.

Now that we know that the limits make sense we can discuss the infinite network further. If the infinite ladder is connected to a source, and the source is modulated in time, then
the modulation propagates out into the ladder like a wave with well-defined group velocity. Thus energy can be delivered by the source out into the infinite network and the energy does not come back, it is absorbed: the infinite network acts like a "black box" whose impedance has a positive real part.

To see this, first use Kirchhoff’s laws to show that $\tilde{V}_n(\omega) = (-\gamma)^n \tilde{V}_s(\omega)$ as done in [1]. Then put $-\gamma = e^{i\delta(\omega)}$. Now describe the source voltage $V_s(t)$ by the modulated signal $V_s(t) = f(t)e^{i\omega_0 t}$, where the modulating function $f$ is slowly varying. The Fourier transform of the source voltage is $\tilde{V}_s(\omega) = \tilde{f}(\nu)$, where $\nu := \omega - \omega_0$, and the spectrum of $f$ vanishes outside a small neighborhood $|\nu| < \nu_0$ of the zero frequency. Then the time-dependence of the voltage on the $n$-th section of the ladder is $V_n(t) = e^{i\omega_0 t} \frac{1}{2\pi} \int_{|\nu|<\nu_0} d\nu e^{i\nu t + in\delta(\omega_0 + \nu)} \tilde{f}(\nu)$. Compare this with the standard representation of a wave packet $g(x, t) := \frac{1}{2\pi} \int_{|\nu|<\nu_0} d\nu e^{i\nu t - ik(\nu)} \tilde{f}(\nu)$, with group velocity $v := \frac{d\delta(\omega)}{d\omega}|_{\omega=\omega_0}$. In our case the role of the variable $x$ is played by parameter $n$ and $k = -\delta(\omega_0 + \nu)$, so $v = -1/\tau$, where $\tau := \frac{d\delta(\omega)}{d\omega}|_{\omega=\omega_0} > 0$. The quantity $\tau$ has the physical meaning of the time needed for the wave to propagate through one section of the ladder.

### III. THE CONTRACTION MAPPING PRINCIPLE

In [2] the authors propose to apply the contraction mapping principle to prove the convergence of $z_n$ for $\omega < \omega_c$, where

$$z_{n+1} = f(z_n), \quad z_1 = Z_1 + Z_2,$$

and

$$f(z) = Z_1 + 1/(1/Z_2 + 1/z).$$

They argue that the sequence converges to a fixed point $\zeta = f(\zeta)$ only if the mapping in equation (7) is a contraction mapping ([2], line 1 below (5)). This is not right. Let us review the contraction mapping principle [5].

The contraction mapping principle says: If there is a closed set $D$ such that (a) $f(D) \subseteq D$ and (b) $|f(z') - f(z)| \leq q|z' - z|$ for all $z, z' \in D$ with $q$ a constant, $0 < q < 1$, then there is in $D$ a unique fixed point $\zeta = f(\zeta)$ of the map $f : D \to D$ and the sequence $z_{n+1} = f(z_n), z_1 \in D,$ converges to $\zeta$. 

[5]
In [2] the authors do not specify a $D$ that is mapped into itself by $f$ nor do they check that their initial approximation $z_1$ generates a sequence that ever reaches $D$. They recognize this gap and point it out in a footnote. They have simply calculated $|f'(\zeta)|$ for the fixed points $\zeta$ and say that only if $|f'(\zeta)| < 1$ will the sequence converge. This is false in general: sequences not generated by contraction maps may converge. A simple example is $f(z) = z^2 + 1/4$. The sequence $z_{n+1} = f(z_n)$ converges to the fixed point $\zeta = 1/2$ for $z_1$ in $D = [0, 1/2]$ but $f'(1/2) = 1$, and although $f(D) \subseteq D$, this $f$ is not a contraction mapping because there is no $q < 1$ that satisfies condition (b) above for this set $D$. Furthermore, fixed points are possible even when $f' > 1$. A nice example of this is $f(z) = \tan(z)$: at all the fixed points $\zeta_j = \tan \zeta_j$ we have $|f'(\zeta_j)| = |\sec^2 \zeta_j| \geq 1$.

In section 2 we proved convergence of the sequence in equation (1) without appeal to the contraction mapping principle. Our proof shows that if the conditions specified there are satisfied, then there exists a closed set $D_\epsilon = \{z : |z - z_+| \leq \epsilon\}$ such that for sufficiently small $\epsilon > 0$ the map $f$ does map $D_\epsilon$ into itself and is a contraction. The initial approximation $z_1 = Z_1 + Z_2$ does not belong to this set, but our proof shows that for any $z_1 \neq z_-$ the approximations do eventually reach $D_\epsilon$, as the authors of [2] hoped.

[1] R. P. Feynman, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964) Vol. II, Chapter 22, 12-15.

[2] H. Krivine and A. Lesne, Am. J. Phys. 71, 31-33 (2003).

[3] J. L. Uretsky, Am. J. Phys. 71, 1320 (2003).

[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition reprinted, (Cambridge University Press, 1958), 154-155.

[5] L. W. Baggett, *Functional Analysis, A Primer*, (Marcel Dekker, 1992), 251-252.