Gravitating Q-tubes and cylindrical spacetime

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Abstract

We consider a model involving a self-interacting complex scalar field minimally coupled to gravity and emphasize the cylindrically symmetric classical solutions. A general ansatz is performed which transforms the field equations into a system of differential equations. In the generic case, the scalar field depends on the four space-time coordinates. The underlying Einstein vacuum equations lead to a family of explicit solutions extending the Kasner space-time. The solutions of the coupled system are -static as well as stationnary-gravitating Q-tubes of scalar matter which deform space-time.

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1 Introduction

Nontopological solitons are known for a long time since the pioneering work of Friedberg, Lee and Sirlin [1], see also [2] for a review. Among the most studied of this type of classical solutions are Q-balls [3]. Such solitons are classical objects build out of scalar field, they are localized in space and stabilized due to a conserved charge associated to an U(1) global symmetry of the underlying field theory. They occur in several contexts of particle theory; for example in the supersymmetric extensions of the Standard Model [4]. Possibilities that Q-balls could play a role in the baryon asymmetry of the Universe and dark matter were emphasized, for example, in [5] and [6] respectively.

While the Q-balls are generically seen as spherically symmetric objects, the classical equations can admit solutions with different types of symmetries and topology, for example solutions with a cylindrical shape (or string shape) can bee looked for [7] : the so called Q-tubes. These solutions can be seen as two-dimensionnal Q-balls embedded in three-dimensional space time. An unified picture of Q-balls and Q-tube is emphasized in [8]. The Q-tube solutions can be made spinning and were first constructed in [9] where the question of spinning Q-balls was adressed in two and three dimensional space-time.

When set in an appropriate context, the Q-balls and Q-tubes can be very large and massive. Accordingly they can be of astronomical size and it becomes necessary to take the gravitational effect into account. The gravitating Q-balls are called boson stars, the patterns of solutions then become very rich and several features of boson stars were reported e.g. in [10]. Further properties of gravitating Q-balls were discussed in [11].

In this paper we address the construction of gravitating Q-tubes including a rotation in the plane perpendicular to the tube and a momentum (or boost) along the axis. We first discuss the metric that incorporates these two effects. The spatial part of the metric describes a rotating-boosted cylinder geometry. It turns out that analytic results can be obtained for the vacuum Einstein equations [12].

For the matter fields, we consider a complex scalar field interacting through an appropriate U(1)-invariant lagrangian. The corresponding globally symmetric lagrangian is coupled minimally to gravity. To implement the
cylindrical symmetry, we use for the fields an extension of the ansatz used in [13, 14]. Non-linear sigma models analogous to the model under investigation were studied namely in [15].

Solving the full equations by means of a numerical technique, we manage to construct several families of Q-tubes and we argue that they exist in some range of the parameters encoding their spin or boost. However it turns out rather difficult to construct Q-tubes in the generic case, i.e. with both rotation and boost.

Let us finally point out that the model under consideration in this paper is a particular case of the more involved superconducting string model [16] which also contains U(1) a gauge symmetry and a second complex scalar field. The equations studied in the present paper correspond to a particular limit of the ones solved in [17]. The particular case corresponding to gravitating Q-tubes and the incorporation of the rotations was, however, not emphasized by these authors.

2 Model and ansatz

We consider a complex scalar field $\phi(x)$ self-interacting through a specific U(1)-invariant potential and minimally coupled to gravity. The corresponding action reads:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + 16\pi G \mathcal{L}_{\text{matter}}),$$  \hspace{1cm} (1)

where $\mathcal{L}_{\text{matter}}$ denotes the matter Lagrangian:

$$\mathcal{L}_{\text{matter}} = \frac{1}{2} \left( \partial_\mu \phi \right)^* \partial^\mu \phi - V(|\phi|), \quad V(z) = m^2 \left( \frac{1}{2} z^2 - \gamma z^4 + \kappa z^6 \right).$$  \hspace{1cm} (2)

Here we employ for the potential the form used in [9] (other forms are also used currently, see e.g. [8]). It is well known that the above action possesses a Noether current $j_\mu = -i(\phi^* \partial_\mu \phi - (\partial_\mu \phi)^*)$ of the corresponding conserved charge is noted $Q$.

The coupled matter and gravity field equations are obtained from the variation of the action with respect to the scalar and metric fields, respectively:

$$\nabla^\mu \nabla_\mu \phi = -m^2 \phi (1 - 4\gamma |\phi|^2 + 6\kappa |\phi|^4); \quad G_{\mu\nu} = 8\pi GT_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \hspace{1cm} (3)$$

and where $T_{\mu\nu}$ is the energy-momentum tensor

$$T_{\mu\nu} = -(g_{\mu\nu} \mathcal{L}_{\text{matter}} - 2 \partial \mathcal{L}_{\text{matter}} \partial g_{\mu\nu}) - \frac{1}{2} \left( \partial_\mu \phi^* \partial_\nu \phi + \partial_\mu \phi^* \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_{\text{matter}} \right).$$  \hspace{1cm} (4)

We want to study regular solitons of the above equations presenting a cylindrical symmetry about the axis $z$. In general, such solutions are characterized by one angular momentum and eventually a linear momentum about the $z$ axis, a boost. The most general metric compatible with the above requirements and the cylindrical symmetry can be parametrized according to

$$ds^2 = g_{ab}(R) dx^a dx^b - dR^2$$

$$= N^2 dt^2 - dR^2 - L^2 (d\varphi - W dt)^2 - K^2 (dz - M d\varphi - v dt)^2,$$  \hspace{1cm} (5, 6)

where $R$ denotes the radial variable in the $x, y$-plane, $\varphi$ the corresponding angle (varying from 0 to $2\pi$) and the functions $N, L, K$ and $W, M, v$ depend on $R$ only. This metric is such that $\sqrt{-g} = KLN$; it reduces to the ansatz used in [13] in the limit $M = v = 0$.

We complete the ansatz by choosing an harmonic dependence on $\varphi$ and $z$ for the complex scalar field:

$$\phi(x) = f(R) \exp(i(\omega t + n\varphi + \lambda z)), \hspace{1cm} (7)$$

where $\omega, \lambda \in \mathbb{R}$, $n \in \mathbb{N}$. Note that the scalar field depends on all the spacetime variables. The form of the scalar field is such that the explicit dependence on the $z, t, \varphi$ variables disappears from the equations. We will refer to the case where $\lambda \neq 0$ as boosted and the case where $n \neq 0$ as rotating.
With the ansatz above, the scalar field equation reduces to the differential equation
\[ f'' + \left( \frac{N'}{N} + \frac{M'}{L} + \frac{K'}{K} \right) f' + \frac{1}{N^2} (W(\lambda M + n) + \nu + \omega)^2 f - \frac{1}{L^2} (\lambda M + n)^2 f - \frac{\lambda^2}{K^2} f = \frac{d}{df} V(f), \] (8)
where the prime here denotes the derivative with respect to \( R \). The resulting Einstein equations (which we do not write explicitly for shortness) can easily be reconstructed from the identities presented in the next section and in Appendix A.1.

In this paper, we mainly focus the discussion to the spinning case; that the boosted case is qualitatively similar. In the two particular cases, the function \( M \) vanishes identically. This function is non-trivial only in the generic case where both rotation and boost are present. The equations for the general system are given in Appendix A. It turns out that switching on both the boost and the rotation leads to a delicate numerical problem. We were able to integrate the full system in this case, but the interpretation of the resulting solutions is still unclear. However, as we will further argue, such solutions must exist and consistently solve the Einstein equations and the matter field equation.

3 Vacuum solutions

We are interested in solutions where the matter field is localized around the symmetry axis of the tube. In this respect, the knowledge of the vacuum solutions is useful.

In the non-rotating and non-boosted case \( W = 0, \nu = 0 \), it is well known that the solutions are given by the family of ‘Kasner-space-time’ labelled by the Kasner parameters \( a, b, c \) :
\[ ds^2 = \gamma_t R^a dt^2 - dR^2 - (1 + \Delta) R^b d\varphi^2 - \gamma_z R^c dz^2 \] (9)
with the conditions \( a + b + c = 1, a^2 + b^2 + c^2 = 1 \). The parameter \( \Delta \) characterizes the angular deficit of the solution; \( \gamma_t \) and \( \gamma_z \) are not intrinsic: they can be rescaled in the variables \( t \) and \( z \) respectively.

Solving the coupled equations (i.e. with matter) leads to specific values of the parameters \( a, b, c \) and \( \Delta \), depending of the coupling constants of the potential and of \( \omega \).

3.1 Integrals of motion

In order to discuss the rotating or boosted case, it is useful to note that six out of the seven independent Einstein vacuum equations lead to first integrals. Indeed, the relevant components of the Ricci tensor can be set in the form
\[ R_t^t = -\frac{K_t'}{2\sqrt{-g}}, \quad R_t^\varphi = -\frac{K_{t\varphi}}{2\sqrt{-g}}, \quad R_t^z = -\frac{K_t'z}{2\sqrt{-g}}, \]
\[ R_{\varphi}^\varphi = -\frac{K_{\varphi\varphi}}{2\sqrt{-g}}, \quad R_{\varphi}^z = -\frac{K_{\varphi z}}{2\sqrt{-g}}, \quad R_z^z = \frac{K_{zz}}{2\sqrt{-g}}, \] (10)
where
\[ K_t = \frac{K^3 M \nu M'}{N} - 2KLN' + \frac{K^3 L \nu N'}{N} + \frac{KL^3 W W'}{N}, \]
\[ K_{\varphi} = \frac{K^3 L M W M'}{N} + \frac{K^3 L M \nu N'}{N} - \frac{K^3 L W W'}{N}, \]
\[ K_z = \frac{K^3 L (W M' + \nu')}{N}, \]
\[ K_{\varphi\varphi} = -K N L' + \frac{K^3 L M W^2 M'}{N} - \frac{K^3 M N N'}{L} + \frac{K^3 L M W \nu'}{N} - \frac{K^3 L W W'}{N}, \]
\[ K_{\varphi z} = \frac{K^3 L W^2 M'}{N} + \frac{K^3 N M'}{L} + \frac{K^3 L W \nu'}{N}, \]
\[ K_{z z} = -2LNK' - \nu \left( \frac{K^3 L M M'}{N} + \frac{K^3 L \nu'}{N} \right) - M \left( \frac{K^3 M' (L^2 W^2 - N^2)}{L N} + \frac{K^3 L W \nu'}{N} \right), \] (11)
We further note that the remaining equation $R_r^r = 0$ reduces to
\[
R_r^r = \frac{K''}{K} + \frac{L''}{L} + \frac{N''}{N} + \frac{K_1 + 2L_k (K N)'}{2K^4 M_k^2 N} - \frac{(L_k^2 + K^2 M_k^2)}{K^4 L^4},
\]
where we defined
\[
K_1 = K_t K_z + K_x K_{xz} + K_\psi K_{\psi z}, M_k = K_\nu - M K_z, L_k = K_z L.
\]
A suitable combination of (11) leads to
\[
(N L K)' = -\frac{1}{2} (K_{zz} + K_{\psi \phi} + K_i),
\]
which imposes that in the general case, at least half of the Kasner relations hold ($a + b + c = 1$, assuming $N, (K, L) \propto r^a, (r^b, r^c)$ resp.)

### 3.2 Solutions in matrix form

Using the formalism developed in [12], it is possible to find the general solution of the system $R_{\nu \mu} = 0$ in a closed form. In fact the solutions of vacuum-Einstein equations corresponding to the metric (6) can be written in the following form involving two $3 \times 3$ matrices $A$ and $C$:
\[
g(R) = C \exp[2A \ln((R - R_0))] , \quad \text{Tr}A = \text{Tr}A^2 = 1 , \quad C^T = C , \quad (CA)^T = CA. \quad (15)
\]
The three eigenvalues of $A$, say $a, b, c$, are the Kasner powers mentioned above. Assuming these eigenvalues to be non-degenerate, we can further write
\[
g(R) = CA \text{ diag}((R - R_0)^2a, (R - R_0)^2b, (R - R_0)^2c) A^{-1} , \quad A = A \text{ diag}(a, b, c) A^{-1} \quad (16)
\]
The extraction of the functions parametrizing the metric (6) leads in general to cumbersome rational functions of $R$.

### 3.3 Rotating case

In the case $W \neq 0$, $v = M = 0$, the expression for $g$ simplifies considerably. If we further require the metric to be a deformation of the Minkowski space-time, i.e. with $|b - 1| \sim 0, a \sim 0, c \sim 0$ the matrices $A, C$ need to be of special form. We find
\[
A = \begin{pmatrix}
1 & -c_{11} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
leading to
\[
g = \begin{pmatrix}
c_{11} R^{2a} & (c_{11} c_{22} - c_{12}^2) R^{2b} + \frac{c_{11} a}{c_{11}} R^{2a} & 0 \\
c_{12} R^{2a} & 0 & 0 \\
0 & 0 & c_{33} R^{2c}
\end{pmatrix}
\]
with obvious definitions for $c_{ij}$ and $c_{13} = c_{23} = 0$.

The diagonal 'main' functions $N, L, K$ obey the Kasner form only asymptotically and the following exact relations hold
\[
W' = -K_\phi \frac{N}{K L^3} , \quad K = K_0 (NL)^{c/(a+b)},
\]
with the ensuing asymptotic form for the function $W$ and the non-diagonal element of the metric:
\[
W(R) \sim -\frac{K_\phi}{R^{2(b-a)}} , \quad g_{02} = L^2 W \sim W_0 R^{2a}
\]
As long as the Kasner powers $b, c$ are such that $2b + c > 1$, the 'W-terms' in the equations are subdominant with respect to terms involving the diagonal function. The solutions that we will report in the next section are mainly of this type.
3.4 Boosted case

In the case where \( v \neq 0 \), \( W = M = 0 \), the result of the previous subsection can be adapted by exchanging the role of the coordinates \( \varphi \) and of \( z \). The relevant integral of motion leads to

\[
v' = K z \frac{N}{K^3}L, \quad v(r) \sim \frac{K z}{R^2(c-a r)}, \tag{21}
\]

The generalisation of the Eq. (17) to the case with boost and rotation is obtained by choosing for \( C \) an arbitrary symmetric matrix anf for \( A \) an upper triangular matrix of the form

\[
A = \begin{pmatrix}
1 & A_{12} & A_{13} \\
0 & 1 & A_{23} \\
0 & 0 & 1
\end{pmatrix}
\tag{22}
\]

where

\[
A_{12} = -\frac{c_{12}}{c_{11}}, \quad A_{13} = \frac{c_{12}c_{23} - c_{12}c_{22}}{c_{11}c_{22} - c_{12}^2}, \quad A_{23} = \frac{c_{12}c_{13} - c_{11}c_{23}}{c_{11}c_{22} - c_{12}^2}
\]

As we will point out in the next section, in the case of spinning, boosted solutions with matter fields the asymptotic metric is not compatible with the form for generic values of the parameters \( \omega, \lambda \). This suggests that only a very small range of parameters lead to boosted and rotating configurations where both Kasner conditions are fulfilled. This deserves further investigations.

From now on, we focus on the rotating case only except if stated explicitly.

4 Equations and physical quantities

4.1 Energy momentum tensor

From now on, we concentrate on the case with no boost (i.e. \( M = v = 0 \)). In order to establish the Einstein equations, we need the energy momentum tensor associated with the scalar field. The non-vanishing components of this tensor read:

\[
T^t_t = V(f) + \omega^2 \frac{f^2}{2N^2} + \frac{(f')^2}{2} + f^2 n^2 \frac{N^2 - L^2 W^2}{2N^2 L^2},
\]

\[
T^r_r = V(f) - \omega^2 \frac{f^2}{2N^2} - \frac{\omega f^2}{N^2} \frac{(f')^2}{2} + f^2 n^2 \frac{N^2 - L^2 W^2}{2N^2 L^2},
\]

\[
T^\varphi_\varphi = V(f) - \omega^2 \frac{f^2}{2N^2} + \frac{(f')^2}{2} - f^2 n^2 \frac{N^2 - L^2 W^2}{2N^2 L^2},
\]

\[
T^z_z = V(f) - \omega^2 \frac{f^2}{2N^2} - \frac{\omega f^2}{N^2} + \frac{(f')^2}{2} + f^2 n^2 \frac{N^2 - L^2 W^2}{2N^2 L^2},
\]

\[
T^t_\varphi = \frac{nf^2 (\omega + nW)}{N^2}
\]

(23)
4.2 Ricci tensor

Correspondingly, the relevant components of the Ricci tensor reduce to
\[
R_{tt} = \frac{(KLN')'}{KLN} + WW'\frac{L(KLN' - 3KNL' - K'NL)}{2KN^3} - ((W')^2 + WW''L^2)\frac{L^2}{2N^2},
\]
\[
R_{tr} = \frac{N'' + K''}{N} + \frac{L''}{L} - (W')^2\frac{L^2}{2N^2},
\]
\[
R_{\varphi\varphi} = \frac{(KLN')'}{KLN} - WW'\frac{L(KLN' - 3KNL' - K'NL)}{2KN^3} + (W''W + (W')^2)\frac{L^2}{2N^2},
\]
\[
R_{zz} = \frac{L^2W''}{2N^2} - LW'(KLN' - 3KL'N - K'LN)\frac{2KN^3}{2N^2}.
\]

4.3 Physical quantities

The solutions can be characterized by their energy \( E \) and tension \( T \) per unit length. Along with \[16, 17\] we adopt for these quantities the definitions
\[
E = \int \int \sqrt{-h} T_{tt}dRd\varphi , \quad T = \int \int \sqrt{-h} T_{zz}dRd\varphi ,
\]
where \( h \) represents the determinant of the induced metric on the \( R, \varphi \)-plane (the integration over the angle \( \varphi \) is trivial). Note that another definition of the mass is used in \[13, 14\]. Solitons like Q-balls and Q-tube can further be characterized by the Noether charge underlying the global symmetry of the model
\[
Q = \int j^0\sqrt{-g}d\varphi dR = 4\pi\int_0^\infty f^2\omega + nW \frac{N}{N^2}NLKdR ,
\]
In the case of spinning soliton the solution can further be characterized by the angular momentum \( J \) (per unit length). The quantities \( J \) and \( Q \) are proportional:
\[
J = \int T_{0r}\sqrt{-g}dRd\varphi = 4\pi n\int_0^\infty NLK[f^2\omega + nW]dR , \quad J = nQ
\]
For \( Q \) and \( J \), we used the definition of the measure which makes them proportional.

5 Solutions for \( n = 0 \)

To our knowledge, the coupled field equations do not admit solutions in an explicit form. We relied on numerical techniques to construct them. The routine Colsys \[18\] was used for this purpose. By an appropriate rescaling of the scalar field, the parameter \( \gamma \) of the potential in \[2\] can be set to one and the mass parameter \( m \) can be absorbed in the gravitation constant, we therefore conveniently use \( \alpha = 16\pi m^2G \). With these scale conventions, the system appears with two intrinsically different coupling constants : \( \alpha \) and \( \kappa \). Families of solutions labelled by \( \omega \) and \( \lambda \) (or alternatively by \( f(0) \)) can then be constructed numerically. We first concentrate on the non-spinning case characterized by \( n = 0 \). Assuming for the moment that there in no boost (i.e. setting \( \lambda = 0 \)), the conditions associated to the non diagonal Einstein equations are solved consistently by setting \( W = v = M = 0 \). The boundary conditions which guarantee the regularity of the configuration on the symmetry axis (i.e. for \( R = 0 \)) and the finiteness of the energy (per unit length with respect to \( z \)) are
\[
N(0) = 1 , \quad N'(0) = 0 , \quad K(0) = 1 , \quad K'(0) = 0 , \quad L(0) = 0 , \quad L'(0) = 1 , \quad f'(0) = 0 ,
\]
Figure 1: The quantities $E, T, Q$ (left); the Kasner power and angular deficit $\Delta$ as functions of $\omega^2$ for $n = 0$, $\alpha = 0.1$ and $\kappa = 0.4$.

while $f(0) \equiv \psi_0$ is a free parameter (note: the conditions $N(0) = K(0) = 1$ result of an appropriate rescaling of the coordinates $t$ and $z$). The asymptotic condition $f(R \to \infty) = 0$ completes the boundary value problem. In the absence of gravity, the scalar field equation implies the following exponential decay of the scalar field

$$f(R) \propto \frac{1}{\sqrt{R}} e^{-\sqrt{m^2 - \omega^2} R}.$$ \hfill (29)

For $\alpha > 0$, the asymptotic form is more involved but the numerical result confirm that the scalar field indeed decays exponentially.

Fixing a value for $\omega$ in principle leads to one (or more, see below) solution with a fixed value $\psi_0$. We found it convenient to supplement the system with the equation $d\omega/dR = 0$ and to take advantage of the extra boundary condition to impose $f(0) = \phi_0$. The corresponding value of $\omega$ being reconstructed numerically. With the potential specified as above, the reasoning of Ref. [9] demonstrates that the solutions exist in the interval

$$\omega_c^2 \leq \omega^2 \leq m^2 \quad , \quad \omega_c^2 = \min\{0, \frac{m^2}{2}(1 - \frac{1}{2\kappa})\}$$ \hfill (30)

The pattern of solutions is therefore different for $\kappa < 1/2$ and $\kappa > 1/2$.

5.1 $\kappa < 0.5$

In this case the soliton can be constructed up to $\omega = 0$. Setting for definiteness $\kappa = 0.4$ and $\alpha = 0.1$, we obtained the family of solutions whose pattern is illustrated by Fig. 1. We see that, for $\omega^2 \to 1$, the scalar field approaches the vacuum (i.e. $\phi(x) = 0$) and Minkowski space-time is approached. For $\omega \to 0$, the matter field in the core becomes important and the angular deficit is large. For the value $\omega = 0$ the metric functions $N(R), K(R)$ become constants for $R \to \infty$ while the angular deficit is maximal. The profiles of the solutions are shown on Fig. 2 and compared for two different values of $\omega$. 

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Figure 2: Comparaison of profiles of the metric functions $N, K$ (left), $L'$ and the scalar field (right) for $\omega = 0.0$ and $\omega = 0.4$ (here $\alpha = 0.1, \kappa = 0.4$).

Figure 3: The quantities $E, T, Q$ (left); the Kasner power and angular deficit $\Delta$ as functions of $\omega^2$ for $n = 0$, $\alpha = 0.1$ and $\kappa = 0.55$.

5.2 $\kappa < 0.5$

The crucial difference with respect to the case $\kappa > 0.5$ is that the solutions stop at a strictly positive value of $\omega$, say $\omega = \omega_c$. In fact it turns out that for small enough values of $\omega^2$ two solutions exist forming two branches that coincide at $\omega = \omega_c$. It is natural to refer to the branch directly connect to Minkowski space-time as to the main branch since, for a fixed $\omega$, it has the lowest mass. When we progress on the second branch (while increasing $\omega$) the so called 'thin wall limit' is approached. In particular, the scalar function varies very slowly for $R < R_c$ for some definite radius $R_c$ and then rapidly reaches its asymptotic value $\phi = 0$ at $R \sim R_c$. At
the same time, the charge and energy (per unit length) become quite large while the tension becomes negative. These properties are illustrated on Fig. 3.

5.3 Boosted solutions

All the solutions available in the case $n = 0$ can be deformed by a boost, i.e. setting $\lambda > 0$ in the equations. The Einstein equation corresponding to $R_{ij}^0$ is no longer trivial leading to a non vanishing function $v(R)$. It turns out that the function $v(R)$ and the parameter $\omega$ enter the equations through a special combination which we conveniently redefine as $\tilde{v}(R) \equiv v(R) + \omega/\lambda$. The relevant boundary to solve the system of five equations consists of the conditions (23) for the metric functions $N, K, L$ supplemented by

$$f(0) = \phi_0, \quad f'(0) = 0, \quad \tilde{v}'(0) = 0, \quad f(R \to \infty) = 0$$

where $\phi_0$ is a constant. This leads to a family of solutions labelled by the two parameters $\lambda$ and $\phi_0$. These calculations are systematic and not reported here.

6 Solutions for $n > 0$

Setting $n > 0$, the above equations lead to spinning solutions since the component $T^0_\nu$ of the energy momentum tensor is non zero. The regularity of the scalar field equation at the origin imposes $f(0) = 0$ while the derivative $f'(0)$ is determined by the numerical result.

6.1 Spinning Q-tubes

Setting $\alpha = 0$, we have constucted families spinning solutions -without gravitation- of the field equations. The two dimensional counterpart was obtained in [9]. In this case, the Einstein equations lead to a Minkowski space-time (i.e. with $W = 0$). Along with the corresponding solutions in the $n = 0$ case, the spinning Q-tubes exist for a finite interval of the parameter $\omega$. The physical parameters of the solutions corresponding $\kappa = 0.4$ and $\kappa = 0.55$ are reported on Fig. 4 When the solutions cannot be constructed in the limit $\omega = 0$, they exist up to a minimal value of $\omega$ (for $\kappa = 0.55$, we find $\omega_m \approx 0.38$). When the minimal value of $\omega$ is approached the solution approaches a thin wall limit configuration, both the energy and the charge become quite large as illustrated by Fig. 4. As pointed out above, the relation $J = Q$ holds for these non gravitating Q-tubes.

6.2 Gravitating, Spinning Q-tubes

Setting $n = 1$ and $\alpha > 0$ in the equations does not lead to any reduction of the system for generic value of $\omega$. Only in the case $\omega = 0$ the equation for the function $W(R)$ becomes linear and is fulfilled by $W(R) = 0$. For the metric fields $N, L, K$, the boundary conditions on the symmetry axis are choosen like in the non spinning case. The regularity of the equations at $R = 0$ further requires $W'(0) = 0$; it is natural to complete the boundary value problem by imposing $W(R \to \infty) = 0$. The numerical results confirm the existence of regular solutions extrapolating between the initial conditions imposed at $R = 0$ and the asymptotic Kasner form.

As pointed out in Sect. 2.2, the scalar field is exponentially small in the asymptotic region. In this region a metric of the form (??) in particular $W$ obeys Eq. (14), where $C$ and $W_0$ can be determined numerically. The three powers $a, b, c$ can be extracted from the numerical solutions and it was confirmed that the obey the Kasner relations within the numerical errors (of order $10^{-6}$ in our case). With the value $\alpha = 0.1$ used to compute the solutions, the Kasner powers are such that the terms depending on $W(R)$ in the Einstein equations are subleading and the effect of the rotation can be considered as a correction with respect to a purely Kasner solution. In particular, it turns out that the Kasner powers are such that the relation $2b + c > 1$ is obeyed for the families of solutions that we have studied. (more precisely $0 \leq a \leq 0.08; 0.9925 \leq b \leq 1; 0 \geq c \geq -0.082$, they are reported on Fig 4). In spite of the fact that the diagonal terms are dominant in the equations, the non-diagonal component of the metric $g_{02}$ does not vanish in the asymptotic region. The spinning Q-ball somehow produces a rotational effect on space-time which persists in the asymptotic region.
The fact that the scalar field approaches the asymptotic value \( f(R \to \infty) = 0 \) exponentially ensures that the integral determining the quantities \( U, T, Q, J \) are convergent. The profiles of \( T_0^0, T_2^0, g_{00}, f(R) \) and \( W(R) \) are shown on Fig. 6 (left side) in the case \( \alpha = 0.1, \omega = 0.5 \). The dependence of the mass, the tension and the angular momentum on the parameter \( \omega \) are reported on the right side of the figure.

Independently of the solutions obeying the boundary condition \( W(R \to \infty) = 0 \), we produced another family of spinning Q-tubes by imposing for the metric function \( W(R) \) the condition \( W(0) = W'(0) = 0 \). In this case also, the scalar field is localized and the asymptotic value \( f = 0 \) is approached exponentially. Outside the core
Figure 6: The metric components $g_{00}$, W(R), f(R), the energy and angular momentum density for spinning, gravitating Q-tube ($n = 1, \alpha = 0.1$) for $\kappa = 0.4$ and $\omega = 0.5$ (left). Mass, Tension, angular momentum and the parameters $a, c$ as functions of $\omega$ (right).

of the string, the different metric fields behave according to

$$N(R) = N_0 R^a, \quad L(R) = \Delta R, \quad K(R) = K_0 R^{-a}, \quad W(R) = W_0 + o(R^{2a-2})$$

where the parameters $N_0, K_0, W_0, \Delta$ and the power $a$ are constants. These results reveal that the time-time component of the metric $g_{00} = N_0 R^{2a} - W_0^2 \Delta^2 R^2$ vanishes for a critical value of the radius $R = R_c$, where $R_c$ depends on the parameters.

Some relevant metric components of the solution of this type corresponding to $\alpha = 0.1, \omega = 0.05$ are presented on Fig. 7; in this case, we have $R_c \approx 176, a \approx 0.0045, \Delta \approx 0.848$. For the domain of the parameters that we have explored, the scalar field reaches its asymptotic value already for $R \ll R_c$; the form (31) therefore corresponds to a vacuum solution. However, it does not belong to the subclass ([??]). At the approach of the limiting point $R = R_c$ the metric takes the form

$$ds^2 = (R_c - R) dt^2 - \Delta^2 R^2 d\varphi^2 - dR^2 - K_0^2 d\varphi^2 - W_0^2 \Delta R dt d\varphi,$$

The solutions under consideration are then regular only for $R < R_c$. The apparent singularity occurring for $R \to R_c$ in fact a coordinate artefact, the Ricci and Kreischmann scalar invariants indeed vanishes in the limit $R \to R_c$ as demonstrated on Fig. 7. It seems that, far away from the center of the lump, the time of space-time becomes degenerate due to the spinning nature of the Q-tube ($W_0 > 0, \omega > 0$). These spinning solutions share the property that $g_{00}(R_c) = 0$ with the ‘singular Kasner’ solutions of the classification reported in [14]. However in the case of the singular Kasner the metric component $g_{22}(R)$ gets singular for $R \to R_c$ while it remain regular (for instance $g_{22}(R) \sim R^2$) in the present case.

The underlying space-time is also different from the space-time of supermassive cosmic strings discussed e.g. [19, 20]. The associated metric function $g_{22}(R)$ indeed gets a zero at some radial radius, corresponding to a maximal angular deficit.

7 Conclusion

In this paper, we constructed numerically the gravitating counterpart of the Q-tubes of [7] including rotation or boost. This was done by using an appropriate ansatz for the fields. In particular, the metric degrees of freedom
are encoded in six independent functions of a coordinate $R$ representing the distance to the axis of symmetry.

As a preliminary setup for the understanding of the solutions, we analyzed the structure of the vacuum Einstein equations underlying a boosted-rotating tube. The construction elaborated in [12] leads to analytical solutions that can be adapted to our parametrization of the metric. We manage to construct -within our ansatz- the six first integrals to the system transforming the original system of six second order equations into a system of six first order equations. Two of these 'constants of motion' can be interpreted as the angular momentum and the linear momentum along the axis of symmetry (in our notations $K_\varphi$ and $K_z$ respectively). The interpretation for the other constants is less clear, although the energy of the configuration must contain the sum $K_t + K_{zz} + K_{\varphi\varphi}$.

Our results demonstrate that the $Q$-tubes constructed in flat space in [7] are smoothly deformed by gravity, leading to a space-time that is asymptotically of Kasner type. The matter field corresponding to the $Q$-tube is localized around the axis of symmetry and quickly approaches its asymptotic value. The solitons with $n = 1$ are spinning in the plane transverse to the axis of symmetry, while the boost corresponds to a flux along the direction of the axis.

It turns out that for the solely rotating or solely boosted solitons the asymptotic metric is a Kasner spacetime corrected by subdominant terms. We obtained several types of spinning, gravitating $Q$-tubes, labelled essentially by $W(0)$, i.e. the value on the axis of the rotating function on the axis. For a subset of the solutions obtained, the time-time component of the metric vanishes at a critical distance, say $R = R_c$, to the axis of symmetry; the value $R_c$ is much higher than the soliton core. We believe that this feature is related to the fact that the string is infinite and that it would be regularized in the case of strings of finite size.

We obtained many examples of regular solitons with both rotation and boost, although they seem to be regular only on a small range of the parameters. When the boost parameter $\lambda$ becomes too large, some components of the metric change sign (or become singular) at finite values of the radius. In this case, the metric seems not to be of Kasner type even up to subdominant terms. Our results shows that, in all cases, the diagonal components of the metric obey one of the two Kasner conditions (the linear one), while the quadratic relation is not fulfilled. The full understanding of the pattern of solutions needs further investigations which we delay to future considerations.

Finally, note that the metric (6) could be suitable to describe the near zone of a solitons with a torus shape (like vorton, see [21]), similar to the black ring constructed in [22] in pure gravity. In this case, the black string

---

Figure 7: Profiles of the metric components, of the Ricci and Kreischmann invariants of a spinning gravitating $Q$-tube ($n = 1, \alpha = 0.1$) for $\kappa = 0.4$ and $\omega = 0.05$. 
with a momentum along the axis describes the near zone of the ring. Intuitively, the momentum along the axis is mapped to the angular momentum of the ring and balances the self gravity tending to collapse the ring. We could expect a similar phenomena here since quantities like tension and angular momentum are naturally defined from the matter fields.

For cosmic string solutions criteria have been developed to decide about the stability of these objects which is crucial for the possible formation of vortons [23]. This uses the macroscopic properties of the strings in the sense that the velocities of the longitudinal and transversal perturbations are determined in terms of the energy per unit length and the tension of the string. This method has been used to decide about the stability of superconducting cosmic strings (see e.g. [16], [17], [23]). It would be interesting to see whether these techniques can also be employed in our case since the criteria used only rely on the macroscopic quantities of the string.

A Stress tensor in the general case

The Einstein tensor in the general case is easily reconstructed from (11). We give here the nonvanishing components of the stress tensor:

\[
T^t_t = \frac{1}{2} f'^2 + \frac{\lambda^2 f'^2}{2 K^2} + \frac{\lambda f^2 M}{L^2} + \frac{\lambda f^2 M^2}{2 L^2} + \frac{n^2 f^2}{2 L^2} - \frac{\lambda n f^2 M W^2}{N^2} - \frac{\lambda^2 f^2 M v W}{2 N^2} - \frac{n^2 f^2 W^2}{2 N^2} - \frac{\lambda f^2 M^2 W^2}{2 N^2} - \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + V(f)
\]

\[
T^r_r = \frac{1}{2} f'^2 + \frac{\lambda^2 f'^2}{2 K^2} + \frac{\lambda f^2 M}{L^2} + \frac{\lambda f^2 M^2}{2 L^2} + \frac{n^2 f^2}{2 L^2} - \frac{\lambda n f^2 M W^2}{N^2} - \frac{\lambda^2 f^2 M v W}{2 N^2} - \frac{n^2 f^2 W^2}{2 N^2} - \frac{\lambda f^2 M^2 W^2}{2 N^2} - \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + V(f)
\]

\[
T^{\varphi \varphi} = \frac{1}{2} f'^2 + \frac{\lambda^2 f'^2}{2 K^2} + \frac{\lambda^2 f^2 M^2}{2 L^2} - \frac{n^2 f^2}{2 L^2} - \frac{\lambda f^2 M v W}{N^2} - \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{n^2 f^2 W^2}{2 N^2} - \frac{\lambda f^2 M^2 W^2}{2 N^2} - \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + V(f)
\]

\[
T_z^z = \frac{1}{2} f'^2 + \frac{\lambda^2 f'^2}{2 K^2} - \frac{\lambda^2 f^2 M^2}{2 L^2} - \frac{n^2 f^2}{2 L^2} + \frac{\lambda f^2 M v W}{N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} - \frac{n^2 f^2 W^2}{2 N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + V(f)
\]

\[
T^{i \varphi} = -\frac{\lambda f^2}{K^2} - \frac{\lambda f^2}{2 L^2} - \frac{n f^2}{L^2} + \frac{\lambda f^2 M W^2}{N^2} + \frac{\lambda f^2 M v W}{N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + \frac{\omega f^2}{N^2}
\]

\[
T^{i z} = -\frac{\lambda f^2}{K^2} - \frac{n f^2}{L^2} - \frac{\lambda f^2}{2 L^2} + \frac{n f^2}{2 L^2} + \frac{\lambda f^2 M W^2}{N^2} + \frac{\lambda f^2 M v W}{N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + \frac{\omega f^2}{N^2}
\]

\[
T^{\varphi z} = -\frac{\lambda f^2}{K^2} - \frac{n^2 f^2}{L^2} - \frac{\lambda f^2}{2 L^2} + \frac{n f^2}{2 L^2} + \frac{\lambda f^2 M W^2}{N^2} + \frac{\lambda f^2 M v W}{N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda f^2 M^2 W^2}{2 N^2} + \frac{\lambda n f^2 W}{2 N^2} + \frac{\omega f^2}{2 N^2} + \frac{\omega f^2}{N^2}
\]

It can easily be shown that the seven equations are not independant. Indeed, if one solves the off diagonal Einstein equations for $W''$, $v''$, $M''$, injects the solution in the $r-r$ component of the Einstein equation and further derives the resulting equation, then this equation is indentically zero when the other equations are satisfied. In other words, the derivative of the $r-r$ equation is a linear combination of the other components.

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