On the general dual Orlicz-Minkowski problem *

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Abstract
For $K \subseteq \mathbb{R}^n$ a convex body with the origin $o$ in its interior, and $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ a continuous function, define the general dual $(L_\phi)$ Orlicz quermassintegral of $K$ by

$$V_\phi(K) = \int_{\mathbb{R}^n \setminus K} \phi(x) \, dx.$$ 

Under certain conditions on $\phi$, we prove a variational formula for the general dual $(L_\phi)$ Orlicz quermassintegral, which motivates the definition of $\tilde{C}_{\phi,\gamma}(K, \cdot)$, the general dual $(L_\phi)$ Orlicz curvature measure of $K$.

We pose the following general dual Orlicz-Minkowski problem: Given a nonzero finite Borel measure $\mu$ defined on $S^{n-1}$ and a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$, can one find a constant $\tau > 0$ and a convex body $K$ (ideally, containing $o$ in its interior), such that,

$$\mu = \tau \tilde{C}_{\phi,\gamma}(K, \cdot)?$$

Based on the method of Lagrange multipliers and the established variational formula for the general dual $(L_\phi)$ Orlicz quermassintegral, a solution to the general dual Orlicz-Minkowski problem is provided. In some special cases, the uniqueness of solutions is proved and the solution for $\mu$ being a discrete measure is characterized.

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1 Introduction
Let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous function and $\mu$ be a nonzero finite Borel measure defined on the unit sphere $S^{n-1}$, the Orlicz-Minkowski problem asks whether there exists a convex body (a convex and compact subset of $\mathbb{R}^n$ with nonempty interior) $K$ and a constant $\tau > 0$, such that,

$$d\mu = \tau \cdot \varphi(h_K) \, dS_K$$

where $h_K$ denotes the support function of $K$ (see (2.2)) and $S_K$ denotes the surface area measure of $K$ (see (3.17)). The Orlicz-Minkowski problem was first investigated by Haberl, Lutwak, Yang and Zhang in their seminal paper [10] for even measure $\mu$. Solutions to the Orlicz-Minkowski problem for $\mu$ being a discrete and/or general (not necessary even) measure were provided by Huang and He [14] and Li [21]. The planar Orlicz-Minkowski problem in the $L_1$-sense was investigated by Sun and Long [37]. The $p$-capacitary Orlicz-Minkowski problem was posed and studied in [12]. The Orlicz-Minkowski problems are central objects in the recent but rapidly developing Orlicz-Brunn-Minkowski theory [8, 25, 29, 30, 40].

The well-studied classical Minkowski problem and its $L_p$ extension are special cases of the Orlicz-Minkowski problem. When $\varphi(t) = 1$, it becomes the classical Minkowski problem back

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to Minkowski at the turn of the 20th century [31, 32]. Please refer to [33, Chapter 8] for details. When \( \varphi(t) = t^{1-p} \) for \( p \in \mathbb{R} \), it becomes the \( L_p \) Minkowski problem back to Lutwak [26] in 1993. Since then, great progress has been made on the \( L_p \) Minkowski problem, see e.g., [6, 7, 13, 15, 17, 18, 27, 28, 38, 49, 50]. In particular, the singular cases for \( p = 0 \) and for \( p = -n \), referred to as the logarithmic (or \( L_0 \)) Minkowski problem and the centro-affine Minkowski problem, respectively, are arguably more challenging than the cases for \( p \neq 0, -n \). Remarkable contributions on the logarithmic (or \( L_0 \)) and centro-affine Minkowski problems can be found in, e.g., [2, 4, 7, 19, 24, 34, 35, 36, 47, 48]. We would like to mention that the \( L_p \) Minkowski problem can be described through a fully nonlinear second-order partial differential equation (i.e., Monge-Ampère type equation) and plays fundamental roles in the development of the \( L_p \) Brunn-Minkowski theory for convex bodies.

The \( L_p \) surface area measure \( h_{K}^{1-p} S_K \) can be obtained through variational formulas [26, 44]. As an example, for \( p > 1 \) and \( K, L \subseteq \mathbb{R}^n \) convex bodies containing the origin \( o \) in their interiors, one has [26]

\[
\int_{S^{n-1}} h_{L}^{p}(u)h_{K}^{1-p}(u) dS_K(u) = p \cdot \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(K)}{\varepsilon} \tag{1.1}
\]

where \( V(K) \) stands for the volume of \( K \) and \( K + \varepsilon \cdot L \) is a convex body determined by the function \( h_{K+\varepsilon \cdot L}^{p} : S^{n-1} \to (0, \infty) \): for any \( \varepsilon > 0 \),

\[
h_{K+\varepsilon \cdot L}^{p} = h_{K}^{p} + \varepsilon h_{L}^{p}.
\]

Livshyts [23] proposed a surface area measure of \( K \) with respect to a measure \( \mu_g \), where \( g \), the density of \( \mu_g \) with respect to the Lebesgue measure, is continuous on its support. A variational formula for \( \mu_g \) similar to (1.1) for \( p = 1 \) was also provided in [23], which gives a variational interpretation of the surface area measure of \( K \) with respect to \( \mu_g \). The related Minkowski problem was posed and a solution to this problem was given under certain conditions on \( \mu_g \) (such as, \( \mu_g \) being a measure with positive degree of concavity and positive degree of homogeneity). An \( L_p \) extension of the theory by Livshyts was obtained by Wu [39], where the \( L_p \) surface area measure with respect to \( \mu_g \) was proposed and related \( L_p \) Minkowski problem was solved under certain conditions on \( \mu_g \). Indeed, our paper was partially motivated by [23, 39].

This paper is also motivated by the recent work of Zhu, Xing and Ye on the dual Orlicz-Minkowski problem [45], which belongs to the recently initiated dual Orlicz-Brunn-Minkowski theory [9, 41, 46] and can be viewed as the “dual” of the Orlicz-Minkowski problem. For a convex body \( K \subseteq \mathbb{R}^n \) with the origin \( o \) in its interior and a continuous function \( \varphi : (0, \infty) \to (0, \infty) \), the authors in [45] defined the dual Orlicz curvature measure \( \tilde{C}_\varphi(K, \cdot) \) and investigated the following dual Orlicz-Minkowski problem: under what conditions on \( \varphi \) and a given nonzero finite Borel measure \( \mu \) on \( S^{n-1} \), there exist a constant \( \tau > 0 \) and a convex body \( K \) (ideally with the origin in its interior) such that \( \mu = \tau \tilde{C}_\varphi(K, \cdot) \)? A solution to the dual Orlicz-Minkowski problem was given under the assumptions: a) the measure \( \mu \) is not concentrated on any closed hemisphere, i.e., \( \mu \) satisfying (5.29); b) the function \( \varphi \) and its companion function

\[
\phi(t) = \int_{t}^{\infty} \frac{\varphi(s)}{s} ds
\]

satisfy conditions A1)-A3) as described in Section 2. We would like to mention that the assumption on \( \mu \), i.e., (5.29), is necessary for the solutions of various Minkowski problems. A special case with \( \varphi(t) = t^q \) for \( q < 0 \) was solved in the remarkable paper [42] by Zhao, as \( \varphi(t) = t^q \) and its companion \( \phi(t) = -t^q/q \) satisfy conditions A1)-A3). The dual Orlicz-Minkowski problem stemmed from the groundbreaking work [16] in 2016 by Huang, Lutwak, Yang and Zhang, where
they provided a very detailed study of the geometric measures (such as the $q$-th dual curvature measures) in the dual Brunn-Minkowski theory and initiated the very promising dual Minkowski problem for the $q$-th dual curvature measures. In particular, they provided a solution to the dual Minkowski problem for the $q$-th dual curvature measures with $q \in (0, n]$ and even measure $\mu$ (plus some additional conditions). Note that the logarithmic Minkowski problem is the case for $q = n$. Since their groundbreaking work [16], there is a growing body of work in this direction, see e.g., [3, 5, 11, 20, 22, 42, 43, 45].

The starting point of this paper is the general dual $(L_\phi)$ Orlicz quermassintegral. For $K \subseteq \mathbb{R}^n$ a convex body with the origin $o$ in its interior, and $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ a continuous function, define the general dual $(L_\phi)$ Orlicz quermassintegral of $K$ by

$$\mathcal{V}_\phi(K) = \int_{\mathbb{R}^n \setminus K} \phi(x) \, dx.$$ 

In order to have $\mathcal{V}_\phi(K)$ well-defined for each convex body $K$ with the origin $o$ in its interior and to solve the general dual Orlicz-Minkowski problem, some conditions on $\phi$ are required and these conditions are described in Section 2 (i.e., conditions C1) and C2) following Definition 2.1. Some special functions satisfying conditions C1) and C2) are discussed. The convergence of $\mathcal{V}_\phi$ is summarized in Lemma 2.1, and it will be used to establish the existence of the solutions to the general dual Orlicz-Minkowski problem.

To formulate the general dual Orlicz-Minkowski problem, the general dual $(L_\phi)$ Orlicz curvature measure is required. For $K \subseteq \mathbb{R}^n$ being a convex body with the origin $o$ in its interior and a subset $\eta \subseteq S^{n-1}$, denote by $\rho_K$ the radial function of $K$ and $\alpha_K^*(\eta) \subseteq S^{n-1}$ the reverse radial Gauss image of $\eta$, respectively. Define $\tilde{C}_\phi, \nu(K, \cdot)$, the general dual $(L_\phi)$ Orlicz curvature measure of $K$ with $\nu : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$ a continuous function satisfying condition C1) (following Definition 2.1 in Section 2), by

$$\tilde{C}_\phi, \nu(K, \eta) = \int_{\alpha_K^*(\eta)} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du$$

for any Borel set $\eta \subseteq S^{n-1}$, where $du$ is the spherical measure of $S^{n-1}$. The properties for the general dual $(L_\phi)$ Orlicz curvature measure are provided in Section 3. In particular, convenient formulas to calculate integrals with respect to $\tilde{C}_\phi, \nu(K, \cdot)$ are given in Lemma 3.1, and the weak convergence of the general dual $(L_\phi)$ Orlicz curvature measure is summarized in Proposition 3.1. These properties are crucial in solving the general dual Orlicz-Minkowski problem: Given a nonzero finite Borel measure $\mu$ defined on $S^{n-1}$ and a continuous function $\phi : \mathbb{R}^n \setminus \{o\} \to (0, \infty)$, can one find a constant $\tau > 0$ and a convex body $K$ (ideally, containing $o$ in its interior), such that,

$$\mu = \tau \tilde{C}_\phi, \nu(K, \cdot)?$$

A basic method to solve various Minkowski problems is the method of Lagrange multipliers, and hence a variational formula related to the general dual $(L_\phi)$ Orlicz curvature measure is essential. Such a variational formula is proved in Theorem 4.1. In fact, this variational formula motivates our definition of $\tilde{C}_\phi, \nu(K, \cdot)$ and provides a variational interpretation for the general dual $(L_\phi)$ Orlicz curvature measure. With the help of the method of Lagrange multipliers and the established variational formula in Theorem 4.1, a solution to the general dual Orlicz-Minkowski problem is provided in Theorem 5.1. Conditions C1) and C2) are listed in Section 2 following Definition 2.1.

**Theorem 5.1.** Let $\mu$ be a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29) and let $\phi$ be a function satisfy conditions C1) and C2). Then there exists a convex body $K$ containing the origin
o in its interior, such that,
\[
\mu = \frac{\bar{C}_\phi(K, \cdot)}{C_\phi(K, S^{n-1})},
\]
where \( |\mu| = \mu(S^{n-1}) \) is the total \( \mu \)-mass of \( S^{n-1} \).

It seems, in general, not possible to obtain the uniqueness of the solutions to the general dual Orlicz-Minkowski problem. However, in some special cases, say \( \phi \) having certain homogeneity, we are able to prove the uniqueness. Moreover, the solution to the general dual Orlicz-Minkowski problem is proved to be a polytope, when \( \mu \) is a discrete measure. The details will be provided in Section 6.

2 The general dual Orlicz quermassintegral

Our setting in this article is the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) equipped with the standard Euclidean norm \( |\cdot| \) induced by the inner product \( \langle \cdot, \cdot \rangle \). The standard notations \( o, B_2^n \) and \( S^{n-1} \) denote the origin, the unit Euclidean ball and the unit sphere in \( \mathbb{R}^n \), respectively. For a set \( K \subseteq \mathbb{R}^n \), the boundary of \( K \) and the interior of \( K \) are denoted by \( \partial K \) and \( \text{int } K \), respectively. The unit vector \( \bar{x} = x/|x| \in S^{n-1} \) refers to the direction vector of \( x \in \mathbb{R}^n \setminus \{o\} \). By \( \text{conv}(A) \), we mean the convex hull of \( A \subseteq \mathbb{R}^n \); namely, \( \text{conv}(A) \) is the smallest convex set containing \( A \).

We consider the measure \( \nu_\phi \), whose density function with respect to the Lebesgue measure \( dx \) is a continuous function \( \phi : \mathbb{R}^n \setminus \{o\} \rightarrow (0, \infty) \).

**Definition 2.1.** For a measurable subset \( E \subseteq \mathbb{R}^n \) with \( o \in \text{int } E \), define \( \nu_\phi(E) \) by
\[
\nu_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x) \, dx.
\]

Clearly, \( \nu_\phi(\cdot) \) is monotone decreasing, that is, if \( E \subseteq F \) with \( o \in \text{int } E \), then \( \mathbb{R}^n \setminus E \supseteq \mathbb{R}^n \setminus F \) and hence
\[
\nu_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x) \, dx \geq \int_{\mathbb{R}^n \setminus F} \phi(x) \, dx = \nu_\phi(F)
\]
due to the positivity of \( \phi \).

When \( E \) is a star-shaped set in \( \mathbb{R}^n \), \( \nu_\phi(E) \) can be reformulated through the radial function of \( E \) and the spherical measure \( du \) on \( S^{n-1} \). Hereafter, \( E \subseteq \mathbb{R}^n \) is said to be a star-shaped set with respect to \( o \), if \( o \in E \) and the line segment \( [o, x] \subseteq E \) for all \( x \in E \). For a star-shaped set \( E \) with respect to \( o \), one can define its radial function \( \rho_E : S^{n-1} \rightarrow [0, \infty] \) by
\[
\rho_E(u) = \sup \{ \lambda > 0 : \lambda u \in E \}
\]
for each \( u \in S^{n-1} \).

Denote by \( \mathcal{S} \) the set of all star-shaped sets in \( \mathbb{R}^n \) with respect to \( o \) whose radial functions are measurable. In fact, we will be working on the set \( \mathcal{K}_o^n \subseteq \mathcal{S} \), the collection of all convex bodies in \( \mathbb{R}^n \) containing the origin \( o \) in their interiors. That is, if \( K \in \mathcal{K}_o^n \), then \( K \subseteq \mathbb{R}^n \) is a convex compact set with the origin \( o \) in its interior. The support function of \( K \in \mathcal{K}_o^n \), \( h_K(u) : S^{n-1} \rightarrow \mathbb{R} \), is defined by
\[
h_K(u) = \sup_{x \in K} \langle x, u \rangle \quad \text{for each } u \in S^{n-1}.
\]

When \( E \in \mathcal{S} \), \( \nu_\phi(E) \) can be calculated by
\[
\nu_\phi(E) = \int_{\mathbb{R}^n \setminus E} \phi(x) \, dx = \int_{S^{n-1}} \left( \int_{\rho_E(u)}^\infty \phi(ru) r^{n-1} \, dr \right) \, du.
\]
For convenience, let
\[ \Phi(t, u) = \int_t^\infty \phi(ru)r^{n-1} \, dr \]
and hence formula (2.3) can be rewritten as
\[ V_\phi(E) = \int_{S^{n-1}} \Phi(\rho_E(u), u) \, du. \] (2.4)

For each \( K \in \mathcal{K}_o^n \), one can define \( K^* \), the polar body of \( K \), by
\[ K^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}. \]
Clearly \( K^* \in \mathcal{K}_o^n \). Moreover, the bipolar theorem asserts that \( (K^*)^* = K \) (see e.g., [33]) and then
\[ \rho_K(u)h_{K^*}(u) = h_K(u)\rho_{K^*}(u) = 1. \]
Hence, for each \( K \in \mathcal{K}_o^n \), one gets, by formula (2.4)
\[ V_\phi(K) = \int_{S^{n-1}} \Phi(\rho_K(u), u) \, du = \int_{S^{n-1}} \Phi(h_{K^*}(u)^{-1}, u) \, du. \] (2.5)

In later context, for each \( K \in \mathcal{K}_o^n \), \( V_\phi(K) \) will be called the general dual \((L_\phi)\) Orlicz quermassintegral of \( K \).

Now we list the basic conditions for function \( \phi \):

C1) \( \phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \) is a continuous function, such that, for any fixed \( t > 0 \), the function
\[ \Phi(t, u) = \int_t^\infty \phi(ru)r^{n-1} \, dr \]
is positive and continuous on \( S^{n-1} \);

C2) for any fixed \( u_0 \in S^{n-1} \) and any fixed positive constant \( b_0 \in (0, 1) \), one has
\[ \lim_{a \to 0^+} V_\phi(\mathbb{R}^n \setminus \mathcal{C}(u_0, a, b_0)) = \infty, \]
where \( \mathcal{C}(u_0, a, b_0) \) is defined by
\[ \mathcal{C}(u_0, a, b_0) = \{ x \in \mathbb{R}^n : \langle \bar{x}, u_0 \rangle \geq b_0 \text{ and } |x| \geq a \}. \]

In fact, condition C1) guarantees that \( V_\phi(K) < \infty \) for each \( K \in \mathcal{K}_o^n \). To see this, as \( o \in \text{int}K \), there exists a constant \( r_0 > 0 \) such that \( r_0B^n_2 \subseteq K \). By formula (2.5) and the fact that \( V_\phi(\cdot) \) is monotone decreasing, one has,
\[ V_\phi(K) \leq V_\phi(r_0B^n_2) = \int_{S^{n-1}} \Phi(r_0, u) \, du < \infty. \]
Condition C2) is for the solution of the general dual Orlicz-Minkowski problem.

A typical function satisfying conditions C1) and C2) is a continuous function \( \phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \) such that
\[ \sup_{|x| > r_1} \phi(x)|x|^{n-\alpha_1-1} \leq C_1 \quad \text{and} \quad \inf_{|x| < r_1} \phi(x)|x|^{n-\alpha_2-1} \geq C_2 \] (2.6)
hold for some constants $0 < r_1 < \infty$, $C_1 < \infty$, $C_2 > 0$ and $-\infty < \alpha_1, \alpha_2 < -1$. In particular, if

$$\lim_{|x| \to \infty} \phi(x)|x|^{n-\alpha_1-1} = C_1 \quad \text{and} \quad \lim_{|x| \to 0} \phi(x)|x|^{n-\alpha_2-1} = C_2$$

for some constants $0 < C_1, C_2 < \infty$ and $-\infty < \alpha_1, \alpha_2 < -1$, then such $\phi$ satisfies (2.6) (for different constants). Now let us check that a continuous function $\phi$ satisfying (2.6) must also satisfy conditions C1) and C2). To this end, let $t > 0$ and $u \in S^{n-1}$ be fixed. It is obvious to have $\Phi(t, u) > 0$. Moreover

$$\Phi(t, u) = \int_t^{r_1} \phi(ru)r^{n-1} \, dr + \int_{r_1}^{\infty} \phi(ru)r^{n-1} \, dr$$

$$\leq \left| \int_t^{r_1} \phi(ru)r^{n-1} \, dr + \int_{r_1}^{\infty} \phi(ru)r^{n-1} \, dr \right|$$

$$\leq \left| \int_t^{r_1} \phi(ru)r^{n-1} \, dr + C_1 \int_{r_1}^{\infty} r^{\alpha_1} \, dr \right|$$

$$= \left| \int_t^{r_1} \phi(ru)r^{n-1} \, dr - \frac{C_1}{\alpha_1 + 1} r_1^{\alpha_1 + 1} \right|.$$

Thus $\Phi(t, u) < \infty$ due to the continuity of $\phi$, and $\Phi(t, u)$ is well defined. Now we claim that $\Phi(t, \cdot)$ is continuous on $S^{n-1}$. For fixed $t$ and for an arbitrary sequence $u_i \to u$ with $u_i, u \in S^{n-1}$, one has, for all $r \geq t$, $\phi(ru_i)r^{n-1} \to \phi(ru)r^{n-1}$ and

$$\phi(ru_i)r^{n-1} \leq C_1 r^{\alpha_1} + M$$

for all $i \geq 1$, where, due to the continuity of $\phi$,

$$M = \max \left\{ \phi(x)|x|^{n-1} : |x| \text{ is between } t \text{ and } r_1 \right\} < \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{i \to \infty} \Phi(t, u_i) = \lim_{i \to \infty} \int_t^{r_1} \phi(ru_i)r^{n-1} \, dr = \int_t^{r_1} \lim_{i \to \infty} \phi(ru_i)r^{n-1} \, dr = \int_t^{r_1} \phi(ru)r^{n-1} \, dr = \Phi(t, u).$$

Hence $\Phi(t, u)$ is continuous on $S^{n-1}$ and C1) is verified. Now let us verify C2) as follows: for any $b_0 \in (0, 1)$,

$$\lim_{a \to 0^+} \mathcal{V}_\phi(\mathbb{R}^n \setminus \mathcal{G}(u_0, a, b_0)) = \lim_{a \to 0^+} \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} \int_a^{\infty} \phi(ru)r^{n-1} \, dr \, du$$

$$\geq \limsup_{a \to 0^+} \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} \int_a^{r_1} \phi(ru)r^{n-1} \, dr \, du$$

$$\geq C_2 \cdot \limsup_{a \to 0^+} \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} \int_a^{r_1} r^{\alpha_2} \, dr \, du$$

$$= C_2 \cdot \left( \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} du \right) \cdot \limsup_{a \to 0^+} \frac{r_1^{1+\alpha_2} - a^{1+\alpha_2}}{1 + \alpha_2}$$

$$= \infty,$$

where we have used (2.3), (2.6) and $\alpha_2 < -1$.

Now let us provide several special cases of functions satisfying conditions C1) and C2).
Case 1: \( \phi(x) = |x| \) for all \( x \in \mathbb{R}^n \setminus \{0\} \) with \( \psi : (0, \infty) \to (0, \infty) \) a continuous function. In this case,
\[
\Phi(t, u) = \int_t^\infty \phi(ru)r^{n-1} dr = \int_t^\infty \psi(r)r^{n-1} dr := \frac{1}{n} \hat{\phi}(t). \tag{2.7}
\]
Equivalently,
\[
\hat{\psi}(t) = -\hat{\phi}'(t)t^{1-n}/n. \tag{2.8}
\]
By formula (2.5), one has, for \( K \in \mathcal{K}_{\phi}^n \),
\[
\mathcal{V}_{\phi}(K) = \int_{S^{n-1}} \Phi(\rho_K(u), u) du = \frac{1}{n} \int_{S^{n-1}} \hat{\phi}(\rho_K(u)) du = V_{\phi}(K),
\]
where \( V_{\phi}(\cdot) \) is the dual \( (L_{\phi}) \) Orlicz quermassintegral proposed in [45], namely
\[
V_{\phi}(K) = \frac{1}{n} \int_{S^{n-1}} \varphi(\rho_K(u))du.
\]
In [45], the dual Orlicz-Minkowski problem is solved under the following conditions:
A1) \( \hat{\phi} : (0, \infty) \to (0, \infty) \) is a strictly decreasing continuous function with
\[
\lim_{t \to 0^+} \hat{\phi}(t) = \infty \quad \text{and} \quad \lim_{t \to \infty} \hat{\phi}(t) = 0;
\]
A2) \( \hat{\phi}' \), the derivative of \( \hat{\phi} \), exists and is strictly negative on \((0, \infty)\);
A3) \( \hat{\phi}(t) = -\hat{\phi}'(t)t : (0, \infty) \to (0, \infty) \) is continuous; hence
\[
\hat{\phi}(t) = \int_t^\infty \frac{\hat{\phi}(s)}{s} ds.
\]
In Case 1, it is obvious that \( \hat{\phi}(t) = n\hat{\psi}(t)t^n \). Now let us check that if \( \hat{\phi} \) and its companion function \( \hat{\varphi} \) satisfy conditions A1)-A3), then \( \phi(x) = |x|/x \) with \( \psi \) given by (2.8) satisfies conditions C1) and C2). In fact, condition C1) can be easily checked by (2.7) and A1). Let us verify condition C2) as follows: for any \( b_0 \in (0, 1) \),
\[
\lim_{a \to 0^+} \mathcal{V}_{\phi}(a, u_0, b_0) = \lim_{a \to 0^+} \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} \int_a^\infty \phi(ru)r^{n-1} dr du
\]
\[
= \frac{1}{n} \cdot \lim_{a \to 0^+} \hat{\phi}(a) \cdot \left( \int_{\{u \in S^{n-1} : (u, u_0) \geq b_0\}} du \right)
\]
\[
= \infty,
\]
where we have used (2.3), (2.7), and condition A1).

Case 2: \( \phi(x) = \psi(|x|) \phi_2(\bar{x}) \) where \( \bar{x} = x/|x| \), \( \psi : (0, \infty) \to (0, \infty) \) is a continuous function on \((0, \infty)\), and \( \phi_2 : S^{n-1} \to (0, \infty) \) is a continuous function on \( S^{n-1} \). In this case, the general dual \( (L_{\phi}) \) Orlicz quermassintegral of \( K \in \mathcal{K}_{\phi}^n \) has the following form:
\[
\mathcal{V}_{\phi}(K) = \int_{S^{n-1}} \int_{\rho_K(u)}^{\infty} \phi(ru)r^{n-1} dr du
\]
\[
= \int_{S^{n-1}} \left( \int_{\rho_K(u)}^{\infty} \psi(r)r^{n-1} dr \right) \phi_2(u) du
\]
\[
= \frac{1}{n} \int_{S^{n-1}} \tilde{\phi}(\rho_K(u))\phi_2(u) du, \tag{2.9}
\]
where \( \hat{\phi} \) is given by (2.7). Again, if \( \hat{\phi} \) and its companion function \( \hat{\psi} \) satisfy conditions A1)-A3), then \( \hat{\phi}(x) = \psi(|x|)\hat{\phi}_2(\bar{x}) \) with \( \psi \) give by (2.8) satisfies conditions C1) and C2); this follows from an argument similar to the one as in Case 1. A typical example in this case is

\[
\phi(x) = \|x\|^{q-n} = |x|^{q-n} \cdot \|\bar{x}\|^{q-n}
\]

where \( q < 0 \) is a constant and \( \| \cdot \| : \mathbb{R}^n \to [0, \infty) \) is any norm on \( \mathbb{R}^n \). (Note that \( \hat{\phi}_2(\bar{x}) = \|\bar{x}\|^{q-n} \) is always positive, due to the equivalence between the two norms \( \| \cdot \| \) and \( | \cdot | \).) Indeed, when \( \phi(x) = \|x\|^{q-n} = |x|^{q-n} \cdot \|\bar{x}\|^{q-n} \), then \( \psi(|x|) = |x|^{q-n} \). Hence

\[
\hat{\phi}(t) = n \int_t^\infty \psi(r)r^{n-1}dr = n \int_t^\infty r^{q-1}dr = -\frac{n}{q} \cdot t^q
\]

and \( \hat{\phi} = nt^q \), which satisfy conditions A1)-A3).

A sequence of convex bodies \( \{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_o^n \) converging to a convex body \( K \in \mathcal{K}_o^n \) in the sense of Hausdorff metric means that

\[
\|h_{K_i} - h_K\|_\infty = \sup_{u \in S^{n-1}} |h_{K_i}(u) - h_K(u)| \to 0 \quad \text{as} \quad i \to \infty. \tag{2.10}
\]

Indeed, this is equivalent to

\[
\|\rho_{K_i} - \rho_K\|_\infty = \sup_{u \in S^{n-1}} |\rho_{K_i}(u) - \rho_K(u)| \to 0 \quad \text{as} \quad i \to \infty. \tag{2.11}
\]

We will need the following convergence result regarding \( \mathcal{V}_\phi(\cdot) \).

**Lemma 2.1.** Assume that \( \phi \) is a function satisfying condition C1). If the sequence \( \{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_o^n \) converges to \( K \in \mathcal{K}_o^n \) in the sense of Hausdorff metric, then

\[
\lim_{i \to \infty} \mathcal{V}_\phi(K_i) = \mathcal{V}_\phi(K).
\]

**Proof.** Let \( \phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \) be a continuous function satisfying C1). It can be checked that, for any fixed \( u \in S^{n-1} \) and for any fixed constant \( t_0 > 0 \),

\[
\lim_{t \to t_0} \Phi(t,u) = \Phi(t_0,u). \tag{2.12}
\]

In fact, for any fixed \( u \in S^{n-1} \), \( \Phi(t,u) \) is a decreasing function on \( t \in (0, \infty) \). Let \( t \to t_0 \), and without loss of generality assume that \( t > t_0/2 \). By condition C1) and the fact that \( \Phi(t,u) \) is decreasing on \( t \), one has,

\[
\Phi(t,u) = \int_t^\infty \phi(ru)r^{n-1}dr \leq \int_{t_0/2}^\infty \phi(ru)r^{n-1}dr = \Phi(t_0/2,u) < \infty.
\]

It follows from the dominated convergence theorem that

\[
\lim_{t \to t_0} \Phi(t,u) = \lim_{t \to t_0} \int_t^\infty \phi(ru)r^{n-1}dr = \int_{t_0}^\infty \phi(ru)r^{n-1}dr = \Phi(t_0,u).
\]

Let \( \{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_o^n \) be a sequence of convex bodies converging to \( K \in \mathcal{K}_o^n \) in the Hausdorff metric. Based on (2.11), \( \rho_{K_i} \) converges to \( \rho_K \) uniformly on \( S^{n-1} \). Moreover, as \( K \in \mathcal{K}_o^n \), one can find a constant \( R_1 > 0 \), such that, for all \( u \in S^{n-1} \) and for all \( i = 1, 2, \ldots \),

\[
R_1 \leq \rho_{K_i}(u) \quad \text{and} \quad R_1 \leq \rho_K(u).
\]
Together with the fact that \( \Phi(t,u) \) is a decreasing function on \( t \in (0, \infty) \), one has

\[
\Phi(\rho_{K_i}(u), u) \leq \Phi(R_1, u) \quad \text{and} \quad \Phi(\rho_{K}(u), u) \leq \Phi(R_1, u) \quad \text{for all} \quad u \in S^{n-1}.
\]

By condition C1), \( \Phi(R_1, u) \) is positive and continuous on \( S^{n-1} \). Hence,

\[
\int_{S^{n-1}} \Phi(R_1, u) \, du < \infty.
\]

It follows from (2.5), (2.12) and the dominated convergence theorem that

\[
\lim_{i \to \infty} V_\phi(K_i) = \lim_{i \to \infty} \int_{S^{n-1}} \Phi(\rho_{K_i}(u), u) \, du = \int_{S^{n-1}} \Phi(\rho_{K}(u), u) \, du = V_\phi(K).
\]

This concludes the proof of Lemma 2.1. \( \square \)

### 3 The general dual Orlicz curvature measure

For \( K \in \mathcal{K}_n \), the supporting hyperplane of \( K \) at the direction \( u \in S^{n-1} \), denoted by \( H(K, u) \), is given by

\[
H(K, u) = \{ x \in \mathbb{R}^n : \langle x, u \rangle = h_K(u) \}.
\]

Denoted by \( \alpha^*_K(\eta) \) the reverse radial Gauss image of \( \eta \subseteq S^{n-1} \), that is,

\[
\alpha^*_K(\eta) = \{ \bar{x} = x/|x| : x \in \partial K \cap H(K, u) \quad \text{for some} \quad u \in \eta \}.
\]

For convenience, let

\[
\Psi_K(u) = \phi(\rho_K(u))|\rho_K(u)|^n \quad \text{for} \quad u \in S^{n-1}.
\]

In fact, for any \( x \in \partial K \), one has \( \Psi_K(\bar{x}) = \phi(x)|x|^n \).

We are ready to give the definition of the general dual Orlicz curvature measure.

**Definition 3.1.** For any \( K \in \mathcal{K}_n \) and for any function \( \phi \) satisfying condition C1), the general dual \( (L_\phi) \) Orlicz curvature measure of \( K \), denoted by \( \tilde{C}_{\phi, \gamma}(K, \cdot) \), is given by

\[
\tilde{C}_{\phi, \gamma}(K, \eta) = \int_{\alpha^*_K(\eta)} \Psi_K(u) \, du
\]

for any Borel set \( \eta \subseteq S^{n-1} \).

Indeed, for each \( K \in \mathcal{K}_n \), \( \tilde{C}_{\phi, \gamma}(K, \cdot) \) does define a Borel measure on \( S^{n-1} \). To this end, we only need to show that \( C_{\phi, \gamma}(K, \cdot) \) satisfies the countable additivity, as \( \tilde{C}_{\phi, \gamma}(K, \emptyset) = 0 \) holds trivially. That is, we need to prove

\[
\tilde{C}_{\phi, \gamma}(K, \bigcup_{i=1}^{\infty} \eta_i) = \sum_{i=1}^{\infty} \tilde{C}_{\phi, \gamma}(K, \eta_i)
\]
for any sequence of pairwise disjoint Borel sets $\eta_1, \eta_2, \cdots \subseteq S^{n-1}$. Recall that $\boldsymbol{\alpha}_K^*\big(\bigcup_{i=1}^\infty \eta_i\big) = \bigcup_{i=1}^\infty \boldsymbol{\alpha}_K^*(\eta_i)$ by [16, Lemma 2.3] and

$$\boldsymbol{\alpha}_K^*(\eta_i) = x_K(\eta_i) = \{\bar{x} : x \in x_K(\eta_i)\} \subseteq S^{n-1}$$

is spherical measurable for each $i \geq 1$ by [16, Lemma 2.1], where $x_K(\eta_i)$ is the reverse spherical image of $\eta_i \subseteq S^{n-1}$ given by

$$x_K(\eta_i) = \{x \in \partial K : x \in H(K,u) \text{ for some } u \in \eta_i\} \subseteq \partial K.$$ 

Therefore,

$$\tilde{C}_{\phi, \nu}(K, \bigcup_{i=1}^\infty \eta_i) = \int_{\bigcup_{i=1}^\infty \boldsymbol{\alpha}_K^*(\eta_i) \setminus \omega_K} \Psi_K(u) du = \int_{\bigcup_{i=1}^\infty \boldsymbol{\alpha}_K^*(\eta_i)} \Psi_K(u) du. \quad (3.13)$$

The countable additivity will follow immediately if $\bigcup_{i=1}^\infty \boldsymbol{\alpha}_K^*(\eta_i)$ is pairwise disjoint. However, by [16, Lemma 2.4], one gets that $\{\boldsymbol{\alpha}_K^*(\eta_j) \setminus \omega_K\}_{j=1}^\infty$ is pairwise disjoint, where

$$\omega_K = \{v \in S^{n-1} : \boldsymbol{\alpha}_K(v) \text{ has more than one element}\}$$

with $\boldsymbol{\alpha}_K(\omega)$, the radial Gauss image of $\omega \subseteq S^{n-1}$, given by

$$\boldsymbol{\alpha}_K(\omega) = \nu_K(\{\rho_K(u)u \in \partial K : u \in \omega\}) \subseteq S^{n-1}$$

and with $\nu_K(\sigma)$, the spherical image of $\sigma \subseteq \partial K$, given by

$$\nu_K(\sigma) = \{u \in S^{n-1} : x \in H(K,u) \text{ for some } x \in \sigma\} \subseteq S^{n-1}.$$ 

Fortunately, the spherical measure of $\omega_K$ turns out to be zero [16, p.339-340] and hence, by (3.13),

$$\tilde{C}_{\phi, \nu}(K, \bigcup_{i=1}^\infty \eta_i) = \int_{\bigcup_{i=1}^\infty \boldsymbol{\alpha}_K^*(\eta_i) \setminus \omega_K} \Psi_K(u) du$$

$$= \sum_{i=1}^\infty \int_{\boldsymbol{\alpha}_K^*(\eta_i) \setminus \omega_K} \Psi_K(u) du$$

$$= \sum_{i=1}^\infty \int_{\boldsymbol{\alpha}_K^*(\eta_i)} \Psi_K(u) du$$

$$= \sum_{i=1}^\infty \tilde{C}_{\phi, \nu}(K, \eta_i).$$

This concludes that $\tilde{C}_{\phi, \nu}$ is a Borel measure.

Note that $\nu_K(x)$ for $x \in \partial K$, $x_K(u)$ and $\boldsymbol{\alpha}_K(u)$ for $u \in S^{n-1}$ may contain more than one element. When they are singleton sets, $\nu_K(x)$ for $x \in \partial K$, $x_K(u)$ and $\boldsymbol{\alpha}_K(u)$ for $u \in S^{n-1}$ are used, respectively, instead of $\nu_K(x)$, $x_K(u)$ and $\boldsymbol{\alpha}_K(u)$. For any $K \in \mathcal{K}_n$, let $\sigma_K \subseteq \partial K$ be the set given by

$$\sigma_K = \{x \in \partial K : \nu_K(x) \text{ has more than one element}\}.$$ 

Denote by $\partial'K = \partial K \setminus \sigma_K$ the set of points on $\partial K$ that have a unique outer unit normal vector and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. According to [33, p.84], $\mathcal{H}^{n-1}(\sigma_K) = 0$ and hence $\mathcal{H}^{n-1}(\partial'K) = \mathcal{H}^{n-1}(\partial K)$.

The following lemma provides convenient formulas to calculate integrals with respect to the measure $\tilde{C}_{\phi, \nu}(K, \cdot)$. Recall that $\Psi_K(u) = \phi(\rho_K(u)u)[\rho_K(u)]^n$ for all $u \in S^{n-1}$. 
Lemma 3.1. Let \( \phi \) be a function satisfying condition \( C1 \). For each \( K \in \mathcal{K}^n \), the following formulas
\[
\int_{S^{n-1}} g(v) d\tilde{C}_{\phi,v}(K,v) = \int_{S^{n-1}} g(\alpha_K(u)) \Psi_K(u) du \tag{3.14}
\]
\[
= \int_{\partial K} \langle x, \nu_K(x) \rangle g(\nu_K(x)) \phi(x) d\mathcal{H}^{n-1}(x) \tag{3.15}
\]
hold for any bounded Borel function \( g : S^{n-1} \rightarrow \mathbb{R} \).

Proof. First, we prove (3.14). Let \( \gamma(v) = \sum_{i=1}^m a_i 1_{\eta_i}(v) \) for any \( v \in S^{n-1} \) be an arbitrary simple function, where \( \eta_i \subseteq S^{n-1} \) are Borel sets and \( 1_A \) denotes the indicator function of the set \( A \). By [16, (2.21)], one has \( u \in \alpha^*_K(\eta) \) if and only if \( \alpha_K(u) \in \eta \), and this further yields that
\[
\int_{S^{n-1}} \gamma(\alpha_K(u)) \Psi_K(u) du = \int_{S^{n-1}} \sum_{i=1}^m a_i 1_{\eta_i}(\alpha_K(u)) \Psi_K(u) du
\]
\[
= \int_{S^{n-1}} \sum_{i=1}^m a_i 1_{\alpha^*_K(\eta_i)}(u) \Psi_K(u) du
\]
\[
= \sum_{i=1}^m a_i \int_{S^{n-1}} 1_{\alpha^*_K(\eta_i)}(u) \Psi_K(u) du.
\]
Together with Definition 3.1, one has
\[
\int_{S^{n-1}} \gamma(\alpha_K(u)) \Psi_K(u) du = \sum_{i=1}^m a_i \int_{S^{n-1}} 1_{\alpha^*_K(\eta_i)}(u) \Psi_K(u) du
\]
\[
= \sum_{i=1}^m a_i C_{\phi,v}(K,\eta_i)
\]
\[
= \sum_{i=1}^m a_i \int_{S^{n-1}} 1_{\eta_i}(v) d\tilde{C}_{\phi,v}(K,v)
\]
\[
= \int_{S^{n-1}} \gamma(v) d\tilde{C}_{\phi,v}(K,v).
\]
That is, (3.14) holds true for simple functions. Following from a standard limit approach by simple functions, one can prove formula (3.14) for general bounded Borel functions \( g : S^{n-1} \rightarrow \mathbb{R} \).

Next we prove (3.15). According to [16, (2.31)], for each bounded integrable function \( f : S^{n-1} \rightarrow \mathbb{R} \), one has
\[
\int_{S^{n-1}} f(u) \phi(\rho_K(u)u) du = \int_{\partial K} \langle x, \nu_K(x) \rangle f(\bar{x}) \frac{\phi(\rho_K(\bar{x})\bar{x})}{\rho_K(\bar{x})} d\mathcal{H}^{n-1}(x)
\]
\[
= \int_{\partial K} \langle x, \nu_K(x) \rangle f(\bar{x}) \frac{\phi(x)}{|x|^n} d\mathcal{H}^{n-1}(x),
\]
where \( \bar{x} = x/|x|, \rho_K(\bar{x})\bar{x} = x, \) and \( \rho_K(\bar{x}) = |x| \). Together with (3.14) and the fact that \( f = g \circ \alpha_K \) is bounded integrable on \( S^{n-1} \), one has
\[
\int_{S^{n-1}} g(v) d\tilde{C}_{\phi,v}(K,v) = \int_{S^{n-1}} g(\alpha_K(u)) \Psi_K(u) du
\]
\[
= \int_{\partial K} \langle x, \nu_K(x) \rangle g(\nu_K(x)) \phi(x) d\mathcal{H}^{n-1}(x).
\]
Hence, (3.15) holds true.
The weak convergence of the general dual Orlicz curvature measure is proved in the following proposition.

**Proposition 3.1.** Let \( \phi \) be a function satisfying condition C1. If the sequence \( \{K_i\}_{i=1}^{\infty} \subseteq X_o^n \) converges to \( K \in X_o^n \) in the Hausdorff metric, then \( \bar{C}_{\phi,\gamma}(K_i,\cdot) \) converges to \( \bar{C}_{\phi,\gamma}(K,\cdot) \) weakly.

**Proof.** As \( \{K_i\}_{i=1}^{\infty} \subseteq X_o^n \) converges to \( K \in X_o^n \), then \( \rho_{K_i} \) converges to \( \rho_K \) uniformly (see (2.11)) and hence one can find constants \( R_1, R_2 > 0 \), such that, for all \( u \in S^{n-1} \) and for all \( i \geq 1 \),

\[
R_1 \leq \rho_{K_i}(u) \leq R_2 \quad \text{and} \quad R_1 \leq \rho_K(u) \leq R_2.
\]

For any fixed \( u \in S^{n-1} \) and for any function \( \phi \) satisfying condition C1), it can be checked that

\[
\Psi_{K_i}(u) = \phi(\rho_{K_i}(u))|\rho_{K_i}(u)| \rightarrow \phi(\rho_K(u))|\rho_K(u)| = \Psi_K(u) \quad \text{uniformly on} \quad S^{n-1}. \quad (3.16)
\]

Note that \( \alpha_{K_i} \rightarrow \alpha_K \) almost everywhere on \( S^{n-1} \) (see [16, Lemma 2.2]). For any continuous function \( g : S^{n-1} \rightarrow \mathbb{R} \), by (3.16), there exists a constant \( M > 0 \), such that, for all \( u \in S^{n-1} \) and for all \( i = 1, 2, \ldots \),

\[
|g(\alpha_{K_i}(u))\Psi_{K_i}(u)| \leq M \quad \text{and} \quad |g(\alpha_K(u))\Psi_K(u)| \leq M.
\]

It follows from the dominated convergence theorem that

\[
\lim_{i \rightarrow \infty} \int_{S^{n-1}} g(\alpha_{K_i}(u))\Psi_{K_i}(u) \, du = \int_{S^{n-1}} \lim_{i \rightarrow \infty} g(\alpha_{K_i}(u))\Psi_{K_i}(u) \, du = \int_{S^{n-1}} g(\alpha_K(u))\Psi_K(u) \, du.
\]

Together with (3.14), then

\[
\lim_{i \rightarrow \infty} \int_{S^{n-1}} g(v) \, d\bar{C}_{\phi,\gamma}(K_i,v) = \int_{S^{n-1}} g(\alpha_K(u))\Psi_K(u) \, du = \int_{S^{n-1}} g(v) \, d\bar{C}_{\phi,\gamma}(K,v),
\]

hold for any continuous function \( g : S^{n-1} \rightarrow \mathbb{R} \). In conclusion, \( \bar{C}_{\phi,\gamma}(K_i,\cdot) \) converges weakly to \( \bar{C}_{\phi,\gamma}(K,\cdot) \) as desired. \( \Box \)

Denote by \( S_K(\cdot) \) the surface area measure of \( K \in X_o^n \), namely, for any Borel set \( \eta \subseteq S^{n-1} \),

\[
S_K(\eta) = \mathcal{H}^{n-1}(\nu_K^{-1}(\eta)), \quad (3.17)
\]

where \( \nu_K^{-1}(\eta) \) is the reverse spherical image of \( \eta \), i.e., \( \nu_K^{-1}(\eta) = \{x \in \partial K : \nu_K(x) \in \eta\} \).

**Proposition 3.2.** Let \( K \in X_o^n \) and \( \phi \) be a function satisfying condition C1). Then, \( \bar{C}_{\phi,\gamma}(K,\cdot) \) is absolutely continuous with respect to the surface area measure \( S_K(\cdot) \).

**Proof.** Let \( \eta \subseteq S^{n-1} \) be a Borel set and \( g = 1_{\eta} \) in (3.15). Then

\[
\bar{C}_{\phi,\gamma}(K,\eta) = \int_{\nu_K^{-1}(\eta)} \langle x, \nu_K(x) \rangle \phi(x) \, d\mathcal{H}^{n-1}(x).
\]
Since $K \in \mathcal{K}_0^n$ and $\phi$ is a function satisfying condition C1), there exists a constant $T < \infty$, such that, $\langle x, \nu_K(x) \rangle \phi(x) \leq T$ for all $x \in \partial K$. Then

$$
\int_{\nu_K^{-1}(\eta)} \langle x, \nu_K(x) \rangle \phi(x) d\mathcal{H}^{n-1}(x) \leq T \int_{\nu_K^{-1}(\eta)} d\mathcal{H}^{n-1}(x).
$$

If $\eta \subseteq S^{n-1}$ is a Borel set such that $S_K(\eta) = 0$, then $\mathcal{H}^{n-1}(\nu_K^{-1}(\eta)) = 0$ and thus

$$
\tilde{C}_{\phi, \gamma}(K, \eta) \leq T \cdot \mathcal{H}^{n-1}(\nu_K^{-1}(\eta)) = 0.
$$

As a result, $\tilde{C}_{\phi, \gamma}(K, \cdot)$ is absolutely continuous with respect to $S_K(\cdot)$. \hfill \Box

Let us discuss the measure $\tilde{C}_{\phi, \gamma}(K, \cdot)$ for $K \in \mathcal{K}_0^n$ under Case 1 and Case 2 given in Section 2. In Case 1, i.e., $\phi(x) = \psi(|x|)$, it follows from Definition 3.1 that for any Borel set $\eta \subseteq S^{n-1},$

$$
\tilde{C}_{\phi, \gamma}(K, \eta) = \int_{\alpha_K^*(\eta)} \phi(\rho_K(u)[\rho_K(u)]^n du
$$

where $\tilde{\phi}(t) = n\psi(t)t^n$. Recall that for $K \in \mathcal{K}_0^n$ and $\varphi : (0, \infty) \to (0, \infty)$ a continuous function, the dual $L_\varphi$ Orlicz curvature measure of $K$, denoted by $\tilde{C}_\varphi(K, \cdot)$, is defined in [45] as follows: for each Borel set $\eta \subseteq S^{n-1},$

$$
\tilde{C}_\varphi(K, \eta) = \frac{1}{n} \int_{\alpha_K^*(\eta)} \varphi(\rho_K(u)) du.
$$

Hence, (3.18) asserts that $\tilde{C}_{\phi, \gamma}(K, \cdot) = \tilde{C}_\varphi(K, \cdot)$, In particular, if

$$
\phi(x) = \frac{|x|^{q-n}}{n}
$$

which leads to $\tilde{\phi}(t) = t^q$, then $\tilde{C}_{\phi, \gamma}(K, \cdot)$ is just the $q$-th dual curvature measure of $K$ [16]; that is, for any Borel set $\eta \subseteq S^{n-1},$

$$
\tilde{C}_{\phi, \gamma}(K, \eta) = \tilde{C}_\varphi(K, \eta) = \frac{1}{n} \int_{\alpha_K^*(\eta)} [\rho_K(u)]^q du.
$$

In Case 2, i.e., $\phi(x) = \psi(|x|)\phi_2(\bar{x})$, one has, for any Borel set $\eta \subseteq S^{n-1},$

$$
\tilde{C}_{\phi, \gamma}(K, \eta) = \int_{\alpha_K^*(\eta)} \phi(\rho_K(u)[\rho_K(u)]^n du
$$

In this case, Lemma 3.1 can be rewritten as follows.
Corollary 3.1. Let $\phi(x) = \psi(|x|)\phi_2(\bar{x})$ satisfy condition C1). For $K \in \mathcal{X}_o^n$, then

\begin{align*}
\int_{S^{n-1}} g(v) d\tilde{C}_{\phi,\nu}(K,v) &= \frac{1}{n} \int_{S^{n-1}} g(\alpha_K(u)) \hat{\phi}(\rho_K(u)) \phi(u) du \\
&= \frac{1}{n} \int_{\partial K} \langle x, \nu_K(x) \rangle \cdot g(\nu_K(x)) \frac{\hat{\phi}(|x|) \phi_2(\bar{x})}{|x|^n} d\mathcal{H}^{n-1}(x) \\
&= \int_{\partial K} \langle x, \nu_K(x) \rangle \cdot g(\nu_K(x)) \psi(|x|) \phi_2(\bar{x}) d\mathcal{H}^{n-1}(x),
\end{align*}

hold for each bounded Borel function $g : S^{n-1} \to \mathbb{R}$.

4 A variational interpretation for the general dual Orlicz curvature measure

Let $f : \Omega \to (0, \infty)$ be a continuous function with $\Omega$ a closed set of $S^{n-1}$ such that $\Omega$ is not contained in any closed hemisphere of $S^{n-1}$. We shall need the Wulff shape and the convex hull of $f$, denoted by $[f]$ and $\langle f \rangle$ respectively, whose definitions are given by

$$[f] = \cap_{u \in \Omega} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u)\} \quad \text{and} \quad \langle f \rangle = \text{conv}\{\{f(u) : u \in \Omega\}\}.$$  

Clearly, $[h_K] = K$ and $\langle \rho_K \rangle = K$ for each $K \in \mathcal{X}_o^n$. It can be easily checked that $[f] \in \mathcal{X}_o^n$ and $\langle f \rangle \in \mathcal{X}_o^n$ for all continuous functions $f : \Omega \to (0, \infty)$, due to the fact that $\Omega \subseteq S^{n-1}$ is not contained in any closed hemisphere of $S^{n-1}$. As stated in [16, Lemma 2.8],

$$[f]^* = \langle 1/f \rangle.$$  

(4.20)

When $\Omega = S^{n-1}$ and $f \in C^+(S^{n-1})$, the set of all positive continuous functions defined on $S^{n-1}$, it is obvious that $h_{[f]}(u) \leq f(u)$ for all $u \in S^{n-1}$. A less trivial fact is that $h_{[f]}(u) = f(u)$ for almost all $u \in S^{n-1}$ with respect to the surface area measure $S_{[f]}(\cdot)$.

The variational interpretation for the general dual Orlicz curvature measure is stated as follows. Let $\Omega$ be a closed set of $S^{n-1}$ such that $\Omega$ is not contained in any closed hemisphere of $S^{n-1}$.

Theorem 4.1. Let $h_0 : \Omega \to (0, \infty)$ and $g : \Omega \to \mathbb{R}$ be two continuous functions. Define $h_t$ by

$$\log(h_t(u)) = \log(h_0(u)) + tg(u) + o(t, u) \quad \text{for all} \quad u \in \Omega,$$  

(4.21)

where $o(t, \cdot) : \Omega \to \mathbb{R}$ is continuous and $o(t, u)/t \to 0$ uniformly on $\Omega$ as $t \to 0$. Let $\phi$ be a function satisfying condition C1). Then

$$\frac{d}{dt} \phi([h_t]) \bigg|_{t=0} = - \int_{\Omega} g(u) d\tilde{C}_{\phi,\nu}([h_0], u).$$  

(4.22)

Remark. An immediate consequence of (4.22) and the chain rule for derivative is the following formula, which will be used in solving the general dual Orlicz-Minkowski problem:

$$\frac{d}{dt} \log \phi([h_t]) \bigg|_{t=0} = - \frac{1}{\phi([h_0])} \int_{\Omega} g(u) d\tilde{C}_{\phi,\nu}([h_0], u).$$  

(4.23)
Proof. Let \( \rho_0 : \Omega \to (0, \infty) \) be a continuous function. For \( \delta > 0 \) and \( t \in (-\delta, \delta) \), let

\[
\log(\rho_t(u)) = \log(\rho_0(u)) + t g(u) + o(t, u) \quad \text{for all } u \in \Omega,
\]

where \( o(t, \cdot) : \Omega \to \mathbb{R} \) is continuous and \( o(t, u)/t \to 0 \) uniformly on \( \Omega \) as \( t \to 0 \).

First of all, let us prove the following formula: for almost every \( u \in S^{n-1} \) (with respect to the spherical measure),

\[
\frac{d}{dt} \Phi(\rho_t, u) \bigg|_{t=0} = \frac{d}{dt} \int_{\rho_t}^{\infty} \phi(ru)r^{n-1}dr \bigg|_{t=0} = \Psi(\rho_0)g(\alpha^*_0(u)). \tag{4.24}
\]

In fact, it follows from the chain rule and \( \rho_t(u) = h^{-1}(u) \) for all \( u \in S^{n-1} \) that

\[
\frac{d}{dt} \Phi(\rho_t, u) \bigg|_{t=0} = \frac{d}{dt} \int_{e^{-\log h(u)}}^{\infty} \phi(ru)r^{n-1}dr \bigg|_{t=0} = \phi(h^{-1}_t(u))h^{-n}_t(u) \cdot \frac{d}{dt} \log h_t(u) \bigg|_{t=0} = \Psi(\rho_0)g(\alpha^*_0(u)),
\]

where the last equality follows from [16, (4.4)], i.e.,

\[
\lim_{t \to 0} \frac{\log h_t(v) - \log h_t(v)}{t} = g(\alpha^*_0(v))
\]

holds for any \( v \in S^{n-1} \setminus \eta_0 \), with \( \eta_0 = \eta(\rho_0) \) the complement of the set of the regular normal vectors of \( \rho_0 \). Note that the spherical measure of \( \eta_0 \) is zero.

We shall need the following argument in order to use the dominated convergence theorem: there exist two constants \( \delta > 0 \) and \( M > 0 \), such that, for all \( t \in (-\delta, \delta) \) and for all \( u \in S^{n-1} \),

\[
\left| \Phi(\rho_t, u) - \Phi(\rho_0, u) \right| \leq M|t|. \tag{4.25}
\]

Note that \( \rho_t \rightarrow \rho_0 \) in the Hausdorff metric; this is a direct consequence of the Aleksandrov’s convergence lemma [1] and formula (2.11). Therefore, \( \rho_t^* \rightarrow \rho_0^* \) uniformly on \( S^{n-1} \). As \( \rho_0^* \in \mathcal{K}^n \), one can find constants \( l_1, l_2, \delta_1 > 0 \) such that, \( l_1 < \rho(t) \cdot (u) < l_2 \) holds for all \( u \in S^{n-1} \) and for all \( t \in (-\delta_1, \delta_1) \). It follows from condition C1) and the continuity of \( \phi \) that

\[
\left| \left[ \log \Phi(e^{-s}, u) \right] \right| = \left| \phi(e^{-s}u)e^{-sn}/\Phi(e^{-s}, u) \right| \leq L_2 \tag{4.26}
\]

holds for some finite constant \( L_2 \) independent of \( u \in S^{n-1} \) and for all \( s \in (-\log l_2, -\log l_1) \). Note that \( \log h_t(u) \in (-\log l_2, -\log l_1) \) and \( \log h_0(u) \in (-\log l_2, -\log l_1) \) for all \( u \in S^{n-1} \) and for all \( t \in (-\delta_1, \delta_1) \). By (4.26) and the mean value theorem, one has, for all \( u \in S^{n-1} \) and for all \( t \in (-\delta, \delta) \) (without loss of generality, we can assume that \( 0 < \delta < \delta_1 \)),

\[
\left| \log \Phi(h^{-1}_t(u), u) - \log \Phi(h^{-1}_0(u), u) \right| \leq L_2 \left| \log h_t(u) - \log h_0(u) \right| \leq L_2 M_1|t|, \tag{4.27}
\]

where the last inequality follows from [16, Lemma 4.1], i.e., there exist constants \( 0 < \delta, M_1 < \infty \) such that, for all \( u \in S^{n-1} \) and for all \( t \in (-\delta, \delta) \),

\[
\left| \log h_t(u) - \log h_0(u) \right| \leq M_1|t|.
\]
It follows from condition C1) that there is a constant $L_1$ (independent of $u \in S^{n-1}$), such that, for all $u \in S^{n-1}$ and for all $t \in (-\delta, \delta)$,

$$0 < s_t = \frac{\Phi(h_{(\rho_1)}^{-1}(u), u)}{\Phi(h_{(\rho_0)}^{-1}(u), u)} = \frac{\Phi(h_{(\rho_1)}^{-1}(u), u)}{\Phi(h_{(\rho_0)}^{-1}(u), u)} < L_1.$$  

Hence $|s_t - 1| \leq L_1 : |\log s_t|$ (see e.g., [16, p.362]). Together with inequality (4.27), one gets, for all $u \in S^{n-1}$ and for all $t \in (-\delta, \delta)$,

$$\left|\Phi(h_{(\rho_1)}^{-1}(u), u) - \Phi(h_{(\rho_0)}^{-1}(u), u)\right| = \left|\Phi(h_{(\rho_1)}^{-1}(u), u) - \Phi(h_{(\rho_0)}^{-1}(u), u)\right|$$

$$\leq \Phi(h_{(\rho_0)}^{-1}(u), u) \cdot L_1 \cdot \left|\log \Phi(h_{(\rho_1)}^{-1}(u), u) - \log \Phi(h_{(\rho_0)}^{-1}(u), u)\right|$$

$$\leq \Phi(h_{(\rho_0)}^{-1}(u), u) \cdot L_1 L_2 M_1 \cdot |t|$$

That is, inequality (4.25) holds by letting $M = L_1 L_2 M_1 \cdot \max_{u \in S^{n-1}} \Phi(l_1, u) < \infty$.

Now we are ready to prove formula (4.22). To this end, let $[h_t]$ be the Wulff shape associated to $h_t$ with $h_t$ given by (4.21). Consider $\kappa_t = 1/h_t$ and then

$$\log \kappa_t = -\log h_t = -\log h_0 - t g - o(t, \cdot) = \log \kappa_0 - t g - o(t, \cdot).$$

Moreover, $[h_t] = (1/h_t)^* = (\kappa_t)^*$ due to the bipolar theorem and (4.20). It follows from (2.5), (4.24), (4.25) and the dominated convergence theorem that

$$\frac{d}{dt} \mathcal{V}_\phi([h_t])\big|_{t=0} = \frac{d}{dt} \mathcal{V}_\phi((\kappa_t)^*)\big|_{t=0}$$

$$= \frac{d}{dt} \int_{S^{n-1}} \Phi(h_{(\rho_t)}^{-1}(u), u) \, du\big|_{t=0}$$

$$= \int_{S^{n-1}} \frac{d}{dt} \Phi(h_{(\rho_t)}^{-1}(u), u)\big|_{t=0} \, du$$

$$= - \int_{S^{n-1}} \Psi_{(\kappa_0)}^* (u) \cdot g(\alpha_{(\rho_0)}^*(u)) \, du.$$

Together with (3.14) and the fact that the spherical measure of $\eta_0$ is zero, one can prove formula (4.22) as follows:

$$\frac{d}{dt} \mathcal{V}_\phi([h_t])\big|_{t=0} = - \int_{S^{n-1} \setminus \eta_0} \Psi_{(\kappa_0)}^*(u) \cdot g(\alpha_{(\rho_0)}^*(u)) \, du$$

$$= - \int_{S^{n-1}} (\tilde{g} \mathbf{1}_\Omega) (\alpha_{(\rho_0)}^*(u)) \Psi_{(\kappa_0)}^*(u) \, du$$

$$= - \int_{S^{n-1}} (\tilde{g} \mathbf{1}_\Omega)(u) \, d\tilde{C}_{\phi, \gamma}(\langle \kappa_0 \rangle^*, u)$$

$$= - \int_{\Omega} g(u) \, d\tilde{C}_{\phi, \gamma}(\eta_0, u),$$

where $\tilde{g} : S^{n-1} \to \mathbb{R}$ is a continuous function, such that, for all $v \in S^{n-1} \setminus \eta_0$,

$$g(\alpha_{(\rho_0)}^*(v)) = (\tilde{g} \mathbf{1}_\Omega)(\alpha_{(\rho_0)}^*(v)).$$

The existence of such $\tilde{g}$ was proved in [16, p.364].
5 A solution of the general dual Orlicz-Minkowski problem

In this section, we provide a solution to the following general dual Orlicz-Minkowski problem.

The general dual Orlicz-Minkowski problem: Given a nonzero finite Borel measure $\mu$ defined on $S^{n-1}$ and a continuous function $\phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, can one find a constant $\tau > 0$ and a convex body $K$ (ideally $K \in \mathcal{K}_0^n$), such that, $\mu = \tau \tilde{C}_{\phi, \gamma}(K, \cdot)$?

Clearly, if the general dual Orlicz-Minkowski problem has solutions, the constant $\tau$ can be calculated by

$$|\mu| = \int_{S^{n-1}} d\mu(v) = \tau \int_{S^{n-1}} d\tilde{C}_{\phi, \gamma}(K, v) = \tau \cdot \tilde{C}_{\phi, \gamma}(K, S^{n-1})$$

and equivalently

$$\tau = \frac{|\mu|}{\tilde{C}_{\phi, \gamma}(K, S^{n-1})}. \quad (5.28)$$

It is well known that, to have the various Minkowski problems solvable, the given measure $\mu$ must satisfy that $\mu$ is not concentrated in any closed hemisphere, i.e.,

$$\int_{S^{n-1}} \langle \xi, \theta \rangle_+ \, d\mu(\theta) > 0 \quad \text{for all } \xi \in S^{n-1}. \quad (5.29)$$

Hereafter, $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.

In fact, (5.29) is also a necessary condition in our setting. That is, if there exists a convex body $K \in \mathcal{K}_0^n$, such that,

$$\frac{\mu}{|\mu|} = \frac{\tilde{C}_{\phi, \gamma}(K, \cdot)}{\tilde{C}_{\phi, \gamma}(K, S^{n-1})},$$

then $\mu$ satisfies (5.29). To this end, let $\xi \in S^{n-1}$ be given. Then

$$\int_{S^{n-1}} \langle \xi, v \rangle_+ \, d\mu(v) = \frac{|\mu|}{\tilde{C}_{\phi, \gamma}(K, S^{n-1})} \int_{S^{n-1}} \langle \xi, v \rangle_+ \, d\tilde{C}_{\phi, \gamma}(K, v). \quad (5.30)$$

Hence, in order to show that $\mu$ satisfies (5.29), it is enough to show that

$$\int_{S^{n-1}} \langle \xi, v \rangle_+ \, d\tilde{C}_{\phi, \gamma}(K, v) > 0.$$

In fact, it follows from (3.15) that

$$\int_{S^{n-1}} \langle \xi, v \rangle_+ \, d\tilde{C}_{\phi, \gamma}(K, v) = \int_{\partial'K} \langle \xi, \nu_K(x) \rangle_+ \cdot \langle x, \nu_K(x) \rangle \phi(x) \, d\mathcal{H}^{n-1}(x).$$

As $K \in \mathcal{K}_0^n$, one can find a constant $M$ such that $\langle x, \nu_K(x) \rangle \phi(x) \geq M$ for all $x \in \partial' K$. Consequently,

$$\int_{S^{n-1}} \langle \xi, v \rangle_+ \, d\tilde{C}_{\phi, \gamma}(K, v) \geq M \int_{\partial'K} \langle \xi, \nu_K(x) \rangle_+ \, d\mathcal{H}^{n-1}(x) > 0, \quad (5.31)$$

as the surface area measure $S_K(\cdot)$ satisfies

$$\int_{\partial'K} \langle \xi, \nu_K(x) \rangle_+ \, d\mathcal{H}^{n-1}(x) = \int_{\partial K} \langle \xi, \nu_K(x) \rangle_+ \, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \langle \xi, u \rangle_+ \, dS_K(u) > 0.$$

The following theorem also shows that (5.29) is a sufficient condition for the general dual Orlicz-Minkowski problem.
Theorem 5.1. Let $\mu$ be a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29) and let $\phi$ be a function satisfying conditions C1) and C2). Then there exists a convex body $K \in \mathcal{K}_o^n$, such that,

$$\frac{\mu}{|\mu|} = \frac{\mathcal{C}_o \gamma(K, \cdot)}{\mathcal{C}_o \gamma(K, S^{n-1})}.$$ 

In order to prove Theorem 5.1, we need the following lemma.

Lemma 5.1. Let $\mu$ be a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29) and let $\phi$ be a function satisfying conditions C1) and C2). Then there exists a convex body $Q_0 \in \mathcal{K}_o^n$ such that $\mathcal{V}_0(Q_0) = |\mu|$ and

$$\mathcal{F}(Q_0) = \sup \{ \mathcal{F}(K) : \mathcal{V}_0(K) = |\mu| \text{ and } K \in \mathcal{K}_o^n \},$$

where $\mathcal{F} : \mathcal{K}_o^n \to \mathbb{R}$ is defined by

$$\mathcal{F}(K) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_K(v) d\mu(v).$$

Proof. Let $\{Q_i\}_{i=1}^{\infty} \subseteq \mathcal{K}_o^n$ be such that $\mathcal{V}_0(Q_i) = |\mu|$ and

$$\lim_{i \to \infty} \mathcal{F}(Q_i) = \sup \left\{ \mathcal{F}(K) : \mathcal{V}_0(K) = |\mu| \text{ and } K \in \mathcal{K}_o^n \right\}.$$  

First of all, we claim that the sequence $\{Q_i\}_{i=1}^{\infty}$ is uniformly bounded. That is, we need to prove that there exists a constant $R > 0$ such that $Q_i \subseteq R B_2^n$ for all $i = 1, 2, \cdots$

Assume not, i.e., there are no finite constants $R$ such that $Q_i \subseteq R B_2^n$ for all $i = 1, 2, \cdots$. Let $v_i \in S^{n-1}$ be such that $\rho_{Q_i}(v_i) = \max_{u \in S^{n-1}} \rho_{Q_i}(u)$ and $R_{Q_i} = \rho_{Q_i}(v_i)$. Without loss of generality, we can assume that $R_{Q_i} \to \infty$ (otherwise, the sequence $\{Q_i\}_{i=1}^{\infty}$ is uniformly bounded) and $v_i \to v_0$ (due to the compactness of $S^{n-1}$) as $i \to \infty$. Consequently, for any $M > 0$, there exists $i_M > 0$ such that $R_{Q_i} \geq M$ for all $i > i_M$. Clearly, for all $i > i_M$,

$$h_{Q_i}(u) \geq \langle u, v_i \rangle_+ + R_{Q_i} \geq M \langle u, v_i \rangle_+.$$ 

Recall that $\Phi(t, u) = \int_t^{\infty} \phi(ru) r^{n-1} dr$ is decreasing on $t$. Then for all $i > i_M$ and for all $u \in S^{n-1}$,

$$\Phi(\rho_{Q_i}(u), u) = \Phi(h_{Q_i}^{-1}(u), u) \geq \Phi([M \langle u, v_i \rangle_+]^{-1}, u),$$

where we let $\Phi([M \langle u, v_i \rangle_+]^{-1}, u) = 0$ if $\langle u, v_i \rangle_+ = 0$.

Fatou’s lemma implies that

$$\liminf_{i \to \infty} \int_{S^{n-1}} \Phi([M \langle u, v_i \rangle_+]^{-1}, u) du = \liminf_{i \to \infty} \int_{S^{n-1}} \int_{M \langle u, v_i \rangle_+}^{\infty} \phi(ru) r^{n-1} dr du,$$

$$\geq \int_{S^{n-1}} \liminf_{i \to \infty} \int_{0}^{\infty} 1_{[M \langle u, v_i \rangle_+]^{-1}, \infty)} \phi(ru) r^{n-1} dr du,$$

$$\geq \int_{S^{n-1}} \liminf_{i \to \infty} \int_{0}^{\infty} 1_{[M \langle u, v_i \rangle_+]^{-1}, \infty)} \phi(ru) r^{n-1} dr du,$$

$$= \int_{S^{n-1}} \int_{M \langle u, v_0 \rangle_+}^{\infty} \phi(ru) r^{n-1} dr du,$$

$$= \int_{S^{n-1}} \Phi([M \langle u, v_0 \rangle_+]^{-1}, u) du.$$
Together with (2.5) and (5.36), one has
\[
|\mu| = \lim_{i \to \infty} \mathcal{Y}_\phi(Q_i)
= \lim_{i \to \infty} \int_{S^{n-1}} \Phi(\rho_{Q_i}(u), u) \, du
\geq \liminf_{i \to \infty} \int_{S^{n-1}} \Phi([M(u, v_i)]^{-1}, u) \, du
\geq \int_{S^{n-1}} \Phi([M(u, v_0)]^{-1}, u) \, du. \quad (5.36)
\]

For all \( j \geq 2 \), let
\[
\Sigma_j(v_0) := \{ u \in S^{n-1} : \langle u, v_0 \rangle > 1/j \}.
\]
It follows from the monotone convergence theorem and the fact \( \Sigma_j(v_0) \subset \Sigma_{j+1}(v_0) \subset \bigcup_{j=1}^\infty \Sigma_j(v_0) = S^{n-1} \setminus \{ u \in S^{n-1} : \langle u, v_0 \rangle = 0 \} \) that
\[
\lim_{j \to \infty} \int_{\Sigma_j(v_0)} \langle u, v_0 \rangle_+ \, du = \int_{\bigcup_{j=1}^\infty \Sigma_j(v_0)} \langle u, v_0 \rangle_+ \, du = \int_{S^{n-1}} \langle u, v_0 \rangle_+ \, du > 0,
\]
where the last inequality is due to the fact that the spherical measure is not concentrated on any closed hemisphere. Hence, there exists \( j_0 \geq 2 \), such that,
\[
\int_{\Sigma_{j_0}(v_0)} du \geq \int_{\Sigma_{j_0}(v_0)} \langle u, v_0 \rangle_+ \, du \geq \frac{1}{2} \int_{S^{n-1}} \langle u, v_0 \rangle_+ \, du > 0.
\]
It can be checked that \([M(u, v_0)]^{-1} \leq j_0/M \) for all \( u \in \Sigma_{j_0}(v_0) \). By (2.4) and (5.36), one gets
\[
|\mu| \geq \int_{S^{n-1}} \Phi([M(u, v_0)]^{-1}, u) \, du \geq \int_{\Sigma_{j_0}(v_0)} \Phi(j_0/M, u) \, du = \mathcal{Y}_\phi(\mathbb{R}^n \setminus \mathcal{C}(v_0, j_0/M, 1/j_0)),
\]
where for any fixed \( u_0 \in S^{n-1}, a > 0 \) and \( b_0 \in (0, 1), \)
\[
\mathcal{C}(u_0, a, b_0) = \{ x \in \mathbb{R}^n : \langle x, u_0 \rangle \geq b_0 \text{ and } |x| \geq a \}.
\]
As \( \phi \) satisfies condition C2), one gets a contradiction as follows:
\[
\infty > |\mu| \geq \lim_{M \to \infty} \mathcal{Y}_\phi(\mathbb{R}^n \setminus \mathcal{C}(v_0, j_0/M, 1/j_0)) = \infty.
\]
Therefore, the sequence \( \{Q_i^*\}_{i=1}^\infty \) is uniformly bounded.

Without loss of generality, we assume that \( Q_i^* \to Q \) (more precisely, a subsequence of \( \{Q_i^*\}_{i=1}^\infty \)) in the Hausdorff metric for some compact convex set \( Q \subset \mathbb{R}^n \), due to the Blaschke selection theorem (see e.g., [33]). Note that \( Q \) may not be a convex body, however, the support function of \( Q \) can be defined as in (2.2) and \( Q_i^* \to Q \) in the Hausdorff metric is defined as in (2.10).

We now show \( Q \in \mathcal{K}_o^n \) and the proof can be obtained by an argument almost identical to those in [42, 45]. In fact, assume that \( Q \notin \mathcal{K}_o^n \) and \( o \in \partial Q \). Then, there exists \( u_0 \in S^{n-1} \) such that \( \lim_{i \to \infty} h_{Q_i^*}(u_0) = h_Q(u_0) = 0 \). Let
\[
\Sigma_{b_0}(u_0) = \{ v \in S^{n-1} : \langle v, u_0 \rangle > b_0 \}.
\]
By (5.33), \( \mathcal{V}_\phi(Q_i) = |\mu| \) and \( Q_i^* \subseteq RB_2^n \) (without loss of generality, let \( R > 1 \)) for all \( i \), one has

\[
\mathcal{F}(Q_i) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{Q_i}(v) \, d\mu(v)
\]

\[
= \frac{1}{|\mu|} \int_{\Sigma_{\delta_0}(u_0)} \log \rho_{Q_i}(v) \, d\mu(v) + \frac{1}{|\mu|} \int_{S^{n-1} \setminus \Sigma_{\delta_0}(u_0)} \log \rho_{Q_i}(v) \, d\mu(v)
\]

\[
\leq \frac{1}{|\mu|} \int_{\Sigma_{\delta_0}(u_0)} \log \rho_{Q_i}(v) \, d\mu(v) + \log R.
\]

It follows from \( \mu(\Sigma_{\delta_0}(u_0)) > 0 \) and \( \rho_{Q_i} \to 0 \) on \( \Sigma_{\delta_0}(u_0) \) uniformly for some \( \delta_0 > 0 \) that

\[
\lim_{i \to \infty} \mathcal{F}(Q_i) = -\infty,
\]

which is impossible. Hence, \( o \in \text{int}Q \) and then \( Q \in \mathcal{K}_o^n \).

Finally, let us check that \( Q_0 = Q^* \in \mathcal{K}_o^n \) satisfies \( \mathcal{V}_\phi(Q_0) = |\mu| \) and (5.32). In fact, as \( Q_i^* \to Q \), one has \( Q_i \to Q^* = Q_0 \) due to the bipolar theorem. Then

\[
\mathcal{V}_\phi(Q_0) = \lim_{i \to \infty} \mathcal{V}_\phi(Q_i) = |\mu|
\]

is an immediate consequence of Lemma 2.1. On the other hand, \( h_{Q_i} \to h_{Q_0} \) uniformly on \( S^{n-1} \) due to \( Q_i \to Q_0 \in \mathcal{K}_o^n \) and (2.10). Moreover, there exist constants \( R_1, R_2 \in (0, \infty) \), such that, for all \( u \in S^{n-1} \) and for all \( i \geq 1 \),

\[
R_1 \leq h_{Q_i}(u) \leq R_2 \quad \text{and} \quad R_1 \leq h_{Q_0}(u) \leq R_2.
\]

These further imply that, for all \( u \in S^{n-1} \) and for all \( i \geq 1 \),

\[
|\log h_{Q_i}(u)| \leq \max\{|\log R_1|, |\log R_2|\} < \infty.
\]

It follows from the dominated convergence theorem that

\[
\lim_{i \to \infty} \mathcal{F}(Q_i) = \lim_{i \to \infty} -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{Q_i}(v) \, d\mu(v)
\]

\[
= -\frac{1}{|\mu|} \int_{S^{n-1}} \lim_{i \to \infty} \log h_{Q_i}(v) \, d\mu(v)
\]

\[
= -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{Q_0}(v) \, d\mu(v)
\]

\[
= \mathcal{F}(Q_0).
\]

Together with (5.34), one can easily get the desired formula (5.32).

Proof of Theorem 5.1. Recall that each \( K \in \mathcal{K}_o^n \) can be uniquely determined by its support function and vice versa. Thus we can let \( \mathcal{V}_\phi(h_{[f]}) = \mathcal{V}_\phi([f]) \) for all \( f \in C^+(S^{n-1}) \). On the other hand, as \( f \geq h_{[f]} \) for all \( f \in C^+(S^{n-1}) \), then

\[
\mathcal{F}(f) := -\frac{1}{|\mu|} \int_{S^{n-1}} \log f(v) \, d\mu(v) \leq \mathcal{F}(h_{[f]}).
\]

Consider the following optimization problem:

\[
\sup \left\{ \mathcal{F}(f) : \mathcal{V}_\phi([f]) = |\mu| \text{ for } f \in C^+(S^{n-1}) \right\}.
\]
According to (5.37) and Lemma 5.1, the support function of convex body \( Q_0 \in \mathcal{X}^n_0 \) found in Lemma 5.1 is an optimizer for the optimization problem (5.38).

On the other hand, the method of Lagrange multipliers can be used to find the necessary conditions for the optimizers for the optimization problem (5.38). In fact, for \( \delta > 0 \) small enough, let \( h_t(v) = h_{Q_0}(v)e^{\phi(v)} \) for \( t \in (-\delta, \delta) \) and for \( v \in S^{n-1} \), where \( g : S^{n-1} \to \mathbb{R} \) is an arbitrary continuous function. Let

\[
\mathcal{L}(t, \tau) = \mathcal{F}(h_t) - \tau \left( \log \mathcal{Y}_\phi([h_t]) - \log |\mu| \right).
\]

As \( h_{Q_0} \) is an optimizer to (5.38), the following equation holds:

\[
\frac{\partial}{\partial t} \mathcal{L}(t, \tau) \bigg|_{t=0} = 0. \tag{5.39}
\]

It is easily checked that

\[
\frac{\partial}{\partial t} \mathcal{F}(h_t) \bigg|_{t=0} = \frac{\partial}{\partial t} \left( -\frac{1}{|\mu|} \int_{S^{n-1}} [\log h_{Q_0}(v) + tg(v)] d\mu(v) \right) \bigg|_{t=0} = -\frac{1}{|\mu|} \int_{S^{n-1}} g(v) d\mu(v).
\]

It follows from (4.23) that

\[
\frac{\partial}{\partial t} \log \mathcal{Y}_\phi([h_t]) \bigg|_{t=0} = -\frac{1}{\mathcal{Y}_\phi(Q_0)} \int_{S^{n-1}} g(v) d\tilde{C}_{\phi,\psi}(Q_0, v).
\]

Due to \( \mathcal{Y}_\phi(Q_0) = |\mu| \), one can rewrite (5.39) as follows:

\[
\int_{S^{n-1}} g(v) d\mu(v) = \tau \int_{S^{n-1}} g(v) d\tilde{C}_{\phi,\psi}(Q_0, v)
\]

holding for arbitrary continuous function \( g : S^{n-1} \to \mathbb{R} \). Consequently, \( \mu = \tau \tilde{C}_{\phi,\psi}(Q_0, \cdot) \) with the constant \( \tau \) given by (5.28), that is,

\[
\tau = \frac{|\mu|}{\tilde{C}_{\phi,\psi}(Q_0, S^{n-1})}.
\]

In summary, a solution to the general dual Orlicz-Minkowski problem has been found. \( \square \)

The following corollary provides a solution to the general dual Orlicz-Minkowski problem under the Case 2 in Section 2, i.e., \( \phi(x) = \psi(|x|)\tilde{\phi}_2(\tilde{x}) \) with \( \psi : (0, \infty) \to (0, \infty) \) and \( \phi_2 : S^{n-1} \to (0, \infty) \) continuous functions. Again, let \( \tilde{\phi} \) and \( \psi \) be given as in (2.7) or (2.8), and \( \tilde{\phi}(t) = n\tilde{\psi}(t)t^n \).

**Corollary 5.1.** Let \( \phi(x) = \psi(|x|)\tilde{\phi}_2(\tilde{x}) \) be a continuous function such that the continuous function \( \tilde{\phi}_2 \) is positive on \( S^{n-1} \), and the functions \( \tilde{\phi} \) and \( \tilde{\phi} \) satisfy conditions A1)-A3). Then the following are equivalent:

i) \( \mu \) is a nonzero finite Borel measure on \( S^{n-1} \) satisfying (5.29); 

ii) there exists a convex body \( K \in \mathcal{X}^n_0 \) such that

\[
\frac{\int_{S^{n-1}} g(v) d\mu(v)}{|\mu|} = \frac{\int_{S^{n-1}} g(v) d\tilde{C}_{\phi,\psi}(K, v)}{\int_{S^{n-1}} d\tilde{C}_{\phi,\psi}(K, v)} = \frac{\int_{S^{n-1}} g(\alpha_K(u))\tilde{\phi}(\rho_K(u))\phi_2(u) du}{\int_{S^{n-1}} \tilde{\phi}(\rho_K(u))\phi_2(u) du}
\]

hold for each bounded Borel function \( g : S^{n-1} \to \mathbb{R} \).
Proof. As explained in Section 2, under the conditions given in Corollary 5.1, \( \phi(x) = \psi(|x|)\phi_2(\tilde{x}) \) satisfies conditions C1) and C2). The argument in ii) is equivalent to

\[
\frac{\mu}{|\mu|} = \frac{\tilde{C}_{\phi, \gamma}(K, \cdot)}{C_{\phi, \gamma}(K, S^{n-1})}.
\]

The equivalence between i) and ii) is an immediate consequence from (5.30), (5.31), Corollary 3.1 and Theorem 5.1.

6 Uniqueness of solutions of the general dual Orlicz-Minkowski problem

It seems very difficult and maybe even impossible to obtain the uniqueness of solutions of the general dual Orlicz-Minkowski problem for general \( \phi \). In this section, the uniqueness will be proved in special cases. In order to get this done, we need the following theorem.

**Theorem 6.1.** Let \( \phi \) be a function satisfying condition C1) and that \( \phi(x)|x|^n \) is strictly radially decreasing on \( \mathbb{R}^n \setminus \{0\} \). If \( K, L \in \mathcal{K}_0^n \) satisfy that \( C_{\phi, \gamma}(K, \cdot) = C_{\phi, \gamma}(L, \cdot) \), then \( K = L \).

The proof of Theorem 6.1 follows an argument similar to those in [42, 45], and heavily relies on [42, Lemma 5.1]. For readers’ convenience, we list [42, Lemma 5.1] below as Lemma 6.1 and provide a brief sketch of the proof of Theorem 6.1.

**Lemma 6.1.** Suppose that \( K' \), \( L \in \mathcal{K}_0^n \). If the following sets

\[
\eta_1 = \{ v \in S^{n-1} : h_{K'}(v) > h_{L}(v) \}, \\
\eta_2 = \{ v \in S^{n-1} : h_{K'}(v) < h_{L}(v) \}, \\
\eta_3 = \{ v \in S^{n-1} : h_{K'}(v) = h_{L}(v) \}
\]

are nonempty, then the following statements are true:

a) if \( u \in \alpha_{K'}^*(\eta_1) \), then \( \rho_{K'}(u) > \rho_{L}(u) \);

b) if \( u \in \alpha_{L}^*(\eta_2 \cup \eta_3) \), then \( \rho_{L}(u) \geq \rho_{K'}(u) \);

c) \( \alpha_{K'}^*(\eta_1) \subset \alpha_{L}^*(\eta_1) \);

d) \( \mathcal{H}^{n-1}(\alpha_{L}^*(\eta_1)) > 0 \) and \( \mathcal{H}^{n-1}(\alpha_{K'}^*(\eta_2)) > 0 \).

**Proof of Theorem 6.1.** Assume that \( K, L \in \mathcal{K}_0^n \) with \( \tilde{C}_{\phi, \gamma}(K, \cdot) = \tilde{C}_{\phi, \gamma}(L, \cdot) \) are not dilates of each other, namely, \( K \neq tL \) for any \( t > 0 \). Hence, there exists some constant \( t_0 > 0 \) such that \( K' = t_0K \) is a convex body with \( \eta_1, \eta_2, \eta_3 \) defined in Lemma 6.1 being nonempty.

Recall that \( \Psi_{K'}(u) = \phi(\rho_{K'}(u)u)[\rho_{K'}(u)]^n \) for \( u \in S^{n-1} \). Due to Lemma 6.1 and the fact that \( \phi(x)|x|^n \) is strictly radially decreasing on \( \mathbb{R}^n \setminus \{0\} \), one has, for all \( u \in \alpha_{K'}^*(\eta_1) \),

\[
0 < \Psi_{K'}(u) = \phi(\rho_{K'}(u)u)[\rho_{K'}(u)]^n < \phi(\rho_{L}(u)u)[\rho_{L}(u)]^n = \Psi_{L}(u).
\]

Now we claim that the spherical measure of \( \alpha_{K'}^*(\eta_1) \) is positive. In fact, this claim follows from Definition 3.1 and Lemma 6.1 as follows:

\[
\int_{\alpha_{K'}^*(\eta_1)} \Psi_{K'}(u) du = \tilde{C}_{\phi, \gamma}(K, \eta_1) = \tilde{C}_{\phi, \gamma}(L, \eta_1) = \int_{\alpha_{L}^*(\eta_1)} \Psi_{L}(u) du > 0.
\]
Moreover, by (6.40) and Lemma 6.1, one has
\[
\tilde{C}_{\phi, \mathcal{Y}}(K, \eta_1) = \int_{\mathcal{A}_L^*(\eta_1)} \Psi_L(u) \, du \geq \int_{\mathcal{A}_K^*(\eta_1)} \Psi_L(u) \, du > \int_{\mathcal{A}_K^*(\eta_1)} \Psi_K(u) \, du > 0.
\]
Due to the easily checked fact \(\mathcal{A}_K^*(\eta_1) = \mathcal{A}_K^*(\eta_1)\) and Definition 3.1, one gets
\[
\tilde{C}_{\phi, \mathcal{Y}}(K, \eta_1) = \int_{\mathcal{A}_K^*(\eta_1)} \Psi_K(u) \, du
= \int_{\mathcal{A}_K^*(\eta_1)} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du
> \int_{\mathcal{A}_K^*(\eta_1)} \Psi_K(u) \, du
= \int_{\mathcal{A}_K^*(\eta_1)} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du > 0.
\]
Together with the fact that \(\phi(x)|x|^n\) is strictly radially decreasing on \(\mathbb{R}^n \setminus \{0\}\), one has \(t_0 > 1\) and moreover
\[
\phi(\rho_K(u)u)[\rho_K(u)]^n > \phi(t_0 \rho_K(u)u)[t_0 \rho_K(u)]^n
\]
holds for all \(u \in S^{n-1}\).

Similarly, one can check that the spherical measure of \(\mathcal{A}_L^*(\eta_2)\) is positive. It follows from Lemma 6.1 that \(\mathcal{A}_L^*(\eta_2) \subseteq \mathcal{A}_K^*(\eta_2)\) and
\[
0 < \tilde{C}_{\phi, \mathcal{Y}}(K, \eta_2) = \tilde{C}_{\phi, \mathcal{Y}}(L, \eta_2) = \int_{\mathcal{A}_L^*(\eta_2)} \Psi_L(u) \, du \leq \int_{\mathcal{A}_K^*(\eta_2)} \Psi_K(u) \, du = \tilde{C}_{\phi, \mathcal{Y}}(K', \eta_2).
\]
Together with (6.41), Definition 3.1, and \(\mathcal{A}_K^*(\eta_2) = \mathcal{A}_K^*(\eta_2)\), one has
\[
\tilde{C}_{\phi, \mathcal{Y}}(K, \eta_2) \leq \tilde{C}_{\phi, \mathcal{Y}}(K', \eta_2) < \tilde{C}_{\phi, \mathcal{Y}}(K, \eta_2).
\]
This is impossible, and hence \(K\) and \(L\) are dilates of each other.

Now we claim that \(K = L\). Assume not, i.e., there exists a constant \(t \neq 1\) such that \(K = tL\). Let \(t > 1\) and hence \(\phi(\rho_L(u)u)[\rho_L(u)]^n > \phi(\rho_K(u)u)[\rho_K(u)]^n\) for all \(u \in S^{n-1}\). We can get a contradiction as follows:
\[
\tilde{C}_{\phi, \mathcal{Y}}(K, S^{n-1}) = \tilde{C}_{\phi, \mathcal{Y}}(L, S^{n-1}) = \int_{S^{n-1}} \phi(\rho_L(u)u)[\rho_L(u)]^n \, du > \int_{S^{n-1}} \phi(\rho_K(u)u)[\rho_K(u)]^n \, du = \tilde{C}_{\phi, \mathcal{Y}}(K, S^{n-1}),
\]
where we have used the assumption that \(\tilde{C}_{\phi, \mathcal{Y}}(K, \cdot) = \tilde{C}_{\phi, \mathcal{Y}}(L, \cdot)\).

Similarly, one can show that \(t < 1\) is not possible, and hence \(K = L\) as desired.

\(\square\)

**Remark.** When \(\phi(x) = \psi(|x|)\phi_2(x)\) as stated in Case 2 in Section 2, then \(\phi(x)|x|^n\) is a strictly radially decreasing function if \(\phi(t) = n\psi(t)t^n\) is a strictly decreasing function on \(t \in (0, \infty)\). For instance, if \(\phi(x) = ||x||^{q-n}\) for \(q < 0\), then
\[
\phi(x)|x|^n = ||x||^{q-n}||x||^{n} = ||x||^q||x||^{q-n}
\]
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is a strictly radially decreasing function on $\mathbb{R}^n \setminus \{o\}$. On the other hand, if $\phi$ is smooth enough, say the gradient of $\phi$ (denoted by $\nabla \phi$) exists on $\mathbb{R}^n \setminus \{o\}$, a typical condition to make $\phi(x)|x|^n$ strictly radially decreasing is $\langle \nabla (\phi(x)|x|^n), x \rangle < 0$ or equivalently $\langle \nabla \phi(x), x \rangle + n\phi(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{o\}$.

We are now ready to state our result regarding the uniqueness of solutions to the general dual Orlicz-Minkowski problem. If $\phi_2(u) = 1$ for all $u \in S^{n-1}$, it goes back to the case proved by Zhao [42].

**Corollary 6.1.** Let $\phi(x) = |x|^{q-n}\phi_2(\bar{x})$ with $q < 0$ and $\phi_2 : S^{n-1} \to (0, \infty)$ a positive continuous function. Then the following statements are equivalent:

i) $\mu$ is a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29);

ii) there exists a unique convex body $K \in \mathcal{K}_o^n$, such that, $\mu = \tilde{C}_{\phi,\gamma}(K,\cdot)$.

**Proof.** The argument from ii) to i) follows along the same lines as the arguments for (5.30) and (5.31). On the other hand, it follows from Theorem 5.1 that, if $\mu$ is a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29), then there is a convex body $\tilde{K} \in \mathcal{K}_o^n$ such that

$$\frac{\mu}{|\mu|} = \frac{\tilde{C}_{\phi,\gamma}(\tilde{K},\cdot)}{\tilde{C}_{\phi,\gamma}(\tilde{K},S^{n-1})}.$$

By Corollary 3.1, $\alpha_{\lambda K}^*(\eta) = \alpha_{K}^*(\eta)$ and $\rho_{\lambda K} = \lambda \rho_K$ for any constant $\lambda > 0$, and the fact that $u \in \alpha_{K}^*(\eta)$ if and only if $\alpha_K(u) \in \eta$ (see [16, (2.21)]), one has, for any $\lambda > 0$ and for any Borel set $\eta \subseteq S^{n-1}$,

$$\tilde{C}_{\phi,\gamma}(\lambda K,\eta) = \int \alpha_{\lambda K}^*(\eta) \left[\rho_{\lambda K}(u)\right]^q \phi_2(u) du = \lambda^q \int \alpha_{K}^*(\eta) \left[\rho_K(u)\right]^q \phi_2(u) du = \lambda^q \tilde{C}_{\phi,\gamma}(K,\eta).$$

Hence, $\tilde{C}_{\phi,\gamma}(\lambda K,\cdot) = \lambda^q \tilde{C}_{\phi,\gamma}(K,\cdot)$ and

$$\mu = \frac{|\mu|}{\tilde{C}_{\phi,\gamma}(\tilde{K},S^{n-1})} \tilde{C}_{\phi,\gamma}(\tilde{K},\cdot) = \tilde{C}_{\phi,\gamma}(K,\cdot),$$

where

$$K = \left(\frac{|\mu|}{\tilde{C}_{\phi,\gamma}(\tilde{K},S^{n-1})}\right)^{\frac{1}{q}} \tilde{K}.$$ 

Hence, $K \in \mathcal{K}_o^n$ is a convex body such that $\mu = \tilde{C}_{\phi,\gamma}(K,\cdot)$, if $\mu$ is a nonzero finite Borel measure on $S^{n-1}$ satisfying (5.29). The uniqueness of $K$ is an immediate consequence of Theorem 6.1 and the remark after its proof.

The solution for $\mu$ being a discrete measure is stated in the following proposition.

**Proposition 6.1.** Let $\phi(x) = |x|^{q-n}\phi_2(\bar{x})$ with $q < 0$ and $\phi_2 : S^{n-1} \to (0, \infty)$ a positive continuous function. Suppose that $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$ with all $\lambda_i > 0$ is a discrete measure not concentrated in any closed hemisphere (i.e., satisfying (5.29)). Then, there exists a unique polytope $P \in \mathcal{K}_o^n$, such that, $\mu = \tilde{C}_{\phi,\gamma}(P,\cdot)$ and $u_1, u_2, \ldots, u_m$ are the unit normal vectors of the faces of $P$. 

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Proof. It follows from Corollary 6.1 that there exists a unique convex body $K_0 \in \mathcal{K}_o^n$, such that, $\mu = \tilde{C}_{\phi, \tau}(K_0, \cdot)$. The desired argument in this proposition follows if we can prove that $K_0$ is a polytope with $u_1, u_2, \cdots, u_m$ being the unit normal vectors of its faces. To this end, let $M \in \mathcal{K}_o^n$ be a polytope circumscribed about $K_0$ whose faces have the unit normal vectors being exactly $u_1, u_2, \cdots, u_m$. Hence $K_0 \subseteq M$ and $h_M(u_i) = h_{K_0}(u_i)$ for all $i = 1, 2, \cdots, m$.

Suppose that $K_0 \neq M$ (as otherwise, nothing to prove). In this case, there exists a set $\eta_M \subseteq S^{n-1}$, such that, the spherical measure of $\eta_M$ is positive and $\rho_M(u) > \rho_{K_0}(u)$ on $\eta_M$. It follows from (2.9) and (3.19) that $\mathcal{V}_\phi(M) < \mathcal{V}_\phi(K_0)$ and $\tilde{C}_{\phi, \tau}(L, S^{n-1}) = -q \mathcal{V}_\phi(L)$ for all $L \in \mathcal{K}_o^n$. Hence, $\tilde{C}_{\phi, \tau}(M, S^{n-1}) < \tilde{C}_{\phi, \tau}(K_0, S^{n-1}) = |\mu|$. By (6.42), there exists a constant $0 < c < 1$, such that

$$
\tilde{C}_{\phi, \tau}(cM, S^{n-1}) = \tilde{C}_{\phi, \tau}(K_0, S^{n-1}) = |\mu|.
$$

On the other hand, from Corollary 6.1 and the proof of Theorem 5.1, the convex body $(-q)^{1/q}K_0 \in \mathcal{K}_o^n$ is the unique convex body such that $\mathcal{V}_\phi((-q)^{1/q}K_0) = |\mu|$ and

$$
\mathcal{F}((-q)^{1/q}K_0) = \sup \{ \mathcal{F}(K) : \mathcal{V}_\phi(K) = |\mu| \text{ and } K \in \mathcal{K}_o^n \}.
$$

However, this is impossible because $\mathcal{V}_\phi((-q)^{1/q}cM) = |\mu|$ and

$$
\mathcal{F}((-q)^{1/q}cM) = -\frac{1}{|\mu|} \int_{S^{n-1}} \left[ \log h_M(v) + \log c + \log(-q)/q \right] d\mu(v)
$$

$$
= -\frac{1}{|\mu|} \int_{S^{n-1}} \left[ \log h_M(v) + \log(-q)/q \right] d\mu(v)
$$

$$
= -\frac{1}{|\mu|} \cdot \sum_{i=1}^m \lambda_i \left[ \log h_M(u_i) + \log(-q)/q \right]
$$

$$
= -\frac{1}{|\mu|} \cdot \sum_{i=1}^m \lambda_i \left[ \log h_{K_0}(u_i) + \log(-q)/q \right]
$$

$$
= \mathcal{F}((-q)^{1/q}K_0),
$$

where the inequality is due to $0 < c < 1$. Hence $M = K_0$ is a polytope. Moreover, it is easy to get the relation between $\lambda_i$ and the polytope $K_0$. In fact,

$$
\lambda_i = \int_{\{u_i\}} d\mu = \int_{\{u_i\}} d\tilde{C}_{\phi, \tau}(K_0, v)
$$

$$
= \int_{\alpha_{K_0}(u_i)} \left[ \rho_{K_0}(v) \right]^q \phi_2(v) dv
$$

$$
= \int_{\nu_{K_0}^{-1}(\{u_i\})} \langle x, \nu_{K_0}(x) \rangle |x|^{q-n} \phi_2(\tilde{x}) d\mathcal{H}^{n-1}(x)
$$

$$
= \int_{\nu_{K_0}^{-1}(\{u_i\})} \langle x, \nu_{K_0}(x) \rangle \phi(x) d\mathcal{H}^{n-1}(x) > 0
$$

where the third equality follows from (6.42) and the fourth equality follows from Corollary 3.1. Let $P = K_0$, and then $P$ is the desired polytope, such that, $\mu = \tilde{C}_{\phi, \tau}(P, \cdot)$ and $u_1, u_2, \cdots, u_m$ are the unit normal vectors of the faces of $P$. \hfill \square

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