ON THE THIRD-ORDER JACOBSTHAL AND THIRD-ORDER JACOBSTHAL-LUCAS SEQUENCES AND THEIR MATRIX REPRESENTATIONS

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Abstract. In this paper, we first give new generalizations for third-order Jacobsthal \( \{J_n^{(3)}\}_{n \in \mathbb{N}} \) and third-order Jacobsthal-Lucas \( \{j_n^{(3)}\}_{n \in \mathbb{N}} \) sequences for Jacobsthal and Jacobsthal-Lucas numbers. Considering these sequences, we define the matrix sequences which have elements of \( \{J_n^{(3)}\}_{n \in \mathbb{N}} \) and \( \{j_n^{(3)}\}_{n \in \mathbb{N}} \). Then we investigate their properties.

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1. Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [1]). The Jacobsthal numbers \( J_n \) are defined by the recurrence relation

\[
J_0 = 0, \quad J_1 = 1, \quad J_{n+1} = J_n + 2J_{n-1}, \quad n \geq 1.
\]

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation \( j_0 = 2, \ j_1 = 1, \ j_{n+1} = j_n + 2j_{n-1}, \ n \geq 1. \) (see, [2]).

In [3] the Jacobsthal recurrence relation (1.1) is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [1] is expanded and extended to several identities for some of the higher order cases. In particular, third order Jacobsthal numbers, \( \{J_n^{(3)}\}_{n \geq 0} \), and third order Jacobsthal-Lucas numbers, \( \{j_n^{(3)}\}_{n \geq 0} \), are defined by

\[
J_n^{(3)} = J_{n+3}^{(3)} + 2J_{n+1}^{(3)} + 3J_n^{(3)} + \begin{cases} \ -2 & \text{if} \ n \equiv 1 \pmod{3} \ \\
1 & \text{if} \ n \not\equiv 1 \pmod{3} \end{cases},
\]

and

\[
j_n^{(3)} = j_{n+3}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)} + \begin{cases} \ -2 & \text{if} \ n \equiv 1 \pmod{3} \ \\
1 & \text{if} \ n \not\equiv 1 \pmod{3} \end{cases},
\]

respectively.

The following properties given for third order Jacobsthal numbers and third order Jacobsthal-Lucas numbers play important roles in this paper (see [2, 3, 6]).

\[
3J_n^{(3)} + j_n^{(3)} = 2^{n+1},
\]

\[
j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)},
\]

and

\[
J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} \ -2 & \text{if} \ n \equiv 1 \pmod{3} \ \\
1 & \text{if} \ n \not\equiv 1 \pmod{3} \end{cases}.
\]
\begin{align}
\tag{1.7} j_n^{(3)} - 4J_n^{(3)} &= \begin{cases} 
2 & \text{if } n \equiv 0 \pmod{3} \\
-3 & \text{if } n \equiv 1 \pmod{3} \\
1 & \text{if } n \equiv 2 \pmod{3}
\end{cases}, \\
\tag{1.8} j_{n+1}^{(3)} + J_n^{(3)} &= 3J_{n+2}^{(3)}, \\
\tag{1.9} j_n^{(3)} - J_n^{(3)} &= \begin{cases} 
1 & \text{if } n \equiv 0 \pmod{3} \\
-1 & \text{if } n \equiv 1 \pmod{3} \\
0 & \text{if } n \equiv 2 \pmod{3}
\end{cases}, \\
\tag{1.10} \left(J_n^{(3)} - 3\right)^2 + 3J_n^{(3)}J_n^{(3)} &= 4^n, \\
\tag{1.11} \sum_{k=0}^n j_k^{(3)} &= \begin{cases} 
J_{n+1}^{(3)} & \text{if } n \equiv 0 \pmod{3} \\
J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \pmod{3}
\end{cases}, \\
\tag{1.12} \left(j_n^{(3)} - 3\right)^2 - 9 \left(j_n^{(3)} - 3\right)^2 &= 2^{n+2}j_{n-3}^{(3)}.
\end{align}

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by
\begin{align}
\tag{1.13} x^3 - x^2 - x - 2 &= 0; \quad x = 2, \quad \text{and } x = \frac{-1 \pm i\sqrt{3}}{2}.
\end{align}

Note that the latter two are the complex conjugate cube roots of unity. Call them \(\omega_1\) and \(\omega_2\), respectively. Thus the Binet formulas can be written as
\begin{align}
\tag{1.14} J_n^{(3)} &= \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21}\omega_1^n - \frac{3 - 2i\sqrt{3}}{21}\omega_2^n = \frac{1}{7} \left(2^{n+1} - V_n^{(3)}\right), \\
\tag{1.15} j_n^{(3)} &= \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7}\omega_1^n + \frac{3 - 2i\sqrt{3}}{7}\omega_2^n = \frac{1}{7} \left(2^{n+3} + 3V_n^{(3)}\right),
\end{align}
respectively. Here, the sequence \(\{V_n^{(3)}\}_{n \geq 0}\) is defined by
\begin{align}
\tag{1.16} V_n^{(3)} &= \begin{cases} 
2 & \text{if } n \equiv 0 \pmod{3} \\
-3 & \text{if } n \equiv 1 \pmod{3} \\
1 & \text{if } n \equiv 2 \pmod{3}
\end{cases}.
\end{align}

In [4, 5], the authors defined a new matrix generalization of the Fibonacci and Lucas numbers, and using essentially a matrix approach they showed properties of these matrix sequences. The main motivation of this article is to study the matrix sequences of third-order Jacobsthal sequence and third-order Jacobsthal sequence.

2. The third-order Jacobsthal, third-order Jacobsthal-Lucas sequences and their matrix sequences

Now, considering these sequences, we define the matrix sequences which have elements of third-order Jacobsthal and third-order Jacobsthal-Lucas sequences.

\textbf{Definition 2.1.} Let \(n \geq 0\). The third-order Jacobsthal matrix sequence \(\{JM_n^{(3)}\}_{n \in \mathbb{N}}\) and third-order Jacobsthal-Lucas matrix sequence \(\{jM_n^{(3)}\}_{n \in \mathbb{N}}\) are defined respectively by
\begin{align}
\tag{2.15} JM_{n+3}^{(3)} &= JM_{n+2}^{(3)} + JM_{n+1}^{(3)} + 2JM_n^{(3)}, \\
\tag{2.16} jM_{n+3}^{(3)} &= jM_{n+2}^{(3)} + jM_{n+1}^{(3)} + 2jM_n^{(3)},
\end{align}
Theorem 2.2. For $n \geq 0$, we have

\[
M_{J,n}^{(3)} = \left( \frac{M_{j,2}^{(3)} + M_{j,1}^{(3)} + M_{j,0}^{(3)}}{(2 - \omega_1)(2 - \omega_2)} \right) 2^n - \left( \frac{M_{j,2}^{(3)} - (2 + \omega_2)M_{j,1}^{(3)} + 2\omega_2 M_{j,0}^{(3)}}{(2 - \omega_1)(\omega_1 - \omega_2)} \right) \omega_1^n
\]

\[
+ \left( \frac{M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)}}{(2 - \omega_2)(\omega_1 - \omega_2)} \right) \omega_2^n.
\]

(2.17)



(2.18)

Proof. (2.17): The solution of Eq. (2.15) is

\[
M_{J,n}^{(3)} = c_1 2^n + c_2 \omega_1^n + c_3 \omega_2^n.
\]

Then, let $M_{J,0}^{(3)} = c_1 + c_2 + c_3$, $M_{J,1}^{(3)} = 2c_1 + c_2 \omega_1 + c_3 \omega_2$ and $M_{J,2}^{(3)} = 4c_1 + c_2 \omega_1^2 + c_3 \omega_2^2$. Therefore, we have $(2 - \omega_1)(2 - \omega_2)c_1 = M_{J,2}^{(3)} - (\omega_1 + \omega_2)M_{J,1}^{(3)} + \omega_1 \omega_2 M_{J,0}^{(3)}$,

$(2 - \omega_1)(\omega_1 - \omega_2)c_2 = M_{J,2}^{(3)} - (2 + \omega_2)M_{J,1}^{(3)} + 2\omega_2 M_{J,0}^{(3)}$, $(2 - \omega_2)(\omega_1 - \omega_2)c_3 = M_{J,2}^{(3)} - (2 + \omega_1)M_{J,1}^{(3)} + 2\omega_1 M_{J,0}^{(3)}$. Using $c_1$, $c_2$ and $c_3$ in Eq. (2.19), we obtain

\[
M_{J,n}^{(3)} = \left( \frac{M_{J,2}^{(3)} + M_{J,1}^{(3)} + M_{J,0}^{(3)}}{(2 - \omega_1)(2 - \omega_2)} \right) 2^n - \left( \frac{M_{J,2}^{(3)} - (2 + \omega_2)M_{J,1}^{(3)} + 2\omega_2 M_{J,0}^{(3)}}{(2 - \omega_1)(\omega_1 - \omega_2)} \right) \omega_1^n
\]

\[
+ \left( \frac{M_{J,2}^{(3)} - (2 + \omega_1)M_{J,1}^{(3)} + 2\omega_1 M_{J,0}^{(3)}}{(2 - \omega_2)(\omega_1 - \omega_2)} \right) \omega_2^n.
\]

(2.18)

(2.19)

The proof is similar to the proof of (2.17). □

The following theorem gives us the $n$-th general term of the sequence given in (2.15) and (2.16).

Theorem 2.3. For $n \geq 3$, we have

\[
M_{J,n}^{(3)} = \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n}^{(3)} \\
J_{n}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-1}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-2}^{(3)}
\end{bmatrix}
\]

(2.20)
Theorem 2.4. □

Also holds for \( n = 0 \) in (1.2). We have

Proof. (2.20): Let use the principle of mathematical induction on \( n \). For equality in (2.20) holds for all \( n \). By iterating this procedure and considering induction steps, let us assume that the equality in (2.20) holds for all \( n = k \in \mathbb{N} \). To finish the proof, we have to show that (2.20) also holds for \( n = k + 1 \) by considering (1.2) and (2.15). Therefore we get

\[
M_{j,k+2}^{(3)} = M_{j,k+1}^{(3)} + M_{j,k}^{(3)} + 2M_{j,k-1}^{(3)}
\]

By iterating this procedure and considering induction steps, let us assume that the equality in (2.20) holds for all \( n = k \in \mathbb{N} \). To finish the proof, we have to show that (2.20) also holds for \( n = k + 1 \) by considering (1.2) and (2.15). Therefore we get

\[
M_{j,k+2}^{(3)} = M_{j,k+1}^{(3)} + M_{j,k}^{(3)} + 2M_{j,k-1}^{(3)}
\]

Hence we obtain the result. If a similar argument is applied to (2.21), the proof is clearly seen. □

Theorem 2.4. Assume that \( x \neq 0 \). We obtain,

(2.22) \[
\sum_{k=0}^{n} \frac{M_{j,k}^{(3)}}{x^k} = \frac{1}{x^n \nu(x)} \left\{ \begin{array}{ll}
2M_{j,n}^{(3)} + \left( M_{j,n+2}^{(3)} - M_{j,n+1}^{(3)} \right) x + M_{j,n+1}^{(3)} x^2 \\
-M_{j,2}^{(3)} - M_{j,1}^{(3)} - M_{j,0}^{(3)} - \left( M_{j,0}^{(3)} - M_{j,1}^{(3)} \right) x + M_{j,0}^{(3)} x^2
\end{array} \right\},
\]

(2.23) \[
\sum_{k=0}^{n} \frac{M_{j,k}^{(3)}}{x^k} = \frac{1}{x^n \nu(x)} \left\{ \begin{array}{ll}
2M_{j,n}^{(3)} + \left( M_{j,n+2}^{(3)} - M_{j,n+1}^{(3)} \right) x + M_{j,n+1}^{(3)} x^2 \\
-M_{j,2}^{(3)} - M_{j,1}^{(3)} - M_{j,0}^{(3)} - \left( M_{j,0}^{(3)} - M_{j,1}^{(3)} \right) x + M_{j,0}^{(3)} x^2
\end{array} \right\},
\]

where \( \nu(x) = x^3 - x^2 - x - 2 \).

Proof. In contrast, here we will just prove (2.23) since the proof of (2.22) can be done in a similar way. From Theorem 2.2 we have

\[
\sum_{k=0}^{n} \frac{M_{j,k}^{(3)}}{x^k} = \left( \frac{M_{j,2}^{(3)} + M_{j,1}^{(3)} + M_{j,0}^{(3)}}{(2 - \omega_1)(2 - \omega_2)} \right) \sum_{k=0}^{n} \left( \frac{2}{x} \right)^k
\]

\[
- \left( \frac{M_{j,2}^{(3)} - (2 + \omega_2)M_{j,1}^{(3)} + 2\omega_2 M_{j,0}^{(3)}}{(2 - \omega_1)(\omega_1 - \omega_2)} \right) \sum_{k=0}^{n} \left( \frac{\omega_1}{x} \right)^k
\]

\[
+ \left( \frac{M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)}}{(2 - \omega_2)(\omega_1 - \omega_2)} \right) \sum_{k=0}^{n} \left( \frac{\omega_2}{x} \right)^k.
\]
By considering the definition of a geometric sequence, we get

\[
\sum_{k=0}^{n} M_{j,k}^{(3)} = \frac{M_{j,2}^{(3)} + M_{j,1}^{(3)} + M_{j,0}^{(3)}}{(2 - \omega_1)(2 - \omega_2)} x^{n+1} - x^{n+1} \frac{2n+1}{x^n(2 - x)} = \left( \frac{M_{j,2}^{(3)} - (2 + \omega_2)M_{j,1}^{(3)} + 2\omega_2 M_{j,0}^{(3)}}{(2 - \omega_1)(\omega_1 - \omega_2)} \right) \frac{\omega_1^{n+1} - x^{n+1}}{x^n(\omega_1 - x)} + \left( \frac{M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)}}{(2 - \omega_2)(\omega_1 - \omega_2)} \right) \frac{\omega_2^{n+1} - x^{n+1}}{x^n(\omega_2 - x)}
\]

\[
= \frac{1}{x^n \nu(x)} \begin{cases} 
\frac{(M_{j,2}^{(3)} - (2 + \omega_2)M_{j,1}^{(3)} + 2\omega_2 M_{j,0}^{(3)})}{(2 - \omega_1)(\omega_1 - \omega_2)} (2n+1 - x^{n+1})(\omega_1 - x)(\omega_2 - x) \\
\frac{(M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)})}{(2 - \omega_2)(\omega_1 - \omega_2)} (2n+1 - x^{n+1})(2 - x)(\omega_1 - x) \\
+ \frac{(M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)})}{(2 - \omega_2)(\omega_1 - \omega_2)} (2n+1 - x^{n+1})(2 - x)(\omega_2 - x)
\end{cases},
\]

where \( \nu(x) = x^3 - x^2 - x - 2 \). If we rearrange the last equality, then we obtain

\[
\sum_{k=0}^{n} M_{j,k}^{(3)} = \frac{1}{x^n \nu(x)} \begin{cases} 
2M_{j,2}^{(3)} + \left( M_{j,n+2}^{(3)} - M_{j,n+1}^{(3)} \right) x + M_{j,0}^{(3)} - M_{j,1}^{(3)} - M_{j,0}^{(3)} x + M_{j,0}^{(3)} x^2
\end{cases}.
\]

So, the proof is completed.

In the following theorem, we give the sum of third-order Jacobsthal and third-order Jacobsthal-Lucas matrix sequences corresponding to different indices.

**Theorem 2.5.** For \( r \geq m \), we have

(2.24)

\[
\sum_{k=0}^{n} M_{j,mk+r}^{(3)} = \frac{1}{\sigma_n} \begin{cases} 
M_{j,(m(n+1)+r)}^{(3)} - M_{j,r}^{(3)} + 2M_{j,(mn+r)}^{(3)} - 2M_{j,(r+m)}^{(3)} \\\n- M_{j,(m(n+1)+r)} M_{j,mn+r}^{(3)} + M_{j,(m+1)r}^{(3)} m + M_{j,(m+2)r}^{(3)} - M_{j,rm}^{(3)}
\end{cases}
\]

(2.25)

\[
\sum_{k=0}^{n} M_{j,mk+r}^{(3)} = \frac{1}{\sigma_n} \begin{cases} 
M_{j,(m(n+1)+r)}^{(3)} - M_{j,r}^{(3)} - 2M_{j,(mn+r)}^{(3)} + 2M_{j,(r+m)}^{(3)} \\\n- M_{j,(m(n+1)+r)} M_{j,mn+r}^{(3)} + M_{j,(m+1)r}^{(3)} m + M_{j,(m+2)r}^{(3)} - M_{j,rm}^{(3)}
\end{cases},
\]

where \( \sigma_n = 2n+1 + (1 - 2m)(\omega_1^m + \omega_2^m) - 2 \) and \( m(\mu) = 2^n + \omega_1^m + \omega_2^m \).

**Proof.** (2.24): Let us take \( A = \frac{M_{j,2}^{(3)} + M_{j,1}^{(3)} + M_{j,0}^{(3)}}{(2 - \omega_1)(2 - \omega_2)} \), \( B = \frac{M_{j,2}^{(3)} - (2 + \omega_1)M_{j,1}^{(3)} + 2\omega_1 M_{j,0}^{(3)}}{(2 - \omega_2)(\omega_1 - \omega_2)} \) and \( C = \frac{M_{j,2}^{(3)} - (2 + \omega_2)M_{j,1}^{(3)} + 2\omega_2 M_{j,0}^{(3)}}{(2 - \omega_1)(\omega_1 - \omega_2)} \). Then, we write

\[
\sum_{k=0}^{n} M_{j,mk+r}^{(3)} = \sum_{k=0}^{n} (A2^m + B_1^{m+1} + C_2^{m+1})
\]

\[
= A2^r \sum_{k=0}^{n} 2^m - B_1^{m+1} \sum_{k=0}^{n} \omega_1^m + C_2^{m+1} \sum_{k=0}^{n} \omega_2^m
\]

\[
= A2^r \frac{2^m - 1}{2^m - 1} - B_1^{m+1} \left( \frac{\omega_1^{m+1} - 1}{\omega_1^m - 1} \right) + C_2^{m+1} \left( \frac{\omega_2^{m+1} - 1}{\omega_2^m - 1} \right)
\]

\[
= \frac{1}{\sigma_n} \begin{cases} 
\left( A2^m + B_1^{m+1} + C_2^{m+1} \right) \left( \omega_1^m - \omega_2^m \right) \left( m(\mu) - 1 \right)
\end{cases},
\]
where $\sigma_n = 2^{m+1} + (1 - 2^m)(\omega_1^m + \omega_2^m) - 2$. After some algebra, we obtain

$$\sum_{k=0}^{n} M_{j,mk+r}^{(3)} = \frac{1}{\sigma_n} \begin{bmatrix} M_{j,m(n+1)+r}^{(3)} - M_{j,r}^{(3)} + 2^m M_{j,m+1+r}^{(3)} & \cdots & -M_{j,m(n+1)+r}\mu(m) + M_{j,r}\mu(m) + M_{j,m(n+2)+r} - M_{j,r+m}^{(3)} \end{bmatrix},$$

where $\mu(m) = 2^m + \omega_1^m + \omega_2^m$.

(2.25): The proof is similar to the proof of (3.24).

□

3. The relationships between matrix sequences $M_{j,n}^{(3)}$ and $M_{j,n}^{(3)}$

Lemma 3.1. For $m, n \in \mathbb{N}$, the third-order Jacobsthal and third-order Jacobsthal-Lucas matrix sequences are commutative. The following results hold.

(3.26) $M_{j,n}^{(3)}M_{j,m}^{(3)} = M_{j,m}^{(3)}M_{j,n}^{(3)},$

(3.27) $M_{j,n}^{(3)}M_{j,m}^{(3)} = M_{j,m}^{(3)}M_{j,n}^{(3)},$

(3.28) $M_{j,1}M_{j,n}^{(3)} = M_{j,n+1}^{(3)}M_{j,1},$

(3.29) $M_{j,1}M_{j,n}^{(3)} = M_{j,n+1}^{(3)}M_{j,1},$

(3.30) $M_{j,1}M_{j,n}^{(3)} = M_{j,2n+1}^{(3)}.$

Proof. Here, we will just prove (3.26) and (3.28) since (3.27), (3.29) and (3.30) can be dealt with in the same manner. To prove Eq. (3.26), let us use the induction on $m$. If $m = 0$, the proof is obvious since that $M_{j,0}^{(3)}$ is the identity matrix of order 3. Let us assume that Eq. (3.26) holds for all values $k$ less than or equal to $m$. Now we have to show that the result is true for $m + 1$:

$$M_{j,n+1}^{(3)} = M_{j,m}^{(3)}M_{j,m}^{(3)} + 2M_{j,m}^{(3)}M_{j,m+1}^{(3)} + 2M_{j,m}^{(3)}M_{j,m+1}^{(3)} = M_{j,m}^{(3)}M_{j,m+1}^{(3)} + 2M_{j,m}^{(3)}M_{j,m+1}^{(3)} = M_{j,m}^{(3)}M_{j,m+1}^{(3)} + 2M_{j,m}^{(3)}M_{j,m+1}^{(3)} = M_{j,m}^{(3)}M_{j,m+1}^{(3)} + 2M_{j,m}^{(3)}M_{j,m+1}^{(3)} = M_{j,m}^{(3)}M_{j,m+1}^{(3)}.$$

It is easy to see that $M_{j,n+1}^{(3)}M_{j,m}^{(3)} = M_{j,m}^{(3)}M_{j,n+1}^{(3)}$. Hence we obtain the result.

(3.28): To prove equation (3.28), we again use induction on $n$. Let $n = 0$, we get $M_{j,1}^{(3)}M_{j,0}^{(3)} = M_{j,1}^{(3)}$. Let us assume that $M_{j,1}^{(3)}M_{j,n}^{(3)} = M_{j,n+1}^{(3)}$ is true for all values $k$ less than or equal to $m$. Then,

$$M_{j,n+1}^{(3)} = \begin{bmatrix} j_{n+2}^{(3)} & j_{n+1}^{(3)} & 2j_n^{(3)} & 2j_{n-1}^{(3)} \\ j_{n+1}^{(3)} & j_n^{(3)} & 2j_{n-1}^{(3)} & 2j_{n-2}^{(3)} \\ j_n^{(3)} & j_{n-1}^{(3)} & 2j_{n-2}^{(3)} & 2j_{n-3}^{(3)} \\ j_{n-1}^{(3)} & j_{n-2}^{(3)} & 2j_{n-3}^{(3)} & 2j_{n-4}^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_{j,n}^{(3)}M_{j,1}^{(3)} = M_{j,1}^{(3)}M_{j,n}^{(3)} = M_{j,1}^{(3)}M_{j,n-1}^{(3)} = M_{j,1}^{(3)}.$$

Hence the result. □
Theorem 3.2. For $m, n \in \mathbb{N}$ the following properties hold.

(3.31) \[ M_{j,n}^{(3)} = M_{j,n}^{(3)} + 4M_{j,n-1}^{(3)} + 4M_{j,n-2}^{(3)}. \]

(3.32) \[ M_{j,n}^{(3)} = 2M_{j,n+1}^{(3)} - M_{j,n}^{(3)} + 2M_{j,n-1}^{(3)}. \]

(3.33) \[ M_{j,n}^{(3)} = M_{j,n+2}^{(3)} + 3M_{j,n}^{(3)} + 2M_{j,n-1}^{(3)}. \]

Proof. First, here, we will just prove (3.31) and (3.33) since (3.32) can be dealt with in the same manner. So, if we consider the right-hand side of equation (3.31) and use Theorem 2.3 we get

\[
M_{j,n}^{(3)} + 4M_{j,n-1}^{(3)} + 4M_{j,n-2}^{(3)} = \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix} + 4 \begin{bmatrix}
J_{n}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-3}^{(3)} + 2J_{n-4}^{(3)} \\
\end{bmatrix}
\
\]

and we get

\[ M_{j,n}^{(3)}, \]

From Eq. (1.5), $j_n^{(3)} = j_n^{(3)} + 4j_{n-1}^{(3)} + 4j_{n-2}^{(3)}$, as required in (3.31).

Second, let us consider the left-hand side of Eq. (3.33). Using Theorem 2.3, we write

\[ M_{j,1}^{(3)}M_{j,n}^{(3)} = \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix} \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix}. \]

From matrix product, we have

\[
M_{j,1}^{(3)}M_{j,n}^{(3)} = \begin{bmatrix}
5 & 5 & 2 \\
1 & 4 & 4 \\
2 & -1 & 2 \\
\end{bmatrix} \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix}
\]

\[
+ 4 \begin{bmatrix}
J_{n}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-3}^{(3)} + 2J_{n-4}^{(3)} \\
\end{bmatrix} \begin{bmatrix}
J_{n}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-3}^{(3)} + 2J_{n-4}^{(3)} \\
\end{bmatrix}
\]

\[
+ 3 \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix}
\]

\[
+ 2 \begin{bmatrix}
J_{n+1}^{(3)} & J_{n}^{(3)} + 2J_{n-1}^{(3)} & 2J_{n-2}^{(3)} \\
J_{n}^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-3}^{(3)} \\
J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-4}^{(3)} \\
\end{bmatrix}
\]

\[ M_{j,n}^{(3)} + 3M_{j,n+1}^{(3)} + 2M_{j,n-1}^{(3)}. \]

Hence the result. □

Theorem 3.3. For $m, n \in \mathbb{N}$, the following properties hold.

(3.34) \[ M_{j,m}^{(3)}M_{j,n+1}^{(3)} = M_{j,n+1}^{(3)}M_{j,m}^{(3)} = M_{j,m+n+1}^{(3)}. \]

(3.35) \[ (M_{j,n+1}^{(3)})^{m} = (M_{j,1}^{(3)})^{m} M_{j,m-n}^{(3)}. \]
Proof. (3.34): Let us consider the left-hand side of equation (3.34) and Lemma 3.1 and Theorem 3.2. We have
\[
M_{j,m}^{(3)} M_{j,n}^{(3)} = M_{j,m}^{(3)} \left(2M_{j,2}^{(3)} - M_{j,1}^{(3)} + 2M_{j,0}^{(3)}\right) M_{j,n}^{(3)}
\]
\[= 2M_{j,m+n+2}^{(3)} - M_{j,m+n+1}^{(3)} + 2M_{j,m+n}^{(3)}
\]
\[= \left(2M_{j,2}^{(3)} - M_{j,1}^{(3)} + 2M_{j,0}^{(3)}\right) M_{j,m+n}^{(3)}.
\]
Moreover, from Eq. (3.32) in Theorem 3.2, we obtain
\[
M_{j,m}^{(3)} M_{j,n}^{(3)} = M_{j,1}^{(3)} M_{j,m}^{(3)} M_{j,n}^{(3)} = M_{j,m+1}^{(3)} M_{j,m}^{(3)}.
\]
Also, from Lemma 3.1, it is seen that
\[
M_{j,m}^{(3)} M_{j,n}^{(3)} = M_{j,1}^{(3)} M_{j,m}^{(3)} M_{j,n}^{(3)} = M_{j,m+1}^{(3)} M_{j,m}^{(3)}.
\]
Therefore, we have to show that it is true for \(m+1\). If we multiply this \(m\)-th step by \(M_{j,n}^{(3)}\) on both sides from the right, then we have
\[
\left(M_{j,n+1}^{(3)}\right)^{m+1} = \left(M_{j,1}^{(3)}\right)^{m} M_{j,m}^{(3)} M_{j,n+1}^{(3)}
\]
\[= \left(M_{j,1}^{(3)}\right)^{m} M_{j,m}^{(3)} M_{j,n}^{(3)}
\]
\[= \left(M_{j,1}^{(3)}\right)^{m} M_{j,m}^{(3)} M_{j,n+1}^{(3)}
\]
\[= \left(M_{j,1}^{(3)}\right)^{m+1} M_{j,m+n}^{(3)}
\]
\[= \left(M_{j,1}^{(3)}\right)^{m+1} M_{j,(m+1)n}^{(3)}
\]
which finishes the induction and gives the proof of (3.35).

□

Corollary 3.4. For \(n \geq 0\), by taking \(m = 2\) and \(m = 3\) in the Eq. (3.35) given in Theorem 3.3, we obtain
\[
(3.36) \quad \left(M_{j,n+1}^{(3)}\right)^{2} = \left(M_{j,1}^{(3)}\right)^{2} M_{j,2n}^{(3)} = M_{j,1}^{(3)} M_{j,2n+1}^{(3)}.
\]
\[
(3.37) \quad \left(M_{j,n+1}^{(3)}\right)^{3} = \left(M_{j,1}^{(3)}\right)^{3} M_{j,3n}^{(3)} = \left(M_{j,1}^{(3)}\right)^{2} M_{j,3n+1}^{(3)}.
\]

Corollary 3.5. For \(n \in \mathbb{N}\), we have the following result
\[
(3.38) \quad \left(j_{n+1}^{(3)}\right)^{2} + \left(j_{n}^{(3)}\right)^{2} + 2j_{n}^{(3)} j_{n-1}^{(3)} = 34J_{2n+1}^{(3)} + 43J_{2n-1}^{(3)} + 34J_{2n}^{(3)}
\]
\[= 5j_{2n+2} + 5j_{2n+1} + 2j_{2n}.
\]
Proof. The proof can be easily seen by the coefficient in the first row and column of the matrix \(M_{j,n}^{(3)}\) in (3.36) and the Eq. (2.16).

□

4. Conclusions

In this paper, we study a generalization of the Jacobsthal and Jacobsthal-Lucas matrix sequences. Particularly, we define the third-order Jacobsthal and third-order
Jacobsthal-Lucas matrix sequences, and we find some combinatorial identities. As seen in [6] one way to generalize the Jacobsthal recursion is as follows

\[ J^{(r)}_{n+r} = \sum_{k=0}^{r-1} J^{(r)}_{n+r-k} + 2J^{(r)}_n, \]

with \( n \geq 0 \) and initial conditions \( J^{(r)}_k \), for \( k = 0, 1, \ldots, r - 2 \) and \( J^{(r)}_{r-1} = 1 \), has characteristic equation \((x - 2)(x^{r-1} + x^{r-2} + \cdots + 1) = 0\) with eigenvalues 2 and \( \omega_k = e^{2\pi i m/r} \), for \( k = 0, 1, \ldots, r - 1 \). It would be interesting to introduce the higher order Jacobsthal and Jacobsthal-Lucas matrix sequences. Further investigations for these and other methods useful in discovering identities for the higher order Jacobsthal and Jacobsthal-Lucas sequences will be addressed in a future paper.

References

[1] P. Barry, Triangle geometry and Jacobsthal numbers, Irish Math. Soc. Bull. 51 (2003), 45–57.
[2] G. Cerda-Morales, Identities for Third Order Jacobsthal Quaternions, Advances in Applied Clifford Algebras 27(2) (2017), 1043–1053.
[3] G. Cerda-Morales, On a Generalization of Tribonacci Quaternions, Mediterranean Journal of Mathematics 14:239 (2017), 1–12.
[4] H. Civciv and R. Türkmen, On the (s, t)-Fibonacci and Fibonacci matrix sequences, Ars Combinatoria 87 (2008), 161–173.
[5] H. Civciv and R. Türkmen, Notes on the (s, t)-Lucas and Lucas matrix sequences, Ars Combinatoria 89 (2008), 271–285.
[6] Ch. K. Cook and M. R. Bacon, Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations, Annales Mathematicae et Informaticae 41 (2013), 27–39.
[7] A. F. Horadam, Jacobsthal representation numbers, Fibonacci Quarterly 34 (1996), 40–54.

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