Induced orders in free monoids of words

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Abstract
A family of partial orders in the free monoid $A^*$ of words, induced from a partial order in alphabet $A$, is presented. The induced orders generalize the chronological posets that have been defined for the two-letter alphabet only, and the morphological order. We show that the induced orders are natural with respect to alphabet homomorphisms.

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1. Motivation

Algebra of orientons $A^*$ is defined in [Ko] as the free monoid of words [Lo] over a two-letter alphabet $A = \{\pi, \varphi\}$ ordered $\pi < \varphi$. Two partial orders [Be] are introduced into the monoid $A^*$: the self-evident morphological order representing complexity of words, and a less apparent chronological order that has been induced from $\pi < \varphi$ assumed in the alphabet $A$. The relations are defined as follows:

Definition 1.1. Let $A^* = \{\pi, \varphi\}^*$ be words over $A = \{\pi, \varphi\}$. For any two elements $v, w \in A^*$ one defines

(a) morphological order: $v < w$ if $w$ may be obtained by inserting some letters of $A$ into $v$.

(b) chronological order: $v < w$ if $w$ may be obtained from $v$ by erasing some letters $\pi$ and/or inserting some letters $\varphi$ into $v$.

If $\ll$ and $\lll$ denote the covering relation (immediate succession) of the respective orders, then the above definitions may be expressed in terms of single insertions:

(a') $v \ll w$ if $\exists x \in A : v' \circ x \circ v'' = w$ for some splitting $v = v' \circ v''$

(b') $v \lll w$ if either $v' \circ \varphi \circ v'' = w$ or $v = w' \circ \pi \circ w''$ for some splitting $v = v' \circ v''$ or $w = w' \circ w''$.

The corresponding posets are denoted respectively Morph $A^* = \{A^*, <\}$ and Chron $A^* = \{A^*, \ll\}$. Figure 1 displays them for words of length $|w| < 4$. Poset Morph $A^*$ has the least element, the empty word $\varepsilon$; Chron $A^*$ is unbounded.
Remark 1.2. Algebra of “orientons” \( \{\pi, \varphi\}^* \) was introduced in [Ko] to model chronological meanings of grammatical tenses. Intuitively, \( \pi \) refers to the “past”, \( \varphi \) — to the “future”, the empty word \( \varepsilon \) — to the “present tense,” and, for instance, \( \pi\pi\varphi \) — to the “future-in-the-past-in-the-past”. The order \( \pi \prec \varphi \) in the alphabet (“past” precedes “future”) induces a “chronological order” among the words of \( \{\pi, \varphi\}^* \). This infinite set contains a homomorphic images of formal tense systems including a mathematical model of the tense system of English.

While the morphological order is a natural relation associated with any free monoid over an alphabet, the chronological order in \( \{\pi, \varphi\}^* \) is not, and the question rises, “How far may this construction be extended beyond the simple two-letter alphabet?” The form of Definition 1 hardly seems to suggest any possible natural generalization.

Problem 1. Let \( A \) be a partially ordered set of a countable cardinality, and let \( A^* \) be the free monoid of words over the alphabet \( A \). Is there any natural order in \( A^* \) which (i) would be an extension of the order in \( A \) (\( A \) is embedded into \( A^* \) as the one-letter words); (ii) would be natural with respect to homomorphisms of alphabets; and (iii) would coincide with the chronological order for the simple case of \( A = \{\pi \prec \varphi\} \)?

The answer to this problem is affirmative, although the construction is not a direct generalization of Definition 1.1b. The following notes describe this construction.

2. Induced order

In this section, we show a principal construction by which every letter of an ordered alphabet defines an induced order among the words over the original alphabet without the chosen letter.

Let \( \{A, \prec\} \) be a partially ordered set. For any \( a \in A \) we denote \( A_a = A - \{a\} \). Consequently,

\[
A_a^* = (A - \{a\})^*
\] (1)
denotes the free monoid over the alphabet $A_a$. Of course, the relation $<$ restricted to $A_a$ makes it a poset. We shall call a word $w' \in A_a^*$ an $a$-extension of a word $w \in A_a^*$ if $w'$ may be obtained by a number of insertions of the letter $a$ into $w$. (For instance mississippi is an $s$-extension of miiipii, which in turn is a $p$-extension of miii).

Now, we define a relation among words of $A_a^*$, induced by the order $<$ in $A$.

**Definition 2.1.** Let $v, w \in A_a^*$ be two words over a partially ordered alphabet $\{A_a, <\}$. We say write

$$v <_a w$$

if $a$-extensions $v'$ and $w'$ of $v$ and $w$ respectively exist, such that they are of the same length $|v'| = |w'| = n$, and $v' < w'$ in $A^n$, i.e., for each letter it is $v'_i < w'_i$ in $A$, $i = 1, \ldots, n$.

It is not clear whether different insertions would not lead to different directions of $<_a$ for the same pair of words. Here is the main assertion of this note:

**Theorem 2.2.** The relation $<_a$ defines a partial order in $A_a^*$.

The relation $<_a$ is clearly reflexive and transitive. The problem is whether it is skew-symmetric. First, we prove the following lemma.

**Lemma 2.3.** Let $w'$ and $w''$ be two $a$-extensions of the same length $|w'| = |w''| = n$ of a word $w \in A_a^*$ such that $w' < w''$ in $A^n$. Then $w' = w''$.

**Proof.** Let $w' = w'_1 \circ w'_2 \circ \ldots \circ w'_{n'}$, where $w'_i \in A$ for each $i = 1, \ldots, n$. Since $w''$ is of the same length as $w'$, it must be composed from the same collection of letters, and the arrangement of letters in $w''$ is a permutation of the arrangement of letters in $w'$:

$$w'' = w'_{\sigma(1)} \circ w'_{\sigma(2)} \circ \ldots \circ w'_{\sigma(n)}$$

for some $\sigma \in S_n$.

Now, by definition the assumed relation $w' < w''$ in $A^n$ means that it holds for the corresponding letters of $w'$ and $w''$ in $A$:

$$w'_1 < w''_1 = w'_{\sigma(1)}$$
$$w'_2 < w''_2 = w'_{\sigma(2)}$$
$$\vdots$$
$$w'_n < w''_n = w'_{\sigma(n)}$$

Each permutation may be uniquely decomposed into a number of cycles. Assume that $\sigma$ has a cycle of order bigger than 1, say $k$. Then, for some $w'_i \in A$ it is

$$w'_i < w'_{\sigma(i)} < w'_{\sigma^2(i)} < \ldots < w'_{\sigma^{k-1}(i)} < w'_{\sigma^k(i)} = w'_i$$

By transitivity both $w'_i < w'_{\sigma(i)}$ and $w'_{\sigma(i)} < w'_i$ hold. This contradicts the partial order of the alphabet, unless the order of each cycle of $\sigma$ is 1. Therefore $w'_i = w'_{\sigma(i)}$ for each $i$, so $w' = w''$, proving the lemma. □
Corollary. Any two $a$-extensions of the same length $n$ of a word $w \in A_n^a$ are either identical or incomparable in the product poset $A^n$.

Now we can prove the theorem.

Proof of Theorem 2.2: In order to show that the induced relation $\prec_a$ in $A_n^a$ is skew-symmetric for different elements, let us assume a contrario that there exist two different insertions of the letter $a$ into a pair of words $v$ and $w$ in $A_n^a$, such that the resulting two pairs of $a$-extensions, $v'$, $w'$, and $v''$, $w''$, lead to opposite relations:

- $v \prec_a w$ by one insertion, for which $v' \prec w'$ in $A^n$, and
- $v \succ_a w$ by the other one, for which $v'' \succ w''$ in $A^k$,

where the lengths of the words are $|v'| = |w'| = n$ and $|v''| = |w''| = k$ for some $n$ and $k$.

In the form of a diagram:

$$
\begin{array}{c}
 v \\
\downarrow \times \downarrow \\
 v' \prec w' \quad v'' \succ w''
\end{array}
$$

(in $A^n$) (in $A^k$)

Assume that $v \neq w$, since Lemma 2.3 proves the theorem for $v = w$.

Notice that additional simultaneous insertion of the letter $a$ into the words $v'$, $w'$ at the same position preserves the original relation, now in $A^{n+1}$. A number of such insertions will be called a coherent $a$-extension of a pair of words (of the same length).

Now, since $w'$ and $w''$ result by insertions of the letter $a$ into the same word $w$, one may find a further minimal $a$-extensions of $w'$ and of $w''$ such that the resulting words $\bar{w}'$ and $\bar{w}''$ will be identical in $A^m$ for some $m$:

$$
\bar{w}' = \bar{w}'' \quad \text{in } A^m
$$

If $\bar{v}'$ is the $a$-extension of $v'$ coherent with that of $w'$, and $\bar{v}''$ the extension coherent with that of $w''$, the relations between the words become:

$$
\begin{array}{c}
 v' \prec w' \\
\bar{w}' \prec \bar{w}'' \prec \bar{v}'',
\end{array}
$$

in $A^m$, and by (2) and transitivity:

$$
\bar{v}' \prec \bar{v}''.
$$

By Lemma 2.3, this implies $\bar{v}' = \bar{v}''$. Therefore, from (3):

$$
\bar{v}' = \bar{w}' = \bar{w}'' = \bar{v}'' \quad \text{(in } A^m)
$$

Removing all $a$’s from these words we get $v = w$, which contradicts the assumption and concludes the proof. □

We obtain a whole family of induced orders, labeled by the elements of $A$. In particular:
Corollary. If $A = \{ \pi, \eta, \varphi \}$ is linearly ordered $\pi < \eta < \varphi$, then the induced order $\prec_\eta$ in $A_\eta^* = \{ \pi, \varphi \}^*$ coincides with the chronological order of the word algebra over the alphabet $\{ \pi, \varphi \}$. (see Definition 1.1).

3. Augmentation

Now the solution to the Problem (Section 1) seems plausible. In order to obtain a relation in the monoid $A^*$ over an ordered alphabet $A$, one has to enrich first the alphabet by one element, say $e$, and to extend the order of $A$ into $A' = A \cup \{ e \}$, and then apply the technique of induced order described in the previous section. We shall call letter $e$ an auxiliary letter. Poset $A'$ will be called an augmented alphabet.

Definition 3.1. An augmentation of a (finite) poset $A$ is an isomorphism of $A$ into a poset $A'$ of cardinality $|A'| = |A| + 1$.

The original alphabet is restored by dropping letter $e$, i.e. as a set $A \equiv (A \cup \{ e \})_e$. The partial order $\prec_e$ defined by Definition 1.1 turns $A^* \equiv (A \cup \{ e \})^*_e$, due to Theorem 2.3, into a poset.

Of course, the order so obtained strongly depends on the particular choice of augmentation.

Example 1: Consider $A = \{ \pi, \varphi \}$ with $\pi < \varphi$. In order to get an induced order in $A^*$, an augmented poset must be constructed with elements $A' = \{ \pi, \varphi, \eta \}$, where $\eta$ is an auxiliary letter. There are three ways to equip $A'$ with a linear order that agrees with the order in $A$:

(a) $\eta < \pi < \varphi$,  
(b) $\pi < \eta < \varphi$,  
(c) $\pi < \varphi < \eta$

Each leads to another partial order in $A^* = \{ \pi, \varphi \}^*$. Figure 2 displays the corresponding induced posets for the words of length $|w| \leq 3$. Case (b) is identical with the chronological order (see Definition 1.1). Case (c) is dual to the case (a) (replace $\varphi$ with $\pi$ and flip the diagram upside down). Notice that in each of these cases, the one-letter words, which may be identified with the elements of the alphabet, preserve their order $\pi < \varphi$ within $A^*$.

![Figure 2: Three posets over $\{ \pi, \varphi \}^*$ induced by linear augmentations of $\pi < \varphi$.](image-url)
The last observation can be generalized:

**Proposition 3.2.** For any \( n \in \mathbb{N} \) and for any augmentation of a poset \( A \), the poset \( A^n \) with the product partial order is isomorphically embedded into the induced word posets \( A^* \).

**Proof.** Identify the Cartesian product \( A^n \) with words in \( A^* \) of the fixed length \( n \), \( A^n \equiv \{ a \in A^* \mid |a| = n \} \). For any augmentation, if two words are related in \( A^n \), so are they, by definition, in \( A^* \); if they are not related in \( A^n \), then by Corollary 2.4 they are not related in \( A^* \).

In particular, the natural embedding of an alphabet \( A \) into the one-letter words in \( A^* \) is an isomorphism of the order structures. For an illustration of \( n = 2 \) and \( n = 3 \), recognize the particular posets of \( A^n \) (Figure 4) in Figure 2 and Figure 3. Notice that the above property may be extended to an embedding of \( A' = A \cup \{ \varepsilon \} \rightarrow A^* \), if the empty word \( \varepsilon \in A^* \) is reinterpreted as the auxiliary letter \( \eta \) in \( A' \).

The following obvious property ensures naturality of induced order, which was sought in Problem 1.2.

**Proposition 3.3.** Let \( f : A \rightarrow B \) be a homomorphism of posets. For any \( e \in A \), the induced map \( f^* : A^*_e \rightarrow B^*_f(e) \), defined letter-wise, is also a poset homomorphism of induced orders.

**Proof.** Proof is straightforward. Let \( f : A \rightarrow B \) be a poset homomorphism, i.e. if \( a < b \) in \( A \), then \( f(a) < f(b) \) in \( B \). Let \( f^* : A^* \rightarrow B^* \) be a letter-wise extension of \( f \). Clearly, it may be restricted to \( f^* : A^*_e \rightarrow B^*_f(e) \). Relation \( v <_e w \) in \( A^*_e \) means that there are \( e \)-extensions of \( v \) and \( w \), such that \( v' <_e w' \) in \( A^*_e \) for some \( n \in \mathbb{N} \), i.e. \( v'_j <_e w'_j \) in \( A_e \) for \( j = 1, \ldots, n \). So, \( f(v'_j) <_{f(e)} f(w'_j) \) in \( B_e \), and therefore \( f^*(v') <_{f(e)} f^*(w') \) in \( B^*_f(e) \). Hence, by definition, \( f^*(v) <_{f(e)} f^*(w) \) in \( B^* \).

In particular,

**Corollary.** (i) If \( B \subset A \) as sets, and the partial order of \( B \) is that of \( A \) restricted to \( B \), then the induced poset \( B^*_e \) is a subposet of \( A^*_e \), for any \( e \in B \).

(ii) If \( \prec' \) be a suborder of a partial order \( \prec \) in \( A \), then for any \( a \in A \) the order \( \prec'_a \) is a suborder of \( \prec_a \) in \( A^*_a \).

As an extremely simple illustration of (i), consider \( \{ \varepsilon \} \) as a one-element subposet of \( A' = \{ \pi, \eta, \varphi \} \) for any of the given examples of augmentation. The word algebra \( \{ \varphi \}^* \) consists of powers \( \varphi^n \). In a particular example either \( \varphi > \eta \) or \( \varphi \neq \eta \), and hence \( \{ \varphi \}^* \) is either linearly ordered or trivial, and so it occurs in the corresponding posets \( \{ \pi, \varphi \}^* \). For an illustration of (ii), compare Figure 3 with Figure 2, where the corresponding Hasse diagrams form subgraphs on the alphabet level, as well as in the word algebras.

**Example 2:** Consider two nonlinear extensions of \( \pi < \varphi \):

\[
\begin{array}{ccc}
\varphi & \nearrow & \varphi \\
\searrow & \pi & \searrow \\
(a) & & (b)
\end{array}
\]

These lead to posets, which are displayed in Figure 3 for words \( |w| \leq 3 \).
Although posets of Example 2 look “strange,” notice that each is a particular suborder of two of the posets considered in Example 1. This is because the orders of A in Example 1 are particular linearizations of the alphabet posets (a) and (b) above).

4. Further examples and applications

Now let us review a few special cases, illustrated by rather simple examples.

**Definition 4.1.** A raising augmentation of a poset A is a poset $A' = A \cup \{e\}$ with the partial order this of A complemented by relation $e < a$ for any $a \in A$. The induced poset will be denoted $\text{Rais}_A^*$. 

For illustration of $\text{Rais} \{\pi < \varphi\}^*$ see Example 1a.

Augmentation may also be applied to mere sets (viewed as posets with the trivial order). In particular:

**Corollary.** The induced order of a trivial poset is the morphological order.

**Example 3:** Let $A = \{\varphi, \pi\}$ be a set. Raising augmentation of $A$ into a poset $A' = \{\pi, \varphi, \eta\}$ with a two-step relation:

```
  $\varphi$  $\pi$
     /    \
   \     / \\
   $\eta$
```

results in morphological order of orientons. (See Definition 1.1 and Figure 1.) Quite surprisingly, both key orders of the ‘algebra of orientons’ are describable in terms of induced orders.

**Definition 4.2.** A trivial augmentation of a poset $A$ is a poset $A' = A \cup \{e\}$ with the auxiliary letter $e$ left unrelated to $A$.

**Corollary.** The poset induced from the trivial augmentation of a poset $A$ is the disjoint sum of product orders in subsets of $A^*$:

$$\{ A^*, < \} = \text{Prod} A^0 + \text{Prod} A^1 + \ldots + \text{Prod} A^k + \ldots$$

where $\text{Prod} A^k$ is the Boolean lattice of the product order among the words of a fixed length. (Clearly, $\text{Prod} A^0 \equiv \{e\}$, and $\text{Prod} A^1 \equiv \{A, <\}$.)
Proof. By Proposition 3.2, two words of the same length, \(|w| = |v| = k\), are related in \(A^*\) in the same way as in Prod \(A^k\). Words of different lengths in \(A^*\) are not related: any \(e\)-extensions of \(w\) and \(v\) resulting in the same length must have different numbers of the letter \(e\), with some of them occurring where a letter of \(A\) appears in the other word. Since \(e \not\in A\), the extended words are incomparable, and therefore so are the original words \(w\) and \(v\).

Example 4: Trivial augmentations of the two-letter poset \(A = \{\pi < \varphi\}\) with auxiliary letter \(\eta\) is left unrelated to \(A\):

\[
\varphi \quad \eta \\
\quad \pi
\]

splits the word algebra \(A^*\) into a family of disconnected Boolean lattices of constant word length, as illustrated in Figure 4.

![Figure 4: Order in \(\{\pi, \varphi\}^*\) induced from trivial augmentation.](image)

Note, the range of the induced orders over the same word monoid: the words of connected pieces of Prod \(A^*\) appear as the horizontal layers in Morph \(A^*\) (Cf. Figure 1 and Figure 4).

Example 5: Consider the poset \(A^*_\eta\) induced from the following augmentation of a trivial poset

\[
\varphi \quad \pi \quad \eta
\]

(letter \(\eta\) related to \(\varphi\) only). The resulting partial order is illustrated in Figure 5.

![Figure 5: The partition poset for \(A = \{\pi, \varphi\}\).](image)
If the letter π is viewed as a separating bar, “|”, then the $i^{th}$ connected piece of the above graph shows the possible distributions of a number of items ϕ into $i$ boxes, including the partial order of such distributions. The $i^{th}$ piece has $π^{i-1}$ as the least element ($i$ empty boxes), and is isomorphic to the product $\mathbb{N}^i$. This suggests the following:

**Definition 4.3.** Let $A$ be a poset. Consider an augmented poset $A'' = A \cup \{ e, \} |$ with the extended partial order: $e < A$, and $|$ unrelated to $A$ or $e$. A partition poset Part $A'$ is the word algebra over the alphabet $(A \cup \{\} |) \equiv A'$ with the order induced from $A''$.

### 5. Summary

Each poset $A$ treated as an alphabet leads to a natural family of well-defined *induced* partial orders in the set of words over this alphabet (Theorem 2.2). The principal construction of the induced order goes via dropping a letter, say $a$, from the alphabet $A$, and considering the new set, $A_a$, as the alphabet for words, $A_a^*$. The choice of the letter to be dropped determines the partial order in $A_a^*$. **Augmentation** allows an induced order to be defined between the words over the initial alphabet $A$ by, first, embedding the alphabet $A$ into a larger poset $A' = A \cup \{ e \}$, and then applying the principal construction by dropping the auxiliary letter $e$.

Natural properties of induced order easily follow (expressed here for augmentation). The induced order is an extension of the order in the alphabet $A$:

$$A \xrightarrow{\text{homo}} A^*$$

The construction is *natural* with respect to the homomorphisms of alphabets (Proposition 3.3), making the following diagram commute:

$$
\begin{array}{ccc}
A^* & \xrightarrow{f^*} & B^* \\
\uparrow & & \uparrow \\
A & \xrightarrow{f} & B
\end{array}
$$

where $f^*$ is a letter-wise homomorphism induced from $f$. The property that contrasts the induced orders with the lexicographical order is that the product posets Prod $A''$ are isomorphically embedded into $A^*$ (Proposition 3.2).

Since the induced order extends that of the alphabetic order, let us denote the induced poset as Ext $A^*$ for augmentation, or Ext $A_e^*$ for the principal construction:

$$\text{Ext } A_e^* = \{ A_a^*, <_a \}$$

A few canonical constructions (by augmentation, $e \notin A$) may be indicated:

- **Morph $A^*$** = Ext $\{ e < A \}^*_e$ (A is a set)
- **Prod $A^*$** = Ext $\{ e \neq A \}^*_e$
- **Rais $A^*$** = Ext $\{ e < A \}^*_e$
- **Part $A^*$** = Ext $\{ e < A, \eta \neq A \}^*_e$ ($\eta \notin A$, $A$ is a poset)
- **Span $A^*$** = Ext $\{ L(A) < e < G(A) \}^*_e$
where in the last poset \( L(A) \) and \( G(A) \) are the least and the greatest elements of \( A \) respectively.

The above augmentations may be illustrated symbolically:

\[
\begin{array}{cccccc}
\text{Morph} & A^* & \text{Rais} & A^* & \text{Prod} & A^* \\
\text{Part} & A^* & \text{Span} & A^* & \text{Chron} & (A \prec B)^* \\
\end{array}
\]

As to the algebra of orientons \( A^* = \{\pi, \varphi\}^* \), surprisingly both the chronological and the morphological order turns out to be induced extensions:

\[
\begin{align*}
\text{Chron} A^* &= \text{Ext} \{\pi < \eta < \varphi\}_\eta^* \\
\text{Morph} A^* &= \text{Ext} \{\pi > \eta < \varphi\}_\eta^*
\end{align*}
\]

Another construction (interesting in the context of discrete models of causal properties of space-time) concerns the union \( A \cup B \cup \{e\} \) of posets \( A \) and \( B \), complemented by \( A < e < B \). The induced order in \((A \cup B)^*\) defines a poset

\[
\text{Chron} (A < B)^* = \text{Ext} \{A < e < B\}_e^*
\]

which may be viewed as a direct generalization of the chronological order. By analogy to relativity theory, it seems natural to define the future cone and the past cone as the image of \( A^* \) and \( B^* \) in \((A \cup B)^*\), respectively, and the elsewhere as \((A \cup B)^* - (A^* \cup B^*)\).

Here are some questions concerning the induced orders: How are particular properties of the ordered alphabet reflected in the induced order of words; How does the induced order relate to the "algebra of products of partial orders" (in the sense of [2]); What is the relationship between algebra of partial orders (treated on the level of alphabets) and that lifted to the words; How is the structure of the alphabet \( A \) reflected in the structure of the family of posets obtained by deleting different letters from \( A \). In particular, notice "non-commutativity" of the construction; although \((A_a)^*_b = (A_b)^*_a = (A - \{a, b\})^*\), but \( \text{Ext} (A_a)_b^* \neq \text{Ext} (A_b)_a^* \).

**References**

[1] Birkhoff, Garret, *Lattice Theory*, AMS, Providence Rhode Island, 1967 (third ed.).

[2] Jónsson, Bjarni, *Arithmetic of Ordered Sets*, in *Ordered Sets*, pp. 3–41, Ivan Rival (ed.), Reidel, Boston, 1981.

[3] Kocik, Jerzy, *Formal Tense Systems*, submitted.

[4] Lothaire, M. (ed.), *Combinatorics on Words*, Addison-Wesley Pub. Comp., London, 1983.