An Efficient Chorin–Temam Projection Proper Orthogonal Decomposition Based Reduced-Order Model for Nonstationary Stokes Equations

Xi Li¹ · Yan Luo² · Minfu Feng¹

Abstract

In this paper, we propose an efficient proper orthogonal decomposition based reduced-order model (POD-ROM) for nonstationary Stokes equations, which combines the classical projection method with POD technique. This new scheme mainly owns two advantages: the first one is low computational costs since the classical projection method decouples the reduced-order velocity variable and reduced-order pressure variable, and POD technique further improves the computational efficiency; the second advantage consists of circumventing the verification of classical LBB/inf-sup condition for mixed reduced spaces with the help of pressure stabilized Petrov–Galerkin (PSPG)-type projection method, where the pressure stabilization term is inherent which allows the use of non inf-sup stable elements without adding extra stabilization terms. We analyze the proposed projection POD-ROM’s stability and convergence, and numerical experiments validate those theoretical results and also the high-efficiency.

Keywords  Projection method · PSPG stabilization · Proper orthogonal decomposition

Xi Li and Minfu Feng: The work of this author was supported by the National Natural Science Foundation of China (Grant No. 11971337). Yan Luo: The work of this author was supported by the Young Scientists Fund of the National Natural Science Foundation of China (Grant No. 11901078) and by the Fundamental Research Funds for the Central Universities (Grant No. ZYGX2020J021).

Minfu Feng
fmf@scu.edu.cn

Xi Li
li_xi@stu.scu.edu.cn

Yan Luo
luoyan_16@126.com

¹ School of Mathematics, Sichuan University, Chengdu 610064, Sichuan, China
² School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, China
1 Introduction

Full Order Models (FOM) based on fluid dynamics problems (governed by incompressible Stokes and Navier–Stokes equations), e.g. Direct Numerical Simulations (DNS), Large Eddy Simulations (LES), and Reynolds-averaged Navier–Stokes simulations (RANS), frequently require high demand for computational time, especially for problems with multi-parameters or with temporal variable which is discretized into huge time steps on a given time interval. These high computational costs makes FOM improper in some multi-query scenarios. This is the case for uncertainty quantification, inverse problems, real-time scenarios and so on. Reduced Order Models (ROM), on the other hand, provides surrogate models which can be utilized to obtain required discrete data with acceptable accuracy but a tiny computational time. In this article, we focus on Reduced Basis Methods (RBM), one of the special ROM, known as low dimensions of the finite-dimensional subspaces. The reduced bases, we considered here, are generated by Proper Orthogonal Decomposition (POD), and other techniques, e.g. greedy algorithm [48], can also be used but it is out of the scope of the current work and will be pursued in future. After obtaining a low-dimensional subspace by POD on the snapshot sets, Galerkin projection is used to get ROM (G-POD-ROM). In this paper, time extrapolation is considered with fixed, moderate parameters, i.e., we aims to improve the time efficiency by calculating FOM on a short time interval and taking snapshots needed by POD, ROM is derived and used to computing the required data on a longer time interval. The case of varying parameters is similar to the one considered here and can be handled by POD-greedy algorithm [24].

Applications of G-POD-ROM on nonstationary Stokes/Navier–Stokes equations, especially with the spatial variables discretized by finite element(FE) method, have a long history, the pioneering work is done by Kunisch and Volkwein in [31, 32] which has laid the foundation of numerical analysis for G-POD-ROM. Other numerical methods, at the FOM level, includes discontinuous Galerkin and hybrid discontinuous Galerkin [20], finite volume method [45, 46], finite difference method [35] etc. Other references about G-POD-ROM can be found in [33, 34, 43, 49] and so on.

Solving nonstationary Stokes/Navier–Stokes equations with finite element FOM (FE-FOM) and getting G-POD-ROM (FE-POD-ROM) will encounter two main problems. One is that the obtained POD-ROM retains the coupled velocity-pressure variables form stem from the original FE-FOM, i.e., it keeps the saddle-point system of the Stokes/Navier–Stokes equations, which is not ideal for fast getting discrete data. The second is that the obtained mixed reduced basis velocity-pressure spaces cannot achieve stability in some sense, even though the mixed FE velocity-pressure spaces are stable in the sense of classical LBB/inf-sup condition. The second problem sets an obstacle for applying FE-POD-ROM directly to get discrete data quickly in scenarios involving pressure variables. To handle this obstacle, some methods have been proposed to recover the reduced basis pressure a posteriori, such as the Pressure Poisson Equation [36], Supremizer stabilized technique [4] and so on. However, just as stated in [8, 19], there are still some deficiencies in these a posteriori pressure recovery techniques, such as the usage of those techniques still need the fulfillment of LBB/inf-sup condition which restricts some flexible pairs of mixed FE spaces, and the un-physical and unclear boundary condition of Pressure Poisson Equation poses a challenge for the utilization of this technique. With those in mind, some researcher have drawn inspiration from the successful applications of stabilization technique in FE community and proposed to utilize pressure stabilization technique to stabilize the mixed reduced basis velocity-pressure, thus allowing a priori stabilization of the reduced pressure, rather than reconstructing it a posteriori.
Caiazzo et al. [8] firstly proposed to combine stabilization techniques, like residual-based streamline-upwind Petrov–Galerkin stabilization, obtain the POD reduced-order pressure stably. From then on, other stabilized techniques were introduced into FE-POD-ROM, like local projection stabilization [37, 40], artificial compressible [19], streamline diffusion [2], residual-based pressure stabilized Petrov–Galerkin methods [1, 3], residual-based variational multiscale [44] and so on. However, the above stabilized FE-POD-ROM for nonstationary Stokes/Navier–Stokes equations still retain the saddle-point structure and thus the potential of FE-POD-ROM to improve time efficiency can be further exploited.

In this paper, our aim is to address both of the above proposed problems, i.e., to propose a decoupled, reduced pressure stable FE-POD-ROM for the non-stationary Stokes problem. The realization of this idea lies in the incorporation of the classical projection method in the equivalent PSPG form and FE-POD-ROM. As one of the fast decoupled numerical algorithms for solving incompressible fluid, projection method has become a classical one after it was proposed by Chorin and Temam [14, 47] and developed by Shen and Guermond [22, 23, 41, 42] etc. Apart from that, as mentioned by Rannacher in [38], another remarkable feature of this method is that the original Chorin–Temam projection method owns an inherent mechanism of pressure stability. This mechanism was proposed by Rannacher [38] and was analyzed by Frutos et al. [18]. These two advantages of the classical projection method are merged here into the POD technique to get our projection POD-ROM. The main contribution of this paper is that a more efficient ROM numerical scheme for solving nonstationary Stokes equations is obtained by combining the advantageous computational efficiency of both projection method and POD technique. We remark that the extension to nonstationary Navier–Stokes equations will not be essentially difficult since Frutos et al. [16] had given the theoretical analysis for nonstationary Navier–Stokes equations in PSPG-type projection method, where the nonlinear convection term is linearized by full-explicit finite difference(FD) method. If the convection term preserves nonlinear after temporal discrete by FD, hyper-reduction technique, such as Gappy-POD [50], GNAT [9], are available to keep the high efficiency of ROM.

The outline of this paper is as follows: In Sect. 2, we introduce some essential notations. In Sect. 3, the classical projection method will be presented, followed by its stability and convergence results. In Sect. 4, we propose the projection POD numerical scheme and prove its stability and convergence. In Sect. 5, some numerical experiments will be investigated to confirm the theoretical results analyzed before. Finally, in Sect. 6, we draw conclusions to complete this paper.

2 Preliminaries and Notations

we consider the nonstationary Stokes equations

\[
\begin{aligned}
\partial_t u - \nu \Delta u + \nabla p &= f, \quad \text{in } \Omega, \\
\nabla \cdot u &= 0, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial \Omega, \\
\end{aligned}
\]

(2.1)

where \( \Omega \) is an open bounded domain in \( \mathbb{R}^2 \) with a sufficiently smooth boundary \( \partial \Omega \). The unknowns are the vector function \( u \) (velocity) and the scalar function \( p \) (pressure).

In order to state the above problem properly, we need some notations. Let \( L^p(\Omega), \ 1 \leq p \leq \infty \), denote the space of standard \( p \)-th power absolutely integrable functions with respect to(w.r.t.) the Lebesgue measure. In particular, \( L^2(\Omega) \) is a Hilbert space endowed with the
scalar product $(\cdot, \cdot)$ and its induced norm $\|\cdot\|_0$. Moreover, we use $\|\cdot\|_{L^2}^2 := \Delta t \sum_{n=1}^{N} \|\cdot\|_0^2$ to denote the discrete integral in time w.r.t. $\|\cdot\|_0^2$. The Sobolev space $W^{m,p}(\Omega)$ and its norm $\|\cdot\|_{m,p}$ are also standard in the sense of [5, p.23]. Furthermore, we use the abbreviation $H^m(\Omega) := W^{m,2}(\Omega)$ and $\|\cdot\|_m := \|\cdot\|_{m,2}$, it’s a Hilbert space with a scalar product. We denote by $H_0^1(\Omega)$ the space of functions of $H^1(\Omega)$ with vanishing trace on $\partial \Omega$ and by $H^{-1}(\Omega)$ its dual space. The symbol $(\cdot, \cdot)$ in general denotes the duality pairing between the space and its dual one. Vector analogues of the Sobolev spaces along with vector valued functions are denoted by boldface letters, for instance $H^m(\Omega) := (H^m(\Omega))^2$. We all know that by Poincaré inequality [5, p.135], the seminorm $|u|_1 = \|\nabla u\|_0$ in $H^1(\Omega)$ is a norm in $H_0^1(\Omega)$.

Throughout this paper, we use $C$ to denote a positive constant independent of $\Delta t, h$, not necessarily the same at each occurrence.

Let us introduce some convenient spaces. The first one is

$$
H(\text{div}; \Omega) := \{ u \in L^2(\Omega) | \nabla \cdot u \in L^2(\Omega) \},
$$

which is a Hilbert space with norm $\|u\|_{\text{div}} := \|u\|_0 + \|\nabla \cdot u\|_0$, and

$$
H_0(\text{div}; \Omega) := \{ u \in H(\text{div}; \Omega) | u \cdot n|_{\partial \Omega} = 0 \}.
$$

For future use, we also need the following spaces, which play a key role in Helmholtz decomposition,

$$
J_0 := \{ v \in H_0^1(\Omega) : \nabla \cdot v = 0 \},
$$

and

$$
J_1 := \{ v \in L^2(\Omega) : \nabla \cdot v = 0, \text{ and } v \cdot n|_{\partial \Omega} = 0 \}.
$$

By means of spaces defined above, we give the classical Helmholtz decomposition [21, p.29] as

$$
L^2(\Omega) = J_1(\Omega) \oplus [J_1(\Omega)]^\perp = J_1(\Omega) \oplus \{ \nabla q : q \in H^1(\Omega) \}. \tag{2.2}
$$

In order to give a variational formulation of problem (2.1), we consider the velocity space $V := H_0^1(\Omega)$ and the pressure space $Q := L^2_0(\Omega) = \{ q \in L^2(\Omega) | \int_{\Omega} q \, dx = 0 \}$. For a given time constant $T$, a weak formulation of the nonstationary Stokes equations are expressed as follows: for almost $\forall t \in (0, T)$, find $(u, p) : (0, T) \to V \times Q$, such that

$$
\begin{cases}
(\partial_t u, v) + \nu(\nabla u, \nabla v) + (\nabla p, v) = (f, v), & \forall v \in V, \\
(\nabla \cdot u, q) = 0, & \forall q \in Q,
\end{cases} \tag{2.3}
$$

$$
u(u(x), 0) = u^0(x).
$$

Let $\{T_h\}$ be a uniformly regular family of triangulation of $\Omega$ (see [15, p.111]) and $h := \max_{K \in T_h} h_K | h_K := \text{diam}(K)$. To show that the pressure in PSPG-type projection method can keep stable in a more general mixed FE spaces, not necessarily the inf-sup stable ones, we choose equal-order mixed FE spaces and then it’s obviously not stable in the sense of the classical inf-sup condition

$$
\exists \beta_h > 0, \text{ s.t. } \beta_h \|q_h\|_0 \leq \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|\nabla v_h\|_0 \cdot \|q_h\|_0}, \forall q_h \in Q_h, \tag{2.4}
$$

where $V_h \subset V$ and $Q_h \subset Q$ are mixed FE spaces defined below. Denoting

$$
Y^l_h := \{ v_h \in C^0(\Omega) | v_h|_K \in P_l(K), \forall K \in T_h \}, \ l \geq 1,
$$

$\Box$ Springer
where \( \mathbb{P}_l(K) \) is the space of polynomials up to degree \( l \) on \( K \). Since there are two types of velocities in the temporal semi-discrete projection scheme, i.e., intermediate velocity \( \bar{u}^{n+1} \) and end-of-step velocity \( u^{n+1} \), we need the third FE spaces to locate the third full discrete unknowns. To this end, we define the equal-order FE spaces for end-of-step velocity, intermediate velocity and pressure

\[
V_h := Y_h^l \cap V, \quad W_h := Y_h^l \cap J_1, \quad Q_h := Y_h^l \cap Q.
\]

Since the triangulations are assumed to be shape regular, the following inverse inequality holds for each \( \psi_h \in V_h \) on each mesh cell \( K \in \{ T_h \} \) (see [5, Theorem 4.5.11]),

\[
\| \psi_h \|_{m,p,K} \leq C h_k^{l-m+2(\frac{1}{l} - \frac{1}{p})} \| \psi_h \|_{l,q,K}, \tag{2.5}
\]

where \( 0 \leq l \leq m \leq 1 \), \( 0 \leq q \leq p \leq \infty \) and \( h_k \) is the size of the mesh cell \( K \in \{ T_h \} \). The above inequality also holds for \( \forall \psi_h \in W_h \) and \( \forall q_h \in Q_h \) and would change to \( \forall \psi_h \in W_h \) and \( q_h \in Q_h \) as needed. The Sobolev inclusion relation \( L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \) implies the following inequality holds:

\[
\| \psi \|_{-1} \leq C \| \psi \|_0, \quad \forall \psi \in L^2(\Omega). \tag{2.6}
\]

### 3 Classical Projection Method

#### 3.1 Classical Projection Scheme

The original projection method consists of firstly finding an intermediate velocity \( \tilde{u}^{n+1} \) from the momentum equation without the pressure term, and then utilizing the classical Helmholtz decomposition (2.2) to get an end-of-step velocity \( u^{n+1} \) which is solenoidal and its normal component \( u^{n+1} \cdot n \) vanishes on boundary. Meanwhile, as a "by-product" of the above decomposition, a discrete pressure \( p^{n+1} \) can be also get for which some investigations have been made to confirm that this "by-product" pressure is indeed the proper approximation of the exact pressure \( p(t_{n+1}) \), see [38].

For a given timestep \( \Delta t \), we consider a uniform discretization of the time interval \([0, T]\) as \( 0 = t_0 < t_1 < \cdots < t_N \) where \( N = \lceil T/\Delta t \rceil \) is the rounding \( T/\Delta t \) down. In mathematics, the above procedure can be interpreted as the following classical Chorin–Temam projection method [13, 47]:

(Scheme 1—Semi-discrete of classical projection method) For \( n = 0, 1, \ldots, N - 1 \),

**Step 1** Given \( u^n \in J_0 \), find \( \tilde{u}^{n+1} \in V \) that satisfies

\[
\bar{u}^{n+1} - u^n - \nu \Delta \tilde{u}^{n+1} = f^{n+1}, \quad \text{in } \Omega.
\]

**Step 2** Given \( \tilde{u}^{n+1} \in V \), compute \( (u^{n+1}, p^{n+1}) \in H_0(\text{div}, \Omega) \times [Q \cap H^1(\Omega)] \) from

\[
\begin{aligned}
\frac{u^{n+1} + \Delta t \nabla p^{n+1} = \tilde{u}^{n+1}}, & \quad \text{in } \Omega, \\
\nabla \cdot u^{n+1} = 0, & \quad \text{in } \Omega, \\
u^{n+1} \cdot n = 0, & \quad \text{on } \partial \Omega.
\end{aligned}
\]

The usage of classical Helmholtz decomposition is reflected in decomposing the intermediate velocity \( \tilde{u}^{n+1} \) from **Step 1** to get a solenoidal velocity \( u^{n+1} \) and a gradient of a \( H^1 \)-function, which can be be stated as:
**Step 2’** Projecting \( \tilde{u}^{n+1} \) onto the space \( J_0 \)

\[
\mathbf{u}^{n+1} = P_{J_0}(\tilde{u}^{n+1}).
\]

**Remark 3.1** Let us explain the relation between **Step 2** and **Step 2’**. Followed by (2.2), decomposing \( \tilde{u}^{n+1} \) would directly gets:

\[
\tilde{u}^{n+1} = u^{n+1} + \nabla \phi^{n+1},
\]

where \( u^{n+1} \in J_1 \) implies \( \nabla \cdot u^{n+1} = 0 \) and \( u^{n+1} \cdot n |_{\Omega} = 0 \), which is exactly the second and third equations in **Step 2**. Multiplying the previous decomposition by 1/\( \Delta t \) and adding it to the first equation in **Step 1**, we get \( (u^{n+1} - u^n)/\Delta t - \nu \Delta \tilde{u}^{n+1} + \nabla (\phi^{n+1}/\Delta t) = f^{n+1} \), which means \( \phi^{n+1}/\Delta t \) can be regarded as a proper approximation of \( p_{(t_n+1)} \), i.e., we can define \( p^{n+1} := \phi^{n+1}/\Delta t \). Thus, on the one hand, one can compute \( \phi^{n+1} \) in the former decomposition (3.1) and using \( p^{n+1} := \phi^{n+1}/\Delta t \) to get \( p^{n+1} \); on the other hand, \( p^{n+1} \) can be also directly obtained by substituting \( \phi^{n+1} \) with \( \Delta t \) \( p^{n+1} \) in (3.1), which is exactly what the first equation in **Step 2** does.

Or, **Step 2** is equivalent to the following pressure Poisson equation using \( \nabla \cdot u^{n+1} = 0 

**Step 2’’** - Given \( \tilde{u}^{n+1} \in V \), compute \( p^{n+1} \in Q \cap H^2(\Omega) \) from

\[
\begin{align*}
\Delta p^{n+1} & = \frac{1}{\Delta t} \nabla \cdot \tilde{u}^{n+1}, & \text{in } \Omega, \\
\frac{\partial p^{n+1}}{\partial n} & = 0, & \text{on } \partial \Omega.
\end{align*}
\]

Based on the previous FE spaces, we derive the fully discrete scheme of projection method.

(Scheme 2—Fully-discrete of classical projection scheme) Let \( u^0_h \) be the Lagrangian interpolation or Ritz projection onto \( V_h \) of \( u_0 \), then for \( n = 0, 1, \ldots, N - 1(\ll N - 1) \), we can compute \( (\tilde{u}^{n+1}_h, u^{n+1}_h, p^{n+1}_h) \in V_h \times W_h \times Q_h \) iteratively by

\[
\begin{align*}
\frac{\tilde{u}^{n+1}_h - u^n_h}{\Delta t} & + \nu(\nabla \tilde{u}^{n+1}_h, \nabla v_h) = \langle f^{n+1}, v_h \rangle, & \forall v_h \in V_h, \\
(\nabla \cdot \tilde{u}^{n+1}_h, q_h) & + \Delta t(\nabla p^{n+1}_h, \nabla q_h) = 0, & \forall q_h \in Q_h,
\end{align*}
\]

As pointed in [38], it is not necessary to compute the projected velocities \( \{u^n_{h} \}_{n=0}^{N} \) since these quantities can be eliminated. To see this, replacing \( u^n_h \) in the first equation with the third equation at \( t_n \) in (Scheme 2), we reach the following Proj-FE-FOM scheme.

(Scheme 3—Proj-FE-FOM) Let \( u^0_h \) be the Lagrangian interpolation or Ritz projection onto \( V_h \) of \( u_0 \) and \( p^0_h \) denote some proper approximation of initial pressure \( p(t_0) \), we set \( \tilde{u}^0_h = u^0_h + \Delta t \nabla p^0_h \) and replace \( u^n_h \) with the third equality in (Scheme 2), we obtain the following equivalent scheme: for \( n = 0, 1, \ldots, N - 1(\ll N - 1) \), find \( (\tilde{u}^{n+1}_h, p^{n+1}_h) \in V_h \times Q_h \)

\[
\begin{align*}
\frac{\tilde{u}^{n+1}_h - u^n_h}{\Delta t} & + \nu(\nabla \tilde{u}^{n+1}_h, \nabla v_h) + (\nabla p^n_h, v_h) = \langle f^{n+1}, v_h \rangle, & \forall v_h \in V_h, \\
(\nabla \cdot \tilde{u}^{n+1}_h, q_h) & + \Delta t(\nabla p^n_h, \nabla q_h) = 0, & \forall q_h \in Q_h.
\end{align*}
\]
Remark 3.2 For the initial discrete pressure \( p_h^0 \), we simply take \( p_h^0 = 0 \) in later numerical experiments. The reasons behind this are twofold: firstly, it can simplify the solving of initial discrete pressure \( p_h^0 = 0 \), compared to other techniques, like pressure Poisson equation used in [25, 28]; secondly, it can also directly get the initial intermediate discrete velocity \( \tilde{u}_h^0 \) since at this point \( \tilde{u}_h^0 = u_h^0 \). We may also observe that this rude treatment would make the first few time steps discrete pressure appear oscillation. Indeed, numerical experiments have showed that the first few time steps discrete pressure values have to be abandoned because of big discrete errors; nevertheless, we will see in later numerical experiments that, only after a few time steps, the discrete pressure keeps stable.

Remark 3.3 We note that although the obtained FE velocity solution in (Scheme 3—Proj-FE-FOM) is \( \tilde{u}_h^{n+1} \), rather than \( u_h^{n+1} \), we will see in latter numerical experiment that when choosing \( V_h \times Q_h = P^1 - P^1 \), the solved \( \tilde{u}_h^{n+1} \) can be used as a proper FE approximation for the exact velocity \( u_h^{n+1} \). If one wants to obtain \( u_h^{n+1} \), we can multiply by the last equation in (Scheme 2) some test function \( w_h \in W_h = Y_h^1 \cap J_1 \) to get

\[
(u_h^{n+1}, w_h) = (\tilde{u}_h^{n+1}, w_h) - \Delta t (\nabla p_h^{n+1}, w_h).
\]

then, solving above equation can finally obtain \( u_h^{n+1} \in W_h = Y_h^1 \cap J_1 \).

Remark 3.4 Frutos et al. [17] extended the above approach to nonstationary Navier–Stokes equations, where the nonlinear convection field \( u \cdot \nabla u \) are linearized as \( u^n \cdot \nabla u^n \) on time interval \( [t_n, t_{n+1}] \).

We observe that, in the classical projection scheme (3.2) or (3.3), if we set \( \Delta t = O(h^2) \), the existence of PSPG-stabilized term \( \Delta t (\nabla p_h^{n+1}, \nabla q_h) \) makes the spatial convergence order of the error does not achieve more than second order in \( L_2 \) norm of the velocity and first order of the error in the \( L_2 \) norm of the pressure, so \( P^1 - P^1 \) for FE spaces and \( (u, p) \in H^2 \times H^1 \) seem to be the best choice concerning about the efficiency of computation and the regularity for the exact solutions. Thus, We set \( l = 1 \) in the definition of \( Y_h^l \), thus the FE spaces for velocity and pressure would be \( V_h = Y_h^1 \cap V \) and \( Q_h = Y_h^1 \cap Q \), respectively.

In the sequel we assume \( \Delta t = O(h^2) \). Specifically, we assume there exist two positive constants \( C_1, C_2 \), such that

\[
C_1 h^2 \leq \Delta t \leq C_2 h^2.
\]

The following modified inf-sup condition relaxes the classical inf-sup condition (2.4) and makes many mixed FE pairs "stable" in the sense of this modified one, whose detailed proof can be found in [7, Lemma 3] or [28, Lemma 2.1].

Lemma 3.1 (Modified inf-sup condition) Assuming (3.5) holds, then for \( n = 0, 1, \ldots, N-1 \), we have the following pressure stability

\[
\exists \beta > 0, \text{ s.t. } \beta \| q_h \|_0 \leq C h \| \nabla q_h \|_0 + \sup_{v_h \in V_h} \frac{(q_h, \nabla \cdot v_h)}{\| \nabla v_h \|_0}, \forall q_h \in Q_h.
\]

3.2 Convergence of Classical Projection Scheme

In this subsection, we will cite some error estimation results of the FE solution in the classical projection scheme and omit those proofs, since those estimates have been analyzed in [18] in detail. The necessity of citing is based on the fact that the following POD snapshots in Sect. 4
are the FE solutions, which means that when we analyzing the discretization error between the exact solutions and the POD-based reduced-order solutions, apart from the ROM truncation error, the discretization error also involve spatial and temporal discretization errors.

**Lemma 3.2** (Error estimates for Proj-FE-FOM) For \( n = 1, \ldots, N \), let \((u^n, p^n) \in V \times Q\) be the solution of continuous variational form (2.3) at discrete time \( t = t_n \), \((\tilde{u}_h^n, \tilde{p}_h^n) \in V_h \times Q_h\) is the solution obtained from the PSPG-like projection scheme (3.3), and assuming (3.5) holds, i.e. \( \Delta t = O(h^2) \), then we have

\[
\begin{align*}
\|u^n - \tilde{u}_h^n\|_0 + h\|p^n - \tilde{p}_h^n\|_0 + h\sqrt{\Delta t}\|\nabla(p^n - \tilde{p}_h^n)\|_0 &\leq C(h^2 + \Delta t). \\
\|\nabla(u - \tilde{u}_h)\|_{L^2(L^2)} + \|p - \tilde{p}_h\|_{L^2(L^2)} &\leq C(h + \sqrt{\Delta t}).
\end{align*}
\] (3.7)

In addition to the error estimates about the intermediate velocity \( \tilde{u}_h^n \) analyzed in [18], we can also prove the \( L^2 \) error estimates about the end-of-step velocity \( u^n_h \) which is not made in [18]. In other words, we can obtain

**Lemma 3.3** For \( n = 1, \ldots, N \), \( u^n_h \) is the velocity solution obtained in (Scheme 2), and \( u^n \) is the velocity solution of continuous variational form (2.3) at discrete time \( t = t_n \), then

\[
\|u^n - u^n_h\|_0 \leq C(h^2 + \Delta t).
\] (3.8)

**Proof** According to the convergent results given by Lemma 3.2, we can prove the stability of \( \|\nabla p^n_h\|_0 \). That is, the first inequality in (3.7) implies

\[
\|\nabla(p^n - \tilde{p}_h^n)\|_0 \leq C(h/\sqrt{\Delta t} + \sqrt{\Delta t}/h) \leq C,
\]
where we have used (3.5). Then

\[
\|\nabla p^n_h\|_0 \leq \|\nabla(p^n - \tilde{p}_h^n)\|_0 + \|\nabla p^n\|_0 \leq C,
\]
where the constant \( C \) can be chose to stay nearly fixed with the decrease of \( h \) and \( \Delta t \). So, based on the result, testing \( w_h = \tilde{u}_h^{n+1} - u_h^{n+1} \) in (3.4) and rearranging to get

\[
\|\tilde{u}_h^{n+1} - u_h^{n+1}\|_0 \leq \Delta t\|\nabla p_h^{n+1}\|_0 \leq C\Delta t.
\]

Finally, utilizing triangle inequality and Lemma 3.2,

\[
\|u^{n+1} - u_h^{n+1}\|_0 \leq \|u^{n+1} - \tilde{u}_h^{n+1}\|_0 + \|\tilde{u}_h^{n+1} - u_h^{n+1}\|_0 \leq C(h^2 + \Delta t).
\]

\[\square\]

### 4 Chorin–Temam Projection POD-ROM

Before getting the efficient, decouple Chorin–Temam projection POD-ROM, we remark that the key requirement for an efficient ROM evaluation is the so-called offline/online decomposition, which ensures the high-cost computation of high-dimensional FOM, aiming for constructing reduced-order spaces, is done only once (offline stage), and then the calculation of varying discrete temporal low-dimensional ROM requires only very low costs (online stage).
4.1 Offline/Online Decomposition

As mentioned before, we do not consider in this paper the case of varying physical and geometric parameters, but rather the time extrapolation, i.e., the time interval where the problems are solved is longer than the one used to sample FOM data to formulate ROM. Thus, the offline/online decomposition is enforced by the following way: during the offline stage, methods of snapshot is used to collect the FE solution by solving FE-FOM (3.3) on \( M \) discrete time points, where \( M \ll N(= \lfloor T / \Delta t \rfloor) \); during the online stage, the obtained Chorin–Temam projection POD-ROM is solved to obtain the discrete data on discrete time point interval \([M + 1, N]\).

4.1.1 Offline Stage: Constructing Reduced-Order Models by POD Technique

In this subsection, we will briefly give some essential ingredients for realizing POD technique, more details for constructing reduced space and POD-based reduced-order models can be found in [33, 49].

Before constructing POD modes, we notice that there are two ways to generate the POD modes: using snapshots with the corresponding difference quotients(DQs) [10, 11, 29, 31, 32]; or using snapshots only(i.e. without the corresponding DQs) [1, 4, 19, 37, 40]. We include DQs in the following snapshot sets both for velocity and pressure since we need to cite optimal pointwise-in-time projection error estimates Lemma 4.2 which was an assumption before and was confirmed recently in [30] in the presence of DQs. We emphasize that adding DQs in POD snapshot sets is necessary only for numerical analysis, and no DQs would not damage the efficiency of our scheme. More benefits of including DQs can be found in [27].

As we remarked in Remark 3.2, the numerical pressure oscillation in the first few time steps caused by setting \( p_h^0 = 0 \) makes us have to discard the FE solutions in the previous time steps, so when using FE solution obtained by solving (Scheme 3-Proj-FE-FOM) to construct reduced spaces, we also need to consider this situations. In view of the fact that the different choice of initial pressure values will affect the discrete error of FE pressure in the previous time steps, we uniformly denote \( n = n_0, n_0 \geq 1 \) as the starting time steps of taking snapshots in (Scheme 3-Proj-FE-FOM); therefore, for a given positive integer \( M \), we choose the FE solution \((\tilde{u}_h, p_h)\), \( n_0 \leq i \leq n_0 + M - 1 \) in (Scheme 3) where \( n_0 + M - 1 \leq \tilde{N} \), and its DQs \((\partial \tilde{u}_h^i, \partial p_h^i)\), \( n_0 + 1 \leq i \leq n_0 + M - 1 \), where the DQs \( \partial f^i \) are defined by \( \partial f^i := (f^i - f^{i-1})/\Delta t \) for some function \( f \) at discrete time \( t = i \), to formulate the snapshot spaces

\[
\tilde{U} := \langle \tilde{u}_h^{n_0}, \tilde{u}_h^{n_0+1}, \ldots, \tilde{u}_h^{M+n_0-1}, \partial \tilde{u}_h^{n_0+1}, \partial \tilde{u}_h^{n_0+2}, \ldots, \partial \tilde{u}_h^{M+n_0-1} \rangle,
\]

\[
\tilde{P} := \langle p_h^{n_0}, p_h^{n_0+1}, \ldots, p_h^{M+n_0-1}, \partial p_h^{n_0+1}, \partial p_h^{n_0+2}, \ldots, \partial p_h^{M+n_0-1} \rangle.
\]

where \( \langle S \rangle \) denotes the space spanned by the set \( S \) and we denote by \( N_s = 2M - 1 \) the number of snapshots. Let \( d_u, d_p \) be the dimensions of the spaces \( \tilde{U} \) and \( \tilde{P} \) resp.. The symbols \( K_{\tilde{U}}, K_p \) are correlation matrices corresponding to the snapshots \( K_{\tilde{U}} = (K_{\tilde{U}}^{i,j}) \in \mathbb{R}^{N_u \times N_s} \) and \( K_p = (K_{\tilde{P}}^{i,j}) \in \mathbb{R}^{N_p \times N_s} \) where

\[
K_{\tilde{U}}^{i,j} := \frac{1}{N_s} \left( \nabla \tilde{u}_h^i, \nabla \tilde{u}_h^j \right), \quad K_{\tilde{P}}^{i,j} := \frac{1}{N_s} \left( p_h^i, p_h^j \right).
\]

Just as in [31], a singular value decomposition(SVD) is carried out and the leading generalized eigenfunctions are chosen as bases, referred to as the POD bases. We denote by \( \lambda_1 \geq \lambda_2 \geq \ldots \)
\[ \varphi_i = \frac{1}{\sqrt{\lambda_i N_s}} \begin{pmatrix} n_0 + M - 1 \sum_{j=n_0}^{n_0 + M - 1} x_i^j \tilde{u}_h^j \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, d_v, \]

\[ \psi_i = \frac{1}{\sqrt{\gamma_i N_s}} \begin{pmatrix} n_0 + M - 1 \sum_{j=n_0}^{n_0 + M - 1} y_i^j p_h^j \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, d_p. \]

In what follows we will denote by

\[ V_r = \langle \varphi_1, \varphi_2, \ldots, \varphi_r \rangle, \quad Q_r = \langle \psi_1, \psi_2, \ldots, \psi_r \rangle, \]

i.e., we choose the first \( r \) POD bases to span the reduced basis spaces. As we will show below, the matrix in POD-ROM can be obtained directly from the corresponding ones in the FE-FOM by some simple operations on matrices. To this end, we write the POD modes obtained above as the vector product w.r.t. the FE basis functions \( \{ \gamma_i, \psi_i \}_{i=1}^{d_p} \) are the eigen-pairs of \( K_p \). Then, the two POD bases can be written explicitly as

\[ \varphi_i = \frac{1}{\sqrt{\lambda_i N_s}} \begin{pmatrix} n_0 + M - 1 \sum_{j=n_0}^{n_0 + M - 1} x_i^j \tilde{u}_h^j \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, d_v, \]

\[ \psi_i = \frac{1}{\sqrt{\gamma_i N_s}} \begin{pmatrix} n_0 + M - 1 \sum_{j=n_0}^{n_0 + M - 1} y_i^j p_h^j \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, d_p. \]

Finally, POD-related matrix which are unrelated to time variable can be assembled offline as

\[ A_v^u := (\nabla \varphi_i (\cdot), \nabla \varphi_j (\cdot)) = \langle \varphi_i, \varphi_j \rangle A_h^u \Phi_r, \]

\[ A_v^p := (\nabla \psi_i (\cdot), \nabla \psi_j (\cdot)) = \langle \psi_i, \psi_j \rangle A_h^p \Psi_r, \]

\[ M_v^p := (\varphi_i (\cdot), \varphi_j (\cdot)) = \langle \varphi_i, \varphi_j \rangle M_h^u \phi_r, \]

\[ B_r := (\psi_i (\cdot), \nabla \psi_j (\cdot)) = \langle \psi_i, \psi_j \rangle B_h \Phi_r, \]

where \( A_v^u := (\nabla \varphi_{hi} (\cdot), \nabla \varphi_{hj} (\cdot)) \), \( A_v^p := (\nabla \psi_{hi} (\cdot), \nabla \psi_{hj} (\cdot)) \), \( M_v^p := (\varphi_{hi} (\cdot), \varphi_{hj} (\cdot)) \), \( B_h := (\nabla \varphi_{hi} (\cdot), \psi_{hj} (\cdot)) \) are stiff matrices for velocity and pressure, mass matrix for velocity and matrix w.r.t. coupled term.
4.1.2 Online Stage: Formulating Chorin–Temam Projection POD-ROM

With the above preparations, we seek for, in this subsection, a reduced-order approximation of both FE-FOM velocity and pressure field \((\tilde{u}_h^{n+1}, p_h^{n+1})\) under the form

\[
\tilde{u}_h^{n+1} \approx \Phi_r \tilde{u}_r^{n+1}, \quad p_h^{n+1} \approx \Psi_r p_r^{n+1}.
\]

\(\tilde{u}_r^{n+1}\) and \(p_r^{n+1}\) are determined variables by solving the following POD-ROM; that is, we get the following Chorin–Temam projection POD-ROM in algebraic form: given \((\tilde{u}_h^n, p_h^n) = (\sum_{i=1}^r (\nabla \tilde{u}_i^n, \nabla \varphi_i) \varphi_i, \sum_{i=1}^r (p_h^n, \psi_i) \psi_i)\) when \(n = M+n_0-1\), for \(\forall n \geq M+n_0-1\), find \(\tilde{u}_r^{n+1} \in \mathbb{R}^r\) and \(p_r^{n+1} \in \mathbb{R}^r\), such that

\[
\begin{bmatrix}
\frac{1}{\Delta t} M^n_r + v A^n_r & 0 \\
-B_r & \Delta t A^n_r
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_r^{n+1} \\
p_r^{n+1}
\end{bmatrix}
= \begin{bmatrix}
F_r^n + \frac{1}{\Delta t} D_r^n - B_r^n \\
0
\end{bmatrix}
\tag{4.2}
\]

where, \(M^n_r, A^n_r, A^n_p\) and \(B_r\) are defined above, and the three time-dependent matrices \(F^n_r, D^n_r, B^n_r\) can be assembled as

\[
F_r^n = (\Phi_r)^T F_h^n, \quad D_r^n = M^n_r \tilde{u}_r^n, \quad B_r^n = B_r p_r^n.
\]

For the purpose of the subsequent numerical analysis, we give the equivalent Chorin–Temam projection POD-ROM in functional form. To this end, we define the traditional Ritz projection and \(L^2\) into the POD velocity space \(V_r\) and POD pressure space \(Q_r\) resp. as

**Definition 4.1** Let \(\Pi^n_r : L^2(\Omega) \to V_r\) and \(\Pi^n_Q : L^2(\Omega) \to Q_r\) such that

\[
(\nabla (u - \Pi^n_r u), \nabla v_r) = 0, \quad \forall v_r \in V_r,
\]

\[
(p - \Pi^n_Q p, q_r) = 0, \quad \forall q_r \in Q_r.
\]

Then, we can get the following Proj-POD-ROM scheme in functional form:

**Scheme 4 - Proj-POD-ROM** When \(n = M+n_0-1\), let \((\tilde{u}_h^n, p_h^n) = (\Pi^n_r \tilde{u}_r^n, \Pi^n_Q p_h^n)\) be the Galerkin projection onto POD-based reduced spaces \(V_r \times Q_r\), then for \(\forall n \geq M+n_0-1\), we can compute \((\tilde{u}_r^{n+1}, p_r^{n+1}) \in V_r \times Q_r\) from

\[
\begin{cases}
(\frac{\tilde{u}_r^{n+1} - \tilde{u}_r^n}{\Delta t}, v_r) + v(\nabla \tilde{u}_r^{n+1}, \nabla v_r) + (\nabla p_r^{n+1}, v_r) = (f_r^{n+1}, v_r), \quad \forall v_r \in V_r, \\
(\nabla \cdot \tilde{u}_r^{n+1}, q_r) + \Delta t (\nabla p_r^{n+1}, \nabla q_r) = 0, \quad \forall q_r \in Q_r.
\end{cases}
\tag{4.3}
\]

We notice that, after setting \(\Delta t = O(h^2)\), the reduced-order pressure-stabilized term \(\Delta t (\nabla p_r^{n+1}, \nabla q_r)\) is equivalent the Brezzi–Pitkäranta stabilization in the FE community, and this term has been used to stabilize POD reduced-order pressure in [1] in the scenario of varying parameters. We point out that the difference lies in four aspects between **Scheme 4-Proj-POD-ROM** and the one proposed in [1]: (i) The stating point is different. **Scheme 4-Proj-POD-ROM** stems from the classical Chorin–Temam projection method and the pressure-stabilized term is derived from the inherited stabilization of the projection method, which however was directly added in the variational equations in [1]; (ii) The purpose is different. We are aiming to handle the time extrapolation issues and Ali et al. considered the multi-parameters problem; (iii) Decoupling or not. The POD-ROM proposed in this paper is decoupled on velocity and pressure variables, but the counterpart in [1] is coupled about two variables and the resulting saddle-point system poses difficulties when solving the algebraic system. (iv) Having theoretical analysis or not. POD reduced-order velocity and pressure, enhanced by Brezzi–Pitkäranta stabilization in [1], are only numerically verified.

\[\text{Springer}\]
about the accuracy and convergence, but it lacked theoretical support. In this paper, we give the rigorous theoretical analysis and obtain optimal error estimates for temporal, spatial and ROM truncation errors, i.e. Theorem 4.2, to theoretically guarantee the stabilization effect of Brezzi–Pitkäranta stabilization on POD reduced-order velocity and pressure fields.

4.2 Stability and Convergence Analysis of Projection POD-ROM

We will carry on some numerical analysis about the newly proposed Proj-POD-ROM to get its stability and convergence. To this end, we recall the classical conclusions concerning about the projection error between the snapshots solutions and POD-based reduced-order solutions (see [31]).

Lemma 4.1 (Time-averaged projection error estimates) We have the following time-averaged projection error estimates.

\[
\frac{1}{N_s} \sum_{j=n_0}^{n_0+M-1} \left\| \tilde{u}_h^j - \sum_{k=1}^{r} \left( \nabla \tilde{u}^j_k, \nabla \phi_k \right) \phi_k \right\|_0^2 \\
+ \frac{1}{N_s} \sum_{j=n_0+1}^{n_0+M-1} \left\| \partial \tilde{u}_h^j - \sum_{k=1}^{r} \left( \nabla \partial \tilde{u}^j_k, \nabla \phi_k \right) \phi_k \right\|_0^2 = \sum_{k=r+1} d_{\tilde{u}} \lambda_k, \\
\frac{1}{N_s} \sum_{j=n_0}^{n_0+M-1} \left\| p_h^j - \sum_{k=1}^{r} \left( p_h^j, \psi_k \right) \phi_k \right\|_0^2 \\
+ \frac{1}{N_s} \sum_{j=n_0+1}^{n_0+M-1} \left\| \partial p_h^j - \sum_{k=1}^{r} \left( \partial p_h^j, \psi_k \right) \phi_k \right\|_0^2 = \sum_{k=r+1} d_{\psi} \gamma_k, 
\]  

(4.4)

Apart from the above time-averaged projection error estimates, we also need the following up-to-data optimal in time error estimates, which is presented in the form of hypothesis in the previous papers and is proved to be valid in the presence of DQs in [30].

Lemma 4.2 (POD Optimal pointwise-in-time error estimate) We have the following optimal in time projection errors

\[
\max_{0 \leq k \leq N} \left\| \tilde{u}_h^k - \Pi_2^k \tilde{u}_h^k \right\|_0^2 \leq C \sum_{i=r+1} d_{\tilde{u}} \lambda_i, \\
\max_{0 \leq k \leq N} \left\| p_h^k - \Pi_2^k p_h^k \right\|_0^2 \leq C \sum_{i=r+1} d_{\psi} \gamma_i, 
\]

where \( C = 6 \max\{1, T^2\} \).

We also need in the later analysis the time-averaged optimal projection error estimate about DQs, which is derived in [31].

Lemma 4.3 (POD Optimal time-averaged error estimate about DQs) For the difference quotients we have the estimate:

\[
\frac{1}{M} \sum_{k=n_0}^{n_0+M-1} \left\| \partial \tilde{u}_h^k - \Pi_2^k \partial \tilde{u}_h^k \right\|_0^2 \leq C \sum_{i=r+1} d_{\tilde{u}} \lambda_i. 
\]
We can prove the unconditional stability of the Proj-POD-ROM scheme.

**Theorem 4.1** (Unconditional stability of Proj-POD-ROM) We have the following unconditional stability

\[
\left\| \tilde{u}_r^{n+1} \right\|_0^2 + \sum_{n=n_0}^{N-1} \left\| \tilde{u}_r^n - \tilde{u}_r^n \right\|_0^2 + \Delta t \sum_{n=n_0}^{N-1} \left( \nu \left\| \nabla \tilde{u}_r^{n+1} \right\|_0^2 + \Delta t \left\| \nabla p_r^{n+1} \right\|_0^2 \right),
\]

\[
\leq C \left( \left\| \tilde{u}_r^0 \right\|_0^2 + \frac{\Delta t}{\nu} \sum_{n=n_0}^{N-1} \left\| f^{n+1} \right\|_{-1} \right).
\]

**Proof** Taking \((\nu_r, q_r) = (\tilde{u}_r^{n+1}, p_r^n)\) in (4.3) and multiply by \(2\Delta t\), adding both equations and integrating by parts, adding and subtracting \(2\Delta t^2 \left\| \nabla p_r^{n+1} \right\|_0^2\), we get

\[
\left\| \tilde{u}_r^{n+1} \right\|_0^2 - \left\| \tilde{u}_r^n \right\|_0^2 + \left\| \tilde{u}_r^{n+1} - \tilde{u}_r^n \right\|_0^2 + 2\nu \Delta t \left\| \nabla \tilde{u}_r^{n+1} \right\|_0^2 + 2\Delta t^2 \left\| \nabla p_r^{n+1} \right\|_0^2,
\]

\[
\leq C \frac{\Delta t}{\nu} \left\| f^{n+1} \right\|_{-1} + \nu \Delta t \left\| \nabla \tilde{u}_r^{n+1} \right\|_0^2 + 2\Delta t^2 \left( \nabla p_r^{n+1}, \nabla (p_r^{n+1} - p_r^n) \right).
\]

For the last term above, we utilize the second equation in (4.3) twice to obtain

\[
\Delta t \left( \nabla p_r^{n+1}, \nabla (p_r^{n+1} - p_r^n) \right) = -\left( \nabla \cdot (\tilde{u}_r^{n+1} - \tilde{u}_r^n), p_r^{n+1} \right) = (\tilde{u}_r^{n+1} - \tilde{u}_r^n, \nabla p_r^{n+1}).
\]

This equality implies

\[
2\Delta t^2 \left( \nabla p_r^{n+1}, \nabla (p_r^{n+1} - p_r^n) \right) \leq \frac{2}{3} \left\| \tilde{u}_r^{n+1} - \tilde{u}_r^n \right\|_0^2 + \frac{3}{2} \Delta t^2 \left\| \nabla p_r^{n+1} \right\|_0^2.
\]

Inserting the above equality into (4.6) and adding \(n\) from 0 to \(N-1\) we finally obtain (4.5). \(\square\)

We have the following truncation error estimate between the POD reduced-order solution and the snapshot data.

**Lemma 4.4** (POD truncation error) For \(n = 1, 2, \ldots, N\), \((\tilde{u}_h^n, p_h^n)\) is the solution of FE projection scheme (3.3) and \((\tilde{u}_h^n, p_h^n)\) is the solution of POD projection scheme (4.3), then we have the following error estimate:

\[
\left\| \tilde{u}_h^N - \tilde{u}_h^N \right\|_0^2 + \nu \Delta t \sum_{n=n_0}^{N-1} \left\| \nabla (\tilde{u}_h^{n+1} - \tilde{u}_r^{n+1}) \right\|_0^2 + \Delta t^2 \sum_{n=n_0}^{N-1} \left\| \nabla (p_h^{n+1} - p_r^{n+1}) \right\|_0^2,
\]

\[
\leq C \left( \sum_{i=r+1}^{d_h} \lambda_i + \sum_{i=r+1}^{d_h} \gamma_i + \Delta t^2 \right).
\]

**Proof** Proj-FE-FOM (3.3) subtracts Proj-POD-ROM (4.3) to obtain:

\[
\begin{cases}
\left( \frac{\tilde{u}_h^{n+1} - \tilde{u}_h^n}{\Delta t} - \frac{\tilde{u}_r^{n+1} - \tilde{u}_r^n}{\Delta t}, \nu_r \right) + \nu (\nabla (\tilde{u}_h^{n+1} - \tilde{u}_r^{n+1}), \nabla \nu_r) + (\nabla (p_h^{n+1} - p_r^n), \nu_r) = 0,
\end{cases}
\]

\[
(\nabla \cdot (\tilde{u}_h^{n+1} - \tilde{u}_r^{n+1}), q_r) + \Delta t (\nabla (p_h^{n+1} - p_r^{n+1}), \nabla q_r) = 0.
\]

\(\square\)
Denoting
\[
\begin{align*}
\tilde{u}^{n+1} &= \tilde{u}^{n+1} - \Pi_r \tilde{u}^{n+1} - \Pi_r \tilde{u}^{n+1} + \Pi_r \tilde{u}^{n+1} = \tilde{n}^{n+1} + \tilde{w}^{n+1}, \\
p^{n+1} - p^{n+1} &= \tilde{p}^{n+1} - \Pi_r \tilde{p}^{n+1} + \Pi_r \tilde{p}^{n+1} = \eta^{n+1} + w^{n+1}.
\end{align*}
\]
Testing \((v_r, q_r) = (\tilde{w}^{n+1}, w^{n+1})\), we obtain
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{(\tilde{w}^{n+1} - \tilde{u}^{n+1}, \tilde{w}^{n+1})}{\Delta t} + v \|
\tilde{w}^{n+1} \|_0^2 + (\nabla \tilde{w}^{n+1}, \tilde{w}^{n+1}) \\
\frac{(\tilde{n}^{n+1} - \tilde{n}^{n}, \tilde{w}^{n+1})}{\Delta t} + v(\nabla \tilde{n}^{n+1}, \nabla \tilde{w}^{n+1}),
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
+ (\nabla (p^{n+1} - p^{n}), \tilde{w}^{n+1}) + (\nabla (p^{n} - p^{n+1}), \tilde{w}^{n+1}) + (\nabla 1, \tilde{w}^{n+1}) + (\nabla \tilde{w}^{n+1}, \nabla 1) + \Delta t(\nabla \tilde{n}^{n+1}, \nabla \tilde{w}^{n+1}).
\end{align*}
\]
Adding both equations we have the following error equation
\[
\begin{align*}
\frac{(\tilde{w}^{n+1} - \tilde{w}^{n}, \tilde{w}^{n+1})}{\Delta t} + v \|
\tilde{w}^{n+1} \|_0^2 + \Delta t \|
\tilde{w}^{n+1} \|_0^2 = \frac{(\tilde{n}^{n+1} - \tilde{n}^{n}, \tilde{w}^{n+1})}{\Delta t} + v(\nabla \tilde{n}^{n+1}, \nabla \tilde{w}^{n+1}),
\end{align*}
\]
\[
\begin{align*}
+ (\nabla (p^{n+1} - p^{n}), \tilde{w}^{n+1}) + (\nabla (p^{n} - p^{n+1}), \tilde{w}^{n+1}) + (\nabla 1, \tilde{w}^{n+1}) + (\nabla \tilde{w}^{n+1}, \nabla 1) + \Delta t(\nabla \tilde{n}^{n+1}, \nabla \tilde{w}^{n+1}),
\end{align*}
\]
\[
\begin{align*}
\equiv A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.
\end{align*}
\]
We will bound the seven terms above separately.
\[
|A_1| = \left| \frac{(\tilde{n}^{n+1} - \tilde{n}^{n}, \tilde{w}^{n+1})}{\Delta t} \right| \leq C \left| \frac{\tilde{n}^{n+1} - \tilde{n}^{n}}{\Delta t} \right|_0^2 + \frac{1}{6} \|
\tilde{w}^{n+1} \|_0^2.
\]
For the first term above, we utilize the expression of \(\tilde{u}^{n}\) and \(\Pi_r \tilde{u}^{n}\) in the form of POD bases \(\varphi\), to get
\[
\tilde{n}^{n+1} = \tilde{u}^{n+1} - \Pi_r \tilde{u}^{n+1} = \sum_{k=r+1}^{d_{\tilde{u}}} (\nabla \tilde{u}^{n+1}, \nabla \varphi_k) \varphi_k.
\]
Then,
\[
\begin{align*}
\left| \frac{\tilde{n}^{n+1} - \tilde{n}^{n}}{\Delta t} \right|_0^2 = \left| \frac{1}{\Delta t} \left( \sum_{k=r+1}^{d_{\tilde{u}}} (\nabla \tilde{u}^{n+1}, \nabla \varphi_k) \varphi_k - \sum_{k=r+1}^{d_{\tilde{u}}} (\nabla \tilde{u}^{n}, \nabla \varphi_k) \varphi_k \right) \right|_0^2,
\end{align*}
\]
\[
\begin{align*}
= \left| \sum_{k=r+1}^{d_{\tilde{u}}} \left( \nabla \tilde{u}^{n+1} - \nabla \tilde{u}^{n}, \nabla \varphi_k \right) \varphi_k \right|_0^2,
\end{align*}
\]
\[
\begin{align*}
= \left| \sum_{k=r+1}^{d_{\tilde{u}}} \left( \nabla \tilde{u}^{n+1} - \nabla \tilde{u}^{n}, \nabla \varphi_k \right) \varphi_k \right|_0^2,
\end{align*}
\]
\[
\begin{align*}
= \left| \nabla \tilde{u}^{n+1} - \nabla \tilde{u}^{n} \right|_0^2.
\end{align*}
\]
For other terms on the right-side hand of (4.9), \( A_2 \) disappears because of the orthogonality of Ritz projection. For the remaining terms,

\[
|A_3| = \left| \nabla (p^{n+1}_h - p^n_h, \tilde{w}^{n+1}_{u,r}) \right|
\leq C \| \nabla (p^{n+1}_h - p^n_h) \|_0^2 + \frac{1}{6} \| \tilde{w}^{n+1}_{u,r} \|_0^2
\leq C \Delta t \int_{t_n}^{t_{n+1}} \| \nabla (\partial_t p_n) \|_0^2 \, dt + \frac{1}{6} \| \tilde{w}^{n+1}_{u,r} \|_0^2.
\]

Similarly, \( A_4 \) can be estimated as

\[
|A_4| = \left| \nabla (p^n_r - p^{n+1}_r, \tilde{w}^{n+1}_{u,r}) \right|
\leq C \| \nabla (p^n_r - p^{n+1}_r) \|_0^2 + \frac{1}{6} \| \tilde{w}^{n+1}_{u,r} \|_0^2
\leq C \Delta t \int_{t_n}^{t_{n+1}} \| \nabla (\partial_t p_r) \|_0^2 \, dt + \frac{1}{6} \| \tilde{w}^{n+1}_{u,r} \|_0^2.
\]

Using integrating by parts and Lemma 4.2, we can estimated \( A_5, A_6 \) as

\[
|A_5| = \left| \nabla \eta^{n+1}_{p,r}, \tilde{w}^{n+1}_{u,r} \right|
\leq C \| \eta^{n+1}_{p,r} \|_0^2 + \frac{1}{4} \nu \| \nabla \tilde{w}^{n+1}_{u,r} \|_0^2
\leq C \sum_{i=r+1}^d \gamma_i + \frac{1}{4} \nu \| \nabla \tilde{w}^{n+1}_{u,r} \|_0^2,
\]

and

\[
|A_6| = \left| \nabla \cdot \tilde{w}^{n+1}_{u,r}, \tilde{w}^{n+1}_{u,r} \right|
\leq C \| \tilde{w}^{n+1}_{u,r} \|_0^2 + \frac{1}{4} \nu \| \nabla \tilde{w}^{n+1}_{u,r} \|_0^2
\leq C \sum_{i=r+1}^{d_i} \lambda_i + \frac{1}{4} \nu \| \nabla \tilde{w}^{n+1}_{u,r} \|_0^2.
\]

For the last term, using inverse inequality (2.5) and (3.5) to obtain

\[
|A_7| = \left| \Delta t (\nabla \eta^{n+1}_{p,r}, \nabla w^{n+1}_{p,r}) \right|
\leq \frac{1}{2} \Delta t \| \nabla \eta^{n+1}_{p,r} \|_0^2 + \frac{1}{2} \Delta t \| \nabla w^{n+1}_{p,r} \|_0^2
\leq C \| \eta^{n+1}_{p,r} \|_0^2 + \frac{1}{2} \Delta t \| \nabla w^{n+1}_{p,r} \|_0^2
\leq C \sum_{i=r+1}^d \gamma_i + \frac{1}{2} \Delta t \| \nabla w^{n+1}_{p,r} \|_0^2.
\]
Combining with all the seven terms’ results, we have
\[
\left( \frac{\mathbf{w}_{u,r}^{n+1} - \mathbf{w}_{u,r}^n}{\Delta t}, \mathbf{w}_{u,r}^{n+1} \right) + \frac{1}{2} \nu \| \nabla \mathbf{w}_{u,r}^{n+1} \|_0^2 + \frac{1}{2} \Delta t \| \nabla w_{u,r}^{n+1} \|_0^2,
\]
\[
\leq \left\| \partial_t \mathbf{w}_{u,r}^{n+1} - \Gamma_{r} \mathbf{w}_{u,r}^{n+1} \right\|_0^2 + C \Delta t \int_{t_n}^{t_{n+1}} \left( \| \nabla \partial_r p_h \|_0^2 + \| \nabla \partial_r p_r \|_0^2 \right) \, dt,
\]
\[
+ C \sum_{i=r+1}^{d_g} \lambda_i + C \sum_{i=r+1}^{d_p} \gamma_i + \frac{1}{2} \| \mathbf{w}_{u,r}^{n+1} \|_0^2.
\]

Multiplying by $\Delta t$, adding with $n$ from $n_0$ to $N - 1$ and rearranging, we get
\[
\| \mathbf{w}_{u,r}^N \|_0^2 + \Delta t \sum_{n=n_0}^{N-1} \| \mathbf{w}_{u,r}^{n+1} - \mathbf{w}_{u,r}^n \|_0^2 + \nu \Delta t \sum_{n=n_0}^{N-1} \| \nabla \mathbf{w}_{u,r}^{n+1} \|_0^2 + \Delta t^2 \sum_{n=n_0}^{N-1} \| \nabla w_{u,r}^{n+1} \|_0^2,
\]
\[
\leq \Delta t \sum_{n=n_0}^{N-1} \left\| \partial_t \mathbf{w}_{u,r}^{n+1} - \Gamma_{r} \mathbf{w}_{u,r}^{n+1} \right\|_0^2 + C \left( \sum_{i=r+1}^{d_g} \lambda_i + \sum_{i=r+1}^{d_p} \gamma_i \right) + \frac{1}{2} \Delta t \sum_{n=n_0}^{N-1} \| \mathbf{w}_{u,r}^{n+1} \|_0^2 + \Delta t^2 \int_{t_n}^{t_{n+1}} \left( \| \nabla \partial_r p_h \|_0^2 + \| \nabla \partial_r p_r \|_0^2 \right) \, dt.
\]

For the first term on the right-hand side above, we use Lemma 4.3 to have
\[
\Delta t \sum_{n=n_0}^{N-1} \left\| \partial_t \mathbf{w}_{u,r}^{n+1} - \Gamma_{r} \mathbf{w}_{u,r}^{n+1} \right\|_0^2 \leq C \frac{1}{M} \sum_{k=1}^{M} \| \partial_t \mathbf{w}_{k}^{n+1} - \Gamma_{r} \partial_t \mathbf{w}_{k}^{n+1} \|_0^2 \leq C \sum_{i=r+1}^{d_g} \lambda_i.
\]

By discrete Gronwall inequality (see [26]) and using stability result to bound the last term above, we get
\[
\| \mathbf{w}_{u,r}^N \|_0^2 + \Delta t \sum_{n=n_0}^{N-1} \| \mathbf{w}_{u,r}^{n+1} - \mathbf{w}_{u,r}^n \|_0^2 + \nu \Delta t \sum_{n=n_0}^{N-1} \| \nabla \mathbf{w}_{u,r}^{n+1} \|_0^2 + \Delta t^2 \sum_{n=n_0}^{N-1} \| \nabla w_{u,r}^{n+1} \|_0^2,
\]
\[
\leq C \left( \sum_{i=r+1}^{d_g} \lambda_i + \sum_{i=r+1}^{d_p} \gamma_i + \Delta t^2 \right).
\]

Finally, by triangle inequality, Lemma 3.2 and Lemma 4.4, we obtain the following theorem which states the convergence between continuous variational form and POD projection scheme.

**Theorem 4.2** (Error estimate for Proj-POD-ROM) For $n = n_0, n_0 + 1, \cdots, N$, let $(\mathbf{u}^n, p^n)$ is the solution of (2.3) at $t = t_n$, $(\mathbf{u}_{r}^n, p_{r}^n)$ denotes the POD-based reduced-order solutions obtained in (4.3), then we have the following convergence estimate:
\[
\| \mathbf{u}^N - \mathbf{w}_{u,r}^N \|_0^2 + \nu \Delta t \sum_{n=n_0}^{N-1} \| \nabla \mathbf{w}_{u,r}^{n+1} - \mathbf{w}_{u,r}^n \|_0^2 + \Delta t^2 \sum_{n=n_0}^{N-1} \| \nabla (p^n - p_{r}^n) \|_0^2,
\]
\[
\leq C \left( \sum_{i=r+1}^{d_g} \lambda_i + \sum_{i=r+1}^{d_p} \gamma_i + \Delta t^2 + h^2 \right).
\]
Corollary 4.1 For \( n = n_0, n_0 + 1, \ldots, N \), let \( (u^n, p^n) \) is the solution of (2.3) at \( t = t_n \). \( (\tilde{u}_r^n, p_r^n) \) denotes the POD-based reduced-order solutions obtained in (4.3), then we have the following optimal convergence estimate:

\[
\max_n \| u^n - \tilde{u}_r^n \|_0^2 \leq C \left( \sum_{i=r+1}^{d_q} \lambda_i + \sum_{i=r+1}^{d_p} \gamma_i + \Delta t^2 + h^4 \right).
\] (4.12)

4.3 Connection with Supremizer Stabilization Technique

In this section, we will provide the theoretical analysis about an issue which was reported by several papers only in numerical experiments. Specifically, we will analyze that the pressure stabilized term in (2.3) with \( \Delta t = O(h^2) \), in (Scheme 4 - Proj-POD-ROM) can imply the supremizer stabilization technique [4, 39]; in other words, we will show that the satisfaction of Brezzi–Pitkäranta stabilization (4.15) [6] can ensure the inf-sup condition in POD-ROM version (4.14) with the help of supremizer. This connection implies that the presence of Brezzi–Pitkäranta stabilization, just as in (Scheme 4—Proj-POD-ROM), makes the addition of supremizers unnecessary in terms of stability of reduced pressure.

This motivation is inspired by the phenomenon, which was reported by several papers, that adding pressure stabilized term which includes \( h^2 (\nabla p_r, \nabla q_r) \) makes the POD reduced-basis velocity and pressure solutions look similar, with or without supremizer. These papers include [44] which added the residual-based VMS method in POD-ROM with/without supremizer, and [1] which compared the Brezzi–Pitkäranta stabilization POD-ROM with/without supremizer. These papers mentioned above only reported this phenomenon numerically, but lack of theoretical explanation.

For completeness, we give brief introduction of supremizer enrichment technique; for detailed derivation, we refer to [4]. For \( \forall q_r \in Q_r \), the corresponding supremizer, that is, the element realizing the supremum in the inf-sup condition of POD-ROM version, is given by the solution \( s_r = s_r(q_r) \) of the following problem:

Given a function \( q_r \in Q_r \), find \( s_r = s_r(q_r) \), such that

\[
(\nabla s_r, \nabla v_r) = - (\nabla \cdot v_r, q_r), \quad \forall v_r \in V_r.
\] (4.13)

Two algorithms available can be used to formulate the supremizer space \( S_r \): exact supremizer enrichment algorithm and approximate supremizer enrichment algorithm [4]. The latter is more practical but only the former is well-established in theory. Solving (4.13) for each basis function \( \{ \psi_i \}_{i=1}^{r} \) and applying Gram-Schmidt orthonormalization procedure to the obtained solutions yields a set of basis function \( \{ \xi_i \}_{i=1}^{r} \) which spanned the supremizer space as

\[
S_r := \text{span}\{ \xi_i \}_{i=1}^{r}.
\]

Enriching the POD velocity \( V_r \) with supremizer space \( S_r \) to form a bigger POD velocity \( \hat{V}_r := V_r + S_r \), the direct sum of base in \( V_r \) and \( S_r \). The following inf-sup stability between \( \hat{V}_r \) and \( Q_r \) holds [4].

Lemma 4.5 The spaces \( \hat{V}_r \) and \( Q_r \) are stable in the sense of

\[
\exists \beta_S > 0, \text{ s.t. } \beta_S \| q_r \|_0 \leq \sup_{\tilde{v}_r \in \hat{V}_r} \frac{h(\tilde{v}_r, q_r)}{\| \nabla \tilde{v}_r \|_0 \cdot \| q_r \|_0}, \quad \forall q_r \in Q_r.
\] (4.14)

The pressure stabilized term \( \Delta t (\nabla p_r^{n+1}, \nabla q_r) \) in (Scheme 4 - Proj-POD-ROM) is actually the Brezzi–Pitkäranta stabilization [6] after setting \( \Delta t = O(h^2) \), which ensures the POD...
pressure \( q_r \in Q_r \) in (Scheme 4 - Proj-POD-ROM) stable. That is, we have the following lemma.

**Lemma 4.6** There exists a constant \( C \) independent of \( h, r \), such that

\[
\exists \beta_M > 0, \text{ s.t. } \beta_M \|q_r\|_0 \leq \sup_{v_r \in V_r} \frac{b(v_r, q_r)}{\|\nabla v_r\|_0 \cdot \|q_r\|_0} + Ch \|\nabla q_r\|_0, \quad \forall q_r \in Q_r. \quad (4.15)
\]

Based on those preparations, we obtain the main result of this section.

**Theorem 4.3** The Brezzi–Pitkäranta stabilized inf-sup condition (4.15) implies the supremizer stabilized inf-sup condition (4.14).

**Proof** By the realization of supremizer stabilization technique, we can write \( \forall \hat{v}_r \in \hat{V}_r = V_r + S_r \) as

\[
\hat{v}_r = \sum_{i=1}^{r} a_i \varphi_i + \sum_{i=1}^{r} b_i \xi_i := v_r + s_r,
\]

then for \( \forall q_r \in Q_r \),

\[
\sup_{\hat{v}_r \in \hat{V}_r} \frac{b(\hat{v}_r, q_r)}{\|\nabla \hat{v}_r\|_0 \cdot \|q_r\|_0} = \sup_{(v_r, s_r) \in V_r + S_r} \frac{b(v_r + s_r, q_r)}{\|\nabla (v_r + s_r)\|_0 \cdot \|q_r\|_0}
\]

\[
= \sup_{(v_r, s_r) \in V_r + S_r} \frac{b(v_r, q_r)}{\|\nabla (v_r + s_r)\|_0 \cdot \|q_r\|_0} + \sup_{s_r \in S_r} \frac{b(s_r, q_r)}{\|\nabla s_r\|_0 \cdot \|q_r\|_0}
\]

\[
= \sup_{v_r \in V_r} \frac{b(v_r, q_r)}{\|\nabla v_r\|_0 \cdot \|q_r\|_0} + \sup_{s_r \in S_r} \frac{b(s_r, q_r)}{\|\nabla s_r\|_0 \cdot \|q_r\|_0},
\]

where in the last equality we used the definition of supremizer which attained the upper bound. It remains to show that

\[
Ch \|\nabla q_r\|_0 \leq \|\nabla s_r\|_0.
\]

To this end, we utilize the (4.13) to get

\[
h \|\nabla q_r\|_0 = h \sup_{v_r} \frac{\|\nabla q_r \cdot v_r\|_0}{\|v_r\|_0} = h \sup_{v_r} \frac{\|\nabla q_r \cdot v_r\|_0}{\|v_r\|_0} = h \sup_{v_r} \frac{\|\nabla q_r \cdot v_r\|_0}{\|v_r\|_0} \leq h \cdot \sup_{v_r} \frac{\|\nabla s_r \cdot v_r\|_0}{\|v_r\|_0} \leq C \sup_{v_r} \frac{\|\nabla s_r \cdot v_r\|_0}{\|v_r\|_0} = C \|\nabla s_r\|_0,
\]

which concludes this theorem.

\( \Box \)

**5 Numerical Tests**

In this section, we present two numerical experiments to confirm the high time efficiency and the convergence results. The first one is mainly aiming to verify the convergence and time...
efficiency since which possess the exact solutions; the second benchmark test is conducted to test the stability of reduced pressure, especially after a long period of time interval. The open-source FE package iFEM [12] has been used to run the numerical experiments.

5.1 Tests 1: With Exact Solutions

5.1.1 Problem Setting

We take \( \Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \) and the time interval \((0, 1] \) with the viscosity coefficient \( \nu = 1 \) and the prescribed solution

\[
\begin{align*}
\mathbf{u} &= \cos(t) \cdot \left( \pi \sin(\pi x)^2 \sin(2\pi y) \\
&\quad - \pi \sin(2\pi x) \sin(\pi y)^2 \right), \\
p &= \cos(t) \cdot 10 \cos(\pi x) \cos(\pi y).
\end{align*}
\]

Homogeneous Dirichlet boundary condition is satisfied, just as in (2.1), on the experimental domain; meanwhile, the initial condition and right-hand side can be obtained after easily calculated as

\[
\begin{align*}
\mathbf{u}_0 &= \left( \begin{array}{c}
\pi \sin(\pi x)^2 \sin(2\pi y) \\
- \pi \sin(2\pi x) \sin(\pi y)^2
\end{array} \right), \\
f_1 &= -2\nu \pi^3 \cos(t) \sin(2\pi y)(\cos(2\pi x) - 2 \sin(\pi x)^2) \\
&\quad - \pi \sin(t) \sin(\pi x)^2 \sin(2\pi y) - 10\pi \cos(t) \sin(\pi x) \cos(\pi y), \\
f_2 &= 2\nu \pi^3 \cos(t) \sin(2\pi x)(\cos(2\pi y) - 2 \sin(\pi y)^2) \\
&\quad \pi \sin(t) \sin(\pi y)^2 \sin(2\pi x) - 10\pi \cos(t) \sin(\pi y) \cos(\pi x).
\end{align*}
\]

We set \( \Delta t = 0.1h^2 \) and all grids are regular \( N \times N \) triangular grids with SWNE diagonals for different \( N \) (i.e., diagonals coming from connecting the southwest and the northeast vertexes on all rectangles), and we take \( N = 4, 8, 16, 32, 64 \) sequentially. For simplicity, we use \( P^1 - P^1 \) element pair for spatial discretization, and we take the snapshots on the finest computational mesh, i.e., \( h = 1/64 \).

When constructing the snapshots spaces, we take into account the fact that the FE projection scheme involves the initial pressure \( p^0_h \) which is not the part of the definition of the problem, so we take \( p^0_h = 0 \) and it might be good to think of the numerical discrete errors of pressure \( \| p^n - p^n_h \|_{L^2} \) in the previous steps \( n \) are so large that it is not suitable to take FE solution on those steps into snapshot spaces, thus we choose the FE solution \((\mathbf{u}^n_h, p^n_h)\) and its difference quotients \((\partial \mathbf{u}^n_h, \partial p^n_h)\) from \( n = n_0 = 6 \) to \( n = 25 \) to formulate snapshots spaces, which means we take \( M = 20 \) and thus the number of snapshots is \( N_s = 2M - 1 = 39 \).

The FE solution \((\mathbf{u}^n_h, \mathbf{u}^n_j, p^n_h)\) and exact solution \((\mathbf{u}^n, p^n)\) at discrete termination time \( t = \Delta t \cdot N \) on finest mesh, and following POD bases are formed from snapshots via \( L^2 \) inner product.

5.1.2 Convergence of the FE and POD-ROM Projection Schemes

In this subsection, numerical tests will be used to numerically verify two theoretical results, that is Lemma 3.2 and Lemma 3.3, with which we utilize to get convergence analysis Theorem 4.2 of POD Scheme 4. Although Lemma 3.2 has been analyzed in [18], it lacks the
certain time in the process of time-stepping iterative, and the L shows in Table 3 the comparison of cumulative time spent by the two schemes to run to some degree of freedoms (DOFs) for velocity and pressure in FE scheme are 8450 and 4225. We refer to the FE-FOM scheme, we take the number of POD modes is 10 which are what the theoretical results in Lemma 3.2 shows. Especially, we can see from Table 1,

\[
\begin{array}{cccc}
123 & 5.1.3 High Efficiency on Computation & \\
\end{array}
\]

In order to confirm the computational efficiency advantage of the POD-ROM scheme over the FE-FOM scheme, we take the number of POD modes is \( r = 4 \), whereas the number of degree of freedoms (DOFs) for velocity and pressure in FE scheme are 8450 and 4225. We show in Table 3 the comparison of cumulative time spent by the two schemes to run to some certain time in the process of time-stepping iterative, and the \( L^2 \) numerical spatial discrete error of velocity and pressure in two schemes at that time. The recorded two types of time errors and convergence orders of FE solution \( \tilde{u}_h^n, u_h^n \) with \( P^1 - P^1 \) pair

| 1/h | \( \max_n \| u^n - \tilde{u}_h^n \|_{L^2} \) | \( \max_n \| u^n - u_h^n \|_{L^2} \) | \( \| \nabla (u - \tilde{u}_h) \|_{L^2} \) |
|-----|-----------------|-----------------|-----------------|
| error | Rate | Error | Rate | Error | Rate |
| 4 | 5.3013e−01 | – | 4.6509e−01 | – | 4.8187e+00 | – |
| 8 | 1.6490e−01 | 1.7078 | 1.4931e−01 | 1.6392 | 2.6626e+00 | 0.8558 |
| 16 | 4.3368e−02 | 1.9259 | 4.0108e−02 | 1.8963 | 1.3785e+00 | 0.9497 |
| 32 | 1.0969e−02 | 1.9282 | 1.0527e−02 | 1.9298 | 7.1098e−01 | 0.9516 |
| 64 | 2.7499e−03 | 1.9960 | 2.8273e−03 | 1.8966 | 3.7409e−01 | 0.9264 |

which are what the theoretical results in Lemma 3.2 shows. Especially, we can see from Table 1, \( \| u^n - u_h^n \|_{L^2} = O(h^2) \), which is in accordance of Lemma 3.3.

\[
\begin{array}{cccc}
\| \nabla (u - \tilde{u}_h) \|_{L^2} & \| p - p_h \|_{L^2} & \sqrt{\Delta t} \| \nabla (p - p_h) \|_{L^2} \\
\| u^n - \tilde{u}_h^n \|_0 & \| p - p_h \|_{L^2} & \sqrt{\Delta t} \| \nabla (p - p_h) \|_{L^2} \\
\end{array}
\]

\[
\begin{array}{cccc}
1/h & \max_n \| p^n - p_h^n \|_{L^2} & \| p - p_h \|_{L^2} & \sqrt{\Delta t} \| \nabla (p - p_h) \|_{L^2} \\
error & Rate & Error | Rate | Error | Rate |
\end{array}
\]

\[
\begin{array}{cccc}
4 & 2.7636e+00 & – & 2.2987e+00 & – & 1.4286e+00 & – \\
8 & 1.1144e+00 & 1.3103 & 8.8892e−01 & 1.3707 | 4.6814e−01 & 1.6096 |
\end{array}
\]

\[
\begin{array}{cccc}
16 & 3.6664e−01 & 1.6038 & 2.7275e−01 & 1.7045 | 1.3827e−01 & 1.7595 \\
32 & 1.2463e−01 & 1.5567 & 8.1260e−02 & 1.7470 | 4.5158e−01 & 1.6144 \\
64 & 4.6335e−02 & 1.4275 | 2.5152e−02 & 1.6919 | 1.5553e−01 & 1.5378 \\
\end{array}
\]

\[
\| \nabla (u - \tilde{u}_h) \|_{L^2} + \| p - p_h \|_{L^2} = O(h), \\
\| u^n - \tilde{u}_h^n \|_0 + h \| p^n - p_h^n \|_0 + h \sqrt{\Delta t} \| \nabla (p^n - p_h^n) \|_0 = O(h^2),
\]

which are what the theoretical results in Lemma 3.2 shows. Especially, we can see from Table 1, \( \| u^n - u_h^n \|_{L^2} = O(h^2) \), which is in accordance of Lemma 3.3.

\section{5.1.3 High Efficiency on Computation}

After having the snapshots spaces, we determine \( d_\tilde{u} = \text{rank}(\tilde{U}) = 20 \) (for which \( \gamma_i < 10^{-15} \), when \( i > d_\tilde{u} \)), \( d_p = \text{rank}(P) = 20 \) (for which \( \epsilon_i < 10^{-13} \), when \( i > d_p \)). Figure 1 shows the decay of POD eigenvalues(left) of velocity \( \gamma_i, i = 1, \cdots, d_\tilde{u} \) and pressure \( \epsilon_i, i = 1, \cdots, d_p \), together with the corresponding captured system’s energy(right) in the form of \( 100 \sum_{i=1}^{r} \gamma_i / \sum_{i=1}^{d_\tilde{u}} \gamma_i \) for velocity and \( 100 \sum_{i=1}^{r} \epsilon_i / \sum_{i=1}^{d_p} \epsilon_i \) for pressure. We note that the first \( r = 4 \) POD modes already capture nearly 99.99\% of the system’s velocity-pressure energy.

In order to confirm the computational efficiency advantage of the POD-ROM scheme over the FE-FOM scheme, we take the number of POD modes is \( r = 4 \), whereas the number of degree of freedoms (DOFs) for velocity and pressure in FE scheme are 8450 and 4225. We show in Table 3 the comparison of cumulative time spent by the two schemes to run to some certain time in the process of time-stepping iterative, and the \( L^2 \) numerical spatial discrete error of velocity and pressure in two schemes at that time. The recorded two types of time...
Fig. 1 POD velocity-pressure eigenvalues (left) and captured system’s velocity-pressure energy (right)

Table 3 Comparison of errors and time consumed by time-stepping iterative between FE-FOM scheme with $P_1-P_1$ pair and POD-ROM scheme with $r = 4$ on the mesh of $h = 1/64$ and $\Delta t = 0.1 h^2$

| n    | FE-FOM scheme | POD-ROM scheme | CPU run time(s) |
|------|---------------|----------------|-----------------|
|      | $\|u^n - \tilde{u}_h^n\|_{L^2}$ | $\|\tilde{u}_r^n - \tilde{u}_h^n\|_{L^2}$ | CPU run time(s) |
|      | $\|p^n - p_h^n\|_{L^2}$ | $\|p^n - p_r^n\|_{L^2}$ |                |
| 2500 | 2.3789e−03    | 2.9458e−02     | 689             |
| 5000 | 2.3929e−03    | 2.9253e−02     | 1338            |
| 7500 | 2.3740e−03    | 2.8975e−02     | 2005            |
| 10000| 2.3452e−03    | 2.8591e−02     | 2668            |
| 20000| 2.1443e−03    | 2.6010e−02     | 5339            |
| 30000| 1.8163e−03    | 2.1886e−02     | 8060            |
| 40000| 1.3805e−03    | 1.6464e−02     | 10746           |

reported in Table 3 for two schemes are coming from the same conditions. To be specific, they report the time consumption of the assembly of the RHS vector, matrix of reaction terms on the current time level (where the FE and POD mass matrix have been prepared), the enforcement of the boundary conditions, and finally, the solving of the linear system both by left division. We can see from Table 3 that, compared with the FE-FOM scheme, POD-ROM scheme can effectively improve computational efficiency in the context of only consuming approximately 1/20 times to get the numerical solution with even higher accuracy.

Figure 2 plots the temporal evolution of the discrete $L^2$ relative error (in semilogarithmic scale) of the reduced-order velocity and pressure with respect to the full order ones: $\|\tilde{u}_r^n - \tilde{u}_h^n\|_{L^2}/\|\tilde{u}_h^n\|_{L^2}$ and $\|p_r^n - p_h^n\|_{L^2}/\|p_h^n\|_{L^2}$ for different number of POD modes. As expected in Theorem 4.2, the errors would decrease as the number of POD modes $r$ increasing. For velocity, the left figure in Fig. 2 numerically demonstrate this result, and for pressure, the error would increase slightly when $r$ from 6 to 8, but when we continue to increase $r$, the error will decrease, which is consistent with the Theorem 4.2 as a whole.
5.2 Tests 2: Without Exact Solutions

To further verify the practical utility of Proj-POD-ROM, especially for the stability of POD pressure after a time interval much longer than the one used to taking snapshots, we give the forthcoming benchmark test. In this lid-driven problem, the computational domain is $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, no-slip boundary conditions apply on all four sides of this cavity. On the bottom and walls $(u_1, u_2) = (0, 0)$, while $(u_1, u_2) = (1, 0)$ on the lid. There is no body force, and the viscosity $\nu = 0.01$. We first obtain the required snapshots for the projection POD-ROM by running a Proj-FE-FOM scheme (3.3) 200 time steps (that is, $M = 200$ and then $N_s = 2M - 1 = 399$), where the $P_1 - P_1$ mixed FE, with $h = 0.01$, is chosen in space and the BDF1 FD in time with $\Delta t = 0.002$ (i.e., $\Delta t = 10h^2$). After obtaining the required snapshots data for the POD-ROM, we can construct the POD reduced base, so that we can use the Proj-POD-ROM scheme to calculate the discrete spatial data. In other words, starting from $n = 201$, we could run the Proj-POD-ROM scheme to obtain the discrete data in the flow field region efficiently and high-accuracy, running until $n = 5000$, 24 times than the time interval used to obtain FOM data to assemble the ROM. Figure 3 gives the POD velocity field streamline and pressure field at $n = 5000$, where $\tau = 21$.

To further confirm more precisely the accuracy of our Proj-POD-ROM scheme in this benchmark tests, we proceeded to run the Proj-FE-FOM scheme to $n = 5000$ and thus calculated the error between the FE-FOM solutions and the POD-ROM solutions in the $L^2$ norm. Figure 4 illustrates the temporal evolution of discrete error $\| \tilde{u}_n^h - \tilde{u}_n^r \|_{L^2}$ (left) and $\| p_n^h - p_n^r \|_{L^2}$ (right) at different $r$ values. We can see from Fig. 4 that, on the one hand, for the different $r$ values, the discrete $L^2$ errors decrease as the $r$ increasing, which is consistent with the previous theoretical analysis; on the other hand, for some fixed $r$ values, the errors gradually keep stable, even up to $n = 5000$, at $10^{-3}$ (for velocity) and $10^{-4}$ (for pressure) after $r$ is bigger than 15, we remark that these errors are comparable with those in [34], where the author also investigated the time extrapolation by POD with lid-driven benchmark tests.
Fig. 3  Velocity streamline and Pressure at $n = 5000$ with $r = 21$

Fig. 4  Temporal evolution of discrete error $\|\tilde{u}_h^n - \tilde{u}_r^n\|_0$ (left) and $\|p_h^n - p_r^n\|_0$ (right) at different $r$ values

6 Conclusions

In this paper, we proposed an efficient projection POD-ROM, which combined the advantages of classical projection method and POD technique.

The main contribution of the present paper consisted of two aspects: the first one was high computational efficiency. Through auxiliary intermediate velocity variable, the classical projection method decoupled the velocity variable and pressure variable, meanwhile decoupled the saddle-point system arose from Stokes equations, so one strength of projection method lied in its high computational efficiency, or low computational costs; Furthermore, POD technique was utilized to get the ROM, which have made the newly proposed projection
POD-ROM had high computational efficiency. The second contribution was based on the fact that, in the fully discrete scheme of the classical projection method, the original scheme could be rewritten into a PSPG-type pressure stabilization scheme by eliminating the end-of-step velocity, where the pressure stabilized term $\Delta t (\nabla p_h^{n+1}, \nabla q_h^{n+1})$, $\Delta t = O(h^2)$, was inherent, so that some flexible mixed FE spaces pairs (for example, $P_1\cdot P_1$ pair) could be used without considering the classical LBB/inf-sup condition for mixed POD-based reduced basis spaces, which was different from other stabilized FE-POD-ROM to overcome LBB/inf-sup condition by adding extra stabilization terms.

Numerical experiments have been conducted to confirm the convergence, the high-efficiency and the stability of reduced basis pressure for projection POD-ROM scheme. In the first test, we first numerically confirmed that the PSPG-type classical projection scheme owned the desired convergence orders which is consistent with theoretical results, and after taking FE solutions as the snapshots to formulate POD bases/modes to formulate reduced-order solutions, we then obtain the experimental convergence between exact solutions and reduced-order solutions as increasing the number of POD modes. Apart from the projection POD-ROM convergence, we also conduct the experiment to compare the error and computational time between projection FE-FOM and projection POD-ROM, and the results revealed the fact that projection POD-ROM not only had less discretization error, but also less computational costs, compared with projection FE FOM. In the second experiment, a benchmark test have been tested to validate the stability of reduced basis pressure in the form of image and $L^2$ error graph.

One future research direction will be the applied of projection POD on nonlinear non-stationary Navier–Stokes equations. Another investigation is validation of fulfillment of LBB/inf-sup condition for mixed reduced spaces.

Acknowledgements The authors thank the anonymous referees for their constructive comments and suggestions which improved the manuscript.

Data Availability Enquiries about data availability should be directed to the authors.

Declarations

Competing interests The authors declare no competing interests.

References

1. Ali, S., Ballarin, F., Rozza, G.: Stabilized reduced basis methods for parametrized steady Stokes and Navier–Stokes equations. Comput. Math. Appl. 80(11), 2399–2416 (2020)
2. Azaïez, M., Chacón Rebollo, T., Rubino, S.: A cure for instabilities due to advection-dominance in POD solution to advection–diffusion–reaction equations. J. Comput. Phys. 425, Paper No. 109916, 27 (2021)
3. Baiges, J., Codina, R., Idelsohn, S.: Explicit reduced-order models for the stabilized finite element approximation of the incompressible Navier-Stokes equations. Int. J. Numer. Methods Fluids 72(12), 1219–1243 (2013)
4. Ballarin, F., Manzoni, A., Quarteroni, A., Rozza, G.: Supremizer stabilization of POD-Galerkin approximation of parametrized steady incompressible Navier-Stokes equations. Int. J. Numer. Methods Eng. 102(5), 1136–1161 (2015)
5. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 3rd edn., vol. 15 of Texts in Applied Mathematics. Springer, New York (2008)
6. Brezzi, F., Pitkäranta, J.: On the stabilization of finite element approximations of the Stokes equations. In: Efficient Solutions of Elliptic Systems (Kiel, 1984), vol. 10 of Notes Numer. Fluid Mech. Friedr. Vieweg, Braunschweig, pp. 11–19 (1984)
7. Burman, E., Fernández, M.A.: Analysis of the PSPG method for the transient Stokes’ problem. Comput. Methods Appl. Mech. Eng. 200(41–44), 2882–2890 (2011)

8. Caiazzo, A., Iliescu, T., John, V., Schyschlowa, S.: A numerical investigation of velocity-pressure reduced order models for incompressible flows. J. Comput. Phys. 259, 598–616 (2014)

9. Carlberg, K., Farhat, C., Cortial, J., Amsallem, D.: The GNAT method for nonlinear model reduction: effective implementation and application to computational fluid dynamics and turbulent flows. J. Comput. Phys. 242, 623–647 (2013)

10. Chaturantabut, S., Sorensen, D.C.: Nonlinear model reduction via discrete empirical interpolation. SIAM J. Sci. Comput. 32(5), 2737–2764 (2010)

11. Chaturantabut, S., Sorensen, D.C.: A numerical investigation of velocity-pressure reduced order models for incompressible flows. J. Comput. Phys. 259, 598–616 (2014)

12. Chaturantabut, S., Sorensen, D.C.: A state space error estimate for POD-DEIM nonlinear model reduction. SIAM J. Numer. Anal. 50(1), 46–63 (2012)

13. Chen, L.: iFEM: an innovative finite element methods package in matlab. University of Maryland, Preprint (2008)

14. Chorin, A.J.: Numerical solution of the Navier–Stokes equations. Math. Comput. 22, 745–762 (1968)

15. Chorin, A.J.: On the convergence of discrete approximations to the Navier–Stokes equations. Math. Comput. 23, 341–353 (1969)

16. Ciarlet, P.G.: The finite element method for elliptic problems, vol. 40 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2002)

17. de Frutos, J., García-Archilla, B., John, V., Novo, J.: Grad-div stabilization for the evolutionary Oseen problem with inf-sup stable finite elements. J. Sci. Comput. 66(3), 991–1024 (2016)

18. de Frutos, J., García-Archilla, B., Novo, J.: Error analysis of projection methods for non inf-sup stable mixed finite elements: the Navier–Stokes equations. J. Sci. Comput. 74(1), 426–455 (2018)

19. DeCaria, V., Iliescu, T., Layton, W., McLaughlin, M., Schneier, M.: An artificial compression reduced order model. SIAM J. Numer. Anal. 58(1), 565–589 (2020)

20. Fu, G., Wang, Z.: POD-(H)DG method for incompressible flow simulations. J. Sci. Comput. 85, Paper No. 24, 20 (2020)

21. Girault, V., Raviart, P.-A.: Finite element methods for Navier–Stokes equations—Theory and algorithms. Springer Series in Computational Mathematics, vol. 5. Springer, Berlin (1986)

22. Guermond, J.L., Minev, P., Shen, J.: An overview of projection methods for incompressible flows. Comput. Methods Appl. Mech. Eng. 195(44–47), 6011–6045 (2006)

23. Guermond, J.-L., Quartapelle, L.: On the approximation of the unsteady Navier–Stokes equations by finite element projection methods. Numer. Math. 80(2), 207–238 (1998)

24. Haasdonk, B., Ohlberger, M.: Reduced basis method for finite volume approximations of parametrized linear evolution equations. M2AN Math. Model. Numer. Anal. 42(2), 277–302 (2008)

25. Heywood, J.G., Rannacher, R.: Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. 19(2), 275–311 (1982)

26. Heywood, J.G., Rannacher, R.: Finite-element approximation of the nonstationary Navier–Stokes problem. IV. Error analysis for second-order time discretization. SIAM J. Numer. Anal. 27(2), 353–384 (1990)

27. Iliescu, T., Wang, Z.: Are the snapshot difference quotients needed in the proper orthogonal decomposition? SIAM J. Sci. Comput. 36(3), A1221–A1250 (2014)

28. John, V., Novo, J.: Analysis of the pressure stabilized Petrov–Galerkin method for the evolutionary Stokes equations avoiding time step restrictions. SIAM J. Numer. Anal. 53(2), 1005–1031 (2015)

29. Kean, K., Schneier, M.: Error analysis of supremizer pressure recovery for POD based reduced-order models of the time-dependent Navier-Stokes equations. SIAM J. Numer. Anal. 58(4), 2235–2264 (2020)

30. Koc, B., Rubino, S., Schneier, M., Singler, J., Iliescu, T.: On optimal pointwise in time error bounds and difference quotients for the proper orthogonal decomposition. SIAM J. Numer. Anal. 59(4), 2163–2196 (2021)

31. Kunisch, K., Volkwein, S.: Galerkin proper orthogonal decomposition methods for parabolic problems. Numer. Math. 90(1), 117–148 (2001)

32. Kunisch, K., Volkwein, S.: Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. SIAM J. Numer. Anal. 40(2), 492–515 (2002)

33. Luo, Z., Chen, G.: Proper Orthogonal Decomposition Methods for Partial Differential Equations. Mathematics in Science and Engineering. Elsevier/Academic Press, London (2019)

34. Luo, Z., Chen, J., Navon, I.M., Yang, X.: Mixed finite element formulation and error estimates based on proper orthogonal decomposition for the nonstationary Navier-Stokes equations. SIAM J. Numer. Anal. 47(1), 1–19 (2008/09)
35. Luo, Z.-D., Ou, Q.-L., Xie, Z.-H.: Reduced finite difference scheme and error estimates based on POD method for non-stationary Stokes equation. Appl. Math. Mech. (English Ed.) 32(7), 847–858 (2011)
36. Noack, B.R., Papas, P., Monkewitz, P.A.: The need for a pressure-term representation in empirical Galerkin models of incompressible shear flows. J. Fluid Mech. 523, 339–365 (2005)
37. Novo, J., Rubino, S.: Error analysis of proper orthogonal decomposition stabilized methods for incompressible flows. SIAM J. Numer. Anal. 59(1), 334–369 (2021)
38. Rannacher, R.: On Chorin’s projection method for the incompressible Navier–Stokes equations. In: The Navier–Stokes equations II—theory and numerical methods (Oberwolfach, 1991): vol. 1530 of Lecture Notes in Math, pp. 167–183. Springer, Berlin (1992)
39. Rozza, G., Huynh, D.B.P., Manzoni, A.: Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants. Numer. Math. 125(1), 115–152 (2013)
40. Rubino, S.: Numerical analysis of a projection-based stabilized POD-ROM for incompressible flows. SIAM J. Numer. Anal. 58(4), 2019–2058 (2020)
41. Shen, J.: On error estimates of projection methods for Navier–Stokes equations: first-order schemes. SIAM J. Numer. Anal. 29(1), 57–77 (1992)
42. Shen, J.: On error estimates of some higher order projection and penalty-projection methods for Navier–Stokes equations. Numer. Math. 62(1), 49–73 (1992)
43. Singler, J.R.: New POD error expressions, error bounds, and asymptotic results for reduced order models of parabolic PDEs. SIAM J. Numer. Anal. 52(2), 852–876 (2014)
44. Stabile, G., Ballarin, F., Zuccarino, G., Rozza, G.: A reduced order variational multiscale approach for turbulent flows. Adv. Comput. Math. 45(5–6), 2349–2368 (2019)
45. Stabile, G., Rozza, G.: Finite volume POD-Galerkin stabilised reduced order methods for the parametrised incompressible Navier-Stokes equations. Comput. Fluids 173, 273–284 (2018)
46. Stabile, G., Zancanaro, M., Rozza, G.: Efficient geometrical parametrization for finite-volume-based reduced order methods. Int. J. Numer. Methods Eng. 121(12), 2655–2682 (2020)
47. Temam, R.: Une méthode d’approximation de la solution des équations de Navier-Stokes. Bull. Soc. Math. France 96, 115–152 (1968)
48. Veroy, K., Prud’homme, C., Rovas, D., Patera, A.: A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In: Proceedings of 16th AIAA Computational Fluid Dynamics Conference (2013)
49. Volkwein, S.: Model reduction using proper orthogonal decomposition. Lecture Notes, Faculty of Mathematics and Statistics, University of Konstanz (2011)
50. Willcox, K.: Unsteady flow sensing and estimation via the Gappy proper orthogonal decomposition. Comput. Fluids 35(2), 208–226 (2006)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.