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Symmetry in Shannon’s Noiseless Coding Theorem

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Abstract Statements of Shannon’s Noiseless Coding Theorem by various authors, including the original, are reviewed and clarified. Traditional statements of the theorem are often unclear as to when it applies. A new notation is introduced and the domain of application is clarified.

An examination of the bounds of the Theorem leads to a new symmetric restatement. It is shown that the extended upper bound is an acheivable upper bound, giving symmetry to the theorem.

The relation of information entropy to the physical entropy of Gibbs and Boltmann is illustrated. Consequently, the study of Shannon Entropy is strongly related to physics and there is a physical theory of information. This paper is the beginning of of an attempt to clarify these relationships.

Keywords Shannon coding · information theory · entropy · coding compression · information metrics · thermodynamics

1 Introduction

An important result of Shannon [1948] shows that it is possible to approach 100% coding efficiency, when no noise is present [1]. This result has various names: Shannon’s noiseless coding theorem, Shannon’s first theorem, the Data compression theorem, and oddly enough is often not even stated as a theorem but only as a formula. This paper refers to it as the Theorem.

Several definitions are needed to make our notation clear. A set of events $S$ is indexed by $i$, and the event $i$ is represented by the symbol $s_i$, which has probability $p_i$. There are two sets of symbols: the input alphabet $S$, of size $q$, and the code alphabet $A$, of size $r$. The length of the code for event $i$ is $l_i$. The expected code length of a code $C$ on input symbols $S$ is $L_C(S)$.

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\[ H_r(S) = \sum_S p_i \log_r \frac{1}{p_i} \]

\[ L_C(S) = \sum_{i=1}^q p_i l_i \]

We call the entropy to the base r the \( r \)-entropy. A code of radix r is an \( r \)-code. The entropy of a set \( S \) in bits is \( H(S) \).

\[ H(S) = \sum_S p_i \log \frac{1}{p_i} \]

1.1 Physical Entropy

Shannon Entropy \( H \) and Gibbs Entropy \( S \) are abstractly related. In a classical system with a discrete set of micro states \( S \) where \( E_i \) is the energy of a microstate with probability \( p_i \), then the physical Gibbs Entropy is:

\[ S(S) = k_B \sum_S p_i \ln \frac{1}{p_i} \text{ J/K} \]

Where \( k_B \) is the Boltzmann constant \( k = 1.3806504(24) \times 10^{-23} \text{ J/K} \).

This reduces to information entropy by algebraic transform.

\[ H = S(S)/(k_B \ln 2) = \sum_S p_i \log \frac{1}{p_i} \text{ bits} \]

Boltzmann Entropy

\[ S = k_B \ln \Omega \]

\( \Omega \) is the number of microstates consistent with the given macrostate. This is the equililke case of maximum entropy.

\[ H = S/(k_B \ln 2) = \log \Omega = \log n \]

The statistical entropy reduces to Boltzmann’s entropy when all the accessible microstates of the system are equally likely. It is the configuration corresponding to the maximum of a system’s entropy for a given set of accessible microstates. Here the lack of information is maximal. As such, Boltzmann’s entropy is the expression of entropy at thermodynamic equilibrium in the micro-canonical ensemble. Consequentially, the study of Shannon Entropy is strongly related to physics [2]. Naturally, the number of information states is much less than in physical entropy except when Quantum Information is studied but that is a future paper.
2 History

In this section, the notation of each of the original authors will be used, when possible, and some care is needed by the reader. To reduce confusion, we will normalize the terminology where necessary in the representations by various authors. Hopefully, the spirit of their representations will show even if not quite always literally faithful.

Essentially, the Theorem says that we can construct a code with an average number of code symbols per input event that is as close to the entropy of \( S \) as we like. First, we look at how the Theorem is stated by a number of authors. A typical statement of the Theorem is given by Hamming.

**The Theorem** (from Hamming\[2\] 1986) \( H_r(S) \leq L < H_r(S) + \frac{1}{n} \)

The code has \( r \) symbols and \( L \) is the average code word length per input symbol or our \( L \).

**The Theorem** (from Abramson\[3\] 1963)

\[
H_r(S) \leq \frac{L_n}{n} < H_r(S) + \frac{1}{n}
\]

Here the use of the \( n \)th extension coding is indicated.

**The Theorem** (from Gallager\[4\] 1968) ... it is possible to assign code words ...

\[
\frac{H(U)}{\log D} \leq \frac{H(U)}{\log D} + \frac{1}{L}
\]

Since the essence of the theorem is that \( L \) can approach the entropy, the Theorem can also be expressed as a limit.

**The Theorem** (from McEliece\[5\] 1977) \( \lim_{m \to \infty} \frac{n(p^m)}{m} = H_s(p) \)

Here the connection that the entropy is measured in the radix of the code is indicated.

**The Theorem** (from Cover\[6\] 1991) \( H(S) \leq L_n^* < H(S) + \frac{1}{n} \)

This assumes binary coding. \( L_n^* \) is the average code word length per input symbol of an optimum code. Here a restriction on the form of coding is indicated and the Theorem restricted to optimum codes.

**A. Shannon’s Original Statement**

If we return to Shannon’s original work, we see a more complex view and one that makes it more difficult to derive. Shannon first expressed the theorem in terms of channel capacity; however, he also gave the essence of the theorem in terms of entropy in an alternate proof that leads to all the statements in this paper. To begin with, Shannon used an estimator for the entropy of the source. He considered all sequences \( S_i \) of \( N \) symbols in the source, thus, determining \( H \) from the statistics of the message sequence.

**The Theorem** (after Shannon 1948)

\[
G_N = \frac{1}{N} \sum_i p(S_i) \log \frac{1}{p(S_i)} \\
\lim_{N \to \infty} G_N = H \\
G_N \leq B < G_N + \frac{1}{N}
\]
Where $G_N$ approaches the entropy as $N$ increases, and $B$ is the average number of binary symbols per symbol of the source.

Except for Gallager, it is usually not clear from the Theorem statement that this is only true for some codes. One must read the proofs to see that Shannon codes always satisfy the relation. While it is true for Shannon codes or better, it is also true of other codes, a fact generally not apparent from the discussion. The Theorem is important enough to have a nice complete compact statement.

3 Analysis

We now examine various ways of restating the Theorem. When no code radix $r$ is specified, entropy will be measured in bits, and all logs are to the base 2, written as $\lg$. It is important to note that the entropy of the Theorem is measured in the radix of the code. Of course, $H_r = H/\lg r$.

A. The Code Bounding Lemma

The lower bound on the average code length of a code is the $r$-entropy of the source set. This interprets entropy as the best possible expected code length for an $r$-code.

$$H_r(S) \leq \mathcal{L}$$

Discrete codes cannot always achieve this bound. For a uniform distribution, the average length of codes with $q = r^k$, where $k$ is an integer greater than zero, is equal to the entropy.

The surprising fact is that we can always find a code better than the entropy plus one. The usual upper bound proof for this assumes Shannon coding $S$ or better. For simplicity, this is often expressed as the bound for Shannon coding.

$$\mathcal{L}_S < H_r(S) + 1$$

When the radix of the code is binary, it is usual to drop the subscript $r$. Assuming binary simplifies notation in proofs, but obscures the fact that the entropy of these relations must be measured in the radix of the code used.

B. Correcting the Upper Bound

Consider the case $H = 0$

$$0 \leq \mathcal{L}_S < 0 + 1$$

An entropy of zero is given by any distribution where exactly one event is certain. What is the value of $\mathcal{L}$ when $H = 0$? Both Shannon and Huffman codes will assign a code of length 0 to the certain event. The average length of the resulting code will be 0 in this special case. Thus we have: $0 \leq 0 < 0 + 1$

However, the Theorem is not restricted to these codes. Define an extended Shannon code as one that assigns 1 to the certain event. The fact that this is less efficient is not important. Thus we have: $0 \leq 1 < 0 + 1$

This is impossible, so the upper bound must be achievable in this special case.
Allowing a probability of zero results in infinite length code words. A simple further modification of the definition of a Shannon code gives finite codes. If \( p_i = 0 \), then \( l_i = 0 \). Our extended Shannon code is denoted by \( s \).

**C. A New Upper Bound**

\[ T_s \leq H_r(S) + 1 \]

There exist bounded codes that achieve the upper bound. Why was this result not observed before this? Certainly, a great many people have seen this relation. The standard proof, for binary codes, begins with:

\[ \log \left( \frac{1}{p_i} \right) \leq l_i < \log \left( \frac{1}{p_i} \right) + 1 \]

because for a Shannon code \( l_i = \lceil \log(1/p_i) \rceil \).

At this point to allow \( p_i = 0 \) is obviously silly and so the boundary case of a certain event is never considered. Proofs are a demonstration of understanding, not a method of discovery.

Our result on the new upper bound for \( T_C \leq T_s \) is simple but apparently not obvious [1],[2],[3],[4],[5],[6]. It resulted from asking why the Theorem did not exhibit symmetry.

Should this special case be included? The argument that can be used in the certain event boundary case is that when an event is certain, no code word need be used in any coding method, thus the average length is zero, and the original statement is correct. However in general for \( n \) codes, we do not know their exact probabilities and these may or may not be zero. In fact we would have to exclude the assignment of a code to the certain event. While using a Shannon code for the proof of the Theorem does in fact do this, codes less efficient than Shannon still satisfy the original theorem. We can conclude that the restatement of the upper bound is correct.

A final objection is that the chance of one event having a probability of one is unlikely and trivial. This may be so, but equilikely events are even less probable as the number of events increases.

**D. General Code Bounding Lemma**

There is no upper bound in general for \( T_C \), but we can define a working upper bound. Any \( S \) can be encoded by a block code \( B \) and \( T_B = \lceil \log_r q \rceil \). The only reason to use a variable length code is to be more efficient than a block code. So it is reasonable to never use codes with average length worse than a block code. Define a good code, \( gC \), as any code where \( T_{gC} \leq T_B \). This gives a general form of the Lemma.

**Lemma: Good Code Bounding**

\[ H_r \leq T_{gC} \leq H_r + k \]

Where \( k = \log_r q \), or \( k=1 \) for all \( T_C \leq T_s \).

Actually, we have \( T_{gC} \leq \lceil \log_r q \rceil \); thus, the lemma bound is not always good.

**E. The \( n \) th Extension of a Code**

The Theorem is a consequence of recoding a source in blocks of source symbols in order to better match the code lengths to code probabilities. The \( n \)th extension of a source is formed by concatenating all sequences of the original source symbols to form a new (compound) symbol or event. This
new event now has a probability that is the product of the probabilities of 
the original source symbols.

The new alphabet is $S^n$, and the average length of the extended code 
in blocks is $\overline{L}(S^n)$. The average length of the extended code in the input 
symbols $S$ is $\overline{L}(S)$.

**Lemma: Extension Average Length** $\overline{L}(S^n) = n\overline{L}(S)$

When it is necessary to remember that the $n$th extension was encoded for $\overline{L}(S)$, we use the power $n$, as in $\overline{L}^n(S)$.

Our new definition, thus, extends nicely and is related to the entropy 
relation for code extensions.

**Lemma: Extension Entropy**

$$H(S^n) = nH(S)$$

4 Results

We now summarize the results of our discussion and obtain a new statement 
of the Theorem.

**Theorem 1: Code Bounding** For extended Shannon codes $H_r(S) \leq \overline{L}_n \leq H_r(S) + 1$

Further symmetries can be observed.
If $H = \overline{L}$, then $H = \overline{L} < H + 1$.
If $H = 0$, then $H < \overline{L} = H + 1$.

The maximum excess of 1 is not always attained. For $q = r^k$ equally 
likely events, the excess is zero but jumps to $(q - 1)/q$ for *almost* equally 
likely events, where all but one are just over probability $1/q$. Shannon code 
excess can be quite sensitive to small changes in probability.

The new result follows from the known technique of coding the $n$th extension 
and the preceding relations. We have renamed it the Code Compression 
Theorem of Shannon, to reflect what it says.

**Theorem 2: Code Compression** For extended Shannon codes or better

$H_r(S) \leq \overline{L}_n(S) \leq H_r(S) + \frac{1}{n}$

*Proof:* Encode the $n$th extension of a source by an extended Shannon 
code and apply the code bounding theorem.

$$H_r(S^n) \leq \overline{L}_n(S^n) \leq H_r(S^n) + 1$$

Substituting for the extension entropy and extension average length gives :

$$nH_r(S) \leq n\overline{L}_n(S) \leq nH_r(S) + 1$$

Dividing by $n$ gives:

$$H_r(S) \leq \overline{L}_n(S) \leq H_r(S) + \frac{1}{n}$$

*Proof Discussion:* We might want to check if the upper bound is at-
tained when the entropy is zero. Only the sequence of $n$ certain events in
$S^n$ has probability one. All other sequences have probability zero. Since $L_n^n(S^n) = 1$, we have $L_n^n(S^n) = 1/n$.

$$0 \leq \frac{1}{n} \leq 0 + \frac{1}{n}$$

For an extended Shannon or better encoding of the nth extension, the excess of the average over the entropy approaches zero as n increases. Another way to look at this is to realize that as we encode larger and larger event sets the excess bound of 1 decreases as a percent of $H_r$. This can be seen directly from the code bounding theorem.

**Theorem : Code Compression Theorem of Shannon** There exist codes $C$ such that $H_r(S) \leq L_C(S) \leq H_r(S) + \frac{1}{n}$

Nothing is free in life. So the down side of code compression is that a sufficiently long message must be sent in order to apply an nth extension code. This means a delay in receiving messages. In computer terms, Shannon’s Noiseless Coding Theorem is a batch processing system. For the short messages of an interactive processing system, the Theorem is is not applicable.

5 Conclusions

How we think about concepts is influenced by the notation used. The importance of notation is not a new idea [8] but does bear repeating. Indeed, more confusion is usually engendered by the notation used than by the actual ideas expressed.

Boundary points often exhibit unusual behaviour, and this paper gives an interesting example of how such behaviour is not at all obvious until after it has been observed.

Information theoretic inequalities are important statements of fundamental patterns [7][8]. The Code Bounding and Code Compression theorems are quite elegant statements of the interrelationship of coding efficiency and entropy. This new found symmetry only adds to their elegance.

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