EXISTENCE OF AN EXTREMAL OF SOBOLEV INEQUALITY ASSOCIATED WITH DUNKL GRADIENT AND OF STEIN-WEISS INEQUALITY FOR D-RIESZ POTENTIAL

SASWATA ADHIKARI †, V. P. ANOOP, AND SANJAY PARUI

Abstract. In this paper, we prove the existence of an extremal for the Dunkl-type Sobolev inequality in case of $p = 2$. Also we prove the existence of an extremal of the Stein-Weiss inequality for the D-Riesz potential in case of $r = 2$.

1. Introduction

The classical Sobolev inequality states that for all $f \in C^\infty_c(\mathbb{R}^d)$,
\[ \|f\|_{L^q(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}, \tag{1.1} \]
where $1 \leq p < d$, $q = \frac{dp}{d-p}$ and the constant $C > 0$ only depends on $d$. This inequality plays an important role in analysis and as such it has been studied by many for e.g. see ([10, 15, 11]). The problem of finding sharp constant to inequality (1.1) was answered in [17].

One can consider inequality (1.1) in the context of Dunkl setting by replacing the Euclidean gradient $\nabla f$ by Dunkl gradient $\nabla_k f$. Sobolev inequality (1.1) associated with Dunkl gradient was derived for $1 < p < d_k$ in [1, 8].

In the Euclidean space $\mathbb{R}^d$, the negative powers of the Laplacian can be defined as an integral representation in terms of the Riesz potential or fractional integral operator as follows:
\[ (-\Delta)^{-\frac{\alpha}{2}}f(x) = I_\alpha f(x) = (c_\alpha)^{-1} \int_{\mathbb{R}^d} f(y)|x-y|^{-d+\alpha}dy, \]
where $0 < \alpha < d$ and $c_\alpha = 2^\alpha \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-\alpha}{2})}$. One fundamental result for the Riesz potential operator is the Stein-Weiss inequality which gives the weighted $(L^r, L^s)$ boundedness:

**Theorem 1.1.** [16] Let $d \in \mathbb{N}, 1 < r \leq s < \infty, \gamma > \frac{d}{s}, \beta \geq \gamma, 0 < \alpha < d, \beta < \frac{d}{r}, \alpha + \gamma - \beta = d(\frac{1}{r} - \frac{1}{s})$. Then
\[ \| |x|^\gamma I_\alpha f(x)\|_{L^s(\mathbb{R}^d)} \leq C\| |x|^\beta f(x)\|_{L^r(\mathbb{R}^d)}, \forall f \in L^r(\mathbb{R}^d, |x|^{\beta r}). \tag{1.2} \]
S. Thangavelu and Y. Xu in [19] defined the D-Riesz potential operator on Schwartz spaces as follows:

\[ I_k^\alpha f(x) = (c_k^\alpha)^{-1} \int_{\mathbb{R}^d} \tau_y^k f(x) |y|^{\alpha-d_k} w_k(y) dy, \]

where \( 0 < \alpha < d_k \) and \( c_k^\alpha = 2^{\alpha-d_k} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d_k-\alpha}{2})} \). In [7], D. V. Gorbachev et al proved the following Stein-Weiss inequality for the D-Riesz potential.

**Theorem 1.2.** Let \( d \in \mathbb{N}, 1 < r \leq s < \infty, \gamma > -\frac{d_k}{s}, \beta \geq \gamma, 0 < \alpha < d_k, \beta < \frac{d_k}{s}, \alpha + \gamma - \beta = d_k(\frac{1}{r} - \frac{1}{s}) \). Then

\[ \| |x|^{\gamma} I_k^\alpha f(x) \|_{L^s(\mathbb{R}^d, w_k)} \leq C_k \| |x|^{\beta} f(x) \|_{L^r(\mathbb{R}^d, w_k)} \quad \forall f \in L^r(\mathbb{R}^d, |x|^{\beta r} w_k). \quad (1.3) \]

In this paper, first we consider the Sobolev inequality (1.1) for the case \( p = 2 \) associated with Dunkl gradient and are interested to find its extremals. Towards this, for \( u \in \dot{H}^1(\mathbb{R}^d, w_k) \), we consider the function

\[ F(u) = \frac{\int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x) dx}{\left( \int_{\mathbb{R}^d} |u|^q w_k(x) dx \right)^{\frac{1}{q}}}, \quad (1.4) \]

where \( q = \frac{2d_k}{d_k - 2} \) and \( \dot{H}^1(\mathbb{R}^d, w_k) = W^{1,2}(\mathbb{R}^d, w_k) \). The aim of this paper is to show that infimum is attained for the function \( F \) when the infimum is taken over all non-vanishing functions \( u \in \dot{H}^1(\mathbb{R}^d, w_k) \). Recently, similar problem has been considered in [20] by A. Velicu, wherein he shows that the function \( F \) defined in (1.3) attains an infimum and have found the best constant on a Weyl chamber. Velicu uses Nash’s inequality to prove the existence of a minimizer whereas we use Dunkl-type refined Sobolev inequality to prove the existence of a minimizer. Towards this we have made explicit use of Plancherel formula of the Dunkl transform so our proof is different. This type of refined Sobolev inequality on \( \mathbb{R}^d \) is proved in more general setting in [9]. Our approach to this problem is mainly based on [5].

Finally, we consider the problem of finding the existence of an extremals for the inequality (1.3) in case of \( r = 2 \). By definition the best constant \( W_k \) in (1.3) is given by

\[ W_k = \sup \frac{\| |x|^{\gamma} I_k^\alpha f \|_{L^r(\mathbb{R}^d, w_k)}}{\| |x|^{\beta} f \|_{L^2(\mathbb{R}^d, w_k)}}, \quad (1.5) \]

where the supremum is taken over all non-vanishing functions \( f \in L^2(\mathbb{R}^d, w_k) \). We first have obtained weighted boundedness for the Dunkl-type heat semi group operator and an improved version of Stein-weiss inequality (1.3) in the Dunkl setting. Then we have proved a compact embedding

\[ \dot{H}^\alpha_{\beta,k}(\mathbb{R}^d) \subset L^s(K, |x|^s), \]

where

\[ \dot{H}^\alpha_{\beta,k}(\mathbb{R}^d) = \{ u = I_k^\alpha f : f \in L^r(\mathbb{R}^d, |x|^{\beta r} w_k) \} \quad (1.6) \]
is the homogenous Sobolev space in the Dunkl setting, which is a Banach space with the norm \( \|u\|_{H^{\alpha,r}_x(R^d)} = \|x|^{\alpha}f\|_{L^r(R^d,w_k)} \). Using the compact embedding, we prove that \( W_k \) defined in (1.5) has a maximizer. Our approach to this problem is based on [12].

We organize the paper as follows. In section 2, we provide a brief introduction to Dunkl theory and some known results. In section 3, we prove a Dunkl-type refined Sobolev inequality. In section 4, we prove the existence of an extremals for the Sobolev inequality associated with Dunkl gradient in case of \( p = 2 \). In section 5, we prove a weighted estimate for the operator \( e^{t\Delta_k} \). In section 6, we prove an improved version of Stein-Weiss inequality for D-Riesz potential. In section 7, we prove the existence of an extremals of Stein-Weiss inequality for the D-Riesz potential in case of \( r = 2 \).

2. Preliminaries

In this section, we shall briefly introduce the theory of Dunkl operators. For more details on Dunkl theory, we refer to [3, 4, 18].

For \( \nu \in \mathbb{R}^d \setminus \{0\} \) let \( \sigma_\nu \) denote the reflection of \( \mathbb{R}^d \) in the hyperplane \( \langle \nu \rangle^\perp \) given by the following formula:

\[
\sigma_\nu(x) = x - 2 \frac{\langle \nu, x \rangle}{|\nu|^2} \nu.
\]

A finite subset \( R \) of \( \mathbb{R}^d \setminus \{0\} \) is said to be a root system if \( R \cap \mathbb{R}_\nu = \{ \nu, -\nu \} \) and \( \sigma_\nu(R) = R \), \( \forall \nu \in R \). The set of reflections \( \{ \sigma_\nu : \nu \in R \} \) generates the subgroup \( G := G(R) \) of the orthogonal group \( O(d, \mathbb{R}) \), which is known as the reflection group associated with \( R \). From now onwards let \( R \) be a fixed root system in \( \mathbb{R}^d \) and \( G \) be the associated reflection group. For simplicity, we assume \( R \) to be normalized in the sense that \( \langle \nu, \nu \rangle = 2 \), \( \forall \nu \in R \).

A function \( k : R \to \mathbb{C} \) is called a multiplicity function on the root system \( R \) if it is invariant under the natural action of \( G \) on \( R \), that is, if \( k(\sigma_\nu g) = k(g) \), \( \nu, g \in R \). The set of all multiplicity functions forms a \( \mathbb{C} \)-vector space and it is denoted by \( K \).

**Definition 2.1.** Associated with \( G \) and \( k \), the Dunkl operator \( T_\xi := T(\xi)(k) \) is defined by (for \( f \in C^1(\mathbb{R}^d) \))

\[
T_\xi f(x) = \partial_\xi f(x) + \sum_{\nu \in R_+} k(\nu) \langle \nu, \xi \rangle \frac{f(x) - f(\sigma_\nu(x))}{\langle \nu, x \rangle}, \quad \xi \in \mathbb{R}^d,
\]

where \( \partial_\xi \) denotes the directional derivative in the direction of \( \xi \) and \( R_+ \) is a fixed positive subsystem of \( R \).

For \( \xi = e_i \), we shall write \( T_i \) for \( T_{e_i} \). We denote Dunkl gradient by \( \nabla_k = (T_1, T_2, \ldots, T_d) \) and Dunkl Laplacian by \( \Delta_k = \sum_{i=1}^d T_i^2 \). Throughout the paper we assume that \( k \geq 0 \). Let \( w_k \) denote the weight function defined by

\[
w_k(x) = \prod_{\nu \in R_+} |\langle \nu, x \rangle|^{2k(\nu)}, \quad x \in \mathbb{R}^d,
\]
which is a \( G \)-invariant homogeneous function of degree \( 2\gamma_k \) with \( \gamma_k = \sum_{\nu \in \mathbb{R}_+} k(\nu) \).

Let \( d_k = d + 2\gamma_k \). Further, we define the constants \( c_k = \int e^{-\frac{|x|^2}{2}} w_k(x)dx \) and \( a_k = \int w_k(x')dx' \). Then \( c_k \) and \( a_k \) are related by the following formula

\[
c_k = 2^{d_k/2} \Gamma \left( \frac{d_k}{2} \right) a_k. \tag{2.2}
\]

There exists a unique linear isomorphism \( V_k \) on polynomials, which intertwines the associated commutative algebra of Dunkl operators and the algebra of usual partial differential operators. Using the function \( V_k \), one can define the Dunkl kernel \( E_k \) as follows:

\[
E_k(x, y) := V_k(e^{i\langle -y, \cdot \rangle})(x), \quad x \in \mathbb{R}^d, y \in \mathbb{C}^d.
\]

For \( k = 0 \), the Dunkl kernel \( E_k \) reduces to the usual exponential function \( e^{ix \cdot y} \).

Alternatively, it is the solution of a joint eigen value problem for the associated Lie algebra elements. We collect few properties of the Dunkl kernel \( E_k \).

**Proposition 2.1.** Let \( k \geq 0, x, y \in \mathbb{C}^d, \lambda \in \mathbb{C}, \alpha \in \mathbb{Z}_+^d \).

\begin{enumerate}[(i)]  
  \item \( E_k(x, y) = E_k(y, x) \)
  \item \( E_k(\lambda x, y) = E_k(x, \lambda y) \)
  \item \( E_k(x, y) = E_k(x, y) \)
  \item \( |\partial^\alpha_y E_k(x, y)| \leq |x|^{\alpha} \max_{g \in G} e^{Re \langle g x, y \rangle} \).
\end{enumerate}

In particular, \( E_k(-ix, y) \leq 1 \) and \( |E_k(x, y)| \leq e^{|x||y|} \), \( \forall \ x, y \in \mathbb{R}^d \).

Using the Dunkl kernel one can define Dunkl transform, which is the generalization of classical Fourier transform. Dunkl transform enjoys similar properties to that classical Fourier transform.

**Definition 2.2.** For a function \( f \in L^1(\mathbb{R}^d, w_k) \), the Dunkl transform associated with \( G \) and \( k \geq 0 \), denoted by \( \mathcal{F}_k f \), is defined as

\[
\mathcal{F}_k(f)(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx, \quad \xi \in \mathbb{R}^d.
\]

When \( k = 0 \), the Dunkl transform reduces to the classical Fourier transform. The Dunkl transform can be extended to an isometric isomorphism between \( L^2(\mathbb{R}^d, w_k) \) and \( L^2(\mathbb{R}^d, w_k) \) i.e., for \( f \in L^2(\mathbb{R}^d, w_k) \), one has

\[
\|f\|_{L^2(\mathbb{R}^d, w_k)}^2 = \|\mathcal{F}_k(f)\|_{L^2(\mathbb{R}^d, w_k)}^2. \tag{2.3}
\]

The usual translation operator \( f \mapsto f(\cdot - y) \) leaves the Lebesgue measure on \( \mathbb{R}^d \) invariant. However the measure \( w_k(x)dx \) is no longer invariant under the usual translation and the Leibniz’s formula \( T_i(fg) = fT_i g + gT_i f \) does not hold in general. So one can introduce the notion of a generalized translation operator defined on the Dunkl transform by the formula

\[
\mathcal{F}_k(\tau^k f)(\xi) = E_k(iy, \xi) \mathcal{F}_k(f)(\xi). \tag{2.4}
\]
Existence of an extremal of Sobolev inequality associated with Dunkl gradient and of Stein-Weiss inequality for D-Riesz potential

In case when $k = 0$, $\tau_y f$ reduces to the usual translation $\tau_y^0 f(x) = f(x + y)$. In general, the explicit expression for $\tau_y f$ is unknown. It is known only when either $f$ is a radial function or $G = \mathbb{Z}_2^d$.

The convolution of two functions $f, g \in L^2(\mathbb{R}^d, w_k)$ is defined as follows:

$$(f *_k g)(x) = \int_{\mathbb{R}^d} f(y) \tau_y^k g(x) w_k(y) dy.$$ 

The convolution operator satisfies the following basic properties:

(i) $\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f) \cdot \mathcal{F}_k(g)$

(ii) $f *_k g = g *_k f$.

Using the convolution operator, the heat semi-group operator $e^{t\Delta_k}$ is defined as follows:

$$e^{t\Delta_k} u = u *_k q_t^k,$$

where $q_t^k(x) = (2t)^{-(\gamma_k + \frac{d}{2})} e^{-\frac{|x|^2}{4t}}, x \in \mathbb{R}^d$. (2.5)

In [13], it has been shown that the function $q_t^k(x)$ satisfies the Dunkl-type heat equation $\Delta_k u - \partial_t u = 0$ on $\mathbb{R}^d \times (0, \infty)$. A short calculation using the properties of Dunkl transform shows that

$$\mathcal{F}_k(q_t^k)(\xi) = e^{-t|\xi|^2},$$

and

$$\tau_y^k q_t^k(x) = (2t)^{-(\gamma_k + \frac{d}{2})} e^{-\frac{|x|^2}{4t} + \frac{|y|^2}{4t}} E_k \left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right).$$ (2.7)

From (2.7), it is observed that $\tau_y^k q_t^k(x) = \tau_y^k q_t^k(y)$.

We recall few results which will be useful in this paper.

**Theorem 2.1.** [6] Let $1 \leq p \leq \infty$ and $g$ is a Schwartz class radial function. Then for any $y \in \mathbb{R}^d$,

$$\|\tau_y^k g\|_{L^p(\mathbb{R}^d, w_k)} \leq \|g\|_{L^p(\mathbb{R}^d, w_k)}. \quad (2.8)$$

**Lemma 2.1.** (Brezis Lieb Lemma) Let $(X, dx)$ be a measure space and $(f_j)$ be a bounded sequence in $L^p(X)$, $0 < p < \infty$, which converges pointwise a.e. to a function $f$. Then

$$\lim_{j \to \infty} \int_X |f_j|^p - |f_j - f|^p - |f|^p dx = 0.$$

**Theorem 2.2.** [2] In Theorem 4.1, J. P. Anker et al. obtained the following estimate.

For any non-negative integer $m$ and for any multi-indices $\alpha, \beta$, there exists constant $C_{m, \alpha, \beta} > 0$ such that for any $t > 0$ and for any $x, y \in \mathbb{R}^d$, the following estimate holds:

$$|\partial^m \partial_x^\alpha \partial_y^\beta h_t(x, y)| \leq C_{m, \alpha, \beta} t^{-m - |\alpha| - |\beta|} h_{2t}(x, y), \quad (2.9)$$

where $h_t(x, y) = \tau_y^k q_t^k(x)$. 

We observe the following properties of the weight function \( w_k \).

**Properties of \( w_k \):**

1. For \( c > 0 \),
   \[
   \int_{\mathbb{R}^d} e^{-c|x|^2} w_k(x) \, dx = c^{-\frac{d_k}{2}} \int_{\mathbb{R}^d} e^{-|y|^2} w_k(y) \, dy.
   \]

   **Proof.** By substituting \( \sqrt{c}x = y \) and using (2.3) we get
   \[
   \int_{\mathbb{R}^d} e^{-c|x|^2} w_k(x) \, dx = c^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-|y|^2} \prod_{\nu \in \mathbb{R}^+} \left| \langle \nu, \frac{y}{\sqrt{c}} \rangle \right|^{2k} \, dy
   = c^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-|y|^2} \prod_{\nu \in \mathbb{R}^+} e^{-k(\nu)} \langle \nu, y \rangle^{2k} \, dy
   = c^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-|y|^2} w_k(y) \, dy
   = c^{-\frac{d_k}{2}} \int_{\mathbb{R}^d} e^{-|y|^2} w_k(y) \, dy.
   \]

2. For \( c \in \mathbb{R} \),
   \[ w_k(cx) = |c|^{2\gamma_k} w_k(x). \]

   **Proof.** The proof follows from the definition of the weight function \( w_k \) defined in (2.4).

3. \( \int_{\mathbb{R}^d} e^{-|y|^2} w_k(y) \, dy = \frac{a_k}{2} \Gamma \left( \frac{d_k}{2} \right). \)

   **Proof.** By substituting \( x = \sqrt{2}y \) in the integral involved in \( c_k \) and then using (2.2) as well as property (2), we obtain property (3).

4. If \( R > 0 \) and \( c < d_k \), then
   \[ \int_{|y| \leq R} |x|^{-c} w_k(x) \, dx = \frac{a_k R^{d_k - c}}{d_k - c}. \]

   **Proof.** Using property (2),
   \[
   \int_{|y| \leq R} |x|^{-c} w_k(x) \, dx = \int_0^R \int_{S^{d-1}} r^{-c} w_k(r x') r^{n-1} \, dx' \, dr
   = \int_0^R \int_{S^{d-1}} r^{-c} r^{2\gamma_k} w_k(x') r^{n-1} \, dx' \, dr
   = \int_0^R r^{d_k - c - 1} dr \int_{S^{d-1}} w_k(x') \, dx' = \frac{a_k R^{d_k - c}}{d_k - c},
   \]
   since the integrability condition at 0 is \( c < d_k \), thus proving property (4).
3. DUNKL-TYPE REFINED SOBOLEV INEQUALITY

The goal of this section is to prove Dunkl-type refined Sobolev inequality (3.1). In order to prove this we first prove the following Pseudo-Poincare inequality in the Dunkl setting for $p = 2$.

**Lemma 3.1.** For $u \in L^2(\mathbb{R}^d, w_k)$, one has

$$\|u - e^{t\Delta_k}u\|_{L^2(\mathbb{R}^d, w_k)}^2 \leq t\|\nabla_k u\|_{L^2(\mathbb{R}^d, w_k)}^2.$$

**Proof.** In order to prove the above Lemma, we shall make use of the following inequality.

$$(1 - e^{-x})^2 \leq 1 - e^{-x} \leq x, \quad \forall \ x \geq 0.$$ 

Now, using the Plancherel formula (2.3) and (2.6), we get

$$\|u - e^{t\Delta_k}u\|_{L^2(\mathbb{R}^d, w_k)}^2 = \|\mathcal{F}_k(u - e^{t\Delta_k}u)\|_{L^2(\mathbb{R}^d, w_k)}^2 = \|\mathcal{F}_k(u) - \mathcal{F}_k(u * q^k_t)\|_{L^2(\mathbb{R}^d, w_k)}^2$$

$$= \int_{\mathbb{R}^d} |\mathcal{F}_k(u)(\xi) - \mathcal{F}_k(u)(\xi)\mathcal{F}_k(q^k_t)(\xi)|^2 w_k(\xi) d\xi$$

$$= \int_{\mathbb{R}^d} |\mathcal{F}_k(u)(\xi)|^2 (1 - e^{-t|\xi|^2})^2 w_k(\xi) d\xi$$

$$\leq t \int_{\mathbb{R}^d} |\mathcal{F}_k(u)(\xi)|^2 |\xi|^2 w_k(\xi) d\xi$$

$$= t\|\mathcal{F}_k(\nabla_k u)\|_{L^2(\mathbb{R}^d, w_k)}^2 = t\|\nabla_k u\|_{L^2(\mathbb{R}^d, w_k)}^2.$$

$$\Box$$

**Theorem 3.1.** For $d \geq 3$, there is a constant $C_{d,k} > 0$ such that for all $u \in \dot{H}^1(\mathbb{R}^d, w_k)$, one has

$$\left( \int_{\mathbb{R}^d} |u|^q(x)w_k(x)dx \right)^{\frac{1}{q}} \leq C_{d,k} \left( \int_{\mathbb{R}^d} |\nabla_k u|^2(x)w_k(x)dx \right)^{\frac{1}{2}} \left( \sup_{t > 0} t^{(\frac{d-k-2}{4})} \|e^{t\Delta_k}u\|_{\infty} \right)^{\frac{2}{q}}, \quad (3.1)$$

with $q = \frac{2d_k}{d_k-2}$.

**Proof.** Consider the function

$$e^{t\Delta_k}u(x) = u * q^k_t(x) = \int_{\mathbb{R}^d} u(y)(\tau_y^k q^k_t)(x)w_k(y)dy.$$
Applying Holder’s inequality to the function $e^{t\Delta_k}u$ with $p = \frac{2d_k}{d_k+2}$ and $q = \frac{2d_k}{d_k-2}$, we get

$$|\langle e^{t\Delta_k}u \rangle (x) | \leq \| u \|_{q,w_k} \| \tau_{x,t}^k \|_{p,w_k}. \tag{3.2}$$

Now, using (2.8),

$$\| \tau_{x,t}^k \|_{p,w_k} = \int_{\mathbb{R}^d} |(\tau_{x,t}^k)(y)|^p w_k(y) dy \leq \int_{\mathbb{R}^d} |q_t^k(y)|^p w_k(y) dy. \tag{3.3}$$

By substituting the value of $q_t^k$ from (2.5) in the last integral and using property (1) as well as property (3), we obtain

$$\| \tau_{x,t}^k \|_{p,w_k} \leq \int_{\mathbb{R}^d} |(2t)^{-(\gamma_k+p)} e^{-\frac{|y|^2}{4t}} |^p w_k(y) dy = (2t)^{-(\gamma_k+p)} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t}} w_k(y) dy \leq A_{d,k} t^{-\frac{d_k}{2} (p-1)}, \tag{3.4}$$

where $A_{d,k} = 2^{\frac{d_k}{2} p(p-1)} \Gamma(\frac{d_k}{2})$ with $p = \frac{2d_k}{d_k+2}$. This implies that

$$\| \tau_{x,t}^k \|_{p,w_k} \leq A_{d,k}^{\frac{1}{p}} t^{-\frac{d_k}{2} (1 - \frac{1}{p})} = A_{d,k}^{\frac{1}{p}} t^{-\frac{d_k}{2} (1 - \frac{1}{d_k})} = A_{d,k}^{\frac{1}{p}} t^{-\frac{d_k-2}{4}}.$$

Then from (3.2), we get

$$\| e^{t\Delta_k}u \|_\infty \leq A_{d,k}^{\frac{1}{p}} \| u \|_{q,w_k} t^{-\frac{d_k-2}{4}} = C_{d,k} \| u \|_{q,w_k} t^{-\frac{d_k-2}{4}}.$$

Let $I[u] = \sup_{t > 0} \frac{t^{(d_k-2)}}{4} \| e^{t\Delta_k}u \|_\infty$. Then $I[u] \leq C_{d,k} \| u \|_{q,w_k}$. Thus by homogeneity, we can assume that $I[u] \leq 1$, that is,

$$t^{\frac{d_k-2}{4}} e^{t\Delta_k}u(x) \leq 1, \forall \ t > 0, \forall \ x \in \mathbb{R}^d, \tag{3.4}$$

and hence in order to prove (3.1), it is enough to show that

$$\int_{\mathbb{R}^d} |u|^q(x) w_k(x) dx \leq C_{d,k} q \int_{\mathbb{R}^d} |\nabla u|^2(x) w_k(x) dx. \tag{3.5}$$

Now we will be using some basic measure theory results in the proof. Recall that

$$|u(x)|^p = \int_0^\infty \chi_{\{|u(x)|^p \leq \lambda\}} d\lambda = q \int_0^\infty \chi_{\{|u(x)|^q \leq \tau\}} \tau^{q-1} d\tau.$$
Existence of an extremal of Sobolev inequality associated with Dunkl gradient and of Stein-Weiss inequality for D-Riesz potential

From this one can easily write that
\[ \int_{\mathbb{R}^d} |u(x)|^q w_k(x)dx = q \int_0^\infty |\{u|u > \tau\}| \tau^{q-1}d\tau \] (3.6)
where \(|\{u|u > \tau\}|\) is the measure given by \(|\{u|u > \tau\}| = \int_{\mathbb{R}^d} \chi_{\{|u(x)|>\tau\}} w_k(x)dx\). If we write \(u = (u - e^{t\Delta_k}u) + e^{t\Delta_k}u\) for some \(t > 0\) chosen later, then
\[ |\{u|u > \tau\}| = |\{u - e^{t\Delta_k}u| > \tau/2\}| + |\{e^{t\Delta_k}u| > \tau/2\}|. \]

Let us now choose \(t = t_\tau\) satisfying \(\tau/2 = t^{\frac{d_k-2}{2}}\), then from (3.4), \(|\{e^{t\Delta_k}u| > \tau/2\}| = 0\). Hence by (3.6) we have
\[ \int_{\mathbb{R}^d} |u|^q w_k(x)dx \leq q \int_0^\infty |\{u - e^{t\Delta_k}u| > \tau/2\}| \tau^{q-1}d\tau. \] (3.7)

For a fixed constant \(b \geq 1/16\) and for any \(\tau > 0\), we define a function \(u_\tau\) on \(\mathbb{R}^d\) as follows:
\[ u_\tau(x) = \begin{cases} 
(b - \frac{1}{16})\tau & \text{if } u(x) > b\tau, \\
(u(x) - \frac{7}{16})\tau & \text{if } b\tau \geq u(x) > -\frac{7}{16}, \\
0 & \text{if } -\frac{7}{16} > u(x) > -\frac{15}{16}, \\
u(x) + \frac{15}{16} & \text{if } -\frac{15}{16} \geq u(x) \geq -b\tau, \\
-(b - \frac{1}{16})\tau & \text{if } u(x) < -b\tau.
\end{cases} \]

Note that \(u_\tau\) is in \(\hat{H}(\mathbb{R}^d, w_k)\) and
\[ \int_{\mathbb{R}^d} |\nabla u_\tau|^2 w_k(x)dx = \int_{\tau/16 \leq |u| \leq b\tau} |\nabla u|^2 w_k(x)dx. \]

The decomposition \(u - e^{t\Delta_k}u = (u_\tau - e^{t\Delta_k}u_\tau) - e^{t\Delta_k}(u - u_\tau) + (u - u_\tau)\) gives
\[ |\{u - e^{t\Delta_k}u| > \frac{\tau}{2}\}| \leq |\{u_\tau - e^{t\Delta_k}u_\tau| > \frac{\tau}{4}\}| + |\{u - u_\tau| > \frac{\tau}{8}\}| \\
+ |\{e^{t\Delta_k}(u - u_\tau)| > \frac{\tau}{8}\}|. \] (3.8)

By using Chebyshev inequality with Lemma 3.1, we get the bound for the first term of the right hand side of (3.8)
\[ |\{u_\tau - e^{t\Delta_k}u_\tau| > \frac{\tau}{4}\}| \leq (\tau/4)^{-2} \|u_\tau - e^{t\Delta_k}u_\tau\|^2_{L^2(\mathbb{R}^d, w_k)} \leq (\tau/4)^{-2} t\|\nabla u_\tau\|^2_{L^2(\mathbb{R}^d, w_k)} \leq 4(\tau/2)^{-q} \int_{\tau/16 \leq |u| \leq b\tau} |\nabla u|^2 w_k(x)dx, \]
which implies that
\[ \int_0^\infty |\{u_\tau - e^{t\Delta_k}u_\tau| > \frac{\tau}{4}\}| \tau^{q-1}d\tau = 2^{q+2} \log(16b) \int_{\mathbb{R}^d} |\nabla u|^2 w_k(x)dx. \] (3.9)
Now we need to obtain the bound for the second and third term of the right hand side of (3.3). Towards this, first we observe that
\[
|u_{\tau} - u| = |u_{\tau} - u|\chi_{\{|u| \leq b\tau\}} + |u_{\tau} - u|\chi_{\{|u| > b\tau\}} \leq \frac{\tau}{16} + |u|\chi_{\{|u| > b\tau\}},
\]
which leads to again by Chebyshev inequality
\[
|\{|u - u_{\tau}| > \frac{\tau}{8}\}| \leq \{|u|\chi_{\{|u| > b\tau\}} > \frac{\tau}{16}\}| \leq (\tau/16)^{-1} \int_{\mathbb{R}^d} |u|\chi_{\{|u| > b\tau\}} w_k(x)dx.
\]
Using the properties of Dunkl heat kernel and (3.10),
\[
|e^{t\Delta_k}u_{\tau} - e^{t\Delta_k}u| \leq e^{t\Delta_k}|u_{\tau} - u| \leq \frac{\tau}{16} \|\tau^k y^k\|_{L^1(\mathbb{R}^d, w_k)} + e^{t\Delta_k}(|u|\chi_{\{|u| > b\tau\}})
\]
\[
= \frac{\tau}{16} c_k + e^{t\Delta_k}(|u|\chi_{\{|u| > b\tau\}}).
\]
Now we assume the $c_k \leq 1$. Then
\[
|e^{t\Delta_k}u_{\tau} - e^{t\Delta_k}u| \leq \frac{\tau}{16} + e^{t\Delta_k}(|u|\chi_{\{|u| > b\tau\}}).
\]
Hence
\[
|\{|e^{t\Delta_k}(u_{\tau} - u)| > \frac{\tau}{8}\}| \leq \{|e^{t\Delta_k}(|u|\chi_{\{|u| > b\tau\}}) > \frac{\tau}{16}\}|
\]
\[
\leq (\tau/16)^{-1} \int_{\mathbb{R}^d} e^{t\Delta_k}(|u|\chi_{\{|u| > b\tau\}}) w_k(x)dx
\]
\[
= (\tau/16)^{-1} c_k \int_{\mathbb{R}^d} |u|\chi_{\{|u| > b\tau\}} w_k(x)dx
\]
\[
\leq (\tau/16)^{-1} \int_{\mathbb{R}^d} |u|\chi_{\{|u| > b\tau\}} w_k(x)dx.
\]
Then using (3.11), we have the estimate
\[
\int_0^\infty (|\{|e^{t\Delta_k}u_{\tau} - e^{t\Delta_k}u| > \frac{\tau}{8}\}| + |\{|u - u_{\tau}| > \frac{\tau}{8}\}|)\tau^{q-1}d\tau
\]
\[
= \frac{32}{q - 1} b^{-q+1} \int_{\mathbb{R}^d} |u|^q w_k(x)dx.
\]
Now from (3.6), using (3.9) and the above estimate we obtain for sufficiently large $b$,
\[
\int_{\mathbb{R}^d} |u|^q w_k(x)dx \leq \frac{q^{2q+2} \log(16b)}{1 - \frac{2q}{q+1}} b^{-q+1} \int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x)dx.
\]
thus proving (3.5).
Now let us assume that $c_k > 1$. Choose $b > \frac{1}{16c_k}$ and for any $\tau > 0$, define the function $u_\tau$ on $\mathbb{R}^d$ as follows:

$$u_\tau(x) = \begin{cases} 
(b - \frac{1}{16c_k})\tau & \text{if } u(x) > b\tau, \\
\frac{u(x) - \tau}{16c_k} & \text{if } b\tau \geq u(x) \geq \frac{\tau}{16c_k}, \\
u(x) + \frac{\tau}{16c_k} & \text{if } -\frac{\tau}{16c_k} \geq u(x) \geq -b\tau, \\
(b - \frac{1}{16c_k})\tau & \text{if } u(x) < -b\tau.
\end{cases}$$

Now proceeding as before we get,

$$\int_0^\infty \left| \{ |u_\tau - e^{-t\Delta_k}u_\tau| > \frac{\tau}{4} \} \right| \tau^{q-1} d\tau = 2^q + 2^q \log(16bc_k) \int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x) dx.$$

Also, in this case

$$|u_\tau - u| \leq \frac{\tau}{16c_k} + |u| \chi_{\{ |u| > b\tau \}} \leq \frac{\tau}{16} + |u| \chi_{\{ |u| > b\tau \}},$$

and

$$|e^{t\Delta_k}u_\tau - e^{t\Delta_k}u| \leq e^{t\Delta_k}|u_\tau - u| \leq \frac{\tau}{16c_k} \| e^{t\Delta_k} \|_{L^1(\mathbb{R}^d, w_k)} + e^{t\Delta_k}(\| u \|_{L^1(\mathbb{R}^d, w_k)})$$

$$= \frac{\tau}{16} + e^{t\Delta_k}(\| u \|_{L^1(\mathbb{R}^d, w_k)}).$$

Then proceeding exactly as before, for sufficiently large $b$, we have

$$\int_{\mathbb{R}^d} |u|^q w_k(x) dx \leq \frac{q2^q + 2^q \log(16bc_k)}{1 - \frac{32q^k b^{q+1}}{q-1}} \int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x) dx,$$

thus proving (3.5). This completes the proof of Theorem 3.1.

4. Existence of extremals for Dunkl-type Sobolev inequality

The aim of this section is to prove the existence of a minimizer for the function $F$ defined in (1.4). Now we prove the following corollary.

**Corollary 4.1.** For $d \geq 3$, let $(u_j)$ be a bounded sequence in $\dot{H}^1(\mathbb{R}^d, w_k)$. Then either one of the following statements holds.

(i) $(u_j)$ converges to $0$ in $L^q(\mathbb{R}^d, w_k)$.

(ii) There exists a subsequence $(u_{jm})$ of $(u_j)$ and sequences $(a_m) \subset \mathbb{R}^d$ and $(b_m) \subset (0, \infty)$ such that

$$v_m(x) = b_m^{\frac{d}{2} - 2}(\tau^k_{a_m} u_{jm})(b_m x)$$

converges weekly in $\dot{H}^1(\mathbb{R}^d, w_k)$ to a function $v \not= 0$. Moreover, $(v_m)$ converges pointwise a.e. to $v$. 

□
Proof. Assume that (i) does not hold. Then there exists \( \epsilon > 0 \) and a subsequence \((u_{j,m})\) of \((u_j)\) such that \( \|u_{j,m}\|_{L^q(w_k)} \geq \epsilon \). We shall denote \( u_{j,m} \) by \( u_j \) itself. Since \( u_j \) is bounded in \( H^1(\mathbb{R}^d, w_k) \), there exists \( A > 0 \) such that \( \|\nabla_k u_j\|_{L^2(\mathbb{R}^d, w_k)} \leq \sqrt{A} \forall j \).

Now applying the Dunkl-type Sobolev inequality (3.1) for the function \( u_j \), we get

\[
\left( \sup_{t>0} t^{\frac{d_k-2}{4} - \frac{1}{q}} \|e^{t\Delta_k} u_j\|_{\infty} \right)^{\frac{2}{d_k}} \geq C_{d,k}^{-1} A^{-\frac{d_k-2}{2d_k}} \epsilon.
\]

Then there exists \( t_j > 0, x_j \in \mathbb{R}^d \) such that

\[
\frac{d_k-2}{4} t_j \|e^{t_j \Delta_k} u_j(x_j)\| \geq \frac{1}{2} C_{d,k}^{-\frac{d_k}{4}} A^{-\frac{d_k-2}{4}} \epsilon. \tag{4.1}
\]

Let \( G(y) = 2^{-\frac{d_k}{4}} e^{-\frac{|y|^2}{4t_j}} \) and \( v_j(y) = t_j^{\frac{d_k-2}{4}} (\tau_{-x_j}^k u_j)(\sqrt{t_j}y) \). Now consider

\[
\left| \int_{\mathbb{R}^d} G(y)v_j(y)w_k(y)dy \right| = \left| \int_{\mathbb{R}^d} 2^{-\frac{d_k}{4}} e^{-\frac{|y|^2}{4t_j}} t_j^{\frac{d_k-2}{4}} (\tau_{-x_j}^k u_j)(\sqrt{t_j}y)w_k(y)dy \right|
\]

\[
= 2^{-\frac{d_k}{4}} t_j^{\frac{d_k-2}{4}} \left| \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t_j}} (\tau_{-x_j}^k u_j)(y)w_k \left( \frac{y}{\sqrt{t_j}} \right) \frac{1}{t_j^{\frac{d_k}{2}}} dy \right|
\]

\[
= 2^{-\frac{d_k}{4}} t_j^{\frac{d_k-2}{4}} t_j^{\frac{d_k-2}{4}} \left| \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t_j}} (\tau_{-x_j}^k u_j)(y)w_k(y)dy \right|,
\]

using property (2) of section 2. Then using (4.1)

\[
\left| \int_{\mathbb{R}^d} G(y)v_j(y)w_k(y)dy \right| = t_j^{\frac{d_k-2}{4}} \left| \int_{\mathbb{R}^d} q_{t_j}(y)(\tau_{-x_j}^k u_j)(y)w_k(y)dy \right|
\]

\[
= t_j^{\frac{d_k-2}{4}} \left| \int_{\mathbb{R}^d} u_j(y)(\tau_{x_j}^k q_{t_j})(y)w_k(y)dy \right|
\]

\[
= t_j^{\frac{d_k-2}{4}} \left| \int_{\mathbb{R}^d} u_j(y)(\tau_{y}^k q_{t_j})(x_j)w_k(y)dy \right|
\]

\[
= t_j^{\frac{d_k-2}{4}} \left| e^{t_j \Delta_k} u_j(x_j) \right|
\]

\[
\geq \frac{1}{2} C_{d,k}^{-\frac{d_k}{4}} A^{-\frac{d_k-2}{4}} \epsilon. \tag{4.2}
\]

Moreover,

\[
\mathcal{F}_k(v_j)(\xi) = t_j^{\frac{d_k+2}{4}} \mathcal{F}_k(\tau_{-x_j}^k u_j)(-\frac{\xi}{\sqrt{t_j}}).
\tag{4.3}
\]
Indeed, by inserting the value of $v_j$ in Definition 2.2 we get

$$\mathcal{F}_k(v_j)(\xi) = c_k^{-1} \int_{\mathbb{R}^d} v_j(y) E_k(-i\xi, y) w_k(y) dy$$

$$= c_k^{-1} \int_{\mathbb{R}^d} t_j^{\frac{dk-2}{2}} \left( \tau_{-x_j}^{k} u_j \right) (\sqrt{t_j} y) E_k(-i\xi, y) w_k(y) dy.$$ 

Now by replacing $y$ by $\frac{y}{\sqrt{t_j}}$ and proceeding as before, we obtain

$$\mathcal{F}_k(v_j)(\xi) = t_j^{\frac{dk-2}{2}} c_k^{-1} \int_{\mathbb{R}^d} \left( \tau_{-x_j}^{k} u_j \right) (\sqrt{t_j} y) E_k(-i\xi, \frac{y}{\sqrt{t_j}}) w_k(y) dy$$

$$= t_j^{-\frac{4k+2}{4}} \mathcal{F}_k(\tau_{x_j}^{k} u_j) \left( \frac{\xi}{\sqrt{t_j}} \right),$$

using Proposition 2.1(ii) and the definition of Dunkl transform, thus proving (4.3). Therefore, using (4.3)

$$\| \nabla_k v_j \|^2_{L^2(\mathbb{R}^d, w_k)} = \| \mathcal{F}_k(\nabla_k v_j) \|^2_{L^2(\mathbb{R}^d, w_k)}$$

$$= \int_{\mathbb{R}^d} |\xi|^2 |\mathcal{F}_k(v_j)(\xi)|^2 w_k(\xi) d\xi$$

$$= t_j^{-\frac{2k+2}{2}} \int_{\mathbb{R}^d} |\xi|^2 \left| \mathcal{F}_k(\tau_{-x_j}^{k} u_j) \left( \frac{\xi}{\sqrt{t_j}} \right) \right|^2 w_k(\xi) d\xi$$

$$= t_j^{-\frac{2k+2}{2}} t_j^{1+\gamma_k + \frac{d}{2}} \int_{\mathbb{R}^d} |\xi|^2 \left| \mathcal{F}_k(\tau_{-x_j}^{k} u_j)(\xi) \right|^2 w_k(\xi) d\xi.$$

Consequently, using (2.4) and Proposition 2.1(iv), we get

$$\| \nabla_k v_j \|^2_{L^2(\mathbb{R}^d, w_k)} \leq \int_{\mathbb{R}^d} |\xi|^2 |E_k(-ix_j, \xi) \mathcal{F}_k(u_j)(\xi)|^2 w_k(\xi) d\xi$$

$$\leq \int_{\mathbb{R}^d} |\xi|^2 |\mathcal{F}_k(u_j)(\xi)|^2 w_k(\xi) d\xi = \| \nabla_k u_j \|^2_{L^2(\mathbb{R}^d, w_k)}.$$

Thus $\| \nabla_k v_j \|^2_{L^2(\mathbb{R}^d, w_k)} \leq A$ for all $j$. By Banach-Alaoglu theorem, $v_j$ has a weakly convergent subsequence in $\dot{H}^1(\mathbb{R}^d, w_k)$ and let it converge to $w$. Since $G \in \dot{H}^1(\mathbb{R}^d, w_k)^*$, $G(v_j)$ converges to $G(w)$. It follows from (1.2) that $G(v_j) \neq 0$, $\forall j$ and therefore, $G(w) \neq 0$, which in turn imply that $w \neq 0$. This completes the proof of the corollary. □
Theorem 4.1. Let $d \geq 3$. Then the infimum

$$S_{d,k} = \inf_{u \in H^1(\mathbb{R}^d,w_k)} \frac{\int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x) dx}{\left( \int_{\mathbb{R}^d} |u|^q w_k(x) dx \right)^{\frac{2}{q}}}$$

is attained.

Proof. Let $(u_j)$ be a minimizing sequence, which we assume to be normalized in $L^q(\mathbb{R}^d, w_k)$. Then $(u_j)$ is bounded in $H^1(\mathbb{R}^d, w_k)$, since the sequence $\|\nabla k u_j\|_{L^2(\mathbb{R}^d,w_k)}^2$ converges to $S_{d,k}$. Moreover, because $(u_j)$ has norm 1 in $L^q(\mathbb{R}^d, w_k)$, from Corollary 4.1 we can say that after a generalized translation and a dilation, $(u_j)$ converges weakly to a non-zero function $u$ a.e. in $H^1(\mathbb{R}^d, w_k)$. This implies that

$$\int_{\mathbb{R}^d} |\nabla_k u_j|^2 w_k(x) dx = \int_{\mathbb{R}^d} |\nabla_k (u_j - u)|^2 w_k(x) dx + \int_{\mathbb{R}^d} |\nabla_k u|^2 w_k(x) dx + o(1).$$

From Lemma 2.1 we have

$$1 = \int_{\mathbb{R}^d} |u_j|^q w_k(x) dx = \int_{\mathbb{R}^d} |u_j - u|^q w_k(x) dx + \int_{\mathbb{R}^d} |u|^q w_k(x) dx + o(1).$$

As a consequence, as $\frac{2}{q} < 1$,

$$1 = \left( \int_{\mathbb{R}^d} |u_j|^q w_k(x) dx \right)^{\frac{2}{q}} \leq \left( \int_{\mathbb{R}^d} |u_j - u|^q w_k(x) dx \right)^{\frac{2}{q}} + \left( \int_{\mathbb{R}^d} |u|^q w_k(x) dx \right)^{\frac{2}{q}} + o(1),$$

since $(a + b)^n \leq a^n + b^n$ for $a, b > 0, 0 \leq u \leq 1$. 
Hence
\[
S_{d,k} + o(1) = \int_{\mathbb{R}^d} |\nabla_k u_j|^2 w_k(x) \, dx \\
\geq \int_{\mathbb{R}^d} |\nabla_k (u_j - u)|^2 w_k(x) \, dx + \int_{\mathbb{R}^d} |\nabla_k u_j|^2 w_k(x) \, dx + o(1) \\
= \left( \int_{\mathbb{R}^d} |u_j - u|^2 w_k(x) \, dx \right)^{\frac{2}{q}} + \left( \int_{\mathbb{R}^d} |u|^2 w_k(x) \, dx \right)^{\frac{2}{q}} + o(1)
\]

which implies that
\[
S_{d,k} + o(1) \geq \left( \int_{\mathbb{R}^d} |u_j - u|^2 w_k(x) \, dx \right)^{\frac{2}{q}} + \left( \int_{\mathbb{R}^d} |u|^2 w_k(x) \, dx \right)^{\frac{2}{q}} + o(1)
\]

from which it follows that \( u \) is a minimizer. \( \square \)

5. A WEIGHTED ESTIMATE FOR THE HEAT SEMI GROUP \( e^{t\Delta_k} \)

In this section, we prove the following Proposition involving a weighted estimate for the operator \( e^{t\Delta_k} \), using which we show that \( e^{t\Delta_k} \) is a compact operator.

**Proposition 5.1.** Let \( d \geq 2, 1 < r < \infty \). Assume that
\[
0 < \beta < \frac{d_k}{r'}
\]
and fix \( t > 0 \). Then

(i) \( \|e^{t\Delta_k} f\|_{L^\infty(\mathbb{R}^d)} \leq C_{d,r,\beta,t,\lambda} \|x\|^{\beta} \|r w_k\|_{r,w_k} \)

(ii) \( \|x|^w e^{t\Delta_k} f\|_{L^\infty(\mathbb{R}^d)} \leq D_{d,r,\beta,t,\lambda} \|x\|^{\beta} \|r w_k\|_{r,w_k} \) for \( \beta \geq w > 0 \).

(iii) \( \|\partial_i e^{t\Delta_k} f\|_{L^\infty(\mathbb{R}^d)} \leq E_{d,r,\beta,t,\lambda} \|x\|^{\beta} \|r w_k\|_{r,w_k}, \quad i = 1, 2, \ldots, n. \)

**Proof.** Consider
\[
|e^{t\Delta_k} f(x)| \leq \int_{\mathbb{R}^d} |f(y)||\tau^k g^k(x)| w_k(y) \, dy \\
\leq \left( \int_{\mathbb{R}^d} |f(y)| r w_k(y) \, dy \right)^{\frac{1}{r'}} \left( \int_{\mathbb{R}^d} |\tau^k g^k(x)| r^{\frac{1}{r'}} \, dy \right)^{\frac{1}{r'}} \\
= \|y|^{\beta} \|r w_k(I_1(x) + I_2(x))\|_{r,w_k} \quad (5.1)
\]
where

\[ I_1(x) = \int_{|y| \leq \sqrt{t}} |\tau_y \varphi_t^{k}(x)| t^\alpha |y|^{-\beta r} w_k(y) dy \]

and

\[ I_2(x) = \int_{|y| > \sqrt{t}} |\tau_y \varphi_t^{k}(x)| t^\alpha |y|^{-\beta r} w_k(y) dy. \]

Substituting the value of \( \tau_y \varphi_t^{k}(x) \) from (2.7) and then using Proposition 2.1

\[
|\tau_y \varphi_t^{k}(x)| = \left| (2t)^{-\frac{\alpha}{2} + \frac{d}{4}} e^{-\frac{|x|^2 + |y|^2}{24t}} E_k \left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right) \right|
\]

\[
\leq (2t)^{-\frac{\alpha}{2} + \frac{d}{4}} e^{-\frac{|x|^2 + |y|^2}{24t}}
\]

\[
= (2t)^{-\frac{d}{4}} e^{-\frac{|x|^2 + |y|^2}{24t}} \quad (5.2)
\]

where \( C = 2^{-\frac{d}{4}} \). Then

\[ I_1(x) \leq C |t|^{-\frac{\alpha}{2}} \int_{|y| \leq \sqrt{t}} |y|^{-\beta r} w_k(y) dy. \]

By property (4) of section 2, the above integral is finite if \( \beta < \frac{d}{4} \) and equals to \( \frac{a_k}{d_k - \beta r} \). Thus \( I_1(x) \leq A_{d,r,\beta,k} t^{-\frac{\alpha}{2} - \frac{d}{4}} t^{\frac{d}{2}(d_k - \beta r)} \), where \( A_{d,r,\beta,k} = \frac{a_k}{d_k - \beta r} \).

On the other hand, over the integral \( I_2 \), since \( |y| > \sqrt{t}, |y|^{-\beta r} \leq t^{-\frac{\beta r}{2}} \) and hence

\[ I_2(x) \leq t^{-\frac{\beta r}{2}} \int_{|y| > \sqrt{t}} |\tau_y \varphi_t^{k}(x)| t^\alpha |y|^{-\beta r} w_k(y) dy \leq t^{-\frac{\beta r}{2}} \int_{\mathbb{R}^d} |\tau_y \varphi_t^{k}(x)| t^\alpha |y|^{-\beta r} w_k(y) dy \]

\[ \leq t^{-\frac{\beta r}{2}} \int_{\mathbb{R}^d} q_k(y) |y|^{-\beta r} w_k(y) dy \]

\[ \leq B_{d,r,k} t^{-\frac{\beta r}{2}} t^{-\frac{d}{4} (r' - 1)}, \quad (5.3) \]

by taking \( p = r' \) in (5.3), where \( B_{d,r,k} = 2^{-\frac{d}{4}} \left( \frac{r'}{4} \right) \frac{a_k}{2} \Gamma \left( \frac{d_k}{2} \right) \). Finally from (5.1) using the estimates of \( I_1 \), \( I_2 \) and the fact that \((a + b)^u \leq a^u + b^u \) for \( a, b > 0, u < 1 \), we obtain

\[
|e^{\Delta_h f(x)}| \leq \|y|^\beta f\|_{L^\infty(\mathbb{R}^d, w_k)} \left( A_{d,r,\beta,k} t^{\frac{d}{4} - \frac{\beta r}{2}} t^{\frac{d_k - \beta r}{4}} + B_{d,r,k} t^{\frac{d}{4} - \frac{\beta r}{2}} t^{\frac{d}{4} (r' - 1)} \right)^\frac{1}{\beta}
\]

\[
\leq C_{d,r,\beta,k} \|y|^\beta f\|_{L^\infty(\mathbb{R}^d, w_k)} \left( t^{\frac{d}{4} - \frac{\beta r}{2}} t^{\frac{d}{4} (d_k - \beta r)} + t^{\frac{d}{4} - \frac{\beta r}{2}} t^{\frac{d}{4} (r' - 1)} \right) \quad (5.4)
\]

\[
\leq C_{d,r,\beta,k} \|y|^\beta f\|_{L^\infty(\mathbb{R}^d, w_k)};
\]
where $C_{d,r,k} = (\max\{A_d,B_d,r,k\})^{\frac{1}{r}}$ and $C_{d,k} = C_{d,r,k} \left(t^{-\frac{d}{2}} t^{-\frac{d}{2(r)}} + t^{-\frac{d}{2}} t^{-\frac{d}{2(r)}} \right)$.

Thus we have proved that
\[
\left\| e^{t\Delta_k} f \right\|_{L^{\infty}(\mathbb{R}^d)} \leq C_{d,r,k} \left\| \tau \right\|_{L^r(\mathbb{R}^d,w_k)},
\]
(5.5)

thus proving (i).

In particular, when $r = 2$, then from (6.3), we get
\[
\left\| e^{t\Delta_k} f \right\|_{L^{\infty}(\mathbb{R}^d)} \leq C_{d,2,k}^{-\frac{1}{2} \left(\frac{1}{2} + \beta\right)} \left\| \tau \right\|_{L^2(\mathbb{R}^d,w_k)},
\]
(5.6)

where $C_{d,2,k} = 2C_{d,2,k}$.

Now we shall prove (ii). since $w > 0$, using (5.5) we observe that, for $|x| \leq 1$,
\[
\left| \left| x \right| w \right| e^{t\Delta_k} f(x) \right| \leq e^{t\Delta_k} f(x) \leq C_{d,r,k} \left\| \tau \right\|_{L^r(\mathbb{R}^d,w_k)}.
\]
(5.7)

So we assume that $|x| > 1$. Consider
\[
\left| \left| x \right| w \right| e^{t\Delta_k} f(x) \right| \leq \left| \left| x \right| w \right| \int_{\mathbb{R}^d} f(y) \left| \tau \right| q_k(y) w_k(y) dy
\]
\[
\leq \left( \int_{\mathbb{R}^d} \left| f(y) \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} \left| x \right| \left| \tau \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy \right)^{\frac{1}{2}}
\]
\[
\leq \left\| \left| x \right| w \right\| \left( \int_{\mathbb{R}^d} \left| x \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy \right)^{\frac{1}{2}}
\]

since $|x| > 1$ and $\beta \geq w$, it implies that $\left| x \right|^{(w-\beta)'} \leq 1$.

Then
\[
\left| \left| x \right| w \right| e^{t\Delta_k} f(x) \right| \leq \left\| \left| x \right| w \right\| \left( \int_{\mathbb{R}^d} \left| x \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy \right)^{\frac{1}{2}}
\]
(5.8)

where $I_1(x) = \int_{\mathbb{R}^d} \left| x \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy$ and $I_2(x) = \int_{\mathbb{R}^d} \left| x \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy$.

Over the first integral $I_1(x)$, $\left( \left| x \right| - \left| y \right| \right)^2 \geq \frac{\left| x \right|^2}{4} \forall \ y$, since $\left| y \right| \leq \frac{\left| x \right|}{2}$. Consequently, from (5.2) we arrive at the bound $\tau \left| g_k(x) \right| \leq (2t)^{-\frac{d}{4}} e^{-\frac{\left| x \right|^2}{8t}}$. Then
\[
I_1(x) \leq (2t)^{-\frac{d}{4}} e^{-\frac{\left| x \right|^2}{8t}} \int_{\mathbb{R}^d} \left| x \right| \left| y \right| \left| \tau \right| q_k(y) w_k(y) dy.
\]

For $\beta < \frac{d}{r}$, the above integral finite and equals to $\frac{1}{\left(2t\right)^{\frac{d}{4}}} \left[ \left(2t\right)^{\frac{d}{4}} \right] \left| \frac{\left| x \right|^2}{8t} \right| \left| \frac{\left| x \right|^2}{8t} \right|$. Therefore,
\[
I_1(x) \leq \left( \left(2t\right)^{\frac{d}{4}} \left| \frac{\left| x \right|^2}{8t} \right| \left| \frac{\left| x \right|^2}{8t} \right| \right) \left| \frac{\left| x \right|^2}{8t} \right| \left| \frac{\left| x \right|^2}{8t} \right|.
\]
Since $\left| \frac{\left| x \right|^2}{8t} \right| \to 0$ as $|x| \to \infty$, it follows that $I_1(x) \leq A'_{d,r,k}$ for some constant $A'_{d,r,k} > 0$. 
Consequently, combining (5.7) and (5.9), we get a constant $D$ using (5.3). Hence for operator $e$ thus proving (ii).

Next we shall prove (iii). Consider

$$I_2(x) \leq 2^{3r'} \int_{|y| > \frac{d}{t}} |\tau^k_{t} q^k_t(x)|^{r'} w_k(y)dy \leq 2^{3r'} \int |q^k_t(y)|^{r'} w_k(y)dy \leq 2^{3r'} B_{d,r,k} t^{\frac{d}{r}(r'-1)} = B'_{d,r,t,k};$$

using (5.3). Hence for $|x| > 1$, from (5.8), we get a constant $C'_{d,r,t,k}$ such that

$$||x|^\alpha e^{\Delta_k} f(x)|| \leq C'_{d,r,t,k} \|y\|^\alpha u \|w_k\|.$$  \hspace{1cm} (5.9)

Consequently, combining (5.7) and (5.9), we get a constant $D_{d,r,t,k}$ such that

$$||x|^\alpha e^{\Delta_k} f||_{L^\infty(\mathbb{R}^d)} \leq D_{d,r,t,k} \|x\|^\alpha \|f\|_{r,w_k},$$

thus proving (ii).

Next we shall prove (iii). Consider

$$\left| \frac{\partial}{\partial x_i} (e^{\Delta_k} f)(x) \right| = \frac{\partial}{\partial x_i} (f \ast_k q^k_t)(x)$$

$$= \left| \frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} f(y) \tau^k_{t} q^k_t(x) w_k(y)dy \right|$$

$$= \left| \int_{\mathbb{R}^d} f(y) \frac{\partial}{\partial x_i} \tau^k_{t} q^k_t(x) w_k(y)dy \right|$$

$$\leq C t^{-\frac{1}{2}} \int_{\mathbb{R}^d} |f(y)||\tau^k_{t} q^k_t(x)| w_k(y)dy,$$

by taking $m = 0, \alpha = e_i, \beta = 0$ in (2.9). Now using (i), there exists constant $E_{d,r,t,k} > 0$ such that

$$\|\partial_{x_i} e^{\Delta_k} f\|_{L^\infty(\mathbb{R}^d)} \leq E_{d,r,t,k} \|x\|^{\alpha} \|f\|_{r,w_k},$$

thus proving (iii).

Using Proposition 5.1 we prove the following theorem.

**Theorem 5.1.** Let $d \geq 2, 1 < r < \infty$. If $0 < \beta < \frac{d}{r}$, then for any fixed $t > 0$, the operator $e^{\Delta_k}$ is a compact operator from $L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$ to $L^\infty(\mathbb{R}^d)$.

**Proof.** Let $(u_j)_{j \in \mathbb{N}} \in L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$ be a bounded sequence so that $|||x|^\beta u_j||_{r,w_k} \leq C_0$ for all $j \in \mathbb{N}$. We will prove that the sequence $(e^{\Delta_k} u_j)_{j \in \mathbb{N}}$ has convergent subsequence in $L^\infty(\mathbb{R}^d)$ and that will imply the theorem. Let $v_j = e^{\Delta_k} u_j$. Since $(u_j)_{j \in \mathbb{N}}$ is a bounded sequence, using Proposition 2.1 (i), we have

$$\|v_j\|_{L^\infty(\mathbb{R}^d)} = \|e^{\Delta_k} u_j\|_{L^\infty(\mathbb{R}^d)} \leq C \|x|^\beta u_j\|_{r,w_k} \leq C_0.$$
This proves that each $v_j \in L^\infty(\mathbb{R}^d)$ and the collection $(v_j)_j$ is equibounded in $\mathbb{R}^d$. Moreover, Proposition \ref{prop:max} (iii) shows that $(v_j)_j$ is also equiconvergent in $\mathbb{R}^d$. Now for each $n \in \mathbb{N}$, we define the compact set $A_n := \{x \in \mathbb{R}^d : |x| \leq n\}$. Then by the Arzelà-Ascoli theorem for each $n \in \mathbb{N}$, there exists a subsequence of $(v_j)_j$ which converges uniformly in $A_n$. Now by applying diagonal argument, we get a subsequence of $(v_j)_j$ which converges uniformly in every $A_n$. Let us call this subsequence also by $(v_j)_j$ and we can write $v_j \to v$ uniformly in each $A_n$ for some $v \in L^\infty(\mathbb{R}^d)$.

Now let $z$ be such that $0 < z < \beta$. By Proposition \ref{prop:max}(ii), we can write

$$
\|x|^z v_j\|_{L^\infty(\mathbb{R}^d)} = \|x|^z e^{t\Delta_k} u_j\|_{L^\infty(\mathbb{R}^d)} \leq C_1 \|x|^\beta u_j\|_{r,w_k}.
$$

(5.10)

Now by using (5.10) we get

$$
\sup_{|x|>n} |v_j| \leq \sup_{|x|>n} \left(\frac{|x|}{n}\right)^z |v_j| \leq n^{-z} \|x|^z v_j\|_{L^\infty(\mathbb{R}^d)} \leq C_1 n^{-z}.
$$

Thus we can easily see that $v_j \to v$ strongly in $L^\infty(\mathbb{R}^d)$ and this proves the theorem. \hfill \Box

6. **Improved Stein-Weiss Inequality for the D-Riesz Potential**

In this section, we will be focusing on deriving an improved version of the Stein-Weiss inequality for the D-Riesz potential i.e., we are interested to generalize Theorem\ref{thm:stein-weiss}. Towards this, for any $\delta > 0$, first we define the Dunkl Besov space as

$$
\dot{B}^{-\delta,k}_{\infty,\infty} := \{f : f \text{ is a tempered distribution on } \mathbb{R}^d \text{ and } \|f\|_{\dot{B}^{-\delta,k}_{\infty,\infty}} < \infty\},
$$

where

$$
\|f\|_{\dot{B}^{-\delta,k}_{\infty,\infty}} := \sup_{t>0} t^{\delta/2} \|e^{t\Delta_k} f\|_{L^\infty}.
$$

(6.1)

**Theorem 6.1.** Let $d \geq 2$ and $0 < \alpha < d_k$. Also let $\beta, \gamma, \mu, \theta, r$ and $s$ be such that $1 < r < s < \infty$, $\gamma > \frac{d_k}{r}$, $\beta < \frac{d_k}{s}$, $\beta \geq \frac{2}{\gamma}$. Also it satisfy that $\mu > 0$, max\{\frac{d_k}{s} - \alpha, \frac{d_k}{r} - \alpha\} < \theta \leq 1$ and

$$
d_k s + \gamma = (\beta + \frac{d_k}{r} - \alpha) \theta + \mu(1 - \theta).
$$

(6.2)

Then for all $f \in L^r(\mathbb{R}^d, |x|^\beta w_k) \cap \dot{B}^{-\mu-\alpha,k}_{\infty,\infty}$, the following inequality holds:

$$
\|\|x|^\gamma I^k_{\alpha} f\|_{L^r(\mathbb{R}^d, w_k)} \leq C_k \|\|x|^\beta f\|_{L^r(\mathbb{R}^d, w_k)}\|^\theta_{\dot{B}^{-\mu-\alpha,k}_{\infty,\infty}}.
$$

(6.3)

**Proof.** The case $\theta = 1$ reduces to Theorem \ref{thm:stein-weiss}. So we will prove the theorem for $\theta < 1$. For $f \in L^r(\mathbb{R}^d, |x|^\beta w_k)$, let $u = I^k_{\alpha} f$. Then $u$ has an integral representation of the following form:

$$
u = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{t\Delta_k} f dt.
$$

Now, we can write $u = H_k f + L_k f$, where

$$
H_k f := \frac{1}{\Gamma(\alpha/2)} \int_0^T t^{\alpha/2-1} e^{t\Delta_k} f dt \quad \text{and} \quad L_k f := \frac{1}{\Gamma(\alpha/2)} \int_T^\infty t^{\alpha/2-1} e^{t\Delta_k} f dt.
$$
Our aim is to find a bound for $u$. To achieve this we will look for the bounds for $L_k f$ and $H_k f$ separately. Using the definition of Besov norm in (6.1), we have $|L_k f(x)| \leq C_k T^{-\alpha/2} \|f\|_{L^{\infty}_{\infty}^{\mu}}$. Now we will find the bound for $H_k f$.

Let

$$
\Phi_{\alpha,T}(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^T t^{\frac{\alpha}{2}-1} q_k(x) dt.
$$

It can be easily verified that $H_k f = \Phi_{\alpha,T} * k f$. Fix $\epsilon = \frac{\mu - \mu - \theta}{2} > 0$. We note that since $\theta > \frac{\mu}{\mu + \alpha}$, $\alpha - 2\epsilon > 0$.

Since for any non-zero real $x$, $e^{-|x|} \leq \frac{C}{|x|^s}$ holds, we can write by taking $u = (d_k - \alpha)/2 + \epsilon > 0$,

$$
\Phi_{\alpha,T}(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^T t^{\frac{\alpha}{2}-1} \left(2t\right)^{\frac{d_k}{2}} e^{-\frac{|x|^2}{4t}} dt \leq C \int_0^T t^{(\alpha - d_k)/2 - 1} \left(\frac{4t}{|x|^2}\right)^{(d_k - \alpha)/2 + \epsilon} dt \leq C_k \frac{1}{|x|^{d_k - \alpha + 2\epsilon}} \int_0^T t^{-1+\epsilon} dt = C_k \frac{T^\epsilon}{|x|^{d_k - \alpha + 2\epsilon}}.
$$

Since Dunkl translation is linear and positivity-preserving for radial functions we can write $\tau_y^k(\Phi_{\alpha,T}(x)) \leq T^\epsilon \tau_y^k(|x|^{-(d_k - \alpha + 2\epsilon)})(x)$. So we obtain $H_k f(x) \leq C_k T^\epsilon I_{\alpha - 2\epsilon} f(x)$.

Choose $T$ such that $T^{-\mu/2} \|f\|_{L^{\infty}_{\infty}^{\mu}} = T^\epsilon I_{\alpha - 2\epsilon} f(x)$, which implies that

$$
T = \left(\frac{\|f\|_{L^{\infty}_{\infty}^{\mu}}}{I_{\alpha - 2\epsilon} f(x)}\right)^{1/(\epsilon + \mu/2)}.
$$

By substituting the value of $T$, we arrive at the point wise bound

$$
|u(x)| = |I_{\alpha}^k f(x)| \leq C_k I_{\alpha - 2\epsilon} f(x)^\theta \|f\|_{L^{\infty}_{\infty}^{\mu}}^{1-\theta}.
$$

Hence

$$
\|x^\gamma I_{\alpha}^k f\|_{L^{\ast}(\mathbb{R}^d, w_k)} \leq C_k \|x^\gamma I_{\alpha - 2\epsilon} f\|_{L^{\ast}(\mathbb{R}^d, w_k)}^{\theta} \|f\|_{L^{\infty}_{\infty}^{\mu}}^{1-\theta}.
$$

If we assume that $\alpha' = \alpha - 2\epsilon$, $\gamma' = \frac{\gamma}{\theta}$, $s' = \frac{s}{\theta}$, then it is easy to see by the hypothesis of the theorem and the choice of $\epsilon$ that $\gamma' > -\frac{\mu}{\theta}$, $\beta \geq \gamma'$, $0 < \alpha' < d_k$, $r \leq s'$, $\beta < \frac{\mu}{\theta}$ and $\alpha' + \gamma' - \beta = d_k \left(\frac{1}{r} - \frac{1}{s'}\right)$. Hence by using Theorem 1.2 we get

$$
\|x^\gamma I_{\alpha - 2\epsilon} f\|_{L^{\ast}(\mathbb{R}^d, w_k)} \leq C_k' \|x^{\beta} f\|_{L^{\ast}(\mathbb{R}^d, w_k)}.
$$

Now from (6.4) and (6.5), we get the desired inequality

$$
\|x^\gamma I_{\alpha}^k f\|_{L^{\ast}(\mathbb{R}^d, w_k)} \leq D_k \|x^{\beta} f\|_{L^{\ast}(\mathbb{R}^d, w_k)}^{\theta} \|f\|_{L^{\infty}_{\infty}^{\mu}}^{1-\theta}.
$$

□

**Remark 6.1.** One can prove Theorem 6.1 for the case $\theta = \frac{\mu}{\mu + \alpha}$ if the weighted $L^\ast$-boundedness of the maximal function $M_k$ is known for $1 < r < \infty$. We recall
that for $f \in S(\mathbb{R}^d)$, S. Thangavelu and Y. Xu [18], defined the maximal function $M_k$ as follows:

$$M_k f(x) = \sup_{r > 0} \frac{\left| \int_{B_r} f(y) \tau_k x \chi_{B_r}(y) w_k(y) dy \right|}{\int_{B_r} w_k(y) dy}.$$ 

When $\theta = \frac{\mu}{\mu + \alpha}$, following [12] and the fact that $|H_k f(x)| \leq C_k T_{\frac{\mu}{\mu + \alpha}} M_k f(x)$, we have proved

$$\| |x|^\gamma I_k f \|_{L^s(\mathbb{R}^d, w_k)} \leq C_k \| |x|^\beta M_k f \|_{L^r(\mathbb{R}^d, w_k)} \| f \|_{L^\infty(\mathbb{R}^d, w_k)}.$$ 

Now in order to obtain (6.3), one has to prove

$$\| |x|^\beta M_k f \|_{L^r(\mathbb{R}^d, w_k)} \leq C_k \| |x|^\beta f \|_{L^r(\mathbb{R}^d, w_k)}, (6.6)$$

which is not known to be true in general. For $\beta = 0$, (6.6) has been proved by S. Thangavelu and Y. Xu in [18].

7. Existence of an extremals for Stein-Weiss inequality associated with Dunkl Laplacian

**Theorem 7.1.** Let $d \in \mathbb{N}, 1 < r \leq s < \infty, \gamma > -\frac{d_k}{s}, \beta \geq \gamma, 0 < \alpha < d_k, \beta < \frac{d_k}{r}$. Further we assume that

$$\beta + \frac{d_k}{r} > \alpha > \frac{d_k}{r} - \frac{d_k}{s} + \beta - \gamma > 0. \quad (7.1)$$

Then if $\mathcal{K} \subset \mathbb{R}^d$ is compact, then one has the compact embedding

$$\dot{H}^\alpha_{\beta,r}(\mathbb{R}^d) \subset L^s(\mathcal{K}, |x|^s), \quad (7.2)$$

where the space $\dot{H}^\alpha_{\beta,r}(\mathbb{R}^d)$ is defined in (1.7).

**Proof.** Let $u \in \dot{H}^\alpha_{\beta,r}(\mathbb{R}^d)$. Then $u = I_k^\alpha f$, for some $f \in L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$. Now we choose $\tilde{s}$ such that

$$\frac{1}{\tilde{s}}(d_k + \gamma s) = \frac{d_k}{r} + \beta - \alpha. \quad (7.3)$$

We define $v = \frac{s}{\tilde{s}}$ and $\tilde{\gamma} = \frac{\tilde{s}}{s}$. From (7.1), it follows that $v > 1$ and $\tilde{\gamma} = \frac{s}{\tilde{s}}$. Then (7.3) can be rewritten as

$$d_k \left( \frac{1}{r} - \frac{1}{s} \right) = \alpha + \tilde{\gamma} - \beta.$$

We replace $\gamma$ and $s$ in Theorem [12] by $\tilde{\gamma}$ and $\tilde{s}$ respectively. Since $v > 1$, $r \leq s$ implies that $r \leq sv = \tilde{s}$ and $\beta \geq \gamma$ implies $\beta \geq \tilde{\gamma}$. Also $\tilde{\gamma} > -\frac{d_k}{s}$ since $\gamma > -\frac{d_k}{s}$. Thus all the conditions of Theorem [12] are satisfied and hence from (1.3), we have

$$\| |x|^\tilde{\gamma} I_k^\alpha f(x) \|_{L^s(\mathbb{R}^d, w_k)} \leq C_k \| |x|^\beta f(x) \|_{L^r(\mathbb{R}^d, w_k)}. \quad (7.4)$$
Applying Holder’s inequality with components $v$ and $v'$ and using the fact that $\gamma > -\frac{d_k}{s}$,

$$\int_{\mathcal{K}} |u|^\gamma |x|^\gamma w_k(x) \, dx = \int_{\mathcal{K}} |u|^\gamma |x|^\frac{\alpha}{s} w_k(x) \, dx$$

$$\leq \left( \int_{\mathcal{K}} |u|^\gamma |x|^\gamma w_k(x) \, dx \right)^{\frac{1}{\gamma}} \left( \int_{\mathcal{K}} |x|^\gamma w_k(x) \, dx \right)^{\frac{1}{\gamma}}$$

$$\leq C_{\mathcal{K}} \left( \int_{\mathcal{K}} |u|^\gamma |x|^\frac{\gamma}{s} w_k(x) \, dx \right)^{\frac{1}{\gamma}},$$

where $C_{\mathcal{K}} = \left( \frac{\alpha_k}{\alpha_k s + \gamma} \right)^{\frac{1}{s}}$, by property (4) of section 2. Then using (1.4),

$$\left( \int_{\mathcal{K}} |u|^\gamma |x|^\gamma w_k(x) \, dx \right)^{\frac{1}{\gamma}} \leq C_{\mathcal{K}} \left( \int_{\mathcal{K}} |I_{\alpha} f|^\gamma |x|^\frac{\gamma}{s} w_k(x) \, dx \right)^{\frac{1}{\gamma}}$$

$$\leq C_{\mathcal{K}} \|x|^\beta f(x)\|_{L^r(\mathbb{R}^4, w_k)} = C_{\mathcal{K}} \|u\|_{\hat{H}^{\alpha,r}_{\delta/\gamma}(\mathbb{R}^4)},$$

proving that the embedding (7.2) is continuous.

Let us define the kernel of the D-Riesz potential as

$$K_{\alpha,k}(x) = (c_k^\alpha)^{-1} |x|^{-(d_k - \alpha)}$$

and for $t > 0$, the truncated kernel as

$$K^t_{\alpha,k}(x) = (c_k^\alpha)^{-1} |x|^{-(d_k - \alpha)} \chi_{\{|x| > t\}}.$$ 

Then using (1.3), we have the following Lemma.

**Lemma 7.1.** With the same conditions as that of Theorem 7.1, let

$$\delta = \alpha - \left( \frac{d_k}{r} - \frac{d_k}{s} + \beta - \gamma \right).$$

Then for any $f \in L^r(\mathbb{R}^d, |x|^{\beta r})$ and for any $t > 0$,

$$\| (K^t_{\alpha,k} * f - K_{\alpha,k} * f) |x|^{\gamma} \|_{s, w_k} \leq C t^\delta \|x|^{\beta} f\|_{r, w_k}.$$ 

Now we shall show that the embedding (7.2) is compact.

Let $\{u_m\}$ be a bounded sequence in $H^{\alpha,r}_{\beta/\gamma}(\mathbb{R}^d)$. Then we can write $u_m = I_{\alpha}^k f_m$, where $\{f_m\}$ is a bounded sequence in $L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$. Since $L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$ is a reflexive space, $\{f_m\}$ has a subsequence, denoted by $f_m$ itself such that $f_m$ converges weakly to a function $f$ in $L^r(\mathbb{R}^d, |x|^{\beta r} w_k)$. Let $u = I_{\alpha}^k f$. It is easy to see that $u = K_{\alpha,k} * f$. Now let us assume that $u_m = K^t_{\alpha,k} * f_m$ and $u^t = K_{\alpha,k} * f$. Consider

$$\|(u_m - u) |x|^{\gamma} \|_{L^r(\mathcal{K}, w_k)} \leq \|(u_m - u_m^t) |x|^{\gamma} \|_{L^r} + \|(u_m^t - u^t) |x|^{\gamma} \|_{L^r} + \|(u^t - u) |x|^{\gamma} \|_{L^r}.$$ 

Using Lemma 7.1

$$\|(u_m - u_m^t) |x|^{\gamma} \|_{L^r(\mathcal{K}, w_k)} = \|(K_{\alpha,k} * f_m - K^t_{\alpha,k} * f_m) |x|^{\gamma} \| \leq C t^\delta \|x|^{\beta} f_m\|_{r, w_k} \leq D t^\delta.$$
Similarly, \[
\|(u^t - u)|x|^\gamma\|_{L^\gamma(K,\mathbb{R}^d)} \leq C t^\beta \|x^\gamma f\| r, w_k \leq D t^\beta.
\]
Choose \(\epsilon > 0\). For very small \(t > 0\), each of the above estimates can be made less that \(\frac{\epsilon}{10}\) for all \(m\). We are left to get bound for \(\|(u^t_m - u^t)|x|^\gamma\|_{L^\gamma(K,\mathbb{R}^d)}\).

For radial functions \(f(x) = f_0(|x|) \in \mathcal{S}(\mathbb{R}^d)\) (see [14]), one has
\[
\tau_y^k f(x) = \int_{\mathbb{R}^d} f_0(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle})d\mu^k_\epsilon(\eta),
\]
(7.5)
where \(d\mu^k_\epsilon(\eta)\) is a probability Borel measure on \(\mathbb{R}^d\), whose support is contained in \(co(GG.x)\), the convex hull of the \(G\)-orbit of \(x\) in \(\mathbb{R}^d\).

Applying (7.5), for the function \(K_{\alpha,k}\),
\[
\tau_y^k K^\epsilon_{\alpha,k}(x) = \int_{\mathbb{R}^d} K^\epsilon_{\alpha,k}(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle})d\mu^k_\epsilon(\eta)
\]
\[
= \int_{\mathbb{R}^d} \frac{(c^k_\epsilon)^{-1}}{|x|^2 + |y|^2 - 2\langle y, \eta\rangle} \chi_{\{|y|^2 + |y|^2 - 2\langle y, \eta\rangle \geq t\}}(\eta)d\mu^k_\epsilon(\eta)
\]
\[
= \int_{\eta : A(x,y,\eta) \geq t} \frac{(c^k_\epsilon)^{-1}}{A(x,y,\eta)^{d_k-\alpha}}d\mu^k_\epsilon(\eta),
\]
(7.6)
where \(A(x,y,\eta) = \sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle}\). Then
\[
\tau_y^k K^\epsilon_{\alpha,k}(x) \leq \frac{(c^k_\epsilon)^{-1}}{t^{d_k-\alpha}} d\mu_\epsilon\{|\eta : A(x,y,\eta) \geq t\}\leq \frac{(c^k_\epsilon)^{-1}}{t^{d_k-\alpha}},
\]
(7.7)
since \(d\mu^k_\epsilon\) is a probability measure.

It is proved in [1] that
\[
\min_{g \in G} |g.x - y| \leq A(x,y,\eta) \leq \max_{g \in G} |g.x - y|, \forall x, y \in \mathbb{R}^d \text{ and } \eta \in co(GG.x).
\]
(7.8)
Let \(y \in K\), where \(K\) is a compact set in \(\mathbb{R}^d\). Then there exists \(R > 0\) such that \(K \subset B(0,R)\).

Consider
\[
\int_{\mathbb{R}^d} |\tau_y^k K^\epsilon_{\alpha,k}(x)|^\gamma |x|^{-\beta r} w_k(x)dx
\]
\[
= \int_{|x| \leq 2R} |\tau_y^k K^\epsilon_{\alpha,k}(x)|^\gamma |x|^{-\beta r} w_k(x)dx + \int_{|x| > 2R} |\tau_y^k K^\epsilon_{\alpha,k}(x)|^\gamma |x|^{-\beta r} w_k(x)dx
\]
\[
= I_1(y) + I_2(y).
\]
(7.9)
Substituting the bound for \(\tau_y^k K^\epsilon_{\alpha,k}\) from (7.7) in \(I_1(y)\), we have
\[
I_1(y) \leq \frac{(c^k_\epsilon)^{-\gamma}}{t^{\gamma(d_k-\alpha)}} \int_{|x| \leq 2R} |x|^{-\beta r} w_k(x)dx = \frac{(c^k_\epsilon)^{-\gamma}}{t^{\gamma(d_k-\alpha)}} (2R)^{d_k-\beta r} = M_1,
\]
using property (4) of section 2 with the condition $\beta r' < d_k$ i.e., $\beta < \frac{d_k}{r'}$. On the other hand, for the integral $I_2(y)$, substituting the value of $\tau_y^k K^t_{\alpha,k}$ from (7.6) and then using (7.8), we get

$$I_2(y) = \int_{|x|>2R} \left| \int_{|y|>2R} \frac{(c_{k}^{-1}) \rho x_k \mu_k(x, \eta)}{d_k} |x|^{-\beta r'} w_k(x) dx \right| |r'|$$

$$\leq \int_{|x|>2R} \left| \int_{|y|>2R} \frac{(c_{k}^{-1}) \rho x_k \mu_k(x, \eta)}{d_k} |x|^{-\beta r'} w_k(x) dx \right| |r'|$$

$$= (c_{k}^{-1})^{-r'} \int_{|x|>2R} \frac{|x|^{-\beta r'}}{|r'(d_k - \alpha)|} w_k(x) dx. \quad (7.10)$$

Since $g$ is a reflection, we observe that

$$|g.x - y| \geq |g.x| - |y| = |x| - |y|, \quad \forall \, g \in G,$$

If $y \in K$ and $x$ is in the integration region of $I_2(y)$, then $|y| \leq R \leq \frac{|x|}{2}$. So $|g.x - y| \geq \frac{|x|}{2}$. Therefore, $\min_{g \in G} |g.x - y| \geq \frac{|x|}{2}$ for all $y \in K$. From (7.10),

$$I_2(y) \leq (c_{k}^{-1})^{-r'} 2^{r'(d_k - \alpha)} \int_{|x|>2R} |x|^{-\beta r'} |x|^{-\beta r'} w_k(x) dx \leq M_2,$$

if $d_k < \beta r' + r'(d_k - \alpha)$ i.e., if $\alpha < \beta + \frac{d_k}{r'}$. Thus from (7.9),

$$\int_{\mathbb{R}^d} |\tau_y^k K^t_{\alpha,k}(x)|^r |x|^{-\beta r'} w_k(x) dx \leq M_1 + M_2, \quad \forall \, y \in \mathbb{R}^d. \quad (7.11)$$

In particular, we have proved that $\tau_y^k K^t_{\alpha,k} \in L^r(\mathbb{R}^d, |x|^{-\beta r'} w_k), \forall \, y \in K$. Since $f_m$ converges weakly to $f$ in $L^r(\mathbb{R}^d, |x|^{-\beta r'} w_k)$, $K^t_{\alpha,k} * f_m(y)$ converges to $K^t_{\alpha,k} * f(y)$ as a sequence of complex numbers. Thus $u^t_m(y)$ converges to $u^t(y)$ for all $y \in K$. Moreover, since

$$u^t_m(y) = K^t_{\alpha,k} * f_m(y) = \int_{\mathbb{R}^d} f_m(x) \tau_y^k K^t_{\alpha,k}(y) w_k(x) dx,$$

using (7.11) and the fact that $f_m$ is a bounded sequence in $L^r(\mathbb{R}^d, |x|^{\gamma} w_k)$,

$$|u^t_m(y)| \leq \left( \int_{\mathbb{R}^d} |f_m(x)|^r |x|^{\beta r'} w_k(x) dx \right)^{\frac{1}{r'}} \left( \int_{\mathbb{R}^d} |\tau_y^k K^t_{\alpha,k}(y)|^r |x|^{-\beta r'} w_k(x) dx \right)^{\frac{1}{r'}} \leq A.$$
Existence of an extremal of Sobolev inequality associated with Dunkl gradient and of Stein-Weiss inequality for D-Riesz potential

property (4) of section 2). Hence we can make it less than \( \varepsilon \) for large \( m \). Thus

\[
\| u_m - u \|_{L^s(K, \| \cdot \|_w)} \leq \varepsilon
\]

for large \( m \), proving that \( u_m \) converges to \( u \) strongly on \( L^s(K, \| \cdot \|_w) \). This completes the proof that the embedding (7.2) is compact. \( \Box \)

Now using Theorem 6.1 and Theorem 7.1, we are ready to prove our main result that (1.5) has a maximizer.

**Theorem 7.2.** Let \( d \geq 2 \), \( 2 < s < \infty \), \( -d_k^2 < \gamma < \beta \), \( 0 < \beta < \frac{d_k^2}{2} \) and the relation

\[
\frac{1}{s} - \frac{1}{2} = \frac{\beta - \gamma - \alpha}{d_k}
\]

holds. Then there exists a maximizer for \( W_k \).

**Proof.** Let \( \{ f_j \} \in \mathbb{N} \) be a minimizing sequence for \( W_k \), which we can take to be normalized i.e.,

\[
\| x^\beta f_j \|_{L^2(\mathbb{R}^d, w_k)} = 1 \quad \text{and} \quad \| x^\gamma I^k_{\alpha} f_j \|_{L^s(\mathbb{R}^d, w_k)} \to W_k.
\]

Now, in Theorem 6.1, we set

\[
\mu = \frac{d_k}{2} + \beta - \alpha,
\]

and choose \( \theta \) such that

\[
\max \left\{ \frac{2}{s}, \frac{\mu}{\mu + \alpha}, \frac{\gamma}{\beta} \right\} < \theta < 1.
\]

Because of relation (7.12), equation (6.2) holds for this particular choice of \( \mu, \theta \). It is also easy to see remaining conditions of Theorem 6.1 hold under the hypothesis of the Theorem and the choice of \( \mu, \theta \). Also since relation (7.14) holds, from (5.6),

\[
\| f_j \|_{\dot{B}^{-\mu-\alpha \cdot k}_\infty, \infty} = \sup_{t > 0} t^{\frac{\mu+\alpha}{s}} \| e^{t \Delta_k} f_j \|_{L^\infty} = \sup_{t > 0} t^{\frac{1}{2} \left( \frac{d_k}{2} + \beta \right)} \| e^{t \Delta_k} f_j \|_{L^\infty} \leq C_{d, \beta, \theta} \| x^\beta f_j \|_{L^2} = C_{d, \beta, \theta},
\]

showing that \( f_j \in \dot{B}^{-\mu-\alpha \cdot k}_\infty \). Hence we apply Theorem 6.1 and use (7.13) to get,

\[
\| f_j \|_{\dot{B}^{-\mu-\alpha \cdot k}_\infty, \infty} \geq C_k > 0.
\]

In other words,

\[
\sup_{t > 0} t^{\frac{\mu+\alpha}{s}} \| e^{t \Delta_k} f_j \|_{L^\infty} \geq C_k > 0.
\]

It then follows that for each \( j \in \mathbb{N} \), there exists \( t_j > 0 \) such that

\[
\frac{t_j^{\mu+\alpha}}{2} \| e^{t_j \Delta_k} f_j \|_{L^\infty} \geq C_k.
\]

Now, we define

\[
\tilde{f}_j(x) := t_j^{-\frac{d_k}{2} + \beta} f_j(t_j^{\frac{1}{2}} x).
\]

Then it is easy to see that

\[
\| x^\beta \tilde{f}_j \|_{L^2(\mathbb{R}^d, w_k)} = \| x^\beta f_j \|_{L^2(\mathbb{R}^d, w_k)}.
\]

Also,

\[
\| x^\gamma I^k_{\alpha} \tilde{f}_j \|_{L^s(\mathbb{R}^d, w_k)} = \| x^\gamma I^k_{\alpha} f_j \|_{L^s(\mathbb{R}^d, w_k)}.
\]
Indeed, we have

\[ I^k_\alpha \tilde{f}_j(x) = (c^k_\alpha)^{-1} \int_{\mathbb{R}^d} \tilde{f}_j(y) \tau^k_y |\alpha - d_k(\alpha - d_k)| w_k(y) dy \]

\[ = (c^k_\alpha)^{-1} t_j^{1/2} (\frac{d_k}{2} + \beta) \int_{\mathbb{R}^d} f_j(t_j^{1/2} y) \Phi(x, y) w_k(y) dy \]

\[ = (c^k_\alpha)^{-1} t_j^{1/2} (\frac{d_k}{2} + \beta) t_j^{-d_k/2} \int_{\mathbb{R}^d} f_j(y) \Phi(x, t_j^{1/2} y) w_k(y) dy \]

\[ = (c^k_\alpha)^{-1} t_j^{1/2} (\frac{d_k}{2} + \beta) t_j^{-d_k/2} \int_{\mathbb{R}^d} f_j(y) \Phi(t_j^{1/2} y, x) w_k(y) dy \]

\[ = (c^k_\alpha)^{-1} t_j^{1/2} (\frac{d_k}{2} + \beta) t_j^{-d_k/2} \int_{\mathbb{R}^d} f_j(y) \Phi(y, t_j^{1/2} x) w_k(y) dy \]

\[ = t_j^{1/2} (\frac{d_k}{2} + \beta) \int_{\mathbb{R}^d} f_j(y) \alpha_s \tilde{\Phi}(x, y) w_k(y) dy \]

Then by substituting the value of \( \gamma \) from (7.12),

\[ ||x|^\gamma I^k_\alpha \tilde{f}_j||_{s,w_k} = t_j^{\frac{d_k}{2} + \beta} \int_{\mathbb{R}^d} |I^k_\alpha f_j(t_j^{1/2} x)|^s w_k(x) dx \]

\[ = t_j^{\frac{d_k}{2} + \beta} t_j^{-d_k/2} ||x|^\gamma I^k_\alpha f_j||_{s,w_k} \]

\[ = ||x|^\gamma I^k_\alpha f_j||_{s,w_k} ^s, \]

thus proving (7.16). As a consequence, from (7.13), we have

\[ ||x|^\beta \tilde{f}_j||_{L^2(\mathbb{R}^d, w_k)} = 1 \text{ and } ||x|^\gamma I^k_\alpha \tilde{f}_j||_{L^s(\mathbb{R}^d, w_k)} \rightarrow W_k, \quad (7.17) \]

which shows that \( \{ \tilde{f}_j \}_j \) is also a maximizing sequence sequence for \( W_k \).

Moreover, using (2.2), we observe that

\[ e^{1.\Delta_k} \tilde{f}_j(x) = \int_{\mathbb{R}^d} \tilde{f}_j(y) \tau^k_y q^k_1(x) w_k(y) dy \]

\[ = t_j^{1/2} (\frac{d_k}{2} + \beta) t_j^{-d_k/2} \int_{\mathbb{R}^d} f_j(y) (\tau^k_y q^k_1)(x) w_k(y) dy \]

\[ = t_j^{1/2} (\frac{d_k}{2} + \beta) t_j^{-d_k/2} \int_{\mathbb{R}^d} f_j(y) 2^{d_k/2} e^{-|x|^2 + \frac{1}{2(\tau^k_y q^k_1)^2}} E_k \left( \frac{x}{\sqrt{2}} \right) w_k(y) dy \]

\[ = t_j^{1/2} (\frac{d_k}{2} + \beta) \int_{\mathbb{R}^d} f_j(y) (\tau^k_y q^k_1)(t_j^{1/2} x) w_k(y) dy \]

\[ = t_j^{1/2} (\frac{d_k}{2} + \beta) (e^{t_j\Delta_k} f_j)(t_j^{1/2} x). \]
Using (7.13),
\[
\|e^{t\Delta_h} f_j\|_{L^\infty} = t_j^{\frac{1}{2} + \beta} \|\alpha_j^\gamma a_j f_j\|_{L^\infty} = t_j^{\frac{1}{2} + \beta} \|\gamma_j^\alpha a_j f_j\| \geq \frac{C_k}{2} > 0.
\]
(7.18)
Since \(\{f_j\}_j\) is a bounded sequence in \(L^2(\mathbb{R}^d, |x|^{2\beta} w_k)\), by reflexivity, it has a subsequence still denoted by \(f_j\) such that \(f_j\) converges weakly to a function \(h\) in \(L^2(\mathbb{R}^d, |x|^{2\beta} w_k)\). Our aim is to show that \(h\) is indeed a maximizer for \(W_k\), that is,
\[
\|\|x|^\beta h\|_{L^2(\mathbb{R}^d,w_k)} = 1 \quad \text{and} \quad \|\|x|^\gamma h\|_{L^s(\mathbb{R}^d,w_k)} = W_k.
\]
(7.19)
Now we set
\[u_j := I_\alpha^k f_j \quad \text{and} \quad v := I_\alpha^k h.\]
Since by Theorem 5.1 \(e^{t\Delta_h}\) is a compact operator from \(L^2(\mathbb{R}^d, |x|^{2\beta} w_k)\) into \(L^\infty(\mathbb{R}^d)\), passing through a subsequence, we have \(e^{t\Delta_h} f_j\) converges strongly to \(e^{t\Delta_h} h\) in \(L^\infty(\mathbb{R}^d)\). Then from (7.13), \(\|e^{t\Delta_h} h\|_{L^\infty} \geq \frac{C_k}{2} > 0\), which implies that \(h \neq 0\). Again by taking \(r = 2, \beta = \gamma\) in Theorem 7.1 we observe that all the conditions of Theorem 7.1 are satisfied under the hypothesis of the given Theorem and therefore, if \(K\) is a compact set in \(\mathbb{R}^d\), we have the compact embedding
\[
\tilde{H}^{\alpha,2}_{\beta,k}(\mathbb{R}^d) \subset L^s(K, |x|^{\beta s} w_k).
\]
By observing that \(u_j\) is a bounded sequence in \(\tilde{H}^{\alpha,2}_{\beta,k}(\mathbb{R}^d)\) and thereafter following the proof of Theorem 7.1 we can show that \(u_j\) converges strongly to \(v\) in \(L^s(K, |x|^{\beta s} w_k)\). Therefore, up to a subsequence, \(u_j\) converges to \(v\) a.e. in \(K\) and by using diagonal argument, further, up to a subsequence, \(u_j\) converges to \(v\) a.e. in \(\mathbb{R}^d\).

Now proceeding exactly, as in the proof of Theorem 5.1 in [12], we can prove that
\[
\|\|x|^\beta h\|_{L^2(\mathbb{R}^d,w_k)} = 1 \quad \text{and} \quad f_j \to h \text{ strongly in } L^2(\mathbb{R}^d, |x|^{2\beta} w_k).
\]
Since by Theorem 1.2 the operator \(I_\alpha^k\) is continuous from \(L^2(\mathbb{R}^d, |x|^{2\beta} w_k)\) into \(L^s(\mathbb{R}^d, |x|^{\gamma s} w_k)\), hence
\[
u_j \to v \text{ strongly in } L^s(\mathbb{R}^d, |x|^{\gamma s} w_k).
\]
This implies that \(\|u_j\| \to \|v\|\) in \(L^s(\mathbb{R}^d, |x|^{\gamma s} w_k)\). Now (7.19) will follow from (7.17), which completes the proof. 

\[\Box\]

REFERENCES

1. B. Amri and M. Sifi, Riesz transforms for Dunkl transform, Ann. Math. Blaise Pascal, 19 (2012), 247-262.
2. J.P. Anker, J. Dziubaski and A. Hejna, Harmonic Functions, Conjugate Harmonic Functions and the Hardy Space \(H^1\) in the Rational Dunkl Setting, Journal of Fourier Analysis and Applications https://doi.org/10.1007/s00041-019-09666-0.
3. C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311,(1989), 167-183.
4. C. F. Dunkl, Integral Kernels with reflection group invariance, Canad. J. Math. 43 (1991), 1213-1227.
5. R. L. Frank, Sobolev inequalities and uncertainty principles in mathematical physics, Part I [http://www.math.caltech.edu/~rlfrank/sobweb1.pdf] pp. 1601-1610.
6. D. V. Gorbachev, V. I. Ivanov, and S. YU. Tikhonov, *Positive $L^p$–bounded Dunkl-type generalized translation operator and its applications*, Constr. Approx. doi.org/10.1007/s00365-018-9435-5.

7. D. V. Gorbachev, V. I. Ivanov, and S. YU. Tikhonov, *Riesz Potential and maximal function for Dunkl transform* CRM Preprint Series number 1238, 2018.

8. S. Hassani, S. Mustapha naand M. Siphi, *Riesz potentials and fractional maximal function for the Dunkl transform*, J. Lie Theory. 19 (2009), 725-734.

9. M. Ledoux, *On improved Sobolev embedding theorems*, Math. Res. Lett. 10 (2003), 659-669.

10. E. H. Lieb and M. Loss, *Analysis, Second edition*, Graduate studies in Mathematics, Volume 14, American Mathematical Society, 2001.

11. V. G. Maz’ya *Sobolev spaces*, Springer, Berlin, 1985.

12. P. D. Napoli, I. Drelichman, and A. Salort *Weighted inequalities for the Fractional Laplacian and the existence of extrema*, Commun. Contemp. Math. doi.org/10.1142/S0219199718500347.

13. M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), 519-542.

14. M. Rösler, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc. 355 (2003), 2413-2438.

15. L. Saloff-Coste *Aspects of Sobolev-type inequalities*, Volume 289, London Mathematical Society, Lecture note, Cambridge University Press, 2002.

16. E. Stein and G. Weiss, *Fractional integrals on n-dimensional euclidean space*, J. Math. Mech. 7 (1958), 503-514.

17. G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura. Appl. 110 (1976), 353-372.

18. S. Thangavelu and Y. Xu, *Convolution operator and maximal function for the Dunkl transform*, Journal dAnalyse Mathmatique, 2005, Volume 97, Issue 1, pp 255-5.

19. S. Thangavelu and Y. Xu, *Riesz transform and Riesz potentials for Dunkl transform*, J. Comput. Appl. Math. 199 (2007), 181-195.

20. A. Velicu, *Sobolev-type inequality for Dunkl operator*, arXiv:1811.11118 [math.FA].

Saswata Adhikari, School of Mathematical Sciences, NISER Bhubaneswar, Odisha-752050, India.

E-mail address: saswata.adhikari@gmail.com

V. P. Anoop, School of Mathematical Sciences, NISER Bhubaneswar, Odisha-752050, India.

E-mail address: anoop.vp@niser.ac.in

Sanjay Parui, School of Mathematical Sciences, NISER Bhubaneswar, Odisha-752050, India.

E-mail address: parui@niser.ac.in