Gauge invariance and amplitudes of two-photon processes

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Abstract

A method to derive the convenient representations for many two-photon amplitudes is suggested. It is based on the use of the gauge in which the photon propagator has only space components. The amplitudes obtained have no any strong numerical cancellations and, therefore, are very convenient for numerical evaluations. Our approach is illustrated by the consideration of the processes $e^+e^-\rightarrow e^+e^-e^+e^-$, $e^+e^-\rightarrow e^+e^-\mu^+\mu^-$, and $e^+e^-\rightarrow \mu^+\mu^-\pi^+\pi^-$. The method is extended on the case of polarized particles. The amplitudes obtained in this approach have been employed for extension of the event generator developed by F.A. Berends, P.H. Daverveldt, and R. Kleiss.

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I. INTRODUCTION

Investigations of processes $e^+e^- \rightarrow e^+e^-X$ (two-photon processes), where $X$ are pairs of leptons ($e, \mu, \tau$) or hadrons with C-parity $C = +1$ give an important experimental information on physics of $\gamma\gamma$ collisions. Such investigations are the important parts of the physical programs of C- and B- factories. Since the leptonic two-photon processes are completely described by QED, they provide firstly a possibility to test QED in higher orders of the perturbation theory and secondly are important for calibration of the experiments and suppression of backgrounds.

There are a few generators that simulate leptonic two-photon processes beyond the equivalent photon approximation [1–5]. The most popular is a generator of F.A. Berends, P.H. Daverveldt, and R. Kleiss (DIAG36 or BDK generator) [1]. It includes all Feynman diagrams and accounts for identity effect. It is worth noting that this effect is not taken into account in other two-photon event generators.

The BDK prediction of the identity effect in $e^+e^- \rightarrow 2(e^+e^-)$ process has been presented in Ref. [6]. The unexpected growth of the effect with the cut on the invariant mass of produced $e^+e^-$ pair motivated us to make calculations with the BDK generator in which the alternative matrix elements expression were embedded. The BDK prediction on the identity effect were confirmed at the increased accuracy of the numerical calculations.

These alternative matrix element expressions are presented below. Unlike to those used in BDK [7], they contain only spatial components of the involved particle momenta which ensure their numerical stability. Another advantage of them is an easiness of the modification for semi-leptonic final states. Simulation of the semi-leptonic processes such as $e^+e^- \rightarrow \mu^+\mu^-\pi^+\pi^-$ became actual in context of BELLE analysis of $e^+e^- \rightarrow \Upsilon(nS)\pi^+\pi^-$ with $\Upsilon(nS) \rightarrow l^+l^-$ [8]. In this case two-photon processes are main sources of the background.

II. METHOD OF CALCULATION

First of all, to obtain the explicit form of the matrix element, it is necessary to introduce some convenient basis for the Dirac spinors. Since the number of the Feynman diagrams is quite large, for numerical calculations it is important to have the most simple expression
for the total amplitude. Then, the following circumstance must be taken into account. In
the two-photon processes, the main contribution to the cross section is given by kinematics
where a virtual photon is emitted almost along the initial electron momentum \( p \). In such
kinematics the corresponding current \( J_\mu \) has large components \( J_0 \) and \( J_\parallel \), i.e., \( J_0 \sim J_\parallel \sim \gamma J_\perp \sim \gamma^2 (J_0 - J_\parallel) \), where \( J_\parallel = J \cdot p/p \) and \( \gamma = \varepsilon/m_e \) (\( m_e \) is the electron mass, we set \( \hbar = c = 1 \)). As a result, using the Feynman gauge leads to a huge numerical cancellation
between \( J_0^2 \) and \( J_\parallel^2 \) at high energies, which is a problem for numerical calculations with high
accuracy. In our work we avoid this problem by means of the gauge in which the photon
propagator \( D_{\mu\nu}(k) \) contains only spatial components:

\[
D_{ij}(k) = -\frac{1}{\omega^2 - k^2 + i 0} \left( \delta_{ij} - \frac{k_i k_j}{\omega^2} \right), \quad D_{0\mu}(k) = 0, \quad k^\mu = (\omega, k).
\] (1)

To calculate the amplitude of the process, we use the following explicit form of the positive-
energy Dirac spinor \( U_{a,\lambda} \) and negative-energy Dirac spinor \( V_{b,\lambda} \):

\[
U_{a,\lambda} = N_a \left( \begin{pmatrix} \phi_{\lambda a} \\ \sigma \cdot P_a \phi_{\lambda a} \end{pmatrix} \right), \quad V_{b,\lambda} = N_b \left( \begin{pmatrix} \sigma \cdot P_b \phi_{\lambda b} \\ \phi_{\lambda a} \end{pmatrix} \right),
\]

\[
P_a = \frac{p_a}{\varepsilon_a + m}, \quad P_b = \frac{p_b}{\varepsilon_b + m}, \quad N_a = \sqrt{\frac{\varepsilon_a + m}{2\varepsilon_a}}, \quad N_b = \sqrt{\frac{\varepsilon_b + m}{2\varepsilon_b}}.
\] (2)

Here \( \sigma \) are Pauli matrices, \( p_a \) and \( \varepsilon_a \) are the electron momentum and energy, \( p_b \) and \( \varepsilon_b \)
are the positron momentum and energy, \( \phi_{\lambda} \) is the two-component spinor, \( \lambda = \pm 1 \) denotes
two possible polarizations, the same basis of spinors for all particles is used. We introduce
three unit orthogonal vectors \( e_x, e_y, e_z \) (so that \( [e_x \times e_y] = e_z \)) and choose \( \phi_{\lambda} \) to be the
eigenstate of the operator \( \sigma \cdot e_z \), i.e., \( (\sigma \cdot e_z) \phi_{\lambda} = \lambda \phi_{\lambda} \). It is convenient to direct \( e_z \) along
the momentum of initial electron. Then we have

\[
\phi_{\lambda a} \phi_{\lambda b}^\dagger = \frac{1}{2} (A_{ab} + \sigma \cdot B_{ab} ),
\]

\[
A_{ab} = \delta_{\lambda a \lambda b}, \quad B_{ab} = \lambda_a \delta_{\lambda a \lambda b} e_z + \delta_{\lambda a \lambda b} (e_x + i\lambda_a e_y), \quad \bar{\lambda} = -\lambda.
\] (3)

Using the quantities \( A_{ab} \) and \( B_{ab} \), we obtain the matrices \( U_{a,\bar{a}} \hat{U}_b, U_{a,\bar{a}} \hat{V}_b, V_{a,\bar{a}} \hat{U}_b, V_{a,\bar{a}} \hat{V}_b \), which are
necessary for further calculations:

\[
U_{a,\bar{a}} \hat{U}_b = \frac{1}{4} N_a N_b [\gamma^0 f^{(0)}_{ab} - f^{(1)}_{ab} + \gamma^0 \gamma^5 f^{(2)}_{ab} + \gamma^5 f^{(3)}_{ab} + \gamma^0 \gamma^\Sigma \cdot g^{(0)}_{ab} - \Sigma \cdot g^{(1)}_{ab} - \gamma \cdot g^{(2)}_{ab} - \alpha \cdot g^{(3)}_{ab}].
\]
\[
V_a \bar{V}_b = \frac{1}{4} N_a N_b \left[ \gamma^0 f^{(0)}_{ab} + f^{(1)}_{ab} + \gamma^5 f^{(2)}_{ab} - \gamma^5 f^{(3)}_{ab} \right] + \gamma^0 \Sigma \cdot g^{(0)}_{ab} + \Sigma \cdot g^{(1)}_{ab} - \gamma \cdot g^{(2)}_{ab} + \alpha \cdot g^{(3)}_{ab} ,
\]
\[
V_a \bar{U}_b = \frac{1}{4} N_a N_b \left[ \gamma^0 \gamma^5 f^{(0)}_{ab} + \gamma^5 f^{(1)}_{ab} + \gamma^0 f^{(2)}_{ab} - \gamma^5 f^{(3)}_{ab} \right] - \gamma \cdot g^{(0)}_{ab} - \alpha \cdot g^{(1)}_{ab} + \gamma^0 \Sigma \cdot g^{(2)}_{ab} - \Sigma \cdot g^{(3)}_{ab} ,
\]
\[
U_a \bar{V}_b = \frac{1}{4} N_a N_b \left[ \gamma^0 \gamma^5 f^{(0)}_{ab} - \gamma^5 f^{(1)}_{ab} + \gamma^0 f^{(2)}_{ab} + \gamma^5 f^{(3)}_{ab} \right] - \gamma \cdot g^{(0)}_{ab} + \alpha \cdot g^{(1)}_{ab} + \gamma^0 \Sigma \cdot g^{(2)}_{ab} + \Sigma \cdot g^{(3)}_{ab} .
\] (4)

Here

\[
f^{(0)}_{ab} = (P_a P_b + 1) A_{ab} - i [P_a \times P_b] \cdot B_{ab} , \quad f^{(2)}_{ab} = (P_a + P_b) \cdot B_{ab} ,
\]
\[
g^{(0)}_{ab} = (B_{ab} \cdot P_a) P_b + (B_{ab} \cdot P_b) P_a - (P_a \cdot P_b - 1) B_{ab} + i [P_a \times P_b] A_{ab} ,
\]
\[
g^{(2)}_{ab} = (P_a + P_b) A_{ab} - i [B_{ab} \times (P_a - P_b)] ,
\]
\[
f^{(1)}_{ab} = -f^{(0)}_{ab} (P_b \rightarrow -P_b) , \quad f^{(3)}_{ab} = -f^{(2)}_{ab} (P_b \rightarrow -P_b) ,
\]
\[
g^{(1)}_{ab} = -g^{(0)}_{ab} (P_b \rightarrow -P_b) , \quad g^{(3)}_{ab} = -g^{(2)}_{ab} (P_b \rightarrow -P_b) .
\] (5)

All Feynman diagrams have a block structure and can be easily expressed via several quantities. For one-photon emission (or absorption) and annihilation these quantities are

\[
\bar{U}_b V_a = \bar{V}_b U_a = N_a N_b g^{(0)}_{ab} , \quad \bar{U}_b V_a = \bar{V}_b U_a = N_a N_b g^{(2)}_{ab} .
\] (6)

For two-photon emission (or absorption) and annihilation, the block terms are

\[
S^{(1)ij}_{ab}(k) \equiv (k^2 + 2 k p_b)^{-1} \bar{U}_b \gamma^i (\hat{p}_b + \hat{k} + m) \gamma^j U_a = N_a N_b (k^2 + 2 k p_b)^{-1} \times \left[ 2 p^j_b g^{(2)ij}_{ab} + (\delta^{ij} f^{(0)}_{ab} - i e^{ijl} g^{(0)lj}_{ab}) \omega + i e^{ijl} k^l f^{(2)}_{ab} + g^{(2)ij} k^i + g^{(2)ij} k^j - \delta^{ij} g^{(2)k}_{ab} \cdot k \right] ,
\]
\[
S^{(2)ij}_{ab}(k) \equiv (k^2 - 2 k p_b)^{-1} \bar{U}_b \gamma^i (\hat{p}_b + \hat{k} + m) \gamma^j U_a = N_a N_b (k^2 - 2 k p_b)^{-1} \times \left[ -2 p^j_b g^{(2)ij}_{ab} + (\delta^{ij} f^{(0)}_{ab} - i e^{ijl} g^{(0)lj}_{ab}) \omega + i e^{ijl} k^l f^{(2)}_{ab} + g^{(2)ij} k^i + g^{(2)ij} k^j - \delta^{ij} g^{(2)k}_{ab} \cdot k \right] ,
\]
\[
S^{(3)ij}_{ab}(k) \equiv (k^2 + 2 k p_b)^{-1} \bar{U}_b \gamma^i (-\hat{p}_b + \hat{k} + m) \gamma^j U_a = N_a N_b (k^2 + 2 k p_b)^{-1} \times \left[ 2 p^j_b g^{(2)ij}_{ab} + (\delta^{ij} f^{(0)}_{ab} - i e^{ijl} g^{(0)lj}_{ab}) \omega + i e^{ijl} k^l f^{(2)}_{ab} + g^{(2)ij} k^i + g^{(2)ij} k^j - \delta^{ij} g^{(2)k}_{ab} \cdot k \right] ,
\]
\[
S^{(4)ij}_{ab}(k) \equiv (k^2 - 2 k p_b)^{-1} \bar{U}_b \gamma^i (-\hat{p}_b + \hat{k} + m) \gamma^j U_a = N_a N_b (k^2 - 2 k p_b)^{-1}
\]
\[
\times \left[-2p_i^j g_{ab}^{(0)i} + (\delta_{ij} f_{ab}^{(2)} - i \epsilon^{ijl} g_{kl}^{(2)})(\omega + i \epsilon^{ijl} k_l) \right. \\
\left. + i \epsilon^{ijl} k_l + g_{ab}^{(0)i} k_l + g_{ab}^{(0)i} k_l - \delta_{ij} g_{ab}^{(0)} \cdot k \right].
\] (7)

By means of Eqs. (3)–(7), one can easily write the explicit expressions for the amplitudes of huge amount of processes. These expressions are rather simple and have no any strong numerical cancellations. Therefore, they are very convenient for numerical evaluations.

Consider, as an example, the process \(e^+e^- \rightarrow e^+e^-e^+e^-\). Let \(p_1, p_2, p_3\) be the momenta of initial electron and two final electrons, and \(p_4, p_5, p_6\) be the momenta of the initial positron and two final positrons, respectively. Typical Feynman diagrams, which contribute to the amplitude of the process, are shown in Fig. 1. The differential cross section, averaged over polarizations of the initial particles and summed up over polarizations of the final particles, reads

\[
d\sigma = \frac{\alpha^4}{4\pi^4} \frac{dp_1 dp_2 dp_3 dp_5 dp_6}{d\omega} \delta(p_2 + p_3 + p_5 + p_6 - p_1 - p_4) \sum_{\lambda_1} \sum_{a=1}^5 T_{\{\lambda_1\}}^{(a)} \Big|\sum_{a=1}^5 T_{\{\lambda_1\}}^{(a)}\Big|^2,
\]

\[
T_{\{\lambda_1\}}^{(a)} = t_{123456}^{(a)} - t_{132456}^{(a)} - t_{123465}^{(a)} + t_{132465}^{(a)}, \quad j = \frac{\sqrt{(p_1 p_4)^2 - m^4}}{\varepsilon_1 \varepsilon_4}.
\] (8)

Here \(\alpha\) is the fine structure constant, \(t^{(1)}\) is the contribution of the diagram shown in Fig. 1(a), \(t^{(2)}\) is the contribution of the diagram shown in Fig. 1(b), \(t^{(3)}\) is similar to \(t^{(2)}\) but with the photon emission from positron line, \(t^{(4)}\) corresponds to one-photon annihilation of initial \(e^+e^-\) pair with subsequent virtual photon decay into two \(e^+e^-\) pairs (Fig. 1(c)), and \(t^{(5)}\) corresponds to two-photon annihilation of initial \(e^+e^-\) pair with subsequent decay of each virtual photon into \(e^+e^-\) pair (Fig. 1(d)).

The amplitudes \(t^{(a)}\) have the following form

\[
t_{123456}^{(1)} = \left(6 \sum_{k=1}^{N_k} \right) \left[ g_{12}^{(2)i} - \frac{(p_2^i - p_1^i)(p_2 - p_1 \cdot g_{12}^{(2)})}{(\varepsilon_2 - \varepsilon_1)^2} \right] \left[ g_{34}^{(2)i} - \frac{(p_3^i - p_4^i)(p_3 - p_4 \cdot g_{34}^{(2)})}{(\varepsilon_5 - \varepsilon_4)^2} \right] \quad \times \frac{1}{(p_2 - p_1)^2(p_5 - p_4)^2} \left[ S_{63}^{(3)ij}(p_2 - p_1) + S_{63}^{(3)ji}(p_5 - p_4) \right],
\]

\[
t_{123456}^{(2)} = \left(6 \sum_{k=1}^{N_k} \right) \left[ g_{63}^{(0)i} - \frac{(p_3^i + p_6^i)(p_3 + p_6 \cdot g_{63}^{(0)})}{(\varepsilon_3 + \varepsilon_6)^2} \right] \left[ g_{54}^{(2)i} - \frac{(p_5^i - p_4^i)(p_5 - p_4 \cdot g_{54}^{(2)})}{(\varepsilon_5 - \varepsilon_4)^2} \right] \quad \times \frac{1}{(p_3 + p_6)^2(p_5 - p_4)^2} \left[ S_{12}^{(1)ij}(p_3 + p_6) + S_{12}^{(1)ji}(p_5 - p_4) \right],
\]

\[
t_{123456}^{(3)} = \left(6 \sum_{k=1}^{N_k} \right) \left[ g_{63}^{(0)i} - \frac{(p_3^i + p_6^i)(p_3 + p_6 \cdot g_{63}^{(0)})}{(\varepsilon_3 + \varepsilon_6)^2} \right] \left[ g_{12}^{(2)i} - \frac{(p_2^i - p_1^i)(p_2 - p_1 \cdot g_{12}^{(2)})}{(\varepsilon_2 - \varepsilon_1)^2} \right] \quad \times \frac{1}{(p_2 - p_1)^2(p_3 + p_6)^2} \left[ S_{54}^{(2)ij}(p_3 + p_6) + S_{54}^{(2)ji}(p_2 - p_1) \right],
\]
FIG. 1: Typical diagrams contributing to the amplitude of the process \( e^+e^- \rightarrow e^+e^-e^+e^- \); wavy lines correspond to photons and straight lines to electrons and positrons. (a) Additional \( e^+e^- \) pair is a result of annihilation of two virtual photons. (b) Emission of photon with its subsequent decay into \( e^+e^- \) pair. (c) One-photon annihilation of initial \( e^+e^- \) pair with subsequent decay of each virtual photon into two \( e^+e^- \) pairs. (d) Two-photon annihilation of initial \( e^+e^- \) pair with subsequent decay of each virtual photon into \( e^+e^- \) pair.

\[
\begin{align*}
\mathcal{I}^{(4)}_{123456} &= - \left( \prod_{k=1}^{6} N_k \right) \left[ \frac{\mathcal{g}_{63}^{(0)j}}{g_{63}} \left( \frac{(p_3^2 + p_6^2)(p_3 + p_6, g_{63}^{(0)})}{(\varepsilon_3 + \varepsilon_6)^2} \right) \right] \left[ \frac{\mathcal{g}_{14}^{(0)j}}{g_{14}} \left( \frac{(p_1^2 + p_4^2)(p_1 + p_4, g_{14}^{(0)})}{(\varepsilon_1 + \varepsilon_4)^2} \right) \right] \\
&\times \frac{1}{(p_3 + p_6)^2(p_1 + p_4)^2} \left[ S_{52}^{(3)ij}(p_3 + p_6) + S_{52}^{(3)ji}(-p_1 - p_4) \right], \\
\mathcal{I}^{(5)}_{123456} &= - \left( \prod_{k=1}^{6} N_k \right) \left[ \frac{\mathcal{g}_{63}^{(0)j}}{g_{63}} \left( \frac{(p_3^2 + p_6^2)(p_3 + p_6, g_{63}^{(0)})}{(\varepsilon_3 + \varepsilon_6)^2} \right) \right] \left[ \frac{\mathcal{g}_{52}^{(0)j}}{g_{52}} \left( \frac{(p_5^2 + p_2^2)(p_5 + p_2, g_{52}^{(0)})}{(\varepsilon_5 + \varepsilon_2)^2} \right) \right] \\
&\times \frac{1}{(p_3 + p_6)^2(p_2 + p_5)^2} \left[ S_{14}^{(4)ij}(p_3 + p_6) + S_{14}^{(4)ji}(p_2 + p_5) \right].
\end{align*}
\]

We now consider the process \( e^+e^- \rightarrow e^+e^-\mu^+\mu^- \). Let \( p_1 \) and \( p_4 \) are the initial electron and positron momenta, \( p_2 \) and \( p_5 \) are momenta of the final electron and positron, \( p_3 \) and \( p_6 \) are momenta of \( \mu^- \) and \( \mu^+ \). The cross section of the processes \( e^+e^- \rightarrow e^+e^-\mu^+\mu^- \) is given by Eq. (8) with the substitution \( T^{(a)} = \mathcal{I}^{(a)}_{123456} \), where \( \mathcal{I}^{(a)}_{123456} \) is given by Eq. (9) with \( p_{1,2,4,5}^2 = m_e^2 \) and \( p_{3,6}^2 = m_\mu^2 \).

It is easy to generalize our formulas to the case of polarized particles, which is actual
in context of the c-tau factory project \cite{10}. Let electron and positron are described by the Dirac spinors \( u_{a,\zeta_a} \) and \( v_{b,\zeta_b} \), where \( \zeta_a \) and \( \zeta_b \) are unit vectors directed along the electron and positron spin, respectively. Then we note that the following relations hold

\[
\sum_{\lambda_a} \frac{\varepsilon_a}{m} U_{a,\lambda_a} \bar{U}_{a,\lambda_a} = \frac{\hat{p}_a + m}{2m}, \quad \sum_{\lambda_b} \frac{\varepsilon_b}{m} V_{b,\lambda_b} \bar{V}_{b,\lambda_b} = \frac{\hat{p}_b - m}{2m}. \tag{10}
\]

Therefore, using the Dirac equation we can make the following transformation

\[
\begin{align*}
\left( \sum_{\lambda_a} \frac{\varepsilon_a}{m} U_{a,\lambda_a} \right) \cdot \sum_{\lambda_a} U_{a,\lambda_a} Z_{\lambda_a,\zeta_a}, \\
\left( \sum_{\lambda_b} \frac{\varepsilon_b}{m} V_{b,\lambda_b} \right) \cdot \sum_{\lambda_b} V_{b,\lambda_b} \tilde{Z}_{\lambda_b,\zeta_b}, \\
\end{align*}
\]

As a result, the cross section for the polarized initial electron and positron has the form

\[
d\sigma = \frac{\alpha^4}{\pi^4 j} d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_5 d\mathbf{p}_6 \delta(p_2 + p_3 + p_5 + p_6 - p_1 - p_4) \\
\times \sum_{\lambda_2,\lambda_3,\lambda_5,\lambda_6} \left| \sum_{\lambda_1,\lambda_4} \left( \sum_{a=1}^{5} T^{(a)}_{\{\lambda_1\}} \right) Z_{\lambda_1,\zeta_1} \tilde{Z}_{\lambda_4,\zeta_4} \right|^2. \tag{12}
\]

Then we use the relations

\[
\phi_{\lambda,\zeta} \phi_{\lambda,\zeta}^\dagger = \frac{1}{2}[1 + (\zeta \cdot \sigma)], \quad \chi_{4,\zeta_1} \chi_{4,\zeta_4}^\dagger = \frac{1}{2}[1 - (\zeta \cdot \sigma)], \\
\phi_{\lambda\lambda'} \phi_{\lambda'}^\dagger = \frac{1}{2}(1 + \lambda \sigma_3) \delta_{\lambda\lambda'} + \frac{1}{2}(\epsilon_{\lambda'} \cdot \sigma) \delta_{\lambda\lambda'} - \lambda', \\
\epsilon_{\lambda} = e_x + i \lambda e_y,
\]

and the definition of \( Z_{\lambda_a,\zeta_a} \) and \( \tilde{Z}_{\lambda_b,\zeta_b} \). We have

\[
\begin{align*}
\mathcal{M}_{\lambda_1,\lambda_4,\lambda'_1,\lambda'_4} &= Z_{\lambda_1,\zeta_1} \tilde{Z}_{\lambda_4,\zeta_4} Z_{\lambda'_1,\zeta_1}^* \tilde{Z}_{\lambda'_4,\zeta_4}^* = \frac{1}{4} \left\{ \delta_{\lambda_1,\lambda'_1} \delta_{\lambda_4,\lambda'_4} [1 + \lambda_1 (\zeta_1 \cdot e_3)] [1 - \lambda_4 (\zeta_4 \cdot e_3)] \\
&- \delta_{\lambda_1,\lambda'_1} \delta_{\lambda_4,-\lambda'_4} [1 + \lambda_1 (\zeta_1 \cdot e_3)] (\zeta_4 \cdot e_{\lambda_4}) + \delta_{\lambda_1,-\lambda'_1} \delta_{\lambda_4,\lambda'_4} (\zeta_1 \cdot e_{\lambda_4}) [1 - \lambda_4 (\zeta_4 \cdot e_3)] \\
&- \delta_{\lambda_1,-\lambda'_1} \delta_{\lambda_4,-\lambda'_4} (\zeta_1 \cdot e_{\lambda_4}) (\zeta_4 \cdot e_{\lambda_4}) \right\}. \tag{14}
\end{align*}
\]

Here \( \zeta_1 \) and \( \zeta_4 \) are the average polarization vectors in the electron and positron beams, respectively, \( |\zeta_{1,4}| \leq 1 \). Finally we obtain the explicit expression for the cross section with the polarized initial particles:

\[
d\sigma = \frac{\alpha^4}{\pi^4 j} d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_5 d\mathbf{p}_6 \delta(p_2 + p_3 + p_5 + p_6 - p_1 - p_4)
\]
Another example is the differential cross section of the process $e^+e^- \rightarrow \mu^+\mu^-\pi^+\pi^-$. Let $p_1$ and $p_4$ be the momenta $e^-$ and $e^+$, $p_2$ and $p_5$ are the momenta of $\mu^-$ and $\mu^+$, $p_3$ and $p_6$ are the momenta of $\pi^-$ and $\pi^+$. In this process, the main contribution to the amplitude is given by the two types of diagrams shown in Fig. 2. Other contributions are suppressed by the pion electromagnetic form factor $F_\pi(q^2)$ with $q^2 = (p_1 + p_4)^2$. As a result, the differential cross section reads

$$d\sigma = \frac{\alpha^4}{16\pi^4 j\varepsilon_3\varepsilon_6} dp_2 dp_3 dp_5 dp_6 \delta(p_2 + p_3 + p_5 + p_6 - p_1 - p_4) \sum_{\lambda_i = \pm 1} \left| t^{(4)}_\pi + i^{(5)}_\pi \right|^2,$$

$$t^{(4)}_\pi = -F_\pi (p_3 + p_6)^2 (p_3' + p_6') \left[ 1 - \frac{(p_3 + p_6)^2}{(\varepsilon_3 + \varepsilon_6)^2} \right] \left[ g_{14}^{(0)i} - \frac{(p_3' + p_6')(p_1 + p_4; g_{14}^{(0)})}{(\varepsilon_1 + \varepsilon_4)^2} \right] \times \frac{N_1 N_2 N_4 N_5}{(p_3 + p_6)^2(p_1 + p_4)^2} \left[ S_{52}^{(3)ij} (p_3 + p_6) + S_{52}^{(3)ij} (-p_1 - p_4) \right],$$

$$t^{(5)}_\pi = -F_\pi (p_3 + p_6)^2 (p_3' + p_6') \left[ 1 - \frac{(p_3 + p_6)^2}{(\varepsilon_3 + \varepsilon_6)^2} \right] \left[ g_{52}^{(0)i} - \frac{(p_3' + p_6')(p_5 + p_2; g_{52}^{(0)})}{(\varepsilon_5 + \varepsilon_2)^2} \right] \times \frac{N_1 N_2 N_4 N_5}{(p_3 + p_6)^2(p_2 + p_5)^2} \left[ S_{14}^{(4)ij} (p_3 + p_6) + S_{14}^{(4)ij} (p_2 + p_5) \right].$$

where a structure of the pion electromagnetic vertex has been taken into account. Similarly, the cross section of the process $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$ can easily be obtained. We do not present the corresponding result here because of its bulkiness.

**FIG. 2:** Two types of diagrams giving the main contribution to the amplitude of the process $e^+e^- \rightarrow \mu^+\mu^-\pi^+\pi^-$. Wavy lines correspond to photons, thin straight lines correspond to electrons and positrons, double lines correspond to muons, and dotted lines correspond to pions.
III. CONCLUSION

In the present work we have suggested a method to write the convenient representations for many two-photon amplitudes. Our approach is based on the use of the gauge in which the photon propagator $D_{\mu\nu}(k)$ has only space components ($D_{\mu0}(k) = 0$). The amplitudes obtained have no any strong numerical cancellations and, therefore, are very convenient for numerical evaluations. We have illustrated our approach on the examples of the processes $e^+e^- \rightarrow e^+e^-e^+e^-$, $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$, and $e^+e^- \rightarrow \mu^+\mu^-\pi^+\pi^-$. It is shown how the results can be extended on the case of polarized particles.

Basing on this approach the extension of BDK generator [6] was developed allowing for the simulation of $e^+e^- \rightarrow 4l$ processes at increased numerical precision and of $e^+e^- \rightarrow l^+l^-\pi^+\pi^-$ processes with $\pi^+\pi^-$ pair in the $1^{--}$ state [11]. The contribution of narrow $1^{--}$ resonances are accounted via the vacuum polarization effects.

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[11] The source code and the detail description of the generator can be obtained by e-mail A.G.Shamov@inp.nsk.su.