Conditions for the uniqueness of the Gately point for cooperative games

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Abstract

We are studying the Gately point, an established solution concept for cooperative games. We point out that there are superadditive games for which the Gately point is not unique, i.e. in general the concept is rather set-valued than an actual point. We derive conditions under which the Gately point is guaranteed to be a unique imputation and provide a geometric interpretation. The Gately point can be understood as the intersection of a line defined by two points with the set of imputations. Our uniqueness conditions guarantee that these two points do not coincide. We provide demonstrative interpretations for negative propensities to disrupt. We briefly show that our uniqueness conditions for the Gately point include quasibalanced games and discuss the relation of the Gately point to the \( \tau \)-value in this context. Finally, we point out relations to cost games and the ACA method and end upon a few remarks on the implementation of the Gately point and an upcoming software package for cooperative game theory.

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\textbf{Keywords: }TU games; solution concept; quasibalanced games; utopia payoff; cost games; ACA method

1 Introduction

Dermot Gately introduced a new solution concept for cooperative games with transferable utility in Gately (1974) based on minimizing the temptation to leave
the grand coalition for individual players. In the original paper (Gately 1974) the problem of sharing the gains from a joint investment in an electric power grid in India between the participating regions is resolved with the help of the concept “equal propensity to disrupt”. Since the publication of (Gately 1974), the so-called Gately point has become a well-established solution concept taught in books by Straffin (1996) and Narahari (2014) and mentioned in highly regarded survey articles, like e.g. Sandler and Tschirhart (1980) and Young (1994). As of 6 January 2019, 211 quotes of Gately (1974) can be found on GoogleScholar. From its name Gately point one is tempted to assume that the solution concept in question was always unique. In this paper we point out that this is not actually the case. We strive to answer the following question: Under which conditions is the Gately point a unique imputation? Along the way, we also discuss what negative propensities to disrupt tell us about a cooperative game.

2 Preliminary definitions

We are studying a transferable utility game (TU game) in characteristic function form consisting of the player set \( N = \{1, \ldots, n\} \) and the characteristic function \( v : 2^N \to \mathbb{R} \) with \( v(\emptyset) = 0 \). We are using the shorthand notations

\[ v_i = v(\{i\}) \quad \text{for} \quad i = 1, \ldots, n, \]

for the worths of the singleton coalitions.

**Definition 1.** (see Branzei et al (2008), p. 20) The so-called utopia payoff of player \( i \) is given by

\[ M_i = v(N) - v(N\setminus\{i\}) \quad \text{for} \quad i = 1, \ldots, n, \]

i.e. \( M_i \) is the marginal contribution of player \( i \) to the grand coalition.

In this article we will only study games satisfying essentiality in the sense of Chakravarty et al (2013), p. 23.

**Definition 2.** (see Chakravarty et al (2013), p. 23) We call a transferable utility game with player set \( N = \{1, \ldots, n\} \) and characteristic function \( v : 2^N \to \mathbb{R} \) essential if

\[ \sum_{j=1}^{n} v_j < v(N). \quad (1) \]

The imputation set of any essential TU game is guaranteed to consist of more than a single point. For a solution concept in cooperative game theory one would normally prefer the solution vector \( x \in \mathbb{R}^n \) to be an imputation, i.e. both individually
rational $x_i \geq v_i$ for all $i = 1, \ldots, n$ and efficient $\sum_{i=1}^{n} x_i = v(N)$. For a formal definition of the imputation set we refer to Peleg and Sudhölter (2007), p. 20, or Narahari (2014), p. 407.

Note that any cooperative game satisfying (1) is strategically equivalent to a 0-1-normalized game, see Maschler et al (2013), p. 670, or Chakravarty et al (2015), p. 24.

**Definition 3.** (see e.g. Peleg and Sudhölter (2007), p. 10) We call a transferable utility game with player set $N = \{1, \ldots, n\}$ and characteristic function $v : 2^N \to \mathbb{R}$ weakly superadditive if

$$v(S \cup \{i\}) \geq v(S) + v_i \quad \text{for all } S \subseteq N \text{ and } i \notin S.$$ (2)

Note that weak superadditivity (2) guarantees

$$v_i \leq M_i \quad \text{for } i = 1, \ldots, n.$$ (3)

For later convenience we repeat the following

**Definition 4.** (see e.g. Straffin (1996), p. 131, or Narahari (2014), p. 408) We call a transferable utility game with player set $N = \{1, \ldots, n\}$ and characteristic function $v : 2^N \to \mathbb{R}$ superadditive if

$$v(S \cup T) \geq v(S) + v(T) \quad \text{for all } S, T \subseteq N \text{ with } S \cap T = \emptyset.$$ (4)

Finally, we would like to introduce the following game property.

**Definition 5.** We call a transferable utility game with player set $N = \{1, \ldots, n\}$ and characteristic function $v : 2^N \to \mathbb{R}$ weakly constant-sum if

$$v_i + v(N \setminus \{i\}) = v(N) \quad \text{for all } i = 1, \ldots, n.$$ (5)

Note that weakly constant-sum games $v$ can equivalently be characterized by

$$v_i = M_i \quad \text{for } i = 1, \ldots, n.$$ (6)

### 3 Nonuniqueness of the Gately point and uniqueness conditions

In this section we will introduce the Gately point as a solution concept for cooperative games along the lines of the article by Littlechild and Vaidya (1976). The following definition is central to understanding the Gately point as a solution concept for cooperative games.
Definition 6. (see Littlechild and Vaidya (1976), p. 152) For a given transferable utility game with player set \( N = \{1, \ldots, n\} \) and characteristic function \( v : 2^N \to \mathbb{R} \) the expression

\[
d(i, x) = \frac{v(N) - v(N\setminus\{i\}) - x_i}{x_i - v_i} = \frac{M_i - x_i}{x_i - v_i}
\]

quantifies the propensity to disrupt of player \( i \) for a payoff vector \( x \in \mathbb{R}^n \) in the interior of the imputation set, i.e. \( \sum_{j=1}^{n} x_j = v(N) \) with \( x_i > v_i \) for all \( i = 1, \ldots, n \).

Expression (7) quantifies the disruption caused if player \( i \) breaks away from the grand coalition. Within (7) the denominator stands for the loss incurred by player \( i \) for breaking away from the grand coalition, whereas the numerator stands for the joint loss of the rest of the players due to the breakup caused by player \( i \).

The original approach in Gately (1974) for three-person games was generalized to \( n \)-person games by Littlechild and Vaidya (1976), p. 152. The idea is simply to find an imputation \( x \in \mathbb{R}^n \) with minimal propensity to disrupt. It can be shown that this minimal propensity to disrupt can be found by equating the propensity to disrupt over all players, i.e.

\[
d(i, x) = d^* \text{ for } i = 1, \ldots, n.
\]

As pointed out by Littlechild and Vaidya (1976), p. 153, using (7) one can easily find the following closed-form expression

\[
d^* = \frac{(n-1)v(N) - \sum_{j=1}^{n} v(N\setminus\{j\})}{v(N) - \sum_{j=1}^{n} v_j} = \frac{\sum_{j=1}^{n} M_j - v(N)}{v(N) - \sum_{j=1}^{n} v_j}
\]

which also highlights the fact that the Gately point is a solution concept depending solely on the values of the coalitions of sizes 1, \( n-1 \) and \( n \).

Looking at (7) one recognizes that for \( d^* = -1 \) we can not solve for the Gately point. This case can indeed occur for games satisfying (1) and (2). We formalize these findings in

**Theorem 1.** For an essential transferable utility game with player set \( N = \{1, \ldots, n\} \) and characteristic function \( v : 2^N \to \mathbb{R} \) the Gately point is well-defined unless the equal propensity to disrupt \( d^* = -1 \). We can find the Gately point as the unique imputation \( x \in \mathbb{R}^n \) with the components

\[
x_i = v_i + \left(v(N) - \sum_{j=1}^{n} v_j\right) \frac{M_i - v_i}{\sum_{j=1}^{n} M_j - \sum_{j=1}^{n} v_j}
\]

for \( i = 1, \ldots, n \), if one of the following two conditions holds:

a) For games satisfying (3) there needs to hold

\[
v_i < M_i \text{ for at least one } i \in \{1, \ldots, n\},
\]
i.e. (3) is satisfied with strict inequality for at least one $i \in \{1, \ldots, n\}$.

b) We also obtain the Gately point $x \in \mathbb{R}^n$ as a unique imputation if

$$v_i \geq M_i \quad \text{for} \quad i = 1, \ldots, n,$$

as long as (11) is satisfied with strict inequality for at least one $i \in \{1, \ldots, n\}$, i.e. as long as the game is not weakly constant-sum (6).

Proof: As long as $d^* \neq -1$ the expression (9) can be found using (8) by simple algebra. When $d^* \neq -1$ it is justified to set $x_i = v_i$ for those $i \in \{1, \ldots, n\}$ with $v_i = M_i$. Looking at the expression (9), essentiality (1) implies that $x \in \mathbb{R}^n$ is an imputation if and only if

$$\frac{M_i - v_i}{\sum_{j=1}^n M_j - \sum_{j=1}^n v_j} \geq 0$$

for all $i \in \{1, \ldots, n\}$. The latter condition is fulfilled for both games satisfying (3) and games satisfying (11) as long as these games are not weakly constant-sum (6).

Remark 1. The case $d^* < 0$ can be interpreted as enthusiasm of each player not to be the one left out of the grand coalition. In other words: $d^* < 0$ indicates that coalitions of size $n-1$ are preferred over the grand coalition. In the case of (11) being satisfied with strict inequality for at least one $i \in \{1, \ldots, n\}$ this fact is particularly striking as there even holds $d^* < -1$.

Remark 2. Geometrically, (9) allows us to interpret the Gately point as the intersection of the imputation set with the half-line drawn from the point $(v_1, \ldots, v_n)$ with directional vector $(M_1 - v_1, \ldots, M_n - v_n)$.

Remark 3. For 0-normalized games (9) simplifies to

$$x_i = v(N) \frac{M_i}{\sum_{j=1}^n M_j}$$

for $i = 1, \ldots, n$.

We finally consider

Example 1. Let the three-person game $v$ be given by

$$v_1 = 3, v_2 = 4, v_3 = 5, v(\{1, 2\}) = 9, v(\{1, 3\}) = 10, v(\{2, 3\}) = 11, v(N) = 14.$$

The above game is clearly superadditive (4) and essential (1), but the propensity to disrupt equals $-1$ for every imputation $x$. In a sense, the Gately point for $v$ would be the complete imputation set. Naturally, one would make the identical observation considering the 0-normalization of $v$, i.e. the coalitional game $w$ with $w_1 = w_2 = w_3 = 0, w(\{1, 2\}) = w(\{1, 3\}) = w(\{2, 3\}) = w(N) = 2$, or the 0-1-normalization of $v$, i.e. the coalitional game $u$ with $u_1 = u_2 = u_3 = 0, u(\{1, 2\}) = u(\{1, 3\}) = u(\{2, 3\}) = u(N) = 1$. Note that the latter could also be interpreted as a weighted voting game.
4 Relations to the $\tau$-value

In the previous section we have seen that the Gately point is the intersection of the imputation set with a line connecting the points $(v_1, \ldots, v_n)$ and $(M_1, \ldots, M_n)$ and pointed out a problem for the case that these two points coincide [6]. There is another well-established solution concept in cooperative game theory computing the intersection of a line connecting two points with the imputation set, i.e. the $\tau$-value proposed by [Tijs (1981)].

Definition 7. (see Branzei et al. (2008), p. 20) The remainder $R(S, i)$ of player $i$ in coalition $S$ is the amount which remains for player $i$ if coalition $S$ forms and the rest of the players in coalition $S$ all obtain their individual utopia payoffs, i.e.

$$R(S, i) = v(S) - \sum_{j \in S, j \neq i} M_j.$$ 

We can define a vector of minimal rights with components

$$m_i = \max_{S \ni i} R(S, i), \quad \text{for} \quad i = 1, \ldots, n,$$

since player $i$ has a justification to ask at least $m_i$ in the grand coalition.

The $\tau$-value is defined only for quasibalanced games.

Definition 8. (see e.g. Branzei et al. (2008), pp. 31) We call a transferable utility game with player set $N = \{1, \ldots, n\}$ and characteristic function $v : 2^N \to \mathbb{R}$ quasibalanced if

$$m_i \leq M_i \quad \text{for all} \quad i \in \{1, \ldots, n\} \quad (13)$$

and

$$\sum_{j=1}^n m_j \leq v(N) \leq \sum_{j=1}^n M_j. \quad (14)$$

For a quasibalanced game $v$ the $\tau$-value is defined as the intersection of the imputation set with the line from the minimal rights vector $(m_1, \ldots, m_n)$ to the utopia payoff vector $(M_1, \ldots, M_n)$.

Remark 4. (see e.g. Branzei et al. (2008), p. 32) We can find the $\tau$-value with the components

$$\tau_i = \alpha m_i + (1 - \alpha) M_i,$$

where $\alpha \in [0, 1]$ is uniquely determined by the condition $\sum_{i=1}^n \tau_i = v(N)$.

Combining (8) and condition (14) we find that for quasibalanced games $d^* \geq 0$ is guaranteed and we arrive at
Corollary 1. The Gately point is always unique for quasibalanced games.

Note that the conditions we formulated for the Gately point to be a unique imputation are more general than quasibalancedness, i.e. there are games for which the τ-value is not defined whereas the Gately point is. Consider

Example 2. Let the three-person game \(v\) be given by

\[
v_1 = 3, v_2 = 4, v_3 = 5, v(\{1, 2\}) = 9, v(\{1, 3\}) = 10, v(\{2, 3\}) = 11, v(N) = 14.5.
\]

The game \(v\) is not quasibalanced and its Gately point can be computed to \(x_1 = 3\frac{5}{6}, x_2 = 4\frac{1}{6}, x_3 = 5\frac{2}{6}\).

We finally observe that the problem we report for the Gately point never occurs for the τ-value which we already mentioned to be the intersection of the imputation set with a line drawn from the point \((m_1, \ldots, m_n)\) to the point \((M_1, \ldots, M_n)\), see Tijs (1981). However, if these two points coincide, then (14) guarantees this point to be an imputation and thus the τ-value of the game. Note that in this special case there is \(d^* = 0\).

5 Application to cost games and relations to the ACA-method

We are looking at cost games in characteristic function form consisting of the set \(N = \{1, \ldots, n\}\) of agents (or purposes, projects or services) and the characteristic function \(c : 2^N \rightarrow \mathbb{R}\) with \(c(\emptyset) = 0\). We are using the shorthand notation

\[c_i = c(\{i\}) \quad \text{for} \quad i = 1, \ldots, n,
\]

for the costs of single agents. The connection to TU games is given by the associated savings game \(v\) for \(N = \{1, \ldots, n\}\) defined by

\[v(S) = \sum_{i \in S} c_i - c(S)
\]

for every coalition \(S\). Note that the associated savings game \(v\) is automatically 0-normalized.

We are now discussing the so-called ACA (Alternate Cost Avoided) method, i.e. an established method for cost allocation going back to Ransmeier (1942), along the lines of Straffin and Heaney (1981). The ACA method has been widely discussed, see also Otten (1993), Tijs and Driessen (1996) and Young (1994).

The ACA method is based on the concept of allocating separable costs

\[SC_i = c(N) - c(N \setminus \{i\}) = c_i - M_i
\]
for each agent $i = 1, \ldots, n$. The remaining nonseparable costs

$$NSC = c(N) - \sum_{j=1}^{n} SC_j = \sum_{j=1}^{n} M_j - v(N) \quad (15)$$

are assigned in proportion to $c_i - SC_i$, i.e. the final cost allocation for an individual agent $i$ is

$$y_i = SC_i + \frac{c_i - SC_i}{\sum_{j=1}^{n} c_j - SC_j} NSC.$$

As pointed out in Straffin and Heaney (1981), p. 40, the corresponding savings allocation $x \in \mathbb{R}^n$ is exactly the Gately point, i.e.

$$x_i = c_i - y_i = v(N) \frac{M_i}{\sum_{j=1}^{n} M_j}$$

as seen in (12).

It is very natural to understand why the problem of nonuniqueness of ACA never came up in the context of cost games. In practice, only subadditive cost games are studied, i.e. the corresponding savings game is superadditive (4), see Young (1994), p. 1197. The ACA method can only fail to deliver a unique cost allocation if $M_i = 0$, or equivalently $c_i = SC_i$, for $i = 1, \ldots, n$. Then (11) implies $NSC < 0$, whereas studies of ACA for good reason assume nonnegativity of nonseparable costs, see Otten (1993), p. 177, and Tijs and Driessen (1996), p. 1019. Practical ACA calculations would normally stop if $NSC < 0$ and this implies $d^* < 0$, see (8) and (15). We finally consider

**Example 3.** Let the subadditive three-agent cost game $c$ be given by

$$c_1 = 7, c_2 = 8, c_3 = 9, c(\{1, 2\}) = 14, c(\{1, 3\}) = 15, c(\{2, 3\}) = 16, c(N) = 23.$$  

The corresponding savings game $u$ is the weighted voting game $u_1 = u_2 = u_3 = 0, u(\{1, 2\}) = u(\{1, 3\}) = u(\{2, 3\}) = u(N) = 1$ we already know from Example 1. The Gately point does not exist and so ACA fails to deliver a unique cost allocation.

In general, ACA can only run into problems if all coalitions of size $n - 1$ and the grand coalition make identical savings. Then we would expect a coalition of size $n - 1$ to form, but we can not use ACA to single out the one agent $i$ to be left out.

6 Final remarks

The main purpose of this article is to answer the question when it is at all sensible to compute the Gately point of a TU game $v$. We derived very general conditions for the Gately point to be a unique imputation and pointed out why weakly
constant-sum games lead to problems. We feel that our analysis underlines the criticism of the Gately point made in Littlechild and Vaidya (1976), p. 153, that the solution concept only makes use of the values of the coalitions of sizes 1, \( n - 1 \) and \( n \) and completely ignores the rest of the information contained in the coalition function \( v \).

The nonuniqueness of the Gately point was first discussed in Anwander (2017) and it was discovered during efforts to implement the Gately point in R. The authors are currently finalizing an R-package named CoopGame (see Staudacher and Anwander (2019)) which the authors hope to make publicly available via CRAN, the Comprehensive R Archive Network. Among various other solution concepts, the package CoopGame will not only provide an implementation of the Gately point, but also provide the user with possibilities to compute the equal propensity to disrupt \( d^* \) of a given cooperative game \( v \). The scope of our Gately point implementation is slightly broader as for an inessential game \( v \) in the sense of Narahari (2014), p. 408, i.e. if \( \sum_{j=1}^{n} v_j = v(N) \), our code will simply return \((v_1, \ldots, v_n)\). Otherwise, we make sure to check the conditions derived in this paper before the computation of the Gately point and to return a meaningful message in the case the user specifies a TU game \( v \) with an equal propensity to disrupt \( d^* = -1 \).

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