A Comprehensive Analysis of Quantum E-voting Protocols

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Abstract. Recent advances at Google, IBM, as well as a number of research groups indicate that quantum computers will soon be reality. Motivated by the ever more realistic threat quantum computers pose to existing classical cryptographic protocols, researchers have developed several schemes to resist “quantum attacks”. In particular, for electronic voting, several e-voting schemes relying on properties of quantum mechanics have been proposed. However, each of these proposals comes with a different and often not well-articulated corruption model, has different objectives, and is accompanied by security claims which are never formalized and are at best justified only against specific attacks. In this paper, we systematize and evaluate the security of suggested e-voting protocols based on quantum technology. We examine the claims of these works concerning privacy, correctness and verifiability, and if they are correctly attributed to the proposed protocols. In all non-trivial cases, we identified specific quantum attacks that violate these properties. We argue that the cause of these failures lies in the absence of formal security models and in a more general lack of reference to the existing cryptographic literature.

Keywords: quantum electronic voting, quantum networks, quantum cryptography, attacks

1 Introduction

Voting is a fundamental procedure in democratic societies. With the technological advances of the computer era, voting processes can also benefit to become more secure and efficient and as a result more democratic. Indeed, compared to previous manual procedures, electronic voting systems can offer more efficient elections with higher voter participation, better accuracy, while also providing enhanced security guarantees, such as vote-privacy and voter-verification even in the face of untrusted election authorities. For this reason, over the last two decades, several cryptographic protocols such as Helios [1], JCJ/Civitas [25], Scantegrity [9], Demos [28], or Prêt-à-Voter [40] to name just a few, have been proposed and implemented for electronic voting.
All these systems base their security on the assumption that certain problems are computationally hard; typical examples include problems such as integer factorization and discrete logarithm, which are well known to be easy to solve with quantum computers. Indeed, Shor’s algorithm \[42\] allows for large integers to be factored efficiently, and also solves the discrete logarithm problem. Although we do not yet have quantum computers, recent technological advances indicate that these will soon be reality. In this context, and motivated by the fact that quantum computers will be a threat to existing classical cryptographic protocols, researchers have proposed to use quantum communication to implement cryptographic primitives like key distribution, bit commitment, coin flipping and oblivious transfer. Unfortunately, perfect security (i.e. where the best adversarial strategy does not provide better winning probabilities than an honest strategy) with no assumptions has proven to be impossible in the quantum setting for any cryptographic primitive \[31,33\]. In order to achieve this level of security, we need to study different corruption models, under which it might be possible to prove security for the honest participants. Typical examples include limiting the number of dishonest participants that the adversary can compromise, introducing different non-collaborating authorities, etc.

A decade of studies on quantum electronic voting has resulted in several protocols that use the properties of quantum mechanical systems in order to achieve specific requirements such as vote privacy, correctness and verifiability, in the presence of quantum adversaries. However, all these new protocols are studied against different and not well-articulated corruption models, and claim security using ad-hoc proofs that are not formalized and backed only against a limited class of quantum attacks. In particular, none of the proposed schemes provides rigorous definitions of privacy and verifiability, nor formal security proofs against specific, well-defined (quantum) attacker models. When it comes to electronic voting schemes, it is particularly hard to ensure that all the, somehow conflicting, properties hold; it is therefore important that these new quantum protocols be rigorously and mathematically studied and the necessary assumptions and limitations formally established.

This is precisely what we set to address in this paper. That is, we systematize and assess the security of existing e-voting protocols based on quantum technology. We examine the claims of each of these solutions concerning privacy, correctness and verifiability, and evaluate them against their intended properties. Unfortunately our analyses uncover vulnerabilities in all the proposed schemes. While some of them suffer from trivial attacks due to inconsistencies in the security definitions, the main contribution of the paper is to argue that sophisticated attacks can exist even in protocols that “seem secure”, if the security is proven ad hoc, and not in a formal framework. Specifically, we show that we can construct efficient quantum attackers that break the assumed properties of these protocols by corrupting a fraction of voters (whatever small or big). We argue that the cause of these failures lies in the absence of formal security proofs in an appropriate security framework.
It is therefore the aim of this work to highlight the importance of formally defining and proving security in the relatively new field of quantum cryptography, following the steps of seminal works like [2,39] and [45]. In general, our work aims at motivating researchers to study cryptographic primitives under a new quantum perspective, in order to help secure future communication networks in the quantum era to come. This also includes studying classical protocols that are secure against computationally unbounded attackers such as the ones proposed in [6], as well as protocols that rely for their security on problems believed to be hard to solve even for quantum computers; one such example of an e-voting protocol adapting based on lattices is presented in [11], based on the work of [1]. However, it is out of the scope of this study to review such classical protocols, as we are focusing on the possible contribution of quantum computers to the security of e-voting.

**Contributions of this paper:** We propose a systematization of quantum electronic voting approaches based on their key technical features. To the best of our knowledge, our study covers all relevant research in the field, identifying four main families of quantum schemes for electronic voting. We evaluate the security of each scheme against privacy, correctness, and verifiability of each scheme, and uncover vulnerabilities in all these schemes:

- **Two measurement bases based protocols** - These protocols rely on two measurement bases to verify the correct distribution of an entangled state. We demonstrate an attack against recent works that breaks the privacy property, when a fraction of voters is corrupted. We specifically prove that the probability that a number of corrupted states are not tested and used later in the protocol, is non-negligible. We then show how this can be exploited to break voters’ vote privacy. Furthermore, even if the states are shared by a trusted authority, we show that privacy can still be violated in case of abort.

- **Traveling ballot protocols** - The main idea of these protocols is that the “ballot box” circulates among all voters which apply a unitary to it in order to add their vote. However, we show how colluding voters can break honest voters’ vote privacy just by measuring the ballot box before and after the victim has cast their ballot. These protocols further suffer as we will see from double voting attacks, whereby a dishonest voter can simply apply multiple times the voting operator.

- **Quantum distributed ballot protocols** - These schemes aim at addressing the shortcomings of traveling ballot protocols by having each voter vote on a distinct although entangled system and combining them homomorphically. We present an attack that allows the adversary to double-vote and therefore change the outcome of the voting process with probability at least 0.25, if the protocol runs fewer than exponentially many rounds in the number of voters. The intuition behind this attack is that an adversary does not need to find exactly how the ballots have been created in order to influence the outcome of the election; it suffices to find a specific relation between them from left-over voting ballots provided by the corrupted voters.
Conjugate coding based protocols - These protocols exploit BB84 states adding some verification mechanism. The main issue with these schemes, as we show, is that ballots are malleable, allowing an attacker to modify the part of the ballot which encodes the candidate choice to their advantage.

Finally, we identify and discuss current challenges, indicating future research directions.

Outline of this paper: In Section 2 we give the necessary background for understanding the rest of the paper. Section 3 deals with protocols based on two measurement bases. In Section 4 we discuss the traveling ballot family of protocols. Section 5 is dedicated to the distributed ballot family of protocols, and in Section 6 we describe and analyze schemes based on conjugate coding. In Section 7, we briefly mention some other less promising proposals. Finally, in Section 8 we discuss possible solutions and observations for quantum e-voting to be of practical use in future quantum communication networks. Due to lack of space, the details of the proofs are omitted, but are available in the Supplementary Material at the end of the paper.

2 Preliminaries

2.1 Quantum Computation

We use the term quantum bit or qubit [41,35] to denote the simplest quantum mechanical object we will use. We say that a qubit is in a pure state if it can be expressed as a linear combination of other pure states:

\[ |x\rangle = \alpha |0\rangle + \beta |1\rangle, \text{ where } |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

where \(|\alpha|^2 + |\beta|^2 = 1\) for any \(\alpha, \beta \in \mathbb{C}\). The states \(|0\rangle\) and \(|1\rangle\) are called the computational basis vectors. Sometimes it is also helpful to think of a qubit as a vector in the two-dimensional Hilbert space \(\mathcal{H}\). If a qubit cannot be written in the above form, then we say it is in a mixed state. The generalization of a qubit to an \(m\)-dimensional quantum system is called qudit:

\[ |y\rangle = \sum_{j=0}^{m-1} a_j |j\rangle, \text{ where } \sum_{j=0}^{m-1} |a_j|^2 = 1 \]

Let’s now suppose that we have two qubits; we can write the state vector as:

\[ |\psi\rangle = \sum_{i,j \in \{0,1\}} \alpha_{ij} |ij\rangle \]

where \(\sum_{i,j \in \{0,1\}} |\alpha_{ij}|^2 = 1\). If the total state vector \(|\psi\rangle\) cannot be written as a tensor product of two qubits (i.e. \(|x_1\rangle \otimes |x_2\rangle\)), then we say that qubits \(|x_1\rangle\) and
|\psi\rangle_{\pm} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)

A quantum system that is in one of the above states is also called an EPR pair, from the famous paradox [17].

The way we obtain information about a quantum system is by performing a measurement using a family of linear operators \{M_j\} acting on the state space of the system, where \(j\) denotes the different outcomes of the measurement. It holds that:

\[
\sum_j M_j^\dagger M_j = I
\]

and for the continuous case:

\[
\int M_j^\dagger M_j \, dj = I
\]

where \(M_j^\dagger\) is the conjugate transpose of matrix \(M_j\), and \(I\) the identity operator.

In the case of qudit \(|y\rangle\) (as defined above), the probability that the measurement outcome is \(w\) is:

\[
Pr(w) = \langle y | M_w^\dagger M_w | y \rangle
\]

and in the continuous case

\[
Pr(w \in [w_1, w_2]) = \int_{w_1}^{w_2} \langle y | M_j^\dagger M_j | y \rangle \, dj
\]

In the case of a single qubit \(|x\rangle = \alpha |0\rangle + \beta |1\rangle\), measurement in the computational basis will give outcome zero with probability \(|\alpha|^2\) and outcome one with probability \(|\beta|^2\). If our state is entangled, a partial measurement (i.e. a measurement in one of the entangled qudits), not only reveals information about the measured qudit, but possibly about the remaining state. For example, let us recall the Bell state \(|\Phi^+\rangle\). A measurement of the first qubit in the computational basis will give measurement outcome 0 or 1 with equal probability and the remaining qubit will collapse to the state \(|0\rangle\) or \(|1\rangle\) respectively.

In quantum cryptography, the correlations in the measurement outcomes of entangled states are frequently exploited. Another entangled state of interest that we will use in Section 3 gives measurement outcomes that sum up to zero when measured in the computational basis, and equal outcomes when measured in the Fourier basis (denoted by \(|\rangle_F\)). In the three-qubit case, the state is the
following:

\[
|D\rangle = \frac{1}{\sqrt{3}} (|0\rangle_F |0\rangle_F |0\rangle_F + |1\rangle_F |1\rangle_F |1\rangle_F )
\]

\[
= \frac{1}{4} \left( \sum_{j_1+j_2+j_3=0 \pmod{2}} \sum_{j_1+j_2+j_3 \neq 0 \pmod{2}} 2 |j_1\rangle |j_2\rangle |j_3\rangle + (1 + e^{\pi i}) |j_1\rangle |j_2\rangle |j_3\rangle \right)
\]

\[
= \frac{1}{2} (|000\rangle + |011\rangle + |101\rangle + |110\rangle)
\]

Finally, the evolution of a closed quantum system can be described by the application of a unitary operator. Unitary operators are reversible and preserve the inner product. Recall our first example, and let’s say we would like to swap the amplitudes on state \( |x\rangle \), then we can apply the operator \( Z \) (also known as the NOT-gate):

\[
Z |x\rangle = \beta |0\rangle + \alpha |1\rangle,
\]

where \( Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

The \( Z \)-gate is one of the Pauli operators (the others being the \( X \) and \( Y \)), which together with the identity operator \( I \), form a basis for the vector space of \( 2 \times 2 \) Hermitian matrices. The Pauli operators are unitaries, therefore as mentioned above, they also preserve the inner product.

A very important difference between quantum and classical information, is that there is no mechanism to create a copy of an unknown quantum state \[41,35\]. This result, known as the no-cloning theorem, is one of the fundamental advantages and at the same time limitations of quantum information. It becomes extremely relevant for cryptography, since brute-force types of attacks cannot be applied on quantum channels that carry unknown information. When verifying quantum resources however, it is necessary to apply a cut-and-choose technique in order to test that the received quantum states are correcting produced. The quantum source would therefore need to send exponentially many copies (with respect to the security parameter of the protocol) of the quantum state \[26\], in order for the verifier to measure most of them and deduce that with high probability, the remaining ones are correct.

2.2 Electronic Voting

In general, electronic voting protocols consist of election authorities, talliers, voters and bulletin boards \[25,28,1\]. In this work, we will be dealing with protocols involving one tallier and/or one election authority (which we will denote with \( T \) and \( EA \) respectively), as well as a set of voters \( \mathcal{V} = \{ V_k \}_{k=1}^{N} \). Their role in the protocol is for \( EA \) to set the parameters of the protocol, \( \mathcal{V} \) to cast their ballots and \( T \) to gather the votes and compute and announce the result of the election. Ideally, an e-voting protocol needs to fulfill the following properties:

- Correctness: The protocol should behave as intended if the adversary doesn’t interfere at all, \( i.e. \) should allow to the parties to carry out an election.
– Accountability / double voting: Only eligible voters are allowed to vote at most once.
– Privacy [4]: The vote of a voter should remain private, i.e. there should not exist an efficient procedure with which an adversary can extract information about the way a voter voted.
– Verifiability [13]: Voter, and/or external entities to the protocol called auditors, should be able to verify that their vote, or in the case of auditors the total votes, have been counted as intended.
– Receipt freeness [15]: A voter should not be able to prove how they voted to avoid vote selling.

For the purpose of this work we do not need formal definitions of these properties as we mainly focus on attacks. As we will see, these attacks violate in an obvious way the intended properties of any electronic voting system and would be captured by any reasonable definition. It should however be noted that coming up with appropriate definitions for the desired properties is not straightforward and remains still a very active area of research [10]. In classical cryptography, there were efforts to tackle this difficulty by using automated provers and model checkers such as EasyCrypt [12], game based definitions (as in the surveys [13,4]), and by employing the Universal Composability Framework [7,20]. However, in quantum cryptography, it remains unclear how a similar approach can be adopted. An interesting approach appears in [18], where the authors provide an automated verification tool, named QMC (Quantum Model) that enables checking properties of systems which can be expressed within the quantum stabilizer formalism. Finally, a recent work by Unruh [46] on quantum relational Hoare logic might open new avenues and help provide a solution to this problem.

3 Dual Basis Measurement Based Protocols

In this section we discuss a family of quantum voting protocols that exploit for their security the dual basis measurement technique. We will specifically study protocols [48] and [24] that use as a blank ballot an entangled state with an interesting property: when measured in the computational basis, the sum of the outcomes is equal to zero, while when measured in the Fourier basis, all the outcomes are equal. Both of these protocols use cut-and-choose techniques in order to verify that the state was distributed correctly. This means that a large amount of states are checked for correctness and a remaining few are kept at the end unmeasured, to proceed with the rest of the protocol. Although a cut-and-choose technique with just one verifying party is secure if the states that are sampled are exponentially many and the remaining ones are constant, it is not clear how this generalizes to a setting where there are multiple verifying parties. In particular, if we consider an adversary who generates the blank ballots and can corrupt an arbitrary fraction of the voters, it turns out that the probability that some corrupted states are not tested, is non-negligible with respect to the security parameter of the protocol. Specifically, we show that if the corrupted
parties sample their states last, then the probability with which the corrupted states are not checked and remain after all the honest parties sample, is at least a constant with respect to the security parameter of the protocol.

In the following, we will present one of the dual basis protocols [48] and show how to construct a polynomial quantum adversary that violates privacy with non-negligible probability with respect to $\delta_0$, without getting detected.

3.1 Protocol Specification

The self-tallying protocol of Wang et al. [48] is based on the classical protocol of [27]. The voters $\{V_k\}_{k=1}^{N}$, without the presence of any trusted authority or tallier, need to verify that they share specific quantum states. At the end of the verification process, the voters share a classical matrix; every cast vote is equal to the sum of the elements of a row in the matrix. The protocol goes as follows:

1. One of the voters, not necessarily trusted, prepares $N + N2^{\delta_0}$ states of the form:

$$|D_1\rangle = \frac{1}{\sqrt{m^{N-1}}} \sum_{i_1=0}^{m-1} \sum_{k=1}^{N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$

where $m$ is the dimension of the qudits’ Hilbert space, $c$ is the number of the possible candidates such that $m \geq c$ and $\delta_0$ the security parameter. The voter also shares $1 + N2^{\delta_0}$ states of the form:

$$|D_2\rangle = \frac{1}{\sqrt{N!}} \sum_{(i_1,i_2,\ldots,i_N) \in \mathcal{P}_N} |i_1\rangle |i_2\rangle \ldots |i_N\rangle$$

where $\mathcal{P}_N$ is the set of all possible permutations with $N$ elements. Each $V_k$ receives the $k^{th}$ particle from each of the states.

2. The voters agree that the states they receive are indeed $|D_1\rangle, |D_2\rangle$ by using a cut-and-choose technique. Specifically, voter $V_k$ chooses at random $2^{\delta_0}$ of the $|D_1\rangle$ states and asks the other voters to measure half of their particles in the computational and half in the Fourier basis. Whenever the chosen basis is the computational, the measurement results need to add up to 0, while when the basis is the Fourier, then the measurement results are all the same. All voters simultaneously broadcast their results and if one of them notices a discrepancy, the protocol aborts. The states $|D_2\rangle$ are checked in a similar way.

3. The voters are left to share $N$ copies of $|D_1\rangle$ states and one $|D_2\rangle$ state. Each voter holds one qudit for each state. They now all measure their qudits in the computational basis. As a result, each $V_k$ holds a “blank ballot” of dimension $N$ with the measurement outcomes corresponding to parts of $|D_1\rangle$ states:

$$B_k = [\xi_1^{\dagger} \cdots \xi_k^{s_{k_1}} \cdots \xi_N^{s_{k_N}}]^\dagger$$

and a unique index, $s_{k_1} \in \{1, \ldots, N\}$, from the measurement outcome of the qudit that belongs to $|D_2\rangle$. The set of all the blank ballots has the property $\sum_{k=1}^{N} \xi_j^{s_{k_j}} = 0 \mod c$ for all $j = 1, \ldots, N$. 


4. Based on $sk_k$, all voters add their vote, $v_k \in \mathbb{Z}_c$, to the corresponding row of their “secret” column. Specifically, $V_k$ applies $\xi_k^{sk_k} \rightarrow \xi_k^{sk_k} + v_k$.

5. All voters simultaneously broadcast their columns, resulting in a public $N \times N$ table, whose $k$-th column encodes $V_k$’s candidate choice.

$$B = \begin{bmatrix}
\xi_1 \\
\vdots \\
\xi_k \\
B_1^{v_1} \cdots \xi_k^{sk_k} + v_k \cdots B_N^{v_N} \\
\vdots \\
\xi_N^k
\end{bmatrix}$$

6. Each $V_k$ can now check that their vote has been counted, by checking that the corresponding row of the matrix adds up to their vote. If the check fails, the protocol aborts.

7. Each voter can tally the final outcome of the election by computing the sum of the elements of each row of the public $N \times N$ table. The resulting $N$ elements are the result of the election.

3.2 Vulnerabilities Of Dual Basis Measurement Protocols

In this section we present an attack on the cut-and-choose technique of the protocol (step 2), that can be used to violate privacy. We consider a static adversary that corrupts $t$ voters, including the one that distributes the states in step 1. Suppose that the adversary corrupts $N$ out of $N + 2^{b_0}$ states $|D_1\rangle$. We denote with $Bad$, the event that all the corrupted voters choose last (i.e., after the honest voters) which states they want to test, and with $Win$, the event that the $N$ corrupted states are not checked.

We want to compute the probability that event $Win$ happens, given event $Bad$, i.e., the probability none of the $N$ corrupted states is checked by the honest voters, and therefore remain intact until the corrupted voters’ turn. The corrupted voters will of course not sample any of the corrupted states and therefore the corrupted states will be accepted as valid.

The number of corrupted states that an honest voter will check, follows a mixture distribution with each mixture component being one of the hypergeometric distributions $\{HG(L_{ik}, b_{ik}, 2^{b_0}) : 0 \leq b_{ik} \leq N\}$, where $L_{ik}$ is the number of states left to sample from the previous voter and $b_{ik}$ the number of the remaining corrupted states.

We can therefore define the random variable $X_{ik}$ that follows the above mixture distribution, where $i_1, \ldots, i_{N-t}$ is a permutation of the honest voters’ indices (by slightly abusing notation, we consider the first $N-t$ voters to be honest). The following lemma is proven by induction:

**Lemma 1.** Let $X_{ik}$ be a random variable that follows the previous mixture distribution. It holds that:
\[
\Pr[\sum_{k=1}^{N-t} X_{i_k} = 0] = \prod_{k=1}^{N-t} \Pr[X^*_i = 0]
\]

where \(X^*_i \sim HG(L_{i_k}, N, 2^{\delta_0})\).

We are now ready to prove the next proposition which establishes that with at least a constant probability, the corrupted states will remain intact until the end of the verification process (the corrupted voters will of course choose not to test any of them).

**Proposition 1.** For \(0 < \varepsilon < 1\), let \(t = \varepsilon N\) be the fraction of voters controlled by the adversary. It holds that :

\[
\Pr[Win | Bad] > \left(\frac{\varepsilon}{2}\right)^N
\]

**Proof.**

\[
\Pr[Win | Bad] = \Pr[\sum_{k=1}^{N-t} X_{i_k} = 0] = \prod_{k=1}^{N-t} \Pr[X^*_i = 0]
\]

\[
= \prod_{k=0}^{N-t-1} \left(\frac{N + N2^{\delta_0} - N - k2^{\delta_0}}{2^{\delta_0}}\right) = \frac{(N + t2^{\delta_0} - N + 1) \ldots \cdot (N + t2^{\delta_0})}{(N + N2^{\delta_0} - N + 1) \ldots \cdot (N + N2^{\delta_0})}
\]

\[
> \left(\frac{t2^{\delta_0} + 1}{N + N2^{\delta_0}}\right)^N = \left(\frac{t2^{\delta_0}}{N + N2^{\delta_0}} + \frac{1}{N + N2^{\delta_0}}\right)^N
\]

\[
> \left(\frac{\varepsilon}{2}\right)^N
\]

The question now is with what probability event \(Bad\) occurs, i.e how likely is the fact that voters controlled by the adversary are asked to sample last? The answer is irrelevant, because this probability depends on \(N\) and \(t\), and are both independent of \(\delta_0\). As a result,

\[
\Pr[Win] > \Pr[Win | Bad] \Pr[Bad] = (\varepsilon/2)^N f(N, t)
\]

where \(f(N, t)\) is a constant function with respect to the security parameter of the protocol \(\delta_0\), making \(\Pr[Win]\) non-negligible in \(\delta_0\). As a matter of fact, a static adversary will corrupt the voters that maximize \(\Pr[Bad]\). Therefore, we can assume that the honest voters sample the states at random, in order to not favor sets of corrupted voters. Now let us examine how this affects the privacy of the scheme.
Theorem 1. Let $\Pi(N, t, \delta_0)$ be an execution of the self-tallying protocol with $N$ voters, $t$ of them corrupted, and $\delta_0$ the security parameter. We can construct an adversary $A$, which with non-negligible probability in $\delta_0$ violates privacy without getting detected by the honest voters.

Proof. Let $C_A$ be the set of indices of the corrupted voters, where $|C_A| = t$. Suppose that the voter who distributes the states is also corrupted, and prepares $1 + N 2^{b_0}$ states of the form $|D_2\rangle$, $N 2^{b_0}$ states of the form $|D_1\rangle$ and $N$ states of the form:

$$|D_{\text{Corrupt}}\rangle = |\xi_1\rangle \otimes \ldots \otimes |\xi_N\rangle$$

where $\xi_k \in_R \{0, \ldots, c-1\}$ for all $k \in \{2, \ldots, N\}$, and $\xi_1 \in \{0, \ldots, c-1\}$ such that:

$$\xi_1 + \ldots + \xi_N = 0 \mod c$$

From Proposition 1 and the previous observations we know that the probability that states $|D_{\text{Corrupt}}\rangle$ remain intact after the verification procedure in step 2 (i.e. event $\text{Win}$), happens with non-negligible probability in the security parameter $\delta_0$. Therefore, with non-negligible probability, the remaining states in step 3 are: one of the form $|D_2\rangle$ and $N$ of the form $|D_{\text{Corrupt}}\rangle$. All honest voters $V_k$ measure their qudits in the computational basis and end up with a secret number $sk_k$ (from measuring the corresponding part of $|D_2\rangle$) and a column

$$B_k = [\xi_1^k \ldots \xi_{sk_k}^k \ldots \xi_N^k]^\top$$

(from measuring states $|D_{\text{Corrupt}}\rangle$), that is known to the adversary. Now all voters apply their vote $v_k$ to the $B_k$ according to $sk_k$. As a result:

$$B_k^{v_k} = [\xi_1^k \ldots \xi_{sk_k}^k + v_k \ldots \xi_N^k]^\top$$

At this point all voters simultaneously broadcast their $B_k^{v_k}$, as the protocol specifies, and they end up with the matrix $B = (B_1^{v_1} \ldots B_N^{v_N})$. Each $V_k, k \notin C_A$ checks that:

$$\sum_{j=1}^N B[sk_k, j] = v_k \mod c$$

which happens with probability 1 from the description of the attack in the previous steps. As a result, each voter accepts the election result. The adversary knowing both the pre-vote matrix and the post-vote matrix can therefore extract the vote of all honest voters.

A similar attack can be mounted if the adversary instead of corrupting $N$ out of $N + N 2^{b_0}$ $|D_1\rangle$ states, corrupts just 1 of the $|D_2\rangle$ states. The attack is pretty similar to the attack mentioned above but in this case the adversary knows the

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4 The symbol $\in_R$ denotes that the element is chosen uniformly at random from a specific domain.
row in which each voter voted instead of the pre-vote matrix. Moreover, the probability of theorem 1 is improved from \( (\epsilon/2)^N \) to \( \epsilon/2 \) (the proof works in a similar way).

So far we have seen how voters’ privacy can be violated if an adversary distributes the quantum states in the protocol. However, even if the sharing of the states is done honestly by a trusted authority, still an adversary \( A \) can violate the privacy of a voter. This is done by replacing one element in a column of one of the players controlled by \( A \) with a random number. As a result, in step 6), the honest voter whose row doesn’t pass the test, will abort the protocol by broadcasting it. \( A \) will therefore know the identity of the voter aborting and their corresponding vote, since it knows the matrix before the modification of the column element. A possible solution might be the use of a classical anonymous broadcast channel, so that the voters can anonymously broadcast abort if they detect any misbehaviour at step 6). However, this might open a path to other types of attacks, like denial-of-service, and requires further study in order to be a valuable solution.

4 Traveling Ballot Based Protocols

In this section we discuss the traveling ballot family of protocols for referendum type elections. Here, the tallier \( T \) also plays the role of \( EA \), as it sets up the parameters of the protocol in addition to producing the election result. Specifically, it prepares two entangled qudits, and sends one of them (the ballot qudit) to travel from voter to voter. When the voters receive the ballot qudit, they apply some unitary operation according to their vote choice and forward the qudit to the next voter. When all voters have voted, the ballot qudit is sent back to \( T \) who can now measure the whole state in order to compute the result of the referendum. The first quantum scheme in this category was introduced by Vaccaro et al. [47] and later improved by others [5,22,30].

4.1 Protocol Specification

We here present variant [22] of the traveling ballot protocol; an alternative form of the protocol [47], encodes the vote in a phase factor rather than in the qudit itself.

1. \( T \) prepares the state:

\[
|\Omega_0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle_V |j\rangle_T,
\]

keeps the second qudit and passes the first (the ballot qudit) to voter \( V_1 \).

2. For \( k = 1, \ldots, N \), \( V_k \) receives the ballot qudit and applies the unitary \( U^{v_k} = \sum_{j=0}^{N-1} |j + 1\rangle \langle j| \), where \( v_k = 1 \) signifies a “yes” vote and \( v_k = 0 \) a “no” vote.
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(i.e. applying the identity operator). Then, $V_k$ forwards the ballot qudit to the next voter $V_{k+1}$ ($V_N$ sends the ballot qudit back to $T$).

3. The global state held by $T$ after all voters have voted, is:

$$|\Omega_N\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j+m\rangle_V |j\rangle_T$$

where $m$ is the number of “yes” votes.

4. $T$ measures the two qudits in the computational basis. By subtracting the two results, $T$ obtains the election result $m$, and announces it to all voters.

4.2 Vulnerabilities Of Traveling Ballot Based Protocols

The first obvious weakness of this type of protocols is that they are subject to double voting. A corrupted voter can apply the “yes” unitary operation many times without being detected (this issue is addressed in the next session, where we study the distributed ballot voting schemes). Furthermore, these protocols are subject to privacy attacks, when several voters are colluding. In what follows, we describe such an attack on privacy, in the case of two colluding voters. Figure 1 depicts this attack.

Let us assume that the adversary corrupts voters $V_{k-1}$ and $V_{k+1}$ for any $k$. Upon receipt of the ballot qudit, instead of applying the appropriate unitary, $V_{k-1}$ performs a measurement on the traveling ballot in the computational basis. As a result the global state becomes:

$$|\Omega_{k-1}\rangle = |h+m\rangle_V \otimes |h\rangle_T$$

where $|h+m\rangle_V$ is one of the possible eigenstates of the observable $O = \sum_{j=0}^{N-1} |j\rangle \langle j|$, and $m$ is the number of “yes” votes cast by the voters $V_1, \ldots, V_{k-2}$ (note that $V_{k-1}$ does not get any other information about the votes of the previous voters, except number $h+m$). Then $V_{k-1}$ passes the ballot qudit $|h+m\rangle_V$ to $V_k$, who applies the respective unitary for voting “yes” or “no”. As a result the ballot qudit is in the state $|h+m+v_k\rangle_V$. Next, the ballot qudit is forwarded to the corrupted voter $V_{k+1}$, who measures it again in the computational basis and gets the result $h+m+v_k$. The adversary can now infer vote $v_k$ from the two measurement results and figure out how $V_{k+1}$ voted.

The same attack can also be applied in the case where there are many voters between the two corrupted parties. In this case the adversary can’t learn the individual votes but only the total votes, which still constitutes a privacy leakage.

One suggestion presented in [47] is to allow $T$ to perform extra measurements to detect a malicious action during the protocol’s execution. However, this only identifies an attack and does not prevent the adversary from learning some of the votes, as described above. Furthermore, the probability of detecting a deviation from the protocol is constant and as such does not depend on the security parameter and does not lead to a substantial improvement of security. It should
also be noted that verifiability of the election result is not addressed in any of these works, since $T$ is assumed to generate the initial state honestly. In the case where $T$ is corrupted, privacy is trivially violated.

All traveling ballot protocols proposed \cite{47,5,22,30} suffer from the above privacy attack. In the next section we will discuss how this issue has been addressed in these works by revisiting the structure of the protocols. Unfortunately, as we will now see, new issues arise.

5 Distributed Ballot Based Protocols

Here we describe the quantum distributed ballot protocols presented in \cite{17,5,22}. In these schemes, $T$ prepares and distributes to each voter a blank ballot, and gathers it back after all voters have cast their vote in order to compute the final outcome. This type of protocols give strong guarantees for privacy against other voters but not against a malicious $T$ which is trusted to prepare correctly specific states. So it is not hard to see that if the states are not the correct ones, then the privacy of a voter can be violated.

A first attempt presented in \cite{17} suffers from double voting similarly to the discussion in the previous section. The same problem also appears in \cite{16}. Later works \cite{5,22} address this issue with a very elaborate countermeasure. The intuition behind the proposed technique is that $T$ chooses a secret number $\delta$ according to which it prepares two different quantum states: the “yes” and the “no” states. This $\delta$ value is hard to predict due to the non-orthogonality of the shared states and the no-cloning theorem. The authors suggest that many rounds of the protocol be executed. As a result, any attempt of the adversary to learn $\delta$ gives rise to a different result in each round. However, the number of required rounds, as well as a rigorous proof are not presented in the study.
More importantly, a careful analysis reveals that the proposed solution is still vulnerable to double voting. As we will see, an adversary can mount what we call a $d$-transfer attack, and transfer $d$ votes for one option of the referendum election to the other. To achieve this attack, the adversary does not need to find the exact value of $\delta$ (as the authors believed), but knowing the difference of the angles used to create the “yes” and “no” states suffices. We construct a polynomial quantum adversary that performs the $d$-transfer attack with probability at least 0.25, if the number of rounds is smaller than exponential in the number of voters. As a result this makes the protocol practically unrealistic for large scale elections.

5.1 Protocol Specification

We first present the protocol from [5,22]:

1. $T$ prepares an $N$-qudit ballot state:
   
   $$|\Phi\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |j\rangle^\otimes N$$

   where the states $|j\rangle$, $j = 0, \ldots, D - 1$, form an orthonormal basis for the $D$-dimensional Hilbert space, and $D > N$. The $k$-th qudit of $|\Phi\rangle$ corresponds to the blank ballot of $V_k$.

2. $T$ sends to $V_k$ the corresponding blank ballot together with two option qudits, one for the “yes” and one for the “no” option:

   yes: $|\psi(\theta_y)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij\theta_y} |j\rangle$,

   no: $|\psi(\theta_n)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij\theta_n} |j\rangle$

   For $v \in \{y, n\}$ we have $\theta_v = (2\pi l_v/D) + \delta$, where $l_v \in \{0, \ldots, D - 1\}$ and $\delta \in [0, 2\pi/D)$. Values $l_y$ and $\delta$ are chosen uniformly at random from their domain and $l_n$ is chosen such that $N(l_y - l_n \mod D) < D$. These values are known only to $T$.

3. Each $V_k$ decides on “yes” or “no” by appending the corresponding option qudit to the blank ballot and performing a 2-qudit measurement $R = \sum_{r=0}^{D-1} rP_r$, where:

   $$P_r = \sum_{j=0}^{D-1} |j + r\rangle \langle j + r| \otimes |j\rangle \langle j|$$

   According to the result $r_k$, $V_k$ performs a unitary correction $U_{r_k} = \mathbb{I} \otimes \sum_{j=0}^{D-1} |j + r_k\rangle \langle j|$ and sends the 2-qudits ballot along with $r_k$ back to $T$.

4. The global state of the system (up to normalization) is:

   $$\frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} \prod_{k=1}^{N} \alpha_{j,r_k} |j\rangle^\otimes 2N$$
where,
\[
\alpha_{j,r_k} = \begin{cases} 
  e^{i(D+j-r_k)\theta_v^k}, & 0 \leq j \leq r_k - 1 \\
  e^{i(j-r_k)\theta_v^k}, & r_k \leq j \leq D - 1 
\end{cases}
\]

5. For every \( k \), using the announced results \( r_k \), \( T \) applies the unitary operator:
\[
W_k = \sum_{j=0}^{r_k-1} e^{-iD\delta} \ket{j} \bra{j} + \sum_{j=r_k}^{D-1} \ket{j} \bra{j}
\]
on one of the qudits in the global state (it is not important on which one, since changes to the phase factor of a qudit that is part of a bigger entangled state take effect globally). Now \( T \) has the state:
\[
|\Omega_m\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij(m\theta_y+(N-m)\theta_n)} \ket{j} \otimes 2^N
\]
where \( m \) is the number of “yes” votes.

6. By applying the unitary operator \( \sum_{j=0}^{D-1} e^{-ijN\theta} \ket{j} \bra{j} \) on one of the qudits and setting \( q = m(l_y - l_n) \), we have:
\[
|\Omega_q\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{2\pi ijq/D} \ket{j} \otimes 2^N
\]

We note here that \( q \) must be between 0 and \( D-1 \), so that the different outcomes of the election be distinguishable. Now with the corresponding measurement \( T \) can retrieve \( q \). Since \( T \) knows the values \( l_y \) and \( l_n \), it can then derive the number \( m \) of “yes” votes. Note that if a voter does not send back a valid ballot, the protocol execution will abort.

5.2 Vulnerabilities Of Distributed Ballot Protocols

In this section, we show how the adversary can perform the \( d \)-transfer attack in favor of the “yes” outcome. We proceed as follows. We first show that this is possible if the adversary knows the difference \( l_y - l_n \). We then show how the adversary can find out this value, and conclude the section with the probabilistic analysis of our attack which establishes that it can be performed with overwhelming probability in the number of voters.

The \( d \)-transfer attack: Given the difference \( l_y - l_n \), a dishonest voter can violate the no-double-voting property. From the definition of \( l_y \) and \( l_n \) it holds that:
\[
2\pi(l_y - l_n)/D = \theta_y - \theta_n \quad (1)
\]
If a corrupted voter (for example \( V_1 \)) knows \( l_y - l_n \), then they will proceed as follows (w.l.o.g. we assume that they want to increase the number of “yes” votes by \( d \)): 
1. $V_1$ applies the unitary operator:

$$C_d = \sum_{j=0}^{D-1} e^{i(jd(\theta_y - \theta_n))} |j\rangle \langle j|$$

to the received option qudit $|\psi(\theta_y)\rangle$. As a result, the state becomes:

$$C_d |\psi(\theta_y)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{i(jd(\theta_y - \theta_n))} e^{ij\theta_y} |j\rangle$$

2. $V_1$ now performs the 2-qudit measurement specified in step 3 of the protocol and obtains the outcome $r_1$.

3. $V_1$ performs the unitary correction $U_{r_1}$. The global state now is:

$$U_{r_1} P_{r_1} \left( |\Phi\rangle \otimes C_d |\psi(\theta_y)\rangle \right) = \frac{1}{\sqrt{D}} \left[ \sum_{j=0}^{r_1-1} e^{i(D+j-r_1)d} |j\rangle \otimes N^+ + \sum_{j=r_1}^{D-1} e^{i(D-j+r_1)d} |j\rangle \otimes N^+ \right]$$

where $\tilde{\theta} = d(\theta_y - \theta_n) + \theta_y$.

4. Before sending the two qudit ballot and the value $r_1$ to $T$, $V_1$ performs the following operation to the option qudit:

$\textbf{Correct}_{r_1} = \begin{cases} e^{-id(\theta_y - \theta_n)} |j\rangle \langle j|, & 0 \leq j \leq r_1 - 1 \\ |j\rangle \langle j|, & r_1 \leq j \leq D - 1 \end{cases}$

5. After all voters have cast their ballots to $T$, the global state of the system (up to normalization) is:

$$\frac{1}{\sqrt{D}} \left( \sum_{j=0}^{r_1-1} e^{i(D-j-r_1)d} e^{i(D+j-r_1)d} \prod_{k=2}^{N} \alpha_{j,r_k} |j\rangle \otimes 2N \right) + \sum_{j=r_1}^{D-1} e^{i(D-j-r_1)d} e^{i(D-j-r_1)d} \prod_{k=2}^{N} \alpha_{j,r_k} |j\rangle \otimes 2N$$

where,

$$\alpha_{j,r_k} = \begin{cases} e^{i(D-j-r_k)d} e_{v}^{k}, & 0 \leq j \leq r_k - 1 \\ e^{i(D-j-r_k)d} e_{v}^{k}, & r_k \leq j \leq D - 1 \end{cases}$$

and $\theta_v$ describes the vote of voter $V_k$, where $v \in \{y, n\}$. $T$ just follows the protocol specification. It applies some corrections on the state given the announced results $r_k$ and finally the state becomes:

$$\frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{i(D-j-r_1)d} e^{i(D-j-r_1)d} \prod_{k=2}^{N} e^{i(j-r_k)d} e_{v}^{k} |j\rangle \otimes 2N$$
which under a global phase factor is equivalent to:

\[ \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij(d-\theta_n)} e^{ij(m(\theta_y+(N-m)\theta_n))} |j\rangle \otimes |2N\rangle \]

6. $T$ removes the unwanted factor $e^{iN\theta_n}$ as prescribed by the protocol, and the final state is:

\[ |\Omega_{m+d}\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ijd(\theta_n-\theta_n)} e^{ijm(\theta_y-\theta_n)} |j\rangle \otimes |2N\rangle \]

7. After measuring the state, the result is $m+d$ instead of $m$.

Finding the difference between $l_y$ and $l_n$: What is missing in order to complete our attack is to find the difference $l_y - l_n$. We now show how an adversary can learn this difference with overwhelming probability in $N$. We assume that the adversary controls a fraction $\varepsilon$ of the voters ($0 < \varepsilon < 1$), who are (all but one) instructed to vote half the times “yes” and the other half “no”.

Instead of destroying the remaining option qudits (exactly $\varepsilon N/2$ “yes” and $\varepsilon N/2$ “no” votes), the adversary keeps them to run Algorithm 1.

**Algorithm 1** Adversary’s algorithm

**Input:** $D, |\psi(\theta_v)\rangle_{1,\ldots,|\psi(\theta_v)\rangle_{\varepsilon N/2}}$

**Output:** $\tilde{l} \in \{0,\ldots,D-1\}$

1. Record $= [0,\ldots,0] \in \mathbb{N}^{1 \times D}$; \hspace{1cm} \(\triangleright\) This vector shows us how many values are observed in each interval
2. Solution $= ["Null","Null"] \in \mathbb{N}^{1 \times 2}$
3. $i, l, m = 0$
4. while $i \leq \varepsilon N/2$ do
5. Measure $|\psi(\theta_v)\rangle_i$ by using POVM operator $E(\theta)$ from Eq. (2), the result is $y_i$;
6. Find the interval for which $\frac{2\pi i}{D} \leq y_i \leq \frac{2\pi (i+1)}{D}$;
7. Record$[j] = +++$;
8. $i++$
9. end while
10. while $l < D$ do
11. if Record$[l] \geq 40\% (\varepsilon N/2)$ then
12. Solution$[m] = l$
13. $m++$
14. end if
15. $l++$
16. end while
17. if Solution$[] = [0, D-1]$ then
18. Solution $= \{\text{Solution}[1], \text{Solution}[0]\}$
19. end if
20. return $\tilde{l} = \text{Solution}[0]$;

In essence, the algorithm is executed twice - once for each set of option qudits $\{|\psi(\theta_v)\rangle\}_{\varepsilon N/2}$, where $v \in \{y, n\}$. It measures the states in each set and
attributes to each one an integer number. After all states have been measured, the algorithm creates a vector \texttt{Record}, which contains the number of times each integer appeared during the measurements. Finally, Algorithm 1 creates a vector called \texttt{Solution} in which it registers the values that appeared at least 40\% of times during the measurements, equivalently the values for which the \texttt{Record} vector assigned a number greater or equal than 40\% of times. The algorithm outputs the first value in the \texttt{Solution} vector. As we see in Figure 2 with high probability the value that algorithm outputs is either \( l_v \) or \( l_v - 1 \), for both values of \( v \). As a result we can find the difference \( l_y - l_n \).

After having acquired knowledge of \( l_y - l_n \), the adversary can instruct the last corrupted voter to change the outcome of the voting process as previously described.

![Fig. 2. The probabilities with which Algorithm 1 records a value in \{\( l_v - 1 \), \( l_v \), \( l_v + 1 \)\} after measuring state \(|\psi(\theta_v)\rangle\) for \( \delta_1 = \frac{\pi}{2\pi}, \delta_2 = \frac{\pi}{3\pi}, \) and \( \delta_3 = \frac{\pi(2^D-1)}{2^\delta} \).](image)

### Probabilistic analysis:

We prove here that the adversary’s algorithm succeeds with overwhelming probability in \( N \), where \( N \) is the number of voters. Therefore (see Theorem 5), the election protocol needs to run at least exponentially many times with respect to \( N \) in order to guarantee that the success probability of the adversary is at most 0.25. We present here the necessary lemmas and give the full proofs in the Supplementary Material.

In order to compute the success probability of the attack, we first need to compute the probability of measuring a value in the interval \((x_l, x_{l+w})\), where \( x_l = \frac{2\pi l}{D} \), \( l \in \{0, 1, \ldots, D - 1\} \).

#### Lemma 2.

Let \( \Theta_{D,\delta} \in [0, 2\pi] \) be the continuous random variable that describes the outcome of the measurement of an option qudit \(|\psi(\theta_v)\rangle\), \( v \in \{y, n\} \) using operators:

\[
E(\theta) = \frac{D}{2\pi} |\Phi(\theta)\rangle \langle \Phi(\theta)|
\]

\[\text{It is convenient to think of } l \text{ as the } D^{th} \text{ roots of unity.}\]
where $|\Phi(\theta)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij\theta} |j\rangle$. It holds that:

$$\Pr[x_l < \Theta_{D,\delta}^v < x_{l+1}] = \frac{1}{2\pi D} \int_{x_l}^{x_{l+1}} \frac{\sin^2[D(\theta - \theta_v)/2]}{\sin^2[(\theta - \theta_v)/2]} d\theta$$

According to Algorithm 1, an option qudit is attributed with the correct value $l_v$ when the result of the measurement is in the interval $[x_{l_v}, x_{l_v+1}]$. Using Lemma 2, we can prove the following:

**Lemma 3.** Let $|\psi(\theta_v)\rangle$ be an option qudit of the protocol. Then it holds:

$$\Pr[x_{l_v} < \Theta_{D,\delta}^v < x_{l_v+1}] > 0.405$$

Lemma 3 shows us that with probability at least 0.405, the result of the measurement is in the interval $(x_{l_v}, x_{l_v+1})$. Since Algorithm 1 inserts an integer to the Solution vector if it corresponds to at least 40% of the total measured values, $l_v$ will most likely be included in the vector (we formally prove it later). Furthermore, we prove in the following lemma that with high probability, there will be no other values to be inserted in the Solution vector, except the neighbours of the value $l_v$ (namely $l_v \pm 1$).

**Lemma 4.** Let $|\psi(\theta_v)\rangle$ be an option qudit of the protocol. Then it holds:

$$\Pr[x_{l_v-1} < \Theta_{D,\delta}^v < x_{l_v+2}] > 0.9$$

Here we need to note that we are aware of the cases $l_v \in \{0, D-1\}$ where the members $x_{l_v-1}$ and $x_{l_v+2}$ are not defined. It turns out not to be a problem and the same thing can be proven for these values (see Supplementary Material).

We have shown that the probability the measurement outcome lies in the interval $(x_{l_v-1}, x_{l_v+2})$, and therefore gets attributed with a value of $l_v - 1$, $l_v$ or $l_v + 1$, is larger than 0.9. If we treat each measurement performed by Algorithm 1 on each option qudit $|\psi(\theta_v)\rangle$, as an independent Bernoulli trial with success probability $p_l = \Pr[x_l < \Theta_{D,\delta}^v < x_{l+1}]$, we can prove the following theorem:

**Theorem 2.** With overwhelming probability in the number of voters $N$, Algorithm 1 includes $l_v$ in the Solution vector.

$$\Pr[\text{Solution}[0] = l_v \lor \text{Solution}[1] = l_v] > 1 - 1/\exp(\Omega(N))$$

We have proven that with overwhelming probability in $N$, integer $l_v$ occupies one of the two positions of vector Solution, but what about the other value? In the next theorem, we show that with overwhelming probability in $N$, the other value is one of the neighbours of $l_v$, namely $l_v + 1$ or $l_v - 1$.

**Theorem 3.** With negligible probability in the number of voters $N$, Algorithm 1 includes a value other than $(l_v - 1, l_v, l_v + 1)$ in the Solution vector, i.e. $\forall w \in \{0, \ldots, l_v - 2, l_v + 2, \ldots, D - 1\}$:

$$\Pr[\text{Solution}[0] = w \lor \text{Solution}[1] = w] < 1/\exp(\Omega(N))$$
The following lemma can be proven:

**Lemma 5.** With overwhelming probability in $N$, the Solution vector in Algorithm 7 is equal to $[l_v - 1, l_v]$, $[l_v, "Null"]$ or $[l_v, l_v + 1]$. Specifically,

$$\Pr[\text{Solution} \in \{[l_v - 1, l_v], [l_v, "Null"], [l_v, l_v + 1]\}] > 1 - 1/\exp(\Omega(N))$$

Now consider we have two executions of the Algorithm 1, one for the “yes” and one for the “no” option qudits. It turns out that the values in the positions $l_y - 1$ and $l_n - 1$ of the vector Record, follow the same Binomial distribution (it is easy to see that $p_{v-1} = p_{n-1}$). Also, each of them can be seen as a function of $\delta$ which is a monotonic decreasing function that takes a maximum value for $\delta = 0$ (the proof technique is similar to Lemma 3). At this point the probability is equal to $p_{v}$, which is at least 0.405 as we have proven in Lemma 6. Armed with this observation we can prove the next theorem.

**Theorem 4.** If we define the event “Cheat” as:

$$\text{Cheat} = [\text{Algo}(y) - \text{Algo}(n) = l_y - l_n]$$

where Algo$(v)$ is the execution of Algorithm 7 with $v \in \{y, n\}$, then it holds that:

$$\Pr[^{\text{"Cheat"}}] > 1 - 1/\exp(\Omega(N))$$

**Proof.** (sketch) Because of the previous observation we know that there exists a $\delta_0$ such that the probability $p_{v-1}$ is equal to 0.4 for both values of $v$. It holds that:

$$\Pr[^{\text{"Cheat"}}] = \Pr[^{\text{"Cheat"}}|\delta \in [0, \delta_0)] \cdot \Pr[\delta \in [0, \delta_0)]$$

$$+ \Pr[^{\text{"Cheat"}}|\delta = \delta_0] \cdot \Pr[\delta = \delta_0]$$

$$+ \Pr[^{\text{"Cheat"}}|\delta \in (\delta_0, 2\pi/D)] \cdot \Pr[\delta \in (\delta_0, 2\pi/D)]$$

For the first interval, for both values of $v$, Algorithm 1 registers Solution = $[l_v - 1, l_v]$ with overwhelming probability in $N$. This holds because of Theorem 5 and the previous observation. Therefore, for both values of $v$ the algorithm outputs the values $l_v - 1$. As a result, $l_y - 1 - (l_n - 1) = l_y - l_n$.

In the second case, $\Pr[\delta = \delta_0] = 0$, because $\delta$ is a continuous random variable.

In the last case, the probability that the algorithm registers Solution = $[l_v - 1, l_v]$ is negligible in $N$, and by theorem 5 Solution has the form $[l_v]$ or $[l_v, l_v + 1]$. So for both values of $v$, the printed values are $l_y$ and $l_n$.

At this point we have proved that the adversary succeeds with overwhelming probability in $N$ to perform the $d$-transfer attack in one round. But how many rounds should the protocol run in order to prevent this attack?

In the next theorem we prove that if the number of rounds $\rho$ is at most $\exp(\Omega(N))$, the adversary succeeds with probability at least 0.25. Although in
a small scale election these numbers might not be big, in a large scale election it is infeasible to run the protocol as many times, making it either inefficient or insecure. We also note that the probabilistic analysis for one round of execution is independent of the value $D$, so it can not be used to improve the security of the protocol.

**Theorem 5.** Let $(|\Phi\rangle, |\psi(\theta_y)\rangle, |\psi(\theta_n)\rangle, \delta, D, N)$ define one round of the protocol. If the protocol runs $\rho$ rounds, where $2 \leq \rho \leq \exp(\Omega(N))$, the $d$-transfer attack succeeds with probability at least $0.25$.

**Proof.** According to theorem 4 the probability that an adversary successfully performs the $d$-transfer attack is:

$$Pr["Cheat"] > 1 - 1/\exp(\Omega(N))$$

Now, if the protocol runs $\rho$ times, where $2 \leq \rho \leq \exp(\Omega(N))$, this probability becomes:

$$(Pr["Cheat"])^\rho > (1 - 1/\exp(\Omega(N)))^\rho \geq (1 - 1/\rho)^\rho > 0.25$$

6 Quantum voting based on conjugate coding

This section looks at protocols based on conjugate coding \([36,50]\). The participants in this family of protocols are one or more election authorities\(^7\), the tallier and the voters. The election authorities are only trusted for the purpose of eligibility in \([36]\); privacy should be guaranteed by the protocol against both malicious $EA$ and $T$. In \([50]\) privacy is not guaranteed against a corrupted election authority. Unlike the previous protocols, here the voters do not share any entangled states with neither $EA$ nor $T$ in order to cast their ballots. One of the main differences between the protocols in \([36]\) and \([50]\) is the fact that protocol \([36]\) does not provide any verification of the election outcome, while \([50]\) does, but at the expense of receipt freeness, which \([36]\) satisfies. Specifically, in \([50]\) each $V_k$ establishes two keys with $T$ in an anonymous way by using part of protocol \([36]\) as a subroutine. It’s worth to mention that in order this keys to be established further interaction between the voters and $EA$ is required and $EA$ assumed trusted for that task else the privacy of a voter can be violated trivially with probability 1. At the end of an execution, $V_k$ encrypts the ballot with one of the keys and sends it to $T$ over a quantum anonymous channel. $T$ announces the result of each ballot accompanied with the second key so that the voters can verify that their ballot has been counted. This makes it also possible for a coercer to verify how a voter voted, by showing them the second key used as a receipt. It is worth mentioning that protocol \([36]\) could easily be made to satisfy the same notion of verifiability without needing to trust the election authority for privacy.

\(^7\) In \([50]\) the authors introduced two election authorities in order to distribute the trust between them.
6.1 Protocol Specification

A brief description of a protocol execution is as follows:

1. EA picks a vector \( \vec{b} = (b_1, \ldots, b_{n+1}) \in \{0,1\}^{n+1} \), where \( n \) the security parameter of the protocol. This vector will by EA for the encoding of the ballots and it will be kept secret from \( T \) until the end of the ballot casting phase.

2. For each \( V_k \), EA prepares \( w = \text{poly}(n) \) blank ballot fragments each of the form:

\[
|\phi_{\vec{a}_j, \vec{b}}\rangle = |\psi_{a_{j}^{1}, b_{j}}\rangle \otimes \ldots \otimes |\psi_{a_{j}^{n+1}, b_{n+1}}\rangle, j \in \{1, \ldots, w\}
\]

where \( \vec{a}_j = (a_1^j, \ldots, a_n^j, a_{n+1}^j) \) such that:

\[
(a_1^j, \ldots, a_n^j) \in \mathbb{R}^{n}, a_{n+1}^j = a_1^j \oplus \ldots \oplus a_n^j
\]

and:

\[
|\psi_{0,0}\rangle = |0\rangle, |\psi_{1,0}\rangle = |1\rangle, |\psi_{0,1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |\psi_{1,1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\]

These \( w \) fragments will constitute a blank ballot (e.g. the first row of Fig. 3 is a blank ballot fragment).

3. EA sends one blank ballot to each voter \( V_k \) over an authenticated channel.

4. After reception of the blank ballot, each \( V_k \) re-randomizes it by picking for each fragment a vector \( \vec{d}_j = (d_1^j, \ldots, d_n^j) \) such that:

\[
(d_1^j, \ldots, d_n^j) \in \mathbb{R}^{n}, d_{n+1}^j = d_1^j \oplus \ldots \oplus d_n^j.
\]

\( \forall j \in \{1, \ldots, w\}, V_k \) applies unitary \( U_{\vec{d}_j} = Y_{d_1^j} \otimes \ldots \otimes Y_{d_{n+1}^j} \) to the blank ballot fragment \( |\phi_{\vec{a}_j, \vec{b}}\rangle \), where:

\[
Y^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Y^0 = I
\]

5. \( V_k \) encodes the candidate of choice in the \((n + 1)^{th}\)-qubit of the last blank ballot fragment. For example, if we assume a referendum type election, \( V_k \) votes for \( c \in \{0,1\} \) by applying to the blank ballot fragment \( |\phi_{\vec{a}_j, \vec{b}}\rangle \) the unitary operations \( U_{\vec{c}_w} \) respectively, where: \( \vec{c} = (0, \ldots, 0, c) \) (see Fig. 3).

6. \( V_k \) sends the ballot to \( T \) over an anonymous channel.

7. Once the ballot casting phase ends, EA announces \( \vec{b} \) to \( T \).

8. With this knowledge, \( T \) can decode each cast ballot in the correct basis. Specifically, \( T \) decodes each ballot fragment by measuring it in the basis described by vector \( \vec{b} \) and XORs the resulting bits. After doing this to each ballot fragment, \( T \) ends up with a string, which is the actual vote cast.

9. \( T \) announces the election result.

8 Candidate choices are encoded in binary format.
Fig. 3. The ballot consisting of $w$ ballot fragments, which encode the binary choice “0...01” in a referendum type election example.

6.2 Vulnerabilities of Conjugate Coding Protocols

The technique underlying this protocol is closely related to the one used in the first quantum key distribution protocols [3,43]. However, it has some limitations in the context of these voting schemes.

Malleable blank ballots: An adversary can change the vote of an eligible voter, by the time the corresponding ballot is cast over the anonymous channel. The adversary proceeds as follows. Assume $V_k$ has applied the appropriate unitary on the blank ballot in order to vote for the candidate of their choice. And let us consider that the last $m$ ballot fragments encode the candidate. When the adversary sees the cast ballot over the quantum anonymous channel, they apply the unitary $U_{-c_r}^{(m-1)}U_{-c_r}^{(m)}$, where $c_r$ is either 0 or 1, depending on their choice to flip the candidate bit or not. As a result the adversary modifies the ballot of $V_k$ such that it decodes to a different candidate than the intended one. This is possible because the adversary is aware of the ballot fragments used to encode the candidate choice. Furthermore, if the adversary has as it is often unavoidable, side channel information about the likely winning candidate (from pre-election polls for instance), they will be able to change the vote encoded in the ballot into one of their desire. This is possible because the adversary is aware of which bits are encoded in the ballot more frequently. As a result, the adversary knows exactly which unitary operator to apply in order to get a specific candidate.

Violation of privacy: The EA can introduce a “serial number” in a blank ballot to identify a voter, i.e. some of the blank ballot fragments in the head of
the ballot decode to “1” instead of “0”. This allows the \( EA \) to decode any ballot cast over the quantum anonymous channel, linking the identity of the voters with their choice. The authors acknowledged this issue and tried to identify a secure audit mechanism for the integrity of the blank ballots without providing a fully developed solution.

**One-more-unforgeability:** The security of the protocol relies on a new hard quantum problem introduced in [36], which they name *one-more-unforgeability*. The security game that illustrates this assumption goes as follows: a challenger encodes \( w \) blank ballot fragments in a basis \( \overline{b} \) and gives them to the adversary. The adversary wins the game if they produce \( w + 1 \) valid blank ballot fragments in the basis \( \overline{b} \). The authors claim that the probability with which the adversary wins this game is at most \( 1/2 + 1/2(\text{negl}(n)) \). The assumption that this new problem is computationally hard for a quantum adversary needs to be further studied, even though it does seem reasonable.

**On the security parameter:** Because of the ballots’ malleability, an adversary could substitute the parts of the corrupted voters’ blank ballot fragments that encode a candidate, with blank ballot fragments in a random base. Of course these ballots would open into random candidates in a specific domain but would still be valid, since the leading zeros would not be affected by this change. This is because blank ballots contain no entanglement. Now the adversary can keep these valid spare blank ballot fragments to create new valid blank ballots. To address this problem, the size of blank ballots needs to be substantially big compared to the number of voters and the size of the candidate space \( (Nm << w) \).

### 7 Other Protocols

Other protocols have also been proposed, however, we do not include them in this review, as they are not fully developed and can therefore not be appropriately analyzed [44,49,23]. The characteristic of these protocols is that \( EA \) controls when ballots get counted. This can be achieved with either the use of shared GHZ states [19] between \( EA, T, \) and \( V_k \) [44,49] or Bell pairs [44] between \( T \) and \( V_k \) with \( EA \) knowing the identity of the holder of each pair particle. We will not present these protocols in detail as they have many and serious flaws making even the correctness arguable.

In [49] the proposed protocol is claimed to provide verifiability of the election outcome, but without explaining how this can be achieved. From our understanding of the protocol this seems unlikely to be the case. From the description of the protocol each voter can change their mind and announce a different vote from the originally cast one. This is possible because the function every voter uses to encode their vote is not committed in any way.

Another two protocols introduced in [44], have similar limitations. For instance, there is no mechanism for verifiability of the election outcome. In addition, privacy against \( T \) is not satisfied in contrast to protocols we saw in section
This is because each voter’s vote is handled individually and not in a homomorphic manner. All of these could be achieved just by a classical secure channel.

Lastly, the protocol appearing in [23] shares many of the limitations of the former protocols as well as some further ones. The method introduced for detecting eavesdropping in the election process is insecure, as trust is put into another voter in order to detect any deviation from the protocol specification. Moreover, the way each voter votes is not well defined. For example, by the description of the voting procedure, privacy can be violated trivially or a voter never votes.

8 Discussion

In this work, we have examined the current state of the art in quantum e-voting, by presenting the most prominent proposals and analyzing their security. What we have found is that all the proposed protocols fail to satisfy the necessary cryptographic standards in order to be implemented in the future.

Despite this, these protocols open the way to new avenues of research, specifically on whether quantum information can solve some long-standing issues in e-voting and cryptography in general. By studying them, we can identify several interesting ideas for further development as well as possible bottlenecks in future quantum protocols. For instance, we have seen that, unless combined with some new technique, the traveling ballot protocols do not seem to provide a viable solution, as double-voting is always possible, and there is no straightforward way to guarantee privacy. On the other hand, the distributed ballot protocols give us very strong privacy guarantees because of the entanglement between the ballot states. It is possible that by using a similar type of protocol, we might be able to prove unconditional privacy in a composable framework.

Unfortunately, verifiability is not guaranteed in distributed ballot protocols, and appears to be very hard to achieve against a malicious tallyer [13]. An equally important problem present in this type of protocols is double voting. In [22], the authors present an interesting solution, which however might still allow an adversary to vote multiple times, by taking advantage of unused ballots. In order to prevent this, the protocol would need to run exponentially many times with respect to the number of voters, which would in practice be inefficient. It might be possible to overcome this specific issue, but in order to prevent other forms of attacks, a formal proof is necessary in an appropriate model [45,21,7].

One plausible idea of how to improve the protocol to be resistant against the attack we presented in Section 5 is to choose different values of $\delta$ for each ballot. If this was possible, then the attack we described would not be applicable as it relies on the fact that all ballot states share the same $\delta$. On the other hand this is not an easy task because if the $\delta$’s are different for each ballot, it is not obvious how the tallyer could compute the correct result, since the protocol’s correctness is based on a homomorphic property that allows getting the correct result out of the combination of all ballots.
The same observation also holds for [36], where the election authority uses the same $k$ in order to encode each ballot. If each voter could use a cut-and-choose technique in the same sense as in [1] (where each voter tests that a ballot is prepared correctly from the Voting Support Device in order to be cast to the Bulletin Board), a corrupted election authority could not violate the privacy or the correctness of the ballot. But again the problem is that different values of $k$ encode different ballots. As a result, like in the previous case, even an execution of the protocol where everyone acts honestly would not work. It is a very intriguing open question whether these two properties can be achieved at the same time.

Regarding the protocols in section 3, the cut-and-choose technique used is both inefficient and insecure. An inherent problem with this type of technique in the case of one prover and one verifier, is that the number of sampled states grows exponentially with respect to the protocol parameter in order to achieve a satisfied level of security. In the case of the protocols in section 3, even if the sampled states are as many as in the single prover/single verifier case, the protocol still it is not secure. However, the verification process is done in a very specific way, and it would be interesting to study if there are other options to explore in order to improve it. A possible solution could be to provide some type of randomness to the voters (in the form of a common random string for example), which will define if a state should be verified or used for the voting phase (a similar testing process has been studied in the past for other types of entangled states, both in theory [37] and experimentally [34]). However, even if the problem with the cut-and-choose technique is addressed in future works, privacy can still be violated as we have seen, and possible corrections might require the use of more advanced techniques. Notwithstanding these limitations, we believe that our analysis opens new research directions for the study of the quantum cut-and-choose technique, which plays a fundamental role in the secure distribution of quantum information, in the multi-party setting.

Concerning verifiability, the main problem is that all definitions seem unlikely to be satisfied by a quantum protocol for many reasons. For instance, every definition of verifiability in [13] assumes a trusted Bulletin Board, $BB$, where each voter can read and write on it. After the end of the election, the auditors in order to verify the integrity of the election procedure, will read the context of the $BB$ and output either accept or reject. A quantum analogy would require a quantum Bulletin Board, $QBB$, on which the voters would be able to read or write. Moreover, at the end of the election procedure, the auditors should have access to the $QBB$ in order to verify the integrity of the election procedure. In general, both voters and auditors must be able to read the contexts of $QBB$ in order to either verify if a ballot has been recorded or the election outcome is correct, as in classical election protocols. But as we know if someone reads a quantum state, in our case a $QBB$, then the state will be disturbed, something that does not happen with classical information. So the following question arises: what definition for verifiability should we adopt, that would capture both the properties of a public verifiable election protocol and the behavior of quantum information.
Finally, in order for a quantum protocol to be of any interest, it should not only provide at least the same security guarantees compared to classical protocols under the same assumptions, but rather propose some improvement on at least one aspect, may it be security or efficiency. In the past, quantum protocols have been proposed [8] and implemented [38], that achieve better-than-classical guarantees. Other interesting settings relevant to quantum implementations, include bounded adversaries (i.e. that have limited storage [14], noisy storage [29], or are bounded by relativistic constraints [32]), and have already achieved information-theoretic security for basic cryptographic primitives like bit commitment and oblivious transfer.

The question whether quantum technology could enhance electronic voting has not yet been answered, and requires further study of both the existing classical and quantum literature. First, bottlenecks in classical election protocols that could potentially be solved by using quantum subroutines, need to be identified. Then, quantum protocols need to be designed, secure in composable frameworks against well articulated definitions of all the required properties. We hope that this review will be the starting point of a critical analysis and further study of quantum electronic voting.

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Supplementary Material

Now we give detailed proofs of the theorems and lemmas that appear in the main text.

**Lemma 2** Let $\Theta_{D,\delta} \in [0, 2\pi]$ be the continuous random variable that describes the outcome of the measurement of a vote state $|\psi(\theta_v)\rangle$, $v \in \{y, n\}$ using operators

$$E(\theta) = \frac{D}{2\pi} |\Phi(\theta)\rangle \langle \Phi(\theta)|$$

where $|\Phi(\theta)\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{ij\theta} |j\rangle$. It holds that:

$$Pr[x_l < \Theta_{D,\delta} < x_{l+w}] = \frac{1}{2\pi D} \int_{x_l}^{x_{l+w}} \frac{\sin^2[D(\theta - \theta_v)/2]}{\sin^2[(\theta - \theta_v)/2]} d\theta$$

(4)
Proof.

\[
Pr[x_l < \Theta_{D,\delta} < x_{l+w}] = \langle \phi(\theta_v) \rangle \int_{x_l}^{x_{l+w}} E(\theta) \, d\theta \, \langle \phi(\theta_v) \rangle \\
= \int_{x_l}^{x_{l+w}} \langle \phi(\theta_v) \rangle \, E(\theta) \, d\theta \\
= \frac{D}{2\pi D^2} \int_{x_l}^{x_{l+w}} \left| \sum_{j=0}^{D-1} e^{(\theta-\theta_v)ij} \right|^2 d\theta \\
= \frac{1}{2\pi D} \int_{x_l}^{x_{l+w}} \left( \sum_{j=0}^{D-1} \cos[(\theta-\theta_v)j] \right)^2 \\
+ \left( \sum_{j=0}^{D-1} \sin[(\theta-\theta_v)j] \right)^2 d\theta
\]

For any \( x \in \mathbb{R} \), the following two equations hold:

\[
\sum_{j=0}^{D-1} \cos[jx] = \frac{\sin[Dx/2]}{\sin[x/2]} \cos[(D-1)x/2]
\]

\[
\sum_{j=0}^{D-1} \sin[jx] = \frac{\sin[Dx/2]}{\sin[x/2]} \sin[(D-1)x/2]
\]

So finally we have:

\[
Pr[x_l < \Theta_{D,\delta} < x_{l+w}] = \frac{1}{2\pi D} \int_{x_l}^{x_{l+w}} \frac{\sin^2[D(\theta-\theta_v)/2]}{\sin^2[(\theta-\theta_v)/2]} \, d\theta
\]

Lemma 3 Let |\psi(\theta_v)\rangle be a voting state of the protocol. Then it holds:

\[
Pr[x_{l_v} < \Theta_{D,\delta} < x_{l_v+1}] > 0.405
\]

Proof. A simple change of variables in Eq.(4) gives us:

\[
Pr[x_{l_v} < \Theta_{D,\delta} < x_{l_v+1}] = \frac{1}{2\pi D} \int_{0}^{2\pi/D} \frac{\sin^2[D(\theta-\delta)/2]}{\sin^2[(\theta-\delta)/2]} \, d\theta
\]

By setting \((\theta-\delta)/2 = y\), we get:

\[
Pr[x_{l_v} < \Theta_{D,\delta} < x_{l_v+1}] = \frac{1}{\pi D} \int_{-\delta/2}^{(2\pi/D-\delta)/2} \frac{\sin^2[2Dy]}{\sin^2[y]} \, dy
\]

The above is just a function of \( \delta \), which we denote as \( F(\delta) \). In order to lower-bound \( F(\delta) \) we need to find its derivative:

\[
\frac{dF(\delta)}{d\delta} = \frac{1}{2\pi D} \left( \frac{\sin^2[D\delta/2]}{\sin^2[\delta/2]} - \frac{\sin^2[2D\delta]}{\sin^2[(2\pi/D-\delta)/2]} \right)
\]
It is easy to check that:

\[
\frac{dF(\delta)}{d\delta} = 0, \quad \text{when } \delta = 0 \text{ or } \delta = \pi/D
\]

\[
\frac{dF(\delta)}{d\delta} > 0, \quad \text{when } 0 < \delta < \pi/D
\]

\[
\frac{dF(\delta)}{d\delta} < 0, \quad \text{when } \pi/D < \delta < 2\pi/D
\]

It also holds that \( F(0) = F(2\pi/D) \), so the minimum extreme points of our function are equal. As a result we have:

\[
F(\delta) \geq \lim_{\delta \to 0^-} F(\delta) = F(0) \tag{5}
\]

From the fact that:

\[
|\sin x| \leq |x|, \forall x \in \mathbb{R}
\]

\[
|\sin x| \geq |(2/\pi)x|, \forall x \in [0, \pi/2]
\]

\[
|\sin x| \geq |-(2/\pi)x + 2|, \forall x \in [\pi/2, \pi]
\]

It follows:

\[
F(0) \geq \frac{1}{\pi D} \int_0^{\pi/2} \left( \frac{2}{\pi Dy} \right)^2 / y^2 \, dy + \int_{\pi/2}^{\pi} \left( \frac{2}{\pi Dy} + 2 \right)^2 / y^2 \, dy
\]

\[
\geq \frac{4}{\pi^2}
\]

\[
> 0.405
\]

Now in order to prove lemma 4, we need the following proposition:

**Proposition 2.** \( \forall x \in [-2\pi, 2\pi] \) it holds that:

\[
\sin^2 x > \sum_{n=1}^{20} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}
\]

\[
(6)
\]

**Proof.** From the Taylor series expansion at point 0 of \( \cos[x] \), we know that:

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \forall x \in \mathbb{R}
\]

Then:

\[
\sin^2 x = \frac{1}{2} - \frac{\cos[2x]}{2} = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}
\]

\[
= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}
\]
Given the above equation, in order to prove Eq. (6), we simply need to show:

\[ \sum_{n=21}^{\infty} \left( -1 \right)^{n+1} \frac{2^{n-1}x^{2n}}{(2n)!} > 0 \]

If we think of the above as a sum of terms \( a_n \) (\( n = 21, \ldots, \infty \)), for integer \( j \geq 10 \), it holds that:

\[ a_n > 0, \text{ when } n = 2j + 1, \]
\[ a_n < 0, \text{ when } n = 2j. \]

We therefore need to prove that \( \sum_{n=21}^{\infty} a_n > 0 \), which in turn is equivalent to proving that:

\[ |a_n| > |a_{n+1}| \iff 2^{2n-1}x^{2n}/(2n)! > 2^{2n+1}x^{2n+2}/(2n + 2)! \]
\[ \iff 1 > 4x^2/((2n + 1)(2n + 2)) \]
\[ \iff (2n + 1)(2n + 2)/4 > x^2 \]

In this case, the above holds, because the minimum value of \( n \) is 21 and the maximum value of \( x^2 \) is \( 4\pi^2 \).

**Lemma 4** Let \(|\psi(\theta_v)\rangle\) be a voting state of the protocol. Then it holds:

\[ Pr[x_{l_{v-1}} < \Theta_{D,\delta}^v < x_{l_v+2}] > 0.9 \]

**Proof.** We follow exactly the same procedure as lemma 3 and get:

\[
Pr[x_{l_{v-1}} < \Theta_{D,\delta}^v < x_{l_v+2}] = \frac{1}{2\pi D} \int_{x_{l_{v-1}}}^{x_{l_v+2}} \frac{\sin^2[D(\theta - \theta_v)/2]}{\sin^2[(\theta - \theta_v)/2]} d\theta
\]
\[
= \frac{1}{2\pi D} \int_{-2\pi/D}^{4\pi/D} \frac{\sin^2[D(\theta - \delta)/2]}{\sin^2[(\theta - \delta)/2]} d\theta
\]
\[
= \frac{1}{\pi D} \int_{-\pi/D-\delta/2}^{2\pi/D-\delta/2} \frac{\sin^2[Dy]}{\sin^2[y]} dy
\]

where \((\theta - \delta)/2 = y\). Again the above probability depends only on \( \delta \) and can therefore be denoted with \( F(\delta) \). In a similar way as before, we can prove that the minimum of this function is at \( \delta = 0 \) and compute \( F(0) \).
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\[ F(0) = \frac{1}{\pi D} \int_{-\pi/D}^{2\pi/D} \frac{\sin^2[Dy]}{\sin^2[y]} dy \]

\[ \geq \frac{1}{\pi D} \int_{-\pi/D}^{2\pi/D} \sum_{n=1}^{20} \frac{(-1)^{n+1} 12^{n-1} (Dy)^{2n}}{(2n)!} dy \]

\[ = \frac{1}{\pi D} \sum_{n=1}^{20} \int_{-\pi/D}^{2\pi/D} \frac{(-1)^{n+1} 12^{n-1} D^{2n} y^{2n}}{y^2 (2n)!} dy \]

\[ = \frac{1}{\pi D} \sum_{n=1}^{20} \frac{(-1)^{n+1} 12^{n-1} D^{2n}}{(2n)!} \int_{-\pi/D}^{2\pi/D} y^{2(n-1)} dy \]

\[ = \frac{1}{\pi D} \sum_{n=1}^{20} \frac{(-1)^{n+1} 12^{n-1} D^{2n}}{(2n)!} \left[ \frac{y^{2n-1}}{2n-1} \right]_{-\pi/D}^{2\pi/D} \]

\[ \approx 0.9263 \quad (9) \]

**Theorem 2** With overwhelming probability in the number of voters \( N \), algorithm 1 includes \( l_v \) in the Solution vector (i.e. it measures a value in the interval \([x_l, x_{l+1}]\) more than 40% of the time).

\[ \Pr[\text{Solution}[0] = l_v \lor \text{Solution}[1] = l_v] > 1-\frac{1}{\exp(\Omega(N))} \]

**Proof.** We can see each measurement that algorithm 1 performs at each vote state \( |\psi(\theta_v)\rangle \), as an independent Bernoulli trial \( X_l \) with probability of success \( p_l = \Pr[x_l < \Theta_{D,\delta} < x_{l+1}] \). Then the value of \( \text{Record}[l] \) follows the binomial distribution:

\[ X_{\text{Record}[l]} \sim B\left(\frac{\varepsilon N}{2}, p_l\right) \]

We can therefore compute:

\[ \Pr[\text{Solution}[0] = l_v \lor \text{Solution}[1] = l_v] \]

\[ = \Pr[\text{Record}[l_v] \geq 0.4\varepsilon N/2] \]

\[ \geq 1 - \Pr[\text{Record}[l_v] \leq 0.4\varepsilon N/2] \]

\[ \geq 1 - \Pr[\text{Record}[l_v] \leq (1 - \gamma) p_l \varepsilon N/2] \]

\[ \geq 1 - \exp(-\gamma^2 p_l \varepsilon N/6) \]

\[ = 1 - (\exp(-\gamma^2 p_l \varepsilon /6))^N \]

\[ = 1 - 1/\exp(\Omega(N)) \]
Theorem 3 With negligible probability in the number of voters $N$, algorithm includes a value other than $(l_v - 1, l_v, l_v + 1)$ in the Solution vector, i.e. $\forall w \in \{0, \ldots, l_v - 2, l_v + 2, \ldots, D - 1\}$:

$$\Pr[\text{Solution}[0] = w \lor \text{Solution}[1] = w] < 1/\exp(\Omega(N))$$

Proof. Let $w \in \{0, \ldots, D - 1\} \setminus \{l_v - 1, l_v, l_v + 1\}$, then it holds:

$$\Pr[\text{Solution}[0] = w \lor \text{Solution}[1] = w] = \Pr[X_{\text{Record}[w]} \geq 0.4\varepsilon N/2]$$

We know from lemma 4 that $p_w < 0.1$, so $\exists \gamma > 0$ such that $p_w < 0.405$.

$$\Pr[X_{\text{Record}[w]} \geq 0.4\varepsilon N/2] = \Pr[X_{\text{Record}[w]} \geq (1 + \gamma)p_w\varepsilon N/2] < \exp(-\gamma p_w\varepsilon N/6) = (\exp(-\gamma p_w\varepsilon/0))^N = 1/\exp(\Omega(N))$$

Lemma 5 With overwhelming probability in $N$, the Solution vector in algorithm is equal to $[l_v - 1, l_v], [l_v, \text{"Null"}], or [l_v, l_v + 1]$. Specifically,

$$\Pr[\text{Solution} \in \{[l_v - 1, l_v], [l_v, \text{"Null"}], [l_v, l_v + 1]\}] > 1 - 1/\exp(\Omega(N))$$

Proof. Let as define the following events:

$$A = [\text{Solution}[0] = w \lor \text{Solution}[1] = w, w \in \{0, \ldots, l_v - 2, l_v + 2, \ldots, D - 1\}]$$

$$B = [\text{Solution}[0] = l_v \lor \text{Solution}[1] = l_v]$$

Since the cases $\text{Solution} = [l_v, l_v - 1]$ and $\text{Solution} = [l_v + 1, l_v]$ are impossible from the construction of the algorithm, from theorems 2 and 3 it holds:

$$\Pr[\text{Solution} \in \{[l_v - 1, l_v], [l_v, \text{"Null"}], [l_v, l_v + 1]\}] = \Pr[B \lor \neg A] = \Pr[B] - \Pr[B \land A] > 1 - 1/\exp(\Omega(N))$$

1 $p_{l_v} > 0.405 \implies \exists \gamma > 0$ s.t $0.4 = (1 - \gamma)p_{l_v}$
2 The Chernoff bound for a random variable $X \sim B(N, p)$ and expected value $E[X] = \mu$ is: $\Pr[X \leq (1 - \gamma)\mu] \leq \exp(-\gamma^2\mu/3)$
3 The Chernoff bound for a random variable $X \sim B(N, p)$ and expected value $E[X] = \mu$ is: $\Pr[X \leq (1 + \gamma)\mu] \leq \exp(-\gamma\mu/3), \gamma > 1$
Lemma 6. Let $|\psi(\theta_v)|$ be a voting state with $\delta \in [0, 2\pi/D)$ and $l_v = D-1$, where $\delta$ is a continuous random variable. Then it holds:

$$\Pr[x_D - 2 < \Theta_{D,\delta}^v < x_D] + \Pr[x_0 < \Theta_{D,\delta}^v < x_1] > 0.9$$

Proof.

$$\Pr[x_0 < \Theta_{D,\delta}^v < x_1] = \frac{1}{(2\pi D)} \int_{x_0}^{x_1} \left(\frac{\sin[D/2(\theta - \theta_v)]}{\sin[1/2(\theta - \theta_v)]}\right)^2 d\theta$$

Now we set $\theta = \theta - x_D$ to (11) and we have:

$$\Pr[x_0 < \Theta_{D,\delta}^v < x_1] = \frac{1}{(2\pi D)} \int_{x_D}^{x_D+x_1} \left(\frac{\sin[-D\pi + D/2(\theta - \theta_v)]}{\sin[-\pi + 1/2(\theta - \theta_v)]}\right)^2 d\theta$$

Finally we have:

$$\Pr[x_D - 2 < \Theta_{D,\delta}^v < x_D] + \Pr[x_0 < \Theta_{D,\delta}^v < x_1]$$

$$= \frac{1}{(2\pi D)} \int_{x_D}^{x_D+x_1} \left(\frac{\sin[D/2(\theta - \theta_v)]}{\sin[1/2(\theta - \theta_v)]}\right)^2 d\theta$$

From lemma 4, this integral is at least 0.9.

The proof is similar for $l_v = 0$.

Lemma 7. Let $\text{Solution}$ be the matrix of algorithm 2, then it holds:

$$\Pr[\text{Solution} \in \{\{l_v - 1, l_v\}, \{l_v\}, \{l_v, l_v + 1\}\}]$$

$$= \Pr[\text{Solution} \in \{\{l_v - 1, l_v\}, \{l_v\}, \{l_v, l_v + 1\}\}]$$

Proof. (sketch) It holds that:

$$\Pr[\text{Solution} \in \{\{l_v - 1, l_v\}, \{l_v\}, \{l_v, l_v + 1\}\}]$$

$$= \Pr[\text{Solution} \in \{\{l_v - 1, l_v\}\}]$$

$$+ \Pr[\text{Solution} \in \{l_v\}]$$

$$+ \Pr[\text{Solution} \in \{l_v, l_v + 1\}]$$

We need to prove that:

$$\Pr[\text{Solution} \in \{l_v - 1, l_v\}] = \Pr[\text{Solution} = [l_v - 1, l_v]]$$
From the construction of the algorithm\textsuperscript{1} we know that:

\[ \Pr[\text{Solution} = [l_v, l_v - 1] | \text{Solution} \in \{ l_v - 1, l_v \}] = 0 \] (23)

This is true because the values of the \text{Solution} are from the matrix \text{Record} in a progressive manner. So under the assumption that both \( l_v, l_v - 1 \) had appeared at least 40\% times, they inserted in a progressive order. The only time they will not is the case in which \( l_v = 0 \) and \( l_v - 1 = D - 1 \). At first the order is \([0, D - 1]\), but because of the special condition we had in our algorithm the order switches to \([D - 1, 0]\).

It holds that:

\[ \Pr[\text{Solution} = [l_v, l_v - 1] | \text{Solution} \in \{ l_v - 1, l_v \}] = \Pr[\text{Solution} = [l_v, l_v - 1]] + \Pr[\emptyset] \] (24)

\[ \Pr[\text{Solution} = [l_v, l_v - 1]] = \Pr[\text{Solution} = [l_v, l_v - 1]] \] (25)

\[ \Pr[\text{Solution} = [l_v, l_v - 1]] = 0 \] (26)

Similar are the other cases.