CONSTRUCTION OF NEW SYMPLECTIC COHOMOLOGY

$S^2 \times S^2$

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Abstract. In this article, we present new symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. The generalization of this construction is given as well, an infinite family of symplectic 4-manifolds cohomology equivalent to $\#(2g-1)(S^2 \times S^2)$ for any $g \geq 2$. We also compute the Seiberg-Witten invariants of these manifolds.

1. Introduction

The aim of this article is to construct new examples of symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. The similar problems have been studied in the algebro-geometric category, i.e. the existence of algebraic surfaces homology equivalent but not isomorphic to $\mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) as an algebraic variety. Mumford [Mu] and Pardini [P] gave the constructions of such fake $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

We study this problem in the symplectic category and prove the following results.

**Theorem 1.1.** Let $K$ be a genus one fibered knot in $S^3$. Then there exist a symplectic 4-manifold $X_K$ cohomology equivalent to $S^2 \times S^2$.

**Theorem 1.2.** Let $K'$ be any genus $g$ fibered knot in $S^3$. Then there exist an infinite family of symplectic 4-manifolds $V_{K'}$ that is cohomology equivalent to $\#(2g-1)(S^2 \times S^2)$ for $g \geq 2$.

This paper is organized as follows: Section 2 contains the basic definitions and formulas that will be important throughout this paper. Section 3 gives quick introduction to Seiberg-Witten invariants. The remaining two sections are devoted to the construction of family of symplectic 4-manifolds cohomology equivalent to $\#(2g-1)(S^2 \times S^2)$ and the fundamental group computation for our examples.

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2. Preliminaries

2.1. Knots and 3-Manifolds. In this section, we give short introduction to the fibered knots, Dehn surgery and state a few facts that we will be used in our construction. We refer the reader to Section 10.H [R] for a more complete treatment.

Definition 2.1. (Torus Knots) The knot which wraps around the solid torus in the longitudinal direction \( p \) times and in the meridional direction \( q \) times called \((p,q)\) torus knot and denoted as \( T_{p,q} \).

Lemma 2.2. [S] a) The group of the torus knot \( T_{p,q} \) can be represented as follows:

\[ \pi_1(S^3/T_{p,q}) = \langle u, v \mid u^p = v^q \rangle \]

b) The elements \( m = u^s v^r \), \( l = w^p m^{-pq} \), where \( pr + qs = 1 \), describe meridian and longitude of the \( T_{p,q} \) for a suitable chosen basepoint.

All torus knots belongs to the larger category of fibered knots.

Definition 2.3. A knot \( K \) in \( S^3 \) is called fibered if there is fibration \( f : S^3 \setminus K \rightarrow S^1 \) behaving “nicely” near \( K \). This means that \( K \) has a neighbourhood framed as \( S^1 \times D^2 \), with \( K \cong S^1 \times 0 \) and restriction of the map \( f \) to \( S^1 \times (D^2 - 0) \) is the map to \( S^1 \) given by \( (t, x) \rightarrow x/|x| \).

It follows from the definition that a preimage for each point \( p \in S^1 \) is Seifert surface for the given knot. The genus of this Seifert surface will be called the genus of the given fibered knot.

The fibered knots form a large class among the all classical knots. Below we state two theorems that can be used to detect if the given knot is fibered or not.

Theorem 2.4. [S] The knot \( K \subset S^3 \) is a fibered knot of genus \( g \) if and only if the commutator subgroup of its knot group \( \pi_1(S^3 \setminus K) \) is finitely-generated and free group of rank \( 2g \).

Theorem 2.5. [BZ] The Alexander polynomial \( \Delta_K(t) \) of a fibered knot in \( S^3 \) is monic i.e. the first and last non-zero coefficients of \( \Delta_K(t) \) are \( \pm 1 \).

If a genus one knot is fibered, then it can be shown by Theorem 1.5 and also by explicit constructon [BZ] that it is either trefoil or figure eight knot. Also, one can construct infinitely many fibered knots for fixed genus \( g \geq 2 \).

Definition 2.6. (Dehn Surgery) Let \( K \) be a knot in the 3-sphere \( S^3 \). A Dehn surgery on \( S^3 \) is the surgery operation that removes tubular neighborhood of knot \( K \) and gluing in the copy of solid torus \( J = S^1 \times D^2 \) using the any diffeomorphism of the boundary torus. Let \( \gamma = pm + ql \) be an essential loop on the boundary torus such that the meridian of the solid torus \( J \) is glued to the curve \( \gamma \), where \((m,l)\) is the standard meridian and longitude.
for knot $K$ and $p, q$ are coprime. The rational number $p/q$ or $\infty$, slope, is assigned to each surgery.

2.2. Generalized fiber sum.

**Definition 2.7.** Let $X$ and $Y$ be closed oriented smooth 4-manifolds each containing a smoothly embedded surface $\Sigma$ of genus $g \geq 1$. Assume $\Sigma$ represents a homology of infinite order and has self-intersection zero in $X$ and $Y$, so that there exists a product tubular neighborhood, say $\nu\Sigma \cong \Sigma \times D^2$, in both $X$ and $Y$. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi : \Sigma \times S^1 \to \Sigma \times S^1$, we can glue $X \setminus \nu\Sigma$ and $Y \setminus \nu\Sigma$ along the boundary $\partial(\nu\Sigma) \cong \Sigma \times S^1$. The resulting closed oriented smooth 4-manifold, denoted $X \#_\psi Y$, is called a *generalized fiber sum* of $X$ and $Y$ along $\Sigma$.

**Definition 2.8.** Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4-manifold $X$, respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$

**Lemma 2.9.** Let $X$ and $Y$ be closed, oriented, smooth 4-manifolds containing an embedded surface $\Sigma$ of self-intersection 0. Then

$$c_1^2(X \#_\psi Y) = c_1^2(X) + c_1^2(Y) + 8(g - 1),$$
$$\chi_h(X \#_\psi Y) = \chi_h(X) + \chi_h(Y) + (g - 1),$$

where $g$ is the genus of the surface $\Sigma$.

**Proof.** The above formulas simply follow from the well-known formulas

$$e(X \#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X \#_\psi Y) = \sigma(X) + \sigma(Y). \quad \square$$

If $X$ and $Y$ are symplectic 4-manifolds and $\Sigma$ is a symplectic submanifold in both, then according to a theorem of Gompf [Go], $X \#_\psi Y$ admits a symplectic structure. In such a case, we will call $X \#_\psi Y$ a *symplectic sum*.

3. Seiberg-Witten Invariants

In this section, we review the basics of Seiberg-Witten invariants introduced by Seiberg and Witten. Let us recall that the Seiberg-Witten invariant of a smooth closed oriented 4-manifold $X$ with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of $\text{spin}^c$ structures over $X$ [W]. For simplicity, we assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then there is a natural identification of the $\text{spin}^c$ structures of $X$ with the characteristic elements of $H^2(X, \mathbb{Z})$ as follows: to each $\text{spin}^c$ structure $s$ over $X$ corresponds a bundle of positive spinors $W^+_s$ over $X$. Let $c(s) \in H_2(X)$ denote the Poincaré dual of $c_1(W^+_s)$. Each $c(s)$ is a characteristic element of $H_2(X, \mathbb{Z})$ (i.e. its Poincaré dual $\hat{c}(s) = c_1(W^+_s)$ reduces mod 2 to $w_2(X)$).
In this set up, we can view the Seiberg-Witten invariant as an integer valued function

\[ \text{SW}_X : \{ K \in H^2(X, \mathbb{Z}) | K \equiv w_2(TX) \pmod{2} \} \to \mathbb{Z}. \]

If \( \text{SW}_X(\beta) \neq 0 \), then we call \( \beta \) a basic class of \( X \). It is a fundamental fact that the set of basic classes is finite. Furthermore, if \( \beta \) is a basic class, then so is \( -\beta \) with

\[ \text{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4} \text{SW}_X(\beta) \]

where \( e(X) \) is the Euler characteristic and \( \sigma(X) \) is the signature of \( X \).

When \( b_2^+(X) > 1 \), then Seiberg-Witten invariant is a diffeomorphism invariant. It does not depend on the choice of generic metric on \( X \) or a generic perturbation of Seiberg-Witten equations.

If \( b_2^+(X) = 1 \), then the Seiberg-Witten invariant depends on the chosen metric and perturbation of Seiberg-Witten equations. Let us recall that the perturbed Seiberg-Witten moduli space \( M_X(\beta, g, h) \) is defined as the solutions of the Seiberg-Witten equations

\[ F^+_A = q(\psi) + ih, \; D_A \psi = 0 \]

divided by the action of the gauge group. Where \( A \) is connection on the line bundle \( L \) with \( c_1(L) = \beta \), \( g \) is riemannian metric on \( X \), \( \psi \) is the section of the positive spin bundle corresponding to the \( \text{spin}^c \) structure determined by \( \beta \), \( F^+_A \) is the self-dual part of the curvature \( F_A \), \( D_A \) is the twisted Dirac operator, \( q \) is a quadratic function, and \( h \) is self-dual 2-form on \( X \). If \( b_2^+(X) \geq 1 \) and \( h \) is generic metric, then Seiberg-Witten moduli space \( M_X(\beta, g, h) \) is closed manifold with dimension \( d = (\beta^2 - 2e(X) - 3\sigma(X))/4 \). The Seiberg-Witten invariant defined as follows:

\[
\begin{align*}
\text{SW}_X(\beta) &= \langle [M_X(\beta, g, h)], \mu^{d/2} \rangle & \text{if } d \geq 0 \text{ and even} \\
\text{SW}_X(\beta) &= 0 & \text{otherwise}
\end{align*}
\]

where \( \mu \in H^2(M_X(\beta, g, h), \mathbb{Z}) \) is the Euler class of the base fibration.

When \( b_2^+(X) = 1 \), the Seiberg-Witten invariant \( \text{SW}_X(\beta, g, h) \) depends on \( g \) and \( h \). Because of this, there are two types of Seiberg-Witten invariants: \( \text{SW}_X^+ \) and \( \text{SW}_X^- \).

**Theorem 3.1.** [KM], [OS] Let \( X \) be closed, oriented, smooth 4-manifold with \( b_1(X) = 0 \) and \( b_2^+(X) = 1 \). Fix a homology orientation of \( H^2_+(X, \mathbb{R}) \). For given riemannian metric \( g \) let \( \omega^g_+ \) be the unique \( g \)-harmonic self-dual 2-form that has norm 1 and is compatible with the orientation of \( H^2_+(X, \mathbb{R}) \). Then for each characteristic element \( \beta \) with \( (\beta^2 - 2e(X) - 3\text{sign}(X))/4 \geq 0 \)
Theorem 3.8. Let the following holds: If \((2\pi\beta + h_1) \cdot \omega_+^0\) and \((2\pi\beta + h_2) \cdot \omega_+^{g_2}\) are not zero and have same sign, then \(SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)\).

Definition 3.2. If \((2\pi\beta + h) \cdot \omega_+^0 > 0\), then write \(SW_X^+(\beta)\) for \(SW_X(\beta, g, h)\). If \((2\pi\beta + h) \cdot \omega_+^0 < 0\), then write \(SW_X^-(\beta)\) for \(SW_X(\beta, g, h)\).

Theorem 3.3. Let \(X\) be closed, oriented, smooth 4-manifold with \(b_1(X) = 0\) and \(b_2^+(X) = 1\) and \(b_2^- \leq 9\). Then for each characteristic element \(\beta\), pairs of riemannian metrics \(g_1, g_2\) and small perturbing 2-forms \(h_1, h_2\) then \(SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)\).

Proof. Let \(\beta\) be a characteristic element for which \(d \geq 0\). Then \(2\epsilon(X) + 3\text{sign}(X) = 4 + 5b_2^+ - b_2^- \geq 0\), which as implies \(\beta^2 \geq 0\). As a corollary, it follows that \(\beta \cdot \omega_+^0\) and \(\beta \cdot \omega_+^{g_2}\) are both non-zero and have same signs. Now using the Theorem 3.1, the result follows.

Theorem 3.4. (Wall crossing formula) Assume that \(X\) is closed, oriented, smooth 4-manifold with \(b_1(X) = 0\) and \(b_2^+(X) = 1\). Then for each characteristic line bundle \(L\) on \(X\) such that the formal dimension of the Seiberg-Witten moduli space is non-negative even integer \(2m\), then \(SW_X(L) - SW_X(-L) = (-1)^m\).

Note that when \(b_2^- \leq 9\), it follows from the above result that there is well defined Seiberg-Witten invariant which will be denoted as \(SW_X^0(\beta)\).

Theorem 3.5. Suppose that \(X\) is closed symplectic 4-manifold with \(b_2^+(X) > 1\) \((b_2^+(X) = 1)\). If \(K_X\) is a canonical class of \(X\), then \(SW_X(\pm K_X) = \pm 1\) \((SW_X(-K_X) = \pm 1)\).

Definition 3.6. The 4-manifold \(X\) is of simple type if each basic class \(\beta\) satisfies the equation \(\beta^2 = c_1^2(X) = 3\sigma(X) + 2\epsilon(X)\). If \(X\) is symplectic manifold of \(b_2^+(X) > 1\) then it has simple type.

Theorem 3.7. (Generalized adjunction formula for \(b_2^+ > 1\)) Assume that \(\Sigma \subset X\) is an embedded, oriented, connected surface of genus \(g(\Sigma)\) with self-intersection \(|\Sigma|^2 \geq 0\) and represents nontrivial homology class. Then for every Seiberg-Witten basic class \(\beta\), \(2g(\Sigma) - 2 \geq |\Sigma|^2 + |\beta(|\Sigma|)|\). If \(X\) is of simple type and \(g(\Sigma) > 0\), then the same inequality holds for \(\Sigma\) with arbitrary square \(|\Sigma|^2\).

Theorem 3.8. (Generalized adjunction formula for \(b_2^+ = 1\)) Let \(\Sigma \subset X\) is an embedded, oriented, connected surface of genus \(g(\Sigma)\) with self-intersection \(|\Sigma|^2 \geq 0\) and represents nontrivial homology class. Then any characteristic class \(\beta\) with \(SW_X^0(\beta) \neq 0\) satisfy the following inequality \(2g(\Sigma) - 2 \geq \Sigma^2 + |\beta(|\Sigma|)|\).
4. Symplectic manifolds cohomology equivalent to $S^2 \times S^2$

To construct our manifolds, we start with symplectic 4-manifolds described below. By applying Gompf’s symplectic fiber sum operation, we will obtain the manifolds $X_K$.

Let $K$ be a genus one fibered knot (i.e. trefoil or figure eight knot) in $S^3$ and $m$ a meridional circle to $K$. Perform 0-framed Dehn surgery on $K$ and denote the resulting 3-manifold by $M_K$. The manifold $M_K$ has same integral homology as $S^2 \times S^1$, where class of $m$ generates $H_1(M_K)$. Since the knot $K$ is genus one fibered knot, it follows that the manifold $M_K \times S^1$ is a torus bundle over the torus which is homology equivalent to $T^2 \times S^2$. Since $K$ is a fibered knot, $M_K \times S^1$ admits a symplectic structure. Note that $m \times S^1$ is the section of this fibration. The first homology of $M_K \times S^1$ is generated by the standard first homology classes of the torus section and the classes $\gamma_1$ and $\gamma_2$ of the fiber $F$ of the given fibration is trivial in the homology. Smoothly embedded torus section $T_m = m \times S^1$ has a self-intersection zero and its neighborhood in $M_K \times S^1$ has a canonical identification with $T_m \times D^2$.

The intermediate building block in our construction will be the twisted fiber sum of the two copies of manifold $M_K \times S^1$, where we identify the fiber of one fibration to the base of other. Let $Y_K$ denote the mentioned twisted fiber sum $Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1$. It follows from Gompf’s theorem $[FS2]$ that $Y_K$ is symplectic.

Let $T_1$ be the section of the first copy of $M_K \times S^1$ in $Y_K$ and $T_2$ be the fiber of the second copy. Then $T_1 \# T_2$ symplectically embeds into $Y_K$ $[FS2]$. Now suppose that $Y_K \# T_2$ symplectically embeds into $Y_K$ $[FS2]$. Let $(m, x, \gamma_1, \gamma_2)$ be the generators of $H_1(\Sigma_2)$ under the inclusion-induced homomorphism. We choose the diffeomorphism $\phi : T_1 \# T_2 \to T_1 \# T_2$ of the $\Sigma_2$ that changes the generators of the first homology according to the following rule: $\phi(m') = \gamma_1$, $\phi(\gamma_1') = m$, $\phi(x') = \gamma_2$ and $\phi(\gamma_2') = x$. Next, we take the fiber sum along the genus two surface $\Sigma_2$ and denote the new symplectic manifold as $X_K$ i.e. $X_K = Y_K \#_\phi Y_K$. We will show that the new manifold $X_K$ has a trivial first Betti number and has same integral cohomology as $S^2 \times S^2$. Furthermore, $e(X_K) = 4$, $\sigma(X_K) = 0$, $c_1^2(X_K) = 8$, and $\chi_b(X_K) = 1$. We will compute $H_1(X_K)$ first by applying Mayer-Vietoris sequence and next by directly computing the fundamental group of $X_K$.

**Lemma 4.1.** $H_1(X_K, \mathbb{Z}) = 0$ and $H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

**Proof.** We use Mayer-Vietoris sequence to compute the homology of $X_K = Y_K \#_\phi Y_K$. Let $Y_1 = Y_2 = Y_K \setminus \nu \Sigma_2$. Note that $Y_1 \cap Y_2$ is homologous to $\Sigma_2 \times S^1$. By applying the reduced Mayer-Vietoris to the triple $(X_K, Y_1, Y_2)$, we have the following long exact sequence
Proof. Using the lemma 2.10, we have
\[
\cdot\cdot\cdot \rightarrow H_2(S^1 \times \Sigma_2) \rightarrow H_2(Y_1) \oplus H_2(Y_2) \rightarrow H_2(X_K) \rightarrow H_1(\Sigma_2 \times S^1) \rightarrow H_1(Y_1) \oplus H_1(Y_2) \rightarrow H_1(X_K) \rightarrow 0
\]

The simple computation by Kunneth formula yields \( H_1(\Sigma_2 \times S^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = < \lambda > \oplus < m > \oplus < x > \oplus < \gamma_1 > \oplus < \gamma_2 > \). Also, we have \( H_1(Y_1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m > \oplus < x > \) and \( H_1(Y_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m' > \oplus < x' > \).

Let \( \phi_* \) and \( \delta \) denote the last two arrows in the long exact sequence above. Because the way map \( \phi \) is defined, the essential homology generators will map to trivial ones, thus \( \phi_*(m) = \phi_*(x) = \phi_*(m') = \phi_*(x') = 0 \). Since \( \text{Im}(\phi_*) = \text{Ker}(\delta) \), we conclude that \( H_1(X_K) = \text{Ker}(\delta) = \text{Im}(\phi_*) = 0 \).

Next, by using the facts that \( b_1 = b_3 = 0 \), \( b_0 = b_4 = 1 \) and symplectic sum formula for Euler characteristics, we compute \( b_2 = e(Y_K) + e(Y_K) + 2 = 0 + 0 + 2 = 2 \).

We conclude that \( H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \). Note that a basis for the second homology are classes of \( SU_2 = S \) and the new genus two surface \( T \) resulting from the last fiber sum operation, where \( S^2 = T^2 = 0 \) and \( S \cdot T = 1 \). Thus, the manifolds obtained by the above construction have intersection form \( H \), so they are spin. \( \square \)

**Lemma 4.2.** \( e(X_K) = 4 \), \( \sigma(X_K) = 0 \), \( c_1^2(X_K) = 8 \), and \( \chi_h(X_K) = 1 \).

**Proof.** Using the lemma 2.10, we have \( e(X_K) = 2e(Y_K) + 4 \), \( \sigma(X_K) = 2\sigma(Y_K) \), \( c_1^2(X_K) = 2c_1^2(Y_K) + 8 \), and \( \chi_h(X_K) = 2\chi_h(Y_K) + 1 \). Since \( e(Y_K) = 0 \), \( \sigma(Y_K) = 0 \), \( c_1^2(Y_K) = 0 \) and \( \chi_h(Y_K) = 0 \) result follows. \( \square \)

### 4.1. Fundamental Group Computation for Trefoil.

#### 4.1.1. Step 1: Fundamental Group of \( M_K \times S^1 \).

In this section we will assume that \( K \) is trefoil. The case when \( K \) is figure eight can be treated similarly.

Let \( a \) and \( b \) denote the Wirtinger generators of the trefoil knot. Then the group of \( K \) has the following presentation

\[
\pi_1(S^3 \setminus \nu K) = \langle a, b \mid aba = bab \rangle = \langle u, v \mid u^2 = v^3 \rangle
\]

where \( u = bab \) and \( v = ab \). By Lemma 2.2, the homotopy classes of the meridian and the longitude of the trefoil are given as follows: \( m = uv^{-1} = b \) and \( l = u^2(vu^{-1})^{-6} = aba^{-1}ab \). Notice that \( \gamma_1 = a^{-1}b \) and \( \gamma_2 = b^{-1}aba^{-1} \) generate the image of the fundamental group of the Seifert surface of \( K \) under the inclusion-induced homomorphism. Let \( M_K \) denote the result of 0-surgery on \( K \).
Lemma 4.3.

\[ \pi_1(M_K \times S^1) = \pi_1(M_K) \oplus \mathbb{Z} \]

\[ = \langle a, b, x \mid aba = bab, ab^2ab^{-4} = 1, [a, x] = [b, x] = 1 \rangle. \]

Proof. Notice that the fundamental group of \( M_K \) is obtained from the knot group of the trefoil by adjoining the relation \( l = u^2(u^{-1})^{-6} = ab^2ab^{-4} = 1 \). Thus, we have the presentation given above. \( \square \)

4.1.2. Step 2: Fundamental Group of \( Y_K \). Next, we take the two copies of the manifold \( M_K \times S^1 \). In the first copy, take a tubular neighborhood of the torus section \( b \times x \), remove it from \( M_K \times S^1 \) and denote the resulting manifold as \( C_S \). In the second copy, we remove a tubular neighborhood of the fiber \( F \) and denote the complement by \( C_F \).

Lemma 4.4. Let \( C_S \) be the complement of a neighborhood of a section in \( M_K \times S^1 \). Then we have

\[ \pi_1(C_S) = \langle a, b, x \mid aba = bab, [a, x] = [b, x] = 1 \rangle. \]

Proof. Note that \( C_S = (M_K \setminus \nu(b)) \times S^1 = (S^3 \setminus \nu K) \times S^1 \). \( \square \)

Lemma 4.5. Let \( C_F \) be the complement of a neighborhood of a fiber in \( M_K \times S^1 \). Then we have

\[ \pi_1(C_F) = \langle \gamma_1', \gamma_2', d, y \mid \gamma_1' \gamma_2' = [y, \gamma_1'] = [y, \gamma_2'] = 1, \]

\[ d\gamma_1' d^{-1} = \gamma_1' \gamma_2', d\gamma_2' d^{-1} = (\gamma_1')^{-1} \rangle. \]

Proof. To compute the fundamental group of \( C_F \), we will use the following observation: \( C_F \) is homotopy equivalent to a torus bundle over a wedge of two circles. Thus the generators \( d \) and \( y \) do not commute in the fundamental group of \( C_F \). Also, the monodromy along the circle \( y \) is trivial whereas the monodromy along the circle \( d \) is the same as the monodromy of \( M_K \). This implies that the fundamental group of \( C_F \) has the presentation given above. \( \square \)

Lemma 4.6. Let \( Y_K \) be the symplectic sum of two copies of \( M_K \times S^1 \), identifying a section in one copy with a fiber in the other copy. If the gluing map \( \psi \) satisfies \( \psi_\ast(x) = \gamma_1' \) and \( \psi_\ast(b) = \gamma_2' \), then

\[ \pi_1(Y_K) = \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab, [x, a] = [x, b] = 1, \]

\[ [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1, d\gamma_1' d^{-1} = \gamma_1' \gamma_2', d\gamma_2' d^{-1} = (\gamma_1')^{-1}, \]

\[ x = \gamma_1', b = \gamma_2', ab^2ab^{-4} = [d, y] \rangle \]

\[ = \langle a, b, x, d, y \mid aba = bab, [x, a] = [x, b] = 1, \]

\[ [y, x] = [y, b] = 1, dx d^{-1} = xb, db d^{-1} = x^{-1}, ab^2ab^{-4} = [d, y] \rangle. \]
Proof. By Van Kampen’s Theorem, \( \pi_1(Y_K) = \pi_1(C_S) \ast \pi_1(C_F)/\pi_1(T^3) \). One circle factor of \( T^3 \) is identified with the longitude of \( K \) on one side and the meridian of the torus fiber in \( M_K \times S^1 \) on the other side. This gives the last relation.

Inside \( Y_K \), we can find a genus 2 symplectic submanifold \( \Sigma_2 \) which is the internal sum of a punctured fiber in \( C_S \) and a punctured section in \( C_F \). The inclusion-induced homomorphism maps the standard generators of \( \pi_1(\Sigma_2) \) to \( ab^{-1}, b^{-1}aba^{-1}, d \) and \( y \).

Lemma 4.7. There are nonnegative integers \( m \) and \( n \) such that

\[
\pi_1(Y_K \setminus \nu \Sigma_2) = \langle a, b, x, d, y; g_1, \ldots, g_m \mid \text{aba = bab}, \text{y,y, b = [y, b] = 1, dxd}^{-1} = xb, \text{dbd}^{-1} = x^{-1}, ab^{2}ab^{-4} = [d, y], r_1 = \cdots = r_n = 1, r_{n+1} = 1 \rangle
\]

where the generators \( g_1, \ldots, g_m \) and relators \( r_1, \ldots, r_n \) all lie in the normal subgroup \( N \) generated by the element \( [x, b] \), and the relator \( r_{n+1} = 1 \) is a word in \( x, a \) and elements of \( N \). Moreover, if we add an extra relation \( [x, b] = 1 \) to \( \square \), then the relation \( r_{n+1} = 1 \) simplifies to \( [x, a] = 1 \).

Proof. This follows from Van Kampen’s Theorem. Note that \( [x, b] \) is a meridian of \( \Sigma_2 \) in \( Y_K \). Hence setting \( [x, b] = 1 \) should turn \( \pi_1(Y_K \setminus \nu \Sigma_2) \) into \( \pi_1(Y_K) \). Also note that \( [x, a] \) is the boundary of a punctured section in \( C_S \setminus \nu(\text{fiber}) \), and is no longer trivial in \( \pi_1(Y_K \setminus \nu \Sigma_2) \). By setting \( [x, b] = 1 \), the relation \( r_{n+1} = 1 \) is to turn into \( [x, a] = 1 \).

It remains to check that the relations in \( \pi_1(Y_K) \) other than \( [x, a] = [x, b] = 1 \) remain the same in \( \pi_1(Y_K \setminus \nu \Sigma_2) \). By choosing a suitable point \( \theta \in S^1 \) away from the image of the fiber that forms part of \( \Sigma_2 \), we obtain an embedding of the knot complement \( (S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber}) \). This means that \( ab\alpha = baab \) holds in \( \pi_1(Y_K \setminus \nu \Sigma_2) \). Since \( [\Sigma_2]^2 = 0 \), there exists a parallel copy of \( \Sigma_2 \) outside \( \nu \Sigma_2 \), wherein the identity \( ab^{2}ab^{-4} = [d, y] \) still holds. The other remaining relations in \( \pi_1(Y_K) \) are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in \( \pi_1(Y_K \setminus \nu \Sigma_2) \).

4.1.3. Step 3: Fundamental Group of \( X_K \). Now take two copies of \( Y_K \). Suppose that the fundamental group of the second copy has \( e, f, z, s, t \) as generators, and the inclusion-induced homomorphism in the second copy maps the generators of \( \pi_1(\Sigma_2) \) to \( e^{-1}f, f^{-1}ef^{-1}, s \) and \( t \). Let \( X_K \) denote the symplectic sum of two copies of \( Y_K \) along \( \Sigma_2 \), where the gluing map \( \psi \) maps the generators as follows:

\[
\psi_*(a^{-1}b) = s, \ \psi_*(b^{-1}aba^{-1}) = t, \ \psi_*(d) = e^{-1}f, \ \psi_*(y) = f^{-1}ef^{-1}.
\]
Lemma 4.8. There are nonnegative integers \( m \) and \( n \) such that

\[
\pi_1(X_K) = \langle a, b, x, d, y; e, f, z, s, t; g_1, \ldots, g_m; h_1, \ldots, h_m \mid \nonumber
\]
\[
aba = bab, [y, x] = [y, b] = 1, \nonumber
\]
\[
dxd^{-1} = xb, dbd^{-1} = x^{-1}, ab^2ab^{-4} = [d, y], \nonumber
\]
\[
r_1 = \cdots = r_n = r_{n+1} = 1, r'_1 = \cdots = r'_{n+1} = 1, \nonumber
\]
\[
efe = fef, [t, z] = [t, f] = 1, \nonumber
\]
\[
szs^{-1} = zf, sfs^{-1} = z^{-1}, ef^2ef^{-4} = [s, t], \nonumber
\]
\[
d = e^{-1}f, y = f^{-1}efe^{-1}, a^{-1}b = s, b^{-1}aba^{-1} = t, [x, b] = [z, f] \rangle, \nonumber
\]

where \( g_i, h_i \) (\( i = 1, \ldots, m \)) and \( r_j, r'_j \) (\( j = 1, \ldots, n \)) all lie in the normal subgroup \( N \) generated by \([x, b] = [z, f]\). Moreover, \( r_{n+1} \) is a word in \( x, a \) and \( e \) and elements of \( N \), and \( r'_{n+1} \) is a word in \( z, e \) and elements of \( N \).

Proof. This is just a straightforward application of Van Kampen’s Theorem and Lemma 4.7. Also, notice that the abelianization of \( \pi_1(X_K) \) is trivial. \( \square \)

The different symplectic cohomology \( S^2 \times S^2 \)'s can be obtained if we use other genus 1 fibered knot, the figure eight knot, or combination of both in our construction. These manifolds can be distinguished by their fundamental groups. In addition, using a family of non fibered genus one twist knots, we also obtain an infinite family of cohomology \( S^2 \times S^2 \). This family of cohomology \( S^2 \times S^2 \)'s will not be symplectic anymore and can be distinguished by Seiberg-Witten invariants.

4.2. Seiberg-Witten invariants for manifold \( X_K \). Let \( C \) be a basic class of the manifold \( X_K \). We can write \( C \) as a linear combination of \( S \) and \( T \), i.e. \( C = aS + bT \). \( X_K \) is a symplectic manifold and has simple type. So for any basic class \( C \), \( C^2 = 3\sigma(X_K) + 2\varepsilon(X_K) = 8 \). It follows that \( 2ab = 8 \). Next we apply the adjunction inequality to \( S \) and \( T \) to get \( 2g(S) - 2 \geq |S|^2 + |C(S)| \) and \( 2g(T) - 2 \geq |T|^2 + |C(T)| \). These gives us two more restriction on \( a \) and \( b \) : \( 2 \geq |b| \) and \( 2 \geq |a| \). Thus, it follows that \( C = \pm(2S + 2T) \), which is \( \pm \) the canonical class \( K_{X_K} = 2S + 2T \) of \( X_K \). Notice that, since \( b_2^c(X_K) \leq 9 \) and \( (-K_{X_K}) \cdot \omega < 0 \), we have well defined \( SW^0_{X_K} \). Now it follows from the theorems of Section 2 that \( SW^0_{X_K}(-K_{X_K}) = SW^-_{X_K}(-K_{X_K}) = \pm 1 \).

5. Symplectic manifolds cohomology equivalent to \( \#_{(2g-1)}(S^2 \times S^2) \)

We will modify our construction in the previous section to get an infinite family of symplectic 4-manifolds cohomology equivalent to \( \#_{(2g-1)}(S^2 \times S^2) \) for any \( g \geq 2 \).

Let \( K' \) denote the genus \( g \) fibered knot in \( S^3 \) and \( m \) a meridional circle to \( K' \). We first perform 0-framed surgery on \( K' \) and denote the resulting
3-manifold by $Z_{K'}$. The 4-manifold $Z_{K'} \times S^1$ is $\Sigma_g$ bundle over the torus and has the same integral homology as $T^2 \times S^2$. Since $K'$ is a fibered knot, $Z_{K'} \times S^1$ is symplectic manifold. Again, there is a torus section $m \times S^1 = T_m$ of this fibration. The first homology of $Z_{K'} \times S^1$ is generated by the standard first homology classes of this torus section. The standard homology generators of the fiber $F$, which we denote as $\alpha_2, \beta_2, \ldots, \alpha_{g+1}$, and $\beta_{g+1}$ of the given bundle is trivial in the homology. The section $T_m$ has zero self-intersection and its neighborhood in $Z_{K'} \times S^1$ has a canonical identification with $T_m \times D^2$.

We form the twisted fiber sum of the manifold $M_K \times S^1$ with $Z_{K'} \times S^1$ where we identify fiber of the first fibration to the section of other. Let $W_{K'}$ denote the new manifold $W_{K'} = M_K \times S^1_{F=T_m} Z_{K'} \times S^1$. Again, it follows from Gompf’s theorem [Go] that $W_{K'}$ is symplectic.

Let $T_1$ be the section of the $M_K \times S^1$ which we discussed earlier and $\Sigma_g = \Sigma_1' \times \Sigma_g$ be the fiber of the $Z_{K'} \times S^1$. Then $T_1 \# \Sigma_g$ embeds in $W_{K'}$ and has self-intersection zero. Now suppose that $W_{K'}$ is the symplectic 4-manifold given, and $\Sigma_{g+1} = T_1 \# \Sigma_g$ is the genus $g + 1$ symplectic submanifold of self-intersection 0 sitting inside of $W_{K'}$. Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{g+1}, \beta_{g+1}$ be the generators of the first homology of the surface $T_1 \# \Sigma_g = \Sigma_{g+1}$. Let $\psi : T^2 \# \Sigma_g \to \Sigma_g \# T^2$ be the diffeomorphism of the genus $g + 1$ surface that changes the generators of the first homology according to the following rule: $\psi(\alpha_1) = \alpha_{g+1}, \psi(\alpha_{g+1}) = \alpha_1, \psi(\beta_1) = \beta_{g+1}$, and $\psi(\beta_{g+1}) = \beta_1$. Take the fiber sum along this genus $g + 1$ surface $\Sigma_{g+1}$ and denote the resulting symplectic manifold as $V_{K'}$. The new manifold $V_{K'} = W_{K'} \# \psi W_{K'}$ has trivial first homology and has same homology of $\#_{(2g-1)} S^2 \times S^2$.

**Lemma 5.1.** $H_1(V_{K'}, \mathbb{Z}) = 0$, $H_2(V_{K'}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \cdots \oplus \mathbb{Z}$ where there are $2(2g - 1)$ copies of $\mathbb{Z}$

*Proof.* The proof is similar to genus one case and can be obtained by applying Mayer-Vietoris sequence.

Note that $H_2(V_{K'}, \mathbb{Z})$ has rank $4g - 2$. The base for second homology are classes of the $\Sigma_{g+1} = S$, the genus two surface $\Sigma_2 = \Sigma$ resulting from the last fiber sum operation, $(2g - 2)$ rim toris $R_i$ and $(2g - 2)$ associated vanishing classes $V_i$ in the $\Sigma'$. [12] In the manifold $V_{K'}$, these classes contribute $2g - 2$ new hyperbolic pairs. Thus, the manifolds obtained by the above construction have intersection form $\oplus_{2g-1} H$. Notice that $b_1^+(V_{K'}) = b_1^-(V_{K'}) = 2g - 1$

**Lemma 5.2.** $e(V_{K'}) = 4g$, $\sigma(V_{K'}) = 0$, $c_1^2(V_{K'}) = 8g$, and $\chi_h(V_{K'}) = g$

*Proof.* Using the lemma 2.9, we have $e(V_{K'}) = 2e(W_{K'}) + 4g$, $\sigma(V_{K'}) = 2\sigma(W_{K'})$, $c_1^2(V_{K'}) = 2c_1^2(W_{K'}) + 8g$ and $\chi_h(V_{K'}) = 2\chi_h(W_{K'}) + g$. Since $e(W_{K'}) = 0$ and $\sigma(W_{K'}) = 0$, our result follows.  

$\square$
5.1. Seiberg-Witten invariants for manifold $V_{K'}$. Let $C$ be a basic class of the manifold $V_{K'}$. Let us write $C$ as a linear combination of $S$, $\Sigma$, rim torus $R_i$, $i = 1,...,2g-2$ and the associated vanishing classes $V_i$, $i = 1,\cdots,2g-2$, $C = aS + b\Sigma + \sum_{i=1}^{2g-2} u_i R_i + v_i V_i$. By adjunction inequality, the intersection number of any basic class with rim torus 0 i.e. $C \cdot R_i = 0$, $i = 1,\cdots,2g-2$. It follows that $Q^T v = 0$ where $Q$ is the intersection matrix and $v = (v_1,\cdots,v_{2g-2})$. Using the fact that $Q$ is invertible, we get $v_1 = \cdots = v_{2g-2} = 0$. Next, by applying the adjunction inequality again, we have $V_i \cdot C = 0$ for $i = 1,\cdots,2g-2$. Later give rise to system $Qu = 0$, which implies $u_1 = \cdots = u_{2g-2} = 0$. This shows that any basic class has form $C = aS + b\Sigma$. Since $V_{K'}$ is a symplectic manifold, it has simple type. So for any basic class $C$, $C^2 = 3\sigma(V_{K'}) + 2\varepsilon(V_{K'}) = 8g$. It follows that $2ab = 8g$. Next we apply the adjunction inequality to $S$ and $\Sigma$ to get $2g(S) - 2 \geq |S|^2 + |C(S)|$ and $2g(\Sigma) - 2 \geq |\Sigma|^2 + |C(\Sigma)|$. This gives us two more restriction on $a$ and $b : 2g \geq |b|$ and $2 \geq |a|$. It follows that $C = \pm(2S + 2g\Sigma)$, which are $\pm$ the canonical class of the given manifold. By applying the Taubes theorem from Section 2, we conclude that the value of Seiberg-Witten invariants on these classes is $\pm 1$.

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CONSTRUCTION OF NEW SYMPLECTIC COHOMOLOGY $S^2 \times S^2$

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