Properties of Lerch Sums and Ramanujan’s Mock Theta Functions

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Abstract

In this article we study properties of Ramanujan’s mock theta functions that can be expressed by Lerch sums. We mainly show that each Lerch sum is actually the integral of a Jacobian theta function (here we show that for $\vartheta_3(t, q)$ and $\vartheta_4(t, q)$) and the sec − function. We also prove some modular relations and evaluate the Fourier coefficients of a class of Lerch sums.

Keywords: Lerch sums; Mock theta functions; $q$-series; Special functions; Modularity; Generalizations; Integrals; Representations;

1 Lerch Sums, Divisor Sums and Mock Theta Functions

We will start with some remarks regarding the mock theta function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}.$$ (1)

The function $f(q)$ is an order 3 mock theta function and if we define the Ramanujan-Dedekind $\eta$ function by

$$\eta(q) := \prod_{n=1}^{\infty} (1 - q^n), \ |q| < 1 ,$$ (2)

then we have a first known result due to Watson. For this and much more see:

https://en.wikipedia.org/wiki/Mock_modular_form

Theorem 1.

$$f(q) = \frac{2}{\eta(q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2/2+n/2}}{1 + q^n}, \ |q| < 1$$ (3)
Many mock theta functions given by Ramanujan can be expressed in the form of Lerch sums. Actually Zwegers in his doctoral thesis [Zw] constructed a systematic way to recover them and produce new mock theta functions from Lerch sums.

The next theorem concerns the range of convergence of $f(q)$ when $q \in \mathbb{C}$ and as we will see $f(q)$ (given by (1)) converges in the whole complex plane where it defines a meromorphic function with only possible poles at the points $q = \zeta_\nu = e^{\pi i \nu}$, $\nu \in \mathbb{Q}_+$. Also we give some representations of this function using divisor sums. For to prove these we first need a lemma.

**Lemma 1.**

i) If $|q_1| > 1$ and $a$ is arbitrary, then it holds

$$ (a; q_1)_n = \frac{\left(\frac{1}{a}, \frac{1}{q_1}\right)_\infty}{\left(\frac{1}{aq_1^n}, \frac{1}{q_1}\right)_\infty} \frac{1}{(-1/a)^n q_1^{n(n-1)/2}}. \quad (4) $$

ii) If $|q| < 1$ and $a$ is arbitrary, then it holds

$$ (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. \quad (5) $$

**Proof.**

Easy consequence of the definition of $(a; q)_n$:

$$ (a; q)_n = \prod_{j=1}^n \left(1 - aq^{j-1}\right). $$

**Theorem 2.**

The function $f(q)$ is well defined for all $q \in \mathbb{C} - D$, where $D = \{z \in \mathbb{C} : |z| = 1\}$. Moreover if $|q| < 1$, then

$$ f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}, \quad (6) $$

$$ f(1/q) = \sum_{n=0}^{\infty} \frac{q^n}{(-q; q)_n^2} \quad (7) $$

and

$$ f(q) = \chi(q)^{-2} \sum_{n=0}^{\infty} \left(-q^{n+1}; q\right)_\infty^2 q^{n^2}, \quad (8) $$

$$ f(1/q) = \chi(q)^{-2} \sum_{n=0}^{\infty} \left(-q^{n+1}; q\right)_\infty^2 q^n \quad (9) $$
and
\[ f(q) = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left( -2 \sum_{s=1}^{\infty} \frac{q^s}{s} \sum_{d|s, d \geq n+1} (-1)^{s/d} d \right) \]

and
\[ f(q)\chi(q)^2 = \sum_{n=0}^{\infty} q^n \exp \left( -2 \sum_{s=0}^{\infty} \sum_{0 < d|s, d \leq s/(n+1)} \frac{(-1)^d}{d} \right) \]. \quad (10)

Also
\[ f(1/q)\chi(q)^2 = \sum_{n=0}^{\infty} q^n \exp \left( -2 \sum_{s=0}^{\infty} \sum_{0 < d|s, d \leq s/(n+1)} \frac{(-1)^d}{d} \right) \], \quad (11)

where \( \chi(q) = (-q, q)_\infty \).

**Proof.**
Set \( q_1 = 1/q \). Then for to prove (7) we use
\[ (-q_1: q_1)_n = \prod_{j=1}^{n} \left( 1 + q_1^j \right) = \prod_{j=1}^{n} q_1^j \left( 1 + q_1^{-j} \right) = q_1^{n(n+1)/2} (-q_1: q)_n \]. \quad (12)

For proving (9) we use (4) of Lemma 1. We have
\[ (-q_1: q)_n = \frac{(-q; q)_\infty}{(-q_1^{n+1}; q)_\infty} q^{-n(n+1)/2}. \quad (13) \]

For to prove (8) we use (12) and (13) in (6). For to prove (10) and (11) we write
for (11))
\[
\begin{align*}
\frac{\sum_{n=0}^{\infty} \left( -q_1^{n+1}; q_1^2 \right)_n^2 q^n}{\left( -q_1; q_1 \right)_\infty^2} = & \quad \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \prod_{m=1}^{\infty} \left( 1 + q^n q^m \right)^2 = \\
& = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ 2 \log \left( \prod_{m=1}^{\infty} \left( 1 + q^n q^m \right) \right) \right] = \\
& = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ 2 \sum_{m=1}^{\infty} \log(1 + q^n q^m) \right] = \\
& = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ -2 \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} q^{n+l} q^{ml} \left( -1 \right)^l \right] = \\
& = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ -2 \sum_{m,l=1}^{\infty} q^{n+m+l} \left( -1 \right)^l \right] = \\
\end{align*}
\]

3
\[
= \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ -2 \sum_{s=1}^{\infty} q^s \sum_{d|s, d \geq n+1} \frac{(-1)^s/d}{s/d} \right]
\]

Hence finally

\[
f(1/q) = \chi(q)^{-2} \sum_{n=0}^{\infty} q^n \exp \left[ -2 \sum_{s=1}^{\infty} \frac{q^s}{s} \sum_{d|s, d \geq n+1} (-1)^{s/d}d \right]
\]

In the same way we get (10). \textit{qed}

2 Representations and Fourier coefficients of some general class of Lerch sums

Theorem 3 below will help us link Lerch sums with the classical Jacobi theta functions. As we will see it turns out that some Lerch sums are just the image of an integral transform of a Jacobian theta function.

\textbf{Theorem 3.}

Let \( q \) be complex number such \( |q| < 1 \) and \( g \) be function with Fourier series expansion

\[
g(\phi, q) = \sum_{n=1}^{\infty} a_n(q) \cos(2n\phi), \tag{14}
\]

then

\[
\int_0^{\pi} g(\phi, q) \log \left( \frac{\vartheta_4(\phi, q)}{\vartheta_4(0, q)} \right) d\phi = -\pi \sum_{n=1}^{\infty} \frac{a_n(q)q^n}{n(1-q^{2n})}, \tag{15}
\]

where

\[
\vartheta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz} \tag{16}
\]

\textbf{Proof.}

See [Bag1].

\textbf{Proposition 1.}

If \( z, q \in \mathbb{C} \), with \( |q| < 1 \) we define

\[
\psi(z, q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2/2 + n^2/2} \sin(2zn). \tag{17}
\]

Then

\[
1 + \frac{2}{\pi} \int_0^{\pi} \vartheta_4(t, q) \log \left( \frac{\vartheta_4(t, q)}{\vartheta_4(0, q)} \right) dt = \eta(q) f(q) =
\]
\[
= 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2/2+n/2}}{1 + q^n}
\]  

(18)

**Proof.**

Proposition 1 is direct application of Theorem 3.

Continuing inspired from Proposition 1 we give the next generalized theorem:

**Theorem 4.**

If \(|q| < 1\) and \(a > 0\) and if

\[
\psi_1(t, q) = \sum_{n=1}^{\infty} (-1)^n n q^{an^2+bn} \cos(2nt),
\]

(19)

then

\[
\int_0^\pi \psi_1(t, q) \log \left( \frac{\vartheta_4(t, q)}{\vartheta_4(0, q)} \right) \, dt = -\pi \sum_{n=1}^{\infty} \frac{(-1)^n q^{an^2+(b+1)n}}{1 - q^{2n}}
\]

(20)

Also if

\[
\psi_2(t, q) = -\sum_{n=1}^{\infty} (-1)^n n q^{an^2-bn} \cos(2nt),
\]

(21)

then

\[
\int_0^\pi \psi_2(t, q) \log \left( \frac{\vartheta_4(t, q)}{\vartheta_4(0, q)} \right) \, dt = \pi \sum_{n=1}^{\infty} \frac{(-1)^n q^{an^2+(-b+1)n}}{1 - q^{2n}}
\]

(22)

Hence if

\[
\psi(t, q) = \psi_1(t, q) + \psi_2(t, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{an^2+bn} \cos(2nt),
\]

(23)

we get

\[
I = \int_0^\pi \psi(t, q) \log \left( \frac{\vartheta_4(t, q)}{\vartheta_4(0, q)} \right) \, dt = \pi \sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^n q^{an^2+bn}}{q^{-n} - q^n}
\]

(24)

and if \(q = e(z)\), \(Im(z) > 0\), then

\[
\frac{I}{\pi} = \frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^n \exp \left( 2\pi i (an^2 + bn)z \right)}{\sinh(2\pi inz)}.
\]

(25)

The next theorem is representation (reformulation) and evaluation of certain cases of Lerch sums using theta functions.
Theorem 5.
For $a, b, c \in \mathbb{R}$, $a > 0$, $c \neq 0$, $q = e(z)$, $\text{Im}(z) > 0$, we define the function

$$f(a, b; c; q) := \sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^ne \left((an^2 + bn)z\right)}{\sinh(2\pi inz)}.$$  \hspace{1cm} (26)

Then:
1) If $a$ positive real, $b \in \mathbb{N}$, $c = 1$, the function $f(a, b; c; q)$ is a finite sum of ordinary theta functions i.e

$$f(a, b; c; q) = \sum_j \left(2 \sum_{n = 1}^{\infty} (-1)^n e(an^2 z) \cos(bj\pi nz)\right),$$  \hspace{1cm} (27)

where the $b_j$ are given from relations (30), (31) below.
2) When $a, b, c$ are real and $a > 0$, $c \neq 0$, then

$$f(a, b; c; q) = \sum_{l, n \in \mathbb{Z}} (-1)^n q^{an^2 + c(2l + 1)n + b} = \sum_{n = 1}^{\infty} (-1)^n q^{an^2 + b} U_{b-1}(\cos(2\pi nz)).$$  \hspace{1cm} (28)

where

$$\epsilon(n, l) := \text{sign}(n)\text{sign}(l) \frac{\text{sign}(n) + \text{sign}(l)}{2}. $$  \hspace{1cm} (29)

Proof.
1) If $b \in \mathbb{N}$ then the next expansion holds

$$\frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^n e \left((an^2 + bn)z\right)}{\sinh(2\pi inz)} = \sum_{n = 1}^{\infty} (-1)^n e(an^2 z) U_{b-1}(\cos(2\pi nz)), $$  \hspace{1cm} (30)

where $U_n(x)$ is the $n$-th degree Chebyshev polynomial. Hence

$$U_{b-1}(\cos(2\pi nz)) = \begin{cases} 1 + 2 \sum_{j=1}^{\left\lfloor\frac{b-1}{2}\right\rfloor} \cos(4j\pi nz), \text{ when } b \text{ odd} \\ 2 \sum_{j=0}^{\left\lfloor\frac{b-1}{2}\right\rfloor} \cos((4j + 2)\pi nz), \text{ when } b \text{ even.} \end{cases} $$  \hspace{1cm} (31)

and the first result follows.
2) The first equality of (28) is obtained easily from

$$\frac{e(bnz) - e(-bnz)}{2 \sinh(2\pi inz)} = -(e(bnz) - e(-bnz)) \sum_{l=0}^{\infty} e(cn(2l + 1)z). $$  \hspace{1cm} (32)

The second equality of (28) is obtained using the symbols $\epsilon(n, l)$. qed
Theorem 6.

If \( a \in \mathbb{Q}_+^* \), \( c = \frac{1}{2^\nu} \), \( \nu \in \mathbb{N} \), \( b = \text{odd} \cdot c \), \( a\nu \) is integer and \( q = e(z) \), \( Im(z) > 0 \), then the function

\[
f(a, b; c; z) = 2^{-1} \sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^n e \left((an^2 + bn)z\right)}{\sinh(2\pi incz)}
\]  

have the following property

\[
f(a, b; c; \nu z) = \sum_{n=1}^{\infty} \left( \sum_{\substack{d \mid n \mid \text{abs}(d) \neq 0}} (-1)^d \epsilon(d, -\nu(c - b + ad) + n/d) \right) q^n + \sum_{n=1}^{\infty} (-1)^n q^{an^2 + (c-b)\nu n}, \ |q| < 1.
\]  

Proof.

Assume the form

\[ Q = n(an + b + cm). \]  

Using the transformation

\[ a = k_1 A + l_1 B, c = k_2 A + l_2 B, A, B \in \mathbb{Z} - \{0\}, (A, B) = 1, \]  

the form \( Q \) becomes

\[
Q = n(nk_1 A + l_1 B) + b + m(k_2 A + l_2 B) = n(A(nk_1 + mk_2) + b + B(nl_1 + ml_2)).
\]

If

\[
nk_1 + mk_2 = n', nl_1 + ml_2 = m'
\]

and if

\[
k_1 l_2 - l_1 k_2 = 1, k_1, k_2, l_1, l_2, \in \mathbb{Z}
\]

we get

\[
Q = n(A'n + b + Bm')
\]

and \( n = n'l_2 - m'k_2 \). By this way \( Q \) becomes

\[
Q = (n'l_2 - m'k_2) (A'n' + Bm' + b).
\]

Hence if we set in \( Q \)

\[
n'_1 = n'l_2 - m'k_2, n'_2 = An' + Bm' + b,
\]

then

\[
Q = n'_1 n'_2.
\]
This case corresponds to $a, b, c \in \mathbb{Z}, a > 0$. Going backwards we get

$$n = n'_1, m = -\frac{b + a'n'_1 - n'_2}{c},$$

and $Q$ becomes

$$Q = n'_1n'_2.$$  \hfill (42)

Hence for $c = 1$ we obtain the next identity

$$g(q) = \sum_{n,m=-\infty}^{\infty} (-1)^n \epsilon(n, m) q^{an+b+m} =$$

$$= \sum_{n'_1, n'_2 \in \mathbb{Z}} (-1)^{n'_1} \epsilon(n'_1, -b - an'_1 - n'_2) q^{n'_1n'_2} =$$

$$= \sum_{n=1}^{\infty} \left( \sum_{\substack{\text{abs}(d)n \quad d \neq 0}} (-1)^d \epsilon(d, -b - ad + n/d) \right) q^n. \quad (44)$$

Consequently assuming $c = \frac{1}{z}, \nu \in \mathbb{N}$ and having in mind the above relations we get the proof of the theorem. qed

Immediate consequence of the above theorem is the next

**Theorem 7.**
If $q = e(z), \text{Im}(z) > 0$, we define

$$F(a, b; c; q) := \sum_{n \in \mathbb{Z}-\{0\}} \frac{(-1)^n q^{an^2 + bn}}{1 - q^{nc}}. \quad (45)$$

In case that $a$ is positive rational, $c = \frac{1}{z}, \nu \in \mathbb{Q}^*, b = \text{odd} \cdot c, a\nu \in \mathbb{N},$ then we have

$$2^{-1} \sum_{n \in \mathbb{Z}-\{0\}} \frac{(-1)^n e (\nu an^2 + bn)z}{\sinh(2\pi\nu cz)} = -F(a, b + c; 2c; q) =$$

$$= \sum_{n=1}^{\infty} \left( \sum_{\substack{\text{abs}(d)n \quad d \neq 0}} (-1)^d \epsilon(d, -\nu(c - b + ad) + n/d) \right) q^{n/\nu} +$$

$$+ \sum_{n=1}^{\infty} (-1)^n q^{an^2 + (c-b)n}. \quad (46)$$

Proof.
Use Theorem 6 and its proof.

**Notes.**

1) Theorems 6,7 give us the Fourier coefficients of some cases of Lerch sums.

2) In case $b \to b - c$ we have the next evaluation:

If $a \nu = \text{integer}$, $b = \text{even} \cdot c$ and $c = \frac{1}{2} \nu$, $\nu \in \mathbb{Q}^*$, then

$$\sum_{n \in \mathbb{Z} - \{0\}} \frac{(-1)^n e \left( (an^2 + (b - c)n)z \right)}{\sinh(2\pi incz)} =$$

$$= \sum_{n=1}^{\infty} \left( \sum_{\substack{d | n \\text{abs}(d) | n \\text{abs}(d) \neq 0}} (-1)^d \epsilon(d, -\nu(2c - b + ad) + n/d) \right) q^{n/\nu} +$$

$$+ \sum_{n=1}^{\infty} (-1)^n q^{an^2 + (2c - b)n}. \quad (47)$$

3 More representations, evaluations and modular relations of Lerch sums

In this section we will use Poisson summation formula to recover properties of Lerch sums. The Poisson summation formula read as (see [Ch]):

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n), \quad (47.1)$$

where

$$\hat{f}(x) = f(.) \wedge (x) = \int_{-\infty}^{\infty} f(t) e^{-itx} dt \quad (48)$$

is the Fourier transform of $f(t)$.

Assume now $q := e(z) = e^{2\pi iz}$. We define

$$f_1(t) = \exp \left( 2\pi iz(at^2 + Bt) \right), \; B = b - 1 \quad (49)$$

and

$$f_2(t) = \frac{1}{\cosh(2\pi itz^*)}, \quad (50)$$

where the asterisc "*" means complex conjugate. Then setting

$$S(a, b, z) := \sum_{n=-\infty}^{\infty} \frac{q^{an^2 + bn}}{1 + q^{2n}}, \quad (51)$$

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we can write according to Poisson summation formula:

\[
S(a, b, z) := \frac{1}{2} \sum_{n = -\infty}^{\infty} \frac{e (z (an^2 + (b - 1)n))}{\cosh(2\pi zn)} = \frac{1}{2} \sum_{n = -\infty}^{\infty} f_1(n) f_2^*(n) =
\]

\[
= \frac{1}{2} \sum_{n = -\infty}^{\infty} \left( \int_{-\infty}^{+\infty} f_1(t) f_2^*(t) e^{-2\pi i nt} dt \right). \tag{52}
\]

If

\[
F_1(t) = f_1(t) e^{-2\pi i nt} \quad \text{and} \quad F_2(t) = f_2(t) \tag{53}
\]

then using Parseval identity we can write:

\[
\xi(n, a, b, z) := \int_{-\infty}^{+\infty} f_1(t) e^{-2\pi i nt} f_2^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}_1(\gamma) \hat{F}_2^*(\gamma) d\gamma. \tag{54}
\]

Since also

\[
\hat{F}_1(\gamma) = \frac{1}{\sqrt{-2iaz}} \exp \left( \frac{-i(2\pi n + \gamma - 2B\pi z)^2}{8a\pi z} \right) \tag{55}
\]

and

\[
\hat{F}_2^*(\gamma) = \frac{i}{2z} \cdot \text{sech} \left( \frac{i\gamma}{4z} \right), \tag{56}
\]

we have

\[
\xi(n, a, b, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}_1(\gamma) \hat{F}_2^*(\gamma) d\gamma =
\]

\[
= \frac{i}{4\pi z \sqrt{-2iaz}} \int_{-\infty}^{+\infty} \exp \left( -\frac{i(\gamma + 2\pi(n - (b - 1)z))^2}{8a\pi z} \right) \sec \left( \frac{\gamma}{4z} \right) d\gamma. \tag{57}
\]

We make the following change of variable \( \gamma = hL \) and (57) becomes

\[
\int_{-\infty}^{+\infty} \exp \left( -\frac{i(hL + 2\pi(n - (b - 1)z))^2}{8a\pi z} \right) dh \cos \left( \frac{hL}{4z} \right) =
\]

\[
= \frac{iL}{4\pi z \sqrt{-2iaz}} \exp \left( -\frac{i\pi(n - (b - 1)z)^2}{2az} \right) \times
\]

\[
\times \int_{-\infty}^{+\infty} \exp \left[ -\frac{iL^2}{8a\pi z} h^2 - \frac{iL}{2az} (n - z(b - 1)) h \right] \frac{dh}{\cosh(ihL/(4z))}. \tag{58}
\]

But also

\[
\xi(n, a, b, z) =
\]

\[
= \int_{-\infty}^{+\infty} \exp \left( 2\pi i z t^2 - 2\pi i (n - z(b - 1)) t \right) \frac{dt}{\cosh(2\pi izt)} \tag{59}
\]

Hence we have the next definition-theorem:
Theorem 8.
If $Re(2\pi ia) < 0$, then
\[
\xi(n, a, b, z) = \int_{-\infty}^{+\infty} \exp\left(2\pi iat^2 - 2\pi i (n - (b - 1)z) t\right) \frac{dt}{\cosh(2\pi izt)} \tag{60}
\]
and if $Re\left(-\frac{iL^2}{8a\pi z}\right) < 0$, then
\[
\xi(n, a, b, z) = i\frac{L}{4\pi z} \exp\left(-\frac{i\pi(n - (b - 1)z)^2}{2az}\right) \times \int_{-\infty}^{+\infty} \exp\left[-\frac{iL^2}{8a\pi z} t^2 - \frac{iL}{2az} (n - (b - 1)z) h\right] \frac{dh}{\cosh(\imath h L/(4z))}. \tag{61}
\]

Now we use a trick to express the sum $S(a, b, z)$ with a single integral. For $Im(w) > 0$ it holds
\[
\sum_{n=-\infty}^{\infty} \exp\left(-\frac{i(t + 2n\pi)^2}{8\pi w}\right) = \sqrt{-2i w} \cdot \vartheta_3\left(t/2, e^{2\pi i w}\right). \tag{62}
\]
Hence if we rearrange summation with integration in (52) having in mind (54),(57), relation (62) give us
\[
S(a, b; z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \xi(n, a, b, z) =
\]
\[
= \frac{i}{4} \int_{-\infty/2z}^{+\infty/2z} \vartheta_3((h + 1 - b)\pi z, e(az)) \sec\left(\frac{h\pi}{2}\right) dh =
\]
\[
= \frac{i}{4} \int_{-\infty/2z}^{+\infty/2z} \vartheta_3(h\pi z, e(az)) \sec\left(\frac{(h + b - 1)\pi}{2}\right) dh =
\]
\[
= \frac{i}{4\pi z} \int_{-\infty}^{+\infty} \vartheta_3(h, e(az)) \sec\left(\frac{h}{2z} + \frac{(b - 1)\pi}{2}\right) dh. \tag{63}
\]
The rearrange of sum and integration needs explanation. One can see from Theorem 8 and relation (62) that for $a > 0$ and $Im(z) > 0$ this is always possible.
Also it is known (see [Arm,Eb]) that for the transformation of variables
\[
a' = 1/a, \ z' = -1/(4z), \ w' = 2wa'z', \tag{64}
\]
holds
\[
\vartheta_3(w', e(a'z')) = \sqrt{-2iaz} \exp\left(iw^2/(2\pi az)\right) \vartheta_3(w, e(az)). \tag{65}
\]
Hence
Theorem 9.

Setting

\[ P_j(a, b, z) := \frac{i}{4\pi z} \int_{-\infty}^{+\infty} \vartheta_3(h, e(az)) \exp \left( \frac{j h^2}{2\pi az} \right) \sec \left( \frac{h}{2z} + \frac{b - 1}{2} \pi \right) \, dh, \]

where \( j = 0, 1 \). Then

\[ P_0(1/a, b, -1/(4z)) = -\sqrt{-2iaz} \cdot P_1(2z, b, a/2), \quad Im(z) > 0 \] (66)

and also

\[ P_0(a, b, z) = \sum_{n=-\infty}^{\infty} \frac{e((an^2 + (b - 1)n)z)}{\cosh(2\pi inz)} = \sum_{n=-\infty}^{\infty} \frac{q^{an^2 + bn}}{1 + q^{2n}}. \] (67)

Proof.

Relation (67) are the evaluations (63). The proof of (66) is a straightforward application of the transformations (64) and (65) of \( \vartheta_3(h, e(az)) \) functions. qed

Note.

I could not find any physical meaning (or where it might leads) relation (66). It seems to me very weird.

Theorem 10.

In the same way as above if \( a > 0, \quad Im(z) > 0, \quad Im(w) < 0, \quad q = e(z) \), we have:

\[ \sum_{n=-\infty}^{\infty} \frac{q^{an^2 + bn}}{\cosh(2\pi inw)} = \frac{i}{2\pi w} \int_{-\infty}^{+\infty} \vartheta_3(h, e(az)) \sec \left( \frac{h + b\pi z}{2w} \right) \, dh \] (68)

and

\[ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{an^2 + bn}}{\cosh(2\pi inw)} = \frac{i}{2\pi w} \int_{-\infty}^{+\infty} \vartheta_4(h, e(az)) \sec \left( \frac{h + b\pi z}{2w} \right) \, dh. \] (69)

For \( j = 0, 1 \), we define more general

\[ P_j(a, b; z, w; x) := \frac{i}{2\pi w} \int_{-\infty}^{+\infty} \vartheta_3(t, e(az)) \exp \left( \frac{j t^2}{2\pi az} \right) \sec \left( \frac{t}{2w} + \frac{b\pi z}{2w} \right) e^{itx} \, dt. \] (70)

From Theorem 10 is

\[ P_0(a, b; z, w; 0) = \sum_{n=-\infty}^{\infty} \frac{q^{an^2 + bn}}{\cosh(2\pi inw)}, \quad a > 0. \] (71)

This function is a Lerch series function. Hence we can say that \( P_j(a, b; z, w; x) \) is a generalization of Lerch series. The Fourier transformation of these generalized
functions possess modular properties. Moreover we can prove the next:

**Theorem 11.**
If $\text{Im}(z) > 0$ and $a > 0$, then

$$P_j(a', b'; z', w'; (\cdot)) \wedge (\gamma') = -2a\sqrt{-2ia} \cdot z^{3/2} \exp\left(-j i\gamma'^2 \over 2\pi az\right) P_j(a, b; z, w; (\cdot)) \wedge (\gamma),$$

where $j = 0, 1$ and

$$a' = 1/a, \quad b' = 2b \cdot a', \quad z' = -1/(4z), \quad \gamma' = 2\gamma a' \cdot z',$$  

$$w' = 2w a' \cdot z'.$$  

(72)

**Proof.**
When $a > 0$ and $j = 0, 1$, we have

$$P_j(a, b; z, w; (\cdot)) \wedge (\gamma) = \frac{i}{2\pi w} \vartheta_3(\gamma, e(az)) \exp\left(j i\gamma^2 \over 2\pi az\right) \sec\left(\frac{\gamma}{2w} + \frac{b\pi z}{2w}\right).$$

Using the transformation property of $\vartheta_3(t, e(az))$ (relation (65)) and relations (73), we get the result. qed

Assume that $a, \text{Im}(z), \text{Im}(w) > 0$, then setting $\gamma = 0$ in relation (72) we have

$$P_j(a, b; z, w; (\cdot)) \wedge (0) = -2a\sqrt{-2ia} \cdot z^{3/2} P_j(a, b; z, w; (\cdot)) \wedge (0).$$

Hence

$$P_j(a, b; z, w; (\cdot)) \wedge (0) = \frac{i}{2\pi w} \vartheta_3(0, e(az)) \sec\left(\frac{b\pi z}{2w}\right).$$

4 The general case of representation of Lerch sums

From [Bag2] we have the next evaluation formula

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{mn^2/2+(p-2a)n/2} = q^{-a^2/4+\frac{a^2}{w^2}} \eta(q^p) W_{\{a, p\}}(m(q)), \quad (74)$$

where

$$W_{\{a, p\}}(m(q)) = A(a, p; q) = q^\frac{\sqrt{a^2 + \frac{a^2}{w^2}}} \prod_{n=0}^{\infty} (1 - q^{np+a})(1 - q^{np+p-a}) = q^{C[a, p; q]} = q^{C(a^2; q^p)}(q^{p-a}; q^p)_\infty. \quad (75)$$
But
\[ \log \left( (q^a; q^p) \neq (q^{p-a}; q^p) \right) = - \sum_{n=1}^{\infty} q^n \sum_{AB = n \atop B \equiv \pm \alpha(p)} \frac{1}{A}. \] (76)

Hence
\[ W_{a,p}^4 \left( m(q) \right) = q^{\frac{a^2}{2} + \frac{t^2}{2}} \exp \left( - \sum_{n=1}^{\infty} q^n \sum_{AB = n \atop B \equiv \pm \alpha(p)} \frac{1}{A} \right). \] (77)

Also in the same way if \(|q| < 1\), then
\[ \sum_{n=-\infty}^{\infty} q^{mn^2/2 + (p-2a)n/2} = q^{-\frac{p^2}{4} + \frac{t^2}{4}} \eta(q^p) W_{a,p}^3 \left( m(q) \right), \] (78)

where
\[ W_{a,p}^3 \left( m(q) \right) = q^{\frac{a^2}{2} + \frac{t^2}{2}} \exp \left( - \sum_{n=1}^{\infty} q^n \sum_{AB = n \atop B \equiv \pm \alpha(p)} \frac{(-1)^A}{A} \right). \] (79)

We want to find a similar expression for \( \vartheta_4 \left( \pi z, e(az) \right) \) (as in (74)). This can be done if we consider the next transformation of variables: \( a \to a - t \) and \( p \to 2a \). Then
\[ \vartheta_4 \left( \pi z, e(az) \right) = q^{\frac{a^2}{2} + \frac{t^2}{2}} \eta(q^2a) Q_{a,t}^4 \left( m(q) \right), \] (80)

where
\[ Q_{a,t}^4 \left( m(q) \right) = q^{-\frac{a^2}{2} + \frac{t^2}{2}} \exp \left( - \sum_{n=1}^{\infty} q^n \sum_{AB = n \atop B \equiv \pm \alpha \left( \frac{a-t}{2} \right)} \frac{1}{A} \right) \] (81)

and
\[ m(q) := \left( \frac{\vartheta_2(0, q)}{\vartheta_3(0, q)} \right)^2, \] \(|q| < 1\).

Also
\[ \vartheta_3 \left( \pi z, e(az) \right) = q^{\frac{a^2}{2} + \frac{t^2}{2}} \eta(q^2a) Q_{a,t}^3 \left( m(q) \right), \] (82)

where
\[ Q_{a,t}^3 \left( m(q) \right) = q^{-\frac{a^2}{2} + \frac{t^2}{2}} \exp \left( - \sum_{n=1}^{\infty} q^n \sum_{AB = n \atop B \equiv \pm \alpha \left( \frac{a-t}{2} \right)} \frac{(-1)^A}{A} \right). \] (83)
But it holds the following modular identity
\[
\vartheta_3(\pi t', e(a'z')) = \sqrt{-2iaz} \exp \left( \frac{i\pi t'^2 z'}{2a} \right) \vartheta_3(\pi t, e(az)),
\]
where
\[
a' = 1/a, \quad z' = -1/(4z), \quad t' = 2tz/a.
\] (84)

Hence in general for the function \( F_3(a, t; z) := Q_{\{a,t\}}^{(3)}(m(q)), \ q = e(z) \) hold
\[
\frac{F_3(a', t'; z')}{F_3(a, t; z)} = \sqrt{-2iaz} \exp \left( -\frac{i\pi t'^2 z'}{2a} \right) \frac{\eta_D(2az)}{\eta_D \left( \frac{1}{2az} \right)},
\]
where \( \eta_D(z), \ \text{Im}(z) > 0 \) is the Dedekind’s eta function. Using the next functional equation:
\[
\eta_D \left( \frac{-1}{z} \right) = \sqrt{-iz} \cdot \eta_D(z),
\]
(86)
we finally arrive to

**Theorem 12.**

1) (Conjecture) If \( a > 0 \) and \( \text{Im}(z) > 0 \)
\[
\vartheta_3(\pi t, e(az)) = q^{\frac{a^2}{12}} \cdot \frac{\eta_D(q^2a)}{\eta_D(q^2a)} F_3(a, t; z),
\]
the function \( F_3(a, t; z) \) takes algebraic values when \( a, t \in \mathbb{Q}^+ \) and \( z = r_1 + i\sqrt{r_2}, \)
with \( r_1 \) rational and \( r_2 \) is positive rational.

2) If \( a, t \) positive integers with \( a > t \), then
\[
F_3(a, t; z) = Q_{\{a,t\}}^{(3)}(m(q)) = q^{\frac{-a^2}{12} + \frac{t^2}{4a}} \exp \left( -\sum_{n=1}^{\infty} q^n \sum_{A\equiv \pm(a-t) \pmod{2a}} (-1)^A \frac{(-1)^A}{A} \right),
\]
(88)

3) For the transformations (85) it holds
\[
F_3(a', t'; z') = \exp \left( -\frac{i\pi t'^2 z'}{2a} \right) F_3(a, t; z).
\]
(89)

**Theorem 13.** (Conjecture)

When \( a > t \) and \( a, t \) positive rationals, the function \( Q_{\{a,t\}}^{(9)}(x), \ y = 3, 4 \) takes algebraic numbers to algebraic numbers.

**Theorem 14.**

If \( q = e(z) \) and \( a > 0, \ \text{Im}(z) > 0, \ \text{Im}(w) < 0 \), then
\[
S = \frac{1}{\eta_D(2az)} \sum_{n=-\infty}^{\infty} q^{an^2+bn} \cosh(2\pi nw) =
\]
\[
\frac{1}{\eta(q^{2a})} \sum_{n=-\infty}^{\infty} \frac{q^{an^2+bn}}{\cosh(2\pi inw)} = 
\]
\[
i \frac{1}{2w} \int_{-\infty}^{+\infty} \frac{dz}{\eta(q^{2a})} \quad 3 \left( \pi h z, e(az) \right) \sec \left( \frac{(h+b)\pi z}{2w} \right) dh = 
\]
\[
i \frac{iz}{2w} \int_{-\infty}^{+\infty} \left[ a - h, 2a ; q \right] \sec \left( \frac{(h+b)\pi z}{2w} \right) dh = 
\]
\[
i \frac{iz}{2w} \int_{-\infty}^{+\infty} q^{\frac{a^2}{4a} - \frac{h^2}{2a}} F_3(a, h z) \sec \left( \frac{(h+b)\pi z}{2w} \right) dh.
\]

Making the change of variable \( h = h_1/z \), we get
\[
S = i \frac{2w}{2w} \int_{-\infty}^{+\infty} F_3(a, h_1/z ; z) \exp \left( -i\frac{h_1^2}{2az} \right) \sec \left( \frac{(h_1+bz)\pi}{2w} \right) dh_1.
\]

Using the modular relation (89) we arrive to the desired result.

In the same way as above we can consider the Fourier transform pairs
\[
\exp \left( 2\pi i (a(\cdot)^2 + b(\cdot)) - 2\pi i n(\cdot) \right) \leftrightarrow \frac{1}{\sqrt{-2ia}} \exp \left( -i(2n\pi + (\cdot) - 2b\pi)^2 \right)
\]
and
\[
\frac{1}{\cosh(2\pi iw(A(\cdot) + B))} \leftrightarrow i \frac{\exp \left( iB(\cdot) \right)}{2Aw} \sec \left( \frac{(\cdot)}{4Aw} \right),
\]
with \( a, A, B, \text{Im}(z), \text{Im}(w) > 0 \). Then we can write \( (q = e(z)) \):
\[
S_1 = \sum_{n=-\infty}^{\infty} \frac{q^{an^2+bn}}{\cosh(2\pi i w^{*}(An + B))} = \sum_{n=-\infty}^{\infty} f_1(n) f_2^{*}(n),
\]
where
\[
f_1(t) = q^{t^2+bt} \quad \text{and} \quad f_2(t) = \frac{1}{\cosh(2\pi iw(At + B))}.
\]

Hence
\[
S_1 = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} f_1(t) e^{-2\pi int} f_2^{*}(t) dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} F_1(t) F_2^{*}(t) dt,
\]
where \( \eta_D(z) \) is the classical Dedekind eta function.
where $F_1(t) = e^{z(at^2 + bt) - nt}$ and $F_2(t) = f_2(t)$. Hence from the Parseval’s identity and relations (91), (92), we get

$$S_1 = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \hat{F}_1(\gamma) \hat{F}_2^*(\gamma) d\gamma =$$

$$= \frac{i}{4\pi Aw^* \sqrt{-2iaz}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} e^{-i(2n\pi + \gamma - 2b\pi z)^2/(8a\pi z)} e^{-iB\gamma/A} \sec \left( \frac{\gamma}{4Aw^*} \right) d\gamma.$$

Rearranging the order of summation and integration and using the formula

$$\sum_{n=-\infty}^{\infty} \exp \left( -i(2n\pi + \gamma - 2b\pi z)^2/8a\pi z \right) = \vartheta_3 \left( \frac{\gamma}{2} - b\pi z, e(az) \right),$$

we get

$$S_1 = \frac{-i}{4\pi Aw^*} \int_{-\infty}^{+\infty} \vartheta_3 \left( \frac{\gamma}{2} - b\pi z, e(az) \right) e^{-iB\gamma/A} \sec \left( \frac{\gamma}{4Aw^*} \right) d\gamma =$$

$$= \frac{1}{2\pi Aw^*} \int_{-\infty}^{+\infty} \vartheta_3 (\gamma - b\pi z, e(az)) e^{-2iB\gamma/A} \sec \left( \frac{\gamma}{2Aw^*} \right) d\gamma =$$

$$= \frac{b\pi z}{2\pi Aw^*} \int_{-\infty}^{+\infty} \vartheta_3 (\gamma - 1 - b\pi z, e(az)) e^{-2iB\gamma b\pi z/A} \sec \left( \frac{\gamma + b\pi z}{2Aw^*} \right) d\gamma =$$

$$= \frac{b\pi z}{2\pi Aw^*} e^{-2iBb\pi z/A} \int_{-\infty}^{+\infty} \vartheta_3 (\gamma, e(az)) e^{-2iB\gamma b\pi z/A} \sec \left( \frac{\gamma + b\pi z}{2Aw^*} \right) d\gamma.$$

Hence we have proven the next

**Theorem 15.**

If $a, Im(z) > 0$ and $Im(w) < 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{q^{an^2+bn}}{\cosh(2\pi iw(An + B))} =$$

$$= \frac{e^{-2iBb\pi z/A}}{2\pi Aw} \int_{-\infty}^{+\infty} \vartheta_3 (\gamma, e(az)) e^{-2iB\gamma/A} \sec \left( \frac{\gamma + b\pi z}{2Aw} \right) d\gamma \quad (93)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{an^2+bn}}{\cosh(2\pi iw(An + B))} =$$
\begin{equation}
\frac{e^{-2iBb\pi z/A}}{2A\pi iw} \int_{-\infty}^{+\infty} \vartheta_4(\gamma, e(az)) e^{-2iB\gamma/A} \sec \left( \frac{\gamma + b\pi z}{2Aw} \right) d\gamma. \tag{94}
\end{equation}

Finally as in Theorem 14 we get

\begin{equation}
\frac{1}{\eta_D(2az)} \sum_{n=-\infty}^{\infty} \frac{q^{n^2 + bn}}{\cosh(2\pi iw(An + B))} =
\end{equation}

\begin{equation}
= \frac{e^{-2iBb\pi z/A}}{2Aiw} \int_{-\infty}^{+\infty} F_3 \left( \frac{1}{a}, \frac{2\gamma}{a}, -\frac{1}{4z} \right) e^{-2i\pi B\gamma/A} \sec \left( \frac{(\gamma + bz)\pi}{2Aw} \right) d\gamma. \tag{95}
\end{equation}

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