Spectral Decimation for Families of Self-Similar Symmetric Laplacians on the Sierpinski Gasket

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January 17, 2018
This talk focuses on the results from SPUR 2017 by
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All of the results are in our paper *Spectral Decimation for Families of Self-Similar Symmetric Laplacians on the Sierpinski Gasket*, arXiv:1709.02031.

Data and code can be found at https://www.math.cornell.edu/~kingda16/.
Outline

- A Brief Recap: energy, Laplacian, spectral decimation
- Our setup on the unit Interval and SG
- Spectral decimation!
- Application: analogue of trigonometric functions on unit Interval
We will take a look at modified versions of two standard fractals: unit Interval \((I)\) and Sierpinski Gasket \((SG)\)
Iterated Function System (IFS): a system of contraction mappings used to define self-similar fractals. In the standard theory, we have:

- For the unit Interval,
  \[
  \{F_i : F_i(x) = \frac{1}{2}x + \frac{i}{2}, \, i = 0, 1\}
  \]

- For SG,
  \[
  \{F_i : F_i(x) = \frac{1}{3}(x - q_i) + q_i, \, i = 0, 1, 2\}
  \]

where \(V_0 = \{q_0, q_1, q_2\}\) denotes the 3 vertices of the triangle.
Let $V_m$ denote the vertex set of the $m$th level graph approximation of $K$, or

$$V_m = \bigcup_{(a_1, \ldots, a_m)} F_{a_1} \circ \cdots \circ F_{a_m}(V_0)$$

where $V_0$ is the endpoints of the interval (for $K = I$) or the vertices of the triangle (for $K = SG$).
With respect to this IFS, recall that we define a self-similar measure, such that

\[ \mu(F_{a_1} \circ \ldots \circ F_{a_m}(I)) = \left(\frac{1}{2}\right)^m \]

\[ \mu(F_{a_1} \circ \ldots \circ F_{a_m}(SG)) = \left(\frac{1}{3}\right)^m \]
Preliminaries

We also consider the energy, a bilinear form on functions on $V_m$.

**Definition**
The bilinear form of energy on pairs of functions $u, v$ on the $m$th level graph approximation is defined

$$E_m(u, v) = \sum_{x \sim_m y \in V_m} (u(x) - u(y))(v(x) - v(y))$$
Energy and Resistance

Given a function $u$ on $V_m$, we can always extend $u$ on $V_m$ to $V_{m+1}$ such that $E_{m+1}(u, u)$ will be minimal among all extensions; this is called the **harmonic extension**.

For harmonic functions extensions $\tilde{u}$ of $u$,

$$E_{m+1}(\tilde{u}, \tilde{u}) = r E_m(u, u)$$

for some constant $r$, so it makes sense for us to define energy on $K$ by

$$E(u, v) = \lim_{m \to \infty} r^{-m} E_m(u, v)$$

We call $r$ the **renormalization factor**. In the standard case, we used this as the uniform resistance on the edges of the graph approximations of $K$. 

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Energy and Resistance

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We call $r$ the renormalization factor. In the standard case, we used this as the uniform resistance on the edges of the graph approximations of $K$. 
Preliminaries: the Laplacian

We combine these notions to define the *Laplacian*.  

**Definition**

If \( f \) is a function on \( K \), \( -\Delta f \) is the function that satisfies 

\[
\mathcal{E}(f, v) = \int (-\Delta f)v \, d\mu \quad \text{for all } v \in \text{dom}_0\mathcal{E}
\]

The *weak formulation* above leads to a pointwise formula 

\[
-\Delta f(x) = \lim_{m \to \infty} \frac{1}{\int \psi_m^{(x)} \, d\mu} \sum_{x \sim y} \frac{1}{r(m)(x, y)} (f(x) - f(y))
\]

for the appropriate *tent function* \( \psi \)
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\]

for the appropriate *tent function* \( \psi \)

The Laplacian can be seen as the analog of the second derivative on a fractal \( K \).
In the standard case, the Laplacian is self-similar, i.e. there is a constant $c$ such that

$$\Delta(u \circ F_i) = c\Delta(u) \circ F_i.$$
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$$\Delta(u \circ F_i) = c\Delta(u) \circ F_i.$$ 

(In the case of $SG$, we have $c = \frac{1}{5}$.)

This is an important property that we would like to exploit later on as well.
Our setup: a one-parameter family of Laplacians

So far, all the calculations done were based on measure and resistance being uniform across $K$. 
Our setup: a one-parameter family of Laplacians

So far, all the calculations done were based on measure and resistance being uniform across $K$.

We vary the measure and resistance in a symmetric, self-similar way on the unit interval and the Sierpinski gasket.
Setup: Unit Interval

Consider a new IFS on $I$, given by twice-iterating the original contractive mappings $F_i$,

$$\{F_i : F_i(x) = \frac{x}{4} + \frac{i}{4}, i = 0, 1, 2, 3\}$$
Setup: Unit Interval

Consider a new IFS on $I$, given by twice-iterating the original contractive mappings $F_i$,

$$\{F_i : F_i(x) = \frac{x}{4} + \frac{i}{4}, i = 0, 1, 2, 3\}$$

Then we can obtain a measure and resistance w.r.t. this IFS given by the parameters $p$ (measure) and $q$ (resistance):
Our Laplacian then satisfies the following scaling property:

\[
\Delta(u \circ F_i) = \begin{cases} 
  \frac{pq}{4} \Delta(u) \circ F_i & \text{if } i = 0, 3 \\
  \frac{(1-p)(1-q)}{4} \Delta(u) \circ F_i & \text{if } i = 1, 2
\end{cases}
\]
Our Laplacian then satisfies the following scaling property:

\[
\Delta(u \circ F_i) = \begin{cases} 
pq \frac{\Delta(u) \circ F_i}{4} & \text{if } i = 0, 3 
\frac{(1-p)(1-q)}{4} \Delta(u) \circ F_i & \text{if } i = 1, 2
\end{cases}
\]

If we want this renormalization factor to be constant across the interval, we need

\[
\frac{pq}{4} = \frac{(1-p)(1-q)}{4} \implies p + q = 1.
\]

This gives a single parameter \( p \).
Setup: SG

with new IFS

\[ \{ F_{jk} : F_{jk} = F_j \circ F_k, j, k = 0, 1, 2 \} \]
Setup: $SG$

with

$$3\mu_0 + 6\mu_1 = 1,$$
and with the restriction that $\rho$, the effective resistances between points in $V_0$ are equal, we obtain

$$\rho = \frac{9r^2 + 26r + 15}{6(r + 2)}, \quad r_0 = \frac{6r(r + 2)}{9r^2 + 26r + 15}, \quad r_1 = \frac{6(r + 2)}{9r^2 + 26r + 15},$$

where $r = r_0', r_1' = 1$. 
Again, we want to make sure the renormalization factor is constant throughout $SG$, i.e. we want

$$\Delta(u \circ F_{jk}) = c\Delta(u) \circ F_{jk} \quad \text{for all } F_{jk}.$$
One-parameter family

Again, we want to make sure the renormalization factor is constant throughout $SG$, i.e. we want

$$\Delta(u \circ F_{jk}) = c\Delta(u) \circ F_{jk} \quad \text{for all } F_{jk}.$$ 

Since

$$-\Delta(u \circ F_{jj}) = \frac{1}{r_0 \mu_0}(-\Delta u) \circ F_{jj}$$

and

$$-\Delta(u \circ F_{jk}) = \frac{1}{r_1 \mu_1}(-\Delta u) \circ F_{jk} \quad \text{if } j \neq k,$$

imposing the condition $r_0 \mu_0 = r_1 \mu_1 = r_1 \left(\frac{1}{6}(1 - 3\mu_0)\right)$ gives a single parameter $r = \frac{r_0}{r_1}$. 

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Goal: We want to find all eigenvalue and eigenfunction pairs $\lambda \in \mathbb{R}$, $f : K \rightarrow \mathbb{R}$, such that

$$-\Delta f(x) = \lambda f(x)$$

for all $x \in K$ and

$$f(y) = 0 \quad \text{for all } y \in \partial K \text{ (the boundary).}$$

We say that such a function $f$ satisfies Dirichlet boundary conditions.
Figure: Eigenfunctions on level $m = 5$ with lowest 8 eigenvalues with $p = 0.1$
Figure: Eigenfunctions on level $m = 5$ with lowest 8 eigenvalues with $p = 0.9$. 

Eigenfunctions

$$
\begin{align*}
\lambda_1 &= 63.34 \\
\lambda_2 &= 94.5539 \\
\lambda_3 &= 127.1695 \\
\lambda_4 &= 2815.1081 \\
\lambda_5 &= 3094.5257 \\
\lambda_6 &= 3242.6908 \\
\lambda_7 &= 3413.0969 \\
\lambda_8 &= 4202.3913
\end{align*}
$$
A pattern?

| Eigenvalues (m=2 p=0.1 q=0.9) | Eigenvalues (m=2 p=0.9 q=0.1) |
|-------------------------------|-------------------------------|
| 4.57471586875                | 63.2815420676                |
| 94.4237055131                | 94.4237055131                |
| 196.748674296                | 126.934089722                |
| 202.732649677                | 2701.32240586                |
| 2719.41974265                | 2957.22725258                |
| 3092.04404312                | 3092.04404312                |
| 3760.55157127                | 3246.34620395                |
| 3950.61728395                | 3950.61728395                |
| 4140.68299663                | 4654.88836395                |
| 4809.19052478                | 4809.19052478                |
| 5181.81482525                | 4944.00731532                |
| 7698.50191822                | 5199.91216204                |
| 7704.48589361                | 7774.30047818                |
| 7806.81086239                | 7806.81086239                |
| 7896.65985203                | 7837.95302583                |
A pattern?

What might explain this pattern?
A pattern?

What might explain this pattern? Spectral decimation!
We aim to describe the spectra of the Laplacian without handling the limiting structure. Instead, we will look at the spectra for graph approximations. Given

\[-\Delta f(x) = \lambda_m f(x) \quad \forall \ x \in V_m\]

Can we extend \( f \) and assign values for \( f \) on \( V_{m+1} \) such that

\[-\Delta f(x) = \lambda_{m+1} f(x) \quad \forall \ x \in V_{m+1}\]
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\[-\Delta f(x) = \lambda_{m+1} f(x) \quad \forall \ x \in V_{m+1}\]

This turns out to be a solvable linear algebra problem, and is the method used by Fukushima & Shima, 1992 [FS].
First, define

\[
\begin{align*}
    i &: \{ \text{cells on level } m \} \to \mathbb{N} \\
    F_{i_1} \circ \ldots \circ F_{i_m}(I) &\mapsto \# \{ i_j : i_j = 0, 3 \}
\end{align*}
\]
First, define

\[ i : \{ \text{cells on level } m \} \rightarrow \mathbb{N} \]

\[ F_{i_1} \circ ... \circ F_{i_m}(I) \leftrightarrow \# \{ i_j : i_j = 0, 3 \} \]

In other words, is the number of “outside” choices during contraction to cell \( F_{i_1} \circ ... \circ F_{i_m}(I) \).
We can simplify the pointwise definition of our Laplacian:

\[-\Delta_m f(z) = \left( \frac{4}{pq} \right)^m (2f(z) - f(y_1) - f(y_2)) \quad \text{if } i(A_0) = i(A_1)\]

\[-\Delta_m f(z) = \left( \frac{4}{pq} \right)^m (2f(z) - 2qf(y_1) - 2pf(y_2)) \quad \text{if } i(A_0) = i(A_1) + 1\]

\[-\Delta_m f(z) = \left( \frac{4}{pq} \right)^m (2f(z) - 2pf(y_1) - 2qf(y_2)) \quad \text{if } i(A_0) = i(A_1) - 1\]

where $A_0$ has vertices $z, y_2$ and $A_1$ has vertices $z, y_1$. 
Given $\lambda_{m+1} \neq 2(1 - \sqrt{q}), 2(1 + \sqrt{q}), 2$, we get

$$\lambda_m(\lambda_{m+1}, p) = \frac{(4 - \lambda_{m+1})(\lambda_{m+1} - 2)^2 \lambda_{m+1}}{4pq}$$

with all eigenvalues scaled by $2 \left( \frac{4}{pq} \right)^m$
As we see above, $2(1 - \sqrt{q}), 2(1 + \sqrt{q}), 2$ are \textit{forbidden eigenvalues}. It turns out, these eigenvalues (scaled appropriately) are born on each level $m$. 
Forbidden eigenvalues

As we see above, $2(1 - \sqrt{q})$, $2(1 + \sqrt{q})$, $2$ are forbidden eigenvalues. It turns out, these eigenvalues (scaled appropriately) are born on each level $m$.

This completely determines the spectrum of $\Delta$. 
Note that $p, q$ enter all equations symmetrically, which leads us to this corollary:

**Corollary**

Let $\lambda^{(m)}_{n,p}, \lambda^{(m)}_{n,q}$ be the $n$th eigenvalues on level $m$ of the Laplacian with measure parameters $p$ and $q$, respectively. Then $\lambda^{(m)}_{n,p} = \lambda^{(m)}_{n,q}$ if $n \equiv \frac{4^a}{2}$ mod $4^a$ for some $a \leq m$. 
The Sturm-Liouville equation is a well-studied second-order homogeneous linear differential equation of the form

\[ \frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + \left[ \lambda \rho(x) - q(x) \right] u = 0 \]
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$$\frac{d}{dx} \left[ p(x) \frac{du}{dx} \right] + \left[ \lambda \rho(x) - q(x) \right] u = 0$$

Consider the unit interval, and let

$$p(x) = 1$$
$$q(x) = 0$$
$$\rho(x) = 1$$
The Sturm-Liouville equation is a well-studied second-order homogeneous linear differential equation of the form

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Consider the unit interval, and let

$$p(x) = 1$$
$$q(x) = 0$$
$$\rho(x) = 1$$

This yields

$$\Delta u + \lambda u = 0$$
Classical Sturm–Liouville Theory

Theorem (Sturm Comparison Theorem, applied to the standard Laplacian)

Let $u_1, u_2$ be eigenfunctions of the standard Laplacian $-\Delta$ on the unit Interval, with eigenvalues $\lambda_1, \lambda_2$ respectively, such that $\lambda_2 > \lambda_1 > 0$. Then between any pair of zeros of $u_2$, $u_1$ will have at least one zero.
Figure: Zeros of eigenfunctions for $p = 0.8$
Analog of Sturm Liouville

Turns out, our eigenfunctions satisfy many similar properties as standard sine curves!
Similar results on related Laplacians were also obtained by Bird, Ngai, Teplyaev in 2003 [BNT].

**Theorem (Fang-King-L-Strichartz, ’17)**

Let $\lambda_i$ be the $i$th eigenvalue and $f_i$ be the eigenfunction for $\lambda_i$.

(a) For any eigenfunction $f$ of the interval, there is exactly one local extremum between two consecutive zeros.

(b) $f_i$ has $i - 1$ zeros.

(c) If $\lambda_i < \lambda_j$ and $x_k, x_{k+1}$ are consecutive zeros of $f_i$, then $f_j$ has at least one zero in $[x_k, x_{k+1}]$. 
Proof.

(b) The result is true for $p = 0.5$ by the Sturm Comparison Theorem. As $p$ varies continuously, $\lambda_{p,i}$ and $f_{p,i}(x)$ ($i$th eigenfunction and eigenvalue) will vary continuously as a function of $p$. Then if the number of zeros of $f_{i,p}$ differs from that of $f_{i,0.5}$ for some $p$, there must exist some $q$ such that $f_{i,q}$ has a zero that is tangent to the $x$-axis. This is a contradiction by part (a).
Recall that we have constructed a one-parameter family of self-similar, symmetric Laplacians on $SG$, with the parameter

$$r = \frac{r_0}{r_1},$$

where $r_0$, $r_1$ are the resistances on the outer and inner edges, respectively.

We reduced the number of parameters to one by imposing that we have a uniform scaling law for the Laplacian throughout $SG$. 
Figure: Eigenfunctions on level $m = 3$ with lowest 4 eigenvalues with $r = 3$
Figure: Eigenfunctions on level $m = 3$ with lowest 4 eigenvalues with $r = 0.5$
As before, we would like to take

\[-\Delta f(x) = \lambda_m f(x) \quad \forall \ x \in V_m\]

as given, and extend \( f \) to \( V_{m+1} \) such that

\[-\Delta f(x) = \lambda_{m+1} f(x) \quad \forall \ x \in V_{m+1}\]

but for \( SG \) instead of \( I \).
Spectral decimation again...

Similarly,

\[-\Delta_m f(z) = \left( \frac{1}{\mu_0 r_0} \right)^m (4f(z) - f(y_1) - f(y_2) - f(y_3) - f(y_4)) \quad \text{if } i(A_0) = i(A_1) \]

\[-\Delta_m f(z) = \left( \frac{1}{\mu_0 r_0} \right)^m \left( \frac{2}{\mu_0 + \mu_1} \ast (\mu_1(2f(z) - f(y_0) - f(y_1)) + \mu_0(2f(z) - f(y_3) - f(y_4))) \right) \quad \text{if } i(A_0) = i(A_1) + 1 \]

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where $A_0$ has vertices $y_1, y_2, z$ and $A_1$ has vertices $y_3, y_4, z$. 
Extension Algorithm

\[
w_0(x_0, x_1, x_2, \lambda_{m+1}, r) = \frac{81x_0(-3 + (2r + r^2)(\lambda_{m+1} - 9) + \lambda_{m+1})}{\gamma(r, \lambda_{m+1})}
+ \frac{9(x_1 + x_2)(-189 + 135\lambda_{m+1} - 30\lambda^2_{m+1} + 2\lambda^3_{m+1})}{\gamma(r, \lambda_{m+1})}
+ \frac{9(x_1 + x_2)(r^2(-81 + 177\lambda_{m+1} - 30\lambda^2_{m+1} + 2\lambda^3_{m+1}))}{\gamma(r, \lambda_{m+1})}
+ \frac{9(x_1 + x_2)(2r(-135 + 135\lambda_{m+1} - 30\lambda^2_{m+1} + 2\lambda^3_{m+1}))}{\gamma(r, \lambda_{m+1})}
\]
Extension Algorithm

\[ z_0(x_0, x_1, x_2, \lambda_{m+1}, r) = \frac{-9(x_1 + x_2)(54 - 27\lambda_{m+1} + 3\lambda^2_{m+1} + r^2(81 - 36\lambda_{m+1} + 3\lambda^2_{m+1}))}{\gamma(r, \lambda_{m+1})} \]

\[ + \frac{-9r(x_1 + x_2)(189 - 63\lambda_{m+1} + 6\lambda^2_{m+1})}{\gamma(r, \lambda_{m+1})} \]

\[ + \frac{9x_0(-297 + 225\lambda_{m+1} + r^2(-81 + 171\lambda_{m+1} - 54\lambda^2_{m+1} + 4\lambda^3_{m+1}))}{\gamma(r, \lambda_{m+1})} \]

\[ + \frac{9x_0(-54\lambda^2_{m+1} + 4\lambda^3_{m+1} + r(-324 + 432\lambda_{m+1} - 108\lambda^2_{m+1} + 8\lambda_{m+1}))}{\gamma(r, \lambda_{m+1})} \]
$$y_{0,1}(x_0, x_1, x_2, \lambda_{m+1}, r) = -\frac{3x_0(\lambda_{m+1} - 3)^2(135 - 48\lambda_{m+1} + 4\lambda_{m+1}^2)}{\gamma(r, \lambda_{m+1})}$$

$$- 3r^2x_0(243 - 756\lambda_{m+1} + 405\lambda_{m+1}^2 - 72\lambda_{m+1}^3 + 4\lambda_{m+1}^4)}{\gamma(r, \lambda_{m+1})}$$

$$- rx_0(1134 - 2106\lambda_{m+1} + 900\lambda_{m+1}^2 - 144\lambda_{m+1}^3 + 8\lambda_{m+1}^4)}{\gamma(r, \lambda_{m+1})}$$

$$- 3rx_2(405 - r(27\lambda_{m+1} - 243) - 81\lambda_{m+1})}{\gamma(r, \lambda_{m+1})}$$

$$- 3rx_1(567 - 189\lambda_{m+1} + 18\lambda_{m+1}^2 + r(243 - 189\lambda_{m+1} + 18\lambda_{m+1}^2)}{\gamma(r, \lambda_{m+1})}$$
Where

\[
\gamma(r, \lambda_{m+1}) = (9 - 3(2 + 3r)\lambda_{m+1} + (1 + r)\lambda_{m+1}^2)(-405 + 279\lambda_{m+1} - 60\lambda_{m+1}^2 + 4\lambda_{m+1}^3)
+ r(9 - 3(2 + 3r)\lambda_{m+1} + (1 + r)\lambda_{m+1}^2)(-702 + 558\lambda_{m+1} - 120\lambda_{m+1}^2 + 8\lambda_{m+1}^3)
+ r^2(9 - 3(2 + 3r)\lambda_{m+1} + (1 + r)\lambda_{m+1}^2)(-243 + 243\lambda_{m+1} - 60\lambda_{m+1}^2 + 4\lambda_{m+1}^3)
\]

This equation is quintic in $\lambda_{m+1}$, but has explicit zeros, referred to as $b_1(r), b_2(r), \cdots, b_5(r)$. 
$$\lambda_m(\lambda_{m+1}, r) = \frac{-\lambda_{m+1}(\lambda_{m+1} - 3)^2(135 - 48\lambda_{m+1} + 4\lambda_{m+1}^2)}{54r(-6 + r(\lambda_{m+1} - 3) + \lambda_{m+1})}$$

$$+ \frac{-r^3\lambda_{m+1}(1458 - 1701\lambda_{m+1} + 603\lambda_{m+1}^2 - 84\lambda_{m+1}^3 + 4\lambda_{m+1}^4)}{54r(-6 + r(\lambda_{m+1} - 3) + \lambda_{m+1})}$$

$$+ \frac{-r^2\lambda_{m+1}(4941 - 5022\lambda_{m+1} + 1701\lambda_{m+1}^2 - 240\lambda_{m+1}^3 + 12\lambda_{m+1}^4)}{54r(-6 + r(\lambda_{m+1} - 3) + \lambda_{m+1})}$$

$$+ \frac{-r\lambda_{m+1}(4536 - 4455\lambda_{m+1} + 1557\lambda_{m+1}^2 - 228\lambda_{m+1}^3 + 12\lambda_{m+1}^4)}{54r(-6 + r(\lambda_{m+1} - 3) + \lambda_{m+1})}$$

This equation is quintic in $\lambda_{m+1}$ and we can only find numerical solutions.
Forbidden eigenvalues born on each level

Similarly as before, the spectral decimation process depends on $\lambda_{m+1}$ not being certain values, and by a similar argument, we can characterize all forbidden eigenvalues that are born on each level $m$ to complete the spectrum:

| $\lambda$ | $m = 1$ | $m = 2$ | $m = 3$ | ... | $m + 1$ |
|-----------|---------|---------|---------|-----|---------|
| $b_1$     | 1       | 0       | 0       | ... | 0       |
| $b_2$     | 1       | 0       | 0       | ... | 0       |
| $b_3$     | 2       | 6       | 42      | ... | $\frac{3^{2m+3}}{2}$ |
| $b_4$     | 2       | 6       | 42      | ... | $\frac{3^{2m+3}}{2}$ |
| $b_5$     | 2       | 6       | 42      | ... | $\frac{3^{2m+3}}{2}$ |
| $b_6$     | 3       | 39      | 363     | ... | $\frac{3^{2m+2}-3}{2}$ |
| $b_7$     | 1       | 9       | 81      | ... | $9^m$   |
Recent developments

Recent results by Loring, Ogden, Sandine, Strichartz (*Polynomials on the Sierpinski Gasket with Respect to Different Laplacians which are Symmetric and Self-Similar*, in preparation, SPUR 2018) generalize the notion of polynomials with respect to this one-parameter family of Laplacians on $SG$, which were studied in the standard case in [NSTY] by Needleman, Strichartz, Teplyaev, and Yung, 2003.

**Definition**

A polynomial $P$ is a function on $SG$ satisfying $\Delta^n + 1 P = 0$ for some $n$, and the minimum such $n$ is called the degree of $P$. 
Recent developments

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References

[BNT] Erik J. Bird, Sze-Man Ngai, and Alexander Teplyaev. Fractal Laplacians on the Unit Interval. Ann. Sci. Math. Quebec 27 (2003), pp. 135168.

[FKLS] Fang, Sizhen, Dylan A. King, Eun Bi Lee, and Robert S. Strichartz. 2017. Spectral Decimation for Families of Self-Similar Symmetric Laplacians on the Sierpinski Gasket. arXiv:1709.02031

[FS] M. Fukushima and T. Shima. On a Spectral Analysis for the Sierpinski Gasket. Potential Analysis 1 (1 1992), pp. 135. DOI: 10.1007/BF00249784.

[LOSS] Loring, Christian, W. Jacob Ogden, Ely Sandine, and Robert S. Strichartz. 2018. Polynomials on the Sierpinski Gasket with Respect to Different Laplacians which are Symmetric and Self-Similar. (in preparation)
[NSTY] Jonathan Needleman, Robert S. Strichartz, Alexander Teplyaev, Po-Lam Yung (2003) Calculus on the Sierpinski Gasket I: polynomials, exponentials, power series Journal of Fractal Analysis 215.

[Str] Strichartz, Robert S. 2006. Differential Equations on Fractals: A Tutorial. Princeton, N.J: Princeton University Press.

[W] Fang, Sizhen, Dylan A. King, and Eun Bi Lee. Spectral Decimation for Families of Self-Similar Symmetric Laplacians on the Sierpinski Gasket. https://www.math.cornell.edu/~kingda16/.
Thank you!