Irreducible Killing Tensors from Third Rank Killing-Yano Tensors

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Abstract

We investigate higher rank Killing-Yano tensors showing that third rank Killing-Yano tensors are not always trivial objects being possible to construct irreducible Killing tensors from them. We give as an example the Kimura IIC metric were from two rank Killing-Yano tensors we obtain a reducible Killing tensor and from third rank Killing-Yano tensors we obtain three Killing tensors, one reducible and two irreducible.

Key words: Killing tensors, Killing-Yano tensors.

1 Introduction

The interest in irreducible Killing tensors is due to their connection with quadratic first integrals, which are not simply linear combinations of the first integrals associated with Killing vectors of geodesic motion. Few examples of irreducible Killing tensors are known explicitly since the direct integration of Killing equation is not easy. Until now, two indirect methods are known in the construction of Killing tensors, some of them expected to be irreducible. In the first one, irreducible Killing tensors can be considered the "square" of Killing-Yano tensors, and in the second method Killing tensors can be constructed from conformal Killing vectors [1, 2, 3]. An illustration of the existence of extra conserved quantities is provided by Kerr-Newmann and Taub-NUT geometry. For the geodesic motion in the Taub-NUT space, the conserved vector analogous to the Runge-Lenz vector of the Kepler type problem is quadratic in 4-velocities, its components are Killing tensors and they can be expressed as symmetrized products of Killing-Yano tensors [4, 5, 6, 7, 8, 9]. It has been shown that in transverse asymptotically flat space-times, a new set of gravitational charges Y-ADM based on Killing-Yano tensors are found [10]. Also,
under some restrictions, pp-wave metrics and Siklos space-times admit non-generic supercharges \cite{11}. On the other hand, there is a relation \cite{12, 13, 14}, called geometric duality, between spaces admitting irreducible Killing tensors of rank two and the spaces whose metrics are specified through those Killing tensors. Further generalizations of Killing tensors and their existence criteria were discussed \cite{15, 16, 17, 18}. Killing-Yano tensors and their corresponding Killing tensors have been studied extensively \cite{19, 20, 21, 22, 23, 24} in the related context of finding solutions of the Dirac-equation in non-trivial curved space-time. Moreover, in the context of generalized Dirac-type operators \cite{25, 26} the Killing-Yano tensors are indispensable tools.

The aim of this paper is to show that irreducible Killing tensors can be constructed from third rank Killing-Yano tensors. On the other hand, we show that the methods, considered until now as independent, used in the construction of Killing tensors, from conformal Killing vectors, (known in literature as Koutras algorithm \cite{1}, and generalizations \cite{2, 3}), and the method consisting in the construction of Killing tensors from Killing-Yano tensors, are interrelated by Hodge duality. This is true at least in the case when the Hodge dual of a third rank Killing-Yano tensor is a conformal Killing vector of gradient type. In Section 2 some known basic definitions concerning Killing and Killing-Yano tensors as well as their interrelation are presented. In Section 3, we give an alternative form for Killing tensors and Killing tensors equation in terms of Hodge dual of third rank Killing-Yano tensors making the connection with Koutras algorithm. In Section 4 we exemplify our results obtained previously on the Kimura IIC metric. In the first part Killing tensors are to be expressed in terms of Killing-Yano tensors of various rank and in the last part Killing tensors are to be constructed from conformal Killing vectors obtained by Hodge duality from third rank Killing-Yano tensors.

2 Killing and Killing-Yano tensors

In general relativity two generalizations of the Killing vector equation have been discussed extensively since the discovery of the Runge-Lenz type vector in the Kerr space-time:

(a) An antisymmetric tensor $f_{\mu_1 \mu_2 \ldots \mu_n}$ is called a Killing-Yano tensor of rank $n$ if the following equation is satisfied

$$f_{\mu_1 \mu_2 \ldots (\mu_n; \lambda)} = 0.$$  \hspace{1cm} (1)

(b) A symmetric tensor field $K_{\mu_1 \mu_2 \ldots \mu_n}$ is said to be a Killing tensor of rank $n$ if:

$$K(\mu_1 \mu_2 \ldots \mu_n; \lambda) = 0.$$  \hspace{1cm} (2)

From \cite{1} the $n-1$ form field $f_{\mu_1 \mu_2 \ldots \mu_{n-1} \nu} p^\nu$ is parallel transported along affine geodesics with tangent field $p^\nu$ and from \cite{2} we observe that $K_{\mu_1 \mu_2 \ldots \mu_n} p^{\mu_1 \mu_2 \ldots \mu_n}$ is a first integral of the geodesic equation.
These two generalizations can be interconnected, constructing a tensor of rank two as a kind of "square" of Killing-Yano tensor $f_{\mu_1\mu_2...\mu_n}$

$$ K_{\mu\nu} = f_{\mu_1\mu_2...\mu_n} f^{\mu_1\mu_2...\mu_n}_{\nu}, $$

which is a Killing tensor because it is symmetric and satisfies the Killing tensor equation (2) with quantity $K_{\mu\nu}p^{\mu}p^{\nu} = f_{\mu_1\mu_2...\mu_n} f^{\mu_1\mu_2...\mu_n}_{\mu} p^{\mu} p^{\nu}$ the quadratic first integral generated by Killing tensor. In general, the Killing-Yano equation (1) has many linear independent solutions $f^{(i)}_{\mu_1\mu_2...\mu_n}$, indexed in what follows by upper latin indices, and a suitable construction of Killing tensor $K^{(ij)}_{\mu\nu}$ in this case, is given by

$$ K^{(ij)}_{\mu\nu} = f^{(i)}_{\mu_1\mu_2...\mu_{n-1}} f^{(j)}_{\mu_2...\mu_{n-1}}. $$

Observing that the metric tensor $g_{\mu\nu}$, as well as all symmetrized products of any Killing vectors $\xi_I$, and in general, a linear combination of them with constant coefficients, are Killing tensors, i.e.

$$ K_{\mu\nu} = a_0 g_{\mu\nu} + \sum_{I=1}^{N} \sum_{J=1}^{N} a_{IJ} \xi_I(\mu) \xi_J(\nu), $$

we can say that all such Killing tensors are called reducible and all other are called irreducible.

Briefly speaking, the rank one and four Killing-Yano tensors can be considered as two trivial cases. In the first case, the Killing-Yano tensor $f^{(i)}_{\mu}$ is a Killing vector and the associated Killing tensor

$$ K^{(ij)}_{\mu\nu} = f^{(i)}_{\mu} f^{(j)}_{\nu}, $$

is obviously reducible. In the second case, the Killing-Yano tensor $f^{(i)}_{\mu\nu\lambda\rho}$, solutions of the equations $f^{(i)}_{\mu
u\lambda(\rho;\theta)} = 0$, is proportional to the alternating pseudotensor $\eta_{\mu\nu\lambda\rho}$. In this case the associated Killing tensor

$$ K^{(ij)}_{\mu\nu} = f^{(i)}_{\alpha\lambda\rho(\mu} f^{(j)}_{\nu)\alpha\lambda\rho} $$

is equal with the metric up to a constant.

The Killing-Yano tensors $f^{(i)}_{\mu\nu}$ of rank two are objects widely discussed in the literature, being indispensable tools used in the construction of new first integrals of the geodesic motion, Dirac type operators and gravitational anomalies investigations. In this case we have twenty independent equations which have to be solved for six independent components. The associated Killing tensors are given by

$$ K^{(ij)}_{\mu\nu} = f^{(i)}_{\alpha(\mu} f^{(j)}_{\nu)\alpha} $$

some of which are expected to be irreducible.
In order to determine the Killing-Yano tensors $f^{(i)}_{\mu\nu\lambda}$ of rank three a system with fifteen independent equations for four independent components $f^{(i)}_{\mu\nu(\lambda;\rho)} = 0$ must to be solved. The associated Killing tensors are given by

$$K^{(ij)}_{\mu\nu} = f^{(i)}_{\alpha\beta(\mu} f^{(j)\alpha\beta}_{\nu)}$$  \hspace{1cm} (5)

checking which of these are irreducible by comparison with the definition of reducibility.

3 Conformal Killing vectors from third rank Killing-Yano tensors

In this section we extend some of the results obtained in [27] for third rank Killing-Yano tensors and conformal Killing vectors. It is well known that if $f^{(i)}_{\mu\nu\lambda}$ is a third rank Killing-Yano tensor then the dual reads

$$f^{* (i)\mu} = \frac{1}{6} \eta^{\mu\alpha\beta\rho} f^{(i)}_{\alpha\beta\rho}.$$  \hspace{1cm} (6)

Contracting this relation with $\eta_{\mu\alpha\beta\rho}$ we obtain

$$f^{(i)}_{\alpha\beta\rho} = -\frac{1}{4} f^{* (i)\mu} \eta_{\mu\alpha\beta\rho},$$

which enables us to express the Killing tensor (5) in terms of $f^{* (i)\mu}$ as follows

$$K^{(ij)}_{\mu\nu} = f^{* (i)}_{(\mu} f^{* (j)}_{\nu)} - f^{* (i)}_{\alpha} f^{* (j)\alpha} g_{\mu\nu}.$$  \hspace{1cm} (7)

with

$$K^{(ij)}_{(\mu\nu;\lambda)} = \vartheta^{(j)} f^{* (i)}_{(\mu} g_{\nu\lambda)} + \vartheta^{(i)} f^{* (j)}_{(\mu} g_{\nu\lambda)} - (f^{* (i)}_{\alpha} f^{* (j)\alpha} g_{\mu\nu\lambda} + f^{* (j)\alpha} f^{* (i)\alpha} g_{\alpha\beta\mu\nu\lambda}).$$

where $\vartheta^{(i)}$ is the conformal factor. The Killing equation $K^{(ij)}_{(\mu\nu;\lambda)} = 0$ will be satisfied if

$$\vartheta^{(j)} f^{* (i)}_{\mu} + \vartheta^{(i)} f^{* (j)}_{\mu} = f^{* (i)}_{\alpha} f^{* (j)\alpha} g_{\mu\nu\lambda} + f^{* (j)\alpha} f^{* (i)\alpha} f_{\alpha\beta\mu\nu\lambda},$$

or, in another form

$$f^{* (i)\alpha} f^{* (j)}_{[\alpha\beta\mu]} + f^{* (j)\alpha} f^{* (i)}_{[\alpha\beta\mu]} = 0.$$

**Case ($i = j$)**

This case was partially investigated in [27] where was showed that if $f^{* \mu}$ is null, then the associated Killing tensor reduces to $K^{* \mu}_{\nu} = f^{* \mu}_{\nu} f^{* \nu}$ with $f^{* \mu}$ a covariantly constant null Killing vector, which implies that Killing-Yano tensor $f^{(i)}_{\mu\nu\lambda}$ is covariant constant. On the other hand, if $f^{* \mu}$ is non-null, after some algebra, was showed that $f^{* \mu}$ is a conformal Killing vector $f^{* \mu}_{(\mu\nu)} = 2 \vartheta g_{\mu\nu}$ with conformal factor $\vartheta$ and a
gradient field, such that \( f^*_{\mu\nu} = \vartheta g_{\mu\nu} \). The associated Killing tensor (7) is in this case given by

\[
K_{\mu\nu} = f^*_{\mu} f^*_{\nu} - f^{*2} g_{\mu\nu},
\]  

(8)

where \( f^{*2} = f^*_\alpha f^{*\alpha} \). If we define \( Q_{\mu\nu} \equiv f^*_{\mu} f^*_{\nu} \), it is easy to see that \( Q_{\mu\nu} \) is a conformal Killing tensor \( Q_{(\mu\nu;\lambda)} = g_{(\mu\nu)p\lambda} \) of gradient type having the associated vector \( p_\lambda = 2\vartheta f^*_{\lambda} \), with \( p_\lambda = p_\lambda \), allowing to write for (8) the following form

\[
K_{\mu\nu} = Q_{\mu\nu} - pg_{\mu\nu}.
\]  

(9)

We note that relation (8) is also valid for homothetic \( (\vartheta = \text{const.}) \) Killing vectors of gradient type.

- Case \((i \neq j)\):

  In this case, two situations are to be considered:

  (a) First, \( f^*(i)_\mu \equiv \xi_\mu \) is a null Killing vector, and \( f^*(j)_\mu \equiv f^{*\mu} \) a conformal Killing vector, both of gradient type. The associated Killing tensor (7) take the form

\[
K_{\mu\nu} = \xi_{(\mu} f^*_{\nu)} - \xi_\alpha f^{*\alpha} g_{\mu\nu}.
\]  

In this case, \( Q_{\mu\nu} \equiv \xi_{(\mu} f^*_{\nu)} \) is a conformal Killing tensor \( Q_{(\mu\nu;\lambda)} = g_{(\mu\nu)q\lambda} \) of gradient type having the associated vector \( q_\lambda = \vartheta \xi_\lambda \) with \( q_\lambda = q_\lambda \). The associated Killing tensor will take the form

\[
K_{\mu\nu} = Q_{\mu\nu} - qg_{\mu\nu}.
\]  

(9)

(b) Secondly, in (7) we consider \( f^*(i)_\mu \) and \( f^*(j)_\mu \) non-null conformal Killing vectors of gradient type. The tensor \( Q^{(ij)}_{\mu\nu} \) defined by the symmetrised product

\[
Q^{(ij)}_{\mu\nu} \equiv f^*(i)_\mu f^*(j)_\nu
\]  

is a conformal Killing tensor \( Q^{(ij)}_{\mu\nu} = g_{(\mu\nu)\epsilon\lambda} \) of gradient type having the associated vector \( \epsilon_\mu = \vartheta f^*(i)_\mu + \vartheta f^*(j)_\mu \) with \( \epsilon_\mu = \epsilon_\mu \) in which case (7) can be written

\[
K^{(ij)}_{\mu\nu} = Q^{(ij)}_{\mu\nu} - \epsilon g_{\mu\nu}.
\]  

(10)

We note that the same discussion can be made in all cases if one or both Killing vectors \( f^*(i)_\mu, f^*(j)_\mu \) are homothetic of gradient type, in which case (7) will take the appropriate form.

4 Killing Tensors for Kimura IIC Metric

Hauser and Malhiot [28] investigated static spherically symmetric space-times and found all metrics for which an irreducible Killing tensor of rank two exists obtaining quadratic first integrals under the assumption that these are independent of time. Without imposing such assumption Kimura [29] generalize the results obtained before by Hauser and Malhiot and found several metrics having quadratic
first integrals. One of them is known in literature as the Kimura IIC metric. This metric is of Petrov type $D$, with non-zero energy-momentum tensor, given by

$$ds^2 = \frac{r^2}{b} dt^2 - \frac{1}{r^2 b^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

From Killing vector equations

$$\xi_{(\mu;\nu)} = 0,$$

we find four Killing vectors $\xi^{(i)} = \xi^{\mu}_{(i)} \partial_\mu$ given by

$$\begin{align*}
\xi^{(1)} &= \frac{\partial}{\partial t} \\
\xi^{(2)} &= \cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi} \\
\xi^{(3)} &= -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \\
\xi^{(4)} &= \frac{\partial}{\partial \phi}
\end{align*}$$

In covariant form the Killing vectors components reads

$$\begin{align*}
\xi^{(1)} &= (\frac{r^2}{b}, 0, 0, 0) \\
\xi^{(2)} &= (0, 0, r^2 \cos \phi, -r^2 \sin \theta \cos \theta \sin \phi) \\
\xi^{(3)} &= (0, 0, r^2 \sin \phi, r^2 \sin \theta \cos \theta \cos \phi) \\
\xi^{(4)} &= (0, 0, 0, r^2 \sin^2 \theta)
\end{align*}$$

### 4.1 Killing tensors from third rank Killing-Yano tensors

In what follows we construct Killing tensors from Killing-Yano tensors of various rank and using (3) we check which of them are irreducible.

For Killing-Yano tensors of rank two the solution of corresponding equations (4) is

$$f_{23} = r^3 \sin \theta,$$

which exist in any spherically symmetric static space-time [16, 18]. The corresponding components of the Killing tensor are

$$K_{22} = -r^4, \quad K_{33} = -r^4 \sin^2 \theta.$$

It is easy to observe that this tensor can be written as a linear combination of symmetrized products of Killing vectors

$$K_{\mu\nu} = -\sum_{i=1}^{3} \xi_{(i)} \xi_{(i)\nu},$$

identified with the square of the angular momentum, which means that this Killing tensor is reducible.

By straightforward calculation we obtain the following general solution for Killing-Yano tensors of rank three,

$$f_{023} = C_2 br^4 \sin \theta + C_1 r^4 \sin \theta, \quad f_{123} = C_2 r^4 \sin \theta$$
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all other components being zero. Therefore, we have two \((i = 1, 2)\), independent solutions \(f_{(i)}^{(l)}\), given by,

\[
f_{023}^{(1)} = r^4 \sin \theta
\]

and

\[
f_{023}^{(2)} = b t r^4 \sin \theta, \quad f_{123}^{(2)} = r \sin \theta
\]

where we chose the integration constants equals with unity.

Taking into account (5) we can construct the following Killing tensors from the Killing-Yano tensors (11) and (12).

• **Case 1**: If we consider only the solution (11) we obtain for Killing tensor \(K_{\mu\nu}^{(11)}\) the following non-null components

\[
K_{00}^{(11)} = 2 r^4, \quad K_{22}^{(11)} = -2 b r^4, \quad K_{33}^{(11)} = -2 b r^4 \sin^2 \theta
\]

which is reducible since can be written as a linear combination of symmetrized products of Killing vectors

\[
K_{\mu\nu}^{(11)} = 2 b^2 \xi\delta_{\mu\xi}^{(0)} + 2 b \sum_{i=1}^{3} \xi_{i(\mu} \xi_{i|\nu)}.
\]

• **Case 2**: If we consider only the solution (12) we obtain for Killing tensor \(K_{\mu\nu}^{(22)}\) the following non-null components

\[
K_{00}^{(22)} = 2 b^2 t^2 r^4, \quad K_{01}^{(22)} = 2 b t r, \quad K_{11}^{(22)} = \frac{2}{r^2},
\]

\[
K_{22}^{(22)} = 2 b^2 r^2 (1 - b t^2 r^2), \quad K_{33}^{(22)} = 2 b^2 r^2 (1 - b t^2 r^2) \sin^2 \theta.
\]

If we define a new Killing tensor \(K_{\mu\nu}^{(22)} \equiv K_{\mu\nu}^{(22)} + a g_{\mu\nu}\), for \(a = 2 b^2\), we obtain the components

\[
K_{00}^{(22)} = 2 b n^2 (b t^2 r^2 + 1), \quad K_{01}^{(22)} = 2 b t r,
\]

\[
K_{22}^{(22)} = -2 b^3 t^2 r^4, \quad K_{33}^{(22)} = -2 b^3 t^2 r^4 \sin^2 \theta
\]

which is irreducible since it is clearly impossible to obtain from metric and Killing vectors terms which are explicit functions of \(t\).

• **Case 3**: Considering both solution (11) and (12) of the third rank Killing-Yano equation, we obtain the following Killing tensor, with components

\[
K_{00}^{(12)} = 2 b t r^4, \quad K_{01}^{(12)} = r, \quad K_{22}^{(12)} = -2 b^2 t r^4, \quad K_{33}^{(12)} = -2 b^2 t r^4 \sin^2 \theta
\]

We observe that this tensor can not be obtained as linear combination of the metric and symmetrized products of Killing vectors and is also irreducible.
For Killing-Yano tensors of rank four the solution of corresponding equation (1) yields a system of 4 equations with one \( f_{0123} \) independent component

\[ f_{0123} = r^2 \sin \theta. \]

We observe that for a four rank Killing-Yano tensor, which is antisymmetric in all indices, the Killing-Yano equations are equivalent with \( f_{0123;\mu} = 0 \), and the corresponding Killing tensor is reducible, proportional with the metric tensor.

### 4.2 Killing tensors from conformal Killing vectors

In this subsection, for Kimura IIC metric, Killing tensors are constructed from conformal Killing vectors which can be obtained as dual of third rank Killing-Yano tensors.

From (11) and (6) we obtain the following conformal Killing vector

\[ f^*(1) = b \sqrt{br^2} \frac{\partial}{\partial r} \]

with associated conformal factor \( \vartheta(1) = b \sqrt{br} \). We observe that \( f^*_\alpha \) is of gradient type, given by \( f^*_\alpha = \left(-r/\sqrt{b}\right)_{\lambda} \). From (8) with \( f^{*2} = -br^2 \) we obtain for the associated Killing tensor components \( K_{\mu\nu} \), the same expressions as in (13). The same results can be obtained using (9), with \( p_{\lambda} = \vartheta(1) f^*_\alpha = \left(-br^2\right)_{\lambda} \).

From (12) and (6) we obtain the following conformal Killing vector

\[ f^*(2) = -b \sqrt{b} \frac{\partial}{\partial t} + b^2 \sqrt{bt} r^2 \frac{\partial}{\partial r} \]

with associated conformal factor \( \vartheta(1) = b^2 \sqrt{bt} r \). Also, \( f^*_\alpha \) is of gradient type, given by \( f^*_\alpha = \left(-\sqrt{b} tr\right)_{\lambda} \). In this case, using (8) with \( f^{*2} = b^2(1 - bt^2 r^2) \), we obtain for Killing tensor \( K_{\mu\nu} \) components the same expressions as in (14). The final form (15) is obtained by adding \( b^2 g_{\mu\nu} \). On the other hand, (15) can be obtained directly by using (9), with \( p_{\lambda} = \vartheta(1) f^*_\alpha = \left(-b^3 t^2 r^2\right)_{\lambda} \).

For the mixt case, from (7) with \( f^{*2} \) given by (11) and \( f^{*2} \) given by (12) were \( f^{*1} f^{*2} = -b^2 t^2 r^2 \), we obtain for Killing tensor components the same expressions as in (16). The same results can be obtained if we use (10) were \( \epsilon_{\lambda} = \vartheta(1) f^*_\alpha + \vartheta(1) f^*_\alpha = \left(-b^2 t^2 r^2\right)_{\lambda} \).

### 5 Conclusions

In this paper, for Kimura IIC metric, higher rank Killing-Yano tensors were investigated. We have shown that even is impossible to construct irreducible Killing tensors from two rank Killing-Yano tensors as in the usual manner, they can be
constructed from third rank Killing-Yano tensors. We obtain, up to some constants, all Killing tensors known in the literature for Kimura IIC metric, one reducible and two irreducible. As to our knowledge the Kimura IIC metric it is the only one metric where irreducible Killing tensors are constructed from third rank Killing-Yano tensors.

Therefore, higher rank Killing-Yano tensors prove to be useful objects for a description of Dirac type operators on a given background.

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