PRODUCTS OF JACOBIANS AS PRYM-TYURIN VARIETIES

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Abstract. Let $X_1,\ldots,X_m$ denote smooth projective curves of genus $g_i \geq 2$ over an algebraically closed field of characteristic 0 and let $n$ denote any integer at least equal to $1 + \max_{i=1}^m g_i$. We show that the product $JX_1 \times \cdots \times JX_m$ of the corresponding Jacobian varieties admits the structure of a Prym-Tyurin variety of exponent $n^{m-1}$. This exponent is considerably smaller than the exponent of the structure of a Prym-Tyurin variety known to exist for an arbitrary principally polarized abelian variety. Moreover it is given by explicit correspondences.

1. Introduction

A Prym-Tyurin variety of exponent $q$ in a Jacobian $J$ is by definition an abelian subvariety of $J$ such that the canonical polarization of $J$ induces the $q$-fold of a principal polarization. According to a theorem of Matsusaka-Ran and Welters’ criterion (see [1, Section 12.2]), Prym-Tyurin varieties of exponent 1 are Jacobians. Welters showed in [9] that, roughly speaking, every Prym-Tyurin variety of exponent 2 is a classical Prym variety, that is the connected component containing 0 of the norm map $J\tilde{X} \to JX$ of an étale double covering $\tilde{X} \to X$.

In [7] Mumford showed that the product of Jacobians of two general hyperelliptic curves occurs as a classical Prym variety. In [2] we generalized his construction by building Prym-Tyurin varieties which are products of two Prym-Tyurin varieties of smaller exponent. Here we generalize Mumford’s result in a different direction: We show that the product of an arbitrary number of Jacobians of curves of genus $g$ occurs as a Prym-Tyurin variety. To be more precise, the following theorem is a special case of our main results.

Theorem 1.1. Let $X_1,\ldots,X_m$ be smooth projective curves of genus $g_i \geq 2$ over an algebraically closed field of characteristic 0 and $n \geq 1 + \max_{i=1}^m g_i$ an integer. Then the product

$$JX_1 \times \cdots \times JX_m$$

occurs as a Prym-Tyurin variety of exponent $n^{m-1}$ in a Jacobian $J$ of dimension

$$\dim J = n^{m-1}(\sum_{i=1}^m g_i + (m-1)n - m) + 1.$$
Our main results are Theorems 4.3 and 4.4 below, which also imply Mumford’s above mentioned theorem as a special case.

For the proof we use a general result of [2] which allows the construction of new Prym-Tyurin varieties out of given ones. We may assume that the base field is the field of complex numbers. It is well known that any curve $X$ of genus $g$ admits simply ramified coverings $X \to \mathbb{P}^1$ of degree $n$, for all $n \geq g + 1$. We use any such covering to construct a structure of a Prym-Tyurin variety of exponent 1 on the Jacobian $JX$. Then one only has to verify the hypotheses of the above mentioned result.

In Section 2 we recall the theory of [2] for the special case needed here. In Section 3 we work out the presentation of the Jacobians as a Prym-Tyurin variety of exponent 1. Section 4 contains the proofs of our main results.

Throughout this paper, for any finite group $G$, $\chi_0$ will denote its trivial representation. Also, for any subgroup $H$ of $G$, $\rho^G_H$ will denote the representation of $G$ induced by the trivial representation of $H$.

2. Presentation of a Prym-Tyurin variety

We want to apply the main result of [2] to the symmetric group $S_n$ and its self-products. Since all rational irreducible representations of these groups are absolutely irreducible, it suffices to recall a special case of this theorem.

Let $G$ be a finite group, which later we assume to be $S_n$. Let $V_1, \ldots, V_r$ denote nontrivial, pairwise non-isomorphic, absolutely irreducible rational representations of the same dimension of the group $G$, and let $H$ be a subgroup of $G$ such that for all $k = 1, \ldots, r$,

$$\dim V_k^H = 1$$

and $H$ is maximal with this property.

Here “maximal” means that for every subgroup $N$ of $G$ with $H \subsetneq N$ there is an index $k$ such that $\dim V_k^N = 0$.

Choose a set of representatives

$$\{g_{ij} \in G \mid i = 1, \ldots, d \text{ and } j = 1, \ldots, n_i\}$$

for both the left cosets and right cosets of $H$ in $G$, and such that

$$G = \bigsqcup_{i=1}^d H g_{i1} H, \quad \text{and} \quad H g_{i1} H = \bigsqcup_{j=1}^{n_i} g_{ij} H = \bigsqcup_{j=1}^{n_i} H g_{ij}$$

are the decompositions of $G$ into double cosets, and of the double cosets into right and left cosets of $H$ in $G$. Moreover, we assume $g_{11} = 1_G$.

Now let $Z$ be a (smooth projective) curve with $G$-action and quotients

$$\Pi : Z \to \mathbb{P}^1 = Z/G \quad \text{and} \quad \pi : Z \to X := Z/H.$$ 

In [2] we defined a correspondence on $X$, which is given by

$$D(x) = \sum_{i=1}^d \sum_{j=1}^{n_i} b_i \pi g_{ij}(z).$$
for all \( x \in X \) and \( z \in Z \) with \( \pi(z) = x \), where
\[
(2.3) \quad b_i := \sum_{k=1}^{r} \sum_{h \in H} \chi_{V_k}(hg_i^{-1})
\]
is an integer for \( i = 1, \ldots, d \). Moreover, denote
\[
(2.4) \quad b := \gcd\{b_1 - b_i \mid 2 \leq i \leq d\}.
\]
Let \( \delta_D \) denote the endomorphism of the Jacobian \( JX \) associated to the correspondence \( D \). We denote by
\[
P_D := \text{Im}(\delta_D)
\]
the image of the endomorphism \( \delta_D \) in the Jacobian \( JX \) and call it the \textit{(generalized) Prym variety} associated to the correspondence \( D \).

Finally, let us recall the \textit{geometric signature} of a Galois covering of curves with Galois group \( G \). Let \( C_1, \ldots, C_t \) be pairwise different nontrivial conjugacy classes of cyclic subgroups of \( G \). Then this is by definition the tuple \([\gamma, (C_1, s_1), \ldots, (C_t, s_t)]\), where \( \gamma \) is the genus of the quotient curve \( Y \), the covering has a total of \( \sum_{j=1}^{t} s_j \) branch points in \( Y \) and exactly \( s_j \) of them are of type \( C_j \) for \( j = 1, \ldots, t \); that is, the corresponding points in its fiber have stabilizer belonging to \( C_j \).

Then [2, Theorem 4.8] can be stated as follows:

\textbf{Theorem 2.1.} Let \( V_1, \ldots, V_r \) denote nontrivial pairwise non-isomorphic absolutely irreducible rational representations of the group \( G \) satisfying (2.1) for a subgroup \( H \) of \( G \).

Suppose that the action of the finite group \( G \) on a curve \( Z \) has geometric signature \([0; (C_1, s_1), \ldots, (C_t, s_t)]\) and satisfies
\[
(2.5) \quad \sum_{j=1}^{t} s_j \left(q \sum_{k=1}^{r} (\dim V_k - \dim V_{G_j}^{G_i}) - ([G : H] - |H\backslash G/G_j|)\right) = 0,
\]
where \( G_j \) is a subgroup of \( G \) of class \( C_j \) and
\[
(2.6) \quad q = \frac{|G|}{b \cdot \dim V_1}.
\]
Then \( P_D \) is a Prym-Tyurin variety of exponent \( q \) in \( JX \), where \( X = Z/H \).

Furthermore, we showed in [2, Section 4.4] that
\[
(2.7) \quad \dim P_D = \sum_{i=1}^{r} [- \dim V_1 + \frac{1}{2} \sum_{j=1}^{t} s_j (\dim V_i - \dim V_{G_j}^{G_i})]
\]
and
\[
(2.8) \quad g_X = 1 - [G : H] + \frac{1}{2} \sum_{j=1}^{t} s_j ([G : H] - |H\backslash G/G_j|).
\]

Observe that \( b \), and hence \( q \), depends only on the group \( G \), and not on its action on a given curve.
In the sequel we will use the following definition: We say that the construction of the Prym-Tyurin variety \( P = P_D \) of Theorem 2.1 is a presentation of \( P \) with respect to the action of the group \( G \), the subgroup \( H \) and the set of representations \( \{V_1, \ldots, V_r\} \).

3. The Jacobian of a curve as a Prym-Tyurin variety

In this section we show that the Jacobian \( JX \) of any curve \( X \) of genus \( g \geq 2 \) has a presentation as a Prym-Tyurin variety of exponent 1 with respect to a symmetric group \( S_n \), a subgroup \( S_{n-1} \) and the standard representation of \( S_n \). First we need the following lemma.

Lemma 3.1. Let \( G \) be a finite group, \( V \) a nontrivial absolutely irreducible rational representation of \( G \), and \( H \) a subgroup of \( G \) such that the representation \( \rho^G_H \) of \( G \) induced by the trivial representation of \( H \) has the isotypical decomposition

\[
\rho^G_H = \chi_0 \oplus V.
\]

Then the numbers \( q \) and \( b \), defined in equations (2.4) and (2.6) respectively, are given by

\[
b = \frac{|G|}{\dim V} \quad \text{and} \quad q = 1.
\]

Proof. The number of double cosets of \( H \) in \( G \) equals the following character product

\[
|H\backslash G / H| = \langle \rho^G_H, \rho^G_H \rangle_G,
\]

and this is equal to two by our assumptions. Therefore we have two double coset representatives \( g_{11} = 1_G \) and \( g_{21} \). The number \( n_2 \) of right cosets in \( Hg_{21}H \) is \( |G : H| - 1 = \dim V \). Using \( \langle \chi_0, V \rangle_G = 0 \), this gives

\[
0 = \sum_{g \in G} \chi_V(g) = b_1 + n_2 b_2,
\]

where the \( b_i \)’s are given in (2.3).

Moreover, using \( b_1 = |H| \) we obtain

\[
b_2 = \frac{-b_1}{n_2} = \frac{-|H|}{\dim V}.
\]

By definition

\[
b = b_1 - b_2 = |H| + \frac{|H|}{\dim V} = \frac{|H|(\dim V + 1)}{\dim V} = \frac{|H||G : H|}{\dim V},
\]

which implies both assertions. \( \square \)

Now let \( X \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field of characteristic 0. Recall that a covering \( f : X \to \mathbb{P}^1 \) of degree \( n \geq 2 \) is called simple if the fibre \( f^{-1}(p) \) over every branch point \( p \in \mathbb{P}^1 \) consists of exactly \( n - 1 \) different points. According to [3, Proposition 8.1], \( X \) admits a simple covering \( f : X \to \mathbb{P}^1 \) of degree \( n \) for any \( n \geq g + 1 \).
Lemma 3.2. Let \( f : X \to \mathbb{P}^1 \) denote a simple covering of degree \( n \geq 2 \). Then

(a): The covering \( f \) does not factorize.

(b): The Galois group of the Galois closure \( \Pi : Z \to \mathbb{P}^1 \) of \( f \) is the symmetric group \( S_n \).

Proof. (a): Suppose \( f = q \circ t \), with \( t : X \to X' \) and \( q : X' \to \mathbb{P}^1 \). If \( q \) is nontrivial, then it has to be ramified as a covering of \( \mathbb{P}^1 \), in which case \( \deg t = 1 \) since otherwise \( f \) would not be a simple covering.

(b): We may assume that the base field is the field of complex numbers so that the monodromy group is well defined. Since the Galois group of the covering \( \Pi \) coincides with the monodromy group of the covering \( f \), it suffices to show that this is \( S_n \).

Since \( X \) is irreducible and \( f \) is simple, the monodromy group of \( f \) is a transitive subgroup of \( S_n \) generated by transpositions. By part (a), it is also primitive (or see \([8, \text{Lemma 4.4.4}]\)). It is a well-known group theoretical theorem (see e.g. \([4, \text{Satz 4.5, p.171}]\)) that any such subgroup coincides with the full group \( S_n \).

We consider the group \( G := S_n \) as the permutation group of the set \( \{1, \ldots, n\} \). As such, it is generated by \( \tau := (1 2) \) and \( \sigma := (1 2 3 \ldots n) \). We denote by \( Z \) the Galois closure of (the simple covering) \( f : X \to \mathbb{P}^1 \). Therefore, \( G \) acts on \( Z \) with geometric signature \([0; (C_\tau, s)]\), where \( C_\tau \) is the conjugacy class of any subgroup generated by a transposition, and \( s \) is the number of branch points of \( f \).

Consider \( H = \langle (1 2 3 \ldots n - 1), (1 2) \rangle \approx S_{n-1} \), the subgroup of \( G \) fixing \( n \), and let \( V \) denote the standard representation of \( S_n \) (of degree \( n - 1 \)) defined by

\[
V : \sigma \mapsto \begin{bmatrix} 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \tau \mapsto \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.
\]

In the following proposition we see that the Jacobian \( JX \) of \( X \) has a presentation as a Prym-Tyurin variety of exponent 1.

Proposition 3.3. Let \( X \) be a smooth projective curve of genus \( g \geq 2 \), and let \( n \) be any positive integer such that there exists an \( n \)-fold simple covering \( f : X \to \mathbb{P}^1 \). Let \( G, H, V \), the curve \( Z \) and the action of \( G \) on \( Z \) be as above.

Then \( X \approx Z/H \) and the Jacobian \( JX \) has a presentation as a Prym-Tyurin variety of exponent 1 with respect to the action of the group \( G \) on \( Z \) with geometric signature \([0; (C_\tau, s)]\), the subgroup \( H \) and the standard representation \( V \) of \( G \).

Proof. First note that up to conjugacy \( H \) is the only subgroup of index \( n \) of \( G \), therefore \( X \approx Z/H \). To prove that \( JX \) has the desired presentation, we use Theorem 2.1.

Certainly \( \dim V^H = 1 \) and \( H \) is maximal with this property. Moreover, it is easy to see that

\[
\rho^G_H = \chi_0 \oplus V.
\]
Hence Lemma 3.1 implies $q = 1$. It remains to show that equation (2.5) is satisfied. Now we have

$$|H \setminus G/\langle \tau \rangle| = \langle \rho^G_H, \rho^G_{\langle \tau \rangle} \rangle_G = \frac{1}{2} [(1 + \chi_V(1_G)) + (1 + \chi_V(\tau))] = n - 1$$

and

$$\dim V^{(\tau)} = \langle V, \rho^G_{\langle \tau \rangle} \rangle_G = \frac{1}{2} [\chi_V(1_G) + \chi_V(\tau)] = n - 2.$$

where the middle equalities in both equations are due to Frobenius reciprocity. Inserting these values in equation (2.5) completes the proof. □

Remark 3.4. With the notation of Proposition 3.3, the correspondence $D$ on $X$ (defined in Equation (2.2)) for this case is

$$D(x) = \sum_{i=1}^{2} \sum_{j=1}^{n_i} a_i \pi g_{ij}(z),$$

where $n_1 = 1$, $n_2 = n - 1$, $a_1 = (n - 1)!$ and $a_2 = -(n - 2)!$. In fact, the $g_{ij}$ may be chosen as $g_{11} = 1_G$, $g_{21} = (1 \ldots, \tau, 1 \ldots)$, $g_{22} = (2 n), \ldots, g_{n-1,n} = (n - 1 \ldots)$.

4. Products of Jacobians

Fix integers $m \geq 2$ and $n \geq 2$. For each $i = 1, \ldots, m$, let $f_i : X_i \to \mathbb{P}^1$ denote a simple covering of degree $n$ with a smooth projective curve $X_i$ of genus $g_i \geq 2$.

If $Z_i$ denotes the Galois closure of $f_i : X_i \to \mathbb{P}^1$, we have a diagram for every $i = 1, \ldots, m$,

$$\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Z_i \\
\pi_i \downarrow & & \downarrow \Pi_i \\
\Pi_i & \xrightarrow{Z_i} & \mathbb{P}^1
\end{array}$$

According to Lemma 3.2 the Galois group of $\Pi_i$ is the group $G := S_n$. Moreover, the simple ramification of $f_i$ means that $G$ acts with geometric signature $[0; (C_\tau, s_i)]$ on $Z_i$, where $C_\tau$ is the conjugacy class of the subgroups generated by a transposition. The Riemann-Hurwitz formula implies

$$s_i = 2(g_i + n - 1)$$

and thus

$$g(Z_i) = \frac{n!}{2} (g_i + n - 3) + 1.$$ 

For every $i$, the curve $X_i \simeq Z_i/H$, with $H \simeq S_{n-1}$ of the previous section.

Consider the direct product group

$$G^m := \times_{i=1}^{m} G$$

and its subgroup $H^m := \times_{i=1}^{m} H$,

and write $C_i$ for the conjugacy class in $G^m$ of the cyclic subgroup generated by $\tau_i := (1, \ldots, \tau, 1, \ldots, 1)$, with $\tau = (1 \ 2)$ in the $i$th coordinate.
Lemma 4.1. Suppose the branch loci of \( f_i : X_i \to \mathbb{P}^1 \) are pairwise disjoint in \( \mathbb{P}^1 \) (or, equivalently, the branch loci of \( \Pi_i : Z_i \to \mathbb{P}^1 \) are pairwise disjoint in \( \mathbb{P}^1 \)). Then for all \( m \geq 1 \) we have

(a): The fibre product

\[ Z := Z_1 \times_{\mathbb{P}^1} Z_2 \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} Z_m \]

is a smooth projective curve of genus

\[ g(Z) = \frac{(n!)^m}{2} \left( \sum_{i=1}^{m} g_i + m(n-1) - 2 \right) + 1; \]

it is a Galois covering of \( \mathbb{P}^1 \) with geometric signature \([0; (C_1, s_1), \ldots, (C_m, s_m)]\) and Galois group \( G^m \).

(b): The curve \( X := Z/H^m \) coincides with the fibre product

\[ X = X_1 \times_{\mathbb{P}^1} X_2 \times_{\mathbb{P}^1} \cdots \times_{\mathbb{P}^1} X_m \]

and the genus of \( X \) is

\[ g(X) = n^{m-1} \left( \sum_{i=1}^{m} g_i + (m-1)n - m \right) + 1. \]

(c): The natural projections \( q_i : X \to X_i \) do not factorize as

\[
\begin{align*}
\begin{array}{ccc}
X & \xrightarrow{q_i} & X_i \\
\downarrow q & & \downarrow q_i^2 \\
X_i & & \\
\end{array}
\end{align*}
\]

with \( q_i^2 \) nontrivial (cyclic) étale covering for \( i = 1, \ldots, m \).

Proof. Using induction one immediately checks that the fact that the branch loci are disjoint implies that \( Z \) is smooth. Classical Galois theory implies that \( Z \) is Galois over \( \mathbb{P}^1 \) with Galois group \( G^m \). The last statement of (a) is clear from the definitions. It is a consequence of the universal property of the fibre product that the curve \( X := Z/H^m \) is the fibre product over \( \mathbb{P}^1 \) of all the \( X_i \), \( i = 1, \ldots, m \). The genera are computed using the Riemann-Hurwitz formula. This completes the proof of (a) and (b).

(c): It suffices to consider \( q_1 \), so assume \( q_1 \) factorizes as in diagram (4.3). Since the covering \( f_1 : X_1 \to \mathbb{P}^1 \) is isomorphic to the covering \( Z/H \times G^{m-1} \to \mathbb{P}^1 \), it follows that the curve \( \tilde{X}_1 \) is given by a quotient \( Z/N \), where \( N \) is a subgroup of \( G^m \) such that

\[
H^m \subseteq N \subseteq H \times G^{m-1}.
\]
The hypothesis that $q_1^2$ is étale implies that the covering $\tilde{X}_1 \to \mathbb{P}^1$ does not ramify over the branch points of type $C_j$ for $j = 2, \ldots, m$. Therefore $N$ contains all these conjugacy classes; since the class $C_j$ generates the corresponding subgroup $\{1_G\}^{j-1} \times G \times \{1_G\}^{m-j}$, we conclude that $H \times G^{m-1} \leq N$, which together with (4.4) imply that $\tilde{X}_1 = X_1$. \hfill \square

For $j = 1, \ldots, m$ we consider the following absolutely irreducible rational representation of $G^m$

\begin{equation}
V_j := \chi_0 \otimes \cdots \otimes \chi_0 \otimes V \otimes \chi_0 \otimes \cdots \otimes \chi_0,
\end{equation}
given by the outer tensor product of the trivial representation $\chi_0$ ($m - 1$ times) with the standard representation $V$ for $G$ in the $j$-th component.

For $i = 1, \ldots, m$, let $D_i \subset X_i \times X_i$ denote the correspondence defined by (2.2) with respect to the action of the group $G$ on $Z_i$, the subgroup $H$ and the representation $V$ of $G$. Similarly let $D \subset Z \times Z$ denote the correspondence defined by (2.2) with respect to the action of the group $G^m$ on $Z$, the subgroup $H^m$ and the representations $V_1, \ldots, V_m$ of $G^m$. If $q_i : X \to X_i$ denotes the natural projection map, we have the following equality of divisors.

**Lemma 4.2.**

$$D = |H|^{m-1} \sum_{i=1}^m q_i^* D_i.$$ 

**Proof.** As in Section 2 we denote $d = |H\backslash G/H|$ and $\{g_{ij} : i = 1, \ldots, d, \ j = 1, \ldots, n_i\}$. Therefore $|H^m \backslash G^m/H^m| = d^n$, $\{(g_{i_1 \ldots i_m}, g_{i_1 \ldots i_m}) : i_k = 1, \ldots, d, \ k = 1, \ldots, m\}$ are representatives for the double cosets of $H^m$ in $G^m$, and $\{(g_{i_1j_1}, \ldots, g_{i_mj_m}) : i_k = 1, \ldots, d, \ j_k = 1, \ldots, n_i, k = 1, \ldots, m\}$ are representatives of both left and right cosets of the subgroup $H^m$ of $G^m$. According to (2.2) as applied to $G$, $H$ and $V$ we have

$$D_\nu (x_\nu) = \sum_{i=1}^d a_i \sum_{j=1}^{n_i} \tau_\nu g_{ij} (z_\nu) =: \sum_{i=1}^d a_i D_\nu i$$

for all $x_\nu \in X_\nu$, where $z_\nu \in Z_\nu$ is a preimage of $x_\nu$, $1 \leq \nu \leq m$, and where

$$a_i := \sum_{h \in H} \chi_V (h g_{i1}^{-1})$$

are the same integers for all $D_\nu$.

Also note that according to (2.2) applied to $G^m$, $H^m$ and $V_1, \ldots, V_m$ we have

$$D(x_1, \ldots, x_m) = \sum_{1 \leq i_1, \ldots, i_m \leq d} b_{i_1, \ldots, i_m} (D_{1i_1}, \ldots, D_{mi_m})$$
where

\[
\sum_{k=1}^{m} \sum_{h \in H^m} V_k(hg_{i_1 \ldots i_m}^{-1})
\]

\[
= \sum_{h \in H^m} V_1(hg_{i_1 \ldots i_m}^{-1}) + \ldots + \sum_{h \in H^m} V_m(hg_{i_1 \ldots i_m}^{-1})
\]

\[
= \sum_{h=(h_1 \ldots h_m) \in H^m} \chi_V(h_1g_{i_1}^{-1}) + \ldots + \sum_{h=(h_1 \ldots h_m) \in H^m} \chi_V(h_mg_{i_m}^{-1})
\]

\[
= |H|^{m-1}a_i + \ldots + |H|^{m-1}a_m.
\]

Therefore

\[
D(x_1, \ldots, x_m) = |H|^{m-1} \sum_{1 \leq i_1, \ldots, i_m \leq d} (a_i + \ldots + a_m)(D_{1i_1}, \ldots, D_{mi_m})
\]

\[
= |H|^{m-1}(\sum_{1 \leq i_1 \leq d} a_{i_1} \sum_{1 \leq i_2, \ldots, i_m \leq d} (D_{1i_1}, \ldots, D_{mi_m}) + \ldots + \sum_{1 \leq i_m \leq d} a_{i_m} \sum_{1 \leq i_1, \ldots, i_{m-1} \leq d} (D_{1i_1}, \ldots, D_{mi_m}))
\]

Now by definition we have \((q_\nu^* D_\nu)(x_1, \ldots, x_m) = q_\nu^{-1}D_\nu q_\nu (x_1, \ldots, x_m) = q_\nu^{-1}D_\nu(x_\nu)\) for \(\nu = 1, \ldots, m\). Therefore

\[
(q_\nu^* D_\nu)(x_1, \ldots, x_m) = q_\nu^{-1}\left(\sum_{i=1}^{d} a_i D_{vi}\right) = \sum_{i=1}^{d} a_i q_\nu^{-1}(D_{vi})
\]

and we see from (4.7) that

\[
D(x_1, \ldots, x_m) = |H|^{m-1}(q_1^{-1}(D_1(x_1)) + \ldots + q_m^{-1}(D_m(x_m))},
\]

from where the result follows.

The following two theorems are the main result of the paper.

**Theorem 4.3.** For each \(i = 1, \ldots, m\), consider a simply ramified covering \(f_i : X_i \to \mathbb{P}^1\) of degree \(n\), with \(X_i\) of genus \(g_i \geq 2\), and with pairwise disjoint branch loci. Let \(\Pi_i : Z_i \to \mathbb{P}^1\) be the Galois closure of \(f_i\) over \(\mathbb{P}^1\). Denote by \(Z\) the fiber product of all the curves \(Z_i\) over \(\mathbb{P}^1\), and by \(X\) the fiber product of the curves \(X_i\) over \(\mathbb{P}^1\).

Then the action of the group \((S_n)^m\) on the curve \(Z\) defines a Prym-Tyurin variety \(P\) in the Jacobian \(JX\), with \(P\) of exponent \(q = n^{m-1}\) and dimension

\[
\dim P = \sum_{i=1}^{m} g_i = \sum_{i=1}^{m} \dim JX_i.
\]

**Proof.** As above we write \(G = S_n\) and consider the subgroup \(H = \langle (1 \ 2 \ 3 \ \ldots \ n - 1), (1 \ 2) \rangle \simeq S_{n-1}\) as the stabilizer of a point in the general fibre of the map \(f_i : X_i \to \mathbb{P}^1\) for \(i = 1, \ldots, m\). Let again \(V\) denote the standard representation of \(G\) and \(V_i\) the
representations of $G^m$ defined in (1.5). First we observe that equation (2.1) is satisfied for the subgroup $H^m := (S_{n-1})^m$ of $G^m$ and the representations $V_1, \ldots, V_m$.

To see this, note that

$$\dim(V_i)^{H^m} = \langle V_i, \rho_{H^m}^G \rangle_{G^m} = \langle V, \rho_{H}^G \rangle = 1$$

for all $i$. The maximality of $H^m$ with respect to this property is a consequence of the fact that every $V_i$ occurs in $\rho_{H^m}^G$.

Then we have to compute the exponent $q$ as defined by equation (2.6) in this case; for this we need to compute the number $b$ of equation (2.4). Using (4.6) of Lemma 4.2 we obtain

$$b_1 = m|H|^{m-1}a_1.$$ 

The rest of the coefficients of $D$ are of the following form

$$|H|^{m-1}(m - \ell)a_1 + \ell a_2$$

for $\ell = 1, \ldots, m$ with $a_1$ and $a_2$ the coefficients of the correspondence $D_i$ as in Remark 3.4. Therefore the differences are of the following type

$$|H|^{m-1}\ell(a_1 - a_2)$$

for $\ell = 1, \ldots, m$, where $a_1 - a_2$ are the corresponding numbers for the correspondence $D_i$. According to Lemma 3.1 we have

$$a_1 - a_2 = \frac{|G|}{\dim V} = (n - 2)!n,$$

which implies

$$b = |H|^{m-1}(n - 2)!n$$

and hence

$$q = \frac{|G^m|}{b \cdot \dim V} = \frac{|S_n|^m}{|S_{n-1}|^{m-1}(n-2)!n \cdot (n-1)} = n^{m-1}.$$ 

Therefore the assertion follows from Theorem 2.1 as soon as we show that

$$\sum_{i=1}^{m} s_i \left( q \sum_{k=1}^{m} \left( \dim V_k - \dim(V_k)^{\langle \tau_i \rangle} \right) - (|G^m : H^m| - |H^m \backslash G^m / \langle \tau_i \rangle|) \right) = 0 $$

with $s_i$ given by (4.2) and where $\tau_i$ was defined just before Lemma 4.1.

To see this, observe that

$$[G^m : H^m] = n^m$$

and

$$|H^m \backslash G^m / \langle \tau_i \rangle| = |G : H|^{m-1}|H \backslash G / \langle \tau \rangle| = n^{m-1}(n-1).$$

Moreover, we have

$$\dim(V_k)^{\langle \tau_i \rangle} = \langle V_k, \rho_{\langle \tau_i \rangle}^G \rangle_{G^m} = \begin{cases} \langle V, \rho_{\langle \tau_i \rangle}^G \rangle = n - 2, & \text{if } k = i; \\ \dim V = n - 1 & \text{otherwise.} \end{cases}$$

Hence the left hand side of (4.9) is equal to

$$\sum_{i=1}^{m} s_i n^{m-1} \left( (\dim V_i - \dim V_i^{\langle \tau_i \rangle}) - 1 \right) = 0.$$ 

Finally, the computation of the dimension is a consequence of equation (2.7) using (4.2).
The fact that the Prym-Tyurin variety $P$ is constructed via a product of groups suggests that it is a product itself. Moreover equation (4.8) indicates that it is the product of the Jacobian varieties $JX_i$. The next theorem shows that this is in fact the case.

**Theorem 4.4.** Let the notation be as in Theorem 4.3. Then the maps $q_i : X \to X_i$ induce an isomorphism

$$JX_1 \times \cdots \times JX_m \cong P$$

of principally polarized abelian varieties.

**Proof.** The map $q_1^* + \cdots + q_m^* : JX_1 \times \cdots \times JX_m \to JX$ is an isogeny onto its image. According to Lemma 4.2 it maps $JX_1 \times \cdots \times JX_m$ into $P$. From Theorem 4.3 we obtain that $q_1^* + \cdots + q_m^*$ induces an isogeny

$$JX_1 \times \cdots \times JX_m \to P.$$

According to Lemmas 3.2 and 4.1(c) the maps $q_i : X \to X_i$ do not factorize via a nontrivial cyclic étale covering. From this we deduce, using [1, Proposition 11.4.3], that the canonical polarization of $JX$ induces a polarization of the same type on $P$ and $JX_1 \times \cdots \times JX_m$, namely the $n^{m-1}$-fold of a principal polarization. This implies that $q_1^* + \cdots + q_m^* : JX_1 \times \cdots \times JX_m \to P$ is an isomorphism. \□

A first consequence of Theorems 4.3 and 4.4 is Theorem 1.1.

**Proof of Theorem 1.1.** Let $X_1, \ldots, X_m$ be smooth projective curves of genus $g_i \geq 2$ for all $i$ and $n$ an integer at least equal to $1 + \max_{i=1}^m g_i$. According to [3, Proposition 8.1] each $X_i$ admits a simple covering $f_i : X_i \to \mathbb{P}^1$ of degree $n$, since $n \geq g_i + 1$. If necessary, we may move the branch points so that they become pairwise disjoint. According to Lemma 3.2 the Galois group of the Galois closure $Z_i \to \mathbb{P}^1$ is the symmetric group $S_n$. Hence the assumptions of Theorems 4.3 and 4.4 are satisfied. The formula for the dimension of $JX$ is a special case of (2.8). \□

The following corollary is a direct consequence of Theorems 4.3 and 4.4. Note that Mumford’s theorem (see [7, p.346]) mentioned in the introduction is just the special case $m = 2$ of it, using Welters Theorem (see [9]) which implies that these Prym-Tyurin varieties of exponent 2 are classical Prym varieties.

**Corollary 4.5.** Let $X_1, \ldots, X_m$ denote hyperelliptic curves of genus $g_i \geq 2$ for $i = 1, \ldots, m$ whose hyperelliptic coverings have pairwise disjoint ramification locus. Then the product

$$JX_1 \times \cdots \times JX_m$$

occurs as a Prym-Tyurin variety of exponent $2^{m-1}$ in a Jacobian of dimension

$$\dim JX = 2^{m-1}(\sum_{i=1}^m g_i + m - 2) + 1.$$
Remark 4.6. An analogous corollary can be stated for any \( k \geq 3 \) in the case of simply ramified \( k \)-gonal curves \( X_1, \ldots, X_m \). In particular, the case \( m = 2, k = 3 \) gives the Prym-Tyurin varieties of exponent 3 which were studied in [5] starting from a completely different geometric set-up (see [5, Theorems 4.1 and 4.2]).

Remark 4.7. It is well-known that any principally polarized abelian variety of dimension \( g \) occurs as a Prym-Tyurin variety of exponent \( 2^{g-1}(g-1)! \) (see [1, Corollary 12.2.4] and use twice the principal polarization). Notice that the exponent \( n^{m-1} \) is considerably smaller than this number. Moreover, here the Prym-Tyurin varieties are given by an explicit correspondence whereas, in the general case, they are given somewhat abstractly by successive hyperplane sections.

Remark 4.8. Our method for constructing Prym-Tyurin structures on products of Jacobians also works for groups different from \( S_n \). We only give one example, namely using the alternating group \( A_n \) and its standard representation. We omit the details of the proof of the following theorem, since they are completely analogous to the proof of Theorem 1.1.

Theorem 4.9. Let \( X_1, \ldots, X_m \) denote general curves of genus \( g_i \geq 3 \) over an algebraically closed field of characteristic 0 and \( n \geq 1 + 2 \max_{i=1}^{m} g_i \) an integer. Then the product

\[
J X_1 \times \cdots \times J X_m
\]

occurs as a Prym-Tyurin variety of exponent \( n^{m-1} \) in a Jacobian \( J \) of dimension

\[
\dim J = 1 + n^{m-1}(n(2m-1) - 2m + 2 \sum_{i=1}^{m} g_i)
\]

as well as in a Jacobian of dimension

\[
\dim J = 1 + n^{m-1}(n(m-1) - m + \sum_{i=1}^{m} g_i).
\]

For the proof we use the following result, proven in [6, Theorem 3.3]. Let \( g \geq 3 \). Then a general curve of genus \( g \) admits a cover to \( \mathbb{P}^1 \) of degree \( n \) with monodromy group \( A_n \) such that all inertia groups are generated by a double transposition if and only if \( n \geq 2g + 1 \). The assertion also holds for three-cycles instead of double transpositions.

So for each \( n \geq 2g+1 \) a general curve \( X \) of genus \( g \) admits two kind of coverings \( X \to \mathbb{P}^1 \). Both are of degree \( n \) and their corresponding Galois cover \( \Pi : Z \to \mathbb{P}^1 \) has as Galois group \( A_n \). In the first case the action of \( A_n \) on \( Z \) has geometric signature \([0, (C_1, s_1)]\), where \( C_1 \) is the conjugacy class in \( A_n \) of the subgroup generated by \( (1 2)(3 4) \), and in the second case geometric signature \([0, (C_2, s_2)]\) where \( C_2 \) is the conjugacy class in \( A_n \) of the subgroup generated by \( (1 2 3) \).

As before, it may be proven that then the Jacobians \( JX_i \) have a presentation as a Prym-Tyurin variety with exponent \( q = 1 \) with respect to the group \( A_n \), the subgroup \( A_{n-1} \) and the standard representation of \( A_n \). Then Theorem 2.1 can be applied to complete the proof of Theorem 4.9.
Observe that the dimension of the Jacobian $J$ in the first case comes from the fact that the geometric signature for the action on all the curves is $[0, (C_1, s_1)]$ and in the second case on all the curves is $[0, (C_2, s_2)]$. Of course, one could also work out a mixed case.

**Remark 4.10.** One could also use non-absolutely irreducible representations for constructing Prym-Tyurin structures on products of Jacobians. The proof is essentially the same. Instead of Theorem 2.1 one needs the more general theory of [2].

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