Cosmological magnetic fields from inflation and backreaction

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Abstract. We study the backreaction problem in a mechanism of magnetogenesis from inflation. In usual analysis, it has been assumed that the backreaction due to electromagnetic fields spoils inflation once it becomes important. However, there exists no justification for this assumption. Hence, we analyze magnetogenesis from inflation by taking into account the backreaction. On the contrary to the naive expectation, we show that inflation still continues even after the backreaction begins to work. Nevertheless, it turns out that creation of primordial magnetic fields is significantly suppressed due to the backreaction.

Keywords: primordial magnetic fields, inflation

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1 Introduction

The origin of cosmological magnetic fields still remains an enigma in modern precision cosmology [1, 2]. In particular, how to make magnetic fields on Mpc scales would be a big challenge. As an attractive possibility, creation of primordial magnetic fields from inflation has been discussed [3, 4]. In particular, a nonminimal kinetic term of vector fields in supergravity can be used to generate primordial cosmological magnetic fields [5].

Recently, the backreaction issue in this mechanism is raised [6]. There, it is pointed out that magnetic fields from inflation may not be a viable mechanism when the backreaction is taken into account. The difficulty to create magnetic fields from inflation originates in the existence of electric fields whose energy density rapidly grows and soon catches up with the energy density of inflaton. Thus, the backreaction begins to work. The crucial assumption here is that the backreaction destroys inflation. Hence, creation of magnetic fields cannot be compatible with a sufficiently long inflation. However, no one has proved the above assumption. Rather, there is a counter example that the backreaction of vector fields does not spoil inflation [7]. Hence, it is worth studying magnetogenesis from inflation by taking into account the backreaction.

In this paper, we reconsider generation of magnetic fields during inflation. We take into account the backreaction of electromagnetic fields in an inflationary scenario. It turns out that the resultant cosmological evolution with backreaction is drastically different from the one without backreaction. Indeed, the dynamics of inflaton is quite different and the universe is anisotropically inflating although anisotropy is very small [7]. In this new background, we consider generation of magnetic fields and conclude that cosmological magnetic fields could be hardly produced during inflation due to the backreaction although inflation still continues in spite of backreaction.

The organization of the paper is as follows. In section II, we present our model and clarify the condition when the backreaction becomes important. In section III, we review a naive perturbative analysis and reveal the necessity of the backreaction. In section IV, we take into account the backreaction and show that the backreaction changes the dynamics of inflaton significantly. In this new inflationary background, we study generation of magnetic fields and calculate the power spectrum of magnetic fields. We find that it is difficult to produce magnetic fields from inflation due to backreaction although inflation still continues. The final section is devoted to conclusion.
2 Models

In this section, we present our models and basic equations for discussing the backreaction [7]. Here, we envisage the chaotic inflation although it is easy to extend analysis to other scenarios. We consider electromagnetic fields in a homogeneous anisotropic universe and explain why it is believed that inflation is spoiled when the backreaction of electromagnetic fields becomes important. Then, treating electromagnetic fields perturbatively, we clarify in which case the backreaction becomes important.

Let us consider the following action for the gravitational field, the inflaton field $\phi$ and the electro-magnetic vector field $A_\mu$ coupled with $\phi$:

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R - \frac{1}{2} (\partial \mu \phi) (\partial \nu \phi) - V(\phi) - \frac{1}{4} f^2(\phi) F_{\mu\nu} F^{\mu\nu} \right],$$

where $M_p$ is the reduced Planck mass, $g$ is the determinant of the metric, $R$ is the Ricci scalar, $V(\phi)$ is the inflaton potential, $f(\phi)$ is the coupling function of the inflaton field to the vector one, respectively. The field strength of the vector field is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Thanks to the gauge invariance, we can choose the gauge $A_0 = 0$. Without loss of generality, we can take $x$-axis in the direction of the vector. Hence, we take the homogeneous fields of the form $A_{\mu} = (0, A_x(t), 0, 0)$ and $\phi = \phi(t)$ . Note that we have assumed the direction of the vector field does not change in time, for simplicity. This field configuration holds plane symmetry in the plane perpendicular to the vector. Then, we take the metric to be

$$ds^2 = -dt^2 + e^{2\alpha(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2),$$

where the cosmic time $t$ is used. Here, $a \equiv e^\alpha$ is an isotropic scale factor and $\sigma$ represents a deviation from the isotropy. It should be noted that we need to consider an anisotropic spacetime from consistency when we treat a background vector field. With above ansatz, one obtains the equation of motion for the vector field which is easily solved as

$$\dot{A}_x = f^{-2}(\phi)e^{-4\alpha - 4\sigma} p_A,$$  \hspace{1cm} (2.3)

where an overdot denotes a derivative with respect to the cosmic time $t$ and $p_A$ denotes a constant of integration. Substituting (2.3) into other equations, we obtain basic equations

$$\dot{\phi}^2 = \phi^2 + \frac{1}{3M_p^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) + \frac{p_A^2}{2} f^{-2}(\phi)e^{-4\alpha - 4\sigma} \right],$$  \hspace{1cm} (2.4)

$$\dot{\alpha} = -3\dot{\phi} + \frac{1}{M_p^2} V(\phi) + \frac{p_A^2}{6M_p^2} f^{-2}(\phi)e^{-4\alpha - 4\sigma},$$  \hspace{1cm} (2.5)

$$\dot{\sigma} = -3\dot{\sigma} + \frac{p_A^2}{3M_p^2} f^{-2}(\phi)e^{-4\alpha - 4\sigma},$$  \hspace{1cm} (2.6)

$$\ddot{\phi} = -3\dot{\dot{\phi}} - V_\phi(\phi) + p_A^2 f^{-3}(\phi)f_\phi(\phi)e^{-4\alpha - 4\sigma},$$  \hspace{1cm} (2.7)

where $V_\phi$ denotes the derivative of $V$ with respect to $\phi$.

From eq. (2.4), we see the effective potential $V_{\text{eff}} = V + p_A^2 f^{-2}e^{-4\alpha - 4\sigma}/2$ determines the inflaton dynamics. As the second term is coming from the vector contribution, we refer
it to the energy density of the vector. Let’s check if inflation occurs in this model. Using eqs. (2.4) and (2.5), equation for acceleration of the universe is given by

\[ \ddot{a} = \ddot{\alpha} + \alpha'^2 = -2\dot{\sigma}^2 + \frac{1}{3M_p^2} \phi' \left[ V - \frac{P_A}{2} f^{-2} e^{-4\alpha - 4\sigma} \right]. \] (2.8)

It is easy to see that if the energy density of the vector is dominant inflation soon ends. This is the reason why it is usually supposed that the backreaction spoils inflation. However, as we will see in section IV, the relevance of backreaction does not necessarily imply the energetic dominance of the electromagnetic fields.

To see when the backreaction of electromagnetic fields becomes relevant, we consider electromagnetic fields in the isotropic universe. From eq. (2.4), it is apparent that the fate of the electric field depends on the behavior of coupling function \( f(\phi) \). By considering the critical case \( f(\phi) \propto e^{-2\alpha} \) for which the energy density of the vector field remains almost constant during inflation, we can determine the functional form of \( f \) under the assumption of slow-roll inflation. We can use conventional slow-roll equations

\[ \dot{\alpha}^2 = \frac{1}{3M_p^2} V(\phi), \quad 3\dot{\alpha} \dot{\phi} = -V_{\phi}(\phi). \] (2.9)

Then, we have an equation \( da/d\phi = \dot{\alpha}/\dot{\phi} = -V(\phi)/M_p^2 V_{\phi}(\phi) \). This can be easily integrated as \( \alpha = -\int V/M_p^2 V_{\phi} d\phi \). Here, we have absorbed a constant of integration into the definition of \( \alpha \). Thus, we obtain

\[ f = e^{-2\alpha} = e^{2\Delta} \int V_{\phi} e^{2\phi}. \] (2.10)

For the polynomial potential \( V \propto \phi^2 \), we have \( f = e^{2\phi^2/2M_p^2} \). Let us consider more general cases by introducing a parameter \( c \) in the following form [5]:

\[ f = e^{2\Delta} \phi^2 = e^{-2\alpha} \propto a^{-2c}, \] (2.11)

where we have used the relation

\[ e^{2\Delta} \phi^2 = e^{-2\alpha}. \] (2.12)

We notice that the energy density of the vector field during inflation would be negligible when \( c < 1 \) and remains constant when \( c = 1 \). While it grows when \( c > 1 \) and the growth rate of the energy density of vector fields can be calculated as

\[ \frac{P_A^2}{2} f^{-2}(\phi)e^{-4\alpha} \propto \exp(4(c - 1)\alpha). \] (2.13)

Apparently, we need to take into account the backreaction for \( c > 1 \). In the next section, we consider generation of magnetic fields in the conventional inflationary background driven by an inflaton. There, we will recognize that \( c > 1 \) is necessary for creating cosmological magnetic fields and therefore the backreaction turns out to be important.

### 3 Magnetic fields from inflation

In this section, we review the arguments made in the paper by Demozzi et al. [6].
Let us consider the Maxwell fields

\[ S = \int d^4 x \sqrt{-g} \left[ -\frac{1}{4} f^2(\phi) F_{\mu\nu} F^{\mu\nu} \right], \]

where we assumed a general coupling function \((2.11)\). We will consider the isotropic inflationary background

\[ ds^2 = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right), \]

where, for convenience, we switched to the conformal time \(\eta\). It is well known that the physical degrees are described by the transverse vector which can be written in Fourier space as

\[ A^T_i(x, \eta) = \sum_{\sigma=1,2} \int \frac{d^3 k}{(2\pi)^{3/2}} A^\sigma_k(\eta) \epsilon^\sigma_i(k) e^{i k \cdot x}, \]

where \(\sigma\) denotes degrees of polarization and polarization vectors \(\epsilon^\sigma_i(k)\) satisfy the relation \(k_i \epsilon^\sigma_i(k) = 0\) and \(\epsilon^\sigma_i(-k) \epsilon^\rho_i(k) = \delta_{\sigma \rho}\). Assuming the isotropic FRW universe \((3.2)\), we can deduce the action for physical modes

\[ S = \frac{1}{2} \sum_{\sigma=1,2} \int d\eta d^3 k f^2(\phi) \left[ A'^\sigma_k A'_{-k} - k^2 A^\sigma_k A^\sigma_{-k} \right], \]

where a prime represents a derivative with respect to the conformal time. Let us quantize the system by promoting \(A^\sigma_k\) and the conjugate momentum \(\pi^\sigma_k = f(\phi)^2 A'_{-k}\) to operators satisfying \([A^\rho_k, \pi^\sigma_k'] = i\delta^\rho \delta(k - k')\), and others are zero. Using creation and annihilation operators satisfying \([a^\sigma_k, a^\rho_{k'}^\dagger] = \delta^\rho \delta(k - k')\), we can expand \(A^\sigma_k\) as

\[ A^\sigma_k = u_k a^\sigma_k + u_k^* a^\sigma_{-k}^\dagger, \]

where mode functions have to satisfy equations of motion

\[ u''_k +\frac{2f'}{f} u'_k + k^2 u_k = 0 \]

and the normalization conditions \(u_k u^\dagger_k - u_k^* u^\dagger_{-k} = i/f^2\). From the correlation function of the vector field

\[ \langle 0| A_i(x) A^i(0)|0 \rangle = \int \frac{dk}{k} \frac{k^3}{\pi^2 a^2} |u_k|^2 e^{i k \cdot x} = \int \frac{dk}{k} \delta^2_A(k, \eta) e^{i k \cdot x}, \]

we can read off the power spectrum

\[ \delta^2_A(k, \eta) = \frac{|u_k|^2 k^3}{\pi^2 a^2}. \]

Similarly, we can deduce the power spectrum of the magnetic fields \(B_i = \epsilon_{ijk} F^{jk}/2\) defined by

\[ \langle 0| B_i(x) B^i(0)|0 \rangle = \int \frac{dk}{k} \delta^2_B(k, \eta) e^{i k \cdot x} \]
\[ \delta_B^2(k, \eta) = \frac{|u_k|^2 k^5}{\pi^2 a^4}, \]  
(3.10)

where we used the formula
\[ B^i B_i = \frac{1}{2a^4} F_{ik} F_{ik} = \frac{1}{a^4} \left( \partial_i A_k \partial_i A_k - \partial_k A_i \partial_i A_k \right). \]  
(3.11)

With this definition, a scale free spectrum corresponds to \( \delta_B^2 \propto k^0 \).

The vacuum state \(|0\rangle\) can be specified by the initial condition for the positive frequency mode at a sufficiently past time \( \eta \)
\[ u_k(\eta) = \frac{1}{f} \sqrt{2k} e^{-ik\eta}. \]  
(3.12)

The above mode function on subhorizon scales connects to the superhorizon solutions
\[ u_k(\eta) = C_1 + C_2 \int \frac{d\eta}{f^2}, \]  
(3.13)

where \( C_1 \) and \( C_2 \) are constants of integration. For the coupling function, we take
\[ f = \left( \frac{a}{a_f} \right)^{-2c_{\text{eff}}}, \]  
(3.14)

where \( c_{\text{eff}} \) is a parameter and \( a_f \) is the scale factor at the end of inflation \( \eta_f \). In the case of the conventional slow roll inflation, following (2.11), we have \( c_{\text{eff}} = c \). Then, the solution (3.13) becomes
\[ u_k = C_1 + C_2 a^{4c-1}, \]  
(3.15)

where we used the relation \( d\eta = da/(H_I a^2) \) and rescaled constants of integration. Here, the Hubble parameter during inflation \( H_I \) is assumed to be constant. There are two branches where magnetic fields can be created. For a negative \( c \), there exists no backreaction problem because the electric fields do not exist in this case. For this case, however, the analysis is not reliable due to the strong coupling problem [6]. Hence, we will not consider this case hereafter. For a positive \( c \), the second term of (3.15) which depends on time is a relevant one. By matching (3.12) and (3.15) at the horizon crossing \( a_k H_I = k \), the constant \( C_2 \) can be determined. Thus, we obtain
\[ u_k = \frac{1}{f_k \sqrt{2k}} \left( \frac{a}{a_f} \right)^{4c-1} = \frac{1}{f_k \sqrt{2k}} \left( \frac{H_I a}{k} \right)^{4c-1}, \]  
(3.16)

where we defined \( f_k = (a_k/a_f)^{-2c} \). Substituting (3.16) into (3.10), we obtain magnetic fields at the end of inflation \( \eta_f \) as
\[ \delta_B(\lambda_p, \eta_f) = \frac{H_I^2}{\sqrt{2\pi}} \left( \frac{\lambda_p}{H_I} \right)^{2c-3}, \]  
(3.17)

where \( \lambda_p = a_f/k \) is the physical wavelength at the end of inflation corresponding to the co-moving wavenumber \( k \). To get the flat spectrum, we need \( c = 3/2 \). Assuming the GUT scale
inflation $H_I \sim 10^{-6} M_p$, we get $\delta_B \sim 10^{-12} M_p^2$ at the end of inflation. Since $1 G \sim 10^{-20} \text{GeV}^2$, this implies $\delta_B \sim 10^{16} G$ at the end of inflation. If we assume the instantaneous reheating and take $(M_p H_I)^{1/2}$ as a reheating temperature, we have

$$\frac{a_0}{a_f} \sim \frac{(M_p H_I)^{1/2}}{T_0} \sim 10^{20} .$$

where we used the temperature $T_0 \sim 10^{-13} \text{GeV}$ at present $a_0$. Taking into account the relation $\delta_B \propto 1/a^2$, we obtain $\delta_B(\lambda_p, \eta_0) \sim (a_f/a_0)^2 \delta_B(\lambda_p, \eta_0) \sim 10^{-12} G$ at present. This is close to the expected value $10^{-9} G$ from observations. In that case, however, the energy density of the electromagnetic fields $\rho_{\text{em}}$ exceeds the energy of the inflaton. In fact, from the formula for the energy-momentum tensor

$$T_{\mu\nu} = f^2(\phi) \left[ F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right],$$

we can calculate the energy density for the electro-magnetic fields

$$\rho_{\text{em}} = -\langle 0 | T^0_0 | 0 \rangle = \frac{1}{4\pi^2 a^4} \int \frac{dk}{k} k^3 f^2 \left[ |u_k|^2 + k^2 |u_k|^2 \right],$$

which can be estimated as

$$\rho_{\text{em}} \sim H_I^4 \left( \frac{a}{a_i} \right)^{4c-4},$$

where $a_i$ is the beginning of inflation. Once the energy density of the electromagnetic fields $\rho_{\text{em}}$ becomes comparable to the energy density of the inflaton $\rho_\phi$, namely, $\rho_{\text{em}} \sim \rho_\phi \sim M_p^2 H_I^2$, we need to consider the backreaction. More precisely, after the time $a/a_i \sim 10^6$ we need to consider the backreaction. Usually, it is believed that once the electromagnetic fields becomes dominant, the backreaction spoils inflation. However, in the next section, we will see inflation continues opposed to the naive expectation. This is because the backreaction affects the dynamics of inflaton before the energy density of the electromagnetic fields dominates that of the inflaton. Thus, it is worth reexamining generation of magnetic fields from inflation with the backreaction.

### 4 Magnetic fields from inflation with backreaction

We consider the potential $V(\phi) = m^2 \phi^2 / 2$ in this section. Hence, we adopt the coupling function $f(\phi) = e^{c\phi^2 / 2M_p^2}$. Since we are going to look into the situation where the electromagnetic field is not negligible, it is natural to consider an anisotropic spacetime with a coherent vector field. However, as the energy density of the vector field should be subdominant during inflation, the anisotropy is negligible to the lowest order. Hence we ignore $\sigma$ in the basic equations and regard it perturbative quantity.

#### 4.1 Inflation with backreaction

The inflaton dynamics described by eqs. (2.4) and (2.7) can be written as

$$\ddot{\alpha} = \frac{1}{3M_p^2} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \frac{\dot{\phi}^2}{M_p^2} \phi^2 - 4a^2 p_A^2 \right],$$

$$\ddot{\phi} = -3 \dot{\alpha} \dot{\phi} - m^2 \phi + \frac{c}{M_p^2} \phi e^{-\frac{c}{M_p^2} \phi^2 - 4a^2 p_A^2} .$$


When the effect of the vector field is comparable with that of the inflaton field as source terms in (4.2), we get the relation $c p_A^2 e^{-c \phi^2/M_p^2} - 4 \alpha / M_p^2 \sim m^2$. If we define the ratio of the energy density of the vector field $\rho_A \equiv p_A^2 e^{-c \phi^2/M_p^2 - 4 \alpha}$ to that of the inflaton $\rho_\phi \equiv m^2 \phi^2 / 2$ as

$$\mathcal{R} \equiv \frac{\rho_A}{\rho_\phi} = \frac{p_A^2 e^{-c \phi^2/M_p^2 - 4 \alpha}}{m^2 \phi^2 / 2},$$

(4.3)

we find the ratio becomes $\mathcal{R} \sim M_p^2 / c \phi^2$ when the above relation holds. Since inflation takes place typically at $\phi \sim O(10) M_p$, the ratio goes $\mathcal{R} \sim 10^{-2}$. Thus we find that the effect of the vector field in (4.1) is negligible even when it is comparable with that of the scalar field in (4.2).

We notice that the above situation is not transient one but an attractor. Suppose that $\rho_A$ is initially negligible, $\mathcal{R}_i \ll 10^{-2}$. In the conventional slow-roll inflationary phase (2.9), the relation $e^{-\phi^2/M_p^2} \propto e^{4 \alpha}$ holds as was shown in (2.10). Hence, the ratio $\mathcal{R}$ varies as $\mathcal{R} \propto e^{4(c-1) \alpha}$. As we now consider $c > 1$, $\rho_A$ increases rapidly during inflation and eventually reaches $\mathcal{R} \sim 10^{-2}$. Whereas, if the ratio is initially $\mathcal{R}_i \gg 10^{-2}$, the inflaton climbs up the potential due to the effect of the vector field in (4.2), hence $\rho_A$ will decrease rapidly and again $\mathcal{R} \sim 10^{-2}$ will be realized soon. Thus irrespective of initial conditions, $\rho_A$ will track $\rho_\phi$.

From these arguments, the inflaton dynamics after tracking is governed by the modified slow-roll equations

$$\dot{\alpha}^2 = \frac{1}{6 M_p^2} m^2 \phi^2,$$

(4.4)

$$3 \dot{\alpha} \dot{\phi} = - m^2 \phi + \frac{c}{M_p^2} \phi p_A^2 e^{- \frac{c}{M_p^2} \phi^2 - 4 \alpha}.$$

(4.5)

We refer to the phase governed by the above equations as the second inflationary phase, compared to the first one governed by the equations (2.9). Using above equations, we can deduce

$$\phi \frac{d \phi}{d \alpha} = -2 M_p^2 + \frac{2 c p_A^2}{m^2} e^{- \frac{c}{M_p^2} \phi^2 - 4 \alpha}.$$

(4.6)

This can be integrated as $e^{-c \phi^2/M_p^2 - 4 \alpha} = m^2 M_p^2 (c-1) / c^2 p_A^2 \left[1 + D e^{-4(c-1) \alpha}\right]^{-1}$, where $D$ is a constant of integration. This solution rapidly converges to

$$e^{- \frac{c}{M_p^2} \phi^2 - 4 \alpha} = \frac{m^2 M_p^2 (c-1)}{c^2 p_A^2}.$$

(4.7)

Thus, we found $\rho_A$ becomes constant during the second inflationary phase [7]. In particular, from (4.7), the relation

$$f = e^{\frac{c}{M_p^2} \phi^2} \propto a^{-2}$$

holds in this phase. Thus, when the backreaction becomes relevant, we effectively have $c_{\text{eff}} = 1$.

We note that the backreaction makes the expansion anisotropic. In fact, as is done in a previous work [7], we can obtain a remarkable result from (2.6) as

$$\frac{\dot{\sigma}}{\dot{\alpha}} = \frac{1}{3} \frac{c-1}{c} \epsilon,$$

(4.9)
where $\epsilon$ is a slow roll parameter and $\Sigma/H$ is the anisotropic expansion rate normalized by the Hubble parameter. Typically, we have a small anisotropy $\Sigma/H \sim 10^{-2}$. However, it is not impossible to detect this small number by PLANCK [8].

4.2 Generation of magnetic fields

In the situation where magnetic fields have a scale free spectrum $c = 3/2$, the formula (3.8) tells us that the electric fields have a red spectrum. Hence, the largest scale has a dominant contribution to the energy density. We can assume the coherent electric fields with a definite direction dominate the energy density of the electromagnetic fields. Then, the result in the previous subsection is applicable.

Now, we will take into account the backreaction. The point is as follows. Before the backreaction becomes important, we have the relation

$$f \propto \left( \frac{a}{a_f} \right)^{-2c}.$$  \hspace{1cm} (4.10)

However, once the backreaction becomes important, we have an attractor behavior

$$f \propto \left( \frac{a}{a_f} \right)^{-2}.$$  \hspace{1cm} (4.11)

That means the effective $c_{\text{eff}}$ changes from $c_{\text{eff}} = c$ to the critical value $c_{\text{eff}} = 1$ due to backreaction. Since we have the formula (see eq. (3.21))

$$\rho_{\text{em}} = H_f^4 \left( \frac{a_b}{a_t} \right)^{4c-4},$$  \hspace{1cm} (4.12)

the transition point $a_b$ occurs at $\rho_{\text{em}} \sim \rho_A \sim 10^{-2} \rho_\phi \sim 10^{-2} M_p^2 H_f^2$, i.e.

$$\left( \frac{a_b}{a_t} \right)^{4c-4} = 10^{-2} H_f^{-2} M_p^2.$$  \hspace{1cm} (4.13)

We are now in a position to calculate the power spectrum of magnetic fields. First, we consider the modes which exit the horizon before $a_b$. The superhorizon evolution of the mode function before $a_b$ is given by

$$u_k(\eta) = \left( \frac{a}{a_k} \right)^{4c-1},$$  \hspace{1cm} (4.14)

where we should note $f_k$ is defined by $f_k = (a_b/a_f)^{2c-2}(a_k/a_f)^{-2c}$ from the continuity at $a_b$. Since the evolution after $a_b$ becomes $u_k \propto a^3$, we obtain the mode function after $a_b$ as

$$u_k = \left( \frac{a}{a_b} \right)^{4c-1} \left( \frac{a_b}{a_f} \right)^3.$$  \hspace{1cm} (4.15)

From the formula (3.10), we obtain magnetic fields

$$\delta_B(\lambda_p, \eta_f) = \frac{H_f^2}{\sqrt{2\pi}} \left( \frac{\lambda_p}{H^{-1}_f} \right)^{2c-3} \left( \frac{a_b}{a_f} \right)^{2c-2}.$$  \hspace{1cm} (4.16)
Compared to the cases with no backreaction (3.17), the amplitude is reduced by the factor
\[
\left( \frac{a_b}{a_f} \right)^{2c-2} = 10^{-1} H_I^{-1} M_p \left( \frac{a_i}{a_f} \right)^{2c-2}.
\] (4.17)

For the flat spectrum \( c = 3/2 \), we can deduce magnetic fields at the end of inflation as
\[
\delta_B(\lambda_p, \eta_f) = 10^{-1} H_I^{-1} M_p \left( \frac{a_i}{a_f} \right) \frac{H_I^2}{\sqrt{2\pi}}.
\] (4.18)

Without backreaction, we anticipated \( 10^{-12} \)G for the scale invariant case \( c = 3/2 \). However, by taking into account the backreaction, we have a suppression factor \( a_b/a_f \) in (4.16) which is about \( 10^{-24} \). Hence, we can expect at most \( 10^{-36} \)G on Mpc scales at present. For modes which exit the horizon after the transition time \( a_b \), by setting \( c = 1 \) in (4.14), we obtain
\[
u_k(\eta) = \frac{1}{f_k \sqrt{2k}} \left( \frac{a}{a_k} \right)^3,
\] (4.19)
where \( f_k \) is now defined by \( f_k = (a_k/a_f)^{-2} \). Thus, we can calculate magnetic fields at the end of inflation as
\[
\delta_B(\lambda_p, \eta_f) = \frac{H_I^2}{\sqrt{2\pi}} \left( \frac{\lambda_p}{H_I^{-1}} \right)^{-1}.
\] (4.20)

The resultant spectrum for \( c = 3/2 \) is schematically depicted in figure 1. As the spectrum is simple, it is easy to deduce numbers from the formula. Using eq. (47), we find \( a_b/a_i \sim 10^5 \) for \( \alpha = 3/2 \). Here the comoving wavenumber at the beginning is around \( 10^4 \)Mpc because it corresponds to the current horizon scale. Thus the flat line as shown in figure 1 continues until around 0.1 Mpc, where we can expect the amplitude of the magnetic field \( 10^{-36} \)G. On smaller scales than that, the magnetic fields scale with \( k \). At kpc scales, for example, we can then expect \( 10^{-34} \)G.

As is expected, we have a flat spectrum on large scales before the backreaction becomes relevant. However, once the backreaction becomes important, the spectrum becomes blue. Thus, after taking into account the backreaction, we realized that primordial magnetic fields on large scales from inflation cannot be expected. Even if we resort to an amplification mechanism through some hydrodynamical and MHD processes [9], it is difficult to explain magnetic fields in clusters. Of course, the produced magnetic fields do not exceed BBN bounds [10] and limits from CMB distortions [11]

5 Conclusion

We studied cosmological magnetic fields in an inflationary scenario. We have explained how the backreaction works in magnetogenesis in the inflationary scenario. In usual analysis, it has been assumed that the backreaction spoils inflation. However, there is no justification for this assumption. Hence, we have incorporated the backreaction into the analysis and reanalyzed the production process of magnetic fields from inflation. As a consequence, we have shown that inflation still continues after the backreaction becomes important. In spite of this fact, it turned out that creation of primordial magnetic fields is still significantly suppressed due to the backreaction.
Figure 1. The magnitude of magnetic fields is schematically depicted as a function of wavenumber $k$ for the case $c = 3/2$. There exists a break at $k_b = a_b H_I$ in the spectrum due to the backreaction. As can be seen, the amplitude of magnetic fields on Mpc scales gets a suppression by $10^{-24}$ due to this break.

The main point is that the energy density of electric fields have to grow in order to have sufficient amplitude for magnetic fields, which causes the backreaction. What we have found is that the backreaction changes the dynamics of inflaton and, as a consequence, makes the genesis of magnetic fields difficult. However, inflation is not spoiled by the backreaction. Instead, an anisotropic inflationary universe has been created due to backreaction. Although magnetic fields are negligible, since the expansion of the universe is anisotropic, other interesting phenomenology such as the statistical anisotropy of primordial curvature perturbations can be expected [12]. The possibility is now under investigation.

Finally, we should note that it is possible to create large enough magnetic fields if we resort to models where the coupling is large during inflation. However, the naive perturbative treatment of electromagnetic fields is not reliable in those models. Of course, the analysis in the present paper does not exclude the possibility of the existence of viable models in the strong coupling regime.

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