Quantum $f$-divergences in von Neumann algebras II.
Maximal $f$-divergences

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Abstract

As a continuation of the paper [20] on standard $f$-divergences, we make a systematic study of maximal $f$-divergences in general von Neumann algebras. For maximal $f$-divergences, apart from their definition based on Haagerup’s $L^1$-space, we present the general integral expression and the variational expression in terms of reverse tests. From these definition and expressions we prove important properties of maximal $f$-divergences, for instance, the monotonicity inequality, the joint convexity, the lower semicontinuity, and the martingale convergence. The inequality between the standard and the maximal $f$-divergences is also given.

Keywords and phrases: Maximal $f$-divergence, standard $f$-divergence, relative entropy, monotonicity inequality, reverse test, von Neumann algebra, standard form, Haagerup’s $L^p$-space, cyclic representation, operator convex function.

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1 Introduction

Quantum divergences play a significant role in quantum information theory. We have mainly two different kinds of quantum $f$-divergences parametrized by convex (often assumed operator convex) functions $f$ on $(0, +\infty)$. The one is the standard $f$-divergence $S_f(\rho \| \sigma)$ and the other is the maximal $f$-divergence $\tilde{S}_f(\rho \| \sigma)$ (also denoted by $S_f^{\text{max}}(\rho \| \sigma)$). When specialized to the finite-dimensional (or the matrix) case, those $f$-divergences are defined for positive operators $\rho, \sigma$ (for simplicity, assumed invertible) as follows:

\begin{align}
\tilde{S}_f(\rho \| \sigma) &:= \text{Tr} \rho^{1/2} f(\sigma^{-1/2} \rho \sigma^{-1/2}) \sigma^{1/2}, \\
S_f(\rho \| \sigma) &:= \text{Tr} \rho^{1/2} f(L_\rho R_{\sigma^{-1}})(\sigma^{1/2}),
\end{align}

where $L_\rho$ and $R_{\sigma^{-1}}$ are the left multiplication by $\rho$ and the right multiplication by $\sigma^{-1}$. (We extend the above definitions to general positive operators $\rho, \sigma$ by convergence, see [22, 21].) The standard $f$-divergence $S_f(\rho \| \sigma)$ was formerly introduced and studied by Petz [39, 40] in a more general form in the von Neumann algebra setting under the name quasi-entropy (which first appeared in [29]). A typical and the most important example is the relative entropy $D(\rho \| \sigma)$ that is $S_f(\rho \| \sigma)$ when $f(t) = t \log t$, introduced first by Umegaki [49] in semifinite von Neumann algebras, and extended to general von Neumann algebras by Araki [1, 2] based on the relative modular operators. (Note that $L_\rho R_{\sigma^{-1}}$ in (1.2) is the form of relative modular operator in the finite-dimensional case.) On the other hand,
the maximal $f$-divergence $\hat{S}_f(\rho\|\sigma)$ for matrices were studied by Matsumoto [33]. A special example of $\hat{S}_f(\rho\|\sigma)$ when $f(t) = t \log t$ is another version of the relative entropy introduced by Belavkin and Staszewski [3], denoted by $D_{BS}(\rho\|\sigma)$. It is also worth noting that the form $\sigma^{1/2}f(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{1/2}$ in (1.1) is used to define some relative operator entropies [13, 14], and recently called the operator perspective [10] having a role in operator theory.

In [22, 21] we gave comprehensive expositions on standard and maximal $f$-divergences in the matrix setting, mostly from the point of view of the monotonicity inequality (often called the data-processing inequality) and the reversibility of quantum operations. Our goal in the next stage is to extend expositions in [22, 21] to the von Neumann algebra setting. We expect that those extensions would be useful in further developments of quantum information, as well as in some mathematical physics subjects such as quantum field theory (see [52] for the appearance of the relative entropy there). In the previous paper [20] we made systematic study of standard $f$-divergences and standard Rényi divergences in von Neumann algebras. The methodological novelty there is to generalize Kosaki’s variational expression of the relative entropy to general $S_f(\rho\|\sigma)$, from which many important properties of $S_f$ follow immediately. The aim of the present paper is to develop maximal $f$-divergences in a similar way in the general von Neumann algebra setting.

In section 2 of this paper we first give the definition of the maximal $f$-divergence $\hat{S}_f(\rho\|\sigma)$ for $\rho, \sigma \in M^+_\pi$ (the positive part of the predual $M_\pi$ of a von Neumann algebra $M$) and for any operator convex function $f$ on $(0, +\infty)$. For this our idea is to use the representatives $h_\rho, h_\sigma$ in Haagerup’s $L^1(M)$ space and the functional trace $\text{Tr}$ on $L^1(M)$ in place of the trace $\text{Tr}$ in the matrix case. When $\delta \sigma \leq \rho \leq \delta^{-1}\sigma$ for some $\delta > 0$, there exists a unique $A \in s(\sigma)M_\pi s(\sigma)$ (as $s(\sigma)$ being the support projection of $\sigma$) such that $h_\rho^{1/2} = Ah_\sigma^{1/2}$, so that we define $\hat{S}_f(\rho\|\sigma) := \sigma(f(A^*A))$.

Writing $A^*A = (h_\rho^{1/2}h_\sigma^{-1/2})^* (h_\rho^{1/2}h_\sigma^{-1/2}) = h_\sigma^{1/2}h_\rho h_\sigma^{-1/2}$ formally, we can write $\hat{S}_f(\rho\|\sigma) = \text{Tr} h_\sigma^{1/2}f(h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2})h_\sigma^{1/2}$, having a complete resemblance to (1.1). We then extend $\hat{S}_f(\rho\|\sigma)$ by convergence to arbitrary $\rho, \sigma \in M^+_\pi$, and prove the monotonicity inequality under unital positive normal maps and the joint convexity (Theorem 2.9).

In Sections 3 and 4, we analyze the case where $\rho$ is strongly absolutely continuous with respect to $\sigma$, and obtain a general integral expression (Theorem 4.2)

$$\hat{S}_f(\rho\|\sigma) = \int_0^1 (1-t)f\left(\frac{t}{1-t}\right) d\|E_{\rho/p+\sigma}(t)\xi_{p+\sigma}\|_2, \quad (1.3)$$

where $E_{\rho/p+\sigma}(\cdot)$ is the spectral measure of the operator $T_{\rho/p+\sigma} \in \pi_{p+\sigma}(M)_\pi^\prime$ such that $\rho(x) = \langle \xi_{p+\sigma}, T_{\rho/p+\sigma}\pi_{p+\sigma}(x)\xi_{p+\sigma} \rangle$, $x \in M$, for the cyclic representation $\pi_{p+\sigma}$ of $M$ associated with $p+\sigma$. In Section 5, the lower semicontinuity in the norm topology and the martingale convergence for $\hat{S}_f(\rho\|\sigma)$ are proved by using the expression in (1.3).

In Section 6, following Matsumoto’s idea in [33], we obtain a variational expression of the form (Theorem 6.3)

$$\hat{S}_f(\rho\|\sigma) = \min \{S_f(p\|q) : (\Psi, p, q) \}, \quad (1.4)$$

where the minimum is attained over reverse tests $(\Psi, p, q)$ for $\rho, \sigma$ consisting of a unital positive normal map $\Psi : M \to L^\infty(X, \mu)$ on a $\sigma$-finite measure space $(X, \mu)$ and $p, q \in L^1(X, \mu)_+$ with $\Psi_*(p) = \rho$ and $\Psi_*(q) = \sigma$, and $S_f(p\|q)$ is the classical $f$-divergence of $p, q$. From the variational expression in (1.4) we have the inequality $S_f(\rho\|\sigma) \leq \hat{S}_f(\rho\|\sigma)$, in particular, $D(\rho\|\sigma) \leq D_{BS}(\rho\|\sigma)$ that was first proved in [24] for matrices, and moreover the equality $S_f(\rho\|\sigma) = \hat{S}_f(\rho\|\sigma)$ is verified when $\rho, \sigma$ commute. In this way, we present three
different expressions of $\tilde{S}_f(\rho \| \sigma)$ – the definition in the beginning, expressions (1.3) and (1.4), each of which is useful in deriving some of different properties of $\tilde{S}_f(\rho \| \sigma)$.

Finally, the extension of $\tilde{S}_f(\rho \| \sigma)$ to general positive linear functionals $\rho, \sigma$ on a unital $C^*$-algebra is discussed in Section 7, and remarks and problems for further investigation are mentioned in Section 8.

2 Definition and basic properties

Let $M$ be a general von Neumann algebra with predual $M_*$ and $M_+^*$ be the positive part of $M_*$ consisting of normal positive linear functionals on $M$. In the present paper it is convenient for us to work in the framework of Haagerup’s $L^p$-spaces associated with $M$, so we first recall Haagerup’s $L^p$-spaces, see [13] for details. Given a faithful normal semifinite weight $\varphi$ on $M$, let $N$ denote the crossed product $M \rtimes_{\sigma \varphi} \mathbb{R}$ of $M$ by the modular automorphism group $\sigma_t^{\varphi}$ ($t \in \mathbb{R}$). Let $\theta_s$ ($s \in \mathbb{R})$ be the dual action of $N$ so that $\tau \circ \theta_s = e^{-s} \tau$ ($s \in \mathbb{R}$), where $\tau$ is the canonical trace on $N$. Let $\tilde{N}$ denote the space of $\tau$-measurable operators affiliated with $N$. For $0 < p \leq \infty$ Haagerup’s $L^p$-space $L^p(M)$ [17,13] is defined by

$$L^p(M) := \{ x \in \tilde{N} : \theta_s(x) = e^{-s/p}x, \ s \in \mathbb{R} \}.$$  

In particular, $L^\infty(M) = M$. Let $L^p(M)_+ = L^p(M) \cap \tilde{N}_+$, where $\tilde{N}_+$ is the positive part of $\tilde{N}$. Then $M_*$ is canonically order-isomorphic to $L^1(M)$ by a linear bijection $\psi \in M_* \mapsto h_\psi \in L^1(M)$, so that the positive linear functional $\text{tr}$ on $L^1(M)$ is defined by $\text{tr}(h_\psi) = \psi(1)$, $\psi \in M_*$. For $1 \leq p < \infty$ the $L^p$-norm $\|x\|_p$ of $x \in L^p(M)$ is given by $\|x\|_p := \text{tr}(|x|^p)^{1/p}$. Also $\| \cdot \|_\infty$ denotes the operator norm on $M$. For $1 \leq p < \infty$, $L^p(M)$ is a Banach space with the norm $\| \cdot \|_p$ and whose dual Banach space is $L^q(M)$ where $1/p + 1/q = 1$ by the duality

$$(x,y) \in L^p(M) \times L^q(M) \longmapsto \text{tr}(xy) (= \text{tr}(yx)).$$

In particular, $L^2(M)$ is a Hilbert space with the inner product $\langle x,y \rangle = \text{tr}(x^*y) (= \text{tr}(yx^*))$. Then

$$(M, L^2(M), J = *, L^2(M)_+)$$

becomes a standard form [16] of $M$, where $M$ is represented on $L^2(M)$ by the left multiplication. By the uniqueness of a standard form of $M$ up to unitary equivalence [16], our discussions in this paper are independent of the choice of a standard form of $M$. But the standard form $(M, L^2(M), *, L^2(M)_+)$ is more convenient since the Haagerup’s $L^p$-space technique is sometimes useful. Each $\sigma \in M_+^*$ is represented as

$$\sigma(x) = \text{tr}(xh_\sigma) = \langle h_\sigma^{1/2}, xh_\sigma^{1/2} \rangle, \quad x \in M,$$

with the vector representative $h_\sigma^{1/2} \in L^2(M)_+$. Note that the support projection $s(\sigma) \in M$ of $\sigma$ coincides with that of $h_\sigma$. We also note from [16] Corollary 2.5, Lemma 2.6] that for every projection $e \in M$, the standard form of the reduced von Neumann algebra $eMe$ is given as

$$(eMe, eL^2(M)e, J = *, eL^2(M)_+e).$$

The next lemma is well-known while we give a proof for completeness.
Lemma 2.1. Let \( \rho, \sigma \in M^+_t \). Assume that \( \rho \leq \alpha \sigma \), i.e., \( h^1_\rho \leq \alpha h^1_\sigma \) for some \( \alpha > 0 \). Then there exists a unique \( A \in s(\sigma)Ms(\sigma) \) such that \( h^1_\rho = Ah^1_\sigma \). The \( A \) satisfies \( \|A\| \leq \alpha^{1/2} \). Moreover, if \( \beta \sigma \leq \rho \leq \alpha \sigma \) for some \( \alpha, \beta > 0 \), then the above \( A \) satisfies \( \beta s(\sigma) \leq A^*A \leq \alpha s(\sigma) \).

Proof. From the assumption, we have \( \|h^1_\rho \xi \| \leq \alpha^{1/2} \|h^1_\sigma \xi \| \) for all \( \xi \in D(h^1_\sigma) \), the domain of \( h^1_\sigma \). Since \( h^1_\rho, h^1_\sigma \in \tilde{N} \) (\( \tau \)-measurable operators), we have a unique operator \( A \in N \) such that \( A(h^1_\sigma \xi) = h^1_\rho \xi \) for \( \xi \in D(h^1_\sigma) \) and \( A(1-s(\sigma)) = 0 \). These imply that \( A = s(\sigma)A = As(\sigma) \) and \( A^*A \leq \alpha s(\sigma) \). Since \( \theta_s(h^1_\rho) = e^{-s/2}h^1_\rho \) and \( \theta_s(h^1_\sigma) = e^{-s/2}h^1_\sigma \), \( \theta_s(h^1_\rho) = \theta_s(As) \) means that \( h^1_\rho = \theta_s(As) \). Hence it follows that \( \theta_s(A) = A \) for all \( s \in \mathbb{R} \), implying that \( A \in N^0 = M \) (where \( N^0 \) is the \( \theta \)-fixed point algebra). Therefore, \( A \in s(\sigma)Ms(\sigma) \).

Next, assume that \( \beta \sigma \leq \rho \) in addition to \( \rho \leq \alpha \sigma \). Then \( s(\rho) = s(\sigma) \), and there is a unique \( B \in s(\sigma)Ms(\sigma) \) such that \( h^1_\sigma = Bh^1_\sigma \). It is easy to see that \( AB = BA = s(\sigma) \), hence \( B = A^{-1} \in s(\sigma)Ms(\sigma) \). Since \( BB^* \leq \beta^{-1}s(\sigma) \), we have \( \beta s(\sigma) \leq A^*A \leq \alpha s(\sigma) \). \( \square \)

Remark 2.2. We have supplied a rather direct proof of Lemma 2.1 for the convenience of the reader. But a more advanced proof can be given with use of the Connes cocycle Radon-Nikodym derivative \( [D\rho : D\sigma]_{\xi} \), as in \([17, \text{Theorem VIII.3.17; see also [20, Lemma A.1]})\). In fact, \( \rho \leq \alpha \sigma \) for some \( \alpha > 0 \) if and only if \( [D\rho : D\sigma]_{\xi} \) extends to a weakly continuous (\( M \)-valued) function \( [D\rho : D\sigma]_{\xi} \) on the strip \( -1/2 \leq \text{Im} \xi \leq 0 \) which is analytic in the interior. In this case, \( \|\{D\rho : D\sigma\}_{\xi} \| \leq \alpha^{1/2} \) and \( h^1_{\rho/\alpha} = [D\rho : D\sigma]_{\xi}h^1_{\alpha} \). So, \( A \in s(\sigma)Ms(\sigma) \) given in Lemma 2.1 is \( [D\rho : D\sigma]_{\xi}A \), which also shows that the operator \( A \) is determined independently of the choice of the standard form of \( M \).

Throughout the paper we assume that \( f \) is an operator convex function on \((0, +\infty)\), i.e., \( f \) is a real function on \((0, +\infty)\) such that the operator inequality

\[
f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B), \quad 0 \leq \lambda \leq 1,
\]

holds for every positive invertible operators \( A, B \) on any Hilbert space. We set

\[
f(0^+) := \lim_{x \searrow 0} f(x), \quad f'(+\infty) := \lim_{x \to +\infty} f(x)/x,
\]

which are in \((-\infty, +\infty]\).

For \( \rho, \sigma \in M^+_t \) we write \( \rho \sim \sigma \) if \( \delta \sigma \leq \rho \leq \delta^{-1} \sigma \) for some \( \delta > 0 \), and we set

\[
(M^+_t \times M^+_t)_\sim := \{ (\rho, \sigma) \in M^+_t \times M^+_t : \rho \sim \sigma \},
\]

\[
(M^+_t \times M^+_t)_{\leq} := \{ (\rho, \sigma) \in M^+_t \times M^+_t : \rho \leq \alpha \sigma \text{ for some } \alpha > 0 \},
\]

\[
(M^+_t \times M^+_t)_{\geq} := \{ (\rho, \sigma) \in M^+_t \times M^+_t : \alpha \rho \geq \sigma \text{ for some } \alpha > 0 \},
\]

which are all convex sets. We first define the maximal \( f \)-divergence for \( (\rho, \sigma) \in (M^+_t \times M^+_t)_\sim \) and then extend it to general \( \rho, \sigma \in M^+_t \).

Definition 2.3. For each \( (\rho, \sigma) \in (M^+_t \times M^+_t)_\sim \) let \( A \in s(\sigma)Ms(\sigma) \) be as given in Lemma 2.1 so that \( h^1_{\rho/\alpha} = Ah^1_{\sigma/\alpha} \). Since \( A^*A \) is a positive invertible operator in \( s(\sigma)Ms(\sigma) \), we define an self-adjoint operator \( f(A^*A) \) in \( s(\sigma)Ms(\sigma) \) via functional calculus. We define the maximal \( f \)-divergence of \( \rho \) with respect to \( \sigma \) by

\[
\hat{S}_f(\rho || \sigma) := \sigma(f(A^*A)) \in \mathbb{R}.
\]
Here the symbol $\hat{S}$ is used to distinguish the maximal $f$-divergence from the standard $f$-divergence $S_f(\rho||\sigma)$ studied in [20]. Since $h_\rho = h_\sigma^{1/2}A^*A h_\sigma^{1/2}$, it is natural to denote $A^*A$ by $h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2}$ though the expression is rather formal. Below we will sometimes use this expression. Then (2.1) is rewritten as

$$\hat{S}_f(\rho||\sigma) = \text{tr}(h_\sigma f(h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2})),$$

(2.2)

which is in the same form as the maximal $f$-divergence in the matrix case [21] if we consider $\text{tr}$ as the usual trace and $h_\rho, h_\sigma$ as the density matrices.

**Lemma 2.4.** Let $M_0$ be another von Neumann algebra and $\Phi : M_0 \to M$ be a unital positive map that is normal (i.e., if $\{x_\alpha\}$ is an increasing net in $M_+$ with $x_\alpha \nearrow x \in M_+$, then $\Phi(x_\alpha) \nearrow \Phi(x)$). Then for every $(\rho, \sigma) \in (M_+^* \times M_+^*)_\sim$,

$$\hat{S}_f(\rho \circ \Phi||\sigma \circ \Phi) \leq \hat{S}_f(\rho||\sigma).$$

**Proof.** One can define the predual map $\Phi_s : L^1(M) \to L^1(M_0)$ of $\Phi : M_0 \to M$ by $\Phi_s(\psi) = h_\psi \psi \circ \Phi$. Since $\text{tr}(h_\psi \psi \circ \Phi(y)) = \psi(\Phi(y)) = (\Phi_s \psi)(y), y \in M_0$. Then $\Phi_s$ is a tr-preserving positive map since

$$\text{tr} \Phi_s(\psi) = \text{tr} h_\psi \psi \circ \Phi = \psi \circ \Phi(1) = \psi(1) = \text{tr} \psi.$$

Let $e := s(\rho) = s(\sigma) \in M$ and $e_0 := s(\rho \circ \Phi) = s(\sigma \circ \Phi) \in M_0$. For every $x \in (eMe)_+$, since $h_\sigma^{1/2}xh_\sigma^{1/2} \in L^1(M_+)$ and $h_\sigma^{1/2}xh_\sigma^{1/2} \leq \|x\|h_\sigma$, we have $\Phi_s(h_\sigma^{1/2}xh_\sigma^{1/2}) \leq \|x\|\Phi_s(h_\sigma)$.

By Lemma 2.1 there is a unique $b \in M_0$ such that $b(1 - e_0) = 0$ and $\Phi_s(h_\sigma^{1/2}xh_\sigma^{1/2})^{1/2} = b\Phi_s(h_\sigma)^{1/2}$. Define $\Psi(x) := b^*b \in (e_0 M_0 e_0)_+$. One can easily find that $\Psi(ax) = a\Psi(x)$ and $\Psi(x_1 + x_2) = \Psi(x_1) + \Psi(x_2)$ for every $x, x_1, x_2 \in M_+$ and $a \geq 0$. In fact, the former is obvious. For the latter, let $b_i \in M_0$ ($i = 1, 2$) be such that $b_i(1 - e_0) = 0$ and $\Phi_s(h_\sigma^{1/2}x_ih_\sigma^{1/2})^{1/2} = b_i\Phi_s(h_\sigma)^{1/2}$. Since $\Phi_s(h_\sigma^{1/2}x_ih_\sigma^{1/2}) = \Phi_s(h_\sigma)^{1/2}b_i^*b_i\Phi_s(h_\sigma)^{1/2}$, one has

$$\Phi_s(h_\sigma^{1/2}(x_1 + x_2)h_\sigma^{1/2}) = \Phi_s(h_\sigma)^{1/2}(b_1^*b_1 + b_2^*b_2)\Phi_s(h_\sigma)^{1/2},$$

which implies that $\Psi(x_1 + x_2) = b_1^*b_1 + b_2^*b_2 = \Psi(x_1) + \Psi(x_2)$. Then $\Psi$ can extend to a positive linear map $\Psi : eMe \to e_0 M_0 e_0$. It is clear that $\Psi$ is unital, i.e., $\Psi(e) = e_0$. By a Jensen inequality due to Choi [7], for $T := h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2}$ (i.e., $T = A^* A$ in Definition 2.3) we have

$$f(\Psi(T)) \leq \Psi(f(T)).$$

Since

$$\Phi_s(h_\sigma)^{1/2}\Psi(T)\Phi_s(h_\sigma)^{1/2} = \Phi_s(h_\sigma^{1/2}T h_\sigma^{1/2}) = \Phi_s(h_\sigma),$$

we have $\Psi(T) = \Phi_s(h_\sigma)^{-1/2}\Phi_s(h_\rho)\Phi_s(h_\sigma)^{-1/2}$ and

$$\Phi_s(h_\sigma)^{1/2}f(\Psi(T))\Phi_s(h_\sigma)^{1/2} \leq \Phi_s(h_\sigma)^{1/2}\Psi(f(T))\Phi_s(h_\sigma)^{1/2} = \Phi_s(h_\sigma^{1/2}f(T)h_\sigma^{1/2}).$$

Therefore,

$$\hat{S}_f(\rho \circ \Phi||\sigma \circ \Phi) = \text{tr}(\Phi_s(h_\sigma)^{1/2}f(\Psi(T))\Phi_s(h_\sigma)^{1/2})$$

$$\leq \text{tr}(\Phi_s(h_\sigma^{1/2}f(T)h_\sigma^{1/2})) = \text{tr}(h_\sigma^{1/2}f(T)h_\sigma^{1/2}) = \hat{S}_f(\rho||\sigma).$$

□
Lemma 2.5. \( \tilde{S}_f(\rho\|\sigma) \) is jointly convex on \((M^+_* \times M^+_* )\). Slightly more strongly, for any \((\rho_i, \sigma_i) \in (M^+_* \times M^+_* )\) and \(\lambda_i \geq 0 \ (1 \leq i \leq n)\) we have

\[
\tilde{S}_f \left( \sum_{i=1}^{n} \lambda_i \rho_i \parallel \sum_{i=1}^{n} \lambda_i \sigma_i \right) \leq \sum_{i=1}^{n} \lambda_i \tilde{S}_f(\rho_i\|\sigma_i).
\]

Proof. Let \( M := \oplus_{i=1}^{n} M \) and \( \Phi : M \to M \) be a unital \(*\)-homomorphism (hence, completely positive) given as \( \Phi(x) := x \oplus \cdots \oplus x, \ x \in M \). Note that the standard form of \( M \) is given as the direct sum \( \oplus_{i=1}^{n} (M, L^2(M), \ast, L^2(M)_+) \). For given \((\rho_i, \sigma_i) \in (M^+_* \times M^+_* )\) and \(\lambda_i\) let \( \rho := \oplus_{i=1}^{n} \lambda_i \rho_i \) and \( \sigma := \oplus_{i=1}^{n} \lambda_i \sigma_i \) in \( M^+_* \), so \((\rho, \sigma) \in (M^+_* \times M^+_* )\). Since \( \rho \circ \Phi = \sum_{i=1}^{n} \lambda_i \rho_i \) and \( \sigma \circ \Phi = \sum_{i=1}^{n} \lambda_i \sigma_i \), Lemma 2.5 yields

\[
\tilde{S}_f \left( \sum_{i=1}^{n} \lambda_i \rho_i \parallel \sum_{i=1}^{n} \lambda_i \sigma_i \right) \leq \tilde{S}_f(\rho\|\sigma).
\]

Since \( h_\rho = \oplus_{i=1}^{n} \lambda_i h_\rho_i \) and \( h_\sigma = \oplus_{i=1}^{n} \lambda_i h_\sigma_i \), it is immediate to see that

\[
\tilde{S}_f(\rho\|\sigma) = \sum_{i=1}^{n} \lambda_i \tilde{S}_f(\rho_i\|\sigma_i),
\]

implying the asserted inequality. \(\square\)

To extend the maximal \( f \)-divergence \( \tilde{S}_f(\rho\|\sigma) \) to arbitrary \( \rho, \sigma \in M^+_* \), we give the following:

Lemma 2.6. Let \( \rho, \sigma \in M^+_* \). For every \( \eta \in M^+_* \) with \( \eta \sim \rho + \sigma \), the limit

\[
\lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon \eta\|\sigma + \varepsilon \eta) \in (-\infty, +\infty)
\]

exists, and moreover the limit is independent of the choice of \( \eta \) as above.

Proof. Let \( \eta \) be given as stated. Since \( \rho + \varepsilon \eta \sim \sigma + \varepsilon \eta \), \( \tilde{S}_f(\rho + \varepsilon \eta\|\sigma + \varepsilon \eta) \) is defined for each \( \varepsilon > 0 \) by Definition 2.3 and \( 0 < \varepsilon \to \tilde{S}_f(\rho + \varepsilon \eta\|\sigma + \varepsilon \eta) \) is convex by Lemma 2.5. Hence the limit in (2.3) exists in \((-\infty, +\infty)\).

To prove the independence of the choice of \( \eta \), let \( \eta_1, \eta_2 \in M^+_* \) be such that \( \eta_1 \sim \rho + \sigma \) \((i = 1, 2)\). Choose a \( \delta > 0 \) such that \( \delta \eta_1 \leq \eta_2 \leq \delta^{-1} \eta_1 \). By Lemma 2.5 we have

\[
\tilde{S}_f(\rho + \varepsilon \eta_1\|\sigma + \varepsilon \eta_1) = \tilde{S}_f(\rho + \varepsilon \delta \eta_2 + \varepsilon(\eta_1 - \delta \eta_2)\|\sigma + \varepsilon \delta \eta_2 + \varepsilon(\eta_1 - \delta \eta_2)) \leq \tilde{S}_f(\rho + \varepsilon \delta \eta_2\|\sigma + \varepsilon \delta \eta_2) + \tilde{S}_f(\varepsilon(\eta_1 - \delta \eta_2)\|\delta(\eta_1 - \delta \eta_2))) = \tilde{S}_f(\rho + \varepsilon \delta \eta_2\|\sigma + \varepsilon \delta \eta_2) + \varepsilon(\eta_1 - \delta \eta_2)(1)f(1).
\]

Therefore,

\[
\lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon \eta_1\|\sigma + \varepsilon \eta_1) \leq \lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon \delta \eta_2\|\sigma + \varepsilon \delta \eta_2) = \lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon \eta_2\|\sigma + \varepsilon \eta_2).
\]

The converse inequality is similar. \(\square\)

Lemma 2.7. If \((\rho, \sigma) \in (M^+_* \times M^+_* )\), then

\[
\tilde{S}_f(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon \sigma\|1 + \varepsilon)\sigma).
\]

The converse inequality is similar. \(\square\)
Proof. We find that

\[
\hat{S}_f(\rho + \varepsilon \sigma \| (1 + \varepsilon)\sigma) = (1 + \varepsilon)\sigma \left( f\left( \frac{h_\sigma^{-1/2}(h_\rho + \varepsilon h_\sigma)h_\sigma^{-1/2}}{1 + \varepsilon} \right) \right) \\
= (1 + \varepsilon)\sigma \left( f\left( \frac{h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2} + \varepsilon s(\sigma)}{1 + \varepsilon} \right) \right) .
\]

Let \( h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2} = \int_{-\delta}^{\delta} t \, dE(t) \) be the spectral decomposition, where \( 0 < \delta < 1 \) and \( \int_{-\delta}^{\delta} dE(t) = s(\sigma) \). Then we see that

\[
f\left( \frac{h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2} + \varepsilon s(\sigma)}{1 + \varepsilon} \right) = \int_{\delta}^{\delta^{-1}} f\left( \frac{t + \varepsilon}{1 + \varepsilon} \right) \, dE(t)
\]
converges to \( \int_{\delta}^{\delta^{-1}} f(t) \, dE(t) = f(h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2}) \) in the operator norm, so that

\[
\lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \sigma \| (1 + \varepsilon)\sigma) = \sigma(f(h_\sigma^{-1/2}h_\rho h_\sigma^{-1/2}) = \hat{S}_f(\rho \| \sigma).
\]

\[
\end{proof}
\]

**Definition 2.8.** For every \( \rho, \sigma \in M^+_\ast \) define the maximal \( f \)-divergence \( \hat{S}_f(\rho \| \sigma) \) by

\[
\hat{S}_f(\rho \| \sigma) := \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \eta \| \sigma + \varepsilon \eta) \in (-\infty, +\infty]
\]

(2.4)

for any \( \eta \in M^+_\ast \) with \( \eta \sim \rho + \sigma \), where \( \hat{S}_f(\rho + \varepsilon \eta \| \sigma + \varepsilon \eta) \) is defined in Definition 2.3. By Lemmas 2.6 and 2.7 the definition is well defined independently of the choice of \( \eta \) and extend Definition 2.3 for the case \( \rho \sim \sigma \).

**Theorem 2.9.** The monotonicity property of Lemma 2.4 and the joint convexity property of Lemma 2.5 hold true for \( \hat{S}_f(\rho \| \sigma) \) for general \( \rho, \sigma \in M^+_\ast \).

**Proof.** Let \( \Phi : M_0 \to M \) be as in Lemma 2.4. For every \( \rho, \sigma \in M^+_\ast \), by Lemma 2.3 we have for every \( \varepsilon > 0 \),

\[
\hat{S}_f((\rho + \varepsilon (\rho + \sigma)) \circ \Phi \| (\sigma + \varepsilon (\rho + \sigma)) \circ \Phi) \leq \hat{S}_f(\rho + \varepsilon (\rho + \sigma) \| \sigma + \varepsilon (\rho + \sigma)).
\]

Thanks to Definition 2.8 letting \( \varepsilon \searrow 0 \) gives \( \hat{S}_f(\rho \circ \Phi \| \sigma \circ \Phi) \leq \hat{S}_f(\rho \| \sigma) \).

For any \( \lambda_i, \sigma_i \in M^+_\ast \) and \( \lambda_i \geq 0 \) (1 \( \leq i \leq n \)), by Lemma 2.5 we have for every \( \varepsilon > 0 \),

\[
\hat{S}_f\left( \sum_i \lambda_i \rho_i + \varepsilon \left( \sum_i \lambda_i \rho_i + \sum_i \lambda_i \sigma_i \right) \right) = \hat{S}_f\left( \sum_i \lambda_i (\rho_i + \varepsilon (\rho_i + \sigma_i)) \right) \leq \sum_i \lambda_i \hat{S}_f(\rho_i + \varepsilon (\rho_i + \sigma_i) \| \sigma_i + \varepsilon (\rho_i + \sigma_i)).
\]

Letting \( \varepsilon \searrow 0 \) gives \( \hat{S}_f(\sum_i \lambda_i \rho_i \| \sum_i \lambda_i \sigma_i) \leq \sum_i \lambda_i \hat{S}_f(\rho_i \| \sigma_i). \)

\[\blacksquare\]
Another significant property of \( \hat{S}_f(\rho\|\sigma) \) is the joint lower semicontinuity, which we will prove later in Section 5 after developing a general integral formula in Section 4.

The transpose \( \bar{f} \) of \( f \) is defined by

\[
\bar{f}(t) := tf(t^{-1}), \quad t \in (0, +\infty),
\]

which is again an operator convex function on \((0, +\infty)\), see [21] Proposition A.1. It is immediate to see that \( \bar{f}(0^+) = f'(+(\infty)) \) and \( \bar{f}'(+(\infty)) = f(0^+) \). The next proposition shows the symmetry of \( \hat{S}_f(\rho\|\sigma) \) between two variables under exchanging \( f \) and \( \bar{f} \).

**Proposition 2.10.** For every \( \rho, \sigma \in M_*^+ \),

\[
\hat{S}_\bar{f}(\rho\|\sigma) = \hat{S}_f(\sigma\|\rho).
\]

**Proof.** Assume first that \( (\rho, \sigma) \in (M_*^+ \times M_*^+) \). Let \( A, B \in s(\sigma)M_\sigma(\sigma) \) be as given in the proof of Lemma 2.1. Then we write

\[
\hat{S}_\bar{f}(\rho\|\sigma) = \text{tr } h_{\sigma}^{1/2} \bar{f}(A^*A)h_{\sigma}^{1/2}
\]

and

\[
\hat{S}_f(\sigma\|\rho) = \text{tr } h_{\rho}^{1/2} f(B^*B)h_{\rho}^{1/2} = \text{tr } h_{\sigma}^{1/2} A^*f(B^*B)Ah_{\sigma}^{1/2}.
\]

Hence it suffices to show that

\[
\bar{f}(A^*A) = A^*f(B^*B)A
\]

for every continuous function \( f \) on \((0, +\infty)\). By approximation we may show (2.5) when \( f(t) = t^m \) for any non-negative integer \( m \). Since \( A = B^{-1} \), we have

\[
A^*f(B^*B)A = B^{*-1}(B^*B)^mB^{-1} = (BB^*)^mB^{-1} = ((A^*A)^{-1})^{-m} = A^*A((A^*A)^{-1})^{-m} = \bar{f}(A^*A),
\]

so that (2.5) is shown. Now, the asserted equality for general \( \rho, \sigma \in M_*^+ \) immediately follows from the above case and Definition 2.8.

**Proposition 2.11.** Let \( \rho_i, \sigma_i \in M_*^+ \) \((i = 1, 2)\). If \( s(\rho_1) \lor s(\sigma_1) \perp s(\rho_2) \lor s(\sigma_2) \), then

\[
\hat{S}_f(\rho_1 + \rho_2\|\sigma_1 + \sigma_2) = \hat{S}_f(\rho_1\|\sigma_1) + \hat{S}_f(\rho_2\|\sigma_2).
\]

**Proof.** In view of Definition 2.8 one may show the identity in the case where \( (\rho_i, \sigma_i) \in (M_*^+ \times M_*^+) \) \((i = 1, 2)\) with \( s(\sigma_1) \perp s(\sigma_2) \). For \( i = 1, 2 \) choose an \( A_i \in s(\sigma_i)M_\sigma(\sigma_i) \) such that \( h_{\rho_i}^{1/2} = A_ih_{\sigma_i}^{1/2} \). Note that

\[
(A_1 + A_2)h_{\sigma_1 + \sigma_2}^{1/2} = (A_1 + A_2)(h_{\sigma_1}^{1/2} + h_{\sigma_2}^{1/2}) = A_1h_{\sigma_1}^{1/2} + A_2h_{\sigma_2}^{1/2} = h_{\rho_1}^{1/2} + h_{\rho_2}^{1/2} = h_{\rho_1 + \rho_2}^{1/2}
\]

and \( f((A_1 + A_2)^*(A_1 + A_2)) = f(A_1^*A_1) + f(A_2^*A_2) \) as operators in \( s(\sigma_1 + \sigma_2)M_\sigma(\sigma_1 + \sigma_2) \). Hence the asserted equality follows.
Example 2.12. When $M$ is semifinite with a faithful normal semifinite trace $\tau_0$, we have the conventional non-commutative $L^p$-space $L^p(M, \tau_0)$ for $1 \leq p < \infty$, the space of $\tau_0$-measurable operators $x$ affiliated with $M$ such that $||x||_p = \tau_0(|x|^p) < +\infty$, see [39]. The explicit relation between $L^p(M, \tau_0)$ and Haagerup's $L^p(M)$ is found in [48] pp. 62–63. In the semifinite case, $M$ is standardly represented on the Hilbert space $L^2(M, \tau_0)$ by the left multiplication, and one can define $\hat{S}_f(\rho||\sigma)$ for $\rho, \sigma$ with the use of Radon-Nikodym derivatives $h_\rho := dp/d\tau_0$, $h_\sigma := d\sigma/d\tau_0$ in $L^1(M, \tau_0)_+$ (so $\rho(x) = \tau_0(xh_\rho)$ for $x \in M$) in place of Haagerup's $h_\rho, h_\sigma$.

In particular, assume that $M$ is the algebra $\mathcal{B}(\mathcal{H})$ of all linear operators on a finite-dimensional Hilbert space $\mathcal{H}$, or $M = \mathbb{M}_d$, the matrix algebra of size $d := \dim \mathcal{H}$. Let $\rho, \sigma \in \mathbb{M}_d^+$, which define positive linear functionals $\rho(X) := \text{Tr}\rho X$, $\sigma(X) := \text{Tr}\sigma X$ for $X \in \mathbb{M}_d$ with the same notations as $\rho, \sigma$, where $\text{Tr}$ is the usual matrix trace. Then $\hat{S}_f(\rho||\sigma)$ coincides with that defined in [21, Definition 3.21]. In fact, when $\rho, \sigma$ are invertible, Definition 2.3 becomes $\hat{S}_f(\rho||\sigma) = \text{Tr}\sigma f(\rho^{-1/2}\rho\sigma^{-1/2})$. For general $\rho, \sigma \in \mathbb{M}_d^+$ let $e$ be the support projection of $\rho + \sigma$. By Proposition 2.11 $\hat{S}_f(\rho||\sigma)$ in [21] is defined as

$$\lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon I||\sigma + \varepsilon I) = \lim_{\varepsilon \searrow 0} \{\hat{S}_f(\rho + \varepsilon \sigma||\sigma + \varepsilon I) + \hat{S}_f(\varepsilon(I - e)||\varepsilon(I - e))\}$$

$$= \lim_{\varepsilon \searrow 0} \{\hat{S}_f(\rho + \varepsilon \sigma||\sigma + \varepsilon I) + \varepsilon f(1)\sigma(I - e)\}$$

$$= \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \sigma||\sigma + \varepsilon I),$$

which is Definition 2.8.

Example 2.13. Let $M$ be an abelian von Neumann algebra such that $M \cong L^\infty(X, \mu)$ on a $\sigma$-finite measure space $(X, X, \mu)$. The standard form of $M \cong L^\infty(X, \mu)$ is $(L^\infty(X, \mu), L^2(X, \mu), \xi \mapsto \overline{\xi}, L^2(X, \mu)_+)$, where $\phi \in L^\infty(X, \mu)$ is represented on $L^2(X, \mu)$ as the multiplication operator $\xi \mapsto \phi\xi$, $\xi \in L^2(X, \mu)$. Let $\rho, \sigma \in M_+^*$, which are identified with functions in $L^1(X, \mu)_+$ (denoted here by the same $\rho, \sigma$ instead of $h_\rho, h_\sigma$) so that $\rho(\phi) = \int_X \phi d\mu$, $\sigma(\phi) = \int_X \phi d\mu$ for $\phi \in L^\infty(X, \mu)$. With $\eta = \rho + \sigma$ note that

$$\hat{S}_f(\rho + \varepsilon \sigma||\sigma + \varepsilon) = \int_X (\sigma(x) + \varepsilon \eta(x)) f \left( \frac{\rho(x) + \varepsilon \eta(x)}{\sigma(x) + \varepsilon \eta(x)} \right) d\mu(x) = S_f(\rho + \varepsilon \sigma||\sigma + \varepsilon)$$

by [20] Example 2.5]. By Definition 2.8 and [20] Corollary 4.4 (3), take the limit of the above as $\varepsilon \searrow 0$ to see that $\hat{S}_f(\rho||\sigma)$ coincides with the classical $f$-divergence $S_f(\rho||\sigma) = \int_X \sigma f(\rho||\sigma) d\mu$.

Example 2.14. Consider a linear function $f(t) = a + bt$ with $a, b \in \mathbb{R}$. For every $(\rho, \sigma) \in (M_+^* \times M_+^*)_\sim$ let $A \in eMe$ be as in Lemma 2.1 where $e = s(\rho) = s(\sigma)$, so that $h_{\rho}^{1/2} = Ah_{\sigma}^{1/2}$. Then

$$\hat{S}_{a+bt}(\rho||\sigma) = \sigma(ae + bA^*A) = a\sigma(e) + b \text{tr}(h_{\sigma}^{1/2}A^*Ah_{\sigma}^{1/2})$$

$$a\sigma(1) + b \text{tr} h_\rho = a\sigma(1) + b\rho(1).$$

This holds for all $\rho, \sigma \in M_+^*$ by Definition 2.8. Hence together with [20] (2.7),

$$\hat{S}_{a+bt}(\rho||\sigma) = S_{a+bt}(\rho||\sigma) = a\sigma(1) + b\rho(1), \quad \rho, \sigma \in M_+^*. \tag{2.6}$$

Next, consider $f(t) = t^2$. Let $(\rho, \sigma) \in (M_+^* \times M_+^*)_\sim$ and $A$ be as above. Then

$$\hat{S}_{t^2}(\rho||\sigma) = \text{tr}(h_{\sigma}^{1/2}(A^*A)^2h_{\sigma}^{1/2}) = \text{tr}(h_{\rho}^{1/2}AA^*h_{\rho}^{1/2}) = ||h_{\rho}^{1/2}A||_2^2.$$
On the other hand, since \( h_\rho = (h_\rho^1 A) h_\rho^{1/2} \), by [20, Lemma 5.2 and Proposition A.4 (2)] we note that \( S_{\rho^2}(\rho||\sigma) = \|h_\rho^{1/2} A\|_2^2 \). Hence, by Definition 2.8, \( \hat{S}_f(\rho||\sigma) = S_{\rho^2}(\rho||\sigma) \) for all \( \rho, \sigma \in M_+^* \) by Definition 2.8 and [20, (4.6)]. Thus, \( \hat{S}_f = S_f \) if \( f \) is a quadratic polynomial.

In the rest of this section we will present more formulas of \( \hat{S}_f(\rho||\sigma) \) in some special situations.

**Proposition 2.15.** For every \((\rho, \sigma) \in (M_+^* \times M_+^*)_\leq\),

\[
\hat{S}_f(\rho||\sigma) = \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \sigma||\sigma).
\]

For every \((\rho, \sigma) \in (M_+^* \times M_+^*)_\geq\),

\[
\hat{S}_f(\rho||\sigma) = \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \rho||\sigma).
\]

**Proof.** To show the first assertion, let \((\rho, \sigma) \in (M_+^* \times M_+^*)_\leq\); then \( \rho \leq \alpha \sigma \) for some \( \alpha > 0 \). Let \( h_\sigma^{-1/2} h_\rho h_\sigma^{-1/2} = \int_0^\alpha t \, dE(t) \) be the spectral decomposition with \( \int_0^\alpha dE(t) = s(\sigma) \). Then, as in the proof of Lemma 2.7, one has

\[
\hat{S}_f(\rho + \varepsilon \sigma||\sigma) = (1 + \varepsilon) \int_0^\alpha f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) d\sigma(E(t)).
\]

Hence, by Definition 2.8

\[
\hat{S}_f(\rho||\sigma) = \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon \sigma||\sigma) = \lim_{\varepsilon \searrow 0} \int_0^\alpha f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) d\sigma(E(t)).
\]

Since \( \hat{S}_f(\rho + \varepsilon \sigma||\sigma) = \int_0^\alpha f(t + \varepsilon) \, d\sigma(E(t)) \), it suffices to show that

\[
\lim_{\varepsilon \searrow 0} \int_0^\alpha f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) d\sigma(E(t)) = \lim_{\varepsilon \searrow 0} \int_0^\alpha f(t + \varepsilon) \, d\sigma(E(t)). \tag{2.7}
\]

Consider \( f \) as a function on \([0, +\infty)\) by letting \( f(0) = f(0^+) \in (-\infty, +\infty] \). When \( f(0^+) < +\infty \), both sides of (2.7) are equal to \( \int_0^\alpha f(t) \, d\sigma(E(t)) \) by the bounded convergence theorem. When \( f(0^+) = +\infty \), choose a \( \delta > 0 \) with \( \delta \leq \min\{\alpha, 1\} \) such that \( f(t) \) is decreasing on \((0, \delta)\).

Since \( f(t + \varepsilon) \) and \( f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) \) are increasing to \( f(t) \) as \( \delta/2 > \varepsilon \searrow 0 \) for any \( t \in [0, \delta/2] \), the monotone convergence theorem gives

\[
\lim_{\varepsilon \searrow 0} \int_0^{\delta/2} f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) d\sigma(E(t)) = \lim_{\varepsilon \searrow 0} \int_0^{\delta/2} f(t + \varepsilon) \, d\sigma(E(t)) = \int_0^{\delta/2} f(t) \, d\sigma(E(t)).
\]

On the other hand, the bounded convergence theorem gives

\[
\lim_{\varepsilon \searrow 0} \int_0^{\delta/2} f \left( \frac{t + \varepsilon}{1 + \varepsilon} \right) d\sigma(E(t)) = \lim_{\varepsilon \searrow 0} \int_0^{\delta/2} f(t + \varepsilon) \, d\sigma(E(t)) = \int_0^{\delta/2} f(t) \, d\sigma(E(t)).
\]

Hence (2.7) follows.

The second assertion is immediate from the first and Proposition 2.10. \( \square \)

**Proposition 2.16.** If \( f(0^+) < +\infty \), then expression (2.11) (or (2.2)) holds for every \((\rho, \sigma) \in (M_+^* \times M_+^*)_\leq\), where \( f(0) = f(0^+) \).
Proof. Let \((\rho, \sigma) \in (M_+^* \times M_+^*) \leq\). When \(f(0^+) < +\infty\), the proof of Proposition 2.15 gives
\[
\hat{S}_f(\rho \| \sigma) = \int_0^\alpha f(t) \, d\sigma(E(t)) = \sigma(f(h_{\sigma}^{-1/2} h_{\rho} h_{\sigma}^{-1/2}),
\]
which shows the assertion. \(\square\)

**Proposition 2.17.** Let \(\rho, \sigma \in M_+^*\). If \(s(\rho) \not\leq s(\sigma)\) and \(f'(+\infty) = +\infty\), then \(\hat{S}_f(\rho \| \sigma) = +\infty\). If \(s(\sigma) \not\leq s(\rho)\) and \(f(0^+) = +\infty\), then \(\hat{S}_f(\rho \| \sigma) = +\infty\).

*Proof.* For the first assertion, let \(e := s(\sigma)\) and define a unital positive map \(\Phi : \mathbb{C}^2 \to M\) given by \(\Phi(a, b) := a e + b(1 - e)\) for \((a, b) \in \mathbb{C}^2\). From the monotonicity property in Theorem 2.9 and Example 2.13 we have
\[
\hat{S}_f(\rho \| \sigma \circ \Phi) \geq \hat{S}_f(\rho \| \sigma) = S_f((\rho(e), \rho(1 - e))(1, 0)).
\]
Since \(\rho(1 - e) > 0\), from [20, Example 2.5] the above right-hand side is
\[
f(\rho(e)) + f'(+\infty) \rho(1 - e) = +\infty,
\]
where \(f(\rho(e))\) means \(f(0^+)\) if \(\rho(e) = 0\). The second assertion follows from the first and Proposition 2.10. \(\square\)

### 3 Strongly absolutely continuous case

Let \(\rho, \sigma \in M_+^*\). We say that \(\rho\) is **strongly absolutely continuous** with respect to \(\sigma\) if \(\lim_n \rho(x_n^* x_n) = 0\) for any sequence \(\{x_n\}\) in \(M\) such that
\[
\lim_n \sigma(x_n^* x_n) = \lim_{n,m} \rho((x_n - x_m)^* (x_n - x_m)) = 0.
\]
In this case we write \(\rho \ll \sigma\) strongly. Obviously, this implies the simple absolute continuity, i.e., \(\sigma(x^* x) = 0\) implies \(\rho(x^* x) = 0\), equivalently \(s(\rho) \leq s(\sigma)\). In terms of \(h_{\rho}\) and \(h_{\sigma}\), we can rewrite the definition of \(\rho \ll \sigma\) strongly as follows: for \(\{x_n\}\) in \(M\),
\[
\lim_n \|x_n h_{\sigma}^{1/2}\|_2 = \lim_{n,m} \|x_n h_{\rho}^{1/2} - x_m h_{\rho}^{1/2}\|_2 = 0 \implies \lim_n \|x_n h_{\rho}^{1/2}\|_2 = 0.
\]
This means that the operator
\[
R = R_{\rho/\sigma} : x h_{\sigma}^{1/2} + \zeta \mapsto x h_{\rho}^{1/2}, \quad x \in M, \quad \zeta \in L^2(M) \text{ with } \epsilon \zeta = 0,
\]
is closable, where \(\epsilon\) is the projection onto \(M h_{\sigma}^{1/2}\). Note that \(\epsilon = J s(\sigma) J \in M'\) is the \(M'\)-support of \(\sigma\) while \(s(\sigma)\) is the projection onto \(M' h_{\sigma}^{1/2}\), the \(M\)-support of \(\sigma\).

We here give the next lemma for completeness; see [35, 15] for similar characterizations in a bit more general settings.

**Lemma 3.1.** The following conditions for \(\rho, \sigma \in M_+^*\) are equivalent:

(i) \(\rho \ll \sigma\) strongly;

(ii) there exists a (unique) positive self-adjoint operator \(T = T_{\rho/\sigma}\) on \(L^2(M)\) affiliated with \(M'\) such that \(M h_{\sigma}^{1/2}\) is a core of \(T^{1/2}\) and
\[
\rho(x) = \langle T^{1/2} h_{\sigma}^{1/2}, x h_{\sigma}^{1/2} \rangle, \quad x \in M.
\]
Proof. (i) \implies (ii). Condition (i) means that $R$ given in (3.1) is closable, so let $\overline{R}$ be its closure and $\overline{R} = VT^{1/2}$ be the polar decomposition, where $T := R^*\overline{R}$. For every $x, u \in M$ with $u$ unitary, since
\[ uRu^* xh_{\sigma}^{1/2} = uu^* xh_{\rho}^{1/2} = xh_{\rho}^{1/2} = Rxh_{\sigma}^{1/2}, \]
we have $uRu^* = R$, so $\overline{R}u^* = \overline{R}$ and hence $uTu^* = T$. Therefore, $T$ is affiliated with $M'$ and $V \in M'$. By definition, $Mh_{\rho}^{1/2}$ is a core of $\overline{R}$. Since any core of $\overline{R}$ is a core of $T^{1/2}$, $Mh_{\rho}^{1/2}$ is a core of $T^{1/2}$. Moreover, for every $x \in M$,
\[ \rho(x) = \langle h_{\rho}^{1/2}, xh_{\rho}^{1/2} \rangle = \langle VT^{1/2}h_{\sigma}^{1/2}, xVT^{1/2}h_{\sigma}^{1/2} \rangle = \langle T^{1/2}h_{\sigma}^{1/2}, xV^*VT^{1/2}h_{\sigma}^{1/2} \rangle = \langle T^{1/2}h_{\sigma}^{1/2}, xT^{1/2}h_{\sigma}^{1/2} \rangle, \]
where the last equality follows from $xT^{1/2}h_{\sigma}^{1/2} = T^{1/2}xh_{\sigma}^{1/2}$ since $T^{1/2}$ is affiliated with $M'$. The uniqueness of $T$ follows since a positive self-adjoint operator $T$ is determined by the quadratic form $\|T^{1/2}xh_{\sigma}^{1/2}\|^2 = \rho(x^*x)$, $x \in M$ (see, e.g., [45, A.7]).

(ii) \implies (i). Since (3.2) means that $\|xh_{\rho}^{1/2}\| = \|T^{1/2}xh_{\sigma}^{1/2}\|$ for all $x \in M$, the implication immediately follows since $T^{1/2}$ is a closed operator.

In the rest of this section, we assume that $f$ is an operator convex function on $(0, +\infty)$ such that $f(0^+) < +\infty$,

so $f$ extends by continuity to an operator convex function on $[0, +\infty)$. We give the next definition, following the spirit of Belavkin and Staszewski’s relative entropy in [31] (also Example 3.5).

**Definition 3.2.** Let $\rho, \sigma \in M_+^*$ be such that $\rho \ll \sigma$ strongly, and $T_{\rho/\sigma}$ be as given in Lemma 3.1.

We then define
\[ \hat{S}_f(\rho||\sigma) := \langle h_{\sigma}^{1/2}, f(T_{\rho/\sigma}h_{\sigma}^{1/2}) \rangle = \int_0^\infty f(t) d\|E_{\rho/\sigma}(t)h_{\sigma}^{1/2}\|^2, \tag{3.3} \]
where $T_{\rho/\sigma} = \int_0^\infty t dE_{\rho/\sigma}(t)$ is the spectral decomposition of $T_{\rho/\sigma}$. The inner product expression in (3.3) should be understood, to be precise, in the sense of a lower-bounded form (see [43]), which equals the integral expression in (3.3). Since $f(t) \geq at + b$ for some $a, b \in \mathbb{R}$ and
\[ \int_0^\infty t d\|E_{\rho/\sigma}(t)h_{\sigma}^{1/2}\|^2 = \|T_{\rho/\sigma}h_{\sigma}^{1/2}\|^2 < +\infty, \]
note that $\hat{S}_f(\rho||\sigma)$ is well defined with value in $(-\infty, +\infty]$.

**Lemma 3.3.** For every $(\rho, \sigma) \in (M_+^* \times M_+^*)_\leq$,
\[ \hat{S}_f(\rho||\sigma) = \hat{S}_f(\rho||\sigma). \]

**Proof.** Let $A \in eMe$ be as given in Lemma 2.1, where $e := s(\sigma)$. Take the polar decomposition $A = v|A|$, and let $v' := JvJ \in M'$ and $B' := J|A|J \in e'M'e'$ with $e' := JeJ$, the projection onto $Mh_{\sigma}^{1/2}$ (the $M'$-support of $\sigma$). Note that $vv^* = s(\rho)$ and so $v'v^* = Js(\rho)J$. For every $x \in M$,
\[ B'xh_{\sigma}^{1/2} = J|A|Jxh_{\sigma}^{1/2} = Jv^*(Ah_{\rho}^{1/2}x^*) = Jv^*(h_{\rho}^{1/2}x^*) = v^*J(h_{\rho}^{1/2}x^*) = v^*xh_{\rho}^{1/2} \]
so that $v'B' = \overline{R}_{\rho/\sigma}$. In particular, $B'h_{\sigma}^{1/2} = v'^*v_{\rho}^{1/2}$ and

$$v'v'^*v_{\rho}^{1/2} = Jv'Jv_{\rho}^{1/2} = Jh_{\rho}^{1/2} = h_{\rho}^{1/2}.$$ 

Hence we have for every $x \in M$,

$$\langle B'h_{\sigma}^{1/2}, B'xh_{\sigma}^{1/2} \rangle = \langle B'h_{\sigma}^{1/2}, xB'h_{\sigma}^{1/2} \rangle = \langle v'^*v_{\rho}^{1/2}, xv'^*v_{\rho}^{1/2} \rangle$$

$$= \langle h_{\rho}^{1/2}, xv'^*v_{\rho}^{1/2} \rangle = \langle h_{\rho}^{1/2}, xh_{\rho}^{1/2} \rangle = \rho(x).$$

From the uniqueness assertion in Lemma 3.1, this implies that $B' = T_{\rho/\sigma}^{1/2}$. Therefore,

$$\tilde{S}'_{\rho}(\rho||\sigma) = \tilde{S}'_{\rho}(\rho||\sigma),$$

where the last equality is due to Proposition 2.16.

**Theorem 3.4.** Let $\rho, \sigma \in M_+^*$ and assume that $\rho \ll \sigma$ strongly. Then

$$\tilde{S}'_{\rho}(\rho||\sigma) = \tilde{S}'_{\rho}(\rho||\sigma).$$

**Proof.** In view of Definition 2.8 and Lemma 3.3, it suffices to show that

$$\tilde{S}'_{\rho}(\rho||\sigma) = \lim_{\varepsilon \searrow 0} \tilde{S}'_{\rho}(\rho + \varepsilon(\rho + \sigma)||\sigma + \varepsilon(\rho + \sigma)).$$

(3.4)

Let $e'$ be the $M'$-support of $\sigma$, and $T = T_{\rho/\sigma}$ be as given in Lemma 3.1. For every $\varepsilon > 0$, since $(1 + \varepsilon)\sigma + \varepsilon\rho \ll \sigma$ strongly and $\sigma + \varepsilon\rho \ll \sigma$ strongly, we have $R_{(1+\varepsilon)\sigma+\varepsilon\rho/\sigma}$ and $R_{\sigma+\varepsilon\rho/\sigma}$ with the polar decompositions

$$R_{(1+\varepsilon)\sigma+\varepsilon\rho/\sigma} = V_1T_{(1+\varepsilon)\sigma+\varepsilon\rho/\sigma}^{1/2}, \quad R_{\sigma+\varepsilon\rho/\sigma} = V_2T_{\sigma+\varepsilon\rho/\sigma}^{1/2},$$

where $V_1, V_2 \in M'$ are partial isometries with $V_1^*V_1 = V_2^*V_2 = e'$. It is easy to verify that for every $x \in M$,

$$(\varepsilon\sigma + (1 + \varepsilon)\rho)(x) = \langle (\varepsilon e' + (1 + \varepsilon)T)^{1/2}h_{\sigma}^{1/2}, \varepsilon e' + (1 + \varepsilon)T\rangle^{1/2}xh_{\sigma}^{1/2},$$

(3.5)

$$(1 + \varepsilon)\sigma + \varepsilon\rho)(x) = \langle ((1 + \varepsilon)e' + \varepsilon T)^{1/2}h_{\sigma}^{1/2}, ((1 + \varepsilon)e' + \varepsilon T)\rangle^{1/2}xh_{\sigma}^{1/2}. \quad (3.6)$$

It follows from (3.6) that $T_{(1+\varepsilon)\sigma+\varepsilon\rho/\sigma} = (1 + \varepsilon)e' + \varepsilon T$ so that $V_1((1 + \varepsilon)e' + \varepsilon T)^{1/2}h_{\sigma}^{1/2} = h_{(1+\varepsilon)\sigma+\varepsilon\rho}^{1/2}$. Therefore,

$$h_{\sigma}^{1/2} = ((1 + \varepsilon)e' + \varepsilon T)^{-1/2}V_1^*h_{(1+\varepsilon)\sigma+\varepsilon\rho}^{1/2},$$

(3.7)

where we note that $(1 + \varepsilon)e' + \varepsilon T$ and $e' + \varepsilon T$ have the bounded inverses in $e'M'e'$.

Inserting (3.7) into (3.5) gives

$$(\varepsilon\sigma + (1 + \varepsilon)\rho)(x) = \langle (\varepsilon e' + (1 + \varepsilon)T)^{1/2}((1 + \varepsilon)e' + \varepsilon T)^{-1/2}V_1^*h_{(1+\varepsilon)\sigma+\varepsilon\rho}^{1/2},$$

$$(\varepsilon e' + (1 + \varepsilon)T)^{1/2}x((1 + \varepsilon)e' + \varepsilon T)^{-1/2}V_1^*h_{(1+\varepsilon)\sigma+\varepsilon\rho}^{1/2} \rangle$$

$$= \langle V_1((\varepsilon e' + (1 + \varepsilon)T)((1 + \varepsilon)e' + \varepsilon T)^{-1/2}V_1^*h_{(1+\varepsilon)\sigma+\varepsilon\rho},$$

which completes the proof. 

\[ \square \]
\[ V_1((\varepsilon e' + (1 + \varepsilon)T)((1 + \varepsilon)e' + \varepsilon T)^{-1})^{-1/2}V_1^* x h_{(1+\varepsilon)\sigma+\varepsilon\rho} \]

which implies that

\[ T_{\varepsilon \sigma + (1+\varepsilon)\rho/(1+\varepsilon)\sigma+\varepsilon\rho} = V_1(\varepsilon e' + (1 + \varepsilon)T)((1 + \varepsilon)e' + \varepsilon T)^{-1} V_1^* . \]

Therefore,

\[
\begin{align*}
\hat{S}_f(\varepsilon \sigma + (1 + \varepsilon)\rho &\| (1 + \varepsilon)\sigma + \varepsilon\rho) \\
&= \left\langle h_{(1+\varepsilon)\sigma+\varepsilon\rho}^1, f(V_1(\varepsilon e' + (1 + \varepsilon)T)((1 + \varepsilon)\sigma + \varepsilon T)^{-1})V_1^* h_{(1+\varepsilon)\sigma+\varepsilon\rho}^1 \right\rangle, \\
&= \left\langle V_1((1 + \varepsilon)e' + \varepsilon T)^{1/2} h_\sigma^{1/2}, V_1 f((\varepsilon e' + (1 + \varepsilon)T)(1 + \varepsilon)\sigma + \varepsilon T)^{-1})V_1^* V_1((1 + \varepsilon)e' + \varepsilon T)^{1/2} h_\sigma^{1/2} \right\rangle \\
&= \int_0^\infty ((1 + \varepsilon) + \varepsilon t) f' \frac{\varepsilon + (1 + \varepsilon) t}{(1 + \varepsilon) + \varepsilon t} d\| E_{\rho/\sigma}(t) h_\sigma^{1/2} \|^2, \\
&= \int_0^\infty ((1 + \varepsilon + (1 + \varepsilon)T(t) + \varepsilon T)^{-1})V_1^* V_1((1 + \varepsilon)e' + \varepsilon T)^{1/2} h_\sigma^{1/2} \right\rangle \]
\]

where \((\varepsilon e' + (1 + \varepsilon)T((1 + \varepsilon)e' + \varepsilon T)^{-1}\) is a bounded operator and \(E_{\rho/\sigma}()\) is the spectral measure of \(T\). Hence, to show \(5.4\), it suffices to prove that

\[
\lim_{\varepsilon \to 0} \int_0^\infty (1 + \varepsilon + t) f' \frac{\varepsilon + (1 + \varepsilon) t}{(1 + \varepsilon) + \varepsilon t} d\nu(t) = \int_0^\infty f(t) d\nu(t), \quad (3.8)
\]

where \(d\nu(t) := d\| E_{\rho/\sigma}(t) h_\sigma^{1/2} \|^2\), a finite positive measure on \([0, \infty)\).

To prove \(5.8\), recall \(22\) Theorem 8.1 that \(f\) admits the integral expression

\[
f(t) = a + bt + ct^2 + \int_{[0, +\infty)} \psi_s(t) d\mu(s), \quad t \in [0, +\infty), \quad (3.9)
\]

where \(a, b, c \in \mathbb{R}, c \geq 0\),

\[
\psi_s(t) := \frac{t}{1 + s} - \frac{t}{t + s}, \quad s \in (0, +\infty), t \in [0, +\infty),
\]

and \(\mu\) is a positive measure on \((0, +\infty)\) satisfying \(\int_{(0, +\infty)} (1 + s)^{-2} d\mu(s) < +\infty\). Now, let \(0 < \varepsilon < 1/2\) and divide the left-hand integral in \(5.8\) into two parts on \([0, 5]\) and \((5, +\infty)\).

Since \((1 + \varepsilon + (1 + t)) f' \left( \frac{\varepsilon + (1 + t)}{1 + \varepsilon} \right)\) is uniformly bounded for \(\varepsilon \in [0, 1/2]\) and \(t \in [0, 5]\) and \((1 + \varepsilon + (1 + t)) f' \left( \frac{\varepsilon + (1 + t)}{1 + \varepsilon} \right) \rightarrow f(t)\) as \(\varepsilon \searrow 0\) for every \(t \in [0, 5]\), the bounded convergence theorem gives

\[
\lim_{\varepsilon \to 0} \int_{[0, 5]} (1 + \varepsilon + t) f' \left( \frac{t + \varepsilon (1 + t)}{1 + \varepsilon (1 + t)} \right) d\nu(t) = \int_{[0, 5]} f(t) d\nu(t).
\]

To deal with the integral on \((5, +\infty)\), note that for every \(\varepsilon \in (0, 1/2)\) and \(t \in (5, +\infty)\),

\[
\frac{d}{d\varepsilon} \left( \frac{(t + \varepsilon (1 + t))^2}{1 + \varepsilon (1 + t)} \right) = \frac{(1 + t)(t + \varepsilon (1 + t))(2 - t + \varepsilon (1 + t))}{(1 + \varepsilon (1 + t))^2} < 0,
\]
so that for any \( t > 5 \),
\[
\frac{(t + \varepsilon(1 + t))^2}{1 + \varepsilon(1 + t)} \to t^2 \quad \text{as } \varepsilon \downarrow 0.
\]
Moreover, we compute
\[
(1 + \varepsilon(1 + t))\psi_s\left(\frac{t + \varepsilon(1 + t)}{1 + \varepsilon(1 + t)}\right) = \frac{t + \varepsilon(1 + t)}{1 + s} - \frac{t + \varepsilon(1 + t)}{1 + \varepsilon(1 + t)} + s = \frac{t - 1}{1 + s} \cdot \frac{t + \varepsilon(1 + t)}{t + s + \varepsilon(1 + t)(1 + s)},
\]
and for any \( s \in (0, +\infty) \) and \( t > 5 \),
\[
\frac{d}{dz} \frac{t + \varepsilon(1 + t)}{t + s + \varepsilon(1 + t)(1 + s)} = \frac{(1 + t)(1 - t)s}{(t + s + \varepsilon(1 + t)(1 + s))^2} < 0,
\]
so that
\[
0 \leq (1 + \varepsilon(1 + t))\psi_s\left(\frac{t + \varepsilon(1 + t)}{1 + \varepsilon(1 + t)}\right) \to \psi_s(t) \quad \text{as } \varepsilon \downarrow 0.
\]
The monotone convergence theorem yields that
\[
\int_{(5, +\infty)} (1 + \varepsilon(1 + t))f\left(\frac{t + \varepsilon(1 + t)}{1 + \varepsilon(1 + t)}\right) d\nu(t)
\]
\[
= (a + (a + b)\varepsilon) \int_{(5, +\infty)} d\nu(t) + (b + (a + b)\varepsilon) \int_{(5, +\infty)} t d\nu(t)
\]
\[
+ c \int_{(5, +\infty)} \frac{(t + \varepsilon(1 + t))^2}{1 + \varepsilon(1 + t)} d\nu(t)
\]
\[
+ \int_{(5, +\infty)} \int_{(0, +\infty)} (1 + \varepsilon(1 + t))\psi_s\left(\frac{t + \varepsilon(1 + t)}{1 + \varepsilon(1 + t)}\right) d\mu(s) d\nu(t)
\]
converges as \( \varepsilon \downarrow 0 \) to
\[
\int_{(5, +\infty)} (a + bt + ct^2 + \int_{(0, +\infty)} \psi_s(t) d\mu(s)) d\nu(t) = \int_{(5, +\infty)} f(t) d\nu(t).
\]
Hence (3.8) follows.

**Example 3.5.** In [3] Belavkin and Staszewski introduced a kind of relative entropy for states on a \( C^* \)-algebra. Here we restrict to the von Neumann algebra setting. Let \( \rho, \sigma \in M_+^+ \) and assume that \( \rho \ll \sigma \) strongly. Let \( R_{\rho/\sigma} \) and \( T = T_{\rho/\sigma} \) be as in (3.1) and Lemma 3.1 so that \( R_{\rho/\sigma} = VT^{1/2} \). Note that
\[
T^{1/2} x h_\sigma^{1/2} = V^* x h_\rho^{1/2} = x V^* h_\rho^{1/2}, \quad x \in M.
\]
Hence, the vector \( \xi \) and the positive self-adjoint operator \( \overline{\rho(\xi)} \) in [3] (3.2)] are
\[
\xi = V^* h_\rho^{1/2} = T^{1/2} h_\sigma^{1/2}, \quad \overline{\rho(\xi)} = T^{1/2}.
\]
Therefore, Belavkin and Staszewski’s relative entropy \( D_{BS}(\rho||\sigma) \) is given as
\[
D_{BS}(\rho||\sigma) = \langle T^{1/2} h_\sigma^{1/2}, (\log T)T^{1/2} h_\sigma^{1/2} \rangle
\]
\[
\begin{align*}
&= \lim_{\delta \to 0} \int_0^{\delta^{-1}} \log t \, d\|E(t)T^{1/2}h_{\sigma}^{1/2}\|_2 \\
&= \lim_{\delta \to 0} \int_0^{\delta^{-1}} \eta(t) \, d\|E(t)h_{\sigma}^{1/2}\|_2 \\
&= \langle h_{\sigma}^{1/2}, \eta(T)h_{\sigma}^{1/2} \rangle = \mathcal{S}_{\eta}(\rho\|\sigma) - \mathcal{S}_{\eta}^t(\rho\|\sigma).
\end{align*}
\]
where \(T = \int_0^\infty t \, dE(t)\) is the spectral decomposition and \(\eta(t) := t \log t\). In this way, when \(\rho \ll \sigma\) strongly, \(D_{BS}(\rho\|\sigma)\) is realized as the maximal \(f\)-divergence \(\mathcal{S}_f(\rho\|\sigma)\) with \(f = \eta\). Thus we may and do define \(D_{BS}(\rho\|\sigma) := \mathcal{S}_{\eta}(\rho\|\sigma)\) for arbitrary \(\rho, \sigma \in M_+^e\).

4 General integral formula

We modify the arguments in the previous section to show the following:

**Proposition 4.1.** Let \(\rho, \sigma \in M_+^e\).

1. If \(f(0^+) < +\infty\), then
   \[
   \mathcal{S}_f(\rho\|\sigma) = \lim_{\epsilon \to 0} \mathcal{S}_f(\rho\|\sigma + \epsilon \rho).
   \]

2. If \(f'(0^+) < +\infty\), then
   \[
   \mathcal{S}_f(\rho\|\sigma) = \lim_{\epsilon \to 0} \mathcal{S}_f(\rho + \epsilon \sigma\|\sigma).
   \]

**Proof.** (1) Assume that \(f(0^+) < +\infty\), and extend \(f\) to \([0, +\infty)\) by \(f(0) = f(0^+)\). Set \(\eta := \rho + \sigma\) and let \(\epsilon > 0\). Since \(\rho, \sigma\) and \(\sigma + \epsilon \rho\) are all dominated by \(\eta\), we have the three (bounded) positive self-adjoint operators \(T_1 := T_{\rho/\eta}, T_2 := T_{\sigma/\eta}\) and \(T_3 := T_{\sigma + \epsilon \rho/\eta}\) as follows:

\[
\begin{align*}
\overline{T}_{\rho/\eta} &= V_1 T_{1/2}^{1/2}, \\
\overline{T}_{\sigma/\eta} &= V_2 T_{2}^{1/2}, \\
\overline{T}_{\sigma + \epsilon \rho/\eta} &= V_3 T_{3}^{1/2},
\end{align*}
\]

(4.1)

where \(V_k\)’s are partial isometries in \(e'M'e'\) with \(V_3^*V_3 = e'\), where \(e'\) is the projection onto \(Mh_{\eta}^{1/2}\). Since

\[
\rho(x) = \langle T_1 h_{\eta}^{1/2}, xh_{\eta}^{1/2} \rangle, \quad \sigma(x) = \langle T_2 h_{\eta}^{1/2}, xh_{\eta}^{1/2} \rangle, \quad x \in M,
\]

we see that \(T_1 + T_2 = e'\). Moreover, since

\[
(\sigma + \epsilon \rho)(x) = \langle (T_2 + \epsilon T_1) h_{\eta}^{1/2}, xh_{\eta}^{1/2} \rangle, \quad x \in M,
\]

we have \(\overline{T}_{\sigma + \epsilon \rho} = V_3(T_2 + \epsilon T_1)^{1/2}\) so that \(V_3(T_2 + \epsilon T_1)^{1/2}h_{\eta}^{1/2} = h_{(\sigma + \epsilon \rho)}^{1/2}\). We find that

\[
(\rho + \epsilon \sigma)(x) = \langle (T_1 + \epsilon T_2) h_{\eta}^{1/2}, xh_{\eta}^{1/2} \rangle
\]

\[
= \langle (T_1 + \epsilon T_2)(T_2 + \epsilon T_1)^{-1/2} V_3^* h_{(\sigma + \epsilon \rho)}^{1/2}, x(T_2 + \epsilon T_1)^{-1/2} V_3^* h_{(\sigma + \epsilon \rho)}^{1/2} \rangle
\]

\[
= \langle V_3(T_1 + \epsilon T_2)(T_2 + \epsilon T_1)^{-1} V_3^* h_{(\sigma + \epsilon \rho)}^{1/2}, xh_{(\sigma + \epsilon \rho)}^{1/2} \rangle, \quad x \in M,
\]

which implies that

\[
T_{\rho + \epsilon \sigma/\sigma + \epsilon \rho} = V_3(T_1 + \epsilon T_2)(T_2 + \epsilon T_1)^{-1} V_3^*.
\]

Therefore,

\[
\mathcal{S}_f(\rho + \epsilon \sigma\|\sigma + \epsilon \rho)
\]

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Furthermore, by Definition 2.8 one sees that
\[
\langle h^{1/2}_{\sigma+\varepsilon\rho}, f(V_3(T_1 + \varepsilon T_2)(T_2 + \varepsilon T_1)^{-1}V_3^*)h^{1/2}_{\sigma+\varepsilon\rho}\rangle 
\]
\[
= \langle V_3(T_2 + \varepsilon T_1)^{1/2}h^{1/2}_{\eta^*}, V_3 f((T_1 + \varepsilon T_2)(T_2 + \varepsilon T_1)^{-1}V_3 V_3(T_2 + \varepsilon T_1)^{1/2}h^{1/2}_{\eta}) \rangle 
\]
\[
= \langle h^{1/2}_{\eta^*}, (T_2 + \varepsilon T_1)f((T_1 + \varepsilon T_2)(T_2 + \varepsilon T_1)^{-1}h^{1/2}_{\eta}) \rangle. 
\]

Now, since $0 \leq T_1 \leq \varepsilon'$ and $T_2 = \varepsilon' - T_1$, taking the spectral decomposition $T_1 = \int_0^1 t\,dE_1(t)$ with $\int_0^1 dE_1(t) = \varepsilon'$, one can write
\[
\hat{S}_f(\rho + \varepsilon\sigma||\sigma + \varepsilon\rho) = \int_0^1 (1 - t + \varepsilon t)f\left(\frac{t + \varepsilon(1 - t)}{1 - t + \varepsilon t}\right) \, d\nu(t), 
\]
where $d\nu(t) := d\|E_1(t)h^{1/2}_{\eta'}\|$, a finite positive measure on $[0, 1]$.

Similarly, one has
\[
\rho(x) = \langle V_3 T_1(T_2 + \varepsilon T_1)^{-1}V_3^* \times h^{1/2}_{\sigma+\varepsilon\rho}, \rangle, \quad x \in M, 
\]
so that
\[
T_{\rho/\sigma+\varepsilon\rho} = V_3 T_1(T_2 + \varepsilon T_1)^{-1}V_3^*. 
\]

Therefore,
\[
\hat{S}_f(\rho||\sigma + \varepsilon\rho) = \langle h^{1/2}_{\eta^*}, (T_2 + \varepsilon T_1)f(T_1(T_2 + \varepsilon T_1)^{-1})h^{1/2}_{\eta} \rangle 
\]
\[
= \int_0^1 (1 - t + \varepsilon t)f\left(\frac{t}{1 - t + \varepsilon t}\right) \, d\nu(t). 
\]

By Lemma 3.3 one has
\[
\hat{S}_f(\rho + \varepsilon\sigma||\sigma + \varepsilon\rho) = \hat{S}_f(\rho + \varepsilon\sigma||\sigma + \varepsilon\rho), \quad \hat{S}_f(\rho||\sigma + \varepsilon\rho) = \hat{S}_f(\rho||\sigma + \varepsilon\rho). 
\]

Furthermore, by Definition 2.8 one sees that
\[
\hat{S}_f(\rho||\sigma) = \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon(\rho + \sigma)||\sigma + \varepsilon(\rho + \sigma)) 
\]
\[
= \lim_{\varepsilon \searrow 0} \hat{S}_f((1 + \varepsilon)\rho + \varepsilon\sigma||(1 + \varepsilon)\sigma + \varepsilon\rho) 
\]
\[
= \lim_{\varepsilon \searrow 0} \hat{S}_f\left(\rho + \frac{\varepsilon}{1 + \varepsilon} \sigma||\sigma + \frac{\varepsilon}{1 + \varepsilon} \rho\right) 
\]
\[
= \lim_{\varepsilon \searrow 0} \hat{S}_f(\rho + \varepsilon\sigma||\sigma + \varepsilon\rho). 
\]

From (4.2)–(4.5) it suffices to prove that the integrals in (4.2) and (4.3) have the same limit as $\varepsilon \searrow 0$. For this we may prove that
\[
\lim_{\varepsilon \searrow 0} \int_0^1 (1 - t + \varepsilon t)f\left(\frac{t + \varepsilon(1 - t)}{1 - t + \varepsilon t}\right) \, d\nu(t) = \int_0^1 (1 - t)f\left(\frac{t}{1 - t}\right) \, d\nu(t), 
\]
\[
\lim_{\varepsilon \searrow 0} \int_0^1 (1 - t + \varepsilon t)f\left(\frac{t}{1 - t + \varepsilon t}\right) \, d\nu(t) = \int_0^1 (1 - t)f\left(\frac{t}{1 - t}\right) \, d\nu(t), 
\]
where $(1 - t)f(t/(1 - t))$ at $t = 1$ is understood as $\lim_{t \searrow 1}(1 - t)f(t/(1 - t)) = f'(+\infty)$. Let us transfer the proofs of (4.6) and (4.7) into Appendix A, which are more or less similar to that of (3.8).

(2) is immediate from (1) and Proposition 2.10.
In the proof of Theorem 4.1 we used the integral expression of an operator convex function \( f \) on \((0, +\infty)\) satisfying \( f(0^+) < +\infty \). For a general operator convex function \( f \) on \((0, +\infty)\), recall [31] (see also [12, Theorem 5.1]) that \( f \) has an integral expression

\[
f(t) = a + b(t - 1) + c(t - 1)^2 + d \frac{(t-1)^2}{t} + \int_{(0, +\infty)} \frac{(t-1)^2}{t + s} d\mu(s), \quad t \in (0, +\infty). \tag{4.8}
\]

where \( a, b, c, d \geq 0 \) and \( \mu \) is a positive measure on \([0, +\infty)\) with \( \int_{[0, +\infty)} (1 + s)^{-1} d\mu(s) < +\infty \), and moreover \( a, b, c, d \) and \( \mu \) are uniquely determined.

Based on the arguments in the proof of Proposition 4.1, we next present a general integral formula of \( \hat{S}_f(\rho||\sigma) \), which can be the second definition of the maximal \( f \)-divergences.

**Theorem 4.2.** For every \( \rho, \sigma \in M^+_+ \) let \( T_{\rho/\rho+\sigma} = \int_0^1 t dE_{\rho/\rho+\sigma}(t) \) be the spectral decomposition. Then for every operator convex function \( f \) on \((0, +\infty)\),

\[
\hat{S}_f(\rho||\sigma) = \int_0^1 (1 - t)f \left( \frac{t}{1 - t} \right) d\|E_{\rho/\rho+\sigma}(t)h^{1/2}_{\rho+\sigma}\|^2, \tag{4.9}
\]

where \((1 - t)f \left( \frac{t}{1 - t} \right)\) is understood as \( f(0^+) \) for \( t = 0 \) and \( f'(+\infty) \) for \( t = 1 \).

**Proof.** In view of the integral expression in (4.8), we can write \( f = f_1 + f_2 \) with operator convex functions \( f_1, f_2 \) on \((0, +\infty)\) such that \( f_1(0^+) < +\infty \) and \( f_2'(+\infty) < +\infty \) and so \( \tilde{f}_2(0^+) < +\infty \). In fact, we may define

\[
f_1(t) := a + b(t - 1) + c(t - 1)^2 + \int_{[1, +\infty)} \frac{(t-1)^2}{t + s} d\mu(s),
\]

\[
f_2(t) := d \frac{(t-1)^2}{t} + \int_{(0,1)} \frac{(t-1)^2}{t + s} d\mu(s).
\]

We then have

\[
\hat{S}_f(\rho||\sigma) = \hat{S}_{f_1}(\rho||\sigma) + \hat{S}_{f_2}(\rho||\sigma) = \hat{S}_{f_1}(\rho||\sigma) + \hat{S}_{\tilde{f}_2}(\sigma||\rho).
\]

It follows from (4.5), (4.2) and (4.6) in the proof of Proposition 4.1(1) that

\[
\hat{S}_{f_1}(\rho||\sigma) = \int_0^1 (1 - t)f_1 \left( \frac{t}{1 - t} \right) d\|E_{\rho/\rho+\sigma}(t)h^{1/2}_{\rho+\sigma}\|^2, \tag{4.10}
\]

and similarly, with \( \rho, \sigma \) interchanged,

\[
\hat{S}_{\tilde{f}_2}(\sigma||\rho) = \int_0^1 (1 - t)\tilde{f}_2 \left( \frac{t}{1 - t} \right) d\|E_{\sigma/\rho+\sigma}(t)h^{1/2}_{\rho+\sigma}\|^2, \tag{4.11}
\]

where \( T_{\sigma/\rho+\sigma} = \int_0^1 t dE_{\sigma/\rho+\sigma}(t) \) is the spectral decomposition. Since \( T_{\sigma/\rho+\sigma} = e' - T_{\rho/\rho+\sigma} \), we find that \( E_{\sigma/\rho+\sigma}([0, t]) = E_{\rho/\rho+\sigma}([1 - t, 1]) \) for all \( t \in [0, 1] \). Applying this to (4.11) gives

\[
\hat{S}_{\tilde{f}_2}(\sigma||\rho) = \int_0^1 t\tilde{f}_2 \left( \frac{1 - t}{t} \right) d\|E_{\rho/\rho+\sigma}(t)h^{1/2}_{\rho+\sigma}\|^2 = \int_0^1 (1 - t)f_2 \left( \frac{t}{1 - t} \right) d\|E_{\rho/\rho+\sigma}(t)h^{1/2}_{\rho+\sigma}\|^2. \tag{4.12}
\]

Hence (4.9) follows by adding (4.10) and (4.12). \( \square \)

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Corollary 4.3. If \( f(0^+) < +\infty \) and \( f'(\infty) < +\infty \), then \( \tilde{S}_f(\rho||\sigma) \) is finite for every \( \rho, \sigma \in M_r^+ \).

Proof. If \( f(0^+) < +\infty \) and \( f'(\infty) < +\infty \), then the function \( (1-t)f\left(\frac{t}{1-t}\right) \) is bounded on \([0,1]\). Hence the result is obvious from expression (4.3).

Remark 4.4. Similarly to the definition of the standard f-divergence \( S_f(\rho||\sigma) \) in [20] (2.6), one can write (4.9) as sum of three terms

\[
\tilde{S}_f(\rho||\sigma) = \int_{(0,1)} (1-t) f\left(\frac{t}{1-t}\right) d\|E_{\rho/\rho+\sigma}(t)h_{\rho+\sigma}^{1/2}\|^2 \\
+ f(0^+)\langle h_{\rho+\sigma}^{1/2}, E_{\rho/\rho+\sigma}(0)h_{\rho+\sigma}^{1/2} \rangle + f'(\infty)\langle h_{\rho+\sigma}^{1/2}, E_{\rho/\rho+\sigma}(\{1\})h_{\rho+\sigma}^{1/2} \rangle,
\]

where \( E_{\rho/\rho+\sigma}(\{0\}) \) and \( E_{\rho/\rho+\sigma}(\{1\}) \) are the spectral projections of \( T_{\rho/\rho+\sigma} \) for \( \{0\} \) and \( \{1\} \). One might expect that the above boundary terms with \( f(0^+) \) and \( f'(\infty) \) are equal to the corresponding terms \( f(0^+)\sigma(1-s(\rho)) \) and \( f'(\infty)\rho(1-s(\sigma)) \), respectively, in [20] (2.6). But it is not true, as will explicitly be seen in Example 4.5 below. Instead, it is not difficult to find that

\[
\langle h_{\rho+\sigma}^{1/2}, V_1E_{\rho/\rho+\sigma}(\{0\})V_1^*h_{\rho+\sigma}^{1/2} \rangle = \sigma(1-s(\rho)),
\]
\[
\langle h_{\rho+\sigma}^{1/2}, V_2E_{\rho/\rho+\sigma}(\{1\})V_2^*h_{\rho+\sigma}^{1/2} \rangle = \rho(1-s(\sigma)),
\]

where \( V_1, V_2 \) are partial isometries in (4.11).

Example 4.5. Although we noted in Example 2.12 [21] that the definition of \( \tilde{S}_f(\rho||\sigma) \) in [24] coincides with [21] Definition 3.21 in the finite-dimensional case, we examine formula (4.9) in the matrix case \( M = M_d \). For \( \rho, \sigma \in M_d^+ \), since \( R_{\rho/\rho+\sigma}(X(\rho + \sigma)^{1/2}) = X^{1/2} \), one has \( R_{\rho/\rho+\sigma}X = X(\rho + \sigma)^{-1/2}\rho^{1/2} \) and hence \( R_{\rho/\rho+\sigma}^*X = X(\rho + \sigma)^{-1/2} \), which is defined in the sense of generalized inverse. Therefore,

\[
T_{\rho/\rho+\sigma}X = R_{\rho/\rho+\sigma}^*R_{\rho/\rho+\sigma}X = X(\rho + \sigma)^{-1/2}(\rho + \sigma)^{-1/2},
\]

so that \( T_{\rho/\rho+\sigma} = R_{(\rho+\sigma)^{-1/2}(-1)}R_{(\rho+\sigma)^{-1/2}} \) and similarly \( T_{\sigma/\rho+\sigma} = R_{(\rho+\sigma)^{-1/2}(-1)}R_{(\rho+\sigma)^{-1/2}} \), where \( R_{A} \) is the right multiplication by \( A \) on \( M_d \). Note that \( T_{\rho/\rho+\sigma} + T_{\sigma/\rho+\sigma} = e^* \) is the right multiplication of the support projection of \( \rho + \sigma \). Here, for simplicity, assume that \( \sigma \) is invertible, and let \( \sigma^{1/2}(\rho + \sigma)^{-1/2} = VQ^{1/2} \) be the polar decomposition where \( Q := (\rho + \sigma)^{-1/2} \). Then formula (4.9) is written as

\[
\tilde{S}_f(\rho||\sigma) = \langle (\rho + \sigma)^{1/2}, R_Qf(R_{I-Q}^{-1})(\rho + \sigma)^{1/2} \rangle \\
= \langle R_Q^{1/2}(\rho + \sigma)^{1/2}, R_{f(Q^{-1})}R_Q^{1/2}(\rho + \sigma)^{1/2} \rangle \\
= \text{Tr} Q^{1/2}(\rho + \sigma)Q^{1/2}f(Q^{-1} - I) = \text{Tr} V^*\sigma Vf(Q^{-1} - I) \\
= \text{Tr} \sigma f(V(Q^{-1} - I)V^*) = \text{Tr} \sigma f((VQV^*)^{-1} - I) \\
= \text{Tr} \sigma f(\sigma^{1/2}(\rho + \sigma)^{-1}\sigma^{1/2} - I) = \text{Tr} \sigma f(\sigma^{1/2}\rho\sigma^{-1/2}),
\]

which coincides with [21] Definition 3.21, as mentioned in Definition 2.8.

When \( \rho, \sigma \) are not invertible, the two boundary terms in (4.13) are

\[
f(0^+)\langle (\rho + \sigma)^{1/2}, R_{E_0}(\rho + \sigma)^{1/2} \rangle, \\
f'(\infty)\langle (\rho + \sigma)^{1/2}, R_{E_1}(\rho + \sigma)^{1/2} \rangle.
\]
where $E_0, E_1$ are the spectral projections of $(\rho + \sigma)^{-1/2} \sigma (\rho + \sigma)^{-1/2}$ for the eigenvalues 0, 1, respectively. For example, consider the $2 \times 2$ matrix case where $\rho := \begin{bmatrix} 3/2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\sigma := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

By direct computations we find that (4.15) and (4.16) are equal to

$$2f(0^+), \quad \frac{3}{2} f'(+\infty),$$

respectively. On the other hand, the corresponding terms in the definition of $S_f(\rho\|\sigma)$ in [20, (2.6)] are equal to

$$f(0^+)\sigma(1 - s(\rho)) = f(0^+), \quad f'(+\infty)\rho(1 - s(\sigma)) = \frac{3}{4} f'(+\infty),$$

in this case. Thus, the two boundary terms with $f(0^+)$ and $f'(+\infty)$ for $S_f(\rho\|\sigma)$ and $\hat{S}_f(\rho\|\sigma)$ are different each other.

## 5 Lower semicontinuity and martingale convergence

For each $n \in \mathbb{N}$, as in [20, (3.6)], we consider the approximation of $f$ in (4.8) with integral on the cut-off interval $[1/n, n]$, that is,

$$f_n(t) := a + b(t - 1) + c \frac{n(t - 1)^2}{t + n} + d \frac{(t - 1)^2}{t + (1/n)} + \int_{[1/n,n]} \frac{(t - 1)^2}{t + s} d\mu(s), \quad d \in (0, +\infty). \quad (5.1)$$

The following lemma is [20, Lemma 3.1].

**Lemma 5.1.** For each $n \in \mathbb{N}$, $f_n$ is operator convex on $(0, +\infty)$, $f_n(0^+) < +\infty$, $f_n'(+\infty) < +\infty$ and

$$f_n(0^+) \nearrow f(0^+), \quad f_n'(+\infty) \nearrow f'(+\infty), \quad f_n(t) \nearrow f(t)$$

for all $t \in (0, +\infty)$ as $n \to \infty$.

**Lemma 5.2.** For every $\rho, \sigma \in M_+^*$,

$$\hat{S}_f(\rho\|\sigma) = \lim_{n \to \infty} \hat{S}_f_n(\rho\|\sigma) \quad \text{increasingly.}$$

Hence $\hat{S}_f(\rho\|\sigma) = \sup_{n \geq 1} \hat{S}_f_n(\rho\|\sigma)$.

**Proof.** By Lemma 5.1 as $n \to \infty$,

$$(1 - t)f_n \left( \frac{t}{1-t} \right) \nearrow (1 - t)f \left( \frac{t}{1-t} \right), \quad t \in [0,1].$$

Hence the result follows from the monotone convergence theorem applied to the integral formula in (4.9) for $f_n$ and $f$.

In addition to Lemma 5.1, it is readily verified that

$$\lim_{t \searrow 0} f_n'(t) = \lim_{t \searrow 0} \frac{f_n(t) - f_n(0^+)}{t} > -\infty.$$
So, to prove the joint lower semicontinuity of \((\rho, \sigma) \mapsto \widehat{S}_{f}(\rho\|\sigma)\), we may and do assume that 
\(f(0^+) < +\infty, f'(+\infty) < +\infty\) and \(\lim_{t \to 0^+} f'(t) > -\infty\). Such an operator monotone function \(f\) on \([0, +\infty)\), with \(f(0) = f(0^+)\), has the integral expression

\[
f(t) = a + bt + \int_{(0, +\infty)} \frac{t^2}{t + s} \, d\nu(s), \quad t \in [0, +\infty),
\]

where \(a, b \in \mathbb{R}\) and \(\nu\) is a finite positive measure on \((0, \infty)\). Indeed, the function \(g(t) := (f(t) - f(0^+))/t\) is operator monotone on \([0, +\infty)\) by [15] Theorem 2.4, so \(g\) has the integral expression

\[
g(t) = c + \int_{(0, +\infty)} \frac{t}{t + s} \, d\nu(s), \quad t \in [0, +\infty),
\]

where \(c \in \mathbb{R}\), \(c \geq 0\) and \(\nu\) is a positive measure on \((0, +\infty)\), see [5] (V.53)] (also [19] Theorem 2.7.11]). Since \(g(+\infty) = f'(+\infty) < +\infty\), it must follow that \(c = 0\) and \(\nu\) is a finite measure. Hence \(f\) has the expression in (5.3). In view of (2.6) we may and do furthermore assume that

\[
f(t) = \int_{(0, +\infty)} \frac{t^2}{t + s} \, d\nu(s), \quad t \in [0, +\infty),
\]

where \(\nu\) is as above.

**Lemma 5.3.** Let \(f\) be given in (5.3). Then for every \(\rho, \sigma \in M^+_\nu\),

\[
\widehat{S}_{f}(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} \widehat{S}_{f}(\rho\|\sigma + \varepsilon\rho) \quad \text{increasingly.}
\]

Hence \(\widehat{S}_{f}(\rho\|\sigma) = \sup_{\varepsilon > 0} \widehat{S}_{f}(\rho\|\sigma + \varepsilon\rho)\).

**Proof.** Since \(f(0^+) < +\infty\), the convergence is in Proposition 4.1(1). So we need to show that \(0 < \varepsilon \mapsto \widehat{S}_{f}(\rho\|\sigma + \varepsilon\rho)\) is decreasing. Let \(\eta := \rho + \sigma\). By Proposition 2.10 we have

\[
\widehat{S}_{f}(\rho\|\sigma + \varepsilon\rho) = \widehat{S}_{f}(\sigma + \varepsilon\rho\|\rho) = \lim_{\delta \searrow 0} \widehat{S}_{f}(\sigma + \varepsilon\rho + \delta\eta\|\rho + \delta\eta).
\]

Hence it suffices to show that for \(0 < \varepsilon_1 < \varepsilon_2\) and \(\delta > 0\),

\[
\widehat{S}_{f}(\sigma + \varepsilon_1\rho + \delta\eta\|\rho + \delta\eta) \geq \widehat{S}_{f}(\sigma + \varepsilon_2\rho + \delta\eta\|\rho + \delta\eta).
\]

Set \(\sigma_i := \sigma + \varepsilon_i\rho + \delta\eta\) and \(\omega := \rho + \delta\eta\); then \(\sigma_1 \leq \sigma_2\) and \(\sigma_1 \sim \sigma_2 \sim \omega\). Let \(e := s(\omega)\). By Lemma 2.1 there is an \(A \in eMe\) such that \(h_{\sigma_2}^{1/2} = Ah_{\omega}^{1/2}\). Also there is a \(B \in eMe\) such that \(\|B\| \leq 1\) and \(h_{\sigma_1}^{1/2} = Bh_{\omega}^{1/2}\). Since \(h_{\sigma_1}^{1/2} = BAh_{\omega}^{1/2}\), one has

\[
\widehat{S}_{f}(\sigma_1\|\omega) = \omega(\tilde{f}(A^*B^*BA)), \quad \widehat{S}_{f}(\sigma_2\|\omega) = \omega(\tilde{f}(A^*A)).
\]

From (5.3) the function \(\tilde{f}\) is expressed as

\[
\tilde{f}(t) = t\tilde{f}(t^{-1}) = \int_{(0, +\infty)} \frac{1}{1 + st} \, d\nu(s), \quad t \in [0, +\infty),
\]

which is operator monotone decreasing on \([0, +\infty)\). Since \(A^*B^*BA \leq A^*A\), it follows that \(\tilde{f}(A^*B^*BA) \geq \tilde{f}(A^*A)\) and hence \(\widehat{S}_{f}(\sigma_1\|\omega) \geq \widehat{S}_{f}(\sigma_2\|\omega)\), as desired. \(\square\)
Lemma 5.4. Let $T,T_n$ ($n \in \mathbb{N}$) be positive bounded linear operators on a Hilbert space $\mathcal{H}$. Assume that $\sup_n \|T_n\| < +\infty$ and $T_n \to T$ in the weak operator topology. Then for any operator convex function $f$ on $[0, +\infty)$ with $f(0) = 0$ and any $\xi \in \mathcal{H}$,
\[
\langle \xi, f(T)\xi \rangle \leq \liminf_{n \to \infty} \langle \xi, f(T_n)\xi \rangle.
\]

Proof. Let $\{E_n\}$ be a net of finite-dimensional orthogonal projections on $\mathcal{H}$ such that $E_n \not\to I$. Since $\|E_n T E_n - E_n T E\| \to 0$ as $n \to \infty$, $\|f(E_n T E_n) - f(E_n T E)\| \to 0$ as $n \to \infty$ for every $\alpha$. By [7, Theorem 2.5] applied to the map $\Phi(A) = E_n A E_n + (I - E_n) A (I - E_n)$ on $B(\mathcal{H})$, we have
\[
f(E_n T E_n) \leq E_n f(T_n) E_n. \tag{5.4}
\]
Therefore,
\[
\langle \xi, f(E_n T E)\xi \rangle = \lim_{n \to \infty} \langle \xi, f(E_n T E_n)\xi \rangle \leq \liminf_{n \to \infty} \langle \xi, E_n f(T_n) E_n \xi \rangle. \tag{5.5}
\]
Since $E_n T E \to T$ in the strong operator topology and the continuous functional calculus is continuous with respect to the strong operator topology (see, e.g., [15, Theorem A.2]), the left-hand side of (5.5) converges to $\langle \xi, f(T)\xi \rangle$. On the other hand, since $K := \sup_n \|f(T_n)\| < +\infty$, note that
\[
\langle \xi, (E_n T E)\xi \rangle - \langle \xi, (E_n T E_n)\xi \rangle \leq 2K \|\xi\| \|E_n \xi - \xi\|,
\]
so $\langle E_n \xi, f(T_n) E_n \xi \rangle$ converges to $\langle \xi, f(T)\xi \rangle$ as $\alpha \to \infty$ uniformly for $n$. This implies that the right-hand side of (5.5) converges to $\liminf_{n \to \infty} \langle \xi, f(T_n)\xi \rangle$ as $\alpha \to \infty$. Hence the result follows.

We are now in a position to prove the joint lower semicontinuity.

Theorem 5.5. The function $(\rho, \sigma) \in M^+ \times M^+ \mapsto \tilde{S}_f(\rho, \sigma)$ is jointly lower semicontinuous in the norm topology.

Proof. By the argument above Lemma 5.3 we may assume that $f$ is given in (5.3). By Lemma 5.3 it suffices to prove that $(\rho, \sigma) \in M^+ \times M^+ \mapsto \tilde{S}_f(\rho, \sigma)$ is continuous in the norm topology for any $\varepsilon > 0$. Let $\rho_n, \rho, \sigma_n, \sigma \in M^+$, $n \in \mathbb{N}$, be such that $\|\rho_n - \rho\| \to 0$ and $\|\sigma_n - \sigma\| \to 0$. Let $\eta_n := \sigma_n + \varepsilon \rho_n$ and $\eta := \sigma + \varepsilon \rho$; then $\rho_n \leq \varepsilon^{-1} \eta_n$, $\rho \leq \varepsilon^{-1} \eta$ and $\|\eta_n - \eta\| \to 0$. By Lemma 2.1 we have $A \in s(\eta) M s(\eta)$ and $A_n \in s(\eta_n) M s(\eta_n)$ such that $h_{\rho_n}^{1/2} = Ah_n^{1/2}$ and $h_{\rho_n}^{1/2} = A_n h_n^{1/2}$ and $\|A\|, \|A_n\| \leq \varepsilon^{-1/2}$. Let $x, y \in M$. Note that
\[
\|h_{\eta_n}^{1/2} x - h_{\eta_n}^{1/2} y\| \leq \|h_{\eta_n}^{1/2} - h_{\eta_n}^{1/2}\| \|x\| + \|\eta_n - \eta\|^{1/2} \|x\|
\]
thanks to [16, Lemma 2.10 (2)] and similarly $\|h_{\rho_n}^{1/2} x - h_{\rho_n}^{1/2} y\| \leq \|\rho_n - \rho\|^{1/2} \|x\|$. Hence,
\[
\langle h_{\rho_n}^{1/2} x, h_{\rho_n}^{1/2} y \rangle \to \langle h_{\rho}^{1/2} x, h_{\rho}^{1/2} y \rangle \text{ as } n \to \infty.
\]
Since $A(h_{\eta_n}^{1/2}) = h_{\rho_n}^{1/2}$ and $A_n(h_{\eta_n}^{1/2}) = h_{\rho_n}^{1/2}$, one can estimate
\[
\langle h_{\eta_n}^{1/2} x, A_n^* A_n(h_{\eta_n}^{1/2} y) \rangle - \langle h_{\eta_n}^{1/2} x, A^* A(h_{\eta_n}^{1/2} y) \rangle \leq \langle h_{\eta_n}^{1/2} x, h_{\eta_n}^{1/2} y \rangle + \langle h_{\eta_n}^{1/2} x, A_n^* A_n(h_{\eta_n}^{1/2} y) \rangle.
\]
+ |⟨h_{1/2}^n x, A_n^* A_n(h_{1/2}^n y)⟩ - ⟨h_{1/2}^n x, A^* A(h_{1/2}^n y)⟩| \leq ||h_{1/2}^n x - h_{1/2}^n y||_2 ||A_n^* A_n(h_{1/2}^n y)||_2 + ||h_{1/2}^n x||_2 ||A_n^* A_n(h_{1/2}^n y - h_{1/2}^n y)||_2 \\
+ |⟨A_n(h_{1/2}^n y), A(h_{1/2}^n y)⟩ - ⟨A(h_{1/2}^n x), A(h_{1/2}^n y)⟩| \\
\leq \varepsilon^{-1}||\eta - \eta||^{1/2} (||\eta||^{1/2} + ||\eta||^{1/2}) ||x|| ||y|| + ||⟨h_{1/2}^n x, h_{1/2}^n y⟩ - ⟨h_{1/2}^n x, h_{1/2}^n y⟩|| \\
\rightarrow 0 \text{ as } n \rightarrow \infty.

Since $e := s(\eta)$ is the projection onto $\overline{M h_{1/2}^n} = \overline{h_{1/2}^n M}$, the above estimate implies that $eA_n^* A_n e \rightarrow A^* A$ in the weak operator topology. Therefore, by Lemma 5.4 one has

$$
\hat{S}_f(\rho||\eta) = \liminf_{n \rightarrow \infty} \langle h_{1/2}^n, f(A^* A)h_{1/2}^n \rangle \leq \liminf_{n \rightarrow \infty} \langle h_{1/2}^n, f(eA_n^* A_n e)h_{1/2}^n \rangle \leq \liminf_{n \rightarrow \infty} \langle h_{1/2}^n, f(A_n^* A_n)h_{1/2}^n \rangle,
$$

(5.7)

where the last inequality follows from $f(T e) \leq e f(T) e$ similarly to (5.4). Moreover, since $\sup_{n \geq 1} ||f(A_n^* A_n)|| < +\infty$, it follows as in (5.6) that

$$
||\langle h_{1/2}^n, f(A_n^* A_n)h_{1/2}^n \rangle - \langle h_{1/2}^n, f(A_n^* A_n)h_{1/2}^n \rangle|| \rightarrow 0 \text{ as } n \rightarrow \infty,
$$

which implies that

$$
\liminf_{n \rightarrow \infty} \hat{S}_f(\rho||\eta_n) = \liminf_{n \rightarrow \infty} \langle h_{1/2}^n, f(A_n^* A_n)h_{1/2}^n \rangle = \liminf_{n \rightarrow \infty} \langle h_{1/2}^n, f(A_n^* A_n)h_{1/2}^n \rangle.
$$

(5.8)

Hence $\hat{S}_f(\rho||\eta_n) \leq \liminf_{n \rightarrow \infty} \hat{S}_f(\rho||\eta_n)$ follows from (5.7) and (5.8). □

In the rest of this section we establish the martingale convergence for $\hat{S}_f(\rho||\sigma)$. Let $\rho, \sigma \in M^+_\sigma$ and $\eta := \rho + \sigma$. Below it will be convenient to work with the cyclic representation $(\mathcal{H}_\eta, \pi_\eta, \xi_\eta)$ of $M$ associated with $\eta$, rather than the standard representation of $M$, that is, $\pi_\eta$ is a representation of $M$ on a Hilbert space $\mathcal{H}_\eta$ with a cyclic vector $\xi_\eta$ for $\pi_\eta(M)$, i.e., $\mathcal{H}_\eta = \pi_\eta(M)\xi_\eta$, such that $\eta(x) = \langle \xi_\eta, \pi_\eta(x)\xi_\eta \rangle$, $x \in M$. Then there exists a unique $T = T(\rho/\eta) \in \pi_\eta(M)_+$ such that

$$
\rho(x) = \langle \xi_\eta, T \pi_\eta(x)\xi_\eta \rangle, \quad x \in M.
$$

(5.9)

See, e.g., [6, Theorems 2.3.16, 2.3.19]. Note that $T$ is independent of the choice of the cyclic representation up to unitary conjugation. That is, let $(\mathcal{H}, \hat{\pi}, \hat{\xi})$ be another cyclic representation of $M$ associated with $\eta$. There is a unitary $U : \mathcal{H}_\eta \rightarrow \hat{\mathcal{H}}$ such that $\hat{\xi} = U\xi_\eta$ and $\hat{\pi}(x) = U \pi_\eta(x) U^*$, $x \in M$; then $\hat{T} \in \hat{\pi}(M)_+$ as in (5.8) for $(\mathcal{H}, \hat{\pi}, \hat{\xi})$ is $\hat{T} = UT(\rho/\eta)U^*$. A particular choice of $(\mathcal{H}_\eta, \pi_\eta, \xi_\eta)$ is taken as $\mathcal{H}_\eta := M h_{1/2}^\eta = L^2(M)s(\eta)$, $\pi_\eta(x)$ is the left multiplication of $x$ on $\mathcal{H}_\eta \subset L^2(M)$, and $\xi_\eta = h_{1/2}^\eta$. In this case, $T = T(\rho/\eta)$ in (5.6) coincides with $T = T_{\rho/\eta}|_{\mathcal{H}_\eta}$, where $T_{\rho/\eta} \in M'_+$ is given in Lemma 3.1. Since $h_{1/2}^\eta \in \mathcal{H}_\eta$, the formula in (4.9) holds as well when $T_{\rho/\eta}$ is replaced with $T_{\rho/\eta}|_{\mathcal{H}_\eta}$. Therefore, (4.9) is rewritten as

$$
\hat{S}_f(\rho||\sigma) = \int_0^1 (1-t) f \left( \frac{t}{1-t} \right) d\|E(t)\xi_\eta\|^2,
$$

(5.10)

for any cyclic representation $(\mathcal{H}_\eta, \pi_\eta, \xi_\eta)$ of $M$ associated with $\eta = \rho + \sigma$ and the spectral decomposition $T(\rho/\eta) = \int_0^1 t dE(t)$.

Our martingale convergence theorem is
Theorem 5.6. Let \( \{M_\alpha\} \) be an increasing net of unital von Neumann subalgebras of \( M \) such that \((\bigcup_\alpha M_\alpha)'' = M\). Then for every \( \rho, \sigma \in M_*^+ \),

\[
\hat{S}_f(\rho|M_\alpha\|\sigma|_{M_\alpha}) \nearrow \hat{S}_f(\rho\|\sigma).
\]

Proof. Let \( \eta := \rho + \sigma, \rho_\alpha := \rho|M_\alpha, \sigma_\alpha := \sigma|M_\alpha \) and \( \eta_\alpha := \eta|M_\alpha \). From the monotonicity property in Theorem 2.9 (applied to injections \( M_\alpha \hookrightarrow M_\beta \hookrightarrow M \) for \( \alpha \leq \beta \)) we see that \( \hat{S}_f(\rho_\alpha\|\sigma_\alpha) \) is increasing and \( \hat{S}_f(\rho_\alpha\|\sigma_\alpha) \leq \hat{S}_f(\rho\|\sigma) \). Hence it suffices to show that \( \hat{S}_f(\rho\|\sigma) \leq \sup_\alpha \hat{S}_f(\rho_\alpha\|\sigma_\alpha) \). Choose a cyclic representation \((\mathcal{H}_\eta, \pi_\eta, \xi_\eta)\) of \( M \) and \( T = T(\rho/\eta) \) associated with \( \eta = \rho + \sigma \). Let \( \mathcal{H}_\alpha := \pi_\eta(M_\alpha)x_\eta \) and \( P_\alpha \) be the orthogonal projection from \( \mathcal{H}_\eta \) onto \( \mathcal{H}_\alpha \).

Since

\[
P_\alpha \pi_\eta(x)\pi_\eta(y)\xi_\eta = \pi_\eta(x)P_\alpha \pi_\eta(y)\xi_\eta, \quad x, y \in M_\alpha,
\]

one has \( P_\alpha \pi_\eta(x)P_\alpha = \pi_\eta(x)P_\alpha \), for any \( x \in M_\alpha \), and hence \( P_\alpha \in \pi_\eta(M_\alpha)' \). So one can define a representation \( \pi_\alpha \) of \( M_\alpha \) on \( \mathcal{H}_\alpha \) by

\[
\pi_\alpha(x) := \pi_\eta(x)|_{\mathcal{H}_\alpha}, \quad x \in M_\alpha.
\]

Since \( \pi_\eta(M_\alpha)x_\eta = \mathcal{H}_\alpha \) and

\[
\langle \xi_\eta, \pi_\alpha(x)\xi_\eta \rangle = \langle \xi_\eta, \pi_\eta(x)\xi_\eta \rangle = \eta(x) = \eta_\alpha(x), \quad x \in M_\alpha,
\]

it follows that \((\mathcal{H}_\alpha, \pi_\alpha, \xi_\eta)\) is the cyclic representation of \( M_\alpha \) associated with \( \eta_\alpha = \rho_\alpha + \sigma_\alpha \). Since

\[
\pi_\alpha(M_\alpha)' = (\pi_\eta(M_\alpha)|_{\mathcal{H}_\alpha})' = P_\alpha \pi_\eta(M_\alpha)'P_\alpha|_{\mathcal{H}_\alpha} \supset P_\alpha \pi_\eta(M_\alpha)'P_\alpha|_{\mathcal{H}_\alpha},
\]

one has \( P_\alpha TP_\alpha|_{\mathcal{H}_\alpha} \in \pi_\alpha(M_\alpha)' \). Moreover, since

\[
\langle \xi_\eta, P_\alpha TP_\alpha \pi_\alpha(x)\xi_\eta \rangle = \langle \xi_\eta, T\pi_\eta(x)\xi_\eta \rangle = \rho(x) = \rho_\alpha(x), \quad x \in M_\alpha,
\]

we find that

\[
T(\rho_\alpha/\eta_\alpha) = P_\alpha TP_\alpha|_{\mathcal{H}_\alpha}. \quad (5.11)
\]

Since \( P_\alpha \nearrow I \) in the strong operator topology, it follows that \( P_\alpha TP_\alpha \to T \) in the strong operator topology.

Now, for each \( n \in \mathbb{N} \) let \( f_n \) be given in (5.11) and set \( k_n(t) := (1 - t)f_n(t) \), \( t \in [0, 1] \). Then \( k_n \) is a continuous function on \([0, 1] \), so from formula (5.10) and (5.11) it follows that

\[
\hat{S}_{f_n}(\rho\|\sigma) = \langle \xi_\eta, k_n(T)\xi_\eta \rangle, \quad \hat{S}_{f_n}(\rho_\alpha\|\sigma_\alpha) = \langle \xi_\eta, k_n(P_\alpha TP_\alpha)\xi_\eta \rangle
\]

for every \( \alpha \). For each \( n \in \mathbb{N} \), since \( k_n(P_\alpha TP_\alpha) \to k_n(T) \) in the strong operator topology, we obtain

\[
\hat{S}_{f_n}(\rho_\alpha\|\sigma_\alpha) \to \hat{S}_{f_n}(\rho\|\sigma) \quad \text{as } \alpha \to \text{“}\infty\text{”}.
\]

From this and Lemma 5.2, we find that

\[
\sup_\alpha \hat{S}_{f_n}(\rho_\alpha\|\sigma_\alpha) = \sup_n \sup_\alpha \hat{S}_{f_n}(\rho_\alpha\|\sigma_\alpha) = \sup_n \hat{S}_{f_n}(\rho\|\sigma) = \hat{S}_{f_n}(\rho\|\sigma) \geq \sup_n \hat{S}_{f_n}(\rho\|\sigma) = \hat{S}_{f}(\rho\|\sigma),
\]

as desired. \( \square \)
Remark 5.7. We have shown \cite{20} Theorem 4.1 (i) that the standard $f$-divergence $S_f(\rho\|\sigma)$ is jointly lower semicontinuous in the $\sigma(M_s, M)$-topology. It follows from Theorem 5.6 that this property stronger than Theorem 5.5 holds for $\tilde{S}_f(\rho\|\sigma)$ as well whenever $M$ is injective, or equivalently, there is an increasing net $\{M_\alpha\}$ of finite-dimensional unital subalgebras of $M$ such that $M = \bigcup M_\alpha$, see \cite{9, 11}. In fact, in this case, $\tilde{S}_f(\rho\|\sigma) = \sup_\alpha \tilde{S}_f(\rho|M_\alpha\|\sigma|M_\alpha)$ by Theorem 5.6 and $(\rho, \sigma) \mapsto \tilde{S}_f(\rho|M_\alpha\|\sigma|M_\alpha)$ is lower semicontinuous in the $\sigma(M_s, M)$-topology. However, it is unknown whether $\tilde{S}_f(\rho\|\sigma)$ is jointly lower semicontinuous in the $\sigma(M_s, M)$-topology for general $M$.

6 Minimal reverse test

Let $\rho, \sigma \in M_+^*$ be arbitrary, $\eta := \rho + \sigma$ and $e := s(\eta)$. Let $A \in eMe$ be such that $h_{\rho}^{1/2} = Ah_{\eta}^{1/2}$ (Lemma 2.1). Since $0 \leq A^*A \leq I$, we take the spectral decomposition $A^*A = \int_0^1 t \, dE(t)$, where $E(\cdot)$ is a spectral measure on $[0, 1]$ with $\int_0^1 dE(t) = e$, and define a finite Borel measure $\nu$ on $[0, 1]$ by

$$\nu := \eta(E(\cdot)) = \text{tr}(h_\eta E(\cdot)) = \|E(\cdot)h_\eta^{1/2}\|^2,$$

and consider an abelian von Neumann algebra $L^\infty([0, 1], \nu) = L^1([0, 1], \nu)^*$. Note that $A^*A = JT_{\rho/\eta}I$ from the proof of Lemma 5.3 so that $\nu = \|E_{\rho/\eta}(\cdot)h_\eta^{1/2}\|^2$ (see Theorem 4.2).

Lemma 6.1. Define $\Phi_0 : M \to L^\infty([0, 1], \nu)$ by

$$\Phi_0(x) = \frac{d\text{tr}(x h_\eta^{1/2}E(\cdot)h_\eta^{1/2})}{d\nu} \quad (\text{the Radon-Nikodym derivative}), \quad x \in M. \quad (6.1)$$

Then $\Phi_0$ is a unital positive normal map and its predual map

$$\Phi_0^* : L^1([0, 1], \nu) \to L^1(M) \cong M_*$$

satisfies

$$\Phi_0^*(\phi) = h_\eta^{1/2}\left(\int_0^1 \phi(t) \, dE(t)\right) h_\eta^{1/2} \quad (6.2)$$

for every $\phi \in L^\infty([0, 1], \nu) \subset L^1([0, 1], \nu)$. In particular, $\Phi_0^*(t) = h_\rho$ and $\Phi_0^*(1-t) = h_\sigma$, where $t$ denotes the identity function $t \mapsto t$ on $[0, 1]$.

Proof. When $x \in M_+$, for any Borel set $S \subset [0, 1]$ we have

$$0 \leq \text{tr}(x h_\eta^{1/2}E(S)h_\eta^{1/2}) \leq \|x\|\text{tr}(h_\eta E(S)) = \|x\|\nu(S),$$

so that the Radon-Nikodym derivative $\Phi_0(x)$ in (6.1) is well defined and $0 \leq \Phi_0(x) \leq \|x\|1$. So $\Phi_0$ extends to a well defined positive linear map from $M$ to $L^\infty([0, 1], \nu)$. To show the normality of $\Phi_0$, let $\{x_\alpha\}$ be a sequence in $M_+$ such that $x_\alpha \nearrow x \in M_+$. Since $\text{tr}(x_\alpha h_\eta^{1/2}E(\cdot)h_\eta^{1/2})$ is increasing and dominated by $\text{tr}(x h_\eta^{1/2}E(\cdot)h_\eta^{1/2})$, we have $0 \leq \Phi_0(x_\alpha) \nearrow \psi \leq \Phi_0(x)$ for some $\psi \in L^\infty([0, 1], \mu)$. For every Borel set $S \subset [0, 1],$

$$\int_S \psi \, d\nu = \lim_\alpha \int_S \Phi_0(x_\alpha) \, d\nu = \lim_\alpha \text{tr}(x_\alpha h_\eta^{1/2}E(S)h_\eta^{1/2}) = \text{tr}(x h_\eta^{1/2}E(S)h_\eta^{1/2}) = \int_S \Phi_0(x) \, d\nu$$

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which implies that \( \psi = \Phi_0(x) \) and so \( \Phi_0(x_\alpha) \not\sim \Phi_0(x) \).

Hence one can take the predual map \( \Phi_{0*} : L^1((0,1],\mu) \to M_* \). When \( \phi \in L^\infty([0,\infty),\mu) \) \((\subset L^1([0,\infty),\mu))\), for every \( x \in M \) one has

\[
\Phi_{0*}(\phi)(x) = \int_0^1 \phi(0) d\nu = \int_0^1 \phi(t) d\text{tr}(xh_\eta^{1/2}E(t)h_\eta^{1/2}) = \text{tr}\left(xh_\eta^{1/2}\left(\int_0^1 \phi(t) dE(t)\right)h_\eta^{1/2}\right),
\]

which implies (6.2) under the identification \( M_* = L^1(M) \). Hence, \( \Phi_{0*}(1) = h_\eta^{1/2}eh_\eta^{1/2} = h_\eta \), \( \Phi_{0*}(t) = h_\eta^{1/2}(A^*A)h_\eta^{1/2} = h_\rho \) and \( \Phi_{0*}(1-t) = h_\eta - h_\rho = h_\sigma \). \( \square \)

Now, following [33], we introduce the notion of reverse tests for \( \rho, \sigma \in M_*^+ \).

**Definition 6.2.** Let \((X, \mathcal{X}, \mu)\) be a \( \sigma \)-finite measure space and \( \Psi : M \to L^\infty(X, \mu) \) be a positive linear map which is unital and normal. Then the predual map \( \Psi_* : L^1(X, \mu) \to M_* \) is trace-preserving in the sense that \( \int_X \phi d\mu = \Psi_*(\phi)(1) = ||\Psi_*(\phi)|| \) for every \( \phi \in L^1(X, \mu)_+ \). We call a triplet \((\Psi, p, q)\) of such a map \( \Psi \) and \( p, q \in L^1(X, \mu)_+ \) a reverse test for \( \rho, \sigma \) if \( \Psi_*(p) = \rho \) and \( \Psi_*(q) = \sigma \).

The next variational formula of \( \tilde{S}_f(\rho||\sigma) \) can be the third definition of the maximal \( f \)-divergences.

**Theorem 6.3.** For every \( \rho, \sigma \in M_*^+ \),

\[
\tilde{S}_f(\rho||\sigma) = \min\{S_f(p||q) : (\Psi, p, q) \text{ a reverse test for } \rho, \sigma\}.
\] (6.3)

Moreover, \( \Psi : M \to L^\infty(X, \mu) \) in (6.3) can be restricted to those with a standard Borel probability space \((X, \mathcal{X}, \mu)\) or more specifically to a Borel probability space on \([0,1]\).

**Proof.** Let \((\Psi, p, q)\) be a reverse test for \( \rho, \sigma \). By the monotonicity property of \( \tilde{S}_f \) in Theorem 2.9 and Example 2.13 we have

\[
\tilde{S}_f(\rho||\sigma) = \tilde{S}_f(\Psi(p)||\Psi(q)) \leq \tilde{S}_f(p||q) = S_f(p||q).
\]

On the other hand, \((\Phi_0, t, 1-t)\) given in Lemma 6.1 is a reverse test, for which we have the equality \( \tilde{S}_f(\rho||\sigma) = S_f(t||1-t) \). In fact, since we set \( \nu = ||E_{\rho/\eta}||h_\eta^{1/2}||^2 \) in Lemma 6.1 it follows from Theorem 4.2 that

\[
\tilde{S}_f(\rho||\sigma) = \int_0^1 (1-t)f\left(\frac{t}{1-t}\right) d\nu(t) = S_f(t||1-t).
\]

Hence expression (6.3) follows. The latter assertion is clear from Lemma 6.1 (see, e.g., [44] for standard Borel spaces). \( \square \)

The reverse test \((\Phi_0, t, 1-t)\) given in Lemma 6.1 is a minimizer for expression (6.3), which is considered as the von Neumann algebra version of Matsumoto’s minimal or optimal reverse test [33] for \( \rho, \sigma \). Apply the monotonicity property of \( S_f \) [20, Theorem 4.1 (iv)] to this \( \Phi_0 \) (that is a unital and completely positive normal map) to find

\[
S_f(\rho||\sigma) \leq S_f(t||1-t) = \tilde{S}_f(\rho||\sigma),
\]

so we have
Theorem 6.4. For every $\rho, \sigma \in M^+_s$,

$$S_f(\rho\|\sigma) \leq \tilde{S}_f(\rho\|\sigma).$$

Here is an abstract approach to quantum $f$-divergences. We say that a function $S_f^\theta : M^+_s \times M^+_s \to (-\infty, +\infty]$ where $M$ varies over all von Neumann algebras is a monotone quantum $f$-divergence if the following are satisfied:

(a) $S_f^\theta(\rho \circ \Phi \| \sigma \circ \Phi) \leq S_f^\theta(\rho \| \sigma)$ for any unital completely positive normal map $\Phi : M_0 \to M$ between von Neumann algebras and for every $\rho, \sigma \in M^+_s$,

(b) when $M$ is an abelian von Neumann algebra with $M = L^\infty(X, \nu)$ on a $\sigma$-finite measure space $(X, \mu)$, $S_f^\theta(\rho \| \sigma)$ coincides with the classical $f$-divergence of $\rho, \sigma \in L^1(X, \mu)_+$ as in Example 2.13.

If $S_f^\theta$ is a monotone quantum $f$-divergence, then it is clear from Theorem 6.3 that

$$S_f^\theta(\rho \| \sigma) \leq \tilde{S}_f(\rho \| \sigma),$$

which justifies the name maximal $f$-divergence for $\tilde{S}_f$.

In the matrix case, it is easy to verify that if $\rho, \omega \in M_{n}^+$ are commuting, then $S_f(\rho \| \omega) = \tilde{S}_f(\rho \| \omega)$ for every operator convex (even simply convex) function on $(0, \infty)$ for all $\rho, \omega \in M^+_s$. A standard way to define this is as follows: Assume that $\omega \in M^+_s$ is faithful and let $\sigma_t^\omega$ be the modular automorphism group associated with $\omega$. Then $\rho \in M^+_s$ is said to commute with $\omega$ if $\rho \circ \sigma_t^\omega = \rho$ for all $t \in \mathbb{R}$. Different conditions equivalent to this were established, e.g., in terms of the Connes cocycle Radon-Nikodym derivative, in [38] (see also [45, §4.10]).

For (not necessarily faithful) $\omega \in M^+_s$ with $e := s(\omega)$ we define $\sigma_t^\omega$ as the modular automorphism group on $s(\omega)Ms(\omega)$ associated with the restriction of $\omega$ to $eMe$. The above notion of commutativity can extend to the case where $s(\rho) \leq s(\omega)$, by replacing $M$ with $eMe$ and considering $\rho, \omega$ as their restrictions to $eMe$. To introduce the notion for general $\rho, \omega \in M^+_s$, we give the next lemma, whose proof is deferred to Appendix B.

Lemma 6.5. For $\rho, \omega \in M^+_s$ the following conditions are equivalent:

(i) $\rho$ commutes with $\rho + \omega$ (i.e., $\rho \circ \sigma_t^{\rho + \omega} = \rho$ on $s(\rho + \omega)Ms(\rho + \omega)$ for all $t \in \mathbb{R}$);

(ii) $\omega$ commutes with $\rho + \omega$;

(iii) $\alpha \rho + \beta \omega$ commutes with $\gamma \rho + \delta \omega$ for any $\alpha, \beta, \gamma, \delta > 0$;

(iv) $h_\rho h_\omega = h_\omega h_\rho$ (as $\tau$-measurable operators affiliated with $N$, see the first paragraph of Section 2).

When $s(\rho) \leq s(\omega)$, the above conditions are also equivalent to that $\rho$ commutes with $\omega$ (i.e., $\rho \circ \sigma_t^\omega = \rho$ on $s(\omega)Ms(\omega)$ for all $t \in \mathbb{R}$).

Definition 6.6. For $\rho, \omega \in M^+_s$ we say that $\rho, \omega$ commute if the equivalent conditions of Lemma 6.5 hold. When $M$ is semifinite with a faithful normal semifinite trace $\tau_0$, the commutativity of $\rho, \omega$ is equivalent to the commutativity of $d\rho/d\tau_0$, $d\omega/d\tau_0 \in L^1(M, \tau_0)$, see Example 2.12.
Proposition 6.7. If $\rho, \omega \in M_+^*$ commute in the above sense, then

$$S_f(\rho\|\omega) = \tilde{S}_f(\rho\|\omega)$$

for any operator convex function $f$ on $(0, +\infty)$.

Proof. By Theorem 6.4 it suffices to prove that $\tilde{S}_f(\rho\|\omega) \leq S_f(\rho\|\omega)$. By (2.4) and [20, Corollary 4.4 (3)] note that

$$\tilde{S}_f(\rho\|\omega) = \lim_{\varepsilon \searrow 0} \tilde{S}_f(\rho + \varepsilon(\rho + \omega)\|\omega + \varepsilon(\rho + \omega)),$$

$$S_f(\rho\|\omega) = \lim_{\varepsilon \searrow 0} S_f(\rho + \varepsilon(\rho + \omega)\|\omega + \varepsilon(\rho + \omega)).$$

Hence, thanks to (iii) of Lemma 6.5 we may assume that both $\rho, \omega$ are faithful. By assumption, $\rho$ is $\sigma_\tau^f$-invariant. Let $M_\omega$ be the centralizer of $\omega$, i.e., $M_\omega := \{x \in M : \sigma^f_\tau(x) = x, t \in \mathbb{R}\}$, and $E_\omega : M \to M_\omega$ be the conditional expectation with respect to $\omega$. Then it follows [23, Theorem 2.2] that $\rho \circ E_\omega = \rho$ as well as $\omega \circ E_\omega = \omega$. Now, since $\omega|_{M_\omega}$ is a faithful normal trace, one can choose the Radon-Nikodym derivative $A := d(\rho|_{M_\omega})/d(\omega|_{M_\omega})$ so that

$$\rho(x) = \omega(Ax) = \lim_{\varepsilon \searrow 0} \omega(A(1 + \varepsilon A)^{-1}x), \quad x \in M_\omega.$$ 

Let $A$ be the abelian von Neumann subalgebra of $M_\omega$ generated by $A$, and $E_A : M_\omega \to A$ be the conditional expectation with respect to $\omega|_{M_\omega}$. For every $x \in M$ one has

$$\omega(x) = \omega(E_\omega(x)) = \omega(E_A \circ E_\omega(x)),$$

$$\rho(x) = \rho(E_\omega(x)) = \lim_{\varepsilon \searrow 0} \omega(A(1 + \varepsilon A)^{-1}E_\omega(x))$$

$$= \lim_{\varepsilon \searrow 0} \omega(A(1 + \varepsilon A)^{-1}E_A \circ E_\omega(x)) = \rho(E_A \circ E_\omega(x)).$$

Therefore,

$$\tilde{S}_f(\rho\|\omega) = \tilde{S}_f(\rho \circ E_A \circ E_\omega\|\omega \circ E_A \circ E_\omega) \leq \tilde{S}_f(\rho|_A\|\omega|_A)$$

$$= S_f(\rho|_A\|\omega|_A) \leq S_f(\rho\|\omega),$$

where the two inequalities are the monotonicity properties in Theorem 4.5 and 20 Theorem 4.1 (iv)], and the second equality is due to Example 2.13.

In particular, when $f(t) = t \log t$, we have the relation between the relative entropy $D$ and Belavkin and Staszewski’s relative entropy $D_{BS}$ (Example 5.5).

Corollary 6.8. For every $\rho, \omega \in M_+^*$,

$$D(\rho\|\omega) \leq D_{BS}(\rho\|\omega),$$

and $D(\rho\|\omega) = D_{BS}(\rho\|\omega)$ if $\rho, \omega$ commute.

We end this section with another martingale type convergence, which is not included in Theorem 5.6 since $e_\alpha M e_\alpha$’s are not unital subalgebras of $M$. Indeed, we can prove this similarly to the proof of [20, Theorem 4.5] with use of Theorem 5.6 and Proposition 2.11 in view of Proposition 6.7.

Proposition 6.9. Let $\{e_\alpha\}$ be an increasing net of projections in $M$ such that $e_\alpha \nearrow 1$. Then for every $\rho, \omega \in M_+^*$,

$$\lim_{\alpha} \tilde{S}_f(e_\alpha \rho e_\alpha\|e_\alpha \omega e_\alpha) = \tilde{S}_f(\rho\|\omega),$$

where $e_\alpha \omega e_\alpha$ is the restriction of $\omega$ to the reduced von Neumann algebra $e_\alpha M e_\alpha$. 28
7 $C^*$-algebra case

Let $f$ be an operator convex function on $(0, +\infty)$ as before. Let $A$ be a unital $C^*$-algebra, and $A_+^*$ be the set of positive linear functionals (automatically bounded) on $A$. In this section we extend the definition of the maximal $f$-divergence to $\rho, \sigma \in A_+^*$. For any $\rho, \sigma \in A_+^*$ set $\eta := \rho + \sigma$, and $(\pi_\eta, \mathcal{H}_\eta, \xi_\eta)$ be the cyclic representation of $A$ associated with $\eta$ so that $\eta(a) = \langle \xi_\eta, \pi_\eta(a)\xi_\eta \rangle$, $a \in A$, and $\mathcal{H}_\eta = \overline{\pi_\eta(A)\xi_\eta}$. Then there exists a $T \in \pi_\eta(A)_+'$ with $0 \leq T \leq I$ such that

$$\rho(a) = \langle \xi_\eta, T\pi_\eta(a)\xi_\eta \rangle, \quad \sigma(a) = \langle \xi_\eta, (I - T)\pi_\eta(a)\xi_\eta \rangle, \quad a \in A.$$  

The normal extensions $\tilde{\rho}, \tilde{\sigma}$ of $\rho, \sigma$ to $\pi_\eta(A)''$ are defined by

$$\tilde{\rho}(x) := \langle \xi_\eta, Tx\xi_\eta \rangle, \quad \tilde{\sigma}(x) := \langle \xi_\eta, (I - T)x\xi_\eta \rangle, \quad x \in \pi_\eta(A)'',$$

so that $\rho = \tilde{\rho} \circ \pi_\eta$ and $\sigma = \tilde{\sigma} \circ \pi_\eta$.

**Definition 7.1.** For every $\rho, \sigma \in A_+^*$, with the normal extensions $\tilde{\rho}, \tilde{\sigma}$ to $\pi_\eta(A)''$ ($\eta = \rho + \sigma$) we define the maximal $f$-divergence of $\rho, \sigma$ by

$$\tilde{S}_f(\rho \| \sigma) := \tilde{S}_f(\tilde{\rho} \| \tilde{\sigma}).$$

In fact, $\tilde{S}_f(\rho \| \sigma)$ has the same expression as (7.10) with the spectral decomposition $T = \int_0^1 t \, dE(t)$.

**Lemma 7.2.** Let $\pi$ be a representation of $A$ on a Hilbert space $\mathcal{H}$. Assume that $\rho, \sigma \in A_+^*$ have the normal extensions $\tilde{\rho}, \tilde{\sigma}$ to $\pi(A)''$, i.e., $\tilde{\rho}, \tilde{\sigma}$ are normal positive linear functionals on $\pi(A)''$ such that $\rho = \tilde{\rho} \circ \pi$ and $\sigma = \tilde{\sigma} \circ \pi$. Then

$$\tilde{S}_f(\rho \| \sigma) = \tilde{S}_f(\tilde{\rho} \| \tilde{\sigma}).$$

**Proof.** Let $\overline{\eta} := \tilde{\rho} + \tilde{\sigma}$, $(\pi_{\overline{\eta}}, \mathcal{H}_{\overline{\eta}}, \xi_{\overline{\eta}})$ be the cyclic representation of $M := \pi(A)''$ associated with $\overline{\eta}$, and $T \in \pi_{\overline{\eta}}(M)_+'$ be such that $\overline{\eta}(x) = \langle \xi_{\overline{\eta}}, T\pi_{\overline{\eta}}(x)\xi_{\overline{\eta}} \rangle$ for all $x \in M$. Then

$$\eta(a) = \overline{\eta}(\pi(a)) = \langle \xi_{\overline{\eta}}, \pi_{\overline{\eta}}(\pi(a))\xi_{\overline{\eta}} \rangle, \quad a \in A,$$

and

$$\mathcal{H}_{\overline{\eta}} = \overline{\pi_{\overline{\eta}}(M)\xi_{\overline{\eta}}} = \overline{\pi_{\overline{\eta}}(\pi(A))\xi_{\overline{\eta}}}.$$

Here, the last equality is seen as follows: for any $x \in M$, by the Kaplansky density theorem [46, Theorem II.4.8], choose a net $a_\alpha \in A$ such that $\sup_{\alpha} \| \pi(a_\alpha) \| < +\infty$ and $\pi(a_\alpha) \to x$ strongly*. Then

$$\| (\pi_{\overline{\eta}}(\pi(a_\alpha) - x)\xi_{\overline{\eta}} \|^2 = \overline{\eta}( (\pi(a_\alpha) - x)^*(\pi(a_\alpha) - x) ) \to 0.$$ 

Therefore, $(\pi_{\overline{\eta}} \circ \pi, \mathcal{H}_{\overline{\eta}}, \xi_{\overline{\eta}})$ is the cyclic representation of $A$ associated with $\eta$. Moreover, note that $\eta(T) \in (\pi_{\overline{\eta}} \circ \pi)(A)_+''$ and $\rho(a) = \langle \xi_{\overline{\eta}}, T\pi_{\overline{\eta}}(a)\xi_{\overline{\eta}} \rangle$ for all $a \in A$. Hence by Definition 7.1, with the spectral decomposition $T = \int_0^1 t \, dE(t)$ we have

$$\tilde{S}_f(\rho \| \sigma) = \int_0^1 (1 - t) f\left( \frac{t}{1 - t} \right) d\| E(t)\xi_{\overline{\eta}} \|^2. \quad (7.1)$$

On the other hand, applying (5.10) to $\pi_{\overline{\eta}} \in M_+^*$ shows that $\tilde{S}_f(\tilde{\rho} \| \tilde{\sigma})$ has the same integral expression as (7.1), so the asserted equality follows. \qed
The above lemma says that $\tilde{S}_{f}(\rho\|\sigma)$ for $\rho, \sigma \in A^{*}_{+}$ can be defined as $\tilde{S}_{f}(\overline{\rho}\|\overline{\sigma})$ via any $(\pi, \overline{\rho}, \overline{\sigma})$ of a representation $\pi$ of $A$ and normal extensions $\overline{\rho}, \overline{\sigma}$ of $\rho, \sigma$ to $\pi(A)^{\prime\prime}$. An example of such representation, besides $\pi_{0}$ in Definition 7.4, is the universal representation $\pi$ of $A$, for which $\pi(A)^{\prime\prime} \cong A^{**}$ (isometric to the second conjugate space of $A$), see [10] §III.2.

In the rest of the section we give some basic properties $\tilde{S}_{f}(\rho\|\sigma)$ for $\rho, \sigma \in A^{*}_{+}$.

**Proposition 7.3.** The function $(\rho, \sigma) \in A^{*}_{+} \times A^{*}_{+} \mapsto \tilde{S}_{f}(\rho\|\sigma)$ is jointly convex and jointly lower semicontinuous in the norm topology.

**Proof.** Let $\pi$ be the universal representation of $A$. For $\rho_{i}, \sigma_{i} \in M_{i}^{+}$ and $\lambda_{i} \geq 0$ $(1 \leq i \leq n)$ let $\overline{\rho}_{i}, \overline{\sigma}_{i}$ be the normal extensions of $\rho_{i}, \sigma_{i}$ to $\pi(A)^{\prime\prime}$. By Lemma 7.2 and the joint convexity property in Theorem 2.9

$$
\tilde{S}_{f}\left(\sum_{i=1}^{n} \lambda_{i} \rho_{i} \parallel \sum_{i=1}^{n} \lambda_{i} \sigma_{i}\right) = \tilde{S}_{f}\left(\sum_{i=1}^{n} \lambda_{i} \overline{\rho}_{i} \parallel \sum_{i=1}^{n} \lambda_{i} \overline{\sigma}_{i}\right) \leq \sum_{i=1}^{n} \lambda_{i} S_{f}(\overline{\rho}_{i}\|\overline{\sigma}_{i}) = \sum_{i=1}^{n} \lambda_{i} S_{f}(\rho_{i}\|\sigma_{i}).
$$

Next, let $\rho_{n}, \rho, \sigma_{n}, \sigma \in A^{*}_{+}$, $n \in \mathbb{N}$, be such that $\|\rho_{n} - \rho\| \to 0$ and $\|\sigma_{n} - \sigma\| \to 0$. From the Kaplansky density theorem and [25] Lemma IV.3.8, it follows that

$$
\|\overline{\rho}_{n} - \rho\| = \sup_{x \in \pi(A)^{\prime\prime}, \|x\| \leq 1} |(\overline{\rho}_{n} - \rho)(x)| = \sup_{a \in A, \|a\| \leq 1} |(\overline{\rho}_{n} - \rho)(\pi(a))| = \|\rho_{n} - \rho\| \to 0,
$$

and similarly $\|\overline{\sigma}_{n} - \sigma\| = \|\sigma_{n} - \sigma\| \to 0$. Hence by Lemma 7.2 and Theorem 5.5

$$
\tilde{S}_{f}(\rho\|\sigma) = \tilde{S}_{f}(\overline{\rho}\|\overline{\sigma}) \leq \liminf_{n \to \infty} \tilde{S}_{f}(\overline{\rho}_{n}\|\overline{\sigma}_{n}) = \liminf_{n \to \infty} \tilde{S}_{f}(\rho_{n}\|\sigma_{n}),
$$

showing the lower semicontinuity in the norm topology. \hfill \square

**Proposition 7.4.** Let $A_{0}$ be another unital $C^{*}$-algebra and $\Phi : A_{0} \to A$ be a unital positive linear map. Then for every $\rho, \sigma \in A^{*}_{+}$,

$$
\tilde{S}_{f}(\rho \circ \Phi\|\sigma \circ \Phi) \leq \tilde{S}_{f}(\rho\|\sigma).
$$

**Proof.** Let $\pi, \pi_{0}$ be the universal representations of $A, A_{0}$, respectively. One can define the unital positive normal map $\overline{\Phi} : \pi_{0}(A_{0})^{\prime\prime} \to \pi(A)^{\prime\prime}$ subject to the commutative diagrams (see [10] Lemma III.2.2):

$$
\begin{array}{ccc}
A_{0} & \xrightarrow{i} & A_{0}^{*} \\
\Phi \downarrow & & \Phi^{**} \\
A & \xrightarrow{i} & A^{**}
\end{array}
\xrightarrow{\cong} \begin{array}{ccc}
\pi_{0}(A_{0})^{\prime\prime} & \xrightarrow{\pi_{0}} & \pi(A)^{\prime\prime} \\
\overline{\Phi} \downarrow & & \overline{\Phi} \\
\pi_{0}(A_{0})^{\prime\prime} & \xrightarrow{\cong} & \pi(A)^{\prime\prime}
\end{array}
$$

where $i$ is the canonical imbedding of $A$ ($A_{0}$) into $A_{0}^{**}$ ($A_{0}^{*}$) and $\Phi^{**}$ is the second conjugate map. Here it is immediate to verify that $\overline{\Phi} \circ \pi_{0}(a) = \pi \circ \Phi(a)$ for all $a \in A_{0}$. Moreover, the positivity of $\overline{\Phi}$ is seen as follows: for any $x \in \pi_{0}(A_{0})^{\prime\prime}$, by the Kaplansky density theorem, choose a net $a_{\alpha} \in A_{0}$ such that $\sup \|\pi_{0}(a_{\alpha})\| < +\infty$ and $\pi_{0}(a_{\alpha}) \to x$ strongly*. Then

$$
\pi_{0}(a_{\alpha}^{*} a_{\alpha}) \to x^{*} x \ \text{strongly},
$$

so that $\overline{\Phi}(\pi_{0}(a_{\alpha}^{*} a_{\alpha})) \to \overline{\Phi}(x^{*} x)$ weakly. Since $\overline{\Phi}(\pi_{0}(a_{\alpha}^{*} a_{\alpha})) = \pi(\Phi(a_{\alpha}^{*} a_{\alpha})) \geq 0$, $\overline{\Phi}(x^{*} x) \geq 0$. For every $\rho \in A_{+}^{*}$, since $\overline{\Phi} \circ \Phi$ is normal on $\pi_{0}(A_{0})^{\prime\prime}$ and

$$
\overline{\Phi} \circ \pi_{0}(a) = \overline{\Phi} \circ \pi \circ \Phi(a) = \rho \circ \Phi(a), \quad a \in A_{0},
$$

one sees that $\overline{\rho} \circ \overline{\Phi}$ is the normal extension of $\rho \circ \Phi$ to $\pi_{0}(A_{0})^{\prime\prime}$. Therefore, by Lemma 7.2 and the monotonicity property in Theorem 2.9

$$
\tilde{S}_{f}(\rho \circ \Phi\|\sigma \circ \Phi) = \tilde{S}_{f}(\overline{\rho} \circ \overline{\Phi}\|\overline{\sigma} \circ \overline{\Phi}) \leq \tilde{S}_{f}(\overline{\rho}\|\overline{\sigma}) = \tilde{S}_{f}(\rho\|\sigma)
$$

for all $\rho, \sigma \in A^{*}_{+}$. \hfill \square
Proposition 7.5. Let \( \{A_\alpha\} \) be an increasing net of unital \( C^* \)-subalgebras of \( A \) such that \( \bigcup_\alpha A_\alpha \) is norm-dense in \( A \). Then for every \( \rho, \sigma \in A_+^* \),
\[
\widehat{S}_f(\rho|A_\alpha \|\sigma|A_\alpha) \nearrow \widehat{S}_f(\rho\|\sigma).
\]

**Proof.** With the universal representation \( \pi \) of \( A \) we have an increasing net \( \{\pi(A_\alpha)''\} \) of unital von Neumann subalgebras of \( \pi(A)'' \) such that \( \bigcup_\alpha (\pi(A_\alpha))'' = \pi(A)'' \). Hence by Lemma 7.2 and Theorem 5.6
\[
\widehat{S}_f(\rho|A_\alpha \|\sigma|A_\alpha) = \widehat{S}_f(\pi(\rho|A_\alpha)\|\pi(\sigma|A_\alpha)) \nearrow \widehat{S}_f(\pi(\rho\|\sigma) = \widehat{S}_f(\rho\|\sigma)
\]
for all \( \rho, \sigma \in A_+^* \). \( \Box \)

8 Closing remarks and problems

In the previous paper [20] we discussed standard \( f \)-divergences \( S_f(\rho\|\sigma) \) in von Neumann algebra setting for general operator convex functions \( f \) on \((0, +\infty)\). In this paper we present a systematic study of another type of quantum \( f \)-divergences \( \widehat{S}_f(\rho\|\sigma) \) called the maximal \( f \)-divergences in the same setting. Starting with a rather abstract definition (Definition 2.8) we present more explicit expressions of \( \widehat{S}_f(\rho\|\sigma) \) in an integral formula (Theorem 4.2) and in a variational formula (Theorem 6.3), from which we can derive several important properties of \( \widehat{S}_f(\rho\|\sigma) \). Properties of \( S_f(\rho\|\sigma) \) and \( \widehat{S}_f(\rho\|\sigma) \) are common in most cases, but there are also small differences between them. For instance, the joint lower semicontinuity of \( S_f(\rho\|\sigma) \) holds in the \( \sigma(M, M) \)-topology, but that of \( \widehat{S}_f(\rho\|\sigma) \) is shown in the norm topology, and it is open whether \( \widehat{S}_f \) has the same property in the \( \sigma(M, M) \)-topology, as mentioned in Remark 6.7. The monotonicity inequality (DPI) of \( S_f \) holds under unital Schwarz normal maps, while that of \( \widehat{S}_f \) is shown more generally under unital simply positive normal maps.

We have the general inequality \( S_f \leq \widehat{S}_f \) (Theorem 6.4). For matrices \( \rho, \sigma \in M_+^d \) with \( s(\rho) \leq s(\sigma) \), it was shown in [21, Theorem 4.3] that \( S_f(\rho\|\sigma) = \widehat{S}_f(\rho\|\sigma) \) holds if and only if \( \rho\sigma = \sigma\rho \), under a mild assumption on the support of the representing measure for the integral expression of \( f \). In particular, for matrices \( \rho, \sigma \) with \( s(\rho) \leq s(\sigma) \), \( D(\rho\|\sigma) = D_{BS}(\rho\|\sigma) \) holds if and only if \( \rho\sigma = \sigma\rho \). An interesting problem is to extend this result to the von Neumann algebra setting. In the proof of [21, Theorem 4.3] we used the reversibility via equality in the monotonicity inequality for \( S_f \). Thus, our next research topic should be the reversibility question under equality in the monotonicity inequality for \( S_f \). Here we say that a unital normal map \( \Phi : M_0 \to M \) (which satisfies a kind of positivity such as complete positivity) is *reversible* for \( \{\rho, \sigma\} \) in \( M_+^* \) if there exists a map \( \Psi : M \to M_0 \) of similar kind such that \( \rho \circ \Phi \circ \Psi = \rho \) and \( \sigma \circ \Phi \circ \Psi = \sigma \). The question says whether \( \Phi \) is reversible for \( \{\rho, \sigma\} \) or not if \( S_f(\rho \circ \Phi\|\sigma \circ \Phi) = S_f(\rho\|\sigma) < +\infty \). In the matrix case, the question was well studied in [22, 21], including discussions on the equality case in the monotonicity inequality for \( \widehat{S}_f \). For reversibility in the von Neumann algebra case, former results in some special cases of relative entropy and the standard Rényi divergences are found in, e.g., [11, 42, 28], and recent results in the case of sandwiched Rényi divergences are obtained in [26, 27].

The notion of quantum \( f \)-divergences in the opposite direction to \( \widehat{S}_f \) is that of measured (or minimal) \( f \)-divergences, whose matrix case was discussed in [21]. For \( \rho, \sigma \in M_+^* \), taking account of Theorem 6.3 one can define the measured \( f \)-divergence \( S_f^{\text{meas}}(\rho\|\sigma) \) of \( \rho \) with respect to \( \sigma \) by
\[
S_f^{\text{meas}}(\rho\|\sigma) := \sup \{S_f(\rho \circ \Phi\|\sigma \circ \Phi) : \Phi : L^\infty([0,1], \nu) \to M\},
\]
where \( \Phi \) runs over unital positive normal map from \( L^\infty([0,1],\nu) \) on a Borel probability space \([0,1],\nu\) to \( M \). When \( M \) is \( \sigma \)-finite so that it has a faithful normal state, note that the above \( S_f^\mathrm{meas}(\rho\|\sigma) \) is the supremum of the classical \( f \)-divergence \( S_f(M(\rho))\|M(\sigma) \) over \( M \)-valued measurements \( M \) on \([0,1] \), i.e., \( M \) is \( \sigma \)-additive \( M_+\)-valued measure with \( M([0,1]) = 1 \), where \( M(\rho) \) denotes a Borel measure \( \rho(M(\cdot)) \) on \([0,1] \). One can also define \( S_f^\mathrm{pr}(\rho\|\sigma) \) by restricting \( M \) in the above to measurements whose values are projections in \( M \). Due to the monotonicity of \( S_f \) in [20, Theorem 4.1(iv)] it is clear that

\[
S_f^\mathrm{pr}(\rho\|\sigma) \leq S_f^\mathrm{meas}(\rho\|\sigma) \leq S_f(\rho\|\sigma) \leq \hat{S}_f(\rho\|\sigma).
\]

It is interesting to characterize the equality case \( S_f(\rho\|\sigma) = S_f^\mathrm{meas}(\rho\|\sigma) \) or \( S_f(\rho\|\sigma) = S_f^\mathrm{pr}(\rho\|\sigma) \) in terms of commutativity of \( \rho,\sigma \), as in [21, Theorem 4.18] in the matrix case.

Apart from the conventional (or standard) Rényi divergences, a new type of Rényi divergences called the sandwiched ones have extensively been developed in these years. For matrices \( \rho,\sigma \in M_+^d \), the sandwiched Rényi divergence \( \tilde{D}_\alpha(\rho\|\sigma) \) for \( \alpha \in (0,\infty) \setminus \{1\} \) is defined by

\[
\tilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{\tilde{\mathcal{Q}}_\alpha(\rho\|\sigma)}{\Tr \rho}, \quad \tilde{\mathcal{Q}}_\alpha(\rho\|\sigma) := \Tr (\sigma^{1/\alpha} \rho^\alpha \sigma^{-1/\alpha}),
\]

while the standard Rényi divergence is

\[
D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{\Tr \rho^\alpha \sigma^{1-\alpha}}{\Tr \rho},
\]

where \( \sigma^\gamma \) for \( \gamma < 0 \) is defined via the generalized inverse. It is widely known [21, 37, 34] that \( D_\alpha(\rho\|\sigma) \) and \( \tilde{D}_\alpha(\rho\|\sigma) \), together with \( D_1(\rho\|\sigma) = D(\rho\|\sigma)/\Tr \rho \), play a significant role in quantum state discrimination, thus enjoying good operational interpretation. The extension of \( \tilde{D}_\alpha(\rho\|\sigma) \) to the von Neumann algebra setting has been made in recent papers [4, 26, 27], and a detailed exposition on \( D_\alpha(\rho\|\sigma) \) in von Neumann algebras has been given in [20]. For \( \rho,\sigma \in M_+^d \), the maximal Rényi divergence \( \hat{D}_\alpha(\rho\|\sigma) \) is defined by

\[
\hat{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{\hat{\mathcal{Q}}_\alpha(\rho\|\sigma)}{\rho(1)}, \quad \hat{\mathcal{Q}}_\alpha(\rho\|\sigma) := \begin{cases} S_{f_\alpha}(\rho\|\sigma) & \text{if } \alpha > 1, \\ -S_{f_\alpha}(\rho\|\sigma) & \text{if } 0 < \alpha < 1, \end{cases}
\]

where \( f_\alpha(t) := t^\alpha \) (\( \alpha > 1 \)) and \( -t^\alpha \) (\( 0 < \alpha < 1 \)). (Although \( f_\alpha \) for \( \alpha > 2 \) is not operator convex on \((0,\infty) \), one can define \( S_{f_\alpha}(\rho\|\sigma) \), for instance, by the integral expression in (4.9) with \( f = f_\alpha \).) For matrices \( \rho,\sigma \in M_+^d \) with \( s(\rho) \leq s(\sigma) \), we have \( \hat{\mathcal{Q}}_\alpha(\rho\|\sigma) = \Tr \sigma (\sigma^{-1/2} \rho \sigma^{-1/2})^\alpha \) by (4.14), and from [21, Remark 4.6] we see that

\[
\tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma) \leq \hat{D}_\alpha(\rho\|\sigma) \quad \text{for } \alpha \in (0,2) \setminus \{1\},
\]

\[
\tilde{D}_\alpha(\rho\|\sigma) \leq \hat{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma) \quad \text{for } \alpha \in [2,\infty).
\]

In the von Neumann algebra setting, it follows from Theorem 6.3 that \( D_\alpha \leq \hat{D}_\alpha \) for \( \alpha \in (0,2) \setminus \{1\} \), while it was shown in [4, 26] that \( \hat{D}_\alpha \leq D_\alpha \) for \( \alpha \in [1/2,\infty) \setminus \{1\} \). But comparison between between \( D_\alpha, \hat{D}_\alpha \) and \( \tilde{D}_\alpha \) in the von Neumann algebra case has not fully been investigated.

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A Proofs of (4.6) and (4.7)

A.1 Proof of (4.7)

We use the integral expression of \( f \) in (3.9). Let \( \varepsilon \in (0, 1/2) \) and we divide the integral on \([0, 1]\) into two parts on \([0, 2/3]\) and \([2/3, 1]\). Since \( 1 - t + \varepsilon t \geq 1/3 \) for \( \varepsilon \in (0, 1/2) \) and \( t \in [0, 2/3] \), \( (1 - t + \varepsilon t)f(\frac{t}{1 - t + \varepsilon t}) \) is uniformly bounded for those \( \varepsilon, t \) and converges to \((1 - t)f(\frac{t}{1 - t})\) as \( 1/2 > \varepsilon \searrow 0 \). Hence the bounded convergence theorem gives

\[
\lim_{\varepsilon \searrow 0} \int_{[0,2/3]} (1 - t + \varepsilon t)f(\frac{t}{1 - t + \varepsilon t}) \, dv(t) = \int_{[0,2/3]} (1 - t)f(\frac{t}{1 - t}) \, dv(t).
\]

Next, we write

\[
(1 - t + \varepsilon t)f(\frac{t}{1 - t + \varepsilon t}) = a + (b - a + \varepsilon)t + c \frac{t^2}{1 - t + \varepsilon t}
\]

\[+ \int_{(0, +\infty)} (1 - t + \varepsilon t)\psi_s\left(\frac{t}{1 - t + \varepsilon t}\right) d\mu(s) \]

Note that for every \( t \in (2/3, 1] \),

\[
0 < \frac{t^2}{1 - t + \varepsilon t} \nearrow \frac{t^2}{1 - t} \quad \text{as} \quad \varepsilon \searrow 0,
\]

where \( \frac{t^2}{1 - t} = +\infty \) for \( t = 1 \). Also, when \( s \in (0, +\infty) \) and \( t \in (2/3, 1] \), since

\[
(1 - t + \varepsilon t)\psi_s\left(\frac{t}{1 - t + \varepsilon t}\right) = \frac{t}{1 + s} - \frac{t}{1 - t + \varepsilon t + s}
\]

and \( \frac{t}{1 - t + \varepsilon t} > 1 \) for every \( \varepsilon \in (0, 1/2) \), we note that

\[
0 < (1 - t + \varepsilon t)\psi_s\left(\frac{t}{1 - t + \varepsilon t}\right) \nearrow (1 - t)\psi_s\left(\frac{t}{1 - t}\right) \quad \text{as} \quad 1/2 > \varepsilon \searrow 0,
\]

where \((1 - t)\psi_s(\frac{1}{1 - t}) = \frac{1}{1 + s}\) for \( t = 1 \). By the monotone convergence theorem we find that

\[
\int_{(2/3,1]} (1 - t + \varepsilon t)f(\frac{t}{1 - t + \varepsilon t}) \, dv(t)
\]

\[= a \int_{(2/3,1]} dv(t) + (b - a + \varepsilon) \int_{(2/3,1]} t \, dv(t) + c \int_{(2/3,1]} \frac{t^2}{1 - t + \varepsilon t} \, dv(t)
\]

\[+ \int_{(2/3,1]} \int_{(0, +\infty)} (1 - t + \varepsilon t)\psi_s\left(\frac{t}{1 - t + \varepsilon t}\right) d\mu(s) \, dv(t)
\]

converges as \( 1/2 > \varepsilon \searrow 0 \) to

\[
a \int_{(2/3,1]} dv(t) + (b - a) \int_{(2/3,1]} t \, dv(t) + c \int_{(2/3,1]} \frac{t^2}{1 - t} \, dv(t)
\]

\[+ \int_{(2/3,1]} \int_{(0, +\infty)} (1 - t)\psi_s\left(\frac{t}{1 - t}\right) d\mu(s) \, dv(t)
\]

\[= \int_{(2/3,1]} \left( a + (b - a)t + c \frac{t^2}{1 - t} + \int_{(0, +\infty)} (1 - t)\psi_s\left(\frac{t}{1 - t}\right) d\mu(s) \right) \, dv(t)
\]

\[= \int_{(2/3,1]} (1 - t)f(\frac{t}{1 - t}) \, dv(t).
\]

Therefore, (4.7) follows. \qed
A.2 Proof of (4.6)

Let \( \varepsilon \in (0,1) \) and we divide the integral on \([0,1]\) into two parts on \([0,2/3]\) and \((2/3,1]\). As in the proof of (4.7) we have

\[
\lim_{\varepsilon \searrow 0} \int_{[0,2/3]} (1 - t + \varepsilon t) f \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) \, d\nu(t) = \int_{[0,2/3]} (1 - t) f \left( \frac{t}{1 - t} \right) \, d\nu(t).
\]

Next, we write

\[
(1 - t + \varepsilon t) f \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) = (a + \varepsilon b) + (b - a)(1 - \varepsilon)t + c \frac{(t + \varepsilon (1 - t))^2}{1 - t + \varepsilon t} + \int_{(0, +\infty)} (1 - t + \varepsilon t) \psi_s \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) \, d\mu(s).
\]

Compute

\[
\frac{d}{d\varepsilon} \frac{(t + \varepsilon (1 - t))^2}{1 - t + \varepsilon t} = \frac{(t + \varepsilon (1 - t))(2 - 4t + t^2 + \varepsilon t(1 - t))}{(1 - t + \varepsilon t)^2}.
\]

When \( \varepsilon \in (0,1) \) and \( t \in (2/3,1] \), since

\[2 - 4t + t^2 + \varepsilon t(1 - t) \leq 2 - 4t + t^2 + t(1 - t) = 2 - 3t < 0,\]

we have

\[
\frac{d}{d\varepsilon} \frac{(t + \varepsilon (1 - t))^2}{1 - t + \varepsilon t} < 0,
\]

so that

\[
\frac{(t + \varepsilon (1 - t))^2}{1 - t + \varepsilon t} \nearrow \frac{t^2}{1 - t} \quad \text{as} \quad \varepsilon \searrow 0.
\]

Also, when \( s \in (0, +\infty) \) and \( t \in (2/3,1] \), since

\[
(1 - t + \varepsilon t) \psi_s \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) = \frac{t + \varepsilon (1 - t)}{1 + s} - \frac{t + \varepsilon (1 - t)}{(t + \varepsilon (1 - t)) + s}
\]

and for every \( \varepsilon \in (0,1) \), \( \frac{t}{1 - t + \varepsilon t} > 1 \) and

\[
\frac{d}{d\varepsilon} \frac{t}{1 - t + \varepsilon t} = \frac{1 - 2t}{(1 - t + \varepsilon t)^2} < 0,
\]

we note that

\[0 < (1 - t + \varepsilon t) \psi_s \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) \nearrow (1 - t) \psi_s \left( \frac{t}{1 - t} \right) \quad \text{as} \quad 1 > \varepsilon \searrow 0.
\]

By the monotone convergence theorem we find that

\[
\int_{(2/3,1]} (1 - t + \varepsilon t) f \left( \frac{t + \varepsilon (1 - t)}{1 - t + \varepsilon t} \right) \, d\nu(t) = (a + \varepsilon b) \int_{(2/3,1]} \, d\nu(t) + (b - a)(1 - \varepsilon) \int_{(2/3,1]} t \, d\nu(t) + c \int_{(2/3,1]} \frac{(t + \varepsilon (1 - t))^2}{1 - t + \varepsilon t} \, d\nu(t).
\]

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\[ + \int_{(2/3,1]} \int_{(0, +\infty)} (1 - t + \varepsilon t) \psi_s \left( \frac{t + \varepsilon(1 - t)}{1 - t + \varepsilon t} \right) d\mu(s) d\nu(t) \]

corverges as \(1 > \varepsilon \searrow 0\) to

\[
a \int_{(2/3,1]} d\nu(t) + (b - a) \int_{(2/3,1]} t d\nu(t) + c \int_{(2/3,1]} \frac{t^2}{1 - t} d\nu(t)
+ \int_{(2/3,1]} \int_{(0, +\infty)} (1 - t) \psi_s \left( \frac{t}{1 - t} \right) d\mu(s) d\nu(t)
= \int_{(2/3,1]} (a + (b - a)t + c \frac{t^2}{1 - t} + \int_{(0, +\infty)} (1 - t) \psi_s \left( \frac{t}{1 - t} \right) d\mu(s)) d\nu(t)
= \int_{(2/3,1]} (1 - t) f \left( \frac{t}{1 - t} \right) d\nu(t).
\]

Therefore, (4.6) follows.

\[ \tag*{\Box} \]

**B Proof of Lemma 6.5**

For the proof below, we may assume that \(\rho + \omega\) is faithful, by replacing \(M\) with \(eMe\), where \(e := s(\rho + \omega)\). Here, concerning (iv), we note that \((eMe)_*\) is identified with \(eM_*e\), where \(e\psi_e(x) := \psi(eixe)\), \(x \in M\), for \(\psi \in M_*^\tau\), and by [48, Theorem 7], \(h_{e\psi_e} = eh_{\psi}e\) for every \(\psi \in M_*\) so that

\((eMe)_* \cong eL^1(M)e = \{eh_{\psi}e : \psi \in M_*\}\).

(i) \(\iff\) (ii). Since \(\rho + \omega\) is invariant under \(\sigma_t^{\rho+\omega}\), (i) implies that

\[ \omega \circ \sigma_t^{\rho+\omega} = (\rho + \omega) \circ \sigma_t^{\rho+\omega} - \rho \circ \sigma_t^{\rho+\omega} = (\rho + \omega) - \rho = \omega. \]

Hence (i) \(\implies\) (ii), and the converse is similar.

(i) \(\iff\) (iii). Assume (i) (hence also (ii)), and let \(\gamma, \delta > 0\) be arbitrary. Since \(\rho, \omega\) are invariant under \(\sigma_t^{\rho+\omega}\), \((\gamma \rho + \delta \omega) \circ \sigma_t^{\rho+\omega} = \gamma \rho + \delta \omega\). Hence by [38],

\[ (\rho + \omega) \circ \sigma_t^{\gamma \rho + \delta \omega} = \rho + \omega. \]

Assume that \(\gamma \neq \delta\). Since \((\gamma \rho + \delta \omega) \circ \sigma_t^{\rho+\delta \omega} = \gamma \rho + \delta \omega\), it follows that \(\rho, \omega\) are invariant under \(\sigma_t^{\gamma \rho + \delta \omega}\) and so is \(\alpha \rho + \beta \gamma\) for any \(\alpha, \beta > 0\). When \(\gamma = \delta\), the same follows from (i) and (ii) immediately. Hence we have (i) \(\implies\) (iii). The converse is easy.

(i) \(\implies\) (iv). Assume (i) and hence (iii). Let \(\alpha, \beta, \gamma, \delta > 0\). Since (see [30 (18)])

\[ \sigma_t^{\gamma \rho + \delta \omega}(x) = h_{\gamma \rho + \delta \omega}^it h_{\gamma \rho + \delta \omega}^{-it}, \quad x \in M, \]

it follows that

\[ \text{tr}(h_{\alpha \rho + \beta \omega} h_{\gamma \rho + \delta \omega}^{-it} h_{\gamma \rho + \delta \omega}^{-it}) = \text{tr}(h_{\alpha \rho + \beta \omega} x), \quad x \in M, \ t \in \mathbb{R}, \]

so that

\[ h_{\gamma \rho + \delta \omega}^{-it} h_{\alpha \rho + \beta \omega} h_{\gamma \rho + \delta \omega}^{-it} = h_{\alpha \rho + \beta \omega}, \quad t \in \mathbb{R}, \]

as elements of \(L^1(M)\) and hence as elements in \(\overline{N}\). This implies that \(h_{\alpha \rho + \beta \omega}\) commutes with \(h_{\gamma \rho + \delta \omega}\) in the sense of [33, p. 271]. Since those are \(\tau\)-measurable operators, we have

\[ h_{\alpha \rho + \beta \omega} h_{\gamma \rho + \delta \omega} = h_{\gamma \rho + \delta \omega} h_{\alpha \rho + \beta \omega}. \]
so that
\[(\alpha h + \beta h\omega)(\gamma h + \delta h\omega) = (\gamma h + \delta h\omega)(\alpha h + \beta h\omega).\]

Choosing \(\alpha = \delta = 1\) and \(\beta, \gamma \to 0\) gives (iv).

(iv) \(\implies\) (i). Since (iv) implies that \(h\rho h_{\rho+\omega} = h_{\rho+\omega} h\rho\), it is easy to verify that \(h\rho\) commutes with \((h_{\rho+\omega} + 1)^{-1}\). This implies that \(h\rho\) commutes with any spectral projection of \(h_{\rho+\omega}\). Therefore, \(h\rho h_{\rho+\omega}^{it} = h_{\rho+\omega}^{it} h\rho, \ t \in \mathbb{R}\), from which we have
\[\text{tr}(h\rho h_{\rho+\omega}^{it} x h_{\rho+\omega}^{-it}) = \text{tr}(h\rho x), \quad x \in M, \ t \in \mathbb{R}.\]

Hence (i) follows.

Next, when \(s(\rho) \leq s(\omega)\) (hence we may assume that \(\omega\) is faithful), it is seen as above (by replacing \(\rho + \omega\) with \(\omega\)) that (iv) is equivalent to that \(h\rho h_{\rho+\omega}^{it} = h_{\rho+\omega}^{it} h\rho\) for any \(t \in \mathbb{R}\), which means that \(\rho \circ \sigma_t^\omega = \rho, \ t \in \mathbb{R}\).

\[\square\]

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