Research Article

Stability and Hopf Bifurcation of a Vector-Borne Disease Model with Saturated Infection Rate and Reinfection

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1. Introduction

Malaria is a vector-borne infectious disease [1], caused by parasites. It is popular in 102 countries and regions, especially in some countries in Africa, southeast Asia, and South America. In the 30s of this century, malaria spread throughout the country. Clinical symptoms and signs of this disease, such as typical periodic onset of malaria, secondary anemia, and spleen, can cause serious consequences, including dangerous malaria, malarial kidney disease, and black urine fever.

The main way of transmission of malaria is the bite of an infected female anopheline mosquito. The mosquitoes would also be infected when uninfected mosquitoes bite infected people, and this transmission process has an incubation period [2]. The important feature of malaria is that the recovered immune system may establish immune memory for such antigens. It is this characteristic that greatly reduces the spread of malaria [3, 4]. Immune process is slow and, however, takes years or even decades [5]. As time goes by, the immune system gradually weakens, and at this time, reinfection likely occurs; therefore, considering the function of delay and immune system is necessary in the study of malaria.

For the vector-borne diseases such as malaria, a large number of mathematical models have been created [2, 6, 7, 8, 9], most of which consider the local immunity and delay of the spread of malaria in the crowd. Different time delay has been used to describe the latent period in the course of disease transmission [7, 8, 9]. Local stability conditions for the equilibrium of a model with two time delays have been considered by Wan and Cui [8]. The global stability of the equilibrium has been studied for a vector-borne disease model with distributed delay by Cai et al. [10].

Based on the above model, this paper considers a delayed vector-borne model with saturated infection rate and partial immunity to reinfection. We prove that the stability of this system can be changed by time delay and produce Hopf bifurcation, calculating the length of delay to preserve stability. Using the center manifold theorem [11] and norm theory, we determine the stability and bifurcation direction.

2. Model Formulation

$N_1(t)$ represented as the host population at time $t$ is divided into three subclasses: the susceptible $S(t)$, the infected $I(t)$, and the recovered $R(t)$. $N_2(t)$ represented as the vector population at time $t$ is divided into two subclasses: the susceptible $T(t)$ and the infected $V(t)$. The Hopf bifurcation was determined in a model with direct infection and delay by Wei et al. [9]. The mathematical formulation still needs improvements. We consider an improved model as follows:
\[
\begin{align*}
\frac{dS}{dt} &= \Lambda_1 - \frac{b_1 S(t) V(t)}{1 + a V(t)} - \mu_1 S(t), \\
\frac{dI}{dt} &= \frac{b_1 S(t) V(t)}{1 + a V(t)} + \sigma b_1 R(t) V(t) - (\mu_1 + \gamma) I(t), \\
\frac{dR}{dt} &= \gamma I(t) - \sigma b_1 R(t) V(t) - \mu_1 R(t), \\
\frac{dT}{dt} &= \Lambda_2 - b_2 T(t - \tau) I(t - \tau) - \mu_2 T(t), \\
\frac{dV}{dt} &= b_2 T(t - \tau) I(t - \tau) - \mu_2 V(t),
\end{align*}
\]

where \(\Lambda_1\) and \(\Lambda_2\) represent the recruitment rate of the host population and vector population, respectively. \(b\) represents the average number of bites per mosquito per day. The incidence rate \(b_1 S(t) V(t) / (1 + a V(t))\) is the number of infections of the susceptible host caused by the infected vector, and \(a\) is the infectivity rate effect caused by the infected vector. \(\mu_1\) and \(\mu_2\) represent the death rates of the host population and vector population, respectively. \(\beta_1\) is the infection rate from vector to human. \(\sigma(0 < \sigma \leq 1)\) represents the degree of partial protection for recovered person given by a primary infection, where \(\sigma = 0\) represents complete protection and \(\sigma = 1\) represents no protection. \(\gamma\) is the per capita recovery rate of the infected host population. \(\beta_2\) represents the infection rate from human to vector. \(\tau\) is the time delay, representing the incubation period in the vector population; that is to say, a susceptible vector that bites an infective host at time \(t - \tau\) will become infective at time \(t\).

The model (1) meets the initial conditions:

\[
\begin{align*}
S(\theta) &= \phi_1(\theta), \\
I(\theta) &= \phi_2(\theta), \\
R(\theta) &= \phi_3(\theta), \\
T(\theta) &= \phi_4(\theta), \\
V(\theta) &= \phi_5(\theta), \\
\phi_i(\theta) &\geq 0, \quad (i = 1, 2, 3, 4, 5), \\
-\tau &\leq \theta \leq 0,
\end{align*}
\]

where \(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta) \in C([-\tau, 0], R_+^5)\) is the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \(R_+^5\) with the topology of uniform convergence. The norm is defined as follows:

\[
\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} \left\{ |\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|, |\phi_4(\theta)|, |\phi_5(\theta)| \right\}.
\]

Based on the fundamental theory of functional differential equations [12], it is easy to show that the solution of the model (1) with the initial condition (2) is unique and is nonnegative for all \(t \geq 0\).

By (1), we know that

\[
N_k'(t) = \Lambda_k - \mu_k N_k(t), \quad (k = 1, 2),
\]

and can solve it by using the integrating factor:

\[
N_k(t) = N_k(0)e^{-\Lambda_t} + \frac{\Lambda_k}{\mu_k}(1 - e^{-\Lambda t}).
\]

That is,

\[
\lim_{t \to \infty} N_k(t) = \frac{\Lambda_k}{\mu_k} \quad (k = 1, 2).
\]

Form the limiting theory of differential equation [13], we can draw that model (1) is the equivalent of the following equation:

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda_1 - \frac{b_1 S(t) V(t)}{1 + a V(t)} - \mu_1 S(t), \\
\frac{dI}{dt} &= \frac{b_1 S(t) V(t)}{1 + a V(t)} + \sigma b_1 \frac{\Lambda_1}{\mu_1} - S(t) - I(t) \\
&\quad \cdot V(t) - (\mu_1 + \gamma) I(t), \\
\frac{dV}{dt} &= b_2 \frac{\Lambda_2}{\mu_2} - V(t - \tau) I(t - \tau) - \mu_2 V(t).
\end{align*}
\]

Next, the model (7) can be studied in the invariant set:

\[
\Omega = \left\{ (S, I, V) \in R_+^3 \mid 0 \leq S + I \leq \frac{\Lambda_1}{\mu_1}, 0 \leq V \leq \frac{\Lambda_2}{\mu_2}, S \geq 0, I \geq 0 \right\}.
\]

Now, let us consider the existence of equilibrium. First, it is easy to show that system (7) always has a disease-free equilibrium \(E_0 = (S_0, I_0, V_0) = (\Lambda_1/\mu_1, 0, 0)\). The endemic equilibrium \(E_1 = (S_1, I_1, V_1)\) satisfies the following equation:

\[
\begin{align*}
\Lambda_1 &= \frac{b_1 S_1 V_1}{1 + a V_1} - \mu_1 S_1 = 0, \\
\frac{b_1 S_1 V_1}{1 + a V_1} + \sigma b_1 \frac{\Lambda_1}{\mu_1} - S_1 - I_1 = 0, \\
\frac{\Lambda_2}{\mu_2} &= V_1 - \mu_2 V_1 = 0.
\end{align*}
\]

Form (9), we have \(S_1 = \Lambda_1 (1 + a V_1) / b_1 V_1 + \mu_1 (1 + a V_1)\) and \(I_1 = \mu_2 V_1 / (\Lambda_2 / \mu_2 - V_1)\), where \(V_1\) satisfies the following equation:

\[
P_2 V^2 + P_1 V + P_0 = 0,
\]

where

\[
P_2 = \sigma b_1 \left[ \Lambda_1 b_1^2 \beta_1 \beta_2 + \mu_1 \mu_2 (b_1 + a \mu_1) \right], \\
P_1 = b_1^2 \beta_1 \beta_2 \Lambda_1 - \sigma b_1 \Lambda_1 b_1^2 \beta_2 \frac{\Lambda_2}{\mu_2} + \sigma b_1 \mu_2^2 \mu_2 \\
+ \mu_1 \mu_2 (\mu_1 + \gamma) (b_1 + a \mu_1), \\
P_0 = \mu_1 \mu_2 (\mu_1 + \gamma) (1 - R_0), \\
R_0 = \frac{b_2^2 \beta_1 \beta_2 \Lambda_2}{\mu_1 \mu_2^2 (\mu_1 + \gamma)}.
\]
where \( V_1 = (-P_1 + \sqrt{P_1^2 - 4P_0P_2})/2P_2 \) and \( V_2 = (-P_1 - \sqrt{P_1^2 - 4P_0P_2})/2P_2 \) are the two roots of (10) since \( \sigma \in [0, 1] \), and we have

\[
P_1 \geq b^2 \beta_1 \beta_2 \lambda \mu_1 + ab \beta_1 \mu_2 + \mu_1 \mu_2 (\mu_1 + \gamma) (b \beta_1 + a \mu_1) \\
- b \beta_1 \lambda \beta_2 \beta_2 \Lambda_2 \mu_2 + b \beta_1 \beta_2 \lambda \mu_1 + ab \beta_1 \mu_2^2 \\
+ b \beta_1 \mu_2 (\mu_1 + \gamma) (1 - R_0).
\]  

(12)

Obviously, \( P_2 > 0, P_1 > 0 \) and \( P_0 > 1 \) if \( R_0 < 1 \). From the relationship between roots and coefficients, we know that \( V_1 \) and \( V_2 \) are both negative if \( R_0 < 1 \) and \( V_1 \) is positive if \( R_0 > 1 \). According to the above discussion, we can obtain the following theorems.

**Theorem 2.1.** System (7) has the disease-free equilibrium \( E_0 \) if \( R_0 < 1 \). System (7) has the disease-free equilibrium \( E_0 \) and an endemic equilibrium \( E_1 \) if \( R_0 > 1 \).

### 3. Stability of Equilibrium and Hopf Bifurcation

In this section, we study the stability of equilibrium and the existence of Hopf bifurcation of system (7).

The characteristic equation of the linear approximate equation of the system (7) at equilibrium \( E = (S, I, V) \) is

\[
\begin{vmatrix}
\lambda + \frac{b \beta_1 V}{1 + a \nu} + \mu_1 & 0 & \frac{b \beta_1 S}{(1 + a \nu)^2} \\
\nu \beta_1 V - \frac{b \beta_1 V}{1 + a \nu} & \lambda + \nu \beta_1 + \nu + \gamma - \frac{b \beta_1 S}{(1 + a \nu)^2} + Q = 0, \\
0 & -b \beta_1 \left( \frac{\Lambda_2}{\nu_2} - V \right) e^{\lambda t} + \nu + b \beta_2 e^{-\lambda t}
\end{vmatrix}
\]  

(13)

where \( Q = -\nu \beta_1 ((\Lambda_1/\mu_1) - S - I) \).

#### 3.1. The Local and Global Stability of the Disease-Free Equilibrium

At the disease-free equilibrium \( E_0 \), equation (13) can be expressed as follows:

\[
(\lambda + \mu_1) \left( \lambda^2 + (\mu_1 + \gamma + \mu_2) \lambda + \mu_2 (\mu_1 + \gamma) - b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} e^{-\lambda t} \right) = 0.
\]  

(14)

Obviously, equation (14) has a negative real root \( \lambda_1 = -\mu_1 \). To discuss the rest of the characteristic roots of (14), we consider the following equation:

\[
\lambda^2 + (\mu_1 + \gamma + \mu_2) \lambda + \mu_2 (\mu_1 + \gamma) - b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} e^{-\lambda t} = 0.
\]  

(15)

When \( \tau = 0 \), equation (15) is equivalent to

\[
\lambda^2 + (\mu_1 + \gamma + \mu_2) \lambda + \mu_2 (\mu_1 + \gamma) (1 - R_0) = 0.
\]  

(16)

By the using Routh–Hurwitz criterion, (16) has two eigenvalues with negative real parts if \( R_0 < 1 \).

When \( \tau > 0 \), then the roots of (15) can enter the right-half plane in the complex plane by crossing the imaginary axis as the delay \( \tau \) increases.

Let \( \lambda = \omega (\omega > 0) \) be a purely imaginary root of equation (15), then separating the real and imaginary parts yields

\[
\begin{align*}
-\omega^2 + \mu_2 (\mu_1 + \gamma) &= b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} \cos \omega \tau, \\
(\mu_2 + \mu_1 + \gamma) \omega &= -b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} \sin \omega \tau.
\end{align*}
\]  

(17)

Squaring and taking the sum of (17) yields

\[
\omega^4 + (\mu_1 + \gamma)^2 \omega^2 + \mu_2 (\mu_1 + \gamma) (1 - R_0) \times \left( \mu_2 (\mu_1 + \gamma) + b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} \right) = 0.
\]  

(18)

Equation (18) has no roots if \( R_0 < 1 \). Therefore, we conclude that all eigenvalues of equation (14) have negative real parts.

If \( R_0 > 1 \), let

\[
f(\lambda) = \lambda^2 + (\mu_1 + \gamma + \mu_2) \lambda + \mu_2 (\mu_1 + \gamma) - b^2 \beta_1 \beta_2 \frac{\Lambda_1 \Lambda_2}{\mu_1 \mu_2} e^{-\lambda t},
\]  

which implies

\[
f(0) = \mu_2 (\mu_1 + \gamma) (1 - R_0) < 0,
\]

\[
\lim_{\lambda \to \infty} f(\lambda) = +\infty.
\]  

(20)

By the continuity of \( f(\lambda) \) and zero point theorem, \( f(\lambda) = 0 \) has at least one positive root. So, the disease-free equilibrium \( E_0 \) is unstable. Based on the results, we can draw the conclusion.

**Theorem 3.1.** For any \( \tau \), the virus-free equilibrium \( E_0 \) of the system (7) is locally asymptotically stable if \( R_0 < 1 \), and it is unstable if \( R_0 > 1 \).

In fact, using a similar approach to the literature [14], we can know that \( E_0 \) is globally asymptotically stable if \( R_0 < 1 \). A detailed proof is given below.

For a continuous and bounded function \( f(t) \), we define

\[
\begin{align*}
f^{\infty} &\triangleq \lim_{t \to -\infty} \sup f(t), \\
f_{\infty} &\triangleq \lim_{t \to -\infty} \inf f(t).
\end{align*}
\]  

(21)

For system (7), any solution with the initial conditions is \( (S(t), I(t), V(t)) \), and we have

\[
0 \leq S_0 \leq S^\infty \leq \infty, \\
0 \leq I_0 \leq I^\infty \leq \infty, \\
0 \leq V_0 \leq V^\infty \leq \infty.
\]  

(22)
By the fluctuation lemma [15], we know that there is a sequence \( \{t_n\} \); when \( t_n \to \infty \), we have \( S(t_n) \to S^\infty \) and \( S'(t_n) \to 0 \) for \( n \to \infty \). Substituting \( t_n \) into the first equation of (7) yields

\[
S'(t_n) = \Lambda_1 - \frac{b_1 \beta_1 S(t_n) V(t_n)}{1 + a V(t_n)} - \mu_1 S(t_n) \leq \Lambda_1 - \mu_1 S(t_n). \tag{23}
\]

Let us take the limits on both sides:

\[
\lim_{n \to \infty} S'(t_n) \leq \Lambda_1 - \mu_1 \lim_{n \to \infty} S(t_n), \quad \text{i.e.,} \quad \mu_1 S^\infty \leq \Lambda_1. \tag{24}
\]

Similarly,

\[
(\mu_1 + \gamma) V^\infty \leq (1 - \sigma) b_1 \beta_1 S^\infty V^\infty + \sigma b_1 \Lambda_1 V^\infty, \tag{25}
\]

Combining (24) and (25), we know that

\[
\mu_1 V^\infty \leq b_2 \beta_2 \Lambda_2 \frac{\Lambda_1 A_2}{\mu_1} (\mu_1 + \gamma) V^\infty. \tag{26}
\]

Since \( V^\infty \) is the supremum of the function \( V(t) \), \( V^\infty \geq 0 \). If \( V^\infty > 0 \), by using (26), \( \mu_1 \leq b_2 \beta_2 \Lambda_2 (\mu_1 \mu_2 (\mu_1 + \gamma)) \), which contradicts \( R_0 < 1 \). That is to say, \( V^\infty = 0 \), which implies that \( \lim_{t \to \infty} V(t) = 0 \). In the same way by using (25), we have \( \lim_{t \to \infty} I(t) = 0 \). According to the limit theorem [13], we have \( \lim_{t \to \infty} S(t) = \Lambda_1 / \mu_1 \). Combined with the local asymptotic stability of \( E_0 \), we can get the following theorem:

**Theorem 3.2.** For any \( \tau \), the virus-free equilibrium \( E_0 \) of system (7) is globally asymptotically stable if \( R_0 < 1 \).

### 3.2. The Local Stability of the Endemic Equilibrium

From (13), the characteristic equation of linear approximate equation of the system (7) at the endemic equilibrium \( E_1 \) is

\[
\lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 + (B_2 \lambda^2 + B_1 \lambda + B_0) e^{-\lambda \tau} = 0, \tag{27}
\]

where

\[
A_2 = \frac{b_1 \beta_1 V_1}{1 + a V_1} + 2 \mu_1 + \sigma b_1 \beta_1 V_1 + \gamma + \mu_2,
\]

\[
A_1 = \frac{b_1 \beta_1 V_1}{1 + a V_1} + \mu_1 (b_1 \beta_1 V_1 + \mu_1 + \gamma + \mu_2) + (b_1 \beta_1 V_1 + \mu_1 + \gamma) \mu_2,
\]

\[
A_0 = \frac{b_1 \beta_1 V_1}{1 + a V_1} + \mu_1 (b_1 \beta_1 V_1 + \mu_1 + \gamma) \mu_2,
\]

\[
B_2 = b_2 \beta_2 I_1,
\]

\[
B_1 = b_2 \beta_2 I_1 \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} + \sigma b_1 \beta_1 V_1 + 2 \mu_1 + \gamma \right) - \frac{b_2 \beta_1 S_1}{1 + a V_1} \left( \frac{1}{1 + a V_1} - 1 \right) \frac{V_1}{I_1} - \mu_2 (\mu_1 + \gamma),
\]

\[
B_0 = b_2 \beta_2 I_1 \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} + \mu_1 \right) (b_1 \beta_1 V_1 + \mu_1 + \gamma) - \frac{b_2 \beta_1 S_1}{1 + a V_1} \left( \frac{1}{1 + a V_1} - 1 \right) \frac{V_1}{I_1} \times \mu_2 (\mu_1 + \gamma) + \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} - \sigma b_1 \beta_1 V_1 \right) \frac{b_1 \beta_1 S_1}{1 + a V_1} \frac{V_1}{I_1} \frac{\mu_2 V_1}{I_1}.
\]

When \( \tau = 0 \), equation (27) is equivalent to

\[
\lambda^3 + H_2 \lambda^2 + H_1 \lambda + H_0 = 0, \tag{29}
\]

where

\[
H_1 = A_1 + B_1 = \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} + \mu_1 \right) \left( b_1 \beta_1 V_1 + \mu_1 + \gamma + \mu_2 \right) + \sigma b_1 \beta_1 V_1 \mu_2 + b_2 \beta_2 I_1 \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} + \sigma b_1 \beta_1 V_1 + 2 \mu_1 + \gamma \right)
\]

\[
+ \frac{b_1 \beta_1 S_1}{1 + a V_1} \left( 1 - \frac{1}{1 + a V_1} \right) \frac{V_1}{I_1} > 0,
\]

\[
H_0 = A_0 + B_0 = \left( \frac{b_1 \beta_1 V_1}{1 + a V_1} + \mu_1 \right) \left( \left( b_1 \beta_1 V_1 + \mu_1 + \gamma \right) b_2 \beta_2 I_1 + \sigma b_1 \beta_1 V_1 \mu_2 \right) + \left( 1 - \sigma b_1 \beta_1 V_1 \right) \frac{b_1 \beta_1 S_1}{1 + a V_1} \frac{V_1}{I_1} \left( 1 + a V_1 \right)^2 \frac{\mu_2 V_1}{I_1}.
\]
Notice that
\[
H_2 H_1 - H_0 = \left( \frac{b \beta_1 V_1}{1 + aV_1} + b \beta_2 I_1 + 2 \mu_1 + \mu_2 + a b \beta_1 V_1 + \gamma \right) \left( \frac{b \beta_1 V_1}{1 + aV_1} + \mu_1 \right) \times \left( a b \beta_1 V_1 + \mu_1 + \gamma + \mu_2 \right) + o b \beta_1 V_1 \mu_2
\]
\[+ \beta_2 I_1 \left( \frac{b \beta_1 V_1}{1 + aV_1} + a b \beta_1 V_1 + 2 \mu_1 + \gamma \right) + \frac{b \beta_1 S_1}{1 + aV_1} \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[- \left( \frac{b \beta_1 V_1}{1 + aV_1} + \mu_1 \right) \left[ \left( a b \beta_1 V_1 + \mu_1 + \gamma \right) \right] \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[- \frac{b \beta_1 S_1}{1 + aV_1} \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[\times \left( a b \beta_1 V_1 + \mu_1 + \gamma \right) + \frac{b \beta_1 S_1}{1 + aV_1} \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[\times \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[\left[ \left( a b \beta_1 V_1 + \mu_1 + \gamma \right) \right] \left( 1 - \frac{1}{1 + aV_1} \right) \mu_2 V_1 I_1 \]
\[\geq \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 V_1 I_1 \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 V_1 I_1 \]
\[\geq \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \mu_2 \]
\[\frac{V_1}{I_1} = \frac{\mu_1 + \gamma}{(b \beta_1 S_1 / (1 + aV_1) + a b \beta_1 ((\mu_1 / \mu_1) - S_1 - I_1)} \leq \frac{\mu_1 + \gamma}{(b \beta_1 S_1 / (1 + aV_1))} = \frac{(\mu_1 + \gamma) (1 + aV_1)}{b \beta_1 S_1} \]
(31)

It follows that
\[
H_2 H_1 - H_0 \geq \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \frac{b \beta_1 S_1}{1 + aV_1} \]
\[\cdot \left( 1 + aV_1 \right) \mu_2 \left( \mu_1 + \gamma \right) \frac{b \beta_1 S_1}{1 + aV_1} \]
\[= \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \mu_2 \]
\[= \frac{b \beta_1 V_1}{1 + aV_1} \mu_2 \left( \mu_1 + \gamma \right) \left( 1 - \frac{1}{1 + aV_1} \right) > 0. \]
(33)

By using the Routh–Hurwitz criterion, equation (29) only has eigenvalues with negative real parts if \( R_0 > 1 \). We can obtain the following theorem:

**Theorem 3.3.** For \( \tau = 0 \), the endemic equilibrium \( E_1 \) of system (7) is locally asymptotically stable if \( R_0 > 1 \).

### 3.3. Hopf Bifurcation
In this subsection, we devote to investigating the stability of the endemic equilibrium and the existence of Hopf bifurcation.

Let \( \lambda = \omega i (\omega > 0) \) be the root of equation (27), substituting it into equation (27) and separating the real and imaginary parts; we can obtain the following equation:

\[
\begin{cases}
\omega^3 - A_1 \omega = (B_0 - B_2 \omega^2) \cos \omega t - B_1 \omega \sin \omega t, \\
\omega^3 - A_1 \omega = B_1 \omega \cos \omega t - (B_0 - B_2 \omega^2) \sin \omega t.
\end{cases}
\]
(34)

Squaring and taking the sum of (34) yields

\[
\omega^6 + p_3 \omega^4 + p_1 \omega^2 + p_0 = 0,
\]
(35)

where

\[
p_2 = A_2^2 - 2A_1 - B_2^2,
\]
\[
p_1 = A_1^2 - 2A_0 A_2 + 2B_0 B_2 - B_1^2,
\]
\[
p_0 = A_0^2 - B_0^2.
\]

Let \( x = \omega^2 \), then equation (35) is equivalent to

\[
f(x) = 3x^3 + 2p_2 x^2 + p_1 x + p_0 = 0,
\]
(37)

then \( f'(x) = 3x^2 + 2p_2 x + p_1 \). The two roots of equation \( 3x^2 + 2p_2 x + p_1 = 0 \) are
Lemma 3.4. For equation (37),

(i) If \( p_0 < 0 \), then equation (37) has at least one positive root.

(ii) If \( p_0 \geq 0 \) and \( p_2^2 \leq 3p_1 \), then equation (37) has no positive root.

\[
x^* = \frac{-p_2 + \sqrt{p_2^2 - 3p_1}}{3}
\]
\[
x^{**} = \frac{-p_2 - \sqrt{p_2^2 - 3p_1}}{3}
\]  

(38)

According to [16], the condition that equation (37) has positive roots is as follows:

\[
\lambda(\tau) = \xi_0(\tau) + i\omega(\tau)
\]

is the root of equation (35) satisfying \( \xi_0(\tau^*) = 0 \) and \( \omega(\tau^*) = \omega^* \).

Define

\[
t_k^{(i)} = \frac{1}{\omega_k} \left\{ \arccos \left( \frac{(B_0 - B_2\omega_k^2)(A_2\omega_k^2 - A_0) + B_1\omega_k(\omega_k^3 - A_1\omega)}{(B_0 - B_2\omega_k^2)^2 + (B_1\omega_k)^2} \right) + 2j\pi \right\},
\]  

(40)

where \( k = 1, 2, 3, \ j = 0, 1, \ldots \). Obviously, \( \pm \omega_k i \) is a pair of pure virtual root of equation (27).

Let

\[
\tau^* = t_k^{(0)} = \min_{k \in \{1, 2, 3\}} \{ t_k^{(i)} \},
\]  

\[
\omega^* = \omega_k.
\]

It follows that \( \lambda(\tau) = \xi_0(\tau) + i\omega(\tau) \) is the root of equation (35) satisfying \( \xi_0(\tau^*) = 0 \) and \( \omega(\tau^*) = \omega^* \).

Next, we verify the transversal condition. Differentiating the two sides of equation (35) with respect to \( \tau \), we have

\[
\frac{d\lambda}{d\tau} = \frac{-2B\lambda + A_1}{\lambda(\lambda^2 + A_2\lambda^2 + A_1\lambda + A_0)}
\]

(42)

then

\[
\text{Re} \left[ \left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau^*} \right] = \frac{3\omega^* + 2(A_2 - 2A_1 - B_2^3)\omega^* + A_1^2 - 2A_0A_2 + 2B_0B_2 - B_1^2}{(B_1\omega^*)^2 + (B_0 - B_2\omega^*^2)^2}
\]

(43)

\[
= \frac{f'(\omega^*)}{(B_1\omega^*)^2 + (B_0 - B_2\omega^*^2)^2}.
\]

Thus,

\[
\text{sign} \left( \left. \frac{d\text{Re}(\lambda)}{d\tau} \right|_{\tau=\tau^*} \right) = \text{sign} \left( \left. \text{Re} \left[ \left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau^*} \right] \right\} \right)
\]

(44)

\[
= \text{sign} \left( f'(\omega^*) \right).
\]

If \( f'(\omega^*) > 0 \), the transversal condition is satisfied. Therefore, according to the above discussion and the Hopf bifurcation theorem of the differential equations [12], we can get the following result:

(iii) If \( p_0 \geq 0 \) and \( p_2^2 > 3p_1 \), then equation (37) has positive roots if and only if \( x^* > 0 \) and \( f(x^*) \leq 0 \).

Based on Lemma 3.4, we concluded that if (ii) is set up, then the stability of \( E_1 \) will not change when \( \tau \) increases. If equation (37) has a positive root, then the stability of \( E_1 \) may change with the change in \( \tau \).

Suppose that equation (37) has three positive roots, written as \( x_1, x_2, \) and \( x_3 \). Then, equation (35) has positive roots \( \omega_k = \sqrt{x_k} \ (k = 1, 2, 3) \). By using (34),

\[
\cos \omega \tau = \frac{(B_0 - B_2\omega^2)(A_2\omega^2 - A_0) + B_1\omega(\omega^3 - A_1\omega)}{(B_0 - B_2\omega^2)^2 + (B_1\omega)^2}
\]

(39)

\[
\text{Re} \left[ \left. \frac{d\lambda}{d\tau} \right|_{\tau=\tau^*} \right] = \frac{3\omega^* + 2(A_2 - 2A_1 - B_2^3)\omega^* + A_1^2 - 2A_0A_2 + 2B_0B_2 - B_1^2}{(B_1\omega^*)^2 + (B_0 - B_2\omega^*^2)^2}
\]

\[
= \frac{f'(\omega^*)}{(B_1\omega^*)^2 + (B_0 - B_2\omega^*^2)^2}.
\]

Theorem 3.5.

(i) If \( p_0 \geq 0 \) and \( \Delta = p_2^3 - 3p_1 \leq 0 \), then the endemic equilibrium \( E_1 \) is locally asymptotically stable for all \( \tau > 0 \).

(ii) If \( p_0 < 0 \) or \( p_0 \geq 0, \Delta > 0, x^* > 0 \), and \( f(x^*) \leq 0 \), then the endemic equilibrium \( E_1 \) is locally asymptotically stable for \( 0 < \tau < \tau^* \), and if \( f'(\omega^*) \neq 0 \), the system (7) undergoes a Hopf bifurcation at \( E_1 \) when \( \tau = \tau^* \), where
\[ \tau^* = \frac{1}{\omega^*} \left\{ \arccos \left( \frac{(B_0 - B_2 \omega^*) (A_2 \omega^* - A_0) + B_1 \omega^* (\omega^2 - A_1 \omega^*)}{(B_0 - B_2 \omega^*)^2 + (B_1 \omega^*)^2} \right) \right\}. \] (45)

4. Estimation of the Length of Delay to Preserve Stability

In this section, we use a Nyquist criterion [17] to calculate the length of delay to preserve stability.

Consider the system (7) and the space of the real continuous functions that is defined in \([-\tau, +\infty]\) and satisfied the initial conditions (2) in the interval \([-\tau, 0]\). Define

\[ S(t) = S_1 + X(t), \]
\[ I(t) = I_1 + Y(t), \]
\[ V(t) = V_1 + Z(t). \] (46)

Linearization system (7) at the endemic equilibrium \(E_1\) is expressed as follows:

\[
\begin{align*}
SL[X] - X(0) &= -\left( \frac{b\beta_1 V_1}{1 + aV_1} + \mu_1 \right) L[X] - \frac{b\beta_1 S_1}{1 + aV_1} L[Z], \\
SL[Y] - Y(0) &= \left( \frac{b\beta_1 V_1}{1 + aV_1} - ab\beta_1 V_1 \right) L[X] - (ab\beta_1 V_1 + \mu_1 + \gamma) L[Y] + \left[ \frac{b\beta_1 S_1}{1 + aV_1} + ab\beta_1 \left( \frac{A_1}{\mu_1} - S_1 - I_1 \right) \right] L[Z], \\
SL[Z] - Z(0) &= -\mu_2 L[Z] + b\beta_2 \left( \frac{A_2}{\mu_2} - V_1 \right) L[Y] - b\beta_2 I_1 L[Z],
\end{align*}
\]

where

\[
L[Y_{\tau}] = \int_{0}^{\infty} e^{-\omega(t-\tau)} Y(t-\tau)dt = \int_{0}^{\tau} e^{-\omega(t-\tau)} Y(t-\tau)dt + \int_{\tau}^{\infty} e^{-\omega(t-\tau)} Y(t-\tau)dt.
\] (49)

Let \( \tau = t_1 + \tau \), then

\[
L[Y_{\tau}] = \int_{-\tau}^{0} e^{-\omega(t_1+\tau)} Y(t_1)dt_1 + \int_{0}^{\infty} e^{-\omega(t_1+\tau)} Y(t_1)dt_1 = M_1 e^{-\omega \tau} + e^{-\omega \tau} L[Y],
\] (50)

where \( M_1 = \int_{-\tau}^{0} e^{-\omega t} Y(t)dt. \)

Similarly,

\[
L[Y_{\tau}] = \int_{-\tau}^{0} e^{-\omega(t_1+\tau)} Y(t_1)dt_1 + \int_{0}^{\infty} e^{-\omega(t_1+\tau)} Y(t_1)dt_1 = M_2 e^{-\omega \tau} + e^{-\omega \tau} L[Y],
\] (51)

where \( M_2 = \int_{-\tau}^{0} e^{-\omega t} Z(t)dt. \)

Thus, (48) can be written as

\[
(A - SI) \begin{pmatrix} L[X] \\ L[Y] \\ L[Z] \end{pmatrix} = B,
\] (52)

where
The inverse Laplace transformation of \( L[X(t)], L[Y(t)], \) and \( L[Z(t)] \) will have terms which exponentially increase with time if \( L[X(t)], L[Y(t)], \) and \( L[Z(t)] \) have poles with positive real parts. Thus, \( E_1 \) is locally asymptotically stable if and only if all the poles of \( L[X(t)], L[Y(t)], \) and \( L[Z(t)] \) have negative real parts.

By the method of [17] and the Nyquist criterion, the local asymptotic stability of \( E_1 \) needs to satisfy the following two conditions:

\[
\begin{align*}
\text{Re } F(i\mu_0) &= 0, \\
\text{Im } F(i\mu_0) &> 0,
\end{align*}
\]

where

\[
F(s) = s^3 + A_1 s^2 + A_2 s + A_0 + (B_2 s^2 + B_1 s + B_0) e^{-\tau s},
\]

\[
A_0 = \left( B_2 - B_0 \right) \cos \mu_0 \tau + B_1 \sin \mu_0 \tau,
\]

\[
-\mu_0^3 + A_1 \mu_0 > \left( B_0 - B_2 \right) \sin \mu_0 \tau - B_1 \mu_0 \cos \mu_0 \tau.
\]

In order to estimate the length of delay to preserve stability, under the premise of ensuring stability, the following conditions need to be satisfied:

\[
A_2 \mu^2 - A_0 = \left( B_0 - B_2 \mu^2 \right) \cos \mu \tau + B_1 \mu \sin \mu \tau,
\]

\[
-\mu^3 + A_1 \mu > \left( B_0 - B_2 \mu^2 \right) \sin \mu \tau - B_1 \mu \cos \mu \tau.
\]

If (58) and (59) are satisfied simultaneously, they are sufficient conditions to guarantee stability. Our aim is to find an upper bound \( \mu \), to \( \mu_0 \), independent of \( \tau \) and then to estimate \( \tau \) so that (59) holds true for all values of \( 0 \leq \mu \leq \mu_+ \) and in particular at \( \mu = \mu_0 \).

Since \( |\cos \mu \tau| \leq 1 \) and \( |\sin \mu \tau| \leq 1 \), from equation (58), we have

\[
A_2 \mu^2 \leq |B_0| + B_2 \mu^2 + |B_1| \mu + A_0,
\]

and

\[
A_1 \mu \geq \left( |B_0 - B_2 \mu^2| \right) \sin \mu \tau - B_1 \mu \cos \mu \tau.
\]

Let

\[
\mu_+ = \frac{|B_1| + \sqrt{B_1^2 + 4(A_2 - B_2)(A_0 + B_0)}}{2(A_2 - B_2)},
\]

obviously \( \mu_+ \) meets (60) and \( \mu_+ \geq \mu_0 \).

From equation (59), we obtain

\[
\mu^2 < B_1 \cos \mu \tau + \left( B_2 \mu - \frac{B_0}{\mu} \right) \sin \mu \tau + A_1.
\]

Since \( E_1 \) is locally asymptotically stable for \( \tau = 0 \), the inequality (62) will continue to hold for sufficiently small \( \tau \) and \( \mu = \mu_0 \).

On the basis of (58) and (62), we have

\[
B_1 \mu - A_2 \mu + A_1 |B_0| \sin \mu \tau + \left( B_2 \mu^2 - A_2 B_1 - B_0 \right)
\]

\[
\cdot (1 - \cos \mu \tau) < B_2 \mu^2 + A_1 A_2 + A_1 B_1 - A_0 - B_0.
\]

Note the left-hand side of (63) is \( \Phi(\tau, \mu) \) and the right-hand side is \( \rho \). By using inequality \( \sin \mu \tau \leq \mu \tau \) and \( 1 - \cos \mu \tau \leq 2 \sin^2 (\mu \tau/2) \), we can obtain

\[
\Phi(\tau, \mu) \leq |\Phi(\tau, \mu)| \leq \left( |B_1 - A_2 B_2| + \frac{|A_1| B_0}{\mu} \right) \mu \tau
\]

\[
+ |B_2 \mu^2 + A_2 B_1 - B_0| \frac{\mu^2 \tau^2}{2}
\]

\[
= \left( |B_1 - A_2 B_2| \mu^2 + |A_1 B_0| \right) \tau
\]

\[
+ |B_2 \mu^2 + A_2 B_1 - B_0| \frac{\mu^2 \tau^2}{2}.
\]

Note the right-hand side of (64) is \( \varphi(\tau, \mu) \). Clearly, \( \Phi(\tau, \mu) \leq \varphi(\tau, \mu) \leq \varphi(\tau, \mu_+) \) when \( \mu \in [0, \mu_+] \). Thus, if \( \varphi(\tau, \mu_+) \leq \rho \leq K_5 \), we have \( \Phi(\tau, \mu) \leq \rho \leq K_5 \). Let \( \tau^* \) be the positive root of \( \varphi(\tau, \mu_+) = K_5 \), that is,

\[
\tau^* = \frac{1}{2K_1} \left( -K_2 + \sqrt{K_2^2 + 4K_1 K_3} \right).
\]
where

\[ K_1 = \frac{|B_2 \mu^2 + A_2 B_1 - B_0| \mu^2}{2}, \]

\[ K_2 = |B_1 - A_1 B_0| \mu^2 + |A_2 B_0|, \]

\[ K_3 = B_2 \mu^2 + A_1 A_2 + A_1 B_1 - A_0 - B_0. \]

Summarizing the above discussions, we have the following theorem.

**Theorem 4.1.** If \( 0 < \tau < \tau^* \), then the Nyquist criterion holds true and \( \tau^* \) estimates the maximum length of the delay preserving the stability, where \( \tau^* \) satisfies (65).

### 5. Direction and Stability of the Hopf Bifurcation

We have obtained the conditions under which the Hopf bifurcation occurs at \( E_1 \) of the system (7). This section will use the normal form theory and the center manifold theory to give the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions of system (7). We suppose that system (7) undergoes Hopf bifurcation at \( E_1 \) for \( \tau = \bar{\tau}(\tau = \tau_k^{(j)}) \). Let \( \pm i \omega \) be a pair of conjugate pure virtual roots at \( E_1 \) when \( \tau = \bar{\tau} \).

Define

\[ \begin{align*}
  x_1(t) &= S(\tau t) - S_1, \\
  x_2(t) &= I(\tau t) - I_1, \\
  x_3(t) &= V(\tau t) - V_1,
\end{align*} \]

\[ \tau = \bar{\tau} + \mu. \]

Thus, system (7) is equivalent to the following functional differential equation in \( C = C([-\tau, 0], R^3) \).

\[ \frac{dx}{dt} = L_\mu(x_t) + f(\mu, x_t), \]

where \( x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3 \). And \( L_\mu : C \longrightarrow R^3 \) and \( f : \mathbb{R} \times C \longrightarrow \mathbb{R}^3 \) satisfy

\[ L_\mu(\phi) = (\bar{\tau} + \mu)B_1\phi(0) + (\bar{\tau} + \mu)B_2\phi(-1), \]

\[ f(\mu, \phi) = (\bar{\tau} + \mu)(f_1, f_2, f_3)^T, \]

where

\[ \phi(\theta) = (\phi_1(\theta) \phi_2(\theta) \phi_3(\theta))^T \in C, \]

\[ f_1 = -\frac{b_2}{(1 + aV_1)} \phi_1(0) \phi_3(0) + \frac{ab}{1 + aV_1} \phi_1(0) \phi_3^2(0) + \frac{ab}{1 + aV_1} \phi_3^3(0), \]

\[ f_2 = -\frac{ab}{1 + aV_1} \phi_1(0) \phi_3(0) - \frac{ab}{1 + aV_1} \phi_1(0) \phi_3^2(0), \]

\[ f_3 = -\frac{ab}{1 + aV_1} \phi_3^2(0) + \frac{ab}{1 + aV_1} \phi_1(0) \phi_3^3(0) + \frac{ab}{1 + aV_1} \phi_3^3(0), \]

\[ f_4 = -\frac{ab}{1 + aV_1} \phi_3^2(0), \]

\[ f_5 = -\frac{ab}{1 + aV_1} \phi_3^2(0), \]

\[ f_6 = -\frac{ab}{1 + aV_1} \phi_3^2(0). \]
Applying the Riesz representation theorem, there exists a $3 \times 3$ matrix-valued function $\eta(\cdot, \mu): [-1, 0] \to \mathbb{R}^{3 \times 3}$, such that $L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta)$, $\phi \in \mathcal{C}$. We choose

$$\eta(\theta, \mu) = (\bar{\tau} + \mu) B_\delta (\bar{\theta} (\theta + 1))$$

where $\delta$ is the Dirac delta function, meeting $\delta(\theta) = 0 (\theta \neq 0)$ and $\int_{-\infty}^{\infty} \delta(\theta) d\theta = 1$.

We define for $\phi \in \mathcal{C}([-1, 0], \mathbb{R})$,

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(s, \mu) \phi(s), & \theta = 0, \end{cases}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Thus, (68) becomes

$$\dot{x}_i = A(\mu)x_i + R(\mu)x_i,$$

where $x_i = x(t + \theta)$, $\theta \in [-1, 0]$.

In order to construct coordinates to describe the integral manifold near the origin, we need to define inner product and the adjoint operator $A^* = A^*(0)$ of $A$ as follows:

$$A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta^T(t, 0) \psi(-t), & s = 0, \end{cases}$$

$$\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

where $\psi \in \mathcal{C}(0, 1, \mathbb{R})$ and $\eta(\theta) = \eta(\theta, 0)$. Form the discussion in Section 3, we know that $\pm i\omega \tau$ are eigenvalues of $A(0)$. Thus they are also eigenvalues of $A^*$. Define $q(\theta) = (q_1, q_2)^T e^{i\omega \tau}$ and $q^* (s) = D(1, q_1, q_2)^T e^{-i\omega \tau}$ to be the eigenvectors of $A(0)$ and $A^*$ corresponding to the eigenvalues $i\omega \tau$ and $-i\omega \tau$, then

$$A(0)q(\theta) = i\omega \tau q(\theta),$$

$$A^* q^*(s) = -i\omega \tau q^*(s).$$

We can calculate that

$$q_1 = -\frac{I_s(\mu_2 + \beta \beta_1 e^{-i\omega \tau} + \omega)}{b \beta_1 \beta_1 V_1 (1 + aV_1) + (1 + aV_1)^2 \omega},$$

$$q_2 = (1 + aV_1) \frac{V_1}{S_1} + (1 + aV_1)^2 \omega, \frac{1}{b \beta_1 S_1 V_1 e^{-i\omega \tau}},$$

$$q_1^* = \frac{b \beta_1 V_1 + (\mu_1 - \omega)}{b \beta_1 V_1 - \omega} \frac{V_1}{(1 + aV_1)},$$

$$q_2^* = I_s(\mu_1 + \beta \beta_1 e^{-i\omega \tau} + \omega) \frac{b \beta_1 V_1 (1 + aV_1)}{\mu_2 V_1 e^{i\omega \tau} [b \beta_1 V_1 - \omega] (1 + aV_1)}$$

According to (75), we know that

$$\langle q^*(s), q(\theta) \rangle = D(1, q_1^*, q_2^*) (1, q_1, q_2)^T$$

$$- \int_{-1}^{0} \int_{\xi=0}^{\theta} D(1, q_1^*, q_2^*) e^{-i(\xi - \theta) \omega \tau} d\eta(\theta) (1, q_1, q_2)^T e^{i\omega \tau} d\xi$$

$$= D\left[ 1 + q_1^* q_1 + q_2^* q_2 - \int_{-1}^{0} (1, q_1^*, q_2^*) e^{i\omega \tau} d\eta(\theta) (1, q_1, q_2)^T \right]$$

$$= D\left[ 1 + q_1^* q_1 + q_2^* q_2 + \tau q_1 q_2 b \beta_1 \left( \frac{\Lambda_2}{\mu_2} - V_1 \right) e^{-i\omega \tau} - \tau q_1 q_2^* b \beta_1 e^{-i\omega \tau} \right]$$

$$= 1,$$
Thus

\[
D = \frac{1}{1 + q_1^* \eta_1 + q_2^* \eta_2 + \tau e^{i\omega t} \left( \tilde{q}_1 q_2^* b_2^* \mu_2 V_l^1 \eta_1 - \tilde{q}_2 q_1^* b_1^* \eta_1 \right)}.
\]

(79)

Next using the same notation as in [11] and a computation process similar to that in [18], we compute the center manifold \( C_0 \) at \( \mu = 0 \).

Let \( x_i \) be the solution of (74).

Define

\[
z(t) = \langle q^*, x_i \rangle,
\]

\[
W(t, \theta) = x_i(\theta) - 2\text{Re}[z(t)q(\theta)].
\]

On the center manifold \( C_0 \), we have

\[
W(t, \theta) = W(z(t), \bar{z}(t), \theta),
\]

(81)

where

\[
W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.
\]

(82)

For the solution \( x_i \in C_0 \) with \( \mu = 0 \), we can get

\[
z'(t) = i\omega \bar{z} + \langle q^*, (0, W(z, \bar{z}, \theta) + 2\text{Re}[zq(\theta)]) \rangle
\]

\[
= i\omega \bar{z} + q^*(0) f(0, W(z, \bar{z}, \theta) + 2\text{Re}[zq(0)])
\]

\[
= i\omega \bar{z} + q^*(0) f_0(z, \bar{z})
\]

\[
\equiv i\omega \bar{z} + g(z, \bar{z}),
\]

(83)

where

\[
g(z, \bar{z}) = q^*(0) f_0(z, \bar{z}) = q^*(0) f(0, x_i)
\]

\[
= \tau \mathcal{D} \left\{ 2(k_1 + \bar{q}_1^* k_2 + \bar{q}_2^* k_3) \frac{z^2}{2} + (l_1 + \bar{q}_1^* l_2 + \bar{q}_2^* l_3)z\bar{z} + 2(m_1 + \bar{q}_1^* m_2 + \bar{q}_2^* m_3)\frac{\bar{z}^2}{2} + 2(n_1 + \bar{q}_1^* n_2 + \bar{q}_2^* n_3)\frac{z\bar{z}^2}{2} \right\}.
\]

(86)

It follows

\[
g_{20} = 2\tau \mathcal{D}(k_1 + \bar{q}_1^* k_2 + \bar{q}_2^* k_3),
\]

\[
g_{11} = \tau \mathcal{D}(l_1 + \bar{q}_1^* l_2 + \bar{q}_2^* l_3),
\]

\[
g_{02} = 2\tau \mathcal{D}(m_1 + \bar{q}_1^* m_2 + \bar{q}_2^* m_3),
\]

\[
g_{21} = 2\tau \mathcal{D}(n_1 + \bar{q}_1^* n_2 + \bar{q}_2^* n_3),
\]

(87)
where

\[ k_1 = -\frac{b_3 q_2}{(1 + a V_1)^2} + \frac{a b \beta_1 S_1}{(1 + a V_1)^2} q_2^2, \]

\[ k_2 = -a b \beta_1 q_2 (1 + q_1) + \frac{b_3 q_2}{(1 + a V_1)^2} - \frac{a b \beta_1 S_1}{(1 + a V_1)^2} q_2^2, \]

\[ k_3 = -b_3 q_1 q_2 e^{-2i\omega\tau}, \]

\[ l_1 = -\frac{b_3}{(1 + a V_1)^2} (\bar{q}_2 + q_2) + \frac{2a b \beta_1 S_1}{(1 + a V_1)^2} q_2 \bar{q}_2, \]

\[ l_2 = -a b \beta_1 (\bar{q}_2 + q_2 + q_1 \bar{q}_2 + \bar{q}_1 q_2) + \frac{b_3}{(1 + a V_1)^2} (\bar{q}_2 + q_2) - \frac{2a b \beta_1 S_1}{(1 + a V_1)^2} q_2 \bar{q}_2, \]

\[ l_3 = -b_3 (q_1 \bar{q}_2 + q_1 q_2), \]

\[ m_1 = -\frac{b_3}{(1 + a V_1)^2} + \frac{a b \beta_1 S_1}{(1 + a V_1)^2} \bar{q}_2, \]

\[ m_2 = -a b \beta_1 \bar{q}_2 (1 + \bar{q}_1) + \frac{b_3}{(1 + a V_1)^2} - \frac{a b \beta_1 S_1}{(1 + a V_1)^2} \bar{q}_2, \]

\[ m_3 = -b_3 \bar{q}_2 \beta_1 e^{2i\omega\tau}, \]

\[ n_1 = -\frac{b_3}{(1 + a V_1)^2} \left( \frac{W^{(1)}(0)}{2} \bar{q}_2 + \frac{W^{(3)}(0)}{2} + q_2 W^{(1)}(0) + W^{(3)}(0) + \frac{W^{(2)}(0)}{2} \bar{q}_2 + \frac{W^{(3)}(0)}{2} \bar{q}_1 + W^{(2)}(0)q_2 \right) + \frac{a b \beta_1 S_1}{(1 + a V_1)^2} (q_2 + 2q_2 \bar{q}_2) + \frac{a b \beta_1 S_1}{(1 + a V_1)^2} \bar{q}_2^2, \]

\[ n_2 = -a b \beta_1 \left( \frac{W^{(1)}(0)}{2} \bar{q}_2 + \frac{W^{(3)}(0)}{2} + q_2 W^{(1)}(0) + W^{(3)}(0) + \frac{W^{(2)}(0)}{2} \bar{q}_2 + \frac{W^{(3)}(0)}{2} \bar{q}_1 + W^{(2)}(0)q_2 \right) + \frac{b_3}{(1 + a V_1)^2} \left( W^{(1)}(0) \bar{q}_2 + \frac{W^{(3)}(0)}{2} \bar{q}_2 + \frac{W^{(3)}(0)}{2} + q_2 W^{(1)}(0) + W^{(3)}(0) - \frac{a b \beta_1 S_1}{(1 + a V_1)^2} (q_2 + 2q_2 \bar{q}_2) \right) + \frac{3a^2 b \beta_1 S_1}{(1 + a V_1)^2} q_2^2 \bar{q}_2, \]

\[ n_3 = -b_3 \left( \frac{W^{(2)}(0)}{2} (q_2 e^{i\omega\tau} + \frac{W^{(3)}(0)}{2} q_2 e^{i\omega\tau} + W^{(2)}(0)(-1)q_2 e^{-i\omega\tau} + W^{(3)}(0)(-1)q_2 e^{-i\omega\tau}) \right). \]

By using (74) and (80), we can know

\[ W' = s' - z' q - \bar{q} \hat{q}, \]

\[ - \hat{W} + H(z, \bar{z}, \theta), \]

where

\[ H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta) \frac{\bar{q}^2}{2} + \cdots. \]

From (81), (89), and (90), we can get

\[ \begin{cases} (A - 2i\omega\tau) W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta). \end{cases} \]

From (89), (91), and the definition of A, we can calculate

\[ W_{20}(\theta) = \frac{ig_{20}}{\omega \tau} q(0)e^{i\omega\tau} + \frac{ig_{20}}{3\omega \tau} \bar{q}(0)e^{-i\omega\tau} + \Gamma_1 e^{2i\omega\tau}, \]

\[ W_{11}(\theta) = -\frac{ig_{11}}{\omega \tau} q(0)e^{i\omega\tau} + \frac{ig_{11}}{\omega \tau} \bar{q}(0)e^{-i\omega\tau} + \Gamma_2, \]

where \( \Gamma_i = (\Gamma_i^{(1)}, \Gamma_i^{(2)}, \Gamma_i^{(3)})' \in \mathbb{R}^3 \) \( (i = 1, 2) \) are the three-dimensional vectors.

Now, we determine \( \Gamma_1 \) and \( \Gamma_2 \). By (91) and the definition of A, we have
\[
\begin{align*}
\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) &= 2i\omega \bar{\tau} W_{20}(\theta) - H_{20}(\theta), \\
\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) &= -H_{11}(\theta),
\end{align*}
\]
where \(\eta(\theta) = \eta(\theta, 0)\). By (89), when \(\theta = 0\),
\[
H(z, \bar{z}, \theta) = -\bar{q}'(0) f_0 q(0) - q(0) \bar{f}_0 q(0) + f_0 = -g(z, \bar{z}, q(0) - \bar{g}(z, \bar{z}) q(0) + f_0.
\]
That leads to
\[
\begin{align*}
H_{20}(0) &= -g_{20, q}(0) - \bar{g}_{0, q}(0) \\
&= \left( \begin{array}{ccc}
-\frac{b\beta_1}{(1 + aV_1)^2} + \frac{ab\beta_1 S_1}{(1 + aV_1)^2} q_2^2 \\
G_0 \end{array} \right), \\
H_{11}(0) &= -g_{11, q}(0) - \bar{g}_{1, q}(0) \\
&= \left( \begin{array}{ccc}
-\frac{b\beta_1}{(1 + aV_1)^2} \text{Re}(q_2) + \frac{ab\beta_1 S_1}{(1 + aV_1)^2} q_2 \bar{q}_2 \\
\frac{G_1}{2} \\
-b\beta_2 q_1 \bar{q}_2 e^{-2i\omega t} \\
-\beta\beta_2 q_1 \bar{q}_2 e^{-2i\omega t}
\end{array} \right),
\end{align*}
\]
where
\[
\begin{align*}
G_0 &= -\alpha b\beta_1 (q_2 + q_1 \bar{q}_2) + \frac{b\beta_1 q_2}{(1 + aV_1)^2} - \frac{ab\beta_1 S_1}{(1 + aV_1)^2} q_2^2, \\
G_1 &= \left( \begin{array}{ccc}
\frac{b\beta_1}{(1 + aV_1)^2} - b\beta_1 \text{Re}(q_2) - \alpha b\beta_1 \text{Re}(q_1 \bar{q}_2) \\
-\frac{ab\beta_1 S_1}{(1 + aV_1)^2} q_2 \bar{q}_2 \\
\end{array} \right).
\end{align*}
\]
Since
\[
\left( i\omega \bar{\tau}I - \int_{-1}^{0} e^{i\theta \omega t} d\eta(\theta) \right) q(0) = 0,
\]
we have
\[
\left( 2i\omega \bar{\tau}I - \int_{-1}^{0} e^{2i\theta \omega t} d\eta(\theta) \right) E_1 = 27 \left( \begin{array}{ccc}
-\frac{b\beta_1}{(1 + aV_1)^2} + \frac{ab\beta_1 S_1}{(1 + aV_1)^2} q_2^2 \\
G_0 \\
-b\beta_2 q_1 \bar{q}_2 e^{-2i\omega t} \\
-\beta\beta_2 q_1 \bar{q}_2 e^{-2i\omega t}
\end{array} \right).
\]
We can calculate
\[
\Gamma_i^{(1)} = \frac{2}{\Delta_1} \\
\Gamma_i^{(2)} = \frac{2}{\Delta_1} \\
\Gamma_i^{(3)} = \frac{2}{\Delta_1}
\]
where

\[ Q_1 = -a\beta_1 \left( \frac{\Lambda_1}{\mu_1} - S_1 - I_1 \right), \]

\[ G_0 = -a\beta_1 (q_2 + q_1, q_2) + \frac{b\beta_1 q_2}{(1 + aV_1)^2} - \frac{a\beta_1 S_1}{(1 + aV_1)^2} q_2^2, \]

\[
\Delta_1 = \begin{vmatrix}
  2i\omega + \frac{b\beta_1 V_1}{1 + aV_1} + \mu_1 & 0 & \frac{b\beta_1 S_1}{(1 + aV_1)^2} \\
  0 & 2i\omega + a\beta_1 V_1 + \mu_1 + \gamma - \frac{b\beta_1 S_1}{(1 + aV_1)^2} - a\beta_1 \left( \frac{\Lambda_1}{\mu_1} - S_1 - I_1 \right) & 0 \\
 0 & -b\beta_2 \left( \frac{\Lambda_2}{\mu_2} - V_1 \right) e^{-2i\omega \tau} & 2i\omega + \mu_2 + b\beta_2 I_1 e^{-2i\omega \tau}
\end{vmatrix}
\]

Similarly, we have

\[
\int_{-1}^{0} d\eta \Theta(\eta) \Gamma_2 = -2\pi \begin{pmatrix}
\frac{-b\beta_1}{(1 + aV_1)^2} \text{Re} (q_2) + \frac{a\beta_1 S_1}{(1 + aV_1)^2} q_2^2 \\
G_1 \\
-b\beta_2 \text{Re} (q_1 q_2)
\end{pmatrix}.
\]

That leads to

\[
\Gamma_2^{(1)} = \frac{2}{\Delta_2} \begin{vmatrix}
-\frac{b\beta_1}{(1 + aV_1)^2} \text{Re} (q_2) + \frac{a\beta_1 S_1}{(1 + aV_1)^2} q_2^2 & 0 & \frac{b\beta_1 S_1}{(1 + aV_1)^2} \\
G_1 & a\beta_1 V_1 + \mu_1 + \gamma - \frac{b\beta_1 S_1}{(1 + aV_1)^2} + Q_1, \\
-b\beta_2 \text{Re} (q_1 q_2) & -b\beta_2 \left( \frac{\Lambda_2}{\mu_2} - V_1 \right) & \mu_2 + b\beta_2 I_1
\end{vmatrix}
\]

\[
\Gamma_2^{(2)} = \frac{2}{\Delta_2} \begin{vmatrix}
\frac{b\beta_1 V_1}{1 + aV_1} + \mu_1 - \frac{b\beta_1}{(1 + aV_1)^2} \text{Re} (q_2) + \frac{a\beta_1 S_1}{(1 + aV_1)^2} q_2^2 & \frac{b\beta_1 S_1}{(1 + aV_1)^2} \\
G_1 & -\frac{b\beta_1 S_1}{(1 + aV_1)^2} + Q_1, \\
0 & -b\beta_2 \text{Re} (q_1 q_2) & \mu_2 + b\beta_2 I_1
\end{vmatrix}
\]

\[
\Gamma_2^{(3)} = \frac{2}{\Delta_2} \begin{vmatrix}
\frac{b\beta_1 V_1}{1 + aV_1} + \mu_1 - \frac{b\beta_1}{(1 + aV_1)^2} \text{Re} (q_2) + \frac{a\beta_1 S_1}{(1 + aV_1)^2} q_2^2 \\
G_1 & 0 \\
0 & -b\beta_2 \left( \frac{\Lambda_2}{\mu_2} - V_1 \right) & -b\beta_2 \text{Re} (q_1 q_2)
\end{vmatrix}
\]
where

\[
G_1 = \left( \frac{b\beta_1}{1 + aV_1} \right)^2 - ab\beta_1 \operatorname{Re}(q_2) - ab\beta_1 \operatorname{Re}(q_1 q_2^*) - \frac{ab\beta_1 S_1}{(1 + aV_1)\tau} q_2 q_2^*,
\]

\[
\Delta_2 = \begin{vmatrix}
\frac{b\beta_1 V_1}{1 + aV_1} + \mu_1 & 0 & \frac{b\beta_1 S_1}{(1 + aV_1)^2} \\
- \frac{b\beta_1 V_1}{1 + aV_1} + ab\beta_1 V_1 + \mu_1 + y & - \frac{b\beta_1 S_1}{(1 + aV_1)^2} - ab\beta_1 \left( \frac{\Lambda_1}{\mu_1} - S_1^{-1} - I_1 \right) & 0 \\
0 & -b\beta_2 \left( \frac{\Delta_2}{\mu_2} - V_1 \right) & \mu_2 + b\beta_2 I_1
\end{vmatrix}
\]

(iii) If \( T_2 > 0 \), then the period of the bifurcation periodic solution increases; if \( T_2 < 0 \), the period decreases.

6. Numerical Simulations

In this section, we use numerical simulations to illustrate our result about the existence of Hopf bifurcation.

The following parameters are selected: \( \Lambda_1 = 8, b = 0.29, \beta_1 = 0.0033, \alpha = 0.06, \mu_1 = 0.0029, \sigma = 0.48, y = 0.56, \Lambda_2 = 9, \beta_2 = 0.0059, \) and \( \mu_2 = 0.03 \). We can calculate that \( R_0 = 80.2459 \) and \( E_1 = (44.9472, 450.8920, 288.7707) \).

Equation (37) has a positive root \( \mu = 0.5779 \). We have \( \tau^* = 2.1386 \). From Theorem 3.4, we know that the endemic equilibrium \( E_1 \) is locally asymptotically stable for \( 0 < \tau < \tau^* \) (Figure 1), and system (7) undergoes a Hopf bifurcation at \( E_1 \) when \( \tau = \tau^* \). At this time, we can calculate \( c_1(0) = -2.3788 - 0.8427i, \mu_2 = 9.8850 > 0, \beta_2 = -4.7576 < 0, \) and \( T_2 = 1.7152 > 0 \). According to Theorem 5.1, system (7) can produce a stable periodic solution, the Hopf bifurcation is supercritical at \( \tau^* \), and the period of the bifurcation periodic solution increases (Figure 2).

7. Conclusions

In this paper, we discuss the dynamics of the vector-borne disease model with delay-saturated infection rate and re-infection. By calculation, we have the basic reproductive number \( R_0 \). Through \( R_0 \), we determined the existence of disease-free equilibrium \( E_0 \) and the endemic equilibrium \( E_1 \). According to the characteristic equation of the equilibrium points and using the Routh–Hurwitz criterion, we obtained that if \( R_0 < 1 \) the disease-free equilibrium will be stable, and the endemic equilibrium is locally asymptotically stable if \( R_0 > 1 \) and in the absence of time delay. Furthermore, by the fluctuation lemma and the limit theory, we analyzed the global stability of the disease-free equilibrium. We find that the time delay does not affect the
stability of the boundary equilibrium but can change the stability of $E_1$ and lead to the occurrence of Hopf bifurcation. Then by using the Nyquist criterion, we get the maximum length of delay to preserve stability. Next, we found that the conditions for determining the direction and stability of bifurcating periodic solutions. Finally, the correctness of the main conclusion is verified by numerical simulation.

Data Availability
The data we selected is only to verify the correctness of the results. These data are not real data.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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