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Classical solvability of multidimensional two-phase Stefan problem for degenerate parabolic equations.

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We prove locally in time the existence of a smooth solution for multidimensional two-phase Stefan problem for degenerate parabolic equations of the porous medium type. We establish also natural Hölder class for the boundary conditions in the Cauchy-Dirichlet problem for a degenerate parabolic equation.

Key words: free boundary, Stefan problem, classical solvability, porous medium equation, degenerate parabolic equations.

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1 Statement of the problem and the main result

Classical solvability of the Stefan problem for uniformly parabolic equations has been well studied - see for example papers [1] - [9] and the references therein. At the same time, as has long been known, the heat transfer model based on uniformly parabolic equations, has some properties which can not be observed in the reality, in particular, the infinite speed of propagation of disturbances. We also know that more accurate model of the heat transfer is the model which is based on degenerate parabolic equations, such as equations of the form

\[ u_t(x,t) = \nabla(|u|^{m-1}\nabla u(x,t)) = f(x,t), \]  

(1.1)

where \( m > 1 \). As it is known, a short formulation of a classical Stefan problem for the equation (1.1) is the equation

\[ (\beta(u))_t = \nabla(|u|^{m-1}\nabla u(x,t)) = 0, \]

(1.2)

where \( \beta(u) \) is a discontinuous function of the form

\[ \beta(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ \beta(u) = \begin{cases} u, & u \leq 0, \\ u + k, & u > 0, \end{cases} \]

where \( k > 0 \), is the latent heat of fusion (crystallization) and the equation (1.2) is considered in the sense of distributions. At that for a quasilinear equation (1.2) the main unknown is, in fact, the interface \( \{ u = 0 \} \), outside of which the solution of (1.2) is smooth in view of the well-known local theory of uniformly parabolic equations.

In its generalised formulation this problem was considered in a number of papers, from which we mention, for example, [10] - [16], and we do not pretend to be complete in this matter, as the subject of our interest in this article is a smooth solution of the problem.

As for the smooth solutions, in the case of one spatial variable such problem for degenerate parabolic equations was considered in [17] - [21], where it was proved the existence of classical solutions. See also [22].

The aim of this paper is the proof of the classical solvability of the Stefan problem of the type (1.2) for a degenerate equation in a multidimensional setting, that is, the proof of the existence of smooth surface which is the interface \( \{ u = 0 \} \), and the proof of the smoothness of the solution up to the interface.

We now formulate a precise statement of the Stefan problem in a more expanded than (1.2) form, as it is custom in the theory of free boundary problems. Let \( \Omega \) is a doubly connected domain in \( \mathbb{R}^N \), whose boundary consists of two smooth connected surfaces \( \Gamma^+ \) and \( \Gamma^- \) without self-intersections, \( \partial \Omega = \Gamma^+ \cup \Gamma^- \). Suppose, further, that \( \Gamma \) is a given smooth surface without self-intersections lying strictly between \( \Gamma^+ \) and \( \Gamma^- \) and separating the domain \( \Omega \) into two doubly connected subdomains \( \Omega^+ \) and \( \Omega^- \), so that \( \partial \Omega^{\pm} = \Gamma^{\mp} \). For a fixed \( T > 0 \) denote \( \Omega_T = \Omega \times (0, T), \Omega_T^{\pm} = \Omega^{\pm} \times (0, T), \Gamma_T = \Gamma \times [0, T], \Gamma_T^{\pm} = \Gamma^{\pm} \times [0, T] \).

Denote by \( S_T \) a smooth surface in the cylindrical domain \( \Omega_T \) in the space \( (y, \tau) \in \mathbb{R}^N \times [0, T] \), such that at \( \tau = 0 \), it coincides with \( \Gamma \), that is, \( S_T \cap \{ \tau = 0 \} = \Gamma \), \( S_T \) does not intersect surfaces \( \Gamma_T^{\mp} \) and divides the \( \Omega_T \) into two subdomains \( Q_T^+ \) and \( Q_T^- \), and the lateral boundaries of these domains consist of \( S_T \) and \( \Gamma_T^{\mp} \) respectively. Surface \( S_T \) is unknown and has to be determined together with the functions \( u^+(y, \tau) \) and \( u^-(y, \tau) \), which are defined in \( Q_T^\pm \) respectively. The triple \( (S_T, u^+, u^-) \) must satisfy the following conditions (we denote the independent variables by the \( (y, \tau) \) in view of the subsequent change of variables):

\[
\frac{\partial u^\pm}{\partial \tau} - \nabla_y (a^\pm |u^\pm|^{m-1} \nabla_y u^\pm) = 0, \quad (y, \tau) \in Q_T^\pm, \tag{1.3}
\]

\[
u^+(y, \tau) = u^-(y, \tau) = 0, \quad (y, \tau) \in S_T, \tag{1.4}
\]
\[ a^+ \sum_{i=1}^{N} \cos(\vec{N}, y_i) |u^+|^m |y_i|^n - a^- \sum_{i=1}^{N} \cos(\vec{N}, y_i) |u^-|^m |y_i|^n = k \cos(\vec{N}, \tau), \ (y, \tau) \in S_T, \]

(1.5)

\[ u^\pm(y, \tau) = g^\pm(y, \tau), \ (y, \tau) \in \Gamma_T^\pm, \]

(1.6)

\[ u^\pm(y, 0) = u^\pm_0(y). \]

(1.7)

Here \( m > 1, k > 0, a^\pm > 0 \) are given constants, \( g^+(y, \tau), g^-(y, \tau), u^+_0(y), u^-_0(y) \) are given functions, at that

\[ \pm g^\pm(y, \tau) \geq \nu > 0, \ (y, \tau) \in \Gamma_T^\pm; \ \pm u^\pm_0(y) > 0, \ y \in \Omega^\pm, \ u^\pm_0(y) = 0, \ y \in \Gamma, \]

(1.8)

where \( \nu \) is some positive constant: here and below we denoted by the same symbols \( \nu, \mu, C \) all absolute constants, or constants that depend only on the given data of the problem. Note that the conditions (1.4), (1.5) are the three independent conditions at unknown boundary \( S_T \) arising from the equation (1.2).

To formulate the smoothness conditions which we impose on the data of the problem, we introduce some weighted function spaces. First of all, we use the standard Hölder space \( H^{l+\delta}(\Omega) \equiv C^{l+\delta}(\Omega), \ \delta \in (0, 1), \ l \in \mathbb{N} \cup \{0\} \), with the norm \( |u|^l_{l+\delta}(\Omega) \), which are introduced in [23], and also spaces \( H^{l+\delta, \frac{1}{2}}(\Omega_T^\pm) \equiv C^{l+\delta, \frac{1}{2}}(\Omega_T^\pm) \) of functions of \( (y, \tau) \) with the norm \( |u|^l_{l+\delta}(\Omega_T^\pm) \). In [23] the surface of the corresponding classes are also defined. We assume that the surfaces \( \Gamma, \Gamma^\pm \) belong to following classes

\[ \Gamma, \ \Gamma^\pm \in H^{4+\gamma} \]

(1.9)

with some \( 0 < \gamma < 1 \). At the same time we suppose that the functions \( g^\pm \) in (1.6) are such that

\[ h^\pm(y, \tau) \equiv |g^\pm(y, \tau)|^m |g^\pm(y, \tau) \in H^{4+\gamma, \frac{1}{2}}(\Gamma_T^\pm). \]

(1.10)

Suppose, further, that \( d^+(y) \) is a given function from \( H^{2+\gamma}(\Omega_T^+) \), which models the distance from a point \( y \in \Omega_T^+ \) to the surface \( \Gamma \), that is

\[ \nu \leq d^+(y)/\text{dist}(y, \Gamma) \leq \nu^{-1}. \]

(1.11)

Note, that such function can be taken, for example, as the solution of the following problem

\[ \Delta d^+(y) = -1, \ y \in \Omega^+, \]

\[ d^+(y) = 0, \ y \in \Gamma, \ \ \ d^+(y) = 1, \ y \in \Gamma^+. \]
Let, further, \( d^- (y) \) is an analogous function for the domain \( \Omega^- \).

Denote here and below
\[
\alpha = \frac{m-1}{m} \in (0,1),
\]
where \( m > 1 \) is the exponent from the equation (1.3).

We will use the spaces \( C^{2+\gamma, \frac{\alpha}{2}}_s (\Omega_T^\pm) \) from the paper [24] (they are analogous to the corresponding spaces from [25]), where \( 0 < \gamma < \alpha \) is some exponent, and we require
\[
0 < \gamma < \alpha.
\]

These spaces are defined in the following way. First we define the spaces \( C^{2+\gamma}_s (\mathbb{R}_N^+ T) \) in the domain
\[
\mathbb{R}^N_+ = \mathbb{R}^N_+ \times [0,T], \quad \mathbb{R}^N_+ = \left\{ x = (x', x_N) : x_N \geq 0, x' \in \mathbb{R}^{N-1} \right\}. \tag{1.14}
\]

Define a distance between points \( x, \overline{x} \in \mathbb{R}^N_+ \) according to the following formula
\[
s(x, \overline{x}) = \left| \frac{x - \overline{x}}{x_N^{\alpha/2} + \overline{x}_N^{\alpha/2} + |x' - \overline{x}'|^{\alpha/2}}. \tag{1.15}\]

Define further a Hölder constant of a function \( u(x, t) \) with respect to the variable \( x \) according to the distance we have introduced
\[
H^\gamma_{s, \mathbb{R}^N_+ T}(u) \equiv \sup_{(x, t), (\overline{x}, t) \in \mathbb{R}^N_+ T} \frac{|u(x, t) - u(\overline{x}, t)|}{s(x, \overline{x})}. \tag{1.16}\]

Denote by \( C^{\gamma/2,s}_s (\mathbb{R}^N_+ T) \) the space of functions \( u(x, t) \) with the finite norm
\[
|u|_{C^{\gamma/2,s}_s (\mathbb{R}^N_+ T)} \equiv |u|^{(\gamma)}_{s, \mathbb{R}^N_+ T} \equiv |u|^{(0)}_{s, \mathbb{R}^N_+ T} + H^\gamma_{s, \mathbb{R}^N_+ T}(u) + \langle u \rangle^{(\gamma/2)}_{t, \mathbb{R}^N_+ T}, \tag{1.17}\]
where \( \langle u \rangle^{(\gamma/2)}_{t, \mathbb{R}^N_+ T} \) is the Hölder constant with respect to \( t \) with the exponent \( \gamma/2 \) of the function \( u(x, t) \).

Define further the space \( C^{2+\gamma, \frac{\alpha}{2}}_s (\mathbb{R}^N_+ T) \) as the Banach space of functions \( u(x, t) \) with the finite norm
\[
|u|_{C^{2+\gamma, \frac{\alpha}{2}}_s (\mathbb{R}^N_+ T)} \equiv |u|^{(2+\gamma)}_{s, \mathbb{R}^N_+ T} \equiv |u|^{(0)}_{s, \mathbb{R}^N_+ T} + \sum_{i=1}^N |u_{x_i}|_{C^{\gamma/2,s}_s (\mathbb{R}^N_+ T)} + |u_t|_{C^{\gamma/2,s}_s (\mathbb{R}^N_+ T)} + \sum_{i,j=1}^N x_N^{\alpha/2} u_{x_i x_j} |_{C^{\gamma/2,s}_s (\mathbb{R}^N_+ T)}. \tag{1.18}\]
Finally, the spaces $C^2_{s,+} (\Omega^+_{T})$ and $C^2_{s,+} \frac{\gamma}{2} (\Omega^+_{T})$ are defined as the spaces of functions $u(x,t)$ with the property, that in some neighborhood of $\Gamma_T$ after the corresponding change of variables functions $u(x,t)$ belong to the space $C^2_{s,+}\frac{\gamma}{2} (R_{+T}^N)$ or to the space $C^2_{s,+} (R_{+T}^N)$ correspondingly, and out of some neighborhood of $\Gamma_T$ the functions $u(x,t)$ belong to the standard spaces $C^2 - \frac{\gamma}{2} (\Omega^+_T)$ or $C^2 + \frac{\gamma}{2} (\Omega^+_T)$. In particular, for a function $u \in C^2_{s,+} (\Omega^+_T)$ the following norm is finite

\[ |(d^\pm)^{i,j} u_{x,x} |_{C^2 - \frac{\gamma}{2} (\Omega^+_T)} < \infty, \quad i,j = 1,N. \]  

(1.19)

In the case of functions $u(x)$ from the variable $x$ only, $x \in \Omega^+$, the spaces $C^2 (\Omega^+)$ and $C^2 + \gamma (\Omega^+)$ are defined in the completely analogous way.

We will use also some standard anisotropic spaces of smooth functions, which are more general than the spaces $H^{1/2} \equiv C^{1/2}$. Namely, we will use the spaces

\[ C^{l_1,l_2} (\Omega^+_T), \quad C^{l_1,l_2} (\Gamma_T), \]

where $l_1, l_2$ are noninteger positive numbers. Such spaces are defined in [26], for example. They consist from the functions, which have Hölder continuous with respect to $x$ withe the exponent $l_1 - \lfloor l_1 \rfloor$ derivatives with respect to $x$ up to the order $\lfloor l_1 \rfloor$, and also these functions have Hölder continuous with respect to $t$ withe the exponent $l_2 - \lfloor l_2 \rfloor$ derivatives with respect to $t$ up to the order $\lfloor l_2 \rfloor$. The norm of such space we denote by

\[ |u|_{C^{l_1,l_2} (\Omega^+_T)} \equiv |u|_{C^{l_1,l_2} (\Omega^+_T)}. \]

In fact, the functions from such spaces possess the property, that their derivatives with respect to $x$ are smooth with respect to $t$ and their derivatives with respect to $t$ are smooth with respect to $x$. More precisely, if $k_1$ is a multiindex, that is, $k_1 = (k_{1,1}, k_{1,2}, ..., k_{1,N})$ where $k_{1,i}$ are nonnegative integers, and $k_2$ is a nonnegative integer, then the function $\partial_x^{k_1} \partial_t^{k_2} u(x,t)$ belongs to the space $C^{m_1,m_2} (\Omega^+_T)$ with the exponents $m_i = \mu l_i$, where

\[ \mu = 1 - \frac{|k_1|}{l_1} - \frac{k_2}{l_2} \]

(see [26]), and also

\[ |\partial_x^{k_1} \partial_t^{k_2} u|_{C^{m_1,m_2} (\Omega^+_T)} \leq C |u|_{C^{l_1,l_2} (\Omega^+_T)}. \]

(1.20)

In addition, we will use the spaces with zero at the bottom of their notation, that is, the spaces (compare [23], Ch.IV)

\[ C^{2+\gamma,2+\gamma} (\Omega^+_T), \quad C^{l_1,l_2} (\Omega^+_T), \quad C^{1,2} (\Omega^+_T), \quad C^{l_1,l_2} (\Gamma_T). \]

(1.21)
Such notations mean closed subspace of the corresponding space, which consists of functions that vanish at $t = 0$ together with all their derivatives with respect to $t$ up to the highest possible order in the corresponding space.

Let here and throughout below the exponent $\beta \in (0, 1)$ is connected to the exponent $\gamma$ by the equality

$$\beta = \gamma \left(1 - \frac{\alpha}{2}\right) \tag{1.22}$$

In particular, we will use the spaces $C^{2 + \beta - \alpha, 1 + \gamma/2}(\Omega_T^\pm)$ and $C^{2 + \beta - \alpha, 2 + \gamma/2}(\Gamma_T)$, which consist of the functions $u(x, t)$ with smoothness with respect to $x$ up to the order $2 + \beta - \alpha$ and with smoothness with respect to $t$ up to the order $1 + \gamma/2$, that is

$$\sum_{i=1}^{N} \langle u_{x_i} \rangle_{x, \Omega_T^\pm}^{(1 + \beta - \alpha)} + \langle u_t \rangle_{t, \Omega_T^\pm}^{(\gamma/2)} < \infty. \tag{1.23}$$

It is easy to calculate, that because of the relation (1.22), we have

$$1 + \frac{\gamma}{2} = \frac{2 + \beta - \alpha}{2 - \alpha}, \tag{1.24}$$

so we also use the following notation for the mentioned above spaces

$$C^{2 + \beta - \alpha, 2 + \beta - \alpha/2}(\Omega_T^\pm), \quad C^{2 + \beta - \alpha, 2 + \beta - \alpha/2}(\Gamma_T). \tag{1.25}$$

Further, we suppose that the initial conditions in (1.7) are such that

$$v_0^\pm(y) \equiv |u_0^\pm(y)|^{m-1}u_0^\pm(y) \in C^{2+\gamma'}(\Omega^\pm), \tag{1.26}$$

where $\gamma' > \gamma$. Besides, we suppose that

$$\frac{\partial v_0^\pm(y)}{\partial n} \geq \nu > 0, \quad y \in \Gamma, \tag{1.27}$$

where $\vec{n}$ is a normal vector to the surface $\Gamma$ which is directed into $\Omega^+$. We will show below, that the free (unknown) boundary $S_T$ in (1.4), (1.5) can be parameterized in terms of its deviation from the given surface $\Gamma_T = \Gamma \times [0, T]$. We follow to [2] to give the strict formulation. Let $\omega = (\omega_1, ..., \omega_{N-1})$ is a local curvilinear coordinates in a domain $\Theta$ on $\Gamma$. In some small enough neighbourhood $\mathcal{N}$ in $R^N$ of the surface $\Gamma$ we introduce the coordinates $(\omega, \lambda)$ in the way that for any $x \in \mathcal{N}$ we have the following unique representation

$$x = x'(x) + \vec{n}(x'(x))\lambda \equiv x(\omega) + \vec{n}(\omega)\lambda, \tag{1.28}$$

where $x'(x) = x(\omega)$ is the point in the domain $\Theta$ on the surface $\Gamma$ with the coordinates $\omega$, $\vec{n}(\omega)$ - normal to $\Gamma$ at the point $x(\omega)$ with the direction into $\Omega^+$, $\lambda \in R$ means, in fact, deviation of the point $x$ from $\Gamma$, at that $\pm \lambda > 0$ for $x \in \Omega^\pm$. 

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We assume that the mentioned above neighbourhood $\mathcal{N}$ of the surface $\Gamma$ is the set

$$\mathcal{N} = \{x \in \Omega : |\lambda(x)| < \gamma_0\},$$

where $\gamma_0$ is small enough and will be chosen below.

Let $\rho(x', t) \equiv \rho(\omega, t)$ is a small and regular function, which is defined on the surface $\Gamma_T$. Let us note, that here and in what follows we use the notation $\rho(\omega, t)$ with the argument $\omega$ instead of $\rho(x', t)$ for all functions on the surface $\Gamma$ if it does not cause ambiguity. We do that just for simplification of the notation, bearing in mind that in each local domain $\Theta$ on $\Gamma$ we can introduce local coordinates $\omega$.

At the same time the coordinate $\lambda$ in (1.28) does not depend on the choice of local coordinates $\omega$.

We parameterize the unknown surface $S_T$ we the help of the unknown function $\rho(\omega, t)$ as follows

$$S_T \equiv \Gamma_{\rho,T} = \{ (x, t) \in \Omega_T : x = x' + \rho(x', t) \overrightarrow{n}(x') = x(\omega) + \rho(\omega, t) \overrightarrow{n}(\omega)\}, \quad (1.29)$$

where $x' \equiv x(\omega) \in \Gamma$. Note, that this definition of the surface $S_T \equiv \Gamma_{\rho,T}$ does not depend on a choice of local coordinates $\omega$ in a particular local domain on $\Gamma$. Thus, the unknown function $\rho(\omega, t)$ means, in fact, deviation of the surface $\Gamma_{\rho,T} = S_T$ from the given surface $\Gamma_T$. Along with $Q^+_T, Q^-_T$ in (1.3) we use the notation $\Omega^+_{\rho,T} = Q^+_T$ and $\Omega^-_{\rho,T} = Q^-_T$ for the subdomains that $\Gamma_{\rho,T} = S_T$ separates the domain $\Omega_T$. Let, further, $\rho(x, t)$ is an extension of the function $\rho(\omega, t)$ from the surface $\Gamma_T$ to the whole domain $\Omega_T$ to a function with the support in the neighborhood $\mathcal{N}_T = \mathcal{N} \times [0, T]$ of the surface $\Gamma_T$, $\rho(x, t) = E\rho(\omega, t)$, $E$ is some fixed extension operator (the way of such an extension will be listed below), at that we will denote $\rho^+ \equiv E\rho|_{\mathcal{N}^+_T} \equiv E^+\rho$.

Define a mapping $e_{\rho}(x, t)$ from the domain $\overline{\Omega_T}$ on itself with the help of the formula $e_{\rho} : (x, t) \rightarrow (y, \tau)$, where, according to the notations of (1.28),

$$y = \begin{cases} x'(x) + \overrightarrow{n}(x')(\lambda(x) + \rho(x, t)), & x \in \mathcal{N}, \\ x, & x \in \overline{\Omega \setminus \mathcal{N}}, \end{cases} \quad \tau = t, \quad (1.30)$$

or, with the help of the local coordinates $\omega$,

$$y = \begin{cases} x'(\omega(x)) + \overrightarrow{n}(\omega(x))(\lambda(x) + \rho(x, t)), & x \in \mathcal{N}, \\ x, & x \in \overline{\Omega \setminus \mathcal{N}}, \end{cases} \quad \tau = t. \quad (1.31)$$

Here $x'(x) \in \Gamma$, $\omega(x)$, $\lambda(x)$ are $(\omega, \lambda)$- coordinates of a point $x$ in the neighbourhood $\mathcal{N}$. Note here, that the definition of the mapping $e_{\rho}$ does not depend on a choice of local coordinates $\omega$ on the surface $\Gamma$.7
We choose $\gamma_0$ small enough so that under the condition
\[ |\rho|^{1+\beta}_T \leq 2\gamma_0 \]  
(1.32)
the mapping $e_\rho$ is a diffeomorphism of $\Omega^\pm_T$ on themselves, and also the mapping $e_\rho$ is a diffeomorphism of the domains $\Omega^\pm_T$ on the domains $\Omega^\pm_{\rho,T}$. Note, that the surface $\Gamma_{\rho,T}$ is exactly the image of the surface $\Gamma_T$ under this mapping, and the mapping $e_\rho(x,t)$ is the identical mapping out of the neighbourhood $N_T$ of $\Gamma_T$.

About the exponents of the Hölder spaces we use we suppose that
\[ 0 < \gamma < \gamma' < 1, \quad \gamma < \min\{\alpha, 1 - \alpha\}. \]  
(1.33)
Note, that under our choice of $\gamma$ the restriction \( \gamma < \alpha \frac{1}{1 - \alpha/2} \) is also fulfilled. The last restriction was introduced in [24] at the studying of the homogeneous Cauchy-Dirichlet problem for degenerate equations. We need the restriction $\gamma < \alpha$ to have the inequality $1 + \beta - \alpha > 1 - \alpha > \gamma$, which implies that the first derivatives with respect to $x$ of the functions from the classes $C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Omega^\pm_T)$, $C^2_{s+\gamma, \frac{2+\gamma}{2-\alpha}}(\Omega^\pm_T)$ are more smooth than $C^{2+\gamma/2, \frac{2+\gamma}{2-\alpha}}(\Omega^\pm_T)$. At last, we need the restriction $\gamma < \alpha$ to expressions of the form $(d^\pm(x))^{\alpha} \eta(x,t)$ with a smooth function $\eta(x,t)$ would belong to the corresponding space $C^{2+\gamma/2, \frac{2+\gamma}{2-\alpha}}(\Omega^\pm_T)$.

Let us formulate now the main result.

**Theorem 1.1** Let the conditions (1.8) - (1.10), (1.33) on the data of the problem (1.3) - (1.7) and the conditions (1.26), (1.27) are fulfilled. Let also the natural adjustment conditions up to the first order at $\tau = 0, y \in \Gamma, \Gamma^\pm$ for the problem (1.3) - (1.7) are fulfilled. Then there exists such $T > 0$, that on the time interval $[0, T]$ the problem (1.3) - (1.7) has the unique smooth solution, at that the unknown boundary can be represented as in (1.29) with the function $\rho(\omega, t)$ with the properties
\[ \rho(\omega, t) \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T), \quad \rho_t(\omega, t) \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T), \]  
(1.34)
\[ \rho^\pm(x, t) = E^\pm \rho(\omega, t) \in C^{2+\gamma, \frac{2+\gamma}{2-\alpha}}(\Omega^\pm_T), \]
where $\rho(x, t) = E\rho(\omega, t)$ is the extension of the function $\rho(\omega, t)$ to the domain $\Omega_T$. The functions $u^\pm(y, \tau)$ in (1.3) are such that
\[ v^\pm(x, t) \equiv (|u^\pm(y, \tau)|^{m-1}u^\pm(y, \tau)) \circ e_\rho(x, t) \in C^{2+\gamma, \frac{2+\gamma}{2-\alpha}}(\Omega^\pm_T). \]  
(1.35)
Thus, in particular, all of the relations of the problem (1.3) - (1.7) are satisfied in the classical sense.

Subsequent sections of the paper devoted to the proof of the theorem 1.1 in accordance with such a plan.
In the section 2 on the basis of equivalent norms in the spaces $C^{\gamma, \gamma/2}_s(\Omega_T^\pm)$ the natural space of the traces on $\Gamma_T$ of the functions from class $C^{2+\gamma, 2+\gamma/2}_s(\Omega_T^\pm)$ is studied. This allows us to extend the results of [24] about the solvability of the homogeneous initial boundary value problem for a degenerate equation to the case of the inhomogeneous problem.

These results are then used in the section 3 to study a model Stefan problem for degenerate equations, which is one of the central points of this paper. In this case, for the Schauder estimates of the model problems the idea of the paper [28] on the application of the maximum principle to obtain the Schauder estimates is used.

In the sections 4 and 5 the initial problem with the unknown boundary is reduced to nonlinear problem in the fixed domain and linearized on functions that expand the initial data to the domain $t > 0$.

The section 6 is devoted to the study of the resulting linear problem for degenerate equations on the basis of the results of the section 3 about the properties of the corresponding model problem. In this case, to prove the solvability of the linear problem, we apply the idea of [1] on parabolic regularity of the Stefan boundary condition. Note that the corresponding model problem in Section 3 is considered in the presence of the regularization.

Finally, the section 7 completes the proof of the theorem 1.1 by the method of [3].

2 Auxiliary results on the spaces $C^{\gamma, \gamma/2}_s(\Omega_T^\pm)$, $C_s^{2+\gamma, 2+\gamma/2}(\Omega_T^\pm)$, $C^{2+\beta-\alpha, 2+\beta-\alpha/2}_s(\Gamma_T)$.

Note first, that for the spaces with zero in (1.21) the following relations are valid. Let $\gamma' > \gamma$, $l'_1 > l_1$, $l'_2 > l_2$. Let also functions $u$ and $v$ belong to one of the mentioned spaces with the exponents of smoothness $\gamma'$, $2+\gamma'$, $l'_1$, $l'_2$. Then

\[ |u|^{(\gamma)}_{s, \Omega_T^\pm} \leq CT^\mu |u|^{(\gamma)}_{s, \Omega_T^\pm}, \]
\[ |u|^{(2+\gamma)}_{s, \Omega_T^\pm} \leq CT^\mu |u|^{(2+\gamma')}_{s, \Omega_T^\pm}, \]
\[ |uv|^{(\gamma)}_{s, \Omega_T^\pm} \leq CT^\mu |u|^{(\gamma)}_{s, \Omega_T^\pm} |v|^{(\gamma)}_{s, \Omega_T^\pm}, \]
\[ |u|^{(l'_1, l'_2)}_{\Omega_T^\pm} \leq CT^\mu |u|^{(l'_1, l'_2)}_{\Omega_T^\pm}, \]

where $\mu$ is some positive constants which depend on $\gamma$, $\gamma'$, $l_i$, $l'_i$.

These inequalities are well known for the spaces $C^{l'_1, l'_2}_0$ (see [23], [27]), and for the spaces $C^{\gamma, \gamma/2}_0$, $C^{2+\gamma, 2+\gamma/2}_0$ these inequalities are completely analogous.
2.1 An equivalent norm for the spaces $C^\gamma_T(\Omega^\pm_T)$, $C^{2+\gamma}_s(\Omega^\pm_T)$.

Along with the seminorm $H^{\gamma}_{s,R^+_T}$ from (1.16) we consider in $R^+_T$ the following weighted seminorm

$$H^{\gamma}_{\alpha,R^+_T}(f) = \sup_{x,\overline{x} \in R^+_N} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\beta} + \sup_{x,\overline{x} \in R^+_N} \tilde{x}_N^{\gamma} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\gamma},$$

(2.5)

where $\tilde{x}_N = \max\{x_N, \overline{x}_N\}$, and here and throughout, we, without loss of generality, assume that $x_N \leq \overline{x}_N$, so $\tilde{x}_N = x_N$. 

Lemma 2.1 The seminorms $H^{\gamma}_{\alpha,R^+_N}(f)$ and $H^{\gamma}_{s,R^+_N}(f)$ are equivalent.

Proof. Let the seminorm $H^{\gamma}_{s,R^+_N}(f)$ is finite. We show, that then

$$H^{\gamma}_{\alpha,R^+_N}(f) \leq CH^{\gamma}_{s,R^+_N}(f).$$

(2.6)

Let $\varepsilon_0 \in (0,1)$ is small and fixed. Let $x = (x',x_N)$ and let first

$$|x' - \overline{x}'| \geq \varepsilon_0 x_N.$$  

(2.7)

Then the more

$$|x - \overline{x}| \geq |x' - \overline{x}'| \geq \varepsilon_0 x_N.$$  

(2.8)

Under this condition

$$s(x,\overline{x}) = \frac{|x - \overline{x}|}{x_N^{\alpha/2} + \overline{x}_N^{\alpha/2} + |x' - \overline{x}'|^{\alpha/2}} \leq \frac{|x - \overline{x}|}{|x - \overline{x}|^{\alpha/2} + |\overline{x} - \overline{x}'|^{\alpha/2}} \leq C|x - \overline{x}|^{1 - \frac{\alpha}{2}}.$$  

Therefore, as $\beta = \gamma(1 - \frac{\alpha}{2})$,

$$\frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\beta} \leq C \frac{|f(x,t) - f(\overline{x},t)|}{s(x,\overline{x})^\gamma} \leq CH^{\gamma}_{s,R^+_N}(f).$$  

(2.9)

Besides, because of (2.8), and then of (2.9),

$$x_N^{\frac{\gamma}{2}} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\gamma} \leq \frac{x_N^{\gamma} |f(x,t) - f(\overline{x},t)|}{(\varepsilon_0 x_N)^{\gamma} |x - \overline{x}|^{\gamma(1 - \alpha/2)}} \leq CH^{\gamma}_{s,R^+_N}(f).$$  

(2.10)

Let now
\[ |x' - x| \leq \varepsilon_0 x_N. \quad (2.11) \]

Under this condition, as it easy to see,

\[ s(x, \overline{x}) \sim C x_N^{-\frac{\alpha}{2}} |x - \overline{x}|. \quad (2.12) \]

Consequently,

\[ x_N^{-\frac{\alpha}{2}} \frac{\left| f(x, t) - f(\overline{x}, t) \right|}{|x - \overline{x}|^{\gamma}} \leq C \frac{\left| f(x, t) - f(\overline{x}, t) \right|}{s(x, \overline{x})^{\gamma}} \leq C H_{s, R_N}^{\gamma}(f). \quad (2.13) \]

To estimate, further, the unweighted Hölder constant in the definition of \( H_{\alpha, R_N}^{\gamma}(f) \), we consider the two cases.

If

\[ |x_N - \overline{x}_N| \geq \varepsilon_0 x_N, \]

then

\[ |x - \overline{x}| \geq |x_N - \overline{x}_N| \geq \varepsilon_0 x_N \]

and therefore, as it was above,

\[ s(x, \overline{x}) \leq \frac{|x - \overline{x}|}{(|x - \overline{x}|/\varepsilon_0)^{\alpha/2}} \leq C |x - \overline{x}|^{1 - \frac{\alpha}{2}}, \]

so that, as above,

\[ \frac{\left| f(x, t) - f(\overline{x}, t) \right|}{|x - \overline{x}|^{\gamma}} \leq C \frac{\left| f(x, t) - f(\overline{x}, t) \right|}{s(x, \overline{x})^{\gamma}} \leq C H_{s, R_N}^{\gamma}(f). \quad (2.14) \]

If now, under the condition \( (2.11) \), we have

\[ |x_N - \overline{x}_N| \leq \varepsilon_0 x_N, \quad (2.15) \]

then in this case

\[ |x - \overline{x}| \leq |x' - \overline{x}| + |x_N - \overline{x}_N| \leq 2\varepsilon_0 x_N. \quad (2.16) \]

Therefore, in the force of \( (2.12) \),

\[ s(x, \overline{x}) \leq C x_N^{-\alpha/2} |x - \overline{x}| \leq \]

\[ \leq C x_N^{-\alpha/2}(2\varepsilon_0 x_N)^{\alpha/2} |x - \overline{x}|^{1 - \alpha/2} = C |x - \overline{x}|^{1 - \alpha/2}. \quad (2.17) \]

Consequently, in this case
\[
\frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq CH_{s,R_N+T}^\gamma (f). \tag{2.18}
\]

The estimate (2.18) follows now from (2.9), (2.10), (2.13), (2.14) and (2.18).

Further, let now the seminorm \( H_{s,R_N+T}^\gamma (f) \) is finite. Let us prove the following estimate

\[
H_{s,R_N+T}^\gamma (f) \leq CH_{\alpha,R_N+T}^\gamma (f). \tag{2.19}
\]

Let first

\[
|x' - \bar{x}| \leq \varepsilon_0 x_N, \quad x_N > 0. \tag{2.20}
\]

Then

\[
s(x, \bar{x}) \geq \nu \frac{|x - \bar{x}|}{x_N^{\alpha/2}},
\]

and consequently

\[
\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq C \frac{x_N^{\alpha/2} |f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\gamma} \leq CH_{\alpha,R_N+T}^\gamma (f). \tag{2.21}
\]

In the particular case \( x_N = 0 \) we have \( \bar{x}_N = 0 \) and therefore

\[
s(x, \bar{x}) = |x' - \bar{x}'|^{1-\alpha/2} = |x - \bar{x}|^{1-\alpha/2},
\]

and so again

\[
\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} = \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq CH_{\alpha,R_N+T}^\gamma (f). \tag{2.22}
\]

Let now we have

\[
|x' - \bar{x}'| \geq \varepsilon_0 x_N. \tag{2.23}
\]

Then

\[
s(x, \bar{x}) \geq \nu \frac{|x - \bar{x}|}{x_N^{\alpha/2}} \geq \nu |x - \bar{x}|^{1-\alpha/2}, \tag{2.24}
\]

and consequently,

\[
\frac{|f(x, t) - f(\bar{x}, t)|}{s(x, \bar{x})^\gamma} \leq C \frac{|f(x, t) - f(\bar{x}, t)|}{|x - \bar{x}|^\beta} \leq CH_{\alpha,R_N+T}^\gamma (f). \tag{2.25}
\]

Thus, (2.19) follows from (2.21), (2.22), (2.24). And so the equivalence of the seminorms \( H_{s,R_N+T}^\gamma (f) \) and \( H_{\alpha,R_N+T}^\gamma (f) \) is proved. ■
In this way, the norm in the space $C^\gamma_s(R^N_{+T})$ may be given in the form

$$
|u|_{C^{\gamma/2}_s(R^N_{+T})} = |u|_{R^N_{+T}}^{(0)} + H^{\gamma}_{\alpha,R^N_{+T}}(u) + \langle u \rangle_{t,R^N_{+T}}^{\gamma/2}.
$$

(2.26)

Bearing in mind the local straightening of the boundary $\Gamma$, for the case of arbitrary domains $\Omega_T$, the norm in the space $C^{\gamma/2}_s(\Omega_T)$ may be explicitly written as

$$
|u|_{C^{\gamma/2}_s(\Omega_T)} = |u|_{\Omega_T}^{(0)} + \langle u \rangle_{x,\Omega_T}^{(\beta)} + \sup_{x,T\in\Omega_T} (d^\pm(x,\bar{T}))^{\gamma/2} \frac{|u(x,t) - u(\bar{T},t)|}{|x - \bar{T}|} + \langle u \rangle_{t,\Omega_T}^{\gamma/2},
$$

(2.27)

where the functions $d^\pm(x)$ were introduced in the previous section in (1.11) and they model the distance to the boundary $\Gamma$, $d^\pm(x,\bar{T}) = \max\{d^+(x),d^-(\bar{T})\}$.

Quite similar in terms of (2.27) and (1.19) we may explicitly define the norm (1.18) in the space $C^{2+\gamma}_s(\Omega_T)$.

### 2.2 The traces of the functions from $C^{2+\gamma}_s(\Omega_T)$ on $\Gamma_T$.

In view of the smoothness of the surface $\Gamma$, we can use local straightening of the surface at consideration locally defined classes $C^{2+\gamma,\frac{2+\gamma}{2}}_s(\Omega_T)$. Therefore it is sufficient to consider the case of the half-space, that is to consider the finite in $R^N_{+T} = R^N_{+T} \times [0, T]$ function $u(x,t)$ from the space $C^{2+\gamma,\frac{2+\gamma}{2}}_s(R^N_{+T})$ and to consider its trace at $x_N = 0$.

**Lemma 2.2** Let the function $u(x,t)$ is finite and $u(x,t) \in C^{2+\gamma,\frac{2+\gamma}{2}}_s(R^N_{+T})$, $0 < \gamma < \alpha$, $\beta = \gamma(1-\alpha/2)$. Then the function $v(x',t) = u(x',0,t) \in C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}_s(R^N_{+T}) = C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}_s(R^N_{+T})$, at that

$$
|v(x',t)|_{C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}_s(R^N_{+T})} = |u(x',0,t)|_{C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}_s(R^N_{+T})} \leq C |u|^{(2+\gamma)}_{s,R^N_{+T}}.
$$

(2.28)

Besides,

$$
|\nabla_{(x',x_N)} u(x',0,t)|_{C^{1+\beta-\alpha,\frac{1+\beta-\alpha}{2}}_s(R^N_{+T})} \leq C_T |u|^{(2+\gamma)}_{s,R^N_{+T}}.
$$

(2.29)

**Proof.** It follows directly from the definition of the space $C^{2+\gamma,\frac{2+\gamma}{2}}_s(R^N_{+T})$ in (1.18) and from the lemma 2.1 that $u_t(x',0, t) \in C^{\beta,\gamma/2}_s(R^N_{+T})$, and therefore $v_t(x',0) = u_t(x',0,t) \in C^{\beta,\gamma/2}_s(R^N_{+T})$, and in addition

$$
|v_t|_{C^{\beta,\gamma/2}_s(R^N_{+T})} \leq C |u|^{(2+\gamma)}_{s,R^N_{+T}}.
$$

(2.30)
Therefore, in the force of (1.20) (see [26]), it is sufficient to prove uniformly in \( t \) the following estimate

\[
|v(\cdot, t)|_{R^{N-1}}^{(2+\beta-\alpha)} \leq C |u|_{s, R^{N+T}}^{(2+\gamma)},
\]  

and for this it is sufficient to prove, that uniformly in \( t \) and in \( x_N \) for all \( i = 1, N \) for the function \( w = u_{x_i} \) we have

\[
\langle w \rangle_{x, R^{N-1}}^{(1+\beta-\alpha)} \leq C |u|_{s, R^{N+T}}^{(2+\gamma)}.
\]  

So, let \( w = u_{x_i}, i = 1, N \). To prove (2.32) it is sufficient, as it follows from [29], to show that for arbitrary \( h > 0 \) the follows inequality holds

\[
\left| \frac{\Delta^2_{h,x'} w(x_N)}{h^{1+\beta}} \right| \leq C |u|_{s, R^{N+T}}^{(2+\gamma)}.
\]  

Here \( \Delta^2_{h,x'} w(x_N) \equiv \Delta^2_{h,x'} w(x', x_N, t) \) is the second difference from the function \( w \) with respect to the variable \( x'_j, j = 1, N-1 \), with the step \( h \), that is

\[
\Delta^2_{h,x'} w(x_N) = u(x' + jh, x_N, t) - 2u(x', x_N, t) + u(x' - jh, x_N, t).
\]  

Consider the two cases. Let first

\[
h \leq x_N.
\]  

Then according to the mean value theorem with some \( \theta_1, \theta_2 \in (0, 1) \) we have

\[
\left| \frac{\Delta^2_{h,x'} w(x_N)}{h^{1+\beta}} \right| \leq \left| \frac{x_{N \theta_1} w, (x' + j \theta_1 h, x_N, t) - w,(x' - j \theta_2 h, x_N, t)}{h^\beta} \right|
\]

\[
= \left| \frac{x_{N \theta_1} u_{x_i} (x' + j \theta_1 h, x_N, t) - x_{N \theta_2} u_{x_i} (x' - j \theta_2 h, x_N, t)}{h^\beta} \right| \leq C |x_{N \theta_1} u_{x_i} (x', x_N + h, t) - x_{N \theta_2} u_{x_i} (x', x_N + h, t)| \leq C \langle x_{N \theta_1} u_{x_i} (x', x_{N + h}, t) \rangle_{x', R^{N+T}}^{(\beta)} \leq C |u|_{s, R^{N+T}}^{(2+\gamma)}. \]  

(2.36)

Let now \( h \geq x_N \). Write the difference \( \Delta^2_{h,x'} w(x_N) \) in the form

\[
\Delta^2_{h,x'} w(x_N) = -\Delta^2_{h,x'} (w(x', x_N + h, t) - w(x', x_N + h, t)) + \Delta^2_{h,x'} w(x', x_N + h, t) \equiv A_1 + A_2.
\]  

(2.37)
In view of the fact that for the expression $A_2$ the condition (2.35) holds, that is $h \leq x_N + h$, completely analogous to (2.36),

$$\frac{|A_2|}{h^{1+\beta-\alpha}} \leq C |u|^{(2+\gamma)}_{s,R_N^T}. \tag{2.38}$$

To estimate the expression $|A_1|/h^{1+\beta-\alpha}$ we use the formula

$$w(x', x_N + h, t) - w(x', x_N, t) = h \int_0^1 w_{x_N}(x', x_N + \theta h, t) d\theta.$$ 

Consequently, in view of $w_{x_N} = u_{x_i x_N}$,

$$\frac{|A_1|}{h^{1+\beta-\alpha}} \leq \int_0^1 \frac{h^\alpha}{(x_N + \theta h)^\alpha} \left| \frac{\Delta^2_{x_i x_N}(x_N + \theta h)\alpha u_{x_i x_N}(x', x_N + \theta h, t)}{h^\beta} \right| d\theta \leq C \left( x_N^\alpha u_{x_i x_N} \right)^{(x_N/h + \theta)^{-\alpha}} \int_0^1 (x_N/h + \theta)^{-\alpha} d\theta \leq C |u|^{(2+\gamma)}_{s,R_N^T}. \tag{2.39}$$

Thus, from (2.38) and (2.39) we obtain (2.33), and so we have also (2.32) and (2.31). Together with (2.30) this completes the proof of (2.28).

We now show the inequality (2.29). Note that for tangential derivatives $u_{x_i}$, $i = 1, N - 1$, this inequality follows from the above estimate (2.28) and from [26], (1.20). However, for $u_{x_N}$ we need a separate proof. We show (2.29) for $u_{x_k}$, $k = 1, N$.

According to [29], it is enough to show that for $h > 0$

$$|\Delta^2_{h,t} u_{x_k}(x, t)| \leq C |u|^{(2+\gamma)}_{s,R_N^T} h^{1+\beta-\alpha}, \tag{2.40}$$

where

$$\Delta^2_{h,t} v(x, t) = \Delta^2_{h,t} v = v(x, t + 2h) - 2v(x, t + h) + v(x, t)$$

is the second difference of a function $v$ with respect to the variable $t$ with the step $h$.

Let first

$$h^{1+\beta-\alpha} \leq x_N. \tag{2.41}$$

Then we use the following interpolation inequality (see, for example, [26], [30], Ch.1)

$$|v|^{(0)}_{\Pi_T(x_N)} \leq C \left( |v|^{(0)}_{\Pi_T(x_N)} \right)^{1/2} \left( |v|^{(2)}_{\Pi_T(x_N)} \right)^{1/2}.$$
Here we denote $\Pi_T(x_N) = \{(y, t) : x_N/2 < y_N < 3x_N/2, 0 < t < T\}$, and $|v|^{(2)}_{\Pi_T(x_N)}$ means $C^2$-norm with respect to $x$-variables over the specified domain. We obtain for $\Delta^2_{h,t}u_{x_k}(x, t)$, that

$$|\Delta^2_{h,t}u_{x_k}(x, t)| \leq C \left( |\Delta^2_{h,t}u|^{(0)}_{\Pi_T(x_N)} \right)^{1/2} \left( |\Delta^2_{h,t}u|^{(2)}_{\Pi_T(x_N)} \right)^{1/2}.$$  

(2.42)

In view of the properties of the space $C^{2+\gamma, 2+\gamma}_{s,R^N_T}$ (см. [26], [29], (1.20)), we have

$$|\Delta^2_{h,t}u|^{(0)}_{\Pi_T(x_N)} \leq C|u|^{(2+\gamma)}_{s,R^N_T} h^{2+\gamma},$$  

(2.43)

$$|\Delta^2_{h,t}u|^{(2)}_{\Pi_T(x_N)} \leq C|u|^{(2+\gamma)}_{s,R^N_T} x^{-\alpha} h^{2+\gamma}.$$  

(2.44)

From (2.43) and (2.44) considering (2.41), we obtain

$$|\Delta^2_{h,t}u_{x_k}(x, t)| \leq C|u|^{(2+\gamma)}_{s,R^N_T} h^{\frac{1+\gamma - \alpha}{(2-\alpha)}},$$  

(2.45)

that is the inequality (2.40).

Let now

$$h^{\frac{1}{2-\alpha}} \geq x_N.$$  

(2.46)

Write $\Delta^2_{h,t}u_{x_k}(x, t)$ as

$$\Delta^2_{h,t}u_{x_k}(x, t) = -\Delta^2_{h,t} \left[ u_{x_k}(x', x_N + h^{\frac{1}{2-\alpha}}, t) - u_{x_k}(x, t) \right] +$$

$$+ \Delta^2_{h,t}u_{x_k}(x', x_N + h^{\frac{1}{2-\alpha}}, t) \equiv A_1 + A_2,$$  

(2.47)

and for $A_2$, by the above case (2.41), the estimate

$$|A_2| \leq C|u|^{(2+\gamma)}_{s,R^N_T} h^{\frac{1+\gamma - \alpha}{(2-\alpha)}}$$

is valid.

To estimate the expression $A_1$, write it as

$$A_1 = -h^{\frac{1}{2-\alpha}} \int_0^1 \Delta^2_{h,t}u_{x_k,x_N}(x', x_N + \theta h^{\frac{1}{2-\alpha}}, t) d\theta.$$  

Thus, we have for $A_1$, that

$$|A_1| \leq C|u|^{(2+\gamma)}_{s,R^N_T} h^{\frac{1}{2-\alpha}} h^\frac{3}{2} \int_0^1 (x_N + \theta h^{\frac{1}{2-\alpha}})^{-\alpha} d\theta \leq$$
\[
\leq C|u|_{s,R^N_T}^{(2+\gamma)} \int_0^1 \theta^{-\alpha} d\theta = C|u|_{s,R^N_T}^{(2+\gamma)} h^{\frac{1+\beta-\alpha}{2-\alpha}},
\]
that is again the inequality (2.40).

The lemma is proved.

Thus, due to the possibility of the local straightening of the boundary, the following is true.

**Lemma 2.3** Let functions \( u^\pm(x,t) \) belong to the spaces \( C^{2+\gamma,2+\gamma,2s}_N(\Omega_T^-) \). Then the functions \( v^\pm(x,t) = u^\pm(x,t)|_{x \in \Gamma} \) belong to the space \( C^{2+\beta-\alpha,2+\beta-\alpha,2s-(\Gamma_T^-)} \), and

\[
|v^\pm|_{C^{2+\beta-\alpha,2+\beta-\alpha,2s-(\Gamma_T^-)}} \leq C |u|_{s,R^N_T}^{(2+\gamma)}. \tag{2.48}
\]

Besides,

\[
|\nabla v^\pm|_{C^{1+\beta-\alpha,1+\gamma/2-(\Gamma_T^-)}} \leq C |u^\pm|_{s,R^N_T}^{(2+\gamma)}. \tag{2.49}
\]

### 2.3 An extension from the surface \( \Gamma_T \) of the functions from the space \( C^{2+\beta-\alpha,2+\beta-\alpha,2s-(\Gamma_T^-)} \).

In this section we prove the converse of Lemma 2.3 that is, we show that any function of the class \( C^{2+\beta-\alpha,2+\beta-\alpha,2s-(\Gamma_T^-)} \) can be extended to all domains \( \Omega_T^+ \) can be extended to all region up to functions of the class \( C^{2+\gamma,2+\gamma,2s}_N(\Omega_T^+) \), and the extension operator is bounded (here, as above \( \beta = \gamma(1 - \alpha/2) \)). Such an extension operator is constructed in the standard way by applying a sufficiently small partition of the unity in the neighborhood of \( \Gamma \) and by the local straightening of the boundary \( \Gamma \) - see [23]. In this case, it is enough to require the \( H^{2+\gamma} \)-smoothness of the boundary \( \Gamma \). Therefore, the existence of the said extension operator follows in the standard way from the following lemma.

**Lemma 2.4** Let in \( R^N_T \) at \( x_N = 0 \) a finite function \( f(x',t) \) from the class \( C^{2+\beta-\alpha,1+\gamma/2}(R^N_{T-1}) \) is given. Then \( f \) can be extended in the domain \( x_N > 0 \) up to the function \( u(x,t) \) from the class \( C^{2+\gamma,2+\gamma,2s}_N(R^N_{T+}) \), and

\[
|u|_{s,R^N_{T+}}^{(2+\gamma)} \leq C |f|_{C^{2+\beta-\alpha,1+\gamma/2}(R^N_{T-1})}. \tag{2.50}
\]

**Proof.** Let \( u(x,t) \) is the solution of the following Dirichlet problem with the parameter \( t \in [0,T] \):

\[
\Delta u = 0, \quad x \in R^N_+(x_N > 0), \tag{2.51}
\]
\( u|_{x_N=0} = f(x', t), \)  
\( (2.52) \)

\( u \to 0, \quad |x| \to \infty. \)  
\( (2.53) \)

As it is well known, the solution of (2.51)-(2.53) is given by the potential of the double layer, defined by the Newton potential.

Note, first, that for the problem (2.51)-(2.53) we have the following maximum principle

\[ |u|^{(0)}_{R^N_1} \leq |f|^{(0)}_{R^{N-1}_1}. \]  
\( (2.54) \)

Indeed, by (2.53), we can choose \( K > 0 \) so large that \( |u| \leq |f|^{(0)}_{R^{N-1}_1} / 2 \) for \( |x| \geq K \), and, by the properties of the double layer potential and the finiteness of \( f \), a \( K \) can be chosen independent of \( t \). Now consider in the domain \( B_K = R^N_+ \cap \{|x| < K\} \) the functions \( v^\pm = \pm u + |f|^{(0)}_{R^{N-1}_1} \). Within this domain we have

\[ \Delta v^\pm = 0, \quad x \in B_K, \]  
\( (2.55) \)

and on the boundary \( \partial B_K = \{x_N = 0, |x'| \leq K\} \cup \{x_N > 0, |x'| = K\} \) the inequality

\[ v^\pm \geq 0, \quad x \in \partial B_K \]  
\( (2.56) \)

holds. It follows from (2.55), (2.56) and from the maximum principle, that \( v^\pm \geq 0 \) in \( B_K \) for all \( t \), and, thus,

\[ |u|^{(0)}_{B_K} \leq |f|^{(0)}_{R^{N-1}_1}, \quad t \in [0, T]. \]

Due to the choice of \( K \), we have the inequality (2.54) on whole domain \( R^N_+ \).

Consider first the properties of the function \( u(x, t) \) with respect to \( t \). Let \( v(x, t) \) is the solution of (2.51)-(2.53) with the boundary condition

\[ v|_{x_N=0} = f_t(x', t), \]  
\( (2.57) \)

instead of (2.52). Consider also for \( h > 0 \) the following function

\[ u_h(x, t) = \frac{u(x, t + h) - u(x, t)}{h}, \]

which satisfies the problem (2.51)-(2.53) with the boundary condition

\[ u_h|_{x_N=0} = f_h(x', t) \equiv \frac{f(x', t + h) - f(x', t)}{h}. \]  
\( (2.58) \)

Let further \( w(x, t) = u_h(x, t) - v(x, t) \), and the function \( w(x, t) \) also satisfies the problem (2.51)-(2.53), but with the boundary condition
Due to the properties of the function $f(x', t)$, we have with some $\theta(x', t, h) \in (0, 1)$ according to the mean value theorem

$$|\varphi_h(x', t)| = |f_h(x', t) - f_t(x', t)| =$$

$$= |f_t(x', t + \theta h) - f_t(x', t)| \leq \langle f_t(x', t) \rangle_{t, R_{N-1}^T}^{(\gamma/2)} h^{\gamma/2} \to 0, \quad h \to 0. \quad (2.60)$$

Consequently, on the base of (2.54),

$$|u_h - v|^{(0)}_{R_{N}^T} \leq C h^{\gamma/2} \to 0, \quad h \to 0. \quad (2.61)$$

This means, that the function $u(x, t)$ has the derivative with respect to the variable $t$ for $x \in R_N^T$, and $u_t(x, t) = v(x, t)$, that is $u_t(x, t)$ satisfies the problem (2.51)-(2.53) with the boundary condition (2.57).

Further, considering the function

$$v_h(x, t) = \frac{u_t(x, t + h) - u_t(x, t)}{h^{\gamma/2}},$$

we see, that it satisfies the same problem with the boundary condition

$$v_h(x, t)|_{x_N=0} = \frac{f_t(x', t + h) - f_t(x', t)}{h^{\gamma/2}} \equiv f_t(x', t). \quad (2.62)$$

Thus, on the base of (2.54) again,

$$\frac{|u_t(x, t + h) - u_t(x, t)|}{h^{\gamma/2}}_{R_{N}^T} \leq \frac{1}{C} \frac{|f_t(x', t + h) - f_t(x', t)|}{h^{\gamma/2}}_{R_{N-1}^T}, \quad (2.63)$$

which, by the arbitrariness of $h$, means that

$$\langle u_t(x, t) \rangle^{(\gamma/2)}_{t, R_{N}^T} \leq C \langle f_t(x', t) \rangle^{(\gamma/2)}_{t, R_{N-1}^T}. \quad (2.64)$$

Consider now the properties of the function $u(x, t)$ with respect to the variables $x$.

First, it follows from the results of [31], [32], that for each $t \in [0, T]$, due to $f \in C^{2+\beta-\alpha}(R_{N-1}^T)$, we have $u \in C^{2+\beta-\alpha}(R_{N}^T)$, and

$$|u|_{C^{2+\beta-\alpha}(R_{N}^T)} \leq C |f|_{C^{2+\beta-\alpha}(R_{N-1}^T)}, \quad (2.65)$$

where the symbol $x$ at the bottom of the space notation means that we consider the smoothness only with respect to $x$. 19
Show that the following estimates

\[ \langle x_N^\alpha D^2 u \rangle_{x,x'R_T^N}^{(\beta)} \leq C |f|_{C^{2+\beta-\alpha}(R_T^{N-1})}, \quad (2.66) \]

\[ \langle x_N^{-\gamma/2} x_N^\alpha D^2 u \rangle_{x,x'R_T^N}^{(\gamma)} \leq C |f|_{C^{2+\beta-\alpha}(R_T^{N-1})}, \quad (2.67) \]

are valid, that is

\[ H^\gamma_{xN} x_N^\alpha D^2 u \leq C |f|_{C^{2+\beta-\alpha}(R_T^{N-1})}. \quad (2.68) \]

We will use the fact that, as it follows from [33], Ch.5.4, the condition 
\[ f \in C^l(R_N^{N-1}) \text{ in } (2.52), \quad l \in (0,2), \]

is equivalent to the condition

\[ |D^k x u| \leq C k x^{-k+l} |f|_{C^l(R_N^{N-1})}, \quad k \geq 2, \quad (2.69) \]

where here and below \( D^k x u = D^k u \) means a derivative of the \( k \)-th order with respect to \( x \) of the function \( u(x,t) \).

Since it is important to prove (2.66) for \( x_N < 1 \) only (for \( x_N > 1 \) such the estimate follows from the local estimates and is well-known), we consider only the case \( x_N < 1 \).

We also use the well-known interpolation inequality

\[ \langle v(x) \rangle_{x,R_T^N}^{(\beta)} \leq C \left( |v(x)|_0^{1-\beta} + |v(x)|_1^{\beta} \right), \quad (2.70) \]

which is valid for the functions \( v(x) \in C^1(\Omega), \Omega \) is a (possibly unbounded) domain with the sufficiently smooth boundary (see, for example, [30], Ch.1).

In addition, at the proof of (2.66), without loss of generality, we prove smoothness of the function \( x_N^\alpha D^2 u \) with respect to \( x' \) and with respect to \( x_N \) separately and we obtain the estimate (2.66) separately for these two cases.

So, let first \( x_N \) is fixed. Then, by (2.70) and (2.69),

\[ \langle x_N^\alpha D^2 u \rangle_{x'}^{(\beta)} \leq C \left( |x_N^\alpha D^2 u|_0^{1-\beta} + |x_N^\alpha D^2 u|_1^{\beta} \right) \leq \]

\[ \leq C |f|_{C^{2+\beta-\alpha}(R_T^{N-1})} \left( x_N^{-2(2+\beta-\alpha)} x_N^{3(2+\beta-\alpha)} \right)^{1-\beta} \left( x_N^{\alpha x_N^{-3(2+\beta-\alpha)}} \right)^{\beta} = C |f|_{C^{2+\beta-\alpha}(R_T^{N-1})}, \quad (2.71) \]

that is estimate (2.66) with respect to \( x' \).

Analogously, using (2.70) and (2.69), we prove (2.67) with respect to \( x' \):
\[ \leq C|f|_{C^2_x^{2+\beta-\alpha}(R_T^N-1)}x_N^{\gamma/2} \left( x_N^{\alpha-2+(2+\beta-\alpha)} \right)^{1-\gamma} \left( x_N^{\alpha-3+(2+\beta-\alpha)} \right)^{\gamma} = C|f|_{C^2_x^{2+\beta-\alpha}(R_T^N-1)}(2.72) \]

We prove now the relations (2.66), (2.67) with respect to the variable \( x_N \). For this we fix some \( \varepsilon_0 \in (0, 1/16) \) and consider the two cases, assuming without loss of generality that \( x_N \leq x_N \).

Let first

\[
|x_N - \bar{x}_N| = (x_N - \bar{x}_N) \geq \varepsilon_0 x_N.
\] (2.73)

Then

\[
\frac{|x_N^{\alpha-\beta}D^2u(x,t) - \bar{x}_N^{\alpha-\beta}D^2u(\bar{x},t)|}{|x_N - \bar{x}_N|^\beta} \leq C \left( |x_N^{\alpha-\beta}D^2u(x,t)| + |\bar{x}_N^{\alpha-\beta}D^2u(\bar{x},t)| \right). (2.74)
\]

In this case, as above

\[
|x_N^{\alpha-\beta}D^2u(x,t)| \leq C|f|_{C^2_x^{2+\beta-\alpha}(R_T^N-1)}x_N^{\alpha-2+(2+\beta-\alpha)} = C|f|_{C^2_x^{2+\beta-\alpha}(R_T^N-1)}, (2.75)
\]

and similarly for \( |\bar{x}_N^{\alpha-\beta}D^2u(\bar{x},t)| \).

In the same way

\[
x_N^{\gamma/2} \frac{|x_N^{\alpha-\beta}D^2u(x,t) - \bar{x}_N^{\alpha-\beta}D^2u(\bar{x},t)|}{|x_N - \bar{x}_N|^\gamma} \leq C \left( |x_N^{\alpha-\beta}D^2u(x,t)| + |\bar{x}_N^{\alpha-\beta}D^2u(\bar{x},t)| \right)
\] (2.76)

and then proceeding as in (2.75).

Let now

\[
0 < (x_N - \bar{x}_N) \leq \varepsilon_0 x_N,
\] (2.77)

and let also

\[
\Pi(x_N) = \{ y \in R_T^N : x_N - 2\varepsilon_0 x_N \leq y_N \leq x_N + 2\varepsilon_0 x_N \},
\]

\[
\Pi_T(x_N) = \Pi(x_N) \times [0, T].
\] (2.78)

Then, taking into account that on \( \Pi_T(x_N) \) we have \( y_N \sim x_N \), as in the previous case
\[
\frac{|x_N^\alpha D^2 u(x, t) - \overline{x}_N^\alpha D^2 u(\overline{x}, t)|}{|x_N - \overline{x}_N|^\beta} \leq \langle y_N^\alpha D^2 u(y, t) \rangle_{y, \Pi_T(x_N)}^{(\beta)} \leq C \left( |y_N^{\alpha-\beta} D^2 u|_{\Pi_T(x_N)}^{(0)} + x_N^\alpha \langle D^2 u(y, t) \rangle_{y, \Pi_T(x_N)}^{(\beta)} \right) \equiv A_1 + A_2. \tag{2.79}
\]

Here \(A_1\) is estimated in the same way as in (2.75), and \(A_2\) - as well as in (2.71), which gives
\[
\frac{|x_N^\alpha D^2 u(x, t) - \overline{x}_N^\alpha D^2 u(\overline{x}, t)|}{|x_N - \overline{x}_N|^\beta} \leq C|f|_{C^{2+\beta-\alpha}(R_N^N)}. \tag{2.80}
\]

The estimate
\[
x_N^{\gamma/2} \frac{|x_N^\alpha D^2 u(x, t) - \overline{x}_N^\alpha D^2 u(\overline{x}, t)|}{|x_N - \overline{x}_N|^\gamma} \leq C|f|_{C^{2+\beta-\alpha}(R_N^N)} \tag{2.81}
\]
is quite similar. This completes the proof of (2.68).

Similarly, we obtain the properties with respect to the variables \(x\) of the derivative \(u_t\), that is,
\[
H_\alpha^\gamma(u_t) \leq C|f|_{C^{2+\beta-\alpha}(R_N^N)} \tag{2.82}
\]
because \(u_t\) satisfies the problem (2.51)-(2.53) with the boundary condition (2.57).

Indeed, since \(u_t|_{x_N=0} = f_t\), so
\[
\langle u_t \rangle_{x,R_N^N}^{(\beta)} \leq C \langle f_t \rangle_{x,R_N^N}^{(\beta)} \leq C|f|_{C^{2+\beta-\alpha,2+\beta-\alpha}(R_N^N)}. \tag{2.83}
\]

Further, for \(x, \overline{x} \in R_N^N, x_N \geq \overline{x}_N\) consider the difference
\[
\Delta(x, \overline{x}) u_t = x_N^{\gamma/2} \frac{|u_t(x, t) - u_t(\overline{x}, t)|}{|x - \overline{x}|^\gamma}. \tag{2.84}
\]

If \(|x - \overline{x}| \geq \varepsilon_0 x_N\), then
\[
\Delta(x, \overline{x}) u_t \leq C x_N^{\gamma/2} \frac{|u_t(x, t) - u_t(\overline{x}, t)|}{|x - \overline{x}|^\beta} \leq C \langle u_t \rangle_{x,R_N^N}^{(\beta)} \leq C|f|_{C^{2+\beta-\alpha,2+\beta-\alpha}(R_N^N)}. \tag{2.85}
\]

If now \(|x - \overline{x}| \leq \varepsilon_0 x_N\), then \(x_N \sim \overline{x}_N\), and then, using (2.69), we obtain that
\[
\Delta(x, \overline{x}) u_t \leq C x_N^{\gamma/2} \frac{|u_t(x, t) - u_t(\overline{x}, t)|}{|x - \overline{x}|} |x - \overline{x}|^{1-\gamma} \leq
\]
\[ C_N^{\gamma/2} \left| \nabla_x u_t(x, t) \right|_{\prod_T(N-x)}^{(0)} \leq C \left| f_t \right|_{C^0_c(R_T^{-1} N - 1)} x_N^{\gamma/2 + 1 - \gamma} x_N^{-1 + \beta} = C \left| f_t \right|_{C^0_c(R_T^{-1} N - 1)}. \]  

(2.86)

Now (2.82) follows from (2.85) and (2.86).

Let us show now the smoothness of the function \( x_N^\alpha D^2 u(x, t) \) with respect to the variable \( t \), that is show that

\[ \left\langle x_N^\alpha D^2 u \right\rangle^{(\gamma/2)}_{t, R_T^{-1} N} \leq C \left| f \right|_{C^{2, \beta - \alpha, 1/2}(R_T^{-1} N - 1)} = C \left| f \right|_{C^{2, \beta - \alpha, 1/2}(R_T^{-1} N - 1)}. \]  

(2.87)

For this we fix some \( h > 0 \) and consider the function

\[ v_h(x, t) = \frac{u(x, t + h) - u(x, t)}{h^{\gamma/2}}, \]  

(2.88)

which satisfies the problem (2.51)-(2.53) with the following boundary condition

\[ v_h(x, t) |_{x_N=0} = \varphi_h(x, t) \equiv \frac{f(x, t + h) - f(x, t)}{h^{\gamma/2}}. \]  

(2.89)

It follows from the results of [26], (1.20), that uniformly with respect to the variable \( t \) the function \( \varphi_h(x, t) \in C^{2, \alpha}_x(R_T^{-1} N - 1) \) with respect to the variables \( x \), and

\[ \max_{t \in [0, T]} \left| \varphi_h(\cdot, t) \right|^{(2-\alpha)}_{R_T^{-1} N - 1} \leq C \left| f \right|_{C^{2, \beta - \alpha, 1/2}(R_T^{-1} N - 1)}. \]  

(2.90)

Note now, that

\[ x_N^\alpha D^2 u(x, t + h) - x_N^\alpha D^2 u(x, t) \]  

\[ \left| \frac{\gamma}{2} \right| \]  

\[ x_N^\alpha D^2 v_h(x, t) = x_N^\alpha D^2 v_h(x, t). \]

Consequently, it follows from (2.69) that

\[ \left( x_N^\alpha D^2 v_h(x, t) \right) \leq C x_N^\alpha \max_{t \in [0, T]} \left| \varphi_h(\cdot, t) \right|^{(2-\alpha)}_{R_T^{-1} N - 1} x_N^{-2 + (2-\alpha)} = C \max_{t \in [0, T]} \left| \varphi_h(\cdot, t) \right|^{(2-\alpha)}_{R_T^{-1} N - 1}. \]  

(2.91)

So, (2.87) follows from (2.91) and (2.90), in view of the definition of \( v(x, t) \).

Multiplying now the function \( u(x, t) \) by a smooth finite function \( \eta(x) \), which is equal to one in a neighborhood of support of \( f(x', t) \), we get a finite extension of \( f(x', t) \) of desired class, and the estimate (2.50).

The lemma 2.4 is proved.

From this lemma in the standard way (see [23]), as it was described in the beginning of this section, we get the following assertion.
Lemma 2.5 There exist bounded extension operators $E^+$ and $E^-$, such that

$$\rho \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T) \rightarrow E\rho \equiv E^+\rho \equiv \rho^+ \in C_s^{2+\gamma, \frac{2+\gamma}{2-\alpha}}(\Omega_T^+),$$

(2.92)

$$|E^\pm\rho|_{C_s^{2+\gamma, \frac{2+\gamma}{2-\alpha}}(\Omega_T^\pm)} \leq C|\rho|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T)},$$

(2.93)

and we can assume that the supports of the extended functions $\rho^\pm$ are included in the sufficiently small neighbourhood $\mathcal{N}_T$ of the surface $\Gamma_T$. We will denote the extended functions $\rho \equiv E\rho \equiv E^\pm\rho$ by the same symbol $\rho$ to not to overload the notation, that is,

$$\rho|_{\Omega_T^\pm} \equiv \rho^\pm \equiv E\rho|_{\Omega_T^\pm}.$$  

(2.94)

Besides, as it follows from the results of [24] and from the lemma 2.4 (as the lemma 2.4 permits to reduce the situation to the homogeneous boundary conditions), the following assertion is valid.

Lemma 2.6 Let functions $f$ and $g$ are finite, and

$$f(x', t) \in C_{0}^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1}), \quad g(x, t) \in C_{x,0}^{7}(R_T^{N}).$$

Then the problem

$$\frac{\partial u}{\partial t} - x_N^\alpha \Delta u = g(x, t), \quad (x, t) \in R_T^{N},$$

(2.95)

$$u(x', 0, t) = f(x', t), \quad x_N = 0, t \in [0, T],$$

(2.96)

$$u(x, 0) = 0, \quad x \in R_T^{N}$$

(2.97)

has the unique solution $u(x, t)$, which satisfies the estimate

$$|u|_{s, R_T^{N}} \leq C \left( |f|_{C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(R_T^{N-1})} + |g|_{s, R_T^{N}} \right).$$

(2.98)

In the same way, with the help of results of [24] and from the lemma 2.4 we get the following theorem.

Consider the Cauchy-Dirichlet problem for the degenerate equations of the form

$$\frac{\partial u^\pm}{\partial t} - (d^\pm(x))^\alpha B^\pm(x, t) \Delta u^\pm = g^\pm(x, t), \quad (x, t) \in \Omega_T^\pm,$$

(2.99)
\[ u^\pm|_{\Gamma_T} = f^\pm(x,t), \quad (2.100) \]
\[ u^\pm|_{\Gamma_T^+} = h^\pm(x,t), \quad (2.101) \]
\[ u^\pm(x,0) = 0, \quad (2.102) \]
where the functions \( d^\pm(x) \) are introduced in (1.11),
\[ B^\pm(x,t) \in C^{\gamma,\gamma/2}(\Omega_T^\pm), \quad \nu \leq B^\pm \leq \nu^{-1}, \]
\[ g^\pm \in C^{\gamma,\gamma/2}(\Omega_T^\pm), \quad f^\pm \in C_0^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T), \quad h^\pm \in C_0^{2+\gamma,\frac{2+\gamma}{2}}(\Gamma_T^\pm). \quad (2.103) \]

**Theorem 2.7** The problem (2.99) - (2.102) has the unique solution from the space \( C_0^{2+\gamma,\frac{2+\gamma}{2}}(\Omega_T^\pm) \) and the following estimate is valid
\[ |u^\pm|^{(2+\gamma)}_{s,\Omega_T^\pm} \leq C \left( |g^\pm|^{(\gamma)}_{s,\Omega_T^\pm} + |f^\pm|^{(2+\beta-\alpha,\frac{2+\beta-\alpha}{2-\alpha})}_{\Gamma_T} + |h^\pm|^{(2+\gamma,\frac{2+\gamma}{2})}_{\Gamma_T^\pm} \right). \quad (2.104) \]

### 3 The model problem for the two phase Stefan problem for the degenerate equations.

Let \( a \geq 0 \) is a fixed number. Denote \( Q_+^N = \{(x,t) : x \in R_+^N, t \geq -a\} \), \( Q_+^{N-1} = \{(x',t) : x' \in R_+^{N-1}, t \geq -a\} \). Denote further \( R_{+T}^N = Q_+^N \cap \{t \leq T\} \), \( R_{T,a}^N = Q_+^{N-1} \cap \{t \leq T\} \). It is convenient to consider the domains with the \( t \geq -a \), as it will allow us to consider the points with \( t = 0 \) as interior points of general position, which will facilitate the further notation. We agree, which is similar to (1.21), that zero at the bottom of the designation of the spaces of functions defined in these domains means the subspace of the corresponding space whose elements vanish at \( t = -a \) together with all its derivatives with respect to \( t \), which are permitted by the space.

Let \( f(x',t) \) is a finite with respect to \( x \) function, which is defined in \( Q_+^{N-1} \) and is such that
\[ f(x',-a) \equiv 0, \quad f \in C_0^{1+\beta-\alpha,\frac{1+\beta-\alpha}{2-\alpha}}(Q_+^{N-1}), \quad (3.1) \]
which allows us to consider \( f \) as the functions, which is defined for \( t \in (-\infty, \infty) \), extending it by identical zero in the domain \( t < -a \) with the preservation of the class.

Let further
Consider the following model problem for the triple of the unknown functions $u^\pm(x, t)$ and $\rho(x', t)$, which are defined in $Q^N$ and $Q^{N-1}$ correspondingly:

$$\frac{\partial u^\pm}{\partial t} - (\pm x_N)^2 \Delta u^\pm = f_1^\pm, \quad (x, t) \in Q^N_+,$$  \hspace{1cm} (3.3)

$$u^\pm + A^\pm \rho = f_2^\pm, \quad x_N = 0, (x', t) \in Q^{N-1},$$  \hspace{1cm} (3.4)

$$\rho_t - \varepsilon \Delta x' \rho + b^+ \frac{\partial u^+}{\partial x_N} - b^- \frac{\partial u^-}{\partial x_N} = f(x', t), \quad x_N = 0, (x', t) \in Q^{N-1},$$  \hspace{1cm} (3.5)

$$u^\pm(x, -a) = 0, \quad \rho(x', -a) = 0,$$  \hspace{1cm} (3.6)

where $A^\pm$, $b^\pm$, $\varepsilon$ are given positive constants and $\varepsilon \in (0, 1)$.

Note that the term with $\varepsilon$ in (3.5) does not apply directly to the Stefan problem and serves as a regularization of the problem, that will be needed in the proof of the solvability of the corresponding linearized Stefan problem in an arbitrary domain. To the author’s knowledge, this regularization of the boundary condition in the Stefan problem was first used in the paper [1].

Below we prove the following a priori estimate of the solution of the problem (3.3) - (3.7).

**Theorem 3.1** Let $u^\pm(x, t) \in C_{s,0}^{2+\gamma, \frac{2+\gamma}{2}}(Q^N)$, $\rho \in C_{0}^{3+\beta-\alpha, 1+\gamma/2}(Q^{N-1})$, $\rho_t \in C_0^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(Q^{N-1})$, (3.7) are a finite solution of the problem (3.3) - (3.7). Then for arbitrary $T > 0$ the following estimate is valid:

$$U(T) \equiv |u^+|_{s, R^N_T}^{(2+\gamma)} + |u^-|_{s, R^N_{-T}}^{(2+\gamma)} + \varepsilon \sum_{i,j=1}^{N-1,a} |\rho x_{ij}|_{s, R^N_{-T}}^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(R_{T}^{N-1,a}) +$$

$$+|\rho|_{s, R^N_{-T}}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} + |\rho_t|_{s, R^N_{-T}}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \leq$$

$$\leq C_T \left( |f_1^+|_{s, R^N_T}^{(\gamma)} + |f_1^-|_{s, R^N_{-T}}^{(\gamma)} + |f_2^+|_{s, R^N_{-T}}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha})} + \right)$$
\[ + |f^-_2|_{C^{2+\beta-\alpha,2+\beta-\alpha}(R_T^{N-1,a})} + |f^+_1|_{C^{1+\beta-\alpha,1+\beta-\alpha}(R_T^{N-1,a})} \equiv C_T \mathcal{M}(T), \quad (3.8) \]

where the constant \( C_T \) in (3.8) does not depend on \( \varepsilon \in (0,1) \).

Subsequent content of this section is the proof of the theorem 3.1.

Note first that by lemmas 2.6 and 2.2 we can without loss of generality assume

\[ f^\pm_1 \equiv 0, \quad f^\pm_2 \equiv 0, \quad (3.9) \]

since the general case can be reduced to the specified one by the change of the unknown functions \( u^\pm = v^\pm + w^\pm \), where \( v^\pm \) are the new unknowns, and \( w^\pm \) satisfy (3.3) with the boundary conditions

\[ w^\pm_{x_N=0} = f^\pm_2(x', t). \quad (3.10) \]

Thus, further we assume that only the function \( f(x', t) \) is nonzero in the righthand sides of (3.3) - (3.5).

In addition, because the right side of the relations (3.3) - (3.5) belong to the classes with zero at the bottom and because of conditions (3.6), (3.7) we can consider that the relation (3.3) - (3.5) are valid and for \( t < -a \), assuming that all the functions are extended by zero to this domain.

An important point of proving (3.8) is to prove the following a priori estimate.

**Lemma 3.2** Under the conditions of the theorem 3.1 and under the condition (3.9) the following estimate is valid

\[ \langle \nabla x' \rho \rangle^{(1+\beta-\alpha)}_{x',R_T^{N-1,a}} \leq C_T \left( |\nabla x' u^+|^{(0)}_{R_T^{N,a}} + |\nabla x' u^-|^{(0)}_{R_T^{N,a}} + |\nabla x' \rho|^{(0)}_{R_T^{N-1,a}} + \mathcal{M}(T) \right) \equiv \]

\[ \equiv C_T (\mathcal{N}(T) + \mathcal{M}(T)) \leq C_T (T + a)^4 \mathcal{U}(T) + C_T \mathcal{M}(T), \quad (3.11) \]

where

\[ \mathcal{N}(T) \equiv |\nabla x' u^+|^{(0)}_{R_T^{N,a}} + |\nabla x' u^-|^{(0)}_{R_T^{N,a}} + |\nabla x' \rho|^{(0)}_{R_T^{N-1,a}}. \]

To obtain the last inequality in (3.11) we use the estimates (2.1) - (2.4).

**Proof.**

Denote for brevity, \( l = 1 + \beta - \alpha \) and fix a point \((x'_0, t_0)\) in the set \( R_T^{N-1,a} \). In order to maintain the succession of the notations with the paper [28], whose method we’re going to apply, without loss of generality, we will assume that \((x'_0, t_0) = (0,0)\) - this choice is not important, as can be seen from the following

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proof. Suppose, further, \( O = (x' = 0, x_N = 0, t = 0) \) is the corresponding point in \( R_{N,T}^{x,a} \). We show that for every \( h \in (0, 1) \) and for any \( i, j = 1, N - 1 \) we have the following inequality

\[
|\rho_{x_i}(\vec{e}_j h, 0) - \rho_{x_i}(-\vec{e}_j h, 0)| \leq C_T (N(T) + M(T)) h^l,
\]

where \( \vec{e}_j \) is the unit vector of the \( Ox_j \)-axis. Since the point \( O \) and the step \( h \in (0, 1) \) in the relation (3.12) are arbitrary, the estimate (3.11) of the lemma follows from the estimate (3.12).

So, let \( y_1, y_2 \in [0, 1] \), \( y \equiv (y_1, y_2) \) and let also \( i, j \in \{1, 2, ..., N\} \) are fixed.

Consider the differences

\[
v^\pm(x, t, y_1, y_2) = \Delta_{i,y_1} \Delta_{j,y_2} u^\pm(x, t) =
\]

\[
= u^\pm(x + y_1 \vec{e}_i + y_2 \vec{e}_j, t) - u^\pm(x - y_1 \vec{e}_i + y_2 \vec{e}_j, t) -
\]

\[
-w^\pm(x + y_1 \vec{e}_i - y_2 \vec{e}_j, t) + w^\pm(x + y_1 \vec{e}_i + y_2 \vec{e}_j, t),
\]

where

\[
\Delta_{k,h} u(x, t) \equiv u(x + h \vec{e}_k, t) - u(x - h \vec{e}_k, t).
\]

Denote also

\[
r(x', t, y_1, y_2) = \Delta_{i,y_1} \Delta_{j,y_2} \rho(x', t).
\]

Note that

\[
\frac{\partial^2 v^\pm}{\partial x_i^2} - \frac{\partial^2 v^\pm}{\partial y_1^2} = 0, \quad \frac{\partial^2 v^\pm}{\partial x_j^2} - \frac{\partial^2 v^\pm}{\partial y_2^2} = 0.
\]

Therefore in domains \( R_{N}^{x} \times \{-\infty < t < T\} \times \{0 < y_1 < 1\} \times \{0 < y_2 < 1\} \) the functions \( v^\pm(x, t, y) \) satisfy the equations

\[
L^* v^\pm \equiv \frac{\partial v^\pm}{\partial t} -
\]

\[
-(\pm x_N)^\alpha \left( \sum_{k \neq i,j} \frac{\partial^2 v^\pm}{\partial x_k^2} + \frac{3}{4} \frac{\partial^2 v^\pm}{\partial x_i^2} + \frac{3}{4} \frac{\partial^2 v^\pm}{\partial x_j^2} + \frac{1}{4} \frac{\partial^2 v^\pm}{\partial y_1^2} + \frac{1}{4} \frac{\partial^2 v^\pm}{\partial y_2^2} \right) = 0,
\]

Note also, that

\[
|v^\pm| = |\Delta_{i,y_1} \Delta_{j,y_2} u^\pm(x, t)| =
\]

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\[ = y_2 \left| \Delta_{x,y_1} \int_{-1}^{1} u_{x y_2}^\pm (x + \omega y_2 \overrightarrow{e}_j) d\omega \right| \leq 4y_2\mathcal{N}(T). \]

Exactly the same way

\[ |v^\pm| \leq 4y_1\mathcal{N}(T), \]

and therefore

\[ |v^\pm| \leq 4y_{\min}\mathcal{N}(T), \tag{3.18} \]

where

\[ y_{\min} = \min \{y_1, y_2\}. \tag{3.19} \]

Similarly, we have

\[ |r| \leq 4y_{\min}\mathcal{N}(T), \tag{3.20} \]

Denote

\[ y = (y_1, y_2), \quad P^\pm = \{(x, t, y) : |x_m| < 1, m = 1, N-1, 0 < \pm x_N < 1, \]

\[ -1 < t < 0, 0 < y_k < 1, k = 1, 2\}. \tag{3.21} \]

Denote also

\[ \Sigma^\pm = \partial P^\pm \setminus (\{t = 0\} \cup \{x_N = 0\}), \quad \Sigma_0 = \partial P^\pm \cap \{x_N = 0\}, \tag{3.22} \]

that is $\Sigma^\pm$ - are parabolic boundaries of the parallelepipeds $P^\pm$ without their common part $\{x_N = 0\}$, and the last will be denoted by $\Sigma_0$.

In the parallelepipeds $P^\pm$ the functions $v^\pm$ and $r$ satisfy the following problem

\[ L^* v^\pm = 0, \quad (x, t, y) \in P^\pm, \tag{3.23} \]

\[ v^\pm|_{\Sigma^\pm} = g^\pm(x, t, y), \tag{3.24} \]

\[ v^\pm + A^\pm r = 0, \quad x_N = 0, \tag{3.25} \]

\[ r_t - \varepsilon \Delta r + b^+ \frac{\partial v^+}{\partial x_N} - b^- \frac{\partial v^-}{\partial x_N} = F(x, y, t), \quad x_N = 0, \tag{3.26} \]
where

\[ F(x, t, y) = \Delta_{i,y} \Delta_{j,y} f, \quad g^\pm(x, t, y) = \Delta_{i,y} \Delta_{j,y} u^\pm, \]  

(3.27)

and, in view of the assumptions (3.1),

\[ |F(x, t, y)| \leq 2 \langle f \rangle (2y_2). \]

A similar inequality with replacing \( j \) by \( i \) and \( y_2 \) by \( y_1 \) gives similar to (3.18)

\[ |F(x, t, y)| \leq CM(T)y_1. \]  

(3.28)

Note also that by (3.18),

\[ |g^\pm(x, t, y)| \leq CN(T)y_2. \]  

(3.29)

To estimate \( v^\pm \) and \( r \), we’re going to apply to the problem (3.23) - (3.26) the maximum principle in the following form.

**Lemma 3.3** Let functions \( H^\pm(x, t, y) \in C^{2,1}(P^\pm) \cap C^{1,0}(\overline{P}^\pm) \), \( S(x', t, y) \in C^{2,1}(\Sigma_0) \) satisfy the conditions

\[ L^* H^\pm \geq 0, \quad (x, t, y) \in P^\pm, \]  

(3.30)

\[ H^\pm|_{\Sigma^\pm} \geq 0, \]  

(3.31)

\[ H^\pm + A^\pm S = 0, \quad x_N = 0, \]  

(3.32)

\[ S_t - \varepsilon \Delta_{x'} S + b^+ \frac{\partial H^+}{\partial x_N} - b^- \frac{\partial H^-}{\partial x_N} \leq 0, \quad x_N = 0. \]  

(3.33)

Then

\[ H^\pm \geq 0, \quad (x, t, y) \in \overline{P}^\pm; \quad S \leq 0, \quad (x', t, y) \in \Sigma_0. \]  

(3.34)

We do not give a detailed proof of this lemma, since it uses standard arguments. We only note that the functions \( H^\pm \) can not reach a negative minimum at \( \{ x_N = 0 \} \), as in this case, by (3.32), they would reached a negative minimum simultaneously and corresponding point would be, again by (3.32), a point of a positive maximum of the function \( S \). All this together in the standard way contradicts the boundary condition (3.33).

We shall need the the auxiliary functions \( w^\pm(x, t) \), defined on

\[ \Pi^\pm = \{ [x_m] \leq 1, m = 1, N - 1, 0 \leq \pm x_N \leq 1, -1 \leq t \leq 0 \} \]
correspondingly, and such that

\[ L_{tx}w^\pm = \frac{\partial w^\pm}{\partial t} - \]

\[-(\pm x_N)^{\alpha} \left( \sum_{k \neq i,j} \frac{\partial^2 w^\pm}{\partial x_k^2} + \frac{3}{4} \frac{\partial^2 w^\pm}{\partial x_i^2} + \frac{3}{4} \frac{\partial^2 w^\pm}{\partial x_j^2} \right) = 0, \quad x_N \neq 0, \quad (3.35)\]

\[ w^\pm|_{\{|x_k| = 1\} \cup \{t = -1\}} \geq \nu > 0, \quad (3.36)\]

\[ w^\pm(0,0) = 0, \quad w^\pm|_{\Pi^\pm} \geq 0, \quad (3.37)\]

\[ w^\pm(x, t) \in C^{2,1}(\Pi^\pm \cap \{x_N = 0\}). \quad (3.38)\]

Such functions can be constructed as follows. Consider for example, \( w^+(x, t) \).

Let \( G^+(x, t) \) is a function from \( C^\infty \) in \( \mathbb{R}^N_+ \times (-\infty, \infty) \), such that \( G^+ \equiv 0 \) for \( \|x\| + |t| \leq 1/4 \) and for \( t \leq -2, \|x\| \geq 2 \) and \( G^+ > 0 \) in the other points of \( \mathbb{R}^N_+ \times (-\infty, \infty) \). Let \( w^+(x, t) \) is the solution of the following initial boundary value problem in half-space

\[ L_{tx}w^+ = 0, \quad x_N > 0, \quad t > -2, \]

\[ w^+|_{x_N = 0} = G^+(x, t) \in C^\infty, \]

\[ w^+|_{t = -2} = 0. \]

Lemma 2.6 implies that the function \( w^+ \) exists in the appropriate class, and

\[ |w^+|^{(2+\gamma)}_{L^2(\mathbb{R}^N_+ \times [-2, 0])} \leq C(G^+). \quad (3.39)\]

Because of the properties of \( G^+(x, t) \) and by the strong maximum principle (see [34]), the function \( w^+ \) has all desirable properties, including (3.36).

Now consider the following comparison functions defined on \( \mathcal{P}^\pm \). Denote

\[ \varphi(y) = y_1 y_2 \left( y_1^l + y_2^l \right)^{-\frac{l}{l+1}}, \quad (3.40)\]

\[ \psi^\pm(x_N, y) = y_1 y_2 \left[ (y_1 \pm x_N)^l + (y_2 \pm x_N)^l \right]^{-\frac{l}{l+1}}, \quad \pm x_N \geq 0, \quad (3.41)\]

\[ \theta^\pm(x, t, y) = \begin{cases} 
(y_1^{-1} + y_2^{-1})^{-1} w^\pm(x, t), & y_{\min} > 0, \\
0, & y_{\min} = 0.
\end{cases} \quad (3.42)\]
The direct verification shows (cf. [28]), that the functions $\varphi$ and $\psi^\pm$ possess properties

\[
\pm \frac{\partial \psi^\pm}{\partial x_N}|_{x_N=0} \leq -\nu y^l_{\min}, \quad (3.43)
\]

\[
|L^*\psi^\pm| \leq C|x_N|^\alpha y^{-1+l}_{\min}, \quad (3.44)
\]

\[
\frac{\partial \varphi}{\partial x_N}|_{x_N=0} = 0, \quad (3.45)
\]

\[
L^*\varphi \geq \nu|x_N|^\alpha y^{-1+l}_{\min}, \quad (3.46)
\]

\[
\varphi|_{y_k=1} \geq \nu y_{\min}. \quad (3.47)
\]

Thus, if we choose a sufficiently large constant $K > 0$, the functions

\[
h^\pm \equiv \psi^\pm + K\varphi \quad (3.48)
\]

will have the properties

\[
\pm \frac{\partial h^\pm}{\partial x_N}|_{x_N=0} \leq -\nu y^l_{\min}, \quad (3.49)
\]

\[
L^*h^\pm \geq \nu|x_N|^\alpha y^{-1+l}_{\min} > 0, \quad (x, t, y) \in P^\pm, \quad (3.50)
\]

\[
h^\pm|_{y_k=1} \geq \nu y_{\min}. \quad (3.51)
\]

At the same time, the functions $\theta^\pm(x, t, y)$ have the properties

\[
L^*\theta^\pm \geq 0, \quad (x, t, y) \in P^\pm, \quad (3.52)
\]

\[
\left| \frac{\partial \theta^\pm}{\partial x_N} \right| |_{x_N=0} \leq C y_{\min}, \quad (3.53)
\]

\[
\theta^\pm|_{\bigcup \{k|x_k=1\}\cup\{t=-1\}} \geq \nu y_{\min}. \quad (3.54)
\]

Consider now the following comparison functions

\[
H^\pm(x, t, y) \equiv A^\pm \left[ L_1 \theta^\pm(x, t, y) + L_2 h^\pm(x_N, y) \right] (N(T) + M(T)), \quad (3.55)
\]

\[
S(x', t, y) = - \left[ L_1 \theta^\pm(x', 0, t, y) + L_2 h^\pm(0, y) \right] (N(T) + M(T)), \quad (3.56)
\]
where $L_1$ и $L_2$ are some positive constants.

Choosing first $L_1$ and then $L_2$ are sufficiently large, and using on one hand (3.28), (3.29), and on the other hand (3.49) - (3.54), we see that the triple of the functions

\[ H^+ \pm v^+(x, t, y), \quad H^- \pm v^-(x, t, y), \quad S \pm r(x', t, y) \]

satisfies in $P^\pm$ to the conditions of the lemma 3.3. Hence,

\[ H^+ \pm v^+ \geq 0, \quad H^- \pm v^- \geq 0, \quad S \pm r \leq 0, \]

that is

\[ |v^\pm| \leq CH^\pm, \quad |r| \leq C|S|, \quad (x, t, y) \in \overline{P}^\pm. \quad (3.57) \]

Taking in (3.57) $x = 0, t = 0$, in view of $\theta^\pm(0, 0, y) = 0$, we obtain, for example, for $v^+$ similarly [28]

\[ |\Delta_{i,y_1} \Delta_{j,y_2} u^+(0, 0)| \leq C [N(T) + M(T)] y_1 y_2 (y_1 + y_2)^{-\theta + 1}. \]

Dividing both sides of this relation by $y_1$ and taking the limit with $y_1 \to 0$, we obtain

\[ \left| \frac{\partial u^+}{\partial x_i} (0 + y_2 \vec{e}_j, 0) - \frac{\partial u^+}{\partial x_i} (0 - y_2 \vec{e}_j, 0) \right| \leq C [N(T) + M(T)] y_2. \quad (3.58) \]

and similarly

\[ \left| \frac{\partial r}{\partial x_i} (0 + y_2 \vec{e}_j, 0) - \frac{\partial r}{\partial x_i} (0 - y_2 \vec{e}_j, 0) \right| \leq C [N(T) + M(T)] y_2. \quad (3.59) \]

Since all of the above arguments are valid, as noted, for any $(x'_0, t_0) \in R^{N-1,a}_T$, by the same token the estimate (3.11) and the lemma 3.2 are proved.

We continue the proof of the theorem. It follows from (3.5), that

\[ \rho_t - \varepsilon \Delta_{x'} \rho = F(x', t) \equiv -b^+ \frac{\partial u^+}{\partial x_N} + b^- \frac{\partial u^-}{\partial x_N} + f, \quad (3.60) \]

Moreover, in view of the inequalities (1.32), (2.1), (2.4)

\[ |F|_{R^{N-1}_T} \leq C \left( |\nabla u^+|_{R^{N-1}_T}^\gamma + |\nabla u^-|_{R^{N-1}_T}^\gamma \right) + CM(T) \leq \]

\[ \leq C(T + a)^\mu U(T) + CM(T), \quad (3.61) \]

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and

$$F(x', 0) \equiv 0.$$  

Making in the problem (3.60), (3.6) the change of variables $x' = \varepsilon^{1/2} y$, we obtain the problem

$$\rho_t - \Delta_y \rho = \tilde{F}(y, t), \quad (y, t) \in R^N_{y, a}, \quad \rho(y, -a) = 0,$$

(3.62)

and

$$\left\langle \tilde{F}(y, t) \right\rangle_{t, R^N_{y, a}} = \left\langle \tilde{F}(x', t) \right\rangle_{t, R^N_{x', a}}.$$  

(3.63)

It follows from the arguments of [23], гл.IV, that

$$\left\langle \rho_{y_i y_j} \right\rangle_{y, R^N_{y, a}} \leq C \left\langle \tilde{F}(y, t) \right\rangle_{y, R^N_{y, a}}.$$  

(3.64)

$$\left\langle \rho_t \right\rangle_{t, R^N_{y, a}} \leq C \left( \left\langle \tilde{F}(y, t) \right\rangle_{y, R^N_{y, a}} + \left\langle \tilde{F}(y, t) \right\rangle_{t, R^N_{y, a}} \right).$$  

(3.65)

Making in (3.64), (3.65) the inverse change of variables, in view of (3.63) we obtain

$$\left\langle \rho_t(x', t) \right\rangle_{t, R^N_{x', a}} + \varepsilon \sum_{i,j} \left\langle \rho_{x_i x_j} \right\rangle_{x', R^N_{x', a}} \leq C |F(x', t)|_{R^N_{x', a}}.$$  

(3.66)

Thus, in view of the estimate (3.11) of the lemma 3.2, it is proved, that

$$|\rho|_{R^N_{x', a}} + (\nabla_x \rho)_{x', R^N_{x', a}} + \varepsilon |\rho|_{R^N_{x', a}} \leq C(T + a)^{\mu} U(T) + CM(T),$$

or, in view of (1.20), [26] and of the finiteness of $\rho$,

$$|\rho|_{R^N_{x', a}} + (\nabla_x \rho)_{x', R^N_{x', a}} + \varepsilon |\rho|_{R^N_{x', a}} \leq C(T + a)^{\mu} U(T) + CM(T),$$

(3.67)

where the constant $C$ does not depend on $\varepsilon > 0$.

Now, considering $u^\pm(x, t)$ as the solution of the Cauchy-Dirichlet problem (3.3), (3.4), by the lemma 2.6 and the estimate (3.67), we conclude that

$$|u^+|_{s, R^N_{x, T}} + |u^-|_{s, R^N_{x, T}} \leq C(T + a)^{\mu} U(T) + CM(T).$$

(3.68)

It follows that in the condition (3.5)
\[ \left| \frac{\partial u^+}{\partial x_N} \right|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(R_T^{N-1,a})} + \left| \frac{\partial u^-}{\partial x_N} \right|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(R_T^{N-1,a})} \leq C(T+a)^\mu \mathcal{U}(T) + C \mathcal{M}(T). \]

Thus, the function \( \rho(x',t) \) satisfies the Cauchy problem (3.60), (3.6) with the right hand side \( F \) and the last has the property

\[ F(x',-a) = 0, \quad |F|_{C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2}}(R_T^{N-1,a})} \leq C(T+a)^\mu \mathcal{U}(T) + C \mathcal{M}(T). \] (3.69)

Making again in (3.60), (3.6) the change of variables \( x = \varepsilon^{1/2} y \), we arrive at the problem of the form (3.62) with \( \tilde{F} \), where the last is such that

\[ \langle \tilde{F} \rangle_{t,R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq C \langle F \rangle_{t,R_T^{N-1,a}}^{(1+\beta-\alpha)} \]

As above, completely similar to [23], Ch.IV, for solutions of the problem (3.62) we have the estimates

\[ \langle \rho_t \rangle_{t,R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq C \langle \tilde{F} \rangle_{t,R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq C(T+a)^\mu \mathcal{U}(T) + C \mathcal{M}(T), \] (3.71)

\[ \sum_{i,j=1}^{N-1} \langle \rho_{y_i y_j} \rangle_{y,R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq \langle \tilde{F} \rangle_{y,R_T^{N-1,a}}^{(1+\beta-\alpha)}, \] (3.72)

and we note that in obtaining the estimate (3.71) the condition \( F(x',-a) = 0 \) is important.

Proceeding as before and going back to the variables \( x' \), we find from (3.72) and (3.70) that

\[ \varepsilon \sum_{i,j=1}^{N-1} \langle \rho_{x_i x_j} \rangle_{x',R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq C \langle F \rangle_{x',R_T^{N-1,a}}^{(1+\beta-\alpha)} \leq C(T+a)^\mu \mathcal{U}(T) + C \mathcal{M}(T). \] (3.73)

Now combining the estimates (3.73), (3.71), (3.68) and (3.67), we find that

\[ \mathcal{U}(T) \leq C(T+a)^\mu \mathcal{U}(T) + C \mathcal{M}(T). \] (3.74)

Taking now in (3.71) \( T = T_0 \), so that the value of \( T_0 + a > 0 \) is sufficiently small, we obtain estimate (3.8) on the interval \([-a, T_0] \). Considering further the problem (3.3) - (3.7) on the interval \([-a + (a + T_0)/2, -a + 3(a + T_0)/2] \) and removing the initial data with the known functions, that is moving along the axis of \( Ot \) up, exactly as in [22], Ch.IV, we obtain the assertion of the theorem 3.1 on an arbitrary time interval \([-a, T] \).

Thus, the theorem 3.1 is proved. □
4 Reduction of the problem (1.3)-(1.7) to the problem in the fixed domain.

Let $\rho(\omega, t)$ be the unknown function defined in Section 4 and parameterizing unknown (free) boundary $\Gamma_{\rho,T}$ ($\rho(\omega,0) \equiv 0$), and let $\rho(x, t) = E\rho(\omega, t)$ be the extension of this function to the whole domain $\Omega_T$ by the extension operator $E$ from (2.91).

We pass in the problem (1.3)-(1.7) from the unknown functions $v^\pm(y, \tau)$ to the unknown functions after this change of variables, that is, lem (4.1)-(4.5) reduces to the following problem in the known fixed domains $\Omega^\pm_T$.

Then, in view of the properties of (1.31). Denote for simplicity by the same symbols $v^\pm(y, \tau)$ the unknown functions defined in Section 1 and parameterizing unknown functions $v^\pm(y, \tau)$, that is,

$$v^\pm(x, t) \equiv v^\pm(y, \tau) \circ e_\rho(x, t).$$

Then, in view of the properties of $(y, \tau) = e_\rho(x, t)$, in the variables $(x, t)$ the problem (4.1)-(4.5) reduces to the following problem in the known fixed domains $\Omega^\pm_T$:

$$L_\rho(v^\pm) v^\pm = \frac{\partial v^\pm}{\partial t} - h_\rho^\pm \rho_t - |v^\pm|^{\alpha} \nabla^2_{\rho} v^\pm(x, t) = 0, \ (x, t) \in \Omega^\pm_T, \quad (4.6)$$

$$v^+(x, t) = v^-(x, t) = 0, \ (x, t) \in \Gamma_T, \quad (4.7)$$

$$(1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_i} \rho_{\omega_j})(a^+ \frac{\partial v^+}{\partial \lambda} - a^- \frac{\partial v^-}{\partial \lambda}) = -k \rho_t (1 + \rho_\lambda), \ (x, t) \in \Gamma_T, \quad (4.8)$$

$$v^\pm(x, t) = h^\pm(x, t), \ (x, t) \in \Gamma^\pm_T, \quad (4.9)$$

$$v^\pm(x, 0) = v_0^\pm(x), \ x \in \Omega^\pm, \ \rho(\omega, 0) \equiv 0, \quad (4.10)$$

$$\rho(x, t) = E\rho(\omega, t), \quad (4.11)$$
where \( \nabla_\rho \equiv E_\rho \nabla_x \), and the matrix \( E_\rho \) is the conjugate and inverse to Jacobi matrix of the mapping (1.31) for \( t = \text{const} \), \( m_{ij}(x, \rho) \) are some given smooth functions of their arguments, and

\[
h^\pm_\rho(x, t) \equiv \frac{\partial v^\pm}{\partial \lambda} \frac{1}{1 + \rho_\lambda}. \tag{4.12}
\]

Note that the last definition is legitimate, since the function \( \rho(x, t) \) is not identically zero only if \( x \in \mathcal{N} \), where the coordinates \((\omega, \lambda) \equiv (\omega_x, \lambda_x)\) of the point \( x \) are defined, and the coordinate \( \lambda \) is independent of the choice of local coordinates \( \omega \) (we use the index \((\omega_x, \lambda_x)\) to distinguish these coordinates for a point \( x \) from the corresponding coordinates \((\omega_y, \lambda_y)\) for a point \( y \)).

Below we explain the derivation of the relations (4.6)-(4.10), here we note the following. The relation (4.8) contains the expression

\[
S_\rho \equiv S_\rho(\omega, \rho, \rho_\omega) \equiv (1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho)\rho_\omega_i \rho_\omega_j),
\]

which is explicitly expressed in the local coordinates \( \omega \). But, in fact, the expression \( S_\rho \) is strictly a function of the points of the surface \( \Gamma_T \) and its values at the points of \( \Gamma_T \) do not depend on a choice of local coordinates \( \omega \). Indeed, first, for any choice of local coordinates \( \omega \) the condition (4.8) is equivalent to (4.3), which is independent of a choice of local coordinates, and, secondly, all the other factors and the terms but \( S_\rho \) in the relation (4.8) are invariant with respect to a choice of \( \omega \) and they are the function of the point of the surface \( \Gamma_T \) only. Hence, the expression \( S_\rho \), as a function of the point of the surface \( \Gamma_T \), is invariant on a choice of local coordinates \( \omega \) as well. And thus, the map \( \rho \rightarrow S_\rho(\omega, \rho, \rho_\omega) \) defines a nonlinear operator, acting on functions defined on \( \Gamma_T \). This operator is invariant under choice of local coordinates \( \omega \), it acts in the space of functions on \( \Gamma_T \) and has a certain expression \( S_\rho(\omega, \rho, \rho_\omega) \) for every particular choice of the local coordinates \( \omega \).

Further, the expression \( \frac{\partial v^\pm}{\partial \tau} - h^\pm_\rho \rho_t \) is the recalculated in the variables \((x, t)\) derivative \( \frac{\partial v^\pm}{\partial \tau} \) after the change of variables (1.31):

\[
\frac{\partial v^\pm}{\partial \tau} = \frac{\partial v^\pm}{\partial t} \frac{\partial t}{\partial \tau} + \sum_{i=1}^{N-1} \frac{\partial v^\pm}{\partial \omega_{xi}} \frac{\partial \omega_{xi}}{\partial \tau} + \frac{\partial v^\pm}{\partial \lambda_x} \frac{\partial \lambda_x}{\partial \tau}.
\]

Here in fact

\[
\frac{\partial t}{\partial \tau} = 1, \quad \frac{\partial \omega_{xi}}{\partial \tau} = 0, \quad \frac{\partial \lambda_x}{\partial \tau} = 0, \tag{4.13}
\]

and for the value of \( \frac{\partial \lambda_x}{\partial \tau} \), due to the relation

\[
\lambda_x = \lambda_y - \rho(x, t) \circ \epsilon_\rho^{-1},
\]

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and taking into account (4.13), we have
\[
\frac{\partial \lambda}{\partial \tau} = -\frac{\partial}{\partial \tau} \left[ \rho(x, t) \circ e_{\rho}(x, t)^{-1} \right] =
\frac{\partial \rho}{\partial t} - \frac{\partial \rho}{\partial \lambda} \frac{\partial \lambda}{\partial \tau} - \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial \omega_i} \frac{\partial \omega_i}{\partial \tau} = -\rho_t - \rho_{\lambda_x} \frac{\partial \lambda}{\partial \tau}.
\]
So in the variables \(x\) and \(t\)
\[
\frac{\partial \lambda}{\partial \tau} = -\rho_t / (1 + \rho_{\lambda_x}). \tag*{(4.14)}
\]
Thus, it follows from (4.13) and (4.14), that
\[
\frac{\partial v^\pm}{\partial \tau} \circ e_{\rho} = \frac{\partial v^\pm}{\partial t} - \frac{\partial v^\pm}{\partial \lambda} / (1 + \rho_{\lambda}) \rho_t = \frac{\partial v^\pm}{\partial t} - h_{\rho}^\pm \rho_t.
\]
We explain further the transition from the condition (4.3) to the condition (4.8) under the change of variables (1.31), as we shall need in the future the exact explicit form of this condition. Define in the neighborhood \(N_T\) of the surface \(\Gamma_T\) the function
\[
\Phi_{\rho}(y, \tau) = \lambda_x \circ e_{\rho}^{-1}(y, \tau) = \lambda_y - \rho(x, t) \circ e_{\rho}^{-1}(y, \tau) = \lambda(y) - \rho(y, \tau), \tag*{(4.15)}
\]
where for simplicity we have retained for the function \(\rho(x, t) \circ e_{\rho}^{-1}(y, \tau)\) the same notation \(\rho(y, \tau)\). By the definition \(\pm \Phi_{\rho}(y, \tau) > 0\) for \((y, \tau) \in \Omega_{\rho, T}^\pm\) and \(\Phi_{\rho}(y, \tau) = 0\) for \((y, \tau) \in \Gamma_{\rho, T}\). Hence in (4.3)
\[
\cos(N^\perp, y_i) = \frac{\Phi_{\rho y_i}}{|\nabla_{(y, \tau)} \Phi_{\rho}|}, \quad \cos(N^\perp, \tau) = \frac{\Phi_{\rho \tau}}{|\nabla_{(y, \tau)} \Phi_{\rho}|}.
\]
Therefore, the relation (4.3) can be written as follows
\[
a^+(\nabla_y v^+, \nabla_y \Phi_{\rho}) - a^-(\nabla_y v^-, \nabla_y \Phi_{\rho}) = k \Phi_{\rho \tau}. \tag*{(4.16)}
\]
Under the change of variables (1.31) the right hand side of (4.16), due to the definition of \(\Phi_{\rho}\), takes the form
\[
k \Phi_{\rho \tau} = k \frac{\partial \lambda}{\partial \tau} = -k \rho_t / (1 + \rho_{\lambda_x}), \tag*{(4.17)}
\]
owing to (4.14).
On the other hand, under the change of variables (1.31)
\[
(\nabla_y v^\pm, \nabla_y \Phi_{\rho}) \circ e_{\rho}(x, t) = (\nabla_{\rho} v^\pm, \nabla_{\rho} \lambda_x). \tag*{(4.18)}
\]
Denote by \(\Lambda(x)\) the transition matrix from the gradient with respect to the variables \(x\) to the gradient with respect to variables \((\omega_x, \lambda_x)\), that is

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\[ \nabla_x = \Lambda(x) \nabla_{(x, \omega_x)} \quad (\nabla_y = \Lambda(y) \nabla_{(y, \omega_y)}), \quad (4.19) \]

where
\[
\Lambda(x) = \left( \begin{array}{ccc}
\frac{\partial \lambda}{\partial x_1} & \frac{\partial \omega_1}{\partial x_1} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_1} \\
\vdots & \vdots & \cdots & \vdots \\
\frac{\partial \lambda}{\partial x_N} & \frac{\partial \omega_1}{\partial x_N} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_N}
\end{array} \right), \quad (4.20)
\]

and similarly for the variables \( y \). Then in the variables \((x, t)\)
\[
(\nabla_x v^\pm, \nabla_x \lambda_x) = (\mathcal{E}_x \Lambda \nabla_{(x, \omega)} v^\pm, \mathcal{E}_x \Lambda \nabla_{(x, \omega)} \lambda_x).
\]

Note that \( \nabla_{(x, \omega_x)} \lambda_x = \{1, 0, \ldots, 0\} \), and also \( v^\pm \equiv 0 \) on \( \Gamma \), hence \( \partial v^\pm / \partial \omega_i = 0 \), and therefore
\[
\nabla_{(x, \omega_x)} v^\pm = \left\{ \frac{\partial v^\pm}{\partial \lambda_x}, 0, \ldots, 0 \right\} = \frac{\partial v^\pm}{\partial \lambda_x} \{1, 0, \ldots, 0\} = \frac{\partial v^\pm}{\partial \lambda_x} \nabla_{(x, \omega_x)} \lambda_x.
\]

Thus we obtain
\[
(\nabla_y v^\pm, \nabla_y \Phi\rho) \circ \mathcal{E}_x(x, t) = (\nabla_x v^\pm, \nabla_x \lambda_x) = \frac{\partial v^\pm}{\partial \lambda_x} (\nabla_x \lambda_x, \nabla_x \lambda_x). \quad (4.21)
\]

On the other hand, due to the definition of \( \Phi\rho(y, \tau) \),
\[
(\nabla_x \lambda_x, \nabla_x \lambda_x) = (\nabla_y (\lambda_x \circ \mathcal{E}_x^{-1}), \nabla_y (\lambda_x \circ \mathcal{E}_x^{-1})) \circ \mathcal{E}_x = (\nabla_y \Phi\rho, \nabla_y \Phi\rho) \circ \mathcal{E}_x, \quad (4.22)
\]

Using introduced in (4.19) matrix \( \Lambda(y) \), we have
\[
(\nabla_y \Phi\rho, \nabla_y \Phi\rho) = (\Lambda(y) \nabla_{(y, \omega_y)} \Phi\rho, \Lambda(y) \nabla_{(y, \omega_y)} \Phi\rho) = (\nabla_{(y, \omega_y)} \Phi\rho, \Lambda(y)^* \Lambda(y) \nabla_{(y, \omega_y)} \Phi\rho). \quad (4.23)
\]

First, by the definition of \( \Phi\rho \),
\[
\frac{\partial \Phi\rho}{\partial \lambda_y} = \frac{\partial}{\partial \lambda_y} (\lambda_y - \rho(y, \tau)) = 1 - \rho_{\lambda_y},
\]
\[
\frac{\partial \Phi\rho}{\partial \omega_{yi}} = \frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) = -\rho_{\omega_{yi}}. \quad (4.24)
\]

In addition, since the coordinate \( \lambda_y \) is counted by the normal to \( \Gamma \), and \( \omega_{yi} \) are coordinates on the surface \( \Gamma \), then
\[
(\nabla_y \lambda(y), \nabla_y \lambda(y)) = 1, \quad (\nabla_y \lambda(y), \nabla_y \omega_{yi}(y)) = 0, \quad i = 1, \ldots, N - 1.
\]
Therefore the matrix $\Lambda^*(y)\Lambda(y)$ has the form

$$
\Lambda^*(y)\Lambda(y) = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & m_{11} & m_{12} & \ldots & m_{1(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & m_{(N-1)1} & m_{(N-1)2} & \ldots & m_{(N-1)(N-1)}
\end{pmatrix},
$$

(4.25)

where

$$
m_{ij} = m_{ji} = (\nabla g\omega_i(y), \nabla g\omega_j(y))
$$

(4.26)

are some smooth functions.

Thus,

$$
(\nabla_{(\lambda,\omega)}\Phi_\rho, \Lambda^*(y)\Lambda(y)\nabla_{(\lambda,\omega)}\Phi_\rho) = (1 - \rho_{\lambda y})^2 + \sum_{i,j=1}^{N-1} m_{ij}(y)\rho_{\omega_i\rho_{\omega_j}},
$$

(4.27)

Make now in (4.27) the change of variables (1.31), and recalculate the derivatives of $\rho$ with respect to $(\lambda,\omega)$ in terms of the derivatives with respect to $(\lambda_x, \omega_x)$. We have

$$
\rho_{\lambda y} \circ e_\rho = \rho_t \frac{\partial t}{\partial \lambda y} + \rho_{\lambda x} \frac{\partial \lambda x}{\partial \lambda y} + \sum_{i=1}^{N-1} \rho_{\omega_xi} \frac{\partial \omega_xi}{\partial \lambda y}.
$$

(4.28)

It follows from the definition of the mapping $e_\rho$ that

$$
\frac{\partial t}{\partial \lambda y} = 0, \quad \frac{\partial \omega_xi}{\partial \lambda y} = 0.
$$

(4.29)

At the same time by (4.28), (4.29)

$$
\frac{\partial \lambda x}{\partial \lambda y} = 1 - \rho_{\lambda y} = 1 - \rho_{\lambda x} \frac{\partial \lambda x}{\partial \lambda y},
$$

that is

$$
\frac{\partial \lambda x}{\partial \lambda y} = \frac{1}{1 + \rho_{\lambda x}}.
$$

(4.30)

Therefore by (4.28), (4.29) and (4.30)

$$
\rho_{\lambda y} \circ e_\rho = \frac{\rho_{\lambda x}}{1 + \rho_{\lambda x}}.
$$

(4.31)

Further,

$$
\rho_{\omega_{yi}} \circ e_\rho = \rho_t \frac{\partial t}{\partial \omega_{yi}} + \rho_{\lambda x} \frac{\partial \lambda x}{\partial \omega_{yi}} + \sum_{j=1}^{N-1} \rho_{\omega_xj} \frac{\partial \omega_xj}{\partial \omega_{yi}},
$$

(4.32)
\[
\frac{\partial t}{\partial \omega_{yi}} = 0, \quad \frac{\partial \omega_{xj}}{\partial \omega_{yi}} = \delta_{ij}, \quad i, j = 1, \ldots, N - 1. \tag{4.33}
\]

At the same time
\[
\frac{\partial (\lambda_x \circ e_\rho)}{\partial \omega_{yi}} = \left[ \frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) \right] \circ e_\rho = -\rho_{\omega_{yi}} \circ e_\rho,
\]

That is by virtue of (4.32) and (4.33),
\[
\rho_{\omega_{yi}} \circ e_\rho = \rho_{\lambda_x} (-\rho_{\omega_{yi}} \circ e_\rho) + \rho_{\omega_x}, \tag{4.34}
\]

hence by (4.34),
\[
\rho_{\omega_{yi}} \circ e_\rho = \frac{\rho_{\omega_x}}{1 + \rho_{\lambda_x}}. \tag{4.35}
\]

Thus, it follows from (4.21), (4.27), (4.31) and (4.35) that in (4.21)
\[
(\nabla \rho \lambda_x, \nabla \rho \lambda_x) = \frac{1}{(1 + \rho_{\lambda_x})^2} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_{xi}} \rho_{\omega_{xj}} \right]. \tag{4.36}
\]

Finally, the relation (4.8) follows from the relations (4.16), (4.17), (4.21) and (4.36).

5 The linearization of the problem.

Our goal in this section is the extraction of the principal linear part of the problem (4.6)-(4.11) in terms of the deviation of the unknown functions \((v^+, v^-, \rho)\) from functions constructed from the initial data and satisfying (4.6)-(4.11) for \(t = 0\), as it was done in [3], [6].

Note that from the equations (4.6), (4.8) and from the initial data (4.10) we can calculate the derivatives with respect to time \(\partial v^+/\partial t\) and \(\partial \rho/\partial t\) at \(t = 0\):
\[
\frac{\partial \rho}{\partial t} (\omega, 0) = \rho_1(\omega) = \frac{1}{k} (a^+ \frac{\partial v^+_0}{\partial \lambda} - a^- \frac{\partial v^-_0}{\partial \lambda}) |_{\Gamma}, \tag{5.1}
\]

\[
\frac{\partial v^\pm}{\partial t} (x, 0) = v^\pm_1(x) = \frac{\partial v^\pm_0}{\partial \lambda} \rho_1 + a^\pm \left| v^\pm_0 (x) \right|^\alpha \nabla^2 v^\pm_0 (x), \tag{5.2}
\]

and, in view of the assumptions (1.10), (1.26),
\[
\rho_1(\omega) \in C^{1+\beta-\alpha}(\Gamma), \beta' \equiv \gamma'(1 - \alpha) > \beta, \quad v^\pm_1(x) \in C_{s}^{r'}(\Omega^\pm). \tag{5.3}
\]

Completely analogous to [23], Ch.IV, on the base of results [24] on the solvability of the Cauchy-Dirichlet problem for degenerate equations we construct such functions \(w^\pm(x, t) \in C_{s}^{2+\gamma',1+\gamma'/2}(\Omega^\pm)\) that
\[ |w^\pm|^{(2+\gamma',1+\gamma'/2)}_{s,\Omega_T} \leq C \left( |v_0^+|^{(2+\gamma')}_{s,\Omega_T^+} + |v_1^+|^{(\gamma')}_{s,\Omega_T^+} \right) \leq C |v_0^+|^{(2+\gamma')}_{s,\Omega_T^+} \] (5.4)

and

\[ w^\pm(x,0) = v_0^+(x), \quad \partial w^\pm/\partial t(x,0) = v_1^+(x), \quad w^\pm(x,t)|_{\Gamma_T} = 0, \quad w^\pm(x,t)|_{\Gamma_T^\pm} = h^\pm. \] (5.5)

In addition, just as described in [23], Chapter 4, there is a such function \( \sigma(\omega,t) \in C^{\beta'-\alpha,1+\frac{1+\beta'-\alpha}{2}}(\Gamma_T) \) that

\[ |\sigma|_{\Gamma_T}^{(3+\beta'-\alpha,\frac{3+\beta'-\alpha}{2})} \leq C(|v_0^+|^{(2+\gamma')}_{s,\Omega_T^+} + |v_0^-|^{(2+\gamma')}_{s,\Omega_T^-}) \] (5.6)

and

\[ \sigma(\omega,0) = \rho(\omega,0) = 0, \quad \partial \sigma/\partial t(\omega,0) = \rho_1(\omega). \] (5.7)

Moreover, by the method described in [23], Ch.IV, the function \( \sigma(\omega,t) \) can be extended with the class and with the inequality (5.6) to a function defined in \( \Omega_T \) which is non-zero only in the neighborhood \( \mathcal{N} \times [0,T] \) of the surface \( \Gamma_T \).

The linearization of the relations (4.6)-(4.11) consists in the following (we describe the general scheme of the arguments - the exact formulations will be given below). We denote the space

\[ P^{2+\beta-\alpha}(\Gamma_T) = \{ \rho : \rho \in C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}(\Gamma_T), \rho_t \in C^{1+\beta-\alpha,\frac{1+\beta-\alpha}{2}}(\Gamma_T) \} \] (5.8)

with the norm

\[ |\rho|_{P^{2+\beta-\alpha}(\Gamma_T)} \equiv |\rho|_{C^{2+\beta-\alpha,\frac{2+\beta-\alpha}{2}}(\Gamma_T)} + |\rho_t|_{C^{1+\beta-\alpha,\frac{1+\beta-\alpha}{2}}(\Gamma_T)}. \] (5.9)

Denote also

\[ \psi = (v^+, v^-, \rho) \in \mathcal{H} \equiv \mathcal{C}_s^{2+\gamma,1+\gamma/2}(\Omega_T^+) \times \mathcal{C}_s^{2+\gamma,1+\gamma/2}(\Omega_T^-) \times P^{2+\beta-\alpha}(\Gamma_T), \] (5.10)

\[ \psi_0 = (w^+, w^-, \sigma), \]

and represent the the relations (4.6)-(4.11) as

\[ F(\psi) = 0 \] (5.11)

with some non-linear operator of \( \psi \). Keeping essentially in mind the application of Newton’s method, we represent the relation (5.11) as
\[ F'(\psi_0)(\psi - \psi_0) = -F(\psi_0) + [F(\psi_0) + F'(\psi_0)(\psi - \psi_0) - F(\psi)] \equiv 
\]
\[ \equiv f_0 + G(\psi - \psi_0), \quad (5.12) \]

where \( F'(\psi_0) \) is the Frechet derivative of \( F(\psi) \) at the point \( \psi_0 \). In this case, as the new unknown we consider the difference

\[ \varphi = \psi - \psi_0 = (v^+ - w^+, v^- - w^-, \rho - \sigma), \quad (5.13) \]

which belongs to the spaces with zero, that is,

\[ \varphi \in H_0 \equiv C^{2+\gamma,1+\gamma/2}(\Omega_T^+)^2 \times C^{2+\gamma,1+\gamma/2}(\Omega_T^-)^2 \times P_0^{2+\gamma}((\Gamma_T)') \quad (5.14) \]

By the construction of the element \( \psi_0 = (w^+, w^-, \sigma) \), it has an increased smoothness \((\gamma' > \gamma)\) and satisfies the relation \( F(\psi_0) = 0 \) for \( t = 0 \). Therefore, using the inequalities \((2.1)-(2.4)\), we can estimate

\[ \|f_0\| = \|-F(\psi_0)\| \leq CT^\mu. \quad (5.15) \]

Below we show that the operator \( F'(\psi_0) \) has the bounded inverse in the appropriate spaces, so that the equation \((5.12)\) can be rewritten as

\[ \varphi = [F'(\psi_0)]^{-1}f_0 + [F'(\psi_0)]^{-1}G(\varphi) \equiv 
\]
\[ \equiv h_0 + H(\varphi) \equiv K(\varphi), \quad (5.16) \]

where by \((5.15)\)

\[ \|h_0\| \leq CT^\mu, \quad (5.17) \]

and the operator \( H(\varphi) \) is the "quadratic" with respect to \( \varphi \) by the smoothness of \( F(\psi) \) in its argument and by the definition of \( G(\psi - \psi_0) = G(\varphi) \) in \((5.12)\):

\[ \|H(\varphi)\| \leq C||\varphi||^2, \|H(\varphi_2) - H(\varphi_1)\| \leq C(||\varphi_1|| + ||\varphi_2||)||\varphi_2 - \varphi_1||. \quad (5.18) \]

For sufficiently small \( T > 0 \) it follows from \((5.17)\) and \((5.18)\) that the operator \( K(\varphi) \) maps some small ball \( B_r \subset H_0 \) with a small \( r \) into itself and \( K(\varphi) \) is a contractive there. The only fixed point of this operator gives, obviously, the solution of the original problem.

Thus, our goal now is to write the problem \((4.6)-(4.11)\) as \((5.12)\).

Denote

\[ \theta^\pm = v^\pm - w^\pm, \quad \delta = \rho - \sigma. \quad (5.19) \]
Lemma 5.1 The problem (4.6)–(4.11) can be represented as a problem for the unknown functions $\theta^\pm$ and $\delta$ as follows

$$\frac{\partial \theta^\pm}{\partial t} - |u_0^\pm|^\alpha \nabla^2 \theta^\pm - \frac{\partial w^\pm}{\partial \lambda} \left( \frac{\partial \delta}{\partial t} - |u_0^\pm|^\alpha \nabla^2 \delta \right) =$$

$$= F_1^\pm(x, t; \theta, \delta) + F_2^\pm(x, t; \theta, \delta) \equiv F_1^\pm(x, t; \varphi) + F_2^\pm(x, t; \varphi), (x, t) \in \Omega_T^\pm, \quad (5.20)$$

$$\theta^+ = \theta^+ = 0, \quad (x, t) \in \Gamma_T,$$  

$$k\delta_t + \left[ a^+ \frac{\partial \theta^+}{\partial \lambda} - a^- \frac{\partial \theta^-}{\partial \lambda} \right] - \delta_\lambda \left[ a^+ \frac{\partial w^+}{\partial \lambda} - a^- \frac{\partial w^-}{\partial \lambda} \right] =$$

$$= F_3(x, t; \varphi) + F_4(x, t; \varphi), \quad (x, t) \in \Gamma_T,$$  

$$\theta^\pm = 0, \quad (x, t) \in \Gamma_T^\pm,$$  

$$\theta^\pm(x, 0) = 0, \quad \delta(\omega, 0) = 0,$$  

$$\delta(x, t) = E\delta(\omega, t),$$

where for arbitrary $\varphi = (\theta^+, \theta^-, \delta) \in \mathcal{B}_r \subset \mathcal{H}_0$, $r < \gamma_0/2$ in the righthand sides $F_i$ of the relations (5.20)–(5.24) all the functions $F_i$ vanish at $t = 0$ and the following estimates are valid

$$|F_1^\pm(x, t; \varphi)|^{(\gamma)}_{s, \Omega_T^\pm} \leq CT^\mu, \quad (5.26)$$

$$|F_1^\pm(x, t; \varphi_2) - F_1^\pm(x, t; \varphi_1)|^{(\gamma)}_{s, \Omega_T^\pm} \leq CT^\mu \|\varphi_2 - \varphi_1\|_{\mathcal{H}}, \quad (5.27)$$

$$|F_2^\pm(x, t; \varphi)|^{(\gamma)}_{s, \Omega_T^\pm} \leq C \|\varphi\|^2_{\mathcal{H}}, \quad (5.28)$$

$$|F_2^\pm(x, t; \varphi_2) - F_2^\pm(x, t; \varphi_1)|^{(\gamma)}_{s, \Omega_T^\pm} \leq C (\|\varphi_2\|_{\mathcal{H}} + \|\varphi_1\|_{\mathcal{H}}) \|\varphi_2 - \varphi_1\|_{\mathcal{H}}, \quad (5.29)$$

$$|F_3(x, t; \varphi)|^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq CT^\mu, \quad (5.30)$$

$$|F_3(x, t; \varphi_2) - F_3^\pm(x, t; \varphi_1)|^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq CT^\mu \|\varphi_2 - \varphi_1\|_{\mathcal{H}}, \quad (5.31)$$
\[ |F_4(x, t; \varphi)|^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq C \|\varphi\|_{H}^2, \quad (5.32) \]

\[ |F_4(x, t; \varphi_2) - F_4(x, t; \varphi_1)|^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq C(\|\varphi_2\|_{H} + \|\varphi_1\|_{H}) \|\varphi_2 - \varphi_1\|_{H}, \quad (5.33) \]

**Proof.**

Meaning of the inequalities (5.26)-(5.33) is that, according to (5.12), the expressions \( F_1^\pm \) and \( F_3 \) contain smoother terms and to evaluate them, we use inequalities (2.1)-(2.4), and the expressions \( F_2^\pm \) and \( F_4 \) are "quadratic" with respect to \( \varphi \).

In the case of a uniformly parabolic equation in (4.1)((1.3)), this lemma is proved in details in [6], Section 2.3. Therefore, we mention only the differences that arise in the case of degenerate equations.

First, in contrast to the [6], we can not expect that the extended function \( \rho(x, t) = E\rho(\omega, t) \) satisfies the condition \( \partial \rho(x, t)/\partial \lambda = 0 \) on \( \Gamma_T \), and so we explain the obtaining of the relation (5.22) from the relation (4.8). The relation (5.22) is obtained from (4.8) explicitly after substitution in (4.8) the expressions \( \vartheta^\pm = \theta^\pm \), \( \rho = \delta + \sigma \) and the transfer of the junior and quadratic terms in the right-hand part. It is easy to verify that (5.22) coincides with (4.8) for \( F_3(x, t, \varphi) \equiv \left\{ \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_{\omega_i} \rho_{\omega_j} (a^+ \partial v^+ / \partial \lambda - a^- \partial v^- / \partial \lambda) \right\} - \left[ k \sigma_t - (a^+ \partial w^+ / \partial \lambda - a^- \partial w^- / \partial \lambda) \right] - \delta [k \sigma_t - (a^+ \partial w^+ / \partial \lambda - a^- \partial w^- / \partial \lambda)] - \left[ k \rho_t \sigma_{\lambda} \right], \quad (5.34) \]

\[ F_4(x, t, \varphi) \equiv -k \delta \rho_t \sigma_{\lambda}, \quad (5.35) \]

where in (5.34) \( \rho = \delta + \sigma \), \( v^\pm = \theta^\pm \).

Using the fact that \( \delta(\omega, 0) = \sigma(\omega, 0) = \rho(\omega, 0) = 0 \), estimating each term in square brackets in (5.34) separately, and using the inequalities (2.1)-(2.4) it is easy to obtain for \( F_3(x, t, \varphi) \) the estimates (5.30), (5.31). For example, since by the construction

\[ k \sigma_t(x, 0) - (a^+ \partial w^+(x, 0) / \partial \lambda - a^- \partial w^-(x, 0) / \partial \lambda) = 0, \quad x \in \Gamma, \]

then

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\[ \delta_\lambda \left[ k\sigma_t - \left( a^+ \frac{\partial w^+}{\partial \lambda} - a^- \frac{\partial w^-}{\partial \lambda} \right) \right] \bigg|_{\Gamma_T}^{(1+\beta-a, \frac{1+\beta-a}{2})} \leq \]

\[ \leq C T^\mu |\delta|_{\Gamma_T}^{(1+\beta-a, \frac{1+\beta-a}{2})} \left[ k\sigma_t - \left( a^+ \frac{\partial w^+}{\partial \lambda} - a^- \frac{\partial w^-}{\partial \lambda} \right) \right] \bigg|_{\Gamma_T}^{(1+\beta-a, \frac{1+\beta-a}{2})} \leq \]

\[ \leq C T^\mu |\delta|_{\Gamma_T}^{(2+\beta-a, \frac{2+\beta-a}{2})} \leq C T^\mu \|\varphi\|_\mathcal{H}. \]

Since this term is linear with respect to \( \varphi \), this yields (5.30), (5.31) for this term. The remaining terms in the definition of \( F_3(x, t, \varphi) \) are treated similarly.

As for \( F_4(x, t, \varphi) \), the estimates (5.32), (5.33) for this expression are obvious because it is quadratic.

Another difference from [6] is the presence of a degenerate factor in the third (elliptical) terms in the left-hand side of (4.6). Represent this term as (we consider only the equation for the sign “+”)

\[ (v^+)^\alpha \nabla^2 \rho v^+ = (v_0^+)^\alpha \nabla^2 \rho v^+ + [(v^+)^\alpha - (v_0^+)^\alpha] \nabla^2 v^+ \equiv \]

\[ \equiv (v_0^+)^\alpha \nabla^2 \rho v^+ + A^+(\varphi). \quad (5.36) \]

and show that \( A^+(\varphi) \) satisfies the inequalities

\[ |A^+(\varphi)|_{s, \Omega_T}^{(\gamma)} \leq C T^\mu, \quad |A^+(\varphi_2) - A^+(\varphi_1)|_{s, \Omega_T}^{(\gamma)} \leq C T^\mu \|\varphi_2 - \varphi_1\|_\mathcal{H}, \quad (5.37) \]

Let us assume that the function \( \lambda = \lambda(x) \) is extended with the preservation of the class from the neighborhood \( \mathcal{N} \) of the surface \( \Gamma \) on all \( \Omega \) to a function satisfying the conditions

\[ \nu d^+(x) \leq \lambda(x) \leq \nu^{-1} d^+(x), \]

retaining for her the same notation. Write further \( A^+(\varphi) \) in the form

\[ A^+(\varphi) = \left[ \left( \frac{v^+}{\lambda} \right)^\alpha - \left( \frac{v_0^+}{\lambda} \right)^\alpha \right] \lambda^\alpha(x) \nabla^2 v^+. \quad (5.38) \]

Since \( v_0, v^+ = 0 \) for \( x \in \Gamma \), then for \( x \in \mathcal{N} \)

\[ \frac{v^+}{\lambda} = \int_0^1 \frac{\partial v^+}{\partial \lambda}(\lambda s, \omega, t) ds, \quad \frac{v_0^+}{\lambda} = \int_0^1 \frac{\partial v_0^+}{\partial \lambda}(\lambda s, \omega) ds, \quad (5.39) \]

where we assume the neighborhood \( \mathcal{N} \) so small that

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\[
\frac{\partial v^+_0}{\partial \lambda} \geq \nu > 0, \quad x \in \mathcal{N},
\]
(5.40)

In addition, we assume that \( T \) is so small that

\[
\frac{\partial w^+}{\partial \lambda}(x, t) \geq \nu > 0, \quad (x, t) \in \mathcal{N} \times [0, T].
\]
Assuming now that the radius \( r = r(\nu) \) of the ball \( \mathcal{B}_r \ni \varphi \) is sufficiently small, we can assume that for \( \varphi \in \mathcal{B}_r \)

\[
\frac{\partial v^+}{\partial \lambda}(x, t) = \frac{\partial w^+}{\partial \lambda}(x, t) + \frac{\partial \theta^+}{\partial \lambda}(x, t) \geq \nu > 0, \quad (x, t) \in \mathcal{N} \times [0, T].
\]
(5.41)

In addition, outside the neighborhood \( \mathcal{N} \cup \mathcal{T} \) holds

\[
v^+_0(x) \geq \nu > 0, \quad x \in \overline{\Omega^+} \setminus \mathcal{N}.
\]
(5.42)

Therefore, assuming as above \( T \) and \( r \) sufficiently small, we can assume that

\[
v^+(x, t) = w^+(x, t) + \theta^+(x, t) \geq \nu > 0, \quad x \in \overline{\Omega^+} \setminus \mathcal{N}.
\]
(5.43)

Thus, we have the representation

\[
\frac{v^+(x, t)}{\lambda(x)} = \Phi^+(x, t, \varphi) = \begin{cases} \int_0^1 \frac{\partial v^+}{\partial \lambda}(\lambda s, \omega, t) \, ds \geq \nu, & x \in \mathcal{N}, \\ v^+_0, & x \in \mathcal{N} \cup \mathcal{T}, \end{cases}
\]
(5.44)

and

\[
\left| \Phi^+(x, t, \varphi) \right| \leq C \left( \left| \frac{\partial v^+}{\partial \lambda} \right| + \left| v^+_0 \right| \right) \leq C (\|\varphi\|_{\mathcal{H}}),
\]
(5.45)

and also

\[
\Phi^+(x, t, \varphi) \geq \nu > 0.
\]
(5.46)

Similarly

\[
\frac{v^+_0(x)}{\lambda(x)} = \Phi^+_0(x) = \begin{cases} \int_0^1 \frac{\partial v^+_0}{\partial \lambda}(\lambda s, \omega) \, ds \geq \nu, & x \in \mathcal{N}, \\ v^+_0, & x \in \mathcal{N} \cup \mathcal{T}, \end{cases}
\]
(5.47)

so

\[
\left| \Phi^+_0(x) \right| \leq C, \quad \Phi^+_0(x) \geq \nu > 0.
\]
(5.48)

In addition, it follows from the representations (5.44) and (5.47) that

\[
\left| \Phi^+(x, t, \varphi) - \Phi^+_0(x) \right| \leq
\]

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\[
\leq C \left( |v^+ - v_0^+|_{s,\Omega_T^+}^{(\gamma)} + \left| \frac{\partial}{\partial \lambda} (v^+ - v_0^+) \right|_{s,\Omega_T^+}^{(\gamma)} \right) \leq C(\|\varphi\|) T^\mu, \tag{5.49}
\]

and similarly
\[
|\Phi^+(x, t, \varphi_1) - \Phi^+(x, t, \varphi_2)|_{s,\Omega_T^+}^{(\gamma)} \leq CT^\mu \|\varphi_2 - \varphi_1\|. \tag{5.50}
\]

By the properties (5.46) and (5.48), the mapping
\[
\varphi \mapsto B_1(\varphi) \equiv \left( \frac{v^+}{\lambda} \right)^\alpha - \left( \frac{v_0^+}{\lambda} \right)^\alpha = [\Phi^+(x, t, \varphi)]^\alpha - [\Phi_0^+(x, \varphi)]^\alpha \tag{5.51}
\]
is smooth, and by (5.49), (5.50)
\[
|B_1(\varphi)|_{s,\Omega_T^+}^{(\gamma)} \leq CT^\mu, \quad |B_1(\varphi) - B_1(\varphi_1)|_{s,\Omega_T^+}^{(\gamma)} \leq CT^\mu \|\varphi_2 - \varphi_1\|_\mathcal{H}. \tag{5.52}
\]

Now from the definition of the expression \(A^+(\varphi)\) in (5.38), from the relations (5.52) and from the smooth dependence of \(\nabla_\rho = E_\rho \nabla\) on \(\rho\), taking into account that the factor \(\lambda^\alpha(x)\) is appropriate for the weighted estimates of the second derivatives of \(v^+\) and \(\theta^+\) in the space \(C_{0, s}^{2+\gamma, 1+\gamma/2}(\Omega_T^+)\), it is easy to see that
\[
|A^+(\varphi)|_{s,\Omega_T^+}^{(\gamma)} \leq CT^\mu, \quad |A^+(\varphi_2) - A^+(\varphi_1)|_{s,\Omega_T^+}^{(\gamma)} \leq CT^\mu \|\varphi_2 - \varphi_1\|_\mathcal{H}. \tag{5.53}
\]

Thus, in view of (5.36) and (5.53) the linearization of the equation (4.6) is reduced to the linearization of a linear on \(v^+\) equation that was done in details in [6].

Note also that the insignificant difference between (5.20) from [6] is still in that we, in fact, leave in the left-hand side of (5.20) only the leading terms, moving all the other to the expression \(F_1^+(x, t, \varphi)\).

This completes the proof. \(\blacksquare\)

### 6 The linear problem corresponding to the problem (5.20)–(5.24).

In this section we consider the linear problem obtained from the problem (5.20)–(5.24) for a given right-hand sides from corresponding classes. In this case, \(|v_0^+(x)|\) is replaced by \(d^+(x)B^+(x, t) \sim \lambda(x)\partial_x v_0^+(x)\). And, as in the previous section, we assume that \(\lambda(x)\) is extended to all \(\Omega\) to a smooth function of the class \(H^{3+\gamma}\).
\[ \nu \leq \lambda(x), d^\pm(x) \leq \nu^{-1}, \quad x \in \overline{\Omega \setminus N}. \]  

(6.1)

Thus, we consider in the domains \( \overline{\Omega^\pm_T} \) the following problem of finding the functions \( v^\pm(x,t) \), defined in the domains \( \overline{\Omega^\pm_T} \), and the function \( \delta(\omega,t) \), defined on \( \Gamma_T \), on the conditions

\[
\frac{\partial v^\pm}{\partial t} - \lambda(x)^{\alpha} B^\pm(x,t) \nabla^2 v^\pm - A^\pm(x,t) \left( \frac{\partial \delta}{\partial t} - \lambda(x)^{\alpha} B^\pm(x,t) \nabla^2 \delta \right) =
\]

\[ = f^\pm_1(x,t), \quad (x,t) \in \Omega^\pm_T, \]  

(6.2)

\[ v^+(x,t) = v^-(x,t) = 0, \quad (x,t) \in \Gamma_T, \]  

(6.3)

\[ k\delta_t - \varepsilon \Delta \Gamma \delta + \left( a^+ \frac{\partial v^+}{\partial \lambda} - a^- \frac{\partial v^-}{\partial \lambda} \right) - \delta_{\chi} (a^+ A^+(x,t) - a^- A^-(x,t)) = \]

\[ = f_2(x,t), \quad (x,t) \in \Gamma^+_T, \]  

(6.4)

\[ v^\pm(x,t) = 0, \quad (x,t) \in \Gamma^\pm_T, \]  

(6.5)

\[ v^\pm(x,0) = 0, \quad \delta(x,0) = 0, \quad x \in \overline{\Omega^\pm}, \]  

(6.6)

\[ \delta(x,t) = E\delta(\omega,t), \]  

(6.7)

where the extension operator \( E \) was defined in the section 2, \( \Delta \Gamma \) is the Laplace-Beltrami operator on the surface \( \Gamma \) (compare \[1\]). We assume that

\[
f^\pm_1(x,t) \in C^\gamma/2_{0,\gamma} (\overline{\Omega_T^\pm}), \quad f_2(x,t) \in C^{1+\beta-\alpha,\frac{1+\beta-\alpha}{2}}_{0} (\Gamma_T), \]  

(6.8)

\( \varepsilon, a^\pm, k \) are given positive constants,

\[ \nu \leq k, a^\pm, B^\pm(x,t), A^\pm(x,t) \leq \nu^{-1}, \]  

(6.9)

\[ A^\pm(x,t), B^\pm(x,t) \in C^\gamma/2_{\gamma} (\overline{\Omega_T^\pm}). \]  

(6.10)

For the problem (6.2)-(6.7) by the standard method of the freezing of coefficients and multiplication by smooth cutting functions we can obtain the Schauder a priori estimates of the solution completely similar to [23] (or [6] in the case of the Stefan problem). At that the model problem, obtained by the freezing of

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the coefficients in points of the boundary \( \Gamma \) at \( t = 0 \), with the subsequent local rectification of the boundary, was studied in the section 3. At considering such a model problem the functions \( B^\pm(x,t) \) and \( A^\pm(x,t) \) are replaced by the constants \( B^\pm \equiv B^\pm(x_0,0) \) and \( A^\pm \equiv A^\pm(x_0,0) \), \( x_0 \in \Gamma \). After this, the change of the unknown function

\[
u^\pm(x,t) = v^\pm(x,t) - A^\pm \delta
\]

(6.11)

reduces the problem (6.2)-(6.7) with the frozen coefficients and with the flat boundary exactly to the problem (3.3)-(3.7).

From these model problems associated with the boundary \( \Gamma \) we get the estimate of the function \( \delta(x,t)|_\Gamma \) and border estimates of the functions \( v^\pm(x,t) \). After that, the rest of the model problems associated with a strictly interior points of \( \Omega^\pm \) are standard because of the condition (6.7), and due to the absence of degeneracy of the equations at these points - see [23].

Thus, the following is true.

Lemma 6.1 Suppose that the conditions \((6.8) - (6.10)\). Then for the solution of the problem \((6.2) - (6.7)\) from the class \( v^\pm \in C^{2+\gamma, \frac{2+\beta-\alpha}{\nu}}(\Omega_T) \), \( \delta \in C^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T) \) the following a priori estimate is valid

\[
|v^\pm|_{s, \Gamma_T}^{(2+\gamma)} + |\delta|_{\Gamma_T}^{(2+\beta-\alpha, \frac{2+\beta-\alpha}{\gamma})} + |\delta_t|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} + \varepsilon \sum_{i,j=1}^{N-1} |\delta_{\omega_i \omega_j}|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \leq C_T \left( |f_1^+|_{s, \Gamma_T}^{(\gamma)} + |f_1^-|_{s, \Gamma_T}^{(\gamma)} + |f_2|_{\Gamma_T}^{(1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha})} \right) \equiv C_T M(T),
\]

(6.12)

where the constant \( C_T \) in (6.12) does not depend on \( \varepsilon \in (0,1) \).

We now show the solvability of the problem (6.2)-(6.7).

Theorem 6.2 Suppose that the conditions \((6.8) - (6.10)\) are satisfied. Then for \( \varepsilon \in (0,1) \) the problem \((6.2) - (6.7)\) is solvable in the space \( v^\pm \in C^{2+\gamma, \frac{2+\gamma}{\nu}}(\Omega_T) \), \( \delta \in C^{3+\beta-\alpha, 1+\frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T) \), and the estimate of the solution (6.12) is valid.

When \( \varepsilon = 0 \) the problem (6.2)-(6.7) is solvable in the space \( v^\pm \in C^{2+\gamma, \frac{2+\gamma}{\nu}}(\Omega_T) \), \( \delta \in C^{1+\beta-\alpha, \frac{1+\beta-\alpha}{2-\alpha}}(\Gamma_T) \), and the estimate (6.12) without the term with \( \varepsilon \) is valid.

Proof.

Define the linear operator \( M : \delta \rightarrow v^\pm \rightarrow M\delta \) which maps a function \( \delta \in C^{2+\beta-\alpha, \frac{2+\beta-\alpha}{2-\alpha}}(\Gamma_T) \) first to the functions \( v^\pm \), as the solution of the problem (6.2),
with the function δ in (6.2), and then the functions \( v^\pm \) the operator \( M \) maps to the function \( M\delta \), which is determined from the condition (6.4) with the given \( v^\pm \) and \( \delta_\lambda \), that is the function \( M\delta \) is the solution of the problem

\[
k(M\delta)_t - \varepsilon \Delta_t (M\delta) = f_2 - \left( a^+ \frac{\partial v^+}{\partial \lambda} - a^- \frac{\partial v^-}{\partial \lambda} \right) + \delta_\lambda (a^+ A^+ - a^- A^-), \quad (6.13)
\]

Thus, the theorem 6.2 is proved.

By the theorem 2.7 this operator is well defined, and with \( \varepsilon > 0 \), by the known properties of the problem (6.13),

\[
|\delta|^{(3+\beta-\alpha,1+\frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq C_{\varepsilon,T} \left( |\delta|^{(2+\beta-\alpha,2+\beta-\alpha)}_{\Gamma_T} + \mathcal{M}(T) \right), \quad (6.14)
\]

\[
|\delta_2 - \delta_1|^{(3+\beta-\alpha,1+\frac{1+\beta-\alpha}{2-\alpha})}_{\Gamma_T} \leq C_{\varepsilon,T} |\delta_2 - \delta_1|^{(2+\beta-\alpha,2+\beta-\alpha)}_{\Gamma_T}. \quad (6.15)
\]

Consequently, by (2.4),

\[
|\delta_2 - \delta_1|^{(2+\beta-\alpha,2+\beta-\alpha)}_{\Gamma_T} \leq C T^\mu |\delta_2 - \delta_1|^{(2+\beta-\alpha,2+\beta-\alpha)}_{\Gamma_T} \leq C T^\mu |\delta_2 - \delta_1|^{(2+\beta-\alpha,2+\beta-\alpha)}_{\Gamma_T}. \quad (6.16)
\]

Thus, for a sufficiently small \( T = T_\varepsilon \) the operator \( M \) is a contraction on \( C_0^{2+\beta-\alpha,2+\beta-\alpha} (\Gamma_T) \) and therefore has a unique fixed point, which by (6.14), belongs also to the space \( C_0^{3+\beta-\alpha,1+\frac{1+\beta-\alpha}{2-\alpha}} (\Gamma_T) \) and together with the corresponding \( v^\pm \) gives the solution of the problem. The estimate of the solution is given by the lemma 6.1. Moving now step by step up the axis \( Ot \) as in [23], we obtain the theorem with \( \varepsilon > 0 \) for any \( T > 0 \).

Further, by the estimate (6.12), considering the sequence of the solutions \( v^\pm_\varepsilon \), \( \delta_\varepsilon \), \( \varepsilon \to 0 \), we see that this sequence is compact in the spaces \( C_0^{2+\gamma,2+\gamma} (\bar{\Omega}_T) \) and \( C_0^{2+\beta-\alpha,2+\beta-\alpha} (\Gamma_T) \) correspondingly for any \( \gamma < \gamma, \bar{\beta} = \bar{\gamma}(1 - \alpha/2) \). The passing to the limit of this sequence in the spaces \( C_0^{2+\gamma,2+\gamma} (\bar{\Omega}_T) \) and \( C_0^{2+\bar{\beta}-\alpha,2+\bar{\beta}-\alpha} (\Gamma_T), \delta_\varepsilon \in C_0^{2+\beta-\alpha,2+\beta-\alpha} (\Gamma_T) \) gives the solution of the problem (6.2)-(6.7) for \( \varepsilon = 0 \). Besides, as it follows from the uniform in \( \varepsilon \) estimate (6.12), the limit function \( v^\pm \) and \( \delta \) belong to the spaces \( C_0^{2+\gamma,2+\gamma} (\bar{\Omega}_T) \) and \( C_0^{2+\beta-\alpha,2+\beta-\alpha} (\Gamma_T) \), \( \delta_\varepsilon \in C_0^{2+\beta-\alpha,2+\beta-\alpha} (\Gamma_T) \) correspondingly.

Thus, the theorem (6.2) is proved.
7 Completion of the proof of the theorem 1.1

We complete the proof of the theorem 1.1 according to the scheme described in the section 5.

Define on the space $H_0$ ($H_0$ is defined in (5.14)) a nonlinear operator $F(\varphi)$, $\varphi = (\theta^+, \theta^-, \delta)$ $\in H_0$ in (5.20) - (5.25), which maps a given $\varphi$ in the righthand sides of the nonlinear relations (5.20), (5.22) to the solution of the linear problem defined by the lefthand sides of these relations. It follows from the theorem 6.2 and the lemma 5.1 that the operator $F(\varphi)$ has the following properties on the ball $B_r = \{ \varphi : \|\varphi\|_H \leq r \} \subset H_0$ with the sufficiently small radius:

$$\|F(\varphi)\|_H \leq C(T^\mu + r)\|\varphi\|_H + CT^\mu, \quad (7.1)$$

$$\|F(\varphi_1) - F(\varphi_2)\|_H \leq C(T^\mu + r)\|\varphi_1 - \varphi_2\|_H. \quad (7.2)$$

It is easy to see that it follows from the relations (7.1) and (7.2) that for sufficiently small $T$ and $r$ the operator $F(\varphi)$ maps the closed ball $B_r$ into itself and is a contraction there. The only fixed point of this operator gives the solution of the original nonlinear problem with free boundary. Thus, the theorem 1.1 is proved. □
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