Black-body radiation shift of atomic energy-levels:

The \((Z\alpha)^2\alpha T^2/m\) correction

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Abstract

The next-to-leading order black-body radiation(BBR) shift to atomic energy-levels, namely \((Z\alpha)^2\alpha T^2/m\) correction, was studied by using the nonrelativistic quantum electrodynamics(NRQED). We also estimate the one-loop contribution of quadrupole and the two-loop contributions of BBR-shift of the thermal(real) photon. These corrections have not been investigated before. The order of magnitude BBR-shift indicates the one-loop contribution of quadrupole is stronger than the previous result. And the two-loop contribution of BBR-shift of the thermal(real) photon is tiny, but this next-to-leading order BBR-shift may be as significant as the leading order in the multi-electron atoms or cold ones.

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I. INTRODUCTION

The blackbody radiation (BBR) can perturb the atomic energy-levels and shorten the lifetime of states. It is well known that the thermal mass-shift induced by BBR is the main part of the one-loop thermal self-energy correction [1]. The thermal mass-shift is proportional to $T^2$ and irrelevant to the atomic energy-level. The leading term of BBR-shift in the atom, which is proportional to $T^4$, is the remainder term in the one-loop thermal self-energy correction. This effect is tiny. So it is always being neglected in the atomic energy-levels calculations. However, it is still necessary to consider BBR in some areas of atomic physics. Firstly, in the precise calculation of the energy for simple atomic system, such as helium and lithium [2, 3], the accuracy of the energy of the ground state in free-field, has been extremely high, which means many higher-order corrections such as quantum electrodynamic (QED) effect, QED-nuclear recoil, QED-nuclear size, even if BBR-shift, could possibly be taken into account [4]. This is foremost to determine the physical fundamental constant, such as the fine-structure constant $\alpha$ [5]. Secondly, in these decades, the BBR effect becomes an obstacle of accuracy in determination of frequency standard [6, 7]. Several works [8–11] have been devoted to study BBR-shift from 1980s. In those works, the classical BBR’s electric field approximation and rotating-wave approximation were introduced. Escobedo [12] and Solovyev [13] studied the BBR-shift in the hydrogen-like atoms by using the finite temperature quantum electrodynamics approach without previous approximations. However, the BBR-shifts in multi-electron atoms, which the atomic clock is based on, haven’t been studied by the same approach. And those results [8–13] are the leading order of the BBR-shift. It is beneficial to make a further step to study the higher corrections, for example, the $(Z\alpha)^2\alpha T^2/m$ correction in this paper.

The bound state is cumbersome to be studied by using the quantum electrodynamics (QED). We present in this work an efficient way, nonrelativistic quantum electrodynamics (NRQED) [14–17], to attain the expected purposes. The NRQED is an effective field theory in which all virtual physics at scales greater than the electron mass have been integrated out. The advantage of using NRQED is that the perturbations can be calculated order by order. Imitating the finite temperature, QED [1, 18], we replaced the photon propagator with the ensemble averaged photon propagator, when we adopted NRQED. We’ve found a $(Z\alpha)^2\alpha T^2/m$ correction, which is important and could be lager.
than the leading term of BBR-shift when the high/mid-Z or cold hydrogen-like atoms are interrogated. This $T^2$-dependent correction has not been investigated by former works yet. We conjecture this effect will be important in multi-electron atoms or cold ones. We also estimate the one-loop contribution of quadrupole and the two-loop contributions of BBR-shift of the thermal(real) photon. The order of magnitude BBR-shift indicates the one-loop contribution of quadrupole is stronger than the previous result\[?\]. And the two-loop contribution of BBR-shift of the thermal(real) photon is tiny. In this paper, we studied BBR-shift by using NRQED approach and obtained the $(Z\alpha)^2\alpha T^2/m$ correction in section 2. The one-loop contribution of quadrupole and two-loop contributions of BBR-shift were estimated in section 3. The section 4 is devoted to discussion and conclusion.

II. THE $(Z\alpha)^2\alpha T^2/m$ BBR-SHIFT TO THE ATOM

The Feynman rules for the finite temperature QED are given in Fig.1, where $n_B(\omega) = \frac{1}{e^{\beta \omega} - 1}$, $d_{ij}(\omega) = \left( \delta_{ij} - \frac{k_i k_j}{\omega^2} \right)$ and $\beta = \frac{1}{kT}$. We adopt a many-body fermion propagator in Fig.1, where $\gamma^\mu$ are Dirac matrices and $|n\rangle$, $E_n$ are eigenfunction and eigenvalue of the many-body Hamiltonian respectively. Additionally, we adopted coulomb gauge to express the thermal(real) photon propagator, which is derived from the thermal(real) photon propagator\[\ll] in Feynman gauge, by gauge transformation.

The leading term of BBR-Shift is induced by the thermal one-loop self-energy correction, which is represented by diagrams in Fig.2. The contribution of virtual particle pairs[Fig.2(b)] is compressed by a $exp(-m/kT) \sim 0$ factor in the room temperature. So all the diagrams...
that have a thermal photon propagator connecting with virtual particle pairs can be neglected. The contribution of diagrams in Fig. 2(a) is

\[ \Delta E_2 = e^2 P \int \frac{n_B(\omega) d^3 k}{(2\pi)^3 2\omega} d_{ij}(\omega) \sum_{a,b} \langle \psi | (\gamma^i e^{-i\mathbf{k} \cdot \mathbf{x}})^a \frac{2(E - H)}{(E - H)^2 - \omega^2} (\gamma^j e^{i\mathbf{k} \cdot \mathbf{x}})^b | \psi \rangle, \]  

where \( P \) denotes the Cauchy principal value, which is required for a proper treatment of resonant contributions. It is coincided with Porsey’s result, and contains bound-state-independent terms, which are called thermal mass-shift. The remainder term is the BBR-shift. The thermal mass-shift also exists in free electron as well and is much larger than BBR-shift value. The contribution of the Fig. 2 of the nonrelativistic approximation is represented as:

\[ \Delta E_2 = \frac{4\alpha}{3\pi} \sum_{a,b} P \int n_B(\omega) d\omega \langle \psi | \left( \frac{p}{m} \right)_a \frac{2(E - H)}{(E - H)^2 - \omega^2} \left( \frac{p}{m} \right)_b | \psi \rangle + \ldots \]  

(2)

where the first term is the thermal mass-shift, the second term is the BBR-shift of electric dipole moment \((E1)\) and the ellipsis is the higher order term such as the contribution of multi-dipole. The \( N_e \) is the number of the electrons, and \( \alpha(= e^2/(4\pi) \simeq 10^{-4}) \) is the fine-structure constant. The contribution of electric dipole \((E1)\) is

\[ \Delta E_{2E1} = \frac{4\alpha}{3\pi} \sum_{a,b} P \int n_B(\omega) \omega^3 d\omega \langle \psi | (\mathbf{r})_a \frac{2(E - H)}{(E - H)^2 - \omega^2} (\mathbf{r})_b | \psi \rangle \]  

(3)

where \((\mathbf{r})_a\) and \((\mathbf{r})_b\) are the position vectors of electron \( a \) and \( b \) respectively.
The BBR’s characteristic wave-length $1/kT \simeq 10^{-5}m$ in the room temperature is much larger than the atoms’ radius, and the electron is nonrelativistic in the light atoms. Additionally, the contributions of multi-polar moment are suppressed by $\alpha^2 \simeq 10^{-4}$ (Magnetic dipole(M1)) or $(kT/ma)^2 \simeq 10^{-10}$ (Eletric-quadrupole(E2)) comparing with the contribution of the electric dipole(E1) in the room temperature[10]. So the Eq.(3) is the leading term of BBR-shift.

The Eq.(3) can be further simplified in the low-lying states in the room temperature (The approximation $E_{ab} \gg kT$ is applied to our work). Thus we have

$$\Delta E_2 \simeq \frac{4\alpha^3}{45\beta^4} \sum_{E_{\psi} \neq E_M} \frac{|\langle M|\vec{r}|\psi\rangle|^2}{E_{\psi}E_M},$$

which is proportional to $T^4$.

The relativistic corrections aren’t included in Eq.(3), and these corrections can be introduced by the nonrelativistic QED approach[16]

$$\Delta E_{2R} = -\frac{4e^2}{3m_am_b} \sum \int \frac{n_B(\omega)d^3k}{(2\pi)^32\omega} d_{ij}(\omega)\langle \psi | (P^i)_a \frac{1}{E - H - \omega} (V - \langle V \rangle) \frac{1}{E - H - \omega} (P^j)_b + 2V \frac{1}{E - H} (P^i)_a \frac{1}{E - H - \omega} (\omega \rightarrow -\omega) \rangle |\psi\rangle,$$

where the inserted vertex $iV$ is relativistic correction and Breit interaction,

$$V = \sum \frac{-P^4}{8m^3} + \sum \left\{ -q_\alpha q_\beta \left( \frac{1}{8m^2} + \frac{1}{8m^2} \right) \delta^3(\vec{r}_{ab}) - \frac{q_\alpha q_\beta}{2m_am_b 4\pi r_{ab}} \left( \vec{P}_a \cdot \vec{P}_b + \vec{r}_{ab} \cdot (\vec{r}_{ab} \cdot \vec{P}_b) \vec{P}_a \right) - \frac{q_\alpha q_\beta}{2m_am_b} \frac{\vec{r}_{ab} \times \vec{P}_a \cdot \vec{r}_{ab}}{4\pi r_{ab}^3} \vec{P}_a \cdot \vec{\sigma}_a - \frac{q_\alpha q_\beta}{2m_am_b} \frac{\vec{r}_{ab} \times \vec{P}_b \cdot \vec{r}_{ab}}{4\pi r_{ab}^3} \vec{P}_b \cdot \vec{\sigma}_b + \frac{q_\alpha q_\beta}{2m_am_b} \frac{3\vec{\sigma}_a \cdot \vec{r}_{ab} \vec{\sigma}_b \cdot \vec{r}_{ab}}{r_{ab}^5} - \frac{2\vec{\sigma}_a \cdot \vec{\sigma}_b \delta^3(\vec{r}_{ab})}{3} \right\}. \tag{6}$$

If $E_{ab} \gg kT$, the correction Eq.(5) can be written as

$$\Delta E_{2R} = -\frac{8\alpha^3}{27\beta^3} \sum_{M_{1,2}} \left( \langle \psi | (\vec{r})_a | M_2 \rangle \cdot \langle M_1 | (\vec{r})_b | \psi \rangle \langle M_2 | (V - \langle V \rangle) | M_1 \rangle + \frac{1}{E_{M_1M_2}/E_{\psi M_2}} \langle \psi | M_2 \rangle \langle M_2 | M_1 \rangle \cdot \langle M_1 | (\vec{r})_b | \psi \rangle \langle M_2 | V | \psi \rangle \frac{E_{M_1M_2}}{E_{\psi M_1}} \right) \tag{7}.$$
which is proportional to $T^2$. Its magnitude is $(Z\alpha)^2\alpha T^2/m$.

The relativistic correction to current induces another correction \[\Delta E_{2J} = 2e^2P \int \frac{n_B(\omega)d^3k}{(2\pi)^32\omega} d_{ij}(\omega) \sum_{a,b} \langle \psi | (\delta j^i) \rangle_a \frac{2(E - H)}{(E - H)^2 - \omega^2} \left( \frac{p^j}{m} \right)_b | \psi \rangle, \tag{8} \]

where the current is

\[(\delta j^i)_a = -\frac{p^j}{2m^3} + \sum_{c \neq a} \left( \frac{\alpha_q r_{ac} \times \sigma}{4m^2 r_{ac}^3} \right) + \frac{i\omega (p \times \sigma)^i}{4m^2} + \frac{(k \times \sigma)^i (k \cdot r)}{2m}, \tag{9} \]

where $q_c$ is the charge number of the particle $c$. The first two terms are proportional to $T^2$, the third term is proportional to $T^3$, and the last term is proportional to $T^4$ in the low-lying atomic states in the room temperature.

The Eq.(5)(8) are the $(Z\alpha)^2\alpha T^2/m$ corrections. These corrections have not been investigated before.

III. THE QUADRUPOLE’S CONTRIBUTION AND TWO-LOOP CONTRIBUTION OF BBR-SHIFT

It is obvious that Eq.(6) doesn’t contain the contribution of multipole of the thermal photon, which can be obtained from Eq.(1). For example, The quadratic term is

\[
\Delta E_{2Q} = \frac{e^2}{m^2} P \int \frac{n_B(\omega)d^3k}{(2\pi)^32\omega} d_{ij}(\omega) \sum_{a,b} \langle \psi | \left\{ p^j, (-i\mathbf{k} \cdot \mathbf{x}) \right\}_a \left( \frac{2(E - H)}{(E - H)^2 - \omega^2} \right) \left\{ p^j, (i\mathbf{k} \cdot \mathbf{x}) \right\}_b | \psi \rangle \\
+ \left\{ p^i, (i\mathbf{k} \cdot \mathbf{x})^2 \right\}_a \left( \frac{2(E - H)}{(E - H)^2 - \omega^2} \right) \left\{ p^j, (i\mathbf{k} \cdot \mathbf{x}) \right\}_b | \psi \rangle \\
= \frac{e^2}{m^2} P \int \frac{n_B(\omega)d\omega}{(2\pi)^2} \frac{4\delta_{ij} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}}{15} \sum_{a,b} \langle \psi | \left\{ p^j, x^k \right\}_a \left( \frac{2(E - H)}{(E - H)^2 - \omega^2} \right) \left\{ p^j, x^l \right\}_b | \psi \rangle.
\]

(10)

If $E_{ab} \gg kT$, this correction of quadrupole is proportional to $T^4$. This correction is also different from the result in ref[10], which is proportional to $T^6$.

In this section, we will study the two-loop contributions of BBR-shift, which can be obtained by add a virtual or thermal photon propagator to the Fig.2(a).
FIG. 3: The thermal two-loop correction (Finite temperature QED).

FIG. 4: The mixing two-loop correction (Finite temperature QED).

The Fig.3 are thermal two-loop diagrams in multi-electron atoms (Both the photon prop-
agators are thermal(real) propagators). Its contribution is

$$\Delta E_3 = e^4 \sum_{n_1 n_2 n_3} \sum_{a,b,c,d} \mathcal{P} \int \frac{n_B(\omega) d^3 k}{(2\pi)^3 2\omega} \frac{n_B(\omega') d^3 k'}{(2\pi)^3 2\omega'} \left( \delta_{ij} - \frac{k_i k_j}{\omega'^2} \right)$$

$$\langle \psi | \left( \gamma^i e^{i\vec{k}\cdot \vec{x}} \right)_d | n_3 \rangle \langle n_3 | \left( \gamma^k e^{i\vec{k}'\cdot \vec{x}} \right)_c | n_2 \rangle \langle n_2 | \left( \gamma^j e^{-i\vec{k}\cdot \vec{x}} \right)_b | n_1 \rangle \langle n_1 | \left( \gamma^e e^{-i\vec{k}'\cdot \vec{x}} \right)_a | \psi \rangle$$

$$\left[ \frac{1}{E_{\psi n_2} - \omega} \frac{2}{E_{\psi n_2} - \omega'} + \frac{1}{E_{\psi n_1} + \omega} \right] +$$

$$e^4 \sum_{n_1 n_2 n_3} \sum_{a,b,c,d} \int \frac{n_B(\omega) d^3 k}{(2\pi)^3 2\omega} \frac{n_B(\omega') d^3 k'}{(2\pi)^3 2\omega'} \left( \delta_{ik} - \frac{k_i k_k}{\omega'^2} \right)$$

$$\langle \psi | \left( \gamma^i e^{i\vec{k}\cdot \vec{x}} \right)_d | n_3 \rangle \langle n_3 | \left( \gamma^k e^{i\vec{k}'\cdot \vec{x}} \right)_c | n_2 \rangle \langle n_2 | \left( \gamma^j e^{-i\vec{k}\cdot \vec{x}} \right)_b | n_1 \rangle \langle n_1 | \left( \gamma^e e^{-i\vec{k}'\cdot \vec{x}} \right)_a | \psi \rangle$$

$$\left[ \frac{1}{E_{\psi n_2} - \omega} \frac{2}{E_{\psi n_2} - \omega'} + \frac{1}{E_{\psi n_1} + \omega} \right] +$$

$$e^4 \sum_{n_1 n_2 n_3} \sum_{a,b,c,d} \int \frac{n_B(\omega) d^3 k}{(2\pi)^3 2\omega} \frac{n_B(\omega') d^3 k'}{(2\pi)^3 2\omega'} \left( \delta_{ij} - \frac{k_i k_j}{\omega'^2} \right)$$

$$\langle \psi | \left( \gamma^i e^{i\vec{k}\cdot \vec{x}} \right)_d | n_3 \rangle \langle n_3 | \left( \gamma^k e^{-i\vec{k}\cdot \vec{x}} \right)_c | n_2 \rangle \langle n_2 | \left( \gamma^j e^{i\vec{k}'\cdot \vec{x}} \right)_b | n_1 \rangle \langle n_1 | \left( \gamma^e e^{i\vec{k}'\cdot \vec{x}} \right)_a | \psi \rangle$$

$$\frac{2E_{\psi n_3}}{E_{\psi n_2}^2 - \omega^2} \frac{1}{2E_{\psi n_1}}$$

$$\frac{1}{2E_{\psi n_2} - \omega}$$

$$\frac{1}{2E_{\psi n_1} - \omega^2};$$

(11)

where $E_{\psi M} = E_{\psi}(1-i0^+) - E_{M}(1-i0^+)$ is the difference between energys. The contribution of the electric dipole of Eq.(11) is the more important then multipoles’ in the room temperature, which can be obtained by $\gamma^i \rightarrow P^i/m$. This integral is finite both in ultraviolet and infrared regions.

Other diagrams two-loop contributions are the mixing two-loop diagram(Fig.4). The Fig.4(a) is the easiest to be obtained,

$$\Delta E_{4a} = e^4 \sum_{n_1 n_2 n_3} \sum_{a,b,c,d} \mathcal{P} \int \frac{d^3 k}{(2\pi)^3 2\omega} \frac{n_B(\omega') d^3 k'}{(2\pi)^3 2\omega'} \left( \delta_{ij} - \frac{k_i k_j}{\omega'^2} \right)$$

$$\langle \psi | \left( \gamma^i e^{i\vec{k}\cdot \vec{x}} \right)_d | n_3 \rangle \langle n_3 | \left( \gamma^k e^{i\vec{k}'\cdot \vec{x}} \right)_c | n_2 \rangle \langle n_2 | \left( \gamma^j e^{-i\vec{k}\cdot \vec{x}} \right)_b | n_1 \rangle \langle n_1 | \left( \gamma^e e^{-i\vec{k}'\cdot \vec{x}} \right)_a | \psi \rangle$$

$$\left[ \frac{1}{E_{\psi n_2} - \omega} \frac{2}{E_{\psi n_2} - \omega'} + \frac{1}{E_{\psi n_1} + \omega} \right]$$

$$\frac{2E_{\psi n_3}}{E_{\psi n_2}^2 - \omega^2} \frac{1}{2E_{\psi n_1}}$$

This integral is also finite. The remainder parts in Fig.4 are divergence. These integrals must subtract a counterterm, and the differences are their contribution of BBR-shift. We haven’t obtained this difference. However, Its order of magnitude can be easily estimated(We will list in the next section). They are $\alpha$ weaker than (5)(8).
IV. DISCUSSION AND CONCLUSION

We have studied the two-loop corrections of BBR-shift in section 3. Although some operators of corrections haven’t been obtained, we could estimate their order of magnitude in the light Hydrogen-like atoms. Attribute to the order counting rules of correction terms $\langle p \rangle \sim 1/\langle r \rangle \sim mZ\alpha, \langle E \rangle \sim m(Z\alpha)^2$ in the light Hydrogen-like atoms, the dimension parts of the BBR-shift of low-lying states are listed in Table I. Two approximations have been applied, which are nonrelativistic and electric-dipole approximation, and the energy-gaps between low-lying states satisfying $\Delta E \gg kT$.

TABLE I: The magnitude of BBR-shift ($Hz$). They are listed by the descending order of $\alpha$ factor. $\langle \Delta E_{2Ji} \rangle$ is energy-shift Eq.(8) originating from the $i$th term in Eq.(9). Two approximations have been applied, such as (1) nonrelativistic electron and electric-dipole approximation of the thermal photon. (2) The energy-gaps between low-lying states $\Delta E \gg kT$.

|                         | The magnitude of BBR-shift |
|-------------------------|----------------------------|
| $\delta m(T) = -\frac{\alpha^2}{3m\beta^2}$ | $2.42 \times 10^4 \frac{(T)^2}{300^2}$ |
| $\langle \Delta E_{2E1} \rangle \sim \frac{1}{Z^4m^3\alpha^3\beta^3}$ | $10^{-3} \frac{T^4}{Z^4300^3}$ |
| $\langle \Delta E_{2Q} \rangle = \langle \Delta E_{2Ji} \rangle \sim \frac{1}{Z^4m^3\alpha^3\beta^3}$ | $10^{-7} \frac{T^4}{Z^4300^3}$ |
| $\langle \Delta E_{3} \rangle \sim \frac{1}{m^4Z^3\beta^3}$ | $10^{-9} \frac{T^4}{Z^4300^3}$ |
| $\langle \Delta E_{3J3} \rangle \sim \frac{\alpha}{m^2\beta^3}$ | $10^{-4} \frac{T^3}{300^3}$ |
| $\langle \Delta E_{2J2} \rangle \sim \frac{(Z\alpha)^2}{m^3\beta^2}$ | $10^{-1} \frac{ZT^2}{300^2}$ |
| $\langle \Delta E_{2R} \rangle = \langle \Delta E_{2J1} \rangle \sim \frac{(Z\alpha)^2}{m^3\beta^2}$ | $10^{-1} \frac{(ZT)^2}{300^2}$ |
| $\langle \Delta E_{4} \rangle \sim \frac{(Z\alpha)^2\alpha^2}{m^3\beta^2}$ | $10^{-3} \frac{(ZT)^2}{300^2}$ |

The results in table I are arranged by the descending order of $\alpha$ factor. One interesting thing is the magnitude of the contribution of quadrupole is proportional to $T^4$. This result is stronger than the previous result[10], which is suppressed by a $(kT/m)^2 = 10^{-6}$ factor in the room temperature. The two-loop correction of the thermal photon is so tiny comparing with other corrections. The Eq.(3) ($\langle \Delta E_{2E1} \rangle$) is the leading term of BBR-shift, although the order of magnitude of Eq.(5) (8) ($\langle \Delta E_{2R} \rangle, \langle \Delta E_{2J1} \rangle$) is larger than the Eq.(3) in the Table I. What most significant is that comparing with leading term, when charge $Z$ is higher or $T$ is lower, the $(Z\alpha)^2\alpha T^2/m$ correction will become more important (Because $\langle \Delta E_{3} \rangle/\langle \Delta E_{2} \rangle \sim Z^6/T^2$).
This correction \((Z\alpha)^2\alpha T^2/m\) must be recalculated seriously in the future. We also conjecture that it may be important in the multi-electron atoms or cold ones.

The reasons (1) why the \(\langle \Delta E_{2R} \rangle, \langle \Delta E_{2J1} \rangle\) are significant, (2) why the magnitude of the contribution of quadrupole is proportional to \(T^4\) rather than \(T^6\) in ref[10], (3) why \(\langle \Delta E_4 \rangle\) is proportional to \(T^4\) are the same: These BBR-shift is mixing thermal mass-shift(\(\propto T^2\)) with counterpart corrections.

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