Dynamics of Born-Infeld membranes

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Abstract. We present a geometrical inspired study of the dynamics of $D_p$-branes. We focus on the usual nonpolynomial Dirac-Born-Infeld action for the worldvolume swept out by the brane in its evolution in general background spacetimes. We emphasize the form of the resulting equations of motion which are quite simple and resemble Newton’s second law, complemented with a conservation law for a worldvolume bicurrent.

1. Introduction

Nowadays $M$/string theory is the best candidate to unify all fundamental interactions. Its non-perturbative approach has revealed certain important higher dimensional extended objects known as $D_p$-branes. $D_p$-branes are enlightening when non-perturbative properties of superstring theory and $M$-theory are studied [1]. The Dirac-Born-Infeld (DBI) action has been proposed as an elegant effective action governing the dynamics of $D_p$-branes at low energy scales [1, 2].

In this note we aim to perform a Lagrangian geometrical study of the DBI action. We obtain a geometrical interpretation of the mechanical properties of $D_p$-branes which constitutes the mathematical backbone of this note. The equations of motion associated to the coordinates are simply the contraction of the worldvolume stress tensor with the extrinsic curvature equated to an external force. This can be interpreted as a generalized Newton’s second law. The corresponding conserved momentum is constructed with two terms: the kinetic momentum and the interaction of the $D_p$-brane with a Neveu-Schwarz (NS) field. This result generalizes the conserved momentum for a point particle interacting with an electromagnetic field. As we will see, the source of the currents and external forces mentioned above resides in the presence of the antisymmetric NS field.

2. DBI action

Consider a $D_p$-brane, $\Sigma$, of dimension $p$ evolving in a $N$-dimensional background spacetime endowed with an arbitrary metric $G_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, \ldots, N - 1$). The trajectory, or worldvolume, $m$, swept out by $\Sigma$ is an oriented timelike manifold of dimension $p + 1$, described by the embedding functions $x^\mu = X^\mu(\xi^a)$ where $x^\mu$ are
local coordinates of the background spacetime, $\xi^a$ are local coordinates of $m$, and $X^\mu$ are the embedding functions ($a, b = 0, 1, 2, \ldots, p$). The metric induced on the worldvolume from the background is given by $g_{ab} = G_{\mu\nu}X^\mu_aX^\nu_b := X_a \cdot X_b$ with $X^\mu_a = \partial_a X^\mu = \partial X^\mu / \partial \xi^a$ being tangent vectors to $m$. In this framework we introduce $N - p - 1$ normal vectors to the worldvolume, denoted by $n^a_i$ ($i = 1, 2, \ldots, N - p - 1$). These are defined implicitly by $n \cdot X_a = 0$ and we choose to normalize them as $n_i \cdot n_j = \delta_{ij}$. We will adopt index-free notation when convenient in order to avoid a cumbersome description.

The effective nonpolynomial DBI action that controls the low energy dynamics of $Dp$-branes is

$$S_{DBI}[X^\mu, A_a] = \beta_p \int_m d^{p+1}\xi \sqrt{-\det(g_{ab} + F_{ab})},$$

(1)

where $\beta_p$ is the tension of the $Dp$-brane, $F_{ab} = \alpha F_{ab} + B_{ab}$ with $F_{ab} = 2\delta_{[a}A_{b]}$ being the electromagnetic field strength associated to a $U(1)$ gauge field $A_a$ living on $m$ and $B_{ab} = B_\mu X^\mu_aX^\nu_b$ is the pullback to the worldvolume of the NS 2-form $B_\mu^\nu$; here $\alpha$ is the BI parameter related to the inverse tension of fundamental strings. Different BI-like theories exist that consider the housing of other gauge fields [3]. In particular, the DBI action (1) is invariant under worldvolume reparametrizations. Further, the action is invariant under a NS gauge transformation $B_{ab} \to B_{ab} - 2\delta_{[a}\lambda_{b]}$ if we shift the $U(1)$ field $A_a \to A_a + \alpha^{-1}\lambda_a$ where $\lambda_a$ is a 1-form; in other words, $F_{ab} = \partial_a A_b$ is the gauge invariant quantity in the presence of NS background field and not the electromagnetic field tensor $F_{ab}$. For the sake of simplicity throughout the paper we will introduce the following notation: $M_{ab} := g_{ab} + F_{ab}$ is the composite matrix, while $(M^{-1})^{ab}$ denotes its inverse and $M := \det(M_{ab})$.

Under an infinitesimal deformation of the embedding functions $X \to X + \delta X$ as well as $A \to A + \delta A$, the first variation of the action (1) casts out the equations of motion associated to the configuration space $C$ [4],

$$\begin{align*}
-\nabla_a \left\{ \beta_p \sqrt{-M} \left[ (M^{-1})^{ab}G_{\mu\nu} - (M^{-1})^{[ab]}B_{\mu\nu} \right] X^\nu_b \right\} \\
+ \frac{\beta_p}{2} \frac{\sqrt{-M}}{\sqrt{-g}} \left[ (M^{-1})^{ab} \partial_\mu G_{\alpha\nu} - (M^{-1})^{[ab]} \partial_\mu B_{\alpha\nu} \right] X^\alpha_a X^\nu_b = 0,
\end{align*}$$

(2)

$$\nabla_a \left[ -\alpha \beta_p \frac{\sqrt{-M}}{\sqrt{-g}} (M^{-1})^{[ab]} \right] = 0,$$

(3)

where $\nabla_a$ is the covariant derivative associated with $g_{ab}$ and $(M^{-1})^{(ab)}$ and $(M^{-1})^{[ab]}$ denote the symmetric and the antisymmetric parts of the matrix $(M^{-1})$, respectively.

The worldvolume stress-energy tensor is given by $T^{ab} := \frac{2}{\sqrt{-g}} \frac{\partial S_{DBI}}{\partial g_{ab}} = \beta_p \frac{\sqrt{-M}}{\sqrt{-g}} (M^{-1})^{(ab)}$, and the worldvolume covariant bivector is $J^{ab} := \frac{2}{\sqrt{-g}} \frac{\partial S_{DBI}}{\partial F_{ab}} = -\alpha \beta_p \frac{\sqrt{-M}}{\sqrt{-g}} (M^{-1})^{[ab]}$ (see [5]). We can rewrite the equations of motion (2) and (3) in terms of the tensors $T^{ab}$ and $J^{ab}$ as

$$\begin{align*}
\nabla_a \left[ (T^{ab}G_{\mu\nu} + \alpha^{-1}J^{ab}B_{\mu\nu}) X^\nu_b \right] - \frac{1}{2} \left( T^{ab} \partial_\mu G_{\alpha\nu} + \alpha^{-1}J^{ab} \partial_\mu B_{\alpha\nu} \right) X^\alpha_a X^\nu_b = 0\nonumber, \\
\n\nabla_a J^{ab} = 0.
\end{align*}$$

(4)

(5)

$\dagger$ $\beta_p = 2\pi/[g_f(2\pi\sqrt{\alpha})^{p+1}]$, where $\alpha'$ is the inverse of the fundamental string tension and $g_f$ is the string coupling.

$\ddagger$ $\alpha = 2\pi\alpha'$ can be thought of as a parameter included in order to gain control over the theory.
The stress tensor $T^{ab}$ is also conserved, $\nabla_a T^{ab} = 0$. Now, based on this result and with the help of the Gauss-Weigarten equations $\nabla_a X^\mu_b = -\Gamma^\mu_{\alpha\beta} X^\alpha_a X^\beta_b - K^i_{ab} n^\nu_i$, where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols associated to $G_{\mu\nu}$, the equations of motion (4) take the form

$$-T^{ab}K^i_{ab}G_{\mu\nu}n^\nu_i + \alpha^{-1} 1/2 J^{ab}(\partial_\alpha B_{\mu\nu} + \partial_\nu B_{\alpha\mu} + \partial_\mu B_{\nu\alpha}).X^\alpha_a X^\nu_b = 0,$$

(6)

where $K^i_{ab} = -n^i \cdot D_a X_b$ is the extrinsic curvature of the worldvolume and $D_a = X^\mu_a D_\mu$ is the pullback to the worldvolume of the covariant derivative compatible with $G_{\mu\nu}$, that is, $D_\mu [6]$. Finally, the equations of motion (4) acquire the geometrical simplified form

$$T^{ab}K^i_{ab} = F^i$$

(7)

where $F^i = -1/2 \alpha^{-1} J^{ab} H_{\alpha\beta\mu} X^\alpha_a X^\beta_b n^\mu_i$ with $H_{\alpha\beta\mu} = \partial_\alpha B_{\mu\nu} + \partial_\nu B_{\alpha\mu} + \partial_\mu B_{\nu\alpha}$ being the NS strength 3-form field which satisfies $\partial_\mu H^{\mu\alpha\beta} = 0$. Note that the form of the equations of motion (7) can be interpreted as a generalization of Newton’s second law for a particle where $T^{ab}$ plays the role of a mass, $K^i_{ab}$ the generalization of the acceleration in higher dimensions, and $F^i$ a force density. This form of the equations of motion was obtained in other contexts, for instance, in the case of superconducting membranes and membranes interacting with external Kalb-Ramond and $U(1)$ fields in [7, 8, 9].

With respect to the conserved quantities, we must assume that the background spacetime has certain symmetries. In general, the response of the action (1) under an infinitesimal deformation of the embedding functions $X \to X + \delta X$ can be expressed by

$$\delta S_{DBI} = \int_m d^{p+1}\xi \sqrt{-g}[\mathcal{E}_\mu \delta X^\mu + \nabla_a Q^a],$$

(8)

where $\mathcal{E}_\mu$ is the Euler-Lagrange derivatives of the action and $Q^a$ is the Noether charge that depends on the infinitesimal deformations $\delta X$ (see [10] for details). For an infinitesimal constant deformation $\delta X^\mu = \epsilon^\mu$, and assuming that the equations of motion are satisfied, the variation of the action becomes

$$\delta S_{DBI} = \epsilon^\mu \int_m d^{p+1}\xi \sqrt{-g} \nabla_a Q^a_{\mu},$$

(9)

where we can set the Noether charge as $Q^a_{\mu} = T^{ab} X^\nu_b G_{\mu\nu} + \frac{1}{\alpha} J^{ab} B_{\mu\nu} X^\nu_b$. By using the divergence theorem, equation (9) reads

$$\delta S_{DBI} = \epsilon^\mu \int_{\partial m} d^{p}u \sqrt{h} \eta_a Q^a_{\mu} = \epsilon^\mu \int_{\Sigma_f} d^{p}u \sqrt{h} \eta_a Q^a_{\mu} - \epsilon^\mu \int_{\Sigma_i} d^{p}u \sqrt{h} \eta_a Q^a_{\mu},$$

(10)

where the integrals are evaluated on the spacelike hypersurface $\Sigma$ at initial and final times. The spacelike hypersurface $\Sigma$ seen as embedded in $m$, has a timelike normal vector $\eta_a$ and the determinant of its induced metric is $h$. If the action is invariant under $\delta X^\mu = \epsilon^\mu$ we have the constant of motion (because $\Sigma_f$ and $\Sigma_i$ are arbitrary),

$$p_{\mu} = \int_{\Sigma} d^{p}u \sqrt{h} \eta_a (T^{ab} X^\nu_b G_{\mu\nu} + \frac{1}{\alpha} J^{ab} B_{\mu\nu} X^\nu_b).$$

(11)

This result is pretty nice since it generalizes the conserved momentum for a relativistic particle interacting with an electromagnetic field. In our case the first term represents the kinetic momentum and the second one is the coupling between the $Dp$-brane and the NS field.
3. Concluding remarks

In this paper we have carried out a geometrical study of the classical DBI action. As a result of the variational process of the action we found out not only the equations of motion for a $Dp$-brane like a generalized Newton’s second law but also a couple of conserved quantities which we identified as the stress-energy tensor and the worldvolume covariant bicurrent associated to the worldvolume electric induction. Furthermore, our formulation allowed us to obtain the equations of motion for systems with arbitrary background spacetimes. Additionally we note that for static backgrounds we obtained the conserved energy of the $Dp$-brane which is a very powerful result for exploring its dynamics.

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