Upper bound for the generalized repetition threshold.

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Abstract

Let \( A \) be an \( a \)-letter alphabet. We consider fractional powers of \( A \)-strings: if \( x \) is a \( n \)-letter string, \( x^r \) is a prefix of \( xxxx \ldots \) having length \( nr \).

Let \( l \) be a positive integer. Ilie, Ochem and Shallit defined \( R(a, l) \) as the infimum of reals \( r > 1 \) such that there exist a sequence of \( A \)-letters without factors (substrings) that are fractional powers \( x^r' \) where \( x \) has length at least \( l \) and \( r' \geq r \).

We prove that \( 1 + \frac{1}{al} \leq R(a, l) \leq 1 + \frac{c}{al} \) for some constant \( c \).

1 Introduction

A fractional power \( x^r \) of a string \( x \) is defined as \( x^r = xxx \ldots xy \) where \( y \) is a prefix of \( x \) and \( |x^r| = r|x| \). (We assume that \( r > 1 \) is a fraction with denominator \( |x| \).)

One may ask whether there exists an infinite sequence of letters that does not contain fractional powers \( x^r \) with large \( r \) and long \( x \). More precisely, for a given alphabet size \( a \), a given integer \( l \) and a given real \( \alpha \) one may ask whether there exists an infinite sequence of letters that does not contain fractional powers \( x^r \) with \( r > \alpha \) and \( |x| \geq l \).

For \( \alpha = 1 \) the answer is evidently negative (each string \( x \) is a fractional power \( x^1 \)). On the other hand, it is easy to see that for any \( a \geq 2 \) and \( l \geq 1 \) the answer is positive if \( \alpha \) is large enough (there exists a binary sequence that does not contain factors \( x^3 \)). The threshold value that separates negative and positive answers is denoted by \( R(a, l) \) in [7]: the authors note that \( 1 < R(a, l) \leq 2 \) and compute exact values of \( R(a, l) \) for some pairs \((a, l)\). Evidently, \( R(a, l) \) decreases when \( a \) or \( l \) increase.

To get a lower bound for \( R(a, l) \), let us apply the pigeonhole principle to \( a + 1 \) letters at positions \( 0, l, 2l, \ldots, al \). Two of them should be equal and...
this creates a fractional power $x^r$ where $|x| \geq l$ and $r \leq 1 + 1/la$ (this power starts and ends with a letter that appears twice). Therefore,

$$R(a, l) \geq 1 + \frac{1}{la}.$$

Francesca Fiorenzi, Pascal Ochem and Elise Vaslet in [8] gave stronger lower bounds and also some upper bounds for $R(a, l)$. In particular, they proved that

$$1 + \frac{1}{1 + \left\lceil \frac{3}{4}(a - 1) \right\rceil} \leq R(a, l) \leq 1 + \frac{2\ln l}{l\ln \lambda} + O\left(\frac{1}{l}\right),$$

where $\lambda = \frac{(a-1) + \sqrt{(a-1)(a+3)}}{2}$ and a constant in $O$ may depend on $a$ but not on $l$.

In this paper we use Lovász local lemma to prove a stronger upper bound for $R(a, l)$. Our upper bound differs from the lower bound only by a constant:

$$R(a, l) \leq 1 + \frac{c}{la}$$

for some $c$ and for all $a \geq 2, l \geq 1$.

## 2 Kolmogorov complexity of subsequences

We present the proof using the notion of Kolmogorov complexity (also called algorithmic complexity or description complexity). We refer the reader to [1] or [10] for the definition and basic properties of Kolmogorov complexity.

For an infinite sequence $\omega$ and finite set $X \subset \mathbb{N}$ let $\omega(X)$ be a string of length $\#X$ formed by $\omega_i$ with $i \in X$ (in the same order as in $\omega$).

We use the following result from [9] that guarantees the existence of a sequence $\omega$ such that strings $\omega(X)$ have high Kolmogorov complexity for all simple $X$:

**Theorem 1.** Let $\alpha$ be a positive real number less than 1. There exists a binary sequence $\omega$ and an integer $N$ such that for any finite set $X$ of cardinality at least $N$ the inequality

$$K(X, \omega(X)|t) \geq \alpha\#X$$

holds for some $t \in A$.

Here $K(X, \omega(X)|t)$ is conditional Kolmogorov complexity of a pair $(X, \omega(X))$ relative to $t$.

We need a slightly more general version of this result (for any alphabet size):
Theorem 2. Let $a \geq 2$ be an integer. Let $\alpha$ be a positive real less than 1. There exists a sequence $\omega$ in $a$-letters alphabet and an integer $N$ such that for any finite set $X$ of cardinality at least $N$ the inequality
\[ K(X, \omega(X)|t) \geq \alpha \#X \log a \]
holds for some $t \in X$.

Proof. Theorem 2 can be proven using exactly the same argument as in [9] (Lovasz local lemma technique). It can also be formally derived from Theorem 1 as follows: we encode $a$ letters of the alphabet by bit blocks of some length $t$ (large enough). This encoding is not bijective (several blocks encode the same letter) but is chosen in such a way that all letters have almost the same number of encodings (about $2^t/a$). Then we take a sequence from Theorem 1, split it into $t$-bit blocks and replace these blocks by corresponding letters. If some subsequence formed by the letters is simple, then the corresponding bit subsequence is simple, too. (Technically we should change $\alpha$ slightly to compensate for “boundary effects”.)

3 Weak upper bound

To illustrate the technique, we first prove a simple generalization of a result obtained by Berk [6] and provide an upper bound for $R(a, l)$ that is weaker than our final bound:

Theorem 3. For every $a \geq 2$ and every real number $b \in (1, a)$ there exists a number $N$ and a sequence $\omega$ in $a$-letters alphabet such that for every $n \geq N$ the distance between any two different occurrences of the same substring of length $n$ in $\omega$ is at least $b^n$.

Proof. Construct a sequence $\omega$ using Theorem 2 with $\alpha$ close enough to 1.

Let $I$ and $J$ ($|I| = |J| = n$) be different intervals where the same substring of length $n$ occurs in $\omega$. Let $X = I \cup J$. Then $n < \#X \leq 2n$ (intervals $I$ and $J$ are not necessarily disjoint) and the first $n$ letters of $\omega(X)$ are equal to the last $n$ letters of $\omega(X)$. It is easy to see that the string $\omega(X)$ is determined by its first $\#X - n$ letters, $n$ and $\#X$, so $K(\omega(X)) \leq (\#X - n) \log a + O(\log n)$.

Assume $t \in X$. Then $X$ is determined by $t$, the number $n$, the distance between $I$ and $J$ and the ordinal number of $t$ in $X$. So if the distance between $I$ and $J$ is less than $b^n$ then $K(\omega(X), X|t) \leq (|X| - n) \log a + n \log b + O(\log n)$ for large enough $n$ and $\alpha$ that is close enough to 1 (because $\log b < \log a$). This contradicts the inequality of Theorem 2. Therefore sequence $\omega$ does not contain a pair of different occurrences of the same substring of sufficiently large length $n$ with distance between them less than $b^n$. 

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In particular, for every integer $a \geq 2$, every real number $b \in (1, a)$ and for large enough $l$ the following inequality holds:

$$R(a, l) < 1 + \frac{\log_b l}{l}.$$ 

4 The final upper bound

In the weak upper bound we used the same sequence for all values of $l$. And now we need different sequences for different values of $l$ but we want the constant $c$ to be the same. To achieve this goal we use the following “$l$-uniform” version of Theorem 1.

**Theorem 4.** Let $\alpha$ be a positive real number less than 1. There exists an integer $N$ such that for every integer $l$ there exists a binary sequence $\omega$ that has the following property: for every finite set $X$ of cardinality at least $N$ the inequality

$$\text{K}(X, \omega(X)|t, l) \geq \alpha \#X$$

holds for some $t \in A$.

Note that $\omega$ may depend on $l$ while $N$ is the same for all values of $l$. (If we allowed $N$ to be dependent on $l$, this would be a standard relativization of Theorem 1.)

**Proof.** Theorem 4 can be proven in the same way as Theorem 1. And it can also be formally derived from it: if a sequence $\tau$ and a number $N$ satisfy the requirements of Theorem 1 and $\bar{z} : \mathbb{N}^2 \to \mathbb{N}$ is a computable bijection, then the sequence $i \mapsto \omega_i = \tau_{\bar{z}(i, l)}$ and the same number $N$ satisfy the requirements of Theorem 4 for the integer $l$. (The bijection adds $O(1)$-term, but this can be compensated by a small change in $\alpha$: the statement is true for every $\alpha < 1$.)

Now we can start proving the upper bound.

**Theorem 5.** There exists a constant $c$ such that for any $a \geq 2$ and $l \geq 1$ the following inequality holds:

$$1 + \frac{1}{al} \leq R(a, l) \leq 1 + \frac{c}{al}.$$ 

**Proof.** The lower bound is easy (as shown in the introduction). Let us prove the upper bound. Let us assume first that $a = 2$ (the general case can be reduced to this special one).

Consider a sequence $\omega$ satisfying the requirements of Theorem 4 for some $\alpha < \frac{1}{2}$. Then the required sequence with long fractional powers will be constructed as

$$\tau_i = \omega f(i)$$
for some mapping \( f : \mathbb{N} \to \mathbb{N} \).

At first let us define \( f \) at the first \( l \) integers (the value of integer constant \( m \) will be chosen later):

1. \( f(i) = i \mod m \) for \( i < l \) and \( (i \mod m) \neq m - 1 \) (we say that these indexes have rank 1).

2. \( f(mi+m-1) = (m-1) + (i \mod m) \) for \( mi + m - 1 < l \) and \( (i \mod m) \neq m - 1 \) (we say that these indexes have rank 2).

3. \( f(m^2i + m^2 - 1) = 2(m - 1) + (i \mod m) \) for \( m^2i + m^2 - 1 < l \) and \( (i \mod m) \neq m - 1 \) (we say that these indexes have rank 3).

(And so on until \( f \) is defined at all first \( l \) integers.)

Then we define \( f \) on other blocks of \( l \) integers in the same way but using fresh bits each time. So if \( f(\{0, 1, \ldots, l-1\}) = \{0, 1, \ldots, L-1\} \) then \( f(i + jl) = f(i) + jL \).

Suppose the sequence \( \tau_i = \omega_{f(i)} \) contains some fractional power \( xyz \) with \( |xy| \geq l \) and the exponent \( \frac{|xy/x|}{|xy|} \geq 1 + \frac{c}{2l} \). Without loss of generality we can assume that the exponent \( 1 + \frac{c}{2l} \) is not greater than 2 (otherwise the statement of the theorem follows from the existence of a binary sequence, called Thue-Morse sequence, that does not contain any fractional power with exponent greater than 2, see [2], [3]). Also we can assume that \( c > 2m \) (increasing \( c \), we make our task easier). So \( l \geq \frac{c}{2} > m \) and \(|x| \geq \frac{c}{2l}|xy| > m \).

First we consider the case when both occurrences of \( x \) in \( xyz \) lie entirely in some blocks of size \( l \) (in two different blocks, because \(|xy| \geq l|\)). Denote by \( n \) the number of \( l \)-sized blocks between these two occurrences of \( x \) and denote by \( k \) the integer number that satisfies the inequality \( m^{k-1} \leq |x| < m^k \). Then \( m^k > \frac{c}{2l}n \) and \( k \geq 2 \) (because \(|x| \geq \frac{c}{2l}|xy| > m \).

Let us denote by \( I \) and \( J \) the sets of values of \( f \) for the first and second occurrences of \( x \) (respectively) whose rank is not greater than \( k \) (obviously there is at most 1 index in each of these occurrences of \( x \) whose rank is greater than \( k \)). The sets \( I \) and \( J \) are disjoint because these occurrences of \( x \) lie in the different \( l \)-sized blocks. Assume \( Z = I \cup J \), then for some \( t \in Z \) we have \( K(Z, \omega(Z)|t, l) \geq \alpha \#Z \) by the statement of Theorem 4 (we need here that \( m > N + 1 \) since \( \#Z \) should be greater than \( N \)).

Obviously,

\[
\frac{1}{2} \#Z = \#I + O(1) = \#J + O(1) = (k - 1)(m - 1) + \frac{|x|}{m^{k-1}} + O(1).
\]

The set \( Z \) is determined by \( t, l, m, n, k, |x| \) and the start/end positions for the two occurrences of the word \( x \) modulo \( m^k \) (and one bit saying whether \( t \) belongs to the first occurrence of \( x \) or to the second one). So \( K(Z \mid t, l) \leq \log n + O(\log(m^k)) = O(k \log m) \) (since \( m^k > \frac{c}{2l}n \)). We can also calculate
\( \omega(Z) \) if \( \omega(I) \) is given (we need at most one extra bit for calculating the entire string \( x \)).

\[
O(k \log m) + \frac{1}{2} \# Z \geq \alpha \# Z,
\]

but \( \alpha > \frac{1}{2} \) and \( \# Z \geq 2(k - 1)(m - 1) + O(1) \geq k(m - 1) + O(1) \). So \( k(m - 1) < O(k \log m) \) that is a contradiction if \( m \) is large enough. (Recall that the choice of \( m \) was postponed.)

Consider now the general case for the position of the two occurrences of \( x \). If length of \( x \) is not large, i.e. \(|x| \leq l\), we can reduce this case to the previous one by splitting \( x \) into parts and choosing the largest part (we must multiply the constant \( c \) by 3). Now let \( x \) be longer than the block size \((|x| > l)\). We can assume that there is no \( l \)-sized block that intersects both occurrences of \( x \) (in the other case we also split the word \( x \) in parts).

Let us denote by \( I \) and \( J \) the sets of values of \( f \) in the first and second occurrences of \( x \) respectively. The sets \( I \) and \( J \) are disjoint. Assume \( Z = I \cup J \). Then for some \( t \in Z \) we have \( K(Z, \omega(Z))|t,l \geq \alpha \# Z \).

The set \( Z \) is determined by \( t, l, m \) and the relative start/end positions of the two occurrence of the word \( x \) with respect to the one of the preimages of \( t \) (for example, the first one). So \( K(Z \mid t,l) \leq \log |x| + O(\log l) = O(\log |x|) \) (since \(|x| \geq l \) and \(|x| \geq \frac{c}{2}\log |x| \)). To compute \( \omega(Z) \), it is enough to know at most a half of it (\( \omega(I) \) or \( \omega(J) \), whichever is smaller). Therefore

\[
O(\log |x|) + \frac{1}{2} \# Z \geq \alpha \# Z,
\]

but \( \alpha > \frac{1}{2} \) and \( \# Z = \Omega \left( \frac{|x|}{l} (m - 1) \log m \right) = \Omega \left( (\log |x|) \frac{m-1}{\log m} \right) \) (here we use that \(|x| > l > m \) and \( \frac{|x|}{\log |x|} \geq \frac{1}{\log l} \)). That is a contradiction if \( m \) is large enough.

This finishes the proof for \( a = 2 \).

Assume now that \( a \geq 6 \) and \( a \) is even. Let \( \omega \) be the sequence constructed for binary alphabet and \( l' = \frac{a-2}{2}l \). To get the required sequence \( \nu \) we will color the terms of \( \omega \) into \( \frac{a}{2} \) colors: the \( i \)-th block of size \( l \) gets color \( i \) mod \( \frac{a}{2} \). Then the size of the alphabet of sequence \( \nu \) (whose terms are now \( \log \) (bit, color) pairs) equals to \( a \) and \( \nu \) does not contain fractional powers \( z^p \) with \( |z| \geq \frac{a-2}{2}l \) and \( p \geq 1 + \frac{a}{a-2} \). And obviously \( \nu \) does not contain any fractional powers \( z^p \) with \( l \leq |z| \leq \frac{a-2}{2}l \) (because it does not contain pairs of equal letters at these distances).

Therefore \( R(a,l) \leq 1 + \frac{a}{a-2} \) if \( a \geq 6 \) and \( a \) is even, and \( R(2,l) \leq 1 + \frac{2}{27} \).

To prove the theorem for arbitrary \( a \) it remains to note that that \( R(a,l) \) is decreasing in \( a \), so \( R(a,l) \leq 1 + \frac{2c}{\alpha} \) for every \( a \geq 2, l \geq 1 \).
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