Quatetion Windowed Linear Canonical Transform of Two-Dimensional Quaternionic Signals

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Abstract

We investigate the 2D quaternion windowed linear canonical transform (QWLCT) in this paper. Firstly, we propose the new definition of the QWLCT, and then several important properties of newly defined QWLCT, such as bounded, shift, modulation, orthogonality relation, are derived based on the spectral representation of the quaternionic linear canonical transform (QLCT). Finally, by the Heisenberg uncertainty principle for the QLCT and the orthogonality relation property for the QWLCT, the Heisenberg uncertainty principle for the QWLCT is established.

Index Terms

Linear canonical transform, Quaternion linear canonical transform, Quaternion windowed linear canonical transform, Window function, Uncertainty principle

1. Introduction.

It is well-known that the linear canonical transform (LCT) has found broad applications in many fields of applied mathematics, signal processing, radar system analysis, filter design, phase retrieval, and pattern recognition and optics [1,3,4,5]. All of these results show that the LCT plays an important role for chirp signal analysis in parameter estimation, sampling and filtering [2-9]. However, the LCT can’t show the local LCT-frequency contents as a result of its global kernel. In [12,26] the LCT has been successfully applied to research the generalized the windowed function. The WLCT is method devised to study signals whose spectral content changes with time. As shown in [11,8] some important properties of the WLCT are discussed. Those include the analogue of the Poisson summation formula, sampling formulas, Paley-Wiener theorem, and uncertainly relations. It presents the time and LCT-frequency information, and is originally a local LCT distribution.

On the other hand, some authors have generalized the linear canonical transform to quaternion-valued signals, known as the quaternionic linear canonical transform (QLCT). The QLCT was firstly studied in [13] including prolate spheroidal wave signals and uncertainty principles [16]. Some useful properties of the QLCT such as linearity, reconstruction formula, continuity, boundedness, positivity inversion formula and the uncertainty principle were established in [14,10,15,17,13,18,27]. An application of the QLCT to study of generalized swept-frequency filters was introduced in [20]. Because of the non-commutative property of multi-plication of quaternions, there are mainly three various types of 2D quaternion linear canonical
transform (QLCTs): Two-sided QLCTs, Left-sided QLCTs and Right-sided QLCTs (refer to [13]). Based on the (two-sided) QLCT, one may extend the WLCT to quaternion algebra while possessing similar properties as in the classical case.

The WLCT has been widely used, but the quaternion windowed linear canonical transform (QWLCT) hasn’t been mentioned yet. This extension helps solve problems in different fields such as signal processing and optics. The QWLCT proposed in the present work is significantly. The main goals of the present paper are to study the properties of the Two-sided QLCTs of 2D quaternionic signals and to derive the novel concept of the quaternion windowed linear canonical transform, and research important properties of the QWLCT such as bounded, shift, modulation, orthogonality relation. Using the Heisenberg uncertainty principle for the QLCT and the orthogonality relation property for the QWLCT, we establish a generalized QWLCT uncertainty principle. This also lays a good foundation for the practical application in the future.

The paper is organized as follows: Section 2 gives a brief introduction to some general definitions and basic properties of quaternion algebra, QLCTs of 2D Quaternion-valued signals. We give the definition and study the properties of the QWLCT in section 3. In Section 4, provides the uncertainty principles associated with the QWLCT. We give some examples of the QWLCT in section 5. In Section 6, some conclusions are drawn.

2. Preliminary.

In this section, we mainly review some basic facts on the quaternion algebra and the QLCT, which will be needed throughout the paper.

2.1 Quaternion algebra.

The quaternion algebra is an extension of the complex number to 4D algebra. It was first invented by W. R. Hamilton in 1843 and classically denoted by \( \mathbb{H} \) in his honor. Every element of \( \mathbb{H} \) has a Cartesian form given by

\[
\mathbb{H} = \{ q | q := [q]_0 + i[q]_1 + j[q]_2 + k[q]_3, [q]_i \in \mathbb{R}, i = 0, 1, 2, 3 \}
\]  

where \( i, j, k \) are imaginary units obeying Hamilton’s multiplication rules (see [13])

\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]  

Let \([q]_0\) and \(q = i[q]_1 + j[q]_2 + k[q]_3\) denote the real scalar part and the vector part of quaternion number \(q = [q]_0 + i[q]_1 + j[q]_2 + k[q]_3\), respectively. Then, from [22], the real scalar part has a cyclic multiplication symmetry

\[
[pql]_0 = [qlp]_0 = [lpq]_0, \quad \forall q, p, t \in \mathbb{H}
\]  

the conjugate of a quaternion \(q\) is defined by \(\overline{q} = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3\), and the norm of \(q \in \mathbb{H}\) defined as

\[
|q| = \sqrt{\overline{q}q} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}
\]  

it is easy to verify that

\[
\overline{qp} = \overline{pq}, |qp| = |q||p|, \quad \forall q, p \in \mathbb{H}
\]
The quaternion modules $L^2(\mathbb{R}^2, \mathbb{H})$ are defined as

$$L^2(\mathbb{R}^2, \mathbb{H}) := \{ f | f : \mathbb{R}^2 \rightarrow \mathbb{H}, \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} |f(x_1, x_2)|^2dx_1dx_2 < \infty \}$$

(6)

Now we introduce an inner product of quaternion functions $f, g$ defined on $L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} f(x)\overline{g(x)}dx, \quad dx = dx_1dx_2$$

(7)

with symmetric real scalar part

$$\langle f, g \rangle = \frac{1}{2} \{ \langle f, g \rangle + \langle g, f \rangle \} = \int_{\mathbb{R}^2} |f(x)\overline{g(x)}|dx$$

(8)

The associated scalar norm of $f(x) \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined by both (7) and (8):

$$\| f \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle f, f \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} |f(x)|^2dx < \infty$$

(9)

**Lemma 2.1.** If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then the Cauchy-Schwarz inequality holds[25]

$$|\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})}|^2 \leq \| f \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} \| g \|^2_{L^2(\mathbb{R}^2, \mathbb{H})}$$

(10)

If and only if $f = -\lambda g$ for some quaternionic parameter $\lambda \in \mathbb{H}$, the equality holds.

### 2.2 The quaternion linear canonical transform

The QLCT is firstly defined by Kou, et., which is a generalization of the LCT in the frame of quaternion algebra[13]. Due to the non-commutativity of quaternion multiplication, there are three different types of the QLCT: the left-sided QLCT, the right-sided QLCT, and the two-sided QLCT. In this paper, we mainly focus on the two-sided QLCT.

**Definition 2.1.** Let $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ be a matrix parameter satisfying $\det(A_i) = 1$, for $i = 1, 2$. The two-sided QLCT of signals $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$L^H_{A_1, A_2} \{ f \} (w) = \int_{\mathbb{R}^2} K^1_{A_1}(x_1, \omega_1)f(x)K^j_{A_2}(x_2, \omega_2)dx$$

(11)

where $w = (\omega_1, \omega_2)$ is regarded as the QLCT domain, and the kernel signals $K^1_{A_1}(x_1, \omega_1), K^j_{A_2}(x_2, \omega_2)$ are respectively given by

$$K^1_{A_1}(x_1, \omega_1) := \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{i}{2d_1} \left( a_1^2x_1^2 - 2b_1\omega_1x_1 + d_1\omega_1^2 \right)}, & b_1 \neq 0 \\ \sqrt{d_1} e^{\frac{i}{2d_1} \omega_1^2} \delta(x_1 - d_1w_1), & b_1 = 0 \end{cases}$$

(12)

and

$$K^j_{A_2}(x_2, \omega_2) := \begin{cases} \frac{1}{\sqrt{2\pi b_2}} e^{\frac{i}{2d_2} \left( a_2^2x_2^2 - 2b_2\omega_2x_2 + d_2\omega_2^2 \right)}, & b_2 \neq 0 \\ \sqrt{d_2} e^{\frac{i}{2d_2} \omega_2^2} \delta(x_2 - d_2w_2), & b_2 = 0 \end{cases}$$

(13)

From above definition, it is noted that for $b_i = 0$, $i = 1, 2$ the QLCT of a signal is essentially a chirp multiplication and
it is of no particular interest for our objective in this work. Hence, without loss of generality, we set \( b_i \neq 0 \) in the following section unless stated otherwise. Under some suitable conditions, the QLCT above is invertible and the inversion is given in the following section.

**Definition 2.2.** Suppose \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), then the inversion of the QLCT of \( f \) is given by

\[
 f(x) = L^{-1}_{A_1, A_2} \left[ L^H_{A_1, A_2} \{ f \} \right](x) = \int_{\mathbb{R}^2} K^{-1}_{A_1}(x_1, \omega_1) L^H_{A_1, A_2} \{ f \}(w) K^{-1}_{A_2}(x_2, \omega_2) dw
\]  

(14)

The Parseval formula of QLCT is vary available as follows:

**Lemma 2.2 (QLCT Parseval)**[14]. Two quaternion functions \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \) are related to their QLCT via the Parseval formula, given as

\[
 \langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \langle L^H_{A_1, A_2} \{ f \}, L^H_{A_1, A_2} \{ g \} \rangle_{L^2(\mathbb{R}^2, \mathbb{H})}
\]  

(15)

For \( f = g \), one have

\[
 \| f \|^2_{L^2(\mathbb{R}^2, \mathbb{H})} = \| L^H_{A_1, A_2} \{ f \} \|^2_{L^2(\mathbb{R}^2, \mathbb{H})}
\]  

(16)

### 3. Quaternionic Windowed Linear Canonical Transform (QWLCT)

In this section, the generalization of window function associated with the QLCT will be discussed, which is denoted by QWLCT. Moreover, several basic properties of them are investigated.

#### 3.1. The definition of the 2D QWLCT

This section leads to the 2D quaternion window function associated with the QLCT. Due to the noncommutative property of multiplication of quaternions, there are three different types of QWLCTs: Two-sided QWLCT, Left-sided QWLCT and Right-sided QWLCT. Alternatively, we use the (Two-sided) QWLCT to define the QWLCT.

**Definition 3.1 (QWLCT)** Let \( A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) be a matrix parameter satisfying \( \det(A_i) = 1 \), for \( i = 1, 2 \). \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a quaternion window function. The two-sided QWLCT of a signal \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) with respect to \( \phi \) is defined by

\[
 G^A_{\phi, A_1, A_2} f(w, u) = \int_{\mathbb{R}^2} K^1_{A_1}(x_1, \omega_1) f(x) \overline{\phi(x - u)} K^j_{A_2}(x_2, \omega_2) dx
\]  

(17)

where \( u = (u_1, u_2) \in \mathbb{R}^2 \), \( K^1_{A_1}(x_1, \omega_1) \) and \( K^j_{A_2}(x_2, \omega_2) \) are given by (12) and (13), respectively.

For a fixed \( u \), we have

\[
 G^A_{\phi, A_1, A_2} f(w, u) = L^H_{A_1, A_2} \{ f(x) \overline{\phi(x - u)} \}(w)
\]  

(18)
Applying the inverse QLCT to (18), we have
\[
f(x)\phi(x-u) = \mathcal{L}^{-1}_{A_1,A_2}\{G^A_{\phi} f(w, u)\}
\]
\[
= \int_{\mathbb{R}^2} K^j_{A_1}(x_1, \omega_1) G^A_{\phi} \{f\}(w, u) K^j_{A_2}(x_2, \omega_2) dw
\]
(19)

3.2. Some properties of QWLCT

In this subsection, we discuss several basic properties of the QWLCT. These properties play important roles in signal representation.

Theorem 3.1 (Boundedness) Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a window function and \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \), then
\[
|G^A_{\phi} f(w, u)| \leq \frac{1}{2\pi \sqrt{|b_1 b_2|}}\|f\|_{L^2(\mathbb{R}^2)}\|\phi\|_{L^2(\mathbb{R}^2)}
\]
(20)

Proof. By using the quaternion Cauchy-Schwarz inequality (10)
\[
|G^A_{\phi} f(w, u)|^2 = \left( \int_{\mathbb{R}^2} K^j_{A_1}(x_1, \omega_1) f(x)\overline{\phi(x-u)} K^j_{A_2}(x_2, \omega_2) dx \right)^2
\]
\[
\leq \left( \int_{\mathbb{R}^2} |K^j_{A_1}(x_1, \omega_1) f(x)| dx \right)^2 \cdot \left( \int_{\mathbb{R}^2} |\phi(x-u)|^2 dx \right)
\]
\[
= \frac{1}{4\pi^2|b_1 b_2|} \left( \int_{\mathbb{R}^2} |f(x)|^2 dx \right) \left( \int_{\mathbb{R}^2} |\phi(x-u)|^2 dx \right)
\]
\[
= \frac{1}{4\pi^2|b_1 b_2|} \|f\|_{L^2(\mathbb{R}^2)}^2 \|\phi\|_{L^2(\mathbb{R}^2)}^2
\]
where applying the change of variables \( x-u = t \) in the last second step. Then we have
\[
|G^A_{\phi} f(w, u)| \leq \frac{1}{2\pi \sqrt{|b_1 b_2|}}\|f\|_{L^2(\mathbb{R}^2)}\|\phi\|_{L^2(\mathbb{R}^2)}
\]
(22)
which completes the proof. \( \square \)

Theorem 3.2 (Linearity). Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a window function and \( f, g \in L^2(\mathbb{R}^2, \mathbb{H}) \), the QWLCT is a linear operator, namely,
\[
[G^A_{\phi} (\lambda f + \mu g)](w, u) = \lambda G^A_{\phi} f(w, u) + \mu G^A_{\phi} g(w, u)
\]
(23)
for arbitrary constants \( \lambda \) and \( \mu \).

Proof. This follows directly from the linearity of the product and the integration involved in Definition 3.1. \( \square \)

Theorem 3.3 (Parity). Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a window function and \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \). Then we have
\[
G^A_{\phi} \{ Pf \}(w, u) = G^A_{\phi} f(-w, -u)
\]
(24)
where $P\phi(x) = \phi(-x)$ for every $\phi \in L^2(\mathbb{R})$.

**Proof.** A direct calculation gives, for every $f \in L^2(\mathbb{R}^2, \mathbb{H})$,

$$G^A_{P\phi} \{ P f \} (w, u) = \int_{\mathbb{R}^2} K^3_{A_1}(x_1, \omega_1) f(-x) \phi(-(x-u)) K^3_{A_2}(x_2, \omega_2) dx$$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{a_1^2}{2} (-x_1^2) - \frac{x_1}{s_1} + \frac{d_1}{s_1} (-\omega_1^2) - \frac{\omega_1}{s_1} \right)} f(-x) \phi(-(x-u)) \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{a_2^2}{2} (-x_2^2) - \frac{x_2}{s_2} + \frac{d_2}{s_2} (-\omega_2^2) - \frac{\omega_2}{s_2} \right)} dx$$

which proves the theorem according to Definition 3.1. □

**Theorem 3.4 (Shift).** Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have

$$G^A_{\phi} \{ T_r f \} (w, u) = e^{i r_1 \omega_1 c_1} e^{-i \frac{a_1^2}{m_1} c_1} G^A_{\phi} \{ f \} (m, n) e^{i r_2 \omega_1 c_2} e^{-i \frac{a_2^2}{m_1} c_2}$$

where $T_r f(x) = f(x - r)$, $r = (r_1, r_2)$, $m = (m_1, m_2)$, $n = (n_1, n_2) \in \mathbb{R}^2$, $m_i = w_i - a_i r_i$, $n_i = u_i - r_i$, $i = 1, 2$.

**Proof.** By Definition 3.1, we have

$$G^A_{\phi} \{ T_r f \} (w, u) = \int_{\mathbb{R}^2} K^3_{A_1}(x_1, \omega_1) f(x - r) \phi(x-u) K^3_{A_2}(x_2, \omega_2) dx$$

By making the change of variable $t = x - r$ in the above expression, we obtain

$$G^A_{\phi} \{ T_r f \} (w, u) = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{a_1^2}{2} (r_1+t_1)^2 - \frac{r_1}{s_1} + \frac{d_1}{s_1} \omega_1^2 - \frac{\omega_1}{s_1} \right)} f(t) \phi(t-(u-r)) \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{a_2^2}{2} (r_2+t_2)^2 - \frac{r_2}{s_2} + \frac{d_2}{s_2} \omega_2^2 - \frac{\omega_2}{s_2} \right)} dt$$

$$= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} e^{i \left( \frac{a_1^2}{2} (r_1+t_1)^2 - \frac{r_1}{s_1} + \frac{d_1}{s_1} (\omega_1+r_1) \omega_1^2 - \frac{\omega_1}{s_1} \right)} f(t) \phi(t-(u-r)) \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{a_2^2}{2} (r_2+t_2)^2 - \frac{r_2}{s_2} + \frac{d_2}{s_2} (\omega_2-r_2) \omega_2^2 - \frac{\omega_2}{s_2} \right)} dt$$

$$= e^{i \frac{a_1^2}{m_1} (2r_1 a_1 (\omega_1-r_1 a_1)+(r_1 a_1)^2)} e^{i \frac{a_2^2}{m_1} r_2^2} e^{-i \frac{a_1^2}{m_1} c_1} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{a_1^2}{2} (r_1+t_1)^2 - \frac{r_1}{s_1} + \frac{d_1}{s_1} (\omega_1+r_1 a_1)^2 - \frac{\omega_1}{s_1} \right)} f(t) \phi(t-(u-r)) \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{a_2^2}{2} (r_2+t_2)^2 - \frac{r_2}{s_2} + \frac{d_2}{s_2} (\omega_2-r_2 a_2)^2 - \frac{\omega_2}{s_2} \right)} dt \times f(t) \phi(t-(u-r)) \frac{1}{\sqrt{2\pi b_2}} e^{i \frac{a_2^2}{m_1} (2r_2 a_2 (\omega_2-r_2 a_2)+(r_2 a_2)^2)} e^{i \frac{a_2^2}{m_1} r_2^2} e^{-i \frac{a_2^2}{m_1} c_2}$$

Applying the definition of the QWLCT, the above expression can be rewritten in the form

$$G^A_{\phi} \{ T_r f \} (w, u) = e^{i \frac{a_1^2}{m_1} (2r_1 a_1 (\omega_1-r_1 a_1)+(r_1 a_1)^2)} e^{i \frac{a_2^2}{m_1} r_2^2} e^{-i \frac{a_1^2}{m_1} c_1} G^A_{\phi} \{ f \} (m, n)$$

(29)

Because $a_i d_i - b_i c_i = 1$, for $i = 1, 2$. We get

$$e^{i \frac{a_1^2}{m_1} (2r_1 a_1 (\omega_1-r_1 a_1)+(r_1 a_1)^2)} e^{i \frac{a_2^2}{m_1} r_2^2} e^{-i \frac{a_1^2}{m_1} c_1} = e^{i r_1 \omega_1 c_1} e^{-i \frac{a_1^2}{m_1} c_1}$$

(30)
We finally arrive at
\begin{equation}
G_{\phi}^{A_1, A_2} \{ T_r f \} (w, u) = e^{i r_1 \omega_1 c_1} e^{-i \frac{r_1^2}{4} c_1} G_{\phi}^{A_1, A_2} \{ f \} (m, n) \times e^{i r_2 \omega_2 c_2} e^{-i \frac{r_2^2}{4} c_2}
\end{equation}
which completes the proof. □

**Theorem 3.5 (Modulation).** Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. $M_s f$ be modulation operator defined by $M_s f (x) = e^{ixs_1} f(x) e^{ixs_2}$ with $s = (s_1, s_2) \in \mathbb{R}^2$ Then we have
\begin{equation}
G_{\phi}^{A_1, A_2} \{ M_s f \} (w, u) = e^{iw_1 s_1 d_1} e^{-i \frac{b_1 d_1^2}{2}} G_{\phi}^{A_1, A_2} \{ f \} (v, u) e^{i w_2 s_2 d_2} e^{-i \frac{b_2 d_2^2}{2}}
\end{equation}
where $v = (v_1, v_2) \in \mathbb{R}^2$, $v_i = w_i - s_i b_i$, $i = 1, 2$.

**Proof.** From Definition 3.1, it follows that
\begin{equation}
G_{\phi}^{A_1, A_2} \{ M_s f \} (w, u) = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{a_1}{2 \pi^2} x_1^2 + \frac{a_1}{2 \pi^2} - \frac{x_1}{2} \right) \frac{b_1}{2} \frac{d_1}{2} \omega_1^2} e^{i x_1 s_1 f(x) \phi(x - u)} \times e^{i \frac{r_2}{2} \omega_2 c_2} e^{i \left( \frac{a_2}{2 \pi^2} x_2^2 + \frac{a_2}{2 \pi^2} - \frac{x_2}{2} \right) \frac{b_2}{2} \frac{d_2}{2} \omega_2^2} \frac{dx}{\sqrt{2\pi b_2}}
\end{equation}
\begin{equation}
= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} e^{i \left( \frac{a_1}{2 \pi^2} x_1^2 + \frac{a_1}{2 \pi^2} - \frac{x_1}{2} \right) \frac{b_1}{2} \frac{d_1}{2} \omega_1^2} \frac{dx}{\sqrt{2\pi b_2}} \times \int_{\mathbb{R}^2} e^{i \left( \frac{a_2}{2 \pi^2} x_2^2 + \frac{a_2}{2 \pi^2} - \frac{x_2}{2} \right) \frac{b_2}{2} \frac{d_2}{2} \omega_2^2} \frac{dx}{\sqrt{2\pi b_2}}
\end{equation}
\begin{equation}
= e^{iw_1 s_1 d_1} e^{-i \frac{b_1 d_1^2}{2}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{a_1}{2 \pi^2} x_1^2 + \frac{a_1}{2 \pi^2} - \frac{x_1}{2} \right) \frac{b_1}{2} \frac{d_1}{2} \omega_1^2} \frac{dx}{\sqrt{2\pi b_2}} \times \int_{\mathbb{R}^2} e^{i \left( \frac{a_2}{2 \pi^2} x_2^2 + \frac{a_2}{2 \pi^2} - \frac{x_2}{2} \right) \frac{b_2}{2} \frac{d_2}{2} \omega_2^2} \frac{dx}{\sqrt{2\pi b_2}}
\end{equation}
Hence,
\begin{equation}
G_{\phi}^{A_1, A_2} \{ M_s f \} (w, u) = e^{iw_1 s_1 d_1} e^{-i \frac{b_1 d_1^2}{2}} G_{\phi}^{A_1, A_2} \{ f \} (v, u) e^{i w_2 s_2 d_2} e^{-i \frac{b_2 d_2^2}{2}}
\end{equation}
which completes the proof. □

**Theorem 3.6 (Inversion formula).** Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ be a window function, $0 < ||\phi||^2 < \infty$ and $f \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have the inversion formula of the QWLCT,
\begin{equation}
f(x) = \frac{1}{||\phi||^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^{A_1, A_2} (x_1, \omega_1) G_{\phi}^{A_1, A_2} \{ f \} (w, u) \bar{K}_{A_2}^{A_1} (x_2, \omega_2) \phi(x - u) dw du
\end{equation}

**Proof.** Multiplying both sides of (19) from the right by $\phi(x - u)$ and integrating with respect to $du$ we get
\begin{equation}
\int_{\mathbb{R}^2} f(x) \phi(x - u) \phi(x - u) du = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_{A_1}^{A_1, A_2} (x_1, \omega_1) G_{\phi}^{A_1, A_2} \{ f \} (w, u) \bar{K}_{A_2}^{A_1} (x_2, \omega_2) \phi(x - u) dw du
\end{equation}
Using (9), we have
\[
f(x) = \frac{1}{\|\phi\|^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K^i_{A_1}(x_1, \omega_1) G^{A_1, A_2}_\phi \{ f \}(w, u) K^j_{A_2}(x_2, \omega_2) \phi(x-u) \, dw \, du \tag{38}
\]
which completes the proof. □

**Theorem 3.7 (Orthogonality relation).** Let \( \phi, \psi \in L^2(\mathbb{R}^2, \mathbb{R}) \setminus \{0\} \) be a window function and \( f, g \in L^2(\mathbb{R}^2, \mathbb{R}) \). Then
\[
\langle G^{A_1, A_2}_\phi \{ f \}(w, u), G^{A_1, A_2}_\psi \{ g \}(w, u) \rangle = \langle (f, g) \langle \phi, \psi \rangle \rangle_0 \tag{39}
\]

**Proof.** Using (3), (8), we get the output as follows:
\[
\langle G^{A_1, A_2}_\phi \{ f \}(w, u), G^{A_1, A_2}_\psi \{ g \}(w, u) \rangle = \int_{\mathbb{R}^4} [G^{A_1, A_2}_\phi \{ f \}(w, u) \overline{G^{A_1, A_2}_\psi \{ g \}(w, u)}] \, dw \, du
\]
\[
= \int_{\mathbb{R}^4} \left[ G^{A_1, A_2}_\phi \{ f \}(w, u) \int_{\mathbb{R}^2} K^i_{A_1}(x_1, \omega_1) g(x) \psi(x-u) K^j_{A_2}(x_2, \omega_2) \, dx \right] \, dw \, du
\]
\[
= \int_{\mathbb{R}^6} \left[ G^{A_1, A_2}_\phi \{ f \}(w, u) K^{-1}_{A_2}(x_2, \omega_2) \psi(x-u) \overline{g(x)} K^{-1}_{A_1}(x_1, \omega_1) \right] \, dw \, dx \, du
\]
\[
= \int_{\mathbb{R}^4} \left[ \int_{\mathbb{R}^2} K^{-1}_{A_1}(x_1, \omega_1) G^{A_1, A_2}_\phi \{ f \}(w, u) K^{-1}_{A_2}(x_2, \omega_2) \psi(x-u) \overline{g(x)} \right] \, dw \, du
\]
\[
= \int_{\mathbb{R}^4} \left[ f(x) \phi(x-u) \overline{\psi(x-u) g(x)} \right] \, dw \, du \tag{40}
\]

Because
\[
K^{-1}_{A_1}(x_1, \omega_1) = K^i_{A_1}(\omega_1, x_1) = K^i_{A_1}(\omega_1, x_1); \quad K^{-1}_{A_2}(x_2, \omega_2) = K^j_{A_2}(\omega_2, x_2) = K^j_{A_2}(\omega_2, x_2) \tag{41}
\]

Using (19), we have
\[
\langle G^{A_1, A_2}_\phi \{ f \}(w, u), G^{A_1, A_2}_\psi \{ g \}(w, u) \rangle = \int_{\mathbb{R}^4} \left[ \int_{\mathbb{R}^2} K^i_{A_1}(\omega_1, x_1) G^{A_1, A_2}_\phi \{ f \}(w, u) K^j_{A_2}(\omega_2, x_2) \psi(x-u) \overline{g(x)} \right] \, dx \, dw \, du
\]
\[
= \int_{\mathbb{R}^4} \left[ f(x) \phi(x-u) \overline{\psi(x-u) g(x)} \right] \, dx \, dw \, du \tag{42}
\]

Using the change of variables \( x-u = y \), the equation becomes
\[
\langle G^{A_1, A_2}_\phi \{ f \}(w, u), G^{A_1, A_2}_\psi \{ g \}(w, u) \rangle = \int_{\mathbb{R}^4} \left[ f(y) \phi(y) \overline{\psi(y) g(y)} \right] \, dy \, dx
\]
\[
= \left[ \int_{\mathbb{R}^2} f(x) \overline{g(x)} \, dx \int_{\mathbb{R}^2} \psi(y) \overline{\phi(y)} \, dy \right]_0
\]
\[
= \langle (f, g) \langle \phi, \psi \rangle \rangle_0 \tag{43}
\]

which completes the proof. □

Based on the above theorem, we may conclude the following important consequences.
(i) If $\phi = \psi$, then
\[
\langle G^{A_1, A_2}_\phi \{f\}(w, u), G^{A_1, A_2}_\psi \{g\}(w, u) \rangle = \|\phi\|^2_{L^2(\mathbb{R}^2)} \langle f, g \rangle
\] (44)

(ii) If $f = g$, then
\[
\langle G^{A_1, A_2}_\phi \{f\}(w, u), G^{A_1, A_2}_\psi \{f\}(w, u) \rangle = \|f\|^2_{L^2(\mathbb{R}^2)} \langle \phi, \psi \rangle
\] (45)

(iii) If $f = g$ and $\phi = \psi$, then
\[
\langle G^{A_1, A_2}_\phi \{f\}(w, u), G^{A_1, A_2}_\phi \{f\}(w, u) \rangle = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G^{A_1, A_2}_\phi \{f\}(w, u)|^2 dw du = \|f\|^2_{L^2(\mathbb{R}^2)} \|\phi\|^2_{L^2(\mathbb{R}^2)}
\] (46)

Table 1: Properties of the QWLCT of $\phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\}$ and $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, where $\lambda, \mu \in \mathbb{H}$ are arbitrary constants, $r = (r_1, r_2), s = (s_1, s_2)$. $m = (m_1, m_2), v = (v_1, v_2), n = (n_1, n_2) \in \mathbb{R}^2, m_i = w_i - a_i r_i, n_i = u_i - r_i, v_i = w_i - s_i b_i, i = 1, 2$.

| Property          | Function                                                                 | QWLCT                                                                 |
|-------------------|--------------------------------------------------------------------------|-----------------------------------------------------------------------|
| Boundedness       | $|G^{A_1, A_2}_\phi \{f\}(w, u)| \leq \frac{1}{2\pi}\|f\|_{L^2(\mathbb{R}^2)} \|\phi\|_{L^2(\mathbb{R}^2)}$ |                                                                       |
| Linearity          | $\lambda f + \mu g$                                                      | $\lambda G^{A_1, A_2}_\phi \{f\}(w, u) + \mu G^{A_1, A_2}_\phi \{g\}(w, u)$ |
| Parity            | $G^{A_1, A_2}_\phi \{P f\}(w, u)$                                       | $G^{A_1, A_2}_\phi \{f\}(-w, -u)$                                    |
| Shift             | $f(x - r)$                                                               | $e^{ir_1 \omega_1 c_1 e^{-i\omega_1^2 c_1}} G^{A_1, A_2}_\phi \{f\}(m, n)$ |
| Modulation        | $f(x - r)$                                                               | $e^{ir_1 \omega_1 c_1 e^{-i\omega_1^2 c_1}} G^{A_1, A_2}_\phi \{f\}(m, n)$ |
| Formula           | $f(x) = \frac{1}{\|\phi\|_{L^2(\mathbb{R}^2)}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K^{A_1}_\phi(x_1, x_2) G^{A_1, A_2}_\phi \{f\}(w, u) G^{A_1, A_2}_\phi \{g\}(w, u) \phi(x - u) dw du$ |                                                                       |
| Orthogonality     | $\langle G^{A_1, A_2}_\phi \{f\}(w, u), G^{A_1, A_2}_\psi \{g\}(w, u) \rangle = \langle f, g \rangle \langle \phi, \psi \rangle_0$ | $\langle f, g \rangle \langle \phi, \psi \rangle_0$ |
| Heisenberg’s Uncertainty Principle for the QWLCT | The Heisenberg uncertainty principle and quaternions are both basic for quantum mechanics. In quantum mechanics an uncertainty principle asserts that one cannot make certain of the position and velocity of an electron (or any particle) at the same time. That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. In signal processing an uncertainty principle states that the product of the variances of the signal in the time and frequency domains has a lower bound. The uncertainty principles of the Fourier transform and the quaternion Fourier transform had been studied in [21,19]. Recently, the authors established the uncertainty principles associated with the LCT in [28,22,29,23]. The uncertainty principles for the WLCT had been discussed in [24]. Theirs uncertainties were generalizations of Lieb’s uncertainty principles in the WLCT domains. Recently, Heisenberg’s uncertainty relations were extended to the quaternion linear canonical transform [14,13]. This uncertainty principle prescribes a lower bound on the |
Lemma 4.1 (QLCT uncertainty principle [13]). Let \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) and \( L^H_{A_1, A_2} \{ f \}(w) \in L^2(\mathbb{R}^2, \mathbb{H}) \) then we have

\[
\int_{\mathbb{R}^2} x_k^2 |f(x)|^2 dx \int_{\mathbb{R}^2} \omega_k^2 |L^H_{A_1, A_2} \{ f \}(w)|^2 dw \geq \frac{b_k^2}{4} \left( \int_{\mathbb{R}^2} |f(x)|^2 dx \right)^2, \quad k = 1, 2 \tag{47}
\]

Substituting the inverse transform for the QLCT (14) into the left-hand side of (47), we obtain

\[
\int_{\mathbb{R}^2} x_k^2 |L^{-1}_{A_1, A_2} \{ L^H_{A_1, A_2} \{ f \}(w) \}|^2 dx \int_{\mathbb{R}^2} \omega_k^2 |L^H_{A_1, A_2} \{ f \}(w)|^2 dw \geq \frac{b_k^2}{4} \left( \int_{\mathbb{R}^2} |f(x)|^2 dx \right)^2, \quad k = 1, 2 \tag{48}
\]

Further, applying Plancherel’s theorem for the QLCT (16) to the right-hand side of (48), we have

\[
\int_{\mathbb{R}^2} x_k^2 |L^{-1}_{A_1, A_2} \{ L^H_{A_1, A_2} \{ f \}(w) \}|^2 dx \int_{\mathbb{R}^2} \omega_k^2 |L^H_{A_1, A_2} \{ f \}(w)|^2 dw \geq \frac{b_k^2}{4} \left( \int_{\mathbb{R}^2} |L^H_{A_1, A_2} \{ f \}(w)|^2 dw \right)^2, \quad k = 1, 2 \tag{49}
\]

Now we arrive at the following important result.

Theorem 4.1 (QLCT uncertainty principle). Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a window function and \( G^A_\phi \{ f \} \in L^2(\mathbb{R}^2, \mathbb{H}) \) be the QWLCT of \( f \). Then for every \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) we have the following inequality

\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega_k^2 |G^A_\phi \{ f \}(w, u)|^2 dw du \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 dx \right)^{\frac{1}{2}} \geq \frac{b_k}{2} \| f \|_{L^2(\mathbb{R}^2)} \| \phi \|_{L^2(\mathbb{R}^2)} \tag{50}
\]

In order to prove this theorem, we need to introduce the following lemma.

Lemma 4.2. Let \( \phi \in L^2(\mathbb{R}^2, \mathbb{H}) \setminus \{0\} \) be a window function and \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \). Then

\[
\| \phi \|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_k^2 |L^{-1}_{A_1, A_2} \{ G^A_\phi \{ f(w, u) \} \}(x)|^2 dx du \tag{51}
\]

Proof.

\[
\| \phi \|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 dx \int_{\mathbb{R}^2} |\phi(x - u)|^2 du = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 |\phi(x - u)|^2 dx du
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 |\phi(x - u)|^2 dx du \tag{52}
\]

In the second equality, we have applied Fubini’s theorem to reverse the integration order. In the fourth equality, we have applied (19). This completes the proof. \( \square \)

Let us begin with the proof of Theorem 4.1.

Proof. Assume that \( L^H_{A_1, A_2} \{ f \} \in L^2(\mathbb{R}^2, \mathbb{H}) \). Since \( G^A_\phi \{ f \} \in L^2(\mathbb{R}^2, \mathbb{H}) \) we can replace the QLCT of \( f \) by the QWLCT of \( f \) on the both sides of (49). We have

\[
\int_{\mathbb{R}^2} x_k^2 |L^{-1}_{A_1, A_2} \{ G^A_\phi \{ f \}(w, u) \}|^2 dx \int_{\mathbb{R}^2} \omega_k^2 |G^A_\phi \{ f \}(w, u)|^2 dw \geq \frac{b_k^2}{4} \left( \int_{\mathbb{R}^2} |G^A_\phi \{ f \}(w, u)|^2 dw \right)^2 \tag{53}
\]
Taking the square root on both sides of (53) and integrating both sides with respect to $du$, we have
\[
\int_{\mathbb{R}^2} \left\{ \left( \int_{\mathbb{R}^2} x_k^2 |L_{A_1,A_2}^{-1}[G_{\phi}^{A_1,A_2} f(w,u)]|^2 \, dx \right)^{\frac{1}{2}} \right\}^2 \left( \int_{\mathbb{R}^2} \omega_k^2 |G_{\phi}^{A_1,A_2} f(w,u)|^2 \, dw \right)^{\frac{1}{2}} \, du \geq \frac{b_k}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1,A_2} f(w,u)|^2 \, dw \, du
\]
(54)

Furthermore, applying the Cauchy-Schwarz inequality to the left-hand side of (54), we have
\[
\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} x_k^2 |L_{A_1,A_2}^{-1}[G_{\phi}^{A_1,A_2} f(w,u)]|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega_k^2 |G_{\phi}^{A_1,A_2} f(w,u)|^2 \, dw \right)^{\frac{1}{2}} \geq \frac{b_k}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_{\phi}^{A_1,A_2} f(w,u)|^2 \, dw \, du
\]
(55)

Inserting Lemma 4.2 into the second term on the left-hand side of (55) and substituting (46) into the right-hand side of this inequality. Then, we have
\[
\left( \|\phi\|^2_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} x_k^2 |f(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega_k^2 |G_{\phi}^{A_1,A_2} f(w,u)|^2 \, dw \right)^{\frac{1}{2}} \geq \frac{b_k}{2} \|\phi\|^2_{L^2(\mathbb{R}^2)} \|\phi\|^2_{L^2(\mathbb{R}^2)}
\]
(56)

Dividing both sides of (56) by $\|\phi\|^2_{L^2(\mathbb{R}^2)}$, we obtain the desired result. \qed

5. Examples of the QWLCT

For illustrative purposes, we shall discuss examples of the QWLCT. We begin with a straightforward example.

**Example.** Given the window function of the two-dimensional Haar function defined by
\[
\phi(x) = \begin{cases} 
1, & 0 \leq x_1 < \frac{1}{2}, 0 \leq x_2 < \frac{1}{2} \\
-1, & \frac{1}{2} \leq x_1 < 1, \frac{1}{2} \leq x_2 < 1 \\
0, & \text{otherwise},
\end{cases}
\]
(57)

Consider a 2D Gaussian quaternionic function of the form $f(x_1,x_2) = e^{-\langle \alpha_1 x_1^2 + \alpha_2 x_2^2 \rangle}$, for $\alpha_1, \alpha_2 \in \mathbb{R}$ are positive real constants.

Then the QWLCT of $f$ is given by
\[
G_{\phi}^{A_1,A_2} f(w,u) = \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{\alpha_1}{2} x_1^2 - \frac{\alpha_1}{2} u_1^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} f(x) \phi(x-u) \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{\alpha_2}{2} x_2^2 - \frac{\alpha_2}{2} u_2^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} \, dx
\]
\[
= \int_{u_1}^{1+u_1} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{\alpha_1}{2} x_1^2 - \frac{\alpha_1}{2} u_1^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} e^{-\alpha_1 x_1^2} \, dx_1 \int_{u_2}^{1+u_2} \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{\alpha_2}{2} x_2^2 - \frac{\alpha_2}{2} u_2^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} e^{-\alpha_2 x_2^2} \, dx_2
\]
\[
- \int_{\frac{1}{2}+u_1}^{1+u_1} \frac{1}{\sqrt{2\pi b_1}} e^{i \left( \frac{\alpha_1}{2} x_1^2 - \frac{\alpha_1}{2} u_1^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} e^{-\alpha_1 x_1^2} \, dx_1 \int_{\frac{1}{2}+u_2}^{1+u_2} \frac{1}{\sqrt{2\pi b_2}} e^{i \left( \frac{\alpha_2}{2} x_2^2 - \frac{\alpha_2}{2} u_2^2 + \frac{\alpha_2}{2} u_2^2 - \frac{\alpha_2}{2} u_2^2 \right)} e^{-\alpha_2 x_2^2} \, dx_2
\]
(58)
By completing squares, we have

\[
G^{A_1, A_2}_\phi f(w, u) = \int_{u_1}^{\frac{1}{2} + u_1} \frac{1}{\sqrt{2\pi b_1}} e^{-\left(\frac{\sqrt{2\omega_1} - \frac{u_1}{b_1} x_1 + \frac{b_1}{\sqrt{2b_1(2\omega_1 - 1 + a_1)}}}{2}ight)^2 e^{-\frac{\omega_1^2}{2b_1(2\omega_1 - 1 + a_1)} + \frac{\omega^2}{2b_1}}} dx_1
\times \int_{u_2}^{\frac{1}{2} + u_2} \frac{1}{\sqrt{2\pi b_2}} e^{-\left(\frac{\sqrt{2\omega_2} - \frac{u_2}{b_2} x_2 + \frac{b_2}{\sqrt{2b_2(2\omega_2 - 1 + a_2)}}}{2}ight)^2 e^{-\frac{\omega_2^2}{2b_2(2\omega_2 - 1 + a_2)} + \frac{\omega^2}{2b_2}}} dx_2
- \int_{\frac{1}{2} + u_1}^{1 + u_1} \frac{1}{\sqrt{2\pi b_1}} e^{-\left(\frac{\sqrt{2\omega_1} - \frac{u_1}{b_1} x_1 + \frac{b_1}{\sqrt{2b_1(2\omega_1 - 1 + a_1)}}}{2}ight)^2 e^{-\frac{\omega_1^2}{2b_1(2\omega_1 - 1 + a_1)} + \frac{\omega^2}{2b_1}}} dx_1
\times \int_{\frac{1}{2} + u_2}^{1 + u_2} \frac{1}{\sqrt{2\pi b_2}} e^{-\left(\frac{\sqrt{2\omega_2} - \frac{u_2}{b_2} x_2 + \frac{b_2}{\sqrt{2b_2(2\omega_2 - 1 + a_2)}}}{2}ight)^2 e^{-\frac{\omega_2^2}{2b_2(2\omega_2 - 1 + a_2)} + \frac{\omega^2}{2b_2}}} dx_2
\]  

(59)

Making the substitutions \(A_1 = \sqrt{\frac{2\omega_1 - 1 + a_1}{2b_1}}, \quad A_2 = \sqrt{\frac{2\omega_2 - 1 + a_2}{2b_2}}, \quad B_1 = \frac{b_1}{\sqrt{2b_1(2\omega_1 - 1 + a_1)}}, \quad B_2 = \frac{b_2}{\sqrt{2b_2(2\omega_2 - 1 + a_2)}}, \quad J_1 = e^{-\frac{\omega_1^2}{2b_1(2\omega_1 - 1 + a_1)} + \frac{\omega^2}{2b_1}} \text{ and } J_2 = e^{-\frac{\omega_2^2}{2b_2(2\omega_2 - 1 + a_2)} + \frac{\omega^2}{2b_2}} \) in the above expression we immediately obtain

\[
G^{A_1, A_2}_\phi f(w, u) = \int_{u_1}^{\frac{1}{2} + u_1} \frac{1}{\sqrt{2\pi b_1}} e^{-(A_1 x_1 + B_1)^2} J_1 dx_1 \int_{u_2}^{\frac{1}{2} + u_2} \frac{1}{\sqrt{2\pi b_2}} e^{-(A_2 x_2 + B_2)^2} J_2 dx_2
- \int_{\frac{1}{2} + u_1}^{1 + u_1} \frac{1}{\sqrt{2\pi b_1}} e^{-(A_1 x_1 + B_1)^2} J_1 dx_1 \int_{\frac{1}{2} + u_2}^{1 + u_2} \frac{1}{\sqrt{2\pi b_2}} e^{-(A_2 x_2 + B_2)^2} J_2 dx_2
\]  

(60)

Substituting \(y_1 = A_1 x_1 + B_1 \) and \(y_2 = A_2 x_2 + B_2 \) in the above equation, we have

\[
G^{A_1, A_2}_\phi f(w, u) = \int_{A_1 u_1 + B_1}^{A_1 (\frac{1}{2} + u_1) + B_1} e^{-y_1^2} dy_1 \times \int_{A_2 u_2 + B_2}^{A_2 (\frac{1}{2} + u_2) + B_2} e^{-y_2^2} dy_2
- \int_{A_1 u_1 + B_1}^{A_1 (\frac{1}{2} + u_1) + B_1} e^{-y_1^2} dy_1 \times \int_{A_2 u_2 + B_2}^{A_2 (\frac{1}{2} + u_2) + B_2} e^{-y_2^2} dy_2
\]

\[
= \frac{J_1}{A_1 \sqrt{2\pi b_1}} \left( \int_{0}^{A_1 u_1 + B_1} (-e^{-y_1^2}) dy_1 + \int_{0}^{A_1 (\frac{1}{2} + u_1) + B_1} e^{-y_1^2} dy_1 \right)
\times \frac{J_2}{A_2 \sqrt{2\pi b_2}} \left( \int_{0}^{A_2 u_2 + B_2} (-e^{-y_2^2}) dy_2 + \int_{0}^{A_2 (\frac{1}{2} + u_2) + B_2} e^{-y_2^2} dy_2 \right)
\]

\[
- \frac{J_1}{A_1 \sqrt{2\pi b_1}} \left( \int_{0}^{A_1 (\frac{1}{2} + u_1) + B_1} (-e^{-y_1^2}) dy_1 + \int_{0}^{A_1 (1 + u_1) + B_1} e^{-y_1^2} dy_1 \right)
\times \frac{J_2}{A_2 \sqrt{2\pi b_2}} \left( \int_{0}^{A_2 (\frac{1}{2} + u_2) + B_2} (-e^{-y_2^2}) dy_2 + \int_{0}^{A_2 (1 + u_2) + B_2} e^{-y_2^2} dy_2 \right)
\]  

(61)

Equation (61) can be written in the form

\[
G^{A_1, A_2}_\phi f(w, u) = \frac{J_1}{2\sqrt{b_1(2\omega_1 - 1 + a_1)}} \left( -erf(A_1 u_1 + B_1) + erf(A_1 \left( \frac{1}{2} + u_1 \right) + B_1) \right)
\times \frac{J_2}{2\sqrt{b_2(2\omega_2 - 1 + a_2)}} \left( -erf(A_2 u_2 + B_2) + erf(A_2 \left( \frac{1}{2} + u_2 \right) + B_2) \right)
\]

\[
- \frac{J_1}{2\sqrt{b_1(2\omega_1 - 1 + a_1)}} \left( -erf(A_1 \left( \frac{1}{2} + u_1 \right) + B_1) + erf(A_1 (1 + u_1) + B_1) \right)
\times \frac{J_2}{2\sqrt{b_2(2\omega_2 - 1 + a_2)}} \left( -erf(A_2 \left( \frac{1}{2} + u_2 \right) + B_2) + erf(A_2 (1 + u_2) + B_2) \right)
\]

(62)
where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \).

6. Conclusions

Due to the non-commutativity of quaternion multiplication, there are three different types of the QLCT: the left-sided QLCT, the right-sided QLCT, and the two-sided QLCT. Using the basic concepts of quaternion algebra and the two-sided QLCT we introduced the the two-sided QWLCT. Important properties of the QWLCT such as boundedness, linearity, parity, shift, modulation, inversion formula and orthogonality relation were derived. Based on the QWLCT properties and the uncertainty principle for the QLCT, a new uncertainty principle for the QWLCT have been established. The results in this paper are new in the literature. Further investigations on this topic are now under investigation such as the left-sided QWLCT, the right-sided QWLCT. They will be reported in a forthcoming paper.

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