Superintegrable quantum $u(3)$–systems and higher rank factorizations

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Abstract

A class of two-dimensional superintegrable systems on a constant curvature surface is considered as the natural generalization of some well known one-dimensional factorized systems. By using standard methods to find the shape-invariant intertwining operators we arrive at a $so(6)$ dynamical algebra and its Hamiltonian hierarchies. We pay attention to those associated to certain unitary irreducible representations that can be displayed by means of three-dimensional polyhedral lattices. We also discuss the role of superpotentials in this new context.

1 Introduction

This work deals with a class of superintegrable Hamiltonians systems, in the framework of the Schrödinger equation of quantum mechanics, and its connections with the factorization method. We will restrict ourselves to a particular case where the underlying symmetry is the Lie algebra $u(3)$, but its main features can be directly implemented to higher dimensional systems.

The main objective of this study is to show a natural extension to higher dimensional spaces of the intertwining (or Darboux) transformations from a well known class of one-dimensional factorized systems. In fact, we want to set the higher rank $u(n)$-systems corresponding to those having as dynamical algebra the Lie algebra of rank one $u(2)$. We will show in detail that the application of procedures familiar in one dimension to a concrete two-dimensional system will lead us to a wide set of operators closing a dynamical Lie algebra. We also consider discrete symmetry operators quite important to perform equivalences. All these operators connect eigenstates that can be drawn as points in a three-dimensional lattice giving rise to polyhedrons representing degenerate series of $u(3)$ irreducible representations. Each of these series corresponds to the same energy and can be embedded in just one representation of the Lie algebra $so(6)$.

The notion of superpotential will also be re-examined inside the higher rank formalism. Thus, the usual procedure to look for solutions with separable variables can be better appreciated under this point of view.
Thus, we try to implement the program of generalization of the factorizable one-dimensional systems involving Lie algebras of rank one as dynamical algebras, as can be seen, for instance, in the classical paper by Infeld and Hull [1]. We also hope that this work will be useful when dealing with other integrable systems, but not necessarily maximally integrable, for instance, not enjoying for such a wealth of factorizations, or even not having a system of separable variables, but still allowing for algebraic methods [2, 3, 4, 5, 6, 7].

The organization of the paper is as follows. In section 2 we will introduce a two-dimensional superintegrable system and find some separable solutions by standard procedures. Although these polynomial solutions are known and can be found in other references, this will serve us to recall some aspects of the usual factorization technique and to precise the operators of the Lie algebra $u(2)$ related with the Lie algebra $so(4)$, and their support spaces. Next, in section 3 we will look for other sets of intertwining operators, corresponding to the Lie algebras $u(3)$ and $so(6)$, taking also into account discrete symmetries. We characterize the eigenfunctions belonging to irreducible representations that will be depicted as the points on octahedrons and the interpretation of some of its planar sections. The analog of the superpotentials and their relation with certain types of solutions will be considered in section 4. Some conclusions and perspective for future work will close the paper.

2 A superintegrable $u(3)$–Hamiltonian system

We will fix our attention on a superintegrable Hamiltonian system defined inside a three-dimensional Euclidean ambient space $[8, 9, 10, 11]$. In fact, our system lives on the 2-sphere $S$

$$S \equiv (s_0)^2 + (s_1)^2 + (s_2)^2 = 1, \quad (s_0, s_1, s_2) \in \mathbb{R}^3$$

In the frame of the Schrödinger equation, this Hamiltonian takes the form

$$H = -\left(J_0^2 + J_1^2 + J_2^2\right) + \frac{l_0^2 - 1}{4(s_0)^2} + \frac{l_1^2 - 1}{4(s_1)^2} + \frac{l_2^2 - 1}{4(s_2)^2}$$  \hspace{1cm} (2.1)

where $(l_0, l_1, l_2) \in \mathbb{R}^3$, and $J_i = -\epsilon_{ijk}s_j \partial_k$ (note that the $J_i$’s operators generate the rotation Lie algebra $so(3)$). We can parametrize $S$ by means of spherical coordinates $(\phi_1, \phi_2)$ around the $s_2$ axis given by

$$s_0 = \cos \phi_2 \cos \phi_1, \quad s_1 = \cos \phi_2 \sin \phi_1, \quad s_2 = \sin \phi_2$$  \hspace{1cm} (2.2)

Then, the eigenvalue problem

$$H \Phi = E \Phi$$

after substituting the coordinates [22], takes the form of a separable differential equation

$$\left[-\partial_{\phi_2}^2 + \tan \phi_2 \partial_{\phi_2} + \frac{l_2^2 - 1}{4\sin^2 \phi_2} + \frac{1}{\cos^2 \phi_2} \left[-\partial_{\phi_1}^2 + \frac{l_0^2 - 1}{4\cos^2 \phi_1} + \frac{l_1^2 - 1}{4\sin^2 \phi_1}\right]\right] \Phi = E \Phi$$  \hspace{1cm} (2.3)

The solutions separated in the variables $\phi_1$ and $\phi_2$, i.e.

$$\Phi(\phi_1, \phi_2) = f(\phi_1)g(\phi_2)$$
after replacing in (2.3) originate the equations

\[
-\partial_2^2 \phi_1 + \left( \frac{l_0^2 - 1/4}{\cos^2 \phi_1} + \frac{l_1^2 - 1/4}{\sin^2 \phi_1} \right) f(\phi_1) = \alpha f(\phi_1) \quad (2.4)
\]

\[
-\partial_2^2 \phi_2 + \tan \phi_2 \partial_2 \phi_2 + \frac{\alpha}{\cos^2 \phi_2} + \left( \frac{l_2^2 - 1/4}{\sin^2 \phi_2} \right) g(\phi_2) = E g(\phi_2) \quad (2.5)
\]

where \( \alpha \) is a separating constant. Next we will solve each of these two equations through standard factorizations giving rise to polynomials. The key point is that the results obtained for the first equation will match in a certain way with those of the second one originating degenerate levels.

### 2.1 The \( \phi_1 \)-factorization

The one-dimensional Hamiltonian (2.4) in the variable \( \phi_1 \) is a well known example in the theory of factorizations \[1\]. So, in the following we will restrict ourselves to give a list of the relevant results. We will see later, in section 3, how to make use of these considerations in a broader context.

The second order operator at the l.h.s. of eq. (2.4) can be cast as a product of first order operators

\[
H_{\phi_1}(0) = A_0^+ A_0^- + \lambda_0
\]

being

\[
A_0^\pm = \pm \partial \phi_1 - (l_0 + 1/2) \tan \phi_1 + (l_1 + 1/2) \cot \phi_1 \quad \lambda_0 = (l_0 + l_1 + 1)^2
\]

These elements are part of a family of operators \( \{ A_m^+, A_m^-, \lambda_m, H_{\phi_1}^{(m)} \} \), \( m \in \mathbb{Z} \), where

\[
A_m^\pm = \pm \partial \phi_1 - (l_0 + m + 1/2) \tan \phi_1 + (l_1 + m + 1/2) \cot \phi_1 \quad \lambda_m = (l_0 + l_1 + 2m + 1)^2
\]

\[
H_{\phi_1}^{(m)} = -\partial_1^2 + \left( \frac{(l_0 + m)^2 - 1/4}{\cos^2 \phi_1} \right) + \left( \frac{(l_1 + m)^2 - 1/4}{\sin^2 \phi_1} \right) \quad (2.7)
\]

They originate the one-dimensional Hamiltonian hierarchy (2.7), starting from \( H_{\phi_1}^{(0)} \). The Hamiltonians \( H_{\phi_1}^{(m)} \) satisfy the fundamental relation

\[
H_{\phi_1}^{(m)} = A_m^+ A_m^- + \lambda_m = A_{m-1}^- A_{m-1}^+ + \lambda_{m-1} \quad (2.8)
\]

so that \( A_m^\pm \) are shape invariant intertwining operators, i.e.

\[
A_{m}^- H_{\phi_1}^{(m)} = H_{\phi_1}^{(m+1)} A_{m}^- , \quad A_{m}^+ H_{\phi_1}^{(m+1)} = H_{\phi_1}^{(m)} A_{m}^+
\]

Hence, from a formal point of view, the operators \( A_m^\pm \) acting on a Hamiltonian eigenfunction will give another eigenfunction of a consecutive Hamiltonian in the hierarchy with the same
eigenvalue. If we design the eigenfunction spaces of $H_{(m)}^{\phi_1}$ (as differential operators) by $H_m^{\phi_1}$, then we have

$$A_m^\pm : H_m^{\phi_1} \to H_{m\pm 1}^{\phi_1}, \quad A_m^0 : H_m^{\phi_1} \to H_m^{\phi_1}$$

In principle, the discrete spectrum and the physical eigenstates of $H^{\phi_1}_{(m)}$ could be obtained from the fundamental states $f_0^{(m)}$ and their eigenvalues of all the Hamiltonians in the hierarchy $\{H_{(m)}^{\phi_1}\}$. These fundamental states are determined by $A_m^0 f_0^{(m)} = 0$, giving the solutions (up to a normalization constant)

$$f_0^{(m)}(\phi_1) = \cos^{l_0+m+1/2} \phi_1 \sin^{l_1+m+1/2} \phi_1$$

with eigenvalues $\lambda_m = (l_0 + l_1 + 2m + 1)^2$. Often the intertwining operators (2.6) are written in the form

$$A_m^\pm = \pm \partial_\phi_1 + \omega_m(\phi_1), \quad \omega_m(\phi_1) = \frac{\partial_\phi_1 f_0^{(m)}(\phi_1)}{f_0^{(m)}(\phi_1)}$$

where $\omega_m(\phi_1)$ is called superpotential function.

In order to go from the ground eigenstate, $f_0^{(m)}$ of $H_{(m)}^{\phi_1}$, up to the excited eigenfunction, $f_m^{(0)}$ of $H^{\phi_1}_{(0)}$, with the same eigenvalue, we apply consecutive operators $A^+$

$$f_m^{(0)} = A_0^+ A_1^+ \cdots A_{m-1}^+ f_0^{(m)}$$

obtaining explicitly

$$f_m^{(0)} = N \sin^{l_1+1/2} \phi_1 \cos^{l_0+1/2} \phi_1 \ P_{m}^{(l_1,l_0)}(\cos(2\phi_1))$$

where $P_{m}^{(a,b)}(x)$ are Jacobi polynomials and $N$ a normalization constant. Therefore, the spectrum of the first separating Hamiltonian (2.4) is given by

$$\alpha = \lambda_m = (l_0 + l_1 + 2m + 1)^2, \quad m \in \mathbb{Z}^+$$

The following two subsections are devoted to characterise the Lie algebras of shape invariant intertwining operators for the one-dimensional Hamiltonian hierarchies. They will constitute a useful pattern for the two-dimensional Hamiltonians of section § 3.

### 2.2 The dynamical algebra $u(2)$

Starting from the operators $A_m^\pm$ let us define free-index operators $A^\pm$ acting inside the total space $\oplus_m H_m$, in the following way [12, 13]:

$$A_+ f_{(m+1)} := \frac{1}{2} A_m^+ f_{(m+1)} \propto \tilde{f}_{(m)}$$

$$A_- f_{(m)} := \frac{1}{2} A_m^- f_{(m)} \propto \tilde{f}_{(m+1)}$$

$$A f_{(m)} := -\frac{1}{2} (l_0 + l_1 + 2m) f_{(m)} \propto f_{(m)}$$

(2.14)
where \( f_{(m)} \) (or \( \tilde{f}_{(m)} \)) denotes an eigenfunction of \( H_{(m)} \). This action can be extended to linear combinations of eigenfunctions by linearity. With this convention we can rewrite (2.8) and (2.13) simply as the commutators

\[
[A, A^\pm] = \pm A^\pm, \quad [A^-, A^+] = -2A \tag{2.15}
\]

assuming that the action is on any (linear combination of) \( f_{(m)} \)’s. The commutators (2.15) close the Lie algebra \( su(2) \), whose Casimir element is given by

\[
\mathcal{C} = A^+ A^- + A(A - 1).
\]

The eigenvalues of \( \mathcal{C} \), labeling the irreducible unitary representations (IUR), are \( j(j + 1) \), where \( 2j \in \mathbb{Z}^+ \). The dimension of the support spaces of these IUR’s is, obviously, \( 2j + 1 \). We make use of the standard notation \( |j, s\rangle \) for an \( A \)-eigenvector with eigenvalue \( s \), inside the ‘\( j \)-representation’.

Now, we can identify the eigenstates of the Hamiltonians \( H_{(m)}^{\phi_1} \) in terms of representation vectors \( |j, s\rangle \). First, let us consider the ground states \( f_{(m)}^0 \) characterized by

\[
A^- f_{(m)}^0 = 0, \quad A f_{(m)}^0 = -[(l_0 + l_1 + 2m)/2] f_{(m)}^0 \tag{2.16}
\]

following the notation (2.14). These relations suggest the identification (up to a normalization constant)

\[
f_{(m)}^0 = |j_m, -j_m\rangle, \quad j_m = (l_0 + l_1 + 2m)/2
\]

To see that indeed this is the case we need to define the whole representation space as well as an inner product. Thus, consider the space \( \mathcal{L}^2[0, \pi/2] \) of square integrable functions in the interval \([0, \pi/2]\). Then, the wavefunctions obtained from the ground state \( f_{(m)}^0 \) by the consecutive action of the operator \( A^+ \) will span the representation space of a \( j_m \)-representation, with \( j_m = (l_0 + l_1 + 2m)/2 \), provided that both \( l_0 + m \) and \( l_1 + m \) belong to \( \mathbb{Z}^+ \). The wavefunctions of the space so generated vanish at the end points (i.e. \( \Psi(0) = \Psi(\pi/2) = 0 \)), and the hermiticity relations \((A^-)^\dagger = A^+, \ A^\dagger = A \) are implemented in all the space. Hence, under these conditions, \((A^+)^k f_{(m)}^0\) can be identified, up to normalization, with the vector state \(|j_m, -j_m + k\rangle\).

As a consequence, the excited states obtained in this way for any Hamiltonian in a factorization hierarchy where \( l_0 \) and \( l_1 \) are positive integers, correspond to IUR-vector states. For instance, the eigenstate of the \( k \)-th excited level of \( H_{(0)}^{\phi_1} \) is

\[
f_{(0)}^k \equiv |j_k + k, -j_k + k\rangle, \quad j_k = (l_0 + l_1 + 2k)/2, \quad k = 0, 1, 2 \ldots
\]

and \( H_{(0)}^{\phi_1} \) (as well as any \( H_{(m)}^{\phi_1} \)) can be expressed in terms of the \( su(2) \)-Casimir \( \mathcal{C} \) acting on such representations

\[
H_{(0)}^{\phi_1} = 4(\mathcal{C} + 1/4)
\]

Therefore, the eigenvalue equation for any of the excited states can be written as follows

\[
H_{(0)}^{\phi_1} f_{(0)}^k = 4(\mathcal{C} + 1/4)(j_0 + k, -j_0 + k)
\]

\[
= 4(j_0 + k + 1/2)^2 |j_0 + k, -j_0 + k\rangle = (l_0 + l_1 + 2k + 1)^2 f_{(0)}^k
\]

with \( k = 0, 1, 2, \ldots \).
It will be convenient to consider a new diagonal operator $D$, to be added to the generators of $su(2)$ (2.14), define by

$$Df(m) := (l_0 - l_1)f(m)$$

It is immediate to see that $D$ commutes with any other operator of $su(2)$ giving rise to the Lie algebra $u(2)$. In this way any eigenstate in the Hamiltonian hierarchy can be characterised completely by an eigenfunction of an $u(2)$-IUR. Without $D$ we would have an ambiguity due to the fact that different fundamental states with values of $l_0$ and $l_1$ giving the same $j_0 = (l_0 + l_1)/2$ would lead to the same $j$-representation of $su(2)$.

It is worthy noting that when $l_0$ or $l_1$ are not in $\mathbb{Z}^+$ the eigenfunctions and spectrum of the Hamiltonian hierarchies are still given by (2.12) and (2.13), but, these states belong to non-unitary representations of $u(2)$.

2.3 The dynamical algebra $so(4)$

As we have just seen in the previous subsection the eigenstates sharing the same energy of the one-dimensional Hamiltonian hierarchies in the variable $\phi_1$ are given in terms of IUR’s of the dynamical algebra $u(2)$. However, in this respect, there is a point not quite satisfactory: different $u(2)$-IUR’s may correspond to states with the same energy. We would prefer a larger dynamical algebra with a simpler correspondence, i.e., such that only one of its IUR’s gives all the eigenstates with the same energy in the hierarchy.

In order to build up a dynamical algebra having these properties, let us introduce the two-dimensional parameter space $(l_0, l_1)$. Any operator with one subindex defined in subsections 2.1 and 2.2 will change to a two-subindex notation in the following way:

1. The one-dimensional Hamiltonian (2.4) will be denoted by $H_{(l_0, l_1)}$

$$H_{(l_0, l_1)}^{\phi_1} = -\partial_{\phi_1}^2 + \frac{l_0^2 - 1/4}{\cos^2 \phi_1} + \frac{l_1^2 - 1/4}{\sin^2 \phi_1}$$

Its eigenfunctions will be designed by $f_{(l_0, l_1)}$.

2. The factor operators $A_{0}^{\pm}$ in (2.6) will be rewritten as $A_{(l_0, l_1)}^{\pm}$

$$A_{(l_0, l_1)}^{\pm} = \pm \partial_{\phi_1} - (l_0 + 1/2) \tan \phi_1 + (l_1 + 1/2) \cot \phi_1, \quad A_{(l_0, l_1)} = -\frac{1}{2}(l_0 + l_1)$$

Now, in this way, relations (2.9) can be expressed as

$$A_{(l_0, l_1)}^{\pm} H_{(l_0, l_1)}^{\phi_1} = H_{(l_0 + 1, l_1 + 1)}^{\phi_1} A_{(l_0, l_1)}^{\pm} \quad A_{(l_0, l_1)}^{\pm} H_{(l_0 + 1, l_1 + 1)}^{\phi_1} = H_{(l_0, l_1)}^{\phi_1} A_{(l_0, l_1)}^{\pm}$$ (2.17)

With this convention we can also define the free-subindex operators $A^{\pm}, A, D$ as in (2.14).

On the other hand, notice that each two-parameter Hamiltonian $H_{(l_0, l_1)}^{\phi_1}$ is invariant under the reflections

$$I_0 : (l_0, l_1) \to (-l_0, l_1), \quad I_1 : (l_0, l_1) \to (l_0, -l_1)$$
This property gives rise to a second factorisation (see also [14, 15, 16]) via conjugation of the operators of the first factorisation by the reflection operators 

\[ I_0 \mathcal{A}^\pm I_0 = \tilde{\mathcal{A}}^\pm, \quad I_0 \mathcal{A} I_0 = \tilde{\mathcal{A}}, \quad I_0 D I_0 = \tilde{D} \]

\[ \tilde{I}_1 \mathcal{A}^\pm I_1 = \tilde{\mathcal{A}}^\mp, \quad I_1 \mathcal{A} I_1 = -\tilde{\mathcal{A}}, \quad I_1 D I_1 = -\tilde{D} \]

Explicitly

\[ \tilde{\mathcal{A}}^\pm_{(l_0, l_1)} = \pm \partial_{\phi_1} + (l_0 - 1/2) \tan \phi_1 + (l_1 + 1/2) \cot \phi_1, \quad \tilde{\mathcal{A}}_{(l_0, l_1)} = -\frac{1}{2} (-l_0 + l_1) \quad (2.18) \]

The above operators \( \{ \tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\pm \} \) generate a Lie algebra isomorphic to \( su(2) \) denoted by \( \tilde{su}(2) \). Since \( su(2) \) and \( \tilde{su}(2) \) commute and, essentially, \( D \) and \( \tilde{D} \) coincide with \( A \) and \( \tilde{A} \), respectively, the complete dynamical algebra has the structure of a direct sum \( su(2) \oplus \tilde{su}(2) \approx so(4) \).

If we allow to act with the \( so(4) \) generators on an Hamiltonian \( H_{(l_0, l_1)} \) we will get a two-dimensional parameter lattice of Hamiltonians which constitute a \( so(4) \)-hierarchy fixed by the initial values \( (l_0, l_1) \): \( \{ H_{l_0-n+1, l_1+n} \}, m, n \in \mathbb{Z} \). Each energy level of this Hamiltonian hierarchy is degenerated and the eigenstates belong to \( so(4) \)-representations.

Let us concentrate on the hierarchies associated to IUR’s of \( so(4) \). Now, these \( so(4) \)-IUR’s are fixed by the fundamental (or lowest weight) states satisfying

\[ \mathcal{A}_{(l_0, l_1)} f_{(l_0, l_1)}^0 = \tilde{\mathcal{A}}_{(l_0, l_1)} f_{(l_0, l_1)}^0 = 0 \quad (2.19) \]

These are realized, up to a constant, by the wavefunctions

\[ f_{(l_0, n)}^0 = \cos^{1/2} \phi_1 \sin^{n+1/2} \phi_1, \quad n \in \mathbb{Z}^+ \quad (2.20) \]

where we have taken \( l_0 = 0 \) and \( l_1 = n \). We see also that the state \( f_{(l_0, l_1)}^0 \) is stable under \( I_0 \) (i.e. \( I_0 f_{(l_0, l_1)}^0 = f_{(l_0, l_1)}^0 \)), and comes into the other fundamental state (annihilated by \( A^+ \) and \( \tilde{A}^+ \): the highest weight) of the same representation. Hence, these representations will be invariant under \( I_0 \) and \( I_1 \). Therefore, the \( so(4) \)-IUR’s obtained from \( f_{(l_0, l_1)}^0 \) are symmetric tensor products that can be denoted by

\[ j \otimes j, \quad j = l_1/2 = n/2, \quad n \in \mathbb{Z}^\geq 0 \]

where ‘\( j \)’ stands for a \( j \)-representation of \( su(2) \). In this way the degenerancy of the \( n \)-th energy level is \( (n+1) \times (n+1) \), which is composed of \( n+1 \) IUR’s of \( u(2) \) each of them of dimension \( n+1 \).

The Hamiltonians in this hierarchy can be expressed in terms of any of the \( su(2) \)(or \( \tilde{su}(2) \)) Casimir operators \( H_{(l_0, l_1)} = 4(C + 1/4) = 4(\tilde{C} + 1/4) \). With the help of all the discrete reflections we get directly its expression also in terms of the \( so(4) \)-Casimir

\[ H_{(l_0, l_1)} = (C + 1/4) + I_0(C + 1/4)I_0 + I_1(C + 1/4)I_1 + I_0I_1(C + 1/4)I_0I_1 \]

\[ = \{ A^+, A^- \} + 2A^2 + \{ \tilde{A}^+, \tilde{A}^- \} + 2\tilde{A}^2 + 1 \]

\[ = \{ A^+, A^- \} + \{ \tilde{A}^+, \tilde{A}^- \} + L_0^2 + L_1^2 + 1 \]
where the diagonal operators $L_0$ and $L_1$ are defined by

$$L_0 f_{(l_0,l_1)} = l_0 f_{(l_0,l_1)}, \quad L_1 f_{(l_0,l_1)} = l_1 f_{(l_0,l_1)}$$

Certainly, some $so(4)$-hierarchies (those corresponding to the IUR’s previously described) may have Hamiltonians whose explicit expressions coincide

$$H_{(l_0,l_1)} = H_{(-l_0+1,l_1)} = H_{(l_0,-l_1+1)}$$

and the same happens with their corresponding eigenstates. But we can not get rid of this multiplicity unless we enlarge the ambient space.

Another natural question is whether there are other intertwining shape-invariant operators inside the $so(4)$-hierarchy. We can build, for instance, other pairs of operators through the composition of those already known

$$X^\pm = A^\pm \tilde{A}^\pm, \quad Y^\pm = A^\pm \tilde{A}^\mp$$

This kind of shape-invariant operators change two units either the parameter $l_0$ or $l_1$ (but not both at the same time). When we restrict to $l_0 = 0$ or $l_1 = 0$ there are also first order intertwining operators changing one unit the nonvanishing parameter. This feature is not so special; it is also shared by the ‘radial oscillator’ hierarchies [17] (which are closely related to the ones presented here).

For other Hamiltonian $so(4)$-hierarchies the physical eigenstates are described by non-unitary representations that are not invariant under both reflections. In this respect, their description becomes more involved, so that one must be very careful in these cases.

### 2.4 The $\phi_2$–factorization

Now, let us return to the separation process started in subsection 2.1. The second equation obtained from the initial separation of variables can be dealt with along the same lines, substituting the eigenvalues obtained from the previous factorization, $\alpha = \lambda_m = (l_0 + l_1 + 2m)^2$. The most relevant fact, here, is that the new factorization leads to a degeneration of the energy levels which suggest that the underlying dynamical symmetry could be larger, as it will be confirmed in the next section. Thus, substituting in (2.5), we have

$$H_{(0)}^{\phi_2} = -\partial_{\phi_2}^2 + \tan(\phi_2) \partial_{\phi_2} + \frac{(l_0 + l_1 + 2m + 1)^2}{\cos^2(\phi_2)} + \frac{l_2^2 - 1/4}{\sin^2(\phi_2)}$$

$$= \{ \partial_{\phi_2} - (l_0 + l_1 + 2m) \tan(\phi_2) + (l_2 + 1/2) \cot(\phi_2) \}$$

$$\times \{ -\partial_{\phi_2} - (l_0 + l_1 + 2m + 1) \tan(\phi_2) + (l_2 + 1/2) \cot(\phi_2) \}$$

$$+ (l_2 + l_0 + l_1 + 2m + 3/2)(l_2 + (l_0 + l_1 + 2m + 5/2)$$

$$\equiv M_0^+ M_0^- + \mu_0.$$  (2.21)

This is the first one of the Hamiltonian hierarchy $H^{\phi_2}_{(n)}$ in the variable $\phi_2$,

$$H^{\phi_2}_{(n)} = M_n^+ M_n^- + \mu_n = M_{m-1}^- M_{m-1}^+ + \mu_{n-1}$$
where
\[
M_n^\pm = \pm \partial_{\phi_2} - (l_0 + l_1 + 2(m + 1) + n) \tan(\phi_2) + (l_2 + n + 1/2) \cot(\phi_2)
\]
\[
\mu_n = (l_1 + l_0 + l_2 + 2n + 2m + 3/2)(l_2 + l_1 + l_0 + 2n + 2m + 5/2)
\]
Now, the values for the energy (following closely the same arguments of section 2.1) are given by
\[
E = \mu_n = (l_1 + l_0 + l_2 + 2n + 2m + 3/2)(l_2 + l_1 + l_0 + 2n + 2m + 5/2)
\]
(2.22)
The fundamental states \(g_{(n)}^0\) for this factorization are
\[
g_{(n)}^0(\phi_2) = N \cos^{l_1+l_0+2m+1} \phi_2 \sin^{l_2+n+1/2} \phi_2
\]
and the eigenfunctions \(g_{(0)}^n\) of the initial Hamiltonian \(\text{(2.21)}\) can be written in the form
\[
g_{(0)}^n(\phi_2) = \cos^{l_1+l_0+2m+1} \phi_2 \sin^{l_2+1/2} \phi_2 \mathcal{P}_{l_2+1/2,l_1+l_0+2m+1}(\cos 2\phi_2).
\]
(2.23)
The commutation relation for the relevant free-index operators \(M^\pm\), defined in a similar way as \(A^\pm\) in \(\text{(2.14)}\), is again that of \(su(2)\),
\[
[M^-, M^+] = -4(l_1 + l_0 + l_2 + 2m + 2n + 1) \equiv -2M
\]
The eigenfunctions \(\text{(2.23)}\) are square-integrable, but the representations are unitary provided that, besides the previous conditions on \(l_0\) and \(l_1\), the parameter \(l_2\) be also a positive integer number.

In summary, if we finally join the results of both factorizations, the square-integrable eigenfunctions of the Hamiltonian \(\text{(2.3)}\) in the separable variables \((\phi_1, \phi_2)\) are given by the products
\[
\Phi_{m,n}(\phi_1, \phi_2) = f_{(0)}^m(\phi_1) g_{(0)}^n(\phi_2), \quad m, n \in \mathbb{Z}^+
\]
(2.24)
where the components have the polynomial expressions \(\text{(2.12)}\) and \(\text{(2.23)}\). The corresponding eigenvalues given in \(\text{(2.22)}\) are degenerated for those values of \(m\) and \(n\) whose sum \(m + n\) keeps constant (see also, for instance, Ref. [11]).

3 Dynamical symmetries

The spectrum obtained by the methods of section 2 suggest the existence of a bigger dynamical algebra of the Hamiltonian hierarchy. This is the point that we want to address here developing exhaustively the concept of intertwining (shape invariant) operators for this kind of Hamiltonians. Such operators will supply us with a more consistent picture of the spectrum and eigenfunctions. Thus, based on the considerations of subsections 2.2 and 2.3, we will introduce three sets of intertwining operators closing the Lie algebra \(u(3)\). Then, in the following subsection, we will enlarge this algebra to \(so(6)\) by means of the relevant reflections.
3.1 The Hamiltonian $u(3)$–hierarchies

3.1.1 The set $\{A^+, A^-, A\}$

As we will use some properties of section 2 in a different direction, it is convenient to introduce another notation more appropriate to rewrite some previous results. The Hamiltonian (3.1) characterized by the parameters $\ell \equiv (l_0, l_1, l_2)$ will be referred to as $H_{(l_0,l_1,l_2)}$, and the operators defined by (2.6) will be taken henceforth with a three-fold subindex

$$A^\pm_{(l_0,l_1,l_2)} = \pm \partial_\phi - (l_0 + 1/2) \tan \phi + (l_1 + 1/2) \cot \phi$$

(3.1)

Since the differential operators (2.6) and (3.1) depend only on the variable $\phi_1$, they do not affect the part in the total Hamiltonian (2.11) depending on the second separable variable $\phi_2$. So that, in the same way as (2.17) we have the intertwining relations

$$A^-_{(l_0,l_1,l_2)} H_{(l_0,l_1,l_2)} = H_{(l_0+1,l_1+1,l_2)} A^-_{(l_0,l_1,l_2)}$$
$$A^+_{(l_0,l_1,l_2)} H_{(l_0,l_1,l_2)} = H_{(l_0,l_1,l_2)} A^+_{(l_0,l_1,l_2)}$$

This means that now $A^-_{(l_0,l_1,l_2)}$ is acting on eigenstates of $H_{(l_0,l_1,l_2)}$ leading to eigenstates of $H_{(l_0+1,l_1+1,l_2)}$, while $A^+_{(l_0,l_1,l_2)}$ does it in the opposite way (later we will comment on the square-integrability conditions through unitary representations).

If we include the normalizing constant just as in (2.14), and define global operators acting on eigenfunctions of this class of Hamiltonians in the form

$$A^+ \Phi_{(l_0+1,l_1+1,l_2)} := \frac{1}{2} A^+_{(l_0,l_1,l_2)} \Phi_{(l_0+1,l_1+1,l_2)} \propto \tilde{\Phi}_{(l_0,l_1,l_2)}$$
$$A^- \Phi_{(l_0,l_1,l_2)} := \frac{1}{2} A^-_{(l_0,l_1,l_2)} \Phi_{(l_0,l_1,l_2)} \propto \tilde{\Phi}_{(l_0+1,l_1+1,l_2)}$$
$$A \Phi_{(l_0,l_1,l_2)} := -\frac{1}{2} (l_0 + l_1) \Phi_{(l_0,l_1,l_2)}$$

we are lead to the standard $su(2)$ commutators (2.15). Here, we want to stress again that now these operators are acting on the total wavefunction of complete Hamiltonians like $H_{(l_0,l_1,l_2)}$, not just on a factor function in only one variable.

In order to introduce other sets of operators we will use the fact that the Hamiltonian (2.11) can be separated in other coordinate systems. Since the axes $(s_0, s_1, s_2)$ play a symmetric role in the Hamiltonian, we will take their cyclic rotations to get two other sets of coordinates and, hence, new sets of intertwining operators.

3.1.2 The set $\{B^+, B^-, B\}$

We will take the spherical coordinates choosing as third axis not $s_2$, but $s_1$, i.e.

$$s_2 = \cos \xi_2 \cos \xi_1, \quad s_0 = \cos \xi_2 \sin \xi_1, \quad s_1 = \sin \xi_2$$

(3.2)

Then, the initial Hamiltonian is also separated in the coordinates $(\xi_1, \xi_2)$. In particular, we can build the operators $B^\pm_{(l_0,l_1,l_2)}$ in a similar way as $A^\pm_{(l_0,l_1,l_2)}$. From the coordinate systems
we easily arrive at the following expressions for the new set in terms of the initial coordinates \((\phi_1, \phi_2)\)

\[
B_{(l_0,l_1,l_2)}^{\pm} = \pm(\sin \phi_1 \tan \phi_2 \partial_{\phi_1} + \cos \phi_1 \partial_{\phi_2}) - (l_2+1/2) \cos \phi_1 \cot \phi_2 + (l_0+1/2) \sec \phi_1 \tan \phi_2
\]

These operators intertwine the pair of Hamiltonians

\[
B_{(l_0,l_1,l_2)}^{-} H_{(l_0,l_1,l_2)} = H_{(l_0+1,l_1,l_2+1)} B_{(l_0,l_1,l_2)}^{-}
\]

\[
B_{(l_0,l_1,l_2)}^{+} H_{(l_0+1,l_1,l_2+1)} = H_{(l_0,l_1,l_2)} B_{(l_0,l_1,l_2)}^{+}
\]

The ‘global’ operators, defined by

\[
B_{(l_0,l_1,l_2)}^{+} \Phi_{(l_0+1,l_1,l_2+1)} := \frac{1}{2} B_{(l_0,l_1,l_2)}^{+} \Phi_{(l_0+1,l_1,l_2+1)} \propto \tilde{\Phi}_{(l_0,l_1,l_2)}
\]

\[
B_{(l_0,l_1,l_2)}^{-} \Phi_{(l_0,l_1,l_2)} := \frac{1}{2} B_{(l_0,l_1,l_2)}^{-} \Phi_{(l_0,l_1,l_2)} \propto \tilde{\Phi}_{(l_0+1,l_1,l_2+1)}
\]

\[
B \Phi_{(l_0,l_1,l_2)} := -\frac{1}{2}(l_0 + l_2) \Phi_{(l_0,l_1,l_2)}
\]

also close a new \(\text{su}(2)\).

### 3.1.3 The set \(\{C^+, C^-, C\}\)

Finally, taking the spherical coordinates around the \(s_0\) axis,

\[
s_1 = \cos \theta_2 \cos \theta_1, \quad s_2 = \cos \theta_2 \sin \theta_1, \quad s_0 = \sin \theta_2
\]

the Hamiltonian is also separated in the variables \(\{\theta_1, \theta_2\}\) and we get a new pair of operators, that written in terms of the initial \(\phi_1\) and \(\phi_2\) variable, take the expression

\[
C_{(l_0,l_1,l_2)}^{\pm} = \pm(\cos \phi_1 \tan \phi_2 \partial_{\phi_1} - \sin \phi_1 \partial_{\phi_2}) + (l_1-1/2) \csc \phi_1 \tan \phi_2 + (l_2+1/2) \sin \phi_1 \cot \phi_2
\]

These operators act as interviners of the Hamiltonians in the following way

\[
C_{(l_0,l_1,l_2)}^{-} H_{(l_0,l_1,l_2)} = H_{(l_0,l_1-1,l_2+1)} C_{(l_0,l_1,l_2)}^{-}
\]

\[
C_{(l_0,l_1,l_2)}^{+} H_{(l_0,l_1-1,l_2+1)} = H_{(l_0,l_1,l_2)} C_{(l_0,l_1,l_2)}^{+}
\]

The ‘global’ operators are defined by

\[
C_{(l_0,l_1,l_2)}^{+} \Phi_{(l_0,l_1-1,l_2+1)} := \frac{1}{2} C_{(l_0,l_1,l_2)}^{+} \Phi_{(l_0,l_1-1,l_2+1)} \propto \tilde{\Phi}_{(l_0,l_1,l_2)}
\]

\[
C_{(l_0,l_1,l_2)}^{-} \Phi_{(l_0,l_1,l_2)} := \frac{1}{2} C_{(l_0,l_1,l_2)}^{-} \Phi_{(l_0,l_1,l_2)} \propto \tilde{\Phi}_{(l_0+1,l_1,l_2+1)}
\]

\[
C \Phi_{(l_0,l_1,l_2)} := -\frac{1}{2}(-l_1 + l_2) \Phi_{(l_0,l_1,l_2)}
\]

closing the third algebra \(\text{su}(2)\). Notice that \(C = B - A\).

In fact, as we saw in section 3.2 each separable system gives rise to two sets of intertwining operators (in that section distinguished by means of the tilde). However, here we have made a ‘good’ choice of the above three sets that will close a Lie algebra (on this point see section 3.2).
3.1.4 The complete algebra $u(3)$

Now, we can join all the transformations above defined, $A^\pm, A, B^\pm, B, C^\pm, C$, and commute any two of them to check that indeed they close a Lie algebra $su(3)$. The nonvanishing commutators are

\[
\begin{align*}
[A^3, A^\pm] &= \pm A^\pm & [A^-, A^+] &= 2A & [A^+, B^-] &= C^- & [A^+, B] &= -A^+/2 \\
[A^+, C^+] &= -B^+ & [A^+, C] &= A^+/2 & [A^-, B^+] &= -C^+ & [A^-, B] &= A^-/2 \\
[A^+, C^-] &= B^- & [A^-, C] &= -A^-/2 & [A, B^+] &= B^+/2 & [A, C^+] &= -C^+/2 \\
[A, C^-] &= C^-/2 & [B, B^\pm] &= \pm B^\pm & [B^-, B^+] &= 2B & [B^+, C^-] &= -A^+ \\
[B^+, C] &= -B^+/2 & [B^-, C^+] &= C^+/2 & [B^-, C] &= B^-/2 & [B, C^+] &= C^+/2 \\
[B, C^-] &= C^-/2 & [C, C^\pm] &= \pm C^\pm & [C^+, C^-] &= 2C & [A^-, C^-] &= B^-
\end{align*}
\]

The Casimir operator is given by

\[ C = A^+ A^- + B^+ B^- + C^+ C^- + \frac{2}{3} A (A - 3/2) + \frac{2}{3} B (B - 3/2) + \frac{2}{3} C (C - 3/2) \] (3.5)

In order to complete an algebra $u(3)$ we can add a diagonal operator $D$ commuting with all the above transformations. It is a central operator, i.e.

\[ D := l_0 - l_1 - l_2, \quad [D, \cdot] = 0 \]

We can also adopt the global operator convention $H$ for the Hamiltonians in the hierarchy by defining its action on the eigenfunctions $\Phi_{(l_1,l_2,l_3)}$ of $H_{(l_1,l_2,l_3)}$ by

\[ H \Phi_{(l_1,l_2,l_3)} := H_{(l_1,l_2,l_3)} \Phi_{(l_1,l_2,l_3)} \]

In this way we can express the Hamiltonian $H$ in terms of both operators $C$ and $D$

\[ H = 4C - \frac{1}{3} D^2 + \frac{15}{4} \] (3.6)

In the case of one-dimensional systems, one (first order) intertwining set $\{A^\pm\}$ for the Hamiltonian gives rise to its factorization. However, for Hamiltonians with more degrees of freedom (more components, or in more dimensions) the relationship of $H$ with these operators, in general, turns out to be more complex. In our case the set $\{A^\pm, B^\pm, C^\pm\}$ according to expressions (3.5) and (3.6) is enough to express the Hamiltonian as a certain quadratic function $H = \hbar (A^+ A^-, B^+ B^-, C^+ C^-)$ generalizing the usual factorization.

In summary, we have built an algebra $u(3)$ of intertwining operators that, once fixed the initial Hamiltonian with parameter values $(l_0, l_1, l_2)$, gives rise to a two-parameter Hamiltonian hierarchy

\[ \{H_{(l_0+m_l_1+m-n_l_2+n)}\}, \quad m, n \in \mathbb{Z} \]

where the points $(l_0 + m, l_1 + m - n, l_2 + n)$ lie on a certain plane $D = d_0$. In this subsection we will consider this special hierarchy, together with its eigenstates, connected to the IUR’s of $u(3)$. The states of such representations are square integrable and, therefore, should take part of the physical eigenfunctions whose energy eignvalues belong to the spectrum.
In order to build an IUR we start from a fundamental state $\Phi$ annihilated by $A^-$ and $C^-$ (two simple roots of $su(3)$)

$$A^- \Phi_\ell = C^- \Phi_\ell = 0$$

(3.7)

with $\ell = (l_0, l_1, l_2)$. Such states exist only when $l_1 = 0$, taking the explicit form

$$\Phi_\ell(\phi_1, \phi_2) = N \cos^{l_0+1/2} \phi_1 \sin^{l_1/2} \phi_1 \cos^{l_2+1/2} \phi_2$$

(3.8)

where $N$ is a normalizing constant. The diagonal operators $A$ and $C$ act on $\Phi_\ell$ as

$$A \Phi_\ell = -l_0/2 \Phi_\ell, \quad A \Phi_\ell = 0, \quad m = 0, 1, 2, \ldots$$

$$C \Phi_\ell = -l_2/2 \Phi_\ell, \quad C \Phi_\ell = 0, \quad n = 0, 1, 2, \ldots$$

(3.9)

This means that $\Phi_\ell$ is a fundamental state of the representations $j_1 = m/2$ of the subalgebra $su(2)$ generated by $\{A^\pm, A\}$, and $j_2 = n/2$ of the corresponding $su(2)$ determined by $\{C^\pm, C\}$. Such a representation of $su(3)$ will be denoted $(m,n)$ with $m,n \in \mathbb{Z}^+$. The points labeling the states of this representation obtained from $\Phi_\ell$ lie on the plane $D = m - n$ inside the $\ell$-parameter space.

The energy for the states of the IUR’s determined by the fundamental state (3.9) with the parameters $(l_0, 0, l_2)$, according to (3.6) is given by

$$E = (l_0 + l_2 + 3/2)(l_0 + l_2 + 5/2) = (m + n + 3/2)(m + n + 5/2)$$

(3.10)

Therefore, the IUR’s fixed by $(m,n)$ with the same value $m+n$ will lead to states with the same energy. We call such IUR’s an iso-energy series and they will be examined under the light of the algebra $so(6)$ in the following section. The values for the energy (3.10) coincide with the ones computed by the method of variable separation of section 2 once the replacement $l_1 = 1/2$ is performed. We can also check that in this case the ground state (2.24) coincides with those fixing an IUR (3.8).

3.2 The $so(6)$–hierarchy

Following the pattern and motivation of section 2.3, we will consider the relevant discrete symmetries in order to find a larger dynamical algebra.

It is obvious that the Hamiltonian $H_{(l_0,l_1,l_2)}$ is invariant under reflections in the parameter space $\{(l_0,l_1,l_2)\}$

$$I_0 : (l_0, l_1, l_2) \rightarrow (-l_0, l_1, l_2), \quad I_1 : (l_0, l_1, l_2) \rightarrow (l_0, -l_1, l_2), \quad I_2 : (l_0, l_1, l_2) \rightarrow (l_0, l_1, -l_2)$$

Each of these symmetries can be directly implemented in the eigenfunction space, leading through conjugation to another set of intertwining operators that close a Lie algebra isomorphic to $u(3)$ and denoted by $i\mathfrak{u}(3)$

$$ix = I_i X I_i, \quad x \in u(3), \quad ix \in \mathfrak{u}(3), \quad i = 0, 1, 2$$

The intertwining operators of $i\mathfrak{u}(3)$ connect eigenstates of Hamiltonians whose parameters $(l_0, l_1, l_2)$ belong to the planes $iD = k_i$, being $k_i$ certain real constants. We will choose the
following convention for the resulting generators

$$
\{A^\pm, B^\pm, C^\pm\} \xrightarrow{I_0} \{\tilde{A}^\pm, \tilde{B}^\pm, C^\pm\}
$$

$$
\{A^\pm, B^\pm, C^\pm\} \xrightarrow{I_1} \{\tilde{A}^\pm, B^\pm, \tilde{C}^\pm\}
$$

$$
\{A^\pm, B^\pm, C^\pm\} \xrightarrow{I_2} \{A^\pm, \tilde{B}^\pm, \tilde{C}^\pm\}
$$

where, for instance, the sets \(\{A^\pm, A\}\) and \(\{\tilde{A}^\pm, \tilde{A}\}\) close the two commuting Lie algebras \(su(2)\) of section \(2\). The explicit expression for the new operators (labelled with a tilde) can be easily obtained in the same way as it was done in \((2.18)\). The set of all the generators obtained in this process close the Lie algebra of rank 3, \(so(6)\). In the eigenfunction space it is enough to consider three independent diagonal operators \(\{L_0, L_1, L_2\}\) defined by

$$
L_i \Psi_{(l_0,l_1,l_2)} = l_i \Psi_{(l_0,l_1,l_2)}
$$

The Hamiltonian can be expressed in terms of the \(so(6)\)-Casimir operator by means of the ‘symmetrization’ of the \(u(3)\)-Hamiltonian \((3.6)\)

$$
H_{so(6)} = \frac{1}{8} \left( H_{u(3)} + \sum_j I_j H_{u(3)} I_j + \sum_{j \neq k} I_j I_k H_{u(3)} I_j I_k + I_0 I_1 I_2 H_{u(3)} I_0 I_1 I_2 \right)
$$

$$
= \{A^+, A^-\} + \{B^+, B^-\} + \{C^+, C^-\}
$$

$$
+ \{\tilde{A}^+, \tilde{A}^-\} + \{\tilde{B}^+, \tilde{B}^-\} + \{\tilde{C}^+, \tilde{C}^-\} + L_0^2 + L_1^2 + L_2^2 + \frac{41}{12}
$$

Henceforth we remove the subindex ‘\(so(6)\)’ of the Hamiltonian.

The intertwining generators of \(so(6)\) give rise to larger three-dimensional Hamiltonian hierarchies

$$
\{H_{(l_0+m+p,l_1+m-n-p,l_2+n)}\}, \quad m, n, p \in \mathbb{Z}
$$

each one including a class of the previous ones coming from \(u(3)\). The eigenstates of these Hamiltonian hierarchies can be classified in terms of \(so(6)\)-representations. Let us fix our attention in those determined by the \(so(6)\) IUR’s. These IUR’s are build from the fundamental states annihilated by the simple roots \(A^-, C^-, \tilde{A}^-\)

$$
A^- \psi_{l}^{(0)} = C^- \psi_{l}^{(0)} = \tilde{A}^- \psi_{l}^{(0)} = 0
$$

The equations for the operators \(A^-\) and \(C^-\) have been used in \((3.7)\), while the one for \(\tilde{A}^-\) were already applied in \((2.19)\). Therefore, the wavefunctions of the highest weight vectors take the form

$$
\Psi_{l}^{(0)}(\phi_1, \phi_2) = N \cos^{1/2} \phi_1 \sin^{1/2} \phi_1 \cos \phi_2 \sin^{l_2+1/2} \phi_2
$$

characterised by the eigenvalues of the diagonal operators,

$$
L_0 \Psi_{l} = L_1 \Psi_{l} = 0, \quad L_2 \Psi_{l} = n \Psi_{l}, \quad n \in \mathbb{Z}^+
$$

This fundamental state is invariant under the inversions \(I_0\) and \(I_1\), and the representation, so obtained, is also invariant under \(I_2\). Thus, in this way we arrive at two classes of symmetric IUR’s of \(so(6)\)
Each of these IUR’s is described in the parameter space by an octahedral lattice of points such that it will include an iso-energy $su(3)$ (or $isu(3)$) series of representations, quoted in the above subsection, which correspond to parallel exterior faces of the octahedron and some of its sections. Such sections are determined by the values of the diagonal operator $D$ (or $D_i$) whose values fix the corresponding $u(3)$-representations.

Figure 1: Plot of the points representing the states of two odd IUR’s with $q = 1$ (left) and $q = 3$ (right). The 6 ($q = 1$)-eigenstates share the energy $E = \frac{5}{2} \cdot \frac{3}{2}$. The 50 ($q = 3$)-eigenstates share the energy $E = \frac{7}{2} \cdot \frac{5}{2}$ (the points corresponding to $q = 3$ include those of $q = 1$, of the inner octahedron, which are doubly degenerated.

For instance, the $so(6)$-representation labelled by $n = 1$, corresponding to the odd hierarchy, includes the first $su(3)$-series, $(1,0)$ and $(0,1)$ described by the opposite faces of an elemental octahedron. The $so(6)$-representation of the even hierarchy fixed by $n = 2$ includes the $su(3)$-series made of $(2,0)$, $(1,1)$, and $(0,2)$. Those associated to $(2,0)$ and $(0,2)$ correspond to opposite triangular faces, while $(1,1)$ is described by the parallel hexagonal section through the origin. These features can be better appreciated in Figures 1 and 2.

In general, the $so(6)$ IUR’s fixed by the parameter $q$ will include the iso-energy series of the $su(3)$-representations labelled by $(m,n)$ with $m + n = q$. This is the degeneration explained by the larger algebra $so(6)$. A similar discussion can be done with respect to the representations of the $su(2) \oplus \tilde{su}(2)$ subalgebra. They can be identified with square sections of the octahedron.

4 Eigenstates and factorizations

Let $H_\ell$ and $H_{\ell'}$ be two Hamiltonians related by means of a differential operator $X$ in the following form

$$X \ H_\ell = H_{\ell'} \ X \implies X^\dagger \ H_{\ell'} = H_\ell \ X^\dagger$$

(4.1)
Figure 2: The figure on the left is for the $q = 1$ IUR of $so(6)$ where the triangular opposite faces correspond to two IUR’s of $su(3)$. The figure on the right correspond to the the points of a $q = 3$ IUR of $so(6)$. The three sections describe three IUR’s of $su(3)$.

where the dagger denotes adjoint differential operators. Then, it is said that $X$ is an intertwining operator connecting $H_\ell$ with $H_{\ell'}$.

In a formal way, the eigenfunctions of $H_\ell$ are transformed by $X$ into eigenfunctions of $H_{\ell'}$, but one must be careful about the behaviour of some properties, such as square-integrability, singularities, or boundary conditions, which might be altered by $X$. The intertwining problem just as introduced in (4.1), which applies to the $u(3)$-system of section 3, takes into account shape invariance, in the sense that the partner Hamiltonian $H_{\ell'}$ differs from the initial $H_\ell$ simply by changing the values of the parameters: $\ell \rightarrow \ell'$. In general, shape invariance leads to an algebraic structure of the intertwining operators as it happens in our present case.

Now, we will discuss in this section the form of the $su(3)$ intertwining operators of section 3 and its relation to certain eigenstates (similar considerations also apply to $so(6)$). First of all, note that we can write such operators (see expressions (3.1), (3.3) and (3.4)) as

$$A_\ell^\pm = a_\ell^\pm + \alpha_\ell, \quad B_\ell^\pm = b_\ell^\pm + \beta_\ell, \quad C_\ell^\pm = c_\ell^\pm + \gamma_\ell$$

(4.2)

where $a_\pm, b_\pm, c_\pm$ stand for vector fields (expressed, for instance, in the variables $\phi_1, \phi_2$) defined on the sphere and $\alpha_\ell, \beta_\ell, \gamma_\ell$ design functions also defined on the sphere. Notice that

$$a_+ = -(a^-)^\dagger = J_2, \quad b_+ = -(b^-)^\dagger = J_1, \quad c^+ = -(c^-)^\dagger = J_0$$

where $J_0, J_1, J_2$ close the rotation algebra $so(3)$. Moreover, taking the hermitian conjugate we have made use of the invariant measure on the sphere. If we write the Hamiltonians in the hierarchy displaying the kinetic (or free) part and the potential as

$$H_\ell = H^{(\text{kin})} + V_\ell$$

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we see that the vector fields originate the kinetic term, i.e.

\[ H^{(\text{kin})} = a^+a^- + b^+b^- + c^+c^- \]

and the components \(\alpha_\ell, \beta_\ell, \gamma_\ell\) (defined on the sphere) give rise to the potential \(V_\ell(\phi_1, \phi_2)\), labelled by the parameters \(\ell \equiv (l_0, l_1, l_2)\). Substituting (4.2) in the Hamiltonian (3.6) and taking into account (3.5), we get the expression

\[ V_\ell(\phi_1, \phi_2) = (\alpha_\ell)^2 + (a^+\alpha_\ell) + (\beta_\ell)^2 + (b^+\beta_\ell) + (\gamma_\ell)^2 + (c^+\gamma_\ell) + \lambda_\ell \quad (4.3) \]

where \(\lambda_\ell\) is a number depending on \(l_0, l_1, l_2\). Equation (4.3) can be considered as a non-linear partial differential equation linking the unknowns \(\{\alpha_\ell, \beta_\ell, \gamma_\ell\}\) with the potential, in a quite similar way to the Riccati equation for the superpotential \(\omega\) in the one-dimensional Schrödinger equation. For this reason, we sometimes will refer to \(\{\alpha_\ell, \beta_\ell, \gamma_\ell\}\) as superpotential functions. This is in agreement with a more general result [2, 3] where the first order intertwining is built by ‘dressing’ the symmetries of the Laplacian operator with certain functions.

The basic property of the one-dimensional superpotential \(\omega\) was that it could be considered as the logarithmic derivative of a Hamiltonian eigenstate (see (2.10)). Here, we have something similar with respect to the superpotentials \(\{\alpha_\ell, \beta_\ell, \gamma_\ell\}\) but first we want to settle this problem in general terms. If we know an intertwining operator \(X\) satisfying (4.1) it can help us in computing certain eigenfunctions of \(H_\ell\). Notice that if we define the kernel, \(K_X\), of \(X\) as the linear manifold of wave-functions annihilated by \(X\),

\[ X \psi = 0, \quad \forall \psi \in K_X \]

then, such a space is invariant under the Hamiltonian operator \(H_\ell\). Thus, we can look for eigenfunctions inside \(K_X\), in general a much simpler problem. But, in the case of \(X\) being a partial differential operator, its kernel includes certain arbitrary functions, so it is still an infinite dimensional space. This is in sharp contrast with ordinary first order differential operators where the kernel is one-dimensional.

Another option we have at hand is the following. The intertwining relation (4.1) implies the commutation

\[ X^\dagger X H_\ell = H_\ell X^\dagger X \]

This means that we can look for eigenfunctions of \(H_\ell\) inside any eigenfunction space of \(X^\dagger X\), not necessarily that one annihilated by \(X\), as was the case just considered above. In this case, however, a similar expression to (2.10) in terms of such eigenfunctions is no longer valid for \(\omega\). When we know several intertwining operators, as in the present case, we can apply them in different ways according to the above comments.

i) Superpotentials associated to a global fundamental eigenstate of \(\{A^-, B^-, C^-\}\). We consider the intersection of the kernels of all the intertwining operators. Assuming that this subspace is one-dimensional we have just one eigenstate (up to a factor) \(\Phi_0\) annihilated by all the lowering operators \(\{A^-, B^-, C^-\}\). So that, we obtain the following expressions quite similar to (2.10)

\[ \alpha_\ell = -\frac{(a^-\Phi_0)}{\Phi_0}, \quad \beta_\ell = -\frac{(b^-\Phi_0)}{\Phi_0}, \quad \gamma_\ell = -\frac{(c^-\Phi_0)}{\Phi_0} \quad (4.4) \]

This mechanism corresponds to the IUR’s characterized in section 3.
ii) **Superpotentials associated to a partial fundamental eigenstate.** If the above subspace is the trivial null space, we can still restrict ourselves to the kernel subspace of anyone of the intertwining operators, for example $A^-$. Thus, let $\Phi$ be an eigenfunction of $H_\ell$ with $\Phi \in K_{A^-}$, i.e. $A^- \Phi = 0$. This allows us to set

$$\alpha_\ell = -(a^- \Phi)/\Phi$$

From this equation we can also separate variables in $\Phi$. So that, the eigenfunction equation $H \Phi = E \Phi$ leads to a second order ordinary differential equation whose solution can be easily obtained.

However, we must outline that in this case the remaining superpotential functions $\beta_\ell, \gamma_\ell$ have not a simultaneous expression (4.4) in terms of the same $\Phi$, they need different eigenfunctions. Under this point of view, section 2 constitutes an illustration of how this option leads to eigenfunctions separated in the variables $\phi_1, \phi_2$.

iii) **Other excited eigenstates.** The second option is to solve, for instance, the eigenvalue problem $A^+ A^- \Phi = \alpha \Phi$, requiring at the same time $\Phi$ to be also a Hamiltonian eigenfunction. In terms of the ambient coordinates $s_0, s_1, s_2$ this equation is (see also [11])

$$\left\{- \left( s_1 \frac{\partial}{\partial s_0} - s_0 \frac{\partial}{\partial s_1} \right)^2 + (l_0 - 1/4) \frac{s_0^2 + s_1^2}{s_0^2} + (l_1 - 1/4) \frac{s_0^2 + s_1^2}{s_1^2} \right\} \Phi = \alpha \Phi$$

The same procedure can be applied with other more general sets of operators commuting with the Hamiltonian. For instance, we can diagonalize $H$ inside the subspace

$$(e_2 A^+ A^- + e_1 B^+ B^- + e_0 C^+ C^-) \Phi = \alpha \Phi$$

where the $e_i$’s are constant coefficients. This leads to eigenfunctions separated in elliptic coordinates, that we do not consider here [11].

## 5 Conclusions

We have shown how to deal with the $u(3)$ (and the general $u(n)$ case [9, 18] follows the same pattern [19]) analog of a class of factorizable one-dimensional potentials with underlying dynamical algebra $u(2)$. The higher rank systems in consideration are well known inside the class of superintegrable Hamiltonians and, of course, our objective was not to compute original eigenfunctions. Our interest was to apply a different point of view to understand some properties in a new context. For instance, the classification of the irreducible representations of $su(3)$ in series corresponding to $so(6)$-octahedrons, and the relations involved in this framework is a non trivial result that could be best appreciated inside the intertwining technique. The relation of the unitary representations with an special form of the superpotential functions, or the separable eigensolutions determined in terms of intertwining operators clarifies some of the known procedures.

We have seen how the elements of one-dimensional factorizations must be adapted to the new context. For example, the relation of superpotentials and a whole class of eigenfunctions (not just one), the expression of the Hamiltonian operator is not just a simple factorization, the lattice of states must be drawn in a three-dimensional space, etc.
There are several problems that can be addressed using the present procedure. The systems underlying noncompact algebras $u(p,q)$, inhomogeneous Lie algebras $\tilde{u}(p,q)$ and contracted algebras [20] are among the first applications that we expect to report in a near future. But, in general, any other integrable Hamiltonian system will allow for this treatment, with or without variable separation. This application would be of most interest.

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