Black Holes in Quasi-topological Gravity

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Abstract: We construct a new gravitational action which includes cubic curvature interactions and which provides a useful toy model for the holographic study of a three parameter family of four- and higher-dimensional CFT’s. We also investigate the black hole solutions of this new gravity theory. Further we examine the equations of motion of quasi-topological gravity. While the full equations in a general background are fourth-order in derivatives, we show that the linearized equations describing gravitons propagating in the AdS vacua match precisely the second-order equations of Einstein gravity.
1. Introduction

Recently, there has been some interest in gravitational actions with higher curvature actions in the context of the AdS/CFT correspondence. For example, Einstein gravity in the AdS bulk defines a universality class of CFT’s in which the ratio of the shear viscosity to entropy density is given by precisely $\eta/s = 1/(4\pi)$ [1, 2]. However, it is understood that adding higher curvature interactions to the bulk gravity action leads to a broader class of CFT’s in which this ratio generally depends on the value of the additional gravitational couplings [3]. Further it is possible with some holographic constructions to violate the famous bound conjectured by Kovtun, Son and Starinets (KSS) [1] producing theories with $\eta/s < 1/(4\pi)$. In certain string theory constructions, the appearance of curvature-squared interactions produces violations of the KSS bound [4] but these models only produce reliable results in a regime where the corresponding gravitational coupling is parametrically small. Hence at present, one can only deviate perturbatively away from the universality class defined by Einstein gravity in these string theory models.
However, it is also of interest to explore situations where the gravitational couplings are finite. For example, holography can yield new consistency conditions for the gravitational theories and their dual CFT’s. One theory which provides a useful toy model in this regard is Gauss-Bonnet (GB) gravity. Even with a finite coupling for the curvature-squared interaction, this theory still provides some calculation control, which has been exploited in several recent holographic studies [5, 6, 7, 8, 9]. However, GB gravity only introduces a single new coupling which limits the range of dual CFT’s which can be studied. A natural generalization would be the further addition of interactions cubic in the curvature, as this allows the investigation of the the full range of parameters in the three-point function of the stress tensor [10]. A straightforward extension of GB gravity would be to include the cubic interaction of Lovelock gravity [14]. However, because of the topological origin of the Lovelock terms the cubic interaction only contributes to the equations of motion when the bulk dimension is seven or greater. In the context of the AdS/CFT correspondence, this means that such a term will be effective in expanding the class of dual CFT’s in six or more dimensions [8, 9]. Our key result in this paper is to construct a new gravitational action with cubic curvature interactions which provides a useful toy model to study a broader class of four (and higher) dimensional CFT’s, involving three independent parameters. In the following, we describe the construction for the new gravitational action and investigate black hole solutions in this theory. We leave the detailed study of the properties of the dual class of CFT’s to a companion paper [10].

An outline of the rest paper is as follows: We begin with a review of black hole solutions and various aspects of these solutions in Gauss-Bonnet (GB) gravity coupled to a negative cosmological constant in section 2. Inspired by the GB equations of motion determining black hole solutions, we construct a new interaction that is cubic in curvatures and yields similar simple solutions in section 3. Again we wish to emphasize that this interaction is not the six-dimensional Euler density as appears in third-order Lovelock gravity. Further, we show in appendix A that the new interaction does not have a topological origin and hence we call the new theory: ‘quasi-topological gravity.’ We turn to a discussion of the asymptotically AdS black hole solutions in section 4. While the focus of this discussion is planar black holes in five dimensions, we generalize the results to curved horizons and higher dimensions in sections 4.2 and 4.3. In section 5, we examine black hole thermodynamics in the new theory, deriving some of the basic thermal properties of the black holes and the corresponding plasmas in the dual CFT. We examine the equations of motion of quasi-topological gravity in section 6. While the full equations in a general background are fourth-order in derivatives, we show that the linearized equations describing gravitons propagating in the AdS vacuum solutions are precisely the second-order equations of Einstein gravity. We conclude with a brief discussion of our results and future directions in section 7.

While we were in the final stages of preparing this paper, two related preprints appeared in which exceptional new theories of curvature-cubed gravity were constructed. Ref. [11] constructs an interesting curvature-cubed theory in three dimensions. Up to a contribution proportional to the six-dimensional Euler density, the curvature-cubed interaction constructed in five dimensions by [12] is identical to that studied here. Refs. [12, 13] are also able to relate our interactions in $D \geq 7$ to Weyl-invariant combinations of curvatures.
combined with the six-dimensional Euler density.

2. Black Holes in Gauss-Bonnet gravity

We begin here with a brief review of black holes in Gauss-Bonnet (GB) gravity. The latter corresponds to a theory of gravity in which a curvature-squared interaction is added with the form of the density for the Euler characteristic of four-dimensional manifolds,

\[ \chi_4 = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2. \]  

(2.1)

Of course, this term will not affect the gravitational equations of motion if the dimension of the spacetime is four (or lower), however, it makes interesting contributions for \( D \geq 5 \). GB gravity can be seen as the simplest example of the Lovelock theories [14] discussed above. As explained, despite having a higher curvature action, the resulting equations of motion are still second-order in (time) derivatives and this produced some interest in early discussions of higher curvature corrections to string theory [15, 16]. These discussions also lead to an extensive study of black hole solutions in this theory [17]. More recently, there has been some renewed interest in asymptotically AdS black hole solutions in GB gravity [18, 19], especially in the context of the AdS/CFT correspondence [5, 6, 7]. In our following, we discuss the black hole solutions focussing on GB gravity with \( D = 5 \) and with a negative cosmological constant:

\[ I = \frac{1}{16 \pi G_5} \int d^5 x \sqrt{-g} \left[ \frac{12}{L^2} + R + \frac{\lambda L^2}{2} \chi_4 \right]. \]  

(2.2)

We add some comments about higher dimensions at the end of this section.

Let us present the ansatz for the metric of five-dimensional planar AdS black holes, which we will be using throughout the paper:

\[ ds^2 = \frac{r^2}{L^2} \left( -N(r)^2 f(r) \, dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{L^2}{r^2 f(r)} \, dr^2. \]  

(2.3)

Inserting this metric ansatz into the action (2.2) (and integrating by parts a number of times) yields

\[ I = \frac{1}{16 \pi G_5} \int d^5 x \frac{3N(r)}{L^5} \left[ r^4 (1 - f + \lambda f^2) \right]' \]  

(2.4)

where the ‘prime’ indicates differentiation with respect to \( r \). Schematically, the equation of motion coming from varying the lapse \( N \) takes the simple form \([r^4 (\cdots)]' = 0\) and so the metric function \( f \) is given by solving for the roots of a quadratic polynomial [18, 19]:

\[ \lambda f(r)^2 - f(r) + 1 - \frac{\omega^4}{r^4} = 0. \]  

(2.5)

The latter yields two solutions

\[ f_{\pm}(r) = \frac{1}{2 \lambda} \left[ 1 \pm \sqrt{1 - 4 \lambda \left( 1 - \frac{\omega^4}{r^4} \right)} \right]. \]  

(2.6)
Now varying $\delta f$ yields a constraint which requires that $N = \text{constant}$, which we leave unspecified for the moment. In the following, we will consider only the solutions with $f_-$, since the other branch with $f_+$ contains ghosts and is unstable [16] — as we will see in later sections. With the choice $f = f_-$, it is easy to verify that the horizon appears at $r = r_h = \omega$.

In fixing the value of $N$, it is convenient to consider the solution with $\omega = 0,$

$$
\text{ds}^2 = \frac{r^2}{L^2} \left( -N(r)^2 f_\infty \text{dt}^2 + \text{dx}^2 + \text{dy}^2 + \text{dz}^2 \right) + \frac{L^2}{r^2 f_\infty} \text{dr}^2.
$$

(2.7)

where we have adopted the notation

$$
f_\infty \equiv \lim_{r \to \infty} f(r) = \frac{1}{2\lambda} \left[ 1 - \sqrt{1 - 4\lambda} \right].
$$

(2.8)

We recognize eq. (2.7) as anti-de Sitter (AdS) space, presented in the Poincaré coordinates. From $g_{rr}$ above, we also see that the AdS curvature scale is given by $\tilde{L} = L/\sqrt{f_\infty}$. This metric also makes apparent a convenient choice for the lapse, namely $N^2 = 1/f_\infty$, which we adopt in the following. In the AdS vacuum (2.7), this ensures that any motions in the brane directions are limited to lie within the standard light cone, i.e., $0 = -\text{dt}^2 + \text{dx}^2 + \text{dy}^2 + \text{dz}^2$. In the black brane solution (2.3), we still have $\lim_{r \to \infty} N^2 f(r) = 1$ and so this comment applies in the asymptotic region. In the context of the AdS/CFT correspondence, the latter means that the speed of light in the boundary CFT is simply $c = 1$.

Examining the solutions in eq. (2.6) or $f_\infty$ in eq. (2.8), we see that there is an upper bound at $\lambda = 1/4$. For larger values of $\lambda$, the gravitational theory does not have an anti-de Sitter vacuum and the interpretation of the solutions (2.6) becomes problematic. In fact, using the AdS/CFT correspondence to demand consistency of the dual CFT, e.g., requiring that the boundary theory is causal, imposes much more stringent constraints on the GB coupling [5, 6]

$$
-\frac{7}{36} \leq \lambda \leq \frac{9}{100}.
$$

(2.9)

We now turn to the thermodynamic properties of these GB black holes. The temperature of the black brane solutions is then given by the simple expression:

$$
T = \frac{1}{4\pi} \frac{r_h^2 f_\prime |_{r_h}}{L^2} N = \frac{\omega}{\pi L^2} N,
$$

(2.10)

which when evaluated with $N = 1/\sqrt{f_\infty}$ becomes:

$$
T = \frac{\omega}{\pi L^2} \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 4\lambda} \right) \right]^{1/2}.
$$

(2.11)

The latter can be calculated by the standard technique of by analytically continuing the metric (2.3) to Euclidean time, $\tau = -it$, and choosing the periodicity of $\tau$ to ensure the geometry is smooth at $r_h = \omega$. Next we evaluate the Euclidean action:

$$
I_E[T] = \frac{1}{16\pi G_5} \frac{V_3 \omega^4 N}{TL^5 \lambda} \left( \frac{r_h^4}{\omega^4} \left( 12\lambda - 5 + 5\sqrt{1 - 4\lambda} \right) - 4\lambda + \frac{2\lambda}{\sqrt{1 - 4\lambda}} \right),
$$

(2.12)
where \( V_{3} \) is the regulator volume obtained by integrating the \((x, y, z)\) directions. Further we have limited the radial integration from \( r = \omega \) to \( r_{+} \) to regulate the asymptotic or UV divergence in \( I_{E} \). The divergent \( r_{+}^{4} \) contribution is removed with background subtraction using the AdS vacuum (2.7), i.e., that is \( I_{E} - I_{E}^{0} \) remains finite in the limit \( r_{+} \to \infty \). Note that in general, such a calculation would include a generalized Gibbons-Hawking term [20], as well as other boundary terms to regulate the divergences in the Euclidean action [21]. However, these surface terms do not contribute to the final result for planar AdS black holes when we use the background subtraction approach. Therefore we identify the free energy density as

\[
\mathcal{F} = \frac{T}{V_{3}} (I_{E} - I_{E}^{0}) = -\frac{(\pi \sqrt{f_{\infty}} L)^{3}}{16G_{5}} T^{4}.
\]

(2.13)

Then we may identify the energy and entropy densities as

\[
\rho = -T^{2} \frac{d}{dT} \left( \frac{\mathcal{F}}{T} \right) = \frac{3(\pi \sqrt{f_{\infty}} L)^{3} T^{4}}{16G_{5}},
\]

(2.14)

\[
s = -\frac{d\mathcal{F}}{dT} = \frac{(\pi \sqrt{f_{\infty}} L T)^{3}}{4G_{5}} = \frac{1}{4G_{5} L^{3}} \omega^{3}.
\]

One can confirm that the last result matches the entropy calculated using Wald’s techniques [22].

These solutions are easily generalized from five to an arbitrary spacetime dimension, \( D \). In this case, the action is conveniently parameterized as

\[
I = \frac{1}{16\pi G_{D}} \int d^{D}x \sqrt{-g} \left[ \frac{(D-1)(D-2)}{L^{2}} + R + \frac{\lambda L^{2}}{(D-3)(D-4)} \mathcal{X}_{4} \right].
\]

(2.15)

We also generalize the metric ansatz to include spherical and hyperbolic, as well as planar, horizons:

\[
ds^{2} = -(k + \frac{r^{2}}{L^{2}} f(r)) N(r)^{2} dt^{2} + \frac{dr^{2}}{k + \frac{r^{2}}{L^{2}} f(r)} + r^{2} d\ell_{k}^{2}
\]

(2.16)

where \( d\ell_{k}^{2} \) is given by

\[
k = +1 : \quad d\Omega_{D-2}^{2} \quad \text{(metric on } S^{D-2})
\]

\[
k = 0 : \quad \frac{1}{L^{2}} \sum_{i=1}^{D-2} (dx^{i})^{2}
\]

\[
k = -1 : \quad d\Sigma_{D-2}^{2} \quad \text{(metric on } H^{D-2})
\]

(2.17)

Note that for \( k = \pm 1 \), the above line element has unit curvature. With this general ansatz incorporating both curved horizons and \( D \) spacetime dimensions, \( f \) is determined by simply solving for the roots of

\[
\lambda f(r)^{2} - f(r) + 1 - \frac{\omega^{D-1}}{r^{D-1}} = 0,
\]

(2.18)

and the solutions take the form given in eq. (2.6) with the replacement \( \omega^{4}/r^{4} \to \omega^{D-1}/r^{D-1} \).

The lapse is again a constant and as above we choose \( N = 1/\sqrt{f_{\infty}} \). In general, the horizon is determined by \( f(r_{h}) = -k \frac{L^{2}}{r_{h}^{2}} \) and so we only have \( r_{h} = \omega \) for the planar horizons,
\[ i.e., \ k = 0. \] In the case of the curved horizons, explicitly evaluating \( r_h \) requires solving a \( (D - 1) \)-order polynomial in \( r_h \). Hence we only have a relatively simple solution in five dimensions where:

\[
r_h = \frac{1}{2} \sqrt{-2k L^2 + 2\sqrt{k^2}L^4 - 4k^2 \lambda L^4 + 4\omega^4}.
\] (2.19)

3. New Curvature-Cubed Interaction

As discussed in the introduction, we are motivated by considerations of the AdS/CFT correspondence to consider a curvature-cubed theory of gravity in five dimensions. GB gravity has a number of features which one might want to reproduce, such as providing second-order equations of motion and a family of exact black hole solutions. A natural candidate to extend these properties to a curvature-cubed theory would be the Lovelock theory where the six-dimensional Euler density is added as a new gravitational interaction. However, this curvature-cubed interaction would only contribute to the equations of motion in seven and higher dimensions and hence will not contribute in the desired five dimensions. While Lovelock’s work then indicates that it should not be possible to find an alternate action which yields second-order equations of motion, we begin by writing the most general interaction including all possible curvature-cubed (or more precisely, six-derivative) interactions in five dimensions and attempt to tune the coefficients to produce a simple equation for the black hole solutions, as discussed for GB gravity in the previous section. We return to the equations of motion in section 6 and we demonstrate that the linearized equations of motion in the AdS vacuum are indeed second-order.

Let us begin by listing a basis of the possible six-derivative interactions:

1. \( R_{ac}^d R_{cd}^e f R_{ef}^{a b} \)
2. \( R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} \)
3. \( R_{abcd} R_{abcd}^e R_{de} \)
4. \( R_{abcd} R_{ab}^{abcd} R \)
5. \( R_{abcd} R_{ac}^{abcd} R_{bd} \)
6. \( R_{ab}^c R_{bc}^e R_{ce}^a \)
7. \( R_{ab}^c R_{bc}^a R \)
8. \( R^3 \)
9. \( \nabla_a R_{bcde} \nabla^a R_{bcde} \)
10. \( \nabla^a \nabla_a R_{abcd} R_{bd} \)
11. \( \nabla_a R_{bc} \nabla^a R_{bc} \)
12. \( \nabla^a R_{ab} \nabla^b R \)
13. \( \nabla_a R \nabla^a R \).

In assembling this list, we have discarded any total derivatives, \( e.g., \nabla^a \nabla_a \nabla^b \nabla^c R_{bc} \) and we have simplified various expressions using the index symmetries of the Ricci and Riemann tensors. In particular, these symmetries allow us to reduce any other index contraction of three Riemann tensors to some combination of terms 1 and 2. Further, term 12 can be reduced to term 13 using \( \nabla^a R_{ab} = \frac{1}{2} \nabla_b R \). Similarly, using the Bianchi identities, terms 9 and 10 can be shown to be reducible to other terms and total derivatives as well. Hence, we are left with a list of 10 independent interactions which are cubic in curvatures. Combining all of these interactions together in a single expression gives:

\[
\sqrt{-g} Z = \sqrt{-g} \left( c_1 R_{a b}^c d R_{cd}^e f R_{ef}^{a b} + c_2 R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} + c_3 R_{abcd} R_{abcd}^e R_{de}^{e} \\
+ c_4 R_{abcd} R_{abcd}^{abcd} R + c_5 R_{abcd} R_{abcd}^{ac} R_{bd}^{bd} + c_6 R_{a b}^c R_{bc}^a R \right) + c_7 R_{a b}^c R_{bc}^a R + c_8 R^3 + c_9 \nabla_a R_{bc} \nabla^a R_{bc} + c_{10} \nabla_a R \nabla^a R \).
\] (3.1)
At this point, we substitute the black brane metric ansatz (2.3) and evaluate eq. (3.1). The next step will be to see if the coefficients \( c_i \) can be tuned to produce a result with the same form as in eq. (2.6). In order to accomplish this task, we integrate by parts repeatedly to put as many terms as possible in the form \( N(r) \times (\cdots) \) where the factor in brackets is independent of the lapse function. The resulting expression then becomes

\[
\sqrt{-g}Z = \frac{N}{4L^9} \left[ (8r^9 (c_{11} + 2c_{13}) f^2 f^{(6)} + (r^8 (236c_{11} + 496c_{13}) f^2 + r^9 (56c_{13} + 28c_{11}) \times f f') f^{(5)} + (r^9 (24c_3 + 48c_8 + 20c_{11} + 48c_9 + 12c_6 + 24c_7 + 12c_5 + 48c_4 + c_{13}) f f'' + 8r^9 (c_{11} + 2c_{13}) (f')^2 + r^7 (5056c_{13} + 384c_4 + 96c_6 + 2264c_{11} + 96c_2 + 96c_3 + 288c_7 + 960c_8 + 72c_5) f^2 + r^8 (72c_5 + 288c_8 + 120c_3 + 1312c_{13} + 192c_2 + 480c_8 + 192c_7 + 608c_{11} + 84c_6) f f') f^{(4)} + (r^8 (48c_4 + 12c_5 + 48c_8 + 24c_7 + 16c_6 + 12c_6 + 8c_{11} + 24c_3 + 48c_2) f (f')^2 + (r^8 (240c_8 + 96c_6 + 60c_3 + 100c_{11} + 96c_2 + 224c_{13} + 36c_5 + 144c_4 + 42c_6) (f')^2 + r^7 (1494c_5 + 6069c_4 + 3054c_2 + 4380c_7 + 4012c_{11} + 9776c_{13} + 12000c_8 + 1794c_6 + 2340c_4 + 18c_1) f f' + (r^9 (6c_5 + 24c_2 + 24c_4 + 12c_2 + 24c_5 + 12c_3 + 6c_6) f f'' + (1296c_2 + 920c_{13} + 384c_5 + 708c_5 + 388c_{11} + 2016c_6 + 1536c_4 + 414c_6 + 888c_7) f f'' + r^6 (1584c_5 + 8140c_{11} + 36c_1 + 1440c_2 + 1212c_5 + 4944c_2 + 1488c_3 + 5952c_4 + 17280c_8 + 19712c_{13}) f^2 f''' - 2r^9 (c_3 + 4c_8 + c_6 + 2c_7 + 4c_4 + 2c_3 + 4c_2) (f')^3 + (r^8 (42c_3 + 84c_8 + 21c_6 + 84c_4 + 42c_7 + 21c_5 + 84c_4) f f' + r^7 (10896c_8 + 4260c_7 + 18c_1 + 6384c_4 + 1308c_{11} + 3696c_{13} + 1608c_5 + 2700c_3 + 4488c_2 + 1842c_6) f (f')^2 + (r^6 (26544c_{13} + 306c_1 + 28092c_7 + 8790c_5 + 84000c_8 + 10740c_6 + 8916c_{11} + 17472c_2 + 12600c_3 + 35088c_4) f f'' + r^5 (324c_1 + 5436c_5 + 564c_2 + 6960c_8 + 82560c_8 + 8676c_{11} + 22608c_7 + 6069c_3 + 24384c_4 + 24192c_{13}) f^2 + r^4 (264c_{11} + 252c_5 + 1056c_4 + 720c_7 + 300c_6 + 624c_2 + 1920c_5 + 672c_{13} + 408c_3) (f')^2 f''' - 16r^3 (8c_3 + 40c_6 + 400c_8 + 80c_7 + 16c_6 + 3c_1 + 16c_4 + 4c_2) f^3 + r^6 (434c_4 + 1240c_4 + 301c_5 + 950c_7 + 592c_2 + 2800c_8 + 9c_1 + 361c_6) (f')^3 + r^5 (10122c_6 + 10236c_3 + 558c_1 + 10080c_{13} + 2160c_{11} + 29040c_7 + 12480c_2 + 8202c_5 + 31296c_4 + 96000) (f')^2 + r^4 (-900c_{11} + 5904c_4 + 24240c_7 + 504c_4 + 23520c_4 + 7248c_6 + 5748c_5 + 91200c_8 + 5232c_{13}) f^2 f' \right] + \cdots.
\]

Note that not all terms can be put in the desired form with further integration by parts and so the ‘\( \cdots \)’ indicates the presence of spurious terms containing factors like \((N')^2/N\), for example. Focusing on the terms appearing explicitly in eq. (3.2), we find that choosing the values of the \( c_i \)'s as

1. \( c_3 = \frac{9}{7} c_1 - \frac{60}{7} c_2 \)
5. \( c_7 = \frac{33}{14} c_1 - \frac{54}{7} c_2 \)
\[
\begin{align*}
2. \ c_4 &= \frac{3}{8} c_1 + \frac{3}{2} c_2 \\
3. \ c_5 &= \frac{15}{7} c_1 + \frac{72}{7} c_2 \\
4. \ c_6 &= \frac{18}{7} c_1 + \frac{64}{7} c_2 \\
5. \ c_7 &= \frac{15}{56} c_1 + \frac{11}{14} c_2 \\
6. \ c_8 &= \frac{15}{56} c_1 + \frac{11}{14} c_2 \\
7. \ c_{11} &= 0 \\
8. \ c_{13} &= 0
\end{align*}
\]

reduces this expression to the following simple form
\[
\sqrt{-g} Z = \frac{12}{7} N(r) \left( c_1 + 2c_2 \right) (r^4 f^3)' .
\] (3.4)

At the same time, the spurious terms denoted by ‘⋅⋅⋅’ in eq. (3.2) also vanish with this choice of coefficients. It is quite remarkable that there is enough freedom in the general action (3.1) to produce this simple result. In fact, we are still free to choose the (relative) values of \(c_1\) and \(c_2\) in constructing this interaction. Explicitly then, if we choose \(c_1 = 1\), \(c_2 = 0\), the new curvature-cubed interaction takes the form
\[
\begin{align*}
Z_5' &= \frac{1}{56} \left( 21 R_{abcd} R^{abcd} - 72 R_{abcd} R^{abcd} e R^{de} + 120 R_{abcd} R^{ac} R^{bd} + 144 R_a^b R_b^c R_c^a - 132 R_a^b R_a^a R + 15 R^3 \right) \\
&+108 R_{abcd} R^{abcd} R^{abcd} \text{ for } c_1 = 0, c_2 = 1,
\end{align*}
\] (3.5)

or with \(c_1 = 0\), \(c_2 = 1\),
\[
\begin{align*}
Z_5' &= \frac{1}{14} \left( 21 R_{abcd} R^{abcd} - 120 R_{abcd} R^{abcd} e R^{de} + 144 R_{abcd} R^{ac} R^{bd} + 128 R_a^b R_b^c R_c^a - 108 R_a^b R_a^a R + 11 R^3 \right).
\] (3.6)

The fact that we do not produce a unique interaction should not be surprising. Any curvature-cubed interaction can be modified by the addition of the six-dimensional Euler density \(\mathcal{X}_6\) without affecting the equations of motion. In fact, we can infer the form of \(\mathcal{X}_6\) by setting \(c_1 = -2c_2\), in which case eq. (3.4) vanishes, as it must if evaluated for the six-dimensional Euler density. A standard normalization for the six-dimensional Euler density is:
\[
\mathcal{X}_6 = \frac{1}{8} \varepsilon_{abcdef} \varepsilon^{ghijkl} R_{ab}^{gh} R_{cd}^{ij} R_{ef}^{kl} R_{abcd}^{ef} R_{e}^{ab} - 8 R_a^c b R_c^d f R_e^a f - 24 R_{abcd} R_{e}^{abcd} e R^{de} + 3 R_{abcd} R_{abcd} R + 24 R_{abcd} R_{abcd} R + R^3 ,
\] (3.7)

where in the first line, \(\varepsilon_{abcdef}\) is the completely antisymmetric tensor in six dimensions and hence the corresponding expression only applies for \(D = 6\). However, the first line also makes clear that this expression should vanish when evaluated in five (or lower) dimensions. This normalization corresponds to the choice \(c_2 = 4\) and \(c_1 = -8\). That is, \(\mathcal{X}_6 = 4 Z_5' - 8 Z_5\).

### 3.1 Generalizing to \(D \geq 5\)

At this point, we turn to generalizing this construction to higher dimensions. Given the freedom discussed above, we begin by setting \(c_1 = 1\) and \(c_2 = 0\). Comparing to eq. (3.7)
shows we are guaranteed that, if a nontrivial interaction exists, the result will be distinct from the six-dimensional Euler density. With this choice, we substitute the $D$-dimensional extension of eq. (2.3) into the action (3.1). One then finds with a judicious choice of the remaining $c_i$’s, the result can be reduced to

$$\sqrt{-g} Z_D \sim \frac{N(r)}{L^D} (r^{D-1} f(r))^\prime.$$  

(3.8)

The required choice of coefficients (with $c_1 = 1$ and $c_2 = 0$) is:

1. $c_3(D) = -\frac{3(D-2)}{(2D-3)(D-4)}$ 
2. $c_4(D) = \frac{3(3D-8)}{8(2D-3)(D-4)}$ 
3. $c_5(D) = \frac{3D}{(2D-3)(D-4)}$ 
4. $c_6(D) = \frac{6(D-2)}{(2D-3)(D-4)}$ 
5. $c_7(D) = -\frac{3(3D-4)}{2(2D-3)(D-4)}$ 
6. $c_8(D) = \frac{3D}{8(2D-3)(D-4)}$ 
7. $c_{11}(D) = 0$ 
8. $c_{13}(D) = 0$. 

(3.9)

In practice, we determined the coefficients separately for $D = 5 \ldots 10$ and found the general expressions above to fit the results in all of these cases. Given these expressions, the general form of $Z_D$ becomes

$$Z_D = R^c_{\ a \ b} R^{e \ f}_{c \ d} R^{a \ b}_{e \ f} + \frac{1}{(2D-3)(D-4)} \left( \frac{3(3D-8)}{8} R_{abcd} R^{abcd} \right) R - 3(D-2) R_{abcd} R^{e \ f} + 3D R_{abcd} R^{ac} R^{bd} + 6(D-2) R^c_{\ a \ b} R^a_{\ b} R + \frac{3(D-4)}{2} R^a_{\ b} R^b_{\ a} R + \frac{3D}{8} R^3.$$  

(3.10)

One easily verifies that this result reduces to eq. (3.5) for $D = 5$.

An important note is that, with the coefficients prescribed above for $D = 6$, the resulting action (3.8) is trivial, i.e., there is an overall factor of zero in this result. Hence $Z_6$ does not actually produce a nontrivial interaction which is cubic in curvatures. One might be tempted to believe instead that $Z_6$ yields another topological invariant in six dimensions. Of course, there is no obvious known invariant to which $Z_6$ might correspond [23]. In appendix A, we demonstrate that $\int d^6 x \sqrt{g} Z_6$ is not a topological invariant by explicitly evaluating this expression for some nontrivial six-dimensional geometries. Hence, we refer to the theory of gravity extended with our new curvature-cubed interaction as ‘quasi-topological gravity.’ At this point, we also note that our construction does not yield a nontrivial curvature-cubed interaction for $D \leq 4$.

Of course, one can generalize the interaction (3.10) for $D > 6$ by adding another component proportional to the six-dimensional Euler character (3.7). This would be equivalent to leaving $c_2$ arbitrary in our analysis above. Hence, we complete the discussion by generalizing eq. (3.6) to higher dimensions using the formula: $Z'_D = 2Z_D + \frac{1}{6} \mathcal{X}_6$. The final
expression can be written as
\[ Z'_d = R^{cd} R_{cd} - \frac{1}{2 (D - 3)(D - 4)} \left( -12 (D^2 - 5D + 5) R_{abcd} R^{ab} R^{de} + \frac{3}{2} (D^2 - 4D + 2) R R_{abcd} R^{abcd} + 12 (D - 2) (D - 3) R_{abcd} R^{ac} R^{bd} + 8 (D - 1) (D - 3) a b c d e f a - 6 (D - 2)^2 R R^a_b R^a_a + \frac{1}{2} (D^2 - 4D + 6) R^3 \right). \] (3.11)

4. Black Hole Solutions

We have thus far constructed a new gravitational action that includes interactions up to cubic order in the curvature and which still yields particularly simple equations to find black hole solutions. In this section, we complete the study of the black holes in this new theory. While we have written an action for the theory in arbitrary number of spacetime dimensions, we will focus on the case \( D = 5 \) here. Further we begin by examining solutions of the form given in eq. (2.3) and hence construct black holes with planar horizons \([24, 18]\). The extension of this analysis to larger \( D \) and curved horizons (with spherical or hyperbolic geometries) is straightforward and will be discussed briefly at the end of this section.

4.1 Planar Black Holes

We begin with the five-dimensional action:
\[ I = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ \frac{12}{L^2} + \frac{\lambda L^2}{2} \chi_4 + \frac{7\mu L^4}{4} Z_5 \right] \] (4.1)

which extends the GB action with the addition of the curvature-cubed interaction \( Z_5 \). Next, as in section 2, we consider the following metric ansatz:
\[ ds^2 = \frac{r^2}{L^2} \left( -N(r)^2 f(r) dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{L^2}{r^2 f(r)} dr^2. \] (4.2)

Evaluating the action (4.1) with this metric then yields
\[ I = \frac{1}{16\pi G_5} \int d^5 x \frac{3N(r)}{L^5} \left[ r^4 (1 - f + \lambda f^2 + \mu f^3) \right]' \] (4.3)

where the prime again denotes a derivative with respect to \( r \). The variation \( \delta N \) now yields
\[ [r^4 (1 - f + \lambda f^2 + \mu f^3)]' = 0 \]
\[ \Rightarrow \quad 1 - f + \lambda f^2 + \mu f^3 = \frac{\omega^4}{r^4}. \] (4.4)

Similarly, satisfying the equation produced by taking the variation of \( f \) requires that either \( N' = 0 \) or \(-1 + 2\lambda f + 3\mu f^2 = 0\). Since the latter is generally inconsistent with eq. (4.4), we arrive at \( N = \text{constant} \). As in section 2, we choose \( N^2 = 1/f_\infty \), where \( f_\infty = \lim_{r \to \infty} f(r) \). This choice ensures that the speed of light at the boundary metric is just one.
We are now left with a cubic equation (4.4) to solve for \( f(r) \). To do so, we first make the substitution \( f = x - \frac{\lambda}{3\mu} \), with which eq. (4.4) becomes:

\[
x^3 - 3\left(\frac{3\mu + \lambda^2}{9\mu^2}\right)x + 2\left(\frac{2\lambda^3 + 9\lambda\mu + 27\mu^2(1 - \frac{\omega^4}{1})}{54\mu^3}\right) = 0.
\]  (4.5)

This expression is further simplified by defining

\[
p = \frac{3\mu + \lambda^2}{9\mu^2} \quad \text{and} \quad q = -\frac{2\lambda^3 + 9\mu\lambda + 27\mu^2(1 - \frac{\omega^4}{1})}{54\mu^3}.
\]  (4.6)

We then arrive at the depressed form of the equation:

\[
x^3 - 3px - 2q = 0.
\]  (4.7)

In the following discussion, note that the \( r \)-dependence is entirely contained in the coefficient \( q \). Before proceeding, we observe that there are three distinct cases depending on the sign of the discriminant, \( D = q^2 - p^3 \), of eq. (4.7) with the following results:

1. \( q^2 - p^3 > 0 \Rightarrow 1 \) real root and 2 complex roots conjugate to one another
2. \( q^2 - p^3 < 0 \Rightarrow 3 \) unequal real roots
3. \( q^2 - p^3 = 0 \Rightarrow 3 \) real roots, at least 2 of which must be equal

Assuming that \( p \neq 0 \), we define

\[
\alpha = \left(q + \sqrt{q^2 - p^3}\right)^{\frac{1}{3}}, \quad \beta = \left(q - \sqrt{q^2 - p^3}\right)^{\frac{1}{3}},
\]  (4.8)

which allows the roots of eq. (4.7) to be written in the simple form using Cardano’s formula. Shifting these roots as above yields the following solutions:

\[
f_1 = \alpha + \beta - \frac{\lambda}{3\mu},
\]

\[
f_2 = -\frac{1}{2}(\alpha + \beta) + \frac{i\sqrt{3}}{2}(\alpha - \beta) - \frac{\lambda}{3\mu},
\]

\[
f_3 = -\frac{1}{2}(\alpha + \beta) - \frac{i\sqrt{3}}{2}(\alpha - \beta) - \frac{\lambda}{3\mu}.
\]  (4.9)

If \( D = q^2 - p^3 > 0 \), \( \alpha \) and \( \beta \) can be taken as real and \( x_1 \) corresponds to the single real root. In this regime, the solution is then given by \( f = x_1 - \frac{\lambda}{3\mu} \). If \( D = q^2 - p^3 < 0 \), \( \alpha \) and \( \beta \) are necessarily complex but implicitly eq. (4.9) still yields three unequal real roots. In this case, \( \alpha \) and \( \beta \) can be chosen to have conjugate phases, i.e.,

\[
\alpha = \sqrt{p} \ e^{i\theta/3} \quad \text{and} \quad \beta = \sqrt{p} \ e^{-i\theta/3}
\]  (4.10)
where \( \cos \theta = q/p^2 \) and \( \sin \theta = \sqrt{p^3 - q^2/p^2} \). Here, we are using the fact that \( p \) is always positive in this domain. The solutions may then be cast in the explicitly real but implicit form:

\[
\begin{align*}
    f_1 &= 2\sqrt{p} \cos \frac{\theta}{3} - \frac{\lambda}{3\mu}, \\
    f_2 &= -\sqrt{p} \left( \cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) - \frac{\lambda}{3\mu}, \\
    f_3 &= -\sqrt{p} \left( \cos \frac{\theta}{3} - \sqrt{3} \sin \frac{\theta}{3} \right) - \frac{\lambda}{3\mu}.
\end{align*}
\] (4.11)

While the precise form of \( f(r) \) is determined by eqs. (4.9) and (4.11), these results offer little insight into the physical properties of the corresponding solutions, e.g., which solutions actually correspond to black holes. However, we will see below that much of the physics can be inferred directly from the cubic equation (4.4).

At this point, it is convenient to consider the AdS vacuum solutions. As discussed in section 2, the latter can be found by setting \( f(r) \) to be constant (i.e., setting \( \omega = 0 \)) or alternatively taking the limit \( r \to \infty \). Setting \( \omega = 0 \) and \( f(r) = f_\infty \) in eq. (4.4) yields:

\[
h(f_\infty) \equiv 1 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0.
\] (4.12)

With the choice \( N^2 = 1/f_\infty \), the five-dimensional metric (2.3) becomes

\[
    ds^2 = \frac{r^2}{L^2} \left( -dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{L^2}{f_\infty} \frac{dr^2}{r^2},
\] (4.13)

which corresponds to the metric for AdS\(_5\) in Poincaré coordinates. Further from the \( g_{rr} \) component, we see that the radius of curvature of the AdS\(_5\) spacetime is

\[
    \tilde{L}^2 = L^2/f_\infty.
\] (4.14)

Implicitly, we have assumed \( f_\infty > 0 \), which is not always the case — see discussion below.

Now let us examine these solutions as functions of the couplings, i.e., in the \( \mu - \lambda \) plane. In particular, if we insert the asymptotic limit \( r \to \infty \), the discriminant of eq. (4.7) is useful in determining the number of vacuum solutions at different points in the parameter space. As seen in eq. (4.6), \( p \) is unchanged in this limit since it is independent of \( r \) but \( q \) is slightly simplified:

\[
p = \frac{3\mu + \lambda^2}{9\mu^2} \quad \quad q_\infty = -\frac{2\lambda^3 + 9\mu\lambda + 27\mu^2}{54\mu^3}.
\] (4.15)

Given the previous discussion, we see that eq. (4.12) yields 3 real solutions for \( \mathcal{D}_\infty = q_\infty^2 - p^3 \leq 0 \) and 1 real solution for \( \mathcal{D}_\infty > 0 \). The vanishing of the discriminant, \( \mathcal{D}_\infty = 0 \), reduces to a quadratic equation for \( \mu \) with solutions

\[
    \mu = \frac{2}{27} - \frac{\lambda}{3} \pm \frac{2}{27} (1 - 3\lambda)^{3/2}.
\] (4.16)

Eq. (4.16) generates the two (upper) curves in the \( \mu-\lambda \) plane shown in figures 1 and 2. In the region bounded by these two curves, \( \mathcal{D}_\infty < 0 \) and there are three vacuum solutions.
Figure 1: The red and blue curves indicate the positive and negative branches of eq. (4.16), respectively, where $D = 0$. The region bounded by these two curves is where $D < 0$. The green curve indicates $p = 0$. The letter labels refer to the various regions described in table 1. The blue shaded region indicates those couplings for which there exist asymptotically AdS black holes.

while outside of these two curves, $D_\infty > 0$ with one vacuum solution — as long as $\mu \neq 0$. Of course, $\mu = 0$ is a special axis in the parameter space corresponding to the Gauss-Bonnet theory discussed in section 2. Note that the positive branch of eq. (4.16) crosses the $\lambda$-axis at $\lambda = 1/4$, which is precisely the critical coupling in the GB gravity. Recall that in GB gravity for $\lambda < 1/4$, there are two vacua while no vacuum solutions exist for $\lambda > 1/4$.

Below we will find that another interesting boundary in the $\mu$-$\lambda$ plane is $p = 0$, which corresponds to the lowermost (green) curve in figures 1 and 2 - note that $p = 0$ is always below both branches of $D = 0$, except at the points $(\lambda, \mu) = (0, 0)$ and $(\frac{1}{3}, -\frac{1}{27})$. From eq. (4.15), we see that $p = 0$ simply corresponds to $\mu = -\frac{\lambda^2}{3}$. A distinguishing feature of $p = 0$ is that eq. (4.5) becomes a perfect cubic equation.

As a point of clarification, we should note that when eq. (4.12) yields real roots, the value of $f_\infty$ may be either positive (as assumed above) or negative. For example, it is easy to see that if $\mu > 0$ one of the roots must be negative (since $h(f_\infty \to -\infty) \simeq \mu f_\infty^3 < 0$ while $h(f_\infty = 0) = 1 > 0$). Consistency demands $N^2 > 0$ and so we can not use the same choice for the lapse when $f_\infty < 0$. If instead we choose $N^2 = 1/|f_\infty|$, the metric (2.3) becomes

$$ds^2 = \frac{r^2}{L^2} \left( dt^2 + dx^2 + dy^2 + dz^2 \right) - \frac{L^2}{|f_\infty|} \frac{dr^2}{r^2}, \quad (4.17)$$

which corresponds to a particular set of coordinates on five-dimensional de Sitter space, where we observe that $r$ plays the role of time. The radius of curvature of this $dS_5$ spacetime is $\tilde{L}^2 = L^2/|f_\infty|$. Even though with $f_\infty < 0$, eqs. (4.9) or (4.11) still yield nontrivial solutions $f(r)$, however, the corresponding metrics should be interpreted as (singular)
cosmological solutions rather than black holes. Hence we will not consider solutions with $f_\infty < 0$ further here.

We summarize our results in table 1, which enumerates the various kinds of vacuum solutions in different regimes of the parameter space. Note that in this table, we have categorized the AdS vacua as either stable or ‘ghosty,’ i.e., whether or not the graviton is a ghost in a particular AdS vacuum. Recall that for GB gravity, we discarded one branch of the solutions in eq. (2.6) because the analysis of [16] showed that the graviton was a ghost in these backgrounds. A similar analysis applies for quasi-topological gravity, as we will see in section 6. The key factor distinguishing this feature of the various vacua is the slope of the cubic equation determining $f_\infty$:

$$h'(f_\infty) \equiv -1 + 2\lambda f_\infty + 3\mu f_\infty^2.$$  \hspace{1cm} (4.18)

In section 6, we show that this expression appears as a pre-factor in the kinetic term for gravitons propagating in a given AdS vacuum. The kinetic term has the usual sign when $h'(f_\infty) < 0$ and the wrong sign when $h'(f_\infty) > 0$. Hence the stable or ghost-free AdS vacua in table 1 are distinguished by having $h'(f_\infty) > 0$. Given that this factor is simply the slope of $h(f_\infty)$, it is easy to see that since $h(f_\infty = 0) = 1$ then if there is one AdS vacuum (i.e., one root with $f_\infty > 0$), it will be ghost-free. Similarly if there is more than one AdS vacuum, one of these will contain ghosts.

To gain some further insight into the black hole solutions, we return to eq. (4.4) which we re-write as

$$\tilde{h}(f) \equiv \left(1 - \frac{\omega^4}{r^4}\right) - f + \lambda f^2 + \mu f^3 = 0.$$  \hspace{1cm} (4.19)
Now in the asymptotic limit $r \to \infty$, we simply recover eq. (4.12) and the roots match the vacuum solutions $f_\infty$. We can regard the effect of $r$ decreasing through finite values as reducing the ‘constant’ term in the cubic polynomial of $f$ and as a result, the roots of $\tilde{h}(f)$ shift away from $f_\infty$.

To illustrate various possibilities, figure 3 plots an example of $\tilde{h}(f)$ in case (g) with three AdS vacua – see table 1. First we consider the smallest root $f_2$, which asymptotically reaches the AdS vacuum solution with the smallest value of $f_\infty$. As shown when $r$ decreases, this root decreases moving monotonically to the left until it reaches $f_2 = 0$ at $r = \omega$. As $r$ shrinks to even smaller values, $f$ becomes negative and the solution becomes singular with $f_2 \to -\infty$ as $r \to 0$. Of course, this behaviour is precisely that of a black hole with a horizon at $r = \omega$.

Next consider the second root $f_3$ in figure 3. In this case, the slope $\tilde{h}'(f)$ is positive and so the corresponding AdS vacuum contains ghosts. Note that as $r$ decreases (or $\omega^4/r^4$ increases), the root now moves to the right, i.e., $f_3$ grows as we move to the interior of the solution. This behaviour is problematic as it corresponds to a negative mass solution and it seems to be connected to the ghost problems. As we discuss below, the solution reaches a naked singularity at $r = r_0 (= 1.150\omega$, in this particular example) where the two roots, $f_1$ and $f_3$, coalesce. One could overcome the problem with negative masses by simply choosing the integration constant $\omega^4 < 0$. In this case, $f_3$ moves to the left with decreasing $r$ but a naked singularity is still produced when the roots $f_2$ and $f_3$ coalesce.

Finally we turn to $f_1$, the largest of the three roots in figure 3. Here again, the root moves to the left as $r$ decreases so that $f_1$ decreases as we move to the interior of the geometry indicating a positive mass. However, as noted above, this root coalesces with $f_3$ at $r = r_0$ and becomes complex for smaller values of $r$. Defining $f_1(r_0) = f_0$ and Taylor expanding eq. (4.19) about this point, we find

$$f_1(r) \simeq f_0 + \frac{2}{\gamma} \frac{\omega^2}{r_0^3} \left(\frac{r - r_0}{r_0}\right)^{1/2} \quad \text{where} \quad \gamma^2 = -\frac{1}{2} \tilde{h}''(f_0) = \frac{3}{2} |\mu| f_0 - \lambda. \quad (4.20)$$

| $D_\infty$ | $\mu$ | $\lambda$ | Stable AdS | Ghosty AdS | dS | BH solution |
|----------|-------|------|-----------|-----------|----|----------|
| a        | +     | +    | 0         | 0         | 1  | -        |
| b        | +     | -    | 0         | 0         | 1  | -        |
| c        | +     | -    | 1         | 0         | 0  | $f_1$ (in $c_0,c_1$) |
| d        | +     | -    | 1         | 0         | 0  | $f_1$ |
| e        | -     | +    | 1         | 1         | 1  | $f_3$ |
| f        | -     | +    | 1         | 1         | 1  | $f_3$ |
| g        | -     | +    | 2         | 1         | 0  | $f_2$ |
| h        | -     | -    | 1         | 0         | 2  | $f_1$ |

Table 1: Table of various vacua and black hole solutions. The column labeled ‘BH solution’ indicates which root (4.9) yields the nonsingular black hole solution. In case (c), the black hole solution is only realized in the regions denoted ($c_0$) and ($c_1$) in figure 2.
Figure 3: Graph of \( \tilde{h}(f_\infty) \) for \( \mu = -0.005 \) and \( \lambda = 0.145 \). The curve shifts down as \( r \) decreases. Here, the three curves correspond to \( \omega^4/r^4 = 0, 1 \) and 1.752 from the top to bottom. The slope is negative for the roots, \( f_1 \) and \( f_2 \), indicating that these are stable solutions while it is negative for \( f_3 \) indicating the graviton is a ghost in this background. At \( \omega^4/r^4 = 1.752 \), the roots \( f_1 \) and \( f_3 \) coalesce and a curvature singularity appears in both solutions.

Further calculating the curvature using this result yields

\[
R_{abcd}R^{abcd} \propto \frac{\omega^4}{r_0L^4} \frac{1}{(r - r_0)^3}
\]

showing that the spacetime has a naked singularity at this point. One could again examine these solutions with \( \omega^4 < 0 \). However, in this case, \( f_1 \) moves to the right, indicating a negative mass, and a naked singularity arises with \( f_1 \rightarrow +\infty \) as \( r \rightarrow 0 \).

This discussion shows that in case (g) from table 1, only the solution \( f_2 \) corresponds to an asymptotically AdS black hole. The other roots, \( f_1 \) and \( f_3 \), are both asymptotically AdS but produce spacetimes with naked singularities. Examining the other cases in the table in a similar way, one finds that in each parameter regime with an AdS vacuum, there is a single black hole solution corresponding to the smallest positive root of eq. (4.19). The only exception is case (c) where we must also be in the regions denoted (c_0) or (c_1). These restrictions are related to the possibility that a naked singularity will arise if the function \( \tilde{h}(f) \) is not monotonic in the range \( f \in [0, f_\infty] \), as explained for the root \( f_1 \) in the example above – see also figure 4. First of all, in the region (c_0), \( p \) is negative and \( \tilde{h}(f) \) has no extrema at all. In regions (c_1) and (c_2), \( p > 0 \) and so one must examine the extrema \( f_0 \) of \( \tilde{h}(f) \). In region (c_1) to the left of (g) where \( D_\infty < 0 \), both of the extrema \( f_0 > f_\infty \) and so \( \tilde{h}(f) \) is monotonic in the desired range. On the other hand, one finds \( 0 < f < f_\infty \) in region (c_2) to the right of (g). Therefore in this parameter regime, the solution develops a naked singularity when \( r \) reaches the value where \( f = f_0 \). However, the solution corresponds to a black hole with smooth event horizon for parameters in the regions (c_0) and (c_1). All of
Figure 4: The function \( \tilde{h}(f) \) plotted for \( \mu = -0.045 \) and \( \lambda = 0.4 \), typical parameters in the region \((c_2)\). The single (real) root \( f_1 \) corresponds to a ghost-free asymptotically AdS solution. However, when the radius reaches a value where \( \tilde{h}'(f) = 0 \) at the root, indicated with the black dot, the geometry becomes singular.

Our results with regards to which root yields a black hole solution are summarized in table 1.

4.2 Curved horizons

As in eq. (2.16), we can again generalize the metric ansatz to include spherical and hyperbolic, as well as planar, horizons:

\[
d s^2 = - \left( k + \frac{r^2}{L^2} f(r) \right) N(r)^2 d^2 t + \frac{d^2 r}{k + \frac{r^2}{L^2} f(r)} + r^2 d\ell_k^2
\]

where \( d\ell_k^2 \) is given by

\[
\begin{align*}
  k = +1 : & \quad d\Omega_3^2, \\
  k = 0 : & \quad \frac{1}{L^2} \left( dx^2 + dy^2 + dz^2 \right), \\
  k = -1 : & \quad d\Sigma_3^2.
\end{align*}
\]

As in eq. (2.17), we have above the metric on a unit three-sphere for \( k = +1 \) and on a three-dimensional hyperbolic plane with unit curvature for \( k = -1 \). The analysis at the beginning of section 4.1 follows through unchanged. Implicitly, we will assume \( N(r)^2 = 1/f_\infty \) but more importantly, the solutions are again determined by eq. (4.4):

\[
1 - f + \lambda f^2 + \mu f^3 = \frac{\omega^4}{r^4}.
\]
Hence one arrives at the same solutions for \( f \) as previously found. The difference between planar and curved horizons is that the usual horizon equation \( g_{tt} = 0 \) now becomes \( f = -kL^2/\omega^2 \). We have not made a complete analysis of the structure of the new spacetimes throughout \( \mu-\lambda \) plane but let us make the following preliminary remarks.

We can develop a qualitative picture of the solutions using the same graphical approach as in the previous section. However, as well as following the behaviour of the \( \tilde{h}(f) \), we must now also keep track of the critical value of \( f \) where a horizon can form when \( k \neq 0 \), i.e., \( f_h = -kL^2/r^2 \). While the mass parameter \( \omega^4 \) controls how quickly \( \tilde{h}(f) \) (and its roots) are shifting as \( r \) varies, the rate of change in \( f_h \) is controlled by the cosmological constant scale \( L \) while the direction is controlled by \( k \). Our first observation then is that if we have a black hole solution with \( k = 0 \), then for large masses \( \omega \gg L \), we will always find the new (curved) horizon equation with \( r_h \sim \omega \). That is, in this regime, the relevant root of \( \tilde{h}(f) \) moves much more quickly as \( r \) decreases than \( f_h \). Hence the root reaches \( f = 0 \) at \( r = \omega \) (as discussed for the planar horizons) while \( |f_h| = L^2/\omega^2 \ll 1 \) and so there should be a nearby solution for the horizon condition. Therefore we expect that there are smooth black hole solutions with spherical or hyperbolic horizons in all of the same regions of the \( \mu-\lambda \) plane where they were found for planar horizons, in the previous section. However, in these regions, one may find that there is a lower bound on the mass of these black hole solutions different from \( \omega^4 = 0 \).

Let us consider small masses first for spherical horizons with \( k = +1 \) and \( f_h = -L^2/r^2 \). As \( r \) decreases, the smallest positive root of eq. (4.19) moves to the left, approaching \( f = 0 \) as \( r \to \omega \). At the same time, \( f_h \) starts at zero at \( r = \infty \) and then moves to the left to negative values as \( r \) decreases. Hence to form a horizon, the root must ‘catch up’ to \( f_h \). If we tune \( \omega^4 \) to smaller and smaller values, slowing down the rate at which the root moves, it becomes clear that it may never coincide with \( f_h \). For example, consider cases (d) with \( p > 0 \), (e), (f) and (h) in table 1. In each of these cases, there will be a value \( f_0 < 0 \) for which \( \tilde{h}'(f_0) = 0 \), where the spacetime develops a singularity as described in section 4.1 – note that this previous discussion does not change for \( k \neq 0 \). Hence if \( f_h(\omega) = -L^2/\omega^2 \leq f_0 \), then the root will not be able to reach \( f_h \) before hitting \( f_0 \). Hence in this situation, the spacetime will contain a naked singularity. In cases (c), (d) with \( p < 0 \) and (g), \( \tilde{h}(f) \) is monotonic for negative values of \( f \). However, now for small \( r \) (i.e., \( r \ll \omega, L \)) the root behaves as \( f \simeq -[\omega^4/(|\mu| r^4)]^{1/3} \). Hence the root is growing much more slowly than \( f_h \propto r^{-2} \) in this regime and again it becomes apparent that the root will never catch up to \( f_h \). Hence the solution will again have a naked singularity as \( r \to 0 \). Thus our final conclusion is that for spherical horizons with \( k = +1 \) there will always be a lower (positive) bound on the mass parameter below which no black hole solutions exist. This is, of course, qualitatively, the same result as found for spherical black holes for Einstein gravity with a negative cosmological constant [25]. We have not calculated the exact value of the lower bound but one must find \( \omega \simeq L \), with the precise proportionality constant determined by the gravitational couplings, \( \lambda \) and \( \mu \).

We might add that since \( f_h < 0 \) with \( k = +1 \), the discussion given in the previous section remains unchanged for the solutions asymptotic to the ghosty vacua, the extra stable AdS vacuum in case (g) and the AdS vacuum in case (c2). That is, there will be no
spherical black hole solutions in any of these cases.

Now let us turn to considering small masses for hyperbolic horizons with \( k = -1 \) and \( f_h = L^2/r^2 \). In this case, \( f_h \) moves to the right to positive values as \( r \) decreases and so more exotic possibilities arise. For example, even if \( \omega = 0 \) and the root does not move, \( f_h \) will increase and eventually coincide with the root. That is, a horizon will appear even if the mass is set to zero and by continuity, we must also find black holes with small negative masses. Note that with \( \omega^4 \), the root moves to the right to values larger than the initial \( f_\infty \). Here, cases (e), (f) and (g), as well as the region denoted (c1), are distinguished because \( \tilde{h}(f) \) is not monotonic for positive values of \( f > f_\infty \). Hence the negative mass solutions will only contain a horizon if \( f_h \) catches up to the root before reaching the point where \( \tilde{h}' = 0 \). This will set a (negative) lower bound on the mass with \( \omega^4 \sim -L^4 \) where again the precise bound will be determined by \( \lambda \) and \( \mu \). Cases (c0), (d) and (h) with \( k = -1 \) are even more striking. For these cases, \( \tilde{h}(f) \) is monotonic for \( f > f_\infty \) and so a singularity only develops as \( r \to 0 \) and \( f \to \infty \). However, in this regime, the root grows as \( f \simeq [\omega^4/|\mu| r^4]^{1/3} \) while \( f_h = L^2/r^2 \). Hence the solution will also contain a horizon with \( r_h^2 \simeq |\mu| L^6/|\omega^4| \) for arbitrary negative values of \( \omega^4 \). Hence we have found that in any of the parameter regimes where planar black holes exist, there will be black holes with hyperbolic horizons with negative masses. In cases (c1), (e), (f) and (g), there is a lower bound on how negative the mass can become but in cases (c0), (d) and (h), the hyperbolic black holes can have an arbitrarily large negative mass. These results are not entirely surprising given that hyperbolic black holes with negative masses exist for for Einstein gravity with a negative cosmological constant [26], although there is a lower bound on the mass there.

We might briefly also consider the effect of setting \( k = -1 \) in the cases where no planar black hole solutions could be found, \( i.e., \), the solutions asymptotic to the ghosty vacua, the extra stable AdS vacuum in case (g) and the AdS vacuum in case (c2). It is clear that even in these cases a smooth horizon forms with \( \omega = 0 \) since \( f_h = L^2/r^2 \) moves to positive values as \( r \) decreases. Again by continuity, black hole solutions will also exist for small positive and negative values of \( \omega^4 \). As above, the largest root will grow as \( f \simeq [\omega^4/(\mu r^4)]^{1/3} \) in a regime where \( r \to 0 \). Since the critical value \( f_h \) grows more quickly, we conclude that the solutions which asymptote to the ghosty vacua in cases (e) and (f) (with \( \mu > 0 \)) will form a horizon for arbitrarily large positive values of \( \omega^4 \). Similarly, the solutions which asymptote to the second stable AdS vacuum in case (g) or to the AdS vacuum in case (c2) will form hyperbolic horizons for arbitrarily large negative masses. Hence we find that exotic hyperbolic black holes also exist in parameter regimes even where no planar black holes formed.

4.3 Higher dimensions

With the construction described in section 3.1, we extended quasi-topological gravity to higher dimensions \( D \geq 7 \) – recall that our new curvature-cubed interaction did not affect the equations of motion in \( D = 6 \). The general action for \( D \geq 7 \) (as well as \( D = 5 \)) is:

\[
I = \frac{1}{16\pi G_D} \int d^Dx \sqrt{-g} \left\{ \frac{(D-1)(D-2)}{L^2} R + \frac{\lambda L^2}{(D-3)(D-4)} X_4 \right\}
\]
\[
\left( -\frac{8(2D - 3)}{(D - 6)(D - 3)(3D^2 - 15D + 16)} \mu L^4 Z_D \right)
\]  
(4.25)

where \(X_4\) and \(Z_D\) are given in eqs. (2.1) and (3.10), respectively. We choose the metric ansatz in eq. (2.16), which can describe \(D\)-dimensional black holes with planar, spherical or hyperbolic horizons (for \(k = 0, +1\) and \(-1\), respectively). The coefficients of the action (4.25) are chosen so that substituting in this metric (2.16) yields

\[
I = \frac{1}{16\pi G_D} \int d^Dx \frac{(D - 2) N(r)}{L^D} \left[ r^{D-1} \left( 1 - f + \lambda f^2 + \mu f^3 \right) \right]' .
\]  
(4.26)

The lapse must again be constant and a convenient choice is \(N(r) = 1/f_\infty\) as described in section 2 – with the implicit assumption that \(f_\infty > 0\). If we wish to consider the vacuum solutions, \(f(r)\) is also fixed to be constant with \(f(r) = f_\infty\) where these solutions are again determined by eq. (4.12). Hence the various vacuum solutions are again distributed as described in table 1. Assuming \(f_\infty > 0\), the \(D\)-dimensional metric becomes

\[
ds^2 = - \left( k + \frac{r^2}{L^2 f_\infty} \right) \frac{dt^2}{f_\infty} + \frac{dr^2}{k + \frac{r^2}{L^2 f_\infty}} + r^2 d\ell^2 .
\]  
(4.27)

where \(d\ell^2\) is given in eq. (2.17). These solutions correspond to a spherical \((k = +1)\), flat \((k = 0)\) or hyperbolic \((k = -1)\) foliation of \(AdS_D\). From the \(g_{rr}\) component, we see that the radius of curvature of the \(AdS_D\) spacetime is

\[
\tilde{L}^2 = L^2/f_\infty .
\]  
(4.28)

In considering the black hole solutions, the only difference from the analysis for \(D = 5\) in the previous sections is that we replace: \(\omega^4/r^4 \rightarrow \omega^{D-1}/r^{D-1}\). In particular, the latter substitution is made in \(q\) in eq. (4.6). Further, eq. (4.19) is replaced with

\[
\tilde{h}(f) \equiv \left( 1 - \frac{\omega^{D-1}}{r^{D-1}} \right) - f + \lambda f^2 + \mu f^3 = 0 .
\]  
(4.29)

The remainder of the analysis and the results in section 4.1 carries over unaltered. In particular, table 1 correctly describes the planar black hole solutions for \(D \geq 7\). Similarly, the discussion of black holes with curved horizons in section 4.2 remains largely unchanged. In certain cases, the discussion for small masses referred to the behaviour of the root, now, of eq. (4.29) as \(r\) approaches zero. In the present case, this behaviour changes to \(f \simeq \left( \omega^{D-1}/(\mu r^{D-1}) \right)^{1/3}\) while the behaviour of the critical value remains \(f_h = L^2/r^2\). Hence the root is grows more slowly than \(f_h\) for \(D = 5\), at the same rate for \(D = 7\) and more quickly for \(D \geq 8\). As a result, one finds, in cases (c), (d) with \(p < 0\) and (g), that there is a positive lower bound for the mass of spherical black holes for \(D = 5\) and \(7\) but the lower bound is simply \(\omega^{D-1} = 0\) for \(D \geq 8\). Further, the conclusion that hyperbolic black holes exits in cases (c), (d) and (h) with arbitrarily large negative masses only applies for \(D = 5\). Similarly, some of the details about the formation of hyperbolic horizons, in regions of the coupling space where planar black holes do not exist, change depending the spacetime dimension \(D\).
As noted above, this discussion applies for $D \geq 7$, but in those dimensions, one already has a cubic order theory in Lovelock gravity, which also reproduces eq. (4.29). Hence for $D \geq 7$, the black holes for quasi-topological gravity discussed here would be the same as those in cubic Lovelock theory, as considered in [8, 27].

5. Black Hole Thermodynamics

We now turn to the thermodynamic properties of the black hole solutions of quasi-topological gravity. Our focus will be on the planar ($k = 0$) black holes in five dimensions. The extension of these results to curved horizons and higher dimensions is straightforward.

First, we use the standard approach to calculate temperature: analytically continue the metric to Euclidean signature with $\tau = -it$ and periodically identify $\tau$ to produce an everywhere smooth Euclidean section. Interpreting the period of $\tau$ as the inverse temperature, we find

$$T = \frac{1}{4\pi} \frac{r_h^2 f'(r_h)}{L^2 \sqrt{f}} \left[ \frac{2k}{r_h} \right],$$

which assumes the lapse is chosen as $N = 1/\sqrt{f}$. For the planar black holes, $r_h = \omega$ and further we will use $f(r = \omega) = 0$. Then we can evaluate the $f'(r_h)$ by differentiating the constraint equation (4.4) and evaluating the result at $r = \omega$. A simple calculation yields $f'(\omega) = \frac{4}{\omega}$, giving:

$$T = \frac{\omega}{\pi L^2 \sqrt{f}}.$$

In passing we note that extending this calculation to general dimensions and curved horizons yields

$$T = \frac{1}{4\pi \sqrt{f}} \left[ \frac{r_h^2 f'(r_h)}{L^2 \sqrt{f}} - \frac{2k}{r_h} \right] = \frac{D - 1}{4\pi \sqrt{f}} \left[ \frac{\omega^{D-1}}{L^2 r_h^{D-2}} \frac{r_h^4}{r_h^4 + 2\lambda k L^2 r_h^2 - 3\mu k^2 L^4} - \frac{2k}{D - 1} \frac{1}{r_h} \right],$$

where the precise location of the horizon $r_h$ must still be determined for a spherical or hyperbolic horizon but, of course, $r_h = \omega$ in the planar case ($k = 0$).

Next, we calculate the entropy and energy densities of the black holes following the Euclidean action approach, as already sketched for GB theory in section 2. That is, we identify the Euclidean action for the black hole solution, as the leading contribution to the free energy, i.e., $I_E \simeq F/T$. Evaluating the Euclidean action yields

$$I_E[T] = -\frac{1}{16\pi G_5} \int_0^{1/T} d\tau \int_0^{R_0} dr \int d^3 x \sqrt{g_E} \left( \frac{12}{L^2} + R + \frac{\lambda L^2}{2} \chi_4 + \frac{7\mu L^4}{4} \right) + r^4 (3 - 5f(r) + 15\lambda f(r)^2 - 15\mu f(r)^3)$$

$$+ r^5 (-1 + 6\lambda f(r) - 9\mu f(r)^2) f'(r)|_{\omega}^{R_0}.$$
Here \( V_3 = \int d^3x \) is the regulator volume for the (spatial) gauge theory directions and as usual, we have set \( N = 1/\sqrt{f_\infty} \). The final result is simplified using the constraint (4.4) to produce the following asymptotic expansion of \( f(r) \):

\[
 f \simeq f_\infty - \frac{\omega^4}{r^4} \left( 1 - 2\lambda f_\infty - 3\mu f_\infty^2 \right) + \cdots .
\]  

(5.5)

Keeping only the divergent and finite terms in the limit \( R_0 \to \infty \), eq. (5.4) then reduces to:

\[
 I_E[T] = \frac{V_3}{8\pi G_5} \frac{\omega^4}{T L^5 \sqrt{f_\infty}} \left[ R_0^4 f_\infty \left( 1 - 6\lambda f_\infty + 9\mu f_\infty^2 \right) - \frac{1 - 4\lambda f_\infty + 3\mu f_\infty^2}{1 - 2\lambda f_\infty - 3\mu f_\infty^2} \right].
\]  

(5.6)

We remove the divergence in this expression by subtracting the action for the AdS vacuum

\[
 I_0'[T'] = \frac{V_3}{8\pi G_5} \frac{R_0^4}{T'L^5 \sqrt{f_\infty}} f_\infty \left( 1 - 6\lambda f_\infty + 9\mu f_\infty^2 \right),
\]  

(5.7)

where \( T' \) is chosen so that the periodicity of the AdS background matches that of the black hole at the regulator surface \( r = R_0 \):

\[
 \frac{1}{T'} = \frac{1}{T} \frac{\sqrt{f(R_0)}}{\sqrt{f_\infty}} \simeq \frac{1}{T} \left( 1 - \frac{\omega^4}{2R_0^4 f_\infty \left( 1 - 2\lambda f_\infty - 3\mu f_\infty^2 \right)} \right).
\]  

(5.8)

Combining these expressions yields a simple expression for the free energy:

\[
 F[T] = T \left( I_E[T] - I_0'[T] \right) = -\frac{V_3 \omega^4}{16\pi G_5 L^5 \sqrt{f_\infty}} = -\frac{(\pi \sqrt{f_\infty} L)^3}{16 G_5} T^4
\]  

(5.9)

where we have implicitly taken the limit \( R_0 \to \infty \) above. As noted in section 2, for planar AdS black holes, we do not have to account for the generalized Gibbons-Hawking term [20] or the boundary counter-terms [21] in this calculation of the free energy. Finally, to eliminate the regulator volume, we work with the free energy density: \( \mathcal{F} = F/V_3 \). Then we calculate the energy and entropy densities as

\[
 \rho = -T^2 \frac{d}{dT} \left( \mathcal{F}/T \right) = \frac{3(\pi \sqrt{f_\infty} L)^3 T^4}{16 G_5},
\]  

(5.10)

\[
 s = -\frac{d\mathcal{F}}{dT} = \frac{(\pi \sqrt{f_\infty} L T)^3}{4 G_5 L^3} = \frac{1}{4 G_5 L^3} \omega^3.
\]  

(5.11)

Note that these expressions are ‘identical’ to those appearing in eq. (2.14) for GB gravity, however, \( f_\infty \) implicitly depends on the additional coupling \( \mu \). We also comment that these results obey the relation \( \rho = \frac{3}{4} s \), as expected for a four-dimensional CFT at finite temperature.

### 5.1 Noether Charge Approach to Entropy Density

In this section, we verify that the entropy density in eq. (5.11) matches the Wald entropy [22]. Of course, we find agreement [28] but the results for the Wald entropy are also readily
extended to higher dimensions and curved horizons. Wald’s prescription for the black hole entropy is

\[ S = -2\pi \oint d^{D-2}x \sqrt{h} \, Y^{abcd} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} \quad \text{where} \quad Y^{abcd} = \frac{\partial L}{\partial R_{abcd}} \]  

(5.12)

\( \mathcal{L} \) is the Lagrangian and \( \hat{\varepsilon}_{ab} \) is the binormal to the horizon. For the static black holes considered here, \( Y = Y^{abcd} \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} \) is constant on the horizon and so the entropy is given simply as

\[ S = -2\pi Y A, \]  

(5.13)

where \( A = \oint d^{D-2}x \sqrt{h} \) is the ‘area’ of the horizon. Let us divide up the terms in the action (4.25) according to their powers of the curvature tensor: the Einstein term, the GB interaction and the quasi-topological interaction. (Of course, one also has the cosmological constant term but it does not contribute in Wald’s formula (5.12).) Following the above prescription, as usual, the Einstein term yields

\[ Y_1 = -\frac{1}{8\pi G_D} \]  

and the resulting entropy is the expected Bekenstein-Hawking entropy

\[ S = \frac{A}{4G_D}. \]  

(5.13)

Applying this formalism to GB terms, we find

\[ Y_2 = -\frac{1}{4\pi G_D} \frac{\lambda L^2}{(D-3)(D-4)} (R - 2(R_t^t + R_r^r) + 2R_{tr}^{tr}), \]  

(5.14)

where this expression applies for a general static black hole metric. One can think that the tensor components above are presented in an orthonormal frame or alternatively in a coordinate frame, as long as the indices are in precisely the raised and lowered positions as shown. Integrating this over the horizon gives the following contribution to the entropy as

\[ S_2 = \frac{A}{2G_D} \frac{\lambda L^2}{(D-3)(D-4)} (R - 2(R_t^t + R_r^r) + 2R_{tr}^{tr}) \]  

\[ = \frac{A}{4G_D} \frac{D-2}{D-4} (-2\lambda f(r_h)), \]  

(5.15)

where we are using the general metric (2.16) with \( N^2 = 1/f_\infty \) to evaluate the second line. Finally turning to the quasi-topological contribution in the action, we find

\[ Y_3 = \frac{1}{4\pi G_D} \frac{8(2D-3)}{(D-6)(D-3)(3D^2 - 15D + 16)} \left[ \frac{3c_1}{2} (R_{tm}^r R_{rn}^m - R_{rm}^t R_{mt}^n) \right. \]  

\[ + 3c_4 R_{tr}^{mn} R_{trmn} + c_3 \left( R_{tm}^r R_{rn}^m - R_{rm}^t R_{tn}^m + \frac{1}{4} (R_{mpqr} R_{mpqr} + R_{mnpt} R_{mnpt}) \right) \]  

\[ + c_4 \left( 2R_{tr}^{tr} + \frac{1}{2} R_{mpnq} R_{mpnq} + \frac{c_5}{2} (R_t^t R_r^r - R_r^r R_t^t + R_{mrn} R_{mrn} + R_{mtm} R_{mtm}) \right. \]  

\[ + 3c_6 R_{rm}^m R_{rm} + R_{tm}^m R_{tm} + \frac{c_7}{2} (R_{mn} R_{mn} + R (R_r^r + R_t^t) + \frac{3}{2} R^2). \]  

(5.16)

We have left the coefficients arbitrary above but it is understood that they are to be fixed
as in eq. (3.9). Now integrating over the horizon yields

\[ S_3 = -\frac{A}{2G_D(D-6)(D-3)(3D^2-15D+16)} \left[ \frac{3c_1}{2} \left( R_{tm} R_{tn} R_{rn} - R_{tm} R_{tn} R_{rn} \right) \right. \\
+ 3c_2 R_{trmn} R_{trmn} + c_3 \left( R_{tm} R_{tn} - R_{tm} R_{tn} \right) + \frac{1}{4} \left( R_{mnpr} R_{mnpr} + R_{mnpq} R_{mnpq} \right) \\
+ c_4 \left( 2R R_{tr} + \frac{3}{2} R_{mnpq} R_{mnpq} \right) + \frac{c_5}{2} \left( R_{trmn} R_{trmn} + R_{tr} R_{tr} \right) \\
+ \frac{3}{4} c_6 \left( R_{m} R_{m} + R_{tr} R_{tr} \right) + \frac{c_7}{2} \left( R_{mn} R_{mn} + R_{tr} R_{tr} \right) + \frac{3}{2} c_8 R^2 \right] \\
= \frac{A}{4G_D} \left( \frac{D-2}{D-6} \right) \left( -3\mu f(r_h)^2 \right). \tag{5.17} \]

Combining all of these expressions, we arrive at the Wald entropy for quasi-topological gravity:

\[ S = \frac{A}{4G_D} \left( 1 + \frac{2(D-2)}{D-4} \lambda f(r_h) - \frac{3(D-2)}{D-6} \mu f(r_h)^2 \right). \tag{5.18} \]

Evaluating this expression on a planar horizon yields the simple result, \( S = A/(4G_D) \), i.e., the higher curvature contributions vanish on planar horizons. If we divide by the regulator volume, this yields the entropy density:

\[ s = \frac{S}{V_{D-2}} = \frac{\omega^{D-2}}{4G_D L^{D-2}}, \tag{5.19} \]

which, of course, agrees with the result in eq. (5.11) for \( D = 5 \). For the case of black holes with curved horizons, as in section 4.3, we find that the entropy is given by

\[ S_k = \frac{A}{4G_D} \left( 1 + \frac{2(D-2)}{D-4} \lambda k^2 \frac{L^2}{r_h^2} - \frac{3(D-2)}{D-6} \mu k^2 \frac{L^4}{r_h^4} \right). \tag{5.20} \]

\section*{6. Equations of Motion}

We have found that the equations of motion for quasi-topological gravity take an incredibly simple form with the ansatz (2.16) for a static AdS black hole – e.g., see eq. (4.26). In this section, we would like to investigate the equations of motion in greater generality to gain some further insight into this simplicity. In particular, we will see the linearized equations of motion of graviton fluctuations also exhibit a certain simplicity. We have already argued that the new cubic interactions constructed in section 3 do not have a topological origin – see appendix A. Hence this cannot be the source of the simplicity noted above.

Let us begin with the cubic-curvature interactions in eq. (3.1). First we set \( c_{11} = 0 = c_{13} \) and then find the general contribution these terms would make to the metric equations of motion.
motion:

\[ \frac{1}{\sqrt{-g}} \delta I = c_a \left( -\frac{3}{2} R^2 \right)_{ab} + 3 \left( R^2 \right)_{ab} + c_7 \left( -R_a^\beta R^\alpha_c d_{ab} - 2 R R_a^c R^b_d + (R_a^d R^c_d)_{ab} - \Box (R R_{ab}) \right) \\
+ 2 (R R_{(a)}^{(d) b})_{c} + c_a \left( -3 R_{ac} R_d^c d_{ab} - \frac{3}{2} \Box (R_{ac} R^c) + 3 (R_a^d R^c_d)_{cd} \right) \\
+ c_5 \left( -3 R_{(a} R_{b)}^d e R_{de} + 2 (R_{(a}^d e R_{ec})_{b d} - \Box (R_{acde} R_{cd}) + (R_a^d R_{(b)}^d)_{cd} \right) \\
- (R^{de} R_{ab})_{cd} + c_4 \left( -2 R_a^{de} R_{b}^{cd e} - R_{ab} R_{cdef} + (R_{cdef})_{ab} \right) \\
+ 4 (R_{ab}^{cd} R_{b}^{cd})_{cd} + c_3 \left( -2 R_{acde} R_{b}^{cd e} - R_{ab}^{de} R_{cdef} + (R_{cdef})_{ab} \right) \\
+ 2 (R_{a}^{de} R_{b}^{ef} R_{f}^{cd g} R_{g b}^c) + 6 (R_{(a}^{e d f} R_{b}^{de} R_{b}^{de} R_{f}^{c d} + 3 (R_{e}^{f d} R_{f}^{e} R_{b}^{d} R_{b}^{e} R_{c})_{cd} \right) \\
- 3 (R_{(a}^e R_{b}^{d e} R_{c}^{d e f})_{d e} + g_{ab} \left( c_8 \left( \frac{1}{2} R^2 - 3 \Box (R^2) \right) + c_7 \left( \frac{1}{2} R R_a^d R_b^e - (R R_{cd})_{cd} \right) \\
- \Box (R_{a}^d R_{b}^e) \right) + c_6 \left( \frac{1}{2} R_a^d R_b^e R_{c}^{d e} - \frac{3}{2} (R_c^d R_{de}^c)_{cd} \right) + c_5 \left( \frac{1}{2} R_{cd}^{de} R_{cdef} \right) \\
- (R_{e}^{d e} R_{b}^{d ef})_{cd} + c_4 \left( \frac{1}{2} R R_{cdef} R_{cdef} - \Box (R_{cdef} R_{cdef}) \right) + c_3 \left( \frac{1}{2} R_{cdef} R_{cdef} R_{cdef} \right) \\
- \frac{1}{2} (R_{a}^{de c} R_{b}^{de f})_{cd} + c_2 \left( \frac{1}{2} R_a^{d e f} R_{b}^{d e f} R_{c}^{d e f} R_{d}^{d e f} R_{e}^{d e f} R_{f}^{d e f} R_{g}^{d e f} R_{h}^{d e f} \right) \right). \tag{6.1} \]

Since we have eliminated the terms involving derivatives of the curvature from the action by setting \( c_{11} = 0 = c_{13} \), the above contributions contain at most terms with four derivatives of the metric, such as in \( \Box (R R_{ab}) \). This may be misleading, however, since we have made no attempt to simplify this expression using, e.g., the Bianchi identities. A useful check is to choose the coefficients above so that eq. (3.1) corresponds to the six-dimensional Euler density (3.7), i.e., choose \( c_1 = -2 c_2, c_2 = 4 \) and the remaining coefficients as in eq. (3.3). In this case, we were able to verify that any terms involving derivatives of curvatures can be eliminated and the expected field equations involving only factors of the curvature were reproduced – e.g., see [27, 29]. Indeed Lovelock’s general discussion [14] dictates that \( X_6 \) is the only gravity Lagrangian which is cubic in curvatures for which the equations of motion do not contain any derivatives of curvatures. Hence one must expect equations of motion that include terms with derivatives of curvatures for any other choice of the coefficients \( c_i \) and, in particular, for quasi-topological gravity with \( c_i \) as in eq. (3.9). While we still made some effort to ‘tidy up’ the derivative terms in eq. (6.1), we did not produce any particularly illuminating results.

As a next step, we examine the linearized equations of motion for a graviton perturbation in quasi-topological gravity. Hence we fixed the coefficients as in eq. (3.9) and then substitute into the above expression (6.1): \( g_{ab} = g_{[0]}^{ab} + h_{ab} \) where \( g_{[0]}^{ab} \) is a solution of the full equations of motion. Again, the resulting expression of a generic fluctuation is rather complicated and so to proceed further, we restrict our attention to transverse traceless gauge with \( \nabla^a h_{ab} = 0 \) and \( h^a_a = 0 \). This choice simplifies the result somewhat and the
four-derivative contribution is proportional to:

\[
(D - 4) R^{def} \, h_{def \, (ab)} + R^{cd} \left( \Box h_{cd \, (ab)} - 2 R^{cd} \left( \Box h_{c(a) \, b} \right) d \right) + \frac{2}{D - 2} R^{cd} \left( \Box h_{ab} \right) : cd \\
+ 2 \left( \Box h^{(a)} \right) : de \ R_{(cd | e f)} + g_{ab} \left( \Box h_{cd} \right) : f + 2 R \left( \Box h_{c(a) \, b} \right) : - \frac{1}{2} g_{ab} \ R_{cd} \Box^2 h^{cd} \\
+ (D - 3) \Box^2 h^{cd} R_{c a b d} - \frac{R}{(D - 2)} \Box^2 h_{ab},
\]

(6.2)

where we use the standard notation \( T_{(ab)} = \frac{1}{2} \left( T_{ab} + T_{ba} \right) \). Hence despite the reduction in the number of terms, we see that for general backgrounds, the linearized equations of motion for (physical) gravitons include four-derivative contributions.

Of course, if we were considering gravitons propagating in flat space, these contributions would all vanish because \( R_{abcd} = 0 \) in the background spacetime. The same result would apply for any interactions which are cubic in curvatures for a flat background. However, in the present context, it is natural to consider gravitons propagating in an AdS\(_D\) background. In this case, of course, the background curvature is nonvanishing and so one might expect these terms (6.2) will still appear in the linearized equations of motion. However, further simplifications can be expected since AdS\(_D\) is a maximally symmetric spacetime with\(^1\)

\[
R_{abcd} = - \frac{1}{L^2} \left( g_{ac} g_{bd} - g_{ad} g_{bc} \right).
\]

(6.3)

Remarkably, one finds that upon substituting this background curvature into eq. (6.2), all of the remaining four-derivative terms cancel! Further, because the background curvature (6.3) is covariantly constant, there are no nontrivial terms with only three derivatives acting on the graviton. Therefore, in quasi-topological gravity, the linearized graviton equation in an AdS\(_D\) background is only a second-order equation.

With this result in hand, we next construct the linearized equation of motion for the graviton in the AdS\(_D\) vacuum solutions. Hence we must consider the full action (4.25) for quasi-topological gravity, including the cosmological constant, Einstein and Gauss-Bonnet terms as well. The equation of motion for the transverse traceless graviton can then be written as:

\[
- \frac{1}{2} \left( 1 - 2 \lambda f_\infty - 3 \mu f_\infty^2 \right) \left[ \nabla^2 h_{ab} + \frac{2 f_\infty}{L^2} h_{ab} \right] = 8 \pi G_D \tilde{T}_{ab}.
\]

(6.4)

We have added a stress-tensor on the right-hand side, as might arise from minimally coupling the metric to additional matter fields or from quadratic or higher order contributions in the graviton. The second bracketed factor on the right-hand side is the standard Einstein equation of motion for gravitons in an AdS background with curvature \( \tilde{L} = L / \sqrt{f_\infty} \) [30]. One can recognize the first bracketed factor as the slope of the cubic equation (4.12) determining \( f_\infty \), i.e., see eq. (4.18). Hence this slope determines the sign of the graviton propagator or alternatively the sign of the coupling of the graviton to the stress tensor.

We see that the appropriate sign for a well-behaved graviton is negative since this factor

\(^1\)Note that here we are distinguishing \( \tilde{L} \), the curvature scale of the AdS background, from \( L \), the AdS length scale appearing in the action. In particular, recall that in the vacuum solutions above, we found \( \tilde{L} = L / \sqrt{f_\infty} \).
reduces to $-1$ when $\lambda = 0 = \mu$. Hence as discussed in section 4.1, the AdS vacua are only stable when the slope is negative while the graviton is a ghost in AdS backgrounds where the slope is positive.

While restricting to transverse traceless gauge is a convenient simplification to determine the linearized equations in an AdS background, we can do better. A straightforward argument shows that the above gauge-fixed equations of motion (6.4) extend to the full linearized Einstein equations \[30\]:

\[
\frac{1}{2} \left( 1 - 2\lambda f_\infty - 3\mu f_\infty^2 \right) \left[ \nabla^2 h_{ab} + \nabla_a \nabla_b h^c_{c} - \nabla_a \nabla^c h_{cb} - \nabla_b \nabla^c h_{ca} \right] - g^{[0]}_{ab} \left( \nabla^2 h^c_{c} - \nabla^c \nabla^d h_{cd} \right) + \frac{2f_\infty}{L^2} h_{ab} - \frac{(D - 3)f_\infty}{L^2} g^{[0]}_{ab} h^c_{c} = 8\pi G_D \hat{T}_{ab},
\]

(6.5)

where $g^{[0]}_{ab}$ is the background AdS metric. We know that the full linearized equations come from a covariant expression and hence they must invariant under the ‘gauge’ transformations: $\delta h_{ab} = \nabla_a e_b + \nabla_b e_a$. Now the linearized Einstein equations (6.5) are certainly invariant under these transformations and reduce to eq. (6.4) upon fixing to transverse traceless gauge. However, there may be additional contributions which are both gauge invariant and completely vanish for transverse traceless modes. For example, the full equations may include an additional contribution proportional to

$$g^{[0]}_{ab} \left( \nabla^2 h^c_{c} - \nabla^c \nabla^d h_{cd} + \frac{(D - 1)f_\infty}{L^2} h^c_{c} \right),$$

(6.6)

which is gauge invariant but would not contribute in eq. (6.4). In this particular case, it is easy to argue that such a contribution could not arise from the variation of an action (quadratic in $h_{ab}$). However, given the action (4.25) for quasi-topological gravity, another approach is to evaluate the quadratic action (using Mathematica) and subsequently examining the equations of motion for some specific trial perturbations which are not transverse or traceless. In every case considered, the latter equations match precisely the results expected from eq. (6.5). This clearly shows that there are no additional terms of the form given in eq. (6.6) but more importantly that there are no additional four-derivative contributions in the full linearized equations without gauge-fixing. The fact that gravitons propagating in an AdS background simply obey the same equations of motion as in Einstein gravity plays an important role in understanding the holographic properties of quasi-topological gravity [10].

7. Discussion

In section 3, we have constructed a new gravitational action which includes terms cubic in the curvature. Our construction was motivated by the simple equations (2.5) arising to determine the black hole solutions in GB gravity. We were able to reproduce a similar structure (4.4) for our new theory. We wish to emphasize how remarkable this result is.

\[\text{We would like to thank Miguel Paulos for discussions on this point and confirming that several trial perturbations satisfied eq. (6.5) with Mathematica.}\]
In section 6, we showed that the full equations are fourth order in derivatives. However, in section 4, we found that once the geometry of the horizon is fixed, the static black hole solutions are fixed by a single integration constant! It seems that the symmetry imposed on the background geometry must play an important role in producing the simplicity of these solutions. While our approach was to substitute the ansatz into the action, it would be interesting to work directly with the equations of motion and formalize this result in terms of a ‘Birkhoff theorem.’ It would also be interesting to see to what extent this simplicity extends to spinning or electrically charged black holes in quasi-topological gravity.

We also saw that the linearized equations of motion for gravitons propagating in the AdS backgrounds reduced to the same second order equations as for Einstein gravity in section 6. There we saw more or less directly that this simplification comes about due to the maximal symmetry of the AdS spacetime. On the other hand, we should not expect that the four-derivative contributions (6.2) to the equations of motion cancel in the black hole backgrounds. Hence the quasinormal spectrum of the black hole solutions should be studied in detail. In particular, this spectrum may reveal that these solutions are unstable for certain values of the gravitational couplings.

The simplicity of the graviton equations in an AdS background has the interesting consequence that the standard holographic rules apply in matching the metric fluctuations to the stress tensor of the CFT. In a general higher derivative theory, implicitly the graviton would be matched with some higher dimension operator, as well as the stress tensor [6]. This interpretation arises because the higher derivative equations allow the metric fluctuations to have more than the standard asymptotic behaviour in the AdS geometry. In any event, this complication is evaded in quasi-topological gravity and so the effect of the higher derivative gravitational terms will only be felt in the higher $n$-point couplings of the stress tensor.

Having motivated the construction of quasi-topological gravity by considerations of the AdS/CFT correspondence, one might ask how the universality class of dual CFT’s has been expanded. The gravitational theory is defined by three independent dimensionless parameters: $\lambda, \mu$ and $L^{D-2}/G_D$. In general, the three-point function of the stress tensor of a CFT in four or higher dimensions is also characterized by three independent (dimensionless) parameters [31]. This match in the counting of these parameters is not a coincidence, as the discussion of [32] that holographically modelling the full range of these CFT parameters requires the introduction of curvature-squared and curvature-cubed interactions in the bulk gravity theory. We make precise the mapping between the gravitational couplings and the dual CFT parameters, as well as exploring other holographic aspects of quasi-topological gravity in [10, 33].

Another natural extension of this work is to consider analogous gravitational interactions with higher powers of the curvature. Of course, Lovelock gravity provides an infinite sequence of $(\text{curvature})^n$ interactions which still allow for a certain calculational control in the gravitational theory. However, because of their topological origin, these Lovelock terms only contribute to the equations of motion for $D \geq 2n + 1$, i.e., in the context of the AdS/CFT correspondence, for dual CFT’s with $d \geq 2n$. However, our expectation is that our construction of five-dimensional quasi-topological gravity with curvature-cubed
interactions can be extended general (curvature)$^n$ interactions and this has been verified by some preliminary calculations [9]. In fact, our conjecture is that in $D \geq 2n + 1$, an independent interaction can be constructed for each independent scalar contraction of (Weyl tensor)$^n$. For $D < 2n + 1$, Schouten identities will reduce the number of independent interactions, as seen in the present analysis of the curvature-cubed interactions. However, our understanding of this issue remains incomplete, as we are still uncertain as to why there was no effective curvature-cubed interaction in six dimensions. In any event, better understanding the number of independent quasi-topological gravity terms with higher powers of the curvature will be an interesting direction of study. Important new insights into this question were given in refs. [12, 13]. There it was shown that the curvature-cubed interactions constructed here can be simply expressed in terms of scalar contractions of the Weyl tensor combined with the six-dimensional Euler density. Their construction immediately generalizes to an infinite family of higher curvature interactions with similar properties.

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A. A New Topological Invariant?

In section 3.1, our construction produced an nontrivial interaction (3.10) for any number of dimensions $D \geq 5$. However, we noted $Z_6$, the six-dimensional expression, did not contribute to the equations of motion for the black hole metric (2.3) (extended to $D = 6$). In particular, this also means that the the AdS vacua are unaffected by the addition of $Z_6$ to the gravitational action. This behaviour is reminiscent of Lovelock gravity where, for example, the Euler density $\mathcal{X}_6$ provides a nontrivial gravitational interaction for $D \geq 7$ but because of its topological origin, it leaves the equations of motion unaffected for $D \leq 6$. Hence one might be tempted to believe that $Z_6$ yields another topological invariant in six dimensions. However, in the following, we demonstrate that $\int d^6x \sqrt{g} Z_6$ is not a topological invariant by explicitly evaluating this expression for certain specific six-dimensional geometries.

As our first test, we evaluate this expression on a deformed six-sphere with metric:

$$
\begin{align*}
  ds^2 &= R^2 \left[ d\theta^2 + \sin^2 \theta \left( 1 + a \sin^2 \theta \right)^n d\Omega_5^2 \right]
\end{align*}
$$

(A.1)
where \( n \) (implicitly an integer) and \( a \) are constants defining the deformation away from the round six-sphere. We then find:

\[
\int_{S^6} \sqrt{g} \ Z_6 = \frac{544}{3} \pi^3, \quad \int_{S^6} \sqrt{g} \ X_6 = 768 \pi^3.
\]  

(A.2)

where we have normalized \( X_6 \) as in eq. (3.7). Hence we see that both of these results are independent of the deformation parameters. Of course, for \( X_6 \), this occurs because the integrated expression is a topological invariant. While again suggestive for \( Z_6 \), this result is by no means conclusive and hence we consider a further test.

Next we consider the following metric in which the spheres in the direct product \( S^2 \times S^4 \) are deformed:

\[
ds^2 = R^2 \left[ d\theta^2 + \sin^2 \theta \left( 1 + a \sin^2 \theta \right)^2 d\phi^2 \right] + L^2 \left[ d\tilde{\theta}^2 + \sin^2 \tilde{\theta} \left( 1 + b \sin^2 \tilde{\theta} \right)^2 d\Omega_3^2 \right]
\]

where the deformation is characterized by the constants \( a \) and \( b \). In this case, we find:

\[
\int_{S^2 \times S^4} \sqrt{g} \ Z_6 = F(a, b, R/L), \quad \int_{S^2 \times S^4} \sqrt{g} \ X_6 = 1536 \pi^3.
\]

(A.4)

where \( F(a, b, R/L) \) is a complicated (and not particularly illuminating) function of both deformation parameters and the relative radius of curvature of the two spheres. Hence this result makes clear that \( Z_6 \) does not yield a topological invariant.

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