Vortex Solutions in the
Chern-Simons Stuekelberg Model

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1 Abstract

Vortex solutions to the classical field equations in a massive, renormalizable $U(1)$ gauge model are considered in $(2+1)$ dimensions. A vector field whose kinetic term consists of a Chern-Simons term plus a Stuekelberg mass term is coupled to a scalar field. If the classical scalar field is set equal to zero, then there are classical configurations of the vector field in which the magnetic flux is non-vanishing and finite. In contrast to the Nielsen-Olesen vortex, the magnetic field vanishes exponentially at large distances and diverges logarithmically at short distances. This divergence, although not so severe as to cause the flux to diverge, results in the Hamiltonian becoming infinite. If the classical scalar field is no longer equal to zero, then the magnetic flux is not only finite, but quantized and the asymptotic behaviour of the field is altered so that the Hamiltonian no longer suffers from a divergence due to the field configuration at the origin. Furthermore, the asymptotic behaviour at infinity is dependent on the magnitude of the Stuekelberg mass term.

2 Introduction

Static vortex solutions of the $3 + 1$ dimensional Abelian Higgs model (1) have been known for some time. The $2 + 1$ dimensional vortex solutions in models with a pure Chern-Simons kinetic term for the $U(1)$ gauge field have also been discussed (2-4) (see also (5)). This motivates us to consider vortex solutions in a $2 + 1$ dimensional model in which there is both a Chern-Simons and Stuekelberg mass term for the $U(1)$ gauge field. This theory has been shown to be a renormalizable model for a massive vector field (6). The matter field to which this vector field couples is taken to be a scalar with quadratic and quartic self-interactions.

We find that even when the scalar field vanishes, this model supports a vortex-like solution. This configuration has the interesting property that at infinity, the vector field $A_\mu(\vec{r})$ dies out exponentially fast provided the Stuekelberg mass is non-zero. Consequently the
contour integral $\oint \vec{A}(\vec{r}) \cdot d\vec{\ell}$ about a circle at spatial infinity is zero, even though the flux $\Phi = \int dS \epsilon_{ij} \partial_i A_j$ is finite and non-zero. This is due to a logarithmic singularity in $\epsilon_{ij} \partial_i A_j$ at the origin, which precludes equating these two integrals using Stokes' theorem. This same singularity at the origin results in the Hamiltonian density behaving like $r^{-2}$ as $r$ approaches zero.

In order to excise this divergence, we allow the scalar field to be non-vanishing. The flux now is quantized and the Hamiltonian free of divergences due to the singular behaviour at the origin, although the vector field exhibits logarithmic behaviour near $r = 0$. Finiteness at infinity is maintained provided the asymptotic behaviour is chosen appropriately, depending on the magnitude of the Stueckelberg mass.

3 The Model

As in (6), we consider the action

$$S = \int d^3x \left\{ \frac{1}{2} \epsilon^{\mu\alpha\nu} A_\mu \partial_\alpha A_\nu - \frac{1}{2} \mu (A_\mu + \partial_\mu \sigma)^2 
+ \frac{1}{2} \left( |(\partial_\mu + ieA_\mu)\phi|^2 + c_2 \phi^* \phi - c_4 (\phi^* \phi)^2 \right) \right\}$$

$$\left( g_{\mu\nu} = (+ + -); \quad \epsilon_{012} = +1; \quad (\mu, c_2, c_4) > 0 \right).$$

In the first instance, we set $\phi = 0$ and make the ansatz

$$A_\mu(\vec{r}) = \left( - \frac{A(r)y}{r^2}, \frac{A(r)x}{r^2}, A_0(r) \right)$$

in the gauge in which the Stueckelberg field $\sigma$ vanishes. The field equations

$$\epsilon_{ij} \partial_i A_j = \mu A_0$$  \hspace{1cm} [3a]  

$$\epsilon_{ij} \partial_i A_0 = -\mu A_i$$  \hspace{1cm} [3b]

then reduce to

$$A'(r) = \mu r A_0(r)$$  \hspace{1cm} [4a]
\[ A'_0(r) = \frac{\mu A(r)}{r} . \]  

From [4] it is easily seen that
\[ A''_0 + \frac{1}{r} A'_0 - \mu^2 A_0 = 0 \]  
whose solution is given in terms of associated Bessel functions
\[ A_0(r) = \alpha K_0(\mu r) + \beta I_0(\mu r) . \]

Setting \( \beta = 0 \) in order to ensure that \( A_0(r) \) vanishes at infinity, we see from [6] and [4b] that
\[ A(r) = -\alpha r K_1(\mu r) . \]

(The integral representation
\[ K_{\nu}(x) = \int_0^{\infty} dt \ e^{-x \cosh t} \cosh \nu t \]  
for \( K_{\nu}(x) \) is useful.) The constant of integration \( \alpha \) in [6] and [7] is not fixed by considerations related to the equations [4]. (In (2-4), the constant of integration analogous to \( \alpha \) is determined by requiring finiteness at \( r = 0 \); this is not possible here.)

The total flux of magnetic field through the two-dimensional space of our model is
\[ \Phi = \int d^2r \ \epsilon_{ij} \partial_i A_j \]  
which by [2], [3a] and [6] becomes
\[ = \mu \alpha \int_0^{2\pi} d\theta \int_0^{\infty} dr \ r \ K_0(\mu r) \]  
\[ = \frac{4\alpha \pi}{\mu} . \]

There consequently is a non-vanishing magnetic flux in this model which is finite provided \( \mu \neq 0 \). However, Stoke’s theorem cannot be used to rewrite [9] in terms of a line integral
\[ \tilde{\Phi} = \oint_c d\ell \cdot \vec{A}(\vec{r}) \]  

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where the contour $C$ in [11] is a circle at infinity, as $\vec{A}(\vec{r})$ is singular at $r = 0$, as follows from [2] and [7]. Indeed, since $K_\nu(x) \to \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$, we see that $\vec{\Phi} = 0$ in [11].

In (6), it is shown that the Hamiltonian density associated with action of [1] is given by

\[
\mathcal{H} = -\frac{\mu}{2} A_0^2 + \frac{1}{2\mu} \left( \pi_\sigma + \frac{\mu}{2} A_0 \right)^2 + A_0 \epsilon_i j A_j + \frac{\mu}{2} (\partial_i \sigma + A_i)^2 \\
+ \frac{1}{2} |\pi_\phi|^2 + \frac{1}{2} |(\partial_i + ieA_i) \phi|^2 - c_2 \phi^* \phi + c_4 (\phi^* \phi)^2 \tag{12}
\]

once the scalar field is treated in the standard fashion. (Here we have $\pi_\phi = (\partial_0 + ie A_0) \phi$ and $\pi_\sigma = -\mu (\partial_0 \sigma + \frac{1}{2} A_0)$.) Taking into account the ansatz of [2] and the equation of motion of [3a], we see that (12) reduces to

\[
\mathcal{H} = \frac{\mu}{2} (A_0^2 + \vec{A}^2) \tag{13}
\]

which, in turn becomes

\[
= \frac{\mu \alpha^2}{2} \left[ K_0^2(\mu r) + K_1^2(\mu r) \right] \tag{14}
\]

as follows from [2], [6] and [7]. Since $K_0(x) \sim -\log x$ and $K_1(x) \sim \frac{1}{x}$ near $x = 0$, we see that the energy

\[
E = \int d^2r \mathcal{H} \tag{15}
\]

diverges for this field configuration.

We now examine classical field configurations in which the scalar field $\phi$ in [1] is non-zero. Suplementing the ansatz of [2] with

\[
\phi(r, \theta) = e^{in\theta} f(r) \quad (n = 0, 1, \ldots, \theta = \tan^{-1} \frac{y}{x}) \tag{16}
\]

then the linear equations of [4] becomes

\[
-e(n + eA) f^2 = r A'_0 - \mu \left( A + \frac{n}{e} \right) \tag{17a}
\]

\[
-e^2 A_0 f^2 = \frac{1}{r} A' - \mu A_0 \tag{17b}
\]
\[ f'' + \frac{1}{r} f' + e^2 A_0^2 f - \frac{(n + eA)^2 f}{r^2} + 2c_2 f - 4c_4 f^3 = 0 \]  \hspace{1cm} [17c]

provided the Stuekelberg field is taken to be

\[ \sigma = \frac{n}{e} \theta. \]  \hspace{1cm} [18]

The boundary conditions to these equations are taken to be

\[ \lim_{r \to \infty} (A(r), A_0(r), f(r)) = \left( -\frac{n}{e}, 0, \sqrt{c_2/4c_4} \right) \]  \hspace{1cm} [19]

and

\[ \lim_{r \to 0} (A(r), A_0(r), f(r)) = \left( \lambda r^\alpha \ln r, \lambda_0 r^{\alpha_0} \ln r, \sigma r^\beta \right). \]  \hspace{1cm} [20]

The logarithmic terms in [20] are normally not encountered in vortex solutions (1-5), but do not affect the finiteness of either the magnetic flux or energy of the field configuration. The asymptotic behaviour that we postulate is, for small \( r \)

\[ (A(r), A_0(r), f(r)) \sim (\lambda r^2 \ln r, \lambda_0 \ln r, \sigma r^\beta). \]  \hspace{1cm} [21]

Substitution of [21] into the field equations [17] and examining leading order terms at the origin shows that

\[ \lambda_0 = \frac{\mu n}{e} \]  \hspace{1cm} [22a]

\[ \lambda = \frac{\mu^2 n}{2e} \]  \hspace{1cm} [22b]

\[ \beta = |n|. \]  \hspace{1cm} [22c]

Similarly, as \( r \to \infty \), the field equations [17] are satisfied to leading order provided either

\[ (A(r), A_0(r), f(r)) \sim \left( -\frac{n}{e} + a_1 e^{-e^2 \left( \frac{c_2}{2c_4} - \frac{e^2}{2} \right) r}, \frac{a_1}{r} e^{-e^2 \left( \frac{c_2}{2c_4} - \frac{e^2}{2} \right) r}, \sqrt{\frac{e^2}{2c_4} + \ldots a_2 e^{-\sqrt{4c_2} r}} \right) \]  \hspace{1cm} [23a]

or

\[ (A(r), A_0(r), f(r)) \sim \left( -\frac{n}{e} + a_1 e^{-e^2 \left( \frac{c_2}{2c_4} - \frac{e^2}{2} \right) r}, -\frac{a_1}{r} e^{-e^2 \left( \frac{c_2}{2c_4} - \frac{e^2}{2} \right) r}, a_2 e^{-\sqrt{4c_2} r} \right) \]
\[
\left( \text{if } \frac{c_2}{2c_4} < \frac{\mu}{e^2} \right). \tag{23b}
\]

The asymptotic behaviour at infinity is consequently dependent on the magnitude of \( \mu \). In either case though, the magnetic flux through space is given by

\[
\Phi = \int d^2x \, \epsilon_{ij} \partial_i A_j \\
= - \int d^2x \frac{1}{r} \frac{dA}{dr} \\
= \frac{2\pi n}{e}. \tag{24}
\]

Consequently the magnetic flux quantized in units of \( \frac{2\pi}{e} \), as in (1-4).

\section{Discussion}

We have considered static, vortex-like solutions in a 2 + 1 dimensional \( U(1) \) gauge theory in which both a Chern-Simons and Stuekelberg term occurs. An explicit solution with finite flux and divergent energy can be constructed when there is no scalar field. In the presence of a non-vanishing scalar field, the asymptotic form of solutions to the field equations can be found if the Stuekelberg mass is non-zero; it differs qualitatively from the form of solutions occurring when this mass vanishes. Nevertheless, these solutions have finite energy and quantized flux. Computing the charge and angular momentum of these solutions would not be straightforward.

It is of interest to see if these solutions have practical application in condensed matter physics.

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