ON THE MAXIMAL OPERATORS OF RIESZ LOGARITHMIC MEANS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate \((H_p, L_p)\) and \((H_p, L_{p,\infty})\) type inequalities for maximal operators of Riesz logarithmic means of one-dimensional Vilenkin-Fourier series.

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1. Introduction

Weak \((1,1)\)-type inequality for the maximal operator of Fejér means \(\sigma^*\) for Walsh-Fourier series was proved by Schipp [13] and for Vilenkin system by Pál, Simon [12]. Fujji [4] and Simon [15] verified that the \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz [22] generalized this result and proved the boundedness of \(\sigma^*\) from the martingale Hardy space \(H_p\) to the space \(L_p\), for \(p > 1/2\). Simon [14] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\). The counterexample for \(p = 1/2\) due to Goginava [3], (see also [8] and [16]).

Weisz [23] proved that following is true:

**Theorem W.** The maximal operator of Fejér means \(\sigma^*\) is bounded from the Hardy space \(H_{1/2}\) to the space \(L_{1/2,\infty}\).

In [17] and [18] it were proved that the maximal operator \(\tilde{\sigma}^*_p\), defined by

\[
\tilde{\sigma}^*_p := \sup_{n \in \mathbb{N}} \frac{|\sigma_n|}{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)},
\]

where \(0 < p \leq 1/2\) and \([1/2 + p]\) denotes integer part of \(1/2 + p\), is bounded from the Hardy space \(H_p\) to the space \(L_p\).

Moreover, for any nondecreasing function \(\varphi : \mathbb{N}_+ \rightarrow [1, \infty)\) satisfying the condition

\[
\lim_{n \to \infty} \frac{(n + 1)^{1/p - 2} \log^{2[1/2+p]} (n + 1)}{\varphi(n)} = +\infty,
\]

there exists a martingale \(f \in H_p\), such that

\[
\sup_{n \in \mathbb{N}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_p = \infty.
\]

For Walsh-Paley system analogical theorem is proved in [9] and for Walsh-Kaczmarz system in [10] and [20].

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Riesz’ s logarithmic means with respect to the Walsh system was studied by Simon [14], Goginava [11], Gát, Nagy [7] and for Vilenkin systems by Gát [6], Blahota, Gát [2], Tephnadze [19]. In this paper it was proved that maximal operator of Riesz logarithmic means of Vilenkin-Fourier series is bounded from the martingale Hardy space $H_p$ to the space $L_p$ when $p > 1/2$ and is not bounded from the martingale Hardy space $H_p$ to the space $L_p$ when $0 < p \leq 1/2$.

The main aim of this paper is to investigate $(H_p, L_p)$ and $(H_p, L_p, \infty)$ type inequalities for weighted maximal operators of Riesz logarithmic means of one-dimensional Vilenkin-Fourier series.

2. Definitions and Notations

Let $P_+$ denote the set of the positive integers, $P := P_+ \cup \{0\}$.

Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers not less than 2.

Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the group $Z_{m_j}$ with the product of the discrete topologies of $Z_{m_j}$’s.

The direct product $\mu$ of the measures $\mu_k (\{j\}) := 1/m_k$ $(j \in Z_{m_k})$

is a Haar measure on $G_m$ with $\mu (G_m) = 1$.

If $\sup_n m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of $G_m$ are represented by sequences

$x := (x_0, x_1, \ldots, x_j, \ldots) \quad (x_k \in Z_{m_k}).$

It is easy to give a base for the neighborhood of $G_m$

$I_0 (x) := G_m; \quad I_n (x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m, \ n \in P).$

Denote $I_n := I_n (0)$ for $n \in P$ and $\overline{I}_n := G_m \setminus I_n$.

Let

$e_n := (0, 0, \ldots, x_n = 1, 0, \ldots) \in G_m \quad (n \in P).$

It is evident

(2) $\overline{I}_M = \left( \bigcup_{k=0}^{M-2m_k-1} \bigcup_{x_k=1}^{M-1} \bigcup_{l=k+1}^{m_k - 1} I_{l+1} (x_ke_k + x_le_l) \right) \bigcup \left( \bigcup_{k=1}^{M-1} \bigcup_{x_k=1}^{m_k - 1} I_M (x_ke_k) \right).$

If we define the so-called generalized number system based on $m$ in the following way:

$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in P).$
then every $n \in \mathbb{P}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$ where $n_j \in \mathbb{Z}_{m_j}$ ($j \in \mathbb{P}$) and only a finite number of $n_j$’s differ from zero. Let $|n| := \max \{j \in \mathbb{P}; n_j \neq 0\}$.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system.

At first, define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp\left(2\pi i x_k / m_k\right) \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{P}).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{P})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{P}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ \cite{1}.

Now, we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system $\psi$ in the usual manner:

$$\hat{f}(k) := \int_{G_m} f \overline{\psi_k} \, d\mu, \quad (k \in \mathbb{P}),$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{P}^+, \ S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f, \quad (n \in \mathbb{P}^+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{P}^+),$$

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k, \quad (n \in \mathbb{P}^+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

It is well-known that

$$\sup_n \int_{G_m} |K_n| \, d\mu \leq c < \infty.$$
The norm (or quasi-norm) of the space $L^p(G_m)$ is defined by
\[ \|f\|_p := \left( \int_{G_m} |f|^p \, d\mu \right)^{1/p} \quad (0 < p < \infty). \]

The space $L^{p,\infty}(G)$ consists of all measurable functions $f$ for which $\|f\|_{L^{p,\infty}(G)} := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty$.

The $\sigma$-algebra generated by the intervals \{I_n(x) : x \in G_m\} will be denoted by $\mathcal{F}_n (n \in \mathbf{P})$. Denote by $f = (f^{(n)}, n \in \mathbf{P})$ a martingale with respect to $\mathcal{F}_n (n \in \mathbf{P})$. (for details see e.g. [21]). The maximal function of a martingale $f$ is defined by
\[ f^* = \sup_{n \in \mathbf{P}} |f^{(n)}|, \]
respectively.

In case $f \in L_1$, the maximal functions are also given by
\[ f^*(x) = \sup_{n \in \mathbf{P}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|. \]

For $0 < p < \infty$ the Hardy martingale spaces $H^p(G_m)$ consist of all martingales for which $\|f\|_{H^p} := \|f^*\|_p < \infty$.

If $f \in L_1$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbf{P})$ is a martingale. If $f = (f^{(n)}, n \in \mathbf{P})$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:
\[ \hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)}(x) \overline{\psi_i(x)} \, d\mu(x). \]

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in \mathbf{P})$ obtained from $f$.

In the literature, there is the notion of Riesz’ s logarithmic means of the Fourier series. The $n$-th Riesz’s logarithmic means of the Fourier series of an integrable function $f$ is defined by
\[ R_n f := \frac{1}{l_n} \sum_{k=1}^n S_k f, \]
where $l_n := \sum_{k=1}^n \frac{1}{k}$.

The kernels of Riesz’s logarithmic means is established by
\[ L_n := \frac{1}{l_n} \sum_{k=1}^n \frac{D_k(x)}{k}. \]

For the martingale $f$ we consider the following maximal operators...
\[ \sigma^* f : = \sup_{n \in P} |\sigma_n f|, \quad R^* f := \sup_{n \in P} |R_n f|, \]
\[ \tilde{R} f : = \sup_{n \in P} \frac{|R_n f|}{\log (n + 1)}, \quad \tilde{R}_p f := \sup_{n \in P} \frac{\log (n + 1) |R_n f|}{(n + 1)^{1/p-2}}. \]

A bounded measurable function \( a \) is \( p \)-atom, if there exist a dyadic interval \( I \), such that
\[ \int_I ad\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp } (a) \subset I. \]

3. Formulation of Main Results

**Theorem 1.** The maximal operator of Riesz logarithmic means \( R^* \) is bounded from the Hardy space \( H^{1/2} \) to the space \( L^{1/2, \infty} \).

Earlier, it was proved that the maximal operator \( R^* \) is not bounded from the Hardy space \( H^{1/2} \) to the space \( L^{1/2, \infty} \). So, it is interesting to discuss that what type weight we have to apply to get back the boundedness of the maximal operator. We found the answer in the next theorem.

**Theorem 2.**

a) The maximal operator \( \tilde{R} \) is bounded from the Hardy space \( H^{1/2} \) to the space \( L^{1/2} \).

b) Let \( \varphi : P_+ \rightarrow [1, \infty) \) be a nondecreasing function satisfying the condition
\[ \lim_{n \to \infty} \frac{\log (n + 1)}{\varphi(n)} = +\infty. \]

Then the maximal operator
\[ \sup_{n \in P} \frac{|R_n f|}{\varphi(n)} \]
is not bounded from the Hardy space \( H^{1/2} \) to the space \( L^{1/2} \).

**Theorem 3.**

a) Let \( 0 < p < 1/2 \). Then the maximal operator \( \tilde{R}_p^* \) is bounded from the Hardy space \( H_p \) to the space \( L_p \).

b) Let \( 0 < p < 1/2 \) and \( \varphi : P_+ \rightarrow [1, \infty) \) be a nondecreasing function satisfying the condition
\[ \frac{(n + 1)^{1/p-2}}{\log (n + 1) \varphi(n)} = \infty. \]

Then the maximal operator
\[ \sup_{n \in P} \frac{|R_n f|}{\varphi(n)} \]
is not bounded from the Hardy space \( H_p \) to the space \( L_{p, \infty} \).
4. AUXILIARY PROPOSITIONS

Lemma 1. [24] (Weisz) A martingale \( f = (f^{(n)}, n \in \mathbb{P}) \) is in \( H_p (0 < p \leq 1) \) if and only if there exist a sequence \( (a_k, k \in \mathbb{P}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{P}) \) of real numbers such that for every \( n \in \mathbb{P} \)

\[
\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)},
\]

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]

Moreover, \( \|f\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p} \), where the infimum is taken over all decomposition of \( f \) of the form (7).

Lemma 2. [5] (Găt) Let \( A > t, t, A \in \mathbb{P} \), \( x \in I_t \setminus I_{t+1} \). Then

\[
K_{2A} (x) = \begin{cases} 
2^{t-1}, & \text{if } x \in I_A (e_t), \\
\frac{(2^A + 1)}{2}, & \text{if } x \in I_A, \\
0, & \text{otherwise}.
\end{cases}
\]

Analogously of Lemma 4 in [18] if we apply Lemma 2 we can prove that following is true:

Lemma 3. Let \( x \in I_N (x_k e_k + x_l e_l), \ 1 \leq x_k \leq m_k - 1, 1 \leq x_l \leq m_l - 1, k = 0, ..., N - 2, l = k + 1, ..., N - 1 \). Then

\[
\int_{I_N} |K_n (x - t)| d\mu (t) \leq \frac{c M_l M_k}{n M_N}, \quad \text{when } n \geq M_N.
\]

Let \( x \in I_N (x_k e_k), 1 \leq x_k \leq m_k - 1, k = 0, ..., N - 1 \). Then

\[
\int_{I_N} |K_n (x - t)| d\mu (t) \leq \frac{c M_k}{M_N}, \quad \text{when } n \geq M_N.
\]

Lemma 4. Let \( x \in I_N (x_k e_k + x_l e_l), \ 1 \leq x_k \leq m_k - 1, 1 \leq x_l \leq m_l - 1, k = 0, ..., N - 2, l = k + 1, ..., N - 1 \). Then

\[
\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j (x - t)|}{j+1} d\mu (t) \leq \frac{c M_k M_l}{M_N^2}.
\]

Let \( x \in I_N (x_k e_k), 1 \leq x_k \leq m_k - 1, k = 0, ..., N - 1 \). Then

\[
\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j (x - t)|}{j+1} d\mu (t) \leq \frac{c M_k M_l}{M_N^2}.
\]

Proof. Let \( x \in I_N (x_k e_k + x_l e_l), \ 1 \leq x_k \leq m_k - 1, 1 \leq x_l \leq m_l - 1, k = 0, ..., N - 2, l = k + 1, ..., N - 1 \). Using Lemma 3 we have
(8) \[
\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j (x-t)|}{j+1} d\mu (t) \leq \sum_{j=M_N+1}^{n} \frac{cM_k M_l}{(j+1) j M_N} \\
\leq \frac{cM_k M_l}{M_N} \sum_{j=M_N+1}^{\infty} \left( \frac{1}{j} - \frac{1}{j+1} \right) \leq \frac{cM_k M_l}{M_N^2}.
\]

Let \( x \in I_N (x_k e_k) \), \( 1 \leq x_k \leq m_k - 1 \), \( k = 0, ..., N - 1 \). Then

(9) \[
\int_{I_N} \sum_{j=M_N+1}^{n} \frac{|K_j (x-t)|}{j+1} d\mu (t) \leq \sum_{j=M_N+1}^{n} \frac{cM_k}{(j+1) M_N} \leq \frac{cM_k}{M_N} l_n.
\]

Combining (8) and (9) we complete the proof of Lemma 4.

5. PROOF OF THE THEOREMS

**Proof of theorem 1. a)** Using Abel transformation we obtain

(10) \[
R_n f = \frac{1}{l_n} \sum_{j=1}^{n-1} \sigma_j f + \frac{\sigma_n f}{l_n}.
\]

Consequently,

(11) \[
R^* f \leq c \sigma^* f.
\]

Using Theorem W and (11) we conclude that \( R^* \) is bounded from the martingale Hardy space \( H_{1/2} \) to the space \( L_{1/2, \infty} \).

**Proof of theorem 2.** From (10) for the kernels of Riesz's logarithmic means we have

(12) \[
L_n = \frac{1}{l_n} \sum_{j=1}^{n-1} \frac{K_j}{j+1} + \frac{K_n}{l_n}.
\]

By Lemma 1, the proof of theorem 2 will be complete, if we show that

\[
\int_{I} \left| \tilde{R} a \right|^{1/2} d\mu \leq c < \infty,
\]

for every 1/2-atom \( a \), where \( I \) denotes the support of the atom.

Let \( a \) be an arbitrary 1/2-atom with support \( I \) and \( \mu (I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( R_n (a) = \sigma_n (a) = 0 \), when \( n \leq M_N \). Therefore we suppose that \( n > M_N \).
Since \( \|a\|_\infty \leq cM_N^2 \) if we apply \([12]\) we can write

\[
\frac{|R_n a(x)|}{\log(n+1)} = \frac{1}{\log(n+1)} \int_{I_N} |a(t)| |L_n(x-t)| \, d\mu(t)
\]

\[
\leq \frac{\|a\|_\infty}{\log(n+1)} \int_{I_N} |L_n(x-t)| \, d\mu(t)
\]

\[
\leq \frac{cM_N^2}{\log(n+1) l_n} \int_{I_N} \sum_{j=M_N+1}^{n-1} |K_j(x-t)| \, \frac{\, d\mu(t)}{j+1}
\]

\[
+ \frac{cM_N^2}{\log(n+1) l_n} \int_{I_N} |K_n(x-t)| \, d\mu(t).
\]

Let \( x \in I_N(x_k e_k + x_l e_l) \), \( 1 \leq x_k \leq m_k - 1 \), \( 1 \leq x_l \leq m_l - 1 \), \( k = 0, \ldots, N-2 \), \( l = k+1, \ldots, N-1 \). From Lemmas 3 and 4 we have

\[
\frac{|R_n(a)|}{\log(n+1)} \leq \frac{cM_kM_k}{N^2}.
\]

Let \( x \in I_N(x_k e_k) \), \( 1 \leq x_k \leq m_k - 1 \), \( k = 0, \ldots, N-1 \). Applying Lemmas 3 and 4 we have

\[
\frac{|R_n a(x)|}{\log(n+1)} \leq \frac{M_N M_k}{N} \leq cM_N M_k.
\]

Combining \([2], [14]\) and \([15]\) we get

\[
\int_{I_N} \left| \int_{I_N} \frac{R a(x)}{\log(n+1)} \right|^{1/2} d\mu(x)
\]

\[
= \sum_{k=0}^{N-2m_k-1} \sum_{x_k=1}^{N-1} \sum_{l=k+1}^{m_l-1} \sum_{x_l=1}^{1} \int_{I_N(x_k e_k + x_l e_l)} \left| \int_{I_N(x_k e_k)} \frac{R a(x)}{\log(n+1)} \right|^{1/2} d\mu(x)
\]

\[
+ \sum_{k=0}^{N-1m_k-1} \sum_{x_k=1}^{N-1} \int_{I_N(x_k e_k)} \left| \int_{I_N(x_k e_k)} \frac{R a(x)}{\log(n+1)} \right|^{1/2} d\mu(x)
\]

\[
\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{1}{M_l} \frac{1}{M_N} + c \sum_{k=0}^{N-1} \frac{1}{M_N} \sqrt{M_N M_k} \leq c < \infty.
\]

It completes the proof of first part of theorem 2.

b) Let \( \{\lambda_k, k \in \mathbb{P}_+\} \) be an increasing sequence of the positive integers, which saisfies condition \((5)\). For every \( \lambda_k \) there exists a positive integers \( \{n_k, k \in \mathbb{P}_+\} \subset \{\lambda_k, k \in \mathbb{P}_+\} \), such that

\[
\lim_{k \to \infty} \frac{n_k}{\varphi(M_{2\lambda_k+1})} = \infty.
\]

Let

\[
f_{n_k}(x) = D_{M_{2\lambda_k+1}}(x) - D_{M_{2\lambda_k}}(x).
\]
It is evident

\[ \hat{f}_{nk}(i) = \begin{cases} 1, & \text{if } i = M_{2nk}, \ldots, M_{2nk+1} - 1, \\ 0, & \text{otherwise}. \end{cases} \]

We can write

\[ S_i f_{nk}(x) = \begin{cases} D_i(x) - D_{M_{2nk}}(x), & \text{if } i = M_{2nk}, \ldots, M_{2nk+1} - 1, \\ \hat{f}_{nk}(x), & \text{if } i \geq M_{2nk+1}, \\ 0, & \text{otherwise}. \end{cases} \]

From (3) we get (see also \([17]\) and \([18]\))

\[ \|f_{nk}(x)\|_{H_p} = \|f_{nk}^*(x)\|_p \leq cM_{2nk}^{1/p}. \]

Let \(q_{nk}^* = M_{2nk} + M_{2s}, s = 0, \ldots, n_k - 1\). By (16) we have

\[ \left| R_{q_{nk}^*} f_{nk}(x) \right| = \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \left| \sum_{j=M_{2nk+1}}^{q_{nk}^*} S_j f_{nk}(x) \right| = \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \left| \sum_{j=M_{2nk+1}}^{q_{nk}^*} \frac{D_j(x) - D_{M_{2nk}}(x)}{j} \right| = \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \left| \sum_{j=1}^{M_{2s}} \frac{D_{j+M_{2nk}}(x) - D_{M_{2nk}}(x)}{j + M_{2nk}} \right|. \]

Since

\[ D_{j+M_{2nk}}(x) - D_{M_{2nk}}(x) = \psi_{M_{2nk}} D_j(x), \quad j = 1, 2, \ldots, M_{2nk} - 1. \]

we obtain

\[ \left| R_{q_{nk}^*} f_{nk}(x) \right| = \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \sum_{j=1}^{M_{2s}} |D_j(x)|. \]

Let \(x \in I_{2s} \setminus I_{2s+1}\). Then

\[ \left| R_{q_{nk}^*} f_{nk}(x) \right| \geq \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \sum_{j=0}^{M_{2s}} j = \frac{1}{\varphi(q_{nk}^*) l_{q_{nk}^*}} \sum_{j=0}^{M_{2s}} j \geq \frac{cM_{2s}^2}{2M_{2nk}^2} \geq \frac{cM_{2s}^2}{2M_{2nk}^2}. \]
Using (21) we have
\[
\int_{G_m} \left| \tilde{R} f(x) \right|^{1/2} d\mu(x) \\
\geq \sum_{s=1}^{n_k-1} \int_{I_{2s}}^{I_{2s+1}} \left| R_{q_{n_k}} f(x) \right|^{1/2} d\mu(x) \\
\geq c \sum_{s=1}^{n_k-1} \frac{1}{\varphi \left( \frac{M_{2n_k+1}}{M_{2n_k}} \right)} \frac{M_{2s} \sqrt{n_k}}{M_{2n_k}}.
\]
From (17) we have
\[
\left( \int_{G_m} \left| \tilde{R} f(x) \right|^{1/2} d\mu(x) \right)^2 \\
\geq \frac{c n_k}{\varphi \left( \frac{M_{2n_k+1}}{M_{2n_k}} \right)} \rightarrow \infty, \quad \text{when } k \rightarrow \infty.
\]

Theorem 2 is proved.

**Proof of theorem 3.** Let \( 0 < p < 1/2 \). By Lemma 1, the proof of theorem 3 will be complete, if we show that
\[ \int \left| \tilde{R}_p a \right|^p d\mu \leq c_p < \infty, \]
for every p-atom \( a \), where \( I \) denotes the support of the atom.

Let \( a \) be an arbitrary p-atom with support \( I \) and \( \mu(I) = M_N^{-1} \). We may assume that \( I = I_N \). It is easy to see that \( R_n(a) = 0 \), when \( n \leq M_N \). Therefore we suppose that \( n > M_N \).

Since \( \|a\|_\infty \leq c M_N^{1/p} \) using (12) we can write
\[
\log \frac{(n+1)}{(n+1)^{1/p-2}} |R_n a(x)| \\
\leq \log \frac{(n+1)}{(n+1)^{1/p-2}} L_n \int_{I_N} \sum_{j=M_N+1}^{n-1} \frac{|K_j(x-t)|}{j+1} d\mu(t) \\
+ \log \frac{(n+1)}{(n+1)^{1/p-2}} L_n \int_{I_N} |K_n(x-t)| d\mu(t).
\]

Let \( x \in I_N(x_k e_k + x_l e_l), \ 1 \leq x_k \leq m_k - 1, \ 1 \leq x_l \leq m_l - 1, \ k = 0, ..., N - 2, \ l = k + 1, ..., N - 1. \) From Lemmas 3 and 4 when \( n > M_N \) we obtain
\[
\log \frac{(n+1)}{(n+1)^{1/p-2}} |R_n a(x)| \leq c_p M_l M_k.
\]

Let \( x \in I_N(x_k e_k), \ 1 \leq x_k \leq m_k - 1, \ k = 0, ..., N - 1. \) Applying Lemmas 3 and 4 we have
(25) \[
\frac{\log (n + 1)}{(n + 1)^{1/p - 2}} |R_n a (x)| \leq c N M_k.
\]

Combining (2), (24) and (25) we get

\[
\int_{I_N} \left| \frac{\sim^*}{R_p a (x)} \right|^p \| \| d\mu(x)
\]
\[
= \sum_{k=0}^{N-1} \sum_{x_k = 1}^{m_k-1} \sum_{x_l = 1}^{m_l-1} \int_{I_N(x_k e_k + x_l e_l)} \left| \frac{\sim^*}{R_p a (x)} \right|^p \| d\mu(x)
\]
\[
+ \sum_{k=0}^{N-1} \sum_{x_k = 1}^{m_k-1} \int_{I_N(x_k e_k)} \left| \frac{\sim^*}{R_p a (x)} \right|^p \| d\mu(x)
\]
\[
\leq c_p \sum_{k=0}^{N-1} \sum_{l=k+1}^{m_k-1} \frac{1}{M_l} (M_1 M_k)^p + c_p \sum_{k=0}^{N-1} \sum_{l=k+1}^{m_k-1} \frac{1}{M_N} (N M N M_k)^p \leq c_p < \infty.
\]

Which complete the proof of first part of Theorem 2.

Let 0 < p < 1/2 and \( \{ \lambda_k, k \in \mathbb{P}_+ \} \) be an increasing sequence of the positive integers, which satisfies condition (6). It is evident that for every \( \lambda_k \) there exists a positive integers \( \{ n_k, k \in \mathbb{P}_+ \} \subset \{ \lambda_k, k \in \mathbb{P}_+ \} \), such that

\[
\lim_{k \to \infty} \frac{N}{M} (M_2 n_k + 1)^{1/p - 2} (M_2 n_k + 1) \log (M_2 n_k + 1) = \infty.
\]

Combining (18-21) we have

\[
\frac{|R_{M_2 n_k + 1} f_{n_k} (x)|}{\varphi (M_2 n_k + 1)} = \frac{|R_{\varphi (M_2 n_k) f_{n_k}} (x)|}{\varphi (M_2 n_k)} \geq \frac{c}{\varphi (M_2 n_k + 1)} \| f_{n_k} (x) \|_{H_p} \]

for \( x \in I_0 \setminus I_1 = G_m \setminus I_1 \).

From (17) we get

\[
\frac{c}{\varphi (M_2 n_k + 1)} \| f_{n_k} (x) \|_{H_p} \geq \frac{c}{\varphi (M_2 n_k + 1)} (M_2 n_k + 1)^{1/p}
\]

which complete the proof of theorem 3.

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