Hopf Term, Fractional Spin and Soliton Operators in the O(3) Nonlinear Sigma Model

MASAOMI KIMURA

Institute for Cosmic Ray Research
University of Tokyo
Midori, Tanashi, Tokyo 188, Japan

HIROYUKI KOBAYASHI and IZUMI TSUTSUI

Institute of Particle and Nuclear Studies
High Energy Accelerator Research Organization (KEK), Tanashi Branch
Midori, Tanashi, Tokyo 188, Japan

Abstract. We re-examine three issues, the Hopf term, fractional spin and the soliton operators, in the 2 + 1 dimensional O(3) nonlinear sigma model based on the adjoint orbit parametrization (AOP) introduced earlier. It is shown that the Hopf term is well-defined for configurations of any soliton charge \( Q \) if we adopt a time independent boundary condition at spatial infinity. We then develop the Hamiltonian formulation of the model in the AOP and thereby argue that the well-known \( Q^2 \)-formula for fractional spin holds only for a restricted class of configurations. Operators which create states of given classical configurations of any soliton number in the (physical) Hilbert space are constructed. Our results clarify some of the points which are crucial for the above three topological issues and yet have remained obscure in the literature.
1. Introduction

The $O(3)$ nonlinear sigma model (NSM) describes physical systems that undergo a spontaneous breakdown of the global symmetry $O(3)$. It is probably one of the most widely-applicable field theory models, being used in fields ranging from condensed matter physics to high energy physics. The model was studied intensively around a decade ago when the theoretical possibility of particles possessing a fractional spin and statistics in $2+1$ dimensions, namely anyons, attracted a lot of attention [1, 2] (see also [3] for a review) in expectation of possible relevance to the (fractional) quantum Hall effect [4, 5, 6] and high-$T_c$ superconductivity. Such a phenomenon in the NSM was originally suggested by Wilczek and Zee [7] under the presence of a topological term, the Hopf term, which endows solitons (Skyrmions) admitted by the model with a nontrivial phase factor under space rotation or interchange. Since then, several authors have examined the NSM in order, for example, to furnish a firmer basis for fractional spin and statistics [8, 9, 10, 11], and to explore its possible extensions/modifications [12, 13, 14] allowing for anyons. Recently, we added to this series of investigations a study [15] of the NSM using the adjoint orbit parametrization (AOP), which has been known [16, 17, 14] for some time but not thoroughly utilized so far, unlike the other familiar $\mathbb{C}P^1$ parametrization [8]. The aim of this paper is to present a complete AOP analysis of the NSM and thereby re-examine the three topological issues, the Hopf term, fractional spin and soliton operators, discussed previously. We shall find that some of the points which are important for the topological phenomenon to occur and yet have remained obscure are clarified in the AOP.

To understand the points we are going to address in this paper, let us recall that the $O(3)$ nonlinear sigma model is a system of spin vectors $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ constrained on the 2-sphere, $\mathbf{n}^2(x) = \sum_a n_a^2(x) = 1$. In three dimensions the system is governed by the action,

$$ I_0 = \int d^3x \frac{1}{2\lambda^2} \partial_\mu \mathbf{n}(x) \cdot \partial^\mu \mathbf{n}(x), \quad (1.1) $$

where $\lambda$ is a coupling constant. We take the spacetime to be $\mathbb{R}^2 \times [0, T]$ and, as usual, assume that the spin vectors approach a constant vector at spatial infinity,

$$ \mathbf{n}(x) = \mathbf{n}(x, t) \to \mathbf{n}(\infty) \quad \text{as} \quad \|x\| \to \infty, \quad (1.2) $$
at all times $t \in [0, T]$. This boundary condition allows us to regard $\mathbf{n}(x)$ as a map $S^2 \times [0, T] \to S^2$ by identifying all points at spatial infinity of $\mathbb{R}^2$ to be the south-pole of the sphere $S^2$ which is compactified from $\mathbb{R}^2$. The configuration space of the model is then given by the space of these maps (at a fixed time),

$$Q = \text{Map}_0(S^2, S^2),$$

(1.3)

where the subscript 0 indicates that the space consists of based maps (i.e., those with the image of the south-pole fixed).

The topological structure of the model may be characterized by the homotopy groups of the configuration space. Using the identity $\pi_n(Q) = \pi_{n+2}(S^2)$ which holds for the space of based maps $Q$ (see, e.g., [15]), we find

$$\pi_0(Q) = \pi_2(S^2) = \mathbb{Z}.$$  

(1.4)

This implies that the space $Q$ splits into disconnected sectors characterized by an integer. This integer is the soliton number $Q := \int_{S^2} d^2x J^0(x)$ which is the charge of the conserved topological current

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} n_a \partial_\nu n_b \partial_\lambda n_c.$$  

(1.5)

We also find

$$\pi_1(Q) = \pi_3(S^2) = \mathbb{Z},$$  

(1.6)

which shows that, in each sector, configurations are characterized by another integer, called the Hopf (or instanton) number. To express the Hopf number as a term in the action, conventionally one considers

$$H = - \int d^3x A_\mu(x) J^\mu(x),$$  

(1.7)

with the vector potential $A_\mu$ being defined from the current by the relation, $J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$. The expression (1.7) is expected to serve as a topological term expressing the Hopf number and — if it is well-defined to any spin vectors — may be added to the action as $I = I_0 + \theta H$ with $\theta$ an angle parameter. (The topological term can also be induced either by interactions with fermions [13] or by a pure quantum mechanical topological effect [18, 15], even though it does not appear at the classical level.) The problem, however,
is that the expression (1.7) can reproduce the Hopf number only for configurations in the zero soliton sector. This is obvious because, if the original map $S^2 \times [0, T] \to S^2$ is to be regarded as a map $S^3 \to S^2$ by means of suspension $S \cdot S^2 \simeq S^3$ (i.e., by contracting the sphere $S^2$ at $t = 0$ and $t = T$), it must be homotopic to a constant map at the both ends of the period $[0, T]$ and hence must have a vanishing soliton number.

In our previous paper [15] we employed the AOP for the spin vectors, and argued a possible definition of the Hopf term to solitons,\footnote{In this paper we use the term ‘soliton’ to indicate a configuration which has a nonvanishing soliton number, not just a soliton (topologically nontrivial) solution of the equations of motion.} that is, the topological term which reproduces the Hopf number to configurations of any soliton numbers. The idea is simply to convert the configuration of non-vanishing soliton number to a corresponding one of vanishing soliton number using a fixed (standard) configuration which has the opposite soliton number. In this paper we first show in sect.2 that, in the AOP, there exists a boundary condition for the field such that the above conversion procedure becomes unnecessary. We shall see that, although the boundary condition appears slightly more restrictive than the one naively expected from (1.2), it is actually the same because of the gauge symmetry inherent to the AOP. This way we are allowed to dispense with the cumbersome converted fields and use the corresponding Hopf term obtained in the AOP as the formula that truly represents the Hopf number to any configurations. With this boundary condition we present in sect.3 the Hamiltonian formulation of the NSM in terms of the AOP and thereby furnish a basis to quantize the model canonically. We then examine in sect.4 the angular momentum of the system based on the canonical quantization scheme. We shall find that the well-known formula [1, 3] for the fractional spin,

$$J_{\text{fractional}} = \frac{\theta}{2\pi}Q^2,$$

(1.8)

holds for a specific class of configurations (including the soliton solutions) but not for generic configurations. In sect.5 we construct soliton operators explicitly in the (physical) Hilbert space. This is just the AOP version of the soliton operators previously proposed [19], but it has an advantage in that the construction is more transparent and that a single operator can create a soliton state whereas in the previous construction we needed two operators defined in different patches on the space $S^2$. Sect.6 is devoted to our conclusions and discussions.
2. Hopf Term

We wish to show in this section that it is possible (without using the conversion procedure) to define the Hopf term that gives the Hopf integer to any configuration. For this, we first introduce the AOP for the NSM.

The adjoint orbit of a group $G$ passing through an element $K \in g$, where $g$ is the Lie algebra of the group $G$, is the subspace of $g$ formed under the adjoint action, $O_K := \{ gKg^{-1} | g \in G \}$. The orbit $O_K$ is isomorphic to the coset space $G/H$ where $H$ is the isotropy group of the action at $K$. For the $O(3)$ NSM we regard the target space $S^2$ of the map $\mathbf{n}(x)$ as the coset $SU(2)/U(1)$ which is obtained as the adjoint orbit of $SU(2)$ passing through, say, $K = T_3$, where $\{ T_a ; a = 1, 2, 3 \}$ is a basis of $g = \mathfrak{su}(2)$. In terms of the AOP our spin vectors read

$$n(x) := n_1(x)T_1 + n_2(x)T_2 + n_3(x)T_3 = g(x)T_3g^{-1}(x). \quad (2.1)$$

Note that the constraint satisfied by the spin vectors, $\text{Tr} n^2(x) = 1$, is automatically fulfilled by the parametrization (2.1). The AOP possesses redundancy with respect to the isotropy group $H$, which in our case is the $U(1)$ group generated by the element $T_3$, as can be seen from the fact that the same $\mathbf{n}(x)$ can be represented by different $G$-valued fields related by ‘gauge transformations’,

$$g(x) \rightarrow g(x) h(x), \quad \text{where} \quad h(x) \in H. \quad (2.2)$$

An important point to note [14, 15] is that the field $g(x)$ satisfying (2.1) becomes singular unless the corresponding spin vector $\mathbf{n}(x)$ belongs to the zero soliton sector, as we shall see shortly. To deal with $g(x)$ everywhere regular, we shall consider a two dimensional disc $D^2$ whose boundary $\partial D^2$ is identified with spatial infinity which is now the south-pole of the sphere $S^2$ (for the details, see [15]). Then, we shall define $g(x)$ as a map $M := D^2 \times [0, T] \rightarrow SU(2)$. With this definition we see that, if we let $g(\infty) \in SU(2)$ be some element satisfying $n(\infty) = g(\infty) T_3 g^{-1}(\infty)$, the boundary condition (1.2) is fulfilled if

$$g(x) = g(\infty) k(x) \quad \text{for} \quad x \in \partial D^2 \times [0, T], \quad (2.3)$$

Convention: The basis of $\mathfrak{su}(2)$ is taken to be in the defining representation $T_a = \frac{\sigma_a}{2i}$, and our trace, ‘$\text{Tr} := (-2)$ times the matrix trace’, is normalized as $\text{Tr}(T_a T_b) = \delta_{ab}$. The epsilon symbol $\epsilon_{\mu\nu\lambda}$ has the sign $\epsilon_{012} = +1$ and we use $\epsilon_{ij} := \epsilon_{0ij}$. 

5
with \( k(x) = e^{\xi(x)}T_3 \in H \) being an arbitrary smooth function over the boundary. Let us observe that the AOP brings the soliton number to the form,

\[
Q(g) = -\frac{1}{4\pi} \int_{\partial D^2} \text{Tr} T_3(g^{-1}(x)dg(x)).
\]  

(2.4)

We then obtain \( Q = -\frac{1}{4\pi} \int_{\partial D^2} d\xi(x) \), which shows that the soliton number is nothing but the winding number of the field \( g(x) \) at the boundary. Note that gauge transformations (2.2) cannot change the soliton number because the function \( h(x) \), being defined over the contractible space \( D^2 \), reduces to a trivial map \( \partial D^2 \simeq S^1 \to S^1 \) at the boundary. It is now clear that, unless \( Q(g) = 0 \), the field \( g(x) \) cannot be regular upon \( S^2 \times [0, T] \), since the identification \( D^2 \) with \( S^2 \) gives rise to a singularity at the south-pole for \( Q(g) \neq 0 \).

Turning our attention to the Hopf term, we first recall that in terms of the AOP the Hopf term (1.7) reads [17, 15]

\[
H(g) = \frac{1}{48\pi^2} \int_M \text{Tr} (g^{-1}(x)dg(x))^3.
\]  

(2.5)

This term gives the Hopf integer for those \( g(x) \) which are \( x \)-independent at the both ends \( t = 0 \) and \( t = T \), since in this case \( M \) can be deformed to \( S^3 \) (by suspension) leading to the familiar formula of the degree of mapping \( S^3 \to SU(2) \simeq S^3 \) as expected from (1.6). To assign the Hopf integer to configurations having non-vanishing soliton numbers, we consider those \( g(x) \) satisfying the periodic condition in time up to a constant \( h_c \in H \),

\[
g(x, T) = g(x, 0) h_c.
\]  

(2.6)

As discussed in our previous paper [15], we may assign the Hopf integer to this \( g \) by using the formula (2.5) with \( g \) replaced by

\[
\bar{g}(x) := g(\infty) g^{-1}(x, 0) g(x, t).
\]  

(2.7)

Indeed, this converted field \( \bar{g}(x) \) becomes constant at the both ends of the period \([0, T]\), and therefore \( H(\bar{g}) \) can be used to provide the Hopf number to any \( g \). (The factor \( g(\infty) \) is inserted in (2.7) so that \( \bar{g}(x) \) still satisfies the same boundary condition as \( g(x) \).)

But the price we pay for the use of the Hopf term with the converted field is that, because of the extra prefactors in (2.7), the Hopf term makes the conventional treatise of
the model, such as the Hamiltonian formulation, quite cumbersome. To find a solution of
the problem, let us compare the two formulae, $H(g)$ and $H(\bar{g})$, and see if the difference
disappears under some condition. The difference can be written as

$$H(\bar{g}) - H(g) = Q(g)P(g), \quad (2.8)$$

where $Q(g)$ is the soliton charge (2.4) whereas

$$P(g) := -\frac{1}{4\pi} \int_0^T \text{Tr} T_3(g^{-1}(x)dg(x))\big|_{\partial D^2}, \quad (2.9)$$

whose integration is along $t$ at fixed $x$ on the boundary, counts the winding number of the
map $k(x)$ during the period $[0, T]$. More explicitly, it is given by $P = -\frac{1}{4\pi} \int_0^T d\xi(x)$, which
is independent of $x$ on account of (2.6) and the smoothness of $k(x)$. Note that $H(g)$ and $P(g)$ are not gauge invariant in general, in contrast to $H(\bar{g})$ which is gauge invariant.

We then notice that, if $k(x)$ is time-independent at the boundary $\partial D^2$, then it follows
that $P(g) = 0$ and hence $H(\bar{g}) = H(g)$, implying that the conversion procedure becomes
unnecessary. Thus all we need to do is to render the boundary condition (2.3) slightly
more restrictive, by insisting that the function $k(x)$ be time-independent,

$$k(x) = k(x) \quad \text{on} \quad \partial D^2. \quad (2.10)$$

In fact, this is not a real restriction for the boundary condition, since the function $k(x)$
can always be made time-independent by a gauge transformation (2.2) with $h(x, t) = k^{-1}(x, t)k(x, 0)$. We therefore conclude that, under the boundary condition (2.3) with
time-independent $k(x)$, we can use the same formula (2.5) for the Hopf term to bestow a
generic configuration $g(x)$ with the Hopf integer, even though the spacetime $M$ may not
be deformable to $S^3$ without rendering the configuration singular. It should be stressed
that, without the periodicity (2.6) and the above mentioned boundary condition, the Hopf
term (2.5) may not be an integer nor gauge invariant, and that this is also true for the
conventional expression (1.7). We also mention that the Hopf term $H(g)$ is gauge invariant
in so far as the gauge transformation preserves the above boundary condition. The $O(3)$

\footnote{Alternatively, one may define an integer-valued Hopf term by imposing, instead of (2.10), the strict periodicity $h_c = 1$ in (2.6). This however leads to a formula which is not gauge invariant due to $P(g)$.}
NSM in the AOP hence possesses the residual gauge symmetry respecting the boundary condition in the presence of the Hopf term.

3. Hamiltonian Formulation

Having introduced the AOP and thereby defined the Hopf term for the $O(3)$ NSM, we now move on to furnish the Hamiltonian formulation of the model. To this end, we first note that in the AOP the action (1.1), supplemented with the Hopf term (2.5), turns out to be

$$I = \frac{1}{2\lambda^2} \int_M d^3x \, \text{Tr} \left( g^{-1}(x) \partial_\mu g(x) \right)_r^2 + \frac{\theta}{48\pi^2} \int_M \text{Tr} \left( g^{-1}(x) dg(x) \right)^3. \quad (3.1)$$

Here the symbol $X|_r$ denotes the projected part of $X \in g$ in the decomposition $X = X|_h + X|_r$ which is performed according to the decomposition $g = h \oplus r$ where $r$ is the orthogonal complement of the Lie subalgebra $h$ of $H$. In the present case we have $r = \text{span}\{T_1, T_2\}$ and hence, e.g., the kinetic term in the action reads $\text{Tr} (g^{-1}\partial_\mu g)_r^2 = (g^{-1}\partial_\mu g)^2_{a'}$, where the primed indices indicate the components $a' = 1, 2$ in the space $r$. This, of course, is due to the fact that the target space of the $O(3)$ NSM is $S^2$ rather than $SU(2)$. In what follows, however, we introduce the canonical structure of the NSM by symmetric reduction from that of the model defined on the $SU(2)$ group manifold ($i.e.$, the $SU(2)$ principal chiral model) regarding the NSM as a constrained system.

To this end, let us introduce a set of local coordinates $\{q^a; a = 1, 2, 3\}$ to parametrize $g = g(q)$ and define the matrix $N_{ab}(q) := (g^{-1} \frac{\partial}{\partial q^a} g)_b(q)$ which is invertible [17]. From the Lagrangian in $I = \int d^3x \, \mathcal{L}$ we find the canonical momentum conjugate to $q^a$,

$$\pi_a := \frac{\partial \mathcal{L}}{\partial (\partial_0 q^a)} = \frac{1}{\lambda^2} N_{ab'}(g^{-1} \partial_0 g)_b' + \frac{\theta}{32\pi^2} \epsilon_{ij} \epsilon_{abcd} N_{ab}(g^{-1} \partial_i g)_c(g^{-1} \partial_j g)_d, \quad (3.2)$$

which is supposed to satisfy the Poisson bracket relation,

$$\{q^a(x), \pi_b(y)\} = \delta^b_a \delta(x-y). \quad (3.3)$$

A much more convenient quantity than the canonical momentum is the ‘right-current’ $R = R_a T_a$ with

$$R_a := -(N^{-1})_{ab} \pi_b = -\frac{1}{\lambda^2} \delta_{ab'}(g^{-1} \partial_0 g)_b' - \frac{\theta}{32\pi^2} \epsilon_{ij} \epsilon_{acd} (g^{-1} \partial_i g)_c(g^{-1} \partial_j g)_d. \quad (3.4)$$
These components fulfill the relations,

\[
\{ R_a(x), R_b(y) \} = \epsilon_{abc} R_c(x) \delta(x - y), \quad \{ R_a(x), g(y) \} = g(x) T_a \delta(x - y),
\]  

(3.5)

where \( g \) is assumed to be in the defining representation. Together with

\[
\{ g(x), g(y) \} = 0,
\]  

(3.6)

the relations (3.5) form the fundamental Poisson bracket of the NSM in the AOP description at the non-reduced level. It is also worth mentioning that we can construct the ‘left-current’,

\[
L := -g R g^{-1},
\]  

(3.7)

whose components \( L = L_a T_a \) satisfy

\[
\{ L_a(x), L_b(y) \} = \epsilon_{abc} L_c(x) \delta(x - y), \quad \{ L_a(x), g(y) \} = -T_a g(x) \delta(x - y). \]  

(3.8)

The bracket relations, (3.5) and (3.8), show that \( R_a \) and \( L_a \) are the generators of the right and left actions on \( g \), respectively, and hence they commute,

\[
\{ R_a(x), L_b(y) \} = 0.
\]  

(3.9)

The canonical structure of the \( O(3) \) NSM is then found by taking into account the constraint,

\[
\Phi := R_3 + \frac{\theta}{32\pi^2} \epsilon_{ij} \epsilon_{i'j'} (g^{-1} \partial_i g)_{c'} (g^{-1} \partial_j g)_{d'} = 0,
\]  

(3.10)

derived from (3.4). To see if any secondary constraints arise from this primary constraint, we consider the Hamiltonian,

\[
\mathcal{H} := \pi_a \partial_0 q^a - \mathcal{L} = \frac{\lambda^2}{2} \left[ R_{a'} + \frac{\theta}{16\pi^2} \epsilon_{ij} \epsilon_{i'j'} (g^{-1} \partial_i g)_{c'} (g^{-1} \partial_j g)_{d'} \right]^2 + \frac{1}{2\lambda^2} (g^{-1} \partial_i g)^2_a,
\]  

(3.11)

where we have used the constraint (3.10) in the second line. Then, using the relation

\[
\{ R_a(x), (g^{-1} \partial_i g)_{b}(y) \} = \epsilon_{abc} (g^{-1} \partial_i g)_{c}(x) \delta(x - y) - \delta_{ab} \partial_i \delta(x - y)
\]  

(3.12)

derived from the fundamental Poisson bracket, we can readily confirm that the constraint (3.10) persists (strongly) in time,

\[
\{ \Phi(x), \mathcal{H}(y) \} = 0,
\]  

(3.13)
and hence no further constraints appear. Combined with the involutive property,

\[ \{ \Phi(x), \Phi(y) \} = 0, \quad (3.14) \]

the persistency (3.13) implies that the constraint (3.10) is first-class, and therefore generates a gauge symmetry — indeed it is the generator for the gauge transformation (2.2). It should be pointed out, however, that due to the second term in the constraint (3.4) the gauge transformation has a \( \theta \)-dependence, which shows up in the transformation of the quantities involving the current.

Models with first-class constraints are often dealt with using the approach in which one finds a set of second-class constraints that contains the original first-class ones and thereby defines the Dirac bracket from the Poisson bracket so as to form the true Poisson bracket of the reduced system. In the present paper we do not adopt this approach, and instead employ another approach where one considers gauge invariant quantities (physical observables) among which the Poisson bracket agrees with the Dirac bracket. Since the distinction between the ‘strong equality’ and the ‘weak equality’ is unnecessary in this approach, we use the ordinary equality symbol ‘=’ throughout this paper, even when the weak equality symbol ‘\( \approx \)’ is used in the former approach.

In this respect, we mention a physically important observable, namely, the ‘magnetic field’,

\[ B = \epsilon_{ij} \partial_i A_j, \quad \text{where} \quad A_j := (g^{-1} \partial_j g)_3, \quad (3.15) \]

which is gauge invariant \( \{ \Phi, B \} = 0 \) and gives rise to the flux penetrating the disc \( D^2 \) proportional to the soliton charge,

\[ \int_{D^2} d\mathbf{x} B = -4\pi Q. \quad (3.16) \]

The magnetic field also appears in the constraint (3.10) which can be rewritten as \( \Phi = R_3 - \frac{\theta}{16\pi^2} B \). More generally, let us consider the fields,

\[ B_a := \epsilon_{ij} \partial_i (g^{-1} \partial_j g)_a, \quad (3.17) \]

and construct the current \( \mathcal{R} = \mathcal{R}_a T_a \) with

\[ \mathcal{R}_a := R_a - \frac{\theta}{16\pi^2} B_a. \quad (3.18) \]
Note that the constraint (3.10) is now the third component of the new right-current,

\[ \mathcal{R}_3 = \Phi = 0. \] (3.19)

Observe also that the current is covariant under the gauge transformation,

\[ \{ \Phi(x), \mathcal{R}_{a'}(y) \} = \epsilon_{a'b'} \mathcal{R}_{b'}(x) \delta(x - y). \] (3.20)

In fact, this is part of the salient property that the covariant right-current (3.18) forms exactly the same Poisson bracket\(^7\) as the original right-current,

\[ \{ \mathcal{R}_a(x), \mathcal{R}_b(y) \} = \epsilon_{abc} \mathcal{R}_c(x) \delta(x - y), \quad \{ \mathcal{R}_a(x), g(y) \} = g(x) T_a \delta(x - y). \] (3.21)

Being both covariant and fundamental (under the Poisson bracket), the current components \( \mathcal{R}_{a'} \), together with the components \((g^{-1} \partial_i g)_{a'}\), can be used as basic building blocks for constructing physical observables.

For instance, since \( \mathcal{R}_{a'} = -\frac{1}{\lambda^2} (g^{-1} \partial^0 g)_{a'} \) the symmetric energy-momentum tensor derived from the action (3.1),

\[ T^{\mu\nu} = \frac{1}{\lambda^2} (g^{-1} \partial^\mu g)_{a'} (g^{-1} \partial^\nu g)_{a'} - \frac{1}{2\lambda^2} \eta^{\mu\nu} (g^{-1} \partial^\rho g)_{a'}^2, \] (3.22)

is found to be built up only with those covariant elements and is hence manifestly gauge invariant. In particular, the components for energy and momentum read

\[ T^{00} = \frac{\lambda^2}{2} \mathcal{R}_{a'}^2 + \frac{1}{2\lambda^2} (g^{-1} \partial_i g)_{a'}^2 = \mathcal{H}, \] (3.23)

and

\[ T^{0i} = \mathcal{R}_{a'} (g^{-1} \partial_i g)_{a'}, \] (3.24)

respectively.

With respect to the covariant right-current \( \mathcal{R} \) we can define the new left-current,

\[ \mathcal{L} := -g \mathcal{R} g^{-1} = L + \frac{\theta}{16\pi^2} \epsilon_{ij} \partial_i (\partial_j g g^{-1}). \] (3.25)

---

\(^7\) Equivalently, one can also say that, although one finds a functional potential in (3.18) which gives rise to a holonomy in the functional space, the corresponding functional curvature vanishes [20, 14].
Clearly, the components of the current $L = L_a T_a$ satisfy the same relations as before,

$$\{L_a(x), L_b(y)\} = \epsilon_{abc} L_c(x) \delta(x - y), \quad \{L_a(x), g(y)\} = -T_a g(x) \delta(x - y), \quad (3.26)$$

and commute with the covariant right-current,

$$\{R_a(x), L_b(y)\} = 0. \quad (3.27)$$

Being gauge invariant, all the components $L_a$ are physical observables. Note that, as a vector, the physical left-current is orthogonal to the spin vector,

$$L_a n_a = \text{Tr}(L n) = \text{Tr}(g^{-1} L g) T_3 = -R_3 = 0, \quad (3.28)$$
on account of the constraint (3.19).

To complete our gauge invariant approach, it is necessary to find a physical observable corresponding to the field $g$. One obvious way to do this is to go back to the spin vector $n(x)$. The defining relation (2.1) of the AOP shows that it is trivially gauge invariant,

$$\{\Phi(x), n_a(y)\} = 0,$$

and that it transforms as isovector with respect to the left-action,

$$\{L_a(x), n_b(y)\} = \epsilon_{abc} n_c(x) \delta(x - y). \quad (3.29)$$

Based on the Hamiltonian formulation of the NSM developed above, we carry out the canonical quantization by replacing the Poisson bracket with the commutator, $\{ , \} \rightarrow \frac{i}{\hbar}[ , ]$, after promoting the classical quantities $A$ to the corresponding quantum operators $\hat{A}$. We shall use the quantum language in sect.5 to construct soliton operators explicitly. Prior to this, however, in the next section we wish to examine the issue of fractional spin which has been argued to occur in the presence of the Hopf term.

4. Fractional Spin

In this section, following the line of argument given in [1], we shall study the angular momentum of the system paying particular attention to the $\theta$-dependence of the angular momentum which is the source of fractional spin. In $(2+1)$-dimensions, the space rotation group $SO(2)$ is Abelian and the generator is ambiguous up to the addition of a constant.
This ambiguity can be removed \cite{1} by embedding the SO(2) in the full ‘Lorentz group’ \(SO(2,1)\) and thereby define the angular momentum as

\[
J := \int dx \, \epsilon_{ij} \, x^i T^{0j} = \int dx \, \epsilon_{ij} \, x^i R_{a'} (g^{-1} \partial_j g)_{a'}. \quad (4.1)
\]

The angular momentum is a physical observable since its gauge invariance, \(\{ \Phi(x), J \} = 0\), follows from the invariance of \(T^{0j}\). It acts as a generator for the infinitesimal spatial rotation,

\[
\{ J, g \} = \epsilon_{ij} \, x^i \partial_j g - \epsilon_{ij} \, x^i (g^{-1} \partial_j g)^3 g T_3, \quad (4.2)
\]

leading to the transformation of the spin vector as scalar, \(\{ J, n_a \} = \epsilon_{ij} \, x^i \partial_j n_a\). We note in passing that in the application for condensed matter systems where, e.g., \(n(x)\) represents the spin of the valence electrons, the field \(n(x)\) transforms as a vector under the rotation about the axis perpendicular to the plane. In our case, however, we assume \(n(x)\) to be a scalar for the reason that we focus on the possible fractional orbital angular momentum which arises in the \(\theta\)-dependent part in \(J\).

In order to discuss the \(\theta\)-dependence of the angular momentum, let us split it into two parts \(J = J_1 + J_2\) with

\[
J_1 := \int dx \, \epsilon_{ij} \, x^i \, R_{a'} (g^{-1} \partial_j g)_{a'}, \quad J_2 := -\frac{\theta}{16\pi^2} \int dx \, \epsilon_{ij} \, x^i \, B_{a'} (g^{-1} \partial_j g)_{a'}. \quad (4.3)
\]

If we use \(B_{a'} = -\epsilon_{kl} \epsilon_{a'b'} (g^{-1} \partial_k g)_{b'} (g^{-1} \partial_l g)^3\) obtained from (3.17), we can rewrite the manifestly \(\theta\)-dependent part \(J_2\) purely in terms of the gauge field \(A_i\) and its curvature \(B\) in (3.15) as

\[
J_2 = \frac{\theta}{16\pi^2} \int dx \, \epsilon_{ij} \, x^i \, \epsilon^{kl} (g^{-1} \partial_k g)^3 \text{Tr} (T_3 [g^{-1} \partial_l g, g^{-1} \partial_j g])
\]

\[
= \frac{\theta}{16\pi^2} \int dx \, \epsilon_{ij} \, x^i A_j B. \quad (4.4)
\]

We stress that, in general, neither \(J_1\) nor \(J_2\) is gauge invariant in itself,

\[
\{ \Phi(x), J_1 \} = -\{ \Phi(x), J_2 \} = \epsilon_{ij} \, x^i \partial_j B(x). \quad (4.5)
\]

The exception occurs when the curvature depends only on the radial distance \(r = |x|\) of the disc \(B(x) = B(r)\), in which case \(J_1\) and \(J_2\) are separately gauge invariant.
It is interesting to observe that the part $J_2$ admits a concise formula in the Coulomb gauge $\partial_i A_i = 0$,

$$J_2 = \frac{\theta}{4\pi} Q^2,$$  

(4.6)

allowing for the interpretation that the fractional ($\theta$-dependent) part is proportional to the square of the soliton number of the configuration under consideration. The derivation of the $Q^2$-formula (4.6) is essentially the same as the one which has been given for evaluating the similar (but not exactly the same; see below) part in the angular momentum [1]. Namely, we first write the potential as $A_i = \epsilon_{ij} \partial_j \rho$, and find that the function $\rho$ is given by $\rho(x) = \int d\mathbf{y} D(\mathbf{x} - \mathbf{y}) B(\mathbf{y})$ with $D(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi} \delta(\mathbf{x} - \mathbf{y}) = -\frac{1}{4\pi} \ln |\mathbf{x} - \mathbf{y}|^2$ being the inverse of the Laplacian $\Delta = \partial_i \partial^i$. We then have

$$A_i(x) = \epsilon_{ij} \partial_j \int d\mathbf{y} \left( \frac{1}{4\pi} \ln |\mathbf{x} - \mathbf{y}|^2 B(\mathbf{y}) \right).$$

(4.7)

Substituting (4.7) into (4.4) we get

$$J_2 = \frac{\theta}{16\pi^2} \left( \frac{1}{4\pi} \right) \int d\mathbf{x} d\mathbf{y} \epsilon_{ij} x^i \epsilon_{jk} B(\mathbf{x}) \left( \partial_k \ln |\mathbf{x} - \mathbf{y}|^2 \right) B(\mathbf{y})$$

(4.8)

$$= \frac{\theta}{32\pi^3} \int d\mathbf{x} d\mathbf{y} \frac{x_i (x_i - y_i)}{|\mathbf{x} - \mathbf{y}|^2} B(\mathbf{x}) B(\mathbf{y}),$$

which proves (4.6) on account of (3.16).

We cannot, however, conclude from this result that the fractional spin of the system always occurs according to the square charge rule (4.6), simply because the value of $J_2$ depends in general on the gauge chosen; in other words, $J_2$ is not a physical observable for a generic configuration. Moreover, as we shall see more explicitly soon, the part $J_1$ also possesses an implicit $\theta$-dependence in view of the relation (3.4). Clearly, the correct $\theta$-dependence of the spin, or the physically meaningful value of the fractional spin, can be obtained only when one succeeds to separate the gauge invariant $\theta$-dependent part from the total $J$. This however seems impossible if one is to use the set of covariant/invariant elements constructed in sect.3 out of the basic variables in the phase space, and has certainly been unaccomplished by anyone so far. Flawed with this problem, the $Q^2$-formula (1.8) for the fractional part of angular momentum obtained earlier [1, 14, 3] (where the factor of the formula is different from ours due to the different split employed) is untenable as a formula for a generic configuration.
Let us now consider a class of specific configurations to illustrate the gauge dependence in the split of the angular momentum and to examine how the angular momentum can be fractionalized through the physical $\theta$-dependence. To this end, we introduce the polar coordinates $x = (r \cos \varphi, r \sin \varphi)$ with $(r, \varphi) \in [0, 1] \times [0, 2\pi]$ to parametrize the disc $D^2$ of unit radius. We then take the configuration,

$$\mathbf{n}(x, t) = \left( \cos (\alpha(x) + \phi(t)) \sin \beta(x), \sin (\alpha(x) + \phi(t)) \sin \beta(x), \cos \beta(x) \right), \quad (4.9)$$

where $\alpha(r, \varphi)$ and $\beta(r, \varphi)$ are time independent functions representing a generic static configuration possessing the soliton number $n$, say. The only dynamical variable $\phi(t)$, on the other hand, is the collective coordinate representing, say, $m$-times the revolution of the static configuration around the origin $x = 0$ during the time period $[0, T]$. These topological requirements will be met if we assume the boundary conditions,

$$\alpha(r, 2\pi) - \alpha(r, 0) = 2n\pi, \quad \beta(1, \varphi) = \pi, \quad \beta(0, \varphi) = 0, \quad \phi(T) - \phi(0) = 2m\pi. \quad (4.10)$$

The spin vector field (4.9) is realized in the AOP by

$$g(x, t) = e^{(\alpha(x) + \phi(t))T_3} e^{\beta(x)T_2} e^{-(\alpha(x) - \phi(t))T_3} e^{\eta(x, t)T_3}, \quad (4.11)$$

where we have introduced the function $\eta(x, t)$ (which is regular over $D^2$) to account for the gauge freedom. This configuration satisfies the condition (2.10) if $\eta(x, t)$ is time independent $\frac{d}{dt} \eta(x, t) = 0$ on the boundary $\partial D^2$. With the above boundary conditions (4.10) one can easily confirm that our configuration (4.11) has the following soliton and Hopf numbers,

$$Q(g) = -\frac{1}{4\pi} \int_{\partial D^2} d\alpha (\cos \beta - 1) = \frac{1}{4\pi} \times 2n\pi \times 2 = n, \quad (4.12)$$

$$H(g) = \frac{Q(g)}{2\pi} \int_0^T d\phi = \frac{1}{2\pi} \times n \times 2m\pi = nm.$$

Under the class of configurations (4.11) we have the potential $A_i = \partial_i \alpha (\cos \beta - 1) + \partial_i \eta$ and hence the curvature,

$$B = \frac{1}{r} \left( \partial_\varphi (\cos \beta) \partial_r \alpha - \partial_\varphi \alpha \partial_r (\cos \beta) \right). \quad (4.13)$$

The gauge dependence of the $J_2$ part can now be explicitly demonstrated by taking, for instance, the configuration, $\alpha(r, \varphi) = \frac{n}{2\pi} \varphi^2, \beta(r, \varphi) = r$ for which we have $B = \frac{\sin(r)}{r} \frac{n \varphi}{\pi}$. 

15
Since the curvature depends on $\varphi$, we learn from our earlier discussion that $J_2$ cannot be gauge invariant. In fact, substituting the potential and the curvature in (4.4) we find $J_2 = \frac{\theta}{3\pi} n^2$ for, e.g., $\eta = 0$, which disagrees with the result (4.6) obtained in the Coulomb gauge. It even departs from the $Q^2$ rule for $\eta = \varphi (\varphi - 2\pi)$ where we get $J_2 = \frac{\theta}{3\pi} n^2 + \frac{\theta}{6} n$.

This disagreement stems, of course, from the fact that our potential $A_i$ does not satisfy the Coulomb gauge. However, behind this lies the important question as to how the fractional part of the angular momentum can be defined gauge invariantly. To examine this issue, let us evaluate the part $J_1$ and see how the gauge non-invariance disappears when combined with $J_2$. Plugging the configuration (4.11) in (4.3), we find

$$J_1 = -N \dot{\phi} - \frac{\theta}{16\pi^2} \int dx \epsilon_{ij} x^i A_j B,$$

(4.14)

with $N = \frac{1}{\lambda^2} \int dx \sin^2 \beta \partial \varphi \alpha$. To express $\dot{\phi}$ in terms of the conjugate momentum $\Pi_\phi$, we obtain from the action $I = \int dt L$ in (3.1) the Lagrangian for the collective mode in (4.11),

$$L = -M + \frac{K}{2} \dot{\phi}^2 + \frac{\theta}{2\pi} n \dot{\phi},$$

(4.15)

where $M = \frac{1}{\lambda^2} \int dx \left( \sin^2 \beta (\partial_i \alpha)^2 + (\partial_i \beta)^2 \right)$ is the mass of the static configuration and $K = \frac{1}{\lambda} \int dx \sin^2 \beta$ is the ‘nonrelativistic mass’ of the collective mode excitation. The momentum is then obtained as

$$\Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = K \ddot{\phi} + \frac{\theta}{2\pi} n.$$

(4.16)

Thus, with the ‘moment of inertia’,

$$I := \frac{N}{K} = \frac{\int dx \sin^2 \beta \partial \varphi \alpha}{\int dx \sin^2 \beta},$$

(4.17)

we arrive at the formula,

$$J_1 = - \left( \Pi_\phi - \frac{\theta}{2\pi n} \right) I - \frac{\theta}{16\pi^2} \int dx \epsilon_{ij} x^i A_j B.$$

(4.18)

It is now clear that the last term in (4.18) is gauge dependent but canceled precisely when combined with $J_2$ in (4.4), leaving the first term as the physical total angular momentum $J$ of the system.
Upon quantization, the dynamical variables $\phi$ and $\Pi_\phi$ are promoted to the operators $\hat{\phi}$ and $\hat{\Pi}_\phi$ satisfying the commutator $[\hat{\Pi}_\phi, \hat{\phi}] = \frac{1}{i}$. In the coordinate representation we have $\hat{\Pi}_\phi = \frac{1}{i} \frac{d}{d\phi}$, and therefore the wave functions $\Psi_k(\phi) = e^{ik\phi}$ for $k \in \mathbb{Z}$ provide the eigenfunctions of the Hamiltonian obtained from the Lagrangian (4.15),

$$\hat{H} = M + \frac{1}{2K} \left( \hat{\Pi}_\phi - \frac{\theta}{2\pi} n \right)^2,$$

(4.19)

with the eigenvalues,

$$E_k = M + \frac{1}{2K} \left( k - \frac{\theta}{2\pi} n \right)^2.$$

(4.20)

The eigenstates have the total angular momentum,

$$J_k = - \left( k - \frac{\theta}{2\pi} n \right) I,$$

(4.21)

which shows that the fractional spin part is independent of the energy level and given by

$$J_{\text{fractional}} = \frac{\theta}{2\pi} n I.$$

(4.22)

At this point we note that, because of the boundary condition (2.3) with (2.10), the only global symmetry of the original action (3.1) is the one under the combined transformation of the spatial $U(1)$ rotation, generated by letting $\varphi \to \varphi + \epsilon$, and the isospin $U(1)$ rotation with respect to $T_3$, generated by $g \to e^{i\delta T_3} g$. Thus we may restrict our attention to configurations which exhibit this symmetry in the class (4.11) we are considering. An obvious case in which this symmetry is realized occurs (with $\delta = -n\epsilon$) when we have the functions $\alpha$ and $\beta$ of the form,

$$\alpha(r, \varphi) = n\varphi, \quad \beta(r, \varphi) = \beta(r).$$

(4.23)

It follows from (4.13) that in this case the magnetic field becomes rotationally invariant $B = B(r)$, and therefore both $J_1$ and $J_2$ are gauge invariant. Thus the $Q^2$ rule (4.6) for $J_2$ indeed holds but, as we have seen above, this alone does not determine the $\theta$-dependence of the angular momentum because $J_1$ is also $\theta$-dependent. The correct formula for the fractional spin part in the present case can be obtained from (4.22) by noticing that the moment of inertia (4.17) coincides with the soliton number

$$I = n,$$

(4.24)

17
for the configurations (4.23), leading to

\[ J_{\text{fractional}} = \frac{\theta}{2\pi} n^2, \]  

in agreement with (1.8).

Thus we conclude that the \( Q^2 \)-formula (1.8) for fractional spin is valid for configurations of the type (4.23) for which the curvature \( B \) depends only on the radial direction. The known \( n \)-soliton solution actually falls into this restricted class, but generic ones having soliton number \( n \) do not, and for those the formula (1.8) does not hold, as seen by (4.22) which is obtained for our class (but not restricted one) of configurations for which \( I \) is not necessarily \( n \). We recall that in the literature one normally splits the angular momentum \( J \) with the intention that the first part \( J_1 \) becomes the ordinary orbital angular momentum implementing the spatial \( U(1) \) rotation for the spin vectors \( n(x) \) while the second part \( J_2 \) represents the additional contribution induced by the Hopf term — hence proportional to the angle parameter \( \theta \) — causing the fractional spin. Our discussion in this section shows that the angular momentum does not seem to admit the split in the way intended; indeed, our \( J_1 \) in (4.3) does meet the requirement and yet is not \( \theta \)-independent nor a physical observable.

5. Soliton Operators

The ‘soliton operator’ which generates a soliton state has previously been constructed in [19] in order to discuss the fractional spin at the quantum level. The operator introduced there creates a single soliton on the ‘classical vacuum state’, i.e., the eigenstate of the spin vector operator \( \hat{n}(x) \) with the constant eigenvalue \( n(\infty) \) (for all \( x \)) specified by the boundary condition (1.2). Below we shall argue that in the AOP the construction of the operator becomes more transparent and further that it can be generalized to operators creating a state concentrated about a generic classical configuration.

To begin with, we note that the Hilbert space of the NSM may be defined as the physical space \( \mathcal{H}_{\text{phys}} \) consisting of states satisfying the physical state condition, \( \hat{\Phi} |\text{phys}\rangle = 0 \), in the entire Hilbert space \( \mathcal{H} \) of the unconstrained model, that is, the principal chiral model. Being a physical observable, the spin vector operator given by

\[ \hat{n}(x) = \hat{n}_a(x) T_a = \hat{g}(x) T_3 \hat{g}^{-1}(x), \]  

(5.1)
admits the coordinate representation within the physical space $H_{\text{phys}}$:
\[
\hat{n}(x) |n(x)\rangle = n(x) |n(x)\rangle. \tag{5.2}
\]

Among the eigenstates is the classical vacuum state mentioned above,
\[
\hat{n}(x) |n(\infty)\rangle = n(\infty) |n(\infty)\rangle. \tag{5.3}
\]

Now, suppose we are given two arbitrary classical configurations, $n_1(x)$ and $n_2(x)$, possessing different soliton numbers in general. Then there are corresponding states in $H_{\text{phys}}$ having these configurations as eigenvalues. What we shall show below is that in the AOP the unitary operator which relates these two states,
\[
|n_2(x)\rangle = \hat{U} |n_1(x)\rangle, \tag{5.4}
\]
can be constructed along the method of Ref.[19] with more ease and transparency. To this end, let $g_1(x)$ and $g_2(x)$ be the fields associated with the two spin vector fields under the AOP, namely, $n_1 = g_1 T_3 g_1^{-1}$ and $n_2 = g_2 T_3 g_2^{-1}$, respectively. We then have the relation,
\[
n_2(x) = g_{21}(x) n_1(x) g_{21}^{-1}(x), \tag{5.5}
\]
with $g_{21} = g_2 g_1^{-1}$. Thus it is clear that the transformation on the spin vector $n_1 \rightarrow n_2$ can be achieved by the transformation $g_1 \rightarrow g_2 = g_{21} g_1$ in the AOP. Since this is a left-action on the field, it can be implemented if we take the unitary operator $\hat{U}$ to be the representation $\hat{U}(g_{21})$ of the group action in the Hilbert space.

Explicitly, if we use, e.g., the Euler angle decomposition for the element $g_{21}$,
\[
g_{21}(x) = e^{\alpha(x) T_3} e^{\beta(x) T_2} e^{\gamma(x) T_3}, \tag{5.6}
\]
then the unitary operator implementing the left-action is given by
\[
\hat{U}(g_{21}) = e^{i \int dx \alpha(x) \hat{L}_3(x)} e^{i \int dx \beta(x) \hat{L}_2(x)} e^{i \int dx \gamma(x) \hat{L}_3(x)}, \tag{5.7}
\]
where $\hat{L}_a$ are the operators corresponding to the gauge invariant left-current components $L_a$. It then follows from their commutation relations implied by the Poisson bracket (3.26),
\[
[\hat{L}_a(x), \hat{L}_b(y)] = i \epsilon_{abc} \hat{L}_c(x) \delta(x - y), \quad [\hat{L}_a(x), \hat{g}(y)] = -i T_a \hat{g}(x) \delta(x - y), \tag{5.8}
\]

19
that the unitary operator (5.7) provides a representation, \( \hat{U}(g_1) \hat{U}(g_2) = \hat{U}(g_1 g_2) \), and that it gives the left-action,

\[
\hat{U}^{-1}(g_{21}) \hat{g}(x) \hat{U}(g_{21}) = g_{21}(x) \hat{g}(x),
\]

(5.9)
as intended. Using (5.1) and (5.9) we obtain

\[
\hat{n}(x) \hat{U}(g_{21}) |n_1(x)\rangle = \hat{U}(g_{21}) g_{21}(x) \hat{n}(x) g_{21}^{-1}(x) |n_1(x)\rangle \\
= g_{21}(x) n_1(x) g_{21}^{-1}(x) \hat{U}(g_{21}) |n_1(x)\rangle \\
= n_2(x) \hat{U}(g_{21}) |n_1(x)\rangle,
\]

(5.10)

which shows that the unitary operator \( \hat{U}(g_{21}) \) does the job required in (5.4) and hence can be regarded as the operator creating the state \( |n_2(x)\rangle \) out of \( |n_1(x)\rangle \). In particular, when the state \( |n_1(x)\rangle \) is the classical vacuum state \( |n(\infty)\rangle \) in (5.3) and the newly created state \( |n_2(x)\rangle \) has a nonvanishing soliton number, the unitary operator \( \hat{U} \) may be thought of as a soliton operator. Note that the unitary operator that has the effect (5.10) is not unique, since the same construction (5.7) using the operators corresponding to the components of the original left-current \( \hat{L}_a \) yields the identical result (5.10). However, in this case the resultant state \( \hat{U}(g_{21}) |n_1(x)\rangle \) cannot belong to the physical Hilbert space \( \mathcal{H}_{\text{phys}} \), because unlike \( \hat{L}_a \) the operators \( \hat{L}_a \) are not gauge invariant and do not commute with \( \hat{\Phi} \). We also note that, since the unitary operator \( \hat{U}(g_{21}) \) is defined with respect to the eigenstates \( |n(x)\rangle \) used for the coordinate representation (5.2), it has an intrinsic ambiguity associated with the choice of the eigenstates which are determined up to the form, \( \hat{U}(e^{\xi(x)n(x)}) |n(x)\rangle \) with \( \xi(x) \) being a function.

Finally, we show that the unitary operator (5.7) does possess the correct soliton number required to create the state \( |n_2(x)\rangle \) from \( |n_1(x)\rangle \). Indeed, from (5.9) we observe that the soliton operator \( \hat{Q}(g) := Q(\hat{g}) \) transforms as

\[
\hat{U}^{-1}(g_{21}) \hat{Q}(g) \hat{U}(g_{21}) = -\frac{1}{4\pi} \int_{\partial D^2} \text{Tr} T_3(\hat{g}^{-1}d\hat{g}) - \frac{1}{4\pi} \int_{\partial D^2} \text{Tr} \hat{n} (g_{21}^{-1}dg_{21}).
\]

(5.11)

Since the two configurations \( g_1 \) and \( g_2 \) share the same boundary value \( g(\infty) \) as \( g_1(x) = g(\infty)k_1(x), g_2(x) = g(\infty)k_2(x) \) with \( k_1, k_2 \in H \), we have at spatial boundary,

\[
\text{Tr} \hat{n} (g_{21}^{-1}dg_{21})|_{\partial D^2} = \text{Tr} T_3(k_2k_1^{-1})^{-1}d(k_2k_1^{-1}) = \text{Tr} T_3(k_2^{-1}dk_2) - \text{Tr} T_3(dk_1 k_1^{-1}).
\]

(5.12)
Hence (5.11) becomes

$$
\hat{U}^{-1}(g_{21}) \hat{Q}(g) \hat{U}(g_{21}) = \hat{Q}(g) + Q(g_{2}) - Q(g_{1}),
$$

that is, the operator \( \hat{U}(g_{21}) \) shifts the soliton number precisely by the amount of the soliton charge difference between the two configurations, \( g_{1} \) and \( g_{2} \). We point out that our formulation of the soliton operator (5.7) does not require the patching procedure needed in the previous construction [19] where one needs two operators defined on two different patches covering the space \( S^{2} \). The reason for this is basically due to the trivialization of the space \( S^{2} \rightarrow D^{2} \), where we traded the topological property of the spin field \( n(x) \) for the behavior of the field \( g(x) \) at the boundary (2.3).

6. Conclusions and Discussions

We have seen in this paper that the AOP, which employs \( SU(2) \) group-valued fields \( g(x) \) defined on the space \( D^{2} \) for describing the spin vectors of the NSM, is useful in analyzing the topological aspect of the model. We first provided a boundary condition at spatial infinity (i.e., the boundary \( \partial D^{2} \)) that renders the conversion procedure unnecessary, which was required earlier to provide the Hopf term for generic configurations. After this, we presented the Hamiltonian formulation of the NSM in the AOP, where we observed that the model can be interpreted as a constrained system of the \( SU(2) \) principal chiral model. The constraint is first-class and hence generates a gauge symmetry that corresponds to the ambiguity in the AOP. Accordingly, as a reduced system the NSM is described by a full set of physical observables consisting of gauge invariant quantities. The gauge invariance turned out to be crucial in sorting out the correct, physically meaningful, fractional spin part in the total angular momentum of the model. We found that the \( Q^{2} \)-formula for fractional spin proposed earlier does not hold in general, although it is correct for a restricted class of configurations which includes the soliton solutions. It should however be noted that the problem of fractional spin can be addressed (and answered) properly only at the quantum level, rather than the classical level [11], and for this we seem to lack a method to analyze the problem on a general basis except that of using a collective coordinate about a specific class of configurations as we did here. The gauge invariance was also used to determine the soliton operator which creates the state concentrated around a
generic classical configuration upon the classical vacuum state. Our construction is based on the observation that the soliton operator is given by the unitary representation of the left-action in the Hilbert space implementing the transition from one configuration to the other. Being group-valued, the AOP is most appropriate to this purpose.

Our AOP description of the NSM may be extended to the general coset nonlinear models over $G/H$ along the line of Ref.[21] where a set of distinct topological charges are allowed, bearing a fractional spin formula bilinear in the charges. Interestingly, such a formula, which is an analogue of the $Q^2$-formula mentioned above, seems to be the norm for soliton solutions in various $2+1$ dimensional models which exhibit fractional spin (see, for example, [22]). We are thus more than curious to see whether the bilinear formula holds universally in $2+1$ dimensions and, if so, to find a topological cause for the universality. We hope to answer this question in our future publications.

Acknowledgements: We are grateful to H. Otsu and S. Tanimura for helpful discussions and K. Horie and M. Barton for valuable comments. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.199707529).
References

[1] M.J. Bowick, D. Karabali, L.C.R. Wijewardhana, *Nucl. Phys.* B271 (1986) 417.
[2] G.W. Semenoff, P. Sodano, *Nucl. Phys.* B328 (1989) 753.
[3] S. Forte, *Rev. Mod. Phys.* 64 (1992) 193.
[4] S.L. Sondhi, A. Karlhede, S.A. Kivelson, E.H. Rezayi, *Phys. Rev.* B47 (1993) 16419.
[5] W. Apel, Y.A. Bychkov, *Phys. Rev. Lett.* 78 (1997) 2188; *Phys. Rev. Lett.* 79 (1997) 3792.
[6] G.E. Volovik, V.M. Yakovenko, *Phys. Rev. Lett.* 79 (1997) 3791; cond-mat/9711076.
[7] F. Wilczek, A. Zee, *Phys. Rev. Lett.* 51 (1983) 2250.
[8] P.K. Panigrahi, S. Roy, W. Scherer, *Phys. Rev. Lett.* 61 (1988) 2827; *Phys. Rev.* D38 (1988) 3199.
[9] M. Bergeron, G.W. Semenoff, R.R. Douglas, *Int. J. Mod. Phys.* A7 (1992) 2417.
[10] H. Otsu, H. Sato, *Prog. Theor. Phys.* 91 (1994) 1199; *Z. Phys.* C64 (1994) 177.
[11] B. Chakraborty, A.S. Majumdar, hep-th/9710028.
[12] T. Lee, C.N. Rao, K.S. Viswanathan, *Phys. Rev.* D39 (1989) 2350.
[13] Z. Hlousek, D. Sénéchal, S.-H.H. Tye, *Phys. Rev.* D41 (1990) 3773.
[14] N.K. Pak, R. Percacci, *Phys. Rev.* D43 (1991) 1375.
[15] H. Kobayashi, S. Tanimura, I. Tsutsui, KEK Preprint 97-19, hep-th/9705183, to appear in *Nucl. Phys.* B.
[16] A.P. Balachandran, A. Stern, G. Trahern, *Phys. Rev.* D19 (1979) 2416.
[17] A.P. Balachandran, G. Marmo, B.S. Skagerstam, A. Stern, “Classical Topology and Quantum States”, World Scientific, Singapore, 1991.
[18] Y.S. Wu, A. Zee, *Phys. Lett.* 147B (1984) 325;
[19] D. Karabali, *Int. J. Mod. Phys.* A6 (1991) 1369.
[20] Y.S. Wu, A. Zee, *Nucl. Phys.* B272 (1986) 322.
[21] E. D’Hoker, *Phys. Lett.* 357B (1995) 539.
[22] R. Jackiw, E.J. Weinberg, *Phys. Rev. Lett.* 64 (1990) 2234.