Thermal Ionization for Short-Range Potentials

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We study a model of an atom in form of a Schrödinger operator with a compactly supported potential coupled to a bosonic field at positive temperature. We show that, under sufficiently small coupling, it exhibits the behavior of thermal ionization. Mathematically, one has to rule out the existence of zero eigenvalues of the self-adjoint generator of time evolution – the Liouvillian. This will be done by using positive commutator methods with dilations on the space of scattering functions.

1. Introduction

The phenomenon of thermal ionization can be viewed as a positive temperature generalization of the photoelectric effect: an atom is exposed to thermal radiation emitted by a black-body of temperature $T > 0$. Then photons of momentum $\omega$ according to PLANCK’s probability distribution of black-body radiation,

$$\frac{1}{e^{\beta\omega} - 1}, \quad \beta \sim \frac{1}{T},$$

will interact with the atoms’ electrons. Since there is a positive probability for arbitrary high energy photons, eventually one with sufficiently high energy will show up exceeding the ionization threshold of the atom.

For the zero temperature situation (photoelectric effect) there can be found qualitative and quantitative statements for different simplified models of atoms with quantized fields in [3, 22, 9]. If one replaces the atom by a finite-dimensional small subsystem, the model usually exhibits the behavior of return to equilibrium,
see for example [2]. Here the existence of a Gibbs state of the atom leads to the existence of an equilibrium state of the whole system. One is confronted with the same mathematical problem – disproving existence of eigenvalues. The most common technique for handling this are complex dilations, or its infinitesimal analogue: positive commutators, which goes back to work of Mourre (cf. [13]). There are a number of papers which also use positive commutators in the context of return to equilibrium, see [8, 11, 12, 13].

A first rigorous treatment of the positive temperature problem was given by Fröhlich and Merkli in [5] and in a subsequent paper [6] by the same authors together with Sigal. The ionization appears mathematically as the absence of an equilibrium state in a suitable von Neumann algebra describing the complete system. The equilibrium states can be shown to be in one-to-one correspondence with the zero eigenvalues of the Liouvillian. For the proof they used a global positive commutator method first established for Liouvillians in [21] with a conjugation operator on the field space as in [12]. Furthermore, they developed a new virial theorem.

In [5] an abstract representation of a Hamiltonian diagonalized with respect to its energy with a single negative eigenvalue was considered. For the proof certain regularity assumptions of the interaction with respect to the energy were imposed. However, it is not so clear how these assumptions translate to a more concrete setting. The reference [6] on the other hand covers the case of a Schrödinger operator with a long-range potential but with only finitely many modes coupled via the interaction. Moreover, only a compact interval away from zero of the continuous spectrum was coupled.

The purpose of this paper is to transfer the results of [5, 6] to a more specific model of a Schrödinger operator with a compactly supported potential with finitely many eigenvalues. We consider a typical coupling term and we have to impose a spatial cutoff. However, we do not need any restrictions with respect to the coupling to the continuous subspace of the atomic operator as in [6]. Moreover, in contrast to [5, 6] our result is uniform in the temperature by considering an approximation of the so-called level shift operator. For the proof we use the same commutator on the space of the bosonic field as in [12, 5, 6], and we also reuse the original virial theorem of [5, 6]. On the other hand we work with a different commutator on the atomic space, namely the generator of dilations in the space of scattering functions.

The organization of the paper is as follows. In Section 2 we introduce the model and define the Liouvillian. In addition, we state the precise form of our main result in Theorem 2.3 and all the necessary assumptions. We also give a more detailed outline of the proof in Subsection 2.6. Section 3 recalls the abstract virial theorem of [5, 6] and some related technical methods. Then, we verify the requirements of
the abstract virial theorem in our setting in Section 4. We repeat the definition of scattering states (Subsection 4.1) and use those for the concrete choice of the commutators. The major difficulty here is to check that the commutators with the interaction terms are bounded. This requires some bounds with respect to the scattering functions and is elaborated in Section 5. The application of the virial theorem then yields a concrete version – Theorem 4.4. This is the first key element for the proof of the main theorem. The second one is the actual proof that the commutator together with some auxiliary term is positive. This and the concluding proof of Theorem 2.3 at the end can be found in Section 6.

2. Model and the Main Result

A model of a small subsystem interacting with a bosonic field at positive temperature is usually represented as a suitable \( C^* \)- or \( W^* \)-dynamical system on a tensor product algebra consisting of the field and the atom, respectively.

The field is defined by a Weyl algebra with infinitely many degrees of freedom. To implement black-body radiation at a specific temperature \( T > 0 \) the GNS representation with respect to a KMS (equilibrium) state depending on \( T \) is considered. For the atom the whole algebra \( \mathcal{L}(H_p) \) for the atomic Hilbert space \( H_p = L^2(\mathbb{R}^3) \) is used and the GNS representation with respect to an arbitrary mixed reference state is performed. The combined representation of the whole system generates a \( W^* \)-algebra where the interacting dynamics can be defined by means of a Dyson series in a canonical way. Its self-adjoint generator – the Liouvillian – is of great interest when studying such systems.

It is shown in [5, 6] that the absence of zero eigenvalues of the Liouvillian implies the absence of time-invariant states of the \( W^* \)-dynamical system. The details of this construction can be found in [5, 6, 16].

In this paper we start directly with the definition of the Liouvillian without repeating its derivation and the algebraic construction. The only difference in our setting to [5, 6] is the coupling term which can be realized as an approximation of step function couplings as considered in their work.

The purpose of this section is the definition of the concrete Liouvillian, the precise statement of the result – absence of zero eigenvalues – and the required conditions. At the end in Subsection 2.6, we explain the basic structure of the proof.

We start with the three ingredients of the model: the atom, the field and the interaction, which we first discuss separately and state the required assumptions.
2.1. The Atom

For the atom we consider a Schrödinger operator on the Hilbert space $\mathcal{H}_p := L^2(\mathbb{R}^3)$,

$$H_p = -\Delta + V,$$

where we assume that

(H1) $V \leq 0$ and $V \in C^\infty_c(\mathbb{R}^3)$,

(H2) zero is not an eigenvalue of $H_p$.

In particular, this implies that $H_p$ is essentially self-adjoint on $C^\infty_c(\mathbb{R}^3)$ with domain $D(\Delta)$ and has essential spectrum $[0, \infty)$ with finitely many discrete eigenvalues which we denote by $\sigma_d(H_p)$. As we are in a statistical physics setting, we have to consider density matrices as states of our system. They form a subset of the Hilbert-Schmidt operators, so we will work a space isomorphic to the latter,

$$\mathcal{H}_p \otimes \mathcal{H}_p.$$

Remark 2.1

For the result and our proof one actually needs just finitely many derivatives of $V$. Therefore, one could weaken (H1). However it seems a bit tedious in the proof to keep track to which order exactly derivatives are required.

2.2. The Quantized Field

The quantized field will be described by operators on Fock spaces. Let $\mathcal{F}(\mathfrak{h})$ denote the bosonic Fock space over a Hilbert space $\mathfrak{h}$, that is,

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h} \otimes_s^n,$$

where $\otimes_s^n$ is the $n$-times symmetric tensor product of Hilbert spaces, and $\mathfrak{h} \otimes_s^0 := \mathbb{C}$. For $\psi \in \mathcal{F}(\mathfrak{h})$, we write $\psi_n$ for the $n$-th element in the direct sum and we use the notation $\psi = (\psi_0, \psi_1, \psi_2, \ldots)$. The vacuum vector is defined as $\Omega := (1, 0, 0, \ldots)$. For a dense subspace $\mathfrak{d} \subseteq \mathfrak{h}$ we define the dense space of finitely many particles in $\mathcal{F}(\mathfrak{h})$ by

$$\mathcal{F}_{\text{fin}}(\mathfrak{d}) := \{(\psi_0, \psi_1, \psi_2, \ldots) \in \mathcal{F}(\mathfrak{h}) : \psi_n \in \mathfrak{d} \otimes_s^n \text{ for all } n \in \mathbb{N}_0 \text{ and there exists } N \in \mathbb{N}_0 : \psi_n = 0 \text{ for } n \geq N\}$$

where $\otimes_s^n$ now represents $n$-times symmetric tensor product of vector spaces. Again we will work on the space of density matrices and thus use the space

$$\mathfrak{F} := \mathcal{F}(L^2(\mathbb{R}^3)) \otimes \mathcal{F}(L^2(\mathbb{R}^3)) \cong \mathcal{F}(L^2(\mathbb{R} \times S^2)),$$
where \( L^2(\mathbb{R} \times S^2) := L^2(\mathbb{R} \times S^2, d\Sigma) \) and \( d\Sigma \) denotes the standard spherical measure on \( S^2 \), see Remark 2.2. This identification is called ‘gluing’ and was first established by Jakšić and Pillet in [11]. For \( \psi \in \mathfrak{F} \) notice that \( \psi_n \) can be understood as an \( L^2 \) function in \( n \) symmetric variables \((u, \Sigma) \in \mathbb{R} \times S^2\).

Let \( \mathcal{H} \) be another Hilbert space. On \( \mathcal{H} \otimes \mathfrak{F} \) we define the annihilation operator for a function \( F \in L^2(\mathbb{R} \times S^2, \mathcal{L}(\mathcal{H})) \) by

\[
(a(F)\psi)_n(u_1, \Sigma_1, \ldots, u_n, \Sigma_n) = \sqrt{n+1} \int F(u, \Sigma)\psi_{n+1}(u, \Sigma, u_1, \ldots, u_n, \Sigma_n) d(u, \Sigma).
\]

Note that \( a(F)\Omega = 0 \). The creation operator \( a^*(F) \) is defined as the adjoint of \( a(F) \). By definition \( F \mapsto a^*(F) \) is linear whereas \( F \mapsto a(F) \) is anti-linear. In the scalar case of \( \mathcal{H} = \mathbb{C} \), \( \mathcal{H} \otimes \mathfrak{F} = \mathfrak{F} \), the creation and annihilation operators satisfy the canonical commutation relations, for \( f, g \in L^2(\mathbb{R} \times S^2) \),

\[
[a(f), a^*(g)] = (f, g), \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0.
\]

We also define the field operators as \( \Phi(F) = 2^{-1/2}(a(F) + a^*(F)) \).

For a measurable function \( M : \mathbb{R} \times S^2 \to \mathbb{R} \) we introduce the second quantization \( d\Gamma(M) \), which is a self-adjoint operator on \( \mathfrak{F} \), given for \( \psi \in \mathfrak{F} \) by

\[
(d\Gamma(M)\psi)_n(u_1, \Sigma_1, \ldots, u_n, \Sigma_n) = \left( \sum_{i=1}^n M(u_i, \Sigma_i) \right) \psi_{n}(u_1, \Sigma_1, \ldots, u_n, \Sigma_n), \quad n \in \mathbb{N}_0.
\]

In particular we will use the number operator,

\[
N_i := d\Gamma(1).
\]

### 2.3. Liouvillian with Interaction

We assume to have an interaction term with a smooth spatial cutoff. For \((\omega, \Sigma) \in \mathbb{R}_+ \times S^2\), we define a bounded multiplication operator on \( \mathcal{H}_p \) by

\[
G(\omega, \Sigma)(x) = \kappa(\omega)\tilde{G}(\omega, \Sigma)(x), \quad x \in \mathbb{R}^3,
\]

where \( \kappa \) is a function on \( \mathbb{R}_+ \) and, for each \((\omega, \Sigma) \), \( \tilde{G}(\omega, \Sigma) \) is a function on \( \mathbb{R}^3 \), satisfying the following conditions.

(11) **Spatial cutoff:** For \( n \in \{0, \ldots, 3\} \) and all \((\omega, \Sigma), x \in \mathbb{R}^3\), the derivative \( \partial^\alpha_x \tilde{G}(\omega, \Sigma)(x) \) exists and the following holds. For all \((\omega, \Sigma) \) we have \( \partial^\alpha_x \tilde{G}(\omega, \Sigma) \in \mathcal{S}(\mathbb{R}^3) \), where \( \mathcal{S}(\mathbb{R}^3) \) denotes the space of Schwartz functions. \((\omega, \Sigma, x) \mapsto \tilde{G}(\omega, \Sigma)(x) \) is measurable, and for all \( \alpha \in \mathbb{N}_0^3 \) there exists a polynomial \( P \) such that

\[
\left| \partial^\alpha_x \partial^\nu_\Sigma \tilde{G}(\omega, \Sigma)(x) \right| \leq P(\omega), \quad (\omega, \Sigma) \in \mathbb{R}_+ \times S^2, \quad x \in \mathbb{R}^3.
\]
(12) **UV cutoff:** \( \kappa \) decays faster than any polynomial, that is, for all \( n \in \mathbb{N} \),

\[
\sup_{\omega \geq 1} \omega^n \kappa(\omega) < \infty.
\]

(13) **Infrared behavior:** \( \kappa |_{(0, \infty)} \in C^3((0, \infty)) \), and either there exist \( 0 < k_1, k_2 < \infty, p > 2 \) such that if \( \omega < k_1 \), we have

\[
\left| \partial_j^\omega \kappa(\omega) \right| \leq k_2 \omega^{p-j},
\]

for \( j \in \{0, \ldots, 3\} \), or

\[
\kappa(\omega) = \omega^{\pm \frac{1}{2}} \tilde{\kappa}(\omega), \quad \omega > 0,
\]

where \( \tilde{\kappa} \in C^3([0, \infty)) \), and all derivatives of \( \tilde{\kappa} \) are bounded.

Let \( \beta > 0 \) be the inverse temperature and let

\[
\rho_\beta(\omega) := \frac{1}{e^{\beta \omega} - 1}
\]

be the probability distribution for black-body radiation. To describe the interaction in the positive temperature setting it is convenient to introduce the map

\[
\tau_\beta : L^2(\mathbb{R}_+ \times S^2, (\omega^2 + \omega) d\omega d\Sigma) \rightarrow L^2(\mathbb{R} \times S^2),
\]

\[
(\tau_\beta f)(u, \Sigma) := \begin{cases} 
    u \sqrt{1 + \rho_\beta(u)} f(u, \Sigma), & u > 0, \\
    -u \sqrt{\rho_\beta(-u)} f(-u, \Sigma), & u < 0.
\end{cases}
\]

This map extends naturally to a map \( L^2(\mathbb{R}_+ \times S^2, (\omega^2 + \omega) d\omega d\Sigma, \mathcal{L}(\mathcal{H}_p)) \rightarrow L^2(\mathbb{R} \times S^2, (\omega^2 + \omega) d\omega d\Sigma, \mathcal{L}(\mathcal{H}_p)) \) which in the following will also be denoted by just \( \tau_\beta \).

Let \( C_p \) be the complex conjugation on \( \mathcal{H}_p \) given by \( C_p \psi(x) := \overline{\psi(x)} \), and for an operator \( T \in \mathcal{L}(\mathcal{H}_p) \) we write also \( \overline{T} := C_p T C_p \). Let

\[
\mathcal{D} := C_c^\infty(\mathbb{R}^3) \otimes C_c^\infty(\mathbb{R}^3) \otimes \mathcal{F}_\text{fin}(C_c^\infty(\mathbb{R} \times S^2)),
\]

which is a dense subspace of the complete Hilbert space

\[
\mathcal{H} := \mathcal{H}_p \otimes \mathcal{H}_p \otimes \mathfrak{F}.
\]

On \( \mathcal{D} \) we define for \( \lambda \in \mathbb{R} \) the Liouvillian by

\[
L_\lambda := (H_p \otimes \text{Id}_p - \text{Id}_p \otimes H_p) \otimes \text{Id}_l + \text{Id}_p \otimes \text{Id}_p \otimes d\Gamma(u) + \lambda \Phi(I),
\]

where

\[
I(u, \Sigma) := I^{(0)}(u, \Sigma) \otimes \text{Id}_p + \text{Id}_p \otimes I^{(t)}(u, \Sigma),
\]

\[
I^{(0)}(u, \Sigma) := \tau_\beta(G)(u, \Sigma), \quad I^{(t)}(u, \Sigma) := -e^{-\beta u/2} \tau_\beta(G')\mathcal{T}(u, \Sigma).
\]

It will be shown below (Proposition 4.3) that \( L_\lambda \) is indeed essentially self-adjoint on \( \mathcal{D} \).
Remark 2.2
We define the unitary transformations
\[ \Psi : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}_+ \times \mathbb{S}^2, d\omega d\Sigma) \]
with \( \Psi(f)(\omega, \Sigma) := \omega f(\omega \Sigma) \), and
\[ \tau : L^2(\mathbb{R}_+ \times \mathbb{S}^2) \oplus L^2(\mathbb{R}_+ \times \mathbb{S}^2) \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^2), \]
with \( \tau(f_1, f_2)(\omega, \Sigma) := f_1(\omega, \Sigma) \) if \( \omega > 0 \) and \( f_2(-\omega, \Sigma) \) if \( \omega < 0 \). Then this induces the following unitary transformation of Fock spaces as follows
\[
\mathfrak{F}(L^2(\mathbb{R}^3)) \otimes \mathfrak{F}(L^2(\mathbb{R}^3)) \cong \mathfrak{F}(L^2(\mathbb{R}_+ \times \mathbb{S}^2)) \otimes \mathfrak{F}(L^2(\mathbb{R}_+ \times \mathbb{S}^2)) \\
\cong \mathfrak{F}(L^2(\mathbb{R}_+ \times \mathbb{S}^2) \oplus L^2(\mathbb{R}_+ \times \mathbb{S}^2)) \\
\cong \mathfrak{F}(L^2(\mathbb{R} \times \mathbb{S}^2)),
\]
where in the second line we made use of a canonical isomorphism. Let \( \tilde{\mathcal{H}}_f := \mathfrak{F}(L^2(\mathbb{R}^3)) \) and let \( H := d\Gamma(| \cdot |) \) be the free field energy operator on \( \tilde{\mathcal{H}}_f \). Then \( L_\lambda \) is unitary equivalent to the following operator in \( \mathcal{H}_{p} \otimes \mathcal{H}_p \otimes \mathcal{H}_f \otimes \mathcal{H}_f \)
\[
\tilde{L}_\lambda = (H_p \otimes \text{Id}_p - \text{Id}_p \otimes H_p) \otimes \text{Id}_{\tilde{\mathcal{H}}_f \otimes \tilde{\mathcal{H}}_f} + \text{Id}_p \otimes \text{Id}_p \otimes (H_f \otimes \text{Id}_{\tilde{\mathcal{H}}_f} - \text{Id}_{\tilde{\mathcal{H}}_f} \otimes H_f) \\
+ \lambda(\Phi_l(\sqrt{1 + \rho_{\beta}G} \otimes \text{Id}_p - \sqrt{\rho_{\beta}} \text{Id}_p \otimes G^*) \\
+ \Phi_r(\sqrt{\rho_{\beta}G^*} \otimes \text{Id}_p - \sqrt{1 + \rho_{\beta}} \text{Id}_p \otimes G)),
\]
where \( \Phi_l := \Phi \otimes \text{Id}_{\tilde{\mathcal{H}}_f}, \Phi_r := \text{Id}_{\tilde{\mathcal{H}}_f} \otimes \Phi \).

2.4. Main Result
For the proof of our main result we need another additional assumption. The instability of the eigenvalues should be visible in second order in perturbation theory with respect to the coupling. This term is also called level shift operator and the corresponding positivity assumption Fermi Golden Rule condition. In Subsection 2.5 an example is provided where this is satisfied.

Fermi Golden Rule Condition
For \( E \in \sigma_d(H_p) \) let \( p_E := I_{\{E\}}(H_p) \) be the spectral projection corresponding to the eigenvalue \( E \) and let \( P_{\text{ess}} \) be the spectral projection to the essential spectrum of \( H_p \).

(F) Let \( \varepsilon > 0 \). For all \( E \in \sigma_d(H_p) \) we assume there is a \( \gamma_\beta(E) > 0 \), such that we have
\[
p_E(F^{(1)}_\varepsilon + F^{(2)}_\varepsilon) p_E \geq \gamma_\beta(E) p_E,
\]
where
\[ F^{(1)}_\varepsilon := \int_0^\infty \int_{S^2} \omega^2 e^{\beta \omega} G(\omega, \Sigma)^* (H_p - E - \omega)^2 + \varepsilon^2 G(\omega, \Sigma) \, d\Sigma \, d\omega, \]
\[ F^{(2)}_\varepsilon := \int_0^\infty \int_{S^2} \omega^2 \frac{1}{1 - e^{-\beta \omega}} G(\omega, \Sigma)^* (H_p - E + \omega)^2 + \varepsilon^2 G(\omega, \Sigma) \, d\Sigma \, d\omega. \]

Furthermore we set \( \gamma_\beta := \inf_{E \in \sigma(H_p)} \gamma_\beta(E) \).

Notice that \( \gamma_\beta \) might depend on \( \beta \). In particular, this is the case if, as in [5, 6] and Proposition 2.6, \( (F) \) is verified using only the term \( F^{(1)}_\varepsilon \) (for \( \varepsilon \to 0 \)), which decays exponentially to zero if \( \beta \to \infty \). However, one can also obtain results uniformly in \( \beta \) for \( \beta \geq \beta_0 > 0 \) (low temperature) by proving that \( F^{(2)}_\varepsilon \) is positive for a fixed \( \varepsilon > 0 \), which will be done in Corollary 2.4.

This allows us to state the main result of this paper.

**Theorem 2.3**
Assume that \( (H1), (H2), (I1), (I2), (I3) \) and \( (F) \) are satisfied. Let \( \beta_0 > 0 \) and let \( \gamma_\beta > 0 \) be the constant appearing in \( (F) \). Then there exists constants \( \lambda_0, C > 0 \) such that for all \( 0 < |\lambda| < \lambda_0 \) with \( |\lambda| < C \gamma_\beta^2 \) and all \( \beta \geq \beta_0 \), the self-adjoint extension of the operator \( L_\lambda \) given in (2) does not have any eigenvalues.

**2.5. Application**

In the following we present an example of a QED system with a linear coupling term (Nelson Model) where the conditions for the main theorem are satisfied. In particular, one can verify the Fermi Golden Rule condition \( (F) \) in this case. We summarize this in the following corollary to Theorem 2.3.

**Corollary 2.4**
Let \( V \in C_c^\infty(\mathbb{R}^3) \) with \( V \leq 0 \) such that \( -\Delta + V \) has one (negative) discrete non-degenerate eigenvalue and no zero eigenvalue. Let \( \chi \in S(\mathbb{R}^3) \) and let \( \chi \) be constant in a neighborhood of zero. Assume that
\[ G(k)(x) = \kappa(|k|) e^{ikx} \chi(\alpha x), \quad k, x \in \mathbb{R}^3, \]  
where \( \alpha > 0 \) and \( \kappa \) is a function on \( \mathbb{R}_+ \) satisfying \( (12), (I3) \). For instance, we can choose
\[ \kappa(\omega) = C \omega^{-\frac{1}{2}} e^{-c\omega}, \quad \omega \geq 0, \]
for some constants \( c \) and \( C \).

Then for any \( \beta_0 > 0 \) there exist constants \( \lambda_0, C > 0 \) such that for all \( 0 < |\lambda| < \lambda_0 \) with \( |\lambda| < C \gamma_\beta^2 \) and all \( \beta \geq \beta_0 \), \( L_\lambda \), or equivalently \( \tilde{L}_\lambda \) given in Remark 2.2, has no zero eigenvalue.
Proof. Writing $G$ in spherical coordinates with the notation in (1),
\[ G(\omega, \Sigma)(x) = \kappa(\omega)\tilde{G}(\omega, \Sigma)(x), \quad \tilde{G}(\omega, \Sigma)(x) = e^{i\omega \Sigma x} \chi(\alpha x), \quad x \in \mathbb{R}^3, \]
one realizes that derivatives of $\tilde{G}$ with respect to $\omega$ still yield a Schwartz function. Furthermore, derivatives with respect to $x$ yield only polynomial growth in $\omega$.

Thus, (I1) is satisfied. (H1), (H2), (I2), (I3) are satisfied by assumption. It remains the verification of the Fermi Golden Rule condition, which is done in Proposition 2.5.

**Proposition 2.5**

*For any $\varepsilon > 0$ there exists $\alpha > 0$ and $\gamma > 0$ (independent of $\beta$) such that*
\[
\langle \varphi_E, F^{(2)}_\varepsilon \varphi_E \rangle = p_E \int_0^\infty \int_{S^2} \frac{\omega^2}{1 - e^{-\beta \omega}} G(\omega, \Sigma) \frac{\varepsilon \rhoess(H_p - E + \omega)^2 + \varepsilon^2}{G(\omega, \Sigma)^* \rho} \varphi_E d\Sigma d\omega \geq \gamma p_E.
\]

*Proof.* First note that
\[
\langle \varphi_E, F^{(2)}_\varepsilon \varphi_E \rangle \geq \int_0^\infty \int_{S^2} \omega^2 \langle \varphi_E, G(\omega, \Sigma)^* \frac{\varepsilon \rhoess(H_p - E + \omega)^2 + \varepsilon^2}{G(\omega, \Sigma) \varphi_E} d\Sigma d\omega.
\]
The integrand is continuous in $(\omega, \Sigma)$ and non-negative. Thus, it suffices to show that for some $(\omega, \Sigma)$ we have
\[
\left\langle G(\omega, \Sigma)\varphi_E, \frac{\varepsilon \rhoess(H_p - E + \omega)^2 + \varepsilon^2}{G(\omega, \Sigma)} \varphi_E \right\rangle \neq 0,
\]
which follows if we can show
\[
\rhoess G(\omega, \Sigma) \varphi_E \neq 0. \quad (4)
\]

We can write
\[
G(\omega, \Sigma) = \kappa(\omega)\chi_\alpha U_{\omega, \Sigma} \varphi_E,
\]
where $\chi_\alpha$ denotes the multiplication operator by $x \mapsto \chi(\alpha x)$ and $U_{\omega, \Sigma}$ denotes the unitary multiplication operator by $x \mapsto e^{i\omega \Sigma x}$. Now for $\omega \neq 0$,
\[
\rhoess U_{\omega, \Sigma} \varphi_E = (\text{Id} - p_E)U_{\omega, \Sigma} \varphi_E \neq 0, \quad (5)
\]
because $U_{\omega, \Sigma}$ has no eigenvalues. Now it follow that there exists $\alpha > 0$ such that (H1) holds: if not, we had
\[
U_{\omega, \Sigma} \varphi_E = \lim_{\alpha \to 0} \chi_\alpha U_{\omega, \Sigma} \varphi_E \in \text{ran} p_E,
\]
which would be a contradiction to (5). \qed
Instead of using $F^{(2)}$ one can also verify $[F]$ with the first term $F^{(1)}$ in the limit $\varepsilon \to 0$. This does not improve the qualitative statement of Corollary 2.4 and has the drawback that $\gamma_\beta \to 0$ as $\beta \to \infty$ as in [5, 6]. However, in certain regimes it still might be the dominant term.

For the proof we use a dipole approximation as in [9], that is, a power series expansion of the coupling term.

**Proposition 2.6**

Assume that

$$G(k)(x) = \kappa(|k|)e^{i\alpha x} \chi(\alpha^\delta x), \quad k, x \in \mathbb{R}^3,$$

where $\delta > 0$, $\alpha > 0$, and $\kappa$ and $\chi$ satisfy the same assumptions as in Corollary 2.4. We have for all $E \in \sigma_d(H_\rho)$,

$$\lim_{\varepsilon \to 0} \left\langle \varphi_E, F^{(1)}_\varepsilon \varphi_E \right\rangle = \int_{S_2} \int_{\mathbb{R}^3} \frac{(k^2 - E)^2}{c^{\beta(k^2 - E)} - 1} \left| \kappa(k^2 - E) \right|^2 (k^2 - E)^2 \alpha^2 \left| \langle \varphi_E, \Sigma \cdot \hat{x} \varphi(k, \cdot) \rangle \right|^2 dkd\Sigma + O(\alpha^3),$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ denotes the vector of multiplication operators in the respective components. In particular, if $V$ is rotationally invariant and there is only one eigenvalue, then $[F]$ is satisfied for $\alpha > 0$ small enough.

**Proof.** Notice that the terms $F^{(1)}$, $F^{(2)}$ are Dirac sequences in the limit $\varepsilon \to 0$. Thus, as $G$ is continuous,

$$\lim_{\varepsilon \to 0} \left\langle \varphi_E, F^{(2)}_\varepsilon \varphi_E \right\rangle = \int_{S_2} \int_{\mathbb{R}^3} e^{\beta(k^2 - E)} - 1 \left| \kappa(k^2 - E) \right|^2 \left| \langle \varphi_E, e^{i\alpha(k^2 - E)\Sigma \cdot \hat{x}} \chi(\alpha^\delta \cdot) \varphi(k, \cdot) \rangle \right|^2 dkd\Sigma.$$

Now we consider an expansion in powers of $\alpha$. There exists a constant $C$ such that

$$\left| e^{is} - 1 - is \right| \leq C|s|^2, \quad s \in \mathbb{R}.$$

This means that

$$\left| \left\langle \varphi_E, e^{i\alpha(k^2 - E)\Sigma \cdot \hat{x}} \chi(\alpha^\delta \cdot) - \chi(\alpha^\delta \cdot) - i\alpha(k^2 - E)\Sigma \cdot \hat{x} \chi(\alpha^\delta \cdot) \right| \varphi(k, \cdot) \right| \leq \alpha^2 C \| \varphi(k, \cdot) \|_\infty \| \chi \|_\infty \int |\varphi_E(x)||k^2 - E)(\Sigma \cdot x)|^2 dx.$$

Now let us estimate the contribution of $\chi(\alpha^\delta \cdot)$. For this suppose $\chi$ is constant on a ball centered at the origin of radius $r > 0$. As the eigenfunctions decay
exponentially (cf. [20]), we may assume that there are constants \( c, C > 0 \) such that

\[
|\varphi_E(x)| \leq C e^{-c|x|}.
\]

Then we find for \( n = 0, 1 \),

\[
|\langle \hat{x}_j^n \varphi_E, (\chi(\alpha^\delta \cdot) - \chi(0)) \varphi(k, \cdot) \rangle| \leq \|\varphi(k, \cdot)\|_\infty \|\chi\|_\infty \int_{a^\delta|x| > r} |x_j|^n |\varphi_E(x)| \, dx
\]

\[
\leq \|\varphi(k, \cdot)\|_\infty \|\chi\|_\infty C \int_{a^\delta \rho > r} e^{-\frac{\rho}{2}} \, d\rho
\]

\[
\leq \|\varphi(k, \cdot)\|_\infty \|\chi\|_\infty Ce^{-\frac{\rho}{2}},
\]

where we used that polynomial functions are exponentially bounded. On the other hand by the orthogonality relation we find

\[
\langle \varphi_E, \chi(\alpha^\delta \cdot) \varphi(k, \cdot) \rangle = \langle \varphi_E, (\chi(\alpha^\delta \cdot) - \chi(0)) \varphi(k, \cdot) \rangle.
\]

Collecting estimates, we arrive at

\[
\left| \langle \varphi_E, e^{i\alpha(k^2 - E)\Sigma \cdot \hat{x}} \chi(\alpha^\delta \cdot) \varphi(k, \cdot) \rangle - \langle \varphi_E, i\alpha(k^2 - E)\Sigma \cdot \hat{x} \varphi(k, \cdot) \rangle \right|
\]

\[
\leq \alpha^2 \|\varphi(k, \cdot)\|_\infty \|\chi\|_\infty C(\varphi_E),
\]

where \( C(\varphi_E) \) denotes a constant depending on the exponential decay of \( \varphi_E \) and expectations of moments.

Thus in the limit \( \alpha \downarrow 0 \) we determined the leading order contribution. Within the explicit model, one can now verify whether the leading order term does not vanish. If \( V \) is rotationally invariant and \( H_p \) has only one eigenvalue \( \varphi_E \), then

\[
\langle \varphi_E, \hat{x}_j \varphi_E \rangle = 0 \text{ by symmetry}.
\]

Thus by the orthogonality relation of the scattering states \( \int \left| \langle \varphi_E(x), x_j \varphi(k, x) \rangle \right|^2 dk = \|\hat{x}_j \varphi_E\|^2 \neq 0 \).

\[
\square
\]

### 2.6. Overview of the Proof

The first step is to find a suitable conjugate operator \( A \) consisting of a part \( A_p \) on the particle space \( \mathcal{H}_p \) and a part \( A_f \) on the field space \( \mathcal{F} \).

For the latter we make the same choice as established for the first time in [12] and later also used by MERKLI and co-authors in [13, 5, 6], namely the second quantization of the generator of translations,

\[
A_f = d\Gamma(i\partial_u).
\]

Let \( P_\Omega \) denote the orthogonal projection onto the subspace generated by the vacuum \( \Omega \) ("vacuum subspace"). Formally, we obtain on \( \mathcal{F} \) that

\[
i[d\Gamma(u), A_f] = d\Gamma(N_f) \geq P_\Omega^\perp,
\]

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which yields a positive contribution on the space not containing the vacuum. The Hilbert space $\mathcal{H}$ can be further decomposed by means of the projection

$$\Pi := 1_{\{0\}}(L_0) = 1_{\{0\}}(L_p) \otimes P_\Omega$$

as

$$\text{ran}(\text{Id}_{H_p} \otimes \nu_\Pi \otimes P_\Omega^\perp) \oplus \Pi \oplus \text{ran}(1_{\mathbb{R}\setminus\{0\}}(L_p) \otimes P_\Omega).$$

To get a positive operator on $\text{ran} \Pi$, we proceed again as in [5, 6] and consider a bounded operator $A_0$ on the whole space $\mathcal{H}$. The Fermi Golden Rule Condition [F] then implies that

$$i\Pi[L_\lambda, A_0] \Pi > 0.$$ 

The details can be found in Subsection 6.2.

Let $P_{\text{disc}}$ denote the spectral projection to the discrete spectrum of $H_p$ and $P_{\text{ess}} := P_{\text{disc}}^\perp$. The third space in (7) can decomposed further by use of

$$1_{\mathbb{R}\setminus\{0\}}(L_p) = (P_{\text{ess}} \otimes P_{\text{ess}}) \oplus (P_{\text{ess}} \otimes P_{\text{disc}}) \oplus (P_{\text{disc}} \otimes P_{\text{ess}}) \oplus 1_{\mathbb{R}\setminus\{0\}}(L_p)(P_{\text{disc}} \otimes P_{\text{disc}}).$$

We start with the space generated by the first projection $P_{\text{ess}} \otimes P_{\text{ess}}$. In contrast to [6] the conjugate operator in the particle space will be defined as follows. We first diagonalize the non-negative part of $H_p$ with general eigenfunctions corresponding to the positive (continuous) spectrum, the \textit{scattering functions}, which we recall in Subsection 4.1. This yields a unitary map $V_c$ between the non-negative eigenspace of $H_p$ and $L^2(\mathbb{R}^3)$ with the property that $V_c^* H_p V_c = \hat{k}^2$, where $\hat{k} = (k_1, k_2, k_3)$ denotes the vector of multiplication operators with the respective components. Let

$$A_D := \frac{1}{4}(\hat{x}\hat{k} + \hat{k}\hat{x})$$

be the generator of dilations, where $\hat{x} := -i\nabla = (-i\partial_1, -i\partial_2, -i\partial_3)$ and the (non-commutative) scalar product of $\hat{x}$ and $\hat{k}$ is to be understood as sum of products of the component operators. Then

$$A_p := V_c^* A_D V_c$$

has the effect that

$$i[H_p, A_p] = V_c^* \hat{k}^2 V_c = P_{\text{ess}} H_p,$$

which is strictly positive on $\text{ran} P_{\text{ess}}$. We combine $A_f$ and $A_p$ to an operator on $\mathcal{H}$ by

$$A = (A_p \otimes \text{Id}_p - \text{Id}_p \otimes A_p) \otimes \text{Id}_f + \text{Id}_p \otimes \text{Id}_p \otimes A_f,$$

which yields

$$i[L_0, A] = (P_{\text{ess}} H_p \otimes \text{Id}_p + \text{Id}_p \otimes P_{\text{ess}} H_p) \otimes \text{Id}_f + \text{Id}_p \otimes \text{Id}_p \otimes N_f.$$
As $A$ is unbounded, it is necessary to use a virial theorem for the positive commutator method to work. We will indeed use the same abstract versions developed in \cite{5,6} which are repeated in Section 3. The exact definition of $A$ and the verification of the conditions for the virial theorems are given in Section 4.

For the space given by the ranges of the remaining projections in \cite{8} we choose an operator $Q$ on $H_p \otimes H_p$ given as a bounded continuous function of $L_p$, which vanishes at the origin. We add a suitable operator $T$ depending on the interaction and $\lambda$ to accomplish

$$\langle \psi, (Q \otimes P_{\Omega} + T)\psi \rangle = 0$$

for all $\psi \in \ker L_\lambda$. Since the distance between the essential and the discrete spectrum is strictly positive as well as the distance between the eigenvalues, $Q \otimes P_{\Omega}$ is also strictly positive on $\text{ran}(\mathbb{1}_{\mathbb{R}\setminus\{0\}}(L_p) \otimes P_{\Omega})$.

The operator $T$ yields an error term which can be estimated by $N_f$.

Finally, there will arise further error terms from the the commutator of the interaction with $A$ and $A_0$, respectively. The general idea is that they can be estimated by $N_f$ on $\mathfrak{F}$ and bounded terms on $H_p \otimes H_p \otimes \text{ran} P_{\Omega}$, respectively. For the latter we can use the decomposition and the corresponding positive operators explained above. On $\text{ran} P_{\text{ess}} \otimes P_{\text{ess}}$ we estimate them by $\hat{\kappa}^{-2}$ and then use that

$$\hat{\kappa}^2 - \lambda \hat{\kappa}^{-2} > 0$$

for $\lambda > 0$ sufficiently small (cf. \cite{18, X.2}).

### 3. Abstract Virial Theorems

In this section we recall the abstract virial theorems of \cite{5,6}. They are based on Nelson’s commutator theorem, which can be used for proving self-adjointness of operators which are not bounded from below. An important notion will be that of a GJN triple.

**Definition 3.1 (GJN triple)**

Let $H$ be a Hilbert space, $D \subset H$ a core for a self-adjoint operator $Y \geq \text{Id}$, and $X$ a symmetric operator on $D$. We say the triple $(X, Y, D)$ satisfies the Glimm-Jamme-Nelson (GJN) condition, or that $(X, Y, D)$ is a GJN-triple, if there is a constant $C < \infty$, such that for all $\psi \in D$:

$$\|X\psi\| \leq C\|Y\psi\|, \quad \text{(10)}$$

$$\pm i\{\langle X\psi, Y\psi \rangle - \langle Y\psi, X\psi \rangle \leq C \langle \psi, Y\psi \rangle \}. \quad \text{(11)}$$

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Theorem 3.2 (GJN commutator theorem, [18, Theorem X.37])
If \((X,Y,D)\) satisfies the GJN condition, then \(X\) determines a self-adjoint operator (again denoted by \(X\)), such that \(D(X) \supset D(Y)\). Moreover, \(X\) is essentially self-adjoint on any core for \(Y\), and \((10)\) is valid for all \(\psi \in D(Y)\).

A consequence is that the unitary group generated by \(X\) leaves the domain of \(Y\) invariant. The concrete formulation is taken from [5].

Theorem 3.3 (Invariance of domain, [7])
Suppose \((X,Y,D)\) satisfies the GJN condition. Then, for all \(t \in \mathbb{R}\), \(e^{itX}\) leaves \(D(Y)\) invariant, and there is a constant \(\kappa \geq 0\) such that
\[\|Ye^{itX}\psi\| \leq e^{\kappa |t|} \|Y\psi\|, \quad \psi \in D(Y).\]

Based on the GJN commutator theorem, we next describe the setting for a general virial theorem. Suppose one is given a self-adjoint operator \(\Lambda \geq \text{Id}\) with core \(D \subset H\), and operators \(L, A, N, D, C_n, n = 0, 1, 2, 3\), all symmetric on \(D\), and satisfying
\[\langle \phi, D\psi \rangle = i(\langle L\phi, N\psi \rangle - \langle N\phi, L\psi \rangle)\] (12)
and
\[C_0 = L\] (13)
\[\langle \phi, C_{n+1}\psi \rangle = i(\langle C_n\phi, A\psi \rangle - \langle A\phi, C_n\psi \rangle), \quad n = 0, 1, 2,\] (14)
where \(\phi, \psi \in D\). Furthermore we shall assume

(V1) \((X, \Lambda, D)\) satisfies the GJN condition, for \(X = L, N, D, C_n, n = 0, \ldots, 3\). Consequently all these operators determine self-adjoint operators, which we denote by the same letters.

(V2) \(A\) is self-adjoint, \(D \subset D(A)\), and \(e^{itA}\) leaves \(D(\Lambda)\) invariant.

Theorem 3.4 (Abstract virial theorem, [5, Theorem 3.2])
Let \(\Lambda \geq \text{Id}\) be a self-adjoint operator in \(\mathcal{H}\) with core \(D \subset \mathcal{H}\), and let \(L, A, N, D, C_n, n = 0, 1, 2, 3\), be symmetric on \(D\) and satisfying the assumptions above. Assume that \(N\) and \(e^{itA}\) commute, for all \(t \in \mathbb{R}\), in the strong sense on \(D\), and that there exist \(0 \leq p < \infty\) and \(C < \infty\) such that
\[\|D\psi\| \leq C \|N^{1/2}\psi\|,\] (15)
\[\|C_1\psi\| \leq C \|N^p\psi\|,\] (16)
\[\|C_3\psi\| \leq C \|N^{1/2}\psi\|,\] (17)
for all \(\psi \in D\). Then, if \(\psi \in D(L)\) is an eigenvector of \(L\), there is a sequence of approximating eigenvectors \((\psi_n)_{n \in \mathbb{N}}\) in \(D(L) \cap D(C_1)\) such that \(\lim_{n \to \infty} \psi_n = \psi\) in \(\mathcal{H}\), and
\[\lim_{n \to \infty} \langle \psi_n, C_1\psi_n \rangle = 0.\]
4. Definition of the Commutator and Verification of the Virial Theorems

With the scattering function formalism we can now present the concrete choice of the abstract operators \(L, A, N, D, C_n, n = 0, 1, 2, 3\), of Section 3. Subsequently, we prove the assumptions of the abstract virial theorem which we use in order to get the concrete virial theorem [Theorem 4.4]. It will be one main ingredient for the proof of the main result of this paper. We give its proof at the end of Subsection 4.3.

4.1. Scattering States

In this part we recall the theory of generalized eigenstates (scattering states) and the corresponding spectral decomposition.

We assume to have a potential \(V\) satisfying [H1] [H2]. Then the scattering states \(\varphi(k, \cdot)\), \(k \in \mathbb{R}^3\), are defined as generalized eigenvectors,

\[
(-\Delta + V)\varphi(k, \cdot) = k^2 \varphi(k, \cdot),
\]

or as solutions of the so-called Lippmann-Schwinger equation,

\[
\varphi(k, x) = e^{ikx} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} V(y)\varphi(k, y) dy.
\]

We discuss their properties in the following proposition which is basically a combination of [19, Theorem XI.41] with the theory given in [10] and [17]. Notice in particular that Hölder continuity of \(V\) implies the existence of a scattering function for \(k = 0\) in the case \(H_p\) does not have a zero eigenstate (by [10]), therefore the absence of so called half-bound states described as in [17].

The scattering functions can be used for a spectral decomposition of the continuous spectrum of \(H_p\). Denote by \(\varphi_n, n = 1, \ldots, N\) the eigenvectors of \(H_p\) and let \(P_{\text{ess}}\) be the projection to the essential spectrum \([0, \infty)\) of \(H_p\).

**Theorem 4.1** (see [19, Theorem XI.41])

Let \(f \in L^2(\mathbb{R}^3)\).

(a) For all \(k \in \mathbb{R}^3\) there exists a unique bounded function \(\varphi(k, \cdot)\) on \(\mathbb{R}^3\) satisfying [18] such that \(\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}, (k, x) \mapsto \varphi(k, x)\) is continuous.

(b) The generalized Fourier transform

\[
(V_c f)(k) := f^\#(k) := (2\pi)^{-3/2} \text{i.m.} \int_{\mathbb{R}^3} \varphi(k, x) f(x) dx,
\]

where \(\text{i.m.} \int g(x) dx := L^2-\lim_{R \to \infty} \int_{|x| < R} g(x) dx\), exists.
(c) The generalized Fourier coefficients

\[ f_n^\# = (2\pi)^{-3/2} \text{i.m.} \int \overline{\varphi_n(x)} f(x) dk \]

exist, and \( \sum_{n=1}^\infty |f_n^\#|^2 < \infty \).

(d) We have \( \text{ran } V_c = L^2(\mathbb{R}^3) \), and \( V_c \) is a partial isometry, because

\[ \|V_c f\| = \|P_{\text{ess}} f\|. \]

In particular, \( V_c|_{\text{ran } P_{\text{ess}}} : \text{ran } P_{\text{ess}} \to L^2(\mathbb{R}^3) \) is a unitary operator, and \( V_c V_c^* = \text{Id} \).

(e) We have the spectral decomposition

\[ (P_{\text{ess}} f)(x) = \text{i.m.} (2\pi)^{-3/2} \int f^\#(k) \varphi(k, x) dk. \]

(f) If \( f \in D(H_p) \), then

\[ (H_p f)^\#(k) = k^2 f^\#(k), \]

in other words, \( V_c H_p V_c^* = \hat{k}^2 \).

Furthermore, we can extend \( V_c \) to a unitary operator by including the eigenfunctions into consideration. We define

\[ V_d : L^2(\mathbb{R}^3) \longrightarrow \ell^2(N), \quad (V \psi)_i = \langle \varphi_i, \psi \rangle. \]

Obviously \( V_d|_{\text{ran } P_{\text{disc}}} : \text{ran } P_{\text{disc}} \to \ell^2(N) \) is a unitary operator and \( V_d|_{\text{ran } P_{\text{ess}}} = 0 \). Thus,

\[ \mathcal{V} := V_d \oplus V_c : L^2(\mathbb{R}^3) \longrightarrow \ell^2(N) \oplus L^2(\mathbb{R}^3) \]

is unitary.

4.2. Setup for the Virial Theorems

First, we describe the setting on the particle space \( \mathcal{H}_p \). We consider a dense subspace given by

\[ \mathcal{D}_p := V_c^* C_c^\infty (\mathbb{R}^3) \oplus \text{ran } P_{\text{disc}}. \]

Note that \( \mathcal{D}_p \) is dense since \( V_c^* C_c^\infty (\mathbb{R}^3) \subseteq \text{ran } P_{\text{ess}} \) is dense in \( \text{ran } P_{\text{ess}} \). Now, based on the definition of the generator of dilations,

\[ A_D = \frac{1}{4}(\hat{k} \hat{x} + \hat{x} \hat{k}), \]
we define on $D_p$ the conjugation operator

$$A_p := V_c^* A_D V_c.$$  

It is clear that both $H_p$ and $A_p$ leave $D_p$ invariant. Thus we can define $\text{ad}^{(n)}_{A_p}(H_p)$ on $D_p$ for all $n \in \mathbb{N}$. Furthermore, the bounding operator is chosen as

$$\Lambda_p := V_c^* (\hat{k}^2 + \hat{x}^2) V_c + \text{Id}.$$  

Next, on the field space we set

$$A_f := d \Gamma(i \partial u),$$  
$$\Lambda_f := d \Gamma(u^2 + 1).$$  

Now, we can define on the dense subspace of the complete space $H$, $D_p = D_p \otimes D_p \otimes \mathcal{F}_{\text{fin}}(C^\infty_c(\mathbb{R}^3))$, the operators

$$\Lambda = \Lambda_p \otimes \text{Id}_p \otimes \text{Id}_f + \text{Id}_p \otimes \Lambda_p \otimes \text{Id}_f + \text{Id}_p \otimes \text{Id}_p \otimes \Lambda_f,$$
$$L = L_\lambda,$$
$$N = \text{Id}_p \otimes \text{Id}_p \otimes N_f,$$
$$A = (A_p \otimes \text{Id}_p - \text{Id}_p \otimes A_p) \otimes \text{Id}_f + \text{Id}_p \otimes \text{Id}_p \otimes \Lambda_f,$$
$$D = i[L_\lambda, \text{Id}_p \otimes \text{Id}_p \otimes N_f].$$

For symmetric operators $X, Y$ with a dense domain $D_0$ we define multiple commutators by ad$^{(0)}(X) = X$ and ad$^{(n+1)}(X) = i[\text{ad}^{(n)}(X), Y]$ in the form sense on $D_0 \times D_0$. If the form is bounded, we denote the corresponding bounded operator by the same symbol. Now we set

$$W_n := \text{ad}^{(n)}_{A_p}(\Phi(I))$$  

$$= \sum_{k=0}^{n} \binom{n}{k} \Phi\bigg((-i\partial_u)^k \tau_\beta (\text{ad}^{(n-k)}_{A_p}(G)) \otimes \text{Id}_p - (-i\partial_u)^k e^{-\beta u/2} \text{Id}_p \otimes \tau_\beta (\text{ad}^{(n-k)}_{A_p}(G^*))\bigg),$$

$$C_n = \delta_{n,1} \text{Id}_p \otimes \text{Id}_p \otimes N_f + \text{ad}^{(n)}_{A_p}(H_p) \otimes \text{Id}_p + (-1)^n \text{Id}_p \otimes \text{ad}^{(n)}_{A_p}(H_p) + \lambda W_n.$$  

Notice that the conditions in Lemma A.2 for $m = 3$ are satisfied due to (12), [13] which guarantees that the expressions in the field operators in (20) are indeed in $L^2(\mathbb{R} \times S^2, \mathcal{L}(H_p))$. Hence, it is relatively $\text{Id}_p \otimes \text{Id}_p \otimes N_f^{1/2}$-bounded, thus, also $\text{Id}_p \otimes \text{Id}_p \otimes \Lambda_f$-bounded. We will see shortly, that $C_n, n = 1, 2, 3, \text{ are }$indeed essentially self-adjoint on $D$ and we denote their self-adjoint extensions by the same symbols. Moreover, it will be shown below in Proposition 6.3 that $C_1$ is actually bounded from below. Thus, we can assign to $C_1$ a quadratic form $q_{C_1}$.  

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4.3. Verification of the Assumptions of the Virial Theorems

In the given setting just described we can now start to prove the assumptions of the virial theorem \textbf{Theorem 3.4}. Above all, we have to check the GJN condition for the different commutators. The most difficult part will be the discussion of interaction terms $W_n$, $n = 1, 2, 3$. Here, the expressions in the field operators need to be sufficiently bounded. These bounds will be collected in the following proposition which is the main result of \textbf{Section 5}.

\textbf{Proposition 4.2}

Let $\partial_u$ denote the derivative of a $\mathcal{L}(H_p)$-valued function in the sense of the strong operator topology. For all $n, m \in \{0, 1, 2, 3\}$, $j \in \{1, 2, 3\}$, and for all $(u, \Sigma)$ the operators

1. $\partial_u^m \text{ad}^{(n)}_{A_p}(\tau_{\beta}(G)(u, \Sigma))$
2. $\partial_u^m \text{ad}_{V_c^{*} V_j V_c}(\text{ad}^{(n)}_{A_p}(\tau_{\beta}(G)(u, \Sigma)))$
3. $\partial_u^m \text{ad}_{V_j V_c}(\text{ad}^{(n)}_{A_p}(\tau_{\beta}(G)(u, \Sigma)))$

and for all $n, m \in \{0, 1\}$,

4. $\partial_u^m V_c^{*} \hat{V}_j V_c \text{ad}^{(n)}_{A_p}(\tau_{\beta}(G)(u, \Sigma))$

are well-defined, and the corresponding functions $\mathbb{R} \times S^2 \to \mathcal{L}(H_p)$ are continuous and in $L^2(\mathbb{R} \times S^2, \mathcal{L}(H_p))$. Moreover there exists a constant $C$ independent of $\beta$ such that for $n, m \in \{0, 1\}$,

$$\left\| \partial_u^m V_c^{*} \hat{V}_j V_c \text{ad}^{(n)}_{A_p}(\tau_{\beta}(G)) \right\|_{L^2(\mathbb{R} \times S^2, \mathcal{L}(H_p))} \leq C(1 + \beta^{-\frac{1}{2}}).$$

Having this we can now verify the necessary GJN conditions.

\textbf{Proposition 4.3}

The following triples are GJN:

1. $(A_p, \Lambda_p, D_p)$,
2. $(H_p, \Lambda_p, D_p)$,
3. $(L, \Lambda, D)$,
4. $(D, \Lambda, D)$,
5. $(C_i, \Lambda, D)$, $i \in \{1, 2, 3\}$.

In particular, $L = L_\lambda$ is essentially self-adjoint on $D$ for any $\lambda \in \mathbb{R}$ due to (3).
Proof. (1) Using that \((A_D, \Lambda_p, C_{c}\infty(\mathbb{R}^3))\) is GJN, we have, for \(\psi \in \mathcal{D}_p\),
\[
\|A_p\psi\| = \|V^*_c A_D V_c\psi\| = \|A_D V_c\psi\| \leq C \|A_p V_c\psi\| = C \|V^*_c \Lambda_p V_c\psi\|
\]
for some constant \(C\), and
\[
\pm i(\langle A_p\psi, \Lambda_p\psi \rangle - \langle \Lambda_p\psi, A_p\psi \rangle) = \pm i \left( \langle A_D V_c\psi, (\hat{k}^2 + \hat{x}^2)V_c\psi \rangle - \langle (\hat{k}^2 + \hat{x}^2)V_c\psi, A_D V_c\psi \rangle \right)
\]
\[
\leq C \langle V_c\psi, (\hat{k}^2 + \hat{x}^2)V_c\psi \rangle
\]
\[
\leq C \langle \psi, \Lambda_p\psi \rangle.
\]
(2) We have, with regard to the first GJN condition,
\[
\|H_p\psi\| = \|V^*_c \hat{k}^2 V_c\psi + H_p P_{\text{disc}}\psi\| \leq C \|V^*_c (\hat{k}^2 + \hat{x}^2)V_c\psi\| + \sup_{\lambda \in \sigma_d(H_p)} |\lambda| \|\psi\|
\]
as \(\hat{k}^2\) is relatively bounded by \(\hat{x}^2 + \hat{k}^2\). Furthermore,
\[
\langle H_p\psi, \Lambda_p\psi \rangle - \langle \Lambda_p\psi, H_p\psi \rangle = \langle V^*_c \hat{k}^2 V_c\psi, V^*_c (\hat{x}^2 + \hat{k}^2)V_c\psi \rangle - \langle V^*_c (\hat{x}^2 + \hat{k}^2)V_c\psi, V^*_c \hat{k}^2 V_c\psi \rangle
\]
\[
= \langle \hat{k}^2 V_c\psi, \hat{x}^2 V_c\psi \rangle - \langle \hat{x}^2 V_c\psi, \hat{k}^2 V_c\psi \rangle
\]
Using now that the commutator \([\hat{x}^2, \hat{k}^2]\) is obviously bounded in the operator sense by a multiple of \(\hat{x}^2 + \hat{k}^2\), we get also the second GJN condition.

(3) As \(H_p\) is relatively bounded by \(\Lambda_p\) (by the previous argument), \(L_0 = (H_p \otimes \text{Id}_p - \text{Id}_p \otimes H_p) \otimes \text{Id}_t\) is relatively bounded by \(\Lambda\). Next, we know that the interaction terms \(I^{(0)}, I^{(1)} \in L^2(\mathbb{R} \times S^2)\). Hence \(W\) is bounded by \(\text{Id}_p \otimes \text{Id}_p \otimes N^1_t\) and thus bounded by \(\text{Id}_p \otimes \text{Id}_p \otimes N^1_t\). Therefore, the first GJN condition is satisfied.

For the second GJN condition, as \((H_p, \Lambda_p, \mathcal{D}_p)\) is GJN, we get that, for \(\psi \in \mathcal{D}\),
\[
\pm i(\langle L_0\psi, \Lambda\psi \rangle - \langle \Lambda\psi, L_0\psi \rangle) \leq C \langle \psi, (\Lambda_p \otimes \text{Id}_p + \text{Id}_p \otimes \Lambda_p) \otimes \text{Id}_t\psi \rangle
\]
for some constant \(C\).

By \((12)\) [(13)] we know that \((u, \Sigma) \mapsto (u^2 + 1)I^{(0)}(u, \Sigma)\) and \((u, \Sigma) \mapsto (u^2 + 1)I^{(0)}(u, \Sigma)\) are in \(L^2(\mathbb{R} \times S^2, L(H_p \otimes H_p))\). We have
\[
\left| \langle \Phi(I^{(0)})\psi, \text{Id}_p \otimes \text{Id}_p \otimes \Lambda_t\psi \rangle - \langle \text{Id}_p \otimes \text{Id}_p \otimes \Lambda_t\psi, \Phi(I^{(0)})\psi \rangle \right|
\]
\[
= 2^{-\frac{1}{2}} \left| \langle \psi, (a((u^2 + 1)I^{(0)}) - a^*((u^2 + 1)I^{(0)}))\psi \rangle \right|
\]
\[
\leq C \left\| \text{Id}_p \otimes \text{Id}_p \otimes N^1_t\psi \right\| \|\psi\|
\]
\[
\leq C \langle \psi, \Lambda\psi \rangle
\]

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for some constants $C$. The same thing can be shown for the commutator with $\Phi(I^{(r)})$.

It remains to consider the commutator of $W$ with the $\Lambda_p$ terms. One has to show that the weak commutators

$$[\Phi(I^{(l)}), V_c^* (\hat{x}^2 + \hat{k}^2)V_c], \quad [\Phi(I^{(r)}), V_c^* (\hat{x}^2 + \hat{k}^2)V_c]$$

are weakly bounded by $\Lambda_p$. This follows directly from Proposition 4.2 as we can write in the weak sense on $D$,

$$[\Phi(I^{(l)}), V_c^* \hat{x}^2 V_c] = \sum_j [\Phi(I^{(l)}), V_c^* \hat{x}_j V_c] V_c^* \hat{x}_j V_c + V_c^* \hat{x}_j V_c [\Phi(I^{(l)}), V_c^* \hat{x}_j V_c]$$

and analogously for $\Phi(I^{(r)})$, and for the commutator with $V_c^* \hat{k}^2 V_c$.

(4) We have

$$D = \frac{i}{\sqrt{2}} (a(I) - a^*(I)).$$

Thus, the proof works as the one of $L$.

(5) We first consider again the free part. We have on $D_p$,

$$[H_p, A_p] = [V_c^* \hat{k}^2 V_c, A_p] = V_c^* [\hat{k}^2, A_p] V_c = iV_c^* \hat{k}^2 V_c$$

Repeating this procedure yields for $n \in \{1, 2, 3\}$,

$$\text{ad}^{(n)}(H_p) = (-1)^n V_c^* \hat{k}^2 V_c.$$

Then we can show as in the proof of $(H_p, \Lambda_p, D_p)$ that also $(\text{ad}^{(n)}(H_p), \Lambda_p, D_p)$, $n \in \{1, 2, 3\}$, is GJN and so is $(\text{ad}^{(n)}(L_0), \Lambda, D)$, $n = 1, 2, 3$.

It remains to verify the GJN conditions for $(W_n, \Lambda, D)$, $n = 1, 2, 3$. Analogously to the proof of the GJN condition of $L$ we have to show that the expressions in the field operators are bounded, and also stay bounded if we commute with the square root of $\Lambda_p$, that is, we have to show that

$$(u, \Sigma) \mapsto \partial^k u \tau_\beta(\text{ad}^{(n-k)}_{\Lambda_p}(G))(u, \Sigma),$$

$$(u, \Sigma) \mapsto \partial^k u [\tau_\beta(\text{ad}^{(n-k)}_{\Lambda_p}(G))(u, \Sigma), V_c^* \hat{x}_j V_c],$$

$$(u, \Sigma) \mapsto \partial^k u [\tau_\beta(\text{ad}^{(n-k)}_{\Lambda_p}(G))(u, \Sigma), V_c^* \hat{k}_j V_c],$$

$n = 1, 2, 3$, $k = 0, \ldots, n$, $j = 1, 2, 3$, are in $L^2(\mathbb{R} \times S^2)$. As $\kappa$ has the right decay behavior, this follows directly from Proposition 4.2 together with Lemma A.2. 

\[\square\]
Now we can prove the main result of this section, the concrete virial theorem of our setting.

**Theorem 4.4 (Concrete virial theorem)**

Let $C_1$ defined as below. Assume that there exists $\psi \in \mathcal{D}(L_\lambda)$ with $L_\lambda \psi = 0$. Then $\psi \in \mathcal{D}(q_{C_1})$ and $q_{C_1}(\psi) \leq 0$.

**Proof.** We have seen in Proposition 4.3 that $L_\lambda, A, N, D, C_n, n = 0, 1, 2, 3$, satisfy the GJN condition on $\mathcal{D}$. Furthermore, it is shown there that also $(A_p, \Lambda_p, \mathcal{D}_{\lambda})$ is a GJN triple. Hence Theorem 3.2 implies that, for all $t \in \mathbb{R}$, $e^{tA_p}$ leaves $\mathcal{D}(\Lambda_p)$ invariant. Moreover, one computes on $\mathcal{F}_{\text{fin}}(C^\infty_c(\mathbb{R}^3))$,

$$\Lambda_t e^{tA_{p}} = e^{tA_{p}}(\Lambda_t + d\Gamma(2ut - t^2)).$$

Thus, for some fixed $t$ there is a constant $C$ such that $\|\Lambda_t e^{tA_{p}}\psi\| \leq C \|\Lambda_t \psi\|$ for all $\psi \in \mathcal{D}$. As $\mathcal{F}_{\text{fin}}(C^\infty_c(\mathbb{R}^3))$ is a core for $\Lambda_t$, we find $e^{tA_{p}}\mathcal{D}(\Lambda_t) \subseteq \mathcal{D}(\Lambda_t)$. Thus, we conclude $\mathcal{D}(\Lambda)$ is invariant under the unitary group associated to $A$. Finally, by definition of $A$ it is clear that $\text{Id}_p \otimes \text{Id}_p \otimes N_t$ and $e^{tA_{p}}$, $t \in \mathbb{R}$, commute in the strong sense on $\mathcal{D}$.

Therefore, by Theorem 3.4 we find a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(C_1) \cap \mathcal{D}(L_\lambda)$ such that $\lim_{n \to \infty} \psi_n = \psi$ and $\lim_{n \to \infty} \langle \psi_n, C_1 \psi_n \rangle = 0$. Now, as $q_{C_1}$ is closed, thus continuous from below, we obtain $\psi \in \mathcal{D}(q_{C_1})$ and

$$q_{C_1}(\psi) \leq \lim_{n \to \infty} q_{C_1}(\psi_n) = 0. \quad \square$$

### 5. Estimates on the Scattering Functions

The aim of this section is to prove that the commutators of the interaction with the dilation in scattering space are sufficiently bounded. To achieve this, we use the Born series expansion of the scattering functions, that is, we expand them using the recursion formula of the Lippmann-Schwinger equation \[18\]. Then we get the Born series terms, and a remainder term since we perform only finitely many recursion steps. The idea is that the remainder term decays fast enough for the momentum $|k| \to \infty$ for sufficiently many recursion steps.

#### 5.1. Born Series Expansion and Technical Preparations

First we show that that the scattering functions as well as their derivatives with respect to the wave vector $k$ are bounded. For that we use the method of modified square integrable scattering functions, which can be found in [10, 19], originally developed by Rollnik. Remember that $\varphi(k, \cdot), k \in \mathbb{R}^3$, denote the continuous scattering functions on $\mathbb{R}^3$ and $V$ a potential satisfying the assumptions [H1]
As $V$ is compactly supported we may assume that $\text{supp} V$ is contained in a ball around the origin of radius $R$. We now set $\tilde{\varphi}(k, x) := |V(x)|^{1/2} \varphi(k, x)$. Then $\tilde{\varphi}(k, \cdot) \in L^2(\mathbb{R}^3)$ for all $k$, and we can write down the Lippmann-Schwinger equation in the modified form

$$
\tilde{\varphi}(k, x) = |V(x)|^{1/2} e^{ikx} + (L_{[k]} \tilde{\varphi}(k, \cdot))(x),
$$

where

$$
L_{\kappa} \psi(x) := -\frac{1}{4\pi} \int \frac{|V(x)|^{1/2} e^{ik|x-y|} V(y)^{1/2}}{|x-y|} \psi(y) dy, \quad \kappa \geq 0,
$$

and $V(y)^{1/2} := |V(y)|^{1/2} \text{sgn} V(y)$. It is shown in [19] that we can recover the original scattering function from the modified one by

$$
\varphi(k, x) = e^{ikx} - \frac{1}{4\pi} \int e^{i|k|x-y|} V(y)^{1/2} \tilde{\varphi}(k, y) dy,
$$

(21)

Now we extend the results of boundedness of the first derivative of the scattering functions in [10, Lemma 1.1.3] to derivatives of arbitrary order.

**Proposition 5.1**

Let $\hat{D}_k = \frac{k}{|k|} \nabla_k$ and let $\partial_{k,j}$ be the derivative with respect to the $j$-th component of $k$. For all $n \in \mathbb{N}_0$, $m \in \{0, 1\}$ there is a polynomial $P$ such that for all $x$ and $k \neq 0$,

$$
\left| \partial_{k,j}^m \hat{D}_k^n \varphi(k, x) \right| \leq P(|x|).
$$

**Proof.** We get by the modified Lippmann-Schwinger equation

$$
\tilde{\varphi}(k, \cdot) = (\text{Id} - L_{|k|})^{-1}(|V|^{1/2} e_k),
$$

where $e_k(x) := e^{ikx}$. One can show (cf. [10, Lemma 1.1.3]) that $(\text{Id} - L_{|k|})^{-1}$ is uniformly bounded in $k$. Observe that $\hat{D}_k |k| = 1$ and $\hat{D}_k (k/|k|) = 0$. Thus, for any $n \in \mathbb{N}_0$, $\hat{D}_k^n (\text{Id} - L_{|k|})^{-1}$ is again uniformly bounded in $k \neq 0$ since differentiation with $\hat{D}_k$ yields just higher powers of $(\text{Id} - L_{|k|})^{-1}$ and radial derivatives of $L_{|k|}$, which are again bounded operators since $V$ decays fast enough. The same is true if one applies another partial derivative in $k$. Note that it is not differentiable for $k = 0$. Furthermore, for any $n \in \mathbb{N}$,

$$
\sup_{k \neq 0} \left\| \hat{D}_k^n (|V|^{1/2} e_k) \right\|_2 < \infty
$$

as $V$ is compactly supported. Thus, we have shown, for all $n \in \mathbb{N}_0$,

$$
\sup_{k \neq 0} \left\| \hat{D}_k^n \tilde{\varphi}(k, \cdot) \right\|_2 < \infty.
$$
Now we can differentiate (21), estimate the integral with Cauchy-Schwarz, and use that
\[
\int |V(y)| \frac{dy}{|x - y|^2} \leq \|V\|_\infty \int \frac{dy}{y^2} \leq \|V\|_\infty \left( \int \frac{dy}{B_{R}(0) y^2} + \frac{|B_{R}(0)|}{R^2} \right) < \infty \tag{22}
\]
is bounded uniformly in \(x\), which can be seen by considering the cases \(|x| < 2R\) and \(|x| \geq 2R\).

Next, we return to the original setting in the space of bounded continuous functions and perform the Born series expansion. Similar to [10] it is convenient to introduce a symbol for the integral operator in the Lippmann-Schwinger equation. We consider a slightly bigger class of operators to cover also derivatives with respect to \(k\). Let \(C_b(\mathbb{R}^3)\) denote the bounded continuous functions on \(\mathbb{R}^3\) and \(C_{\text{poly}}(\mathbb{R}^3)\) the polynomially bounded continuous functions, that is,
\[
C_{\text{poly}}(\mathbb{R}^3) = \{ \psi \in C(\mathbb{R}^3) : \exists n \in \mathbb{N}_0 : \exists C > 0 : \forall x \in \mathbb{R}^3 : |\psi(x)| \leq C(1 + |x|)^n \}.
\]

**Definition 5.2**
For \(\kappa \geq 0\), \(\psi \in C_{\text{poly}}(\mathbb{R}^3)\) and \(n \in \mathbb{N}_0\), we define
\[
T_{V,\kappa}^{(n)} \psi(x) = \int \frac{e^{i\kappa|x-y|}}{|x - y|^{1-n}} V(y) \psi(y) dy = \int \frac{e^{i\kappa|v|}}{|v|^{1-n}} V(v + x) \psi(v + x) dv.
\]
Furthermore, we write \(T_{V,\kappa} := T_{V,\kappa}^{(0)}\). They have the following elementary properties.

**Proposition 5.3**
For all \(\kappa \geq 0\),

(a) \(T_{V,\kappa}, T_{V,\kappa}^{(-1)}\) are bounded operators from \(C_b(\mathbb{R}^3)\) to \(C_b(\mathbb{R}^3)\),

(b) for all \(n \in \mathbb{N}_0\), \(T_{V,\kappa}^{(n)}\) maps \(C_{\text{poly}}(\mathbb{R}^3)\) to \(C_{\text{poly}}(\mathbb{R}^3)\).

**Proof.** (a) It follows that for \(\psi \in C_b(\mathbb{R}^3)\), \(x \in \mathbb{R}^3\), \(\kappa \geq 0\),
\[
|T_{V,\kappa}^{-1} \psi(x)| \leq \|\psi\|_\infty \int \frac{|V(y)|}{|x - y|^2} dy \leq \|V\|_\infty \|\psi\|_\infty \int_{B_R(0)} \frac{1}{|x - y|^2} dy,
\]
\[
|T_{V,\kappa} \psi(x)| \leq \|\psi\|_\infty \int \frac{|V(y)|}{|x - y|} dy \leq \|V\|_2 \|\psi\|_\infty \left( \int_{B_R(0)} \frac{1}{|x - y|^2} dy \right)^{1/2}.
\]
The integral is bounded independent of \(x\), see [22].

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(b) There exists a constant $C > 0$ such that for all $x \in \mathbb{R}^3$, $n \in \mathbb{N}_0$, $\kappa \geq 0$,

$$|T_{V,n}^{(n)}\psi(x)| \leq C \int_{B_R(0)} \frac{|V(y)|}{|x - y|^{1-\kappa}}(1 + |y|^m)dy$$

$$\leq C(1 + |R|^m) \|V\|_2 \left( \int_{B_R(x)} \frac{1}{|y|^{2-2\kappa}}dy \right)^{1/2}.$$

The last integral can be estimated by

$$\int_{r=0}^{R+|x|} r^{2n} dr,$$

which is bounded by a polynomial in $|x|$.

Using the previous notation and iterating the Lippmann-Schwinger equation (18) we arrive at the following.

**Proposition 5.4**
For all $N \in \mathbb{N}_0$, $k, x \in \mathbb{R}^3$, we have

$$\varphi(k, x) = \sum_{n=0}^{N} \varphi_0^{(n)}(k, x) + \varphi_R^{(N+1)}(k, x),$$

where we set, for $n \in \mathbb{N}_0$,

$$\varphi_0^{(n)}(k, x) := (-4\pi)^{-n} T^n_{V,|k|} e_k(x),$$

$$\varphi_R^{(n)}(k, x) := (-4\pi)^{-n} T^n_{V,|k|} \varphi(k, \cdot)(x),$$

where $e_k(x) = e^{ikx}$.

### 5.2. Estimates of the Born Series Terms

In this subsection we prove decay estimates for the inner products of an abstract coupling function $\chi \in \mathcal{S}(\mathbb{R}^3)$ with (derivatives with respect to $k$ of) the functions $\varphi_0^{(n)}(k, \cdot)$, $k \in \mathbb{R}^3$, $n \in \mathbb{N}_0$, which will be collected in Proposition 5.8 and Proposition 5.9. One can actually show an arbitrary fast decay for any $n \in \mathbb{N}$. The main tool will be a standard stationary phase argument as given below.

**Lemma 5.5** (Stationary phase)
For any $n \in \mathbb{N}_0$ there exists a constant $C$ such that for all $g \in C^\infty_c(\mathbb{R}^3)$ and $k \in \mathbb{R}^3$, we have

$$\left| \int e^{ikx} g(x) dx \right| \leq \frac{C}{\langle k \rangle^n} \sup_{|\alpha| \leq n} \|D^\alpha g\|_1.$$
Proof. For all \( k \in \mathbb{R}^3, j = 1, \ldots, n, \) 
\[
\imath k_j \int e^{i x_k} g(x) dx = \int \partial_j e^{i x_k} g(x) dx \\
= \lim_{R \to \infty} \int_{S_R(0)} e^{i x_k} g(x) dx - \int e^{i x_k} \partial_j g(x) dx.
\]
The first term clearly vanishes. Now we can repeat this procedure \( n \) times. \( \square \)

We proceed by computing the derivatives and the effect of a multiple application of the dilation operator in \( k \) on the Born series terms. The idea is that the application of \( k \nabla_k \) or \( \nabla_k \) on terms of the form 
\[
T_{V_1, |k| \cdots T_{V_p, |k|} e_k}, \quad V_1, \ldots, V_p \in C^\infty_c(\mathbb{R}^3), \quad p \in \mathbb{N},
\]
yields again a linear combination of such terms multiplied with polynomials in \( x \) and \( k \) (Lemma 5.6 and Lemma 5.7). In particular, remember the Born series terms can be written in the form (23). This procedure can be repeated multiple times and the resulting expressions can then be estimated with the stationary phase argument.

**Lemma 5.6**
Assume \( V_1, \ldots, V_p \in C^\infty_c(\mathbb{R}^3). \) Then we can write for all \( k \in \mathbb{R}^3, \)
\[
k \nabla_k \left( T_{V_1, |k| \cdots T_{V_p, |k|} e_k} \right)
\]
as a sum of 
\[
i(k \hat{x}) T_{V_1, |k| \cdots T_{V_p, |k|} e_k}, \quad (24)
\]
where \( \hat{x} \) denotes the multiplication in \( x, \) and terms of the form 
\[
\sum_{l=1}^{p} QT_{W_{l}, |k| \cdots T_{W_p, |k|} e_k}, \quad (25)
\]
where \( Q \) denotes the multiplication in \( x \) with a polynomial of maximal degree one, and \( W_l \in C^\infty_c(\mathbb{R}^3), \) \( l = 1, \ldots, p. \)

**Proof.** We have 
\[
(T_{V_1, |k| \cdots T_{V_p, |k|} e_k})(x) = e^{i k x} \int \left\{ \prod_{l=1}^{p} e^{i(k|u_t| + k u_l)} \right\} V_l \left( x + \sum_{s=1}^{l} u_s \right) d(u_1, \ldots, u_p). \quad (26)
\]

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Differentiation with respect to the first factor yields (24). Under the integral we find
\[ k \nabla_k \frac{p \prod_{l=1}^{p} e^{i(ku_l + k|u_l|)}}{|u_l|} = i \sum_{l'}^{p} (ku_{l'} + |k| |u_{l'}|) \prod_{l=1}^{p} e^{i(ku_l + |k|u_l)} = \sum_{l'}^{p} (u_{l'} \nabla_{u_{l'}} + 1) \prod_{l=1}^{p} \frac{e^{i(ku_l + |k|u_l)}}{|u_l|}. \] (27)

We can now use partial integration in (26) to shift the derivatives to the \( V_l \) terms. Any boundary terms vanish as we consider compactly supported functions. Thus, we arrive at
\[
k \nabla_k \int \left\{ \prod_{l=1}^{p} \frac{e^{i(|k|u_l + ku_l)}}{|u_l|} \right\} \int^{\infty}_{0} V_l \left( x + \sum_{s=1}^{l} u_s \right) d(u_1, \ldots, u_p)
= - \int \left\{ \prod_{l=1}^{p} \frac{e^{i(|k|u_l + ku_l)}}{|u_l|} \right\} \int^{\infty}_{0} \sum_{s=1}^{p} u_s \nabla \left( x + \sum_{s=1}^{p} u_s \right) d(u_1, \ldots, u_p)
= - \int \left\{ \prod_{l=1}^{p} \frac{e^{i(|k|u_l + ku_l)}}{|u_l|} \right\} \int^{\infty}_{0} \left( \prod_{l \neq l'}^{p} \right) V_{l'} \left( x + \sum_{s=1}^{p} u_s \right) d(u_1, \ldots, u_p).
\]

We can now write, with \( W_{l'}(y) := y \nabla V_{l'}(y) \),
\[
\left( \sum_{s=1}^{p} u_s \nabla \right) V_{l'} \left( x + \sum_{s=1}^{p} u_s \right) = W_{l'} \left( x + \sum_{s=1}^{p} u_s \right) - x \nabla W_{l'} \left( x + \sum_{s=1}^{p} u_s \right).
\]

As \( W_{l'} \) and the derivatives of \( V_{l'} \) are again in \( C_c^\infty(\mathbb{R}^3) \) for all \( l' \), we obtain expressions of the form (25).

**Lemma 5.7**

Let \( p \in \mathbb{N} \). Assume that \( V_1, \ldots, V_p \in C_c^\infty(\mathbb{R}^3) \), \( n_1, \ldots, n_p \in \mathbb{N}_0 \). Then for \( j \in \{1, 2, 3\} \),
\[
\partial_{k,j} \left( T^{(n_1)}_{V_1, |k|} \cdots T^{(n_p)}_{V_p, |k|} e_k \right) = i \delta_j T^{(n_1)}_{V_1, |k|} \cdots T^{(n_p)}_{V_p, |k|} e_k - \frac{1}{|k|} \sum_{i=1}^{p} X^{(i)}_1 \cdots X^{(i)}_p e_k,
\]
where
\[
X^{(i)}_l := \begin{cases} T^{(n_i)}_{V_l, |k|}, & i \neq l, \\ T^{(n_l)}_{V_l, |k|}, & i = l. \end{cases}
\]
Proof. We proceed analogously to the proof of Lemma 5.6 but instead of (27) for the integration by parts we use
\[ \nabla_k e^{i(ku_1 + |k||u_1|)} = i \left( \frac{u_1}{|u_1|} + \frac{k}{|k|} \right) e^{i(ku_1 + |k||u_1|)} = \nabla_{u_1} e^{i(ku_1 + |k||u_1|)} \]
\[ \nabla_k e^{i(ku_1 + |k||u_1|)} = i \left( \frac{u_1}{|u_1|} + \frac{k}{|k|} \right) e^{i(ku_1 + |k||u_1|)} = \nabla_{u_1} e^{i(ku_1 + |k||u_1|)} \]
\[ \nabla_k e^{i(ku_1 + |k||u_1|)} = i \left( \frac{u_1}{|u_1|} + \frac{k}{|k|} \right) e^{i(ku_1 + |k||u_1|)} = \nabla_{u_1} e^{i(ku_1 + |k||u_1|)} \]

Finally, we summarize the previous estimates in the following two propositions.

**Proposition 5.8**
Let \( \chi \in S(\mathbb{R}^3) \). For all \( p, m, n \in \mathbb{N}, s \in \{0, 1, 2, 3\}, X \in \{\text{Id}, (\nabla_k + \nabla_{k'}, \nabla_k\} \), there are constants \( n_1, n_2, C, \) such that for all \( k, k' \),
\[ |X(k\nabla_k + k'\nabla_{k'})^s \langle \varphi_0^{(p)}(k, \cdot), \chi \varphi_0^{(m)}(k', \cdot) \rangle| \]
\[ \leq \frac{C}{1 + |k - k'|^n} \sup_{|\alpha| \leq n_1} \|\langle \cdot \rangle^{n_2} D^\alpha \chi\|_1. \]

**Proof.** The \( s \)-fold application of Lemma 5.6 yields that we can write
\[ (k\nabla_k + k'\nabla_{k'})^s \langle \varphi_0^{(p)}(k, \cdot), \chi \varphi_0^{(m)}(k', \cdot) \rangle \]
as linear combination of terms of the form
\[ (k - k')^\alpha \langle T_{V_1,|k|} \cdots T_{V_p,|k|} e_k, P \chi T_{W_1,|k'|} \cdots T_{W_m,|k'|} e_k \rangle, \]for some polynomial \( P \), multi-index \( \alpha, V_1, \ldots, V_p, W_1, \ldots, W_m \in C^\infty_c(\mathbb{R}^3) \). Then we obtain the desired estimate for \( X = \text{Id} \) by the stationary phase argument (Lemma 5.5), which can easily be seen from (26). For \( X = \nabla_k + \nabla_{k'} \) and \( X = \nabla_k \) we apply in addition Lemma 5.7 which yields that the application of \( X \) does not change the structure of the expressions (28).

**Proposition 5.9**
Let \( \chi \in S(\mathbb{R}^3) \). For all \( p, n \in \mathbb{N}, \) there exist constants \( n_1, n_2, C, \) such that for all \( k, k', s \in \{0, 1, 2, 3\}, X \in \{\text{Id}, \nabla_k\} \), we have
\[ |X(k\nabla_k)^s \langle \varphi_0^{(p)}(k, \cdot), \chi \rangle| \leq \frac{C}{1 + |k|^{n_1}} \sup_{|\alpha| \leq n_1} \|\langle \cdot \rangle^{n_2} D^\alpha \chi\|_1. \]

**Proof.** Analogously as above we apply \( s \) times Lemma 5.6 to compute that
\[ (k\nabla_k)^s \langle \varphi_0^{(p)}(k, \cdot), \chi \rangle \]
can be written as
\[ k^\alpha \langle T_{V_1,|k|} \cdots T_{V_p,|k|} e_k, P \chi \rangle, \]for some polynomial \( P \), multi-index \( \alpha, V_1, \ldots, V_p \in C^\infty_c(\mathbb{R}^3) \). Then again the stationary phase argument and Lemma 5.7 for \( X = \nabla_k \) yield the desired estimate.

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5.3. Estimates of the Remainder Terms

Now we prove arbitrary fast polynomial decay for the remainder terms of sufficiently high order. We obtain results for remainder terms in Proposition 5.12 and scalar products of Born series terms with remainder terms (Proposition 5.14). The main tool will be the following lemma, where the basic idea is due to [21]. It is essentially a stationary phase argument together with a suitable coordinate transformation.

**Lemma 5.10 (Klein, Zemach)**

For \( V \in C^\infty_c(\mathbb{R}^3) \), \( n_1, n_2 \in \mathbb{N}_0 \), \( R > 0 \), there exists a constant \( C \) such that for all \( \kappa \geq 0 \),

\[
\sup_{|x|,|x'| \leq R} \left| \int \frac{e^{i|x-y|}}{|x-y|^{1-n_1}} V(y) \frac{e^{i|x'-y|}}{|x'-y|^{1-n_2}} dy \right| \leq \frac{C}{1 + \kappa}. \tag{30}
\]

**Proof.** For the proof we use Prolate Spheroidal coordinates, see [21, appendix] and [14, p. 661]. Let \( D = \frac{1}{2} |x - x'| \). For \( \xi \in [D, \infty) \), \( \eta \in [-1, 1] \), \( \varphi \in [0, 2\pi) \),

we set

\[
\Phi(\xi, \eta, \varphi) = \frac{1}{2} (x + x') + R \begin{pmatrix} \sqrt{(\xi^2 - D^2)(1 - \eta^2)} \cos \varphi \\ \sqrt{(\xi^2 - D^2)(1 - \eta^2)} \sin \varphi \\ \xi \eta \end{pmatrix},
\]

where \( R \) is the rotation matrix transforming \( e_3 \) into \( \frac{x - x'}{|x - x'|} \). A straightforward computation then shows that

\[
\Phi(\xi, \eta, \varphi) = \frac{1}{2} (|x - \Phi(\xi, \eta, \varphi)| + |x' - \Phi(\xi, \eta, \varphi)|),
\]

\[
\eta = \frac{1}{2D} (|x - \Phi(\xi, \eta, \varphi)| - |x' - \Phi(\xi, \eta, \varphi)|),
\]

\[
det \Phi(\xi, \eta, \varphi) = (\xi + D\eta)(\xi - D\eta).
\]

Thus, by change of coordinates,

\[
\int \frac{e^{i|x-y|} V(y) e^{i|x'-y|}}{|x-y|^{1-n_1} |x'-y|^{1-n_2}} dy = \int e^{2i\xi} V(\Phi(\xi, \eta, \varphi))(\xi + D\eta)^{n_1}(\xi - D\eta)^{n_2} d(\xi, \eta, \varphi).
\]

\[
= \int_{D}^{\infty} e^{2i\xi} h(\xi) d\xi,
\]

where \( h(\xi) := \int V(\Phi(\xi, \eta, \varphi))(\xi + D\eta)^{n_1}(\xi - D\eta)^{n_2} d(\eta, \varphi) \). Let \( E := \frac{1}{2} |x + x'| \). Notice that by direct computation, for \( \xi \geq D + E \),

\[
|\Phi(\xi, \eta, \varphi)| \geq \xi - D - E.
\]

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Thus, we get that $h(\xi) = 0$ for $\xi \geq R + D + E$. Then, by integration by parts,

$$\int_D^{\infty} e^{2i\kappa \xi} h(\xi) d\xi = \frac{1}{2i\kappa} \int_D^{R + D + E} \partial_\xi \left( e^{2i\kappa \xi} h(\xi) \right) d\xi$$

$$= \frac{1}{2i\kappa} \left( -h(D) e^{2i\kappa D} - \int_D^{R + D + E} e^{2i\kappa \xi} \partial_\xi h(\xi) d\xi \right).$$

As $D, E \leq R$ are bounded, so is the first term. For the second one notice that

$$\partial_\xi h(\xi) = \int \langle \nabla V(\Phi(\xi, \eta, \varphi)), \partial_\xi \Phi(\xi, \eta, \varphi) \rangle (\xi - D\eta) d(\eta, \varphi) \quad (31)$$

$$+ \int V(\Phi(\xi, \eta, \varphi)) \partial_\xi ((\xi - D\eta)) d(\eta, \varphi) \quad (32)$$

The term (32) is clearly bounded in $R$. The term (31) is bounded up to a constant by

$$\sup_{\eta, \varphi} |\partial_\xi \Phi(\xi, \eta, \varphi)| \leq C \left( 1 + \frac{\xi}{\sqrt{\xi^2 - D^2}} \right),$$

for some constant $C$. This is integrable and the integral is also bounded by a constant only depending on $R$:

$$\int_D^{R + D + E} \frac{\xi}{\sqrt{\xi^2 - D^2}} d\xi = \sqrt{(R + D + E)^2 - D^2}.$$  

**Lemma 5.11**

Let $V_1, \ldots, V_p \in C^\infty_c(\mathbb{R}^3)$ and $n_1, \ldots, n_p \in \mathbb{N}_0$. Then there exists a constant $C$ such that for all $k, x \in \mathbb{R}^3$, continuous bounded functions $\psi$ on $\mathbb{R}^3$,

$$\left| (T^{(n_1)}_{V_1, |k|} \cdots T^{(n_p)}_{V_p, |k|} \psi)(x) \right| \leq \frac{C (1 + \langle x \rangle^{n_1-1} \|\psi\|_\infty)}{1 + |k|^{\frac{n_1-1}{2}}}.$$  

**Proof.** First we assume that $p = 2p^* + 1$. Then

$$(T^{(n_1)}_{V_1, |k|} \cdots T^{(n_p)}_{V_p, |k|} \psi)(x)$$

$$= \int \frac{e^{i|x-y_1|}}{|x-y_1|^{1-n_1}} V_1(y_1) \psi(y_p) d(y_1, y_2, \ldots, y_p).$$  

$$\left\{ \prod_{l=1}^{p^*} \frac{e^{i|y_{2l-1} - y_{2l}|}}{|y_{2l-1} - y_{2l}|^{1-n_{2l}}} V_{2l}(y_{2l}) \right\} \frac{e^{i|y_{2p^*+1}}}{|y_{2l} - y_{2l+1}|^{1-n_{2l+1}}} V_{2l+1}(y_{2l+1}) \psi(y_p) d(y_1, y_2, \ldots, y_p).$$  

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In the following let $C$ denote different constants depending only on $V_l$ and $n_l$, $l = 1, \ldots, p$. We estimate the terms in (34) for $l = 1, \ldots, p^*$ by

\[ \left| \int \frac{e^{i|k||y_{2l-1}-y_{2l}|}}{|y_{2l-1} - y_{2l}|^{1-n_{2l}}} V_{2l}(y_{2l}) e^{i|k||y_{2l+1}-y_{2l+2}|} |y_{2l} - y_{2l+1}|^{1-n_{2l+1}} dy_{2l} \right| \leq \frac{C}{1 + |k|} \]

using Lemma 5.10, the term (33) by

\[ \left| \int \frac{e^{i|k||x-y|}}{|x - y|^{1-n_1}} V_1(y_1) dy_1 \right| \leq C(1 + (x)^{n_1-1}), \]

and thus we find

\[ \left| \left( T_{V_1,|k|}^{(n_1)} \cdots T_{V_p,|k|,\psi}^{(n_p)} \right)(x) \right| \leq C(1 + \langle x \rangle^{n_1-1}) \int \left\{ \prod_{l=0}^{p^*-1} |V_{2l+2}(y_{2l+2})| \right\} \psi(y_p) d(y_2, y_4, \ldots, y_p) \]

\[ \leq C(1 + \langle x \rangle^{n_1-1}) \| \psi \|_{\infty}. \]

One can show the same for odd $p$ by using the standard estimates of Proposition 5.3 for $n_p \leq 1$, and

\[ \int |V_p(y_p)| (1 + \langle y_p \rangle^{n_p-1}) dy_p < \infty \]

for $n_p > 1$.

**Proposition 5.12**

For any $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m \in \{0,1\}$ there exists a constant $C$ such that we have for all $k \neq 0$, $x \in \mathbb{R}^3$,

\[ \left| \partial_{k,j}^m \hat{D}_k^n \mathcal{R}^{(p)}(k, x) \right| \leq \frac{C(1 + \langle x \rangle^{n+m-1})}{1 + |k|^{\frac{m+1}{2}}} \]

**Proof.** For $k \neq 0$, evaluating

\[ \partial_{k,j}^m \hat{D}_k^n T_{V,|k|} \cdots T_{V,|k|} \psi(k, \cdot), \]

yields a linear combination of terms

\[ f(k) T_{V,|k|}^{(n_1)} \cdots T_{V,|k|}^{(n_p)} \partial_{k,j}^m \hat{D}_k^n \psi(k, \cdot), \]

where $f$ is a bounded function on $\mathbb{R}^3$, $0 \leq n' \leq n$, $0 \leq m' \leq m$, and $n_1 + \cdots + n_p + m = n' + m'$. Then we use Lemma 5.11 and Proposition 5.1 to prove the claim. \qed
Lemma 5.13

Let \( p, m, r \in \mathbb{N} \) with \( m \geq 2r+1, V_1, \ldots, V_p, W_1, \ldots, W_m \in C^\infty_c(\mathbb{R}^3) \), and \( n_1, \ldots, n_p, n'_1, \ldots, n'_m \in \mathbb{N}_0 \). Then there exists a constant \( C, n_0 \in \mathbb{N}_0 \), such that for all \( k, k' \), continuous bounded functions \( \psi \) and \( \chi \in \mathcal{S}(\mathbb{R}^3) \),

\[
\left| \left\langle T^{(n_1)}_{V_1, \ldots, V_p, \ldots} \xi, \chi T^{(n'_1)}_{W_1, \ldots, W_m, \ldots} \psi \right\rangle \right| \leq \frac{C\left( \| (1 + \langle \cdot \rangle^{n_0}) \varphi \|_1 + \| (1 + \langle \cdot \rangle^{n'_0}) \varphi' \|_1 \| \psi \|_{\infty} \right)}{(1 + |k|^{\frac{3l_1}{2}+r})(1 + |k'|^{\frac{3l_2}{2}+r})}. \tag{36}
\]

Proof. To shorten the following notation we assume that \( n_1 = \cdots = n_p = n'_1 = \cdots = n'_m = 0 \). The proof works analogously for the other cases.

First we consider \( r = 1 \) and multiply the left-hand side of (36) with \( k \) and show that it has the required bound with \( r = 0 \):

\[
k \left\langle T^{(n_1)}_{V_1, \ldots, V_p, \ldots} \xi, \chi T^{(n'_1)}_{W_1, \ldots, W_m, \ldots} \psi \right\rangle = \int \frac{e^{-i|k|\varepsilon_1}}{|v_1|} V_1(v_1 + x) \frac{e^{-i|k|\varepsilon_2}}{|v_2|} V_2(v_2 + v_1 + x) \ldots \frac{e^{-i|k|\varepsilon_p}}{|v_p|} V_p \left( \sum_{l=1}^p v_l + x \right)
\]

\[
e^{-ik\sum_{l=1}^p v_l} (i\nabla_x e^{-ikx}) \chi(x) \frac{e^{i|k||x-x_1|}}{|x-x_1|} W_1(x_1) \frac{e^{i|k'||x_1-x_2|}}{|x_1-x_2|} W_2(x_2)
\]

\[
\ldots \frac{e^{i|k'||x_{m-1}-x_m|}}{|x_{m-1}-x_m|} W_m(x_m) \psi(x_m) d(v_1, \ldots, v_p, x, x_1, \ldots, x_m).
\]

We now use integration by parts with respect to \( x \). By the product rule, we have a linear combination of several different terms. The ones with derivatives of the potentials \( V_1, \ldots, V_p \) and \( \chi \) can be estimated using [Lemma 5.11] by the bound (36) with \( r = 0 \). For the last term containing \( x \) we have

\[
\nabla_x e^{i|k||x-x_1|} = \nabla_{x_1} e^{i|k||x-x_1|}.
\]

Then we use again partial integration and get a term involving \( \nabla W_1 \), which can be estimated as the first terms, and

\[
\nabla_{x_1} e^{i|k'||x_1-x_2|} = \nabla_{x_2} e^{i|k'||x_1-x_2|}.
\]

Repeating this trick we obtain expressions with derivatives of the \( W \)'s and finally the term where we take the derivative of the last fraction,

\[
\nabla_{x_{m-1}} e^{i|k'||x_{m-1}-x_m|} = K(x_{m-1}, x_m) \frac{x_{m-1} - x_m}{|x_{m-1} - x_m|},
\]

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where
\[ K(x_{m-1}, x_m) := \frac{e^{ik'\|x_{m-1}-x_m\|}}{|x_{m-1} - x_m|} \left( |k'| - \frac{1}{|x_{m-1} - x_m|} \right). \]

The integral operator corresponding to \((x_{m-1}, x_m) \mapsto K(x_{m-1}, x_m)W_m(x_m)\) can be written as
\[ i |k'| T_{W_m,k'} - T_{W_m,k'}^{(-1)}. \]

Both operators are bounded due to Proposition 5.3. Then we can again use Lemma 5.11 but notice that we lose one order in the decay in \(k'\) by the multiplication by \(|k'|\) and by the fact that we cannot gain from the last operator \(T_{W_m,k'}^{(-1)}\) in Lemma 5.11.

Now we repeat the whole procedure \(r\) times and each time we lose one order of decay in \(k'\) and obtain one in \(k\).

\[ \square \]

**Proposition 5.14**

For all \(n \in \mathbb{N}\), there exist constants \(m, n_1, n_2 \in \mathbb{N}\), \(C\), such that for all \(k, k' \neq 0\), \(s \in \{0, 1, 2, 3\}\), \(p \in \mathbb{N}_0\), \(\chi \in \mathcal{S}(\mathbb{R}^3)\), \(X \in \{\text{Id}, \nabla_k + \nabla_{k'}, \nabla_k\}\), we have
\[ \left| X(k\nabla_k + k'\nabla_{k'})^s \left\langle \varphi_0^{(p)}(k, \cdot), \chi \varphi_R^{(m)}(k', \cdot) \right\rangle \right| \leq C \frac{1}{(1 + |k|^s)(1 + |k'|^s)} \sup_{|\alpha| \leq n_1} \|\langle \cdot \rangle^{n_2} D^\alpha \chi\|_1. \]

**Proof.** By applying Lemma 5.6 and (35) for the left and right part of the inner product, respectively, we can write
\[ X(k\nabla_k + k'\nabla_{k'})^s \left\langle \varphi_0^{(p)}(k, \cdot), \chi \varphi_R^{(m)}(k', \cdot) \right\rangle \]
for all given \(X\) and \(s\) as a linear combination of expressions
\[ k^\alpha (k')^\beta f(k') \left\langle T_{V_1,k}^{(n_1)}, \ldots, T_{V_p,k}^{(n_p)} e_k, \chi T_{W_1,k'}^{(n_1)} \cdots T_{W_m,k'}^{(n_m)} \varphi(k', \cdot) \right\rangle, \]
where \(\alpha, \beta\) are multi-indices with \(|\alpha|, |\beta| \leq s\), \(f\) is a bounded function on \(\mathbb{R}^3\), \(V_1, \ldots, V_p, W_1, \ldots, W_m \in C_\infty(\mathbb{R}^3)\), and \(n_1, \ldots, n_p, n_1', \ldots, n_m' \in \mathbb{N}_0\). Now we can estimate these expressions with Lemma 5.13 \(\square\)

**5.4. Commutator with the Interaction**

This part provides the key for the proof of Proposition 4.2. In the following we omit for the moment the regularity function \(k\) of the coupling and work with multiplication operators \(H(\omega, \Sigma), (\omega, \Sigma) \in \mathbb{R}_+ \times S^2\), which are assumed to satisfy the same conditions as \(\partial^\alpha_\omega \tilde{G}(\omega, \Sigma): \) For all \((\omega, \Sigma)\) let \(H(\omega, \Sigma) \in \mathcal{S}(\mathbb{R}^3)\). \((\omega, \Sigma, x) \mapsto H(\omega, \Sigma)(x)\) is measurable, and for all \(m \in \mathbb{N}\) there exists a polynomial \(P\) such that
\[ |\partial^m_x H(\omega, \Sigma)(x)| \leq P(\omega), \quad (\omega, \Sigma) \in \mathbb{R}_+ \times S^2, \quad x \in \mathbb{R}^3. \]

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To show that a commutator $[H, A_p]$, for an operator $H$ on $L^2(\mathbb{R}^3)$, is bounded, we use the following decomposition on $\ell^2(N) \oplus L^2(\mathbb{R}^3)$,

$$\mathcal{V}[H, A_p]\mathcal{V}^* = \begin{pmatrix} 0 & -A_D V_c H V_d^* \\ V_d H V_d^* A_D & [V_c H V_d^*, A_D] \end{pmatrix},$$

where $\mathcal{V}$ is the unitary operator defined in (19) and $N \in \mathbb{N}$ is the number of eigenfunctions of $H_p$. We treat the term on the diagonal in Lemma 5.18 and the off-diagonal terms in Proposition 5.16.

**Lemma 5.15**

Let $n \in \mathbb{N}$ and $(\omega, \Sigma) \in \mathbb{R} \times S^2$. Assume that $\text{ad}_{A_p}^{(n)}(H(\omega, \Sigma))$, $\text{ad}_{A_0}^{(n)}(V_c H(\omega, \Sigma)V_c^*)$, $1 \leq n \leq n_0$, exist as bounded operators. Let $\psi_d \in \text{ran} P_{\text{disc}}$ and $\psi_c \in C^\infty(\mathbb{R}^3)$. Then we have for all $1 \leq n \leq n_0$, $V_c H(\omega, \Sigma)\psi_d \in D(A_0^n)$, and

(a) $V_c \text{ad}_{A_p}^{(n)}(H(\omega, \Sigma))\psi_d = A_D^n V_c H(\omega, \Sigma)\psi_d,$

(b) $V_c \text{ad}_{A_p}^{(n)}(H(\omega, \Sigma))V_c^* \psi_c = \text{ad}_{A_0}^{(n)}(V_c H(\omega, \Sigma)V_c^*) \psi_c.$

**Proof.** This follows by induction over $n$ by the definition of $A_p$. \hfill \Box

**Proposition 5.16**

For all $n \in \{0, 1, 2, 3\}$, $j \in \{1, 2, 3\}$, $(\omega, \Sigma)$, the operators

1. $A_0^n V_c H(\omega, \Sigma) P_{\text{disc}},$
2. $k_j A_0^n V_c H(\omega, \Sigma) P_{\text{disc}},$
3. $\hat{k}_j A_0^n V_c H(\omega, \Sigma) P_{\text{disc}},$

are well-defined and their norms can be estimated uniformly in $(\omega, \Sigma)$ by a polynomial in $\omega$. Furthermore, they are strongly continuous in $(\omega, \Sigma)$.

**Proof.** Choose $N$ big enough so that we find by means of Proposition 5.12 a constant $C$ such that for all $0 \leq n \leq 3$, $m \in \{0, 1\}$, $\psi_d \in \text{ran} P_{\text{disc}}$, $k \neq 0$,

$$\left| \hat{\phi}^{m_n}_{k,j} D_k \hat{\phi}^{(N)}_{R}(k, x) \chi(x) \psi_d(x)dx \right| \leq \frac{C \|\psi_d\|}{1 + |k|^m}. \tag{37}$$

Expanding $\varphi(k, x)$, we obtain for $\psi_d \in \text{ran} P_{\text{disc}}$, $k \neq 0$,

$$V_c H(\omega, \Sigma) \psi_d(k) = (2\pi)^{-3/2} \int \varphi(k, x) H(\omega, \Sigma)(x) \psi_d(x) dx$$

$$= (2\pi)^{-3/2} \left\{ \sum_{n=0}^{N} \int \varphi^{(n)}_{0}(k, x) H(\omega, \Sigma)(x) \psi_d(x) dx \right\}$$

$$+ \int \varphi^{(N)}_{R}(k, x) H(\omega, \Sigma)(x) \psi_d(x) dx \tag{38}$$

$$= (2\pi)^{-3/2} \left\{ \sum_{n=0}^{N} \int \varphi^{(n)}_{0}(k, x) H(\omega, \Sigma)(x) \psi_d(x) dx \right\}$$

$$+ \int \varphi^{(N)}_{R}(k, x) H(\omega, \Sigma)(x) \psi_d(x) dx \tag{39}.$$
Then the terms which come out if we apply $A_n, \hat{k}_j, \hat{x}_j, n \in \{0, 1, 2, 3\}, j \in \{1, 2, 3\}$, respectively, to (38), can be estimated with Proposition 5.9 by

$$P(\omega) \frac{\|\psi_\delta\|}{1 + |k|^2},$$

where $P$ is some polynomial not depending on $\psi_\delta$ and $k$. The ones coming from (39) can be estimated with (37) by

$$C \frac{\|\psi_\delta\|}{1 + |k|^2},$$

with a constant $C$ independent of $\psi_\delta$ and $k$. This proves that the operators (1), (2) and (3) are well-defined and their norm can be estimated by a polynomial in $\omega$.

Finally, strong continuity in $(\omega, \Sigma)$ follows from dominated convergence as we obviously have point-wise convergence. \hfill \Box

**Lemma 5.17**

For all $(\omega, \Sigma)$ and $k, k' \in \mathbb{R}^3$ let

$$K_{\omega, \Sigma}(k, k') := \int H(\omega, \Sigma)(x) \varphi(k, x) dx.$$

There exists a polynomial $P$ such that for all $s \in \{0, 1, 2, 3\}$, $j \in \{1, 2, 3\}$, $m \in \{0, 1\}$, $(\omega, \Sigma)$, $k, k' \neq 0$ we can estimate the absolute value of

1. $(k \nabla_k + k' \nabla_{k'})^* K_{\omega, \Sigma}(k, k')$,
2. $(\partial_{k,j} + \partial_{k',j})(k \nabla_k + k' \nabla_{k'})^* K_{\omega, \Sigma}(k, k')$,
3. $(k_j - k_j') (k \nabla_k + k' \nabla_{k'})^* K_{\omega, \Sigma}(k, k')$,
4. $\partial_{k,j}(k \nabla_k + k' \nabla_{k'})^m K_{\omega, \Sigma}(k, k')$,

respectively, from above by

$$P(\omega) \left( \frac{1}{(1 + |k|^2)(1 + |k'|^2)} + \frac{1}{1 + |k - k'|^4} \right).$$

Moreover, for fixed $k, k' \neq 0$, the functions $\mathbb{R}_+ \times S^2 \to \mathbb{C}$ mapping $(\omega, \Sigma)$ to the expressions (1)-(4) are continuous.
Proof. We choose $N$ big enough such that Proposition 5.14 holds for $m = N$ and $n = 3$. Then

$$K_{\omega, \Sigma}(k, k') = \int \varphi(k', x)H(\omega, \Sigma)(x)\varphi(k, x)dx$$

$$= \sum_{n,n'=0}^{N} \int \varphi_0^{(n')} (k', x)H(\omega, \Sigma)(x)\varphi_0^{(n)} (k, x)dx$$

$$+ \sum_{n=0}^{N} \int \varphi_R^{(n)} (k', x)H(\omega, \Sigma)(x)\varphi_0^{(n)} (k, x)dx$$

$$+ \int \varphi_0^{(n)} (k', x)H(\omega, \Sigma)(x)\varphi_R^{(n)} (k, x)dx$$

$$+ \int \varphi_R^{(n)} (k', x)H(\omega, \Sigma)(x)\varphi_R^{(n)} (k, x)dx.$$ (40)

(41)

(42)

(43)

By (11), for all $\alpha \in \mathbb{N}_0^3$, there exists a polynomial $P$ such that

$$\|\partial_\alpha^0 H(\omega, \Sigma)(x)\|_1 \leq P(\omega), \quad (\omega, \Sigma) \in \mathbb{R}_+ \times S^2, \ x \in \mathbb{R}^3.$$

We now apply the derivatives given above and use Proposition 5.8 to estimate the terms which come from (40) by

$$\frac{P(\omega)}{1 + |k - k'|^4}$$

for some polynomial $P$. Similarly, we use Proposition 5.14 for (41) and (42), and Proposition 5.12 for (43).

The continuity property follows by dominated convergence as $H(\omega, \Sigma)$ is bounded by the Schwartz function $\chi$ and the derivatives of the scattering functions are bounded by polynomials (Proposition 5.1).

Lemma 5.18

For all $n \in \{0, 1, 2, 3\}$, $m \in \{0, 1\}$, $j \in \{1, 2, 3\}$, $(\omega, \Sigma)$, the operators

(1) $\text{ad}^{(n)}_{\text{ad}_b^0} (V_c H(\omega, \Sigma)V^*_c)$,

(2) $\text{ad}_{k_j} (\text{ad}^{(n)}_{\text{ad}_b^0} (V_c H(\omega, \Sigma)V^*_c))$,

(3) $\hat{x}_j \text{ad}^{(n)}_{\text{ad}_b^0} (V_c H(\omega, \Sigma)V^*_c)$,

(4) $\hat{x}_j \text{ad}^{(n)}_{\text{ad}_b^0} (V_c H(\omega, \Sigma)V^*_c)$,

are well-defined bounded operators and we can estimate their norms uniformly in $(\omega, \Sigma)$ by a polynomial in $\omega$. Moreover, the functions $\mathbb{R}_+ \times S^2 \to \mathcal{L}(H_p)$ mapping $(\omega, \Sigma)$ to the expressions above, are continuous.

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Proof. The operators can be written as integral operators with integral kernels which can be estimated as in Lemma 5.17. An integral operator with integral kernel
\[(k, k') \mapsto \frac{1}{(1 + |k|^2)(1 + |k'|^2)}\]
is Hilbert-Schmidt. An operator with integral kernel
\[(k, k') \mapsto \frac{1}{1 + |k - k'|^4}\]
is bounded by Young’s inequality for convolutions: \[\|K \ast \psi\|_2 \leq \|K\|_1 \|\psi\|_2,\quad K \in L^1(\mathbb{R}^3), \quad \psi \in L^2(\mathbb{R}^3).\]

With regard to the continuity in \((\omega, \Sigma)\) notice that the integral kernels are point-wise continuous for almost all \(k, k'\) by Lemma 5.17. Finally, dominated convergence yields the continuity in norm for these operators.

The following central proposition can be thought of as a version of Proposition 4.2 without \(\kappa\).

**Proposition 5.19**
For all \(n \in \{0, 1, 2, 3\}, m \in \{0, 1\}, \ j \in \{1, 2, 3\}, \ (\omega, \Sigma), \) the operators

1. \(\text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\),
2. \(\text{ad}_{\hat{V}^\ast j} V_c (\text{ad}^{(n)}_{A_p}(H(\omega, \Sigma)))\),
3. \(\text{ad}_{\hat{V}^\ast j} V_c (\text{ad}^{(n)}_{A_p}(H(\omega, \Sigma)))\),
4. \(V_c^\ast \hat{j} V_c \text{ad}^{(m)}_{A_p}(H(\omega, \Sigma))\),

are well-defined and we can estimate their norms uniformly in \((\omega, \Sigma)\) by a polynomial in \(\omega\). The corresponding functions \(\mathbb{R}_+ \times S^2 \rightarrow \mathcal{L}(H_p)\) are continuous and in \(L^2(\mathbb{R}_+ \times S^2, \mathcal{L}(H_p))\)

Proof. Let \(\psi \in \mathcal{D}_p\), then by definition of \(\mathcal{D}_p\) there exist \(\psi_c \in C_c^\infty(\mathbb{R}^3), \psi_d \in \text{ran} P_{\text{disc}}\) such that
\[\psi = V_c^\ast \psi_c + \psi_d.\]

We now first prove that [1] is bounded by induction over \(n\). Assume for \((\omega, \Sigma)\) and \(n \in \mathbb{N}\) that we have shown that \(\text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\) is bounded. Then we compute
In Proposition 5.19 we plug in using Lemma 5.15:
\[
\langle \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\psi, A_p\psi \rangle - \langle A_p\psi, \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\psi \rangle
\]
\[
= \langle \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))V_c^*\psi_c, V_c^*A_D\psi_c \rangle - \langle V_c^*A_D\psi_c, \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))V_c^*\psi_c \rangle
\]
\[
+ \langle V_c \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\psi_d, A_D\psi_c \rangle - \langle A_D\psi_c, V_c \text{ad}^{(n)}_{A_p}(H(\omega, \Sigma))\psi_d \rangle
\]
\[
= \langle \text{ad}^{(n)}_{A_p}(V_cH(\omega, \Sigma)V_c^*)\psi_c, A_D\psi_c \rangle - \langle A_D\psi_c, \text{ad}^{(n)}_{A_p}(V_cH(\omega, \Sigma)V_c^*)\psi_c \rangle
\]
\[
+ \langle A_p^0V_cH(\omega, \Sigma)\psi_d, A_D\psi_c \rangle - \langle A_D\psi_c, A_p^0V_cH(\omega, \Sigma)\psi_d \rangle.
\]

The first two terms are bounded by Lemma 5.18 and the last two terms are bounded by a polynomial in \(\omega\) by Proposition 5.16. The continuity in \((\omega, \Sigma)\) follows from the continuity of the operator-valued function \((\omega, \Sigma) \mapsto \text{ad}^{(n)}_{A_p}(V_cH(\omega, \Sigma)V_c^*)\) and of \((\omega, \Sigma) \mapsto A_p^0V_cH(\omega, \Sigma)\psi_d\). The proof of (2) and (3) is completely analogous. Finally, for \(m \in \{0, 1\}\), we note that
\[
V_c \text{ad}^{(m)}_{A_p}(H(\omega, \Sigma))\psi = \text{ad}^{(m)}_{A_p}(V_cH(\omega, \Sigma)V_c^*)\psi_c + A_p^0V_cH(\omega, \Sigma)\psi_d
\]
is in \(D(|p|)\) by Lemma 5.18 and Proposition 5.16 and we can bound it by a polynomial in \(\omega\).

**Proof of Proposition 4.2.** We write
\[
\partial^m_{\omega} \text{ad}^{(n)}_{A_p}(G(\omega, \Sigma)) = \sum_{l=0}^{m} \binom{m}{l} \partial^l_{\omega} \kappa(\omega) \text{ad}^{(n)}_{A_p}(\partial^{m-l}_{\omega} \tilde{G}(\omega, \Sigma)),
\]
In Proposition 5.19 we plug in \(H(\omega, \Sigma)(x) := \partial^{m-l}_{\omega} \tilde{G}(\omega, \Sigma)(x)\), and we get a polynomial \(P\) such that for all \(l = 0, \ldots, m, (\omega, \Sigma)\),
\[
\left\| \text{ad}^{(n)}_{A_p}(\partial^{m-l}_{\omega} \tilde{G}(\omega, \Sigma)) \right\| \leq P(\omega).
\]
Then it follows by Lemma A.2 with the decay behavior of \(\kappa\) that the corresponding operator-valued functions are indeed in \(L^2(\mathbb{R} \times \mathbb{S}^2, \mathcal{L}(H_p))\). In the other two cases one can proceed analogously.

6. Proof of Positivity and of the Main Theorem

In this section the main estimates and proofs of positivity of the commutator are discussed. First we introduce the two terms \(A_0\) and \(C_Q\) which we add to \(C_1\) as already mentioned in the overview of the proof. Then we show how the main theorem is proven given that we know that the sum of all three terms is positive (Proposition 6.2). Then, in Subsection 6.2 we prove how we estimate the three
terms separately and which error terms occur. With that we conclude by proving Proposition 6.2.

In this section the conditions of Theorem 2.3 are assumed, that is, that (H1), (H2), (I1), (I2), (I3) and (F) are satisfied.

6.1. Putting Things together, Proof of the Main Theorem

In the following we use for \( m \in \mathbb{N}_0 \) the short-hand notation

\[
\hat{N}_f := \text{Id}_p \otimes \text{Id}_p \otimes N_f, \quad \hat{P}_\Omega := \text{Id}_p \otimes \text{Id}_p \otimes P_\Omega.
\]

Remember that we have on \( D, C_1 = V_c^* k^2 V_c \otimes \text{Id}_p \otimes \text{Id}_f + \text{Id}_p \otimes V_c^* k^2 V_c \otimes \text{Id}_f + \hat{N}_f + \lambda W_1, \) (44)

where \( W_1 \) was the commutator with the interaction, cf. (20). Obviously, \( C_1 \) is strictly positive on the orthogonal complement of the vacuum subspace for \( \lambda = 0 \) and its first two terms are positive on \( (\text{ran}(P_{\text{disc}} \otimes P_{\text{disc}}))^\perp \otimes \mathfrak{F}. \)

On the space \( \text{ran} \Pi \) we use the Fermi Golden Rule and introduce the corresponding conjugation operator \( A_0 \) in order to obtain a positive expression in Proposition B.2 as a commutator with \( L_\lambda. \) It was first developed for zero temperature systems in [1] and later adapted to the positive temperature case in [13]. It is a bounded self-adjoint operator on \( \mathcal{H}, \) given by

\[
A_0 := i \lambda (\Pi_\Omega W R_\varepsilon^2 \Pi_\perp - \Pi_\perp R_\varepsilon^2 W \Pi),
\]

where \( R_\varepsilon^2 := (L_\lambda + \varepsilon^2)^{-1} \) and \( \varepsilon > 0. \) We will choose \( \varepsilon > 0 \) such that (F) holds. One can show that \( \text{ran} A_0 \subseteq \mathcal{D}(L_\lambda) \) since \( \text{ran} A_0 \subseteq \mathcal{H}_p \otimes \mathcal{H}_p \otimes \mathfrak{F}_\text{fin} \) (cf. Lemma B.1).

Furthermore,

\[
i[L_\lambda, A_0] = -\lambda [L_\lambda, \Pi_\Omega W R_\varepsilon^2 \Pi_\perp - \Pi_\perp R_\varepsilon^2 W \Pi] \quad (45)
\]

extends to a bounded self-adjoint operator as well and we have \( \varepsilon \Pi_\Omega i[L_\lambda, A_0] \Pi > 0 \) for small \( \varepsilon > 0, \) see Proposition B.2.

To obtain positivity on the remaining space \( (\ker L_p)^\perp \otimes \text{ran} P_\Omega \) we introduce the bounded operator

\[
C_Q := Q \otimes P_\Omega + \frac{\lambda}{2} W(L_p^{-1} Q \otimes P_\Omega) + \frac{\lambda}{2} (W(L_p^{-1} Q \otimes P_\Omega))^*
\]

on \( \mathcal{H}, \) where \( L_p^{-1} \) is to be understood in the sense of functional calculus as an unbounded operator, and

\[
Q := L_p^2 1_{[-1,1]}(L_p) + 1_{(-\infty,-1)\cup(1,\infty)}(L_p).
\]

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Notice that by construction \( \text{ran} \, Q \subseteq D(L_p^{-1}) \), \( L_p^{-1}Q \) is bounded and self-adjoint, so the definition of \( C_Q \) makes sense. Furthermore, the first summand \( Q \otimes P_\Omega \) is indeed positive on \( (\ker L_p)^\perp \otimes \text{ran} \, P_\Omega \).

The idea will be to show that the sum of the three operators (44), (45) and (46) is positive and has zero expectation with any element of the kernel of \( L_\lambda \). To this end we define for \( \psi \in D(q_{C_1}) \cap D(L_\lambda) \) and some \( \theta > 0 \),

\[
q_{\text{tot}}(\psi) := q_{C_1}(\psi) + \theta \langle \psi, i[L_\lambda, A_0] \psi \rangle + \langle \psi, C_Q \psi \rangle.
\]

By the virial theorem for \( C_1 \) and by construction of the other two terms, this form is actually non-positive for any \( \psi \in \ker L_\lambda \). This is the content of the following proposition.

**Proposition 6.1**

*For arbitrary \( \lambda, \varepsilon, \delta \), and \( \psi \in \ker L_\lambda \) we have \( \psi \in D(q_{C_1}) \) and \( q_{\text{tot}}(\psi) \leq 0 \).*

**Proof.** Let \( \psi \in \ker L_\lambda \). By Theorem 4.4 we know that \( \psi \in D(q_{C_1}) \), and

\[
q_{C_1}(\psi) \leq 0.
\]

It is clear that \( \langle \psi, i[L_\lambda, A_0] \psi \rangle = 0 \). Furthermore, by construction, the last term vanishes:

\[
0 = \langle (L_p^{-1}Q \otimes P_\Omega)L_\lambda \psi, \psi \rangle = \langle \psi, L_\lambda(L_p^{-1}Q \otimes P_\Omega) \psi \rangle = \langle \psi, (L_p \otimes \Id + \lambda W)(L_p^{-1}Q \otimes P_\Omega) \psi \rangle + \langle \psi, (Q \otimes P_\Omega) \psi + \lambda W(L_p^{-1}Q \otimes P_\Omega) \psi \rangle.
\]

Then \( \langle \psi, C_Q \psi \rangle = 0 \) follows by adding the complex conjugate term. Thus,

\[
q_{\text{tot}}(\psi) = q_{C_1}(\psi) \leq 0.
\]

We prove in the next subsection the following proposition which states that \( q_{\text{tot}} \) is in fact positive. Remember that \( \gamma_\beta \) is the constant appearing in (F) which might depend on \( \beta \).

**Proposition 6.2**

*Let \( \beta_0 > 0 \). Then there exists constant \( \lambda_0, C > 0 \) such that for all \( 0 < |\lambda| < \lambda_0 \) with \( |\lambda| < C \gamma_\beta^2 \) and all \( \beta \geq \beta_0 \), we have \( q_{\text{tot}} > 0 \). That is, \( q_{\text{tot}} \geq 0 \) and \( q_{\text{tot}}(\psi) = 0 \) for some \( \psi \in D(q_{C_1}) \cap D(L_\lambda) \) implies \( \psi = 0 \).*

This positivity result together with Proposition 6.1 now implies the main theorem Theorem 2.3.
6.2. Error Estimates

In the following proposition we prove separate estimates from below of the three operators (44), (45) and (46).

Proposition 6.3
(a) There exist constants \( c_1, c_2 > 0 \) such that, for all \( \lambda \in \mathbb{R} \), we have in the sense of quadratic forms on \( D(qc_1) \),

\[
C_1 \geq \left[ V_c^* \hat{k}^2 V_c \otimes \text{Id}_p + \text{Id}_p \otimes V_c^* \hat{k}^2 V_c - c_1 (1 + \beta^{-1}) \lambda^2 \left( V_c^* \langle \hat{x} \rangle^{-2} V_c \otimes \text{Id}_p + \text{Id}_p \otimes V_c^* \langle \hat{x} \rangle^{-2} V_c + (P_{\text{ess}} \otimes P_{\text{ess}})^\perp \right) \right] \otimes \text{Id}_f + c_2 \hat{P}_\Omega^\perp.
\]

(b) For all \( \varepsilon > 0 \) there exist constants \( c_1, c_2, c_3 > 0 \) (depending on \( \varepsilon \)) such that for \( |\lambda| < 1 \),

\[
i[L_\lambda, A_0] \geq (1 - c_1 |\lambda|) 2\lambda^2 \Pi W R_q^2 \Pi^\perp W - c_2 |\lambda| \text{Id}_f \otimes \text{Id}_f \otimes P_{\Omega}^\perp
\]

\[
- c_3 (1 + \beta^{-1}) \lambda^2 \Pi_{L_\mu \neq 0} \left( V_c^* \langle \hat{x} \rangle^{-2} V_c \otimes \text{Id}_p + \text{Id}_p \otimes V_c^* \langle \hat{x} \rangle^{-2} V_c + (P_{\text{ess}} \otimes P_{\text{ess}})^\perp \right) \otimes \text{Id}_f + c_2 \hat{P}_\Omega^\perp.
\]

(c) There exists a constant \( c_1 > 0 \) such that, for all \( \lambda \in \mathbb{R} \),

\[
C_Q \geq (1 - c_1 |\lambda| (1 + \beta^{-1})) Q \otimes P_\Omega - |\lambda| \hat{P}_\Omega^\perp.
\]

Before we can give the proof of Proposition 6.3 we need some preparatory lemmas. It is convenient to introduce some further notation for the interaction and the commuted interaction. We separate them into parts which act on the left and right of the particle space tensor product, respectively,

\[
I_1(u, \Sigma) := I_1^{(l)}(u, \Sigma) \otimes \text{Id}_p + \text{Id}_p \otimes I_1^{(r)}(u, \Sigma),
\]

\[
I_1^{(l)}(u, \Sigma) := (-i \partial_u) \tau_\beta(G)(u, \Sigma) + \tau_\beta(\text{ad}_{A_p}^{(l)}(G))(u, \Sigma),
\]

\[
I_1^{(r)}(u, \Sigma) := (-i \partial_u) e^{-\beta u/2} \tau_\beta(\tilde{G}^\perp)(u, \Sigma) - e^{\beta u/2} \tau_\beta(\text{ad}_{A_p}^{(l)}(G^\perp))(u, \Sigma).
\]

We introduce also integrated versions which will be used in the further estimates.

\[
W_\mu := \int I(u, \Sigma)^* I(u, \Sigma) d(u, \Sigma),
\]

\[
W_{1, \mu} := \int I_1(u, \Sigma)^* I_1(u, \Sigma) d(u, \Sigma),
\]

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and the left and right parts, for $\alpha = l, r$,

$$W_p^{(\alpha)} := \int I^{(\alpha)}(u, \Sigma)^* I^{(\alpha)}(u, \Sigma) d(u, \Sigma),$$

$$W_{1,p}^{(\alpha)} := \int I_1^{(\alpha)}(u, \Sigma)^* I_1^{(\alpha)}(u, \Sigma) d(u, \Sigma).$$

By construction $W_1 = \Phi(I_1)$ and we have the decomposition

$$I_1(u, \Sigma) = I_1^{(1)}(u, \Sigma) \otimes \text{Id}_p + \text{Id}_p \otimes I_1^{(r)}(u, \Sigma).$$

(49)

First, we estimate the commuted interaction term $W_1$ appearing in (48).

**Lemma 6.4**

For any $\delta > 0$ we have

$$\pm \lambda W_1 \leq \delta \tilde{N}_f + \frac{1}{\delta} \lambda^2 W_{1,p} \otimes \text{Id}_f$$

(50)

in the sense of forms.

**Proof.** By [11] and [12] we know that $I_1 \in L^2(\mathbb{R} \times S^2, \mathcal{L}(H_p))$. The standard estimates of the creation and annihilation operators lead to the inequality above. \hfill \Box

Next, we prove a bound for $W_p$ and $W_{1,p}$.

**Lemma 6.5**

There exists constants $C$ independent of $\beta$ such that, for $\alpha = l, r$,

(a) $W_p^{(\alpha)} \leq C(1 + \beta^{-1})$,

(b) $W_{1,p}^{(\alpha)} \leq C(1 + \beta^{-1})$,

(c) $\text{P} \text{ess } W_p^{(\alpha)} \leq C(1 + \beta^{-1})V_c^* \langle \hat{x} \rangle^{-2} V_c,$

(d) $\text{P} \text{ess } W_{1,p}^{(\alpha)} \leq C(1 + \beta^{-1})V_c^* \langle \hat{x} \rangle^{-2} V_c.$

**Proof.** By Proposition 4.2 we have that, for all $j \in \{1, 2, 3\}$, $\alpha = l, r$,

$$I^{(\alpha)}, I_1^{(\alpha)}, I^{(\alpha)} V_c^* \hat{x}_j V_c, I_1^{(\alpha)} V_c^* \hat{x}_j V_c \in L^2(\mathbb{R} \times S^2, \mathcal{H}_p),$$

and there is a constant $C$ independent of $\beta$ such that we can estimate the norm of these expressions by

$$C(1 + \beta^{-1}).$$
Thus, the same applies for $I^{(\alpha)} V_\epsilon^* \langle \hat{x} \rangle V_\epsilon, I^{(\alpha)} V_\epsilon^* \langle \hat{x} \rangle V_\epsilon \in L^2(\mathbb{R} \times \mathbb{S}^2, \mathcal{H}_p)$. Consequently, we obtain, for $\alpha = l, r$, and some constant $C > 0$ not depending on $\beta$,

$$P_{\text{ess}} W_{\text{p}}^{(\alpha)} P_{\text{ess}}$$

$$= V_\epsilon^* \langle \hat{x} \rangle V_\epsilon \int (I^{(\alpha)}(u, \Sigma)V_\epsilon^* \langle \hat{x} \rangle V_\epsilon)^* I^{(\alpha)}(u, \Sigma)V_\epsilon^* \langle \hat{x} \rangle V_\epsilon d(u, \Sigma)V_\epsilon^* \langle \hat{x} \rangle V_\epsilon^{-1} V_\epsilon$$

$$\geq C(1 + \beta^{-1}) V_\epsilon^* \langle \hat{x} \rangle V_\epsilon^{-2} V_\epsilon.$$

The proof for $W_{\text{p}}^{(\alpha)}$ is analogous. \qed

Now, we estimate the second term involving $A_0$. We have

$$i[L_\lambda, A_0] = -\lambda[L_\lambda, \Pi W R_\epsilon^2 \Pi^\perp - \Pi^\perp R_\epsilon^2 W \Pi].$$

With respect to the different subspaces we obtain

$$\Pi i[L_\lambda, A_0] \Pi = 2\lambda^2 \Pi W R_\epsilon^2 \Pi^\perp W \Pi, \quad (51)$$

$$\Pi^\perp i[L_\lambda, A_0] \Pi^\perp = -\lambda^2 (\Pi^\perp \Pi W R_\epsilon^2 \Pi^\perp + \Pi^\perp R_\epsilon^2 W \Pi \Pi^\perp), \quad (52)$$

$$\Pi i[L_\lambda, A_0] \Pi^\perp = \lambda \Pi W R_\epsilon^2 \Pi^\perp L_\lambda \Pi^\perp. \quad (53)$$

The Fermi Golden Rule condition implies strict positivity of $\Pi i[L_\lambda, A_0] \Pi$, see Proposition B.2. The other expressions can be potentially negative and are estimated in the following lemma. It contains sharper estimates than [6] which we use later for a Birman-Schwinger argument.

**Lemma 6.6**

*For all $\varepsilon$, and all $\lambda$ the following holds.*

(a) *We have $\Pi^\perp i[L_\lambda, A_0] \Pi^\perp = \mathbf{1}_N \sum_{\hat{N}=1} \Pi^\perp i[L_\lambda, A_0] \Pi^\perp \mathbf{1}_N$. Moreover,*

$$\left\| \Pi^\perp i[L_\lambda, A_0] \Pi^\perp \right\| \leq 2 \frac{\lambda^2}{\varepsilon^2} \left\| W_\text{p} \right\|.$$

(b) *For arbitrary $\delta_1, \delta_2 > 0$,*

$$\Pi^\perp i[L_\lambda, A_0] \Pi + \Pi i[L_\lambda, A_0] \Pi^\perp \leq (|\lambda| \delta_1 + \delta_2 \lambda^2) \Pi W R_\epsilon^2 \Pi^\perp W \Pi + \frac{|\lambda|}{\delta_1} \Pi \overline{P}_\Omega$$

$$+ \frac{\lambda^2}{\delta_2} \left( \Pi^\perp a(I)^\perp R_\epsilon^2 a(I) \Pi^\perp \Pi \overline{P}_\Omega \right.$$

$$+ \Pi^\perp \int \frac{I(u, \Sigma)^* \Pi^\perp I(u, \Sigma)}{u^2 + \varepsilon^2} d(u, \Sigma) \Pi^\perp \Pi \overline{P}_\Omega \right).$$

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Proof. (a) Consider the first term in (52),
\[ \Pi \perp W \Pi \Pi R^2_{\varepsilon} \Pi \perp = a^\ast (I) \Pi a(I) R^2_{\varepsilon} \Pi \perp. \]

Clearly, this operator vanishes everywhere except on \( \text{ran} \tilde{N}_i = 1 \). By standard estimates of creation annihilation operators we obtain
\[ \| a^\ast (I) \Pi a(I) \| \leq \int I^\ast(k) I(k) dk, \]
thus
\[ \| \Pi \perp W \Pi \Pi R^2_{\varepsilon} \Pi \perp \| \leq \frac{\int I^\ast(k) I(k) dk}{\varepsilon^2}, \]
which proves the first inequality, as the second term in (52) is just the adjoint of the first one.

(b) Using the formulas above, and the operator inequality
\[ A^\ast B + B^\ast A \leq A^\ast A + B^\ast B, \quad (54) \]
we get
\[
\Pi \perp \lambda [L_\lambda, A_0] \Pi + \Pi \Pi [L_\lambda, A_0] \Pi \perp \\
= \lambda (\Pi W R^2_{\varepsilon} \Pi \perp L_0 \Pi \perp \text{1}_{\tilde{N}_i = 1} + \text{1}_{\tilde{N}_i = 1} \Pi \perp L_0 R^2_{\varepsilon} \Pi \perp) + \\
\lambda^2 (\Pi W R^2_{\varepsilon} \Pi \perp \text{1}_{\tilde{N}_i = 1} \Pi \perp W \Pi \perp + \Pi \perp W \text{1}_{\tilde{N}_i = 1} \Pi \perp R^2_{\varepsilon} \Pi \perp) \\
\leq |\lambda| (\delta_1 \Pi W R^2_{\varepsilon} \Pi \perp W \Pi \perp + \delta_1^\ast \text{1}_{\tilde{N}_i = 1} \Pi \perp L_0 \Pi \perp R^2_{\varepsilon} L_0 \Pi \perp \text{1}_{\tilde{N}_i = 1}) \\
+ \lambda^2 (\delta_2 \Pi W R^2_{\varepsilon} \Pi \perp W \Pi \perp + \delta_2^\ast \Pi \perp W \text{1}_{\tilde{N}_i = 1} \Pi \perp R^2_{\varepsilon} \text{1}_{\tilde{N}_i = 1} W \Pi \perp). \quad (55) \]

We have
\[ \| \text{1}_{\tilde{N}_i = 1} \Pi \perp L_0 \Pi \perp R^2_{\varepsilon} L_0 \Pi \perp \text{1}_{\tilde{N}_i = 1} \| \leq 1, \]
which yields a bound for the second operator in (55). The second one (56) only operates on the space \( \text{ran}(\tilde{P}_\Omega + \text{1}_{\tilde{N}_i = 2}) \), so we can write
\[
\Pi \perp W \text{1}_{\tilde{N}_i = 1} \Pi \perp R^2_{\varepsilon} \text{1}_{\tilde{N}_i = 1} W \Pi \perp \\
= (\tilde{P}_\Omega + \text{1}_{\tilde{N}_i = 2}) \Pi \perp W \text{1}_{\tilde{N}_i = 1} \Pi \perp R^2_{\varepsilon} \text{1}_{\tilde{N}_i = 1} W \Pi \perp (\tilde{P}_\Omega + \text{1}_{\tilde{N}_i = 2}). \quad (57) \]

Now, we can use the same operator inequality as above to bound (57) by
\[ 2(\Pi \perp a^\ast (I) \Pi \perp R^2_{\varepsilon} a(I) \Pi \perp \text{1}_{\tilde{N}_i = 2} + \Pi \perp \int \frac{I^\ast(u, \Sigma) \Pi \perp I(u, \Sigma)}{u^2 + \varepsilon^2} d(u, \Sigma) \Pi \perp \tilde{P}_\Omega). \]
Having all this we can give the proof for the concrete error estimates we stated.

**Proof of Proposition 6.3.**

(a) First, we use **Lemma 6.4** and the explicit form of $C_1$ to obtain on $D$, for any $\delta_1 > 0$,

\[ C_1 = (V_c^p \hat{k}^2 V_c \otimes \text{Id}_p + \text{Id}_p \otimes V_c^p \hat{k}^2 V_c) \otimes \text{Id}_f + N_f + \lambda W_1 \]

\[ \geq (V_c^p \hat{k}^2 V_c \otimes \text{Id}_p + \text{Id}_p \otimes V_c^p \hat{k}^2 V_c) \otimes \text{Id}_f + (1 - \delta_1) \hat{P}_\Omega - \frac{\lambda^2}{\delta_1} W_{1,p}. \]

Next, note that the operator inequality (54) yields

\[ W_{1,p} \leq 2(W_{1,p}^{(l)} \otimes \text{Id}_p + \text{Id}_p \otimes W_{1,p}^{(r)}). \]

Then, using (54) again, and subsequently **Lemma 6.5**, a decomposition into $\text{ran } P_{\text{ess}}$ and $\text{ran } P_{\text{disc}}$ gives, for $\alpha = l, r$,

\[ W_{1,p}^{(\alpha)} \leq 2(P_{\text{ess}} W_{1,p}^{(\alpha)} P_{\text{ess}} + P_{\text{disc}} W_{1,p}^{(\alpha)} P_{\text{disc}}) \leq C(1 + \beta^{-1})(V_c^\ast \langle \hat{x} \rangle^{-2} V_c + P_{\text{disc}}), \]

where $C > 0$ is a constant not depending on $\beta$. Choosing any $0 < \delta_1 < 1$ yields (48) on $D$. As $D$ is a core for $C_1$, it is also a form core for $q_{C_1}$, so the operator inequality can be extended to the corresponding forms in the form sense on $D(q_{C_1})$.

(b) We have for all $\delta_1, \delta_2 > 0$, by **Lemma 6.6**

\[ i[L, A_0] = \Pi i[L, A_0] \Pi + \Pi^\perp i[L, A_0] \Pi^\perp + \Pi^\perp i[L, A_0] \Pi + \Pi i[L, A_0] \Pi^\perp \]

\[ \geq (1 - (|\lambda| \delta_1 + \delta_2 \lambda^2)) \Pi i[L, A_0] \Pi - 2 \lambda^2 \varepsilon_2 \| W_p \| \Pi \hat{N}_{l=2} \]

\[ - \frac{|\lambda|}{\delta_1} \Pi \hat{N}_{l=1} - 2 \lambda^2 \delta_2 \Pi^\perp a^\ast(I) \Pi^\perp R^2_c a(I) \Pi^\perp \Pi \hat{N}_{l=2} \]

\[ - 2 \lambda^2 \delta_2 \varepsilon_2 \Pi^\perp W_p \Pi^\perp \hat{P}_\Omega. \]

Notice that the operator $\Pi^\perp a^\ast(I) \Pi^\perp R^2_c a(I) \Pi^\perp \Pi \hat{N}_{l=2}$ is in fact bounded. The last term can be decomposed and estimated as above in the proof of (a).

(c) We have

\[ W(L_p^{-1} Q \otimes P_{\Omega}) = a^\ast(I)(L_p^{-1} Q \otimes P_{\Omega}) = a^\ast(I L_p^{-1} Q) \hat{P}_\Omega. \]
Thus, by the standard estimates for creation and annihilation operators, we obtain for all \( \delta > 0 \) on \( D(\hat{N}_f) \),

\[
W(L_p^{-1}Q \otimes P_\Omega) + (W(L_p^{-1}Q \otimes P_\Omega))^* \\
\leq \delta \hat{N}_f + \delta^{-1} \left( \int L_p^{-1}QI(u, \Sigma)^*I(u, \Sigma)L_p^{-1}Qd(u, \Sigma) \right) \otimes P_\Omega.
\]

Hence, we get

\[
C_Q \geq Q \otimes P_\Omega - |\lambda| \delta \hat{N}_f - |\lambda| \|W_p\| \delta^{-1}(L_p^{-1}Q)^2 \otimes P_\Omega
\geq (1 - 2\lambda(1 + \beta^{-1})\delta^{-1})Q \otimes P_\Omega - |\lambda| \delta \hat{N}_f,
\]

where we used that the concrete choice of \( q \) as in (17) yields

\[
(L_p^{-1}Q)^2 = (L_p^2I_{[-1,1]}(L_p) + L_p^{-2}I_{(-\infty,-1)\cup(1,\infty)}(L_p)) \leq 2 \text{Id}_p \otimes \text{Id}_p. \quad \square
\]

After the preparations we are now able to put all the estimates of this chapter together in order to prove positivity of \( q_{\text{tot}} \).

**Proof of Proposition 6.2.** First choose \( \varepsilon > 0 \) such that Proposition B.2 holds. Using Proposition 6.3, we obtain in the sense of forms on \( D(q_1) \),

\[
q_{\text{tot}} \geq \dot{V}_\varepsilon^* \left( \hat{k}^2 - c_1 \lambda^2(1 + \theta)(1 + \beta^{-1})\langle \hat{x} \rangle^{-2} \right) \dot{V}_\varepsilon \otimes \text{Id}_p \otimes \text{Id}_f \\
+ \text{Id}_p \otimes \dot{V}_\varepsilon^* \left( \hat{k}^2 - c_2 \lambda^2(1 + \theta)(1 + \beta^{-1})\langle \hat{x} \rangle^{-2} \right) \dot{V}_\varepsilon \otimes \text{Id}_f \\
+ \left( c_3 - c_4(1 + \theta)|\lambda|(1 + \beta^{-1}) \right) \hat{P}_\Omega^{-1} \\
+ 2\theta \lambda^2(1 - c_5|\lambda|)\gamma_\beta \Pi - c_6 \lambda^2(1 + \beta^{-1})\Pi \\
+ \left[ \left( 1 - c_7|\lambda|(1 + \beta^{-1}) \right) Q \\
- c_8 \lambda^2(1 + \theta)(1 + \beta^{-1})(P_{\text{ess}} \otimes P_{\text{ess}})^{\perp}1_{E \setminus \{0\}}(L_p) \right] \otimes P_\Omega,
\]

for constants \( c_i > 0, i \in \{1, \ldots, 8\} \), independent of \( \lambda \) and \( \beta \). Now we set \( \theta = |\lambda|^{-1/2} \) in order to have a positive term in (60) of higher order. Next, we make \( |\lambda| > 0 \) sufficiently small in the following sense: First we make it so small such that, by the uncertainty principle lemma (cf. [18, X.2]),

\[
\hat{k}^2 - \max\{c_1, c_2\}\lambda^2(1 + |\lambda|^{-2})(1 + \beta^{-1})\langle \hat{x} \rangle^{-2} > 0.
\]

Furthermore, we can make it small enough such that we get strictly positive operators in (59) and (60) on ran \( P_\Omega^{\perp} \) and ran \( \Pi \), respectively. Note that we have to choose \( |\lambda| \) small enough proportional to \( \gamma_\beta^2 \) due to (60). For the last term, we have

\[
(P_{\text{ess}} \otimes P_{\text{ess}})^{\perp}1_{E \setminus \{0\}}(L_p) \leq 1_{E \setminus \{0\}}(L_p^2),
\]

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where
\[ \Xi := \inf_{\lambda \in \sigma_{\text{disc}}(H_p), \mu \in \sigma(H_p)} (\lambda - \mu)^2 > 0. \]

Now we can plug in \( Q = L_p^2 \mathbb{1}_{[0,1]}(L_p^2) + \mathbb{1}_{(1,\infty)}(L_p^2) \), and use that
\[ Q - \delta \mathbb{1}_{[\Xi,\infty)}(L_p^2) = L_p^2 \mathbb{1}_{[0,\Xi]}(L_p^2) + (L_p^2 - \delta) \mathbb{1}_{[\Xi,1]}(L_p^2) + (1 - \delta) \mathbb{1}_{(1,\infty)}(L_p^2) > 0 \]
on \( \text{ran} \mathbb{1}_{\mathbb{R}\setminus\{0\}}(L_p) \), for \( \delta < \min\{\Xi,1\} \). Thus we can achieve that (61) is positive on \( \text{ran}(\mathbb{1}_{\mathbb{R}\setminus\{0\}}(L_p) \otimes P_0) \) for \( |\lambda| \) small enough. The claim now follows in view of the decomposition (7). \( \Box \)

A. Decay in the Glued Positive Temperature Space

In Lemma A.2 the decay behavior of functions and their derivatives in the positive temperature gluing representation is discussed. That is, sufficient decay conditions of a function are stated such that the derivatives of the transformation by \( \tau_\beta \), defined as in Subsection 2.3, are integrable. This is necessary for the proof that the commuted interaction is sufficiently bounded.

Before that we state some properties of the decay behavior of the Planck distribution and its first derivative.

Lemma A.1 (Properties of \( \rho_\beta \))
There exists constants \( C \) such that for all \( \omega \geq 0 \),
\[ \begin{align*}
(a) & \quad \sqrt{\rho_\beta(\omega)} \leq \frac{1}{\sqrt{\beta \omega}}, \\
(b) & \quad \sqrt{1 + \rho_\beta(\omega)} \leq 1 + \frac{1}{\sqrt{\beta \omega}}, \\
(c) & \quad \partial_\omega \sqrt{\rho_\beta(\omega)} \leq C(\omega^{-1} + \beta^{-\frac{1}{2}} \omega^{-\frac{3}{2}}), \\
(d) & \quad \partial_\omega \sqrt{1 + \rho_\beta(\omega)} \leq C(\omega^{-1} + \beta^{-\frac{1}{2}} \omega^{-\frac{3}{2}}).
\end{align*} \]
Proof. Elementary calculation. \( \Box \)

Lemma A.2
Assume that \( f \in L^2(\mathbb{R}^3) \), \( m \in \mathbb{N} \), and for all \( \Sigma \in \mathcal{S} \), we have that \( \omega \mapsto f(\omega \Sigma) \) is in \( C^m(\mathbb{R}_+) \). Furthermore assume that there exist constants \( \varepsilon > 0 \) and \( 0 < k_1, k_2, K_1, K_2 < \infty \) such that for all \( \Sigma \in \mathcal{S}^2 \),
\[ (1) \ |\partial_\omega^j f(\omega \Sigma)| \leq k_2 \omega^{m-1+\varepsilon-j}, \text{ for } \omega < k_1, \]
\[ |\partial_u^j f(\omega, \Sigma)| \leq K_2 \omega^{-3/2-\varepsilon}, \text{ for } \omega > K_1, \]

Then \( \partial_u^j \tau_\beta(f) \in L^2(\mathbb{R} \times \mathbb{S}^2) \) for \( j = 0, \ldots, m \). Moreover, for \( j = 0, 1 \), there exists a constant \( k_3 \) independent of \( \beta \) such that

\[ \left\| \partial_u^j \tau_\beta(f) \right\|_2 \leq k_3 (1 + \beta^{-1}). \quad (62) \]

Instead of condition (1) we can also require

\[ (1') \quad f(\omega, \Sigma) = \omega^{\pm \frac{1}{2}} \tilde{f}(\omega, \Sigma), \text{ for } \omega < k_1, \]

for a function \( \tilde{f} \) such that, for all \( \Sigma \in \mathbb{S}^2 \),

\[ [0, \infty) \rightarrow \mathbb{C}, \; \omega \mapsto \tilde{f}(\omega, \Sigma), \]

is in \( C^m(\mathbb{R}_+) \) and all derivatives are bounded uniformly in \( \omega \) and \( \Sigma \).

**Proof.** For \( u > 0 \), we have

\[ \partial_u^j (\tau_\beta f)(u, \Sigma) = \sum_{l=0}^j \binom{j}{l} \partial_u^l \left( u \sqrt{1 + \rho_\beta(u)} \right) \partial_u^{j-l} f(u \Sigma). \quad (63) \]

We first check the sufficient decay behavior near zero. The first factor in the sum \( (63) \), \( \partial_u^l \left( u \sqrt{1 + \rho_\beta(u)} \right) \), is in \( O\left( u^{\frac{j}{2}-l} \right) \) for \( u \rightarrow 0 \). The same can be shown for \( u < 0 \) as \( \sqrt{\rho_\beta} \) has the same decay behavior. The second factor is in \( O\left( u^{m-1-(j-l)+\varepsilon} \right) \), hence there exist constants \( k'_1, k'_2 \), such that, for \( 0 < |u| < k'_1 \),

\[ \left| \partial_u^l (\tau_\beta f)(u, \Sigma) \right| \leq k'_2 |u|^{-\frac{j}{2}+m-j+\varepsilon}. \]

This yields that the function \( \partial_u^j (\tau_\beta f)(\cdot, \Sigma) \) has a weak derivative for all \( \Sigma \in \mathbb{S}^2 \) and \( j = 0, \ldots, m-1 \), and

\[ u \mapsto \mathbb{1}_{[-1, 1]}(u) \partial_u^j (\tau_\beta f)(u, \Sigma) \]

is in \( L^2(\mathbb{R}) \) for all \( \Sigma \in \mathbb{S}^2 \) and \( j = 0, \ldots, m \).

In the case of the alternative condition \( (1') \) we get for \( f(\omega, \Sigma) := \omega^{\pm \frac{1}{2}} \tilde{f}(\omega, \Sigma) \),

\[ (\tau_\beta f)(u, \Sigma) = \begin{cases} \sqrt{u + u \rho_\beta(u)} \tilde{f}(u \Sigma), & u > 0, \\ \sqrt{-u \rho_\beta(-u)} \tilde{f}(-u \Sigma), & u < 0. \end{cases} \]

It is easy to check that \( u \mapsto (\tau_\beta f)(u, \Sigma) \) is in \( C^m(\mathbb{R}) \) for all \( \Sigma \) and all derivatives are bounded uniformly in \( u \) and \( \Sigma \). This also yields the desired infrared regularity. The same is true if we choose \( f(\omega, \Sigma) := \omega^{\pm \frac{1}{2}} \tilde{f}(\omega, \Sigma) \).
Second, we discuss the decay if \( u \) tends to infinity. The first factor in (63) is in \( O(u) \) for \( u \to \infty \). This means there exist constants \( K_1', K_2' \), such that, for all \( j = 0, \ldots, m \) and all \( u \) with \( |u| > K_1' \),

\[
\left| \partial_u^j (\tau f)(u, \Sigma) \right| \leq K_2' |u|^{-\frac{1}{2} - p - \varepsilon}.
\]

This yields that \( u \mapsto 1_{\mathbb{R}\setminus[-1,1]}(u) \partial_u^j (\tau f)(u, \Sigma) \) is in \( L^2(\mathbb{R}) \) for all \( \Sigma \in S^2 \) and \( j = 0, \ldots, m \).

Finally, note that (62) follows immediately from Lemma A.1.

\[ \Box \]

**B. Fermi Golden Rule**

In this part we review the result [6, Proposition 3.2] – how the Fermi Golden Rule condition \([F]\) implies the positivity of the commutator with \( A_0 \) – and generalize it with the obvious modifications to the coupling considered in this paper.

First we state some elementary properties of the conjugation operator \( A_0 \).

**Lemma B.1**

The operator

\[ A_0 = i\lambda(\Pi \mathcal{W} R^2 \Pi \perp - \Pi \perp R^2 \mathcal{W} \Pi), \]

is bounded, self-adjoint and \( \text{ran} \ A_0 \subseteq \mathcal{D}(L_\lambda) \) for any \( \lambda \) and \( \varepsilon > 0 \).

**Proof.** Note that \( \Pi \) contains the projection to the vacuum subspace, so the creation operators are bounded and the annihilation operators vanish. Thus, the operator is indeed bounded and self-adjoint by construction.

Furthermore, the range of the first summand of \( A_0 \) equals \( \text{ran} \Pi \) and of the second one \( \mathcal{D}(L_0^2) \cap \mathfrak{F} \), which are clearly subsets of \( \mathcal{D}(L_\lambda) \).

\[ \Box \]

**Proposition B.2**

Assume that \([F]\) is satisfied. Then

\[ \varepsilon \Pi \mathcal{W} \Pi \perp R^2 \Pi \perp \Pi \mathcal{W} \Pi \geq \gamma_\beta \Pi \]

on \( \text{ran} \ Pi \).

**Proof.** We have \( \Pi \mathcal{W} \Pi = 0 \) and thus \( \Pi \mathcal{W} \Pi \perp R^2 \Pi \perp \Pi \mathcal{W} \Pi = \Pi \mathcal{W} R^2 \Pi \mathcal{W} \). Notice that

\[ \Pi = P_0 \otimes P_\Omega = \sum_{E \in \sigma(H_\nu)} p_E \otimes p_E \otimes P_\Omega, \]

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where \( P_0 := 1_{[0]}(L_p) \). Then we compute

\[
\Pi W R_2^2 \Pi \geq \Pi W R_2^2 (P_{\text{ess}} \otimes P_{\text{disc}} \otimes \text{Id}_f) W \Pi \\
= \Pi (a(\tau_\beta(G \otimes \text{Id}_p)) - a(e^{-\beta/2} \tau_\beta(\text{Id}_p \otimes \overline{G^\ast})) \frac{P_{\text{ess}} \otimes P_{\text{disc}} \otimes \text{Id}_f}{L_0^2 + \varepsilon^2} \\
(a^*(\tau_\beta(G \otimes \text{Id}_p)) - a^*(e^{-\beta/2} \tau_\beta(\text{Id}_p \otimes \overline{G^\ast}))) \Pi \\
= \Pi a(\tau_\beta(G \otimes \text{Id}_p)) \frac{P_{\text{ess}} \otimes P_{\text{disc}} \otimes \text{Id}_f}{L_0^2 + \varepsilon^2} a^*(\tau_\beta(G \otimes \text{Id}_p)) \Pi \\
= \sum_{E \in \sigma_d(H_p)} \Pi a(\tau_\beta(G \otimes \text{Id}_p)) \frac{P_{\text{ess}} \otimes P_{\text{E}} \otimes \text{Id}_f}{(H_p \otimes \text{Id}_p \otimes \text{Id}_f - E + d\Gamma(u))^2 + \varepsilon^2} a^*(\tau_\beta(G \otimes \text{Id}_p)) \Pi,
\]

with \( d\Gamma(u) := \text{Id}_p \otimes \text{Id}_p \otimes d\Gamma(u) \), and where we used the pull through formula in the last step. Evaluating \( \Pi \) and using the definition of \( \tau_\beta \), we arrive at

\[
\Pi W R_2^2 \Pi \geq (P_0 \sum_{E \in \sigma_d(H_p)} \int_\mathbb{R} \int_{\mathbb{S}^2} \tau_\beta(G^\ast \otimes \text{Id}_p)(u, \Sigma) \frac{P_{\text{ess}}}{(H_p - E + u)^2 + \varepsilon^2} \otimes P_{\text{E}} + \int_\mathbb{R} \int_{\mathbb{S}^2} \tau_\beta(G \otimes \text{Id}_p)(u, \Sigma) d\Sigma du P_0) \otimes P_{\Omega} \\
= p_{\text{E}}(F_\varepsilon^{(1)} + F_\varepsilon^{(2)}) p_{\text{E}} \otimes p_{\text{E}} \otimes P_{\Omega},
\]

with \( F_\varepsilon^{(1)}, F_\varepsilon^{(2)} \) defined as in [F].

\[\Box\]

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