Evaluation of Effective Elastic Moduli Using Micromechanics

Mrs. Misba Mehdi(1), Atul Ramesh Bhagat(2), G.R Selokar(3).

Mrs. Misba Mehdi, Sri Satya Sai University of Technology and Medical Sciences, Sehore, Bhopal, M.P.

Dr. Atul Ramesh Bhagat, ASL, Kanchanbagh, Hyderabad, Telangana.

Dr. G.R. Selokar, Mechanical Engineering Head of the Department, Sri Satya Sai University of Technology and Medical Sciences, Sehore, Bhopal, M.P.

misbamehdi@gmail.com

Abstract. In this work, a direct approach is done to predict the accurate elastic properties of particulate composites containing and fibrous composites. To predict the elastic modulus, poisson’s ratio and shear modulus, the empirical formulas like rule of mixtures, Halpin T-sai equation were taken into consideration. Properties were also determined using analytical methods like Eshelby scheme, Mori-Tanaka scheme, Self-Consistent scheme and combined approach of Mori-Tanaka and Self-Consistent schemes. The predicted properties were compared with the experimental data available from literature. Simulation was carried out for aluminium alumina particulate composites and also for boron epoxy unidirectional composites. Combined approach given by Peng et al [1]. has been used and its effectiveness has been shown.

1. Introduction:

A material is said to be composite when the combination of two or more materials are mixed to obtain a third material with higher effective properties. Properties of the composite differ significantly from the constituents from which they are made. Detailed data of properties are required for carrying out the analysis of the structures made of these composites. It was evitable that during twentieth century, usage of fibres reinforcement was tremendously more because fibres have superior mechanical property. The aerospace industry started using reinforced fibrous laminated plastics instead of metallic material for making the aeroplane parts. E.g. glass, carbon, boron, and Kevlar, plastics etc epoxies and polyesters etc was tremendously used by composite designers. In order to maintain the efficient weight with higher modulus and high strength properties while designing the metal was replaced by fibre reinforce plastic. Because of the heterogeneous composition of the composites, one could derive any characteristic material behaviour. Because of such a unique quality of flexibility many different streams were attracted as it is easy to manufacture, moulding to any desire shape, repair ability, corrosion resistance, durability, adaptability, cost effectiveness, etc.. As we are all aware that, the composite materials bear high strength physical properties and stiffness to density ratios the aviation and aerospace industry was very keen on the usage of the composite materials. The composite materials mostly used are carbon and glass fiber reinforce plastic as it holds the characteristic of high strength and stiffness for their density. In polymer matrix it is brittle which is not stiff or strong either.

In civil engineering, e.g. the GFRP dome structure in Benghajj was constructed in 1968. And several GFRP shells were constructed in seventies, the use of glass fibre reinforced in buildings was
initiated in early sixties. In civil engineering, composite bagged its place because of the material cost and in automotive engineering it was in early seventies. All the automotive components like exterior body panels, frameworks/chassis, bumpers, drive shafts, suspension systems, wheels, steering wheel columns and instrument panels of automotive vehicles are being manufacture in composites. Composites materials, was an attraction in marine industry because of its strong, stiff and light characteristics. The composites are highly used in transportation systems also to build bogeys and compartments. Even in fabrication of energy the composites are used to build wind-mill rotor blades and flywheels. In electrical and electronics also, composites were used in building composite antennas instead of metallic antennas. Even in sports industry the badminton rackets, fishing rods, vaulting poles, hockey sticks, surf boards, etc are all made by composite material [2].

The particulate composites and conventional metallic materials are perfect examples for isotropic material as their physical properties are same in all directions and doesn’t depend on its orientation. Whereas, the fibrous composites are considered as anisotropic material as their physical properties depends on different directions [3]. As properties can be determined experimentally with certain limitations, micromechanics is seen as an effective tool to determine these properties. To evaluate or to know the exact or accurate effective property of the composites certain approaches available are homogenization approach using FEA techniques and micromechanics approach with analytical formulae [3]. Elastic constants are used in engineering analysis to obtain stress and strain relationships in the structure during service. To determine the precise value of elastic moduli certain approaches were considered, which are Eshelby equivalent inclusion theory, in which Mori-Tanaka [4],[5],[3], and Self consistent [6],[7] and combined method of Mori-Tanaka and Self-Consistent were considered. In all these above-mentioned methods Eshelby Tensor along with the geometry of inclusions are considered [8]. The term Eshelby’s inclusion in continuum mechanics is defined as a problem in which an ellipsoidal elastic inclusion is involve in an elastic body and analytical solutions to this type of problems was first given by John. D. Eshelby in 1957[8]. Mori-Tanaka, in 1973 proposed an approach in relation to both the stress and strain of the fiber and matrix composite material. Again Benveniste [9] in 1987, reformulated the Mori-Tanaka equation by introducing a tensor parameter as an equivalent inclusion which is called as Mori-Tanaka Tensor. This tensor depends only on Eshelby Tensor [8], which determines the accurate elastic property of the unidirectional composite. With the Eshelby-Mori-Tanaka’s method, we could study the mechanical behaviour of the composites containing various kind of inclusion shapes, including ellipsoidal, penny-shape disc, spherical inclusion and non-circular cylinder reinforcement[10], [11],[12],[13],[14],[15], and [16], and non-ellipsoidal fillers [17], [18] [19],[20].

2. **EMPIRICAL TECHNIQUES USED IN THIS PAPER ARE GIVEN BELOW:**

2.1 **RULE OF MIXTURE:**

With the help of micro-mechanics approach rule of mixture, the specific stiffness of the composite can be determined. It is assumed that the bonding between the fiber and matrix is perfect, where fiber is distributed uniformly and matrix are free from voids. Before applying the loads on the lamina, it is assumed that lamina is stress free in initial stage. The loads are always applied parallel or normal to the fiber direction and finally the fiber and matrix are linearly elastic materials.

\[
E_i = E_f V_f + E_m V_m \quad (1)
\]

Where,

- \(E_i\) = Elastic moduli of composite,
- \(E_f\) = Elastic moduli of fibre,
- \(V_f\) = Volume fraction of fibre,
- \(E_m\) = Elastic moduli of matrix,
- \(V_m\) = Volume fraction of matrix

Above formula applied to both particulate and unidirectional composites.

2.2 **HALPHIN-TSAI:**
The Halpin-Tsai model is a mathematical model, which predicts the elastic property of composite material by taking into account the geometry and orientation of fibre and matrix of the filler. The specimen is based on the self-consistent method which is considered as empirical [4]. Halpin and Tsai expressed the composite property $P_c$ in terms of $P_m$ for matrix property and $P_f$ the reinforcing phase or fibre by using the following relationships as shown below:

$$P_c = P_m \left(1 + \frac{\xi \eta V_f}{1 - \eta V_f}\right)$$

Where, $\xi$ is the curve fitting geometry parameter, it is different for different properties of same composite material. For e.g. If matrix were isotropic and incompressible $\xi_{max} = \frac{1.5}{1.5} = 1$; $\xi \leq 1.5$

$\eta$ is the Length correction factor for the rule of mixture. The values of $\eta$ are limited to:

a) Rigid inclusions (i.e., $M_r = \infty, \eta = 1$;  

b) Homogeneous material $M_r = 1, \eta = 0$; and 

c) Voids $M_r = 0, \eta = \frac{-1}{\xi}$.

Therefore, the limiting values are $\xi = 0 ; \eta = 1 ; \xi = \infty ; \eta=0$ [21]

In Halpin-Tsai, the elastic modulus of shear modulus is given by;

$$\frac{1}{G_{12}} = V_f / G_f + V_m / G_m, \quad \eta$$

Shear Moduli of Fibre $G_f = E_f / (2(1+\nu_f))$

Shear Moduli of Matrix $G_m = E_m / (2(1+\nu_m))$

$\nu_f = $ Poison’s ration of fibre

$\nu_m = $ Poison’s ratio of matrix.

$G_{12} / G_m = (1 + \xi \eta V_f) / (1 - \eta V_f)$

Length correction factor for the rule of mixture ($\eta$) 

$\eta = ((G_f / G_m) - 1) / ((G_f / G_m) + \xi)$

Curve fitting parameter $\xi = 1 + 40V_f$ \(10^9\)

$\xi = 2L/d$

$L$ is length of fibres.

$d$ is diameter of the fibres.

3. ANALYTICAL METHODS FOR PROPERTIES PREDICTION:

The Elastic constants (as stiffness and strength) of composite material are predicted with the help of various mathematical expressions in analytical method. And to evaluate the elastic properties of the composite material different methodologies are considered, as mentioned below

(1) Eshelbeey Method;  
(2) Mori-Tanaka Method;  
(3) Self-Consistent Method;  
(4) Combined Mori-Tanaka and Self-Consistent.

3.1 Eshelbeey Method:

In a composite material, if the inhomogeneities are spread at far apart, then interactions are neglected. The interaction is considered when inhomogeneity is present in a homogeneous matrix without mixing with other inhomogeneities. The below figure explains the original homogeneous matrix with $r^0$-inhomogeneity, which has stiffness $L_0$, and uniform strain $\varepsilon^0$. It is assumed in $r^0$-inhomogeneity, an ellipsoidal inhomogeneity was imbedded in uniform strain matrix.
(a) Composite with \( r \)th inhomogeneity.
(b) The original matrix with stiffness \( L_0 \) is put through the uniform strain \( \epsilon^0 \) in \( r \)th inhomogeneity. The composite accurate stiffness tensor is given as,

\[
\tilde{L} = [L_0 + \sum_{r=1}^{N} C_r (L_r - L_0) T_r] 
\]

- (4)

\[
\tilde{T}_r = [I + S_r L_0^{-1} (L_r - L_0)]^{-1} \]

- (5)

Where, \( \tilde{L} \) is the effective stiffness tensor,
\( L_0 \) is Stiffness tensor of the original matrix
\( L_r \) is the stiffness tensor in \( r \)th inhomogeneity,
\( T_r \) is the global strain concentration tensor,
\( C_r \) is the volume fraction of the \( r \)th inhomogeneity,
\( I \) is the Identity matrix,
\( S_r \) is the Eshelby Tensor.

When the inclusions are imbedded in the matrix there was stress strain and displacement in a linear elastic body, due to this change Eshelbey initiated his work, and concluded that the stress and strain inside the inclusion is uniform throughout the distribution and it has a closed form regardless of material property. The initial transformation of strain is called Eigen Strain in Mechanics of Composites [8].

3.2 MORI-TANAKA METHOD:

The Eshelbey and Mori-Tanaka scheme differ from each other by using the strain in the matrix \( \tilde{\epsilon}^0 \), which is taking care of other inhomogeneities effect. In Eshelbey method, we assume that when no other homogeneities are present the strain \( \tilde{\epsilon}^0 = \epsilon^0 \) in the matrix is considered. The Mori-Tanaka method is assuming the absence of all inhomogeneities by taking \( \tilde{\epsilon}^0 = \overline{\epsilon}^0 \) into consideration when inhomogeneities are present, where \( \overline{\epsilon}^0 \) is the average strain in the matrix. As shown in the below figures, without the inhomogeneity added to the composite with the \( r \)th inhomogeneity and the original matrix with stiffness \( L_0 \) is put through the uniform strain \( \tilde{\epsilon}^0 \). Hence, because of the above reason Mori-Tanaka method result is better for accurate elastic moduli when compare to Eshelbey Method.
(a) Composite with r\textsuperscript{th} inhomogeneity.

(b) The actual matrix with stiffness $L_0$ is put through the uniform strain $\bar{\varepsilon}$ without the inhomogeneity added.

$$
\bar{L} = [L_0 + \sum_{r=1}^{N} C_r (L_r - L_0) A_r] \quad \text{(6)}
$$

$$
A_r = T_r [C_0 I + \sum_{r=1}^{N} C_r T_r]^{-1} \quad \text{(7)}
$$

Where,

- $\bar{L}$ is the effective stiffness tensor,
- $L_0$ is Stiffness of the original matrix,
- $C_r$ is the volume fraction of the r\textsuperscript{th} inhomogeneity,
- $L_r$ is the stiffness tensor in r\textsuperscript{th} inhomogeneity,
- $T_r$ and $A_r$ is the global strain concentration tensor,
- $I$ is the Identity matrix,
- $C_0$ is the volume fraction when $r = 0$.

3.3 SELF–CONSISTENT METHOD:

In Self–Consistent method, the effect of one inhomogeneities on other inhomogeneity is taken into account. This method is based on the auxiliary inclusion, in which when a single ellipsoidal inclusion is imbedded in a composite, the bonding between the composite and the inclusion is considered to be perfect. The elastic property can be determined by considering the relation between the stress and strain in the homogeneous material and stress strain in the inclusion or the stress strain concentration factors. To determine the effective elastic property of this type of system Hill [22] and Budiansky [23] did a depth study. In actual when compare with Eshelbey and Mori–Tanaka method, the Self–Consistent method gives us more accurate or exact value of effective stiffness tensor. As shown in the below figures the composite with r\textsuperscript{th} inhomogeneity is put through the traction boundary conditions or displacement boundary conditions. It is clearly understood, in spite of adding the loads to the composite the influence on other inhomogeneity is within homogeneous matrix of stiffness $\bar{L}$ which is put through to $\bar{\varepsilon}$ even before inhomogeneity is added.

(a) Original composite with r\textsuperscript{th} inhomogeneity.  
(b) Homogeneous matrix with effective stiffness $\bar{L}$ is put through the uniform strain $\bar{\varepsilon}$ in the r\textsuperscript{th} inhomogeneity even before the inhomogeneity added.

$$
\bar{L} = L_0 + \sum_{r=1}^{N} C_r (L_r - L_0) \bar{T}_r \quad \text{(8)}
$$

$$
\bar{T}_r = \left[I + S_r L_r^{-1}(L_r - \bar{L}) \right]^{-1} \quad \text{(9)}
$$

Where, the Eshelbey tensor with overbar indicates that the element should be assess based on the accurate elastic constants of the composite.
3.4 COMBINED MORI-TANAKA AND SELFCONSISTENT METHOD:

One can achieve the accurate or exact elastic property of composites through combined method of Mori-Tanaka and Self-Consistent proposed approach. It is considered, a representative volume element (RVE) has fibre reinforcement of a composite, matrix and volume fractions c as inclusions, the dot inclusions could be sphere, ellipsoidal, circular cylinder etc of same size distributed randomly in the composite. In the proposed approach the dot inclusions are divided into two groups, where an arbitrary parameter \( \gamma \) is consider where \( (\gamma \geq 1) \). In group 1, let reinforcement or fibre of a composite has mechanical property of the matrix \( L_m \) and the dot inclusion mechanical property be \( L_c \) with total volume fraction \( c \), then dot inclusion total volume fraction be \( \frac{1}{\gamma} \) over the fibre or reinforcement, and in the group 2 the dot inclusion with total volume fraction \( \frac{c}{\gamma} \) over the fibre or reinforcement.

In the group 1 the dot inclusions are already added to the actual matrix after which the material becomes fictitious matrix \( L_m \) and the accurate elastic property of composite is determined with Self-Consistent method. The accurate elastic property of fictitious matrix \( L_{\text{m-f}} \) of the group II composite is determined by Mori-Tanaka method. Consider the volume of fibre is \( V_f \), the volume of the actual matrix is \( V_m = (1-c)V_f \) and the volume of the dot inclusions in group 1 is \( (1-\gamma^{-1})V_f \), the volume of the fictitious matrix is \( V_{m-f} = V_m + (1-\gamma^{-1})V_f = \frac{1-\gamma^{-1}}{\gamma}V_f \) and the identical volume fraction of the dot inclusions in the fictitious matrix is \( c = (1-\gamma^{-1})V_f / V_m = \frac{\gamma-1}{\gamma-c} \).

In both the cases the limit of an arbitrary parameter \( \gamma \) is \( \gamma = 1 \) to \( \gamma \to \infty \). As the value of \( \gamma \) is increased the value of effective elastic property of composite is evaluated and which varies between both the methods (Mori-Tanaka and Self-Consistent). These results are then compared with the
experimental results of some type of composite. The arbitrary parameter $\gamma$ which is introduced could be varied accordingly to achieve accurate effective elastic property of the composites experimentally.

![Diagram of separating dots inclusion into two groups](image1)

**Fig 3.4.1:** Separating dots inclusion into two group

- **RVE with dot**
- **Group 1**
- **Group 2 (m dots)**

![Diagram of fictitious matrix composed of actual matrix and the dots in Group 1](image2)

**Fig 3.4.2:** Fictitious matrix composed of actual matrix and the dots in Group 1

- **Actual Matrix**
- **Dots in Group 1**
- **Fictitious Matrix**

![Diagram of fictitious matrix composed of fictitious matrix and the dots in Group 2](image3)

**Fig 3.4.3:** Composite material composed of fictitious matrix and the dots in Group 2.

**Fig 3.5:** Flowchart: Group 1 and Group 2.

4. **EXAMPLES AND VERIFICATION:**
Validation of code is verified from evaluation of mechanical properties of particulate composites with a Self-Consistent method and Mori-Tanaka method. In this work, the accurate elastic property of particulate composites evaluated with the proposed approach, by assuming the inclusions as isotropic spherical particle, circular cylinder and elliptical cylinder particle. The inclusions are analysed by
varying the effective Young’s modulus of Al/Al₂O₃ composite against the volume fraction of Al₂O₃ particle. The Young’s modulus Eₐ=69 GPa and the poison’s ratio vₐ = 0.345 the matrix and the Young’s modulus Eₐ= 400 GPa and the poison’s ratio vₐ = 0.25 the Al₂O₃ particle inclusion respectively [16].

The above graph shows the variation of the effective young’s modulus and volume fraction of Al₂O₃ particles inclusion with the proposed approach and the curves are marked and compared with the experimental results also [16]. When the volume fraction increases the poisson’s ratio decreases till it reaches its minimum value. Hence, the nature of the graph is non-linear.

Simulation is also carried out for unidirectional boron epoxy composite. The values which are considered for calculation of the elastic properties of matrix and fibre material are taken from literature and are presented in below Table 1 [24]

**Table 1. Material Properties**

| Elastic properties | Matrix material (Epoxy) | Fiber material (Boron) |
|--------------------|------------------------|-----------------------|
| Young’s modulus (E) in GPa | 4.14 | 414 |
| Poisson’s ratio (v) | 0.35 | 0.2 |

Fig 4.1 Variations of E (Al/Al₂O₃) Vs Vf (Al₂O₃)

Fig 4.2 Variations v Vs Vf (Al₂O₃)

Fig 4.3: Transverse modulus Vs volume fraction for Boron/Epoxy

Fig 4.4: Zoomed in view of Transverse modulus Vs volume fraction for Boron/ Epoxy.
The above graph which is plotted shows the prediction for unidirectional composite boron epoxy. Figure 4.3 and 4.4 show the transverse modulus of boron epoxy using all methods. The prediction is compared with experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970. It is found that Mori-Tanaka method predicts the transverse modulus more accurately than other methods. Figure 4.5 shows variation of poisson’s ratio with volume fraction, where the Poisson’s ratio decreases with increase in volume fraction.

5. CONCLUSION:
In this work, effective properties of particulate and unidirectional composites have been predicted. Particulate composite with aluminium alumina constituents and unidirectional composites with boron fibre and epoxy matrix are considered. Properties are predicted with rule of mixtures, Hills method and analytical methods, Eshelby, Mori-Tanaka, Self-consistent and Combined method of Mori-Tanaka and self-consistent are used [25]. Eshelby gives better prediction only at lower volume fraction but fails at higher volume fraction. Self-consistent approach gives the better prediction compared to Mori-Tanaka and Eshelby. Combined method gives better prediction for particulate composite and Mori Tanaka gives better estimation for transverse modulus in unidirectional composite. Properties predicted by above methods have been compared with the experimental data from literature. Predicted values of elastic moduli found to agree well with the experimental data from literature.

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7. ANNEXURE:

(1) Sphere \((a_1=a_2=a_3)\)

\[
\begin{align*}
I_1 &= I_2 = I_3 = 4\pi / 3 \\
I_{11} &= I_{22} = I_{33} = I_{12} = I_{23} = I_{31} = 4\pi / 5a^2 \\
s_{1111} &= s_{2222} = s_{3333} = (7-5\nu)/(15(1-\nu)) \\
s_{1122} &= s_{2233} = s_{3311} = s_{2211} = s_{3322} = (5\nu-1)/15(1-\nu) \\
s_{4444} &= s_{5555} = s_{6666} = (4-5\nu)/15(1-\nu)
\end{align*}
\]

(2) Circular Cylinder

\[
\begin{align*}
s_{1111} &= s_{1122} = s_{1133} = 0.00000001 \\
s_{2222} &= s_{3333} = (5-4\nu)/8(1-\nu) \\
s_{2233} &= s_{3322} = (4\nu-1)/8(1-\nu) \\
s_{2211} &= s_{3311} = \nu/2(1-\nu) \\
c &= (3-4\nu)/8(1-\nu) \\
s_{1212} &= s_{1313} = 1/4
\end{align*}
\]

(3) Elliptical Cylinder \((a_3=10000000)\)

\[
\begin{align*}
I_1 &= 4\pi a_3/(a_1+a_2) \\
I_2 &= 4\pi a_3/(a_1+a_2) \\
I_3 &= 10000000 \\
I_{12} &= 4\pi/(a_1+a_2)^2 \\
I_{13} &= 1 \\
I_{23} &= 0 \\
s_{1111} &= 1/2(1-\nu)\{a_2^2+2a_1a_2/(a_1+a_2)^2+(1-2\nu)/(a_1+a_2)+2(a_1+a_2)^2\} \\
s_{2222} &= 1/2(1-\nu)\{a_1^2+2a_2/(a_1+a_2)^2+(1-2\nu)/(a_1+a_2)+2(a_1+a_2)^2\} \\
s_{3333} &= 0.00000001 \\
s_{1122} &= 1/2(1-\nu)\{a_2^2/(a_1+a_2)^2+(1-2\nu)/(a_1+a_2)+2(a_1+a_2)^2\} \\
s_{2233} &= 1/2(1-\nu)(2a_1a_2/a_3+a_2) \\
s_{3311} &= 0.00000001 \\
s_{1133} &= 1/2(1-\nu)(2a_3/a_1+a_2) \\
s_{2211} &= 1/2(1-\nu)(a_1^2/(a_1+a_2)^2+(1-2\nu)/(a_1+a_2)+2(a_1+a_2)^2) \\
s_{3322} &= 0.00000001 \\
s_{4444} &= 1/2(1-\nu)\{a_1^2/(a_3/a_1+a_2)^2+(1-2\nu)/2\} \\
s_{5555} &= a_1/(a_1+a_2) \\
s_{6666} &= a_2/(2a_1+a_2)
\end{align*}
\]