Polyakov Effective Action from Functional Renormalization Group Equation

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Abstract

We discuss the Polyakov effective action for a minimally coupled scalar field on a two dimensional curved space by considering a non-local covariant truncation of the effective average action. We derive the flow equation for the form factor in $\int \sqrt{g} R c_k(\Delta) R$, and we show how the standard result is obtained when we integrate the flow from the ultraviolet to the infrared.

1 Introduction

In recent years several groups have investigated the possibility that a quantum theory of gravity can be constructed along the lines of Asymptotic Safety [1 2 3 4 5]. In this scenario metric degrees of freedom can still be used to construct quantum gravity if a non trivial (other than Gaussian) fixed point of the renormalization group (RG) flow with finite

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dimensional ultraviolet (UV) critical surface can be found. Only after the introduction, in this context, of an exact RG flow equation [1] it has become possible to properly investigate this issue. The evidence for the existence of such a fixed point has gradually improved but still much work has to be done to firmly establish this scenario and to be able to make reliable predictions, especially at the IR scales. Present day truncations have to be extended mainly in the direction of including an infinite number of terms [7, 8] and to include non-local invariants. Actually the two issues are related as to study non-local terms it is necessary to consider the RG flow of functions of the covariant Laplacian. As a first step in this direction, we will show how it is possible to write the flow equation for a, possibly non-local, structure function in the simple case of the effective average action for a minimally coupled scalar field on a two dimensional manifold. This analysis is worth doing also because we already know the answer: by integrating the conformal anomaly, Polyakov [6] was able to calculate the effective action for this system. This is one of the few cases in which the full non-local structure of the effective action in known exactly. Reproducing Polyakov’s result is thus an important test for the exact RG flow equation approach, as an example of a complete RG trajectory [9], and as an essential first step towards understanding more complicated non-local invariants.

The exact RG flow equation is a non-linear functional differential equation that, in the simplest cases, involves a functional trace over functions of the covariant Laplacian. The computation of these traces is a technically rather difficult task, actually it is one of the major obstacles in the development of the field. Heat kernel techniques are intensively used. To be able to handle non-local terms in truncations of the effective average action calls for the use of the non-local heat kernel expansion developed by Barvinsky and Vilkovisky [14].

In the second section of the paper we briefly review the standard derivation of the conformal anomaly and of the Polyakov effective action. In the third we introduce the exact RG flow equation for a minimally coupled scalar field on a $d$-dimensional manifold. In the fourth section we show how to derive the flow equation for a non-local truncation of the effective average action, and we use it to rederive Polyakov’s result.

2 Conformal anomaly and Polyakov effective action

We consider a minimally coupled scalar field $\phi$ on a two dimensional background manifold $(\mathcal{M}, g)$ where $g$ is a Riemannian metric. We will work in Euclidean signature. The classical
action (CA) is
\[ S[\phi, g] = \frac{1}{2} \int d^2 x \sqrt{g} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi = \frac{1}{2} \int d^2 x \sqrt{g} \Delta \phi , \]
where we introduced the Laplacian operator \( \Delta \) acting on scalar fields:
\[ \Delta \phi = -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu \nu} \partial_\nu \phi) . \]

As the action (1) is invariant under diffeomorphisms, it follows from Noether’s theorem that the (classical) energy momentum tensor, defined by
\[ T^{\mu \nu} = \frac{2}{\sqrt{g}} \frac{\delta S[\phi, g]}{\delta g_{\mu \nu}} , \]
is conserved, \( \nabla_\mu T^{\mu \nu} = 0 \). In the case of the minimally coupled scalar we readily find
\[ T^{\mu \nu} = \frac{1}{2} g^{\mu \nu} (\partial \phi)^2 - \partial_\mu \phi \partial_\nu \phi . \]
It is easy to check that the energy momentum tensor is traceless if \( d = 2 \). This follows from the fact that the action (1) is additionally conformal invariant in the special case of a two dimensiona manifold. Under a Weyl rescaling \( g^{\mu \nu} \rightarrow e^{\sigma} g^{\mu \nu} \) the covariant Laplacian, in \( d \) dimensions, transforms as
\[ \Delta \phi \rightarrow e^{-\sigma} \left[ \Delta \phi - \left( \frac{d}{2} - 1 \right) g^{\mu \nu} \partial_\mu \sigma \partial_\nu \phi \right] , \]
and we see that only in \( d = 2 \) the term \( \sqrt{g} \Delta \) and the action (1) are invariant.

The generating functional of correlation functions for a scalar field on the manifold \((\mathcal{M}, g)\) is defined as usual by
\[ Z[J, g] = \int D_g \phi \exp \left( -S[\phi, g] + \int d^d x \sqrt{g} J \phi \right) , \]
where \( J \) is a scalar source field. Following \[\text{(5)}\] the functional integral measure can be defined by the requirements
\[ \int D_g \phi e^{-\langle \phi, \phi \rangle_g} = 1 \quad \langle \phi, \phi \rangle_g = \frac{1}{2} \int d^d x \sqrt{g} \phi^2 \]
and is invariant under diffeomorphisms.
To find the expectation value of the energy momentum tensor we take a functional derivative of (4) with respect to the metric at $J = 0$:

$$\frac{\delta Z[0, g]}{\delta g_{\mu\nu}} = -\int D\phi \frac{\delta S[\phi, g]}{\delta g_{\mu\nu}} e^{-S[\phi, g]} = -Z[0, g] \sqrt{g} \frac{\sqrt{g}}{2} \langle T^{\mu\nu} \rangle .$$

(6)

Note that we did not differentiate the measure since we assumed it is invariant under diffeomorphisms and has thus zero variation with respect to the metric. We are assuming that there are no diffeomorphisms anomalies. Recalling the definition of the generating functional of connected correlation functions

$$W[J, g] = \log Z[J, g],$$

we can write

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta W[0, g]}{\delta g_{\mu\nu}} .$$

(7)

This is the quantum energy momentum tensor. We can rewrite it in terms of the effective action (EA) which is the Legendre transform of $W[J, g]$ with respect to the first argument,

$$\Gamma[\phi, g] = \int d^2x \sqrt{g} J(\phi) \varphi - W[J(\phi), g],$$

(8)

where $J = J(\phi)$ is to be obtained by solving $\frac{\delta W}{\delta J} = \varphi$ for $J$. Using (7) and (8) at $J = 0$ gives

$$\langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{g}} \frac{\delta \Gamma[\phi, g]}{\delta g_{\mu\nu}} .$$

(9)

The quantum energy momentum tensor (9) is the functional derivative, with respect to the metric, of the EA, as the classical energy momentum tensor is the functional derivative, with respect to the metric, of the CA.

Symmetries of the CA imply relations between the correlation functions which are described by Ward-Takahashi (WT) identities. To derive the WT identities related to diffeomorphism invariance, we consider an infinitesimal diffeomorphism,

$$\delta \epsilon_{\mu\nu} = \nabla_{\mu} \epsilon_{\nu} + \nabla_{\nu} \epsilon_{\mu} .$$
The EA transforms as
\[ \delta \epsilon \Gamma[0, g] = \int d^2 x \delta \epsilon g_{\mu \nu} \frac{\delta \Gamma[0, g]}{\delta g_{\mu \nu}} = - \int d^2 x \sqrt{g} \epsilon_\nu \nabla_\mu \langle T^{\mu \nu} \rangle, \tag{10} \]
where we used (9) and integrated by parts. Being the EA invariant under reparametrizations of the metric \( \delta \epsilon \Gamma[0, g] = 0 \), equation (10) implies \( \nabla_\mu \langle T^{\mu \nu} \rangle = 0 \). In the case of Weyl rescalings the EA transforms as
\[ \delta \sigma \Gamma[0, g] = \int d^2 x \delta \sigma g_{\mu \nu} \frac{\delta \Gamma[0, g]}{\delta g_{\mu \nu}} = \frac{1}{2} \int d^2 x \sqrt{g} \delta \sigma \langle T^\mu \rangle. \tag{11} \]
In this case we expect \( \langle T^\mu \rangle \neq 0 \) because the functional measure we chose is not Weyl invariant. To see this consider an infinitesimal Weyl rescaling \( \delta \sigma g_{\mu \nu} = g_{\mu \nu} \delta \sigma \). The functional measure transforms as:
\[ \delta \sigma (\phi, \phi)_g = \frac{1}{2} \int d^2 x \delta \sigma \sqrt{g} \phi^2 = \frac{1}{2} \int d^2 x \sqrt{g} \delta \sigma \phi^2 \neq 0. \]
We thus expect an anomaly in the WT identity related to Weyl rescalings (11).

The functional integral (4) with the CA (1) is Gaussian. Hence the exact EA is given by the one-loop formula
\[ \Gamma[0, g] = S[0, g] + \frac{1}{2} \text{Tr} \log S^{(2)}[0, g] \]
\[ = -\frac{1}{2} \int_{1/\Lambda^2}^\infty \frac{dt}{t} \text{Tr} e^{-t \Delta}. \tag{12} \]
We have regularized the UV divergences of the parameter integral introducing an UV cutoff \( \Lambda \) and we used the simple relation \( S^{(2)}[0, g] = \Delta \). The EA is independent of the mean value of the scalar field \( \varphi \) and is a purely geometrical functional constructed with invariants of the metric tensor \( g \).

We will look for a possible Weyl anomaly by inserting equation (12) into the WT relation (11). Under a Weyl rescaling the EA changes as
\[ \delta \sigma \Gamma[0, g] = -\frac{1}{2} \int_{1/\Lambda^2}^\infty \frac{dt}{t} \text{Tr} \delta \sigma e^{-t \Delta}. \]
Some simple manipulations
\[
\frac{1}{t} \text{Tr} \left[ \delta \sigma e^{-t \Delta} \right] = - \text{Tr} \left[ \delta \sigma \Delta e^{-t \Delta} \right] = \text{Tr} \left[ \delta \sigma \Delta e^{-t \Delta} \right] = - \frac{d}{dt} \text{Tr} \left[ \delta \sigma e^{-t \Delta} \right]
\]
give
\[
\delta \sigma \Gamma[0, g] = \frac{1}{2} \text{Tr} \left[ \delta \sigma e^{-\Delta / \Lambda^2} \right].
\]
The trace in the last expression can be calculated by the heat kernel using the local expansion (40) derived in the appendix. We find
\[
\text{Tr} \left[ \delta \sigma e^{-\Delta / \Lambda^2} \right] = \frac{\Lambda^2}{4\pi} \int d^2x \sqrt{g} \delta \sigma + \frac{1}{24\pi} \int d^2x \sqrt{g} R \delta \sigma + O \left( \frac{1}{\Lambda^2} \right). \tag{13}
\]
The first, divergent term is the Weyl variation of $\int \sqrt{g}$ (in $d = 2$) and can be renormalized by introducing in $S[\phi, g]$ a conformal symmetry breaking “cosmological term” $a_\Lambda \int \sqrt{g}$ and choosing $a_\Lambda = -\frac{\Lambda^2}{8\pi}$. The renormalized cosmological constant is zero then. We cannot renormalize the second term in equation (13) because it is topological and thus its Weyl variation is a total derivative:
\[
\delta \sigma \int d^2x (\sqrt{g} R) = \int d^2x \sqrt{g} [\delta \sigma R + \sqrt{g} (-R \delta \sigma + \Delta \delta \sigma)] = \int d^2x \sqrt{g} \Delta \delta \sigma.
\]
Comparing with (11) we find the two dimensional conformal anomaly induced by a minimally coupled scalar field:
\[
\langle T^\mu_\mu \rangle = - \frac{R}{24\pi}. \tag{14}
\]
More generally, if we consider $N_S$ scalar fields and $N_F$ Dirac fermions we find
\[
\langle T^\mu_\mu \rangle = - \frac{c}{24\pi} R \quad c = N_S + N_F,
\]
where $c$ is the conformal anomaly coefficient.

Using the fact that all metrics on a two dimensional manifold are locally conformally flat, Polyakov [6] showed that it is possible to calculate the full EA by functionally integrating the conformal anomaly. To do this we write the metric as conformally equivalent to the flat metric: $g_{\mu\nu} = e^\sigma \delta_{\mu\nu}$. The Ricci scalar becomes
\[
R = -e^{-\sigma} \partial^2 \sigma \tag{15}
\]
and using the anomaly equation (14) together with equation (9) we find

\[ \frac{\delta \Gamma[0, e^\sigma \delta]}{\delta \sigma} = -\frac{1}{48\pi}(-\partial^2)\sigma. \]

This relation is trivially integrated to give

\[ \Gamma[0, e^\sigma \delta] = -\frac{1}{96\pi} \int d^2 x \sigma(-\partial^2)\sigma + \text{const}, \]

where the constant can be set to be zero. Using \( \Delta = -e^{-\sigma}\partial^2 \), and equation (15) in the reverse way as before, we finally find

\[ \Gamma[0, g] = -\frac{1}{96\pi} \int d^2 x \sqrt{g} \frac{1}{\Delta} R. \] (16)

This is the Polyakov EA for a minimally coupled scalar field on the manifold \( (\mathcal{M}, g) \) [6]. Considering that \( \sqrt{g_x} \Delta_x G_{xy} = \delta_{xy} \), we actually have to write

\[ \Gamma[0, g] = -\frac{1}{96\pi} \int d^2 x d^2 y \sqrt{g_x} \sqrt{g_y} R_x G_{xy} R_y. \] (17)

Note that EA (16) is non-local since it involves the two point correlation function \( G_{xy} = (\Delta^{-1})_{xy} \) evaluated at different points of \( \mathcal{M} \), but it is analytical in the curvatures.

Having at our disposal the complete quantum effective action (16) we can, for example, calculate the quantum energy momentum tensor using equation (9). After a few manipulations we find [15]

\[ \langle T^{\mu\nu} \rangle = \frac{1}{48\pi} \left[ -2\nabla^\mu \nabla^\nu \frac{1}{\Delta} R - \left( \nabla^\mu \frac{1}{\Delta} R \right) \left( \nabla^\nu \frac{1}{\Delta} R \right) + 2g^{\mu\nu} \left( \nabla_\alpha \frac{1}{\Delta} R \right) \left( \nabla_\nu \frac{1}{\Delta} R \right) \right]. \] (18)

If we take the trace of equation (18) we can check that the conformal anomaly (14) is correctly reproduced. Expression (18) is exact, non-local and contains polarization as well as particle production effects; the Hawking radiation in \( d = 2 \) can be derived starting from this expression for the energy momentum tensor [15]. Note that in the limit of flat space we have \( \langle T^{\mu\nu} \rangle = 0 \) since the scalar field becomes free.

\(^1\text{Here we use the compact notation } \delta_{xy} \equiv \delta^{(2)}(x-y), \ G_{xy} \equiv G(x-y) \text{ and so on.} \)
3 Effective average action

The effective average action (EAA) $\Gamma_k[\varphi, g]$ is a functional that interpolates smoothly between the CA and the EA. We will construct the EAA for a scalar on the $d$-dimensional manifold $(g, M)$. More details on the EAA on curved backgrounds and in presence of quantized gravity can be found in [3].

Starting from the functional integral (4) we add to the CA an infrared (IR) cutoff or “regulator” term $\Delta S_k[\phi, g]$ of the form

$$\Delta S_k[\phi, g] = \frac{1}{2} \int d^d x \sqrt{g} \phi R_k(\Delta) \phi,$$

where the kernel $R_k(\Delta)$ is chosen so to suppress the field modes $\phi_n$, eigenfunctions of the Laplacian $\Delta \phi_n = \lambda_n \phi_n$, with eigenvalues smaller than the cutoff scale $\lambda_n < k^2$. The functional form of $R_k(z)$ is arbitrary except for the requirements that it should be a monotonically decreasing function in both $z$ and $k$, that $R_k(z) \to 0$ for $z \gg k^2$ and that $R_k(z) \to k^2$ for $z \ll k^2$.

The scale dependent generating functional of correlation functions that generalizes equation (4) is

$$Z_k[J, g] = \int \mathcal{D} g \phi \exp \left( -S[\phi, g] - \Delta S_k[\phi, g] + \int \sqrt{g} J \phi \right). \quad (19)$$

The scale dependent EA, referred to as the effective average action (EAA) is defined by the relation

$$\Gamma_k[\varphi, g] + \Delta S_k[\varphi, g] = \int \sqrt{g} J(\varphi) \varphi - W_k[J(\varphi), g]. \quad (20)$$

Note that the Legendre transform of the generating functional of the connected correlation functions is $\Gamma_k[\varphi, g] + \Delta S_k[\varphi, g]$ and not the EAA; thus $\Gamma_k[\varphi, g]$ need not to be a convex functional for non zero $k$.

In the limits $k \to 0$, and $k \to \Lambda$ the EAA approaches, respectively, the EA and the bare CA [9]:

$$\lim_{k \to 0} \Gamma_k[\varphi, g] = \Gamma[\varphi, g] \quad \lim_{k \to \Lambda} \Gamma_k[\varphi, g] = S[\varphi, g].$$

The EAA and its RG flow offer a different approach to quantization. In theory space, the space of “all” action functionals, the bare action represents the UV starting point of a RG trajectory which reaches the EA in the IR. The integration of successive modes is done step by step when lowering the cutoff scale $k$.

There is a considerable freedom in the choice of the functional form of the regulator kernel.
Figure 1: Dimensionless bare propagator (continuous) together with regularized propagators using optimized cutoff (thick), exponential cutoff (long dashed) and mass-type cutoff (short dashed) plotted as functions of $z/k^2$.

We will use the following three examples here, the “optimized” \cite{11}, the exponential, and the mass-type cutoff, respectively:

\[
R_{k}^{\text{opt}}(z) = (k^2 - z)\theta(k^2 - z)
\]

\[
R_{k}^{\text{exp}}(z) = \frac{z}{e^{z/k^2} - 1}
\]

\[
R_{k}^{\text{mass}}(z) = k^2.
\]

The mass cutoff is not properly a cutoff because for $z \gg k^2$ it does not go to zero. Hence it does not guarantee the UV finiteness of the flow. Still it is useful since it often allows for analytical computations of UV finite quantities.

In the case we will study in this paper, where the bare propagator goes like $\frac{1}{p^2} \equiv \frac{1}{z}$, it is easy to see how the cutoff acts as an IR regulator. The regularized propagator $\frac{1}{z + R_k(z)}$ is shown in Figure 1 for the three different cutoff shapes in (21) plotted together with the bare one. For modes of momentum eigenvalues greater then the RG scale $z \gg k^2$ the propagation is unaffected while starting at the cutoff scale their propagation is successively suppressed as if they were massive particles of mass $k$.

It is possible to derive an exact RG flow equation for the EAA by differentiating equation (20) with respect to the “RG time” $t = \log k/k_0$ ($k_0$ is a reference scale) as first done in \cite{12} for the case of a scalar field on a $d$ dimensional flat background. The flow equation for the
situation we are considering of a $d$ dimensional scalar field on a curved background \[3\] reads
\[
\partial_t \Gamma_k[\varphi, g] = \frac{1}{2} \text{Tr} \frac{\partial_t R_k[g]}{\Gamma_k^{(2,0)}[\varphi, g] + R_k[g]}.
\] (22)

Here $\Gamma_k^{(2,0)}[\varphi, g]$ is the second functional derivative of the EAA with respect to $\varphi$. The trace in (22) is a sum over the eigenvalues of the covariant Laplacian and is usually evaluated using heat kernel techniques as explained in the Appendix. Equation (22) is exact and is both UV and IR finite.

The flow equation (22) can be seen as a RG improvement of the one-loop effective action derived from the bare action $S[\varphi, g] + \Delta S_k[\varphi, g]$. Using the definition of the EAA (20) and the standard one-loop relation for the EA we find
\[
\Gamma_k[\varphi, g] = S[\varphi, g] + \frac{1}{2} \text{Tr} \log \left( S^{(2,0)}[\varphi, g] + R_k[g] \right),
\] (23)

Differentiating with respect to the RG time gives
\[
\partial_t \Gamma_k[\varphi, g] = \frac{1}{2} \text{Tr} \frac{\partial_t R_k[g]}{S^{(2,0)}[\varphi, g] + R_k[g]}.
\] (24)

If now we “RG improve” equation (24) in the sense of replacing in the right side the Hessian of the bare action with the Hessian of the EAA we recover the exact flow equation (22).

4 Flow equations

For a minimally coupled scalar field the one-loop EAA (23) is already exact and it is thus enough to consider the flow equation (24). The right hand side of (24) is independent of $\varphi$ so the flow will not generate any non trivial dependence of the EAA on $\varphi$. We will thus concentrate on the flow of $\Gamma_k[0, g]$ which is the non trivial part of the EAA.

The Hessian of the bare action (11) is just the covariant Laplacian $S^{(2,0)}[0, g] = \Delta$ and equation (24) reduces to
\[
\partial_t \Gamma_k[0, g] = \frac{1}{2} \text{Tr} \frac{\partial_t R_k(\Delta)}{\Delta + R_k(\Delta)},
\] (25)

We can evaluate the trace in equation (25) using the technology developed in the Appendix to which the reader could turn at this point. Defining the function $h(z) = \frac{\partial_t R_k(z)}{z + R_k(z)}$ and using
the heat kernel non-local expansion (44) in equation (39) from the Appendix we find

$$\text{Tr } h(\Delta) = \frac{1}{4\pi} Q_1[h] \int d^2 x \sqrt{g} + \frac{1}{24\pi} Q_0[h] \int d^2 x \sqrt{g} R +$$

$$+ \frac{1}{4\pi} \int d^2 x \sqrt{g} R \left[ \int_0^\infty ds \tilde{h}(s) s f_R(s \Delta) \right] R + O(R^3),$$

(26)

Here $\tilde{h}(s)$ is the Laplace transform of the function $h(x)$ and the $Q$-functional are defined in (46). To make progress we need to devise a truncation of the EAA to insert into the left hand side of (25). We will consider an ansatz where the EAA is local in the curvature (analytical in the metric) but non-local in the covariant momentum squared, i.e. in $\Delta$. We are lead to the following truncation ansatz which comprises the first terms of the curvature expansion:

$$\Gamma_k[0,g] = \int d^2 x \sqrt{g} (a_k + b_k R + R c_k(\Delta) R) + O(R^3).$$

(27)

Here $c_k(x)$ is any function of the covariant Laplacian. We are working in two dimensions so this is the only structure function at second order in the curvature to be considered.

By comparing equation (26) to (27), the beta functions for the first two couplings in (27) are immediately found:

$$\partial_t a_k = \frac{1}{8\pi} Q_1[h]$$

$$\partial_t b_k = \frac{1}{48\pi} Q_0[h].$$

(28)

From the curvature squared term we find the flow of the non-local structure function

$$\partial_t c_k(x) = \frac{1}{8\pi} \int_0^\infty ds \tilde{h}_k(s) s f_R(s x).$$

(29)

If we now insert the explicit form of the heat kernel structure function $f_R(x)$ from equation (45) of the Appendix, written in terms of the basic parameter integral (43), and use relation (46), we find:

$$8\pi \partial_t c_k(x) = \frac{1}{32} \int_0^1 d\xi Q_{-1} [h(z + x \xi(1 - \xi))] +$$

$$+ \frac{1}{8x} \int_0^1 d\xi Q_0 [h(z + x \xi(1 - \xi))] - \frac{1}{16x} Q_0[h] +$$
\[ + \frac{3}{8x^2} \int_0^1 d\xi Q_1 [h(z + x\xi(1 - \xi))] - \frac{3}{8x^2} Q_1[h]. \] (30)

In the last equation the dummy index \( z \) is shown to indicate that the \( Q \)-functionals are to be evaluated at the shifted point \( z + x\xi(1 - \xi) \). The next step is to use (47) to calculate the \( Q \)-functionals:

\[
8\pi \partial_t c_k(x) = -\frac{1}{32} \int_0^1 d\xi h'(x\xi(1 - \xi)) + \frac{1}{8x} \int_0^1 d\xi h(x\xi(1 - \xi)) +
- \frac{1}{16x} h(0) - \frac{3}{8x^2} \int_0^1 d\xi \int_{x\xi(1-\xi)} d\xi h(z). \quad (31)
\]

Note that we combined the last two terms of equation (30) into a single \( z \) integral.

Equation (31) is the explicit flow equation for the structure function \( c_k(x) \). It should be possible to integrate equation (31) from the UV to the IR scale to recover the Polyakov EA (16) without specifying the cutoff shape function \( R_k(z) \). Here we will show how this can be done by explicitly using the cutoff shapes in (21).

First we use the “optimized” cutoff to evaluate the beta functions (28):

\[
\partial_t a_k = \frac{k^2}{4\pi}, \quad \partial_t b_k = \frac{1}{24\pi}. \quad (32)
\]

After collecting the overall power \( k^{-2} \) and writing the parameter integrals in terms of the dimensionless variable \( x/k^2 \), the flow equation (31) can be rewritten in the form

\[
\partial_t c_k(x) = \frac{1}{8\pi k^2} f \left( \frac{x}{k^2} \right). \quad (33)
\]

The function \( f(u) \) depends explicitly on the cutoff shape function used. In the case of the optimized and mass cutoffs we find after some elementary integrations, respectively,

\[
f_{\text{opt}}(u) = \frac{1}{8u} \left[ \sqrt{\frac{u}{u-4}} - \frac{u+4}{u} \sqrt{\frac{u-4}{u}} \right] \theta(u-4)
\]

\[
f_{\text{mass}}(u) = \frac{\sqrt{u(u+4)(u+6)} + 8(u+3) \text{artanh} \sqrt{\frac{u}{u+4}}}{(u+4)^{3/2} u^{5/2}}. \quad (34)
\]

The parameter integrals in equation (31) cannot be evaluated analytically for the exponential cutoff, but this can still be done numerically. The functions \( f(u) \) evaluated for the three
different cutoffs are plotted in Figure 2. Note that they are all analytic in a neighborhood of the origin, \( f_{\text{opt}}(u) \) is even zero in the entire interval \([0, 4)\).

If we were to interpret \( f(u) \) as a power series in \( u \) about \( u = 0 \), it follows that we have a non-zero running of local terms of the form \( c_k^{(n)} \int \sqrt{g} R \Delta^n R \) only for the exponential and the mass cutoff. For example, we can expand for small \( u \)

\[
\begin{align*}
  f_{\text{mass}}(u) &= \frac{1}{30} - \frac{u}{70} + \frac{u^2}{210} + O(u^3),
\end{align*}
\]

and read off the resulting beta functions for the couplings \( c_k^{(n)} \) in the mass cutoff case. For the optimized cutoff none of the couplings \( c_k^{(n)} \) has a non-zero beta function. Any finite truncation of the EAA containing some of the couplings \( c_k^{(n)} \) will never reproduce the correct IR behavior and will lead to IR divergences. More importantly, the running of the couplings \( c_k^{(-n)} \), \( n > 0 \), which multiply non-local terms involving inverse powers of \( \Delta \) is zero for all three cutoff choices. In particular, the beta function of the coupling \( c_k^{(-1)} \) pertaining to the operator \( \int \sqrt{g} R \frac{1}{\Delta} R \) is zero, even if this is the form the EAA is expected to reach at \( k = 0 \)! We conclude that, at least for the cutoff shapes we considered, to capture the non-local features of the EAA we need to consider the running of the whole structure function.

We now integrate the flow equations from the UV scale \( \Lambda \), where we impose the initial conditions \( \Gamma_\Lambda[\phi, g] = S_\Lambda[\phi, g] \), to the IR scale \( k \). We will see that imposing the initial conditions not only selects which theory we are quantizing, but also implements the renormalization
conditions. In the limit $k \to 0$ we will find the full EA.

We start solving the differential equations (32). Integrating from $k$ to $\Lambda$ gives

$$a_k = a_\Lambda - \frac{1}{4\pi}(\Lambda^2 - k^2)$$

$$b_k = b_\Lambda - \frac{1}{24\pi} \log \frac{\Lambda}{k}.$$  \hspace{1cm} \text{(35)}

The coupling $a_k$ has to be renormalized, this can be done by setting $a_\Lambda = \frac{\Lambda^2}{4\pi}$ so that the renormalized coupling vanishes in the IR, $a_0 = 0$. This preserves conformal invariance of the EA.

Integrating the RG equation (33) of the structure function gives

$$c_k(x) = c_\Lambda(x) - \frac{1}{8\pi} \int_k^\Lambda \frac{dk'}{k'^3} f \left( \frac{x}{k'^2} \right).$$

If we use the variable $y = x/k^2$ we have $dk/k^3 = -dy/2x$ and we come to

$$c_k(x) = c_\Lambda(x) - \frac{1}{16\pi x} \int_{x/\Lambda^2}^{x/k^2} dy f(y).$$ \hspace{1cm} \text{(36)}

If the integral in (36) is convergent at both the lower and upper limit, it becomes a pure number in the limit $\Lambda \to \infty$ and $k \to 0$. The functional form of $c_0(x)$ will be in agreement with the Polyakov effective action (16) if

$$\int_0^\infty dy f(y) = \frac{1}{6}.$$  

This condition should be met for any cutoff choice. A simple integration of (34) shows that this is so for the optimized and mass cutoffs. A numerical evaluation shows that also the exponential cutoff gives the desired result.

Imposing the boundary condition $c_\infty(x) = 0$, we thus recover Polyakov’s non-local EA, equation (16), as the result of the integration of the flow:

$$c_0(x) = -\frac{1}{96\pi x}.$$  

We are now in a position to write down the EAA, within truncation (27) and using the
optimized cutoff \( (21) \), at any given scale \( k \) we have:

\[
\Gamma_k[0, g] = \frac{k^2}{4\pi} \int d^2x \sqrt{g} - \frac{1}{24\pi} \log \frac{\Lambda}{k} \int d^2x \sqrt{g} R + \\
\quad - \frac{1}{96\pi} \int d^2x \sqrt{g} R \left[ \frac{\sqrt{\Delta/k^2} - 4(\Delta/k^2 + 2) \theta(\Delta/k^2 - 4)}{\Delta (\Delta/k^2)^{3/2}} \right] R + O(R^3). \tag{37}
\]

This relation is our main result. It shows how the EAA interpolates smoothly between the classical action at the scale \( k = \Lambda \) and the EA at the scale \( k = 0 \).

Note that in principle we still have to show that all higher terms that would extend the truncation \( (27) \) to higher curvature terms, and are in principle present in the EAA, vanish at \( k = 0 \). Only then we would have completely recovered Polyakov’s result. We shall not embark on such a proof since this issue is special to two dimensions and, contrary to the above discussion, does not generalize to higher dimensions \cite{16}.

5 Conclusion

In this paper we explained how the Polyakov effective action for a minimally coupled scalar field on a curved two dimensional manifold emerges within the functional RG approach. To do this we calculated the RG flow of the structure function \( c_k(\Delta) \) using the non-local heat kernel expansion. We learned that in order to be able to recover, at the IR scale, special non-local terms in the EAA, \( \int \sqrt{g} R \frac{1}{\Delta} R \) in our example, it is necessary to include the running of the complete structure function which allows for an arbitrary dependence on \( \Delta \). We also saw that, quite remarkably, individual non-local terms in a Laurent series expansion, \( \int \sqrt{g} R \Delta^{-n} R, n > 0 \), have no RG running, even though the \( k \to 0 \) limit of the EAA is precisely of this type. This is an important observation in view of future physical applications when we move on to consider quantized gravity, especially in four dimensions. In that case the flow equations for the corresponding structure functions will be a non-linear integro-differential equation since the flow will no longer be one-loop like as in the simple case treated in this paper. This strategy seems promising for finding the IR completion of the RG flow of Quantum Einstein Gravity which was computed within local truncations and found to terminate at unphysical singularities in the IR \cite{13}. 

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A Non local Heat Kernel expansion

In this appendix we review the basic relations related to the heat kernel expansion. We will consider both the local and the non-local expansion of the heat kernel for the covariant Laplacian \( \Delta \) on a general \( d \)-dimensional manifold.

The heat kernel \( K_{xy}(s) \) satisfies the following differential equation with boundary condition:

\[
(\partial_s + \Delta_x) K_{xy}(s) = 0 \quad \text{and} \quad K_{xy}(0) = \delta_{xy}.
\]

This equation describes the diffusion of a test particle on the manifold. Here \( s \) is the heat kernel “propertime” parameter which is related to the diffusion constant \( D \) and to time \( t \) as \( s = Dt \). We use the compact notation \( K_{xy}(s) \equiv K(s; x, y) \) and \( \delta_{xy} \equiv \delta^{(d)}(x - y) \).

Equation (38) is solved by \( K_{xy}(s) = e^{-s\Delta} \delta_{xy} \) and the trace of the heat kernel is thus equal to \( \text{Tr} K(s) = \text{Tr} e^{-s\Delta} \).

The usefulness of the heat kernel expansion stems from the fact that every functional trace of a function \( h(\Delta) \) of the covariant Laplacian can be related to it by a Laplace transform:

\[
\text{Tr} h(\Delta) = \int_0^\infty ds \tilde{h}(s) \text{Tr} e^{-s\Delta}.
\]

\( \tilde{h}(s) \) is the Laplace transform of \( h(x) \). To compute such a functional trace one just need to know the expansion of the heat kernel trace to the desired accuracy.

For the trace there exists a standard asymptotic series in local curvature polynomials,

\[
\text{Tr} K(s) = s^{-d/2} \left(B_0[\Delta] + sB_2[\Delta] + s^2B_4[\Delta] + \ldots \right),
\]

where the \( B_{2n} \) are the integrated heat kernel coefficients. They are related to the un-integrated coefficients \( b_{2n} \) by the relation

\[
B_{2n}[\Delta] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} \text{tr} b_{2n}[\Delta](x).
\]
For the Laplacian \( \Delta \) operator acting on scalars the first few are

\[
b_0[\Delta] = 1 \quad \quad b_2[\Delta] = \frac{R}{6} \quad \quad b_4[\Delta] = \frac{1}{180} \left( R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu} + \frac{5}{2} R^2 - 6 \nabla^2 R \right). (41)\]

For the purposes of this paper we need a more sophisticated version of the heat kernel expansion developed by Barvinsky and Vilkovinsky [14] which includes an infinite number of terms in the form of non-local structure factors. The generalized non-local expansion that replaces (40) reads

\[
\text{Tr} \ K(s) = \frac{1}{(4\pi s)d/2} \int d^d x \sqrt{g} \left[ 1 + s \frac{R}{6} + s^2 R_{\mu\nu} f_a(s\Delta) R_{\mu\nu} + s^2 R f_b(s\Delta) R + O(R^3) \right], (42)\]

where \( f_a(x) \) and \( f_b(x) \) are two particular linear combinations [14] of the basic structure factor

\[
f(x) = \int_0^1 d\xi e^{-x\xi(1-\xi)}. (43)\]

In equation (42) only two of the three possible curvature square terms appear, the third one has been eliminated by using Bianchi’s identities and discarding boundary terms. For this reason the total derivative term in the coefficient \( B_4 \) is not present and a straightforward series expansion of the structure functions \( f_a(x) \) and \( f_b(x) \) will not give the same numerical coefficients as in (41). This is because the form factor expansion (42) gives directly the integrated coefficients \( B_{2n} \) while the standard expansion gives the un-integrated coefficients \( b_{2n} \). (See [14] for more details.) The series (42) if expanded with respect to the variable \( s\Delta \) gives, subject to the remark just made, the coefficients of \( \int \sqrt{g} R_{\mu\nu} \Delta^n R_{\mu\nu} \) and \( \int \sqrt{g} \Delta^n R \) contributing to the coefficients \( B_{2n} \).

We are interested in the \( d = 2 \) case where \( R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \) and so equation (42) becomes:

\[
\text{Tr} \ K(s) = \frac{1}{4\pi s} \int d^2 x \sqrt{g} \left[ 1 + s \frac{R}{6} + s^2 R f_R(s\Delta) R + O(R^3) \right], (44)\]

where

\[
f_R(x) = \frac{1}{32} f(x) + \frac{1}{8x} f(x) - \frac{1}{16x} + \frac{3}{8x^2} f(x) - \frac{3}{8x^2}. (45)\]

As we mentioned before, the first coefficient in a series expansion of \( f_R(x) = \frac{1}{60} + \ldots \) does not
match the coefficient $B_4$ evaluated in $d = 2$ where we find the coefficient $\frac{1}{72}$. To find agreement between the coefficients, terms of third order in the curvatures have to be considered in the non-local expansion.

The last technical tool we need in order to evaluate functional traces are the integrals

$$Q_n[h] = \int_0^\infty ds \tilde{h}(s) s^{-n}$$

(46)

that occur when we insert in (39) the heat kernel expansions (40) or (44). Standard result about Mellin transforms [3] give the formulas

$$Q_n[h] = \begin{cases} \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} h(z) & n > 0 \\ (-1)^n h^{(n)}(0) & n \leq 0 \end{cases}$$

(47)

If in addition a factor $e^{-sa}$ is present in the integral (46) then the function $h$ in (47) has to evaluated at $z + a$. For more details refer to Appendix A of [3].

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