On closures of cycle spaces of flag domains

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Abstract

Open orbits $D$ of noncompact real forms $G_0$ acting on flag manifolds $Z = G/Q$ of their semisimple complexifications $G$ are considered. Given $D$ and a maximal compact subgroup $K_0$ of $G_0$, there is a unique complex $K_0$–orbit in $D$ which is regarded as a point $C_0 \in C_q(D)$ in the space of $q$-dimensional cycles in $D$. The group theoretical cycle space $\mathcal{M}_D$ is defined to be the connected component containing $C_0$ of the intersection of the $G$–orbit $G(C_0)$ with $C_q(D)$. The main result of the present article is that $\mathcal{M}_D$ is closed in $C_q(D)$. This follows from an analysis of the closure of the universal domain $U$ in any $G$-equivariant compactification of the affine symmetric space $G/K$, where $K$ is the complexification of $K_0$ in $G$.

1 Background and notation

Throughout this article $G_0$ denotes a noncompact simple Lie group of adjoint type. Generalizations of our results to the semisimple case require only formal adjustments and will not be discussed. Having fixed a maximal compact subgroup $K_0$ of $G_0$, we will make use of Iwasawa decompositions $G_0 = K_0A_0N_0$.

We regard $G_0$ as a closed subgroup of its universal complexification $G$. Therefore the complexification $K$ of $K_0$ is a closed complex subgroup of $G$ and we consider the affine symmetric space $\Omega = G/K$. Certain $G$-equivariant projective algebraic compactifications $X$ of $\Omega$
play an important role in our work. These arise as follows.

Let $Z = G/Q$ be a $G$-flag manifold, i.e., $Q$ is a complex parabolic subgroup of $G$. Starting with the basic work ([W1]) there has been substantial interest in complex geometric objects related to the $G_0$-action on $Z$ (see [ETHW] for a systematic presentation). In particular there are only finitely many $G_0$-orbits in $Z$ and therefore there are open orbits $D$ which merit study from the complex geometric viewpoint.

Each such $D$ contains a unique $K_0$-orbit $C_0$ which is a complex submanifold of $Z$. We let $q := \dim_{\mathbb{C}} C_0$ be the dimension of this base cycle and regard it as a point $C_0 \in C_q(D)$ in the full cycle space of $D$. Without further notation we replace $C_q(D)$ by its connected component containing $C_0$ in the irreducible component which contains $C_0$ in the full cycle space $C_q(Z)$.

In ([WeW]) a group theoretical cycle space $\mathcal{M}_D$ was introduced. For this consider the $G$-orbit $\mathcal{M}_Z := G(C_0)$ of the base cycle in $C_q(Z)$. Since the induced $G$–action on $C_q(Z)$ is algebraic, $\mathcal{M}_Z$ is Zariski open in its closure and its intersection $\mathcal{M}_Z \cap C_q(D)$ with the semialgebraic open set $C_q(D)$ consists of at most finitely many components. The cycle space $\mathcal{M}_D$ is defined to be the connected component of this intersection which contains $C_0$.

2 The main result

Our goal here is to present a proof of the following result. It will be reliant on the more technical results of the following sections.

**Theorem 2.1.** The group theoretical cycle space $\mathcal{M}_D$ is closed in the full cycle space $C_q(D)$.

Let us attempt to put this in perspective. First of all $\mathcal{M}_D$ is a locally closed complex submanifold of $C_q(D)$. The representation of the $G$–isotropy group on the tangent space of $C_q(D)$ at the base cycle $C_0$ has been calculated in detail (see Part IV in [ETHW]). In particular, even for a fixed $G_0$, there is a great variety of representations depending on $D$ and the flag manifold $Z$, and, for example, the codimension of $\mathcal{M}_D$ in $C_q(D)$ can vary wildly.
In the case where $\mathcal{M}_D$ is open in $C_q(D)$ Theorem 2.1 states that $\mathcal{M}_D = C_q(D)$. This is particularly useful in situations where the cycles have meaning in complex geometry, e.g., in the case of period domains such as the moduli space of marked K3-surfaces: Any two cycles differ only by a transformation in the complex group $G$ and no degeneration is possible.

In the future we hope that the full cycle space $C_q(D)$ can be explicitly computed and that it will be of use in representation theory. We already know that in many cases it is a Stein space and it is very likely that it is Kobayashi hyperbolic. Since $\mathcal{M}_D$ is closed, it is quite possible that recently developed $G_0$–invariant theory (see [HSch, HSt]) can be applied to show that $C_q(D)$ is a Luna-slice type bundle over $\mathcal{M}_D$. Since $\mathcal{M}_D$ has already been described with great precision (see below) and there are good Ansätze for describing the fiber, this could very well lead to the desired precise description of $C_q(D)$.

Now let us recall the description of $\mathcal{M}_D$. For a certain well–understood special class of domains which are said to be of Hermitian holomorphic type, where in particular $G_0$ is of Hermitian type, $\mathcal{M}_D$ is just the associated bounded symmetric domain $\mathcal{B}$. In this case the stabilizer of $C_0$ in $G$ is a parabolic group $P$ so that $\mathcal{M}_Z = G/P$ is the compact dual of $\mathcal{B}$ (see e.g. [FHW]). Since $\mathcal{M}_Z$ compact, it is a direct consequence of the definitions that $\mathcal{M}_D$ is closed in $C_q(D)$. Thus we may assume that $D$ is not of Hermitian holomorphic type, and the following result is applicable ([HW, FH], see also [FHW]).

**Theorem 2.2.** If $D$ is not of Hermitian holomorphic type, then $\mathcal{M}_D$ is naturally biholomorphic to the universal domain $\mathcal{U}$ contained in the affine symmetric space $\Omega = G/K$.

Before going into the details of the definition of $\mathcal{U}$ (first introduced in a representation theoretical context by Akhiezer and Gindikin), we emphasize that $\mathcal{U}$ is defined independent of which flag manifold $Z$ and domain $D$ is under consideration. For this reason and since $\mathcal{U}$ occurs in a number of contexts, several of which are important in this article, we refer to it as being universal.

The domain $\mathcal{U}$ is defined as follows. If $g_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is an Iwasawa decomposition at the Lie algebra level, then one defines the
polytope $\omega_0 \subset a_0$ as follows:

$$\omega_0 = \bigcap_\alpha \{ \xi \in a_0 : |\alpha(\xi)| < \frac{\pi}{2} \},$$

where $\alpha$ runs over the restricted roots. Then

$$\mathcal{U} := G_0 \exp(i\omega_0)(x_0),$$

where $x_0$ is a base point with isotropy group $K$.

Since we have eliminated the Hermitian holomorphic case from discussion, the $G$-isotropy group $\tilde{K}$ at $C_0$ is just a finite extension of the connected group $K$, and the cycle space $M_D$ lifts biholomorphically to $\mathcal{U}$ in $\Omega = G/K$ (FHW). In our discussion of closures the finite cover $G/K \to G/\tilde{K}$ plays no role. Hence, for notational convenience we simply assume that $M_D \subset \Omega$. Since it is quite difficult to know anything specific about the closure of $M_Z$, we consider all possible situations

$$M_D = \mathcal{U} \subset \Omega \subset X = \text{cl}(\Omega),$$

where $X$ is an arbitrary projective algebraic $G$-equivariant compactification of $\Omega = G/K$.

One of the main methods used in proving Theorem 2.2 is that of Schubert incidence geometry. We make strong use of this in the present article, and therefore we now sketch the basics (for details see FHW or the original papers HW, FH). Given a Borel subgroup $B \subset G$, a $B$-Schubert variety $S$ in $Z$ is the closure $S = O \cup Y$ of a $B$-orbit $O$ in $Z$. Here $q$-codimensional Schubert varieties of Iwasawa-Borel subgroups, i.e., those which contain a component $A_0N_0$ of an Iwasawa-decomposition $G_0 = K_0A_0N_0$, are important.

If $B$ is an Iwasawa-Borel subgroup and $S$ is an associated $q$-codimensional Schubert variety with $S \cap C_0 \neq \emptyset$, then $S \cap C \neq \emptyset$ for every $C \in M_D$. Furthermore, the complement $Y$ of the open orbit $O$ in $S$ is contained in the complement of $D$. In particular, the incidence variety

$$I_Y := \{ C \in \mathcal{C}_q(Z) : C \cap Y \neq \emptyset \}$$

is contained in the complement of the cycle space $M_D$. 

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The intersection $H_{Y,\Omega} := I_Y \cap \Omega$ is a $B$-invariant complex algebraic hypersurface which is contained in the complement of $\mathcal{M}_D = \mathcal{U}$ in $\Omega$. A final step in the proof of Theorem 2.2 can be formulated as follows. For this, in the Hermitian case the Schubert variety $S$ must be chosen appropriately, but otherwise it only required to have the properties described above.

**Theorem 2.3.** For every point $p$ in the boundary $\text{bd}_\Omega(\mathcal{M}_D) = \text{bd}_\Omega(\mathcal{U})$ there exists $k \in K_0$ with $p \in k(H_{Y,\Omega})$.

We now state our main technical result which is proved in \[4\,2\].

**Theorem 2.4.** Let $X$ be an arbitrary $G$-equivariant compactification of the affine symmetric space $\Omega = G/K$. Then the interior of the closure $\text{cl}_X(\mathcal{U})$ of the universal domain $\mathcal{U}$ in $X$ is $\mathcal{U}$ itself and

$$\text{bd}_X(\mathcal{U}) = \text{cl}_X(\text{bd}_\Omega(\mathcal{U})).$$

The essential point of this theorem is that the interior of $\text{cl}_X(\mathcal{U})$ is $\mathcal{U}$. The second statement, $\text{bd}_X(\mathcal{U}) = \text{cl}_X(\text{bd}_\Omega(\mathcal{U}))$, is a direct consequence of this fact. To see this, note that $\text{cl}_X(\mathcal{U}) \setminus \text{cl}_X(\text{bd}_\Omega(\mathcal{U}))$ is open in $\text{cl}_X(\mathcal{U})$ and contains $\mathcal{U}$. Therefore, by the first statement in the theorem this open set is exactly $\mathcal{U}$ and we have have the decomposition

$$\text{cl}_X(\mathcal{U}) = \mathcal{U} \cup \text{cl}_X(\text{bd}_\Omega(\mathcal{U}))$$

which is equivalent to the desired result.

Using Theorem 2.4 we can now give the

**Proof of Theorem 2.1.** Applying Theorem 2.4 if $H_X$ denotes the closure in $X$ of the hypersurface $H_{Y,\Omega}$ of Theorem 2.3 then, since $K_0$ is compact, it follows that every $p \in \text{bd}_X(\mathcal{U})$ is contained in some translate $k(H_X)$.

Since $Y$ is closed, every cycle in $k(H_X)$ also has nonempty intersection with $k(Y)$. Thus if $p \in \text{bd}_X(\mathcal{U})$ is regarded as a cycle $C$, then $C \not\subset D$. Therefore $\mathcal{M}_D = \mathcal{U}$ is closed in $\mathcal{C}_q(D)$. \[Q.E.D.\]

From this proof one sees that the main new ingredients for this result are to be found in Theorem 2.4. The proof of this result is in turn heavily reliant on particular properties of special $G$-equivariant compactifications $X$ of $\Omega$. In the Hermitian case we choose $X = X_+ \times X_-$. 

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to be the product of the two compact Hermitian symmetric spaces (see §3) and in the nonhermitian case we make strong use of the DeConcini-Procesi compactification (see §4.1 and §4.2).

The desired result for an arbitrary equivariant compactification of $\Omega$ follows from the fact that any two such compactifications are equivariantly birationally equivalent. In the case where $M_D$ is actually contained in a finite (algebraic) quotient $\tilde{\Omega} = G/\tilde{K}$ of $\Omega$, the quotient map extends to an equivariant rational map of the special compactification of $\Omega$ under consideration to the closure $\tilde{X}$ of $M_Z$ in the cycle space $C_q(Z)$. The arguments that show that the birational maps which arise from the various compactifications of $\Omega$ play no role in the discussion show that such generically finite rational maps also play no role. Thus, as stated above, we simply assume that $M_Z = \Omega = G/K$ from the beginning.

### 3 The Hermitian case

In this section it is assumed that $G_0$ is of Hermitian type. In this case the parabolic subgroups of $G$ containing $K_0$ are $P_+ = KS_+$ and $P_- = KS_-$, where $S_+$ and $S_-$ are unipotent part of $P_+$ and $P_-$. They correspond to Hermitian symmetric spaces $X_+ = G/P_+$ and $X_- = G/P_-$ with the base point $x_+$ and $x_-$. Consider the diagonal action of $G$ on $X_+ \times X_-$. Then the isotropy group at $x = (x_+, x_-)$ is $K$ and the affine symmetric space $\Omega = G(x_+, x_-) = G/K$ is open dense in $X_+ \times X_-$. Write $E = X_+ \times X_- \setminus \Omega$. It is known that the universal domain $U$ is equal to $B_+ \times B_- = G_0 x_+ \times G_0 x_-$. The following Lemma is proved in the proof of Theorem 3.8 of [WZ].

**Lemma 3.1.** bd$_{X_+}(B_+) \times B_-$ and $B_+ \times$ bd$_{X_-}(B_-)$ are contained in $G/K = G(x_+, x_-)$.

**Proposition 3.2.** For any $G$-orbit $\mathcal{O}$ in $E$, cl$_{X_+ \times X_-}(B_+ \times B_-) \cap \mathcal{O}$ has no interior point in $\mathcal{O}$.

**Proof.** The boundary bd$_{X_+ \times X_-}(B_+ \times B_-)$ is the union (bd$_{X_+}(B_+) \times B_-) \cup (B_+ \times$ bd$_{X_-}(B_-)) \cup (bd_{X_+}(B_+) \times$ bd$_{X_-}(B_-))$. By Lemma 3.1 the first two subsets bd$_{X_+}(B_+) \times B_-$ and $B_+ \times$ bd$_{X_-}(B_-)$ are contained in $G/K = G(x_+, x_-)$. So cl$_{X_+ \times X_-}(B_+ \times B_-) \cap \mathcal{O}$ is contained in bd$_{X_+}(B_+) \times$ bd$_{X_-}(B_-)$.

If cl$_{X_+ \times X_-}(B_+ \times B_-) \cap \mathcal{O}$ has an interior point in $\mathcal{O}$, then the image $\pi$ (cl$_{X_+ \times X_-}(B_+ \times B_-) \cap \mathcal{O}$) under the projection $\pi : \mathcal{O} \to X_+$ would...
have an interior point in $X_+$ because $\pi$ is $G$-equivariant and surjective. But $\pi(\cl_{X_+ \times X_-}(\mathcal{B}_+ \times \mathcal{B}_-) \cap \mathcal{O})$ is contained in $\pi(\bd_{X_+}(\mathcal{B}_+)) = \bd_{X_+}(\mathcal{B}_+)$ which has no interior point in $X_+$. \hfill \Box$

Let us now turn to the

Proof of Theorem 2.4 in the case where $G_0$ is of Hermitian type. Let $E = X \setminus \Omega$. We will show that $\cl_X(U) \cap E$ has no interior point in $E$. Let $X_0 = X_+ \times X_-$ be the $G$-equivariant compactification of $\Omega$ considered above and put $E_0 = X_0 \setminus \Omega$. Since $X$ is $G$-equivariantly birationally equivalent to $X_0$, we have the following diagram:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & \mathcal{X}_0 \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{X} & \xrightarrow{\pi} & \mathcal{X}_0 \\
\end{array}
$$

Let $\pi : \mathcal{X} \to \mathcal{X}_0$ and $p : \mathcal{X} \to \mathcal{X}$ denote the respective proper modifications.

Assume that $\bd_X(U) \cap E$ has an interior point in $E$. Then $\bd_X(U) \cap \mathcal{O}$ has an interior point in $\mathcal{O}$ for some $G$-orbit $\mathcal{O}$ in $E$ of codimension 1 in $X$. Therefore the restriction $p : \mathcal{O} := p^{-1}(\mathcal{O}) \to \mathcal{O}$ is biholomorphic, because the indeterminant locus of $p$ has codimension $\geq 2$. The other projection $\pi(\mathcal{O}) : \mathcal{O} \to \mathcal{X}_0 \subset \mathcal{X}. Since $\pi(\mathcal{O})$ is an open map, $\pi(\bd_X(U) \cap \mathcal{O})$ has an interior point in $\mathcal{O}_0$, contrary to Proposition 3.2 \hfill \Box$

4 Non-Hermitian case

For the remainder of this paper we assume that $G_0$ is not of Hermitian type. Our work is devoted to proving Theorem 2.4 in that case. Here we let $X := X^W$ be the DeConcini-Procesi compactification of $\Omega = G/K$ (DeCP). Orbits in the boundary $E := X^W \setminus \Omega$ are denoted by $O_I$, where $I$ is a subset of $\{1, \ldots, r\}$, and $S_I := \cl(O_I)$. Recall that for every such $I$ the compactification $X^W$ is realized in $\mathbb{P}(V_I) \times P(V_J)$ where $J = \{1, \ldots, r\} \setminus I$, and that the projection on the first factor defines a $G$-equivariant morphism $\pi_I : X \to \mathbb{P}(V_I)$ to the projective space of the irreducible representation space $V_I$. 

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The restriction $\pi_I|S_I : S_I \to C_I = G/P_I$ is a fiber bundle whose fiber is the DeConcini-Procesi compactification of an affine symmetric space of a root theoretically distinguished Levi-factor of $P_I$.

Since we have supposed that $G_0$ is not of Hermitian type, the image $X_I := \text{Im}(\pi_I)$ is another $G$-equivariant compactification of the affine symmetric space $G/K$. In this case we denote by $\Omega_I$ the open $G$-orbit in $X_I$.

### 4.1 Extending sections

Let $H_I$ be the restriction to $X_I$ of the hyperplane bundle $H$ of $\mathbb{P}(V_I)$. Since the vector space of sections of $H$ is an irreducible $G$-representation space, it follows that the restriction map defines isomorphisms

$$\Gamma(\mathbb{P}(V_I), H) \cong \Gamma(X_I, H_I) \cong \Gamma(C_I, H|C_I).$$

If $L_I := \pi_I^*H_I$, then, using the isomorphism $\Gamma(S_I, L_I|S_I) \cong \Gamma(C_I, H_I|C_I)$, we see that the restriction map

$$R_I : \Gamma(X, L_I) \to \Gamma(S_I, L_I|S_I)$$

is surjective. We note that $R_I$ is $G$-equivariant. Therefore we may choose a $G$-invariant irreducible representation subspace of $\Gamma(X, L_I)$ which is mapped isomorphically onto $\Gamma(S_I, L_I|S_I)$. In particular, if $B$ is a Borel subgroup of $G$ and $s_0$ is a $B$-eigenvector in $\Gamma(S_I, L_I|S_I)$, then there is a $B$-eigenvector $t_0 \in \Gamma(X, L_I)$ with $R_I(t_0) = s_0$.

**Proposition 4.1.** If $s \neq 0 \in \Gamma(S_I, L_I|S_I)$ is the restriction $s = R_I(t)$, then the intersection of the support $|t|$ with $\Omega$ is not empty.

**Proof.** If not, then $|t|$ is the union of certain irreducible components of $X \setminus \Omega$. But $t|S_I = \pi_I^*(t_I)$ and $t_I$ is the restriction to $C_I$ of a unique section $\tilde{t}_I \in \Gamma(X_I, H_I)$. However, $\pi_I^*(\tilde{t}_I) = t$, and since $\pi_I|\Omega : \Omega \to \Omega_I$ is an isomorphism, it follows that $|t_I|$ is a union of certain components of $X_I \setminus \Omega_I$. Now, each such component contains the closed orbit $C_I = G/P_I$. Therefore $t_I = 0$ and consequently $s = 0$, contrary to assumption. \qed
4.2 Iwasawa-envelopes

Here we complete the proof of our main Theorem 2.1 by using properties of the Iwasawa-envelope of the universal domain. In order to this in $\Omega = G/K$, we fix an Iwasawa-Borel subgroup $B$ of $G$ and let $x_0 \in \Omega$ be a base point with $\Omega_0 := G_0(x_0) = G_0/K_0$ being the real symmetric space of basic interest.

Let $H_\Omega$ be the complement in $\Omega$ of the open $B$-orbit $B(x_0)$, consider the closed $G_0$-invariant set

$$F_\Omega := \bigcup_{k \in K_0} k(H_\Omega) = \bigcup_{g \in G_0} g(H_\Omega), \quad (1)$$

and define the Iwasawa-envelope $E_I(\Omega)$ to be the connected component containing $x_0$ of the complement of $F_\Omega$ in $\Omega$. We regard $E_I(\Omega)$ as an envelope of $U$ in $\Omega$, because every hypersurface $k(H_\Omega)$ is contained in its complement (11). In fact, the opposite inclusion also holds (13) and we have the following alternative description of $U$ which in fact holds even if $G_0$ is of Hermitian type.

**Proposition 4.2.** The Iwasawa-envelope $E_I(\Omega)$ agrees with the universal domain $U$.

Now we give the analogous definition of the Iwasawa-envelope for an arbitrary (algebraic) $G$-compactification $X$ of $\Omega$. For this let $H_X$ be the closure in $X$ of the hypersurface $H_\Omega$, and, replacing $H_\Omega$ by $H_X$, define $F_X$ in the same way as $F_\Omega$. Then $E_I(X)$ is defined to be the connected component containing $x_0$ of the complement of $F_X$ in $X$.

**Theorem 4.3.** If $X$ is an arbitrary $G$-equivariant compactification of $\Omega$, then $E_I(X) = E_I(\Omega)$.

This follows from Theorem 2.4 and Proposition 4.2 in the same way that Theorem 2.1 follows from Theorem 2.4 and Theorem 2.3. But when $G_0$ is not of Hermitian type, Theorem 4.3 is an immediate consequence of the following result.

**Theorem 4.4.** If $X$ is an arbitrary equivariant compactification of $\Omega$ and $G_0$ is not of Hermitian type, then

$$F_X = \bigcup_{k \in K_0} k(H_X) = \bigcup_{g \in G_0} g(H_X)$$

contains the full complement $E = X \setminus \Omega$.
Before turning to the proof, let us first prove a preparatory result which strongly uses the assumption that $G_0$ is not of Hermitian type.

**Lemma 4.5.** Assume that $G_0$ is not of Hermitian type and let $G/P$ be a $G$-flag manifold. If $B$ is an Iwasawa-Borel subgroup of $G$ and $H$ is the complement of the open $B$-orbit in $G/P$, then

$$F := \bigcup_{k \in K_0} k(H) = \bigcup_{g \in G_0} g(H)$$

is equal to $G/P$.

**Proof.** Recall that in every open $G_0$–orbit $\gamma$ in $G/P$ there exists a unique complex $K_0$–orbit $C$. Since $G_0$ is not of Hermitian type, such cycles $C$ are positive-dimensional.

Now the complement of $H$ in $G/P$ is algebraically equivalent to an affine space $\mathbb{C}^n$ which contains no positive dimensional subvarieties. Thus $H \cap C \neq \emptyset$ for every base cycle $C$ in every open $G_0$–orbit $\gamma$. The desired result then follows from the facts that the union of the open $G_0$–orbits is dense and $F$ is closed. 

The DeConcini-Procesi compactification $X^W$ of $\Omega = G/K$ plays a special role in the proof of Theorem 4.4.

**Proof of Theorem 4.4 for $X = X^W$.** Given $I$ as in §4.1, we show that $F_{X^W} \supset S_I$. For this let $H$ be the complement of the open $B$-orbit in $C_I = G/P_I$ as in the above Lemma. By Proposition 4.1 (and the brief discussion previous to it) its pullback $H_I$ to $S_I$ is the zero-set of the restriction of a $B$-eigensection $t \in \Gamma(X^W, L_I)$ with $|t| \cap \Omega \neq \emptyset$.

Since $|t| \cap \Omega$ is contained in $H_\Omega$, it follows that $H_I \subset H_{X^W}$. Now by Lemma 4.5 we know that $\bigcup_{k \in K_0} k(H) = C_I$. Thus it is immediate that

$$F_{X^W} = \bigcup_{k \in K_0} k(H_{X^W}) \supset \bigcup_{k \in K_0} k(H_I) = S_I$$

which completes the proof for $X^W$. 

Now let $X$ be any (algebraic) $G$-equivariant compactification of $\Omega$. Since it is $G$-equivariantly birationally equivalent to $X^W$, we have the following diagram:

$$\begin{array}{ccc}
\hat{X} & \leftarrow & X^W \\
\downarrow & & \downarrow \\
X & & X
\end{array}$$
Let \( \pi : \tilde{X} \to X^W \) and \( p : \tilde{X} \to X \) denote the respective proper modifications.

**Proof of Theorem 4.4 for arbitrary \( X \).** It is enough to prove this for \( \tilde{X} \), because \( p(F_{\tilde{X}}) = F_X \). Now \( \pi : \tilde{X} \to X^W \) is a \( G \)-equivariant proper modification. Since every Borel subgroup \( B \) in \( G \) also has an open orbit in \( \tilde{X} \), there are also only finitely many \( G \)-orbits in \( \tilde{X} \) and it follows that the preimage \( \tilde{S}_I = \pi^{-1}(S_I) \) is also the closure of a \( G \)-orbit.

Furthermore, if \( H_I = |t| \cap S_I \) is a \( B \)-invariant hypersurface which is defined by a \( B \)-eigenvector \( t \in \Gamma(X^W, L_I) \) as in the proof for \( X^W \), then the corresponding section \( |\tilde{t}| \in \Gamma(\tilde{X}, \tilde{L}_I) \) of the pullback bundle \( \tilde{L}_I := \pi^*L_I \) has the analogous properties. Namely, the intersection \( |\tilde{t}| \cap \tilde{\Omega} \) of its support with the open \( G \)-orbit in \( \tilde{\Omega} \) is nonempty, and \( |\tilde{t}| \cap \tilde{S}_I \) is a \( B \)-invariant hypersurface \( \tilde{H}_I \) in \( \tilde{S}_I \). Thus \( \tilde{H}_I \) is a subset of the hypersurface \( \tilde{H}_I \) which defines the Iwasawa-envelope in \( \tilde{\Omega} \).

Finally, since the hypersurfaces \( k(H_I) \) cover \( O_I \) (and therefore \( S_I \)) as \( k \) runs over \( K_0 \) and the hypersurfaces \( k(\tilde{H}_I) \) are just their \( \pi \)-preimages, it follows that the hypersurfaces \( k(\tilde{H}_I), k \in K_0 \), cover \( \tilde{S}_I \). This shows that \( F_{\tilde{X}} \supset \tilde{S}_I \) which completes the proof in the case of an arbitrary compactification \( X \). \( \square \)

As a consequence of Theorem 4.4 we are now able to give the

**Proof of Theorem 2.4 in the case where \( G_0 \) is not of Hermitian type.** Let \( V \) be the interior of the closure \( \text{cl}_X(\mathcal{U}) \) and observe that \( V \cap \Omega = \mathcal{U} \). In particular, the intersection \( V \cap k(H_X) = \emptyset \) for all \( k \in K_0 \). On the other hand, if \( p \in V \cap \Omega \), it follows from Theorem 4.4 that \( p \) is in some such \( k(H_X) \) and \( V \cap k(H_X) \neq \emptyset \). Thus no point of \( V \) is in \( E \) and the first statement of Theorem 2.4 follows. As we explained directly after the statement of Theorem 2.4, the second statement is an immediate consequence of the first. \( \square \)

As we explained in \( \square \) Theorem 2.4 implies that for every open \( G_0 \)-orbit \( D \) of an arbitrary real form \( G_0 \) in an arbitrary \( G \)-flag manifold \( Z \) the group theoretically defined cycle space \( \mathcal{M}_D \) is closed in the full cycle space \( \mathcal{C}_q(D) \) and therefore the proof of our main result is complete.

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