Quantum state engineering in arrays of nonlinear waveguides

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The optical continuous variable (CV) framework is a true contender for quantum communication and quantum information processing [1]. Demonstrations of protocols putting the field fluctuations to good use have been achieved since 1992 [2–7] towards large-scale entangled states for quantum computing [8, 9]. However, all these major advances were achieved with table-top experiments and for research purposes only. Integrated optics can fill the gap towards the development of real-world quantum technologies [10]. Entanglement and superposition underpin the advantage of quantum protocols. Thus, an integrated synthesizer of multimode entangled states is a key component for the boost of quantum technologies. Entanglement on chip has been demonstrated for two parties [11], in an integrated version of the bulk optics implementations that cascade squeezers and beamsplitters [12]. We discuss here the versatility of a monolithic device—without any bends in the active region nor specific functionalized regions—where nonlinearity and coupling act simultaneously: the array of nonlinear waveguides (ANW) [13].

Such arrays have been used in the discrete variable domain [14, 15] and have been proposed to achieve specific multimode states in the CV domain with \( N = 2 \) waveguides [16, 17] or scaling up the number of waveguides [18–20]. Recently, the ANW has been proposed as a versatile source to engineer tailored cluster states for measurement-based quantum computing [21]. Here we provide a complementary perspective by exploring both the mathematical backbone of these results and the physical insight that can be obtained from it. We show how entanglement and squeezing manifests in optical modes that are produced by the ANW, and we derive possible tuning parameters, derive analytical and semi-analytical solutions, and harness them to find a good working point for the generation of linear clusters.

Our framework is based on the choice and use of different sets of available eigenmodes of the ANW which simplify the dynamics of the system and open up new possibilities to encode quantum information. The use of eigenmodes is the key to demonstrate tunable multiparticle entanglement in the frequency domain [22, 23]. We establish parallels between the ANW and such frequency combs, thus connecting spatial and frequency encoding through a joint mathematical framework. We further show that full engineering of multimode squeezed states can be achieved in the spatial domain i) by specific design of the ANW shaping the nonlinearity and coupling, and designing a suitable phase matching (as suggested before [21] and further exemplified here); and externally, ii) by adjusting the pumping profile and by adapting the measurement strategies of the output fields.

The main conceptual, practical and technological assets of the ANW in comparison with other platforms are i) the large number of degrees of freedom available that enable to reconfigure its operation, ii) a number of analytical solutions based on symmetries present in arrays of waveguides that are a guide to develop specific quantum protocols, iii) the possibility to encode quantum information in the individual mode basis or in any other basis based on linear combinations of individual modes, iv) the small footprint, from few millimeters to centimeters, and v) the simplicity of the pumping-detection optical setup that can be based on available telecom fiber-optic components. Notably, quantum information encoded in the individual spatial modes can be distributed to different locations of a quantum network in a natural way, which is harder to implement in the frequency domain [24].

The article is organized as follows: we first derive
the governing equations of propagation and squeezing in ANW in section [II]. We then detail and illustrate an inventory of the tuning parameters for squeezing and entanglement engineering in section [III]. Notably, we give in this section [III] various sets of analytical solutions that provide considerable insight on the impact of the pumping phase profile on the generation of entanglement. We build on this formalism and intuition to demonstrate multimode squeezing in section [IV] and detail the possibilities to produce linear cluster states in ANW in section [V].

II. THE ARRAY OF NONLINEAR WAVEGUIDES: POSSIBLE ENCODINGS AND GENERAL SOLUTIONS

A. The array of nonlinear waveguides

The array of nonlinear waveguides consists of $N$ identical $\chi^{(2)}$ waveguides in which degenerate spontaneous downconversion (SPDC) and evanescent coupling between the generated fields take place. The array can be made up of, for instance, periodically poled lithium niobate (PPLN) waveguides as sketched in Figure 1a. In each waveguide, an input harmonic field at frequency $\omega_h$ is type-0 downconverted into a signal field at frequency $\omega_s$. We consider that the phase matching condition $\Delta \beta \equiv \beta(\omega_h) - 2\beta(\omega_s) = 0$, with $\beta(\omega_{h,s})$ the propagation constant at frequency $\omega_{h,s}$, is fulfilled all along the coupling zone and in the coupling zone only. The energy of the signal modes propagating in each waveguide is exchanged between the coupled waveguides through evanescent waves, whereas the interplay of the second harmonic waves is negligible for the considered propagation lengths due to their high confinement into the guiding region. We set our calculation in the reasonable regime of pump undepletion [17]. We consider a general array of $N$ identical waveguides and continuous-wave propagating fields. The physical processes involved are then described by the [24]

$$\frac{d\hat{A}_j}{dz} = iC_0 (f_j - \hat{A}_{j-1} + f_j \hat{A}_{j+1}) + 2i\eta_j \hat{A}_j^{\dagger},$$

where $A_0 = 0$ and $\hat{A}_{N+1} = 0$, $f_0 = f_N = 0$ and $j = 1, \ldots, N$ is the individual mode index. $\hat{A}_j \equiv \hat{A}_j(z, \omega_s)$ are monochromatic slowly-varying amplitude annihilation operators of signal (s) photons corresponding to the $j$th waveguide – in the individual mode basis – $[\hat{A}_j(z, \omega), \hat{A}^{\dagger}_j(z', \omega')] = \delta(z - z')\delta(\omega - \omega')\delta_{j,j'}$. The effective nonlinear coupling constant corresponding to the $j$th waveguide is given by $\eta_j = g \alpha_{h,j}$, where $g$ is the nonlinear constant – proportional to $\chi^{(2)}$ – and to the spatial overlap of the signal and harmonic fields in each waveguide and $\alpha_{h,j}$ is the strong coherent undepleted pump field propagating in the $j$th waveguide. The parameters $\eta_j$ can be tuned by means of a suitable set of pump phases and amplitudes at each waveguide. $C_j = C_0 f_j$ is the linear coupling constant between modes $j$ and $j+1$, and $z$ is the coordinate along the direction of propagation. Both the coupling and nonlinear constants depend on the set signal frequency, $C_0 \equiv C_0(\omega_s)$ and $g \equiv g(\omega_s)$, and they are taken as real without loss of generality.

Since we are interested in CV squeezing and entanglement, we will also use along the paper the field quadratures $\hat{x}_j$ and $\hat{y}_j$, where $\hat{x}_j = (\hat{A}_j + \hat{A}^{\dagger}_j)$ and $\hat{y}_j = i(\hat{A}^{\dagger}_j - \hat{A}_j)$ are, respectively, the amplitude and phase quadratures corresponding to a signal optical mode $A_j$ (Figure 1a). The system of equations (1) in terms of the individual-modes quadratures can be rewritten in compact form as

$$\frac{d\hat{\xi}}{dz} = \Delta(z) \hat{\xi},$$

where $\Delta(z)$ is a $2N \times 2N$ matrix of coefficients and $\hat{\xi} = (\hat{x}_1, \ldots, \hat{x}_N, \hat{y}_1, \ldots, \hat{y}_N)^T$.

In general, either Equation (1) or Equation (2) can be solved numerically for a specific set of parameters $(C_j, \eta_j, N)$, or even analytically if $N$ is small. However, it is difficult to gain physical insight from numerical or low-dimension analytical solutions due to the increasing complexity of the system with the number of waveguides. We propose below and use throughout the paper two modal approaches – complementary to the individual-mode approach – that enlighten the problem of propagation in ANWs. We thus use

i) **Linear (propagation) supermodes**, i.e. the eigenmodes of the corresponding linear array of waveguides assuming $\eta_j \propto g = 0$. In the actual array of nonlinear waveguides where $\eta_j \propto g \neq 0$, these modes are squeezed and coupled through the nonlinearity. This basis has analytical solutions independently of the number $N$ of waveguides for specific pump-field distributions. We show these solutions in section [III].

ii) **Nonlinear (squeezing) supermodes**, i.e. the eigenmodes of the full nonlinear system. These modes are squeezed and by construction fully decoupled but $z$-dependent. We point out that, in some cases (see section [III.B.1]), both linear and nonlinear supermodes are degenerate up to local phases.

![Figure 1](image-url)

FIG. 1. a) Sketch of an array of nonlinear waveguides based on a PPLN waveguide array made up of nine waveguides working in a SPDC configuration pumping the central waveguide. Propagating pump field in blue. Evanescently-coupled SPDC signal fields in red. Quantum noise variances and correlations are measured by multimode balanced homodyne detection. b) Example of quadrature correlations between two individual SPDC signal modes $j$ and $j'$. 

[17]
and the diagonal basis of nonlinear supermodes is obtained directly from them. In contrast, in arrays of nonlinear waveguides the generator shown here for the sake of comparison. In frequency combs, the coupling between the individual modes is nonlinear

\[ \hat{L}_{j,j'} = \text{sinc}[(\omega_j + \omega_{j'})]n(\omega_j + \omega_{j'}) \]

and nonlinear supermode bases, work out the correspond-

\[ M = \hbar \sum_{j=1}^{N} \left\{ C_0(f_{j-1}A_{j-1}A_{j}^\dagger + f_{j+1}A_{j+1}A_{j}^\dagger) + \eta_j A_{j}^\dagger A_{j}^2 + H.c. \right\} \]

\[ \hat{B}_{S,k} = \hat{A}_{S,k} e^{-i\lambda_k z} = \sum_{j=1}^{N} M_{k,j} \hat{A}_j e^{-i\lambda_k z} \]

\[ \hat{M}_S^Z = -i \hbar^2 \sum_{k,k'=1}^{N} \hat{L}_{k,k'}(z) \hat{B}_{S,k}^\dagger \hat{B}_{S,k'}^\dagger + H.c. \]

\[ \hat{C}_{S,m} = \sum_{k,j=1}^{N} (T_{m,k}^z M_{k,j} e^{i(\delta_j - \lambda_k + \lambda_{k'}) z}) \]

These two complementary approaches connect our work and our result of spatial multimode squeezed states exhibited in sections [4] and [5] to the spectral [22, 23, 26], spatial [27, 29] or temporal [30, 31] modes of previous works. In the next section [11], we introduce both linear and nonlinear supermode bases, work out the corresponding propagation equations, and give the general solution to the propagation problem. Furthermore, we use the relationship between the two bases to draw mathematical parallels with SPDC frequency modes [24, 32, 33].

B. Propagation equations

The general solutions to the propagation in ANWs have been recently introduced in [21]. Below we present a detailed calculation of those solutions in both the complex and quadratures representation of the optical fields, and compare our spatial domain solutions with those obtained in similar physical systems working with frequency modes.

1. Complex optical fields

Considering coupling only between nearest-neighbor waveguides, a linear waveguide array – i.e. Equation (1) with \( \eta_j = 0 \) – presents supermodes \( \hat{A}_{S,k} \), i.e. propagation eigenmodes [34]. In general, any linear waveguide array is represented by a Hermitian tridiagonal matrix – Jacobi matrix – with non-negative entries and thus by a set of non-degenerate eigenvalues and eigenvectors given in terms of orthogonal polynomials [35]. These eigenvectors, that we call the linear supermodes, form a basis and are represented by an orthogonal matrix \( M \) with real elements \( M_{k,j} \). The individual modes of the waveguides and the linear supermode basis are thus related by

\[ \hat{A}_{S,k} = \sum_{j=1}^{N} M_{k,j} \hat{A}_j. \]

The supermodes are orthonormal

\[ \sum_{j=1}^{N} M_{k,j} M_{k',j}^* = \delta_{k,k'}, \]  

with a spectrum of eigenvalues \( \lambda_k \). We consider here a constant coupling along propagation \( C_j(z) = C_j \). Note that in the case of z-dependent coupling strength \( C_j(z) \), the eigenmode basis becomes local with \( M \equiv M(z) \). Equation (1) for the nonlinear waveguide array can be written as

\[ \frac{d\hat{A}_{S,k}}{dz} = i \sum_{j=1}^{N} \sum_{k'=1}^{N} M_{k,j} M_{k',j}^* (\lambda_k \hat{A}_{S,k'} + 2\eta_j \hat{A}_{S,k'}^\dagger), \]

in the supermode basis, where we have used the eigenvalue condition \( C_0(f_{j-1}M_{k',j-1} + f_{j+1}M_{k',j+1}) = \lambda_k M_{k',j} \). Using slowly-varying supermode amplitudes \( \hat{B}_{S,k} = \hat{A}_{S,k} e^{-i\lambda_k z} \), and the orthogonality of the supermodes Equation (3), the following propagation equation is obtained

\[ \frac{d\hat{B}_{S,k}}{dz} = 2i \sum_{j=1}^{N} \sum_{k'=1}^{N} \eta_j M_{k,j} M_{k',j}^* \hat{B}_{S,k'}^\dagger e^{-i(\lambda_k + \lambda_{k'}) z}. \]  

The momentum operator in the interaction picture which produces Equation (4) by means of the Heisenberg equa-
tions $d\hat{B}_{S,k}/dz = (i/\hbar)\hat{\mathcal{M}}_{S,k}^{\dagger} \hat{B}_{S,k}$ is thus

$$\mathcal{M}_{S}^{\dagger} = -i\hbar \frac{\tilde{\eta}}{2} \sum_{k,k'=1}^{N} \mathcal{L}_{k,k'}(z) \hat{B}_{S,k}^{\dagger} \hat{B}_{S,k'} + H.c.$$

The coupling matrix $\mathcal{L}(z)$ is the local joint-spatial supermode distribution of the ANW and its elements are given by

$$\mathcal{L}_{k,k'}(z) = 2\pi \sum_{j=1}^{N} |\eta_j| \hat{M}_{k,j} \hat{M}_{k',j} e^{i\phi_j - (\lambda_k + \lambda_{k'})z}, \quad (5)$$

with $\eta_j = |\eta_j| e^{i\phi_j}$ and $\tilde{\eta}$ an arbitrary parameter, e.g. the highest $|\eta_j|$. $\eta_j$ can also be a function of the position $\eta_j \equiv \eta_j(z) = g(z) \alpha_{n,j}$ [36]. $\mathcal{L}(z)$ is a complex symmetric matrix which gathers all the information about the spatial shape of the pump, i.e. amplitudes and phases in each waveguide, and the signal supermodes coupling. Finally, the Heisenberg equations in the linear supermode basis can be simply written as

$$\frac{d\hat{B}_{S,k}}{dz} = \tilde{\eta} \sum_{k'=1}^{N} \mathcal{L}_{k,k'}(z) \hat{B}_{S,k'}. \quad (6)$$

The formal solution to Equation (6) is given by

$$\begin{pmatrix} \hat{B}_{S}(z) \\ \hat{B}_{S}^{\dagger}(z) \end{pmatrix} = \exp \left( \tilde{\eta} \int_{0}^{z} \mathcal{L}(z')dz' \right) \begin{pmatrix} \hat{B}_{S}(0) \\ \hat{B}_{S}^{\dagger}(0) \end{pmatrix}, \quad (7)$$

with $\hat{B}_S = (\hat{B}_{S,1}, \ldots, \hat{B}_{S,N})^T$. This solution displays the effect of the nonlinearity on the linear supermodes: z-dependent amplification and coupling. A simpler solution can be obtained diagonalizing the matrix argument of the exponential in Equation (7) through the nonlinear supermodes. In general, the linear supermode basis does not diagonalize the propagation in the ANWs and the solution of Equation (7) is configuration-dependent. However, Equation (6) presents analytical solutions independently of the dimension $N$ for specific pump-field distributions as we show in section III.

A feature of the ANWs is that the evanescent coupling produces a phase mismatch between the pump and the generated signal waves which results in a z-dependent interaction, in such a way that the eigenmodes of the full nonlinear system –the nonlinear supermodes– are local. This coupling-based phase mismatch affects the amount of squeezing and entanglement generated in the ANWs. The local nonlinear supermode basis displays independently squeezed modes and helps to quantify the amount of nonclassicality generated in the array at different propagation distances. The local nonlinear supermodes $\hat{C}_S$ basis is such that

$$\hat{C}_{S,m} = \sum_{k=1}^{N} \mathcal{T}_{m,k}(z) \hat{B}_{S,k}, \quad (8)$$

where $[\hat{C}_{S,m}(z), \hat{C}_{S,m'}(z')] = \delta(z - z') \delta_{m,m'}$, and $\mathcal{Y}(z)$ is an unitary matrix which diagonalizes the complex symmetric matrix $\int_{0}^{z} \mathcal{L}(z')dz'$ by a congruence transformation – the Autonne-Takagi transformation [37], such that

$$\mathcal{Y}(z) \int_{0}^{z} \mathcal{L}(z')dz' \mathcal{Y}^T(z) = \hat{\Lambda}(z), \quad (9)$$

with $\hat{\Lambda}(z)$ a local diagonal matrix with non-negative real entries. Applying Equations (8) and (9) on Equation (7) we obtain a simple solution in the diagonal local basis

$$\hat{C}_{S,m}(z) = \cosh[\tilde{r}_m(z)] \hat{C}_{S,m}(0) + \sinh[\tilde{r}_m(z)] \hat{C}_{S,m}^\dagger(0). \quad (10)$$

Each local nonlinear supermode is a single-mode squeezed state. The $\tilde{r}_m(z)$ are $\Lambda_{m,m}(z)$ the down-conversion gains at a propagation distance $z$ and quantify the available nonlinearity and thus squeezing. The relation between the nonlinear supermodes and the individual modes is

$$\hat{C}_{S,m} = \sum_{k=1}^{N} \sum_{j=1}^{N} (\mathcal{Y}_{m,k}(z) \hat{M}_{k,j} e^{-i\lambda_k z}) \hat{A}_j. \quad (11)$$

This expression encapsulates the mechanisms at play in the ANWs: the evanescent coupling generates the linear supermodes ($\hat{M}_{k,j}$) which get a phase due to propagation ($\lambda_k z$) and the nonlinearity couples them locally ($\mathcal{Y}_{m,k}(z)$). In terms of the individual modes, the solution to the nonlinear system is

$$\hat{A}_j(z) = \sum_{k,m,j'=1}^{N} (\hat{M}_{j,k} \mathcal{Y}_{k,m}(z) \hat{M}_{m,j'} e^{i\lambda_{k} z}) \cosh[\tilde{r}_m(z)] \hat{A}_j(0) + \sinh[\tilde{r}_m(z)] \hat{A}_j^\dagger(0). \quad (11)$$

Equations (7), (10) and (11) are the general solutions to the propagation problem in ANW in the linear supermodes, nonlinear supermodes and individual mode bases, respectively. These three solutions represent a resource for encoding quantum information. Particularly, Equation (11) is a useful tool in the DV framework to explore further, for instance, driven quantum walks [38].

Remarkably, the kind of equations that we find here for spatial modes are formally similar to those that appear in the context of SPDC in frequency combs [21][32][33]. We draw parallels between the spectral approach that is used in ANW in Table I. In frequency combs, the interaction, in such a way that the eigenmodes of the full nonlinear system –the nonlinear supermodes– are local. This coupling-based phase mismatch affects the amount of squeezing and entanglement generated in the ANWs. The local nonlinear supermode basis displays independently squeezed modes and helps to quantify the amount of nonclassicality generated in the array at different propagation distances. The local nonlinear supermodes $\hat{C}_S$ basis is such that

$$\hat{C}_{S,m} = \sum_{k=1}^{N} \mathcal{T}_{m,k}(z) \hat{B}_{S,k}, \quad (8)$$

where $[\hat{C}_{S,m}(z), \hat{C}_{S,m'}(z')] = \delta(z - z') \delta_{m,m'}$, and $\mathcal{Y}(z)$ is an unitary matrix which diagonalizes the complex symmetric matrix $\int_{0}^{z} \mathcal{L}(z')dz'$ by a congruence transformation – the Autonne-Takagi transformation [37], such that

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Each local nonlinear supermode is a single-mode squeezed state. The $\tilde{r}_m(z)$ are $\Lambda_{m,m}(z)$ the down-conversion gains at a propagation distance $z$ and quantify the available nonlinearity and thus squeezing. The relation between the nonlinear supermodes and the individual modes is

$$\hat{C}_{S,m} = \sum_{k=1}^{N} \sum_{j=1}^{N} (\mathcal{Y}_{m,k}(z) \hat{M}_{k,j} e^{-i\lambda_k z}) \hat{A}_j. \quad (11)$$

This expression encapsulates the mechanisms at play in the ANWs: the evanescent coupling generates the linear supermodes ($\hat{M}_{k,j}$) which get a phase due to propagation ($\lambda_k z$) and the nonlinearity couples them locally ($\mathcal{Y}_{m,k}(z)$). In terms of the individual modes, the solution to the nonlinear system is

$$\hat{A}_j(z) = \sum_{k,m,j'=1}^{N} (\hat{M}_{j,k} \mathcal{Y}_{k,m}(z) \hat{M}_{m,j'} e^{i\lambda_{k} z}) \cosh[\tilde{r}_m(z)] \hat{A}_j(0) + \sinh[\tilde{r}_m(z)] \hat{A}_j^\dagger(0). \quad (11)$$

Equations (7), (10) and (11) are the general solutions to the propagation problem in ANW in the linear supermodes, nonlinear supermodes and individual mode bases, respectively. These three solutions represent a resource for encoding quantum information. Particularly, Equation (11) is a useful tool in the DV framework to explore further, for instance, driven quantum walks [38].

Remarkably, the kind of equations that we find here for spatial modes are formally similar to those that appear in the context of SPDC in frequency combs [21][32][33]. We draw parallels between the spectral approach that leads to multimode entanglement and our spatial approach in ANW in Table I. In frequency combs, the individual modes are a discrete set of $N$ frequency modes $\tilde{A}_j$ that are nonlinearly coupled in a bulk crystal with a quadratic nonlinearity. The diagonalization of the corresponding coupling matrix $\hat{L}$ produces a set of nonlinear supermodes $\tilde{b}_m$, whose eigenvalues $\eta \Lambda_{m,m}$ are proportional to SPDC gains. Table I (left) shows the main elements involved in frequency-comb SPDC and the related Hamiltonian in the individual $\hat{H}$ and nonlinear supermode $\hat{H}_S$ basis. In ANWs the evanescent coupling
between the individual modes $\hat{A}_j$ generates the linear
supermodes $\hat{A}_{S,k}$ and the nonlinear coupling mediated
by the pump fields mixes them. Table I (right) shows the
main elements involved in spatial ANWs and the related
momenta in the individual $\mathcal{M}_i$ linear supermode
basis $\mathcal{M}_{S_i}$, and nonlinear supermode $\mathcal{M}_S$ basis.
Note that $\mathcal{M}_S$ does not represent a real dynamical generator.
It is indeed a formal squeezing momentum with singular
values $\Lambda_{m,m}(z) = d^\dagger_{z} \Lambda_{m,m}(z)$ defined only at a set
$z$ that we define for the sake of comparison. The coupling
matrix $L(z)$ is defined here in the linear supermode
basis and the diagonalization of $\int_0^\infty L(z') dz'$ produces a
set of nonlinear supermodes $\hat{C}_{S,m}$. The consequence of
this diagonalization in two steps to get to nonlinear cou-
pling (see Table I) is that the nonlinear supermodes are
$z$-dependent – i.e. local. At each propagation plane $z$ a
different set of nonlinear supermodes diagonalizes Equation (7)
with SPDC gains $\tilde{r}_m(z)$. This feature is the main
conceptual difference between frequency combs and spa-
tial (7) with SPDC gains $\tilde{r}_m(z)$. The formal solution of this equation is
the full evolution of the system is obtained by solving
Equation (2). The formal solution of this equation is
given by [39]. The nonlinear supermodes are obtained from the complex rep-
presentation of $\hat{A}_1$. The most interesting observables in Gaussian CV are the
second-order moments of the quadrature operators, prop-
erly arranged in the covariance matrix $V$ [40]. The ele-
ments of this matrix can be efficiently measured by means of
homodyne detection. For a quantum state initially in
vacuum, the covariance matrix at any plane $z$ is given by
$V(z) = S(z) S^T(z)$, with $S$ the value of the shot noise
related to each quadrature in our notation. Evolution of variances $V(\xi, \xi)$ and quantum correlations $V(\xi, \xi)$
can be obtained at any length from the elements of this matrix.
The covariance matrix can also be computed from the Bloch-Messiah decomposition as
$$V(z) = R_1(z) K^2(z) R_1^T(z).$$
Thus, $K^2(z)$ is the covariance matrix in the nonlinear su-
permode basis and $R_1(z)$ the symplectic transformation matrix between the individual and nonlinear supermode
basis [equivalent to Equation (11) for complex fields]. The $n$th nonlinear supermode is squeezed and thus non-
classical if $K^2_{N+n+m}(z) = e^{-2r_m(z)} < 1$, and the small-
est value of $K^2_{N+n+m}(z)$ is called the generalized squeezed variance and it is a measure of the nonclassicality of the quantum state [11].
Note that the complex and real approaches are equiv-
alent [37]. The first method is applied to the complex
joint-spatial supermode distribution and is numerically
easier to compute. It gives the relative downconversion
rates and therefore the amount of squeezing available in
the ANW. The second method is applied to the propa-
gator in the symplectic form and it enables to work out
directly the noise properties of the quantum state. We
detail how these noise properties can be engineered in the
following sections.

III. ENGINEERING TOOLBOX FOR
PRODUCTION AND DETECTION OF
MULTIMODE SQUEEZING

To operate the ANW, several knobs are accessible experi-
mentally. A reconfigurable multimode shaper at pump
frequency inputs the desired profile $(\tilde{\Omega}, \phi)$ in the
array through a V-groove fiber array. Bent waveguides
conduct the pump modes to the periodically poled ANWs
where signal modes are generated and evanescently coupled.
The coupling profile, wavevector phasematching
poling period and coupling phasematching poling period
$(\tilde{f}, \Lambda_{AB}, \Lambda_{C})$ can be suitably engineered for a specific op-
eration mode. The output light is collected by V-groove
fiber arrays and directed to a multimode balanced homodyne
detector (BHD) where modes are measured using adapted local-oscillator (LO) phase and electronic gain
profiles $(\tilde{G}, \hat{G})$.

The class of ANWs which we introduced in section
IIIA thus presents a number of parameters that can be
engineered for a desired operation. The evanescent
coupling profile $\tilde{f} = (f_1, \ldots, f_N)$, the length of the sam-
ple $L$, the number of waveguides $N$ –and notably its

In terms of individual modes quadratures $\hat{x}_j$ and $\hat{y}_j$,
the full evolution of the system is obtained by solving
Equation (2). The formal solution of this equation is
given by
$$\hat{\xi}(z) = S(z) \hat{\xi}(0),$$
with $S(z) = \exp\left\{\int_0^z \Delta(z') \, dz'\right\}$. The propagator $S(z)$ is
a symplectic matrix which contains all the information
about the propagation of the quantum state of the sys-
tem. We can apply on it a Bloch-Messiah decomposition as follows [39]
$$S(z) = R_1(z) K(z) R_2(z),$$
where $R_1(z)$ and $R_2(z)$ are both ortho-
gonal and symplectic matrices and $K(z) = \text{diag}\{e^{r_1(z)}, e^{r_2(z)}, \ldots, e^{r_N(z)}, e^{-r_1(z)}, e^{-r_2(z)}, \ldots, e^{-r_N(z)}\}$
is a phase-squeezed diagonal matrix. The nonlinear supermodes are the same as those obtained equation [3]
by the Autonne-Takagi factorization and thus $K(z)$ is
given by [37]
$$K(z) = \exp\left\{\tilde{\eta} \begin{pmatrix} \hat{\Lambda}(z) & 0 \\ 0 & -\hat{\Lambda}(z) \end{pmatrix} \right\}.$$ Therefore we can identify the squeezing parameters as
$\tilde{r}_m(z) = \tilde{r}_m(z) \equiv \tilde{\eta} \hat{A}_{m,m}(z)$. The spatial profiles of the
nonlinear supermodes are obtained from the complex rep-
resentation of $R_1$. The quantum states generated in ANWs are Gaussian.
parity– and the poling periods \[36\] are built-in and cannot be tuned once the sample is fabricated. In contrast, the power and phase profile, given respectively by \[\vec{\eta} = (|\eta_1|, \ldots, |\eta_N|)\] and \[\vec{\phi} = (\arg(\eta_1), \ldots, \arg(\eta_N))\], the coupling strength \(C_0\) and the basis of detection can be set for a required operation or encoding of information. \(C_0 \equiv C_0(\omega_p)\) can indeed be adjusted by tuning the phase matching \(\Delta \beta \equiv \Delta \beta(T)\) with the temperature \(T\) of the sample and adjusting the frequency \(\omega_p\) of the pump laser accordingly to recover the degeneracy point \[33\]. We introduce below a number of engineering strategies related to the coupling, pumping, phase matching and detection parameters that can be used to produce and detect a desired multimode squeezed state:

i) In section III.A we review and extend the analytical expressions for the linear supermodes and the propagation constant for three specific coupling \(\vec{f}\) profiles. These coupling profiles are put to good use in section IV to exemplify different squeezing behaviors.

ii) In section III.B we establish analytical joint spatial supermode distributions for specific pumping \((\vec{\eta}, \vec{\phi})\) configurations, we deduce in three limiting cases analytical expressions for the covariance matrices that are valid for any ANW - any \(f\)-, for any number of waveguides \(N\) and any propagation distance \(z\). We display covariance matrices in these limiting cases and in an intermediate scenario. We further use and comment these propagation results on the modes when discussing the generation of squeezing and entangled states in section IV.

iii) In section III.C we describe engineering of the propagation solutions using dedicated phase matching in ANW (\(\Lambda_C\)) to favor specific supermodes towards entanglement generation.

iv) In section III.D we recall LO shaping in multimode BHD used to detect squeezing and entanglement in a given basis.

### A. Coupling profile engineering

As introduced section III.B, every set of nearest-neighbor coupled waveguides has a family of propagation supermodes given by a matrix \(M\). The slowly varying amplitude corresponding to the \(k\)th supermode propagates along the array with a propagation constant \(\lambda_k\). Each family of linear supermodes depends on the coupling profile \(\vec{f}\). The engineering of this profile enables a specific operation or logic gate \[33\]. A number of demonstrations with optical lattices has been exhibited over the last years \[44–46\]. Very recently, the production of topologically protected quantum states in a Su-Schrieffer-Heeger lattice has been demonstrated \[37\].

A summary of properties of the supermodes can be found in ref. \[48\]. Particularly, every family of supermodes corresponding to an array of identical waveguides fulfill the following relations

\[
\lambda_k = -\lambda_{N+1-k}, \quad (13)
\]

\[
M_{N+1-k,j} = (-1)^{j+1} M_{k,j}. \quad (14)
\]

We label the supermodes connected two by two by Equation \[13\] as side supermodes \((k, N+1-k\). In arrays with an odd number of waveguides there is also a central supermode \(k = (N+1)/2 \equiv l\) with propagation constant \(\lambda_l = 0\). We thus refer to it as the zero supermode.

Applying the above relations in the orthonormalization condition Equation \[3\], we find the following modified orthonormality conditions

\[
\sum_{j=1}^{N} (-1)^j M_{k,j} M_{k',j} = \delta_{k,N+1-k'}, \quad (15)
\]

\[
\sum_{2 \leq 2j \leq N} M_{k,2j} M_{k',2j} = \frac{1}{2} (\delta_{k,k'} - \delta_{k,N+1-k'}), \quad (16)
\]

\[
\sum_{1 \leq 2j-1 \leq N} M_{k,2j-1} M_{k',2j-1} = \frac{1}{2} (\delta_{k,k'} + \delta_{k,N+1-k'}). \quad (17)
\]

These relations are general and, notably, they are instrumental to configure the pump to obtain simple analytical solutions through Equation \[5\]. We derive and give such solutions in section III.B.

We exhibit below three paradigmatic examples of coupling profile engineering: the homogeneous profile array, the parabolic profile array and the square root profile array. We display the supermodes that each array produces and their respective propagation constants.

1. **Homogeneous profile array**

The homogeneous linear array exhibits a constant coupling between waveguides \(f_j = 1\). It is thus a symmetric lattice. The supermodes are orthonormal Chebyshev polynomials that can be written in terms of simple trigonometric functions as \[49\]

\[
M_{k,j} = M_{j,k} \equiv \frac{\sin\left(\frac{jk\pi}{N+1}\right)}{\sqrt{\sum_{j'=1}^{N} \sin^2\left(\frac{j'k\pi}{N+1}\right)}}.
\]

The Chebyshev supermodes for \(N = 5\) waveguides are sketched in Figure 2a. The spectrum of its eigenvalues is given by

\[
\lambda_k = 2C_0 \cos\left(\frac{k\pi}{N+1}\right),
\]

which are the propagation constants related to each supermode.

2. **Parabolic profile array**

The parabolic linear array exhibits a coupling between waveguides given by the profile \(f_j = \sqrt{j(N-j)/2}\). It
is a symmetric lattice. The supermodes are orthonormal Krawtchouk polynomials that can be written in terms of Jacobi polynomials as \[ M_{k,j} = 2^{j \cdot \frac{N+1}{2}} \sqrt{\frac{(j-1)!(N-j)!}{(k-1)!(N-k)!}} \cdot j^{N-k+1-j-k-j}(0) \]

\[ = M_{j,k}. \]

The Krawtchouk supermodes for \( N = 5 \) waveguides are sketched in Figure 2b. Note that for a small number of polynomials evaluated at these roots the eigenvalues \( \lambda_k \) are in this case equally spaced as given by \[ \lambda_k = \frac{N - 2k + 1}{2} C_0. \]

The continuous limit (\( N \to \infty \)) of these discrete eigenfunctions are the Hermite-Gaussian functions \([46]\). Remarkably, a parametric generalization of this set of supermodes, so-called para-Krawtchouk supermodes, allows fractional revivals and thus generalizes beam splitters – or directional couplers – to \( N \) dimensions \([35]\). The square-root or Glauber-Fock linear array exhibits \( \sqrt{f_a} \) coupling between waveguides given by the profile \( \sqrt{f_a} \equiv \frac{\sqrt{2}}{\sqrt{2} C_0} \cdot \sqrt{\lambda_k} \).

3. Square-root profile array

The square-root or Glauber-Fock linear array exhibits a coupling between waveguides given by the profile \( f_j = \sqrt{f} \). It is an asymmetric lattice that can be symmetrized. The eigenvalues \( \lambda_k \) are obtained as the roots of the \( N \)th Hermite polynomial given by \([51, 52]\)

\[ H_N[\sqrt{\lambda_k} / \sqrt{2} C_0)] = 0, \]

with \( k' \equiv k - 1 = 0, \ldots N - 1 \). The Glauber-Fock supermodes can be written in terms of normalized Hermite polynomials evaluated at these roots

\[ M_{k,j} = \frac{H_j \cdot \sqrt{\lambda_k} / \sqrt{2} C_0)}{\sqrt{2} N_{k'}}. \]

where \( N_{k'} = \sum_{j=0}^{N-1} H_j \cdot [\lambda_k / \sqrt{2} C_0)]^2 / (2j!) \). The Glauber-Fock supermodes for \( N = 5 \) waveguides are sketched in Figure 2b. Emulation of a driven quantum harmonic oscillator has been demonstrated in this lattice \([53]\).

B. Pump profile engineering

Suitable manipulation of individual power and phase pump fields by means of off-the-shelf elements as fiber attenuators and phase shifters, followed by input into the ANWs through V-groove arrays, enables an on-demand pump distribution engineering.

The pump profile couples the propagation supermodes generating the joint-spatial supermode distribution Equation 5. In general, this generates complicated connections between the linear supermodes. However, the orthogonality and symmetry properties of the linear supermodes \([Equations (3) and (13) to (17)]\) lead to simple analytical solutions in some cases. An outstanding simplification of the system is obtained when pumping all the waveguides with the same power \( |\eta| = \text{constant} \). From now on, we refer to this as a flat pump profile. Another simplified solution is obtained when pumping only the even or odd waveguides, or when pumping only the central waveguide in an odd ANWs. Below we give the joint-spatial supermode distributions obtained with these input configurations and the analytical solutions to the Heisenberg Equations 6 in the simplest cases.

1. Flat pump profile: uniform phase

When all waveguides are equally pumped such that \( |\eta| = |\eta| = \tilde{\eta} \) and \( \phi_j = \phi \), the local joint-spatial supermode distribution Equation 5 is notably simplified to

\[ \mathcal{L}_{k,k'}(z) = 2i \delta_{k,k'} e^{i\phi - (\lambda_k + \lambda_{k'})z}, \]

where we have used the orthonormality of the linear supermodes Equation 3. This pump configuration diagonalizes the momentum and the following Heisenberg equations are obtained

\[ \frac{d\hat{B}_{S,k}}{dz} = 2i \tilde{\eta} e^{i\phi - 2\lambda_k z} \hat{B}_{S,k}^\dagger. \]

The solutions are

\[ \hat{B}_{S,k} = \{ \cos(F_k z) \hat{B}_{S,k}(0) + i \sin(F_k z) \hat{B}_{S,k}(0) + 2|\eta|e^{i\phi} \hat{B}_{S,k}(0) \} e^{-i\lambda_k z}, \]

(18)

with \( F_k = \sqrt{\lambda_k^2 - 4|\eta|^2} \). For typical evanescent coupling, nonlinearities and pump powers found in quadratic ANWs \(|\lambda_k| > 2|\eta| \) and thus \( F_k \in \mathbb{R} \). We consider cases only in this power regime in the remainder of the article. Equation 18 simplifies into Equations (6)-(7) of ref. 17 for the nonlinear directional coupler (\( N = 2 \)). The supermodes evolution is similar to the one found there for the individual modes: the power of the SPDC supermode periodically oscillates between a maximum and zero with oscillation periods \( L_k = \pi / (2F_k) \). It is interesting to note that waveguide arrays with odd number of waveguides \( N \) exhibit a zero supermode \( l = (N + 1) / 2 \). As introduced in section II A, this is a propagation eigenmode with zero eigenvalue \( \lambda_l = 0 \) in the slowly-varying amplitude approximation \([48]\). The oscillation period of the zero-supermode is imaginary.
Consider $L_i = \pi/(4|\eta|)$, thus leading to hyperbolic solutions as
\begin{equation}
\hat{B}_{S,i}(z) = \cosh(2|\eta|z)\hat{B}_{S,i}(0) + i e^{i\phi} \sinh(2|\eta|z)\hat{B}_{S,i}^\dagger(0),
\end{equation}
\begin{equation}
\hat{A}_j(z) = \sum_{j'=1}^{N} [\tilde{U}_{j,j'}(z)\hat{A}_{j'}(0) + \tilde{V}_{j,j'}(z)\hat{A}_{j'}^\dagger(0)],
\end{equation}
where
\begin{align*}
\tilde{U}_{j,j'}(z) &= \sum_{k=1}^{N} M_{j,k} M_{k,j'}[\cos(F_k z) + i \frac{\lambda_k}{F_k} \sin(F_k z)], \\
\tilde{V}_{j,j'}(z) &= \sum_{k=1}^{N} M_{j,k} M_{k,j'}[\frac{2|\eta| e^{i\phi}}{F_k} \sin(F_k z)],
\end{align*}
with $\sum_{k=1}^{N} |\tilde{U}_{j,j'}(z)|^2 - |\tilde{V}_{j,j'}(z)|^2 = 1$. Note that for $|\eta| = 0$, $\tilde{U}_{j,j'}(z) = U_{j,j'}(z)$ and $\tilde{V}_{j,j'}(z) = 0$, with $U_{j,j'}(z) \equiv \sum_{k=1}^{N} M_{j,k} M_{k,j'} e^{i\lambda_k z}$, we recover the solution corresponding to the linear array.

From these equations, it is straightforward to obtain the elements of the covariance matrix $V(z)$, which read
\begin{align}
V(x_i, x_j) &= \sum_{k=1}^{N} \frac{M_{i,k} M_{j,k}}{F_k^2} \{\lambda_k^2 - 4|\eta|^2 \cos(2F_k z) \\
&\quad - 4|\eta| \sin(F_k z)[F_k \sin(\phi) \cos(F_k z) + \lambda_k \cos(\phi) \sin(F_k z)]\}, \\
V(y_i, y_j) &= \sum_{k=1}^{N} \frac{M_{i,k} M_{j,k}}{F_k^2} \{\lambda_k^2 - 4|\eta|^2 \cos(2F_k z) \\
&\quad + 4|\eta| \sin(F_k z)[F_k \sin(\phi) \cos(F_k z) + \lambda_k \cos(\phi) \sin(F_k z)]\}, \\
V(x_i, y_j) &= \sum_{k=1}^{N} \frac{M_{i,k} M_{j,k}}{F_k^2} \times \\
&\quad 4|\eta| \sin(F_k z)[F_k \cos(\phi) \cos(F_k z) - \lambda_k \sin(\phi) \sin(F_k z)].
\end{align}

This configuration generates quantum correlations between the individual modes –off-diagonal components of the covariance matrix (as shown in Figure 3(b)), and hence entanglement is possible in that basis. Likewise, the mean number of signal photons generated in the $j$th waveguide at any propagation length can be directly calculated from Equations (21) as $\bar{N}_j = V(x_j, x_j) + V(y_j, y_j) - 2$.

Remarkably, the results displayed in this section are general for any ANWs –any evanescent coupling profile $f$– since they are based only on the orthonormality of the supermodes. Equations (21) remain valid for any number of waveguides $N$ or propagation distance $z$, thus they are a valuable tool which we use in section V to engineer linear clusters.
2. Flat pump profile: alternating $\pi$ phase

When all waveguides are equally pumped such that $|\eta_j| = |\eta| = \tilde{\eta}$ with an alternating phase $\phi_j = (j + 1)\pi + \phi$, the joint-spatial supermode matrix $\mathcal{L}_{k,k'}(z)$ is notably simplified to

$$\mathcal{L}_{k,k'}(z) = 2i\delta_{k,N+1-k'}e^{i(\phi_k + \phi_{k'})}z$$

via Equation (15). This pump configuration diagonalizes the momentum and the following Heisenberg equations are obtained

$$d\hat{B}_{S,k}(z) = 2i\tilde{\eta}e^{i\phi}\hat{B}_{S,N+1-k}'(0)$$

The downconversion gains are proportional to $2|\eta|$. The solution to the above equation is

$$\hat{B}_{S,k}(z) = \cosh(2|\eta|z)\hat{B}_{S,k}(0) + ie^{i\phi}\sinh(2|\eta|z)\hat{B}_{S,N+1-k}'(0), (22)$$

Note that this is the solution of a perfectly phase-matched nondegenerate parametric amplifier [55]. The supermode solution Equation (22) can be written in the individual mode basis as the following transformation

$$\hat{A}_j(z) = \sum_{j'=1}^{N} U_{j,j'}(z) \times$$

$$[\cosh(2|\eta|z)\hat{A}_{j'}(0) + (-1)^{j'+1}ie^{i\phi}\sinh(2|\eta|z)\hat{A}_{j'}(0)], (23)$$

where we have used Equations (14), (15) and the propagator related to the linear array $U_{j,j'}(z)$ introduced above. The solution is thus decoupled in this configuration: input single-mode squeezed states of light squeezed along the axis $(j'+1)\pi + \phi$ propagate in the corresponding linear array with propagation matrix $U_{j,j'}(z)$. From this equation, after a long but straightforward calculation, we obtain the elements of the covariance matrix $V(z)$, which read as follows

$$V(x_i,x_j) = [\cosh(4|\eta|z) + (-1)^j\sin(\phi)\sinh(4|\eta|z)]\delta_{i,j},$$

$$V(y_i,y_j) = [\cosh(4|\eta|z) + (-1)^j\sin(\phi)\sinh(4|\eta|z)]\delta_{i,j},$$

$$V(x_i,y_j) = (-1)^j\cos(\phi)\sinh(4|\eta|z)\delta_{i,j}. (24)$$

Then, in this case quantum correlations are efficiently generated in the supermode basis but they disappear in the individual mode basis—no off-diagonal elements of the covariance matrix (Figure 1b)–. The device thus produces independent squeezed fields. The results obtained in this section are general for any coupling profile $\tilde{f}$ since they rely on Equations (13) - (15) only. Equations (24) remain valid for any number of waveguides $N$ or propagation distance $z$. Notably, this is an interesting regime for discrete variables since $N$-dimensional two-photon NOON states are generated [30].

3. Flat pump profile: any alternating phase

Both cases analyzed in sections III.B.1 and III.B.2 are encompassed through the use of Equations (16) - (17). In the case of an array composed of $N$ waveguides equally pumped such that $|\eta_j| = |\eta| = \tilde{\eta}$ and alternating phases $\phi_{2j}$ and $\phi_{2j-1}$, the joint-spatial supermode matrix Equation (5) is notably simplified to

$$\mathcal{L}_{k,k'}(z) = 2ie^{i\phi}\cosh(\Delta\phi\tilde{z})e^{-2i\lambda_k\tilde{z}\delta_{k,k'}} - i\sin(\Delta\phi\tilde{z})\delta_{k,N+1-k'}], (25)$$

with $\Delta\phi^\pm = (\phi_{2j} \pm \phi_{2j-1})/2$. Thus the solution of the system oscillates between Equations (18) and (22) for a general phase difference $\Delta\phi\tilde{z}$. Particularly, for $\phi_{2j} = \phi + \pi/2$ and $\phi_{2j-1} = \phi$ both the diagonal and antidiagonal terms have the same weight such as

$$\mathcal{L}_{k,k'}(z) = \sqrt{2}ie^{i(\phi + \pi/4)}[e^{-2i\lambda_k\tilde{z}\delta_{k,k'}} - i\delta_{k,N+1-k'}].$$

The solution will present then both oscillatory and hyperbolic terms. More light is shed on the features that this configuration produces in section IV.

4. Pumping only the even or odd waveguides

Another simplified joint-spatial supermode matrix is obtained if either even waveguides only ($|\eta_{2j}| = \tilde{\eta}$, $|\eta_{2j-1}| = 0$ and $\phi_{2j} = \phi$) or odd waveguides only ($|\eta_{2j-1}| = \tilde{\eta}$, $|\eta_{2j}| = 0$ and $\phi_{2j-1} = \phi$) are pumped, such that

$$\mathcal{L}_{k,k'}(z) = ie^{i\phi}[e^{-2i\lambda_k\tilde{z}\delta_{k,k'}} \pm \delta_{k,N+1-k'}],$$

with plus for odd and minus for an even pump profile through Equations (16) - (17). The solutions are here more complex than those of cases III.B.1 and III.B.2. For instance, pumping only the odd waveguides we have the following solution to Equation (6)

$$\hat{B}_{S,k}(z) = [\cosh(|\eta|z)\cos(\tilde{F}_k z) + i\frac{\lambda_k}{\tilde{F}_k} \sin(\tilde{F}_k z)]\hat{B}_{S,k}(0)$$

$$+ \frac{|\eta|}{\tilde{F}_k} \sin(\tilde{F}_k z)[\cosh(|\eta|z)\hat{B}_{S,k}'(0) - i\sin(|\eta|z)\hat{B}_{S,N+1-k}(0)]$$

$$+ i\sin(|\eta|z)\cos(\tilde{F}_k z) + i\frac{\lambda_k}{\tilde{F}_k} \sin(\tilde{F}_k z)\hat{B}_{S,N+1-k}(0)]e^{-i\lambda_k\tilde{z}}, (26)$$

with $\tilde{F}_k = \sqrt{\lambda_k^2 - |\eta|^2}$ and where we have set $\phi = 0$ for the sake of simplicity. This solution shows that the side linear supermodes are symmetrically coupled two by two $-k$ with $N+1-k'$, being the zero supermode the only one independently squeezed. The solution in the individual mode basis
can be written as Equation (20) with
\[ \tilde{U}_{j,j'}(z) = \sum_{k=1}^{N} M_{j,k} M_{k,j'} |\cosh(|\eta| z)\cos(\tilde{F}_k z) + i \frac{\lambda_k |\eta| \sin(|\eta| z) \sin(\tilde{F}_k z)}{\tilde{F}_k}, \]
\[ \tilde{V}_{j,j'}(z) = \sum_{k=1}^{N} M_{j,k} M_{k,j'} (-1)^{j'+1} |\sin(|\eta| z)\cos(\tilde{F}_k z) + i \frac{\lambda_k |\eta| \cos(|\eta| z) \sin(\tilde{F}_k z)}{\tilde{F}_k}, \]
where we have used the property of the supermodes Equation (14). The elements of the covariance matrix \( V(z) \) are in this case the following
\[ V(x_i, x_j) = \sum_{k=1}^{N} M_{i,k} M_{j,k} |\cosh(2|\eta| z)\cos(2\tilde{F}_k z) + (-1)^j \frac{\lambda_k |\eta| \sin(2|\eta| z) \sin(2\tilde{F}_k z) + (-1)^j \frac{\lambda_k |\eta| \cos(2|\eta| z) \sin(2\tilde{F}_k z)}{\tilde{F}_k}, \]
\[ V(y_i, y_j) = \sum_{k=1}^{N} M_{i,k} M_{j,k} |\cosh(2|\eta| z)\cos(2\tilde{F}_k z) - (-1)^j \frac{\lambda_k |\eta| \sin(2|\eta| z) \sin(2\tilde{F}_k z) - (-1)^j \frac{\lambda_k |\eta| \cos(2|\eta| z) \sin(2\tilde{F}_k z)}{\tilde{F}_k}, \]
\[ V(x_i, y_j) = \sum_{k=1}^{N} M_{i,k} M_{j,k} |(-1)^{j+1} \sin(2|\eta| z)\cos(2\tilde{F}_k z) + \frac{|\eta|}{\tilde{F}_k} \cos(2|\eta| z) \sin(2\tilde{F}_k z). \]
Thus, quantum correlations between the individual modes are generated. These solutions generalize to \( N \) dimensions the paradigmatic example of pumping one waveguide in a nonlinear directional coupler [56]. Similar solutions are obtained when pumping only the even waveguides.

### 5. Pumping the central waveguide in an odd ANW

A common and simple way of pumping an odd ANWs is to inject the pump only in the central waveguide \( j = l \equiv (N + 1)/2 \) (see Figure 11) [13]. The following joint-spatial supermode distribution is then obtained
\[ \mathcal{L}_{k,k'}(z) = 2i e^{i\phi_l} M_{k,l} M_{k',l} e^{-i(\lambda_k + \lambda_{k'}) z}. \]
Notably, in the case of symmetric coupling profile arrays like the homogeneous or the parabolic profiles shown above, the elements of the zero supermode have zeros in the even elements, i.e. \( M_{k,l} = 0 \) for \( k \) even. Thus, only odd supermodes are produced in the ANWs under this configuration. For instance, for \( N=5 \) and a homogeneous coupling profile we obtain as approximated solutions Equation (19) for the zero supermode \( l = 3 \) and Equation (22) for the \( k = 1, 5 \) side supermodes after rescaling \( |\eta| \) to \( |\eta|/l \). Figure 3 shows the covariance matrix in the individual mode basis related to this pump configuration in an ANWs with a homogeneous coupling profile.

The above five cases exhibit the versatility of the ANWs through pump engineering and shed light on propagation in these devices. We further discuss the relationship between linear and nonlinear supermodes, and the generated squeezing along propagation in section IV. The generation of multiparticle entanglement in ANW has been recently tackled in [20] [21] and we further present in section V an efficient protocol for the generation of linear cluster states based on the analytical solution obtained in section III B.1.

### C. Phase-matching engineering

A common phase-matching technique for efficient frequency conversion in \( \chi^{(2)} \) nonlinear waveguides is obtained through wavevector quasi-phase matching (\( \Delta\beta\)-QPM). A standard implementation of \( \Delta\beta\)-QPM is periodical inversion of the second-order susceptibility \( \chi^{(2)} \) with period \( \Lambda_{\Delta\beta} = 2\pi/\Delta\beta \), like for instance in PPLN waveguides [57]. However, in the case of waveguide arrays, a second cause of phase mismatch – the coupling– is present, as shown in Equation (5). In this case a similar strategy can be used to phase match specific supermodes through a second periodical inversion \( \Delta\kappa(k') \)–coupling quasi-phase matching (C-QPM)– [36]. This slow modulation will match the propagation constant \( \lambda_{k'} \) of the \( k' \)th slowly varying supermode amplitude. We consider, for instance, a homogeneous coupling profile where \( \lambda_{k'} = -\lambda_N + k' \equiv 2C_0 \cos(k'\pi/(N+1)) \). In this case, the periodical inversion–coupling period– can be set as \( \Lambda_{\kappa}(k') = |\pi/\lambda_{k'}| \), thus phase matching the \( k' \)th and \( (N+1-k') \)th side supermodes. Equation (5) is then written as
\[ \mathcal{L}_{k,k'}(z) \approx \frac{8i}{\pi} \sum_{j=1}^{N} \frac{|\eta_l|}{|\eta|} M_{k,j} M_{k',j} \cos(2\lambda_{k'} z) e^{i(\phi_j - (\lambda_k + \lambda_{k'}) z)}, \]
where we have used the first-order Fourier series of square-wave C-QPM domains with duty cycles of 50\%. Thus, using a flat pump profile, Equation (20) is simplified to
\[ \mathcal{L}_{k,k'}(z) \approx \frac{8i}{\pi} \cos(2\lambda_{k'} z) \delta_{k,k'} e^{i(\phi - (\lambda_k + \lambda_{k'}) z)}, \]
and the Heisenberg equations read
\[ \frac{d\tilde{B}_{S,k}}{dz} \approx \frac{4i\tilde{\eta}}{\pi} e^{i\phi} \tilde{B}_{S,k}, \quad k = k', N + 1 - k', \]
\[ \frac{d\tilde{B}_{S,k}}{dz} \approx \frac{4i\tilde{\eta}}{\pi} e^{i(\phi - 2\lambda_k z)} \tilde{B}_{S,k'}, \quad k \neq k', N + 1 - k'. \]
Hyperbolic solutions as Equation (19) are obtained for the \( k' \)th and \( (N+1-k') \)th supermodes and oscillatory solutions like Equation (18) for the other supermodes. Note that the gains are reduced by a factor 2/\( \pi \) in comparison with the no C-QPM case which can be compensated with a propagation distance \( \pi/2 \) longer.

This powerful technique allows to control the supermodes efficiently building up. In terms of individual-modes entanglement it would be interesting to build up supermodes but with light in all the individual modes. Remarkably, in the case of parabolic arrays with an even number of waveguides all the supermodes can efficiently build up. This interesting case will be presented elsewhere.
entanglement in a new way, accessible only to guided-wave nonlinear components. As shown above, the different methods of detection available enable the encoding of quantum information in three ways: individual modes, linear and nonlinear supermodes. We now investigate multimode squeezing focusing on the nonlinear supermodes, which maximize the resources available at every propagation length in terms of squeezing and can generate entanglement through LO shaping or emulation \[21\]. We thus display in this section the outcomes in terms of squeezing obtained from numerical solutions of Equation (12) using the general method of Bloch-Messiah. The large parameter space of the ANW enables an infinite number of configurations. We focus on the configurations displayed in section \[11\] in terms of pumping profile \(-\) in amplitude and phase \(-\), propagation length and coupling profile. This allows us to discuss the relationship of these numerical results in terms of nonlinear supermodes with the analytical solutions obtained in section \[11\] in terms of individual modes and linear supermodes.

For the purpose of assessing multimode squeezing, i) we start displaying the connection between the individual and nonlinear supermode bases through the covariance matrices in both bases at a fixed propagation length. We then move to the evolution of squeezing in the nonlinear supermode basis. ii) We first focus on the case of flat pumping to ii.a) analyze the evolution of squeezing along propagation with any alternating pumping phase, and to ii.b) display the influence of different coupling profiles and strengths on the squeezing. Finally, we discuss ii.c) the connection between the nonlinear supermodes and the linear supermodes for an uniform phase. iii) We finish exhibiting the squeezing obtained for other pumping profiles: iii.a) pumping only the odd waveguides, and iii.b) the simplest configuration, pumping only the central waveguide in an ANW with an odd number of waveguides.

### D. Balanced homodyne detection

The measurement of quantum noise variances and correlations is carried out by multimode balanced homodyne detection (BHD) \[33\]. In a fully fibered approach the multimode squeezed state generated in the array can be collected in optical fibers through a V-groove array. A laser at signal frequency is demultiplexed into a number of individual optical fibers with fiber attenuators and phase shifters and individually mixed with the output SPDC through 3 dB fibered beam splitters as sketched in Figure 3. Each pair of mixed signals is sent to a BHD where the current of each photodiode is subtracted and suitably amplified.

We point out that the spatial profile of the LO in the multimode BHD has to be adapted to the strategy of entanglement generation and the entangled state basis. Access to the quantum information encoded in the individual or any of the supermode bases will indeed depend on a suitable BHD \[55\]. Figure 3 displays the possible knobs at the measurement stage. The local oscillator can be tuned to detect correlations between the individual output signals of the array \(-\) in the individual mode basis \(\hat{A}\) \(-\) or shaped to match any supermode \(-\) in the linear supermode basis \(A_S\), nonlinear supermode basis \(C_g\) or any other basis. This LO shaping can be accomplished using attenuator and phase shifters or can be emulated with electronic gains. Remarkably, LO shaping enables the measurement of entangled states encoded in bases based on nonlinear supermodes. We have discussed these issues elsewhere \[21\] and mentioned the existing implementations in other domains \[23, 28\].

### IV. MULTIMODE SQUEEZING

The ANW is a natural platform for generating multimode squeezing due to the distributed coupling and nonlinearity. Such a distributed configuration combines the necessary squeezing and coupling instrumental to produce multimode squeezing and coupling instrumental to produce multimode.
with a similar topology of quantum correlations, but different strength and sign. Figures 4a and 4d display the diagonal covariance matrix $K^2(z)$ and the transformation between bases $R_1(z)$, making more obvious the difference with the flat pump case of Figures 4i and 4k. Overall, these figures show the versatility of our approach, yielding different multimode squeezing features for different input pump profiles. We analyze in more depth the obtained squeezing $K^2_{N+m}(z)$, exploring further the parameter space along propagation $z$, coupling $C_j$ and pumping $\eta_j$.

2. Evolution of multimode squeezing for a flat pump profile

a. Alternating pumping phase $\Delta \phi^-$ and nonlinear supermodes squeezing behavior. Figure 5 shows the evolution of noise squeezing ($K^2_{N+m}(z) < 1$) of the five nonlinear supermodes for a flat pump profile in a $N = 5$ ANW. We show the effect of the coupling profile $f$, the value of the coupling constant $C_0$ and the relative pump phase $\Delta \phi^-$ [Equation (28)] on $K^2(z)$. Figures 5a, 5c and 5e show the result for a homogeneous, parabolic and square-root coupling profile, respec-
FIG. 5. Evolution of nonlinear supermode squeezing $K_{N,m}^2(z)$ in a five-waveguides (a) homogeneous, (b) parabolic and (c) square-root coupling-profile nonlinear array. The zero supermode ($m=l=3$) is always efficiently squeezed independently of $\Delta \phi^-$ (solid, green). $\Delta \phi^- = 0$ (solid, blue and orange), $\Delta \phi^- = \pi/2$ (solid, green) and intermediate cases: $\Delta \phi^- = \pi/8$ (dotted), $\Delta \phi^- = \pi/4$ (dashed) and $\Delta \phi^- = 3\pi/8$ (dot-dashed). 3 dB squeezing level in dotted, gray. $C_0 = 0.24 \text{ mm}^{-1}$ for (a)-(b). $C_0 = 0.08 \text{ mm}^{-1}$ for (c). $\eta = 0.015 \text{ mm}^{-1}$.

 independantly. Figure 5c shows the result for a coupling strength three times lower than that used in Figures 5a and 5b. The squeezed eigenvalues are degenerate two by two for the $m$th and $(N+1-m)$th nonlinear supermodes. We refer to them as side nonlinear supermodes. Likewise, the zero nonlinear supermode [$m=l \equiv (N+1)/2$] is the only nondegenerate supermode and it is always efficiently built-up and squeezed, independently of the value of $\Delta \phi^-$ (solid, green). The oscillatory and hyperbolic limit cases we pointed out for propagation in Equations (18) and (19) explain and match the squeezing behaviors displayed in each case. Full degeneracy and efficient squeezing –hyperbolic– is obtained for all the supermodes for $\Delta \phi^- = \pi/2$ (solid, green). $\Delta \phi^- = 0$ produces oscillatory squeezing (solid, blue and orange) in the side nonlinear supermodes which decreases as the coupling strength $C_0$ increases (Figures 5a-5c). Notably, for intermediate cases $\Delta \phi^- = \pi/8$ (dotted), $\pi/4$ (dashed), $3\pi/8$ (dot-dashed), squeezing builds up smoothly for the side nonlinear supermodes and it approaches degeneracy for long propagation distances, whereas at short distances it is disturbed by the oscillatory part of Equation (25). However, this disturbance is important since it mixes the individual downconverted modes and thus triggers quantum correlations and entanglement in the individual basis as we show in section V.

b. Influence of the coupling strength and coupling profile on multimode squeezing. In addition to the effect of increased homogeneous coupling on oscillatory behavior exemplified by the comparison of Figures 5a and 5b, we display the differ-
ent features obtained for homogeneous and parabolic coupling profiles (Fig. 5a-5b). For $\Delta \phi^\circ = 0$, there are certain lengths for the parabolic coupling profile where only the zero nonlinear supermode survives due to the equal spacing between the supermode propagation constants. Remarkably, for even number of waveguides and a parabolic coupling profile (not shown), there are propagation distances where destructive interference destroys all the SPDC generated light due to a evolving phase mismatch that periodically switches the system from downconversion to upconversion. Recently, bipartite entanglement between non-coupled pump fields has been demonstrated through this effect for two waveguides in the optical parametric amplification and second harmonic generation regimes $^{53,54}$ $^{59}$. Thus, this effect can also produce multipartite entanglement between non-interacting fields. We outline that the parabolic-coupling profile excited with a flat pump profile represents the spatial analogous case to the case of a frequency comb pumped with a Gaussian spectral shape since the Krawtchouk supermodes are Hermite-Gaussian functions in the continuous limit $^{52}$.

c. Linear vs nonlinear supermodes for a flat pump profile with uniform phase. We would like to end this section with a small discussion on the connection between solutions in terms of linear and nonlinear supermodes. To that end we use the flat pump configuration with uniform phase $\Delta \phi^\circ = 0$. The nonlinear supermodes diagonalize the covariance matrix as shown in Figures 4 and 5. However, in that case the linear supermodes also diagonalize the covariance matrix as shown in section IV.B.4. Both basis exhibit the same levels of squeezing, but different spatial profile evolution $^{59}$. The spatial profile related to the zero nonlinear supermode $|m = l \equiv (N + 1)/2\rangle$ obtained from $R_l(z)$ coincides with that calculated with Equation (19), but the side nonlinear supermodes ($m \neq l$) are slightly different from the side linear supermodes ($k \neq l$) obtained through Equation (15) and change with propagation. The cause of this disagreement is that the flat pump configuration diagonalizes the system up to a local phase rotation – the covariance matrix in the linear supermode basis is block-diagonal, i.e. the quadratures of the linear supermodes are not at the maximum and minimum of the squeezing ellipse, whereas the Bloch-Messiah decomposition yields a fully diagonal covariance matrix. From the point of view of the experiment, this phase does not make any difference since the local oscillator of the balanced homodyne detector will sweep the entire squeezing ellipse. However, the linear supermode approach is here far more insightful and practical than Bloch-Messiah’s one since the spatial profile is invariant along propagation and thus the $k$th supermode squeezing can be measured with a fixed LO profile $\theta_k = \{M_{k,1}, M_{k,2}, \ldots, M_{k,N}\}$, whereas the LO profile used to measure the $n$th nonlinear supermode squeezing would depend indeed on the length of the sample, the pump power and the coupling strength. More details on this are found in appendix A

3. Evolution of multimode squeezing with non-flat pumping profile

a. Pumping every other waveguide. Figure 6a shows the evolution of noise squeezing when pumping only the odd waveguides ($|\eta_{2j-1} = |\eta\rangle, |\eta_{2j} = 0\rangle$) of a $N = 5$ waveguides homogeneous coupling-profile nonlinear array. This pump configuration indeed excites the five nonlinear supermodes, with a zero nonlinear supermode efficiently squeezed (green) and side nonlinear supermodes squeezing building up hyperbolically with an oscillatory modulation. The analytical solution obtained for the linear supermodes Equation (20) anticipated this feature, since the side supermodes ($k, N + 1 - k$) are coupled two by two in that basis. We outline that in the case of pumping the even waveguides ($|\eta_{2j} = |\eta\rangle, |\eta_{2j-1} = 0\rangle$), we obtain the same solution for the side nonlinear supermodes but with the zero nonlinear supermode in vacuum state. This is due to the inability to excite a supermode composed of odd elements when pumping the even waveguides in arrays with symmetric coupling profiles. Thus, in terms of multimode squeezing as a resource for quantum information the odd pumping is more efficient. Note that this does not happen in arrays with asymmetric coupling profiles like the square-root coupling profile (see Figure 2a). In the case of an ANW made up of an even number of waveguides we excite all the supermodes independently on the parity of the total number of waveguides since there is no zero supermode.

b. Pumping the central waveguide in an odd ANW. Figure 6b shows the evolution of noise squeezing when pumping only the central waveguide ($|\eta_{2j} = |\eta\rangle, |\eta_{2j-1} = 0\rangle$) of a $N = 5$ waveguides homogeneous coupling-profile nonlinear array. This pump configuration leads to the excitation of only three out of five nonlinear supermodes – the odd ones – with the other two in vacuum state along propagation. The comparison between Figure 5a and Figure 6b thus further sheds light on the difference between Figure 5a and Figure 6b. The squeezing increases hyperbolically with an oscillatory modulation. Note that in this pumping configuration there is no direct correlation between linear and nonlinear supermodes anymore. However, the nonlinear supermode squeezing exhibited in this case can be also explained in terms of the linear supermodes as in Section III.B.5. The leading terms of the zero ($k = 1, 3$) and side ($k = 1, 5$) linear supermode equations correspond to degenerate and nondegenerate parametric amplifiers, respectively, leading to hyperbolic squeezing. First-order terms introduce a $z$-dependent coupling between the zero and the side supermodes with period $z_p \approx 2\pi/|\lambda_{1(5)}|$ for $|\lambda_{1(5)}| \gg 2|\eta|$.

4. Evolution of multimode squeezing with flat pumping profile

a. Pumping every other waveguide. Figure 6c shows the evolution of noise squeezing when pumping only the odd waveguides ($|\eta_{2j-1} = |\eta\rangle, |\eta_{2j} = 0\rangle$) of a $N = 5$ waveguides homogeneous coupling-profile nonlinear array. This pump configuration indeed excites the five nonlinear supermodes, with a zero nonlinear supermode efficiently squeezed (green) and side nonlinear supermodes squeezing building up hyperbolically with an oscillatory modulation. The analytical solution obtained for the linear supermodes Equation (20) anticipated this feature, since the side supermodes ($k, N + 1 - k$) are coupled two by two in that basis. We outline that in the case of pumping the even waveguides ($|\eta_{2j} = |\eta\rangle, |\eta_{2j-1} = 0\rangle$), we obtain the same solution for the side nonlinear supermodes but with the zero nonlinear supermode in vacuum state. This is due to the inability to excite a supermode composed of odd elements when pumping the even waveguides in arrays with symmetric coupling profiles. Thus, in terms of multimode squeezing as a resource for quantum information the odd pumping is more efficient. Note that this does not happen in arrays with asymmetric coupling profiles like the square-root coupling profile (see Figure 2a). In the case of an ANW made up of an even number of waveguides we excite all the supermodes independently on the parity of the total number of waveguides since there is no zero supermode.
ANW. The tuning of the pumping and coupling parameters, together with suitable encoding leads to different configurations of squeezing. The multimode squeezing presented above is a resource for multimode entanglement. We have indeed demonstrated very recently a protocol for the generation of large multimode entangled states for quantum networks and the versatile production of cluster states for quantum computing in ANW. In the next section we focus on a specific class of entangled states useful for quantum computing—the cluster states. We show that pumping with a flat profile as introduced in section III is a good strategy to generate large linear cluster states. In particular we demonstrate how to choose a good working point in an analytically and semi-analytically scanned parameter space and how to further numerically optimize the parameters.

V. EFFICIENT GENERATION OF LINEAR CLUSTER STATES

An ideal CV cluster state is a simultaneous eigenstate of specific quadrature combinations called nullifiers. Cluster states are associated with a graph or adjacency matrix $B$. The nodes of the graph represent the modes of the cluster state in a given basis, and the edges the entanglement connections among the nodes. Moreover, the label of the modes that are part of the cluster can be suitably set to maximize the entanglement between nodes. The nullifiers are given by

$$\hat{\delta}_i \equiv \hat{x}_i(\theta_i + \pi/2) - \sum_{l=1}^{N} B_{i,l'} \hat{x}_{l'}(\theta_{l'}) \quad \forall i = 1, \ldots, N,$$

where $B$ is the graph associated to the cluster and $\hat{x}_i(\theta_i) = \hat{x}_i \cos(\theta_i) + \hat{y}_i \sin(\theta_i)$ is the $i$th generalized quadrature in a given basis. We consider unit-weight cluster states with $B_{i,i'} = 1$ for modes $i$ and $i'$ being nearest neighbors in the graph and all the other entries of $B$ are zero.

Cluster states are the resource of CV measurement-based quantum computing (MBQC). The computation relies in this framework on the availability of a large multimode entangled state on which a specific sequence of measurements is performed. The choice of basis widens the range of application in MBQC. The nullifier variances tend to zero in the ideal limit of infinite squeezing. Experimentally, a cluster state can be certified if two conditions are satisfied: i) the noise of a set of normalized nullifiers lies below shot noise

$$V(\hat{\delta}_{i}) < 1 \quad \forall i = 1, \ldots, N,$$

where $\bar{\delta}_i \equiv \delta_i/\sqrt{1 + n(i)}$ is the normalized nullifier and $n(i)$ is the number of nearest neighbors to the $i$th node of the cluster, and ii) the cluster state is fully inseparable, i.e., it violates a set of multipartite entanglement inequalities.

We exhibit here how linear cluster states encoded in the individual mode basis are produced naturally in the flat pump configuration—introduced in section III and explored from a squeezing point of view in section IV. Notably, in the context of MBQC, a linear 4-mode cluster state is a sufficient resource for an arbitrary single-mode Gaussian unitary. Hence, linear cluster states represent key resources in this domain. The adjacency matrix $B_{iim}$ corresponding to a linear cluster is the same as that related to the coupling in an homogeneous array when the encoding $i$th node = $j$th mode is used. Thus, we further numerically optimize the parameters.

FIG. 8. Nullifier variances $V(\hat{\delta}_{i})$ for a $N = 5$ linear cluster state generated in an ANWs with homogeneous coupling and flat pump profiles as a function of the coupling strength $C \equiv C_0$ and the pump power via $\eta \equiv |\eta|$. Simultaneous values of $V(\hat{\delta}_{1}) < 2/3$ are signature of cluster production. a) $V(\hat{\delta}_{1}) = V(\hat{\delta}_{2}) < 2/3$, b) $V(\hat{\delta}_{2}) = V(\hat{\delta}_{4})$, and c) $V(\hat{\delta}_{3})$. The white areas stand for $V(\hat{\delta}_{i}) \gg 1$. The black dot marks the point $(C, \eta) = (0.08, 0.033) \text{ mm}^{-1}$. $\phi_j = -\pi/2$. $z = 20 \text{ mm}$. 
the ANWs can be a natural platform for the generation of this class of cluster states. The normalized nullifiers for a linear cluster composed of \( N \) modes are given by

\[
\delta_i = \frac{y_i(\theta_i) - x_{i+1}(\theta_{i+1}) + x_{i+1}(\theta_{i+1})}{\sqrt{1 + n(i)}},
\]

with \( x_0(\theta_0) = x_{N+1}(\theta_{N+1}) = 0 \) and where we have related the \( i \)th node of the cluster with the \( j \)th individual mode of the ANWs \((i = j = 1, \ldots, N)\). The full inseparability of the cluster nodes can also be assessed by means of the Van Loock - Furusawa multipartite entanglement witness (VLF) \([65]\). For a linear cluster the VLF is given in terms of the following \( N - 1 \) inequalities \([12]\)

\[
V(\delta_i) + V(\delta_{i+1}) \geq \begin{cases} \frac{\sqrt{2}}{3} & \text{for } i = 1, N - 1, \\ \frac{4}{3} & \text{for } i = 2, \ldots, N - 2. \end{cases}
\]

Thus, simultaneous values of \( V(\delta_i) < 2/3 \) ensure the production of a linear cluster. Below we exhibit the use of the analytical solutions Equations \([21]\) in the generation of linear cluster states as shown in Figure 4. These solutions are a suitable initial working point in the huge space of parameters for the production of a linear cluster. This is very important in order to design a sample: we have a starting point from which optimization via pumping profile, LO phases and electronic gains can improve the result. Remarkably, we have found that these solutions are not very far from the best working point to produce this kind of states \([21]\).

Figure 5 maps the nullifier variances characterizing a \( N = 5 \) linear cluster state produced in an ANWs with homogeneous coupling and propagation length \( z = 20 \) mm. Due to the symmetry of the system the nullifiers are degenerate two by two, but the \( i \)th nullifier: \( V(\delta_i) = V(\delta_{i+1}) \) (Figure 3a), \( V(\delta_1) = V(\delta_2) \) (Figure 3b), and \( V(\delta_3) \) (Figure 3c). The contour plots display common areas fulfilling the condition \( V(\delta_i) < 2/3 \) (blue areas). For instance, for \( |\eta| = 0.06 \text{ mm}^{-1} \) and \( C_0 = 0.16 \text{ mm}^{-1} \), we get \( V(\delta_1(5)) = 0.34 \), \( V(\delta_3(4)) = 0.42 \), and \( V(\delta_5) = 0.40 \). These values are of the order of those obtained in the frequency domain with frequency combs \([27]\).

In order to gain insight about the scalability of this configuration, Figure 6 pictures the evolution along propagation of the nullifier variances related to linear cluster states made up of \( N = 5 \) (Figure 6a) and \( N = 15 \) (Figure 6b) modes. Now, we optimize the amount of power per waveguide \( \eta \) used for a given coupling constant. We use the sum of the five (fifteen) nullifier variances \( F_C(\eta) = \sum_{i=1}^{5(15)} V(\delta_i) \) at each \( z \) as the fitness function to optimize. We use an evolution-strategy algorithm to tackle this optimization \([8]\). As commented above, the nullifier variances are degenerate due to the symmetry of the system. Remarkably, the linear cluster condition \( V(\delta_i) < 2/3 \) is fulfilled in both cases for a large range of distances. In order to connect Figures 5 and 6 we have marked as a black dot in Figures 6a, b, and c, the coordinates \((C, \eta) = (0.08, 0.033) \text{ mm}^{-1}\) corresponding to the variances of the nullifiers at \( z = 20 \) mm shown in Figure 7. The maxima values of \( \eta \) used in the optimization are 0.038 and 0.035 mm\(^{-1}\) for \( N = 5 \) and 15, respectively. These values are attainable with current technology \([20, 11]\). Note that the coupling constant is wavelength dependent \( C_0 = C_0(\omega_5) \) \([50]\). Thus, for a fixed ANW length, modifying the operating wavelength \( \lambda \) and the temperature of the sample \( T \), we can access to more favorable conditions to obtain multipartite entanglement. This is clearly shown in Figure 8 when fixing the value of nonlinear strength \( |\eta| \) and checking the value of the nullifiers for different values of coupling strength \( C_0 \).

We have demonstrated the production of linear cluster states with our analytical solutions Equations \([21]\). However, the parameter space of the full approach is much larger than that corresponding to this special case. This enables the optimized generation of linear and other classes of cluster states in the individual mode basis or any other basis \([21]\).

### VI. CONCLUSIONS

The ANW is a versatile system for quantum state engineering, as presented in [D. Barral et al., arXiv:1912.11154v3]. Here, we complemented this claim, based on optimization of parameters for the generation of specific multimode entangled states, with a more fundamental perspective through a comprehensive analysis of scalable analytical solutions. As
such, we build an intuition for the possibilities of the system in terms of the available tuning parameters. This analytical approach provides insight into the particular features of the multimode squeezed states, produced in a given parameter configuration, and into working points in the parameter-space which maximize given properties of the generated state.

In particular, we have unfolded how the available internal and external parameters of the ANW affect the generated multimode squeezing, which is at the root of multimode entanglement. We have especially detailed how the fields propagate and how their fluctuations are squeezed in three relevant mode bases, related to three encodings of quantum information. The practical individual mode bases, where each mode corresponds to an individual waveguide, naturally provides individual outputs which are useful in quantum networks architectures. The linear supermode basis provides insight through analytical solutions and simple detection with a constant LO profile. Finally, the local nonlinear supermode basis maximizes the squeezing resources and is instrumental for multimode entanglement through LO shaping or postprocessing. We have provided insights on the engineering choices that can be made in terms of coupling strength and profile, measurement strategy and pumping geometry in phase and amplitude. We have exemplified the usefulness of our analytical solutions and further numerical optimization providing working points in the parameter space to produce linear cluster and assess their scalability. The quantum information-encoding strategy and pumping geometry in phase and amplitude. We have provided insights on the engineering choices that can be made in terms of coupling strength and profile, measurement strategy and pumping geometry in phase and amplitude. The quantum information-encoding strategy and pumping geometry in phase and amplitude.

The covariance matrix in the linear supermode basis $V_S$ obtained directly from Equation (18) is block diagonal. Suitable rotations (phase shifts) in the phase space related to each supermode can however diagonalize fully the linear supermode covariance matrix $V_S$. From Equation (18), we can straightforwardly calculate the covariance matrix $V_S$ related to the uncoupled $k$th linear supermode. A rotation in the $k$th supermode phase space of an angle

$$\vartheta_k = \frac{1}{2} \arctan \left( \frac{2V(x_{S,k}, y_{S,k})}{V(y_{S,k}, y_{S,k}) - V(x_{S,k}, x_{S,k})} \right) + \frac{\pi}{2},$$

diagonalizes the covariance matrix $V_S$ yielding

$$V(x_{S,k}, x_{S,k}) = \frac{V(x_{S,k}, x_{S,k}) + V(y_{S,k}, y_{S,k})}{2} + \sqrt{(V(y_{S,k}, y_{S,k}) - V(x_{S,k}, x_{S,k}))^2 + 4V(x_{S,k}, y_{S,k})^2},$$

$$V(y_{S,k}, y_{S,k}) = \frac{V(x_{S,k}, x_{S,k}) + V(y_{S,k}, y_{S,k})}{2} - \sqrt{(V(y_{S,k}, y_{S,k}) - V(x_{S,k}, x_{S,k}))^2 + 4V(x_{S,k}, y_{S,k})^2}.$$

This diagonal matrix is the same as $K^2(z)$ obtained by the Bloch-Messiah decomposition Equation (12). $R_1(z)$ can be factorized thus as $R_1(z) = MR(\vec{\vartheta})$, with

$$R(\vec{\vartheta}) = \begin{pmatrix} \cos (\vec{\vartheta}) & \sin (\vec{\vartheta}) \\ -\sin (\vec{\vartheta}) & \cos (\vec{\vartheta}) \end{pmatrix},$$

and $\cos (\vec{\vartheta}) = \text{diag}(\cos (\vartheta_1), \ldots, \cos (\vartheta_n))$, [equally for $\sin (\vec{\vartheta})$].

We show an example for the sake of clarification. For a pump phase profile $\Delta \phi = 0$ with $\phi = 0$, the covariance matrix elements in the propagation supermode basis are

$$V(x_{S,k}, x_{S,k}) = [\cosh (r_k) + \sinh (r_k) \cos (2F_k z)] e^{-r_k},$$

$$V(y_{S,k}, y_{S,k}) = [\cosh (r_k) - \sinh (r_k) \cos (2F_k z)] e^{r_k},$$

$$V(x_{S,k}, y_{S,k}) = \sinh (r_k) \sin (2F_k z),$$

with $r_k = (1/2) \ln [|\lambda_k + 2|]/(\lambda_k - 2|)]$. The squeezing phase is given by $\vartheta_k = \pi/2 - (1/2) \arctan ((\cosh (r_k) \tan (F_k z))^{-1})$. It depends on the pump power via $|\gamma|$, the coupling strength $C_0$ and the propagation length $z$ – and thus $R_1(z)$. The larger squeezing is obtained periodically at distances $z_k = (2n + 1)\pi/(2F_k)$ different for each $k$th supermode, with $n$ any positive integer. The diagonalized variances at those distances are $V(x_{S,k\neq l}, x_{S,k\neq l}) = e^{2r_k}$, $V(y_{S,k\neq l}, y_{S,k\neq l}) = e^{-2r_k}$. These are the same values as the minima of the blue and orange curves in Figure 2. The $k$th mode squeezing disappears at periodic distances $z_k = n\pi/F_k$, the maxima of blue and orange curves in Figure 2. In the case of an odd number of waveguides, a $\vartheta_l = \pi/4$ rotation in phase space diagonalizes the covariance matrix corresponding to the zero supermode independently of $z$, with $V(x_{S,l}, x_{S,l}) = e^{4|\gamma|}$, $V(y_{S,l}, y_{S,l}) = e^{-4|\gamma|}$ (green curves in Figure 2). The zero supermode is therefore the same for both bases.

In summary, the total available squeezing of the linear and nonlinear supermodes is the same, but it is distributed in a different way.

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**Appendix A**

The covariance matrix in the linear supermode basis $V_S$ obtained directly from Equation (18) is block diagonal. Suitable rotations (phase shifts) in the phase space related to each supermode can however diagonalize fully the linear supermode covariance matrix $V_S$. From Equation (18), we can straightforwardly calculate the covariance matrix $V_S$ related to the uncoupled $k$th linear supermode. A rotation in the $k$th supermode phase space of an angle

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