POLYNOMIALS, SIGN PATTERNS AND DESCARTES’ RULE OF SIGNS

VLADIMIR PETROV KOSTOV

ABSTRACT. By Descartes’ rule of signs, a real degree $d$ polynomial $P$ with all nonvanishing coefficients, with $c$ sign changes and $p$ sign preservations in the sequence of its coefficients ($c + p = d$) has $\text{pos} \leq c$ positive and $\text{neg} \leq p$ negative roots, where $\text{pos} \equiv c \pmod{2}$ and $\text{neg} \equiv p \pmod{2}$. For $1 \leq d \leq 3$, for every possible choice of the sequence of signs of coefficients of $P$ (called sign pattern) and for every pair $(\text{pos}, \text{neg})$ satisfying these conditions there exists a polynomial $P$ with exactly $\text{pos}$ positive and exactly $\text{neg}$ negative roots (all of them simple). For $d \geq 4$ this is not so. It was observed that for $4 \leq d \leq 10$, in all nonrealizable cases either $\text{pos} = 0$ or $\text{neg} = 0$. It was conjectured that this is the case for any $d \geq 4$. We show a counterexample to this conjecture for $d = 11$.

Key words: real polynomial in one variable; sign pattern; Descartes’ rule of signs

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1. Introduction

The classical Descartes’ rule of signs says that the real polynomial $P(x, a) := x^d + a_{d-1}x^{d-1} + \cdots + a_0$ has not more positive roots than the number $c$ of sign changes in the sequence of its coefficients. This rule has been announced by René Descartes (1596-1650) in his work La Géométrie published in 1637. In 1828 Carl Friedrich Gauss (1777-1855) has shown that if the roots are counted with multiplicity, then the number of positive roots has the same parity as $c$. When applied to $P(-x)$, these results give an upper bound on the number of negative roots of $P$. It is proved in [1] that all possible cases (i.e. of $c$, $c - 2$, $c - 4$, \ldots positive roots) are realizable by suitably chosen polynomials $P$ with $c$ sign changes. Notice that here we do not impose restrictions on the number of negative roots.

In what follows we consider polynomials $P$ without zero coefficients. Denoting by $p$ the number of sign preservations in the sequence of coefficients of $P$, and by $\text{pos}_P$ (resp. $\text{neg}_P$) the number of positive and negative roots of $P$, one can write:

\begin{equation}
\text{pos}_P \leq c, \quad \text{pos}_P \equiv c \pmod{2}, \quad \text{neg}_P \leq p, \quad \text{neg}_P \equiv p \pmod{2}.
\end{equation}

We call sign pattern a finite sequence $\sigma$ of $\pm$-signs; we assume that the leading sign of $\sigma$ is $+$. For a given sign pattern of length $d + 1$ with $c$ sign changes and $p$ sign preservations we call $(c, p)$ its Descartes pair, $c + p = d$. For a given sign pattern $\sigma$ with Descartes pair $(c, p)$ we call $(\text{pos}, \text{neg})$ an admissible pair for $\sigma$ if conditions (1.1), with $\text{pos}_P = \text{pos}$ and $\text{neg}_P = \text{neg}$, are satisfied.
One could ask the question whether given a sign pattern \( \sigma \) of length \( d + 1 \) and an admissible pair \((\text{pos}, \text{neg})\) one can find a degree \( d \) real monic polynomial the signs of whose coefficients define the sign pattern \( \sigma \) and which has exactly \text{pos} simple positive and exactly \text{neg} simple negative roots. In such a case we say that the couple \((\sigma, (\text{pos}, \text{neg}))\) is \textit{realizable}.

It turns out that for \( d = 1, 2 \) and 3 the answer is positive, but for \( d = 4 \) the answer is negative; this is due to D. J. Grabiner, see [4]. Namely, for the sign pattern \( \sigma^* := (+, +, -, -, +) \) (with Descartes pair \((2, 2)\)), the pair \((2, 0)\) is admissible, see [1], but the couple \((\sigma^*, (2, 0))\) is not realizable. The proof of this is easy — for a monic polynomial \( P_5 := x^5 + a_4 x^4 + \cdots + a_0 \) with signs of the coefficients defined by \( \sigma^* \) and having exactly two positive roots \( u < v \) one has \( a_j > 0 \) for \( j \neq 2, a_2 < 0 \) and \( P_5((u+v)/2) < 0 \). Hence \( P_j((u+v)/2) < 0 \) because \( a_j((u+v)/2)^j = a_j(-((u+v)/2)^j, j = 0, 2, 4 \) and \( 0 < a_j((u+v)/2)^j = -a_j(-((u+v)/2)^j, j = 1, 3 \). As \( P(0) = a_0 > 0 \), there are two negative roots \( \xi < (u+v)/2 < \eta \) as well.

modulo the standard \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action described below, Grabiner’s example is the only one of nonrealizable couple \((\text{sign pattern}, \text{admissible pair})\) for \( d = 4 \). The \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action is defined on such couples by two generators. Denote by \( \sigma(j) \) the \( j \)th component of the sign pattern \( \sigma \). The first of the generators replaces the sign pattern \( \sigma \) by \( \sigma^* \), where \( \sigma^* \) stands for the reverted (i.e. read from the back) sign pattern multiplied by \( \sigma(0) \), and keeps the same pair \((\text{pos}, \text{neg})\). This generator corresponds to the fact that the polynomials \( P(x) \) and \( x^d P(1/x)/P(0) \) are both monic and have the same numbers of positive and negative roots. The second generator exchanges \( \text{pos} \) with \( \text{neg} \) and changes the signs of \( \sigma \) corresponding to the monomials of odd (resp. even) powers if \( d \) is even (resp. odd); the rest of the signs are preserved. We denote the new sign pattern by \( \sigma_m \). The generator corresponds to the fact that the roots of the polynomials \((\text{both monic}) P(x) \) and \( (-1)^d P(-x) \) are mutually opposite, and if \( \sigma \) is the sign pattern of \( P \), then \( \sigma_m \) is the one of \((-1)^d P(-x) \). For a given sign pattern \( \sigma \) and an admissible pair \((\text{pos}, \text{neg})\) the couples \((\sigma, (\text{pos}, \text{neg}))\), \((\sigma^*, (\text{pos}, \text{neg}))\), \((\sigma_m, (\text{neg}, \text{pos}))\) and \((\sigma_m^*, (\text{neg}, \text{pos}))\) are simultaneously realizable or not. (One has \((\sigma_m)^r = (\sigma^*)_m\).)

All cases of couples (sign pattern, admissible pair) for \( d = 5 \) and 6 which are not realizable are described in [1]. For \( d = 7 \) this is done in [3] and for \( d = 8 \) in [3] and [7]. For \( d = 5 \) there is a single nonrealizable case (up to the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action). The sign pattern is \((+, +, -, +, -, -)\) and the admissible pair is \((3, 0)\). For \( n = 6 \), 7 and 8 there are respectively 4, 6, and 19 nonrealizable cases. In all of them one of the numbers \text{pos} or \text{neg} is 0. It is conjectured in [3] that this is the case for any \( d \). The conjecture is based on the fact that, using a computer, J. Forsgård has shown that this is the case also for \( d = 9 \) and 10.

In the present paper we show that the conjecture fails for \( d = 11 \).

\textbf{Notation 1.} For \( d = 11 \) we denote by \( \sigma^0 \) the following sign pattern (we give on the first and third lines below respectively the sign patterns \( \sigma^0 \) and \( \sigma^0_m \) while the line in the middle indicates the positions of the monomials of odd powers):

\[
\sigma^0 = ( + - - - - - + + + + + ) \\
11 9 7 5 3 1 \\
\sigma^0_m = ( + + - - + + - - + + + ) \\
\]

In a sense \( \sigma^0 \) is centre-antisymmetric – it consists of one plus, five minuses, five pluses and one minus.
Theorem 1. The sign pattern \( \sigma^0 \) is not realizable with the admissible pair \((1, 8)\).

The next section contains comments concerning the above result and realizability of sign patterns and admissible pairs in general. Section 3 contains some technical lemmas which allow to simplify the proof of Theorem 1. The method of proof is explained in Section 4. Section 5 contains the proofs of lemmas used in Section 4.

2. Comments

Theorem 1 shows that the problem of classifying all nonrealizable cases (sign pattern, admissible pair), for any degree \( d \), is a difficult one. At present, an exhaustive answer in a closed form of a theorem is not known. One could try to find sufficient conditions for realizability expressed, say, in terms of the ratios between \( d \), \( c \) and \( p \). In papers [3] and [9] series of nonrealizable cases were found (defined either for every degree \( d \) or for every odd or even degree sufficiently large). In all of them either \( \text{pos} = 0 \) or \( \text{neg} = 0 \). The construction of such series with \( \text{pos} \neq 0 \neq \text{neg} \) and the proof of their nonrealizability seems to be sufficiently hard given that \( d \geq 11 \).

One of the series of nonrealizable cases considered in [3] concerns sign patterns with exactly two sign changes, consisting of \( m \) pluses followed by \( n \) minuses followed by \( q \) pluses, \( m + n + q = d + 1 \). Set

\[ \kappa := \frac{d - m - 1}{m} \cdot \frac{d - q - 1}{q}. \]

Lemma 1. For \( \kappa \geq 4 \) such a sign pattern is not realizable with the admissible pair \((0, d - 2)\). The sign pattern is realizable with any admissible pair of the form \((2, v)\).

The lemma is Proposition 6 of [3]. One of the tools for constructing new realizable cases is the following concatenation lemma (proved in [3]):

Lemma 2. Suppose that the monic polynomials \( P_j \) of degrees \( d_j \) and with sign patterns of the form \((+, \sigma_j), j = 1, 2 \) (where \( \sigma_j \) contains the last \( d_j \) components of the corresponding sign pattern) realize the pairs \((\text{pos}_j, \text{neg}_j)\). Then

1. if the last position of \( \sigma_1 \) is +, then for any \( \varepsilon > 0 \) small enough the polynomial \( \varepsilon d_2 P_1(x)P_2(x/\varepsilon) \) realizes the sign pattern \((+, \sigma_1, \sigma_2)\) and the pair \((\text{pos}_1 + \text{pos}_2, \text{neg}_1 + \text{neg}_2)\);

2. if the last position of \( \sigma_1 \) is -, then for any \( \varepsilon > 0 \) small enough the polynomial \( \varepsilon d_2 P_1(x)P_2(x/\varepsilon) \) realizes the sign pattern \((+, \sigma_1, -\sigma_2)\) and the pair \((\text{pos}_1 + \text{pos}_2, \text{neg}_1 + \text{neg}_2)\) (here \(-\sigma_2\) is obtained from \(\sigma_2\) by changing each + by – and vice versa).

It is clear that if Theorem 1 is true, then one should not be able to deduce with the help of Lemma 2 the realizability of the sign pattern \( \sigma^0 \) with the admissible pair \((1, 8)\). Now we show that this is indeed impossible. It suffices to check the cases \( \deg P_1 \geq 6, \deg P_2 \leq 5 \) due to the centre-antisymmetry of \( \sigma^0 \) and the possibility to use the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action.

In all these cases the sign pattern of the polynomial \( P_1 \) has exactly two sign changes (it comprises the first sign +, the five minuses that follow and the next between one and five pluses). These cases are (we use the notation from Lemma 1) \( m = 1, n = 5, q = 1, \ldots, 5 \). The values of \( \kappa \) are respectively 16, 10, 8, 7 and \( 32/5 \), all of them are > 4. By Descartes’ rule the polynomial \( P_1 \) can have either 0 or 2 positive roots. Should it have 2, then its concatenation with \( P_2 \) should have
at least 2 positive roots (by Lemma 2) which is impossible. So \( P_1 \) has no positive roots. The sign patterns defined respectively by \( P_1 \) and \( P_2 \) have \( 4 + (q - 1) \) and \( 5 - q \) sign preservations. By Lemma 1 the polynomial \( P_1 \) has \( \leq 2 + (q - 1) \) negative roots, and as \( P_2 \) has \( \leq 5 - q \) ones, the concatenation of \( P_1 \) and \( P_2 \) has \( \leq 6 \) negative roots. Therefore a polynomial (if it exists) realizing the couple \((\sigma^0, (1, 8))\) cannot be represented as a concatenation of two polynomials \( P_1 \) and \( P_2 \).

This still does not exclude the existence of such a polynomial. In [3] certain examples of polynomials realizing given sign patterns and admissible pairs had to be constructed directly. Before passing to the proof of Theorem 1 we explain the role that the concatenation lemma could play in solving the problem of realizability of sign patterns with admissible pairs.

If in the process of solving this problem one arrives to a situation when there exists \( d_0 \in \mathbb{N} \) such that for \( d \geq d_0 \) the realizability of all realizable cases can be deduced from some general statements and from the concatenation lemma, then it would be sufficient to find the exhaustive list of realizable cases for \( d < d_0 \) and the problem would be solved. One could qualify as a general statement Lemma 1 or the fact that, for even \( d \), a sign pattern consisting of \( d + 1 \) pluses is realizable with the pair \((0, 0)\), see [3], etc. The (non)existence of such a degree \( d_0 \) is not self-evident, and if it exists, it is not a priori clear how many new general statements of (non)realizability should have to be proved.

3. Preliminaries

Notation 2. We denote by \( S \) the subset of \( \mathbb{R}^{11} \) such that if \( a \in S \), then the signs of the coefficients of the polynomial \( P(x, a) = x^{11} + a_{10}x^{10} + \cdots + a_0 \) define the sign pattern \( \sigma^0 \) and the polynomial \( P \) realizes the pair \((1, 8)\).

By \( T \) we define the subset of \( S \) for which one has \( a_{10} = -1 \). For a polynomial from \( S \) one can obtain the conditions \( a_{11} = 1 \), \( a_{10} = -1 \) by rescaling and multiplication by a nonzero constant (\( a_{11} \) stands for the leading coefficient).

Lemma 3. For \( a \in \bar{S} \) one has \( a_j \neq 0 \) for \( j = 9, 8, 7, 4, 3, 2 \), and one does not have \( a_6 = 0 \) and \( a_5 = 0 \) simultaneously.

Indeed, for \( a_j = 0 \) (where \( j \) is one of the indices \( 9, 8, 7, 4, 3, 2 \)) there are less than 8 sign changes in the sign pattern \( \sigma^0_m \) hence by Descartes’ rule of signs the polynomial \( P(., a) \) has less than 8 negative roots counted with multiplicity. The same is true for \( a_5 = a_6 = 0 \).

Lemma 4. For \( a \in \bar{S} \) one has \( a_0 \neq 0 \).

Remark 1. A priori the set \( \bar{S} \) can contain polynomials with all roots real and nonzero. The positive ones can be either a triple one or a double and a simple ones (but not three simple ones).

Proof of Lemma 4. Consider first the case \( a_j \neq 0 \) (\( j \neq 0 \)), \( a_0 = 0 \). Hence the polynomial \( P \) has a root at 0, either 0 or 2 positive roots and 8 negative roots. Suppose that \( P \) has no positive roots. Then the degree 10 polynomial \( P/x \) defines a sign pattern with exactly two sign changes and has 8 negative roots. There exists no such polynomial. Indeed, if it is with distinct negative roots and with no positive roots, then this would contradict Lemma 1 (with the notation of Lemma 1 one has \( \kappa = 32/5 > 4 \)). If it has 8 negative roots counted with multiplicity, then one can make them distinct by a series of perturbations which do not change the signs of
the coefficients of the polynomial, which increase the number of distinct negative roots while keeping their total multiplicity equal to 8 and which do not introduce new positive roots as follows.

Suppose that $P$ has a negative root $-b$ of multiplicity $r$, $1 < r \leq 8$. Set $P \mapsto P + \varepsilon P_1$, where $\varepsilon \in (\mathbb{R}, 0)$, $\varepsilon > 0$ and if $P = (x + b)^r x Q_1 Q_2$, where $Q_1, Q_2$ are polynomials, $Q_2$ having a complex conjugate pair of roots, $Q_1$ having $8 - r$ negative roots counted with multiplicity, then $P_1 = (x + b)^{r - 1} x Q_1$ (this decreases by 1 the multiplicity of the root $-b$ and introduces a new simple negative root).

If the polynomial $P/x$ has two positive roots, then in fact this must be a double positive root $g$ because $a \in \mathcal{S}$. In this case the perturbations are with $P_1$ of the form $(x + b)^{r - 1} x Q_1 (x - g)^2$; after having thus obtained $P$ with 8 negative simple roots and a double root at $g$ one makes another perturbation $P \mapsto P \pm \varepsilon x$ (the sign before $\varepsilon$ depends on whether $P$ has a minimum or maximum at $g$) after which the degree 10 polynomial $P/x$ is with 8 negative simple roots and no other real root which is a contradiction with Lemma 1.

Suppose now that $a_j \neq 0$ ($j \geq 2$) and $a_1 = a_0 = 0$. One considers in the same way the degree 9 polynomial $P/x^2$ to obtain a contradiction with Lemma 1. In this case one has $\kappa = 7$.

Suppose now that exactly one of the coefficients $a_5$ or $a_6$ is 0 (we assume that this is $a_5$, for $a_6$ the reasoning is analogous) and either $a_1 \neq 0$, $a_0 = 0$ or $a_1 = a_0 = 0$ (all other coefficients $a_j$ being nonzero). Then in the perturbations we set $P_1 = (x + b)^{r - 1} x (h_1 + h_2) Q_1$, where the real numbers $h_i$ are distinct, different from any of the roots of $P$ and chosen such that the coefficient $\delta$ of $x^2$ of $P_1$ is 0. The choice is possible because all coefficients of the polynomial $(x + b)^{r - 1} Q_1$ are positive hence $\delta$ is of the form $A + (h_1 + h_2)B + C h_1 h_2$, where $A > 0$, $B > 0$ and $C > 0$.

From now on we consider mainly $T$ (and not $S$) in order not to take into account the possibility $a_{10}$ to vanish at some points of $\mathcal{S}$.

**Remark 2.** It results from Lemmas 3 and 4 that for a polynomial in $\mathcal{T}$ exactly one of the following possibilities exists: 1) all its coefficients are nonvanishing; 2) exactly one of them is vanishing, and this is either $a_1$ or $a_5$ or $a_6$; 3) exactly two of them are vanishing, and these are either $a_1$ and $a_5$ or $a_1$ and $a_6$.

**Lemma 5.** There exists no real degree 11 polynomial the signs of whose coefficients define the sign pattern $\sigma^0$ and which has a single positive simple root, negative roots of total multiplicity 8 and a complex conjugate pair with nonpositive real part.

**Proof.** Suppose that such a monic polynomial exists. One can represent it in the form $P = P_1 P_2 P_3$, where $\deg P_1 = 8$, all roots of $P_1$ are negative hence $P_1 = \sum_{j=0}^{8} a_j x^j$, $a_j > 0$, $a_8 = 1$; $P_2 = x - w$, $w > 0$; $P_3 = x^2 + \beta_1 x + \beta_0$, $\beta_j \geq 0$, $\beta_1^2 - 4 \beta_0 < 0$.

By Descartes’ rule of signs the polynomial $P_1 P_2 = \sum_{j=0}^{9} \gamma_j x^j$, $\gamma_9 = 1$, has exactly one sign change in the sequence of its coefficients. It is clear that as $0 > a_{10} = \gamma_8 + \beta_1$, and as $\beta_1 \geq 0$, one must have $\gamma_8 < 0$. But then $\gamma_j < 0$ for $j = 0, \ldots, 8$. For $j = 4, \ldots, 8$ one has $a_j = \gamma_{j-2} + \beta_1 \gamma_{j-1} + \beta_0 \gamma_j < 0$ which means that the signs of $a_j$ do not define the sign pattern $\sigma^0$. □

**Remark 3.** Lemma 5 implies that the set $\mathcal{T}$ can contain only polynomials with negative roots of total multiplicity 8 and positive roots of total multiplicity 1 or 3.
Suppose that there exists a monic polynomial $P$ with no root at 0 (Lemma 4). Indeed, when approaching the border of $T$, the complex conjugate pair can coalesce into a double positive (but never nonpositive) root; the latter might eventually coincide with the simple positive root.

4. The method of proof

Consider $\mathbb{R}^{10}$ as the space of the coefficients of the polynomial $P(x, a)|_{a_{10}=-1}$. Suppose that there exists a monic polynomial $P(x, a^*)$ with signs of its coefficients as defined by the sign pattern $\sigma^0$ (with $a_{10} = -1$), with 8 distinct negative, a simple positive and two complex conjugate roots. Then for $a$ close to $a^* \in \mathbb{R}^{10}$ all polynomials $P(x, a)$ share with $P(x, a^*)$ these properties. Therefore the interior of the set $T$ is nonempty. In what follows we denote by $\Gamma$ the connected component of $T$ to which $a^*$ belongs. Denote by $-\delta$ the value of $a_9$ for $a = a^*$ (recall that this value is negative).

Lemma 6. There exists a compact set $K \subset \bar{\Gamma}$ containing all points of $\bar{\Gamma}$ with $a_9 \in [-\delta, 0)$. Hence there exists $\delta_0 > 0$ such that for every point of $\bar{\Gamma}$ one has $a_9 \leq -\delta_0$, and for at least one point of $K$ and for no point of $\bar{\Gamma} \setminus K$ does one have $a_9 = -\delta_0$.

Proof. Suppose that there exists an unbounded sequence $\{a^n\}$ of values of $a \in \bar{\Gamma}$ with $a_9^n \in [-\delta, 0)$. Hence one can perform rescalings $x \mapsto \beta_n x$, $\beta_n > 0$, such that the largest of the moduli of the coefficients of the monic polynomials $Q_n := (\beta_n)^{-11}P(\beta_n x, a^n)$ equals 1. These polynomials belong to $\bar{S}$, not necessarily to $\bar{T}$ because $a_{10}$ after the rescalings, in general, is not equal to $-1$. The coefficient of $x^9$ in $Q_n$ equals $a_9^n/\beta_n^2$. The sequence $\{a^n\}$ being unbounded there exists a subsequence $\beta_{n_k}$ tending to $\infty$. This means that the sequence of monic polynomials $Q_{n_k} \in \bar{S}$ with bounded coefficients has as one of its limit points a polynomial in $\bar{S}$ with $a_9 = 0$ which contradicts Lemma 3.

Hence the tuple of coefficients $a_j$ of $P(x, a) \in \bar{\Gamma}$ with $a_9 \in [-\delta, 0)$ remains bounded (hence the same holds true for the moduli of the roots of $P$) from which the existence of $K$ and $\delta_0$ follows.

The above lemma implies the existence of a polynomial $P_0 \in \bar{\Gamma}$ with $a_9 = -\delta_0$. We say that $P_0$ is $a_9$-maximal. Our aim is to show that no polynomial of $\bar{\Gamma}$ is $a_9$-maximal which contradiction will be the proof of Theorem 4.

Definition 1. A real univariate polynomial is hyperbolic if it has only real (not necessarily simple) roots. We denote by $H \subset \bar{\Gamma}$ the set of hyperbolic polynomials in $\bar{\Gamma}$. Hence these are monic degree 11 polynomials having positive and negative roots of respective total multiplicities 3 and 8 (zero roots are impossible by Lemma 3). By $U \subset \bar{\Gamma}$ we denote the set of polynomials in $\bar{\Gamma}$ having a complex conjugate pair, a simple positive root and negative roots of total multiplicity 8. Thus $\bar{\Gamma} = H \cup U$ and $H \cap U = \emptyset$. We denote by $U_0, U_2, U_{2.2}, U_3$ and $U_4$ the subsets of $U$ for which the polynomial $P \in U$ has respectively 8 simple negative roots, one double and 6 simple negative roots, at least two negative roots of multiplicity $\geq 2$, one triple and 5 simple negative roots and a negative root of multiplicity $\geq 4$.

The following lemma on hyperbolic polynomials will be used further in the proofs.

Lemma 7. Suppose that $V$ is a degree $d \geq 2$ hyperbolic polynomial with no root at 0. Then:
(1) $V$ does not have two or more consecutive vanishing coefficients.
(2) If $V$ has a vanishing coefficient, then the signs of its surrounding two coefficients are opposite.
(3) The number of positive (of negative) roots of $V$ is equal to the number of sign changes in the sequence of its coefficients (in the one of $V(-x)$).

The proofs of the lemmas of this section except Lemma 6 are given in Sections 5 (Lemmas 7–12), 6 (Lemma 13) and 7 (Lemmas 14–16).

**Lemma 8.**
(1) No polynomial of $U_{2,2} \cup U_4$ is $a_9$-maximal.
(2) For each polynomial of $U_3$ there exists a polynomial of $U_0$ with the same values of $a_9, a_6, a_5$ and $a_1$.
(3) For each polynomial of $U_0 \cup U_2$ there exists a polynomial of $H \cup U_{2,2}$ with the same values of $a_9, a_6, a_5$ and $a_1$.

The lemma implies that if there exists an $a_9$-maximal polynomial in $\bar{\Gamma}$, then there exists such a polynomial in $H$. So from now on we aim at proving that $H$ contains no such polynomial hence $H$ and $\bar{\Gamma}$ are empty.

**Lemma 9.** There exists no polynomial in $H$ having exactly two distinct real roots.

**Lemma 10.** The set $H$ contains no polynomial having one triple positive root and negative roots of total multiplicity 8.

Hence a polynomial in $H$ (if any) has a double and a simple positive roots and negative roots of total multiplicity 8.

**Lemma 11.** There exists no polynomial $P \in H$ having exactly three distinct real roots and satisfying the conditions $\{a_1 = 0, a_5 = 0\}$ or $\{a_1 = 0, a_6 = 0\}$.

It follows from the lemma and from Lemma 3 that a polynomial $P \in H$ having exactly three distinct roots can satisfy at most one of the conditions $a_1 = 0, a_5 = 0$ and $a_6 = 0$.

**Lemma 12.** No polynomial in $H$ having exactly three distinct real roots is $a_9$-maximal.

Thus an $a_9$-maximal polynomial in $H$ (if any) must have at least four distinct real roots.

**Lemma 13.** The set $H$ contains no polynomial having a double and a simple positive roots and exactly two distinct negative roots of total multiplicity 8, and which satisfies either the conditions $\{a_1 = a_5 = 0\}$ or $\{a_1 = a_6 = 0\}$.

**Lemma 14.** The set $H$ contains no $a_9$-maximal polynomial having exactly four distinct real roots and satisfying exactly one or none of the conditions $a_1 = 0$, $a_5 = 0$ or $a_6 = 0$.

Therefore an $a_9$-maximal polynomial in $H$ (if any) must have at least five distinct real roots.

**Lemma 15.** The set $H$ contains no $a_9$-maximal polynomial having exactly five distinct real roots.

**Lemma 16.** The set $H$ contains no $a_9$-maximal polynomial having at least six distinct real roots.

Hence the set $H$ contains no $a_9$-maximal polynomial at all. It results from Lemma 8 that there is no such polynomial in $\bar{\Gamma}$. Hence $\bar{\Gamma} = \emptyset$.
5. PROOFS OF LEMMAS 7, 8, 9, 11, 12 AND 13

Proof of Lemma 7: Prove part (1). Suppose that a hyperbolic polynomial $V$ with two or more vanishing coefficients exists. If $V$ is degree $d$ hyperbolic, then $V^{(k)}$ is also hyperbolic for $1 \leq k < d$. Therefore we can assume that $V$ is of the form $x^d + c$, where $\deg L = d$, $\ell \geq 3$, $L(0) \neq 0$ and $c = V(0) \neq 0$. If $V$ is hyperbolic and $V(0) \neq 0$, then such is also $W := x^d + x^d L(1/x)$ and also $W^{(d-\ell)}$ which is of the form $ax^\ell + b$, $a \neq 0 \neq b$. However given that $\ell \geq 3$ this polynomial is not hyperbolic.

For the proof of part (2) we use exactly the same reasoning, but with $\ell = 2$. The polynomial $ax^2 + b$, $a \neq 0 \neq b$, is hyperbolic if and only if $ab < 0$.

To prove part (3) we consider the sequence of coefficients of $V := \sum_{j=0}^d v_j x^j$, $v_0 \neq 0 \neq v_d$. Set $\Phi := \{k | v_k \neq 0 \neq v_{k-1}, v_{k-1} < 0\}$, $\Psi := \{k | v_k \neq 0 \neq v_{k-1} \geq 0\}$ and $\Lambda := \{k | v_k = 0\}$. Then $\Phi + \Psi + 2\Lambda = d$. By Descartes’ rule of signs the number of positive (of negative) roots of $V$ is $\text{pos}_V \leq \Phi + \Lambda$ (resp. $\text{neg}_V \leq \Psi + \Lambda$). As $\text{pos}_V + \text{neg}_V = d$, one must have $\text{pos}_V = \Phi + \Lambda$ and $\text{neg}_V = \Psi + \Lambda$. There remains to notice that $\Phi + \Lambda$ is the number of sign changes in the sequence of coefficients of $V$ (and $\Psi + \Lambda$ of $V(-x)$), see part (2) of the lemma.

Proof of Lemma 8: Prove part (1). A polynomial of $U_2, 2$ or $U_4$ respectively is representable in the form:

$$P := (x + u)^2(x + v)^2S^2$$

where $\Delta := (x^2 - \xi x + \eta)(x - w)$ and $S := x^4 + Ax^3 + Bx^2 + Cx + D$. All coefficients $\eta$, $\xi$, $\eta$, $A$, $B$, $C$, $D$ are positive and $\xi^2 - 4\eta < 0$ (see Lemma 5): for $A$, $B$, $C$ and $D$ this follows from all roots of $P^2/\Delta$ and $P^4/\Delta$ being negative. (The roots of $x^4 + Ax^3 + Bx^2 + Cx + D$ are not necessarily different from $-u$ and $-v$.) We consider the two Jacobian matrices

$$J_1 := (\partial(a_{10}, a_9, a_1, a_5)/\partial(\xi, \eta, w, u))$$
$$J_2 := (\partial(a_{10}, a_9, a_1, a_6)/\partial(\xi, \eta, w, u))$$

In the case of $P^1$ their determinants equal

$$\det J_1 = \Pi(CDV + 2CDu + C^2uv + 2BDuv + 4BDuv$$
$$+ 2BDu + 2BCu^2v + BCU^2v + ADuv + 3ADuv + ACuv^2 + 2ACu^2v^2)$$
$$\det J_2 = \Pi(BDV + 2BDu + Du^2 + 2Du^2 + 3Du^2 + BCuv + 2ADuv + 4ADuv + 4ADuv + 2ADuv + Cuv^2 + 2u^2v^2 C + 2ACuv^2 + ACu^2v^2)$$
$$\Pi := -2v(u + w)(-\eta - w^2 + w\xi)(\xi u + \eta + w^2)$$

These determinants are nonzero. Indeed, each of the factors is either a sum of positive terms or equals $-\eta - w^2 + w\xi < -\xi^2/4 - w^2 + w\xi = -(\xi^2/2 - w)^2 \leq 0$. Thus one can choose values of $(\xi, \eta, w, v)$ close to the initial one $(u, A, B, C$ and $D$ remain fixed) to obtain any values of $(a_{10}, a_9, a_1, a_5)$ or $(a_{10}, a_9, a_1, a_6)$ close to the initial one. In particular, with $a_{10} = -1$, $a_1 = 5 = 0$ or $a_{10} = -1$, $a_1 = a_6 = 0$ while $a_9$ can have values larger than the initial one. Hence this is not an $a_9$-maximal polynomial. (The values of the coefficients $a_j$, $j = 0, 2, 3, 4, 6$ or $5, 7$ and $8$ can change, but their signs remain the same if the change of the value of $(\xi, \eta, w, v)$ is small enough.) The same reasoning is valid for $P^4$ as well in which case one has
\[ \det J_1 = M(3CD + C^2u + 8BDu + 3BCu^2 + 6ADu^2 + u^3C + 3ACu^3), \]
\[ \det J_2 = M(3BD + 6u^2D + BCu + 8ADu + 3u^3C + 3ACu^2), \]
\[ M := -4u^2(w + u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2). \]

To prove part (2) we observe that if the triple root of \( P \in U_3 \) is at \(-u < 0\), then in case when \( P \) is increasing (resp. decreasing) in a neighbourhood of \(-u\) the polynomial \( P - \varepsilon x^2(x + u) \) (resp. \( P + \varepsilon x^2(x + u) \)), where \( \varepsilon > 0 \) is small enough, has three simple roots close to \(-u\); it belongs to \( \Gamma \), its coefficients \( a_j, 2 \neq j \neq 3 \), are the same as the ones of \( P \), the signs of \( a_2 \) and \( a_3 \) are also the same.

For the proof of part (3) we observe first that 1) for \( x < 0 \) the polynomial \( P \) has four maxima and four minima and 2) for \( x > 0 \) one of the following three things holds true: one has \( P' > 0 \), or there is a double positive root \( \gamma \) of \( P' \), or \( P' \) has two positive roots \( \gamma_1 < \gamma_2 \) (they are both either smaller or greater than the positive root of \( P \)). Suppose first that \( P \in U_0 \). Consider the family of polynomials \( P - t \), \( t \geq 0 \). Denote by \( t_0 \) the smallest value of \( t \) for which one of the three things happens: \( P - t \) has a double negative root \( v \) (hence a local maximum), \( P - t \) has a triple positive root \( \gamma \) or \( P - t \) has a double and a simple positive roots (the double one is at \( \gamma_1 \) or \( \gamma_2 \)). In the second and third case one has \( P - t_0 \in H \). In the first case, if \( P - t_0 \) has another double negative root, then \( P - t_0 \in U_{2,2} \) and we are done. If not, then consider the family of polynomials

\[ P_s := P - t_0 - s(x^2 - v^2)^2(x^2 + v^2)^2 = P - t_0 - s(x^8 - 2v^4x^4 + v^8), \quad s \geq 0. \]

The polynomial \( -(x^8 - 2v^4x^4 + v^8) \) has double real roots at \( \pm v \) and a double complex conjugate pair. It has the same signs of the coefficients of \( x^8 \), \( x^4 \) and 1 as \( P - t_0 \) and \( P \). The rest of the coefficients of \( P - t_0 \) and \( P_s \) are the same. As \( s \) increases, the value of \( P_s \) for every \( x \neq \pm v \) decreases, so for some \( s = s_0 \) \( 0 \) for the first time one has either \( P_s \in U_{2,2} \) (another local maximum of \( P_s \) becomes a double negative root) or \( P_s \in H \) (\( P_s \) has positive roots of total multiplicity 3, but not three simple ones). This proves part (3) for \( P \in U_0 \).

If \( P \in U_2 \) and the double negative root is a local minimum, then the proof of part (3) is just the same. If this is a local maximum, then one skips the construction of the family \( P - t \) and starts constructing directly the family \( P_s \).

**Proof of Lemma** Suppose that such a polynomial exists. Then it must be of the form \( P := (x + u)^3(x - w)^3, \ u > 0, \ w > 0 \). The conditions \( a_{10} = -1 \) and \( a_1 > 0 \) read:

\[ 8u - 3w = -1 \quad \text{and} \quad u^7w^2(3u - 8w) > 0. \]

In the plane of the variables \( (u, w) \) the domain \( \{u > 0, \ w > 0, \ 3u - 8w > 0\} \) does not intersect the line \( 8u - 3w = -1 \) which proves the lemma.

**Proof of Lemma** Represent the polynomial in the form \( P = (x + u_1) \cdots (x + u_8)(x - \xi)^3 \), where \( u_j > 0 \) and \( \xi > 0 \). The numbers \( u_j \) are not necessarily distinct. The coefficient \( a_{10} \) then equals \( u_1 + \cdots + u_8 - 3\xi \). The condition \( a_{10} = -1 \) implies \( \xi = \xi_* := (u_1 + \cdots + u_8 + 1)/3 \). Denote by \( \tilde{a}_1 \) the coefficient \( a_1 \) expressed as a function of \( (u_1, \ldots, u_8, \xi) \). Using computer algebra (say, MAPLE) one finds \( 27\tilde{a}_1|_{\xi = \xi_*} \):

\[ 27\tilde{a}_1|_{\xi = \xi_*} = -(-u_1 \cdots u_8 + X + Y)(u_1 + \cdots + u_8 + 1)^2, \]
where \( Y := u_1 \cdots u_8(1/u_1 + \cdots + 1/u_8) \) and \( X := u_1 \cdots u_8 \sum_{1 \leq i,j \leq 8, i \neq j} u_i/u_j \) (the sum \( X \) contains 56 terms). We show that \( a_1 < 0 \) which contradiction proves the lemma. The factor \((u_1+\cdots+u_8+1)^2\) is positive. The factor \( \Xi := -u_1 \cdots u_8 + X + Y \) contains a single monomial with a negative coefficient, namely, \(-u_1 \cdots u_8\). Consider the sum

\[
-u_1 \cdots u_8 + u_1^2 u_3 u_4 u_5 u_6 u_7 u_8 + u_2^2 u_3 u_4 u_5 u_6 u_7 u_8
\]

(the second and third monomials are in \( X \)). Hence \( \Xi \) is representable as a sum of positive quantities, so \( \Xi > 0 \) and \( a_1 < 0 \).

**Proof of Lemma 11:** Suppose that such a polynomial exists. Then it must be of the form \((x + u)^8(x - w)^2(x - \xi)\), where \( u > 0, w > 0, \xi > 0, w \neq \xi \). One checks numerically (say, using MAPLE), for each of the two systems of algebraic equations

\[
a_{10} = -1, \quad a_1 = 0, \quad a_5 = 0 \quad \text{and} \quad a_{10} = -1, \quad a_1 = 0, \quad a_6 = 0,
\]

that each real solution \((u, w, \xi)\) or \((u, v, w)\) contains a nonpositive component.

**Proof of Lemma 12:** Making use of Lemma 10 we consider only polynomials of the form \((x+u)^8(x-w)^2(x-\xi)\). Consider the Jacobian matrix \( J^*_5 := (\partial(a_{10}, a_9, a_1)/\partial(u, w, \xi)) \). Its determinant equals \( 6u^6(u+w)(u-7w)(\xi-w)(k+u) \). All factors except \( u-7w \) are nonzero. Thus for \( u \neq 7w \) one has \( \det J_1 \neq 0 \), so one can fix the values of \( a_{10} \) and \( a_1 \) and vary the one of \( a_9 \) arbitrarily close to the initial one by choosing suitable values of \( u, w \) and \( \xi \). Hence the polynomial is not \( a_9 \)-maximal. For \( u = 7w \) one has \( a_3 = -117649u^2(35w + 8\xi) < 0 \) which is impossible. Hence there exist no \( a_9 \)-maximal polynomials which satisfy only the condition \( a_1 = 0 \) or none of the conditions \( a_1 = 0, a_5 = 0 \) or \( a_6 = 0 \). To see that there exist such polynomials satisfying only the condition \( a_5 = 0 \) or \( a_6 = 0 \) one can consider the matrices \( J^*_5 := (\partial(a_{10}, a_9, a_5)/\partial(u, w, \xi)) \) and \( J^*_6 := (\partial(a_{10}, a_9, a_6)/\partial(u, w, \xi)) \). Their determinants equal respectively

\[
112u^2(u+w)(5u-3w)(\xi-w)(\xi+u) \quad \text{and} \quad 112(u+w)(3u-w)(\xi-w)(\xi+u)
\]

and are nonzero respectively for \( 5u \neq 3w \) and \( 3u \neq w \), in which cases in the same way we conclude that the polynomial is not \( w_3 \)-maximal. If \( u = 3w/5 \), then \( a_1 = -(2187/390625)w^2(-3w+34\xi) \) and \( a_{10} = -\xi + 14w/5 \). As \( a_1 > 0 \) and \( a_{10} < 0 \), one has \( w > 34\xi/3 \) and \( \xi > 14w/5 > (34/3)(14/5)\xi \) which is a contradiction. If \( w = 3u \), then \( a_6 = 14u^2(10u+\xi) > 0 \) which is again a contradiction.

\[
6. \text{Proof of Lemma 13}
\]

The multiplicities of the negative roots of \( P \) define the following a priori possible cases:

A) (7, 1), B) (6, 2), C) (5, 3) and D) (4, 4).

In all of them the proof is carried out simultaneously for the two possibilities \( \{a_1 = a_5 = 0\} \) and \( \{a_1 = a_6 = 0\} \). In order to simplify the proof we fix one of the roots to be equal to \(-1\) (this can be achieved by a change \( x \mapsto \beta x, \beta > 0 \), followed by \( P \mapsto \beta^{-11}P \)). This allows to deal with one parameter less. By doing so we can no longer require that \( a_{10} = -1 \), but only that \( a_{10} < 0 \).
Proof in Case A): We use the following parametrisation: $P = (x+1)^7(sx+1)(tx-1)^2(wx-1)$, $s > 0$, $t > 0$, $w > 0$, $t \neq w$, i.e. the negative roots of $P$ are at $-1$ and $-1/s$ and the positive ones at $1/t$ and $1/w$.

The condition $a_1 = w + 2t - s - 7 = 0$ yields $s = w + 2t - 7$. For $s = w + 2t - 7$ one has

\[
\begin{align*}
a_3 &= a_{32}w^2 + a_{31}w + a_{30} & a_4 &= a_{42}w^2 + a_{41}w + a_{40}, \\
a_{31} &= -(2t - 7)^2 & a_{32} &= -2t + 7, \\
a_{30} &= -2t^3 + 28t^2 - 98t + 112 & a_{40} &= \frac{-2t - 14}{2} + 1, \\
&= 2t^3 - 35t^2 + 140t - 147 & a_{42} &= t^2 - 14t + 21
\end{align*}
\]

The coefficient $a_{30}$ has a single real root $9.436\ldots$ hence $a_{30} < 0$ for $t > 9.436\ldots$. On the other hand

\[
a_{32}w^2 + a_{31}w = w(-2t + 7)(w + 2t - 7) = w(-2t + 7)s
\]

which is $< 0$ for $t > 9.436\ldots$. Thus the inequality $a_3 > 0$ fails for $t > 9.436\ldots$. Observing that $a_{41} = (2t - 7)a_{42}$ one can write

\[a_4 = (w + 2t - 7)wa_{42} + a_{40} = swa_{42} + a_{40}.
\]

The real roots of $a_{42}$ (resp. $a_{40}$) equal $1.708\ldots$ and $12.291\ldots$ (resp. $1.136\ldots$). Hence for $t \in [1.708\ldots, 12.291\ldots]$ the inequality $a_4 > 0$ fails. There remains to consider the possibility $t \in (0, 1.708\ldots)$.

It is to be checked directly that for $s = w + 2t - 7$ one has

\[a_{10}/t = (7t - 2)w(w + 2t - 7) + t(7 - 2t) = (7t - 2)ws + t(7 - 2t)
\]

which is $\geq 0$ (hence $a_{10} < 0$ fails) for $t \in [2/7, 7/2]$. Similarly

\[
\begin{align*}
a_6 &= a_6 w^2 + a_7 + \frac{1}{7} w^3 + a_8 + \frac{1}{7} w^3, & \quad a_6 &= \frac{21t^2 - 70t + 35}{2}, & \quad a_8 &= \frac{-70t^3 + 350t^2 - 490t + 140}{7},
\end{align*}
\]

The real roots of $a_6$ (resp. $a_8$) equal $0.612\ldots$ and $0.612\ldots > 2/7 = 0.285\ldots$ and $2.720\ldots$ (resp. $0.381\ldots > 2/7, 2, 2.618\ldots$) hence for $t \in (0, 2/7)$ one has $a_8^* > 0$ and $a_8^* > 0$, i.e. $a_6 > 0$ and the equality $a_6 = 0$ or the inequality $a_6 < 0$ is impossible.

\[\square\]

Proof in Case B): We parametrise $P$ as follows: $P = (x+1)^6(Tx^2 + Sx - 1)^2(wx - 1)$, $T > 0$, $w > 0$. In this case we presume $S$ to be real, not necessarily positive. The factor $(Tx^2 + Sx - 1)^2$ contains the double positive and negative roots of $P$.

From $a_1 = w + 2S - 6 = 0$ one finds $S = (6 - w)/2$. For $S = (6 - w)/2$ one has

\[
\begin{align*}
a_{10}/T &= (6w - 1)T^2 + 6w - w^2, & \quad a_7 &= a_7T^2 + a_7T^2, & \quad a_7T^2 &= 15w - 20, \\
a_{72} &= 15w - 20, & \quad a_{71} &= -20w^2 + 105w - 78, & \quad 4a_{70} &= 15w^3 - 162w^2 + 468w - 192.
\end{align*}
\]

Suppose first that $w > 1/6$. The inequality $a_{10} < 0$ is equivalent to $T < (w^2 - 6w)/(6w - 1)$. As $T > 0$, this implies $w > 6$. 

\[\square\]
For $T = (w^2 - 6w)/(6w - 1)$ one obtains $a_7 = 3C/(6w - 1)^2$, where the numerator
$C := 40w^5 - 444w^4 + 1345w^3 - 502w^2 + 300w - 64$ has a single real root $0.253\ldots$.
Hence for $t > 6$ one has $C > 0$ and $a_7|_{T=(w^2 - 6w)/(6w - 1)} > 0$. On the other hand
$a_7|_{T=0}$ has roots $0.489\ldots$, $4.504\ldots$ and $5.805\ldots$, so for $w > 6$ one has
$a_7|_{T=0} > 0$. For $w > 6$ fixed, and for $T \in [0, (w^2 - 6w)/(6w - 1)]$, the value of the derivative

$$\frac{\partial a_7}{\partial T} = (30w - 40)T - 20w^2 + 105w - 78$$

is maximal for $T = (w^2 - 6w)/(6w - 1)$; this value equals

$$-(90w^3 - 430w^2 + 333w - 78)/(6w - 1)$$

which is $< 0$ because the only real root of the numerator is $3.882\ldots$. Thus
$\frac{\partial a_7}{\partial T} < 0$ and $a_7$ is minimal for $T = (w^2 - 6w)/(6w - 1)$. Hence the inequality
$a_7 < 0$ fails for $w > 1/6$. For $w = 1/6$ one has $a_{10} = 35T/36 > 0$.

So suppose that $w \in (0, 1/6)$. In this case the condition $a_{10} < 0$ implies $T > (w^2 - 6w)/(6w - 1)$. For $T = (w^2 - 6w)/(6w - 1)$ one gets

$$a_4 = 3D/(6w - 1)^2,$$

where $D := 64w^5 - 300w^4 + 502w^3 - 1345w^2 + 444w - 40$ has a single real root $3.939\ldots$. Hence for $w \in (0, 1/6)$ one has $D < 0$ and
$a_4|_{T=(w^2 - 6w)/(6w - 1)} < 0$. The derivative $\frac{\partial a_4}{\partial T} = -w^2 - 2T - 6$ being negative
one has $a_4 < 0$ for $w \in (0, 1/6)$, i.e. the inequality $a_4 > 0$ fails. □

Proof in Case C): We use the following parametrisation: $P = (x+1)^3(xs+1)^3(x^2 - 1)^2(xw - 1)$.

From $a_1 = w + 2t - 5 - 3s = 0$ one gets $s = (w + 2t - 5)/3$. For $s = (w + 2t - 5)/3$
one has $27a_{10} = tS(w + 2t - 5)^2$, where

$$S := 10wt^2 - 2t^2 + 5w^2t - 21wt + 5t - 2w^2 + 10w.$$

The factor $S$ can be represented as a polynomial in $w$ or in $t$; for each of the cases
we give its discriminant (and the latter’s real roots) as well:

$$S = (5t - 2)w^2 + (10 - 21t + 10t^2)w + 5t - 2t^2,$$

$$D_1 = 5(t-2)(2t-1)(10t^2 - 13t + 10), \quad 0.5, 2$$

$$S = (10w - 2)w^2 + (5w^2 - 21w + 5)t - 2w^2 + 10w,$$

$$D_2 = 5(w^2 - 5w + 1)(5w^2 - w + 5), \quad 0.208\ldots, 4.791\ldots$$

Hence for $t \in [0.5, 2]$ or for $w \in [0.208\ldots, 4.791\ldots]$ one has respectively $D_1 \leq 0$
and $D_2 \leq 0$ hence $S \geq 0$ and the inequality $a_{10} < 0$ fails. The partial derivative

$$\frac{\partial S}{\partial t} = 5w^2 - 21w + 20wt - 4t + 5 = 5w(w - 4.2) + (20w - 4)t + 5$$

is positive for $t > 2$ and $w > 4.791\ldots$. Hence $S > 0$ for $t > 2$ and $w > 4.791\ldots$.

For $(t, w) \in (0, 0.5) \times (0, 0.208\ldots)$ one has $w + 2t - 5 < 0$, i.e. $s < 0$. Thus Case C) is impossible outside the two semi-strips

$$\Sigma_1 := \{(t, w) \in (0, 0.5) \times (4.791\ldots, \infty)\} \quad \text{and} \quad \Sigma_2 := \{(t, w) \in (2, \infty) \times (0, 0.208\ldots)\}.$$

**Lemma 17.** The inequality $a_4 > 0$ fails on $\Sigma_2$. 

Proof. Indeed,

\[ 27a_4 = w^4 + s_3w^3 + s_2w^2 + s_1w + s_0 \]

where

\[
\begin{align*}
s_3 &= -10t + 25 \\
s_1 &= -22t^3 + 75t^2 - 120t + 175 \\
s_0 &= -20t^4 + 110t^3 - 300t^2 + 350t - 410
\end{align*}
\]

Indeed, its derivative has no real roots) hence in \( \Sigma_2 \), so in \( \Sigma_1 \).

One can observe that \( \Sigma \) is decreasing for \( t \geq 2 \) (because the only real root of its derivative equals 1), so in \( \Sigma_2 \) one has \( s_0 < s_0 \mid t=2 = -350 \). Finally, the quantity \( s_1 \) is decreasing (its derivative has no real roots) hence in \( \Sigma_2 \) the term \( s_1w \) is less than \( s_1 \mid t=2w \leq 59 \times 0.208 \ldots < 13 \). Thus \( a_4 < 0.05 - 350 + 13 < 0 \) in \( \Sigma_2 \).

We define the sets

\[
\begin{align*}
\Sigma_3 &:= \{(t, w) \in [0, 0.5] \times [6.75, \infty)\}, \\
\Sigma_4 &:= \{(t, w) \in [0.25, 0.5] \times [4.79, 6.75]\}, \\
\Sigma_5 &:= \{(t, w) \in [0, 0.25] \times [5, 6.75]\} \quad \text{and} \\
\Sigma_6 &:= \{(t, w) \in [0, 0.25] \times [4.79, 5]\}.
\end{align*}
\]

One can observe that \( \Sigma_1 \subset (\Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6) \). For \( w = 6.75 \) one has

\[ 27a_6 = 14t^5 + 511.77t^4 - 44.09375t^3 - 6341.949214t^2 - 4336.44531t + 3760.50781 \]

Its real roots are \(-36.303 \ldots, -3.058 \ldots, -1.324 \ldots, 0.503 \ldots \text{ and } 3.629 \ldots\). Hence for \( t \in (0, 0.5) \), \( w = 6.75 \) one has \( a_6 > 0 \). One can represent \( 27 \partial a_6 / \partial w \) in the form \((4w - 5 + 2t)g\), where

\[ g := 4t^4 + 4t^3w + t^2w^2 - 35t^2 - 20wt^2 + 90t - 10w^2t + 20wt - 5 - 40w + 10w^2. \]

Hence \( g \mid w=6.75 = 4t^4 + 27t^3 - 124.4375t^2 - 230.625t + 180.625 \), with real roots \(-9.360 \ldots, -1.982 \ldots, 0.610 \ldots \text{ and } 3.982 \ldots\), so \( g \mid w=6.75 > 0 \) for \( t \in (0, 0.5) \).

Lemma 18. The derivative \( \partial g / \partial w = (2t^2 - 20t + 20)w + 4t^3 - 20t^2 + 20t - 40 \) is positive on \( \Sigma_3 \).

Hence this is the case of \( \partial a_6 / \partial w \) and \( a_6 \) as well, so the inequality \( a_6 < 0 \) or the equality \( a_6 = 0 \) fails of \( \Sigma_3 \).

Proof. On \( \Sigma_3 \) one has

\[
\begin{align*}
(2t^2 - 20t + 20)w > (-20t + 20)w > 10 \times 6.75 = 67.5 \quad \text{and} \\
4t^3 - 20t^2 + 20t - 40 > 4t^3 - 40 > -40
\end{align*}
\]

so \( \partial a_6 / \partial w > 0 \).

Lemma 19. One has \( a_{10} \geq 0 \) on \( \Sigma_4 \).

Proof. One has \( a_{10} = (t/27)(w^2 + 2t - 5)^2S \), see \( (6.2) \), hence \( S \mid t=0.25 = -0.75w^2 + 5.375w + 1.125 \) which is positive for \( w \in [4.79, 6.75] \). The lemma follows from \( \partial S / \partial t = (20w - 4)t + 5w^2 - 21w + 5 \) being positive for \( (t, w) \in \Sigma_4 \).
Lemma 20. One has $a_6 > 0$ in $\Sigma_5$.

Proof. We use the following expression for $27a_6$:

$$27a_6 = h_4w^4 + h_3w^3 + h_2w^2 + h_1w + h_0,$$

for $h_4 = t^2 - 10t + 10$, $h_3 = 6t^3 - 35t^2 + 50t - 70$, $h_2 = 12t^4 - 30t^3 + 90t + 90$, $h_1 = 8t^5 - 20t^4 - 70t^3 + 355t^2 - 460t + 25$, $h_0 = -40t^5 + 100t^4 - 50t^3 - 50t^2 + 50t + 260$.

Hence the values for $w = 5$ of the derivatives $27\partial^*a_6/\partial w^*$ are the following polynomials:

$$27\partial^0a_6/\partial w^0 = 300t^4 - 400t^3 - 2025t^2 + 135$$

$$27\partial^1a_6/\partial w^1 = 8t^5 + 100t^4 + 80t^3 - 1770t^2 - 810t + 675$$

$$27\partial^2a_6/\partial w^2 = 24t^4 + 120t^3 - 750t^2 - 1320t + 1080$$

$$27\partial^3a_6/\partial w^3 = 36t^3 - 90t^2 - 900t + 780$$

$$27\partial^4a_6/\partial w^4 = 24t^2 - 240t + 240$$

All of them are positive for $t \in [0, 0.25]$ from which and from the Taylor series of $a_6$ w.r.t. the variable $w$ the lemma follows.

Lemma 21. One has $a_{10} \geq 0$ on $\Sigma_6$.

Proof. Recall that the quantity $S$ was defined by (6.2). The values for $t = 0$ of the derivatives $\partial^*S/\partial t^*$ are:

$$\partial^0S/\partial t^0 = -2w^2 + 10w$$

$$\partial^1S/\partial t^1 = 5w^2 - 21w + 5$$

$$\partial^2S/\partial t^2 = 20w - 4.$$ 

They are all nonnegative for $w \in [4.791, 5]$ from which and from the Taylor series of $S$ w.r.t. the variable $t$ one gets $S \geq 0$ in $\Sigma_6$ and the lemma follows.

Proof in Case D): $P = (x + 1)^4(sx + 1)^4(tx - 1)^2(wx - 1)$.

The condition $a_1 = w + 2t - 4s - 4 = 0$ implies $s = (w + 2t - 4)/4$. For $s = (w + 2t - 4)/4$ one has $256a_{10} = t(w + 2t - 4)^3H^*$, where

$$H^* := 8wt^2 - 2t^2 + 4w^2t - 5wt + 4t + 8w - 2w^2.$$ 

Lemma 22. The inequality $H^* \geq 0$ (hence $a_{10} \geq 0$) holds in each of the two cases $t \in [1/2, 2]$ and $w \in [1/4, 4]$. It holds also for $(t, w) \in [2, \infty) \times [4, \infty)$, for $(t, w) \in (0, 1/2] \times (0, 1/4]$ and for $(t, w) \in [0, 3/4] \times [4, 6.71]$.

Remark 4. In other words, for $t > 0$, $w > 0$, the inequality $a_{10} < 0$ fails outside the domain $\Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 := (2, \infty) \times (0, 1/4), \quad \Omega_2 := (0, 1/2) \times (6.71, \infty), \quad \Omega_3 := (0, 0.3) \times (4, 6.71).$$

We set $\Omega_3 = \Omega_3^- \cup \Omega_3^+$, where

$$\Omega_3^- := (0, 0.3) \times (4, 5), \quad \Omega_3^+ := (0, 0.3) \times (5, 6.71).$$
Proof of Lemma 22. We represent $H^*$ in two ways:

$$H^* = H_{2w}w^2 + H_{3w}w + H_{0w}, \quad H_{2w} = 4t - 2, \quad H_{3w} = 8t^2 - 5t + 8, \quad H_{0w} = -2t^2 + 4t$$

and

$$H^* = H_{2t}t^2 + H_{1t}t + H_{0t}, \quad H_{2t} = 8w - 2, \quad H_{1t} = 4w^2 - 5w + 4, \quad H_{0t} = -2w^2 + 8w.$$  

The first statement of the lemma follows from $H_{jw} \geq 0$, $j = 1, 2, 3$ for $t \in [1/2, 2]$ and $H_{jt} \geq 0$, $j = 1, 2, 3$ for $w \in [1/4, 4]$. The quantity $H^*$ is a degree 2 polynomial in $t$. For $t = 2$ and $w \in [4, \infty)$ one has

$$H^* = 30w + 6w^2 > 0, \quad \partial^2 H^*/\partial t^2 = 16w - 4 \geq 0 \quad \text{and} \quad \partial H^*/\partial t = 16wt - 4t + 4w^2 - 5w + 4 = (16t - 4)w(w - 5) + 4 > 0,$$

so by representing $H^*$ as a Taylor series in the variable $t$ we see again that $H^* > 0$ for $(t, w) \in [2, \infty) \times [4, \infty)$. Next, for $(t, w) \in (0, 1/2] \times (0, 1/4)$ one can write

$$H^* = t(4 - 2t - 5w) + 2w(4 - w) + 8wt^2 + 4w^2 t > 0.$$  

Finally, as $\partial H^*/\partial t = (16w - 4)t + 4w^2 - 5w + 4$, where the polynomial $4w^2 - 5w + 4$ has no real roots, one has $\partial H^*/\partial t > 0$ in $[0.3, 1/2] \times [4, 6.71]$. On the other hand for $t = 0.3$ the polynomial $H^*$ equals $w(7.22 - 0.8w) + 1.02$ which is positive for $w \in [4, 6.71]$. Hence $H^* > 0$ in $[0.3, 1/2] \times [4, 6.71]$. □

Lemma 23. The inequality $a_5 \geq 0$ fails for $(t, w) \in [2, \infty) \times (0, 1/4] \supset \Omega_1$.

Proof. The quantity $a_5^* := 256a_5$ equals

$$1536t + 768w - 1536t^2 - 384w^2 - 1536wt + 768w^2 t + 1280w^2 t^2 - 32w^3 - 416w^2 t^2 - 384wt^3 - 16t^3 w^2 + 16t^4 w - 72t^2 w^3 - 22tw^4 - 128w^3 + 512w^3 + 44w^4 - 64t^4 - 96t^5 + w^5.$$  

The values $v_j$ for $t = 2$ of its partial derivatives $\partial^j a_5^*/\partial t^j$, $j = 0, \ldots, 5$, equal respectively

$$v_0 = -3072 - 640w^2 - 480w^3 + w^5,$$

$$v_1 = -8192 - 512w - 1088w^2 - 320w^3 - 22w^4,$$

$$v_2 = -15360 - 1280w - 1024w^2 - 144w^3,$$

$$v_3 = -23040 - 1536w - 96w^2,$$

$$v_4 = -24576 + 384w,$$

$$v_5 = -11520.$$  

They are all negative for $w \in (0, 1/4]$. Hence all coefficients of the Taylor series w.r.t. $t$ of the coefficient $a_5$ for $t = 2$, $w \in (0, 1/4]$, are negative and such is $a_5$ for $(t, w) \in [2, \infty)$ as well. □

Lemma 24. The inequality $a_6 \leq 0$ fails for $(t, w) \in (0, 1/2] \times [6.71, \infty) \supset \Omega_2$ and for $(t, w) \in (0, 0.3] \times [5, \infty) \supset \Omega_3^+$.  

Thus after Lemmas 22, 23 and 24 there remains to prove that for $(t, w) \in \Omega_1^-$ the sign(s) of some (of the) coefficient(s) $a_j$ is/are not the one(s) prescribed by the sign pattern.
Proof of Lemma 24. One has

\[
256a_6 = 1024 - 768w - 1536t - 576w^2t + 1920t^2 + 864w^2 - 352w^3 \\
-1280t^3 + 800t^4 - 256t^5 + 26w^4 + 4w^5 - 16t^6 + 384wt \\
-384w^2t + 400w^3t + 720w^2t^2 + 448wt^3 - 352t^3w^2 - 256t^4w + 40t^3w^3 \\
+104t^4w^2 + 64t^5w - 272t^2w^3 - t^2w^4 - 56tw^4 - 2tw^5.
\]

We list below the values of the functions \(u_j := 256 \frac{\partial^j a_6}{\partial w^j}, \ j = 0, \ldots, 5\), for \(w = 6.71\). They are all positive for \(t \in (0,1/2)\) (this can be checked numerically). From the Taylor series of \(a_6\) for \(w = 6.71\) one concludes that \(a_6 > 0\) for \((t, w) \in (0,1/2) \times [6.71, \infty)\). Here’s the list:

\[
\begin{align*}
  u_0 & := -16t^6 + 173.44t^5 + 3764.7464t^4 - 2037.93476t^3 \\
  & \quad -52440.84297t^2 - 44774.66948t + 35543.86077 \\
  u_1 & := 64t^5 + 1139.68t^4 + 1127.0520t^3 - 28669.71244t^2 \\
  & \quad -41261.71907t + 35244.43996 \\
  u_2 & := 208t^4 + 906.40t^3 - 10051.0092t^2 - 27388.66364t + 25772.93608 \\
  u_3 & := 240t^3 - 1793.04t^2 - 12021.1320t + 12880.8240 \\
  u_4 & := -24t^2 - 2954.40t + 3844.80 \\
  u_5 & := 240(2 - t)
\end{align*}
\]

In the same way we consider the values for \(w = 5\) of these same functions, see the list below. One can check that they are all positive for \(t \in (0,0.3)\) and by analogy we conclude that \(a_6 > 0\) for \((t, w) \in (0,0.3) \times [5, \infty)\).

\[
\begin{align*}
  u_0 & := -16t^6 + 64t^5 + 2120t^4 - 2840t^3 - 16625t^2 - 5266t + 3534 \\
  u_1 & := 64t^5 + 784t^4 - 72t^3 - 14084t^2 - 9626t + 6972 \\
  u_2 & := 208t^4 + 496t^3 - 7020t^2 - 10952t + 8968 \\
  u_3 & := 240t^3 - 1752t^2 - 7320t + 7008 \\
  u_4 & := -24t^2 - 2544t + 3024 \\
  u_5 & := 240(2 - t)
\end{align*}
\]

\[\Box\]

Lemma 25. For \((t, w) \in (0,1/2) \times [4,6.71] \supset \Omega_4^-\) the coefficient \(a_6\) is a decreasing function in \(t\). For \(t = 0\), \(w \in [4,6.71]\) one has \(a_6 \geq 0\) with equality only for \(w = 4\).

Proof. The second claim of the lemma follows from

\[
256a_6|_{t=0} = 4w^5 + 26w^4 - 352w^3 + 864w^2 - 768w + 1024
\]

whose real roots are \(-13.978\ldots, 3.110\ldots, 4\). To prove the first claim we list the derivatives \(\eta_j := 256 \frac{\partial^j a_6}{\partial \theta^j}|_{t=0}, \ j = 1, \ldots, 6\), and their real roots (\(\eta_4\) has no real roots):
\[\eta_1 := -2w^5 - 56w^4 + 400w^3 - 576w^2 + 384w - 1536 - 34.115 \ldots, 2.782 \ldots, 4\]
\[\eta_2 := -2w^4 - 544w^3 + 1440w^2 - 768w + 3840 - 274.626 \ldots, 2.948 \ldots\]
\[\eta_3 := 240w^3 - 2112w^2 + 2688w - 7680 7.894 \ldots\]
\[\eta_4 := 2496w^2 - 6144w + 19200\]
\[\eta_5 := 7680w - 30720 4\]
\[\eta_6 := -11520 .\]

As we see, for \(w \in [4, 6.71]\) one has \(\eta_1 \leq 0, \eta_2 < 0, \eta_3 < 0, \eta_4 > 0, \eta_5 \geq 0\) and \(\eta_6 < 0\). One can majorize the Taylor series for \(t = 0\) of

\[256 \partial a_6/\partial t = \eta_1 + t(\eta_2 + t\eta_3/2 + t^2\eta_4/6 + t^3\eta_5/24 + t^4\eta_6/120)\]

by omitting the nonpositive terms \(\eta_1, t^2\eta_3/2\) and \(t^4\eta_6/120\) and by giving to \(t\) inside the brackets its maximal value 1/2. This gives the polynomial

\[t(\eta_2 + \eta_4/24 + \eta_5/192) = t(-2w^4 - 544w^3 + 1544w^2 - 984w + 4480) ,\]
with real roots \(-274.815 \ldots, 3.083 \ldots\), hence negative for \(w \in [4, 6.71]\). \(\square\)

**Lemma 26.** Consider the quantity \(H^*\) (see (6.3)) as a polynomial in \(t\). For \(w \in [4, 6.71]\) it has a single root \(\tau(w) \in [0, 1/2]\),

\[\tau = (-4w^2 + 5w - 4 + \sqrt{(4w^2 + 19w + 4)(4w^2 - 13w + 4)})/4(4w - 1) .\]

One has \(H^* < 0\) (hence \(a_{10} < 0\)) for \(t < \tau\) and \(H^* > 0, a_{10} > 0\) for \(t > \tau\). The equality \(\tau = 0\) takes place only for \(w = 4\).

**Proof.** The statements about \(\tau\) are to be checked directly. The signs of \(H^*\) follow easily from \(H^*|_{t=0} = 2w(4 - w) \leq 0\) with equality only for \(w = 4\). \(\square\)

**Lemma 27.** Consider \(a_6\) as a function in \((t, w)\). Then with \(\tau\) as defined in Lemma 26 one has \(a_6(\tau, w) \geq 0\) for \(w \in [4, 5]\) with equality only for \(w = 4\).

**Remark 5.** The lemma implies that at least one of the inequalities \(a_6 < 0\) and \(a_{10} < 0\) fails in \(\Omega_4\). Indeed, for \(t \geq \tau\) this is \(a_{10} < 0\) (see Lemma 26), for \(t < \tau\) this is \(a_6 < 0\) (see Lemmas 25 and 27).

**Proof of Lemma 27.** Set \(Y := \sqrt{(4w^2 + 19w + 4)(4w^2 - 13w + 4)}\). One checks numerically that

\[256 a_6(\tau, w) = (wC_0 + (4w^2 + 19w + 4)C_1Y)/(4w - 1)^6 ,\]

where

\[C_0 := 6144w^{10} - 6144w^9 - 224512w^8 + 2284416w^7 - 6369192w^6 + 6270368w^5 - 3922014w^4 + 1993629w^3 - 860272w^2 + 234384w - 25728\]
\[C_1 := 384w^7 - 2496w^6 + 632w^5 - 4064w^4 + 4730w^3 - 1355w^2 - 136w + 64 .\]

(With \(t = \tau(w)\), \(a_6\) becomes a degree 6 polynomial in \(Y\) with coefficients in \(\mathbb{R}(t)\). Using the fact that \(Y^2\) is a polynomial in \(t\) one obtains the above form of 256 \(a_6\).)
All real roots of $C_9$ are smaller than 4, so $C_9 > 0$ for $w \in [4, 5]$. The real roots of $C_9$ equal $-0.192.0, 0.269, \ldots$ and 6.455, so $C_9$ is negative for $w \in [4, 5]$. Hence $wC_9 - (4w^2 + 19w + 4)C_1Y > 0$ and the inequality $wC_9 + (4w^2 + 19w + 4)C_1Y > 0$ is equivalent to $w^2C_9^2 - (4w^2 + 19w + 4)^2C_1^2Y^2 > 0$. The left-hand side of the last inequality equals $128(w - 4)C_2(4w - 1)^6$ with

$$C_2 := 55296w^{12} + 82944w^{11} - 1638912w^{10} + 6310368w^9 - 13847224w^8 + 10530920w^7 - 8336710w^6 + 5520431w^5 - 2256796w^4 + 758480w^3 - 378304w^2 + 63488w + 2048.$$

The largest real root of $C_2$ equals 3.045, so $C_2 > 0$ for $w \in [4, 5]$ and the lemma is proved.

7. Proofs of Lemmas 14, 15 and 16

Proof of Lemma 14.

**Notation 3.** If $\zeta_1, \zeta_2, \ldots, \zeta_k$ are distinct roots of the polynomial $P$ (not necessarily simple), then by $P_{\zeta_1}, P_{\zeta_2}, \ldots, P_{\zeta_2}, \ldots, P_{\zeta_k}$ we denote the polynomials $P/(x - \zeta_1)$, $P/(x - \zeta_2), \ldots, P/(x - \zeta_1)(x - \zeta_2) \ldots (x - \zeta_k)$.

Denote by $u, v, w$ and $t$ the four distinct roots of $P$ (all nonzero). Hence $P(x) = (x - u)^m(x - v)^n(x - w)^p(x - t)^q, m + n + p + q = 11$. For $j = 1, 5$ or 6 we show that the Jacobian $3 \times 4$-matrix $J := (\partial(a_{10}, a_{9}, a_j)/\partial(u, v, w, t))$ (where $a_{10}, a_9, a_j$ are the corresponding coefficients of $P$ expressed as functions of $(u, v, w, t)$) is of rank 3. (The entry in position $(2, 3)$ of $J$ is $\partial a_9/\partial w$.) Hence one can vary the values of $(u, v, w, t)$ in such a way that $a_{10}$ and $a_j$ remain fixed (the value of $a_{10}$ being $-1$) and $a_9$ takes all possible nearby values. Hence the polynomial is not $a_9$-maximal.

The columns of $J$ are defined by the coefficients of the polynomials $-mP_u = \partial P/\partial u$, $-nP_v, -pP_w$ and $-qP_t$. By abuse of language we say that the linear space $F$ spanned by the columns of $J$ is generated by the polynomials $P_u, P_v, P_w$ and $P_t$. As $P_{u,v} = (P_u - P_v)/(v - u), P_{u,w} = (P_u - P_w)/(w - u)$ and $P_{u,t} = (P_u - P_t)/(t - u)$, one can choose as generators of $F$ the quadruple $(P_u, P_{u,v}, P_{u,w}, P_{u,t})$; in the same way $(P_u, P_{u,v}, P_{u,v,w}, P_{u,v,w,t})$ or $(P_u, P_{u,v}, P_{u,v,w}, P_{u,v,w,t})$ the latter polynomials are of respective degrees 10, 9, 8 and 7. As $(x - t)P_{u,v,w,t} = P_{u,v,w}, (x - w)P_{u,v} = P_{u,v,w,t}$ etc. one can choose as generators the quadruple $x^3P_{u,v,w,t}, x^2P_{u,v,w,t}, xP_{u,v,w,t}, P_{u,v,w,t})$. Set $P_{u,v,w,t} := x^7 + Ax^6 + \cdots + G$. The coefficients of $x^{10}$, $x^9$ and $x^8$ of the last quadruple define the matrix $J^* := \begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 0 \\ D & C & B & A \end{pmatrix}$. Its columns span the space $F$ hence rank $J^* =$ rank $J$. As at least one of the coefficients $B$ and $A$ is nonzero (Lemma 7) one has rank $J^* = 3$ and the lemma follows (for the case $j = 6$). In the cases $j = 5$ and $j = 1$ the last row of $J^*$ equals respectively $(E, D, C, B)$ and $(0, 0, 0, G, F)$ and in the same way rank $J^* = 3$.

Proof of Lemma 15. We are using Notation 3 and the method of proof of Lemma 14.

Denote by $u, v, w, t, h$ the five distinct real roots of $P$ (not necessarily simple). Thus using Lemma 10 one can assume that
We require to hold only the inequality
\[ (x-u)^{\ell}(x-v)^{m}(x-w)^{n}(x-t)^{2}(x-h), \quad u,v,w,t,h > 0, \quad \ell+m+n = 8. \]

Set \( J := (\partial(a_{10}, a_{0}, a_{j}, a_{1})/\partial(u,v,w,t,h))^j, \quad j = 5 \) or 6. The columns of \( J \) span a linear space \( L \) defined by analogy with the space \( F \) of the proof of Lemma 14, but spanned by 4-vector-columns.

Set \( P_{u,v,w,t,h} := x^6 + ax^5 + bx^4 + cx^3 + dx^2 + fx + g. \) Consider the vector-column
\[ (0, 0, 0, 1, a, b, c, d, f, g)^t. \]

The similar vector-columns defined after the polynomials \( x^s P_{u,v,w,t,h}, \quad s \leq 4, \) are obtained from this one by successive shifts by one position upward. To obtain generators of \( L \) one has to restrict these vector-columns to the rows corresponding to \( x^{10} \) (first), \( x^9 \) (second), \( x^8 \) \((11-j)\)th and \( x \) (tenth row).

Further we assume that \( a_1 = 0. \) If this is not the case, then at most one of the conditions \( a_5 = 0 \) and \( a_0 = 0 \) is fulfilled and the proof of the lemma can be finished by analogy with the proof of Lemma 14.

Consider first the case \( j = 6. \) Hence the rank of \( J \) is the same as the rank of the matrix
\[
M := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
d & c & b & a & 1 \\
0 & 0 & 0 & g & f
\end{pmatrix}
\begin{pmatrix}
x^{10} \\
x^{9} \\
x^{8} \\
x^{7} \\
x
\end{pmatrix}.
\]

One has rank \( M = 2+\)rank \( N, \) where \( N = \begin{pmatrix}
b & a & 1 \\
a & 0 & g \\
0 & g & f
\end{pmatrix}. \) Given that \( g \neq 0, \) one can have rank \( N < 2 \) only if \( b = 0 \) and \( af = g. \) We show that the condition \( b = 0 \) leads to the contradiction \( a_{10} > 0. \) We set \( u = 1 \) to reduce the number of parameters, so we require to hold only the inequality \( a_{10} < 0, \) but not the equality \( a_{10} = -1. \) We have to consider the following cases for the values of the triple \((\ell,m,n)\) (see (7.41): 1) (6, 1, 1), 2) (5, 2, 1), 3) (4, 3, 1), 4) (4, 2, 2) and 5) (3, 3, 2). Notice that
\[
P_{u,v,w,t,h} = (x+1)^{\ell-1}(x+v)^{m-1}(x+w)^{n-1}(x-t).
\]

In case 1) one has \( b = 10 - 5t, \) so \( t = 2. \) For \( t = 2 \) one has \( a_1 = 4vw - 20vw - 4hw + 4hw \) and the condition \( a_1 = 0 \) yields \( h = h_1 := vw/(5vw + v + w) < 1/5. \) Notice that \( a_{10} = 2 + v + w - h \) which for \( h = h_1 \) is positive – a contradiction.

In case 2) we obtain \( b = 6a^2 + 4w - 4ut - tv \) hence \( t = t_2 := (3+2v)/(4+v). \) One has \( a_1 = -tv(\text{tv}-2vhw+twv+3thw+2thw) \) and for \( t = t_2 \) the condition \( a_1 = 0 \) gives
\[
h = h_2 := vw(3+2v)/(9v^2w + 3 + 2v^2 + 15vw + 6w) < w.
\]

Observe that \( a_{10} = 5 + 2v - 2t + (w - h) > 5 + 2v - 2t. \) However for \( t = t_2 \) one has \( 5 + 2v - 2t_2 = (8 + 5v + 2v^2)/(4 + v) > 0. \)

In case 3) one gets \( b = 3 + 6v + v^2 - 3t - 2t v = 0, \) so \( t = t_3 := (3 + 6v + v^2)/(3 + 2v). \) As \( a_1 = -tv(\text{tv}-2vhw+twv+3thw+4thw) = 0, \) for \( t = t_3 \) one obtains
\[
h = h_3 := vw(3+6v+v^2)/(24vw + 23v^2w + 3v + 6v^2 + v^3 + 4vw^3 + 9w) < w.
\]

One has \( a_{10} = 4 + 3v - 2t + (w - h) > 4 + 3v - 2t. \) For \( t = t_3 \) one checks directly that
4 + 3v - 2t_3 = (6 + 5v + 4v^2)/(3 + 2v) > 0, i.e. \( a_1 > 0 \).

In case 4) one has \( b = 3 + 3v + 3w + vw - 3t - tv - tw \), therefore \( t = t_4 := (3 + 3v + 3w + vw)/(3 + v + w) \). As \( a_1 = -tvw(-vwt - 2vwh + 4thw + 2thw + 2thw) \), for \( t = t_4 \) it follows from \( a_1 = 0 \) that

\[
h = h_4 := \frac{vw(3 + 3v + 3w + vw)}{2(9vw + 6v^2w + 6vw^2 + 2v^2w^2 + 3v + 3v^2 + 3w + 3w^2)}
\]

which is \( < w/2 \). One has \( a_{10} = 4 + 2v + 2w - 2t - h \) which for \( h = h_4 \) and \( t = t_4 \) is

\[
> 4 + 2v + 3w/2 - 2t_4 = (1/2)(12 + 8v + 5w + 4v^2 + 3vw + 3w^2)/(3 + v + w) > 0.
\]

In case 5) one has \( b = 1 + 4v + v^2 + 2w + 2vw - 2t - 2tv - tw \), therefore

\[
t = t_5 := (1 + 4v + v^2 + 2w + 2vw)/(2 + 2v + w).
\]

As \( a_1 = -tvw(-vwt - 2vwh + 3thw + 2thv + 3thw) \), the condition \( a_1 = 0 \) yields

\[
h = h_5 := \frac{vw(1 + 4v + v^2 + 2w + 2vw)}{15vw + 15v^2w + 10vw^2 + 3v^3 + 6v^2w^2 + 2v + 8v^2 + 2v^3 + 3w + 6w^2}
\]

which is \( < w/2 \). One has \( a_{10} = 3 + 3v + 2w - 2t - h \) which for \( t = t_5 \), \( h = h_5 \) is

\[
> 3 + 3v + 3w/2 - 2t_5 = (1/2)(8 + 8v + 4w + 8v^2 + 4vw + 3w^2)/(2 + 2v + w) > 0.
\]

Now consider the case \( j = 5 \). The matrices \( M \) and \( N \) equal respectively

\[
M := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ f & d & c & b \\ 0 & 0 & 0 & g & f \end{pmatrix}, \quad N = \begin{pmatrix} c & b & a \\ 0 & g & f \end{pmatrix}.
\]

One has rank \( N < 2 \) only for \( c = 0 \) and \( bf = ag \). Similarly to the case \( j = 6 \) we show that the equality \( c = 0 \) leads to the contradiction \( a_{10} > 0 \). We define the cases 1) – 5) in the same way as above.

In case 1) one has \( c = 10 - 10t \), so \( t = 1 \). As \( a_1 = vw - 4vwh - hv - hw \), the equality \( a_1 = 0 \) implies \( h = h^1 := vw/(4w + v + w) < 1/4 \). One has \( a_{10} = 4 + v + w - h \) which for \( h = h^1 \) is positive – a contradiction.

In case 2) one gets \( c = -2w(-2w^3 - 3w + 3a + 2t) \), so \( c = 0 \) implies \( t = t^2 := (2 + 3v)/(3 + 2v) \). From \( a_1 = -vw(-vwh - 2vwh + thv + 4thw + 2thw) = 0 \) one gets (for \( t = t^2 \))

\[
h = h^2 := vw(2 + 3v)/(11v^2w + 2v + 3v^2 + 10vw + 4w) < w.
\]

From \( a_{10} = 5 + 2v + w - 2t - h \) one sees that for \( h = h^2 \), \( t = t^2 \) it is true that

\[
a_{10} > 5 + 2v - 2t^2 = (11 + 10v + 4v^2)/(3 + 2v) > 0.
\]

In case 3) one obtains \( c = 1 + 6v + 3v^2 - 3t - 6tv - v^2t \), so \( t = t^3 := (1 + 6v + 3v^2)/(3 + 6w + v^2) \). The condition \( a_1 = -tv^2(-vwt - 2vwh + thv + 4thw + 3thw) = 0 \) with \( t = t^3 \) implies

\[
h = h^3 := vw(1 + 6v + 3v^2)/(16vw + 21v^2w + 10vw^3 + v + 6v^2 + 3v^3 + 3w) < w.
\]
But then from \( a_{10} = 4 + 3v + w - 2t - h \) with \( t = t^3, h = h^3 \) follows

\[
a_{10} > 4 + 3v - 2t^3 = (10 + 21v + 16v^2 + 3v^3)/(3 + 6v + v^2) > 0.
\]

In case 4) one has \( c = 1 + 3v + 3w + 3vw - 3t - 3tv - 3tw - vwt \), so \( c = 0 \), implies \( t = t^4 := (1 + 3v + 3w + 3vw)/(3 + 3v + 3w + v) \). For \( t = t^4 \) the condition

\[
a_1 = -tvw(-vwt - 2vwh + 4thw + 2th + 2thw) = 0
\]

\( h = h^4 := (1/2)vw(1+3v+3w+3vw)/(5vw+6v^2w+6vw^2+5v^2w^2+v+3v^2+w+3w^2) \)

which is \(< w/2 \). Thus \( a_{10} = 4 + 2v + 2w - 2t - h \) with \( t = t^4, h = h^4 \) implies

\[
a_{10} > 4 + 2v + 3w/2 - 2t^4
\]

\[= \frac{20 + 24v + 21w + 17vw + 12v^2 + 4v^2w + 9w^2 + 3v^2w}{2(3 + 3v + 3w + vw)} > 0.
\]

In case 5) we get \( c = 2v + 2v^2 + w + 4vw + v^2w - t - 4tv - v^2t - 2tw - 2vwt \) and \( c = 0 \) implies

\[
t = t^5 := (2v + 2v^2 + w + 4vw + v^2w)/(1 + 4v + v^2 + 2w + 2vw) .
\]

For \( t = t^5 \) the equalities \( a_1 = -tv^2w(-vwt - 2vwh + 3thw + 2thw + 3thw) = 0 \)

\( h = h^5 := \frac{vw(2v + 2v^2 + w + 4vw + v^2w)}{6vw + 12v^2w + 6v^3 + 11vw^2 + 11v^2w^2 + 3w^2v^3 + 4v^2 + 4v^3 + 3w^2} \)

which is \(< w/2 \). Hence \( a_{10} = 3 + 3v + 2w - 2t - h \) with \( t = t^5, h = h^5 \) implies

\[
a_{10} > 3 + 3v + 3w/2 - 2t^5
\]

\[= \frac{6 + 22v + 22v^2 + 11w + 20vw + 6v^3 + 11v^2w + 6w^2 + 6vw^2}{2(1 + 4v + v^2 + 2w + 2vw)} > 0.
\]

\[\Box\]

**Proof of Lemma 16:** We use the same ideas and notation as in the proof of Lemma 15. Six of the six or more real roots of \( P \) are denoted by \((u, v, w, t, h, q)\). The space \( L \) is defined by analogy with the one of the proof of Lemma 15. The Jacobian matrix \( J \) is of the form

\[
J := (\partial(a_{10}, a_9, a_j, a_1)/\partial(u, v, w, t, h, q))^t .
\]

Set \( P_{u,v,w,t,h,q} := x^5 + ax^4 + bx^3 + cx^2 + dx + f \) and consider the vector-column

\[
(0, 0, 0, 0, 1, a, b, c, d, f)^t
\]

Its successive shifts by one position upward correspond to the polynomials \( x^s P_{u,v,w,t,h,q} \), \( s \leq 5 \). In the case \( j = 6 \) the matrices \( M \) and \( N \) look like this:

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
d & c & b & a & 1 \\
0 & 0 & 0 & 0 & f \\
0 & 0 & 0 & 0 & f \\
\end{pmatrix}
\]

and

\[
N = \begin{pmatrix}
b & a & 1 & 0 \\
0 & 0 & f & d
\end{pmatrix}.
\]
One has rank $M = 2+\text{rank } N$ and rank $N = 2$ because $f \neq 0$ and at least one of the two coefficients $b$ and $a$ is nonzero (Lemma 7). Hence rank $M = 4$ and the lemma is proved by analogy with Lemmas 14 and 15. In the case $j = 5$ the third row of $M$ equals $(f \ d \ c \ b \ a \ 1)$, the first row of $N$ equals $(c \ b \ a \ 1)$, at least one of the two coefficients $c$ and $b$ is nonzero and again rank $M = 4$.

□

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