Fluctuations in two-band superconductors in strong magnetic field

A E Koshelev 1 and A A Varlamov 1,2

1 Materials Science Division, Argonne National Laboratory, 9700 S. Cass Ave., Argonne, IL 60439, USA
2 CNR-SPIN, viale del Politecnico, 1 Roma I-00133, Italia

E-mail: koshelev@anl.gov

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Abstract
We consider the behavior of the fluctuating specific heat and conductivity in the vicinity of the upper critical field line for a two-band superconductor. Multiple-band effects are pronounced when the bands have very different coherence lengths. The transition to superconductive state is mainly determined by the properties of the rigid condensate of the ‘strong’ band, while the ‘weak’ band with a large coherence length of the Cooper pairs causes the nonlocality in fluctuation behavior and break down of the simple Ginzburg–Landau picture. As expected, the multiple-band electronic structure does not change the functional forms of dominating divergencies of the fluctuating corrections when the magnetic field approaches the upper critical field. The temperature dependence of the coefficients, however, is modified. The large in-plane coherence length sets the field scale at which the upper critical field has upward curvature. The amplitude of fluctuations and fluctuation width enhances at this field scale due to reduction of the effective z-axis coherence length. We also observe that the apparent transport transition displaces to lower temperatures with respect to the thermodynamic transition. Even though this effect exists already in a single-band case at sufficiently high fields, it may be strongly enhanced in multiband materials.

Keywords: multiband superconductors, superconducting fluctuations, iron-based superconductors

(Some figures may appear in colour only in the online journal)

1. Introduction

Strong fluctuations is an important general feature of superconductors with high transition temperature, small coherence length and high anisotropy. Thermal fluctuations in superconductors is a mature field which has been developing for almost 50 years. Quantitative analysis of fluctuations remains one of the best ways to access the microscopic parameters of superconductors [1, 2]. The recent microscopic theoretical studies of transport fluctuation phenomena for arbitrary temperatures and magnetic fields [3–9] allow us to analyze the experimental findings for conventional and cuprate superconductors [10–13] in their phase diagrams along the upper-critical field line $H_{c2}(T)$, in its periphery, and close to the quantum phase transition at the point $H_{c2}(0)$. Another important recent advance in the field is development of quantitative theory accounting for strong fluctuations which allows for accurate calculation of thermodynamic quantities in the vicinity of the transition line, see review [14].

Since the discovery of superconductivity in MgB2 (see review [15]) the properties of multiband superconductors returned to the spotlight of attention after half a century of oblivion [16]. Further discovery of multiband high-temperature superconductivity in the iron pnictides and chalcogenides gave an even stronger boost to this field, see recent experimental [17] and theoretical [18] reviews.

Superconducting properties of the magnesium diboride are strongly influenced by multiband effects. Among the anomalies found in MgB2 was the unusually narrow temperature range of applicability of the standard Ginzburg–Landau (GL) theory [19]. The Cooper pairs of different kinds, formed by the...
carriers of the $\pi$ band and by the carriers of the $\sigma$ band respectively, behave themselves as the unique condensate only very close to $T_c$. Due to the large difference in the $c$-axis coherence lengths of $\sigma$ and $\pi$ bands, the condensates of different kinds are already split at temperatures parametrically close to $T_c$: $T - T_c \approx \xi_{\sigma}/\xi_{\pi}^2 + \xi_{\sigma} \ll 1$ (here $\xi_{\sigma} \ll 1$ is the relative interband interaction constant).\(^3\) Evidently, this particularity has to manifest itself in fluctuations. The expected dominating effect is renormalization of the effective $c$-axis coherence length which enters all fluctuation properties. While near $T_c$ this length is given by the band-averaged value, above the crossover temperature it is mostly determined by the more anisotropic $\sigma$ band and therefore becomes much smaller.

Corresponding theory generalizing the microscopic theory of fluctuations on a two-band superconductor and deriving the related nonlocal GL functional has been developed in \([20]\). It was strongly focused on the applications to the magnesium diboride in which the main differences between the bands are the strength of intraband coupling constants and the values of the $c$-axis coherence length. In result, the very early manifestation of the short wavelength fluctuations in the $\pi$ band (where superconducting interaction is weaker) were predicted. These predictions of the theory were actually never confirmed experimentally, and, in general, an experimental situation concerning fluctuations in MgB$_2$ is somewhat uncertain. Early papers on the fluctuation conductivity \([21]\), magnetization \([22, 23]\), and specific heat \([24]\) described the experimental data using conventional single-band theory. However, for analysis of the fluctuating magnetization very close to the transition temperature $T_c$ in \([22]\), an additional artificial assumption on granular structure of the material was needed to describe the data. In the follow up paper \([25]\) the double-peak structure of the fluctuation magnetization slightly above $T_c$ has been attributed to the two field scales of the $\sigma$ and $\pi$ bands. To our knowledge, this is the only observed fluctuation behavior attributed to the two-band effects. The fluctuation conductivity $\delta \sigma$ analyzed in \([26]\) did not follow theoretical predictions \([20]\). Instead, it had the typical two-dimensional behavior, $\delta \sigma \propto T_c/(T - T_c)$, in contradiction with the three-dimensional (3D) electronic structure of the material.

The difficulties of detection of fluctuation multiband effects predicted by the theory are probably related to the small amplitude of fluctuations in MgB$_2$. The Ginzburg number in the clean limit is very small for this material, $\Gamma(3) \approx 1.5 \times 10^{-6}$ \([20]\). Therefore it is challenging to fabricate uniform samples with intrinsic transition width limited by fluctuations. Note that no published fluctuation measurement has been done on a single crystal. Studies of the fluctuation magnetization \([22, 23, 25]\) and specific heat \([24]\) have been performed on misoriented powders or polycrystalline samples and the fluctuation conductivity have been studied on thin films \([21, 26]\).

In contrast to magnesium diboride, the iron pnictides are multiband semimetals and, as a consequence, are characterized by quite strong fluctuations. Depending on compound, the estimates for the Ginzburg number range from $3 \times 10^{-5}$ to $5 \times 10^{-3}$\(^4\). It is likely that behavior of superconducting fluctuations in the iron-based superconductors at sufficiently low temperatures and high magnetic fields is influenced by the multiple-band effects. These effects are mostly pronounces when there is a large difference between the gaps and coherence lengths in different bands. Unfortunately, the partial coherence lengths for different bands are not known in present. Behavior of fluctuations has been analyzed for several iron-based superconductors using conventional theory \([27]\) and no obvious multiband effects have been reported so far. On the other hand, noticeable upward curvature in the temperature dependence of the upper critical field observed in some compounds \([28]\) has been attributed to multiband effects, see also reviews \([29]\). Also, it has been observed that in FeSe$_{0.5}$Te$_{0.5}$ \([30]\) and BaFe$_2$(As$_{0.7}$P$_{0.3}$)$_2$ \([31]\) in strong magnetic field the apparent transport transition is displaced to the lower temperatures relative to the thermodynamic transition measured by the specific heat. It is plausible that this displacement is also caused by multiband effects.

Usually the effect of a magnetic field on fluctuations becomes essential when the magnetic length $\ell_H$ reaches the value of the fluctuation Cooper pair size. Since the coherence lengths of different bands together with the gaps in a multiplet superconductor can differ strongly, one can expect that the short-wavelength fluctuation modes in them will be excited at very different fields, like it was found in the temperature dependencies of the paracoconductivity and fluctuation heat capacity for MgB$_2$ \([20]\).

In this article we consider thermal fluctuations in a two-band superconductor near the upper critical field $H_{c2}(T)$ for arbitrary relation between the coherence lengths and general structure of the coupling-constant matrix. Remaining in the region of relatively strong magnetic fields along the line $H_{c2}(T)$, suitable for analysis of experimental data, we will derive the general formulas for the fluctuation corrections to the specific heat and conductivity.

2. Fluctuations near the upper critical field for a single-band superconductor

Manifestations of fluctuations in the phase diagram of a superconductor are very rich and diverse \([1, 2, 5, 7]\). The line $H_{c2}(T)$ can be approached in its initial part where $H_{c2}(T) \ll H_{c2}(0)$ within the frameworks of the GL theory. In this section we summarize well-known results for the fluctuation specific heat and conductivity near the upper critical field line. In the immediate vicinity of the transition temperature these results are also valid for multiband superconductors. Since the main modification of the theory for the

\(^3\) For parameters of MgB$_2$, $\xi_{\sigma}/\xi_{\pi}^2 + \xi_{\sigma} \approx 0.02-0.05$.

\(^4\) These estimates are obtained for two iron-pnictide compounds using the definition (5): BaFe$_2$(As$_{0.7}$P$_{0.3}$)$_2$, $T_c = 30$ K, $\lambda_\sigma = 274$ nm, and $\xi_\sigma = 0.8$ nm gives $\Gamma(3) \approx 3 \times 10^{-4}$ and SmFeAsO$_{0.85}$Fe$_{0.15}$, $T_c = 50$ K, $\lambda_\sigma = 136$ nm, and $\xi_\sigma = 0.17$ nm gives $\Gamma(3) \approx 5 \times 10^{-3}$. 

two-band case includes a proper treatment of fluctuations with the length scales below one of the microscopic coherence length, we also briefly overview known manifestations of such short-scale fluctuations in conventional single-band superconductors.

2.1. Specific heat

Fluctuation correction to the specific heat of the superconductor with the axial symmetry of the spectrum placed in the magnetic field \( H \) directed along the axis of symmetry can be easily calculated in the frameworks of the GL scheme [2]. One can write the fluctuation contribution to the free energy in the Landau representation as

\[
F_{\text{GL}}(\epsilon, H) = -\frac{HV}{\Phi_0} \sum_{n=0}^\infty \int \frac{d\epsilon}{2\pi} \ln \frac{A}{\epsilon + h(2n + 1) + \xi_\tau^2 \xi_\zeta^2}.
\]

(1)

Here \( V \) is the sample volume, \( \Phi_0 = \pi \hbar / e \) is the magnetic flux quantum, \( \epsilon = \ln T/\xi_\tau \) is the magnetic field, \( H_{c2}(0) = \Phi_0 / (2\pi \xi_\tau^2) \) is the linear extrapolation of the second critical field at zero temperature, \( \xi_\tau \) and \( \xi_\zeta \) are the transversal and longitudinal coherence lengths.

The sum in equation (1) is evidently divergent and in order to regularize it one should introduce a formal cut-off parameter, the number of the last Landau level, at which the summation is interrupted (we address the curious reader to [2]). Since we are interested here in the parameter, the number of the last Landau level, at which the order parameter becomes strong and non-Gaussian, the problem of regularization does not arise and one can write

\[
\delta C(\epsilon, H) \approx -\frac{1}{VT} \frac{\partial^2}{\partial \epsilon^2} F_{\text{GL}}(\epsilon, H)
= \frac{h}{8\pi \xi_\tau^2 \xi_\zeta^2} \sum_{n=0}^\infty \frac{1}{(\epsilon + h(2n + 1))^{3/2}}.
\]

(2)

Along the line \( H_{c2}(T) \) the most singular contribution in the sum (2) arises from the lowest Landau level \( n = 0 \),

\[
\delta C = \frac{1}{8\pi \xi_\tau^2 \xi_\zeta^2} \frac{h}{(h - h_{c2})^{3/2}},
\]

(3)

where \( h_{c2}(T) = H_{c2}(T)/H_{c2}(0) = -\epsilon \approx (T_c - T)/T_c \). Comparing it to the mean-field value of the heat capacity jump \( \Delta C = 8\pi^2 T_c/(7\zeta(3)) \), one finds

\[
\frac{\delta C}{\Delta C} = \sqrt{G_{i(3)}} \frac{h_{c2}}{(h - h_{c2})^{3/2}},
\]

(4)

with the 3D Ginzburg number

\[
G_{i(3)} = \left[ \frac{\zeta(3)}{64\pi^2 \xi_\tau^2 \xi_\zeta^2} \right]^2 = \frac{16\pi^4 \lambda_{xy}^4 T_c}{\xi_\zeta^2 \Phi_0^2},
\]

(5)

where \( \lambda_{xy} \) is the in-plane London penetration depth, \( \nu \) is the normal-state density of states per spin, and \( \zeta(3) \approx 1.202 \). The results (3) and (4) are valid for magnetic fields exceeding the typical value \( H_0 = H_{c2}(0)G_{i(3)} \). As follows from (4), the fluctuation width of the superconducting transition in magnetic field grows as \( G_{i(3)}(H) = G_{i(3)}(H_{c2}^0)^{1/3} \), see figure 1(a).

2.2. Paraconductivity

Time-dependent GL theory allows to write near the initial part of the line \( H_{c2}(T) \) also the expression for paraconductivity, see, e.g., [2],

\[
\sigma_{\text{par}} = \frac{\pi^2}{14\zeta(3)} \frac{h}{\sqrt{h - h_{c2}}},
\]

(6)

Being normalized on the Drude conductivity of the normal phase, \( \sigma_n = 2e^2/\nu D \), where \( D \) is the diffusion constant, this result can be represented as

\[
\frac{\sigma_{\text{par}}}{\sigma_n} = \frac{\pi^4}{14\zeta(3)} \left( \frac{G_{i(3)}(H)}{|e|} \right)^{1/3}
\]

(7)

Comparing equations (4) and (7), we see that the apparent widths of transition are different for thermodynamics and transport. We can define the field \( h^* \) at which the fluctuation correction to heat capacity becomes of the order of the jump, corresponding to the boundary of the fluctuation region, \( h^* - h_{c2} = G_{i(3)}^{1/3} |e|^{3/2} \). At this field we have

\[
\frac{\sigma_{\text{par}}}{\sigma_n}(h^*) = \frac{\pi^4}{14\zeta(3)} \left( \frac{G_{i(3)}}{|e|} \right)^{1/3}
\]

(8)

with \( \pi^4/[14\zeta(3)] \approx 5.8 \). We see that this ratio remains small at \( h = h^* \) for temperatures \( |e| > (1/14\zeta(3))^3 G_{i(3)} \sim 194 G_{i(3)} \) corresponding to fields \( H > 194H_{d2} \). This result implies that at sufficiently high fields the relative fluctuation correction to conductivity remains small even when fluctuations of the order parameter become strong and non-Gaussian.

2.3. Short-scale fluctuations in conventional superconductors

Historically, the first manifestation of the short-wave-length fluctuations in conventional superconductors has been revealed in the field dependence of the fluctuation magnetization above \( T_c \) about forty year ago [1]. The GL theory for a 3D superconductor [32–35] predicts that the fluctuation magnetization grows linearly with the magnetic field for \( H < H_{c2}(0) \) and crosses over to the \( \sqrt{H} \) dependence for higher fields. Such a limitless growth of the fluctuation magnetization with field is somewhat counter-intuitive, because one would expect that the magnetic field should eventually suppress superconducting fluctuations. Indeed, the experimental fluctuation magnetization shows nonmonotonic field dependence and at high fields its values occur to be significantly lower than the GL prediction [36, 37]. This apparent contradiction has been resolved within the microscopic theory which properly accounts for the short-wave-length and dynamical fluctuations [38–40]. It occurred that in the clean case the GL theory breaks
down very early, at magnetic fields of the order of 0.05\(H_{c2}(0)\).

The reviewed above GL results for the fluctuation specific heat and conductivity are only valid in the vicinity of the transition temperature for fields \(H \ll H_{c2}(0)\). In order to describe fluctuations close to the line \(H_{c2}(T)\) in its upper part, in the vicinity of the field \(H_{c2}(0)\) one has to develop the microscopic theory accounting for short-wave-length and dynamical fluctuations [3–9]. Both approaches for the conductivity and heat capacity match in the wide domain of temperatures and magnetic fields along the line \(H_{c2}(T)\) in its central part. In contrast, for multiple-band superconductors, due to the possible diversity in the partial coherence length \(s\), the GL description of fluctuations may break down at magnetic fields significantly smaller than \(H_{c2}(0)\) at temperature quite close to \(T_c\). In this case a proper microscopic description of the short-wave-length fluctuation is necessary which we present in the next section.

3. Fluctuation corrections in a two-band superconductor near the upper critical field

3.1. Model

We consider a two-band superconductor described by the \(2 \times 2\) coupling-constant matrix \(\lambda_{\alpha\beta}\). We assume strong scattering of quasiparticles inside the bands (dirty limit) but neglect the interband scattering. In the case of very different coherence lengths in different bands, the nonlocal effects become important at moderate magnetic fields, significantly smaller than the low-temperature critical field \(H_{c2}(0)\). In this case the standard GL model breaks down quite close to the transition temperature and more general description is required. Fluctuation thermodynamics of a two-band superconductor is described by the following quadratic nonlocal energy functional for the band gap parameters \(\Delta_{\alpha}\)

\[
\mathcal{R}[\Delta_{\alpha}] = \min_{\nu_{\alpha}} \int d^3r \left\{ \sum_{\alpha,\beta} \nu_{\alpha} \left( w_{\alpha\beta} + \delta_{\alpha\beta} \epsilon \right) \Delta_{\alpha}^{\alpha} \Delta_{\beta}^{\beta} + 2\pi T \sum_{a,\omega > 0} \nu_{a} \left[ \omega \left| F_{a} - \frac{\Delta_{\alpha}}{\omega} \right|^{2} + \sum_{j} \frac{D_{a}}{2} |D_{j}|^{2} \right] \right\}. \tag{9}
\]

Here \(F_{a}\) are the anomalous Green’s functions, \(\omega = \pi T (2m + 1)\) are the Matsubara frequencies, \(a = 1, 2\) is the band index, \(j\) is the coordinate index, \(D_{a,j}\) are the diffusion coefficients (we assume that \(D_{a,a} = D_{\kappa\kappa}\)), and \(D_{j} \equiv \nabla_{j} - (2\pi T / \rho_{0})A\). The degenerate matrix \(w_{\alpha\beta}\) is defined as [20]

\[
\begin{align*}
    w_{11} &= -\frac{\lambda_{12}}{2} + \sqrt{\frac{\lambda_{12}^{2}}{4} + \lambda_{12} \lambda_{21}}, \\
    w_{22} &= \frac{\lambda_{12}}{2} + \sqrt{\frac{\lambda_{12}^{2}}{4} + \lambda_{12} \lambda_{21}}, \\
    w_{12} &= -\frac{\lambda_{12}}{2}, \\
    w_{21} &= -\frac{\lambda_{12}}{2},
\end{align*}
\tag{10}
\]

with \(\lambda_{1} = \lambda_{11} - \lambda_{22}\). The functional (9) takes into account possible nonlocality when the spatial scale of the order

\[\text{Figure 1.} \text{ Field dependences for fluctuation region for (a) single-band and (b) two-band superconductors.} \text{ } \text{Gi}(H)\text{ is the width of fluctuation region given by (24). The shaded area near } T_c \text{ shows applicability region of the conventional GL theory.}\]
parameter variations becomes shorter than the one of the microscopic band coherence lengths.

The anomalous Green’s functions $F_{\alpha}$ rigidly follow fluctuations of the gap parameters. Variation of the functional (9) with respect to $F_{\alpha}$ gives the linear Usadel equations

$$\alpha F_{\alpha} - \sum_j D_{\alpha j}^2 F_{\alpha} = \Delta_{\alpha}. \quad (11)$$

Substitution of solution of this equation into (9) gives the nonlocal functional for the gap parameters $\Delta_{\alpha}$.

To explore fluctuations in magnetic field, we will assume the Landau gauge, $A = (0, Hx, 0)$ and introduce the eigenstates of the operator $\sum_j D_{\alpha j}^2$ (Landau levels) as

$$\left[ -\frac{d^2}{dx^2} + \left(q_x^2 - \frac{2\pi}{\phi_0} Hx + q_z^2 \right) / f_{\alpha}^2 \right] \nu_{\alpha q} = a_{\alpha n q}, \nu_{\alpha q} \quad (12)$$

with the eigenvalues

$$a_{\alpha n q} = \frac{2\pi H}{\phi_0} (2n + 1) + q_z^2 / f_{\alpha}^2 \quad (13)$$

and the band anisotropy parameters $\nu_{\alpha q}^2 = D_{\alpha x} / D_{\alpha z}$. Expanding $F_{\alpha}$ and $\Delta_{\alpha}$ with respect to these eigenstates, $F_{\alpha}(r) = \sum_n \int \frac{dq_y}{(2\pi)^2} F_{\alpha n q} \Psi_{\alpha n q}$, $\Delta_{\alpha}(r) = \sum_n \int \frac{dq_y}{(2\pi)^2} \Delta_{\alpha n q} \Psi_{\alpha n q}^*$, we immediately obtain from (11) $F_{\alpha n q} = \frac{w_{\alpha q}}{\omega + (D_{\alpha z} / 2\phi_0) \nu_{\alpha q}}. \quad (14)$

Substituting these expansions into the functional (9), we arrive to the Gaussian energy functional presented in terms of the fluctuating Landau-level amplitudes of the order parameter

$$\mathcal{F} = \frac{1}{L_\perp} \sum_n \int_0^{\frac{2\pi H \phi_0}{\pi}} dq_y \int_{-\infty}^{\infty} dq_z$$

$$\nu_{\alpha q} \left[f_{\alpha q} + \delta f_{\alpha q} (e + \beta_{\alpha n q}) \right] \Delta_{\alpha n q}^* \nu_{\alpha q}, \quad (15)$$

where

$$\beta_{\alpha n q} = \beta \left[ \frac{2\pi q_z^2}{\phi_0} H / (2n + 1) + q_z^2 / f_{\alpha}^2 \right], \quad \beta \equiv \frac{8D_{\alpha i}^2}{\mathcal{A}}. \quad (16)$$

$\nu_{\alpha q}$ is the Euler constant, and $\psi(x)$ is the digamma function.

Equation for the upper critical field [42], $H_{c2}$, is determined by the condition of degeneracy of the matrix $w_{\alpha q} + \delta w_{\alpha q} (e + \beta_{\alpha n q})$ with $\beta_{\alpha n q} \equiv \beta_{\alpha n q} = \beta \left[ 2\pi q_z^2 / H / \phi_0 \right]$ giving

$$\left[e + \beta_{0,0}\right] w_{11} + \left[e + \beta_{1,0}\right] w_{22} + \left[e + \beta_{2,0}\right] = 0. \quad (17)$$

The shape of the dependence $H_{c2}(T)$ following from this equation depends on the relation between the bands coherence lengths $\xi_{\alpha}$ and relative strength of superconductivity described by the matrix $w_{\alpha q}$.

### 3.2. Single-mode approximation

In zero magnetic field the superconducting instability develops in one channel corresponding to the fixed relation between the gap band parameters, $\Delta_{\alpha} \propto \psi_{\alpha}^{(1)}$, where $\psi_{\alpha}^{(1)}$ is the eigenvector corresponding to zero eigenvalue of the matrix $w_{\alpha q}$, $w_{11}^{(1)} = 0$.

$$w_{\alpha q} \psi_{\alpha}^{(1)} = 0$$

or, explicitly

$$\psi_{1}^{(1)} = \frac{w_{12}}{\sqrt{w_{11}^2 + w_{12}^2}} = \text{sign}(w_{12}) \frac{\sqrt{w_{11}^2 + w_{12}^2}}{w_{11}}, \quad \psi_{2}^{(1)} = -\frac{w_{11}}{\sqrt{w_{11}^2 + w_{12}^2}} = -\frac{\sqrt{w_{11}^2 + w_{12}^2}}{w_{11}}.$$

Here we used the relations $\nu_{12} w_{12} = \nu_{21} w_{21}$ and $w_{11}^{(1)} w_{22}^{(1)} = w_{21}^{(1)} w_{11}^{(1)}$. In typical situation the second eigenvalue of this matrix, $w_{22} = w_{11} + w_{22}$, is large meaning that this mode is strongly gapped, and fluctuations in this channel can be neglected. That is why in this case one can only take into account fluctuations in the unstable channel and use the projections of the order parameter to the eigenstate of this channel $\psi_{\alpha}^{(1)}$. As our purpose is to illustrate the multiband effects on fluctuations in the simplest situation, we will limit ourselves with the single-mode approximation. In the following we skip index (1) in $\psi_{\alpha}^{(1)}$, $\psi_{\alpha} \rightarrow \psi_{\alpha}$.

Equation for $H_{c2}$ becomes $\sum_{\alpha = 1, 2} \nu_{\alpha} (e + \beta_{\alpha,0}) \psi_{\alpha}^2 = 0$. One can show that for arbitrary band-space vector $A_{\alpha}$ the identity

$$\sum_{\alpha = 1, 2} \nu_{\alpha} A_{\alpha} \psi_{\alpha}^2 = \nu_{12} A_{12} + \nu_{21} A_{21}$$

is valid. Being applied to the equation for $H_{c2}$, this identity allows to transform it to the following simple form

$$e + \beta_{0} = 0, \quad (18)$$

where we introduced the band average $\mathcal{A}$ for the arbitrary $A_{\alpha}$ as

$$\mathcal{A} \equiv \frac{A_{12}}{\nu_{11} + w_{22}}. \quad (19)$$

The weights with which the bands contribute to the averages are determined by the strength of the superconducting pairing in them. For example, for stronger second band $A_{22} \gg A_{11}$ we have $w_{11} \gg w_{22}$. Differentiating (18) with respect to the temperature, we obtain the useful relation

$$\frac{\partial \psi_{\alpha}^2}{\partial T} \frac{2\pi}{\phi_0} \left[H_{c2} + \mathcal{T}_{c2}\right] = 1. \quad (20)$$

with $H_{c2} \equiv |dH_{c2}/dT|$. The temperature dependence of the upper critical field becomes unconventional when the bands have very different coherence lengths $\xi_{\alpha}$, $\xi_{1,1} \gg \xi_{2,2}$. In this
case in the range \( H_1 = \frac{\Phi_0}{2\pi \xi_{\perp,1}} \ll H_2 \ll \Phi_0 = \frac{\Phi_0}{2\pi \xi_{\perp,2}} \), equation for \( H_2 \) takes the form

\[
\epsilon + \frac{\ln \left( \frac{\epsilon H_2}{\pi \xi_2} \right) w_{22} + H_2 \nu_1 w_{11} + w_{22}}{w_{11} + w_{22}} = 0.
\]

The characteristic two-band feature is a noticeable upward curvature of the temperature dependence of the upper critical field of \( H_{2c}(T) \sim H \), see figure \( 1(b) \).

Taking the projection \( \Delta_{a,n,q} = \Delta_{a,q} \psi_n \), the nonlocal GL functional (14) can be transformed to the following form

\[
F = \frac{\nu_2}{L_2} \sum_n \int_0^{2\pi} \frac{d\delta}{2\pi} \left( \epsilon + \tilde{\rho}_{hq} \right) |\Delta_{a,q}|^2
\]

with

\[
\nu_2 = \frac{\nu_2 (w_{11} + w_{22})}{\nu_1 w_{11} + \nu_2 w_{22}}.
\]

### 3.3. Specific heat

The fluctuation correction to the free energy following from (19) is given by

\[
\delta F = -T \frac{H}{\Phi_0} \sum_n \int_0^{2\pi} \frac{d\delta}{2\pi} \ln \frac{A}{\epsilon + \tilde{\rho}_{n,q}}.
\]

This gives the fluctuating specific heat \( \delta C = -T \partial^2 \delta F / \partial T^2 \)

\[
\delta C = \frac{H}{\Phi_0} \sum_n \int_0^{2\pi} \frac{d\delta}{2\pi} \left( 1 + T \tilde{\rho}_{n,q} / \partial T \right)^2 \left( \epsilon + \tilde{\rho}_{n,q} \right)^2.
\]

The temperature derivative in the numerator can be evaluated as \( T \tilde{\rho}_{n,q} / \partial T = -2\pi \tilde{\rho}_{n,q} \xi_{\perp,2}^2 \). As in the case of a single-band superconductor, the dominating divergency in the specific heat for \( H \rightarrow H_{c2} \) is given by the \( n = 0 \) term. Using the expansion

\[
\epsilon + \tilde{\rho}_{0q} \approx 2\pi \tilde{\rho}_{0q} \xi_{\perp,1}^2 (H - H_{c2}) / \Phi_0 + \tilde{\rho}_{0q} \xi_{\perp,1}^2,
\]

and relation (18), we find

\[
\delta C \approx \frac{H}{\Phi_0} \frac{\left( TH_{c2} \right)^2}{4 \tilde{\rho}_{0q} \xi_{\perp,1}^2 \left( H - H_{c2} \right) + \left( TH_{c2} \right)^2} \approx \frac{H / \Phi_0}{4 \tilde{\rho}_{0q} \xi_{\perp,1}^2 \left( 1 + H_{c2} / (TH_{c2}) \right) \left( (T - T_{c2}) / T \right)^{3/2}},
\]

where \( T_{c2} \equiv T_{c2}(H) \) is defined by the relation \( H_{c2}(T_{c2}) = H \). Multiband effects manifest themselves in this result via nonlinear temperature dependence of the upper critical field and renormalization of the \( c \)-axis coherence length

\[
\xi_{\perp,1} \rightarrow \xi_{\perp,1,}\text{eff}(H) = \frac{\tilde{\rho}_{0q} \xi_{\perp,1}^2}{\Phi_0}.
\]

In particular, in the range \( H_1 = \frac{\Phi_0}{2\pi \xi_{\perp,1}} \ll H \ll H_2 = \frac{\Phi_0}{2\pi \xi_{\perp,2}} \), we can explicitly write

\[
\xi_{\perp,1,}\text{eff}(H) = \left( H / H_{c2} \right) w_{22} \xi_{\perp,1}^2 + w_{11} \xi_{\perp,2}^2.
\]

We can see that the contribution to \( \xi_{\perp,1,}\text{eff}(H) \) from the large coherence length decays in this region as \( 1 / H \).

To evaluate the width of fluctuation region we have to compare the fluctuating specific heat with the mean-field heat capacity jump. In the two-band case it is given by \( \Delta C = 8\pi^2 n_2 T_c / (\xi_{\perp,1}(3)) \), with the renormalization factor [43]

\[
n_2 = \frac{\nu_1 \nu_2 (w_{11} + w_{22})^2}{\nu (\nu_1 w_{11} + \nu_2 w_{22})^2} \lesssim 1.
\]

Finally, we obtain

\[
\frac{\delta C}{\Delta C} \approx \sqrt{\frac{G_{l,3}(H)}{H_{c2}(0)}} \frac{\xi_{\perp,1}}{\xi_{\perp,1,}\text{eff}(H)} \times \frac{1}{\sqrt{1 + H_{c2} / (TH_{c2}) \left( (T - T_{c2}) / T \right)^{3/2}}},
\]

where \( \xi_{\perp,1} \) is the GL coherence length defined by

\[
\xi_{\perp,1}^2 = \frac{w_{22} \xi_{\perp,1}^2 + w_{11} \xi_{\perp,2}^2}{w_{22} + w_{11}}.
\]

Note that we absorbed the factor \( r_{12} \) in the microscopic definition of \( G_{l,3}(H) \) while the definition of \( G_{l,3}(H) \) in terms of the phenomenological parameters remains unchanged.

Equation (23) allows us to evaluate the field dependent width of the fluctuation region defined by the criterion

\[
\delta C(T, H) / \Delta C \sim 1
\]

\[
G_{l,3}(H) \approx G_{l,3}(H) / H_{c2}(0) \left( \frac{\xi_{\perp,1}}{\xi_{\perp,1,}\text{eff}(H)} \right)^{2/3}.
\]

One can see that, in addition to the single-band broadening \( G_{l,3}(H) \propto H^{2/3} \), a further smearing of the fluctuation region is caused by the reduction of the \( z \)-axis coherence length. The fluctuation region for a two-band superconductor is illustrated in figure 1(b).

### 3.4. Conductivity

The calculation of the fluctuation corrections to conductivity at arbitrary temperatures and fields is a highly nontrivial problem due to pronounced nonlocal and delay effects [3, 4, 6–9]. In contrast to the GL region near the transition temperature, consensus on accurate results for all contribution to conductivity is not achieved yet. In most part of the phase diagram (except very low temperatures) dominating fluctuation contribution to conductivity near the upper critical field is given by the Aslamazov–Larkin term which was computed in the whole temperature range [7]. This result can be
straightforwardly generalized to multiband case as

\[
\delta\sigma_{\text{cl}}^{\text{AL}}(T, H) = \frac{e^2}{(2\pi)^2} \sum_{n=0}^\infty \sum_{\nu, \pi} \int_{-\infty}^\infty \frac{dq_z}{2\pi} e^{\frac{-n\pi}{2}H} \frac{d\xi}{2\pi} \times \left\{ 2 \Re \left[ \Psi_{n,\nu,\pi} \Phi_{n,\nu,\pi} \right] \right\} \left[ L_{\text{eff}}^R \left( n, \nu, \pi \right) \right]
\]

where

\[
\Psi_{n,\nu,\pi} = \beta_{n,\nu,\pi} (-iz) - \beta_{n,\nu,\pi} (-iz),
\]

\[
\left( L_{\text{eff}}^R \right)^{-1} = -\nu_0 \left[ \omega_{\text{pH}} + \left( \epsilon + \beta_{\text{eff}} \right) (-iz) \right] \delta\sigma_{\text{eff}}.
\]

and

\[
\beta_{n,\nu,\pi} (z) = \psi_0 \left( \frac{1}{2} + \frac{z}{4\pi T} \right) + \frac{4e^2}{\mu_0 H(2n+1)} \frac{\sin^2(\theta/2)}{\pi}\]

In the single-mode approximation we have

\[
L_{\text{eff}}^R (z) = \psi_0 \left( \frac{1}{2} + \frac{z}{4\pi T} \right) - \psi_0 \left( \frac{1}{2} \right).
\]

In the classical regime we can use the small-frequency expansions, \(\sin(z/2T) \approx z/2T, \beta_{\text{eff}}(z) \approx \eta_0 z + \beta_{\text{eff}}\), with \(\eta_0 = \frac{e^2}{\mu_0 H} \). Performing the frequency integration in (25) one finds

\[
\delta\sigma_{\text{cl}}^{\text{AL}} = \frac{2T e^2}{\pi} \sum_{n=0}^\infty \left( n + 1 \right) \int_{-\infty}^\infty \frac{dq_z}{2\pi} \times \left\{ \psi_0 (\epsilon + \beta_{\text{eff}}) \left( \eta_0 z + \beta_{\text{eff}} \right) \right\} \times \left\{ \frac{(\epsilon + \beta_{\text{eff}})}{\eta_0 z + \beta_{\text{eff}}} \right\} \left( \frac{\epsilon + \beta_{\text{eff}}}{\eta_0 z + \beta_{\text{eff}}} \right)^2.
\]

In the vicinity of the \(H_{c2}(T) \) line, as usual, we can keep only the \(n=0\) term and approximately obtain

\[
\delta\sigma_{\text{cl}}^{\text{AL}} \approx \frac{e^2}{4} \int_{-\infty}^\infty \frac{dq_z}{2\pi} \frac{\beta_0'}{\epsilon + \beta_0 q_z}.
\]

Expanding \(\beta_{\text{eff}} \approx \beta_0 + \frac{e^2}{\mu_0 H} q_z^2 \) and performing \(q_z\) integration, we finally obtain

\[
\delta\sigma_{\text{cl}}^{\text{AL}} \approx \frac{e^2}{4} \int_{-\infty}^\infty \frac{dq_z}{2\pi} \frac{\beta_0'}{\epsilon + \beta_0 q_z}.
\]

We can see that, similar to the specific-heat correction (22a, 22b), the multiband effects influence these results via nonlinear temperature dependence of \(H_{c2}\) and renormalization of the \(c\)-axis coherence length. An additional factor \(\bar{P}_0 \) appears due to renormalization of the dynamics coefficient \(\eta_0\).

To evaluate the apparent width of the transport transition, we normalize \(\delta\sigma_{\text{cl}}^{\text{AL}} \) to the normal-state conductivity \(\sigma_n = 2e^2(\nu_D1 + \nu_D2)\) and obtain

\[
\frac{\delta\sigma_{\text{cl}}^{\text{AL}}}{\sigma_n} \approx \frac{\pi^2}{14\zeta(3)} \frac{\nu_D^2}{\nu_D1 + \nu_D2} \frac{\xi_0^2}{\xi_0^2,1 + \nu_D^2} \left( \frac{T}{T - T_c} \right).
\]

We can see that, in comparison to the similar single-band result (8), this ratio contains several additional factors, \(\bar{P}_0^2, \xi_0^2,1 + \nu_D^2, \xi_0^2 \xi_0,1 + \nu_D^2 \). All these factors are smaller than one. The most important among them is the last factor. The GL coherence length in its numerator \(\xi_0\) is determined by the bands contributions weighted by the relative strength of superconductivity, \(\xi_0^2 = (\nu_D^2 + \nu_D^2) / (\nu_D^2 + \nu_D^2)\), while the normal-state average in the denominator \(\nu_D^2 + \nu_D^2\) is just determined by the partial densities of states. This ratio may be very small if the band with large coherence length has weak superconductivity. In other words, the relative fluctuation correction reduces because the band with strong superconductivity may dominate the fluctuation correction, while the band with largest mobility dominates the normal-state conductivity. In this case the apparent width of the transport transition narrows down and shifts to lower temperatures in comparison with the thermodynamic transition. Even though this effect exists already in a single-band case at sufficiently high fields, as discussed in section 2.2, we see that it may be strongly enhanced in the multiband case.

4. Conclusions

We considered the fluctuating specific heat and conductivity in the vicinity of the upper critical field line for a two-band superconductor in dirty limit. Multiple-band effects strongly influence superconducting properties when the bands have very different coherence lengths. In the case of strongly different bands the transition to superconductive state is mainly determined by the properties of the rigid condensate of the ‘strong’ band, while the large coherence length of the Cooper pairs from the ‘weak’ band leads to the nonlocality in fluctuation behavior and break down of the simple GL picture. This large coherence length \(\xi_0\) sets the field scale \(H_f\) at which the upper critical field has upward curvature. The contribution of the ‘weak’ band to the fluctuation corrections rapidly
decreases above $H_1$. As expected, the multiple-band effects do not change the functional forms of dominating divergencies of the fluctuating corrections for $H \to H_2(T)$, $\delta c \propto (H - H_c2)^{-1/2}$ and $\delta C \propto (H - H_c2)^{-3/2}$. The corresponding coefficients are modified by the local slope of the upper critical field, $dH_c2/dT$, and renormalization of the $z$-axis coherence length.

Analyzing known results for fluctuating specific heat and conductivity in a single-band superconductor, we observed that for strong enough fields $H \gg 194H_c0$ the relative fluctuation correction to conductivity remains small even when fluctuations of the order parameter become strong and non-Gaussian. Qualitatively, this theoretical finding corresponds to the experimental observation that in some iron-based superconductors the resistive transition at high magnetic field takes place at apparently lower temperatures in comparison with the specific-heat transition [30, 31]. For example, for the compound $\text{BaFe}_2(\text{As}_{0.67}\text{P}_{0.33})_2$ using $H_c2(0) \approx 67.5\text{T}$ and $G(\omega) \approx 3 \times 10^{-5}$, we obtain that such behavior is expected above the field $194H_c0 \sim 0.37\text{T}$. This shift of the apparent transport transition to lower temperatures may be further enhanced by the multiple-band effects, because the ‘weak’ band may dominate the normal-state conductivity, while its contribution to the fluctuation correction is reduced due to weakness of superconductivity.

In this paper we considered the dirty-limit case which is realized when scattering inside the bands is strong and the microscopic theory has the simplest form. In fact, due to very small coherence lengths, most iron-based superconductors are actually in clean limit which is not quantitatively described by the presented theory. Nevertheless, we believe that this circumstance does not change our results qualitatively. Indeed, as it was pointed above, along the line $H_c2(T)$, for not very low temperatures the dominant contribution to fluctuation conductivity is the Aslamazov–Larkin one. In the framework of the standard local theory it can be obtained in the hydrodynamic limit and all information concerning the impurities concentration is included in the coherence lengths (see equations (3) and (6)). Transition to the clean case results in decrease of fluctuation effects inversely proportional to growing coherence lengths. What concerns the two-band superconductors, in the case of large difference between the coherence lengths in different bands, a general picture for the clean case remains similar to the dirty one. Namely, in this case strong nonlocality in the band with the large coherence length also will lead to breaking of the conventional behavior of fluctuations above the field scale set by this length.

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