Majorana-Like Representation Spaces:
Construction and Physical Interpretation *

D. V. Ahluwalia and M. B. Johnson
MP-9, MS H-846, Los Alamos National Laboratory,
Los Alamos, New Mexico 87545, USA

T. Goldman
T-5, MS B-283, Los Alamos National Laboratory,
Los Alamos, New Mexico 87545, USA

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Abstract

We present a formalism that extends the Majorana-construction to arbitrary spin \((j,0) \oplus (0,j)\) representation spaces. For the example case of spin-1, a wave equation satisfied by the Majorana-like \((1,0) \oplus (0,1)\) spinors is constructed and its physical content explored. The \((j,0) \oplus (0,j)\) Majorana-construct is found to possess an unusual classical and quantum field theoretic structure. Relevance of our formalism to parity violation, hadronic phenomenologies, and grand unified field theories is briefly pointed out.

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1. Introduction

A recent careful analysis [1] of the \((j,0) \oplus (0,j)\) representation space has resulted in a rather surprising conclusion that bosons and antibosons in this representation space have opposite intrinsic parity (and as such provides a hitherto unknown realization of a class of quantum field theories which Bargmann, Wightman, and Wigner had classified many years ago [2]). The physical origin of this result lies in the fact, for the example case of spin-1 studied in Ref. [1], that in the \((1,0) \oplus (0,1)\) representation space \(C\) and \(P\) anticommute. This gives a prime motive to further investigate the \((j,0) \oplus (0,j)\) representation space. In this paper, we study and extend the Majorana construction [3] for the \((\frac{1}{2},0) \oplus (0,\frac{1}{2})\) representation space to the \((j,0) \oplus (0,j)\) representation space of arbitrary spin. In addition to the above indicated motivation, there is significant theoretical [4, 5] and experimental [6] interest in the subject of Majorana field and further investigation of the Majorana-construct and its relationship with space-time symmetries should provide useful insights for quantum field theories of truly neutral particles. It should be noted that a few years ago, within the context of Rarita-Schwinger/Bargmann-Wigner field [7], Radescu [8] extended the concept of the Majorana field to fermions of arbitrary spin. Recently, Boudjema et al. [9, 10] generalized Radescu’s work to bosons. As is well known, and as the authors of Ref. [10] indeed note, the massless limit of the RS/BW formalism has inherent difficulties for spins \(j \geq \frac{3}{2}\). In contrast, due to a theorem of Weinberg (see Sec. III of Ref. [11]), the \((j,0) \oplus (0,j)\) representation spaces have well-defined massless limits. This has been recently confirmed explicitly in Refs. [12, 13, 14, 15, 16, 17, 18] in various contexts.

The rest of the paper is composed as follows. In the next section we begin with introducing necessary conventions and definitions by reviewing the the Dirac-like \((j,0) \oplus (0,j)\) spinors. This is followed, in Section 3, by generalizing the concept of spin-\(\frac{1}{2}\) Majorana-construction to the \((j,0) \oplus (0,j)\) representation space. Section 4 is then devoted to the explicit construction of the Majorana-like \((1,0) \oplus (0,1)\) spinors. Section 5 presents the wave equation satisfied by the Majorana-like \((1,0) \oplus (0,1)\) spinors. Section 6 constructs the associated field operator, and in Section 7 we present some concluding remarks.

The formalism that we develop is valid for massive as well as massless particles. To facilitate the study of the massless limit, we work in the front-form Weinberg-Soper formalism [19, 20, 21] recently developed in Ref. [12].
2. Dirac-Like \((j, 0) \oplus (0, j)\) Spinors

The front-form Dirac-like \((j, 0) \oplus (0, j)\) covariant spinors in the Weinberg-Soper formalism (in the chiral representation) are defined as:

\[
\psi\{p^\mu\} = \begin{bmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{bmatrix}.
\]  
(1)

The argument \(p^\mu\) of chiral-representation spinors will be enclosed in curly brackets \(\{\}\). The Lorentz transformation of the front-form \((j, 0)\) spinors is given \([12]\) by

\[
\phi_R(p^\mu) = \Lambda_R(p^\mu) \phi_R(\tilde{p}^\mu) = \exp (\beta \cdot J) \phi_R(\tilde{p}^\mu)
\] ,
(2)

and the front-form \((0, j)\) spinors transform as

\[
\phi_L(p^\mu) = \Lambda_L(p^\mu) \phi_L(\tilde{p}^\mu) = \exp (-\beta^* \cdot J) \phi_L(\tilde{p}^\mu).
\]  
(3)

The \(\tilde{p}^\mu\) represents the front-form four momentum for a particle at rest: \(\tilde{p}^\mu \equiv (p^+ = m, p^1 = 0, p^2 = 0, p^- = m)\). The \(J\) are the standard \((2j + 1) \times (2j + 1)\) spin matrices, and \(\beta\) is the boost parameter introduced in Ref. \([12]\)

\[
\beta = \eta (\alpha v^r, -i \alpha v^\ell, 1)
\]  
(4)

where \(\alpha = [1 - \exp(-\eta)]^{-1}\), \(v^r = v_x + i v_y\) (and \(v^\ell = v_x - i v_y\)). In terms of the front-form variable \(p^+ \equiv E + p_z\), one can show that

\[
cosh(\eta/2) = \Omega \left( p^+ + m \right), \quad \sinh(\eta/2) = \Omega \left( p^+ - m \right),
\]  
(5)

with \(\Omega = [1/(2m)] \sqrt{m/p^+}\). The norm \(\overline{\psi}\{p^\mu\} \psi\{p^\mu\}\), with

\[
\overline{\psi}\{p^\mu\} \equiv \overline{\psi}^\dagger\{p^\mu\} \Gamma^0, \quad \Gamma^0 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]  
(6)

is so chosen that in the massless limit: a. The Dirac-like \((j, 0) \oplus (0, j)\) rest spinors identically vanish (there can be no massless particles at rest); and b. Only the Dirac-like \((j, 0) \oplus (0, j)\) spinors associated with \(h = \pm j\) front-form helicity \([12]\) degrees of freedom survive.

These requirements uniquely determine (up to a constant factor, which we choose to be
the $(2j+1)$-element-column form of $\phi_R(\not{p}^\mu)$ and $\phi_L(\not{p}^\mu)$ to be

\[
\phi_j^R(\not{p}^\mu) = \frac{m_j}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \phi_{j-1}^R(\not{p}^\mu) = \frac{m_j}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \phi_{-j}^R(\not{p}^\mu) = \frac{m_j}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},
\]

(7)

[with similar expressions for $\phi_L(\not{p}^\mu)$] in a representation in which $J_z$ is diagonal. The sub-
scripts $h = j, j-1, \cdots, -j$ on $\phi^R_h(\not{p}^\mu)$ in Eq. (7) refer to the front-form helicity degree of freedom. The reader should refer to Sec. 2.5 of Ref. [22] for an alternate discussion of the non-trivial nature (even though it appears as a “normalization factor”) of the factor $m_j$ in Eq. (7).

3. Majorana-Like $(j, 0) \oplus (0, j)$ Spinors

Following Ramond’s work on spin-$\frac{1}{2}$, we define the front-form $(j, 0) \oplus (0, j)$ $\theta$-conjugate spinor

\[
\psi^\theta\{p^\mu\} \equiv \begin{bmatrix} (\xi \Theta_{[j]} \phi_L(p^\mu) \\ (\xi \Theta_{[j]} \phi_R(p^\mu) \end{bmatrix},
\]

(8)

where $\xi$ is a c-number, and $\Theta_{[j]}$ is the Wigner’s time-reversal operator (see Refs.: p. 61 of [24], Eqs. 6.7 and 6.8 of the first reference in [20], and Ch. 26 of [25])

\[
\Theta_{[j]} \mathbf{J} \Theta_{[j]}^{-1} = -\mathbf{J}^*,
\]

(9)

and $^*$ denotes the operation of algebraic complex conjugation. The parameter $\xi$ is fixed by imposing the constraint:

\[
\left[\psi^\theta\{p^\mu\}\right]^\theta = \psi\{p^\mu\}.
\]

(10)

The time-reversal operator $\Theta_{[j]}$ is defined as: $\Theta_{[j]} = (-1)^{j+\sigma} \delta_{\sigma', -\sigma}$. It has the properties:

$\Theta_{[j]}^* \Theta_{[j]} = (-1)^{2j}, \quad \Theta_{[j]}^* = \Theta_{[j]}$. In the definition of $\Theta_{[j]}$, $\sigma$ and $\sigma'$ represent eigenvalues of $\mathbf{J}$. For $j = \frac{1}{2}$ and $j = 1$, the $\Theta_{[j]}$ have the explicit forms:

\[
\Theta_{[1/2]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Theta_{[1]} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

(11)

The properties of the Wigner’s $\Theta_{[j]}$ operator allow the parameter $\xi$ involved in the definition of $\theta$-conjugation to be fixed as $\pm i$ for fermions and $\pm 1$ for bosons. However, without loss
of generality, we can ignore the minus sign [which contributes an overall phase factor to the \( \theta \)-conjugated spinors \( \psi^\theta \{ p^\mu \} \)] and fix \( \xi \) as

\[
\xi = \begin{cases} 
  i, & \text{for fermions} \\
  1, & \text{for bosons}
\end{cases}
\]

(12)

The existence of the Majorana spinors for the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space is usually (see, e.g., p.16 of Ref. [23]) associated with the “magic of Pauli matrices,” \( \sigma \). The reader may have already noticed that \( i \Theta \frac{1}{2} \) is identically equal to \( \sigma_y \); and it is precisely this matrix that enters into the CP-conjugation of the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) spinors.

The reason that the Majorana-like \((j, 0) \oplus (0, j)\) representation spaces, as opposed to the \((j, 0) \oplus (0, j)\) spaces spanned by Dirac-like spinors defined by Eq. (1), can be constructed for arbitrary spins hinges upon two observations: 1. Independent of spin, the front-form boosts for the \((j, 0)\) and \((0, j)\) spinors have the property that \([\Lambda_R(p^\mu)]^{-1} = [\Lambda_L(p^\mu)]^\dagger\) and \([\Lambda_L(p^\mu)]^{-1} = [\Lambda_R(p^\mu)]^\dagger\); and 2. Existence of the Wigner’s time-reversal matrix \( \Theta \) for any spin.

These two observations when coupled with the transformation properties of the right- and left-handed spinors, Eqs. (2,3), imply that if \( \phi_R(p^\mu) \) transforms as \((j, 0)\), then \((\zeta \Theta_{[j]} )^* \phi_R(p^\mu) \) transforms as \((0, j)\) spinor. Similarly, if \( \phi_L(p^\mu) \) transforms as \((0, j)\), then \((\zeta \Theta_{[j]} )^* \phi_L(p^\mu) \) transforms as \((j, 0)\) spinor. Here, \( \zeta = \exp(i \vartheta) \) is an arbitrary phase factor. As such we introduce \((j, 0) \oplus (0, j)\) Majorana-like spinors

\[
(j, 0) \mapsto \rho \{ p^\mu \} = \begin{bmatrix} \phi_R(p^\mu) \\
(\zeta \Theta_{[j]} )^* \phi_R(p^\mu) \end{bmatrix},
\]

(13)

\[
(0, j) \mapsto \lambda \{ p^\mu \} = \begin{bmatrix} (\zeta \Theta_{[j]} ) \phi_L(p^\mu) \\
\phi_L(p^\mu) \end{bmatrix}.
\]

For formal reasons, the operator multiplying \( \phi_L^*(p^\mu) \), in the definition of \( \lambda \{ p^\mu \} \), is written as \( \zeta \Theta \) rather than \( (\zeta \Theta_{[j]} )^* \): What we have done, in fact, is exploited the property \( \Theta^* = \Theta \) and chosen \( \zeta \Theta = \zeta \Theta^* \). Since \( \zeta \Theta \) is yet to be determined, this introduces no loss of generality. The advantage of all this is that \( \rho \{ p^\mu \} \) and \( \lambda \{ p^\mu \} \) can now be seen as nothing but Weyl spinors (in the \( 2(2j + 1) \)-element form)

\[
\psi_R \{ p^\mu \} = \begin{bmatrix} \phi_R(p^\mu) \\
0 \end{bmatrix}, \psi_L \{ p^\mu \} = \begin{bmatrix} 0 \\
\phi_L(p^\mu) \end{bmatrix},
\]

(14)
added to their respective $\theta$-conjugates. The condition (10) is satisfied not only by Majorana-like self-$\theta$-conjugate spinors but also by antiself-$\theta$-conjugate spinors. Allowing for this freedom, we now fix $\zeta_\rho$ and $\zeta_\lambda$ by demanding (the defining property of the Majorana-like spinors):

$$\rho^\theta\{p^\mu\} = \pm \rho\{p^\mu\} \quad \text{and} \quad \lambda^\theta\{p^\mu\} = \pm \lambda\{p^\mu\}, \quad (15)$$

and find:

$$\zeta_\rho = \pm \xi \quad \text{and} \quad \zeta_\lambda = \pm \xi. \quad (16)$$

The choice $\zeta_\rho = \zeta_\lambda = +\xi$ yields self-$\theta$-conjugate spinors $\rho^{S_a}\{p^\mu\}$ and $\lambda^{S_b}\{p^\mu\}$; while $\zeta_\rho = \zeta_\lambda = -\xi$ corresponds to antiself-$\theta$-conjugate spinors $\rho^{A_b}\{p^\mu\}$ and $\lambda^{A_b}\{p^\mu\}$.

It may be noted that the Dirac-like spinors, Eq. (1), and the Majorana-like spinors, Eqs. (13), are the only spinors that can be introduced in any $P$-covariant theory [1] in the $(j,0) \oplus (0,j)$ representation space. The former describe particles with a conserved charge (which may be zero), while the latter are inherently for the description of neutral particles.

Before we proceed further, we make a few observations on the definition of $\theta$-conjugation. For the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ case, the definition (8) of $\theta$-conjugation can be verified to coincide with $CP$-conjugation. The reader may wish to note that what Raymond (see Ref. [23], p. 20) calls a “charge conjugate spinor,” in the context of spin-$\frac{1}{2}$, is actually a $CP$-conjugate spinor. This, we suspect, remains true for fermions of higher spins also. Surprisingly, for the $(1,0) \oplus (0,1)$ spinors (and presumably for bosons of higher spins also) $\theta$-conjugation equals $\Gamma^5 C$ within a phase factor [2] of $-(-1)^{|h|}$. The mathematical origin of this fact may be traced to the constraint (11) and the property $\Theta^*_{[j]} \Theta_{[j]} = (-1)^{2j}$ of the Wigner’s time-reversal operator $\Theta$.

One may ask if one changes the constraint (11) to read $[\psi^\theta\{p^\mu\}]^\theta = -\psi^\theta\{p^\mu\}$ for bosons, whether one can obtain an alternate definition of $\theta$-conjugation (so that $\theta$-conjugation equals $CP$ for bosons also) to construct self/antiself-$\theta$-conjugate objects. A simple exercise reveals that no such construction yields self/antiself-$\theta$-conjugate objects. The reader may wish to note parenthetically that when the result (12) is coupled with the definition of $\theta$-conjugation, Eq. (8), we discover that the operation of $\theta$-conjugation treats the right-handed and left-handed spinors in a fundamentally asymmetric fashion for fermions. This is readily inferred

[1] A theory that is covariant under the operation of parity is not necessarily a parity non-violating theory. See Sec. 7 for a brief discussion of this point.

[2] The charge conjugation operator $C$, along with $P$ and $T$, for the $(1,0) \oplus (0,1)$ spinors and fields was recently obtained in Ref. [1].
by studying the relative phases with which \( \xi \Theta_{[j]} \) \( \phi_L^*(p^\mu) \) and \( \xi \Theta_{[j]} \) \( \phi_R^*(p^\mu) \) enter in Eq. (8).

4. Explicit Construction of Majorana-Like \((1, 0) \oplus (0, 1)\) Spinors

We now cast these formal considerations into more concrete form by studying the \((1, 0) \oplus (0, 1)\) Majorana-like representation space as an example. As in Ref. [12], we introduce [3] the generalized canonical representation in the front form:

\[
\psi[p^\mu] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \psi\{p^\mu\} .
\]

(17)

The argument \( p^\mu \) of canonical-representation spinors will be enclosed in square brackets \([ \cdot ]\). Here \( \mathbb{1} \) is the \((2j+1) \times (2j+1)\) identity matrix. The boost \( M(p^\mu) \), which connects the rest-spinors \( \psi[p^\mu] \) with the spinors associated with front-form four momentum \( p^\mu \), \( \psi[p^\mu] \), is determined from Eqs. (2), (3), and (17):

\[
\psi[p^\mu] = M(p^\mu) \psi[p^\mu] , \quad M(p^\mu) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{bmatrix} ,
\]

(18)

with \( \mathcal{A} = \Lambda_R(p^\mu) + \Lambda_L(p^\mu) \), \( \mathcal{B} = \Lambda_R(p^\mu) - \Lambda_L(p^\mu) \).

Using the identities (needed to evaluate \( M(p^\mu) \) explicitly) given in Ref. [12] we first obtain the spin-1 \( \rho^{S_0}[p^\mu] \). These are tabulated in Table I. The \( \rho^{S_0}[p^\mu] \) spinors satisfy the following orthonormality relations: \( \mathcal{T}_h^{S_0}[p^\mu] \rho_h^{S_0}[p^\mu] = m^2 \Theta_{hh'} \), where

\[
\mathcal{T}_h[p^\mu] \equiv (\rho_h[p^\mu])^\dagger \Gamma^0 , \quad \Gamma^0 \equiv \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} .
\]

(19)

The front-form \((1, 0) \oplus (0, 1)\) Majorana-like spinors, Table I, should be compared with the front-form \((1, 0) \oplus (0, 1)\) Dirac-like spinors obtained in our recent work [12]. For instance, in the massless limit for the Dirac-like spinors, the \( h = \pm 1 \) degrees of freedom are non-vanishing and the \( h = 0 \) degree of freedom identically vanishes. On the other hand, in the massless limit, for the spin-1 Majorana-like spinors \( \rho^{S_0}[p^\mu] \), it is only the \( h = +1 \) degree of freedom that is non-vanishing, while the \( h = 0 \) and \( h = -1 \) degrees of freedom identically vanish.

[3] The generalized canonical representation is introduced here for no other reason except to be able to compare the results of the present work with our earlier work of Ref. [12].
The origin of the above observation lies in the fact [4] that the (1, 0) and (0, 1) boosts, $\Lambda_R(p^\mu)$ and $\Lambda_L(p^\mu)$, essentially become projectors of the $\phi_{+1}^R(p^\mu)$ and $\phi_{-1}^L(p^\mu)$ as $m \to 0$. To see this, introduce

$$Q_R(m) \equiv \left( \frac{m}{p^+} \right) \Lambda_R(p^\mu) = \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{2} p^r/p^+ & m/p^+ & 0 \\ (p^r/p^+)^2 & \sqrt{2} m p^r/(p^+)^2 & m^2/(p^+)^2 \end{bmatrix}, \quad (20)$$

$$Q_L(m) \equiv \left( \frac{m}{p^+} \right) \Lambda_L(p^\mu) = \begin{bmatrix} m^2/(p^+)^2 - \sqrt{2} m p^r/(p^+)^2 & (p^r/p^+)^2 \\ 0 & m/p^+ & -\sqrt{2} p^r/p^+ \\ 0 & 0 & 1 \end{bmatrix}. \quad (21)$$

The quasi-projector nature of $Q_R(m \to 0)$ and $Q_L(m \to 0)$ is immediately observed by verifying that: $Q_R^2(m \to 0) = Q_R(m \to 0)$ and $Q_L^2(m \to 0) = Q_L(m \to 0)$; but in general $Q_R(m \to 0) + Q_L(m \to 0) \neq 1$ and $Q_{R,L}(m \to 0) \neq Q_{R,L}(m \to 0)$.

To incorporate the $h = -1$ degree of freedom in the massless limit, and to be able to treat the massive particles without introducing manifest parity violation, we now repeat the above procedure for the $\lambda^{S_h}[p^\mu]$ (and $\rho^{A_h}[p^\mu]$ and $\lambda^{A_h}[p^\mu]$ for the sake of completeness) spinors. We find:

$$\lambda^{S_h}_{-h}[p^\mu] = -(-1)^{|h|} \rho^{S_h}_h[p^\mu], \quad (22)$$

$$\rho^{A_h}_h[p^\mu] = \Gamma^5 \rho^{S_h}_h[p^\mu], \quad \Gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (23)$$

$$\lambda^{A_h}_h[p^\mu] = -(-1)^{|h|} \Gamma^5 \rho^{S_h}_h[p^\mu] = (-1)^{|h|} \rho^{A_h}_h[p^\mu], \quad (24)$$

with $1 = 3 \times 3$ identity matrix and

$$\overline{\rho}^{S_h}[p^\mu] \lambda^{S_h}_{-h}[p^\mu] = m^2 \delta_{hh'} = \overline{\rho}^{A_h}[p^\mu] \lambda^{A_h}_{h'}[p^\mu], \quad (25)$$

$$\overline{\rho}^{A_h}[p^\mu] \rho^{S_h}_h[p^\mu] = 0 = \overline{\rho}^{S_h}[p^\mu] \rho^{A_h}_h[p^\mu]. \quad (26)$$

As we will see in Sec. 6, the bi-orthogonal nature of the $\rho[p^\mu]$ and $\lambda[p^\mu]$ spinors results in a rather unusual quantum field theoretic structure for the $(1, 0) \oplus (0, 1)$ Majorana-like field. Similar results hold true for other spins (including spin $\frac{1}{2}$). The bi-orthogonal nature

[4] Even though we make these observations for spin-1, results similar to those that follow are true for all spins (including spin $\frac{1}{2}$).
of the Majorana-like spinors is forced upon us by self/antiself-θ-conjugacy condition \[^{115}\] and cannot be changed as long as we require that the basis-spinors correspond to definite spin-projections (front-form helicity-basis in our case).

It should now be recalled that for the Dirac-like \((1,0) \oplus (0,1)\) spinors \(u_\sigma\{p^\mu\}\) and \(v_\sigma\{p^\mu\}\), we know \[^{1}\] from the associated wave equation that the \(u_\sigma\{p^\mu\}\) spinors are associated with the forward-in-time propagating solutions (the “positive energy solutions”) \(u_\sigma\{p^\mu\}\exp[-i(Et - \mathbf{p} \cdot \mathbf{x})]\), and \(v_\sigma\{p^\mu\}\) spinors are associated with the backward-in-time propagating \[^{5}\] solutions (the “negative energy solutions”) \(v_\sigma\{p^\mu\}\exp[+i(Et - \mathbf{p} \cdot \mathbf{x})]\).

Can one infer similar results by studying the wave equation associated with the front-form \((1,0) \oplus (0,1)\) Majorana-like spinors?

5. Wave Equation for Majorana-Like \((1,0) \oplus (0,1)\) Spinors

Combining the Lorentz transformation properties for the \(\phi_R(p^\mu)\) and \(\phi_L(p^\mu)\), given by Eqs. \(^{2}\) and \(^{3}\), with the definitions \(^{13}\) of Majorana-like spinors, we obtain the wave equations satisfied by the \((1,0) \oplus (0,1)\) Majorana-like spinors. For the \(\rho\{p^\mu\}\) spinors, the wave equation we obtain reads (in chiral representation, where it takes its simplest form):

\[
\left[\begin{array}{cc}
-\zeta_\rho m^2 \Theta_{[1]} & \mathcal{O}_1 \\
\mathcal{O}_2 & -\zeta_\rho m^2 \Theta_{[1]}
\end{array}\right] \rho\{p^\mu\} = 0, \quad (27)
\]

where in the front form the operators \(\mathcal{O}_1\) and \(\mathcal{O}_2\) are defined as: \(\mathcal{O}_1 = g_{\mu\nu} p^\mu p^\nu \exp(-\beta \cdot \mathbf{J}^*) \exp(\beta^* \cdot \mathbf{J})\) and \(\mathcal{O}_2 = g_{\mu\nu} p^\mu p^\nu \exp(\beta^* \cdot \mathbf{J}^*) \exp(-\beta \cdot \mathbf{J})\). The non-zero elements of the front form (the flat space time) metric \(g_{\mu\nu}\) are: \(g_{+-} = \frac{1}{2} = g_{-+}\) and \(g_{11} = -1 = g_{22}\). The wave equation for the \(\lambda\{p^\mu\}\) spinors is the same as Eq. \(^{27}\) with \(\zeta_\lambda\) being replaced by \(\zeta_\rho\). The dispersion relations associated with the solutions of Eq. \(^{27}\) are obtained by setting the determinant of the square bracket in Eq. \(^{27}\) equal to zero.

A simple, though somewhat lengthy, algebra transforms the resulting equation into (true for all spin-1 Majorana-like spinors, hence all reference to a specific spinor is dropped below): \(-\left(p^\tau p^\tau - p^+ p^- - \zeta m^2\right)^3 \left(p^\tau p^\tau - p^+ p^- + \zeta m^2\right)^3 = 0\). As a result, the associated

\[^{5}\] Recall that the usual interpretation of the “negative energy” states as antiparticles fails (see p. 66 of Ref. \(^{27}\)) for bosons. On the other hand, the Stuckelberg-Feynman framework \(^{28}\) applies equally to fermions and bosons.
dispersion relations read:

\[
p^+ = \frac{p^\ell p^r + \zeta m^2}{p^-}, \quad p^+ = \frac{p^\ell p^r - \zeta m^2}{p^-},
\]

(28)
each with a multiplicity 3 (for a given \(\zeta\)). Again, as seen in Refs. [15, 18, 29], like the case for the Dirac-like \((1, 0) \oplus (0, 1)\) spinors, the wave equation for the Majorana-like \((1, 0) \oplus (0, 1)\) spinors contains tachyonic degeneracy. For the Dirac-like \((1, 0) \oplus (0, 1)\) spinors, we find that the tachyonic solutions can be reinterpreted as physical solutions within the context of a quartic self interaction and spontaneous symmetry breaking [29]. Here, we concentrate on the physically acceptable dispersion relations \(p^+ = (p^\ell p^r + m^2)/p^-\); or equivalently \(E^2 = p^2 + m^2\).

The wave equation satisfied by the plane wave solutions \(\rho\{x\} = \rho\{p^\mu\} \exp(-i\epsilon p^\mu x^\mu)\) and \(\lambda\{x\} = \lambda\{p^\mu\} \exp(-i\epsilon p^\mu x^\mu)\) is obtained by first expanding the exponentials in Eq. (27), in accordance with the identities given in Appendix A of Ref. [12], and then letting \(p^\mu \to i\partial^\mu\). Next, to determine \(\epsilon\) we study the resulting equation for the plane-wave solutions associated with the rest spinors. It is easily verified that for \(\rho\{p^\mu\}\) as well as \(\lambda\{p^\mu\}\), it is not \(\epsilon\) (directly) but \(\epsilon^2\) that is constrained by the relation: \(\epsilon^2 = 1\), giving \(\epsilon = \pm 1\). This is consistent with the intuitive understanding in that we cannot distinguish between the forward-in-time propagating ("particles") and the backward-in-time propagating ("antiparticles") Majorana-like objects. The above arguments are independent of which representation we choose within the \((1, 0) \oplus (0, 1)\) representation space.

6. Majorana-Like \((1, 0) \oplus (0, 1)\) Field Operator

We now exploit the above considerations on the Majorana-like spinors to construct the associated field operator. Generalizing the spin-\(\frac{1}{2}\) definition for a Majorana particle of Ref. [5], we define a general \((j, 0) \oplus (0, j)\) Majorana-like field operator \(\Xi(x)\)

\[
U(C_{\theta}) \Xi(x) U^{-1}(C_{\theta}) = \pm \Xi(x) \quad .
\]

(29)

In Eq. (29), the "+" sign defines the self-\(\theta\)-conjugate and the "−" sign defines the antiself-\(\theta\)-conjugate field operator. The explicit chiral-representation expression for \(\theta\)-conjugation operator \(C_{\theta}\) as contained in Eq. (8) is

\[
C_{\theta} = C_{\theta} K = \begin{bmatrix}
0 & \xi \Theta_{[j]} \\
(\xi \Theta_{[j]})^* & 0
\end{bmatrix} K
\]

(30)
where $K$ complex conjugates (on the right) the objects in the Majorana-like $(j, 0) \oplus (0, j)$ representation space. For the example case of spin-1, when Eqs. (29) are coupled with the additional physical requirement that all helicity degrees of freedom be treated symmetrically for manifest $P$-covariance, the field operator $\Xi(x)$ is determined to be

\[
\Xi^S_{h}(x) = \sum_{h=0,\pm 1} \int d^4p \left[ S^S_h(p^\mu) \rho^S_h[p^\mu] \exp(-ip \cdot x) + \eta_{GK} S^S_{h}^{(\lambda)}(p^\mu) \lambda^S_h[p^\mu] \exp(+ip \cdot x) \right],
\]

\[
\Xi^{A}_{h}(x) = \sum_{h=0,\pm 1} \int d^4p \left[ A^A_h(p^\mu) \rho^A_h[p^\mu] \exp(-ip \cdot x) + \eta_{GK} A^A_{h}^{(\lambda)}(p^\mu) \lambda^A_h[p^\mu] \exp(+ip \cdot x) \right],
\]

where $\eta_{GK}$ is the generalized [6] Goldhaber-Kayser phase factor; and

\[
\begin{align*}
\left[ S^S_h(p^\mu), S^S_{h'}(p'^{\mu}) \right] &= -(-1)^{|h|} (2\pi)^3 2E(\vec{p}) \delta_{h,-h'} \delta(\vec{p} - \vec{p}') , \\
\left[ S^S_h(p^\mu), S^{(\lambda)}_{h'}(p'^{\mu}) \right] &= -(-1)^{|h|} (2\pi)^3 2E(\vec{p}) \delta_{h,-h'} \delta(\vec{p} - \vec{p}') ,
\end{align*}
\]

with similar expression for the creation and annihilation operators of the $\Xi^{A}_{h}(x)$ field. Several unusual features of expressions for the field operators $\Xi(x)$, Eqs. (31) and (32), and commutators (33) and (34) should be explicitly noted: I. The factor $-(1)^{|h|} \delta_{h,-h'}$, rather than the usual $\delta_{hh'}$, in the r.h.s. of Eqs. (33) and (34) arises from the bi-orthogonal [26] nature of $\rho[p^\mu]$ and $\lambda[p^\mu]$ spinors. II. The creation operator $S^{(\lambda)}_{h}(p^\mu)$ for the plane wave $\lambda^S_h[p^\mu] \exp(ip \cdot x)$ is identical (within a phase factor) to the creation operator for the plane wave $\rho^S_{-h}[p^\mu] \exp(-ip \cdot x)$

\[
S^{(\lambda)}_{h}(p^\mu) = -(-1)^{|h|} S_{-h}^{(\rho)}(p^\mu) ,
\]

with similar comments applicable to $A^{(\lambda)}_{h}(p^\mu)$ and $A^{(\rho)}_{h}(p^\mu)$.

In addition, in view of our results of Sec. 5, the association of the $\rho[p^\mu]$ spinors with the forward-in-time propagating solutions and $\lambda[p^\mu]$ spinors with backward-in-time propagating solutions in the explicit expressions of $\Xi(x)$ above is purely a convention.

Finally, we wish to emphasize that the field operators we arrive at differ from similar expressions [7] found in literature for the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ Majorana field. Unlike the field operators $\Xi(x)$, these expressions [even though they satisfy Eq. (29)] do not exploit the Majorana-construction in the $(j, 0) \oplus (0, j)$ representation space and as a result cannot be expected to contain full physical content of a truly neutral particle.

[6] See footnote 19 of Ref. [5].
[7] See, for example, Eq. (3.25) of Ref. [5]; and Eq. (2.5) of Gluza and Zralek’s paper in Ref. [4].
7. Concluding Remarks

We have succeeded in extending the Majorana-construction for the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space to all \((j, 0) \oplus (0, j)\) representation spaces despite the general impression that Majorana’s original construction was due to certain “magic of Pauli matrices.” We studied the \((1, 0) \oplus (0, 1)\) Majorana-like representation space in some detail and presented an associated wave equation. Since nature has a host of neutral “fundamental particles” of spin-1 and -2 and composite hadronic structures of even higher spins, the existence of the Majorana-like \((j, 0) \oplus (0, j)\) representation spaces introduced in this work may have some physical relevance for the unification beyond the electroweak theory and hadronic phenomenologies.

In the massless limit, \((j, 0) \oplus (0, j)\) fields, independent of spin and independent of whether they are Dirac-like or Majorana-like, contain only two helicity degrees of freedom. This observation allows the construction of higher-spin field theories without introducing or imposing any auxiliary fields, negative-norm states, or constraints. This fact may have some significance for theories involving supersymmetric transformations, which transform between fermions and bosons, and which are normally rife with non-physical, additional fields. It should be explicitly noted that even though the construction of the \((j, 0) \oplus (0, j)\) Majorana-like fields is manifestly covariant under parity, in general massive Majorana-like particles carry imaginary intrinsic parity \cite{30} and hence these particles in interactions with Dirac-like/Dirac particles (like charged leptons and quarks) naturally lead to non-conservation of parity.

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    For spins \( j \geq 1 \), these results will be reported in a forthcoming publication.
TABLE I: Spin-1 self-θ-conjugate Majorana-like $\rho^S_\theta[p^\mu]$ spinors. Here $p^\pm = E \pm p_z$, $p^r = p_x + i p_y$ and $p^\ell = p_x - i p_y$. The subscript $h = 0, \pm 1$ on $\rho^S_\theta[p^\mu]$ refers to the front-form helicity \[12\] degree of freedom. The remaining spinors $\lambda^S_\theta[p^\mu]$, $\rho^A_\theta[p^\mu]$ and $\lambda^A_\theta[p^\mu]$ are related to $\rho^S_\theta[p^\mu]$ via Eqs. \[22\] to \[24\].

|                  | $\rho^S_\theta[p^\mu]$ | $\rho^S_\theta[p^\mu]$ | $\rho^S_\theta[p^\mu]$ |
|------------------|--------------------------|--------------------------|--------------------------|
| $\frac{1}{2}$    | $p^+ + (p^r/2/p^\ell)$  | $p^\ell/p^+$             | $1/p^+$                  |
|                  | $\sqrt{2}(p^r - p^\ell)$| 0                        | 0                        |
| $p^+ + (p^r/2/p^\ell)$ | $p^\ell/p^+$             | $p^\ell/p^+$             | $1/p^+$                  |
| $p^+ - (p^r/2/p^\ell)$ | $-p^\ell/p^+$            | $-1/p^+$                 |
| $\sqrt{2}(p^r + p^\ell)$ | $\sqrt{2}$              | 0                        |
| $-p^+ + (p^r/2/p^\ell)$ | $p^\ell/p^+$             | $1/p^+$                  |