HEISENBERG REALIZATIONS, EIGENFUNCTIONS AND PROOF OF THE KURLBERG-RUDNICK SUPREMUM CONJECTURE

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Preliminary version

Abstract

In this paper, proof of the Kurlberg-Rudnick supremum conjecture for the quantum Hannay-Berry model is presented. This conjecture was stated in P. Kurlberg’s lectures at Bologna 2001 and Tel-Aviv 2003. The proof is a primer application of a fundamental solution: all the Hecke eigenfunctions of the quantum system are constructed. The main tool in our construction is the categorification of the compatible system of realizations of the Heisenberg representation over a finite field. This enables us to construct certain ”perverse sheaves” that stands motivically prior to the Hecke eigenfunctions.

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\section{Introduction}

In most branches of modern science, systems are described by mathematical equations. For example, in quantum mechanics one has to consider the eigenfunction and eigenvalue problems of the well-known \textit{Schrödinger} equation:

\[ (-\frac{\hbar^2}{2m} \Delta + V) \Psi = \lambda \Psi, \quad (0.0.1) \]

on some configuration space \(X\). One of the main problems is to obtain an effective description of the solutions of such equations. This problem of course becomes quite intractable as the equation becomes more complicated.

\subsection{Quantum chaos}

A classical mechanical system is modelled by a phase space \(M\). A point of the space records the position and momenta of all the particles in the system. The evolution of the system in time defines a path \(\gamma(t)\) in \(M\). Newton's laws say that this path is the solution of the differential equation:

\[ \frac{d\gamma}{dt} = \xi_H(\gamma(t)), \quad (0.1.1) \]

where \(\xi_H\) is called the Hamiltonian vector field and \(H\) is a function on \(M\) which corresponds to the energy of the system. A physical observable (temperature, position, momentum,...) corresponds to a function \(f\) on \(M\). If the system is in a state \(\gamma(t)\), then the outcome of the observation corresponding to \(f\) is a single value \(f(\gamma(t))\).

A quantum mechanical system is modelled by an Hilbert space \(\mathcal{H}\). A vector \(\Psi\) in \(\mathcal{H}\) records the state of the system. The evolution of the system in time therefore corresponds to a path \(\Psi(t)\) in \(\mathcal{H}\). The physical laws of the system are encoded by a self-adjoint operator \(H_\hbar\). The parameter \(\hbar\) is called the Planck constant. These laws state that if \(\Psi(t)\) is a vector that represents the system at time \(t\), then it satisfies an analogue of (0.1.1):

\[ \partial_t \Psi(t) = H_\hbar \Psi(t). \quad (0.1.2) \]

A solution \(\Psi\) of equation (0.1.2) does not represents a classical trajectory. Instead, it is interpreted as a statistical entity in the following sense. In general, a physical observable is an operator \(\pi_\hbar(f)\) on \(\mathcal{H}\). If the system is in the state \(\Psi\) then the observation that corresponds to \(\pi_\hbar(f)\) cannot be predicted exactly. There is a probability distribution of possible outcomes which is determined by \(\Psi\) and the average is given by the matrix coefficient \(\langle \pi_\hbar(f) \Psi, \Psi \rangle\).
Consider a chaotic classical dynamical system. The main objective (cf. [M, S]) of quantum chaos theory is to explain how chaotic behavior is manifested at the quantum-mechanical level, or at least in the semi-classical limit as $\hbar$ tends to 0. During the 70’s and the 80’s Berry, Berry-Tabor, Bohigas, Gianonni and Schmidt obtained (cf. [Be, H, M, S]) very accurate conjectural descriptions of the behavior of eigenfunctions and eigenvalues for generic chaotic systems, albeit not much has been confirmed mathematically.

0.2 The Hannay-Berry Model

It is for the above reasons that the physicists J. Hannay and M.V. Berry proposed around 1980 [HB] a simple mathematical model for quantum mechanics on the two-dimensional torus $\mathbb{T}$. In this model, one considers an ergodic (discrete) dynamical system generated by a single linear map:

$$A : \mathbb{T} \rightarrow \mathbb{T}.$$

The associated quantum system (cf. [GH1, GH2, KR1]) is an operator $\rho_\hbar(A)$ acting on a finite dimensional Hilbert space $\mathcal{H}$.

0.3 The Kurlberg-Rudnick conjectures

The Kurlberg-Rudnick conjectures is a set of conjectural statements [Ku1, Ku2, KR3, K1, R2] describing the operator $\rho_\hbar(A)$. They were motivated by a series of fundamental papers (cf. [DGI, KR1, KR2, KR3]). These conjectures describe the behavior of the common eigenstates $\Psi$ of the Hecke symmetries of the operator $\rho_\hbar(A)$, i.e.,:

$$\rho_\hbar(B)\Psi = \lambda(B)\Psi, \quad (0.3.1)$$

for every symmetry $B$.

0.4 Supremum and value distribution conjecture

This is a conjecture [Ku1, Ku2] on the size and on the value distribution of the Hecke eigenstates $\Psi$, $\Psi \leq 2$. The conjecture says that for $\hbar$ of the form $\hbar = \frac{1}{p}$, where $p$ is a prime number, the Hecke eigenstates are uniformly bounded, i.e.,:

$$\|\Psi\|_\infty \leq 2 \quad (0.4.1)$$

for any Hecke eigenstate $\Psi$. Moreover, the conjecture claims that the value distribution of each of these $\Psi$’s (suitably twisted [K2]) behaves like the trace of a random matrix in SU(2).
Special cases of this conjecture were obtained in [KR2] using results established in a work of N. Katz [K2]. However, in general only the bound:

$$\|\Psi\|_\infty \leq p^{3/8},$$

was available [KR2]. In the current work, proof of the supremum conjecture (0.4.1) is presented. Moreover, we show that this bound holds true in any realization of the quantum system (corresponds to measuring different observables).

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1 The Hannay-Berry Model

1.1 Classical phase space

Our classical phase space is the 2n-dimensional symplectic torus \((\mathbb{T}, \omega)\). It is equipped with an action of the group \(\Gamma \simeq \text{Sp}(2n, \mathbb{Z})\) acting by linear symplectomorphisms.

In more detail, consider the torus \(\mathbb{T} := W/\Lambda\), where \(W\) is a 2n-dimensional real vector space, i.e., \(W \simeq \mathbb{R}^{2n}\), and \(\Lambda \subset W\) is a full rank lattice, i.e., \(\Lambda \simeq \mathbb{Z}^{2n}\). The symplectic structure on \(\mathbb{T}\) is obtained from a skew-symmetric bilinear form \(\omega : W \times W \to \mathbb{R}\). We require \(\omega\) to be integral, i.e., \(\omega(\Lambda \times \Lambda) \subset \mathbb{Z}\) and normalized, i.e., \(\text{Volume}(\mathbb{T}) = 1\). Denote by \(\Gamma \subset \text{Sp}(W, \omega)\) the subgroup of elements preserving the lattice \(\Lambda\). We have \(\Gamma \simeq \text{Sp}(2n, \mathbb{Z})\). The group \(\Gamma\) naturally acts as the group of linear symplectomorphisms of \((\mathbb{T}, \omega)\).
1.2 Quantization

Our quantum object will be a pair of compatible representations of a quantum algebra $A_{\hbar}$ and the group $\Gamma$.

In more detail, consider the algebra $A$ of trigonometric polynomials on the torus, namely functions which can be presented as a finite linear combination of characters. Note that the algebra $A$ has as a basis the lattice of characters $T^\vee := \text{Hom}(T, \mathbb{C}^*)$.

Next, we want to define a family of quantum algebras. Consider the dual lattice $\Lambda^* \subset W^*$, defined by $\Lambda^* = \{ \xi \in W^* : \xi(\Lambda) \subset \mathbb{Z} \}$. Note that we can identify $\Lambda^*$ with the lattice of characters on $T$ by the following map:

$$\xi \in \Lambda^* \longmapsto e^{2\pi i <\xi, \cdot>} \in T^\vee.$$

We construct a family of (star, $*$-) algebras $A_{\hbar}$, i.e., for each $\hbar \in \mathbb{R}$ we consider the algebra generated over $\mathbb{C}$ by the symbols $\{ s(\xi) : \xi \in \Lambda^* \}$ satisfying the relations $s(\xi + \eta) = e^{\pi i h \omega(\xi, \eta)} s(\xi) s(\eta)$. The family $A_{\hbar}$ form a one-parametric deformation of the commutative algebra $A$, that is, $A_0 = A$. The parameter $\hbar$ is called the Planck constant. The symbols $\{ s(\xi) : \xi \in \Lambda^* \}$ form a basis of $A_{\hbar}$. This allows us to identify, as vector spaces, the algebras $A_{\hbar}$ and $A$ for every value of $\hbar$. We will often identify the symbol $s(\xi)$ with the element $\xi$ itself, in order to save notation. Following [Ri], we call the algebra $A_{\hbar}$ the Rieffel’s (algebraic) quantum torus.

The group $\Gamma$ acts by automorphism on the algebra $A_{\hbar}$ via the formula $B \cdot s(\xi) = s(B\xi)$ for every $B \in \Gamma$. This induces an action of $\Gamma$ on the category $\text{Rep}(A_{\hbar})$ of ($*$-) representation of $A_{\hbar}$, and hence on the set $\text{Irr}(A_{\hbar})$ of isomorphism classes of irreducible algebraic representations of $A_{\hbar}$. More concretely, given a representation $\pi : A_{\hbar} \to \text{End}(\mathcal{H})$ and an element $B \in \Gamma$, we define a new representation $\pi^B : A_{\hbar} \to \text{End}(\mathcal{H})$ by $\pi^B(f) := \pi(f \circ B)$.

For the remainder of this section, we fix $\hbar = \frac{1}{p}$, where $p$ is an odd prime number. We have the following basic theorem:

**Theorem 1.1 ([GH2])** There exists a unique (up to isomorphism) irreducible representation $(\pi, A_{\hbar}, \mathcal{H})$ so that its isomorphism class $[\pi] \in \text{Irr}(A_{\hbar})$ is fixed by $\Gamma$.

Let $(\pi, A_{\hbar}, \mathcal{H}) \in \text{Irr}(A_{\hbar})$, be a representation such that $B \cdot [\pi] = [\pi]$ for all $B \in \Gamma$. This is equivalent to having a projective representation $\rho : \Gamma \to \text{PGL}(\mathcal{H})$ and compatibility condition:

$$\rho(B)\pi(f)\rho(B)^{-1} = \pi(f \circ B) , \quad (1.2.1)$$
for every $B \in \Gamma$ and $f \in \mathcal{A}_h$. Condition (1.2.1) is called, traditionally, the Egorov identity.

In fact one can do even better. The projective representation $\rho$ can be linearized. There exists a canonical linearization $\rho : \Gamma \to \text{GL}(\mathcal{H})$, which factors through the finite quotient group $G \cong \text{Sp}(2n, \mathbb{F}_p)$. Altogether we can take our quantum object to consist of a pair:

$$\pi : \mathcal{A}_h \rightarrow \text{End}(\mathcal{H}),$$
$$\rho : G \rightarrow \text{GL}(\mathcal{H}),$$

satisfying the Egorov identity (1.2.1).

Comments.

1. All irreducible representations $\pi \in \text{Irr}(\mathcal{A}_h)$ are of similar nature. They are all finite dimensional, more precisely $p$-dimensional. The set of equivalence classes of irreducible representations is a manifold, which is a principal homogeneous space over the torus $\mathbb{T}$ (cf. [GH2]).

2. The representation $\rho : G \to \text{GL}(\mathcal{H})$ is the celebrated Weil representation [W1] of the finite symplectic group. This representation is obtained here via quantization of the torus. This approach is different from the classical constructions [Ge] and extend an earlier work carried out by the authors in the two-dimensional setting [GH3].

1.3 Classical and quantum dynamical systems

Here we restrict our attention to the case $n = 2$, namely we deal with the two-dimensional symplectic torus $(\mathbb{T}, \omega)$.

**Classical system.** We fix an hyperbolic element $A \in \Gamma$, namely, an element for which all eigenvalues in $\overline{\mathbb{Q}}$ are not roots of unity. Consider the corresponding automorphism

$$A : \mathbb{T} \longrightarrow \mathbb{T}.$$  \hspace{1cm} (1.3.1)

It is well-known that, via the iterations of the action (1.3.1), the element $A$ generates a discrete ergodic dynamical system on the torus $\mathbb{T}$.

**Quantum system.** Taking the element $A$, now considered as an element of the finite group $G$, we obtain a unitary operator:

$$\rho(A) : \mathcal{H} \rightarrow \mathcal{H}.$$  \hspace{1cm} (1.3.2)

The operator (1.3.2) is considered to be the quantization of the classical dynamical system $A : \mathbb{T} \to \mathbb{T}$. 

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2 Universal Supremum Bound Conjecture

It is the main meta-question in the area of quantum chaos to investigate: What manifestations of the chaotic behavior of A are seen at the quantum mechanical level? In what follows we are going to study a specific aspect of this question \[\text{Be, H, KR3}\].

2.1 Hecke eigenvectors

Denote by \(< A > \subset G\) the cyclic group generated by the element A. It is contained inside a “slightly” bigger group \(T_A\), namely, the centralizer of the element A in G. The group \(T_A\) is an algebraic group, more precisely it consists of the rational points of an algebraic torus. We follow \([KR1]\) and call \(T_A\) the Hecke torus.

The torus \(T_A\) acts on \(H\) in a semi-simple fashion. Hence, we obtain a decomposition into character spaces:

\[
H = \bigoplus_{\chi: T_A \to \mathbb{C}^\ast} H_{\chi}.
\]

The group \(T_A\) acts on the subspace \(H_{\chi}\) by the character \(\chi\). The subspace \(H_{\chi}\) is called the \(\chi\)-Hecke eigenspace, and a unit vector \(v_\chi \in H_{\chi}, \|v_\chi\| = 1\), is called a (normalized) \(\chi\)-Hecke eigenvector.

2.1.1 Universal supremum bound

Fix a multiplicative character \(\chi: T_A \to \mathbb{C}^\ast\), and let \(v_\chi \in H_{\chi}\) be a (normalized) Hecke eigenvector. The elements \(\xi \in \Lambda^\ast\), gives a standard generating set of observables \(\{\pi(\xi)\}_{\xi \in \Lambda^\ast}\). Choosing an observable \(\pi(\xi)\), we can realize \(H \simeq H_{\xi} := \Gamma(\sigma(\xi), H_{\xi})\), i.e. the global section of an Hermitian line bundle \(H_{\xi}\), on the spectrum \(\sigma(\xi)\) of the operator \(\pi(\xi)\). This line bundle is described by having fibers \((H_{\xi})_x = H_x\) for every \(x \in \sigma(\xi)\) on which we have the action \(\pi(\xi)|_{H_x} = x \cdot \text{Id}\). The vector \(v_\chi\) gives a global section \(\Psi_\chi \in H_{\xi}\). We denote by \(\|\Psi_\chi\|_\infty\) its supremum:

\[
\|\Psi_\chi\|_\infty := \sup_{x \in \sigma(\xi)} |\Psi_\chi(x)|.
\]

In this paper we are going to prove the following extended version of the Kurlberg-Rudnick supremum conjecture (cf. \([Ku1, Ku2, KR3]\)):

**Theorem 2.1 (Universal supremum bound)** We have:

\[
\|\Psi_\chi\|_\infty \leq 2.
\]
Note, that the bound is independent of the character $\chi$, the observable $\xi$ and the Planck constant $\hbar = \frac{1}{p}$.

Comments.

1. **Physical interpretation.** The unit sphere $S(\mathcal{H}) = \{ v \in \mathcal{H} : \|v\| = 1 \}$ constitutes the set of (pure) quantum states. Measuring a quantum observable $\pi(\xi)$ on a quantum state $v \in S(\mathcal{H})$, amounts to realizing $v$ as a section $\Psi_v \in \mathcal{H}_{\xi}$. The possible outcomes of the measurement procedure are elements $x \in \sigma(\xi)$, with probabilities $|\Psi_v(x)|^2$. The statement of Theorem 2.1 may be interpreted as saying that an Hecke eigenvector $v_\chi$ is a special quantum state, on which measurement of any standard observable gives outcomes in fairly uniform distribution.

2. **Mathematical interpretation.** Suppose we are looking at a specific vector $v \in \mathcal{H}$ in two different realizations $\mathcal{H}_{\xi}$ and $\mathcal{H}_{\eta}$ corresponding to the observables $\xi, \eta \in \Lambda^*$, that are assumed to be non-proportional. The associated sections $\Psi^\xi_v \in \mathcal{H}_{\xi}$ and $\Psi^\eta_v \in \mathcal{H}_{\eta}$ are related by a certain kind of Fourier transform. Hence, Theorem 2.1 may be interpreted as saying that $\Psi_v$ satisfies $|\Psi_v(x)| \leq 2$, and also $|\widehat{\Psi}_v(x)| \leq 2$, where $\widehat{\Psi}_v$ is the Fourier transform of $\Psi_v$. This is of-course not a trivial property.

3. **Implication of the Kurlberg-Rudnick supremum conjecture.** Fix a prime $p$, and choose a standard observable $\xi$. The realization $(\rho, G, \mathcal{H}_\xi)$ amounts to the standard realization of the Weil representation on the space of functions $L^2(F_p, \mathbb{C})$. Consider an Hecke eigenvector $\Psi_\chi \in L^2(F_p, \mathbb{C})$. The Kurlberg-Rudnick conjecture states that $|\Psi_\chi(x)| \leq 2$ for every $x \in F_p$. Note, that this bound is independent of the prime $p$. This is of-course a particular case of Theorem 2.1. The element $A$ being hyperbolic implies that for half of the primes $p$, the torus $T_A$ is inert (does not split). We want to mention here that the main problem is to compute $\Psi_\chi$ for $p$ where $T_A$ is inert.

3 Canonical Hilbert Space

Let $(V, \omega)$ be a $2n$-dimensional symplectic vector space over the finite field $\mathbb{F}_q$. Fix a non-trivial additive character $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^*$. We have the following theorem (cf. [GH3]):

**Theorem 3.1 (Canonical Hilbert space)** There exists a canonical Hilbert space $\mathcal{H}_V$ associated to the pair $(V, \psi)$. 

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An immediate consequence of this theorem is that all symmetries of \((V, \omega)\) automatically act on \(\mathcal{H}_V\). In particular, we obtain a (linear) representation of the group \(G := \text{Sp}(V, \omega)\) of linear symplectomorphisms.

### 3.1 Construction

#### 3.1.1 Heisenberg group

There exists a two-step nilpotent group \(H = H(V, \omega)\), called the Heisenberg group, associated to the symplectic vector space \((V, \omega)\). As a set \(H = V \times \mathbb{F}_q\), with the following multiplication rule:

\[
(v, z) \cdot (v', z') = (v + v', z + z' + \frac{i}{2} \omega(v, v')).
\]

The center of \(H\) is \(Z(H) = \{(0, z) : z \in \mathbb{F}_q\}\). Identifying \(Z(H) = \mathbb{F}_q\), we consider the character \(\psi\) to be a character of the center \(Z(H)\). We have the following fundamental theorem:

**Theorem 3.2 (Stone-Von Neumann)** There exists a unique (up to isomorphism) irreducible representation \((\pi, H, \mathcal{H})\) with central character \(\psi\), i.e., \(\pi|_{Z(H)} = \psi\).

The representation \((\pi, H, \mathcal{H})\) is called the Heisenberg representation.

#### 3.1.2 Models

Although the Heisenberg representation is unique, it admits a multitude of different models (realizations). Here we construct a specific family of such models. A fundamental ingredient in our construction is the notion of enhanced Lagrangian\(^1\) suggested to us by J. Bernstein [BG, GH3, GH]. Consider the set \(\text{Lag} := \text{LGr}(V, \omega)\) of all Lagrangian subspaces in \(V\). The set \(\text{Lag}\) is called the Lagrangian Grassmannian associated to \(V\).

**Definition 3.3** An enhanced Lagrangian is a pair \((L, \sigma_L)\), where \(L \in \text{Lag}\), and \(0 \neq \sigma_L \in \Lambda^a L\).

The set of enhanced Lagrangians is denoted by \(\text{Lag}^o\). Consider an element \(L^o \in \text{Lag}^o\), \(L^o = (L, \sigma_L)\). Let \(\tilde{L} = pr^{-1}(L) = L \times Z(H)\), where \(pr : H \rightarrow V\) is the standard projection. The set \(\tilde{L} \subset H\) is an abelian subgroup of \(H\). Define the character \(\psi_{L^o} : \tilde{L} \rightarrow \mathbb{C}^*\) as an extension of \(\psi: \psi_{L^o}(l, z) = \psi(z)\). Associated to

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\(^1\)We thank A. Polishchuk for pointing out to us that this should be thought of as an \(\mathbb{F}_q\)-analogue of well-known considerations with usual oriented Lagrangians giving explicitly the metaplectic covering of \(\text{Sp}(2n, \mathbb{R})\) (cf. [LV]).
this data we construct an Hilbert space $\mathcal{H}_L = \mathcal{H}(H, L^0, \psi) = \{ f : H \to \mathbb{C} : f(\tilde{l} \cdot h) = \psi_L(\tilde{l}) f(h) \}$. This is a particular case of induction, equivalently we can write $\mathcal{H}_L = \text{Ind}^H_L(\psi_L)$. The Hilbert space $\mathcal{H}_L$ admits a representation of $H$, acting via right multiplication. We denote this representation by $(\pi_L, H, \mathcal{H}_L)$. It is not hard to see that $\pi_L$ is an irreducible representation, with central character $\psi$, thus it constitutes a model of the Heisenberg representation.

3.1.3 Canonical intertwining operators

Given a pair $(L^0, M^0) \in \text{Lag}^0 \times \text{Lag}^0$, we have two models $(\pi_L, H, \mathcal{H}_L)$ and $(\pi_M, H, \mathcal{H}_M)$. Consider the space $\text{Int}_{M^0,L^0}$ of intertwining operators:

$$\text{Int}_{M^0,L^0} := \text{Hom}_H(\mathcal{H}_L, \mathcal{H}_M).$$

Both models are irreducible representations of $H$ so we have $\dim \text{Int}_{M^0,L^0} = 1$. Every element $F \in \text{Int}_{M^0,L^0}$ is proportional to an averaging operator:

$$F(f)(h) = g_F \sum_{m \in M} f(m \cdot h),$$

for $f \in \mathcal{H}_L$ and $h \in H$. Here $g_F$ is the proportionality coefficient. It turns out that one can choose the elements $F_{M^0,L^0}$ in a canonical fashion, and we shall discuss this next.

Let $G_m$ denote the (finite) multiplicative group, $G_m = \mathbb{F}_q^*$. Given a Lagrangian $L \in \text{Lag}$, we have the group $G_m$ acting on the 1-dimensional vector space $\wedge^n L$ by homoteties. This induces an action of $G_m$ on the space $\text{Lag}^0$. It is given by the formula $a \cdot (L, \sigma_L) = (L, a \cdot \sigma_L)$ for $a \in G_m$. Denote by $\chi_q : G_m \to \mathbb{C}^*$ the Legendre quadratic character. We have the following theorem:

**Theorem 3.4 (Canonical intertwining operators)** There exists a canonical family $\{ F_{M^0,L^0} \in \text{Int}_{M^0,L^0} \}$ characterized by the following properties:

1. **Normalization.** $F_{L^0,L^0} = 1$.
2. **Invariance.** $F_{M^0,L^0} = F_{gM^0,gL^0}$ for every element $g \in G$.
3. **Convolution.** $F_{N^0,M^0} \circ F_{M^0,L^0} = F_{N^0,L^0}$.
4. **Sign rule.** $F_{aM^0,L^0} = \chi_q(a)F_{M^0,L^0}$, and $F_{M^0,aL^0} = \chi_q(a)F_{M^0,L^0}$ for every $a \in G_m$.

**Comment.** We elaborate on the formal meaning of Property 2 in Theorem 3.4. The group $G$ acts on all structures involved. The action on the space of enhanced Lagrangians $\text{Lag}^0$ is tautological. Also $G$ acts on the Heisenberg
group by automorphism. This action is the standard action on the vector space \( V \), and it is trivial on the center \( Z(H) \). This induces an action of \( G \) on the space of functions \( L^2(H, \mathbb{C}) \), given by \( g \cdot f(h) = f(g^{-1}h) \). It is easy to verify that \( g \) sends the space \( \mathcal{H}_L^o \) isomorphically to the space \( \mathcal{H}_{gL^o} \). The notation \( F^g_{M^o,L^o} \) stands for the composition \( g \circ F_{M^o,L^o} \circ g^{-1} \). Now the interpretation of Property 2 is clear.

### 3.1.4 Canonical Hilbert space

The canonical Hilbert space \( \mathcal{H}_V \) is defined as the subspace \( \mathcal{H}_V \subset \bigoplus_{L^o \in \text{Lag}^o} \mathcal{H}_{L^o} \), with:

\[
\mathcal{H}_V := \{(v_{L^o})_{L^o \in \text{Lag}^o} : v_{M^o} = F_{M^o,L^o}(v_{L^o}) \text{ for every pair } (M^o, L^o) \in \text{Lag}^o \times \text{Lag}^o \}.
\]

In other words, the space \( \mathcal{H}_V \) consists of compatible systems of vectors. The existence of such systems is a consequence of the convolution property satisfied by the canonical intertwining operators (see Property 3 in Theorem 3.4).

### 3.2 Weil representation

The vector space \( \mathcal{H}_V \), being canonical, admits automatically a representation of the group \( G \), called the Weil representation. We give here two explicit descriptions of the Weil representation:

**(I) Invariant form.** Consider the map \( \rho_V : G \to \text{GL}(\mathcal{H}_V) \), which is defined by the action on a compatible system \( \vec{v} = (v_{L^o})_{L^o \in \text{Lag}^o} \):

\[
[\rho_V(g)](\vec{v}) := v^g_{g^{-1}L^o},
\]

where the function \( v^g_{g^{-1}L^o} \in \mathcal{H}_{L^o} \) is defined by the formula:

\[
v^g_{g^{-1}L^o}(h) := v_{g^{-1}L^o}(g^{-1}h),
\]

for every \( h \in H \). It is a direct consequence of the properties 1-3 in Theorem 3.4 that formula (3.2.1) gives a representation of \( G \).

**(II) Models.** Choosing an enhanced Lagrangian \( L^o \in \text{Lag}^o \), we can identify \( \mathcal{H}_{L^o} \simeq \mathcal{H}_V \) via the inclusion \( \mathcal{H}_{L^o} \hookrightarrow \bigoplus_{M^o \in \text{Lag}^o} \mathcal{H}_{M^o} \). Under this identification, the representation \( (\rho_V, G, \mathcal{H}_V) \) is realized as a representation \( (\rho_{L^o}, G, \mathcal{H}_{L^o}) \), which is given by:

\[
\rho_{L^o}(g)(v_{L^o}) = F_{L^o, gL^o}(v^g_{L^o}).
\]
3.3 Back to quantum mechanics

We would like to establish a dictionary between quantum mechanics on the two-dimensional torus \((T, \omega)\) (see Section 2), and the formalism of the canonical Hilbert space. This type of equivalence is valid when the planck constant \(\hbar\) takes rational values - rational quantization. We will be even more particular, assuming that \(\hbar = \frac{1}{p}\), where \(p\) is an odd prime.

Consider the lattice \(\Lambda^* = \text{Hom}(T, \mathbb{C}^*)\) of characters of \(T\). Let \(V = \Lambda^*/p\Lambda^*\) be the quotient abelian group. The group \(V\) has a natural structure of a two-dimensional vector space over the finite field \(\mathbb{F}_p\). The form \(\omega\) induces a symplectic form on \(V\), which we denote also by \(\omega : V \times V \to \mathbb{F}_p\). In the definition of the algebra \(A_\hbar\), for \(\hbar = \frac{1}{p}\), a character \(e^{\frac{2\pi i x}{p}}\) appeared. We denote it by \(\psi\), and consider it as a character of the finite field \(\mathbb{F}_p\).

The dictionary works as follows:

- The algebra \(A_\hbar\) is replaced by the Heisenberg group \(H = H(V, \omega)\).
- The representation \((\pi, A_\hbar, H)\) (see Theorem 1.1) corresponds to the Heisenberg representation, realized on the canonical Hilbert space \((\pi_V, H, H_V)\).
- The Weil representation \((\rho, G, H)\) is taken in its invariant form on the canonical Hilbert space \((\rho_V, G, H_V)\).
- Choosing a standard observable \(\xi \in \Lambda^*\), and identifying \(H \simeq H_\xi\) corresponds to considering the enhanced Lagrangian \(L_\xi = (L_\xi, \sigma_\xi) := (\mathbb{F}_p, \xi, \xi)\) and identifying \(H_V \simeq H_{L_\xi}\). The vector space \(H_{L_\xi}\) is obtained via induction so it is the space \(\Gamma(X_{L_\xi}, H_{L_\xi})\) of global sections of an Hermitian line bundle \(H_{L_\xi}\) on the set \(X_{L_\xi} = L_\xi \setminus H\). An Hecke eigenvector \(v_\chi \in H_V\) is realized as a global section \(\Psi_\chi \in H_{L_\xi}\).

Having the above dictionary, Theorem 2.1 can be equivalently formulated as follows:

**Theorem 3.5 (Universal supremum bound - reformulation)** We have

\[
\sup_{x \in X_{L_\xi}} |\Psi_\chi(x)| \leq 2,
\]

for every character \(\chi\), every enhanced Lagrangian \(L^0\) and every prime \(p\).

An interesting question is whether one can compute in some effective manner the Hecke eigenvector \(v_\chi\). We will elaborate on this issue. Assume first that there exists an enhanced Lagrangian \(L^0 \in \text{Lag}^0\), \(L^0 = (L, \sigma)\), such that \(L\) is
fixed by the Hecke torus $T_A$, that is, $gL = L$ for every $g \in T_A$. In this case, one can compute the Hecke eigenvector $v_\chi$ in the realization $H_{L^\sigma}$. In more detail, let $L'' = (L', \sigma')$ be an enhanced Lagrangian such that $V = L \oplus L'$, and $L'$ is also fixed by $T_A$ (such a choice always exists). Using $L'$ as a cross section $s : X_{L^\sigma} \to H$ we identify $H_{L^\sigma} = L^2(L', \mathbb{C})$. It is easy to deduce that in this case the torus $T_A$ splits. Moreover, it is possible to choose the identification $T_A \cong G_m$ so that the restriction of the Weil representation to $G_m$ acting in the realization $L^2(L', \mathbb{C})$ is given by the following formula:

$$[\rho_{L^\sigma}(a)f](l') = \chi(a)f(al').$$

Now, identifying $L' \cong \mathbb{F}_p$, by $x \mapsto x\sigma'$, we take the function $\Psi_\chi \in L^2(\mathbb{F}_p, \mathbb{C})$, with:

$$\Psi_\chi(x) := \chi(x)\chi(x).$$

(3.3.1)

It is clear that formula (3.3.1) describes a $\chi$-eigenvector for $T_A$.

**Summary.** We found a realization $H_{L^\sigma}$ in which the Hecke torus $T_A$ acts in a geometric manner. In this realization we are able to compute precisely the eigenvector $v_\chi$.

**Problem.** What to do when the Hecke torus is inert, i.e., does not split? In this case there exists no Lagrangian $L \in \text{Lag}$, which is fixed by $T_A$. Hence, we will approach the problem from a more abstract perspective.

## 4 Geometrization

Geometrization is a general methodology, invented by Grothendieck, by which sets are replaced by algebraic varieties (over the finite field $\mathbb{F}_q$) and functions are replaced by sheaf theoretic objects ($\ell$-adic Weil sheaves). In this section we are going to apply this methodology to obtain a geometric analogue of the canonical Hilbert space. In particular, we obtain the following: the canonical intertwining operators (CIO) $\{F_{M^\sigma,L^\sigma}\}$ are replaced by a single (shifted) perverse Weil sheaf. Each model $(\rho_{L^\sigma}, G, \pi_{L^\sigma}, H, H_{L^\sigma})$ is replaced by a category $\mathcal{D}_{L^\sigma}$ of Weil sheaves, equipped with a compatible action of the groups $H$ and $G$. Moreover, there is a formal procedure to reconstruct the representation $(\rho_{L^\sigma}, G, \pi_{L^\sigma}, H, H_{L^\sigma})$ from the category $\mathcal{D}_{L^\sigma}$.

---

2This is a generalization of the geometrization of the Weil representation proposed by Deligne in [D] and presented by the authors in [GH3].

3Similar to, and very much influenced by the procedure that appears in the work of Braverman-Polishchuk [BP].
4.1 Functorial description of the CIO

For every pair \((M^\circ, L^\circ) \in \text{Lag}^\circ \times \text{Lag}^\circ\), the intertwining operator \(F_{M^\circ, L^\circ} : \mathcal{H}_{L^\circ} \to \mathcal{H}_{M^\circ}\) is given by a kernel function \(F_{M^\circ, L^\circ} : H \times H \to \mathbb{C}\) satisfying the following properties:

1. **Intertwining.**
   \[
   F_{M^\circ, L^\circ}(m \cdot h_1, l \cdot h_2) = \psi_{M^\circ} (m) \psi_{L^\circ}^{-1}(l) F_{M^\circ, L^\circ}(h_1, h_2),
   \]
   \[
   F_{M^\circ, L^\circ}(h_1 \cdot h, h_2 \cdot h) = F_{M^\circ, L^\circ}(h_1, h_2),
   \]
   for every \(m \in \tilde{M}, l \in \tilde{L}\) and \(h_i \in H\).

2. **Normalization.**
   \[
   F_{L^\circ, L^\circ}(h_1, h_2) = \begin{cases} 
   \psi_{L^\circ}(l) & \text{if } h_1 = l \cdot h_2, \\
   0 & \text{otherwise}.
   \end{cases}
   \]
   for every \(l \in \tilde{L}\) and \(h_i \in H\).

3. **Invariance.**
   \[
   F_{g M^\circ, g L^\circ}(g(h_1), g(h_2)) = F_{M^\circ, L^\circ}(h_1, h_2),
   \]
   for every \(g \in G\) and \(h_i \in H\).

4. **Convolution.**
   \[
   F_{N^\circ, M^\circ} * F_{M^\circ, L^\circ} = F_{N^\circ, L^\circ},
   \]
   for every triple \(N^\circ, M^\circ, L^\circ \in \text{Lag}^\circ\). Here \(*\) is the convolution of kernels:
   \[
   F_{N^\circ, M^\circ} * F_{M^\circ, L^\circ}(h_1, h_2) := \sum_{h \in M \setminus H} F_{N^\circ, M^\circ}(h_1, h) F_{M^\circ, L^\circ}(h, h_2).
   \]

5. **Sign Rule.**
   \[
   F_{a M^\circ, a L^\circ}(h_1, h_2) = \chi_q(a) F_{M^\circ, L^\circ}(h_1, h_2),
   \]
   for every \(a \in G_m\) and \(h_i \in H\).

Note that Property 1 states that the kernel \(F_{M^\circ, L^\circ}\) represents an intertwining operator, namely, an operator from \(\mathcal{H}_{L^\circ}\) to \(\mathcal{H}_{M^\circ}\), which commutes with the action of the Heisenberg group. The other properties 2-5 are only a reformulation of properties 1-4 from Theorem 3.4.
4.1.1 Diagrammatic description

Our next step will be to obtain a diagrammatic description of properties 1-5 above. In order to do this, we have to fix some additional notations.

In general, when we have a set $X$, and another set $S$, which is called a base, we use the notation $X_S := X \times S$. We always (unless explicitly stated otherwise) use the notation $pr : X_S \to S$ for the standard projection on the base.

Consider the set $\text{Lag} = \text{LGr}(V, \omega)$. Let $C \to \text{Lag}^o$ be the pull-back of the canonical vector bundle on $\text{Lag}$ via the forgetful map $\text{Lag}^o \to \text{Lag}$, i.e., $C_{L^o} = L$ for every $L^o \in \text{Lag}^o$. We also define the extended vector bundle $\tilde{C} \to \text{Lag}^o$ by $\tilde{C}_{L^o} = pr^{-1}(L)$, where $pr : H \to V$ is the standard projection. Finally define $\psi_{\tilde{C}} : \tilde{C} \to \mathbb{C}$ as $\psi_{\tilde{C}} = p^*\psi$, where $p$ is the total projection on the center, i.e., $p : \tilde{C} \to Z_{\text{Lag}^o} = Z \times \text{Lag}^o \to Z$.

**Actions.**

- Denote by $pr_1 : \text{Lag}^o \times \text{Lag}^o \to \text{Lag}^o$ and $pr_2 : \text{Lag}^o \times \text{Lag}^o \to \text{Lag}^o$ the projections on the first and the second coordinate correspondingly. We define the action:
  \[ a : (pr_1^*\tilde{C} \times pr_2^*\tilde{C}) \times_{\text{Lag}^o} H^2_{\text{Lag}^o} \to H^2_{\text{Lag}^o}, \]  
  given (fiberwise) by:
  \[ a_{M^o,L^o}(\tilde{m}, \tilde{l}, h_1, h_2) = (\tilde{m} \cdot h_1, \tilde{l} \cdot h_2). \]

- We define the action:
  \[ b : H^2_{\text{Lag}^o} \times_{\text{Lag}^o} H^2_{\text{Lag}^o} \to H^2_{\text{Lag}^o}, \]  
  given (fiberwise) by:
  \[ b_{M^o,L^o}(h_1, h_2, h) = (h_1 \cdot h, h_2 \cdot h). \]

- We define the action:
  \[ m : G \times H^2_{\text{Lag}^o} \to H^2_{\text{Lag}^o}, \]  
  given by:
  \[ m(g, h_1, h_2, M^o, L^o) = (gh_1, gh_2, gM^o, gL^o). \]

- We define the action:
  \[ h : G_m \times H^2_{\text{Lag}^o} \to H^2_{\text{Lag}^o}, \]  
  given by:
  \[ h(a, h_1, h_2, M^o, L^o) = (h_1, h_2, aM^o, aL^o). \]
The collection of kernels \( \{F_{M^L_o}\} \) forms a single function \( F : H^2_{\text{Lag}^o} \to \mathbb{C} \), satisfying the following properties:

1. \( a^*F = \psi \cdot \psi^{-1} \cdot pr^*F, \) where \( pr = pr^2_{H^0_{\text{Lag}^o}}. \)
2. \( b^*F = pr^*F, \) where \( pr = pr^2_{H^0_{\text{Lag}^o}}. \)
3. **Invariance.** \( m^*F = pr^*F, \)
   where \( pr = pr^2_{H^0_{\text{Lag}^o}}. \)
4. **Convolution.** \( pr^*_{12}F \ast pr^*_{23}F = pr^*_{13}F. \)

In more detail, consider the projections \( pr_{ij} : \text{Lag}^o \times \text{Lag}^o \times \text{Lag}^o \to \text{Lag}^o \times \text{Lag}^o, \) that are given for \( i \neq j \) by

\[
pr_{ij}(L^o_1, L^o_2, L^o_3) = L^o_k, \quad k \neq i, j.
\]

Define:

\[
pr^*_{12}F \ast pr^*_{23}F = pr^*_{13}(\Delta^*pr^*_{12}F \cdot pr^*_{23}F) \tag{4.1.3}
\]

where:

\[
\Delta : H^3_{\text{Lag}^o} \to pr^*_{12}H^2_{\text{Lag}^o} \times pr^*_{23}H^2_{\text{Lag}^o},
\]

is the diagonal map:

\[
\Delta_{M^0,L^0}(h_1, h_2, h_3) = (h_1, h_2, h_3).
\]

The function \( \Delta^*(pr^*_{12}F \cdot pr^*_{23}F) \), which lives on the set \( H^3_{\text{Lag}^o} \) is \( pr^*_{23}\tilde{C} \) invariant, so it descends to a function \( \Delta^*(pr^*_{12}F \cdot pr^*_{23}F) \) on \( pr^*_{23}\tilde{C} \backslash H^3_{\text{Lag}^o}. \)

We use the notation \( pr_{13} \) for the operation of taking the sum of values over the fibers of the map \( pr_{13} \) (fiberwise integration). The choice of this notation will become clear when we translate to the geometric setting.

5. **Sign Rule.** \( h^*F = \chi \otimes pr^*F, \) where \( pr = pr^2_{H^0_{\text{Lag}^o}}. \)
4.2 Geometric canonical intertwining operators

In the sequel, we are going to translate back and forth between algebraic varieties defined over the finite field $\mathbb{F}_q$, and their corresponding sets of rational points. In order to prevent confusion between the two, we use bold-face letters for denoting a variety $X$, and normal letters for denoting its corresponding set of rational points $X = X(\mathbb{F}_q)$. We recall that an algebraic variety $X$, namely a variety in the usual sense - over the algebraically closed field $\mathbb{F}_q$, is said to be defined over $\mathbb{F}_q$ if it is equipped with the Frobenius endomorphism $\text{Fr} : X \to X$. The set $X = X(\mathbb{F}_q)$ is the set of points, fixed by the Frobenius, $X(\mathbb{F}_q) = \{x \in X : \text{Fr}(x) = x\}$. The Frobenius structure is also called rational structure. We denote by $D^w(X) = D^{w,b}(X)$ the bounded derived category of $\ell$-adic Weil sheaves. We choose once an identification $\mathbb{Q}_\ell \simeq \mathbb{C}$, so all sheaves are considered over the complex numbers. Given an object $G \in D^w(X)$, one can associate to $G$ a function $f^G : X \to \mathbb{C}$, as follows:

$$f^G(x) := \sum_i (-1)^i \text{Tr}(\text{Fr}|_{H^i(G_x)}).$$

This procedure is called Grothendieck’s sheaf-to-function correspondence [Gr]. We also use the notation $\chi_{\psi}(G) := f^G$, and call it the Euler characteristic of the sheaf $G$. Hence, we can start geometrizing our constructions.

The symplectic space $(V, \omega)$ can be naturally identified as the set $V = V(\mathbb{F}_q)$, where $V$ is an algebraic variety, defined over $\mathbb{F}_q$, $V \simeq \mathbb{A}^{2n}$, equipped with a skew symmetric form $\omega : V \times V \to \mathbb{A}^1$, respecting the rational structure of both sides. We have $\omega = \omega|_V$. The Heisenberg group $H$, can be naturally identified as the set $H = H(\mathbb{F}_q)$, where $H = V \times \mathbb{A}^1$ with the same multiplication formulas. We have the center $Z = Z(H) = \{(0, z) : z \in \mathbb{A}^1\}$. We have the Artin-Schreier sheaf $\mathcal{L}_\psi$ associated with the central character $\psi : Z \to \mathbb{C}^*$, that is, $f^{\mathcal{L}_\psi} = \psi$. The group $G = \text{Sp}(V, \omega)$ is identified as $G = G(\mathbb{F}_q)$, where $G = \text{Sp}(V, \omega)$. Next, we replace the sets $\text{Lag}^o, C$ and $\bar{C}$ by the corresponding algebraic varieties $\text{Lag}^o, C$ and $\bar{C}$. We have the sheaf $\mathcal{L}_{\psi, \bar{C}} = p^* \mathcal{L}_\psi$, where $p : \bar{C} \to Z$ is the total projection on the center. The sheaf $\mathcal{L}_{\psi, \bar{C}}$ is associated via sheaf-to-function correspondence to the function $\psi_{\bar{C}}$, $f^{\mathcal{L}_{\psi, \bar{C}}} = \psi_{\bar{C}}$. Finally, we can define the actions $a, b, m$ and $h$ between the corresponding algebraic varieties. All actions respect rational structure, and reduce to give the old formulas between sets of rational points. We have the following fundamental theorem:

**Theorem 4.1 (Geometric canonical intertwining operators)** There exists a unique (up to unique isomorphism) (shifted) Weil perverse sheaf $\mathcal{F}$ on the variety $H^2_{\text{Lag}^o}$, equipped with the following data:

1. $a^* \mathcal{F} \simeq \mathcal{L}_{\psi, \bar{C}} \boxtimes \mathcal{L}_{\psi^1, \bar{C}} \otimes pr^* \mathcal{F}$, where $pr = pr_{H^2_{\text{Lag}^o}}$. 

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b^*F \simeq pr^*F, \text{ where } pr = pr_{H^2_{Lag^0}}.

2. **Normalization.** \(\Delta^*F \simeq (\mathbb{C})_{Lag^0}, \text{ where } \Delta : Lag^0 \to H^2_{Lag^0}, \Delta(L^0) = (L^0, L^0, 0, 0), \text{ and } 0 \text{ denote the unit element in } H.

3. **Equivariance.** \(m^*F \simeq pr^*F, \text{ where } pr = pr_{H^2_{Lag^0}}.

4. **Convolution.** \(pr_{12}^*F \ast pr_{23}^*F \simeq pr_{13}^*F, \text{ where }

\quad pr_{12}^*F \ast pr_{23}^*F = pr_{13}((pr_{12}^*F \otimes pr_{23}^*F)).

We use the same terminology as in (4.1.3), noting that the sheaf \(\Delta^*(pr_{12}^*F \otimes pr_{23}^*F)\) is \(pr_{2}^*\widetilde{C}\) equivariant so it descent to a sheaf \(\Delta^*(pr_{12}^*F \otimes pr_{23}^*F)\) on \(pr_{2}^*\widetilde{C}\mid H^2_{Lag^0}.

5. **Sign rule.** \(h^*F \simeq L_{\chi_q} \boxtimes pr^*F, \text{ where } pr = pr_{H^2_{Lag^0}}, \text{ and } L_{\chi_q} \text{ denotes the Kummer sheaf on } \mathbb{G}_m \text{ associated to the quadratic character } \chi_q.

The isomorphisms 1-5 satisfy some obvious compatibility conditions which we will omit here.

### 4.3 Geometric models

We fix an element \(L^0 \in Lag^0\) of the form \(L^0 = (L, \sigma_L)\). We denote by \(D(H) := Db(H)\) the bounded derived category of \(\ell\)-adic sheaves on \(H\).

Recall the notation \(\widetilde{L} = pr^{-1}(L)\), where \(pr : H \to V\) is the standard projection. Consider the sheaf \(\mathcal{L}_{\psi,L^0} := p^*\mathcal{L}_{\psi}, \text{ where } p : \widetilde{L} = L \times Z \to Z\).

We denote by \(D_{L^0} = Db_{L^0,\psi}(H)\) the bounded derived category of \(\mathcal{L}_{\psi,L^0}\)-equivariant sheaves. In more detail, the subgroup \(\widetilde{L} \subset H\) acts on the group \(H\) via left multiplication. We denote this action by \(a_{L^0} : \widetilde{L} \times H \to H\) and we define:

**Definition 4.2** An \(\mathcal{L}_{\psi,L^0}\)-equivariant structure on a sheaf \(G \in D_{L^0}\) is given by an isomorphism: \(\alpha : a_{L^0}'.G \simeq G\), satisfying a cocycle condition.

**Comment.** Note that the subgroup \(\widetilde{L}\) acts freely on \(H\). Hence, in the definition of \(D_{L^0}\) we do not have to use the elaborate notion of a ”real” equivariant derived category [BL].

The triangulated category \(D_{L^0}\) admits a perverse t-structure, and we denote its heart by \(\text{Perv}_{L^0} := \text{Perv}_{L^0,\psi}(H)\) - the abelian category of \(\mathcal{L}_{\psi,L^0}\)-equivariant perverse sheaves.
4.3.1 Intertwining functors

Given a pair \((M^\circ, L^\circ) \in \text{Lag}^\circ \times \text{Lag}^\circ\), we consider the sheaf \(\mathcal{F}_{M^\circ, L^\circ}\) - the fiber of the sheaf \(\mathcal{F}\) at the point \((M^\circ, L^\circ)\). Using the sheaf \(\mathcal{F}_{M^\circ, L^\circ}\) we construct a functor that for simplicity we will denote by the same notation:

\[
\mathcal{F}_{M^\circ, L^\circ} : \mathcal{D}_{L^\circ} \to \mathcal{D}_{M^\circ},
\]

given by convolution:

\[
\mathcal{G} \mapsto \mathcal{F}_{M^\circ, L^\circ} \ast \mathcal{G},
\]

where \(\mathcal{G} \in \mathcal{D}_{L^\circ}\). It is an important fact, which follows from a deep theorem of Katz-Laumon on the \(\ell\)-adic Fourier transform [KL], that the functor \(\mathcal{F}_{M^\circ, L^\circ}\) respects the perverse t-structure on both sides, \(\mathcal{F}_{M^\circ, L^\circ}(\text{Perv}_{L^\circ}) \subset \text{Perv}_{M^\circ}\).

4.3.2 Group actions

We have actions of the groups \(H\) and \(G\) on the category \(\mathcal{D}_{L^\circ}\). These constitute a categorification of the Heisenberg representation \(\pi\), and the Weil representation \(\rho\) correspondingly.

**Heisenberg action.** The Heisenberg group \(H\) acts on itself by right multiplication. We denote this action by \(R : H \times H \to H\). This induces an action of \(H\) on the category \(\mathcal{D}_{L^\circ}\). Namely, associated to every element \(h \in H\) we have a functor \(\mathcal{K}_\pi(h) = \mathcal{K}_{\pi, L^\circ}(h) : \mathcal{D}_{L^\circ} \to \mathcal{D}_{L^\circ}\) given by:

\[
\mathcal{K}_\pi(h) := R_h^*,
\]

the pull-back by the map \(R_h : H \to H\).

**Claim 4.3** We have a canonical isomorphism of functors:

\[
\mathcal{K}_\pi(h_1) \circ \mathcal{K}_\pi(h_2) \simeq \mathcal{K}_\pi(h_1 \cdot h_2),
\]

for every \(h_1, h_2 \in H\).

**Symplectic action.** The symplectic group \(G\) acts on \(H\) via its tautological action on the vector space \(V\). This induces an action of \(G\) on the category \(\mathcal{D}(H)\). In more detail, associated to every element \(g \in G\) we have a functor \(\tilde{\mathcal{K}}_\rho(g) : \mathcal{D}(H) \to \mathcal{D}(H)\), given by:

\[
\tilde{\mathcal{K}}_\rho(g)(\mathcal{G}) = (g^{-1})^* \mathcal{G}.
\]

We have a canonical isomorphism of functors:

\[
\tilde{\mathcal{K}}_\rho(g_1) \circ \tilde{\mathcal{K}}_\rho(g_2) \simeq \tilde{\mathcal{K}}_\rho(g_1 \cdot g_2) 
\quad (4.3.1)
\]
for every $g_1, g_2 \in G$.

However, the functor $\tilde{\mathcal{K}}_\rho(g)$ does not preserve the category $\mathcal{D}_{\mathcal{L}^\circ}$. In fact, we have $\tilde{\mathcal{K}}_\rho(g) : \mathcal{D}_{\mathcal{L}^\circ} \to \mathcal{D}_{\mathcal{gL}^\circ}$. In order to arrive back to the category $\mathcal{D}_{\mathcal{L}^\circ}$, we apply next the intertwining functor $\mathcal{F}_{\mathcal{L}^\circ, \mathcal{gL}^\circ}$, so we define $\mathcal{K}_\rho(g) = \mathcal{K}_{\rho, \mathcal{L}^\circ}(g) : \mathcal{D}_{\mathcal{L}^\circ} \to \mathcal{D}_{\mathcal{L}^\circ}$, by:

$$\mathcal{K}_\rho(g) := \mathcal{F}_{\mathcal{L}^\circ, \mathcal{gL}^\circ} \circ \tilde{\mathcal{K}}_\rho(g).$$

We have an isomorphism:

$$\mathcal{K}_\rho(g_1) \circ \mathcal{K}_\rho(g_2) \simeq \mathcal{K}_\rho(g_1 \cdot g_2),$$

for every $g_1, g_2 \in G$. The last isomorphism is a formal consequence of isomorphism (4.3.1), and the convolution property of the sheaf of intertwining kernels $\mathcal{F}$ (see Property 4, in Theorem 4.1).

### 4.4 Deligne’s Weil representation sheaf

The correct point of view is to construct a single sheaf, $\mathcal{K}_\rho = \mathcal{K}_{\rho, \mathcal{L}^\circ}$, on the variety $G \times H \times H$. Consider the map:

$$m_{\mathcal{L}^\circ} : G \times H \times H \to H_{\text{Lag}}^2,$$

given by:

$$m_{\mathcal{L}^\circ}(g, h_1, h_2) = (h_1, g(h_2), \mathcal{L}^\circ, g\mathcal{L}^\circ).$$

We define:

$$\mathcal{K}_\rho := m_{\mathcal{L}^\circ}^* \mathcal{F},$$

where $\mathcal{F}$ is the sheaf of intertwining kernels (cf. Theorem 4.1).

Now we have the following actions:

$$a : (\mathcal{L}_G \times_G \mathcal{L}_G) \times_G H_G^2 \to H_G^2,$$

given (fiberwise) by:

$$a_g(\tilde{t}_1, \tilde{t}_2, h_1, h_2) := (\tilde{t}_1 \cdot h_1, \tilde{t}_2 \cdot h_2),$$

and:

$$b : H_G^2 \times_G H_G \to H_G^2,$$

given (fiberwise) by:

$$b_g(h_1, h_2, h) := (h_1 \cdot h, h_2 \cdot h).$$

Note that we use the same notations as in (4.1.1), and (4.1.2), because we deal here essentially with the same actions.
Claim 4.4 (Weil representation sheaf) The sheaf $\mathcal{K}_\rho$ satisfies the following properties:

1. $a^* \mathcal{K}_\rho \simeq (\mathcal{L}_\psi L_\mathbb{G} \boxtimes L_\psi^{-1} L_\mathbb{G}) \boxtimes \mathcal{K}_\rho$, where $\mathcal{L}_\psi L_\mathbb{G} = \tilde{L} \times G \to \tilde{L} \to \mathbb{Z}$.
   
   $b^* \mathcal{K}_\rho \simeq pr^* \mathcal{F}_\rho$, where $pr = pr_{H^2}$.

2. Convolution. $pr_1^* \mathcal{K}_\rho * pr_2^* \mathcal{K}_\rho \simeq m^* \mathcal{K}_\rho$. In more detail, $pr_1, pr_2 : G \times G \to G$ are projections on the left and right coordinate correspondingly, $m : G \times G \to G$ is the multiplication map. Considering the following sequence of maps:

   $$pr_1^* H^2_G \times_G \mathbb{H} \times_G pr_2^* H^2_G \xrightarrow{\Delta} H^3_G \to \tilde{L}_G \backslash H^2_G \xrightarrow{pr_{13}} m^* H^2_G$$

   We define $pr_1^* \mathcal{K}_\rho * pr_2^* \mathcal{K}_\rho = pr_{13}(\Delta^*(pr_1^* \mathcal{K}_\rho * pr_2^* \mathcal{F}_\rho))$.

Comment. The sheaf $\mathcal{K}_\rho$ should be considered as a sheaf of kernels, which constitutes a geometric analogue of the Weil representation $\rho$, realized in a model associated with the enhanced Lagrangian $L\psi$. Indeed, in the case where $L\psi$ is rational, namely $Fr \circ L\psi = L\psi$, the sheaf $\mathcal{K}_\rho$ admits a Frobenius (Weil) structure, and taking the corresponding function $f^\mathcal{K}_\rho$ gives the kernels of the representation $(\rho_{L\psi}, G, H_{L\psi}, L\psi = L^o(\mathbb{F}_q))$. We would like to complement that $\mathcal{K}_\rho$ is the sheaf appearing in Deligne’s letter to Kazhdan from 1982 [Di1, Ci1].

4.5 Twisted Weil structure

Consider the Frobenius endomorphism $Fr = Fr_H : H \to H$. Assume we are given an object $\mathcal{G} \in \mathcal{D}_{L^o, \psi}$. Then we have:

$$Fr^* \mathcal{G} \in \mathcal{D}_{Fr^{-1}L^o, \psi},$$

where $Fr^{-1}L^o = (Fr^{-1}L, Fr^{-1}\sigma_L)$. We use here the fact that $\mathcal{L}_\psi$ is a Weil sheaf, $Fr^* \mathcal{L}_\psi \simeq \mathcal{L}_\psi$.

Now, in order to arrive back to the category $\mathcal{D}_{L^o, \psi}$, we apply next the intertwining functor to obtain:

$$\mathcal{F}_{L^o, Fr^{-1}L^o}(Fr^* \mathcal{G}) \in \mathcal{D}_{L^o, \psi}.$$  

In fact, this procedure defines a functor

$$Fr_{L^o} := \mathcal{F}_{L^o, Fr^{-1}L^o} \circ Fr^* : \mathcal{D}_{L^o, \psi} \to \mathcal{D}_{L^o, \psi}.$$  

Moreover, we have $Fr_{L^o}(Perv_{L^o, \psi}) \subset Perv_{L^o, \psi}$. The functor $Fr_{L^o}$ is thought of as a generalized Frobenius.
We define the triangulated category $\mathcal{D}^w_{L^\circ,\psi} = \mathcal{D}^w(H, L^\circ, \psi)$ of $L^\circ$-Weil sheaves. An object in $\mathcal{D}^w_{L^\circ,\psi}$ is a pair $(F, \alpha)$, where $G \in \mathcal{D}^w_{L^\circ,\psi}$, and $\alpha$ is an isomorphism $\alpha: \text{Fr}^*L^\circ G \simeq G$. The category $\mathcal{D}^w_{L^\circ,\psi}$ inherits a perverse t-structure. We denote by $\text{Perv}^w_{L^\circ,\psi} = \text{Perv}^w(H, L^\circ, \psi)$ the abelian heart of perverse $L^\circ$-Weil sheaves.

Given a pair $(M^\circ, L^\circ)$, the intertwining functor $\mathcal{F}_{M^\circ,L^\circ}$ respects the Weil structures on both sides, therefore it gives a functor between corresponding Weil categories: $\mathcal{D}^w_{L^\circ}$ and $\mathcal{D}^w_{M^\circ}$. We denote this functor also by $\mathcal{F}_{M^\circ,L^\circ} : \mathcal{D}^w_{L^\circ} \to \mathcal{D}^w_{M^\circ}$.

We have $\mathcal{F}_{M^\circ,L^\circ}(\text{Perv}^w_{L^\circ}) \subset \text{Perv}^w_{M^\circ}$.

**Comment.** In the case where $L^\circ$ is rational, i.e., $\text{Fr}L^\circ = L^\circ$ we have $\mathcal{F}_{L^\circ,\text{Fr}^{-1}L^\circ} = \text{Id}$. Hence, the category $\mathcal{D}^w_{L^\circ}$ becomes only the category of traditional Weil sheaves.

The action functors $\mathcal{K}_\pi$ and $\mathcal{K}_\rho$ restrict to give actions of the finite groups $H$ and $G$ on $\mathcal{D}^w_{L^\circ}$. The intertwining functor $\mathcal{F}_{M^\circ,L^\circ}$ commutes with these actions.

## 4.6 From a category to an Hilbert space

We denote by $K_{L^\circ}$ and $K_{M^\circ}^w$ the Grothendieck $K$ groups of the categories $\mathcal{D}_{L^\circ}$ and $\mathcal{D}_{M^\circ}^w$ correspondingly. These are infinite dimensional (complex) vector spaces (at least after tensoring with $\mathbb{C}$). In particular, the actions $\mathcal{K}_\pi$ and $\mathcal{F}_\rho$ of the groups $H$ and $G$ factor through, and give the vector space $K_{L^\circ}^w$ a structure of an infinite dimensional representation. The same argument works for the intertwining functor $\mathcal{F}_{M^\circ,L^\circ}$, giving an intertwining homomorphism $\mathcal{F}_{M^\circ,L^\circ} : K_{L^\circ}^w \to K_{M^\circ}^w$, for every pair $(L^\circ, M^\circ)$.

Next we are going to define a canonical finite dimensional quotient. Considering $G \in \mathcal{D}^w_{L^\circ,\psi}$, we have $\mathbb{D}(G) \in \mathcal{D}_{L^\circ,\psi^{-1}}$, where we denote by $\mathbb{D}$ the functor of Vardier duality.

**Matrix coefficient.** Let $pr_1, pr_2 : H \times H \to H$, be the projections on the left and right coordinates correspondingly. Let $R : H \times H \to H$ denote the right action of $H$ on itself. Given a pair $\mathcal{E}, G \in \mathcal{D}_{L^\circ}^w$. Considering the product $pr_1^*\mathbb{D}(\mathcal{E}) \otimes r^*G$, it is an equivariant sheaf with respect to the diagonal action of $L$ on $H \times H$, therefore it descends to a sheaf $pr_1^*\mathbb{D}(\mathcal{E}) \otimes r^*G$ on the quotient $L \setminus (H \times H)$. Define the sheaf:

$$m(\mathcal{E}, G) := pr_2^*(pr_1^*\mathbb{D}(\mathcal{E}) \otimes r^*G)$$

(4.6.1)

The sheaf $m(\mathcal{E}, G)$ is called the *matrix coefficient* of $\mathcal{E}$ and $G$. We have the following fundamental statement:
Proposition 4.5 The sheaf $m(\mathcal{E}, \mathcal{G})$ admits a Frobenius structure, namely $m(\mathcal{E}, \mathcal{G}) \in D^w(H)$.

Using the previous result, we next apply sheaf-to-function correspondence obtaining a function $f^m(\mathcal{E}, \mathcal{G}) : H \to \mathbb{C}$. To a pair $\mathcal{E}, \mathcal{G} \in D^w_L$, we can associate a function $f^m(\mathcal{E}, \mathcal{G})$ on $H$.

This procedure clearly factorizes to the level of $K$ groups, giving a $(H, G)$ invariant pairing:

$$\langle \cdot, \cdot \rangle_L : K^w_L \times K^w_L \to L^2(H, \mathbb{C}).$$

Denoting by $N^w_L \subset K^w_L$ the radical of $\langle \cdot, \cdot \rangle_L$, we define $H_L^\circ = K^w_L/N^w_L$.

Following the work [BP], we will call the quotient map:

$$K^w_L \twoheadrightarrow H_L^\circ, \quad (4.6.2)$$

which assigns to a sheaf $\mathcal{G}$ the vector $[\mathcal{G}] \in H_L^\circ$, the Braverman-Polishchuk’s Correspondence$^4$.

This correspondence is compatible with the actions of the groups $H$ and $G$, hence the vector space $H_L^\circ$ inherits the actions of $H$ and $G$ also. We denote these representations by $(\pi_L^\circ, H, H_L^\circ)$, and $(\rho_L^\circ, G, H_L^\circ)$. We have the following theorem:

**Theorem 4.6** The vector space $H_L^\circ$ is finite dimensional. Moreover,

1. The representation $(\pi_L^\circ, H, H_L^\circ)$ is isomorphic to the Heisenberg representation.

2. The representation $(\rho_L^\circ, G, H_L^\circ)$ is isomorphic to the Weil representation.

Finally, we have that $F_{M^\circ, L^\circ}$ sends $N^w_L \to N^w_M$, giving an intertwining operator between the models $H_L^\circ$ and $H_M^\circ$.

**Summary.** To every enhanced Lagrangian $L^\circ$, not necessarily rational, we associate a model $H_L^\circ$ of the Heisenberg-Weil representations. Underlying every such model lies a category $D^w_L$ of (twisted) Weil sheaves.

**Comment.** In the case where $L^\circ$ is rational, i.e., $Fr L^\circ = L^\circ$, we get $H_L^\circ = H_L^\circ$, where $L^\circ = L^\circ(\mathbb{F}_q)$, and $H_L^\circ$ is the vector space constructed in Section 3.1.2.

---

$^4$This correspondence becomes the usual Grothendieck’s sheaf-to-function correspondence in the case where the enhanced Lagrangian $L^\circ$ is rational.
5 Hecke Eigenvectors

In the remainder of this section we consider the case \( n = 2 \). Let \((V, \omega)\) be a two-dimensional symplectic vector space over the finite field \( \mathbb{F}_q \). We consider a maximal torus \( T \subset G := \text{Sp}(V, \omega) \), which we will call the Hecke torus, and a multiplicative character \( \chi : T \rightarrow \mathbb{C}^* \). Our goal is to construct effectively a \( \chi \)-Hecke-eigenvector \( v_\chi \in H_V \). We denote by \( L_\chi \) the Kummer sheaf on the variety \( T \) associated to the character \( \chi \).

5.1 Construction

The idea behind the construction is to use a ”good realization” \( \mathcal{H}_{L^0} \) in which the Hecke torus acts in a geometric fashion and to construct first an Hecke eigensheaf. Then, the second step is to apply the Braverman-Polishchuk correspondence (4.6.2) to obtain the desired eigenvector.

5.1.1 Good realization

There exists an enhanced Lagrangian \( L^0 = (L, \sigma_L) \) such that \( L \) is fixed by \( T \), that is, \( gL = L \) for every \( g \in T \). Applying the Frobenius we obtain another enhanced Lagrangian \( L'' = (L', \sigma_{L'}) := (\text{Fr}L, \text{Fr} \sigma_L) \). We can choose \( L'' \) such that \( \omega(\sigma_L, \sigma_{L'}) = 1 \). We consider the decompositions: \( V = L' \oplus L \) and \( H = L' \times L \times \mathbb{Z} \). Note, that \( \mathbb{Z} = \mathbb{Z}(H) = \mathbb{A}_1 \).

We have \( X_{L^0} := \tilde{L}\backslash H \simeq L' \), therefore we can identify:

\[
\mathcal{D}_{L^0, \psi} \simeq \mathcal{D}(L'), \quad \text{Perv}_{L^0, \psi} \simeq \text{Perv}(L'). \tag{5.1.1}
\]

Next, we compute the twisted Frobenius \( \text{Fr}_{L^0} \) in the realization \( \mathcal{D}(L') \):

\[
\text{Fr}_{L^0}(\mathcal{G})(l') = \text{Fr}_{L^0}(\mathcal{G})(l', 0, 0) = F_{L^0, \text{Fr}^{-1}L^0} \circ \text{Fr}^* \mathcal{G}(l', 0, 0)
\]

\[
= \star \int_{l \in L} \text{Fr}^* \mathcal{G}((0, l, 0) \cdot (l', 0, 0))
\]

\[
= \star \int_{l \in L} \text{Fr}^* \mathcal{G}((l', l, \frac{1}{2} \omega(l.l')))
\]

\[
= \star \int_{l \in L} \mathcal{G}(\text{Fr}l', \text{Fr}l, \frac{1}{2} \text{Fr} \omega(l.l'))
\]

\[
\overset{(1)}{=} \star \int_{l \in L} \mathcal{L}_\psi(\frac{1}{2} \omega(l, l') + \frac{1}{2} \text{Fr} \omega(l, l')) \otimes \mathcal{G}(\text{Fr}l)
\]

\[
\overset{(2)}{=} \int_{l \in L} \mathcal{L}_\psi(\omega(l, l')) \otimes \mathcal{G}(\text{Fr}l)
\]
In (1) we use the identity: 
\[(\text{Fr}_l, \text{Fr}_{l'}, \frac{1}{2} \omega(l, l')) = (0, \text{Fr}_{l'}, \frac{1}{2} \omega(l, l') - \frac{1}{2} \text{Fr} \omega(l, l')) \cdot (\text{Fr}_l, 0, 0)\]
and in (2) we use the fact that \(L_\psi\) is a Frobenius sheaf, i.e., \(\text{Fr}^* L_\psi \simeq L_\psi\). The symbol \(*\) stands for a normalization coefficient which appears in the canonical intertwining operator. This coefficient, although it has a deep meaning, does not play any role in our arguments, therefore we disregard its explicit formula.

Identifying further: \(L \simeq \mathbb{A}^1\), via \(x \mapsto \sigma^{-1}_L x\), and \(L' \simeq \mathbb{A}^1\), via \(x \mapsto \sigma^{-1}_{L'} x\), we get \(V \simeq \mathbb{A}^1 \times \mathbb{A}^1\). In these coordinates the Frobenius morphism is given by: \(\text{Fr}(x, y) = (y^p, -x^p)\). Now we can write \(\text{Fr}_L\) in coordinates:

\[
\text{Fr}_L(G)(x) = \int_{y \in \mathbb{A}^1} L_\psi(xy) \otimes G(y^p).
\]

### 5.1.2 Hecke eigensheaf

We consider the sheaf of Deligne kernels \(K_\rho = K_{\rho, L}\) (Section 4.3). Restricting to the torus \(T\), and using (5.1.1) we may consider \(K_\rho\) as a sheaf on \(T \times L' \times L'\). It is a direct computation to obtain the explicit formula:

\[
K_\rho(g, x, y) = \mathcal{L}_{\chi_q}(g) \otimes \delta_{x-g^{-1}y},
\]

where \(\mathcal{L}_{\chi_q}\) is the Kummer sheaf associated to the quadratic character \(\chi_q: T \to \mathbb{C}^*\). The sheaf \(\delta_{x-g^{-1}y}\) is obtained as \(\delta_{x-g^{-1}y} = iL_{\chi_{L'}}\), where \(i: L' \to L' \times L'\), \(i(x, y) = (x, g^{-1}y)\).

Consider the imbedding \(j: T \hookrightarrow L', j(g) = g\sigma_{L'}\) and the sheaf:

\[
S_{\psi, \chi} := j!(\mathcal{L}_{\chi_q} \otimes \mathcal{L}_\chi).
\]

We have the following proposition:

**Proposition 5.1** The sheaf \(S_{\psi, \chi}\) is perverse, i.e., \(S_{\psi, \chi} \in \text{Perv}(L')\). Moreover,

1. It admits an \(L^\circ\)-Weil structure, i.e., \(\text{Fr}_{L^\circ}(S_{\psi, \chi}) \simeq S_{\psi, \chi}\).
2. It admits an \(\mathcal{L}_\chi\)-\(T\) equivariant structure, i.e., \(K_\rho * S_{\psi, \chi} \simeq \mathcal{L}_\chi \boxtimes S_{\psi, \chi}\).

### 5.1.3 Hecke eigenfunction

Apply the Braverman-Polishchuk correspondence (4.6.2) and obtain the function:

\[
\Psi_{\chi} := [S_{\psi, \chi}] \in \mathcal{H}_{L^\circ}.
\]

It is clear that this gives a \(\chi\)-\(T\)-eigenfunction.
6 Proof of the Supremum Conjecture

Let \( L^\circ = (L, \sigma_L) \) be a rational enhanced Lagrangian, i.e., \( \text{Fr } L^\circ = L^\circ \). Let \( L^\circ = L^\circ(\mathbb{F}_p) \). Consider the associated model \( \mathcal{H}_{L^\circ} \). We have \( \mathcal{H}_{L^\circ} = \mathcal{H}_{L^\circ} = \Gamma(X_{L^\circ}, \mathcal{H}_{L^\circ}) \), where \( \mathcal{H}_{L^\circ} \) is an Hermitian line bundle on \( X_{L^\circ} = \overline{L} \setminus H \). Let \( \Psi_{\chi}^{L^\circ} \in \mathcal{H}_{L^\circ} \) be a \( \chi \)-T-eigenvector. It will be convenient for us to take \( \Psi_{\chi}^{L^\circ} \) with the normalization \( \| \Psi_{\chi}^{L^\circ} \|^2 = q \).

Fix a point \( x \in X_{L^\circ} \). Denote by \( a_x \) the quantity:

\[
a_x := |\Psi_{\chi}^{L^\circ}(x)|^2.
\]

We would like to study the quantity \( a_x \). In particular, we will prove that independent of \( p, L^\circ \) and \( \chi \) we have:

**Proposition 6.1 (Supremum conjecture - restated)** The following bound holds:

\[
a_x \leq 4,
\]

for every \( x \in X_{L^\circ} \).

First we give the quantity \( a_x \) a representation theoretic interpretation. Consider the abelian subgroup \( L \subset H \). It acts semisimply on the space \( \mathcal{H}_{L^\circ} \), therefore we obtain a decomposition:

\[
\mathcal{H}_{L^\circ} = \bigoplus_{\psi: L \to \mathbb{C}^*} \mathcal{H}_{\psi},
\]

where \( \psi \) runs over the set of characters of the group \( L \). The point \( x \) corresponds to a specific character \( \psi_x : L \to \mathbb{C}^* \). Denote by \( P_x = P_{x,\psi}^{L^\circ} = \frac{1}{|L|} \sum_{l \in L} \psi_x(l) \pi_{L^\circ}(l) \) the orthogonal projector on the space \( \mathcal{H}_{\psi_x} \). We have:

\[
a_x = \langle P_x \Psi_{\chi}^{L^\circ}, \Psi_{\chi}^{L^\circ} \rangle.
\]

The main observation is, that the scalar \( a_x \), which is defined completely in representation theoretic terms, does not depend on the specific model \( \mathcal{H}_{L^\circ} \). We are free to use a different model for the computation. We choose a model in which \( \Psi_{\chi} \) has a convenient form. In more detail, let \( M^\circ = (M, \sigma_M) \) be such that \( M \) is fixed by \( T \). Let \( \Psi_{\chi}^{M^\circ} = \mathcal{F}_{M^\circ}(\Psi_{\chi}^{L^\circ}) \). Then \( \Psi_{\chi}^{M^\circ} \in \mathcal{H}_{M^\circ} \) and clearly \( \Psi_{\chi}^{M^\circ} \) is a \( \chi \)-T-eigenvector. We have:

\[
a_x = \langle P_x \Psi_{\chi}^{M^\circ}, \Psi_{\chi}^{M^\circ} \rangle, \tag{6.0.2}
\]

where now \( P_x = P_{x,\psi}^{M^\circ} = \frac{1}{|L|} \sum_{l \in L} \psi_x(l) \pi_{M^\circ}(l) \).
Developing formula (6.0.2) we obtain:
\[
a_x = \frac{1}{|L|} \sum_{l \in L} \psi_x(l) \langle \pi_{M^0} (l) \Psi^M, \Psi^M \rangle.
\]

Next we study the function: \( \langle \pi_{M^0} (\cdot) \Psi^M, \Psi^M \rangle : H \to \mathbb{C} \). This is only a matrix coefficient. We will analyze it using geometry.

Let \( S_{\psi_x} \in D^{M^0} \) be an \( L \)-eigensheaf (see Proposition 5.1). We can assume that \( \Psi^M = [S_{\psi_x}] \) (this is the reason why we choose the specific normalization for \( \Psi^M \)). We have \( \langle \pi_{M^0} (\cdot) \Psi^M, \Psi^M \rangle = f^{m(S_{\psi_x}, S_{\psi_x})} \). It is enough to study the Weil sheaf \( m(S_{\psi_x}, S_{\psi_x}) \in D_w(H) \). We can in fact compute \( m(S_{\psi_x}, S_{\psi_x}) \) explicitly: Let \( M' = (M', \sigma_M) = (F_{M'}, F_{\sigma_M}) \). We can assume that \( \omega(\sigma_M, \sigma_M) = 1 \). We have the decompositions: \( V = M' \times M \), and \( H = M' \times M \times Z \). Recalling the definition of a matrix coefficient (4.6.1), we obtain:
\[
m(S_{\psi_x}, S_{\psi_x})(m', m, z) = \int_{t \in M'} \mathcal{L}_\psi(z + \frac{1}{2} \omega(m', m) + \omega(t, m)) \otimes S_{\psi_x}^{M^0}(t + m') \otimes D(S_{\psi_x}^M)(t).
\] (6.0.3)

As was previously done, we can identify \( D_{M^0, \psi} = D(M') \) as plain triangulated categories, but Weil structure is realized by the functor \( Fr_{M^0} : D(M') \to D(M) \). Taking coordinates \( M' \approx \mathbb{A}^1 \), \( x \mapsto x \cdot \sigma_M \), and \( M \approx \mathbb{A}^1 \), \( x \mapsto x \sigma_M \), we can further identify \( D(M') = D(A^1) \). In these coordinates we have \( S_{\psi_x}^M = L \). Writing formula (6.0.3) translates to:
\[
m(S_{\psi_x}, S_{\psi_x})(x', x, z) = \int_{t \in \mathbb{A}^1} \mathcal{L}_\psi(z - \frac{1}{2} x'x + xt) \otimes \mathcal{L}_\chi(t + x) \otimes \mathcal{L}_\chi^{-1}(x').
\]

We have the following proposition:

**Proposition 6.2** The sheaf \( m(S_{\psi_x}, S_{\psi_x}) \in \text{Perv}^{w, 1}(H) \), irreducible of pure weight 0, and it is equivariant with respect to the action of \( T_A \). Moreover, \( m(S_{\psi_x}, S_{\psi_x}) \) is smooth on the open set \( U = H \setminus (M, 0) \cup (0, M) \).

**Proof of proposition 6.1** We have:
\[
\sum_{l \in L} \psi_x(l) \langle \pi_{M^0} (l) \Psi^M, \Psi^M \rangle = g + \sum_{l \in L^x} \psi_x(l) \langle \pi_{M^0} (l) \Psi^M, \Psi^M \rangle.
\]

It is enough to estimate \( \sum_{l \in L^x} \psi_x(l) \langle \pi_{M^0} (l) \Psi^M, \Psi^M \rangle \). Consider the function:
\[
g = \psi_x(\cdot) \langle \pi_{M^0} (\cdot) \Psi^M, \Psi^M \rangle : L \to \mathbb{C}.
\]
This function is obtained as \( g = f^\mathcal{G} \), where

\[
\mathcal{G} := \mathcal{L}_\psi \otimes m(S_{\psi,\chi}^{M^e}, S_{\psi,\chi}^{M^e})_{|L^\times}.
\]

From proposition 6.2 we deduce that \( m(S_{\psi,\chi}^{M^e}, S_{\psi,\chi}^{M^e})_{|L^\times} \) is smooth, therefore \( \mathcal{G} \) is non-trivial and smooth. This implies that the sheaf \( \pi_! \mathcal{G} \) on the point, i.e., \( \pi_! \mathcal{G} \in \mathcal{D}(pt) \), where \( \pi \) denotes the projection \( \pi : L^\times \to pt \), is supported at degree 2, namely:

\[
H^i(\pi_! \mathcal{G}) = 0, \quad \text{for every } i \neq 2.
\]

Moreover, \( \pi_! \mathcal{G} \) is of weight \( \leq 0 \). This means that the eigenvalues of Frobenius are bounded by:

\[
|e.v.(Fr|H^2(\pi_! \mathcal{G}))| \leq q.
\]

Now, denoting by \( d \) the dimension \( d := \dim H^2(\pi_! \mathcal{G}) \), we get:

\[
\sum_{l \in L^\times} \psi_x(l) \langle \pi_{M^e}(l) \Psi_{\chi}^{M^e}, \Psi_{\chi}^{M^e} \rangle = f^\pi \mathcal{G} \leq dq.
\]

Hence,

\[
a_x \leq d + 1.
\]

Therefore, by a direct computation, using the explicit formulas of the sheaf \( \mathcal{G} \), we show that \( d = 3 \). This completes the proof. \( \blacksquare \)

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