1 Abstract

We present a parallel optimization algorithm for a convex function $f$ on a Hilbert Space, $\mathbb{H}$, under $r \in \mathbb{N}$ linear inequality constraints by finding optimal points over sets of equality constraints. Given enough threads, and strict convexity, the complexity is $O(r\langle\cdot,\cdot\rangle_{\min\mathbb{H}} f)$. The method works on constrained spaces with empty interiors, furthermore no feasible point is required, and the algorithm recognizes when the feasible space is empty.

Key Words

Convex Optimization, Hilbert Space, Strict Convexity, Linear Complexity, Parallel Optimization

2 Introduction

For years, the field of constrained convex optimization has been dominated by interior point methods [11, 14]. These methods inherently suffer from a number of weaknesses: Without additional criteria, they can not be implemented in parallel, they often require a feasible point to start, and even when they do not, they require the feasible space be nonempty [7], and they typically terminate when they are within an $\epsilon$ distance of the true optimal point and their complexity is a function of their accuracy.

*The authors are with the Electrical and Computer Engineering Department, Stevens Institute of Technology, Hoboken, NJ 07030 USA, e-mails: {eneimand, ssabau}@stevens.edu.
The algorithm developed here, for convex functions over linear inequality constrained spaces, offers a simple solution to these challenges. It utilizes an arbitrary optimization algorithm for unconstrained spaces, \( \arg \min g f \) and preserves many of its properties. When \( \arg \min g f \) finds an exact answer, without the need for an epsilon approximation or non asymptotic finite convergence, so too does the method presented here. This is because the method does not utilize an iterative minimization sequence. Such an algorithm, given some \( x_i \in \mathbb{H} \), generates an \( x_{i+1} \) with \( f(x_i) \rightarrow \min f \). By evading that approach, we circumvent one of the chief difficulties with work in Hilbert spaces: closed and bounded does not mean compact.

Text books and papers on optimization in \( \mathbb{R}^n \) and Hilbert spaces are now ubiquitous, [2], [1], [4] to name a few. We have unconstrained methods for finding optimal points in Hilbert spaces, recent examples of minimization sequence methods include [5], and [8].

In finite dimensions, we have faster and more exact linear inequality convex optimization methods [10]. Their method requires the feasible space be non empty. With regards to complexity, [9] found that they could get it as low as

The closest progenitor of the method presented here can be found at [13]. In this projection method, the optimal point is found over every subset inequality-turned-equality constraints, each of those affine-subspace optimal points is then checked for membership in the polyhedron, and of those that are members, the closest one to the point being projected, is also the optimal point over the entire polyhedron. The method presented here, besides broadening from projections to more general convex functions, uses necessary and sufficient criteria to significantly reduce the number of affine spaces that need to be searched for an optimal point, and once an affine space containing the polyhedral optimal point is found, there’s no need to compare it to optimal points found over other affine spaces.

The distributed complexity of the method presented here, along with the lack of common assumptions like non empty feasible spaces, and the simplicity of the method, will likely lead to the common usage of this algorithm on systems capable of large scale multi threading.

When \( r \gg n \) the complexity of our method is \( O(r^{n+1}\langle \cdot, \cdot \rangle \min_A f) \) for some affine space \( A \). When \( n \gg r \) we achieve a complexity of \( O(2^{r+1}\langle \cdot, \cdot \rangle \min_A f) \). Both of these complexity results are weaker than the polynomial time of interior point methods [11], however when a large number of threads are available to process the problem in parallel, the time complexity of the algorithm becomes \( O(\min\{r,n\}\langle \cdot, \cdot \rangle \min_A f) \).

In section 3 we introduce some important definitions and use them to achieve some preliminary results. In section 4 we state and prove the propositions that build the algorithm. In section 5 we present the algorithm and discuss the value of strictly convex functions. And in section 6 we analyze the complexity of the algorithm.
3 Some Definitions and Preliminary Lemmas

We present a handful of polyhedral properties that enable my optimization algorithm.

Definition 1. For some polyhedron \( P \) we define \( \mathcal{H}_P \) to be a finite collection of \( r \in \mathbb{N} \) closed half-spaces in \( \mathbb{H} \), an \( n \in \mathbb{N} \cup \{\infty\} \)-dimensional Hilbert Space. \( \forall H \in \mathcal{H}_P \) we define the boundary hyperplane \( \partial H \), the vector \( n_H \in \mathbb{H} \) normal to \( \partial H \), and \( b_H \in \mathbb{R} \) such that \( H = \{ x \in \mathbb{H} \mid \langle n_H, x \rangle \leq b_H \} \). For any \( H \in \mathcal{H}_P \) we say that \( H \) is a half-space of \( P \) and \( \partial H \) a hyperplane of \( P \).

Definition 2. We define the convex polyhedron \( P := \bigcap_{H \in \mathcal{H}_P} H \). For finite dimensional \( \mathbb{H} = \mathbb{R}^n \), this gives us \( P = \{ x \in \mathbb{R}^n \mid N x \leq b \} \) where \( N \) is a \( r \times n \) matrix whose rows are the vectors \( \{n_H\}_{H \in \mathcal{H}_P} \) and \( b \in \mathbb{R}^r \) elements are \( \{b_H\}_{H \in \mathcal{H}_P} \) aligned with the corresponding \( n_H \) rows in \( N \).

Example 3. Examples of polyhedrons include \( \mathbb{H}, \emptyset, \{42\} \), a rectangle, and a set we’ll call the ‘A’ polyhedron. ‘A’ := \( \{(x, y) \in \mathbb{R}^2 \mid y \leq \frac{1}{2} \land x + y \leq 1 \land -x + y \leq 1 \} \).

We have \( \mathcal{H}_A := \{ \tilde{F}, \tilde{G}, \tilde{H} \} \) with \( \tilde{F} := \{(x, y) \in \mathbb{R}^2 \mid y \leq \frac{1}{2} \} \), \( \tilde{G} := \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1 \} \), and \( \tilde{H} := \{(x, y) \in \mathbb{R}^2 \mid -x + y \leq 1 \} \). We’ll come back to this in future examples.

For a convex function \( f \mid \mathbb{H} \to \mathbb{R} \), the optimization algorithm below determines if \( P \) is empty, or finds a point in the set \( \arg \min_P f \). We assume the underlying optimization method is able to find the entire optimal set, and given that set is able to find its subset in a given half-space.

Example 4. Given some \( y \in \mathbb{H} \), let \( f(x) = \|x - y\| \) we may solve the projection problem \( \Pi_P(y) := \arg \min_P f \). Here, the optimal set \( \arg \min_P f \) will always have a unique value.

Unconstrained convex functions can often be optimized quickly. Optimization over affine spaces and half-spaces is easily reduced to the unconstrained optimization problem \([2]\). Some functions, like projection functions \([10]\) can be optimized as quickly as \( O(n^3) \) over a finite dimensional affine space.

Given a polyhedron \( P \) and a convex function \( f \), the algorithm constructs in parallel a large set of affine spaces that might contain the optimal point of \( f \), then uses Necessary Criteria to quickly filter out many of those affine spaces. Those that pass the fast Necessary Criteria filter are then checked for the slightly more cumbersome sufficient criteria test. The successful affine space, \( A^* \), will have \( \min_{A^*} f = \min_P f \).

Definition 5. We define \( P \)'s affine spaces as \( A_P := \{ \cap_{H \in \mathcal{H}_P} \partial H \mid \eta \in 2^{\mathcal{H}_P} \} \setminus \emptyset \) be the set of all the affine spaces - intersections of the surfaces - of \( P \). Note that \( A_P \) has at most \( \sum_{i=1}^{n} \binom{n}{i} \leq \min(r^n, 2^r) \) elements since the intersection of more than \( n \) distinct hyperplanes will be an empty set, or redundant with an intersection of fewer hyperplanes. If \( A \in A_P \) we shall say that \( A \) is one of \( P \)'s affine spaces.
Example 6. If \( \mathcal{H}_P = \{ F, G, H \} \) then \( \mathcal{A}_P = \{ H, \partial H, \partial G, \partial F, \partial H \cap \partial G, \partial H \cap \partial F, \partial F \cap \partial G, \partial H \cap \partial G \cap \partial F \} \). If this \( P \) is in \( \mathbb{R}^3 \), \( \partial H \) might be a plane, \( \partial H \cap \partial G \) a line, and \( \partial H \cap \partial G \cap \partial F \) a single point. However, if any of those intersections are empty then they are not included in \( \mathcal{A}_P \).

Example 7. Consider the ‘A’ polyhedron from example 6. There are two things to notice. First is that the intersection of the 3 hyperplanes in \( \mathbb{R}^2 \) is empty, \( \partial H \cap \partial G \cap \partial F = \emptyset \in \mathcal{A}_P^x \). The second thing to notice is that \( P \) can have affine spaces, in this case the point \( \partial G \cap \partial H \), that are disjoint with \( P \). The affine space that is a point at the top of the ‘A’ is outside of our polyhedron.

The boundary, \( \partial P \), of a polyhedron \( P \) is the union of subsets of the affine spaces of \( P \). If we solve the unconstrained problem and find \( \text{arg min}_{\mathcal{H}} f \cap P \neq \emptyset \) then we have a solution for \( \text{arg min}_{\mathcal{P}} f \), otherwise, we will find the solution for \( \text{arg min}_{\mathcal{P}} f \) in \( \partial P \), lest \( f \) have distinct local and global minimums. Previous attempts to find an affine space \( A \in \mathcal{A}_P \) such that \( \text{min}_A f = \text{min}_P f \) may have faltered because \( \mathcal{A}_P \) is very big and, at first glance, it’s not immediately clear that such an \( A \) even exists; if one is found, how would it be recognized?

Even for fast affine optimizations like projection, the additional difficulty of computing the projection function, \( O(n^3) \) [10], gives us \( O(n^3 |\mathcal{A}_P|) \) to simply compute all the projections.

We present below novel necessary and sufficient conditions for the affine space \( A \) to have \( \text{min}_A f = \text{min}_P f \) and guarantee its existence. While The Sufficient Criteria requires the computation \( \text{min}_A f \), the necessary condition often does not. This reduces the amount of time it takes to find such an \( A \).

Definition 8. For \( a, b \in \mathbb{H} \) we define \( \overline{a, b} := \{(1 - t)a + tb \mid t \in [0, 1]\} \) as the line segment from \( a \) to \( b \) and \( \overline{a, b} := \{(1 - t)a + tb \mid t \in \mathbb{R}\} \) as the line containing \( a \) and \( b \).

We include lemmas 9 and 10 for the reader’s convenience.

Lemma 9. If \( a, b \in \mathbb{H} \) and \( H \) is half-space such that \( a \in H \) and \( b \in H^c \) then \( \partial H \cap \overline{a, b} \) has exactly one point.

Proof. Let \( g[0, 1] \xrightarrow{\text{cont.}} \mathbb{R} \) with \( g(t) = \langle n_H, (1 - t)a + tb \rangle \). Since \( a \in H \Rightarrow g(0) = \langle n_H, a \rangle < b_H \) and similarly \( g(1) = \langle n_H, b \rangle > b_H \). By the intermediate value theorem, \( \exists t_H \in (0, 1) \) such that \( g(t_H) = b_H \). This gives us \( (1 - t_H)a + t_H b \in \partial H \cap \overline{a, b} \). The intersection is nonempty.

Let’s falsely assume \( \exists x_1, x_2 \in \overline{a, b} \cap \partial H \) with \( x_1 \neq x_2 \). It is a property of affine spaces, including \( \partial H \), that for any two points in the space, the line containing those two points is also in the space. It follows that \( a, b \in \partial H \) in contradiction to \( b \in H^c \Rightarrow x_1 = x_2 \).

Lemma 10. Let \( a, b, \) and \( c \) be distinct points in \( \mathbb{H} \) with \( b \in \overline{a, c} \).
1. $\|a - b\| + \|b - c\| = \|a - c\|$
2. $\|a - b\| < \|a - c\|.$
3. If $f : \mathbb{H} \to \mathbb{R}$ is convex and $f(a) < f(c)$ then $f(b) < f(c).$
4. If $f : \mathbb{H} \to \mathbb{R}$ is convex and $f(a) \leq f(c)$ then $f(b) \leq f(c).$

Proof (10.1). Let $a, b, c$ be as described above. Then for some $t_b \in (0, 1)$ we have $b = (1 - t_b)a + t_b c.$

With substitution we then get $\|a - b\| = -t_b\|a + c\|$ and $\|b - c\| = (1 - t_b)\|a - c\|$

Consider the following:

$\|a + c\| + \|a - c\| \geq 0$ since it is the sum of positive numbers.

$-t_b\|a + c\| + -t_b\|a - c\| \leq 0$ since $-t_b < 0.$

$-t_b\|a + c\| + (1 - t_b)\|a - c\| \leq \|a - c\|$

$\|a - b\| + \|b - c\| \leq \|a - c\|.$

But by the triangle inequality $\|a - b\| + \|b - c\| \geq \|a - c\|.$

$\Rightarrow \|a - b\| + \|b - c\| = \|a - c\|. \hfill \Box$

Proof (10.2). Given $a, b, c$ as described above, from (10.1) we have $\|a - b\| + \|b - c\| = \|a - c\|.$ Since $b + c \Rightarrow \|b - c\| > 0.$ The other elements of the equation are non negative and the desired result is immediate. $\hfill \Box$

Proof (10.3). Let $a, b, c$ and $t_b$ be as described above. $f(b) = f((1 - t_b)a + t_b c) \leq (1 - t_b)f(a) + t_b f(c)$ which is in the interval $(f(a), f(c))$ since $t_b \in (0, 1).$ It follows that $f(b) < f(c). \hfill \Box$

Proof (10.4). If $f(a) = f(c)$ then $f(b) = f((1 - t_b)a + t_b c) \leq (1 - t_b)f(a) + t_b f(c) = (1 - t_b)f(a) + t_b f(a) = f(a). \hfill \Box$

**Definition 11.** For any set $X \subseteq \mathbb{H}$ we define the **affine hull**, $\text{aff}(X),$ as the smallest affine space containing $X$ with regards to inclusion.

**Definition 12.** We use $B_r(y) := \{x \in \mathbb{H} \mid \|x - y\| < r\}$ to denote the **ball** centered at $y$ with a radius of $r.$

**Definition 13.** For any $X \subseteq \mathbb{H}$ we define the **relative interior** $\text{relint}(X) = \{x \mid \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \cap \text{aff}(X) \subseteq X\}.$

**Lemma 14.** Let $K \subseteq P$ be a convex set and $A$ be the the smallest space, with regards to inclusion, in $\mathcal{L}P$ such that $K \subseteq A$ and let $y \in \text{relint}(K),$ then for some $H \in \mathcal{H},$ if $y \in \partial H$ then $A \subseteq \partial H.$
Proof. Let \( H \in \mathcal{H}_P \) such that \( y \in \partial H \cap \text{relint}(K) \). By the definition of relative interior, there exists an \( N := B_r(y) \cap \text{aff}(K) \), such that \( N \subset P \cap A \).

Let us falsely assume \( A \) is not a subset of \( \partial H \). If \( K \not\subset \partial H \) then by the definition of \( A \), \( A \not\subset \partial H \) in contradiction to the false assumption we just made. Therefore, there exists an \( a \in K \setminus \partial H \) and the line \( \overline{a, y} := \{ta + (1-t)y \mid t \in \mathbb{R}\} \) intersects \( \partial H \), as a result of lemma 9.

Let \( y_1, y_2 \) be distinct points in \( N \cap \overline{a, y} \), each one a distance of \( \epsilon/2 \) from \( y \). Such points exist because any line containing two points in an affine space is entirely in that affine space, \( \overline{a, y} \subset \text{aff}(K) \).

Either \( y_1 \) or \( y_2 \) is not in \( H^o \). Otherwise, by the convexity of \( H^o \) we would have \( y \in H^o \), a contradiction to \( y \in \partial H \).

We also have either \( y_1 \) or \( y_2 \) in \( \partial H^c \) since, by the definition of an affine space, we would otherwise have the whole line \( \overline{a, y} \) in \( \partial H \) including \( a \), which would be a contradiction. So Without loss of generality we may choose \( y_1 \in H^c \). But this is a contradiction since \( N \subset P \).

Definition 15. For \( A \in \mathcal{A}_P \) we define the \textbf{P-cone} of \( A \) and \( P \) as \( P_A := \bigcap \{ H \in \mathcal{H}_P \mid A \subset H \} \) be the polyhedron whose hyperplanes, a subset of the hyperplanes of \( P \), intersect to equal \( A \). We will call this \( P \)-cone of \( A \) where \( P \) is implied.

Example 16. If we use the 'A' polyhedron then the \( P \)-cone of the top point \( P_{F \cap G} = F \cap G \). Note that \( F \cap G \cap H = P \subset P_{F \cap G} \).

Example 17. Note that by our definition, \( P_{\emptyset} = \mathbb{H} \).

Remark 18. If \( A, B \in \mathcal{A}_P \) and \( B \not\supset A \) then the \( P \)-cone of \( B \) with regards to \( P_A \) is the \( P \)-cone of \( B \) with regards to \( P \). Since \( B \) is a superspace of \( A \), any hyperplane \( \partial H \) of \( P \) that contains \( B \) also contains \( A \). It follows that \( H \in \mathcal{H}_{P_A} \).

Proposition 19. Let \( K \subset P \) be a convex set and \( A \) be the smallest set, with regards to inclusion, in \( \mathcal{A}_P \) such that \( K \not\subset A \). Then for any \( x \in \text{relint} K \) there exists an \( \epsilon > 0 \) such that \( P_A \cap B_{\epsilon}(x) = P \cap B_{\epsilon}(x) \).

Proof. Let \( x \in \text{relint} K \). Let \( Q \subset \mathbb{H} \) be a polyhedron such that \( \mathcal{H}_Q = \mathcal{H}_P \setminus \mathcal{H}_{P_A} \). Then we can define \( \epsilon := \min_{y \in \partial Q} \|y - x\| \). If we falsely assume \( \epsilon = 0 \) then there exists an \( H \in \mathcal{H}_Q \) with \( x \in \partial H \cap P \). Since \( x \in \text{relint} K \), we may conclude from lemma 12 that \( A \subset \partial H \) and that \( H \in \mathcal{H}_{P_A} \), a contradiction. We may conclude \( \epsilon > 0 \).

(\( \subseteq \)) Let \( y \in B_{\epsilon}(x) \cap P_A \). Let’s falsely assume \( y \in P^c \). There exists an \( H \in \mathcal{H}_P \) such that \( y \in H^c \). We have \( \mathcal{H}_P = \mathcal{H}_Q \cup \mathcal{H}_{P_A} \). Since \( y \in P_A \) it follows that \( H \in \mathcal{H}_Q \). Since \( x \in P \subset H \), by lemma 9 we may consider the unique \( \partial H \cap \overline{a, b} \) and from lemma 10 we conclude that \( \| \partial H \cap \overline{a, b} - x \| < \|y - x\| < \epsilon \), a contradiction to our choice of epsilon. We may conclude that \( P_A \cap B_{\epsilon}(x) \subseteq P \cap B_{\epsilon}(x) \).

(\( \supseteq \)) Since \( P \Subset P_A \), we may conclude that \( P_A \cap B_{\epsilon}(x) \supseteq P \cap B_{\epsilon}(x) \).
4 Properties of Optimization over Polyhedrons

For each of the propositions in this section and to come, we use an arbitrary convex polyhedron \( P \) and convex function \( f : \mathbb{R} \to \mathbb{R} \) as defined above.

**Lemma 20.** For any convex \( K \subset \mathbb{H} \), the set \( \operatorname{arg min}_K f \) is convex.

**Proof.** Let \( a, b \in \operatorname{arg min}_K f \). Since \( K \) is convex, \( a, b \in K \). It follows that for any \( t \in (0, 1) \) we have \( f(ta + (1-t)b) \geq f(a) = f(b) = \min_K f \).

By the convexity of \( f \), we have \( f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) = tf(a) + (1-t)f(a) = f(a) \).

The double inequality renders \( f(ta + (1-t)b) = f(a) \Rightarrow ta + (1-t)b \in \operatorname{arg min}_K f \). \( \square \)

**Definition 21.** Let \( A := \cap \{ \partial H \mid H \in \mathcal{H}_P, \operatorname{arg min}_P f \subset \partial H \} \) be the intersection of all the hyperplanes of \( P \) that contain \( \operatorname{arg min}_P f \). We call \( A \), a **min space** of \( f \) on \( P \), or where \( f \) is implied, we may omit it.

**Remark 22.** The min space exists and is unique. If there are no hyperplanes of \( P \) that contain \( \operatorname{arg min}_P f \), because \( \operatorname{arg min}_P f = \operatorname{arg min}_H f \in P^0 \), then the min space is \( \mathbb{H} \).

**Remark 23.** The min space is the smallest affine space of \( P \) containing \( \operatorname{relint} P \), with regards to set inclusion.

**Remark 24.** For a min space \( A \) we have \( \operatorname{arg min}_P f \subset A \).

**Proposition 25 (The Necessary Criteria).** Let \( A \) be the min space for some \( f \) on \( P \), then \( A \) meets the Necessary Criteria which are as follows:

1. \( \operatorname{arg min}_P f \subset \operatorname{arg min}_A f \)
2. \( \operatorname{arg min}_A f = \operatorname{arg min}_{P_A} f \)

**Proof (25.1).** From remark \( 24 \) we have \( \min_A f \leq \min_P f \).

Let’s falsely assume there exists a \( x \in A \) such that \( f(x) < \min_P f \) and let \( y \in \operatorname{relint} \operatorname{arg min}_P f \). By proposition \( 19 \) there exists an \( \epsilon \) such that \( B_\epsilon(y) \cap P = B_\epsilon(y) \cap P_A \). The line segment \( \overline{x,y} \) is entirely in \( A \subset P_A \), so we may choose a \( z \in \overline{x,y} \cap B_\epsilon(y) \cap P_A \). Since \( z \in \overline{x,y} \) by lemma \( 10 \) we have \( f(z) < f(y) = \min_P f \).

By proposition \( 19 \) we have \( z \in P \), a contradiction. \( \square \)

**Proof (25.2).** Let’s falsely assume that there exists an \( x \in P_A \setminus A \) such that \( f(x) \leq \min_P f \), which by remark \( 24 \) has \( \operatorname{arg min}_P f \subset A \), and let \( y \in \operatorname{relint} \operatorname{arg min}_P f \). Then by proposition \( 19 \) we can let \( \epsilon > 0 \) such that \( B_\epsilon(y) \cap P = B_\epsilon(y) \cap P_A \).
Since \(x \in P_A \setminus A\), it follows that \(\bar{x}, \bar{y}, y \subset P_A \setminus A\). If there was a second point beside \(y\) in \(A\) then by the definition of an affine space, \(x\) would be in \(A\) as well.

We may choose a \(z \in \bar{x}, \bar{y} \cap \partial D_r(y) \subset P \cap P_A\) with a distance of \(\frac{r}{2}\) from \(y\). We have \(z \in P \setminus A\) and by lemma 10 we have \(f(z) \leq f(y)\) in contradiction to remark 24 if \(f(z) = f(y)\) and in contradiction to \(y \in \arg \min_{P} f\) if the inequality is strict.

We may conclude that for all \(x \in P_A \setminus A\) we have \(f(x) > \min_{P} f\). From 25.1 we have if \(x \in A\) then \(f(x) \geq \min_{P} f\). Combining these two, and remark 24 we get the desired result.

\[\square\]

**Example 26.** Consider \(\Pi_A(\frac{1}{3}, 1)\) (recall examples 3 and 4). We can check each of the affine spaces to see if they meet the necessary criteria. The \(P\)-cone of \(\partial H\) is \(H\). Since \(\Pi_{\partial H}(y) \in \partial H\) and \(-\frac{1}{3} + 1 < 1\) giving \(\Pi_{\partial H}(y) = y \in \partial H^c\) we may conclude from the necessary criteria (proposition 25.2) that \(\partial H\) is not the min space. Since \(y \in P^c \cap \hat{G}^c\) we have \(\Pi_{\partial \hat{G}}(y) = \Pi_{\hat{G}}(y)\) and \(\Pi_{\partial \hat{F}}(y) = \Pi_{\hat{F}}(y)\) so they pass the necessary criteria. We’ll now consider the affine spaces of \(\text{codim} 2\). \(\Pi_{\partial \hat{G} \cap \partial \hat{H}}(y) = \partial \hat{G} \cap \partial \hat{H} = (0, 1)\), but its \(P\)-cone has \(\Pi_{\partial \hat{G} \cap \partial \hat{H}}(y) = (\frac{1}{8}, \frac{7}{8}) + (0, 1)\). The other two \(\text{codim} 2\) affine spaces similarly fail the necessary criteria test. The Hilbert space, \(\mathbb{H}\), also meets the necessary criteria.

Note that ‘\(A\)’ is a simple problem and it’s easy to more or less eyeball both where the projection is and which affine spaces meet the necessary criteria. For more complex problems, computing the optimal point over a \(P\)-cone may be nearly as complex as computing the optimal point over the original polyhedron. This challenge is resolved by the algorithm below.

For those affine spaces that meet the necessary criteria, we next consider The Sufficient Criteria.

**Proposition 27** (The Sufficient Criteria). If for some \(A \in A_P\) there exists an \(x \in \arg \min_{P_A} f \cap P\) then \(x \in \arg \min_{P} f\).

**Proof.** Let \(A \in A_P\) such that there exists an \(x \in \arg \min_{P_A} f \cap P\). Since \(x \in P \subset P_A \Rightarrow x \in \arg \min_{P} f\). \(\square\)

**Example 28.** If we consider those affine spaces from example 26 which met the necessary criteria, we will now check The Sufficient Criteria. For \(\mathbb{H}\) we had \(\Pi_{\mathbb{H}}(y) = y \in P^c\), so it is not our min space. We also have \(\Pi_{\hat{G}} \in P^c\) but \(\Pi_{\hat{F}} \in P\). That is, \(\partial \hat{F}\) meets both the necessary and sufficient criteria and we may conclude it is the min space.

**Corollary 29.** If \(A\) meet the Necessary Criteria (25) and there exists an \(x \in \arg \min_{A} f \cap P\) then \(x \in \arg \min_{P} f\).

**Definition 30.** Let \(A, B \in A_P\), be nonempty with \(A \subset B\). We can say that \(B\) disqualifies \(A\) from \(P\), with regards to \(f\), if \(B\) is the min space of \(f\) on \(P_A\). If
there is no such $B$, then we say $A$ is a candidate for $f$ on $P$. Where $f$ and $P$ are implied, they are omitted.

**Corollary 31.** If some $A \in A_P$, is disqualified from $P$ then $A$ is not the min space on $P$.

**Proof.** Let $A, B \in A_P$ such that $B$ disqualifies $A$. We have $B$ is the min space of $P_A$ and there exists an $x \in \arg \min_{P_A} f \setminus A$, otherwise $A$ would be the min space over $P_A$ and not $B$. But this is a contradiction to proposition 25.

Given $A, B \in A_P$, and with $A \not\subseteq B$ we can use disqualification to determine that $A$ is not the min space without expensively computing $\min_A f$.

**Corollary 32.** If $A$ is a candidate for $P$ and there exists an $x \in \arg \min_A f \cap P$ then $x \in \arg \min_P f$.

**Proof.** Let $A$ be a candidate for $P$ and $x \in \arg \min_A f \cap P$.

By definition 21 there exists the min space $B \in A_P$ with regards to $P_A$. Since $A$ is a candidate, $B$ can not be a strict superspace of $A$. Since all the affine spaces of $P_A$ are superspaces of $A \Rightarrow B = A$ and $A$ meets criteria 252 $\arg \min_A f \subseteq \arg \min_{P_A} f$.

That is to say, $x \in \arg \min_{P_A} f \cap P$. By proposition 27 The Sufficient Criteria, we have the desired result.

**Corollary 33.** If $A, B \in A_P$ and $B$ disqualifies $A$, then it is the unique affine space of $P$ that does so.

**Proof.** Let $A, B \in A_P$, such that $B$ disqualifies $A$ from $P$. $B$ is the min space on $P_A$. The uniqueness of the min space follows directly from the definition 21.

**Corollary 34.** Let $A, B \in A_P$ with $B$ disqualifying $A$. Then $\arg \min_{P_A} f \subseteq \arg \min_{P_B} f$.

**Proof.** Let $A, B \in A_P$ with $B$ disqualifying $A$. By definition 30 $B$ is the min space of $P_A$. From proposition 25 the Necessary Criteria, we have $\arg \min_{P_A} f \subseteq \arg \min_B f \subseteq \arg \min_{P_B} f$.

**Lemma 35.** For $A, B \in A_P$ with $A \subseteq B$, if $\arg \min_{P_A} f \cap P_B$ is nonempty then $\arg \min_{P_A} f \subseteq \arg \min_{P_B} f$.

**Proof.** Let $A, B \in A_P$ with $A \subseteq B$. Since $P_A \subseteq P_B$ we have $\min_{P_B} f \leq \min_{P_A} f$. If $\arg \min_{P_B} f \in P_A$ then $\arg \min_{P_A} f \subseteq \arg \min_{P_B} f$. 

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Definition 36. For $A, B \in A_P$ we say that $B$ is an **immediate superspace** of $A$ if $B \supseteq A$ and there exists an $H \in \mathcal{H}_P$ such that $A = \partial H \cap B$. We will also say that $A$ is an **immediate subspace** of $B$.

**Example 37.** In the ‘$A$’ example (3). The immediate superspaces of $\partial G \cap \partial H$ are $\partial G$ and $\partial H$.

**Proposition 38.** If $A, B \in A_P$ with $B$ an immediate superspace of $A$ and $\arg \min_{P_A} f \in B \setminus A$, then $B$ is the min space of $P_A$.

**Proof.** Let $A, B \in A_P$ with $B \supseteq A$ and $H \in \mathcal{H}_P$ such that $A \equiv B \cap \partial H$.

Let’s falsely assume that $B$ is not a min space of $P_A$. Then there exists a $G \in \mathcal{H}_{P_A}$ with $\partial G$ not a superspace of $B$, such that $\arg \min_{P_A} f \subseteq B \cap \partial G$ rendering $B$ not the min space.

If $G = H$ we have a contradiction to $\arg \min_{P_A} f \subset A^c$.

If $G \neq H$, and since it’s not a superspace of $B$ then $A \in A_P \Rightarrow A = \cap_{E \in \mathcal{H}_{P_A}} \partial E$

it follows $A = \partial H \cap B \cap \partial G \neq \partial H \cap B$ in contradiction to the definition of $A$. □

**Remark 39.** If $A$ is the intersection of $n$ hyperplanes of $P$, then it has $n$ immediate superspaces, each can be generated by taking the intersection of $n - 1$ of the hyperplanes that intersect to make $A$.

**Proposition 40.** Let $A \in A_P$, then $A$ is disqualified from $P$ if and only if there exists an immediate superspace, $B$, such that $\arg \min_{P_B} f \cap (P_A \setminus A)$ is nonempty.

**Proof.** ($\Rightarrow$) Let $A \in A_P$, such that $A$ is disqualified from $P$.

If $A$ is disqualified by some $B \in A_P$, an immediate superspace of $A$, then there exists an $x \in \arg \min_B f \cap P_A \cap A^c$. Since $B$ is a min space for $P_A$ we have $\arg \min_B f = \arg \min_{P_B} f$ giving us the desired result.

If $A$ is disqualified by some $C$ that is not an immediate superspace of $A$, from lemma [Lemma], we have $\arg \min_{P_A} f \subset \arg \min_{P_C} f = \arg \min_{P_C} f$. Let $B$ be an immediate superspace of $A$ such that $A \subseteq B \subset C$. Since $P_A \subseteq P_B \Rightarrow \arg \min_{P_A} f \subseteq P_B$.

Since $P_B \subset P_C$ and $\arg \min_C f = \arg \min_{P_B} f \subseteq P_B \Rightarrow \arg \min_{P_B} f = \arg \min_{P_B} f$ giving is the desired result $\arg \min_{P_B} f = \arg \min_{P_A} f$. Note that their exists a $x \in \arg \min_{P_A} f \setminus A$ since $C$ is the min space.

($\Leftarrow$) Let $B$ be an immediate superspace of $A$ and let $x \in \arg \min_{P_B} f \cap (P_A \setminus A)$.

Since $P_A \subseteq P_B$ and $x \in \arg \min_{P_A} f$ then $x \in \arg \min_{P_A} f$. The min space $P_A$ contains $x \in A^c$, so that space is not $A$, and therefore disqualifies $A$. □

**Remark 41.** For $A$ and $B$ with $B$ an immediate superspace of $A$, then for some $H \in \mathcal{H}_P$ we have $A = \partial H \cap B$, and $\arg \min_{P_B} f \cap (H \setminus A)$ is non empty then $\arg \min_{P_B} f \cap P_A$ is non empty and $A$ is disqualified.
Proposition 42. Let $A \in \mathcal{A}_P$. If for all immediate superspaces, $B$, we have \( \arg\min_{P_B} f \subset P_A \), then $A$ is the min space of $P_A$ and a candidate.

Proof. Let $A \in \mathcal{A}_P$ and assume that for all immediate superspaces, $B$, we have \( \min_{P_B} f \subset P_A \).

By proposition 40, $A$ is a candidate for $P_A$. Since, from the definition of the min space, every polyhedron must have one, and all superspaces of $A$ in $\mathcal{A}_P$, are out, then only $A$ itself is left.

Proposition 38 and its results allow us to determine if an affine space, $A \in \mathcal{A}_P$, meets the necessary criteria by looking exclusively at its immediate superspaces, $B$, and their $P$-cones. Checking the optima, \( \arg\min_{P_B} f \), for each $B \in \mathcal{B}$ for membership $P_A$ yields either the optimal point of the $P$-cone of the current affine space, or the information that \( \arg\min_{P_A} f \) can be found in $A$. Either way, by knowing \( \arg\min_{P_B} f \) for all $B \in \mathcal{B}$, we can easily find \( \arg\min_{P_A} f \). This result lends itself to the following algorithm, wherein we begin by finding the optimum $H$, then at each iteration find the optimum of all the $P$-cones of the immediate sub-spaces, until one of those sup-spaces meets the necessary and sufficient criteria.

This method eliminates the need to find the optimal points over $P$-cones for any the affine spaces that do not meet the necessary criteria, because those optima have already been computed by one of the immediate super-spaces.

5 The Optimization Algorithm

We will begin with an auxiliary algorithm. In this algorithm we use the definitions for affine spaces of the polyhedron (definition 5), the $P$-cone of an affine
space (definition \[15\]), and the immediate super-space (definition \[36\]).

**Algorithm 1:** Checks Proposition \[40\]

**Input:**
- \( f \xrightarrow{\text{conv.}} \mathbb{R} \)
- An affine space \( A \in \mathcal{A}_P \)
- \( A \)'s immediate superspaces, \( B \)
- For each \( B \in B \) we get a \( m_B := \arg \min_B f \).

**Output:**
- \textbf{true} if \( A \) is a candidate and \( y \in P_A^S \) and \textbf{false} otherwise.
- \( p_A \), the saved minimum onto \( P_A \)

\begin{algorithm}
\begin{algorithmic}
\For {\( B \in B \)}
\If {\( m_B \cap P_A \neq \emptyset \)} // prop. \[40\]
\State \( m_A \leftarrow m_B \) is saved as the minimum onto \( P_A \). // lemma \[35\]
\State \textbf{return false} // Remark \[31\]
\EndIf
\EndFor
\State \( m_A \leftarrow \arg \min_A f \) is computed and saved // prop. \[42\]
\State \textbf{return true} // prop. \[40\]
\end{algorithmic}
\end{algorithm}

For the reader’s convenience we recall the definition of strictly convex and the following lemma.

**Definition 43.** A function \( f \mid \mathbb{H} \rightarrow \mathbb{R} \) is \textbf{strictly convex} if for all \( x, y \in \mathbb{H} \) and \( t \in (0, 1) \) we have \( f((1-t)x + ty) < (1-t)f(x) + tf(y) \).

**Lemma 44.** If \( f \) is strictly convex, then for any convex \( K \), \( \arg \min_K f \) has at most one element.

**Proof.** For some affine \( K \) Let’s falsely assume there exists distinct \( x, y \in \arg \min_K f \). Since \( \arg \min_K f \) is convex (proposition \[20\]), for all \( t \in (0, 1) \) we have \( (1-t)x + ty \in \arg \min_A f \Rightarrow f(x) = f((1-t)x + ty) \). Since \( f \) is strictly convex, and \( f(x) = f(y) \), we also have \( f((1-t)x + ty) < (1-t)f(x) + tf(y) = (1-t)f(x) + tf(x) = f(x) \). Combining the two gives \( f(x) = f((1-t)x + ty) < (1-t)f(x) + tf(y) = (1-t)f(x) + tf(x) = f(x) \Rightarrow f(x) < f(x) \), a contradiction.

Observe that if \( m_B \) is a single point then checking line \[2\] in algorithm \[1\] is \( \mathcal{O}((,) \rangle) \), since by remark \[41\] we only need to check if \( m_B \in H \) for the unique \( H \in \mathcal{H}_A \cap \mathcal{H}_B \). We already know that \( m_B \in B \) and similarly weather or not \( m_B \in B \). If \( m_B \) is a larger set, then checking \( m_B \cap P_A \) is a more complex operation and outside the scope of this complexity analysis. We can ensure that \( m_B \) be a single point by requiring that \( f \) be strictly convex.

If \( m_B \) is more than a single point, and \( m_B \cap P_A \) can be computed quickly, then line \[3\] should read \( m_A \leftarrow m_B \cap P_A \) to later speed up the check of the intersection
of $m_A$ (which at that future iteration will be called $m_B$) and other half-spaces.

**Algorithm 2:** Finds $\arg \min_P f$.

**Input:** A set of half-spaces $\mathcal{H}_P$ and a function $f: \mathbb{H} \xrightarrow{\text{conv}} \mathbb{R}$

**Output:** $\min_P f$

1. if $\arg \min_{\mathcal{H}} f \in P$ then
   2. return $m_{\mathcal{H}}$
3. for $i \leftarrow 1$ to $n$ do
   4. for $A \in A_P$ with codim$(A) = i$ in parallel do
   5. if Alg. 1 on $A$ then
   6. if $m_A \cap P \neq \emptyset$ then // prop. 27
   7. return $m_A$ // Remark 32
8. return $P$ is empty.

**Example 45.** We will revisit example 3 one last time by running this algorithm on the following problem: $\Pi_A'(1, 1)$.

We will first compute $\Pi_H(1, 1) = (1, 1)$ and check The Sufficient Criteria (27). We have $(1, 1) \in P^c$ so we continue to all the affine spaces of our polyhedron with codim$(1)$. Let’s first look at $\partial H$. Since $(1, 1) \in H = P_{\partial H}$ we conclude two things: $\partial H$ is not the min space since it does not meet the necessary criteria, and $\Pi_H(1, 1) = (1, 1)$, the optimal point over the $P$-cone of the immediate superspace. For both $\partial \hat{G}$ and $\partial \hat{F}$ we have $(1, 1)$ outside their $P$-cones $\hat{F}$ and $\hat{G}$. It follows that $\partial \hat{F}$ and $\partial \hat{G}$ are candidates. We will compute $\Pi_{\partial \hat{F}} = (1, \frac{1}{2})$ and $\Pi_{\partial \hat{G}} = (\frac{1}{2}, \frac{1}{2})$. We now know the minimums over the $P$-cones preparing us for the next iteration which will use this information, but first We can check both those points for membership in $P$. The point $(1, \frac{1}{2}) \in P^c$ but the point $(\frac{1}{2}, \frac{1}{2}) \in P$, and meets The Sufficient Criteria giving $\Pi_A'(1, \frac{1}{2})$.

An important thing to note in this example is that $\partial \hat{G}$ is not the min space, $\partial \hat{G} \cap \partial \hat{F}$ is. However, The Sufficient Criteria are met.

### 6 Analysis and complexity

Using the propositions above, We make a lot of complexity shortcuts over the course of the algorithm, but ultimately, for all $i < n$ the algorithm reviews $|\{A \in A_P \mid \text{codim}A = i\}| \leq \binom{r}{i}$ affine spaces. For each affine space, $A$, the space is disqualified by one of its immediate superspaces, or it needs to, more expensively if the affine space is a candidate, compute and check $\arg \min_A f \in P$. Checking membership in a polyhedron is $O(r(\cdot, \cdot))$.

If $n < \log_2(r)^{1-r}$ we have a total computational complexity $O(\sum_{i=1}^n \binom{r}{i}(\cdot, \cdot)r \min_A f) \leq O(r^n(\cdot, \cdot) \min_A f) = O(r^{n+1}(\cdot, \cdot) \min_A f)$, which is to say, polynomial as a function of $r$. 

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If \( n > \log_2(r)^{-1}r \) then instead we have \( O(\sum_{i=1}^{r} \binom{r}{i} r(\cdot, \cdot) \min_A f) \leq O(2^{r+1}(\cdot, \cdot) \min_A f) \). The impact of \( n \) here is hidden inside the inner product and \( \min_A f \). But if \( \min_A f \) is polynomial as a function of \( n \) then so to is our algorithm.

However, if multi threading is available, so that each of the affine spaces in \( A_P \) with \( \text{codim} \ i \) can be processed in parallel, then \( \sum_{i=1}^{r} \binom{r}{i} \) simplifies to \( \min(n, r) \) sets of \( O \) over any thread path to line 6, by checking all the half-spaces at once, we can further reduce complexity.

If we also use parallelization in algorithm 2 to check The Sufficient Criteria on line 1, by checking all the half-spaces at once, we can further reduce complexity over any thread path to \( O(\min\{r, n\} r(\cdot, \cdot) \min_A f) \).

Within algorithm 1 the filter algorithm, there exists an additional opportunity for distributing the work load by running the tasks in the loop, line 1 in parallel. If we also use parallelization in algorithm 2 to check The Sufficient Criteria on line 3, by checking all the half-spaces at once, we can further reduce complexity over any thread path to \( O(\min\{r, n\} r(\cdot, \cdot) \min_A f) \).

To determine how many threads we need to achieve optimal performance for the worst case scenario, we may consider \( \max_{i \leq n} \binom{i}{r} \). If \( n > \frac{r}{2} \) then Pascal’s triangle tells us that we have the maximum at \( i = \frac{n}{2} \), and Stirling’s formula \( \binom{n}{r} \approx (\pi n^{2-1} r)^{1/2} 2^r \) tells us that we could use roughly \( (\pi n^{2-1} r)^{1/2} 2^r \) threads for algorithm 2. If we also multi thread algorithm 1 then at each stage we can multiply by an additional \( \min(n, r) \) threads yielding optimal performance with \( \min(n, r)(\pi 2^{2-1} r)^{1/2} 2^r \) threads. If \( n < \frac{r}{2} \) then the optimal number of threads goes down to \( n \binom{n}{r} \). By comparison, threads on modern GPU’s run in the tens of thousands \( \text{[15]} \), and there’s every indication that number will continue to increase exponentially \( \text{[6]} \).

Additional efforts at parallelization in computing the inner product and \( \min A \) may yield even better results, but this is situation dependent and outside the scope of this paper.

Remark 46. If we consider the projection function from example \( \text{[1]} \) \( O(n^3) \) to project onto an affine space, and the inner product that is scalar product \( O(n) \), if \( r >> n \) we have \( O(n^{n+1} n^4) \) and if we have ample threads \( O(n^3) \) over each thread.

Note that unlike many interior point methods, the complexity is not a function of accuracy; there is no \( \epsilon \) term that compromises speed with the desired distance from the correct answer.
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