C, PT, CPT invariance of pseudo-Hermitian Hamiltonians

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Abstract

We propose construction of a unique and definite metric ($\eta + \eta$), time-reversal operator (T) and an inner product such that the pseudo-Hermitian matrix Hamiltonians are C, PT, CPT invariant and PT(CPT)-norm is definite (definite). Here, P and C denote the generalized symmetries: parity and charge-conjugation respectively. The limitations of the other current approaches have been brought out.

I. INTRODUCTION : PT-SYMMETRY AND PSEUDO-HERMITICITY

Last few years have witnessed a remarkable development wherein the discrete symmetries of a Hamiltonian seem to decide if the eigenspectrum will be real. It has been conjectured [1] that Hamiltonians possessing symmetry under the combined transformation of parity (P: $x \to -x$) and time-reversal (T: $i \to -i$) will have real discrete spectrum provided the eigenstates are also simultaneous eigenstates of PT. Interesting situations are those where P and T are individually broken. An overwhelming number of evidences supporting the conjecture are available. [1-7].

The real eigenvalues of a PT symmetric Hamiltonian are found connected with a more general property of the Hamiltonian namely the pseudo-Hermiticity. The concept of pseudo-Hermiticity was developed in 50s-60s [9] following definition of a distorted definition of inner product $\langle \eta \rangle$ [8], $\eta$ is called a metric. A Hamiltonian is called pseudo-Hermitian, if it is such that

$$\eta H \eta^{-1} = H^\dagger.$$ (1)
The eigenstates corresponding to real eigenvalues are $\eta$-orthogonal and eigenstates corresponding to complex eigenvalues have zero $\eta$-norm (2). Identifying $\eta$ for a non-Hermitian Hamiltonian when it has real eigenvalues is very crucial. Most of the PT-symmetric Hamiltonians having real eigenvalues have recently been claimed to be P-pseudo-Hermitian, and several other interesting results have been derived [10]. Several non-Hermitian Hamiltonians of both types PT-symmetric and non-PT-symmetric possessing real spectrum have been identified as pseudo-Hermitian under $\eta = e^{-\theta_p}$ and $e^{-\phi(x)}$ [11]. Some more interesting developments relate to weak-pseudo-Hermiticity [12], pseudo-anti-Hermiticity [13] and construction non-PT-symmetric (pseudo-Hermitian) complex potential potentials having real eigenvalues [14] A new pseudo-unitary group and Gaussian-random pseudo-unitary ensembles of pseudo-Hermitian matrices have been proposed [15]. This development gives rise to new energy-level distributions which are expected to represent the spectral fluctuations of PT-symmetric systems.

Most interesting feature of the eigenstates of such Hamiltonians is the indefiniteness [5-8] (positivity-negativity) of the norm which is the consequence of the $\eta$-inner product [8]

$$\langle \Psi_m | \eta \Psi_n \rangle = \epsilon_n \delta_{m,n}, \quad (2)$$

where $\epsilon_n (= \pm 1)$ is indefinite (positive-negative). Recall, that the usual norm in Hermiticity is $\langle \Psi_n | \Psi_n \rangle$ positive-definite. Currently, the negativity of the PT-norm has been proposed to indicate the presence of a hidden symmetry called $C$ which mimics charge-conjugation symmetry ($C$) [17]. It has been claimed that CPT-norm will be positive definite. An interesting scope for PT-symmetric quantum field theory has been argued. The construction of the new involutary operator $C$ has been discussed. A $2 \times 2$ matrix Hamiltonian which is actually pseudo-Hermitian with real eigenvalues has been employed and by constructing $P=\eta$, $T=K_0$ and a CPT-norm the, the novel proposal has been illustrated [16]. $K_0$ represents complex-conjugation operator; e.g. $K_0(AB) = A^*B^*$. Though sufficient and consistent for their assumed model of non-Hermitian Hamiltonian, let us remark that these constructions are too simple to work in general.

**II. CURRENT DEVELOPMENTS AND MOTIVATION**

The next related development [18] caters to the construction of generalized involutary operators $C,P,T$ from the bi-orthonormal [8,14] basis ($\Psi, \Phi$) of the pseudo-Hermitian Hamiltonian with real eigenvalues. In doing so, the well developed machinery of pseudo-Hermiticity
has been aptly utilized. This development, however, does not dwell upon the negativity of the PT-norm and invoking of C for the positive-definiteness of the CPT-norm. In this approach, the search of various symmetries of H and their identification as C, PT or CPT has been proposed. Despite, obtaining a curiously different definition of T other than the simple $K_0$ [16], this dichotomy has neither been remarked nor resolved. Also, despite this incompatibility a similar definition of the CPT-inner product [16] has been adopted [18].

Further, when a Hamiltonian is Hermitian and of the type $\mathcal{H} = p^2/(2m) + V(x)$ by adopting the definition of C [16] which becomes P now due to Hermiticity, it has been claimed that Hermitian operators, $\mathcal{H}$ have parity P and they are PT-invariant. It may be noted that, the definition of P proposed in [19] is identical to the definition of generalized parity proposed in [18] when the Hamiltonian becomes Hermitian.

While following these developments one very strongly feels that a Hermitian Hamiltonian ought to be P, T, PT, and CPT invariant. The PT (CPT)-norm ought to be indefinite (definite). Also the eigenstates of H should display the orthonormality consistent with the definition of norm under the same inner product. These primary contentions do however not meet in either of the approaches [16,18]

In fact, these expectations have been met lately, not without incorporating a generalized definition of T [20] a la, discarding $T = K_0$ [16] and proposing an inner product [20]. In this Letter, we propose further extension of these [20] definitions so as to bring consistency in proposing the C, PT, and CPT invariance of a pseudo-Hermitian Hamiltonian (real eigenvalues) definiteness of CPT-norm and the indefiniteness of PT-norm. In our studies, we prefer the use of matrix notations and matrix models of Hamiltonians. Recall that in case of Hermiticity, for the usual stationary states the three modifications $\Psi(x), \Psi^*(x)$ and $\Psi^\dagger(x)$ usually coincide. However, in matrix notations, we have four distinct modifications of the state these are $\Psi, \Psi^*$ (complex-conjugate) ,$\Psi'$ (transpose), $\Psi^\dagger$ (transpose and complex-conjugate). This makes the matrix notations more general, unambiguous and unmistakable.

III. PSEUDO-HERMITIAN MATRICES : A UNIQUE AND DEFINITE METRIC

Let us notice the non-Hermitian complex matrix, $H$, given below admitting real eigenvalues $E_{0,1} = a \pm \sqrt{bc}$, when $bc > 0$. We find that there exist four metrics $\eta_i$ under which $H$ is pseudo-Hermitian.
\[ H = \begin{bmatrix} a & -ib \\ ic & a \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} r^2 & -s \\ s & 1 \end{bmatrix}, \quad \eta_3 = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix}, \quad \eta_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3} \]

Here \( r = \sqrt{c/b} \) and \( s \) is in general an arbitrary complex number, indicating that a metric need not necessarily be Hermitian. These \( \eta_1 \) (Pauli’s \( \sigma_x \)) and \( \eta_{2,3,4} \) have, in fact, been found by crude algebraic manipulations demonstrating that metric \( \eta \) is non-unique as informed earlier [10]. Furthermore, if \( \eta_1 \) and \( \eta_2 \) are found then infinitely many metrics can be constructed as \( \eta = (c_1 \eta_1 + c_2 \eta_2) \) provided \( \eta \) is invertible. On one hand, the four metrics given above (3) do provide several operators \( F_{i,j} = \eta_i \eta_j^{-1}, i \neq j = 1,4 \) which by commuting with \( H \) bring out its hidden symmetries [10]. In fact, the currently discussed C, PT, and CPT symmetries shall be seen connected to \( F_{i,j} \) in the examples to follow in the sequel. On the other hand, the non-uniqueness of \( \eta \) apart from its indefiniteness may be undesirable as the metric determines the expectation values of various operators as \( \langle \Psi | A \eta \rangle \). We state and prove the following theorem which helps us in fixing a unique and definite metric. This could be seen as a method to find at least one metric under which a given matrix is pseudo-Hermitian.

**Theorem**: If a diagonalizable complex matrix \( H \) admits real eigenvalues \( (E_1, E_2, ..., E_n) \) and \( D \) is its diagonalizing matrix then \( H \) is \( \eta \)-pseudo-Hermitian, where \( \eta = (DD^\dagger)^{-1} \). Converse of this also holds.

**Proof**: Let

\[ D^{-1}HD = \text{Diag}[E_1, E_2, ..., E_n] \tag{4a} \]

\[ \Rightarrow D^{-1}\eta^{-1}\eta H \eta^{-1}\eta D = \text{Diag}[E_1, E_2, ..., E_n] \tag{4b} \]

Invoking the pseudo-Hermiticity (1), we write

\[ D^{-1}\eta^{-1}H^\dagger\eta D = \text{Diag}[E_1, E_2, ..., E_n] \tag{4c} \]

The transpose-conjugation of Eq. 3(a) yields

\[ D^\dagger H^\dagger(D^{-1})^\dagger = \text{Diag}[E_1, E_2, ..., E_n] \tag{4d} \]

Upon comparing last two equations, we get \( D^{-1}\eta^{-1} = D^\dagger \) and \( \eta D = (D^{-1})^\dagger \) which imply \( \eta = (DD^\dagger)^{-1} \). □

When \( H \) is Hermitian, \( D \) will be unitary and we get \( \eta = I \) as a special case. In general, \( D \)
will be pseudo-unitary: \( D^\dagger = \delta D^{-1} \delta^{-1} \) \([8,15]\). w.r.t. some metric \( \delta \) which may not be same as \( \eta \).

**Proof (Converse):** Let

\[
D^{-1}HD = \text{Diag}[E_1, E_2, E_3, \ldots, E_n]. \tag{5a}
\]

and

\[
(DD^\dagger)^{-1}H(DD^\dagger) = H^\dagger \tag{5b}
\]

\( \Rightarrow (D^\dagger)^{-1}(D^{-1}HD)D^\dagger = H^\dagger. \tag{5c} \)

\( \Rightarrow (D^\dagger)^{-1}(\text{Diag}[E_1, E_2, E_3, \ldots, E_n])D^\dagger = H^\dagger. \tag{5d} \)

By taking transpose-conjugate on both the sides, we have

\[
\Rightarrow D(\text{Diag}[E_1, E_2, E_3, \ldots, E_n])^\dagger D^{-1} = H. \tag{5e} \]

By left (right) multiplying by \( D \) \( (D^{-1}) \) on both the sides, we get

\[
\Rightarrow (\text{Diag}[E_1^*, E_2^*, E_3^*, \ldots, E_n^*]) = D^{-1}HD. \tag{5f} \]

Eq. (5e) and (5f) imply nothing but the reality of eigenvalues. \( \square \)

Similarly, when all the eigenvalues are complex conjugate and \( D \) is the diagonalizing arranged such that complex conjugate pairs remain together then it can be proved that \( \bar{\eta} = (DSD^\dagger)^{-1}, \)

where \( S \) is Pauli’s \( \sigma_x \), when \( H \) is \( 2 \times 2 \) otherwise when \( H \) is \( 2n \times 2n \), \( S \) is block-diagonal matrix: \( S = \text{Diag}[\sigma_x, \sigma_x, \sigma_x, \ldots, \sigma_x] \). We now denote and state thus obtained metric as

\[
\eta_+ = (DD^\dagger)^{-1}, \tag{6} \]

to actually see that the indefinite norm (2)

\[
N_{\eta^+} = \Psi^\dagger \eta_+ \Psi = \Psi^\dagger (DD^\dagger)^{-1} \Psi = \Psi^\dagger D^{-1}D^\dagger \Psi = (D^{-1}\Psi)^\dagger (D^{-1}\Psi) = \chi^\dagger \chi > 0. \tag{7} \]

is now positive definite. Finding eigenvalues, eigenvectors and diagonalizing matrix is a standard exercise. In that the theorem stated and proved above is indeed an attractive proposal to find the metric for a given complex non-Hermitian matrix admitting real eigenvalues under which it is pseudo-Hermitian. However, by multiplying the columns (rows) by arbitrary constants we can get many diagonalizing matrices say \( D_j \) and this would give rise to as many metrics say \( \eta_j \) under which \( H \) will be pseudo-Hermitian. For the sake of uniqueness, one may only use \( \eta \)-normalized (2) eigen-vectors to construct \( D \). Earlier, it has been proved that if a pseudo-Hermitian Hamiltonian, \( H \), has real eigenvalues then there exists and operator \( O \) such that \( H \) is pseudo Hermitian under: \( (OO^\dagger) \) \([10]\) and \( (OO^\dagger)^{-1} \) \([12]\). Another, form for \( \eta_+ \) in terms of the eigenvectors has also been proposed \([18]\).
IV. CONSTRUCTION OF C.P.T AND PROPOSAL OF AN INNER PRODUCT

When pseudo-Hermitian Hamiltonian (1) has real eigenvalues, we have [8]

\[ H \Psi_N = E_n \Psi, \quad H^\dagger \Phi_n = E_n \Phi_n, \quad \Phi = \eta \Psi, \]  

(8)

(\Psi_n, \Phi_n) are called bi-orthonormal basis and \( \Phi = \eta \Psi \). We have also witnessed in the example above (3) that several metrics could be obtained under which a given \( H \) is pseudo-Hermitian. Let us stress that this interesting practical experience remains elusive in several formal definitions. Let us examine the properties of the metrics obtained in (3). The metric \( \eta_1 \) is involutary \((U^2 = 1)\). The metrics \( \eta_1, \eta_3, \eta_4 \) are Hermitian, unitary and simple \((\text{det}U = 1)\). The metrics \( \eta_3, \eta_4 \) are real-symmetric. The metric \( \eta_2 \) very importantly is non-Hermitian in general. The metrics \( \eta_1, \eta_4 \) are (constant) disentangled with the elements of \( H \) and we call them as \textit{secular} [15]. It will be very interesting to investigate whether or not one can always find an involutary and \textit{secular} metric for an arbitrary pseudo-Hermitian matrix. The interesting exposition [10] that most of the known PT-symmetric Hamiltonians are actually P-pseudo-Hermitian is very valuable in order to connect pseudo-Hermiticity with P and T and hence to possible physical situations [15]. Once, the involutary metric is found it will be fixed for the definition of orthonormality (2) and we will assume it to represent the generalized P. This \textit{ad-hoc} strategy also seems to have been adopted in [16]. Therefore, the question of a definition to construct P again, from the bi-orthonormal basis \((\Psi, \Phi)\) either does not arise or will yield \( P=\eta \), eventually.

Here, one very important remark is in order: in the recent works on pseudo-Hermiticity, the indefiniteness of the \( \eta \)-norm (or orthonormality) has not been realized and this has given rise to an assumption that \textit{somehow} \( \Phi_+^\dagger \Psi_+ \) is positive-definite (e.g., Eqs. (11,12) in [10], Eqs. (5,6) in [12], Eq. (7) in [13]). Consequently, representations of \( I \) (the completeness) in terms of \((\Psi, \Phi)\), for instance, for two level matrix Hamiltonian, has been given as \((\Psi_0 \Phi_0^\dagger + \Psi_1 \Phi_1^\dagger)\). Though, known earlier [3-9], however, the indefiniteness of the norm is centrally consequent to the novel identification of charge-conjugation symmetry by Bender et al.[16].

Thus having fixed \( \eta \) for \( H \), we find \( \eta \)-normalized (2) eigenvectors \( \Psi_n \). These, normalized eigenvectors are used to construct the diagonalizing matrix \( \mathbf{D} \) and \( \eta_+ \) (6) which are unique only under the fixed \( \eta \). We obtain another basis \( \{ \Upsilon_n \} \) as

\[ \Upsilon_n = \eta_+ \Psi_n, \]  

(9)

which by construction (see (7)) is such that
\[ \Psi_m^\dagger \Upsilon_n = \delta_{m,n}. \]  

In the spirit of [18], we propose to construct \( P \) as

\[ P = \sum_{n=0}^{N} (-1)^n \Psi_n \Psi_n^\dagger, \]  

such that \( P \Upsilon_n = (-)^n \Psi_n \), implying that neither of \( \Psi_n, \Upsilon_n \) are the eigenstates of parity as it should be. We define the anti-linear time-reversal operator \( T \) as

\[ T = \left( \sum_{n=0}^{N} \Upsilon_n \Upsilon_n' \right) K_0 \]  

such that \( T \Psi_n = \Upsilon_n \) and we further have

\[ PT = \left( \sum_{n=0}^{N} (-)^n \Psi_n \Upsilon_n' \right) K_0, \]  

such that \( PT \Psi_n = (-)^n \Psi_n \). We adopt the definition of \( C \) as proposed in [18]

\[ C = \sum_{n=0}^{N} (-1)^n \Psi_n \Upsilon_n^\dagger, \quad \text{where} \quad \sum_{n=0}^{N} \Psi_n \Upsilon_n^\dagger = 1 \]  

such that \( C \Psi_n = (-)^n \Psi \). Next using (13) and (14) the symmetry operator \( CPT \) takes the form

\[ CPT = \left( \sum_{n=0}^{N} \Psi_n \Upsilon_n' \right) K_0, \]  

such that \( CPT \Psi_n = \Psi_n \). The following involutions

\[ (CPT)^2 = (PT)^2 = C^2 = 1 \]  

always hold. However, we get

\[ T^2 = P^2, \quad \text{iff} \quad (-)^{m+n} \Psi_m^\dagger \Psi_n = \Upsilon_m^\dagger \Upsilon_n. \]  

When the Hamiltonian is Hermitian, \( P \) and \( T \) have been proved to be involutary [20]. However, for pseudo-Hermitian Hamiltonian this becomes conditional. In Eq. (87) of [18], the above condition is suggested to be ensuring that \( P \) and \( T \) are involutary. Let us remark that this condition only ensures that \( P^2 = T^2 \). Further, since we choose \( P \) to be involutary and so will \( T \) be. We find that the following commutation relations

\[ [H, C] = [H, PT] = [H, CPT] = 0, \quad \text{and} \quad [H, P] \neq 0 \neq [H, P] \]
displaying the invariance and non-invariance of the Hamiltonian. We now define a X-inner product as

\[(X\Psi_m)^\dagger \Upsilon_n = (X\Psi_m)^\dagger \eta^+_n \Psi_n = \epsilon_n \delta_{m,n},\]  

(19)

where \(\epsilon_n (= \pm 1)\) is indefinite. Consequently, the X-norm as

\[N_{X,n} = (X\Psi_n)^\dagger \Upsilon_n = (X\Psi_n)^\dagger \eta^+_n \Psi_n.\]  

(20)

Here \(X\) represents the symmetry operators such as C, PT, and CPT constructed above, such that \([H, X] = 0\). Since \(X\Psi_n = \epsilon_n \Psi\), \(\epsilon_n\) is real, the X-inner product in view of (7) will be real-definite.

V. EXAMINATION OF THE OTHER CURRENT APPROACHES

Let us examine the inner products defined in [16] and [18]. The inner product (Eqs.(5,12,22) in [16]) in our notations reads as

\[(X\Psi_m)^\dagger \Psi_n,\]  

(21)

which is not real-definite in general, noting the fact that \(\Psi_n\) are eigenvectors over a complex field (the elements of these vectors are complex). The same shortcoming of not being real-definite applies to the inner product analysed and proposed in (Eq. (75) in [18]) which would read as

\[(X\Upsilon_m)^\dagger \Psi_n\]  

(22)

We have earlier [20] proved and illustrated that the definition of the inner product (21) [16] does not let the energy-eigenstates of the Hermitian \(H\) to be orthogonal. We would like to claim that our definition of the X-inner product proposed here is most general and consistent so far [3-9,16,18-20], for the PT-symmetric or pseudo-Hermitian Hamiltonians.

Let us now appreciate how despite the inner product (21) not being real-definite in general, the physically intriguing and also consistent claims of C, PT, and CPT invariance of \(H\) and definiteness of CPT-norm could have been made. The eigenvectors of a pseudo Hermitian matrix are naturally \(\eta\)-orthogonal (2) let us remark that \(H\) in [16] is pseudo-Hermitian under \(\eta = \sigma_x\), which has been chosen to be P. In fact, \(H\) (Eq.(14) in [16]) is a special example, where the elementary \((\psi_n)\) eigenvectors are also \textit{incidentally} orthogonal.
as $\psi_0^\dagger \psi_1 = 0$, in addition to the $\eta$-orthogonality: $\psi_0^\dagger \eta \psi_1 = 0$. Therefore, the concept and method of pseudo-Hermiticity which promises generality could be relaxed here [16]. Next, these eigenvectors are to be multiplied by suitable factors to obtain the relevant useful basis, $\{\Psi_n\}$, such that $P T \psi_n = \eta K_0 \Psi_n = (-)^n \Psi_n$. We would like to add one more such instance. where this method could succeed again is the following

$$H = \begin{bmatrix} a-c & ib \\ ib & a+c \end{bmatrix}, \quad \eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P, \quad \psi_0 = \begin{bmatrix} 1 \\ -i r \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} 1 \\ -i/r \end{bmatrix}, \quad (23)$$

where, we again have $\psi_0^\dagger \psi_1 = 0$, besides the $\eta$-orthogonality (2). The eigenvalues are $E_{0,1} = a \pm \sqrt{c^2-b^2}$, $r = \frac{c+\sqrt{c^2-b^2}}{b}$ these are real as long as $c^2 > b^2$. The illustrations $I_1, I_2$ given below are also aimed at citing examples where the approach taken in [16] does not work. It is, however, worth mentioning that the prescriptions suggested in Section IV, which are in keeping with the spirit of the approach in [18] sans the inner-product defined there and T, works for both the examples: one in [16] and the other discussed above in (23). The most notable failure of the approach in [16] has already been reported in [20] when it is applied back to Hermiticity.

VI. ILLUSTRATIONS

The definitions for the construction of $P, T, C$, though general, certain features can still not be proved. For instance whether $C$ and $P$ will always not commute. Whether $P$ and $T$ will always commute. When a complex (non-Hermitian) matrix Hamiltonian having real eigenvalues has $P$, which is not involutary will we get an involutary $T$? In this regard, simple doable examples are desirable. In the following we present two illustrations to throw some more light for the un-answered questions stated here.

Without loss of generality, we take $2 \times 2$ matrix Hamiltonians [15] and construct $P, T, C$ as per Eqs. (11), (12) and (14) as

$$P = \Psi_0 \Psi_0 - \Psi_1 \Psi_1, \quad T = (\Upsilon_0 \Upsilon_0^\dagger + \Upsilon_1 \Upsilon_1^\dagger) K_0, \quad C = \Psi_0 \Upsilon_0 - \Psi_1 \Upsilon_1, \quad (24)$$

for short. In illustration : $I_1$, we take up the same Hamiltonian as given in (3), here the fundamental metric ($P$) is involutary and in illustration : $I_2$, it is kept non-involutary.

$I_1$:
We take pseudo-Hermitian Hamiltonian, $H$, and the fundamental metric, $\eta (= \eta_1)$, from (3). The $\eta$-normalized eigenvectors are
\[ \Psi_0 = \sqrt{\frac{2}{r}} \begin{bmatrix} -i/r \\ 1 \end{bmatrix}, \quad \Psi_1 = \sqrt{\frac{2}{r}} \begin{bmatrix} 1/r \\ -i \end{bmatrix} \]  

(25)

One can readily check that \( \Psi_0^\dagger \eta \Psi_1 = 0 \), but \( \Psi_0^\dagger \Psi_1 = -\frac{i}{2r} \) is not for the approach [16] to work here. Following section IV, we obtain \( P, T, \) and \( \eta_+ \) as

\[ P = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} K_0, \quad \eta_+ = \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix} K_0, \]  

(26)

Notice that \( P \) turns out to be the same as \( \eta_1 \)-the chosen fundamental metric. The symmetry operators \( C, PT \) and \( CPT \) are

\[ C = \begin{bmatrix} 0 & -i/r \\ ir & 0 \end{bmatrix}, \quad PT = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} K_0, \quad CPT = \begin{bmatrix} 0 & -i/r \\ -ir & 0 \end{bmatrix} K_0, \]  

(27)

The symmetry operator \( C \) could be checked to be identical to \( \eta_1 \eta_3^{-1} \). (see (3)), demonstrating how two distinct metrics combine to yield a hidden symmetry of the Hamiltonian. In addition to the general results stated above, we get \( (CP)^{-1} = PC = \eta_+ \), in actual \( CPT \)-invariance \( C, P \) do commute [17]. We also confirm the commutation of \( P \) and \( T \) and the involutions \( T^2 = P^2 = 1 \). Similar, experience can be had by studying the model of [16] and (23).

Interestingly, the fundamental metrics in all these cases are the Pauli’s matrices which are involutory, Hermitian, unitary, simple and also \emph{secular}.

\( \mathbf{I}_2 \):

In the following, let us now take an example where the fundamental metric is only Hermitian and \emph{secular} as it does not affect the eigenvalues: \( E_{0,1} = \frac{1}{2} [(a + b) \pm \sqrt{(a - b)^2 + 4e^2}] \). We introduce \( \theta = \frac{1}{2} \tan^{-1} \frac{2e}{a - b} \).

\[ H = \begin{bmatrix} a & -ic/x \\ icx & b \end{bmatrix}, \quad \eta = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}, \quad \Psi_0 = \sqrt{x} \begin{bmatrix} \cos \theta/x \\ i \sin \theta \end{bmatrix}, \quad \Psi_1 = \frac{1}{\sqrt{x}} \begin{bmatrix} i \sin \theta \\ x \cos \theta \end{bmatrix}, \]  

(28)

Check that the states are only \( \eta \)-orthogonal and the condition \( \Psi_0^\dagger \Psi_1 = i \sin \theta (1 + x^2)/x \neq 0 \) like in \( \mathbf{I}_1 \) and unlike in Section V, is not met here. We construct \( P, T, C \) as

\[ P = \begin{bmatrix} \cos 2\theta & -i \sin 2\theta \\ i \sin 2\theta & -x \cos 2\theta \end{bmatrix}, \quad T = \begin{bmatrix} x \cos 2\theta & i \sin 2\theta \\ i \sin 2\theta & \cos 2\theta \end{bmatrix} K_0, \quad C = \begin{bmatrix} \cos 2\theta & -i \sin 2\theta \\ i x \sin 2\theta & -\cos 2\theta \end{bmatrix}, \]  

(29)

and \( \eta \) is returned as \( \eta_+ \). Very interestingly, \( P \) is different from the fundamental metric \( \eta \). Since this fundamental metric is definite giving \( \Psi_n^\dagger \eta \Psi_n = +1 \), the construction of \( \eta_+ \).
as per (6) yields it back. Unlike other examples here we have $T^2 \neq P^2 \neq 1$, whereas the results (16) are met. We find that $P$ and $T$ commute; $C$ and $P$ do not commute. We get $PC \neq (CP)^{-1} = \eta_+ = \eta$. When $x = 1$, the scenario for Hermiticity can be observed.

VII. CONCLUSIONS

The theorem stated and proved in section III adds an important result in matrix algebra [8] for constructing a metric(s) $\eta_+ = (DD^\dagger)^{-1}$ (6) where $D$ is the diagonalizing matrix for the pseudo-Hermitian matrix which has real eigenvalues. The proven positive definiteness (7) of this metric is of utility while constructing the generalized $P$, $T$, $C$ and an inner product for a matrix-Hamiltonian which possesses a real spectrum.

If $X$ is a symmetry operator for the Hamiltonian $H$, i.e. $[X, H] = 0$ then the proposed definition of the inner product as $\langle X\Psi|\eta_+\Psi \rangle$ (19) or even $\langle X\Psi|\eta\Psi \rangle$ is the most general definition proposed so far [3-9,16,18-20] when Hamiltonians are PT-symmetric or $\eta$-pseudo-Hermitian.

We have examined the approach in [16] to be too simple to work in general. The approach in [18], sans its inner product, is found to be correct and more general. However, our modification of the definition of $T$ makes it compatible with the proposed indefiniteness of PT-norm and definiteness of CPT-norm [16]. The examples using several matrix Hamiltonians drawn from our recent [15] studies on pseudo-Hermiticity have illustrated various contentions explicitly. The works using non-matrix Hamiltonians and yet making similar claims could be desirable further.

Admittedly, the only properties possessed by $C$, $PT$, and CPT are their involutions (16) various commutations (18) inner product (19), to strike their correspondance with the actual $C, PT, \lnot\lnot CPT$ of Hermitian field theory. Much deeper connections and agruments would be required to make claims a la the conventional $CPT$ invariance [17]. One point that requires emphasis is that in pseudo-Hermiticity, we are able to construct only three distinct involutary operators, which we have designated as $P$, $T$ and $C$ as against the conventional $\mathcal{P}, \mathcal{T}, C$ [17]. In this regard, our matrix Hamiltonians could be useful for further refinements in the theory of $C$, $PT$, and CPT invariance. Also these may be taken as toy models of a futuristic pseudo-Hermitian field theory.
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