THE HARTREE AND HARTREE-FOCK EQUATIONS IN LEBESGUE $L^p$ AND FOURIER-LEBESGUE $\hat{L}^p$ SPACES

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Abstract. We establish some local and global well-posedness for Hartree-Fock equations of $N$ particles (HFP) with Cauchy data in Lebesgue spaces $L^p \cap L^2$ for $1 \leq p \leq \infty$. Similar results are proven for fractional HFP in Fourier-Lebesgue spaces $\hat{L}^p \cap L^2$ ($1 \leq p \leq \infty$). On the other hand, we show that the Cauchy problem for HFP is mildly ill-posed if we simply work in $\hat{L}^p$ ($2 < p \leq \infty$). Analogue results hold for reduced HFP. In the process, we prove the boundedness of various trilinear estimates for Hartree type non linearity in these spaces which may be of independent interest. As a consequence, we get natural $L^p$ and $\hat{L}^p$ extension of classical well-posedness theories of Hartree and Hartree-Fock equations with Cauchy data in just $L^2$-based Sobolev spaces.

1. Introduction

1.1. Motivation and physical context. The Hartree-Fock equation (HFE), defined in (1.1), is a key effective equation of quantum physics. It plays a role similar to that of the Boltzmann equation in classical physics. The HFE describes large systems of identical fermions by taking into account the self-interactions of charged fermions as well as an exchange term resulting from Pauli’s principle. A semirelativistic version of the HFE was developed in [16] for modeling white dwarfs. The HFE model [26] leads to the Kohn-Sham equation underlying the density functional theory which is exceptionally effective in computations in quantum chemistry and in particular, of the electronic structure of matter. The HFE is used for several applications in many-particle physics [28]. For detailed background and recent developments on HFE and beyond, we refer to the excellent survey [11] and the references therein.

In [24] fractional Laplacians have been applied to model physical phenomena. This was formulated by Laskin [24] as a result of extending the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. Specifically, when $\alpha = 1$, the fractional Hartree equation, defined in (1.2), can be used to describe the dynamics of pseudo-relativistic boson stars in the mean-field limit, and when $\alpha = 2$ the Lévy motion becomes Brownian motion. The Hartree equation also arises in the nonlinear optics of nonlocal, nonlinear optical media [29].

1.2. Hartree-Fock equations. The Hartree-Fock equations of $N$ particles is given by

$$
\begin{cases}
    i\partial_t \psi_k - (-\Delta)^{\alpha/2} \psi_k + \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^{\gamma}} \ast |\psi_l|^2 \right) \psi_k - \kappa \sum_{l=1}^{N} \psi_l \left( \frac{e^{-a|x|}}{|x|^{\gamma}} \ast (\overline{\psi_l} \psi_k) \right) = 0, \\
    \psi_k|_{t=0} = \psi_{0,k}
\end{cases}
$$

(1.1)

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where $a \geq 0$, $t \in \mathbb{R}$, $\psi_k : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$, $k = 1, 2, \ldots, N$, $0 < \gamma < d$, $\kappa$ is constant, and $\ast$ denotes the convolution in $\mathbb{R}^d$. The fractional Laplacian is defined as

$$\mathcal{F}[(-\Delta)^{\alpha/2}u](\xi) = (2\pi)^{\alpha}|\xi|^{\alpha}\mathcal{F}u(\xi), \quad 0 < \alpha < \infty$$

where $\mathcal{F}$ denotes the Fourier transform. The Hartree factor

$$H_\psi = \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^{\gamma}} \ast |\psi_l|^2 \right)$$

describes the self-interaction between charged particles as a repulsive force if $\kappa > 0$, and an attractive force if $\kappa < 0$. In $H_\psi$ the cases $a = \gamma = 1$ and $a = 0, \gamma = 1$ corresponds to, well-known, Yukawa and Coulomb potentials respectively. The last term on the left side of (1.1) is the so-called “exchange term (Fock term)”

$$F_\psi(\psi_k) = \kappa \sum_{l=1}^{N} \psi_l \left( \frac{e^{-a|x|}}{|x|^{\gamma}} \ast (\overline{\psi_l} \psi_k) \right)$$

which is a consequence of the Pauli principle and thus applies to fermions. In the mean-field limit ($N \to \infty$), this term is negligible compared to the Hartree factor. In this case, (1.1) is replaced by the $N$ coupled equations, the so-called reduced Hartree-Fock equations:

$$i\partial_t \psi_k - (-\Delta)^{\alpha/2} \psi_k + \kappa \sum_{l=1}^{N} \left( \frac{e^{-a|x|}}{|x|^{\gamma}} \ast |\psi_l|^2 \right) \psi_k = 0, \quad \psi_k|_{t=0} = \psi_{0,k}. \quad (1.2)$$

In particular, when $a = 0, N = 1$, and $\alpha = 2$, (1.2) is the classical Hartree equation. We denote by (#) either (1.1) with $N \geq 2$ or (1.2) with $N \geq 1$.

Fröhlich-Lenzmann [16, Theorem 2.1] proved that (#) with Coulomb type self-interaction is locally well-posed in $H^s(\mathbb{R}^3)$ ($s \geq 1/2$). Moreover, they [16, Theorem 2.2] proved global existence for sufficiently small initial data. Carles-Lucha-Moulay [7, Section IV] studied global well-posedness of (1.1) for Coulomb type self-interaction and with an external potential, and obtained some $H^s(\mathbb{R}^3)$ regularity. Lenzmann [25, Theorems 1, 2 and 3] proved some local and global well-posedness for Hartree equation with Yukawa type self-interaction in $H^s(\mathbb{R}^3)$ with $s \geq 1/2$. See also [12, 13, 31].

Thus most authors have studied well-posedness for the Cauchy problem of (#) in $L^2$-based Sobolev spaces. This is of course very crucial from the physical point of view. Also from the mathematical point of view, the major reason behind this is the fact that the free Schrödinger propagator $U(t) := e^{it\Delta} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ if and only if $p = 2$. This raises a natural question: Can we expect well-posedness theory for (#) in function spaces—which are not just based on $L^2$-integrability? The fantastic progress has been made for this in the last decade. In fact, Zhou [34] proved some well-posedness for nonlinear Schrödinger equation (NLS) in some $L^p$-Sobolev spaces for $p < 2$. Then local well-posedness for Hartree equation was studied in [19–22] with Cauchy data in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 \leq p \leq 2$ (cf. [5, 30, 33]). The method of proofs employed in these works heavily rely on the Zhou spaces [34] functional frame work—which is interesting but quite technical in itself. And there is a subtle flaw in the proof in [21], see Remark 1.2 (2) for detail. Besides, this raises question: can we extend these works in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $2 < p \leq \infty$? This has been widely open question since the pioneering work of Zhou for NLS in [34]. The novelty of this paper is that we could establish local well-posedness for (#) in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ without using Zhou
spaces functional framework. This new approach simplifies the method of proofs and enables us to cover wider range $1 \leq p \leq \infty$. Theorem 1.1 thus completes the study of local well-posedness for Hartree equation in Lebesgue spaces $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$. Specifically, we have following theorem.

**Theorem 1.1** (Local well-posedness in $L^p \cap L^2$). Let $\gamma$ satisfy one of the followings
- $0 < \gamma < \min\{1, 2d(\frac{1}{p} - \frac{1}{2})\}$ for $1 \leq p \leq \frac{4}{3}$
- $0 < \gamma < \min\{1, \frac{4}{3}\}$ for $\frac{4}{3} \leq p \leq \infty$.

Assume that $\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N}) \in (L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N$ and $\alpha = 2$. Then there exists $T > 0$ and a unique local solution $(\psi_1, \ldots, \psi_N)$ to (#) such that

$$(U(-t)\psi_1(t), \ldots, U(-t)\psi_N(t)) \in (C([0,T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N).$$

The linear counterpart problem of (#) (free Schrödinger equation) is ill-posed in $L^p(\mathbb{R}^d)$ for $p \neq 2$. But Theorem 1.1 reveals that after a linear transformation using the semigroup $U(-t)$ generated by the linear problem, (#) is locally well-posed in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We will give two different proofs of Theorem 1.1. In the first proof Zhou spaces will play no role, this contrasts with the proof given in [21] for the Hartree equation. In the second proof we make use of Zhou spaces to get the local existence. (Our approach differs from [21], see Remark 3.7 for detail.) In this case, local solutions enjoy some Zhou spaces regularity.

**Remark 1.2.**
1. The proof of Theorem 1.1 relies on a factorization formula for $U(t)$ stated in Lemma 2.6, trilinear estimates (Subsection 2.2), and Strichartz estimates. For the detailed proof strategy, see Remark 3.1.
2. Local well-posedness established for Hartree equation in [21, Theorem 1.3] in $L^p \cap L^2$ for $1 < p < 2$ used Zhou spaces approach together with some clever ideas. However, there is a flaw in the proof, see Remark 3.6 for detail. We overcome this issue by not decomposing Duhamel type operator $D_{a,\gamma}$, see (3.20). See the proof of Lemma 3.5 and Remark 3.7.
3. We do not know factorization formula for fractional Schrödinger propagator $e^{-it(-\Delta)^{\alpha/2}} (\alpha \neq 2)$, and so the analogue of Theorem 1.1 for $\alpha \neq 2$, remains an open question.

The local solutions of Theorem 1.1 can be extended globally, under an additional assumption on $\gamma$. Specifically, we have the following theorem:

**Theorem 1.3** (Global well-posedness in $L^p \cap L^2$). Let $\alpha = 2$ and $0 < \gamma < \min\{1, \frac{4}{3}\}$. Then the local solution to (#) given by Theorem 1.1 extends to a global one such that

$$(U(-t)\psi_1(t), \ldots, U(-t)\psi_N(t)) \in (C(\mathbb{R}, L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N).$$

Moreover, it follows that $(\psi_1(t), \ldots, \psi_N(t)) \in (C(\mathbb{R}, L^2(\mathbb{R}^d)))^N$ and that if $1 \leq p \leq 2$ then the global solution enjoys the following smoothing effect in terms of special integrability:

$$(\psi_1(t), \ldots, \psi_N(t))|_{(\mathbb{R}\setminus\{0\} \times \mathbb{R}^d)} \in (C(\mathbb{R} \setminus \{0\}, L^{p'}(\mathbb{R}^d)))^N.$$
Exploiting this mass conservation law, Proposition 2.4 below, Strichartz estimates, and blow-up alternative, we prove the above global existence.

**Remark 1.4.** The sign of $\kappa$ in Hartree and Fock terms determines the defocusing and focusing character of the nonlinearity. We shall see that this will play no role in our analysis, as we do not use the conservation of energy of $(\#)$ to achieve global existence. This contrasts with well-posedness scenarios in $H^{1/2}(\mathbb{R}^3)$. For example, in [16, Theorem 2.3] it is proved that radially symmetric data with negative energy lead to blow-up solutions in finite time for (1.2) with $\alpha = \gamma = 1$.

For higher order Hartree-Fock equation Carles et. al. in [7, Corollary 4.7] obtained propagation of $H^s(\mathbb{R}^3)$-regularity with $s \in \mathbb{N}$. We plan to address similar result in the spaces involving $L^p(\mathbb{R}^d)$-integrability.

We now turn our attention to the well-posedness of $(\#)$ in the Fourier-Lebesgue spaces $\hat{L}^p(\mathbb{R}^d)$ (with $1 \leq p \leq \infty$) defined by

$$\hat{L}^p(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : \| f \|_{\hat{L}^p} := \| \mathcal{F} f \|_{L^p} < \infty \}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We note that by Hausdroff-Young inequality, $L^p(\mathbb{R}^d) \subset \hat{L}^p(\mathbb{R}^d)$ for $1 \leq p \leq 2$ and $\hat{L}^p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for $2 \leq p \leq \infty$. We denote by $L^2_{rad}(\mathbb{R}^d)$, space of radial functions in $L^2(\mathbb{R}^d)$.

Now we state the following theorem:

**Theorem 1.5** (Local well-posedness in $\hat{L}^p \cap L^2$). Let

$$X = \begin{cases} 
\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a \geq 0, \alpha = 2, 0 < \gamma < \min\{2, \frac{d}{2}\}, p \in [1, \infty] \\
\hat{L}^p(\mathbb{R}^d) \cap L^2_{rad}(\mathbb{R}^d) & \text{if } a \geq 0, d \geq 2, \frac{2d}{d-1} < \alpha < 2, 0 < \gamma < \min\{\alpha, \frac{d}{2}\}, p \in [1, \infty] \\
\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a = 0, 0 < \alpha < \infty, 0 < \gamma < 2d\left(\frac{1}{2} - \frac{1}{p}\right), p \in (2, \infty] \\
\hat{L}^p(\mathbb{R}^d) & \text{if } a > 0, 0 < \alpha < \infty, 0 < \gamma < 2d\left(\frac{1}{2} - \frac{1}{p}\right), p \in (2, \infty].
\end{cases}$$

Assume that $(\psi_{0,1}, ..., \psi_{0,N}) \in X^N$. Then there exist $T > 0$ and a unique local solution $(\psi_1, ..., \psi_N)$ to $(\#)$ such that

$$(\psi_1(t), ..., \psi_N(t)) \in (C([0, T], X))^N.$$ 

Moreover, the map $(\psi_{0,1}, ..., \psi_{0,N}) \mapsto (\psi_1, ..., \psi_N)$ is locally Lipschitz from $\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $(C([0, T], \hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)))^N$.

The fractional Schrödinger propagator $U_\alpha(t) := e^{-it(-\Delta)^{\alpha/2}}$ is bounded in $\hat{L}^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$ (see Lemma 3.8). Hence, we do not need to transfer $(\#)$ using the semigroup $U_\alpha(t)$ to establish local existence in $\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. This contrasts with local solutions of Theorem 1.1. In [8, Proposition 3.3], Carles-Mouzaoui proved that the Hartree equation is locally well-posed in $\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $p = \infty$ and Bhimani [2, Proposition 4.5] proved this result for the fractional Hartree equation. Hyakuma [21, Theorem 1.8] proved local well-posedness for the Hartree equation in $\hat{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $2 \leq p \leq \infty$. In fact, he used Zhou spaces to prove local existence. The particular case of Theorem 1.5 establishes these results for any $1 \leq p \leq \infty$. In [20, Theorem 1.2] it is proved that the Hartree equation is locally well-posed in $\hat{L}^p(\mathbb{R}^d)$ ($2 \leq p < \infty$) if $2d(\frac{1}{2} - \frac{1}{p}) \leq \gamma < \min\{2, d\}$. The particular case of Theorem 1.5 reveals that this result is also true for any $2 \leq p \leq \infty$ and $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$ with Yukawa type self interaction.
Remark 1.6. (1) In Theorem 1.5 the radial symmetry assumption for initial data comes due to use of fractional Strichartz estimate (Proposition 2.3) in the proof.

(2) We have the local existence result (Theorem 1.5) in $\tilde{L}^p(\mathbb{R}^d)$ for $2 < p \leq \infty$, without any radial assumption on initial data, if $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$. The analogue of this result for $1 \leq p < 2$ is not known to us.

Theorem 1.7 (Global well-posedness in $L^2 \cap \tilde{L}^p$). Let

\[
X = \begin{cases}
\tilde{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) & \text{if } a \geq 0, \quad \alpha = 2, 0 < \gamma < \min\{2, \frac{d}{2}\}, \quad p \in [1, \infty] \\
\tilde{L}^p(\mathbb{R}^d) \cap L^2_{\text{rad}}(\mathbb{R}^d) & \text{if } a \geq 0, \quad d \geq 2, \quad \frac{2d}{d+1} < \alpha < 2, 0 < \gamma < \min\{\alpha, \frac{d}{2}\}, \quad p \in [1, \infty].
\end{cases}
\]

Assume that $(\psi_{0,1}, \ldots, \psi_{0,N}) \in X^N$. Then the local solution to (\#) given by Theorem 1.5 extends to a global one such that

\[
(\psi_1(t), \ldots, \psi_N(t)) \in \left(C(\mathbb{R}, X) \cap L^4_{\text{loc}}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d))\right)^N.
\]

Carles-Mouzaoui [8, Theorem 1.1] proved that the Hartree equation is globally well-posed in $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and Hyakuna [21, Theorem 1.9] generalized this result to $L^2(\mathbb{R}^d) \cap \tilde{L}^p(\mathbb{R}^d)$ for $2 \leq p < \infty$. On the other hand, Bhimani [2, Theorem 1.2] generalized the Carles-Mouzaoui result to the fractional Hartree equation in $L^2(\mathbb{R}^d) \cap \tilde{L}^\infty(\mathbb{R}^d)$. The particular case of Theorem 1.7 establishes these results for any $1 \leq p \leq \infty$.

Remark 1.8. (1) To extend the local existence (proved in $\tilde{L}^p(\mathbb{R}^d)$ for $\alpha \neq 2$) globally, first we prove that (\#) is globally well-posed (see Proposition 2.4) in $L^2_{\text{rad}}(\mathbb{R}^d)$ via Strichartz estimates for the fractional Schrödinger equation (see Proposition 2.3) – where we need initial data to be a radial, $\alpha \in (2d/(2d-1), 2)$ and dimension $d \geq 2$ (see [17, p.26-27]). Invoking Proposition 2.4, we get the global existence in $\tilde{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Thus we notice in the proof that, to take the advantage of Proposition 2.4, the hypothesis, initial data to be radial of Theorem 1.7 is necessary.

(2) The analogue for Theorem 1.7 without radial assumption on initial data remains an interesting open question.

(3) Due to lack of appropriate Strichartz estimates for the fractional Schrödinger equation with $\alpha > 2$, we do not know whether (\#) with $\alpha > 2$ is globally well-posed in $L^2(\mathbb{R}^d)$ and also whether the solution is in some mixed $L^p_{\text{loc}}(\mathbb{R}, L^{q(\gamma)}(\mathbb{R}^d))$ spaces (the analogue of Proposition 2.4). In view of this, the analogue of Theorem 1.7 for $\alpha > 2$ remains another open question.

Theorem 1.9 (Improved well-posedness in 1D). Let $\alpha = 2, 0 < \gamma < 1$, and

\[
X = \begin{cases}
L^p(\mathbb{R}) \cap L^2(\mathbb{R}) & \text{if } p \in (4/3, 2] \\
\tilde{L}^p(\mathbb{R}) \cap L^2(\mathbb{R}) & \text{if } p \in [2, 4].
\end{cases}
\]

Assume that $\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N}) \in X^N$. Then there exists a unique global solution of (\#) such that $(U(-t)\psi_1(t), \ldots, U(-t)\psi_N(t)) \in (C(\mathbb{R}, X))^N$ when $p \in (4/3, 2]$ and $(\psi_1(t), \ldots, \psi_N(t)) \in (C(\mathbb{R}, X))^N$ when $p \in [2, 4]$.

The proof of Theorem 1.9 relies on generalized Strichartz estimates (see Lemma 3.15) in 1D and following the strategy in [21]. Specifically, we shall see that this enables us to estimate the integral nonlinear part of (\#) (see (3.7) and (3.38)), and as a consequence we can improve the range of $\gamma$. 
Table 1. Results Summary

| $a$     | $\alpha$ | Space          | $p$            | $\gamma$       | Result          |
|---------|----------|----------------|----------------|----------------|-----------------|
| $[0, \infty)$ | $\{2\}$ | $L^p \cap L^2$ | $[1, \infty)$  | $(0, 1 \land \frac{d}{2})$ | global          |
|         |          |                | $[1, \frac{4}{3})$ | $(0, 1 \land 2d(\frac{1}{p} - \frac{1}{2}))$ | local           |
|         |          |                | $(\frac{4}{3}, 2)$ | $(0, 1)$       | global if $d = 1$ |
|         |          | $\tilde{L}^p \cap L^2$ | $[1, \infty)$  | $(0, 2 \land \frac{d}{2})$ | global          |
|         |          |                | $(2, \infty)$  | $(0, 2d(\frac{1}{2} - \frac{1}{p}))$ | local           |
|         |          |                | $(2, 4)$        | $(0, 1)$       | global if $d = 1$ |
| $(0, \infty)$ | $(0, \infty)$ | $\tilde{L}^p \cap L^2$ | $(2, \infty)$  | $(0, 2d(\frac{1}{2} - \frac{1}{p}))$ | local           |
|         |          |                | $(\frac{2d}{2d - 1}, 2)$ | $(0, 1)$       | global if $d \geq 2$ |
| $(0, \infty)$ | $(0, \infty)$ | $\tilde{L}^p$ | $(2, \infty)$  | $(0, 2d(\frac{1}{2} - \frac{1}{p}))$ | mild ill-posed |

We next show that (#) with Coulomb type potential shows a mild form of ill-posedness in the mere $\hat{L}^p(\mathbb{R}^d)^N$ ($2 < p \leq \infty$) spaces for $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$. Specifically, we have the following result:

**Theorem 1.10** (Failure of $C^3$—smoothness in $\hat{L}^p$ ). Let $a = 0$, $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$, $2 < p \leq \infty$ and fix $0 < t \leq T$. Denote the solution map of (#) by $U(t) : \psi_0 \mapsto \psi(t)$. Assume that $U(t)$ is well-defined as a map acting in $(\hat{L}^p(\mathbb{R}^d))^N$, then $U(t)$ is not $C^3$—smooth at $\psi_0 = 0$ in $(\hat{L}^p(\mathbb{R}^d))^N$.

The study of mild ill-posedness (failure of $C^3$—smoothness) for the Hartree-Fock equation is new as far as we are aware. This type of mild ill-posedness was initiated by Bourgain in [6] for KdV and mKdV, see also [32]. This essentially involves showing “unboundedness” of the third Picard iterate associated with (#), see (4.1) and Subsection 4.2. Since then, many authors have used this approach, see, for example, [14, Proposition 4.1] for cubic nonlinear half-wave equation. We exploit this approach to prove Theorem 1.10. See point (3) below for further comments on the proof.

Theorem 1.10 states that (#) experience a mild form of ill-posedness in the sense that the solution map fails to be $C^3$—smooth. However, we do not know whether the solution map fails to be continuous at origin in $(\hat{L}^p(\mathbb{R}^d))^N$. This remains an interesting open question. On the other hand, we could show that (#) is not quantitatively well-posed in $(\hat{L}^p(\mathbb{R}^d))^N$. See Subsection 4.2 for detailed discussion.

We summarize our findings in Table 1. We write $x \land y = \min\{x, y\}$.

1.3. Several comments on the paper.

(1) We define **trilinear operator** associated to Hartree-type nonlinearity by

$$\hat{H}_{a, \gamma, t}(f, g, h) = \left[\left(S_{a, t} \ast |\cdot|^{-d}\right)(f \ast \bar{g})\right] \ast h$$
where

\[ S_{a,t} = \begin{cases} \frac{\delta_0}{(4a^2t^2+|t|^2)^{(d+1)/2}} & \text{if } a = 0 \\ \frac{a|t|}{(4a^2t^2+|t|^2)^{(d+1)/2}} & \text{if } a > 0 \end{cases} \quad (1.3) \]

with \( t \in \mathbb{R} \) and \( \delta_0 \) being the Dirac distribution with mass at origin in \( \mathbb{R}^d \). We shall briefly discuss our main ideas and techniques to establish local well-posedness of (\#) in \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for \( 2 < p \leq \infty \):

- The proof of local well-posedness crucially depends on a suitable availability of \( \hat{H}_{a,\gamma,t} \) estimates. (See also Remark 3.1 below.) Specifically, we have following trilinear estimates (to be established in Subsection 2.2 below):

\[ \| \hat{H}_{a,\gamma,t}(f,g,h) \|_{L^p \cap L^2} \lesssim \| f \|_{L^p \cap L^2} \| g \|_{L^p \cap L^2} \| h \|_{L^p \cap L^2}, \quad \forall p \in [1,2). \quad (1.4) \]

- Using (1.4) one can establish local well-posedness in \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for \( 1 \leq p < 2 \). Up to now we cannot know the validity of (1.4) for \( p \in (2,\infty] \). Hence, we cannot follow previously employed ideas to deal with the case \( 2 < p \leq \infty \).

- To overcome the issue mentioned above, we introduced a ball \( \mathcal{V}_T^a \), (see Case I in the first Proof of Theorem 1.1 in Subsection 3.1) involving the “twisting” free Schrödinger propagator (unlike the usual choice). We then effectively use Strichartz estimates to establish the contraction of twisted Duhamel operator \( \Phi_{\psi_0} \) (see (3.5)) on \( \mathcal{V}_T^a \). This leads to local existence. The advantage of this approach is that, though we do not know (1.4) for \( 2 < p \leq \infty \), we could establish local well-posedness for \( 2 < p \leq \infty \).

In fact, this approach works for all \( p \in [1,\infty] \).

(2) It might be tempting to think that well-posedness for (\#) with Coulomb type self-interaction immediately implies well-posedness for (\#) with Yukawa type self interaction as \( e^{-a|\cdot|} |\cdot|^{-\gamma} \) (Yukawa type: \( a > 0 \)) has faster decay at infinity compared to \( |\cdot|^{-\gamma} \) (Coulomb type: \( a = 0 \)). However, it is not the case. The local existence in \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) heavily rely on factorization Lemma 2.6. Besides we shall notice that for \( a > 0 \) the trilinear operator \( \hat{H}_{a,\gamma,t} \) depends on a time parameter \( t \). On the other hand, \( \hat{H}_{a,\gamma,t} \) is independent of time parameter in the case of \( a = 0 \). Specifically, we cannot straight-way claim

\[ \| \hat{H}_{a,\gamma,t}(f,g,h) \|_{L^p \cap L^2} \leq c \| \hat{H}_{0,\gamma,t}(f,g,h) \|_{L^p \cap L^2} \]

and apply estimates for \( a = 0 \). This new time parameter makes analysis more delicate while dealing with Yukawa type self interaction. (Also, cf. Theorems 1.5 and 1.10 for \( a = 0 \) and \( a > 0 \)).

(3) In order to prove unboundedness of third Picard iterate in the proof of Theorem 1.10, we have carefully adopted the technique form [8] (where the unboundedness is proved for classical Hartree equation in \( \hat{L}^\infty \)). In (1.1) due to the presence of Fock term (exchange term) \( F_\psi(\psi_k) \), the computation of the third iterate is more subtle compared to the classical Hartree case and thus required careful analysis to prove Theorem 1.10. We also note that the proof of Theorem 1.10 relies on the fact that the Fourier transform of the Coulomb type potential is homogeneous. On the other hand, the Fourier transform of the Yukawa
type potential is not homogeneous, see Lemma 2.1. Therefore the proof does not work for Yukawa type potential.

(4) For \( s > 0 \) we can choose \( 2 < p \leq \infty \) so that \( s > d(\frac{1}{2} - \frac{1}{p}) \). In this case, by Hölder inequality, we have \( H^s(\mathbb{R}^d) \subset \tilde{L}^p(\mathbb{R}^d) \) and so \( H^s(\mathbb{R}^d) \subset \tilde{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \). Thus Theorems 1.5 and 1.7 reveals that we can solve (\#) with Cauchy data in the larger space \( \tilde{L}^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) compared to \( H^s(\mathbb{R}^d) \). In particular, this complements Lenzmann’s work [25, Theorems 1, 2 and 3] on Hartree equation with Yukawa type self interaction, especially for \( 0 < s < 1/2 \) in dimension 3 and for all \( s > 0 \) in other dimensions.

(5) We note that

\[
H^{1/2}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3) \cap L^2(\mathbb{R}^3).
\]

In [16, Theorem 2.3] Fröhlich-Lenzmann proved that radially symmetric data with negative energy lead to blow-up solutions for (\#) with \( \gamma = \alpha = 1 \) and \( a = 0 \) in finite time in \( H^{1/2}(\mathbb{R}^3) \)-norm. On the other hand, Theorem 1.7 ensures that (\#) with \( 3/5 < \alpha < 2, 0 < \gamma < \min\{\alpha, 3/2\} \) is globally well-posed in \( \tilde{L}^6(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \).

This paper is organized as follows. In Section 2, we introduce preliminaries and establish key ingredients which will be used in the sequel. Specifically, in Subsections 2.1 and 2.2 we prove factorization Lemma 2.6 and various trilinear estimates respectively. We shall see this will play a vital role in proving main results of the paper. In Subsections 3.1, 3.2 we prove Theorems 1.1 and 1.5 respectively. In Subsections 3.3, 3.4 we prove Theorems 1.3 and 1.7 respectively. In Subsection 3.5, we prove Theorem 1.9. In Section 4 we prove Theorem 1.10.

2. PRELIMINARIES AND KEY INGREDIENT

Notations and known results. For real numbers \( A, B \) the notation \( A \lesssim B \) means \( A \leq cB \) for some universal constant \( c > 0 \), whereas \( A \asymp B \) means \( c^{-1}A \leq B \leq cA \) for some \( c \geq 1 \). Also \( A \gtrsim B \) means \( B \lesssim A \). For complex \( A, B \) by \( A \asymp B \) means \( A = cB \) for some universal constant \( c \neq 0 \). The characteristic function of a set \( E \subset \mathbb{R}^d \) is \( \chi_E(x) = 1 \) if \( x \in E \) and \( \chi_E(x) = 0 \) if \( x \notin E \). Let \( I \subset \mathbb{R} \) be an interval and \( X \) be a Banach space of functions. Then the norm of the space-time Lebesgue space \( L^q(I, X) \) is defined by \( \|u\|_{L^q(I, X)} = \left( \int_I \|u(t)\|_X^q dt \right)^{1/q} \) and when \( I = [0, T], T > 0 \) we denote \( L^q(I, X) \) by \( L^q_T(X) \). For \( p \in [1, \infty] \), its Hölder conjugate, denoted by \( p' \), is given by \( \frac{1}{p} + \frac{1}{p'} = 1 \). The norm on \( N \)-fold product \( X^N \) of Banach space \( (X, \| \cdot \|_X) \) is given by

\[
\|\psi\|_{X^N} = \max_{1 \leq j \leq N} \|\psi_j\|_X, \quad \psi = (\psi_1, \cdots, \psi_N) \in X^N.
\]

The Schwartz space is denoted by \( \mathcal{S}(\mathbb{R}^d) \) (with its usual topology), and the space of tempered distributions is denoted by \( \mathcal{S}'(\mathbb{R}^d) \). For two Banach spaces of functions \( A, B \) in \( \mathcal{S}'(\mathbb{R}^d) \) we note that \( A \cap B \) is also a Banach space with the norm \( \| \cdot \|_{A \cap B} = \max\{\| \cdot \|_A, \| \cdot \|_B\} \). For \( x = (x_1, \cdots, x_d), y = (y_1, \cdots, y_d) \in \mathbb{R}^d \), we put \( x \cdot y = \sum_{i=1}^d x_i y_i \). Let \( \mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d) \) be the Fourier transform defined by \( \mathcal{F}_f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx, \xi \in \mathbb{R}^d \). Then \( \mathcal{F} \) is a bijection and the inverse Fourier transform is given by \( \mathcal{F}^{-1} f(x) = F^\gamma(x) = F f(-x) \) for \( x \in \mathbb{R}^d \), and this Fourier transform can be
uniquely extended to $\mathcal{F} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ such that for each $u \in S'(\mathbb{R}^d)$ one has $\langle \mathcal{F} u, \varphi \rangle = \langle u, \mathcal{F} \varphi \rangle$ for all $\varphi \in S(\mathbb{R}^d)$.

**Lemma 2.1.**

1. For $f(x) = e^{-2\pi|x|}$, we have $\hat{f}(\xi) = \frac{c_d \xi^d}{(1+|\xi|^2)^{(d+1)/2}}$ with $c_d = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma$ is the Gamma function and $\hat{f} \in L^p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$.

2. Let $0 < \gamma < d$. Then for $f(x) = |x|^{-\gamma}$, we have $\hat{f}(\xi) = \frac{C_d \xi^\gamma}{|\xi|^d}$. 

For $f \in S(\mathbb{R}^d)$, we define the **fractional Schrödinger propagator** $e^{-it(-\Delta)^{\alpha/2}}$ for $t \in \mathbb{R}, \alpha > 0$ as follows:

$$
[U_\alpha(t)f](x) = \left[ e^{-it(-\Delta)^{\alpha/2}} f \right](x) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{2\pi i \xi \cdot t} \hat{f}(\xi) d\xi. 
$$

(2.1)

For $\alpha = 2$, we simply write $U_2 = U$. In this case we have (see [9, Lemma 2.2.4])

$$
[U(t)f](x) = \left[ e^{it\Delta} f \right](x) = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{i|y|^2/4t} f(y) dy.
$$

(2.2)

**Definition 2.2.** A pair $(q, r)$ is $\alpha$-fractional admissible if $q \geq 2, r \geq 2$ and $\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{d}$, $(q, r, d) \neq (\infty, 2, 2)$.

**Proposition 2.3** (Strichartz estimates). Denote

$$
DF(t, x) = U_\alpha \phi(x) + \int_0^t U_\alpha(t-s)F(s, x)ds.
$$

1. Let $\phi \in L^2(\mathbb{R}^d), d \in \mathbb{N}$ and $\alpha = 2$. Then for any time interval $I \ni 0$ and $2$-admissible pairs $(q_j, r_j), j = 1, 2$, there exists a constant $C = C(r_1, r_2)$ such that

$$
\|D(F)\|_{L^q(I, L^{r_j}(\mathbb{R}^d))} \leq C\|\phi\|_{L^2} + C\|F\|_{L^{q_j}(I, L^r(\mathbb{R}^d))}, \quad \forall F \in L^{q_j}(I, L^r(\mathbb{R}^d))
$$

where $q_j$ and $r_j$ are Hölder conjugates of $q_j$ and $r_j$ respectively [23].

2. Let $\phi \in L^2_{rad}(\mathbb{R}^d), d \geq 2$, and $\frac{2d}{2d-1} < \alpha < 2$. Then for any time interval $I \ni 0$ and $\alpha$-fractional admissible pairs $(q_j, r_j), j = 1, 2$, there exists a constant $C = C(r_1, r_2)$ such that

$$
\|D(F)\|_{L^q(I, L^{r_j}(\mathbb{R}^d))} \leq C\|\phi\|_{L^2} + C\|F\|_{L^{q_j}(I, L^r_{rad}(\mathbb{R}^d))}, \quad \forall F \in L^{q_j}(I, L^r_{rad}(\mathbb{R}^d))
$$

where $q_j$ and $r_j$ are Hölder conjugates of $q_j$ and $r_j$ respectively [17, Corollary 3.10].

For the sake of completeness, we recall the following standard existence result. We shall see that this result will play a vital role to prove global existence (Theorems 1.3, 1.7, and 1.9). Specifically, we have the following:

**Proposition 2.4.** Let $\alpha > 0, 0 < \gamma < \min\{\alpha, d\}$ and

$$
X = \begin{cases} 
L^2(\mathbb{R}^d) & \text{if } \alpha = 2, d \geq 1 \\
L^2_{rad}(\mathbb{R}^d) & \text{if } \frac{2d}{2d-1} < \alpha < 2, d \geq 2.
\end{cases}
$$

If $(\psi_{0,1}, \ldots, \psi_{0,N}) \in X^N$ then $(\#)$ has a unique global solution

$$
(\psi_1, \ldots, \psi_N) \in \left(C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{4\alpha/\gamma}_{\text{loc}}(\mathbb{R}, L^{4d/(2d-\gamma)}(\mathbb{R}^d))\right)^N.
$$

In addition, its $L^2$-norm is conserved, $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \forall t \in \mathbb{R}, k = 1, 2, \ldots, N$ and for all $\alpha$-fractional admissible pairs $(q, r)$, one has $(\psi_1, \ldots, \psi_N) \in \left(L^q_{\text{loc}}(\mathbb{R}, L^r(\mathbb{R}^d))\right)^N$. 

Proof. For the proof of case $a = 0$, that is, (#) with Coulomb type potential, see [3, Propositions 4.2 and 4.3]. The proof of case $a > 0$, that is, (#) with Yukawa type potential is similar. Hence, we omit the details. See also [8, Proposition 2.3] and [7, Theorem 4.9].

2.1. Factorization formula for Schrödinger propagator. For $t \neq 0$, we define multiplication, dilation and reflection operators (for functions $w$ on $\mathbb{R}^d$) and their inverses as follows:

- multiplication: $M_t w(x) = e^{i|x|^2/4t}w(x), M_t^{-1} w(x) = e^{-i|x|^2/4t}w(x)$
- dilation: $D_t w(x) = \frac{1}{(4\pi it)^{d/2}} w\left(\frac{x}{4\pi t}\right)$ and $D_t^{-1} w(x) = (4\pi it)^{d/2} w(4\pi tx)$
- reflection: $R w(x) = w(-x)$ and $R^{-1} w(x) = w(-x)$.

Lemma 2.5. Let $0 \neq t \in \mathbb{R}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then we have $U(t)\varphi = M_t D_t \mathcal{F} M_t \varphi$ and $U(-t)\varphi = M_t^{-1} \mathcal{F}^{-1} D_t^{-1} M_t^{-1} \varphi$.

Proof. Follows from formula (2.2) and the above definitions, see [18, p.372].

Now, for $f, g, h \in \mathcal{S}(\mathbb{R}^d), a \geq 0, t \in \mathbb{R},$ and $0 < \gamma < d$, we define trilinear operators associated to Hartree-type nonlinearity as follows

$$\mathcal{H}_{a,\gamma}(f, g, h) = \left[ \frac{e^{-a|\cdot|}}{|\cdot|^\gamma} \ast (fg) \right] h, \quad \widehat{\mathcal{H}}_{a,\gamma,t}(f, g, h) = \left[ (S_{a,t} \ast |\cdot|^{-\gamma})(f \ast \bar{g}) \right] \ast h \quad (2.3)$$

where $S_{a,t}$ is as in (1.3). Now we decompose $\widehat{\mathcal{H}}_{a,\gamma,t}$ in the following way

$$\widehat{\mathcal{H}}_{a,\gamma,t}^j(f, g, h) := [(S_{a,t} \ast k_j)(f \ast \bar{g})] \ast h, \quad j = 1, 2 \quad (2.4)$$

where $k_1, k_2$ are given by

$$k_1(x) = \chi_{\{|x| \leq 1\}}(x)|x|^\gamma - d, \quad k_2(x) = \chi_{\{|x| > 1\}}(x)|x|^\gamma - d. \quad (2.5)$$

Note that $k_1 \in L^p(\mathbb{R}^d)$ for $1 \leq p < \frac{d}{d-\gamma}$, $k_2 \in L^q(\mathbb{R}^d)$ for $\frac{d}{d-\gamma} < q \leq \infty$ and $\widehat{\mathcal{H}}_{a,\gamma,t} = \widehat{\mathcal{H}}_{a,\gamma,t}^1 + \widehat{\mathcal{H}}_{a,\gamma,t}^2$.

Lemma 2.6. Let $0 \neq t \in \mathbb{R}, 0 < \gamma < d, a \geq 0,$ and $v_j(t) = U(-t) u_j(t) \in \mathcal{S}(\mathbb{R}^d)$ with $j = 1, 2, 3$. Then we have $U(-t)\mathcal{H}_{a,\gamma}(u_1, u_2, u_3) \propto |t|^{-\gamma} M_t^{-1} \mathcal{H}_{a,\gamma,t}(M_t v_1, R M_t v_2, M_t v_3)$.

Proof. For the case $a = 0$, see [21, Lemma 2.1]. So it remains to prove the case $a > 0$. Note that $D_t^{-1}(fg) = (4\pi it)^{-d/2}(D_t^{-1}f)(D_t^{-1}g), \mathcal{F}^{-1}D_t^{-1} = D_{-t}\mathcal{F} = cRD_t\mathcal{F}$ and $U(t)\bar{u} = U(-t)u$. Using these equalities and performing the change of variable, we may rewrite

$$D_t^{-1}\left[ (|\cdot|^{-\gamma}e^{-a|\cdot|}) \ast (fg) \right](x) = (4\pi it)^{d/2}\left( (|\cdot|^{-\gamma}e^{-a|\cdot|}) \ast (fg) \right)(4\pi tx)$$

$$= (4\pi it)^{d/2}(4\pi t)^d \int_{\mathbb{R}^d} |4\pi ty|^{-\gamma}e^{-4\pi |ty|}(fg)(4\pi t(x-y))dy$$

$$= (-4\pi it)^{d/2}(4\pi |t|)^{-\gamma}\left( (|\cdot|^{-\gamma}e^{-4\pi |\cdot|}) \ast (D_t^{-1}fD_t^{-1}g) \right)(x).$$
Using the above equalities and Lemma 2.5, we obtain
\[ M_t U(-t) \mathcal{H}_{a,\gamma}(u_1, u_2, u_3) = \mathcal{F}^{-1} D_t^{-1} M_t^{-1} \mathcal{H}_{a,\gamma}(u_1, u_2, u_3) \]
\[ = \mathcal{F}^{-1} D_t^{-1} \left( \left( \left( | \cdot |^{-\gamma} e^{-a|\cdot|} \right) * (M_t^{-1} u_1)(M_t \tilde{u}_2) \right) M_t^{-1} u_3 \right) \]
\[ \propto t^{-d/2} \mathcal{F}^{-1} \left[ D_t^{-1} \left( \left( | \cdot |^{-\gamma} e^{-a|\cdot|} \right) * (M_t^{-1} u_1)(M_t \tilde{u}_2) \right) D_t^{-1} M_t^{-1} u_3 \right] \]
\[ \propto |t|^{-\gamma} \mathcal{F}^{-1} \left[ \left( \left( | \cdot |^{-\gamma} e^{-4a|\cdot|} \right) * (D_t^{-1} M_t^{-1} u_1)(D_t^{-1} M_t \tilde{u}_2) \right) D_t^{-1} M_t^{-1} u_3 \right] \]
\[ \propto |t|^{-\gamma} \left[ \left( \left( | \cdot |^{-\gamma} * \mathcal{F}^{-1} e^{-4a|\cdot|} \right) \right) \left( (D_t^{-1} M_t^{-1} u_1) * (D_t^{-1} M_t \tilde{u}_2) \right) D_t^{-1} M_t^{-1} u_3 \right] \]

Since, by Lemma 2.1,
\[ \left( e^{-4a|\cdot|} \right)(\xi) = \frac{c_d}{(2a|\xi|)^d} \left( 1 + \frac{|\xi|^2}{4a^2t^2} \right)^{-(d+1)/2} \frac{a|t|}{(4a^2t^2 + |\xi|^2)^{(d+1)/2}} =: S_{a,t} \quad (a > 0), \]
it follows that
\[ M_t U(-t) \mathcal{H}_{a,\gamma}(u_1, u_2, u_3) \propto |t|^{-\gamma} \left[ \left( | \cdot |^{-\gamma} * S_{a,t} \right) \left( (M_t U(-t) u_1) * (R D_t F M_t \tilde{u}_2) \right) \right] * M_t U(-t) u_3 \]
\[ \propto |t|^{-\gamma} \left[ \left( | \cdot |^{-\gamma} * S_{a,t} \right) \left( (M_t U(-t) u_1) * (R M_t^{-1} U(t) \tilde{u}_2) \right) \right] * M_t U(-t) u_3 \]
\[ \propto |t|^{-\gamma} \left[ \left( | \cdot |^{-\gamma} * S_{a,t} \right) \left( (M_t U(-t) u_1) * (R M_t U(-t) \tilde{u}_2) \right) \right] * M_t U(-t) u_3 \]
\[ = |t|^{-\gamma} \widehat{\mathcal{H}}_{a,\gamma,t}(M_t v_1, R M_t v_2, M_t v_3). \]

This completes the proof. \( \square \)

2.2. Trilinear estimates. In this subsection we prove some useful trilinear estimates for \( \widehat{\mathcal{H}}_{a,\gamma,t} \) and \( \mathcal{H}_{a,\gamma} \) (see (2.3)). We start with the following:

**Lemma 2.7.** Assume \( 0 < \gamma < d \). Let \( k_j \) \( (j = 1, 2) \) and \( S_{a,t} \) be given by (2.5) and (1.3) respectively. Then we have \( \|k_1 * S_{a,t}\|_{L_1} \lesssim \|k_1\|_{L_1} \) and \( \|k_2 * S_{a,t}\|_{L_2} \lesssim \|k_2\|_{L_2} \) for all \( r_1 \in [1, \frac{d}{d-\gamma}] \) and for all \( r_2 \in (\frac{d}{d-\gamma}, \infty]. \)

**Proof.** The case \( a = 0 \) being trivial assume that \( a > 0 \). Note that for \( d = 1 \), we have
\[ \|S_{a,t}\|_{L_1} = \int_{\mathbb{R}} \frac{a|t|}{4a^2 t^2 + |\xi|^2} d\xi \propto \frac{1}{a|t|} \int_0^\infty \frac{1}{1 + (r/2a|t|)^2} dr \int_0^\infty \frac{1}{1 + s^2} ds \propto 1. \]
For \( d \geq 2 \), we obtain
\[ \|S_{a,t}\|_{L_1} \propto \int_0^\infty \frac{a|t|r^{d-1}}{(4a^2 t^2 + r^2)^{(d+1)/2}} dr \propto a|t| \int_0^\infty \frac{(s - 4a^2 t^2)^{(d-2)/2}}{s^{d+1/2}} ds \leq a|t| \int_0^\infty \frac{s^{(d-2)/2}}{s^{d+1/2}} ds = 1. \]
Now Young inequality, gives the desired inequalities. \( \square \)

**Remark 2.8.** Note that we separate the computation of \( L^1 \)-norm for \( S_{a,t} \) in two cases as the third step in the proof of case \( d \geq 2 \) does not work for \( d = 1 \).

**Proposition 2.9** \( (L^p \text{-estimates}). \) Let \( 0 < \gamma < d \), \( f_j \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) \( (j = 1, 2, 3) \) and \( \widehat{\mathcal{H}}_{a,\gamma,t}, \widehat{\mathcal{H}}_{a,\gamma,t}^2 \) are given by (2.3) and (2.4) respectively.
(1) Assume that $1 \leq p < 2$ and $0 < \gamma < 2d(\frac{1}{p} - \frac{1}{2})$. Then we have

$$||\hat{\mathcal{H}}^j_{a,\gamma,t}(f_1, f_2, f_3)||_{L^2} \lesssim \begin{cases} ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^2} & \text{if } j = 1 \\ ||f_1||_{L^p} ||f_2||_{L^p} ||f_3||_{L^p} & \text{if } j = 2 \end{cases}$$

and

$$||\hat{\mathcal{H}}^j_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p} \lesssim \begin{cases} ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^p} & \text{if } j = 1 \\ ||f_1||_{L^p} ||f_2||_{L^p} ||f_3||_{L^p} & \text{if } j = 2 \end{cases}$$

As a consequence, we have $||\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p \cap L^2} \lesssim \prod_{j=1}^3 ||f_j||_{L^p \cap L^2}$.

(2) Assume that $2 < p \leq \infty$ and $0 < \gamma < d(\frac{1}{2} - \frac{1}{p})$. Then we have $||\hat{\mathcal{H}}_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p} \lesssim ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^p \cap L^2}$.

**Proof.** (1) By Young, Hölder, Hausdorff-Young inequalities and Lemma 2.7, for $1 \leq p \leq \infty$, we have

$$||(k_1 * S_{a,t})(f_1 * f_2)) * f_3||_{L^p} \leq ||(k_1 * S_{a,t})(f_1 * f_2)||_{L^1} ||f_3||_{L^p} \leq ||k_1 * S_{a,t}||_{L^1} ||f_1 * f_2||_{L^\infty} ||f_3||_{L^p} \leq ||k_1 * S_{a,t}||_{L^1} ||f_1||_{L^\infty} ||f_2||_{L^p} ||f_3||_{L^p} \lesssim ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^p}.$$  

Since $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1 + \frac{1}{p'/(2-p)}$ and $\frac{p}{2(p-1)} > \frac{d}{d-\gamma}$, Hölder and Young inequalities and Lemma 2.7 imply

$$||\hat{\mathcal{H}}^2_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p} = ||(k_2 * S_{a,t})(f_1 * f_2)) * f_3||_{L^p} \leq ||(k_2 * S_{a,t})(f_1 * f_2)||_{L^1} ||f_3||_{L^p} \lesssim \prod_{l=1}^3 ||f_l||_{L^p}.$$  

Similarly, $||\hat{\mathcal{H}}^2_{a,\gamma,t}(f_1, f_2, f_3)||_{L^2} \leq ||(k_2 * S_{a,t})(f_1 * f_2)) * f_3||_{L^2} \leq ||f_1||_{L^p} ||f_2||_{L^p} ||f_3||_{L^2} \lesssim ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^2} \cdot$

(2) Since $\frac{1}{p} + \frac{1}{p'} = 1 + \frac{1}{(2p)/(p+2)}$ and $\frac{2p}{p+2} > \frac{d}{d-\gamma}$, Young inequality and Lemma 2.7 give

$$||\hat{\mathcal{H}}^2_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p} = ||(k_2 * S_{a,t})(f_1 * f_2)) * f_3||_{L^p} \leq ||(k_2 * S_{a,t})(f_1 * f_2)||_{L^p} \lesssim \prod_{l=1}^3 ||f_l||_{L^p}.$$  

Combining the above inequality with (2.6), we get the desired estimate. \hfill \Box

**Remark 2.10.** By (2.6) it is clear that for all $p \in [1, \infty]$, $||\hat{\mathcal{H}}^1_{a,\gamma,t}(f_1, f_2, f_3)||_{L^p} \lesssim ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^p}$ holds for $\gamma$ in the wider range $0 < \gamma < d$.

**Proposition 2.11 (L^p-estimates).** (1) Assume that $1 \leq p < 2$ and $0 < \gamma < d(\frac{1}{p} - \frac{1}{2})$. Then we have $||\mathcal{H}_{a,\gamma}(f_1, f_2, f_3)||_{L^p} \lesssim ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^p \cap L^2}$.

(2) Assume that $2 < p \leq \infty$ and $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$, and let

$$X = \begin{cases} L^2(\mathbb{R}^d) \cap \tilde{L}^p(\mathbb{R}^d) & \text{if } a \geq 0 \\ \tilde{L}^p(\mathbb{R}^d) & \text{if } a > 0. \end{cases}$$

Then we have $||\mathcal{H}_{a,\gamma}(f_1, f_2, f_3)||_{X} \lesssim \prod_{j=1}^3 ||f_j||_{X}.$
Remark 3.1 (Strategy of proof for local well-posedness). It is known that \( U(t) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) if and only if \( p = 2 \). For this reason, it is believed that, one cannot expect to so lve NLS with initial data in \( L^p(\mathbb{R}^d) \) \((p \neq 2)\) as the linear counterpart of NLS is ill-posed in \( L^p(\mathbb{R}^d) \). However, we can overcome this difficulty via the following strategy:

(i) Apply \( U(-t) \) to the integral form of \((\#)\), that is to \((3.3)\), and search for solution \( \psi \) so that 
\[
\phi(t) = U(-t)\psi(t) \in X_T = (C([0, T], L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)))^N.
\]
Now notice that the linear counterpart of \((3.4)\) is well-posed in \( L^p(\mathbb{R}^d) \). This idea is inspired by the work of Zhou \([34]\) for the NLS in \( L^p(\mathbb{R}) \) \((1 < p < 2)\).

(ii) Invoke factorization formula (Lemma 2.6) to obtain transformed integral operator, say \( \Phi \) (see \((3.5)\)).

(iii) Choose closed ball of radius \( b \), and centered at the origin, say \( \mathcal{V}_b^T \), in \( X_T \) (we note that the choice of \( \mathcal{V}_b^T \) varies as the Lebesgue space exponent \( p \) varies).

(iv) Apply trilinear (Subsection 2.2) and Strichartz estimates to obtain \( \Phi : \mathcal{V}_b^T \to \mathcal{V}_b^T \) is contraction, and hence the local existence.

Remark 3.2. We shall give the proof only for the Hartree-Fock equation \((1.1)\). The proof for the reduced Hartree-Fock equation \((1.2)\) can be proved similarly and hence we shall omit the details.

In this section we shall prove our existence theorems (Theorem 1.1 to Theorem 1.9). To this end, we start with the following technical lemma.
\textbf{Proof.} Note that $F\mathcal{L}_\varphi = \mathcal{L}_\varphi$. Then $F((\Omega(u_1, u_2))(t)) = (FM_t U(-t)u_1(t)) \left( FM_t U(-t)u_2(t) \right)$. Therefore by Hörder inequality, we have

$$\|F((\Omega(u_1, u_2))(t))\|_{L^p} \leq \|FM_t U(-t)u_1(t)\|_{L^{2p}} \|FM_t U(-t)u_2(t)\|_{L^{2p}}.$$ (3.2)

By Lemma 2.5, we have

$$\|FM_t U(-t)u_1(t)\|_{L^{2p}} = \left( \int_{\mathbb{R}^d} |D_t^{-1}M_t^{-1}u_1(t)|^{2p} dx \right)^{1/2p} = \left( (4\pi|t|)^d \int_{\mathbb{R}^d} |u_1(t)|^{2p} dx \right)^{1/2p}$$

(here by $u_1(t)(y)$ we mean $u_1(t, y)$). Using this in (3.2), we obtain the desired inequality. \hfill \square

3.1. Local well-posedness in $L^p \cap L^2$.

**First proof of Theorem 1.1.** By Duhamel’s formula, we rewrite (1.1) as

$$\psi_k(t) = U(t)\psi_{0,k} + i \int_0^t U(t-s)(H_s\psi_k)(s)ds - i \int_0^t U(t-s)(F_s\psi_k)(s)ds.$$ (3.3)

Writing $\psi_k(t) = U(t)\phi_k(t)$, we have

$$\phi_k(t) = \psi_{0,k} + i \int_0^t U(-s)(H_s\psi_k)(s)ds - i \int_0^t U(-s)(F_s\psi_k)(s)ds.$$ (3.4)

Let $\psi_0 = (\psi_0,0, \psi_0,1, \cdots, \psi_0,N)$. Using Lemma 2.6, we have

$$\phi_k(t) = \psi_{0,k} + \sum_{i=1}^N \sum_{j=1}^2 \int_0^t s^{-\gamma} M_s^{-1} H_{a,\gamma,s}^{ij}(M_s\phi_i(s), RM_s\phi_i(s), M_s\phi_k(s))ds$$

$$- c\sum_{i=1}^N \sum_{j=1}^2 \int_0^t s^{-\gamma} M_s^{-1} H_{a,\gamma,s}^{ij}(M_s\phi_i(s), RM_s\phi_i(s), M_s\phi_k(s))ds := \Phi_{\psi_0,k}(\phi)(t).$$ (3.5)

- \textbf{Case I:} $0 < \gamma < \min\{1, \frac{4}{7}\}$ ($1 \leq p \leq \infty$).

Let $q_1 = \frac{8}{7}, r = \frac{8d-4}{d-2}$, and introduce the space

$$V^T_b = \{ v \in L^\infty_T(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) : \|v\|_{L^\infty_T(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))} \leq b, \quad \|U(t)v(t)\|_{L^2_T(L^{p})} \leq b, \quad \|U(t)v(t)\|_{L^2_T(L^{p})} \leq b \},$$

where $q_2, \rho$ to be chosen later. We set $V^T_b = (V^T_b)^N$ and define the distance on it by

$$d(u, v) = \max \{ \|u_j - v_j\|_{L^p_T(L^2 \cap L^p)}, \|U(t)[u_j(t) - v_j(t)]\|_{L^p_T(L^2)}, \|U(t)[u_j(t) - v_j(t)]\|_{L^p_T(L^2)} : j = 1, 2, \cdots, N \}.$$ (3.6)

Then $(V^T_b, d)$ is a complete metric space. Next, we show that $\Phi_{\psi_0} := (\Phi_{\psi_0,1}, \cdots, \Phi_{\psi_0,N})$, defined by (3.5), takes $V^T_b$ into itself for suitable choices of $b$ and $T > 0$. Let $\phi = (\phi_1, \cdots, \phi_N) \in V^T_b$. Denote $\phi_k(t) = \sum_{i=1}^N \sum_{j=1}^2 \int_0^t s^{-\gamma} M_s^{-1} H_{a,\gamma,s}^{ij}(M_s\phi_i(s), RM_s\phi_i(s), M_s\phi_k(s))ds$, $j = 1, 2$ (3.6)

and

$$I_{k,l,m}^1(t) = I_{k,l,m}^2(t) + I_{k,l,m}^3(t).$$ (3.7)
By Remark 2.10, we have
\[ \| I_{k,l,m}(t) \|_{L^p} = \left\| \int_0^t s^{-\gamma} M_s^{-1} \tilde{H}^l_{a,\gamma,s}(M_s \phi_k(s), R M_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^p} \]
\[ \lesssim \int_0^t s^{-\gamma} \| \phi_k(s) \|_{L^2} \| \phi_l(s) \|_{L^2} \| \phi_m(s) \|_{L^p} ds \lesssim b^3 T^{1-\gamma}. \quad (3.8) \]

In view of (2.4) and (3.1), we note that
\[ \tilde{H}^2_{a,\gamma}(M_s \phi_k(s), R M_s \phi_l(s), M_s \phi_k(s)) = [(k_2 * S_{a,s})(\Omega(\psi_k, \psi_l))(s)] * (M_s \phi_k(s)). \]

By Young and Hölder inequalities, we have
\[ \| I_{k,l,m}(t) \|_{L^p} \lesssim \int_0^t s^{-\gamma} \| (k_2 * S_{a,s})(\Omega(\psi_k, \psi_l))(s) \|_{L^2} \| \phi_m(s) \|_{L^p} ds \]
\[ \lesssim \int_0^t s^{-\gamma} \| (k_2 * S_{a,s}) \|_{L^p} \| \Omega(\psi_k, \psi_l)(s) \|_{L^{p'}} \| \phi_m(s) \|_{L^p} ds. \]

Here we choose \( \rho \) such that
\[ \frac{d}{d-\gamma} < \rho < 2. \quad (3.9) \]

Note that we are able to choose such \( \rho \) as \( \gamma < d/2 \). By Lemmas 2.7 and 3.3 and Hausdorff-Young inequality, we have
\[ \| I_{k,l,m}(t) \|_{L^p} \lesssim \int_0^t s^{d-\gamma-d/\rho} \| \psi_k(s) \|_{L^{2p}} \| \psi_l(s) \|_{L^{2p}} \| \phi_m(s) \|_{L^p} ds. \]

Note that \( d - \gamma - \frac{d}{\rho} > 0 \). Choose \( q_2, q_3 \) so that
\[ \frac{1}{q_2} = \frac{d}{4 \left( 1 - \frac{1}{\rho} \right)} \quad \text{and} \quad \frac{1}{q_3} = 1 - \frac{d}{2 \left( 1 - \frac{1}{\rho} \right)}. \quad (3.10) \]

By Hölder inequality, we have
\[ \| I_{k,l,m}(t) \|_{L^p} \lesssim T^{d-\gamma-d/\rho} \int_0^t \| \psi_k(s) \|_{L^{2p}} \| \psi_l(s) \|_{L^{2p}} \| \phi_m(s) \|_{L^p} ds \]
\[ \lesssim T^{d-\gamma-d/\rho} \| \psi_k \|_{L^{2p}(L^{2p})} \| \psi_l \|_{L^{2p}(L^{2p})} \| \phi_m \|_{L^{2p}(L^p)} \lesssim T^{d-\gamma-d/\rho} + \frac{1}{q_2} ||U(t)\phi_k||_{L^{2p}(L^{2p})} ||U(t)\phi_l||_{L^{2p}(L^{2p})} ||\phi_m||_{L^{2p}(L^p)} \lesssim T^{d-\gamma-d/\rho} + \frac{1}{q_2} b^3. \quad (3.11) \]

Combining (3.5), (3.8) and the above inequality, we have
\[ \| \Phi_{\psi_0,\phi} \|_{L^{2p}(L^p)} \lesssim \| \psi_{0,k} \|_{L^p} + N b^3 (T^{1-\gamma} + T^{d-\gamma-d/\rho} + \frac{1}{q_2}). \quad (3.12) \]

For \((q, L) \in \{(q_1, r), (q_2, 2p), (\infty, 2)\}\) and \( K = \frac{\xi_{-n}}{|t|^n} \), by Proposition 2.3 we have
\[ \| U(t)I_{k,l,m} \|_{L^2(L^q)} \lesssim \| (K * (\psi_k \psi_l)) \psi_m \|_{L^{2p}(L^{q_2})}. \]
Note that \( \frac{1}{q_1^*} = \frac{4 - \gamma}{4} + \frac{1}{q_1^*}, \ \frac{1}{r} = \frac{\gamma}{2\alpha} + \frac{1}{r}, \ \frac{4 - \gamma}{4} = \frac{2}{q_1} + \frac{2\gamma}{2^*} \). By Hölder and Hardy-Littlewood-Sobolev inequalities, we have

\[
\| (K \ast (\psi_k \overline{\psi_l})) \psi_m \|_{L_p^q(L^r')} \leq \left\| \left( K \ast (\psi_k \overline{\psi_l}) \right) \right\|_{L_p^q} \left\| \psi_m \right\|_{L^q_p} 
\leq \left\| \left( | \cdot |^{-\gamma} \ast \psi_k \overline{\psi_l} \right) \right\|_{L_p^q} \left\| \psi_m \right\|_{L^q_p(L^r')}
\leq \left\| \left( | \psi_k \overline{\psi_l} | \right) \right\|_{L_p^q} \left\| \psi_m \right\|_{L^q_p(L^r')}
\leq T^{1 - \frac{2q}{q_1}} \left\| \psi_k \right\|_{L_p^q(L^r')} \left\| \psi_l \right\|_{L_p^q(L^r')} \left\| \psi_m \right\|_{L^q_p(L^r')}
\tag{3.13}
\]

and hence

\[
\| U(t)I_{k,l,m}(t) \|_{L_p^q(L^r')} \lesssim T^{1 - \gamma/2} \| \psi_k \|_{L_p^q(L^r')} \| \psi_l \|_{L_p^q(L^r')} \| \psi_m \|_{L^q_p(L^r')}.
\tag{3.14}
\]

Therefore by (3.5) and Proposition 2.3, we have

\[
\| U(t) \Phi_{\psi_0,k}(\phi) \|_{L_p^q(L^r')} \lesssim \| \psi_0 \|_{L^2} + Nb^2 T^{1 - \gamma/2}.
\tag{3.15}
\]

Choose \( b = 2\| \psi_0 \|_{L^2} \) and \( T > 0 \) small enough so that (3.12), (3.15) imply \( \Phi_{\psi_0}(\phi) \in V_b^T \).

Note that by tri-linearity of \( \hat{H}_{a_1, \gamma, l} \), we have

\[
\hat{H}_{a_1, \gamma, l}(f_1, f_2, f_3) - \hat{H}_{a_1, \gamma, l}(g_1, g_2, g_3) = \hat{H}_{a_1, \gamma, l}(f_1 - g_1, f_2, f_3) + \hat{H}_{a_1, \gamma, l}(g_1, f_2 - g_2, f_3) + \hat{H}_{a_1, \gamma, l}(g_1, g_2, f_3 - g_3).
\tag{3.16}
\]

Using this, for \( u, v \in V_b^T \), arguing as above in (3.12), we have

\[
\| \Phi_{\psi_0,k}(u) - \Phi_{\psi_0,k}(v) \|_{L_p^q(L^r')} \lesssim Nb^2 (T^{1 - \gamma} + T^{d - \gamma - \frac{d}{p} + \frac{1}{q_1}}) d(u, v)
\tag{3.17}
\]

and on the other hand as in (3.14)

\[
\| U(t)(\Phi_{\psi_0,k}(u)(t) - \Phi_{\psi_0,k}(v)(t)) \|_{L_p^q(L^r')} \lesssim T^{1 - \gamma/2} Nb^2 d(u, v).
\tag{3.18}
\]

Using (3.17) and (3.18), we may conclude that \( \Phi_{\psi_0} : V_b^T \to V_b^T \) is a contraction provided \( T \) is sufficiently small (depending on \( \| \psi_{0,1} \|_{L^p \cap L^2}, ..., \| \psi_{0,N} \|_{L^p \cap L^2}, d, \gamma, N \)). Then, by Banach contraction principle, there exists a unique \( (\phi_1, ..., \phi_N) \in V_b^T \) solving (3.5).

- **Case II:** \( 0 < \gamma < \min\{1, 2d(\frac{1}{p} - \frac{1}{2})\} \) and \( 1 \leq p < 2 \) (improves Case I when \( 1 \leq p < \frac{4}{3} \)).

For \( b, T > 0 \), let

\[
V_b^T = \{ v \in L^\infty((0, T), L^p \cap L^2(\mathbb{R}^d)) : \| v \|_{L_p^q(L^r')} \leq b \}.
\]

We set \( V_b^T = (V_b^T)^N \) and define the distance on it by

\[
d(u, v) = \max \left\{ \| u_j - v_j \|_{L_p^q(L^r')} : j = 1, 2, \cdots, N \right\},
\]

where \( u = (u_1, u_2, \cdots, u_N), v = (v_1, v_2, \cdots, v_N) \in V_b^T \). Then \( (V_b^T, d) \) is a complete metric space. Next, we show that the mapping \( \Phi_{\psi_0} \), defined by (3.5), takes \( V_b^T \) into itself for a suitable choice of
b and small $T > 0$. Let $\phi = (\phi_1, \ldots, \phi_N) \in V_b^T$. Choose $b$ so that $\|\psi_0\|_{(L^p \cap L^2)^N} = b/2$. Then, by Proposition 2.9(1), for $0 < t < T$, we obtain
\[
\|\Phi_{\psi_0,k}(\phi)(t)\|_{L^p \cap L^2} \leq \frac{b}{2} + cN \int_0^t s^{-\gamma} \|M_\phi \phi_1(s)\|_{L^p \cap L^2}^2 \|M_\phi \phi_k(s)\|_{L^p \cap L^2} ds \\
\leq \frac{b}{2} + cNb^3 \int_0^t s^{-\gamma} ds = \frac{b}{2} + \frac{2cNb^3}{1-\gamma}T^{1-\gamma}
\]
Now we choose $T > 0$ small enough so that $\frac{2cNb^3}{1-\gamma}T^{1-\gamma} \leq \frac{1}{2}$ to achieve $\|\Phi_{\psi_0,k}(\phi)(t)\|_{L^p \cap L^2} \leq b$ for all $k = 1, \ldots, N$. Hence $\Phi_{\psi_0}$ is a map from $V_b^T$ to itself with the above choices of $b$ and $T$. For $u, v \in V_b^T$, by Proposition 2.9 (1), (3.16) and arguing as above, we obtain
\[
d(\Phi_{\psi_0}(u), \Phi_{\psi_0}(v)) \lesssim Nb^2 \int_0^t s^{-\gamma} \|u_1(s) - v_1(s)\|_{L^p \cap L^2} ds \lesssim Nb^2 T^{1-\gamma} d(u, v).
\]
Thus $\Phi_{\psi_0} : V_b^T \to V_b^T$ is a contraction map provided that $T$ is sufficiently small (depending on $\|\psi_0.1\|_{L^p \cap L^2}, \ldots, \|\psi_0.N\|_{L^p \cap L^2}, d, \gamma, N$). Then, by Banach contraction principle, there exists a unique $(\phi_1, \ldots, \phi_N) \in V_b^T$ solving (3.5).

In [34], Zhou proved local existence for the cubic NLS in $L^p(\mathbb{R})$ by introducing a function space (to be defined below) based on the fundamental theorem of calculus and the Schrödinger propagator. Specifically, for $T > 0, 1 \leq p, q \leq \infty$ and $\theta \geq 0$, the Zhou spaces $\tilde{X}^p_{q,\theta}(T), \tilde{Y}^p_{q,\theta}(T), \text{ and } Y^p_{q,\theta}(T)$ are given by
\[
\tilde{X}^p_{q,\theta}(T) = \{ v : [0, T] \times \mathbb{R}^d \to \mathbb{C} : \|v\|_{\tilde{X}^p_{q,\theta}(T)} := \|v\|_{(L^p \cap L^2)}(t : \mathbb{R}^d) < \infty \}.
\]
\[
\tilde{Y}^p_{q,\theta}(T) = \{ v \in \tilde{X}^p_{q,\theta}(T) : \|v\|_{\tilde{Y}^p_{q,\theta}(T)} := \|v(0)\|_{L^p} + \|v\|_{\tilde{Y}^p_{q,\theta}(T)} < \infty \}
\]
and
\[
Y^p_{q,\theta}(T) = \{ u : [0, T] \times \mathbb{R}^d \to \mathbb{C} | U(-t)u(t) \in \tilde{Y}^p_{q,\theta}(T) \}. \quad (3.19)
\]
Now using the Zhou spaces, we will briefly give a different proof of local existence in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ($p \in [1, \infty) \setminus \{2\}$) for (#) with certain $\gamma$’s. Strictly speaking, in this case, we will prove the local existence in $\tilde{Y}^p_{\infty,\gamma}(T)$ which is continuously embedded in $C([0, T], L^p(\mathbb{R}^d))$:

**Lemma 3.4 (Lemma 2.1 [22]).** Let $0 < T < \infty, p \geq 1$ and $0 < \gamma < 1$. Then $\tilde{Y}^p_{\infty,\gamma}(T) \hookrightarrow C([0, T], L^p(\mathbb{R}^d))$.

The key estimates for $\tilde{Y}^p_{\infty,\gamma}(T)$-regularity of the local existence is:

**Lemma 3.5.** Denote the Duhamel type operator by
\[
D_{a,\gamma}(v_1, v_2, v_3)(t) = \int_0^t M_{a}^{-1}s^{-\gamma}\tilde{H}_{a,\gamma,s}(M_{a}v_1(s), RM_{a}v_2(s), M_{a}v_3(s)) ds \quad (3.20)
\]
(1) Assume $0 < \gamma < 1$. Then $\|D_{a,\gamma}(v_1, v_2, v_3)\|_{\tilde{Y}^p_{\infty,\gamma}(T)} \lesssim \prod_{j=1}^{3} \|v_j\|_{\tilde{Y}^p_{\infty,\gamma}(T)}$.
(2) Assume $0 < \gamma < \min\{1, 2d(\frac{1}{p} - \frac{1}{2})\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{1, d(\frac{1}{p} - \frac{1}{2})\}$ when $2 < p \leq \infty$. Then $\|D_{a,\gamma}(v_1, v_2, v_3)\|_{\tilde{X}^p_{\infty,\gamma}(T)} \lesssim \prod_{j=1}^{3} \|v_j\|_{\tilde{Y}^p_{\infty,\gamma}(T) \cap \tilde{Y}^p_{\infty,\gamma}(T)}.$
Proof. (1) In view of (2.3), we have
\[ \widehat{H}_{a,\gamma,t}(M_t v_1(t), R M_t v_2(t), M_t v_3(t)) = \left( \left( S_{a,t} * | \cdot |^{-\delta} \right) (M_t v_1(t) \ast R M_t v_2(t)) \right) \ast M_t v_3(t). \]

Then using Lemma 2.6 we get
\[
I := \| \partial_t D_{a,\gamma}(v_1, v_2, v_3) \|_{L^{2/\gamma}(L^2)} \lesssim \left\| t^{-\gamma} \widehat{H}_{a,\gamma,t}(M_t v_1(t), R M_t v_2(t), M_t v_3(t)) \right\|_{L^{2/\gamma}(L^2)} \\
\lesssim \| U(-t) H_{a,\gamma} U(t) v_1(t), U(t) v_2(t), U(t) v_3(t) \|_{L^{2/\gamma}(L^2)} \\
= \| H_{a,\gamma}(U(t) v_1(t), U(t) v_2(t), U(t) v_3(t) \|_{L^{2/\gamma}(L^2)}. \]

By Hölder and Hardy-Littlewood-Sobolev inequalities, we have
\[
I \lesssim \left\| \left( \cdot | \cdot |^{-\gamma} e^{-\alpha | |} \right) \ast \left( (U(t) v_1(t)) \overline{(U(t) v_2(t))} \right) \right\|_{L^{\frac{4d}{3d-2\gamma}}} \left\| U(t) v_3(t) \right\|_{L^{\frac{6d-4d}{3d-2\gamma}}} \lesssim \left\| (U(t) v_1(t)) \overline{(U(t) v_2(t))} \right\|_{L^{\frac{3d}{3d-2\gamma}}} \left\| U(t) v_3(t) \right\|_{L^{\frac{6d}{3d-2\gamma}}}, \]
\[
\lesssim \left\| U(t) v_1(t) \right\|_{L^{\frac{3d}{3d-2\gamma}}} \left\| U(t) v_3(t) \right\|_{L^{\frac{6d-4d}{3d-2\gamma}}} \leq \prod_{l=1}^{3} \left\| U(t) v_l(t) \right\|_{L^{\frac{6d}{3d-2\gamma}}}. \]

Using the fundamental theorem of calculus, we have
\[
v_l(t) = v_l(0) + \int_0^t \partial_s v_l(s) ds \quad \Longrightarrow \quad U(t) v_l(t) = U(t) v_l(0) + \int_0^t U(t) \partial_s v_l(s) ds. \quad (3.21)\]

Put \( q = \frac{2}{\gamma} \) and \( r = \frac{6d}{3d-2\gamma} \). Then by the above equality we have
\[
\| U(t) v_l(t) \|_{L^{3q}(L^r)} \lesssim \| U(t) v_l(0) \|_{L^{3q}(L^r)} + \left\| \int_0^t U(t) \partial_s v_l(s) ds \right\|_{L^{3q}(L^r)} \\
\lesssim \| U(t) v_l(0) \|_{L^{2q}(L^r)} + \left\| \int_0^t \| U(t) \partial_s v_l(s) \|_{L^r} ds \right\|_{L^{3q}}, \]

By Minkowski inequality, we have
\[
\left\| \int_0^t \| U(t) \partial_s v_l(s) \|_{L^r} ds \right\|_{L^{3q}} \lesssim \left( \int_0^T \left( \int_0^T \| U(t) \partial_s v_l(s) \|_{L^r}^3 dt \right)^{\frac{1}{3q}} \right)^{\frac{1}{3}} \lesssim \int_0^T \| U(t) \partial_s v_l(s) \|_{L^{3q}} ds. \]

Therefore, \( (3q, r) \) being a 2-admissible pair, we get
\[
\| U(t) v_l(t) \|_{L^{3q}(L^r)} \lesssim \| v_l(0) \|_{L^2} + \int_0^T \| \partial_s v_l(s) \|_{L^2} ds = \| v_l \|_{L^2(T)}^{\frac{3q}{2}}. \]

(2) In view of Proposition 2.9, we obtain
\[
\| D_{a,\gamma}(v_1, v_2, v_3) \|_{L^{p}} \lesssim \sup_{s \in [0, T]} \| \partial_s D_{a,\gamma}(v_1, v_2, v_3)(s) \|_{L^p} \\
\lesssim \sup_{s \in [0, T]} \| \widehat{H}_{a,\gamma,s}(M_s v_1(s), R M_s v_2(s), M_s v_3(s)) \|_{L^p} \\
\lesssim \sup_{s \in [0, T]} \prod_{l=1}^{3} \| v_l(s) \|_{L^2 \cap L^p} \lesssim \prod_{l=1}^{3} \| v_l \|_{L^2 \cap L^p}, \]

where the last inequality follows from (3.21) by taking \( L^2, L^p \)-norms on both sides of it. \( \Box \)
Remark 3.6. The local well-posedness proved in [21, Theorem 1.3] for Hartree equation in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for $1 < p \leq 2$ depend on [21, Proposition 4.1]. We found that there is a subtle flaw in it. Specifically, while proving a result similar to Lemma 3.5 (1) with $a = 0, 1 < p \leq 2$ in [21, see eq. (4.6), p.1093], the following inequality

$$
\|t^{-\gamma} \hat{H}_{0,\gamma,t}^j(v_1, v_2, v_3)\|_{L^q_t(L^2)} \leq c \|t^{-\gamma} \hat{H}_{0,\gamma,t}(v_1, v_2, v_3)\|_{L^q_t(L^2)}
$$

(3.22)

where $\hat{H}_{0,\gamma,t}$ and $\hat{H}_{0,\gamma,t}^j$ are as in (2.3) and (2.4) respectively, has been crucially used. However, we find that (3.22) is not true.

We shall justify our claim by producing a counter example. Note that (3.22) implies

$$
\|\hat{H}_{0,\gamma,t}(v_1, v_2, v_3)\|_{L^2} \leq c \|\hat{H}_{0,\gamma,t}(v_1, v_2, v_3)\|_{L^2}
$$

(3.23)

for all $t$ in some positive measured subset of $[0, T]$. If possible, we assume that (3.22) is true. Let $k_1$ and $k_2$ are as in (2.5) and $k := k_1 + k_2$. Then, by Plancherel theorem, we have

$$
\|\hat{H}_{0,\gamma,t}^j(f, g, h)\|_{L^2} \leq c \|\hat{H}_{0,\gamma,t}(f, g, h)\|_{L^2} \iff \|[k_j(f * \bar{g})] * h\|_{L^2} \leq c \|[k(f * \bar{g})] * h\|_{L^2}
$$

$$
\iff \|\mathcal{F}[k_j(f * \bar{g})]\mathcal{F}h\|_{L^2} \leq c \|\mathcal{F}[k(f * \bar{g})]\mathcal{F}h\|_{L^2}.
$$

Thus, we have

$$
\int_{\mathbb{R}^d} \left(|\mathcal{F}[k_j(f * \bar{g})]|^2 - c^2 |\mathcal{F}[k(f * \bar{g})]|^2\right) |\mathcal{F}h|^2 \leq 0.
$$

Since $h$ is arbitrary we conclude $|\mathcal{F}[k_j(f * \bar{g})](\xi)| \leq c |\mathcal{F}[k(f * \bar{g})](\xi)|$ for a.e. $\xi \in \mathbb{R}^d$. If we choose compactly supported $f * \bar{g}$ in the space $L^\infty(\mathbb{R}^d)$, so that $\mathcal{F}[k_j(f * \bar{g})], \mathcal{F}[k(f * \bar{g})]$ are continuous, then we would have

$$
|\mathcal{F}[k_j(f * \bar{g})](\xi)| \leq c |\mathcal{F}[k(f * \bar{g})](\xi)| \quad \text{for all } \xi \in \mathbb{R}^d, \ j = 1, 2.
$$

(3.24)

For simplicity we take dimension $d = 1$ and so $0 < \gamma < 1$. Let $0 < a < 1, b > 0$ (to be chosen later). Define $H_{a,b} : \mathbb{R} \to \mathbb{R}$ as follows

$$
H_{a,b}(x) = \begin{cases} 
  a - x, & \text{for } 0 \leq x < a \\
  0, & \text{for } a \leq x < 2 \text{ and } 2 + 2b \leq x < \infty \\
  2 - x, & \text{for } 2 \leq x < 2 + b \\
  x - 2 - 2b, & \text{for } 2 + b \leq x < 2 + 2b \\
  H_{a,b}(-x), & \text{for } -\infty < x < 0.
\end{cases}
$$
The graph of $H_{a,b}$ is given in Figure 1 and it is clear that $H_{a,b} \in L^\infty(\mathbb{R})$ is compactly supported. Note that $H_{a,b}$ is a finite linear combinations of triangle functions. So $H_{a,b}$ can also be viewed as the finite linear combination of functions of the form $\chi_A * \chi_B$ where $A, B$ are finite intervals in $\mathbb{R}$. Thus, we have $H_{a,b} \in \tilde{L}^\infty(\mathbb{R})$. We recall Riesz factorization criterion: $\tilde{L}^\infty(\mathbb{R}^d) = L^2(\mathbb{R}^d) * L^2(\mathbb{R}^d)$, see for e.g., [27, Theorem B]. In view of this, there exists $f, g \in L^2(\mathbb{R})$ such that $H_{a,b} = f * g$. We shall now compute the LHS and RHS of (3.24) at point 0. In fact,

$$\mathcal{F}[k(f * \bar{g})](0) = \int_{\mathbb{R}} |x|^{-1} (f * \bar{g})(x) dx$$

$$= 2 \int_0^a x^{-1}(a-x) dx - 2 \int_2^{2+b} x^{-1}(x-2) dx - 2 \int_{2+b}^{2+2b} x^{-1}(2 + 2b - x) dx$$

$$= \frac{2}{\gamma(\gamma + 1)} [(a^{\gamma+1} - (2 + 2b)^{\gamma+1} + 2(2 + b)^{\gamma+1} + 2b + 1)] - \frac{2a^{\gamma+1}}{\gamma(\gamma + 1)} > 0,$$

where $G(b) = (2 + 2b)^{\gamma+1} - 2(2 + b)^{\gamma+1} + 2b + 1$. Note that $G(0) = 0$ and by using strict convexity of $x \mapsto x^{1+\gamma}$ on $(0, \infty)$ for $\gamma > 0$, we have $G(b) > 0$ for $b > 0$. Choose $0 < a < 1$ small and adjust $b > 0$ so that $a^{\gamma+1} = G(b)$. Hence, we obtain $\mathcal{F}[k(f * \bar{g})](0) = 0$. But note that

$$|\mathcal{F}[k_1(f * \bar{g})](0)| = \int_{|x| \leq 1} |x|^{-1} (f * \bar{g})(x) = 2 \int_0^a x^{-1}(a-x) dx = \frac{2a^{\gamma+1}}{\gamma(\gamma + 1)}$$

and this contradicts (3.24) as the RHS is zero in (3.24) for $\xi = 0$. Thus (3.22) is not true.

**Remark 3.7.** Because of the issue discussed in Remark 3.6 we do not decompose Duhamel type operator $D_{a,\gamma}$ (defined in (3.20) above) in two parts. We rather work on $D_{a,\gamma}$ without decomposing into two parts but still get the nonlinear estimates of Lemma 3.5.

**Second proof of Theorem 1.1.** Here we assume $0 < \gamma < \min\{1, 2d(\frac{1}{p} - \frac{1}{2})\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{1, d(\frac{1}{2} - \frac{1}{p})\}$ when $2 < p \leq \infty$.

For $b, T > 0$, let $V^b_T(v_0) = \{v \in \bar{Y}^p_{2,\gamma,0}(T) \cap \bar{Y}^p_{\infty,\gamma}(T) : \|v\|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} \leq b, v(0) = v_0\}$. For $\psi_0 = (\psi_{0,1}, \ldots, \psi_{0,N}) \in (L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N$, introduce the space $V^b_T(\psi_0) = V^b_T(\psi_{0,1}) \times V^b_T(\psi_{0,2}) \times \cdots \times V^b_T(\psi_{0,N})$ and define the distance on it by $d(u, v) = \max \{\| u_j - v_j \|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} : j = 1, 2, \ldots, N\}$. Next, we show that the mapping $\Phi_{\psi_0}$, defined by (3.5), takes $V^b_T(\psi_0)$ into itself for suitable choice of $b$ and small $T > 0$. Let $\phi = (\phi_1, ..., \phi_N) \in V^b_T(\psi_0)$. Since $\| \psi_k,0 \|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} = 0$, by Lemma 3.5, we have

$$\| \Phi_{\psi_0,k}(\phi) \|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)}$$

$$\leq \sum_{l=1}^N \| D_{a,\gamma}(\phi_l, \phi_l, k) \|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)} + \sum_{l=1}^N \| D_{a,\gamma}(\phi_k, \phi_l, k) \|_{\tilde{X}^2_{2,\gamma,0}(T) \cap \tilde{X}^p_{\infty,\gamma}(T)}$$

$$\leq \| \phi_k \|_{\tilde{Y}^2_{1,0}(T) \cap \tilde{Y}^p_{1,0}(T)} \sum_{l=1}^N \| \phi_l \|_{\tilde{Y}^2_{1,0}(T) \cap \tilde{Y}^p_{1,0}(T)}.$$
Hence, by taking $0 < T < 1$, we obtain
\[
\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{H}^{2,\gamma}_0(T) \cap \tilde{H}^{2,\gamma}_0(T)} \lesssim N \left( \|\psi_0\|_{L^2 \cap L^p}^3 + T^{3(1-\gamma)} b^3 \right).
\]
We set $b = 2eN\|\psi_0\|_{(L^p \cap L^2)^N}^3$ and $T > 0$ small so that we have $\|\Phi_{\psi_0,k}(\phi)\|_{\tilde{H}^{2,\gamma}_0(T) \cap \tilde{H}^{2,\gamma}_0(T)} \leq b$.
Consequently, we have $\Phi_{\psi_0}(\phi) \in \mathcal{V}_T^T(\psi_0)$. To check $\Phi_{\psi_0}$ is a contraction, for $u, v \in \mathcal{V}_T^T(\psi_0)$, using (3.16) together with Lemma 3.5, we can conclude
\[
d(\Phi_{\psi_0}(u), \Phi_{\psi_0}(v)) \lesssim \|u - v\|_{L^2 \cap L^p}^3 \sum_{i=1}^{N} \|u_i - v_i\|_{\tilde{H}^{2,\gamma}_0(T) \cap \tilde{H}^{2,\gamma}_0(T)} \lesssim NT^{1-\gamma}\|u\|_{L^2 \cap L^p} + T^{2(1-\gamma)} b^2 d(u, v).
\]
Thus $\Phi_{\psi_0} : \mathcal{V}_T^T(\psi_0) \rightarrow \mathcal{V}_T^T(\psi_0)$ is a contraction provided $T > 0$ is small enough.
Finally, we note that by Proposition 2.4, we have $\psi \in C([0, T], L^2(\mathbb{R}^d))^N$, and consequently $\phi \in C([0, T], L^2(\mathbb{R}^d))^N$. By Lemma 3.4 we have $\phi \in C([0, T], L^p(\mathbb{R}^d))^N$ for $0 < \gamma < 1$. □

3.2. Local well-posedness in $\tilde{L}^p \cap L^2$.

Lemma 3.8. For all $t \in \mathbb{R}$ and $0 < \alpha < \infty$ the fractional Schrödinger propagator $U_{\alpha}(t) = e^{-it(-\Delta)^{\alpha/2}}$ is an isometry on $\tilde{L}^p(\mathbb{R}^d)$ ($1 \leq p \leq \infty$), that is, $\|e^{-it(-\Delta)^{\alpha/2}} f\|_{\tilde{L}^p} = \|f\|_{\tilde{L}^p}$.

First proof of Theorem 1.5. By Duhamel’s formula, we rewrite (1.1) as
\[
\psi_k(t) = U_{\alpha}(t)\psi_{\psi_0,k} + i \int_0^t U_{\alpha}(t-s)(H_\psi\psi_k)(s)ds - i \int_0^t U_{\alpha}(t-s)(F_\psi\psi_k)(s)ds \quad (3.25)
\]
\[
\begin{aligned}
\Psi_{\psi_0,k}(\psi)(t) = \\
\end{aligned}
\]

• Case I: $0 < \gamma < \min\{\alpha, \frac{d}{2}\}$ ($1 \leq p \leq \infty$).

Hereafter, for $\alpha \in (\frac{2d}{d-1}, 2)$, we assume initial data are radial and $d \geq 2$. In fact, in this case, the members of $U_{\alpha}^T$, to be defined below, are radial functions. For the notational convenience, we omit mentioning this explicitly in the proof below. Let $s = \alpha/2$ and $q_1 = \frac{8\alpha}{\gamma}$, $r = \frac{8d}{2d-\gamma}$, and for $T, b > 0$, introduce the space
\[
U_{\alpha}^T = \{v \in L^\infty_T(\tilde{L}^p_\infty(\mathbb{R}^d)) : \|v\|_{L^\infty_T(\tilde{L}^p(\mathbb{R}^d))} \leq b, \|v\|_{L^q_1(L^r)} \leq b, \|v\|_{L^q_2(L^2r)} \leq b \},
\]
where $q_2, \rho$ to be chosen later. We set $U_{\alpha}^T = \left(U_{\alpha}^T\right)^N$ and the distance on it by
\[
d(u, v) = \max \{\|u_j - v_j\|_{L^\infty_T(\tilde{L}^p(\mathbb{R}^d))}, \|u_j - v_j\|_{L^q_1(L^r)} \leq b, \|v\|_{L^q_2(L^2r)} \leq b \},
\]
where $u = (u_1, u_2, \ldots, u_N), v = (v_1, v_2, \ldots, v_N) \in U_{\alpha}^T$. Next, we show that the mapping $\Psi_{\psi_0} = (\Psi_{\psi_0,1}, \ldots, \Psi_{\psi_0,N})$, defined by (3.25), takes $U_{\alpha}^T$ into itself for suitable choice of $b$ and small $T > 0$. Let $\psi = (\psi_1, \ldots, \psi_N) \in U_{\alpha}^T$. Denote
\[
J_{k,l,m}(t) = \int_0^t U_{\alpha}(t-s)\mathcal{H}_{\alpha,\gamma}(\psi_k(s), \psi_l(s), \psi_m(s))ds.
\]
Let $\frac{d}{d-\gamma} < \rho \leq 2$, $h_\alpha(\xi) = (a^2 + \xi^2)^{-\alpha/2}$ (see (2.5) and Lemma 2.1) and $K = \frac{e^{-a|\xi|}}{|\xi|^{\alpha}}$. We have
\[
\|\mathcal{H}_{\alpha,\gamma}(f, g, h)\|_{\tilde{L}^p} = \|F([K * (f \overline{g})])h\|_{L^p} \leq \|F[K * (f \overline{g})]\|_{L^1} \|Fh\|_{L^p} = \|FKF(f \overline{g})\|_{L^1} \|h\|_{\tilde{L}^p} \leq \left(\|K \ast h_\alpha\|_{L^1} \|F(f \overline{g})\|_{L^\infty} + \|K \ast h_\alpha\|_{L^p} \|F(f \overline{g})\|_{L^r}\right) \|h\|_{\tilde{L}^p} \leq \left(\|f\|_{L^2} \|g\|_{L^2} + \|f\|_{L^2r} \|f\|_{L^2r}\right) \|h\|_{\tilde{L}^p} \|h\|_{\tilde{L}^p} \|h\|_{\tilde{L}^p}.
\]
Choose \( q_2 \) as \( \frac{\alpha}{2q_2} = d\left(\frac{1}{2} - \frac{1}{2p}\right) \) so that \( (2q_2, 2p) \) is an \( \alpha \)-fractional admissible pair. Then

\[
\|J_{k,l,m}(t)\|_{\tilde{L}^p} \lesssim \int_0^t \left( \|\psi_k(s)\|_{L^2} \|\psi_l(s)\|_{L^2} + \|\psi_k(s)\|_{L^{2p}} \|\psi_l(s)\|_{L^{2p}} \right) \|\psi_m(s)\|_{\tilde{L}^p} \, ds
\]

\[
\lesssim t \|\psi_k\|_{L^\infty(L^2)} \|\psi_l\|_{L^\infty(L^2)} \|\psi_m\|_{L^\infty(\tilde{L}^p)} + \|\psi_k\|_{L^{2q_2}(L^2)} \|\psi_l\|_{L^{2q_2}(L^2)} \|\psi_m\|_{L^{2q_2}(\tilde{L}^p)}
\]

using Hölder inequality. Therefore

\[
\|\Psi_{\phi_0,k}(\psi)(t)\|_{\tilde{L}^p} \lesssim \|\psi_0,k\|_{\tilde{L}^p} + Nb^3(T + T^{\frac{1}{q_2}}).
\]

For \( (q, r) \in \{(q_1, r), (2q_2, 2p), (\infty, 2)\} \), by Proposition 2.3 we have

\[
\|\Psi_{\psi_0,k}(\psi)(t)\|_{L^q(L^r)} \lesssim \|\psi_0,k\|_{L^2} + \sum_{l=1}^N \|(K * |\psi_l|^2)\psi_k\|_{L^{q_1}(L^{r_1})} + \sum_{l=1}^N \|(K * (\bar{\psi}_l\psi_l))\psi_k\|_{L^{q_1}(L^{r_1})}.
\]

Now we have \( \frac{1}{q_1} = \frac{4s-\gamma}{4s} + \frac{1}{q_1}, \frac{1}{r_1} = \frac{\gamma}{2d} + \frac{1}{r} \) and \( \frac{4s-\gamma}{4s} = \frac{2}{q_1} + \frac{2s-\gamma}{2s} \). By Hölder and Hardy-Littlewood-Sobolev inequalities, as in the calculation (3.13) we have

\[
\left\|(K * (\bar{\psi}_l\psi_l))\psi_k\right\|_{L^{q_1}(L^{r_1})} \leq \left\||\psi_k\|_{L^{q_1}(L^{r_1})}\right\|_{L^{q_1}(L^{r_1})} \|\psi_m\|_{L^{q_1}(L^{r_1})}
\]

\[
\lesssim \left\||\psi_k\|_{L^{q_1}(L^{r_1})}\right\|_{L^{q_1}(L^{r_1})} \|\psi_m\|_{L^{q_1}(L^{r_1})}
\]

\[
\leq T^{1-\frac{\gamma}{2d}} \|\psi_k\|_{L^{q_1}(L^{r_1})} \|\psi_l\|_{L^{q_1}(L^{r_1})} \|\psi_m\|_{L^{q_1}(L^{r_1})}.
\]

Combining the above two inequalities, we obtain

\[
\|\Psi_{\psi_0,k}(\psi)(t)\|_{L^q(L^r)} \lesssim \|\psi_0,k\|_{L^2} + T^{1-\frac{\gamma}{2d}} \sum_{l=1}^N \|\psi_l\|_{L^{q_1}(L^{r_1})}^2 \|\psi_k\|_{L^{q_1}(L^{r_1})} \lesssim \|\psi_0,k\|_{L^2} + T^{1-\frac{\gamma}{2d}} Nb^3.
\]

Choose \( b = 2c\|\psi_0\|_{(L^\infty(L^p))} \) and \( T > 0 \) small enough so that (3.28) and the above inequality imply \( \Psi_{\psi_0}(\psi) \in \mathcal{U}_b^T \). On the other hand for \( u, v \in \mathcal{V}_b^T \), using trilinearity of \( \mathcal{H}_{a,\gamma} \), we have

\[
\|\Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t)\|_{\tilde{L}^p} \lesssim \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(u_l, u_l, u_l) - \mathcal{H}_{a,\gamma}(v_l, v_l, v_l)\|_{\tilde{L}^p} \, ds
\]

\[
+ \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(u_l, u_l, u_l) - \mathcal{H}_{a,\gamma}(v_l, v_l, v_l)\|_{\tilde{L}^p} \, ds \lesssim N(T + T^{\frac{1}{q_2}})b^2 d(u, v).
\]

On the other hand again by using Proposition 2.3 and trilinearity we obtain

\[
\|\Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t)\|_{L^\infty(L^2)} \lesssim T^{1-\frac{\gamma}{2d}} Nb^2 d(u, v).
\]

Choose \( T > 0 \) further small so that (3.29) and (3.30) imply that \( \Psi_{\psi_0} \) is a contraction.

• **Case II:** \( 0 < \gamma < 2d(\frac{1}{2} - \frac{1}{2p}), 2 < p \leq \infty \).

Let \( X \) be as in Proposition 2.11. For \( b, T > 0 \), let \( U^T_b = \{ v \in L^\infty(T^\infty(X) : \|v\|_{L^\infty(T^\infty(X)} \leq b \}. \) We set \( \mathcal{U}_b^T = (U^T_b)^N \) and the distance on it by \( d(u, v) = \max \{ \|u_j - v_j\|_{L^\infty(T^\infty(X)} : j = 1, 2, \cdots, N \} \), where \( u, v \in \mathcal{U}_b^T \). Next, we show that the mapping \( \Psi_{\psi_0} \), defined by (3.25), takes \( \mathcal{U}_b^T \) into itself for suitable
choice of $b$ and small $T > 0$. Let $\psi = (\psi_1, ..., \psi_N) \in U_0^T$. We note that
\[
\|\Psi_{\psi_0,k}(\psi)(t)\|_X \lesssim \|\psi_0, k\|_X + \sum_{l=1}^{N} \int_0^t \|H_{\alpha, \gamma}(\psi_l(s), \psi_l(s), \psi_k(s))\|_X ds \\
+ \sum_{l=1}^{N} \int_0^t \|H_{\alpha, \gamma}(\psi_k(s), \psi_l(s), \psi_l(s))\|_X ds.
\]
Therefore taking $b = 2 \|\psi_0\|_X^N$ and using Proposition 2.11 (2), we have
\[
\|\Psi_{\psi_0,k}(\psi)(t)\|_X \leq \frac{b}{2} + c \sum_{l=1}^{N} \int_0^t \|\psi_l\|_X^2 \|\psi_k\|_X \leq \frac{b}{2} + cNTb^3 \leq b.
\]
for $T > 0$ small enough. For $u, v \in U_0^T$, by tri-linearity of $H_{\alpha, \gamma}$ we have
\[
\|\Psi_{\psi_0,k}(u)(t) - \Psi_{\psi_0,k}(v)(t)\|_X \lesssim Nb^2Td(u, v).
\]
Thus $\Psi : U_0^T \rightarrow U_0^T$ is a contraction provided $T > 0$ is small enough. \hfill \Box

We introduce the function space $\tilde{W}^p_{q,0}(T)$ ($1 \leq p \leq \infty$) which is similar to $\tilde{Y}^p_{q,0}(T)$ to get local well-posedness. Specifically, we define
\[
\tilde{W}^p_{q,0}(T) = \{v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{C} : \|v\|_{\tilde{W}^p_{q,0}(T)} = \|v^\theta\|_{L^p(T)} < \infty\}
\]
and
\[
\tilde{Z}^p_{q,0}(T) = \{v \in \tilde{W}^p_{q,0}(T) : \|v\|_{\tilde{Z}^p_{q,0}(T)} := \|v(0)\|_{L^p} + \|v\|_{\tilde{W}^p_{q,0}(T)} < \infty\}.
\]
Now we state the required inclusion result.

Lemma 3.9. Let $T > 0$, $p \geq 1$ and $0 < \gamma < 1$. Then $\tilde{Z}^p_{q,0}(T) \hookrightarrow C([0, T], \tilde{L}^p(\mathbb{R}^d))$.

Proof. The proof is similar to the proof of Lemma 3.4. See [22, Lemma 2.1] for details. \hfill \Box

Lemma 3.10. Denote
\[
D_{\alpha, \gamma}(v_1, v_2, v_3)(t) = \int_0^t U_{\alpha}(-s)H_{\alpha, \gamma}(U_{\alpha}(s)v_1(s), U_{\alpha}(s)v_2(s), U_{\alpha}(s)v_3(s))ds.
\]
(1) Assume that $0 < \gamma < \min\{\alpha, d(\frac{1}{p} - \frac{1}{2})\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{\alpha, 2d(\frac{1}{2} - \frac{1}{p})\}$ when $2 < p \leq \infty$. Then $\|D_{\alpha, \gamma}(v_1, v_2, v_3)\|_{\tilde{W}^2_{1,0}(T) \cap \tilde{Z}^2_{1,0}(T)} \lesssim \prod_{l=1}^{3} \|v_l\|_{\tilde{Z}^p_{1,0}(T) \cap \tilde{Z}^p_{1,0}(T)}$.

(2) Assume that $2 < p \leq \infty$ and $0 < \gamma < 2d(\frac{1}{2} - \frac{1}{p})$. Then $\|D_{\alpha, \gamma}(v_1, v_2, v_3)\|_{\tilde{W}^2_{1,0}(T) \cap \tilde{Z}^2_{1,0}(T)} \lesssim \prod_{l=1}^{3} \|v_l\|_{\tilde{Z}^p_{1,0}(T) \cap \tilde{Z}^p_{1,0}(T)}$.

Proof. (1) Set $q = \frac{2}{\gamma}, r = \frac{6d}{3d - 2\gamma}$ so that $(3q, r)$ becomes an $\alpha$-fractional admissible pair. By a similar argument as in the proof of Lemma 3.5 (1), we obtain
\[
\|\partial_t D_{\alpha, \gamma}(v_1, v_2, v_3)\|_{L^p_t(L^2)} \times \|U_{\alpha}(-t)H_{\alpha, \gamma}(U_{\alpha}(t)v_1(t), U_{\alpha}(t)v_2(t), U_{\alpha}(t)v_3(t))\|_{L^p_t(L^2)} \\
= \|H_{\alpha, \gamma}(U_{\alpha}(t)v_1(t), U_{\alpha}(t)v_2(t), U_{\alpha}(t)v_3(t))\|_{L^p_t(L^2)} \\
\lesssim \prod_{l=1}^{3} \|U(t)v_l\|_{L^p_t(L^p)} \lesssim \prod_{l=1}^{3} \|v_l\|_{\tilde{Z}^p_{1,0}(T)}.
\]
In view of Proposition 2.11 and Lemma 3.8, we obtain
\[
\| \mathcal{D}_{a, \gamma}^{\alpha} (v_1, v_2, v_3) \|_{\widetilde{W}_{\infty, 0}^{p}(T)} = \sup_{t \in [0, T]} \| \partial_t \mathcal{D}_{a, \gamma}^{\alpha} (v_1, v_2, v_3)(t) \|_{\widetilde{L}^p} \\
\lesssim \sup_{t \in [0, T]} \| \mathcal{H}_{a, \gamma} (U_\alpha(t)v_1(t), U_\alpha(t)v_2(t), U_\alpha(t)v_3(t)) \|_{\widetilde{L}^p} \\
\lesssim \sup_{t \in [0, T]} \prod_{\ell=1}^{3} \| \ell(t) \|_{L^2 \cap \widetilde{L}^p} \lesssim \prod_{\ell=1}^{3} \| \ell(t) \|_{\widetilde{Z}_{t, 0}^{\alpha, \gamma}(T) \cap \widetilde{Z}_{t, 0}^{1, \gamma}(T)}.
\]
(2) Here it remains to estimate the $\widetilde{W}_{\infty, 0}^{2}(T)$-semi norm which follows in a similar way as the $\widetilde{W}_{\infty, 0}^{p}(T)$ estimate above.

\section*{Second proof of Theorem 1.5.} For $\alpha \in (\frac{2d}{2d-1}, 2)$, we assume $d \geq 2$ and initial data to be radial.
\begin{itemize}
\item \textbf{Case I:} $0 < \gamma < \min\{\alpha, d(\frac{1}{p} - \frac{1}{t})\}$ when $1 \leq p < 2$ and $0 < \gamma < \min\{\alpha, 2d(\frac{1}{p} - \frac{1}{t})\}$ when $2 < p \leq \infty$.
\end{itemize}

Applying $U_\alpha(-t)$ to the Duhamel’s formula, we rewrite (1.1) as
\begin{equation}
\phi_k(t) = \psi_0,k + \int_{0}^{t} U_\alpha(-s)(H_\psi \psi_k)(s)ds - \int_{0}^{t} U_\alpha(-s)(F_\psi \psi_k)(s)ds =: \Phi_{\psi_0,k}(\phi).
\end{equation}
For $b > b, T > 0$, let $V_b^{T}(v_0) = \{ v \in \tilde{Z}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{Z}_{\infty, 0}^{p}(T) : \| v \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} \leq b, v(0) = v_0 \}$. We set $V_b^{T}(\psi_0) = V_b^{T}(\psi_0, 1) \times V_b^{T}(\psi_0, 2) \times \cdots \times V_b^{T}(\psi_0, N)$, and the distance on it by $d(u, v) = \max \{ \| u_j - v_j \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} : j = 1, 2, \cdots, N \}$. Next, we show that the mapping $\Phi_{\psi_0}$ defined by (3.31) takes $V_b^{T}(\psi_0)$ into itself for suitable choice of $b > 0$ and small $T > 0$. In fact, taking $0 < T \leq 1$ and as the terms with integral sign in (3.31) is combination of $D_{a, \gamma}^{\alpha}(\phi_k, \phi_l, \phi_m)$’s, by Lemma 3.10, we obtain
\begin{equation}
\| \Phi_{\psi_0,k}(\phi) \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} \lesssim \sum_{l=1}^{N} \| \phi_k \|_{\tilde{Z}_{t, 0}^{1, \gamma}(T) \cap \tilde{Z}_{t, 0}^{1, \gamma}(T)}^2 \lesssim \sum_{l=1}^{N} \| \phi_l \|_{\tilde{Z}_{t, 0}^{1, \gamma}(T) \cap \tilde{Z}_{t, 0}^{1, \gamma}(T)}^2.
\end{equation}
By Hölder inequality, we have $\| v_1 \|_{\tilde{W}_{2, 0}^{1}(T)} \leq T^{1 - \frac{\gamma}{p}} \| v_1 \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T)}$ and $\| v_1 \|_{\tilde{W}_{2, 0}^{1}(T)} \leq T \| v_1 \|_{\tilde{W}_{\infty, 0}^{p}(T)}$. Therefore, for $0 < T < 1$, we have
\begin{equation}
\| v_1 \|_{\tilde{Z}_{t, 0}^{1, \gamma}(T) \cap \tilde{Z}_{t, 0}^{1, \gamma}(T)} \lesssim \| v_1(0) \|_{\tilde{L}^p \cap \tilde{L}^p} + T^{\frac{1 - \gamma}{\alpha}} \| v_1 \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)}.
\end{equation}
Hence
\begin{equation}
\| \Phi_{\psi_0,k}(\phi) \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} \lesssim \sum_{l=1}^{N} \| \phi_l \|_{\tilde{Z}_{t, 0}^{1, \gamma}(T) \cap \tilde{Z}_{t, 0}^{1, \gamma}(T)}^2 \lesssim \sum_{l=1}^{N} \| \phi_l \|_{\tilde{Z}_{t, 0}^{1, \gamma}(T) \cap \tilde{Z}_{t, 0}^{1, \gamma}(T)}^2.
\end{equation}
Set $b = 2cN \| \psi_0 \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)}^3$ then choose $T > 0$ small enough to get $\| \Phi_{\psi_0,k}(\phi) \|_{\tilde{W}_{\alpha, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} \leq b$. It follows that $\Phi(\phi) \in V_b^{T}(\psi_0)$. For $u, v \in V_b^{T}(\psi_0)$ by trilinearity, Lemma 3.10 and (3.32)
\begin{equation}
d(\Phi_{\psi_0,k}(u), \Phi_{\psi_0,k}(v)) \lesssim T^{1 - \frac{\gamma}{d} \left( \| \psi_0 \|_{\tilde{L}^p \cap \tilde{L}^p} + T^{2(1 - \frac{\gamma}{\alpha}) \beta} \right)} \sum_{l=1}^{N} \| u_l - v_l \|_{\tilde{W}_{2, \gamma, 0}^{2}(T) \cap \tilde{W}_{\infty, 0}^{p}(T)} \lesssim T^{1 - \frac{\gamma}{d} N \left( \| \psi_0 \|_{\tilde{L}^p \cap \tilde{L}^p} + T^{2(1 - \frac{\gamma}{\alpha}) \beta} \right)} d(u, v).
\end{equation}
Then $\Phi_{\psi_0} : V_b^{T}(\psi_0) \to V_b^{T}(\psi_0)$ is a contraction for further small enough $T > 0$ if needed.
\begin{itemize}
\item \textbf{Case II:} $0 < \gamma < 2d \left( \frac{1}{2} - \frac{1}{p} \right)$ with $2 < p \leq \infty$.
\end{itemize}
Lemma 3.10 (2) and similar argument as in Case I give the solution \( \phi \in (\tilde{Z}^2_{\infty,0}(T) \cap \tilde{Z}^p_{\infty,0}(T))^N \) to (3.31).

Remark 3.11. Note that the second proof gives the existence of solution in Zhou spaces. So it adds Zhou space regularity to the solutions given by the first proof. For \( 0 < \gamma < 1 \) such solutions are in \( C([0,T],\tilde{L}^p(\mathbb{R}^d)) \) (see Lemma 3.9).

3.3. Global well-posedness in \( L^p \cap L^2 \). We extend the local solution established in Theorem 1.1 globally. Let \( \phi = (\phi_1,\phi_2, \cdots, \phi_N) \) be the local solution (given by Theorem 1.1) to (3.4) which is in 
\( (C([0,T],L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)))^N \) for any \( 0 < T < T_0 \). We start with the following lemma.

Lemma 3.12. On the time interval \([0,T_0)\), the local solution (given by Theorem 1.1) \( \psi(t) = (U(t)\phi_1(t), \cdots, U(t)\phi_N(t)) \) coincides with the global \( L^2 \)-solution for the initial datum \( \psi_0 = \psi(0) \) given by Proposition 2.4.

Proof. The assertion follows from uniqueness of local solution given by Theorem 1.1 (particularly see the metric space defined in the Case I of the First proof of Theorem 1.1 in Subsection 3.1) and Proposition 2.4.

Proposition 3.13. Assume \( 0 < \gamma < d/2 \). Let \( T_0 > 0 \) be such that for any \( 0 < T < T_0 \) the local solution \( \phi \) of (3.5) exists in \( C([0,T],L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N \). Then \( \sup_{t \in [0,T_0]} \|\phi(t)\|_{(LP)^N} < \infty. \)

Proof. We fix \( T \in (0,T_0) \) and \( t \in [0,T] \). Taking (3.5) into account, we obtain
\[
\|\phi_k(t)\|_{L^p} \lesssim \|\psi_{0,k}\|_{L^p} + \sum_{l=1}^N \sum_{j=1}^2 \|I^j_{l,k,l}(t)\|_{L^p} + \sum_{l=1}^N \sum_{j=1}^2 \|I^j_{l,k,l}(t)\|_{L^p},
\]
where \( I^j_{k,l,k} \) are given by (3.6). Since the solution of (5.16) in \( C([0,T],L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))^N \) for all \( 0 < T < T_0 \), we have a conservation of \( L^2 \)-norm i.e. \( \|\phi_k(t)\|_{L^2} = \|\phi_k(0)\|_{L^2} \). Using Proposition 2.9(1), we have
\[
\|I^j_{k,l,m}(t)\|_{L^p} \lesssim \int_0^t s^{-\gamma}\|\phi_k(s)\|_{L^2}\|\phi_l(s)\|_{L^2}\|\phi_m(s)\|_{L^p} ds \\
\quad \lesssim T_0^{-1-\gamma-1/q_4}\|\phi_k\|_{L^p_{T_0}(L^2)}\|\phi_l\|_{L^p_{T_0}(L^2)}\|\phi_m\|_{L^p_{T_0}(L^2)},
\]
where \( q_4 \) is chosen so that \( q_4 > \frac{1}{1-\gamma} \). Let \( q_2 \) and \( \rho \) be given as in (3.9) and (3.10) respectively. By (3.11), we have
\[
\|I^2_{k,l,m}(t)\|_{L^p} \lesssim T_0^{-d-\gamma-d/\rho}\|\psi_k\|_{L^p_{T_0}(L^{2q_2} \cap L^2)}\|\psi_l\|_{L^p_{T_0}(L^{2q_2} \cap L^2)}\|\phi_m\|_{L^p_{T_0}(L^{2q_2} \cap L^2)}.
\]
Note that \( d-\gamma-d/\rho > 0 \). Since \( (q_2,2\rho) \) is an admissible pair, in view of Lemma 3.12 and Proposition 2.4, we have \( \|\psi_j\|_{L^p_{T_0}(L^{2q_2} \cap L^2)} < \infty. \) It follows that
\[
\|I^2_{k,l,m}(t)\|_{L^p} \leq C_{T_0}\|\phi_m\|_{L^p_{T_0}(L^{2q_2} \cap L^2)},
\]
Thus we have from (3.33), (3.34) and (3.35) that
\[
\|\phi(t)\|_{(LP)^N} \lesssim C_{\psi_0,T_0} + NC_{\psi_0,T_0}\|\phi\|_{L^p_{T_0}(LP)^N},
\]
where \(q = \max\{q_3, q_4\}\). Therefore
\[
\|\phi(t)\|_{(L^p)^N}^q \lesssim C_{\psi_0,T_0}^q + N^q C_{\psi_0,T_0}^q \int_0^t \|\phi(t)\|_{(L^p)^N}^q \, dt.
\]
By Gronwall’s lemma \(\|\phi(t)\|_{(L^p)^N}^q \lesssim C_{\psi_0,T_0}^q (1 + N^q C_{\psi_0,T_0}^q t e^{C_{\psi_0,T_0} N \cdot t})\) which is desired. \(\square\)

Let \(\psi(t) = (\psi_1(t), \cdots, \psi_N(t))\) be a global \(L^2\)-solution given by Proposition 2.4. We define
\[
T_+(\psi_0) = \sup \left\{ T > 0 : U(-t)\psi(t)|_{[0,T] \times \mathbb{R}^d} \in C([0,T], L^p(\mathbb{R}^d))^N \right\}
\]
where \(U(-t)\psi(t) = (U(-t)\psi_1(t), \cdots, U(-t)\psi_N(t))\). By Theorem 1.1, we have \(T_+(\psi_0) > 0\).

**Proposition 3.14.** Assume \(T_+(\psi_0) < \infty\). Then \(\lim_{t \to T_+(\psi_0)} \|U(-t)\psi(t)\|_{(L^p)^N} = \infty\).

**Proof.** We point out that the assertion relies on the fact that the local existence time \(T\), from Theorem 1.1, depends only on \(\|\psi_0\|_{(L^2 \cap L^p)^N}, \gamma, d, N\). Now the proof is standard, see e.g., [21, Lemma 5.4] for the Hartree equation, and so we omit the details. \(\square\)

**Proof of Theorem 1.3.** It is enough to prove that \(T_+(\psi_0) = \infty\). If not, Proposition 3.14 implies \(\lim_{t \to T_+(\psi_0)} \|U(-t)\psi(t)\|_{(L^p)^N} = \infty\) contradicting Proposition 3.13 as \(T_+(\psi_0) > 0\). The last assertion of the theorem follows from Proposition 2.4 and Hausdorff-Young inequality. \(\square\)

### 3.4. Global well-posedness in \(\hat{L}^p \cap L^2\).

**Proof of Theorem 1.7.** The proof strategy is similar to the proof of Theorem 1.3. Specifically, taking Theorem 1.5 and Proposition 2.4 into account, to prove Theorem 1.7, it is enough to show that the \((\hat{L}^p)^N\)-norm of the solution remains bounded in finite time. Let \(t \in [0, T]\).

- **Case I:** \(0 < \gamma < \min\{\alpha, \frac{d}{2}\}\), \(1 \leq p < 2\).

  By (3.25), we have
  \[
  \|\psi_k(t)\|_{\hat{L}^p} \leq \|\psi_{0,k}\|_{\hat{L}^p} + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_l(s), \psi_l(s), \psi_k(s))\|_{\hat{L}^p} \, ds \tag{3.37}
  \]
  \[
  + \sum_{l=1}^N \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_l(s))\|_{\hat{L}^p} \, ds.
  \]

  By Propositions 2.11(1) and 2.4, we have
  \[
  \int_0^t \|\mathcal{H}_{a,\gamma}(\psi_k(s), \psi_l(s), \psi_m(s))\|_{\hat{L}^p} \, ds \lesssim \int_0^t \|\psi_k(s)\|_{L^2} \|\psi_l(s)\|_{L^2} \|\psi_m(s)\|_{\hat{L}^p \cap L^2} \, ds
  \]
  \[
  \lesssim \int_0^t \|\psi_m(s)\|_{\hat{L}^p \cap L^2} \, ds = T \|\psi_{0,m}\|_{L^2} + \int_0^t \|\psi_m(s)\|_{\hat{L}^p} \, ds.
  \]

  Using this and (3.37), we have
  \[
  \|\psi(t)\|_{(\hat{L}^p)^N} \lesssim \|\psi_0\|_{(\hat{L}^p)^N} + NT \|\psi_0\|_{(L^2)^N} + N \int_0^t \|\psi(s)\|_{(\hat{L}^p)^N} \, ds.
  \]

  Now the result follows by Gronwall’s lemma.

- **Case II:** \(0 < \gamma < \min\{\alpha, \frac{d}{2}\}\) \(1 \leq p \leq \infty\).
For $\alpha \in (\frac{2d}{2d-1}, 2)$, we assume $d \geq 2$ and initial data is radial. By (3.26) and (3.27), we have
\[
\|J_{k,l,m}(t)\|_{L^p} \leq T^{\frac{1}{12}} \|\psi_k\|_{L_\infty^\infty(L^2)} \|\psi_l\|_{L_\infty^\infty(L^2)} \|\psi_m\|_{L_\infty^\infty'(L^p)} + \|\psi_k\|_{L_{T, T}^2(L^{2\theta'})} \|\psi_l\|_{L_{T, T}^2(L^{2\theta'})} \|\psi_m\|_{L_{T, T}^2'(L^p)}.
\]
By (3.37) and Strichartz estimates, we have $\|\psi_k(t)\|_{L^p} \lesssim \|\psi_{0,k}\|_{L^p} + (1 + T^{\frac{1}{12}}) \sum_{l=1}^N \|\psi_l\|_{L_{T, T}^2'(L^p)}$ and so $\|\psi(t)\|_{(L^p)^N} \lesssim C_{\psi_0, T} (1 + N \|\psi\|_{L_{T, T}^2'(L^p)^N})$. Gronwall’s lemma gives the bound.

\[\square\]

3.5. **Improved well-posedness in 1D.** We have proved the result (local and global) if $0 < \gamma < \frac{1}{2}$ for $d = 1$, see Theorems 1.1 - 1.7. Now we improve it to $0 < \gamma < 1$ for global existence. To do this it is enough to prove it for $\frac{1}{2} \leq \gamma < 1$. In the proof, we impose the condition $\frac{2}{3\theta} \leq \gamma < 1$ for $p \in (\frac{4}{3}, 2]$ and $\frac{2}{5\theta} < \gamma < 1$ for $p \in [2, 4)$. Note that $\frac{2}{3\theta} < \frac{1}{2}$ for $p \in (\frac{4}{3}, 2]$ and $\frac{2}{5\theta} < \frac{1}{2}$ for $p \in [2, 4)$. The extra ingredient we use here is Lemma 3.15 below.

**Lemma 3.15** (Generalized Strichartz estimate [10, 15]). $\|U(t)\phi\|_{L^{3\theta}(R \times R)} \lesssim \|\phi\|_{L^p(R)}$ for $\frac{4}{3} < p \leq 2$. As a consequence, by the duality argument, for $2 \leq p < 4$, $\sup_{J \subset J} \|\int_J U(-s)F(s)ds\|_{L^p} \leq \|F\|_{L^{3\theta'}(J \times R)}$.

**Proof of Theorem 1.9.** As an application of Lemma 3.15, we shall obtain some improved estimate for $\|I_{k,l,m}\|_{L^p}$ (see (3.7)). Specifically, compare the estimate (3.8) to (3.38) below. We shall see that this will play a vital role to improve the range of the exponent $\gamma$ in the Hartree factor.

- **Step A I:** Improving the local result for $L^p$ space.

Note that $\|\varphi\|_{L^p} = \|F^{-1}\varphi\|_{L^p} = \|\overline{F}\varphi\|_{L^{p'}} = \|\overline{F}\varphi\|_{L^{p'}}$ and by (2.1) $FM_sF^{-1} = U(-1/16\pi^2 s)$ as
\[
FM_sF^{-1}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} e^{i|x|^2/4s} F^{-1}(\varphi)(x)dx = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{i|x|^2/4s} F\varphi(x)dx = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} e^{-4\pi^2 i |x|^2 (-1/16\pi^2 s)} F\varphi(x)dx = [U(-1/16\pi^2 s)\varphi](\xi).
\]
In view of these and (3.7), we obtain $\|I_{k,l,m}(t)\|_{L^p}$ is less or equal to
\[
\left\| \int_0^t s^{-\gamma} M_s^{-1} \mathcal{H}_{a,\gamma, s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^p}
= \left\| \int_0^t s^{-\gamma} FM_s^{-1} \mathcal{H}_{a,\gamma, s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^{p'}}
= \left\| \int_0^t s^{-\gamma} FM_s \mathcal{H}_{a,\gamma, s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^{p'}}
= \left\| \int_0^t s^{-\gamma} U(-1/16\pi^2 s) FM_s \mathcal{H}_{a,\gamma, s}(M_s \phi_k(s), RM_s \phi_l(s), M_s \phi_m(s)) ds \right\|_{L^{p'}}
= \left\| \int_{1/8}^\infty s^{\gamma - 2} U \left( \frac{-s}{16\pi^2} \right) FM_s \mathcal{H}_{a,\gamma, 1/s}(M_{1/s} \phi_k(1/s), RM_{1/s} \phi_l(1/s), M_{1/s} \phi_m(1/s)) ds \right\|_{L^{p'}}.
\]
Using Lemma 3.15 and changing the s-variable, we get that
\[
\|I_{k,l,m}(t)\|_{L^p} \lesssim \left\| s^{-\gamma} F\mathcal{H}_{a,\gamma,1/s}(M_{1/s}\phi_k(1/s), R M_{1/s}\phi_l(1/s), M_{1/s}\phi_m(1/s)) \right\|_{L^\infty([16\pi^2/t,\infty) \times \mathbb{R})}
\]
\[
\leq \left\| s^{-\gamma} F\mathcal{H}_{a,\gamma,1/s}(M_{1/s}\phi_k(1/s), R M_{1/s}\phi_l(1/s), M_{1/s}\phi_m(1/s)) \right\|_{L^\infty([1/t,\infty) \times \mathbb{R})}
\]
\[
= \left\| s^{2-\gamma-2/\tilde{r}} F\mathcal{H}_{a,\gamma,s}(M_{s}\phi_k(s), R M_{s}\phi_l(s), M_{s}\phi_m(s)) \right\|_{L^\infty((0,t] \times \mathbb{R})},
\]
where \(\tilde{r} = (3p')\). Since \(U(-t)\psi_k(t) = \phi_k(t)\), by (2.3) and (3.1), we have
\[
\tilde{H}_{a,\gamma,s}(M_{s}\phi_k(s), R M_{s}\phi_l(s), M_{s}\phi_m(s)) = \left[ (|\cdot|^{-\gamma} * S_{a,s})\Omega(\psi_k(s), \psi_l(s)) \right] * M_{s}\phi_m(s).
\]
In view of this, we may obtain
\[
\| F\mathcal{H}_{a,\gamma,s}(M_{s}\phi_l(s), R M_{s}\phi_l(s), M_{s}\phi_m(s)) \right\|_{L^\tilde{r}}
\]
\[
\leq \| F \left[ (|\cdot|^{-\gamma} * S_{a,s})\Omega(\psi_k(s), \psi_l(s)) \right] \|_{L^{3p'/2}} \| F M_{s}\phi_m(s) \|_{L^{p'}}
\]
\[
\lesssim \| |\cdot|^{-\gamma} * F \Omega(\psi_k(s), \psi_l(s)) \|_{L^{3p'/2}} \| \phi_m(s) \|_{L^p} \lesssim \| F \Omega(\psi_k(s), \psi_l(s)) \|_{L^\tilde{r}} \| \phi_k(s) \|_{L^p},
\]
where \(\tilde{R} = \left(1 + \frac{2}{3p'} - \gamma \right)^{-1}\). Using Lemma 3.3 we have
\[
\| F\mathcal{H}_{a,\gamma,s}(M_{s}\phi_l(s), R M_{s}\phi_l(s), M_{s}\phi_m(s)) \right\|_{L^\tilde{r}} \leq |s|^{-1/\tilde{R}} \| \psi_k(s) \|_{L^2\tilde{R}} \| \psi_l(s) \|_{L^2\tilde{R}} \| \phi_m(s) \|_{L^p}.
\]
Note that \(3 - \gamma - 2/\tilde{r} - 1/\tilde{R} = 0\) and hence by Hölder’s inequality
\[
\|I_{k,l,m}(t)\|_{L^p} \lesssim \| \psi_k \|_{L^\infty_t(L^2\tilde{R})} \| \psi_l \|_{L^\infty_t(L^2\tilde{R})} \| \phi_m \|_{L^{2/(2-\gamma)}(L^p)}
\]
\[
\lesssim T^{1-\frac{2}{\tilde{r}}} \| \psi_k \|_{L^\infty_t(L^2\tilde{R})} \| \psi_l \|_{L^\infty_t(L^2\tilde{R})} \| \phi_m \|_{L^{\infty_t}(L^p)},
\]
where \(\tilde{Q} = \left(\frac{2}{1} - \frac{1}{6p'}\right)^{-1}\). Note that \(\tilde{Q}, \tilde{R} \geq 2\) by the conditions imposed on \(\gamma\). Let \(q_1 = \frac{8}{7}\) and \(r = \frac{4}{2-\gamma}\). We define
\[
V_b^T = \{ v \in L^\infty_t(L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)) : \| v \|_{L^\infty_t(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)} \leq b, \| U(t)v(t) \|_{L^\infty_t(L^2(\mathbb{R}^d))} \leq b, \| U(t)v(t) \|_{L^q_1(L^r)} \leq b \}
\]
and \(V_b^T = (V_b^T)^N\). Now arguing as in Case I of the proof of Theorem 1.1, we can establish the local well-posedness of (1.1) with \(0 < \gamma < 2\) in \(L^p(\mathbb{R}) \cap L^2(\mathbb{R})\).

**Step A II:** Improving the global result for \(0 < \gamma < 1\) for \(L^p\) space.

Note that from (3.5) and (3.38) we have
\[
\| \phi_k(t) \|_{L^p} \lesssim \| \psi_0,k \|_{L^p} + \sum_{l=1}^N \| \psi_l \|_{L^\infty_t(L^2\tilde{R})} \| \psi_l \|_{L^\infty_t(L^2\tilde{R})} \| \phi_l \|_{L^{2/(2-\gamma)}(L^p)}
\]
\[
+ \sum_{l=1}^N \| \psi_k \|_{L^\infty_t(L^2\tilde{R})} \| \psi_l \|_{L^\infty_t(L^2\tilde{R})} \| \phi_l \|_{L^{2/(2-\gamma)}(L^p)}.
\]
Now \((\tilde{Q}, 2\tilde{R})\) being admissible, Strichartz estimate gives
\[
\| \phi_k(t) \|_{L^p} \lesssim \| \psi_0,k \|_{L^p} + \sum_{l=1}^N \| \phi_l \|_{L^{2/(2-\gamma)}(L^p)}.
\]
Now we can proceed as before in Subsection 3.3.

**Step B I:** Improving the local result for \(\tilde{L}^p\)-space.
Using Lemma 3.15 we have that
\[
\left\| \int_0^t U(t-s)H_{\gamma,\psi}(\psi_k)(s) \, ds \right\|_{\hat{L}^p} \lesssim \sum_{l=1}^N \| H_{\alpha,\gamma}(\psi_l, \psi_k) \|_{L^r([0,t] \times \mathbb{R})}.
\]

Now using Hölder, Hausdorff-Young and Hardy-Littlewood-Sobolev
\[
\| H_{\alpha,\gamma}(\psi_k(s), \psi_l(s), \psi_m(s)) \|_{L^r} \leq \left\| | \cdot |^{-\gamma} * | \psi_k(s) \psi_l(s)| \right\|_{L^\bar{R}} \| \psi_m \|_{\hat{L}^p}
\]
\[
\lesssim \| \psi_k(s) \psi_l(s) \|_{L^\bar{R}} \| \psi_m \|_{\hat{L}^p} \leq \| \psi_k(s) \|_{L^2} \| \psi_l(s) \|_{L^2} \| \psi_m \|_{\hat{L}^p},
\]
where \( \bar{R} = \frac{3\alpha}{2}, \) \( R = \left( \frac{\bar{R}}{2} - \gamma - \frac{2}{mp} \right)^{-1} \). Therefore Hölder’s inequality in t-variable we have (recall \( J_{k,l,m} \) from (3.26)), we have
\[
\| J_{k,l,m}(t) \|_{\hat{L}^p} \lesssim \| \psi_k \|_{L^Q([0,T_0], L^{2R})} \| \psi_l \|_{L^Q([0,T_0], L^{2R})} \| \psi_m \|_{L^{2/(2-\gamma)}([0,t], \hat{L}^p)}
\]
\[
\lesssim T^{1-\frac{1}{2}} \| \psi_k \|_{L^Q([0,T_0], L^{2R})} \| \psi_l \|_{L^Q([0,T_0], L^{2R})} \| \psi_m \|_{L^{2/(2-\gamma)}([0,t], \hat{L}^p)},
\]
where \( Q = \left( \frac{\bar{R}}{2} + \frac{1}{mp} - \frac{1}{b} \right)^{-1} \). Note that \( Q, R \geq 2 \) by the conditions imposed on \( \gamma \). Let \( q_1 = \frac{2}{\gamma} \) and \( r = \frac{4}{\gamma} - 1 \), and for \( T, b > 0 \), introduce the space
\[
U_b^Q = \left\{ v \in L_T^\infty(L^2(\mathbb{R}^d) \cap \hat{L}^p(\mathbb{R}^d)) : \| v \|_{L_T^\infty(\hat{L}^p)} \leq b, \| v \|_{L_T^{q_1}([0,T])} \leq b, \| v \|_{L_T^Q(\hat{L}^p)} \leq b \right\}.
\]

Now we proceed as in Case I in Subsection 3.2.

**Step B II:** Improving the global result for \( \hat{L}^p \)-space.

By (3.37) and (3.39) we have
\[
\| \psi_k(t) \|_{\hat{L}^p} \lesssim \| \psi_{0,k} \|_{\hat{L}^p} + \sum_{l=1}^N \| \psi_l \|^2_{L^Q([0,T], L^{2R})} \| \psi_k \|_{L^{2/(2-\gamma)}([0,T], \hat{L}^p)}
\]
\[
+ \sum_{l=1}^N \| \psi_k \|_{L^Q([0,T], L^{2R})} \| \psi_l \|_{L^Q([0,T], L^{2R})} \| \psi_l \|_{L^{2/(2-\gamma)}([0,T], \hat{L}^p)}
\]

Strichartz estimate gives \( \| \psi_k(t) \|_{\hat{L}^p} \lesssim \| \psi_{0,k} \|_{\hat{L}^p} + \sum_{l=1}^N \| \psi_l \|_{L^{2/(2-\gamma)}([0,T], \hat{L}^p)} \) as \((Q, 2R)\) is admissible.

Now we can proceed as before in Subsection 3.4. \( \square \)

4. Failure of \( C^3 \)-smoothness in mere \( \hat{L}^p \) spaces

4.1. Proof of Theorem 1.10.

**Proof of Theorem 1.10.** It is known\(^1\) that if the map \( U(t) \) is \( C^3 \)-smooth at zero in \((\hat{L}^p(\mathbb{R}^d))^N\), then the necessary condition is that there exist \( C > 0 \) such that
\[
\left\| \frac{\partial^3 [U(t)(\delta \psi_0)]}{\partial \delta^3} \right\|_{(\hat{L}^p)^N} \leq C \| \psi_0 \|^3_{(\hat{L}^p)^N} \text{ for all } \psi_0 \in (\hat{L}^p(\mathbb{R}^d))^N.
\]

\(^1\)In [6], Bourgain introduced this approach to establish failure of \( C^3 \)-smoothness for the solution map of KdV and mKdV, see also [32]. Since then, many authors have used this approach, see for example, [14, Proposition 4.1] for cubic nonlinear half-wave equation.
In the following we shall show that estimate (4.1) cannot hold with a constant $C$ independent of $\psi_0 \in (\hat{L}^p(\mathbb{R}^d))^N$. To do this, consider the problem (with $K = | \cdot |^{-\gamma}$)

$$i \partial_t \psi_k - (-\Delta)^{\alpha/2} \psi = \sum_{l=1}^N (K * |\psi_l|^2) \psi_k - (K * (\psi_k \bar{\psi}_l)) \psi_l, \quad \psi_k(0) = \delta \psi_{0,k}$$

where $\delta \geq 0$, $\psi_0 = (\psi_{0,1}, \cdots, \psi_{0,N})$, $k = 1, 2, \ldots, N$. By Duhamel’s formula, we have

$$\psi_k(\delta)(t) = U_\alpha(t) \delta \psi_{0,k} - i \int_0^t U_\alpha(t - \tau) [(K * (\psi_l(\delta) \bar{\psi}_l(\delta))) \psi_k(\delta) - (K * (\psi_k(\delta) \bar{\psi}_l(\delta))) \psi_l(\delta)](\tau)d\tau$$

(4.2)

where Einstein’s convention is used for summation. Note that one has

$$U(t)(\delta \psi_0) = \psi(\delta, t) = (\psi_1(\delta)(t), \cdots, \psi_N(\delta)(t)) := \psi(\delta)(t).$$

Therefore, in order to show an estimate (4.1) fail (and hence failure of $C^1-$smoothness), it is suffice to show that the following estimate fails

$$\left\| \frac{\partial^3 \psi_k}{\partial \delta^3}(0,t) \right\|_{\hat{L}^p} \leq C \|\psi_0\|^3_{(\hat{L}^p)^N}$$

(4.3)

for at least one $k$. To this end, we shall first compute $\frac{\partial^3 \psi_k}{\partial \delta^3}(0,t)$. Put $I_{k,l,m}(\delta, t, \cdot) = \int_0^t U_\alpha(t - \tau)[(K * (\psi_k(\delta) \bar{\psi}_l(\delta))) \psi_m(\delta)](\tau)d\tau$. By straightforward calculations, we get

$$\frac{\partial I_{k,l,m}}{\partial \delta} = \int_0^t U_\alpha(t - \tau)[(K * (\psi_k \bar{\psi}_l + \psi_k \bar{\psi}_l)) \psi_m + (K * (\psi_k \bar{\psi}_l)) \frac{\partial \psi_m}{\partial \delta}] (\tau)d\tau,$$

$$\frac{\partial^2 I_{k,l,m}}{\partial \delta^2} = \int_0^t U_\alpha(t - \tau)[(K * (\psi_k \frac{\partial^2 \bar{\psi}_l}{\partial \delta^2} + 2 \psi_k \frac{\partial \bar{\psi}_l}{\partial \delta} + \frac{\partial^2 \psi_k}{\partial \delta^2}) \psi_m + 2(K * (\psi_k \bar{\psi}_l)) \frac{\partial \psi_m}{\partial \delta} + (K * (\psi_k \bar{\psi}_l)) \frac{\partial^2 \psi_m}{\partial \delta^2}] (\tau)d\tau,$$

$$\frac{\partial^3 I_{k,l,m}}{\partial \delta^3} = \int_0^t U_\alpha(t - \tau)[(K * (\psi_k \frac{\partial^3 \bar{\psi}_l}{\partial \delta^3} + 3 \psi_k \frac{\partial^2 \bar{\psi}_l}{\partial \delta^2} + 3 \frac{\partial^2 \psi_k}{\partial \delta^2} \frac{\partial \bar{\psi}_l}{\partial \delta} + \frac{\partial^3 \psi_k}{\partial \delta^3}) \psi_m + 3(K * (\psi_k \bar{\psi}_l)) \frac{\partial^2 \psi_m}{\partial \delta^2} + (K * (\psi_k \bar{\psi}_l)) \frac{\partial^3 \psi_m}{\partial \delta^3}] (\tau)d\tau.$$
Then (4.3) will fail for \( k = 1 \) if
\[
\| \mathcal{A}_1(t) \psi_0 \|_{L^p} \leq C \| \psi_0 \|^3_{(L^p)^N} \tag{4.4}
\]
fails. Below we adopt the technique from [8] (where the Hartree case is treated with \( p = \infty \)) to our scenario.

Let \( \psi_0 = (\psi_{0,0}, \psi_{0,2}, 0, \cdots 0) \in \mathcal{S}(\mathbb{R}^d)^N \) so that
\[
\mathcal{A}_1(t) \psi_0 = i \int_0^t U(t - \tau) \left[ (K*|U_\alpha(\tau)\psi_{0,2}|^2) U_\alpha(\tau)\psi_0,1 \right] - (K* (U_\alpha(\tau)\psi_{0,1} U_\alpha(\tau) \psi_{0,2})) U_\alpha(\tau)\psi_{0,2}] d\tau.
\]
Define a family \( \{ \psi^h_0 \}_{h > 0} \) of functions by
\[
\psi^h_0(x) = h^\lambda \psi_0(hx), \quad (x \in \mathbb{R}^d, \lambda > 0).
\]
For all \( h > 0, \)
\[
\| \psi^h_0 \|_{L^p} = h^{\lambda - d/p} \| \psi_{0,2} \|_{L^p}.
\]
Choose \( \lambda = d/p \) so that
\[
\| \psi^h_0 \|_{L^p} = \| \psi_{0,2} \|_{L^p} \quad \text{for all } \ h > 0,
\]
and thus RHS of (4.4) remains constant for all \( h > 0. \)

Next, we develop the expression of \( \| \mathcal{A}_1(\psi_0) (t) \|_{L^p} \) and will show that \( \| \mathcal{A}_1(\psi_0)(t) \|_{L^p} \to \infty \) as \( h \to 0, \) contradicting (4.4). We note that (with \( c_\alpha = -(2\pi)^\alpha \))
\[
\mathcal{F} \left[ \left( K* \left( U_\alpha(s) \psi^h_{0,2} \right) U_\alpha(s) \psi^h_{0,1} \right) \right] (\xi)
\]
\[
= \int_{\mathbb{R}^d} \mathcal{K}(\xi - y) \left[ \mathcal{F} \left( U_\alpha(s) \psi^h_{0,2} \right) \mathcal{F} \left( U_\alpha(s) \psi^h_{0,1} \right) \right] (\xi - y) dy
\]
\[
= \int_{\mathbb{R}^2} e^{i\alpha s(y + |\xi - y| - |\alpha|)} \mathcal{K}(\xi - y) \mathcal{F} \psi^h_{0,2}(y - z) \mathcal{F} \psi^h_{0,1}(z) (\xi - y - z) dy dz
\]
\[
= h^{3(\lambda - d)} \int_{\mathbb{R}^2} e^{i\alpha s(y + |\xi - y| - |\alpha|)} \mathcal{K}(\xi - y) \mathcal{F} \psi^h_{0,2}(y - z) \mathcal{F} \psi^h_{0,1}(z) (\xi - y - z) dy dz.
\]
We rewrite that
\[
\mathcal{A}_1(\psi_0)(t) = \int_0^t U_\alpha(t - s) g(s) ds =: F(t) \tag{4.6}
\]
with
\[
g(s) = -i \left( K* \left( U_\alpha(s) \psi_{0,2} \right) \right) U_\alpha(s) \psi_{0,1} + i \left( K* \left( U_\alpha(s) \psi_{0,1} \psi_0,2 \right) \right) U_\alpha(s) \psi_{0,2} \tag{4.7}
\]
Then
\[
\| \mathcal{A}_1(\psi^h_0)(t) \|_{L^p} = \| \mathcal{F} \mathcal{A}_1(\psi^h_0)(t) \|_{L^{p'}}
\]
\[
= \int_{\mathbb{R}^d} \left| \mathcal{F} \left[ U_\alpha(t - s) \left[ \left( K* \left( U_\alpha(s) \psi_{0,2} \right) \right) U_\alpha(s) \psi_{0,1} \right] \right] (\xi) \right| ds
\]
\[
- \int_0^t \mathcal{F} \left[ U_\alpha(t - s) \left[ \left( K* \left( U_\alpha(s) \psi^h_{0,2} \right) U_\alpha(s) \psi^h_{0,1} \right) \right] (\xi) \right] ds \]d\xi
\]
\[
= \int_{\mathbb{R}^d} \left| \int_0^t e^{i\alpha (t-s) |\xi|} \mathcal{F} \left[ \left( K* \left( U_\alpha(s) \psi_{0,2} \right) \right) U_\alpha(s) \psi_{0,1} \right] (\xi) ds \right| d\xi
\]
\[
- \int_0^t e^{i\alpha (t-s) |\xi|} \mathcal{F} \left[ \left( K* \left( U_\alpha(s) \psi^h_{0,1} \right) U_\alpha(s) \psi^h_{0,2} \right) \right] (\xi) ds \]d\xi =: I.
Performing the change of variables \((y \mapsto hy, z \mapsto hz, s \mapsto s/h^a)\), we obtain
\[
\int_0^t e^{ic_\alpha(t-s)|\xi|} \mathcal{F} \left( \left[ (K \ast \left( U_\alpha(s) \psi^h_{0,k} U_\alpha(s) \psi^h_{0,m} \right) \right) U_\alpha(s) \psi^h_{0,m} \right] (\xi) ds
\]
\[
= \int_0^t e^{ic_\alpha(t-s)|\xi|} h^3(\lambda - d) \int_{\mathbb{R}^2d} e^{ic_\alpha s(|y|+|\xi-y-z|)|\xi|} \widetilde{K}(\xi - y)
\]
\[
\widetilde{\psi}_{0,k}(\xi/h - y - z) \widetilde{\psi}_{0,l}(z/\xi) d\xi dy dz ds
\]
\[
= h^{3\lambda - d - \alpha} \int_0^{th^\alpha} e^{ic_\alpha(t-h^\alpha s)|\xi|} \int_{\mathbb{R}^2d} e^{ic_\alpha s(|y|+h^\alpha|\xi-h(y+z)|)|\xi|} \widetilde{K}(\xi - hy)
\]
\[
\widetilde{\psi}_{0,k}(\xi/h - y - z) \widetilde{\psi}_{0,l}(z/\xi) d\xi dy dz ds.
\]
In view of this we may rewrite
\[
I = \int_{\mathbb{R}^d} \left| h^{3\lambda - d - \alpha} \int_0^{th^\alpha} e^{ic_\alpha(t-h^\alpha s)|\xi|} \int_{\mathbb{R}^2d} e^{ic_\alpha s(|y|+h^\alpha|\xi-h(y+z)|)|\xi|} \widetilde{K}(\xi - hy)
\]
\[
\left(\widetilde{\psi}_{0,2}(\xi/h - y - z) \widetilde{\psi}_{0,1}(y) - \widetilde{\psi}_{0,1}(\xi/h - y - z) \widetilde{\psi}_{0,2}(y)\right) \widetilde{\psi}_{0,2}(z) d\xi dy dz ds \right| d\xi
\]
\[
= h^{d+(3\lambda - d - \alpha)p'} \int_{\mathbb{R}^d} \left| \int_0^{th^\alpha} e^{ic_\alpha(s-h^\alpha s)|\xi|} \int_{\mathbb{R}^2d} e^{ic_\alpha s(|y|+|\xi-y-z|)|\xi|} \widetilde{K}(h(\xi - y))
\]
\[
\left(\widetilde{\psi}_{0,2}(\xi - y - z) \widetilde{\psi}_{0,1}(y) - \widetilde{\psi}_{0,1}(\xi - y - z) \widetilde{\psi}_{0,2}(y)\right) \widetilde{\psi}_{0,2}(z) d\xi dy dz ds \right| d\xi
\]
\[
= h^{d+(3\lambda - d - \alpha + (\gamma - d))p'} \int_{\mathbb{R}^d} \left| \int_0^{th^\alpha} e^{ic_\alpha(s-h^\alpha s)|\xi|} \int_{\mathbb{R}^2d} e^{ic_\alpha s(|y|+|\xi-y-z|)|\xi|} \widetilde{K}(\xi - y)
\]
\[
\left(\widetilde{\psi}_{0,2}(\xi - y - z) \widetilde{\psi}_{0,1}(y) - \widetilde{\psi}_{0,1}(\xi - y - z) \widetilde{\psi}_{0,2}(y)\right) \widetilde{\psi}_{0,2}(z) d\xi dy dz ds \right| d\xi
\]
as the kernel \(\widetilde{K}\) is homogeneous of degree \(-\gamma\) (as \(a = 0\)). Using (4.5), we have
\[
I = h^{d+(3\lambda - 2d - \alpha + \gamma)p'} \left| \int_{\mathbb{R}^d} \left| \int_0^{th^\alpha} e^{ic_\alpha(s-h^\alpha s)|\xi|} \mathcal{F} \left( (K \ast \left( U_\alpha\psi^h_{0,2}(s)^2 \right) U_\alpha(s)\psi^h_{0,1}(s) \right) \right) (\xi) ds -
\]
\[
- \int_0^{th^\alpha} e^{ic_\alpha(s-h^\alpha s)|\xi|} \mathcal{F} \left( (K \ast \left( U_\alpha\psi^h_{0,1} U_\alpha(s)\psi^h_{0,2} \right) \right) (\xi) ds \right| d\xi
\]
\[
= h^{d+(3\lambda - 2d - \alpha + \gamma)p'} \left| \mathcal{A}_1(\psi_0^h)(th^\alpha) \right|_{L^p}^{p'}.
\]
Since \(\lambda = d/p\), combining the above equalities, we obtain
\[
\left| \mathcal{A}_1(\psi_0^h)(t) \right|_{L^p} \asymp h^{2d/p - d - \alpha + \gamma} \left| \mathcal{A}_1(\psi_0)(th^\alpha) \right|_{L^p}.
\]
(4.8)
Next we investigate more closely the quantity \(F\) given by (4.6). Taylor formula gives
\[
F(t) = F(0) + F'(0)t + \frac{t^2}{2} \int_0^1 (1 - \theta) F''(t\theta) d\theta.
\]
Note that \(F(0) = 0\) and hence for \(0 \leq t \leq 1\), we have
\[
\left| F(t) - F'(0)t \right|_{L^p} \leq t^2 \int_0^1 \left| F''(s\theta) \right|_{L^p} d\theta \leq t^2 \left| F'' \right|_{L^\infty([0,1])}.
\]
(4.9)
By Leibniz integral rule and Lemma 4.3 below, the first derivative of $F$ is given by

$$F'(t) = U_\alpha(0)g(t) + \int_0^t \frac{\partial}{\partial t}U_\alpha(t-s)g(s)ds = g(t) - i \int_0^t U_\alpha(t-s)(-\Delta)^{\alpha/2}g(s)ds$$

and similarly the second derivative of $F$ is given by

$$F''(t) = g'(t) - i(-\Delta)^{\alpha/2}g(t) - \int_0^t U_\alpha(t-s)(-\Delta)^\alpha g(s)ds.$$ 

Hence, we have

$$\|F''\|_{L^\infty([0,1],\mathbb{L}^p)} \leq \|g'\|_{L^\infty([0,1],\mathbb{L}^p)} + \|(-\Delta)^{\alpha/2}g\|_{L^\infty([0,1],\mathbb{L}^p)} + \|(-\Delta)^\alpha g\|_{L^\infty([0,1],\mathbb{L}^p)} < \infty,$$

as $\psi_0 \in \mathcal{S}(\mathbb{R}^d)^N$. Using (4.9) and the above, we have

$$\|F(t) - F'(0)t\|_{\mathbb{L}^p} \lesssim t^2$$

Using this and $F'(0) = g(0)$, we obtain $\|g(0)\|_{\mathbb{L}^p} \lesssim \|F(t)\|_{\mathbb{L}^p} + t^2$. Hence in particular

$$th^\alpha\|g(0)\|_{\mathbb{L}^p} \lesssim \|A_1(\psi_0)(th^\alpha)\|_{\mathbb{L}^p} + t^2h^{2\alpha}$$

and so by (4.8)

$$\|A_1(\psi_0^h)(t)\|_{\mathbb{L}^p} \gtrsim h^{2d/p-d-\alpha+\gamma}\|A_1(\psi_0)(th^\alpha)\|_{\mathbb{L}^p} \gtrsim th^{2d/p-d+\gamma}\|g(0)\|_{\mathbb{L}^p} - t^2h^{2d/p-d+\alpha+\gamma}.$$

Now $2d/p - d + \gamma < 0$ if $\gamma < d - 2d/p = 2d(\frac{1}{2} - \frac{1}{p})$. Now using Lemma 4.2 below choose $\psi_{0,1}, \psi_{0,2} \in \mathcal{S}(\mathbb{R}^d)$ such that $\|g(0)\|_{\mathbb{L}^p} \neq 0$. Then since $\alpha > 0$, we have

$$\|A_1(\psi_0^h)(t)\|_{\mathbb{L}^p} \gtrsim th^{2d/p-d+\gamma}\|g(0)\|_{\mathbb{L}^p} - t^2h^{2d/p-d+\gamma+\alpha} \longrightarrow \infty$$

as $h \to 0$. This completes the proof. \hfill \Box

**Remark 4.1.** In the above we presented proof for Hartree Fock case. For the reduced Hartree Fock case fixing $\psi_{0,2} = 0$, the $g$ defined in (4.7) would be replaced by $g(s) = -i\left(K * |U_\alpha(s)\psi_{0,1}|^2\right)U_\alpha(s)\psi_{0,1}$ and hence Lemma 4.2 below would be fulfilled for any $\psi_{0,1} \neq 0$.

**Lemma 4.2.** Let $g$ be as in (4.7). Then there exist $\psi_{0,1}, \psi_{0,2} \in \mathcal{S}(\mathbb{R}^d)$ such that $\|g(0)\|_{\mathbb{L}^p} \neq 0$.

**Proof.** Note that $ig(0) = \left(K * |\psi_{0,2}|^2\right)\psi_{0,1} - K * ((\psi_{0,1}\overline{\psi_{0,2}})) \psi_{0,2} \in \mathcal{S}(\mathbb{R}^d)$ for all $\psi_{0,1}, \psi_{0,2} \in \mathcal{S}(\mathbb{R}^d)$. Let us choose real valued $\psi_{0,1}, \psi_{0,2} \in \mathcal{S}(\mathbb{R}^d)$ (see Figure 2) such that $\psi_{0,1}(0) = \psi_{0,2}(0) = 1$, $0 \leq \psi_{0,1} \leq \psi_{0,2}$, $\psi_{0,1} < \psi_{0,2}$ on a set of positive measure in $\mathbb{R}^d$. Then

$$ig(0)(0) = \int_{\mathbb{R}^d} K(-y)(\psi_{0,2}(y) - \psi_{0,1}(y))\psi_{0,2}(y)dy > 0.$$ 

Thus $g(0) \neq 0$ and hence $Fg(0) \neq 0$ in $\mathcal{S}(\mathbb{R}^d)$. This completes the proof. \hfill \Box
Lemma 4.3. Let \( f \in S(\mathbb{R}^d \times \mathbb{R}) \). Then for all \( s, t \in \mathbb{R} \), we have \( \frac{\partial}{\partial t} U_\alpha(t-s) f(s) = -iU_\alpha(t-s) (-\Delta)^{\alpha/2} f(s) \).

Proof. Let \( v_s(t) = U_\alpha(t)f(s) \) \((s \in \mathbb{R})\). Then \( v_s \) solves

\[
\partial_t v_s + i(-\Delta)^{\alpha/2} v_s = 0, \quad v_s|_{t=0} = f(s).
\]

Hence \( \partial_t v_s(t) = -i(-\Delta)^{\alpha/2} v_s(t) \) for all \( t \in \mathbb{R} \). Therefore

\[
\frac{\partial}{\partial t} U_\alpha(t-s) f(s) = \partial_t v_s(t-s) = -i(-\Delta)^{\alpha/2} v_s(t-s) = -i(-\Delta)^{\alpha/2} U_\alpha(t-s) f(s).
\]

Note that the operators \((-\Delta)^{\alpha/2}\) and \( U_\alpha \) commute. Indeed, for \( h \in S(\mathbb{R}^d) \), we have

\[
\mathcal{F}[(-\Delta)^{\alpha/2} U_\alpha(t) h] = (2\pi)^{\alpha/2} \mathcal{F}[U_\alpha(t) h] = (2\pi)^{\alpha/2} \mathcal{F}[U_\alpha(t)(-\Delta)^{\alpha/2} h] = \mathcal{F}[U_\alpha(t)(-\Delta)^{\alpha/2} h]
\]

which implies \((-\Delta)^{\alpha/2} U_\alpha(t) h = U_\alpha(t)(-\Delta)^{\alpha/2} h\). Now (4.10) gives the result. \(\square\)

4.2. Further Remarks. In this subsection we aim to discuss several issues (as promised in Subsection 1.2) for the ill-posedness for \((\#)\). For the convenience of the reader, we briefly recall ill-posedness strategy developed in [1]. We consider the abstract equation

\[
u = Lf + \mathcal{N}(u, u, u).
\] (4.11)

Here, \( f \in D \) (initial data space) and \( u \in S \) (solution space). The linear operator \( L : D \to S \) is densely defined, and tri-linear operator \( \mathcal{N} : S \times S \times S \to S \) is also densely defined.

The Hartree-Fock equation (1.1) maybe written as

\[
\psi_k(t) = U_\alpha(t)\psi_{0,k} + i \int_0^t U_\alpha(t-s)(H\psi_k(s))ds - i \int_0^t U_\alpha(t-s)(F\psi_k(s))ds
\]

for \( k = 1, 2, ..., n \). In this case, we may take

\[
L = (U_{\alpha}, \cdots, U_{\alpha}) \quad \text{and} \quad \mathcal{N} = (\mathcal{N}^1, \cdots, \mathcal{N}^N)
\]

where, for \( k = 1, 2, ..., N \), we put

\[
\mathcal{N}^k(\psi, \psi, \psi) = i \int_0^t U_\alpha(t-s)(H\psi_k(s))ds - i \int_0^t U_\alpha(t-s)(F\psi_k(s))ds.
\]

Thus, (1.1) can be rewritten as in the form (4.11) namely

\[
\psi = L\psi_0 + \mathcal{N}(\psi, \psi, \psi)
\]

where \( \psi = (\psi_1, \cdots, \psi_N) \) and \( \psi_0 = (\psi_{0,1}, \cdots, \psi_{0,N}) \). We say (4.11) is quantitative well-posed in the Banach spaces \( D, S \) if the following estimates

\[
\|L(f)\|_S \lesssim \|f\|_D \quad \text{and} \quad \|\mathcal{N}(u_1, u_2, u_3)\|_S \lesssim \|u_1\|_S \|u_2\|_S \|u_3\|_S.
\]

We define the non-linear maps (Picard’s iterations) \( A_n : D \to S \) for \( n = 1, 2, ... \) by the recursive formulæ

\[
A_1(f) = L(f)
\]

\[
A_n(f) = \sum_{n_1, n_2, n_3 \geq 1 \atop n_1 + n_2 + n_3 = n} \mathcal{N}(A_{n_1}(f), A_{n_2}(f), A_{n_3}(f)) \quad \text{for} \ n > 1.
\]
The power series $\sum_{n=1}^{\infty} A_n(f)$ gives solution to (4.11), i.e., $u[f] = \sum_{n=1}^{\infty} A_n(f)$ whenever (4.11) is quantitative well-posed in the Banach spaces $D, S$. See [1, Section 3] for details.

**Proposition 4.4** (Proposition 1 in [1]). Suppose that (4.11) is quantitatively well-posed in the Banach spaces $D, S$, with a solution map $f \mapsto u[f]$ from a ball $B_D$ in $D$ to a ball $B_S$ in $S$. Suppose that these spaces are then given other norms $D'$ and $S'$, which are weaker than $D$ and $S$ in the sense that

$$\|f\|_{D'} \lesssim \|f\|_D \quad \text{and} \quad \|u\|_{S'} \lesssim \|u\|_S.$$ 

Suppose that the solution map $f \mapsto u[f]$ is continuous from $(B_D, \|\cdot\|_{D'})$ (i.e. the ball $B_D$ equipped with the $D'$ topology) to $(B_S, \|\cdot\|_{S'})$. Then for each $n$, the non-linear operator $A_n : D \to S$ is continuous from $(B_D, \|\cdot\|_{D'})$ to $(S, \|\cdot\|_{S'})$.

It’s worth noting that if (4.11) is quantitatively well-posed from $D$ to $S$, Proposition 4.4 allows us to prove ill-posedness in coarse topologies $D', S'$ by demonstrating that at least one of the $A_n$ is discontinuous from $D'$ to $S'$. We are now ready make several comments on proof of Theorem 1.10.

1. In the proof of Theorem 1.10, for $k = 1, 2$, $\|\psi_{0,k}\|_{L^2} = h^{d/d-2}/2 \|\psi_{0,k}\|_{L^2} \to \infty$ as $h \to 0$. Hence data sequence $\{\psi_{0,k}\}$ leaves any ball in $\hat{L}^p \cap L^2(\mathbb{R}^d)$. Therefore, one cannot apply Proposition 4.4 with this choice of data sequence to conclude that the solution map is discontinuous in a ball in $(\hat{L}^p \cap L^2(\mathbb{R}^d))^N$ with $\|\cdot\|_{(\hat{L}^p)^N}$-topology. However, recently we have studied on stronger form of ill-posedness (norm inflation with infinite loss of regularity) in Fourier Lebesgue spaces with negative regularity, see [4].

2. In the proof of Theorem 1.10, $A = (A_1, \cdots, A_N)$ is actually the third Picard iterate $A_3$ (up to a nonzero constant) and we showed that estimate (4.4) fails. Hence, by the method of contradiction and using Theorem 3 in [1], it follows that (#) is not quantitatively well-posed form $(\hat{L}^p)^N$ to $S$ for any subspace $S \hookrightarrow C([0,T), (\hat{L}^p)^N)$ for any $T > 0$. In particular, the estimates $\|\mathcal{N}(\psi_1, \psi_2, \psi_3)\|_{C([0,T), (\hat{L}^p)^N)} \lesssim \prod_{j=1}^{2} \|\psi_j\|_{C([0,T), (\hat{L}^p)^N)}$ fails.

3. In view of Lemma 2.11 (2), (#) is qualitatively well-posed from $(\hat{L}^p \cap L^2)^N$ to $C([0,T), (\hat{L}^p \cap L^2)^N)$. On the other hand, above point (2) says this fails if one works with mere $\hat{L}^p$ instead of $\hat{L}^p \cap L^2$. Thus we find some form of sharpness at the level of quantitative well-posedness between the spaces $\hat{L}^p \cap L^2$ and $\hat{L}^p$ for (#). Similar sharpness at the level of well-posedness (in the sense of Hadamard) remains open, i.e. whether one has a failure of continuity of the solution map from $(\hat{L}^p)^N$ to $C([0,T), (\hat{L}^p)^N)$.

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[31] M. Tarulli and G. Venkov, Decay and scattering in energy space for the solution of weakly coupled Schrödinger-Choquard and Hartree-Fock equations, J. Evol. Equ., 21 (2021), pp. 1149–1178.

[32] N. Tzvetkov, Remark on the local ill-posedness for kdv equation, Comptes Rendus de l’Académie des Sciences-Series I-Mathematics, 329 (1999), pp. 1043–1047.

[33] B. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, Journal of Differential Equations, 232 (2007), pp. 36–73.

[34] Y. Zhou, Cauchy problem of nonlinear Schrödinger equation with initial data in Sobolev space $W^{k,p}$ for $p < 2$, Transactions of the American Mathematical Society, 362 (2010), pp. 4683–4694.

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