Small-threshold behaviour of two-loop self-energy diagrams: some special cases$^*$

F. A. Berends$^a$, A. I. Davydychev$^{a,b}$ and V. A. Smirnov$^{a,b}$

$^a$Instituut-Lorentz, University of Leiden, P.O.B. 9506, 2300 RA Leiden, The Netherlands

$^b$Institute for Nuclear Physics, Moscow State University, 119899 Moscow, Russia

Abstract

An algorithm to construct analytic approximations to two-loop diagrams describing their behaviour at small non-zero thresholds is discussed. For some special cases (involving two different-scale mass parameters), several terms of the expansion are obtained.

$^*$ Presented at QCD and QED in Higher Order, 1996 Zeuthen Workshop on Elementary Particle Theory, Rheinsberg, April 21–26, 1996, to appear in Nucl. Phys. B (Proc. Suppl.).

$^†$ This research was supported by the EU under contract number INTAS-93-0744.
1 INTRODUCTION

The calculation of two-loop Feynman diagrams with (different) masses meets serious technical difficulties, even for two-point functions. After the tensor decomposition of two-loop self-energy diagrams [1], the problem is reduced to scalar integrals. When no exact analytic expressions are available, two main strategies can be chosen: either numerical or approximative analytical. As to the numerical strategy, there exist various integral representations [2] which provide results for given masses via numerical integration. A semi-numerical approach was considered in [3].

The analytical approach uses explicit formulae for the asymptotic expansion of Feynman diagrams in momenta and masses and is based on general theorems on asymptotic expansions [4] (see also [5] for review). For two-loop self-energy diagrams with general masses, the first results have been obtained outside the particle thresholds: a small momentum expansion below the lowest threshold [6] and a large momentum expansion above the highest threshold [7] (some special cases were considered in [8]). However, small and large momentum expansions do not describe the behaviour between the lowest and the highest physical thresholds. In the special case of a zero mass threshold the small momentum expansion can be extended to the lowest non-vanishing threshold [8] (see also in [10]). The expansion coefficients involve then powers of \( \ln(-k^2) \) (where \( k \) is the external momentum).

Our investigation considers cases when one (or two) two-particle threshold(s) is (are) small with respect to the other thresholds, but not anymore zero. Using asymptotic expansions in the large mass limit, a series converging above the small threshold can be found. The expansion coefficients now contain the two-particle cut(s) associated with the small threshold(s) and therefore the non-regular behaviour around the threshold(s) is described. Here we restrict the applications to diagrams involving one large (\( M \)) and one small (\( m \)) mass parameter, i.e. some masses are equal. In [11] a more detailed description of the approach and of the expressions for general masses is given.

2 THE APPROACH

Consider the diagram shown in Fig. 1. The corresponding scalar Feynman integral reads

\[
J \{\nu_i;\{m_i\};k\} = \int \int \frac{d^n p \, d^n q}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}},
\]

(1)

where \( n = 4 - 2\epsilon \) is the space-time dimension [12], and \( (D_i)^{\nu_i} \equiv (p_i^2 - m_i^2 + i0)^{\nu_i} \) are the powers of the propagators corresponding to the internal lines. The momenta \( p_i \) are constructed from \( k \) and the loop momenta.

In general, the diagram in Fig. 1 has two two-particle thresholds, at \( k^2 = (m_1 + m_4)^2 \) and \( k^2 = (m_2 + m_5)^2 \), and two three-particle thresholds, at \( k^2 = (m_1 + m_3 + m_5)^2 \) and \( k^2 = (m_2 + m_3 + m_4)^2 \). Let us consider some of the masses to be large, while the other masses and the external momentum are small. We shall denote the large masses with capital letters, \( M_i \). The classification of all different small-threshold configurations has been given in refs. [3, 11] (see also in Table 1).

We now need to introduce some notation. Let \( \Gamma \) be the original graph (Fig. 1), its subgraphs are denoted as \( \gamma \), and the corresponding reduced graph \( \Gamma/\gamma \) is obtained from \( \Gamma \) by shrinking \( \gamma \) to a point. Furthermore, \( J_{\gamma} \) is the dimensionally-regularized Feynman integral with the denominators corresponding

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{The master two-loop two-point diagram}
\end{figure}
to a graph $\gamma$. In particular, $J_\Gamma$ corresponds to $\{\}$ itself. Then the general theorem \[\] yields

$$J_\Gamma \sim \sum_{k, m_i \to 0} \sum_{\gamma} J_{\Gamma/\gamma} \circ T_{k, m_i, q_i} J_\gamma,$$

(2)

where the sum goes over all subgraphs $\gamma$ which (i) contain all the lines with the large masses $M$, and (ii) are one-particle irreducible with respect to the light lines (with the masses $m_i$). The Taylor operator $T_{k, m_i, q_i}$ expands the integrand of $J_\gamma$ in small masses $m_i$, the external momentum $k$ and the loop momenta $q_i$ which are “external” for a given subgraph $\gamma$. The symbol “$\circ$” means that the polynomial in $q_i$, which appears as a result of applying $T$ to $J_\gamma$, should be inserted in the numerator of the integrand of $J_{\Gamma/\gamma}$.

Let us consider which subgraphs $\gamma$ contribute to the sum (2) for the different small-threshold configurations, using the numbering in Fig.1. For example, the “full” graph $\{12345\}$ includes all five lines, $\{134\}$ corresponds to a subgraph without the lines 2 and 5, etc. The $\gamma$’s contributing to the expansion are shown in Table 1.

After partial fractioning, the following types of contributions arise in the coefficients of the small-threshold expansion (in situations with two-particle small thresholds): (a) two-loop vacuum diagrams with two (or one) large-mass lines and one (or two) massless lines; (b) products of a one-loop massive diagram (with small masses and external momentum $k$) and a one-loop massive tadpole; (c) products of two one-loop massive diagrams with external momentum $k$. For the diagrams containing a small three-particle threshold, we also need results for the diagrams obtained by shrinking the lines corresponding to large masses.

The contributions of type (a) are discussed in \[\] . The one-loop two-point functions involving small masses of internal lines (occurring in the contributions (b) and (c)) are “responsible” for describing the two-particle threshold irregularities (some relevant results are collected in \[\] ).

### 3 SOME RESULTS

As an example, consider the diagram in Fig. 1 with $\nu_1 = \ldots = \nu_5 = 1$ and $n=4$. Denote the corresponding integral \[\] as (here, $m_i$ may correspond to either small or large masses)

$$J(m_1, m_2, m_3, m_4, m_5; k) = -\pi^4 \sum_{j=0}^{\infty} S_j,$$

(3)

where $S_j$ are the terms of our expansion. For a given $j$, the term $S_j$ is a sum of all contributions of the order $(k^2)^{j_0} \prod_i (m_i^2)^{j_i}$ (where the product is taken over all lines with small masses) with $j_0 + \sum j_i = j$. The $S_j$ may contain logarithms of the ratios of the masses and some functions which are described below.

The contributions of type (a) involve a function $\mathcal{H}(M_1^2, M_2^2)$ which can be expressed in terms of $\text{Li}_2$ (see in \[\] ). It is antisymmetric and therefore vanishes when $M_1^2 = M_2^2$.

The contributions of types (b) and (c) are expressed through the function $\tau(m_1, m_2; k^2)$ which is related to the finite part of the one-loop self energy and can be expressed in terms of elementary functions.
Table 2: Terms of the expansion for the cases 1 and 1a

| Case 1a: | $J(M,m,m,M,m;k)$ |
|----------|------------------|
| $M^2S_0$ | $-\frac{\tau}{72} + \frac{L_m}{2} - \frac{\tau}{2}$ |
| $M^4S_1$ | $k^2\left(-\frac{\tau}{24720} + \frac{L_m}{2} + \frac{L_m}{360} + \frac{\tau}{120}\right) + m^2\left(-\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^6S_2$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^8S_3$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^{10}S_4$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |

| Case 1b: | $J(M,m,m,m,M,m;k)$ |
|----------|------------------|
| $M^2S_0$ | $-\frac{\tau}{72} + \frac{L_m}{2} - \frac{\tau}{2}$ |
| $M^4S_1$ | $k^2\left(-\frac{\tau}{24720} + \frac{L_m}{2} + \frac{L_m}{360} + \frac{\tau}{120}\right) + m^2\left(-\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^6S_2$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^8S_3$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
| $M^{10}S_4$ | $\frac{k^2}{2}(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3) + m^2\left(\frac{\tau}{120} + \frac{L_m}{2} - \frac{\tau}{2} + 2L_m + 2\xi_2 - 3\right)$ |
Table 3: Terms of the expansion for the case 1b

| Case 1b | \( J(M, m, M, m; k) \) |
|---------|-------------------|
| \( M^4 S_0 \) | \(-\tau + L_m + \zeta_2 - 4\) |
| \( M^4 S_1 \) | \((k^2)^2 (-\frac{4}{3} \tau + \frac{1}{3} L_m + \frac{1}{3} \zeta_2 - \frac{11}{15}) + m^2 (-\tau L_m - \frac{4}{3} \tau + L_m^2 + \frac{2}{3} L_m + 3 \zeta_2 - \frac{107}{18})\) |
| \( M^6 S_2 \) | \((k^2)^2 (-\frac{1}{3} \tau + \frac{1}{3} L_m + \frac{1}{3} \zeta_2 - 1 \frac{77}{36} + L_m^2 + \frac{19}{10} L_m + 4 \zeta_2 - \frac{1297}{1800})\) |
| \( M^8 S_3 \) | \((k^2)^3 (-\frac{1}{15} \tau + \frac{1}{15} L_m + \frac{1}{7} \zeta_2 - \frac{1487}{960}) + (k^2)^2 m^2 (-\tau L_m - \frac{329}{120} \tau + L_m^2 + \frac{183}{20} L_m + 5 \zeta_2 - \frac{75407}{5400})\) |
| \( M^{10} S_4 \) | \((k^2)^4 (-\frac{1}{45} \tau + \frac{1}{45} L_m + \frac{1}{9} \zeta_2 - \frac{86039}{252000}) + (k^2)^3 m^2 (-\tau L_m - \frac{2267}{10} \tau + L_m^2 + \frac{49103}{1180} L_m + 7 \zeta_2 - \frac{4537092669}{841999200})\) |
| \( M^{12} S_5 \) | \((k^2)^5 (-\frac{1}{63} \tau + \frac{1}{63} L_m + \frac{1}{9} \zeta_2 - \frac{2822857}{999200}) + (k^2)^2 m^4 (-\tau L_m - \frac{2367}{10} \tau + L_m^2 + \frac{49103}{1180} L_m + 7 \zeta_2 - \frac{4537092669}{841999200})\) |
| \( M^{14} S_6 \) | \((k^2)^6 (-\frac{1}{49} \tau + \frac{1}{49} L_m + \frac{1}{7} \zeta_2 - \frac{85046848}{35315200}) + (k^2)^5 m^3 (-\tau L_m - \frac{6007}{20} \tau + \frac{32}{9} L_m^2 + \frac{18179}{560} L_m + 6 \zeta_2 - \frac{10105059}{560})\) |
| \( \Delta = 4m_1^2 m_2^2 - (k^2 - m_1^2 - m_2^2)^2 \) | (4) |

we get

\[
\tau(m_1, m_2; k^2) = \frac{1}{|k^2|} \left\{ \sqrt{-\Delta} \ln \frac{k^2 - m_1^2 - m_2^2 - \sqrt{-\Delta}}{k^2 - m_1^2 - m_2^2 + \sqrt{-\Delta}} + (m_1^2 - m_2^2) \ln \frac{m_2^2}{m_1^2} + \pi \sqrt{-\Delta} \theta (k^2 - (m_1 + m_2)^2) \right\}.
\] (5)

In our contributions, the \( \tau \) function depends on the small masses only. In fact, it contains the main information about the small-threshold behavior at two-particle thresholds. The \( \theta \) term in the braces yields an imaginary part in the region beyond the physical threshold (i.e. for \( k^2 > (m_1 + m_2)^2 \)). Further properties of the \( \tau \) function are discussed in [1].

In Tables 2, 3, 4 we present results for the terms of the small-threshold expansion (3) for the cases 1, 1a, 1b and 2 (cf. Table 1), provided that all large masses are equal to \( M \) and all small masses are equal to \( m \). We use the notation

\[
\tau \equiv \tau(m, m; k^2), \quad L_m \equiv \ln(m^2/M^2), \quad \zeta_2 = \frac{1}{\pi^2}.
\]
by the functions corresponding to the sunset diagram (with three propagators) and the diagram with 

Some of the diagrams considered are interesting from the physical point of view, since they may occur in 

general results presented in [11], here we listed the contributions (from conditions on relative values of the external momentum squared and small masses. In addition to the asymptotic expansions in the large mass limit, we presented an analytic approach to calculating these 

We have studied the behaviour of two-loop self-energy diagrams when the external momentum and some 

Table 4: Terms of the expansion for the case 2

| Case 2 | $J(m,m,M,m;m;k)$ |
|--------|--------------------|
| $M^4 S_0 = -\tau^2 + 2\tau L_m - 2\tau - L_m^2 + 2L_m - 2\zeta_2$ |
| $M^4 S_1 = k^2 (\frac{1}{3} \tau^2 - \tau L_m + \frac{1}{3} \tau + \frac{1}{2} L_m^2 - \frac{1}{2} L_m + \zeta_2 - \frac{1}{3}) + m^2 (-2\tau^2 + 6\tau L_m - \tau - 4L_m^2 - 8\zeta_2 + 10)$ |
| $M^6 S_2 = (k^2)^2 (-\frac{1}{9} \tau^2 + \frac{2}{3} \tau L_m - \frac{2}{9} \tau + \frac{1}{3} L_m^2 + \frac{1}{3} L_m - \frac{1}{3} \zeta_2 + 1) + k^2 m^2 \left( \frac{7}{2} \tau^2 - \frac{20}{3} \tau L_m + \frac{8}{3} \tau + 4L_m^2 - 2L_m + 8\zeta_2 - 14 \right) + m^4 \left( -\frac{4}{9} \tau^2 + 20\tau L_m + \frac{4}{9} \tau - 18L_m^2 + 14L_m + 36\zeta_2 + 47 \right)$ |
| $M^8 S_3 = (k^2)^3 \left( \frac{4}{3} \tau^2 - \frac{1}{3} \tau L_m \right) + \left( \frac{1}{3} \tau^2 + \frac{1}{2} L_m^2 - \frac{1}{3} L_m + \frac{1}{3} \zeta_2 - \frac{1}{3} \right) + m^6 \left( -16\tau^2 + 70L_m \right) + \left( \frac{25}{3} \tau^2 - \frac{290}{3} \tau L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) + m^8 \left( \frac{25}{9} \tau^2 - 252L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) + m^{10} \left( \frac{25}{9} \tau^2 - 252L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) + m^{12} \left( \frac{25}{9} \tau^2 - 252L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) |
| $M^{10} S_4 = (k^2)^4 \left( -\frac{1}{9} \tau^2 + \frac{2}{3} \tau L_m - \frac{1}{3} \tau + \frac{1}{3} L_m^2 + \frac{1}{3} L_m - \frac{1}{3} \zeta_2 + \frac{1}{3} \right) + m^{12} \left( \frac{25}{9} \tau^2 - 252L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) + m^{14} \left( \frac{25}{9} \tau^2 - 252L_m \right) + \left( \frac{1}{3} \tau^2 + 4L_m^2 - \frac{2}{3} L_m - 8\zeta_2 + \frac{217}{18} \right) |

In the limit $m \to 0$, $\tau \to (L_m - L)$, where $L \equiv \ln(-k^2/M^2)$. In this limit the results presented in 

Tables 2–4 reproduce the first four columns of Table 1 from ref. [1]. To perform the calculations, we used 

the REDUCE system [14].

4 CONCLUSIONS

We have studied the behaviour of two-loop self-energy diagrams when the external momentum and some 

of the masses are small with respect to the large masses. By use of explicit formulae for the terms of asymptotic expansions in the large mass limit, we presented an analytic approach to calculating these 

diagrams by keeping the first few terms of the expansion. The main idea was to avoid putting any 

conditions on relative values of the external momentum squared and small masses. In addition to the 

general results presented in [1], here we listed the contributions (from $S_0$ to $S_6$) for some special cases. 

Some of the diagrams considered are interesting from the physical point of view, since they may occur in 

the Standard Model calculations.

In cases 3 and 4, three-particle small thresholds arise. The small-threshold behaviour is then defined 

by the functions corresponding to the sunset diagram (with three propagators) and the diagram with
four propagators (e.g. eq. (1) with $\nu_1 = 0$). Unfortunately, sufficient analytic information about these diagrams is not yet available.

The algorithm can also be extended to the three-point two-loop diagrams with small thresholds (for zero-mass thresholds, see ref. [15]).

Acknowledgements. A. D. is grateful to the organizers (DESY-Zeuthen) for support of his participation in the Rheinsberg workshop. A. D. and V. S. were partly supported by the RFBR grant 96-01-00654.

References

[1] G. Weiglein, R. Scharf and M. Böhm, Nucl. Phys. B416 (1994) 606; D. Kreimer, Mod. Phys. Lett. A9 (1994) 1105.

[2] D. Kreimer, Phys. Lett. B273 (1991) 277; J. Fujimoto, Y. Shimizu, K. Kato and Y. Oyanagi, KEK preprint 92-213; F.A. Berends and J.B. Tausk, Nucl. Phys. B421 (1994) 456; A. Czarnecki, U. Kilian and D. Kreimer, Nucl. Phys. B433 (1995) 259; S. Bauberger, F.A. Berends, M. Böhm and M. Buza, Nucl. Phys. B434 (1995) 383; A. Ghinculov and J.J. van der Bij, Nucl. Phys. B436 (1995) 30; S. Bauberger and M. Böhm, Nucl. Phys. B445 (1995) 25.

[3] J. Fleischer and O.V. Tarasov, Z. Phys. C64 (1994) 413.

[4] S.G. Gorishny, S.A. Larin and F.A. Tkachov, Phys. Lett. B124 (1983) 217; G.B. Pivovarov and F.V. Tkachov, Preprints INR P-0370, II-549 (Moscow, 1984); S.G. Gorishny and S.A. Larin, Nucl. Phys. B283 (1987) 452; K.G. Chetyrkin, Teor. Mat. Fiz. 75 (1988) 26; 76 (1988) 207; preprint MPI-PAE/PTh 13/91; S.G. Gorishny, Nucl. Phys. B319 (1989) 633; V.A. Smirnov, Commun. Math. Phys. 134 (1990) 109.

[5] V.A. Smirnov, Renormalization and asymptotic expansions (Birkhäuser, Basel, 1991); Mod. Phys. Lett. A10 (1995) 1485.

[6] A.I. Davydychev and J.B. Tausk, Nucl. Phys. B397 (1993) 123.

[7] A.I. Davydychev, V.A. Smirnov and J.B. Tausk, Nucl. Phys. B410 (1993) 325.

[8] D.J. Broadhurst, Z. Phys. C47 (1990) 115; D.J. Broadhurst, J. Fleischer and O.V. Tarasov, Z. Phys. C60 (1993) 287; R. Scharf and J.B. Tausk, Nucl. Phys. B412 (1994) 523; D.T. Gegelia, G.Sh. Japaridze and K.Sh. Turashvili, Teor. Mat. Fiz. 101 (1994) 225.

[9] F.A. Berends, A.I. Davydychev, V.A. Smirnov and J.B. Tausk, Nucl. Phys. B439 (1995) 536.

[10] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B438 (1995) 278.

[11] F.A. Berends, A.I. Davydychev and V.A. Smirnov, Leiden preprint INLO-PUB–21/95 (hep-ph/9602390), to appear in Nucl. Phys. B.

[12] G. ’t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189; C.G. Bollini and J.J. Giambiagi, Nuovo Cim. 12B (1972) 20.

[13] G. ’t Hooft and M. Veltman, Nucl. Phys. B153 (1979) 365.

[14] A.C. Hearn, REDUCE user’s manual, RAND publication CP78 (Santa Monica, 1987).

[15] J. Fleischer, V.A. Smirnov and O.V. Tarasov, Bielefeld preprint BI-TP–95-39 (hep-ph/9605392).