LOCAL HADAMARD WELL–POSEDNESS AND BLOW–UP FOR REACTION–DIFFUSION EQUATIONS WITH NON–LINEAR DYNAMICAL BOUNDARY CONDITIONS

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Dedicated to Jerry Goldstein on the occasion of his 70th birthday

Abstract. The paper deals with local well–posedness, global existence and blow–up results for reaction–diffusion equations coupled with nonlinear dynamical boundary conditions. The typical problem studied is

\[
\begin{align*}
  u_t - \Delta u &= |u|^{p-2} u & \text{in} & \quad (0, \infty) \times \Omega, \\
  u &= 0 & \text{on} & \quad [0, \infty) \times \Gamma_0, \\
  \frac{\partial u}{\partial \nu} &= -|u_t|^{m-2} u_t & \text{on} & \quad [0, \infty) \times \Gamma_1, \\
  u(0, x) &= u_0(x) & \text{in} & \quad \Omega
\end{align*}
\]

where \( \Omega \) is a bounded open regular domain of \( \mathbb{R}^n \) (\( n \geq 1 \)), \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( 2 \leq p \leq 1+2^*/2 \), \( m > 1 \) and \( u_0 \in H^1(\Omega) \), \( u_0|_{\Gamma_0} = 0 \). After showing local well–posedness in the Hadamard sense we give global existence and blow–up results when \( \Gamma_0 \) has positive surface measure. Moreover we discuss the generalization of the above mentioned results to more general problems where the terms \( |u|^{p-2} u \) and \( |u_t|^{m-2} u_t \) are respectively replaced by \( f(x, u) \) and \( Q(t, x, u_t) \) under suitable assumptions on them.

1. Introduction and main results. We consider the problem

\[
\begin{align*}
  u_t - \Delta u &= f(x, u) & \text{in} & \quad (0, \infty) \times \Omega, \\
  u &= 0 & \text{on} & \quad [0, \infty) \times \Gamma_0, \\
  \frac{\partial u}{\partial \nu} &= -Q(t, x, u_t) & \text{on} & \quad [0, \infty) \times \Gamma_1, \\
  u(0, x) &= u_0(x) & \text{in} & \quad \Omega
\end{align*}
\]

where \( u = u(t, x) \), \( t \geq 0 \), \( x \in \Omega \), \( \Delta = \Delta_x \) denotes the Laplacian operator with respect to the \( x \) variable, \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) (\( n \geq 1 \)) of class \( C^1 \) (see [10]), with \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \), \( \Gamma_0 \) and \( \Gamma_1 \) are measurable over \( \partial \Omega \), endowed

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with \((n - 1)\)-dimensional surface measure \(\sigma\). These properties of \(\Omega, \Gamma_0\) and \(\Gamma_1\) will be assumed, without further comments, throughout the paper. The initial datum \(u_0\) belongs to the energy space \(H^1(\Omega)\), with the compatibility condition \(u_0 = 0\) on \(\Gamma_0\). Moreover \(Q\) represents a nonlinear dynamical term such that \(Q(t, x, v)v \geq 0\), and \(f\) represents a nonlinear internal reaction (or source) term, i.e. \(f(x, u)u \geq 0\).

When \(Q \equiv 0\) problem (1) is an initial–boundary value problem related to a semilinear reaction–diffusion equation with homogeneous Dirichlet–Neumann boundary conditions. In this case local well–posedness, under suitable assumptions on \(f\), can be obtained in a standard way using semigroup theory. See for example \([43, 55]\) or \([2]\) combined with \([15, \text{Appendix}]\). There is also a wide literature on global existence and blow–up for such type of problems, starting from the classical paper of Levine \([33]\). See for example \([12, 23, 29, 34, 35, 46]\), \([62, \text{Section 5}]\) and \([18, 30, 47, 48, 49]\).

In this case the concavity method of H. Levine is effective in getting blow–up results.

When \(Q(t, x, u_t) = \alpha(t, x)u_t\) problem (1) consists in a reaction–diffusion equation coupled with a linear dynamical boundary condition. For well–posedness results, obtained by semigroup and interpolation theories we refer to \([2, 19, 20, 27, 28]\), while blow–up results were proven in \([21, 31]\). We also refer to \([7]\) for a physical motivation of dynamical boundary conditions, and to the recent papers \([22, 63, 64]\). Also in this case the concavity method applies (see \([50]\)) in order to establish blow–up.

In this paper we study problem (1) when, roughly, \(Q(t, x, u_t) \approx |u_t|^{m - 2}u_t\) as \(|u_t| \geq 1, m > 1\), and \(f(x, u) \approx |u|^{p - 2}u, p \geq 2\), as \(|u| \geq 1\). The interest in considering superlinear terms \((m > 2)\) is mainly of theoretical nature. However, a physical model involving \(Q(t, x, u_t) = u_t + |u_t|^{m - 2}u_t, m > 2\), is given in Appendix A.

In order to state and prove our results in the simplest possible way we shall first consider the model problem

\[
\begin{cases}
  u_t - \Delta u = |u|^{p-2}u & \text{in } (0, \infty) \times \Omega, \\
  u = 0 & \text{on } [0, \infty) \times \Gamma_0, \\
  \frac{\partial u}{\partial \nu} = -|u_t|^{m-2}u_t & \text{on } [0, \infty) \times \Gamma_1, \\
  u(0, x) = u_0(x) & \text{in } \Omega
\end{cases}
\]

where \(m > 1, p \geq 2\). We denote by \(2^*\) the critical exponent of Sobolev embedding \(H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)\), i.e. \(2^* = 2n/(n - 2)\) when \(n \geq 3\) while \(2^* = \infty\) when \(n = 1, 2\).
Moreover we denote \( \| \cdot \|_q = \| \cdot \|_{L^q(\Omega)} \), \( \| \cdot \|_{q, \Gamma_1} = \| \cdot \|_{L^q(\Gamma_1)} \) for \( 1 \leq q \leq \infty \), and the Hilbert space \( H^1_{\Gamma_0}(\Omega) := \{ u \in L^1(\Omega) : u|_{\Gamma_0} = 0 \} \), \( \| u \|^2_{H^1_{\Gamma_0}} := \| u \|^2_2 + \| \nabla u \|^2_2 \), where \( u|_{\Gamma_0} \) stands for the restriction of the trace of \( u \) on \( \partial \Omega \) to \( \Gamma_0 \). The first aim of the paper is to show that problem (2) is well–posed in \( H^1_{\Gamma_0}(\Omega) \). The first step in this direction is given by the following result.

**Theorem 1.1. (Local existence and uniqueness)** Let \( m > 1 \) and

\[
2 \leq p \leq 1 + \frac{2^*}{2}.
\]

Then, given \( u_0 \in H^1_{\Gamma_0}(\Omega) \), there is a \( T^* = T^*(\| u_0 \|_{H^1_{\Gamma_0}}, m, p, \Omega, \Gamma_1) \in (0, 1] \), decreasing in the first variable, such that problem (2) has a unique weak solution \( u \) in \( [0, T^*) \times \Omega \). Moreover

\[
u \in C([0, T^*] \cap H^1_{\Gamma_0}(\Omega)),
\]

\[
u_t \in L^m((0, T^*) \times \Gamma_1) \cap L^2((0, T^*) \times \Omega)
\]

and the energy identity

\[
\frac{1}{2} \| \nabla u \|^2_2 \bigg|_s^t + \int_s^t \| u_t \|^m_{m, \Gamma_1} + \| u_t \|^2_2 = \int_s^t \int_\Omega |u|^{p-2} u u_t
\]

holds for \( 0 \leq s \leq t \leq T^* \). Finally

\[
\| u \|_{C([0, T^*] \cap H^1_{\Gamma_0}(\Omega))} \leq 4 \| u_0 \|_{H^1_{\Gamma_0}}.
\]

**Remark 1.** The assumption \( p \leq 1 + 2^*/2 \) in Theorem 1.1 is quite restrictive when \( n \geq 3 \), although it appears often in the literature quoted above. Clearly it expresses the assumption that the Nemitski operator \( u \mapsto |u|^{p-2} u \) is locally Lipschitz from \( H^1(\Omega) \) to \( L^2(\Omega) \). Such type of assumptions has been overcome, in the author's knowledge, either by getting additional a–priori estimates, as done for example in [8, 52], or using linear semigroup and interpolation theories, as done for example in [2, 20]. While in this case the nonlinear term \( Q \) does not give useful estimates, being active on the boundary, it prevents to use linear theory and interpolation of semigroups. Nonlinear semigroup theory can be used, as in [17], but in this case one still needs to assume that the Nemitski operator above is locally Lipschitz, as in [15]. To prove Theorem 1.1 we found simpler to first use the monotonicity method of J. L. Lions and then to use a contraction argument.

By using the same energy estimates used to prove Theorem 1.1 we complete our well–posedness analysis as follows.

**Theorem 1.2. (Continuation and local Hadamard well–posedness)** Under the assumption of Theorem 1.1, problem (2) has a unique weak maximal solution \( u \) in \( [0, T_{\text{max}}) \times \Omega \). Moreover \( u \in C([0, T_{\text{max}}); H^1_{\Gamma_0}(\Omega)) \),

\[
u_t \in L^m((0, T) \times \Gamma_1) \cap L^2((0, T) \times \Omega)
\]

for any \( T \in (0, T_{\text{max}}) \), and the following alternative holds:

(i) either \( T_{\text{max}} = \infty \);

(ii) or \( T_{\text{max}} < \infty \) and

\[
\lim_{t \to T_{\text{max}}} \| u(t) \|_{H^1_{\Gamma_0}} = +\infty.
\]

\(^{1}\)see Definition 3.1 below for the precise meaning of weak solutions, which are essentially distributional solutions enjoying a suitable regularity
Finally $u$ depends continuously on the initial datum $u_0$, that is given any $T \in (0, T_{max})$ and any sequence $\{u_n\}_n$ in $H^1_{Γ_0}(Ω)$ such that $u_n \rightarrow u_0$ in $H^1_{Γ_0}(Ω)$, the corresponding weak solution $u^n$ is defined in $[0, T] \times Ω$ and $u^n \rightarrow u$ in $C([0, T]; H^1_{Γ_0}(Ω))$.

The second aim of the paper is to study the alternative (i)–(ii) in previous Theorem by giving global existence versus blow–up results. When $p = 2$ it is straightforward to prove that $u$ is global (see Theorem B.1 in Appendix B), so we focus on the more interesting case $p > 2$. Although we are not able to give a complete answer, as usual for nonlinear problems, we give two partial answers when

$$\sigma(Γ_0) > 0,$$

so a Poincarè–type inequality holds (see [65]) and consequently $\|\nabla u\|_2$ is an equivalent norm in $H^1_{Γ_0}(Ω)$. This assumption allows us to use potential–well arguments.

In order to state our next results we need to recall the stable and unstable sets introduced in [60]. When $p > 2$ and (3) holds we introduce the functionals

$$J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p} \|u\|^p, \quad K(u) = \|\nabla u\|_2^2 - \|u\|^p$$

defined for $u \in H^1_{Γ_0}(Ω)$, and the number

$$d = \inf_{u \in H^1_{Γ_0}(Ω) \setminus \{0\}} \sup_{λ > 0} J(λu).$$

When $p > 2$ and (3), (9) hold true it is easy to see that $d > 0$. See Lemma 4.1 below, where two different characterizations of $d$ are given. We define the stable and unstable sets as

$$W_s = \{u_0 \in H^1_{Γ_0}(Ω) : K(u_0) \geq 0 \text{ and } J(u_0) < d\}$$

(12)

$$W_u = \{u_0 \in H^1_{Γ_0}(Ω) : K(u_0) \leq 0 \text{ and } J(u_0) < d\}.$$  

(13)

As an application of Theorem 1.2 and of a potential–well estimate we give the following global existence result.

**Theorem 1.3. (Global existence)** Under the assumptions of Theorem 1.1 and the further assumptions (9) and $p > 2$, if $u_0 \in W_s$ then $T_{max} = ∞$ and $u(t) \in W_s$ for all $t \geq 0$.

While Theorem 1.3 can be seen as a simple application of Theorem 1.2, to recognize that solutions of problem (2) starting in the unstable set blow–up is a more difficult task. When $m = 2$ this result can be proved by a concavity argument (see [58]), which cannot be applied when $m \neq 2$, making this case more interesting. By combining the main technique of [35] with an estimate used in [60] for wave equation we are able to prove the following result.

**Theorem 1.4. (Blow–up)** Under the assumptions of Theorem 1.1 and the further assumptions (9), $p > 2$ and

$$m < m_0(p) := \frac{2(n + 1)p - 4(n - 1)}{n(p - 2) + 4},$$

(14)

if $u_0 \in W_u$ then $T_{max} < ∞$, $u(t) \in W_u$ for all $t \in [0, T_{max})$, and

$$\lim_{t \to T_{max}} \|u(t)\|_p = +∞.$$
Remark 3. Clearly assumption (14) yields $m < p$ since it is trivial to prove that $m_0(p) \leq p$ for $p \geq 2$. It strongly reduces the applicability of Theorem 1.4, as shown by Figure 1 which illustrates the set of the couples $(p, m)$ satisfying (3) and (14). As $m_0(p) > 2$ for $p > 2$, the result is rather sharp in the sublinear case $1 < m \leq 2$, while (3) and (14) force that $m < 4$ when $n = 1$, $m < 3$ when $n = 2$ and $m < 2 + \frac{2}{m-1}$ when $n \geq 3$. This assumption, which looks to be a technical one, comes directly from [60], where it was introduced, and is due to the difficulty in comparing the effect of high order polynomial dissipation, which is related to the $L^m$ norm on $\Gamma_1$, with the effect of the source, related to the $L^p$ norm on $\Omega$. After nine years from its use, the authors are not aware of any improvement.

Remark 4. The set $W_u$ is not optimal, since as it is shown in [4, 6, 5] one can consider an unstable set $\tilde{\Sigma}$ properly including $W_u$, the set of initial data such that $J(u_0) = d$ and $\|u_0\|_p > \tilde{\lambda}_1$ (see (76) below) already considered in [57]. Here we preferred to use this (a bit more restrictive) definition of unstable set for the sake of simplicity.

As a preliminary step in the proof of Theorem 1.1 we give a well-posedness result for the problem

\[
\begin{align*}
&u_t - \Delta u = g(t, x) \quad \text{in } (0, T) \times \Omega, \\
&u = 0 \quad \text{on } [0, T) \times \Gamma_0, \\
&\frac{\partial u}{\partial \nu} = -|u_t|^{m-2}u_t \quad \text{on } [0, T) \times \Gamma_1, \\
&u(0, x) = u_0(x) \quad \text{in } \Omega
\end{align*}
\]
where $m > 1$, $T > 0$ is arbitrary and $g$ is a given forcing term acting on $\Omega$. Although problem (15) can be studied using the analysis of [17], it is not trivial in that way to get the following result.

**Theorem 1.5. (Well–posedness for an auxiliary problem)** Suppose that $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. Then there is a unique weak solution $u$ of (15) in $[0, T] \times \Omega$. Moreover

$$u \in C([0, T]; H^1_{\Gamma_0}(\Omega)),$$

$$u_t \in L^2((0, T) \times \Omega) \cap L^m((0, T) \times \Gamma_1)$$

and the energy identity

$$\frac{1}{2} \left\| \nabla u \right\|^2_{L^2(\Omega)} \Big|^{t}_{s} + \int_{s}^{t} \left\| u_t \right\|^2_{2} + \left\| u_t \right\|^m_{m, \Gamma_1} = \int_{s}^{t} \int_{\Omega} g u_t$$

holds for $0 \leq s \leq t \leq T$. Finally, given any couple of initial data $u_{01}, u_{02} \in H^1_{\Gamma_0}(\Omega)$ and any couple of forcing terms $g_1, g_2 \in L^2((0, T) \times \Omega)$, respectively denoting by $u^1$ and $u^2$ the solutions of (15) corresponding to $u_{01}$, $g_1$ and to $u_{02}$, $g_2$, the following estimate holds

$$\left\| u^1 - u^2 \right\|^2_{C([0, T]; H^1_{\Gamma_0}(\Omega))} \leq 2(1 + T) \left( \left\| u_{01} - u_{02} \right\|^2_{H^1_{\Gamma_0}(\Omega)} + \left\| g_1 - g_2 \right\|^2_{L^2((0, T) \times \Omega)} \right).$$

**Remark 5.** A short comparison with the results which can be obtained by directly applying the abstract results in [17] is in order. Assumptions (A1–2) in [17, Theorem 1] force to restrict to the case $m = 2$, while the assumption $D(B) \subset V$ in [17, Theorems 2–3] implies $m \leq 2(n - 1)/(n - 2)$ when $n \geq 3$. Next one can apply [17, Theorem 4] only when $g$ is more regular in time. Finally, [17, Theorem 5] can be applied only when $m = 2$.

In order to explain the main difficulties arising in the proofs of our main results we now make some comparison with the arguments used by the second author in [61]. Theorem 1.5 is essentially proved as [61, Theorem 1.5], even if the necessary adaptations require some care. Theorem 1.1 is proved by a contraction argument instead that a compactness one. Theorem 1.2 has no counterpart in [61]. Finally the proof of Theorem 1.4 requires an untrivial mixing of the technique of [35] with the estimate used in [60], so the authors consider it as the main contribution in the present paper.

The paper is organized as follows. Section 2 deals with some notation and preliminary material, including the proof of Theorem 1.5, Section 3 is devoted to local well–posedness theory for problem (2) while in Section 4 we study global existence and blow–up for it. Finally the results presented in this introduction are generalized in Section 5 to problem (1), under suitable assumptions on the nonlinearities $f$ and $Q$. For the sake of simplicity we first present the proofs for the model problem (2) and then we give in (5) the generalizations needed to handle with (1). This section is naturally addressed to a more specialized audience and consequently an higher lever of mathematical expertise of the reader is supposed. In particular most proofs are only sketched.

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\[ \text{See Definition 2.1 below for the precise meaning of weak solution} \]
2. Notation and preliminaries. We introduce the notations
\( C^\infty_c(\mathcal{O}) \) space of compactly supported real–valued \( C^\infty \) functions on any open set \( \mathcal{O} \subset \mathbb{R}^n \),
\( C^\infty((a, b); X) \) space of \( C^\infty \) \( X \)-valued functions in \( (a, b) \), \( X \) Banach space,
\( C([a, b]; X) \) space of norm continuous \( X \)-valued functions in \([a, b]\),
\( C_w([a, b]; X) \) space of weakly continuous \( X \)-valued functions in \([a, b]\),
\( q' \) Hölder conjugate of \( q \geq 1 \), i.e. \( 1/q + 1/q' = 1 \),
\( X' \) the dual space of \( X \),
\( (\cdot, \cdot) \) scalar product in \( L^2(\Omega) \).
Moreover we call the trace theorem the existence of the continuous trace mapping \( H^1_{\Gamma_0}(\Omega) \rightarrow L^2(\partial\Omega) \). Moreover the trace of \( u \) on \( \Omega \) will be denoted by \( u_{|\partial\Omega} \). We also call the Sobolev Embedding Theorem the existence of the continuous embedding \( H^1_{\Gamma_0}(\Omega) \rightarrow L^p(\Omega) \) for \( 2 \leq p < 2^* \).
We start by setting the Banach space
\[ X = \{ u \in H^1_{\Gamma_0}(\Omega) : u|_{\Gamma_1} \in L^m(\Gamma_1) \} \tag{20} \]
endowed with the norm \( \|u\|_X = \|u\|_{H^1_{\Gamma_0}} + \|u|_{\Gamma_1}\|_{m, \Gamma_1}. \) For elements \( u \in X \) we shall use the simpler notation \( \|u\|_{m, \Gamma_1} \) to mean \( \|u|_{\Gamma_1}\|_{m, \Gamma_1}. \) We now give the precise meaning of weak solution of (15).

**Definition 2.1.** Let \( u_0 \in H^1_{\Gamma_0}(\Omega) \) and \( g \in L^2((0, T) \times \Omega) \). We say that \( u \) is a weak solution of (15) in \([0, T] \times \Omega\) if
\( (a) \) \( u \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega)), \) \( u_t \in L^2(0, T) \times \Omega; \)
\( (b) \) the spatial trace of \( u \) on \((0, T) \times \partial\Omega\) (which exists by the trace theorem) has a distributional time derivative on \((0, T) \times \partial\Omega\),
\( (c) \) for all \( \phi \in X \) and for almost all \( t \in [0, T] \) the distribution identity
\[ \int_\Omega u_t(t)\phi + \int_\Omega \nabla u(t)\nabla \phi + \int_{\Gamma_1} |u_t(t)|^{m-2} u_t(t)\phi = \int_\Omega g(t)\phi \tag{21} \]
holds true;
\( (d) \) \( u(0) = u_0. \)

Note that, in (d), \( u(0) \) makes sense since, by (a),
\[ u \in H^1(0, T; L^2(\Omega)) \rightarrow C([0, T]; L^2(\Omega)). \]
In order to prove Theorem 1.5 we need the following Lemma, which extends [54, Theorems 3.1 and 3.2] to the present situation. Its proof consists in a rather technical application of the arguments in [54] which is given in Appendix C for the reader convenience.

**Lemma 2.2.** Let \( 0 < T < \infty, m > 1, \)
\[ u_0 \in H^1_{\Gamma_0}(\Omega), \quad g \in L^2((0, T) \times \Omega), \quad \zeta \in L^{m'}((0, T) \times \Gamma_1) \tag{22} \]
and suppose that \( u \) is a weak solution of
\[ \begin{cases} u_t - \Delta u = g(t, x) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial n} = \zeta & \text{on } [0, T) \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{23} \]
i.e. a function
\[ u \in L^\infty(0, T; H^1_{\Gamma_0}(\Omega)) \tag{24} \]
such that
\[ u_t \in L^2((0, T) \times \Omega), \] (25)
the spatial trace of \( u \) on \((0, T) \times \partial \Omega \) (which exists by the trace theorem) has a distributional time derivative on \((0, T) \times \partial \Omega\) belonging to \( L^m((0, T) \times \partial \Omega) \), and, for all \( \phi \in X \) and almost all \( t \in [0, T] \) the function \( u \) satisfies
\[ \int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi - \int_{\Gamma_1} \zeta(t) \phi = \int_{\Omega} g(t) \phi. \] (26)
Then
\[ u \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \] (27)
and the energy identity
\[ \frac{1}{2} \|\nabla u\|^2_2 \bigg|_s^t + \int_s^t \|u_t\|^2_2 - \int_s^t \int_{\Gamma_1} \zeta u_t = \int_s^t \int_{\Omega} g u_t \] (28)
holds for \( 0 \leq s \leq t \leq T \).

**Proof of Theorem 1.5.** To prove the existence of a weak solution of (15) we apply the Faedo–Galerkin procedure. Let \((w_k)_k\) be a sequence of linearly independent vectors in the space \( X \), which was defined in (20), whose finite linear combinations are dense in it. By using the Graham–Schmidt orthonormalization process, we can take \((w_k)_k\) to be orthonormal in \( L^2(\Omega) \). Since (see [59, Lemma A1, Appendix A]) \( X \) is dense in \( H^1_{\Gamma_0}(\Omega) \) for all \( k \in \mathbb{N} \) there are real numbers \( y_{0k}^j \), \( j = 1, \ldots, k \), such that
\[ u_{0k} = \sum_{j=1}^k y_{0k}^j w_j \to u_0 \text{ in } H^1_{\Gamma_0}(\Omega). \] (29)

For any fixed \( k \in \mathbb{N} \) we look for approximate solutions of (15), that is for solutions
\[ u^k(t) = \sum_{j=1}^k y_j^k(t) w_j, \] of the finite–dimensional problem
\[ \begin{cases} (u^k, w_j) + (\nabla u^k, \nabla w_j) + \int_{\Gamma_1} |u^k|^{m-2} u^k w_j = \int_{\Omega} g w_j, & j = 1, \ldots, k, \\ u^k(0) = u_{0k}. \end{cases} \] (30)

In order to recognize that (30) has a local solution, we set
\[ y_{0k} = (y_{0k}^1, \ldots, y_{0k}^k)^T, \quad y_k = (y_k^1, \ldots, y_k^k)^T, \] (31)
\[ A_k = ((\nabla w_i, \nabla w_j))_{i,j=1,\ldots,k}, \quad B_k(x) = (w_1(x), \ldots, w_k(x))^T, \] (32)
\[ G_k(y) = y + \int_{\Gamma_1} |B_k(x) \cdot y|^{m-2} B_k(x) \cdot y B_k(x) dx, \quad y \in \mathbb{R}^k, \] (33)
and \( H_k(t) = \int_{\Omega} g(t,x) B_k(x) dx \), so problem (30) can be rewritten as
\[ \begin{cases} G_k(y_k(t)) + A_k y_k(t) = H_k(t), \\ y_k(0) = y_{0k}. \end{cases} \] (34)

Then, using the arguments in [61, Proof of Theorem 1.5] we get that \( G_k \) is an homeomorphism from \( \mathbb{R}^k \) into itself, with inverse \( G_k^{-1} \), and that (34) has a solution \( y_k \in W^{1,1}(0, t_k) \) for some \( t_k \in (0, T] \), and consequently (30) has a solution \( u^k \in W^{1,1}(0, t_k; X) \). Moreover, since \( G_k(y) y \geq |y|^2 \) for all \( y \in \mathbb{R}^k \), by the Schwartz inequality it follows that \(|y| \leq |G_k(y)|\). Then \( |G_k^{-1}(y)| \leq |y| \) for all \( y \in \mathbb{R}^k \), so that
\[ |G_k^{-1}(H_k(t) - A_k y_k(t))| \leq |H_k(t)| + \|A_k\| |y_k|. \] (35)
Multiplying (30) by \((g_{ik})'\) and summing for \(j = 1, \ldots, k\), we obtain the energy identity (here and in the sequel, explicit dependence on \(t\) will be omitted, when clear)

\[
\frac{d}{dt} \left( \frac{1}{2} \|\nabla u^k\|_2^2 + \|u^k_t\|_2^2 + \|u^k_t\|_{m, \Gamma_1}^m \right) = \int_{\Omega} gu^k.
\]

(36)

Integrating over \((0, t), 0 < t < t_k\), and using Young inequality, we get

\[
\frac{1}{2} \|\nabla u^k\|_2^2 + \int_0^t \left( \|u^k_t\|_2^2 + \|u^k_t\|_{m, \Gamma_1}^m \right) \leq \frac{1}{2} \|\nabla u_{0k}\|_2^2 + \frac{1}{2} \|g\|_{L^2((0, T) \times \Omega)}^2 + \frac{1}{2} \int_0^t \|u^k_t\|_2^2.
\]

Then, using (29), there exists \(C = C \left( \|\nabla u_0\|_2, \|g\|_{L^2((0, T) \times \Omega)} \right) > 0\) such that

\[
\|\nabla u^k\|_{L^\infty(0, t_k; L^2(\Omega))} \leq C, \\
\|u^k_t\|_{L^2((0, t_k) \times \Omega)} \leq C, \\
\|u^k_t\|_{L^m((0, t_k) \times \Gamma_1)} \leq C, \\
\|u^k_t|^{m-2} u^k\|_{L^{m'}((0, t_k) \times \Gamma_1)} \leq C,
\]

(37)

for \(k \in \mathbb{N}\). By (29), (37) and Hölder inequality in time it follows that

\[
\|u^k\|_2^2 \leq \|u_{0k}\|_2^2 + \int_0^t \|u^k_t\|_2^2 \leq \|u_{0k}\|_2^2 + T^{1/2} \left( \int_0^t \|u^k_t\|_2^2 \right)^{1/2} \leq C'
\]

(38)

for some \(C' = C' \left( \|u_0\|_{H^1_0}, \|g\|_{L^2((0, T) \times \Omega)}, T \right) > 0\). Since \((w_k)_k\) is orthonormal in \(L^2(\Omega)\), we have \(|y_k(t)| = \|u^k(t)\|_2\), so (38) yields that \(|y_k(t)| \leq C'\). Then, by (35)

\[
|G_k^{-1}(H_k(t) - A_k y_k(t))| \leq |H_k(t)| + C'\|A_k\| \in L^1(0, T),
\]

We can then apply [16, Theorem 1.3, Chapter 2] to conclude that \(t_k = T\) for \(k = 1, \ldots, n\). Next, by (37) and (38), it follows that, up to a subsequence,

\[
\begin{align*}
\frac{1}{2} \|\nabla u^k\|_2^2 + \int_0^t \|u^k_t\|_2^2 + \int_{\Gamma_1} \chi w - \int_{\Omega} gw - C'\|A_k\| \leq C',
\end{align*}
\]

(39)

A consequence of the convergences (39) and of Aubin–Lions compactness Lemma (see [11, 3, 53]) is that \(u^k \rightharpoonup u\) strongly in \(C([0, T]; L^2(\Omega))\), so that \(u(0) = u_0\). It follows in a standard way (see, for example, [59, p. 272]) that \(\varphi\) is the distribution time derivative of \(u\) on \((0, T) \times \partial\Omega\), i.e. \(\varphi = u_t\).

Next, multiplying (30) by \(\phi \in C_c^\infty(0, T)\), integrating on \((0, T)\), passing to the limit as \(k \to \infty\) (using (39)) and finally using the density of the finite linear combinations of \((w_k)_k\) in \(X\), we obtain

\[
\int_0^T \left[ (u_t, w) + (\nabla u_t, \nabla w) + \int_{\Gamma_1} \chi w - \int_{\Omega} gw \right] \phi = 0
\]

for all \(w \in X, \phi \in C_c^\infty(0, T)\). Consequently \((u_t, w) + (\nabla u_t, \nabla w) + \int_{\Gamma_1} \chi w - \int_{\Omega} gw\) almost everywhere in \((0, T)\). Then to prove that \(u\) is a weak solution of (15) we have only to show that

\[
\chi = |u_t|^{m-2} u_t \text{ a.e. on } (0, T) \times \Gamma_1.
\]

(40)

By Lemma 2.2 we obtain (27) and the energy identity

\[
\frac{1}{2} \|\nabla u\|_2^2 \bigg|_0^T + \int_0^T \|u_t\|_2^2 + \int_0^T \int_{\Gamma_1} \chi u_t = \int_0^T \int_{\Omega} gu_t.
\]

(41)
The classical monotonicity method (see [42] or [61, p. 186]) then allows us to prove (40).

Finally, to prove the estimate (19), which also yields the uniqueness of the solution, we recognize that \( v = u_1 - u_2 \) is a weak solution of problem (23) with \( g = g_1 - g_2 \), \( \xi = -|u_1|^m u_1 - |u_2|^m u_2 \), and \( u_0 = u_{01} - u_{02} \). Then, by Lemma 2.2, using the monotonicity of the map \( x \rightarrow |x|^{m-2} x \) we get the estimate

\[
\frac{1}{2} \| \nabla v(t) \|^2 + \int_0^t \| v_t \|^2 \leq \int_0^t g v_t + \frac{1}{2} \| \nabla u_0 \|^2 \quad \text{for all } t \in [0, T).
\]

By Young inequality

\[
\| \nabla v(t) \|^2 + \int_0^t \| v_t \|^2 \leq 2 \| g \|^2_{L^2([0,T] \times \Omega)} + \| \nabla u_0 \|^2 \quad \text{for all } t \in [0, T).
\]

Moreover \( \| v(t) \|^2 \leq \left( \| u_0 \|^2 + f_0 \| v_t \|^2 \right)^2 \leq 2 \| u_0 \|^2 + 2T \int_0^T \| v_t \|^2 \). By combining the last two estimates we get (19) and conclude the proof. \( \square \)

3. Proofs of Theorems 1.1 and 1.2. This section is devoted to prove our main well–posedness Theorems 1.1 and 1.2. We first precise the meaning of weak solution of (42) makes sense due to the Sobolev Embedding Theorem.

**Definition 3.1.** Let \( u_0 \in H^1_{Y_0}(\Omega) \). When assumption (3) holds we say that \( u \) is a weak solution of problem (2) in \([0,T] \times \Omega \) if (a–d) of Definition 2.1 hold, with the distribution identity (21) being replaced by

\[
\int_\Omega u(t) \phi + \int_\Omega \nabla u(t) \nabla \phi + \int_{\Gamma_1} |u_t(t)|^{m-2} u_t(t) \phi = \int_\Omega \phi(t)(u(t)|^{p-2} u(t)) \phi \tag{42}
\]

Moreover we say that \( u \) is a weak solution of problem (2) in \([0,T] \times \Omega \) if \( u \) is a weak solution of (2) in \([0,T'] \times \Omega \) for all \( T' \in (0,T) \).

**Remark 6.** Since \( p \leq 1 + 2^*/2 \) and \( \phi \in H^1_{Y_0}(\Omega) \) the integral in the right–hand side of (42) makes sense due to the Sobolev Embedding Theorem.

**Proof of Theorem 1.1.** We set, for any \( 0 < T < \infty \), the Banach space \( Y_T = C([0,T]; H^1_{Y_0}(\Omega)) \) endowed with the usual norm \( \| u \|_{Y_T} = \| u \|_{L^\infty([0,T];H^1_{Y_0}(\Omega))} \), and the closed convex set \( X_T = \{ u \in Y_T : u(0) = u_0 \} \). Let \( u \in X_T \). By (3) we have \( 2(p-1) \leq 2^* \) and then, by the Sobolev Embedding Theorem,

\[
\| u(t) \|^p_{2(p-1)} \leq K_0 \| u(t) \|^p_{H^1_{Y_0}}, \quad \forall t \in [0,T], \tag{43}
\]

for some \( K_0 = K_0(\Omega) > 0 \) (in the sequel of the proof \( K_i, i \in \mathbb{N} \), will denote suitable positive constants depending on \( p, n \) and \( \Omega \)). Hence \( \| u \|^p_{L^2} \leq \| u \|^p_{L^\infty} \). Then by Theorem 1.5 there is a unique weak solution \( v \) of the problem

\[
\begin{aligned}
&v_t - \Delta v = |u|^{p-2} u, & &\text{in } (0,T) \times \Omega, \\
v & = 0 & &\text{on } [0,T) \times \Gamma_0, \\
\frac{\partial v}{\partial \nu} & = -|v_t|^{m-2} v_t & &\text{on } [0,T) \times \Gamma_1, \\
v(0,x) & = u_0(x) & &\text{in } \Omega.
\end{aligned}
\tag{44}
\]

Moreover \( v \in C([0,T]; H^1_{\Gamma_0}(\Omega)), v_t \in L^m((0,T) \times \Gamma_1) \cap L^2((0,T) \times \Omega) \) and the energy identity

\[
\frac{1}{2} \| \nabla v(t) \|^2 + \int_0^t \left( \| v_t \|^m_{\Gamma_1} + \| v_t \|^2 \right) = \int_0^t \int_\Omega |u|^{p-2} u v_t \tag{45}
\]
holds for all $t \in [0, T]$. We define $\Phi : X_T \to X_T$ by $\Phi(u) = v$, where $v$ denotes the solution of (44) that corresponds to $u$. We are going to prove that we can apply the Banach Contraction Theorem to $\Phi : B_R \to B_R$ where $B_R = \{ u \in X_T : \| u \|_{Y_T} \leq R \}$, provided that $R$ is sufficiently large and $T$ is sufficiently small. Note that $B_R$ is non-empty for

$$R \geq R_0 := \| u_0 \|_{H^{\frac{1}{2}}_0}. \quad (46)$$

We first claim that $\Phi$ maps $B_R$ into itself for $R$ sufficiently large and $T$ small enough. Let $u \in B_R$. By (45) and (43) we get, for $t \in [0, T]$,

$$\frac{1}{2} \| \nabla v(t) \|^2 + \int_0^t \| v_t \|^2 \leq \frac{1}{2} \| \nabla u_0 \|^2 + K_0 2^{(p-1)} \int_0^t \| u \|^p \| v_t \|^2. \quad (47)$$

Now using Young inequality it follows that, for all $t \in [0, T]$,

$$\frac{1}{2} \| \nabla v(t) \|^2 + \int_0^t \| v_t \|^2 \leq \frac{1}{2} \| \nabla u_0 \|^2 + K_1 R^{p-1} \int_0^t \| v_t \|^2 \leq \frac{1}{2} \| \nabla u_0 \|^2 + \frac{1}{2} K_1^2 R^{2(p-1)} T + \frac{1}{2} \int_0^t \| v_t \|^2. \quad (48)$$

Consequently, by (46),

$$\| \nabla v \|^2_{L^\infty(0, T; L^2(\Omega))} \leq R_0^2 + 2K_2 R^{2(p-1)} T. \quad (49)$$

Using Hölder inequality we have $\| v(t) \|^2 \leq \| u_0 + \int_0^t v_t(s) ds \|^2 \leq \| u_0 \|^2 + T^\frac{1}{2} \left( \int_0^t \| v_t \|^2 \right)^{\frac{1}{2}}$ and so, by (49),

$$\| v(t) \|^2 \leq 2 \| u_0 \|^2 + 2T \left( \int_0^t \| v_t \|^2 \right)^{\frac{1}{2}} \leq 2(1 + T) R_0^2 + 4K_2 R^{2(p-1)} T. \quad (50)$$

Now restricting to $T \leq 1$ we have $T^2 \leq T$ and so combining (48) and (50) we get

$$\| v \|^2_{Y_T} \leq (3 + 2T) R_0^2 + 6K_2 R^{2(p-1)} T \leq 5R_0^2 + 6K_2 R^{2(p-1)} T. \quad (51)$$

By (51) in order to prove that $v \in B_R$, it is enough to show that $5R_0^2 \leq \frac{1}{2} R^2$ and $6K_2 R^{2(p-1)} T \leq \frac{1}{2} R^2$. Hence our claim holds for

$$R = 4R_0 \quad \text{and} \quad T \leq \min \left\{ 1, K_3 R_0^2(2-p) \right\}. \quad (52)$$

In the sequel we shall assume that (52) holds.

We now claim that, for $T$ small enough, the map $\Phi$ is a contraction. Let $u, \bar{u} \in B_R$, and denote $v = \Phi(u), \bar{v} = \Phi(\bar{u}), w = v - \bar{v}$. Clearly, $w$ is a weak solution (in the sense of Lemma 2.2) of the problem

$$\begin{align*}
  w_t - \Delta w &= |u|^{p-2} u - |\bar{u}|^{p-2} \bar{u} \quad \text{in } (0, T) \times \Omega, \\
  w &= 0 \quad \text{on } [0, T) \times \Gamma_0, \\
  \frac{\partial w}{\partial \nu} &= -|v_t|^{m-2} v_t + |\bar{v}_t|^{m-2} \bar{v}_t \quad \text{on } [0, T) \times \Gamma_1, \\
  w(0, x) &= 0 \quad \text{in } \Omega. 
\end{align*} \quad (53)$$

By (51)
Since \( v_t, \bar{v}_t \in L^m((0, T) \times \Gamma_1) \), we also know that \(|v_t|^{m-2} v_t\) and \(|\bar{v}_t|^{m-2} \bar{v}_t\) belong to \( L^m((0, T) \times \Gamma_1) \). Moreover, by (3), the functions \(|u|^{p-2} u\) and \(|\bar{u}|^{p-2} \bar{u}\) belong to \( L^2((0, T) \times \Omega) \). Then we can apply Lemma 2.2 so that, for \( t \in [0, T] \),

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \int_0^t \| w_t \|_2^2 + \int_0^t \int_{\Gamma_1} \left[ |v_t|^{m-2} v_t - |\bar{v}_t|^{m-2} \bar{v}_t \right] w_t
\]

\[= \int_0^t \int_{\Omega} \left[ |u|^{p-2} u - |\bar{u}|^{p-2} \bar{u} \right] w_t. \tag{54}
\]

Using the monotonicity of the map \( x \to |x|^{m-2} x \) and the elementary inequality

\[|A|^{p-2} A - |B|^{p-2} B| \leq K_4 |A - B| \left( |A|^{p-2} + |B|^{p-2} \right), \tag{55}\]

for \( A, B \in \mathbb{R}, p \geq 2 \), we get

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \int_0^t \| w_t \|_2^2 \leq K_4 \int_0^t \int_{\Omega} \left( |u|^{p-2} + |\bar{u}|^{p-2} \right) |u - \bar{u}| \| w_t \|_2. \tag{56}
\]

We now set \( r = 2^* \) if \( n \in \mathbb{N}, n \neq 2 \), while \( r = 2p \) when \( n = 2 \), so that \( 2 \leq p \leq 1 + r/2 \leq 1 + 2^*/2 \) and \( r > 2 \). We also fix \( s > 2 \) such that \( \frac{1}{2} + \frac{r}{s} + \frac{r}{s} = 1 \), that is \( s = \frac{2r}{r-2} \). By applying triple Hölder inequality and the elementary inequality

\[(A + B)^r \leq \max \{1, 2^{r-1} \} (A^r + B^r) \quad \text{for } A, B \geq 0, \quad \tau \geq 0, \tag{57}\]

from (56) we get

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \int_0^t \| w_t \|_2^2 \leq K_5 \int_0^t \left( \int_{\Omega} \left( |u|^{s(p-2)} + |\bar{u}|^{s(p-2)} \right) \right)^{\frac{1}{s}} \| u - \bar{u} \|_r \| w_t \|_2.
\]

But \( s(p-2) \leq r \) since \( p \leq 1 + \frac{r}{2} \), so by the Sobolev Embedding Theorem and weighted Young inequality we obtain, for any \( \varepsilon > 0 \),

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \int_0^t \| w_t \|_2^2 \leq K_5 \int_0^t \left[ \| u \|_r^{s(p-2)} + \| \bar{u} \|_r^{s(p-2)} \right]^{\frac{1}{s}} \| u - \bar{u} \|_r \| w_t \|_2
\]

\[\leq K_7 R^{p-2} \int_0^t \| u - \bar{u} \|_r \| w_t \|_2 \tag{58}
\]

\[\leq K_8 R^{p-2} \int_0^t \| u - \bar{u} \|_{H^1_{\Omega}} \| w_t \|_2
\]

\[\leq K_8 R^{p-2} \left[ \frac{1}{2\varepsilon} \int_0^t \| u - \bar{u} \|_{H^1_{\Omega}}^2 + \frac{\varepsilon}{2} \int_0^t \| w_t \|_2^2 \right]
\]

and consequently

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \int_0^t \| w_t \|_2^2 \leq \frac{K_9 R^{p-2}}{\varepsilon} \| u - \bar{u} \|_{L^\infty(0,T;H^1_{\Omega}(\Omega))}^2 + K_9 R^{p-2} \varepsilon \int_0^t \| w_t \|_2^2.
\]

Now we choose \( \varepsilon = 1/(2K_9 R^{p-2}) \) so previous estimate reads as

\[
\frac{1}{2} \| \nabla w(t) \|_2^2 + \frac{1}{2} \int_0^t \| w_t \|_2^2 \leq 2K_9^2 R^{2(p-2)} T \| u - \bar{u} \|_{L^\infty(0,T;H^1_{\Omega}(\Omega))}^2 \tag{59}
\]

and consequently

\[
\| \nabla w(t) \|_2 \leq K_{10} R^{p-2} \sqrt{T} \| u - \bar{u} \|_{L^\infty(0,T;H^1_{\Omega}(\Omega))} \quad \text{for all } t \in [0, T] \tag{60}
\]
Hence, solutions of (2) on \([0,T]\) are unique weak solutions in \([0,T]\). By combining (60) and (62) we consequently get (as \(T \leq 1\))
\[
\|w(t)\|_{H_{10}^{1}}^{2} \leq K_{13}^{2}(2^{-p})T \|u - \tilde{u}\|_{L_{x}^{\infty}(0,T;H_{10}^{1}(\Omega))}^{2}.
\]
Then \(\Phi\) is a contraction provided \(K_{13}R^{2-p}\sqrt{T} < 1\), that is, by \((52)\), provided
\[
T < K_{13}^{2}(4R)2^{(2-p)}.
\]
We can finally choose \(T' = \min\left\{1, K_{3}R_{0}^{2(2-p)} + \frac{1}{2} K_{13}^{2}(4R_{0})2^{(2-p)}\right\}\) which is decreasing in \(R_{0}\). So, by applying Banach Contraction Theorem with \(T = T'\), there is a weak solution of (2) on \([0,T']\) \times \(\Omega\) satisfying \((4)-(6)\). Moreover \((7)\) follows by \((52)\).

In order to prove that the solution is unique we use a standard procedure of ODEs, using previous claims, which is briefly outlined as follows. Let \(u, \tilde{u}\) be two weak solutions of (2) on \([0,T']\) \times \(\Omega\). By Lemma 2.2 we have \(u, \tilde{u} \in C([0,T'] ; H_{10}^{1}(\Omega))\). Suppose by contradiction that \(u \neq \tilde{u}\). Then
\[
T' = \sup\{\tau > 0 : u = \tilde{u} \text{ on } [0,\tau]\} < T' \quad \text{and} \quad u(T') = \tilde{u}(T') \text{ by continuity}. \tag{65}
\]
Setting \(u_{1}(t) = u(t+T')\), \(\tilde{u}_{1}(t) = \tilde{u}(t+T')\) we have \(u_{1}, \tilde{u}_{1} \in C([0,T' - T'] ; H_{10}^{1}(\Omega))\) and \(\tilde{u}_{0} := u_{1}(0) = \tilde{u}_{1}(0)\). Then \(u_{1}, \tilde{u}_{1}\) are weak solutions of (2) with initial datum \(\tilde{u}_{0}\). By continuity there is \(0 < T'' < T' - T'\) such that
\[
\max\{\|u_{1}\|_{C([0,T''] ; H_{10}^{1}(\Omega))}, \|\tilde{u}_{1}\|_{C([0,T''] ; H_{10}^{1}(\Omega))}\} \leq 4 \|\tilde{u}_{0}\|_{H_{10}^{1}}.
\]
Hence \(u_{1}\) and \(\tilde{u}_{1}\) are fixed points for \(\Phi\) in \(B_{4\|\tilde{u}_{0}\|_{H_{10}^{1}}}\) when \(T = T''\), so by previous claim \(u_{1} = \tilde{u}_{1}\) on \([0,T]\), contradicting \((65)\).

**Proof of Theorem 1.2.** The existence of the unique maximal solution \(u\) of (2) follows by Theorem 1.1 in a standard way: first one sets \(U\) to be the set of all weak solutions of (2), then one proves that any two elements of \(U\) must coincide on the intersection of their domains, arguing as at the end of previous proof, finally one defines \(u(t)\) to coincide with any of these solution for \(t\) in the union of the domains.

Next, in order to prove that the alternative (i)–(ii) holds, let us suppose, by contradiction, that
\[
T_{\text{max}} < \infty \quad \text{and} \quad \lim_{t \rightarrow T_{\text{max}}} \|u(t)\|_{H_{10}^{1}} < \infty. \tag{66}
\]
Then there is a sequence \(T_{n} \rightarrow T_{\text{max}}\) such that \(\|u(T_{n})\|_{H_{10}^{1}}\) is bounded. Thus, by Theorem 1.1, the Cauchy problem (2) with initial time \(T_{n}\) and initial datum \(u(T_{n})\) as a unique weak solution in \([T_{n}, T_{n} + T']\), where \(T' = T'_{*}(\sup_{n \in \mathbb{N}} \|u(T_{n})\|_{H_{10}^{1}} , m, p, \Omega, \Gamma_{1})\) is independent on \(n\). This leads to a contradiction, since, in this way, we can continue the solution to the right of \(T_{\text{max}}\).

Now, in order to prove that \(u\) depends continuously on the initial datum, we fix \(T \in (0,T_{\text{max}})\) and we denote \(M = \|u\|_{C([0,T] ; H_{10}^{1}(\Omega))}\). Since \(u_{0n} \rightarrow u_{0}\) in \(H_{10}^{1}(\Omega)\) there is \(n_{1} \in \mathbb{N}\) such that \(\|u_{0n}\|_{H_{10}^{1}} \leq \|u_{0}\|_{H_{10}^{1}} + 1 \leq M + 1\). Then, by Theorem
1.1, problem (2) with initial datum $u_{0n}$ has an unique solution $u^n$ in $[0, T^*) \times \Omega$, with $T^* = T^*(M + 1, m, p, \Omega, \Gamma_1) \in (0, 1]$ and

$$
\|u^n\|_{C([0,T^*]; H^1_{\text{loc}}(\Omega))} \leq 4 \|u_{0n}\|_{H^1_{\text{loc}}} \leq 4(M + 1)
$$

(67)

for all $n \in \mathbb{N}$. Now we define $w^n = u^n - u$, which is a weak solution of the problem

\[
\begin{cases}
    w^n_p - \Delta w^n = |u^n|^{p-2} u^n - |u|^{p-2} u & \text{in } (0, T^*) \times \Omega, \\
    w^n = 0 & \text{on } [0, T^*) \times \Gamma_0, \\
    \frac{\partial w^n}{\partial \nu} = -|u^n|^{m-2} u^n + |u|^m u_t & \text{on } [0, T^*) \times \Gamma_1, \\
    w^n(0) = u_{0n} - u_0 & \text{in } \Omega
\end{cases}
\]

in the sense of Lemma 2.2. Consequently

\[
\frac{1}{2} \| \nabla w^n(t) \|^2 + \int_0^t \| w^n_r \|^2 dt 
\leq 4^{p-2}(M + 1)^{p-2} K_8 \left[ \frac{1}{2\varepsilon} \int_0^t \| w^n \|_{H^1_{\text{loc}}}^2 + \frac{\varepsilon}{2} \int_0^t \| w^n_r \|^2 \right] + \frac{1}{2} \| \nabla (u_{0n} - u_0) \|^2
\]

(69)

for any $\varepsilon > 0$. Consequently, for $\varepsilon > 0$ sufficiently small we have

\[
\frac{1}{2} \| \nabla w^n(t) \|^2 + \int_0^t \| w^n_r \|^2 dt 
\leq C_3 \int_0^t \| w^n \|_{H^1_{\text{loc}}}^2 + \frac{1}{2} \| \nabla (u_{0n} - u_0) \|^2
\]

(70)

where $C_3 = C_3(p, n, \Omega, u_0, T) > 0$. Moreover, since $T^* \leq 1$, by using H"older inequality we get $\|w^n(t)\|_2 \leq \|u_{0n} - u_0\|_2 + \left( \int_0^t \| w^n_r \|^2 \right)^{1/2}$ and so by (70)

\[
\|w^n(t)\|^2_2 \leq 2 \|u_{0n} - u_0\|^2_2 + 4C_3 \int_0^t \| w^n \|^2_{H^1_{\text{loc}}}.
\]

(71)

Combining (70) and (71) we get

\[
\|w^n(t)\|^2_{H^1_{\text{loc}}} \leq 2 \|u_{0n} - u_0\|^2_{H^1_{\text{loc}}} + C_4 \int_0^t \| w^n \|^2_{H^1_{\text{loc}}}
\]

(72)

where $C_4 = C_4(p, n, \Omega, u_0, T) > 0$. By Gronwall inequality the estimate

\[
\|w^n(t)\|_{H^1_{\text{loc}}} \leq \sqrt{2} \|u_{0n} - u_0\|_{H^1_{\text{loc}}} e^{\frac{C_4}{2} t}, \quad \text{for all } t \in [0, T^*],
\]

(73)

follows. In particular we have

\[
\|w^n(T^*)\|_{H^1_{\text{loc}}} \leq \sqrt{2} \|u_{0n} - u_0\|_{H^1_{\text{loc}}} e^{\frac{C_4}{2} T^*}.
\]

(74)

Then, since $u_{0n} \to u_0$ as $n \to \infty$, for $n \geq n_2$, with $n_2$ sufficiently large, we have $\|w^n(T^*)\|_{H^1_{\text{loc}}} \leq \|u(T^*)\|_{H^1_{\text{loc}}} + 1 \leq M + 1$. Hence we get that $w^n$ is defined in $[T^*, 2T^*)$. Moreover, by repeating previous argument for $t \in [T^*, 2T^*)$ and using (74), we get $\|w^n(t)\|_{H^1_{\text{loc}}} \leq \sqrt{2} \|w^n(T^*)\|_{H^1_{\text{loc}}} e^{\frac{C_4}{2} (t-T^*)} \leq 2 \|u_{0n} - u_0\|_{H^1_{\text{loc}}} e^{\frac{C_4}{2} t}$.

After a finite number $k = \left\lceil \frac{T^*}{T} \right\rceil$ of iterations we get that for $n$ large enough $u^n$
is defined in $[0,T]$ and $\|u^n(t)-u(t)\|_{H^1_{\Gamma_0}} \leq 2^{\frac{1}{\sigma}} \|u_{0n}-u_0\|_{H^1_{\Gamma_0}} e^{\frac{C}{\sigma}t}$ for $t \in [0,T]$, concluding the proof.

4. **Proofs of Theorems 1.3 and 1.4.** When $\sigma(\Gamma_0) > 0$ a Poincarè type inequality holds (see [65, Corollary 4.5.3]) and we can take $\|\nabla u\|_2$ as an equivalent norm in $H^1_{\Gamma_0}(\Omega)$. Then using the Sobolev’s Embedding Theorem, since $p \leq 1+2^*/2 \leq 2^*$, we have

$$B_1 := \sup_{u \in H^1_{\Gamma_0}(\Omega), u \neq 0} \frac{\|u\|_p}{\|\nabla u\|_2} < +\infty. \quad (75)$$

We denote, when $2 < p \leq 1+2^*/2$,

$$\lambda_1 = B_1^{-\frac{2}{p-2}}, \quad \tilde{\lambda}_1 = B_1^{-\frac{2}{p-2}}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2. \quad (76)$$

$$W_1 = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : J(u_0) < E_1 \ and \ \|\nabla u_0\|_2 < \lambda_1 \}, \quad (77)$$

$$\tilde{W}_1 = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : J(u_0) < E_1 \ and \ \|u_0\|_p < \tilde{\lambda}_1 \}, \quad (78)$$

$$W_2 = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : J(u_0) < E_1 \ and \ \|\nabla u_0\|_2 > \lambda_1 \}, \quad (79)$$

and

$$\tilde{W}_2 = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : J(u_0) < E_1 \ and \ \|u_0\|_p > \tilde{\lambda}_1 \}. \quad (80)$$

At first we give the following useful characterization of $d$, $W_s$ and $W_u$.

**Lemma 4.1.** Suppose $2 < p \leq 1+2^*/2$, $\sigma(\Gamma_0) > 0$ and let $d$, $W_s$ and $W_u$ be respectively defined by (11), (12) and (13). Then $E_1 = d$, $W_s = W_1 = \tilde{W}_1$ and $W_u = W_2 = \tilde{W}_2$.

**Proof.** An easy calculation shows that for any $u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}$ we have max $J(\lambda u) = J(\lambda(u) u) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\nabla u\|_2}{\|u\|_p}\right)^{2p/(p-2)}$, where $\lambda(u) = \frac{\|\nabla u\|_2^{2/(p-2)}}{\|u\|_p^{2/(p-2)}}$. Hence, by (75), $d = E_1$. In order to show that $W_s = W_1 = \tilde{W}_1$, we first prove that $W_1 \subset W_2$. Let $u_0 \in W_s$ and suppose, by contradiction, that $\|\nabla u_0\| \geq \lambda_1$. Since $J(u_0) < d = E_1$ and $\|u_0\|_p \leq \|\nabla u_0\|_2^2$ it follows that

$$E_1 > J(u_0) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u_0\|_2^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

which contradicts (76). By (75), since $\tilde{\lambda}_1 = B_1 \lambda_1$, one immediately gets that $W_1 \subset \tilde{W}_1$. To prove that $\tilde{W}_1 \subset W_1$, let $u_0 \in \tilde{W}_1$. By (75), (78) and (76) we have $\|u_0\|_p < \tilde{\lambda}_1^{-p-2} \|u_0\|_2^p = B_1^{-p-2} \|u_0\|_2^2 \leq \|\nabla u_0\|_2^2$ and so $K(u_0) \geq 0$.

In order to show that $W_u = W_2 = W_2$ we first prove that $W_2 \subset W_u$. Let $u_0 \in W_2$ and suppose, by contradiction, that $K(u_0) > 0$. So $\|u_0\|_p \leq \|\nabla u_0\|_2^2$ by (10). Moreover, $J(u_0) < d = E_1$ and $\|\nabla u_0\|_2 > \lambda_1$. Then it follows that

$$E_1 > \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u_0\|_2^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

which contradicts (76). By (75) one immediately gets that $\tilde{W}_2 \subset W_2$. To prove that $W_u \subset \tilde{W}_2$ and conclude the proof, we take $u_0 \in W_u$. We note that, by (75), we have $J(v) \geq h(\|v\|_p)$ for all $v \in H^1_{\Gamma_0}(\Omega)$, where $h$ is defined by $h(\lambda) = \frac{1}{2} B_1^{-2} \lambda^2 - \frac{1}{p} \lambda^p$ for
\( \lambda \geq 0 \). Moreover one easily verify that \( h(\tilde{\lambda}_1) = E_1 \). The, since \( J(u_0) < E_1 \), we have \( \|u_0\|_p \neq \tilde{\lambda}_1 \). Moreover, since \( K(u_0) \leq 0 \), by (75) we have \( B_1^{-2} \|u_0\|_p^2 \leq \|\nabla u_0\|_2^2 \leq \|u_0\|_p^p \) and so \( \|u_0\|_p \geq B_1^{-p/(p-2)} = \lambda_1 \), concluding the proof. \( \square \)

In what follows we shall use the following derivation formula, which is proved here for the sake of completeness only.

**Lemma 4.2.** Under the assumptions of Theorem 1.1, let \( u \) be a weak solution of problem (2) in \([0,T] \times \Omega \). Then

\[
\frac{d}{dt}\|u(t)\|_p^p = p \int_{\Omega} |u(t)|^{p-2} u(t) u_t(t) \quad \text{for almost all } t \in (0,T). \tag{81}
\]

**Proof.** By Definition 2.1–(a) and (3) we have \( |u| \in L^\infty(0,T;L^1(\Omega)) \), and consequently \( \int_{\Omega} |u|^p \in L^\infty(0,T) \subset L^2(0,T) \). It also follows that \( u \in H^1((0,T) \times \Omega) \).

Since the real function \( x \mapsto |x|^p \) in locally Lipschitz continuous, by the chain rule in Sobolev spaces (see [45]) the function \( t \to |u(t,x)|^p \) is absolutely continuous for almost all \( x \in \Omega \) and \( \frac{\partial}{\partial t} |u|^p = p|u|^{p-2}u_t \in L^2(0,T;L^1(\Omega)) \to L^1((0,T) \times \Omega) \), where assumption (3) was used again. It follows that for all \( \varphi \in C_c^\infty(\Omega) \) and \( \chi \in C_c^\infty(0,T) \) we have \( \int_{(0,T) \times \Omega} |u|^p \varphi \chi' = -\int_{(0,T) \times \Omega} p|u|^{p-2} u_t \varphi \chi \). Using Fubini’s Theorem, since \( \varphi \) is arbitrary it follows that \( \int_0^T |u|^p \chi' = -\int_0^T p|u|^{p-2} u_t \chi \in L^1(\Omega) \).

Since \( \int_{\Omega} |u|^p u_t \in L^2(0,T) \) it follows from last formula that \( \|u\|_p \in H^1(0,T) \) and (81) holds in the weak sense. By [10, Theorem 8.2] we see that this holds also almost everywhere in \((0,T)\), concluding the proof. \( \square \)

We now show that \( W_s \) and \( W_u \) are invariant under the flow generated by (2).

**Lemma 4.3.** Under the assumptions of Theorem 1.1, let \( u \) be the weak maximal solution of problem (2). Also assume that (9) holds. Then

(i) if \( u_0 \in W_s \) we have \( u(t) \in W_s \) for all \( t \in [0,T_{\max}) \);
(ii) if \( u_0 \in W_u \) we have \( u(t) \in W_u \) for all \( t \in [0,T_{\max}) \).

**Proof.** By Lemma 4.2, the energy identity (6) can be written as

\[
J(u(t))^{\frac{t}{s}} = -\int_s^t (\|u_t(\tau)\|_{m,1}^m + \|u(\tau)\|_2^2) \, d\tau. \tag{82}
\]

Consequently \( t \mapsto J(u(t)) \) is decreasing in \([0,T_{\max})\) and by Lemma 4.1

\[
J(u(t)) \leq J(u_0) < E_1 \quad \text{for all } t \in [0,T_{\max}). \tag{83}
\]

On the other hand, by (75) we have the inequality \( J(u(t)) \geq g(\|\nabla u(t)\|_2) \), where \( g(\lambda) = \lambda^2/2 - B_1^2 \lambda^p/p \) for \( \lambda \geq 0 \). It is straightforward to verify that \( g \) is increasing in \([0, \lambda_1) \) and decreasing in \([\lambda_1, \infty) \), so \( \lambda_1 \) is the maximum point for \( g \), and that \( g(\lambda_1) = E_1 \). Consequently, by (83) we have \( \|\nabla u(t)\|_2 \neq \lambda_1 \) for all \( t \in [0,T_{\max}) \). Since the function \( t \mapsto \|\nabla u(t)\|_2 \) is continuous, by Lemma 4.1 the proof is complete. \( \square \)

**Proof of Theorem 1.2.** By Theorem 1.2 we just have to prove that when \( u_0 \in W_s \), the alternative (8) in Theorem 1.2 leads to a contradiction, which is obtained by combining Lemma 4.3–(i) with the Poincaré type inequality recalled at the beginning of the section. \( \square \)
Proof of Theorem 1.4. By Theorem 1.2 it is enough to prove that there are no solutions in the whole $(0, \infty) \times \Omega$. We argue by contradiction. Since $J(u_0) < E_1$, we can fix $E_2 \in (J(u_0), E_1)$. We set
\[ H(t) := E_2 - J(u(t)). \] (84)
By using (77), Lemma 4.1 and Lemma 4.3 we get
\[ H(t) < E_1 - \frac{1}{2} \|\nabla u(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p \leq E_1 - \frac{1}{2} \lambda_1^2 + \frac{1}{p} \|u(t)\|_p^p \leq \frac{1}{p} \|u(t)\|_p^p. \] (85)
By (82) we have
\[ H'(t) = \|u_t(t)\|_{m, \Gamma_1}^m + \|u_t(t)\|_2^2 \geq 0, \] (86)
so that
\[ H(t) \geq H(0) = E_2 - J(u_0) > 0. \] (87)
Since, as claimed in Remark 3, it is trivial to verify that $m_0(p) \leq p$ for $p \geq 2$, by (3) and (14) we have $m < 1 + 2^* / 2$, which is nothing but the Sobolev critical exponent for the trace embedding $H^{1}(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see [1, Theorem 5.22, p. 114]). Hence we have that $u(t)|_{\Gamma_1} \in L^m(\Gamma_1)$ for all $t \in [0, T_{\text{max}})$, so we can take $\phi = u(t)$ in (42). In this way (here and in the sequel of the proof explicit dependence on $t$ will be omitted) we obtain the identity
\[ \|u\|_p^p - \|\nabla u\|_2^2 = \int_{\Gamma_1} |u_t|^{m-2} u_t u + (u_t, u). \] (88)
We estimate the two terms in right-hand side of (88) separately. By H"older inequality we get
\[ \int_{\Gamma_1} |u_t|^{m-2} u_t u \leq \|u_t\|_{m, \Gamma_1}^{m-1} \|u\|_{m, \Gamma_1}. \] (89)
To estimate the $L^m(\Gamma_1)$ norm of $u|_{\Gamma_1}$ we first recall the trace embedding for Sobolev space of fractional order (see [1, Theorem 7.58, p. 218] and [56]) $H^s(\mathbb{R}^n) \hookrightarrow W^{s,1}(\mathbb{R}^{n-1})$ when $2 \leq l < \infty$, $\chi = s - \frac{n}{2} + \frac{n-1}{l} > 0$. Since $W^{s,1}(\mathbb{R}^{n-1}) \hookrightarrow L^1(\mathbb{R}^{n-1})$, using the $C^1$ regularity of $\Omega$ and a standard partition of the unity we have the trace embedding $H^s(\Omega) \hookrightarrow L^1(\partial\Omega)$ when $2 \leq l < \infty$, $s - \frac{n}{2} + \frac{n-1}{l} > 0$ and $0 < s \leq 1$. Using the last embedding with $l = \max\{2, m\}$, the fact that $\partial\Omega$ has finite surface measure and H"older inequality we get
\[ \|u\|_{m, \Gamma_1} \leq C_1 \|u\|_{H^s(\Omega)} \] (90)
with $C_1 = C_1(m, s, \Omega) > 0$, when
\[ \max \left\{ \frac{1}{2}, \frac{n}{2} - \frac{n-1}{m} \right\} < s < 1. \] (91)
Next, by the interpolation inequality (see [41, p.49] 3 ) and the already quoted Poincaré type inequality, we have
\[ \|u\|_{H^s(\Omega)} \leq C_2 \|u\|_2^{1-s} \|\nabla u\|_2^s \] (92)
$C_2 = C_2(s, \Omega, \Gamma_0) > 0$. By combining (90) and (92) we get
\[ \|u\|_{m, \Gamma_1} \leq C_3 \|u\|_2^{1-s} \|\nabla u\|_2^s \] (93)
3Actually interpolation inequality is stated in the quoted reference only for $C^\infty$ domains $\Omega$, but as explicitly remarked there this assumption is not optimal. In particular, since $0 < s \leq 1$, the $C^1$ regularity assumed here is sufficient to prove the result. Unfortunately the authors were not able to find a reference where interpolation inequality is stated under optimal regularity assumptions.
for some $C_3 = C_3(m,s,\Omega,\Gamma_0) > 0$. By (89) and (93)
\[ \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \left\| u_t \right\|_{m,\Gamma_1}^{m-1} \left\| u \right\|_2^{1-s} \left\| \nabla u \right\|_2^s. \] (94)

By weighted Young inequality, if
\[ s < \frac{2}{m}, \] (95)

for any $\delta > 0$ we have the estimate
\[ \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \left[ C_4(\delta) \left\| u_t \right\|_{m,\Gamma_1}^m + \delta \left\| \nabla u \right\|_2^2 + \delta \left\| u \right\|_p^p \right] \left\| u \right\|_p^{1-s-p(1/m-s/2)} \]
where $C_4(\delta) = C_4(\delta, m, s) > 0$. Consequently, if $1 - s - p \left( \frac{1}{m} - \frac{s}{2} \right) < 0$, that is if
\[ s < \left( \frac{p}{m} - 1 \right) / \left( \frac{p}{2} - 1 \right), \] (96)

setting $\alpha_s = - \left[ 1 - s - p(1/m - s/2) \right] / p > 0$ we obtain
\[ \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \left[ C_4(\delta) \left\| u_t \right\|_{m,\Gamma_1}^m + \delta \left\| \nabla u \right\|_2^2 + \delta \left\| u \right\|_p^p \right] \left\| u \right\|_p^{-p\beta_s}. \] (97)

Now we have to show the existence of a value of the parameter $s$ satisfying (91), (95) and (96). When $1 < m < 2$ we have $\frac{p}{m} - 1 > \frac{p}{2} - 1$ and $\frac{2}{m} > 1$, so (91), (95) and (96) reduce to $\frac{1}{2} < s < 1$. When $m > 2$ we have $\left( \frac{p}{m} - 1 \right) / \left( \frac{p}{2} - 1 \right) \leq \frac{2}{m} \leq 1$ and $\frac{2}{2} - \frac{n-1}{m} > \frac{1}{2}$, so (91), (95) and (96) reduce to $\frac{p}{2} - \frac{n-1}{m} < s < \left( \frac{p}{m} - 1 \right) / \left( \frac{p}{2} - 1 \right)$. Clearly such an $s$ does exist by assumption (14). We fix it.

Now we consider the second term in the right hand side of (88). Since $p > 2$ and $\Omega$ is bounded, applying Hölder inequality we easily get
\[ |(u_t, u)| \leq \left\| u_t \right\|_2 \left\| u \right\|_2 \leq C_5 \left\| u_t \right\|_2 \left\| u \right\|_p = C_5 \left\| u_t \right\|_2 \left\| u \right\|_p^{\frac{p}{p}} \left\| u \right\|_p^{1-\frac{p}{p}}, \]
where $C_5 = C_5(\Omega, p) > 0$. By weighted Young inequality, for any $\delta > 0$ we obtain
\[ \int_{\Omega} u_t u \leq C_5 \left[ \frac{1}{4\delta} \left\| u_t \right\|_2^2 + \delta \left\| u \right\|_p^p \right] \left\| u \right\|_p^{1-\frac{p}{p}}. \] (98)

Now we set
\[ \bar{\beta}_s = \min \left\{ \alpha_s, - \frac{1}{p} + \frac{1}{2} \right\}. \] (99)

Since $p > 2$ we have $\bar{\beta}_s > 0$. Since, by (85) and (87) we have
\[ \left\| u \right\|_p \geq \left[ pH(0) \right]^{1/p}, \] (100)

we can combine (97) and (98) (by also using (99)) to obtain
\[ \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| + |(u_t, u)| \leq C_7 \left[ C_6(\delta) \left\| u_t \right\|_{m,\Gamma_1}^m + \left\| u_t \right\|_2^2 \right] + \delta \left\| \nabla u \right\|_2^2 + 2\delta \left\| u \right\|_p^p \left\| u \right\|_p^{-p\bar{\beta}_s}, \] (101)

where $C_7 = C_7(m, p, \Omega, H(0)) > 0$ and $C_6(\delta) = C_6(\delta, p, m) > 0$. By combining (88) with (101) we get
\[ \left\| u \right\|_p^p - \left\| \nabla u \right\|_2^2 \leq C_7 \left[ C_6(\delta) \left\| u_t \right\|_{m,\Gamma_1}^m + \left\| u_t \right\|_2^2 \right] + \delta \left\| \nabla u \right\|_2^2 + 2\delta \left\| u \right\|_p^p \left\| u \right\|_p^{-p\bar{\beta}_s}. \] (102)
Consequently, by (86) and (100)
\[
\|u\|_p^p - \|\nabla u\|_2^2 \leq C_u(\delta) H(\delta) H'(t) \|u\|_p^{p\beta} - C_T(pH(0))^{-\beta} \delta \left[\|\nabla u\|_2^2 + 2\|u\|_p^p\right] - C_u(\delta) H'(t) \|u\|_p^{-p\beta} + C_\theta \delta \left[\|\nabla u\|_2^2 + \|u\|_p^p\right]
\]
where \( C_u(\delta) = C_u(\delta, m, p, H(0), \Omega) > 0 \) and \( C_\theta = C_\theta(m, p, H(0), \Omega) > 0 \). Consequently
\[
2(1 + C_\theta) \left[\frac{\|u\|_p^2}{2}\right] + p(1 - C_\theta) \frac{1}{p} \|u\|_p^p \leq C_u(\delta) H'(t) \|u\|_p^{-p\beta}.
\]
By (84) the last estimate can be rewritten as
\[
2(1 + C_\theta) H(t) - 2(1 + C_\theta) E_2 + \left[p(1 - C_\theta) - 2(1 + C_\theta)\right] \frac{1}{p} \|u\|_p^p \leq C_u(\delta) H'(t) \|u\|_p^{-p\beta}, \quad (103)
\]
Now, by Lemma 4.3, (76) and (80) we have \( \|u\|_p^p \geq \bar{\lambda}_1^p = \lambda_1^2\), so previous estimates yields
\[
2(1 + C_\theta) H(t) - 2(1 + C_\theta) E_2 + \lambda_1^2 \left[(1 - C_\theta) - \frac{2}{p} (1 + C_\theta)\right] \leq C_u(\delta) H'(t) \|u\|_p^{-p\beta}, \quad (104)
\]
Now, since \( E_2 < E_1 \), using (76) and the fact that \( C_\theta \) is independent on \( \delta \), as \( \delta \to 0^+ \) we have
\[
-2(1 + C_\theta) E_2 + \lambda_1^2 \left[(1 - C_\theta) - \frac{2}{p} (1 + C_\theta)\right] \to -2E_2 + \lambda_1^2 \frac{2p - 2}{p} \to -2E_1 + \lambda_1^2 \frac{2p - 2}{p} = 0
\]
Hence, by fixing \( \delta > 0 \) sufficiently small, there exists two positive constants \( C_{10} \) and \( C_{11} \) dependent on \( m, p, H(0) \) and \( \Omega \) such that
\[
C_{10} H(t) \leq C_{11} H'(t) \|u\|_p^{-p\beta}. \quad (105)
\]
By (85) the last estimate implies that
\[
H'(t) \geq C_{12} H_{p}^{1+\beta}(t), \quad (106)
\]
where \( C_{12} = C_{12}(m, p, H(0), \Omega) > 0 \), which by integration yields the required contradiction, concluding the proof. \[ \square \]

5. More general results. This section is devoted to generalize our results to problem (1), where \( Q \) and \( f \) satisfy suitable assumptions which generalize the specific behaviour of \( |u_t|^{m-2} u_t \) and \( |u|^{p-2} u \). Our assumptions on \( Q \) are the following ones.

(Q1) \( Q \) is a Carathéodory real function defined on \( (0, \Theta) \times \Gamma_1 \times \mathbb{R} \) for some \( \Theta > 0 \), \( Q(t, x, 0) = 0 \) for almost all \( (t, x) \in (0, \Theta) \times \Gamma_1 \), and there exist an exponent \( m > 1 \) and positive constants \( c_1, c_2, c_3, c_4 \) and \( c_4 \), possibly dependent on \( \Theta \), such that
\[
\begin{align*}
c_1 |v|^{m-1} &\leq |Q(t, x, v)| \leq c_2 |v|^{m-1} \quad \text{when } |v| \geq 1 \\
c_3 |v|^{m-1} &\leq |Q(t, x, v)| \leq c_4 \quad \text{when } |v| \leq 1
\end{align*}
\]
for almost all \( (t, x) \in (0, \Theta) \times \Gamma_1 \) and all \( v \in \mathbb{R} \).
(Q2) The function $Q(t, x, \cdot)$ is increasing for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$.

**Remark 7.** When $Q = Q(v)$ assumptions (Q1)–(Q2) reduce (independently on $\Theta$) to assume that $Q \in C(\mathbb{R})$ is increasing and such that

$$Q(0) = 0, \quad \lim_{v \to 0} \frac{|Q(v)|}{|v|^{m-1}} > 0, \quad 0 < \lim_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} \leq \lim_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty,$$

as for example $Q = Q_0(v) = a|v|^{\mu - 2} v + b|v|^{m-2} v$, $a \geq 0$, $b > 0$, $1 < \mu \leq m$. Moreover (Q1–2) are also satisfied for any $\Theta > 0$ by $Q = Q_1(t, v) = d(t)Q_0(v)$, where $d \in L^\infty_{loc}([0, \infty)), d > 0, 1/d \in L^\infty_{loc}([0, \infty))$.

**Remark 8.** Let us note, for a future use, that (Q1)–(Q2) yield the existence of positive constants $c_5$ and $c_6$ (possibly dependent on $\Theta$) such that

$$|Q(t, x, v)| \leq c_5(1 + |v|^{m-1})$$

and

$$Q(t, x, v)v \geq c_6 |v|^m$$

for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$ and all $v \in \mathbb{R}$.

5.1. **Forced heat equation.** We first present our generalization of Theorem 1.5 to the problem

$$\begin{aligned}
&u_t - \Delta u = g(t, x) \quad \text{in} \ (0, T) \times \Omega, \\
u = 0 \quad \text{on} \ [0, T) \times \Gamma_0, \\
&\frac{\partial u}{\partial n} = -Q(t, x, u_t) \quad \text{on} \ [0, T) \times \Gamma_1, \\
u(0, x) = u_0(x) \quad \text{in} \ \Omega,
\end{aligned}$$

where $g$ is a given term acting on $\Omega$ and $T > 0$ is fixed.

**Definition 5.1.** Let $u_0 \in H^1_{	ext{loc}}(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. We say that $u$ is a weak solution of (109) in $[0, T) \times \Omega$ if (a–d) of Definition 2.1 hold, with the distribution identity (21) being replaced by

$$\int_\Omega u_t(t)\phi + \int_\Omega \nabla u(t)\nabla \phi + \int_{\Gamma_1} Q(t, \cdot, u_t(t)) u_t(t)\phi = \int_\Omega g(t)\phi,$$

which makes sense due to (107).

**Theorem 5.2.** Suppose that (Q1) and (Q2) hold with $\Theta = T$ and that $g \in L^2((0, T) \times \Omega)$. Then, given any initial datum $u_0 \in H^1_{\text{loc}}(\Omega)$, there is a unique weak solution $u$ of (109) in $[0, T) \times \Omega$. Moreover (16) and (17) hold and $u$ satisfies the energy identity

$$\frac{1}{2} \left\| \nabla u \right\|_2^2 |_s^t + \int_s^t \left\| u_t \right\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t)u_t = \int_s^t \int_{\Omega} gu$$

for $0 \leq s \leq t \leq T$. Finally, given any couple of forcing terms $g_1, g_2 \in L^2((0, T) \times \Omega)$ and any couple of forcing terms $g_1, g_2 \in L^2((0, T) \times \Omega)$, denoting by $u^1$ and $u^2$ the solutions of (109) respectively corresponding to $u_{01}$ and to $u_{02}$, $g_2$, the estimate (19) holds true.

**Sketch of the proof of Theorem 5.2.** Using (107), (108) and (Q2) we can repeat almost verbatim the proof of Theorem 1.5 by replacing everywhere $|u^k|^{m-2}u^k$ with $Q(t, x, u^k_t)$, so starting from the problem

$$\begin{aligned}
&\left\{ \begin{array}{l}
(u^k_t, w_j) + (\nabla u^k_t, \nabla w_j) + \int_{\Gamma_1} Q(\cdot, \cdot, u^k_t)w_j = \int_\Omega gw, \quad j = 1, \ldots, k, \\
u^k(0) = u_{0k},
\end{array} \right.
\end{aligned}$$

for 0 ≤ s ≤ t ≤ T.
The definition (33) is now replaced by $G_k(t, y) = y + \int_{\Gamma_1} Q(t, \cdot, B_k(x) \cdot y) B_k(x) dx$, $t \in (0, T)$, $y \in \mathbb{R}^k$, so in the generalization of problem (34) now $G_k$ explicitly depends on $t$. By using assumption (Q2) the arguments of [61, Proof of Theorem 1.5] continue to work in this more general situation for any fixed $t \in (0, T)$, while all the other estimates keep unchanged. The energy identity (36) continues to hold provided the term $\|u^k_t\|_{m, \Gamma_1}$ is replaced by the term $\int_{\Gamma_1} Q(t, x, u^k_t) u^k_t$. By using (108) and (Q2) we still get (37) with $|u^k_t|^m - 2 u^k_t$ being replaced by $Q(t, x, u^k_t)$ in the forth line, where now $C^r$ depends also on $c_1 - c_4$. Finally, to apply the monotonicity method we use (Q2), which is also used in the proof of estimate (19).

5.2. Local well-posedness. We generalize Theorem 1.1 to problem (1) under the following assumption on $f$:

(F1) $f$ is a Carathéodory real function defined on $\Omega \times \mathbb{R}$, $f(x, 0) = 0$ for almost all $x \in \Omega$ and there is an exponent $p \geq 2$ and a positive constant $c_7$ such that for almost all $x \in \Omega$ and all $u_1, u_2 \in \mathbb{R}$

$$|f(x, u_1) - f(x, u_2)| \leq c_7 |u_1 - u_2| (1 + |u_1|^{p-2} + |u_2|^{p-2}).$$

An explicit example of a function $f$ which satisfies (F1) (use (55)) is given by

$$f = f_0(x, u) = a(x)|u|^{p-2} u + b(x)|u|^{q-2} u, \quad 2 \leq q \leq p, \quad a, b \in L^\infty(\Omega).$$

When $f$ is independent on $x$ assumption (F1) can be equivalently written as follows:

$$f \in W^{1,\infty}_{loc}(\mathbb{R}), \quad f(0) = 0, \quad |f'(u)| = O \left( |u|^{p-2} \right) \text{ as } |u| \to \infty.$$  (113)

A further example of a non-algebraic nonlinearity satisfying (113) is given by $f = \pm f_1$, where $f_1(u) = |u|^{p-2} u$, $p \geq 2$, when $|u| \geq 1$ while $f_1(u) = u$ when $|u| \leq 1$.

Remark 9. We note that an immediate consequence of (F1) is the existence of a positive constant $c_8$ such that

$$|f(x, u)| \leq c_8(|u| + |u|^{p-1})$$  (114)

for almost all $x \in \Omega$ and all $u \in \mathbb{R}$.

Definition 5.3. Let $u_0 \in H^1_{\Gamma_1}(\Omega)$ and suppose that (Q1-2), (F1) and assumption (3) hold. We say that $u$ is a weak solution of (1) in $[0, T] \times \Omega$ if (a–d) of Definition 2.1 hold, with the distribution identity (21) being replaced by

$$\int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi + \int_{\Gamma_1} Q(t, x, u_t(t)) \phi = \int_{\Omega} f(x, u(t)) \phi.$$  (115)

Moreover we say that $u$ is a weak solution of problem (1) in $[0, T'] \times \Omega$ if it is a weak solution of (1) in $[0, T'] \times \Omega$ for all $T' \in (0, T)$.

Theorem 1.1 is generalized as follows.

Theorem 5.4. Suppose that (Q1), (Q2) and (F1) hold together with (3). Then given any initial datum $u_0 \in H^1_{\Gamma_1}(\Omega)$ there is $T^* = T^*(\|u_0\|_{H^1_{\Gamma_1}}, m, p, \Omega, \Gamma_1, c_1, c_2, c_3, c_4)$ in $(0, \min\{1, \Theta\}]$, decreasing in the first variable, such that problem (1) has a unique weak solution in $[0, T^*] \times \Omega$. Moreover (4), (5) and (7) hold, together with the energy identity

$$\frac{1}{2} \|\nabla u\|_2^2 + \int_s^t \|u_t\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t) u_t = \int_s^t \int_{\Omega} f(\cdot, u) u_t$$  (116)

for $0 \leq s \leq t \leq T^*$. 
Sketch of the proof. We repeat the proof of Theorem 1.1. We take $T \leq \Theta$ and $u \in X_T$. We note that, by (114) and (3), we have $f(\cdot, u) \in L^\infty(0, T; L^2(\Omega))$. So by Theorem 5.2 there is a unique solution $v$ of the problem
\[
\begin{align*}
  &v_t - \Delta v = f(x, u) \quad \text{in } (0, T^*) \times \Omega, \\
  &v(\cdot) = 0 \quad \text{on } [0, T^*) \times \Gamma_0, \\
  &\frac{\partial v}{\partial v} = -Q(t, x, v_t) \quad \text{on } [0, T^*) \times \Gamma_1, \\
  &v(0, x) = u_0(x) \quad \text{in } \Omega.
\end{align*}
\]
(117)

We set $\Phi : X_T \to X_T$ by $\Phi(u) = v$. By using the same arguments in the proof of Theorem 1.1 together with assumptions (Q2) and (F1) we get for any $T$ the estimate
\[
\frac{1}{2} \left\| \nabla v(t) \right\|_2^2 + \frac{1}{2} \int_0^T \left\| v_t \right\|_2^2 \leq \frac{1}{2} \left\| \nabla u_0 \right\|_2^2 + K_2(R^2 + R^{2(p-1)}) T
\]
which generalizes (47) to this more general situation, where now the constants $K_i$ depends also on $c_T$. Then we proceed as in the quoted proof with $(R^2 + R^{2(p-1)})$ replacing $R^{2(p-1)}$. Consequently we get that $\Phi(B_R) \subset B_R$ provided that
\[
R = 4R_0, \quad \text{and } T \leq \min \left\{ 1, \Theta, K_3(16 + 16^{p-1}R_0^{2(p-2)})^{-1} \right\},
\]
(119)
generalizing (52). In order to show that, for suitable $T$, $\Phi$ is a contraction in $B_R$ we proceed exactly as in the quoted proof by taking $u, \bar{u} \in B_R$, $v = \Phi(u)$, $\bar{v} = \Phi(\bar{u})$, $w = v - \bar{v}$. Clearly, $w$ is a weak solution of the problem
\[
\begin{align*}
  &w_t - \Delta w = f(x, u) - f(x, \bar{u}) \quad \text{in } (0, T) \times \Omega, \\
  &w(\cdot) = 0 \quad \text{on } [0, T) \times \Gamma_0, \\
  &\frac{\partial w}{\partial v} = -Q(t, x, v_t) + Q(t, x, \bar{v}_t) \quad \text{on } [0, T) \times \Gamma_1, \\
  &w(0, x) = 0 \quad \text{in } \Omega.
\end{align*}
\]
(120)
generalizing (53). Since by (114) we have $f(\cdot, u), f(\cdot, \bar{u}) \in L^\infty(0, T; L^2(\Omega))$ and by (107) we have $Q(\cdot, \cdot, u), Q(\cdot, \cdot, \bar{v}) \in L^{m'}((0, T) \times \Gamma_1)$ we can apply Lemma 2.2 to get
\[
\frac{1}{2} \left\| \nabla w(t) \right\|_2^2 + \int_0^t \left\| w_t \right\|_2^2 + \int_0^t \int_{\Gamma_1} [Q(\cdot, \cdot, v_t) - Q(\cdot, \cdot, \bar{v}_t)] w_t \\
= \int_0^t \int_{\Omega} [f(\cdot, u) - f(\cdot, \bar{u})] w_t.
\]
Using (F1) and (Q2) we generalize the estimate (56) to the following one
\[
\frac{1}{2} \left\| \nabla w(t) \right\|_2^2 + \int_0^t \left\| w_t \right\|_2^2 \leq c_T \int_0^T \int_{\Omega} \left( \left| u - \bar{u} \right| (1 + |u|^{p-2} + |\bar{u}|^{p-2}) \right) |w_t|.
\]
(121)
Consequently exactly the same arguments used in the quoted proof allow to prove the estimate
\[
\left\| w(t) \right\|_{H^1_0}^2 \leq K_{13}^2 (1 + R^{2(p-2)})^2 T \left\| u - \bar{u} \right\|_{L^\infty(0, T; H^1_0(\Omega))}^2.
\]
(122)
replacing (63), so by (119), $\Phi$ is a contraction provided $T < K_{13}^{-2} (1 + 4^{p-2}R_0^{2(p-2)})^{-2}$. We can the finally fix $T^*$ and complete the proof.

The following result is nothing but the generalization of Theorem 1.2.

**Theorem 5.5.** Suppose that (Q1–2) hold for all $\Theta > 0$, together with (F1) and (3). Then the assertions of Theorem 1.2 hold when problem (2) is replaced by (1).
Sketch of the proof. We describe the adaptations needed to cover this more general situation with respect to the arguments used in the proof of Theorem 1.2. The existence of a unique weak maximal solution of (1) follows exactly in the same way. When proving the alternative (i–ii), since the equation is not autonomous (as the term \( Q \) is explicitly time–dependent) a more detailed explanation is needed. Let us suppose by contradiction that (66) holds, so there is a sequence \( T_n \to T_{\text{max}}^- < \infty \) such that \( \|u(T_n)\|_{H_{\Gamma_0}^1} \) is bounded. Since \( Q \) satisfies assumptions (Q1–2) for all positive \( \Theta \), we can choose \( \Theta = T_{\text{max}}^- + 1 \). We set for any \( n \in \mathbb{N} \) the time–shifted nonlinear term \( Q_n(t,x,v) = Q(t + T_n, x, v) \), which satisfies assumptions (Q1–2) with \( \Theta = \Theta_n := T_{\text{max}}^- - T_n + 1 \geq 1 \), so that \( Q_n \) satisfies the same assumptions for \( \Theta = 1 \) for all \( n \in \mathbb{N} \). It follows that the existence time \( T^* \) assured by Theorem 5.4 is independent on \( n \), so problem (1) with initial time \( T_n \) and initial datum \( u(T_n) \) has a unique weak solution in \( [T_n, T_n + T^*] \times \Omega \), which leads to the desired contradiction.

When proving the continuous dependence of the solution \( u \) on the initial datum we get the energy identity

\[
\frac{1}{2} \left\| \nabla w^n \right\|^2_{L^2} + \int_0^t \left\| w^n_t \right\|^2_{L^2} + \int_0^t \int_{\Gamma_1} [Q(\cdot, \cdot, u^n) - Q(\cdot, \cdot, u)] w^n_t = \int_0^t \int_{\Omega} [f(\cdot, u^n) - f(\cdot, u)] w^n_t
\]

generalizing (68). Using assumptions (Q2), (F1) and (3) we then get the estimate (69) again, so we can conclude the proof exactly as in Theorem 1.2.

5.3. Global existence versus blow–up. In order to generalize Theorems 1.3 and 1.4 to problem (1) we first generalize Lemma 4.2. We introduce the notation

\[
F(x,u) = \int_0^u f(x,s)ds.
\]

Lemma 5.6. Under the assumptions of Theorem 5.4, let \( u \) be a weak solution of problem (1) in \([0,T] \times \Omega\). Then

\[
\frac{d}{dt} F(\cdot, u(t)) = \int_{\Omega} f(\cdot, u(t)) u_t(t) \quad \text{for almost all } t \in (0,T).
\]

Proof. We first note that an immediate consequence of (114) is that |\( F(x,u) \)\| \( \leq c_0(1 + |u|^p) \) for a positive constant \( c_0 \). Hence \( \int_{\Omega} F(\cdot, u) \in L^\infty(0,T) \subset L^2(0,T) \). Consequently exactly the same arguments used in the proof of Lemma 4.2 apply to this more general case. 

To extend in a suitable way the definition of the stable and unstable sets we need to introduce a second structural assumption on the nonlinearity \( f \).

(F2) There is \( c_{10} \geq 0 \) such that \( F(x,u) \leq \frac{c_{10}}{p} |u|^p \) for almost all \( x \in \Omega \) and all \( u \in \mathbb{R} \).

We remark that the model nonlinearity \( f_0 \) defined in (112) satisfies (F2) if and only if \( a \leq 0 \).

When \( \sigma(\Gamma_0) > 0, p > 2 \) and (F2) holds we set

\[
D_1 := \sup_{u \in H_{\Gamma_0}^1(\Omega), u \neq 0} \frac{\int_{\Omega} F(\cdot, u)}{\| \nabla u \|^2_{L^2}} \leq \frac{c_{10}}{p} B_{1}^p.
\]
When $D_1 > 0$ we also set
\[ \lambda_1 = (pD_1)^{-1/(p-2)}, \quad E_1 = \left( \frac{1}{2} - \frac{1}{p} \right) \lambda_1^2, \quad (125) \]
while $\lambda_1 = E_1 = +\infty$ when $D_1 \leq 0$. Moreover we denote
\[ J(u) = \frac{1}{2} \|\nabla u_0\|^2_2 - \int_{\Omega} F(\cdot, u) \quad \text{for any } u \in H^1_{\Gamma_0}(\Omega), \quad (126) \]
\[ W_s = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : \|\nabla u_0\| < \lambda_1 \text{ and } J(u_0) < E_1 \}, \quad (127) \]
\[ W_u = \{ u_0 \in H^1_{\Gamma_0}(\Omega) : \|\nabla u_0\| > \lambda_1 \text{ and } J(u_0) < E_1 \}. \quad (128) \]

Clearly due to Lemma 4.1 when $f = |u|^{p-2}u$ definitions (127) and (128) coincide with (12) and (13), even if they are inspired from (77) and (79).

We now generalize the potential–well argument contained in Lemma 4.3.

**Lemma 5.7.** Suppose that (Q1–2) hold for all $\Theta > 0$, together with (F1–2) and (3). Suppose moreover that $\sigma(\Gamma_0) > 0$ and $p > 2$. Then the conclusion of Lemma 4.3 continue to hold.

**Proof.** By Lemma 5.6 the energy identity (116) can be written as
\[ J(u(\tau)) \big|_s^t = -\int_s^t \int_{\Gamma_1} Q(\cdot, u(\tau)) u(t) d\tau - \int_s^t \|u_t\|^2_2 d\tau. \quad (129) \]

By (129) and (Q2) the energy function $E(t) := J(u(t))$ is decreasing in $[0, T_{\text{max}}]$. Hence (83) continue to hold. By (124) we have $J(u(t)) \geq \tilde{g}(\|\nabla u(t)\|_2)$, where $\tilde{g} = \frac{\lambda_1^2}{2} - D_1 \lambda_1^p$ if $D_1 > 0$, while $\tilde{g} = \frac{\lambda_1^2}{2}$ if $D_1 \leq 0$. Then, when $D_1 > 0$, the same arguments used in the proof of Lemma 4.3 apply, while there is nothing to prove when $D_1 \leq 0$. \(\square\)

We can now state the generalization of Theorem 1.3.

**Theorem 5.8.** Under the assumptions of Lemma 5.7 if $u_0 \in W_s$ then $T_{\text{max}} = \infty$ and $u(t) \in W_s$ for all $t \geq 0$.

**Proof.** When $D_1 > 0$ we can exactly repeat the proof of Theorem 1.3 by using Lemma 5.7. When $D_1 \leq 0$ the same argument applies since in this case we have $J(u) \geq \frac{1}{2} \|\nabla u\|^2_2$ so $W_s$ is bounded. \(\square\)

In order to generalize Theorem 1.4 we need to strengthen assumption (Q1–2) to the following ones.

(Q1') $Q$ is a Carathéodory real function defined on $(0, \infty) \times \Gamma_1 \times \mathbb{R}$, $Q(t, x, 0) = 0$ for almost all $(t, x) \in (0, \infty) \times \Gamma_1$, and there exists exponents $1 < \mu \leq m$, a positive function $d(t)$ such that $d, 1/d \in L^\infty_{\text{loc}}([0, \infty))$, and positive constants $c_0', c_2', c_3'$ and $c_4'$ such that
\[ c_0' d(t) \max\{|Q(t, x, v)|, |Q(t, x, v)|^\mu \} \leq c_2' d(t) \max\{|Q(t, x, v)|, |Q(t, x, v)|^\mu \} \text{ when } |v| \geq 1, \quad \text{and} \]
\[ c_3' d(t) \max\{|Q(t, x, v)|, |Q(t, x, v)|^\mu \} \leq c_4' d(t) \max\{|Q(t, x, v)|, |Q(t, x, v)|^\mu \} \text{ when } |v| \leq 1 \]
for almost all $(t, x) \in (0, \infty) \times \Gamma_1$ and all $v \in \mathbb{R}$.

(Q2') The function $Q(t, x, \cdot)$ is increasing for almost all $(t, x) \in (0, \infty) \times \Gamma_1$. 


Remark 10. We remark that the nonlinearities \( Q_0 \) and \( Q_1 \) defined in Remark 7 satisfy as well assumption \((Q1’-2’). Moreover when \( Q = Q(v) \) these assumptions reduce to assume that \( Q \in C(\mathbb{R}) \) is increasing and
\[
0 < \lim_{v \to 0} \frac{|Q(v)|}{|v|^{m-1}} \leq \lim_{v \to 0} \frac{|Q(v)|}{|v|^{m-1}} < \infty, \quad 0 < \lim_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} \leq \lim_{|v| \to \infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty.
\]
We also note, for future use, some further consequences of \((Q1 \) and \((Q2’). Since \( Q(0) = 0 \), by \((Q2) \) we have \( Q(t,x,v) \geq 0 \), so \( Q(t,x,v) = |Q(t,x,v)||v| \). Hence, when \(|v| \geq 1 \) we have
\[
|Q(t,x,v)| \leq \epsilon_d' d^{1/m}(t)|Q(t,x,v)|^{1/m'} \quad (130)
\]
while when \(|v| \leq 1 \)
\[
|Q(t,x,v)| \leq \epsilon_d' d^{1/m}(t)|Q(t,x,v)|^{1/m'} \quad (131)
\]
for almost all \((t,x) \in (0,\infty) \times \Gamma_1 \), where \( \epsilon_d \) and \( \epsilon_d' \) are positive constants.

In order to state our blow–up result for problem (1) we need a further specific structural assumption on \( f \).

(F3) There is \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0,\epsilon_0) \) there exists \( c_{11} = c_{11}(\epsilon) > 0 \) such that
\[
f(x,u)u - (p-\epsilon)F(x,u) \geq c_{11} |u|^p
\]
for almost all \( x \in \Omega \) and all \( u \in \mathbb{R} \).

Clearly the model nonlinearity \( f_0 \) defined in (112) satisfies \((F2–3) \) if and only if \( a \leq 0 \). We can finally state

**Theorem 5.9.** Suppose that \((Q1’-Q2’), (F1–3), (3) \) and \((14) \) hold. Moreover suppose that \( \sigma(\Gamma_0) > 0, p > 2 \),
\[
\int_{\infty}^{\infty} \frac{dt}{d^{1/(m-1)} + d^{1/(\mu-1)}} = \infty \quad (132)
\]
and \( u_0 \in W_\nu \). Then the conclusions of Theorem 1.4 hold.

Remark 11. Assumption (132) needs some comment, as it express the possible time–behavior of \( Q \). When \( d(t) = (1+t)^\beta, \beta \in \mathbb{R} \), it reduces to \( \beta \leq \mu - 1 \), and in particular when \( \mu = m \) in assumption \((Q1) \) (what happens for example when \( Q(v) = d(t)|v|^{m-2}v \)), it reduces to \( \beta \leq m - 1 \), which is a well–known optimal assumption to prevent over–damping for time dependent damping terms in ordinary differential systems.

**Proof.** As in the proof of Theorem 1.4 we prove, by contradiction, that there are no solutions in the whole \((0,\infty) \times \Omega \). We fix \( E_2 \in (J(u_0), E_1) \) and set \( H \) by (84). By using Lemma 5.7 and (124) we get a slightly generalized version of (85), that is
\[
H(t) \leq \frac{c_{10}}{p} \|u(t)\|_p^p \quad (133)
\]
By \((Q2) \) formula (86) is now generalized to
\[
H'(t) = \int_{\Gamma_1} Q(t, \cdot, u_t)u_t + \|u_t(t)\|_2^2 \geq \int_{\Gamma_1} Q(t, \cdot, u_t)u_t \geq 0 \quad (134)
\]
so that \((87) \) holds true. By \((107) \) we can again take \( \phi = u_t \) in the distribution identity (115) so getting the following generalized version of (88)
\[
\int_{\Omega} f(\cdot, u)u - \|\nabla u\|_2^2 = \int_{\Gamma_1} Q(\cdot, \cdot, u_t)u + (u_t, u) \quad (135)
\]
The estimate (98) of the second term in the right hand side of (135) keeps unchanged, while the estimate the first term in it needs a more detailed explanation. We use (130), (131) and Hölder inequality twice to get
\[
I_1 := \left\| \int_{\Gamma_1} Q(\cdot, u_t) u \right\|
\]
\[
\leq c_5 d^{1/m} \int_{\{x \in \Gamma_1; |u_t| \leq 1\}} [Q(t, \cdot, u_t) u_t]^{1/m'} |u| + c_4 d^{1/\mu} \int_{\{x \in \Gamma_1; |u_t| \leq 1\}} [Q(t, \cdot, u_t) u_t]^{1/\mu'} |u|
\]
\[
\leq (c_5 + c_4) \left( d^{1/m} \left( \int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/m'} + d^{1/\mu} \left( \int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/\mu'} \right) \|u\|_{m, \Gamma_1}
\]
which generalizes (89). Now we estimate \(\|u\|_{m, \Gamma_1}\) in previous formula by using (93). In this way we obtain
\[
I_1 \leq C_3 \left[ d^{1/m} \left( \int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/m'} + d^{1/\mu} \left( \int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/\mu'} \right] \|u\|^{-s} \|\nabla u\|_2^s
\]
for exponents \(s\) satisfying (91) (generalizing (94)). The same arguments used in the proof of Theorem 1.4 then give, for any \(\delta > 0\),
\[
I_1 \leq C_3 \left[ C_4(\delta) d^{1/(m-1)} \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right] \|u\|^{1-s-p(1/m-s/2)}_p + C_3 \left[ C_4(\delta) d^{1/(\mu-1)} \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right] \|u\|^{1-s-p(1/\mu-s/2)}_p
\]
provided also (95) (and consequently \(s < 2/\mu\) as well) holds. By (87) and (133) we have \(\|u\|_p \geq \left( \frac{p}{c_{10}} H(0) \right)^{1/p} \), so from previous formula we derive, as \(\mu \leq m\),
\[
I_1 \leq C_3' \left\{ C_4(\delta) \left[ d^{1/m} + d^{1/\mu} \right] \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right\} \|u\|^{p\beta_s}_p,
\]
where \(C_3' = C_3'(p, m, s, \Omega, H(0)) > 0\) and \(\beta_s = -\left[ 1 - s - p(1/m-s/2) \right]/p > 0\), generalizing (97). By plugging the last estimate and (98) in (135) and using (134) we get
\[
\int_{\Omega} f(\cdot, u) u - \|\nabla u\|_2^2 \leq C_8(\delta) \left[ d^{1/m} + d^{1/\mu} \right] H(t) \|u\|^{-p\beta_s}_p + C_9 \delta \left( \|\nabla u\|_2^2 + \|u\|_p^p \right)
\]
so generalizing (102). Consequently, by (84) and (126) we have, for any \(\varepsilon > 0\),
\[
\int_{\Omega} f(\cdot, u) u + (p - \varepsilon) H(t) - (p - \varepsilon) E_2 + \left[ \frac{p - \varepsilon}{2} - (1 + C_9 \delta) \right] \|\nabla u\|_2^2
\]
\[
- (p - \varepsilon) \int_{\Omega} F(\cdot, u) - C_9 \delta \|u\|_p^p \leq C_8(\delta) \left[ d^{1/m} + d^{1/\mu} \right] H(t) \|u\|^{p\beta_s}_p
\]
where \(\beta_s\) is given by (99). Then, using assumption (F3) for \(\varepsilon \in (0, \varepsilon_0]\) we have
\[
[c_{11}(\varepsilon) - C_9 \delta] \|u\|_p^p - \left[ \frac{p - \varepsilon}{2} - (1 + C_9 \delta) \right] \|\nabla u\|_2^2 + (p - \varepsilon) H - (p - \varepsilon) E_2
\]
\[
\leq C_8(\delta) \left[ d^{1/m} + d^{1/\mu} \right] H(t) \|u\|^{p\beta_s}_p.
\]
By Lemma 5.7 we have \( \|\nabla u\|_2 \geq \lambda_1 \), so by (125) we get
\[
\left[ \frac{p - \varepsilon}{2} - (1 + C_9\delta) \right] \|\nabla u\|_2^2 - (p - \varepsilon)E_2 \geq \left(-C_9\delta - \frac{\varepsilon}{p}\right)\lambda_1^2 + (p - \varepsilon)(E_1 - E_2).
\] (137)
We fix \( \varepsilon = \varepsilon_1 \) small enough in order to have \( \frac{\varepsilon}{p}\lambda_1^2 < \frac{E_2 - E_1}{2} \). After that we fix \( \delta = \delta_1 \) such that \( C_9\delta = \frac{E_2 - E_1}{2} \) and \( c_{11}(\varepsilon_1) - C_9\delta > 0 \). Consequently, from (136) and (137) we obtain
\[
(p - \varepsilon_1)H(t) \leq C_9(\delta_1) \left[ d^{\frac{1}{1-\rho}} + d^{\frac{1}{\rho}} \right] H'(t)\|u\|_p^{p\beta},
\]
generalizing (105). By (133) we finally obtain \( H'(t) \geq C_9 \left[ d^{\frac{1}{1-\rho}} + d^{\frac{1}{\rho}} \right] H^{1+\beta}(t) \)
which generalizes (106). By integrating and using assumption (132) we get the desired contradiction and conclude the proof. \( \square \)

Appendix A. A physical model. This section is devoted to describe a physical model which motivates problem (1). Let \( \Omega \) represent a solid body surrounded by a fluid denoted by \( A \), whit contact \( \Gamma_1 \) and (possibly) having an internal cavity with contact boundary \( \Gamma_0 \). We suppose that a heat reaction-diffusion process occurs inside \( \Omega \) such that, if \( u = u(t, x) \) represents the temperature at point \( x \) and time \( t \), the quantity of heat produced by the reaction is proportional to a superlinear power of the rate of change of the temperature, i.e. to \( u^{p-1} \) with \( p > 2 \). Thus the process can be modelled by the heat equation with source
\[
u_t - \rho \Delta u = |u|^{p-2} u \text{ in } (0, T) \times \Omega \] (138)
where the thermal conductivity \( \rho > 0 \) is taken to be 1 for simplicity. The surrounding fluid is supposed to be a perfect conductor of heat, so the temperature in \( A \) is spatially homogeneous and can be described by a number \( v = v(t) \) for any \( t \geq 0 \). In particular, there is no diffusion in the fluid. Such assumption is realistic if the fluid is well stirred. Moreover, we introduce a refrigerating process in the fluid with the help of which one tries to control the reaction inside the solid \( \Omega \). We assume that the refrigerating system is controlled in such a way that the heat absorbed from the fluid is proportional to a power of the rate of change of the temperature, as \( |v'(t)|^{m-2}v'(t) \). Let \( j = j(t, x) \) be the heat flux from \( \Omega \) to \( A \). Then the rate of change of the temperature \( v'(t) \) is given by \( v'(t) = -|v'(t)|^{m-2}v(t) + \int_{\Gamma_1} j(t, x) dS \)
On the other hand, the heat flux \( j(t, x) \) is given by the classical conductivity rule by \( j(t, x) = -\frac{\partial u}{\partial v} \), since \( \rho = 1 \). Finally, the thermal contact of the fluid at \( \Gamma_1 \) yields the continuity condition \( u(t, x) = v(t) \), \( x \in \Gamma_1, t \geq 0 \), while the temperature on \( \Gamma_0 \) is assumed to be constant (for simplicity constantly vanishing), that is
\[
u(t, x) = 0, \ x \in \Gamma_0, \ t \geq 0 \] (139)
Combining (138)-(139), we obtain (1) with \( f = |u|^{p-2} u \) and \( Q = u_t + |u|^m u_t \). These nonlinear terms are included in theory developed in Section 5. In particular Theorem 5.9 shows that the refrigerating system cannot avoid the internal explosion with this conditions.

Appendix B. Global existence for problem (2) when \( p = 2 \). This section is devoted to state and prove the global existence result for problem (2) when \( p = 2 \) mentioned in the Introduction. For the sake of generality we actually shall prove a more general version of it dealing with problem (1).
Theorem B.1. Under the assumptions of Theorem 5.5 if \( p = 2 \) then \( T_{\text{max}} = \infty \).

Proof. We suppose by contradiction that \( T_{\text{max}} < \infty \). By (116) together with assumptions (Q1–2) and (114) we have

\[
\frac{1}{2} \| \nabla u \|_2^2 + \int_0^t \| u_t \|_2^2 \leq \frac{1}{2} \| \nabla u_0 \|_2^2 + 2c_8 \int_0^t \int_\Omega |u| |u_t|.
\]

By Hölder and weighted Young inequalities we consequently get

\[
\frac{1}{2} \| \nabla u \|_2^2 + \int_0^t \| u_t \|_2^2 \leq \frac{1}{2} \| \nabla u_0 \|_2^2 + 2c_8^2 \int_0^t \| u \|_2 \int_0^t \| u_t \|_2^2
\]

and consequently

\[
\| \nabla u \|_2^2 + \int_0^t \| u_t \|_2^2 \leq \| \nabla u_0 \|_2^2 + 4c_8^2 \int_0^t \| u \|_2^2.
\]

Moreover, by integrating and using Hölder inequality in time we have

\[
\| u \|_2^2 \leq \left( \| u_0 \|_2 + \int_0^t \| u_t \|_2 \right)^2 \leq 2 \| u_0 \|_2^2 + 2T_{\text{max}} \int_0^t \| u_t \|_2^2.
\]

Combining (140) and (141) we get

\[
\int_0^t \| u_t \|_2^2 \leq \| \nabla u_0 \|_2^2 + 8T_{\text{max}} c_8^2 \left( \| u_0 \|_2^2 + \int_0^t ds \int_0^s \| u_t (\tau) \|_2^2 d\tau \right).
\]

By Gronwall inequality we then get that \( \int_0^t \| u_t \|_2^2 \) is bounded up to \( T_{\text{max}} \). By (141) we consequently get that also \( \| u \|_2 \) is bounded. Hence, by (140) also \( \| \nabla u \|_2 \) is bounded. So we contradict (8) and conclude the proof. \( \square \)

Appendix C. Proof of Lemma 2.2. At first we denote \( H = L^2 (\Omega) \), \( V = H^1_0 (\Omega) \) and \( W = L^m (\Gamma_1) \). Since \( V \) is dense in \( H \), using [54, Theorem 2.1] and (24), (25), we obtain that

\[
u \in C_w ([0, T ]; V ).
\]

The key point is to show that the energy identity holds. With this aim and fixed \( 0 \leq s \leq t \leq T \), we set \( \theta_0 \) to be the characteristic function of the interval \([s, t]\).

For small \( \delta > 0 \), let \( \theta_\delta (\tau) = \theta_\delta (\tau) \) be 1 for \( \tau \in [s + \delta, t - \delta] \), zero for \( \tau \notin (s, t) \) and linear in the intervals \([s, s + \delta] \) and \([t - \delta, t]\). Next let \( \eta \) be a standard mollifying sequence, that is, \( \eta = \eta \in C^\infty (\mathbb{R}) \), supp \( \eta \in (-\varepsilon, \varepsilon) \), \( \int_{-\infty}^{\infty} \eta = 1 \), \( \eta \) even and nonnegative, and \( \eta_{\varepsilon} = \varepsilon^{-1} \eta (\cdot / \varepsilon) \). Let \( * \) denote time convolution. We approximate \( u \), extended as zero outside \([0, T]\), with \( v = \eta * (\theta u) \in C^\infty_c (\mathbb{R}; V) \). Then

\[
0 = \int_{-\infty}^{+\infty} \frac{1}{2} \frac{d}{dt} \| \nabla v \|_2^2 = \int_{-\infty}^{+\infty} (\nabla v, \nabla v_1).
\]

Using standard convolution properties and the Leibnitz rule, we see that \( v_t = \eta * (\theta' u) + \eta * (\theta u_t) \) in \( H \), so that \( \eta * (\theta u_t) \in C^\infty_c (\mathbb{R}; V) \). Then, by (143),

\[
0 = \int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \eta * (\theta' \nabla u)) + \int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \nabla (\eta * (\theta u_t))).
\]

Using (25) and the fact that \( u_t \in L^m ([0, T ] \times \partial \Omega) \) we can take \( \phi = \eta * \eta * (\theta u_t) \) in (26). Then, multiplying by \( \theta \), integrating from \(-\infty \) to \( \infty \) and using standard
properties of convolution, we can evaluate the second term in (144) in the following way:

$$
\int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \nabla(\eta * (\theta u_t))) = \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta \zeta) \eta * (\theta u_t)
+ \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta g) \eta * (\theta u_t) - \int_{-\infty}^{+\infty} \|\eta * (\theta u_t)\|_2^2 \tag{145}
$$

Combining (144) and (145), and recalling that $\theta = \theta_\varepsilon$, we obtain the first approximate energy identity

$$
0 = \int_{-\infty}^{+\infty} (\eta * (\theta_\varepsilon \nabla u), \eta * (\theta_\varepsilon' \nabla u)) - \int_{-\infty}^{+\infty} \|\eta * (\theta_\varepsilon u_t)\|_2^2
+ \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta_\varepsilon \zeta) \eta * (\theta_\varepsilon u_t) + \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta_\varepsilon g) \eta * (\theta_\varepsilon u_t)
=: I_1 + I_2 + I_3 + I_4. \tag{146}
$$

Now we examine each term in (146) separately as $\varepsilon \to 0$ and $\varepsilon$ (i.e. $\eta$) is fixed. Since $\theta_\varepsilon \to \theta_0$ a.e. and

$$
\|\eta * (\theta_\varepsilon \zeta)\|_{m', \Gamma_1} \leq \|\zeta\|_{m', \Gamma_1}, \quad \|\eta * (\theta_\varepsilon u_t)\|_{m', \Gamma_1} \leq \|u_t\|_{m', \Gamma_1},
\|\eta * (\theta_\varepsilon g)\|_2 \leq \|g\|_2, \quad \|\eta * (\theta_\varepsilon u_t)\|_2 \leq \|u_t\|_2,
$$

using (22), (25) and Lebesgue Dominated Convergence Theorem we get the convergences

$$
-I_2 \to \int_{-\infty}^{+\infty} \|\eta * (\theta_0 u_t)\|_2^2
I_3 \to \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta_0 \zeta) \eta * (\theta_0 u_t)
I_4 \to \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta_0 g) \eta * (\theta_0 u_t). \tag{147}
$$

Next we decompose the term $I_1$ as

$$
I_1 = \int_{-\infty}^{+\infty} (\eta * (\theta_0 \nabla u), \eta * (\theta_0' \nabla u)) + \int_{-\infty}^{+\infty} \eta * ((\theta_\varepsilon - \theta_0) \nabla u), \eta * (\theta_\varepsilon' \nabla u))
:= I_5 + I_6
$$

Since $\theta_\varepsilon \to \theta_0$ in $L^1(\mathbb{R})$, by (24) we have that $\eta * ((\theta_\varepsilon - \theta_0) \nabla u) \to 0$ strongly in $L^\infty(0, T; H)$. Moreover, by (24),

$$
\|\eta * (\theta_\varepsilon' \nabla u)\|_{L^1(0, T; H)} \leq \|\theta_\varepsilon'\|_{L^1(\mathbb{R})} \|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0, T; H)} 
\leq 2 \|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0, T; H)},
$$

so that $I_6 \to 0$ as $\varepsilon \to 0$. Next we note that, by the properties of convolution and the specific form of $\theta_\varepsilon$,

$$
I_5 = \int_{-\infty}^{+\infty} \theta_\varepsilon'(\eta * \eta * (\theta_0 \nabla u), \nabla u)
= \frac{1}{\delta} \int_{-\delta}^{+\delta} (\eta * \eta * (\theta_0 \nabla u), \nabla u) - \frac{1}{\delta} \int_{t-\delta}^{t} (\eta * \eta * (\theta_0 \nabla u), \nabla u).
$$
By (142), the function \((\eta \ast \eta \ast (\theta_0 \nabla u), \nabla u)\) is continuous, so

\[
I_5 \to (\eta \ast \eta \ast (\theta_0 \nabla u)(s), \nabla u(s)) \to (\eta \ast \eta \ast (\theta_0 \nabla u)(t), \nabla u(t)) \quad \text{as} \quad \delta \to 0. \tag{148}
\]

Combining the convergences (147)-(148) with (146), recalling that \(\eta = \eta_\varepsilon\) and letting \(\rho_\varepsilon = \eta_\varepsilon \ast \eta_\varepsilon\), we obtain the second approximate energy identity

\[
\begin{align*}
(\rho_\varepsilon \ast (\theta_0 \nabla u), \nabla u), \quad (\rho_\varepsilon \ast (\theta_0 \nabla u))_t &= \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta \ast (\theta_0 \zeta) \eta_\varepsilon \ast (\theta_0 u_t) \\
&\quad + \int_{-\infty}^{+\infty} \int_{\Omega} \eta \ast (\theta_0 \varrho) \eta_\varepsilon \ast (\theta_0 u_t) - \int_{-\infty}^{+\infty} \| \eta_\varepsilon \ast (\theta_0 u_t) \|^2. \tag{149}
\end{align*}
\]

Now we consider the convergence of the two sides of (149) as \(\varepsilon \to 0\). By standard arguments, using (25) and the fact that \(u_t \in L^2((0,T) \times \Omega)\) we get that \(\eta_\varepsilon \ast (\theta_0 u_t) \to \theta_0 u_t\) strongly in \(L^2((0,T) \times \Gamma_1)\) and in \(L^2((0,T) \times \Omega)\). Hence, using (22) and remembering that \(g \in L^2((0,T) \times \Omega)\), the right-hand side of (149) goes to

\[
\begin{align*}
\int_{-\infty}^{+\infty} \int_{\Gamma_1} \theta_0^2 \zeta u_t + \int_{-\infty}^{+\infty} \int_{\Omega} \theta_0^2 g u_t - \int_{-\infty}^{+\infty} \| \theta_0 u_t \|^2 \\
= \int_s^t \int_{\Gamma_1} \zeta u_t + \int_s^t \int_{\Omega} g u_t - \int_s^t \| u_t \|^2.
\end{align*}
\]

Concerning the left-hand side of (149), we note that \(\sup \rho_\varepsilon \subset (-2 \varepsilon, 2 \varepsilon)\), \(0 \leq \rho_\varepsilon = O(\varepsilon^{-1})\) and \(\int_0^1 \rho_\varepsilon = \frac{1}{2} \int_0^1 \rho_\varepsilon = \frac{1}{2}\). Therefore, for sufficiently small \(\varepsilon\),

\[
(\rho_\varepsilon \ast (\theta_0 \nabla u)(t), \nabla u(t)) - \frac{1}{2} \| \nabla u(t) \|^2 = \int_0^t \rho_\varepsilon(\sigma) (\nabla u(t - \sigma) - \nabla u(t), \nabla u(t)) \, d\sigma.
\]

Since, by (142), \(\sigma \to (\nabla u(t - \sigma) - \nabla u(t), \nabla u(t))\) is continuous and vanishes when \(\sigma = 0\), we conclude that, as \(\varepsilon \to 0\), \((\rho_\varepsilon \ast (\theta_0 \nabla u)(t), \nabla u(t)) \to \frac{1}{2} \| \nabla u(t) \|^2\). The same result, of course, continues to hold when \(t\) is replaced by \(s\). Then we can pass to the limit in (149) and conclude the proof of (28). To show that (27) holds, we note that, by (28), it follows that \(t \to \| \nabla u(t) \|^2\) is continuous. Now we fix \(t \in [0,T]\) and let \(t_k \to t\). Using (142), we have \(\| u(t_k) - u(t) \|^2 = \| u(t_k) \|^2 + \| u(t) \|^2 - 2(u(t_k), u(t)) \to 0\) as \(k \to \infty\), concluding the proof.

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