A LINEAR OPERATOR ASSOCIATED WITH THE MITTAG-LEFFLER FUNCTION AND RELATED CONFORMAL MAPPINGS

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Abstract In the present paper, we introduce a linear operator associated with the Mittag-Leffler function. Some convolution properties of meromorphic functions involving this operator are given.

Keywords Meromorphic functions, conformal mapping, Mittag-Leffler function, second-order differential subordination, Hadamard product (or convolution), convex functions, univalent functions.

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1. Introduction

The familiar Mittag-Leffler function \(E_\alpha(z)\) introduced by Mittag-Leffler [5] and its generalization \(E_{\alpha,\beta}(z)\) introduced by Wiman [12] are defined by the following series:

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z, \alpha \in \mathbb{C}; \ \Re(\alpha) > 0)
\]

(1.1)

and

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C}; \ \Re(\alpha) > 0),
\]

(1.2)

respectively. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

\[
E_1(z) = E_{1,1}(z) = e^z, \quad E_2(z^2) = E_{2,1}(z^2) = \cosh z \quad \text{and}
\]

\[
E_2(-z^2) = E_{2,1}(-z^2) = \cos z.
\]

The above-defined functions \(E_\alpha(z)\) and \(E_{\alpha,\beta}(z)\), as well as their various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems

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and in the study of complex systems. In particular, the Mittag-Leffler function is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. Several properties of the Mittag-Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$, together with their generalizations, can be found in a number of recent works (see [1–3] and [7–11]).

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p}z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}),$$

which are analytic in the punctured open unit disk

$$\mathbb{U}_0 = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}.$$ 

The class $\Sigma(p)$ is closed under the Hadamard product (or convolution):

$$(f_1 \ast f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,1}a_{n-p,2}z^{n-p} = (f_2 \ast f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j}z^{n-p} \in \Sigma(p) \quad (j = 1, 2).$$

For $f \in \Sigma(p)$, we consider the following operator $T_{\alpha,\beta} : \Sigma(p) \to \Sigma(p)$ associated with the Mittag-Leffler function:

$$T_{\alpha,\beta}f(z) = \left(\Gamma(\beta)z^{-p}E_{\alpha,\beta}(z)\right) \ast f(z)$$

$$= z^{-p} + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha_n + \beta)}a_{n-p}z^{n-p},$$

where $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Let $\mathcal{P}$ be the class of functions $h$ with $h(0) = 1$, which are analytic and convex univalent in the open unit disk $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$.

For functions $f$ and $g$ analytic in $\mathbb{U}$, we say that $f$ is subordinate to $g$, written $f \prec g$, if $g$ is univalent in $\mathbb{U}$, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now we introduce the following new subclass of $\Sigma(p)$.

**Definition 1.1.** A function $f \in \Sigma(p)$ is said to be in the class $\mathcal{M}_{\alpha,\beta}(\lambda; h)$ if it satisfies the second order differential subordination:

$$\frac{\lambda - 1}{p}z^{p+1}(T_{\alpha,\beta}f(z))^\prime + \frac{\lambda}{p(p+1)}z^{p+2}(T_{\alpha,\beta}f(z))^\prime\prime \prec h(z),$$

where $\lambda, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $h \in \mathcal{P}$.

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_nz^n,$$

which are analytic in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\gamma)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{U})$$

(1.7)
for some \( \gamma \) \((\gamma < 1)\). When \( 0 \leq \gamma < 1 \), \( S^*(\gamma) \) is the class of starlike functions of order \( \gamma \) in \( \mathbb{U} \). A function \( f \in \mathcal{A} \) is said to be prestarlike of order \( \gamma \) in \( \mathbb{U} \) if

\[
\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma) \quad (\gamma < 1).
\]  

(1.8)

We denote this class by \( R(\gamma) \) (see [6]). It is obvious that a function \( f \in \mathcal{A} \) is in the class \( R(0) \) if and only if \( f \) is convex univalent in \( \mathbb{U} \) and \( R(\frac{1}{2}) = S^*(\frac{1}{2}) \).

The study of the Mittag-Leffler functions \( E_\alpha(z) \) and \( E_{\alpha,\beta}(z) \) is a recent interesting topic in geometric function theory. In the present paper we shall make a further contribution to the subject by showing some convolution properties for meromorphic functions involving the Mittag-Leffler functions.

The following lemmas will be used in our investigation.

**Lemma 1.1** ([6]). Let \( \gamma < 1 \), \( f \in S^*(\gamma) \) and \( g \in R(\gamma) \). Then, for analytic function \( F \) in \( \mathbb{U} \),

\[
\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(\mathbb{U})),
\]

where \( \overline{co}(F(\mathbb{U})) \) denotes the closed convex hull of \( F(\mathbb{U}) \).

**Lemma 1.2** ([4]). Let \( g(z) = 1 + \sum_{n=m}^{\infty} b_n z^n \) (\( m \in \mathbb{N} \)) be analytic in \( \mathbb{U} \). If \( \Re(g(z)) > 0 \) \((z \in \mathbb{U})\), then

\[
\Re(g(z)) \geq \frac{1 - |z|^m}{1 + |z|^m} \quad (z \in \mathbb{U}).
\]

2. Hadamard product properties

In this section we shall derive several Hadamard product properties for functions in the class \( M_{\alpha,\beta}(\lambda; h) \).

**Theorem 2.1.** Let \( f \in M_{\alpha,\beta}(\lambda; h) \), \( g \in \Sigma(p) \) and \( \Re(z^p g(z)) > \frac{1}{2} \) \((z \in \mathbb{U})\). Then \( f * g \in M_{\alpha,\beta}(\lambda; h) \).

**Proof.** For \( f \in M_{\alpha,\beta}(\lambda; h) \) and \( g \in \Sigma(p) \), we have

\[
\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta}(f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f * g)(z))''
\]

\[
= \frac{\lambda - 1}{p} (z^p g(z)) * (z^{p+1} (T_{\alpha,\beta} f(z))') + \frac{\lambda}{p(p+1)} (z^p g(z)) * (z^{p+2} (T_{\alpha,\beta} f(z)))''
\]

\[
= (z^p g(z)) * \psi(z),
\]

(2.1)

where

\[
\psi(z) = \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta} f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta} f(z))'' < h(z).
\]

(2.2)

In view of the conditions of Theorem 2.1, the function \( z^p g(z) \) has the Herglotz representation:

\[
z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in \mathbb{U}),
\]

(2.3)
where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$. Since the function $h$ is convex univalent in $U$, it follows from (2.1) to (2.3) that
\[
\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta}(f \ast g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f \ast g)(z))''
= \int_{|x|=1} \psi(xz) d\mu(x) < h(z).
\]
This shows that $f \ast g \in M_{\alpha,\beta}(\lambda; h)$. The proof of Theorem 2.1 is completed.

**Theorem 2.2.** Let $f \in M_{\alpha,\beta}(\lambda; h)$, $g \in \Sigma(p)$ and $z^{p+1}g(z) \in R(\gamma)$ ($\gamma < 1$). Then $f \ast g \in M_{\alpha,\beta}(\lambda; h)$.

**Proof.** From (2.1) we can write
\[
\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta}(f \ast g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f \ast g)(z))''
= \frac{(z^{p+1}g(z)) \ast (z\psi(z))}{(z^{p+1}g(z)) \ast z},
\]
where the function $\psi$ is defined as in (2.2).

Since the function $h$ is convex univalent in $U$,
\[
\psi(z) < h(z), \quad z^{p+1}g(z) \in R(\gamma) \quad \text{and} \quad z \in S^*(\gamma) \quad (\gamma < 1),
\]
from (2.4) and Lemma 1.1, we obtain the desired result. The proof of Theorem 2.2 is completed.

Taking $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 2.2, we have the following consequence.

**Corollary 2.1.** Let $f \in M_{\alpha,\beta}(\lambda; h)$. Also let $g \in \Sigma(p)$ satisfy either of the following conditions:

(i) $z^{p+1}g(z)$ is convex univalent in $U$

or

(ii) $z^{p+1}g(z) \in S^*(\frac{1}{2})$.

Then $f \ast g \in M_{\alpha,\beta}(\lambda; h)$.

**Theorem 2.3.** Let $\lambda \leq 0$ and
\[
f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in M_{\alpha,\beta}(\lambda; h_j) \quad (j = 1, 2),
\]
where
\[
h_j(z) = \frac{1 + A_j z}{1 + B_j z} \quad \text{and} \quad -1 \leq B_j < A_j \leq 1.
\]
If $f \in \Sigma(p)$ is defined by
\[
(T_{\alpha,\beta}f(z))' = -\frac{1}{p} \left( (T_{\alpha,\beta}f_1(z))' \ast (T_{\alpha,\beta}f_2(z))' \right),
\]
then $f \in M_{\alpha,\beta}(\lambda; h)$, where
\[
h(z) = \gamma + (1 - \gamma) \frac{1 + z}{1 - z}
\]
and \( \gamma \) is given by

\[
\gamma = \begin{cases} 
1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 + \frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{2}} - 1 - u^{-\frac{1}{2}} - 1 \, du \right) & (\lambda < 0) \\
1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} & (\lambda = 0).
\end{cases}
\tag{2.9}
\]

The bound \( \gamma \) is sharp when \( B_1 = B_2 = -1 \).

**Proof.** We consider the case when \( \lambda < 0 \). By setting

\[
H_j(z) = \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha, \beta} f_j(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha, \beta} f_j(z))'' \quad (j = 1, 2)
\]

for \( f_j \) \((j = 1, 2)\) given by (2.5), we find that

\[
H_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j} z^n < 1 + \frac{A_j z}{1 + B_j z} \quad (j = 1, 2)
\tag{2.10}
\]

and

\[
(T_{\alpha, \beta} f_j(z))' = \frac{p(p+1)}{\lambda} z^{(1-\lambda)(p+1)} \int_0^z t^{\frac{2}{p+1}} - 1 H_j(t) dt \quad (j = 1, 2).
\tag{2.11}
\]

Now, if \( f \in \Sigma(p) \) is defined by (2.7), we find from (2.11) that

\[
(T_{\alpha, \beta} f(z))' = -\frac{1}{p} \left( (T_{\alpha, \beta} f_1(z))' * (T_{\alpha, \beta} f_2(z))' \right)
\]

\[
= -\frac{1}{p} \left( \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{2}} - 1 H_1(uz) du \right) * \left( \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{2}} - 1 H_2(uz) du \right)
\]

\[
= \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{2}} - 1 H(uz) du,
\tag{2.12}
\]

where

\[
H(z) = -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{2}} - 1 (H_1 * H_2)(uz) du.
\tag{2.13}
\]

Also, by using (2.10) and the Herglotz theorem, we see that

\[
\Re \left\{ \left( \frac{H_1(z) - \gamma_1}{1 - \gamma_1} \right) * \left( \frac{1}{2} + \frac{H_2(z) - \gamma_2}{2(1 - \gamma_2)} \right) \right\} > 0 \quad (z \in \mathbb{U}),
\]

which leads to

\[
\Re \{ (H_1 * H_2)(z) \} > \gamma_0 = 1 - 2(1 - \gamma_1)(1 - \gamma_2) \quad (z \in \mathbb{U}),
\]

where

\[
0 \leq \gamma_j = \frac{1 - A_j}{1 - B_j} < 1 \quad (j = 1, 2).
\]

According to Lemma 1.2, we have

\[
\Re \{ (H_1 * H_2)(z) \} \geq \gamma_0 + (1 - \gamma_0) \frac{1 - |z|}{1 + |z|} \quad (z \in \mathbb{U}).
\tag{2.14}
\]
Now it follows from (2.12) to (2.14) that
\[
\Re \left\{ \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta} f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta} f(z))'' \right\} = \Re \{H(z)\} = -\lambda - 1 \int_0^1 \frac{\gamma_0 + (1 - \gamma_0) 1 - u|z|}{1 + u|z|} \, du = \gamma,
\]
which proves that \( f \in M_{\alpha,\beta}(\lambda; h) \) for the function \( h \) given by (2.8).

When \( B_1 = B_2 = -1 \), we consider the functions \( f_j (j = 1, 2) \) defined by
\[
(T_{\alpha,\beta} f_j(z))' = \frac{p(p+1)}{\lambda} z^{\frac{\lambda - 1}{p}} \int_0^z t^{-\frac{\lambda + 1}{p} - 1} 1 + A_j t 1 - t \, dt \quad (j = 1, 2),
\]
for which we have
\[
H_j(z) = \frac{1 + A_j z}{1 - z} \quad (j = 1, 2)
\]
and
\[
(H_1 \ast H_2)(z) = \frac{1 + A_1 z}{1 - z} \ast \frac{1 + A_2 z}{1 - z} = 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - z}.
\]
Hence, for the function \( f \) given by (2.7), we have
\[
\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta} f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta} f(z))'' = \frac{p + 1}{\lambda} \int_0^1 u^{-\frac{\lambda + 1}{p} - 1} \left( 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - u z} \right) \, du
\]
as \( z \to -1 \).

Finally, for the case when \( \lambda = 0 \), the proof of Theorem 2.3 is simple, and we choose to omit the details involved. Now the proof of Theorem 2.3 is completed.
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