Abstract. We investigate the electromagnetic response of a periodic checkerboard consisting of alternating rectangular cells of positive refractive index ($\varepsilon = +1$, $\mu = +1$) and negative refractive index ($\varepsilon = -1$, $\mu = -1$). We show that the system has peculiar imaging properties in that it reproduces images of a source in one cell in every other cell. Using coordinate transformations, we map this system into a class of imaging systems in three dimensions consisting of three orthogonal planes delimiting eight alternating cubical regions of positive and negative index media sharing the same vertex. We also generalize these results to more general checkerboards that are inhomogeneous and anisotropic that can then be used to generate a class of three-dimensional (3D) corner imaging systems.
In 1967, Veselago [1] speculated that negative values for the dielectric permittivity ($\varepsilon$) and the magnetic permeability ($\mu$) would result in a negative refractive index (NRI). Negative refractive index materials (NRMs) consisting of structured metamaterials have now been proposed [2, 3] and demonstrated [4, 5] at microwave and radio frequencies creating excitement in the scientific community due to the intriguing possibilities these metamaterials offer (see [6] for a recent review). Perhaps one of the most interesting applications that these metamaterials appear to offer is the possibility of a new class of perfect lenses which do not suffer the diffraction limitation of conventional lenses [7]. A slab of NRM, with $n = -1$, can image both the far-field propagating modes and the near-field evanescent modes of a source, and thus in principle can act as a perfect lens. The resolution of this system is not limited by wavelength, but only by the extent of dissipation [8]–[10], dispersion [11, 12] and imperfections in the constituent materials [6]. In any physical realization, the materials are always dissipative, spatially dispersive at short lengthscales [13], and at very short lengthscales even the assumption of an effective medium breaks down. Thus there is always a cut-off for the smallest (but sub-wavelength) lengthscale that can be imaged by such a lens. In most physical realizations, this is primarily set by the levels of dissipation and next by the intrinsic lengthscale of the structure of the metamaterial. Hence, although a perfect image would not be attainable, substantial sub-wavelength image resolution is possible [6, 14, 19].

In fact, the perfect slab lens is only one member of a whole category of systems which satisfy a generalized lens theorem [15]. A negatively refracting slab is in some sense complementary to an equal thickness of vacuum, and cancels its presence for both propagating and evanescent near-field modes. In fact, the materials involved do not even have to be homogeneous and could have an arbitrary variation (within the approximation of a metamaterial) in the directions transverse to the imaging axis. Now consider the more general situation where the dielectric permittivity and the magnetic permeability are arbitrary functions of the transverse spatial co-ordinates (see figure 1):

$$\varepsilon_1 = +\varepsilon(x, y), \quad \mu_1 = +\mu(x, y) \quad \forall -d < z < 0,$$

$$\varepsilon_2 = -\varepsilon(x, y), \quad \mu_2 = -\mu(x, y) \quad \forall 0 < z < d.$$
Figure 1. A pair of complementary optical media nullifies the effect of each other for the passage of radiation. Positive and negative refractive index are schematically depicted by white and coloured regions. The paths in the spatially varying media are in general not straight lines.

Figure 2. A pair of 2D corners of negative refractive index can focus a source back on to itself. This system can be mapped on to a layered system with a periodic set of sources.

We will consider the imaging axis to be the $z$ axis. Thus we see that the system is anti-symmetric with respect to the $z = 0$ plane. It turns out that such a system also transfers the image of a source placed at the $z = -d$ to the $z = d$ plane in the same exact sense that it includes both the propagating and evanescent components \cite{15}. Thus to an observer on the right-hand side, it would appear as if the region between $z = -d$ and $z = d$ did not exist. We will refer to such media with the same sense of transverse spatial variation but with opposite signs as optical complementary media, and the effect of any such pairs of complementary media on radiation is null.

Using a general method of co-ordinate mapping, one can map Maxwell’s equations to other geometries and obtain perfect lenses in other geometries. For instance, using a mapping from a layered structure consisting of layers of complementary media together with a periodic set of line sources, it was observed in \cite{15,16} that two negative 2D corners sharing the same corner combine to make an optical system within which light radiating from a line source is bent around a closed trajectory and is refocused back on to the line source (see figure 2).
In this paper, we first present a proof of the generalized lens theorem based on the symmetries of Maxwell’s equations. Then we explore imaging effects through negative refraction in checkerboard structures of positive–negative refraction index materials, and later map them on to 3D corner lenses which focus a source back on to itself.

2. Proof of the generalized lens theorem

Consider Maxwell’s equations in a material medium

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3) \]

\[ \nabla \times (\mu^{-1} \mathbf{B}) = \frac{\partial (\varepsilon \mathbf{E})}{\partial t}. \quad (4) \]

Let us list the transformations which leave these equations invariant. The transformations fall into the following categories

S1 (Generalized conformal invariance): \( \mathbf{E} \rightarrow \mathcal{A}\mathbf{E}, \mathbf{B} \rightarrow \mathcal{A}\mathbf{B}, \mu^{-1} \rightarrow \mathcal{A}\mu^{-1}\mathcal{A}^{-1}, \varepsilon \rightarrow \mathcal{A}\varepsilon\mathcal{A}^{-1} \), where \( \mathcal{A} \) is invertible and an element of \( GL_3(\mathbb{R}) \) (a group of \( 3 \times 3 \) linear operators).

S2 (Generalized duality): \( \mathbf{E} \rightarrow -\varepsilon^{-1}\mathbf{B}, \mathbf{B} \rightarrow \mu\mathbf{E}, \) or \( \mathbf{E} \rightarrow \mu^{-1}\mathbf{B}, \mathbf{B} \rightarrow -\varepsilon\mathbf{E} \) (iff \( \mu = \varepsilon \)).

S3: \( \mu \rightarrow \alpha\mu, \varepsilon \rightarrow \varepsilon/\alpha \), where \( \alpha \) is a non-zero scalar.

S4 (Time reversal): \( t \rightarrow -t, \mathbf{B} \rightarrow -\mathbf{B} \).

S5 (Parity invariance): \( \mathbf{r} \rightarrow -\mathbf{r}, \) where \( \mathbf{r} = [x, y, z] \), \( \mathbf{E} \rightarrow -\mathbf{E} \).

S6: Any additional space-time symmetries.

The combination of any of these symmetries is again a symmetry of the system of equations. Then we can assert that ‘if the fields in a particular region of space can be mapped on to another region of space through the symmetry transformations S1–S6 while preserving the respective boundary conditions, then the transformed fields solve the field equations whenever the original fields do’.

Now consider a homogeneous slab of medium for \(-d < z < 0\) with dielectric permittivity and magnetic permeability tensors

\[ \tilde{\varepsilon}_1 = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}, \quad \tilde{\mu}_1 = \begin{pmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix}. \quad (5) \]

For propagation along the \( z \) direction and origin at the interface, let us use the symmetry operations S5 and S3(\( \alpha = -1 \)), followed by S1(\( \mathcal{A} \)) with

\[ \mathcal{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6) \]
Figure 3. An infinite checkerboard with optically complementary cells (two phases) recreates images of a source in one cell in every other cell. The ray picture can reproduce only some of the images as shown by arrows.

We will call this sequence of operations a mirror operation. This choice of $A$ preserves the continuity of $E_i$ and $\mu^{-1}B_i$ across the boundary. Then the resulting complementary medium on the right for $0 < z < d$ is then

$$\tilde{\varepsilon}_2 = \begin{pmatrix} -\varepsilon_{xx} & -\varepsilon_{xy} & +\varepsilon_{xz} \\ -\varepsilon_{yx} & -\varepsilon_{yy} & +\varepsilon_{yz} \\ +\varepsilon_{zx} & +\varepsilon_{zy} & -\varepsilon_{zz} \end{pmatrix}, \quad \tilde{\mu}_2 = \begin{pmatrix} -\mu_{xx} & -\mu_{xy} & +\mu_{xz} \\ -\mu_{yx} & -\mu_{yy} & +\mu_{yz} \\ +\mu_{zx} & +\mu_{zy} & -\mu_{zz} \end{pmatrix}, \quad (7)$$

which is the result obtained in [15]. The entries in $\tilde{\varepsilon}$ and $\tilde{\mu}$ could also be spatially varying along $x$ and $y$.

3. Checkerboards of homogeneous, isotropic media

Let us consider a checkerboard system of alternating rectangular blocks of negative and positive refractive index media as shown in figure 3. Let us call the axes along the checkerboard directions $\theta$ and $\Phi$, and the direction normal to the plane $\ell$ and define the checkerboard by (see figure 3),

$$\varepsilon(\theta, \Phi) = \mu(\theta, \Phi) = +1, \quad \forall (\theta, \Phi) \in \text{positive,} \quad (8)$$

where

$$\text{positive} = (0, \pi/2) \times (0, \pi/2) \cup (\pi/2, \pi) \times (\pi/2, \pi) \quad (9)$$

and

$$\text{negative} = (0, \pi/2) \times (\pi/2, \pi) \cup (\pi/2, \pi) \times (0, \pi/2). \quad (10)$$

This checkerboard is invariant along the $\ell$ direction, periodic of period $\pi$ in the $\theta$ and $\Phi$ directions.
First of all, it is clear that a single source generates an image inside every other cell (positive or negative) of the checkerboard (see figure 3). This can be easily shown by considering the imaging along, say, the \( \theta \) direction and using the generalized lens theorem \([15]\), whereby the condition of complementarity is satisfied for the layers along the imaging \( \theta \) direction with varying refractive index \( n \) in the \( \Phi \) direction transverse to the imaging direction. Thus for a source at \( \theta = \theta_1, \Phi = \phi_1 \) in the first positive cell, we have a set of images along a \( \Phi = \phi_1 \) line at \( \theta = \pm n\pi \pm \theta_1 \) and \( -\theta_1 \), where \( n \) is a positive integer. Similarly, applying the generalized lens theorem along the \( \Phi \) direction, we can show now that the entire set of image points would be reproduced along the \( \Phi \) direction. Thus we have an image point in every cell of the checkerboard structure corresponding to the source placed in any one cell.

Since the \( \epsilon \) and \( \mu \) are periodic with period \( \pi \) in the \( \theta \) and \( \Phi \) directions, the system has an additional invariance under a translation by a lattice vector \( (n\pi \text{ and } n\pi, \text{ where } n \text{ is an integer}) \). The corresponding symmetry operations for the imaging in the checkerboard are a translation by a lattice vector followed by a mirror operation \( S1(A) \cdot S5 \cdot S3(\alpha = -1) \) as in the case of the slab lens. Then it follows that the fields in every pair of cells or pairs of blocks of cells are the transformed versions of the fields in the cell with the source, thus proving the imaging properties of the checkerboard system.

The eigenfunctions of the checkerboard can easily be written down. These are the periodic functions with periodicity of \([\pi, \pi]\)

\[
E(\theta, \Phi, \ell) = \exp(\pm \kappa_\theta |\theta| \pm \kappa_\Phi |\Phi|) \exp[i(\pm k_\ell \ell - \omega t)]
\]  
(11)

in \( \theta \in [-\pi/2, \pi/2], \Phi \in [-\pi/2, \pi/2] \) and where \([k_\ell, \kappa_\theta, \kappa_\Phi]\) are the eigenvalues satisfying \( k_\ell^2 - \kappa_\theta^2 - \kappa_\Phi^2 = \omega^2/c^2 \). \( k_\ell \) is obtained by box-normalization conditions in the \( \ell \) direction. Thus we have two conditions and three variables to be determined. Hence all modes with \( \kappa_\theta \) and \( \kappa_\Phi \) leading to the same \( k_\ell \) are degenerate at a given frequency and the density of modes diverges. Note that propagating modes are included here as well for imaginary \( \kappa_\theta \) and \( \kappa_\Phi \).

The imaging properties of the checkerboard are actually quite counter-intuitive. A plain ray analysis (see figure 3) reveals only the images along the line \( \Phi = \phi_1 \) and \( \theta = \theta_1 \) (imaging across slabs) and two (depending on the position of the source within the cell) of the four possible images in the diagonal neighbouring cells (imaging by the 2D corner lens). But it will not reveal the rest of the images. This is one more instance of the failure of the ray picture in the context of such resonant positive–negative systems as pointed out earlier by Pendry \([17]\). But the generalized lens theorem is an exact statement and transcends any such ray analysis.

Note here that we make no claims about the temporal order in which the images form when the source is switched on sharply. These are the single-frequency solutions to Maxwell’s equations for steady sources. These checkerboard systems are extremely singular and contain a very large number of corners between positive and negative cells where the density of plasmon states actually diverges \([15]\). This has also been numerically verified \([16]\). Any source that is sharply switched on would have a frequency spread which would result in exciting resonant surface plasmon modes of the system that would ring continually in the absence of dissipation \([18]\) and a perfect image would form only asymptotically at infinite time.

The presence of dissipation is well known to affect the imaging badly and the singularity in the density of modes at the corners only makes it worse \([16]\). The ratio of the wavelength to the smallest resolvable lengthscale in the image depends logarithmically on the magnitude of the absorption coefficient \([8]\) and the extent of sub-wavelength resolution obtained for the slab lens in experimental realizations is about \( \lambda/2.5 \) to \( \lambda/6 \) \([20]–\[22]\). However, there is also
the possibility to compensate for the dissipation in the regions with negative refractive index by using media with optical gain (or active elements) in the regions with positive refractive index [19]. Although it is yet to be implemented experimentally, this combination of amplifying media and negative material parameters can also lead to the stimulated emission of surface plasmons [23, 24] (termed as Spason) and is independent of the dimensionality of the system. Indeed it is this possibility that makes the discussion of such singular checkerboard systems meaningful and interesting.

3.1. Inhomogeneous and anisotropic checkerboards

Firstly, we note that the variation of $\varepsilon$ and $\mu$ along the $\ell$ direction is irrelevant to the imaging. We could also have the $\varepsilon$ and $\mu$ varying with $\theta$ and $\Phi$ in the checkerboard cells. Provided the positive and negative cells are optically complementary, we would retain the image transfer properties of the homogeneous checkerboard system. As an example, consider a checkerboard with $\varepsilon_i(\theta, \Phi) = +\varepsilon_i \sin \theta$ or $+\varepsilon_i / \sin \theta$ in positive and $\varepsilon_i(\theta, \Phi) = -\varepsilon_i \sin \theta$ or $-\varepsilon_i / \sin \theta$ in negative, and similarly for $\mu_i(\theta, \Phi)$. It is clear that such a system satisfies the condition of complementarity and the mirror anti-symmetry, and therefore it should also image in a similar manner. We will call such a system a sine–cosecant checkerboard for our later use. Similarly, we can generate a checkerboard structure with as many complementary phases as we wish, provided that they respect mirror symmetries along the main axes of the cells.

3.2. Mapping of homogeneous cubic corners on to sine–cosecant checkerboards

Now we will proceed to prove that eight 3D cubic corners with alternating refractive indices of $n = \pm 1$ (see figure 4) and sharing of the common vertex will behave analogously to the 2D corner lens proposed in [15]. Consider the mapping of coordinates:

$$x_1 = r_0 e^{\ell/\ell_0} \sin \theta \cos \Phi, \quad x_2 = r_0 e^{\ell/\ell_0} \sin \theta \sin \Phi, \quad x_3 = r_0 e^{\ell/\ell_0} \cos \theta. \quad (12)$$

In the spherical (orthogonal) frame $(\ell, \theta, \Phi)$, we denote by $\tilde{\varepsilon}_i$ (respectively $\tilde{\mu}_i$), $i = l, \theta, \Phi$, the three non-zero components of the 'diagonal' tensors $\tilde{\varepsilon}$ (respectively $\tilde{\mu}$) associated with the 2D spatially varying anisotropic checkerboards. These are related to the piecewise constant function $\varepsilon$ (respectively $\mu$) for the 3D system of $(x_1, x_2, x_3)$ as in [25]

$$\tilde{\varepsilon}_i = \varepsilon (Q \ell Q \theta Q \Phi)^2, \quad \tilde{\mu}_i = \mu (Q \ell Q \theta Q \Phi)^2, \quad i = l, \theta, \Phi, \quad (13)$$

with

$$Q_i^2 = \left( \frac{\partial x_1}{\partial i} \right)^2 + \left( \frac{\partial x_2}{\partial i} \right)^2 + \left( \frac{\partial x_3}{\partial i} \right)^2. \quad (14)$$

From (12), (14) can be recast as

$$Q_\ell = r_0 e^{\ell/\ell_0}, \quad Q_\theta = r_0 e^{\ell/\ell_0}, \quad Q_\Phi = r_0 e^{\ell/\ell_0} \sin \theta, \quad Q_\ell Q_\theta Q_\Phi = r_0^3 e^{3\ell/\ell_0} \sin \theta. \quad (15)$$

Upon making the choice $\ell_0 = 1$, we deduce from (13) and (15) that the transformed medium consists of eight anisotropic heterogeneous complementary regions whose permittivity and...
permeability are given by

\[
\tilde{\varepsilon}_\ell = \varepsilon_\ell r_0 e^\ell \sin \theta, \quad \tilde{\mu}_\ell = \mu_\ell r_0 e^\ell \sin \theta,
\]
\[
\tilde{\varepsilon}_\theta = \varepsilon_\theta r_0 e^\ell \sin \theta, \quad \tilde{\mu}_\theta = \mu_\theta r_0 e^\ell \sin \theta,
\]
\[
\tilde{\varepsilon}_\phi = \varepsilon_\phi r_0 e^\ell \frac{1}{\sin \theta}, \quad \tilde{\mu}_\phi = \mu_\phi r_0 e^\ell \frac{1}{\sin \theta}.
\] (16)

Variation along the transverse \( r \) direction is irrelevant as before. We want for the set of eight homogeneous 3D corners

\[
\varepsilon_\ell = \mu_\ell = \varepsilon_\theta = \mu_\theta = \varepsilon_\phi = \mu_\phi = \pm 1.
\] (17)

This simply amounts to taking

\[
\tilde{\varepsilon}_\ell = \tilde{\mu}_\ell = \tilde{\varepsilon}_\theta = \tilde{\mu}_\theta = \pm e^\ell r_0 \sin \theta, \quad \text{and} \quad \tilde{\varepsilon}_\phi = \tilde{\mu}_\phi = \pm e^\ell r_0 \frac{1}{\sin \theta}.
\] (18)

which is the sine–cosecant checkerboard, but with a doubly periodic set of sources with period \( 2\pi \) along \( \Phi \) and period \( \pi \) along \( \theta \) as shown in figure 4. Hence, the cubic corner with the homogeneous materials with alternating signs for the refractive index will also form an imaging device with an image point inside every cubic corner. Thus, we have now generalized the result of [15] for a 2D corner to a 3D corner. Analogously, there should be an infinite degeneracy of the associated surface states in this case also. We can also map a multiphase checkerboard that satisfies the mirror anti-symmetry condition to generate a multiphase cubic corner. Figure 5 schematically shows such a mapping from a four-phase checkerboard to a four-phase cubic-corner lens.

**Figure 4.** (a) Two-phase checkerboard with positive (white) and negative (blue) squares and a doubly periodic array of point sources. (b) The two-phase cubic corner lens which can be generated from a sine–cosecant checkerboard: each octant although represented by a cube extends to infinities. The density of modes diverges at the origin.
Figure 5. Unit cell of a four-phase checkerboard (left) having the mirror anti-symmetry can be mapped on to a four-phase cubic corner. The colours green and yellow are taken to represent the regions with the complementary negative parameters of those regions represented by grey and red respectively. The 3D corner generated is composed of the regions formed by the intersections of the cones of $\theta = \pm \pi/4$ and the planes $\theta = \pi/2$ and $\Phi = \pi/2, \pi$. The regions extend out to infinity.

The electric and magnetic fields within this 3D corner reflector can be expressed as:

$$E_i = Q_i^{-1} \tilde{E}_i, \quad H_i = Q_i^{-1} \tilde{H}_i, \quad i \in \{\ell, \theta, \Phi\},$$

(19)

where $Q_i$ is given by (15).

3.3. Mapping on to spatially varying anisotropic cubic corners

In the previous section, we mapped a cubic system into a checkerboard system which satisfied the conditions of optical complementarity. But, in general we can generate a large variety of inhomogeneous, anisotropic cubic corners that behave as imaging systems, from some corresponding checkerboard system containing a doubly periodic set of point sources. Mapping our system of generalized coordinates $[\ell, \theta, \Phi]$ on to Cartesian coordinates $[x_1, x_2, x_3]$ as follows:

$$\ell = \frac{\ell_0}{2} \ln \left( \frac{x_1^2 + x_2^2 + x_3^2}{r_0^2} \right),$$

$$\theta = \arccos \left( \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right) \left[ \pi \right],$$

$$\Phi = \arctan \left( \frac{x_2}{x_1} \right) \left[ \pi \right],$$

(20)

where $r_0$ is a scale factor and can be taken to be the radial position of the source. Here, $\ell$ denotes the radial (logarithmic) coordinate and $\theta$ and $\Phi$ denote the longitudinal and azimuthal coordinates; we can now generate the corresponding cubic corner systems in $[x_1, x_2, x_3]$. 

New Journal of Physics 7 (2005) 164 (http://www.njp.org/)
For example, if we map a checkerboard system with homogeneous and isotropic cells, then we deduce that

\[ Q_1 = \sqrt{\frac{\ell_0 x_1^2}{r^4} + \frac{(x_1 x_3)^2}{r^4 (x_1^2 + x_2^2)} + \frac{x_2^2}{(x_1^2 + x_2^2)^2}}, \]

\[ Q_2 = \sqrt{\frac{\ell_0 x_2^2}{r^4} + \frac{(x_2 x_3)^2}{r^4 (x_1^2 + x_2^2)} + \frac{x_1^2}{(x_1^2 + x_2^2)^2}}, \]

\[ Q_3 = \sqrt{\frac{\ell_0 x_3^2}{r^4} + \frac{(1 - x_3^2/r^2)^2}{x_1^2 + x_2^2}}, \]

\hfill (21)

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \) Upon making the choice \( \ell_0 = 1 \) and the change of variable

\[ x_1 = r \sin \theta \cos \Phi, \quad x_2 = r \sin \theta \sin \Phi, \quad x_3 = r \cos \theta, \]

\hfill (22)

\( (21) \) can be recast as

\[ Q_1 = \frac{1}{r} \sqrt{\cos^2 \Phi + \frac{\sin^2 \Phi}{\sin^2 \theta}}, \quad Q_2 = \frac{1}{r} \sqrt{\sin^2 \Phi + \frac{\cos^2 \Phi}{\sin^2 \theta}}, \quad Q_3 = \frac{1}{r}. \]

\hfill (23)

We deduce from (13) (with \( i = 1, 2, 3 \)) and (23) that the transformed medium consists of eight anisotropic spatially varying complementary regions whose diagonal tensors of permittivity and permeability are given by

\[ \tilde{\nu}_1 = \frac{\nu_1}{r} \left( \frac{\cos^2 \theta \sin^2 \Phi + \cos^2 \Phi}{\sin^2 \theta \cos^2 \Phi + \sin^2 \Phi} \right), \]

\[ \tilde{\nu}_2 = \frac{\nu_2}{r} \left( \frac{\sin^2 \theta \cos^2 \Phi + \sin^2 \Phi}{\cos^2 \theta \sin^2 \Phi + \cos^2 \Phi} \right), \]

\[ \tilde{\nu}_3 = \frac{\nu_3}{r \sin^2 \theta} \sqrt{\sin^2 \theta \cos^2 \Phi + \sin^2 \Phi} \sqrt{\sin^2 \theta \sin^2 \Phi + \cos^2 \Phi}, \]

\hfill (24)

where \( \nu = \epsilon, \mu. \) Firstly, note that the transverse \( r \) (or \( \ell \)) dependence is irrelevant to the imaging. Now, choosing

\[ \epsilon_1 = \mu_1 = r \left( \frac{\sin^2 \theta \cos^2 \Phi + \sin^2 \Phi}{\cos^2 \theta \sin^2 \Phi + \cos^2 \Phi} \right), \]

\[ \epsilon_2 = \mu_2 = \epsilon_1^{-1} = \mu_1^{-1}, \]

\[ \epsilon_3 = \mu_3 = r \sin^2 \theta \sqrt{\sin^2 \theta \cos^2 \Phi + \sin^2 \Phi} \frac{1}{\sqrt{\sin^2 \theta \sin^2 \Phi + \cos^2 \Phi}}, \]

we get \((\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}_3)\) and \((\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)\) to be identical to the checkerboard made of homogeneous, isotropic materials. This is the cubical corner lens of anisotropic, inhomogeneous materials that we end up with by mapping from a homogeneous checkerboard.
4. Homogeneous isotropic 3D checkercubes

It can be easily shown by using the generalized lens theorem that a periodic homogeneous isotropic checkercube medium with adjacent cubical regions having $n = +1$ and $n = -1$ respectively will also make a similar imaging system as it satisfies the prerequisite optical complementarity conditions along all three imaging directions. A point source located within a cell will be imaged into every other cell. In this system of checkercubes of complementary media, all surface plasma modes are degenerate at frequency $\omega_p/\sqrt{2}$. At this frequency, the density of states must become infinite.

5. Conclusion

In conclusion, we have demonstrated that by mapping complementary checkerboard media, it is possible to design imaging configurations that are 3D. We have generalized the result of a 2D corner lens [15] to 3D corner structures which can either consist of homogeneous isotropic regions of space or be anisotropic and spatially varying. The only condition is that the eight cubical corners must satisfy the condition of mirror anti-symmetry across the interface between the positive and negative regions. These are very interesting singular configurations for which all the surface states are infinitely degenerate. In fact, this can also be taken as a technique to develop degenerate plasmonic 3D surfaces. We believe that such metamaterials may be usefully engineered for microwave frequencies if active gain elements are incorporated into the regions of positive material parameters.

Acknowledgments

Part of this work was undertaken while SG received support from Department of Defense (Office of Naval Research, Multidisciplinary University Research Initiative grant N00014-01-1-0803). SAR thanks Professor J B Pendry for hospitality during part of the time when this work was carried out and acknowledges support from the Department of Science and Technology (India) under grant no SR/S2/CMP-54/2003.

References

[1] Veselago V G 1967 Sov. Phys.—Solid State 8 2854
    Veselago V G 1968 Sov. Phys.—Usp. 10 509
[2] Pendry J B, Holden A J, Stewart W J and Youngs I 1996 Phys. Rev. Lett. 76 4773
    Pendry J B, Holden A J, Robbins D J and Stewart W J 1998 J. Phys.: Condens. Matter 10 4785
[3] Pendry J B, Holden A J, Robbins D J and Stewart W J 1999 IEEE Trans. Microwave Theory Tech. 47 2075
[4] Shelby R, Smith D R and Schultz S 2001 Science 92 297
[5] Parazzoli C G, Gregor R B, Li K, Koltenbah B E C and Tanielan N 2003 Phys. Rev. Lett. 90 107401
[6] Ramakrishna S A 2005 Rep. Prog. Phys. 68 449
[7] Pendry J B 2000 Phys. Rev. Lett. 85 3966
[8] Smith D R, Schurig D, Rosenbluth M, Schultz S, Ramakrishna S A and Pendry J B 2003 Appl. Phys. Lett. 82 1506
[9] Shamonina E, Kalinin V A, Ringhofer K H and Solymar L 2002 Electron. Lett. 37 1243

New Journal of Physics 7 (2005) 164 (http://www.njp.org/)
[10] Fang N and Zhang X 2003 Appl. Phys. Lett. 82 161
[11] Shen J T and Platzmann P M 2002 Appl. Phys. Lett. 80 3286
[12] Cummer S 2003 Appl. Phys. Lett. 82 1503
[13] Larkin I A and Stockman M I 2005 Nano Lett. 5 339
[14] Ramakrishna S A, Pendry J B, Wiltshire M C K and Stewart W J 2003 J. Mod. Opt. 50 1419
[15] Pendry J B and Ramakrishna S A 2003 J. Phys.: Condens. Matter 15 6345
[16] Guenneau S, Gralak B and Pendry J B 2005 Opt. Lett. 30 1204
[17] Pendry J B 2004 Contemp. Phys. 45 191
[18] Gomez-Santos G 2003 Phys. Rev. Lett. 90 077401
[19] Ramakrishna S A and Pendry J B 2003 Phys. Rev. B 67 201101 (R)
[20] Grbic A and Eleftheriades G V 2004 Phys. Rev. Lett. 92 117403
[21] Melville D O S and Blaikie R J 2005 Opt. Express 13 2127–34
[22] Fang N, Lee H, Sun C and Zhang X 2005 Science 308 534
[23] Bergman D J and Stockman M I 2002 Phys. Rev. Lett. 90 027402
[24] Lawandy N M 2004 Appl. Phys. Lett. 85 5040
[25] Ward A J and Pendry J B 1996 J. Mod. Opt. 43 73