An Extended Discrete Hardy-Littlewood-Sobolev Inequality

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Abstract

Hardy-Littlewood-Sobolev (HLS) Inequality fails in the “critical” case: $\mu = n$. However, for discrete HLS, we can derive a finite form of HLS inequality with logarithm correction for a critical case: $\mu = n$ and $p = q$, by limiting the inequality on a finite domain. The best constant in the inequality and its corresponding solution, the optimizer, are studied. First, we obtain a sharp estimate for the best constant. Then for the optimizer, we prove the uniqueness and a symmetry property. This is achieved by proving that the corresponding Euler-Lagrange equation has a unique nontrivial nonnegative critical point. Also, by using a discrete version of maximum principle, we prove certain monotonicity of this optimizer.

1 Introduction

The well-known Hardy-Littlewood-Sobolev (HLS) inequality states that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x - y|^{\mu}} \, dx \, dy \leq \mathcal{C}_{p,\mu,n} \|f\|_p \|g\|_q$$

for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ provided that

$$0 < \mu < n, 1 < p, q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} + \frac{\mu}{n} = 2.$$

$\mathcal{C}_{p,\mu,n}$ is the best constant for (1.1), and proved by Lieb [9] that, such $\mathcal{C}_{p,\mu,n}$ and corresponding maximizing pair $(f, g)$ exists. In particular, Lieb also gave the explicit $f$ and $\mathcal{C}_{p,\mu,n}$ in the case $p = q$. The method Lieb used was to examine the Euler-Lagrange equation that the maximizing pair $(f, g)$ satisfies with some techniques to exploit the symmetry of $f$. This idea is inherited in [8] and here to find the sharp estimate of best constant of a finite form of HLS in a critical case: $p = q = 2$, and hence $\mu = n$.

Following the idea that the maximizer of HLS satisfies corresponding E-L equations, the study of the HLS inequality and weighted inequality later generalized by Stein and Weiss [11] is naturally related to the studies of various of integral equations. For recent results, see [2, 12, 4, 3] and a brief summary can be found in [1]. These works have studied regularity and radial symmetry of solutions of such integral systems, and introduced a method of moving plane in an integral form which is proved to be a powerful tool. In [5], the result of integral system corresponding to HLS (1.1) is improved to all cases, i.e. the condition $p, q \geq 1$ is removed. In this paper, we do not use...
the method of moving plane directly, but borrowing its idea, we use a maximum principle to deal with a discrete problem and prove the symmetry of the solution.

First, let’s have a look at the discrete and 1-dimensional version of HLS inequality (1.1), the Hardy-Littlewood-Pólya (HLP) Inequality [6]: if $a \in l^p(\mathbb{Z})$ and $b \in l^q(\mathbb{Z})$ and $0 < \mu < 1, 1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} + \mu = 2$,

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^\mu} \leq C \|a\|_p \|b\|_q$$ (1.2)

where $r, s \in \mathbb{Z}$ and the constant $C$ depends on $p$ and $q$ only.

For this HLP inequality (1.2), let’s consider the critical case: $p = q = 2$ and $\mu = 2 - \frac{1}{p} - \frac{1}{q} = 1$, for which the original HLP fails, but we can compromise and get a finite form of HLP. In [8], the inequality is extended to the critical case as: If $a, b \in l^p(\mathbb{Z})$, then

$$\sum_{r \neq s, 1 \leq r, s \leq N} \frac{a_r b_s}{|r-s|} \leq \lambda_N \|a\|_2 \|b\|_2.$$ (1.3)

where $\lambda_N$ is the best constant for (1.3), and $\lambda_N = 2 \ln N + O(1)$.

**Remark 1.** One of the reasons that we consider discrete version of HLS instead of the original inequality is, when $\mu = 1$ the integrand on the left side of HLS (1.1) is not always integrable on a finite domain for $L^p$ functions. So it is not as convenient to extend 1-dimension HLS inequality (1.1) to the critical case in a similar finite form as to extend HLP (1.2) to (1.3).

As for the high dimensional discrete HLS, if $a, b \in l^p(\mathbb{Z}^n)$, and

$$0 < \mu < n, 1 < p, q < \infty$$ with $\frac{1}{p} + \frac{1}{q} + \frac{\mu}{n} = 2$,

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^\mu} \leq C \|a\|_p \|b\|_q$$ (1.4)

where $r, s \in \mathbb{R}^n$ and the constant $C$ depends on $p$ and $q$ only. We can extend (1.4) to a finite form in the corresponding critical case: $p = q = 2$ and $\mu = n$, in the following way:

**Theorem 2.** If $r, s \in \mathbb{R}^n$ and $1 \leq r_i, s_i \leq N$ where $r_i, s_i$ are integers and $1 \leq i \leq n$, then $a_r, b_s \in \mathbb{R}^L$, where $L = N^n$. let

$$\lambda_N = \max_{\|a\|_2 = \|b\|_2 = 1} \sum_{r \neq s} \frac{a_r b_s}{|r-s|^n}$$ (1.5)

So, we have an extension of HLS inequality

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^n} \leq \lambda_N \|a_r\|_2 \|b_s\|_2$$ (1.6)

where the two statements below holds

(i) $|S^{n-1}| \ln N - o(\ln N) < \lambda_N < |S^{n-1}| \ln N + o(\ln N)$. 

2
(ii) $\exists a^N = b^N$ and $\|a^N\|_2 = 1$ such that the equality in (1.6) holds, and $a^N \in \mathbb{R}^L_+$ where $L = N^n$.

Let's call the triplet $(a^N, \lambda_N)$ the optimizer of (1.6) since it is unique, and there are some properties of the optimizer. First, as a consequence of the uniqueness, we have symmetry property of the optimizer in the following sense,

**Theorem 3.** Let $(a^N, \lambda_N)$ be the optimizer. $\Phi : S \to S$ is an isometric map, where $S = \{r \in \mathbb{R}^n_+ | 1 \leq r_i \leq N \}$. Then $\overline{a^N_{\Phi(r)}} = a^N_r$.

Second, the optimizer has certain monotone decaying property. For convenience of writing, let's change the range of $r_i$ from $[1, N]$ to $[-N, N]$, which makes no essential change to the results above, and we have the monotone decaying property for this special case,

**Theorem 4.** If $(a^N, \lambda_N)$ is the optimizer and $r \in \mathbb{R}^n$, $-N \leq r_i \leq N$ for $1 \leq i \leq n$, then $a^N \in \mathbb{R}^L_+$, where $L = (2N + 1)^n$, and $a^N$ has a monotone decaying property from its central element: For $1 \leq i \leq n$,

$$
\begin{align*}
\frac{a^N_{(r_i, r')}}{a^N_{(r_{i-1}, r')}} & \leq 1 \leq N \\
\frac{a^N_{(r_i, r')}}{a^N_{(r_{i-1}, r')}} & \geq 1 \leq N + 1 \leq 0
\end{align*}
$$

To prove theorem 4 we use the following maximum principle,

**Theorem 5 (Maximum Principle).** Let $\mathbb{R}^L_+$ be the positive cone in $\mathbb{R}^L$, i.e., if $a \in \mathbb{R}^L_+$ then every element of $a$ is positive. Suppose a linear equation:

$$
u = Au + f$$

where $A : \mathbb{R}^L_+ \to \mathbb{R}^L_+$ with $\|A\|_2 < 1$, and $f \in \mathbb{R}^L_+$, then $\exists u$ satisfies (1.8) and $u \in \mathbb{R}^L_+$. In other words, $(I - A)^{-1} \in \mathbb{R}^{L \times L}$.

This Maximum Principle follows directly from standard contracting mapping iteration. It is a discrete version of maximum principle analogous to the usual versions in PDE. To see this, let's look at a typical maximum principle: let $\Omega \subset \mathbb{R}$ be an open bounded and connected domain with smooth boundary $\partial \Omega$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of following equation,

$$
\begin{align*}
-\Delta u & = f \geq 0 \text{ in } \Omega \\
u & = 0 \text{ on } \partial \Omega
\end{align*}
$$

Then by maximum principle $u \geq 0$ in $\Omega$. Actually, by strong maximum principle, $u > 0$ or $u \equiv 0$ in $\Omega$.

So, theorem 5 is indeed saying that if $(I - A)u = f \in \mathbb{R}^L_+$, then $u \in \mathbb{R}^L_+$. Corresponding to strong maximum principle, in theorem 5 if every entry of $A$ is strictly positive, it is easy to see that $u \in \mathbb{R}^L_+$. For more general symmetric linear operators, there is also maximum principle, and one can check [7] for details.

## 2 Best Constant Estimate in High Dimension Space

**Proof of part (i) of theorem 2** Step 1. $\lambda_N \geq |S^n| \ln N - o(\ln N)$.

Let $a = b$, and

$$a_r = N^{-\frac{n}{2}}$, 1 \leq r_i \leq N, 1 \leq i \leq n \tag{2.1}$$
So, $\|a\|_2 = 1$.
By the definition of $\lambda_N$, we have

$$\lambda_N \geq \sum_{r \neq s} \frac{a_r a_s}{|s - r|^n} = N^{-n} \sum_{r \neq s} \frac{1}{|s - r|^n}$$

$$= N^{-n} \left\{ 2^n \sum_{x_n = 1}^{N-1} \cdots \sum_{x_1 = 1}^{N-1} \frac{(N - x_1) \cdots (N - x_1)}{(x_1^2 + \cdots + x_n^2)^{\frac{n}{2}}} - o(N^n \ln N) \right\}$$

$$\geq \left( \frac{N}{2} \right)^n \frac{2}{\pi} \cdots \frac{2}{\pi} \int_0^1 \frac{r}{r^n} (N - r \cos \phi_1) \cdots |r - s|^{n-1} dr d\phi_1 \cdots d\phi_n - o(\ln N)$$

$$= \left( \frac{2}{N} \right)^n |S^{n-1}| 2^{-n} N^n \ln N - o(\ln N)$$

$$= |S^{n-1}| \ln N - o(\ln N)$$

**Step 1.** $\lambda_N \leq |S^{n-1}| \ln N + o(\ln N)$
Let $J(a, b) = \sum_{r \neq s} \frac{a_r b_s}{|r - s|^n}$. Hence, $\lambda_N = \max \|a\|_2 = \|b\|_2 = 1$ $J(a, b)$, i.e. we will maximize $J(a, b)$ under the constraints $\|a\|_2 = \|b\|_2 = 1$ (in fact, we use $\frac{1}{2}\|a\|_2^2 = \frac{1}{2}\|b\|_2^2 = \frac{1}{2}$). Therefore, we conduct Euler-Lagrange equations and by compactness: $\exists \|a_N\|_2 = \|b_N\|_2 = 1$ such that $\lambda_N = J(a_N, b_N)$ and,

$$\begin{align*}
\lambda_1 a_N &= \sum_{s \neq r} b_s^N |s - r|^n \\
\lambda_2 b_N &= \sum_{r \neq s} a_r^N |r - s|^n
\end{align*}$$

where $r, s \in \mathbb{R}^n$ and $1 \leq r_i, s_i \leq N$.

For convenience, write (2.2) in matrix form,

$$\begin{cases}
\lambda_1 a_N = A b_N \\
\lambda_2 b_N = A a_N
\end{cases}$$

(2.3)

Left multiply the first equation of (2.3) by $a^T$, the second equation by $b^T$, and by the fact that $A$ is symmetric and $\|a_N\|_2 = \|b_N\|_2 = 1$, one sees that

$$\lambda_1 = \lambda_1 \|a_N\|_2^2 = a_N^T A b_N = J(a_N, b_N)$$

$$= b_N^T A^T a_N = \lambda_2 \|b_N\|_2^2 = \lambda_2$$

and since $\lambda_N = J(a_N, b_N)$, we have $\lambda_1 = \lambda_2 = \lambda_N$. 

4
Now, let \( b_{s_0} = \max |a^N_r|, |b^N_s| > 0 \), so, \( b_{s_0}^N \lambda_N = \sum_{r \neq s_0} \frac{a^N_r}{|r - s_0|^n} \), which leads to

\[
\lambda_N = \sum_{r \neq s_0} \frac{a^N_r}{b_{s_0} |r - s_0|^n} \leq \sum_{r \neq s_0} \frac{1}{|r - s_0|^n}
\]

\[
\leq \frac{\sum e^{N - 1} dr d\sigma}{\sum e^{N - 1} dr d\sigma} \leq |S^{n-1}|( \ln \sqrt{2N} ) = |S^{n-1}|( \ln + 1 + \frac{2}{2} \ln 2 )
\]

\[
= |S^{n-1}| \ln N + o( \ln N )
\]

Part (ii) will be shown later in section 3. □

**Lemma 1.** If \((a, b, \lambda_N)\) satisfies \( \|a\|_2 = \|b\|_2 = 1 \) and makes the equality of (1.6) hold, then \( a, b \in \mathbb{R}^L_+ \cup \mathbb{R}^L_- \).

Notice that if there is a sign change among the elements of \( a \) and \( b \), \((a, b)\) must not be an optimizer since \( |\sum a_i b_i| < \sum |a_i| |b_i| \). So the lemma holds, and it means that we can assume the triplet \((a^N, b^N, \lambda_N)\) above to satisfy \( a^N, b^N \in \mathbb{R}^L_+ \).

Now, let’s introduce a notation,

**Definition 1.** \((a, b, \lambda_N)\) such that

- \( \|a\|_2 = \|b\|_2 = 1 \)
- \( a, b \in \mathbb{R}^L_+ \)
- The equality of (1.6) holds

is called an optimizer or solution of optimization of (1.6).

Obviously, \((a^N, b^N, \lambda_N)\) is an optimizer. Next, we are going to prove part(ii) of theorem 2, i.e., the optimizer is unique in positive cone and \( a^N = b^N \).

## 3 Uniqueness of The Optimizer

From previous discussion we see that, an optimizer of (1.6), \((a^N, b^N, \lambda_N)\), satisfies Euler-Lagrange equations (2.2). We are going to show the optimizer is unique in positive cone by showing the solution of the Euler-Lagrange equations in the positive cone \( \mathbb{R}^L_+ \) where \( L = N^n \) is unique. Considering the following equations,

\[
\begin{align*}
\lambda_1 a_r &= \sum_{s \neq r} \frac{b_s}{|s - r|^n} \\
\lambda_2 b_s &= \sum_{r \neq s} \frac{a_r}{|r - s|^n}
\end{align*}
\]

(3.1)

where \( \|a\|_2 = \|b\|_2 = 1 \), \( r = (r_i) \in \mathbb{R}^n \), and \( 1 \leq r_i \leq N, 1 \leq i \leq n \). \( a, b \in \mathbb{R}^L \), where \( L = N^n \). By lemma 1, we only need to study solution of (3.1) in the positive cone \( \mathbb{R}^L_+ \).

In the proof, we will use the following simple map,
Definition 2. Let $T : \mathbb{R}^L \to \mathbb{R}^L_+$ such that $(Ta)_i = |a_i|$ for $1 \leq i \leq L$.

Theorem 6. If $(a, b, \lambda_1, \lambda_2)$ is a solution of (3.1), where $a, b \in \mathbb{R}^L_+$, then $\lambda_1 = \lambda_2 = \lambda_N$, and $a = b \in \mathbb{R}^L_+$ is unique.

Proof. Step 1. $\lambda_1 = \lambda_2$.
This is similar to step 2 of theorem 2. So, let $\lambda = \lambda_1 = \lambda_2$.

Step 2. $a, b \in \mathbb{R}^L_+$.

Since

$$\lambda a_r = \sum_{s \neq r} b_s \frac{|s - r|^n}{n}$$

$$= \frac{1}{\lambda} \sum_{t} \sum_{s \neq r, t} \frac{|r - s|^n}{|t - s|^n} a_t$$

$$= \frac{1}{\lambda} \sum_{t} C(r, t)a_t$$

we have $\lambda^2 a = Ca$, where $C = A^T A$ and $A$ is a symmetric matrix. So $C$ is non-negative definite. Since $C(r, t) > 0$, $a \in \mathbb{R}^L_+$ and $a \neq 0$ for $\|a\| = 1$, the last term above is strictly positive. Therefore, $a, b \in \mathbb{R}^L_+$.

Let $0 \leq \mu_1 \leq \mu_2 \cdots \leq \mu_L$ be the eigenvalues of $C$. Then $\exists \xi \in \mathbb{R}^L_+$, s.t. $C\xi = \mu L\xi$, and $\|\xi\| = 1$, and $\xi \notin \mathbb{R}^L_+$. We can assume the last property because eigenvectors appear in pairs with opposite signs. Also, by theory of adjoint operators, $\mu_L = \max_{\|x\|=1} \langle x, Cx \rangle = \langle \xi, C\xi \rangle$.

Step 3. $\exists \xi \in \mathbb{R}^L_+$, $\|\xi\| = 1$, and $\mu_{L-1} < \mu_L$.

First, $\exists \xi \in \mathbb{R}^L_+$. If not, then $\xi \notin \mathbb{R}^L_+ \cup \mathbb{R}^L_-$.

Then we have

$$\mu_L = \langle \xi^T, C\xi \rangle = \langle T\xi^T, C(T\xi) \rangle \leq \max_{\|x\|=1} \xi^T C\xi = \mu_L$$

(3.2)

where $T$ is defined in definition 2. A contradiction. So, $\exists \xi \in \mathbb{R}^L_+$, and since $C\xi = \mu L\xi$, $\xi \in \mathbb{R}^L_+$.

The argument above also shows that $\mu_{L-1} < \mu_L$. If not, $\mu_{L-1} = \mu_L$, then by a similar argument as above $\exists \xi_{L-1} \in \mathbb{R}^L_+$, s.t. $C\xi_{L-1} = \mu_L\xi_{L-1}$, and moreover $\xi_{L-1} \perp L\xi_{L-1}$ which is impossible.

Step 4. $a = b = \xi_L$, $\lambda = \lambda_N = \sqrt{\mu_L}$.

Considering $\lambda^2 a = Ca$,

1. If $\lambda^2 \neq \mu_L$, then $a \perp L\xi_L$. Since $a \in \mathbb{R}^L_+$ by step 2, this is impossible. So, $\lambda^2 = \mu_L$.

2. Since $Ca = \mu_L a, C\xi_L = \mu_L \xi_L$, and by the fact that $\mu_{L-1} < \mu_L$ and $\|a\| = \|\xi_L\| = 1, a = \xi_L$.

Similarly, $b = \xi_L$.

3. If $(a_N, b_N, \lambda_N)$ is an optimizer of (1.6) in the positive cone, it is a solution of (3.1). So, $a = a_N = b = b_N$, $\lambda^2 N = \lambda^2 = \mu_L$, and $\lambda, \lambda_N > 0$, so $\lambda = \lambda_N$. □

Proof of part (ii) of theorem 2. The same as the 3rd argument of step 4 above, since an optimizer $(a_N, b_N, \lambda_N)$ is a solution of (3.1), part (ii) follows from theorem 6.
Remark 7. At the time of this writing, thanks to Professor Dongsheng Li of Jiaotong University in Xi’an, we find that uniqueness follows directly from Perron’s theorem [10]. So the proof above can be much simplified.

Corollary 1. \( \lambda \) is increasing as \( N \) increases.

Proof. Let \( \lambda_N \) and \( A_N \) be a solution and coefficient matrix of (3.1). So, \( \lambda_N = \max_{\|\xi\|=1} \xi^T A_N \xi = \overline{\xi_N^T A_N \xi_N} \)

\[
= (\overline{\xi_N}, 0)^T A_{N+1} (\overline{\xi_N}, 0)
\]

\[
< \max_{\|\xi\|=1} \xi^T A_{N+1} \xi = \lambda_{N+1}
\]

where \((\overline{\xi_N}, 0)\) means \((\overline{\xi_N}, 0) \in \mathbb{R}^L\) and \(L = (N+1)^n\), and arranging \(\overline{\xi_N}\) to take the first \(N^n\) entries and stuffing the rest with zeros. Then calculate in blocks of matrices. \(\square\)

4 Symmetry of The Optimizer

From section 2 we see the uniqueness of the optimizer of (1.6) in positive cone. So, from this point, if it is clear in context, we use \((a = b, \lambda)\) instead of \((a^N, b^N, \lambda_N)\) when referring the optimizer of (1.6) for simplicity.

Proof of Theorem 3. From (3.1) we have

\[
\lambda a_r = \sum_{s \neq r} \frac{a_s}{|s - r|^n}
\]

then

\[
\lambda a_{\Phi(r)} = \sum_{s \neq \Phi(r)} \frac{a_s}{|s - \Phi(r)|^n} = \sum_{t \neq r} \frac{a_{\Phi(t)}}{|\Phi(t) - \Phi(r)|^n} = \sum_{t \neq r} \frac{a_{\Phi(t)}}{|t - r|^n}
\]

So, \(\bar{a} = (a)_{\Phi(r)}\) is also a solution to (3.1). Then, by uniqueness of the solution, \(\bar{a} = a\). So, \(a_{\Phi(r)} = a_r\). \(\square\)

Example 4.1. If \(a\) is an optimizer, then \(a_{(r_i, r')} = a_{(N-r_i+1, r')}\) for \(1 \leq i \leq N\).

5 Monotone Property of The Optimizer

For convenience of writing, we change the range of \(r_i\)’s from \(1 \leq r_i \leq N\) to \(-N \leq r_i \leq N\) which makes no change to the results above essentially.

Proof of Theorem 4. We are only going to show (1.7) is true for \(i = 1\) for simplicity. Consider \(d_r^{(1)} = a_{(r_1-1, r')} - a_{(r_1, r')},\) where \(1 \leq r_1 \leq N\) and \(-N \leq r_i \leq N, 2 \leq i \leq n\). So \(d^{(1)} \in \mathbb{R}^{N(2N+1)(n-1)}\).
Then by applying theorem 3 we have

\[
d^{(1)}_r = \frac{1}{\lambda} \sum_{s \neq (r_1 - 1, r')} \frac{a_s}{|s - (r_1 - 1, r')|^n} - \sum_{s \neq (r_1, r')} \frac{a_s}{|s - (r_1, r')|^n}
\]

\[
= \frac{1}{\lambda} \sum_{t = (t_1, t') \neq (r_1, r'), \frac{1}{N+1} \leq t_1 \leq N+1} \frac{a(t_1 - 1, t')}{|t - (r_1, r')|^n} - \sum_{s \neq (r_1, r')} \frac{a_s}{|s - (r_1, r')|^n}
\]

\[
= \frac{1}{\lambda} \sum_{t = (t_1, t') \neq (r_1, r'), \frac{1}{N+1} \leq t_1 \leq N+1} \frac{d^{(1)}_t}{|t - (r_1, r')|^n} + \sum_{t = (-t_1 + 1, t') \neq (r_1, r'), \frac{1}{N+1} \leq t_1 \leq N} \frac{-d^{(1)}_t}{|(-t_1 + 1, t') - (r_1, r')|^n}
\]

\[
+ \sum_{t = (N+1, t') \neq (r_1, r')} \frac{a(N, t')}{|t - (r_1, r')|^n} - \sum_{t = (-N, t') \neq (r_1, r')} \frac{a(-N, t')}{|t - (r_1, r')|^n}
\]

\[
= \frac{1}{\lambda} \left( \sum_{(t_1, t') \neq r, \frac{1}{N+1} \leq t_1 \leq N} \frac{|(t_1, t') - r|^n}{|t - (r_1, r')|^n} - \frac{1}{|(-t_1 + 1, t') - r|^n} d^{(1)}_t + \frac{-d^{(1)}_t}{2r_1 - 1^n} + f(r) \right)
\]

Also by theorem 3, \(a(N, t') = a(-N, t')\), easily one sees that \(f(r) \geq 0\).

So, for \(1 \leq r_1 \leq N\)

\[
(\lambda + \frac{1}{2r_1 - 1^n})d^{(1)}_r = \sum_{(t_1, t') \neq r, \frac{1}{N+1} \leq t_1 \leq N} \frac{1}{|t_1, t'_1 - r|^n} - \frac{1}{|(-t_1 + 1, t') - r|^n} d^{(1)}_t + f(r)
\]

Write the above equations in matrix form,

\[
d^{(1)} = Ad^{(1)} + F
\]

where \((F)_r = \frac{1}{(\lambda + \frac{1}{2r_1 - 1^n})} f(r)\), and

\[
A(r, t) = \begin{cases} 
\lambda \frac{1}{2r_1 - 1^n} \frac{1}{|(t_1, t'_1) - r|^n} - \frac{1}{|(-t_1 + 1, t') - r|^n}, & r \neq t \\
0, & r = t
\end{cases}
\]

It is easy to see that entries of \(A\) and \(F\) are non-negative. So, \(A : \mathbb{R}_+^L \to \mathbb{R}_+^L\), where \(L = N(2N + 1)(n-1)\), and \(F \in \mathbb{R}_+^L\). Therefore, provided \(|A| < 1\), then by Theorem 3 (Maximum Principle) we get \(d^{(1)} \in \mathbb{R}_+^L\), hence (1.7) is proved. So, the only thing left to prove is \(|A| < 1\).

Notice that if \(C, D\) are symmetric matrices such that \(C, D : \mathbb{R}_+^L \to \mathbb{R}_+^L\), for some positive integer \(L\), then \(|C| \leq |C + D|\), because

\[
|C| = \max_{||\xi||=1} \xi^T C \xi = \xi^T C \xi \leq \xi^T (C + D) \xi \leq \max_{||\xi||=1} \xi^T (C + D) \xi = ||C + D||
\]

Let

\[
C(r, t) = \begin{cases} 
\lambda \frac{1}{2r_1 - 1^n} \frac{1}{|t_1, t'_1 - r|^n}, & r \neq t \\
0, & r = t
\end{cases}
\]
and

\[ D(r, t) = \begin{cases} 
\frac{1}{n} \left( \frac{1}{(2r_1-1)^n} \right) \left( -t_1 + 1, t' \right) - r \frac{|r_1 - 1|}{n}, & r \neq t \\
0, & r = t
\end{cases} \]

So,

\[ \|A\| \leq \|A + D\| = \|C\| \leq \frac{1}{\lambda + \delta(N)} \|A_N\| \]

where \( A_N \) is the matrix of (3.1) of the case that \(-N \leq r_i \leq N, 1 \leq i \leq n\), so \( \|A_N\| = \lambda \). So, \( \|A\| < 1. \) □

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