GEOMETRIC PREQUANTIZATION OF THE MODULI SPACE OF THE VORTEX EQUATIONS ON A RIEMANN SURFACE

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Abstract. The moduli space of solutions to the vortex equations on a Riemann surface are well known to have a symplectic (in fact Kähler) structure. We show this symplectic structure explicitly and proceed to show a family of symplectic (in fact, Kähler) structures $\Omega_{\Psi_0}$ on the moduli space, parametrised by $\Psi_0$, a section of a line bundle on the Riemann surface. Next we show that corresponding to these there is a family of prequantum line bundles $P_{\Psi_0}$ on the moduli space whose curvature is proportional to the symplectic forms $\Omega_{\Psi_0}$.

1. Introduction

Geometric prequantization is a construction, if possible, of a prequantum line bundle $L$ on a symplectic manifold, $(\mathcal{M}, \Omega)$ whose curvature is proportional to the symplectic form. The Hilbert space of the quantization is the space of the square integrable sections of $L$. To every $f \in C^\infty(\mathcal{M})$ we associate an operator acting on the Hilbert space, namely, $\hat{f} = -i\hbar[X_f - \frac{i}{\hbar}\theta(X_f)] + f$ where $X_f$ is the vector field defined by $\Omega(X_f, \cdot) = -df$ and $\theta$ is a symplectic potential corresponding to $\Omega$. Then if $f_1, f_2 \in C^\infty(\mathcal{M})$ and $f_3 = \{f_1, f_2\}$, Poisson bracket of the two induced by the symplectic form, then $[\hat{f}_1, \hat{f}_2] = -i\hbar f_3$. [22]

The motivating example in our context would be the geometric quantization of the moduli space of flat connections on a principal $G$-bundle $P$ on a compact Riemann surface $\Sigma$. [21], [2]. Let $\mathcal{A}$ be the space of Lie-algebra valued connections on the principal bundle $P$. Let $\mathcal{N}$ be the moduli space of flat connections (i.e. the space of flat connections modulo the gauge group). One can construct the determinant line bundle of the Cauchy-Riemann operator, namely, $L = \wedge^{\text{top}}(\text{Ker} \bar{\partial}_A)^* \otimes \wedge^{\text{top}}(\text{Coker} \bar{\partial}_A)$ on $\mathcal{A}$. [18]. The curvature induced by the Quillen metric on this bundle coincides with the natural Kähler form $\omega_A$ namely, $-\text{Tr} \int_\Sigma \alpha \wedge \beta$, where $\alpha, \beta \in T_A^* \mathcal{A} = \Omega^1(M, \text{ad}P)$. It can be shown, using a moment map construction, that this symplectic form descends to the moduli space of flat connections $\mathcal{N}$. The determinant line bundle is also well-defined on $\mathcal{N}$ and is the candidate for the prequantum line bundle of the geometric quantization.

Inspired by this construction, we constructed three prequantum line bundles on the moduli space of solutions to the self-duality equations over a Riemann surface [8], [9] corresponding to the three symplectic forms which give rise to the hyperKähler structure of the moduli space.

In this paper we geometrically quantize the moduli space of vortex equations. Geometric quantization of the vortex moduli space has been done before in [6] and [19]. In the first paper the authors use algebraic geometry and in the second paper, the author uses the special form of the moduli space when the Riemann surface is a sphere. It would be interesting see what is the relation of the present
quantization to the ones in [6] and [19]. The relation may not be straightforward since in the present quantization we find a whole family of (topologically equivalent, but perhaps holomorphically non-equivalent) prequantum line bundles $\mathcal{P}_{\Psi_0}$ whose curvatures correspond to a family of symplectic forms $\Omega_{\Psi_0}$ parametrised by $\Psi_0$, a section of a line bundle on the Riemann surface, as explained later. This symplectic form $\Omega_{\Psi_0}$ is a variant of the standard symplectic form $\Omega$ on the vortex moduli space.

The vortex equations are as follows. Let $M$ be a compact Riemann surface and let $\omega = h^2 dz \wedge d\bar{z}$ be the purely imaginary volume form on it, (i.e. $h$ is real). Let $A$ be a unitary connection on a principal $U(1)$ bundle $P$ i.e. $A$ is a purely imaginary valued one form i.e. $A = A^{(1,0)} + A^{(0,1)}$ such that $A^{(1,0)} = -\overline{A^{(0,1)}}$. Let $L$ be a complex line bundle associated to $P$ by the defining representation. Let $\Psi$ be a section of $L$, i.e. $\Psi \in \Gamma(M, L)$ and $\Psi$ be a section of its dual, $\overline{\Psi}$. There is a Hermitian metric $H$ on $L$, i.e. the inner product $<\Psi_1, \Psi_2>_{H} = \Psi_1 H \overline{\Psi}_2$ is a smooth function on $M$. (Here $H$ is real).

The pair $(A, \Psi)$ will be said to satisfy the vortex equations if
\begin{align*}
(1) \quad & F(A) = (1 - |\Psi|^2_H) \omega, \\
(2) \quad & \partial_A \Psi = 0,
\end{align*}
where $F(A)$ is the curvature of the connection $A$ and $d_A = \partial_A + \bar{\partial}_A$ is the decomposition of the covariant derivative operator into $(1,0)$ and $(0,1)$ pieces. Let $\mathcal{S}$ be the space of solutions to (1) and (2). There is a gauge group $G$ acting on the space of $(A, \Psi)$ which leaves the equations invariant. We take the group $G$ to be abelian and locally it looks like Maps$(M, U(1))$. If $g$ is an $U(1)$ gauge transformation then $(A_1, \Psi_1)$ and $(A_2, \Psi_2)$ are gauge equivalent if $A_2 = g^{-1} d g + A_1$ and $\Psi_2 = g^{-1} \Psi_1$. Taking the quotient by the gauge group of $\mathcal{S}$ gives the moduli space of solutions to these equations and is denoted by $\mathcal{M}$. It is well known that there is a natural metric on the moduli space $\mathcal{M}$ and in fact the metric is Kähler, see [20], [15], [14], [19], [3], [7], [5] and the references there.

In this paper, we show the metric explicitly and write down the symplectic (in fact, the Kähler form) $\Omega$ arising from this metric and the complex structure. This is because some modification of this symplectic form gives us a whole family of symplectic forms $\Omega_{\Psi_0}$ parametrised by a fixed section $\Psi_0$ of the line bundle $L$ which vanishes on a set of measure zero. $\Omega_{\Psi_0}$ coincide with $\Omega$ when $L$ is a trivial bundle with $|\Psi_0|_H = 1$. In fact $\Omega_{\Psi_0}$ is a Kähler form on the moduli space. We show that there exists a holomorphic prequantum line bundle, namely, a determinant line bundle, whose Quillen curvature is proportional to the symplectic form $\Omega_{\Psi_0}$. Thus as $\Psi_0$ varies, we get a whole family of prequantum line bundles which are topologically equivalent, but perhaps not holomorphically equivalent.

2. METRIC AND SYMPLECTIC FORMS

Let $\mathcal{A}$ be the space of all unitary connections on $P$ and $\Gamma(M, L)$ be sections of $L$. Let $\mathcal{C} = \mathcal{A} \times \Gamma(M, L)$ be the configuration space on which equations (1) and (2) are imposed. Let $p = (A, \Psi) \in \mathcal{C}$, $X = (\alpha_1, \beta)$, $Y = (\alpha_2, \eta) \in T_p \mathcal{C} \equiv \Omega^1(M, i\mathbb{R}) \times \Gamma(M, L)$ i.e. $\alpha_i = \alpha^{(0,1)}_i + \alpha^{(1,0)}_i$ such that $\alpha^{(0,1)}_i = -\overline{\alpha^{(1,0)}_i}$, $i = 1, 2$. On $\mathcal{C}$ one can define a metric
\[ G(X, Y) = \int_M *_1 \alpha_1 \wedge \alpha_2 + 2i \int_M \text{Re} \beta, \eta >_H \omega \]
and an almost complex structure $\mathcal{I} = \left[ \begin{array}{cc} *1 & 0 \\ 0 & i \end{array} \right] : T_p \mathcal{C} \to T_p \mathcal{C}$ where $*1 : \Omega^1 \to \Omega^1$
is the Hodge star operator on $M$ such that $*1(\eta dz) = -i\eta dz$ and $*1(\bar{\eta} d\bar{z}) = i\bar{\eta} d\bar{z}$.

It is easy to check that $\mathcal{G}$ is positive definite. In fact, if $\alpha_1 = \alpha^{(1,0)} + \alpha^{(0,1)} = adz - \bar{a}d\bar{z}$ is an imaginary valued 1-form, $*1(\alpha_1) = -i(adz + \bar{a}d\bar{z})$ and $\mathcal{G}(X, X) = 4 \int_M |a|^2 dx \wedge dy + 4 \int_M |\beta|^2_H h^2 dx \wedge dy$ where $\omega = h^2 dz \wedge d\bar{z} = -2ih^2 dx \wedge dy$.

The symplectic form $\Omega$

We define

$$\Omega(X, Y) = \int_M \alpha_1 \wedge \alpha_2 + 2i \int_M \text{Re} \langle i\beta, \eta \rangle \omega$$

such that $\mathcal{G}(X, Y) = \Omega(X, Y)$. Moreover, we have the following:

**Proposition 2.1.** The metrics $\mathcal{G}$, the symplectic form $\Omega$, and the almost complex structure $\mathcal{I}$ are invariant under the gauge group action on $\mathcal{C}$.

**Proof.** Let $p = (A, \Psi) \in \mathcal{C}$ and $g \in G$, the gauge group, where $g \cdot p = (A + g^{-1}dg, g^{-1}\Psi)$.

Then $g_\ast : T_p \mathcal{C} \to T_{g \cdot p} \mathcal{C}$ is given by the mapping $(Id, g^{-1})$ and it is now easy to check that $g$ and $\Omega$ are invariant and $\mathcal{I}$ commutes with $g_\ast$. \qed

**Proposition 2.2.** The equation (1) can be realised as a moment map $\mu = 0$ with respect to the action of the gauge group and the symplectic form $\Omega$.

**Proof.** Let $\zeta \in \Omega(M, i\mathbb{R})$ be the Lie algebra of the gauge group (the gauge group element being $g = e^{\zeta}$); note that $\zeta$ is purely imaginary. It generates a vector field $X_\zeta$ on $\mathcal{C}$ as follows:

$$X_\zeta(A, \Psi) = (d\zeta, -\zeta\Psi) \in T_p \mathcal{C}$$

where $p = (A, \Psi) \in \mathcal{C}$.

We show next that $X_\zeta$ is Hamiltonian. Namely, define $H_\zeta : \mathcal{C} \to \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta \cdot (F_A - (1 - |\Psi|^2_H)\omega).$$

Then for $X = (\alpha, \beta) \in T_p \mathcal{C}$,

$$dH_\zeta(X) = \int_M \zeta d\alpha + \int_M \zeta (\Psi H^\beta + \bar{\Psi} H \beta) \omega$$

$$= -\int_M (d\zeta) \wedge \alpha + 2i \int_M \text{Re}(i(-\zeta\Psi)H^\beta) \omega$$

$$= \Omega(X_\zeta, X),$$

where we use that $\bar{\zeta} = -\zeta$.

Thus we can define the moment map $\mu : \mathcal{C} \to \Omega^2(M, i\mathbb{R}) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A, \Psi) = (F(A) - (1 - |\Psi|^2_H)\omega).$$

Thus equation (1) is $\mu = 0$. \qed

**Lemma 2.3.** Let $\mathcal{S}$ be the solution spaces to equation (1) and (2), $X \in T_p \mathcal{S}$. Then $\mathcal{I}X \in T_p \mathcal{S}$ if and only if $X$ is $\mathcal{G}$-orthogonal to the gauge orbit $O_p = G \cdot p$. 

Proof. Let $X_\zeta \in T_pO_p$, where $\zeta \in \Omega^0(M, i\mathbb{R})$, $G(X, X_\zeta) = -\Omega(I\mathcal{L}X, X_\zeta) = -\int_M \zeta \cdot d\mu(I\mathcal{L}X)$, and therefore $I\mathcal{L}X$ satisfies the linearization of equation (1) iff $d\mu(I\mathcal{L}X) = 0$, i.e., iff $G(X, X_\zeta) = 0$ for all $\zeta$. Second, it is easy to check that $I\mathcal{L}X$ satisfies the linearization of equation (2) whenever $X$ does. \hfill \square

Theorem 2.4. $\mathcal{M}$ has a natural symplectic structure and an almost complex structure compatible with the symplectic form $\Omega$ and the metric $G$.

Proof. First we show that the almost complex structure descends to $\mathcal{M}$. Then using this and the symplectic quotient construction we will show that $\Omega$ gives a symplectic structure on $\mathcal{M}$.

(a) To show that $I$ descends as an almost complex structure we let $pr : S \to S/G = \mathcal{M}$ be the projection map and set $[p] = pr(p)$. Then we can naturally identify $T_{[p]}\mathcal{M}$ with the quotient space $T_pS/T_pO_p$, where $O_p = G \cdot p$ is the gauge orbit. Using the metric $G$ on $S$ we can realize $T_{[p]}\mathcal{M}$ as a subspace in $T_pS$, $G$-orthogonal to $T_pO_p$. Then by lemma 2.3 this subspace is invariant under $I$. Thus $I_{[p]} = I|_{T_p(O_p)^+}$, gives the desired almost complex structure. This construction does not depend on the choice of $p$ since $I$ is $G$-invariant.

(b) The symplectic structure $\Omega$ descends to $\mu^{-1}(0)/G$, (by proposition 2.2 and by the Marsden-Weinstein symplectic quotient construction, since the leaves of the characteristic foliation are the gauge orbits). Now, as a 2-form $\Omega$ descends to $\mathcal{M}$, due to proposition (2.1) so does the metric $G$. Closure of $\Omega$ is easy. We check that equation (2) does not give rise to new degeneracy of $\Omega$ (i.e. the only degeneracy of $\Omega$ is due to (1) but along gauge orbits). Thus $\Omega$ is symplectic on $\mathcal{M}$. Since $G$ and $I$ descend to $\mathcal{M}$ the latter is symplectic and almost complex. \hfill \square

The family of symplectic forms $\Omega_{\Psi_0}$

Choose a fixed $\Psi_0 \in \Gamma(M, L)$ such that $|\Psi_0|_H = 0$ only on a set of measure zero on $M$. (This $\Psi_0$ has nothing to do with $\Psi$.)

Define a symplectic form on $C$ as

$$\Omega_{\Psi_0}(X, Y) = -\int_M \alpha_1 \cdot \alpha_2 + 2i \int_M \text{Re} < i\beta, \eta > H|\Psi_0|^2_H \omega$$

$$= -\int_M \alpha_1 \cdot \alpha_2 - \int_M (\beta H \bar{\eta} - \bar{\beta} H \eta)|\Psi_0|^2_H \omega$$

$|\Psi_0|^2_H$ plays the role of a conformal rescaling of the volume form $\omega$ on $M$ which appears in $\Omega$, where we allow the conformal factor to have zeroes on sets of measure zero.

Theorem 2.5. $\Omega_{\Psi_0}$ descends to $\mathcal{M}$ as a symplectic form.

Proof. Let $p = (A, \Psi)$.

It is easy to show that $\Omega_{\Psi_0}$ is closed (this follows from the fact that on $C$ it is a constant form – does not depend on $(A, \Psi)$). We have to show it is non-degenerate.

Suppose there exists $(\alpha_1, \beta) \in T_{[p]}(\mathcal{M})$ s.t.

$$\Omega_{\Psi_0}((\alpha_2, \eta), (\alpha_1, \beta)) = 0$$

$\forall (\alpha_2, \eta) \in T_{[p]}(\mathcal{M})$. Using the metric $G$ we identify $T_{[p]}\mathcal{M}$ with the subspace in $T_pS$, $G$-orthogonal to $T_pO_p$, (i.e. the tangent space to the moduli space is identified to the tangent space to solutions which are orthogonal to the gauge orbits, the orthogonality is with respect to the metric $G$.) Thus $(\alpha_1, \beta), (\alpha_2, \eta)$ satisfy the
linearization of equation (1) and (2) and \( \mathcal{G}((\alpha_1, \beta), X_\zeta) = 0 \) and \( \mathcal{G}((\alpha_2, \eta), X_\zeta) = 0 \) for all \( \zeta \).

Now, by (2.3) \( \mathcal{I}(\alpha_2, \eta) \in T_pS \). Also,

\[
\mathcal{G}(\mathcal{I}(\alpha_1, \beta), X_\zeta) = \Omega((\alpha_1, \beta), X_\zeta) = -\int_M \zeta d\mu((\alpha_1, \beta)) = 0
\]

since \( d\mu((\alpha_1, \beta)) = 0 \) is precisely one of the equations saying that \( (\alpha_1, \beta) \in T_pS \). Thus \( \mathcal{I}(\alpha_1, \beta) \in T_p^1M \), (since it is in \( T_pS \) and \( \mathcal{G} \)-orthogonal to gauge orbits).

Take \( (\alpha_2, \eta) = \mathcal{I}(\alpha_1, \beta) = (\#_1\alpha_1, i\beta) \). Then

\[
0 = \Omega_{\Psi_0}(\mathcal{I}(\alpha_1, \beta), (\alpha_1, \beta)) = -\int_M (\#_1\alpha_1 \wedge \alpha_1) + 2i \int M \Re < i\beta, \beta >_H |\Psi_0|_H^2 \omega
\]

\[
= -4 \int_M |a|^2 dx \wedge dy - 4 \int_M |\beta|^2_H |\Psi_0|_H^2 h^2 dx \wedge dy
\]

where \( \omega = -2ih^2 dx \wedge dy \) and \( \alpha_1 = adz - \bar{a}d\bar{z} \in \Omega^1(M, i\mathbb{R}) \) and \( \#_1\alpha_1 = -i(adz + \bar{a}d\bar{z}) \). By negativity of both the terms and the fact that \( \Psi_0 \) has zero on a set of measure zero on \( M \), \( (\alpha_1, \beta) = 0 \) a.e. Thus \( \Omega_{\Psi_0} \) is symplectic.

\[
\square
\]

3. PREQUANTUM LINE BUNDLE

In this section we briefly review the Quillen construction of the determinant line bundle of the Cauchy Riemann operator \( \partial_A = \bar{\partial} + A^{(0,1)} \), [18], which enables us to construct prequantum line bundle on the vortex moduli space.

First let us note that a connection \( A \) on a \( U(1) \)-principal bundle induces a connection on any associated line bundle \( L \). We will denote this connection also by \( A \) since the same "Lie-algebra valued 1-form" \( A \) (modu representations) gives a covariant derivative operator enabling you to take derivatives of sections of \( L \) [17], page 348. A very clear description of the determinant line bundle can be found in [18] and [4]. Here we mention the formula for the Quillen curvature of the determinant line bundle \( \Lambda^{top}(\operatorname{Ker}\partial_A)^\star \otimes \Lambda^{top}(\operatorname{Coker}\partial_A) = \det(\partial_A) \), given the canonical unitary connection \( \nabla_Q \), induced by the Quillen metric, [18]. Recall that the affine space \( \mathcal{A} \) (notation as in [18]) is an infinite-dimensional Kähler manifold. Here each connection is identified with its \( (0,1) \) part which is the holomorphic part. Since the connection \( A \) is unitary (i.e. \( A = A^{(1,0)} + A^{(0,1)} \) s.t. \( A^{(1,0)} = -A^{(0,1)} \)) this identification is easy. In fact, for every \( A \in \mathcal{A}, \mathcal{T}_A^\prime(\mathcal{A}) = \Omega^{0,1}(M, i\mathbb{R}) \) and the corresponding Kähler form is given by

\[
F(\alpha^{(0,1)}_1, \alpha^{(0,1)}_2) = \Re \int_M (\alpha^{(0,1)}_1 \wedge \#_2 \alpha^{(0,1)}_2),
\]

\[
= -\frac{1}{2} \int_M \alpha_1 \wedge \alpha_2
\]

where \( \alpha^{(0,1)}_1, \beta^{(0,1)}_1 \in \Omega^{0,1}(M, i\mathbb{R}) \) and \( \#_2 \) is the Hodge-star operator such that \( \#_2(\eta dz) = -\bar{\eta}d\bar{z} \) and \( \#_2(\bar{\eta}d\bar{z}) = \eta dz \) and we have used \( \alpha^{(0,1)}_1 = -\alpha^{(1,0)}_i, i = 1, 2 \). Let \( \nabla_Q \) be the conection induced from the Quillen metric. Then the Quillen
Let us denote by $\Phi(M, L)$ the direct sum of eigenspaces of the operator $\Delta$ of eigenvalues $(1, 0)$ parts of a connection defined by $A = A^{(0,1)} + B^{(0,1)}$. Let $\Phi$ be the same dimension. Let $\Delta$ denote the Laplacian corresponding to $A = A^{(0,1)} + B^{(0,1)}$, where $\Delta$ is defined to be $\Delta = \Delta(A)$. Recall $\Delta = \Delta(A)$ is a well-defined line bundle over $\mathcal{M}$. We can show that the operators $D$ and $D_g$ have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let $\Delta$ denote the Laplacian corresponding to $D$ and $\Delta_g$ that corresponding to $D_g$. The Laplacian is $\Delta = \Delta(D)$ where $\Delta = \Delta(A^{(0,1)} + B^{(0,1)})$, where recall $A^{(0,1)} = -A^{(1,0)}$ and $B^{(1,0)} = -B^{(0,1)}$. Note that $\Delta = \Delta(D_g) = g\Delta g^{-1}$ under gauge transformation. Then $\Delta_g = g\Delta g^{-1}$. Thus the isomorphism of eigenspaces is $s \rightarrow gs$. We describe here how to define the line bundle on the moduli space. Let $K^a(\Delta)$ be the direct sum of eigenspaces of the operator $\Delta$ of eigenvalues $< a$, over the open subset $U^a = \{ A^{(0,1)} + B^{(0,1)}| a \notin \text{Spec} \Delta \}$ of the affine space $\mathcal{J}_+$. The determinant line bundle is defined using the exact sequence

$$0 \rightarrow \text{Ker}D \rightarrow K^a(\Delta) \rightarrow D(K^a(\Delta)) \rightarrow \text{Coker}D \rightarrow 0$$

Thus one identifies

$$\wedge^\text{top}(\text{Ker}D)^* \otimes \wedge^\text{top}(\text{Coker}D)$$

with $\wedge^\text{top}(K^a(\Delta))^* \otimes \wedge^\text{top}(D(K^a(\Delta)))$ (see [4], for more details) and there is an isomorphism of the fibers as $D \rightarrow D_g$. Thus one can identify

$$\wedge^\text{top}(K^a(\Delta))^* \otimes \wedge^\text{top}(D(K^a(\Delta))) \equiv \wedge^\text{top}(K^a(\Delta))^* \otimes \wedge^\text{top}(D(K^a(\Delta_g))).$$

By extending this definition from $U^a$ to $V^a = \{(A, \Psi)| a \notin \text{Spec} \Delta \}$, an open subset of $\mathcal{C}$, we can define the fiber over the quotient space $V^a/G$ to be the equivalence class of this fiber. Covering $\mathcal{C}$ with open sets of the type $V^a$, we can define it on $\mathcal{C}/G$. Then we can restrict it to $\mathcal{M} \subset \mathcal{C}/G$. 

4. Prequantum bundle on $\mathcal{M}$

First we note that to the connection $A$ we can add any one form and still obtain a derivative operator.

Let $\omega = h^2 dz \wedge d\bar{z}$ where recall $h$ is real. Let $\theta = h dz , \bar{\theta} = h d\bar{z}$ be 1-forms (page 28) such that $\omega = \theta \wedge \bar{\theta} = h^2 dz \wedge d\bar{z}$. Let $\Psi_0$ be the same fixed section used to define $\Omega_{\Psi_0}$. Recall $\Psi_0$ has zero on a set of measure zero on $M$. Note $\Psi H \Psi_0$ is a smooth function on $M$. Thus $B^{(0,1)} = \Psi H \Psi_0 \theta$ is a $(0, 1)$-form we would like to add to the connection $A^{(0,1)}$ to make another connection form. Note that $B^{(0,1)}$ is gauge invariant, since $\Psi$ and $\Psi_0$ gauge transform in the same way. Note that $A^{(0,1)} \pm B^{(0,1)}$ are the $(0, 1)$ parts of a connection defined by $A \pm B = A^{(0,1)} \pm B^{(0,1)} = A^{(0,1)} \pm B^{(0,1)}$ where $B^{(1,0)}$ is defined to be $B^{(1,0)} = -B^{(0,1)}$.

**Definitions:** Let us denote by $L_+ = \text{det}(\bar{\partial} + A^{(0,1)} \pm B^{(0,1)})$ a determinant bundle on $\mathcal{J}_+ = \{ A^{(0,1)} \pm \Psi H \Psi_0 \theta | A \in \mathcal{A}, \Psi \in \Gamma(M, L) \}$ which is isomorphic to $C = \mathcal{A} \times \Gamma(M, L)$.

Thus $\mathcal{P}_{\Psi_0} = L_+ \otimes L_-$ well-defined line bundle on $\mathcal{C}$.

**Lemma 4.1.** $\mathcal{P}_{\Psi_0}$ is a well-defined line bundle over $\mathcal{M} \subset \mathcal{C}/G$, where $G$ is the gauge group.

**Proof.** First consider the Cauchy-Riemann operator $D = \bar{\partial} + A^{(0,1)} + B^{(0,1)}$. Under gauge transformation $D = \bar{\partial} + A^{(0,1)} + B^{(0,1)} \rightarrow D_g = g(\bar{\partial} + A^{(0,1)} + B^{(0,1)}) g^{-1}$. We can show that the operators $D$ and $D_g$ have isomorphic kernel and cokernel and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. Let $\Delta$ denote the Laplacian corresponding to $D$ and $\Delta_g$ that corresponding to $D_g$. The Laplacian is $\Delta = \Delta(D)$ where $\Delta = \Delta(A^{(0,1)} + B^{(0,1)})$, where recall $A^{(0,1)} = -A^{(1,0)}$ and $B^{(1,0)} = -B^{(0,1)}$. Note that $\Delta = \Delta(D_g) = g\Delta g^{-1}$ under gauge transformation. Then $\Delta_g = g\Delta g^{-1}$.
Similarly one can deal with the other case of $\bar{\partial} + A^{(0,1)} - B^{(0,1)}$. Let $([A], [\Psi]) \in \mathcal{C}/G$, where $[A], [\Psi]$ are gauge equivalence classes of $A, \Psi$, respectively. Then associated to the equivalence class $([A], [\Psi])$ in the base space, there is an equivalence class of fibers coming from the identifications of $\det(\bar{\partial} + A^{(0,1)} - B^{(0,1)})$ with $\det(g(\bar{\partial} + A^{(0,1)} - B^{(0,1)}))^{-1}$ as mentioned in the previous case.

This way one can prove that $\mathcal{P}_{\Psi_0}$ is well defined on $\mathcal{C}/G$. Then we restrict it to $\mathcal{M} \subset \mathcal{C}/G$.

\textbf{Curvature and symplectic form:}

Let $p = (A, \Psi) \in S$. Let $X, Y \in T_p[\mathcal{M}]$. Since $T_p\mathcal{M}$ can be identified with a subspace in $T_pS$ orthogonal to $T_pO_p$, if we write $X = (\alpha_1, \beta)$ and $Y = (\alpha_2, \eta)$, $\alpha_1, \alpha_2 \in T_\mathcal{M}A = \Omega^I(M, i\mathbb{R})$, and $\beta, \eta \in T_p\Gamma(\mathcal{M}, \mathcal{L}) = \Gamma(M, L)$, then $X, Y$ can be said to satisfy a) $X, Y \in T_pS$ and b) $X, Y$ are $G$-orthogonal to $T_pO_p$, the tangent space to the gauge orbit.

Let $\mathcal{F}_{\mathcal{L}_\pm}$ denote the Quillen curvatures of the determinant line bundles $\mathcal{L}_\pm$, respectively. $\mathcal{L}_\pm$ are determinants of Cauchy-Riemann operators of the connections $A^{(0,1)} \pm H\bar{\Psi}_\theta \Psi_\theta$. Thus in the curvature, we will have $\alpha_1^{(0,1)} \pm \beta H\bar{\Psi}_\theta \Psi_\theta$ and $\alpha_2^{(0,1)} \pm \eta H\bar{\Psi}_\theta \Psi_\theta$, (see Quillen’s formula in the section above).

\[
\mathcal{F}_{\mathcal{L}_\pm}(X, Y) = \frac{i}{\pi} \text{Re} \int_M (\alpha_1^{(0,1)} \pm \beta H\bar{\Psi}_\theta \Psi_\theta) \wedge (\alpha_2^{(0,1)} \pm \eta H\bar{\Psi}_\theta \Psi_\theta) = \frac{i}{\pi} \text{Re} \int_M (\alpha_1^{(0,1)} \pm \beta H\bar{\Psi}_\theta \Psi_\theta) \wedge (\alpha_2^{(0,1)} \pm \eta H\bar{\Psi}_\theta \Psi_\theta)
\]

Note that $\text{Re} \int_M \alpha_1^{(0,1)} \wedge \alpha_2^{(0,1)} = \frac{1}{2} \int_M \alpha_1 \wedge \alpha_2$ where we have used the fact that $\alpha_i = \alpha_i^{(0,1)} + \alpha_i^{(1,0)}$ s.t. $\alpha_i^{(0,1)} = -\alpha_i^{(1,0)}$, $i = 1, 2$. We have also used that $\bar{\partial} \wedge \theta = -\omega = 2ih^2dx \wedge dy$, is purely imaginary. One can easily compute that

\[
\langle \mathcal{F}_{\mathcal{L}_+} + \mathcal{F}_{\mathcal{L}_-} \rangle(X, Y) = \frac{i}{\pi} \int_M \alpha_1 \wedge \alpha_2 - \int_M (\beta H\bar{\eta} - \bar{\beta}H\eta)\Psi_\theta \bar{\Psi}_\theta \Omega\theta \omega = \frac{i}{\pi} \Omega_{\Psi_0}(X, Y)
\]

Now $A^{(0,1)}$ is holomorphic w.r.t. the complex structure $*_1$ and $\Psi$ is holomorphic w.r.t. multiplying by $i$, $A^{(0,1)} \pm B^{(0,1)}$ is holomorphic w.r.t. the complex structure $\iota$. Thus $\mathcal{L}_+, \mathcal{L}_-$ and $\mathcal{P}_{\Psi_0}$ are holomorphic, (same argument as in [15]).

Thus, we have proven the following theorem:

\textbf{Theorem 4.2.} $\mathcal{P}_{\Psi_0} = \mathcal{L}_+ \otimes \mathcal{L}_-$ is a well-defined holomorphic line bundle on $\mathcal{M}$ whose Quillen curvature is $\mathcal{F}_{\mathcal{L}_+} + \mathcal{F}_{\mathcal{L}_-}$ which is $\frac{i}{\pi} \Omega_{\Psi_0}$. Thus $\mathcal{P}_{\Psi_0}$ is a prequantum bundle on $\mathcal{M}$.

\textbf{Polarization:} In passing from prequantization to quantization, one needs a polarization. It can be shown that the almost complex structure $\iota$ is integrable on $\mathcal{M}$, (see, for example, Ruback’s argument mentioned in [20] or matscin et review of [14], [15]). In fact, $\Omega_{\Psi_0}$ is a Kähler form and $\mathcal{G}_{\Psi_0}(X, Y) = \Omega_{\Psi_0}(X, \iota Y)$ is a Kähler metric on the moduli space (since it is positive definite). $\mathcal{P}_{\Psi_0}$ is a holomorphic line bundle on $\mathcal{M}$. Thus we can take holomorphic square integrable sections of $\mathcal{P}_{\Psi_0}$ as our Hilbert space. The dimension of the Hilbert space is not easy to compute. (For instance, the holomorphic sections of the determinant line bundle on the moduli
space of flat connections for $SU(2)$ gauge group is the Verlinde dimension of the space of conformal blocks in a certain conformal field theory). This would be a topic for future work.

**Remark:**

As $\Psi_0$ varies, the corresponding line bundles are all topologically equivalent since the curvature forms have to be of integral cohomology and that would be constant. Thus they have the same Chern class. However they may not be holomorphically equivalent.

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