The Blackbody Radiation in $D$-Dimensional Universes\textsuperscript{1}

(A Radiação de Corpo Negro em Universos $D$-Dimensionais)

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Abstract

The blackbody radiation is analyzed in universes with $D$ spatial dimensions. With the classical electrodynamics suited to the universe in focus and recurring to the hyperspherical coordinates, it is shown that the spectral energy density as well as the total energy density are sensible to the dimensionality of the universe. Wien’s displacement law and the Stefan-Boltzmann law are properly generalized.

Keywords: blackbody, $D$-dimensions, extra dimensions, hyperspherical coordinates

Resumo

A radiação de corpo negro é analisada em universos com $D$ dimensões espaciais. Usando a eletrodinâmica clássica consonante com a dimensionalidade do universo, e recorrendo ao sistema de coordenadas hiperesféricas, mostra-se que tanto a densidade de energia espectral quanto a densidade de energia total são sensíveis ao número de dimensões espaciais. A lei do deslocamento de Wien e a lei de Stefan-Boltzmann são apropriadamente generalizadas.

Palavras-chave: corpo negro, $D$ dimensões, dimensões extras, coordenadas hiperesféricas
1 Introduction

The dimensionality of space can be specified by the number $D$, where $D + 1$ is the maximum number of points which can be mutually equidistant. The dimensionality of the space we live, the physical space, is not derived from any physical law but we perceive that we can move ourselves in three distinct directions: to the left and to the right, forward and backward, upward and downward. Therefore, it is convenient for most of the purposes to take for granted that we live in a 3-dimensional universe. Obviously this perception of dimensionality is conditioned by the limitations of our sensations, and a possible world with more than three dimensions might be properly detected by means of experiments realized in laboratories.

A number of theories involving the unification of the fundamental physical interactions, such as Kaluza-Klein theories [1] and superstring theories [2]-[3], demand physical universes encompassing extra spatial dimensions. It is argued that those extra dimensions do not manifest themselves neither to our sensory experiences nor to the laboratory experiments because such extra dimensions are curled up into dimensions of the order of $10^{-33}\text{cm}$, an extremely small size to be aware of through the senses or even to be detected by experiments. Physical theories in lower dimensions are also interesting not only as simpler models for the “realistic” three-dimensional world but also for exhibiting some peculiarities that turn out them different from the 3-dimensional physical theories [4]-[5].

The growing interest in the physics of multidimensional universes calls for a pedagogical approach of a few topics of physics that are usually explored only in the three-dimensional space. The classical electrodynamics in universes with 1 and 2 spatial dimensions has already received a didactic approach by Lapidus [6]. The author also speculated about the classical electrodynamics in $D$-dimensional universes. Influenced by Lapidus’ work [6], the present paper approaches the influence of the dimensionality of space in the blackbody radiation, one of the first topics the students face at their beginning studies on quantum theory.

The present analysis of the blackbody radiation in $D$-dimensional universes follows the standard procedure for counting the number of standing wave modes in a cavity (see, e.g., [7]-[8]). Nevertheless, the classical electrodynamics consonant with the dimensionality of the universe in question is used. As a part of the discourse, a few criticisms to the literature is also presented. Recurring to the hyperspherical coordinates, it is shown that
both the spectral energy density and the total energy density are sensible to the dimensionality of the universe. Wien’s displacement law as well as the Stefan-Boltzmann law are properly generalized. Finally, the readers are instigated to generalize the calculation of the specific heat capacity of a crystalline solid and the calculation of the average energy of the particles of an ideal Bose gas in $D$-dimensional universes.

2 Preliminary precautions

Lapidus [6] stated that the electric field in a one-dimensional universe obeys the wave equation. That statement induces unwary readers to guess about the propagation of electric waves in such a universe. To tell the truth, neither there are electric waves nor electromagnetic waves. In such an one-dimensional universe the magnetic field is absent and “Maxwell’s equations” are given by

$$\frac{\partial E}{\partial x} = \frac{\rho}{\varepsilon_0}, \quad \frac{\partial E}{\partial t} = -\frac{J}{\varepsilon_0}$$

(1)

where $\varepsilon_0$ is the permittivity of free space. The charge and current densities satisfy the equation of continuity: $\partial \rho / \partial t + \partial J / \partial x = 0$. In the absence of sources the only possible configuration for the electric field, $E$, is to be homogeneous and static. A homogeneous and static electric field always satisfies the wave equation in a trivial way but it is clear there is no electromagnetic wave in an one-dimensional universe. The absence of magnetic fields in such an one-dimensional universe reinforces our argument. This initial observation naturally restricts the considerations of the cavity radiation for universes with $D > 1$.

It is true that Refs. [7]-[8] take into account the electromagnetic modes in a cubic cavity, though, as a didactic appeal, they first consider the modes in a one-dimensional cavity. It is worthwhile to mention that such an one-dimensional cavity does not consist in an one-dimensional world, but only an one-dimensional cavity embedded in a three-dimensional world. This argument is strengthen by noting that the authors consider two possible directions for the polarization of the electric field. This observation refuses the statement of the authors of Ref. [8], when they remark that the number of modes in one-dimensional and three-dimensional cavities depend on different powers for the frequencies just because the dimensionality of the corresponding universes differ.
The observations in the previous paragraphs illustrate the subtle and crucial role of the dimensionality of the universe concerning the propagation of electromagnetic waves.

3 The cavity radiation in D-dimensional universes

The law of the blackbody radiation is customarily formulated as the electromagnetic energy density inside a cavity into the frequency range between \( \nu \) and \( \nu + d\nu \). Thus, one must know the density of states in the interval \( d\nu \) and multiply it by the average energy.

Let us consider the electromagnetic radiation in thermal equilibrium in a metallic cavity with the shape of a \( D \)-dimensional cube (hypercube\(^1\)) whose walls are maintained at the absolute temperature \( T \). Choosing a system of orthogonal Cartesian coordinates with origin at one of the vertex of the cube and axes parallel to their edges, one can write the components of the electric field corresponding to the standing waves in the cavity as

\[
E_a(x_a, t) = E_a^{(0)} \sin(|\kappa_a|x_a) \sin(2\pi \nu t) \tag{2}
\]

where \( E_a^{(0)} \) is the amplitude of the \( a \)-th component of the electric field oscillating with frequency \( \nu \), \( \kappa_a \) is the component of the wave number, and \( a = 1, 2, \ldots D \). The electric field must vanish at the walls of the cavity so that the components of the wave number satisfy the conditions

\[
|\kappa_a| = \frac{\pi}{L} n_a, \quad n_a = 1, 2, 3, \ldots \tag{3}
\]

where \( L \) is the length of the edge of the cube. Therefore, the number of possible modes for the electric field in a given state of polarization, into the interval between \( \kappa_a \) and \( \kappa_a + d\kappa_a \), is given by

\[
N_P(\kappa_a)d\kappa_a = \frac{L}{2\pi} d\kappa_a \tag{4}
\]

The presence of the factor 2 in the denominator of (4) allows us to consider that the components of the vector wave number have positive as well as negative values.

\(^1\)Square in the case \( D = 2 \), cube in the case \( D = 3 \).
The total number of modes in the cavity is the product of the possible modes in each Cartesian axis:

$$N_P(|\vec{\kappa}|)d|\vec{\kappa}| = \prod_{a=1}^{D} N_P(\kappa_a) d\kappa_a = V \frac{dV_\kappa}{(2\pi)^D}$$  \hspace{1cm} (5)

where $V = L^D$ is the volume of the cavity and $dV_\kappa$ is the infinitesimal $D$-dimensional element of volume in the $\kappa$-space, viz.

$$dV_\kappa = \prod_{a=1}^{D} d\kappa_a$$  \hspace{1cm} (6)

This element of volume is, as a matter of fact, a volume which fill the space between the $(D - 1)$-dimensional hyperspherical shells centered about the origin with radii $|\vec{\kappa}|$ and $|\vec{\kappa}| + d|\vec{\kappa}|$.

In a $D$-dimensional world the electric field has $D$ components and the magnetic field has $D(D - 1)/2$ components \[6\]. Since the electromagnetic waves are transverse, one can conclude that the electric field has $D - 1$ mutually perpendicular components standing at right angles to the direction of the wave propagation. In other words, there are $D - 1$ possible directions of polarizations for the electric field\[3\]. In this way, the number of possible modes, including all possible polarizations of the electric field, is

$$N(|\vec{\kappa}|)d|\vec{\kappa}| = (D - 1) \frac{V dV_\kappa}{(2\pi)^D}$$  \hspace{1cm} (7)

To evaluate $dV_\kappa$ it is convenient to recur to the hyperspherical coordinates. The hyperspherical coordinates in a space with $D$ dimensions ($D > 1$) are related to the Cartesian coordinates by \[9\]:

\[x_1 = r \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1}\]

\[x_2 = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1}\]

\[\text{Circles in the case } D = 2, \text{ spherical surfaces in the case } D = 3.\]

\[\text{If the precautions foreseen in Section 2 were not considered, one might conclude that there are no electromagnetic waves in an one-dimensional universe but there might be longitudinal electric waves. This last possibility, though, only can be excluded by the analysis of Maxwell’s equations expressed by (1).}\]
\[ x_a = r \cos \theta_{a-1} \sin \theta_a \ldots \sin \theta_{D-1}, \quad \text{for} \quad 3 \leq a \leq D - 1 \]

\[ x_D = r \cos \theta_{D-1} \]

in such a way that the distance of a point to the origin is expressed in terms of the Cartesian coordinates as \( r = \sqrt{\sum_{a=1}^{D} x_a^2} \). In addition,

\[ 0 \leq \theta_1 \leq 2\pi, \quad \text{and} \quad 0 \leq \theta_a \leq \pi \quad \text{for} \quad 2 \leq a \leq D - 1 \]  

The \( D \)-dimensional element of volume\(^4\) and the \((D-1)\)-dimensional element of solid angle\(^5\) are given by

\[ \prod_{a=1}^{D} dx_a = r^{D-1} dr \, d\Omega, \quad d\Omega = \prod_{a=1}^{D-1} (\sin \theta_a)^{a-1} d\theta_a \]  

(10)

For calculating the total solid angle, instead of proceeding with the calculation of the integrals by brute force, some special functions are used. The beta function, \( B(z, w) \), can be defined as \[ B(z, w) = 2 \int_{0}^{\pi/2} (\sin \theta)^{2z-1} (\cos \theta)^{2w-1} \, d\theta \]  

\[ = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(z) > 0, \quad \Re(w) > 0 \]  

(11)

where \( \Gamma(z) \) is the gamma function

\[ \Gamma(z) = \int_{0}^{\infty} y^{z-1}e^{-y} \, dy \]  

(12)

Then, knowing that \( \Gamma(1/2) = \sqrt{\pi} \), one can write

\[ \int_{0}^{\pi} (\sin \theta_a)^{a-1} \, d\theta_a = \sqrt{\pi} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)} \]  

(13)

\(^4\)Element of area in the case \( D = 2 \).

\(^5\)Element of plane angle in the case \( D = 2 \).
in such a manner that the total solid angle is given by

\[ \Omega = 2\pi^{D/2} \times \begin{cases} 
1, & \text{for } D = 2 \\
\prod_{a=2}^{D-1} \frac{\Gamma(a)}{\Gamma(\frac{a}{2})}, & \text{for } D \geq 3 
\end{cases} \]  

(14)

Writing \( \Omega \) in the compact form

\[ \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)} \]  

(15)

one can finally express \( dV_\kappa \) as

\[ dV_\kappa = \frac{2\pi^{D/2}}{\Gamma(D/2)} |\vec{\kappa}|^{D-1}d|\vec{\kappa}| \]  

(16)

Note that if the signs of de \( \kappa_a \) were restricted only to positive values, the factors 2 and \( 2^D \) in the denominators of (4) and (7), respectively, would be absent. In this circumstance only the infinitesimal element of volume in the space \( \kappa \) corresponding to \( \kappa_a > 0 \) should be considered, implying that the result expressed by (16) should be divided by \( 2^D \) for restrict itself to a \( 2^D \)-ant\(^6\).

Since \( |\vec{\kappa}| = 2\pi\nu/c \), where \( c \) is the speed of the electromagnetic wave, the number of modes with frequencies between \( \nu \) and \( \nu + d\nu \) can be written as

\[ N(\nu)d\nu = (D - 1) V \frac{2}{\Gamma(D/2)} \left( \frac{\sqrt{\pi}}{c} \right)^D \nu^{D-1}d\nu \]  

(17)

Now, each mode has an average energy given by Plank’s radiation formula independently of the dimensionality of the universe \([7,8]\), i.e.,

\[ E_{med} = \frac{h\nu}{\exp(h\nu/k_B T) - 1} \]  

(18)

where \( h \) and \( k_B \) are the Planck and Boltzmann constants, respectively, in such a way that the energy density\(^7\) into the interval between \( \nu \) and \( \nu + d\nu \) is given by

\[ \rho_T(\nu) d\nu = 2 \left( \frac{\sqrt{\pi}}{c} \right)^D \frac{D - 1}{\Gamma(D/2)} \frac{h\nu^D}{\exp(h\nu/k_B T) - 1} d\nu \]  

(19)

\(^6\)A quadrant in the case \( D = 2 \), an octant in the case \( D = 3 \), and so on.

\(^7\)Energy per unit area in the case \( D = 2 \), energy per unit volume in the case \( D = 3 \), and so on.
This is the mathematical expression for the Plank blackbody spectrum in a $D$-dimensional universe. It gives the spectral distribution for the energy in the cavity. Fig. 1 illustrates the spectral energy density (energy density per unit frequency) for the particular cases $D = 2, 3$ and $4$, for $T = 1500$ K$^8$.

The result known as Wien’s displacement law, concerned to the proportionality between the temperature and the frequency that maximizes the spectrum, i.e. $\nu_{\text{max}}/T = \alpha = \text{const}$, is licit whatsoever the dimensionality of the universe. Nevertheless, the constant $\alpha$ increases as the number of the dimensions of the universe increase. Indeed, substituting $x = h\nu_{\text{max}}/k_BT$ in (19) and making $d\rho_T(x)/dx = 0$, one finds

$$e^{-x} = 1 - \frac{x}{D}$$

Eq. (20) has as positive solution$^9$

$$x = D + W\left(-De^{-D}\right)$$

where $W$ is the Lambert W function defined as

$$W(z) + e^{W(z)} = z$$

Note that if $x$ is to be seen as a function of the dimensionality of the space for $D > 1$, it is a monotonic increasing function. This happens because the second term at the right hand side of (21) is a monotonic increasing function for $D > 1$ and tends asymptotically to zero as $D \to \infty$. Of course, the very same conclusion concerning the behaviour of $\alpha$ as a function of $D$ could be obtained in a simpler manner by plotting Eq. (20). It should be mentioned, though, that the numerical solution of (20) becomes extremely easy by using computer algebra softwares which have the Lambert W function built in their libraries.

The total energy density in the cavity is obtained by integrating the spectral energy density over all the frequencies, $\rho_T = \int_0^\infty \rho_T(\nu) d\nu$, resulting in

$$\rho_T = a_DT^{D+1}$$

$^8$Due to errors on the scales of the figures 1-1 and 1-8 in the Ref. $^8$, both at the abscissa and at the ordinate axes, any attempt to compare with the Fig. 1 is disabled.

$^9$x = 0 is another possible solution.
Using the identities [10]

\[ \Gamma(z + 1) = z \Gamma(z) \quad \text{and} \quad \Gamma(2z) = \frac{2^{2z-1/2}}{\sqrt{2\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \tag{24} \]

the constant of proportionality \( a_D \) can be written as

\[ a_D = \left(\frac{2}{hc}\right)^D \left(\sqrt{\pi}\right)^{D-1} k_B^{D+1} D (D - 1) \left(\frac{D + 1}{2}\right) \zeta(D + 1) \tag{25} \]

where \( \zeta(z) \) is the Riemann zeta function [10]:

\[ \zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \quad \text{Re}(z) > 1 \tag{26} \]

which can also be written in the form

\[ \zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{y^{z-1}}{e^y - 1} \, dy \tag{27} \]

The rate of radiation of energy per unit area, known as radiancy, is defined as the total energy emitted per unit time and per unit area of the cavity surface. The radiancy and the energy density in the cavity are related by purely geometric factors. It results that such quantities are proportional, as one will see.

The energy emitted by an infinitesimal element of area \( dA \) of the surface of the cavity in the interval of time \( dt \) occupies a hemisphere \((D-1)\)-dimensional with radius \( cdt \) centered about \( dA \). The energy contained in a cylinder\(^{10}\) of length \( cdt \), with inclination \( \theta_{D-1} \) with respect to the Cartesian axis \( x_D \) perpendicular to the element of area, is

\[ dU_T = \rho_T cdt \, dA \cos \theta_{D-1} \tag{28} \]

Nevertheless, only the fraction \( d\Omega/\Omega \) of the energy contained in the cylinder propagates in the direction specified by the angle \( \theta_{D-1} \). Therefore, the energy emitted in the direction specified by the angle \( \theta_{D-1} \) is

\[ R_T(\theta_{D-1}) = \rho_T c \frac{d\Omega}{\Omega} \cos \theta_{D-1} \tag{29} \]

\(^{10}\)In the case \( D = 2 \) the cylinder is a rectangle, the element of area is an element of line and the hemisphere is a semicircle.
In order to obtain the energy emitted all over possible directions one must integrate the previous expression over all the angles that sweeps the hemisphere. There results

\[ R_T = \frac{\rho_T c}{\Omega} \times \begin{cases} \frac{2}{\pi}, & \text{for } D = 2 \\ \frac{2(\sqrt{\pi})^{D-1}}{D-1} \prod_{a=2}^{D-2} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}, & \text{for } D \geq 4 \end{cases} \]  

that can even be written as

\[ R_T = \left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{D+1}{2}\right)}\right)^{D-1} \frac{\rho_T c}{\Omega} = \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)} \frac{c}{2\sqrt{\pi}} \rho_T \]

Now one are ready to proclaim the law which relates the radiancy with the temperature, the generalized Stefan-Boltzmann law:

\[ R_T = \sigma_D T^{D+1} \]  

where the factor \( \sigma_D \) is the generalized Stefan-Boltzmann constant, given by

\[ \sigma_D = \left(\frac{2}{c}\right)^{D-1} \left(\sqrt{\pi}\right)^{D-2} \frac{k_B^{D+1}}{\hbar^D} D (D - 1) \Gamma\left(\frac{D}{2}\right) \zeta(D + 1) \]

As a function dependent of the number of the dimensions of space, the generalized Stefan-Boltzmann constant is a monotonic increasing function.

\section{Conclusions}

Thanks to the hyperspherical coordinates, the analytical treatment of the blackbody radiation in \( D \)-dimensional universes has been successful. The case of one-dimensional cavities was excluded from the context by taking into account the possible independent directions for the polarization of an electromagnetic wave.

The experimental results are consistent with the blackbody radiation in a three-dimensional world. Since higher dimensional universes with such extra dimensions properly compactified were not considered, the possibility of
higher dimensional universes can not be discarded by contrasting the experimental data with the analysis done in this work.

It is worthwhile to observe that the results presented in this paper would be unchanged if one had considered the blackbody radiation as a photon gas obeying the usual distribution of Bose. This is so because one would had to consider that each photon with energy $h\nu$ can have $D - 1$ possible states of polarization.

The formalism developed in this work can be used to generalize other topics of the quantum theory in an easy way. For instance, the calculation of the vibration modes of the atoms in an elastic solid, or the number of states of a phonon gas, for the generalization of the Debye theory of the specific heat capacity of a solid. In this scenario, diversely from the case of electromagnetic waves, longitudinal acoustic waves are feasible in a one-dimensional universe. It is anticipated that the specific heat capacity at high temperatures behaves as $C_V = DR$ (the generalized Dulong e Petit law), where $R$ is the universal gas constant, and that at low temperatures it behaves as $C_V \sim T^D$. Another interesting topic is that one related to the Bose condensation. In this last topic the number of quantum states of the ideal gas with energy into the interval between $E$ and $E + dE$ is proportional to $E^{D/2 - 1} dE$, a result divergent at $E = 0$ in the case $D = 1$ and independent of $E$ in the case $D = 2$. The calculations which lead to these results, as well as the calculation of the average energy of the particles of an ideal Bose gas (with a promising problem in the calculation of the number of particles for $D < 3$), are left to the readers.

**Note added in proof**

After the paper was submitted, we become aware of the more technical and quasi-homonymous “Black-body radiation in extra dimensions” by H. Alnes, F. Ravndal and K. Wehus (arXiv: quant-ph/0506131), uploaded to the web seventeen days later.

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Figure 1: Spectral energy density as a function of $\nu$ for $T = 1500$ K. Using the fundamental physical constants in the international system of units, the frequency must be multiplied by $10^{14}$ in order to be expressed in Hz, whereas the spectral energy density must be multiplied by $10^{-23}$, $10^{-17}$ and $10^{-11}$ in order to have the units $W\cdot m^{-1}\cdot Hz^{-1}$, $W\cdot m^{-2}\cdot Hz^{-1}$ and $W\cdot m^{-3}\cdot Hz^{-1}$, in the cases $D = 2$, 3 and 4, respectively.