Systematic Stabilization of Constrained Piecewise Affine Systems

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Abstract—This paper presents an efficient, offline method to simultaneously synthesize controllers and seek closed-loop Lyapunov functions for constrained piecewise affine systems on triangulated subsets of the admissible states. Triangulation refinements explore a rich class of controllers and Lyapunov functions. Since an explicit Lipschitz Lyapunov function is found, an invariant subset of the closed-loop region of attraction is obtained. Moreover, it is a control Lyapunov function, so minimum-norm controllers can be realized through online quadratic programming. It is formulated as a sequence of semi-definite programs. The method avoids computationally burdensome non-convex optimizations and a-priori design choices that are typical of similar existing methods.

I. INTRODUCTION

Piecewise affine (PWA) state-space models can approximate a large class of nonlinear systems by partitioning the state-space into regions with distinct, affine dynamics [1]. For instance, this can be done by linearizing a smooth nonlinear system around some operating points and selecting switching surfaces. Moreover, many hybrid systems have equivalent PWA representations [2], [3]. For most physical systems, respecting the state and input constraints must be ensured in control design. Consequently, systematic means to design stabilizing controllers would be broadly applicable and has garnered continual attention [4]–[15]. This work builds upon the triangulation-based methods of [16], [17] to create a novel, computationally efficient synthesis method for this important class of systems.

Stabilizing, piecewise linear state-feedback controllers and a single quadratic Lyapunov function can be sought for unconstrained PWA systems by solving linear matrix inequalities (LMIs) using ellipsoidal approximations for the regions [4], [18]. If unsuccessful, using exact region descriptions or searching in a richer class of functions is possible, however, the search often can no longer be formulated using LMIs. For instance, using exact polytopic descriptions trades-off against added conservatism due to the S-procedure [19, Ch. 3] and formulating a conservative convex optimization. Similar conservatism and restrictive convex-relaxations arise and formulating a conservative convex optimization.

The finite-time optimal controller has to circumvent initialization issues, demands a known control Lyapunov function (CLF).

When respecting input and state constraints is critical, the problem deepens. The finite-time optimal controller has to account for possible switches and the set of initial states is generally non-convex, making it costly to find explicit offline solutions [7], [8], [14], [23]. Guaranteeing recursive feasibility and Lyapunov stability in the receding-horizon implementation requires a terminal invariant set and a CLF on that set [13], [24], [25]. If the closed-loop equilibrium is a shared point of some regions, finding such a CLF is non-trivial and needs the methods of [4], [5], [9], [12], [22], bringing once again the issues of the S-procedure, and initialization,

II. PRELIMINARIES

Notation. The interior, boundary, and closure of $\Omega \in \mathbb{R}^n$ are denoted by $\Omega^\circ$, $\partial \Omega$, and $\overline{\Omega}$, respectively. The set of real-valued functions with $r$ times continuously differentiable partial derivatives over their domain is denoted by $C^r$. The $i$th element of a vector $x$ is denoted by $x^{(i)}$. The preimage of a function $f$ with respect to a subset $\Omega$ of its codomain is defined by $f^{-1}(\Omega) \equiv \{x | f(x) \in \Omega\}$. The transpose and Euclidean norm of $x \in \mathbb{R}^n$ are denoted by $x^T$ and $\|x\|$, respectively. The set of all compact subsets $\Omega \subset \mathbb{R}^n$ satisfying i) $\Omega^\circ$ is

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connected and contains the origin, and ii) $\Omega = \overline{D}$, is denoted by $\mathbb{R}^n$. The vector of ones in $\mathbb{R}^n$ is denoted by $1_n$.

In this paper, the Lyapunov functions and controllers are defined on a triangulated subset of the state space. The required definitions are given next.

**Definition 1 (Affine independence [16]):** A collection of vectors $\{x_0, \ldots, x_n\} \in \mathbb{R}^n$ is called affinely independent if $x_1 - x_0, \ldots, x_n - x_0$ are linearly independent.

**Definition 2 (n-simplex [16]):** An $n$-simplex is the convex combination of $n+1$ affinely independent vectors in $\mathbb{R}^n$, denoted $\sigma = \text{co}(\{x_j\}_{j=0}^n)$, where $x_j$’s are called vertices.

In this paper, simplex always refers to $n$-simplex. By abuse of notation, $T$ will refer to both a collection of simplexes and the sets in all the simplexes of the collection.

**Definition 3 (Triangulation [16]):** A set $T \in \mathbb{R}^n$ is called a triangulation if it is a finite collection of $m_T$ simplexes, denoted $T = \{\sigma_i\}_{i=1}^{m_T}$, and the intersection of any of the two simplexes in $T$ is either a face or the empty set.

The following two conventions are used throughout this paper for triangulations and their simplexes. Let $T = \{\sigma_i\}_{i=1}^{m_T}$. Further, let $\{x_{i,j}\}_{j=0}^{n} = \sigma_i$’s vertices, making $\sigma_i = \text{co}(\{x_{i,j}\}_{j=0}^{n})$. The choice of $x_{i,0}$ in $\sigma_i$ is arbitrary unless $0 \in \sigma_i$, in which case $x_{i,0} = 0$. The vertices of the triangulation $T$ that are in $\Omega \subseteq T$ is denoted by $\Omega_T$.

**Definition 4 (Triangulable Set):** A compact, connected subset of $\mathbb{R}^n$ that has no isolated points, and can be exactly covered by a finite number of simplexes.

**Definition 5 (Constraint Surfaces of a Triangulation):** Let $\mathcal{T}$ be the triangulation of a triangulable set $\Omega \subseteq \mathbb{R}^n$. The surface $\mathcal{H} \subset \mathcal{T}$ is called a constraint surface in $\mathcal{T}$ if it is exactly covered by the faces of some simplexes in $\mathcal{T}$.

**Lemma 1 ([16, Rem. 9]):** Consider the triangulation $T = \{\sigma_i\}_{i=1}^{m_T}$, where $\sigma_i = \text{co}(\{x_{i,j}\}_{j=0}^{n})$, and a set $W = \{W_x\}_{x \in \mathcal{T}} \subset \mathbb{R}$. Let $\sigma_i = \text{co}(\{x_{i,j}\}_{j=0}^{n})$, and $X_0 \in \mathbb{R}^n$ be a matrix that has $x_{i,j} - x_{i,0}$ as its $j$-th row, and $W_i \in \mathbb{R}^n$ be a vector that has $W_{x_{i,j}} - W_{x_{i,0}}$ as its $j$-th component.

The function $W(x) = x_{i,j}^T X_i - W_i + \omega_i$ is the unique CPA interpolation of $W$ on $\mathcal{T}$, satisfying $W(x) = W_{x_i}$, $\forall x \in \mathcal{T}$.

Note that since the elements of $\{x_{i,j}\}_{j=0}^{n}$ are affinely independent, $X_i$ in Lemma 1 is invertible. The Dini derivative of a CPA $W$ at $x$ is defined as $D^+ W(x) = \limsup_{h \to 0^+} \frac{(W(x+h) - W(x))/h}{W(x)}$, which equals $W'(x)$ where $W \in C^1$.

Also, a continuous function $g(x) \in \mathbb{R}^n$ is piecewise in $C^2$ on a triangulation $T = \{\sigma_i\}_{i=1}^{m_T}$, denoted $g \in C^2(T)$, if it is in $C^2$ on $\sigma_i$ for all $i \in \mathbb{Z}_{m_T}^+$. The following theorem gives a general stabilization criteria for constrained systems using CPA Lyapunov functions.

**Theorem 1 ([17, Thm. 3]):** Consider the system

$$
\dot{x} = g(x, u), \quad x \in \mathcal{X} \subseteq \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \mathbb{R}^m, \quad g(0, 0) = 0. \quad (1)
$$

Consider the following nonlinear program.

$$
[V^*, L^*, x^*, a^*, b^*] = \argmin_{V, L, x, a, b} J(V, L, x, a, b) \quad \text{s.t.} \quad V_0 = 0, \quad a \geq 1, \quad b_i > 0, \quad \|x(t_i)\| \leq V_i, \quad \forall x \in \mathcal{E}_T \setminus \{0\}, \quad (2a)
$$

$$
\frac{\partial^2 g(x)}{\partial x^i \partial x^j} \bigg|_{x=0} \leq l_i, \quad \forall x \in \mathcal{Z}_{m_T}^+, \quad (2b)
$$

$$
\frac{\partial^2 g(x)}{\partial x^i \partial x^j} \bigg|_{x=0} \leq l_i, \quad \forall x \in \mathcal{Z}_{m_T}^+, \quad (2c)
$$

where $V_0 = \lambda \sigma_i$, $V_0 = \lambda \sigma_i$, $V_0 = \lambda \sigma_i$, $V_0 = \lambda \sigma_i$, and $V = \{V_x\}_{x \in \mathcal{T}} \subset \mathbb{R}$, $L = \{l_i\}_{i=1}^{m_T} \subset \mathbb{R}^n$, and $\mathcal{U} = \{u_{i,j}\}_{x \in \mathcal{T}} \subset \mathbb{R}$, and $J$ is a cost function, and for $u(\cdot, \lambda)$ satisfying (23).

$$
\beta_i \geq \max_{p,q,r \in \mathbb{E}_T} \max_{\lambda \in \mathbb{E}_T} \frac{\partial^2 g(x)}{\partial x^i \partial x^j} \bigg|_{x=0} \leq \frac{\max_{k \in \mathbb{E}_T} \|x_{i,k} - x_{i,0}\| + \|x_{i,j} - x_{i,0}\|}{\|x(t)\|} \leq a^* \cdot \beta_i \cdot e^{-(b_i / a^*)}, \quad \text{if} \quad x(t_0) \in \mathcal{A}^0. \quad (3)
$$

**III. MAIN RESULTS**

By using CPA functions for both the Lyapunov function and the controller, a method for stabilizing control-affine systems, $\dot{x} = f(x) + G(x)u$ with state and input constraints, was developed in [17]. If $f(x)$ and $u$ are affine functions of $x$ on each simplex and $G(x)$ is constant, the $\beta_i$ term in (3) vanishes since $g(x)$ is affine with respect to $x$.

This observation leads to the following theorem that gives sufficient conditions for stabilization of PWA systems.

**Theorem 2:** Consider the constrained control system

$$
\dot{x} = A_n x + B_n u + e_s, \quad x \in \mathcal{X} \subseteq \mathbb{R}^n, \quad u \in \mathcal{U} \subseteq \mathbb{R}^m, \quad (4)
$$

where $\mathcal{U} = \{u \in \mathbb{R}^m \mid H u \leq h_c\}$, $\mathcal{X} \subseteq \mathbb{R}^n$, and $\{R_i\}_{i=1}^M$ be a partition of $\Omega$, where $A_n, B_n, s$ are constant on each $R_i$. In all $R_i$’s containing the origin, $e_s = 0$. Suppose that $\mathcal{T}$ is a triangulation of $\Omega$, comprised of triangulations of $\{R_i\}_{i=1}^M$. Let $u$ be CPA on $\mathcal{T}$, and

$$
V = \{V_x\}_{x \in \mathcal{T}} \subset \mathbb{R}^n, \quad U = \{u_{i,j}\}_{x \in \mathcal{T}} \subset \mathbb{R}^m, \quad a \in \mathbb{R}, \quad b = \{b_1, b_2\} \subset \mathbb{R} \text{ be the unknowns. Consider}
$$

$$
y^* = \arg\min_{y} J(y) \quad \text{s.t.} \quad V_0 = 0, \quad a \geq 1, \quad b_i > 0, \quad \|x(t_i)\| \leq V_i \quad \forall x \in \mathcal{E}_T \setminus \{0\}, \quad (5a)
$$

$$
u_0 = 0, \quad H u_x \leq h_c, \quad \forall x \in \mathcal{E}_T \setminus \{0\}, \quad (5b)
$$

$$
D_{i,j}^+ V \leq -b_2 x_{i,j}, \quad \forall x \in \mathcal{Z}_{m_T}^+, \quad (5c)
$$

where $D_{i,j}^+ V = \{A_n x_{i,j} + e_s \lambda \sigma_i \} \in \mathcal{V}_T \cup \{0\}$, and $J(\cdot)$ is a cost function. If $b_2 > 0$ in (5), then the CPA function $V^* : \mathcal{T} \to \mathbb{R}$ constructed from the elements of $V^*$ is a Lyapunov function of $\dot{x} = A_n x + B_n u^*(x) + e_s$.
where \( u^*(\cdot) \) is the CPA function constructed from \( U^* \). Let \( A = V^{-1} \{ 0, r \} \) be in \( \mathbb{R}^n \) for some \( r > 0 \). Then \( x = 0 \) is locally exponentially stable for the closed-loop system with \( \|x(t)\| \leq e^{\sum_{i=1}^n b_i^0} \|V_x(\cdot)\| \) if \( x(t_0) \in A^0 \).

**Proof:** We show that (5) verifies (2). By requiring \( \mathcal{T} \) to be comprised of triangulations of \( \{R_k\} \), the switching surfaces of (4) are constraint surfaces. So, on each simplex in \( \mathcal{T} \) with \( \mathcal{A} \) to be locally exponentially stable for the closed-loop system from (4), and the surfaces of (4) are constraint surfaces. So, on each simplex in \( \mathcal{T} \) with \( \mathcal{A} \) and \( \mathcal{B} \) satisfying the condition (2e) without needing (2c). Since \( \mathcal{A} \) is CPA, \( \mathcal{C} \) implies (2d). Lastly, (5d)–(5b) are the same as (2a)–(2b). Thus the proof follows from that of Theorem 1.

Even if \( b_2^0 \leq 0 \) in (5), a connected subset of \( \mathcal{T} \) that has \( D_i^+ \mathcal{V}^* > 0 \) on its vertices might exist, using the following.

**Corollary 1 ([17, Cor. 1]):** Suppose that \( b_2^0 \leq 0 \) in Theorem 2. Let \( \mathcal{T} = \{ \{ \sigma_i \} \} \cup \{ \sigma_i \} \), \( i \in I \), \( j \in J \) is feasible for (5), and \( \{ \sigma_i \} \) are constraint surfaces. So, on each simplex in \( \mathcal{T} \) with \( \mathcal{A} \) and \( \mathcal{B} \) satisfying the condition (2e) without needing (2c). Since \( \mathcal{A} \) is CPA, \( \mathcal{C} \) implies (2d). Lastly, (5d)–(5b) are the same as (2a)–(2b). Thus the proof follows from that of Theorem 1.

Even if \( b_2^0 \leq 0 \) in (5), a connected subset of \( \mathcal{T} \) that has \( D_i^+ \mathcal{V}^* > 0 \) on its vertices might exist, using the following.

**Corollary 2:** Suppose that \( a > 0 \) is a fixed, known number in (5). Let \( y = [V, U, a, b] \) satisfy (5a)–(5d). Consider the following optimization.

\[
\delta y^* = \arg\min_{\delta y} J(y + \delta y) \quad \text{s.t. } \quad \delta V_0 = 0, \quad \delta b_1 + \delta b_2 > 0, \\
(\delta b_1 + \delta b_2) ||y|| \leq \mathcal{V}_y^0 + \delta V_x, \forall x \in \mathbb{E}_T \backslash \{0\}, \\
\delta u_0 = 0, \quad H(u_x + \delta u_x) \leq h_c, \forall x \in \mathbb{E}_T, \\
P_{i,j} \leq 0, \quad \forall i \in \mathbb{Z}_{i}^{m_T}, \quad j \in \mathbb{Z}_{i}^{m_T}, \quad \text{and} \quad \delta V_{i,j} = \mathcal{X}_i^{-1}\mathcal{V}_i', \delta \mathcal{V}_{i,j} = \mathcal{X}_i^{-1}\delta \mathcal{V}_i, \text{as in Lemma 1}.
\]

**Proof:** To see (6a)’s feasibility, observe that \( \delta y = 0 \) satisfies (6) since in this case, (6) is equivalent to (5) with \( y = y \). Substitution reveals that (6a)–(6c) implies (5a)–(5c) for \( y = y + \delta y \). To see that (6d) implies (5d), note that \( w^T v \leq 1/2(\mathcal{V} w + \mathcal{V} v) \) for any two same-dimension vectors. Applying this fact with \( (v, w) = (\delta \mathcal{V}_x, B_x \delta u_{x,i,j}) \) and \( (v, w) = (\delta \mathcal{V}_x, \delta b_2) \) shows that Schur Complement [27, Ch 2], (6d) is implied. Finally, \( J(y + \delta y) \leq J(y) \) because otherwise \( \delta y = 0 \) would be a better, feasible solution.

Given a triangulation and a linear or quadratic cost function \( J(V, U, b_1) \), a method of searching for a stabilizing CPA controller is given in Algorithm 1. It iteratively increases \( b_2 \) until it is positive. This can continue until a desired decay rate is ensured. Then, by fixing \( b_2 \)’s value, \( J(\cdot) \) is iteratively minimized. If finding a positive \( b_2 \) is not successful, triangulation refinement, discussed later, is needed.

Once Algorithm 2 returns \( y \), can serve as the initial guess for (5) with a desired cost function \( J(\cdot) \) to improve performance offline. Moreover, since the corresponding Lyapunov function of the returned controller is also a Lipschitz CLF, a minimum-norm controller can be formulated as an online quadratic programming (QP) [28]. Suppose that \( b_2^0 > 0 \) is found by Algorithm 2 and \( V^* \) is the corresponding CPA Lyapunov function. Let \( A = V^{-1} \{ 0, r \} \), \( r > 0 \), where
Algorithm 1 CPA control design on a fixed triangulation

**Inputs:** The PWA system (4), and a triangulation $T \subseteq \mathcal{X}$ that has the switching surfaces as constraint surfaces, and a linear or quadratic $\hat{J}(V, U, b_1)$

**Outputs:** $u(x)$, and a positive-invariant set $A$

1. $y :=$ a feasible point of (5) (using Initialization[1] or [2])
2. $\hat{J} := -b_2 > 0$ since $b_2$ is to be maximized
3. repeat
4. Use Theorem[2]
5. until $b_2 > 0$ is large enough OR $b_2$ is not changing
6. if $b_2 > 0$ is found then
7. Fix $b_2$, and let $J := \hat{J}(\cdot)$
8. repeat
9. Use Theorem[2]
10. until $J$ is sufficiently small OR $J$ is not changing
11. Return $u(x)$ and find a set $A = V^{-1}(\{0, r\}, r > 0$,
12. where $A \subseteq T$ and $A \in \Omega^n$
13. end if

$A \subseteq T$ and $A \in \Omega^n$. Starting at any $x \in A$, the minimum-norm controller can be written as

$$u^*(x) = \arg \min_u u^T \hat{H}(x) u + \hat{h}(x)^T u$$

s.t. $H u \leq h_c, \nabla V^T(x) + b_2 V^*(x) \leq 0, \forall i \in I,$

where $I = \{i \in Z_1^m \mid x \in \sigma_i\},$ and $\hat{H}(x)$ is positive definite.

The optimization is feasible for all $x \in A$, because the corresponding CPA controller of $V^*$ is a feasible point for it.

B. Triangulation Refinement

If finding a $b > 0$ in Theorem[2] or Algorithm[1] fails, the triangulation can be refined. This introduces more vertices and thus controller parameters, increasing the possibility of finding a stabilizing one. These refinements can be local by tracking the value of $D_{ij}^+ V$ on the simplices in $T$. An important assumption of Theorem[2] was that the switching surfaces of the PWA system are included in constraint surfaces of the triangulation. Let $T$ be the triangulation of $\mathcal{X}$ in which $H_i$ its constraint surfaces, include the switching surfaces of [4], and let $\rho : \Omega \rightarrow \mathbb{R}_+$, where $\Omega \subseteq \mathcal{X}$, be a function representing simplex sizes in a region of interest. Algorithm[2] describes a simple way of refining triangulations.

V. Numerical Simulation

An example is adopted here from [29, Sc 5.4] with slight modifications, including additional constraints, to compare the introduced method, referred to as ‘CPA’, with two other well-established ones, EMPC and the PWA method of [5]. All the computations were carried out in MATLAB on a desktop computer with an AMD Ryzen 5 CPU and 8 GB DDR4 RAM. To solve semi-definite programs (SDPs), SeDuMi [30] with YALMIP [31] were used. The LQR cost for the initializations was $2x^T x + u^T u$. The toolbox Mesh2D [32] was used for triangulation generation, where the maximum element size function was used for refinements.

Algorithm 2 Control design with triangulation refinement

**Inputs:** System (4), cost function, simplex size function $\rho : \mathcal{X} \rightarrow \mathbb{R}_+$, where $\Omega \subseteq \mathcal{X}$, minimum simplex size $\rho_{\min}$, constraint surfaces $H$, $0 < \gamma < 1$

**Outputs:** $y = [V, U, a, b]$

1. repeat
2. Generate $T$, the $\mathcal{X}$’s triangulation respecting $\rho$ and $H$, including the switching surfaces
3. Solve (5) or use Algorithm[1]
4. if desired objectives are met then
5. Return $y$
6. end if
7. $\rho := \gamma \rho$
8. until $\rho_{\min}$ is reached

Consider the PWA system (4) with $s \in \mathbb{R}_+^3$, where $A_s = \{0.1, 1.1; p_s = -1\}, p_1 = 0.1, p_2 = -0.9, p_3 = -1.9$, and $B_s = \{0, 1\}^T, \forall s \in \mathbb{Z}_3$, and $c_2 = 0, c_1 = c_3 = \{0, 1\}^T$. The set $\mathcal{X}$, depicted in Fig. 1a, includes the polytopic regions $\{R_s\}_{s=1}^3$, where $R_1 = \mathcal{X} \cap \{x \in \mathbb{R}^2 \mid x(1) < -1\}$, $R_2 = \mathcal{X} \cap \{x \in \mathbb{R}^2 \mid -1 \leq x(1) \leq 0\}$, and $R_3 = \mathcal{X} \cap \{x \in \mathbb{R}^2 \mid x(1) > 1\}$. The input constraint is $|u| \leq u_{\max}$. The problem is solved for the two cases, $u_{\max} = 1$ and $u_{\max} = 2$. The required offline time to synthesize stabilizing controllers, referred as synthesis time, denoted $t_{\text{syn}}$, and the settling times of closed-loop systems to $||x|| \leq 0.01$, denoted $t_{\text{settle}}$, and the ratio of the obtained region of attraction (ROA)’s area over $\mathcal{X}$’s area, denote $A_{\text{ROA}}$, were compared using this paper’s method and the two following ones.

EMPC: The system was discretized using Euler’s method with a 0.1 s sampling time. Since the origin is only in $R_2$, the terminal set was selected as the maximal positive-invariant set in $R_2$ associated with the terminal cost obtained from the solution of the Ricatti equation for the LQR cost $2x^T x + u^T u$, which was used as the running cost. Note that the terminal choices are not easy-to-find when the origin is shared between some regions. Thus, this example gives a significant advantage to EMPC’s synthesis time. EMPC’s horizon is denoted by $N$. For synthesis, MPT3 [33] was used.

PWA [5]: This method searches for a quadratic Lyapunov function and a PWA state-feedback, $u = K_s x + w_s$, in each $R_s$, while maintaining continuity across the switching surfaces. We augmented it with $|K_s x_p + w_s| \leq u_{\max}$, where $x_p$ denotes the vertices of $R_s$, to enforce input constraints. Since [5] did not address input constraints and would be computationally burdensome with more complex regions, this gives both additional functionality and an advantage to PWA’s synthesis time. To solve the BMIs, [5] alternates between fixing and seeking between finding Lyapunov functions versus controllers at each iteration. The function ‘fmincon’ was used to find a feasible initialization for the controllers and the equilibria. Although finding suitable parameters to make [5]’s method work involved some trial-and-error, they were not included in the synthesis time. The iterations increase a uniform decay rate for the Lyapunov functions.
TABLE I: Comparing the CPA controller on a fine triangulation to an EMPC that has \( N = 5 \), \( t_{\text{syn}}^{\text{EMPC}} = 73.2 \) s, \( A_X^{\text{EMPC}} = 0.64 \)

| \( b_2 \) | \( t_{\text{sett}}^{\text{CPA}} / t_{\text{sett}}^{\text{EMPC}} \) | \( A_X^{\text{CPA}} / A_X^{\text{EMPC}} \) | \( t_{\text{sett}}^{\text{CPA}} / t_{\text{sett}}^{\text{EMPC}} \) |
|---|---|---|---|
| 0.07 | 0.88 | 1.00 | 1.17 |

* Average settling times over the shared ROA

TABLE II: Comparing the CPA controller on a coarse triangulation to a PWA one that has \( t_{\text{syn}}^{\text{PWA}} = 4.9 \) s (stagnation), \( A_X^{\text{PWA}} = 0.61 \)

| \( b_2 \) | \( t_{\text{sett}}^{\text{CPA}} / t_{\text{sett}}^{\text{PWA}} \) | \( A_X^{\text{CPA}} / A_X^{\text{PWA}} \) | \( t_{\text{sett}}^{\text{CPA}} / t_{\text{sett}}^{\text{PWA}} \) |
|---|---|---|---|
| 0.18 | 0.46 | 0.56 | 0.61 | 0.64 |

* Average settling times over the shared ROA

B. Case 2: \( u_{\text{max}} = 2 \)

Here, PWA was also able to synthesize a controller as the input constraint was looser. Since it took only 4.9 s for PWA iterations to stagnate, we allowed the same synthesis time to CPA and EMPC. With \( N = 2 \) EMPC synthesized a controller with \( N = 2 \) in 2.8 s. Using the coarsest possible triangulation, depicted in Fig. 2a the CPA controller achieved \( b_2 = 0.6 \) in 3.8 s. The values obtained by the CPA controller are compared to the PWA one in Table II. The first CPA controller, obtained in only 0.6 s, had comparable average settling time over the shared ROA to the PWA’s but its ROA’s area is almost 25% smaller. As \( b_2 \) increased, the CPA gained advantage in average settling time and expanded its ROA till stagnation at 6.5 s. The obtained controller after 3.9 s that had \( b_2 = 0.61 \) and 12% settling time advantage over the PWA is visually compared to the PWA controller in Fig. 2a.

C. Discussion

Although significant advantage was given to EMPC by including the origin in the interior of only one mode, and also to the PWA method by not accounting the time spent on trial-and-error, the CPA method was competitive to both and achieved significant improvements in terms of synthesis time and settling time. It always initializes feasibly, and removes a priori design choices. Moreover, triangulation refinement allows searching a rich class of Lyapunov functions and controllers as their defining vertices increase. This happens with no added conservatism and complexity, contrasting it with refinements that are also allowed in PWA method at the expense of added conservatism due to the S-procedudre and more complexity in finding a feasible start. Both CPA and EMPC have rigorous ways of respecting input constraints. However, EMPC has a more sophisticated way of finding large ROAs. Using the CPA method to find terminal choices for EMPC when they are not trivial is appealing because of the discussed design convenience. Enlarging the ROAs using the CPA method will be considered in future.

VI. Conclusion

In this paper, a systematic, offline stabilization method for constrained PWA systems was proposed that searches CPA
Lyapunov functions and controllers on triangulated subsets of the admissible states via iterative SDPs. The method returns an invariant subset of ROA and explicit Lyapunov functions. The settling time is compared for the CPA and PWA controllers only.

**Fig. 2:** Comparison of ROAs of the three controllers, EMPC with $N=2$, CPA with $b_2=0.61$, and PWA, for Case 2: $u_{\text{max}}=1$. The settling time is compared for the CPA and PWA controllers only.

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