A LOCAL EXISTENCE RESULT FOR SYSTEM
OF VISCOELASTICITY WITH PHYSICAL VISCOSITY

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ABSTRACT. We prove the local in time existence of the classical solutions to the system of equations of isothermal viscoelasticity with clamped boundary conditions. We deal with a general form of viscous stress tensor $\mathcal{Z}(F, \dot{F})$, assuming a Korn-type condition on its derivative $D_p \mathcal{Z}(F, \dot{F})$. This condition is compatible with the balance of angular momentum, frame invariance and the Claussius-Duhem inequality. We give examples of linear and nonlinear (in $\dot{F}$) tensors $\mathcal{Z}$ satisfying these required conditions.

1. Introduction and the main results

In this paper, we are concerned with the local in time existence of the classical solutions to the system of equations of isothermal viscoelasticity. The system we study is given through the balance of linear momentum:

\begin{align}
\xi_{tt} - \text{div}
\left(DW(\nabla \xi) + \mathcal{Z}(\nabla \xi, \nabla \xi_t)\right) = 0 & \quad \text{in } \Omega \times \mathbb{R}^+,
\end{align}

and it is subject to initial data:

\begin{align}
\xi(0, \cdot) = \xi_0 & \quad \text{and } \xi_t(0, \cdot) = \xi_1 & \quad \text{in } \Omega,
\end{align}

the clamped boundary conditions:

\begin{align}
\xi(\cdot, X) = X & \quad \forall X \in \partial \Omega,
\end{align}

and the non-interpenetration ansatz:

\begin{align}
det \nabla \xi > 0 & \quad \text{in } \Omega.
\end{align}

Here, $\xi : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ denotes the deformation of a reference configuration $\Omega \subset \mathbb{R}^n$ which models a viscoelastic body with constant temperature and density. A typical point in $\Omega$ is denoted by $X$, and the deformation gradient, the velocity and velocity gradient are given as:

\begin{align}
F = \nabla \xi \in \mathbb{R}^{n \times n}, & \quad v = \xi_t \in \mathbb{R}^n, & \quad Q = \nabla \xi_t = \nabla v = F_t \in \mathbb{R}^{n \times n}.
\end{align}

In (1.1) the operator $\text{div}$ stands for the spacial divergence of an appropriate field. We use the convention that the divergence of a matrix field is taken row-wise. In what follows, we shall also use the matrix norm $|F| = (\text{tr}(F^T F))^{1/2}$, which is induced by the inner product: $F_1 : F_2 = \text{tr}(F_1^T F_2)$. To avoid notational confusion, we will often write $\langle F_1 : F_2 \rangle$ instead of $F_1 : F_2$. 

1.1. The elastic energy density $W$. The mapping $DW : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ in (1.1) is the Piola-Kirchhoff stress tensor which, in agreement with the second law of thermodynamics [8], is expressed as the derivative of an elastic energy density $W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}_+$. The principles of material frame invariance, material consistency, and normalisation impose the following conditions on $W$, valid for all $F \in \mathbb{R}^{n \times n}$ and all proper rotations $R \in SO(n)$:

\begin{itemize}
  \item[(i)] $W(RF) = W(F)$,
  \item[(ii)] $W(F) \to +\infty$ as $\det F \to 0$,
  \item[(iii)] $W(Id) = 0$.
\end{itemize}

Examples of $W$ satisfying the above conditions are:

\begin{align*}
W_1(F) &= |(F^T F)^{1/2} - Id|^2 + |\log \det F|^q,
W_2(F) &= |(F^T F)^{1/2} - Id|^2 + \left| \frac{1}{\det F} - 1 \right|^q \text{ for } \det F > 0,
\end{align*}

where $q > 1$ and $W$ is intended to be $+\infty$ if $\det F \leq 0$ [22]. Another case-study example, satisfying (i) and (iii) is: $W_0(F) = |F^T F - Id|^2 = (|F^T F|^2 - 2|F|^2 + n)$.

We will assume that $W$ is smooth in a neighborhood of $SO(n)$. Since $\text{div}(DW(\nabla \xi))$ is a lower order term in (1.1), it follows that other properties of $W$ play actually no role in the proof of our main Theorem 1.1 and 1.2. We hence remark that the same results are valid when $\text{div}(DW(\nabla \xi))$ is replaced by $\text{div}(DW((\nabla \xi)A(X)^{-1}))$. Such term corresponds to the so-called non-Euclidean elasticity, where the deformation $\xi$ of the reference configuration strives to achieve a prescribed Riemannian metric $g = A^T A$ on $\Omega$. This model pertains to the description of prestrained materials and morphogenesis of growing tissues [19, 18].

1.2. The viscous stress tensor $Z$. The viscous stress tensor is given by the mapping $Z : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, depending on the deformation gradient $F$ and the velocity gradient $Q$. It should be compatible with the following principles of continuum mechanics: balance of angular momentum, frame invariance, and the Clausius-Duhem inequality [3]. That is, for every $F, Q \in \mathbb{R}^{3 \times 3}$ with $\det F > 0$, we require that:

\begin{itemize}
  \item[(i)] $\text{skew}(F^{-1}Z(F, Q)) = 0$, i.e. $Z = FS$ with $S$ symmetric.
  \item[(ii)] $Z(RF, R_t F + RQ) = RZ(F, Q)$ for every path of rotations $R : \mathbb{R}_+ \rightarrow SO(n)$, i.e. in view of (i): $S(RF, RKF + RQ) = S(F, Q) \forall R \in SO(n)$ \forall $K \in \text{skew}$.
  \item[(iii)] $Z(F, Q) : Q \geq 0$, i.e. in view of (i): $S : \text{sym}(F^T Q) \geq 0$.
\end{itemize}

Examples of $Z$ satisfying the above are:

\begin{align*}
Z_m(F, Q) &= [\text{sym}(QF^{-1})]^{2m+1}F^{-1,T},
Z'_0(F, Q) &= 2(\det F)\text{sym}(QF^{-1})F^{-1,T},
Z'_0(F, Q) &= 2F\text{sym}(F^T Q).
\end{align*}
We note that in the case of $Z_0'$, the related Cauchy stress tensor $T_0' = 2(det F)^{-1} Z_2 F^T = 2\text{sym}(QF^{-1})$ is the Lagrangean version of the stress tensor $2\text{sym}\nabla v$ written in the Eulerian coordinates. For incompressible fluids $2\text{div}(\text{sym}\nabla v) = \Delta v$, giving the usual parabolic viscous regularization of the fluid dynamics evolutionary system.

1.3. The main results. Our main assumption implying the dissipative properties of (1.1) will be expressed in terms of the following condition on a (constant coefficient) linear operator $M : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$:

(1.8) \[ \|\nabla \zeta\|_{L_2(\mathbb{R}^n)}^2 \leq \gamma \int_{\mathbb{R}^n} (M \nabla \zeta) : \nabla \zeta \quad \forall \zeta \in W^{1,2}(\mathbb{R}^n, \mathbb{R}^n). \]

Note that (1.8) is a Korn-type estimate, reducing to the classical Korn inequality for $M(F) = \text{sym}F$ and $\gamma = 2\sqrt{n}$. Naturally, (1.8) is equivalent to (2.1) which is the same estimate valid for all $\zeta \in W^{2,1}(U, \mathbb{R}^n)$ with $\zeta|_{\partial U} = 0$, on any fixed open, bounded $U \subset \mathbb{R}^n$.

It can be shown, via Fourier transform (see Lemma 2.2), that (1.8) is also equivalent to the strict positive definiteness of $M$ when restricted to the space of rank-one matrices $Q = a \otimes b$:

(1.9) \[ \forall a, b \in \mathbb{R}^n \quad |a|^2 |b|^2 = |a \otimes b|^2 \leq \gamma (M(a \otimes b) : a \otimes b). \]

We point out that the above condition resembles, naturally, the local well-posedness criterion for the inviscid elasticity system [15], where the validity of (1.9) for $M = DW(\nabla \xi_0)$ is equivalent to the hyperbolicity of the first-order system (1.1) with $Z = 0$.

The main result of this paper is the following:

**Theorem 1.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and let:

(1.10) \[ \xi_0 \in W^{2,p}(\Omega), \quad \xi_1 \in W^{2-2/p,p}(\Omega) \quad \text{for some} \quad p > n + 2, \]

satisfy:

\[ \inf_{X \in \Omega} \det \nabla \xi_0(X) > 0, \quad \xi_0(X) = X, \quad \xi_1(X) = 0 \quad \forall X \in \partial \Omega. \]

Assume that the viscous tensor $Z$ has the property that:

(1.11) \[ \forall X \in \Omega, \quad (1.8) \text{ holds with } M = D_Q Z(\nabla \xi_0(X), \nabla \xi_1(X)) \] and $\gamma$ independent of $X$.

Then there exists $T_{max} > 0$ such that the problem (1.1), (1.2), (1.3), (1.4) admits a unique regular solution $\xi \in W^{2,1}_p(\Omega \times (0, T)) \cap L^\infty((0, T), W^{2,p}_p(\Omega))$ with $\xi_t \in W^{2,1}_p(\Omega \times (0, T))$ for all $T < T_{max}$.

The proof of Theorem 1.1 will be given in sections 2, 3, 4. In section 5 we show that viscous stress tensors in (1.7) satisfy (1.11): for any initial data $\xi_0, \xi_1$ in case of the linear in $Q$ tensors $Z_0, Z'_0, Z''_0$, and for initial data enjoying additionally $\det \text{sym}(\nabla \xi_1(\nabla \xi_0)^{-1}) \neq 0$ in case of the nonlinear (in $Q$) tensors $Z_1, Z_2$, see Lemma 2.3. Thanks to this observation, Theorem 1.1 proves the mathematical well-posedness of a class of physically well-posed models.
With the same techniques of proof of Theorem 1.1, one can show that:

**Theorem 1.2.** Let $S$ be the solution operator of the problem (1.1) - (1.4) as described in Theorem 1.1, given by:

$$S(\xi_0, \xi_1) = (\xi, \xi_t),$$

$S : W^2_p(\Omega) \times W^{2-2/p}_p(\Omega) \to \left( W^{2,2}_p(\Omega \times (0,T)) \cap L_\infty((0,T), W^2_p(\Omega)) \right) \times W^{2,1}_p(\Omega \times (0,T)).$

Then $S$ is continuous.

We omit the proof and refer instead to standard texts [1, 17, 20, 21], or to an application of the same methods in the more current context as in Theorem 1.2 [9].

1.4. **Relation to previous works.** The dynamical viscoelasticity (1.1) has been the subject of vast studies in the last decades. For $Z(F, Q) = Q$ conflicting with the frame invariance (1.6) (ii), various results on existence, asymptotics and stability have been obtained in [2, 23, 24, 12]. For dimension $n = 1$, existence of solutions to (1.1) has been shown in [7, 4] for $Z$ depending nonlinearly on $Q$.

Existence and stability of viscoelastic shock profiles for a large class of models originating from (1.1) has been studied, among others, in [3, 5].

Existence of Young measure solutions to system (1.1) was shown in [10], without any additional assumptions on $Z$, but with condition (1.6) (iii) strengthened to the uniform dissipativity i.e: $Z(F, Q) \geq \gamma|Q|^2$. These measure-valued solutions were shown to be the unique classical weak solutions under the extra monotonicity assumption:

$$(1.12) \quad \langle Z(F_1, Q_1) - Z(F_2, Q_2) : Q_1 - Q_2 \rangle \geq \kappa |Q_1 - Q_2|^2 - l |F_1 - F_2|^2,$$

see also [25] for a treatment of slightly more general type of PDEs under the same condition. As noted in [10], (1.12) is incompatible with the balance of angular momentum (1.6) (i). In particular, (1.12) is not satisfied by any of the examples in (1.7), even $Z_0, Z'_0, Z''_0$ which enjoy condition (1.11) for any invertible $F = \nabla \xi_0(X)$ and any $Q = \nabla \xi_1(X)$.

From the theory of PDEs viewpoint, our present result is a rather straightforward application of the theory of nonlinear (quasilinear) parabolic systems. Namely, we apply the maximal regularity estimates to control the nonlinearities of the system (1.1).

We choose the $L_p$-framework in order to avoid technical difficulties, but a similar result and estimates are expected in the Besov spaces framework [9]. In a sense, our result is hence a consequence of the classical works of Ladyzhenskaya, Solonnikov and Uralceva [17], which has been further developed in [11, 20], and which is a powerful tool in the study of the parabolic-elliptic systems.

1.5. **Notation.** By $L_p(\Omega)$ we denote the space of functions integrable with respect to the Lebesgue measure, with $p$-th power. By $W^{k,l}_p(\Omega \times (0,T))$ for $k, l \in \mathbb{N}$ we denote the anisotropic Sobolev space defined by the norm:

$$\|u\|_{W^{k,l}_p(\Omega \times (0,T))} = \|u\|_{L_p(\Omega \times (0,T))} + \|\nabla^k u, \partial_t^l u\|_{L_p(\Omega \times (0,T))}.$$
where $\nabla^k$ is the $k$-th space derivative and $\partial_t$ is the time derivative. The isotropic version is given by:

$$\|u\|_{W^k_p(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla^k u\|_{L^p(\Omega)}.$$  

The space $W^{2-2/p}_p(\Omega)$ is the trace (in time) space of $W^{2,1}_p(\Omega \times (0,T))$. For further details we refer to [6].

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## 2. The Constant Coefficient Problem

The following auxiliary result will be needed in the proof of Theorem 1.1:

**Lemma 2.1.** Assume that $M : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ is a linear map satisfying the Korn-type inequality:

$$\|\nabla \zeta\|^2_{L^2(U)} \leq \gamma \int_U (M \nabla \zeta) : \nabla \zeta \quad \forall \zeta \in W^1_2(U, \mathbb{R}^n) \text{ with } \zeta|_{\partial U} = 0.

Then the solution to:

$$\begin{cases}
\zeta_t - \text{div} (M \nabla \zeta) = f & \text{in } U \times (0,T), \\
\zeta = 0 & \text{on } \partial U \times (0,T), \\
\zeta(0,\cdot) = \zeta_0 & \text{in } U
\end{cases}
$$

admits the following maximal regularity estimate:

$$\|\zeta_t, \nabla^2 \zeta\|_{L^p(U \times (0,T))} \leq C_{\gamma,p,U} \left( \|f\|_{L^p(U \times (0,T))} + \|\zeta_0\|_{W^{2-2/p}_p(U)} \right),
$$

where the dependence of $C$ on $U$ is uniform for any family of domains which are uniformly bilipschitz homeomorphic to each other after appropriate dilations.

Towards a proof of Lemma 2.1 note first that for $M = \text{Id}$, i.e. when (2.1) holds trivially, (2.3) is a classical maximal regularity parabolic estimate for the heat equation. When $M(F) = \text{sym} F$, i.e. when (2.1) reduces to Korn’s inequality, the proof of (2.3) is also immediate. For, take $\text{div}$ of the equation in (2.2), and note that $\text{div}^T \text{div}(\text{sym} \nabla \zeta) = \text{div} \left( \frac{1}{2} \Delta \zeta + \frac{1}{2} \nabla \text{div} \zeta \right) = \frac{1}{2} (\text{div} \Delta \zeta + \frac{1}{2} \Delta \text{div} \zeta) = \Delta \text{div} \zeta$ so that:

$$(\text{div} \zeta)_t - \Delta (\text{div} \zeta) = \text{div} f \quad \text{in } U \times (0,T).$$

By the maximal regularity estimate for the heat equation:

$$\|\nabla \text{div} \zeta\|_{L^p(U \times (0,T))} \leq C_{p,U} \left( \|f\|_{L^p(U \times (0,T))} + \|\text{div} \zeta_0\|_{W^{1-1/p}_p(U)} \right).$$
Now, (2.2) can be written as:
\[ \zeta_t - \frac{1}{2} \Delta \zeta = f + \frac{1}{2} \nabla \text{div} \zeta \quad \text{in} \ U \times (0, T). \]

We hence obtain:
\[ \| \zeta_t, \nabla^2 \zeta \|_{L^p(U \times (0,T))} \leq C_{p,U} \left( \| f + \frac{1}{2} \nabla \text{div} \zeta \|_{L^p(U \times (0,T))} + \| \text{div} \zeta_0 \|_{W^{2-1/p}_p(U)} \right), \]
which combined with (2.4) yields (2.3).

In the general case, Lemma 2.1 follows from the maximal regularity theory developed for parabolic initial-boundary value problems in [11]. Under the ellipticity condition (b) on page 98 in there (see also Definition 5.1), the estimate (2.3) is a consequence of Theorem 7.11. We now prove that condition (2.1) implies that the constant coefficient operator \(-\text{div} (M \nabla \zeta)\) has its spectrum contained in the proper sector of the complex plane, which immediately gives ellipticity in the sense of [11].

**Lemma 2.2.** Conditions (1.8), (2.1) and (1.9) are equivalent. Moreover, under any of these conditions the operator \(-\text{div} M \nabla (\cdot)\) is elliptic, i.e:

(2.5) \[ \text{spec}(-\text{div} M \nabla (\cdot)) \subset \{ z \in \mathbb{C} : \text{Re} \ z > 0, \text{ and arg } z \in [\alpha_*, \alpha^*] \text{ with } -\frac{\pi}{2} < \alpha_* < \alpha^* < \frac{\pi}{2} \}. \]

**Proof.** 1. Conditions (1.8) and (2.1) are equivalent in view of the density of \(C^\infty_c(\mathbb{R}^n)\) in \(W^{2,1}_1(\mathbb{R}^n)\). To include (1.9), we use linearity of Fourier transform and Plancherel’s identity:

\[ \| \nabla \zeta \|_{L^2(\mathbb{R}^n)}^2 = \| (\nabla \zeta)^\wedge \|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\hat{\zeta}(k)| \otimes k|^2 \, dk \]
\[ \int_{\mathbb{R}^n} \langle M(\nabla \zeta) : (\nabla \zeta)^\wedge \rangle = \int_{\mathbb{R}^n} \langle M((\nabla \zeta)^\wedge) : (\overline{\nabla \zeta}) \wedge \rangle = \int_{\mathbb{R}^n} \langle M(\hat{\zeta}(k) \otimes k) : (\overline{\hat{\zeta}(k) \otimes k}) \rangle \, dk. \]

Hence, (1.8) is equivalent to:

(2.6) \[ \forall \zeta \in W^{1}_1(\mathbb{R}^n) \quad \| \hat{\zeta} \otimes k \|_{L^2(\mathbb{R}^n)}^2 \leq \gamma \int_{\mathbb{R}^n} \langle M(\hat{\zeta} \otimes k) : (\overline{\hat{\zeta} \otimes k}) \rangle \, dk. \]

It is therefore clear that (1.9) implies (1.8). On the other hand, given \(k_0, a_0 \in \mathbb{R}^n\), consider: \(\hat{\zeta}_{\text{m}}(k) = \left( \rho_m^{1/2}(k - k_0) + \rho_m^{1/2}(k + k_0) \right) a_0\), where \(\rho_m\) is the standard radially symmetric mollifier supported in the ball \(B(0, 1/m)\). Applying (2.6) to \(\xi_n \in W^{1}_1(\mathbb{R}^n, \mathbb{R}^n)\) and passing to the limit \(m \to \infty\), yield (1.9) for the matrix \(Q = a_0 \otimes k_0\).

2. To prove (2.5), consider the eigenvalue problem:

\[ \lambda \zeta - \text{div}(M(\nabla \zeta)) = 0 \quad \text{in} \ \mathbb{R}^n, \]
which after passing to the Fourier variable \(k \in \mathbb{R}^n\) becomes:

\[
\lambda \hat{\zeta}(k) = M(\hat{\zeta}(k) \otimes k)k.
\]

Upon writing \(\lambda = \sigma|k|^2\), the problem (2.7) is equivalent to locating the eigenvalues \(\sigma\) of the family of linear operators \(\{M_k\}_{|k|=1}\), \(M_k : \mathbb{R}^n \to \mathbb{R}^n\) given by: \(M_k(a) = M(a \otimes k)k\). Recalling (1.9) we see that each \(M_k\) is strictly positive definite:

\[
M_k(a) \cdot a = \langle M(a \otimes k) : (a \otimes k) \rangle \geq \frac{|k|^2}{\gamma} |a|^2.
\]

Consequently, spectrum of every \(M_k\) satisfies \(\text{Re } \sigma > 0\). By continuity with respect to \(k\) which varies in the compact set \(|k| = 1\), we obtain the inclusion (2.5).

Finally, we have the following:

**Lemma 2.3.** The viscous stress tensors \(Z\) in (1.7) satisfy (2.1) with \(M = D_QZ(F_0, Q_0)\), for every \(F_0, Q_0\) with \(\det F_0 > 0\), in the following manner:

(i) \(Z''_0\) with \(\gamma = |F_0^{-1, T}|^2\).

(ii) \(Z'_0\) with \(\gamma = |F_0|^2(\det F_0)^{-1}\).

(iii) \(Z_0\) with \(\gamma = \frac{1}{2}|F_0|^2\).

If we additionally assume that \(\det \text{sym}(Q_0F_0^{-1}) \neq 0\) then we also have:

(iv) \(Z_1\) with \(\gamma = 2|F_0|^2|\text{sym}(Q_0F_0^{-1})|^{-1}|^2\).

(v) \(Z_2\) with \(\gamma = 2|F_0|^2|\text{sym}(Q_0F_0^{-1})|^{-1}|^4\).

The proof of Lemma 2.3 will be given in section 5. We now remark that in the proof of the main Theorem 1.1, Lemma 2.1 will be used to the operators \(M = MX = D_QZ(F_0, Q_0)\), at finitely many spacial points \(X \in \Omega\), where \(F_0 = \nabla \xi_0(X)\) and \(Q_0 = \nabla \xi_1(X)\). It is clear that when the initial data \(\xi_0, \xi_1\) with regularity (1.10) satisfy \(\det \nabla \xi > 0\) (or the two conditions \(\det \nabla \xi > 0\) and \(\det \text{sym}(\nabla \xi_1(\nabla \xi_0^{-1})^{-1}) \neq 0\) whenever required) then the constants \(\gamma\) in Lemma 2.3 have a common upper and lower bounds, independent of \(X\). Therefore, Lemma 2.1 and the estimate (2.3) may be used with a uniform constant \(C_{p,U}\), also independent of \(X\).

3. The main a-priori estimate

Given \(\xi_0, \xi_1\) as in Theorem 1.1 let \(\bar{\xi}_1 \in W^{2, 1}_p(\Omega \times \mathbb{R}_+\) be the solution to:

\[
\begin{cases}
(\bar{\xi}_1)_t - \Delta \bar{\xi}_1 = 0 & \text{in } \Omega \times \mathbb{R}_+,
\bar{\xi}_1 = 0 & \text{on } \partial \Omega \times \mathbb{R}_+,
\bar{\xi}_1(0, \cdot) = \xi_1 & \text{in } \Omega,
\end{cases}
\]

Define the extension \(\tilde{\xi}\) of \(\xi_0\), so that \(\partial_t \tilde{\xi} = \bar{\xi}_1\):

\[
\tilde{\xi}(t, x) = \xi_0(x) + \int_0^t \bar{\xi}_1(s, x) \, ds.
\]
By continuity, it is clear that: \( \inf_{\Omega \times (0,T)} \det \nabla \xi > 0 \) for \( T \) sufficiently small. We define:
\[
D = D(T) = \| \xi_{tt}, \nabla^2 \xi_t \|_{L^p(\Omega \times (0,T))}
\]
and note that:
\[
\lim_{T \to 0} D(T) = 0.
\]

**Lemma 3.1.** Let \( \xi_0, \xi_1 \) be as in Theorem 1.1 and assume that:

\[
\text{for every } X \in \Omega \text{ (1.8) holds with } M = D_Q \mathcal{Z}(\nabla \xi_0(X), \nabla \xi_1(X)).
\]

Let \( \xi \in W^{2,2}_p(\Omega \times (0,T)) \) with \( \xi_t \in W^{2,1}_p(\Omega \times (0,T)) \) be a solution to the problem (1.1), (1.2), (1.3), (1.4), and denote:
\[
\Theta = \Theta(T) = \| (\xi - \bar{\xi})_{tt}, \nabla^2 (\xi - \bar{\xi})_t \|_{L^p(\Omega \times (0,T))},
\]
where \( \bar{\xi} \) is as in (3.1). Then, there exists \( T_{00} < T_0 \) and a constant \( C \), both depending only on \( \xi_0 \) and \( \xi_1 \) (and, naturally, on \( \bar{\xi}_0 \) and \( \bar{\xi}_1 )) such that for every \( T < T_{00} \) we have:
\[
\Theta \leq C(T^{1/p} + D + (T^{1/p} + D)\Theta + \Theta^2 + \Theta^4).
\]
In particular:
\[
\Theta(T) \leq C \quad \forall T < T_{00}.
\]

Before we give the proof of the lemma, we gather below some standard inequalities that will be frequently used for different functions: \( u \) defined on \( \Omega \times (0,T) \), and \( w \) defined on \( \Omega \). We always assume that \( T < 1 \).

\[
\| w \|_{W^2_p(\Omega)} \leq C_p,\Omega \| \Delta w \|_{L^p(\Omega)} \quad \text{when } w|_{\partial\Omega} = 0,
\]
\[
\sup_{t \in (0,T)} \| u(t,\cdot) \|_{W^{2-2/p}_p(\Omega)} \leq C_p,\Omega \left( \| u_t - \Delta u \|_{L^p(\Omega \times (0,T))} + \| u(0,\cdot) \|_{W^{2-2/p}_p(\Omega)} \right)
\]
when \( u|_{\partial\Omega \times (0,T)} = 0 \),
\[
\| w \|_{L^\infty(\Omega)} \leq C_p,\Omega \| w \|_{W^{1-1/p}_p(\Omega)}, \quad \text{in fact: } \| w \|_{C^{0,\alpha}(\Omega)} \leq C_{\alpha,p,\Omega} \| w \|_{W^{1-1/p}_p(\Omega)},
\]
\[
\| \nabla u \|_{C^{0,\alpha/2}} \leq C_{\alpha,p,\Omega} \| u \|_{W^{2,1}_p(U \times (0,T))}.
\]
The inequality (3.5) is the usual elliptic estimate [14], and (3.6) is the parabolic estimate from [6]. The Morrey embedding gives (3.7) for \( p > n + 2 \) [14], while (3.8) follows from the embedding \( \nabla W^{2,1}_p(\Omega \times (0,T)) \subset L^\infty(\Omega \times (0,T)) \), also valid for \( p > n + 2 \) [17]. We stress that the constants \( C \) in all the above bounds are universal, i.e. they are independent of \( T \). Additionally, the dependence of \( C \) in (3.8) on \( U \subset \Omega \) is uniform for any family of domains which are uniformly bilipschitz homeomorphic to each other after appropriate dilations.
We further remark the following simple bound:

\[
\|\nabla^2 u\|_{L_p(\Omega \times (0, T))} = \left( \int_{\Omega} \int_0^T \left| \int_0^t \nabla^2 u_t \, ds + \nabla^2 u(0, \cdot) \right|^p \, dt \, dX \right)^{1/p} 
\]

(3.9)

\[
\leq T^{1/p} \left( \int_{\Omega} \int_0^T T^{p'/p} \left| \nabla^2 u_t \right|^p \, dt \, dX \right)^{1/p} + T^{1/p} \|\nabla^2 u(0, \cdot)\|_{L_p(\Omega)}.
\]

Let now \( \xi \) and \( \bar{\xi} \) be as in Lemma 3.1. Using (3.6) to \( (\xi - \bar{\xi})_t \), we obtain:

\[
\sup_{t \in (0, T)} \| (\xi - \bar{\xi})_t (t, \cdot) \|_{L_p(\Omega)} + \sup_{t \in (0, T)} \| \nabla (\xi - \bar{\xi})_t (t, \cdot) \|_{L_p(\Omega)} 
\]

(3.10)

\[
\leq C \left( \| (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} + \| \nabla^2 (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} \right) \leq C \Theta,
\]

and consequently:

\[
\| (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} + \| \nabla (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} \leq C T^{1/p} \Theta.
\]

By (3.7), (3.9) used to \( \xi - \bar{\xi} \), and (3.9), (3.11) we get:

\[
\| \nabla (\xi - \bar{\xi}) \|_{L^\infty(\Omega \times (0, T))} = \sup_{t \in (0, T)} \| \nabla (\xi - \bar{\xi})_t (t, \cdot) \|_{L_\infty(\Omega)} \leq C \left( \| (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} + \| \nabla^2 (\xi - \bar{\xi})_t \|_{L_p(\Omega \times (0, T))} \right) \leq C T^{1/p} \Theta.
\]

Let now \( \xi \) and \( \bar{\xi} \) be as in Lemma 3.1. Using (3.6) to \( (\xi - \bar{\xi})_t \), we directly obtain:

\[
\| (\xi - \bar{\xi})_t \|_{L^\infty(\Omega \times (0, T))} + \| \nabla (\xi - \bar{\xi})_t \|_{L^\infty(\Omega \times (0, T))} \leq C \Theta.
\]

In all the above inequalities (3.10) - (3.13), we write \( \Theta = \Theta(T) \). The constant \( C \) depends only on the initial data of the problem \( \xi_0, \xi_1 \) (in addition to its dependence on \( \Omega \) and \( p \)).

**Proof of Lemma 3.1**

We will always assume that \( T < 1 \). Note that for \( T < T_0 \) sufficiently small, the constraint (1.4) is a consequence of the same constraint on the initial data \( \xi_0 \), by continuity. Likewise:

\[
\| DZ(\nabla \xi, \nabla \bar{\xi})_t, D^2 Z(\nabla \xi, \nabla \bar{\xi}), D^3 Z(\nabla \xi, \nabla \bar{\xi}) \|_{L^\infty(\Omega \times (0, T))} \leq C.
\]

(3.14)

1. The system (1.1) can be rewritten as:

\[
(\xi - \bar{\xi})_t - \text{div} \left( Z(\nabla \xi, \nabla \bar{\xi}) - Z(\nabla \xi, \nabla \bar{\xi}) \right) = \text{div} \left( DW(\nabla \xi) \right) + \text{div} \left( Z(\nabla \bar{\xi}, \nabla \bar{\xi}) \right) - \bar{\xi}_t
\]

and further, it has the form:

\[
(\xi - \bar{\xi})_t - \text{div} \left( DQ Z(\nabla \xi, \nabla \bar{\xi}) \right) \nabla (\xi - \bar{\xi})_t = F[\xi, \bar{\xi}],
\]

(3.15)
We shall now prove the bound:

\begin{equation}
(F[\xi, \bar{\xi}] = \text{div}(DW(\nabla \xi)) + \text{div}(Z(\nabla \bar{\xi}, \nabla \tilde{\xi})) - \bar{\xi}_{tt}
+ \text{div}(Z(\nabla \xi, \nabla \xi_t) - Z(\nabla \bar{\xi}, \nabla \bar{\xi}_t))
+ \text{div}(Z(\nabla \xi, \nabla \xi_t) - Z(\nabla \tilde{\xi}, \nabla \tilde{\xi}_t) - D_Q Z(\nabla \xi, \nabla \xi_t) \nabla(\xi - \tilde{\xi}_t))
= \text{div}(DW(\nabla \xi)) + \text{div}(Z(\nabla \xi, \nabla \xi_t)) - \bar{\xi}_{tt}
+ \text{div}(D_F Z(\nabla \bar{\xi}, \nabla \bar{\xi}_t) \nabla(\xi - \tilde{\xi}))
+ \text{div}\left(\int_0^1 (1 - s) D_{FF}^2 Z(s \nabla \xi + (1 - s) \nabla \xi_t) (\nabla(\xi - \tilde{\xi}) \otimes \nabla(\xi - \tilde{\xi})) \, ds\right)
+ \text{div}\left(\int_0^1 (1 - s) D_{QQ}^2 Z(\nabla \bar{\xi}, s \nabla \xi_t + (1 - s) \nabla \tilde{\xi}_t) (\nabla(\xi - \tilde{\xi})_t \otimes \nabla(\xi - \tilde{\xi}_t)) \, ds\right).\end{equation}

We shall now prove the bound:

\begin{equation}
\|F[\xi, \bar{\xi}]\|_{L_p(\Omega \times (0, T)} \leq C \left(T^{1/p} + D + (T^{1/p} + D) \Theta + \Theta^2 + \Theta^4\right).
\end{equation}

By (3.9) and (3.12) it follows that:

\begin{equation}
\|\text{div}(DW(\nabla \xi))\|_{L_p(\Omega \times (0,T)} \leq \|D^2 W(\nabla \xi)\|_{L\infty(\Omega \times (0,T)} \|\nabla^2 \xi\|_{L_p(\Omega \times (0,T)}
\leq (\|D^2 W(\nabla \xi)\|_{L\infty} + C \|\nabla(\xi - \bar{\xi})\|_{L\infty}) T^{1/p}.
\end{equation}

\begin{equation}
\leq C(1 + T^{1/p} \Theta) + \|\nabla^2 \xi_t\|_{L_p} + \|\nabla^2 \xi_0\|_{L_p(\Omega)}
\leq C(1 + T^{1/p} \Theta)(1 + \Theta + D) \leq CT^{1/p}(1 + \Theta) + C(1 + \Theta + D).
\end{equation}

Using (3.11) and (3.9) to \(\tilde{\xi}\), we obtain:

\begin{equation}
\|\text{div}(Z(\nabla \bar{\xi}, \nabla \bar{\xi}_t))\|_{L_p(\Omega \times (0,T)}
\leq \|D Z(\nabla \bar{\xi}, \nabla \bar{\xi}_t)\|_{L\infty(\Omega \times (0,T)} \left(\|\nabla^2 \bar{\xi}\|_{L_p(\Omega \times (0,T)} + \|\nabla^2 \bar{\xi}_t\|_{L_p(\Omega \times (0,T)}\right)
\leq C(T^{1/p} \|\nabla^2 \bar{\xi}_t\|_{L_p} + T^{1/p} \|\nabla^2 \xi_0\|_{L_p(\Omega)} + \|\nabla^2 \bar{\xi}_t\|_{L_p}) \leq C(T^{1/p} + D).
\end{equation}
By (3.14), (3.13), (3.9), (3.12) we get:

$$
\begin{align*}
(3.20) \quad & \| \text{div} \left( D F Z (\nabla \xi, \nabla \xi_t) \nabla (\xi - \tilde{\xi}) \right) \|_{L_p(\Omega \times (0,T))} \\
& \leq \| D_{FF}^2 Z (\nabla \xi, \nabla \xi_t) \|_{L_{\infty}} \left( \| \nabla^2 \xi \|_{L_p(\Omega \times (0,T))} + \| \nabla^2 \xi_t \|_{L_p(\Omega \times (0,T))} \right) \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \\
& \quad + \| D_F Z (\nabla \xi, \nabla \xi_t) \|_{L_{\infty}} \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p(\Omega \times (0,T))} \\
& \leq C \left( 1 + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \right) \left( \| \nabla^2 \xi \|_{L_p} + \| \nabla^2 \xi_t \|_{L_p} + \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} \right) \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \\
& \quad + C \left( 1 + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \right) \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} \\
& \leq C(1 + \Theta) (T^{1/p} + \Theta + D) T^{1/p} \Theta + C(1 + \Theta) T^{1/p} \Theta \\
& \leq C T^{1/p} (1 + \Theta + D) \Theta (1 + \Theta).
\end{align*}
$$

and:

$$
(3.21) \quad \| \text{div} \left( \int_0^1 (1-s) D_{FF}^2 Z (s \nabla \xi + (1-s) \nabla \tilde{\xi}, \nabla \xi_t) (\nabla (\xi - \tilde{\xi}) \otimes \nabla (\xi - \tilde{\xi})) \, ds \right) \|_{L_p(\Omega \times (0,T))}
\leq \sup_{s \in [0,1]} \| \text{div} \left( D_{FF}^2 Z (s \nabla \xi + (1-s) \nabla \tilde{\xi}, \nabla \xi_t) (\nabla (\xi - \tilde{\xi}) \otimes \nabla (\xi - \tilde{\xi})) \right) \|_{L_p(\Omega \times (0,T))}
\leq \sup_{s \in [0,1]} \left[ \| D^3 Z (s \nabla \xi + (1-s) \nabla \tilde{\xi}, \nabla \xi_t) \|_{L_{\infty}} \left( \| \nabla^2 \xi \|_{L_p} + \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} + \| \nabla^2 \xi_t \|_{L_p} \right) \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}}^2 \\
+ \| D^2 Z (s \nabla \xi + (1-s) \nabla \tilde{\xi}, \nabla \xi_t) \|_{L_{\infty}} \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \right] \\
\leq C \left( 1 + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \right) \left( T^{1/p} + \Theta + D \right) T^{2/p} \Theta^2 \\
& \quad + C \left( 1 + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} + \| \nabla (\xi - \tilde{\xi}) \|_{L_{\infty}} \right) T^{2/p} \Theta^2 \\
\leq C T^{1/p} (1 + \Theta + D) \Theta^2 (1 + \Theta).
\end{align*}
$$

In the same manner, we see that:

$$
(3.22) \quad \| \text{div} \left( \int_0^1 (1-s) D_{QQ}^2 Z (s \nabla \xi_t + (1-s) \nabla \tilde{\xi}_t) (\nabla (\xi - \tilde{\xi})_t \otimes \nabla (\xi - \tilde{\xi})_t) \, ds \right) \|_{L_p(\Omega \times (0,T))}
\leq C \left( 1 + \| \nabla (\xi - \tilde{\xi})_t \|_{L_{\infty}} \right) \left( \| \nabla^2 \xi \|_{L_p} + \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} + \| \nabla^2 \xi \|_{L_p} \right) \| \nabla (\xi - \tilde{\xi})_t \|_{L_{\infty}}^2 \\
& \quad + C \left( 1 + \| \nabla (\xi - \tilde{\xi})_t \|_{L_{\infty}} \right) \| \nabla^2 (\xi - \tilde{\xi}) \|_{L_p} \| \nabla (\xi - \tilde{\xi})_t \|_{L_{\infty}} \\
\leq C(1 + \Theta) (T^{1/p} + \Theta + D) \Theta^2 + C(1 + \Theta) \Theta \\
\leq C(T^{1/p} + \Theta + D) \Theta (1 + \Theta)^2.
\end{align*}
$$
Combining (3.18) – (3.22), the bound (3.17) follows if only $T < 1$, ensuring $D(T) < 1$ by (3.2).

2. We will now work with the localizations of the system (3.15). Let $\{B_k\}_{k=1}^N$ be a covering of $\Omega$ by a finite number $N = N(r)$ of balls $B_k = B(X_k, r)$ with centers $X_k \in \Omega$ and radius $r < 1$. This family of coverings (parametrized by $r$) should such that all the sets $2B_k \cap \Omega$ are uniformly bilipschitz homeomorphic to each other after appropriate dilations and that the covering numbers of $\{2B_k \cap \Omega\}_k$ are independent of $r$.

Let $\pi_k : \mathbb{R}^n \to [0, 1]$ be smooth cut-off functions satisfying: $\pi_k = 1$ on $B_k$, and $\pi_k = 0$ on $\mathbb{R}^n \setminus 2B_k$ where $2B_k = B(X_k, 2r)$, and $\|\nabla^a \pi_k\|_{L_\infty} \leq C r^{-|\alpha|}$. After multiplying (3.15) by $\pi_k$, we obtain:

$$(3.23) \quad (\pi_k (\xi - \bar{\xi}))_{tt} - \text{div} \left( D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \nabla (\pi_k (\xi - \bar{\xi})_t) \right) = \pi_k F[\xi, \bar{\xi}] + G_k[\xi, \bar{\xi}],$$

where:

$$G_k[\xi, \bar{\xi}] = \pi_k \text{div} \left( [D_Q Z (\nabla \xi, \nabla \xi_t) - D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k))] \nabla (\xi - \bar{\xi})_t \right)$$

$$- \left( D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \nabla (\xi - \bar{\xi})_t \right) \nabla \pi_k$$

$$- \text{div} \left( D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k)) ((\xi - \bar{\xi})_t \otimes \nabla \pi_k) \right).$$

We shall now prove the bound:

$$(3.24) \quad \|G_k[\xi, \bar{\xi}]\|_{L_p(2B_k \times (0,T))} \leq C (r^\alpha + T^{1/2}) \|\nabla (\xi - \bar{\xi})_t\|_{L_p(2B_k \times (0,T))}$$

$$+ C \left( 1 + \frac{1}{r^2} \right) (T^{1/p} + D) \Theta(1 + \Theta).$$

Using (3.11), we obtain:

$$(3.25) \quad \| (D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \nabla (\xi - \bar{\xi})_t) \nabla \pi_k \|_{L_p(2B_k \times (0,T))} \leq \frac{C}{r} \|\nabla (\xi - \bar{\xi})_t\|_{L_p} \leq \frac{C}{r} T^{1/p} \Theta^2.$$

Likewise:

$$(3.26) \quad \| \text{div} (D_Q Z (\nabla \xi_0(X_k), \nabla \xi_1(X_k)) ((\xi - \bar{\xi})_t \otimes \nabla \pi_k)) \|_{L_p(2B_k \times (0,T))}$$

$$\leq \frac{C}{r^2} \|\nabla (\xi - \bar{\xi})_t\|_{L_p} + \frac{C}{r^2} \|\xi - \bar{\xi}\|_{L_p} \leq \frac{C}{r^2} T^{1/p} \Theta.$$
Finally, by (3.14), (3.9), (3.13) and (3.8) we have:

\[ (3.27) \]
\[ \| \pi_k \text{ div} \left( [D_Q Z(\nabla \xi, \nabla \xi_t) - D_Q Z(\nabla \xi_0(X_k), \nabla \xi_1(X_k))] \nabla (\xi - \xi)_t \right) \|_{L^p(2B_k \times (0,T))} \]
\[ \leq C \left( \| \nabla^2 \xi \|_{L^p} + \| \nabla^2 \xi_t \|_{L^p} \right) \| \nabla (\xi - \xi)_t \|_{L^\infty} \]
\[ + \| D_Q Z(\nabla \xi, \nabla \xi_t) - D_Q Z(\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \|_{L^\infty(2B_k \times (0,T))} \| \pi_k \nabla^2 (\xi - \xi)_t \|_{L^p(2B_k \times (0,T))} \]
\[ \leq C(T^{1/p} + D)\Theta \]
\[ + C \left( \| \nabla \xi - \nabla \xi_0(X_k) \|_{L^\infty(2B_k \times (0,T))} + \| \nabla \xi_t - \nabla \xi_1(X_k) \|_{L^\infty(2B_k \times (0,T))} \right) \cdot \| \pi_k \nabla^2 (\xi - \xi)_t \|_{L^p(2B_k \times (0,T))} \]
\[ \leq C(T^{1/p} + D)\Theta + C(r^\alpha + T^{\alpha/2})\| \xi \|_{W^{2,1}_p(\Omega \times (0,T))} \]
\[ \cdot \| \pi_k \nabla^2 (\xi - \xi)_t \|_{L^p(2B_k \times (0,T))} \]

Combining (3.26) – (3.27) and noting that \[ \| \xi \|_{W^{2,1}_p(\Omega \times (0,T))} \leq C(\| \xi_0 \|_{W^{p}(\Omega)} + \| \xi_1 \|_{W^{2,2-2/p}(\Omega)}) \]
for \( T \) small enough we conclude (3.24) in view of (3.2).

We now use Lemma 2.1 to the problem (3.23), i.e. we set \( \zeta = \pi_k(\xi - \xi)_t, M = D_Q Z(\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \) where \( U = 2B_k \cap \Omega \) is the uniform constant from the assumption (1.11). Indeed, it is easy to notice that if (2.1) holds for some set \( \gamma \) then it holds with the same constant \( \gamma \) on every open subset \( U_1 \subset U \). By (2.3) we now obtain:

\[ \| \pi_k(\xi - \xi)_t, \nabla^2(\pi_k(\xi - \xi)_t) \|_{L^p(2B_k \times (0,T))} \leq C\| \pi_k F[\xi, \xi], G_k[\xi, \xi] \|_{L^p(2B_k \times (0,T))}. \]

Summing over finitely many \( k : 1 \ldots N \), we get in view of (3.24):

\[ \| (\xi - \xi)_t, \nabla^2(\xi - \xi)_t \|_{L^p(\Omega \times (0,T))} \leq C(r^\alpha + T^{\alpha/2})\| \nabla^2(\xi - \xi)_t \|_{L^p(\Omega \times (0,T))} \]
\[ + CN^{1/p} \left( (1 + 1/r)(T^{1/p} + D)\Theta + 1 \right) \| F[\xi, \xi] \|_{L^p(\Omega \times (0,T))} \]

where, again, \( C \) depends only on the covering number of \( \{ B_k \}_{k=1}^N \), on \( \bar{\xi} \), \( p \) and \( \Omega \), but not on \( r, N, T \) or \( \Theta \). Consequently, for \( r \) and \( T \) sufficiently small, we arrive at:

\[ \Theta \leq CN^{1/p} \left( 1 + \frac{1}{r} \right) (T^{1/p} + D + T^{1/p} + D)\Theta + \Theta^2 + \Theta^4 \]
in virtue of (3.17). This concludes the proof of (3.3).

3. To prove (3.4), consider the functions:

\[ g(\Theta) = \Theta \quad \text{and} \quad g_\epsilon(\Theta) = C(\epsilon + \epsilon\Theta + \Theta^2 + \Theta^4), \]

where \( C \) is a given constant and \( \epsilon > 0 \) is a small parameter.

Clearly, \( g(0) < g_\epsilon(0) \) for every \( \epsilon \). Take now:

\[ (3.28) \]
\[ \epsilon < \min\left\{ \frac{1}{16C^2}, \frac{1}{4C}, 1 \right\} \]
and let $\Theta_0 \in (4C_\epsilon, \frac{1}{\epsilon})$ with $\Theta_0 < 1$. Then: $\max\{C_\epsilon, C_\epsilon^2\Theta_0, C\Theta_0^2, C\Theta_0^4\} < \frac{\Theta_0}{4}$ and hence $g(\Theta_0) > g_1(\Theta_0)$.

Taking now $T_{00}$ so small that, in addition to other requirements imposed in the course of the proof, $\epsilon = T^{1/p} + D$ satisfies (3.28), we obtain that for every $T \in [0, T_{00})$ the quantity $\Theta(T)$ must stay below $\Theta_0$, in virtue of continuity of the function $T \mapsto \Theta(T)$ and $\Theta(0) = 0$. This ends the proof of (3.4) and of Lemma 3.1. 

4. A proof of Theorem 1.1

We only outline the proof of Theorem 1.1 which is standard, and we point to its most important steps. Let $\bar{\xi}$ be as in (3.1). Recall that the system (1.1) can be rewritten as:

\begin{equation}
(\xi - \bar{\xi})_{tt} - \text{div} \left(DQZ(\nabla \bar{\xi}, \nabla \xi)(\nabla (\xi - \bar{\xi})_t)\right) = F[\xi, \bar{\xi}],
\end{equation}

where the right hand side $F[\xi, \bar{\xi}]$ is given in (3.16). We shall seek a solution $\xi$ as the fixed point of the operator:

\begin{equation}
T(\tilde{\xi} - \bar{\xi}) = \xi - \bar{\xi}, \quad \xi \text{ is a solution to:}
\end{equation}

\begin{equation}
(\xi - \bar{\xi})_{tt} - \text{div} \left(DQZ(\nabla \bar{\xi}, \nabla \xi)(\nabla (\xi - \bar{\xi})_t)\right) = \tilde{F}[\tilde{\xi}, \bar{\xi}],
\end{equation}
in the Banach space:

\begin{equation}
E_{\Omega, T} = \left\{ u \in L_p(\Omega \times (0, T)); u(0, \cdot) = 0, u_t(0, \cdot) = 0, u|_{\partial \Omega \times (0, T)} = 0, \right. \\
u_{tt}, \nabla^2 u_t \in L_p(\Omega \times (0, T)) \},
\end{equation}

equipped with the norm:

$$
\|u\|_{E_{\Omega, T}} = \Theta[u](T) = \|u_{tt}, \nabla^2 u_t\|_{L_p(\Omega \times (0, T))}.
$$

1. Following calculations as in the proof of Lemma 3.1 it results that:

$$
\forall \tilde{\xi} - \bar{\xi} \in E_{\Omega, T} \quad F[\tilde{\xi}, \bar{\xi}] \in L^p(\Omega \times (0, T)).
$$

2. Integrating (4.2) against $(\xi - \bar{\xi})_t$ on $\Omega \times (0, T)$ and using the estimate (4.5) in Lemma 4.1 below with $\zeta = (\xi - \bar{\xi})_t$, we obtain:

$$
sup_{t \in (0, T)} \|(\xi - \bar{\xi})_t(t, \cdot)\|_{L_2(\Omega)}^2 + \|\nabla (\xi - \bar{\xi})_t\|_{L_2(\Omega \times (0, T))}^2
\leq C \int_0^T \int_{\Omega} F^2[\tilde{\xi}, \bar{\xi}] \, dx \, dt + C \|(\xi - \bar{\xi})_t\|_{L_2(\Omega \times (0, T))}^2
\leq C \int_0^T \int_{\Omega} F^2[\tilde{\xi}, \bar{\xi}] \, dx \, dt + CT \sup_{t \in (0, T)} \|(\xi - \bar{\xi})_t(t, \cdot)\|_{L_2(\Omega)}^2,
$$
which implies the following energy estimate, for $T$ small:

\[
\sup_{t \in (0, T)} \| (\xi - \tilde{\xi})_t(t, \cdot) \|_{L^2(\Omega)}^2 + \| \nabla (\xi - \tilde{\xi})_t \|_{L^2(\Omega \times (0, T))}^2 \leq C \int_0^T \int_{\Omega} F^2(\tilde{\xi}, \tilde{\xi}) \, dx \, dt.
\]

In virtue of (4.4), the Galerkin construction of the approximants:

\[
(\xi_N - \tilde{\xi})_t = \sum_{k=1}^N a^k_N(t) w_l(x),
\]

where \( \{w_l\}_{l=1}^\infty \) is an orthonormal base of \( W^1_2(\Omega) \), yields existence of a weak solution \( \xi - \tilde{\xi} = \lim_{N \to \infty} (\xi_N - \tilde{\xi}) \) of the problem (4.2), with: \( (\xi - \tilde{\xi})_t \in L^\infty((0, T), L^2(\Omega)) \) and \( \nabla (\xi - \tilde{\xi})_t \in L^2(\Omega \times (0, T)) \).

3. A modification of arguments in section 3 implies that the weak solution \( \xi \) is actually regular in the class determined by (4.3), i.e:

\[
\xi - \tilde{\xi} \in E_{\Omega, T}.
\]

Moreover, for every small \( \epsilon > 0 \):

\[
\text{if } \Theta[\tilde{\xi} - \tilde{\xi}](T) \leq \epsilon \quad \text{then } \quad \Theta[\xi - \tilde{\xi}](T) \leq \epsilon.
\]

4. In now suffices to show that the map \( T \) is a contraction in some ball \( B_\epsilon \subset E_{\Omega, T} \).

This is done by applying methods of [3] to the system:

\[
(\xi_1 - \xi_2)_t - \text{div} (D_Q Z(\nabla \tilde{\xi}, \nabla \tilde{\xi}) \nabla (\xi_1 - \xi_2)) = F[\tilde{\xi}_1, \tilde{\xi}] - F[\tilde{\xi}_2, \tilde{\xi}],
\]

where \( T(\tilde{\xi}_1 - \tilde{\xi}) = \xi_1 - \tilde{\xi} \). For \( \epsilon > 0 \) sufficiently small it follows that:

\[
\Theta[\xi_1 - \xi_2](T) \leq \frac{1}{2} \Theta[\tilde{\xi}_1 - \tilde{\xi}_2](T),
\]

which completes the proof. 

\[\blacksquare\]

The key role above was played by the following estimate:

**Lemma 4.1.** Let \( T < T_0 \) be sufficiently small and assume that \( Z \) satisfies (1.11). Then for every \( \zeta \in W^{2,1}_2(\Omega \times (0, T)) \) such that \( \zeta(0, \cdot) = 0 \) and \( \zeta_{|\partial \Omega \times (0, T)} = 0 \), there holds:

\[
\| \nabla \zeta \|^2_{L^2(\Omega \times (0, T))} \leq 4\gamma \int_0^T \int_{\Omega} D_Q Z(\nabla \tilde{\xi}, \nabla \tilde{\xi}) \nabla \zeta : \nabla \zeta \, dx \, dt + C\| \zeta \|^2_{L^2(\Omega \times (0, T))},
\]

with constant \( C \) independent of \( \zeta \).

**Proof.** Consider a covering \( \{B_k\}_{k=1}^N \) of \( \Omega \) by a finite number \( N = N(r) \) of balls \( B_k = B(X_k, r) \) with centers \( X_k \in \Omega \) and radius \( r > 0 \). This family of coverings (parametrized by \( r \)) should be such that their covering numbers are uniform in \( r \). Let \( \{\pi_k\}_{k=1}^N \) be a partition of unity subject to \( \{B_k\} \).
For a fixed $t \in (0, T)$, with a slight abuse of notation, we shall still write $\zeta = \zeta(t, \cdot) \in W^1_2(\Omega)$. By (1.11) it follows that:

$$
\int_{\Omega} \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \nabla \zeta : \nabla \zeta \rangle
$$

(4.6) where we accumulated the error terms in:

$$
\sum_{k=1}^N \int_{B_k} \langle DQ Z(\nabla \xi_0(X_k), \nabla \xi_1(X_k)) \nabla (\pi_k^{1/2} \zeta) : \nabla (\pi_k^{1/2} \zeta) \rangle \, dx + \sum_{B_k} E_k[\xi, \bar{\xi}]
$$

$$
\geq \frac{1}{\gamma} \sum_{k=1}^N \|\nabla (\pi_k^{1/2} \zeta)\|_{L^2(B_k)}^2 + \sum_{B_k} E_k[\xi, \bar{\xi}],
$$

where we accumulated the error terms in:

$$
E_k[\xi, \bar{\xi}] = \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \pi_k^{1/2} \nabla \zeta : \pi_k^{1/2} \nabla \zeta \rangle
$$

$$
- \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \nabla (\pi_k^{1/2} \zeta) : \nabla (\pi_k^{1/2} \zeta) \rangle
$$

$$
+ \langle [DQ Z(\nabla \xi_0, \nabla \xi_1) - DQ Z(\nabla \xi_0(X_k), \nabla \xi_1(X_k))] \nabla (\pi_k^{1/2} \zeta) : \nabla (\pi_k^{1/2} \zeta) \rangle.
$$

Hence:

$$
\left| \int_{B_k} E_k[\xi, \bar{\xi}] \, dx \right| \leq \left| \int_{B_k} \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \pi_k^{1/2} \nabla \zeta : (\zeta \otimes \nabla \pi_k^{1/2}) \rangle \right|
$$

$$
+ \left| \int_{B_k} \langle DQ Z(\nabla \xi_0, \nabla \xi_1)(\zeta \otimes \nabla \pi_k^{1/2}) : \nabla (\pi_k^{1/2} \zeta) \rangle + C \|\nabla (\pi_k^{1/2} \zeta)\|_{L^2(B_k)}^2 \right|
$$

$$
\leq C \|\zeta\|_{W^2_2(\Omega)} \|\zeta\|_{L^2(\Omega)} + C \|\nabla (\pi_k^{1/2} \zeta)\|_{L^2(\Omega)}^2,
$$

where $C$ is a universal constant depending only on the initial data and $Z$, while the constant $C_r$ depends on the covering $\{B_k\}$. Taking $r$ small, so that $C_r < 1/(2\gamma)$, by (4.6) we now arrive at:

$$
\int_{\Omega} \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \nabla \zeta : \nabla \zeta \rangle
$$

$$
\geq \frac{1}{2\gamma} \sum_{k=1}^N \|\nabla (\pi_k^{1/2} \zeta)\|_{L^2(B_k)}^2 - C \|\zeta\|_{W^2_2(\Omega)} \|\zeta\|_{L^2(\Omega)}
$$

$$
\geq \frac{1}{2\gamma} \sum_{k=1}^N \|\pi_k^{1/2} \nabla \zeta\|_{L^2(B_k)}^2 - C \|\zeta\|_{W^2_2(\Omega)} \|\zeta\|_{L^2(\Omega)}
$$

$$
\geq \frac{1}{2\gamma} \|\nabla \zeta\|_{L^2(\Omega)}^2 - C \left( \epsilon \|\nabla \zeta\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|\nabla \zeta\|_{L^2(\Omega)}^2 \right),
$$

by Young’s inequality. With $\epsilon$ sufficiently small, it yields:

$$
\|\nabla \zeta\|_{L^2(\Omega)}^2 \leq 3\gamma \int_{\Omega} \langle DQ Z(\nabla \xi_0, \nabla \xi_1) \nabla \zeta : \nabla \zeta \rangle + C \|\zeta\|_{L^2(\Omega)}^2.
$$

(4.7)
Integrating in $t$, we eventually arrive at:

$$
\| \nabla \zeta \|_{L^2(\Omega \times (0,T))}^2 \leq 3\gamma \int_0^T \int_\Omega \langle D_\xi Z(\nabla \xi_0, \nabla \xi_1) \nabla \zeta \colon \nabla \zeta \rangle \, dx \, dt + C \| \zeta \|_{L^2(\Omega \times (0,T))}^2
$$

$$
\leq 3\gamma \int_0^T \int_\Omega \langle D_\xi Z(\nabla \bar{\xi}, \nabla \bar{\xi}_t) \nabla \zeta \colon \nabla \zeta \rangle \, dx \, dt
$$

$$
+ CT \| \nabla \zeta \|_{L^2(\Omega \times (0,T))}^2 + C \| \zeta \|_{L^2(\Omega \times (0,T))}^2,
$$

which for $T$ small enough implies (4.5). ■

5. A proof of Lemma \[2.3\]

1. To prove (i), note that $D_Q Z_0''(F_0, Q_0)Q = 2F_0 \text{sym}(F_0^T Q)$ so that:

$$
\forall Q \in \mathbb{R}^{n \times n} \quad \langle D_Q Z_0''(F_0, Q_0)Q : Q \rangle = 2\langle \text{sym}(F_0^T Q) : F_0 Q \rangle = |\text{sym}(F_0^T Q)|^2.
$$

Take $\zeta \in W_2^1(\Omega, \mathbb{R}^n)$ with trace 0 on the boundary $\partial \Omega$. We have:

$$
\int_\Omega |\nabla \zeta|^2 \leq |F_0^{-1,T}|^2 \int_\Omega |\nabla (F_0^T \zeta)|^2 \leq 2|F_0^{-1,T}|^2 \int_\Omega |\text{sym}\nabla (F_0^T \zeta)|^2
$$

$$
= |F_0^{-1,T}|^2 \int_\Omega \langle D_Q Z_0''(F_0, Q_0) \nabla \zeta \colon \nabla \zeta \rangle,
$$

where we applied Korn’s inequality to the map $x \mapsto F_0^T \zeta(x)$.

2. To prove (ii), observe that $D_Q Z_0'(F_0, Q_0)Q = 2(\text{det } F_0) \text{sym}(Q F_0^{-1}) F_0^{-1,T}$ so that:

$$
\langle D_Q Z_0'(F_0, Q_0)Q : Q \rangle = 2(\text{det } F_0) |\text{sym}(Q F_0^{-1})|^2.
$$

Then, for any test function $\zeta$ as above, we have:

$$
\int_\Omega |\nabla \zeta|^2 \leq |F_0|^2 \int_\Omega |(\nabla \zeta) F_0^{-1}|^2 \, dx = |F_0|^2 \int_{F_0 \Omega} |\nabla (\zeta \circ (F_0^{-1} y))|^2 (\text{det } F_0^{-1}) \, dy
$$

$$
\leq 2|F_0|^2 (\text{det } F_0^{-1}) \int_{F_0 \Omega} |\text{sym}((\nabla \zeta) \circ F_0^{-1}) F_0^{-1})|^2 \, dy
$$

$$
= 2|F_0|^2 (\text{det } F_0^{-1}) (\text{det } F_0) \int_{F_0 \Omega} |\text{sym}((\nabla \zeta) F_0^{-1})|^2 \, dx
$$

$$
= |F_0|^2 (\text{det } F_0)^{-1} \int_\Omega \langle D_Q Z_0'(F_0, Q_0) \nabla \zeta \colon \nabla \zeta \rangle,
$$

where we applied Korn’s inequality to the map $y \mapsto \zeta(F_0^{-1} y)$ on the open domain $F_0 \Omega$. 

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3. To prove (iii) – (v), observe that:

\[ \langle D_Q Z_m(F_0, Q_0) Q : Q \rangle = \left\langle \sum_{j=0}^{2m} (\text{sym}(Q_0 F_0^{-1}))^j \text{sym}(Q F_0^{-1}) (\text{sym}(Q_0 F_0^{-1}))^{2m-j} : Q F_0^{-1} \right\rangle \]

\[ = \left\langle \sum_{j=0}^{2m} A^j B A^{2m-j} : Q F_0^{-1} \right\rangle , \]

where we denoted:

\[ A = \text{sym}(Q_0 F_0^{-1}), \quad B = \text{sym}(Q F_0^{-1}). \]

Since the matrix \( \sum_{j=0}^{2m} A^j B A^{2m-j} \) is symmetric, it follows that:

\[ \langle D_Q Z_m(F_0, Q_0) Q : Q \rangle = \left\langle \sum_{j=0}^{2m} A^j B A^{2m-j} : B \right\rangle \]

Let \( \zeta \) be a test function as in Lemma 2.1. By calculations similar to (5.1) we get:

\[ \hat{\Omega} \left\| \nabla \zeta \right\|^2 \leq \frac{1}{2} |F_0|^2 \int_{\Omega} \langle D_Q Z_0(F_0, Q_0) \nabla \zeta : \nabla \zeta \rangle, \]

proving (iii). To prove (iv), we compute:

\[ \langle D_Q Z_1(F_0, Q_0) Q : Q \rangle = \langle A^2 B : B \rangle + \langle ABA : B \rangle + \langle BA^2 : B \rangle \]

\[ = \langle AB : AB \rangle \langle BA : AB \rangle + |AB|^2 = 2(\text{sym}(AB) : AB) + |AB|^2 \]

\[ = 2|\text{sym}(AB)|^2 + |AB|^2 \geq |AB|^2. \]

Therefore, by calculations similar to (5.1):

\[ \int_{\Omega} \left\| \nabla \zeta \right\|^2 \leq 2|F_0|^2 \int_{\Omega} \left\| \text{sym}(\nabla \zeta) F_0^{-1} \right\|^2 \leq 2|F_0|^2 |A^{-1}|^2 \int_{\Omega} |AB|^2 \]

\[ \leq 2|F_0|^2 |\text{sym}(Q_0 F_0^{-1})|^{-1}|^2 \int_{\Omega} \langle D_Q Z_1(F_0, Q_0) \nabla \zeta : \nabla \zeta \rangle. \]

Finally, in order to prove (v) we derive:

\[ \langle D_Q Z_2(F_0, Q_0) Q : Q \rangle = \langle A^4 B + A^3 BA + A^2 BA^2 + ABA^3 + BA^4 : B \rangle \]

\[ = |A^2 B + ABA|^2 + |A^2 B|^2 \geq |A^2 B|^2, \]

which, in the same manner as in (5.2) yields:

\[ \int_{\Omega} \left\| \nabla \zeta \right\|^2 \leq 2|F_0|^2 |A^{-2}|^2 \int_{\Omega} |A^2 B|^2 \]

\[ \leq 2|F_0|^2 |\text{sym}(Q_0 F_0^{-1})|^{-1}|^4 \int_{\Omega} \langle D_Q Z_2(F_0, Q_0) \nabla \zeta : \nabla \zeta \rangle. \]

The proof of Lemma 2.3 is done.
References

[1] H. Amann, *Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory*, Monographs in Mathematics 89. Birkhauser Boston 1995.

[2] G. Andrews, *On the existence of solutions to the equation utt = u_{xxt} + (u_x)_x*, J. Diff. Eqs. 35, 200231, 1980.

[3] S. Antman and R. Malek-Madani, *Travelling waves in nonlinearly viscoelastic media and shock structure in elastic media*, Quart. Appl. Math. 46, 7793, 1988.

[4] S. Antman and T. Seidman, *Quasilinear hyperbolic-parabolic equations of one-dimensional viscoelasticity*, J. Diff. Eqs. 124, 132184, 1996.

[5] B. Barker, M. Lewicka, and K. Zumbrun, *Existence and stability of viscoelastic shock profiles*, Arch. Rational Mech. Anal. 200, Number 2, (2011) 491–532.

[6] O. Besov, V. Il’i’in, and S. Nikolski, *Integral representations of functions and imbedding theorems. Vol. I.*, Translated from the Russian. Scripta Series in Mathematics, Washington, D.C., Halsted Press 1978.

[7] C. Dafermos, *The mixed initial-boundary value problem for the equations of one-dimensional nonlinear viscoelasticity*, J. Diff. Eqs. 6, 7186, 1969.

[8] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag 1999.

[9] R. Danchin and P.B. Mucha, *A Lagrangian approach for the incompressible Navier-Stokes equations with variable density*, Comm. Pure Appl. Math. 65 (10), 1458–1480, 2012.

[10] S. Demoulini, *Weak solutions for a class of nonlinear systems of viscoelasticity*, Arch. Rat. Mech. Anal. 155 (4), 299-334, 2000.

[11] R. Denk, M. Hieber, and J. Prüss, *R-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Memoirs of the AMS, 166, Springer, 2003.

[12] G. Friesecke and G. Dolzmann, *Implicit time discretization and global existence for a quasilinear evolution equation with nonconvex energy*, SIAM J. Math. Anal. 28, 363380, 1997.

[13] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, Springer Tracts in Natural Philosophy, 38, 39, Springer, 1998.

[14] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag, Berlin, 2001.

[15] T. Hughes, T. Kato and J. Marsden, *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rational Mech. Anal. 63 no. 3, 273294 (1977).

[16] A. Korn, *Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen*, Bull. Int. Cracovie Akademie Umiejet, Classe des Sci. Math. Nat., (1909) 705–724.

[17] O. Ladyzhenskaya, V. Solonnikov and N. Uralceva, *Linear and quasilinear eqs of parabolic type* Translation of Mathematical Monographs 23, AMS 1968.

[18] M. Lewicka, L. Mahadevan and M. Pakzad, *The Foppl-von Karman equations for plates with incompatible strains*, Proceedings of the Royal Society A 467, 402–426, 2011.

[19] M. Lewicka and M. Pakzad, *Scaling laws for non-Euclidean plates and the W^{2,2} isometric immersions of Riemannian metrics*, ESAIM: Control, Optimisation and Calculus of Variations, doi:10.1051/cocv/2010039

[20] G.M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., NJ, 1996.

[21] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, 16, Birkhuser Verlag, Basel, 1995.
[22] M.G. Mora and L. Scardia, *Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density*, J. Differential Equations 252 (2012), 35-55.

[23] R. Pego, *Phase transitions in one-dimensional nonlinear viscoelasticity*, Arch. Rational Mech. Anal. 97, 353394, 1987.

[24] P. Rybka, *Dynamical modeling of phase transitions by means of viscoelasticity in many dimensions*, Proc. Roy. Soc. Edin. A 121, 101138, 1992.

[25] B. Tvedt, *Quasilinear equations for viscoelasticity of strain-rate type*, Arch. Rat. Mech. Anal. 189 (2), 237-281, 2008.

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