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On the structure of dg-categories of relative singularities

Massimo Pippi

Abstract

In this paper we show that every object in the dg-category of relative singularities Sing(B, f) associated to a pair (B, f), where B is a ring and f ∈ B, is equivalent to a retract of a K(B, f)-dg module concentrated in n + 1 degrees. When n = 1, we show that Orlov’s comparison theorem, which relates the dg-category of relative singularities and that of matrix factorizations of an LG-model, holds true without any regularity assumption on the potential.

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Introduction

A matrix factorizations of a pair (B, f), where B is a ring and f ∈ B is the datum of two projective finitely-generated modules (E0, E1) together with two morphisms d0 : E0 → E1, d1 : E1 → E0 such that d1 ∗ d0 = f ∗ idE0 and d0 ∗ d1 = f ∗ idE1. These objects, introduced by D. Eisenbud in [Eis80], can be organized in a 2-periodic dg-category MF(B, f) in a natural way. On the other hand, given such a pair (B, f), we can define another dg-category Sing(B, f), called the dg-category of relative singularities of the pair. The pushforward along the inclusion i : Spec(B) ×_{A^h} S → Spec(B) induces a dg-functor

\[ i_* : \text{Sing}(\text{Spec}(B)) \times_{A^h} S \to \text{Sing}(\text{Spec}(B)) \]

where Sing(Z) stands for Coh^1(Z)/Perf(Z). Then Sing(B, f) is defined as the fiber of this dg-functor. The connection between dg-categories of relative singularities and dg-categories of matrix factorizations has been first envisioned by R.O. Buchweitz and D. Orlov (see [Buch87] and [Orl94]), who showed that if B is regular and f is a regular section, then (the homotopy-categories of) MF(B, f) and Sing(B, f) are equivalent. Notice under these hypothesis Spec(B) ×_{A^h} S = Spec(B/f) and Sing(B, f) ≃ Sing(B/f)\[\text{sing}\]. The dg-category of relative singularities was first introduced by J. Burke and M. Walker in [BW12] in order to remove the regularity hypothesis on B.

In the recent paper [BRTV] the authors show, along the way, that these equivalences are part of a lax-monoidal ∞-natural transformation

\[ \text{Or}^{-1,\circ} : \text{Sing}(\bullet, \bullet) \to \text{MF}(\bullet, \bullet) : \text{LG}(1)_{\text{op,\, perf}} \to \text{dgCat}^\text{sing,\, perf}_{\text{per}} \]

and suggest that, in order to remove the regularity hypothesis on f, one should consider the derived zero locus Spec(B) ×_{A^h} S instead of classical. This remark comes from the observation that if f is regular the two notion coincide and that if B is regular and f = 0, one can compute that both MF(B, 0) and Sing(B, 0) ≃ Sing(Spec(B) ×_{A^h} S) are equivalent to Perf(B[u, u^{-1}]), where u sits in cohomological degree 2, while the classical zero locus of f coincides with B and thus the associated dg-category of singularities is zero.

More generally, one can consider the dg-categories of relative singularities of any pair (B, f), where f ∈ B with n ≥ 1, defined analogously to the case where n = 1:

\[ \text{Sing}(B, f) := \text{fiber}(i_* : \text{Sing}(\text{Spec}(B)) \times_{A^h} S) \to \text{Sing}(\text{Spec}(B)) \]

There exists an algorithm which shows that this dg-category is built up from K(B, f)-dg modules concentrated in n + 1 degrees:

\[ \text{Sing}(X) = 0 \text{ if and only if } X \text{ is regular.} \]

1Indeed, if X is an underived (Noetherian) scheme, Sing(X) = 0 if and only if X is regular.
Theorem. (2.5) Let \((\text{Spec}(B), f)\) be a \(n\)-dimensional affine Landau-Ginzburg model over \(S\). Then every object in the dg-category of relative singularities \(\text{Sing}(B, f)\) is a retract of an object represented by a \(K(B, f)\)-dg module concentrated in \(n + 1\) degrees.

Moreover, when \(n = 1\), the algorithm mentioned above can be used to show that

\[
(E, d, h) = 0 \rightarrow E_{m} \xrightarrow{h_{m+1}/d_{m}} E_{m+1} \leftarrow \cdots \leftarrow E_{m' - 1} \xleftarrow{h_{0}/d_{-1}} E_{m'} \rightarrow 0
\]

be an object in \(\text{Coh}^*(B, f)\). Then the following equivalence holds in \(\text{Sing}(B, f)\):

\[
(E, d, h) \simeq \bigoplus_{i \in \mathbb{Z}} E_{2i - 1} \xleftarrow{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i} \xrightarrow{d + h}
\]

Moreover, it is natural in \((E, d, h)\).

It is then possible to deduce the following

Corollary. (3.11) The lax monoidal \(\infty\)-natural transformation

\[
\text{Orl}^{-1, \circ} : \text{Sing}(\bullet, \bullet) \rightarrow \text{MF}(\bullet, \bullet) : \text{LG}_{S}^{1}(\text{op, 1}) \rightarrow \text{dgCat}^{idm, \circ}_{S}
\]

constructed in [BRTV, §2.4] defines a lax-monoidal \(\infty\)-natural equivalence.

Recall that, for an affine LG model \((\text{Spec}(B), f)\),

\[
\text{Orl}^{-1}_{(B, f)} : \text{Sing}(B, f) \xrightarrow{\simeq} \text{MF}(B, f)
\]

is defined as the dg-functor

\[
(E, d, h) \xrightarrow{\phi} \bigoplus_{i \in \mathbb{Z}} E_{2i - 1} \xleftarrow{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i}
\]

\[
(E', d', h') \xrightarrow{\phi} \bigoplus_{i \in \mathbb{Z}} E'_{2i - 1} \xrightarrow{d' + h'} \bigoplus_{i \in \mathbb{Z}} E'_{2i}
\]

The corollary above improves all the previous results on the equivalence between the dg-categories of singularities and the dg-category of matrix factorizations as it removes the regularity assumption on the potential.

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1 Preliminaries

In this section we will introduce notation and recall some well known facts about the theory of dg-categories.

For us, all rings will be commutative with an identity element. Moreover, we will always assume the Noetherianity assumption, even when not explicitly mentioned.

Notation 1.1. We fix a base ring \(A\). We will refer to its prime spectrum \(\text{Spec}(A)\) by \(S\) and to the category of \(S\)-schemes of finite type by \(\text{Sch}_S\).

We will usually identify every ordinary category with its nerve. We will therefore avoid to write \(N(\mathcal{C})\) to refer to the nerve of the ordinary category \(\mathcal{C}\).
Reminders on dg-categories

Remark 1.2. For more details on the theory of dg-categories, we invite the reader to consult [To11] and/or [Go14].

Consider the ordinary category $\text{dgCat}_S$ of small $A$-linear dg-categories together with $A$-linear dg-functors. Recall that a dg-functor is a Dwyer-Kan (DK for short) is a dg-functor which induces quasi-isomorphisms on the hom-complexes and such that the functor induced on the homotopy categories is essentially surjective. It is a crucial fact in the theory of dg-categories the existence of a combinatorial model category structure on $\text{dgCat}_S$ whose weak equivalences are exactly DK-equivalences (see [Tab05]). The underlying $\infty$-category of this model category is the $\infty$-localization of $\text{dgCat}_S$ with respect to the class of DK-equivalences. We will denote this $\infty$-category by $\text{dgCat}^\text{idm}_S$.

Another crucial class of dg-functors is that of Morita-equivalences: a dg-functor $T \to T'$ is a Morita equivalence if it induces a DK-equivalence on the associated derived categories of perfect dg-modules. The class of DK-equivalences is contained in that of Morita equivalences. Therefore, using the theory of Bousfield localization if it induces a DK-equivalence on the associated derived categories of perfect dg-modules. The underlying $\infty$-category, that we will label $\text{dgCat}^\text{idm}_S$, coincides with the $\infty$-localization of $\text{dgCat}_S$ with respect to Morita equivalences. In particular we have the following couple of composable $\infty$-functors;

$$\text{dgCat}_S \to \text{dgCat}_S \to \text{dgCat}^\text{idm}_S$$  \hspace{1cm} (1.2.1)

The $\infty$-category $\text{dgCat}^\text{idm}_S$ can be identified with the full subcategory of $\text{dgCat}_S$ of dg-categories $T$ for which the Yoneda embedding $T \mapsto \hat{T}$ is a DK-equivalence. Here, $\hat{T}$ stands for the dg-category of compact (i.e. perfect) $T^{op}$-modules. Then the $\infty$-functor $\text{dgCat}_S \to \text{dgCat}^\text{idm}_S$ is a left adjoint to the inclusion $\infty$-functor, which can be informally described by the assignment $T \mapsto \hat{T}$.

We can enhance both $\text{dgCat}_S$ and $\text{dgCat}^\text{idm}_S$ with a symmetric monoidal structure in such a way that, if we restrict to the full subcategory $\text{dgCat}^\text{idm}_S \subseteq \text{dgCat}_S$ of locally-flat (small) dg-categories, there we get two composable symmetric monoidal $\infty$-functors

$$\text{dgCat}^\text{idm}_S \to \text{dgCat}^\text{idm}_S \to \text{dgCat}^\text{idm, }$$  \hspace{1cm} (1.2.2)

For more on Morita theory of dg-categories, we refer to [To07].

Of major relevance in the following is the definition of quotient of dg-categories. Given a dg-category $T$ together with a full sub dg-category $T'$, both of them in $\text{dgCat}^\text{idm}_S$, we will consider the dg-quotient $T/T'$ which is defined as the pushout $T \amalg_{T'} 0$ in $\text{dgCat}^\text{idm}_S$. Here $0$ stands for the final object in $\text{dgCat}^\text{idm}_S$ i.e. the dg-category with only one object and the zero hom-complex. More generally, we can define the dg-quotient of any morphism $T' \to T$ in $\text{dgCat}^\text{idm}_S$ as the pushout above. A fundamental fact is that the homotopy category of $T/T'$ coincides with the Verdier quotient of $T$ by the full subcategory generated by the image of $T'$ (see [Dg3]). The dg-category $T/T'$ can also be obtained as the image in $\text{dgCat}^\text{idm}_S$ of the pushout $T \amalg_{T'} 0$ calculated in $\text{dgCat}_S$.

We conclude this section by recalling that compact objects in $\text{dgCat}^\text{idm}_S$ coincide with dg-categories of finite-type over $A$, as defined in [TV07]. In particular,

$$\text{Ind}(\text{dgCat}^\text{ft}_S) \simeq \text{dgCat}^\text{idm}_S$$  \hspace{1cm} (1.2.3)

Higher dimensional Landau-Ginzburg models

Context 1.3. Assume that $A$ is a local, Noetherian regular ring of finite dimension.

Recall that the category of Landau-Ginzburg models over $S$ is the category of flat $S$-schemes of finite type together with a potential (i.e. a map to $\mathbb{A}_S^1$). The morphisms are those morphisms of $S$-schemes which are compatible with the potential. Moreover, this category has a natural symmetric monoidal enhancement due to the fact that $\mathbb{A}_S^1$ is a scheme in abelian groups. It is very easy to generalize this category to the case where schemes are provided with multipotentials, i.e. with maps to $\mathbb{A}_S^n$, for any $n \geq 1$.

Definition 1.4. Fix $n \geq 1$. Define the category of $n$-dimensional Landau-Ginzburg models over $S$ ($n$-LG models over $S$ for brevity) to be the full subcategory of $\text{Sch}_{S/\mathbb{A}_S^n}$ spanned by those objects

$$X \xrightarrow{f = (f_1, \ldots, f_n)} \mathbb{A}_S^n$$

where $p$ is a flat morphism. Denote this category by $\text{LG}_S(n)$ and its objects by $(X, f)$.

For convenience, we also introduce the following (full) subcategories of $\text{LG}_S(n)$:
\[ + : \mathbb{A}^n_S \times_S \mathbb{A}^n_S \to \mathbb{A}^n_S \] on \( \mathbb{A}^n_S \), corresponding to

\[ A[T_1, \ldots, T_n] \to A[X_1, \ldots, X_n] \otimes_A A[Y_1, \ldots, Y_n] \]

\[ T_i \mapsto X_i \otimes 1 + 1 \otimes Y_i \quad i = 1, \ldots, n \]

Then define

\[ \boxplus : \text{LG}_S(n) \times \text{LG}_S(n) \to \text{LG}_S(n) \]

by the formula

\[ ((X, f), (Y, g)) \mapsto (X \times_S Y, f \boxplus g) \]

Here, \( f \boxplus g \) is the following composition

\[ X \times_S Y \xrightarrow{f \times g} \mathbb{A}^n_S \times_S \mathbb{A}^n_S \xrightarrow{+} \mathbb{A}^n_S \]

Notice that \( X \times_S Y \) is still flat over \( S \), whence this functor is well defined. It is also easy to remark that \( \boxplus \) is associative - i.e. there exist natural isomorphisms \( ((X, f) \boxplus (Y, g)) \boxplus (Z, h) \cong (X, f) \boxplus ((Y, g) \boxplus (Z, h)) \) - and that for any object \((X, f), (S, \emptyset) \boxplus (X, f) \cong (X, f) \boxplus (S, \emptyset)\). More briefly, \( \text{LG}_S(n, \boxplus, (S, \emptyset)) \) is a symmetric monoidal category. It is not hard to see that this construction works on \( \text{LG}_S(n)^{\text{fl}} \), \( \text{LG}_S(n)^{\text{aff}} \) and \( \text{LG}_S(n)^{\text{aff,fl}} \) too. Indeed, this is clear for \( \text{LG}_S(n)^{\text{aff}} \) and if \( f \) and \( g \) are flat morphisms, so is \( f \times g \) and therefore \( f \boxplus g \) is a composition of flat morphisms.

**Notation 1.6.** We will denote by \( \text{LG}_S(n)^{\boxplus} \) (resp. \( \text{LG}_S(n)^{\text{fl}, \boxplus} \), \( \text{LG}_S(n)^{\text{aff}, \boxplus} \), \( \text{LG}_S(n)^{\text{aff,fl}, \boxplus} \) ) these symmetric monoidal categories.

**Remark 1.7.** Notice that \( \text{LG}_S(1)^{\boxplus} \) is exactly the symmetric monoidal category \( \text{LG}_S^{\boxplus} \) defined in \([BRTV] \S2\).

**Remark 1.8.** Fix \( n \geq 1 \). Notice that the symmetric group \( S_n \) acts on the category of \( n\)-LG models over \( S \). Indeed, for any \( \sigma \in S \) and for any \((X, f) \in \text{LG}_S(n)\), we can define

\[ \sigma \cdot (X, f) := (X, \sigma \cdot f) \]

**Dg-categories of singularities**

It is a classic theorem due to Serre that a Noetherian local ring \( R \) is regular if and only if it has finite global dimension. This extremely important fact can be rephrased by saying that the every object in \( \text{Coh}^b(R) \) is equivalent to an object in \( \text{Perf}(R) \). In particular, \( R \) is regular if and only if \( \text{Coh}^b(R)/\text{Perf}(R) \) is zero. This explains why the quotient above is called **category of singularities**.

Before going on with the precise definitions, let us fix some notation.

Let \((X, f)\) be a \( n\)-LG model over \( S \). Then consider the (derived) zero locus of \( f \), i.e. the (derived) fiber product

\[
\begin{array}{ccc}
X_0 & \xrightarrow{i} & X \\
\downarrow & & \downarrow f \\
S & \xrightarrow{\text{zero}} & \mathbb{A}^n_S \\
\end{array}
\]

\[ ^2 \text{notice that it is flat} \]
Remark 1.9. Notice that \( X_0 \simeq X \times_{\mathbb{A}^n} S \) coincides with the classical zero locus whenever \( (X, f) \) belongs to \( \text{LG}_S(n) \). In general, we always have a closed embedding \( t : X \times_{\mathbb{A}^n} S = \pi_0(X_0) \rightarrow X_0 \).

Remark 1.10. As \( S \rightarrow \mathbb{A}^n_S \) is lci and this class of morphism is closed under derived fiber products, we get that \( i : X_0 \rightarrow X \) is a lci morphism of derived schemes.

We will consider the following (\( A \)-linear) dg-categories:

- \( \text{QCoh}(X) \) (resp. \( \text{QCoh}(X_0) \)), the \( A \)-linear dg-categories of complexes of quasi-coherent complexes on \( X \) (resp. \( X_0 \));
- \( \text{Perf}(X) \) (resp. \( \text{Perf}(X_0) \)), the full sub-dg-category of \( \text{QCoh}(X) \) (resp. \( \text{QCoh}(X_0) \)) spanned by perfect complexes. Recall that, for a derived scheme \( Z \), an object \( E \in \text{QCoh}(Z) \) is perfect if, locally, it belongs to the thick sub-dg-category of \( \text{QCoh}(Z) \) spanned by \( \mathcal{O}_Z \). Perfect complexes are exactly dualizable objects. In our case, they coincide with compact objects in \( \text{QCoh}(Z) \) too (see [BZFN]);
- \( \text{Coh}^b(X) \) (resp \( \text{Coh}^b(X_0) \)), the full sub-dg-category of \( \text{QCoh}(X) \) (resp. \( \text{QCoh}(X_0) \)) spanned by those cohomologically bounded complexes \( E \) such that \( H^i(E) \) is a coherent \( H^0(\mathcal{O}_X) \) (resp. \( H^0(\mathcal{O}_{X_0}) \)) module;
- \( \text{Coh}^-(X) \) (resp \( \text{Coh}^-(X_0) \)), the full sub-dg-category of \( \text{QCoh}(X) \) (resp. \( \text{QCoh}(X_0) \)) spanned by those cohomologically bounded above complexes \( E \) such that \( H^i(E) \) is a coherent \( H^0(\mathcal{O}_X) \) (resp. \( H^0(\mathcal{O}_{X_0}) \)) module;
- \( \text{Coh}^b(X_0) \text{Perf}(X) \), the full sub-dg-category of \( \text{Coh}^b(X_0) \) spanned by those objects \( E \) such that \( i_* E \) belongs to \( \text{Perf}(X) \).

Remark 1.11. Analogously to [BRTV] §2, we have the following inclusions

\[
\text{Perf}(X) \subseteq \text{Coh}^b(X) \subseteq \text{Coh}^-(X) \subseteq \text{QCoh}(X)
\]

Indeed, being \( X \) and \( X_0 \) eventually coconnective (see [CRT] §4, Definition 1.1.6), we have the inclusions \( \text{Perf}(X) \subseteq \text{Coh}^b(X) \) and \( \text{Perf}(X_0) \subseteq \text{Coh}^b(X_0) \). Moreover, as \( i \) is lci, by [To12], we have that \( i_* \) preserves perfect complexes. Thus, the inclusion \( \text{Perf}(X_0) \subseteq \text{Coh}^b(X_0) \text{Perf}(X) \) holds.

Remark 1.12. As it is explained in [BRTV] Remark 2.14, the dg categories \( \text{Perf}(X), \text{Perf}(X_0), \text{Coh}^b(X), \text{Coh}^b(X_0) \text{Perf}(X) \) are idempotent complete. Indeed, the same argument provided in loc. cit. for the case \( n = 1 \) works in general.

Notice that all the results in [BRTV] §2.3.1 are not specific of the monopotential case and they remain valid in our situation. We will recall these statements for the reader’s convenience and refer to loc. cit. for the proofs, which remain untouched.

Proposition 1.13. Let \( (X, f) \in \text{LG}_S(n) \). Then the inclusion functor induces an equivalence

\[
\text{Coh}^b(X_0) \text{Perf}(X) \simeq \text{Coh}^-(X_0) \text{Perf}(X)
\]

(1.13.1)

In particular, the following square is cartesian in \( \text{dgCat}^\text{idm}_S \)

\[
\begin{array}{ccc}
\text{Coh}^-(X_0) & \xrightarrow{i_*} & \text{Coh}^-(X) \\
\downarrow & & \downarrow \\
\text{Coh}^b(X_0) \text{Perf}(X) & \xrightarrow{i_*} & \text{Perf}(X)
\end{array}
\]

(1.13.2)

We now give definitions for the relevant dg-categories of singularities. The reader should be aware that there are plenty of this objects that one can consider, and we will define some of them later on. The following category, known as category of absolute singularities, first appeared in [Orl04]. The following is a dg-enhancement of the original definition, as it appears in [BRTV].

Definition 1.14. Let \( Z \) be a derived scheme of finite type over \( S \) whose structure sheaf is cohomologically bounded. The \( \text{dg-category of absolute singularities of } Z \) is the dg-quotient (in \( \text{dgCat}^\text{idm}_S \))

\[
\text{Sing}(Z) := \text{Coh}^b(Z)/\text{Perf}(Z)
\]

(1.14.1)
Remark 1.15. Notice that the finiteness hypothesis on $Z$ in Definition (1.14) are absolutely indispensable, as otherwise $\text{Perf}(Z)$ may not contained in $\text{Coh}^b(Z)$.

Remark 1.16. It is well known that, for an underived (Noetherian) scheme $Z$, the dg-category $\text{Sing}(Z)$ is zero if and only if the scheme is regular. On the other hand, when we allow $Z$ to be a derived scheme, $\text{Sing}(Z)$ may be non trivial even if the underlying scheme is regular. For example, consider $Z = \text{Spec}(A \otimes_{A[T]} A)$.

Following [BRTV] we next consider the dg-category of singularity associated to an n-dimensional LG-model.

Definition 1.17. Let $(X, f) \in \text{LG}_S(n)$. The dg-category of singularities of $(X, f)$ is the following fiber in $\text{dgCat}^\text{ldm}_{S}$

$$\text{Sing}(X, f) := \text{Ker}(i_* : \text{Sing}(X_0) \to \text{Sing}(X))$$

(1.17.1)

Remark 1.18. Notice that $\text{Sing}(X, f)$ is a full sub-dg-category of $\text{Sing}(X_0)$ (see [BRTV] Remark 2.24]). Moreover, these two dg-categories coincide whenever $X$ is a regular $S$-scheme.

Proposition 1.19. (See [BRTV] Proposition 2.25]) Let $(X, f)$ be a n-dimensional LG model over $S$. Then there is a canonical equivalence

$$\text{Coh}^b(X_0)_{\text{Perf}(X)}/\text{Perf}(X) \simeq \text{Sing}(X, f)$$

(1.19.1)

where the quotient on the left is taken in $\text{dgCat}^\text{ldm}_{S}$.

We shall now re-propose, for the multi-potential case, the strict model for $\text{Coh}^b(X_0)_{\text{Perf}(X)}$ which was first introduced in [BRTV].

Construction 1.20. Let $(\text{Spec}(B), f) \in \text{LG}_S(n)_{\text{aff}}$. Consider the Koszul complex $K(B, f)$

$$0 \to \bigwedge^n(B\varepsilon_1 \oplus \cdots \oplus B\varepsilon_n) \to \cdots \to \bigwedge^2(B\varepsilon_1 \oplus \cdots \oplus B\varepsilon_n) \to (B\varepsilon_1 \oplus \cdots \oplus B\varepsilon_n) \to B \to 0$$

(1.20.1)

cenerated in degrees $[-n, 0]$. The differential is given by

$$\bigwedge^k(B\varepsilon_1 \oplus \cdots \oplus B\varepsilon_n) \to \bigwedge^{k-1}(B\varepsilon_1 \oplus \cdots \oplus B\varepsilon_n)$$

where $\phi : B^n \to B$ is the matrix $[f_1 \ldots f_n]$. Multiplication is given by concatenation. Notice that $K(B, f)$ is a cofibrant $B$-module and that we always have a truncation morphism $K(B, f) \to B/f$, which is a quasi-isomorphism whenever $f$ is a regular sequence.

Therefore, we can present $K(B, f)$ as the cdga $B[\varepsilon_1, \ldots, \varepsilon_n]$, where the $\varepsilon_i$’s sit in degree $-1$ and are subject to the following conditions:

$$d(\varepsilon_i) = f_i \quad i = 1, \ldots, n$$

$$\varepsilon_i^2 = 0 \quad i = 1, \ldots, n$$

$$\varepsilon_i \ldots \varepsilon_k = (-1)^{\sigma} \varepsilon_{i_{\sigma(1)}} \ldots \varepsilon_{i_{\sigma(k)}} \quad \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}, \sigma \in S_k$$

Example 1.21. For instance, when $n = 1$, $K(B, f)$ is the cdga $B \overset{f}{\to} B$ concentrated in degrees $[-1, 0]$ and, when $n = 2$, $K(B, (f_1, f_2))$ is the cdga $B \overset{f_1}{\oplus} B \overset{f_2}{\to} B$ concentrated in degrees $[-2, 0]$.

Remark 1.22. Notice that $K(B, f)$ provides a model for the cdga associated to the simplicial commutative algebra $B \otimes_{A[T_1, \ldots, T_n]} A$. Indeed, this can be computed explicitly for $n = 1$ and the general case follows from the compatibility of the Dold-Kan correspondence with (derived) tensor products.

This strict model for the derived zero locus of an affine LG model of order $n$ over $S$ gives us strict models for the relevant categories of modules too. Following [BRTV]:

• There is an equivalence of $A$-linear dg-categories between $\text{QCoh}(X_0)$ and the dg-category (over $A$) of cofibrant $K(B, f)$-dg-modules, which we will denote $K^b(B, f)$. A $K^b(B, f)$-dg-module is the datum of a cochain complex of $B$-modules $(E, d)$, together with $n$ morphisms $h_1, \ldots, h_n : E \to E[1]$ of degree $-1$ such that

$$\begin{align*}
  h_i^2 &= 0 & i = 1, \ldots, n \\
  [d, h_i] &= f_i & i = 1, \ldots, n \\
  [h_i, h_j] &= 0 & i, j = 1, \ldots, n
\end{align*}$$

(1.22.1)
• \( \text{Coh}^h(X_0) \subseteq \text{QCoh}(X_0) \) corresponds to the full sub-dg-category of \( \overline{K(B,f)} \) spanned by those modules of cohomologically bounded amplitude and whose cohomology is coherent over \( B/\underline{f} \);

• \( \text{Perf}(X_0) \subseteq \text{QCoh}(X_0) \) corresponds to the full sub-dg-category of \( \overline{K(B,f)} \) spanned by those modules which are homotopically finitely presented.

**Remark 1.23.** Notice that, for any \( K(B,f) \)-dg module, for any \( 1 \leq k \leq n \) and for any \( \{i_1, \ldots, i_k\} \subseteq \{1,\ldots,n\} \) (where the \( i_j \)’s are pairwise distinguished), the following formula holds:

\[
[d, h_{i_1} \ldots h_{i_k}] = \sum_{j=1}^k (-1)^j f_{i_j} h_{i_1} \circ \ldots \circ h_{i_j} \circ \ldots \circ h_{i_k}
\]

**Remark 1.24.** As in the mono-potential case (see [BRTV, Remark 2.30]), \( \iota_s : \text{Coh}(X_0) \to \text{QCoh}(X) \) corresponds, under these equivalences, to the forgetful functor \( K(B,f) \to \overline{B} (K(B,f)) \) is a cofibrant \( B \)-module).

We propose the following straightforward generalization of [BRTV, Construction 2.31] as a strict model for \( \text{Coh}^h(X_0)_{\text{perf}(X)} \):

**Construction 1.25.** Let \( \text{Coh}^s(B,f) \) be the \( A \)-linear sub-dg-category of \( K(B,f) \) spanned by those modules whose image along the forgetful functor \( K(B,f) - \text{dgmod} \to B - \text{dgmod} \) is a strictly perfect complex of \( B \)-modules. In particular, the morphism \( E \) in \( \text{Coh}^s(B,f) \) is a degree-wise projective cochain complex of \( B \)-modules together with \( n \) morphisms \( h_1, \ldots, h_n \) of degree \(-1\) satisfying the identities \([1,22,1]\). As \( A \) is a local ring, it is clear that \( \text{Coh}^s(B,f) \) is a locally flat \( A \)-linear dg-category.

**Lemma 1.26.** Let \( (X,f) = (\text{Spec}(B),f) \) be a \( n \)-dimensional affine LG model over \( S \). Then the cofibrant replacement dg-functor induces an equivalence

\[
\text{Coh}^s(B,f)[q.iso^{-1}] \simeq \text{Coh}^s(B,f)/\text{Coh}^{s,acy}(B,f) \simeq \text{Coh}^h(X_0)_{\text{perf}(X)}
\]

where \( \text{Coh}^{s,acy}(B,f) \) is the full sub-dg-category of \( \text{Coh}^s(B,f) \) spanned by acyclic complexes. In particular, this implies that we have equivalences of dg-categories

\[
\text{Coh}^s(B,f)/\text{Perf}^s(B,f) \simeq \text{Coh}^h(X_0)_{\text{perf}(X)}/\text{Perf}(X_0) \simeq \text{Sing}(X,f)
\]

where \( \text{Perf}^s(B,f) \) is the full sub-dg-category of \( \text{Coh}^s(B,f) \) spanned by those modules which are perfect over \( K(B,f) \).

**Proof.** See [BRTV, Lemma 2.33]. The same proof holds true in our situation too. \( \square \)

We now exhibit the functorial properties of \( \text{Coh}^s(\bullet, \bullet) \). Let \( u : (\text{Spec}(C),g) \to (\text{Spec}(B,f)) \) be a morphism in \( \text{LG}_{S}(n)^{\text{aff}} \). Define the dg-functor

\[
u^* : \text{Coh}^s(B,f) \to \text{Coh}^s(C,g)
\]

by the law

\( E \mapsto E \otimes_B C \)

Notice that this dg-functor is well defined as \( E \otimes_B C \) is strictly bounded and degree-wise \( C \)-projective. It is clear that if two composable morphisms

\[
(Spec(B,f),f) \overset{u}{\to} (Spec(B',f'),f') \overset{u'}{\to} (Spec(B'',f''),f'')
\]

are given, \( u^* \circ u'^* \simeq (u' \circ u)^* \) are equivalent dg-functors \( \text{Coh}^s(B'',f'') \to \text{Coh}^s(B,f) \). It is also clear that \( id_{(Spec(B),f)} \simeq id_{\text{Coh}^s(B,f)} \) and that this law is (weakly) associative and (weakly) unital. In other words,

\[
\text{Coh}^s(\bullet, \bullet) : \text{LG}_{S}(S)^{\text{aff}, \text{op}} \to \text{dgCat}^H_S
\]

has the structure of a pseudo-functor. We next produce a lax-monoidal structure on this pseudo-functor. We begin by producing a map

\[
\text{Coh}^s(B,f) \otimes \text{Coh}^s(C,g) \to \text{Coh}^s(B \otimes_A C, f \otimes_B g)
\]
Consider the following diagram

\[
\begin{array}{ccc}
\text{Spec}(K(B,f)) \times_{B} \text{Spec}(C,g) & \xrightarrow{\phi} & \text{Spec}(K(B \otimes_{A} C, f \boxtimes g)) \\
\downarrow & & \downarrow \\
S & \xrightarrow{\text{zero}} & \mathbb{A}_{S}^{n} \\
\text{Spec}(K(B,f)) \times_{B} \text{Spec}(C,g) & \xrightarrow{\psi} & \text{Spec}(B \otimes_{A} C) \\
\downarrow & & \downarrow \\
S & \xrightarrow{\text{zero}} & \mathbb{A}_{S}^{n} \times_{S} \mathbb{A}_{S}^{n} \\
\end{array}
\]

(1.26.6)

Notice that all the squares in this diagram are (homotopy) cartesian and that all the horizontal maps are lci morphisms of (derived) schemes. Write

\[K(B, f) = B[\epsilon_{1}, \ldots, \epsilon_{n}]\]
\[K(C, g) = C[\delta_{1}, \ldots, \delta_{n}]\]
\[K(B \otimes_{A} C, f \boxtimes g) = B \otimes_{A} C[\gamma_{1}, \ldots, \gamma_{n}]\]

where all the \(\epsilon_{i}\)'s, \(\delta_{i}\)'s and \(\gamma_{i}\)'s sit in degree \(-1\) and are subject to the relations \((1.20)\). Then \(\phi\) corresponds to the morphism of cdga's

\[K(B \otimes_{A} C, f \boxtimes g) \to K(B, f) \otimes_{A} K(C, g)\]
\[\gamma_{i} \mapsto \epsilon_{i} \otimes 1 + 1 \otimes \delta_{i} \quad i = 1, \ldots, n\]

which is the identity in degree zero, while \(\psi\) and \(\psi \circ \phi\) are just the obvious inclusion of \(B \otimes_{A} C\).

Then, we define \((1.26.5)\) by

\[(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G} := \phi_{*}(pr_{1*} \mathcal{F} \otimes K(B, f) \otimes_{A} K(C, g) pr_{2*} \mathcal{G})\]

where \(pr_{1}\) and \(pr_{2}\) are the projections from \(\text{Spec}(K(B, f)) \times_{B} \text{Spec}(K(C, g))\) to \(\text{Spec}(K(B, f))\) and \(\text{Spec}(K(C, g))\) respectively. We need to show that \(\mathcal{F} \boxtimes \mathcal{G}\) lies in \(\text{Coh}^{h}(B \otimes_{A} C, f \boxtimes g)\). This is equivalent to the statement that the underlying complex of \(pr_{1*} \mathcal{F} \otimes K(B, f) \otimes_{A} K(C, g) pr_{2*} \mathcal{G}\) is perfect over \(B \otimes_{A} C\). Consider the (homotopy) cartesian square

\[
\text{Spec}(K(B, f)) \times_{B} \text{Spec}(K(C, g)) \xrightarrow{pr_{2}} \text{Spec}(K(C, g)) \xrightarrow{pr_{1}} \text{Spec}(K(B, f)) \xrightarrow{p} S
\]

and notice that we have the following chain of equivalences

\[
p_{*} pr_{1*} (pr_{2*} \mathcal{F} \otimes K(B, f) \otimes_{A} K(C, g) pr_{2*} \mathcal{G}) \xrightarrow{\text{proj. form.}} p_{*} (\mathcal{F} \otimes K(B, f) pr_{1*} pr_{2*} \mathcal{G})
\]

As \(\mathcal{F} \in \text{Coh}^{h}(B, f)\), \(\mathcal{G} \in \text{Coh}^{h}(C, g)\) and perfect complexes are stable under tensor product, we conclude.

We next exhibit the lax unit \(\Delta\) by

\[
\Delta \rightarrow \text{Coh}^{h}(A, \emptyset)
\]

(1.26.9)

This is simply the dg-functor defined by

\[\bullet \mapsto A\]

where \(A\) (concentrated in degree 0) is seen as a module over \(K(A, \emptyset)\) in the obvious way, i.e. the \(\epsilon_{i}\)'s act via zero.

This defines a (right) lax monoidal structure on \(\text{Coh}^{h}(\bullet, \bullet) : \text{LG}^{\otimes}_{S}(\text{aff,op}, \otimes) \rightarrow \text{dgCat}^{h}_{S, \otimes}\)

(1.26.10)
Remark 1.27. Notice that the same structure defines a lax monoidal structure on the functor
\[
\text{Perf}^\otimes(\bullet, \bullet) : \text{LG}_S(n)^{\text{aff,op,}} \to \text{dgCat}_S^\otimes
\] (1.27.1)

By the same technical arguments of [BRTV] Construction 2.34, Construction 2.37] we produce a (right) lax monoidal \(\infty\)-functor
\[
\text{Coh}^\otimes(\bullet)_{\text{Perf}(\bullet)} : \text{LG}_S(n)^{\text{aff,op,}} \to \text{dgCat}_S^{\text{idm,}\otimes}
\] (1.27.2)

In order to define the lax monoidal \(\infty\)-functor
\[
\text{Sing}(\bullet, \bullet)^\otimes : \text{LG}_S(n)^{\text{aff,op,}} \to \text{dgCat}_S^{\text{idm,}\otimes}
\] (1.27.3)

consider the category Pairs-dgCat^lf S whose objects are pairs \((T, S)\), where \(T\) is an \(A\)-linear dg-category and \(S\) a class of morphisms in \(T\). Given two objects \((T, S)\) and \((T', S')\), morphisms \((T, S) \to (T', S')\) are those \(dg\)-functors \(F : T \to T'\) such that \(S\) is sent into \(S'\). Composition and identities are defined in the obvious way. Given a morphism \((T, S) \to (T', S')\), we say that it is a Dwyer-Kan equivalence if the underlying \(dg\)-functor is so (i.e. is a quasi-equivalence). We denote the class of Dwyer-Kan equivalences in Pairs-dgCat^lf S by \(W_{DK}\).

Notice that Pairs-dgCat^lf S inherits a symmetric monoidal structure from \(dgCat^\otimes_{S}\) by setting \((T, S) \otimes (T', S') = (T \otimes T', S \otimes S')\). We will refer to this symmetric monoidal category by Pairs-dgCat^lf S. As we are considering locally flat \(dg\)-categories, it is immediate that this tensor structure is compatible with \(DK\) equivalences. For any \(n\)-dimensional affine LG model \((\text{Spec}(B), f)\) over \(S = \text{Spec}(A)\), define \(W_{\text{Perf}(B, f)}\) as the class of morphisms \((0 \to E)_{E \in \text{Perf}(B, f)}\) in \(\text{Coh}^\otimes(B, f)\). Consider the functor
\[
\text{LG}_S(n)^{\text{aff,op}} \to \text{Pairs-dgCat}_S^\otimes
\] (1.27.4)

\[(\text{Spec}(B, f)) \mapsto (\text{Coh}^\otimes(B, f), W_{\text{Perf}(B, f)})\]

If \(E \in \text{Perf}^\otimes(B, f)\) and \(F \in \text{Perf}^\otimes(C, g)\), then \((0 \to E) \otimes (0 \to F) \in W_{\text{Perf}(B, f)} \otimes W_{\text{Perf}(C, g)}\) is sent to \(0 \to E \otimes F\) via \((1.26.5)\), which belongs to \(W_{\text{Perf}(B \otimes A, C \otimes B_2)}\). Then the functor \((1.27.4)\) has a lax monoidal enhancement
\[
\text{LG}_S(n)^{\text{aff,op}} \to \text{Pairs-dgCat}_S^\otimes
\] (1.27.5)

By [BRTV] Construction 2.34 and Construction 2.37, there is a strongly monoidal \(\infty\)-functor
\[
\text{loc}^\otimes_{dg} : \text{Pairs-dgCat}_S^\otimes[\text{W}^{-1}_{DK}] \to \text{dgCat}^\otimes_S
\] (1.27.6)

sending a pair \((T, S)\) to the \(dg\)-localization \(T[S^{-1}]_{dg}\).

We finally define \((1.27.3)\) as the following composition
\[
\text{LG}_S(n)^{\text{aff,op}} \xrightarrow{\text{loc}^\otimes_{dg}(1.27.5)} \text{Pairs-dgCat}_S^\otimes[\text{W}^{-1}_{DK}] \xrightarrow{(1.27.6)} \text{dgCat}^\otimes_S \xrightarrow{\text{loc}} \text{dgCat}_S^{\text{idm,}\otimes}
\] (1.27.7)

Notice that \((\text{Spec}(B), f) \in \text{LG}_S(n)^{\text{aff}}\) is sent to \(\text{Sing}(B, f)\) by Lemma \((1.26)\) and by the fact that the quotient \(\text{Coh}^\otimes(B, f)/\text{Perf}^\otimes(B, f)\) is, by definition, the \(dg\)-localization \(\text{Coh}^\otimes(B, f)[\text{W}^{-1}_{\text{Perf}(B, f)}]\) (see [Lo87] §8.2).

Remark 1.28. If \(n = 1\), the lax monoidal structure on the \(\infty\)-functor \(\text{Sing}(\bullet, \bullet)^\otimes\) identifies with the lax monoidal structure on the \(\infty\)-functor defined in [BRTV] Proposition 2.45.

2 The structure of \(\text{Sing}(B, f)\)

In this section we will prove that, in the category of relative singularities \(\text{Sing}(B, f)\) associated to a \(n\)-dimensional affine Landau-Ginzburg model over \(S\), every object is a retract of an object that can be represented by a \(K(B, f)\)-dg module concentrated in \(n + 1\)-degrees. We begin with the following observation:

Lemma 2.1. Let \(\phi : (E, d, h) \to (E', d', h')\) be a cocycle-morphism of \(K(B, f)\)-dg-module.\(^{4}\) Then the cone of \(\phi\) is given by
\[
\begin{align*}
E_{n+1} \oplus E'_{n+1} & \xrightarrow{\begin{bmatrix} -h_{n+2} & 0 \\ 0 & h'_{n+1} \end{bmatrix}} E_{n+2} \oplus E'_{n+1} \\
E_{n+1} \oplus E'_{n+1} & \xrightarrow{\begin{bmatrix} -h_{n+3} & 0 \\ 0 & h'_{n+2} \end{bmatrix}} E_{n+3} \oplus E'_{n+2}
\end{align*}
\] (2.1.1)

\(^{4}\)Here \(d\) (resp. \(d'\)) stands for the differential and \(h^i\) (resp. \(h'^i\)) stands for the action of \(e_i\), where
\[
K(B, f) = 0 \to B_{-1} \to \cdots \to B_{-n} \oplus \cdots \oplus B_{-n} \to 0
\]
Proof. Note that the underlying complex of $B$-modules is the classical cone. It only remains to check that all the morphisms involved in the proof of the fact that this complex of $B$ modules is the cone are compatible with the action of $\varepsilon$. This is a tedious but elementary verification.

Consider an object $(E,d,\{h^i\}_{i\in\{1,...,n\}}) \in \text{Coh}^s(B,f)$. Then its underlying $B$-dg module $(E,d)$ is strictly perfect. As the (derived) pullback preserves perfect complexes, $(E,d) \otimes_B K(B,f)$ lies in $\text{Perf}^s(B,f)$. This is the $K(B,f)$-dg module which, in degree $m$ and $m+1$ has the shape

\[
\sum_{k=0}^{n} E_{m+k} \otimes_B \bigwedge^{B}(B_{\varepsilon_1} \oplus \cdots \oplus B_{\varepsilon_n}) \xrightarrow{\partial_m} \sum_{k=0}^{n} E_{m+k+1} \otimes_B \bigwedge^{B}(B_{\varepsilon_1} \oplus \cdots \oplus B_{\varepsilon_n})
\]  

(2.1.2)

Moreover, $\partial_m$ is defined as follows: for any $x \in E_{m+k}$

\[
\partial_m(x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k) = (-1)^{k} d_{m+k}(x) \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k + \sum_{j=1}^{n} ((-1)^j f_{ij} x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_{ij} \wedge \cdots \wedge \varepsilon_k)
\]

(2.1.3)

The $-1$ degree morphisms

\[
\eta_{m}^{i} : \sum_{k=0}^{n} E_{m+k} \otimes_B \bigwedge^{B}(B_{\varepsilon_1} \oplus \cdots \oplus B_{\varepsilon_n}) \rightarrow \sum_{k=0}^{n} E_{m+k-1} \otimes_B \bigwedge^{B}(B_{\varepsilon_1} \oplus \cdots \oplus B_{\varepsilon_n})
\]

are defined, for $x \in E_{m+k}$, by

\[
\eta_{m}^{i}(x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k) = x \otimes \varepsilon_j \wedge \varepsilon_i \wedge \cdots \wedge \varepsilon_k
\]

(2.1.5)

Notice that we have a morphism of $B$-dg modules $\phi : (E,d) \otimes_B K(B,f) \rightarrow (E,d,\{h^i\}_{i\in\{1,...,n\}})$ which is defined in degree $m$ by $(x \in E_{m+k})$

\[
\bigwedge_{k=0}^{m} E_{m+k} \otimes_B \bigwedge^{B}(B_{\varepsilon_1} \oplus \cdots \oplus B_{\varepsilon_n}) \rightarrow E_{m}
\]

(2.1.6)

\[
x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k \mapsto h_{m-1}^{i_1} \circ \cdots \circ h_{m+k}^{i_k}(x)
\]

where with this notation, when $k=0$, we just mean the identity morphism.

**Lemma 2.2.** $\phi$ is a cocycle morphism of $K(B,f)$-dg modules.

*Proof.* It is clear that $\phi$ is a morphism of $(K,B,f)$-dg modules, i.e. that $\phi \circ \eta^i = h^i \circ \phi$

We then only need to show that $\phi$ commutes with the differentials too. Pick $x \in E_{m+k}$. Then

\[
d_m(\phi_m(x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k)) = d_m(h_{m+1}^{i_1} \circ \cdots \circ h_{m+k}^{i_k}(x)) = \sum_{j=1}^{k} (-1)^{j+1} f_{ij} h_{m+2}^{i_1} \circ \cdots \circ h_{m+k}^{i_k}(x) + (-1)^k h_{m+1}^{i_1} \circ \cdots \circ h_{m+k}^{i_k}(x)
\]

On the other hand, we have that

\[
\phi_{m+1}(\partial_m(x \otimes \varepsilon_i \wedge \cdots \varepsilon_k)) = \phi_{m+1}((-1)^k d_{m+k}(x) \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_k + \sum_{j=1}^{n} ((-1)^j f_{ij} x \otimes \varepsilon_i \wedge \cdots \wedge \varepsilon_{ij} \wedge \cdots \wedge \varepsilon_k))
\]

\[
= (-1)^k h_{m+2}^{i_1} \circ \cdots \circ h_{m+k+1}^{i_k}(x) \circ d_{m+k}(x) + \sum_{j=1}^{k} (-1)^{j+1} f_{ij} h_{m+2}^{i_1} \circ \cdots \circ h_{m+k}^{i_k}(x)
\]

If $k = 0$, then $\phi_m(x) = x$ and there is nothing to show.

**Remark 2.3.** As the source of $\phi$ is a perfect $K(B,f)$-dg module, it follows that $(E,d,\{h^i\}_{i\in\{1,...,n\}}$ and $\text{cone}(\phi)$ are equivalent in Sing$(B,f)$.

**Proposition 2.4.** Assume that $(E,d,\{h^i\}_{i\in\{1,...,n\}}$ as above is concentrated in degrees $[m',m]$, where $m - m' \geq n + 1$ (i.e. the dg-module is concentrated in at least $n + 2$-degrees). Then $\text{cone}(\phi)$ is equivalent, in Sing$(B,f)$, to a $K(B,f)$-dg module concentrated in degrees $[m',m-1]$.
Proof. We claim that we can exhibit \( \text{cone}(\phi) \) as the cone of a cocycle morphism of \( (K(B, f)) \)-dg modules whose domain is
\[
- ((E_{m'} \xrightarrow{d_{m'}} \cdots \xrightarrow{d_{m'+n}} E_{m'+n}) \otimes_B K(B, f))
\]
(2.4.1)
which is a perfect \( (K(B, f)) \)-dg module concentrated in degrees \([m' - n, m' + n]\). The \(-\) above means that we change the sign of all the \( \delta_i \)'s and \( \mu_i \)'s. Notice that it is a \( (K(B, f)) \)-sub-dg module of \( \text{cone}(\phi) \) and that \( \delta \) and the \( \mu_i \)'s coincide with the ones induced by this inclusion.

Now consider the \( (K(B, f)) \)-sub-dg module of \( \text{cone}(\phi) \), in degree \( s \) is the projective \( B \)-module
\[
E_s \oplus \left( \bigoplus_{j = 0}^{n} E_{j+s+1} \otimes_B \left( B \varepsilon_1 \oplus \cdots \oplus B \varepsilon_n \right) \right) \subseteq (\text{cone}(\phi))_s
\]
(2.4.2)
which we will refer to as \( (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \). This is still a \( (K(B, f)) \)-dg module as \( \partial \) and \( \{\eta^s\}_{s=1,\ldots,n} \) are well defined on it, i.e. \( \partial_s(F_s) \subseteq F_{s+1} \) and \( \eta^s(F_s) \subseteq F_{s-1} \). Notice that, for \( s \leq m' - 1 \), \( F_s = 0 \) and, for \( s \geq m' + n \), \( F_s = \text{cone}(\phi)_s \). This means that \( (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \) is a \( (K(B, f)) \)-dg module concentrated in degrees \([m',m] \), as \( \text{cone}(\phi)_s = 0 \) if \( s > m \). Label \( \varepsilon : (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \rightarrow \text{cone}(\phi) \) the canonical inclusion and \( \pi : \text{cone}(\phi) \rightarrow (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \) the canonical projection.

Notice that, for any \( s \), we have that
\[
F_s \oplus \left(-((E_{m'} \xrightarrow{d_{m'}} \cdots \xrightarrow{d_{m'+n}} E_{m'+n}) \otimes_B K(B, f))_{s+1} = \right.
\]
\[
E_s \oplus \left( \bigoplus_{j = 0}^{n} E_{j+s+1} \otimes_B \left( B \varepsilon_1 \oplus \cdots \oplus B \varepsilon_n \right) \right) \oplus \left( \bigoplus_{j = 0}^{n} E_{j+s+1} \otimes_B \left( B \varepsilon_1 \oplus \cdots \oplus B \varepsilon_n \right) \right)
\]
\[
\simeq E_s \oplus \left( \bigoplus_{j = 0}^{n} E_{j+s+1} \otimes_B \left( B \varepsilon_1 \oplus \cdots \oplus B \varepsilon_n \right) \right) = \text{cone}(\phi)_s
\]
Define
\[
\psi : -\left((E_{m'} \xrightarrow{d_{m'}} \cdots \xrightarrow{d_{m'+n}} E_{m'+n-1}) \otimes_B K(B, f)\right) \rightarrow (F, \partial, \{\eta^s\}_{s=1,\ldots,n})
\]
(2.4.4)
in every degree as the composition
\[
\left( \bigoplus_{j = 0}^{n} E_{j+s} \otimes_B \left( B \varepsilon_1 \oplus \cdots \oplus B \varepsilon_n \right) \right) \subseteq \text{cone}(\phi)_{s+1} \xrightarrow{\partial_s} \text{cone}(\phi) \xrightarrow{\pi} F_s
\]
This is a cocycle morphism by construction. Notice that, as \( \eta^s(F_s) \subseteq F_{s+1} \) and \( \eta^s((E_{m'} \xrightarrow{d_{m'}} \cdots \xrightarrow{d_{m'+n}} E_{m'+n-1}) \otimes_B K(B, f)) \subseteq \left((E_{m'} \xrightarrow{d_{m'}} \cdots \xrightarrow{d_{m'+n}} E_{m'+n}) \otimes_B K(B, f)\right)_{s+1} \), by Lemma 2.1 we find that \( \text{cone}(\psi) = \text{cone}(\phi) \).

To conclude, notice that \( (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \) coincides, in degrees \( m - 1 \) and \( m \), with
\[
E_{m-1} \oplus E_m \xrightarrow{[d_{m-1}]} E_m
\]
Therefore, \( (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \) is quasi-isomorphic to
\[
F_{m'} \xrightarrow{\partial_{m'}} \cdots \xrightarrow{\partial_{m-2}} F_{m-2} \xrightarrow{\partial_{m-3}} \cdots \xrightarrow{\partial_{m-2}} \text{Ker}([-d_{m-1}]) \simeq E_{m-1}
\]
As \( (F, \partial, \{\eta^s\}_{s=1,\ldots,n}) \) is equivalent to \( (E, d(h)_{i=1,\ldots,n}) \) in \( \text{Sing}(B, f) \), we have proved the proposition.

Then the following structure theorem holds:

**Theorem 2.5.** Let \( (\text{Spec}(B), f) \) be a \( n \)-dimensional affine Landau-Ginzburg model over \( S \). Then every object in the dg-category of relative singularities \( \text{Sing}(B, f) \) is a retract of an object represented by a \( (K(B, f)) \)-dg module concentrated in \( n + 1 \) degrees.

**Proof.** Let \( (E, d, \{h^s\}_{s=1,\ldots,n}) \) be an object in \( \text{Coh}^n(B, f) \) concentrated in degrees \([m',m]\). We produce an inductive argument on amplitude \( a = m - m' + 1 \) of the interval where \( (E, d, \{h^s\}_{s=1,\ldots,n}) \) is nonzero. If \( m - m' \leq n \) there is nothing to prove. Otherwise, apply the previous proposition. Now, as the homotopy category of \( \text{Sing}(B, f) \) coincides with the idempotent completion of the Verdier quotient of the homotopy category of \( \text{Coh}^n(\text{Spec}(K(B, f))) \), we conclude.

\(^3\)Clearly, \( \partial \) and \( \{\eta^s\}_{s=1,\ldots,n} \) are induced by \( \text{cone}(\phi) \)
3 Orlov’s theorem

Matrix factorizations

It is well known (see [Orl04], [BW12], [EPo15], [BRTV]) that the dg-category of relative singularities $\text{Sing}(B, f)$ associated to a 1-dimensional affine flat Landau-Ginzburg model over a regular local ring is equivalent to the dg-category of matrix factorizations $\text{MF}(B, f)$ introduced by Eisenbud ([Eis80]). In this section we shall recall what matrix factorizations are.

Context 3.1. In this section we will always work in the context of 1-dimensional LG models. Therefore, we will omit to say it explicitly.

Let $(\text{Spec}(B), f)$ be an affine LG model over $S$.

Definition 3.2. A matrix factorization over $(B, f)$ is the datum of a pair of projective $B$-modules of finite type $E_0, E_1$ together with $B$-linear morphisms $E_0 \xrightarrow{p_0} E_1$ and $E_1 \xrightarrow{p_1} E_0$ such that $p_1 \circ p_0 = f$ and $p_0 \circ p_1 = f$.

We can naturally organize matrix factorizations in a $\mathbb{Z}/2\mathbb{Z}$-graded dg-category $\text{MF}(B, f)$ as follows:

- the objects of $\text{MF}(B, f)$ are matrix factorizations over $(B, f)$;
- given two matrix factorizations $(E, p)$ and $(F, q)$ over $(B, f)$, we define the morphisms in degree 0 (resp. 1) $\text{Hom}^0((E, p), (F, q))$ (resp. $\text{Hom}^1((E, p), (F, q))$) as the $B$-module of pairs of $B$-linear morphisms $(\phi_0 : E_0 \to F_0, \phi_1 : E_1 \to F_1)$ (resp. $(\psi_0 : E_0 \to F_1, \psi_1 : E_0 \to F_1)$);
- given a map $(\chi_0, \chi_1) : (E, p) \to (F, q)$ of degree $i$ ($i = 0, 1$), we define $\delta((\chi_0, \chi_1)) := q \circ \chi - (-1)^i \chi \circ p$;
- composition and identities are defined in the obvious way.

Then we can view $\text{MF}(B, f)$ as an $A$-linear dg category by means of the structure morphism $A \to B$.

Remark 3.3. Notice that since we are considering projective $B$-modules and $B$ is flat over $A$, $\text{MF}(B, f)$ is a locally flat $A$-linear dg-category.

The homotopy category of $MF(B, f)$ has a triangulated structure: the suspension is defined as

$$(E_0 \xrightarrow{p_1} E_1)[1] = E_1 \xrightarrow{-p_0} E_0 \quad (3.3.1)$$

and the cone of a closed morphism $(\phi) : (E, p) \to (F, q)$ is defined by

$$F_0 \oplus E_1 \xrightarrow{\begin{bmatrix} q_1 & 0 \\ q_0 & \phi_1 \end{bmatrix}} F_1 \oplus E_0 \quad (3.3.2)$$

See [Orl04] for more details. Moreover, $\text{MF}(B, f)$ has a symmetric monoidal structure, defined by

$$(E, p) \otimes (F, q) = (E_0 \otimes_B F_0) \oplus (E_1 \otimes_B F_1) \xrightarrow{\begin{bmatrix} p \otimes q \\ 0 \end{bmatrix}} (E_0 \otimes_B F_1) \oplus (E_1 \otimes_B F_0) \quad (3.3.3)$$

As explained in [BRTV], it is possible to define a lax monoidal $\infty$-functor

$$\text{MF}(\bullet, \bullet) : \text{LG}(1)_{\text{aff,op,} \Bbb{Z}} \to \text{dgcat}_{A}^{\text{idem,} \otimes} \quad (3.3.4)$$

It is then possible to extend it to $\text{LG}(1)_{\text{op,} \Bbb{Z}}$ by Kan extension. With a little abuse of notation, we still denote this extension by

$$\text{MF}(\bullet, \bullet)^{\otimes} : \text{LG}(1)_{\text{op,} \Bbb{Z}} \to \text{dgcat}_{A}^{\text{idem,} \otimes} \quad (3.3.5)$$

We refer to [BRTV] for more details.

Remark 3.4. There exists a second definition of matrix factorizations for non-affine LG-models $(X, f)$, see [BW12], [Efi18], [Orl12]. If $X$ is a separated scheme with enough vector bundles, the two definitions agree.

Remark 3.5. Being a lax monoidal $\infty$-functor, $(3.3.5)$ factors through $\text{Mod}_{\text{MF}(A, 0)}(\text{dgCat}_{S}^{\text{idem,} \otimes})$. 

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More on the structure of Sing$(B, f)$

As the Koszul algebra $K(B, f)$ is particularly simple, in the case $n = 1$ it is possible to give a more detailed description of the objects of Sing$(B, f)$. This is what we will do in the following. Our first remark concerns the periodicity of the dg-category Sing$(B, f)$.

**Lemma 3.6.** Let $\frac{E_{n-1}}{E_{n-2}} \xleftarrow{\frac{q}{p}} \frac{E_n}{F_n}$ be an object in Coh$(B, f)$. Then it is equivalent to $\frac{E_{n-1}}{E_{n-2}} \xrightarrow{\frac{p}{q}} \frac{E_n}{F_n}$ in Sing$(B, f)$.

**Proof.** Consider $\left( E \xrightarrow{p} F \right) \otimes_B K(B, f) \in \text{Perf}^*(B, f)$. This is the $K(B, f)$ dg-module

\[
\begin{pmatrix}
0 & 1 \\
-p & f
\end{pmatrix} \xleftarrow{\begin{pmatrix}
1 \\
0
\end{pmatrix}}
\begin{pmatrix}
E_{n-2} \\
E_{n-1}
\end{pmatrix} \xleftarrow{\begin{pmatrix}
1 & 0 \\
f & p
\end{pmatrix}}
\frac{E_n}{F_n}
\tag{3.6.1}
\]

Then let $\phi$ be the following morphism of $K(B, f)$ dg modules:

\[
\begin{pmatrix}
0 & 1 \\
-p & f
\end{pmatrix} \xleftarrow{\begin{pmatrix}
1 & 0 \\
f & p
\end{pmatrix}}
\begin{pmatrix}
E_{n-2} \\
E_{n-1}
\end{pmatrix} \xleftarrow{\begin{pmatrix}
1 \\
0
\end{pmatrix}}
\frac{E_n}{F_n}
\tag{3.6.2}
\]

This morphism exhibits an equivalence in Sing$(B, f)$ between $E \xleftarrow{\frac{q}{p}} F$ and cone$(\phi)$, which is

\[
\begin{pmatrix}
0 & -1 \\
p & -f
\end{pmatrix} \xleftarrow{\begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix}}
\begin{pmatrix}
E_{n-2} \\
E_{n-1}
\end{pmatrix} \xleftarrow{\begin{pmatrix}
0 & q \\
1 & p
\end{pmatrix}}
\frac{E_n}{F_n}
\tag{3.6.3}
\]

and can be written as the cone of the following morphism of $K(B, f)$ dg-modules

\[
\begin{pmatrix}
0 & -1 \\
p & -f
\end{pmatrix} \xleftarrow{\begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix}}
\begin{pmatrix}
E_{n-2} \\
E_{n-1}
\end{pmatrix} \xleftarrow{\begin{pmatrix}
0 & q \\
1 & p
\end{pmatrix}}
\frac{E_n}{F_n}
\tag{3.6.4}
\]

Notice that the source of this morphism is $E \otimes_B K(B, f)$, where $E$ is a complex concentrated in degree $n - 1$. In particular, as $E$ is a projective $B$-module, it is a perfect $K(B, f)$ dg-module. Therefore, in Sing$(B, f)$, the target of this morphism is equivalent to $E \xleftarrow{\frac{q}{p}} F$. Then consider the following morphism of $K(B, f)$ dg-modules:

\[
\begin{pmatrix}
-1 & 0 \\
-f & q
\end{pmatrix} \xleftarrow{\begin{pmatrix}
0 & q \\
1 & p
\end{pmatrix}}
\begin{pmatrix}
E_{n-2} \\
E_{n-1}
\end{pmatrix} \xleftarrow{\begin{pmatrix}
0 & q \\
1 & p
\end{pmatrix}}
\frac{E_n}{F_n}
\tag{3.6.5}
\]
It is not hard to verify that this is a quasi-isomorphism. Following the chain of equivalences in \( \text{Sing}(B, f) \) we get that

\[
\begin{array}{c}
E_{n-1} \xrightarrow{q} E_{n} \xrightarrow{p} E_{n-1}
\end{array}
\]

\[\square\]

**Corollary 3.7.** Let \( E_{n-1} \xleftarrow{q} E_{n} \xrightarrow{p} E_{n-1} \) be in \( \text{Coh}^*(B, f) \). Then

\[
\left( E_{n-1} \xrightarrow{q} E_{n} \right) [1] \simeq \left( E_{n-1} \xrightarrow{-p} E_{n} \right)
\]

in \( \text{Sing}(B, f) \).

**Proof.** In \( \text{Coh}^*(B, f) \), we know that \( \left( E_{n-1} \xrightarrow{q} E_{n} \right) [1] \) is equivalent to \( \left( E_{n-2} \xleftarrow{-p} E_{n-1} \right) \). Then, by **Lemma** (3.6) we get

\[
\left( E_{n-1} \xrightarrow{q} E_{n} \right) [1] \simeq \left( E_{n-2} \xrightarrow{-p} E_{n-1} \right) \simeq \left( E_{n-1} \xrightarrow{-p} E_{n} \right)
\]

\[\square\]

We will now provide an explicit description of the image of an object via the quotient functor

\[
\text{Coh}^*(B, f) \to \text{Sing}(B, f)
\]

**Theorem 3.8.** Let

\[
(E, d, h) = 0 \longrightarrow E_m \xrightarrow{h_{m+1}} E_{m+1} \xleftarrow{d_m} \cdots \xrightarrow{h_{-1}} E_{m-1} \xrightarrow{h_0} E_{m'} \longrightarrow 0
\]

be an object in \( \text{Coh}^*(B, f) \). Then the following equivalence holds in \( \text{Sing}(B, f) \):

\[
(E, d, h) \simeq \bigoplus_{i \in \mathbb{Z}} E_{2i-1} \xleftarrow{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i}
\]

Moreover, it is natural in \((E, d, h)\).

**Proof.** The first part of the proof is the same as the one of **Theorem 2.5** but we rewrite it in an explicit manner for the reader’s convenience. Moreover, we will assume that \( m = -2n + 1 \) for some \( n > 0 \) (if \( m = -2n + 2 \), just put \( E_{-2n+1} = 0 \) and that \( m' = 0 \). It is clear that this does not compromise the generality of the proof.

Consider the perfect \( K(B, f) \)-dg module \((E, d) \otimes_B K(B, f)\) and the following morphism \( \phi : (E, d) \otimes_B K(B, f) \to (E, d, h) \) of \((B, f)\)-dg modules:

\[
\begin{array}{c}
E_{-2n+1} \xrightarrow{0} E_{-2n+2} @\xleftarrow{1} E_{-2n+1} \xrightarrow{0} E_{-2n+3} @\xleftarrow{0} E_{-2n+2} \xrightarrow{0} \cdots \xrightarrow{0} E_0 @\xleftarrow{1} E_{-1} \xrightarrow{0} E_0 \\
\end{array}
\]

\[
\begin{array}{c}
\begin{bmatrix} f \\ h_{-2n+2} \end{bmatrix} \begin{bmatrix} f \\ d_{-2n+1} \end{bmatrix} \begin{bmatrix} f \\ d_{-2n+2} \end{bmatrix} \begin{bmatrix} f \\ d_{-2n+3} \end{bmatrix} \begin{bmatrix} f \\ \cdots \end{bmatrix} \begin{bmatrix} f \\ d_{-1} \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} \begin{bmatrix} f \\ 1 \end{bmatrix} \begin{bmatrix} f \\ 1 \end{bmatrix} \end{array}
\]

\[\square\]

\[\text{recall that } (E, d) \text{ is a perfect } B\text{-dg module}\]
Then, \( \text{Cone}(\phi) \) is equivalent to \((E, d, h)\) in \( \text{Sing}(B, f) \). \( \text{Cone}(\phi) \) is the \( K(B, f) \)-dg module

\[
\begin{bmatrix}
0 & -1 \\
-2n & -f \\
-2n & -2n+2 & 0 \\
-2n+1 & h_{-2n+2} & 1 \\
0 & 0 & h_{-2n+2} \\
0 & 0 & d_{-2n+1} \\
-2n+1 & d_{-2n+3} & 0 \\
-2n+2 & -f & 0 \\
-2n+2 & h_{-2n+3} & d_{-2n+1}
\end{bmatrix}
\]

\[
(3.8.4)
\]

which can be seen as the cone of the following morphism (call it \( \varphi \))

\[
\begin{bmatrix}
0 & 1 \\
-2n & -d_{-2n+1} \\
-2n+1 & f \\
0 & 0 & h_{-2n+2} \\
0 & 0 & -d_{-2n+2} \\
0 & 0 & -d_{-2n+2} \\
0 & 0 & 1 \\
0 & -f & 0 \\
0 & h_{-2n+3} & d_{-2n+1}
\end{bmatrix}
\]

\[
(3.8.5)
\]

As the source of this morphism is \( (E_{-2n+1} \xrightarrow{d_{-2n+1}} E_{-2n+2}) \mathcal{O}_B K(B, f) \), it is a perfect \( K(B, f) \)-dg module.

Therefore, in \( \text{Sing}(B, f) \) we have that

\[
(E, d, h) \simeq \text{cone}(\phi) = \text{cone}(\varphi) \simeq \text{target}(\varphi)
\]

The cohomology groups in degree \(-1\) and \(0\) of \( \text{target}(\varphi) \) vanish. Therefore, we have found that in \( \text{Sing}(B, f) \) \((E, d, h)\) is equivalent to

\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & h_{-2n+2} \\
0 & 0 & -d_{-2n+2} \\
0 & 0 & 1 \\
-d_{-2n+1} & 0 & 0 \\
-d_{-2n+2} & -f & 0 \\
-d_{-2n+2} & h_{-2n+3} & d_{-2n+1}
\end{bmatrix}
\]

\[
(3.8.6)
\]

where

\[
K = \text{Ker}\left( \begin{bmatrix} -f & -d_{-1} & 0 \\ h_0 & 1 & d_{-2} \end{bmatrix} \right)
\]

This is still an element in \( \text{Coh}'(B, f) \). Indeed, from the short exact sequence of \( B \)-modules

\[
0 \rightarrow \text{Ker}(\begin{bmatrix} 1 & d_{-1} \end{bmatrix}) \rightarrow E_0 \oplus E_{-1} \xrightarrow{\begin{bmatrix} 1 & d_{-1} \end{bmatrix}} E_0 \rightarrow 0
\]

since \( E_0 \) and \( E_{-1} \) are \( B \)-projective, we conclude that \( \text{Ker}(\begin{bmatrix} 1 & d_{-1} \end{bmatrix}) \) is \( B \)-projective too. As the complex \( \text{cone}(\varphi) \) is exact in degree \(-1\), we also have the following short exact sequence of \( B \)-modules:

\[
0 \rightarrow K \xrightarrow{\begin{bmatrix} -f & -d_{-1} & 0 \\ h_0 & 1 & d_{-2} \end{bmatrix}} \text{Im}(\begin{bmatrix} -f & -d_{-1} & 0 \\ h_0 & 1 & d_{-2} \end{bmatrix}) \rightarrow 0
\]

\[
= \text{Ker}\left( \begin{bmatrix} f & d_{-1} \end{bmatrix} \right)
\]
As $E_0$, $E_{-1}$, $E_{-2}$ and $Ker\left[\begin{array}{cc} f & d_{-1}\end{array}\right]$ are projective $B$-modules, we conclude.

Notice that we have found, in $\text{Sing}(B,f)$, an equivalence between $(E,d,h)$ (which is concentrated in degrees $[-2n+1, 0]$) and an object represented by a complex concentrated in degrees $[−2n+1, −2]$. Therefore, by induction, we have proved that the image of $(E,d,h)$ is equivalent, in $\text{Sing}(B,f)$, to a $K(B,f)$-dg module concentrated in degrees 0 and −1 (i.e. by a matrix factorization). Nevertheless, we can do better than this. Indeed, notice that the $K(B,f)$-dg module (3.8.6) can be written as the cone of the following morphism of $K(B,f)$-dg modules:

$$
\begin{array}{ccc}
E_{-2n+1} & \xleftarrow{1} & E_{-2n+3} \\
-2n+1 & & -2n+3 \\
\end{array}
\begin{array}{ccc}
& f & \\
& & -d_{-2n+3}\
\end{array}
\begin{array}{ccc}
\left[\begin{array}{c}
0 \\
h_{-2n+2} \\
\end{array}\right] & \xrightarrow{\left[\begin{array}{c}
0 \\
0 \\
h_{-2n+3} \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
h_{-2n+3} \\
\end{array}\right] & \xleftarrow{\left[\begin{array}{c}
0 \\
0 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+3} \\
-2n+2 \\
\end{array}\right] \\
E_{-2n+1} & \xleftarrow{\left[\begin{array}{c}
0 \\
h_{-2n+2} \\
h_{-2n+3} \\
\end{array}\right] + E_{-2n+1} \oplus E_{-2n+2} + E_{-2n+3}} & E_{-2n+5} & \xleftarrow{\left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+3} \\
-2n+2 \\
\end{array}\right] \\
-2n+4 & \xleftarrow{\left[\begin{array}{c}
0 \\
h_{-2n+4} \\
h_{-2n+5} \\
\end{array}\right] + E_{-2n+4} \oplus E_{-2n+5} \oplus E_{-2n+3}} & E_{-2n+9} & \xleftarrow{\left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+3} \\
-2n+2 \\
\end{array}\right] \\
\ldots & \xleftarrow{\ldots} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] & \xleftarrow{\ldots} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] \\
K_{-2} & & & & \\
\end{array}
$$

As the source of this morphism is $E_{-2n+3} \otimes_B K(B,f)$, and $E_{-2n+3}$ is a perfect $B$-module, this morphism provides an equivalence between $(E,d,h)$ and the target in $\text{Sing}(B,f)$. Moreover, we can iterate this procedure: the target of this morphism can be written as the cone of the following morphism:

$$
\begin{array}{ccc}
E_{-2n+4} & \xleftarrow{1} & E_{-2n+4} \\
-2n+4 & & -2n+4 \\
\end{array}
\begin{array}{ccc}
& f & \\
& & -d_{-2n+4}\
\end{array}
\begin{array}{ccc}
\left[\begin{array}{c}
0 \\
h_{-2n+2} \\
h_{-2n+3} \\
\end{array}\right] & \xrightarrow{\left[\begin{array}{c}
0 \\
0 \\
h_{-2n+4} \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
h_{-2n+4} \\
\end{array}\right] & \xleftarrow{\left[\begin{array}{c}
0 \\
0 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+4} \\
-2n+2 \\
\end{array}\right] \\
E_{-2n+1} & \xleftarrow{\left[\begin{array}{c}
0 \\
h_{-2n+2} \\
h_{-2n+3} \\
\end{array}\right] + E_{-2n+1} \oplus E_{-2n+2} + E_{-2n+3}} & E_{-2n+5} & \xleftarrow{\left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+4} \\
-2n+2 \\
\end{array}\right] \\
-2n+4 & \xleftarrow{\left[\begin{array}{c}
0 \\
h_{-2n+4} \\
h_{-2n+5} \\
\end{array}\right] + E_{-2n+4} \oplus E_{-2n+5} \oplus E_{-2n+3}} & E_{-2n+9} & \xleftarrow{\left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right]} & \left[\begin{array}{c}
0 \\
d_{-2n+4} \\
-2n+2 \\
\end{array}\right] \\
\ldots & \xleftarrow{\ldots} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] & \xleftarrow{\ldots} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] \\
K_{-2} & & & & \\
\end{array}
$$

Once again, as the source of this morphism of $K(B,f)$-dg modules is perfect, we obtain an equivalence between $(E,d,h)$ and the target of the morphism in $\text{Sing}(B,f)$. Proceeding this way, we obtain a chain of equivalences between our initial $K(B,f)$-dg module and the following:

$$
\begin{array}{ccc}
E_{-2n+1} & \xleftarrow{h_{-2n+2}} & E_{-2n+1} \\
-2n+1 & \xleftarrow{d_{-2n+1} - 2n+2} & -2n+2 \\
\end{array}
\begin{array}{ccc}
& \xrightarrow{h_{-2n+3}} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] \\
& & \xleftarrow{\ldots} \\
\end{array}
\begin{array}{ccc}
E_{-4} & \xleftarrow{h_{-4}} & E_{-2n+1} \oplus E_{-2n+2} \oplus E_{-2n+3} \\
-4 & \xleftarrow{d_{-4}} & \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] \\
\end{array}
\ldots \xrightarrow{\ldots} \left[\begin{array}{c}
0 \\
-1 \\
0 \\
-1 \\
1 \\
\end{array}\right] \\
K_{-2} & & \\
\end{array}
$$

By an induction argument, this $K(B,f)$-dg module is equivalent, in $\text{Sing}(B,f)$, to

$$
\bigoplus_{i\in\mathbb{Z}} E_{2n-1} \xleftarrow{\left[\begin{array}{cccc}
h_{-2n+2} & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 \\
\vdots & & & \vdots \\
0 & \ldots & 0 & h_{-2} \\
0 & \ldots & -1 & 0 \\
\end{array}\right]} E_{-2n+2} \oplus E_{-2n+4} \oplus \cdots \oplus E_{-4} \oplus K \\
$$

(3.8.10)
We can finally consider the following morphism of $K(B, f)$-dg modules concentrated in degrees $-1$ and $0$

$$
\bigoplus_{i \in \mathbb{Z}} E_{2i-1} \xrightarrow{id} \bigoplus_{i \in \mathbb{Z}} E_{2i-1} \\
\oplus \quad \oplus
\uparrow \quad \uparrow \quad \uparrow \\
\oplus \quad \oplus
\downarrow \quad \downarrow \quad \downarrow \\
\bigoplus_{i \in \mathbb{Z}} E_{2i} \xrightarrow{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i-1}
$$

It is not hard to check that morphism $\text{(3.8.11)}$ is a quasi-isomorphism. Notice that the target of $\text{(3.8.11)}$ is equivalent in $\text{Sing}(B, f)$ to the $K(B, f)$-dg module $(E, d, h)$.

Also notice that since all the passages above are functorial, the equivalence is natural in $\text{Sing}(B, f)$.

In particular, a morphism of $K(B, f)$-dg modules $\phi : (E, d, h) \to (E', d', h')$ corresponds, under this equivalence, to

$$
\begin{align*}
\bigoplus_{i \in \mathbb{Z}} E_{2i-1} & \xrightarrow{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i-1} \\
\oplus \phi_{2i-1} & \quad \oplus \phi_{2i} \\
\bigoplus_{i \in \mathbb{Z}} E_{2i-1} & \xrightarrow{d' + h'} \bigoplus_{i \in \mathbb{Z}} E_{2i-1}
\end{align*}
$$

Remark 3.9. The algorithm we have provided actually puts the final $K(B, f)$-dg module

$$
E_{-2n+1} \oplus E_{-2n+3} \oplus \cdots \oplus E_{-3} \oplus E_{-1} \xrightarrow{d + h} E_{-2n+2} \oplus E_{-2n+4} \oplus \cdots \oplus E_{-2} \oplus E_0
$$

in degrees $-2n + 1$ and $-2n + 2$. However, thanks to Lemma 3.6, this is equivalent in $\text{Sing}(B, f)$ to the same dg-module concentrated in degrees $-1$ and $0$

Corollary 3.10. Let $\phi : E_{-1} \xrightarrow{d} E_0 \to E'_{-1}$ be a closed morphism in $\text{Coh}^+(B, f)$. Then $\text{cone}(\phi)$ is equivalent to $E'_{-1} \oplus E_0 \xrightarrow{d'} E'_0 \oplus E_{-1}$ in $\text{Sing}(B, f)$.

Proof. This is a straightforward consequence of the computation of $\text{Cone}(\phi)$ in $\text{Coh}^+(B, f)$ and of the previous theorem.

Corollary 3.11. The lax monoidal $\infty$-natural transformation

$$
\text{O}r l^{1, \otimes} : \text{Sing}(\bullet, \bullet) \to \text{MF}(\bullet, \bullet) : \text{LG}_S(1)^{op, \otimes} \to \text{dgCat}_S^\text{sym, \otimes}
$$

constructed in [BRTV, §2.4] defines a lax-monoidal $\infty$-natural equivalence.
Proof. By Kan extension and descent, it is sufficient to consider the affine case. Let $(\text{Spec}(B), f) \in LG_{S(1)^{\text{aff.op}}}$. As the dg-categories $\text{Sing}(B, f)$ and $\text{MF}(B, f_s)$ are triangulated, it is sufficient to show that the induced functor

$$[Orl^{-1}] : [\text{Sing}(B, f)] \to [\text{MF}(B, f)]$$

$$(E, d, h) \mapsto \bigoplus_{i \in \mathbb{Z}} E_{2i-1} \mapsto \frac{d + h}{d + h} \bigoplus_{i \in \mathbb{Z}} E_{2i},$$

is an equivalence. Consider

$$Orl : [\text{MF}(B, f)] \to [\text{Sing}(B, f)]$$

$$E \xleftarrow{\frac{q}{p}} F \mapsto E_{-1} \xleftarrow{\frac{q}{p}} F_{0}$$

This is an exact functor between triangulated categories by Corollary 3.7 and by Corollary 3.10. It is clear that $[Orl^{-1}] \circ Orl$ is the identity functor. By Theorem 3.8, $Orl \circ [Orl^{-1}]$ is equivalent to the identity functor too. \hfill \square

Remark 3.12. Notice that $Orl$ is a derived version of the "Cok" functor introduced in [Orl04]. Indeed, when $f$ is flat, the $K(B, f)$-dg module $\text{coker}(p)$ concentrated in degree 0 is quasi-isomorphic to $E_{-1} \xleftarrow{\frac{q}{p}} F_{0}.$

Remark 3.13. In [EfPo15], the authors also introduced a coherent version of $\text{MF}(B, f)$. When $f$ is flat, they proved it to be equivalent to another category of singularities, defined as the Verdier quotient

$$\text{Sing}(B, f)_{\text{coh}} = \text{Coh}^b(B/f)/E$$

(3.13.1)

where $E$ is the thick subcategory of $\text{Coh}^b(B/f)$ generated by the image of the pullback $\iota^* : \text{Coh}^b(B) \to \text{Coh}^b(B/f)$.

Our proof of Theorem 3.8 also tells us that, for any $f$, all objects in this triangulated category can be represented by $K(B, f)$-dg modules concentrated in degrees $[-1, 0]$. This can be used to show that the equivalence proven in [EfPo15] holds for any potential $f$, provided that we consider the derived fiber instead of $B/f$.

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