Spectral Densities
and Borel Transforms
in Compton scattering

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ABSTRACT

We show that the leading double spectral density in sum rules for Compton-like processes can be obtained by simple properties of the Borel transform, extending an approach widely used in the literature on sum rules, and known to be valid only for the spectral densities of form factors. The extension is illustrated in the scalar case, where it is shown to be consistent with Cutkosky rules. Using arguments based on the analyticity properties of the vertex and the box diagram, we show that Compton scattering is, however, a favourable case and indeed possible disagreements between the two methods are likely to be encountered in more general situations.
1 Introduction

There are two main methods, developed in the past for 2 and 3 point functions \([2, 3]\) in QCD sum rules, which allow a calculation of the spectral function, for any specific choice of the interpolating currents. One of them is based on the cutting rules for the propagator \([3]\), while a second method employs the Borel transform in a rather original way \([1]\). While for 2-point functions the calculation of the spectral density is rather straightforward and the cutting rules are probably the simplest way to approach the problem, for 3-point functions the calculations are not so obvious and an independent check of the result obtained in one way, by some other independent method, is most welcomed. These checks have been done in the past in the case of 3-point functions – to lowest order – and the technical value of these results cannot be underestimated since the analytic structure of vertex functions, for on-shell external legs, even to lowest order, is non-trivial. We remind that in the sum rule approach to exclusive reactions the first contribution to the spectral density for form factors comes from a diagram which does not contain any gluon and the double discontinuity of it – the spectral density – is calculated from a simple triangle diagram. When we consider 4-point functions, it is almost mandatory to check results obtained by one method (the cutting rules) with those obtained by the "Borel method" of Ref. \([1]\) and show that indeed there is agreement between the two. The example we have in mind is pion Compton scattering. It has been discussed elsewhere \([2]\) that this process can be described, in the region of moderate \(s; t; u\) invariants by sum rule methods. The approach proposed is formulated in full agreement with the usual canons of QCD sum rules, but the larger complexity of the method requires, at various stages, a consolidation of the methods employed in the derivation of the spectral density of this process and a clear comprehension of its relation to the form factor case. One of the most striking features of the leading spectral density for Compton scattering at fixed angle is the independence of such a function from the virtualities of the two pions, a fact which is completely new compared to the case of the form factors. In this note we show that the agreement between the Borel method of Ref. \([1]\) and the cutting rules of \([3]\) in the derivation of the hadronic spectral density...
density can be understood from a rather simple perspective, related
to the analyticity of the spectral function in the momentum transfer.
Similar arguments can be formulated for pion Compton scattering. A
simple analytic continuation of the box diagram - in the case of Com-
pton scattering - gives us the possibility to apply the Borel method to
this more complex case directly, and gives a result which can be easily
understood in terms of the Cutkosky rules. Our discussion here is of
technical nature, we refer to [2] for a more comprehensive treat men t.

2 The Borel method

Let's focus our attention on Fig. 1b. This diagram gives the leading
spectral function for form factors to lowest order in \( q^2 \) [3]. The dashed
lines describe the usual cutting rules for the propagators. If we assume a
scalar fermions in the loop, the spectral density for such a diagram
takes the form (with \( p_2^2 = s_2 \); \( p_1^2 = s_1 \))

\[
Z(s_1; s_2; t) = \int \frac{d^4k}{(2\pi)^4} \frac{(k^2)}{((p_1 - k)^2 + (p_1 + q)^2)};
\]  

where the \(^+\) in the delta functions characterize the momentum \(^\text{cm}\) ow.
A direct calculation of this diagram gives

\[
Z(s_1; s_2; t) = \frac{1}{2} \frac{(s_1 + s_2 - t)^2}{(s_1 + s_2 - t)^2 + 4s_1s_2}^{(1/2)}
\]

The evaluation can be easily carried out in the Breit frame of the
two (pion) lines characterized by the virtualities \( p_1^2 \) and \( p_2^2 \). It easy to
recognize that \( [2] \) does not develop any additional singularity for positive \( s_1 \); \( s_2 \), for \( t = (p_2 - p_1)^2 \) in the physical region, which is the main
reason why the same result can be obtained by a completely different
method \([1]\). Another important observation, in this context, is the
independence of the discontinuity from any mass in the propagator
(see also \([1]\)). Since similar analyticity properties of the spectral den-
sity turn out to be still valid for Compton scattering (in the massless
\( q^2 \) case), we are going to illustrate in some more detail how the "Borel
method" \([1]\) works first in the case of 3-point functions and, later on,
for Compton scattering. In our rederivation of the spectral densities
for the pion we hope to clarify the connections between the cutting rules and the latter approach.

We start from two-point functions. Let's consider the dispersion relation of the polarization operator

\[
(q_i^2) = \frac{1}{Z} \int_0^1 \frac{(s)ds}{(s - q_i^2)} + \text{subtractions};
\]

(3)

with a singularity cut starting at \( q_i^2 = 0 \). Eq. (3) can also be written in the form

\[
(q_i^2) = \frac{1}{Z} \int_0^1 ds \, d(s) e^{(s - q_i^2)}; \quad q_i^2 < 0;
\]

(4)

where we have used the exponential parametrization of the denominator.

The Borel transform in one variable is defined in its differential version by the operator \([5]\]

\[
B(Q^2; M^2) = \lim_{Q^2 \to M^2} \frac{1}{(n+1)!} (Q^2)^n \left( \frac{d^m}{dQ^2} \right)^n;
\]

(5)

\( M^2 \) denotes the Borel mass.

It satisfies the identity

\[
B(Q^2; M^2) e^{Q^2} = (1 - M^2) Q^2 > 0;
\]

(6)

Acting on the polarization operator \((q_i^2)\) with the Borel transform we get the usual exponential suppression of the higher states

\[
M^2 B(q_i^2; M^2) (q_i^2) = \frac{1}{Z} \int_0^1 ds \, (s) e^{s-M^2};
\]

(7)

At this point we can Borel transform once again eq. (7), with respect to the inverse Borel mass \( l = M^2 \), in order to obtain

\[
B(l = M^2) (M^2) B(q_i^2; M^2) (q_i^2) = \frac{1}{l = M^2};
\]

(8)

Eq. (8) shows that by applying in a sequence Borel transform \(s\) on \((q_i^2)\) we obtain an expression from which we can easily identify the
spectral weight \((s)\) of eq. (3). Let's now come to 3-point functions. The Borel transformed amplitude for the pion form factor is given by \([1]\]

\[
\langle M^2_1 ; M^2_2 ; \vec{q}^2 \rangle = \frac{1}{2} \int_0^1 ds_1 \int_0^1 ds_2 \exp \left( -\frac{s_1}{M^2_1} - \frac{s_2}{M^2_2} \right)
\]

\[
= \frac{3}{2^2 (M^2_1 + M^2_2)} \int_0^1 dx (1 \times x) \exp \left( \frac{x\vec{q}^2}{M^2_1 + M^2_2} \right) \quad (9)
\]

To isolate the spectral function from eq. (9) we need to use Borel transforms and act on it with the differential operator

\[
B \quad (1= M^2_1 \quad 1= M^2_2 \quad 1= M^2_1 M^2_2) : (10)
\]

Here we discuss a possible way of doing this. The inversion of (9) can be obtained by taking inverse Laplace transforms twice, respect to \(1= M^2_1\) of this equation. In fact, for a given function \(F \left( M^2 \right)\), the following identity

\[
L \quad (1= M^2 \quad 1= M^2 \quad 1= M^2 \quad 1= M^2 \quad 1= M^2) = \quad (l= )B \quad (l= M^2 \quad 1= )F \quad (M^2) \quad (11)
\]

relates the differential operator given in (5) to the inverse Laplace transform

\[
L \quad (1= M^2 \quad 1= M^2 \quad 1= M^2 \quad 1= M^2 \quad 1= M^2) = \frac{1}{2} \int_0^1 dx \times (1 \times x) \exp \quad \exp \quad \exp \quad \exp \quad \exp
\]

Defining \(1= M^2_1 = 1; 1= M^2_2 = 2\), then we are required to act with the operator defined in eq. (10) on the integral function

\[
\langle 1; 2; \vec{q}^2 \rangle = \frac{1}{1+2} \int_0^1 dx \times (x+1) \exp \quad \exp \quad \exp \quad \exp
\]

Using the Gaussian relation

\[
\exp \quad \exp \quad \exp
\]

we can rewrite eq. (5) into the form

\[
\exp \quad \exp
\]

we can rewrite eq. (5) into the form
By using the relation

\[ L^1 (1! 1) \frac{x \exp x q^2}{(1 + 2)^{1-2}} = \exp 1 + 2^2 = \frac{1}{(1 2)} (1 2) \]  

on (15) we finally get

\[ L^1 (2! 2) L^1 (1! 1) (1; 2q^2) = L^1 (2! 2) L^1 (1! 1) (1; 2q^2) = \]

\[ = \frac{Z_1}{Z_1} \frac{Z_1}{Z_1} \frac{x \exp x q^2 (1 + 2)^2 2x^{1-2} q}{(x + 1)^4 (1 + 2))^{1-2}} : (15) \]

By redenning \( i = j \); \( i = 1; 2 \), and after doing the explicit integration of in eq. (15), it is easy to relate this last result to the 3-particle cut integral for the triangle diagram (Fig. 1b), in the scalar case,

\[ Y (q^2; s_1; s_2) = \frac{Z_1}{Z_1} \frac{Z_1}{Z_1} \frac{d}{f^{1-2} (1 + 2)^2 ((f + 1-2)^2 + q^2)} ; \]

\[ f = 2 \quad 1 \quad 2 ; \]

(18)

The derivative with respect to \( q^2 \) in eq. (15) takes into account the fermionic character of the propagators in Fig. 1b compared to the
The spectral function for the pion form factor can be expressed in the form \[ \text{pert} \frac{3}{3}(s_1, s_2; t) \]

\[
= \frac{3}{2} t^2 \frac{d^1}{d t^12} + \frac{t^2}{3} \frac{d^1}{d t^13} \left( \frac{1}{(s_1 + s_2 - t)} \frac{1}{4s_1 s_2} \right) \text{ for } t = q^2.
\]

For four-point functions this procedure simplifies considerably. The method can be applied exactly as in the form factor case, although it is necessary to work in the Euclidean region from the beginning.

To be specific let's consider the full contribution to the box diagram (not just its cut) related to Fig. 1 and, for simplicity, let's restrict our considerations to the scalar case. We consider therefore the following 4-point function

\[
T_4 = \frac{Z}{k^2(p_1 - k)^2(p_1 + q_1)^2(p_2 - k)^2} \frac{d^4k}{k^2(p_1 - k)^2(p_1 + q_1)^2(p_2 - k)^2} \text{ for } m \text{ massless propagators. At fixed angle and with } s + t + u = p_1^2 + p_2^2 \text{ where } s > 0; t < 0; u < 0
\]

and moderately large \( s; t; u \), using Cauchy's theorem on \( T_4 \), we can assume the validity of a spectral representation for such integral of the form

\[
T_4 = \frac{1}{4} \frac{Z}{2} ds_1 \frac{Z}{2} ds_2 \frac{(s_1; s_2; s; t)}{(s_1; p_1^2);(s_2; p_2^2)} \text{ for } s + t + u = p_1^2 + p_2^2 \text{ where } s > 0; t < 0; u < 0
\]

where the contours \( s_1; s_2 \) are again chosen as in Fig. 2. If we introduce a double spectral function \( (s_1; s_2; s; t) \) we can rewrite \( T_4 \) as

\[
T_4(p_1^2; p_2^2; s; t) = \frac{1}{4^2} \frac{Z}{2} ds_1 \frac{Z}{2} ds_2 \frac{(s_1; s_2; s; t)}{(s_1; p_1^2);(s_2; p_2^2)} + \cdots
\]

where the neglected pieces involve a complex part of the contour. As we've already discussed the usual definition of the transform given by \( \left[ \text{\frac{5}{2}} \right] \) does not apply to this case since the virtualities in \( \left[ \text{\frac{5}{2}} \right] p_1^2 \)
and $p_2^2$ are forced to stay inside the area delimited by the contour in Fig. 4. In Ref. [2], using Cutkosky rules, it has been shown that the leading perturbative spectral function can be obtained, for this diagram, within the conditions on $s$ and $t$ given by (22), by the 3-cut integral (see Fig. 1a)

$$J(p_1^2; p_2^2; s; t; m = 0) = \frac{Z}{d^4 k} \frac{(k^2 + (p_1 \cdot k)^2 + (p_2 \cdot k)^2)}{(p_1 \cdot k + q_1)^2}.$$  (25)

The evaluation of (58) has also been discussed in [2] and the answer, for the associated spectral density, turns out to be rather simple

$$(s_1; s_2; s; t) = (2 \cdot 3) ^3 J(p_1^2; p_2^2; s; t)$$

$$= \frac{4i}{s t}.$$  (26)

Compared to the form factor case (see eq. (24)) the discontinuity along the cut as given by (26) is $s_1$ and $s_2$ independent. However such a simplification comes from having neglected the quark masses in eq. (24) and having not considered other subleading cuts in the evaluation of $(s_1; s_2; s; t)$, which are suppressed by power of $s; t$ or $u$, compared to the leading result. Here we intend to show that the result in (26) can be reobtained by the "Borel method" discussed in the previous section. Let's first notice that the spectral density, if we allow massive scalar propagators in eq. (24), even in the scalar case, is rather different from the expression given by eq. (26). Again, in this case as in the massless one, the leading spectral function is in fact obtained from the 3-particle cut integral

$$J(p_1^2; p_2^2; s; t; m) = \frac{Z}{d^4 k} \frac{(k^2 + m^2) + (p_1 \cdot k)^2 + (p_2 \cdot k)^2}{(p_1 \cdot k + q_1)^2}.$$  (27)

and it can be expressed in the form

$$(s_1; s_2; s; t; m) = (2 \cdot i) ^3 J(s_1; s_2; s; t; m)$$

$$= \frac{4i}{t(4m^2 s_1 s_2 + 4m^2 s s_2 + 4m^2 s s_1 + 4m^2 s^2 + 4m^2 s t s^2 t)}.$$  (28)
which reproduces eq. \((26)\) in the \(m = 0\) case. Compared to \((26)\), eq. \((28)\) shows that the spectral function develops a singularity at \(s_2\) dependent positions in the \(s_1\) plane, as it is expected in general. This shows that the simple result given by eq. \((28)\) is due to the approximation of massless quarks that we have considered. However, given the fact that the leading discontinuity of \(T_4\), for positive \(p_1^2\) and \(p_2^2\), is constant at \(\pi\) ed angle, eq. \((28)\) gives us the indication that we can extend the dispersion integral in \((24)\), with suitable subtractions, to in nity, without worrying about eventual additional thresholds which might be encountered for large positive virtualities of the pions in the dispersion relation. This more general feature of the spectral density (its mass dependence), as we mentioned above, is new compared to the form factor case. This also explains why both Cutkosky rules and the "Borel method", illustrated above, converge toward a unique answer in the case of vertex functions.

3 Application to Compton scattering

In order to confirm our reasoning, let's show that the scalar \((s_1; s_2; s; t)\) given by eq \((28)\) can be reobtained by the "Borel method" described above, a method whose validity relies primarily on the assumption that a given amplitude can be described by a suitable spectral representation. Once this fact is accepted, we act with Borel transforms on \((24)\) with \(\bar{z}\) going to in nity, exactly as in the form factor case.

Let's work in the euclidean region of \(T_4(p_1^2; p_2^2; s; t)\) with spacelike external invariants \((q_1^2; p_2^2 < 0, s = (p_1 + q_1)^2 < 0, t = (q_2 - q_1)^2 < 0)\). It is then possible to relate \(T_4\) to its euclidean continuation \(T_{4E}\) and use the Schwinger parametrization for the latter.

\[
T_{4E} = \int d^4k \frac{1}{Z} \frac{Z_1}{Z} \exp \int_0^\infty \frac{k^2}{2} (p_1^0 k)^2 (p_1^0 k + q_1^0)^2 (p_2^0 k)^2
\]

where \(\frac{1}{Z}\) is a short notation for the proper time parameters. Divergences, in this representation, reappear when we move into the physical \(s; t\) region. We perform the integration over the loop momentum.
in eq. \( (29) \) to get

\[
T_{4E} = \int_0^{Z_1} dx_1 dx_2 dx_3 dx_4 \; (l_1 x_1 x_2 x_3 x_4)^{Z_1} \; e^{Z_1} \; \text{deg}
\]  \( (30) \)

where

\[
Z_1 \; = \; (A_1(x_1)s_1 \; A_2(x_2)s_2 \; A_3(x_3; t))
\]  \( (31) \)

with suitable expression for \( A_1; A_2; A_3 \). \( T_{4E} \) is related to \( T_4 \) by the analytic continuation

\[
T_4(p_1^2; p_2^2; s; t) = iT_{4E}(p_1^0; p_2^0; s^0; t^0)
\]  \( (32) \)

in the region where \( p_1^2 = p_1^0 \) and, in general, \( p_1^2 \neq p_1^0 \). As we have discussed before, we can send the radius \( \epsilon \) of eq. \( (24) \) to infinity and introduce suitable subtractions in the corresponding dispersion integral in order to get

\[
T_4 = \frac{1}{4} \int_0^{Z_1} dx_1 dx_2 dx_3 dx_4 \; (l_1 \; x_1 x_2 x_3 x_4)^{Z_1} \; e^{Z_1} \; \text{deg} + \text{subtr:}
\]  \( (33) \)

which, in the euclidean region, becomes

\[
T_{4E} = \frac{1}{4} \int_0^{Z_1} dx_1 dx_2 dx_3 dx_4 \; (l_1 \; x_1 x_2 x_3 x_4)^{Z_1} \; e^{Z_1} \; \text{deg} + \text{subtr:}
\]  \( (34) \)

Exactly as in the case of the polarization operator (eqs. \( (23) \) and \( (24) \)), we can now apply Borel transforms on eq. \( (34) \) to get

\[
\epsilon = (M_1^2; M_2^2; s^0; t^0)
\]  \( (35) \)

\[
= B(p_1^0; 1 = M_1^2) B(p_2^0; 1 = M_2^2) T_{4E}(p_1^0; p_2^0; s^0; t^0)
\]  \( (36) \)
where we have defined

\[ b_1 = \frac{s(l \times 1 \times 2)}{x_2}, \]
\[ b_2 = \frac{(sx_1 + tx_2)}{x_1}; \quad (37) \]

Notice that \( B(p_{2i}^0 \! M_1^2) = B(p_{2i}^2 \! M_1^2), \) since \( p_{2i}^2 < 0 \) by assumption.

By acting similarly on \( T_4 \) as given by \((33)\) — with \( B(p_{2i}^2 \! l = M_1^2) \) — we also get

\[
M_1^2!M_2^2!s!t! = \frac{1}{4} \int_0^1 \int_0^1 ds_1 ds_2 (s_1; s_2; s; t)e^{s_1 M_1^2} e^{s_2 M_2^2} = B(p_{2i}^2 \! M_1^2) B(p_{2i}^2 \! M_2^2) T_4(p_{2i}^2; p_{2i}^2; s; t): \quad (38)
\]

Reapplying the Borel transform (this time with respect to \( l = M_1^2 \)) sequentially on both \( E \) and using eq. \((9)\) we get respectively

\[
B(1 = M_1^2 \! l = M_1^2 \! l = M_2^2 \! s_0 \! t_0) = \frac{1}{4} \int_0^1 \int_0^1 dx_1 dx_2 (l \ b_1 \ l) (l \ b_2 \ l) = \frac{1}{4} \int_0^1 \int_0^1 E(l = 1; l = 2; s_0^0; t_0^0) \quad (39)
\]

and

\[
B(1 = M_1^2 \! l = M_1^2 \! l = M_2^2 \! s_0 \! t_0) = \frac{1}{4} \int_0^1 \int_0^1 (l = 1; l = 2; s; t) = (40)
\]

As we can see from eq. \((33)\), the integral on the two parameters \( x_1 \) and \( x_2 \) is now trivial and can be performed straightforwardly. A simple calculation then gives

\[
E(l = 1; l = 2; s_0^0; t_0^0) = \frac{4}{1 \times 2 \times s_0^0 t_0^0}; \quad (41)
\]
Using the identification $l_1 = 1; s_1; l_2 = 1; s_2$, analytically continuing back with the prescriptions $s^0; s^0; t^0; t$ and using eq. (32) we finally get

$$ (p^2_1; p^2_2; s; t) = 4i \frac{4}{s t}; $$

which agrees with the one obtained from the cutting rules eq. (26). Within the Borel method difficulties related to the evaluation of discontinuity integrals are bypassed. The application of the Borel method to higher point functions, as we have seen, is simple, but, in principle, one should expect additional singularities in the perturbative spectral functions which do not allow a dispersion relation extending up to infinity in the plane of the two pion virtualities.

4 Conclusions

We have seen that there is a clear correspondence between two different methods for the evaluation of the perturbative spectral functions, both of them largely employed in the context of QCD sum rules. We have shown that their agreement can be extended to Compton-like processes and we have also pointed out some possible shortcomings. In the massless case a comparison between the two approaches turns out to be possible.

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Figure Captions

1a. Contribution to the spectral density for pion Compton scattering.

1b. The vertex diagram in the spectral density for the pion form factor.

2. The integration contours for the scalar amplitude in the $p_1^2$ plane, which includes possible additional thresholds.