VIABILITY PROBLEM WITH PERTURBATION IN HILBERT SPACE

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Abstract. This paper deals with the existence result of viable solutions of the differential inclusion
\[\dot{x}(t) \in f(t, x(t)) + F(x(t))\]
where \(K\) is a locally compact subset in separable Hilbert space \(H\), \((f(s, \cdot))_s\) is an equicontinuous family of measurable functions with respect to \(s\) and \(F\) is an upper semi-continuous set-valued mapping with compact values contained in the Clarke subdifferential \(\partial c V(x)\) of an uniformly regular function \(V\).

Key words: Regularity, upper semi-continuous, equicontinuous perturbation, Clarke subdifferential.

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1. Introduction

Existence result of local solution for differential inclusion with upper semi-continuous and cyclically monotone right hand-side whose values in finite-dimensional space, was first established by Bressan, Cellina and Colombo (see [6]). The authors exploited rich properties of subdifferential of convex lower semi-continuous function; in order to overcome the weakly convergence of derivatives of approximate solutions, they used the basic relation (see [7])
\[\frac{d}{dt} V(x(t)) = ||\dot{x}(t)||^2.\]

Later, Ancona, Cellina and Colombo (see [1]), under the same hypotheses as the above paper, extend this result to the perturbed problem
\[\dot{x}(t) \in f(t, x(t)) + F(x(t))\]
where \(f(\cdot, \cdot)\) is a Carathéodory function.

This program of research was pursued by a series of works. In the first one (see [9]), Truong proved a viability result for similar problem, where the perturbation \(f\) is replaced by a globally continuous set-valued mapping \(G\) with values in finite-dimensional space. This result was extended by Bounkkel (see [4]) for a similar EJQTDE, 2007 No. 7, p. 1
problem, where $F$ is not cyclically monotone but contained in the Clarke subdifferential of locally Lipschitz uniformly regular function. However under very strong assumptions namely, the space of states is finite-dimensional and the following tangential condition

$$\left( G(t, x) + F(x) \right) \subset T_K(x)$$

where $T_K(x)$ is the contingent cone at $x$ to $K$.

Recently, Morchadi and Sajid (see [8]) proved an exact viability version of the work of Ancona and Colombo assuming the same hypotheses and the following tangential condition

$$\forall (t, x) \in \mathbb{R} \times K, \exists v \in F(x) \text{ such that } \lim_{h \to 0^+} \inf \frac{1}{h} d_K \left( x + hv + \int_t^{t+h} f(s, x) ds \right) = 0.$$  (1.1)

Remark that in all the above works, the convexity assumption of $V$ and/or the finite-dimensional hypothesis of the space of states were widely used in the proof.

This paper is devoted to establish a local solution of the problem

$$\dot{x}(t) \in f(t, x(t)) + F(x(t)), \quad F(x(t)) \subset \partial_c V(x(t)),$$

where $K$ is a locally compact subset of a separable Hilbert space $H$, $F$ is an upper semi-continuous multifunction, $\partial_c V$ denotes the Clarke subdifferential of a locally lipschitz function $V$ and the set $\{f(s, .) : s \in \mathbb{R}\}$ is equicontinuous, where for each $x \in K, s \mapsto f(s, x)$ is measurable and the same tangential condition (1.1). One case deserves mentioning: when $f$ is globally continuous, the condition (1.1) is weaker than the following

$$\left( f(t, x) + F(x) \right) \cap T_K(x) \neq \emptyset.$$

To remove the convexity assumption of $V$ and the finite-dimensional hypothesis of $H$, we rely on some properties of Clarke subdifferential of uniformly regular function and the local compactness of $K$.

2. Preliminaries and statement of the main result

Let $H$ be a real separable Hilbert space with the norm $\| \cdot \|$ and the scalar product $\langle \cdot, \cdot \rangle$. For $x \in H$ and $r > 0$ let $B(x, r)$ be the open ball centered at $x$ with radius $r$ and $\bar{B}(x, r)$ be its closure and put $B = B(0, 1)$.

Let us recall the definition of the Clarke subdifferential and the concept of regularity that will be used in the sequel.

**Definition 2.1.** Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and $x$ be any point where $V$ is finite. The Clarke subdifferential of $V$ at $x$ is defined by

$$\partial_c V(x) := \{y \in H : \langle y, h \rangle \leq V^\dagger(x, h), \text{ for all } h \in H\},$$

where $V^\dagger(x, h)$ is the generalized Rockafellar directional derivative given by

$$V^\dagger(x, h) := \limsup_{x' \to x, V(x') \to V(x), t \to 0^+} \frac{V(x' + th') - V(x')}{t}.$$
Definition 2.2. Let $V : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $U \subset \text{Dom} V$ be a nonempty open subset. We will say that $V$ is uniformly regular over $U$ if there exists a positive number $\beta$ such that for all $x \in U$ and for all $\xi \in \partial_p V(x)$ one has

$$<\xi, x' - x> \leq V(x') - V(x) + \beta\|x' - x\|^2$$

for all $x' \in U$.

$\partial_p V(x)$ denotes the proximal subdifferential of $V$ at $x$ which is the set of all $y \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \bar{\delta}B$

$$<y, x' - x> \leq V(x') - V(x) + \sigma\|x' - x\|^2.$$

We say that $V$ is uniformly regular over closed set $S$ if there exists an open set $U$ containing $S$ such that $V$ is uniformly regular over $U$. For more details on the concept of regularity, we refer the reader to [4].

Proposition 2.3. [3, 4] Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and $S$ a nonempty closed set. If $V$ is uniformly regular over $S$, then the following conditions holds:

(a) The proximal subdifferential of $V$ is closed over $S$, that is, for every $x_n \to x \in S$ with $x_n \in S$ and every $\xi_n \to \xi$ with $\xi_n \in \partial_p V(x_n)$ one has $\xi \in \partial_p V(x)$.

(b) The proximal subdifferential of $V$ coincides with the Clarke subdifferential of $V$ for any point $x$.

(c) The proximal subdifferential of $V$ is upper hemicontinuous over $S$, that is, the support function $x \mapsto \sigma(v, \partial_p V(x))$ is u.s.c. over $S$ for every $v \in H$.

Now let us state the main result.

Let $V : H \to \mathbb{R}$ be a locally Lipschitz function and $\beta$-uniformly regular over $K \subset H$. Assume that

(H1) $K$ is a nonempty locally compact subset in $H$;

(H2) $F : K \to 2^H$ is an upper semi-continuous set valued map with compact values satisfying

$$F(x) \subset \partial_c V(x) \quad \text{for all } x \in K;$$

(H3) $f : \mathbb{R} \times H \to H$ is a function with the following properties:

(1) For all $x \in H$, $t \mapsto f(t, x)$ is measurable,

(2) The family $\{f(s, \cdot) : s \in \mathbb{R}\}$ is equicontinuous,

(3) For all bounded subset $S$ of $H$, there exists $M > 0$ such that

$$\|f(t, x)\| \leq M, \quad \forall (t, x) \in \mathbb{R} \times S;$$

(H4) (Tangential condition) $\forall (t, x) \in \mathbb{R} \times K, \exists v \in F(x)$ such that

$$\lim_{h \to 0^+} \inf \frac{1}{h}d_K\left(x + hv + \int_t^{t+h} f(s, x)ds\right) = 0.$$

EJQTDE, 2007 No. 7, p. 3
For any \( x_0 \in K \), consider the problem:

\[
\begin{align*}
\dot{x}(t) & \in f(t, x(t)) + F(x(t)) \quad \text{a.e;} \\
x(0) & = x_0; \\
x(t) & \in K.
\end{align*}
\]  
(2.1)

**Theorem 2.4.** If assumptions (H1)-(H4) are satisfied, then there exists \( T > 0 \) such that the problem (2.1) admits a solution on \([0, T]\).

### 3. Proof of the main result

Choose \( r > 0 \) such that \( K_0 = K \cap (x_0 + r\bar{B}) \) is compact and \( V \) is Lipschitz continuous on \( x_0 + r\bar{B} \) with Lipschitz constant \( \lambda > 0 \). Then \( \partial_e V(x) \subset \lambda\bar{B} \) for every \( x \in K_0 \). Let \( M > 0 \) such that

\[
\| f(t, x) \| \leq M, \forall (t, x) \in \mathbb{R} \times (x_0 + r\bar{B}).
\]  
(3.1)

Set

\[
T = \frac{r}{2(\lambda + 1 + M)}.
\]  
(3.2)

In the sequel, we will use the following important Lemma. It will play a crucial role in the proof of the main result.

**Lemma 3.1.** If assumptions (H1)-(H4) are satisfied, then for all \( 0 < \varepsilon < \inf(T, 1) \), there exists \( \eta > 0 \) such that \( \forall (t, x) \in [0, T] \times K_0, \exists h_{t,x} \in [\eta, \varepsilon], u \in F(x) + \frac{1}{h_{t,x}} \int_{t}^{t+h_{t,x}} f(s, x) ds + \frac{\varepsilon}{2T} \bar{B}, y_{t,x} \in K_0 \) and \( v \in F(y_{t,x}) \) such that

\[
(x + h_{t,x}u) \in K \cap \bar{B}(x + h_{t,x}v + \int_{t}^{t+h_{t,x}} f(s, x) ds, \lambda + M + 1).
\]

**Proof.** Let \((t, x) \in [0, T] \times K_0\), be fixed, let \( 0 < \varepsilon < \inf(T, 1) \). Since \( F \) is u.s.c on \( x \), then there exists \( \delta_x > 0 \) such that

\[
F(y) \subset F(x) + \frac{\varepsilon}{2T} \bar{B}, \quad \text{for all } y \in B(x, \delta_x).
\]

Let \((s, y) \in [0, T] \times K_0 \). By the tangential condition, there exists \( v \in F(y) \) and \( h_{s,y} \in [0, \varepsilon] \) such that

\[
d_K \left( y + h_{s,y}v + \int_{t}^{t+h_{s,y}} f(\tau, y) d\tau \right) < h_{s,y} \frac{\varepsilon}{4T}.
\]

Consider the subset

\[
N(s, y) = \left\{ (t, z) \in \mathbb{R} \times H/d_K (z + h_{s,y}v + \int_{t}^{t+h_{s,y}} f(\tau, z) d\tau) < h_{s,y} \frac{\varepsilon}{4T} \right\}.
\]

Since

\[
\| f(\tau, z) \| \leq M, \forall (\tau, z) \in \mathbb{R} \times \bar{B}(x_0, r),
\]

EJQTDE, 2007 No. 7, p. 4
then the dominated convergence theorem applied to the sequence \((\chi_{[t,t+h_{\tau,u}]}f(\cdot))\)_t of functions shows that the function
\[
(l, z) \mapsto z + h_{\tau,u}v + \int_t^{t+h_{\tau,u}} f(\tau, z) d\tau
\]
is continuous. So that, the function
\[
(l, z) \mapsto d_K \left( z + h_{\tau,u}v + \int_t^{t+h_{\tau,u}} f(\tau, z) d\tau \right)
\]
is continuous and consequently \(N(s, y)\) is open. Moreover, since \((s, y)\) belongs to \(N(s, y)\), there exists a ball \(B((s, y), \eta_{s,y})\) of radius \(\eta_{s,y} < \delta_x\) contained in \(N(s, y)\), therefore, the compact subset \([0, T] \times K_0\) can be covered by \(q\) such balls \(B((s_i, y_i), \eta_{s_i,y_i})\). For simplicity, we set
\[
h_{s_i,y_i} := h_i \quad \text{and} \quad \eta_i := \eta_{s_i,y_i}, \ i = 1, \ldots, q.
\]
Put \(\eta = \min\{h_i/1 \leq i \leq q\}\) and let \(i \in \{1, \ldots, q\}\) such that \((t, x) \in B((s_i, y_i), \eta_i)\), hence \((t, x) \in N(s_i, y_i)\). Then there exists \(v_i \in F(y_i)\) such that
\[
d_K \left( x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right) < \frac{\varepsilon}{4T}.
\]
Let \(x_i \in K\) such that
\[
\frac{1}{h_i} \left\| x_i - \left( x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right) \right\| \leq \frac{1}{h_i} d_K \left( x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right) + \frac{\varepsilon}{4T}.
\]
Hence
\[
\left\| \frac{x_i - x}{h_i} - \left( v_i + \frac{1}{h_i} \int_t^{t+h_i} f(\tau, x) d\tau \right) \right\| < \frac{\varepsilon}{2T}.
\]
Set
\[
u = \frac{x_i - x}{h_i},
\]
then \(x_i = x + h_i u \in K\) and
\[
u \in \left( \frac{1}{h_i} \int_t^{t+h_i} f(\tau, x) d\tau + F(y_i) + \frac{\varepsilon}{2T} B \right).
\]
Since \(\| x - y_i \| < \eta_i < \delta_x\) we have
\[
F(y_i) \subset F(x) + \frac{\varepsilon}{2T} B,
\]
then
\[
u \in \left( \frac{1}{h_i} \int_t^{t+h_i} f(\tau, x) d\tau + F(x) + \frac{\varepsilon}{T} B \right).
\]
On the other hand, since \(x \in K\), we have
\[
\left\| x_i - \left( x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right) \right\| \leq d_K \left( x + h_i v_i + \int_t^{t+h_i} f(\tau, x) d\tau \right) + \frac{\varepsilon}{4T}.
\]
Lemma 3.1, there exist and step, from available estimates that a subsequence converges to a solution of (2.1). Thus we have constructed the sequences \((h_p)_p \subseteq [\eta, \varepsilon] \), \(((x_p)_p, (y_p)_p) \subseteq K_0 \times K_0\) and \(((u_p)_p, (v_p)_p) \subseteq H \times H\) such that

\[
\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i.
\]

Then we have constructed the sequences \((h_p)_p \subseteq [\eta, \varepsilon] \), \(((x_p)_p, (y_p)_p) \subseteq K_0 \times K_0\) and \(((u_p)_p, (v_p)_p) \subseteq H \times H\) such that for every \(p = 1, \ldots, s\), we have

(i) \(x_p = x_{p-1} + h_{p-1}u_{p-1};\)

(ii) \(x_p \in \bigcap B \left( x_{p-1} + h_{p-1}v_{p-1} + \frac{\sum_{i=0}^{p-1} h_i}{\sum_{i=0}^{p-2} h_i} f(s, x_{p-1})ds, \lambda + M + 1 \right) ;\)

\(\frac{\varepsilon}{4T}\)

Thus \(x_i \in B \left( x + h_i v_i + \int_{t}^{t+h_i} f(\tau, x) d\tau, \lambda + M + 1 \right) .\) \(\square\)

Now, we are able to prove the main result. Our approach consists of constructing, in a first step, a sequence of approximate solutions and deduce, in a second step, from available estimates that a subsequence converges to a solution of (2.1).

**Step 1. Approximate solutions.** Let \(x_0 \in K_0\) and \(0 < \varepsilon < \inf(T, 1).\) By Lemma 3.1, there exist \(\eta > 0, h_0 \in [\eta, \varepsilon], u_0 \in \left( \frac{1}{\varepsilon} \int_0^{h_0} f(s, x_0)ds + F(x_0) + \frac{\varepsilon}{4T} B \right),\) \(y_0 \in K_0\) and \(v_0 \in F(y_0)\) such that

\[
x_1 = x_0 + h_0 u_0 \in K \bigcap B \left( x_0 + h_0 v_0 + \int_0^{h_0} f(s, x_0)ds, \lambda + M + 1 \right) .
\]

Then by (H2), (3.1) and (3.2), we have

\[
\| x_1 - x_0 \| = \| h_0 u_0 \| \leq (\lambda + 1 + M)T < r
\]

and thus \(x_1 \in K_0.\) Set \(h_{-1} = 0.\) By induction, for \(q \geq 2\) and for every \(p = 1, \ldots, q - 1\), we construct the sequences \((h_p)_p \subseteq [\eta, \varepsilon] \), \(((x_p)_p, (y_p)_p) \subseteq K_0 \times K_0\) and \(((u_p)_p, (v_p)_p) \subseteq H \times H\) such that \(\sum_{p=1}^{q-1} h_p \leq T\) and

\[
x_p = x_{p-1} + h_{p-1} u_{p-1};
\]

\[
x_p \in K \bigcap B \left( x_{p-1} + h_{p-1} v_{p-1} + \frac{\sum_{i=0}^{p-1} h_i}{\sum_{i=0}^{p-2} h_i} f(s, x_{p-1})ds, \lambda + M + 1 \right) ;
\]

\[
u_p \in \left( \frac{1}{\varepsilon} \int_0^{h_p} f(s, x_p)ds + F(x_p) + \frac{\varepsilon}{4T} B \right).
\]

Since \(h_i \geq \eta > 0\) there exists an integer \(s\) such that

\[
\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i.
\]

Then we have constructed the sequences \((h_p)_p \subseteq [\eta, \varepsilon] \), \(((x_p)_p, (y_p)_p) \subseteq K_0 \times K_0\) and \(((u_p)_p, (v_p)_p) \subseteq H \times H\) such that for every \(p = 1, \ldots, s\), we have

(i) \(x_p = x_{p-1} + h_{p-1} u_{p-1};\)

(ii) \(x_p \in K \bigcap B \left( x_{p-1} + h_{p-1} v_{p-1} + \frac{\sum_{i=0}^{p-1} h_i}{\sum_{i=0}^{p-2} h_i} f(s, x_{p-1})ds, \lambda + M + 1 \right) ;\)
(iii) \( u_p \in F(x_p) + \frac{1}{h_p} \sum_{i=0}^{p-1} h_i f(s, x_{p}) ds + \tilde{\nu} B; \)

(iv) \( v_p \in F(y_p). \)

By induction, for all \( p = 1, \ldots, s \) we have

\[
x_p = x_0 + \sum_{i=0}^{p-1} h_i u_i.
\]

Moreover by (iii), (H2), (3.1), (3.2) and because \( \sum_{i=0}^{p-1} h_i < T \), we have

\[
\|x_p - x_0\| = \left\| \sum_{i=0}^{p-1} h_i u_i \right\| \leq \sum_{i=0}^{p-1} h_i \|u_i\| \leq \sum_{i=0}^{p-1} h_i (\lambda + 1 + M) < r, \quad (3.3)
\]

hence \( x_p \in K_0. \)

For any nonzero integer \( k \) and for every integer \( q = 0, \ldots, s - 1 \) denote by \( h^k_q \) a real associated to \( \varepsilon = \frac{1}{k} \) and \( x = x_q \) given by Lemma 3.1. Consider the sequence \((\tau^k)\) defined as the following

\[
\begin{align*}
\tau^k_0 &= 0, \quad \tau^k_{q+1} = T; \\
\tau^k_q &= h^k_0 + \ldots + h^k_{q-1} & \text{if } 1 \leq q \leq s,
\end{align*}
\]

and define on \([0, T]\) the sequence of functions \((x_k(\cdot))\) by

\[
\begin{align*}
x_k(t) &= x_{q-1} + (t - \tau^k_{q-1}) u_{q-1}, \quad \forall t \in [\tau^k_{q-1}, \tau^k_q]; \\
x_k(0) &= x_0.
\end{align*}
\]

**Step 2. Convergence of approximate solutions.** By definition of \( x_k(\cdot) \), for all \( t \in [\tau^k_{q-1}, \tau^k_q] \) we have \( \dot{x}_k(t) = u_{q-1} \). By (iii), (H2), (3.1), for a.e. \( t \in [0, T] \), we have

\[
\|\dot{x}_k(t)\| \leq \lambda + 1 + M.
\]

On the other hand, by (ii), (iv), (H2), (3.1) and (3.3) we have

\[
\begin{align*}
\|x_q\| &\leq \left\| x_q - (x_{q-1} + h^k_{q-1} u_{q-1} + \int_{\tau^k_{q-1}}^{\tau^k_q} f(s, x_{q-1}) ds) \right\| \\
&\quad + \left\| x_{q-1} + h^k_{q-1} u_{q-1} + \int_{\tau^k_{q-1}}^{\tau^k_q} f(s, x_{q-1}) ds \right\| \\
&\leq \lambda + M + 1 \left\| x_0 - (x_0 - x_{q-1}) + h^k_{q-1} u_{q-1} + \int_{\tau^k_{q-1}}^{\tau^k_q} f(s, x_{q-1}) ds \right\| \\
&\leq \lambda + M + 1 \left\| x_0 \right\| + \left\| x_0 - x_{q-1} \right\| + h^k_{q-1} \|u_{q-1}\| + h^k_{q-1} M \\
&\leq \lambda + M + 1 \left\| x_0 \right\| + r + \lambda + M \\
&< 2(\lambda + M + 1) + \left\| x_0 \right\| + r = R.
\end{align*}
\]

Then \( x_q \in K_0 \cap \bar{B}(0, R) = K_1 \). By construction, for all \( t \in [\tau^k_{q-1}, \tau^k_q] \) we have

\[
x_k(t) = x_{q-1} + (t - \tau^k_{q-1}) u_{q-1} = x_{q-1} + \frac{(t - \tau^k_{q-1})}{h^k_{q-1}} (x_q - x_{q-1}).
\]

EJQTDE, 2007 No. 7, p. 7
Also since $0 \leq t - \tau_k^{q-1} \leq \tau_k^q - \tau_k^{q-1} = h_k^{q-1}$, we have
\[
0 \leq \frac{(t - \tau_k^{q-1})}{h_k^{q-1}} \leq 1.
\]

Then
\[
\frac{(t - \tau_k^{q-1})}{h_k^{q-1}}(x_q - x_{q-1}) \in co\{0 \cup (K_1 - K_0)\},
\]
hence $x_k(t) \in K_0 + co\{0 \cup (K_1 - K_0)\}$ which is compact. Therefore, we can select a subsequence, again denoted by $(x_k(.)_k$ which converges uniformly to an absolutely continuous function $x(.)$ on $[0, T]$, moreover $\dot{x}_k(.)$ converges weakly to $\dot{x}(.)$ in $L^2([0, T], H)$. The family of approximate solution $x_k(.)$ satisfies the following property.

**Proposition 3.2.** For every $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that
\[
\lim_{k \to +\infty} d_{grF} \left( (x_k(t), \dot{x}_k(t)) - \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \right) = 0.
\]

**Proof.** Let $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and $\lim_{k \to +\infty} \tau_k^{q-1} = t$. Since
\[
\dot{x}_k(t) - \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \in F(x_{q-1}) + \frac{1}{kT} B,
\]
we have
\[
d_{grF} \left( (x_k(t), \dot{x}_k(t)) - \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \right) \leq \|x_k(t) - x_k(\tau_k^{q-1})\| + \frac{1}{kT},
\]
hence
\[
\lim_{k \to +\infty} d_{grF} \left( (x_k(t), \dot{x}_k(t)) - \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds \right) = 0.
\]

**Claim 3.3.**
\[
\lim_{k \to +\infty} \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds = f(t, x(t)).
\]

**Proof.** Fix any $t \in [0, T]$, there exists $q \in \{1, \ldots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$, $\lim_{k \to +\infty} \tau_k^{q-1} = \lim_{k \to +\infty} \tau_k^q = t$ and $\lim_{k \to +\infty} x_k(\tau_k^{q-1}) = x(t)$. Put
\[
G(t, y) = \int_0^t f(s, y) ds.
\]
Note that the function $G$ is differentiable on $t$ and
\[
\frac{dG}{dt}(t, y) = f(t, y).
\]

EJQTDE, 2007 No. 7, p. 8
We have
\[
\begin{align*}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\
&\leq \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\
&\quad + \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|.
\end{align*}
\]
On the other hand
\[
\begin{align*}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} \right\| \\
&= \left\| \frac{\tau_k^q - t}{\tau_k^q - \tau_k^{q-1}} \left( \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} \right) \right\| \\
&\leq \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x) \right\| \\
&\quad + \left\| \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} - f(t, x(t)) \right\|.
\end{align*}
\]
Hence
\[
\begin{align*}
&\left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \\
&\leq \left\| \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} - f(t, x) \right\| \\
&\quad + 2 \left\| \frac{G(t, x(t)) - G(\tau_k^{q-1}, x(t))}{t - \tau_k^{q-1}} - f(t, x(t)) \right\|.
\end{align*}
\]
As
\[
\lim_{k \to +\infty} \frac{G(\tau_k^q, x(t)) - G(t, x(t))}{\tau_k^q - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t))
\]
and
\[
\lim_{k \to +\infty} \frac{G(\tau_k^{q-1}, x(t)) - G(t, x(t))}{\tau_k^{q-1} - t} = \frac{dG}{dt}(t, x(t)) = f(t, x(t)),
\]
we have
\[
\lim_{k \to +\infty} \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} = f(t, x(t)). \tag{3.5}
\]
Put
\[
\rho_k = \left\| \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\|.
\]
EJQTDE, 2007 No. 7, p. 9
On the other hand we have
\[
\left\| \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_k(\tau_k^{q-1})) ds - f(t, x(t)) \right\|
\]
\[
= \left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\|
\]
\[
\leq \left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - \frac{G(\tau_k^q, x(t)) - G(\tau_k^{q-1}, x(t))}{\tau_k^q - \tau_k^{q-1}} \right\| + \rho_k
\]
\[
= \left\| \frac{1}{\tau_k^q - \tau_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} (f(s, x_k(\tau_k^{q-1}))-f(s, x(t))) ds \right\| + \rho_k.
\]
Since the family \( \{f(s, \cdot) : s \in \mathbb{R}\} \) is equicontinuous, then there exists \( k_0 \) such that
\[
\|f(s, x_k(\tau_k^{q-1}))-f(s, x(t))\| \leq \frac{1}{k} \text{ for all } k \geq k_0 \text{ and for all } s \in \mathbb{R},
\]
consequently we have for \( k \geq k_0 \)
\[
\left\| \frac{G(\tau_k^q, x_k(\tau_k^{q-1})) - G(\tau_k^{q-1}, x_k(\tau_k^{q-1}))}{\tau_k^q - \tau_k^{q-1}} - f(t, x(t)) \right\| \leq \frac{1}{k} + \rho_k.
\]
By (3.5), the last term converges to 0. This completes the proof of the Claim.

The function \( x(.) \) has the following property

**Proposition 3.4.** For all \( t \in [0, T] \), we have \( \dot{x}(t) - f(t, x(t)) \in \partial_t V(x(t)) \).

**Proof.** The weak convergence of \( \dot{x}_k(.) \) to \( \dot{x}(.) \) in \( L^2([0, T], H) \) and the Mazur’s Lemma entail
\[
\dot{x}(t) \in \bigcap_k \text{co}\{\dot{x}_m(t) : m \geq k\}, \text{ for a.e. on } [0, T].
\]
Fix any \( t \in [0, T] \), there exists \( q \in \{1, \ldots, s + 1\} \) such that \( t \in [\tau_k^{q-1}, \tau_k^q] \) and \( \lim_{k \to +\infty} \tau_k^{q-1} = t \). Then for all \( y \in H \)
\[
<y, \dot{x}(t) > \leq \inf_{m \geq k} \sup_{m \geq k} < y, \dot{x}_k(t) > .
\]
Since \( F(x) \subset \partial_t V(x) \), then by (3.4), one has
\[
\dot{x}_k(t) \in \partial_t V(x_{q-1}) + \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B.
\]
Thus for all \( m \)
\[
<y, \dot{x}(t) > \leq \sup_{k \geq m} \sigma \left( y, \partial_t V(x_{q-1}) + \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B \right),
\]
from which we deduce that
\[
<y, \dot{x}(t) > \leq \limsup_{k \to +\infty} \sigma \left( y, \partial_t V(x_{q-1}) + \frac{1}{h_k^{q-1}} \int_{\tau_k^{q-1}}^{\tau_k^q} f(s, x_{q-1}) ds + \frac{1}{kT} B \right).
\]
By Proposition 2.3, the function \( x \mapsto \sigma(y, \partial_k V(x)) \) is u.s.c and hence we get
\[
<y, \dot{x}(t)> \leq \sigma(y, \partial_k V(x(t)) + f(t, x(t))).
\]
So, the convexity and the closedness of the set \( \partial_k V(x(t)) \) ensure
\[
\dot{x}(t) - f(t, x(t)) \in \partial_e V(x(t)).
\]
\[\square\]

**Proposition 3.5.** The application \( x(\cdot) \) is a solution of the problem (2.1).

**Proof.** As \( x(\cdot) \) is an absolutely continuous function and \( V \) is uniformly regular locally Lipschitz function over \( K \) (hence directionally regular over \( K \) (see [5])), by Theorem 2 in Valadier [10, 11] and by Proposition 3.4, we obtain
\[
\frac{d}{dt} V(x(t)) = < \dot{x}(t), \dot{x}(t) - f(t, x(t)) > \hspace{0.05in} \text{a.e. on} \ [0, T],
\]
therefore,
\[
V(x(T)) - V(x_0) = \int_0^T \| \dot{x}(s) \|^2 ds - \int_0^T < \dot{x}(s), f(s, x(s)) > ds \hspace{0.05in} \text{(3.6)}
\]
On the other hand, by construction, for all \( q = 1, \ldots, s + 1 \), we have
\[
\dot{x}_k(t) = \frac{1}{h_{q-1}^k} \int_{\tau_{q-1}^k}^{\tau_q^k} f(s, x_{q-1}) ds \in \partial_e V(x_{q-1}) + \frac{1}{kT} B.
\]
Let \( b_q \) such that
\[
\dot{x}_k(t) = \frac{1}{h_{q-1}^k} \int_{\tau_{q-1}^k}^{\tau_q^k} f(s, x_{q-1}) ds + \frac{1}{kT} b_q \in \partial_e V(x_{q-1}).
\]
Since \( V \) is \( \beta \)-uniformly regular over \( K \), we have
\[
V(x_k(\tau_q^k)) - V(x_k(\tau_{q-1}^k)) \geq < x_k(\tau_q^k) - x_k(\tau_{q-1}^k), \dot{x}_k(t) >
\]
\[
- \frac{1}{h_{q-1}^k} \int_{\tau_{q-1}^k}^{\tau_q^k} f(s, x_{q-1}) ds + \frac{1}{kT} b_q >
\]
\[
- \beta \| x_k(\tau_q^k) - x_k(\tau_{q-1}^k) \|^2
\]
\[
= < \int_{\tau_{q-1}^k}^{\tau_q^k} \dot{x}_k(s) ds, \dot{x}_k(t) >
\]
\[
- \frac{1}{h_{q-1}^k} \int_{\tau_{q-1}^k}^{\tau_q^k} f(s, x_{q-1}) ds + \frac{1}{kT} b_q >
\]
\[
- \beta \| x_k(\tau_q^k) - x_k(\tau_{q-1}^k) \|^2
\]
\[
= \int_{\tau_{q-1}^k}^{\tau_q^k} < \dot{x}_k(s), \dot{x}_k(s) > ds
\]
\[
- \frac{1}{h_{q-1}^k} \int_{\tau_{q-1}^k}^{\tau_q^k} f(s, x_{q-1}) ds > ds
\]
EJQTDE, 2007 No. 7, p. 11
Claim 3.6.

\[ V(x_k(T)) - V(x_0) \geq \int_0^T \| \dot{x}_k(s) \|^2 \, ds - \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > ds \]

(3.7)

Claim 3.6.

\[ \lim_{k \to +\infty} \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > ds = \int_0^T < \dot{x}(s), f(s, x(s)) > ds. \]

Proof. We have

\[ \left\| \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > ds - \int_0^T < \dot{x}(s), f(s, x(s)) > ds \right\| \]

\[ = \left\| \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > - < \dot{x}(s), f(s, x(s)) > ds \right\| \]

\[ \leq \left\| \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > - < \dot{x}(s), f(s, x(s)) > ds \right\| \]

\[ + \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} < \dot{x}_k(s), f(s, x(s)) > - < \dot{x}(s), f(s, x(s)) > ds \]

\[ \leq \sum_{q=1}^{s+1} \int_{\tau_{k-1}^q}^{\tau_k^q} \| < \dot{x}_k(s), \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(\tau, x_{q-1}) \, d\tau > - < \dot{x}_k(s), f(s, x(s)) > ds \]

\[ + \int_0^T < \dot{x}(s), f(s, x(s)) > - < \dot{x}(s), f(s, x(s)) > ds \right\| \]

Since

\[ \| \dot{x}_k(t) \| \leq \lambda + M + 1, \quad \lim_{k \to +\infty} \frac{1}{h_{q-1}} \int_{\tau_{k-1}^q}^{\tau_k^q} f(s, x_{q-1}) \, ds = f(t, x(t)) \]

and \( \dot{x}_k(.) \) converges weakly to \( \dot{x}(.) \), the last term converges to 0. This completes the proof of the Claim. \( \square \)

Claim 3.7.

\[ \lim_{k \to +\infty} \sum_{q=1}^{s+1} \beta \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 = 0. \]

EJQTDE, 2007 No. 7, p. 12
**Proof.** By construction we have

\[ \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \| = \| (\tau_k^q - \tau_k^{q-1}) u_{q-1} \| \]

\[ \leq (\tau_k^q - \tau_k^{q-1}) \| u_{q-1} \| \]

\[ \leq (\tau_k^q - \tau_k^{q-1})(\lambda + 1 + M). \]

Hence

\[ \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 \leq (\tau_k^q - \tau_k^{q-1})^2(\lambda + 1 + M)^2 \]

\[ \leq (\tau_k^q - \tau_k^{q-1})h_{q-1}(\lambda + 1 + M)^2 \]

Then

\[ \sum_{q=1}^{s+1} \beta \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 \leq \frac{\beta T(\lambda + 1 + M)^2}{k} \]

hence

\[ \lim_{k \to +\infty} \sum_{q=1}^{s+1} \beta \| x_k(\tau_k^q) - x_k(\tau_k^{q-1}) \|^2 = 0. \]

Note that

\[ \lim_{k \to +\infty} \frac{1}{kT} \sum_{q=1}^{s+1} \int_{\tau_k^{q-1}}^{\tau_k^q} < \dot{x}_k(s), b_q > ds = 0. \]

By passing to the limit for \( k \to \infty \) in (3.7) and using the continuity of the function \( V \) on the ball \( B(x_0, r) \), we obtain

\[ V(x(T)) - V(x_0) \geq \limsup_{k \to +\infty} \int_0^T \| \dot{x}_k(s) \|^2 ds - \int_0^T < \dot{x}(s), f(s, x(s)) > ds. \]

Moreover, by (3.6), we have

\[ \| \dot{x} \|^2 \geq \limsup_{k \to +\infty} \| \dot{x}_k \|^2 \]

and by the weak l.s.c of the norm ensures

\[ \| \dot{x} \|^2 \leq \liminf_{k \to +\infty} \| \dot{x}_k \|^2. \]

Hence we get

\[ \| \dot{x} \|^2 = \lim_{k \to +\infty} \| \dot{x}_k \|^2. \]

Finally, there exists a subsequence of \( (\dot{x}_k(.))_k \) (still denoted \( (\dot{x}_k(.))_k \)) converges pointwisely to \( \dot{x}(.) \). In view of Proposition (3.2), we conclude that

\[ d_{grF}(x(t), \dot{x}(t) - f(t, x(t)))) = 0 \]

and as \( F \) has a closed graph, we obtain

\[ \dot{x}(t) \in f(t, x(t)) + F(x(t)) \text{ a.e on } [0, T]. \]

*EJQTDE, 2007 No. 7, p. 13*
Now, let $t \in [0,T]$, there exists $q \in \{1, \ldots, s+1\}$ such that $t \in [\tau_k^{q-1}, \tau_k^q]$ and
\[
\lim_{k \to +\infty} \tau_k^{q-1} = t.
\]
Since
\[
\lim_{k \to +\infty} \| x(t) - x_k(\tau_k^{q-1}) \| = 0,
\]
$x_k(\tau_k^{q-1}) \in K_0$ and $K_0$ is closed we obtain $x(t) \in K_0 \subset K$. The proof is complete.

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