Nonlinear electrodynamics is skilled with knots

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Abstract – The aim of this letter is threefold: First is to show that nonlinear generalizations of electrodynamics support various types of knotted solutions in vacuum. The solutions are universal in the sense that they do not depend on the specific Lagrangian density, at least if the latter gives rise to a well-posed theory. Second, is to describe the interaction between probe waves and knotted background configurations. We show that the qualitative behaviour of this interaction may be described in terms of Robinson congruences, which appear explicitly in the causal structure of the theory. Finally, we argue that optical arrangements endowed with intense background fields could be the natural place to look for the knots experimentally.

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Introduction. – Knot theory is a fascinating branch of topology with many potential applications to physics [1]. Roughly speaking, a knot is a closed loop of string, with no free ends. Mathematically, we can define it as an embedding of a circle in 3-dimensional Euclidean space $\mathbb{R}^3$ which cannot be untied without cutting or permitting the curve to pass through itself. Two of such embeddings are said to represent the same knot if there exists a deformation of $\mathbb{R}^3$ upon itself taking one of the knots into the other smoothly (ambient isotopy). A link, in its turn, is a collection of knotted loops which do not intersect, all tangled up together.

Thought-provoking is the existence of space-filling knots/links in some field theories with physical interest, both quantum and classical. Recently, these exotic configurations have been investigated in contexts as diverse as hydrodynamics [2], Bose-Einstein condensates [3], ferromagnetism [4] and non-Abelian gauge theories, where they are supposed to describe stable excitations such as glueballs [5]. Little-known, however, is the unexpected result of Rañada that Maxwell’s linear equations admit solutions such that all the lines of force are closed and any pair of magnetic (electric) lines are linked [6–9]. Rañada’s method is based on maps from Minkowski space-time to the complex plane $$(\phi, \theta): \mathbb{R}^{1+3} \to \mathbb{C}$$ which are required to be single-valued at spatial infinity. After identifying the complex plane with the unit sphere via stereographic projection, he effectively obtains $$(\phi, \theta): \mathbb{R} \times S^3 \to S^2$$ which are nontrivial since the third homotopy group of the sphere is $\pi_3(S^2) \in \mathbb{Z}$. The linked lines are interpreted as the pre-images of the maps and fulfill the whole of space with a fibered structure which coincides with the Hopf fibration [10] for an instant of time. The corresponding solutions have been dubbed electromagnetic knots and received considerable attention in the last few years. Importantly, it was shown by Irvine and Bowmeester that approximate knots of light could be generated using tightly focused circularly polarized laser beams [11].

This paper deals with electromagnetic knots in the context of nonlinear electrodynamics [12,13]. Specifically, we show that, if the theory arises from a Lagrangian and the latter gives rise to well-behaved equations, knots and links appear as universal exact solutions. This means that closed knotted/linked loops made of electric and magnetic fields are consistent with the nonlinear evolution equations whilst their topology are preserved in time, very much in the same way as they do in Maxwell’s theory. As is well known, nonlinear theories of this type are relevant for several reasons. In QED, the polarization of the vacuum leads naturally to nonlinear effects (such as the light-light scattering) which are effectively described by Euler-Heisenberg’s Lagrangian [14]. In dielectrics, the interaction between molecules and external fields can be described by an effective nonlinear theory, which is typically observed at very high light intensities such as those provided by pulsed lasers [15]. The Born-Infeld model has mathematical connections to string theory, for its Lagrangian appears in relation with the gauge fields on a
D-brane [16]. Finally, the increasing availability of multi-hundred TW and PW lasers brings the confirmation of long-predicted nonlinear phenomena closer [17–19].

Our construction is grounded on null fields and employs some of the machinery recently discussed in [20–22]. However, due to the nonlinearity of the underlying equations, there is a striking novelty in our approach: small-amplitude/high-frequency waves arriving from the vacuum may interact with the knotted fields in a nontrivial way. This result promises to open up the possibility of detection using ordinary radiation, which will receive an imprint of knottedness due to nonlinear effects if the background fields are sufficiently intense.

**Nonlinear electrodynamics.** – Consider a Minkowski space-time \((\mathbf{M},g)\) with signature convention \(+,−,−,−\) and write \(F_{ab}\) for the electromagnetic field tensor. Let

\[
F := F_{ab}F^{ab}, \quad G := *F_{ab}F^{ab}
\]

(1)
de note the field invariants, with \(*\) the Hodge operator defined by

\[
*F_{ab} := \frac{1}{2} \eta^{abcd} F_{cd},
\]

(2)
where \(\eta^{abcd}\) is the completely antisymmetric Levi-Civita tensor. We are interested in nonlinear theories in empty space provided by the action

\[
S = \int \mathcal{L}(F,G) \sqrt{-g} \, d^4x,
\]

(3)
with the Lagrangian density an arbitrary smooth function of the invariants. The equations of motion read as

\[
\nabla_a (\mathcal{L}_F F^{ab} + \mathcal{L}_G * F^{ab}) = 0, \quad \nabla[a F_{bc}] = 0,
\]

(4)
with \(\mathcal{L}_F\) and \(\mathcal{L}_G\) the partial derivatives with respect to the invariants, for conciseness. As is well known, eqs. (4) constitute a system of first-order quasilinear PDEs with constraints and can be considerably difficult to solve in general situations. For the resulting theory to have predictive power, however, we assume henceforth: i) Well-posedness of the Cauchy problem\(^1\); ii) Maxwell’s electrodynamics is recovered for sufficiently weak fields. A well-known example fulfilling these requirements for sufficiently small fields is provided by the Euler-Heisenberg effective action

\[
\mathcal{L}_{EH} = -\frac{1}{4} F + \frac{\alpha^2}{90m_e^4} \left[ F^2 + \frac{7}{4} G^2 \right]
\]

(5)
with \(\alpha\) the fine-structure constant and \(m_e\) the electron mass [14]. This nonlinear Lagrangian takes into account one-loop quantum corrections to Maxwell’s equations implying in the violation of the principle of superposition for electromagnetic fields in vacuum.

\(^1\)Well-posedness is at the roots of physics, for it amounts to the predictability power of the theory, asserting that solutions exist, are unique and depend continuously on the data (see, for instance, [23]).

In order to discuss about electric and magnetic fields in a covariant fashion, it is convenient to introduce a space-time foliation induced by the level sets \(\Sigma_t\) of a scalar \(t(x)\). Here, the gradient \(t_a := \partial_a t\) is taken to be time-like, future-directed and normalized everywhere, i.e. \(\sqrt{g^{tt}} t_t = 1\). This foliation introduces a local 1+3 “threading” of the space-time into time and space and automatically defines a family of fundamental observers via the vector field \(\lambda^a(x)\) (see [24] for details). With these conventions, the observers uniquely decompose the electromagnetic field tensor as

\[
F^{ab} = g^{cd} \lambda_c \lambda_d + \eta^{ab} \lambda_c \lambda_d, \quad \lambda^a := g_{ac} \lambda_d - g_{ad} \lambda_c.
\]

(6)
The electric and magnetic fields experienced by the observer are then given by

\[
E^a = F^{a}_{\ b} \lambda^b, \quad B^a = *F^{a}_{\ b} \lambda^b.
\]

(7)
Note that \(E_{\lambda t}^a = B_{\lambda t}^a = 0\, ensuring that, at each space-time point, the electromagnetic fields are space-like vectors living in the observer 3-dimensional rest-space at that point. A direct calculation reveals that, in terms of the electromagnetic components, the field invariants read as

\[
F = 2(B^2 - E^2), \quad G = 4\mathbf{E} \cdot \mathbf{B}.
\]

(8)

The concept of space-filling lines of force thus appears as the integral curves in \(\Sigma_t\) satisfying the system

\[
\frac{dx^a}{d\lambda} = E^a(\lambda), \quad \frac{dx^a}{d\tau} = B^a(\tau),
\]

with \(\tau\) and \(\lambda\) real parameters. Knowledge of the geometry/topology of these lines for all \(t\) is sufficient to qualitatively describe the fields satisfying (4) in time. This framework naturally induces the question: Are there solutions of eqs. (4) such that the corresponding field lines form knotted loops filling the hyper-surfaces \(\Sigma_t\) with a nontrivial topology? Surprisingly, we will see that such solutions exist, at least for fields satisfying the null condition \(F = G = 0\) (fig. 1).
From complex mappings to null fields. – A powerful (though not very well known) mechanism to generate null electromagnetic fields dates back to Bateman [25]. This approach has been recently employed by Besieris and Shaarawi [20], Kedia et al. [21] and Hoyos et al. [22]. Let us, very briefly, review some aspects of the formalism and describe it in a more covariant fashion. Consider, to begin with, complex maps $\alpha, \beta : (M, g) \rightarrow \mathbb{C}$ satisfying the fully nonlinear system of first-order PDEs

$$
(g^{abcd} + i\eta^{abcd}) \partial_c \alpha \partial_d \beta = 0. \tag{9}
$$

It is known that a sufficient condition for (9) to admit nontrivial solutions is that

$$
(\partial_a \alpha \partial^a \alpha)(\partial_b \beta \partial^b \beta) = 0, \quad \partial_a \alpha \partial^a \beta = 0. \tag{10}
$$

A pair $\alpha, \beta$ solving (9) and (10) is called a Bateman pair and induces the antisymmetric complex tensor

$$
\mathcal{M}_{ab} := \eta_{cd} \partial_c \alpha \partial_d \beta, \tag{11}
$$

called the Riemann-Silberstein 2-form. With this definition one can easily verify that the imaginary part of the Riemann-Silberstein 2-form is minus the dual of its real part, i.e.

$$
\mathcal{M}^{ab} = F^{ab} - i * F^{ab}. \tag{12}
$$

A field $F_{ab}$ specified in this way is necessarily null (self-conjugate in Bateman’s terminology), for if we contract (12) with itself we obtain

$$
\mathcal{M}^{ab} \mathcal{M}_{ab} = 2(F - iG) = 0, \tag{13}
$$

which identically vanishes as a consequence of (9).

A remarkable property of this construction is that it automatically generates null solutions to Maxwell’s linear equations in vacuum. Indeed, applying the covariant divergence to (12) and using (9), there follows

$$
\nabla_a \mathcal{M}^{ab} = \nabla_a F^{ab} - i \nabla_a * F^{ab} = 0.
$$

From this analysis it is clear that if a pair $\alpha, \beta$ is known, it is always possible to obtain algorithmically a null solution of Maxwell’s equations in vacuum. What is more, once the pair is obtained, a whole family of new pairs $f, g$ result, where $f$ and $g$ can be taken as arbitrary holomorphic functions of $\alpha$ and $\beta$ (see ref. [21] for further details).

What about the equations of nonlinear electrodynamics? Can we extend the technique discussed so far so as to find solutions of (4)? The answer is provided by the following lemma: Every Bateman pair automatically induces null solutions of the nonlinear equations (4). The proof is straightforward since if the field is null everywhere, the quantities $L_F$ and $L_G$ in eqs. (4) can be treated as constants and, therefore, drop out from the covariant derivatives, implying again

$$
\nabla_a F^{ab} = 0, \quad \nabla_a * F^{ab} = 0, \tag{14}
$$

which are nothing but Maxwell’s equations in free-space.

Knotted and linked solutions. – An interesting consequence of the lemma is that a class of well-known knotted solutions discussed in the literature may be directly extrapolated to the nonlinear case. Let us review how the simplest knots emerge in this context. Consider, for instance, the pair

$$
\alpha = \frac{r^2 - t^2 - 1 + 2iz}{r^2 - (t - i)^2}, \quad \beta = \frac{2(x - iy)}{r^2 - (t - i)^2}, \tag{15}
$$

which satisfies eqs. (9) and (10). Here $(t, x, y, z)$ denotes usual Cartesian coordinates in $M$ and $r^2 = x^2 + y^2 + z^2$. In ref. [21] it is shown that the choice $f = \alpha^p$ and $g = \beta^q$ with $p, q \in \mathbb{Z}$ gives rise to fields whose electric and magnetic field lines are grouped into knotted and linked tori, nested one inside the other with $(p, q)$-torus knots at the core of the fibration. These being null fields satisfying Maxwell’s equations, they automatically satisfy the quasilinear PDEs (4).

Also, the topological structure of the solutions is preserved for all times since, as shown by Irvine in [26], the field lines satisfy the “frozen field” condition and evolve as if they were unbreakable filaments embedded in a fluid, stretching and deforming smoothly. In other words, the field as a whole evolve as an ambient isotopy of the space-like hyper-surface $\Sigma_t$. As an electromagnetic field has an infinite number of magnetic/electric field lines it is common to use two averaged quantities in order to quantify how much the field lines are knotted and linked. They are the electromagnetic helicities, and are given explicitly in terms of the Whitehead integrals [27]:

$$
\mathcal{H}_m := \int * F^{ab} A_t b_a d^3x, \quad \mathcal{H}_e := \int F^{ab} C_b t_a d^3x
$$

with $F_{ab} = \partial_a A_b$ and $* F_{ab} = \partial_a C_b$. These quantities are gauge invariant when the integral is over all space and one can show using the null condition that they do not depend on the choice of foliation. Therefore, the helicities are topological invariants and, for the above choice of $f$ and $g$, they read as $\mathcal{H}_m = \mathcal{H}_e = (p + q)^{-1}$ in appropriate units. The simplest solution $p = 1$ is equivalent to a solution previously considered by Trautman and generalized by Bialynicki-Birula [28]. The lines of force associated to the $(p, q)$-solutions may be obtained by computing the integral curves of the Riemann-Silberstein vector. These curves lie on the isosurfaces of the complex scalar [22]

$$
\Phi_E + i \Phi_B = \alpha^p \beta^q \tag{16}
$$

and consist of knotted tori when $p \neq 1, q \neq 1$ are co-prime integers. We note that the solutions hold as far as the nonlinear theory is well defined for null fields, i.e. $F = G = 0$. Therefore, their universality do not depend on the specific choice of the Lagrangian.

Probe waves and effective metrics. – A direct consequence of nonlinear electrodynamics is the violation of the principle of superposition. This property implies that
linearized wavy disturbances about a smooth background solution propagate non-trivially. Borrowing from the terminology of laser optics, we may say that probe fields (waves) interact with pump fields (knots) and are scattered by the latter. In this vein, a multitude of effects on the polarization, wave covektor, frequency and velocity of “photons” that interact with the background knots may take place. This is in contrast with previous results due to Arrayás and Trueba, who investigated the classical relativistic motion of charged particles in a knotted electromagnetic field within the context of the linear theory [29]. They conclude that a deeper understanding of the interaction between electromagnetic knots and charged test particles could be useful to design experiments to produce knots in the laboratory. Interestingly, if the knots were produced in a nonlinear regime, we could naturally use incoming light from a distant source in order to detect their presence.

In order to further elaborate on this possibility we restrict ourselves to the regime of small-amplitude/high-frequency probe fields propagating about a given background pump field. Within this regime light propagation is governed by a modified dispersion relation which depends implicitly on the pump fields. The core of this result was first presented by Boillat [12] and Plebanski [13] in the early 70s using Hadamard’s method of discontinuities. More recently, similar results were re-derived by Novello et al. [30] and by Obhukov and Rubilar [31] using the technique of the effective metrics (also called optical pump fields). In the server’s rest frame (or, equivalently, of the magnetic field), a space-like vector playing the role of the Poynting vector. From these expressions we expect that, for a generic background field, both $A_\pm$ and $F^{ac}F_c^b$ will be quite complicated functions of $E^a$ and $B^a$. However, in the special case of a null field ($F = G = 0$), two important simplifications emerge:

- the coefficients $A_\pm$ do not depend on space-time position since all relevant quantities such as $P, Q, R$ are evaluated on top of a constant state;
- eq. (19) reduces to the simple form

$$F^{ac}F_c^b = -\sigma^a\sigma^b \quad \text{(21)}$$

with

$$\sigma^a = |E| (t^a + \eta^{abcd}e_b e_c d) \quad \text{(22)}$$

$e_b, b$, denoting space-like orthonormal vectors in the directions of the electric and magnetic fields, respectively, i.e.

$$e_a e^a = -1 \quad b_b b^b = -1 \quad e_a b^a = 0$$

and $|E|$ the modulus of the electric field in the observer’s rest frame (or, equivalently, of the magnetic field).

We note that, according to these definitions, the vector field $\sigma^a$ is null with respect to the background metric, i.e. $g_{ab}\sigma^a\sigma^b = 0$ and automatically satisfies the algebraic relations

$$F_{[kl}\sigma_{m]} = 0, \quad F_{km}\sigma^m = 0. \quad \text{(23)}$$

A vector field satisfying these equations is known in the literature as a Robinson congruence. Roughly speaking, Robinson’s theorem [33] states that, for any real, nonzero null bi-vector $F_{kl}$, the conditions eq. (23) determine a geodesic and shear-free null congruence in spacetime. Conversely, if $\sigma^m$ is given, $F_{kl}$ can be determined up to a change of amplitude and polarization. We are
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now able to describe the interaction between a small-amplitude/high-frequency probe wave and a background pump \((p, q)\)-knot. Since the latter is defined in terms of a null field, it turns out that the reciprocal effective metrics (18) can be written as

\[
\hat{g}^{ab} = g^{ab} - A_{\pm} \sigma^a \sigma^b.
\]

with \(A_{\pm}\) and \(\sigma^a\) evaluated using the \((p, q)\) ansatz for the Bateman pair discussed in the last section. Now, hyperbolicity of the equations of motion eqs. (4) guarantees that eqs. (17) describe algebraic varieties with the topologies of convex cones. With these cones in hand, we can readily predict the behaviour of incoming rays using the results presented in [30]. They are given as the null geodesics of the covariant effective metrics

\[
\hat{g}_{\pm ab} = g_{ab} + A_{\pm} \sigma_a \sigma_b,
\]

each satisfying the relations \(\hat{g}_{\pm ab} \hat{g}^{\pm cd} = \delta^d_b\) and \(\hat{g}_{\pm ab} \hat{g}^{\pm cd} = \delta^d_a\). This result suggests that the rays may receive an imprint of knottedness due to the interaction and, therefore, could be experimentally detected with future generations of super high power lasers such as the XFEL laser at DESY. Importantly, as different kinds of knots cannot be deformed continuously into each other, the causal structure associated to each of them carries implicitly the topological invariant, which may be used to detect different knot signatures.

**Summary and prospects.** — The theoretical prediction of knotted states of light within the context of Maxwell’s theory and their possible experimental verification have received considerable attention in the last few years. In this paper, we have shown that several types of known electromagnetic knots can be extended to nonlinear regimes in quite a simple way. Our result implies that probe waves arriving from a distant source would interact with the background knots nontrivially, at least if the latter were made of sufficiently intense fields. Given the fact that the behaviour of the rays are controlled by the Robinson congruence, it is fairly possible that distinct knots will imprint different degrees of knottedness to the incoming beam. A particularly exciting prospect is to investigate the behaviour of such rays in the context of the Euler-Heisenberg Lagrangian, which takes into account one loop corrections due to QED. We shall discuss in a forthcoming communication the powers at which these nonlinear electromagnetic interactions arise, comparing these powers with the power of the XFEL laser.

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