BOUNDS ON ACCUMULATION RATES OF EIGENVALUES ON MANIFOLDS WITH DEGENERATING METRICS

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ABSTRACT. We consider a family of manifolds with a class of degenerating warped product metrics
\[ g_\epsilon = \rho(\epsilon, t)^{2a} dt^2 + \rho(\epsilon, t)^{2b} ds_M^2, \]
with \( M \) compact, \( \rho \) homogeneous degree one, \( a \leq -1 \) and \( b > 0 \). We study the Laplace operator acting on \( L^2 \) differential \( p \)-forms and give sharp accumulation rates for eigenvalues near the bottom of the essential spectrum of the limit manifold with metric \( g_0 \).

1. INTRODUCTION

There are many examples of non-compact manifolds which can be thought of as a ‘limit’ of a sequence of compact manifolds. Particularly nice examples are hyperbolic manifolds in dimensions 2 and 3; the cusp closing theorem of Thurston [17] then says that every complete, non-compact manifold \( M_0 \) is the limit of a sequence of hyperbolic manifolds \( M_k \to M_0 \). Since the Laplacian on \( M_0 \) has continuous spectrum, one expects the eigenvalues of \( M_k \) to accumulate. In dimension 2, Ji, Zworski, and Wolpert ([10, 11, 19, 20]) have given bounds for the accumulation rate of eigenvalues near the bottom of the essential spectrum in the hyperbolic case, while in dimension 3 analogous results were obtained by Chavel and Dodziuk ([3]). Dodziuk and McGowan obtained similar results for the Laplacian acting on differential forms ([9]).

Colbois and Courtois considered convergence of eigenvalues below the bottom of the essential spectrum in a much more general setting [6]. The accumulation rate for eigenvalues of the Laplacian on functions for manifolds \( N = \tilde{N} \cup (M^n \times I) \) with ’pseudo-hyperbolic’ metrics on \( (M^n \times I) \) was given by Judge [12]. Judge also computes the essential spectrum for a more general class of degenerating metrics, and investigates the convergence of eigenfunctions.

We will consider manifolds \( N_\epsilon = \tilde{N} \cup (M^n \times I) \), \( \tilde{N} \) and \( M^n \) compact, with \( n = dim(M) \), and a family of metrics
\[ g_\epsilon = \rho(\epsilon, t)^{2a} dt^2 + \rho(\epsilon, t)^{2b} ds_M^2 \]
on \( M^n \times I \). Here \( \rho = c_1 \epsilon + c_2 t \), \( c_1, c_2 > 0 \), \( t \in I = [0, 1] \), \( a \leq -1 \), \( b > 0 \), and \( ds_M^2 \) is the metric on \( M^n \). We identify the boundary of \( \tilde{N} \) with \( M^n \times 1 \). These are the metrics discussed by Melrose in [16] and considered by Judge in [12]. We consider only non-negative values of \( t \) with \( t \in [0, 1] \), which simplifies
the statements of the results, although we must consider manifolds with boundary. The condition $a \leq -1$ means that the limiting manifold $N_0$ is complete.

We study the accumulation rate for eigenvalues near the bottom of the essential spectrum of the Laplacian acting on both functions and differential forms. Our main results are

**Theorem 1.** Suppose $N_\epsilon = \tilde{N} \cup (M^n \times I)$, $\tilde{N}$ and $M^n$ compact, with metric

\[(2)\quad g_\epsilon = \rho(\epsilon, t)^{2a} dt^2 + \rho(\epsilon, t)^{2b} ds_M^2\]
on $M^n \times I$, with $\rho$ as above. Let

\[R = \int_0^1 \rho(\epsilon, s)^a ds\]

be the geodesic distance from the boundary $0 \times M^n$ of $N_\epsilon$ to $\tilde{N}$. Let $\Xi_\epsilon(x^2)$ be the number of eigenvalues of the Laplacian acting on coexact $p$-forms (satisfying absolute boundary conditions on the boundary of $N_\epsilon$) in $[\sigma, \sigma + x^2)$ where $\sigma$ is the bottom of the essential spectrum for coexact forms of degree $p$ and $0 < p < n$. Then

\[\Xi_\epsilon(x^2) = \frac{dxR}{\pi} + O_x(1)\]

where $d$ is the dimension of the space of harmonic forms of degree $p$ on $M$.

This agrees with the results of Judge [12], Chavel and Dodziuk [3] and Dodziuk and McGowan [9] in the special cases they considered.

**Theorem 2.** Suppose $N_\epsilon$ is as in Theorem 2. Then the essential spectrum of the Laplacian acting on coexact $p$-forms, $0 \leq p \leq n$ on $N_0$ is

\[\left[\left(\frac{n-2p}{2}\right)^2 c_2^2 b^2, \infty\right) \quad a = -1\]

\[[0, \infty) \quad a < -1\]

Note that this agrees with Judges results ([12]) for functions when $p = 0$, and with Mazzeo and Phillips results for the essential spectrum on geometrically finite hyperbolic manifolds ([14], with $c_2 = b = 1$ and $a = -1$). We have recently learned that these results for the essential spectrum have been obtained independently by Antoci (??).

This paper is organized as follows. In Section 2 we discuss the geometry of the manifolds under consideration, and rewrite the metric (1) in a way which makes the geometry more evident. In Section 3 we illustrate our techniques by computing the essential spectrum and accumulation rates for eigenvalues as $\epsilon \to 0$ in the case of functions ($p = 0$). In Section 4 we compute the essential spectrum and give lower bounds on the accumulation rate in the $p \neq 0$ case. Finally, in Section 5 we give upper bounds on the accumulation rate for $p \neq 0$, completing the proof of Theorem 2.
We wish to thank Józef Dodziuk for many helpful conversations.

2. The Geometry

Metrics of the type (1) are discussed by Melrose in [16]. When \( a \leq -1 \) such metrics are complete on the limit manifold \( N \). Melrose classifies metrics where \( a = -1, b = 1 \) as 'hc', or hyperbolic cusp metrics, and metrics where \( a = -1, b = 0 \) as 'boundary', or metrics with cylindrical end. Since we will consider metrics where \( a \leq -1, b > 0 \) we rewrite the metric to make the geometry more evident.

Let \( \tau \) be the geodesic distance from \( t = 0 \) to \( t = 1 \), in other words the geodesic distance from a point \((0, p \in M)\) to \( \tilde{N} \). Then

\[
\tau = \int_0^1 \rho(\epsilon, s)^a \, ds = \int_0^1 (c_1 \epsilon + c_2 s)^a \, ds
\]

and we have two distinct cases,

\[
\tau = \frac{1}{c_2} \left( \ln \left( \frac{c_1 \epsilon + c_2 t}{c_1 \epsilon} \right) \right) \quad a = -1
\]

\[
\tau = \frac{1}{c_2} \left( \frac{(c_1 \epsilon + c_2 t)^{a+1} - (c_1 \epsilon)^{a+1}}{c_2(a+1)} \right) \quad a < -1
\]

Solving for \( t \) and substituting into the metric (1) we get

\[
ds^2 = d\tau^2 + (c_1 \epsilon)^{2b} e^{2bc_2 \tau} ds_M^2 \quad a = -1
\]

\[
ds^2 = d\tau^2 + (c_2(a + 1) \tau + (c_1 \epsilon)^{a+1})^{2b} \tau ds_M^2 \quad a < -1
\]

which is of the form \( ds^2 = d\tau^2 + f_\epsilon(\tau) ds_M^2 \) in both cases. As \( \epsilon \to 0, \tau \to \infty \), and we have a warped product \( I \times f_\epsilon M \), with the length of the interval given by

\[
\tau(1) = \begin{cases} 
R = \frac{1}{c_2} \ln \left( \frac{c_1 \epsilon + c_2}{c_1 \epsilon} \right) & a = -1 \\
R = \frac{1}{c_2} \left( \frac{(c_1 \epsilon + c_2)^{a+1} - (c_1 \epsilon)^{a+1}}{c_2(a+1)} \right) & a < -1
\end{cases}
\]

When \( a = -1 \), \( f_\epsilon(\tau) \) gives an essentially hyperbolic metric; one thinks of pinching off a closed geodesic, with \( \epsilon \) the length of that geodesic. When \( a < -1 \), the cross sections \( M \) shrink at a slower rate as one recedes from \( \tilde{N} \), and the warped product \( I \times f_\epsilon M \) is intermediate between a hyperbolic cusp and a flat cylinder.

Clearly, for any fixed \( \epsilon \) and \( \tau \), the cross section \( \{ \tau \} \times f_\epsilon(\tau) M \) has injectivity radius bounded below by some constant. Moreover, \( f_\epsilon(\tau) \) is an increasing function of \( \tau \) for all \( a \leq -1 \). Since \( M \) is compact, a scaling argument shows that the first non-zero eigenvalue of the Laplacian acting on coexact forms of degree \( p \) on \( M_{\epsilon, \tau} \), say \( \nu_{p, \epsilon}(\tau) \), is a decreasing function of \( \tau \). Hence, as the geodesic distance of a given cross section from \( \tilde{N} \) increases, \( \nu_{p, \epsilon}(\tau) \) increases. This allows us, for technical reasons, to restructure our decomposition of \( N \) as follows:

\[
N = \tilde{N}' \cup (M \times [0, R - r_0 + 1]),
\]
where \( \tilde{N}' = \tilde{N} \cup M \times [R-r_0, R] \). \( r_0 \) will be chosen so that \( \nu_{p,\epsilon}(r_0) \) is relatively large.

3. Functions

We follow essentially the argument in [9]. First, we choose a function \( f \) whose restriction to \( \tilde{N}' \) is orthogonal to a basis of eigenfunctions with eigenvalues less than or equal to \( \sigma + x^2 \). This will only change the counting function \( N_\epsilon(x^2) \) by a bounded amount which can be absorbed into the \( O(1) \) term ([3, Lemma 3.6]). Next, we decompose \( f \) on \( M \times [0, R-r_0] \) as \( f = \bar{f} + \bar{\bar{f}} \), where \( \bar{f} \) depends only on \( \tau \) and \( \bar{\bar{f}} \) is orthogonal to constants on \( M \). \( \bar{f} \) is computed by averaging over each cross section.

Now, if we choose \( r_0 \) so that \( \nu_{p,\epsilon}(r_0) > \sigma + x^2 \), then \( \bar{\bar{f}} \) does not contribute to the counting function \( N_\epsilon(x^2) \). Concentrating on \( \bar{f} \), a straightforward calculation shows that

\[
\Delta \bar{f} = *d * d \bar{f} = \frac{1}{f^2_\epsilon(\tau)} \frac{d}{d\tau} \left( \frac{d \bar{f}}{d\tau} \frac{\partial}{\partial \tau} f^2_\epsilon(\tau) \right)
\]

is a classical Sturm-Liouville problem, and we can convert to the form (see [7])

\[
u'' - ru = \lambda u
\]

with

\[
u = f^2_\epsilon(\tau) \bar{f}, \quad r = \left( \frac{f^2_\epsilon(\tau)}{f^2_\epsilon(\tau)} \right)''
\]

When \( a = -1 \),

\[
r = \left( \frac{ncy^2}{2} \right)^2,
\]

and (6) becomes

\[
u'' = \left( \lambda + \left( \frac{ncy^2}{2} \right)^2 \right) u.
\]

We get

**Proposition 3.** Suppose \( N_\epsilon = \tilde{N} \cup (M^n \times I) \), \( \tilde{N} \) and \( M^n \) compact, with metric

\[
g_\epsilon = \rho(\epsilon, t)^{-2} dt^2 + \rho(\epsilon, t)^{2b} ds^2_M
\]

on \( M^n \times I \), with \( \rho = c_1 \epsilon + c_2 t \). Then the essential spectrum of the Laplacian acting on functions on \( N_0 \) is

\[
\left[ \left( \frac{ncy^2 b}{2} \right)^2 = \sigma, \infty \right).
\]
Let $R$ be as in (5) and let $N_\epsilon(x^2)$ be the number of eigenvalues of the Laplacian acting on function in $[\sigma, \sigma + x^2)$. Then

$$N_\epsilon(x^2) = \frac{xR}{\pi} + O_x(1).$$

This is as in [12], with slightly different notation. When $a < -1$, (8)

$$r = \frac{(a + 1)bn(bn - 2a - 2)c_2^2}{2(c_2(a+1)\tau + (c_1\epsilon)^{a+1})^2}.$$  

The potential $\phi$ is integrable, and [3, Theorem 4.1] tells us the counting function for the corresponding Sturm-Liouville problem has the same asymptotics as if the potential were identically 0. Hence,

**Proposition 4.** Suppose $N$ is as in Proposition 3 with metric

$$g_\epsilon = \rho(\epsilon,t)^{2a} dt^2 + \rho(\epsilon,t)^{2b} ds_M^2$$

on $M^n \times I$, where $a < -1$ and $\rho$ as above. Then the essential spectrum of the Laplacian acting on functions on $N_0$ is $[0, \infty)$. Let $R$ be as in [9] and let $N_\epsilon(x^2)$ be the number of eigenvalues of the Laplacian acting on function in $[0, x^2)$. Then

$$N_\epsilon(x^2) = \frac{xR}{\pi} + O_x(1).$$

The essential spectrum in this case was given by Judge ([12]). The accumulation estimate (10) can also be obtained using Judge’s techniques ([13]).

4. Upper eigenvalue bounds for forms

We consider the sequence of eigenvalues of the Laplacian acting on coexact forms of degree $p$, $0 < \nu_1 \leq \nu_2 \leq \cdots \to \infty$. If we can give an upper bound $y \geq \nu_j$ for some $j$, we will obtain a lower bound for the counting function $\Xi_\epsilon(y) \leq j$. We work in the space $E$ of $C^\infty$ coexact forms of degree $p$ on $N_\epsilon$ with support contained in $\{x | 1 \leq d(x, \check{N}) \leq R\}$, with coefficients which depend only on $\tau$. Any form $\omega \in E$ is zero on $\check{N}$, and we choose forms

$$\omega = \sum_{i=1}^{d} b_i d\tau \wedge H_i$$

where $d$ is the dimension of the space of harmonic $p$ forms on the cross section $M$ and $H_i, i = 1, 2, \ldots, d$ is a basis of harmonic $p$ forms on $M$.

Using Courant’s min-max principle the eigenvalues $\nu_j$ are no greater than the critical values of the Rayleigh-Ritz quotient $(\Delta \omega, \omega)/(\omega, \omega)$ with $\omega \in E$. Since $\omega$ is coexact, we have

$$\frac{(\Delta \omega, \omega)}{(\omega, \omega)} = \frac{(d\omega, d\omega)}{(\omega, \omega)}$$
Since the $b_i$ depend only on $\tau$, and an application of $d$ to the sum in (11) involves only derivatives with respect to other basis elements, we compute

$$d\omega = \sum_{i=1}^{d} b_i' d\tau \wedge H_i$$

where the prime indicates differentiation with respect to $\tau$. Computing the respective $L^2$ norms we have

$$(\omega, \omega) = C_M \sum_{i=1}^{d} \int_{1}^{R} b_i^2 f_\epsilon \frac{n-2p}{2} (\tau) \, d\tau$$

(12)

$$(d\omega, d\omega) = C_M \sum_{i=1}^{d} \int_{1}^{R} (b_i')^2 f_\epsilon \frac{n-2p}{2} (\tau) \, d\tau$$

(13)

$C_M$ can be computed by integrating the basis elements of the cross section $M$.

Using integration by parts in the numerator of the Rayleigh-Ritz quotient, we get $d$ copies of a Sturm-Liouville problem very similar to the one in section 3,

$$-\frac{1}{f_\epsilon^{n-2p}} \left( b_i f_\epsilon^{\frac{n-2p}{2}} \right)' = \lambda b_i.$$

As in Section 3 we reduce to the form

$$u'' - ru = \lambda u.$$ 

In the pseudo-hyperbolic case, when $a = -1$, we have

$$r = \left( \frac{n-2p}{2} \right)^2 c_2^2 b^2$$

and we get

**Proposition 5.** Suppose $N_\epsilon = \tilde{N} \cup (M^n \times I)$, $\tilde{N}$ and $M^n$ compact, with metric

$$g_\epsilon = \rho(\epsilon, t)^{-2} dt^2 + \rho(\epsilon, t)^{2b} ds_M^2$$

on $M^n \times I$, with $\rho = c_1 \epsilon + c_2 t$. If $N_\epsilon(x^2)$ is the number of eigenvalues of the Laplacian acting on coexact $p$-forms in $[\left( \frac{n-2p}{2} \right)^2 c_2^2 b^2, \left( \frac{n-2p}{2} \right)^2 c_2^2 b^2 + 2^2]$, then

$$N_\epsilon(x^2) \geq \frac{dxR}{\pi} + O_x(1)$$

where $d$ is the dimension of the space of harmonic forms of degree $p$ on $M$. In this case, $R = \frac{1}{c_2} \ln \left( \frac{c_1 + c_2}{c_1} \right)$, and letting $\epsilon \to 0$, we see that the essential spectrum of the Laplacian acting on coexact $p$-forms, $0 \leq p \leq n$ on $N_0$ is $\left[ \left( \frac{n-2p}{2} \right)^2 c_2^2 b^2, \infty \right)$ if $d \neq 0$. 

When $a < -1$ a messy but straightforward calculation give

$$r = \frac{c_2^2 b(n-2p)(\frac{b(n-2p)}{2}) - 1}{2(c_2(a+1)\tau + (c_1\epsilon)^{a+1})^2}.$$  

This is an integrable potential, and we get

**Proposition 6.** Suppose $N_\epsilon = \tilde{N} \cup (M^n \times I)$, $\tilde{N}$ and $M^n$ compact, with metric

\begin{equation}
(15) \quad g_\epsilon = \rho(\epsilon, t)^{2a} dt^2 + \rho(\epsilon, t)^{2b} ds^2_M
\end{equation}

on $M^n \times I$, with $\rho = c_1\epsilon + c_2 t$. If $N_\epsilon(x^2)$ is the number of eigenvalues of the Laplacian acting on coexact $p$-forms in $[0, x^2)$, then

$$N_\epsilon(x^2) \geq \frac{dxR}{\pi} + O_x(1)$$

where $d$ is the dimension of the space of harmonic forms of degree $p$ on $M$. In this case, $R = \frac{(c_1\epsilon + c_2)^{a+1} - (c_1\epsilon)^{a+1}}{c_2(a+1)}$, and letting $\epsilon \to 0$, we see that the essential spectrum of the Laplacian acting on coexact $p$-forms, $0 \leq p \leq n$ on $N_0$ is $[0, \infty)$ if $d \neq 0$.

5. **Lower eigenvalue bounds for forms**

We will use the method of [15, Lemma ?] to get global lower eigenvalue bounds for forms on $N_\epsilon$ based on lower eigenvalue bounds on local eigenvalue bounds on (overlapping) pieces of $N_\epsilon$. In particular, we use the idea of constructing a globally defined form while keeping control of the Rayleigh-Ritz quotient as in [9]. For details on the underlying Čech-de Rham formalism see [2, Chapter 2]. The pieces we will consider might have mildly singular boundaries, but all the familiar results of Hodge theory hold ([3, 5], see also [15, Section ?] and [9, Section 4]). We omit many details here, but refer the reader especially to [9] if they wish to fill in the blanks.

First, we pick a simple open cover of $N_\epsilon$ consisting of two pieces; $U_1 = \tilde{N}' \setminus \partial N'$, and $U_2 = M \times [0, R - r_0 + 1]$. Recall that $\tilde{N}' = \tilde{N} \cup M \times [R - r_0, R]$, so $U_1$ and $U_2$ overlap, with $U_1 \cap U_2 = M \times [R - r_0, R - r_0 + 1]$. Next, we choose a coexact $p$ form $\phi$ so that the restriction $\phi|_{U_1} = \phi_1$ is orthogonal to the finite dimensional space of exact eigenforms (on $U_1$) of degree $p+1$ with eigenvalue less than or equal to $y^2$. This is possible using [9, Proposition 5.1]; the proof must be modified somewhat to account for the more general setting here, but the modifications are simple if messy. We will specify values for $r_0$ and $y^2$ later.

Now, since $\phi_1$ is assumed to be exact with eigenvalue greater than or equal to $y^2$, there exists a unique coexact form $\psi_1$ of degree $p$ on $U_1$ with $d\psi_1 = \phi_1$ and

$$\frac{(\phi_1, \phi_1)}{(\psi_1, \psi_1)} = \frac{(d\psi_1, d\psi_1)}{(\psi_1, \psi_1)} \geq y^2.$$
Likewise, by exactness, there exists a unique coexact form $\psi$ on $U$ with $d\psi = \phi$, but we do not yet have any bounds on the Rayleigh-Ritz quotient (and hence on eigenvalues)

$$\frac{(\phi, \phi)}{(\psi, \psi)} = \frac{(d\psi, d\psi)}{(\psi, \psi)}.$$  \hfill (16)

Next, we wish to decompose $\phi_i$, $i = 1, 2$ on $M \times [0, R]$ in a similar fashion to our decomposition for functions at the beginning of Section 3. We will model ourselves on the argument in [9], but since the cross section $\tau \times M^n$ is arbitrary here, we cannot just average coefficients. Rather, we use a harmonic projection. First, decompose $\phi_i = \alpha \wedge d\tau + \beta$, where $\beta$ does not contain $d\tau$. Next, use harmonic projection on $\alpha$ to get,

$$\alpha = \sum_{i=1}^{d} a_i d\tau \wedge H_i + \gamma$$  \hfill (17)

where $d$ is the dimension of the space of harmonic $p$ forms on $M$, $H_i$ is a basis of harmonic $p$ forms on $M$, and the $a_i$ depend only on $\tau$.

Now, we can write $\phi_i = \bar{\phi}_i + \bar{\psi}_i$, with

$$\bar{\phi}_i = \sum_{i=1}^{d} a_i d\tau \wedge H_i$$  \hfill (17)

$$\bar{\psi}_i = \phi_i - \bar{\phi}_i = \beta + \gamma$$  \hfill (18)

We do the same for $\psi_i$. By construction, the coefficients of $\bar{\phi}_i$ and $\bar{\psi}_i$ depend only on $\tau$. A straightforward calculation (see, for example, [8]), shows that as the metric on the cross sections scales by a factor $f_{\tau}(\tau)$, the Rayleigh-Ritz quotient scales as

$$\frac{(d\bar{\phi}_i, d\bar{\phi}_i)|_{g_\tau}}{(\bar{\phi}_i, \bar{\phi}_i)|_{g_\tau}} = \frac{1}{f_{\tau}(\tau)} \frac{(d\bar{\phi}_i, d\bar{\phi}_i)|_{g_1}}{(\bar{\phi}_i, \bar{\phi}_i)|_{g_1}}.$$  \hfill (19)

For small $\tau$, $f_{\tau}(\tau)$ is small, and thus if $r_0$ is chosen appropriately, $\bar{\phi}_i$ will not contribute to any accumulation of eigenvalues.

So far, we have put only a finite number of conditions, depending only on $x$, on our original selection of $\phi$. These conditions guarantee that $\phi_1$ is orthogonal to the finite dimensional space of exact eigenforms (on $U_1$) of degree $p + 1$ with eigenvalue less than or equal to $y^2$. We still need to determine how many additional choices we must make to gain control of the Rayleigh-Ritz quotient \hfill (16). By construction, we can write

$$\bar{\phi}_2 = \sum_{i=1}^{d} a_i d\tau \wedge H_i$$

where $d$ is the dimension of the space of harmonic $p$ forms on $M$, $H_i$ is a basis of harmonic $p$ forms on $M$, and the $a_i$ depend only on $\tau$. Consequently,
we can write
\[ \bar{\psi}_2 = \sum_{i=1}^{\zeta} f_i \, d\tau \wedge \alpha_{i,p-1} + \sum_{i=1}^{d} b_i H_i \]
with \( d\bar{\psi}_2 = \bar{\phi}_2, a_i = b'_i \), and the prime denoting differentiation with respect to \( \tau \).

To evaluate the Rayleigh-Ritz quotient on \( U_2 \), we use
\[ (\bar{\psi}_2, \bar{\psi}_2) = C_M \sum_{i=1}^{d} \int_{0}^{R-r_0} b^2_i f_\epsilon \frac{n-2\mu}{\pi} (\tau) \, d\tau \]  
\[ (\bar{\phi}_2, \bar{\phi}_2) = C_M \sum_{i=1}^{d} \int_{0}^{R-r_0} (b'_i)^2 f_\epsilon \frac{n-2\mu}{\pi} (\tau) \, d\tau \]
and we again have \( d \) copies of
\[ \frac{1}{f_\epsilon^n} \left( b'_i f_\epsilon \frac{n-2\mu}{\pi} \right)' = \lambda b_i. \]

Letting \( \sigma \) be the bottom of the essential spectrum for \( N_0 \) we see that the number of eigenvalues in the interval \( [\sigma, \sigma + x^2] \) for the equation \( \Delta_p \psi_2 = \nu \psi_2 \) is given by \( \frac{dx}{\pi} + O(x) \). Thus, we can choose \( \phi \) in such a way that \( \psi_2 \) is orthogonal in \( L^2 \) to the basis of eigenforms with eigenvalues less than \( \sigma + x^2 \) on \( U_2 \) by imposing \( \frac{dx}{\pi} + O(x) \) conditions. The number of conditions imposed on the choice of \( \phi \) depends only on \( x \).

We have chosen \( \phi \) in such a way that we have control over the relevant Rayleigh-Ritz quotients on both \( U_1 \) and \( U_2 \), but it is not the case that \( \psi_1 \) and \( \psi_2 \) must match on \( U_1 \cap U_2 \). Since \( d\psi_1 = \phi|_{U_1} \) and \( d\psi_2 = \phi|_{U_2} \) it is clear that the difference \( \psi_2 - \psi_1 \) must be exact, and we use the method of \[15\] Section 7 to build a globally defined form \( \psi \) with \( d\psi = \phi \) and with control over the Rayleigh-Ritz quotient. Since our open cover has only two pieces, this produces no difficulties. If we choose \( y^2 \) above in such a way that summing the relevant Rayleigh-Ritz inequalities gives the correct lower eigenvalue bound, we have

**Theorem 7.** Suppose \( N_\epsilon = \tilde{N} \cup (M^n \times I), \tilde{N} \) and \( M^n \) compact, with metric
\[ (21) \quad g_\epsilon = \rho(\epsilon, t)^{2\alpha} dt^2 + \frac{\rho(\epsilon, t)^{2\beta}}{s_M^n} ds \]
on \( M^n \times I \), with \( \rho \) as above, and \( d \) the dimension of the space of harmonic \( p \)-forms on \( M^n \). Let
\[ R = \int_{0}^{1} \rho(\epsilon, s)^{a} ds. \]

Let \( N_\epsilon(x^2) \) be the number of eigenvalues of the Laplacian acting on coexact \( p \)-forms in \( [\sigma, \sigma + x^2] \) where \( \sigma \) is the bottom of the essential spectrum for
coexact forms of degree \( p \) and \( 0 < p < n \). Then

\[
N_e(x^2) = \frac{dx R}{\pi} + O_x(1)
\]

where \( d \) is as in theorem 2.

In the special case when \( d = 0 \), i.e. when there are no harmonic forms on the cross section, we have the following corollary,

**Corollary 8.** Suppose \( N \) is as above, with \( d = 0 \) for some \( p \). Then the essential spectrum of the Laplacian acting on exact forms of degree \( p \) is empty.

**References**

[1] F. Antoci *On the spectrum of the Laplace-Beltrami operator for \( p \)-forms for a class of warped product metrics.* to appear, Advances in Mathematics.

[2] R. Bott and L. W. Tu. *Differential forms in algebraic topology,* Graduate Texts in Math., no. 82 Springer Verlag, New York, 1982.

[3] I. Chavel and J. Dózziuk. *The spectrum of degenerating hyperbolic 3-manifolds.* J. Diff. Geometry 39 (1994), 123–137.

[4] J. Cheeger. *On the Hodge theory of Riemannian pseudomanifolds,* Geometry of the Laplace Operator (R. Osserman and A. Weinstein, eds.), Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, RI, 1980, pp. 91–146.

[5] J. Cheeger. *Spectral geometry of singular Riemannian spaces.* J. Diff. Geometry 18 (1983), 575-657.

[6] B. Colbois and G. Courtois. *Convergence de varités et convergence du spectre du Laplacien.* Ann. Sci. École Norm. Sup. 24, (4) (1991),507–518.

[7] R. Courant and D. Hilbert. *Methods of Mathematical Physics Volume I.* John Wiley Sons, New York, 1937.

[8] J. Dózziuk *Eigenvalues of the Laplacian on forms.* Proc. Amer. Math Soc. 85 (1982), 438-443.

[9] J. Dózziuk and J. McGowan. *The spectrum of the Hodge Laplacian for a degenerating family of hyperbolic three manifolds.* Trans. Amer. Math Soc. 347, 6 (1995), 1981–1995.

[10] L. Ji *Spectral degeneration of hyperbolic surfaces.* J. Geom. Anal. 38, (1993), 263–313.

[11] L. Ji and M. Zworski *The remainder estimate in spectral accumulation for degenerating surfaces.* J. Functional Anal. 38, (1993), 263–313.

[12] C. Judge. *Tracking eigenvalues to the frontiers of moduli space, I.* J. Functional Anal. 184, (2001),273–290.

[13] C. Judge. *Private correspondence.*

[14] R. Mazzeo and R. Phillips. *Hodge theory on hyperbolic manifolds.* Duke Math. J. 60, (1990), 509-559.

[15] J. McGowan. *The p-spectrum of compact hyperbolic three manifolds.* Math. Ann. 279, (1993), 725–745.

[16] R. Melrose. *Geometric scattering theory.* Cambridge Univ. Press, Cambridge, 1995.

[17] W. Thurston. *Three-dimensional geometry and topology. Vol. I.* Edited by Silvio Levy, Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[18] E. Titchmarsh. *Eigenfunction expansions associated with second order differential equations, I.* Cambridge Univ. Press, London, 1946.

[19] S. Wolpert. *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces.* Comm. Math. Phys. 112, (1987),283–315.
[20] S. Wolpert. *Spectral limits for hyperbolic surfaces, II*. Invent. Math. **108**, (1992), 91-129.