Isothermal plasma waves in a gravitomagnetic planar analog

M Sharif and Umber Sheikh

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus Lahore-54590, Pakistan
E-mail: msharif@math.pu.edu.pk

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Abstract

We investigate the wave properties of the Kerr black hole with isothermal plasma using 3+1 ADM formalism. The corresponding Fourier-analyzed perturbed GRMHD equations are used to obtain the dispersion relations. These relations lead to the real values of the components of the wave vector $k$ which are used to evaluate quantities such as phase and group velocities etc. These have been discussed graphically in the neighborhood of the pair production region. The results obtained verify the conclusion of Mackay et al according to which rotation of a black hole is required for negative phase velocity propagation.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The study of gravitomagnetic waves in rotating black holes is important because the existence of black holes can be ultimately verified with the help of the infalling plasma radiation and super-radiance of gravitomagnetic waves. Information from the magnetosphere can be transmitted from one region to another only by means of propagation allowed by the plasma state. The gravitational field and the wind bring perturbations to the external fluid dynamics. The response of black holes to the external perturbations can be explored with the help of the wave scattering method.

General relativity is the theory of four-dimensional spacetime but we experience a three-dimensional space evolved in time. It is easier to split the spacetime into three-dimensional space and one-dimensional time to develop a better understanding of the physical phenomenon. The split we usually use in understanding the general relativistic physics of black holes and plasmas is the 3+1 ADM split introduced by Arnowitt et al [1]. This split in the formulation of general relativity is particularly appropriate for applications to the black hole theory, as...
described by Thorne et al [2–4]. Using this formalism, the wave propagation theory in the Friedmann universe was investigated by Holcomb and Tajima [5]. Holcomb [6] and Dettmann et al [7]. Komissarov [8] discussed the famous Blandford–Znajek solution.

Blandford and Znajek [9] found a process which describes the extraction of rotational energy in the form of Poynting flux. A black hole with a force free magnetosphere behaves as a battery with internal resistivity in a circuit made by poloidal current. This current system is considered to be equivalent to incoming and outgoing waves. The incoming waves transport energy in a direction opposite to the Poynting flux. Penrose [10] was the pioneer who provided the idea of extraction of energy from the rotating black hole by a specific process called the Penrose process. In the wavelength analog of the Penrose process [11] an incoming wave with positive energy splits up into a transmitted wave with negative energy and a refracted wave with enhanced positive energy. The negative energy wave propagates into the black hole equivalent to a positive Poynting flux coming out of the horizon [12, 13].

The key features of the Kerr black hole were beautifully summarized by Müller [14] who investigated the accretion physics in the plasma regime of the general relativistic magnetohydrodynamics (GRMHD). Punsley et al [15] considered black hole magnetohydrodynamics in a broader sense. Musil and Karas [16] observed the evolution of disturbances originated in the outer parts of the accretion disk and developed a numerical scheme to show the transmission and reflection of waves. Koide et al [17] modeled the GRMHD behavior of plasma flowing into a rotating black hole in a magnetic field. They showed (numerical simulations) that the energy of a spinning black hole can be extracted magnetically. Zhang [18, 19] formulated the black hole theory for stationary symmetric GRMHD with its applications in Kerr geometry. He discussed wave modes for cold plasma with specific interface conditions. Buzzi et al [20, 21] provided a linearized treatment of transverse and longitudinal plasma waves in general relativistic two component plasma (3+1 ADM formalism) propagating in the radial direction close to the Schwarzschild horizon.

Mackey et al [22] provided the idea that a negative phase velocity plane wave propagates in the ergosphere of a rotating black hole. They verify that the rotation of a black hole is required for negative phase velocity propagation which is a characteristic of the Veselago medium. This medium was hypothetically mentioned by Veselago [23] and later formed experimentally [24] as a material (called metamaterial or left-handed material). Much work has been carried out to investigate the characteristics of this medium [25]. Woodley and Mojahedi [26] showed (using full wave simulations and analytical techniques) that in left-handed materials, the group velocity can be either positive (backwards wave propagation) or negative. Sharif and Umber [27, 28] investigated some properties of plasma waves by investigating real wave numbers. The analysis has been done for the cold as well as isothermal plasmas living in the neighborhood of the event horizon by using the Rindler approximation of the Schwarzschild spacetime. In a recent paper, the same authors [29] have found some interesting properties of cold plasma waves using perturbation wave analysis to the GRMHD equations in the vicinity of the Kerr black hole. They have also discussed the existence of a Veselago medium near the pair production region.

This paper has been extended to investigate the wave properties for the isothermal plasma. We have focused this work to investigate the following three main objectives:

1. The behavior of gravitomagnetic waves under the influence of gravity and magnetospheric wind is analyzed. This helps us to detect the response of the black hole magnetospheric plasma oscillations to gravitomagnetic perturbations near the pair production region. The pair production region lies near the event horizon of the black hole.

2. The existence of a Veselago medium in the black hole regime is checked.
(3) The negative phase velocity propagation regions are investigated and the results are compared with those obtained by Mackay et al [22].

To this end, we derive the GRMHD equations in 3+1 formalism using the isothermal equation of state. The component form of the equations for specific background assumptions is obtained by using perturbations. We consider the perturbed quantities as plane harmonic waves produced by gravity and wind due to black hole rotation. The Fourier analysis method for waves is applied and dispersion relations are derived. These relations lead to the \( x \)-component of the wave vector from which the relevant quantities are investigated to analyze the wave properties near the pair production region.

The paper is organized as follows. The next section is oriented with the description of the Kerr analog spacetime and mathematical laws in 3+1 formalism for this model. Section 3 is devoted to the assumptions corresponding to the background flow. In section 4, the GRMHD equations along with their Fourier analyzed perturbed form for the isothermal equation of state of plasma are given. Section 5 provides the solutions of the dispersion relations. In the last section, we shall discuss the results.

2. Mathematical framework

This section contains the line element for a general spacetime model. The electrodynamics corresponding to the Kerr planar analog in 3+1 formalism is also considered.

2.1. Description of model spacetime

The line element of the spacetime in 3+1 formalism can be written as

\[
d s^2 = -\alpha^2 \, dt^2 + \gamma_{ij} (dx^i + \beta^i \, dt) (dx^j + \beta^j \, dt),
\]

(2.1)

where the lapse function (\( \alpha \)), shift vector (\( \beta \)) and spatial metric (\( \gamma \)) are functions of time and space coordinates.

We consider the planar analog of Kerr spacetime [19], i.e.,

\[
d s^2 = -dt^2 + (dx + \beta(z) \, dt)^2 + dy^2 + dz^2.
\]

(2.2)

Here \( z, x, y \) and \( t \) correspond to Kerr’s radial \( r \), axial \( \phi \), poloidal \( \theta \) and time \( t \) coordinates. The Kerr metric depends non-trivially on both \( r \) and \( \theta \), whereas this model metric depends on \( z \) only. The lapse function \( \alpha \) is taken to be unity to avoid the effects of horizon and redshift. The value of the shift function \( \beta \) (analog to the Kerr-type gravitomagnetic potential) decreases monotonically from 0 (\( z \to \infty \)) to some constant value (\( z \to -\infty \)). We have assumed the direction of \( \beta \) along the \( x \)-axis. This shift function derives an MHD wind which extracts translational energy analogous to the rotational energy for the Kerr metric. The shift vector in three dimensions will be denoted by the Greek letter \( \beta \). The Kerr-type horizon has been pushed off to \( z = -\infty \). The pair production region lies at \( z = 0 \) where the plasma is created. The newly created particles are then driven up to relativistic velocities by magnetic–gravitomagnetic coupling as they flow off to infinity and down towards the horizon. Geometrized units will be used throughout the paper.

2.2. Electrodynamics in the Kerr planar analog

We consider the magnetosphere filled with MHD fluid and take the perfect MHD flow condition in the fluid’s rest frame,

\[
E + V \times B = 0,
\]

(2.3)
with $\mathbf{V}$, $\mathbf{B}$ and $\mathbf{E}$ being the fiducial observer (FIDO) measured fluid velocity, magnetic and electric fields, respectively. For perfect MHD flow in (2.1) with $\alpha = 1$, the differential form of Faraday’s law in 3+1 formalism [18] turns out to be

$$\frac{d\mathbf{B}}{d\tau} + (\mathbf{B} \cdot \nabla)\beta - (\nabla \cdot \beta)\mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}),$$

(2.4)

where $\frac{d}{d\tau} \equiv \frac{d}{dt} - \beta \cdot \nabla$ is the FIDO measured rate of change of any three-dimensional vector in absolute space. The Gauss law of magnetism, according to FIDO, can be written as [18]

$$\nabla \cdot \mathbf{B} = 0.$$

(2.5)

For (2.1) with $\alpha = 1$, the local conservation law of rest mass [18] according to FIDO is

$$\frac{D\rho_0}{D\tau} + \rho_0 \gamma^2 \mathbf{V} \cdot \frac{DV}{D\tau} + \rho_0 \nabla \cdot (\mathbf{V} - \beta) = 0,$$

(2.6)

where $\rho_0$ is the rest-mass density, $\gamma$ is the Lorentz factor and $\frac{D}{D\tau} \equiv \frac{d}{dt} + \mathbf{V} \cdot \nabla$ is the time derivative moving along the fluid. The FIDO measured law of the force balance equation [18] for the spacetime, given by equation (2.1) with $\alpha = 1$, takes the form

$$\left\{ \left( \rho_0 \gamma^2 \mu + \frac{B^2}{4\pi} \right) \gamma_{ij} + \rho_0 \gamma^4 \mu \mathbf{V}_i \mathbf{V}_j - \frac{1}{4\pi} B_i B_j \right\} \frac{DV_j}{D\tau} + \rho_0 \gamma^2 \mathbf{V}_i \frac{D\mu}{D\tau}$$

$$- \left( \frac{B^2}{4\pi} \gamma_{ij} - \frac{1}{4\pi} B_i B_j \right) V^{l,k} V^{j,k} = -\rho_0 \gamma^2 \mu \beta_{j,i} V^j - p^i$$

$$+ \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi} (\mathbf{B})_j^2 + \frac{1}{4\pi} B_{i,j} B^j$$

$$- \frac{1}{4\pi} \left[ (\mathbf{B} \times (\mathbf{V} \times \nabla \times (\mathbf{V} \times \mathbf{B}) - (\mathbf{B} \cdot \nabla)\beta) + (\mathbf{V} \times \mathbf{B}) \cdot \nabla \beta \right]|,$$

(2.7)

where $\mu$ is the specific enthalpy and $p$ is the pressure of the fluid. The FIDO measured local energy conservation law (equation (2.4) of [28]), for equation (2.1) with $\alpha = 1$, is given as follows:

$$\rho_0 \gamma^2 \frac{D\mu}{D\tau} + \mu \gamma^2 \frac{D\rho_0}{D\tau} + 2\rho_0 \mu \gamma^4 \mathbf{V} \cdot \frac{DV}{D\tau} - \frac{dp}{d\tau} - \mu \rho_0 \gamma^2 \nabla \cdot \beta + \rho_0 \mu \gamma^2 \nabla \cdot \mathbf{V}$$

$$- \rho_0 \mu \gamma^2 \nabla \cdot (\mathbf{V} \cdot \nabla)\beta + \frac{1}{4\pi} \left( (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) + (\mathbf{V} \times \mathbf{B}) \cdot \frac{d}{d\tau} (\mathbf{V} \times \mathbf{B}) \right)$$

$$+ (\mathbf{V} \times \mathbf{B}) \cdot [(\mathbf{V} \times \mathbf{B}) \cdot \nabla] \beta - (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \mathbf{B}) (\mathbf{V} \cdot \beta) \right) = 0.$$

(2.8)

Equations (2.4)–(2.8) give the perfect GRMHD equations.

3. Specialization of background flow for model spacetime

In this section, we give the background flow and relative assumptions which will be used to simplify the problem.

3.1. Description of flow quantities

The FIDO measured 4-velocity of fluid can be described by a spatial vector field lying in the $xz$-plane [19]:

$$\mathbf{V} = V(z)e_x + u(z)e_z.$$

Here the Lorentz factor takes the form $\gamma = \frac{1}{\sqrt{1 - u^2}}$. The magnetic field measured by FIDO is also assumed to be in the $xz$-direction:

$$\mathbf{B} = B[z(z)e_x + e_z].$$
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where \( B \) is constant. The corresponding Poynting vector becomes

\[
S = \frac{1}{4\pi} (E \times B).
\]

We have considered an example of the stationary flow of an isothermal MHD fluid in our model spacetime (2.2). These flows are used as stationary model magnetospheres whose dynamical perturbations are to be studied. The plasma is moving in the \( xz \)-direction. The perturbed flow is along the \( z \)-direction due to the black hole’s gravity and along the \( x \)-direction due to rotation of the black hole (in the direction of the shift vector of our analog spacetime). This flow will be analyzed to seek the properties of plasma waves.

3.2. Perturbations and wave propagation

The perturbed flow in the magnetosphere (which is in the \( xz \)-plane) can be characterized by velocity \( \mathbf{v} \), magnetic field \( \mathbf{B} \), the fluid density \( \rho \) and pressure \( p \). We denote the unperturbed quantities by a superscript zero and the following dimensionless notations are used for perturbations \((\delta \mathbf{V}, \delta \mathbf{B}, \delta p, \delta \rho)\):

\[
\tilde{\rho} \equiv \frac{\delta \rho}{\rho} = \tilde{\rho}(t, x, z), \quad \tilde{p} \equiv \frac{\delta p}{p} = \tilde{p}(t, x, z),
\]

\[
\mathbf{v} \equiv \delta \mathbf{V} = v_x(t, x, z)\mathbf{e}_x + v_z(t, x, z)\mathbf{e}_z,
\]

\[
\mathbf{b} \equiv \frac{\delta \mathbf{B}}{B} = b_x(t, x, z)\mathbf{e}_x + b_z(t, x, z)\mathbf{e}_z.
\]

The perturbed variables take the following form:

\[
\rho = \rho^0 + \rho\tilde{\rho}, \quad p = p^0 + p\tilde{p},
\]

\[
\mathbf{V} = \mathbf{V}^0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}^0 + \mathbf{b}.
\]

It is also assumed that the perturbations have the sinusoidal dependence of \( t, x \) and \( z \). Thus

\[
\tilde{\rho}(t, x, z) = c_1 e^{-i(\omega t - k_x x - k_z z)}, \quad \tilde{p}(t, x, z) = c_6 e^{-i(\omega t - k_x x - k_z z)},
\]

\[
v_x(t, x, z) = c_2 e^{-i(\omega t - k_x x - k_z z)}, \quad v_z(t, x, z) = c_3 e^{-i(\omega t - k_x x - k_z z)},
\]

\[
b_x(t, x, z) = c_4 e^{-i(\omega t - k_x x - k_z z)}, \quad b_z(t, x, z) = c_5 e^{-i(\omega t - k_x x - k_z z)}.
\]

Using the values of components of \( \mathbf{k} \), we can discuss quantities such as the phase velocity vector, group velocity vector, refractive index and its change with respect to angular frequency. These quantities would help to investigate the wave behavior of the Kerr black hole magnetosphere and the properties of the Veselago medium.

4. GRMHD equations for Kerr spacetime in the isothermal state of plasma

The isothermal equation of state means that there is no exchange of energy between the plasma and the magnetic field. This state can be expressed by the following equation:

\[
\mu = \frac{\rho + p}{\rho_0} = \text{constant}.
\]

When we use this equation of state, the set of GRMHD equations (2.4)–(2.8) take the following form for the spacetime given in equation (2.2), i.e., \( \beta = (\beta_x, 0, 0) \):

\[
\frac{d\mathbf{B}}{d\tau} + (\mathbf{B} \cdot \nabla)\beta = \nabla \times (\mathbf{V} \times \mathbf{B}),
\]

\[
\nabla \cdot \mathbf{B} = 0.
\]
\[
\frac{D(\rho + p)}{D\tau} + (\rho + p)\gamma^2 V \cdot \frac{DV}{D\tau} + (\rho + p)\nabla \cdot V = 0, \tag{4.3}
\]

\[
\left[ (\rho + p)\gamma^2 + \frac{B^2}{4\pi} \right] \delta_{ij} + (\rho + p)\gamma^2 \{ \frac{\partial}{\partial \tau} \}, B_j \left\{ \frac{dV_j}{d\tau} \right\},
\]

\[
+ (\rho + p)\gamma^2 V_{i,k} V^k + (\rho + p)\gamma^2 \{ \frac{\partial}{\partial \tau} \}, V_{j,k} V^k = (\rho + p)\gamma^2 \beta_{j,i} V^j
\]

\[
= - p_j + \frac{1}{4\pi} (B_{i,j} - B_{j,i}) B^j - \frac{1}{4\pi} \left[ B \times \left( \frac{\partial}{\partial \tau} \right) (V \times dB) \right] \], \tag{4.4}
\]

\[
\gamma^2 V \cdot \frac{D}{D\tau} (\rho + p) + 2(\rho + p)\gamma^2 V \cdot \frac{DV}{D\tau} - \frac{dp}{d\tau} + (\rho + p)\gamma^2 (\nabla \cdot V)
\]

\[
+ (V \times B) \cdot \frac{d}{d\tau} (V \times B) + (V \times B) \cdot \left[ (V \times B) \cdot \nabla \right] \beta = 0. \tag{4.5}
\]

These equations proceed in a similar way as used in [27, 28]. Equations (3.1) and (3.2)

as well as the restrictions for the velocity and magnetic fields, given in section 3.1, lead to the

perturbed form of equations (4.1)-(4.5) given in appendix A. When we use equation (3.3),

the Fourier analyzed perturbed equations take the following form:

\[
\begin{align*}
\dot{k}_x c_2 - ik_x \lambda' c_2 - c_2 (k_x u - \omega - \omega') + c_5 \{ (V - \beta) ik_x + (V - \beta') \} &= 0, \\
\dot{k}_x c_2 - ik_x \lambda c_3 + c_5 \{ (V - \beta) k_x + k_x u - \omega \} &= 0, \\
k_x c_4 &= - k_x c_5,
\end{align*}
\]

\[
\begin{align*}
c_1 [p(-\omega + (V - \beta) k_x + u k_x] - [p'u + p'u' + pu \gamma^2 (V' V + uu')] + \\
c_2 (\rho + p) [-\omega + \beta k_x] \gamma^2 V + ik_x \gamma^2 u V + ik_x (1 + \gamma^2 V^2) + \\
+ \gamma^2 u [(1 + 2 \gamma^2 V^2) V' + 2 \gamma^2 u V u'] + c_3 (\rho + p) [-\omega + \beta k_x] \gamma^2 u V + \\
+ ik_x (1 + \gamma^2 u^2) - (1 - 2 \gamma^2 u^2) (1 + \gamma^2 u^2) u' + 2 \gamma^2 u V V' + c_6 [p(-\omega + \\
+ (V - \beta) k_x + u k_x] + [p'u + p'u' + pu \gamma^2 (V' V + uu')] &= 0, \\
c_1 \rho \gamma^2 u [(1 + \gamma^2 V^2) V' + \gamma^2 u V u'] + c_6 [p \gamma^2 u [(1 + \gamma^2 V^2) V' + \gamma^2 u V u'] + ik_x] + c_2 \left[ \begin{array}{l}
\frac{(\rho + p) \gamma^2 V + \frac{B^2}{4\pi}}{1 + \gamma^2 V^2} + (k_x V + k_x u) \\
\times \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) - \frac{B^2}{4\pi} \right\} + (\rho + p) \gamma^2 u u' \\
+ 4(1 + \gamma^2 V^2) V V' \left\{ (\rho + p) \gamma^2 u V - \frac{\lambda B^2}{4\pi} \right\} + (k_x V + k_x u) \\
\times \left\{ (\rho + p) \gamma^2 u V + \frac{\lambda B^2}{4\pi} \right\} + (\rho + p) \gamma^2 [2 \gamma^2 (1 + \gamma^2 u^2) u V u'] \\
+ [(1 + 2 \gamma^2 u^2) (1 + 2 \gamma^2 V^2) - \gamma^2 V^2 V'] + \frac{B^2 \omega'}{4\pi} \right\} + \frac{B^2}{4\pi} c_4 [-ik_x (1 - u^2) + uu'] \\
+ \frac{B^2}{4\pi} c_5 \left\{ -\omega' - u (V - \beta') + ik_x (1 - V^2) - 2 u V k_x \right\} &= 0. \tag{4.10}
\end{array} \right.
\end{align*}
\]
Equation (4.8) gives the relation between the x- and z-components of the wave vector, i.e. $k_z = -\frac{\omega}{c}k_x$, which will be used in the next section.

5. Numerical solutions

This section is devoted to the numerical solutions of the dispersion equations. The following subsection contains the relative assumptions which make the equations easier to deal with.
5.1. Relative assumptions

In our stationary symmetric background, the $x$-component of the velocity vector can be written in the form \[ V = C + \lambda u, \] where \( C \equiv \beta + V_F \) with \( V_F \) as an integration constant. We assume the value of the shift function \[ \beta = \tanh(z) - 1 \] with \( V_F = 1 \). Further, \( B^2 = 8\pi \) and \( \lambda = 1 \) are taken so that the magnetic field becomes constant. Thus the $x$-component takes the form

\[ V = 1 + \beta + u = \tanh(z) + u. \]

Substituting the value of $V$ in the conservation law of rest mass for the three-dimensional hypersurface, i.e.,

\[ \rho_0 \gamma u = \mu (\rho + p) \gamma u = A \text{ (constant)} \quad (5.1) \]

with the assumption that rest-mass density is constant, we get an equation of the form

\[ 3u^2 + 2u \tanh(z) + \tanh^2(z) - 1 = 0 \]

quadratic in $u$ with the assumption that $A/\rho_0 = 1$. The solutions of this equation and the corresponding values of $V$ are given as follows:

\[ u_1 = -\frac{1}{3} \tanh(z) - \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}, \quad V_1 = \frac{2}{3} \tanh(z) - \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}. \quad (5.2) \]

\[ u_2 = -\frac{1}{3} \tanh(z) + \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}, \quad V_2 = \frac{2}{3} \tanh(z) + \frac{1}{3} \sqrt{3 - 2 \tanh^2(z)}. \quad (5.3) \]

We shall use these values to solve the dispersion relations. The Poynting vector for these values takes the following form:

\[ S = 2 \tanh(z)(\mathbf{e}_x - \mathbf{e}_z). \]

These quantities are valid for the region outside the pair production region. We consider the region \(-5 \leq z \leq 5\) and omit the region \(-1 \leq z \leq 1\) due to large variations in the background flow quantities. In the rest of the region, these quantities become constant and the Fourier analyzed procedure is valid for this region. Further, we use the relation $k_z = -k_x$ which reduces the wave vector to $(k_x, 0, -k_z)$.

Computer programming (using Mathematica) is used to evaluate a root of the dispersion relation for the plasma moving towards the black hole with the velocity components given by equation (5.2). This is given as a separate file with all the required codes. Other roots can be obtained in a similar manner.

It is observed that the sextic equation has all roots admitting imaginary values at several points. The quintic equation gives one real root of the positive $z$ region for both the values of velocity (equations (5.2) and (5.3)), shown in figures 1 and 2. For the negative $z$ region, the velocity, given by equation (5.2), gives one real root shown in figure 3. The approximated root becomes imaginary at $z = -5$ which we omit and our mesh reduces to $-4.8 \leq z \leq -1, 0 \leq \omega \leq 10$ for the interpolating function. The values at $z = -5$ are extrapolated afterwards. The velocity components, given by equation (5.3) for the negative $z$ region, leads to three real roots shown in figures 4, 5 and 6. The real data values for the root give a real interpolation function.

It is clear that figures 1 and 2 represent the neighborhood of the pair production region towards the outer end (as the wave number is found for the positive values of $z$) whereas figures 3, 4, 5 and 6 show the neighborhood of the pair production region towards the event horizon (as the wave number is determined for the negative values of $z$).
5.2. Results

First, we obtain $k_x$ for the velocity components, given by equations (5.2) and (5.3) in the positive $z$ region. These are shown in figures 1 and 2, respectively.

In figure 1, the $x$-component of the wave vector is negative near the pair production region and for the waves with negligible angular frequency. For each angular frequency, the waves grow monotonically in number when they move away from the event horizon. There is a sudden increase in the $x$-component of the wave vector, it admits positive values for a particular value of $z$, then decreases slightly and smoothly increases afterwards. The fluid near the pair production
Figure 2. Veselago medium exists near the pair production region for the velocity components given by equation (5.3). In the same region, negative phase and group velocity propagation regions are observed. Most of the region in the neighborhood of the pair production region shows anomalous dispersion.

region (region with the negative $x$-component of the wave vector) possesses negative values for the $x$-components of the phase and group velocities. For this region, the wave vector is in the opposite direction of the Poynting vector and hence it shows the existence of the Veselago medium there [23]. For the same region, the phase and group velocity vectors are in the direction opposite to the Poynting flux and hence the regions are of negative phase and group velocity propagation. The change in the refractive index with respect to the angular frequency is positive for the regions (i) $1.6 \leq z < 2$, $0.4 \leq \omega \leq 1$, (ii) $2 \leq z < 3$, $0.42 \leq \omega \leq 2.92$, (iii) $3 \leq z < 4$, $0.225 \leq \omega \leq 10$ and (iv) $4 \leq z \leq 5$, $0.14 \leq \omega \leq 10$ for which the dispersion is normal [26, 30]. In the rest of the region, most of the points admit anomalous dispersion.
Near the pair production region, the dispersion is normal whereas the rest of the region admits normal as well as anomalous points of dispersion for velocity components given in equation (5.2).

Figure 2 shows that the $x$-component of the wave vector is negative for the region $1.0 \leq z \leq 1.89$. It is large near the event horizon and decreases up to $z = 1.75$ after which it increases and fluctuations occur in the values. In the region $1.89 < z \leq 10$, it takes random values. The negative values of $k_x$ in the region imply that the wave vector is in the opposite direction to the Poynting vector, which indicates the properties of the Veselago medium. In the region $0 \leq z \leq 1.89$, the $x$-components of the phase and group velocities take negative values and hence this is of the negative phase and group velocity propagation region. Both these quantities admit random values in the region $1.89 < z < 10$. For the region $1 \leq z \leq 1.6$, $0 < \omega \leq 0.079$, the change in the refractive index with respect to the angular frequency is positive and hence the dispersion is found to be normal. In the region
For the velocity components given by equation (5.3), dispersion is normal except for the waves with very low angular frequency. For the negative \( z \) region, i.e., the region towards the event horizon of the black hole in the neighborhood of the pair production region, we obtain one value of \( k_x \) for the velocity components given by equation (5.2) and three for the velocity components given by equation (5.3). These values are shown, respectively, by figures 3, 4, 5 and 6.

In figure 3, the \( x \)-component of the wave vector is negative for the region \(-1.925 \leq z \leq -1.0\) where the Poynting vector is parallel to the wave vector and hence the medium is usual. The refractive index greater than 1 and positive change in the refractive index with respect to the angular frequency indicate normal dispersion. In the rest of the region, all the three
For the velocity components given by equation (5.3), a Veselago medium exists in the whole region with negative phase and group velocity propagation properties. The region of normal dispersion extends as the waves move away from the pair production region. The waves with negligible angular frequency do not disperse normally.

Quantities admit random values and hence there are normal as well as anomalous points of dispersion. For the region $-1.4 \leq z \leq -1.0$, $0.5 \leq \omega \leq 10$, the $x$-components of phase and group velocities are negative such that $v_{px} > v_{gx}$. These velocity components admit random values in the rest of the region.

Figure 4 indicates that the $x$-component of the wave vector is negative for the whole region and hence the wave vector and the Poynting vector are in the same direction showing the existence of the usual medium. As the values of $z$ and $\omega$ grow, $k_x$ decreases and hence $v_{px}$ and $v_{gx}$ are negative in this region. Although $v_{px} > v_{gx}$ for the region $0 \leq \omega \leq 10^{-15}$, yet they are nearly equal for the rest of the region. The refractive index is greater than 1 in the whole region. The refraction increases as the waves move towards the pair production region.
The change in the refractive index with respect to the angular frequency is negative for the region $0 < \omega \leq 0.6$ which shows anomalous dispersion of waves. For the rest of the region, it is positive and thus indicates normal dispersion.

Figure 5 shows that $k_z$ is positive for the whole region. Thus the wave vector is in the opposite direction to the Poynting vector which indicates the presence of the Veselago medium. The quantity $k_z$ increases with the increase in $z$ and $\omega$ except for the waves with negligible angular frequency. $v_{px}$ and $v_{gx}$ are nearly equal and admit positive values which show negative phase and group velocity propagation in the whole region. The refractive index is negative and decreases as the values of $z$ and $\omega$ increase. The change in the refractive index with respect to the angular frequency is negative for the regions (i) $-2 \leq z \leq -1$, $2.15 \leq \omega \leq 10$
(ii) $-3 \lesssim z < -2, 5.5 \lesssim \omega \lesssim 10$ and (iii) $-4 \lesssim z < -3, 9.5 \lesssim \omega \lesssim 10$, which indicates anomalous dispersion in these regions. It is positive for the rest of the region which indicates normal dispersion except for the waves with negligible angular frequency.

In figure 6, the $x$-component of the wave vector is positive for the whole region and increases with the increase in the angular frequency. As the waves move away from the pair production region, the $x$-component of the phase velocity decreases slightly and then increases. In contrast, the $x$-component of the group velocity increases a little and then decreases. The refractive index is negative due to the fact that the Poynting vector is in the opposite direction to the wave vector which shows the existence of the Veselago medium. The positive values of $x$-components of phase and group velocities show the existence of negative phase and group velocity propagation regions. The change in the refractive index with respect to the angular frequency is negative throughout the region and hence shows anomalous dispersion except for the waves with negligible angular frequency.

6. Conclusion

It is well known that charged particles are created in the pair production region. Some of these particles which are pushed onto orbits with negative energy by the Lorentz force move towards the event horizon and the others move towards the outer end of the magnetosphere. These particles would reach their destinations if the plasma existing in the neighborhood of the pair production region allows them to do so. The generation of plasma is necessary to support the MHD magnetically dominated flow. Due to particle generation, waves are produced in the neighborhood of the pair production region. The dispersion relations of waves lead to the understanding of how much the surrounding medium lets the waves disperse through.

This paper studies the wave properties for the isothermal plasma moving with velocity $V$ and admits a constant magnetic field which thread the Kerr black hole magnetosphere. Gravitomagnetic waves and a pair of particles are produced in the $z = 0$ region. If the medium living around the pair production region allows the particles and waves to pass through, energy extraction from the black hole is possible. This can be well understood by investigating the properties of the waves in this region.

We have considered a black hole immersed in a rarefied plasma with a uniform magnetic field which seems to provide support for carrying currents flowing across the magnetic field lines. Due to the strength of the magnetic field, much energy can be extracted due to the fact that the plasma particles that fall into the black hole’s horizon have negative energy. The 3+1 GRMHD equations are derived for the neighborhood of the pair production region and two-dimensional perturbations are discussed in the context of perfect MHD conditions. We assume that the rotation is in the $x$-direction and the horizon is at $z = -\infty$. The perturbations are taken only in the $xz$-direction. The dispersion relations are formulated by assuming the perturbations as plane waves. We solve these relations by taking the wave vector as $(k_x, 0, -k_z)$ and obtain the $x$-component of the wave vector. This component leads to the properties of the isothermal plasma in the neighborhood of the pair production region.

We have discussed these relations for the regions $1 \lesssim z \lesssim 5$ and $-5 \lesssim z \lesssim -1$. The dispersion relations for the region $1 \lesssim z \lesssim 5$ are shown in the figures 1 and 2. These figures indicate that near the pair production region, the plasma admits the properties of the Veselago medium. The region which is nearer to the pair production region does not allow the waves to pass through. Thus the particles and waves cannot get out of this region. The small regions far away from the pair production region admit normal dispersion of waves which indicate that the waves pass through them. As we go away from the pair production region, normal dispersion exists frequently, as shown in figure 1.
The region \(-5 \leq z \leq -1\) allows us to investigate whether there is a possibility for the waves to move towards the black hole event horizon or not. The dispersion relations for this region are shown in figures 3, 4, 5 and 6. From figures 3, 4 and 5, we find that there are chances for the waves to pass through the neighborhood of the pair production region when the plasma admits the properties of the usual or Veselago medium. Figure 6 indicates that there can be a situation when plasma admits the properties of the Veselago medium in the neighborhood of the pair production region, it may not allow the waves to pass through the region.

It is interesting to note that figures 2 and 3 show the irregular dependence of wave vectors on the angular frequency and \(z\). Mathematically, this irregularity is due to the nature of the roots obtained for these graphs. The irregular behavior may be due to a disturbance of the equilibrium between outward- and inward-directed forces. The outward-directed forces are caused by the particle pressure and the curvature drift due to the non-uniform magnetic field and inward-directed forces are exerted by the tangential stress of the magnetic field lines for the low frequency regime.

For the high frequency regime, there is a class of MHD instabilities which sometimes develop in a thin plasma column carrying a strong current. If a kink begins to develop in such a column, the magnetic forces on the inside of the kink become larger than those on the outside, which leads to the growth of perturbation. The column then becomes unstable and causes a disruption. Both the ballooning and kink modes are ideal MHD instabilities.

In figures 1, 2, 5 and 6, we obtain the properties of the Veselago medium. The phase and group velocity vectors propagate in the direction opposite to the Poynting vector which verifies the results of Mackay et al [22] according to which rotation of a black hole is required for negative phase velocity propagation.

We can conclude that waves produced in the pair production region due to pair creation cannot get out of its neighborhood towards the outer end of the magnetosphere. The same result has been obtained for the cold plasma case [29]. We obtain some cases where favorable conditions are present to allow energy to move towards the black hole horizon. For the cold plasma, these conditions are present for the usual medium whereas for the isothermal plasma, these situations occur for the usual as well as the Veselago medium. For the plasmas with pressure, the black hole can suck particles and waves for both the usual and Veselago medium, whereas for the cold plasmas, this situation holds for the usual medium.

Strong magnetic coupling enforces the accreting particles to fall into the black hole with negative energy and negative angular momentum. This indicates that energy and angular momentum flow from the black hole into the disk. When the particles fall into the black hole with negative energy, energy is extracted from the black hole [31]. When the particle with positive energy and positive angular momentum leaves the pair production region and goes towards the event horizon, much energy and momentum are lost and ultimately the particle has negative energy and angular momentum [32]. Thus if the particle either with negative or positive energy leaves the pair production region and gets a chance to reach the event horizon, the result is the extraction of energy from the black hole transmitted to the accretion disk. This shows that when the magnetosphere is filled with isothermal plasma admitting the properties of the Veselago as well as the usual medium, our results indicate that energy extraction is possible.

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Appendix

This appendix includes the details to reach at the perturbed form of the GRMHD equations (4.1)–(4.5). The component forms of these equations are also given.

When we introduce the perturbations from equation (3.2), the linearized GRMHD equations (4.1)–(4.5) become

\[
\left\{ \left( \frac{\partial}{\partial t} - \beta \cdot \nabla \right) (\delta B) \right\} = \nabla \times (\nabla \times B) + \nabla \times (\nabla \times \delta B) - (\delta B \cdot \nabla)\beta, \tag{A.1}
\]

\[
\nabla \cdot (\delta B) = 0, \tag{A.2}
\]

\[
\left\{ \left( \frac{\partial}{\partial t} + (V - \beta) \cdot \nabla \right) (\delta \rho + \delta p) + (\rho + p)\gamma^2 V \cdot \left[ \frac{\partial}{\partial t} + (V - \beta) \cdot \nabla \right] V \right. \]
\[
\left. + (\rho + p)(\nabla \cdot v) + (\delta \rho + \delta p)(\nabla \cdot V) + (\delta \rho + \delta p)\gamma^2 V \cdot (\nabla \cdot V) \nabla \gamma - (\rho + p)\gamma^2 (V \cdot \nabla V) \cdot V \right. \]
\[
\left. + (\rho + p)v \cdot \nabla \ln u, \right. \tag{A.3}
\]

\[
\left[ (\rho + p)\gamma^2 + \frac{B^2}{4\pi} \right] \delta_{ij} + (\rho + p)\gamma^2 V_i V_j - \frac{1}{4\pi} B_i B_j \left( \frac{\partial}{\partial t} - \beta \cdot \nabla \right) V^j \]
\[
- \frac{1}{4\pi} \left( (\delta B_i)_{,j} - (\delta B_j)_{,i} \right) B^j = - (\delta p)_{,j} + \gamma^2 (\delta \rho + \delta p) V^j \]
\[
+ 2(\rho + p)\gamma^2 (V \cdot v)V^j + (\rho + p)v^j \beta_{j,i} + \frac{1}{4\pi} (B_{i,j} - B_{j,i})\delta B^j \]
\[
- (\rho + p)\gamma^2 (V_i V^j + v_j V^i) V^j \]
\[
- \gamma^2 V_i ((\delta \rho + \delta p) V^j + 2(\rho + p)\gamma^2 (V \cdot v)V^j + (\rho + p)v^j V_i^j) \]
\[
- \gamma^2 V_i ((\delta \rho + \delta p) V^j + 4(\rho + p)\gamma^2 (V \cdot v)V^j + (\rho + p)v^j V_i^j) V_{,jk} V^k, \tag{A.4}
\]

\[
\gamma^2 \left\{ \frac{\partial}{\partial t} + (V - \beta) \cdot \nabla \right\} (\delta \rho + \delta p) + (\rho + p)\gamma^2 V \cdot \left[ \frac{\partial}{\partial t} + (V - \beta) \cdot \nabla \right] V \]
\[
- (\rho + p)\gamma^2 V \cdot \nabla \ln u + (\rho + p)\gamma^2 V \cdot (V \cdot \nabla) V + 6(\rho + p)\gamma^2 (V \cdot v) V \cdot (V \cdot \nabla) V \]
\[
+ 2(\delta \rho + \delta p)\gamma^2 V \cdot (V \cdot \nabla) V + 2(\rho + p)\gamma^2 V \cdot (V \cdot \nabla) V \]
\[
- 2(\rho + p)\gamma^2 (V \cdot v) V \cdot (V \cdot \nabla) V + 2(\delta \rho + \delta p)\gamma^2 (V \cdot v) V \cdot (V \cdot \nabla) V \]
\[
+ (\delta \rho + \delta p)\gamma^2 (V \cdot \nabla) V + (\rho + p)\gamma^2 (V \cdot v) - \gamma^2 (\beta \cdot \nabla) (\delta \rho + \delta p) \]
\[
+ 2(\rho + p)\gamma^2 (V \cdot v)(\beta \cdot \nabla \ln u) - 6(\rho + p)\gamma^2 (V \cdot v) V \cdot (\beta \cdot \nabla) V \]
\[
- 2(\rho + p)\gamma^2 V \cdot (\beta \cdot \nabla) V - 2(\delta \rho + \delta p)\gamma^2 V \cdot (\beta \cdot \nabla) V \]
\[
- (\delta \rho + \delta p)\gamma^2 V \cdot (\nabla \cdot V) \beta - (\rho + p)\gamma^2 V \cdot (v \cdot \nabla) \beta \]
\[
- 2(\rho + p)\gamma^2 (V \cdot v)V \cdot (V \cdot \nabla) \beta + \frac{1}{4\pi} \left. \left( V \times B \right) \cdot \left( V \times \delta B \right) \right. \]
\[
+ \left. \left( V \times \delta B \right) \cdot (V \times B) + \left( V \times B \right) \cdot (V \times \delta B) \right. \]
\[
+ \left. (V \times B) \cdot \left( \frac{dv}{dt} \times B + V \times \frac{dB}{dt} \right) \right. \tag{A.5}
\]
The component form of these equations are
\[
\begin{align*}
\frac{db_z}{d\tau} + Vb_{z,x} + ub_{z,z} &= -u'b_z + (V - \beta')b_z + v_{z,z} - \lambda v_{z,z} - \lambda'v_z, \quad (A.6) \\
\frac{db_x}{d\tau} + Vb_{x,z} + ub_{x,z} &= \lambda v_{x,z} - v_{x,z}, \quad (A.7) \\
b_{x,x} + b_{z,z} &= 0, \quad (A.8)
\end{align*}
\]
\[
\begin{align*}
\rho \frac{d\rho}{d\tau} + p \frac{dp}{d\tau} + \rho V\dot{\rho}_{z,x} + pV\dot{\rho}_{z,z} + \rho u\dot{\rho}_{z,z} + p\dot{u} - (\rho - \tilde{\rho})(p' u + pu') + p\rho u''(V' V + uu') + (\rho + p)\gamma'2V\left(V\frac{dv_z}{d\tau} + u\frac{dv_z}{d\tau}\right) + (\rho + p)(1 + \gamma^2 V^2) v_{z,z} \\
+ (\rho + p)(1 + \gamma^2 u^2) v_{z,z} + (\rho + p)uV\gamma'(v_{z,z} + v_{z,x}) \\
= - (\rho + p)\gamma^2 u[(1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V'u] v_x \\
+ (\rho + p)\left[(1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2)\frac{u'}{u} - 2\gamma^4 u^2 V^2\right] v_z, \quad (A.9)
\end{align*}
\]
\[
\begin{align*}
\left\{ (\rho + p)\gamma^2(1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \frac{dv_z}{d\tau} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{dv_z}{d\tau} \\
+ \left\{ (\rho + p)\gamma^2(1 + \gamma^2 V^2) - \frac{B^2}{4\pi} \right\} (Vv_{z,x} + uv_{z,z}) - \frac{B^2}{2\pi} uVb_{z,z} \\
+ \left\{ (\rho + p)\gamma^4 u V + \frac{\lambda B^2}{4\pi} \right\} (Vv_{z,x} + uv_{z,z}) + \frac{B^2}{4\pi} [(1 - V^2)b_{z,x} \\
- (1 - u^2)b_{z,z}] = - \frac{B^2}{4\pi} uu'b_z + \frac{B^2}{4\pi} (\lambda' + u(V - \beta'))b_z - p\tilde{\rho}_{z,x} \\
- (\rho\tilde{\rho} + p\tilde{\rho}')\gamma^2 u[(1 + \gamma^2 V^2) V' + 2\gamma^2 u V'u] - (\rho + p)\gamma^2 u[(1 + 4\gamma^2 V^2) uu'] \\
+ 4(1 + \gamma^2 V^2) VV' v_x - \left\{ (\rho + p)\gamma^2 u[(1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2) - \gamma^2 V^2] V' \\
+ 2\gamma^2(1 + 2\gamma^2 u^2) uVu'] + \frac{B^2}{4\pi} u\lambda'\right\} v_z, \quad (A.10)
\end{align*}
\]
\[
\begin{align*}
\left\{ (\rho + p)\gamma^2(1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{dv_z}{d\tau} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{dv_z}{d\tau} \\
+ \left\{ (\rho + p)\gamma^2(1 + \gamma^2 u^2) - \frac{\lambda^2 B^2}{4\pi} \right\} (Vv_{z,x} + uv_{z,z}) + \frac{\lambda B^2}{2\pi} uVb_{z,z} \\
+ \left\{ (\rho + p)\gamma^4 u V + \frac{\lambda B^2}{4\pi} \right\} (Vv_{z,x} + uv_{z,z}) - \frac{\lambda B^2}{4\pi} [(1 - V^2)b_{z,x} \\
- (1 - u^2)b_{z,z}] = - \frac{B^2}{4\pi} (\lambda' - \lambda uu')b_z - \frac{\lambda B^2}{4\pi} u(V - \beta')b_z \\
- \rho\tilde{\rho}'uu'(1 + \gamma^2 u^2) + \gamma^2 u V V' - V\beta' - [p\tilde{\rho}_{z,x} + p'\tilde{\rho}] \\
+ p\tilde{\rho}'uu'(u + 2\gamma^2 u^2 V V' - V\beta')] - (\rho + p)\gamma^2 u^2(1 + 4\gamma^2 V^2) V' \\
- (1 + 2\gamma^2 V^2)\beta' + 2\gamma^2 u V(1 + 2\gamma^2 u^2)u'v_x - \left\{ (\rho + p)\gamma^2 u - 2\gamma^2 u V\beta' \\
+ (1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2)u' + 2\gamma^2(1 + 2\gamma^2 u^2)u V V' - \frac{\lambda B^2}{4\pi} u\lambda'\right\} v_z, \quad (A.11)
\end{align*}
\]
\[ \begin{align*}
gamma^2 \rho \frac{\partial \tilde{\rho}}{\partial t} + p(\gamma^2 - 1) \frac{\partial \tilde{p}}{\partial t} + \tilde{p} \gamma^2 (\rho' u + \rho u' + 2 \rho u \gamma^2 (V V' + uu')) \\
- \rho \mu V \beta' + \tilde{p} \gamma^2 (u p' + u' p + 2 \rho u \gamma^2 (V V' + uu')) - \rho \gamma^2 u V \beta' \\
+ \gamma^2 \rho \tilde{p}_x (V - \beta) + \gamma^2 u \rho \tilde{\rho}_z + \gamma^2 \tilde{p} \rho (V - \beta) + \gamma^2 u \tilde{p}_x \tilde{p}_z \\
+ \frac{\partial v_x}{\partial t} \left[ 2(\rho + p) \gamma^4 V - \frac{B^2}{4 \pi} (u \lambda - V) \right] + \frac{\partial v_x}{\partial t} \left[ 2(\rho + p) \gamma^4 u + \frac{\lambda B^2}{4 \pi} (u \lambda - V) \right] \\
+ v_x u \left[ 2(\rho + p) \gamma^4 V + \frac{B^2}{4 \pi} (u \lambda - V) \right] + v_{x,z} (V - \beta) \left[ 2(\rho + p) \gamma^4 u \right] \\
+ \frac{B^2 \lambda}{4 \pi} (u \lambda - V) \right] + v_{x,z} \left[ (\rho + p) (1 + 2 \gamma^2 u^2) - \frac{B^2 \lambda}{4 \pi} u (u \lambda - V) \right] \\
+ \frac{B^2}{4 \pi} u b_x \left[ \lambda' - (u \lambda - V) u' \right] + \frac{B^2}{4 \pi} b_z \left[ - \lambda' V + u (u \lambda - V) (V - \beta) \right] \\
+ v_x \left[ (\rho + p) \gamma^4 u \left[ 2 \gamma^2 V' + 6 \gamma^4 V (V V' + uu') - \beta' (1 + 2 \gamma^2 V^2) \right] - \frac{B^2 \lambda'}{4 \pi} \right] \\
+ v_z \left[ \frac{B^2 \lambda'}{4 \pi} \left( \lambda - u (u \lambda - V) \right) + (\rho + p) \gamma^2 \left\{ \frac{u'}{u} + 2 \gamma^2 uu' + 6 \gamma^4 u^2 (V V' + uu') \right\} \right] \\
+ \gamma^2 (V V' + uu') - V \beta' (1 + \gamma^2 u^2) \right] = 0. \quad (A.12)
\end{align*} \]

We have used the conservation law of rest mass for the three-dimensional hypersurface, i.e., given by equation (5.1) to simplify equation (A.9). The same equation will be used to simplify the Fourier-analyzed form of equations (4.11)–(4.17).

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