The Determination of the Metric by the Weyl and Energy-Momentum Tensors

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Abstract

This brief paper investigates the consequences for the metric tensor of space-time when the Weyl tensor (in its conformally invariant form) and the energy-momentum tensor is specified. It is shown that, unless rather special conditions hold, the metric is uniquely determined up to a constant conformal factor.

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1 Introduction

The object of this paper is to re-discuss some ideas that were first published some time ago. In [1] and with more detail in [2] the problem was raised as to how tightly the metric tensor of space-time was determined by specifying the physical sources of the gravitational field in the form of the energy-momentum tensor together with the "free" gravitational field sources through the Weyl tensor. In [3,4] a similar problem was raised but in a different and more restricted form.

To establish notation let \((M, g)\) be a space-time with Lorentz metric \(g\) and Weyl tensor \(C\) in its (tensor type) \((1, 3)\) form with components \(C^{abcd}\). Let the curvature tensor in its (1, 3) form be denoted by \(R\) with components \(R^{abcd}\) and the associated energy-momentum tensor in its type \((0, 2)\) form be denoted by \(T\) with components \(T_{ab}\). For \(m \in M\), \(T_m M\) denotes the tangent space to \(M\) at \(m\), a comma denotes a partial derivative and \(\mathcal{L}\) a Lie derivative. The metric \(g\) is assumed to satisfy the Einstein field equations. It is remarked that the Petrov type (which is a statement involving \(C\) and \(g\)) is actually uniquely determined by the tensor \(C\) in its \((1, 3)\) form (see, e.g. [5,6]). This result is required in what is to follow.

2 Metric Ambiguities

Let \(g'\) be another Lorentz metric on \(M\) which gives rise to the same tensors \(C\) and \(T\) as does \(g\). Suppose also that the Petrov type is not \(O\) or \(N\) over a non-empty open subset of \(M\) and that there are no non-trivial solutions for \(k \in T_m M\) of the equation \(R_{abcd}k^d = 0\) for each \(m\) in a non-empty open subset of \(M\), where the curvature components are computed from \(g\). Then \(g' = \alpha g\) for some positive constant \(\alpha\) [1] (see also [2,5,6]). It should be added that the clauses in this statement do not prevent the result that \(C\) and \(T\) determine the metric up to a constant factor (i.e. up to units of measurement) being generically true (in a well defined sense) as the results in [7] show. However, one should view this result as being somewhat formal since it relies on the fact that \(C\) and \(T\) are specified in their \((1, 3)\) and \((0, 2)\) tensor types, respectively. One could possibly make a case for specifying the Weyl tensor in its conformally invariant \((1, 3)\) form (and which, with the above clauses on \(C\), actually fixes the conformal class of the metric - but not without them [6]) but there seems to be no such case for the type \((0, 2)\)
specification for $\mathcal{T}$.

The results in the previous paragraph involved an application of the Bianchi identity to the curvature tensor $\mathcal{R}$ and relied heavily on the fact that this tensor, because of the clauses stated, is actually the same whether it is computed using $g$ or $g'$ [1,6]. If one tries the same argument but with $\mathcal{T}$ specified in either its $(1,1)$ (with components $T^a{}_b$) or its $(2,0)$ (with components $T^{ab}$) form the curvature tensors are no longer necessarily equal and complications arise. In [3] the metrics $g$ and $g'$ were assumed conformally related and $\mathcal{T}$ was specified in its $(1,1)$ form. The Bianchi identity (conservation law) was then applied to the components $T^a{}_b$ and shown to lead to a restriction on the eigenvalues of this tensor. In fact this method works no matters which tensor type of the energy-momentum tensor is specified. To see this suppose the $(2,0)$ type is specified so that $g$ and $g'$ each have type $(2,0)$ energy-momentum tensor $T^{ab}$. If $g$ and $g'$ lead to the same Weyl tensor $\mathcal{C}$ and if $\mathcal{C}$ is not of Petrov type $O$ or $N$ over any non-empty open subset of $M$ then $g' = e^{2\sigma}g$ for $\sigma : M \rightarrow \mathbb{R}$ [5]. If the covariant derivatives with respect to the Levi-Civita connections arising from $g$ and $g'$ are denoted by a semi-colon and a stroke and the associated Christoffel symbols by $\Gamma^a_{bc}$ and $\Gamma'^a_{bc}$, respectively, then on subtracting the equations $T^{ab}{}_{;b} = 0$ and $T^{ab}{}_{|b} = 0$ one finds

$$T^{ac}p^b_{cb} + T^{cb}p^a_{cb} = 0 \quad (1)$$

$$P^a_{bc} = \Gamma^a_{bc} - \Gamma'_a_{bc} = \sigma,_{b}^{\delta}c + \sigma,_{c}^{\delta}b - \sigma,_{d}g^{da}g_{bc} \quad (2)$$

On substituting (2) into (1) one finds

$$T^{ab}\sigma,_{b} = \frac{1}{6} T g^{ab}\sigma,_{b} \quad (T = T^{ab}g_{ab})$$

$$\Leftrightarrow T^{ab}\sigma,_{b} = \frac{1}{6} T' g'^{ab}\sigma,_{b} \quad T' = T^{ab}g'_{ab} = e^{2\sigma}T \quad (3)$$

Thus, whichever metric is used to compute the eigenvalues and the trace, unless one sixth of that trace is an eigenvalue of the associated energy-momentum tensor one is forced to conclude that $\sigma,_{a} \equiv 0$ on $M$ and hence that $\sigma$ is constant on $M$.

Similarly, if one assumes that the type $(1,1)$ form of the energy-momentum tensor is the same for $g$ and $g'$ as originally done in [3] (respectively the type $(0,2)$ form) one uses the equations $T^a{}_{;b} = T^a{}_{|b} = 0$ (respectively $T^{ab}{}_{;b} = T^{ab}{}_{|b} = 0$ and noting in this latter case that $T'' = T_{ab}g'^{ab} = e^{-2\sigma}T_{ab}g^{ab}$ =
$e^{-2\sigma T}$) to find

$$T_a^b \sigma_b = \frac{1}{4} T \sigma, \quad \text{type (1, 1)},$$

$$T_{ab} \sigma^b = \frac{1}{2} T \sigma, \quad \leftrightarrow \quad T_{ab} \sigma'^b = \frac{1}{2} T'' \sigma, \quad \text{type (0, 2)} \quad (4)$$

where $\sigma^b = g^{ba} \sigma, a$ and $\sigma'^b = g'^{ba} \sigma, a$. Again it is easily seen in each case that the condition that the appropriate fraction of the trace is an eigenvalue is independent of the metric used and that if this fraction of the trace is not an eigenvalue then $\sigma$ is constant on $M$.

### 3 Specific Forms for Matter Distribution

One can now check specific forms for the energy-momentum distribution to see if this appropriate multiple of $T$ could be an eigenvalue. For a perfect fluid with unit flow vector $u$, density $\rho$ and pressure $p$ one has

$$T_{ab} = (p + \rho) u_a u_b + pg_{ab} \quad (5)$$

and so the eigenvalues are $p$ and $-\rho$ and $T = 3p - \rho$. The condition that either of these eigenvalues equals $\frac{1}{6} T$ ($(2, 0)$ case) are $3p + 5\rho = 0$ and $3p + \rho = 0$, the condition that either equals $\frac{1}{4} T$ ($(1, 1)$ case) is $p + \rho = 0$ (the same condition in each case) and the conditions that either equals $\frac{1}{2} T$ ($(0, 2)$ case) are $3p + \rho = 0$ and $p - \rho = 0$. The dominant energy conditions [8] for the tensor (5), assumed nowhere zero on $M$, require $\rho > 0$ and $p + \rho \geq 0$ and so the condition $3p + 5\rho = 0$ is ruled out.

For any Einstein-Maxwell (Electrovac) field, $T = 0$. But if the Maxwell field is non-null, the energy-momentum tensor $T$ (assumed nowhere zero) is non-degenerate, being of Segre type $\{(1, 1)(11)\}$ at each point with two distinct nowhere zero eigenvalues differing only in sign. Since in this case $T$ has no vanishing eigenvalues at any $m \in M$ a contradiction is achieved and $\sigma$ is constant on $M$ so that $g'$ is a constant multiple of $g$. No such contradiction, however, is obtained if the Maxwell field is null for then $T$ has Segre type $\{(211)\}$ with all eigenvalues zero.

If the energy-momentum content of $M$ is described by two non-interacting perfect fluids with energy-momentum tensors of the type (5) and pressure and density functions $p_1$, $\rho_1$ and $p_2$, $\rho_2$ and fluid flow vectors $u$ and $v$, 4
respectively, the energy-momentum tensor is

\[ T_{ab} = (p_1 + \rho_1)u_au_b + (p_2 + \rho_2)v_av_b + (p_1 + p_2)g_{ab} \]  

and has Segre type \{(1,1)1\} with eigenvalues \(-(\rho_1 + \rho_2) - \epsilon, p_1 + p_2 + \epsilon \) and \(p_1 + p_2 \) (repeated) and where \( \epsilon \) is an easily calculated positive function of \(p_1, p_2, \rho_1, \rho_2\) and \(u^a v_a\) [9]. In this case \( T = 3(p_1 + p_2) - (\rho_1 + \rho_2) \) and the conditions derived in the previous section are easily written down. For example, if \( T \) is prescribed in its \((1,1)\) form, they are either \(3K = -4\epsilon < 0\) or \(K = -4\epsilon < 0\) or \(K = 0\) where \(K = p_1 + p_2 + \rho_1 + \rho_2\).

Similar calculations can be performed for other combinations of energy-momentum tensors using the algebraic results in [9] from which it is clear that, unless rather special conditions hold, the space-time metric is determined up to the choice of unit of measurement. More precisely, one has the following general result. Suppose that the type \((1,3)\) tensor \(C\) is specified and is not type \(O\) or \(N\) over any non-empty open subset of \(M\). Then if

(i) the energy-momentum tensor is specified in its \((0,2)\) form and there are no non-trivial solutions for \(k \in T_mM\) of the equation \(R^a_{bcd}k^d = 0\) over any non-empty open subset of \(M\), or,

(ii) the energy-momentum tensor is specified in its \((0,2)\) form (respectively, its \((1,1)\) or its \((2,0)\) form) and if one half (respectively, one quarter or one sixth) of the trace is not an eigenvalue of it,

then it follows that, unless the rather special conditions described above hold, the space-time metric is determined up to a constant conformal factor.
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