Uncertainty Principle for the Cantor Dyadic Group

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Abstract

We introduce a notion of localization for dyadic functions, i.e. functions defined on the Cantor group. Localization is characterized by functional $UC_d$ that is similar to the Heisenberg uncertainty constant for real-line functions. We are looking for dyadic analogs of quantitative uncertainty principles. To justify our definition we use some test functions including dyadic scaling and wavelet functions.

Keywords Localization; dyadic analysis; Cantor group; uncertainty constant; uncertainty principle; scaling function; wavelet.

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1 Introduction

Good time-frequency localization of function $f : \mathbb{R} \rightarrow \mathbb{C}$ means that both function $f$ and its Fourier transform $Ff$ have sufficiently fast decay at infinity. The functional called the Heisenberg uncertainty constant (UC) serves as a quantitative characteristic of this property. Smaller UCs correspond to more localized functions. The uncertainty principle (UP) expresses a fundamental property of nature and can be stated as follows. If $f \neq 0$ then it is impossible for $f$ and $Ff$ to be sharply concentrated simultaneously. In terms of the UC it means that there exists an absolute lower bound for the UC.

There are numerous analogs and extensions of this framework for different algebraic and topological structures. For example, the localization of periodic functions is measured by means of the Breitenberger UC [1]. For some particular cases of locally compact groups (namely a euclidean motion groups, non-compact semisimple Lie groups, Heisenberg groups)
a counterpart of the UC is suggested in \[9\]. A generalization of operator interpretation for the UC is discussed in \[12\]. These and many others related topics are described in the excellent survey \[4\]. But to our knowledge, the question of a quantitative UC for the Cantor dyadic group has not been addressed in the literature. In this paper we try to understand what "good localization" means for functions defined on the Cantor dyadic group. So, a notion of the dyadic UC is suggested and justified. The existence of a lower bound is proven for the dyadic UC. We calculate this functional for dyadic scaling and wavelet functions and find good localized dyadic wavelet frames.

We do not discuss qualitative UPs in this paper. There exists a qualitative UP for a wide class of groups and the Cantor group belongs to the class (see p.224 (7.1) \[4\]). It is easy to see that dyadic function \(f_0 = \chi_{[0,1)} = \hat{f}_0\), where \(\hat{f}\) is a Walsh-Fourier transform of \(f\) (see the definition in Section 2), satisfies the extremal equality in this UP. There are a lot of results in this direction (see \[7\], \[6\] and references therein).

The paper is organized as follows. First, we introduce necessary notations and auxiliary results. In section 3, we formulate the definition of the dyadic UC, prove a dyadic UP, answer the question how to calculate the dyadic UC in some particular important cases. In section 4, we calculate the dyadic UC for Lang’s wavelet and looking for wavelet frames with small dyadic UCs.

2 Notations and Auxiliary Results

Let \(x = \sum_{j \in \mathbb{Z}} x_j 2^{-j-1}\) be a dyadic expansion of \(x \in [0, \infty) = \mathbb{R}_+\), where \(x_j \in \{0, 1\}\). For \(x = p2^n\), \(p \in \mathbb{N}, n \in \mathbb{Z}\), there are two possible expansions, one terminates in 0’s and another does in 1’s. We choose the first one, that is \(x_j \to 0\) as \(j \to \infty\). The dyadic sum of \(x\) and \(y\) is defined by

\[x \oplus y := \sum_{j \in \mathbb{Z}} |x_j - y_j|2^{-j-1}\]

Then \([0, \infty)\) is metrizable with the distance between \(x, y\) defined to be \(x \oplus y\). A function that is continuous from the \(\oplus\)-topology to the usual topology is called \(w\)-continuous. It is well known (see \[10\] sections 1.3, 9.1], \[5\] sections 1.1, 1.2] that this framework is a representation of the Cantor dyadic group, i.e. the Cartesian product of countably many copies of \(\mathbb{Z}_2\), the discrete cyclic group of order 2 (the set \(\{0, 1\}\) with discrete topology and modulo 2 addition).

The Walsh-Fourier transform of \(f \in L_1(\mathbb{R}_+)\) is defined by

\[\hat{f}(t) := \int_{\mathbb{R}_+} f(x)w(t, x)\,dx\]

where the function \(w(t, x) := (-1)^{\sum_{j \in \mathbb{Z}} t_j x_j^{-1}}\) is the representation for a character of the dyadic group. The Walsh-Fourier transform inherits many properties from the Fourier transform (see \[10\] sections 9.2, 9.3]). For example, the Plancherel theorem holds

\[\int_{\mathbb{R}_+} f(x)\,g(x)\,dx = \int_{\mathbb{R}_+} \hat{f}(x)\overline{\hat{g}(x)}\,dx\]
for \( f, g, \hat{f}, \hat{g} \in L_1(\mathbb{R}_+) \) with standard extension to \( L_2(\mathbb{R}_+) \). Functions \( w(n, x) \), where \( n = 0, 1, 2, \ldots \) are called the Walsh functions. They form an orthonormal basis for \( L_2([0, 1]) \). The Walsh system is a dyadic analog of the trigonometric system.

The fast Walsh–Fourier transform of \( x = (x_k)_{k=0,2^{n-1}} \in \mathbb{R}^{2^n} \) is defined by \( c = xW \), where \( W = 2^{-\frac{j}{2}}(w(m, k/2^n))^{2^n-1}_{k,m=0} \) is the normalized Walsh matrix (see [10, section 9.7] accurate within the normalization). The matrix \( W \) is orthogonal, symmetric, and unitary \( W^{-1} = W \).

The concept of a dyadic derivative is quite different from its classical counterpart (see [10 section 1.7], [13 section 6.3]). The function

\[
 f^{[1]}(x) := \sum_{j \in \mathbb{Z}} 2^{j-1}(f(x) - f(x \oplus 2^{-j-1}))
\]

is called the dyadic derivative of \( f \) at \( x \). The inherited properties are the following

\[
 w^{[1]}(n, x) = nw(n, x), \quad \hat{f}^{[1]}(t) = t\hat{f}(t).
\]

But unfortunately the dyadic derivative does not support some natural properties such as the chain rule and the rule \((fg)' = fg' + f'g\).

Let \( H \) be a separable Hilbert space. If there exist constants \( A, B > 0 \) such that for any \( f \in H \) the following inequality holds \( A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \), then the sequence \( (f_n)_{n \in \mathbb{N}} \) is called a frame for \( H \). If \( A = B (= 1) \), then the sequence \( (f_n)_{n \in \mathbb{N}} \) is called a (normalized) tight frame for \( H \).

If the set of functions \( \psi_{j,k}(x) := 2^{j/2}\psi(2^jx \oplus k) \) forms a frame or a basis of \( L_2(\mathbb{R}_+) \), then it is called a dyadic wavelet frame or basis. Using the routine procedure, it can be generated from multiresolution analysis starting with an auxiliary function, that is a scaling function \( \varphi \).

The foundation of the dyadic (Walsh) analysis is contained in [10, 5]. The concept of a dyadic wavelet function and elements of multiresolution analysis theory for the Cantor dyadic group is developed in [8] and later in [3, 2].

### 3 Localization of Dyadic Functions

The quantitative characteristic of the time-frequency localization is the uncertainty constant (UC). Originally, the concept of an uncertainty constant and principle was introduced for the real line case in 1927. The Heisenberg uncertainty constant of \( f \in L_2(\mathbb{R}) \) is the functional

\[
 \Delta_H^2(f) := \Delta_f \Delta_{Ff} \text{ such that }
\]

\[
 \Delta_f^2 := \frac{1}{\|f\|_{L^2(\mathbb{R})}^2} \int_\mathbb{R} (x - x_f)^2|f(x)|^2 \, dx, \quad \Delta_{Ff}^2 := \frac{1}{\|Ff\|_{L^2(\mathbb{R})}^2} \int_\mathbb{R} (t - t_{Ff})^2|Ff(t)|^2 \, dt,
\]

\[
 x_f := \frac{1}{\|f\|_{L^2(\mathbb{R})}^2} \int_\mathbb{R} x|f(x)|^2 \, dx, \quad t_{Ff} := \frac{1}{\|Ff\|_{L^2(\mathbb{R})}^2} \int_\mathbb{R} t|Ff(t)|^2 \, dt,
\]

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where \( Ff \) denotes the Fourier transform of \( f \). It is well known that \( UC_H(f) \geq 1/2 \) for a function \( f \in L_2(\mathbb{R}) \) and the minimum is attained on the Gaussian. Let us make some preliminary remarks to motivate the definition of a localization characteristic for the dyadic case.

**Remark 1** It is easy to see that \( x_f \) is the solution of the minimization problem

\[
\min_{\tilde{x}} \int_{\mathbb{R}} (x - \tilde{x})^2 |f(x)|^2 \, dx.
\]

Hence, the squared \( UC_H \) takes the form

\[
\frac{1}{\|f\|_{L_2(\mathbb{R})}^2} \min_{\tilde{x}} \int_{\mathbb{R}} (x - \tilde{x})^2 |f(x)|^2 \, dx \quad \frac{1}{\|Ff\|_{L_2(\mathbb{R})}^2} \min_{\tilde{t}} \int_{\mathbb{R}} (t - \tilde{t})^2 |Ff(t)|^2 \, dt.
\]

**Remark 2** It is well known that \( x_f \) equals to the integral mean value of the function \( f \), while \( \Delta_f \) means the dispersion with respect to the \( x_f \). The sense of the sign "-" in the definition of \( \Delta_f \) is the distance between \( x \) and \( x_f \). Thus, we have

\[
UC_H^2 = \frac{1}{\|f\|_{L_2(\mathbb{R})}^2} \min_{\tilde{x}} \int_{\mathbb{R}} \text{dist}^2(x, \tilde{x}) |f(x)|^2 \, dx \quad \frac{1}{\|Ff\|_{L_2(\mathbb{R})}^2} \min_{\tilde{t}} \int_{\mathbb{R}} \text{dist}^2(t, \tilde{t}) |Ff(t)|^2 \, dt.
\]

Now we are ready to introduce the definition of a localization characteristic for the dyadic setup.

**Definition 1** Suppose \( f \in L_2(\mathbb{R}_+) \) is a complex valued dyadic function, then the functional

\[
UC_d(f) := V(f)\hat{V}(\hat{f}),
\]

where

\[
V(f) := \frac{1}{\|f\|_{L_2(\mathbb{R}_+)}^2} \min_{\tilde{x}} \int_{\mathbb{R}_+} (x \oplus \tilde{x})^2 |f(x)|^2 \, dx,
\]

\[
V(\hat{f}) := \frac{1}{\|\hat{f}\|_{L_2(\mathbb{R}_+)}^2} \min_{\tilde{t}} \int_{\mathbb{R}_+} (t \oplus \tilde{t})^2 |\hat{f}(t)|^2 \, dt
\]

is called the **dyadic uncertainty constant (the dyadic UC)** of the function \( f \).

**Remark 3** Suppose \( g \) is a bounded dyadic complex-valued function, \( g(x), \, xg(x) \in L_2(\mathbb{R}_+) \). We denote \( G(y) := \int_{\mathbb{R}_+} (x \oplus y)^2 |g(x)|^2 \, dx \). Since \( g(x), \, xg(x) \in L_2(\mathbb{R}_+) \) and \( x \oplus y < x + y \) it follows that \( G(y) \) is finite for \( y \in \mathbb{R}_+ \). Then there exists a point \( y^* \) such that \( \min_y G(y) = G(y^*) \). Indeed, it is clear that \( y^* \) can not be outside the interval \([0, 2^n] \) for some probably large \( n \in \mathbb{N} \) depending on \( g \). It can be checked that \([0, 2^n]\) is compact in the dyadic topology. The function \( x \oplus y \) is \( W \)-continuous, therefore \( G \) is \( W \)-continuous. It is well known that under these conditions, the image \( G([0, 2^n]) \) is compact. Finally, since \( G([0, 2^n]) \subset \mathbb{C} \), it follows that \( G([0, 2^n]) \) is bounded and closed.
**Example 1.** Let \( \chi_M \) be a characteristic function of a set \( M \). Denote \( f_1(x) = \chi_{[0,1/4]}(x) \) and \( g_1(x) = \chi_{[3/4,1]}(x) \). Then it is easy to calculate their Walsh-Fourier transforms 
\[ \hat{f}_1 = \chi_{[0,4]}/4 \quad \text{and} \quad \hat{g}_1 = w(3,1/4) \chi_{[0,4]}/4. \]
It is natural to characterize ”the dispersion” of these functions by means of the diameters of their supports. Thus, \( \text{diam}[0, 1/4] := \sup_{x,y \in [0,1/4]}(x+y) = 1/4 \), \( \text{diam}[3/4, 1] = 1/4 \), and \( \text{diam}[0, 4] = 4 \). So, these functions should have the same localization. On the other side, let us consider the functions \( f_2(x) = \chi_{[0,3/8]}(x) \) and \( g_2(x) = \chi_{[3/4,9/8]}(x) \). Their Walsh-Fourier transforms are 
\[ \hat{f}_2 = \chi_{[0,4]}/4 + w(1,3/4) \chi_{[0,8]/8} \quad \text{and} \quad \hat{g}_2 = w(3,1/4) \chi_{[0,4]}/4 + w(1,3/4) \chi_{[0,8]/8}. \]
Calculating the diameters we get \( \text{diam}[0, 3/8] = 1/2 \), \( \text{diam}[3/4, 9/8] = 2 \), and \( \text{diam}[0, 8] = 8 \). So, the first function should be more localized. Indeed, Table 1 shows that our suppositions are correct.

Columns named \( \tilde{x}_0(f) \) and \( \tilde{t}_0(f) \) mean sets of \( \tilde{x} \) and \( \tilde{t} \) minimizing the functionals \( \int_{\mathbb{R}^+} (x+\tilde{x})^2 |f(x)|^2 \, dx \) and \( \int_{\mathbb{R}^+} (t+\tilde{t})^2 |f(t)|^2 \, dt \) respectively.

**Table 1: The dyadic uncertainty constants: Example 1.**

| \( f \) | \( |f| \|^2 \) | \( \tilde{x}_0(f) \) | \( \tilde{t}_0(f) \) | \( V(f) \) | \( V(f) \) | \( UC_d(f) \) |
|---|---|---|---|---|---|---|
| \( f_1 \) | 1/4 | [0, 1/4] | [0, 1/4] | 1/48 | 16/3 | 1/9 |
| \( g_1 \) | 1/4 | [3/4, 1] | [0, 4] | 1/48 | 16/3 | 1/9 |
| \( f_2 \) | 3/8 | [0, 1/8] | [0, 2] | 3/64 | 8 | 3/8 |
| \( g_2 \) | 3/8 | [3/4, 7/8] | [0, 4] | 71/64 | 32/3 | 71/6 |

**Remark 4** The operator interpretation of the UC does not work for the dyadic setup. Let \( P \) and \( M \) be self-adjoint, symmetric or normal operators defined on a Hilbert space, \( [P, M]_- := PM - MP \) be a commutator of \( P \) and \( M \), and \( [P, M]_+ := PM + MP \) be an anticommutator of \( P \) and \( M \). The following inequality named the Schrödinger uncertainty principle (see [11]) is a simple consequence of the Cauchy-Buniakowski-Schwarz inequality

\[
\|Mf - \beta f\|^2 \|Pf - \alpha f\| \geq \frac{1}{4} \left( \|([P, M]_-, f)\|^2 + \|([P, M]_+, f) - 2\alpha \beta \|f\|^2 \right),
\]

where \( \beta := (Mf, f)/\|f\|^2 \), \( \alpha := (Pf, f)/\|f\|^2 \). It gives two functionals both used as the UCs: the first one is more traditional, but some authors (see [12]) exploit the second one as well

\[
UC_-(f) := \frac{\|Mf - \beta f\| \|Pf - \alpha f\|}{\|([P, M]_, f)\|} \geq 1/2 \tag{1}
\]

\[
UC_+(f) := \frac{\|Mf - \beta f\| \|Pf - \alpha f\|}{\|([P, M]_+, f) - 2\alpha \beta \|f\|^2 \|} \geq 1/2. \tag{2}
\]

Defying in [11] \( Pf(x) = if'(x) \) and \( Mf(x) = x f(x) \), one get the Heisenberg UC in \( L^2(\mathbb{R}) \). The dyadic extension of this framework has the following trouble. If the inner product \( \langle P_H f, M_H f \rangle \) is real-valued then the mean value of the commutator \( \langle [P, M]_-, f \rangle = 2i \Im (P_H f, M_H f) \) vanishes. In classical setup the inner product is pure imaginary for a real-valued \( f \). But for natural choice of dyadic operators on \( L^2(\mathbb{R}_+) \), namely \( Pf(x) = f^{(1)}(x) \) and \( Mf(x) = xf(x) \), it turns out to be real-valued. Thus, one get identical zero in the denominator of [11]. The reason of the trouble is the difference between the operators \( if' \) and \( f^{(1)} \). It is caused by the definitions of respective characters and the properties of derivatives, namely
(e^{it})' = ie^{it} and (w(n, t))^{[1]} = nw(n, t), the imaginary unit appears only in the classical case.

A dyadic counterpart of (2) does not give an adequate characteristic of localization. Indeed, it equals to infinity for the very well localized function \( f_0 := \chi_{[0, 1)} \), \( \hat{f}_0 = \hat{f}_0 \), while, \( UC_d(f_3) = 1/9 \).

There is a lower bound for \( UC_d \), so we get an uncertainty principle for the dyadic Cantor group.

**Theorem 1** For any function \( f \in L_2(\mathbb{R}_+) \), the following inequality holds

\[
UC_d(f) \geq C, \text{ where } C \simeq 8.5 \times 10^{-5}.
\]

**Proof.** Suppose \( f_1(x) := w(\hat{t}, x)f(x \oplus \hat{x}) \), then \( \hat{f}_1(t) := w(t, \hat{x})\hat{f}(t \oplus \hat{t}) \), and it is straightforward calculation to see that

\[
\int_{\mathbb{R}_+} (t \oplus \hat{t})^2 |\hat{f}(t)|^2 \, dt = \int_{\mathbb{R}_+} t^2 |\hat{f}_1(t)|^2 \, dt,
\]

\[
\int_{\mathbb{R}_+} (x \oplus \hat{x})^2 |f(x)|^2 \, dx = \int_{\mathbb{R}_+} x^2 |f_1(x)|^2 \, dx. \tag{3}
\]

So, it is sufficient to prove

\[
\|xg(x)\| \|t\hat{g}(t)\| \geq \sqrt{C}\|g\|^2.
\]

It can be done in the same manner as its classical counterpart (see [9] Theorem 1.1, Corollaries 1.2, 1.3)).

1. Let \( E \) be a measurable subset of \( \mathbb{R}_+ \), \( |E| \) be a Lebesgue measure of \( E \), and \( 0 < \theta < 1/2 \). Then

\[
\left( \int_E |\hat{f}|^2 \right)^{1/2} \leq K_1(\theta)|E|^\theta \|x^0 f(x)\|_2, \text{ where } K_1(\theta) = (2 \theta)^{-2\theta(1 - 2\theta)^{\theta - 1}}.
\]

Indeed, suppose \( B = [0, b) \), \( B' = [b, \infty) \). Then

\[
\left( \int_E |\hat{f}|^2 \right)^{1/2} \leq \left( \int_E |\hat{f}_{\chi_B}|^2 \right)^{1/2} + \left( \int_E |\hat{f}_{\chi_{B'}}|^2 \right)^{1/2}.
\]

Using definition of the Walsh-Fourier transform, the Cauchy-Bunyakovsky-Schwarz inequality, and elementary properties of integrals we get for the first and the second summands

\[
\left( \int_E |\hat{f}_{\chi_B}|^2 \right)^{1/2} \leq \left| E \right|^{1/2} \sup_{E} |\hat{f}_{\chi_B}| \leq \left| E \right|^{1/2} \|f_{\chi_B}\|_1 \leq \left| E \right|^{1/2} \|x^{-\theta} \chi_B(x)\|_2 \|x^0 f(x)\|_2 = \left| E \right|^{1/2} (1 - 2\theta)^{-1/2} b^{-\theta+1/2} \|x^0 f(x)\|_2.
\]

\[
\left( \int_E |\hat{f}_{\chi_{B'}}|^2 \right)^{1/2} \leq \|f_{\chi_B}\|_2 \leq \sup_{B'} x^{-\theta} \|x^0 f(x)\|_2 \leq b^{-\theta} \|x^0 f(x)\|_2.
\]
So,
\[
\left( \int_E |\hat{f}|^2 \right)^{1/2} \leq \left( |E|^{1/2}(1 - 2\theta)^{-1/2}b^{-\theta+1/2} + b^{-\theta} \right) \|x^\theta f(x)\|_2.
\]

It remains to minimize the right side over \( b \) (\( b_{\min} = 4\theta^2|E|^{-1}(1 - 2\theta)^{-1} \)) to get the desired inequality.

2. Let us prove \( \|f\|^2_2 \leq 2K_1(\theta)\|x^\theta f(x)\|_2t^\theta \hat{f}(t)\|_2 \) for \( 0 < \theta < 1/2 \). Denote \( E = [0, r) \), \( E' = [r, \infty) \). Then using the first item, we obtain
\[
\|f\|^2_2 = \|\hat{f}\|^2_2 = \int_E |\hat{f}|^2 + \int_{E'} |\hat{f}|^2 \leq K_1^2(\theta)r^{2\theta}\|x^\theta f(x)\|^2_2 + r^{-2\theta}t^\theta \hat{f}(t)\|_2^2.
\]

Minimizing the last expression over \( r \) (\( r_{\min} = \|t^\theta \hat{f}(t)\|^{1/(4\theta)}_2(K_1^2(\theta)\|x^\theta f(x)\|_2)^{-1/(4\theta)} \)) we get the necessary inequality.

3. Since the function \( g(\alpha) := \left( \|x^\alpha f(x)\|_2\|f\|_2^{-1/\alpha} \right) \) decreases for \( \alpha > 0 \) \( (g'_{\alpha} > 0) \), then
\[
\|x^\alpha f(x)\|_2 \leq \|f\|^{1-\alpha/\beta}_2 \|x^\beta f(x)\|^{\alpha/\beta}_2
\]
for \( 0 < \alpha < \beta \).

4. Applying the last inequality (\( \alpha = \theta \)) to item 2 we obtain
\[
\|f\|^2_2 \leq 2K_1(\theta)\|x^\theta f(x)\|_2t^\theta \hat{f}(t)\|_2 \leq 2K_1(\theta)\|f\|^{2-2\theta/\beta}_2 \|x^\beta f(x)\|^{\theta/\beta}_2 \|t^\beta \hat{f}(t)\|^{\theta/\beta}_2,
\]
thus
\[
\|f\|^2_2 \leq (2K_1(\theta))^{\beta/\theta}\|x^\beta f(x)\|_2\|t^\beta \hat{f}(t)\|_2.
\]

So, choosing \( \beta = 1 \) we have
\[
\|xf(x)\|_2\|t\hat{f}(t)\|_2 \geq C(\theta)\|f\|^2_2, \text{ where } C(\theta) = (2K_1(\theta))^{-1/\theta}.
\]

To get the dyadic uncertainty principle it remains to maximize \( C^2(\theta) \) over \( \theta \), \( \max_\theta C^2(\theta) \approx C^2(0.382) \approx 8.5 \times 10^{-5} \). \( \square \)

It is not easy to calculate \( UC_d \) for an arbitrary function because of the dyadic minimization problem underlying in the definition of \( UC_d \). The following result gives a possible way to calculate the dyadic UC on a wide class of functions. The minimization problem adds up to exhaustive search among \( 2^n \) variants.

**Lemma 1** Let \( f(x) = \chi_{[0,1]}(x) \sum_{k=0}^\infty a_kw(k, x) \) be a uniformly convergent series restricted on \([0, 1]\), \( f_n(x) = \chi_{[0,1]}(x) \sum_{k=0}^{2^n-1} a_kw(k, x) \) be its partial sum, \( V(f) < +\infty \), \( V(\hat{f}) < +\infty \). Then the dyadic \( UC \) takes the form
\[
UC_d(f) = \lim_{n \to \infty} V(f_n)V(\hat{f}_n), \text{ where}
\]
The Walsh-Fourier coefficient of $\xi$ interval, $f$ representation of support of Indeed, where $c := (c_k)_{k=0,2^n-1}$ is the fast Walsh-Fourier transform of $a := (a_k)_{k=0,2^n-1}$.

**Proof.** Suppose $\Delta_{k,n} := [k2^{-n}, (k+1)2^{-n})$, $k = 0, \ldots, 2^n - 1$, $n = 0, 1, \ldots$ is a dyadic interval, $\xi_{k,n} := \chi_{\Delta_{k,n}}$ is the characteristic function of $\Delta_{k,n}$, and $f_n(x) = \sum_{k=0}^{2^n-1} b_k \xi_{k,n}(x)$ is a representation of $f_n$ with respect to the orthogonal system $\{\xi_{k,n} : k = 0, \ldots, 2^n - 1, n = 0, 1, \ldots\}$. It is easy to find a connection between $a = (a_k)_{k=0,2^n-1}$ and $b = (b_k)_{k=0,2^n-1}$.

Indeed,

$$\sum_{k=0}^{2^n-1} a_k w(k, x) = f_n(x) = \sum_{k=0}^{2^n-1} b_k \xi_{k,n}(x).$$

The Walsh-Fourier coefficient of $f_n$ is

$$a_k = \int_{[0,1]} f_n(x) w(k, x) dx = \int_{[0,1]} \sum_{m=0}^{2^n-1} b_m \xi_{m,n}(x) w(k, x) dx = \sum_{m=0}^{2^n-1} b_m \int_{\Delta_{m,n}} w(k, x) dx = \sum_{m=0}^{2^n-1} b_m \frac{\omega_{k,m}^n}{2^n},$$

where $\omega_{k,m}^n$ is a value of $w(k, \cdot)$ on $\Delta_{m,n}$. Let us denote $c_k := b_k 2^{-n/2}$, $\bar{\omega}_{k,m}^n := \omega_{k,m}^n 2^{-n/2}$. Then $a_k = \sum_{m=0}^{2^n-1} c_m \bar{\omega}_{k,m}^n$, that is $a = cW$. Thus, $c$ is the fast Walsh-Fourier transform of $a$.

If $\tilde{x}_n$ minimizes the functional $\int_{\mathbb{R}_+} (x + \tilde{x})^2 |f_n(x)|^2 dx$ then $\tilde{x}_n$ cannot be outside the support of $f_n$. So, $\tilde{x} \in [0, 1) = \cup_{k=0,2^n-1} \Delta_{k,n}$. Then, for $\tilde{x} \in \Delta_{k_0,n}$ we have

$$\int_{\mathbb{R}_+} (x + \tilde{x})^2 |f_n(x)|^2 dx = \int_{[0,1]} (x + \tilde{x})^2 \left| \sum_{k=0}^{2^n-1} b_k \xi_{k,n}(x) \right|^2 dx = \int_{[0,1]} (x + \tilde{x})^2 \sum_{k=0}^{2^n-1} b_k^2 \xi_{k,n}(x) dx = \sum_{k=0}^{2^n-1} b_k^2 \int_{\Delta_{k,n}} (x + \tilde{x})^2 dx = \sum_{k=0}^{2^n-1} b_k^2 \frac{x^3}{3} \bigg|_{\Delta_{k,n} \cap \tilde{x}} = \sum_{k=0}^{2^n-1} b_{k \oplus k_0}^2 \frac{x^3}{3} \bigg|_{\Delta_{k,n}} = \sum_{k=0}^{2^n-1} c_{k \oplus k_0}^2 \frac{3k^2 + 3k + 1}{3 \times 2^{2n}}.$$ 

So, recalling Definition [IV] we get

$$V(f_n) := \frac{1}{\|f_n\|_{L_2(\mathbb{R}_+)}^2} \min \int_{\mathbb{R}_+} (x+\tilde{x})^2 |f(x)|^2 dx = \frac{1}{\sum_{k=0}^{2^n-1} |a_k|^2} \min \sum_{k=0}^{2^n-1} c_{k \oplus k_0}^2 \frac{3k^2 + 3k + 1}{3 \times 2^{2n}}.$$
The Walsh-Fourier transform of \( f_n \) is
\[
\hat{f}_n(t) = \sum_{k=0}^{2^n-1} a_k \int_{[0,1)} w(x, t) w(x, k) \, dx = \sum_{k=0}^{2^n-1} a_k \chi_{[k, k+1)}(t).
\] (4)

Then repeating the above calculations, we have
\[
V(\hat{f}_n) := \frac{1}{\|f_n\|_{L_2(\mathbb{R}_+)}^2} \min_{t} \int_{\mathbb{R}_+} (t \oplus \tilde{t})^2 |\hat{f}(t)|^2 \, dt = \frac{1}{\sum_{k=0}^{2^n-1} |c_k|^2} \min_{k_1} \sum_{k=0}^{2^n-1} a_k^2 \frac{3k^2 + 3k + 1}{3}.
\]

To conclude the proof, it remains to show that \( UC_d(f) = \lim_{n \to \infty} UC_d(f_n) \). We denote \( V_0(g) := \|g\|^2_{L_2(\mathbb{R}_+)} V(g) = \min_{x} \int_{\mathbb{R}_+} (x \oplus \tilde{x})^2 |g(x)|^2 \, dx \).

Firstly, we prove \( \lim_{n \to \infty} V_0(f_n) = V_0(f) \). Assume that the minimum of the functional \( V_0(f_n) \) is achieved at the point \( \tilde{x}_n^* \), the minimum of the functional \( V_0(f) \) is achieved at the point \( \tilde{x}^* \). The functions \( f_n \) converge uniformly on \([0, 1]\) to \( f \), i.e. for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) and for all \( x \in [0, 1] \) we have \( \|f(x) - f_n(x)\| \leq |f(x) - f_n(x)| < \varepsilon \).

Then
\[
|f(x)|^2 - |f_n(x)|^2 \leq 2|f(x)| |f(x) - f_n(x)| + |f(x) - f_n(x)|^2 \leq 2|f_n(x)| \varepsilon + \varepsilon^2 \leq 2(|f(x)| + \varepsilon)\varepsilon + \varepsilon^2.
\]

After multiplication by \((x + y)^2\) and integration over \([0, 1]\) both sides of the above inequality, for all \( y \in [0, 1] \) and for all \( n \geq N \) we get
\[
\int_{[0,1]} (x \oplus y)^2 |f_n(x)|^2 \, dx - \int_{[0,1]} (x \oplus y)^2 |f_n(x)|^2 \, dx \leq \varepsilon C
\]
where \( C = \max_{y \in [0,1]} \int_{[0,1]} (x \oplus y)^2 (2|f(x)| + 3\varepsilon) \, dx \). The last inequality should be valid for \( y = \tilde{x}_n^* \)
\[
\int_{[0,1]} (x \oplus \tilde{x}_n^*)^2 |f(x)|^2 \, dx - V_0(f_n) \leq \varepsilon C \quad \forall n \geq N.
\]

Finally, we can decrease the left-hand side of the inequality by taking minimum of the functional over \( \tilde{x}_n^* \)
\[ V_0(f) - V_0(f_n) \leq \varepsilon C. \]
Similarly, we can prove the following inequality
\[ V_0(f_n) - V_0(f) \leq \varepsilon C. \]
But it requires to start with
\[
|f_n(x)|^2 - |f(x)|^2 \leq 2|f(x)| |f(x) - f_n(x)| + |f(x) - f_n(x)|^2 \leq 2|f(x)| \varepsilon + \varepsilon^2 \quad \forall n \geq N, \forall x \in [0, 1]
\]
and after the integration take \( y = \tilde{x}^* \). As a result, we get \( \lim_{n \to \infty} V_0(f_n) = V_0(f) \).

Now, let us prove \( \lim_{n \to \infty} V_0(\hat{f}_n) = V_0(\hat{f}) \). Assume that the minimum of the functional \( V_0(\hat{f}_n) \) is achieved at the point \( \tilde{f}_n^* \), the minimum of the functional \( V_0(\hat{f}) \) is achieved at the
point $\tilde{t}^*$. By (11) we conclude that $|\hat{f}_{n+1}(t)|^2 \geq |\hat{f}_n(t)|^2$ for all $t \in \mathbb{R}_+$. After multiplication by $(t \oplus y)^2$ and integration over $\mathbb{R}_+$ both sides of the above inequality, we get

$$\int_{\mathbb{R}_+} (t \oplus y)^2 |\hat{f}_{n+1}(t)|^2 dt \geq \int_{\mathbb{R}_+} (t \oplus y)^2 |\hat{f}_n(t)|^2 dt \quad \forall y \in \mathbb{R}_+.$$ 

Thus, the last inequality should be valid for $y = \tilde{t}_{n+1}$

$$V_0(\hat{f}_{n+1}) = \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{n+1})^2 |\hat{f}_{n+1}(t)|^2 dt \geq \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{n+1})^2 |\hat{f}_n(t)|^2 dt \geq V_0(\hat{f}_n).$$

Therefore, $V_0(\hat{f}_{n+1}) \geq V_0(\hat{f}_n)$ for all $n \in \mathbb{N}$, in particular, $V_0(\hat{f}) \geq V_0(\hat{f}_n)$. Let us consider the difference

$$V_0(\hat{f}) - V_0(\hat{f}_n) = \min \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{m})^2 |\hat{f}(t)|^2 dt - \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{n})^2 |\hat{f}_n(t)|^2 dt \leq \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{m})^2 |\hat{f}(t)|^2 - |\hat{f}_n(t)|^2 \cdot dt = \int_{\mathbb{R}_+} (t \oplus \tilde{t}_{m})^2 \sum_{k=0}^{\infty} |a_k|^2 \chi_{|k,k+1)}(t) dt.$$ 

There exists $N \in \mathbb{N}$ such that $\tilde{t}^* \in [0,2^N)$ and $\tilde{t}^* \in [0,2^N)$ for all $n \in \mathbb{N}$ simultaneously. It can be shown by contradiction. Indeed, assume that for any $N \in \mathbb{N}$ there exists $m > N$ such that $\tilde{t}^*_m \geq 2^N$. Then the following inequalities

$$V_0(\hat{f}) \geq V_0(\hat{f}_m) = \int_{[0,2^N)} (t \oplus \tilde{t}_{m})^2 |\hat{f}_m(t)|^2 dt \geq \int_{[2^N,2^m)} (t \oplus \tilde{t}_{m})^2 |\hat{f}_m(t)|^2 dt \geq 2^N \sum_{k=0}^{2^N-1} |a_k|^2$$

should be valid for all $N$. This leads to a contradiction. The function $\int_{\mathbb{R}_+} (t \oplus y)^2 |\hat{f}(t)|^2 dt$ is bounded on $[0,2^N)$ (see Remark 5). Therefore, for all $\varepsilon > 0$ there exists $M$ such that for all $m > M$, $m \in \mathbb{N}$

$$\int_{[m,\infty)} (t \oplus y)^2 |\hat{f}(t)|^2 dt = \int_{\mathbb{R}_+} (t \oplus y)^2 \sum_{k=m}^{\infty} |a_k|^2 \chi_{|k,k+1)}(t) dt < \varepsilon.$$ 

Then for such that $2^n > m$ we have $V_0(\hat{f}) - V_0(\hat{f}_n) < \varepsilon$. Hence, $\lim_{n \to \infty} V_0(\hat{f}_n) = V_0(\hat{f})$. Together with $\lim_{n \to \infty} V_0(f_n) = V_0(f)$, $\lim_{n \to \infty} \|f_n\|^2_{L_2(\mathbb{R}_+)} = \|f\|^2_{L_2(\mathbb{R}_+)}$, and $\lim_{n \to \infty} \|\hat{f}_n\|^2_{L_2(\mathbb{R}_+)} = \|\hat{f}\|^2_{L_2(\mathbb{R}_+)}$ we get the required statement for UC_d. □

Remark 5 It is easy to extend Lemma 4 to the functions of the form

$$g(x) := \chi_{[0,2^N)}(x) \sum_{k=0}^{\infty} a_k w_k(x/2^N).$$ 

Indeed, let $g_n(x) := \chi_{[0,2^N)}(x) \sum_{k=0}^{2^n-1} a_k w_k(x/2^N)$ be a partial sum of the above function $g$, $f_n(x) = g_n(2^N x)$ the function defined in Lemma 4. Then standard calculations show that $\|g_n\|^2_2 = 2^N \|f\|^2_{L_2(\mathbb{R}_+)}$, $\|\hat{g}_n\|^2_2 = 2^N \|\hat{f}_n\|^2_2$, $\int_{\mathbb{R}_+} (x \oplus (\hat{\tilde{f}}^2))^2 |g_n(x)|^2 dx = 2^N \int_{\mathbb{R}_+} (x \oplus (\hat{\tilde{f}}^2))^2 |f_n(x)|^2 dx$ and $\int_{\mathbb{R}_+} (t \oplus \tilde{t})^2 |\hat{g}_n(t)|^2 dt = 2^{-N} \int_{\mathbb{R}_+} (t \oplus (\hat{\tilde{f}}^2))^2 |\hat{f}_n(t)|^2 dt$. Hence, recalling the definition of UC_d we get UC_d(g_n) = UC_d(f_n). The class of the functions of the form $g$ is rather large and important as any orthogonal compactly supported dyadic scaling and wavelet functions belong to this set (see 3, section 5).
We denote \( q_k := \frac{3k^2 + 3k + 1}{8 \times 2^n} \) and suppose \( \|a\| = 1 \), then \( ||c|| = ||aW|| = 1 \) and the \( UC_d(f_n) \) takes the form

\[
UC_d(f_n) = \min_{k_1=0, 2^n-1} \sum_{k=0}^{2^n-1} a_k^2 q_k \min_{k_0=0, 2^n-1} \sum_{k=0}^{2^n-1} c_{k+k_0}^2 q_k.
\]

Let us fix \( n \). It follows from (3) that the minimization problem

\[
\begin{cases}
UC_d(f_n) \to \min \\
\|a\| = 1
\end{cases}
\]

is equivalent to the following one

\[
\begin{cases}
\sum_{k=0}^{2^n-1} a_k^2 q_k \sum_{k=0}^{2^n-1} c_k^2 q_k \to \min \\
\|a\| = 1
\end{cases}
\]

Using Wolfram Mathematica 8.0 we solve numerically the last minimization problem for \( n = 2; 3; 4; 5; 6 \). The result is demonstrated in Table 2.

| \( n \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) |
|---|---|---|---|---|---|
| \( \min f_n UC_d(f_n) \) | 0.0891 | 0.0882 | 0.0873 | 0.0881 | 0.0872 |

### 4 Examples

#### 4.1 Lang’s wavelet and scaling function

To examine and illustrate the definition of the dyadic UC we use the first nontrivial example of orthogonal wavelets on the Cantor dyadic group (see [8]). The dyadic scaling function is defined by

\[
\varphi_a(x) = \frac{1}{2} \chi_{[0, 1)} \left( \frac{x}{2} \right) \left( 1 + a \sum_{j=0}^{\infty} b^j \left( 2^{j+1} - 1, \frac{x}{2} \right) \right), \quad \hat{\varphi}_a = \chi_{[0, 1/2)} + a \sum_{j=0}^{\infty} b^j \chi_{[2^{j+1/2}, 2^{j+1})},
\]

where \( 0 < a \leq 1, a^2 + b^2 = 1, a, b \in \mathbb{R} \). The corresponding wavelet is defined by

\[
\psi_a(x) = 2a_0 \varphi_a(2x + 1) - 2a_1 \varphi_a(2x) + 2a_2 \varphi_a(2x + 3) - 2a_3 \varphi_a(2x + 2),
\]

where \( a_0 = (1 + a + b)/4, \quad a_1 = (1 + a - b)/4, \quad a_2 = (1 - a - b)/4, \quad a_3 = (1 - a + b)/4 \). Then the wavelet system \( \{\psi_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{R}_+} \) forms an orthonormal basis in \( L_2(\mathbb{R}_+) \).
The integrals defining the dyadic UC for the scaling and wavelet functions are

\[
\begin{align*}
\int_{\mathbb{R}_+} (x \oplus \bar{x})^2 |\varphi_a(x)|^2 \, dx &= \frac{4}{3} + \frac{1}{4} w \left(1, \frac{\bar{x}}{2}\right) (-4a + abw(1, \bar{x})) \\
&- \frac{a^2b}{2} \sum_{j=0}^{\infty} \left(\frac{b^2}{2}\right)^j w(2^j, \bar{x}) + \frac{a^2b^2}{16} \sum_{j=0}^{\infty} \left(\frac{b^2}{4}\right)^j w(2^j \oplus 2^{j+1}, \bar{x}) \\
\int_{\mathbb{R}_+} (t \oplus \bar{t})^2 |\tilde{\varphi}_a(t)|^2 \, dt &= A(0, \bar{t}) + a^2 \sum_{j=0}^{\infty} b^{2j} A(2^j - 1/2, \bar{t}) \\
&+ \frac{b^2}{16} \sum_{j=0}^{\infty} \left(\frac{b^2}{4}\right)^j w(2^{j+1} \oplus 2^{j+2}, \bar{x}) + \frac{a^3}{4} w \left(1, \frac{\bar{x}}{2}\right) b \sum_{j=0}^{\infty} \left(\frac{b^2}{2}\right)^j w(2^{j+1}, \bar{x}) \\
\int_{\mathbb{R}_+} (t \oplus \bar{t})^2 |\tilde{\psi}_a(t)|^2 \, dt &= b^2 A(1/2, \bar{t}) + a^2 \sum_{j=0}^{\infty} b^{2j} A(2^j - 1, \bar{t}) + a^4 \sum_{j=1}^{\infty} b^{2j} A(2^j - 1/2, \bar{t}),
\end{align*}
\]

where \(A(\xi, \eta) = \frac{1}{3}((\inf \{[\xi, \xi + 1/2] \oplus \eta\}) + 1/2)^3 - \frac{1}{3}((\inf \{[\xi, \xi + 1/2] \oplus \eta\})^3.\) It turns out that

\[
UC_d(\varphi_a), UC_d(\psi_a) < \infty \iff \sqrt{3}/2 < a \leq 1.
\]

The dyadic UCs for the different values of the parameter \(a\) are collected in Table 3 and Table 4. The best localized function here is the Haar scaling function. It corresponds to the case \(a = 1.\)

| \(a\) | \(V(\varphi_a)\) | \(x_0(\varphi_a)\) | \(V(\tilde{\varphi}_a)\) | \(t_0(\varphi_a)\) | \(UC_d(\varphi_a)\) |
|----|----------------|----------------|----------------|----------------|----------------|
| 0.9 | 0.346          | 0              | 1.29           | [1/2, 1)       | 0.446          |
| 0.95| 0.315          | 0              | 0.482          | [1/2, 1)       | 0.152          |
| 1  | 1/3            | [0, 1)         | 1/3            | [0, 1)         | 1/9            |

### 4.2 Dyadic wavelet frames with good localization

1. Let us consider generators of normalized tight frames [2 Example 3.2] for \(L_2(\mathbb{R}_+)\):

\[
g_{l,s}(x) = 2^{-s} \chi_{[0,2^s)} w(l, 2^{-s} x),
\]
Table 4: The dyadic uncertainty constants for $\psi_a$.

| $a$  | $\Delta^2_d(\psi_a)$ | $\hat{x}_0(\psi_a)$ | $\Delta^2_d(\hat{\psi}_a)$ | $t_0(\varphi_a)$ | $UC^2_d(\psi_a)$ |
|------|----------------------|----------------------|-----------------------------|------------------|------------------|
| 0.9  | 0.280                | 0.5                  | 7.438                       | 3/2, 2          | 2.083            |
| 0.95 | 0.254                | 0.5                  | 1.546                       | 3/2, 2          | 0.393            |
| 1    | 1/3                  | [0, 1)               | 1/3                         | [0, 1)          | 1/9              |

where $l \in \mathbb{N}$, $s \in \mathbb{Z}_+$. The Walsh-Fourier transform of $g_{l,s}$ is $\widehat{g_{l,s}} = \chi_{U_{l,s}}$, where $U_{l,s} = 2^{-s}(l \oplus [0, 1))$. Suppose that $\psi = g_{l,s}$. Then $\{\psi_{j,\alpha}\}$ is a normalized tight frame for $L_2(\mathbb{R}^+)$. For all $l \in \mathbb{N}, s \in \mathbb{Z}_+$ the dyadic UC is $UC^2_d(g_{l,s}) = \frac{1}{9}$.

2. As it was noted in Table 2 numerically $\min UC^2_d(f_n) \simeq 0.0891$ for $n = 2$. Let us try to find a frame generator such that its dyadic UC is close to this value. Let $\psi = \chi_{[0, 1)}(x) \sum_{k=0}^{3} a_k w(k, x)$. From the frame criteria, we should provide zero moment for the frame generator $\psi$ or, equivalently, $\widehat{\psi}(0) = 0$. Thus, we assume that $a_0 = 0$. Using Wolfram Mathematica 8.0 we solve numerically the minimization problem (5). The coefficients are $(a_0, a_1, a_2, a_3) = (0, 0.094206, 0.551564, 0.828796)$. Using Theorem 3.2 in [2], we compute the frame bounds for the frame $\{\psi_{j,k}\}$, namely $A = 0.313098$, $B = 0.695777$. The dyadic UC is $UC^2_d(\psi) = 0.091286$ and it is close to the minimal possible constant for $n = 2$.

The same computations can be done for the case $n = 3$. Let $\psi = \chi_{[0, 1)}(x) \sum_{k=0}^{7} a_k w(k, x)$. The minimum for $UC^2_d(\psi)$ is delivered by the coefficients $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (0, 0.001335, -0.009155, -0.022170, -0.067567, -0.138436, -0.601657, -0.783391).$ The frame bounds for the frame $\{\psi_{j,k}\}$ are $A = 0.004649$, $B = 0.614194$. The dyadic UC is $UC^2_d(\psi) = 0.0882147$.

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