ANOMALOUS POLARIZATION-CURVATURE INTERACTION IN A GRAVITATIONAL-WAVE FIELD

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An exact solution to the dynamic equations for a massive boson traveling in a pp-wave gravitational background under the influence of the force induced by curvature, is presented. We focus on the effect of anomalous polarization-curvature interaction and consider models in which the coupling constant of such an interaction is treated to be either a deterministic quantity or a random variable.

1 Introduction

The motion of a point particle with intrinsic structure has become a subject of detailed studies about a century ago. M. Abraham\(^1\) was the first to consider in detail the motion of an electron as a particle possessing a supplementary vector degree of freedom. In the twenties of the past century L.H. Thomas\(^2\), J. Frenkel\(^3\) and I. Tamm\(^4\) studied different aspects of the classical spinning particle motion. The equations of motion for particles with arbitrary multipole moments in the framework of General Relativity were obtained by M. Mathisson\(^5\) and then revived by A. Papapetrou\(^6\) and W.G. Dixon\(^7\). Studying the effect of spin precession in an external electromagnetic field in the framework of special relativity, V. Bargmann, L. Michel and V.L. Telegdi\(^8\) introduced the term “anomalous magnetic moment of an electron”. In the absence of an anomalous moment, the Bargmann-Michel-Telegdi (BMT) equations coincide with Bloch’s equations\(^9\). In the last three decades, the interest in studying the dynamic equations for particles with intrinsic degrees of freedom, being derived in the framework of Lagrangian or Hamiltonian formalism, has grown considerably\(^10\)-\(^33\). Thereupon, the BMT-model has gained the status of a test model for verification of new models of dynamics of a point particle with spin or polarization in external fields of various kinds. The history of studying the dynamics of spinning particles is presented in the book\(^34\): alternative approaches to derivation of motion equations for charge and spin, the problem of non-collinearity of momentum and velocity four-vectors and the origin and interpretation of the Thomas precession are examined in detail there.

The most important thing while analyzing the dynamic equations for particles with intrinsic structure in the gravitational field is the studying of a spin-curvature (polarization-curvature) interaction. Per se, the Mathisson-Papapetrou-Dixon equations are a non-minimal generalization of the classical dynamic equations because they take into consideration the Riemann curvature tensor and its covariant derivatives. Formally, the introduction of curvature-dependent forces (or “tidal” forces by the conventional terminology) breaks the free fall universality principle and thus contradicts Einstein’s equivalence principle. Thereupon, the spin (polarization)-curvature interaction is now considered to be a theoretical prediction whose experimental verification may have an interesting application to the problem of testing the metric theories of gravity. Following I.B. Khriplovich\(^20\)-\(^26\) and M. Bander and K. Yee\(^24\), using an analogy with the anomalous interaction of a spinning particle and the electromagnetic field, introduced in the BMT theory, it seems reasonable to consider also the anomalous interaction of spin (polarization) with the gravitational field. Such an analysis involves a new parameter \(Q\) as a constant of anomalous spin (polarization)-curvature interaction, by analogy with the phenomenological parameter \(g−2\), introduced in the BMT theory\(^8\) as a constant of anomalous interaction of spinning particle with the electromagnetic field. Under a generalization of such kind, spin (polarization) dynamics is predetermined by a Lorentz-type force, one addend of which is proportional to the Riemann curvature tensor\(^24\).

In this paper we suggest an exactly integrable model of the dynamics of a relativistic particle with an arbitrary directed polarization vector in the gravitational radiation field. The model is based on the known dynamic equations for particles with supplementary degrees of freedom and takes into consideration an anomalous interaction of polarization with curvature. The paper develops our recent works\(^35\)-\(^38\), in which, in particular, the effect of parametric oscillations of Faraday rotation of the polarization vector was predicted. The results ob-
tained in this paper demonstrate a possibility of new type for the precession of the polarization vector, indicated as ‘hyperbolic rotation’. We consider two types of models: first, a model with a deterministic coupling constant of anomalous interaction between the space-time curvature and particle polarization, and, second, the model with a random coupling constant. One of the purposes of this paper is to compare the behaviour of particle in two different cases: first, when the anomalous polarization-curvature interaction is absent (the coupling constant is zero identically), second, when the random coupling constant is zero on the average.

The paper is organized as follows. In Sec. 2, evolution equations for the velocity four-vector and the spin (polarization) four-vector are established and compared with the known dynamic equations for particles with an intrinsic structure. In Sec. 3, exact solutions to the resulting evolution equations are found for the case when a boson travels in a pp-wave gravitational background under the influence of a deterministic force induced by curvature. In Sec. 4, we discuss the specific features of boson motion in the case when the coupling constant of the polarization-curvature interaction is considered as a random variable with zero mean value. The last section contains conclusions.

2 Evolution equations

2.1 General model

It is well-known that, for a point particle with supplementary tensor degrees of freedom, the generalized momentum four-vector \( P^i \) is not parallel to the velocity four-vector \( U^i \equiv \frac{dx^i}{d\tau} \) (see, e.g., [34])

\[
P^i = mcU^i + q^i, \quad q^i U_i = 0. \tag{1}
\]

The part of the momentum \( q^i \), which is orthogonal to the velocity four-vector, may include the angular momentum tensor \( S^{ij} \), the quadrupole momentum tensor \( J^{ijmn} \) as well as the tensors describing higher multipole moments \( \delta^{ijkl} \). \[ 28 \]

\[
q^i \equiv U_j \delta^{ij} - \frac{4}{3} R_{nmij} J^{ijmn} U_j + \ldots. \tag{2}
\]

The dot denotes the derivative \( D/d\tau = U^j \nabla_j \), where \( \nabla_i \) is the covariant derivative. The Riemann curvature tensor \( R_{nmij} \) and its covariant derivatives happen to be included into the formulae for the generalized momentum as necessary structure elements. As usual, the scalar product \( U_i U^i = \text{const} \) is set equal to unity identically,

\[
U_i U^i = 1. \tag{3}
\]

The scalar product \( P^i P_i = m^2 c^2 + q^i q_i \) does not remain constant along the particle word-line, or, in other words, it is not an integral of motion. In our opinion, it is then more convenient to construct the dynamic model for particles with intrinsic structure using the velocity four-vector then. Starting from the relation

\[
U^i U_i - \frac{1}{2} \frac{D}{D\tau} (U^i U_i) = 0, \tag{4}
\]

one can find the generic form of an evolution equation for the velocity four-vector:

\[
\dot{U}^i = f^i, \quad f^i U_i = 0. \tag{5}
\]

Using the identity:

\[
f_i \equiv U_i (f_k U^k) + \Delta^k U_k, \quad \Delta^k \equiv \delta^k - U_k U^k, \tag{6}
\]

and the orthogonality condition \( f_k U^k = 0 \), one can represent the effective force four-vector \( f^i \) in the form

\[
f^i = \Delta^k U_k (x, U, S) \equiv (V_k U_k - U_i V_k) U^k. \tag{7}
\]

Here \( V^k (x, U, S) \) is an arbitrary vector function depending on the coordinates, the velocity four-vector and spin (polarization) variables. Introducing the new antisymmetric tensor \( \Omega_{ik} (x, U, S) \equiv V_k U_i - U_i V_k \), one can represent the dynamic equation \( \dot{U} \) in two equivalent forms

\[
\dot{U}^i = \Delta^k U_k (x, U, S) \leftarrow \rightarrow \dot{U}_i = \Omega_{ik} (x, U, S) U^k. \tag{9}
\]

To characterize the state of a particle which possesses a vector degree of freedom (spin or polarization), one uses the well-known decomposition of the anti-symmetric tensor of total moment

\[
S_{ik} = L_k U_i - \epsilon_{iklm} S^l U^m. \tag{10}
\]

Here \( L^i \) is the four-vector of orbital moment and \( S^i \) is the spin (polarization) four-vector, defined as follows:

\[
L_i = S_{ik} U^k, \quad S_i = \frac{1}{2} \epsilon_{iklm} S^l U^m U^k \equiv S^k U_k. \tag{11}
\]

where \( \epsilon_{iklm} \) is the Levi-Civita tensor

\[
\epsilon_{iklm} = \sqrt{-g} E_{iklm}, \quad E_{0123} = -1, \tag{12}
\]

and the asterisk in \( [11] \) introduces the corresponding duality symbol. The four-vectors \( L_i \) and \( S_i \) are orthogonal to the velocity four-vector automatically by the definition \( [11] \)

\[
L_i U_i = 0, \quad S^k U_k = 0. \tag{13}
\]

In this sense the so-called Tamm condition \( S^k U_k = 0 \) is not an additional requirement. One can find an obvious analogy between the decomposition \( [11] \) and the decomposition of the Maxwell tensor \( F_{ik} \) in electrodynamics \( [39] \)

\[
F_{ik} = E_i U_k - E_k U_i - \epsilon_{iklm} B^l U^m. \tag{14}
\]

Following this analogy, one can say, that \( S^i \) plays the same role as the four-vector of magnetic induction \( B^i \).
(and is in fact a pseudo-vector), while \( L^i \) plays the role of electric field four-vector \( E^i \) (and is a true vector). The four-vector \( S^i \) is considered to be a vector parameter of the intrinsic state of a particle. The scalar square of this four-vector is assumed to be constant along the particle world-line, i.e.,

\[
S^k S_k = -E^2 = \text{const} \neq 0 .
\]

This requirement results in

\[
S^k \dot{S}_k = 0 ,
\]

and, following the previous discussion concerning the \( \dot{U}^i \) decomposition, one can state that the corresponding evolution equation has two equivalent forms

\[
\dot{S}^i = \left( \delta^i_k - \frac{S_i S_k}{S^i S^k} \right) W_k(x, U, S) \leftrightarrow \dot{\hat{S}}_i = \omega_{ik}(x, U, S) S^k ,
\]

where

\[
\omega_{ik} = \frac{W_i S_k - S_i W_k}{S^i S^k} .
\]

From the orthogonality condition \( S_i U^i = 0 \) it follows that

\[
U^i \dot{S}_i + S^i U_i = 0 ,
\]

consequently, the four-vectors \( V_i \) and \( W_i \) as well as the tensors \( \Omega_{ik} \) and \( \omega_{ik} \) are connected by the requirements

\[
U^i W_i + S^i V_i = 0 , \quad (\omega_{ik} - \Omega_{ik}) U^i S^k = 0 .
\]

This means, in particular, that

\[
\omega_{ik} S^k = \Omega_{ik} S^k + \Omega_{ik}(x, U, S) S^k ,
\]

with

\[
\tilde{\Omega}_{ik} = -\tilde{\Omega}_{ki} , \quad \tilde{\Omega}_{ik} U^i S^k = 0 .
\]

Thus, we can represent the evolution equations of a particle with spin or polarization in the following generic form:

\[
\dot{U}_i = \Omega_{ik} U^k , \quad \dot{\hat{S}}_i = \Omega_{ik} S^k + \dot{\hat{\Omega}}_{ik} U^k .
\]

### 2.2 Comparison with well-known models

#### 2.2.1 The Bargmann-Michel-Telegdi model

To recover the BMT results \[9\]

\[
\dot{U}_i = \frac{e}{mc^2} F_{ik} U^k ,
\]

\[
\dot{\hat{S}}_i = \frac{e}{mc^2} \left[ \frac{g}{2} F_{ik} S^k + U_i \left( \frac{g}{2} - 1 \right) F_{kl} S^k U^l \right] ,
\]

where \( g \) is a gyro-magnetic ratio, one has to suppose that \( \Omega_{ik} \) is proportional to the Maxwell tensor \( F_{ik} \):

\[
\Omega_{ik} = \frac{e}{mc^2} F_{ik} ,
\]

and \( \Omega_{ik} \) has to be equal to

\[
\tilde{\Omega}_{ik} = \frac{e}{mc^2} \left( \frac{g}{2} - 1 \right) (U_k F_{il} - U_i F_{kl}) S^l .
\]

#### 2.2.2 The Bander - Yee model

Bander and Yee \[21\] obtained the following spin evolution equation:

\[
\frac{DS_{ab}}{D\tau} + (U_b S_{ac} - U_a S_{bc}) \frac{DU_c}{D\tau} = - \left( \frac{eg}{4mc^2} \epsilon_{abcd} + \frac{\kappa}{8mc} R^{cdef}_{ik} S_f \right) \times (S_{ac} \Delta_{bd} + S_{bd} \Delta_{ac}) .
\]

The dynamic equations derived in \[21\] have terms which are second-order in the spin, terms with derivatives of the Maxwell tensor and a term with the second order derivative of \( U^i \). In the approximation where we neglect all these terms, we have the following equation \[21\]:

\[
\dot{U}_i = \left[ \frac{e}{mc^2} F_{ik} + \frac{1}{2mc} R_{ikab} S^{ab} \right] U^k .
\]

Let us consider a model with vanishing orbital moment \( L_i = 0 \). Then, in terms of the spin (polarization) four-vector \( S^a \), the dynamic equations \[28\] have the form \[23\] if

\[
\Omega_{ik} = \left[ \frac{e}{mc^2} F_{ik} - \frac{1}{mc} R_{ikab} S^a U^b \right] ,
\]

where

\[
R_{ikab} = \frac{1}{2} R_{ikcd} \epsilon_{cdab} .
\]

is the right-dual Riemann tensor. This quantity is a pseudo-tensor, but the product \( R_{ikab} S^a \) happens to be a true tensor. Eqs. \[29\] can be represented in the form \[28\] when

\[
\tilde{\Omega}_{ik} = \left[ U_k \tilde{\Omega}_{il} - U_l \tilde{\Omega}_{ki} \right] S^l ,
\]

with

\[
\tilde{\Omega}_{ik} = \frac{e}{mc^2} F_{ik} \left( \frac{g}{2} - 1 \right) - \frac{\kappa - 1}{mc} R_{ikab} S^a U^b .
\]

### 2.3 Equations of boson dynamics

Let us consider the dynamics of vector bosons possessing zero charge \( e = 0 \) and polarization four-vector \( S^i \). The dynamic equation for such a particle can be immediately obtained from \[28\]:

\[
\dot{U}_i = -\frac{1}{mc^2} R_{iklm}^{*} U^k S^l U^m .
\]

The equation of polarization evolution, obtained from \[28\] with \[29\], \[31\] and \[32\] yields

\[
\dot{S}_i = -\frac{1}{mc^2} \left[ \kappa R_{iklm}^{*} S^k S^l U^m + (\kappa - 1) U_i R_{iklm} S^k U^l S^m U^n \right] .
\]

Eqs. \[33\] and \[34\] can be also obtained from the BMT equations by a formal substitution discussed by Khriplovich in \[30\]:

\[
\frac{e}{c} F_{ik} \rightarrow \frac{1}{2} R_{ikab} S^{ab} = -R_{ikab}^{*} S^a U^b ,
\]
3 Boson dynamics in a pp-wave gravitational field. Deterministic model

3.1 Gravitational wave background

The well-known pp-wave solution of the vacuum Einstein equations can be represented by the metric \[ ds^2 = 2 du dv - L^2 \left[ e^{2\beta (dx^2)^2} + e^{-2\beta (dx^3)^2} \right] \cosh 2\gamma + 2 \sinh 2\gamma dx^2 dx^3, \]

where \( u = \frac{1}{\sqrt{2}}(ct - x^1), \quad v = \frac{1}{\sqrt{2}}(ct + x^1) \)

are the retarded and advanced time, respectively. The functions \( \beta(u) \) and \( \gamma(u) \) are considered to be arbitrary functions of the retarded time, satisfying the following conditions on the wave front plane \( u = 0 \):

\[ \beta(0) = \gamma(0) = 0, \quad \beta'(0) = \gamma'(0) = 0. \]

The background factor \( L(u) \) satisfies the equation \[ L'' + L \left( \beta'^2 \cosh^2 2\gamma + \gamma'^2 \right) = 0 \]

and the initial data \[ L(0) = 1, \quad L'(0) = 0. \]

The gravitational wave (GW) is indicated as a GW with the first polarization when \( \beta(u) \neq 0, \gamma(u) = 0 \), and as a GW with the second polarization when \( \beta(u) = 0, \gamma(u) \neq 0 \). For the pp-wave solution \([34]\), the right-dual Riemann tensors \( R_{ijkl}^{*} \) and the left-dual one \( R^{*}_{ijkl} \equiv \frac{1}{2} \epsilon_{jkl}^{pq} R_{pqim} \) coincide and have the following nonvanishing components:

\[ R^{*}_{2u2u} = - \left[ L^2 e^{2\beta} \left( \gamma' - \frac{1}{2} \beta' \sinh 4\gamma \right) \right], \]
\[ R^{*}_{3u3u} = \left[ L^2 e^{-2\beta} \left( \gamma' + \frac{1}{2} \beta' \sinh 4\gamma \right) \right], \]
\[ R^{*}_{2u3u} = R^{*}_{1u2u} = \left[ L^2 \beta' \cosh^2 2\gamma \right]. \]

Here and in what follows, Latin indices run over four values: \( u, v, 2, 3 \), while Greek indices run over two values: \( 2, 3 \).

3.2 Exact solutions to the evolution equations

3.2.1 Key subsystem of the evolution equations

The GW metric \([37]\) admits two spacelike Killing vectors and one null covariantly constant Killing vector \([31]\)

\[ \xi^i = \delta^i_2, \quad \xi^i = \delta^i_3, \quad \xi^i = \delta^i_v, \]

\[ g_{ik} \xi^i \xi^k = 0, \quad \nabla_k \xi^i = 0. \]

The projection of the dynamic equation \([33]\) onto the direction given by \( \xi^i \) yields

\[ \frac{dU_v}{dt} = 0 \rightarrow U_v = \text{const} = C_v. \]

It follows from \([45]\) that the parameter \( \tau \) is linearly connected with the \( u \) coordinate since

\[ \frac{du}{d\tau} \equiv U_u = \frac{mc}{C_v} U_v = C_v. \]

When \( C_v \neq 0 \) (this condition does not hold only for a massless particle co-moving the GW), one obtains

\[ \tau = \tau_0 + \frac{u}{C_v}. \]

Since \( U_i U_i = 1 \), the \( U_u \) component of the velocity four-vector can be found immediately:

\[ U_u = \frac{1}{2C_v} \left[ 1 - g^{\alpha\sigma} U_{\alpha} U_{\sigma} \right]. \]

Similarly, from \( S_i U_i = 0 \), one obtains an expression for the \( S_u \) component of the polarization four-vector:

\[ S_u = -\frac{1}{C_v} \left[ S_v U_u + g^{\alpha\sigma} U_{\alpha} S_{\sigma} \right]. \]

Thus the \( U_u \) and \( S_u \) quantities are found as soon as \( U_2, U_3, S_2, S_3 \) and \( S_v \) become known. For the remaining five components of the velocity and polarization four-vectors, one has five equations:

\[ \frac{dU_\alpha}{du} = -\frac{1}{mc} R^{*}_{\alpha u\sigma} (C_v S^\sigma - S_v U^\sigma), \]
\[ C_v \frac{dS_\alpha}{du} = (C_v S^\sigma - S_v U^\sigma) \]
\[ \times \left[ \frac{g_{\alpha\sigma}}{2} - \frac{Q + 1}{mc} S_v R^{*}_{\alpha u\sigma} \right] - \frac{Q}{mc} U_\alpha R^{*}_{\alpha u\sigma} (C_v S^\sigma - S_v U^\sigma), \]
\[ \frac{dS_v}{du} = -\frac{Q}{mc} R^{*}_{\alpha u\sigma} (C_v S^\alpha - S_v U^\alpha) \]
\[ \times (C_v S^\sigma - S_v U^\sigma) \]

with the following initial conditions:

\[ U_{\alpha}(0) = C_{\alpha}, \quad S_{\alpha}(0) = E_{\alpha}, \quad S_v(0) = E_v. \]
After introducing the new variables

\[ X_\alpha \equiv C_v S_\alpha - S_v U_\alpha \]  

(52)
a closed subset of three equations emerges from the set (51)

\[
\frac{dX_\alpha}{du} = -\frac{1}{mc} R^*_\alpha \sigma u u(u) X_\sigma, \tag{53}
\]

\[
\frac{dS_\alpha}{du} = \frac{Q}{mc} R^*_\alpha \sigma u u(u) X_\sigma, \tag{54}
\]

where

\[ a'_\alpha(u) \equiv \frac{1}{\sqrt{2}} g^\sigma (u) g^\rho (u). \tag{55} \]

The prime denotes a derivative with respect to the retarded time \( u \). For the transversal components of the velocity four-vector \( U_\alpha \) we obtain two decoupled equations

\[
\frac{dU_\alpha}{du} = -\frac{1}{mc} R^*_\alpha \sigma u u(u) X_\sigma, \tag{56}
\]

which give the following formal solutions for \( U_\alpha \):

\[
U_\alpha(u) = C_\alpha - \frac{1}{mc} \int_0^u R^*_\alpha \sigma u u(\tilde{u}) X_\sigma(\tilde{u}) \, d\tilde{u}. \tag{57}
\]

The unknowns \( S_\alpha(u) \) can be found immediately from the relation (52):

\[
S_\alpha(u) = \frac{1}{C_v} [X_\alpha(u) + S_v U_\alpha(u)]. \tag{58}
\]

Thus all the solutions of Eqs. (51) will be represented by quadratures as soon as solutions to the system (53) are known. We will designate the latter as the key subsystem.

### 3.3 Exact solutions to the key subsystem for the case \( Q \equiv 0 \)

One of the purposes of the paper is to compare the behaviour of a vector boson in the GW background in two cases. The first one takes place when the coupling constant \( Q \) is zero identically, i.e., when there is no anomalous interaction of polarization with curvature. The second case is connected with the suggestion that \( Q \) effectively depends on the environment of the boson and is a random variable with zero mean value. Thus we first consider the evolution equations with \( Q \equiv 0 \). From (58) it directly follows that

\[ S_v(u) = \text{const} = S_v(0) \equiv E_v, \tag{59} \]

and Eqs. (53) result in

\[ X'_\alpha = a'_\alpha(u) X_\sigma. \tag{60} \]

Such a system have been solved in [10] and discussed in [35-38,41,42], and here we use the result:

\[
\begin{pmatrix} X_2(u) \\ X_3(u) \end{pmatrix} = L \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix} \begin{pmatrix} \cosh \gamma & \sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} \cos \psi - \sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} C_v E_2 - E_v C_2 \\ C_v E_3 - E_v C_3 \end{pmatrix}, \tag{61}
\]

where

\[
\psi(u) = \int_0^u \beta'(\tilde{u}) \sinh 2\gamma(\tilde{u}) \, d\tilde{u}. \tag{62}
\]

The precession of the polarization four-vector, described by Eqs. (61), was discussed in detail in Ref. [37].

### 3.4 Exact solutions to the key subsystem for the case \( Q \neq 0 \)

To gain analytic progress in the case of an anomalous interaction of polarization with curvature, let us consider the GW to have the first polarization (\( \gamma = 0 \)). The right-dual curvature tensor \( R^*_{\alpha \beta \gamma \delta} \) has only two nonzero components:

\[ R^*_{2u\alpha u} = R^*_{3u\alpha u} = (L^2 \beta')', \tag{63} \]

and thus the equations for \( X_\alpha \) yield:

\[
X_2' = \left( \frac{L'}{L} + \beta' \right) X_2 - \frac{Q}{mc} S_v e^{2\beta} (L^2 \beta')' \tag{64}
\]

\[
X_3' = \left( \frac{L'}{L} - \beta' \right) X_3 - \frac{Q}{mc} S_v e^{-2\beta} (L^2 \beta')', \tag{65}
\]

The substitution

\[ X_2 = L e^\beta Y_2, \quad X_3 = L e^{-\beta} Y_3 \tag{66} \]

results in the following equations for the new variables \( Y_2 \) and \( Y_3 \):

\[
Y_2' = \Omega(u) Y_3, \quad Y_3' = \Omega(u) Y_2, \tag{67} \]

\[
\Omega(u) = -\frac{Q}{mc} S_v(u) \frac{[L^2(u) \beta'(u)]'}{L^2(u)}. \tag{68} \]

Solutions to this system can be represented in the elementary functions:

\[
Y_2(u) = Y_2(0) \cosh \Psi(u) + Y_3(0) \sinh \Psi(u), \tag{69} \]

\[
Y_3(u) = Y_3(0) \cosh \Psi(u) + Y_2(0) \sinh \Psi(u), \tag{69}
\]

where

\[ \Psi \equiv \int_0^u \Omega(\tilde{u}) \tilde{u}, \quad \Psi(0) \equiv 0, \tag{70} \]

As for the unknown \( S_v(u) \), using the expressions for quadratic integrals of motion \( U^iU_i=1 \), \( S^iU_i=0 \), \( S^iS_i=-E^2_0 \) as well as the definition of \( X_s(u) \) \( \text{[12]} \), and the expressions for \( Y_s(u) \) \( \text{[34]} \), one can show that the following relations take place:

\[
S^2_v(u) = C^2_v E^2_0 - \left[ Y^2_v(0) + Y^2_3(0) \right] \cosh 2\Psi(u)
- 2Y_2(0)Y_3(0) \sinh 2\Psi(u) \equiv B^2(\Psi),
\]

\[
E^2_v = C^2_v E^2_0 - Y^2_v(0) - Y^2_3(0).
\]

The relations \( \text{[68], [70], [72]} \) make it possible to find \( \Psi(u) \) by quadratures:

\[
\pm \int_0^u \frac{d\Psi}{\mathcal{E}(\Psi)} = -\frac{Q}{mc} r^{-1}(u),
\]

where

\[
r^{-1}(u) \equiv \int_0^u R_{u3u}(\tilde{u}) d\tilde{u} = \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} \right\} (\tilde{u}) d\tilde{u},
\]

The function \( r(u) \) has the dimension of distance; it may be considered as some effective curvature radius.

Generally, the solution for \( \Psi(u) \) can be represented in terms of elliptic functions. Below we consider some approximations to such solutions. When the function \( \Psi(u) \) is known, the transversal components of the velocity four-vector and the polarization four-vector take the form

\[
U_2(u) = C_2 + \frac{Y_3(0)}{mc} \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} e^{\beta(u)} \sinh \Psi \right\} (\tilde{u}) \tilde{u} + \frac{Y_2(0)}{mc} \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} e^{\beta(u)} \cosh \Psi \right\} (\tilde{u}) \tilde{u} + \frac{Y_3(0)}{mc} \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} e^{-\beta(u)} \sinh \Psi \right\} (\tilde{u}) \tilde{u},
\]

\[
U_3(u) = C_3 + \frac{Y_3(0)}{mc} \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} e^{-\beta(u)} \cosh \Psi \right\} (\tilde{u}) \tilde{u} + \frac{Y_2(0)}{mc} \int_0^u \left\{ \frac{(L^2\beta'^2)}{L^2} e^{-\beta(u)} \sinh \Psi \right\} (\tilde{u}) \tilde{u},
\]

\[
S_2(u) = \frac{Y_2(0)}{C_v} L(u) e^{\beta(u)} \cosh \Psi(u)
+ \frac{Y_2(0)}{C_v} L(u) e^{-\beta(u)} \sinh \Psi(u) + \frac{S_v(u)}{C_v} U_2(u),
\]

\[
S_3(u) = \frac{Y_3(0)}{C_v} L(u) e^{-\beta(u)} \cosh \Psi(u)
+ \frac{Y_2(0)}{C_v} L(u) e^{-\beta(u)} \sinh \Psi(u) + \frac{S_v(u)}{C_v} U_3(u).
\]

Eqs. \( \text{[68]-[70]} \) display a new type of precession of the polarization vector, which can be designated as \textit{hyperbolic rotation} of polarization under the influence of an anomalous curvature force. The above quantities depend non-linearly on the \( Q \) parameter via the function \( \Psi(u) \) given by Eqs. \( \text{[71], [72]} \). When \( Q \equiv 0 \), \( \Psi(u) = 0 \), and the integrals in \( \text{[66] and [67]} \) take an explicit form:

\[
U_2|_{Q=0}(u) = C_2 + \frac{1}{mc} \left( C_v E_3 - E_v C_3 \right) \left[ L(u) e^{\beta(u)} \right],
\]

\[
U_3|_{Q=0}(u) = C_3 - \frac{1}{mc} \left( C_v E_2 - E_v C_2 \right) \left[ L(u) e^{-\beta(u)} \right],
\]

\[
S_2|_{Q=0}(u) = E_v \frac{C_2}{C_v} + \left( E_2 - E_v \frac{C_2}{C_v} \right) \left[ L(u) e^{\beta(u)} \right],
\]

\[
S_3|_{Q=0}(u) = E_v \frac{C_3}{C_v} + \left( E_3 - E_v \frac{C_3}{C_v} \right) \left[ L(u) e^{-\beta(u)} \right].
\]

Thus Eqs. \( \text{[66], [68], [69], [70], [71]-[74]} \) describe the evolution of a relativistic polarized boson in a pp-wave gravitational background in the framework of a model with the deterministic parameter \( Q \) (coupling constant of the polarization-curvature interaction). Eqs. \( \text{[80]-[83]} \), describing the particular case \( Q = 0 \), will be used directly in the next section.

4 Random polarization-curvature interaction

We suppose that the anomalous interaction of polarization with curvature has a random nature and can be attributed to the particle interaction with its environment mediated by the space-time curvature. Mathematically, this means that the parameter \( Q \) can be treated as a random variable. Basing on such an ansatz, one can calculate the mean values of the velocity and polarization four-vectors and compare the results with those for \( Q \equiv 0 \), given by Eqs. \( \text{[80]-[83]} \). The distribution function for the \( Q \) variable is supposed to be Gaussian:

\[
f(Q) = \frac{1}{D \sqrt{\pi}} \exp \left( -\frac{Q^2}{D^2} \right),
\]

where \( D \) is a dispersion parameter.

4.1 First special solution: \( Y_2(0) = Y_3(0) \equiv 0 \)

To obtain a starting exact result, let us suppose that both initial values \( Y_2(0) \) and \( Y_3(0) \) are equal to zero. In this case it follows from \( \text{[72]} \) that \( S_v(u) = E_v = \mathcal{E}(\Psi) \), and \( \Psi(u) \) can be found explicitly:

\[
\Psi(u) = -\frac{Q}{mc} \frac{E_v}{r^{-1}(u)}.
\]
On the other hand, the $\Psi(u)$ function happens to be a hidden one, since in this degenerate case $U_\alpha$, $S_\nu$ and $S_\alpha$ remain constant:

$$X_\alpha(u) = 0, \quad U_\alpha(u) = C_\alpha,$$

$$S_\alpha(u) = \frac{E_e}{C_v}C_\alpha, \quad S_\nu(u) = E_e = C_vE_0.$$  \hfill (86)

The quantities $S_u(u)$ and $U_u(u)$

$$S_u(u) = -\frac{E_e}{2C_v} [1 + g^{\alpha\beta}(u)C_\alpha C_\beta],$$

$$U_u(u) = \frac{1}{2C_v} [1 - g^{\alpha\beta}(u)C_\alpha C_\beta]$$  \hfill (87)

do not contain $\Psi(u)$ and consequently do not depend on $Q$. This special type of particle motion is characterized by the following feature: the projection of the polarization four-vector onto the GW front plane is parallel to the projection of the velocity four-vector not only at the initial moment, but during the whole period of evolution, too.

### 4.2 Second special solution: $E_e = 0$,

$Y_2(0) \cdot Y_3(0) = 0$

According to (88), Eqs. (61), (62) describe an exact solution to the evolution equations also in the case when $Q \neq 0$, but $S_u(u) \equiv 0$. It is possible, when, first, $S_u(0) = E_u = 0$, second, $X_2(0) \cdot X_3(0) = C_v^2E_2E_3 = 0$, as can be seen from (61) and (88). In particular, it is possible, when the initial polarization four-vector $S^\nu(0)$ has a vanishing projection onto the null Killing vector $\xi^i$ as well as onto one of the spacelike Killing vectors $\xi^i$ or $\xi^j$. The polarized boson with such characteristics, traveling in the GW background, does not feel the influence of the curvature-induced force. Both in first and the second special cases, exact results of averaging over $Q$ coincide with those for $Q \equiv 0$.

### 4.3 Third special solution: $Y_2(0) = -Y_3(0) \neq 0$

Eqs. (89) show that if $Y_2(0) = -Y_3(0) \equiv Y(0)$, then at an arbitrary retarded time instant

$$Y_2(u) = -Y_3(u) = Y(0) \cdot e^{-\Psi(u)}.$$  \hfill (88)

Then Eq. (72) takes the form

$$\int_0^{\Psi(u)} \frac{d\tilde{\Psi}}{\sqrt{C_v^2E_0^2 - 2Y^2(0)e^{-2\Psi}}} = -\frac{Q}{mc} r^{-1}(u).$$  \hfill (89)

Integration in (89) yields

$$e^{-\Psi(u)} = \frac{C_vE_0}{\sqrt{2Y(0)}} \times \cosh \left[ \arccosh \frac{C_vE_0}{\sqrt{2Y(0)}} \right] \frac{QC_vE_0}{mc} r^{-1}(u).$$  \hfill (90)

At the initial instant $u = 0$, when $r^{-1}(0) = 0$, one obtains $\Psi(0) = 0$. Then at the instant $u^*$, when

$$\frac{C_vE_0}{\sqrt{2Y(0)}} = \cosh \left[ \frac{QC_vE_0}{mc} r^{-1}(u^*) \right],$$  \hfill (91)

the function $e^{-\Psi}$ obviously reaches its maximum value

$$e^{-\Psi(u^*)} = \frac{C_vE_0}{\sqrt{2Y(0)}}.$$  \hfill (92)

The instant $u^*$ is a stopping point for the hyperbolic rotation since the quantity $S_u(u^*)$ vanishes (see, (73)), and we obtain $\Omega(u^*) = 0$ and $\frac{d\Omega}{du}|_{u=u^*} = 0$. Using (89) in Eqs. (73), (74), we obtain an example of an explicit representation of an exact solution by elementary functions.

#### 4.4 Fourth special solution: $Y_2(0) = Y_3(0) \neq 0$

Exact results for this case can be obtained from the results of the Sec. 4.3, by the formal substitution $\Psi \rightarrow -\Psi$.

#### 4.5 Statistical averaging in the general case

Eqs. (72) and (74) show that, generally, the dependence of $\Psi$ on the parameter $Q$ can be represented in terms of elliptic function. For special initial data (see Sec. 4.1-4.4) one can obtain the results in elementary functions. Taking into account the relation (72), one can state, that the growth of $\Psi(u)$ is restricted by the requirement $S^2_u \geq 0$. When $S^2_u = 0$, the function $\Psi$ reaches its extreme values

$$\exp\{2\Psi_{1,2}\} = \frac{C_v^2E_0^2 + \sqrt{C_v^4E_0^4 - (Y_2^2(0) - Y_3^2(0))^2}}{(Y_2(0) + Y_3(0))^2}.$$  \hfill (93)

Both extreme values $\Psi_1$ and $\Psi_2$ correspond to the relations

$$\Omega(u^*) = 0, \quad \frac{d\Psi}{du}|_{u=u^*} = 0.$$  \hfill (94)

When $Y_2(0) \rightarrow -Y_3(0)$, the expression (94) gives (74) if we choose the minus sign in the numerator; the plus sign gives the solution for the case $Y_2(0) \rightarrow Y_3(0)$. Thus the function $\Psi(u)$ remains finite. To perform averaging over $Q$, we consider the leading-order terms in the expressions for $U_\alpha$ and $S_\alpha$. The dimensionless parameter $(E_e/mc)r^{-1}(u)$ is considered to be small. Since $E_e$ in quasiclassical theory is supposed to be proportional to the Planck constant, the inequality $(E_e/mc)r^{-1}(u) << 1$ does not require, in general, that the GW is weak. When $Y_2(0)$ and $Y_3(0)$ are not zero simultaneously, the leading-order term in the expansion of the function $\Psi(u)$ has exactly the same form as in (80). Averaging with the Gaussian function (84) yields

$$\langle U_2\rangle_Q - U_{2,Q=0}$$

$$= D^2 \int_0^u \left\{ \frac{e^{\beta (L_2^2 \beta')^2}}{L} r^{-2} \right\} (\tilde{u})d\tilde{u},$$  \hfill (95)
\[ \langle U_3 \rangle_Q - U_{3, Q=0} = D^2 \frac{E_3^2 Y_2(0)}{4m^3c^3} \int_{0}^{u} \left\{ e^{-\beta \left( L^2 \beta^2 \right)^{y} r^{-2}} \right\} (\hat{u}) d\hat{u}, \quad (96) \]

\[
(S_v)_{Q} - S_{v, Q=0} = -\frac{D^2}{2E_{v}m^3c^2} \cdot r^{-2}(u) \times \{2Y_2(0)Y_3^2(0) + E_3^2[Y_2^2(0) + Y_3^2(0)]\}, \quad (97)
\]

\[
(S_2)_{Q} - S_{2, Q=0} = \frac{D^2 Y_2(0) E_3}{4m^3c^2 C_v} \int_{0}^{u} \left\{ \left( L^2 \beta^2 \right)^{y} r^{-2} \right\} (\hat{u}) d\hat{u}
\]

\[
\frac{D^2 Y_2(0) Y_3(0) E_v}{m^3c^2 C_v} r^{-1}(u) \int_{0}^{u} \left\{ \left( L^2 \beta^2 \right)^{y} r^{-1} \right\} (\hat{u}) d\hat{u}
\]

\[
\frac{D^2 [2Y_2^2(0) Y_3^2(0) + E_3^2[Y_2^2(0) + Y_3^2(0)]]}{2m^2c^2 E_c C_v}
\]

\[
\times \left[ C_2 + \frac{Y_3(0)}{mc} (L^2 \beta^2)^{y}(u) \right] r^{-2}(u), \quad (98)
\]

\[
(S_3)_{Q} - S_{3, Q=0} = \frac{D^2 Y_3(0) E_3}{4m^2c^2 C_v} \cdot L^{-\beta} r^{-2}(u)
\]

\[
+ \frac{D^2 Y_2(0) E_3}{4m^3c^2 C_v} \int_{0}^{u} \left\{ \left( L^2 \beta^2 \right)^{y} r^{-2} \right\} (\hat{u}) d\hat{u}
\]

\[
\frac{D^2 Y_2(0) Y_2^3(0) E_v}{m^3c^2 C_v} r^{-1}(u) \int_{0}^{u} \left\{ \left( L^2 \beta^2 \right)^{y} r^{-1} \right\} (\hat{u}) d\hat{u}
\]

\[
- \frac{D^2 [2Y_2^2(0) Y_3^2(0) + E_3^2[Y_2^2(0) + Y_3^2(0)]]}{2m^2c^2 E_c C_v}
\]

\[
\times \left[ C_3 - \frac{Y_2(0)}{mc} (L^2 \beta^2)^{y}(u) r^{-2}(u) \right]. \quad (99)
\]

We have used the notation

\[ \langle A \rangle_Q \equiv \int_{-\infty}^{\infty} dQ f(Q) A(Q) . \quad (100) \]

Thus the effect of random polarization-curvature force can lead to essential changes in the particle state when the dispersion parameter \( D \) is large enough.

5 Conclusions

1. We have formulated the master equations governing the dynamics of a massive relativistic boson with an arbitrary directed polarization four-vector, traveling in the gravity field under the influence of an anomalous polarization-curvature interaction (see Eqs. (95) and (98)). The master equations constitute an analogue of the Bargmann-Michel-Telegdi model [5], describing the evolution of an electron with an anomalous magnetic moment.

They use the direct analogy between the electromagnetic and curvature interactions proposed by Khriplovich [30], and represent a submodel of the Bander and Yee general model [24].

2. The master equations have been solved exactly for the model of boson dynamics in a pp-wave gravitational background. We have distinguished three exactly solvable submodels. The first submodel describes the deterministic anomalous polarization-curvature interaction (see, (45), (48), (49), (74), (76)-(79)). The second one represents its particular case with a vanishing coupling constant of the anomalous polarization-curvature interaction (see, [30]-[33]). The third submodel introduces a random anomalous polarization-curvature interaction.

3. The exact solutions of the master equations obtained here demonstrate a specific type of the polarization four-vector behaviour, designated as “hyperbolic rotation” induced by anomalous interaction of polarization with the space-time curvature. “Hyperbolic rotation” of the polarization four-vector affects the boson dynamics; the velocity four-vector depends non-linearly on the initial data describing the direction of the polarization four-vector.

4. The master equations happen to be nonlinear with respect to the velocity four-vector \( U^i \) as well as the polarization four-vector \( S^i \). This feature makes essentially different the two models: the first one, where the deterministic anomalous polarization-curvature interaction vanishes, and the second one, where the random coupling constant of the anomalous interaction vanishes on the average. The corresponding distinction is demonstrated by Eqs. (95)-(99).

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