Ternary Social Networks: Dynamic Balance and Self-Organized Criticality

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Antal et al. [Phys. Rev. E 72, 036121 (2005)] have studied the balance dynamics on the social networks. In this paper, based on the model proposed by Antal et al., we improve it and generalize the binary social networks to the ternary social networks. When the social networks get dynamically balanced, we obtain the distributions of each relation and the time needed for dynamic balance. Besides, we study the self-organized criticality on the ternary social networks based on our model. For the ternary social networks evolving to the sensitive state, any small disturbance may result in an avalanche. The occurrence of the avalanche satisfies the power-law form both spatially and temporally. Numerical results verify our theoretical expectations.

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I. INTRODUCTION

Antal et al. [1] have studied the balance dynamics on the social networks based on the notion of social balance [2,3]. In the networks, each node is connected to all the others, representing each person knows all the others in the society. Each edge in the networks has two values, +1 and -1. If the edge is +1, it means the two persons are friendly with each other. If the edge is -1, it means the two persons are hostile towards each other. At every step, they choose a triangle randomly from the network. If the product of the three edges is +1, the triangle is stable. Otherwise, if the product is -1, the triangle is unstable. For the stable triangle, it satisfies (i) the friend of my friend being my friend; (ii) the enemy of my friend being my enemy; (iii) the friend of my enemy being my enemy; and (iv) the enemy of my enemy being my friend. The unstable triangles always try to be stable, but the final state of the network depends on the edge flipping probability $p$, which is set manually. If $p \geq \frac{1}{2}$, the network will reach the state of "paradise", with all relations being friendly. Several studies around the balance dynamics have been carried out, including the studies of the university class of triad dynamics [1] and the satisfiability problem of computer science [2], etc.

In the first half of this paper, at first based on Ref. [1], we map the edge relations to the node relations with some relaxation, so there are two opposing opinions in the network. We suppose for each triad relation in the social networks, it will change from stable to unstable or the reverse, caused by the change of some persons’ opinions in it. While for the change, the person’s opinion depending both on the other opinions in the triad relation and the opinions all around him. Next, considering the social networks, there being two opposing opinions is extreme, we introduce the neutral opinion. So the values of each node are generalized to +1, 0 and -1, and the metastable triad relations appear. We find the densities of each triad relation associated with the Hamiltonian and the geometrical temperature which is determined by the structure of the network without manual setting. Based on our model, we obtain the distributions of each triad relation and the time needed for dynamic balance both for the binary and ternary cases.

In the latter half of this paper, we study the self-organized criticality (SOC) [4,5] on the ternary social networks. The previous studies of the SOC on complex networks are mainly based on the BTW model [6,9–14]. In this paper, we establish our model with some differences. As we know, in the society, if one relation changes from one state into another, it may affect other relations associating with it. The effect may be big or small, depending on many internal factors. So in this paper, we simplify this phenomenon, with an eye to the triad balance dynamics, and make two modifications compared to Sec. II. We find under specific conditions the SOC to be observable. Besides, for the small-world network [15,16], we find the way of construction have an influence on the occurrence of the avalanche. We analyze it theoretically. Numerical results verify our theoretical expectations.

This paper is organized as follows. In Sec. III we map the edge relations to the node relations and generalize the binary networks to the ternary ones. For both the binary and the ternary networks, we obtain the distributions of each triad relation and the time needed for dynamic balance. In Sec. IV we study the SOC on the regular network and the small-world network respectively, and find out the small-world effect on the occurrence of the avalanche. We end this paper with conclusions in Sec. IV.
II. DYNAMIC BALANCE

A. The Node Model

First we define the node relations in analogy with Ref. [1] with some relaxation. We propose each node in the network has two values, +1 and -1, standing for each person’s two opposite standpoints in the society, or two opposing opinions. For a triad relation, if all nodes have the same value, i.e., all persons have the same opinion, the relation is stable. If there are different opinions in a triad relation, conflicts are easily to occur, so the relation is unstable. The stable and unstable triad relations are given in Fig. 1. In our model, the stable and unstable triad relations have the probabilities to change into the other, which is more realistic for the social networks. As we know in the society, stable relations may become unstable because of conflicts and the surroundings, and unstable relations may become stable because of the same interest and the surroundings. Considering each node has two values which have the probabilities to change into the other, we study the social networks in association with the spin systems. In the following context, we establish our model in association with the Ising model and the generalized Glauber dynamics [17, 18].

When associated with the spin systems, each node in the network standing for a spin. We propose the Hamiltonian of each triangle to be

$$H_\Delta = -\alpha (\sigma_i \sigma_j + \sigma_j \sigma_k + \sigma_i \sigma_k),$$

where \(\sigma_i, \sigma_j\) and \(\sigma_k\) stand for the three spins of the triangle. In the binary case, \(\sigma_i, \sigma_j\) and \(\sigma_k\) can take either of the two values, +1 and -1. Besides, \(\alpha > 0\) is a real parameter, standing for the strength of coupling. So the Hamiltonian is the form of the Ising model. The evolving rules are as follows. At every step we choose a triangle from the network randomly, and choose one of the three nodes from the selected triangle at random, then let the selected node do spin flipping. The spin flipping probability is defined as

$$p(\sigma_i \rightarrow \sigma'_i) = \frac{e^{-\beta H'_\Delta}}{Z},$$

where \(H'_\Delta = -\alpha (\sigma'_i \sigma_j + \sigma_j \sigma_k + \sigma'_i \sigma_k)\) stands for the final Hamiltonian after the spin flipping, with \(\sigma'_i\) the final value of the selected spin, and \(\sigma_j, \sigma_k\) the values of the other spins unchanged. It is noted that the spin flipping probability depends on the final value, independent of the initial one [18]. In addition, \(Z = \sum_{\sigma'_i = +1, -1} e^{-\beta H'_\Delta}\) is the normalizing factor and \(\beta\) is a geometrical temperature determined by the structure of the network. Since \(\beta\) is not a real temperature, and in order to embody the majority principle, we propose

$$\beta = 1/|\sum_{j \neq i} \sigma_j/(N-1) - \sigma'_i| \approx 1/|M - \sigma'_i|,$$

where \(N\) is the size of the network, and \(M = \sum_{i} \langle \sigma_i(t) \rangle / N\) is the average value of all spins. So the spin flipping probability depends both on the triad relation and the surroundings [18, 19].

The numbers of positive and negative nodes in the network are as follows,

$$N^+ = \frac{C_N^3}{C_N^2} (3a_1 + 2a_2 + a_3) = \frac{N}{3} (3a_1 + 2a_2 + a_3),$$

$$N^- = \frac{C_N^3}{C_N^2} (a_2 + 2a_3 + 3a_4) = \frac{N}{3} (a_2 + 2a_3 + 3a_4)$$

where \(N\) is the number of total nodes and \(a_i\) stands for the triangle density, satisfying \(\sum_{i=1}^{3} a_i = 1\). The densities of each triangle attached to a positive node are

$$n^+_{a_i} = \frac{N}{3} C_i a_i = \frac{C_i a_i}{3a_1 + 2a_2 + a_3},$$

where \(i = 1, 2, 3\), and \(C_i\) is the number of positive nodes in \(a_i\). In a similar way, the densities of each triangle attached to a negative node are

$$n^-_{a_j} = \frac{N}{a_2 + 2a_3 + 3a_4},$$

where \(j = 2, 3, 4\), with \(D_j\) the number of negative nodes in \(a_j\).

For each node, the spin flipping probabilities from one value to the other are given by Eq. (2), i.e.,

$$p(\sigma_i \rightarrow \sigma'_i) = \frac{e^{-\beta H'_\Delta}}{Z},$$

So the total flipping probabilities for each node from one value to the other are as follows,
\[ p(1 \rightarrow -1) = \sum_{i=1}^{4} \frac{C_i a_i}{3} p(1 \rightarrow -1) = a_1 \frac{e^{-\alpha/(1+M)}}{e^{-\alpha/(1+M)} + e^{3\alpha/(1-M)}} + \frac{2a_2}{3} \frac{e^{-\alpha/(1+M)}}{e^{-\alpha/(1+M)} + e^{3\alpha/(1-M)}} + \frac{a_3}{3} \frac{e^{3\alpha/(1+M)}}{e^{3\alpha/(1+M)} + e^{-\alpha/(1-M)}}, \]  

\[ P(-1 \rightarrow 1) = \sum_{i=1}^{4} \frac{D_i a_i}{3} p(-1 \rightarrow 1) = a_2 \frac{e^{3\alpha/(1-M)}}{e^{3\alpha/(1-M)} + e^{-\alpha/(1+M)}} + \frac{2a_3}{3} \frac{e^{-\alpha/(1-M)}}{e^{-\alpha/(1-M)} + e^{3\alpha/(1-M)}} + \frac{a_4}{3} \frac{e^{-\alpha/(1-M)}}{e^{-\alpha/(1-M)} + e^{3\alpha/(1+M)}}, \]

where \( a_1 \) to \( a_4 \) stand for each triangle density, and \( C_i \) and \( D_i \) the number of positive and negative nodes in \( a_i \) respectively. When the network gets dynamically balanced, the densities of each triad relation are as follows,

\[ a_1 = \frac{1}{8}, \quad a_2 = \frac{3}{8}, \quad a_3 = \frac{3}{8}, \quad a_4 = \frac{1}{8}. \]  

We note the network reaches an explicit state, independent of the parameter \( \alpha \). And in the dynamically balanced state \( n^+ = n^- \), where \( n^+ \) and \( n^- \) stand for the densities of positive and negative nodes respectively.

The time needed for dynamic balance is

\[ T_N \sim N^{C(\alpha)}, \]  

where \( N \) is the size of the network, and \( C(\alpha) > 0 \), being a function of \( \alpha \). The numerical results are given in Fig. 2. More details are given in Appendix A.1.

**B. The Generalized Node Model**

There being only two opposing opinions in the social networks is extreme. We propose the neutral opinion should exist, which means the person holding this opinion is indifferent to other opinions. So we propose each node should have three values, \(+1\), \(0\) and \(-1\), standing for the three opinions. If there are the same opinion in the triad relation except the neutral opinion, the relation is stable. If there are different opinions in the triad relation except the neutral opinion, the relation is unstable. Otherwise the relation is metastable. The simplest way is by judging the Hamiltonian (see Eq. \( (1) \)). If \( H_\Delta < 0 \), it is stable. If \( H_\Delta > 0 \), it is unstable. If \( H_\Delta = 0 \), it is metastable. There are four stable relations, three unstable relations, and three metastable relations in the network, as shown by Fig. 1 and Fig. 3.

Compared with Eq. \( (4) \), the numbers of positive, neu-

![Fig. 2: A plot of \( T_N \) with \( N \) in the log-log coordinate. \( M(0) = -1 \), \( N \) is the size of the network, and \( T_N \) is the time needed for dynamic balance. The numerical results are obtained by taking the average of \( 5 \times 10^5 \) simulations on the networks with size from \( 2 \times 10^3 \) to \( 2 \times 10^4 \).](image)

![Fig. 3: \( a_5 \) and \( a_7 \) represent the stable relations, \( a_6 \) the unstable relation, while \( a_8 - a_{10} \) the metastable relations.](image)
tral and negative nodes in the network are as follows,

\[ N^+ = \frac{N}{3} (3a_1 + 2a_2 + a_3 + 2a_5 + a_6 + a_8), \]
\[ N^0 = \frac{N}{3} (a_5 + a_6 + a_7 + 2a_8 + 2a_9 + 3a_{10}), \]
\[ N^- = \frac{N}{3} (a_2 + 2a_4 + 3a_4 + a_6 + 2a_7 + a_9), \]

where \( a_1-a_{10} \) stand for the densities of each triangle. The densities of each triangle attached to a positive node are as follows,

\[ n^+_a = \frac{E_i a_i}{3a_1 + 2a_2 + a_3 + 2a_5 + a_6 + a_8}, \]

where \( i = 1, 2, 3, 5, 6, 8, \) and \( E_i \) is the number of positive nodes in \( a_i \). In a similar way, the densities of each triangle attached to a neutral node are as follows,

\[ n^0_{a_j} = \frac{F_j a_j}{a_5 + a_6 + a_7 + 2a_8 + 2a_9 + 3a_{10}}, \]

where \( j = 5, 6, 7, 8, 9, 10, \) and \( F_j \) is the number of neutral nodes in \( a_j \). The densities of each triangle attached to a negative node are as follows,

\[ n^-_{a_k} = \frac{G_k a_k}{a_2 + 2a_4 + 3a_4 + a_6 + 2a_7 + a_9}, \]

where \( k = 2, 3, 4, 6, 7, 9, \) with \( G_k \) being the number of negative nodes in \( a_k \).

We study the ternary network in association with the Potts spin systems. Since the values of each spin are more than two, the spin flipping mechanism changes into the spin transition mechanism. For the spin transition mechanism, the spin takes one of the three values, +1, 0 and -1, depending both on the triad relation and the surroundings [18, 19]. The Hamiltonian, the spin transition probabilities and the geometrical temperature are illustrated by Eqs. (1), (2) and (3) respectively. The total probabilities from one value to the others for each node are as follows,

\[ p(\text{1} \rightarrow \text{0}) = \frac{a_2}{3} e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_5}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

\[ p(\text{1} \rightarrow \text{1}) = \frac{a_2}{3} e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_5}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

\[ p(\text{0} \rightarrow \text{1}) = \frac{a_5}{3} e^{-\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_8}{3} e^{-\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

\[ p(\text{0} \rightarrow \text{0}) = \frac{a_5}{3} e^{-\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_8}{3} e^{-\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

\[ p(\text{1} \rightarrow \text{1}) = \frac{a_1}{3} e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_5}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

\[ p(\text{1} \rightarrow \text{0}) = \frac{a_1}{3} e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{2a_5}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}} + \frac{a_4}{3} e^{\frac{a}{3M}} + e^{\frac{a}{3M}} + \frac{e^{\frac{a}{3M}}}{3} + e^{-\frac{a}{3M}}, \]

But Eq. (10) is correct only for \( M \neq 0 \). When \( M \rightarrow 0 \), the transition probabilities are defined as (the explana-
sections are given in Appendix A1.

\[
\begin{align*}
\mathcal{P}(-1 \rightarrow 0) &= 0, \quad \mathcal{P}(1 \rightarrow 0) = 0, \\
\mathcal{P}(0 \rightarrow 1) &= \frac{1}{2} \left( \frac{a_5}{3} + \frac{a_6}{3} + \frac{a_7}{3} + \frac{2a_8}{3} + \frac{2a_9}{3} + a_{10} \right), \\
\mathcal{P}(0 \rightarrow -1) &= \frac{1}{2} \left( \frac{a_5}{3} + \frac{a_6}{3} + \frac{a_7}{3} + \frac{2a_8}{3} + \frac{2a_9}{3} + a_{10} \right), \\
\mathcal{P}(1 \rightarrow -1) &= \frac{1}{2} \left( a_1 + \frac{2a_2}{3} + \frac{a_3}{3} + \frac{2a_5}{3} + \frac{a_6}{3} + \frac{a_8}{3} \right), \\
\mathcal{P}(-1 \rightarrow 1) &= \frac{1}{2} \left( \frac{a_2}{3} + \frac{2a_3}{3} + a_4 + \frac{a_6}{3} + \frac{2a_7}{3} + \frac{a_9}{3} \right).
\end{align*}
\]  

(17)

Since the transition probabilities are not continued at \( M = 0 \), we cannot give the analytical results of \( n^+, n^-, n^0 \) with \( \alpha \), so we turn to the numerical method. The results are shown by Fig. 4. We note when \( \alpha \rightarrow 0 \), \( n^+ = n^- = n^0 = 1/3 \). And when \( \alpha \rightarrow +\infty \), \( n^0 = 0 \), which means the neutral opinion disappears, and the ternary network comes back to the binary network.

When the network gets dynamically balanced, the distributions of each triangle are as follows,

\[ a_i \sim C_{i mn} x^i y^m, \]

where \( n^+ \sim n^- = x \) and \( n^0 = y \). Besides, \( i = 1, 2, 3, \ldots, 10 \), and \( l, m, n \) are the respective numbers of positive, neutral, negative nodes in \( a_i \), along with \( C_{i mn} \) the corresponding combinatorial number. For example \( C_{201} = 3 \).

For the time needed for dynamic balance, we expect it to satisfy

\[ T_N \sim N^{D(\alpha)}, \]

where \( N \) is the size of the network and \( D(\alpha) > 0 \), being a function of \( \alpha \).

The numerical results of the time needed for dynamic balance are given in Fig. 5. More details are given in Appendix A2.

III. SELF-ORGANIZED CRITICALITY

As is well known, SOC [6–8] is studied on many complex systems, including on the random graph [9], the scale-free network [10–12] and the small-world network [13], etc, which are mainly based on the BTW model [6]. Besides, there are the studies of the SOC on complex networks based on other models, which we refer the readers to Ref. [14] for details. In this paper, we study the SOC on the ternary social networks based on our model and the evolving rules, along with two modifications compared to Sec. III. The first modification is the structure of the network. Because for a completely connected network, any disturbance is globe, with no propagation, so the occurrence of the avalanche will not follow the power-law form either spatially or temporally. Besides, compared to the completely connected network, the regular network or the small-world network [15, 16] is closer to the real structure of the society. So we take the latter forms for study. The second modification is the judgment of a triad relation. In order to embody the majority principle, we take the judgement as the gauge invariance or the gauge variance [19–22], which is explained as follows. Under this judgement, the triad relation evolves from one state to the other by changing node’s value.

If the value of a node being +1, it means the person shows the friendly face in the triad relation, and always tries to keep the relation stable. If the value being -1, it means the person shows the hostile face, and always tries to destroy the stability of the triad relation. While the value being 0, it means the person does not care whether the triad relation is stable or not. So we propose if the
friendly faces are in the majority, the triad relation is in one state, or the gauge invariance state. While the hostile faces are in the majority, the triad relation is in the other state, or the gauge variance state. The explicit expressions of the gauge invariance and the gauge variance states [18, 22] are as follows.

For each triad relation, we propose if

$$\sigma_1 + \sigma_2 + \sigma_3 \geq 0 \text{ or } \sigma_1 = \sigma_2 = \sigma_3 = 0,$$  

it satisfies the gauge invariance. If

$$\sigma_1 + \sigma_2 + \sigma_3 < 0 \text{ or } \sigma_1 \neq \sigma_2 \neq \sigma_3 = 0,$$  

the triad relation does not satisfy the gauge invariance. Here $\sigma_1, \sigma_2$ and $\sigma_3$ are the values of the three nodes of the triangle. So $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}$ are gauge invariance states, while $a_{3}, a_4, a_6, a_7, a_9$ are gauge variance ones, as shown by Fig. 6 and Fig. 7.

We study the ternary network in association with the Potts spin systems. Since each node has three values, the way of spin changing is the spin transition [18]. Besides, we propose the spin transition depends both on itself and the surroundings [18, 19]. Considering the SOC is determined by the structure of the system, we set the geometric temperature as $\beta = \frac{1}{T} > 0$, where $T = |\sum_{<i,j>} \sigma_j / n_i - \sigma_j'|$, with $\sigma_j'$ the final value of $i$, and $\sum_{<i,j>} \sigma_j / n_i$ the average value of all neighboring spins of $i$. Based on these considerations, we give the spin transition probability of each node as follows,

$$W(\sigma_i \rightarrow \sigma_i') = \frac{1}{Z} e^{-\beta H(\sigma_i', \Sigma_{<i,j>}, \sigma_j)}.$$  

In Eq. (22), $H(\sigma_i', \sum_{<i,j>} \sigma_j) = -\sigma_i' \sum_{<i,j>} \sigma_j$, where $\sum_{<i,j>} \sigma_j$ is the sum of values of all neighboring spins of $i$, and $Z = \sum_{\sigma_i'} e^{-\beta (\sum_{<i,j>} \sigma_j)}$ is the normalizing factor. In order to avoid the denominator of $\beta$ being zero, we propose if the final value of the spin results in $T = 0$, it happens with probability zero, while the other values happen with probability one. In this way, whatever the initial condition is, the network will evolve to the sensitive state.

The evolving rules are as follows. We set all triad relations in the network to satisfy gauge invariance initially. The simplest way is to set all spins to be zero, which stands for all persons in the society being neutral with each other initially. After a long enough time of evolution, the network will reach the sensitive state. Then any small disturbance may result in an avalanche.

For any small disturbance:

1. Choose a triangle at random. If it is gauge variance, nothing happens. If it is gauge invariance, we choose one of the three nodes at random, and let the node do spin transition according to Eq. (22). If after the spin transition, the triangle is still gauge invariance, nothing happens. Otherwise, we store all the gauge invariance triangles attached to the selected triangle, and goto step (2).

2. Choose all the stored triangles successively at random. Because of the interaction, first we judge whether the triangle is gauge invariance. If it is gauge variance, nothing happens. Otherwise, let one of the three nodes do spin transition. If the final triangle is gauge invariance, nothing happens, otherwise we find out all the gauge invariance triangles attached to the selected triangle.

3. Find out all the gauge invariance triangles by the combined action of the stored triangles in step (2). If the number of gauge invariance triangles to store is nonzero, goto step (2), otherwise goto step (1).

For each avalanche, we define the size as the number of triangles stored in step (2) changed from the gauge invariance state to the other state, and plus the one changed in step (1). Besides, we define the time of avalanche as follows. If the number of stored triangles in step (2) changed from the gauge invariance state to the other state is nonzero, the time is increased by one, and plus the one changed in step (1).

At first, we study the SOC on the regular network. The structure of the regular network is given in Fig. 6. The size distribution and the time distribution are given in Fig. 7(a) and 7(b) respectively. In the following content, we keep the number of nodes in both the regular network and the small-world network constant as $9 \times 10^4$. Besides, the numerical results are obtained by taking the average of 100 simulations. For each simulation, we execute $9 \times 10^6$ time of disturbance.

In Fig. 7(a) and Fig. 7(b), the diagrams are curved when $s$ and $t$ are small. Because we set all spins to be zero initially. It takes a long time for the network to reach the sensitive state. While the diagrams curved at the ends is caused by the finite size effect.

Next, we study the SOC on the small-world network [15, 16]. The structure of the small-world network is given in Fig. 8.

When a triangle changes from gauge invariance to gauge variance, the value of the selected node is de-
creased. Because of the small-world effect, it may result in the triangles far away compared to the regular network becoming of gauge variance. If the triangles far away compared to the regular network become of gauge variance, it will block the spread. So we expect the size of avalanche to decrease, and the exponent of the power-law distribution to decrease accordingly. Based on these analysis, we set the edge adding probability $q$ of the small-world from $q = 0.05$ to $q = 0.3$, and observe the exponent of the size distribution decrease from -1.91 to -2.79, as given in Fig. 8 (a) and Fig. 8 (b) respectively.

As is well known, for the small-world network [15, 16], when the edge adding probability $q$ is big enough, the structure of the network becomes random. We note the diagrams are curved in Fig. 9 (a) and Fig. 9 (b) when $s$ is small, but they take different forms. It means that the random networks are easier for the spread in the initial conditions.

Then there is the time distribution of the avalanche on the small-world network, which is similar to the size distribution of the avalanche, as given in Fig. 10 (a) and Fig. 10 (b) for comparison. But we note, when $q$ is big enough, with the structure of the network becoming random, the time distribution no longer satisfies the power-law form. Besides, there is condensation at the end of the diagram, which we want to study in the later work.

At last, for the construction of the small-world net-
work, when considering the factor of distance, the small-world effect on the triangles far away compared to the regular network is weakened, which may be easier for the spread. Besides, the number of local triangles are increased. So we expect the size of avalanche to increase, and the exponent of the size distribution to increase accordingly. For the small-world network, when considering the distance effect, it is constructed as follows. From a regular network, as shown by Fig. 6 for each node in the network, it sends out an edge with probability \( q \) and attaches to another node in the network. The probability of attachment between nodes \( i \) and \( j \) is

\[
p_{ij} = \frac{\ell_{i,j} - \delta}{\sum_{m<n} \ell_{m,n} - \delta}.
\]

Here \( \delta > 0 \), and \( \ell_{i,j} \) stands for the smallest distance between \( i \) and \( j \) on the regular network. Then it attaches to one neighbor of the selected node at random. Multi-connection and self-connection are forbidden. In this manner, more triangles are created. For the small-world network, we keep \( q = 0.05 \) constant, and set \( \delta = 1.0 \) and \( \delta = 4.0 \) for comparison. For the size distribution of the avalanche, the exponent increases from -1.85 to -1.69, as shown by Fig. 11(a) and Fig. 11(b). Meanwhile, for the time distribution of the avalanche, the exponent increases from -2.56 to -2.22, as shown by Fig. 12(a) and Fig. 12(b).

In a word, the size distribution of the avalanche always satisfies the power-law form, no matter on the regular network, the small-world network or the random network. But for the time distribution of the avalanche, it deviates from the power-law form on the random network, which we want to do further study in the future work.

### IV. CONCLUSIONS

We have studied the social networks based on our model and obtained meaningful results. First, in Sec. III of this paper, based on Ref. [1], we map the edge relations to the node relations with some relaxation, along with introducing the neutral relation. So the social networks change from the binary networks into the ternary ones. We suppose the change of each triad relation depending both on itself and the surroundings. Based on our model, when the social networks get dynamically balanced, we obtain the distributions of each triad relation and the time needed for dynamic balance. Besides, we find under the extreme condition \( \alpha \to +\infty \), the ternary networks come back to the binary ones. Second, in Sec. III of this paper, we study the SOC on the ternary social networks based on our model. Because we suppose, if one triad relation changes, it will affect other triad relations associating with it. The effect may be big or small, depending on many internal factors. While compared with Sec. III of this paper, we make two modifications for our node model as follows. One is the structure of the social networks, which takes the form of regular or small-world network. The other is the judgement of a triad relation, which takes the form of the gauge invariance or the gauge variance. When the ternary social networks evolving to the sensitive state, any small disturbance may result in an avalanche. Since the number of triad relations changed in the avalanche being no scale preferred, it should satisfy the power-law distributions both spatially and temporally. Third, for the small-world network, we find out the small-world effect on the occurrence of the avalanche both theoretically and numerically.
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Appendix A: Distributions of each triad relation and the time needed for dynamic balance

1. The node model

For the node model, each node has two spin values, +1 and -1. We define one step of time as executing $N$ spin flipping successively at random [1]. Because each spin flipping is linked up with $(N-1)(N-2)/2$ variations of the triangles, the variations of the densities of each triad relation with time are as follows,

$$
\frac{da_1}{dt} = \phi(-1 \to 1) n_{a_2}^- - \phi(1 \to -1) n_{a_1}^+,
$$

$$
\frac{da_2}{dt} = \phi(1 \to -1) n_{a_1}^- + \phi(-1 \to 1) n_{a_3}^- - \phi(-1 \to -1) n_{a_4}^+,
$$

$$
\frac{da_3}{dt} = \phi(1 \to -1) n_{a_4}^- + \phi(-1 \to 1) n_{a_3}^+ - \phi(-1 \to -1) n_{a_5}^-,
$$

$$
\frac{da_4}{dt} = \phi(1 \to -1) n_{a_5}^- - \phi(-1 \to 1) n_{a_4}^+.
$$

(A1)

When the network gets dynamically balanced, we have

$$
\frac{da_1}{dt} = \frac{da_2}{dt} = \frac{da_3}{dt} = \frac{da_4}{dt} = 0, \quad \text{(A2)}
$$

and

$$
\phi(-1 \to 1) = \phi(1 \to -1). \quad \text{(A3)}
$$

From Eqs. (A1), (A2) and (A3), along with the normalization $\sum_{i=1}^{4} a_i = 1$, we obtain

$$
a_1 = \frac{1}{8}, a_2 = \frac{3}{8}, a_3 = \frac{3}{8}, a_4 = \frac{1}{8}. \quad \text{(A4)}
$$

We note the distributions of each triad relation independent of the parameter $\alpha$, and $n^+ = n^-$ in the dynamically balanced state.

Defining at any time the density of positive nodes to be $\rho$, we obtain

$$
a_i = C_3^j \rho^j (1 - \rho)^{3-j},
$$

$$
M = n^+ - n^- = 2\rho - 1,
$$

$$
\rho = (1 + M)/2, \rho(\infty) = \frac{1}{2}, \quad \text{(A5)}
$$

where $j$ is the number of positive nodes in $a_i$ and $C_3^j$ is the combinatorial number. We have the new equations as follows,

$$
\frac{dn^+}{dt} = \phi(-1 \to 1) - \phi(1 \to -1),
$$

$$
\frac{dn^-}{dt} = \phi(1 \to -1) - \phi(-1 \to 1),
$$

$$
\frac{dM}{dt} = \frac{dn^+}{dt} - \frac{dn^-}{dt} = 2[\phi(-1 \to 1) - \phi(1 \to -1)]. \quad \text{(A6)}
$$

Substitute Eqs. (8), (9) and (A5) into Eq. (A6), we obtain

$$
\frac{dM}{dt} = 2 \left[ \left(1 + \frac{M}{2}\right)^2 \left(1 - \frac{M}{2}\right) \frac{e^{3\alpha/(1-M)}}{e^{3\alpha/(1-M)} + e^{-\alpha/(1+M)}} + \left(1 + \frac{M}{2}\right)^3 \frac{e^{-\alpha/(1+M)}}{e^{3\alpha/(1+M)} + e^{-\alpha/(1+M)}} - \left(1 + \frac{M}{2}\right)^3 \frac{e^{-\alpha/(1-M)}}{e^{3\alpha/(1-M)} + e^{-\alpha/(1+M)}} - 2 \left(1 + \frac{M}{2}\right)^2 \left(1 - \frac{M}{2}\right) \frac{e^{-\alpha/(1+M)}}{e^{3\alpha/(1+M)} + e^{-\alpha/(1+M)}} \right]. \quad \text{(A7)}
$$

For Eq. (A7), when $M(t) \sim 0$, after some calculations, we obtain the approximate result,

$$
M(t) \sim t^{-1/C(\alpha)}, \quad \text{(A8)}
$$

where $C(\alpha) > 0$, being a function of $\alpha$.

For the dynamic balance of the network, we propose

$$
|M(T_N)| \sim O \left(\frac{1}{N}\right). \quad \text{(A9)}
$$
So the time needed for dynamic balance is

\[ T_N \sim N^{C(\alpha)}. \quad (A10) \]

In the simulations, both for the binary and the ternary cases, if the final value \( \sigma_i' \) results in the denominator of \( \beta \) (see Eq. 3) being zero, we propose it happens with probability zero, while the other values happen with probability one (see Eq. (17) for example). In this manner, no matter what \( M(0) \) (or \( M(0) \) and \( \rho(0) \)) is, the network will reach the dynamically balanced state.

2. The generalized model

When each node has three spin values, the variations of the densities of each triad relation with time are as follows,

\[
\begin{align*}
\frac{da_1}{dt} &= \mathcal{P}(-1 \to 1) n_{a2}^- + \mathcal{P}(0 \to 1) n_{a5}^0 - [\mathcal{P}(1 \to 0) + \mathcal{P}(1 \to -1)] n_{a1}^+ , \\
\frac{da_2}{dt} &= \mathcal{P}(1 \to -1) n_{a1}^+ + \mathcal{P}(-1 \to 1) n_{a3}^- + \mathcal{P}(0 \to -1) n_{a5}^0 + \mathcal{P}(0 \to 1) n_{a6}^0 - [\mathcal{P}(1 \to 0) + \mathcal{P}(1 \to -1)] n_{a2}^+ \\
&\quad - [\mathcal{P}(-1 \to 0) + \mathcal{P}(-1 \to 1)] n_{a2}^- , \\
\vdots \\
\frac{da_9}{dt} &= \mathcal{P}(1 \to 0) n_{a6}^+ + \mathcal{P}(-1 \to 0) n_{a7}^- + \mathcal{P}(1 \to -1) n_{a8}^+ + \mathcal{P}(0 \to -1) n_{a10}^0 - [\mathcal{P}(1 \to 0) + \mathcal{P}(1 \to -1)] n_{a9}^+ \\
&\quad - [\mathcal{P}(-1 \to 0) + \mathcal{P}(-1 \to 1)] n_{a9}^- , \\
\frac{da_{10}}{dt} &= \mathcal{P}(1 \to 0) n_{a8}^+ + \mathcal{P}(1 \to 0) n_{a9}^- - [\mathcal{P}(0 \to -1) + \mathcal{P}(0 \to 1)] n_{a10}^0 .
\end{align*}
\]  
\[(A11)\]

When the network gets dynamically balanced, it satisfies

\[
\frac{da_i}{dt} = 0, \quad (A12)
\]

where \( i = 1, 2, 3, \ldots, 10 \), along with the normalization

\[
\sum_{i=1}^{10} a_i = 1. \quad (A13)
\]

Although Eqs. (A11) are very hard to solve, from appendix A and the symmetric spin transition mechanism, we expect \( n^+ \simeq n^- \) and \( M(\infty) \simeq 0 \) in the dynamically balanced state. Defining \( n^+ \simeq n^- = x \) and \( n^0 = y \) in the dynamically balanced state, the densities of each type of the triangles are

\[
a_i \simeq C_{lmn} x^l y^m, \quad (A14)
\]

where \( l, m \) and \( n \) are the numbers of positive, neutral and negative nodes in \( a_i \) respectively. \( C_{lmn} \) is the corresponding combinatorial number. For example \( C_{201} = 3 \). Because near \( M = 0 \), there are fluctuations. The spin transition probabilities are \( (M \neq 0) \)

\[
\begin{align*}
\mathcal{P}(-1 \to 0) &\simeq x^3 + x^3 + \frac{2x^2 y}{1 + e^\alpha + e^{-\alpha}} + \frac{2x^2 y}{1 + e^\alpha + e^{-\alpha}} + xy^2, \\
\mathcal{P}(-1 \to 1) &\simeq x^3 + \frac{2x^2 ye^\alpha}{1 + e^\alpha + e^{-\alpha}} + \frac{2x^2 ye^{-\alpha}}{1 + e^\alpha + e^{-\alpha}} + \frac{xy^2}{3}, \\
\mathcal{P}(0 \to -1) &\simeq x^2 y + \frac{2xy^2 e^{-\alpha}}{1 + e^\alpha + e^{-\alpha}} + \frac{2xy^2 e^\alpha}{1 + e^\alpha + e^{-\alpha}} + \frac{y^3}{3}, \\
\mathcal{P}(0 \to 1) &\simeq x^2 y + \frac{2xy^2 e^\alpha}{1 + e^\alpha + e^{-\alpha}} + \frac{2xy^2 e^{-\alpha}}{1 + e^\alpha + e^{-\alpha}} + \frac{y^3}{3}, \\
\mathcal{P}(1 \to -1) &\simeq x^3 + \frac{2x^2 ye^{-\alpha}}{1 + e^\alpha + e^{-\alpha}} + \frac{2x^2 ye^\alpha}{1 + e^\alpha + e^{-\alpha}} + \frac{xy^2}{3}, \\
\mathcal{P}(1 \to 0) &\simeq x^3 + x^3 + \frac{2x^2 y}{1 + e^\alpha + e^{-\alpha}} + \frac{2x^2 y}{1 + e^\alpha + e^{-\alpha}} + \frac{xy^2}{3}. \\
\end{align*}
\]  
\[(A15)\]
and \((M \to 0)\)

\[
\begin{align*}
\mathbf{p}(-1 \to 0) &= 0, \quad \mathbf{p}(1 \to 0) = 0, \\
\mathbf{p}(0 \to 1) &= 2x^2y + 2xy^2 + \frac{y^3}{2}, \\
\mathbf{p}(0 \to -1) &= 2x^2y + 2xy^2 + \frac{y^3}{2}, \\
\mathbf{p}(1 \to -1) &= 2x^3 + 2x^2y + \frac{xy^2}{2}, \\
\mathbf{p}(-1 \to 1) &= 2x^3 + 2x^2y + \frac{xy^2}{2}. \tag{A16}
\end{align*}
\]

The densities of each triangle attached to a positive node are

\[
\begin{align*}
n^+_i &= \frac{c_1 x^2 + 2c_2 x^2 + 2c_3 xy + c_4 y^2}{(2x + y)^2}, \tag{A17} \\
n^+_a &\approx \frac{c_1 x^2 + 2c_2 x^2 + 2c_3 xy + c_4 y^2}{(2x + y)^2}.
\end{align*}
\]

where \(i = 1, 2, 3, 5, 6, 8, \) and \(\sum_{j=1}^{4} c_j = 1, c_j = 0, 1\). That is, only one of the coefficient \(c_j\) is nonzero. In a similar way, the densities of each triangle attached to a neutral node are

\[
\begin{align*}
n^0 &= \frac{c_1 x^2 + 2c_2 x^2 + 2c_3 xy + c_4 y^2}{(2x + y)^2}, \tag{A18} \\
n^0 &\approx \frac{c_1 x^2 + 2c_2 x^2 + 2c_3 xy + c_4 y^2}{(2x + y)^2}.
\end{align*}
\]

where \(j = 5, 6, 7, 8, 9, 10\). And the densities of each triangle attached to a negative node are

\[
n^+ = \frac{c_1 x^2 + 2c_2 x^2 + 2c_3 xy + c_4 y^2}{(2x + y)^2}, \tag{A19}
\]

where \(k = 2, 3, 4, 6, 7, 9\).

Because the transition probabilities (see Eq. [16] and Eq. [17]) are not continued at \(M = 0\), we cannot obtain the analytical results for the distributions of \(n^+, n^-\) and \(n^0\) with \(\alpha\), so we turn to the numerical method, as shown by Fig. \(\text{f}1\). We note when \(\alpha \to +\infty\), the neutral opinion disappears, and the ternary network comes back to the binary network. That can be explained as follows. In the triad relation, the more people try to keep it stable, i.e., to show the same opinion with the others, the less likely the neutral opinion will exist.

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