\(L^p\)-solutions of deterministic and stochastic convective Brinkman–Forchheimer equations

Manil T. Mohan

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Abstract
In the first part of this work, we establish the existence and uniqueness of a local mild solution to deterministic convective Brinkman–Forchheimer (CBF) equations defined on the whole space, by using properties of the heat semigroup and fixed point arguments based on an iterative technique. Moreover, we prove that the solution exists globally. The second part is devoted for establishing the existence and uniqueness of a pathwise mild solution upto a random time to the stochastic CBF equations perturbed by Lévy noise by exploiting the contraction mapping principle. Then by using stopping time arguments, we show that the pathwise mild solution exists globally. We also discuss the local and global solvability of the stochastic CBF equations forced by fractional Brownian noise.

Keywords Convective Brinkman–Forchheimer equations · Lévy noise · Fractional Brownian motion · Mild solution

Mathematics Subject Classification Primary 76D06; Secondary 35Q30, 76D03, 47D03

1 Introduction

1.1 Deterministic convective Brinkman–Forchheimer equations

The Cauchy problem for convective Brinkman–Forchheimer equations (CBF) in \(\mathbb{R}^d, d \geq 2\) can be written as
\[
\frac{\partial u(t, x)}{\partial t} - \mu \Delta u(t, x) + (u(t, x) \cdot \nabla) u(t, x) + \alpha u(t, x) + \beta |u(t, x)|^{r-1} u(t, x) + \nabla p(t, x) = f(t, x), \quad \text{in} \ (0, T) \times \mathbb{R}^d,
\]

with the conditions

\[
\begin{align*}
\nabla \cdot u(t, x) &= 0, \quad \text{in} \ (0, T) \times \mathbb{R}^d, \\
u(0, x) &= u^0(x), \quad \text{in} \ {0} \times \mathbb{R}^d, \\
|u(t, x)| \to 0 \quad &\text{as} \ |x| \to \infty, \ t \in [0, T).
\end{align*}
\]

In (1.1), \(u(t, x) \in \mathbb{R}^d\) stands for the velocity field at time \(t\) and position \(x\), \(p(t, x) \in \mathbb{R}\) represents the pressure field, \(f(t, x) \in \mathbb{R}^d\) is an external forcing. The constant \(\mu > 0\) denotes the positive Brinkman coefficient (effective viscosity), the positive constants \(\alpha\) and \(\beta\) represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. For \(\alpha = \beta = 0\), we obtain the classical Navier–Stokes equations (NSE). The absorption exponent \(r \in [1, \infty)\) and the case \(r = 3\) is known as the critical exponent. The critical homogeneous CBF equations (1.1) have the same scaling as NSE only when \(\alpha = 0\) (see Proposition 1.1, [18] and no scale invariance property for other values of \(\alpha\) and \(r\)). Since \(\alpha\) does not play a major role in our analysis, we fix \(\alpha = 0\) in the rest of our analysis.

The Cauchy problem for (1.1) is considered in [5], and the authors proved the existence of global weak solutions, for any \(r \geq 1\), global strong solutions, for any \(r \geq 7/2\) and that the strong solution is unique, for any \(7/2 \leq r \leq 5\). An improvement for this result is made in [38] and the authors proved that the Cauchy problem for (1.1) possesses global strong solutions, for any \(r > 3\) and the strong solution is unique, when \(3 < r \leq 5\). Later, the authors in [39] proved that the strong solution exists globally for \(r \geq 3\), and they established two regularity criteria, for \(1 \leq r < 3\). Moreover, for any \(r \geq 1\), they established that the strong solution is unique even among weak solutions.

The global well-posedness of a 3D Brinkman–Forchheimer-extended Darcy model with periodic boundary conditions is obtained in [26]. A simple proof of the existence of global-in-time smooth solutions for the CBF equations (1.1)–(1.2) with \(r > 3\) is given in [18]. Using the monotonicity property of the linear and nonlinear operators and Minty-Browder technique, the existence of global weak as well as strong solutions of the CBF equations (1.1)–(1.2) with \(r \geq 3\) (2\(\beta\mu \geq 1\) for \(r = 3\)) in bounded and periodic domains is obtained in [28]. Our goal is to prove the existence of a unique local mild solution and its global existence for the CBF equations (1.1)–(1.2).

### 1.1.1 Abstract formulation and mild solution

The \textit{Helmholtz-Hodge projection} denoted by \(\mathcal{P}\) is a bounded linear operator from \(L^p(\mathbb{R}^d)\) to \(J_p := \mathcal{P}L^p(\mathbb{R}^d)\). Note that the space \(J_p\) is a separable Banach space with \(L^p(\mathbb{R}^d)\)-norm denoted by \(\| \cdot \|_p\) and the operator \(\mathcal{P}\) is an orthogonal projection of \(L^2(\mathbb{R}^d)\) onto the subspace \(\mathcal{H} := J_2\). Remember that \(\mathcal{P}\) can be expressed in terms of the Riesz transform (cf. [27] for more details). We use the notation \(\mathcal{L}(\mathcal{H}, J_p)\) for the space
of all bounded linear operators from $\mathbb{H}$ to $\mathbb{J}_p$. Let us denote $G_{k,p} := \mathcal{P}H^k_p(\mathbb{R}^d)$, $k \geq 0, 1 < p < \infty$. We apply the projection operator $\mathcal{P}$ to the system (1.1) to obtain

$$\begin{cases}
\frac{du(t)}{dt} + \mu Au(t) + B(u(t)) + \beta C(u(t)) = \mathcal{P}f(t), & t \in (0, T], \\
u(0) = x,
\end{cases}$$

(1.3)

where

$$Au = -\mathcal{P}\Delta u,$$

with domain $D_p(A) = D(-\Delta) \cap \mathbb{J}_p,$

$$B(u) = B(u, u),$$

with $B(u, v) = \mathcal{P}[(u \cdot \nabla)v] = \mathcal{P}[\nabla \cdot (u \otimes v)],$

$$C(u) = \mathcal{P}[|u|^{-1}u],$$

for all $u, v \in \mathbb{G}_{2,2}$ and $x \in \mathbb{J}_p$. For $r \geq 1$, the operator $C(\cdot)$ is Gateaux differentiable with the Gateaux derivative

$$C'(u)v = \begin{cases}
\mathcal{P}(v), & \text{for } r = 1, \\
\mathcal{P}(|u|v) + (r-1)\mathcal{P}\left(\frac{u}{|u|} - (u \cdot v)\right), & \text{if } u \neq 0, \text{ for } 1 < r < 3, \\
0, & \text{if } u = 0, \\
\mathcal{P}(|u|^{-1}v) + (r-1)\mathcal{P}(u|u|^{-3}(u \cdot v)), & \text{for } r \geq 3,
\end{cases}$$

(1.4)

for all $u, v \in \mathbb{L}^p(\mathbb{R}^d)$, for $p \in [2, \infty)$. It should be remembered that $\mathcal{P}\Delta = \Delta \mathcal{P}$ (cf. [27]), and hence the operator $A$ is essentially equal to $-\Delta$ and $e^{-\mu tA}$ is substantially the heat semigroup (Gauss-Weierstrass semigroup, [21]) and is given by

$$(e^{-\mu tA}u)(x) = \int_{\mathbb{R}^d} \psi(t, x - y)u(y)dy, \quad \text{where } \psi(t, x) = \frac{1}{(4\pi \mu t)^d} e^{-\frac{|x|^2}{4\mu t}},$$

for $t > 0$, $x \in \mathbb{R}^d$ and $u \in \mathbb{L}^q(\mathbb{R}^d), q \in [1, \infty)$. Thus, the operator system (1.3) can be transformed into a nonlinear integral equation as follows:

$$u(t) = e^{-\mu tA}x - \int_0^t e^{-\mu (t-s)A}B(u(s)) + \beta C(u(s))ds + \int_0^t e^{-\mu (t-s)A}\mathcal{P}f(s)ds,$$

(1.5)

for all $t \in [0, T]$. For a given $x \in \mathbb{J}_p$ and $f \in L^1(0, T; \mathbb{J}_p)$, a function $u \in C([0, T]; \mathbb{J}_p)$, for max $\left\{d, \frac{d(r-1)}{2}\right\} < p < \infty$ satisfying (1.5) is called a mild solution to the system (1.3).

Since $e^{-\mu tA}$ is an analytic semigroup, we infer that $e^{-\mu tA} : \mathbb{L}^p \to \mathbb{L}^q$ is a bounded map whenever $1 < p \leq q < \infty$ and $t > 0$, and there exists a constant $C$ depending on $p$ and $q$ such that (see [21])

$$\|e^{-\mu tA}g\|_q \leq Ct^{-\frac{d}{2}}\left(\frac{1}{p} - \frac{1}{q}\right)\|g\|_p,$$

(1.6)
\begin{align}
\|\nabla e^{-\mu tA}g\|_q & \leq Ct^{-\frac{1}{2} - \frac{d}{2p}(\frac{1}{p} - \frac{1}{q})}\|g\|_p, \quad (1.7)
\end{align}
for all $t \in (0, T]$ and $g \in L^p(\mathbb{R}^d)$. Using the estimates (1.6)–(1.7), one can estimate $\|e^{-\mu tA}[u, v]\|_p$ as
\begin{align}
\|e^{-\mu tA}[u, v]\|_p & \leq Ct^{-\left(\frac{1}{2} + \frac{d}{2p}\right)}\|u\|_p\|v\|_p, \quad (1.8)
\end{align}
for all $t \in (0, T]$ and $u, v \in L_p$. Furthermore, using the estimate (1.6), we calculate $\|e^{-\mu tA}C(u)\|_p$ and $\|e^{-\mu tA}C'(u)v\|_p$ as
\begin{align}
\|e^{-\mu tA}C(u)\|_p & \leq Ct^{-\frac{d(e-1)}{2p}}\|u\|'^p, \quad (1.9)
\end{align}
\begin{align}
\|e^{-\mu tA}C'(u)v\|_p & \leq Ct^{-\frac{d(e-1)}{2p}}\|u\|'^{p-1}\|v\|_p, \quad (1.10)
\end{align}
for all $t \in (0, T]$ and $u, v \in L_p$. For the existence of local mild solutions in $L^p$ to 3D NSE on the whole space and bounded domains, the interested readers are referred to see [13,17,21,37], etc.

In the first part of this work, we prove the following result:

**Theorem 1.1** For $\max\left\{d, \frac{d(r-1)}{2}\right\} < p < \infty$, let $x \in L_p$ and $f \in L^1(0, T; \mathbb{R}^d)$ be given. Then, there exists a time $0 < T_* < T$ such that (1.3) has a unique mild solution given by (1.5) in $C([0, T_*]; \mathbb{R}^d)$. For $r \geq 3 \,(4\beta \mu \geq 1)$ for $r = 3)$ and $f \in L^p(0, T; \mathbb{R}^d)$, the solution is global in time with $u \in C([0, T]; \mathbb{R}^d)$, for $\frac{d(r-1)}{2} < p < \infty$.

For $r \geq 3$ and $f \in L^2(0, T; \mathbb{H})$, the global existence holds for $d = 2$, $r \in [1, 3]$ with $u \in L^{\frac{2p}{p+2}}(0, T; \mathbb{R})$ for $2 < p < \infty$ and for $d = 3$, $r \in [1, 3]$, with $u \in L^{\frac{2p}{3(p-3)}}(0, T; \mathbb{R})$, for $3 < p \leq 6$.

### 1.2 Stochastic CBF equations perturbed by Lévy noise

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with an increasing family of sub-sigma fields $(\mathcal{F}_t)_{t \geq 0}$ of $\mathcal{F}$ satisfying the usual conditions. On taking the external forcing as Lévy noise, one can rewrite the stochastic counterpart of the problem (1.3) for $t \in (0, T]$ as
\begin{align}
\begin{cases}
    du(t) + [Au(t) + B(u(t)) + \beta C(u(t))]dt = \Phi dW(t) + \int_Z \xi(t, \varepsilon)\tilde{\mathcal{F}}(dt, d\varepsilon), \\
    u(0) = x,
\end{cases} \quad (1.11)
\end{align}
where $x \in \mathbb{R}_q$, $\mathbb{P}$-a.s. In (1.11), $W = \{W(t)\}_{t \geq 0}$ is a cylindrical Wiener process and for an orthonormal basis $\{e_j(x)\}_{j=1}^\infty$ and $W(\cdot)$ can be represented as $W(t) = \sum_{j=1}^\infty e_j(x)\beta_j(t)$, where $\{\beta_j(\cdot)\}_{j=1}^\infty$ is a sequence of one-dimensional mutually independent Brownian motions [9]. The bounded linear operator $\Phi : \mathbb{H} \to \mathbb{R}_q$, for some
$q \in [2, \infty)$ is a $\gamma$-radonifying operator in $\mathbb{J}_q$ such that

$$\Phi dW(t) = \sum_{j=1}^{\infty} \Phi e_j(x) d\beta_j(t) = \sum_{j=1}^{\infty} \int_{\mathbb{R}^d} \mathcal{H}(x, y) e_j(y) dy d\beta_j(t),$$

where $\mathcal{H} (\cdot, \cdot)$ is the kernel of the operator $\Phi$ (Theorem 2.2, [3]). In particular, the operator $\Phi \in \gamma(\mathbb{H}; \mathbb{J}_q)$ satisfies

$$\|\Phi\|_{\gamma(\mathbb{H}; \mathbb{J}_q)} = \left\{ \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} |\mathcal{H}(x, y)|^2 \, dy \right]^{q/2} \, dx \right\}^{1/q} < +\infty,$$

where $\gamma(\mathbb{H}; \mathbb{J}_q)$ is the space of all $\gamma$-radonifying operators from $\mathbb{H}$ to $\mathbb{J}_q$.\footnote{Let $\mathbb{U}$ be a real separable Hilbert space and $\mathcal{X}$ be a Banach space. A bounded linear operator $R \in \mathcal{L}(\mathbb{U}, \mathcal{X})$ is $\gamma$-radonifying provided that there exists a centered Gaussian probability $\nu$ on $\mathcal{X}$ such that $\int_{\mathcal{X}} \varphi(x) d\nu(x) = \| R^* \varphi \|_{\mathbb{U}}, \quad \varphi \in \mathcal{X}'$. Such a measure is at most one, and hence we set $\| R \|_{\gamma(\mathbb{U}, \mathcal{X})} = \int_{\mathcal{X}} \| x \|_{\mathcal{X}}^2 d\nu(x)$. We denote $\gamma(\mathbb{U}; \mathcal{X})$ for the space of $\gamma$-radonifying operators, and $\gamma(\mathbb{U}, \mathcal{X})$ equipped with the norm $\| \cdot \|_{\gamma(\mathbb{U}, \mathcal{X})}$ is a separable Banach space.}

Let us denote by $Z$, a measurable subspace of some Hilbert space (for example measurable subspaces of $\mathbb{R}^d$, $L^2(\mathbb{R}^d)$, etc) and $\lambda(dz)$, a $\sigma$-finite Lévy measure on $Z$ with an associated Poisson random measure $\pi(dt, dz) = \pi(dt, dz) - \lambda(dz)dt$ as the compensated Poisson random measure. The jump noise coefficient $\xi(t, z) := \xi(t, z, x)$ is such that $\xi : [0, T] \times Z \times \mathbb{R}^d \to \mathbb{J}_q$, for some $q \in [2, \infty)$ and in particular, $\xi$ satisfies

$$\int_0^T \int_{\mathbb{R}^d} \|\xi(t, z)\|_{\mathbb{J}_q}^q \lambda(dz) dt < +\infty,$$  \hspace{1cm} (1.12)$$

for some $\ell \in [2, \infty)$. We denote the space of all functions $\xi : [0, T] \times Z \times \mathbb{R}^d \to \mathbb{J}_q$ satisfying (1.12) by $L^\ell(0, T; L^\ell(\mathbb{J}_q))$. The processes $W(\cdot)$ and $\pi(\cdot, \cdot)$ are mutually independent.

The existence of a unique strong solution to stochastic tamed 3D NSE perturbed by Gaussian noise in the whole space as well as in the periodic boundary case is obtained in [34]. Recently, the authors in [4] improved their results for a slightly simplified system. The local and global existence and uniqueness of solutions for general deterministic and stochastic nonlinear evolution equations with coefficients satisfying some local monotonicity and generalized coercivity conditions is established in [23]. The author in [24] showed the existence and uniqueness of strong solutions for a large class of SPDEs, where the coefficients satisfy the local monotonicity and Lyapunov condition, and he provided stochastic tamed 3D NSE as an example. By adding a term of Brinkman–Forchheimer type to Navier–Stokes equations in the whole space, the existence and uniqueness of global strong solutions are proved in [2]. The existence and uniqueness of a pathwise strong solution to 3D tamed Navier–Stokes equations driven by multiplicative Lévy noise in periodic domains is examined in [11], and stochastic CBF equations perturbed by Gaussian as well as jump noises in...
periodic domains are available in [29,31], etc. Our aim is to establish the existence of a unique local pathwise mild solution and its global existence for stochastic CBF equations perturbed by Lévy noise.

We transform the operator system (1.3) into a stochastic nonlinear integral equation as follows:

\[
\begin{align*}
    u(t) &= e^{-\mu t}x - \int_0^t e^{-\mu(t-s)}[B(u(s)) + \beta C(u(s))]ds + \int_0^t e^{-\mu(t-s)}\Phi dW(s) \\
    &\quad + \int_0^t \int_\mathbb{R} e^{-\mu(t-s)}\xi(s, z)\tilde{\pi}(ds, dz),
\end{align*}
\]

(1.13)

for all \( t \in [0, T] \). The existence of pathwise mild solutions for 2D and 3D NSE perturbed by Gaussian as well as jump noise is available in [10,15,27,41], etc and the references therein.

**Definition 1.2** A \( \mathbb{J}_p \)-valued and \( \mathcal{F}_t \)-adapted stochastic process \( u : [0, T) \times \mathbb{R}^d \times \Omega \to \mathbb{R} \) with \( \mathbb{P} \)-a.s. càdlàg trajectories for \( t \in [0, T] \), is a mild solution to the system (1.11), if for any \( T > 0, u(t) := u(t, \cdot, \cdot) \) satisfies the integral equation (1.13) \( \mathbb{P} \)-a.s., for each \( t \in [0, T] \).

The second aim of this work is to establish the following result:

**Theorem 1.3** For \( \Phi \in \gamma(\mathbb{H}; \mathbb{J}_p), \xi \in L^2(0, T; L^2_\lambda(\mathbb{J}_p)) \cap L^p(0, T; L^p_\lambda(\mathbb{J}_p)) \) and \( \max\left\{ d, \frac{d(r-1)}{2}\right\} < p < \infty \), let the \( \mathcal{F}_0 \)-measurable initial data \( x \in \mathbb{J}_p, \mathbb{P} \)-a.s. be given. Then there exists a random time \( 0 < \tilde{T} < T \) such that (1.11) has a unique pathwise mild solution \( u \) with an \( \mathcal{F}_t \)-adapted càdlàg modification in \( L^\infty(0, \tilde{T}; \mathbb{J}_p), \mathbb{P} \)-a.s.

For \( d = 2, 3 \), \( r \in [1, 3] \), \( \Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \cap \gamma(\mathbb{H}; \mathbb{J}_{p+r-1}), \xi \in L^2(0, T; L^2_\lambda(\mathbb{J}_p \cap \mathbb{J}_{p+r-1})) \cap L^p(0, T; L^p_\lambda(\mathbb{J}_p)) \cap L^{p+r-1}(0, T; L^p_{\lambda,r}(\mathbb{J}_{p+r-1})) \) and \( \frac{d(r-1)}{2} < p < \infty \), the solution is global in time with \( u \in L^\infty(0, T; \mathbb{J}_p), \mathbb{P} \)-a.s. and \( \xi \in L^2(0, T; L^2_\lambda(\mathbb{J}_p \cap \mathbb{J}_{r+1} \cap \mathbb{J}_4)) \cap L^2_{\lambda,p}(0, T; L^2_{\lambda}(\mathbb{J}_p)) \cap L^{2p-2}(0, T; L^2_{\lambda}(\mathbb{J}_p \cap \mathbb{J}_{r+1})) \cap L^4(0, T; L^4_{\lambda}(\mathbb{J}_4)), \) the solution is global with \( u \in L^2_{\lambda,p}(0, T; \mathbb{J}_p), 2 < p < \infty, \mathbb{P} \)-a.s., for \( d = 2 \) and \( u \in L^{\frac{4p}{(p-2)}}(0, T; \mathbb{J}_p), 3 < p \leq 6, \mathbb{P} \)-a.s. for \( d = 3 \).

**1.3 Stochastic CBF equations perturbed by fractional Brownian noise**

Let us now consider the stochastic CBF equations perturbed by fractional Brownian noise as

\[
\begin{align*}
    &\begin{cases}
        du(t) + [Au(t) + B(u(t)) + \beta C(u(t)))]dt = \Phi dW^H(t), \\
        u(0) = x,
    \end{cases}
\end{align*}
\]

(1.14)

where \( \Phi \in \mathcal{L}(\mathbb{H}, \mathbb{J}_p) \) and \( W^H(\cdot) \) is the cylindrical fractional Brownian motion with Hurst parameter \( H \in (0, 1) \) (see Sect. 4 for more details). One can transform the
operator system (1.14) into a stochastic nonlinear integral equation as

\[ u(t) = e^{-\mu t}A x - \int_0^t e^{-\mu(t-s)}A [B(u(s)) + \beta C(u(s))] ds + \int_0^t e^{-\mu(t-s)}\Phi dW^H(s), \]

(1.15)

for all \( t \in [0, T] \). For the well-posedness of 2D stochastic NSE, existence of density for 2D stochastic NSE and local solvability of 3D stochastic NSE perturbed by fractional Brownian noise, we refer the interested readers to [14,16,19], respectively. We prove the existence of a unique local pathwise mild solution and its global existence for the Brownian noise, we refer the interested readers to [14,16,19], respectively. We prove the existence of a unique local pathwise mild solution and its global existence for the system (1.14). One can prove the following theorem in a similar way as in Theorem 1.3.

**Theorem 1.4** For \( \Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \) and \( \max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty \), let the \( \mathcal{F}_0 \)-measurable initial data \( x \in \mathbb{J}_p, \mathbb{P}\text{-a.s.} \) be given. Then there exists a random time \( 0 < \overline{T} < T \) such that (1.11) has a unique pathwise mild solution \( u \) with an \( \mathcal{F}_1 \)-adapted continuous modification in \( C([0, \overline{T}]; \mathbb{J}_p), \mathbb{P}\text{-a.s.} \).

For \( r \geq 3 (\beta \mu > 4 \text{ for } r = 3) \), \( \Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \cap \gamma(\mathbb{H}; \mathbb{J}_{p+r-1}) \) and \( \frac{d(r-1)}{2} < p < \infty \), the solution is global in time with \( u \in C([0, T]; \mathbb{J}_p), \mathbb{P}\text{-a.s.} \).

For \( d = 2, 3, r \in [1, 3] \), \( \Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \cap \gamma(\mathbb{H}; \mathbb{J}_{r+1}) \cap \gamma(\mathbb{H}; \mathbb{J}_4) \), the solution is global with \( u \in L^2_p(0, T; \mathbb{J}_p), 2 < p < \infty, \mathbb{P}\text{-a.s.} \), for \( d = 2 \) and \( u \in L^{6p/(3p-2)}(0, T; \mathbb{J}_p), 3 < p \leq 6, \mathbb{P}\text{-a.s.} \) for \( d = 3 \).

### 1.4 Major challenges and organization of the paper

The main difficulty in establishing the existence of local mild solutions to the deterministic CBF equations (1.3) lies under estimating the nonlinear terms, which we successfully overcame using the estimates (1.8)–(1.10). On the stochastic counterpart, along with these difficulties, additional complications arise due to the presence of noise term (proper regularity of stochastic convolution is needed). In the case of Lévy noise, we handle this obstacle by using the stochastic convolution estimates obtained in [40]. For fractional Brownian noise and \( \alpha \)-regular Volterra processes, we overcame this hurdle by using the stochastic convolution results established in [7]. Thus, making use of the estimates (1.6)–(1.10) and fixed point arguments (iterative technique) or contraction mapping principle, we achieve our goals. It can be easily seen that in the sub-critical and critical cases (that is, for \( r \in [1, 3] \)), the condition on \( p \) is \( d < p < \infty \), which is same as that of NSE (cf. [13,15], etc) and for the super-critical case (that is, \( r \in (3, \infty) \)), the condition on \( p \) becomes \( \frac{d(r-1)}{2} < p < \infty \).

The rest of the paper is organized as follows: In the next section, we prove Theorem 1.1 by using fixed point arguments based on an iterative technique. Theorem 1.3 is proved in Sect. 3 by transforming the equation (1.11) into a pathwise deterministic equation (see 3.4 below) and then showing that the system (3.4) has a unique (local as well as global) mild solution by using the contraction mapping principle. By using stopping time arguments, we conclude the proof of Theorem 1.3. Section 4 is devoted
for examining the existence of a unique pathwise mild solution (local and global) to
the system (1.14).

2 Existence and uniqueness of deterministic CBF equations

In this section, we present the existence and uniqueness of local and global mild
solution to the problem (1.3) (Theorem 1.1). We use fixed point arguments (by using
a simple iterative technique) to obtain the required result. Let us now prove Theorem
1.1.

Proof of Theorem 1.1 We prove the theorem in the following steps:

Step I. Existence: As discussed in [13,21], etc, in order to prove the theorem, we use
an iterative technique. Let us set

\[ u_0(t) = e^{-\mu t} x, \]

\[ u_{n+1}(t) = u_0 + G(u_n)(t), \quad n = 0, 1, 2, \ldots, \]

where

\[ G(u)(t) = -\int_0^t e^{-\mu(t-s)} [B(u(s)) + \beta C(u(s))] \, ds + \int_0^t e^{-\mu(t-s)} \mathcal{P} f(s) \, ds, \]

which is continuous for all \( t \in [0, T] \). Since \( e^{-\mu t} \) is a contraction semigroup on \( L^p(\mathbb{R}^d) \), first we note that

\[ \|u_0(t)\|_p = \|e^{-\mu t} x\|_p \leq \|x\|_p. \]

Using the estimates (1.6)–(1.10), we find

\[
\|u_{n+1}(t)\|_p \\
\leq \|x\|_p + \int_0^t \|e^{-\mu(t-s)} A B(u_n(s))\|_p \, ds + \beta \int_0^t \|e^{-\mu(t-s)} C(u_n(s))\|_p \, ds \\
\quad + \int_0^t \|e^{-\mu(t-s)} \mathcal{P} f(s)\|_p \, ds \\
\leq \|x\|_p + C \int_0^t (t-s)^{-\left(\frac{1}{2} + \frac{d}{2p}\right)} \|u_n(s)\|_p^2 \, ds + C \int_0^t (t-s)^{-\frac{d(r-1)}{2p}} \|u_n(s)\|_p^r \, ds \\
\quad + C \int_0^t \|f(s)\|_p \, ds \\
\leq \left\{ \|x\|_p + C \int_0^t \|f(s)\|_p \, ds \right\} + Ct^{\frac{1}{2} - \frac{d}{2p}} \sup_{s \in [0,t]} \|u_n(s)\|_p^2 \\
\quad + Ct^{\frac{1}{2} - \frac{d(r-1)}{2p}} \sup_{s \in [0,t]} \|u_n(s)\|_p^r.
\]
\[ \leq \left\{ \|x\|_p + C \int_0^T \|f(s)\|_p ds \right\} + CT^{\frac{1}{2} - \frac{d}{2p}} f_n^2 + CT^{1 - \frac{d(r-1)}{2p}} f_n^r, \]  

(2.3)

for all \( t \in [0, T] \), where

\[ f_n = \sup_{t \in [0,T]} \|u_n(t)\|_p. \]

For \( f_0 = \left\{ \|x\|_p + C \int_0^T \|f(s)\|_p ds \right\} \), from the above relation, it is immediate that

\[ f_{n+1} \leq f_0 + CT^{\frac{1}{2} - \frac{d}{2p}} f_n^2 + CT^{1 - \frac{d(r-1)}{2p}} f_n^r, \quad n = 0, 1, 2, \ldots, \]

(2.4)

which is a nonlinear recurrence relation. One can easily show by induction that if

\[ \frac{1}{2} \min \left\{ \left( \frac{1}{4CT^{\frac{1}{2} - \frac{d}{2p}}} \right)^{\frac{1}{1-r}}, \left( \frac{1}{4CT^{1 - \frac{d(r-1)}{2p}}} \right)^{\frac{1}{r-1}} \right\} > f_0, \]

then

\[ f_n \leq \min \left\{ \left( \frac{1}{4CT^{\frac{1}{2} - \frac{d}{2p}}} \right)^{\frac{1}{1-r}}, \left( \frac{1}{4CT^{1 - \frac{d(r-1)}{2p}}} \right)^{\frac{1}{r-1}} \right\} =: K, \quad \text{for all } n = 1, 2, 3, \ldots, \]

so that the sequence \( \{f_n\} \) is uniformly bounded.

Let us now consider

\[ v_{n+2}(t) = u_{n+2}(t) - u_{n+1}(t) \]

\[ = - \int_0^t e^{-\mu(t-s)A} [B(u_{n+1}(s)) - B(u_n(s))] ds \]

\[ - \beta \int_0^t e^{-\mu(t-s)A} [C(u_{n+1}(s)) - C(u_n(s))] ds, \]

(2.5)

for all \( t \in [0, T] \). Once again using the estimates (1.6)–(1.10), we obtain

\[ \|v_{n+2}(t)\|_p \]

\[ \leq \|u_{n+2}(t) - u_{n+1}(t)\|_p \]

\[ \leq \int_0^t \|e^{-\mu(t-s)A} B(u_{n+1}(s)) - B(u_n(s))\|_p ds \]

\[ + \int_0^t \|e^{-\mu(t-s)A} B(u_n(s), u_{n+1}(s) - u_n(s))\|_p ds \]

\[ + \beta \int_0^t \|e^{-\mu(t-s)A} \int_0^1 C'(\theta u_{n+1}(s) + (1 - \theta)u_n(s))(u_{n+1}(s) - u_n(s)) d\theta \|_p ds. \]
From the inequality (2.7), we have
\[
\frac{1}{2} - \frac{d}{2p} \left( \| u_{n+1}(s) \|_p + \| u_n(s) \|_p \right) \| u_{n+1}(s) - u_n(s) \|_p ds
\]
\[
+ C \int_0^t (t-s) \frac{d(t-1)}{2p} \left( \| u_{n+1}(s) \|_p^{r-1} + \| u_n(s) \|_p^{r-1} \right) \| u_{n+1}(s) - u_n(s) \|_p ds
\]
\[
\leq C \frac{1}{2} \frac{d}{2p} \sup_{s \in [0,t]} \left( \| u_{n+1}(s) \|_p + \| u_n(s) \|_p \right) \sup_{s \in [0,t]} \| v_{n+1}(s) \|_p
\]
\[
+ C t \frac{1}{2} \frac{d}{2p} \left( \| u_{n+1}(s) \|_p^{r-1} + \| u_n(s) \|_p^{r-1} \right) \sup_{s \in [0,t]} \| v_{n+1}(s) \|_p.
\]
for all \( t \in [0, T] \). Therefore, we deduce that
\[
\sup_{t \in [0,T]} \| v_{n+2}(t) \|_p \leq C \left( K T \frac{1}{2} - \frac{d}{2p} + K^{r-1} T^{1-\frac{d(d-1)}{2p}} \right) \sup_{t \in [0,T]} \| v_{n+1}(t) \|_p
\]
\[
\leq C^{n+1} \left( K T \frac{1}{2} - \frac{d}{2p} + K^{r-1} T^{1-\frac{d(d-1)}{2p}} \right)^{n+1} \sup_{t \in [0,T]} \| v_1(t) \|_p
\]
\[
\leq 2 K C^{n+1} \left( K T \frac{1}{2} - \frac{d}{2p} + K^{r-1} T^{1-\frac{d(d-1)}{2p}} \right)^{n+1}, \quad n = 0, 1, 2, \ldots.
\]

Let us now consider the infinite series of the form
\[
u_0(t) + v_1(t) + v_2(t) + \cdots + v_n(t) + \cdots.
\]

The \( n^{th} \) partial sum of the series is \( u_n(t) \), that is,
\[
u_n(t) = u_0(t) + \sum_{m=0}^{n-1} v_{m+1}(t).
\]

Therefore, the sequence \( \{ u_n(t) \} \) converges if and only if the series (2.8) converges. From the inequality (2.7), we have
\[
\sup_{t \in [0,T]} \| u_0(t) \|_p + \sum_{m=0}^{\infty} \sup_{t \in [0,T]} \| v_{m+1}(t) \|_p
\]
\[
\leq \frac{K}{2} + \sum_{m=0}^{\infty} 2 K C^m \left( K T \frac{1}{2} - \frac{d}{2p} + K^{r-1} T^{1-\frac{d(d-1)}{2p}} \right)^m
\]
\[
= \frac{K}{2} + \frac{2K}{1 - C \left( K T \frac{1}{2} - \frac{d}{2p} + K^{r-1} T^{1-\frac{d(d-1)}{2p}} \right)} < +\infty,
\]
provided
\[ C \left( K T^{\frac{1}{2}} - \frac{d}{2p} + K^{r-1} T^{1 - \frac{d(r-1)}{2p}} \right) < 1. \]

Thus, we can choose a time \( 0 < T_* < T \) in such a way that the above condition is satisfied. Therefore the series (2.8) converges uniformly in \([0, T_*]\) and we denote the sum of the series by \( u(t) \). Then, the relation (2.9) provides
\[ u(t) = \lim_{n \to \infty} u_n(t). \]

The uniform convergence of \( u_n(t) \) to \( u(t) \) and the continuity of the operator \( B(\cdot) + \beta C(\cdot) \) gives us
\[ u(t) = u_0 + G(u)(t), \]

which is a mild solution to the problem (1.3) in the interval \([0, T_*]\). The continuity of the function \( u(\cdot) \) follows from the uniform convergence and the continuity of the sequence \( \{u_n(\cdot)\}_{n=0}^\infty \).

**Step II. Uniqueness:** Let us now show the uniqueness. Let \( u_1(\cdot) \) and \( u_2(\cdot) \) be two local mild solutions of the problem (1.3). Then \( u = u_1 - u_2 \) satisfies:
\[ u(t) = -\int_0^t e^{-\mu(t-s)A}[B(u_1(s)) - B(u_2(s))]ds - \beta \int_0^t e^{-\mu(t-s)A}[C(u_1(s)) - C(u_2(s))]ds. \quad (2.11) \]

A calculation similar to (2.6) yields
\[ \|u(t)\|_p \leq C T_*^{\frac{1}{2}} - \frac{d}{2p} \sup_{t \in [0, T_*]} \left( \|u_1(s)\|_p + \|u_2(s)\|_p \right) \sup_{t \in [0, T_*]} \|u(s)\|_p 
\]
\[ + C T_*^{1 - \frac{d(r-1)}{2p}} \sup_{t \in [0, T_*]} \left( \|u_1(s)\|_p^{r-1} + \|u_2(s)\|_p^{r-1} \right) \sup_{t \in [0, T_*]} \|u(s)\|_p \]
\[ \leq C \left( K T_*^{\frac{1}{2}} - \frac{d}{2p} + K^{r-1} T_*^{1 - \frac{d(r-1)}{2p}} \right) \sup_{t \in [0, T_*]} \|u(s)\|_p, \quad (2.12) \]

for all \( t \in [0, T_*] \). One can choose a \( T_* \) such that
\[ C \left( K T_*^{\frac{1}{2}} - \frac{d}{2p} + K^{r-1} T_*^{1 - \frac{d(r-1)}{2p}} \right) < 1 \]
and hence the uniqueness of \( u \in C([0, T_*]; \mathbb{P}_p) \) follows.

**Step III. Global existence:** For \( 3 \leq r < \infty \) \((4\beta\mu \geq 1 \text{ for } r = 3)\) and \( f \in L^\infty(0, T; \mathbb{P}_p) \), one can show that the \( u(\cdot) \) is a global mild solution. Let \( \mathcal{V} := \{ u \in C^\infty_0(\mathbb{R}^d) : \text{div } u = 0 \} \). Note that \( \mathcal{V} \) is dense in \( \mathbb{P}_p \) and \( \mathbb{G}_{k,p}, k \geq 0, p \in [2, \infty) \). Since \( \mathbb{G}_{2,2} \) is a separable Hilbert space and \( \mathbb{G}_{2,2} \) is dense in \( \mathbb{H} \), there exists a set \( \{w_1, w_2, \cdots, w_n, \cdots\} \subset \mathbb{G}_{2,2} \) which is a complete orthonormal basis of \( \mathbb{H} \). Let
$\mathbb{H}_n$ be the $n$-dimensional subspace of $\mathbb{H}$ defined by $\mathbb{H}_n = \text{span}\{w_1, w_2, \cdots, w_n\}$ with the norm inherited from $\mathbb{H}$. Let us denote the orthogonal projection from $\mathbb{H}$ onto $\mathbb{H}_n$ by $P_n$, that is, $h_n = P_n h = \sum_{j=1}^n (h, w_j) w_j$, for $h \in \mathbb{H}$. Remember that an application of Sobolev’s inequality yields that the embedding of $\mathbb{G}_{2, 2} \subset \mathbb{J}_p$ is continuous for all $2 \leq p < \infty$, so that $\|P_n \|_{\mathcal{L}(\mathbb{J}_p)} \leq C \|P_n \|_{\mathcal{L}(\mathbb{G}_{2, 2})} \leq C$ and $\|P_n - I\|_{\mathcal{L}(\mathbb{J}_p)} \leq C \|P_n - I\|_{\mathcal{L}(\mathbb{G}_{2, 2})} \to 0$ as $n \to \infty$. Let us consider the following approximate equations for the system (1.3) on the finite dimensional space $\mathbb{H}_n$:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{du^n(t)}{dt} + \mu A u^n(t) + B_n(u^n(t)) + \beta C_n(u^n(t)) = \mathcal{P} f^n(t), \quad t \in (0, T],
\end{array} \right.
\end{align*}
$$

where $B_n(\cdot) = P_n B(\cdot)$ and $C_n = P_n C(\cdot)$. Note that $\|x_n - x\|_p, \|f_n - f\|_p \to 0$ as $n \to \infty$. There exists a unique solution $u^n \in C([0, T^*]; \mathbb{H}_n)$ to the system (2.13), which can be represented as

$$
\begin{align*}
u^n(t) &= e^{-\mu t A} x_n - \int_0^t e^{-\mu (t-s) A} [B_n(u^n(s)) + \beta C_n(u^n(s))] ds + \int_0^t e^{-\mu (t-s) A} \mathcal{P} f_n(s) ds,
\end{align*}
$$

for all $t \in [0, T^*]$. By using a calculation similar to (2.12), one can easily show that

$$\sup_{t \in [0, T^* \wedge T_n]} \|\nu^n(t) - u(t)\|_p \to 0 \quad \text{as} \quad n \to \infty. \quad (2.15)$$

Remember that $\Delta$ and $\mathcal{P}$ commutes. Taking the inner product with $|\nu^n|^p - 2 \nu^n$, for $\frac{d(r-1)}{2} < p < \infty$ to the first equation in (2.13), we find

$$
\begin{align*}
\frac{1}{p} \frac{d}{dt} \|\nu^n(t)\|_p^p + \mu \|\nu^n(t)\|_p^{p-2} \nabla \nu^n(t) \cdot \nabla \nu^n(t) + 4 \mu \left( \frac{p-2}{p^2} \right) \|\nabla \nu^n(t)\|_2^2 &+ \beta \|\nu^n(t)\|_{r+p-1}^{r+p-1} \\
&= - (B(\nu^n(t), |\nu^n(t)|^{p-2} \nu^n(t))) + (\mathcal{P} f_n(t), |\nu^n(t)|^{p-2} \nu^n(t)),
\end{align*}
$$

for a.e. $t \in [0, T]$. Using Hölder’s, interpolation and Young’s inequalities, we estimate $(B(\nu^n), |\nu^n|^{p-2} \nu^n)$ as

$$
|B(\nu^n), |\nu^n|^{p-2} \nu^n)| \\
\leq \|\nu^n\|_p^{p-2} \|\nabla \nu^n\|_2 \|\nu^n\|_p^{p+2} \\
\leq \|\nu^n\|_p^{p-2} \|\nabla \nu^n\|_2 \|\nu^n\|_{r+p-1}^{r+p-1} \|\nu^n\|_p^{p(r-3)/(2(r-1))}
$$
Thus, for all $t > 3$. Furthermore, we have

$$|(\mathcal{P} f_n, |u^n|^{p-2} u^n)| \leq \|f\|_p \|u^n\|_p \leq \frac{1}{p} \|u^n\|_p^p + \frac{p-1}{p} \|f\|_p^p. \quad (2.18)$$

Therefore using the above estimates in (2.16), we deduce that

$$\|u^n(t)\|_p^p + \frac{\mu_p}{2} \int_0^t \|u^n(s)|^{\frac{p-2}{2}} \nabla u^n(s)\|_2^2 ds + \frac{\beta_p}{2} \int_0^t \|u^n(s)\|_{r+p-1}^{r+p-1} ds$$

$$\leq \|x_n\|_p^p + \left[1 + \frac{p}{2\mu} \left(\frac{r-3}{r-1}\right) \left(\frac{2}{\beta\mu(r-1)}\right)^\frac{2}{r-3}\right] \int_0^t \|u^n(s)\|_p^p ds$$

$$+ (p-1) \int_0^t \|f(s)\|_p^p ds, \quad (2.19)$$

for all $t \in [0, T]$. An application of Gronwall’s inequality in (2.19) yields

$$\sup_{t \in [0,T]} \|u^n(t)\|_p^p \leq \left\{\|x\|_p^p + \frac{p-1}{p} \int_0^T \|f(s)\|_p^p ds\right\} e^{(\eta_p+1)T}, \quad (2.20)$$

where

$$\eta_p = \frac{p}{2\mu} \left(\frac{r-3}{r-1}\right) \left(\frac{2}{\beta\mu(r-1)}\right)^\frac{2}{r-3}. \quad (2.21)$$

Thus, for $r > 3$ and $\frac{d(e-1)}{2} < p < \infty$ and $f \in L^p(0, T; \|\cdot\|_p)$, the time $T^*$ can be extended to $T$, so that $u^{n^*} \in C([0, T]; \|\cdot\|_p^p)$. Since $u^n \to u$ in $C([0, T^* \wedge T_n]; \|\cdot\|_p)$, one can easily obtain that $u \in C([0, T]; \|\cdot\|_p)$. For $r = 3$, we estimate $(B(u^n), |u^n|^{p-2} u^n)$ as

$$|(B(u^n), |u^n|^{p-2} u^n)| \leq \|u^n\|^{\frac{p-2}{2}} \nabla u^n\|_2\|u^n\|_{p+2}^{\frac{p+2}{2}}$$

$$\leq \theta \mu \|u^n\|^{\frac{p-2}{2}} \nabla u^n\|_2^2 + \frac{1}{4\theta \mu} \|u^n\|_{p+2}^{p+2}, \quad (2.22)$$
for $0 < \theta \leq 1$. Thus, from (2.16), we get

$$
\|u^n(t)\|_p^p + 2\mu(1 - \theta) \int_0^t \|\nabla u^n(s)\|_2^2 \, ds + 2(\beta - \frac{1}{4\theta \mu}) \int_0^t \|u^n(s)\|_p^{p+2} \, ds
\leq \|x\|_p^p + (p - 1) \int_0^t \|f(s)\|_p \, ds.
$$

(2.23)

For $4\beta \mu \geq 1$ and $d < p < \infty$, we obtain the required result.

Let us now consider the case $d = 2$ and $r \in [1, 3]$. Taking the inner product with $u^n(\cdot)$ to the first equation in (2.13), we find

$$
\frac{1}{2} \frac{d}{dt} \|u^n(t)\|_2^2 + \mu \|\nabla u^n(t)\|_2^2 + \beta \|u^n(t)\|_{r+1}^{r+1}
= (P f_n(t), u^n(t)) \leq \|f_n(t)\|_2 \|u^n(t)\|_2 \leq \frac{1}{2} \|f(t)\|_2^2 + \frac{1}{2} \|u^n(t)\|_2^2,
$$

(2.24)

for a.e. $t \in [0, T]$. Thus it is immediate that

$$
\sup_{t \in [0, T]} \|u^n(t)\|_2^2 + 2\mu \int_0^T \|\nabla u^n(t)\|_2^2 \, dt + 2\beta \int_0^T \|u^n(t)\|_{r+1}^{r+1} \, dt
\leq \left\{ \|x\|_2^2 + \int_0^T \|f(t)\|_2^2 \, dt \right\} e^T.
$$

(2.25)

Therefore, we have $u^n \in L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{G}_{1,2}) \cap L^{r+1}(0, T; \mathbb{J}_{r+1})$. By using Sobolev’s embedding, it is clear that $L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{G}_{1,2}) \subset L^{r+1}(0, T; \mathbb{J}_{r+1})$, so that $u^n \in L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{G}_{1,2})$. An application of Gagliardo-Nirenberg’s interpolation inequality yields $\|u^n\|_p \leq C \|\nabla u^n\|_2^{\frac{p-2}{p}} \|u^n\|_2^{\frac{2}{p}}$, so that

$$
\int_0^T \|u^n(t)\|_p^{\frac{2p}{p-2}} \, dt \leq C \sup_{t \in [0, T]} \|u^n(t)\|_2^{\frac{4}{2p-2}} \int_0^T \|\nabla u^n(t)\|_2^2 \, dt,
$$

and $L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{G}_{1,2}) \subset L^{\frac{2p}{p+2}}(0, T; \mathbb{J}_p)$, for $2 < p < \infty$. Therefore, we get $u^n \in L^{\frac{2p}{p+2}}(0, T; \mathbb{J}_p)$ and since $u^n \to u$ in $C([0, T^* \wedge T_*]; \mathbb{J}_p)$, one can conclude that $u \in L^{\frac{2p}{p+2}}(0, T; \mathbb{J}_p)$, for $2 < p < \infty$.

Finally, we consider the case $d = 3$ and $r \in [1, 3]$. We have from (2.25) that $u^n \in J := L^{\infty}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{G}_{1,2}) \cap L^{r+1}(0, T; \mathbb{J}_{r+1})$. Once again using Gagliardo-Nirenberg’s interpolation inequality, we get $\|u^n\|_p \leq C \|\nabla u^n\|_2^{\frac{3(p-2)}{2p}} \|u^n\|_2^{\frac{6-p}{2p}}$, for $2 \leq p \leq 6$. Thus, we have

$$
\int_0^T \|u^n(t)\|_p^{\frac{4p}{p(p-2)}} \, dt \leq C \sup_{t \in [0, T]} \|u^n(t)\|_2^{\frac{2(p-2)}{p(p-2)}} \int_0^T \|\nabla u^n(t)\|_2^2 \, dt,
$$
so that \( \mathcal{J} \subset L^\infty(0, T; H) \cap L^2(0, T; G_{1,2}) \subset L^{4p/(p-2)}(0, T; \mathbb{J}_p) \), for \( 2 < p \leq 6 \). Therefore, we have \( u^n \in L^{\frac{4p}{p-2}}(0, T; \mathbb{J}_p) \) and since \( u^n \to u \) in \( C([0, T^* \wedge T_*]; \mathbb{J}_p) \), we arrive at \( u \in L^{\frac{4p}{p-2}}(0, T; \mathbb{J}_p) \), for \( 3 < p \leq 6 \).

\[ \blacksquare \]

3 Existence and uniqueness of stochastic CBF equations

This section is devoted for establishing the existence and uniqueness of a local and global pathwise mild solution to the system (1.11) (Theorem 1.3). We use the contraction mapping principle to obtain the local existence and uniqueness result and stopping times arguments to establish the global existence.

3.1 The linear problem

For any \( q \in [2, \infty) \), we know that \( e^{-\mu tA} \) is a \( C_0 \)-contraction semigroup on \( L^q(\mathbb{R}^d) \), and \( L^q(\mathbb{R}^d) \) is a martingale type-2 Banach space and also a 2-smooth Banach space. Let us now consider the stochastic Stokes equation:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\text{d}w(t) + A w(t) \text{d}t = \Phi \text{d}W(t) + \int_Z \xi(t, z) \tilde{\pi} (\text{d}r, \text{d}z), \\
w(0) = 0,
\end{array} \right.
\end{aligned}
\]

(3.1)

where \( \Phi \in \gamma(H; \mathbb{J}_q) \) and \( \gamma \in L^2(0, T; L^2(\mathbb{J}_q)) \cap L^\ell(0, T; L^\ell(\mathbb{J}_q)) \), for some \( \ell, q \in [2, \infty) \). Making use of Theorem 3.6, [40], the unique solution of the problem (3.1) with paths in \( L^\infty(0, T; \mathbb{J}_q), q \in [2, \infty), \mathbb{P}\text{-a.s.} \), can be represented by the stochastic convolution

\[
w(t) = \int_0^t e^{-\mu(t-s)A} \Phi \text{d}W(s) + \int_0^t \int_Z e^{-\mu(t-s)A} \xi(s-, z) \tilde{\pi} (\text{d}s, \text{d}z),
\]

(3.2)

for all \( t \in [0, T] \), and (3.2) has an \( \mathcal{F}_t \)-adapted càdlàg modification such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|w(t)\|_q^\ell \right]
\leq C \left( \|\Phi\|_{\gamma(H; \mathbb{J}_q)} T + \left( \int_0^T \int_Z \|\xi(t, z)\|_{L^2(\mathbb{J}_q)} \text{d}z \text{d}t \right)^{\frac{\ell}{2}} + \int_0^T \int_Z \|\xi(t, z)\|_{L^\ell(\mathbb{J}_q)} \text{d}z \text{d}t \right),
\]

(3.3)

and \( \sup_{0 \leq t \leq T} \|w(t)\|_q < \infty, \mathbb{P}\text{-a.s.} \).
3.2 The nonlinear problem

Let us now establish the existence of a local as well as global pathwise mild solution to the stochastic CBF system (1.11).

Let us set \( \mathbf{v} = \mathbf{u} - \mathbf{w} \). Then, \( \mathbf{v}(\cdot) \) satisfies the following system \( \mathbb{P}\)-a.s.:

\[
\frac{d\mathbf{v}(t)}{dt} + [A\mathbf{v}(t) + B(\mathbf{v}(t) + \mathbf{w}(t)) + \beta C(\mathbf{v}(t) + \mathbf{w}(t))] = 0, \quad t \in (0, T],
\]

\[
\mathbf{v}(0) = \mathbf{x}.
\]

Note that for each fixed \( \omega \in \Omega \), (3.4) is a deterministic system. The operator system (3.4) can be transformed into an nonlinear integral equation as

\[
\mathbf{v}(t) = e^{-\mu t}Ax - \int_0^t e^{-\mu(t-s)A}[B(\mathbf{v}(s) + \mathbf{w}(s)) + \beta C(\mathbf{v}(s) + \mathbf{w}(s))]ds,
\]

for all \( t \in [0, T] \). As in the case of deterministic CBF equations, we obtain the existence of a unique local mild solution to the system (3.4) by using the contraction mapping principle in the space \( C([0, \tilde{T}]; \mathcal{J}_p) \), \( \mathbb{P}\)-a.s., for \( \max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty \), where \( 0 < \tilde{T} < T \) is a random time. We mainly follow the works [8,30], etc to obtain main results of this section. Let us set

\[
\Sigma(M, \tilde{T}) = \{ \mathbf{v} \in C([0, \tilde{T}]; \mathcal{J}_p) : \| \mathbf{v}(t) \|_p \leq M, \quad \mathbb{P}\text{-a.s., for all } t \in [0, \tilde{T}] \}.
\]

Clearly the space \( \Sigma(M, \tilde{T}) \) equipped with supremum topology is a complete metric space.

**Theorem 3.1** For \( \max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty \), let \( \Phi \in \gamma(\mathcal{H}; \mathcal{J}_p) \) and \( \xi \in L^2(0, T; L^2_\lambda(\mathcal{J}_p)) \cap L^p(0, T; \mathcal{J}_p), \) and let the \( \mathcal{F}_0\)-measurable initial data \( \mathbf{x} \in \mathcal{J}_p \), \( \mathbb{P}\text{-a.s.} \) be given. Then, for \( M > \| x \|_p \), there exists a random time \( \tilde{T} \) such that (3.4) has a unique mild solution in \( \Sigma(M, \tilde{T}) \).

For \( r \geq 3 \) (\( \beta \mu > 4 \) for \( r = 3 \)), \( \Phi \in \gamma(\mathcal{H}; \mathcal{J}_p) \cap \gamma(\mathcal{H}; \mathcal{J}_{p+r-1}), \) \( \xi \in L^2(0, T; L^2_\lambda(\mathcal{J}_p)) \cap L^p(0, T; L^p_\lambda(\mathcal{J}_p)) \cap L^{p+r-1}(0, T; L^{p+r-1}_\lambda(\mathcal{J}_{p+r-1})) \) and \( \frac{d(r-1)}{2} < p < \infty \), the solution is global in time with \( \mathbf{v} \in C([0, T]; \mathcal{J}_p), \) \( \mathbb{P}\text{-a.s.} \).

For \( d = 2, 3, \) \( r \in [1, 3] \), \( \Phi \in \gamma(\mathcal{H}; \mathcal{J}_p) \cap \gamma(\mathcal{H}; \mathcal{J}_{r+1}) \cap \gamma(\mathcal{H}; \mathcal{J}_p), \) and \( \xi \in L^2(0, T; L^2_\lambda(\mathcal{J}_p)) \cap L^{p+r-1}(0, T; L^{p+r-1}_\lambda(\mathcal{J}_{p+r-1})) \) and \( \frac{d(r-1)}{2} < p < \infty \), the solution is global with \( \mathbf{v} \in L^{\frac{2p}{2p-(r+1)p}}(0, T; \mathcal{J}_p), \) \( 2 < p < \infty \), for \( d = 2 \) and \( \mathbf{v} \in L^{\frac{4p}{2p-(r+1)p}}(0, T; \mathcal{J}_p), \) \( 3 < p \leq 6 \), for \( d = 3 \).
Proof We prove the Theorem in the following steps:

**Step I. Existence and Uniqueness:** Let us take any \( v \in \Sigma(M, \tilde{T}) \) and define \( y(t) = F(v)(t) \) by

\[
y(t) = F(v)(t) := e^{-\mu t}A x - \int_0^t e^{-\mu(t-s)}A[B(v(s) + w(s)) + \beta C(v(s) + w(s))]ds,
\]

(3.7)

for all \( t \in [0, \tilde{T}] \). Let us first establish that \( F : \Sigma(M, \tilde{T}) \rightarrow \Sigma(M, \tilde{T}) \). Making use of the estimates (1.6)–(1.10), we find

\[
\|y(t)\|_p \leq \|e^{-\mu t}A x\|_p + \int_0^t \|e^{-\mu(t-s)}A[B(v(s) + w(s)) + \beta C(v(s) + w(s))]\|_p ds \\
\leq \|x\|_p + C \int_0^t (t-s)^{-\left(\frac{d}{2p} + \frac{1}{2}\right)}\|v(s) + w(s)\|_p^2 ds \\
+ C \int_0^t (t-s)^{-\frac{d(r-1)}{2p}}\|v(s) + w(s)\|_p ds \\
\leq \|x\|_p + C T^{-\frac{1}{2}} \sup_{s \in [0,t]}\|v(s) + w(s)\|_p^2 + C T^{-\frac{1}{2}} \sup_{s \in [0,t]}\|v(s) + w(s)\|_p^r \\
+ C T^{-\frac{1}{2}} \frac{d}{2p} \sup_{s \in [0,t]}\|v(s) + w(s)\|_p^r \\
\leq \|x\|_p + C T^{-\frac{1}{2}} \frac{d}{2p} (M^2 + \kappa_p^2) + C T^{-\frac{1}{2}} \frac{d(r-1)}{2p} (M^r + \kappa_p^r),
\]

(3.8)

\( \mathbb{P}\)-a.s., for all \( t \in [0, \tilde{T}] \), where

\[
\kappa_p = \sup_{t \in [0,T]} \|w(t)\|_p.
\]

Now, since \( M > \|x\|_p, \mathbb{P}\)-a.s., and \( \max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty \), one can choose \( 0 < \tilde{T} < T \) in such a way that \( \|y(t)\|_p \leq M \), for all \( t \in [0, \tilde{T}] \), provided

\[
\|x\|_p + C \tilde{T}^{-\frac{1}{2}} \frac{d}{2p} (M^2 + \kappa_p^2) + C \tilde{T}^{-\frac{1}{2}} \frac{d(r-1)}{2p} (M^r + \kappa_p^r) \leq M.
\]

Therefore \( y \in \Sigma(M, \tilde{T}) \).

Our next aim is to show that \( F : \Sigma(M, \tilde{T}) \rightarrow \Sigma(M, \tilde{T}) \) is a contraction. Let us consider \( v_1, v_2 \in \Sigma(M, \tilde{T}) \) and set \( y_i(t) = F(v_i)(t) \), for all \( t \in [0, \tilde{T}] \) and \( i \in \{1, 2\} \) and \( y = y_1 - y_2 \). Then \( y(\cdot) \) satisfies
\[
y(t) = - \int_0^t e^{-\mu(t-s)A} \left[ B(v_1(s) + w(s)) - B(v_2 + w(s)) \right] ds
\]
\[
- \int_0^t e^{-\mu(t-s)A} \beta C(v_1(s) + w(s)) - C(v_2(s) + w(s)) ds,
\]

\(\mathbb{P}\)-a.s., for all \(t \in [0, \tilde{T}]\). Once again using the bilinearity of \(B(\cdot)\) and Taylor’s formula, we find

\[
\|y(t)\|_p \leq \int_0^t \left\| e^{-\mu(t-s)A} B(v_1(s) - v_2(s), v_1(s) + w(s)) \right\|_p ds
\]
\[
+ \int_0^t \left\| e^{-\mu(t-s)A} B(v_2(s) + w(s), v_1(s) - v_2(s)) \right\|_p ds
\]
\[
+ \beta \int_0^t \left\| e^{-\mu(t-s)A} \int_0^1 C'(\theta v_1(s) + (1 - \theta)v_2(s) + w(s))(v_1(s) - v_2(s)) d\theta \right\|_p ds
\]
\[
\leq C \int_0^t (t - s)^{-\left(\frac{1}{2} + \frac{d}{2p}\right)} \|v_1(s) + w(s)\|_p \|v_1(s) - v_2(s)\|_p ds
\]
\[
+ C \int_0^t (t - s)^{-\left(\frac{1}{2} + \frac{d}{2p}\right)} \|v_2(s) + w(s)\|_p \|v_1(s) - v_2(s)\|_p ds
\]
\[
+ C \int_0^t (t - s)^{-\frac{(r-1)}{2p}} \left( \|v_1(s)\|_p + \|v_2(s)\|_p + \|w(s)\|_p \right)^{r-1} \|v_1(s) - v_2(s)\|_p ds
\]
\[
\leq C t^{1 - \frac{d}{2p}} \left( \sup_{s \in [0, t]} \left( \|v_1(s)\|_p + \|v_2(s)\|_p + \|w(s)\|_p \right) \sup_{s \in [0, t]} \|y(s)\|_p \right)
\]
\[
+ C t^{1 - \frac{(r-1)}{2p}} \left( \sup_{s \in [0, t]} \left( \|v_1(s)\|_p^{r-1} + \|v_2(s)\|_p^{r-1} + \|w(s)\|_p^{r-1} \right) \sup_{s \in [0, t]} \|y(s)\|_p \right)
\]
\[
\leq C \left( \tilde{T}^{\frac{1}{2} - \frac{d}{2p}} (M + \kappa_p) + \tilde{T}^{1 - \frac{(r-1)}{2p}} (M^{r-1} + \kappa_p^{r-1}) \right) \sup_{t \in [0, \tilde{T}]} \|y(t)\|_p, \tag{3.9}
\]

for all \(t \in [0, \tilde{T}]\). For \(\max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty\), one can choose \(0 < \tilde{T} < T\) in such a way that

\[
C \left( \tilde{T}^{\frac{1}{2} - \frac{d}{2p}} (M + \kappa_p) + \tilde{T}^{1 - \frac{(r-1)}{2p}} (M^{r-1} + \kappa_p^{r-1}) \right) < 1.
\]

Hence, \(F\) is a strict contraction in \(\Sigma(M, \tilde{T})\) and an application of the contraction mapping principle provides the existence of mild solution to the problem (3.4) up to a time \(0 < \tilde{T} < T\). Uniqueness follows form the representation (3.5) and (3.9).

Remember that the results obtained above is valid \(\mathbb{P}\)-a.s. for \(\omega \in \Omega\), in particular \(\kappa_p\) and \(\tilde{T}\) depends on \(\omega\). Let us now prove that \(\tilde{T} = T\), \(\mathbb{P}\)-a.s., and hence we remove the dependence on \(\omega\) for the time interval up to which the solution exists.
Step II. Global Existence: As in the proof of Theorem 1.1, we consider a finite dimensional approximation of the system (3.4). Let us take the following approximate equations of the system (1.3) on the finite dimensional space $\mathbb{H}_n$:

$$
\begin{align*}
\frac{dv^n(t)}{dt} + [Av^n(t) + B_n(v^n(t) + w_n(t))] + \beta C_n(v^n(t) + w_n(t)) = 0, \\
v^n(0) = x_n,
\end{align*}
$$

where $w_n = P_n w$. It can be easily shown that $\sup_{t \in [0, T]} \|w_n(t) - w(t)\|_q \to 0$, $\mathbb{P}$-a.s. as $n \to \infty$, for $q = p, p + r - 1, p + 2, r + 1$, under appropriate assumptions on the noise coefficient. As in the previous discussion, one can show that the system (3.10) has a unique local solution

$$
v^n(t) = e^{-\mu t A}x_n - \int_0^t e^{-\mu(t-s)A}[B_n(v^n(s) + w_n(s)) + \beta C_n(u^n(s) + w_n(s))]ds,
$$

with $v^n \in C([0, \hat{T}]; \mathbb{H}_n)$, for some $0 < \hat{T} < T$. Moreover, one can establish that $\sup_{t \in [0, \hat{T} \wedge \hat{T}]} \|v^n(t) - v(t)\|_p \to 0$ as $n \to \infty$ (see (3.9)).

Case I: $d = 2, 3, r > 3$. Taking the inner product with $\|v^n\|^{p-2}v^n$ to the first equation in (3.4), we find

$$
\begin{align*}
\frac{1}{p} \frac{d}{dt} \|v^n(t)\|_p^p + \mu \|v^n(t)\|_2^2 + 4\mu \left(\frac{p-2}{p^2}\right) \|\nabla v^n(t)\|_2^2 & = -\langle B(v^n(t) + w_n(t)), |v^n(t)|^{p-2}v^n(t) \rangle \\
& \quad - \beta(C(v^n(t) + w_n(t)), |v^n(t)|^{p-2}v^n(t)),
\end{align*}
$$

for a.e. $t \in [0, T]$. We estimate $-\beta(C(v^n + w_n), |v^n|^{p-2}v^n)$ using Taylor’s formula, Hölder’s and Young’s inequalities as

$$
\begin{align*}
- \beta(C(v^n + w_n), |v^n|^{p-2}v^n) & = - \beta(C(v^n + w_n) - C(v^n), |v^n|^{p-2}v^n) - \beta(C(v^n), |v^n|^{p-2}v^n) \\
& = - \beta \left( \int_0^1 C'(\theta v^n + (1 - \theta)w_n) d\theta w_n, |v^n|^{p-2}v^n \right) - \beta \|v^n\|_{r+p-1}^{r+p-1} \\
& \leq r\beta \langle |v^n| + |w_n| \rangle^{r-1} |w_n|, \|v^n\|^{p-1} - \beta \|v^n\|_{r+p-1}^{r+p-1} \\
& \leq C \beta \left( \|w_n\|_{r+p-1}^{r+p-1} + \|w_n\|_{r+p-1}^{r+p-1} \|v^n\|^{p-1} \right) - \beta \|v^n\|_{r+p-1}^{r+p-1} \\
& \leq - \frac{3\beta}{4} |v^n|^{r+p-1} + C \|w_n\|_{r+p-1}^{r+p-1}.
\end{align*}
$$

(3.12)
The term \(-(B(v^n + w_n), |v^n|^{|p-2}v^n)\) can be written as

\[-(B(v^n + w_n), |v^n|^{|p-2}v^n)\]

\[= -(B(v^n, v^n), |v^n|^{|p-2}v^n) - (B(v^n, w_n), |v^n|^{|p-2}v^n) - (B(w_n, v^n), |v^n|^{|p-2}v^n) - (B(w_n, w_n), |v^n|^{|p-2}v^n) = \sum_{i=1}^{4} I_i.\]  

(3.13)

We estimate \(I_1\) in a similar way as in (2.17) as

\[I_1 \leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + C ||v^n||_p^p.\]  

(3.14)

Remember that \(\mathcal{B}\) and \(\nabla\) commutes. Using an integration by parts, Hölder’s and Young’s inequalities, we estimate \(I_2\) as

\[I_2 \leq (p - 1) ||v^n||_p^{\frac{p-2}{2}} \nabla v^n||_2 ||v^n||_p^{\frac{p-2}{2}} w_n v^n||_2 \]

\[\leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + C ||v^n||_p^p \]

\[\leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + C ||w_n||_{r+p-1}^p \]

\[\leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + C ||w_n||_{r+p-1}^p + C \]

\[\leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + \frac{1}{p} ||v^n||_p^p + C ||w_n||_{r+p-1}^p + C \]

\[\leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + \frac{1}{p} ||v^n||_p^p + \frac{1}{p} ||w_n||_{r+p-1}^p + C ||w_n||_p^p,\]

(3.15)

for \(r > 3\). A calculation similar to (3.15) yields

\[I_3 \leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + \frac{1}{p} ||v^n||_p^p + \frac{1}{p} ||w_n||_{r+p-1}^p + C ||w_n||_p^p,\]

(3.16)

\[I_4 \leq \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{2(p - 1)^2}{\mu} ||v^n||_{r+p-1}^p + \frac{\mu}{8} ||v^n||_2^{\frac{p-2}{2}} \nabla v^n||^2 + \frac{\beta}{8} ||v^n||_{r+p-1}^p + \frac{1}{p} ||v^n||_p^p + \frac{1}{p} ||w_n||_{r+p-1}^p + C ||w_n||_p^p,\]

(3.17)
for $r > 3$. Combining (3.15)–(3.17) and substituting it in (3.14), we get

$$\begin{align*}
- (B(v^n + w_n), |v^n|^{p-2}v^n) & \leq \frac{\mu}{2} \| |v^n|^{\frac{p-2}{2}} \nabla v^n \|_2^2 + \frac{\beta}{2} \| v^n \|_{r+p-1}^{r+p-1} + C \| v^n \|_p^p + C \| w \|_{r+p-1}^{r+p-1} + C \| w \|_p^p.
\end{align*}$$

(3.18)

Using (3.12) and (3.18) in (3.11) and then integrating from 0 to $t$, we deduce that

$$\begin{align*}
\| v^n(t) \|_p^p + \frac{\mu P}{2} \int_0^t \| v^n(s) \|^{\frac{p-2}{2}} \nabla v^n(s) \|_2^2 ds + \frac{\beta P}{4} \int_0^t \| v^n(t) \|_{r+p-1}^{r+p-1} ds
\leq \| x_n \|_p^p + C \int_0^t \| w(s) \|_p^p ds + C \int_0^t \| w(s) \|_{r+p-1}^{r+p-1} ds + \int_0^t \| w(s) \|_{r+p-1}^{r+p-1} ds,
\end{align*}$$

for all $t \in [0, T]$. An application of Gronwall’s inequality in (3.19) yields

$$\begin{align*}
\sup_{0 \leq t \leq T} \| v^n(t) \|_p^p + \frac{\mu P}{2} \int_0^T \| v^n(t) \|^{\frac{p-2}{2}} \nabla v^n(t) \|_2^2 dt + \frac{\beta P}{4} \int_0^T \| v^n(t) \|_{r+p-1}^{r+p-1} dt
\leq \left\{ \| x \|_p^p + C \int_0^T \| w(t) \|_p^p dt + \int_0^T \| w(t) \|_{r+p-1}^{r+p-1} dt \right\} e^{CT}.
\end{align*}$$

(3.20)

for $r > 3$. Since $w \in L^\infty(0, T; \|_{r+p-1}, \mathbb{P})$, we obtain $v^n \in C([0, T]; \|_{r+p-1})$. Since $v^n \to v$ in $C([0, T]; \|_{r+p-1})$, one can conclude that $v \in C([0, T]; \|_{r+p-1})$.

Case II: $d = 2, 3, r = 3, \beta \mu > 4$. For $r = 3$ and $\beta \mu > 4$, a calculation similar to (2.22) provides

$$\begin{align*}
I_1 & \leq \theta_1 \mu \| |v^n|^{\frac{p-2}{2}} \nabla v^n \|_2^2 + \frac{1}{4 \theta_1 \mu} \| v^n \|_{p+2}^{p+2}, \\
I_2 & \leq \theta_2 \mu \| |v^n|^{\frac{p-2}{2}} \nabla v^n \|_2^2 + \frac{1}{4 \theta_2 \mu} \| v^n \|_{p+2}^{p+2} + C \| w \|_{p+2}^{p+2}, \\
I_3 & \leq \theta_3 \mu \| |v^n|^{\frac{p-2}{2}} \nabla v^n \|_2^2 + \frac{1}{4 \theta_3 \mu} \| v^n \|_{p+2}^{p+2} + C \| w \|_{p+2}^{p+2}, \\
I_4 & \leq \theta_4 \mu \| |v^n|^{\frac{p-2}{2}} \nabla v^n \|_2^2 + \frac{1}{4 \theta_4 \mu} \| v^n \|_{p+2}^{p+2} + C \| w \|_{p+2}^{p+2},
\end{align*}$$

for $0 < \theta_1, \theta_2, \theta_3, \theta_4 \leq 1$. We estimate $-\beta (C(v^n + w_n), |v^n|^{p-2}v^n)$ by applying a similar calculation as in (3.12) to get

$$\begin{align*}
-\beta (C(v^n + w_n), |v^n|^{p-2}v^n) \leq -\beta (1 - \theta_5) \| v^n \|_{p+2}^{p+2} + \frac{C}{\theta_5} \| w \|_{p+2}^{p+2},
\end{align*}$$

(3.21)
for $0 < \theta_5 \leq 1$. Thus, from (3.11), we have

$$
\|v^n(t)\|_p^p + p\mu\left(1 - (\theta_1 + \theta_2 + \theta_3 + \theta_4)\right) \int_0^t \|v^n(s)\|_2^{p-2} \nabla v^n(s) \cdot 2d_s
$$

$$
+ p \left[\beta - \left(\theta_5 + \frac{1}{4\theta_1\mu} + \frac{1}{4\theta_2\mu} + \frac{1}{4\theta_3\mu} + \frac{1}{4\theta_4\mu}\right)\right] \int_0^t \|v^n(s)\|_2^{p+2}ds
$$

$$
\leq \|x\|_p^p + C \int_0^t \|w(s)\|_2^{p+2}ds,
$$

(3.22)

for all $t \in [0, T]$. Thus, for $w \in L^\infty(0, T; \mathbb{P})$, $\mathbb{P}$-a.s., we obtain $v^n \in C([0, T]; \mathbb{P})$, which also implies $v \in C([0, T]; \mathbb{P})$.

**Case III:** $d = 2, r \in [1, 3]$. For $d = 2$ and $r \in [1, 3]$, taking the inner product with $v^n$ to the first equation in (3.4), we find

$$
\frac{1}{2} \frac{d}{dt} \|v^n(t)\|_2^2 + \mu \|\nabla v^n(t)\|_2^2 + \beta \|v^n(t)\|_{r+1}^{r+1}
$$

$$
= -(B(v^n(t) + v_n(t)), v^n(t)) - \beta (C(v^n(t) + v_n(t)) - C(v^n(t)), v^n(t))
$$

$$
= -(B(v^n(t), v_n(t)), v^n(t)) - (B(v_n(t), v^n(t)), v^n(t))
$$

$$
- (B(w_n(t), v_n(t)), v^n(t)) - \beta \left(\int_0^1 C'(\theta v^n(t) + (1-\theta)w_n(t))d\theta w_n(t), v^n(t)\right)
$$

$$
\leq 2 \|\nabla v^n(t)\|_2^2 \|w_n(t)\|_4 \|v^n(t)\|_4^4 + \|\nabla v^n(t)\|_2 \|w_n(t)\|_4^2
$$

$$
+ \beta r \left(\|v^n(t)\|_{r+1} + \|w_n(t)\|_{r+1}\right)^{r-1} \|w_n(t)\|_{r+1} \|v^n(t)\|_{r+1}
$$

$$
\leq \frac{\mu}{2} \|\nabla v^n(t)\|_2^2 + \frac{27}{2\mu^3} \|w_n(t)\|_4^4 \|v^n(t)\|_2^2 + \frac{\beta}{2} \|v^n(t)\|_{r+1}^{r+1} + C \|w_n(t)\|_{r+1}^{r+1},
$$

(3.23)

for a.e. $t \in [0, T]$, where we have used Taylor’s formula, Hölder’s Gagliardo-Nirenberg’s and Young’s inequalities. Integrating the above inequality from 0 to $t$, we find

$$
\|v^n(t)\|_2^2 + \mu \int_0^t \|\nabla v^n(s)\|_2^2ds + \beta \int_0^t \|v^n(s)\|_{r+1}^{r+1}ds
$$

$$
\leq \|x\|_2^2 + C \int_0^t \|w_n(s)\|_4^4 \|v^n(s)\|_2^2ds + C \int_0^t \|w_n(s)\|_{r+1}^{r+1}ds,
$$

(3.24)

for all $t \in [0, T]$. An application of Grownall’s inequality in (3.24) gives

$$
\sup_{t \in [0, T]} \|v^n(t)\|_2^2 + \mu \int_0^T \|\nabla v^n(s)\|_2^2ds + \beta \int_0^T \|v^n(s)\|_{r+1}^{r+1}ds
$$

$$
\leq \left\{\|x\|_2^2 + C \int_0^T \|w(t)\|_{r+1}^{r+1}dt\right\} \exp \left(C \int_0^T \|w(t)\|_4^4dt\right),
$$

(3.25)
Since \( \mathbf{w} \in L^\infty(0, T; \mathbb{J}_p \cap \mathbb{J}_4 \cap \mathbb{J}_{r+1}) \), \( \mathbb{P}\)-a.s., the discussion in the proof of Theorem 1.1 yields \( \mathbf{v}^n \in L^{\frac{2p}{p-2}}(0, T; \mathbb{J}_p) \), \( 2 < p < \infty \), which implies \( \mathbf{v} \in L^{\frac{2p}{p-2}}(0, T; \mathbb{J}_p) \), \( 2 < p < \infty \).

**Case IV:** \( d = 3, r \in [1, 3] \). For \( d = 3 \) and \( r \in [1, 3] \), we just need to change the estimate \(- (\mathbf{B}(\mathbf{v}^n, \mathbf{w}_n), \mathbf{v}^n) - (\mathbf{B}(\mathbf{w}_n, \mathbf{v}^n), \mathbf{v}^n) \) only. Using Hölder’s Gagliardo-Nirenberg’s and Young’s inequalities, we obtain

\[
(B(v^n, w_n) - (B(w_n, v^n), v^n))
\leq ||\nabla v^n||_2 ||v^n||_4 ||w_n||_4 \\
\leq \frac{1}{2} \mu ||\nabla v^n||_2^2 + \frac{1}{8} \left( \frac{7}{4\mu} \right)^7 ||w_n||^8 ||v^n||^2. \tag{3.26}
\]

Thus, one can establish a result similar to (3.25) as

\[
\sup_{t \in [0, T]} ||v^n(t)||_2^2 + \mu \int_0^T ||\nabla v^n(s)||_2^2 ds + \beta \int_0^T ||v^n(s)||_r^{r+1} ds \leq \left( ||x||_2^2 + CT \sup_{t \in [0, T]} ||w(t)||_r^{r+1} \right) \exp \left( C \sup_{t \in [0, T]} ||w(t)||_4^4 \right). \tag{3.27}
\]

Since \( \mathbf{w} \in L^\infty(0, T; \mathbb{J}_p \cap \mathbb{J}_4 \cap \mathbb{J}_{r+1}) \), \( \mathbb{P}\)-a.s., the discussion in the proof of Theorem 1.1 yields \( \mathbf{v}^n \in L^{\frac{4p}{3(p-2)}}(0, T; \mathbb{J}_p) \), \( 3 < p \leq 6 \), which also implies \( \mathbf{v} \in L^{\frac{4p}{3(p-2)}}(0, T; \mathbb{J}_p) \), \( 3 < p \leq 6 \).

Let us now prove Theorem 1.3 using Theorem 3.1.

**Proof of Theorem 1.3** Since \( \mathbf{u} = \mathbf{v} + \mathbf{w} \), the existence of a unique local mild solution follows from Theorem 3.1 and the fact that \( \mathbf{w} \) is the unique mild solution to (3.2). It is now left to prove the global existence only.

**Case I:** \( d = 2, 3, r \geq 3 \): We first consider the case \( d = 2, 3 \) and \( r > 3 \). The existence of a local mild solution up to a stopping time \( 0 < \tau \leq T \) and uniform bounds for the \( \mathbb{J}_p \)-norm in \([0, T]\) has been established in Theorem 3.1. For the approximate sequence \( \{\mathbf{u}^n\} \), let us now define a sequence of stopping times by

\[
\tau_m^n := \inf_{t \geq 0} \left\{ t : \|u^n(t)\|_p > m \right\}, \tag{3.28}
\]

for \( m \in \mathbb{N} \). Therefore, the mild solution for \( \mathbf{u}^n \) exists up to a time \( T \wedge \tau_m^n \) and \( \tau_m^n \rightarrow T \) as \( n \rightarrow \infty \), by using the estimate (3.20). Note that \( \tau_m^n \leq \tau_k^n \), whenever \( m \leq k \). Thus, \( \tau_m^n \) is an increasing sequence and let us define \( \tau^n_\infty := \lim_{m \rightarrow \infty} \tau_m^n \). Our aim is to show that \( \tau^n_\infty = T \), \( \mathbb{P}\)-a.s. Or in other words, one has to show that \( \mathbb{P} \{ \omega \in \Omega : \tau^n_\infty(\omega) < T \} = 0 \). Since \( v^n = u^n + w^n \), from (3.20), we infer that...
\[
E \left[ \sup_{t \in [0, T]} \| u^n(t) \|_p^p \right] \\
\leq \left\{ E \left[ \| x \|_p^p \right] + C T E \left[ \sup_{t \in [0, T]} \| w(t) \|_p^p \right] + C T E \left[ \sup_{t \in [0, T]} \| w(t) \|_{p^{r+1}}^{p^{r+1}-1} \right] \right\} e^{C T} \\
+ E \left[ \sup_{t \in [0, T]} \| w(t) \|_p^p \right] =: MT.
\]

(3.29)

From the stopping time definition given in (3.28), we have

\[
P \{ \omega \in \Omega : \tau_\infty^n(\omega) < T \} \leq \frac{1}{m^p} E \left[ \sup_{t \in [0, T]} \| u(t) \|_p^p \right] = \frac{MT}{m^p} \to 0 \text{ as } m \to \infty,
\]

(3.30)

and hence \( P \{ \omega \in \Omega : \tau_\infty^n(\omega) < T \} = 0 \). Thus, we have \( \tau_\infty^n = T \), \( \mathbb{P}\text{-a.s.} \), and since \( T \leq \tau_\infty^n \to T \) as \( n \to \infty \), we finally obtain that the mild solution for (1.11) exists globally. The case of \( r = 3 \) and \( \beta \mu > 4 \) can be established in a similar way.

**Case II:** \( d = 2, 3, r \in [1, 3] \). Let us now define a sequence of stopping times by

\[
\tilde{\tau}_m^n := \inf_{t \geq 0} \left\{ t : \int_0^t \| u^n(s) \|_{p^{r-2}}^{2p^{r-2}} ds > m \right\},
\]

(3.31)

for \( m \in \mathbb{N} \). For \( d = 2 \) and \( r \in [1, 3] \), from (3.25), we infer that

\[
\int_0^T \| u^n(t) \|_{p^{r-2}}^{2p^{r-2}} dt \leq C \int_0^T \| v^n(t) \|_{p^{r-2}}^{2p^{r-2}} dt + C \int_0^T \| w^n(t) \|_{p^{r-2}}^{2p^{r-2}} dt \\
\leq C \left\{ \sup_{t \in [0, T]} \| v(t) \|_2^2 \right\} \int_0^T \| v(t) \|_2^2 dt + C \int_0^T \| w(t) \|_{p^{r-2}}^{2p^{r-2}} dt \\
\leq C \left\{ \| x \|_2^2 + C T \sup_{t \in [0, T]} \| w(t) \|_{p^{r+1}}^{(r+1)p} \right\} \exp \left( C \sup_{t \in [0, T]} \| w(t) \|_4^4 \right) \\
+ C T \sup_{t \in [0, T]} \| w(t) \|_{p^{r-2}}^{2p^{r-2}},
\]

(3.32)
for $2 < p < \infty$. Thus there exists a constant $C_1 > 0$ such that

$$\log \left\{ \int_0^T \| u^n(t) \|_{\frac{2p}{p-2}} \, dt \right\}$$

$$\leq \log \left\{ C \left[ \| x \|_{\frac{2p}{p-2}}^2 + CT \sup_{t \in [0,T]} \| w(t) \|_{\frac{(r+1)p}{r+1}} \right] \right\} + C \sup_{t \in [0,T]} \| w(t) \|_4^4$$

$$+ \log \left\{ CT \sup_{t \in [0,T]} \| w(t) \|_{\frac{2p}{p-2}} \right\} + C_1. \quad (3.33)$$

By an application of Jensen’s inequality, we get

$$\mathbb{E} \left\{ \log \left[ \int_0^T \| u^n(t) \|_{\frac{2p}{p-2}} \, dt \right] \right\}$$

$$\leq \log \left\{ C \mathbb{E} \left[ \| x \|_{\frac{2p}{p-2}}^2 \right] + CT \mathbb{E} \left[ \sup_{t \in [0,T]} \| w(t) \|_{\frac{(r+1)p}{r+1}} \right] \right\} + C \mathbb{E} \left[ \sup_{t \in [0,T]} \| w(t) \|_4^4 \right]$$

$$+ \log \left\{ CT \mathbb{E} \left[ \sup_{t \in [0,T]} \| w(t) \|_{\frac{2p}{p-2}} \right] \right\} + C_1 =: \tilde{M}_T. \quad (3.34)$$

For $\tilde{\tau}_n \supseteq \lim_{m \to \infty} \tilde{\tau}_m$, an application of Markov’s inequality yields

$$\mathbb{P} \left\{ \omega \in \Omega : \tau_n^\infty(\omega) < T \right\}$$

$$\leq \mathbb{P} \left\{ \omega \in \Omega : \tau_m^\infty(\omega) < T \right\} \leq \mathbb{P} \left\{ \log \left[ \int_0^T \| u^n(t) \|_{\frac{2p}{p-2}} \, dt \right] \leq \log m \right\}$$

$$\leq \frac{1}{\log m} \mathbb{E} \left\{ \log \left[ \int_0^T \| u^n(t) \|_{\frac{2p}{p-2}} \, dt \right] \right\} = \frac{\tilde{M}_T}{\log m} \to 0 \text{ as } m \to \infty, \quad (3.35)$$

and hence $\mathbb{P} \left\{ \omega \in \Omega : \tau_n^\infty(\omega) < T \right\} = 0$. Therefore, we get $\tau_n^\infty = T$, $\mathbb{P}$-a.s., and hence the mild solution for the system (1.11) exists globally.

The case of $d = 3$ and $r \in [1, 3]$ can be established in a similar way by defining a sequence of stopping times $\tilde{\tau}_m \supseteq \inf_{t \geq 0} \left\{ t : \int_0^t \| u^n(s) \|_{\frac{2p}{p-2}} \, ds > m \right\}$, for $m \in \mathbb{N}$ and $3 < p \leq 6$ and using the estimate (3.27). \hfill \square

### 4 Stochastic CBF equations forced by fractional Brownian motion

In this section, we obtain the existence of a unique local mild solution to the stochastic CBF equations (1.14), for $d = 2, 3$. Furthermore, we discuss the existence of a global mild solution. Before going to the main results, we first discuss some basics of fractional Brownian motion.
4.1 Fractional Brownian motion

The first study on fractional Brownian motion (fBm) within the Hilbertian framework is reported in [22]. Due to various practical applications, the stochastic analysis of fBm has been intensively developed starting from the nineties. For a comprehensive study, the interested readers are referred to see [1,32,33], etc. In this subsection, we provide a brief description of fBm and its stochastic integral representation in separable Hilbert spaces (cf. Sects. 4 and 5, [20] for fBm defined in separable Banach spaces). Let us consider a time interval \([0, T]\), where \(T\) is an arbitrary fixed time horizon.

**Definition 4.1** A fractional Brownian motion (fBm) with Hurst parameter \(H \in (0, 1)\) is a centered Gaussian process \(W^H\) with covariance

\[
R_H(t, s) := \mathbb{E}\left[ W^H(t)W^H(s) \right] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right),
\]

where \(s, t \in [0, T]\).

Note that if \(H = \frac{1}{2}\), then \(W^{\frac{1}{2}}\) is the standard Brownian motion. It should be recalled that fBm is not a Markov process except in the case \(H = \frac{1}{2}\). The fBm is the only \(H\)-self-similar Gaussian process (that is, for any constant \(a > 0\), the processes \(\{a^{-H}W^H(at)\}_{0 \leq t \leq T}\) and \(W^H = \{W^H(t)\}_{0 \leq t \leq T}\) have the same distribution) with stationary increments (Proposition 1.1, [36]) and

\[
\mathbb{E}\left[ (W^H(t) - W^H(s))^2 \right] = |t - s|^{2H}.
\]

Furthermore, the process \(W^H\) admits the Wiener integral representation of the form

\[
W^H(t) = \int_0^t K_H(t, s)dW(s),
\]

where \(W = \{W(t)\}_{t \in [0,T]}\) is a Wiener process, and \(K_H(\cdot, \cdot)\) is the kernel given by

\[
K_H(t, s) = d_H(t - s)^{H - \frac{1}{2}} + s^{H - \frac{1}{2}}F\left(\frac{t}{s}\right),
\]

where \(d_H\) is a constant and

\[
F(z) = d_H \left( \frac{1}{2} - H \right) \int_0^{z-1} \theta^{H - \frac{3}{2}} \left( 1 - (\theta + 1)^{H - \frac{1}{2}} \right) d\theta.
\]

For \(H > \frac{1}{2}\), the kernel \(K_H(\cdot, \cdot)\) has the simpler expression

\[
K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du,
\]
where $t > s$ and $c_H = \left(\frac{H(H-1)}{\beta(2-2H,H-H)}\right)^{\frac{1}{2}}$, $\beta(\cdot, \cdot)$ being the beta function. The fact that the process defined by (4.1) is an fBm follows from the equality

$$\int_0^{t \wedge s} K_H(t, u)K_H(s, u)du = R_H(t, s).$$

Moreover, the kernel $K_H(\cdot, \cdot)$ satisfies the condition

$$\frac{\partial}{\partial t} K_H(t, s) = d_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{\frac{1}{2} - H} (t - s)^{H - \frac{3}{2}}.$$

Note that the fBm is an $\alpha$-regular Volterra process for $\alpha = H - \frac{1}{2}$, where $H > \frac{1}{2}$ (see Remark 2.2, [7] for more details).

Let $U$ be a separable Hilbert space with scalar product $(\cdot, \cdot)$. Let $E_H$ denote the linear space of $U$-valued step functions on $[0, T]$ of the form

$$\varphi(t) = \sum_{i=0}^{m-1} x_i \mathbb{1}_{[t_i, t_{i+1})}(t), \quad (4.2)$$

where $0 = t_0, t_1, t_2, \ldots, t_m \in [0, T], m \in \mathbb{N}, x_i \in U$. The space $E_H$ is equipped with the inner product

$$\left( \sum_{i=0}^{m-1} x_i \mathbb{1}_{[0, t_i)}, \sum_{j=0}^{n-1} y_j \mathbb{1}_{[0, s_j)} \right)_{\mathcal{H}} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (x_i, y_j) R_H(t_i, s_j).$$

Note that $E_H$ is a pre-Hilbert space and we denote the completion of $E_H$ with respect to $(\cdot, \cdot)_{\mathcal{H}}$ by $\mathcal{H}$. For $\varphi \in E_H$ of the form (4.2), let us define its Wiener integral with respect to the fBm as

$$\int_0^T \varphi(s)dW^H(s) = \sum_{i=0}^{m-1} x_i (W^H(t_{i+1}) - W^H(t_i)).$$

It is clear that the mapping $\varphi = \sum_{i=1}^m x_i \mathbb{1}_{(t_i, t_{i+1}]} \mapsto \int_0^T \varphi(s)dW^H(s)$ is an isometry between $E_H$ and the linear space $\text{span}\{W^H(t) : t \in [0, T]\}$ viewed as a subspace of $L^2(\Omega; U)$, since

$$\mathbb{E} \left[ \left\| \int_0^T \varphi(s)dW^H(s) \right\|_U^2 \right] = \|\varphi\|_{\mathcal{H}}^2.$$

The image of an element $\varphi \in \mathcal{H}$ under this isometry is called the Wiener integral of $\varphi$ with respect to the fBm $W^H$. For $0 < s < T$, we consider the operator $K^* : \mathcal{E}_H \to L^2(0, T; U)$ as
\[(K^* \varphi)(s) = K(T, s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r, s) dr.\]

For $H > \frac{1}{2}$, the operator $K^*$ has the simpler expression
\[(K^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.\]

The integrals appearing on the right-hand side are both Bochner integrals. Since the operator $K^*$ satisfies $(K^* \varphi, K^* \psi)_{L^2(0, T; \mathbb{U})} = (\varphi, \psi)_{\mathcal{H}}$, for all $\varphi, \psi \in \mathcal{E}_H$, $K^*$ can be extended to an isometry between $\mathcal{H}$ and $L^2(0, T; \mathbb{U})$ in the sense that
\[
\mathbb{E} \left[ \left\| \int_0^T \varphi(s) dW^H(s) \right\|^2 \right] = \|K^* \varphi\|_{L^2(0, T; \mathbb{U})}^2 = \|\varphi\|_{\mathcal{H}}^2, \text{ for all } \varphi \in \mathcal{H}.
\]

Hence we have the following connection with the Wiener process $W$
\[
\int_0^t \varphi(s) dW^H(s) = \int_0^t (K^* \varphi)(s) dW(s), \quad (4.3)
\]
for every $t \in [0, T]$, and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K^* \varphi \in L^2(0, T; \mathbb{U})$. Furthermore, if $\varphi, \psi \in \mathcal{H}$ are such that $\int_0^T \int_0^T |\varphi(t)||\psi(t)||t-s|^{2H-2} ds dt < \infty$, then their scalar product in $\mathcal{H}$ is given by
\[
(\varphi, \psi)_{\mathcal{H}} = \int_0^T \int_0^T \varphi(t) \psi(t)|t-s|^{2H-2} ds dt.
\]

In general, careful justification is needed for the existence of right hand side of (4.3) (cf. Sect. 5.1, [32]). As we are discussing the case of Wiener integrals over the Hilbert space $\mathbb{U}$, we point out that if $\varphi \in L^2(0, T; \mathbb{U})$ is a deterministic function, then the relation (4.3) holds, and the right hand is well defined in $L^2(\Omega; \mathbb{U})$ if $K^* \varphi \in L^2(0, T; \mathbb{U})$.

### 4.2 Cylindrical Brownian motion

For a Hilbert space $\mathbb{U}$, let us now define the standard cylindrical fractional Brownian motion in $\mathbb{U}$ as the formal series (cf. [12,35])
\[
W^H(t) = \sum_{n=0}^{\infty} e_n W_n^H(t), \quad (4.4)
\]
where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in $\mathbb{U}$ and $W_n^H$ is an one-dimensional fBm. It should be noted that the series (4.4) does not converge in $L^2(\Omega; \mathbb{U})$, and thus $W^H(t)$ is not a well-defined $\mathbb{U}$-valued random variable. But, one can consider
a Hilbert space $\mathbb{U}_1$ such that $\mathbb{U} \subset \mathbb{U}_1$, the linear embedding is a Hilbert-Schmidt operator, therefore, the series (4.4) defines a $\mathbb{U}_1$-valued Gaussian random variable and $\{W^H(t)\}_{t \geq 0}$ is a $\mathbb{U}_1$-valued cylindrical fBm.

Let $\mathbb{V}$ be another real and separable Hilbert space and $L_2(\mathbb{U}, \mathbb{V})$ denote the space of Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{V}$. As discussed in [9], it is possible to define a stochastic integral of the form:

$$\int_0^T \varphi(t) dW^H(t),$$

where $\varphi : [0, T] \mapsto L(\mathbb{U}, \mathbb{V})$, and the integral (4.5) is a $\mathbb{V}$-valued random variable, which is independent of choice of $\mathbb{U}_1$. Let $\varphi$ be a deterministic function with values in $L_2(\mathbb{U}, \mathbb{V})$ satisfying:

(i) for each $x \in \mathbb{U}$, $\varphi(\cdot)x \in L^p(0, T; \mathbb{V})$, for $p > \frac{1}{H}$,
(ii) $\int_0^T \int_0^T \|\varphi(s)\|_{L_2(\mathbb{U}, \mathbb{V})} \|\varphi(t)\|_{L_2(\mathbb{U}, \mathbb{V})} |s-t|^{2H-2} ds dt < \infty$.

Then the stochastic integral (4.5) can be expressed as

$$\int_0^T \varphi(t) dW^H(t) := \sum_{n=1}^{\infty} \int_0^T \varphi(t) e_n dW_n^H(t) = \sum_{n=1}^{\infty} \int_0^T (K^n_H \varphi e_n) dW_n(t),$$

(4.6)

where $W_n$ is the standard Brownian motion connected to fBm $W^H_n$ by the representation formula (4.1). If $H \in (\frac{1}{2}, 1)$, then $\varphi e_n \in L^2(0, T; \mathbb{V})$, for each $n \in \mathbb{N}$, so that the terms of the series (4.6) are well-defined. For $\varphi = -\infty$, we obtain that (4.6) is the well-defined for $H \in (0, \frac{1}{2})$ also. Moreover, the sequence of random variables $\left\{\int_0^T \varphi(s) e_n dW^H_n(s)\right\}_{n=1}^{\infty}$ are mutually independent Gaussian random variables (cf. [12]).

For cylindrical Brownian motions in a separable Banach space $\mathbb{V}$, the interested readers are referred to see Sects. 4 and 5, [20]. For stochastic integrals in $\mathbb{V}$, a series expansion similar to (4.6) is available, where the Hilbert–Schmidt operators from $\mathbb{U}$ to $\mathbb{V}$ are replaced by $\gamma$-radonifying operators from $\mathbb{U}$ to $\mathbb{V}$ (see [20] for more details). One can refer [6,25], etc for the local solvability in $L^p$-spaces for some mathematical models like semilinear heat equation, Hardy–Hénon parabolic equations, etc perturbed by fBm.

### 4.3 SCBF equations perturbed by fractional Brownian motion

We consider $\mathbb{U} = \mathbb{H} = \mathbb{J}_2$, $\{e_j\}_{j=1}^{\infty}$ as the complete orthonormal basis of $\mathbb{J}_2$, and we take $d = 2, 3$. Next, we consider the following stochastic Stokes equation perturbed by fractional Brownian noise as

\[
\begin{cases}
\frac{d}{dt}w(t) + Aw(t)dt = \Phi dW^H(t), \\
w(0) = 0,
\end{cases}
\]

(4.7)
where $\Phi \in \mathcal{L}(\mathbb{H}, \mathbb{J}_p)$ determines the space correlation of the noise process and $W^H_t = \{W^H(t)\}_{t \in [0, T]}$ is a cylindrical fractional Brownian process. Since the operator $A$ generates an analytic semigroup on $\mathbb{J}_p$, we have

$$\|e^{-\mu t} \Phi\|_{\gamma(\mathbb{H}; \mathbb{J}_p)} \leq C \|\Phi\|_{\gamma(\mathbb{H}; \mathbb{J}_p)}, \quad \text{for } t \geq 0. \quad (4.8)$$

Therefore, there exists a unique solution of the problem (3.1) with paths in $C([0, T]; \mathbb{J}_p), \ p \in [2, \infty), \mathbb{P}$-a.s., which can be represented by the stochastic convolution

$$w(t) = \int_0^t e^{-\mu(t-s)} \Phi dW^H(s), \quad (4.9)$$

for all $t \in [0, T]$, and has a modification such that

$$\sup_{t \in [0, T]} \left\| \int_0^t e^{-\mu(t-s)} \Phi dW^H(s) \right\|_p < \infty, \mathbb{P}$-a.s., \quad (4.10)$$

for $H \in (0, 1)$. Then the local existence and uniqueness as in Theorem 1.4 can be established in a similar way as that of Theorems 3.1 and 1.3. For the global existence, one needs to assume $\Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \cap \gamma(\mathbb{H}; \mathbb{J}_p+r-1), \ for \ r \geq 3 (\beta \mu > 4 \ for \ r = 3)$ and $\Phi \in \gamma(\mathbb{H}; \mathbb{J}_p) \cap \gamma(\mathbb{H}; \mathbb{J}_p+r+1) \cap \gamma(\mathbb{H}; \mathbb{J}_{4}), \ for \ d = 2, 3, r \in [1, 3]$.

**Remark 4.2** One can also consider the stochastic CBF equations perturbed by distributed $\alpha$-regular Volterra processes as

$$\left\{ \begin{array}{l}
du(t) + [Au(t) + B(u(t))] + \beta C(u(t))] dt = \Phi dB(t), \\
u(0) = x,
\end{array} \right. \quad (4.11)$$

where $\Phi \in \gamma(\mathbb{H}; \mathbb{J}_p)$ satisfies (4.8) and $B$ is an infinite-dimensional $\alpha$-regular cylindrical Volterra process with $\alpha \in (0, \frac{1}{2})$, which belongs to a finite Wiener chaos (see [7] for more details on $\alpha$-regular Volterra processes). If $\|\Phi\|_{\gamma(\mathbb{H}; \mathbb{J}_p)} < \infty$, then the process

$$w(t) = \int_0^t e^{-\mu(t-s)} \Phi dB(s),$$

has a modification in $C([0, T]; \mathbb{J}_p), \ p \in [\frac{2}{1+2\alpha}, \infty), \mathbb{P}$-a.s. Thus a result similar to Theorem 1.4 can be obtained in this case also for the system (4.11), that is, the existence and uniqueness of a local mild solution

$$u(t) = e^{-\mu t} x - \int_0^t e^{-\mu(t-s)} [B(u(s)) + \beta C(u(s))] ds + \int_0^t e^{-\mu(t-s)} \Phi dB(s),$$
for $t \in [0, \tilde{T})$, where $0 < \tilde{T} < T$ is a random time, to the system (4.11) with $\mathbb{P}$-a.s. continuous modification with trajectories in $\mathbb{J}_p$, for $\max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty$. Under further assumptions on $\Phi$, one can obtain the global existence also.

5 Conclusions and future plans

The existence of a local mild solution and the existence of a global mild solution in $\mathbb{L}^p(\mathbb{R}^d)$ with $\max \left\{ d, \frac{d(r-1)}{2} \right\} < p < \infty$ for deterministic and stochastic CBF equations in $\mathbb{R}^d$ (for various kinds of noises) are established in this work. The case of $p = \max \left\{ d, \frac{d(r-1)}{2} \right\}$ is an interesting problem and it will be addressed in a future work (for similar works, see [21] for the deterministic NSE and [27] for stochastic NSE).

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