Partial entropy production in heat transport

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Received 6 February 2018
Accepted for publication 15 April 2018
Published 5 June 2018

Online at stacks.iop.org/JSTAT/2018/063203
https://doi.org/10.1088/1742-5468/aabfca

Abstract. We consider a system of two Brownian particles (say A and B),
coupled to each other via harmonic potential of stiffness constant $k$. Particle-A is
connected to two heat baths of constant temperatures $T_1$ and $T_2$, and particle-B is
connected to a single heat bath of a constant temperature $T_3$. In the steady
state, the total entropy production for both particles obeys the fluctuation
theorem. We compute the total entropy production due to particle-A in the
coupled system (partial and apparent entropy production) in the steady state
for a time segment $\tau$. When both particles are weakly interacting with each
other, the fluctuation theorem for partial and apparent entropy production
is studied. We find a significant deviation from the fluctuation theorem. The
analytical results are also verified using numerical simulations. Furthermore,
we investigate the effect of hidden fast degrees of freedom on the steady state
fluctuation theorem for a system of a single Brownian particle coupled to two
heat baths of distinct temperatures and dissipation constants, and coupled to a
third heat bath of vanishing temperature and dissipation constant.

Keywords: fluctuation phenomena, large deviations in non-equilibrium
systems, heat conduction
1. Introduction

Understanding the properties of driven small systems has challenged both theoretical and experimental techniques. Examples of small scale systems include biopolymers (such as DNA, RNA and proteins), enzymes, Brownian particles, Brownian motors, and nanoscale engines. These systems always remain in contact with a noisy environment, called as heat bath, that satisfies fluctuation-dissipation theorem [1]. To drive these systems away from equilibrium, an external source of energy is needed. The general framework of equilibrium statistical mechanics [2] is not applicable for these driven systems. When the system is close to equilibrium, linear response of it in the presence of a small field, is related to the fluctuation properties of the given system (in the absence of field) in the equilibrium. The fluctuation relations go beyond the linear response theory and are valid for the system driven arbitrarily far from equilibrium. These relations are particularly useful for small scale systems where fluctuations are predominant. Within this context, Evans et al [3] used numerical simulation to understand the distribution of entropy production of a shear fluid interacting with a thermal...
Partial entropy production in heat transport

They found an interesting symmetry relation between the probability of positive entropy production to that of negative one in a steady state, which is referred as steady state fluctuation theorem. Gallavotti and Cohen [4] proved the fluctuation theorem for Hamiltonian system using chaotic assumption for the dynamics. Subsequently, Kurchan [5] derived the fluctuation theorem for Langevin dynamics which was later extended by Lebowitz and Spohn [6] for general Markov process. Jarzynski’s equality [7, 8] provides a method to estimate the difference in the free energies between two equilibrium states using nonequilibrium work done. Hatano–Sasa relation [9] which is analogous to Jarzynski’s equality, deals with the transition between steady states. Evans and Searles [10] proved the fluctuation theorem in transient regime (i.e. when the initial distribution is the equilibrium one). Crooks work fluctuation theorem [11, 12] measures the relative probabilities of work done in forward and reverse processes starting from respective equilibrium distributions.

In the realm of stochastic thermodynamics [13], thermodynamical quantities such as heat, work, power dissipation, power injection, etc, are defined at the single trajectory level of a non-equilibrium process. These observables have probability distribution function rather than having a unique value. When such distribution functions satisfy large deviation principle [14], those are characterized by a large deviation function. With the interest of obtaining these distributions and large deviation functions, several efforts have been put to test the corresponding fluctuation relations for the observables mentioned above. In contrast to initial studies where the sole contribution to the entropy production was considered from the heat dissipation by the system in the thermal bath, Seifert [15, 16] identified entropy production along the stochastic trajectory using a colloidal particle and for general stochastic dynamics obeying master equation. This entropy production consists of two parts: the entropy production in the system $\Delta S_{\text{sys}}$ and the entropy production in the medium $\Delta S_{\text{med}}$. The sum of these two is called as total entropy production $\Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{med}}$. The system entropy production remains bounded for system having bounded energy. For such systems, the medium entropy production has major contribution towards the total entropy production in the long time limit. Thus, the corresponding fluctuation theorem holds in the long time limit. When both of these parts are considered, the total entropy production $\Delta S_{\text{tot}}$ in the steady state, satisfies the fluctuation theorem given as

$$\frac{P(\Delta S_{\text{tot}} = s\tau)}{P(\Delta S_{\text{tot}} = -s\tau)} = e^{s\tau},$$

for all time, where $\Delta S_{\text{tot}}$ is an extensive quantity with the observation time $\tau$. Though, total entropy production $\Delta S_{\text{tot}}$ satisfies steady state fluctuation time theorem in general, this relation may not hold for other stochastic observables such as work done, heat flow, power flux, etc, [17–26].

Consider a Brownian particle connected to two heat reservoirs at different temperatures $T_1$ and $T_2$ (where $T_1 > T_2$). According to standard thermodynamics, the heat flows from a hot to a cold reservoir. However, in this small system, due to thermal fluctuations, once in a while, the heat may flow in reverse direction, although on an average, the heat current follows the sign of $(T_1 - T_2)$ in accordance with second law of thermodynamics. The entropy production in the baths is $\Delta S_{\text{med}} = -(Q_1/T_1 + Q_2/T_2)$ (see section 5), where $Q_i$ is the heat energy transferred by $i$th bath to the Brownian reservoir. They found an interesting symmetry relation between the probability of positive entropy production to that of negative one in a steady state, which is referred as steady state fluctuation theorem. Gallavotti and Cohen [4] proved the fluctuation theorem for Hamiltonian system using chaotic assumption for the dynamics. Subsequently, Kurchan [5] derived the fluctuation theorem for Langevin dynamics which was later extended by Lebowitz and Spohn [6] for general Markov process. Jarzynski’s equality [7, 8] provides a method to estimate the difference in the free energies between two equilibrium states using nonequilibrium work done. Hatano–Sasa relation [9] which is analogous to Jarzynski’s equality, deals with the transition between steady states. Evans and Searles [10] proved the fluctuation theorem in transient regime (i.e. when the initial distribution is the equilibrium one). Crooks work fluctuation theorem [11, 12] measures the relative probabilities of work done in forward and reverse processes starting from respective equilibrium distributions.

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Partial entropy production in heat transport

This quantity may not always satisfy the fluctuation theorem (1) as shown in section 5. However, once the system entropy production $\Delta S_{\text{sys}}$ is added to medium one, resulted total entropy production $\Delta S_{\text{tot}}$ obeys the relation given in (1).

In all of above the examples, it is assumed that all the relevant degrees of freedom (DOFs) of the system are considered. However, there may arise a situation where the complete description of the system may not possible. The cause of incomplete information may be because the system is coarse grained or there may be hidden DOFs which affect the observed system. Recently, there has been lot of excitement in understanding the system properties when the partial information of a given system is available. For example, Rahav et al [27] considered a Markov jump process on a set of finite number of states. The jump from a given state to another state is described by transition rates. Assuming certain transition rates higher than other ones, the whole network of states is then mapped to a group of clusters. This type of coarse graining modified the entropy production and they showed that the coarse grained entropy satisfied the fluctuation theorem provided the transition rates of the system to jump within clusters are sufficiently high. Similar results found in [28] where authors have discussed the projection of Markov process with constant transition rates to smaller number of observable aggregated states, and the resulting entropy production on the set of all aggregated state satisfied both detailed and integral fluctuation theorems. Puglisi et al [29] showed an example where decimation of certain fast states with respect to a given threshold time does not affect the entropy production, provided it does not entirely remove the loops carrying net probability current. In all of these references, it is shown that coarse graining based on time-scale separation did not alter the underlying physics of the problem. There are some other studies which differ from the above mentioned ones where relaxation time of all relevant DOFs is much larger than that of bath DOFs. For example, in [30], two paramagnetic colloidal particles of same size were trapped in separate non-overlapping toroidal traps. Whole system was in contact of the heat bath. Tangential forces were applied on each particle using laser field. Consequently, both particles reached in the non-equilibrium stationary state. A static magnetic field perpendicular to the plane of toroidal traps, was used to set an interaction among these particles. The total entropy production due to one of the particles in this coupled system (i.e. partial entropy production as defined below in our paper) was measured and the deviation from fluctuation theorem with increasing coupling strength was observed. In a theoretical model [31], fluctuation theorem for the entropy production of a single electron box is studied in a coupled electron box system. In an another example [32], authors studied molecular motors which are modeled by flashing ratchet, and found that Gallavotti–Cohen symmetry is preserved only when both chemical and mechanical DOFs are considered in the theory. There are also some examples where partial information is utilized to get full system properties. For example, in [33], Ribezzi-Crivellari and Ritort have shown a method to infer full work distribution from the partial work measurement using Crooks fluctuation theorem. Amann et al [34] have derived a criterion to describe a non-equilibrium steady state of a Markov system of three states using the data from sufficiently long two states trajectories. Some other studies related to the area of partial observation of a complete system can also be seen in [35–42]. The main conclusion of these
studies is, when a system consists of DOFs having relaxation time much larger than that of bath DOFs, then the partial system or subsystem may not behave like a complete system for large coupling strength. Recently, Gupta et al [43] have discussed a fluctuation theorem for partial and apparent entropy production in weak coupling limit in a general scenario, and shown that deviation from the fluctuation theorem can be seen even in the limit of coupling strength tending to zero. In [44], authors have shown a technique to diminish the effect of weak coupling on the observed DOFs from the hidden variables. In both of the previous setups, the authors have chosen external stochastic Gaussian forces to drive the system into nonequilibrium state. In this paper, we consider a system of a coupled Brownian particle where one of the particles (say particle-A) is connected to two heat baths at different temperatures while the other one (say particle-B) is attached to a single heat bath. We take the interaction between these two particles to be harmonic. Here, we focus on the total entropy production in the steady state by one of the particles (say particle-A) in the coupled system. We give two definitions of entropy production of partial system of the complete system: (1) partial and (2) apparent entropy production. In the limit of vanishing coupling, deviation from the fluctuation theorem (see (1)) is studied. There are two important features of this paper: (1) In contrast to [43, 44], we use thermal gradient to drive the system into nonequilibrium steady state, and (2) The asymmetry function given in (76), has negative slope which was not observed in the earlier studies.

The paper is organized as follows. We describe the model system and give two definitions of entropy production of partial system of a coupled system in section 2. Section 3 contains the Fokker–Planck equation for the restricted moment generating function of functional $W$ given in (22), and its general solution at the large time. Moreover, the moment generating function $\langle e^{-\lambda \Delta S_A^{tot}} \rangle \sim g(\lambda)e^{\tau \mu(\lambda)}$ for the generalized entropy production given in (18), is shown in the large time limit ($\tau \to \infty$), and then, invert it using saddle point method which yields the probability density function for generalized partial entropy production given in (18) (section 4). In section 5, we compute the medium entropy production by a single Brownian particle connected to a thermal gradient, and show that for the entropy production to satisfy the fluctuation theorem for large but finite time, it is necessary to incorporate the system entropy production in the total entropy production. Since we are interested in the steady state fluctuation theorem for partial and apparent entropy production in the weak coupling limit, we discuss the assumption to approximate the prefactor term $g(\lambda) \approx g_0(\lambda)$ in section 6. The cumulant generating function $\mu(\lambda)$ is analyzed in section 7. In section 7.1, we discuss the Gallavotti–Cohen symmetry for the cumulant generating function $\mu(\lambda)$. The asymmetry function which measures the deviation from the fluctuation theorem is discussed in section 8. Section 9 contains the comparison of the analytical predictions with the numerical simulations. In section 10, we show how the hidden fast DOFs effect the fluctuation theorem in the weak coupling limit for a single Brownian particle (slow DOF) coupled to three baths of different temperatures. We summarize our paper in section 11. The detailed calculation for the moment generating function $\langle e^{-\lambda \Delta S_A^{tot}} \rangle \sim g(\lambda)e^{\tau \mu(\lambda)}$ in the large time limit ($\tau \to \infty$) for both definitions of entropy production is given in appendix.
2. Model

Consider a Brownian particle (say particle-A) of mass $m$, in contact with two heat baths at temperatures $T_1$ and $T_2$ ($T_1 > T_2$). Let $\gamma_1$ and $\gamma_2$ are the dissipation constants of the baths with temperatures $T_1$ and $T_2$, respectively. Suppose the given Brownian particle-A is coupled harmonically with another Brownian particle (say particle-B) of mass $m$. The particle-B is in contact with a single heat bath of a constant temperature $T_3$ and the dissipation constant $\gamma_3$. The schematic diagram of the coupled Brownian particle system is shown in figure 1. The Hamiltonian of the system is given as

$$H(y, v_A, v_B) = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}ky^2,$$  

(2)

where $y = x_A - x_B$ is the relative separation between particle-A and particle-B, $k$ is the stiffness constant, $v_A$ and $v_B$ are the velocities of particle-A and particle-B, respectively.

The evolution of the given system is described by following Langevin equations

$$\dot{y} = v_A(t) - v_B(t),$$  

(3)

$$m\dot{v}_A = -\gamma_A v_A(t) + \eta_A(t) - ky(t),$$  

(4)

$$m\dot{v}_B = -\gamma_B v_B(t) + \eta_B(t) + ky(t),$$  

(5)

where $\eta_A(t) = \eta_1(t) + \eta_2(t)$, $\eta_B(t) = \eta_3(t)$, $\gamma_A = \gamma_1 + \gamma_2$, and $\gamma_B = \gamma_3$. The thermal noises $\eta_1(t)$, $\eta_2(t)$, and $\eta_3(t)$ are from the heat baths, with mean zero and correlations $\langle \eta_i(t)\eta_j(t') \rangle = 2T_i\gamma_i\delta_{ij}\delta(t - t')$, where $\{i, j\} = \{1, 2, 3\}$. We set Boltzmann’s constant $k_B = 1$ throughout the calculations.

The aim of this paper is to understand the validity of the fluctuation theorem for the total entropy production of a subsystem or partial system of the complete system in the steady state. In the following subsections, we give two definitions of entropy production of partial system.

2.1. Partial entropy production

Suppose we are interested in the total entropy production due to particle-A in a coupled Brownian particle system as shown in figure 1. The dynamics of particle-A is given by (4). Multiplying (4) by $v_A(t)$ on both sides and integrating over time $t$ from 0 to $\tau$ yields

$$\frac{1}{2}m[v_A^2(\tau) - v_A^2(0)] = Q_1(t) + Q_2(t) - k\int_0^\tau dt \ y(t)v_A(t),$$  

(6)

where $Q_i = \int_0^\tau [\eta_i(t) - \gamma_i v_A(t)]v_A(t)dt$, is the heat absorbed by Brownian particle-A from the $i$th heat bath. The term on the left hand side is the change in the kinetic energy of the Brownian particle-A from time $t = 0$ to $t = \tau$. The third term on the right hand side is energy change due to interaction among the particles. These baths are of infinite size and have infinite heat capacity. Therefore, these are assumed to be always in thermal equilibrium. Using standard thermodynamics, the entropy production in the baths due to Brownian particle-A can be written as

https://doi.org/10.1088/1742-5468/aabfca
\[
\Delta \bar{S}_A^{\text{med}} = -\left(\frac{Q_1}{T_1} + \frac{Q_2}{T_2}\right) = \Delta \beta Q_1 - \frac{k}{T_2} \int_0^\tau dt \, y(t)v_A(t) - \frac{m}{2T_2} [v_A^2(\tau) - v_A^2(0)],
\]

where \(\Delta \beta = \frac{1}{T_2} - \frac{1}{T_1}\) is the difference of inverse temperatures.

Total entropy production \(\Delta \bar{S}_A^{\text{sys}}\) of Brownian particle-A of system shown in figure 1 is the sum of medium entropy production \(\Delta \bar{S}_A^{\text{med}}\) and system entropy production \(\Delta \bar{S}_A^{\text{sys}}\) of the particle-A. In the steady state, the system entropy production of the particle-A from time \(t = 0\) to \(t = \tau\) is [15, 16]

\[
\Delta \bar{S}_A^{\text{sys}} = -\ln P_{ss}(v_A(\tau)) + \ln P_{ss}(v_A(0)),
\]

where \(P_{ss}(v_A)\) is the steady state distribution obtained after integrating the joint steady state distribution \(P_{ss}^{\text{full}}(y, v_A, v_B)\) over \(y\) and \(v_B\)

\[
P_{ss}(v_A) = \frac{1}{\sqrt{2\pi H_P}} \exp \left[ -\frac{v_A^2}{2H_P} \right].
\]

In the above equation, \(H_P\) is given by

\[
H_P = \lim_{\tau \to \infty} \left\langle \left[ v_A(\tau) - \langle v_A(\tau) \rangle \right]^2 \right\rangle = \frac{\gamma_3(\gamma_1 + \gamma_2 + \gamma_3)(\gamma_1 T_1 + \gamma_2 T_2) + mk(\gamma_1 T_1 + \gamma_2 T_2 + \gamma_3 T_3)}{m(\gamma_1 + \gamma_2 + \gamma_3)(mk + \gamma_1 \gamma_3 + \gamma_2 \gamma_3)}.
\]

Therefore, total entropy production due to particle-A in the coupled system (i.e. partial entropy production) is given as

\[
\Delta \bar{S}_A^{\text{tot}} = \Delta \beta Q_1 - \frac{k}{T_2} \int_0^\tau dt \, y(t)v_A(t) - \frac{1}{2} \left(\frac{m}{T_2} - \frac{1}{H_P}\right) [v_A^2(\tau) - v_A^2(0)].
\]
2.2. Apparent entropy production

Consider an experiment where we want to find the entropy production for a single Brownian particle (say particle-A) in the contact with two heat baths of temperatures (dissipation constants) \( T_1, \gamma_1 \) and \( T_2, \gamma_2 \) (see figure 1 with \( \delta = 0 \)). The Langevin equation for Brownian particle-A is

\[
m \dot{v}_A = - (\gamma_1 + \gamma_2) v_A(t) + \eta_1(t) + \eta_2(t).
\]

Multiplying above equation by \( v_A \) on both sides and integrating over time \( t \) from 0 to \( \tau \) gives (6) with \( k = 0 \). Therefore, one can write the entropy production in the baths due to Brownian particle-A as

\[
\Delta \tilde{S}_{\text{med}}^A = - \left( \frac{Q_1}{T_1} + \frac{Q_2}{T_2} \right) = \Delta \beta Q_1 - \frac{m}{2T_2} [v_A^2(\tau) - v_A^2(0)].
\]

The system entropy production of the Brownian particle-A in steady state is given as

\[
\Delta \tilde{S}_{\text{sys}}^A = - \ln \tilde{P}_{ss}(v_A(\tau)) + \ln \tilde{P}_{ss}(v_A(0)).
\]

In the above equation, \( \tilde{P}_{ss}(v_A) \) is the steady state distribution obtained from (12)

\[
\tilde{P}_{ss}(v_A) = \frac{1}{\sqrt{2\pi H_A}} \exp \left( - \frac{v_A^2}{2H_A} \right),
\]

where

\[
H_A = \lim_{\tau \to \infty} \langle [v_A(\tau) - \langle v_A(\tau) \rangle]^2 \rangle = \frac{\gamma_1 T_1 + \gamma_2 T_2}{m(\gamma_1 + \gamma_2)}.
\]

Therefore, total entropy production of particle-A can be written as

\[
\Delta \tilde{S}_{\text{tot}}^A = \Delta \beta Q_1 - \frac{1}{2} \left( \frac{m}{T_2} - \frac{1}{H_A} \right) [v_A^2(\tau) - v_A^2(0)].
\]

It is important to note that equation (16) is written by assuming that there is no other particle is coupled to the given particle-A. Therefore, an experimentalist naively uses (16) to compute the entropy production due to a single Brownian particle-A coupled to two heat baths of distinct temperatures. If there is one more particle (say particle-B) coupled to a heat bath, is present and interacting harmonically with given particle-A, then the actual dynamics of particle-A will be given by (3)–(5). To understand what an experimentalist observes without the prior knowledge of particle-B, we use (16) for entropy production with the actual dynamics given in (3)–(5), and then we compute the distribution of total entropy production of particle-A. This definition of entropy production we call apparent entropy production.

In fact, we can combine both definitions of entropy production (i.e. (11) and (16)) using following parameter

\[
\Pi = \begin{cases} 
1 & \text{Partial entropy production,} \\
0 & \text{Apparent entropy production.}
\end{cases}
\]

\[ (17) \]
Therefore, the generalized partial entropy production reads as
\[ \Delta S_{\text{tot}}^A = \Delta \beta Q_1 - \frac{\Pi k}{T_2} \int_0^\tau dt \ y(t) v_A(t) - \frac{1}{2} \left[ \frac{m}{T_2} - \frac{1}{H} \right] \left[ v_A^2(\tau) - v_A^2(0) \right], \tag{18} \]
where \( H = \Pi H_P + (1 - \Pi) H_A. \)

Our goal is to compute the distribution of generalized partial entropy production, i.e. \( P(\Delta S_{\text{tot}}^A) \), in the non-equilibrium steady state. From (18), we see that \( \Delta S_{\text{tot}}^A \) depends on thermal noises quadratically. Therefore, the distribution of generalized partial entropy production will be non-Gaussian.

The generalized partial entropy production \( (\Delta S_{\text{tot}}^A) \) given in (18), is a stochastic quantity whose value depends upon both initial state of the system and thermal noises. Therefore, probability distribution function of it is obtained by inverting the moment generating function defined as
\[ Z(\lambda) = \langle \exp(-\lambda \Delta S_{\text{tot}}^A) \rangle, \tag{19} \]
where the angular brackets indicate the average over both initial configuration of the system and set of all paths. Instead of computing \( Z(\lambda) \) directly, it is useful to first compute the restricted moment generating function for \( \Delta S_{\text{tot}}^A \) defined as
\[ Z(\lambda, U, \tau|U_0) = \langle \exp(-\lambda \Delta S_{\text{tot}}^A) \delta[U - U(\tau)] \rangle_{U,U_0}, \tag{20} \]
where the angular brackets represent the average over trajectories starting from initial variable \( U_0 = [y(0), v_A(0), v_B(0)]^T \) to final variable \( U(\tau) = [y(\tau), v_A(\tau), v_B(\tau)]^T \). Substituting \( \Delta S_{\text{tot}}^A \) from (18) in (20), we get
\[ Z(\lambda, U, \tau|U_0) = \exp \left[ \lambda/2(mT_2^{-1} - H^{-1}) \left( U^T \Sigma U - U_0^T \Sigma U_0 \right) \right] Z_W(\lambda, U, \tau|U_0), \tag{21} \]
where \( \Sigma_{ij} = \delta_{i,j} \delta_{2,j} \) with \( \{i, j\} = \{1, 2, 3\} \), \( W \) is
\[ W = \Delta \beta Q_1 - \frac{\Pi k}{T_2} \int_0^\tau dt \ y(t) v_A(t), \tag{22} \]
and
\[ Z_W(\lambda, U, \tau|U_0) = \langle e^{-\lambda W} \delta[U - U(\tau)] \rangle_{U,U_0}. \tag{23} \]
Since the boundary terms do not contribute in the averaging process in (21), we have taken them outside from the angular brackets. In the following, we write the evolution equation satisfied by \( Z_W(\lambda, U, \tau|U_0) \).

3. Fokker–Planck equation

The restricted moment generating function \( Z_W(\lambda, U, \tau|U_0) \) follows the Fokker–Planck equation [25, 45–47]
\[ \frac{\partial Z_W(\lambda, U, \tau|U_0)}{\partial \tau} = {\mathcal{L}}_\lambda Z_W(\lambda, U, \tau|U_0), \tag{24} \]
https://doi.org/10.1088/1742-5468/aabfca
where the differential operator $\mathcal{L}_\lambda$ has the following form
\[
\mathcal{L}_\lambda = \frac{1}{m} \sum_{i=A,B} \left[ \frac{\partial H}{\partial x_i} \frac{\partial}{\partial v_i} - \frac{\partial H}{\partial v_i} \frac{\partial}{\partial x_i} \right] + \frac{\gamma_e v_B}{m} \frac{\partial}{\partial v_B} + \frac{\gamma_1 T_1 + \gamma_2 T_2}{m^2} \frac{\partial^2}{\partial v_A^2} + \frac{\gamma_3 T_3}{m^2} \frac{\partial^2}{\partial v_B^2} + \sum_{i=1}^{n} \frac{\gamma_i}{m} \right] + \lambda \left[ \gamma_1 \Delta \beta \left( v_A^2 + \frac{T_1}{m} \right) + \frac{\Pi k y v_A}{T_2} \right] + \lambda^2 \Delta \beta^2 v_A^2 \gamma_1 T_1 + \frac{v_A}{m} (\gamma_1 + 2 \lambda \gamma_1 \Delta \beta) \frac{\partial}{\partial v_A}.
\]

The differential equation (24) is subjected to the initial condition $Z_W(\lambda, U, 0|U_0) = Z(\lambda, U, 0|U_0) = \delta(U - U_0)$.

We do not know whether the above differential equation can be solved to get full solution. Nevertheless, one can write the general solution of the differential equation as
\[
Z_W(\lambda, U, \tau|U_0) = \sum_n e^{\tau \mu_n(\lambda)} \chi_n(U_0, \lambda) \Psi_n(U, \lambda).
\] (26)

In the above equation, $\chi_n(U_0, \lambda)$ and $\Psi_n(U, \lambda)$ are the $n$th left and right eigenfunctions, respectively, corresponding to the eigenvalue $\mu_n(\lambda)$ of the differential operator $\mathcal{L}_\lambda$. These eigenfunctions satisfy orthonormality condition
\[
\int \chi_n(U, \lambda) \Psi_m(U, \lambda) \, dU = \delta_{nm}.
\] (27)

In most of the physical situations, one is interested in the large time solution of the differential equation. In such cases, the solution of differential equation (24), is dominated by the largest eigenvalue of the full spectrum, i.e. $\mu(\lambda) = \max\{\mu_n(\lambda)\}$. Thus, for large time
\[
Z_W(\lambda, U, \tau|U_0) = e^{\tau \mu(\lambda)} \chi(U_0, \lambda) \Psi(U, \lambda) + \ldots,
\] (28)

where $\mu(\lambda)$ is the largest eigenvalue of the differential operator $\mathcal{L}_\lambda$ and the corresponding left and right eigenfunctions are $\chi(U_0, \lambda)$ and $\Psi(U, \lambda)$, respectively. The steady state distribution one can obtain from the restricted moment generating function by substituting $\lambda = 0$ and taking large time limit: $P_{ss}^{\text{full}}(U) = Z_W(0, U, \tau \to \infty|U_0) = \Psi(U, 0)$ by identifying $\mu(0) = 0$ and $\chi(U_0, 0) = 1$. Therefore, integrating $Z(\lambda, U, \tau|U_0)$ given in (21), over the initial steady state distribution $P_{ss}^{\text{full}}(U_0)$ and the final state $U$ yields
\[
Z(\lambda) = \int \, dU \int \, dU_0 \, P_{ss}^{\text{full}}(U_0) Z(\lambda, U, \tau|U_0) = g(\lambda) e^{\tau \mu(\lambda)} + \ldots,
\] (29)

where $g(\lambda)$ is the prefactor given as
\[
g(\lambda) = \int \, dU \int \, dU_0 \, P_{ss}^{\text{full}}(U_0) \chi(U_0, \lambda) \Psi(U, \lambda) e^{\lambda/2(mT_2^{-1} - H^{-1})(U^T \Sigma U - U_0^T \Sigma U_0)} .
\] (30)

The largest eigenvalues $\mu(\lambda)$ and corresponding left eigenfunction $\chi(U_0, \lambda)$ and right eigenfunction $\Psi(U, \lambda)$ can be found using a technique developed in [48].

The cumulant generating function $\mu(\lambda)$ obtained as (see (A.31))
\[
\mu(\lambda) = -\frac{1}{4\pi \gamma} \int_{-\infty}^{\infty} du \, \ln \left[ 1 + \frac{h(u, \lambda)}{q(u)} \right],
\] (31)
in which

https://doi.org/10.1088/1742-5468/aabfca
\[ h(u, \lambda) = 4\lambda(1 - \lambda)\beta_{12}[\alpha_{12}(1 - \beta_{12})^2\{u^4 + u^2(\alpha_{13} - \delta) + \delta^2/4\} + \delta^2\alpha_{13}\beta_{12}(\beta_{12} + \Pi - 1)/4] \\
- \lambda\delta^2\alpha_{13}[(\beta_{12} +\beta_{13}\Pi - \beta_{13}\lambda)(\beta_{12} + \Pi - 1) + \alpha_{12}\beta_{13}\Pi(\beta_{12} - \beta_{13} + \beta_{13}\Pi)]. \] (32)

\[ q(u) = \beta_{12}^2[u^6 + u^4\{(1 + \alpha_{12})^2 + \alpha_{13}^2 - 2\delta\} + u^2\{(1 + \alpha_{12})^2 - \delta\}(\alpha_{13}^2 - \delta) + \delta^2(1 + \alpha_{12} + \alpha_{13})^2/4]. \] (33)

Here \( \beta_{1j} = \frac{\tau_j}{\tau_1} \) and \( \alpha_{1j} = \frac{\tau_j}{\tau_1} \) with \( j = 2, 3 \), the coupling parameter \( \delta = \frac{2\kappa m}{\gamma_1} \), and \( \tau_\gamma = m/\gamma_1 \) is the viscous relaxation time.

The analysis for \( g(\lambda) \) and \( \mu(\lambda) \) are given in sections 6 and 7, respectively. In appendix, we give the detailed calculation for computation of the moment generating function \( Z(\lambda) \sim g(\lambda)e^{s\mu(\lambda)} \), in the large time limit.

4. Probability distribution function

The probability distribution function for generalized partial entropy production \( \Delta S_{\text{tot}}^A \) or any observable whose moment generating function \( Z(\lambda) \) is of the form given by (29), is obtained by inverting it using inverse transformation

\[ P(\Delta S_{\text{tot}}^A = s\tau/\tau_\gamma) = \int_{-\infty}^{+i\infty} \frac{d\lambda}{2\pi i} Z(\lambda)e^{s\mu(\lambda)} \approx \int_{-\infty}^{+i\infty} \frac{d\lambda}{2\pi i} g(\lambda)e^{(\tau/\tau_\gamma)\mu(\lambda) + \lambda s}, \] (34)

where \( \tau_\gamma \) is the characteristic time-scale, \( \mu(\lambda) = \tau_\gamma \mu(\lambda) \) is the scaled cumulant generating function, and the contour of integration is taken along the direction of imaginary axis passing through the origin of the complex \( \lambda \)-plane. If both \( \mu(\lambda) \) and \( g(\lambda) \) are analytic functions of \( \lambda \), for large time \( (\tau \gg \tau_\gamma) \), we can approximate the above integral using saddle-point method. Therefore, we get \([14, 20–23]\)

\[ P(\Delta S_{\text{tot}}^A = s\tau/\tau_\gamma) \approx \frac{g(\lambda^*)e^{(\tau/\tau_\gamma)K(\lambda^*)}}{\sqrt{2\pi(\tau/\tau_\gamma)|K''(\lambda^*)|}}, \] (35)

where the saddle point \( \lambda^*(s) \) is calculated by solving the following equation

\[ \frac{\partial \mu(\lambda)}{\partial \lambda} \bigg|_{\lambda = \lambda^*(s)} = -s \] (36)

and the function \( K(\lambda^*) = \mu(\lambda^*) + \lambda^*s \).

In (35),

\[ K''(\lambda^*) = \frac{\partial^2}{\partial \lambda^2} [\mu(\lambda) + \lambda s] \bigg|_{\lambda = \lambda^*(s)}. \] (37)

Now, assume that both \( g(\lambda) \) and \( \mu(\lambda) \) satisfy Gallavotti–Cohen symmetry (condition I) \([6]\), i.e. \( g(\lambda) = g(1 - \lambda) \) and \( \mu(\lambda) = \mu(1 - \lambda) \). Therefore, we can write the probability distribution function for negative entropy production as

\[ P(\Delta S_{\text{tot}}^A = -s\tau/\tau_\gamma) \approx \int_{-\infty}^{+i\infty} \frac{d\lambda}{2\pi i} (1 - \lambda)e^{(\tau/\tau_\gamma)\mu(1 - \lambda) + (1 - \lambda)s + s} \\
\approx e^{-s\tau/\tau_\gamma} \int_{-\infty}^{+i\infty} \frac{d\lambda}{2\pi i} g(\lambda)e^{(\tau/\tau_\gamma)\mu(\lambda) + \lambda s}. \] (38)
If both \( g(\lambda) \) and \( \tilde{\mu}(\lambda) \) do not have singularities between \( \lambda \in [0, 1] \) (condition II), we can easily shift the contour of integration from \((1 - i\infty, 1 + i\infty)\) to \((-i\infty, +i\infty)\). Therefore, from (34) and (38), we get

\[
P(\Delta S_{\text{tot}}^{A} = s\tau / \tau_{\gamma}) \approx e^{s\tau / \tau_{\gamma}}, \tag{39}
\]

The equation (39) is the fluctuation theorem for \( \Delta S_{\text{tot}}^{A} \) in steady state. If \( g(\lambda) \) and \( \tilde{\mu}(\lambda) \) satisfy both conditions I and II, then the corresponding observable (for instance, in our case, the observable is \( \Delta S_{\text{tot}}^{A} \)) will satisfy fluctuation theorem. Note that we have proved this result for large but finite time.

In the following section, we discuss the result for the entropy production for a single Brownian particle connected to a thermal gradient.

## 5. Single Brownian particle in contact with two heat baths (\( \delta = 0 \))

In this section, we consider only one Brownian particle (say particle-A) in the contact with two heat baths at temperatures \( T_{1} \) and \( T_{2} < T_{1} \) (see figure 1 with \( \delta = 0 \)). Note that this case is different from the case \( \delta \to 0 \). Here, we raise two important points for this model: (1) the medium entropy production will not obey fluctuation theorem in the steady state, and (2) when the system entropy production is added to the medium entropy production, the total entropy production satisfies the fluctuation theorem in the steady state.

The Langevin equation for Brownian particle-A is

\[
m \dot{v}_{A} = -(\gamma_{1} + \gamma_{2})v_{A}(t) + \eta_{1}(t) + \eta_{2}(t). \tag{40}
\]

Entropy production in both baths due to particle-A is given as

\[
\Delta S_{\text{med}} = -\left[ \frac{Q_{1}}{T_{1}} + \frac{Q_{2}}{T_{2}} \right] = \Delta \beta Q_{1} - \frac{m}{2T_{2}} [v_{A}^{2}(\tau) - v_{A}^{2}(0)], \tag{41}
\]

where \( Q_{1} = \int_{0}^{\tau} dt [\eta_{1} - \gamma_{1}v_{A}]v_{A}(t) \), is the heat energy given by the bath of the temperature \( T_{1} \) and dissipation constant \( \gamma_{1} \) to the Brownian particle-A. Therefore, the restricted moment generating function for medium entropy production is given as

\[
Z(\lambda, v_{A}, \tau|v_{A}(0)) = \exp \left[ \frac{\lambda m}{2T_{2}} \{v_{A}^{2} - v_{A}^{2}(0)\} \right] \left\langle e^{-\lambda \Delta \beta Q_{1} \delta [v_{A} - v_{A}(\tau)]} \right\rangle_{v_{A}, v_{A}(0)}
\]

\[
= \exp \left[ \frac{\lambda m}{2T_{2}} \{v_{A}^{2} - v_{A}^{2}(0)\} \right] \tilde{Z}(\lambda, v_{A}, \tau|v_{A}(0)), \tag{42}
\]

where \( \tilde{Z}(\lambda, v_{A}, \tau|v_{A}(0)) \) satisfies the following differential equation \([45, 46]\)

\[
\frac{\partial \tilde{Z}(\lambda, v_{A}, \tau|v_{A}(0))}{\partial \tau} = \mathcal{L}_{A}^{\lambda} \tilde{Z}(\lambda, v_{A}, \tau|v_{A}(0)). \tag{43}
\]

In the above equation, the differential operator \( \mathcal{L}_{A}^{\lambda} \) is given as

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https://doi.org/10.1088/1742-5468/aabfca
Partial entropy production in heat transport

\[ \mathcal{L}_{\lambda}^A = \frac{\gamma_1 T_1 + \gamma_2 T_2}{m^2} \frac{\partial^2}{\partial v_A^2} + \frac{v_A}{m} (\gamma_1 + \gamma_2 + 2\lambda \Delta \beta \gamma_1 T_1) \frac{\partial}{\partial v_A} + \left[ \frac{\gamma_1 + \gamma_2}{m} + \lambda \Delta \beta \left( \frac{\gamma_1 v_A^2}{m} + \frac{\gamma_1 T_1}{m} \right) + \lambda^2 \Delta \beta^2 v_A^2 \gamma_1 T_1 \right]. \]  

(44)

The differential equation (43) is subjected to initial condition $\tilde{Z}(\lambda, v_A, \tau|v_A(0)) = \delta[v_A - v_A(0)]$. One can solve this differential equation exactly [25, 47]. In the limit of large $\tau$, we get

\[ Z(\lambda, v_A, \tau|v_A(0)) \approx e^{(\tau/t_\gamma)\mu_0(\lambda)} \sqrt{\frac{m(\gamma_1 + \gamma_2)\nu(\lambda)}{2\pi(\gamma_1 T_1 + \gamma_2 T_2)}} \times \exp \left[-\frac{m(\gamma_1 + \gamma_2)v_A^2}{4(\gamma_1 T_1 + \gamma_2 T_2)} \{\nu(\lambda) - 2\lambda - 1\} - \frac{m(\gamma_1 + \gamma_2)v_A^2(0)}{4(\gamma_1 T_1 + \gamma_2 T_2)} \{\nu(\lambda) + 2\lambda - 1\} \right], \]

where $t_\gamma = 2m/(\gamma_1 + \gamma_2)$. Integrating the above equation over the initial steady state distribution $Z(0, v_A, \tau \to \infty|v_A(0)) = \tilde{P}_{ss}(v_A(0))$ given in (15), and final variables $v_A$, we get

\[ Z(\lambda) \approx e^{(\tau/t_\gamma)\mu_0(\lambda)} g_0(\lambda), \]

where

\[ \mu_0(\lambda) = 1 - \nu(\lambda), \]

\[ g_0(\lambda) = \frac{2\sqrt{\nu(\lambda)}}{\sqrt{1 + \nu(\lambda) - 2\lambda} \sqrt{1 + \nu(\lambda) + 2\lambda}}. \]

(48)

(49)

In the above equation, $\lambda_\pm = 1/2[1 \pm \sqrt{1 + 1/\alpha}]$ in which $\alpha = \gamma_1 \gamma_2 T_1 T_2 \Delta \beta^2/(\gamma_1 + \gamma_2)^2$. In (48), the first factor in the denominator comes from the integration over final variable $v_A$ while the second one comes from the integration over the initial state $v_A(0)$ with respect to the distribution $\tilde{P}_{ss}(v_A(0))$. Here, $\mu_0(\lambda)$ is analytic function when $\lambda \in (\lambda_-, \lambda_+)$. First denominator in $g_0(\lambda)$ has one branch point at $\lambda = \lambda_n = 1$ for all $\alpha \in (0, \infty)$ where $\lambda_n \in (\lambda_-, \lambda_+)$ when $1 - 2\lambda_+ < 0$ while second denominator has a branch point at $\lambda = \lambda_b = (\alpha - 1)/(\alpha + 1)$ for $\alpha \leq 1/3$ where $\lambda_- \leq \lambda_b < \lambda_+ \leq 1 + 2\lambda_- \leq 0$.

The probability distribution function of the medium entropy production $\Delta S_{med}$ can be obtained by inverting $Z(\lambda)$ given in (46) (see section 4). Here, the saddle point $\lambda^*_0(s)$ is given by solving the following equation

\[ \frac{\partial \mu_0(\lambda)}{\partial \lambda} \bigg|_{\lambda = \lambda^*_0(s)} = -s. \]

(50)

When $\alpha > 1/3$, $g_0(\lambda)$ has only one singularity, i.e. at $\lambda = \lambda_n$. Therefore, the saddle point $\lambda^*_0(s)$ moves from $\lambda_- \to \lambda_n$ as $s$ decreases from $s = +\infty$ to $s_n$, and gets stuck at $\lambda = \lambda_n$, where $s_n$ is

https://doi.org/10.1088/1742-5468/aabfca

J. Stat. Mech. (2018) 063203
the solution of $\lambda_0'(s_a) = \lambda_a$. Therefore, $P(\Delta S_{\text{med}} = s\tau/t_\gamma) \sim e^{(\tau/t_\gamma)[\mu_0(\lambda_-) + \lambda_- s]}$ as $s \to +\infty$. Around $s = s_a$, i.e. $\lambda_0' = \lambda_a - \epsilon_a$, where $0 < \epsilon_a \ll 1$, $P(\Delta S_{\text{med}} = s\tau/t_\gamma) \sim e^{(\tau/t_\gamma)[\mu_0(\lambda_a) + \lambda_a s]}$. This implies for large $s$, we find [20–23, 25, 43, 44]

$$\lim_{(\tau/t_\gamma) \to \infty} \frac{t_\gamma}{\tau} \ln \frac{P(\Delta S_{\text{med}} = s\tau/t_\gamma)}{P(\Delta S_{\text{med}} = -s\tau/t_\gamma)} = \mu_0(\lambda_-) - \mu_0(\lambda_a) + [\lambda_- + \lambda_a] s. \quad (51)$$

Similarly, for $\alpha < 1/3$, $g_0(\lambda)$ has both singularities, i.e. $\lambda_a$ and $\lambda_b$. In this case, (51) modifies as

$$\lim_{(\tau/t_\gamma) \to \infty} \frac{t_\gamma}{\tau} \ln \frac{P(\Delta S_{\text{med}} = s\tau/t_\gamma)}{P(\Delta S_{\text{med}} = -s\tau/t_\gamma)} = \mu_0(\lambda_b) - \mu_0(\lambda_a) + [\lambda_b + \lambda_a] s. \quad (52)$$

Therefore, we find that entropy production in the medium will not satisfy fluctuation theorem for large scaled parameter $s$ for any $\alpha \in (0, \infty)$. It is interesting to note that once we incorporate the entropy production of the system into entropy production in the medium, i.e. $\Delta S_{\text{tot}} = \Delta S_{\text{med}} + \Delta S_{\text{sys}}$, where $\Delta S_{\text{sys}} = -\ln \tilde{P}(v_\Lambda(\tau)) + \ln \tilde{P}(v_\Lambda(0))$, in which $\tilde{P}_{ss}(v_\Lambda(\tau))$ is given in (15), the prefactor term $g_0(\lambda)$ corresponding to $\Delta S_{\text{tot}}$ modifies to

$$g_0(\lambda) = \frac{2\sqrt{\nu(\lambda)}}{1 + \nu(\lambda)}, \quad (53)$$

and it is analytic function within the same domain as that of $\mu_0(\lambda)$. Therefore, both $\mu_0(\lambda)$ and $g_0(\lambda)$ satisfy condition I and II. Consequently, in this case, total entropy production satisfies fluctuation theorem for all $\alpha$.

6. Analysis for prefactor $g(\lambda)$ for $\delta \neq 0$

In our problem, it is difficult to obtain the analytical expression for the prefactor term $g(\lambda)$ given in (A.38), as it requires the computation of matrices $H_1(\lambda)$, $H_2(\lambda)$ and $H_3(\lambda)$ (see (A.32)–(A.34)) which is a non-trivial problem. Since we are interested in the weak coupling limit ($\delta \to 0$), we can write

$$g(\lambda) \approx g_0(\lambda) + \delta^a g_1(\lambda), \quad \text{where} \quad a > 0. \quad (54)$$

The term $g_0(\lambda)$ is the same prefactor term as given in (53). The term $g_1(\lambda)$ may have singularities, but in the limit $\delta \to 0$, we can approximate the correction term as [43, 44]

$$g(\lambda) \approx g_0(\lambda). \quad (55)$$

7. Analysis for cumulant generating function $\mu(\lambda)$ for $\delta \neq 0$

In this section, we show how to compute the cumulant generating function. For simplicity, we assume $\alpha_{12} = \alpha_{13} = 1$, i.e. $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. Therefore, $h(u, \lambda)$ and $q(u)$ in (31) become

https://doi.org/10.1088/1742-5468/aabfca
The roots of the quadratic equation (56) are given as
\[ u = (1 \pm \delta)(1 - \delta - \frac{4}{3}) \pm \sqrt{(1 - \delta - \frac{4}{3})^2 - \frac{4}{9}}. \]

We see that arguments of the logarithm of the integrand in (31) are quite involved and not very illuminating to us. Therefore, we compute \( h(u, \lambda) + q(u) \) as
\[ h(u, \lambda) + q(u) = -p_1(u)\lambda^2 + p_2(u)\lambda + q(u) = 0, \]
where
\[ p_1(u) = 4\beta_{12}[(1 - \beta_{12})^2\{u^4 + u^2(1 - \delta) + \delta^2/4\} + \delta^2\beta_{13}(\beta_{12} + \Pi - 1)/4] \]
\[ + \delta^2\beta_{13}[(\Pi - 1)(\beta_{12} + \Pi - 1) + \beta_{12}\Pi^2], \]
and
\[ p_2(u) = 4\beta_{12}[(1 - \beta_{12})^2\{u^4 + u^2(1 - \delta) + \delta^2/4\} + \delta^2\beta_{13}(\beta_{12} + \Pi - 1)/4] \]
\[ - \delta^2\beta_{12}[\beta_{12} + \Pi - 1 + \Pi(\beta_{12} - \beta_{13})]. \]

The roots of the quadratic equation (59) are given as
\[ \lambda^\delta_{\pm}(u) = \frac{p_2(u) \pm \sqrt{p_2(u)^2 + 4p_1(u)q(u)}}{2p_1(u)}. \]
where

\[ r_{\pm}^P(\beta_{12}, \beta_{13}, \delta) = (1 - \beta_{12})^2[4x_1 \mp (1 + 6\beta_{12} + \beta_{12}^2)] \pm \beta_{13}(1 + \beta_{12})[3(1 - \beta_{12})^2
\] 
\[ + 4\beta_{13}(1 + \beta_{12})] \mp [(1 - \beta_{12})^2 + \beta_{13}(1 + \beta_{12})^2\delta]. \]

In (64), we take \( 0 < \delta < 1 \), and function \( x_1 \) is given by

\[ x_1 = \sqrt{(1 + \beta_{12})(1 + \beta_{12} + \beta_{13})[(\beta_{12} - 1/2)^2 + \beta_{13}(1 + \beta_{12}) + 3/4]}, \]

where \( x_1 \) is a positive functions of \( \beta_{12} \) and \( \beta_{13} \).

While the function \( r_{P}^+(\beta_{12}, \beta_{13}, \delta) > 0 \) for all \( \beta_{12}, \beta_{13} \) at \( 0 < \delta < 1 \) which indicates that the function \( \lambda_{\delta}^+(u) \) has similar behavior as shown in figure 2(b), the function \( r_{P}^-(\beta_{12}, \beta_{13}, \delta) \) changes sign depending upon the choice of parameters \( \beta_{12}, \beta_{13} \) at \( 0 < \delta < 1 \). Therefore, the contour separating these two regions is given by

\[ r_{P}^-(\beta_{12}, \beta_{13}, \delta) = 0. \]  

(65)

We plot the phase diagram as shown in figure 3(a) in the limit \( \delta \to 0 \), in which red dashed contour corresponds to above equation. Thus, \( \lambda_{\delta}^-(u) \) has similar behaviours as shown in figures 2(a) and (c) for region I and II, respectively, of figure 3(a).
In the case of apparent entropy production ($\Pi = 0$), we find

$$\frac{\partial^2 \lambda_\pm(u)}{\partial u^2} \bigg|_{u=0} = \frac{2\beta_{12}(1 - \delta)r_\pm^A(\beta_{12}, \beta_{13}, \delta)}{\delta^2 x_2(1 - \beta_{12})(\beta_{12} + \beta_{13})^2},$$

(66)

where

$$r_\pm^A(\beta_{12}, \beta_{13}, \delta) = \mp[(\beta_{12} - \beta_{13})(4 \pm 2x_2) + \beta_{12}\beta_{13}(1 + 2\delta) + \beta_{13}^2(3 + \delta) - \beta_{12}^2(2 - \delta)].$$

In (66), we take $0 < \delta < 1$, and function $x_2$ is given by

$$x_2 = \sqrt{(1 + \beta_{12} + \beta_{13})(4 + \beta_{12} + \beta_{13})},$$

(67)

where $x_2$ is positive function of $\beta_{12}$ and $\beta_{13}$.

The function $r_\pm(\beta_{12}, \beta_{13}, \delta) > 0$ for all $\beta_{12}, \beta_{13}$ and $0 < \delta < 1$ which suggests that the function $\lambda_\pm(u)$ has variation with respect to $u$ as shown in figure 2(a). The function $r_\pm(\beta_{12}, \beta_{13}, \delta)$ can be either positive or negative depending upon the choice of $\beta_{12}, \beta_{13}$ at $0 < \delta < 1$. Therefore, the equation of contour separating these two regions is given as

$$r_\pm^A(\beta_{12}, \beta_{13}, \delta) = 0.$$  

(68)

In the limit $\delta \to 0$, the phase diagram is shown in figure 3(b) where red dashed contour is given by above equation. Thus, the root $\lambda_\pm(u)$ has the variations as shown in figures 2(d) and (b) in region I and II, respectively, of figure 3(b). Note that in both phase diagram, the axis (not shown) corresponds to $\delta$ is perpendicular to the plane of paper.

The extrema of $\lambda_\pm(u)$ shown in figure 2, give the cut-off on the real line of the complex $\lambda$-plane within which the cumulant generating function $\mu(\lambda)$ is a real function.
Note that the saddle point \( \lambda^*(s) \) also lies within this domain. The equation for extremum is given by
\[
\frac{\partial \lambda^\delta(u)}{\partial u} \bigg|_{u=u^*_\pm} = 0,
\]
where \( u^*_\pm \) is the solution of the following equation
\[
\sqrt{p_2^\delta(u^*_\pm) + 4p_1(u^*_\pm)q(u^*_\pm) \pm p_2(u^*_\pm)[p_1(u^*_\pm)p_2'(u^*_\pm) - p_2(u^*_\pm)p_1'(u^*_\pm)]} \\
\pm 2p_1(u^*_\pm)[p_1(u^*_\pm)q'(u^*_\pm) - q(u^*_\pm)p_1'(u^*_\pm)] = 0.
\]
In the above equation, \( \lambda \) represents the derivative with respect to \( u \). In the weak coupling limit (\( \delta \to 0 \)), we compute \( u^*_\pm \) (upto leading order in \( \delta \)) using perturbation theory for both definitions of entropy production. In the case of partial entropy production (\( \Pi = 1 \)),
\[
u^*_{\pm} = \begin{cases} 
\pm \sqrt{\frac{5-3\beta_{13} - 4\beta_{12}(1+2\beta_{13})}{2(1-\beta_{12})}} + o(\sqrt{\delta}) & \text{for region I of figure 3(a)}, \\
0 & \text{for region II of figure 3(a)}, \\
\end{cases}
\]
\( u^*_+ = 0 \) for region I and II of figure 3(a). Thus, \( \lambda^\delta(u^*_-) \to \lambda_- \) in the region I whereas \( \lambda^\delta(u^*_+) \to \tilde{\lambda}_- \) in the region II of figure 3(a) in the limit of \( \delta \to 0 \). Similarly, \( \lambda^\delta_+(u^*_+) \to \tilde{\lambda}_+ \) for both regions I and II of figure 3(a) in the limit \( \delta \to 0 \).

In the case of apparent entropy production (\( \Pi = 0 \)),
\[
u^*_{\pm} = \begin{cases} 
\pm \sqrt{\frac{5\beta_{12} + 4\beta_{12}(1-\beta_{13}) - 4\beta_{13}}{2\beta_{12}}} + o(\sqrt{\delta}) & \text{for region I of figure 3(b)}, \\
0 & \text{for region II of figure 3(b)}, \\
\end{cases}
\]
\( u^*_+ = \sqrt{\frac{\delta}{2}} + o(\sqrt{\delta}) \) for region I and II of figure 3(b). In the weak coupling limit, \( \lambda^\delta_+(u^*_+) \to \lambda_+ \) in the region I and \( \lambda^\delta_+(u^*_+) \to \tilde{\lambda}_+ \) in the region II of figure 3(b). Similarly, \( \lambda^\delta_+(u^*_+) \to \lambda_- \) for both regions I and II of figure 3(b) in the limit \( \delta \to 0 \).

When \( \Pi = 1 \), \( \tilde{\lambda}_\pm \) are given by
\[
\tilde{\lambda}_\pm = \frac{1 + \beta_{13} + \beta_{12}(-4 + \beta_{12} + \beta_{13}) \pm x_1}{2[1 + \beta_{13} + \beta_{12}(-2 + \beta_{12} + \beta_{13})]},
\]
whereas \( \tilde{\lambda}_+ \) for \( \Pi = 0 \) is
\[
\tilde{\lambda}_+ = \frac{\beta_{12}(2 - \beta_{12} - \beta_{13} + x_2)}{2(1 - \beta_{12})(\beta_{12} + \beta_{13})}.
\]
One can find \( \lambda_\pm \) in section 5.

7.1. Gallavotti–Cohen symmetry for cumulant generating function \( \mu(\lambda) \)

It can be seen from (31) that \( \mu(\lambda) \) does not satisfy Gallavotti–Cohen symmetry, i.e. \( \mu(\lambda) \neq \mu(1 - \lambda) \) for large \( \delta \), which is also a signature of partial measurement. In figure 4, we have plotted the cumulant generating function \( \mu(\lambda) \) (blue solid line) and \( \mu(1 - \lambda) \) (red dashed line) against \( \lambda \) for: (a) region I of figures 3(a) and (b) region II of figures 3(a) and (c) region I of figures 3(b) and (d) region II of figure 3(b). All of the figures are
plotted for fixed coupling parameter $\delta = 10^{-10}$. Except for region I of figure 3(b), the cumulant generating function does not satisfy the Gallavotti–Cohen symmetry even in the limit $\delta \to 0$.

Figure 4. The cumulant generating function $\mu(\lambda)$ (blue solid line) and $\mu(1-\lambda)$ (red dashed line) are plotted against $\lambda$ for: (a) region I of figures 3(a) and (b) region II of figures 3(a) and (c) region I of figures 3(b) and (d) region II of figure 3(b). All of the above figures are plotted for fixed coupling parameter $\delta = 10^{-10}$. Except for region I of figure 3(b), the cumulant generating function $\mu(\lambda)$ does not satisfy the Gallavotti–Cohen symmetry even in the limit $\delta \to 0$.

8. Large deviation function, fluctuation theorem and asymmetry function

Both $\mu(\lambda)$ and $g_0(\lambda)$ are real functions within any pair of singularities, i.e. $\lambda_+^\delta (u_+^\delta)$ and $\lambda_-^\delta (u_-^\delta)$ depending upon the choice of $\beta_{12}$ and $\beta_{13}$ in the weak coupling limit. The saddle...
Point \( \lambda^*(s) \) moves from \( \lambda_\delta^-(u_-^*) \) to \( \lambda_\delta^+(u_+^*) \) as \( s \) decreases from \( +\infty \) to \( -\infty \). Therefore, the probability density function \( p(s) \) has the following large deviation form [14]

\[
p(s) \sim e^{(\tau/\tau_\gamma) I(s)}
\]

(73)

where the probability density function is (see (35))

\[
p(s) = (\tau/\tau_\gamma)P(\Delta S_{\text{tot}}^A = s\tau/\tau_\gamma),
\]

(74)

and \( I(s) = K(\lambda^*(s)) \) is the large deviation function [14]. We define an asymmetry function \( f(s) \) as

\[
f(s) = \frac{\tau_\gamma}{\tau} \ln \left[ \frac{p(s)}{p(-s)} \right].
\]

(75)

Thus, in the large time limit \( \tau/\tau_\gamma \to \infty \), the asymmetry function is given by

\[
f(s) = \lim_{\tau/\tau_\gamma \to \infty} \frac{\tau_\gamma}{\tau} \ln \left[ \frac{p(s)}{p(-s)} \right] = I(s) - I(-s).
\]

(76)

The quantity of interest is variation of the asymmetry function \( f(s) \) with the scaled parameter \( s = \Delta S_{\text{tot}}^A \tau/\tau_\gamma \), and deviation from \( f(s) = s \) will indicate the violation of steady state fluctuation theorem. In the limit \( \delta \to 0 \), the asymptotic behavior of \( f(s) \) in the case of partial entropy production for \( s \to \infty \) is given by [43]

\[
f(s) = \begin{cases} 
\mu_0(\lambda_-) - \mu_0(\lambda_+) + (\lambda_- + \lambda_+)s, & \text{for region I of figure 3(a)}, \\
\mu_0(\lambda_-) - \mu_0(\lambda_+) + (\lambda_- + \lambda_+)s, & \text{for region II of figure 3(a)},
\end{cases}
\]

(77)

whereas for apparent entropy production

\[
f(s) = \begin{cases} 
s, & \text{for region I of figure 3(b)}, \\
\mu_0(\lambda_-) - \mu_0(\lambda_+) + (\lambda_- + \lambda_+)s, & \text{for region II of figure 3(b)},
\end{cases}
\]

(78)

and \( f(-s) = -f(s) \). In (77) and (78), the cumulant generating function \( \mu_0(\lambda) \) corresponds to the case \( \delta = 0 \), is given in (47).

We plot analytical asymmetry function \( f(s) \) given in (76) against \( s \) in figures 6(a)–(d) and 7(a)–(c) for partial and apparent entropy production, respectively, for \( \delta = 0.1 \) (red dashed line) and \( \delta = 0.01 \) (black solid line). The asymmetry function \( f(s) \) in the limit \( \delta \to 0 \) (orange dotdashed line) is also plotted for each case [43]. In these figures, magenta tiny dashed lines correspond to the asymptotic expression of asymmetry function \( f(s) \) given by (77) and (78) for partial and apparent entropy production, respectively. One can see, as the coupling parameter \( \delta \) reduces, the asymmetry function \( f(s) \) converges to that of \( \delta \to 0 \) case.

9. Numerical simulation

In figures 5(a)–(d), we show the comparison of theoretical predictions (red dashed line) of probability density function \( p(s) \) given by (74) and the asymmetry function \( f(s) \) given by (75) for partial entropy production for coupling parameter \( \delta = 0.1 \) with the

https://doi.org/10.1088/1742-5468/aabfca
numerical simulation results (blue dots) with time $\tau = 50.0$ (figures 5(a) and (b)) and $\tau = 150.0$ (figures 5(c) and (d)). This comparison indicates that as the observation time $\tau/\tau_γ$ increases, the agreement between theoretical predictions and numerical simulation results gets better.

We compare the analytical asymmetry functions $f(s)$ given in (75) (red dashed line) with the numerical simulation results (blue dots) for $\delta = 0.1$ in figures 6(e)–(h) and 7(d)–(f) for partial and apparent entropy production, respectively. The probability density function $p(s)$ (red dashed line) given in (74) is also compared with numerical simulation results (blue dots) for $\delta = 0.1$ in figures 6(i)–(l) and 7(g)–(i) for partial and apparent entropy production, respectively. All of these results are compared at time $\tau/\tau_γ = 150$, and show that there is nice agreement between theoretical predictions and numerical simulations.
Figure 6. The analytically evaluated asymmetry function $f(s)$ given in (76) for partial entropy production is plotted against the scaled variable $s = \Delta S_A^{\text{tot}} / \tau$ in figures (a)–(d) for respective $\beta_{12}$ and $\beta_{13}$ of phase diagram shown in figure 3(a). These plots are obtained for $\delta = 0.1$ (red dashed line) and $\delta = 0.01$ (black solid line). The asymmetry functions in the limit $\delta \to 0$ (orange dotdashed line) are also plotted for respective cases [43]. The asymptotic behaviours for asymmetry function $f(s)$ for partial entropy production given in (77) are shown by magenta tiny dashed lines. The comparison of analytical results (red dashed line) for asymmetry function $f(s)$ given by (75) and probability density function $p(s)$ given by (74) with the numerical simulations (blue dots) for partial entropy production are shown in figures (e)–(h) and (i)–(l), respectively. These comparisons are shown for fixed $\delta = 0.1$ and $\tau / \gamma = 150.0$. 

https://doi.org/10.1088/1742-5468/aabfca
10. Single Brownian particle in contact with three heat baths

In the previous sections (except section 5), we computed the partial and apparent entropy production of a system composed of DOFs having time scale of relaxation much larger than that of bath DOFs. The system and the bath DOFs can be called as slow and fast DOFs. In this section, we consider a single Brownian particle of mass $m$, in contact with three heat reservoirs of temperatures $T_1, T_2, T_3$ and respective dissipation constants $\gamma_1, \gamma_2, \gamma_3$. Here, our aim is to understand the effect of hidden fast DOFs on the fluctuation theorem in the steady state in the weak coupling limit ($\delta \to 0$).
The velocity \( v(t) \) of the Brownian particle evolves according to underdamped Langevin equation
\[
m \ddot{v} = [-\gamma_1 v(t) + \eta_1(t)] + [-\gamma_2 v(t) + \eta_2(t)] + \delta[-\gamma_3 v(t) + \eta_3(t)].
\]
(79)

The thermal noises \( \eta_1, \eta_2 \) and \( \eta_3 \) acting on the Brownian particle, have mean zero and correlations \( \langle \eta_i(t) \eta_j(t') \rangle = 2\gamma_i \delta(t - t') \). Since the velocity is linear in thermal noises, the steady state distribution is given by
\[
P_{\text{ss}}(v(\tau)) = \frac{1}{\sqrt{2\pi \sigma_v^2}} \exp \left[ -\frac{v^2(\tau)}{2\sigma_v^2} \right],
\]
where
\[
\sigma_v^2 = \frac{\gamma_1 T_1 + \gamma_2 T_2 + \delta^2 \gamma_3 T_3}{m(\gamma_1 + \gamma_2 + \delta \gamma_3)}.
\]

Here the observable is the partial entropy production \( \Delta S_{\text{tot}}^{12} \) of the Brownian particle due to two heat baths with temperatures (dissipation constants) \( T_1 (\gamma_1) \) and \( T_2 (\gamma_2) \) in the steady state for a duration \( \tau \)
\[
\Delta S_{\text{tot}}^{12} = -\left( \frac{Q_1}{T_1} + \frac{Q_2}{T_2} \right) - \ln P_{\text{ss}}(v(\tau)) + \ln P_{\text{ss}}(v(0)) = Q + \frac{1}{2\sigma_v^2}[v^2(\tau) - v^2(0)],
\]
(81)

where
\[
Q = a_0 Q_1 + b_0 Q_2,
\]
(82)
in which \( a_0 = -1/T_1 \) and \( b_0 = -1/T_2 \).

The Fokker–Planck equation for the restricted moment generating function of \( Q \) is given by [45, 46]
\[
\frac{\partial Z_{\text{Q}}(\lambda, v, \tau|v_0)}{\partial \tau} = \left[ \frac{\gamma_1 T_1 + \gamma_2 T_2 + \delta^2 \gamma_3 T_3}{m^2} \frac{\partial^2}{\partial v^2} + \frac{\gamma_1 + \gamma_2 + \delta \gamma_3 + 2\lambda(a_0 \gamma_1 T_1 + b_0 \gamma_2 T_2)}{m} \right] v \frac{\partial Z_{\text{Q}}(\lambda, v, \tau|v_0)}{\partial v} + \lambda[(a_0 \gamma_1 + b_0 \gamma_2)v^2
\]
\[- (a_0 \gamma_1 T_1 + b_0 \gamma_2 T_2)/m] + (a_0^2 \gamma_1 T_1 + b_0^2 \gamma_2 T_2)\lambda^2 v^2] Z_{\text{Q}}(\lambda, v, \tau|v_0),
\]
(83)

where subscript \( Q \) represents that the restricted moment generating function \( Z_{\text{Q}}(\lambda, v, \tau|v_0) \) is obtained from \( P(Q, v, \tau|v_0) \) using Fourier transform over \( Q \). The differential equation given above is subjected to initial condition \( Z_{\text{Q}}(\lambda, v, \tau = 0|v_0) = \delta(v - v_0) \).

For simplicity, we choose \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma \). In the large time limit \( (\tau \to \infty) \), the solution of (83) is given by [25, 47]
\[
Z_{\text{Q}}(\lambda, v, \tau|v_0) = e^{(v/v_0)\mu(\lambda)} \sqrt{\frac{m \nu(\lambda)}{2\pi \Theta}} \exp \left[ -\frac{mv_v^2}{4\Theta}[2 + \delta - 4\lambda + \nu(\lambda)] \right] 
\]
\[ \times \exp \left[ -\frac{mv_v^2}{4\Theta}[-2 - \delta + 4\lambda + \nu(\lambda)] \right] + \ldots.
\]
(84)

In the above equation, the characteristic time scale \( \tau_\gamma = m/\gamma, \Theta = T_1 + T_2 + \delta^2 T_3, \) and
\[
\bar{\mu}(\lambda) = \frac{1}{2}[2 + \delta - \bar{\nu}(\lambda)],
\]

with
\[
\bar{\nu}(\lambda) = \sqrt{(2 + \delta - 4\lambda)^2 - 4\Theta(1 - \lambda)(a_0 + b_0)} = \sqrt{(\bar{\alpha}_0 - 16)(\bar{\lambda}_+^\delta - \lambda - \bar{\lambda}_+^\delta)}.
\]

In the above equation, the branch points \(\bar{\lambda}_+^\delta\) are
\[
\bar{\lambda}_+^\delta = \frac{1}{2(\bar{\alpha}_0 - 16)}[\bar{\alpha}_0 - 8(2 + \delta) \pm \sqrt{\bar{\alpha}_0 - 16 + 4\delta^2}],
\]
where \(\bar{\alpha}_0 = 4(1 + 1/\beta_{12})(1 + \beta_{12} + \delta^2 \beta_{13})\) in which \(\beta_{1j} = T_j/T_1\).

Therefore, the restricted moment generating function for partial entropy production \(\Delta S_{12}^{\text{tot}}\) can be written as
\[
\bar{Z}(\lambda, v, \tau|v_0) = \exp \left[ -\frac{\lambda}{2\sigma_v^2}[\nu^2(\tau) - \nu^2(0)] \right] Z_Q(\lambda, v, \tau|v_0)
\]
\[
= e^{(\tau/\tau_\gamma)\bar{\mu}(\lambda)} \frac{m\bar{\nu}(\lambda)}{2\pi\Theta} \exp \left[ -\frac{mv^2}{4\Theta}[2 + \delta + 2\lambda\delta + \bar{\nu}(\lambda)] \right]
\]
\[
\times \exp \left[ -\frac{mv^2}{4\Theta}[-2 - \delta - 2\lambda\delta + \bar{\nu}(\lambda)] \right] + \ldots.
\]

The moment generating function for the partial entropy production is given by
\[
\bar{Z}(\lambda) = \int dv \int dv_0 \ P_{ss}(v_0) \ \bar{Z}(\lambda, v, \tau|v_0) = e^{(\tau/\tau_\gamma)\bar{\mu}(\lambda)} \bar{g}(\lambda) + \ldots,
\]
where the prefactor is
\[
\bar{g}(\lambda) = \frac{2\sqrt{(2 + \delta)\nu(\lambda)}}{\sqrt{2 + \delta + 2\lambda\delta + \bar{\nu}(\lambda)} \sqrt{2 + \delta - 2\lambda\delta + \bar{\nu}(\lambda)}}.
\]

Here, first term is the denominator comes from integrating the restricted moment generating function of partial entropy production over the final variable \(v\) while the second term in the denominator arises from integrating the restricted moment generating function over the initial steady state ensemble \(P_{ss}(v_0)\).

In \(\bar{g}(\lambda)\), both of the denominators have one branch point each for particular choice of \(\beta_{12}, \beta_{13}\) and \(\delta\). Corresponding to first denominator, \(\lambda = \lambda_c\) is the zero of \([2 + \delta + 2\lambda\delta + \bar{\nu}(\lambda)]\), and \(\lambda_c \in [\bar{\lambda}_c^\delta, \bar{\lambda}_c^\delta]\) when \([2 + \delta + 2\delta\bar{\lambda}_c^\delta] \leq 0\) (condition 1). On the other hand, \(\lambda = \lambda_d\) is the zero of second denominator having \([2 + \delta - 2\lambda\delta + \bar{\nu}(\lambda)]\), and \(\lambda_d \in (\bar{\lambda}_d^\delta, \bar{\lambda}_d^\delta]\) only when \([2 + \delta - 2\delta\bar{\lambda}_d^\delta] \leq 0\) (condition 2). Condition 2 is simply given by \(\delta \geq 2\).

\(\lambda_{c,d}\) are given as
\[
\lambda_c = \frac{\bar{\alpha}_0 - 16 - 4\delta(\delta + 4)}{\bar{\alpha}_0 - 16 + 4\delta^2},
\]
\[
\lambda_d = 1.
\]
Since we are interested in the weak coupling limit \((\delta \to 0)\), using condition 1, we plot the phase diagram as shown in figure 8 in \((\beta_{12}, \beta_{13})\) plane for various values of \(\delta\). In the phase diagram, condition 1 does not hold to the left side of contours for respective \(\delta\). From the figure 8, it is clear that there is no singularity present in the prefactor \(\bar{g}(\lambda)\) in the weak coupling limit, i.e. \(\delta \to 0\). Hence, \(\bar{g}(\lambda)\) is analytic function of \(\lambda\) in the weak coupling limit. Therefore, in the limit \(\delta \to 0\), the probability density function \(\bar{p}(s)\) is given by

\[
\bar{p}(s) \sim e^{(\tau/\gamma)\bar{I}(s)},
\]

where \(\bar{I}(s) = \bar{\mu}(\bar{\lambda}^*) + \bar{\lambda}^* s\) is the large deviation function \([14]\). The saddle point \(\bar{\lambda}^*(s)\) is the solution of \(\bar{\mu}'(\bar{\lambda}^*) + s = 0\) which gives

\[
\bar{\lambda}^*(s) = \frac{1}{2(\bar{\alpha}_0 - 16)} \left[ \bar{\alpha}_0 - 8(2 + \delta) - 2s \sqrt{\frac{\bar{\alpha}_0(\bar{\alpha}_0 - 16 + 4\delta^2)}{\bar{\alpha}_0 - 16 + 4s^2}} \right],
\]

Therefore,

\[
\bar{I}(s) = 1 + \frac{\delta}{2} + s[\bar{\alpha}_0 - 8(2 + \delta)] - \frac{1}{4} \sqrt{\frac{\bar{\alpha}_0(\bar{\alpha}_0 - 16 + 4\delta^2)}{\bar{\alpha}_0 - 16 + 4s^2}} \left( 1 + \frac{4s^2}{\bar{\alpha}_0 - 16} \right).
\]

The asymmetry function \(f(s)\) in this case is

\[
f(s) = \bar{I}(s) - \bar{I}(-s) = \frac{s[\bar{\alpha}_0 - 16 - 8\delta]}{\bar{\alpha}_0 - 16}.
\]

Thus, in the weak coupling limit \((\delta \to 0)\), the fluctuation theorem is satisfied \((f(s) = s)\).
11. Summary

We have considered a coupled Brownian particle system. Both particles are connected by a harmonic spring of stiffness $k$. One of the particles is connected to a thermal gradient and the other one is connected to a single bath of a constant temperature. The main goal of this paper is to understand the deviation of fluctuation theorem for total entropy production of one of the particles (say particle-A) in the coupled system in the steady state when the interaction between the particles is weak. We have given two definitions of entropy production of partial system: partial and apparent entropy production. For convenience, we defined five dimensionless parameters:

1. $\delta = 2km/\gamma^2$
2. $\beta_{12} = T_2/T_1$
3. $\beta_{13} = T_3/T_1$
4. $\alpha_{12} = \gamma_2/\gamma_1$
5. $\alpha_{13} = \gamma_3/\gamma_1$

When $\alpha_{12} = \alpha_{13} = 1$, we plotted phase diagrams in ($\beta_{12}$, $\beta_{13}$) plane, for both definitions of entropy production, in the limit $\delta \to 0$. In the weak coupling limit ($\delta \to 0$), we have found that fluctuation theorem for apparent entropy production in the steady state is satisfied only in the region I of the phase diagram shown in figure 3(b). The results given above are also supported by the numerical simulations, and they have very nice agreement.

In a different model system where a single Brownian particle is connected with three heat baths of distinct temperatures (see section 10) in which the coupling $\delta$ of one of the baths with the particle is assumed to be very weak, we have computed the total entropy production of Brownian particle due to two heat baths (i.e. partial entropy production) in this system (see (81)). This model is exactly solvable. In the limit of $\delta \to 0$, we have seen that the fluctuation theorem for partial entropy production is restored in the steady state. The results given so far in the paper sound contradictory with the result obtained from this model. This is because, in the paper, we considered the coupling between slow variables (particles A and B) whereas in this example, we considered the coupling between slow DOF (Brownian particle) and fast DOFs (heat bath). Therefore, in the weak coupling limit, one may see the violation of fluctuation theorem for partial and apparent entropy production when slow DOFs are being coupled.

Appendix. Calculation for moment generating function $Z(\lambda)$

In this section, we give the complete calculation for the moment generating function for generalized partial entropy production in the large time limit. The system shown in figure 1 is described by coupled Langevin equation given in (3)–(5) and can be rewritten in the matrix form

$$\dot{y} = A^T V(t), \quad \text{(A.1)}$$

$$m \dot{V} = -\Gamma V(t) - kAy(t) + \xi(t), \quad \text{(A.2)}$$

where $A = (1, -1)^T$, $V = (v_A, v_B)^T$, $\xi = (\eta_A, \eta_B)^T$, and $\Gamma_{ij} = \delta_{ij}(\delta_{1j}\gamma_A + \delta_{2j}\gamma_B)$ with $\{i, j\} = \{1, 2\}$.

Consider a time dependent quantity $F(t)$. The finite time Fourier transform and its inverse are given as

https://doi.org/10.1088/1742-5468/aabfca

27
where $\omega_n = 2\pi n/\tau$. Using (A.3), one can write (A.1) and (A.2) in the frequency domain as

$$\tilde{y}(\omega_n) = A^T G \tilde{\xi}(\omega_n) - \frac{1}{\tau} \left[ (\gamma_3 + i\omega_n)(G_{22} - G_{12})\Delta y + mA^T G \Delta V \right],$$

(A.5)

$$\tilde{V}(\omega_n) = i\omega_n G \tilde{\xi}(\omega_n) + \frac{G}{\tau} [k A \Delta y - i m \omega_n \Delta V].$$

(A.6)

In the above equations, $\Delta y = y(\tau) - y(0)$, $\Delta V = V(\tau) - V(0)$, and the Green’s function matrix is $G(\omega_n) = [-m\omega_n^2 + i\omega_n\Gamma + \Phi]^{-1}$, where $\Phi_{rl} = k(2\delta_{i,r} - 1)$ with $\{l, r\} = \{1, 2\}$. Therefore, we can write $\tilde{y}(\omega_n)$ and $\tilde{v}_A(\omega_n)$ as

$$\tilde{y}(\omega_n) = (G_{11} - G_{12})[\tilde{y}_1(\omega_n) + \tilde{y}_2(\omega_n)] + (G_{12} - G_{22})\tilde{y}_3 - \frac{1}{\tau} \Delta U^T q_1,$$

(A.7)

$$\tilde{v}_A(\omega_n) = i\omega_n [G_{11}\{\tilde{y}_1(\omega_n) + \tilde{y}_2(\omega_n)\} + G_{12}\tilde{y}_3(\omega_n)] + \frac{1}{\tau} \Delta U^T q_2,$$

(A.8)

where

$$q_1^T = [(\gamma_3 + i\omega_n m)(G_{22} - G_{12}), m(G_{11} - G_{12}), m(G_{12} - G_{22})],$$

(A.9)

$$q_2^T = [k(G_{11} - G_{12}), -i m \omega_n G_{11}, -i m \omega_n G_{12}].$$

(A.10)

In the above equations, the matrix elements $G_{ij} = [G(\omega_n)]_{ij}$.

The row vector $U^T(\tau) = [y(\tau), V^T(\tau)]$ is given as

$$U^T(\tau) = \lim_{\epsilon \to 0} \sum_{n=\infty}^{\infty} e^{-i\epsilon \omega_n} [\tilde{y}(\omega_n), \tilde{V}^T(\omega_n)].$$

(A.11)

Substituting (A.5) and (A.6) in the above equation, we find that the terms

$$\lim_{\epsilon \to 0} \sum_{n=\infty}^{\infty} e^{-i\epsilon \omega_n} [(\gamma_3 + i\omega_n m)(G_{22} - G_{12})\Delta y + mA^T G \Delta V],$$

$$\lim_{\epsilon \to 0} \sum_{n=\infty}^{\infty} e^{-i\epsilon \omega_n} [k A^T \Delta y - i m \omega_n \Delta V] G^T,$$

go to zero. This is because in the limit of large $\tau$, we convert the summation into integration, and these terms have poles in the upper half of the complex $\omega$-plane. Therefore, using the calculus of residue, one can find that the contribution from these terms vanishes identically. This implies
\[ U^T(\tau) = \lim_{\epsilon \to 0} \sum_{n=-\infty}^{\infty} e^{-i\omega_n \left[ \{ \tilde{\eta}_1(\omega_n) + \tilde{\eta}_2(\omega_n) \} q_3^T + \tilde{\eta}_3(\omega_n) q_4^T \right]}, \]  

(A.12)

where

\[ q_3^T = (G_{11} - G_{12}, i\omega_n G_{11}, i\omega_n G_{12}), \]  

(A.13)

\[ q_4^T = (G_{12} - G_{22}, i\omega_n G_{12}, i\omega_n G_{22}). \]  

(A.14)

The mean and variance of \( U(\tau) \) are

\[ \langle U(\tau) \rangle = 0, \]  

(A.15)

\[ \langle U(\tau) U^T(\tau) \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \left[ (T_1 \gamma_1 + T_2 \gamma_2) q_3 q_3^\dagger + T_3 \gamma_3 q_4 q_4^\dagger \right], \]  

(A.16)

respectively. In (A.16), we have used the definition of correlation function of noises in the frequency domain as given by

\[ \langle \tilde{\eta}_i(\omega) \tilde{\eta}_j(\omega') \rangle = \frac{2T_i \gamma_i}{\tau} \delta(\omega + \omega') \delta_{ij}. \]  

(A.17)

From (A.12), it is clear that the steady state distribution of \( U = (y, v_A, v_B)^T \) is Gaussian distribution whose mean and variance are given in (A.15) and (A.16), respectively:

\[ P_{ss}(U) = \frac{1}{\sqrt{(2\pi)^{3} \text{det} M}} \exp \left[ -\frac{1}{2} U^T M^{-1} U \right], \]  

(A.18)

where \( M_{ij} = \langle U(\tau) U^T(\tau) \rangle_{ij} \).

The quantity \( W \) is non-linear in thermal noises (see (22)). Therefore, we write it in the frequency domain using (A.3) and (A.4) as

\[ W = \frac{\tau}{2} \sum_{n=-\infty}^{\infty} \left[ \Delta \beta I_{1n} - \frac{\Pi k}{T_2} I_{2n} \right], \]  

(A.19)

where

\[ I_{1n} = \tilde{\eta}_1(\omega_n) \tilde{v}_A(-\omega_n) + \tilde{\eta}_1(-\omega_n) \tilde{v}_A(\omega_n) - 2\gamma_1 \tilde{v}_A(\omega_n) \tilde{v}_A(-\omega_n), \]  

(A.20)

\[ I_{2n} = \tilde{y}(\omega_n) \tilde{v}_A(-\omega_n) + \tilde{y}(-\omega_n) \tilde{v}_A(\omega_n). \]  

(A.21)

Substituting \( \tilde{y}(\omega_n) \) and \( \tilde{v}_A(\omega_n) \) from (A.7) and (A.8) in (A.20) and (A.21) yields

\[ I_{1n} = \iota \omega_n [G_{11} (\tilde{\eta}_1 + \tilde{\eta}_2) \tilde{v}_A + G_{12} \tilde{\eta}_3 \tilde{v}_A - G_{11}^* \tilde{\eta}_1 \tilde{v}_A^* - G_{12}^* \tilde{\eta}_3 \tilde{v}_A^*] + \frac{\Delta U^T q_2}{\tau} \tilde{v}_A \]  

\[ -2\gamma_1 \omega_n^3 \left[ |G_{11}|^2 (\tilde{\eta}_1 + \tilde{\eta}_2) (\tilde{v}_A^* + \tilde{v}_A) + |G_{12}|^2 \tilde{\eta}_3 \tilde{v}_A^* + G_{11} G_{12}^* (\tilde{\eta}_1 + \tilde{\eta}_2) \tilde{v}_A^* \right] \]  

\[ + G_{12} G_{11}^* \tilde{\eta}_1 \tilde{v}_A^* - \frac{2\gamma_1}{\tau^2} \Delta U^T q_2 q_2^\dagger \Delta U - 2i\gamma_1 \omega_n \frac{q_2^\dagger \Delta U}{\tau} [G_{11} (\tilde{\eta}_1 + \tilde{\eta}_2) + G_{12} \tilde{\eta}_3] \]  

\[ + \frac{q_2^\dagger \Delta U}{\tau} \tilde{v}_A + 2i\gamma_1 \omega_n \frac{\Delta U^T q_2}{\tau} [G_{11} (\tilde{\eta}_1 + \tilde{\eta}_2) + G_{12} \tilde{\eta}_3], \]  

(A.22)
and

\[ I_{2n} = \omega_n(\bar{\eta}_1 + \bar{\eta}_2)(\bar{\eta}_1^* + \bar{\eta}_2^*)(G_{11}G_{11}^* - G_{11}G_{12}^*) + \omega_n(\bar{\eta}_1 + \bar{\eta}_2)^2[G_{11}(G_{12} - G_{22}^*) - G_{12}(G_{11} - G_{12}^*)]
\]

[\text{and if } G_{ij} = [G(-\omega_n)]_{ij}, \tilde{\eta}_r = \tilde{\eta}_r(\omega_n), \text{ and } \tilde{\eta}_r^* = \tilde{\eta}_r(-\omega_n), \text{ where } r = 1, 2, 3.]

Now, the restricted moment generating function for \( W \) is given by

\[ Z_W(\lambda, U, \tau|U_0) = \langle e^{-\lambda W} \delta[U - U(\tau)] \rangle_{U,U_0} = \int \frac{d^3\sigma}{(2\pi)^3} e^{i\sigma^T U} \langle e^{E(\tau)} \rangle_{U,U_0}, \quad (A.24) \]

where we have used the integral representation of Dirac delta function, and \( E(\tau) \) is given by

\[ E(\tau) = -\lambda W - i\sigma^T U(\tau). \quad (A.25) \]

Using (A.12) and (A.19), we write \( E(\tau) \) as

\[ E(\tau) = \sum_{n=1}^{\infty} \left[ -\lambda \tau \sigma_n^T C_n \sigma_n + \sigma_0^T \sigma_n + \sigma_0^T \sigma_n^* + \frac{\lambda}{\tau} |f_n|^2 \right] - \frac{\lambda \tau}{2} \sigma_0^T C_0 \sigma_0 + \sigma_0^T \sigma_0 + \frac{\lambda}{2\tau} f_0^2, \quad (A.26) \]

where \( C_n = \Delta \beta C_n - \frac{\hbar}{2\tau} C_n^\| \), and the row vector containing thermal noises in the frequency domain is \( \sigma_n^T = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \). Here the matrices \( C_n^\| \) and \( C_n^\| \) are

\[ C_n^\| = \begin{bmatrix} C_{11}^\| & C_{12}^\| & C_{13}^\| \\ C_{21}^\| & C_{22}^\| & C_{23}^\| \\ C_{31}^\| & C_{32}^\| & C_{33}^\| \end{bmatrix} \quad \text{and} \quad C_n^\| = \begin{bmatrix} C_{11}^\| & C_{12}^\| & C_{13}^\| \\ C_{21}^\| & C_{22}^\| & C_{23}^\| \\ C_{31}^\| & C_{32}^\| & C_{33}^\| \end{bmatrix}, \]

where \( C_{ij}^\| (\omega_n) = C_{ij}^\| (-\omega_n) \) in which \{i, j\} = \{1, 2, 3\}.

The matrix elements of \( C_n^\| \) and \( C_n^\| \) are

\[ C_{11}^\| = \omega_n(G_{11} - G_{11}^*), \quad C_{12}^\| = -i\omega_nG_{11}^*, \quad C_{13}^\| = -i\omega_nG_{12}^*, \]

\[ C_{22}^\| = -2\gamma_1\omega_n^2|G_{11}|^2, \quad C_{23}^\| = -2\gamma_1\omega_n^2|G_{11}G_{12}|, \quad C_{33}^\| = -2\gamma_1\omega_n^2|G_{12}|^2, \]

\[ C_{12}^\| = C_{21}^\| = C_{22}^\| = i\omega_n[G_{12}G_{11}^* - G_{11}G_{12}^*], \quad C_{13}^\| = C_{23}^\| = \omega_n[G_{11}(G_{12}^* - G_{22}^*) - G_{12}(G_{11}^* - G_{12}^*)], \]

\[ C_{33}^\| = i\omega_n[G_{22}G_{12}^* - G_{12}G_{22}^*]. \]
The column vector $\tilde{\alpha}_n$ is

$$\tilde{\alpha}_n = -\lambda \begin{bmatrix} a_{11}^T \Delta U \\ a_{21}^T \Delta U \\ a_{31}^T \Delta U \end{bmatrix} - i e^{-i\omega_n} \begin{bmatrix} q_{11}^T \sigma \\ q_{12}^T \sigma \\ q_{13}^T \sigma \end{bmatrix}, \quad (A.27)$$

in which

$$a_{11}^T = [\Delta(1 - 2i\gamma_n G_{11}) - \frac{\Pi k}{T_2} (G_{11} - G_{12})] q_2^\dagger + \frac{i\omega_n \Pi k}{T_2} G_{11} q_1^\dagger,$$

$$a_{21}^T = [-2i\gamma_n \Delta \beta G_{11} - \frac{\Pi k}{T_2} (G_{11} - G_{12})] q_2^\dagger + \frac{i\omega_n \Pi k}{T_2} G_{11} q_1^\dagger,$$

$$a_{31}^T = [-2i\gamma_n \Delta \beta G_{12} - \frac{\Pi k}{T_2} (G_{12} - G_{22})] q_2^\dagger + \frac{i\omega_n \Pi k}{T_2} G_{12} q_1^\dagger.$$

In (A.26), $|f_n|^2$ is given by

$$|f_n|^2 = \Delta U^T \left[ 2\Delta \beta \gamma q_2 q_2^\dagger - \frac{\Pi k}{T_2} (q_1 q_2^\dagger + q_2 q_1^\dagger) \right] \Delta U.$$

Therefore,

$$\langle e^{E(\tau)} \rangle_{U, U_0} = \prod_{n=1}^N \left\langle \exp \left[ -\lambda \tau \zeta_n^T C_n \zeta_n^* + \zeta_n^T \tilde{\alpha}_n + \tilde{\alpha}_n^T \zeta_n^* + \frac{\lambda}{\tau} |f_n|^2 \right] \right\rangle \times \left\langle \exp \left[ -\frac{\lambda \tau}{2} \zeta_0^T C_0 \zeta_0 + \zeta_0^T \tilde{\alpha}_0 + \frac{\lambda}{2\tau} f_0^2 \right] \right\rangle, \quad (A.28)$$

where the angular brackets represent the average over the noise distribution. For $n = 0$, the average is over the distribution $P(\zeta_0) = (2\pi)^{-3/2}(\det \Lambda)^{-1/2} \exp[-\frac{1}{2} \zeta_0^T \Lambda^{-1} \zeta_0]$.

In which $\Lambda = \text{diag} \left( \frac{2\gamma_1}{\tau}, \frac{2\gamma_2}{\tau}, \frac{2\gamma_3}{\tau} \right)$ whereas for each $n \geq 1$, average is over distribution $P(\zeta_n) = \pi^{-3} (\det \Lambda)^{-1} \exp[-\frac{1}{2} \zeta_n^T \Lambda^{-1} \zeta_n^*]$. After some simplification, (A.28) becomes

$$\langle e^{E(\tau)} \rangle_{U, U_0} = e^{\tau \mu(\lambda)} \exp \left[ \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \tilde{\alpha}_{-n}^T \Omega_n^{-1} \tilde{\alpha}_n + \frac{\lambda |f_n|^2}{\tau} \right) \right], \quad (A.29)$$

where $\Omega_n = [\Lambda^{-1} + \lambda \tau C_n]^{-1}$.

In the large time limit ($\tau \rightarrow \infty$), we convert the summation in the above equation into integration. Therefore, we get

$$\langle e^{E(\tau)} \rangle_{U, U_0} \approx e^{\tau \mu(\lambda)} \exp \left[ -\frac{1}{2} \sigma^T H_1(\lambda) \sigma + i \Delta U^T H_2(\lambda) \sigma + \frac{1}{2} \Delta U^T H_3(\lambda) \Delta U \right], \quad (A.30)$$

where one can identify $\mu(\lambda)$, $H_1(\lambda)$, $H_2(\lambda)$, and $H_3(\lambda)$ as

$$\mu(\lambda) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \ln[\det(\Lambda \Omega)], \quad (A.31)$$

$$H_1(\lambda) = \frac{\tau}{2\pi} \int_{-\infty}^{\infty} d\omega \rho^T \Omega^{-1} \phi, \quad (A.32)$$

https://doi.org/10.1088/1742-5468/aabfca
Therefore, we write \( \tilde{b}_1 = -d \).

The column vectors \( b_j \) are given as

\[
\begin{align*}
b_{11} &= q_2 \left[ \Delta\beta(1 + 2i\gamma_1 \omega G_{11}^*) - \frac{\Pi k}{T_2} (G_{11}^* - G_{12}^*) \right] - \frac{i\omega \Pi k}{T_2} q_1 G_{11}^*, \\
b_{12} &= q_2 \left[ 2i\gamma_1 \omega \Delta\beta G_{11}^* - \frac{\Pi k}{T_2} (G_{11}^* - G_{12}^*) \right] - \frac{i\omega \Pi k}{T_2} q_1 G_{11}^*, \\
b_{13} &= q_2 \left[ 2i\gamma_1 \omega \Delta\beta G_{12}^* - \frac{\Pi k}{T_2} (G_{12}^* - G_{22}^*) \right] - \frac{i\omega \Pi k}{T_2} G_{12}^* q_1.
\end{align*}
\]

Therefore, the moment generating function \( Z_W(\lambda, U, \tau | U_0) \) can be rewritten as

\[
Z_W(\lambda, U, \tau | U_0) = \int \frac{d^3\sigma}{(2\pi)^3} e^{i\sigma^T U \Phi} \langle e^{E(\tau)} \rangle_{U_0} \approx \frac{e^{\tau \mu(\lambda)} e^{\frac{1}{2} \Delta U^T H_3 \Delta U} e^{-\frac{i}{2} \Delta U^T H_2 (H_1^{-1} (U + H_2^T \Delta U))} \sqrt{(2\pi)^3 \det H_1(\lambda)}}{\sqrt{(2\pi)^3 \det H_1(\lambda)}}.
\]

We can factorize the restricted moment generating function \( Z_W(\lambda, U, \tau | U_0) \) using (28) into left and right eigenfunctions. Consequently, the matrices \( H_1(\lambda), H_2(\lambda), H_3(\lambda) \) satisfy the condition \( H_3 - H_2 H_1^{-1} H_2^T - H_1^{-1} H_2^T = 0 \). Therefore, we write

\[
Z_W(\lambda, U, \tau | U_0) \approx \frac{e^{\tau \mu(\lambda)} e^{-\frac{i}{2} \Delta U^T L_1(\lambda) U} e^{-\frac{i}{2} \Delta U^T L_2(\lambda) U_0} \sqrt{(2\pi)^3 \det H_1(\lambda)}}{\sqrt{(2\pi)^3 \det H_1(\lambda)}}.
\]

where the matrices \( L_1(\lambda) = H_1^{-1} + H_1^{-1} H_2^T \) and \( L_2 = -H_1^{-1} H_2^T \).

Using (21), we can write the moment generating function for generalized partial entropy production

\[
Z(\lambda, U, \tau | U_0) \approx \frac{e^{\tau \mu(\lambda)} e^{-\frac{i}{2} \Delta U^T \tilde{L}_1(\lambda) U} e^{-\frac{i}{2} \Delta U^T \tilde{L}_2(\lambda) U_0} \sqrt{(2\pi)^3 \det H_1(\lambda)}}{\sqrt{(2\pi)^3 \det H_1(\lambda)}}
\]

where the matrices \( \tilde{L}_1(\lambda) \) and \( \tilde{L}_2(\lambda) \) modify as

\[
\tilde{L}_1(\lambda) = L_1(\lambda) - \lambda (mT_2^{-1} - H^{-1}) \Sigma, \\
\tilde{L}_2(\lambda) = L_2(\lambda) + \lambda (mT_2^{-1} - H^{-1}) \Sigma.
\]

The moment generating function \( Z(\lambda) \) as given in (29), for generalized partial entropy production \( \Delta S_{\text{tot}}^A \) can be obtained by integrating the restricted moment generating function \( Z(\lambda, U, \tau | U_0) \) over the initial steady state distribution \( P_{\text{ss}}^{\text{full}}(U_0) \) and the final variable \( U \). Therefore, the prefactor given in (30) reduces to

\[
g(\lambda) = |\det H_1(0) \det H_1(\lambda) \det \tilde{L}_1(\lambda) \det \tilde{L}_2(\lambda + H_1^{-1}(0))|^{-1/2}.
\]

https://doi.org/10.1088/1742-5468/aabfca
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