PRODUCTS OF LIPSCHITZ-FREE SPACES AND APPLICATIONS

PEDRO L. KAUFMANN

Abstract. We show that, given a Banach space \( X \), the Lipschitz-free space over \( X \), denoted by \( F(X) \), is isomorphic to \((\sum_{n=1}^{\infty} F(X))_{\ell_1}\). Some applications are presented, including a non-linear version of Pełczyński’s decomposition method for Lipschitz-free spaces and the identification up to isomorphism between \( F(R^n) \) and the Lipschitz-free space over any compact metric space which is locally bi-Lipschitz embeddable into \( R^n \) and which contains a subset that is Lipschitz equivalent to the unit ball of \( R^n \). We also show that \( F(M) \) is isomorphic to \( F(c_0) \) for all separable metric spaces \( M \) which are absolute Lipschitz retracts and contain a subset which is Lipschitz equivalent to the unit ball of \( c_0 \). This class contains all \( C(K) \) spaces with \( K \) infinite compact metric (Dutrieux and Ferenczi had already proved that \( F(C(K)) \) is isomorphic to \( F(c_0) \) for those \( K \) using a different method). Finally we study Lipschitz-free spaces over certain unions and quotients of metric spaces, extending a result by Godard.

1. Introduction

Let \((M,d,0)\) be a pointed metric space (that is, a distinguished point 0 in \( M \), called base point, is chosen), and consider the Banach space \( Lip_0(M) \) of all real-valued Lipschitz functions on \( M \) which vanish at 0, equipped with the norm

\[ \|f\|_{Lip} := \inf_{x,y \in M, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}. \]

On the closed unit ball of \( Lip_0(M) \), the topology of pointwise convergence is compact, so \( Lip_0(M) \) admits a canonical predual, which is called the Lipschitz-free space over \( M \) and denoted by \( \mathcal{F}(M) \). This space is the closure in \( Lip_0(M)^* \) of \( \operatorname{span}\{\delta_x : x \in M\} \), where \( \delta_x \) are the evaluation functionals defined by \( \delta_x(f) = f(x) \). It is readily verified that \( \delta : x \mapsto \delta_x \) is an isometry from \( M \) into \( \mathcal{F}(M) \). Given \( 0' \), it is clear that \( T : Lip_0(M) \to Lip_{0'}(M) \) defined by \( T(f) := f - f(0') \) is a weak*-to-weak* continuous isometric isomorphism, thus the choice of different base points yields isometrically isomorphic Lipschitz-free spaces. We refer to [14] for a study on Lipschitz functions spaces, and to [14] and [6] for an introduction to Lipschitz-free spaces and its basic properties.

One of the main properties of the Lipschitz-free spaces is that it permits to interpret Lipschitz maps between metric spaces from a linear point of view:

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Proposition 1.1. Let $M$ and $N$ be pointed metric spaces, consider $\delta^M$ and $\delta^N$ the isometries that assign each $x \in M$ (respectively, each $x \in N$) to the corresponding evaluation functional $\delta^M_x$ in $\mathcal{F}(M)$ (respectively, $\delta^N_x$ in $\mathcal{F}(N)$), and suppose that $L : M \to N$ is a Lipschitz function such that $L(0_M) = 0_N$. Then there is an unique linear map $\hat{L} : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\hat{L} \circ \delta^M = \delta^N \circ L$, that is, such that the following diagram commutes:

![Diagram](attachment:image.png)

Moreover, $\|\hat{L}\| = \|L\|_{\text{Lip}}$.

In particular, if $M$ and $N$ Lipschitz equivalent (that is, there is a bi-Lipschitz map between $M$ and $N$) then $\mathcal{F}(M)$ and $\mathcal{F}(N)$ are isomorphic. The converse is not true, even if $M$ and $N$ are assumed to be Banach spaces: if $K$ is an infinite compact metric space, then $\mathcal{F}(C(K))$ is isometric to $\mathcal{F}(c_0)$, even though $C(K)$ is not Lipschitz equivalent to $c_0$ in general (recall that, if $C(K)$ is uniformly homeomorphic to $c_0$, then it is isomorphic to $c_0$ c.f. [8]). This first counterexample for the Banach space case was presented by Dutrieux and Ferenczi in [3].

Despite of the simplicity of the definition of the Lipschitz-free spaces, many fundamental questions about their structure remain unanswered. Godard [5] characterized the metric spaces $M$ such that $\mathcal{F}(M)$ is isometrically isomorphic to a subspace of $L^1$ as exactly those who are isometrically embeddable into $\mathbb{R}$-trees (that is, connected graphs with no cycles with the graph distance); on the other hand, Naor and Schechtman [13] have shown that $\mathcal{F}(\mathbb{Z}^2)$ (thus also $\mathcal{F}(\mathbb{R}^2)$) is not isomorphic to any subspace of $L^1$. From this arises the natural question of characterizing the metric spaces $M$ such that $\mathcal{F}(M)$ is (nonisometrically) isomorphic to $L^1$. Godefroy and Kalton [6] have shown that, given a Banach space $X$, $X$ has the bounded approximation property if and only if $\mathcal{F}(X)$ has that property, and recently Hájek and Pernecká [4] have shown that $\mathcal{F}(\mathbb{R}^n)$ admits a Schauder basis, refining a result from [10]. However, it is not known whether $\mathcal{F}(F)$ admits a Schauder basis for any given closed subset $F \subset \mathbb{R}^n$. It is also not of this author’s knowledge whether $\mathcal{F}(\mathbb{R}^n)$ is isomorphic to $\mathcal{F}(\mathbb{R}^m)$ or not, for distinct $m,n \geq 2$. Even the study of Lipschitz-free spaces over very simple subsets of $\mathbb{R}^2$ can present difficulties (see the question posed after Proposition 5.1).

In this context, we continue the exploration of what could be considered basic properties of Lipschitz-free spaces and their relation with the underlying metric spaces. We will show, for instance, that for any given Banach space $X$, we have that $\mathcal{F}(X)$ is isomorphic to $(\sum_{n=1}^{\infty} \mathcal{F}(X))_{\ell_1}$ (see Theorem 3.1). This provides in particular a kind of non-linear version for Pelczyński’s decomposition method (see Corollary 3.2), which in turn can be used to obtain the mentioned example by Dutrieux and Ferenczi of non-Lipschitz equivalent Banach spaces sharing the same Lipschitz-free space. In fact, we show that $\mathcal{F}(M)$ is isomorphic
to $\mathcal{F}(c_0)$ for a wider class of metric spaces (see Corollary 3.5). We will also show that, for compact metric spaces $M$ which are locally bi-Lipschitz embeddable in $\mathbb{R}^n$, we have that $\mathcal{F}(M)$ admits a complemented copy in $\mathcal{F}(\mathbb{R}^n)$; when moreover the euclidean ball $B_{\mathbb{R}^n}$ is bi-Lipschitz embeddable in $M$, we have that $\mathcal{F}(M)$ and $\mathcal{F}(\mathbb{R}^n)$ are actually isomorphic (see Theorem 3.8). The class of metric spaces satisfying both properties includes all $n$-dimensional compact Riemannian manifolds. Independently, we study the behavior of Lipschitz-free spaces with respect to certain gluings of metric spaces, expanding an initial idea presented by Godard [3].

1.1. Notation. We say that two metric spaces $M$ and $N$ are $C$-Lipschitz equivalent, for some constant $C > 0$, if there a bi-Lipschitz onto map $\varphi : M \to N$ satisfying $\|\varphi\|_{\text{Lip}} \cdot \|\varphi^{-1}\|_{\text{Lip}} \leq C$. $M$ and $N$ are then Lipschitz equivalent if they are $C$-Lipschitz equivalent for some $C > 0$; in that case we also write $M \sim \sim N$. Given two Banach spaces $X$ and $Y$, we write $X \cong Y$ when $X$ and $Y$ are isometrically isomorphic, $X \sim \sim Y$ when there is a complemented copy of $X$ in $Y$, and $X \simeq Y$ when $X$ and $Y$ are isomorphic. If $X$ and $Y$ are isomorphic, the Banach-Mazur distance between $X$ and $Y$ is defined by

$$d_{BM}(X,Y) := \inf\{\|T\|,\|T^{-1}\| : T \text{ is an isomorphism from } X \text{ onto } Y\}.$$ 

$\|T\|,\|T^{-1}\|$ is called the (linear) distortion of $T$. When $d_{BM}(X,Y) \leq C$ for some $C > 0$, we say that $X$ is isomorphic to $Y$ with distortion bounded by $C$.

$Ext_0(F,M)$ denotes the set of linear extension operators for Lipschitz functions and $Ext_0^1(F,M)$ is the set of pointwise-to-pointwise continuous elements of $Ext_0(F,M)$ (see Subsection 2.1).

1.2. Structure of this work. In Section 2 we present some background results on linear extension operators for Lipschitz functions and consequent ways to decompose the Lipschitz-free space over a metric space using metric quotients. In Section 3 we show that, for every Banach space $X$, $\mathcal{F}(X) \cong (\sum_{n=1}^{\infty} \mathcal{F}(X))_t$, and derive some consequences.

Sections 4 and 5 are independent of the results presented in Section 3. In Section 4 we show that, for every Banach space $X$, $d_{BM}(\mathcal{F}(X),\mathcal{F}(X) \oplus_1 \mathcal{F}(X)) \leq 4$. In Section 5 we provide formulas for Lipschitz-free spaces over certain unions of metric spaces.

2. Linear extensions of Lipschitz functions and the Lipschitz-free space over metric quotients

2.1. Linear extensions of Lipschitz functions. Given a pointed metric space $(M,d,0)$ and a subset $F$ containing 0, let us denote by $Ext_0(F,M)$ the set of all extensions $E : Lip_0(F) \to Lip_0(M)$ which are linear and continuous ($E$ being an extension means that $E(f)|_F = f$ for all $f \in Lip_0(F)$). It is immediate to see that, if we choose another base point $0'$ contained in $F$, to each element $E \in Ext_0(F,M)$ there is a corresponding $E' \in Ext_0(F,M)$, defined by $E'(f) := E(f-f(0')) + f(0')$, which satisfies $\|E'\| = \|E\|$, so generally it is not important which base point is chosen. Recall that there are always
continuous but not necessarily linear extensions from $\text{Lip}_0(F)$ to $\text{Lip}_0(M)$; for example the infimum convolution
$$E(f)(x) := \inf_{y \in F} \{ f(y) + \| f \|_{\text{Lip}} d(x, y) \}$$
is such an extension and it is an isometry, although in most cases it fails to be linear. It is possible, though, to have $\text{Ext}_0(F, M) = \emptyset$; Brudnyi and Brudnyi provide us with an example of a two-dimensional Riemannian manifold $M$, equipped with its geodesic metric, which admits a subset $F$ satisfying that condition (see Theorem 2.18 in [2]).

We will be particularly interested in the subset $\text{Ext}_0^{pt}(F, M)$ of $\text{Ext}_0(F, M)$ consisting of the pointwise-to-pointwise continuous elements. The fact that on bounded sets of $\text{Lip}_0(F)$ the weak* and the pointwise topologies coincide implies that any element of $\text{Ext}_0(F, M)$ is weak*-to-weak* continuous if and only if it belongs to $\text{Ext}_0^{pt}(F, M)$. Therefore, any $E \in \text{Ext}_0^{pt}(F, M)$ admits a preadjoint $P : \mathcal{F}(M) \to \mathcal{F}(F)$, which is a (continuous) canonical projection, in the sense that $P(\mu) = \mu|_F$ for all finitely supported $\mu \in \mathcal{F}(M)$. In particular, $\mathcal{F}(F)$ is complemented in $\mathcal{F}(M)$. Reciprocally, given a continuous projection $P : \mathcal{F}(M) \to \mathcal{F}(F)$ such that $P(\mu) = \mu|_F$ for all finitely supported $\mu \in \mathcal{F}(M)$, we have that $P^* \in \text{Ext}_0^{pt}(F, M)$.

Even in the context where $M$ is a Banach space and $F$ is a closed linear subspace, we might not get this complementability condition. Consider, for example, $c_0$ and let $X$ be a subspace of $c_0$ which fails to have the bounded approximation property. As we mentioned in the introduction, given a Banach space $Y$, $Y$ has the bounded approximation property if and only if $\mathcal{F}(Y)$ has that property. Since this property is inherited by complemented subspaces, it follows that $\mathcal{F}(X)$ cannot be isomorphic to a complemented subspace of $\mathcal{F}(c_0)$. One can still pose the question of whether or not $\text{Ext}_0(X, c_0)$ is empty.

On the other hand, we have the following positive example:

**Proposition 2.1** (Lancien, Pernecká [10]). There exists $C > 0$ such that, for each $n \in \mathbb{N}$ and each subset $F$ of $\mathbb{R}^n$ containing 0, there exists $E$ in $\text{Ext}_0^{pt}(F, \mathbb{R}^n)$ satisfying $\|E\| \leq C \sqrt{n}$.

This result appears as part of the proof of Proposition 2.3 of [10], which states that $\mathcal{F}(F)$ has the $C \sqrt{n}$-bounded approximation property. It involves a construction by Lee and Naor from [12], and the fact that $\mathbb{R}^n$ admits a so-called $K$-gentle partition of the unity with respect to $F$, which in turn induces the mentioned extension $E$.

2.2. **Metric quotients and Lipschitz-free spaces.** We turn our attention to a special kind of metric quotient. Given a pointed metric space $(M, d, 0)$ and a subset $F$ containing 0, let $\sim$ be the equivalence relation which collapses $F$ to a point (that is, the equivalence classes are either singletons or $F$). We define the metric quotient of $M$ by $F$, denoted by $M/F$, as the pointed metric space $(M/\sim, \tilde{d}, [0])$, where $\tilde{d}$ is defined by
$$\tilde{d}([x], [y]) = \min \{ d(x, y), d(x, F) + d(y, F) \}. \tag{2.1}$$

The space $\text{Lip}_0(M/F)$ can be interpreted as the closed linear subspace of $\text{Lip}_0(M)$ consisting of all of its functions which are null in $F$. Depending on how $F$ is placed in $M$, we can have the following decomposition for $\mathcal{F}(M)$:
Lemma 2.2. Let \((M,d,0)\) be a pointed metric space and \(F\) be a subset containing 0, and suppose that there exists \(E \in Ext_{0}^{pl}(F,M)\). Then
\[
\mathcal{F}(M) \simeq \mathcal{F}(F) \oplus_{1} \mathcal{F}(M/F),
\]
with distortion bounded by \((\|E\| + 1)^{2}\).

Proof. Define \(\Phi : Lip_{0}(F) \oplus_{\infty} Lip_{0}(M/F) \to Lip_{0}(M)\) by \(\Phi(f,g) = E(f) + g\). It is straightforward that \(\Phi\) is an onto isomorphism with \(\|\Phi\| \leq \|E\| + 1\), that \(\Phi\) is pointwise-to-pointwise continuous and that its inverse \(\Phi^{-1} h \mapsto (h|_{F}, T(h|_{F}) - h)\) has norm also bounded by \(\|E\| + 1\). It follows that \(\Phi\) is the adjoint of an isomorphism \(\Psi\) between \(\mathcal{F}(M)\) and \(\mathcal{F}(F) \oplus_{1} \mathcal{F}(M/F)\) satisfying the desired distortion bound. \(\square\)

3. Products of Lipschitz-free spaces

In this section we will show that \(\mathcal{F}(X) \simeq (\sum_{n=1}^{\infty} \mathcal{F}(X))_{\ell_1}\) for any Banach space \(X\), and derive some consequences. With that purpose we will use the following construction by Kalton [9]. Let \((M,d,0)\) be a pointed metric space, denote by \(B_{r}\) the closed balls centered at 0 and with radius \(r > 0\) and consider, for each \(k \in \mathbb{Z}\), the linear operator \(T_{k} : \mathcal{F}(M) \to \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})\) defined by
\[
T_{k} \delta_{x} := \begin{cases} 
0, & \text{if } x \in B_{2^{k-1}}; \\
(\log_{2} d(x,0) - k + 1) \delta_{x}, & \text{if } x \in B_{2^{k}} \setminus B_{2^{k-1}}; \\
(k + 1 - \log_{2} d(x,0)) \delta_{x}, & \text{if } x \in B_{2^{k+1}} \setminus B_{2^{k}}; \\
0, & \text{if } x \notin B_{2^{k+1}}.
\end{cases}
\]

Lemma 4.2 from [9] says that for each \(\gamma \in \mathcal{F}(M)\) we have that \(\gamma = \sum_{k \in \mathbb{Z}} T_{k} \gamma\) unconditionally and
\[
\sum_{k \in \mathbb{Z}} \|T_{k} \gamma\|_{F} \leq 72 \|\gamma\|_{F}. \tag{3.1}
\]

Another result from that same paper that we will use is Lemma 4.2, which states that, given \(r_{1}, \ldots, r_{n}, s_{1}, \ldots, s_{n} \in \mathbb{Z}, r_{1} < s_{1} < r_{2} < \cdots < s_{n}\) and \(\gamma_{k} \in \mathcal{F}(B_{2^{r_{k}}} \setminus B_{2^{s_{k}}})\) and writing \(\theta := \min_{k=1,\ldots,n-1}(r_{k+1} - s_{k})\), then
\[
\|\gamma_{1} + \cdots + \gamma_{n}\|_{F} \geq \frac{2^{\theta} - 1}{2^{\theta} + 1} \sum_{k=1}^{n} \|\gamma_{k}\|_{F}. \tag{3.2}
\]

Theorem 3.1. Let \(X\) be a Banach space. Then
\[
\mathcal{F}(X) \simeq \left(\sum_{n=1}^{\infty} \mathcal{F}(X)\right)_{\ell_1}.
\]

Proof. Note that \(S : (\gamma_{k}) \in \left(\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})\right)_{\ell_1} \mapsto \sum_{k \in \mathbb{Z}} \gamma_{k} \in \mathcal{F}(X)\) is linear, continuous and onto, and from \((3.1)\) we get that \(T : \gamma \in \mathcal{F}(X) \mapsto (T_{k} \gamma) \in \left(\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})\right)_{\ell_1}\) is a well defined one-to-one linear continuous operator. Thus \(T \circ S\) is a continuous projection from \((\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}}))_{\ell_1}\) onto the isomorphic copy \(T(\mathcal{F}(X))\) of \(\mathcal{F}(X)\).
Denote \( M := \cup_{k \in \mathbb{Z}} (B_{2^{2k+1}} \setminus B_{2^{2k}}) \), and consider \( E \in Ext_0(M \cup \{0\}, X) \) which extends each element of \( Lip_0(M \cup \{0\}) \) linearly on each radial segment \([2^{2k-1}, 2^{2k}] x, k \in \mathbb{Z}, x \in S_X\). Clearly \( E \) is pointwise-to-pointwise continuous, thus it is the adjoint of some \( P : \mathcal{F}(X) \to \mathcal{F}(M \cup \{0\}) \), which is a projection which satisfies \( P(\mu) = \mu|_M \) for all finitely supported \( \mu \in \mathcal{F}(X) \). Note that \( \mathcal{F}(M \cup \{0\}) \cong \mathcal{F}(M) \), since \( 0 \in M \). Now by (3.2), the natural identification \( \text{Id} : \mathcal{F}(M) \to (\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1} \) is an isomorphism. So there is a complemented copy of \((\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}\) in \( \mathcal{F}(X) \).

Note that, by Proposition 1.1, re-scalings of any metric space give rise to isometrically isomorphic Lipschitz-free spaces. Thus all spaces \( \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}), k \in \mathbb{Z} \) are isometrically isomorphic to \( \mathcal{F}(B_2 \setminus B_1) \) and all spaces \( \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}}), k \in \mathbb{Z} \) are isometrically isomorphic to \( \mathcal{F}(B_4 \setminus B_1) \), which in turn is isomorphic to \( \mathcal{F}(B_2 \setminus B_1) \). It follows that

\[
\mathcal{F}(X) \overset{c}{\hookrightarrow} \left( \sum_{j=1}^{\infty} \mathcal{F}(B_2 \setminus B_1) \right)_{\ell_1} \quad \text{and} \quad \left( \sum_{j=1}^{\infty} \mathcal{F}(B_2 \setminus B_1) \right)_{\ell_1} \overset{c}{\hookrightarrow} \mathcal{F}(X).
\]

Since \( \left( \sum_{j=1}^{\infty} \mathcal{F}(B_2 \setminus B_1) \right)_{\ell_1} \) is isomorphic to its \( \ell_1 \)-sum, by a standard Pelczyński’s decomposition method we have

\[
\mathcal{F}(X) \simeq \left( \sum_{j=1}^{\infty} \mathcal{F}(B_2 \setminus B_1) \right)_{\ell_1}
\]

and the conclusion follows immediately. \( \square \)

As a direct consequence of Theorem 3.1 and Proposition 1.1 we get the following nonlinear version of Pelczyński’s decomposition method for Lipschitz-free spaces.

**Corollary 3.2.** Let \( X \) be a Banach space and \( M \) be a metric space, and suppose that \( X \) and \( M \) admit Lipschitz retracts \( N_1 \) and \( N_2 \), respectively, such that \( X \) is Lipschitz equivalent to \( N_2 \) and \( M \) is Lipschitz equivalent to \( N_1 \). Then \( \mathcal{F}(X) \simeq \mathcal{F}(Y) \).

**Proof.** \( \mathcal{F}(X) \) is isomorphic to \( \mathcal{F}(N_2) \), which in turn is a complemented subspace of \( \mathcal{F}(M) \). Analogously, \( \mathcal{F}(M) \) is isomorphic to a complemented subspace of \( \mathcal{F}(N_1) \). The conclusion follows by applying the standard Pelczyński’s decomposition method. \( \square \)

**Corollary 3.3.** Let \( X \) be a Banach space. Then

\[
\mathcal{F}(X) \simeq \mathcal{F}(B_1).
\]

**Proof.** Since \( B_1 \) is a Lipschitz retract of \( X \), it follows that \( \mathcal{F}(X) \) contains a complemented copy of \( \mathcal{F}(B_1) \). In the proof of Theorem 3.1 we have shown that \( \mathcal{F}(X) \) is isomorphic to \((\sum_{k \in \mathbb{Z}} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}\), which is clearly isomorphic to \((\sum_{k<0} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}\) since all summands are isometrically isomorphic. Let \( N := \cup_{k<0} (B_{2^{2k+1}} \setminus B_{2^{2k}}) \) Again by (3.2), \((\sum_{k<0} \mathcal{F}(B_{2^{2k+1}} \setminus B_{2^{2k}}))_{\ell_1}\) is isomorphic to \( \mathcal{F}(N) \), which is complemented in \( \mathcal{F}(B_1) \) since there is a pointwise-to-pointwise continuous element in \( Ext_0(N \cup \{0\}, B_1) \). The conclusion
follows by an application of Pełczyński’s decomposition method.

**Proposition 3.4.** Let $M$ be a metric space and $X$ be a Banach space. Suppose that $M$ contains a subset which is Lipschitz equivalent to the unit ball of $X$. Then $\mathcal{F}(M)$ contains a complemented copy of $\mathcal{F}(X)$.

**Proof.** Consider $N \subset M$ and $L : N \to B_X$ a bi-Lipschitz onto map. Write $N_1 := L^{-1}(\frac{1}{2}B_X)$, and note that, by Corollary 3.3, $\mathcal{F}(N_1) \simeq \mathcal{F}(X)$. It suffices then to show that $\text{Ext}^0_t(N_1, M)$ is nonempty for some $0' \in N_1$.

Consider $E \in \text{Ext}_t(\frac{1}{2}B_X, B_X)$ such that, for all $f \in \text{Lip}_0(\frac{1}{2}B_X)$ and all $x \in S_X$, $E(f)(x) = 0$ and $E(f)$ is linear in the radial segment $[\frac{1}{2}x, x]$. Then $F : \text{Lip}_{L^{-1}(0)}(N_1) \to \text{Lip}_{L^{-1}(0)}(M)$ defined by

$$F(f)(x) := \begin{cases} E(f \circ L^{-1})(L(x)) & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

belongs to $\text{Ext}^0_t(N_1, M)$. □

Recall that a subset $F$ of a metric space $M$ is called a **Lipschitz retract** of $M$ if there is a Lipschitz map from $M$ onto $F$ which coincides with the identity on $F$. A metric space is said to be an **absolute Lipschitz retract** if it is a Lipschitz retract of every metric space containing it. Given any metric space $M$, the space $C_u(M)$ of real-valued bounded and uniformly continuous functions on $M$, equipped with the uniform norm, is an example of Banach space which is an absolute Lipschitz retract (see e.g. [1], Theorem 1.6). This class includes all $C(K)$ spaces for $K$ compact metric space, in particular it includes $c_0$. Since all separable metric spaces are bi-Lipschitz embeddable in $c_0$ ([1], Theorem 7.11), we obtain the following class of metric spaces $M$ with $\mathcal{F}(M) \simeq \mathcal{F}(c_0)$:

**Corollary 3.5.** Let $M$ separable metric space which is an absolute Lipschitz retract and such that the unit ball of $c_0$ is bi-Lipschitz embeddable into $M$. Then $\mathcal{F}(M) \simeq \mathcal{F}(c_0)$.

In particular, if $K$ is an infinite compact metric space, then $\mathcal{F}(C(K)) \simeq \mathcal{F}(c_0)$.

**Proof.** By Corollary 3.4 there is a complemented copy of $\mathcal{F}(c_0)$ in $\mathcal{F}(M)$, $M$ is Lipschitz equivalent to some subset $F$ of $c_0$, and $F$ is an absolute Lipschitz retract since this property is preserved by Lipschitz equivalences. Thus $F$ is a Lipschitz retract of $c_0$, and by Proposition 1.1 this implies that $\mathcal{F}(F)$ (and thus $\mathcal{F}(M)$) admits a complemented copy in $\mathcal{F}(c_0)$. The conclusion follows from Theorem 3.1 and an application of Pełczyński’s decomposition method. □

**Corollary 3.6.** Let $F$ be a subset of $\mathbb{R}^n$ with nonempty interior. Then $\mathcal{F}(F) \simeq \mathcal{F}(\mathbb{R}^n)$.

**Proof.** By Proposition 2.1 there is a complemented copy of $\mathcal{F}(F)$ in $\mathcal{F}(\mathbb{R}^n)$; by Proposition 3.4, there is also a complemented copy of $\mathcal{F}(\mathbb{R}^n)$ in $\mathcal{F}(F)$. The result follows from Theorem 3.1 and an application of Pełczyński’s decomposition method. □
Remark. In [4], Hájek and Pernecká have shown that \( F(\mathbb{R}^n) \) admits a Schauder basis, and rose the natural question of whether or not the same holds true for \( F(F) \), being \( F \) any closed subset of \( \mathbb{R}^n \). Note that, by Corollary 3.6, the problem is reduced to the case where \( F \) has empty interior.

In order to study Lipschitz-free spaces of locally euclidean metric spaces, alongside the corollaries of Theorem 3.1, the following result becomes handy:

**Theorem 3.7** (Lang, Plaut [11]). Let \( M \) be a compact metric space such that each point of \( M \) admits a neighborhood which is bi-Lipschitz embeddable in \( \mathbb{R}^n \). Then \( M \) is bi-Lipschitz embeddable in \( \mathbb{R}^n \).

**Theorem 3.8.** Let \( M \) be a compact metric space such that each \( x \in M \) admits a neighborhood which is bi-Lipschitz embeddable in \( \mathbb{R}^n \). Then there is a complemented copy of \( F(M) \) in \( F(\mathbb{R}^n) \).

If moreover the unit ball of \( \mathbb{R}^n \) is bi-Lipschitz embeddable into \( M \), then \( F(M) \cong F(\mathbb{R}^n) \).

In particular, the Lipschitz-free space over any \( n \)-dimensional compact Riemannian manifold equipped with its geodesic metric is isomorphic to \( F(\mathbb{R}^n) \).

**Proof.** The first part follows directly from Lang and Plaut’s result and the fact that the Lipschitz-free space over any subset of \( \mathbb{R}^n \) admits a complemented copy in \( F(\mathbb{R}^n) \). The second part follows from Corollary 3.6, Theorem 3.1 and the application of Pełczyński’s decomposition method. □

**Remark.** Note that the compactness condition in Theorem 3.8 is necessary, even if have uniformity on the embeddings into \( \mathbb{R}^n \). For example, \( \mathbb{Z} \times \mathbb{R} \) is locally isometric to line segments, but \( F(\mathbb{Z} \times \mathbb{R}) \) is not isomorphic to a subspace of \( F(\mathbb{R}) \cong L^1 \), by Naor and Schechtman’s result mentioned in the introduction.

4. \( F(X) \cong F(X)^2 \) with low distortion

Let \( X \) be a Banach space. By Theorem 3.1, \( F(X) \cong F(X)^2 \). In this Section we will show that we have the uniform bound \( d_{BM}(F(X), F(X) \oplus_1 F(X)) \leq 4 \); we will do this via an elementary construction based on metric properties of \( X \).

We start by recalling some definitions and results on quotient metric spaces which are of a more general kind than the ones presented in Section 2. For details and more on that subject, we refer to Weaver’s book [14]. Let \((M, d)\) be a complete metric space, and let \( \sim \) be an equivalence relation on \( M \). The element of \( M/\sim \) containing \( x \in M \) will be denoted by either \( \tilde{x} \) or \([x]_\sim\). Define a pseudometric \( \tilde{d} \) on \( M/\sim \) by

\[
\tilde{d}(\tilde{x}, \tilde{y}) := \inf \left\{ \sum_{j=1}^{n} d(x_j, y_j) : n \in \mathbb{N}, x \sim x_1, y_j \sim x_{j+1}, y_n \right\}.
\]

(4.1)

This pseudometric can be roughly interpreted in the following way: it is the length of the shortest discrete path from \( x \) to \( y \) when we are allowed to teleport between equivalent
elements. An equivalent way to define \( \tilde{d} \), that will be useful for further constructions, is the following:

\[
\tilde{d}(\tilde{x}, \tilde{y}) = \sup \left\{ |f(x) - f(y)| : f : M \to \mathbb{R} \text{ is constant in each } \tilde{z} \in \tilde{M}, \|f\|_{\text{Lip}} \leq 1 \right\},
\]

where \( \|f\|_{\text{Lip}} \) is the Lipschitz constant of \( f \).

On \( M \) we define yet the equivalence relation \( \approx \) which identifies all \( x, y \in M \) satisfying \( \tilde{d}(x, y) = 0 \), and on \( M/\approx \) we define the metric \( \tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(x, y) \). We define \( X_\approx \), the metric quotient (or just quotient) of \( X \) with respect to \( \sim \), as the completion of \( M/\approx \). Note that, for a given complete metric space \( (M, d, 0) \) and an equivalence relation \( \sim \) on \( M \), by (4.2) there is a canonical isometric isomorphism between \( \text{Lip}_0(M_\approx) \) and the closed subspace of \( \text{Lip}_0(M) \) consisting of all functions that are constant in each class \( \tilde{x} \in M/\approx \).

We recall some definitions concerning path metric spaces. Let \( (M, d) \) be a pseudometric space, and let \( \varphi : I \to M \) be a curve (that is, \( I \) is an interval and \( \varphi \) is continuous). The length of \( \varphi \) is \( \ell(\varphi) := \sup \{ \sum_{j=1}^{n} d(\varphi(x_{j-1}), \varphi(x_j)) \} \), where the supremum is taken over \( n \in \mathbb{N} \) and \( x_j \in I, x_0 < \cdots < x_n \). \( (M, d) \) is said to be a path metric space if \( d \) is a metric and \( d(x, y) = \inf \{ \ell(\varphi) : \varphi \text{ is a curve in } M \text{ having endpoints } x \text{ and } y \} \). A minimizing geodesic in a path metric space is any curve \( \varphi : I \to M \) such that \( d(\varphi(t), \varphi(s)) = |t - s| \) for all \( t, s \in I \); \( (M, d) \) is said to be geodesic if any two elements of \( M \) are joined by a minimizing geodesic.

**Proposition 4.1.** Let \( (M, d) \) be a path metric space. Then each metric quotient of \( M \) is a path metric space.

**Proof.** Fix an equivalence relation \( \sim \) on \( M \). Let \( x, y \in M \), and for each \( k \in \mathbb{N} \) consider pairs \( (x_1^k, y_1^k), \ldots, (x_{n_k}^k, y_{n_k}^k) \) of elements of \( M \) such that

\[
x \sim x_1^k, y_j^k \sim x_{j+1}^k (j = 1, \ldots, n_k - 1), y_{n_k}^k \sim y
\]

and \( \sum_{j=1}^{n_k} d(x_j^k, y_j^k) \xrightarrow{k} \tilde{d}(\tilde{x}, \tilde{y}) \). Since \( (M, d) \) is a path metric space, there exist, for each \( k \in \mathbb{N} \) and \( j = 1, \ldots, n_k \), curves \( \varphi_j^k \) with endpoints \( x_j^k \) and \( y_j^k \), respectively, and such that

\[
\sum_{j=1}^{n_k} \ell(\varphi_j^k) < \tilde{d}(\tilde{x}, \tilde{y}) + \frac{1}{k}.
\]

Concatenating we get a curve \( \varphi^k \) in \( M/\sim \) with endpoints \( \tilde{x} \) and \( \tilde{y} \) satisfying \( \ell(\varphi^k) < \tilde{d}(\tilde{x}, \tilde{y}) + \frac{1}{k} \). Since for any curve \( \tilde{\varphi} \) in \( M/\sim \) with endpoints \( \tilde{x} \) and \( \tilde{y} \) we have \( \tilde{d}(\tilde{x}, \tilde{y}) \leq \ell(\tilde{\varphi}) \), it follows that

\[
\tilde{d}(\tilde{x}, \tilde{y}) = \inf \{ \ell(\tilde{\varphi}) : \tilde{\varphi} \text{ is a curve in } M/\sim \text{ having endpoints } \tilde{x} \text{ and } \tilde{y} \},
\]

and then clearly the same holds for \( M/\approx \) and thus for \( M_\approx \). \( \square \)

**Remark:** In Proposition 4.1 we cannot substitute path metric space by geodesic metric space:
**Proposition 4.2.** There is a geodesic metric space \( M \) which admits a metric quotient that is a non-geodesic path metric space.

**Proof.** Let \( e_j \) be the standard unit vectors of \( \ell_1 \) and consider the metric subspace of \( \ell_1 \) defined by \( M := \bigcup_{j=1}^{\infty} [0, 1] e_j / \bigcup_{j=1}^{\infty} \{ e_j \} \). Let \( F := \bigcup_{j=1}^{\infty} \{ e_j \} \) and suppose that \( \sim \) is the equivalence relation which collapses \( F \) to a point. Note that, in this case, \( M / \sim = M_{\infty} \), that the \( M_{\infty} \)-distance between \( \tilde{0} \) and \( \tilde{e}_1 \) is \( 1 \) and that there are minimizing geodesics with endpoints \( \tilde{0} \) and \( \tilde{e}_1 \) going through each segment \([0, 1] e_j \).

Let \( F_j := \left[ \frac{1}{4} + \frac{1}{2^{2+j}}, \frac{3}{4} - \frac{1}{2^{2+j}} \right] e_j \) be interpreted as a subset of \( M \), and consider on \( M \) the equivalence relation \( \equiv \) that collapses each \( F_j \) to a point, and the respective quotient metric space \(( M / \equiv, d) \) (again, in this case we have \( M / \equiv = M / \equiv \)). Then \( d([\tilde{0}]_\equiv, [\tilde{e}_1]_\equiv) = \frac{1}{2} \) and there are curves \( \varphi_j \) in \( M / \equiv \) with endpoints \([\tilde{0}]_\equiv \) and \([\tilde{e}_1]_\equiv \) with \( \ell(\varphi_j) \xrightarrow{j \to \infty} \frac{1}{2} \), even though there is no minimizing geodesic in \( M / \equiv \) with endpoints \([\tilde{0}]_\equiv \) and \([\tilde{e}_1]_\equiv \). One verifies the path metric property for all pairs of elements of \( M / \equiv \). □

**Lemma 4.3.** Consider \( M \) a path metric space, \( N \) a metric space, \( f : M \to N \) and \( C > 0 \). Then \( f \) is \( C \)-Lipschitz if and only if it is locally \( C \)-Lipschitz.

**Proof.** To prove the nontrivial implication, fix \( \delta > 0 \), let \( x, y \in M \) and let \( \varphi : I \to M \) be a curve with endpoints \( x \) and \( y \) satisfying

\[
\ell(\varphi) < d_M(x, y) + \delta.
\]

For each \( t \in I \), there exists by hypothesis \( \epsilon_t > 0 \) such that \( f|_{\varphi([t-\epsilon_t, t+\epsilon_t])} \) is \( C \)-Lipschitz. Since \( I \) is compact, there are \( t_1 < \cdots < t_n \) such that \( \bigcup_{j=1}^{\infty} [t - \epsilon_t, t + \epsilon_t] \subseteq I \). We can then easily find \( \varphi \)-consecutive points \( z_1, \ldots, z_m \) in \( \varphi(I) \) satisfying

\[
d_N(f(x), f(y)) \leq d_N(f(x), f(z_1)) + d_N(f(z_1), f(z_2)) + \cdots + d_N(f(z_m), f(y))
\]

\[
\leq C(d_M(x, z_1) + d_M(z_1, z_2) + \cdots + d_M(z_m, y))
\]

\[
\leq C(d_M(x, y) + \delta).
\]

Since \( \delta \) was arbitrary, the conclusion follows. □

Let \((X, \| \cdot \|)\) be a Banach space. We now proceed with the construction of a pair of metric quotients of \( X \), namely \( X_L \) and \( X_R \), which have properties useful for studying products of \( Lip_0(X) \) (see Proposition 4.4). Let \( \alpha, \beta : [0, +\infty) \to [0, +\infty) \) be the continuous functions defined in each \([2^m, 2^{m+1}], m \in \mathbb{Z} \) by

\[
\alpha(t) := \begin{cases} 
2^m, & \text{if } 2^m \leq t < 2^m + 2^m; \\
2^m + 2^m, & \text{if } 2^m + 2^m \leq t < 2^m + 2^{m+1};
\end{cases}
\]

and

\[
\beta(t) := \begin{cases} 
2^{m-1}, & \text{if } 2^m \leq t < 2^m + 2^m; \\
t - 2^m, & \text{if } 2^m + 2^m \leq t < 2^m + 2^{m+1}.
\end{cases}
\]

Consider the equivalence relations \( \sim_L \) and \( \sim_R \) on \( X \) defined by

\[
x \sim_L y \iff x = y \text{ or } (x = \lambda y \text{ with } \lambda > 0, \text{ and } \alpha \text{ is constant in } [\|x\|, \|y\|])
\]

\[
x \sim_R y \iff x = y \text{ or } (x = \lambda y \text{ with } \lambda > 0, \text{ and } \beta \text{ is constant in } [\|x\|, \|y\|])
\]
and

\[ x \sim_R y \iff x = y \text{ or } (x = \lambda y \text{ with } \lambda > 0, \text{ and } \beta \text{ is constant in } [\|x\|,\|y\|]) \]

and denote by \( X_L = (X_L, d_L) \) and \( X_R = (X_R, d_R) \) the corresponding quotient metric spaces.

To prove the next lemma we use Hopf-Rinow’s Theorem which states that in a complete and locally compact path metric space, each pair of points are joined by a minimizing geodesic (see e.g. [7]).

**Lemma 4.4.** \( X_L \) and \( X_R \) are geodesic.

**Proof.** For any \( x, y \in X \), note that the metric space \((\text{span}_X \{x, y\}/ \sim_R, d_R)\) satisfies the conditions in Hopf-Rinow’s Theorem, thus there is a minimizing geodesic \( \gamma \) in \((\text{span}_X \{x, y\}/ \sim_R, d_R)\) (thus also in \( X_R \)) with endpoints \( \bar{x} \) and \( \bar{y} \). The same argument holds for \( X_L \). \( \square \)

**Lemma 4.5.** There exist onto bi-Lipschitz mappings \( L : X \to X_L \) and \( R : X \to X_R \) with \( \|L\|_{\text{Lip}} \leq 1, \|L^{-1}\|_{\text{Lip}} \leq 4/3, \|R\|_{\text{Lip}} \leq 3/2 \) and \( \|R^{-1}\|_{\text{Lip}} \leq 1 \).
Similarly, let \( \tilde{z} \) be the intersection point between the line segment \([x, y]\) and \(S_{2m+1}\). Then the pairs \(x, z\) and \(y\) satisfy (1), and by (4.4) we have
\[
d(R(x), R(y)) \leq d(R(x), R(z)) + d(R(z), R(y)) \leq \frac{3}{2} \|x - z\| + \|z - y\| = \frac{3}{2} \|x - y\|.
\]
Similarly, let \(\tilde{z}\) be the intersection of \(R(S_{2m+1})\) with a minimizing geodesic with endpoints \(R(x)\) and \(R(y)\). Then \(d(R(x), \tilde{z}) = \|R(x) - \tilde{z}\|\) and \(d(R(\tilde{z}), R(y)) = \|\tilde{z} - R(y)\|\), and thus by (4.4) we have that
\[
\|x - y\| \leq \|x - R^{-1}(\tilde{z})\| + \|R^{-1}(\tilde{z}) - y\| \leq d(R(x), \tilde{z}) + d(R(\tilde{z}), R(y)) = d(R(x), R(y)).
\]
For the remainder case (3), we can obtain the desired inequalities by taking a convenient point in a minimizing geodesic with endpoints \(R(x)\) and \(R(y)\) and reducing the problem
to the case (2).

The Lipschitz equivalence between \( X \) and \( X_L \) is given by the mapping \( L : X \setminus \{0\} \to X_L \setminus \{0\} \) defined by

\[
L(x) := \left( \frac{1}{2} + \frac{2^{m_x - 1}}{\|x\|} \right)^{\sim_L},
\]

which squeezes each crown \( C_m \) onto the thinner crown \( L(C_m) = (B_{2^{m+2^{m-1}}} \setminus B_{2^{m-1}})^{\sim_L} \). To show this and obtain the Lipschitz constants, one simply must follow the same steps taken to do that for \( R \). The only difference will appear when getting to (4.3), which will read

\[
\left( \frac{1}{2} + \frac{2^{m_x - 1}}{\|y\|} \right) \|x - y\| \leq \|L(x) - L(y)\| \leq \left( \frac{1}{2} + \frac{2^{m_x - 1}}{\|x\|} \right) \|x - y\|,
\]

and thus (4.4) will read

\[
\frac{3}{4} \|x - y\| \leq d_L(L(x), L(y)) \leq \|x - y\|.
\]

Following analogous steps, we get to the conclusion. \( \square \)

**Proposition 4.6.** Let \( X \) be a Banach space. Then \( d_{BM}(F(X), F(X) \oplus_1 F(X)) \leq 4 \).

**Proof.** Recall that, by (4.2), \( \text{Lip}_0(X_L) \cong Y_L \) and \( \text{Lip}_0(X_R) \cong Y_R \), where \( Y_L \) and \( Y_R \) are the closed subspaces of \( \text{Lip}_0(X) \) defined by

\[
Y_L := \{ f \in \text{Lip}_0(X) : f \text{ is constant in each equivalence class of } X_L \}
\]

and

\[
Y_R := \{ f \in \text{Lip}_0(X) : f \text{ is constant in each equivalence class of } X_R \}.
\]

Let \( \Phi : Y_L \oplus Y_R \to \text{Lip}_0(X) \) be defined by \( \Phi(f, g) := f + g \). Then \( \Phi \) is linear with norm \( \|\Phi\| \leq 2 \). Moreover, \( \Phi \) admits an inverse defined by

\[
(\Phi^{-1}h)(x) = \left( \frac{\alpha(\|h(x)\|)}{\|h(x)\|} h(x), \frac{\beta(\|h(x)\|)}{\|h(x)\|} h(x) \right).
\]

Since the functions \( x \mapsto \frac{\alpha(\|x\|)}{\|x\|} x \) and \( x \mapsto \frac{\beta(\|x\|)}{\|x\|} x \) are 1-Lipschitz, it follows that \( \|\Phi^{-1}\| \leq 1 \). Now from Lemma 4.5 it follows that there is an isomorphism \( \Psi \) from \( \text{Lip}_0(X) \oplus_\infty \text{Lip}_0(X) \) onto \( Y_L \oplus Y_R \) satisfying \( \|\Psi\| \|\Psi^{-1}\| \leq \frac{4}{3} \cdot \frac{3}{2} = 2 \). Then \( \Phi \circ \Psi \) is an isomorphism from \( \text{Lip}_0(X) \oplus_\infty \text{Lip}_0(X) \) onto \( \text{Lip}_0(X) \) satisfying \( \|\Phi \circ \Psi\| \|\Phi \circ \Psi^{-1}\| \leq 4 \). Since \( \Phi \) and \( \Psi \) are pointwise-to-pointwise continuous, \( \Phi \circ \Psi \) induces an isomorphism \( \tilde{T} : F(X) \to F(X) \oplus_1 F(X) \) satisfying \( T^* = \Phi \circ \Psi \) and \( \|T\| \|T^{-1}\| \leq 4 \). \( \square \)
5. On Lipschitz-free spaces over unions of metric spaces

In this section we will provide a couple of formulas for computing the Lipschitz-free space over certain unions of metric spaces from the Lipschitz-free spaces of the original metric spaces, provided that we have an orthogonal placement of the metric spaces involved (in a sense to be made precise). The first one generalizes in particular a first step taken in this direction by Godard [5] (see Proposition 5.2 below). The idea of studying the behavior of the Lipschitz-free spaces with respect to unions is motivated, for example, by the problem of characterizing the metric spaces such that the corresponding Lipschitz-free spaces admit isomorphic embeddings into $L^1$. We start by studying how taking certain orthogonal unions of metric spaces have effect on the corresponding Lipschitz-free spaces:

**Proposition 5.1.** Suppose that $M = \bigcup_{\gamma \in \Gamma} M_\gamma$ is a metric space with metric $d$, and suppose that there exists $0 \in M$ satisfying

1. $M_\gamma \cap M_\eta = \{0\}$ for $\gamma \neq \eta$, and
2. (orthogonality) there exists $C \geq 1$ such that, for all $\gamma \neq \eta$, $x \in M_\gamma$ and $y \in M_\eta$, $d(x,0) + d(y,0) \leq C d(x,y)$.

Then

$$F(\bigcup_{\gamma \in \Gamma} M_\gamma) \simeq \left(\sum_{\gamma \in \Gamma} F(M_\gamma)\right)_{\ell_1}$$

with distortion bounded by $C$.

**Proof.** We can assume $0 = 0_M = 0_{M_\gamma}$, for all $\gamma$. Consider

$$\Phi : (f_\gamma) \in \left(\sum_{\gamma \in \Gamma} Lip_0(M_\gamma)\right)_{\ell_\infty} \mapsto h \in Lip_0(\bigcup_{\gamma \in \Gamma} M_\gamma),$$

(5.1)

where $h|_{M_\gamma} = f_\gamma$. $h = \Phi(f_\gamma)$ is well defined since, for each $x \in M_\zeta$ and $y \in M_\eta$ with $\zeta \neq \eta$,

$$|\Phi((f_\gamma))(x) - \Phi((f_\gamma))(y)| = |f_\zeta(x) - f_\eta(y)| \leq \|f_\zeta\|d(x,0) + \|f_\eta\|d(y,0) \leq C \max\{\|f_\zeta\|, \|f_\eta\|\}d(x,y) \leq C\|f_\gamma\|d(x,y).$$

Since $\Phi$ is linear, we have from the inequality above that

$$\|\Phi\| \leq \max\{1, C\}. \quad (5.2)$$

It is clear that $\Phi$ is surjective, and that $\|\Phi^{-1}\| \leq 1$. $\Phi$ is weak*-to-weak* continuous, thus it is the adjoint of an isomorphism from $F(\bigcup_{\gamma \in \Gamma} M_\gamma)$ onto $\left(\sum_{\gamma \in \Gamma} F(M_\gamma)\right)_{\ell_1}$. It is clear from (5.2) that the distortion of $\Phi$ is bounded by $C$. □

Note that, if we exclude the orthogonality condition, the concatenating operator $\Phi$ defined in (5.1) does not need to take values in $Lip_0(M)$. We do not know if it is possible to remove this condition, even for finite unions. In fact, in particular we do not now the answer to the following apparently simple question:
**Problem.** Consider the set $Cusp := \{(x, 0) : x \geq 0\} \cup \{(x, x^2) : x \geq 0\}$ endowed with the euclidean metric. Is $\mathcal{F}(Cusp)$ isomorphic to a subspace of $L^1$?

Note that $Cusp$ is not Lipschitz equivalent to the real line, so we cannot appeal to Proposition 1.1 to have a positive answer. A negative answer would imply that there is no isomorphic copy of $\mathcal{F}(\mathbb{R}^2)$ in $L^1$, which as we mentioned is one of the main results from [13].

The mentioned result by Godard can be seen as a particular case of Proposition 5.1:

**Proposition 5.2 (Godard [5], Proposition 5.1).** Let $\Gamma$ be a set with a distinguished point 0, denote $\Gamma^* := \Gamma \setminus \{0\}$, and let $M = \bigcup_{\gamma \in \Gamma} M_\gamma$ be a metric space with metric $d$. Suppose that there exist $A, B > 0$ such that $A \leq d(x, y) \leq B$ whenever $x$ and $y$ belong to different $M_\gamma$’s. Then

$$\mathcal{F}(M) \simeq \left( \sum_{\gamma \in \Gamma} \mathcal{F}(M_\gamma) \right) \oplus \ell_1(\Gamma^*).$$

**(Alternative) proof.** Assume that $A \leq 1 \leq B$. Fix a point $p \in M_0$, and for each $\gamma \neq 0$ fix $0_\gamma \in M_\gamma$ and define $\Phi_\gamma : Lip_p(M_\gamma \cup \{p\}) \to Lip_{0_\gamma}(M_\gamma) \oplus_\infty \mathbb{R}$ by $\Phi_\gamma(f) := \Phi_\gamma^1(f) + \Phi_\gamma^2(f)(f - f(0_\gamma), f(0_\gamma))$. It is easily seen that $\|\Phi_\gamma\| \leq B$. $\Phi_\gamma$ admits an inverse $\Phi_\gamma^{-1}$ which is defined by $\Phi_\gamma^{-1}(f, r)(x) = f(x) + r$, if $x \in M_\gamma$ and $\Phi_\gamma^{-1}(f, r)(p) = 0$. We have the bound $\|\Phi_\gamma^{-1}\| \leq \frac{B + 1}{A}$; to see this, let $f \in B_{Lip_{0_\gamma}(M_\gamma)}$ and $r \in \mathbb{R}$ with $|r| \leq 1$, and let $x \in M_\gamma$. Then

$$\frac{|\Phi_\gamma^{-1}(f, r)(x) - \Phi_\gamma^{-1}(f, r)(p)|}{d(x, p)} \leq \frac{|f(x) - r|}{d(x, p)} \leq \frac{|f(x)| + |r|}{A} \leq \frac{B + 1}{A}.$$ 

It follows that $\Phi : Lip_p(M_0) \oplus_\infty \left( \sum_{\gamma \in \Gamma^*} Lip_p(M_\gamma \cup \{p\}) \right)_{\ell_1} \to \left( \sum_{\gamma \in \Gamma} Lip_{0_\gamma}(M_\gamma) \right)_{\ell_\infty} \oplus_\infty \ell_\infty(\Gamma^*)$ defined by

$$\Phi(f, (f_\gamma)_{\gamma \in \Gamma^*}) := ((f, (\Phi_\gamma^1(f_\gamma))_{\gamma \in \Gamma^*}), (\Phi_\gamma^2(f_\gamma))_{\gamma \in \Gamma^*})$$

is a pointwise-to-pointwise continuous isomorphism with $\|\Phi\|, \|\Phi^{-1}\| \leq \frac{B(B + 1)}{A}$. It is therefore the adjoint of an isomorphism

$$\mathcal{F}(M_0) \oplus_1 \left( \sum_{\gamma \in \Gamma^*} \mathcal{F}(M_\gamma \cup \{p\}) \right)_{\ell_1} \simeq \left( \sum_{\gamma \in \Gamma} \mathcal{F}(M_\gamma) \right) \oplus_1 \ell_1(\Gamma^*). \quad (5.3)$$

Now the spaces $M_0, (M_\gamma \cup \{p\}), \gamma \in \Gamma^*$ satisfy the orthogonality condition (2) of Proposition 5.1, thus $\mathcal{F}(M) = \mathcal{F}(M_0 \cup (\cup_{\gamma \in \Gamma^*}(M_\gamma \cup \{p\})) \simeq \mathcal{F}(M_0) \oplus_1 \left( \sum_{\gamma \in \Gamma^*} \mathcal{F}(M_\gamma \cup \{p\}) \right)_{\ell_1}$, and the conclusion follows by (5.3). □

Note that, using Lemma 2.2 we can also have some control when the intersection of the metric spaces involved is nontrivial. We present a version for the union of two metric spaces.
Proposition 5.3. Let \( M \cup N \) be a metric space with metric \( d \), suppose that \( F := M \cap N \) is closed and nonempty, and choose a base point \( 0 \) in \( F \). Assume that

1. there exists \( E \) in \( \text{Ext}_0^d(F, M \cup N) \) satisfying \( \| E \| \leq C \), and
2. (orthogonality) there exists \( C \geq 1 \) such that, for all \( x \in M \) and \( y \in N \),
   \[
   d(x, F) + d(y, F) \leq C d(x, y).
   \]

Then

\[
\mathcal{F}(M \cup N) \simeq \mathcal{F}(M/F) \oplus_1 \mathcal{F}(N/F) \oplus_1 \mathcal{F}(F)
\]

with distortion bounded by \( C(\| E \| + 1)^2 \).

**Proof.** Denote by \( \tilde{d} \) the metric of \( (M \cup N)/F \), and note that \( M/F \) and \( N/F \) are subsets of \( (M \cup N)/F \). For each \( \tilde{x} \in M/F \) and each \( \tilde{y} \in N/F \),

\[
\tilde{d}(\tilde{x}, 0) + \tilde{d}(\tilde{y}, 0) = \min \{d(x, 0), d(x, F)\} + \min \{d(y, 0), d(y, F)\}
= d(x, F) + d(y, F) \leq C \min \{d(x, y), d(x, F) + d(y, F)\} = C \tilde{d}(\tilde{x}, \tilde{y}),
\]

and thus we can apply Proposition 5.1 and get that \( d_{BM}(\mathcal{F}(M \cup N), \mathcal{F}(M/F) \oplus_1 \mathcal{F}(N/F)) \leq C \). Since \( \mathcal{F}(M \cup N) \) is \((\| E \| + 1)^2\)-isomorphic to \( \mathcal{F}((M \cup N)/F) \oplus_1 \mathcal{F}(F) \) by Lemma 2.2, the result follows. \( \square \)

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CAPES Foundation, Ministry of Education of Brazil, Brasília/DF 70040-020, Brazil and Institute de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 Place Jussieu, 75005 Paris, France

E-mail address: plkaufmann@gmail.com