ON SPECTRAL PROPERTIES OF HIGH-DIMENSIONAL SPATIAL-SIGN COVARIANCE MATRICES IN ELLIPTICAL DISTRIBUTIONS WITH APPLICATIONS

WEIMING LI WANG ZHOU

Abstract. Spatial-sign covariance matrix (SSCM) is an important substitute of sample covariance matrix (SCM) in robust statistics. This paper investigates the SSCM on its asymptotic spectral behaviors under high-dimensional elliptical populations, where both the dimension $p$ of observations and the sample size $n$ tend to infinity with their ratio $p/n \to c \in (0, \infty)$. The empirical spectral distribution of this nonparametric scatter matrix is shown to converge in distribution to a generalized Marčenko-Pastur law. Beyond this, a new central limit theorem (CLT) for general linear spectral statistics of the SSCM is also established. For polynomial spectral statistics, explicit formulae of the limiting mean and covariance functions in the CLT are provided. The derived results are then applied to an estimation procedure and a test procedure for the spectrum of the shape component of population covariance matrices.

1. Introduction

Elliptical family of distributions, originally introduced in [20], is an important extension of the multivariate normal distribution and has been broadly applied in biology, finance and economics, signal and image processing, etc. [14, 17]. A random vector $\mathbf{x}$ with zero mean is said to be elliptically distributed if it has a stochastic representation [14]:

\begin{equation}
\mathbf{x} = w\mathbf{A}\mathbf{u},
\end{equation}

where $\mathbf{A}$ is a $p \times p$ matrix with $\text{rank}(\mathbf{A}) = p$, $w \geq 0$ is a scalar random variable representing the radius of $\mathbf{x}$, and $\mathbf{u} \in \mathbb{R}^p$ is the random direction, independent of $w$ and uniformly distributed on the unit sphere in $\mathbb{R}^p$. Besides the normal distribution, this family includes many other celebrated distributions, such as multivariate $t$-distribution, Kotz-type distributions, and Gaussian scale mixture. In general, the radius $w$ needs not be independent of the direction $\mathbf{u}$ but can be a function of the chosen direction [35].

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a sequence of independent and identically distributed (i.i.d.) random vectors from the elliptical model in (1.1). Many statistical procedures for this model prefer to transform the original observations into spatial-sign samples for the
eigenvectors and their eigenvalues have a one-to-one correspondence. Powerful tools to recover spectral features of the population SSCM, i.e. $\Sigma = \mathbf{T}$ and then those of the shape matrix $\mathbf{B}$, where $\mathbf{T}$ has the form
\[
\mathbf{T} = \frac{1}{p} \sum_{j=1}^{p} \mathbf{y}_j \mathbf{y}_j',
\]
which is actually the sample covariance matrix (SCM) of $(\mathbf{y}_j)$. As a robust alternative to the SCM $\mathbf{S}_n = \sum_{j=1}^{n} \mathbf{x}_j \mathbf{x}_j'/n$, this nonparametric scatter matrix $\mathbf{B}_n$ is a fast computed and orthogonally equivariant statistic with high breakdown point, and thus is highly recommended in applications, such as principle component analysis and structural test for covariance matrices, see [23], [16], [39], [31], to name a few. Despite its merits, the SSCM is also a controversial statistic in “small $p$, large $n$” scenarios due to its lack of affine equivariance [27]. However, the pursuit of this property seems not advisable for high-dimensional situations, as claimed in [38] that any well-defined affine equivariant scatter matrix must be proportional to the SCM $\mathbf{S}_n$ whenever $p > n$. Therefore, it is of great interest to discover behaviors of the SSCM in high-dimensional robust statistics.

In this paper, using tools of random matrix theory, we investigate asymptotic spectral behaviors of the SSCM $\mathbf{B}_n$ in high-dimensional frameworks where both the dimension $p$ and the sample size $n$ tend to infinity with their ratio $p/n \to c$, a positive constant in $(0, \infty)$. Specifically, let $(\lambda_j)_{1 \leq j \leq p}$ be the eigenvalues of $\mathbf{B}_n$, then the empirical spectral distribution (ESD) of $\mathbf{B}_n$ is by definition
\[
F_{\mathbf{B}_n} = \frac{1}{p} \sum_{j=1}^{p} \delta_{\lambda_j},
\]
where $\delta_b$ denotes the Dirac mass at $b$. Our aim is to study the limiting properties of $F_n$ and the central limit theorem (CLT) for linear spectral statistics (LSS) of the form $\int f(x) dF_n(x)$ for a class of smooth test functions $f$. These properties may become powerful tools to recover spectral features of the population SSCM, i.e. $\Sigma = pE(\mathbf{x}\mathbf{x}'/||\mathbf{x}||^2)$, and then those of the shape matrix $\mathbf{T}$ since the matrices $\Sigma$ and $\mathbf{T}$ share the same eigenvectors and their eigenvalues have a one-to-one correspondence [9]. Moreover, as $p \to \infty$, the two matrices coincide in the sense that the spectral norm $||\Sigma - \mathbf{T}|| \to 0$, as long as $||\Sigma||$ (or $||\mathbf{T}||$) is uniformly bounded, see Lemma 4.1.

Spectral properties of high-dimensional SCM have been extensively studied in random matrix theory since the pioneer work of [25]. The standard model in the literature has the form
\[
(1.2) \quad \tilde{\mathbf{x}} = \sigma \mathbf{A} \mathbf{z},
\]
where $\mathbf{A}$ is as before, $\sigma$ is a constant, and $\mathbf{z} = (z_1, \ldots, z_p)' \in \mathbb{R}^p$ is a set of i.i.d. random variables satisfying $E(z_1) = 0$, $E(z_1^2) = 1$, and $E(z_1^4) < \infty$. Let $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$ be $n$ i.i.d. copies of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{S}}_n = \sum_{j=1}^{n} \tilde{\mathbf{x}}_j \tilde{\mathbf{x}}_j'/n$ be the corresponding SCM. It has been known that the ESD of $\tilde{\mathbf{S}}_n$ converges to the celebrated Marčenko-Pastur (MP) law when $\mathbf{A} = I_p$, and generalized MP law for general matrix $\mathbf{A}$, as $(n, p) \to \infty$ with $p/n \to c > 0$. One
can refer to [25] and [36]. The CLT for LSS of $\tilde{S}_n$ was first studied in [19] by assuming the population to be standard multivariate normal. One breakthrough on the CLT was obtained by [3], where the population is allowed to be general with $E(z_1^4) = 3$. This fourth moment condition was then weakened to be $E(z_1^4) < \infty$ in [30]. For more references, one can refer to [4], [2], [15], and references therein. However, these results do not apply to general elliptical populations since the two underlying models in (1.1) and (1.2) have little in common, except for normal distributions. In fact, for general elliptical populations, it has been reported that the ESD of the SCM $S_n$ converges to a deterministic distribution that is not a generalized MP law, but has to be characterized by both the distribution of $w$ and the limiting spectrum of $T$ through a system of implicit equations [11, 24]. The involvement of $w$ seriously interferes with our understanding of the spectrum of $T$ from the ESD of $S_n$. This again motivates us to shift our attention to the SSCM $B_n$ which discards the random radiuses ($w_j$) and focus only on the directions ($A u_j$).

The main contributions of this paper are as follows. First in Section 2, asymptotic results on the eigenvalues of $B_n$ are derived, including the limit of the ESD $F_{B_n}$ and a new CLT for LSS of $B_n$. As a corollary, polynomial spectral statistics are fully addressed with explicit limiting mean and covariance functions in the CLT. Then in Section 3, relying on these results, we develop two statistical applications on the spectrum of $\Sigma$, the population SSCM, under a setting that the spectrum forms a discrete distribution with finite support. One is to estimate the spectrum of $\Sigma$ through moment methods and the other is to test the hypothesis that there are no more than $d_0$ distinct eigenvalues of $\Sigma$. Technical proofs of the main theorems are gathered in Section 4. Some lemmas and their necessary proofs are postponed to the last section.

2. HIGH-DIMENSIONAL THEORY FOR EIGENVALUES OF $B_n$

2.1. Limiting spectral distribution of $B_n$. We consider here the limit of the ESD sequence $(F_{B_n})$ in high-dimensional regimes, namely limiting spectral distribution (LSD). Our main assumptions are listed below.

Assumption (a). Both the sample size and population dimension $n, p$ tend to infinity in such a way that $c_n = p/n \rightarrow c \in (0, \infty)$.

Assumption (b). Sample observations are $y_j = \sqrt{p} A u_j / ||A u_j||$, $j = 1, \ldots, n$, where $A$ is a $p \times p$ matrix with $A A' = T$ and $(u_j)$ are i.i.d. random vectors, uniformly distributed on the unit sphere in $\mathbb{R}^p$.

Assumption (c). The spectral norm of $\Sigma = E(y_1 y_1')$ is bounded and its spectral distribution $H_p$ converges weakly to a probability distribution $H$, called population spectral distribution (PSD).

From Lemma 4.1, it is clear that the spectral distributions of $\Sigma$ and $T$ are asymptotically identical. So one can certainly replace $\Sigma$ with $T$ in Assumption (c), which does not affect the LSD of $F_{B_n}$. However we keep $\Sigma$ because it is easy to describe the CLT for LSS using the spectral distribution $H_p$ of $\Sigma$.

For the characterization of the LSD of $F_{B_n}$, we need to introduce the Stieltjes transform of a measure $G$ on the real line, which is defined as

$$m_G(z) = \int \frac{1}{x - z} dG(x), \quad z \in \mathbb{C} \setminus S_G,$$

where $S_G \subset \mathbb{R}$ denotes the support of $G$. 
**Theorem 2.1.** Suppose that Assumptions (a)-(c) hold. Then, almost surely, the empirical spectral distribution $F_{\mathbf{B}_n}$ converges weakly to a probability distribution $F^{c,H}$, whose Stieltjes transform $m = m(z)$ is the unique solution to the equation

$$m = \frac{1}{t(1-c-czm)-z}dH(t), \quad z \in \mathbb{C}^+, \quad \text{(2.1)}$$

in the set $\{m \in \mathbb{C} : -(1-c)/z + cm \in \mathbb{C}^+ \}$ where $\mathbb{C}^+ \equiv \{ z \in \mathbb{C} : \Im(z) > 0 \}$.

The LSD $F^{c,H}$ defined in (2.1) agrees with that in [25]. Let $m = m(z)$ denote the Stieltjes transform of $F_{\mathbf{B}_n}^{c,H} = cF^{c,H} + (1-c)\delta_0$. Then (2.1) can also be represented as

$$z = -\frac{1}{m} + c\int \frac{t}{1+tm}dH(t), \quad z \in \mathbb{C}^+. \quad \text{(2.2)}$$

See [36]. For procedures on finding the density function and the support set of $F^{c,H}$ from (2.1) and (2.2), one is referred to [4].

### 2.2. CLT for linear spectral statistics of $\mathbf{B}_n$.

Let $F_{\mathbf{B}_n}^{c_n,H_p}$ be the LSD as defined in (2.2) with the parameters $(c, H)$ replaced by $(c_n, H_p)$. Writing $G_n = F_{\mathbf{B}_n} - F_{\mathbf{B}_n}^{c_n,H_p}$, we next study the fluctuation of

$$\int f(x)dG_n(x) = \int f(x)[F_{\mathbf{B}_n}(x) - F_{\mathbf{B}_n}^{c_n,H_p}(x)],$$

which is a centralized linear spectral statistic with analytic $f$.

**Theorem 2.2.** Suppose that Assumptions (a)-(c) hold. Let $f_1, \ldots, f_k$ be $k$ functions analytic on an open interval containing

$$\left[ \liminf_{p \to \infty} \lambda_{\min}^2 \delta(x)(1 - \sqrt{c})^2, \quad \limsup_{p \to \infty} \lambda_{\max}^2 (1 + \sqrt{c})^2 \right].$$

Then the random vector

$$p \left( \int f_1(x)dG_n(x), \ldots, \int f_k(x)dG_n(x) \right)$$

converges weakly to a Gaussian vector $(X_{f_1}, \ldots, X_{f_k})$, whose mean function is

$$\mathrm{EX}_f = -\frac{1}{2\pi i} \int_{c_1} f(z) \int \frac{c(m'(z)t)^2dH(t)}{m(z)(1+m(z)t)^3}dz - \frac{cm(z)m'(z)}{\pi i} \int_{c_1} f(z)$$

$$\times \left[ \int \frac{\gamma_2 t - t^2dH(t)}{1+m(z)t} \int \frac{tdH(t)}{(1+m(z)t)^2} - \int \frac{t^2dH(t)}{1+m(z)t} \int \frac{tdH(t)}{(1+m(z)t)^2} \right]dz$$

and covariance function is

$$\mathrm{Cov}(X_f, X_g) = -\frac{1}{2\pi i} \int_{c_1} \int_{c_2} \frac{f(z_1)g(z_2)m'(z_1)m'(z_2)}{m(z_1)m(z_2)}dz_1dz_2$$

$$+ 2\gamma_2c \int xf'(x)dF(x) \int xg'(x)dF^{c,H}(x)$$

$$- \frac{1}{\pi i} \int_{c_1} \frac{f(z)m'(z)}{m^2(z)}dz \int xg'(x)dF^{c,H}(x)$$

$$- \frac{1}{\pi i} \int_{c_1} \frac{g(z)m'(z)}{m^2(z)}dz \int xf'(x)dF^{c,H}(x).$$
(f, g \in \{f_1, \cdots, f_k\}), where the contours \(C_1\) and \(C_2\) are non-overlapping, closed, counter-clockwise orientated in the complex plane, and each encloses the support of the LSD \(F^{c_H}\).

When the underlying population is multivariate normal, the elliptical model in (1.1) and the linear transformation model in (1.2) hold simultaneously. In this case, it is interesting to compare the limiting distribution in Theorem 2.2 based on SSCM with the classical result in [3] based on SCM. It turns out that there are some additional terms in our new CLT: the second contour integral in the mean function and the second to fourth summands in the covariance function.

Among all LSS, polynomial spectral statistics are of fundamental importance. The bases of these statistics are moments of ESD \(F^{B_n}\), i.e.

\[
\hat{\beta}_{nj} = \frac{1}{p} \text{tr}(B_n^j) = \int x^j dF^{B_n}(x), \quad j = 1, 2, \ldots.
\]

The first order moment \(\hat{\beta}_{n1}\) is 1 since \(\text{tr}(B_n) \equiv \text{tr}(\Sigma) \equiv p\). Other moments \((\hat{\beta}_{nj})\), \(j \geq 2\), are random. Their limiting behavior can be described through the following two quantities

\[
\beta_{nj} = \int x^j dF^{c_n,H_F}(x) \quad \text{and} \quad \gamma_{nj} = \int t^j dH_P(t),
\]

as well as their limits, denoted by \(\beta_j\) and \(\gamma_j\), respectively, \(j = 1, 2, \ldots\). From [28], the quantities \((\beta_{nj})\) and \((\gamma_{nj})\) are connected through the recursive formulæ:

\[
\beta_{nj} = \sum c_{n,j}^{i_1+\cdots+i_j-1}(\gamma_{n2})^{i_2} \cdots (\gamma_{nj})^{i_j}\phi(i_1, \ldots, i_j), \quad j \geq 2,
\]

and \(\beta_{n1} = \gamma_{n1} \equiv 1\), where the sum runs over the following partitions of \(j\):

\[(i_1, \ldots, i_j) : j = i_1 + 2i_2 + \cdots + ji_j, \quad i_t \in \mathbb{N},\]

and \(\phi(i_1, \ldots, i_j) = j!/|i_1! \cdots i_j!(j + 1 - i_1 - \cdots - i_j)!|\). The joint limiting distribution of moments \((\hat{\beta}_{nj})_{2 \leq j \leq k}\) can be derived from Theorem 2.2 by taking functions \(f_j(x) = x^j, j = 2, \ldots, k\). For this particular case, the mean and covariance functions in the limiting distribution can be explicitly formulated.

**Corollary 2.1.** Suppose that Assumptions (a)-(c) hold. Then the random vector

\[
p \left( \hat{\beta}_{n2} - \beta_{n2}, \ldots, \hat{\beta}_{nk} - \beta_{nk} \right) \xrightarrow{D} N_{k-1}(v, \Psi).
\]

The mean vector \(v = (v_j)_{2 \leq j \leq k}\) satisfies

\[
v_j = \frac{cP^j}{(j-2)!} \left[ \frac{P_{2,3}}{1-cz^2P_{2,2}} + 2\gamma_2P_{1,1}P_{1,2} - 2P_{2,1}P_{1,2} - 2P_{1,1}P_{2,2} \right]^{(j-2)} \bigg|_{z=0},
\]

where \(P_{s,t} = \int x^s(1+xz)^{-t}dH(x), P = (czP_{1,1} - 1)\), and \(g^{(\ell)}(z)\) denotes the \(\ell\)th derivative of \(g(z)\) with respect to \(z\). The covariance matrix \(\Psi = (\psi_{ij})_{2 \leq i,j \leq k}\) has entries

\[
\psi_{ij} = \sum_{\ell=0}^{i-1} (i - \ell) u_{i,\ell}u_{j,i+\ell} + 2c\gamma_2ij\beta_1\beta_j + 2j\beta_ju_{i,i+1} + 2i\beta_iu_{j,j+1},
\]

where \(u_{s,t} = [P^s]^{(t)}/t!|_{z=0} \).
3. Applications to spectral inference

Inference on PSD is fundamentally important in many high-dimensional statistical analysis, such as the principal component analysis \([18, 8, 40]\), factor models \([12, 13]\), and covariance matrix estimation \([21]\).

In this section, we illustrate two statistical applications of the theoretical results developed in Section 2: one is estimating a PSD and the other is testing the order of a PSD. The family of PSDs under study is a class of parameterized discrete distributions developed in Section \(3.1\), i.e. \((c_n, H^p) \equiv (c, H)\) for all \((n, p)\) large.

### 3.1. Estimation of a PSD

For the model in \((3.1)\), \([1]\) introduced a moment method for the PSD estimation. By assuming the order \(d\) to be known, their method first estimates the moments \((\gamma_j)\) of \(H\) through the recursive formulae in \((2.3)\), and then solve a system of moment equations, \(\{\hat{\gamma}_j = \sum_{i=1}^{d} a^j_i w_i, \ j = 0, \ldots, 2d - 1\}\), to get a consistent estimator of \(\theta\).

In our situation, with notation \(\beta_j = (\beta_2, \ldots, \beta_j)'\) and \(\gamma_j = (\gamma_2, \ldots, \gamma_j)'\) for \(j \geq 2\), we denote

\[g_1 : \gamma_{2d-1} \rightarrow \theta \quad \text{and} \quad g_{2,j} : \beta_j \rightarrow \gamma_j\]

as the mappings between the corresponding vectors. These two mappings are both one-to-one and the determinants of their Jacobian matrices are all nonzero. See \([1]\). Therefore, applying Theorem 2.1, \(\hat{\beta}_j := (\hat{\beta}_n2, \ldots, \hat{\beta}_{nj})' \overset{a.s.}{\rightarrow} \beta_j\) which is followed by \(\hat{\theta}_n := g_1 \circ g_{2,2d-1}(\hat{\beta}_{2d-1}) \overset{a.s.}{\rightarrow} \theta\), as \((n, p) \rightarrow \infty\). However, as shown by the CLT in Corollary 2.1, the estimator \(\hat{\beta}_j\) is biased by the order of \(O(1/p)\). So it’s natural to modify \(\hat{\beta}_j\) by subtracting its limiting mean in the CLT to obtain a better estimator of \(\theta\). Beyond this correction, the CLT can also provide confidence regions for the parameter \(\theta\).

Denote the modified estimators of \(\beta_j\), \(\gamma_j\), and \(\theta\) by

\[(3.2) \quad \hat{\beta}_j^* = \hat{\beta}_j - \frac{1}{p}(\hat{v}_2, \ldots, \hat{v}_j)', \quad \hat{\gamma}_j^* = g_{2,j}(\hat{\beta}_j^*), \quad \text{and} \quad \hat{\theta}_n^* = g_1(\hat{\gamma}_{2d-1}^*),\]

respectively, where \(\hat{v}_\ell = v_\ell(\hat{\beta}_\ell)\) with \(v_\ell\) defined in Corollary 2.1 for \(\ell = 2, \ldots, j\). From Theorem 2.1, Corollary 2.1, and a standard application of the Delta method, one may easily get asymptotic properties of these estimators.
**Theorem 3.1.** Suppose that Assumptions (a)-(c) hold and the true value \( \theta \) is an inner point of \( \Theta \). Then we have \( \hat{\beta}_j \overset{a.s.}{\longrightarrow} \beta_j \), \( \hat{\gamma}_j \overset{a.s.}{\longrightarrow} \gamma_j \), \( \hat{\theta}_n \overset{a.s.}{\longrightarrow} \theta \), and moreover

\[
(3.3) \quad p(\hat{\gamma}_j - \gamma_j) \xrightarrow{D} N_{j-1}(0, J_{2j} \Psi_{j,j}' J_{2j}'), \quad p(\hat{\theta}_n - \theta) \xrightarrow{D} N_{2k-2}(0, J_{1} J_{2,2d-1} \Psi_{2d-1} J_{2,2d-1}'),
\]

where \( J_1 \) and \( J_{2,\ell} \) represent the Jacobian matrices \( \partial g_1 / \partial \gamma_{2d-1} \) and \( \partial g_{2,\ell} / \partial \beta_\ell \), respectively, and \( \Psi_{\ell} \) is defined in Corollary 2.1 with \( k = \ell \).

**3.2. Test for the order of a PSD.** The aforementioned estimation procedure requires that the order \( d \) of the PSD be pre-specified. In general, this prior knowledge should be testified in advance. To deal with this problem, we consider the hypotheses

\[
(3.4) \quad H_0 : d \leq d_0 \quad v.s. \quad H_1 : d > d_0,
\]

where \( d_0 \geq 1 \) is a known constant. These hypotheses can also be regarded as a generalization of the well-known sphericity hypotheses on covariance matrices, i.e. the case \( d_0 = 1 \).

In [32], a test procedure was outlined based on a moment matrix \( \Gamma \) and its estimator \( \hat{\Gamma} \) which can be formulated as

\[
\Gamma = \begin{pmatrix}
1 & \gamma_1 & \cdots & \gamma_{d_0} \\
\gamma_1 & \gamma_2 & \cdots & \gamma_{d_0+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{d_0} & \gamma_{d_0+1} & \cdots & \gamma_{2d_0}
\end{pmatrix}
\quad \text{and} \quad \hat{\Gamma} = \begin{pmatrix}
\hat{\gamma}_1 & \cdots & \hat{\gamma}_{d_0} \\
\hat{\gamma}_1 & \cdots & \hat{\gamma}_{d_0+1} \\
\vdots & \ddots & \vdots \\
\hat{\gamma}_{d_0} & \cdots & \hat{\gamma}_{2d_0}
\end{pmatrix}.
\]

Here we set \( \hat{\gamma}_1 = 1 \) and \( \hat{\gamma}_j = \hat{\gamma}_j^{*} \), as defined in (3.2), for \( j \geq 2 \). It has been proved that the determinant \( \det(\hat{\Gamma}) \) of \( \hat{\Gamma} \) is zero if the null hypothesis in (3.4) holds, otherwise \( \det(\Gamma) \) is strictly positive [22]. Therefore, the determinant \( \det(\hat{\Gamma}) \) can serve as a test statistic for (3.4) and the null hypothesis shall be rejected if the statistic is significantly greater than zero. Applying Theorem 3.1 and the main theorem in [32], the asymptotic distribution of \( \det(\hat{\Gamma}) \) is obtained immediately.

**Theorem 3.2.** Suppose that Assumptions (a)-(c) hold. Then the statistic \( \det(\hat{\Gamma}) \) is asymptotically normal, i.e.

\[
(3.5) \quad p \left( \det(\hat{\Gamma}) - \det(\Gamma) \right) \xrightarrow{D} N(0, \sigma^2),
\]

where \( \sigma^2 = \alpha' \Omega V' \alpha \) with \( \alpha = \text{vec}(\text{adj}(\Gamma)) \), the vectorization of the adjugate matrix of \( \Gamma \). The first two rows and columns of the \((2d_0 + 1) \times (2d_0 + 1)\) matrix \( \Omega \) consist of zero and the remaining submatrix \( J_{2,2d_0} \Psi_{2d_0} J_{2,2d_0}' \) is defined in (3.3). The \((d_0 + 1)^2 \times (d_0 + 1)^2\) matrix \( V = (v_{ij}) \) is a 0-1 matrix with only \( v_{i,i} = 1, a_i = i - (i - 1)/(d_0 + 1)d_0, i = 1, \ldots, (d_0 + 1)^2 \), where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \).

From Theorem 3.1, the limiting variance \( \sigma^2 \) in (3.5) is a continuous function of \( \gamma_{4d_0} \).

While, under the null hypothesis, this variance is a function of \( \gamma_{2d_0-1} \), denoted by \( \sigma^2_{H_0}(\gamma_{2d_0-1}) \). Let \( \hat{\sigma}^2_{H_0} = \hat{\sigma}^2_{H_0}(\hat{\gamma}_{2d_0-1}) \). Then it is a strongly consistent estimator of \( \sigma^2_{H_0}(\gamma_{2d_0-1}) \).

**Corollary 3.1.** Suppose that Assumptions (a)-(c) hold. Then, under the null hypothesis,

\[
T_n := \frac{p \det(\hat{\Gamma})}{\hat{\sigma}_{H_0}} \xrightarrow{D} N(0, 1),
\]
Table 1. Estimation for Model 1 with sample size $n = 100, 200, 400$ and $c = 2$. The number of independent replications is 10,000 and the nominal coverage probability (C. P.) is fixed at 95%.

| $\theta$ | $n = 100$ |   | $n = 200$ |   | $n = 400$ |   |
|----------|-----------|---|-----------|---|-----------|---|
|         | Mean      | St. D. | C. P.    |   | Mean      | St. D. | C. P.    |   | Mean      | St. D. | C. P.    |
| $a_1 = 0.5$ | 0.4839    | 0.1145 | 0.9375   |   | 0.4960    | 0.0550 | 0.9491   |   | 0.5000    | 0.0269 | 0.9486   |
| $w_1 = 0.5$ | 0.4915    | 0.1135 | 0.9137   |   | 0.4968    | 0.0588 | 0.9423   |   | 0.4997    | 0.0292 | 0.9488   |
| $a_2 = 1.5$ | 1.5030    | 0.1330 | 0.9288   |   | 1.4990    | 0.0668 | 0.9426   |   | 1.4998    | 0.0329 | 0.9487   |
| $w_2 = 0.5$ | 0.5085    | 0.1135 | 0.9137   |   | 0.5032    | 0.0588 | 0.9423   |   | 0.5003    | 0.0292 | 0.9488   |

Table 2. Estimation for Model 2 with sample size $n = 400, 800, 1600$ and $c = 1/4$. The number of independent replications is 10,000 and the nominal coverage probability (C. P.) is fixed at 95%.

| $\theta$ | $n = 400$ |   | $n = 800$ |   | $n = 1600$ |   |
|----------|-----------|---|-----------|---|-----------|---|
|         | Mean      | St. D. | C. P.    |   | Mean      | St. D. | C. P.    |   | Mean      | St. D. | C. P.    |
| $a_1 = 0.2$ | 0.1887    | 0.0429 | 0.9227   |   | 0.1988    | 0.0147 | 0.9358   |   | 0.2003    | 0.0071 | 0.9367   |
| $w_1 = 0.3$ | 0.2824    | 0.0447 | 0.9403   |   | 0.2956    | 0.0184 | 0.9525   |   | 0.2990    | 0.0090 | 0.9483   |
| $a_2 = 1.0$ | 0.9660    | 0.1347 | 0.9345   |   | 0.9924    | 0.0661 | 0.9486   |   | 0.9991    | 0.0337 | 0.9433   |
| $w_2 = 0.4$ | 0.4064    | 0.0373 | 0.9453   |   | 0.4012    | 0.0209 | 0.9239   |   | 0.4002    | 0.0110 | 0.9351   |
| $a_3 = 1.8$ | 1.7824    | 0.0856 | 0.9236   |   | 1.7919    | 0.0440 | 0.9413   |   | 1.7960    | 0.0227 | 0.9392   |
| $w_3 = 0.3$ | 0.3113    | 0.0696 | 0.9221   |   | 0.3031    | 0.0365 | 0.9429   |   | 0.3008    | 0.0189 | 0.9420   |

as $n \to \infty$. In addition, the asymptotic power of $T_n$ tends to 1.

Corollary 3.1 follows directly from Theorem 3.2 and its proof is thus omitted. This corollary includes as a particular case the sphericity test. For this case, the test statistic reduces to $T_n = n(\hat{\gamma}^2 - 1)/2$ and its null distribution is consistent with that in [31].

3.3. Simulation experiments. Simulations are carried out to evaluate the performance of proposed estimation and test for discrete PSDs in (3.1). Samples of $(z_{ij})$ are drawn from $N(0, 1)$ and all statistics are calculated from 10,000 independent replications.

The estimation procedure are conducted for two PSDs, Models 1 and 2: Model 1 is of order 2 with the dimension to sample size ratio $c = 2$ and Model 2 is of order 3 with the ratio $c = 1/4$.

- Model 1: $H_1 = 0.5\delta_{0.5} + 0.5\delta_{1.5}$ and $c = 2$.
- Model 2: $H_2 = 0.3\delta_{0.2} + 0.4\delta_{1} + 0.3\delta_{1.8}$ and $c = 1/4$.

The sample size is $n = 100, 200, 400$ for Model 1 and $n = 400, 800, 1600$ for Model 2, respectively. In addition to empirical means and standard deviations of all estimators, we also calculate 95% confidence intervals for all parameters and report their coverage probabilities. Results are collected in Tables 1 and 2, which clearly demonstrate the consistency of all estimators as the sample size $n$ become large.

Next we examine the test for the order of a PSD. Two models are employed for this experiment:

- Model 3: $H_3 = 0.5\delta_{1-x} + 0.5\delta_{1+x}$,
- Model 4: $H_4 = 0.25\delta_{0.5-x} + 0.25\delta_{0.5+x} + 0.25\delta_{1.5-x} + 0.25\delta_{1.5+x}$,

where the parameter $x \in [0, 0.5)$ represents the distance between the null and alternative hypotheses. In particular, Model 3 is used for testing $H_0 : d \leq 1$ (sphericity
Table 3. Empirical size and power of $T_n$ in percentage under Model 3 and Model 4 with the sample size $n = 400$. The number of independent replications is 10,000 and the nominal significance level is 0.05.

| $H_0 : d \leq 1$ under Model 3 | $H_0 : d \leq 2$ under Model 4 |
|--------------------------------|--------------------------------|
| $x$                           | $c = \frac{1}{2}$ |
| 0                             | 5.24 5.81 9.13 |
| 0.02                          | 17.91 34.86 62.30 |
| 0.04                          | 87.31 98.01 99.90 |
| 0.06                          | 100 100 |
| 0.08                          | 0.10 0.12 0.14 0.16 0.18 |
| $c = 1$                       | 5.33 5.92 8.43 |
| 5.92                          | 18.09 35.62 63.12 |
| 8.43                          | 88.14 98.69 99.96 |
| 12.22                         | 100 100 100 |
| $c = 2$                       | 4.76 6.39 9.69 |
| 5.65                          | 16.33 30.09 49.17 |
| 8.56                          | 71.60 86.54 95.20 |
| 16.33                         | 98.67 99.97 100 |

4. Proofs

4.1. Some key lemmas. We present three lemmas which form the core basis for the proofs of Theorems 2.1 and 2.2.

Lemma 4.1. Let $\mathbf{x} = (x_1, \ldots, x_p)' \sim N_p(0, \mathbf{T})$ where $\mathbf{T} = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2)$ is a diagonal matrix with the spectral norm $||\mathbf{T}||$ bounded. Write $r_k = \sum_{i=1}^p \sigma_i^{2k}/p$, $k = 1, 2$. Then we have for $1 \leq i \neq j \leq p$,

$E\left(\frac{x_i^2}{\sum_{i=1}^p x_i^2/p}\right) = \frac{\sigma_i^2}{r_1} + \frac{2\sigma_i^2 r_2 - 2\sigma_i^4 r_1}{p r_1^3} + o\left(\frac{1}{p}\right),$

$E\left(\frac{x_i^2 x_j^2}{(\sum_{i=1}^p x_i^2/p)^2}\right) = \frac{\sigma_i^2 \sigma_j^2}{r_1^2} + \frac{6 \sigma_i^2 \sigma_j^2 r_2 - 4 \sigma_i^2 \sigma_j^2 (\sigma_i^2 + \sigma_j^2) r_1}{p r_1^4} + o\left(\frac{1}{p}\right),$

$E\left(\frac{x_i^4}{(\sum_{i=1}^p x_i^2/p)^2}\right) = \frac{3 \sigma_i^4}{r_1^2} + \frac{18 \sigma_i^4 r_2 - 24 \sigma_i^6 r_1}{p r_1^4} + o\left(\frac{1}{p}\right).$

Proof. As three expectations can be evaluated through a similar way, we only present the details for the second one as an illustration. Replacing the denominator of the quantity inside the expectation by $r_1^2$ and making their difference yields

$$
\frac{x_i^2 x_j^2}{(\sum_{i=1}^p x_i^2/p)^2} - \frac{x_i^2 x_j^2}{r_1^2} = \frac{x_i^2 x_j^2 [p^2 r_1^2 - (\sum_{i=1}^p x_i^2)^2]}{p^2 r_1^4} + \frac{x_i^2 x_j^2 [p^2 r_1^2 - (\sum_{i=1}^p x_i^2)^2]^2}{p^4 r_1^8} + o_p\left(\frac{1}{p}\right)
$$

(4.1) $\quad := \frac{A_p}{r_1^4} + \frac{B_p}{r_1^6} + o_p\left(\frac{1}{p}\right),$
where

\[ A_p = \frac{x_i^2 x_j^2}{p^2} \left[ \left( \sum_{k \neq i,j} \sigma_k^2 \right)^2 - \left( \sum_{k \neq i,j} x_k^2 \right)^2 \right] + o_p \left( \frac{1}{p^2} \right) \]

\[ B_p = \frac{x_i^2 x_j^2}{p^4} \left[ \left( \sum_{k \neq i,j} \sigma_k^2 \right)^2 - \left( \sum_{k \neq i,j} x_k^2 \right)^2 \right]^2 + o_p \left( \frac{1}{p} \right). \]

Taking expectations of \( A_p \) and \( B_p \), we get

\[
E(A_p) = \frac{-2\sigma_i^2 \sigma_j^2 r_2 + 4\sigma_i^2 \sigma_j^2 (\sigma_i^2 + \sigma_j^2) r_1}{p} + o \left( \frac{1}{p} \right),
\]

\[
E(B_p) = \frac{8\sigma_i^2 \sigma_j^2 r_2 r_1^2}{p} + o \left( \frac{1}{p} \right),
\]

which combined with (4.1) gives

\[
E \left( \frac{x_i^2 x_j^2}{(\sum_{i=1}^p x_i^2/p)^2} \right) = \frac{E(x_i^2 x_j^2)}{r_1^2} + \frac{E(A_p)}{r_1^2} + \frac{E(B_p)}{r_1^2} + o \left( \frac{1}{p} \right)
= \frac{\sigma_i^2 \sigma_j^2}{r_1^2} + \frac{6\sigma_i^2 \sigma_j^2 r_2 - 4\sigma_i^2 \sigma_j^2 (\sigma_i^2 + \sigma_j^2) r_1}{pr_1^4} + o \left( \frac{1}{p} \right).
\]

**Lemma 4.2.** Let \( y = \sqrt{p}x/\|x\| \) where \( x \) is as defined in Lemma 4.1 such that \( E(yy') = \Sigma \). For any \( p \times p \) complex matrices \( C \) and \( \tilde{C} \) with bounded spectral norms,

\[
E(y'Cy - \text{tr}\Sigma C) (y'\tilde{C}y - \text{tr}\Sigma \tilde{C})
= \text{tr}\Sigma C \Sigma \tilde{C}' + \text{tr}\Sigma C \Sigma \tilde{C} + \frac{2}{p} \left( \gamma_2 \text{tr}\Sigma C \text{tr}\Sigma \tilde{C} - \text{tr}\Sigma^2 C \text{tr}\Sigma \tilde{C} - \text{tr}\Sigma C \text{tr}\Sigma^2 \tilde{C} \right) + o(p),
\]

where \( \gamma_2 = \text{tr}\Sigma^2/p \).

**Proof.** By symmetry, \( E(y_i^2 y_j) = E(y_i^2 y_j y_k) = E(y_i y_j y_k y_l) = 0 \) for \( 1 \leq i \neq j \neq k \neq l \leq p \).

Write \( C = (C_{ij}) \) and \( \tilde{C} = (\tilde{C}_{ij}) \), we thus get

\[
(4.2) \quad E(y'Cy)(y'\tilde{C}y) = \sum_{i=1}^p C_{ii} \tilde{C}_{ii} E(y_i^4) + \sum_{i \neq j} (C_{ij} \tilde{C}_{jj} + C_{ij} \tilde{C}_{ij} + C_{ij} \tilde{C}_{ji}) E(y_i^2 y_j^2).
\]

From Lemma 1, we have

\[
\sum_{i \neq j} C_{ij} \tilde{C}_{jj} E(y_i^2 y_j^2) = \frac{\text{tr}\Sigma C \text{tr} \tilde{C}}{r_1^2} + \frac{6r_2 \text{tr}\Sigma C \text{tr} \tilde{C}}{pr_1^4} - \frac{4r_1 \left( \text{tr} \Sigma^2 C \text{tr} \tilde{C} + \text{tr} C \text{tr} \Sigma \tilde{C} \right) + o(p)}{pr_1^4}
= -\frac{1}{3} \sum_{i=1}^p C_{ii} \tilde{C}_{ii} E(y_i^4) + o(p),
\]

\[
\sum_{i \neq j} C_{ij} \tilde{C}_{ij} E(y_i^2 y_j^2) = \frac{\text{tr} \Sigma C \text{tr} \tilde{C}}{r_1^2} - \frac{1}{3} \sum_{i=1}^p C_{ii} \tilde{C}_{ii} E(y_i^4) + o(p),
\]

\[
\sum_{i \neq j} C_{ij} \tilde{C}_{ji} E(y_i^2 y_j^2) = \frac{\text{tr} \Sigma C \text{tr} \tilde{C}}{r_1^2} - \frac{1}{3} \sum_{i=1}^p C_{ii} \tilde{C}_{ii} E(y_i^4) + o(p).
\]
Therefore, our next aim is to study the fluctuation of the random process $x$ and $\tilde{C}$ from the above quantities and (4.5). Let $v > 0$ be arbitrary, $x_r$ any number greater than $\limsup_{p \to \infty} \lambda_{\max}^\Sigma (1 + \sqrt{c})^2$, and $x_l$ any negative number if $\liminf_{p \to \infty} \lambda_{\min}^\Sigma (1 - \sqrt{c})^2 I(0,1) (\cdot) = 0$, otherwise choose $x_l \in (0, \liminf_{p \to \infty} \lambda_{\min}^\Sigma (1 - \sqrt{c})^2)$. Define a contour $C$ as

$$C = \{ x + iv : x \in [x_l, x_r] \} \cup \{ x + iv : x \in \{x_r, x_l\}, v \in [-v_0, v_0] \}.$$  

Let $m_0(z)$ and $\overline{m}_n(z)$ be the Stieltjes transforms of $F_{c_n, H_p}$ and $c_n F_{c_n, H_p} + (1 - c_n) \delta_0$. Our next aim is to study the fluctuation of the random process

$$M_n(z) = p[m_n(z) - m_0(z)] = n[\overline{m}_n(z) - \overline{m}_n(z)], \quad z \in C.$$  

For this, we define a truncated version $\widehat{M}_n(z)$ of $M_n(z)$ as

$$\widehat{M}_n(z) = \begin{cases} M_n(z) & z \in C_n, \\ M_n(x + in^{-1} \varepsilon_n) & x \in \{x_l, x_r\} \text{ and } v \in [0, n^{-1} \varepsilon_n], \\ M_n(x - in^{-1} \varepsilon_n) & x \in \{x_l, x_r\} \text{ and } v \in [-n^{-1} \varepsilon_n, 0], \end{cases}$$

where $C_n = \{ x + iv_0 : x \in [x_l, x_r] \} \cup \{ x \pm iv : x \in \{x_l, x_r\}, v \in [n^{-1} \varepsilon_n, v_0] \}$ and the sequence $(\varepsilon_n)$ decreasing to zero satisfying $\varepsilon_n > n^{-a}$ for some $a \in (0, 1)$. 

Finally, from (4.3), we may replace $\mathbf{T}$ with $r_1 \Sigma$ and replace $r_2/r_1^2$ with $\text{tr}(\Sigma^2)/p$ in the above expression and then obtain the result of the Lemma. □
Lemma 4.3. Under Assumptions (a)-(c), the random process \( \hat{M}_n(\cdot) \) converges weakly to a two-dimensional Gaussian process \( M(\cdot) \) satisfying for \( z, z_1, z_2 \in \mathcal{C} \),

\[
\text{EM}(z) = \int \frac{c(m'(z)t^2dH(t)}{m(z)(1 + m(z)t)^3} + 2cm(z)m'(z)\left[ \int \frac{\gamma_2 t - t^2dH(t)}{1 + m(z)t} \right] \int \frac{tdH(t)}{(1 + m(z)t)^2}
\]

(4.6) \hspace{1cm} \int \frac{tdH(t)}{1 + m(z)t} \int \frac{t^2dH(t)}{(1 + m(z)t)^2}

and covariance function

\[
\text{Cov}(M(z_1), M(z_2)) = \frac{2m'(z_1)m'(z_2)}{(m(z_1) - m(z_2))^2} - \frac{2}{(z_1 - z_2)^2} + \frac{2\gamma_2}{c} (m(z_1) + z_1m'(z_1)) (m(z_2) + z_2m'(z_2))
\]

(4.7) \hspace{1cm} - \frac{2}{c} \left( \frac{m'(z_1)}{m^2(z_1)} - 1 \right) (m(z_2) + z_2m'(z_2)) - \frac{2}{c} \left( \frac{m'(z_2)}{m^2(z_2)} - 1 \right) (m(z_1) + z_1m'(z_1)).

Proof. Split \( \hat{M}_n(z) \) into two parts, \( \hat{M}_n(z) = M^{(1)}_n(z) + M^{(2)}_n(z) \), where

\[
M^{(1)}_n(z) = p[m_n(z) - \text{Em}_n(z)] \quad \text{and} \quad M^{(2)}_n(z) = p[\text{Em}_n(z) - m_0(z)].
\]

Following the strategy in [3], we prove the convergence of \( \hat{M}_n(z) \) by three steps:

Step 1: Finite dimensional convergence of \( M^{(1)}_n(z) \) in distribution;

Step 2: Tightness of \( M^{(1)}_n(z) \) on \( \mathcal{C}_n \);

Step 3: Convergence of \( M^{(2)}_n(z) \).

Without loss of generality, we assume \( ||\Sigma|| \leq 1 \) for all \( p \). Constants appearing in inequalities will be denoted by \( K \) which may take different values from one expression to the next.

Step 1: Finite dimensional convergence of \( M^{(1)}_n(z) \) in distribution. We show in this part, for any \( w \) complex numbers \( z_1, \ldots, z_w \in \mathcal{C}_n \), the random vector

\[
[M^{(1)}_n(z_1), \ldots, M^{(1)}_n(z_w)]
\]

(4.8) converges in distribution to a Gaussian vector. We begin with introducing some notation which will be frequently used in the sequel.

\[
\begin{align*}
\mathbf{r}_j &= (1/\sqrt{n})\mathbf{y}_j, \quad \mathbf{D}(z) = \mathbf{B}_n - z\mathbf{I}, \\
\mathbf{D}_j(z) &= \mathbf{D}(z) - \mathbf{r}_j'\mathbf{r}_j, \quad \mathbf{D}_{ij}(z) = \mathbf{D}(z) - \mathbf{r}_i - \mathbf{r}_j'\mathbf{r}_j, \\
\varepsilon_j(z) &= \mathbf{r}_j'\mathbf{D}_j^{-1}(z)\mathbf{r}_j - \frac{1}{n}\text{tr}\Sigma\mathbf{D}_j^{-1}(z), \quad \delta_j(z) = \mathbf{r}_j'\mathbf{D}_j^{-2}(z)\mathbf{r}_j - \frac{1}{n}\text{tr}\Sigma\mathbf{D}_j^{-2}(z), \\
\beta_j(z) &= \frac{1}{1 + \mathbf{r}_j'\mathbf{D}_j^{-1}(z)\mathbf{r}_j}, \quad \tilde{\beta}_j(z) = \frac{1}{1 + n^{-1}\text{tr}\Sigma\mathbf{D}_j^{-1}(z)}, \\
b_n(z) &= \frac{1}{1 + n^{-1}\text{Etr}\Sigma\mathbf{D}_j^{-1}(z)}.
\end{align*}
\]

Note that, for any \( z = u + iv \in \mathbb{C}^+ \), the last three quantities are bounded in absolute value by \( |z|/v \).

Let \( E_0(\cdot) \) denote expectation and \( E_j(\cdot) \) denote conditional expectation with respect to the \( \sigma \)-field generated by \( \mathbf{r}_1, \ldots, \mathbf{r}_j \). From the martingale decomposition and the identity

\[
\mathbf{D}^{-1}(z) - \mathbf{D}_j^{-1}(z) = -\mathbf{D}_j^{-1}(z)\mathbf{r}_j\mathbf{D}_j^{-1}(z)\beta_j(z),
\]

(4.9)
we have
\[
M_n^{(1)}(z) = \text{tr}(D^{-1}(z) - ED^{-1}(z))
\]
\[
= \sum_{j=1}^{n} \text{tr}E_j D^{-1}(z) - \text{tr}E_{j-1} D^{-1}(z)
\]
\[
= \sum_{j=1}^{n} \text{tr}E_j[D^{-1}(z) - D_j^{-1}(z)] - \text{tr}E_{j-1}[D^{-1}(z) - D_j^{-1}(z)]
\]
\[
(4.10) \quad = - \sum_{j=1}^{n} (E_j - E_{j-1})\beta_j(z)r_j'D_j^{-2}r_j.
\]

Writing \( \beta_j(z) = \bar{\beta}_j(z) - \bar{\beta}_j(z)\beta_j(z)\epsilon_j(z) = \bar{\beta}_j(z) - \bar{\beta}_j^2\epsilon_j(z) + \bar{\beta}_j^2 \beta_j(z)\epsilon_j^2(z) \), we have
\[
(E_j - E_{j-1})\beta_j(z)r_j'D_j^{-2}r_j
\]
\[
= (E_j - E_{j-1}) (\bar{\beta}_j(z)\delta_j(z) - \bar{\beta}_j^2(z)\epsilon_j(z)r_j'D_j^{-2}(z)r_j + \bar{\beta}_j(z)\beta_j(z)\epsilon_j^2(z)r_j'D_j^{-2}(z)r_j)
\]
\[
= \frac{d}{dz}E_j\bar{\beta}_j(z)\epsilon_j(z) - (E_j - E_{j-1})\bar{\beta}_j^2(z) (\epsilon_j(z)\delta_j(z) - \beta_j(z)\epsilon_j^2(z)r_j'D_j^{-2}(z)r_j).
\]

Note that
\[
E \left| \sum_{j=1}^{n} (E_j - E_{j-1})\bar{\beta}_j^2(z)\epsilon_j(z)\delta_j(z) \right|^2 = \sum_{j=1}^{n} E|\sum_{j=1}^{n} (E_j - E_{j-1})\bar{\beta}_j^2(z)\epsilon_j(z)\delta_j(z)|^2
\]
\[
\leq 4 \sum_{j=1}^{n} E|\bar{\beta}_j^2(z)\epsilon_j(z)\delta_j(z)|^2
\]
\[
\leq \frac{4|z|^4}{\nu^4} \sum_{j=1}^{n} E^{\frac{1}{2}}|\epsilon_j(z)|^4 E^{\frac{1}{2}}|\delta_j(z)|^4
\]
which is \( o(1) \) from Lemma 5.1. Similarly, \( E \left| \sum_{j=1}^{n} (E_j - E_{j-1})\bar{\beta}_j^2(z)\beta_j(z)\epsilon_j^2(z)r_j'D_j^{-2}(z)r_j \right|^2 = o(1) \). Thus we get
\[
\sum_{j=1}^{n} (E_j - E_{j-1})\bar{\beta}_j^2(z) (\epsilon_j(z)\delta_j(z) + \beta_j(z)\epsilon_j^2(z)r_j'D_j^{-2}(z)r_j) = o_p(1)
\]
which implies that we need only to consider the limiting distribution of
\[
-\frac{d}{dz} \sum_{j=1}^{n} E_j\bar{\beta}_j(z)\epsilon_j(z) = -\frac{d}{dz} \sum_{j=1}^{n} (E_j - E_{j-1})\bar{\beta}_j(z)\epsilon_j(z)
\]
in finite dimensional situations. For any \( \epsilon > 0 \),
\[
\sum_{j=1}^{n} E \left| E_j \frac{d}{dz}\epsilon_j(z)\bar{\beta}_j(z) \right|^2 I(|E_j \frac{d}{dz}\epsilon_j(z)\beta_j(z)| \geq \epsilon)
\]
\[
\leq \frac{1}{\epsilon^2} \sum_{j=1}^{n} E \left| E_j \frac{d}{dz}\epsilon_j(z)\bar{\beta}_j(z) \right|^4 \leq \frac{K}{\epsilon^2} \sum_{j=1}^{n} \left( \frac{|z|^4 E|\delta_j(z)|^4}{\nu^4} + \frac{|z|^8 p^4 E|\epsilon_j(z)|^4}{\nu^{16} n^4} \right)
\]
which tends to zero according to Lemma 5.1 and thus verifies the Lyapunov condition. Therefore, from the martingale CLT (Lemma 5.4), the random vector in (4.8) will tend
to a Gaussian vector \((M^{(1)}(z_1), \ldots, M^{(1)}(z_w))\) with covariance function (4.11)
\[
\text{Cov}(M^{(1)}(z_1), M^{(1)}(z_2)) = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{\partial^2}{\partial z_1 \partial z_2} E_{j-1} \left( E_{j} \varepsilon_{j}(z_1) \beta_j(z_1) \cdot E_{j} \varepsilon_{j}(z_2) \beta_j(z_2) \right),
\]
provided this limit exists. By the same arguments in page 571 of [3], it is sufficient to show that
\[
\sum_{j=1}^{n} E_{j-1} \prod_{k=1}^{2} E_{j} \beta_j(z_k) \varepsilon_j(z_k)
\]
converges in probability. Since
\[
E[\beta_j(z) - b_n(z)]^2 = |b_n(z)|^2 n^{-2} E[\beta_1(z) (\text{tr} \Sigma D_1^{-1}(z) - E \text{tr} \Sigma D_1^{-1}(z))]^2 
\leq \frac{|z|^4}{n^2 v^4} E \sum_{k=2}^{n} (E_k - E_{k-1}) \text{tr}(D_1^{-1}(z) - D_{1k}^{-1}(z))^2 
\leq \frac{K |z|^4}{v^6} E \sum_{k=2}^{n} \text{tr}(D_1^{-1}(z) - D_{1k}^{-1}(z))^2 
\leq \frac{K |z|^4}{v^6 n},
\]
where the last inequality is from
\[
|\text{tr}(D^{-1}(z) - D_j^{-1}(z)) M| \leq \frac{||M||}{v},
\]
for any \(p \times p\) matrix \(M\), see Lemma 2.6 in [37]. Moreover, from the definition of \(m_0(z)\) and discussions in Page 439 in [5], we also have
\[
b_n(z) + zm_0(z) \to 0.
\]
It is hence sufficient to study the convergence of
\[
z_1 z_2 m_0(z_1) m_0(z_2) \sum_{j=1}^{n} E_{j-1} \left( E_{j} \varepsilon_{j}(z_1) E_{j} \varepsilon_{j}(z_2) \right),
\]
whose second mixed partial derivative yields the limit of (4.11). From Lemma 2, we know that
\[
(4.16) = 2(T_1 + \gamma_n T_2 - T_3 - T_4),
\]
where
\[
T_1 = \frac{z_1 z_2 m_0(z_1) m_0(z_2)}{n^2} \sum_{j=1}^{n} \text{tr} \left[ E_{j} \Sigma D_j^{-1}(z_1) E_{j} (\Sigma D_j^{-1}(z_2)) \right],
\]
\[
T_2 = \frac{z_1 z_2 m_0(z_1) m_0(z_2)}{pn^2} \sum_{j=1}^{n} \text{tr} \left[ E_{j} \Sigma D_j^{-1}(z_1) \right] \text{tr} \left[ E_{j} \Sigma D_j^{-1}(z_2) \right],
\]
\[
T_3 = \frac{z_1 z_2 m_0(z_1) m_0(z_2)}{pn^2} \sum_{j=1}^{n} \text{tr} \left[ E_{j} \Sigma^2 D_j^{-1}(z_1) \right] \text{tr} \left[ E_{j} \Sigma D_j^{-1}(z_2) \right],
\]
\[
T_4 = \frac{z_1 z_2 m_0(z_1) m_0(z_2)}{pn^2} \sum_{j=1}^{n} \text{tr} \left[ E_{j} \Sigma D_j^{-1}(z_1) \right] \text{tr} \left[ E_{j} \Sigma^2 D_j^{-1}(z_2) \right].
\]
Now we consider the limit of $T_1$. Let
\[
\beta_{ij}(z) = \frac{1}{1 + r_i^j D_{ij}^{-1}(z) r_i}, \quad b_1(z) = \frac{1}{1 + n^{-1} \text{tr} \Sigma D_{ij}^{-1}(z)}, \quad L(z) = zI - \frac{n-1}{n} b_1(z) \Sigma.
\]

Note that
\[
||L(z)||^{-1} \leq \frac{|b_1^{-1}(z)|}{\Im(z b_1^{-1}(z))} \leq \frac{|b_1^{-1}(z)|}{\Im(z)} \leq \frac{1 + p/(nv)}{v}. \tag{4.18}
\]

From the equality $r_i^j D_{ij}^{-1}(z) = \beta_{ij}(z) r_i^j D_{ij}^{-1}(z)$, we get
\[
D_j^{-1}(z) + L^{-1}(z) = L^{-1}(z) (D_j(z) + L(z)) D_j^{-1}(z) = L^{-1}(z) \left( \sum_{i \neq j} r_i r_i' - \frac{n-1}{n} b_1(z) \Sigma \right) D_j^{-1}(z) = L^{-1}(z) \left( \sum_{i \neq j} r_i \beta_{ij}(z) r_i' D_{ij}^{-1}(z) - \frac{n-1}{n} b_1(z) \Sigma D_j^{-1}(z) \right) = b_1(z) R_1(z) + R_2(z) + R_3(z), \tag{4.19}
\]

where
\[
R_1(z) = \sum_{i \neq j} L^{-1}(z) (r_i r_i' - n^{-1} \Sigma) D_{ij}^{-1}(z),
R_2(z) = \sum_{i \neq j} (\beta_{ij}(z) - b_1(z)) L^{-1}(z) r_i r_i' D_{ij}^{-1}(z),
R_3(z) = n^{-1} b_1(z) L^{-1}(z) \Sigma \sum_{i \neq j} (D_{ij}^{-1}(z) - D_j^{-1}(z)).
\]

For any $p \times p$ matrix $M$, let $|\|\|M\||$ denote a non-random upper bound for the spectral norm of $M$. From Lemma 5.1, (4.14), and (4.18), we get
\[
E|\text{tr} R_1(z) M| \leq n E^{1/2} |r_i D_{12}^{-1}(z) ML^{-1}(z) r_1 - n^{-1} \text{tr} \Sigma D_{12}^{-1}(z) ML^{-1}(z)|^2 \leq n^{1/2} K|\|M\|| \frac{(1 + p/(nv))}{v^2}, \tag{4.20}
\]
\[
E|\text{tr} R_2(z) M| \leq n E^{1/2} |\beta_{12}(z) - b_1(z)|^2 |E^{1/2} |r_i^j D_{12}^{-1} ML^{-1}(z) r_1|^2 \leq n^{1/2} K|\|M\|| |z|^2 (1 + p/(nv)) v^2, \tag{4.21}
\]
\[
|\text{tr} R_3(z) M| \leq |\|M\|| |z| (1 + p/(nv)) v^3, \tag{4.22}
\]

where the matrix $M$ in the first two inequalities is assumed nonrandom.

Using the equality (4.9) we write
\[
\text{tr} E_j( R_1(z_1)) \Sigma D_j^{-1}(z_2) \Sigma = R_{11}(z_1, z_2) + R_{12}(z_1, z_2) + R_{13}(z_1, z_2), \tag{4.23}
\]
where

\[
R_{11}(z_1, z_2) = -\sum_{i<j} \beta_{ij}(z_2)r_i'E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)r_i' \Sigma L^{-1}(z_1)r_i,
\]

\[
R_{12}(z_1, z_2) = -\text{tr} \sum_{i<j} L^{-1}(z_1)n^{-1} \Sigma E_j(D_{ij}^{-1}(z_1))\Sigma (D^{-1}_j(z_2) - D_{ij}^{-1}(z_2))\Sigma,
\]

\[
R_{13}(z_1, z_2) = \text{tr} \sum_{i<j} L^{-1}(z_1)(r_ir_i' - n^{-1}\Sigma)E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma.
\]

From (4.14) and (4.18) we get \(|R_{12}(z_1, z_2)| \leq (1 + p/(nv))/v^3\) and \(E|R_{13}(z_1, z_2)| \leq n^{1/2}(1 + p/(nv))/v^3\). Using Lemma 5.1 we have, for \(i < j\),

\[
E \left| \beta_{ij}(z_2)r_i'E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)r_i' \Sigma L^{-1}(z_1)r_i 
- b_1(z_2)n^{-2}\text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \text{tr} (D_{ij}^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) \right| \leq Kn^{-1/2},
\]

and by (4.14),

\[
\left| \text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \text{tr} (D_{ij}^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) 
- \text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \text{tr} (D_{ij}^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) \right| \leq Kn.
\]

These imply that

\[
(4.24) \quad E \left| R_{11}(z_1, z_2) + \frac{j-1}{n^2}b_1(z_2)\text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \text{tr} (D_{ij}^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) \right| 
\leq Kn^{1/2}.
\]

Therefore, from (4.19)-(4.24),

\[
\text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \left(1 + \frac{j-1}{n^2}b_1(z_1)b_1(z_2)\text{tr} (D_{ij}^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) \right) 
= -\text{tr} L^{-1}(z_1)\Sigma D_{ij}^{-1}(z_2)\Sigma + R_{14}(z_1, z_2),
\]

where \(E|R_{14}(z_1, z_2)| \leq Kn^{1/2}\). From this and applying (4.19)-(4.24) again, we get

\[
(4.25) \quad \text{tr} (E_j(D_{ij}^{-1}(z_1))\Sigma D_{ij}^{-1}(z_2)\Sigma) \left(1 - \frac{j-1}{n^2}b_1(z_1)b_1(z_2)\text{tr} (L^{-1}(z_2)\Sigma L^{-1}(z_1)\Sigma) \right) 
= \text{tr} L^{-1}(z_1)\Sigma L^{-1}(z_2)\Sigma + R_{15}(z_1, z_2),
\]

where \(E|R_{15}(z_1, z_2)| \leq Kn^{1/2}\).
From (4.15) and (4.25), we obtain that
\[
\text{tr} \left( E_j(D_j^{-1}(z_1)) \Sigma D_j^{-1}(z_2) \Sigma \right) \left( 1 - \frac{j - 1}{n^2} m_0(z_1) m_0(z_2) \right)
\times \text{tr} \left( (I + m_0(z_2)) \Sigma^{-1} (I + m_0(z_1)) \Sigma^{-1} \right)
= \text{tr} \left( E_j(D_j^{-1}(z_1)) \Sigma D_j^{-1}(z_2) \Sigma \right) \left( 1 - \frac{j - 1}{n} \int \frac{c_j m_0(z_1) m_0(z_2) t^2 dH_p(t)}{(1 + t m_0(z_1))(1 + t m_0(z_2))} \right)
= \frac{nc_n}{z_1 z_2} \int \frac{t^2 dH_p(t)}{(1 + t m_0(z_1))(1 + t m_0(z_2))} + R_{16}(z_1, z_2).
\]
Here \( E|R_{16}(z_1, z_2)| \leq Kn^{1/2} \). Letting
\[
a_n(z_1, z_2) = \int \frac{c_n m_0(z_1) m_0(z_2) t^2 dH_p(t)}{(1 + t m_0(z_1))(1 + t m_0(z_2))},
\]
we get
\[
T_1 = \frac{1}{n} \sum_{j=1}^n a_n(z_1, z_2) + o_p(1) \overset{i.p.}{\longrightarrow} \int_0^1 \frac{1}{1 - z} dz,
\]
where
\[
a(z_1, z_2) = \int \frac{c m(z_1) m(z_2) t^2 dH(t)}{(1 + t m(z_1))(1 + t m(z_2))} = 1 + \frac{m(z_1) m(z_2)(z - z_2)}{m(z_2) - m(z_1)}.
\]
Elementary calculations reveal that
\[
\frac{\partial^2 T_1}{\partial z_1 \partial z_2} = \frac{m'(z_1) m'(z_2)}{(m(z_1) - m(z_2))^2} - \frac{1}{(z_1 - z_2)^2}.
\]

Now we derive the limits of \( T_2, T_3, T_4 \) and their second mixed partial derivatives. From (4.15), (4.19)-(4.22), it's easy to show that
\[
\text{tr} E_j D_j^{-1}(z_1) \Sigma^2 \text{tr} E_j D_j^{-1}(z_2) \Sigma = \frac{1}{z_1 z_2} \int \frac{t dH_p(t)}{1 + t m_0(z_1)} \int \frac{t dH_p(t)}{1 + t m_0(z_2)} + R_{17}(z_1, z_2),
\]
\[
\text{tr} E_j D_j^{-1}(z_1) \Sigma \text{tr} E_j D_j^{-1}(z_2) \Sigma = \frac{1}{z_1 z_2} \int \frac{t^2 dH_p(t)}{1 + t m_0(z_1)} \int \frac{t dH_p(t)}{1 + t m_0(z_2)} + R_{18}(z_1, z_2),
\]
where \( E|R_{17}(z_1, z_2)| \leq Kn \) and \( E|R_{18}(z_1, z_2)| \leq Kn \). We thus get
\[
T_2 = c_n \int \frac{t m_0(z_1) dH_p(t)}{1 + t m_0(z_1)} \int \frac{t m_0(z_2) dH_p(t)}{1 + t m_0(z_2)} + o_p(1) \overset{i.p.}{\longrightarrow} \frac{1}{c} \int_0^1 (1 + z_1 m(z_1))(1 + z_2 m(z_2)) dH_p(t),
\]
\[
T_3 = c_n \int \frac{t^2 m_0(z_1) dH_p(t)}{1 + t^2 m_0(z_1)} \int \frac{t m_0(z_2) dH_p(t)}{1 + t m_0(z_2)} + o_p(1) \overset{i.p.}{\longrightarrow} \int \frac{t^2 m(z_1)(1 + z_2 m(z_2)) dH_p(t)}{1 + t m(z_1)}
\]
\[
T_4 = c_n \int \frac{t m_0(z_1) dH_p(t)}{1 + t m_0(z_1)} \int \frac{t m_0(z_2) dH_p(t)}{1 + t m_0(z_2)} + o_p(1) \overset{i.p.}{\longrightarrow} \int \frac{t^2 m(z_2)(1 + z_1 m(z_1)) dH_p(t)}{1 + t m(z_2)}.
\]
Their corresponding derivatives are

\[
\frac{\partial^2 T_2}{\partial z_1 \partial z_2} = \frac{1}{c} \left( m(z_1) + z_1 m'(z_1) \right) \left( m(z_2) + z_2 m'(z_2) \right),
\]

\[
\frac{\partial^2 T_3}{\partial z_1 \partial z_2} = \int \frac{t^2 m'(z_1)(m(z_2) + z_2 m'(z_2)) dH_p(t)}{(1 + t m(z_1))^2},
\]

\[
\frac{\partial^2 T_4}{\partial z_1 \partial z_2} = \frac{1}{c} \left( \frac{m'(z_1)}{m^2(z_1)} - 1 \right) \left( m(z_2) + z_2 m'(z_2) \right),
\]

\[
\frac{\partial^2 T_5}{\partial z_1 \partial z_2} = \int \frac{t^2 m'(z_2)(m(z_1) + z_1 m'(z_1)) dH_p(t)}{(1 + t m(z_2))^2},
\]

\[
\frac{\partial^2 T_6}{\partial z_1 \partial z_2} = \frac{1}{c} \left( \frac{m'(z_2)}{m^2(z_2)} - 1 \right) \left( m(z_1) + z_1 m'(z_1) \right),
\]

respectively.

Collecting results in (4.17), (4.27)-(4.30), we finally get the covariance function in the lemma.

**Step 2: Tightness of \( M_n^{(1)}(z) \).** From the arguments in [3], the tightness of \( M_n^{(1)}(z) \) can be established by verifying the moment condition:

\[
\sup_{n,z_1,z_2 \in C_n} \frac{E|\Delta_n^{(1)}(z_1) - \Delta_n^{(1)}(z_2)|^2}{|z_1 - z_2|^2} < \infty.
\]

We first claim that moments of \( D^{-1}(z) \), \( D_j^{-1}(z) \) and \( D_{ij}^{-1}(z) \) are all bounded in \( n \) and \( z \in C_n \). Taking \( D^{-1}(z) \) for example, it’s clear that \( E|\Delta_n^{-1}(z)|^q < 1/v_0^q \) for \( z \in C_n \). For \( z \in C_l \cup C_r \), applying Lemma 5.5 with suitably large \( s \),

\[
E|\Delta_n^{-1}(z)|^q \leq K_1 + \frac{1}{v_0^q} P(\|B_n\| > \eta_r \text{ or } \lambda_{\text{min}}^B < \eta_l)
\]

\[
\leq K_1 + K_2 n^q \varepsilon^{-q} n^{-s} \leq K,
\]

where the two constant \( \eta_r \) and \( \eta_l \) satisfy \( \limsup_{n,p \to \infty} \lambda_{\text{max}}^I(1 + \sqrt{c})^2 < \eta_r < c \) and \( x_l < \eta_l < \liminf_{n,p \to \infty} \lambda_{\text{min}}^I(0,1)(c)(1 - \sqrt{c})^2 \). Therefore for any positive \( q \), we may assume that

\[
\max\{|E|\Delta_n^{-1}(z)|^q, E|\Delta_j^{-1}(z)|^q, E|\Delta_{ij}^{-1}(z)|^q\} \leq K_q.
\]

Using the above argument, we can extend the inequality in Lemma 5.1 to

\[
E \left[ a(v) \prod_{l=1}^q (y_l^r B_l(v) y_1 - tr \Sigma B_l(v)) \right] \leq K p^{q/2},
\]

where the matrices \( B_l(v) \) are independent of \( u_1 \) and

\[
\max\{|a(v)|, ||B_l(v)||\} \leq K \left[ 1 + p^s I \left( ||B_n|| \geq \eta_r \text{ or } \lambda_{\text{min}}^B \leq \eta_l \right) \right]
\]

for some positive \( s \), where \( \tilde{B} \) is \( B_n \) or \( B_n \) with some \( r_j \)'s removed. In applications of (4.33), \( a(v) \) can be a product of factors of \( \beta_1(z) \) or \( r_j^T D_j^{-1}(z_1) D_j^{-1}(z_2) r_1 \) or similar terms. It’s easy to verify that these terms satisfy (4.34), see pages 579 and 580 in [3] for details.

Let

\[
\gamma_j(z) = r_j^T D_j^{-1}(z) r_j - \frac{1}{n} E r_j^T D_j^{-1}(z).
\]
We first handle moments of \( \gamma_j(z) \). By a similar decomposition in (4.10), we may get
\[
E|\gamma_j(z) - \varepsilon_j(z)|^q = E \left| \frac{1}{n} \sum_{i \neq j} (E_i - E_{i-1}) \beta_{ij}(z) r'_i D_{ij}^{-1}(z) \Sigma D_i^{-1}(z) r_i \right|^q.
\]
Applying Lemma 5.3 and the Hölder inequality to the above expression we then get, for even \( q \),
\[
E|\gamma_j(z) - \varepsilon_j(z)|^q \leq \frac{K}{n^q} E \left[ \sum_{i \neq j} |(E_i - E_{i-1}) \beta_{ij}(z) r'_i D_{ij}^{-1}(z) \Sigma D_i^{-1}(z) r_i|^2 \right]^{q/2}
\leq \frac{K}{n^{q/2}} \sum_{i \neq j} E|E_i - E_{i-1}) \beta_{ij}(z) r'_i D_{ij}^{-1}(z) \Sigma D_i^{-1}(z) r_i|^q
\leq \frac{K}{n^{q/2}}.
\]
(4.35)
where the last inequality uses the boundedness of \( E|\beta_{ij}(z)|^q \) and \( E|r'_i D_{ij}^{-1}(z) \Sigma D_i^{-1}(z) r_i|^q \). From (4.33) and (4.35), we get
\[
E|\varepsilon_j(z)|^q \leq K n^{-q/2} \quad \text{and} \quad E|\gamma_j(z)|^q \leq K n^{-q/2},
\]
for even \( q \).

Next we show that \( b_n(z) \) is bounded for all \( n \). By the equality \( b_n(z) - \beta_j(z) = b_n(z) \beta_j(z) \gamma_j(z) \) and the boundedness of \( E|\beta_j(z)|^q \) and \( E|\gamma_j|^q \), we have
\[
|b_n(z)| = |E \beta_j(z) + E \beta_j(z)b_j(z) \gamma_j(z)| \leq K_1 + K_2 |b_n(z)| n^{-1/2}
\]
and thus, for all \( n \) large enough,
\[
|b_j(z)| \leq \frac{K_1}{1 - K_2 n^{-1/2}} < K.
\]
(4.37)

Now we prove (4.31). From the martingale decomposition and (4.9), we have
\[
\frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{z_1 - z_2} = \sum_{j=1}^n (E_j - E_{j-1}) \tr D^{-1}(z_1)D^{-1}(z_2)
\]
\[
= \sum_{j=1}^n (E_j - E_{j-1}) \left( \tr D^{-1}(z_1)D^{-1}(z_2) - \tr D_j^{-1}(z_1)D_j^{-1}(z_2) \right)
\]
\[
= \sum_{j=1}^n (E_j - E_{j-1}) \beta_j(z_1) \beta_j(z_2) \left( r'_j D_j^{-1}(z_1)D_j^{-1}(z_2) r_j \right)^2
\]
\[
- \sum_{j=1}^n (E_j - E_{j-1}) \beta_j(z_1) r'_j D_j^{-2}(z_1)D_j^{-1}(z_2) r_j
\]
\[
- \sum_{j=1}^n (E_j - E_{j-1}) \beta_j(z_2) r'_j D_j^{-1}(z_1)D_j^{-2}(z_2) r_j
\]
\[
:= A_1 + A_2 + A_3.
\]
It is then enough to show \( E|A_1|^2 \), \( E|A_2|^2 \), and \( E|A_3|^2 \) are all bounded. The arguments for the boundedness are all similar to those in pages 582 and 583 in [3], and hence we only present the details for \( E|A_1|^2 \) for illustration.
Replacing $\beta_j(z)$ in $R_1$ with $\beta_j(z) = b_n(z) - b_n(z)\beta_j(z)\gamma_j(z)$, we may obtain $A_1 = A_{11} - A_{12} - A_{13}$ where

$$A_{11} = \sum_{j=1}^{n} b_n(z_1)b_n(z_2)(E_j - E_{j-1}) (r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j)^2,$$

$$A_{12} = \sum_{j=1}^{n} b_n(z_1)b_n(z_2)(E_j - E_{j-1})\beta_j(z_1)\gamma_j(z_1) (r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j)^2,$$

$$A_{13} = \sum_{j=1}^{n} b_n(z_2)(E_j - E_{j-1})\beta_j(z_1)\beta_j(z_2)\gamma_j(z_2) (r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j)^2.$$

From (4.33), (4.34), and (4.37),

$$E|A_{11}|^2 = E \left| \sum_{j=1}^{n} b_n(z_1)b_n(z_2)(E_j - E_{j-1}) \left[ (r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j)^2 \right. \right.$$

$$\left. \left. - \frac{1}{n^2} \left( \text{tr} \Sigma D_j^{-1}(z_1)D_j^{-1}(z_2) \right)^2 \right] \right|^2,$$

$$\leq K \sum_{j=1}^{n} E \left| r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j - \frac{1}{n} \text{tr} \Sigma D_j^{-1}(z_1)D_j^{-1}(z_2) \right|^2,$$

$$\leq K.$$

Using (4.33), (4.34), (4.36), and (4.37),

$$E|A_{12}|^2 = \sum_{j=1}^{n} b_n^2(z_1)b_n^2(z_2) E \left| (E_j - E_{j-1})\beta_j(z_1)\gamma_j(z_1) (r'_jD_j^{-1}(z_1)D_j^{-1}(z_2)r_j)^2 \right|^2,$$

$$\leq K \sum_{j=1}^{n} \left[ E|\gamma_j(z_1)|^2 + v^{-10}p^2P \left( ||B_n|| \geq \eta \right. \right.$$

$$\left. \left. \text{or} \lambda_{\text{min}} \leq \eta \right) \right],$$

$$\leq K.$$

Similarly, we may get $E|A_{13}|^2 < K$. Hence the tightness of $M_n^{(1)}(z)$ is obtained.

**Step 3: Convergence of $M_n^{(2)}(z)$.** To finish the proof, it is enough to show that the sequence of $M_n^{(2)}(z)$ is bounded and equicontinuous, and converges to the mean function of the lemma for $z \in C_n$. The boundedness and equicontinuity can be verified following the arguments on pages 592 and 593 of [3], and thus we only focus on the convergence of $M_n^{(2)}(z)$.

We first list some results that will be used in the sequel:

(4.38) $\sup_{n,z \in C_n} |b_n(z) + zm_0(z)| \to 0, \quad \sup_{n,z \in C_n} ||zI - b_n(z)\Sigma|| < \infty,$

(4.39) $\sup_{n,z \in C_n} E|\text{tr}D^{-1}(z)M - \text{Etr}D^{-1}(z)M|^2 \leq K||M||^2,$

where $M$ is any nonrandom $p \times p$ matrix. These results can be verified step by step following similar discussions in [3] and we omit the details.

Writing $V(z) = zI - b_n(z)\Sigma$, we decompose $M_n^{(2)}(z)$ as

(4.40) $M_n^{(2)}(z) = [pEm_n(z) + \text{tr}V^{-1}(z)] - [\text{tr}V^{-1}(z) + pm_0(z)] := S_n(z) - T_n(z)$

(4.41) $= [nEm_n(z) + nb_n(z)/z] - [nb_n(z)/z + nm_0(z)] := S_n(z) - T_n(z).$
Notice that

\[ T_n(z) = p \int \frac{dH_p(t)}{z - b_n(z)t} - p \int \frac{dH_p(t)}{z + zm_n(z)t} \]
\[ = p [b_n(z) + zm_n(z)] \int \frac{tdH_p(t)}{(z - b_n(z)t)(z + zm_n(z)t)} \]
\[ = c_n T_n(z) \int \frac{tdH_p(t)}{(z - b_n(z)t)(1 + m_n(z)t)}. \]

We have

\[ \frac{M_n^{(2)}(z) - S_n(z)}{M_n^{(2)}(z) - S_n(z)} = \frac{T_n(z)}{T_n(z)} = \frac{c_n}{z} \int \frac{tdH_p(t)}{(1 + m_n(z)t)^2} + o(1), \]

where the second equality uses the convergence in (4.38).

Our next task is to study the limits of \( S_n(z) \) and \( S_n(z) - S_n(z) \). For simplicity, we suppress the expression \( z \) when it is served as independent variables of some functions in the sequel. All expressions and convergence statements hold uniformly for \( z \in C_n \).

We first simplify the expression of \( S_n \). Using the identity \( r_j' D_j^{-1} = r_j' D_j^{-1} \beta_j \), we have

\[ S_n = \text{Etr}(D^{-1} + V^{-1}) \]
\[ = \text{Etr} \left[ V^{-1} \left( \sum_{j=1}^{n} r_j' (D_j^{-1} - b_n \Sigma) \right) D^{-1} \right] \]
\[ = nE\beta_1 r_1' D_1^{-1} V^{-1} r_1 - b_n \text{Etr} \Sigma D^{-1} V^{-1}. \] (4.42)

From (4.9) and \( \beta_1 = b_n - b_n \beta_1 \gamma_1 \),

\[ \text{Etr} V^{-1} \Sigma (D_1^{-1} - D^{-1}) = \text{Etr} V^{-1} \Sigma D_1^{-1} r_1 r_1' D_1^{-1} \beta_1 \]
\[ = b_n E(1 - \beta_1 \gamma_1) r_1' D_1^{-1} V^{-1} \Sigma D_1^{-1} r_1, \]

where \(|E \beta_1 \gamma_1 r_1' D_1^{-1} V^{-1} \Sigma D_1^{-1} r_1| \leq Kn^{-1/2} \). From this and (4.42), we get

\[ S_n = nE\beta_1 r_1' D_1^{-1} V^{-1} r_1 - b_n \text{Etr} \Sigma D_1^{-1} V^{-1} + \frac{1}{n} b_n^2 \text{Etr} D_1^{-1} V^{-1} \Sigma D_1^{-1} \Sigma + o(1). \]

Plugging \( \beta_1 = b_n - b_n^2 \gamma_1 + b_n^3 \gamma_2 - \beta_1 b_n^2 \gamma_1^3 \) into the first term in the above equation, we obtain

\[ nE\beta_1 r_1' D_1^{-1} V^{-1} r_1 = b_n \text{Etr} D_1^{-1} V^{-1} \Sigma - nb_n^2 \text{Etr} \gamma_1 r_1' D_1^{-1} V^{-1} r_1 \]
\[ + nb_n^3 \text{Etr} \gamma_1^2 r_1' D_1^{-1} V^{-1} r_1 - nb_n^3 \text{Etr} \gamma_1 r_1' D_1^{-1} V^{-1} r_1. \]
Note that, from (4.33), (4.36), and (4.39),

\[ E\gamma_1' D_1^{-1} V^{-1} r_1 = E\left[ r_1' D_1^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} \right] \left[ r_1' D_1^{-1} V^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} V^{-1} \right] + \frac{1}{n} \text{Cov}(\text{tr} D_1^{-1}, \text{tr} D_1^{-1} V^{-1}) = E\left[ r_1' D_1^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} \right] \left[ r_1' D_1^{-1} V^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} V^{-1} \right] + o\left(\frac{1}{n}\right), \]

\[ E\gamma_2' D_1^{-1} V^{-1} r_1 = E\gamma_2' \left[ r_1' D_1^{-1} V^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} V^{-1} \right] + \frac{1}{n} \text{Cov}(\gamma_2, \text{tr} D_1^{-1} V^{-1}) + \frac{1}{n} E\gamma_1'E\text{tr} D_1^{-1} V^{-1} = \frac{1}{n} E\gamma_1'E\text{tr} D_1^{-1} V^{-1} + o\left(\frac{1}{n}\right), \]

\[ E\gamma_3' D_1^{-1} V^{-1} r_1 = o\left(\frac{1}{n}\right). \]

We thus arrive at

\[ S_n = -nb_n^2 E\left[ r_1' D_1^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} \right] \left[ r_1' D_1^{-1} V^{-1} r_1 - \frac{1}{n} \text{tr} D_1^{-1} V^{-1} \right] + nb_n^2 E\gamma_1'E\text{tr} D_1^{-1} V^{-1} + \frac{1}{n} b_n^2 E\text{tr} D_1^{-1} V^{-1} D_1^{-1} + o(1). \]

On the other hand, by the identity \( r_j' D^{-1} = r_j' D_j^{-1} \beta_j \), we have

\[ p + z\text{tr} D^{-1} = \text{tr}(B_n D^{-1}) = \sum_{j=1}^n \beta_j r_j' D_j^{-1} r_j = n - \sum_{j=1}^n \beta_j, \]

which implies \( n z\text{tr} \Sigma_n = -\sum_{j=1}^n \beta_j \). From this, together with \( \beta_1 = b_n - b_n^2 \gamma_1 + b_n^3 \gamma_1^2 - \beta_1 b_n^3 \gamma_1^3 \), (4.33), we get

\[ S_n = -\frac{n}{z} E (\beta_1 - b_n) = -\frac{n}{z} b_n^3 E\gamma_1^2 + o(1). \]
Applying Lemma 2 to the simplified $S_n$ and $\bar{S}_n$, and then replacing $D_j$ with $D$ in the derived results yield

\[
S_n = -\frac{b_3^2}{n^2} \left[ \text{Etr} D^{-1} \Sigma D^{-1} D^{-1} \Sigma + \frac{1}{p} \left( \frac{\gamma_2 \text{Etr} D^{-1} \Sigma D^{-1} \Sigma + 1}{p} \right) \right]
\]

Using (4.43) and (4.44), we have

\[
S_n = \frac{-2b_3}{zn} \left[ \text{Etr} D^{-1} \Sigma D^{-1} D^{-1} \Sigma \right] + o(1),
\]

To study the limits of $S_n$ and $\bar{S}_n$, we compare the difference between $D^{-1}$ and $V^{-1}$. Similar to (4.19)-(4.22), we have

\[
D^{-1} + V^{-1} = b_n \bar{R}_1 + \bar{R}_2 + \bar{R}_3,
\]

where $\bar{R}_1 = \sum_{j=1}^{n} V^{-1}(r_j r_j' - n^{-1} \Sigma)D_j^{-1}$ and, for any $p \times p$ matrix $M$,

\[
|\text{Etr} \bar{R}_2 M| \leq n^{1/2} K \text{tr} \text{tr} M||M||^4 \text{tr} M^{-1}, \quad |\text{tr} \bar{R}_3 M| \leq K \text{tr} \text{tr} M||M||^2 \text{tr} M^{-1}.
\]

Moreover, for nonrandom $M$ with bounded norm,

\[
|\text{Etr} \bar{R}_1 M| \leq n^{1/2} K.
\]

Similar to (4.23), we write

\[
\text{tr} \bar{R}_1 \Sigma D^{-1} M = \bar{R}_{11} + \bar{R}_{12} + \bar{R}_{13},
\]

where $\bar{R}_{11} = \text{tr} \sum_{j=1}^{n} V^{-1} r_j r_j' D_j^{-1} \Sigma (D^{-1} - D_j^{-1}) M$, $E\bar{R}_{12} = 0$, and $|E\bar{R}_{13}| \leq K$. Using (4.32), (4.33), and (4.39), we get

\[
E\bar{R}_{11} = -nE \beta_1 r_1 D_1^{-1} \Sigma D_1^{-1} r_1 r_1' D_1^{-1} MV^{-1} r_1
\]

Using (4.32), (4.33), and (4.39), we get

\[
E\bar{R}_{11} = -b_n n^{-1} E(\text{tr} D_1^{-1} \Sigma D_1^{-1} \Sigma)(\text{tr} D_1^{-1} MV^{-1} \Sigma) + o(1)
\]

Using (4.40), (4.41), and (4.42), we get

\[
E\bar{R}_{11} = -b_n n^{-1} E(\text{tr} D_1^{-1} \Sigma D_1^{-1} \Sigma)(\text{tr} D_1^{-1} MV^{-1} \Sigma) + o(1).
\]
From (4.45)-(4.49) we get

\[
\frac{1}{n} \text{Etr} D^{-1} \Sigma D^{-1} \Sigma \\
= - \frac{1}{n} \text{Etr} V^{-1} \Sigma D^{-1} \Sigma \left( 1 + \frac{b_n^2}{n} \text{Etr} V^{-1} \Sigma D^{-1} \Sigma \right) + o(1)
\]

(4.50)

\[
\frac{1}{n} \text{Etr} D^{-1} \Sigma D^{-1} V^{-1} \Sigma \\
= - \frac{1}{n} \text{Etr} V^{-1} \Sigma D^{-1} V^{-1} \Sigma \left( 1 + \frac{b_n^2}{n} \text{Etr} V^{-1} \Sigma D^{-1} \Sigma \right) + o(1)
\]

(4.51)

From (4.15), (4.45)-(4.51) we get

\[
\frac{1}{n} \text{Etr} D^{-1} \Sigma^k \\
= - \int \frac{c_n t^k dH_p(t)}{z(1 + m_0 t)} + o(1), \quad k = 1, 2,
\]

\[
\frac{1}{n} \text{Etr} D^{-1} V^{-1} \Sigma^k \\
= - \int \frac{c_n t^k dH_p(t)}{z^2(1 + m_0 t)^2} + o(1), \quad k = 1, 2,
\]

\[
\frac{1}{n} \text{Etr} D^{-1} \Sigma D^{-1} \Sigma \\
= \int \frac{c_n t^2 dH_p(t)}{z^3(1 + m_0 t)^3} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1} + o(1),
\]

\[
\frac{1}{n} \text{Etr} D^{-1} \Sigma D^{-1} V^{-1} \Sigma \\
= \int \frac{c_n t^2 dH_p(t)}{z^3(1 + m_0 t)^3} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1} + o(1).
\]

Combining the above results with (4.43) and (4.44), we obtain

\[
S_n = - \int \frac{c_n m_0^2 t^2 dH_p(t)}{z(1 + m_0 t)^3} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1}
\]

\[
- \frac{2}{z} \left\{ \int \frac{t^2 dH_p(t)}{(1 + m_0 t)^2} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1} \right. \\
+ \frac{2 c_n m_0^3}{z} \left\{ \int \frac{t^2 dH_p(t)}{(1 + m_0 t)^2} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1} \right. \\
+ \gamma_2 \left\{ \int \frac{t^2 dH_p(t)}{1 + m_0 t} \right\} - 2 \int \frac{t^2 dH_p(t)}{1 + m_0 t} \int \frac{t^2 dH_p(t)}{1 + m_0 t} \int \frac{t^2 dH_p(t)}{1 + m_0 t} \int \frac{t^2 dH_p(t)}{1 + m_0 t} \right\} \right. + o(1),
\]

\[
\bar{S}_n = 2 c_n m_0^3 \left\{ \int \frac{t^2 dH_p(t)}{(1 + m_0 t)^2} \left[ 1 - \int \frac{c_n m_0^2 t^2 dH_p(t)}{(1 + m_0 t)^2} \right]^{-1} + \gamma_2 \left\{ \int \frac{t^2 dH_p(t)}{1 + m_0 t} \right\} - 2 \int \frac{t^2 dH_p(t)}{1 + m_0 t} \int \frac{t^2 dH_p(t)}{1 + m_0 t} \right\} + o(1).
\]
Therefore we get
\[
M^{(2)}_n(z) = \frac{S_n - S_n T_n / T_n}{1 - T_n / T_n}
\]
\[
\rightarrow \left[ 1 - \frac{\text{ctd} dH(t)}{z(1 + mt)^2} \right]^{-1} \left\{ \int \frac{\text{cm}^2 t^2 dH(t)}{z(1 + mt)^3} \left[ 1 - \frac{\text{cm}^2 t^2 dH(t)}{(1 + mt)^2} \right]^{-1} - \frac{2\text{cm}^2}{z} \left[ \int \frac{\gamma_2 t - t^2 dH(t)}{1 + mt} \right] \right\},
\]
as \(n \to \infty\). Using the identity
\[
(4.52) \quad \left[ 1 - \frac{\text{ctd} dH(t)}{z(1 + mt)^2} \right]^{-1} = -z m \left[ 1 - \frac{\text{cm}^2 t^2 dH(t)}{(1 + mt)^2} \right]^{-1} = -z m' \frac{m}{m}
\]
we finally obtain the mean function of the lemma.

\[
\square
\]

4.2. Proof of Theorem 2.1. Following Theorem 1.1 in [5], it is sufficient to show that, for any bounded sequence of symmetric matrices \(\{C_p\}\),
\[
(4.53) \quad \text{Var}(y' C_p y) = o(p^2).
\]
Write \(y = \sqrt{p} A u / ||A u|| = \sqrt{p} A z / ||A z||\) where \(z \sim N(0, I_p)\). Since the eigenvalues of the SSCM \(B_n\) are invariant under orthogonal transformation, it’s enough to consider the diagonal matrix \(A\). Therefore, by taking \(C = \tilde{C} = C_p\) in Lemma 4.2, one can verify the condition (4.53).

4.3. Proof of Theorem 2.2. For any distribution function \(G\) and function \(f\) analytic on a simple connected domain \(D\) containing the support of \(G\), it holds that
\[
(4.54) \quad \int f(x) dG(x) = -\frac{1}{2\pi i} \oint_{C} f(z)m_G(z) dz
\]
where \(m_G(z)\) denotes the Stieltjes transform of \(G\) and \(C \subset D\) is a simple, closed, and positively oriented contour enclosing the support of \(G\). Similar to (4.4), we choose \(v_0, x_r, x_l\) such that \(f_1, \ldots, f_k\) are all analytic on and inside the contour \(C\). We denote by \(K\) a common upper bound of these functions on \(C\). Therefore, almost surely, for all \(n\) large, \(\{f_1, \ldots, f_k\}\) satisfy the equation in (4.54) with \(G = F^{B_n}\) and moreover,
\[
\left| \int f_i(z)(M_n(z) - \widehat{M}_n(z)) dz \right| \leq 4K \varepsilon_n \left( \max \{ \lambda^{\Sigma}_{\max}(1 + \sqrt{c_n})^2, \lambda^{B_n}_{\max} \} - x_r \right)^{-1}
\]
\[
+ |\min \{ \lambda^{\Sigma}_{\min} l_{(0,1)}(c_n)(1 - \sqrt{c_n})^2, \lambda^{B_n}_{\min} \} - x_l |^{-1}
\]
which converges to zero as \(n \to \infty\). Since
\[
\widehat{M}_n(\cdot) \to \left( -\frac{1}{2\pi i} \int_{C_1} f_1(z) \widehat{M}_n(z) dz, \ldots, -\frac{1}{2\pi i} \int_{C_k} f_k(z) \widehat{M}_n(z) dz \right)
\]
is a continuous mapping of \(C(C, \mathbb{R}^2)\) into \(\mathbb{R}^k\), it follows from Lemma 4.3 that the above random vector converges to a multivariate Gaussian vector \((X_{f_1}, \ldots, X_{f_k})\) whose mean and covariance functions are
\[
\text{E}(X_f) = -\frac{1}{2\pi i} \oint_{C_1} f(z) \text{E}[M(z)] dz,
\]
\[
\text{Cov}(X_f, X_g) = -\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f(z_1) g(z_2) \text{Cov}[M(z_1), M(z_2)] dz_1 dz_2,
\]
where \( f, g \in \{f_1, \ldots, f_k\} \) and \( \{C_1, C_2\} \) are two non-overlapping analogues of the contour \( C \).

From the following two identities
\[
\frac{1}{2\pi i} \oint_{C_1} f(z) (m(z) + zm'(z)) \, dz = -\frac{1}{2\pi i} \oint_{C_1} zf'(z)m(z) \, dz = c \int x f'(x) dF(x),
\]
\[
\frac{1}{2\pi i} \oint_{C_1} f(z) \left( \frac{m'(z)}{m^2(z)} - 1 \right) \, dz = \frac{1}{2\pi i} \oint_{C_1} f(z) \frac{m'(z)}{m^2(z)} \, dz,
\]
we obtain the form of the limiting covariance function in the theorem.

4.4. **Proof of Corollary 2.1.** Choose a contour \( C \) for the integrals such that \( \max_{t \in S_H, z \in C} |tm(z)| < 1 \), where \( S_H \) is the support of \( H \). Let \( m(C) = \{m(z) : z \in C\} \) denote the image of \( C \) under \( m(z) \). Then \( m(C) \) is a simple and closed contour having clockwise direction and enclosing zero [33].

By the identity in (2.2), the integral in the mean function of Theorem 2.2 becomes
\[
v_j = -\frac{c}{2\pi i} \int_{m(C)} \frac{P_j(m) P_{2,3}(m)}{m^{-1} (1 - cm^2 P_{2,2}(m))} \, dm - \frac{c}{\pi i} \int_{m(C)} \frac{P_j(m) P_{1,1}(m) P_{1,2}(m)}{m^{j-1}} \, dm
+ \frac{c}{\pi i} \int_{m(C)} \frac{P_j(m) P_{1,1}(m) P_{2,2}(m)}{m^{j-1}} \, dm + \frac{c}{\pi i} \int_{m(C)} \frac{P_j(m) P_{1,1}(m) P_{2,2}(m)}{m^{j-1}} \, dm.
\]

From this and the Cauchy integral theorem, we get the mean function. The covariance function can be obtained following the proof of Theorem 1 in [33].

5. **APPENDIX**

**Lemma 5.1.** For any \( p \times p \) complex matrix \( C \) and \( y = \sqrt{p} \mathbf{x} / \|\mathbf{x}\| \) with \( \mathbf{x} \sim N(0, \Sigma) \) and \( \|\Sigma\| \leq 1 \),
\[
\mathbb{E} |y' Cy - \text{tr} \Sigma C|^q \leq K_q \|C\|^q p^{q/2}, \quad q \geq 2,
\]
where \( K_q \) is a positive constant depending only on \( q \).

**Proof.** This lemma follows from Lemma 2.2 in [3] and similar arguments in the proof of Lemma 5 in [15]. \( \square \)

**Lemma 5.2 ([7]).** Let \( \{X_k\} \) be a complex martingale difference sequence with respect to the increasing \( \sigma \)-field \( \{\mathcal{F}_k\} \). Then, for \( q \geq 2 \),
\[
\mathbb{E} \left| \sum X_k \right|^q \leq K_q \left\{ \mathbb{E} \left( \sum \mathbb{E} \left( |X_k|^2 |\mathcal{F}_{k-1} \right) \right)^{q/2} + \mathbb{E} \left( \sum |X_k|^q \right) \right\}.
\]

**Lemma 5.3 ([7]).** Let \( \{X_k\} \) be a complex martingale difference sequence with respect to the increasing \( \sigma \)-field \( \{\mathcal{F}_k\} \). Then, for \( q > 1 \),
\[
\mathbb{E} \left| \sum X_k \right|^q \leq K_q \mathbb{E} \left( \sum |X_k|^2 \right)^{q/2}.
\]

**Lemma 5.4** (Theorem 35.12 of [6]). Suppose for each \( n \) \( Y_{n1}, Y_{n2}, \ldots, Y_{nr_n} \) is a real martingale difference sequence with respect to the increasing \( \sigma \)-field \( \{\mathcal{F}_{nj}\} \) having second moments. If for each \( \varepsilon > 0 \),
\[
\sum_{j=1}^{r_n} \mathbb{E} (Y_{nj}^2 I_{|Y_{nj}| \geq \varepsilon}) \to 0 \quad \text{and} \quad \sum_{j=1}^{r_n} \mathbb{E} (Y_{nj}^2 |\mathcal{F}_{n,j-1}\}) \overset{i.p.}{\to} \sigma^2,
\]

then \( \sum_{j=1}^{r_n} \mathbb{E} (Y_{nj}^2) \overset{i.p.}{\to} \sigma^2 \).
as \( n \to \infty \), where \( \sigma^2 \) is a positive constant, then
\[
\sum_{j=1}^{r_n} Y_{nj} \xrightarrow{D} N(0, \sigma^2).
\]

**Lemma 5.5.** Suppose that Assumptions (a)-(c) hold. Then, for any \( s \) positive,
\[
P(||B_n|| > \eta_r) = o(n^{-s}),
\]
whenever \( \eta_r > \limsup_{p \to \infty} ||\Sigma||(1 + \sqrt{c})^2 \). If \( 0 < \liminf_{p \to \infty} \lambda_{\min}^{\Sigma} I_{(0,1)}(c) \) then,
\[
P(\lambda_{\min}^{B_n} < \eta_r) = o(n^{-s}),
\]
whenever \( 0 < \eta_r < \liminf_{p \to \infty} \lambda_{\min}^{\Sigma} I_{(0,1)}(c)(1 - \sqrt{c})^2 \).

**Proof.** Let \( x_j = AZ_j \) where \( AA' = T \) and \( z_j \sim N(0, I_p), \) \( j = 1, \ldots, n \). Also let \( B_n^{(0)} = (1/n) \sum_{j=1}^{n} AZ_j z_j'A' \). From [3], the conclusions of this lemma hold when \( (B_n, \Sigma) \) are replaced with \( (B_n^{(0)}, T) \). Choose \( \eta_r^{(0)} \) and \( \eta_r^{(0)} \) satisfying
\[
\eta_l < r_1^{-1} \eta_r^{(0)} < \liminf_{p \to \infty} \lambda_{\min}^{\Sigma} I_{(0,1)}(c)(1 - \sqrt{c})^2 \quad \text{and} \quad \limsup_{p \to \infty} ||\Sigma||(1 + \sqrt{c})^2 < r_1^{-1} \eta_r^{(0)} < \eta_r,
\]
where \( r_1 = \text{tr}(T)/p \). From Lemma 1, we have
\[
\eta_r^{(0)} < \liminf_{p \to \infty} \lambda_{\min}^{T} I_{(0,1)}(c)(1 - \sqrt{c})^2 \quad \text{and} \quad \limsup_{p \to \infty} ||T||(1 + \sqrt{c})^2 < \eta_r^{(0)}.
\]

Using inequalities
\[
\min_{1 \leq j \leq n} \frac{p}{||Az_j||^2} \lambda_{\min}^{B_n^{(0)}} \leq \lambda_{\min}^{B_n} \leq \max_{1 \leq j \leq n} \frac{p}{||Az_j||^2} ||B_n^{(0)}||,
\]
we may get
\[
P(||B_n|| > \eta_r) \leq P(||B_n^{(0)}|| > \eta_r^{(0)}) + P\left( \max_{1 \leq j \leq n} \frac{p}{||Az_j||^2} ||B_n^{(0)}|| > \eta_r, ||B_n^{(0)}|| \leq \eta_r^{(0)} \right)
\]
\[
\leq P\left( \max_{1 \leq j \leq n} \frac{p}{||Az_j||^2} > \frac{\eta_r}{\eta_r^{(0)}} \right) + o(n^{-s})
\]
\[
\leq nP\left( \frac{||Az_1||^2}{p} - r_1 > \frac{\eta_r^{(0)}}{\eta_r} \right) + o(n^{-s}),
\]
\[
= o(n^{-s}),
\]
where the last equality is from the Chebyshev inequality and the fact \( r_1 > \eta_r^{(0)}/\eta_r \). Similarly, \( P(\lambda_{\min}^{B_n} < \eta_l) = o(n^{-s}) \).

\[\square\]

**References**

[1] Bai, Z. D., Chen, J. Q., and Yao, J. F. (2010). On estimation of the population spectral distribution from a high-dimensional sample covariance matrix. *Aust. N. Z. J. Stat.* 52, 423–437.

[2] Bai, Z. D., Hu, J., Pan, G. M., and Zhou, W. (2015). Convergence of the empirical spectral distribution function of Beta matrices. *Bernoulli*, 21, 1538–1574.

[3] Bai, Z. D. and Silverstein, J. W. (2004). CLT for linear spectral statistics of large-dimensional sample covariance matrices. *Ann. Probab.*, 32, 553–605.
[4] **BAI, Z. D. AND SILVERSTEIN, J. W.** (2010). *Spectral analysis of large dimensional random matrices*, 2nd ed., Springer, New York.

[5] **BAI, Z. D. AND ZHOU, W.** (2008). Large sample covariance matrices without independence structures in columns. *Statist. Sinica, 18*, 425–442.

[6] **BILLINGSLEY, P.** (1995). *Probability and Measure*, 3rd ed., Wiley, New York.

[7] **BURKHOLDER, D. L.** (1973). Distribution function inequalities for martingales. *Ann. Probab., 1*, 19–42.

[8] **CAI, TONY, MA, Z. M., AND WU, Y. H.** (2013) Sparse PCA: Optimal rates and adaptive estimation. *Ann. Statist. 41*, 3074-3110.

[9] **DURRE, A, TYLER D. E., VOGEL, D.** (2016). On the eigenvalues of the spatial sign covariance matrix in more than two dimensions. *Statist. Probab. Lett., 111*, 80–85.

[10] **EL KAROUI, N.** (2008). Spectrum estimation for large dimensional covariance matrices using random matrix theory. *Ann. Statist. 36*, 2757–2790.

[11] **EL KAROUI, N.** (2009). Concentration of measure and spectra of random matrices: applications to correlation matrices, elliptical distributions and beyond. *Ann. Appl. Probab., 19*, 2362–2405.

[12] **FAN, J., FAN, Y., AND LV, J.** (2008). High dimensional covariance matrix estimation using a factor model. *Journal of Econometrics, 147*, 186–197.

[13] **FAN, J., LIAO, Y., AND MINCHEVA, M.** (2013). Large covariance estimation by thresholding principal orthogonal complements. *J. Roy. Stat. Soc. Ser. B, 75*, 603–680.

[14] **FANG, K. T., ZHANG, Y. T.** (1990). *Generalized multivariate analysis*. Springer-Verlag, Berlin; Science Press, Beijing.

[15] **GAO, J. T., HAN, X., PAN, G. M., AND YANG, Y. R.** (2016). High dimensional correlation matrices: the central limit theorem and its applications. *J. Roy. Stat. Soc. Ser. B*, doi: 10.1111/rssb.12189.

[16] **GERVINI, D.** (2008). Robust functional estimation using the median and spherical principal components. *Biometrika, 95*, 587–600.

[17] **GUPTA, A. K., VARGA, T., BODNAR, T.** (2013). *Elliptically contoured models in statistics and portfolio theory*, 2nd ed., Springer, New York.

[18] **JOHNSTONE, I. M.** (2001). On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statist. 29*, 295-327.

[19] **JONSSON, D.** (1982). Some limit theorems for the eigenvalues of a sample covariance matrix. *J. Multivariate Anal., 12*, 1–38.

[20] **KELKER, D.** (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhyā Ser. A, 32*, 419–430.

[21] **LEDoit, O. AND WOLF, M.** (2012). Nonlinear shrinkage estimation of large-dimensional covariance matrices. *Ann. Statist., 40*, 1024–1060.

[22] **LI, W. M. AND YAO, J. F.** (2014). A local moment estimator of the spectrum of a large dimensional covariance matrix. *Statist. Sinica, 24*, 919–936.

[23] **LOCANTORE, N., MARRON, J. S., SIMPSON, D. G., TRIPOLI, N., ZHANG, J. T., AND COHEN, K. L.** (1999). Robust principal component for functional data. *Test, 8*, 1–73.

[24] **LI, W. M. AND YAO, J. F.** (2016). On structure testing for component covariance matrices of a high-dimensional mixture. *Manuscript*.

[25] **MARČENKO, V. A. AND PASTUR, L. A.** (1967). Distribution of eigenvalues in certain sets of random matrices. *Mat. Sb. (N.S.), 72*, 507–536.
Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). Robust statistics: theory and methods. John Wiley & Sons Ltd., Chichester.

Magyar A. F., Tyler D. E. (2014). The asymptotic inadmissibility of the spatial sign covariance matrix for elliptically symmetric distributions. *Biometrika*, **101**, 673–688.

Nica, A. and Speicher, R. (2006). *Lectures on the Combinatorics of Free Probability*. Cambridge University Press, New York.

Oja, H. (2010). *Multivariate Nonparametric Methods with R. An approach based on spatial signs and ranks*. Springer-Verlag, New York.

Pan, G. M. and Zhou, W. (2008). Central limit theorem for signal-to-interference ratio of reduced rank linear receiver. *Ann. Appl. Probab.*, **18**, 1232–1270.

Paindaveine, D. and Verdebout, T. (2016) On high-dimensional sign tests. *Bernoulli*, **22**, 1745–1769.

Qin, Y. L. and Li, W. M. (2016). Testing the order of a population spectral distribution for high-dimensional data. *Comput. Stat. & Data An.*, **95**, 75–82.

Qin, Y. L. and Li, W. M. (2017). “Bias-reduced estimators of moments of a population spectral distribution and their applications”, In *Big and Complex Data Analysis: Statistical Methodologies and Applications* (Ejaz Ahmed ed.), Springer.

Rao, N. R., Mingo, J. A., Speicher, R., and Edelman, A. (2008). Statistical eigen-inference from large Wishart matrices. *Ann. Statist.*, **36**, 2850–2885.

Randles, R. H. (1989). A distribution-free multivariate sign test based on inter-directions. *J. Amer. Statist. Assoc.*, **84**, 1045–1050.

Silverstein, J. W. (1995). Strong convergence of the empirical distribution of eigenvalues of large-dimensional random matrices. *J. Multivariate Anal.*, **55**, 331–339.

Silverstein, J. W. and Bai, Z. D. (1995). On the empirical distribution of eigenvalues of a class of large dimensional random matrices. *J. Multivariate Anal.*, **54**, 175–192.

Tyler, D. E. (2010). A note on multivariate location and scatter statistics for sparse data sets. *Statist. Probab. Lett.*, **80**, 1409–1413.

Visuri, S., Oja, H., Koivunen, V. (2001). Subspace-based direction-of-arrival estimation using nonparamatric statistics. *IEEE Trans. Signal Process.*, **49**, 2060–2073.

Wang, W. and Fan, J. (2017). Asymptotics of empirical eigen-structure for ultra-high dimensional spiked covariance model. *Ann. Statist.*, to appear.

**E-mail address:** li.weiming@shufe.edu.cn

**E-mail address:** stazw@nus.edu.sg