Mass Matrix Models: The Sequel

Miriam Leurer\textsuperscript{a}, Yosef Nir\textsuperscript{a} and Nathan Seiberg\textsuperscript{b}

\textsuperscript{a}Department of Particle Physics
\textit{Weizmann Institute of Science, Rehovot 76100, Israel}

\textsuperscript{b}Department of Physics and Astronomy
\textit{Rutgers University, Piscataway, NJ 08855-0849}

The smallness of the quark sector parameters and the hierarchy between them could be the result of a horizontal symmetry broken by a small parameter. Such an explicitly broken symmetry can arise from an exact symmetry which is spontaneously broken. Constraints on the scales of new physics arise from new flavor changing interactions and from Landau poles, but do not exclude the possibility of observable signatures at the TeV scale. Such a horizontal symmetry could also lead to many interesting results: (i) quark – squark alignment that would suppress, without squark degeneracy, flavor changing neutral currents induced by supersymmetric particles, (ii) exact relations between mass ratios and mixing angles, (iii) a solution of the $\mu$-problem and (iv) a natural mechanism for obtaining hierarchy among various symmetry breaking scales.
1. Introduction

Quark mass ratios and mixing angles have two intriguing features: The smallness of most of these parameters and the hierarchy among them. The hierarchy in the quark mixing angles is clearly presented in the Wolfenstein’s parameterization \[1\] of the CKM matrix:

\[
V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & \lambda^3 A (\rho + i \eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & \lambda^2 A \\
\lambda^3 A (1 - \rho + i \eta) & -\lambda^2 A & 1
\end{pmatrix}
\]  

The hierarchy is reflected in the dependence of the various entries on different powers of \(\lambda \sim 0.2\): all other quantities are experimentally determined to be of order one, \(i.e.\ A\) is of order 1 while \(\rho\) and \(\eta\) are between \(\lambda\) and 1. The order of magnitude of the three mixing angles is therefore given in powers of \(\lambda\):

\[
|V_{us}| \sim \lambda, \quad |V_{cb}| \sim \lambda^2, \quad |V_{ub}| \sim \lambda^3 - \lambda^4.
\]  

The hierarchy among the quark masses can also be expressed in powers of \(\lambda\). The low energy (\(\sim 100 \text{ GeV}\)) values are

\[
\frac{m_u}{m_c} \sim \lambda^3 - \lambda^4, \quad \frac{m_c}{m_t} \sim \lambda^3, \\
\frac{m_d}{m_s} \sim \lambda^2, \quad \frac{m_s}{m_b} \sim \lambda^2, \\
\frac{m_b}{m_t} \sim \lambda^2 - \lambda^3, \quad \frac{m_t}{\langle \phi_u \rangle} \sim 1.
\]  

The exact power of \(\lambda\) for each of the mass ratios may vary a little, depending the top mass, and on the exact value one chooses for \(\lambda\) in the range 0.20 – 0.22. Furthermore, all the parameters run under the renormalization group and therefore, the order of magnitude estimates (1.2) and (1.3) depend on the scale. Assuming that only the top quark Yukawa coupling is large and using the one loop renormalization group with the particle content of the minimal supersymmetric standard model, we find that the only changes in our order of magnitude estimates at a high (\(\sim 10^{15} \text{ GeV}\)) scale are:

\[
\frac{m_c}{m_t} \sim \lambda^3 - \lambda^4, \quad \frac{m_b}{m_t} \sim \lambda^3.
\]  

We would like to make two comments about the estimates in (1.3):
(i) It is possible that $m_u$ as deduced from first order chiral Lagrangian predictions reflects non-perturbative strong interaction effects rather than the value of the high energy parameter $\mu$. In particular, it could be that the bare $m_u$ vanishes, thus providing a solution (which is natural in our framework $\mathcal{B}$) to the strong CP problem. The phenomenological viability of this scenario is controversial.

(ii) The small ratio $\frac{m_b}{m_t}$ may be a result of a large ratio of VEVs, $\tan\beta = \frac{\langle \phi_u \rangle}{\langle \phi_d \rangle} \sim \lambda^{-2} - \lambda^{-3}$, or it may be the result of a small ratio of Yukawa couplings when $\tan\beta \sim 1$.

For the large part of our study, these two points are not crucial, and we use $m_u/m_c \sim \lambda^3$ and $\tan\beta \sim 1$.

As articulated by 'tHooft $\mathcal{B}$, small numbers are natural only if an exact symmetry is acquired when they are set to zero ("naturalness"). Therefore, both the smallness of the quark sector parameters and the hierarchy among them may be related to a symmetry – a horizontal symmetry $\mathcal{H}$ that acts on the quarks (for recent discussions, see e.g. $\mathcal{F}$). Such a horizontal symmetry may be responsible for the hierarchy, if it is explicitly broken by an operator in the Lagrangian whose coefficient is the small parameter $\lambda$. The transformation laws of $\lambda$ under $\mathcal{H}$ control the order in perturbation theory of the various elements in the quark mass matrices and, consequently, some parameters depend on powers of $\lambda$ higher than others, namely a hierarchy can be generated. This phenomenon is common in atomic physics, nuclear physics and particle physics and is known as "selection rules."

The next step is to understand the origin of this explicitly broken $\mathcal{H}$. Several different mechanisms can exist. Here we focus on the possibility that $\lambda$ is promoted to a quantum field which has an expectation value. This expectation value spontaneously breaks the exact symmetry $\mathcal{H}$. More precisely, we add a scalar field $S$ whose expectation value $\langle S \rangle$ breaks $\mathcal{H}$. The small numbers in the Lagrangian appear then as powers of the ratio $\lambda = \frac{\langle S \rangle}{M}$ where $M$ is a higher energy scale at which the information about $\mathcal{H}$-breaking is communicated to the light fermions.

The organization of this paper follows this logic. First in sections 2 and 3 we consider an explicitly broken $\mathcal{H}$ and examine its consequences. The simplest framework, that of an Abelian group $\mathcal{B}$, $\mathcal{H} \subset U(1)$, and a single small breaking parameter $\lambda$, is described in

1 We do not study non-Abelian symmetries. An example of a non-Abelian model is presented
subsection 2.1. There is an essentially unique model in this framework. The next-to-simplest symmetry, \( \mathcal{H} \subset U(1) \times U(1) \) with two small breaking parameters (one for each \( U(1) \)) opens up interesting possibilities: First, the hierarchy in the quark parameters can be achieved with lower powers of the small parameters. This is demonstrated in an example in subsection 2.2, that is the basis for an interesting high-energy model to be presented later. Second, we can acquire highly suppressed entries in the quark mass matrices. This may lead to suppression of FCNC induced by squark-gluino diagrams. We explain this mechanism (‘Quark – Squark Alignment’) \([7]\) in subsection 2.3. Phenomenological constraints are discussed in subsection 2.4.

The possibility of acquiring zero or highly suppressed entries in the quark mass matrices allows relations that go beyond the naive order of magnitude estimates. The possibility that \( |V_{ub}| \ll |V_{us}V_{cb}| \) (rather than of the same order of magnitude) is discussed in subsection 3.1, while close to exact relations between mixing angles and mass ratios are obtained in subsection 3.2.

Sections 4 — 6 are concerned with the embedding of the low energy effective theory with explicitly broken horizontal symmetry in a more fundamental theory. As mentioned above, the most natural theory takes \( \mathcal{H} \) to be an exact symmetry, spontaneously broken by the VEVs of scalar fields. The mechanism, its phenomenological consequences and constraints on the scale of spontaneous symmetry breaking are described in section 4. A very plausible explanation of the physics at the scale \( M \), where the information about the spontaneous breaking is communicated to the light fermions, was suggested by Froggatt and Nielsen \([8]\) and was further studied in references \([9]\) and \([10]\): \( M \) is the mass scale for heavy mirror quarks. The scalars responsible for the breaking couple the heavy sector to the light one. This mechanism, its phenomenological consequences and constraints on the scale \( M \) are studied in section 5. The existence of extra supermultiplets affects the running of the various gauge couplings. In section 6 we investigate the constraints that follow if we assume that there is no further new physics between \( M \) and the Planck scale \( M_P \) and require the absence of Landau poles. This analysis allows us to address the important question of whether flavor physics may be directly observed in future experiments.

---

\[\text{in reference } [6].\]
Our full framework requires that there are several new scales of physics between the SUSY breaking scale and the Planck scale. In section 7 we study the $H$-invariant Higgs potential and suggest a mechanism that can naturally generate the required hierarchy. Furthermore, this mechanism automatically solves the “$\mu$-problem.” A discussion of our results is given in section 8.

Our discussion proceeds from low energies to high energies. At every step describing the physics at higher energies we add more assumptions which lead to more constraints. It is possible that our ideas about some energy scale will turn out to be correct while the speculations about higher energies will turn out to be wrong.

2. Models of Explicitly Broken Horizontal Symmetry

2.1. The Master Model

The simplest framework to explain the order of magnitude relations (1.2) and (1.3) is that of a horizontal symmetry $H = U(1)_H$, with a small breaking parameter $\lambda \sim 0.2$. Terms that break $H$ by $n (> 0)$ units of charge are suppressed by $\lambda^n$. In the full Lagrangian, we require that $H$ is a discrete subgroup, $Z_N \subset U(1)$. Then terms that break $H$ by $n > N$ units of charge are suppressed only by $\lambda^l$ for $l = n \mod N$. In most of the cases that we are interested in, $N$ is large enough and this does not happen. Therefore, for convenience, we will treat $H$ as a continuous $U(1)$ symmetry. In the few cases where the discreteness of the symmetry does play an important role, we explicitly point out its effects.

We will usually restrict ourselves to supersymmetric theories (with supersymmetry broken by soft terms). Then terms in the fermion mass matrices that break the horizontal symmetry by $n < 0$ units of charge vanish. This is due to our assumption that $\lambda$ is a (single) coefficient of an operator which explicitly breaks $H$ and the fact that the effective superpotential is holomorphic in the coupling constants of the theory (no powers of $\lambda^\dagger$). This is a special case of the non-renormalization theorem of reference [11]. If more than one small $\lambda$ which break the same symmetry exist, say one with negative charge and another with positive charge, then such terms need not vanish. The consequences of this fact will be discussed below.
We use the following notation for the various fields and their interactions. \( Q_i \) denote the left-handed quark doublets and \( \bar{d}_i \) (\( \bar{u}_i \)) denote the left handed anti-down (anti-up) \( SU(2) \) singlets. \( \phi_d \) and \( \phi_u \) are the Higgs fields of hypercharge \(-1/2\) and \(+1/2\), respectively. The Yukawa couplings are denoted by \( Y^d \) and \( Y^u \), and the mass matrices by \( M^d \) and \( M^u \). The Yukawa interactions are then given by

\[
L_Y = Y^d_{ij} \phi_d Q_i \bar{d}_j + Y^u_{ij} \phi_u Q_i \bar{u}_j.
\]

The interaction (2.1) has an accidental \( U(1) \) symmetry, which we call \( U(1)_X \). Under this symmetry \( \phi_d \) carries charge \(-1\), the \( \bar{d}_i \) fields carry charge \(+1\), while all other fields carry vanishing \( X \) charges. The \( U(1)_X \) symmetry must be explicitly broken in other sectors of the Lagrangian, since otherwise its spontaneous breaking at the weak scale implies the existence of an unwanted axion. Despite being broken, \( U(1)_X \) turns out to be extremely useful. First, using \( U(1)_X \), hypercharge and baryon number symmetries, we can set the horizontal charges of \( \phi_u \), \( \phi_d \) and \( Q_3 \) to zero. As long as we restrict our discussion to \( U(1)_X \) invariant terms in the Lagrangian, such a horizontal charge redefinition is justified and simplifies the analysis considerably. Second, the existence of \( U(1)_X \) in the Yukawa sector implies that QCD anomalies pose no problem to our horizontal symmetries. It is always possible to find a horizontal symmetry \( \tilde{\mathcal{H}} \subset \mathcal{H} \times U(1)_X \) that restricts the quark mass matrices in precisely the same way as does \( \mathcal{H} \) but is free of QCD anomaly. We discuss this point in detail in subsection 4.3.

The order of magnitude of the Yukawa couplings is determined by their horizontal quantum numbers. Assuming positive (or vanishing) horizontal charges to all quark fields one finds [8]:

\[
Y^d_{ij} \sim \lambda^{H(Q_i)+H(\bar{d}_j)}, \quad Y^u_{ij} \sim \lambda^{H(Q_i)+H(\bar{u}_j)}.
\]

(2.2)

The mixing angles and mass ratios can then be estimated:

\[
|V_{ij}| \sim \lambda^{H(Q_i)-H(Q_j)},
\]

\[
\frac{m_{d_i}}{m_{d_j}} \sim \lambda^{H(Q_i)-H(Q_j)+H(\bar{d}_i)-H(\bar{d}_j)}, \quad \frac{m_{u_i}}{m_{u_j}} \sim \lambda^{H(Q_i)-H(Q_j)+H(\bar{u}_i)-H(\bar{u}_j)}.
\]

(2.3) (2.4)

(Both (2.3) and (2.4) are given here for \( i < j \).)
The order of magnitude estimates (1.2) and (1.3) then determine a *unique* set of 
$H$-charges for all quark fields (when we take $m_u/m_c \sim \lambda^3$, $m_b/m_t \sim \lambda^2$, $\tan \beta \sim 1$):

\[
\begin{array}{cccccccc}
Q_1 & Q_2 & Q_3 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\
(3) & (2) & (0) & (3) & (2) & (2) & (3) & (1) & (0)
\end{array}
\]

With these charges the mass matrices have the order of magnitude entries

\[
M^d \sim \langle \phi_d \rangle \begin{pmatrix}
\lambda^6 & \lambda^5 & \lambda^5 \\
\lambda^5 & \lambda^4 & \lambda^4 \\
\lambda^3 & \lambda^2 & \lambda^2
\end{pmatrix}, \quad M^u \sim \langle \phi_u \rangle \begin{pmatrix}
\lambda^6 & \lambda^4 & \lambda^3 \\
\lambda^5 & \lambda^3 & \lambda^2 \\
\lambda^3 & \lambda & 1
\end{pmatrix}.
\]

It is straightforward to check that these mass matrices indeed lead to mixing angles and mass ratios as given in (1.2) and (1.3). We also note that the determinants of the mass matrices are

\[
\det M^d \sim \langle \phi_d \rangle^3 \lambda^{12}, \quad \det M^u \sim \langle \phi_u \rangle^3 \lambda^9.
\]

The powers of $\lambda$ in these determinants will be of importance in the discussion of the embedding of the model in a high energy theory.

How predictive is this framework? We have made eight discrete choices of charges and explained the order of magnitude of nine physical (continuous) parameters ($m_t/\langle \phi_u \rangle \sim 1$, $m_b/m_t \sim \lambda^2$, the four mass ratios of eq. (2.4), and the three mixing angles of equation (2.3)). This means that the model predicts a single order of magnitude relation. Indeed, we find

\[
|V_{ub}/V_{cb}| \sim |V_{us}|
\]

independent of the choice of charges. The Particle Data Group quotes [12]

\[
|V_{ub}/V_{cb}| = 0.10 \pm 0.03, \quad |V_{us}| = 0.22,
\]

consistent with (2.8). Recently, however, the CLEO collaboration announced a new measurement [13]:

\[
|V_{ub}/V_{cb}| \sim 0.05 - 0.10.
\]

We remind the reader that there is a strong theoretical model dependence in the extraction of $|V_{ub}/V_{cb}|$ from the experimental data. However, if it eventually turns out that

\[2\] It is trivial to modify the charges for the other possibilities.
$|V_{ub}/V_{cb}| \sim \lambda^2$, it will pose a problem to the naive master model presented here. In subsection 3.1 we show how the relation (2.3) can be avoided in a more sophisticated model within our framework.

We take $\mathcal{H}$ to commute with Supersymmetry. The alternative, that $\mathcal{H}$ is an R symmetry will be briefly discussed below. Then the charges (2.5) are common to complete supermultiplets. This determines the form of the squark mass-squared matrices as well. We denote these by $\tilde{M}^d_2$ and $\tilde{M}^u_2$, and divide each to $3 \times 3$ sub-matrices:

\[
\tilde{M}^d_2 = \begin{pmatrix}
\tilde{M}^d_{LL} & \tilde{M}^d_{LR} \\
(\tilde{M}^d_{LR})^\dagger & \tilde{M}^d_{RR}
\end{pmatrix}, \quad \tilde{M}^u_2 = \begin{pmatrix}
\tilde{M}^u_{LL} & \tilde{M}^u_{LR} \\
(\tilde{M}^u_{LR})^\dagger & \tilde{M}^u_{RR}
\end{pmatrix}.
\]

The sub-index $L$ ($R$) refers to the scalar partners of the quark doublets $Q_i$ (singlets $\bar{d}_i$ or $\bar{u}_i$). The leading contributions to the diagonal blocks arise from $A$-type SUSY breaking terms, while the leading contributions to the off-diagonal blocks arise from soft SUSY breaking terms analytical in the fields. This leads to two important differences between the diagonal and the off-diagonal blocks: (i) Entries in the diagonal blocks that break $\mathcal{H}$ by $n$ units of charge are suppressed by $\lambda^{|n|}$, whether $n$ is negative or positive. In the off-diagonal blocks such entries are suppressed by $\lambda^n$ for $n > 0$, but vanish when $n < 0$. (ii) The entries in the diagonal blocks are proportional to $\tilde{m}^2$ ($\tilde{m}$ is the scale of SUSY breaking) while those in the off-diagonal blocks are proportional to $\tilde{m}\langle \phi_q \rangle$. If $\tilde{m} \gg \langle \phi_q \rangle$, the off-diagonal blocks are negligible to all our purposes.

The charge assignments (2.3) give

\[
\tilde{M}^d_{LL} \approx \tilde{M}^u_{LL} \sim \tilde{m}^2 \begin{pmatrix}
1 & \lambda & \lambda^3 \\
\lambda & 1 & \lambda^2 \\
\lambda^3 & \lambda^2 & 1
\end{pmatrix},
\]

\[
\tilde{M}^d_{RR} \sim \tilde{m}^2 \begin{pmatrix}
1 & \lambda & \lambda \\
\lambda & 1 & 1 \\
\lambda & 1 & 1
\end{pmatrix}, \quad \tilde{M}^u_{RR} \sim \tilde{m}^2 \begin{pmatrix}
1 & \lambda^2 & \lambda^3 \\
\lambda^2 & 1 & \lambda \\
\lambda^3 & \lambda & 1
\end{pmatrix},
\]

\[
(\tilde{M}^d_{LR})_{ij} \sim \tilde{m}M^d_{ij}, \quad (\tilde{M}^u_{LR})_{ij} \sim \tilde{m}M^u_{ij}.
\]

We remind the reader that all entries are order of magnitude estimates and not exact numbers. (In particular, the diagonal elements in each of $\tilde{M}^d_2$ and $\tilde{M}^u_2$ are all of order $\tilde{m}^2$ but not equal to each other.) However, for $\tilde{m}$ considerably higher than the electroweak
breaking scale, the approximate equality $\tilde{M}_{dL}^2 \approx \tilde{M}_{uL}^2$ holds to $O(\langle \phi \rangle^2 / \tilde{m}^2)$ and not just as an order of magnitude estimate.

If $\mathcal{H}$ is an $R$ symmetry, the diagonal blocks $\tilde{M}_{dL}^2, \tilde{M}_{dR}^2, \tilde{M}_{uL}^2$ and $\tilde{M}_{uR}^2$ are unchanged. The off-diagonal blocks are different and, unlike eq. (2.14), their suppression by powers of $\lambda$ is not the same as for the corresponding elements in the quark mass matrices. Then their effects on FCNC can be significant (even though they are suppressed by $\langle \phi \rangle / \tilde{m}$). Thus, our discussion of quark – squark alignment in subsection 2.3 does not apply in general to horizontal $R$ symmetries.

2.2. Models with $U(1)_{H_1} \times U(1)_{H_2}$ Symmetry

Models with a more complicated symmetry structure than a simple $U(1)$ offer new possibilities in constructing mass matrices. All the important features of such symmetries are already present in the simplest extension,

$$\mathcal{H} = U(1)_{H_1} \times U(1)_{H_2}, \quad (2.15)$$

with two small breaking parameters:

$$\epsilon_1 \sim \lambda^p, \quad \epsilon_2 \sim \lambda^q, \quad (2.16)$$

where $q > p \geq 1$.

Let us compare a model with the horizontal symmetry (2.15) to the master model of the previous section. A quark field that carries charge $H$ in the master model, must carry charges $(H_1, H_2)$ under (2.13) such that

$$H = pH_1 + qH_2. \quad (2.17)$$

We note the following points:

(i) The choice of $(H_1, H_2)$ is, in general, not unique. Thus, unlike the master model, for each horizontal symmetry of the type (2.13), there are several models (namely, sets of charge assignments for the quark fields) that produce the same hierarchy in mixing angles and mass ratios.
(ii) Consider the determinants of the mass matrices. For example,

$$\det M^d = \epsilon_1 \sum_i (H_1(Q_i) + H_1(\bar{d}_i)) \epsilon_2 \sum_i (H_2(Q_i) + H_2(\bar{d}_i)),$$  \hspace{1cm} (2.18)

to be compared with the master model

$$\det M^d(\text{master}) = \lambda \sum_i (H(Q_i) + H(\bar{d}_i)).$$  \hspace{1cm} (2.19)

Since $q > 1$, it is trivial to show that the sum of powers of the $\epsilon_i$’s in the new model is smaller than the power of $\lambda$ in the master model. When we later study the underlying theory, we will find that the lower the power of $\epsilon$, the weaker is the lower bound on the horizontal physics scale.

(iii) In the master model, all entries in the mass matrices have their naively expected values. In the model of eq. (2.15), some entries are suppressed and would vanish if the horizontal symmetry were continuous. For example, $M^d_{ij}$ would vanish if either $H_1(Q_i) + H_1(\bar{d}_j) < 0$ or $H_2(Q_i) + H_2(\bar{d}_j) < 0$. Such suppressed entries open interesting possibilities. In particular, we find that it is possible to solve the problem of the squark-gluino box diagram contribution to neutral meson mixing without requiring squark degeneracy (see next subsection). Another interesting consequence is the possibility of relations between mixing angles and mass ratios (see section 3).

Let us present an explicit example. The symmetry is of the type (2.15) with breaking parameters

$$\epsilon_1 \sim \lambda^2, \quad \epsilon_2 \sim \lambda^3. \hspace{1cm} (2.20)$$

(To explain $|V_{us}| \sim \lambda$, we always need either $p = 1$ or $q - p = 1$.) There are four models that reproduce the order of magnitude relations of the master model. This is a result of two possible choices for the charges of each of $\bar{d}_1$ and $\bar{u}_1$: (0, 1) or (3, $-1$). In one of the four models the charges are:

$$\begin{array}{cccccccc}
Q_1 & Q_2 & Q_3 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\
(0, 1) & (1, 0) & (0, 0) & (3, -1) & (1, 0) & (1, 0) & (0, 1) & (-1, 1) & (0, 0)
\end{array} \hspace{1cm} (2.21)$$

The mass matrices have the order of magnitude entries

$$M^d \sim \langle \phi_d \rangle \begin{pmatrix}
\epsilon_1^3 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_2 \\
0 & \epsilon_1^2 & \epsilon_1^2 \\
0 & \epsilon_1 & \epsilon_1
\end{pmatrix}, \quad M^u \sim \langle \phi_u \rangle \begin{pmatrix}
\epsilon_2^2 & 0 & \epsilon_2 \\
\epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 \\
\epsilon_2 & 0 & 1
\end{pmatrix}. \hspace{1cm} (2.22)$$
We see that each entry in the mass matrix is either of the same order of magnitude as in the master model, or zero. When we take into account the fact that \( \mathcal{H} \) is discrete and not continuous, we find that the vanishing entries are modified but still very suppressed relative to their values in the master model. It is again straightforward to check that these mass matrices lead to mixing angles and mass ratios as given in (1.2) and (1.3).

The order of magnitude estimates of the squark mass-squared matrices in this model are:

\[
\tilde{M}^d_{LL} \approx \tilde{M}^u_{LL} \sim \tilde{m}^2 \begin{pmatrix} 1 & \epsilon_1 \epsilon_2 & \epsilon_2 \\ \epsilon_1 \epsilon_2 & 1 & \epsilon_1 \\ \epsilon_2 & \epsilon_1 & 1 \end{pmatrix},
\]

(2.23)

\[
\tilde{M}^d_{RR} \sim \tilde{m}^2 \begin{pmatrix} 1 & \epsilon_1^2 \epsilon_2 & \epsilon_1 \epsilon_2 \\ \epsilon_1 \epsilon_2 & 1 & \epsilon_1 \epsilon_2 \\ \epsilon_1^2 \epsilon_2 & \epsilon_1 \epsilon_2 & 1 \end{pmatrix}, \quad \tilde{M}^u_{RR} \sim \tilde{m}^2 \begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 \\ \epsilon_1 & 1 & \epsilon_1 \epsilon_2 \\ \epsilon_2 & \epsilon_1 \epsilon_2 & 1 \end{pmatrix},
\]

(2.24)

\[
(\tilde{M}^d_{LR})_{ij} \sim \tilde{m} M^d_{ij}, \quad (\tilde{M}^u_{LR})_{ij} \sim \tilde{m} M^u_{ij}.
\]

(2.25)

### 2.3. Quark Squark Alignment

For generic squark masses, box diagrams with squarks and gluinos give unacceptably large contributions to neutral meson (\( K, B \) and \( D \)) mixing [14]. The standard solution to this problem is to assume that squarks are degenerate to a very good approximation. This is not motivated in generic supergravity models or string theory, though it may hold under special conditions [15]. Both squark degeneracy and proportionality of trilinear Higgs–squark couplings to Yukawa couplings can be natural if supersymmetry breaking is communicated to the light particles by gauge interactions [16], or in models with a non-Abelian horizontal symmetry [17] [6].

In reference [6] an alternative mechanism to suppress squark contributions to FCNC was suggested: the approximate alignment of quark mass matrices with squark mass-squared matrices. The idea is that a horizontal symmetry, of the type discussed in this work, forces both \( M^q \) and \( \tilde{M}^{q2} \) to be approximately diagonal in the basis where the horizontal charges are well defined. Consequently, the mixing matrix for quark – squark – gluino couplings is close to a unit matrix and FCNC are suppressed, regardless of whether squarks are degenerate or not.
To make the discussion concrete, we define the diagonalizing matrices for quarks,

\[ V_L^d M^d V_R^{d\dagger} = \text{diag}(m_d, m_s, m_b), \]
\[ V_L^u M^u V_R^{u\dagger} = \text{diag}(m_u, m_c, m_t), \]  
(2.26)

for down squarks,

\[ \tilde{V}_L^d \tilde{M}_L^d \tilde{V}_L^{d\dagger} = \text{diag}(m_{\tilde{d}_L}^2, m_{\tilde{s}_L}^2, m_{\tilde{b}_L}^2), \]
\[ \tilde{V}_R^d \tilde{M}_R^d \tilde{V}_R^{d\dagger} = \text{diag}(m_{\tilde{d}_R}^2, m_{\tilde{s}_R}^2, m_{\tilde{b}_R}^2), \]  
(2.27)

and similarly for up squarks. Here we assume \( \tilde{m} \gg \langle \phi_{u,d} \rangle \). Then, the CKM matrix is

\[ V = V_L^u V_L^{d\dagger}, \]

while the mixing matrices for gluino interactions are

\[ K_L^d = V_L^d \tilde{V}_L^{d\dagger}, \quad K_R^d = V_R^d \tilde{V}_R^{d\dagger}, \]
\[ K_L^u = V_L^u \tilde{V}_L^{u\dagger}, \quad K_R^u = V_R^u \tilde{V}_R^{u\dagger}. \]  
(2.28)

Various FCNC processes, and in particular the mixing of neutral mesons, put upper bounds on elements of the \( K_M^q \) matrices (\( M = L, R; q = d, u \)). The bounds are particularly strong on

\[ \langle K_{ij}^q \rangle = \sqrt{(K_{ij}^L)^2 + (K_{ij}^R)^2}. \]  
(2.29)

For \( m_3 = \tilde{m} = 1 \, \text{TeV} \), the bounds are

\[ \Delta m_K \implies (K_{12}^d)^2 \leq 0.05, \quad \langle K_{12}^d \rangle \leq 0.006, \]
\[ \epsilon_K \implies (K_{12}^d)^2 \leq 0.004, \quad \langle K_{12}^d \rangle \leq 0.0005, \]
\[ \Delta m_D \implies (K_{12}^u)^2 \leq 0.1, \quad \langle K_{12}^u \rangle \leq 0.04, \]
\[ \Delta m_B \implies (K_{13}^d)^2 \leq 0.1, \quad \langle K_{13}^d \rangle \leq 0.04. \]  
(2.30)

A few points are in order regarding these constraints:

(i) There are also bounds on the mixing matrices \( K_{LR}^q = V_{LR}^q \tilde{V}_R^{q\dagger} \) and \( K_{RL}^q = V_{RL}^q \tilde{V}_L^{q\dagger} \).

However, these bounds are easily satisfied in our framework and we do not present them here.

---

3 Usually, these bounds are applied to the off diagonal entries in the squark mass matrices in the basis where the quark mass matrices are diagonal. When the squarks are not even approximately degenerate, as is the case in our discussion, the bounds are on the matrix elements of \( K \).
(ii) The bound on $\epsilon_K$ is valid only when we assume that all CP violating phases are arbitrary and of order one.

(iii) We emphasize that there is an ambiguity of a factor of a few in these bounds, coming from the exact value of the $\tilde{m}$ scale; from possible differences between the gluino mass $m_{\tilde{g}}$ and the average squark mass $\tilde{m}$; and from hadronic uncertainties in matrix elements of quark operators.

Generically, the horizontal symmetries employed in our various models give

\[
(K^d_L)_{12} \lesssim \lambda, \quad (K^d_R)_{12} \lesssim \lambda, \quad (K^d)_{12} \lesssim \lambda,
\]

\[
(K^u_L)_{12} \lesssim \lambda^2, \quad (K^u_R)_{12} \lesssim \lambda^2, \quad (K^u)_{12} \lesssim \lambda^2,
\]

\[
(K^d_L)_{13} \lesssim \lambda^3, \quad (K^d_R)_{13} \lesssim \lambda, \quad (K^d)_{13} \lesssim \lambda^2.
\]

This means that the main problem is the suppression of the squark contributions to $\Delta m_K$ and $\epsilon_K$; the contributions to $\Delta m_D$ and $\Delta m_B$ are generically suppressed to just an acceptable level. We now describe a class of models that we call “Quark-Squark-Alignment” (QSA) models, in which squark contributions to $\Delta M_K$ and to $\epsilon_K$ are highly suppressed and (2.30) is satisfied.

We again focus on models with $\tan \beta \sim 1$ and $m_b/m_t \sim \lambda^2$, but models with satisfactory quark-squark alignment exist also for $\tan \beta \sim m_t/m_b$ or for $m_b/m_t \sim \lambda^3$. The main problem is to avoid $(V^d_L)_{12} \sim \lambda$ and $(V^d_R)_{12} \sim \lambda$, while keeping the CKM values $|V_{us}| \sim \lambda$ and $|V_{ub}| \sim \lambda^3$. The expressions for the elements of the diagonalizing matrices in terms of elements of the mass matrices are given in Appendix A. Using these expressions, we find that, to satisfy (2.30), the following entries in $M^d$ have to vanish: $M^d_{12}$, $M^d_{21}$, either $M^d_{13}$ or $M^d_{32}$ and either $M^d_{31}$ or $M^d_{23}$. We stress again that when we say that a particular $M^d_{ij}$ vanishes, we actually mean that it would vanish if $H$ were continuous. However, as $H$ is discrete, $M^d_{ij}$ is not zero but only highly suppressed compared to its value in (2.6).

It is impossible to get vanishing (or suppressed) entries in models with $H = U(1)$. We therefore scanned models with $H = U(1) \times U(1)$. Some models with $\epsilon_1 = \lambda^2$ and $\epsilon_2 = \lambda^3$ give satisfactory quark-squark alignment (such a model was presented in ref. [7]), but run into other phenomenological problems. Specifically, $|V_{td}|$ is highly suppressed, leading to an unacceptably small $B - \bar{B}$ mixing. On the other hand, we found many acceptable models
with $\varepsilon_1 = \lambda$ and $\varepsilon_2 = \lambda^2$ that lead to satisfactory suppression of the squark contributions to $\Delta m_K$ and $\epsilon_K$. The down quarks mass matrix in these models is always of the form

$$M^d \sim \langle \phi_d \rangle \begin{pmatrix} \lambda^4 & 0 & \lambda^3 \\ 0 & \lambda^2 & \lambda^2 \\ 0 & 0 & 1 \end{pmatrix},$$

(2.33)

which leads to the following order of magnitude estimates:

$$(V^d_L)_{12} \sim \lambda^5, \quad (V^d_R)_{12} \sim \lambda^7.$$  (2.34)

Let us present one explicit example, which has also interesting implications for CP asymmetries in $B$ decays:

$$Q_1 \quad Q_2 \quad Q_3 \quad \bar{d}_1 \quad \bar{d}_2 \quad \bar{d}_3 \quad (3, 0) \quad (0, 1) \quad (0, 0) \quad (-1, 2) \quad (4, -1) \quad (0, 1).$$  (2.35)

For the down squark masses, we find in this model

$$\tilde{M}^d_{LL} \sim \tilde{m}^2 \begin{pmatrix} 1 & \varepsilon_1 \varepsilon_2^3 & \varepsilon_1^3 \\ \varepsilon_1^3 & 1 & \varepsilon_2 \\ \varepsilon_1 \varepsilon_2 & \varepsilon_2 & 1 \end{pmatrix},$$

(2.36)

$$\tilde{M}^d_{RR} \sim \tilde{m}^2 \begin{pmatrix} 1 & \varepsilon_1^5 \varepsilon_2^3 & \varepsilon_1 \varepsilon_2 \\ \varepsilon_1^3 \varepsilon_2^3 & 1 & \varepsilon_1^4 \varepsilon_2^2 \\ \varepsilon_1^3 \varepsilon_2 & \varepsilon_1^3 \varepsilon_2^2 & 1 \end{pmatrix},$$

(2.37)

$$(\tilde{M}^d_{LR})_{ij} \sim \tilde{m} M^d_{ij}.$$  (2.38)

This gives

$$\tilde{(V^d_L)}_{12} \sim \lambda^5, \quad \tilde{(V^d_R)}_{12} \sim \lambda^{11},$$  (2.39)

which, together with eq. (2.34), leads to

$$\tilde{(K^d_L)}_{12} \sim \lambda^5, \quad \tilde{(K^d_R)}_{12} \sim \lambda^7,$$  (2.40)

consistent with (2.30).

Let us now see how the constraint on $\Delta m_D$ in eq. (2.31) is satisfied in this model. For that, we have to assign horizontal charges to the $\bar{u}_i$ fields. Take as an example

$$\bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_3 \quad (-1, 2) \quad (1, 0) \quad (0, 0).$$  (2.41)
The constraints on \((K^u_R)_{12}\) and \(\langle K^u_r \rangle_{12}\) are easily satisfied, but the constraint on \((K^u_L)_{12}\) is only barely so. Then the model predicts that \(\Delta m_D\) is very close to the experimental upper bound. This is actually not just a feature of the model presented here, but a crucial test of the quark–squark alignment idea: in all QSA models, \((V^d_L)_{12}\) is highly suppressed and, therefore, \((V^u_L)_{12}\) must be equal to the Cabibbo angle, namely \((V^u_L)_{12} \sim \lambda\). This gives

\[
(K^u_L)_{12} \sim \lambda,
\]

which is at the order of the upper bound. The conclusion is that in all QSA models, \(D - \bar{D}\) mixing is orders of magnitude above the Standard Model and should be very close to its present upper bound.

The QSA model presented here has also interesting implications for \(B\)-physics. A rough estimate of the ratio between the SUSY contribution to \(B - \bar{B}\) mixing and the Standard Model one gives

\[
\frac{|M_{12}(B^0)|_{\text{SUSY}}}{|M_{12}(B^0)|_{\text{SM}}} \approx 250[(K^d_L)_{13}^2 + (K^d_R)_{13}^2] + 2500\langle K^d_{13} \rangle^2,
\]

where we used vacuum-insertion approximation for the various matrix elements and a scale \(\tilde{m} \sim 1\, \text{TeV}\). From equations \((2.33), (2.36), (2.37)\) and \((2.38)\) we find

\[
(K^d_L)_{13} \sim \lambda^3, \quad (K^d_R)_{13} \sim \lambda^3,
\]

which gives \(\frac{|M_{12}(B^0)|_{\text{SUSY}}}{|M_{12}(B^0)|_{\text{SM}}} \approx 0.15\). While this contribution is small enough to satisfy the \(\Delta m_B\) constraint in \((2.31)\), it may lead to observable effects in CP asymmetries in \(B\) decays. It is important here that the quark–squark alignment is precise enough to satisfy the \(\epsilon_K\) constraint: this means that we have no reason to assume that the new CP violating phases in the \(K^q_M\) matrices are small. With new phases of \(\mathcal{O}(1)\), and with magnitude which is \(\mathcal{O}(0.15)\) of the Standard Model one, the shift from the Standard Model predictions in CP asymmetries in the decays of neutral \(B\) into final CP eigenstates may be as large as \(\mathcal{O}(0.3)\).

The potentially large effect on CP asymmetries in \(B^0\) decays is not a generic feature of quark–squark alignment models. Actually, it is possible to show that, while \((K^d_L)_{13} \sim \lambda^3\) in all our models of quark–squark alignment, the order of magnitude estimate \((2.44)\) is
an upper bound on \((K^d_R)_{13}\) in this framework. The fact that the specific model presented in this subsection saturates this bound and, therefore, gives interesting effects was our reason to present it in the first place. In most other models \((K^d_R)_{13} \sim \lambda^5\) or even \(\lambda^7\), so that the shift from the Standard Model predictions for CP asymmetries is of \(\mathcal{O}(0.01)\) or even less. A measurement of the asymmetries may then distinguish among our various models.

A similar investigation can be made for CP asymmetries in \(B_s\) decays:

\[
\frac{|M_{12}(B_s)|_{\text{SUSY}}}{|M_{12}(B_s)|_{\text{SM}}} \approx 10[(K^d_L)_{23}^2 + (K^d_R)_{23}^2] + 100\langle K^d_{23}\rangle^2. \quad (2.45)
\]

For models of quark–squark alignment

\[
(K^d_L)_{23} \sim \lambda^2, \quad (K^d_R)_{13} \lesssim \lambda^4. \quad (2.46)
\]

The shift from the Standard Model predictions for CP asymmetries in \(B_s\) decays is then of \(\mathcal{O}(0.01)\). This is probably too small to be experimentally observed. It also leads to the interesting situation [18] that the angles \(\alpha, \beta\) and \(\gamma\) as deduced from the CP asymmetries in \(B \to \pi \pi, B \to \psi K_S\) and \(B_s \to \rho K_S\), respectively, would sum up to \(\pi\) even if there is new physics in \(B^0\) mixing (this could be precisely the SUSY contributions discussed in this subsection!) such that the deduced values of \(\alpha\) and \(\beta\) do not really correspond to angles of the unitarity triangle.

Summarizing the phenomenological tests of the quark–squark alignment mechanism:

(i) Squarks are not necessarily degenerate;
(ii) \(D - \bar{D}\) mixing is close to the experimental bound;
(iii) CP asymmetries in \(B^0\) (but not \(B_s\)) decays may differ by up to \(\mathcal{O}(0.3)\) from their Standard Model values.

2.4. Higher Order Terms

So far, we have considered only the dimension three mass terms that arise when we assume a low energy effective model with an explicitly broken horizontal symmetry, such that terms that break the symmetry by \(n \geq 0\) \((n < 0)\) units of charge are suppressed by \(\lambda^n\) (are forbidden). With the same minimal set of assumptions, the effective Lagrangian at low energies would contain also higher order terms that obey similar selection rules.
These terms may have important effects. In particular, they may induce FCNC and affect quark–squark alignment. We now discuss these effects.

First, we consider constraints on four quark operators from FCNC. Take, for example, a \( \Delta s = 2 \) four quark operator \( \mathcal{O}_K \). It would appear in the effective Lagrangian in the generic form

\[
\frac{F_K}{M^2} \mathcal{O}_K,
\]

where \( F_K \) is a dimensionless coefficient that includes all suppression factors, such as powers of \( \lambda^n \) or powers of \( 1/(4\pi)^\ell \) if it first appears at the \( \ell \)-loop level in the full theory, and \( M \) is the scale below which the effective Lagrangian description holds. Let us further define

\[
X_K \equiv \frac{\langle K^0|\mathcal{O}_K|\bar{K}^0\rangle}{\langle K^0|(|\bar{s}_L\gamma_\mu d_L|^2)|K^0\rangle}.
\]

We similarly define the coefficients and matrix elements for \( \Delta c = 2 \) and \( \Delta b = 2 \) operators. Then the requirement that four quark operators do not contribute more than the experimental values of (or bounds on) neutral meson mixing yields

\[
M \gtrsim \begin{cases} 
1600 \text{ TeV} & \Delta m_K, \\
20000 \text{ TeV} & \epsilon_K, \\
570 \text{ TeV} & \Delta m_D, \\
530 \text{ TeV} & \Delta m_B.
\end{cases}
\]

Since our framework incorporates Abelian horizontal symmetry, there is no symmetry reason to forbid four quark operators of e.g. the form \((Q_i\gamma^\mu Q_i^\dagger)(Q_j\gamma_\mu Q_j^\dagger)\), \( i, j = 1, 2, 3 \) (and similarly for \( \bar{d}_i \) and \( \bar{u}_i \)). These terms are neutral under \( \mathcal{H} \), and therefore are not suppressed by powers of \( \lambda \). When rotating to the mass eigenbasis, FCNC operators are induced. In particular, some combination of \( \Delta s = 2 \) and \( \Delta c = 2 \) operators is unavoidable. The weakest constraint corresponds to the case \((V^d_{L12}) = 0, (V^u_{L12}) \sim \lambda \) (as in models of quark–squark alignment). It comes from \( \Delta m_D \) with \( X_D = 1 \) and \( F_D = \lambda^2 \) (assuming that the operator arises at tree level in the full theory):

\[
M \gtrsim 100 \text{ TeV}.
\]

If \((V^d_{L12}) \sim \lambda \), as is the case in many of our models, then a stronger bound from \( \Delta m_K \) holds:

\[
M \gtrsim 300 \text{ TeV}.
\]
If, in addition, \( \text{Im}(F_K) \) is comparable to the real part, then an even stronger bound from 
\( \epsilon_K \) holds:
\[
M \gtrsim 4000 \text{TeV}.
\] (2.52)

Additional and potentially stronger bounds arise if non-diagonal four quark operators are not horizontally suppressed. For example, there are four operators that may contribute to \( K - \bar{K} \) mixing (we take into account only \( SU(2)_L \times U(1)_Y \) invariant terms):
\[
\begin{align*}
\mathcal{O}_{1K} &= Q_2 \gamma^\mu Q_1^\dagger Q_2 \gamma^\mu Q_1^\dagger, \quad (X_K^{V+A} = 1), \\
\mathcal{O}_{2K} &= \bar{d}_2^\dagger \gamma^\mu \bar{d}_1 \bar{d}_2^\dagger \gamma^\mu \bar{d}_1, \quad (X_K^{V+A} = 1), \\
\mathcal{O}_{3K} &= Q_2 \gamma^\mu Q_1^\dagger \bar{d}_2^\dagger \gamma^\mu \bar{d}_1, \quad (X_K^{V-A} \approx 3.6), \\
\mathcal{O}_{4K} &= Q_2 \bar{d}_1 \bar{d}_2^\dagger Q_1^\dagger, \quad (X_K^{S-P} \approx 4.4).
\end{align*}
\] (2.53)

However, only \( \mathcal{O}_{3K} \) and \( \mathcal{O}_{4K} \) could avoid horizontal suppression. This would happen if
\[
H_i(Q_1) - H_i(Q_2) = H_i(\bar{d}_1) - H_i(\bar{d}_2).
\] (2.54)

\( \mathcal{O}_{1K} \) and \( \mathcal{O}_{2K} \) are always horizontally suppressed and therefore give bounds that are not stronger than (2.51). Also, the analogous \( \Delta c = 2 \) and \( \Delta b = 2 \) operators are always horizontally suppressed. (This is simple to see, as a necessary result of (2.54) is \( m_i/m_j \sim V_{ij}^2 \).) In models where the operators \( \mathcal{O}_{3K} \) and \( \mathcal{O}_{4K} \) are not suppressed, the following bounds apply (again, assuming that they are induced by tree diagrams in the full theory):
\[
M \gtrsim 3000 \text{TeV},
\] (2.55)

and with CP violating phases of order one,
\[
M \gtrsim 40000 \text{TeV}.
\] (2.56)

For the various explicit models presented in this section, we find

(i) In the master model, (2.54) is fulfilled (see (2.5)) so that \( M \gtrsim 3000 \text{TeV} \).

(ii) In the model of subsection 2.2, (2.54) is not fulfilled (see (2.21)) so that \( M \gtrsim 300 \text{TeV} \).

However, with a different choice of charge, \( \bar{d}_1(0, 1) \), as made in [10], (2.54) is fulfilled and \( M \gtrsim 3000 \text{TeV} \).
(iii) In the quark–squark alignment model of subsection 2.3, (2.54) is not fulfilled (see (2.35)) so that $M \gtrsim 100 \text{ TeV}$.

Next we discuss bounds on the scale $M$ that arise from terms involving higher powers of the scalar fields:

$$Q_i \tilde{d}_j \phi_d \left( \frac{\phi_u \phi_d}{M^2} \right)^n; \quad Q_i \tilde{u}_j \phi_u \left( \frac{\phi_u \phi_d}{M^2} \right)^m. \tag{2.57}$$

These terms violate Natural Flavor Conservation and would contribute to FCNC. In particular, for $i, j = 1, 2$, there would be scalar-mediated tree diagrams contributing to $K - \bar{K}$ and $D - \bar{D}$ mixing and therefore leading to bounds on $M$.

Note that the terms in (2.57) break $U(1)_X$ and therefore one cannot use the simplified horizontal charge assignments achieved by $U(1)_X \times U(1)_Y \times U(1)_B$ transformations. The allowed powers $n$ and $m$ in (2.57) will depend on the true horizontal symmetry. We consider here the case $n, m = 1$. This leads to the strongest possible bounds on the scale $M$ but in many of our models $n, m > 1$ and the bounds are consequently weaker.

The contributions depend on the intermediate scalar mass. We take the upper bound on the mass of the lightest neutral scalar in SUSY, $m_\phi \lesssim 150 \text{ GeV}$ [19]. For $\tan \beta \sim 1$, the strongest bound comes from $\Delta m_K$ (with $F_K \sim \frac{9}{64} \frac{v^4}{M^4}$), while for $\tan \beta \sim \frac{m_t}{m_b}$, the strongest bound comes from $\Delta m_D$ (with $F_D \sim \frac{9}{8} \frac{v^4}{M^4} \frac{m_b^2}{m_t^2}$):

$$M \gtrsim \begin{cases} 26 \text{ TeV} & \tan \beta \sim 1, \\ 4 \text{ TeV} & \tan \beta \sim \frac{m_t}{m_b}. \end{cases} \tag{2.58}$$

The contributions to $M^d_{12}$, $M^d_{21}$ from terms of the form (2.57) may spoil the precise alignment in our models of quark–squark alignment (QSA). Requiring that this should not happen gives, in this class of models and if $n = 1$ is allowed,

$$M(\text{QSA}) \gtrsim 40 \text{ TeV}. \tag{2.59}$$

When the bounds from four quark operators are taken into account, we see that the effects of the $U(1)_X$ breaking terms on our mass matrices, on FCNC and on quark–squark alignment are always unimportant.
3. Beyond the Naive Relations

3.1. $|V_{ub}/V_{cb}| \ll |V_{us}|$

As mentioned above, it may turn out that $|V_{ub}/V_{cb}|$ is not of the same order of magnitude as $|V_{us}|$, namely that the naive model-independent prediction of our framework fails. We now show that, under special circumstances this naive prediction can be modified while all other order of magnitude relations are maintained. We will present a model where

$$|V_{us}| \sim \lambda, \quad |V_{cb}| \sim \lambda^2, \quad |V_{ub}| \sim \lambda^4. \quad (3.1)$$

We will see that to produce (3.1), we need a specific discrete symmetry.

To suppress $|V_{ub}|$ below $\lambda^3$, certain entries in the mass matrices have to be suppressed relative to their naive values. Let us first consider the (unrealistic) case of a continuous horizontal symmetry, for which each entry in the mass matrix can either get its naive value or vanish. Using appendix A we find that the mass matrices have to take the following form

$$M^d \sim \langle \phi_d \rangle \begin{pmatrix} \lambda^6 & \lambda^5 & 0 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ \lambda^3 & 0 & \lambda^2 \end{pmatrix}, \quad M^u \sim \langle \phi_u \rangle \begin{pmatrix} \lambda^6 & 0 & 0 \\ \lambda^5 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda & 1 \end{pmatrix}. \quad (3.2)$$

Some additional entries may vanish, but not all of them. In particular, to produce $|V_{us}| \sim \lambda$, we need $M^d_{12} \sim \lambda$.

As long as the effects of the discrete symmetry are negligible, the mass matrices (3.2) give $|V_{ub}| \sim \lambda^3$. If additional entries are zero, the value of $|V_{ub}|$ may be even further suppressed, but it will always be an odd power of $\lambda$. To obtain (3.1), the effects of the discrete symmetry have to play a role. In particular, it must allow at least one of the following four options:

(i) $M^u_{13} \sim \lambda^4$; (ii) $M^u_{12} \sim \lambda^5$; (iii) $M^d_{13} \sim \lambda^6$; (iv) $M^d_{32} \sim \lambda^3$.

It is impossible to produce the required structure within a model with a single $U(1)$. Considering models with $\mathcal{H} = U(1) \times U(1)$, we find that the required structure cannot be produced in models with $\epsilon_1 \sim \lambda^2$ and $\epsilon_2 \sim \lambda^3$, but it can in models of with $\epsilon_1 \sim \lambda$ and $\epsilon_2 \sim \lambda^2$. There is a very large number of models with this pattern of symmetry breaking that produce (3.2) and (3.3). The charges of all fields except $Q_3$ and $\bar{u}_3$ have more than
one option. For example, the charge of $\bar{d}_3$ could be either $(2, 0)$ or $(0, 1)$. Quite a few of these models give mass matrices of the form (3.2). We choose to present one example, to demonstrate that the desired suppression of $|V_{ub}|$ is possible.

The model that we choose as an example has the following set of charges:

$$
\begin{align*}
Q_1 & = (-5, 4) \\
Q_2 & = (-2, 2) \\
Q_3 & = (0, 0) \\
\bar{d}_1 & = (7, -2) \\
\bar{d}_2 & = (6, -2) \\
\bar{d}_3 & = (2, 0) \\
\bar{u}_1 & = (11, -4) \\
\bar{u}_2 & = (3, -1) \\
\bar{u}_3 & = (0, 0).
\end{align*}
$$

Actually, for $Q_1(-5, 4)$ there is only a single choice for the charges of all other fields except for $\bar{d}_1$ and $\bar{u}_1$. The latter ones do not affect the required zeros. The choice of their charges in (3.3) is correlated with the choice of discrete symmetry below, and is motivated by considerations that go beyond the low energy framework – we explain this in subsection 4.3.

We would like to introduce a discrete subgroup of the above symmetry such that $|V_{ub}| \sim \lambda^4$. As mentioned above, one of the ways to do it is to lift the zero in $M_{13}^u$ and have $M_{13}^u \sim \lambda^4$ instead. As the charge of $M_{13}^u$ under $U(1)_{H_1} \times U(1)_{H_2}$ is $(-5, 4)$, there are two discrete subgroups that would do precisely that: $Z_9 \times Z_4$ and $Z_7 \times Z_3$. With the latter symmetry (and the choice of charges for $\bar{d}_1$ and $\bar{u}_1$ made in (3.3)) we get the following mass matrices:

$$
M^d \sim \langle \phi_d \rangle \begin{pmatrix}
\epsilon_1^2 \epsilon_2^2 & \epsilon_1 \epsilon_2 \epsilon_2^2 & \epsilon_1^4 \epsilon_2 \\
\epsilon_1^5 \epsilon_2^2 & \epsilon_1^4 \epsilon_2 \epsilon_2^2 & \epsilon_1^2 \epsilon_2 \\
\epsilon_2 & \epsilon_1^6 \epsilon_2 & \epsilon_1^2 \epsilon_2
\end{pmatrix},
M^u \sim \langle \phi_u \rangle \begin{pmatrix}
\epsilon_1^6 & \epsilon_1^5 & \epsilon_1^2 \epsilon_2 \\
\epsilon_1^5 \epsilon_2 & \epsilon_1^5 & \epsilon_1^2 \epsilon_2 \\
\epsilon_1^4 \epsilon_2 & \epsilon_1^4 \epsilon_2 \epsilon_2^2 & 1
\end{pmatrix}.
$$

(3.4)

It is easy to check that these mass matrices produce the order of magnitude relations (1.3) and (3.1).

### 3.2. Exact Relations between Quark Parameters

The fact that an Abelian horizontal symmetry could produce zeros (or highly suppressed terms) in the quark mass matrices, opens up the interesting possibility of (close to) exact relations between various, otherwise independent, parameters of the quark sector.

---

4 Clearly, with a non-Abelian horizontal symmetry it is also possible to find exact relations.\[4]
The model in the previous subsection is an example. In general, \(|V_{td}|, |V_{us}|\) and \(|V_{cb}|\) are independent parameters. Unitarity of the CKM matrix requires (see (1.1))

\[
V_{td} = V_{us}^* V_{cb}^* - V_{ub}^*.
\]

In the previous subsection we presented models where \(|V_{ub}| \sim \lambda |V_{us}V_{cb}|\). Then, the following relation arises:

\[
|V_{td}| = |V_{us}V_{cb}|[1 + \mathcal{O}(\lambda)].
\]

This relation is, of course, consistent with present constraints. (This is a rather trivial statement because the best upper bound on \(|V_{td}|\) at present comes from CKM unitarity.) It is actually the only phenomenologically acceptable relation that involves only mixing parameters.

We searched for close-to-exact relations that involve both mass ratios and mixing angles. Our basic assumption is that each entry has either its “naive” value as in eq. (2.6), or it vanishes. (In case that a discrete symmetry replaces a zero entry with one that is suppressed compared to the naive one, the same relation would hold but potentially with lesser accuracy.) One can find some general rules. For example, no exact relation can involve masses of first quark generation. The proof for that is very simple: \(m_d\) and \(m_u\) depend on elements of the first column in \(M_d\) and \(M_u\), respectively, but none of the mixing angles depends on these elements to leading order (see Appendix A for the dependence of the mixing angles on mass matrix elements). Similar considerations lead to the following conclusion:

Only a single exact relation could arise in our framework of supersymmetric Abelian horizontal symmetry. It requires six entries in the quark mass matrices to vanish:

\[
M_d \sim \langle \phi_d \rangle \lambda^2 \begin{pmatrix} \lambda^6 & 0 & \lambda^3 \\ \lambda^5 & \lambda^2 & 0 \\ \lambda^3 & 1 & 1 \end{pmatrix}, \quad M_u \sim \langle \phi_u \rangle \begin{pmatrix} \lambda^6 & 0 & 0 \\ \lambda^5 & \lambda^3 & 0 \\ \lambda^3 & 0 & 1 \end{pmatrix}.
\]

(3.7)

It is possible to exchange the second and third columns of \(M_d\) without changing the results. Note also that elements of the first columns do not affect the relation, so some of the entries there might vanish as well.
The exact relation that follows (to order $\lambda^2$) from (3.7) is

$$\frac{m_s^2}{m_b^2} = \left| \frac{V_{cb}V_{ub}}{V_{us}} \right|. \quad (3.8)$$

Using the values of mixing angles from [12] and mass ratios from [20], we have

$$\frac{m_s^2}{m_b^2} = 0.0010^{+0.0009}_{-0.0006}, \quad \left| \frac{V_{cb}V_{ub}}{V_{us}} \right| = 0.0007^{+0.0006}_{-0.0004}, \quad (3.9)$$

so that the present accuracy in determining the various parameters does not allow a test of the idea that an Abelian horizontal symmetry might lead to exact relations.

4. Spontaneously Broken $\mathcal{H}$

4.1. Extending the Scalar Sector

The low energy models described in the previous section can arise naturally if $\mathcal{H}$ is an exact symmetry of the Lagrangian, and is broken by the VEV of a scalar that is a singlet of the Standard Model gauge group and carries one unit of horizontal charge [8]. For example, the master model requires the existence of a single complex scalar field $S(-1)$, with

$$\lambda = \frac{\langle S \rangle}{M}, \quad (4.1)$$

($M$ is the scale at which the information about the spontaneous symmetry breaking is communicated to the light fermions.) The selection rule becomes obvious now. For

$$H(Q_i) + H(\bar{d}_j) = l \geq 0,$$

$$H(Q_i) + H(\bar{u}_j) = k \geq 0, \quad (4.2)$$

the exact horizontal symmetry allows only Yukawa terms of the form

$$\mathcal{L}_Y = \frac{\Gamma^d_{ij}}{M_i^l} S^l \phi_d Q_i \bar{d}_j + \frac{\Gamma^u_{ij}}{M_k^k} S^k \phi_u Q_i \bar{u}_j, \quad (4.3)$$

(where the dimensionless Yukawa couplings $\Gamma^q_{ij} = \mathcal{O}(1)$). Supersymmetry requires that the Yukawa terms are analytic in $S$. Consequently, $S^\dagger$ cannot take part in the Yukawa sector and terms with $l < 0$ or $k < 0$ are forbidden (except for very highly suppressed non-supersymmetric contributions).
Models with $\mathcal{H} = U(1)_{H_1} \times U(1)_{H_2}$ need the introduction of two Standard Model singlet scalars, $S_1$ and $S_2$, with horizontal charges

$$S_1(-1, 0), \quad S_2(0, -1),$$

and vacuum expectation values

$$\frac{\langle S_1 \rangle}{M} \sim \lambda^p, \quad \frac{\langle S_2 \rangle}{M} \sim \lambda^q,$$

where $(p, q) = (2, 3)$ in the models of subsection 2.2 or $(p, q) = (1, 2)$ in the quark–squark alignment models of subsection 2.3. We always assume that two separate scales of VEVs should break two different symmetries, namely that all VEVs that break the same symmetry should be at a single scale. Equations (4.4) and (4.5) are consistent with this assumption, as each VEV breaks a different $U(1)$ factor in $\mathcal{H}$.

4.2. Bounds from FCNC

The $S_i$ scalars couple non-diagonally to quarks. Consequently, they mediate FCNC through tree diagrams. Specifically, they induce four quark operators of the type $\mathcal{O}_4$ of equation (2.53). As the masses of the scalars is of the order of their vacuum expectation values, the scale $M$ of equation (2.47) should be replaced by $\langle S \rangle$. The factor $F$ is model dependent. Let us examine a few examples.

(i) In the master model (see (2.6)), $F_K \sim \frac{m_d m_s}{\langle S \rangle^2}$, $F_B \sim \frac{m_d m_b}{\langle S \rangle^2}$ and $F_D \sim \frac{m_u m_c}{\langle S \rangle^2}$. Then the strongest bounds come from the $K$ system,

$$\langle S \rangle \gtrsim 0.4 \, \text{TeV} \quad \Rightarrow \quad M \gtrsim 2 \, \text{TeV}, \quad (\Delta m_K);$$

$$\langle S \rangle \gtrsim 1.4 \, \text{TeV} \quad \Rightarrow \quad M \gtrsim 7 \, \text{TeV}, \quad (\epsilon_K).$$

(ii) In the model of subsection 2.2 (see (2.22)), $F_K \sim \frac{\lambda^2 m_d m_s}{\langle S_1 \rangle^2}$, $F_B \sim \frac{\lambda^4 m_d m_b}{\langle S_1 \rangle^2}$ and $F_D \sim \frac{\lambda^4 m_u m_c}{\langle S_2 \rangle^2}$. Then the only bound on $\langle S_i \rangle$ which is above the electroweak scale comes from $\epsilon_K$,

$$\langle S_1 \rangle \gtrsim 0.6 \, \text{TeV} \quad \Rightarrow \quad M \gtrsim 13 \, \text{TeV}, \quad (\epsilon_K).$$

(iii) In the model of reference [10], all elements of $M^d$ assume their naive values, so that $F_K$ and $F_B$ are similar to the master model. On the other hand, $M_{12}^u$ is suppressed
and consequently so is $F_D$. The resulting bounds are then
\[ \langle S_2 \rangle \gtrsim 0.4 \, \text{TeV} \implies M \gtrsim 50 \, \text{TeV}, \quad (\Delta m_K); \]
\[ \langle S_2 \rangle \gtrsim 1.4 \, \text{TeV} \implies M \gtrsim 170 \, \text{TeV}, \quad (\epsilon_K). \]  

(iv) In the quark–squark alignment models of subsection 2.3, both $F_K$ and $F_B$ are always highly suppressed (see (2.33)). $F_D$ depends on the charge assignments of the $\bar{u}_i$ fields, but in all cases $F_D \lesssim \frac{m_{\text{max}}}{\langle S_2 \rangle^2}$. If the bound is saturated, then $\langle S_2 \rangle$ cannot be lower than the electroweak scale. If $F_D$ is further suppressed (as is the case in the example given in eq. (2.41)), then no useful bound arises.

4.3. QCD Anomalies and the Breaking of $U(1)_X$

Now, that we have extended our framework to exact horizontal symmetries that are only spontaneously broken, we should discuss in more detail the subject of QCD anomalies. As mentioned in section 2, QCD anomalies pose no problem because of the $U(1)_X$ symmetry of the Yukawa sector. Furthermore, we mentioned that $U(1)_X$ must be broken in some sector in the Lagrangian or else an axion will be generated. We now discuss this in more detail.

We first discuss the case where $H \subset U(1)$ with the simplified charges one gets by $U(1)_X \times U(1)_Y \times U(1)_B$ transformations. This simplified horizontal symmetry may have a nonvanishing QCD anomaly $A_H$:
\[ A_H = \sum_i [H(Q_i) + H(\bar{u}_i) + H(\bar{d}_i)]. \]  

The true horizontal symmetry is a $Z_n \subset U(1)_{\tilde{H}} \subset U(1)_H \times U(1)_X$ and it must not have QCD anomaly,
\[ A_{\tilde{H}} = 0(\mod n). \]  

With $\tilde{H}$ charges given by
\[ \tilde{H} = aH + bX, \]  

\begin{footnote}
We do not discuss $SU(2) \times U(1)$ anomalies because they depend on the charges in the lepton sector. Since the mixing angles in that sector are not known, these charges are not constrained significantly and anomalies can be easily avoided.
\end{footnote}
it is easy to see that $\tilde{H}$ and $H$ constrain the quark mass matrices in precisely the same way. However, now

$$A_{\tilde{H}} = a A_H + 3b.$$  \hfill (4.12)

We can always find appropriate values for $a$ and $b$ so that (4.10) is satisfied.

Having shown that $U(1)_X$ can always be used to avoid QCD anomalies, we now discuss its breaking. We will assume that $U(1)_X$ is broken by terms of the form

$$(\phi_d \phi_u)^p S^q.$$ \hfill (4.13)

The $U(1)_H \times U(1)_X$ assignments of the scalar fields are $\phi_d(0,-1)$, $\phi_u(0,0)$ and $S(-1,0)$. The interaction (4.13) should conserve only $Z_n \subset U(1)_{\tilde{H}}$. We therefore require

$$\tilde{H}((\phi_d \phi_u)^p S^q) = -aq - bp = 0 \text{(mod n)}. \hfill (4.14)$$

Clearly, there are always solutions $(p,q)$ to the requirement (4.14) (take, for example $q = A_H \text{(mod n)}$ and $p = 3$).

To summarize, QCD anomalies do not pose a problem in our framework: even if $U(1)_H$ is anomalous, there is always a $Z_n \subset U(1)_{\tilde{H}} \subset U(1)_H \times U(1)_X$ which is free of QCD anomaly and should be considered as “the true horizontal symmetry.” Note that the anomaly constraint restricts the $U(1)_X$ breaking terms, as these should be $Z_n$ invariant.

The extension of this mechanism to models where $\mathcal{H} \subset U(1) \times U(1)$ is straightforward. The horizontal symmetry is an anomaly free $Z_m \times Z_n$ and the $U(1)_X$ symmetry is broken by a $(\phi_d \phi_u)^p S_1^q S_2^r$ term. For example, consider the model presented in subsection 3.1. This model was constructed to give $|V_{ub}| \sim \lambda^4$. The charge assignments under the $U(1)_{H_1} \times U(1)_{H_2} \times U(1)_X$ of the Yukawa sector are

$$
\begin{align*}
Q_1 & \quad Q_2 & \quad Q_3 & \quad \bar{d}_1 & \quad \bar{d}_2 & \quad \bar{d}_3 & \quad \bar{u}_1 & \quad \bar{u}_2 & \quad \bar{u}_3 \\
(-5, 4, 0) & \quad (-2, 2, 0) & \quad (0, 0, 0) & \quad (7, -2, 1) & \quad (6, -2, 1) & \quad (2, 0, 1) & \quad (11, -4, 0) & \quad (3, -1, 0) & \quad (0, 0, 0)
\end{align*}
$$ \hfill (4.15)

We add a term

$$(\phi_d \phi_u)S^5_1 S^5_2$$ \hfill (4.16)

25
which breaks $U(1)_H \times U(1)_H \times U(1)_X$ to $Z_7 \times Z_3$. Under this symmetry quarks transform as

$$
\begin{array}{cccccccc}
Q_1 & Q_2 & Q_3 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\
(2,1) & (5,2) & (0,0) & (2,0) & (1,0) & (4,2) & (4,2) & (3,2) & (0,0)
\end{array}.
$$

(4.17)

and the scalar fields as

$$
\begin{array}{cccc}
\phi_d & \phi_u & S_1 & S_2 \\
(-2,-2) & (0,0) & (-1,0) & (0,-1)
\end{array}.
$$

(4.18)

The resulting mass matrices are those of equations (3.4). It is straightforward to verify that the discrete symmetries are free of QCD anomalies.

Note that in this model we could not choose an arbitrary discrete symmetry – the horizontal symmetry must be $Z_7 \times Z_3$ to give the desired value for $|V_{ub}|$. However, we used the freedom in choosing the horizontal charges of $\bar{d}_1$ and $\bar{u}_1$ and the choice of $p,q,r$ in equation (4.16), to find solutions to the anomaly equations.

5. Physics at the Scale $M$

5.1. Extending the Quark Sector

In previous sections we have described the scale $M$ as the scale at which the information about the breaking of the horizontal symmetry $H$ is communicated to the light quarks, but we have not given any explicit mechanism that would do that. In this section we make yet another layer of assumptions: we use the mechanism suggested by Froggatt and Nielsen (FN) [8].

The FN mechanism assumes that there are additional quarks that transform non-trivially under $H$. These extra quarks come in mirror representations, namely they may appear in any of the following representations of $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_H$:

$$
\begin{align*}
P(3, 2, +1/6, H) & \quad \text{and} \quad \bar{P}(\bar{3}, 2, -1/6, -H); \\
D(3, 1, -1/3, H) & \quad \text{and} \quad \bar{D}(\bar{3}, 1, +1/3, -H); \\
U(3, 1, +2/3, H) & \quad \text{and} \quad \bar{U}(\bar{3}, 1, -2/3, -H).
\end{align*}
$$

(5.1)

Obviously, the new quarks can acquire heavy masses at a scale $M$ that is much higher than the electroweak breaking scale.
At the scale $M$ we consider the most general $\mathcal{H}$-invariant renormalizable Yukawa terms. As an example, we show how $M^u$ of equation (2.22),

$$M^u_{\text{light}} \sim \langle \phi_u \rangle \begin{pmatrix} \epsilon_2^2 & 0 & \epsilon_2 \\ \epsilon_1 \epsilon_2 & \epsilon_2 & \epsilon_1 \\ \epsilon_2 & 0 & 1 \end{pmatrix},$$

(5.2)
can be produced in the full theory \cite{10}. We add to the light quarks listed in eq. (2.21) three $SU(2)$-singlet charge $+2/3$ mirror quarks,

$$U_1 \quad U_2 \quad U_3 \quad (1,0) \quad (0,1) \quad (0,0),$$

(5.3)
and $\bar{U}_i$ with opposite charges. Then the $6 \times 6$ matrix $M^u_{\text{full}}$ with columns corresponding to $(\bar{u}_i, U_i)$ and rows to $(Q_i, U_i)$ has order of magnitude entries

$$M^u_{\text{full}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \langle \phi_u \rangle & 0 \\ 0 & 0 & 0 & \langle \phi_u \rangle & 0 & 0 \\ 0 & 0 & \langle \phi_u \rangle & 0 & 0 & \langle \phi_u \rangle \\ 0 & \langle S_2 \rangle & \langle S_1 \rangle & M & 0 & \langle S_1 \rangle \\ 0 & \langle S_2 \rangle & 0 & M & \langle S_2 \rangle & 0 \\ \langle S_2 \rangle & 0 & 0 & 0 & 0 & M \end{pmatrix}.$$ 

(5.4)

When the heavy quarks at the scale $M$ are integrated out, the resulting $M^u_{\text{light}}$ for the three observed generations of up quarks is $M^u$ of (5.2).

We had to add three $U + \bar{U}$ fields. This could have been foreseen by the following “determinant argument”: The determinant of the light fermions mass matrix is

$$\det M^u_{\text{light}} \sim \langle \phi_u \rangle^3 \epsilon_2^3,$$

(5.5)
which now means

$$\det M^u_{\text{light}} \sim \frac{\langle \phi_u \rangle^3 \langle S_2 \rangle^3}{M^3}.$$ 

(5.6)

However, an examination of the structure of $M^u_{\text{full}}$ shows that $\det M^u_{\text{full}}$ is a polynomial in $\langle \phi_u \rangle$, $\langle S_i \rangle$ and $M$. As $\det M^u_{\text{full}} = \det M^u_{\text{light}} \times \det M^u_{\text{heavy}}$, we deduce that $\det M^u_{\text{heavy}} \sim M^k$ with $k \geq 3$, so that at least three $U + \bar{U}$ are required.

The general rule is then: if $\det M^q_{\text{light}} \sim \langle \phi_q \rangle^3 \prod_i \epsilon_i^{m_i}$, then the minimal number of massive $q$-quarks required is $\sum_i m_i$. Thus, for example, as for the same model (see (2.22)) $\det M^d_{\text{light}} \sim \langle \phi_d \rangle^3 \epsilon_1^6$, at least six massive $D + \bar{D}$ (or $P + \bar{P}$) are required. An explicit
realization is given in ref. [10]. As another example, in the master model \( \det M^d_{\text{light}} \sim \langle \phi_d \rangle \lambda^{12} \), so at least twelve \( D + \bar{D} \) are required, and \( \det M^u_{\text{light}} \sim \langle \phi_u \rangle \lambda^9 \), so at least nine \( U + \bar{U} \) are required. The number of massive quarks will be important in our discussion of Landau poles in subsection 6.1.

5.2. Bounds on \( M \) from FCNC

With a full theory for physics at the scale \( M \), we can check whether the four quark operators discussed in subsection 2.4 indeed arise and calculate the \( F \) coefficients. We find that, if baryon number is conserved, the massive colored supermultiplets cannot contribute to neutral meson mixing in tree diagrams. Instead, the leading contributions come from box diagrams with intermediate heavy \( D \) or \( U \) quarks and \( S_i \) and \( \phi_q \) scalars. An explicit calculation gives that the \( F \) coefficients are suppressed by a factor \( \sim 1/4\pi \) compared to the estimates in subsection 2.4.

The following bounds then hold on the scale \( M \) of extra heavy quarks:

(i) Cabibbo mixing shows that the weakest bound that applies to all models (coming from \( \Delta m_D \)) is

\[
M \gtrsim 10 \ TeV. \tag{5.7}
\]

(ii) In models where \( (V^d_L)_{12} \sim \lambda \), a stronger bound (coming from \( \Delta m_K \)) holds,

\[
M \gtrsim 25 \ TeV. \tag{5.8}
\]

(iii) In models where \( \text{Im}[(V^d_L)_{11}(V^d_L)^*_{12}] \sim \lambda \), an even stronger bound (coming from \( \epsilon_K \)) holds,

\[
M \gtrsim 300 \ TeV. \tag{5.9}
\]

(iv) In models where \( H(Q_1) - H(Q_2) = H(\bar{d}_1) - H(\bar{d}_2) \), a bound from \( \Delta m_K \) stronger than (5.8) holds,

\[
M \gtrsim 250 \ TeV. \tag{5.10}
\]

(v) In models where \( H(Q_1) - H(Q_2) = H(\bar{d}_1) - H(\bar{d}_2) \), and there are CP violating phases of order one, a bound from \( \epsilon_K \) stronger than (5.9) holds,

\[
M \gtrsim 3000 \ TeV. \tag{5.11}
\]
This can be easily applied to the explicit models of section 2. The master model is constrained by (5.10) and possibly (5.11). The model of subsection 2.2 is constrained by (5.8) and possibly (5.9) (but its version presented in [10] has the same constraints as the master model). Models of quark–squark alignment are constrained by (5.7).

6. Physics Above $M$

6.1. Landau Poles

The explanation of the physics responsible for the hierarchy in the quark sector parameters is now complete. It involves two scales (beyond the electroweak breaking scale): the scale of spontaneous $\mathcal{H}$-breaking, $\langle S \rangle$ (this might happen in several scales), and the higher scale at which the information is communicated to the observed quarks, $M$. Physics above the scale $M$ has no direct bearing on the quark parameters. It may, however, further constrain the scale $M$.

These constraints on the scale $M$ are a result of the running of the coupling constants: we do not allow Landau poles below the Planck scale $M_P$. Landau poles may arise when we add massive supermultiplets that transform non-trivially under $SU(3)_C \times SU(2)_L \times U(1)_Y$. In our full framework, as described in sections 2–5, we have, in addition to the representations of the minimal supersymmetric Standard Model, the extra heavy quarks required for the FN mechanism. To calculate the running of the coupling constants up to the Planck scale, we need to know also the particle representations and the gauge structure between $M$ and $M_P$. If we adopt the most conservative approach, namely that the gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$ up to $M_P$, we get a lower bound on the scale $M$, that we denote by $M_{\text{min}}$. Alternatively, for a given scale $M$ we can get an upper bound on the scale at which the gauge symmetry has to increase – below the location of the Landau pole. Below, we present this upper bound corresponding to $M \sim 250$ TeV and denote it by $M^G_L$.

In our framework, the one-loop running of the three gauge couplings (neglecting
threshold effects) is given by

\[
[\alpha_s(M_P)]^{-1} - [\alpha_s(m_Z)]^{-1} + \frac{7}{2\pi} \ln \frac{M_{\text{SUSY}}}{m_Z} + \frac{3}{2\pi} \ln \frac{M}{M_{\text{SUSY}}} + \frac{3 - N_3}{2\pi} \ln \frac{M_P}{M},
\]
\[
[\alpha_2(M_P)]^{-1} - [\alpha_2(m_Z)]^{-1} + \frac{3}{2\pi} \ln \frac{M_{\text{SUSY}}}{m_Z} - \frac{1}{2\pi} \ln \frac{M}{M_{\text{SUSY}}} - \frac{1 + N_2}{2\pi} \ln \frac{M_P}{M},
\]
\[
[\alpha_1(M_P)]^{-1} - [\alpha_1(m_Z)]^{-1} - \frac{41}{20\pi} \ln \frac{M_{\text{SUSY}}}{m_Z} - \frac{33}{10\pi} \ln \frac{M}{M_{\text{SUSY}}} - \frac{33 + N_1}{10\pi} \ln \frac{M_P}{M},
\]  

(6.1)

where

\[
N_3 = 2N_P + N_U + N_D,
\]
\[
N_2 = 3N_P + N_L,
\]
\[
N_1 = N_P + 8N_U + 2N_D + 3N_L + 6N_E,
\]  

(6.2)

with \(N_P\) the number of mirror quark doublets, \(N_U\) the number of mirror up-quark singlets, \(N_D\) the number of mirror down-quark singlets, \(N_L\) the number of mirror lepton doublets and \(N_E\) the number of charged lepton singlets. For the gauge couplings at the scale \(m_Z\), we take

\[
[\alpha_s(m_Z)]^{-1} \approx 9, \quad [\alpha_2(m_Z)]^{-1} \approx 30, \quad [\alpha_1(m_Z)]^{-1} \approx 59.
\]  

(6.3)

(Note that \([\alpha_1(m_Z)]^{-1} = \frac{3}{5} [\alpha(m_Z)]^{-1} \cos^2 \theta_W\) is defined differently from \(\alpha'\).)

The requirement that there is no Landau pole below \(M_P\) gives the following bounds:

| \(N_3\) | \(M_{\text{min}}[\text{TeV}]\) | \(M_L^{SU(3)}[\text{TeV}]\) |
|---|---|---|
| 5 | 1 |  |
| 6 | \(5 \cdot 10^2\) | \(2 \cdot 10^{15}\) |
| 7 | \(4 \cdot 10^4\) | \(1 \cdot 10^{12}\) |
| 8 | \(1 \cdot 10^6\) | \(1 \cdot 10^{10}\) |
| 9 | \(1 \cdot 10^7\) | \(7 \cdot 10^8\) |
2. No Landau Poles in $\alpha_2$

| $N_2$ | $M_{\text{min}}[\text{TeV}]$ | $M_{L}^{SU(2)}[\text{TeV}]$ |
|-------|-----------------|-----------------|
| 5     | $3 \cdot 10^2$  | $1 \cdot 10^{16}$ |
| 6     | $5 \cdot 10^4$  | $1 \cdot 10^{14}$ |
| 7     | $2 \cdot 10^6$  | $5 \cdot 10^{12}$ |
| 8     | $3 \cdot 10^7$  | $4 \cdot 10^{11}$ |
| 9     | $3 \cdot 10^8$  | $4 \cdot 10^{10}$ |

3. No Landau Poles in $\alpha_1$

| $N_1$ | $M_{\text{min}}[\text{TeV}]$ | $M_{L}^{U(1)}[\text{TeV}]$ |
|-------|-----------------|-----------------|
| 15    | 0.3             |                 |
| 16    | 3               |                 |
| 17    | 26              |                 |
| 18    | $2 \cdot 10^2$  |                 |
| 19    | $9 \cdot 10^2$  | $8 \cdot 10^{15}$ |
| 20    | $4 \cdot 10^3$  | $4 \cdot 10^{15}$ |
| 25    | $1 \cdot 10^6$  | $3 \cdot 10^{14}$ |
| 30    | $6 \cdot 10^7$  | $3 \cdot 10^{13}$ |

6.2. Could There Be Low Energy Flavor Physics?

The reason that we have studied the various constraints on the scales so carefully is that we consider the following question as highly important: Could the flavor physics, responsible for the hierarchy in the quark sector parameters, be directly observable? Without going into a detailed discussion of the possible signatures, let us just estimate for now that, in order that the New Physics is observed, at least the lowest of the new scales (typically the scale below which the horizontal symmetry is completely broken) should be at the few TeV region. Explicitly, we ask whether we could have $\langle S_2 \rangle \lesssim 2$ $\text{TeV}$ ($\langle S \rangle \lesssim 2$ $\text{TeV}$ for a single $Z_n$). As $\langle S_2 \rangle / M \gtrsim \lambda^3$ in our various models, we should check whether $M \lesssim 250$ $\text{TeV}$
Examining the three Tables above we see that to have low energy flavor physics (and no modified gauge symmetry below $M_P$), we need $N_3 \leq 6$, $N_2 \leq 5$ and $N_1 \leq 18$. These constraints are very difficult to satisfy. For example, we can only allow $N_P \leq 1$ and $N_U \leq 2$. It is straightforward to see that in all our models, to explain $m_u \ll m_c \ll m_t$, at least three massive up quarks are required; to explain $m_d \ll m_s \ll m_b \ll m_t$ with $\tan \beta \sim 1$ at least five massive down quarks are required; and if we wish to explain the hierarchy in lepton masses in a similar way, a large number of massive charged leptons (at least six with $\tan \beta \sim 1$) is required as well. Then it is hopeless, under our assumptions, to have low energy flavor physics.

One way out of this grim conclusion is to give up the most speculative part in our theory, for example, allow a change in the gauge structure not too far above $M$. Then, we should not worry about Landau poles. The constraints from FCNC are much weaker and allow many of our models to have flavor physics at the TeV scale. For example, in all models of quark–squark alignment, if we ignore the Landau poles constraints, $M$ of order $10 \, \text{TeV}$ and $\langle S \rangle$ below $\text{TeV}$ are allowed.

But for now, let us adopt our full framework, namely spontaneously broken $\mathcal{H}$ at a scale $\langle S_i \rangle$, mirror quarks at $M$, and neither new particles nor new gauge structure above $M$ and up to $M_P$. Then, to allow low energy flavor physics, we have to give up some of the ingredients in our models that necessitated the large number of massive supermultiplets. First, we should better work with $\mathcal{H} = Z_m \times Z_n$ symmetry and $\epsilon_1 \sim \lambda^2$, $\epsilon_2 \sim \lambda^3$ breaking parameters. This, as explained in subsection 5.1, allows lower powers of $\epsilon_i$ in the determinant and hence fewer massive quarks. However, in addition we have to modify our estimates of two parameters:

(i) $m_u = 0$.

What we mean here is not that the bare mass of the up quark is highly suppressed but finite - that would require many more massive $U$s. We need a bare mass that is exactly zero (up to non-perturbative QCD effects that would generate its value as determined from meson masses). Then, the FN mechanism should account for $m_c \ll m_t$ only, which can be done with a single massive $U$ or $P$. 

32
(ii) \( \tan \beta \sim m_t/m_b \).

For a large \( \tan \beta \), the FN mechanism is not responsible for \( m_b/m_t \). To account for \( m_d \ll m_s \ll m_b \), only three massive Ds or Ps are needed. (For recent discussions of how to naturally produce \( \tan \beta \gg 1 \), see [21].)

Let us give an explicit example [10]. Take the model presented in subsection 2.2 and modify it to the case \( m_u = 0, \tan \beta \sim m_t/m_b \). This can be done, for example, by modifying the charges in (2.21) to

\[
\begin{array}{cccccccc}
Q_1 & Q_2 & Q_3 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\
(0,1) & (1,0) & (0,0) & (2,-1) & (0,0) & (0,0) & (-1,1) & (-1,1) & (0,0)
\end{array}
\] (6.4)

The mass matrices have the order of magnitude entries

\[
M^d \sim \langle \phi_d \rangle \begin{pmatrix}
\epsilon_1^2 & \epsilon_2 & \epsilon_2 \\
0 & \epsilon_1 & \epsilon_1 \\
0 & 1 & 1
\end{pmatrix},
M^u \sim \langle \phi_u \rangle \begin{pmatrix}
0 & 0 & \epsilon_2 \\
\epsilon_2 & \epsilon_2 & \epsilon_1 \\
0 & 0 & 1
\end{pmatrix}.
\] (6.5)

Note that \( M^u \) has a zero eigenvalue. Then

\[
\det M^d \sim \langle \phi_d \rangle^3 \epsilon_1^3, \quad m_c m_t \sim \langle \phi_u \rangle^2 \epsilon_2,
\] (6.6)

imply that a full theory could be constructed with a single \( U \) and three \( D \). We choose to construct the massive sector with one doublet \( P \) and three \( D \):

\[
P (1, -1) \quad D_1 (1, 0) \quad D_2 (0, 1) \quad D_3 (-1, 1).
\] (6.7)

The 4 \times 4 matrix \( M^u_{\text{full}} \) with the fourth row (column) corresponding to \( P(\bar{P}) \) is

\[
M^u_{\text{light}} \sim \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \langle S_2 \rangle \\
0 & 0 & \langle \phi_u \rangle & 0 \\
0 & \langle \phi_u \rangle & 0 & M
\end{pmatrix} \implies M^u_{\text{light}} \sim \langle \phi_u \rangle \begin{pmatrix}
0 & 0 & 0 \\
0 & \epsilon_2 & 0 \\
0 & 0 & 1
\end{pmatrix},
\] (6.8)

while the 6 \times 6 matrix \( M^d_{\text{full}} \) with rows corresponding to \( (Q_i, D_i) \) and columns to \( (\bar{d}_i, \bar{D}_i) \) is

\[
M^d_{\text{full}} \sim \begin{pmatrix}
0 & 0 & 0 & 0 & \langle \phi_d \rangle & 0 \\
0 & 0 & 0 & \langle \phi_d \rangle & 0 & 0 \\
0 & \langle \phi_d \rangle & \langle \phi_d \rangle & 0 & 0 & 0 \\
0 & \langle S_1 \rangle & \langle S_1 \rangle & M & 0 & 0 \\
0 & \langle S_2 \rangle & \langle S_2 \rangle & 0 & M & \langle S_1 \rangle \\
\langle S_1 \rangle & 0 & 0 & 0 & 0 & M
\end{pmatrix},
\] (6.9)
leading to $M_{\text{light}}^d$ of eq. (6.5).

In this model,

$$N_3 = 5, \quad N_2 = 3 + N_L, \quad N_1 = 7 + 3N_L + 6N_E. \quad (6.10)$$

The constraints from the Landau poles do not exclude $M$ in the few hundreds TeV region, though with $N_L = 2$ and $N_E = 1$ the bound would rise to 900 TeV. The Landau poles constraints are even weaker for the model of [10] with $m_u = 0$ and $\tan \beta \sim m_t/m_b$: there we could use the FN mechanism with one massive doublet and two massive down singlets, which would allow the addition of three massive charged leptons with $M \sim 300$ TeV. However, in the latter case the bounds from four Fermi operators are between 250 TeV (from $\Delta m_K$) and 3000 TeV (from $\epsilon_K$), depending on the phase.

To summarize, models with $H \subset U(1)_H \times U(1)_X$ and breaking parameters $\epsilon_1 \sim \lambda^2$, $\epsilon_2 \sim \lambda^3$, are viable candidates for low energy flavor physics ($\langle S_2 \rangle \sim 2$ TeV) provided that (i) $m_u = 0$ (at high energies) and (ii) $\tan \beta \sim m_t/m_b$. Scanning all such models, we found only two examples where, under special circumstances, $\langle S_2 \rangle$ can be at the TeV scale: the model of subsection 2.2, if lepton masses do not all come from FN mechanism, and the model of reference [10], if CP violating phases in box diagrams involving the massive quarks are suppressed. Also, if the gauge structure changes above $M$, then many of our models could, in principle, have their flavor physics at low energy.

7. Small Hierarchy From Large Hierarchy

Let us assume that the horizontal symmetry is

$$Z_n \subset U(1)_H \times U(1)_X. \quad (7.1)$$

If $U(1)_X$ is broken by a term $S^{n-2}(\phi_u \phi_d)/M_p^{n-3}$ in the superpotential $W$, then all the following terms will also appear in $W$:

$$W = \frac{1}{M_p^{n-3}} \sum_{m=0}^{[n/2]} A_m S^{n-2m}(\phi_u \phi_d)^m, \quad (7.2)$$

34
where $A_m$ are dimensionless coefficients of order one. This contributes to the Higgs potential, $V^W = |\partial W/\partial S|^2 + |\partial W/\partial \phi_u|^2 + |\partial W/\partial \phi_d|^2$:

$$V^W = \frac{1}{M_P^{2n-6}} \sum_{m,k=0}^{[n/2]} A_mA_k \left\{ (n-2m)(n-2k)S^{2(n-m-k-1)}(\phi_u\phi_d)^{m+k} + mkS^{2(n-m-k)}(\phi_u\phi_d)^{m+k-2}(\phi_u^2 + \phi_d^2) \right\}. \tag{7.3}$$

In addition there are SUSY $D$-terms,

$$V^D = D_d|\phi_d|^4 + D_u|\phi_u|^4 + D_{ud}|\phi_u|^2|\phi_d|^2, \tag{7.4}$$

soft SUSY breaking terms analytic in the fields,

$$V^\tilde{m} = \frac{\tilde{m}}{M_P^{n-3}} \sum_{m=0}^{[n/2]} B_m S^{n-2m} (\phi_u\phi_d)^m. \tag{7.5}$$

and $A$-terms,

$$V^A = \tilde{m}^2(C_s|S|^2 + C_u|\phi_u|^2 + C_d|\phi_d|^2). \tag{7.6}$$

$D_i, B_i$ and $C_i$ are all dimensionless coefficients of order one, and $\tilde{m}$ is the SUSY breaking scale.

The extremum equations,

$$\frac{\partial V}{\partial S} = 0, \quad \frac{\partial V}{\partial \phi_u} = 0, \quad \frac{\partial V}{\partial \phi_d} = 0. \tag{7.7}$$

have a solution of the form:

$$\langle S \rangle \sim M_P \left( \frac{\tilde{m}}{M_P} \right)^{-\frac{1}{n-2}}, \tag{7.8}$$

$$\langle \phi_u \rangle \sim \langle \phi_d \rangle \sim \tilde{m}. \tag{7.9}$$

It is easy to check that, even though the potential for the various scalars is very flat, all scalars acquire masses larger than or of order $\tilde{m}$. There might, however, be cosmological problems with such scalars whose couplings are very weak.

This solution has some attractive features:

(i) The so-called $\mu$-problem in SUSY is solved. The horizontal symmetry forbids a term $\phi_u\phi_d$ in the superpotential. Equation (7.3) implies that the electroweak breaking scale is naturally at the SUSY breaking scale.
(ii) Out of the large hierarchy between the SUSY breaking scale \( \tilde{m} \) and the Planck scale \( M_P \), we can naturally produce a smaller hierarchy of scales \( \langle S_i \rangle \), as can be seen from (7.8).

As an example, consider the models of reference [10], described in subsection 2.2. There we need hierarchy of scales:

\[
\langle S_2 \rangle : \langle S_1 \rangle : M \sim 1 : 5 : 125.
\]

(7.10)

Let us assume that the scale \( M \) is the spontaneous breaking scale of some additional discrete symmetry by a vacuum expectation value of a scalar field \( \langle S_3 \rangle \). Then, if the full discrete symmetry is \( Z_7 \times Z_6 \times Z_{10} \), we get from (7.8)

\[
\begin{align*}
\langle S_2 \rangle & \sim M_P \left( \frac{\tilde{m}}{M_P} \right)^{1/4} \sim 10^{15} \text{GeV}, \\
\langle S_1 \rangle & \sim M_P \left( \frac{\tilde{m}}{M_P} \right)^{1/5} \sim 6 \times 10^{15} \text{GeV}, \\
\langle S_3 \rangle & \sim M_P \left( \frac{\tilde{m}}{M_P} \right)^{1/8} \sim 10^{17} \text{GeV},
\end{align*}
\]

(7.11)

consistent with (7.10).

Note that if the relevant large scale is indeed \( M_P \), as we assumed in this subsection, it is very difficult to produce a low energy horizontal symmetry. With \( n = 3 \) we get \( \langle S \rangle = \tilde{m} \), but for \( n \geq 4 \) we get \( \langle S \rangle \geq \sqrt{\tilde{m} M_P} \sim 10^8 \text{ TeV} \).

It could in principle be that the scale in the denominator of the non-renormalizable terms of the Higgs potential is lower than \( M_P \). It could even be that there is a ladder of scales, one for each broken symmetry. This would of course enable one to produce a hierarchy of relatively low scales.

8. Conclusions

Abelian horizontal symmetries could explain in a simple and natural way the smallness of the quark sector parameters and the hierarchy among them. The master model, presented in subsection 2.1, demonstrates that a simple Abelian group, with a single not-so-small breaking parameter, and “reasonable” horizontal charges, could account for the fact that the hierarchy in the quark sector parameters spans five orders of magnitude.
It would be much more difficult, however, to have convincing evidence that such a symmetry is indeed responsible for the hierarchy. As far as the quark mass ratios and mixing angles are concerned, the symmetry explains eight order of magnitude relations but predicts only one. The fact that this single prediction is indeed consistent with present measurements is encouraging but can hardly be taken as evidence for the horizontal symmetry idea.

The simplest way in which the existence of a horizontal symmetry could be revealed is by direct observation of new particles related to flavor physics. Though not absolutely necessary, we find that it is likely that the mechanism that produces the hierarchy in the quark parameters requires the existence of extra scalars with flavor changing couplings and massive mirror quarks. A crucial question is then whether these new particles could have masses at scales accessible to experiment, say a few $TeV$. We find that this is not a very likely possibility but not impossible. If we are fortunate to have flavor physics at low energies, it probably means that the bare mass of the up quark vanishes (thus providing a solution to the strong CP problem) and that $\tan \beta$ is large.

Whatever scale we associate with the New Physics, it may have many other interesting consequences:

(i) A horizontal symmetry could align quark mass matrices with squark mass-squared matrices in a precise enough way to suppress SUSY contributions to neutral meson mixing. If squarks are found, and if they are non-degenerate, a horizontal symmetry is almost unavoidable. Another crucial test to the quark – squark alignment mechanism is that $D \to \bar{D}$ mixing should be close to the present experimental upper bound.

(ii) An Abelian horizontal symmetry could lead to an exact relation between the parameters, $m_s^2/m_b^2 = |V_{cb}V_{ub}/V_{us}|$.

(iii) A horizontal discrete symmetry has a natural mechanism to generate the hierarchy among scales of spontaneous symmetry breaking. It can solve in a simple way the $\mu$-problem of supersymmetry and provide a natural explanation to the fact that the electroweak breaking scale is close to the SUSY breaking scale.

---

6 We should note here that if $H$ is spontaneously broken at low energies, there might be cosmological problems with domain walls.
While none of these possibilities is a necessary consequence of a horizontal symmetry, they may provide support to the idea that the hierarchy in the quark sector parameters is a result of such a symmetry.

Acknowledgements

It is a pleasure to thank T. Banks, A. Dabholkar, M. Dine, K. Intriligator, D. Kaplan, A. Nelson, P. Pouliot and S. Shenker for several useful discussions. This work was supported in part by DOE grant #DE-FG05-90ER40559. YN is an incumbent of the Ruth E. Recu Career Development chair, and is supported in part by the Israel Commission for Basic Research, by the United States – Israel Binational Science Foundation (BSF), and by the Minerva Foundation.

Appendix A. Diagonalizing the Quark Mass Matrices

We would like to estimate the elements of the diagonalizing matrices $V_M^q$ ($M = L$ or $R$, $q = u$ or $d$) of the mass matrices as defined in (2.26). We follow the formalism of reference [22] with some modifications which are appropriate for our case. We present the $V_L^q$ matrices as

$$V_L^q = \begin{pmatrix}
1 & -s_{12}^q & 0 \\
s_{12}^{q*} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -s_{13}^q \\
0 & 1 & 0 \\
s_{13}^{q*} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -s_{23}^q \\
0 & s_{23}^{q*} & 1
\end{pmatrix}. \quad (A.1)
$$

$V_R^u$ is given by a similar formula, with $s_{12}^u$ replaced by $s_{12}^{u*}$. $V_R^d$ is

$$V_R^d = \begin{pmatrix}
1 & -s_{12}^d & 0 \\
s_{12}^{d*} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -s_{13}^d \\
0 & 1 & 0 \\
s_{13}^{d*} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23}^d & -s_{23}^d \\
0 & s_{23}^{d*} & c_{23}^d
\end{pmatrix}. \quad (A.2)
$$

The difference between $V_R^d$ and the other $V$’s stems from the fact that we allow $M_{i2}^d \sim M_{i3}^d$ ($i = 1, 2, 3$), as is the case in many of our models. Below we give the leading contributions to the diagonalizing parameters $s_{jk}^1$ and $s_{jk}^2$ in terms of the mass matrices.

Starting with the up sector, we assume that there is a hierarchy between all rows and all columns and define:

$$y_{ij}^u = \frac{M_{ij}^u}{M_{33}^u}. \quad (A.3)$$

38
Following [22], we also introduce the notation:

\[
\tilde{y}_{22}^u = y_{22}^u y_{33}^u - y_{23}^u y_{32}^u, \quad (A.4)
\]

\(|\tilde{y}_{22}| = m_e/m_t\). Then, the \(s_{ij}^u\) mixing angles are

\[
\begin{align*}
    s_{12}^u &= \frac{y_{12}^u}{\tilde{y}_{22}^u} + \frac{y_{11}^u y_{21}^u}{|\tilde{y}_{22}^u|^2} - \frac{y_{13}^u (y_{32}^u + y_{23}^u y_{22}^u)}{|\tilde{y}_{22}^u|^2} - \frac{y_{11}^u y_{31}^u (y_{32}^u + y_{32}^u y_{22}^u)}{|\tilde{y}_{22}^u|^2}, \\
    s_{13}^u &= y_{13}^u + y_{11}^u y_{31}^u + y_{12}^u (y_{32}^u + y_{22}^u y_{23}^u) + y_{11}^u y_{21}^u (y_{23}^u + y_{22}^u y_{32}^u), \\
    s_{23}^u &= y_{23}^u + y_{22}^u y_{32}^u.
\end{align*}
\]

The \(s_{ij}^u\) mixing angles are given by formulae similar to \((A.5)\), with the replacement \(y_{ij}^u \leftrightarrow y_{ji}^u\).

Turning to the down sector, we define

\[
\begin{align*}
    y_{i1}^d &= \frac{M_{i1}^d}{\sqrt{|M_{22}^d|^2 + |M_{33}^d|^2}}, \\
    y_{i2}^d &= \frac{M_{i2}^d M_{33}^d - M_{i3}^d M_{32}^d}{|M_{22}^d|^2 + |M_{33}^d|^2}, \\
    y_{i3}^d &= \frac{M_{i3}^d M_{33}^d + M_{i2}^d M_{32}^d}{|M_{22}^d|^2 + |M_{33}^d|^2}. \quad (A.6)
\end{align*}
\]

Then, the \(s_{ij}^d\) mixing angles are

\[
\begin{align*}
    s_{12}^d &= \frac{y_{12}^d}{y_{22}^d} + \frac{y_{11}^d y_{21}^d}{y_{22}^d} - \frac{y_{13}^d y_{23}^d}{y_{22}^d} - \frac{y_{11}^d y_{31}^d y_{23}^d}{y_{22}^d}, \\
    s_{13}^d &= y_{13}^d + y_{11}^d y_{31}^d + y_{12}^d y_{22}^d y_{23}^d + y_{11}^d y_{21}^d y_{23}^d, \\
    s_{23}^d &= y_{23}^d.
\end{align*}
\]

The \(s_{ij}^d\) mixing angles are

\[
\begin{align*}
    s_{12}^d &= \frac{y_{12}^d}{y_{22}^d} + \frac{y_{11}^d y_{21}^d}{|y_{22}^d|^2} - \frac{y_{13}^d y_{23}^d}{|y_{22}^d|^2} - \frac{y_{11}^d y_{31}^d y_{23}^d}{|y_{22}^d|^2}, \\
    s_{13}^d &= y_{13}^d + y_{11}^d y_{31}^d + y_{12}^d y_{22}^d y_{23}^d + y_{11}^d y_{21}^d y_{23}^d, \\
    s_{23}^d &= \frac{M_{32}^d}{\sqrt{|M_{32}^d|^2 + |M_{33}^d|^2}} + y_{22}^d y_{23}^d, \\
    c_{23}^d &= \frac{M_{33}^d}{\sqrt{|M_{32}^d|^2 + |M_{33}^d|^2}}. \quad (A.8)
\end{align*}
\]
If \( M_{32} = 0 \) (as happens in most of our models), one may alternatively define: 
\[
y_{ij}^d = M_{ij}^d / M_{33}^d.
\]
Then (A.7) and (A.8) still hold, except the last equation in (A.8) which should be replaced by \( c_{23}^d = 1 \). We stress that the above formulae give the leading contribution to the mixing parameters (when expanding in powers of \( \lambda \sim 0.2 \)), and should not be used to extract the next to leading terms, as these may have further contributions.

The CKM matrix elements are then given, to leading order, by:
\[
|V_{us}| = |s_{12}^d - s_{12}^u|, \\
|V_{cb}| = |s_{23}^d - s_{23}^u|, \\
|V_{ub}| = |s_{13}^d - s_{13}^u - s_{12}^u(s_{23}^d - s_{23}^u)|.
\]

(A.9)

Appendix B. Subtlety in the kinetic terms

So far we ignored the potential renormalization of the kinetic terms.\(^7\) The canonical kinetic terms can be modified to
\[
\sum_{q,i,j} R_{ij}^q q_i^\dagger \gamma^\mu \partial_\mu q_j
\]
(B.1)

\((q = Q, \bar{u}, \bar{d} \text{ and } i = 1, 2, 3)\) in a way consistent with the horizontal symmetry (we assume \( \mathcal{H} = U(1)_{H_1} \times U(1)_{H_2} \) with two explicit breaking parameters \( \epsilon_1 \) and \( \epsilon_2 \)):
\[
R_{ij}^q \sim \epsilon_1 |H_1(q_i) - H_1(q_j)| \epsilon_2 |H_2(q_i) - H_2(q_j)|.
\]

(B.2)

The mass terms in the Lagrangian, \( M_{ij}^u \) and \( M_{ij}^d \), are constrained by the symmetries:
\[
M_{ij}^d \sim \epsilon_1 H_1(Q_i) + H_1(\bar{d}_i) \epsilon_2 H_2(Q_i) + H_2(\bar{d}_i)
\]
(B.3)

when both \( H_1(Q_i) + H_1(\bar{d}_i) \geq 0 \) and \( H_2(Q_i) + H_2(\bar{d}_i) \geq 0 \) and zero otherwise and
\[
M_{ij}^u \sim \epsilon_1 H_1(Q_i) + H_1(\bar{u}_i) \epsilon_2 H_2(Q_i) + H_2(\bar{u}_i)
\]
(B.4)

when both \( H_1(Q_i) + H_1(\bar{u}_i) \geq 0 \) and \( H_2(Q_i) + H_2(\bar{u}_i) \geq 0 \) and zero otherwise.

\(^7\) We thank A. Dabholkar for raising this subject.
We can rescale the $q_i$’s and mix different $q_i$’s with the same horizontal charges to set all diagonal elements of $R^q$ to one and all off diagonal elements with $H_1(q_i) - H_1(q_j) = H_2(q_i) - H_2(q_j) = 0$ to zero. Then

$$R^q_{ij} = \delta_{ij} + r^q_{ij} \quad \text{(B.5)}$$

where $r^q_{ij} = O(\epsilon_1, \epsilon_2)$.

In order to find the true mass matrices, the fields $q_i$ should be redefined:

$$q_i = V^q_{ij} q'_i \quad \text{(B.6)}$$

where $V^q$ satisfy

$$V^q V^q\dagger = (R^q)^{-1}. \quad \text{(B.7)}$$

(The matrix $R^q$ is hermitian and positive definite. Therefore, eq. (B.7) has a solution. The ambiguity in the solution under multiplication of $V^q$ from the right by a unitary transformation can be fixed by imposing that $V^q$ is hermitian.) The true mass matrices are then

$$M'^d = (V^Q)^T M^d V^\dagger; \quad M'^u = (V^Q)^T M^u V^\dagger. \quad \text{(B.8)}$$

We should now ask: What are the consequences of the distinctions between $M$ and $M'$?

To answer the question we have to analyze the situation more carefully. Clearly,

$$V^q = (\sqrt{1 + r^q})^{-1} = 1 - \frac{1}{2} r^q + O((r^q)^2). \quad \text{(B.9)}$$

The matrix elements of $V^q$ are of the same order of magnitude as those of $R^q$. To show that, we should make sure that no element of a power of $r^q$ is larger than that of $r^q$. This fact follows from

$$\langle r^q \rangle_{ij}^2 \sim \sum_k \epsilon_1 |H_1(q_i) - H_1(q_k)| \epsilon_2 |H_2(q_i) - H_2(q_k)| \epsilon_1 |H_1(q_k) - H_1(q_j)| \epsilon_2 |H_2(q_k) - H_2(q_j)|$$

$$\lesssim \sum_k \epsilon_1 |H_1(q_i) - H_1(q_j)| \epsilon_2 |H_2(q_i) - H_2(q_j)| \sim r^q_{ij}. \quad \text{(B.10)}$$

We conclude that

$$R^q_{ij} \sim V^q_{ij}. \quad \text{(B.11)}$$
Consider now the master model with only a single $U(1)$ and all charges non-negative. Without loss of generality we can limit ourselves to $M^d$:

\[ M^d_{ij} = \sum_{lk} V^Q_{li} M^d_{lk} V_{kj} \sim \sum_{lk} \lambda^{|H(Q_i) - H(Q_l)|} \lambda^{H(Q_i) + H(\bar{d}_k)} \lambda^{|H(\bar{d}_j) - H(\bar{d}_k)|} \sim \lambda^{H(Q_i) + H(\bar{d}_j)}, \tag{B.12} \]

where in the last step only terms in the sum over $l, k$ with both $H(Q_i) - H(Q_l) \geq 0$ and $H(\bar{d}_j) - H(\bar{d}_k) \geq 0$ contribute. We conclude that the numbers of order one in $M$ can change but the order of magnitude is unchanged.

Next, consider more complicated models with two $U(1)$’s. By expressing $\epsilon_1$ and $\epsilon_2$ in terms of $\lambda$ and using (B.12), it is clear that the order of magnitude of the various entries in $M$ cannot be modified. The only danger is that we might lift some of the zeros in $M$. Again, without loss of generality we can limit ourselves to $M^d$:

\[ M^d_{ij} = \sum_{lk} V^Q_{li} M^d_{lk} V_{kj} \sim \sum_{lk} \epsilon_1^{H_1(Q_i) + H_1(\bar{d}_k)} \epsilon_2^{H_2(Q_i) + H_2(\bar{d}_k)} \epsilon_1^{H_1(\bar{d}_j) - H_1(\bar{d}_k)} \epsilon_2^{H_2(\bar{d}_j) - H_2(\bar{d}_k)}, \tag{B.13} \]

where the sum over $l, k$ is restricted to terms with both

\[ H_1(Q_i) + H_1(\bar{d}_k) \geq 0; \quad \text{and} \quad H_2(Q_i) + H_2(\bar{d}_k) \geq 0. \tag{B.14} \]

Every term in the sum is smaller than or equal to $\epsilon_1^{H_1(Q_i) + H_1(\bar{d}_j)} \epsilon_2^{H_2(Q_i) + H_2(\bar{d}_j)}$ with equality only when

\[ H_1(Q_i) - H_1(Q_l) \geq 0, \]
\[ H_2(Q_i) - H_2(Q_l) \geq 0, \]
\[ H_1(\bar{d}_j) - H_1(\bar{d}_k) \geq 0, \]
\[ H_2(\bar{d}_j) - H_2(\bar{d}_k) \geq 0. \tag{B.15} \]

Using (B.14) and (B.13) it is easy to see that this is possible only when both $H_1(Q_i) + H_1(\bar{d}_j) \geq 0$ and $H_2(Q_i) + H_2(\bar{d}_j) \geq 0$, i.e. only when $M^d_{ij} \neq 0$. However, in the case of interest with $M^d_{ij} = 0$, all the terms in the sum in (B.13) are smaller than the value in the master model $\epsilon_1^{H_1(Q_i) + H_1(\bar{d}_j)} \epsilon_2^{H_2(Q_i) + H_2(\bar{d}_j)}$ by at least by one power of $\max(\epsilon_1, \epsilon_2)$.

We conclude that the renormalization of the kinetic terms can modify the numbers of order one but cannot lift the zeros in the mass matrix to their master model value.
In the aligned model we needed much better accuracy. The zeros were not allowed to be lifted to a high power of $\lambda$. Fortunately, no more work is needed in these cases. The matrices $R^d$ are of the same order as the squark mass matrices $\frac{1}{m}\tilde{M}^2$, and the matrices $V^q$ are of the order of the matrices which diagonalize the squark mass matrices, $\tilde{V}^q$. Therefore, the true $V_L^q$ and $V_R^q$ which diagonalize the quark mass matrices are of the same order as the $K$ matrices of equation (2.28) and the discussion in subsection 2.3 is not modified.

Clearly, when the $U(1)$ symmetries are replaced by discrete symmetries which lift the zeros a more careful analysis is needed.

Finally, we mention that if the high energy theory is of the FN type, then further suppression of the deviation of $R^d$ from a unit matrix may occur. Detailed examination of the tree diagrams that lead to renormalization of the kinetic terms reveals that it is proportional to at least one power of $\epsilon_i$ (it could be either $\epsilon_1$ or $\epsilon_2$) and at least one power of $\epsilon_j^\dagger$ (which, again, could be either $\epsilon_1^\dagger$ or $\epsilon_2^\dagger$). This suppression may go well beyond the naive horizontal suppression that we assumed in this appendix, making the effects of renormalization of kinetic terms entirely negligible.
References

[1] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.
[2] H. Georgi and I.N. McArthur, Harvard preprint HUTP-81/A011 (1981), unpublished; D.B. Kaplan and A.V. Manohar, Phys. Rev. Lett. 56 (1986) 2004.
[3] T. Banks, Y. Nir and N. Seiberg, unpublished.
[4] G. ’tHooft, Lecture at the Cargese Summer Institute, (1979)
[5] A. Antaramian, L.J. Hall and A. Rasin, Phys. Rev. Lett. 69 (1992) 1871; L.J. Hall and S. Weinberg, Phys. Rev. D48 (1993) R979.
[6] P. Pouliot and N. Seiberg, Rutgers preprint RU-93-39 (1993), hep-ph/9308363, Phys. Lett. in press.
[7] Y. Nir and N. Seiberg, Phys. Lett. B309 (1993) 337.
[8] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B147 (1979) 277.
[9] S. Dimopoulos, Phys. Lett. 129B (1983) 417; J. Bagger, S. Dimopoulos, E. Masso and M. Reno, Nucl. Phys. B258 (1985) 565; J. Bagger, S. Dimopoulos, H. Georgi and S. Raby, In: Proc. Fifth Workshop on Grand Unification. Eds. Kang, K., Fried, H. and Frampton, P., Singapore, World Scientific (1984); Z.G. Berezhiani, Phys. Lett. B129 (1983) 99, B150 (1985) 177; A. Davidson, V.P. Nair and K.C. Wali, Phys. Rev. D29 (1984) 1505; A. Davidson and K.C. Wali, Phys. Rev. Lett. 60 (1988) 1313; A. Davidson, S. Ranfone and K.C. Wali, Phys. Rev. D41 (1990) 208.
[10] M. Leurer, Y. Nir and N. Seiberg, Nucl. Phys. B398 (1993) 319.
[11] N. Seiberg, Rutgers preprint RU-93-45 (1993), hep-ph/9309335.
[12] K. Hikasa et al., Particle Data Group, Phys. Rev. D45 (1992) S1.
[13] I.P.J. Shipsey (CLEO Collaboration), a talk given in the International Europhysics Conference on High Energy Physics, Marseille 1993.
[14] R. Barbieri and R. Gatto, Phys. Lett. 110B (1982) 211; J. Ellis and D.V. Nanopoulos, Phys. Lett. 110B (1982) 44; H.P. Nilles, Phys. Rep. 110 (1984) 1; F. Gabbiani and A. Masiero, Nucl. Phys. B322 (1989) 235.
[15] V.S. Kaplunovsky and J. Louis, Phys. Lett. B306 (1993) 269; R. Barbieri, J. Louis and M. Moretti, Phys. Lett. B312 (1993) 451.
[16] M. Dine, A. Kagan and S. Samuel, Phys. Lett. B243 (1990) 250; M. Dine and A. Nelson, Phys. Rev. D48 (1993) 1277.
[17] M. Dine, A. Kagan and R. Leigh, SCIIPP-93/04 (1993), hep-ph/9304299.
[18] Y. Nir and D. Silverman, Nucl. Phys. B345 (1990) 301.
[19] H.E. Haber and R. Hempfling, Phys. Rev. Lett. 66 (1991) 1815; Y. Okada, M. Yamaguchi and T. Yanagida, Prog. Theo. Phys. 85 (1991) 1; J. Ellis, G. Ridolfi and F. Zwirner, Phys. Lett. B257 (1991) 83.
[20] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1992) 77.
[21] L.J. Hall, R. Rattazzi and U. Sarid, LBL preprint LBL-33997 (1993), hep-ph-9306309; A. Nelson and L. Randall, San Diego preprint UCSD-PTH-93-24 (1993), hep-ph-9308277.

[22] L.J. Hall and A. Rasin, Phys. Lett. **B315** (1993) 164.