Density of States of a d-wave Superconductor in the Presence of Strong Impurity Scatterers: a Non Perturbative Result

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Abstract

We present a method to compute the density of states induced by a finite density of non magnetic impurities in a d-wave superconductor, in the unitary limit of very strong scattering centers. For frequencies very small as compared to the superconducting gap ($\omega \ll \Delta_0$) the additional density of states has the leading divergence $\delta \rho(\omega) \simeq n_i / \left( |2\omega| \ln^2 |\omega/\Delta_0| \right)$. This result is non perturbative.

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As a consequence of the breakdown of Anderson’s theorem, when impurity scattering violates the symmetry of the superconducting state, the superconducting energy gap is depleted and impurities act as strong pair breakers. This is the case in s-wave superconductors with magnetic impurities for which there is creation of bound states in the gap. Since they are not in the same symmetry representation, non magnetic impurities act as pair breakers in unconventional superconductors with higher orbital momentum such as d-wave superconductors. As a result, the measured low temperature properties of YBaCu$_2$O$_{7-4}$ displays a remarkable sensitivity to the presence of non magnetic impurities: the critical temperature, for example, is suppressed even for very small density of impurities. Furthermore the d-wave superconductor is special, due to the presence of gap nodes which prevents the complete freezing of scattering processes at low energy.

The standard method to treat these problems of disorder combines the T-matrix approximation with standard impurity averaging techniques. For three dimensional systems such as polar superconductors or heavy Fermion superconductors the standard perturbative approach is reliable. In the limit of low impurity concentration $n_i$, a perturbative expansion in $n_i$ leads to a finite density of states at the chemical potential. For two dimensional systems (to which it is believed the high-$T_C$ cuprates belong) the standard procedure of averaging over impurities may be complicated by the appearance of logarithmic singularities in the perturbative expansion of the single electron self-energy. Such a situation appears in a variety of two dimensional systems characterized by a Dirac-like canonical spectrum. This remark has cast some doubt upon the validity of perturbative expansions in $n_i$ because at each loop-level logarithmic divergences prevent the series to converge. Nevertheless self-consistent versions of the standard averaging technique have been performed showing a finite density of states at zero energy. On the other hand, some non perturbative methods have been used to treat a weak disorder potential. The solution then depends on the symmetry of the pure system. In the special case of the d-wave superconductor the density of states still vanishes at low energy in the presence of disorder. Recently Ref. concluded that the density of states is finite above a very low energy scale (essentially the level spacing of a localization volume), below which a pseudo-gap appears. Early numerical simulations seem to confirm finite density of states at zero energy.

The issue of finite density of states is crucial for the conduction properties in the disordered compound. Indeed, if there exist states in the gap, possible anomalous overlaps between well separated impurities can induce a new conduction mechanism entirely through impurity wave functions in a so-called ‘impurity band’ and may lead to delocalization. On the other hand, the results of Ref. concluded that the states are localized.

Here, we reexamine the issue of the density of states in a dirty d-wave superconductor. We consider the limiting case of a dilute concentration $n_i$ of identical impurities in the unitary limit. In other words, each impurity is a strong s-wave scatterer which is represented by an infinitely strong point-like repulsive potential (infinite scattering potential $V_0$) whose position is random. Using standard perturbative techniques, previous studies concluded that the density of states was finite at the Fermi energy. In contrast, using non-perturbative techniques, we find that the density of states is singular at the Fermi energy, $\rho(\omega) = n_i/\left(2|\omega|\ln^2|\omega/\Delta_0|\right)$. The origin of this singular density of states is the existence of impurity bound states at the Fermi energy ($E_F = 0$). It has been shown that one single impurity in the unitary limit creates a bound state at $E = 0$, with a spatial envelope that
decays as $1/(R \ln R)$. For many impurities these states overlap, but our result indicates that a singular density of states remains at zero energy of the form $\rho(\omega) = n_i/\left(2|\omega| \ln^2 |\omega/\Delta_0|\right)$.

In this paper, we introduce a new method to calculate the leading divergence in the density of states induced by $N$ non magnetic impurities in a system of two dimensional Dirac fermions in the unitary limit. We will outline the general features of the proof and present a different derivation than in our previous work. The question of averaging will be discussed in details and we show how the result is exact provide the average over the different configurations of impurities is performed at the end of the calculation. Our result can be applied to several systems of Dirac fermions in two dimensions. We explicitly treat the d-wave superconductor.

The paper is organized as follows. In section I we introduce the model of BCS d-wave superconductor, diagonalize it and define notations. In section II we establish the T-matrix equation starting from the equations of motion. We also introduce the matrix $\hat{M}$ as the inverse of the T-matrix. In section III we prove a sum-rule useful to calculate the density of states. Section IV is dedicated to a detailed study of the structure and matrix elements of $\hat{M}$. Section V contains the heart of the proof. We first discuss the effect of taking the unitary limit. Then we indicate which are the quantities where the divergence appears, leading to the singular density of state that we found. In the conclusion we discuss how our result relates to the different theories of disordered superconductors.

I. THE MODEL

The generic Hamiltonian for a d-wave superconductor can be written

$$H_0 = \sum_k \phi_k^\dagger [\varepsilon_k \sigma_3 + \Delta_k \sigma_1] \phi_k.$$  
(1)

It describes BCS quasiparticles with the kinetic energy $\varepsilon_k = W (\cos k_x + \cos k_y) - \mu$ ($\mu$ is the chemical potential) in the presence of the spin singlet superconducting order parameter $\Delta_k = \Delta_0 (\cos k_x - \cos k_y)$. Distances are measured in units of the lattice constant. The $\sigma_i$ are the Pauli matrices in the particle-hole space

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

The spinor $\phi_k^\dagger = (c_{k,\uparrow}^\dagger, c_{-k,\downarrow})$ creates a particle and a hole with momenta $k$ and $-k$, respectively. We shall present our results in terms of a $d_{x^2-y^2}$ state even though our conclusions apply more generally to any state where $\Delta_k$ vanishes linearly along a direction parallel to the Fermi surface.

Instead of using the Nambu formalism, we work with the diagonalized version of (1) in order to access directly the properties of quasiparticles. The Bogoliubov transformation that diagonalizes $H_0$ is given by

$$c_{k,\uparrow} = u_k \alpha_k - v_k \beta_k, \quad \alpha_k = u_k c_{k,\uparrow}^\dagger + v_k c_{-k,\downarrow}^\dagger,$$

$$c_{-k,\downarrow} = v_k \alpha_k + u_k \beta_k, \quad \beta_k = -v_k c_{k,\uparrow}^\dagger + u_k c_{-k,\downarrow}^\dagger.$$  
(2)
where $\alpha_k$ and $\beta_k$ create a particle and a hole with momentum $k$ and the coefficients $u_k$ and $v_k$ satisfy

\begin{align*}
    u_k^2 &= \frac{1}{2} \left(1 + \frac{\varepsilon_k}{\omega_k}\right), \\
    v_k^2 &= \frac{1}{2} \left(1 - \frac{\varepsilon_k}{\omega_k}\right), \\
    u_k v_k &= \frac{\Delta_k}{2\omega_k},
\end{align*}

with $\omega_k = \sqrt{\varepsilon_k^2 + \Delta_k^2}$. Given the short-hand notation

\begin{align*}
    \psi_{k,0}^\dagger &\equiv \alpha_k^\dagger, \\
    \psi_{k,1}^\dagger &\equiv \beta_k^\dagger,
\end{align*}

the BCS Hamiltonian can now be rewritten

$$H_0 = \sum_k \sum_{\nu=0,1} \omega_k (-1)^{\nu} \psi_{k,\nu}^\dagger \psi_{k,\nu}.$$

The disorder is introduced through $N$ repulsive scalar potentials $V_0$ located at random positions in the lattice:

$$H_I = V_0 \sum_{i=1}^N \sum_{\sigma=\uparrow,\downarrow} c_{i\sigma}^\dagger c_{i\sigma}.$$

The full BCS Hamiltonian $H = H_0 + H_I$ describes a dirty d-wave superconductor.

II. THE T-MATRIX EQUATION

A. The Hamiltonian

With the help of a Fourier transformation to the reciprocal lattice, the impurity potential becomes

$$H_I = \frac{V_0}{\mathcal{V}} \sum_{i,k,k'} e^{i(k-k') \cdot R_i} \left(c_{i\sigma}^\dagger c_{i\sigma}^\dagger + c_{i\sigma}^\dagger c_{i\sigma}^\dagger\right),$$

where $\mathcal{V}$ is the volume of the system. Rewriting the impurity term in terms of quasiparticles gives

$$c_{k\sigma}^\dagger c_{k'\sigma}^\dagger = u_k u_k' \alpha_k^\dagger \alpha_k' - v_k u_k' \beta_k^\dagger \alpha_k' - u_k v_k' \alpha_k^\dagger \beta_k' + v_k v_k' \beta_k^\dagger \beta_k'.$$

Thus

$$c_{k\sigma}^\dagger c_{k'\sigma}^\dagger = \sum_{\nu,\nu'} (-1)^\nu (-1)^{\nu'} t_{k\sigma} \psi_{k\nu}^\dagger \psi_{k'\nu'}^\dagger,$$

where we have introduced the short-hand notation.
Similarly,
\[ c_{-k}^\dagger c_{-k'} = - \left( v_k v_{k'} \alpha_k^\dagger \alpha_{k'} + v_k u_{k'} \beta_k^\dagger \alpha_{k'} + u_k v_{k'} \alpha_k^\dagger \beta_{k'} + u_k u_{k'} \beta_k^\dagger \beta_{k'} \right) + \text{const.} , \]
and, neglecting the constant term, we get
\[ c_{-k}^\dagger c_{-k'} = - \sum_{\nu, \nu'} t_{k\nu+1} t_{k'\nu'+1} \psi_{k\nu}^\dagger \psi_{k'\nu'} . \]

In summary, the random BCS Hamiltonian can be written
\[ H = \sum_{k, \nu} \omega_k (-1)\nu \psi_{k, \nu}^\dagger \psi_{k, \nu} + \frac{V_0}{V} \sum_i \sum_{k, k', \nu, \nu'} e^{i(k-k') \cdot R_i} \left[ (-1)\nu (-1)\nu' t_{k\nu} t_{k'\nu'} - t_{k\nu+1} t_{k'\nu'+1} \right] \psi_{k\nu}^\dagger \psi_{k'\nu'} . \]

**B. Equations of motion**

As the impurities break translation invariance, the anomalous two-point function
\[ G_{kq}^{\nu \nu'}(\tau) = \langle T_\tau [\psi_{k\nu}(\tau) \psi_{q\nu'}^\dagger(0)] \rangle \]
depends on two momenta. The equations of motion are
\[ - \sum_{q, m} L_{kq}^{\nu m} G_{qk}^{m \nu'} = \delta_{kk'} \delta_{\nu \nu'} , \]
where \( L_{kk'}^{\nu \nu'} \) is
\[ L_{kk'}^{\nu \nu'} = \left[ \partial_\tau + (-1)\nu \omega_k \right] \delta_{kk'} \delta_{\nu \nu'} + \frac{V_0}{V} \sum_i e^{i(k-k') \cdot R_i} \left[ (-1)\nu (-1)\nu' t_{k\nu} t_{k'\nu'} - t_{k\nu+1} t_{k'\nu'+1} \right] \psi_{k\nu}^\dagger \psi_{k'\nu'} . \]

We are dealing with a problem of non-interacting particles scattered by the static potential generated by \( N \) impurities. As we shall see, the anomalous Green function in Eq. (11) can be solved by inverting a \( 2N \times 2N \) matrix. To this end, define the one-point functions
\[ G_{k\nu}^0(\tau) \equiv \frac{-1}{\partial_\tau + (-1)\nu \omega_k} , \]
\[ g_{k\nu}^1(R_i) \equiv (-1)\nu e^{-i k \cdot R_i} t_{k\nu} G_{k\nu}^0 , \]
\[ g_{k\nu}^2(R_i) \equiv e^{-i k \cdot R_i} t_{k\nu+1} G_{k\nu}^0 , \]
together with

\[
A_{ij} = \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot (R_i - R_j)} (t_{k\nu})^2 G_{k\nu}^0 ,
\]

(16)

\[
C_{ij} = \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot (R_i - R_j)} (t_{k\nu+1})^2 G_{k\nu}^0 ,
\]

(17)

\[
B_{ij} = \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot (R_i - R_j)} (-1)^n t_{k\nu} t_{k\nu+1} G_{k\nu}^0 .
\]

(18)

The integration over \( k \) is to be performed over the first Brillouin zone. Equation (13) is inserted into the equations of motion (12). We then solve a \( 2N \times 2N \) system of linear equations (see appendix [3]). This gives

\[
G_{kk'}^{\nu\nu'} = G_{kk'}^0 - \frac{V_0}{V} N_{-k\nu}^T \hat{M}^{-1} \cdot N_{k\nu'} ,
\]

(19)

with \( N_{k\nu} \) a vector made of the 2\( N \) components

\[
N_{k\nu} \equiv \left( N_{k\nu}^1 \right) , \quad N_{k\nu}^1 \equiv \begin{pmatrix} g_{k\nu}(R_1) \\ \vdots \\ g_{k\nu}(R_N) \end{pmatrix} , \quad N_{k\nu}^2 \equiv \begin{pmatrix} g_{k\nu}(R_1) \\ \vdots \\ g_{k\nu}(R_N) \end{pmatrix} ,
\]

(20)

and \( \hat{M} \) is a \( 2N \times 2N \) matrix defined by

\[
\hat{M} = \begin{bmatrix} -\hat{I} + V_0 \hat{A} & V_0 \hat{B} \\ V_0 \hat{B} & \hat{I} + V_0 \hat{C} \end{bmatrix} ,
\]

(21)

with \( \hat{A}, \hat{B}, \hat{C} \) are \( N \times N \) matrices whose matrix elements are respectively \( A_{ij}, B_{ij} \) and \( C_{ij} \). \( \hat{I} \) is the identity matrix. With these definitions the T-matrix equation can be written

\[
G_{kk'}^{\nu\nu'} = G_{kk'}^0 + N_{-k\nu}^T \hat{T} N_{k\nu'} ,
\]

(22)

with \( \hat{T} = -\frac{V_0}{V} \hat{M}^{-1} \).

III. A SUM RULE FOR THE DENSITY OF STATES

The increment in the density of states induced by the impurities is

\[
\delta \rho(\omega) = -\frac{1}{\pi} Im \sum_{k\nu} \delta G_{kk}^{\nu\nu}(\omega + i0^+) ,
\]

(23)

where

\[
\delta G_{kk}^{\nu\nu} = -\frac{V_0}{V} N_{-k\nu}^T \hat{M}^{-1} \cdot N_{k\nu} .
\]

(24)

Recall that the summation over \( k \) is restricted to the first Brillouin zone. We can rewrite
\[ \delta G_{kk}^{\nu\nu} = \frac{V_0}{V} \sum_{i,j} N_{-k\nu}^i M_{ij}^{-1} N_{k\nu}^j . \]

If we go to Matsubara frequencies and define

\[ G_{k\nu}^0 = \frac{1}{i\omega_n - (-1)^\nu \omega_k} , \]
we notice that

\[ \sum_{k\nu} (-V_0 N_{k\nu}^j N_{-k\nu}^i) = \frac{\partial}{\partial i\omega_n} \hat{M}_{ij} , \]

and then

\[ \sum_{k\nu} \delta G_{kk}^{\nu\nu}(i\omega_n) = \frac{1}{V} Tr \left[ \hat{M}^{-1} \frac{\partial \hat{M}}{\partial i\omega_n} \right] , \]

where the Trace is assumed to run over \( k, \nu \) as well as over the matrix indices. By convention we call

\[ \delta G \equiv \sum_{k\nu} \delta G_{kk}^{\nu\nu} . \]

The equation (26) can also be written \((\ln \text{Det} = \text{Tr} \ln)\)

\[ \delta G(i\omega_n) = \frac{1}{V} \frac{\partial \left( \ln \text{Det} \hat{M} \right)}{\partial i\omega_n} . \]

We used this expression in a previous paper\(^2\) in order to derive the additional density of states. In this paper we will prefer to use the following expression

\[ \delta G(i\omega_n) = \frac{1}{V} Tr \left[ \hat{M}^{-2} \frac{\partial \hat{M}^2}{2 \partial i\omega_n} \right] . \]

**IV. THE MATRIX ELEMENTS OF \( \hat{M} \)**

As seen previously the evaluation of the density of states relies on calculating the determinant of \( \hat{M} \). We begin by evaluating its matrix elements. In this section, we just quote the result; the explicit calculation being given in Appendix \[^3\]. In order to perform this calculation we make the assumption that the energy \( \omega \ll \Delta_0 \). This will enable us to linearize the spectrum for small energies. Second we assume that \( W = \Delta_0 \) and the chemical potential \( \mu = 0 \) in order to simplify the calculation. We will see later what happens when these two conditions are relaxed. Under these conditions there are four nodes in the Brillouin zone (cf. figure \[^1\]) located at \((\pm \frac{\pi}{2}, \pm \frac{\pi}{2})\). With \( k' \equiv \left( \frac{\pi}{2}, \frac{\pi}{2} \right) + k \) we get
FIG. 1. Linearization of the energy spectrum at very low energy. In the first Brillouin zone we have four nodes centered at \((\pm \pi/2, \pm \pi/2)\) around which the energy spectrum is linearized. There are some symmetry relations between the four sub-zones called BZ 1, BZ 2, BZ 3 and BZ 4. The dotted line represents the points where \(\varepsilon_k = \cos k_x + \cos k_y = 0\). The linearized spectrum around the nodes is represented in a third dimension in the plane \((k_x, k_y)\).

\[
\omega_{k'}^2 = W^2 \left[ \left( \cos k'_x + \cos k'_y - \mu \right)^2 + \left( \cos k'_x - \cos k'_y \right)^2 \right] \\
\approx 2W^2k^2 .
\]  

(30)

Thus, \(\omega_k = Dk\) with \(D = \sqrt{2}W\). after integrating over the four nodes in the Brillouin zone, we find

\[
A_{ij} = A_0^0(R_{ij}) + A_1^1(R_{ij}) , \\
C_{ij} = A_0^0(R_{ij}) - A_1^1(R_{ij}) , \\
B_{ij} = B_1^1(R_{ij}) ,
\]

(31)

where

\[
A_0^0(R) \equiv i\omega_n \sum_k \frac{1}{(i\omega_n)^2 - \omega_k^2} e^{-ik \mathbf{R}} \\
= -F_0^0(R) \frac{i\omega_n}{2\pi D^2} K_0 \left( \frac{|R\omega_n|}{D} \right) ,
\]

(32)

whereby

\[
F_0^0(R) = 2 \cos \frac{\pi}{2}(R_x + R_y) + 2 \cos \frac{\pi}{2}(R_x - R_y) ,
\]

(33)

and \(K_0\) is the Bessel function of rank zero. Note that \(|\omega_n| = \sqrt{-(i\omega_n)^2}\). As shown in Appendix B, Eq. (B16),

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\[ A^1(R) \equiv \sum_k \frac{\varepsilon_k}{(\omega_n)^2 - \omega_k^2} e^{-ikR} \]
\[ = \mathcal{F}^1(R) \frac{\omega_n}{2\sqrt{2\pi D^2}} K_1 \left( \left| \frac{R\omega_n}{D} \right| \right), \tag{34} \]

with
\[ \mathcal{F}^1(R) = 2 \sin \frac{\pi}{2} (R_x + R_y)(\cos \varphi + \sin \varphi) + 2 \cos \frac{\pi}{2} (R_x - R_y)(\cos \varphi - \sin \varphi), \tag{35} \]

where \( \varphi \) is the angle between \( R \) and the x-axis and \( K_1 \) is the Bessel function of rank one.

In the same manner
\[ B^1(R) \equiv \sum_k \frac{\Delta_k}{(\omega_n)^2 - \omega_k^2} e^{-ikR} \]
\[ = \mathcal{F}^2(R) \frac{\omega_n}{2\sqrt{2\pi D^2}} K_1 \left( \left| \frac{R\omega_n}{D} \right| \right), \tag{36} \]

with
\[ \mathcal{F}^2(R) = 2 \sin \frac{\pi}{2} (R_x + R_y)(\cos \varphi - \sin \varphi) + 2 \cos \frac{\pi}{2} (R_x - R_y)(\cos \varphi + \sin \varphi). \tag{37} \]

Note that the point \( R = 0 \) is rather special with \( A^0(0) = \frac{4}{\pi D^2} \log |\omega_n/D| \) and with \( A^1(0) = B^1(0) = 0. \)

### V. EVALUATION OF THE DENSITY OF STATES

#### A. Unitary limit and low energies

In this section we will evaluate the leading term in the density of states in the limit of low frequencies. When \( 0 < |R|\omega_n| \ll D \) we have
\[ K_0 \left( \frac{|R\omega_n|}{D} \right) \simeq \ln \left( \frac{|R\omega_n|}{D} \right), \quad K_1 \left( \frac{|R\omega_n|}{D} \right) \simeq \frac{D}{|\omega_n|R}, \tag{38} \]
so that in the limit \( |\omega_n|/D \to 0 \) we have
\[ \left| A^0(R) \right| \ll \left| A^1(R) \right|. \tag{39} \]

In the limit of low frequencies and for \( R \neq 0 \), this enables us to neglect \( A^0(R) \) as compared to \( A^1(R) \) in the evaluation of the matrix \( \hat{M} \) in Eq. [24]. For \( R = 0 \), \( A^0(0) \) and \( A^1(0) \) are negligible as compared to \( A^1(R) \) for \( R \neq 0 \). In the sequel we can safely avoid the point \( R = 0 \) in summations over \( R \) that occur during the evaluation of \( \hat{M}^2 \). Second we notice that the Bessel function \( K_0 \) and \( K_1 \) have an exponential cut-off at \( R_{\text{max}} = D/|\omega_n| \) so that we can safely use the approximation:
\[ A^1(R) \simeq \begin{cases} \frac{\mathcal{F}^1(R)}{2\sqrt{2\pi D R_{ij}}} & \text{if } R < D/|\omega_n|, \\ 0 & \text{elsewhere,} \end{cases} \tag{40} \]
\[
B^1(R_{ij}) \simeq \begin{cases} 
\frac{\mathcal{F}^2(R_{ij})}{2\sqrt{2\pi}DR_{ij}}, & \text{if } R < D/|\omega_n|, \\
0, & \text{elsewhere}.
\end{cases}
\tag{41}
\]

An important remark to make is that the \(\omega\)-dependence of the matrix elements of \(\hat{M}\) appears only through the upper cut-off of the Bessel functions. In what follows, we make the crucial assumption of the unitary limit, i.e., that \(V_0 \to \infty\). Recalling the form of \(\hat{M}\) in equation (21) we see that in this limit the identity matrix in \(\hat{M}\) becomes negligible when compared to \(A_{ij}\) and \(C_{ij}\).

**B. The divergences appear**

Our aim is now to factorize the leading divergences in this problem. In order to make the divergence apparent, it is more convenient to work with

\[
\delta G(i\omega_n) = Tr \left[ \hat{M}^{-2} \frac{\partial \hat{M}^2}{\partial i\omega_n} \right].
\tag{42}
\]

From section V A, we expect logarithmic factors \(\ln |D/\omega_n|\) to appear. An important point to stress is that for any configuration of the impurities, we will always find some factors \(\ln |D/\omega_n|\) in \(\hat{M}^2\). Within the unitary approximation we have

\[
\hat{M}^2 = \left[ \hat{A}^2 + \hat{B}^2 \quad \hat{A}\hat{B} - \hat{B}\hat{A} \right].
\tag{43}
\]

To see that \(\ln |D/\omega_n|\) is necessarily present in the diagonal terms of \(\hat{M}^2\), we estimate

\[
M_{ii}^2 = \sum_j \left( A_{ij}A_{ji} + B_{ij}B_{ji} \right)
= \sum_j \frac{1}{(2\pi D)^2} \left[ (\mathcal{F}^1)^2 + (\mathcal{F}^2)^2 \right],
\tag{44}
\]

where the summation over \(j\) is restricted to \(0 < R_{ij} < D/|\omega_n|\). Provided the impurities are rather homogeneously scattered in the system (around each impurity site \(R_i\) one can find a macroscopic amount of impurities inside a circle of radius \(D/|\omega_n|\), cf. figure 2), we can take the continuous limit,

\[
\hat{M}_{ii}^2 \sim \frac{2\pi V_0}{(2\pi D)^2} \int_{D/|\omega_n|}^{D/|\omega_n|} \frac{(\mathcal{F}^1)^2 + (\mathcal{F}^2)^2}{R} \, dR.
\tag{45}
\]

Now \((\mathcal{F}^1)^2 + (\mathcal{F}^2)^2\) are oscillatory but always positive so for all impurity at site \(R_i\) we have

\[
\hat{M}_{ii}^2 = C \ln \frac{D}{|\omega_n|},
\tag{46}
\]

where \(C\) is a constant.
FIG. 2. The point $\mathbf{R}_i$ and $\mathbf{R}_j$ where the impurities are located are represented in this figure. The integrand in Eq. (45) centered around the point $\mathbf{R}_i$ is nonzero only when $R_{ij} < D/|\omega_n|$.

The biggest coefficients in $\hat{M}^2$ are situated on the diagonal. To see it, we distinguish between the off-diagonal elements that are diagonal in the particle-hole grading and those that are not. The magnitude of off-diagonal elements that are diagonal in the particle-hole grading can be written

$$\hat{M}^2_{ik} = \sum_j \frac{1}{2\pi D} \left[ \frac{F_{ij}^1 F_{jk}^1 + F_{ij}^2 F_{jk}^2}{R_{ij} R_{jk}} \right].$$

(47)

As $i \neq k$, $\hat{M}^2_{ik}$ picks up an oscillatory prefactor (a combination of $e^{\pm i\pi/2 R_{ik}} e^{\pm \varphi}$ as in eqns. (33), (35), (37)). Furthermore, the logarithmic divergence gets rescaled by $R_{ik}$:

$$\hat{M}^2_{ik} = C \text{Osc}(\mathbf{R}_{ik}) \ln \left| \frac{D R_{ik}}{\omega_n} \right|,$$

(48)

where $|\text{Osc}(\mathbf{R}_{ik})| \leq 1$. Hence for all sites $k \neq i$ we have

$$\left| \frac{\hat{M}^2_{ik}}{\hat{M}^2_{ii}} \right| \leq 1.$$

(49)

In turn, the terms in the off-diagonal blocks of $\hat{M}^2$ with respect to the particle-hole vanish. Indeed, if we denote the off-diagonal elements of $\hat{M}^2$ with respect to the particle-hole grading by

$$T_{ik} \equiv A_{ij} B_{jk} - B_{ij} A_{jk},$$

(50)

we have
FIG. 3. Vanishing of the element $T_{ik}$. The summation zone is the surface of intersection of the two disks. The first term in (51) is represented in the upper drawing whereas the second one is represented in the lower drawing. For each summation point $R_j$ in the upper drawing there is a symmetric one $\tilde{R}_j$ in the lower drawing such that $R_{ij} = \tilde{R}_{jk}$ and $\tilde{R}_{ij} = R_{jk}$.

$$T_{ik} = \sum_j \frac{1}{(2\pi D)^2} \left( \frac{F^1(R_{ij}) F^2(R_{jk})}{R_{ij} R_{jk}} - \frac{F^2(R_{ij}) F^1(R_{jk})}{R_{ij} R_{jk}} \right) ,$$

(51)

where here the sum runs over the points $R_j$ such that $R_{ij} < D/|\omega_n|$ and $R_{jk} < |\omega_n|$. Noticing the symmetry of $F^1$ and $F^2$ under the transformation $\phi \rightarrow \phi + \pi$, we can show that the two terms on the r. h. s. of Eq. (51) cancel identically. Indeed the integrands $\frac{F^1(R_{ij})}{R_{ij}}$ and $\frac{F^2(R_{ik})}{R_{ik}}$ are represented respectively within each circle of figure 3 (both of these terms have a cut-off at $R_{max} = D/|\omega_n|$). The summation zone is the surface of intersection of the two disks. The first term in (51) is represented in the upper drawing whereas the second one is represented in the lower drawing. For each summation point $R_j$ in the upper drawing there is a symmetric one $\tilde{R}_j$ in the lower drawing such that $R_{ij} = \tilde{R}_{jk}$ and $\tilde{R}_{ij} = R_{jk}$. Thus $T_{ik} = 0$.

In conclusion we find it convenient to define a matrix $\hat{S}$ by

$$\hat{M}^2 = C \ln \left| \frac{D}{\omega_n} \right| \hat{S} ,$$

(52)

where $C$ is the constant defined in (46), independent of disorder. The matrix $\hat{S}$ depends on the particular configuration of the impurities in the system. It satisfies

$$|S_{ij}| \leq 1 .$$

(53)
C. Asymptotic value of the density of states

Substituting the value of $\hat{M}^2$ from (52) into the equation (42) we find

$$\delta G(i\omega_n) = T_{\text{div}}(i\omega_n) + R(i\omega_n),$$

where $\langle \cdot \rangle$ denotes the average over disorder, $\hat{I}$ is the $2N \times 2N$ density matrix, and

$$T_{\text{div}}(i\omega_n) \equiv \frac{1}{2V} \frac{\partial}{\partial i\omega_n} \ln \ln \left( \frac{D}{|\omega_n|} \right) Tr \hat{I}, \quad |\text{cal}R(i\omega_n)| \equiv \frac{1}{2V} \langle Tr\hat{S}^{-1} \frac{\partial}{\partial i\omega_n} \hat{S} \rangle.$$  (55)

Then the first term in Eq. (54) $T_{\text{div}}$ is responsible for the singular density of states that we obtain. Indeed it gives rise to

$$T_{\text{div}}(i\omega_n) = \frac{N}{V} \frac{1}{i\omega_n} \ln \frac{|\omega_n|}{D},$$

where $N$ is the number of impurities. After analytic continuation (remember that $|\omega_n| = \sqrt{-(i\omega_n)^2}$, Eq. (B16) ) and assuming that the reminder $R$ in Eq. (54) is negligible, we get

$$\delta \rho(\omega) \simeq -\frac{1}{\pi} n_i Im \left[ \frac{1}{(\omega + i\delta) \ln \left( \frac{\omega+\delta}{D} \right)} \right],$$

and thus

$$\delta \rho(\omega) \simeq \frac{n_i}{2} \frac{1}{|\omega| \left[ \ln^2(|\omega|/D) + (\pi/2)^2 \right]},$$

where $n_i$ is the density of impurities in the system. We note that this expression is normalizable:

$$\int_{-D}^{D} \delta \rho(\omega)d\omega = 2n_i.$$  (59)

In order to prove the result (58) we still have to show that the reminder $R$ in (42) is negligible as compared to $T_{\text{div}}$. In order to do this we have to give some insight about the form of $\hat{S}^{-1}$.

D. The form of $\hat{S}^{-1}$

The matrix $\hat{S}$ is invertible (since $\hat{M}$ is invertible) and we will find a reasonable candidate to the inverse of $\hat{S}$ in order to give an estimation of the reminder $R$. Inverting $\hat{S}$ means we can find a matrix $\hat{S}^{-1}$ such that for any given pair of sites $i$ and $k$

$$\sum_j \hat{S}_{ij} \hat{S}_{jk}^{-1} = \delta_{ik}. \quad (60)$$

We introduce a pictorial representation of $\hat{S}$ by drawing a disk ( called $\hat{S}$-disk) of radius $|D/\omega_n|$ centered at $\mathbf{R}_i$. To each location $\mathbf{R}_j$ of an impurity there corresponds a matrix element $S_{ij}$ which depends on the vector $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$. From Eqs. (10,11) we recall that
FIG. 4. Volume of summation of $\sum_j \hat{S}_{ij} \hat{S}_{jk}^{-1}$ in two configurations. In (a) we have $i = k$ and in (b) we have $i \neq k$ but $i$ still very close to $k$.

\[
\begin{aligned}
|S_{ij}| &\leq 1, \quad \text{for } R_{ij} < |D/\omega_n|, \\
S_{ij} &= 0, \quad \text{for } R_{ij} \geq |D/\omega_n|,
\end{aligned}
\]

(61)

inside the disk, $\hat{S}$ has some non vanishing matrix elements $|S_{ij}| < 1$. Outside the disk, $S_{ij} = 0$. It’s important to note at this point that the only dependence on $\omega$ in the $\hat{S}$ matrix comes from the cut-off. In order to satisfy (60) we must presume that $\hat{S}^{-1}$ has the same cut-off $|D/\omega_n|$ as $\hat{S}$. Hence we represent again $S_{ik}^{-1}$ by a disk (called $\hat{S}^{-1}$-disk), but centered around $R_k$ this time. Now the summation $\sum_j \hat{S}_{ij} \hat{S}_{jk}^{-1}$ runs over the intersection of the $\hat{S}$-disk and the $\hat{S}^{-1}$-disk. The key difficulty in order to invert $\hat{S}$ is to find a matrix $\hat{S}^{-1}$, such that when the two disks have the same center we have

\[
\sum_j \hat{S}_{ij} \hat{S}_{ji}^{-1} = 1,
\]

whereas when the two centers differ, even by a small amount, we have

\[
\sum_j \hat{S}_{ij} \hat{S}_{jk}^{-1} = 0, \quad k \neq i.
\]

This is illustrated on figure 4 where on the left side (case (a)) the two circles are centered at the same point and on the right side (case (b)) the two centers differ by a tiny amount. In both cases the intersecting area of the two disks is almost identical, but in case (a) the result has to be 1 whereas in case (b) it has to be 0.

The matrix elements of $\hat{S}$ inside the disk are random, but the condition $|\hat{S}_{ij}| \leq 1$ is independent of the realization of disorder. The worst possible situation for differentiating between cases (a) and (b) is when all the matrix elements inside the $\hat{S}$-disk have their maximum value 1.

We define the external boundary of the $\hat{S}^{-1}$-disk. By external boundary we mean the circle exactly adjacent to the disk and external to it. Upon the external boundary the matrix elements of $\hat{S}^{-1}$ are defined as non vanishing and negative, so that the summation over it compensates the summation over the intersection of the $\hat{S}$-disk and the $\hat{S}^{-1}$-disk. In case (a) the external boundary doesn’t touch the $\hat{S}$-disk and thus has no effect upon the summation over the intersection of the $\hat{S}$-disk and the $\hat{S}^{-1}$-disk. Alternatively in case (b)
the summations over the intersection of the \( \hat{S} \)-disk and the \( \hat{S}^{-1} \)-disk and over the external boundary cancel out.

Let’s take an explicit example where \( S_{ij} = 1 \) if \( R_{ij} < |D/\omega_n| \) and \( S_{ij} = 0 \) elsewhere. We define \( \hat{S}^{-1}_{ij} \) in the following way. For \( R_{ij} < |D/\omega_n| \), \( \hat{S}^{-1}_{ij} \) is proportional to a random configuration of \( \pm 1 \) such that after integration over a disk of volume \( V = \pi |D/\omega_n|^2 \) we get

\[
\sum_{V} \pm 1 = \sqrt{V} \\
= \sqrt{\pi} \frac{D}{|\omega_n|} ,
\]

(62)

By the central limit theorem, there are many random configurations of \( \pm 1 \) that verify this condition. We then take the external boundary to be proportional to \((-1/\sqrt{\pi})\) with the same proportionality constant. Thus

\[
\begin{cases}
\hat{S}^{-1}_{ij} = A(\pm 1) , & \text{if } R_{ij} < |D/\omega_n| , \\
\hat{S}^{-1}_{ij} = -A \frac{1}{\sqrt{\pi}} , & \text{if } R_{ij} = |D/\omega_n| , \\
\hat{S}^{-1}_{ij} = 0 , & \text{if } R_{ij} > |D/\omega_n| ,
\end{cases}
\]

(63)

where \( A \) is a constant. We notice that when \( \mathbf{R}_i \) and \( \mathbf{R}_k \) are infinitely close, then

\[
\sum_{\text{boundary}} \hat{S}^{-1}_{ij} = -A \sqrt{\pi} \frac{D}{|\omega_n|} ,
\]

and exactly compensates the summation over the volume inside. The proportionality constant \( A \) is fixed so that

\[
\sum_j \hat{S}_{ij} \hat{S}^{-1}_{ij} = 1 ,
\]

thus \( A = \frac{|\omega_n|}{\sqrt{\pi}D} \).

In summary

\[
\begin{cases}
\hat{S}^{-1}_{ij} = \frac{|\omega_n|}{\sqrt{\pi}D} (\pm 1) , & \text{if } R_{ij} < |D/\omega_n| , \\
\hat{S}^{-1}_{ij} = -\frac{|\omega_n|}{\pi D} , & \text{if } R_{ij} = |D/\omega_n| , \\
\hat{S}^{-1}_{ij} = 0 , & \text{if } R_{ij} > |D/\omega_n| .
\end{cases}
\]

(64)

Now for each intermediate case where \( k \neq i \) (cf. figure 3) we want to be sure that the intersection of the \( \hat{S} \)-disk and the \( \hat{S}^{-1} \)-disk compensates the sum over the external boundary of \( \hat{S}^{-1} \) which crosses the \( \hat{S} \)-disk. This is obviously not the case for any configuration of random \( \pm 1 \) in \( \hat{S}^{-1} \) but we believe there is one (and actually only one because there is only one inverse for \( \hat{S} \)!) configuration which satisfies it for all positions of \( \mathbf{R}_i \) and \( \mathbf{R}_k \). As represented in figure 3, we call \( \theta \) the angle made by the center \( i \) with the two points \( A \) and \( B \),
intersection between the two circles. The area of summation is given by $A_\theta = \theta R^2 - 4R^2 \sin \theta$ and the intersecting arc’s length by $L_\theta = \theta R$. They both scale in respectively $R^2$ and $R$ with varying prefactors. So the desired configuration of random $\pm 1$ in the $\hat{S}^{-1}$-disk has to be “denser” towards the center of the circle than towards the boundary. We are not able to write down explicitly this configuration of random $\pm 1$ inside the area of the $\hat{S}^{-1}$-disk, but actually it doesn’t matter, because as we will see in the next paragraph the evaluation of the reminder doesn’t depend on it.

**E. Evaluation of the reminder $\mathcal{R}$**

Now we evaluate

$$\mathcal{R} = \frac{1}{2V} \langle Tr \hat{S}^{-1} \frac{\partial}{\partial i \omega_n} \hat{S} \rangle .$$

(65)

An important point is that as $\hat{S}$ depends on $\omega_n$ only via its boundary, $\partial \hat{S} / \partial \omega_n$ is a matrix with non zero values only on the external boundary of the $\hat{S}$-disk. explicitly, taking our example where $\hat{S}_{ij} = 1$ if $R_{ij} < D/|\omega_n|$ we get
FIG. 6. Summation over $j$ in $\frac{\partial S}{\partial \omega_n}$.

\[
\begin{cases}
- \frac{\partial \hat{S}}{\partial \omega_n} = 1, & \text{if } R_{ij} = D/|\omega_n|, \\
\frac{\partial S}{\partial \omega_n} = 0, & \text{elsewhere}.
\end{cases}
\]

(66)

Since the matrix elements in the $\hat{S}^{-1}$-disk have random (positive and negative) signs whereas upon the external boundary they have constant sign (negative here), the maximum value of $|\hat{S}^{-1}\partial \hat{S}/\partial \omega_n|$ is reached when $i = k$, that is when the external boundary of the $\hat{S}$-disk (where the matrix elements of $\partial \hat{S}/\partial \omega_n$ are non-vanishing) matches the external boundary of the $\hat{S}^{-1}$-disk (cf. Fig. 6).

In our special case we get

\[
|\hat{S}^{-1}\partial \hat{S}/\partial \omega_n| = \left(\frac{|\omega_n|}{\pi D}\right) \left(\frac{2\pi D}{|\omega_n|}\right),
\]

(67)

\[
= 2.
\]

Since $|\partial \hat{S}/\partial \omega_n|$ has its maximum value when all the matrix elements of $\hat{S}$ inside $\hat{S}$-disk equal 1, we have indeed evaluated an upper bound for the reminder $R$. The reminder $R$ is thus negligible compared to the leading divergence in the density of states in the limit $|\omega_n|/D \ll 1$.

VI. DISCUSSION

The first question to address is what happens when the different conditions under which our calculation was performed are relaxed. First consider the more realistic situation where the bandwidth $W$ and the superconducting gap $\Delta_0$ are not equal (experimentally, we have $\Delta_0 \approx 0.10 W$). According to Ref. 25 the power law dependence of the matrix elements of $\hat{M}$ (cf. equations (40) and (41)) is still preserved, but gets an overall prefactor of $\Delta_0/W$. The
scale under which our calculation is valid is now $|\omega| \ll V_0\Delta_0/W$. In addition, the hopping matrix $\hat{M}$ will show a strong spatial anisotropy. This anisotropy can be absorbed into the overall prefactors $\mathcal{F}^0$, $\mathcal{F}^1$ and $\mathcal{F}^2$ (resp. eqns. (33), (35) and (37)) entering the definition of the matrix elements $A_{ij}$, $C_{ij}$ and $B_{ij}$. Since only the square of these factors enter the leading divergence (cf. Eq. (44)), we believe the result stays unchanged.

What happens if the bandwidth are not symmetric anymore, that is if $\mu \neq 0$ but still $\mu \ll \Delta_0$?

First the nodes are moved away from the point $(\pm \pi/2, \pm \pi/2)$ so that transversal nodes are now separated by the vectors $\mathbf{Q} = (\pi(1 - \delta), \pi(1 - \delta))$ and $\mathbf{Q}^* = (-\pi(1 - \delta), \pi(1 - \delta))$, where $\delta = \mu/\Delta_0$ and $\mu$ is the increase in the chemical potential. This leads to a change of the phase factors in $A_{ij}$ and $B_{ij}$. Namely we get

$$\mathcal{F}^0(R) = 2 [\cos (\mathbf{Q} \cdot \mathbf{R}/2) + \cos (\mathbf{Q}^* \cdot \mathbf{R}/2)];$$

$$\mathcal{F}^1(R) = 2 [\sin (\mathbf{Q} \cdot \mathbf{R}/2) (\cos \varphi + \sin \varphi), - \sin (\mathbf{Q}^* \cdot \mathbf{R}/2) (\cos \varphi - \sin \varphi)];$$

and

$$\mathcal{F}^2(R) = 2 [\sin (\mathbf{Q} \cdot \mathbf{R}/2) (\cos \varphi - \sin \varphi), - \sin (\mathbf{Q}^* \cdot \mathbf{R}/2) (\cos \varphi + \sin \varphi)].$$

As $\delta$ is a small parameter, this change in the phase won’t affect the existence of the logarithmic divergence in $\hat{M}^2$. Additionally, away from half filling, the bands of quasi particles and quasi holes become asymmetric to account for the removing of particles in the system. The difference induced in $A_{ij}$, $B_{ij}$ and $C_{ij}$ comes from the highest part of the energy spectrum, where $k \approx 1$ and $\omega \approx D$ and shouldn’t affect our result.

Our solution is valid under the assumption of unitary limit, meaning that $V_0$ is the largest scale in the problem. As shown in ref. 22 to non interacting single impurities is associated the creation of a bound states decaying as $1/(R \ln R)$. Following Balatsky et al. (20) this would lead to delocalization due to the formation of an impurity band. The result we find for the density of states is indeed reminiscent of a Dyson-like singularity (in d=1, a Dyson-like singularity corresponds to a density of state diverging in $1/\left(|\omega| \ln^3(|\omega|/D)\right)$), associated in one dimension with delocalization.

What happens when the condition of unitarity is relaxed is still very much an open problem. In the case of weak disorder, some $\sigma$-model analysis have been performed concluding to a vanishing density of state under an energy scale $E_2 = D_F/\xi^2$, where $D_F$ is the bare diffusion constant and $\xi$ is the localization length. This result is strongly supported by symmetry considerations. Indeed, a disordered d-wave superconductor belongs to class $C_I$, according to the classification of ref. 23, meaning that the Hamiltonian is invariant under time-reversal symmetry as well as spin rotation symmetry. According to random matrix theory a universal behavior is expected, inducing a vanishing density of states on the scale of the level spacing induced by the finite localization length.

Consider one impurity with scattering potential $V_0$. The effective potential at the impurity site can be evaluated exactly and is given by $\tilde{V} = \frac{V_0}{1 - V_0 |\omega| \ln D/|\omega|}$. In the unitary limit ($V_0 \to \infty$) we get $\tilde{V} = \frac{V_0}{|\omega| \ln D/|\omega|}$. This effective potential diverges when $\omega$ goes to zero. On the other hand, we notice that the derivation of non linear sigma models for disordered systems requires Gaussian disorder, and especially require that the average effective disorder
potential vanishes $\langle V \rangle = 0$ (the second moment is nonzero). If $\langle V \rangle$ is nonzero but is a constant as a function of energy, it can be absorbed as a redefinition of the chemical potential, but the case where the effective potential would diverge as $\omega$ goes to zero belongs to another universality class: the energy of an effective non-linear sigma model would renormalized to $\omega - \Sigma(\omega)$, where $\Sigma(\omega)$ diverges when $\omega$ goes to zero. The energy scale under which the non-linear sigma model describes the diffusive modes then becomes inaccessible since the effective energy never gets close to zero. In our problem the result obtained on the density of states indicates that the self-energy is of the form $\Sigma(\omega) = \frac{1}{\omega \ln^2(\omega/D)}$, diverging as $\omega$ goes to zero, but still different from the one impurity case. The Feynmann diagrams leading to such this self-energy will be studied in a future publication. We believe the method presented here, using the T-matrix equation takes care in a non perturbative way of the leading divergence in the unitary limit. One possible scenario which would reconcile the two limits of weak and strong disorder is that the unitarity limit fixed point is unstable (as soon as $V_0$ becomes finite, the effective potential saturates), but the cross-over regime close to unitarity is still very much influenced by the strong disorder fixed point.

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APPENDIX A: DERIVATION OF THE T-MATRIX EQUATION

Starting from equation (12), replacing the lagrangian of (13) into it and multiplying from the left by $G^0$ gives

$$G_{kk'}^{
u
u'} + \frac{V_0}{\mathcal{V}} \sum_i e^{ikR_i} (-1)^\nu t_{kk'} G_{k\nu}^0 G_{k'\nu'}^1 (R_i)$$

$$+ \frac{V_0}{\mathcal{V}} \sum_i e^{ikR_i} t_{kk'} G_{k\nu}^0 G_{k'\nu'}^2 = G_{kk'}^0 \delta_{kk'} \delta_{\nu\nu'} , \quad (A1)$$

where

$$G_{k\nu}^1 (R_i) \equiv - \sum_q e^{-iqR_i} (-1)^m t_{qm} G_{qk}^{m\nu} ,$$

$$G_{k\nu}^2 (R_i) \equiv \sum_q e^{-iqR_i} t_{qm} G_{qk}^{m\nu} . \quad (A2)$$

Now we have two unknown functions $G^1$ and $G^2$ that we evaluate using formula (A1):

$$- G_{k\nu}^1 (R_j) + \frac{V_0}{\mathcal{V}} \sum_i \sum_q e^{iq(R_i-R_j)} (t_{qm})^2 G_{qm}^0 G_{k\nu}^1 (R_i)$$

$$+ \frac{V_0}{\mathcal{V}} \sum_i \sum_q e^{iq(R_i+R_j)} (-1)^m t_{qm} G_{k\nu}^0 G_{k\nu}^1 (R_i) = (-1)^\nu e^{-ikR_j} t_{kk'} G_{k\nu}^0 ; \quad (A3)$$

$$G_{k\nu}^2 (R_j) + \frac{V_0}{\mathcal{V}} \sum_i \sum_q e^{iq(R_i-R_j)} (-1)^m t_{qm} G_{k\nu}^0 G_{k\nu}^1 (R_i)$$

$$+ \frac{V_0}{\mathcal{V}} \sum_i \sum_q e^{iq(R_i+R_j)} (t_{qm})^2 G_{qm}^0 G_{k\nu}^2 (R_i) = e^{-ikR_j} t_{kk'} G_{k\nu}^0 . \quad (A4)$$

These two equations can be rewritten matricially as

$$(-\delta_{ij} + V_0 A_{ij}) G_{k\nu}^1 (R_j) + V_0 B_{ij} G_{k\nu}^2 (R_j) = N_{k\nu}^1 (R_j) ,$$

$$(\delta_{ij} + V_0 C_{ij}) G_{k\nu}^2 (R_j) + V_0 B_{ij} G_{k\nu}^1 (R_j) = N_{k\nu}^2 (R_j) . \quad (A5)$$

Define the $2N$ vector $V$

$$V \equiv \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} , \quad V_{k\nu}^1 \equiv \begin{pmatrix} G_{k\nu}^1 (R_1) \\ \vdots \\ G_{k\nu}^1 (R_N) \end{pmatrix} , \quad V_{k\nu}^2 \equiv \begin{pmatrix} G_{k\nu}^2 (R_1) \\ \vdots \\ G_{k\nu}^2 (R_N) \end{pmatrix} , \quad (A6)$$

and the equation (A5) can be written

$$\dot{M}V_{k\nu} = N_{k\nu} . \quad (A7)$$

But then equation (A1) becomes

$$G_{kk'}^{
u
u'} + \frac{V_0}{\mathcal{V}} N_{k\nu} \cdot V_{k\nu} = G_{k\nu}^0 \delta_{kk'} \delta_{\nu\nu'} . \quad (A8)$$

Insertion of (A7) yields

$$G_{kk'}^{
u
u'} = G_{kk'}^0 - \frac{V_0}{\mathcal{V}} N_{k\nu}^T \cdot \dot{M}^{-1} \cdot N_{k'\nu'} . \quad (A9)$$
APPENDIX B: CALCULATION OF THE MATRIX ELEMENTS \( A_{ij}, B_{ij} \) AND \( C_{ij} \)

\[
A_{ij} = \sum_n \int \frac{d^2 k}{(2\pi)^2} e^{-i \mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)} (t_{kn})^2 G_{kn}^0
\]

\[= \int \frac{d^2 k}{(2\pi)^2} e^{-i \mathbf{k} \cdot \mathbf{R}_{ij}} \left( \frac{u_k^2}{i\omega_n - \omega_k} + \frac{v_k^2}{i\omega_n + \omega_k} \right),
\]

where the integration runs over the first Brillouin zone. Since

\[u_k^2 = \frac{1}{2} \left( 1 + \frac{e_k}{\omega_k} \right),
\]

\[v_k^2 = \frac{1}{2} \left( 1 - \frac{e_k}{\omega_k} \right),
\]

we get

\[A_{ij} = A^0(\mathbf{R}_{ij}) + A^1(\mathbf{R}_{ij}),
\]

with

\[A^0(\mathbf{R}_{ij}) \equiv i\omega_n \int \frac{d^2 k}{(2\pi)^2} \frac{e^{-i \mathbf{k} \cdot \mathbf{R}_{ij}}}{(i\omega_n)^2 - \omega_k^2},
\]

\[A^1(\mathbf{R}_{ij}) \equiv \int \frac{d^2 k}{(2\pi)^2} \frac{\varepsilon_k}{(i\omega_n)^2 - \omega_k^2} e^{-i \mathbf{k} \cdot \mathbf{R}_{ij}}.
\]

Similarly

\[C_{ij} = A^0(\mathbf{R}_{ij}) - A^1(\mathbf{R}_{ij}),
\]

and

\[B_{ij} = \int \frac{d^2 k}{(2\pi)^2} \frac{\Delta_k}{(i\omega_n)^2 - \omega_k^2} e^{-i \mathbf{k} \cdot \mathbf{R}_{ij}}
\]

\[\equiv B^1(\mathbf{R}_{ji}).
\]

1. Evaluation of \( A^0(\mathbf{R}_{ji}) \)

As we can see on figure 1, the spectrum has four nodes at the points

\[P_1 = \left( \frac{\pi}{2}, \frac{\pi}{2} \right), \quad P_2 = \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad P_3 = \left( -\frac{\pi}{2}, -\frac{\pi}{2} \right), \quad P_4 = \left( \frac{\pi}{2}, -\frac{\pi}{2} \right).
\]

Under the assumptions \( \mu = 0 \) and \( \Delta_0 = W \), we can linearize the spectrum around each node in the following way:

\[\mathbf{k}' \equiv \left( \frac{\pi}{2}, \frac{\pi}{2} \right) + \mathbf{k}, \quad \omega_{k'}^2 = W^2 \left[ (\cos k'_x + \cos k'_y)^2 + (\cos k'_x - \cos k'_y)^2 \right],
\]

and we get

\[\omega_{k'}^2 \simeq D^2 k^2, \quad \text{with} \quad D = \sqrt{2}W.
\]
Similarly,
\begin{align}
\varepsilon_k' & \simeq W(k_x + k_y) , \\
\Delta_k' & \simeq (k_x - k_y) \Delta_0 .
\end{align} \tag{B11}

In order to evaluate $A^0(R)$ we divide the integral into a sum of four integrals around each node:
\begin{align}
A^0(R) &= \sum_{k' \in BZ_1} i\omega_n \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} + \sum_{k' \in BZ_2} i\omega_n \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} \\
&\quad + \sum_{k' \in BZ_3} i\omega_n \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} + \sum_{k' \in BZ_4} i\omega_n \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} . \tag{B12}
\end{align}

In each term, the fact of developing around a particular node gives a specific prefactor, so that we have
\begin{align}
A^0(R) &= F^0(R) \frac{e^{-i \varepsilon_k R}}{(i\omega_n)^2 - D^2 k^2} , \tag{B13}
\end{align}
with
\begin{align}
F^0(R) &= e^{-i \frac{\pi}{2} (R_x + R_y)} + e^{-i \frac{\pi}{2} (R_x - R_y)} + e^{i \frac{\pi}{2} (R_x + R_y)} + e^{i \frac{\pi}{2} (R_x - R_y)} . \tag{B14}
\end{align}

Thus
\begin{align}
F^0(R) &= 2 \cos \frac{\pi}{2} (R_x + R_y) + 2 \cos \frac{\pi}{2} (R_x - R_y) . \tag{B15}
\end{align}

Now calling $\theta$ the angle between $k$ and $R$ (cf. figure 3),
\begin{align}
A^0(R) &= F^0(R) (i\omega_n) \int_0^1 k \, dk \int_0^{2\pi} d\theta \frac{e^{-i k R \cos \theta}}{(i\omega_n)^2 - D^2 k^2} \\
&= -F^0(R) \frac{(i\omega_n)}{2\pi D^2} \int_0^1 k \, dk \frac{J_0(k R)}{(\omega_n / D)^2 + k^2} \\
&= -F^0(R) \frac{i\omega_n}{2\pi D^2} K_0 \left(R |\omega_n| / D \right) , \tag{B16}
\end{align}
where $K_0$ is the Bessel function of rank zero. Note that we have defined $|\omega_n| = \sqrt{-(i\omega_n)^2}$.

**2. Calculation of $A^1(R)$**

As previously, we can decompose the summation in the Brillouin zone into four parts:
\begin{align}
A^1(R) &= \sum_{k' \in BZ_1} \varepsilon_{k'} e^{-i \varepsilon_{k'} R} \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} + \sum_{k' \in BZ_2} \varepsilon_{k'} e^{-i \varepsilon_{k'} R} \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} \\
&\quad + \sum_{k' \in BZ_3} \varepsilon_{k'} e^{-i \varepsilon_{k'} R} \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} + \sum_{k' \in BZ_4} \varepsilon_{k'} e^{-i \varepsilon_{k'} R} \frac{e^{-i \varepsilon_k' R}}{(i\omega_n)^2 - \omega_{k'}^2} . \tag{B17}
\end{align}
If we call $\theta$ the angle between $\mathbf{k}$ and $\mathbf{R}$ and $\varphi$ the angle between $\mathbf{R}$ and the $x$-axis as represented on figure [7], the first term in (B17) can be written

$$T_1 \equiv \sum_{k' \in BZ_1} \varepsilon_{k'} e^{-i \mathbf{k'} \cdot \mathbf{R}} \frac{e^{-i \mathbf{k'} \cdot \mathbf{R}}}{(i \omega_n)^2 - \omega_{k'}^2} \int_0^{2\pi} d\theta \int_0^1 k \, dk \frac{D}{\sqrt{2(2\pi)^2}} \frac{(k_x + k_y)e^{-i k R \cos \theta}}{(i \omega_n)^2 - D^2 k^2}.$$  \hspace{1cm} (B18)

Using

$$k_x = k(\cos \theta \cos \varphi - \sin \theta \sin \varphi),$$
$$k_y = k(\cos \theta \sin \varphi + \sin \theta \sin \varphi),$$ \hspace{1cm} (B19)

we get

$$T_1 = e^{-i \frac{\pi}{4}(R_x + R_y)}(\cos \varphi + \sin \varphi) \frac{D}{\sqrt{2(2\pi)^2}} \int_0^{2\pi} d\theta \int_0^1 k \, dk \frac{e^{-i k R \cos \theta}}{(i \omega_n)^2 - D^2 k^2}.$$ \hspace{1cm} (B20)

Thus, after summing over the four nodes we get,

$$A^1(\mathbf{R}) = F^1(\mathbf{R}) \frac{\omega_n}{2\sqrt{2\pi}} \frac{1}{D^2} K_1 \left( \frac{R|\omega_n|}{D} \right),$$ \hspace{1cm} (B21)

with

$$F^1 = 2 \left[ \sin \frac{\pi}{2}(R_x + R_y)(\cos \varphi + \sin \varphi) + \sin \frac{\pi}{2}(R_x - R_y)(\cos \varphi - \sin \varphi) \right].$$ \hspace{1cm} (B22)

The evaluation of $B^1(\mathbf{R})$ is done in the same way as the one of $A^1(\mathbf{R})$. 

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