ADDITIVITY AND LINEABILITY IN VECTOR SPACES

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Abstract. Gámez-Merino, Munoz-Fernández and Seoane-Sepúlveda proved that if additivity $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable where $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$. They asked if $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$ can be weakened. We answer this question in negative. Moreover, we introduce and study the notions of homogeneous lineability number and lineability number of subsets of linear spaces.

1. Introduction

A subset $M$ of a linear space $V$ is $\kappa$-lineable if $M \cup \{0\}$ contains a linear subspace of dimension $\kappa$ (see [1], [2], [3], [22], [23], [26]). If additionally $V$ is a linear algebra, then in a similar way one can define algebraability of subsets of $V$ (see [2], [4], [5], [6], [7], [9], [10], [11], [12], [24], [25], [26]). If $V$ is a linear topological space, then $M \subseteq V$ is called spaceable (dense-lineable) if $M$ contains a closed infinitely dimensional subspace (dense subspace) (see [13], [14], [27]). The lineability problem of subsets of linear spaces of functions or sequences have been studied by many authors. The most common way of proving $\kappa$-lineability is to construct a set of cardinality $\kappa$ of linearly independent elements of $V$ and to show that any linear combination of them is in $M$.

We will concentrate on a non-constructive method in lineability. Following the paper [21], we will consider a connection between lineability and additivity. This method does not give a specific large linear space, but ensures that such a space exists.

Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$. The additivity of $\mathcal{F}$ is defined as the following cardinal number

$$\mathcal{A}(\mathcal{F}) = \min\{|F| : F \subseteq \mathbb{R}^\mathbb{R}, \varphi + F \not\subseteq F \text{ for every } \varphi \in \mathbb{R}^\mathbb{R}\} \cup \{(2^\mathfrak{c})^+\}.$$ 

The notion of additivity was introduced by Natkaniec in [29] and then studied by several authors [17], [16], [18], [19] and [30].

A family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ is called star-like if $a\mathcal{F} \subseteq \mathcal{F}$ for all $a \in \mathbb{R} \setminus \{0\}$. Gámez-Merino, Munoz-Fernández and Seoane-Sepúlveda proved the in [21] following result which connects the lineability and the additivity of star-like families.

Theorem 1.1. Let $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ be star-like. If $\mathfrak{c} < \mathcal{A}(\mathcal{F}) \leq 2^\mathfrak{c}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable.

The authors noted that there is a star-like family $\mathcal{F}$ such that $\mathcal{A}(\mathcal{F}) = 2$ and $\mathcal{F}$ is not 2-lineable. They asked if the above result is true for $2 < \mathcal{A}(\mathcal{F}) \leq \mathfrak{c}$. We will answer this question in negative. This will show that Theorem 1.1 is sharp.
Let us observe that the notion of additivity can be stated for abelian groups as follows. If \((G, +)\) is an abelian group and \(F \subseteq G\), then the additivity of \(F\) is the cardinal number 
\[
A(F) = \min(\{|F| : F \subseteq G \text{ and } \forall \varphi \in G(\varphi + F \not\subseteq F)\} \cup \{|G|\}).
\]

On the other hand, the lineability is a natural notion in vector spaces. We say that a set \(F \subseteq V\), where \(V\) is a linear space, is \(\lambda\)-lineable if there exists a subspace \(W\) of \(V\) such that \(W \subseteq F \cup \{0\}\) and \(\dim W = \lambda\). Now, the lineability \(L(F)\) is the cardinal number 
\[
L(F) = \min\{\lambda : F \text{ is not } \lambda\text{-lineable}\}.
\]
Clearly \(L(F)\) is a cardinal number less or equal to \((\dim V)^+\) and it can take any value between 1 and \((\dim V)^+ - \text{ see Proposition 2.4.}\)

2. Results

The following Lemma 2.1 and Theorem 2.2 are generalizations of [21, Lemma 2.2 and Theorem 2.4] in the settings of abelian groups and vector spaces over infinite fields, respectively. Short proofs of this facts are essentially the same as those in [21], but our presentation is more general and from the proof of Theorem 2.2 we extract a new notion of lineability, namely homogeneous lineability. Moreover the authors of [21] claimed that Theorem 1.1 held true also in the case \(c = A(F)\). However we will show (see Theorem 2.5 and Theorem 2.6) that this is not true. Let us remark that all examples of families \(F \subseteq \mathbb{R}^R\) discussed in [21] have additivity \(A(F)\) greater than \(c\), so the described mistake has almost no impact on the value of this nice paper.

**Lemma 2.1.** Let \((G, +)\) be an abelian group. Assume that \(F\) is a subgroup of \(G\) and \(F \subseteq G\) is such that

\((1)\) \[2|F| < A(F).\]

Then there is \(g \in F \setminus F\) with \(g + F \subseteq F\). That means actually that some coset of \(F\) different from \(F\) is contained in \(F\).

**Proof.** Let \(h \in G \setminus F\) and put \(F_h = (h + F) \cup F\). Then \(|F_h| = 2|F|\). By \((1)\) there is \(g \in G\) such that \(g + F_h \subseteq F\). Thus \(g + F \subseteq F, (g + h) + F \subseteq F\) and \(0 \in F\), and consequently \(g \in F\) and \(g + h \in F\). It is enough to show that \(g \not\in F\) or \(g + h \not\in F\). Suppose to the contrary that \(g, g + h \in F\). Then \(h = (g + h) - g \in F\) which is a contradiction. \(\Box\)

Let us remark that if \(A(F)\) is an infinite cardinal, then the condition \(|F| < A(F)\) implies condition \((1)\).

Assume that \(V\) is a vector space, \(A \subseteq V\) and \(f_1, \ldots, f_n \in V\). Fix the following notation \([A] = \text{span}(A)\) and \([f_1, \ldots, f_n] = \text{span}(\{f_1, \ldots, f_n\})\).

**Theorem 2.2.** Let \(V\) be a vector space over a field \(\mathbb{K}\) with \(\omega \leq |\mathbb{K}| = \mu < \dim V\). Assume that \(F \subseteq V\) is star-like, that is \(aF \subseteq F\) for every \(a \in \mathbb{K} \setminus \{0\}\), and

\((2)\) \[\mu < A(F) \leq \dim V.\]
Then $F \cup \{0\}$ is $\mathcal{A}(F)$-lineable, in symbols $\mathcal{L}(F) > \mathcal{A}(F)$. Moreover, any linear subspace $Y$ of $V$ contained in $F$ of dimension less than $\mathcal{A}(F)$ can be extended to $\mathcal{A}(F)$-dimensional subspace also contained in $F$.

Proof. Let $Y$ be a linear subspace of $V$ with $Y \subseteq F \cup \{0\}$. Let $X$ be a maximal element, with respect to inclusion, of the family

$$\{X : Y \subseteq X \subseteq F \cup \{0\}, \; X \text{ is a linear subspace of } V\}.$$ 

Suppose to the contrary that $|X| < \mathcal{A}(F)$. Then by Lemma 2.1 there is $g \in F \setminus X$ with $g + X \subseteq F$. Let $Z = [g] + X$ and take any $z \in Z \setminus X$. Then there is $x \in X$ and nonzero $a \in \mathbb{K}$ with $z = ag + x = a(g + x/a) \in F$. Since $X \subseteq F$ we obtain that $Z \subseteq F$. This contradicts the maximality of $X$. Therefore $\mathcal{A}(F) \leq |X| < \mathcal{L}(F)$. □

The assertion of Theorem 2.2 leads us to the following definition of a new cardinal function. We define the **homogeneous lineability number** of $F$ as the following cardinal number

$$\mathcal{H}\mathcal{L}(F) = \min(\{\lambda : \text{there is linear space } Y \subseteq F \cup \{0\} \text{ with } \dim Y < \lambda \}$$

which cannot be extended to a linear space $X \subseteq F \cup \{0\}$ with $\dim X = \lambda \cup \{|V|\}$. Now the assertion of Theorem 2.2 can be stated briefly.

**Corollary 2.3.** Let $V$ be a vector space over a field $\mathbb{K}$ with $\omega \leq |\mathbb{K}| = \mu < \dim V$. Assume that $\mathcal{F}$ is star-like and $\mu < \mathcal{A}(\mathcal{F}) \leq \dim V$. Then $\mathcal{H}\mathcal{L}(\mathcal{F}) > \mathcal{A}(\mathcal{F})$.

Now we will show basic connections between $\mathcal{H}\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(\mathcal{F})$.

**Proposition 2.4.** Let $V$ be a vector space. Then

(i) $\mathcal{H}\mathcal{L}(\mathcal{F}) \leq \mathcal{L}(\mathcal{F})$ for every $\mathcal{F} \subseteq V$;

(ii) For every successor cardinal $\kappa \leq (\dim V)^+$ there is $\mathcal{F} \subseteq V$ with $\mathcal{H}\mathcal{L}(\mathcal{F}) = \mathcal{L}(\mathcal{F}) = \kappa$;

(iii) For every $\lambda, \kappa \leq (\dim V)^+$ such that $\lambda^+ < \kappa$ there is $\mathcal{F} \subseteq V$ with $\mathcal{H}\mathcal{L}(\mathcal{F}) = \lambda^+$ and $\mathcal{L}(\mathcal{F}) = \kappa$;

(iv) $\mathcal{H}\mathcal{L}(\mathcal{F})$ is a successor cardinal.

Proof. Note that the cardinal number $\mathcal{L}(\mathcal{F})$ can be defined in the similar terms as it was done for $\mathcal{H}\mathcal{L}(\mathcal{F})$, namely $\mathcal{L}(\mathcal{F})$ is the smallest cardinal $\lambda$ such that the trivial linear space $Y = \{0\}$ cannot be extended to a linear space $X \subseteq \mathcal{F} \cup \{0\}$ with $\dim V = \lambda$. Therefore $\mathcal{H}\mathcal{L}(\mathcal{F}) \leq \mathcal{L}(\mathcal{F})$.

To see (ii) take any successor cardinal $\kappa \leq |V|^+$. There is $\lambda \leq |V|$ with $\lambda^+ = \kappa$. Let $\mathcal{F}$ be a linear subspace of $V$ of dimension $\lambda$. Then $\mathcal{F}$ is $\lambda$-lineable but not $\lambda^+$-lineable. Thus $\mathcal{L}(\mathcal{F}) = \kappa$. Note that any linear subspace of $\mathcal{F}$ can be extended to a $\lambda$-dimensional space $\mathcal{F}$, but cannot be extended to a $\kappa$-dimensional space. Thus $\mathcal{H}\mathcal{L}(\mathcal{F}) = \kappa$.

Let $\text{Card}[\lambda, \kappa] = \{\nu : \lambda \leq \nu < \kappa \text{ and } \nu \text{ is a cardinal number}\}$. For every $\nu \in \text{Card}[\lambda, \kappa)$ we find $B_\nu$ of cardinality $\nu$ such that $B = \bigcup\{B_\nu : \nu \in \text{Card}[\lambda, \kappa)\}$ is a linearly independent subset of $V$. Let $W_\nu = [B_\nu]$. Let $\mathcal{F} = \bigcup_{\nu \in \text{Card}[\lambda, \kappa)} W_\nu$. Clearly $\mathcal{F}$ is $\nu$-lineable for any $\nu < \kappa$. Take any linear space $W \subseteq \mathcal{F}$. Since $[\mathcal{F}] = [B]$, then any element $x$ of $W$ is of the form $\sum_{i=1}^{k} \sum_{j=1}^{n_i} \alpha_{ij} w_{ij}$ where $\alpha_{ij} \in \mathbb{K}$.
and \( w_{i1}, \ldots, w_{in} \) are distinct elements of \( B_{\nu}, \lambda \leq \nu_1 < \cdots < \nu_k < \kappa \). If \( x \in W_\nu \) for some \( \nu \), then \( x = \sum_{p=1}^{m} \alpha_p w_p, \alpha_p \in \mathbb{K}, w_{1}, \ldots, w_{m} \in B_\nu \) are distinct. Then

\[
\sum_{i=1}^{k} \sum_{j=1}^{n_i} \alpha_{ij} w_{ij} = \sum_{p=1}^{m} \alpha_p w_p.
\]

Now, if \( \nu \notin \{\nu_1, \ldots, \nu_k\} \), then \( \alpha_{ij} = 0 = \alpha_p \) for every \( i, j, p \). Thus \( x = 0 \). If \( \nu = \nu_1 \), then \( \alpha_{ij} = 0 \) for every \( i \neq l \). Thus \( x \in W_{\nu_1} \). This shows that \( W \) does not contain any nontrivial linear combination of \( \alpha_1 x_1 + \cdots + \alpha_n x_n \) with \( x_i \in W_{\nu_i} \) for distinct \( \nu_1, \ldots, \nu_n \). Therefore \( W \subseteq W_\nu \) for some \( \nu \). Consequently \( \dim W < \kappa \), which means that \( F \) is not \( \kappa \)-lineable. Hence \( \mathcal{L}(F) = \kappa \).

Take any linear space \( Y \subseteq F \) of dimension less than \( \lambda \). As before we obtain that \( Y \) is a subset of some \( W_\nu \). Therefore \( Y \) can be extended to a linear subspace of \( F \) of dimension \( \lambda^+ \). Hence \( \mathcal{H}_\lambda(F) = \lambda^+ \).

To prove (iv) assume that \( \mathcal{H}_\lambda(F) = \kappa \). Then for any \( \lambda < \kappa \) and any linear space \( Y \subseteq F \cup \{0\} \) of dimension less than \( \lambda \) there is a linear space \( X \supset Y \) contained in \( F \cup \{0\} \) of dimension \( \lambda \). Suppose to the contrary that \( \kappa \) is a limit cardinal. There are cardinals \( \tau_\mu < \kappa, \mu < \text{cf}(\kappa) \leq \kappa \) with \( \bigcup_{\nu<\text{cf}(\kappa)} \tau_\nu = \kappa \).

Since \( \mathcal{H}_\lambda(F) = \kappa \), then for any linear space \( Y \subseteq F \) of dimension less than \( \kappa \) we can inductively define an increasing chain \( \{Y_\nu : \dim Y < \nu < \text{cf}(\kappa)\} \) of linear spaces with \( \dim Y_\nu = \tau_\nu \) and \( Y_\nu \subseteq F \cup \{0\} \). Then \( Y' = \bigcup Y_\nu \) is a linear space of dimension \( \kappa \) such that \( Y \subseteq Y' \subseteq F \). Hence \( \mathcal{H}_\lambda(F) \geq \kappa^+ \) which is a contradiction. \( \square \)

**Theorem 2.5.** Assume that \( 3 \leq \kappa \leq \mu, \mathbb{K} \) is a field of cardinality \( \mu \) and \( V \) is a linear space over \( \mathbb{K} \) with \( \dim V = 2^\mu \). Then there is a star-like family \( F \subseteq V \) with \( \mathcal{A}(F) = \kappa \) which is not \( 2 \)-lineable.

**Proof.** Let \( \{G_\xi : \xi < 2^\mu\} \) be an enumeration of all subsets of \( V \) of cardinality less than \( \kappa \). Let \( I \subseteq V \) be a linearly independent set of cardinality \( \kappa \). Inductively for any \( \xi < 2^\mu \) we construct \( \varphi_\xi \in V \) and \( X_\xi \subseteq V \) such that

- \( \varphi_\xi + f \notin Y_\xi := |I \cup \bigcup_{\beta < \xi} X_\beta| \) for any \( f \in G_\xi \);
- \( X_\xi = \bigcup_{f \in G_\xi} [\varphi_\xi + f] \).

Suppose that we have already constructed \( \varphi_\xi \) and \( X_\xi \) for every \( \xi < \alpha \). Let \( Y_\alpha = |I \cup \bigcup_{\xi < \alpha} X_\xi| \). Since \( \dim |X_\xi| < \kappa \), then \( \dim Y_\alpha \leq |\alpha| \kappa + \kappa \). Thus \( |Y_\alpha| < 2^\mu \) and we can choose \( \varphi_\alpha \notin Y_\alpha - G_\alpha \) (equivalently \( \varphi_\alpha + f \notin Y_\alpha \) for every \( f \in G_\alpha \)). Define \( X_\alpha = \bigcup_{f \in G_\alpha} [\varphi_\alpha + f] \).

Observe that \( X_\xi \cap X_\xi' = \{0\} \) and \( Y_\xi \subseteq Y_\xi' \) for \( \xi < \xi' \). Moreover \( Y_\xi \cap X_\xi' = \{0\} \) and \( X_\xi \cap |I| = \{0\} \) for \( \xi \leq \xi' \). Define \( F = \bigcup_{\xi < 2^\mu} X_\xi \). Take any \( G \subseteq V \) with \( |G| < \kappa \). There is \( \xi \) with \( G = G_\xi \). Then \( \varphi_\xi + G_\xi \subseteq X_\xi \subseteq F \). Therefore \( \mathcal{A}(F) \geq \kappa \).

Now, we will show that for any \( \varphi \in V \) there is \( i \in I \) with \( \varphi + i \notin F \). Suppose to the contrary that it is not the case, that is there is \( \varphi \in V \) such that for any \( i \in I \) we have \( \varphi + i \in F \). Then there are distinct \( i, i' \in I \) with \( \varphi + i, \varphi + i' \in F \). Suppose first that \( \varphi + i \in X_\xi \) and \( \varphi + i' \in X_{\xi'} \) with \( \xi < \xi' \). Then

\[
X_{\xi'} \ni \varphi + i' = \varphi + i + (i' - i) \in [X_\xi \cup I] \subseteq Y_\xi.
\]

Thus \( \varphi + i' = 0 \). Therefore \( \varphi \in |I| \) and consequently \( \varphi + I \subseteq |I| \). Since \( i \neq i' \), then \( X_\xi \ni \varphi + i = \varphi + i' + (i - i') = i - i' \in |I| \) which contradicts the fact that \( X_\xi \cap |I| = \{0\} \). Hence there is \( \xi \) such that \( \varphi + i \in X_\xi \) for every \( i \in I \). Since \( |G_\xi| < \kappa \) and \( |I| = \kappa \), there are two distinct \( i, i' \in I \) such that...
\[ \varphi + i = a(\varphi + f) \text{ and } \varphi + i' = a'((\varphi + f) \text{ for some } a, a' \in \mathbb{K} \text{ and } f \in G_\xi. \text{ Thus } i - i' = (a - a')(\varphi + f) \text{ and therefore } \varphi + f \in [Z] \text{ which is a contradiction.} \]

Finally, we obtain that \( \varphi + I \not\in \mathcal{F} \) for every \( \varphi \in V \), which means that \( A(\mathcal{F}) \leq \kappa \). Hence \( A(\mathcal{F}) = \kappa \).

Let \( U = [h, h'] \) for two linearly independent elements \( h, h' \in \mathcal{F} \). Then \( h \in X_\xi \) and \( h' \in X_{\xi'} \) for some \( \xi \) and \( \xi' \). If \( \xi < \xi' \), then \( h \in Y_{\xi'} \) and \( h' \not\in Y_{\xi'} \). Let \( f \in U \setminus ([h] \cup [h']) \). Then \( f = ah + a'h' \) for some \( a, a' \in \mathbb{K} \setminus \{0\} \). If \( f \in Y_{\xi'} \), then \( h' = (f - ah)/a' \in Y_{\xi'} \) which leads to contradiction. Thus \( f \not\in Y_{\xi'} \), which means that \( U \cap Y_{\xi'} = [h] \). Since two-dimensional space \( U \) cannot be covered by less than \( \mu \) many sets of the form \( [g] \), then \( U \not\subset Y_{\xi'} \cup X_{\xi'} \). However \( U \not\subseteq Y_{\xi'} \cup X_{\xi} \). Therefore \( U \cap X_\alpha = \{0\} \) for every \( \alpha > \xi' \). Hence \( U \not\subseteq \mathcal{F} \).

If \( h, h' \in X_\xi \), then \( U \cup \bigcup_{\beta \neq \xi} X_\beta = \{0\} \) and \( U \not\subseteq X_\xi \). That implies that \( \mathcal{F} \) does not contain two-dimensional vector space.

Finally, note that \( \mathcal{F} \) is star-like. \(\square\)

The next result, which is a modification of Theorem 2.5 shows that if \( A(\mathcal{F}) \leq |\mathbb{K}| \), then \( \mathcal{L}(\mathcal{F}) \) can be any cardinal not greater than \((2^\mu)^+ \).

**Theorem 2.6.** Assume that \( 3 \leq \kappa \leq \mu \), \( \mathbb{K} \) is a field of cardinality \( \mu \) and \( V \) is a linear space over \( \mathbb{K} \) with \( \dim V = 2^\mu \). Let \( 1 < \lambda \leq (2^\mu)^+ \). There is star-like family \( \mathcal{F} \subseteq V \) such that \( \kappa \leq A(\mathcal{F}) \leq \kappa + 1 \) and \( \mathcal{L}(\mathcal{F}) = \lambda \).

**Proof.** Let \( \{G_\xi : \xi < 2^\mu \} \) be an enumeration of all subsets of \( V \) of cardinality less than \( \kappa \). Write \( V \) as a direct sum \( V_1 \oplus V_2 \) of two vector spaces \( V_1 \) and \( V_2 \) with \( \dim V_1 = \dim V_2 = 2^\mu \). Let \( \text{Card}(\lambda) = \{\nu : \nu \text{ is a cardinal number}\} \). As in the proof of Proposition 2.4(iii) we can find vector spaces \( W_\nu \subseteq V_1 \), \( \nu \in \text{Card}(\lambda) \) such that \( \dim W_\nu = \nu \) and the union of bases of all \( W_\nu \)'s forms a linearly independent set. Put \( Z = \bigcup_{\nu \in \text{Card}(\lambda)} W_\nu \subseteq V_1 \) and note that \( \mathcal{L}(Z) = \lambda \). Let \( I \subseteq V_2 \) be a linearly independent subset of cardinality \( \kappa + 1 \). Inductively for any \( \xi < 2^\mu \) we construct \( \varphi_\xi \in V \) and \( X_\xi \subseteq V \) such that \( \varphi_\xi \) and \( X_\xi \) satisfy the formulas (a) and (b) from the proof of Theorem 2.5. Define \( \mathcal{F} = Z \cup \bigcup_{\xi < 2^\mu} X_\xi \). Then \( A(\mathcal{F}) \geq \kappa \).

Now, we will show that for any \( \varphi \in V \) there is \( i \in I \) with \( \varphi + i \not\in \mathcal{F} \). Suppose to the contrary that it is not the case, that is there is \( \varphi \in V \) such that for any \( i \in I \) we have \( \varphi + i \in \mathcal{F} \). Then there are \( i, i' \in I \) with \( \varphi + i, \varphi + i' \in \mathcal{F} \). Suppose first that \( \varphi + i \in Z \) and \( \varphi + i' \in Z \). Then \( i - i' \in V_1 \). Since \( [I] \cap V_1 = \{0\} \), then \( i = i' \). Thus there is at most one element \( i \in I \) with \( \varphi + i \in Z \). Now there are at least \( \kappa \geq 3 \) elements \( i \in I \) such that \( \varphi + i \in \bigcup_{\xi < 2^\mu} X_\xi \). Using the argument as in the proof of Theorem 2.5 we reach a contradiction and we obtain that \( \varphi + I \not\subset \mathcal{F} \) for every \( \varphi \in V \), which means that \( A(\mathcal{F}) \leq \kappa + 1 \). Hence \( \kappa \leq A(\mathcal{F}) \leq \kappa + 1 \).

Since \( Z \subseteq \mathcal{F} \) and \( \mathcal{L}(Z) = \lambda \), then \( \mathcal{L}(\mathcal{F}) \geq \lambda \). From the proof of Theorem 2.5 we obtain that \( \bigcup_{\xi < 2^\mu} X_\xi \) does not contain 2-dimensional vector space. To show that \( \mathcal{L}(\mathcal{F}) = \lambda \) it suffices to show that each 2-dimensional space \( W \) contained in \( \mathcal{F} \) must be a subset of \( Z \).

Let \( W \) be a 2-dimensional space which is not contained in \( Z \). In the proof of Theorem 2.5 we have shown that the cardinality of the family of one-dimensional subspaces of \( W \) is less than \( \mu \). Moreover, by the construction of \( Z \), \( W \) has at most two one-dimensional subspaces contained in \( Z \). Consequently \( W \) is not a subset of \( \mathcal{F} \).

\( \square \)

Note that \( \mathcal{H}(\mathcal{F}) = 2 \) for \( \mathcal{F} \) constructed in the proofs of Theorem 2.5 and Theorem 2.6.
3. lineability of residual star-like subsets of Banach spaces

Let us present the following example. Let $\mathcal{J}$ be an ideal of subsets of some set $X$ which does not contain $X$. By add($\mathcal{J}$) we denote the cardinal number defined as \( \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{J} \text{ and } \bigcup \mathcal{G} \notin \mathcal{J}\} \) where $|\mathcal{G}|$ stands for cardinality of $\mathcal{G}$. Let $\mathcal{N}$ and $\mathcal{M}$ stands for $\sigma$-ideal of null and meager subsets of the real line, respectively. Then $\omega_1 \leq \text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M}) \leq \mathfrak{c}$. Moreover, if $X$ is an uncountable complete separable metric space and $\mathcal{M}_X$ is an ideal of meager subsets of $X$, then add($\mathcal{M}_X$) = add($\mathcal{M}$).

Let $V = \mathbb{R}$ be a linear space over $\mathbb{K} = \mathbb{Q}$. Let $\mathcal{J}$ be a translation invariant proper $\sigma$-ideal of subsets of $\mathbb{R}$ and let $\mathcal{F}$ be a $\mathcal{J}$-residual subset of $\mathbb{R}$, i.e. $\mathcal{F}^c \in \mathcal{J}$. It turns out that $\mathcal{A}(\mathcal{F}) \geq \text{add}(\mathcal{J})$. To see it fix $F \subseteq \mathbb{R}$ with $|F| < \text{add}(\mathcal{J})$. For any $f \in F$ consider a set

\[
T_f = \{ t \in \mathbb{R} : t + f \notin \mathcal{F} \} = \{ t \in \mathbb{R} : \exists g \in \mathcal{F}^c (t = g - f) \} \subseteq \mathcal{F}^c - f.
\]

Thus $T_f \in \mathcal{J}$. Since $|F| < \text{add}(\mathcal{J})$, then also $\bigcup_{f \in F} T_f \in \mathcal{J}$. Therefore there is $t \in \mathbb{R}$ such that $t + f \in \mathcal{F}$ for any $f \in F$.

If we additionally assume that $\mathcal{F}$ is star-like, then by Theorem 2.2 we obtain that $\mathcal{H}_{\mathcal{L}}(\mathcal{F}) > \text{add}(\mathcal{J})$. In particular if $A$ is positive Lebesgue measure (non-meager with Baire property), then $QA = \{ qa : q \in \mathbb{Q}, a \in A \}$ is $\mathbb{Q}$-star-like of full measure (residual) and therefore $\mathcal{H}_{\mathcal{L}}(\mathcal{F}) > \text{add}(\mathcal{N}) (> \text{add}(\mathcal{M}))$.

Using a similar reasoning one can prove the following.

**Theorem 3.1.** Assume that $X$ is a separable Banach space. Let $\mathcal{F} \subseteq X$ be residual and star-like, and let $\mathbb{K} \subseteq \mathbb{R}$ be a field of cardinality less than add($\mathcal{M}$). Consider $X$ as a linear space over $\mathbb{K}$. Then $\mathcal{H}_{\mathcal{L}}(\mathcal{F}) > \text{add}(\mathcal{M})$. In particular $\mathcal{F}$ contains an uncountably dimensional vector space over $\mathbb{K}$.

Let $\hat{C}[0,1]$ stand for the family of functions from $C[0,1]$ which attain the maximum only at one point. Then $\hat{C}[0,1]$ is star-like and residual but not 2-lineable, see [15] and [28] for details. This shows that Theorem 3.1 would be false for $\mathbb{K} = \mathbb{R}$. On the other hand, Theorem 3.1 shows that for any field $\mathbb{K} \subseteq \mathbb{R}$ of cardinality less than add($\mathcal{M}$) there is uncountable family $\mathcal{F} \subseteq \hat{C}[0,1]$ such that any nontrivial linear combination of elements from $\mathcal{F}$ with coefficients from $\mathbb{K}$ attains the maximum only at one point.

4. Homogeneous lineability and lineability numbers of some subsets of $\mathbb{R}^\mathbb{R}$

In this section we will apply Theorem 2.2 to obtain homogeneous lineability of families of functions from $\mathbb{R}^\mathbb{R}$. We will consider those families for which the additivity has been already computed.

Let $f \in \mathbb{R}^\mathbb{R}$. We will say that

1. $f \in D(\mathbb{R})$ ($f$ is Darboux) if $f$ maps connected sets onto connected sets.
2. $f \in ES(\mathbb{R})$ ($f$ is everywhere surjective) if $f(U) = \mathbb{R}$ for every nonempty open set $U$;
3. $f \in SES(\mathbb{R})$ ($f$ is strongly everywhere surjective) if $f$ takes each real value $\mathfrak{c}$ many times in each interval;
4. $f \in PES(\mathbb{R})$ ($f$ is perfectly everywhere surjective) if $f(P) = \mathbb{R}$ for every perfect set $P$;
5. $f \in J(\mathbb{R})$ ($f$ is Jones function) if the graph of $f$ intersects every closed subset of $\mathbb{R}^2$ with uncountable projection on the $x$-axis.
6. $f \in AC(\mathbb{R})$ ($f$ is almost continuous, in the sense of J. Stallings) if every open set containing the graph of $f$ contains also the graph of some continuous function.
(7) If \( h : X \to \mathbb{R} \), where \( X \) is a topological space, \( h \in \text{Conn}(X) \) (\( h \) is a connectivity function) if the graph of \( h|_C \) is connected for every connected set \( C \subseteq X \).

(8) \( f \in \text{Ext}(\mathbb{R}) \) (\( f \) is extendable) if there is a connectivity function \( g : \mathbb{R} \times [0, 1] \to \mathbb{R} \) such that \( f(x) = g(x, 0) \) for every \( x \in \mathbb{R} \).

(9) \( f \in \text{PR}(\mathbb{R}) \) (\( f \) is a perfect road function) if for every \( x \in \mathbb{R} \) there is a perfect set \( P \subseteq \mathbb{R} \) such that \( x \) is a bilateral limit point of \( P \) and \( f|_P \) is continuous at \( x \).

(10) \( f \in \text{PC}(\mathbb{R}) \) (\( f \) is peripherally continuous) if for every \( x \in \mathbb{R} \) and pair of open sets \( U, V \subseteq \mathbb{R} \) such that \( x \in U \) and \( f(x) \in V \) there is an open neighborhood \( W \) of \( x \) with \( W \subseteq U \) and \( f(\text{bd}(W)) \subseteq V \).

(11) \( f \in \text{SZ}(\mathbb{R}) \) (\( f \) is a Sierpiński–Zygmund function) if \( f \) is not continuous on any subset of the real line of cardinality \( c \).

We start from recalling two cardinal numbers:

\[
e_\kappa = \min\{|F| : F \subseteq \mathbb{R}, \forall \varphi \in \mathbb{R} \exists f \in F(\text{card}(f \cap \varphi) < c)\},
\]

\[
d_\kappa = \min\{|F| : F \subseteq \mathbb{R}, \forall \varphi \in \mathbb{R} \exists f \in F(\text{card}(f \cap \varphi) = c)\}.
\]

It was proved in [17] that \( \mathcal{A}(D(\mathbb{R})) = \mathcal{A}(\text{AC}(\mathbb{R})) = e_\kappa \). Therefore \( \mathcal{H}(D(\mathbb{R})), \mathcal{H}(\text{AC}(\mathbb{R})) \geq e_\kappa^+ \). More recently in [21] it was proved that \( \mathcal{A}(J(\mathbb{R})) = e_\kappa \). Thus \( \mathcal{H}(J(\mathbb{R})) \geq e_\kappa^+ \). On the other hand, by the result of [20], \( \mathcal{L}(J(\mathbb{R})) = (2^\kappa)^+ \). Since \( J(\mathbb{R}) \subseteq \text{PES}(\mathbb{R}) \subseteq \text{SES}(\mathbb{R}) \subseteq \text{ES}(\mathbb{R}) \subseteq D(\mathbb{R}) \), then additivity number for the classes \( \text{PES}(\mathbb{R}), \text{SES}(\mathbb{R}), \text{ES}(\mathbb{R}) \) is \( e_\kappa \) while their lineability number is largest possible.

Since in some model of ZFC we have \( e_\kappa < 2^\kappa \), our method does not give optimal solution for lineability number in this cases.

It was proved in [19] that \( \mathcal{A}(\text{Ext}(\mathbb{R})) = \mathcal{A}(\text{PR}(\mathbb{R})) = c^+ \). Thus \( \mathcal{H}(\text{Ext}(\mathbb{R})), \mathcal{H}(\text{Ext}(\mathbb{R})) \geq c^{++} \).

These two classes were not considered in the context of lineability.

In [18] it was proved that \( \mathcal{A}(\text{SZ}(\mathbb{R})) = d_\kappa \). Thus \( \mathcal{H}(\text{SZ}(\mathbb{R})) \geq d_\kappa^+ \). In [7] it was shown that \( \text{SZ}(\mathbb{R}) \) is \( \kappa \)-lineable if there exists a family of cardinality \( \kappa \) consisting of almost disjoint subsets of \( c \). On the other hand in [24] the authors proved that if there is no family of cardinality \( \kappa \) consisting of almost disjoint subsets of \( c \), then \( \text{SZ}(\mathbb{R}) \) is not \( \kappa \)-lineable. In [18] the authors proved that it is consistent with ZFC+CH and \( \mathcal{A}(\text{SZ}(\mathbb{R})) = c^+ < 2^\kappa \). However, if CH holds, then there is a family of cardinality \( 2^\kappa \) consisting of almost disjoint subsets of \( c \) and therefore \( \text{SZ}(\mathbb{R}) \) is \( 2^\kappa \)-lineable. Consequently, as in previous examples, consistently \( \mathcal{A}(\text{SZ}(\mathbb{R}))^+ < L(\text{SZ}(\mathbb{R})) \).

It was proved in [19] that \( \mathcal{A}(\text{PC}(\mathbb{R})) = 2^\kappa \). Therefore \( \mathcal{H}(\text{PC}(\mathbb{R})) = (2^\kappa)^+ \) is the largest possible. Note that Darboux functions are peripherally continuous. Since the set of all functions which everywhere discontinuous and Darboux are strongly \( 2^\kappa \)-algebrable, see [8], then so is \( \text{PC}(\mathbb{R}) \), which is much stronger property that \( 2^\kappa \)-lineability.

If \( V \) is a vector space over \( \mathbb{K} \) with \( |\mathbb{K}| = \mu \geq \omega \), \( \mathcal{F} \) is star-like, \( \mathcal{F} \subseteq V \) and \( \mathcal{A}(\mathcal{F}) = \kappa > \mu \), then \( \mathcal{F} \cup \{0\} = \bigcup \mathcal{B} \), where every member \( B \) of \( \mathcal{B} \) is a linear space of dimension \( \dim B \geq \kappa \). It follows from the fact that for any \( f \in \mathcal{F} \setminus \{0\} \) and any maximal vector space \( B \) contained in \( \mathcal{F} \) such that \( [f] \subseteq B \) by Theorem 2.2 we have \( \dim B \geq \kappa \).

**Theorem 4.1.** Let \( \mathcal{F} \subseteq V \) be such that \( \mathcal{A}(\mathcal{F}) = \kappa > |\mathbb{K}| \). Assume that there is a vector space \( X \subseteq V \) such that \( X \cap \mathcal{F} \subseteq \{0\} \) with \( \dim X = \tau \leq \kappa \). Then \( \mathcal{B} \) contains at least \( \tau \) many pairwise distinct elements.
Proof. Assume first that dim $X = \kappa$. Let $X = \bigcup_{\xi < \kappa} X_\xi$ be such that $X_\xi \subseteq X_{\xi'}$ provided $\xi < \xi'$ and $X_\xi$ is a linear space with dim $X_\xi = |\xi|$ for every $\xi < \kappa$. Since $|X_\xi| < A(\mathcal{F})$, there is $\varphi_\xi \in \mathcal{F}$ with $\varphi_\xi + X_\xi \subseteq \mathcal{F}$. Take any two distinct elements $x, y \in X$. There is $\xi$ such that $x, y \in X_\xi$ and $\varphi_\xi + x, \varphi_\xi + y \in \mathcal{F}$. Thus $x - y \in X$. Hence $x, y$ are not in the same $B$ from $\mathcal{B}$. Consequently $|\mathcal{B}| \geq \kappa$. If dim $X < \kappa$, the proof is similar and a bit simpler. \hfill \Box

The fact that $\mathcal{F} \cup \{0\}$ can be represented as a union of at least $\tau$ linear spaces, each of dimension at least $\kappa$ we denote by saying that $\mathcal{F}$ has property $B(\kappa, \tau)$. Surprisingly families of strange function defined by non-linear properties can be written as unions of large linear spaces.

Corollary 4.2. (1) $D(\mathbb{R})$, $ES(\mathbb{R})$, $SES(\mathbb{R})$, $PES(\mathbb{R})$ and $J(\mathbb{R})$ have $B(\epsilon, \epsilon)$.

(2) $AC(\mathbb{R})$ has $B(\epsilon, c)$.

(3) $PC(\mathbb{R})$ has $B(2^\epsilon, c)$.

(4) $PR(\mathbb{R})$ have $B(c^+, c^+)$.

(5) $SZ(\mathbb{R})$ has $B(d_\epsilon, d_\epsilon)$.

Proof. We need only to show that for each of the given families of functions there is a large linear space disjoint from it. This is in fact the same as saying that complements of these families are $\tau$-lineable for an appropriate $\tau$. Note that $X_1 = \{ f \in \mathbb{R} : f(x) = 0 \text{ for every } x \leq 0 \}$ is disjoint from $ES(\mathbb{R})$, $SES(\mathbb{R})$, $PES(\mathbb{R})$, $J(\mathbb{R})$ and $SZ(\mathbb{R})$. Let $X_2$ be a space of dimension $2^\epsilon$ such that $X_2 \setminus \{0\}$ consists of nowhere continuous functions with a finite range. Such a space $X_2$ was constructed in [23]. Clearly, $X_2$ is disjoint from $D(\mathbb{R})$. Let $X_3$ be a linear space of dimension $c$ such that $X_3 \setminus \{0\}$ consists of functions which has dense set of jump discontinuities. Such a space $X_3$ was constructed in [27]. Then $X_3 \cap AC(\mathbb{R}) = X_3 \cap PC(\mathbb{R}) = \{0\}$. Let $X_4$ be a linear space of dimension $2^\epsilon$ such that $X_4 \setminus \{0\} \subseteq PES(\mathbb{R})$. Such a space $X_4$ was constructed in [22]. Then $X_4 \cap PR(\mathbb{R}) = \{0\}$. \hfill \Box

We end the paper with the list of open questions:

1. Is it true that $A(\mathcal{F})^+ \geq H\mathcal{L}(\mathcal{F})$ for any family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$?

2. What is are homogeneous lineability numbers of the following families $D(\mathbb{R})$, $AC(\mathbb{R})$, $PES(\mathbb{R})$, $J(\mathbb{R})$, $Ext(\mathbb{R})$, $PR(\mathbb{R})$ and $SZ(\mathbb{R})$?

3. Are the families $Ext(\mathbb{R})$ and $PR(\mathbb{R}) 2^\epsilon$-lineable in $ZFC$?

4. Are the complements of the families $Ext(\mathbb{R})$, $PC(\mathbb{R})$ and $AC(\mathbb{R}) 2^\epsilon$-lineable?

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