Dimension of the Moduli Space of curves with an involution.

Luis Fuentes García* Manuel Pedreira Pérez

Authors' address: Departamento de Algebra, Universidad de Santiago de Compostela. 15706 Santiago de Compostela. Galicia. Spain. e-mail: pedreira@zmat.usc.es; luisfg@usc.es

Abstract: Given a smooth curve $X$ of genus $g$ we compute the dimension of the family of curves $C$ which have an involution over $X$. Moreover we distinguish when the curve $C$ is hyperelliptic.

Mathematics Subject Classifications (1991): Primary, 14H10; secondary, 14H30, 14H37.

Key Words: Curves, involution.

Introduction. Let $\mathcal{M}_π$ be the Moduli of smooth curves of genus $π$. It is well known that a generic smooth curve $C ∈ \mathcal{M}_π$ with $π ≥ 3$ does not have nontrivial automorphism. In particular a generic smooth curve does not have involutions.

However, given a smooth curve $X$ of genus $g$ and a divisor $b ∈ \text{Div}(X)$ verifying that $2(b − K)$ is smooth we can construct a double cover $γ : C → X$, where $C$ is a smooth curve of genus $π ≥ 2g − 1$. $C$ is not generic because it has an involution induced by $γ$. Moreover, we can characterize the divisors $b$ which provides double covers $γ : C → X$ with $C$ hyperelliptic (see [2]). Thus if we denote by $C^g_π$ the family of curves of genus $π$ with an involution of genus $g$ and by $\text{Ch}^g_π$ the family of hyperelliptic curves of genus $π$ with an involution of genus $g$, we have that:

$$\text{Ch}^g_π ⊂ C^g_π ⊂ \mathcal{M}_π.$$ 

In this paper we compute the dimension of $C^g_π$ and $\text{Ch}^g_π$. Let $r = 2(π − 1) − 4(g − 1)$. The main results are:

1. If $r ≥ 0$ then $\dim(C^g_π) = 2π − g − 1$. Moreover, the dimension of the family of curves with an involution over a hyperelliptic curve of genus $g$ is $2π − 2g + 1$.

2. If $r < 0$ then $\dim(C^g_π) = \emptyset$.

*Supported by an F.P.U. fellowship of Spanish Government
3. If \( r = 0, 2, 4 \) then \( \dim(C_{\mathcal{H}^u}) = \pi \).

4. If \( r < 0 \) or \( r > 4 \) then \( C_{\mathcal{H}^u} = \emptyset \).

Moreover, we proof that \( C_{\mathcal{H}^u} = C_{\mathcal{H}^{\pi-g}} \). In fact we prove that given an
hyperelliptic curve \( C \) of genus \( \pi \geq 2 \), an involution of \( C \) of genus \( g \) composed
with the canonical involution provides an involution of genus \( \pi - g \).

1 Preliminaries.

Let \( C, X \) be two smooth curves of genus \( \pi \) and \( g \) respectively. Let \( \gamma : C \rightarrow X \)
be a double cover. We know the following facts (see \cite{2}):

1. \( \gamma_* \mathcal{O}_C \cong \mathcal{O}_X \oplus \mathcal{O}_X(K - b) = \mathcal{S}_b \) is a decomposable geometrically ruled
   surface over the curve \( X \). We call it canonical geometrically ruled surface. \( b \) is a nonspecial divisor on \( X \)
   verifying \( 2b - 2K \sim B \), where \( B \) is the branch divisor. If \( X_0 \) is the curve of minimum self-intersection of \( S_b \)
   and \( X_1 \sim X_0 + (b - K)f \) then \( C \sim 2X_1 \). In particular \( C \in \langle 2X_0 + BF, 2X_1 \rangle \).

2. Conversely, let \( \mathcal{O}_X \oplus \mathcal{O}_X(K - b) \) be a decomposable geometrically ruled
   surface over the curve \( X \), such that \( 2b - 2K \) is a smooth divisor. Then
   the generic curve \( C \) in the linear system \( |2X_1| \) is smooth and then we
   have a double cover \( C \rightarrow X \). Moreover, if \( C \in \langle 2X_0 + BF, 2X_1 \rangle \), with
   \( B \sim 2b - 2K \) then the branch divisor of the cover is \( B \).

In order to compute the dimension of the curves with an involution over a
fixed curve \( X \), we have to study when two curves in the linear system \( |2X_1| \) are
isomorphic. In this way we have the following proposition:

**Proposition 1.1** Let \( X \) be a smooth curve of genus \( g \). Let \( b \) be a nonspecial
divisor of degree \( b \geq 2g - 2 \) defining a canonical ruled surface \( \mathcal{S}_b \). Let \( B \sim
2b - 2K \) be different points. Then, there is a unique curve \( C \in |2X_1| \) up to
isomorphism with a \( 2 : 1 \) map \( \gamma : C \rightarrow X \) whose ramification points over \( X \)
are the points of \( B \).

**Proof:** We know that given a curve \( X_1 \in |X_0 + (b - K)f| \), a generic curve \( C \) of
the pencil \( L = \langle 2X_0 + BF, 2X_1 \rangle \) is a curve with an involution \( \gamma : C \rightarrow X \)
and ramification points at \( B \) (see Lemma 1.9 and Theorem 1.10 in \cite{3}). This curve
is invariant by the unique involution of \( \mathcal{S}_b \) that fixes the curves \( X_0 \) and \( X_1 \). In
this way, \( C \) meets each generator in two points and these points are related by
the involution.
Let $C$ and $C'$ be two curves of the pencil $L$. Let $Pf$ be a generic generator. We can define an automorphism of $S_b$ that fixes $X_0 \cap Pf$, $X_1 \cap Pf$ and takes a point of $C \cap Pf$ into a point of $C' \cap Pf$. If we consider the restriction of this automorphism to the pencil $L$ we see that it takes $C$ into $C'$ and the two curves are isomorphic.

Now, let $X_1$ and $X'_1$ two irreducible curves of the linear system $|X_0 + (b - K)f|$. Since $S_b$ is a decomposable ruled surface we can define an automorphism of $S_b$ that takes $X_1$ into $X'_1$. In this way the curves of the pencils $L = \langle 2X_0 + Bf, 2X_1 \rangle$ and $L' = \langle 2X_0 + B'f, 2X'_1 \rangle$ are isomorphic and our claim follows.

### 2 Computing the dimensions.

**Proposition 2.1** Let $C^X_\pi$ be the family of smooth curves of genus $\pi \geq 1$ with an involution over a smooth curve $X$ of genus $g > 0$. Let $r = 2(\pi - 1) - 4(g - 1)$.

1. If $r > 0$ then $\dim(C^X_\pi) = r - \dim(\text{Aut}(X))$.
2. If $r = 0$ then $\dim(C^X_\pi) = 0$.
3. If $r < 0$ then $C^X_\pi = \emptyset$.

**Proof:** Let $C$ a smooth curve with an involution over the curve $X$ of genus $g$. Let $\gamma : C \rightarrow X$ be the $2 : 1$ map. By Hurwitz Theorem we know that the map $X$ has $r = 2(\pi - 1) - 4(g - 1)$ ramifications. If $r$ is negative $C^X_\pi = \emptyset$.

1. Suppose that $r > 0$. Consider the following incidence variety:

$$J = \{(C, B) \in C^X_\pi \times U^r/C \text{ has an involution branched at } B \in X\}$$

where $U_r \subset S^rX$ are the open set of $r$ unordered different points. We have two projection maps: $p : J \rightarrow C^X_\pi$ and $q : J \rightarrow U^r$.

Given a curve $C \in C^X_\pi$ there is a $2 : 1$ map $C \rightarrow X$ with $r$ ramifications. From this, $p$ is a surjection. Moreover, since $\pi > 1$ the group of automorphisms of $C$ is finite, so there are a finite number of involutions of $C$ over $X$. In this way we obtain a finite number of possible ramification points of $X$ up to automorphisms of $X$. But if $g = 1$, $\dim(\text{Aut}(X)) = 1$ and $r \geq 1$ and if $g > 1$, $\dim(\text{Aut}(X)) = 0$. From this there is at most a finite number of automorphism fixing $r$ generic points. Therefore, $\dim(p^{-1}(C)) = \dim(\text{Aut}(X))$.

On the other hand, given a set $\beta \in X$ of $r$ different points we can take a divisor $b \in \text{Div}(X)$ such that $2b - 2K \sim \beta$. Let $S_b$ be the corresponding
canonical ruled surface. By Proposition 1.1, there is a unique curve $C \in |2X_1| \subset S_b$ with an involution over $X$ with branch points over the set $\beta$, so $q$ is a surjection. Furthermore, we know that a curve $C$ with an involution over the curve $X$ lays on the linear system $|2X_1|$ of a canonical ruled surface. Since there are a finite number of divisors $b$ satisfying $2b - 2K \sim b$ (see [2]), we see that $\dim(q^{-1}(C)) = 0$ and $\dim(J) = \dim(U^r) = r$.

Thus, we have:

$$\dim(C_X^\pi) = \dim(J) - \dim(p^{-1}(C)) = r - \dim(Aut(X)).$$

2. Suppose that $r = 0$. In this case there are not ramification points. Given a curve $C \in C_X^\pi$ we know that lays on the canonical system $|2X_1|$ of a canonical ruled surface $S_b$ with $2b - 2K \sim 0$. All curves of this system are isomorphic (Proposition 1.1). Moreover, there are a finite number of divisors $b$ verifying $2b \sim 2K$. Thus $\dim(C_X^\pi) = 0$.

**Proposition 2.2** Let $Ch_\pi^X$ be the family of smooth hyperelliptic curves of genus $\pi \geq 1$ with an involution over a smooth curve $X$ of genus $g$. Let $r = 2(\pi - 1) - 4(g - 1)$. Then:

1. If $X$ is neither elliptic nor hyperelliptic then $Ch_\pi^X = \emptyset$.

2. If $X$ is elliptic or hyperelliptic then:

   (a) If $r > 4 (\pi > 2g + 1)$ or $r < 0 (\pi < 2g - 1)$ then $Ch_\pi^X = \emptyset$.

   (b) If $r = 4 (\pi = 2g + 1)$ then $\dim(Ch_\pi^X) = 2$

   (c) If $r = 2 (\pi = 2g)$ then $\dim(Ch_\pi^X) = 1$.

   (d) If $r = 0 (\pi = 2g - 1)$ then $\dim(Ch_\pi^X) = 0$.

**Proof:** We apply Theorem 3.6 of [2]. We see that $Ch_\pi^X = \emptyset$ except when $X$ is elliptic or hyperelliptic and $r = 0, 2, 4$.

1. Suppose that $r = 2$ or $r = 4$. By Theorem 3.6 of [2], the branch divisor $\beta$ verifies:

   (a) If $X$ is hyperelliptic and $r = 4$ then $\beta \sim 2g_2^1$.

   (b) If $X$ is elliptic and $r = 4$ then $\beta \sim a_1 + a_2 + a_3 + a_4$ with $a_1 + a_2 \sim a_3 + a_4$.

   (c) If $X$ is hyperelliptic and $r = 2$ then $\beta \sim g_2^1$.

   (d) If $X$ is elliptic and $r = 2$ then $\beta \sim a_1 + a_2$ for any $a_1, a_2 \in X$, $a_1 \neq a_2$. 

4
Thus, if we consider the incidence variety:

$$J_h = \{(C, B) \in \text{Ch}^X_\pi \times U^r / C \text{ has an involution branched at } B \in X\}$$

in this case the projection map $q_h : J_h \longrightarrow U^r$ is not a surjection. In fact we have:

(a) If $X$ is hyperelliptic and $r = 4$ then $\dim(\text{Im}(q_h)) = 2$.
(b) If $X$ is elliptic and $r = 4$ then $\dim(\text{Im}(q_h)) = 3$.
(c) If $X$ is hyperelliptic and $r = 2$ then $\dim(\text{Im}(q_h)) = 1$.
(d) If $X$ is elliptic and $r = 2$ then $\dim(\text{Im}(q_h)) = 2$.

Now, reasoning as in the proposition above we obtain:

$$\dim(\text{Ch}^X_\pi) = \dim(J_h) - \dim(p^{-1}(C)) = \dim(\text{Im}(q_h)) - \dim(\text{Aut}(X)).$$

and the result follows.

2. Suppose that $r = 0$. By Theorem 3.6 of [2], $\text{Ch}^X_2 \neq \emptyset$ when $X$ is elliptic or hyperelliptic and then $\dim(\text{Ch}^X_2) = 0$.

**Proposition 2.3** Let $C^g$ be the family of smooth curves of genus $\pi > 1$ with an involution over a curve of $g \geq 0$. Let $r = 2(\pi - 1) - 4(g - 1)$.

1. If $r > 0$ ($\pi > 2g - 1$) then $\dim(C^g) = r - \dim(\text{Aut}(X)) + \dim(M_g)$.
2. If $r = 0$ ($\pi = 2g - 1$) then $\dim(C^g) = \dim(M_g)$.
3. If $r < 0$ ($\pi < 2g - 1$) then $\dim(C^g) = 0$.

From this, $\dim(C^g) = 2\pi - g - 1$. Moreover, the dimension of the family of curves with an involution over a hyperelliptic curve of genus $g$ is $2\pi - 2g + 1$.

**Proof:** Since a curve of genus $\pi > 1$ has at most a finite number of involutions, such curve only has involutions over a finite number of curves of genus $g$. Therefore, $\dim(C^g) = \dim(C^g_{\pi}) + \dim(M_g)$.

Moreover, we know that:

When $g = 0$ then $\dim(M_g) = 0$ and $\dim(\text{Aut}(X)) = 3$.

When $g = 1$ then $\dim(M_g) = 1$ and $\dim(\text{Aut}(X)) = 1$.

When $g > 1$ then $\dim(M_g) = 3(g - 1)$ and $\dim(\text{Aut}(X)) = 0$.

If $g \geq 2$ the dimension of the family of hyperelliptic curves of genus $g$ is $\dim(C^g_2) = 2g - 1$. 

5
We have supposed that \( \pi > 1 \) so \( r > 0 \) when \( g = 1 \) or \( g = 0 \). Now, applying Proposition 2.1 the result follows.

**Proposition 2.4** Let \( C_h^g \) be the family of smooth hyperelliptic curves of genus \( \pi > 1 \) with an involution over a curve of \( g \geq 0 \). Let \( r = 2(\pi - 1) - 4(g - 1) \).

1. If \( r = 0, 2, 4 \) (\( \pi = 2g - 1, 2g, 2g + 1 \)) then \( \dim(C_h^g) = \pi \).

2. If \( r < 0 \) (\( \pi < 2g - 1 \)) or \( r > 4 \) (\( \pi > 2g + 1 \)) then \( C_h^g = \emptyset \).

**Proof:** Since a curve of genus \( \pi > 1 \) has at most a finite number of involutions, such curve only has involutions over a finite number of curves of genus \( g \). Therefore, \( \dim(C_h^g) = \dim(C_h^X) + \dim(C_0^g) \). Moreover, we know that \( \dim(C_0^g) = 2g - 1 \). Applying the Proposition 2.2 the result follows.

**Remark 2.5** Note that the dimension of \( C_h^g \) does not depend of \( g \). The reason is that an involution of a hyperelliptic curve \( C \) of genus \( g \) provides an involution of genus \( \pi - g \), if we compose it with the canonical involution. We will study this situation in next section.

### 3 The hyperelliptic case.

Let \( C \) be an hyperelliptic curve of genus \( \pi \geq 2 \). \( C \) has a canonical involution defined by its unique \( \pi_2 \). We will denote it by \( \delta : C \rightarrow C \), with \( \delta(P) = \pi_2(P) \). Note that any automorphism \( \nu : C \rightarrow C \) of \( C \) verifies that \( \nu(\pi_2) = \pi_2 \), so it commutes with \( \delta \).

Let \( \sigma : C \rightarrow C \) be an involution of genus \( g \) and \( \gamma : C \rightarrow X \) the corresponding double cover. Let \( b \) the divisor of \( X \) such that \( \gamma_*\mathcal{O}_C \sim \mathcal{O}_X \oplus \mathcal{O}_X(\mathcal{K} - b) \).

**Lemma 3.1** Let \( P \) be a point of \( C \). Then \( \delta(P) = \sigma(P) \) if and only if \( x = \gamma(P) \) is a base point of \( b \).

**Proof:** Let \( x = \gamma(P) \). Then \( \delta(P) = \sigma(P) \) if and only if \( \gamma^*(x) \sim \pi_2 \). But

\[
\gamma^*(x) \sim \pi_2 \iff h^0(\mathcal{O}_C(\mathcal{K}_C - \gamma^*(x))) = h^0(\mathcal{O}_C(\mathcal{K}_C)) - 1
\]

and

\[
h^0(\mathcal{O}_C(\mathcal{K}_C - \gamma^*(x))) = h^0(\mathcal{O}_{S_b}(X_0 + (b - x)f) = h^0(\mathcal{O}_X(\mathcal{K} - x)) + h^0(\mathcal{O}_X(b - x))
\]

\[
h^0(\mathcal{O}_C(\mathcal{K}_C)) = h^0(\mathcal{O}_{S_b}(X_0 + bf) = h^0(\mathcal{O}_X(\mathcal{K})) + h^0(\mathcal{O}_X(b))
\]

Because \( \mathcal{K}_X \) is base-point-free the conclusion follows.
Theorem 3.2 Let \( C \) an hyperelliptic curve of genus \( \pi \geq 2 \). Let \( \sigma : C \to C \) be an involution of genus \( g \). Then \( \delta \sigma \) is an involution of \( C \) of genus \( \pi - g \).

Proof: Note that \( \delta \sigma \) is an involution, because \( \sigma \) commutes with any automorphism of \( C \).

Let us study the ramifications points of \( \delta \sigma \). We have that \( \delta \sigma(P) = P \iff \sigma(P) = \delta(P) \). By Lemma 3.1, this happens when \( \gamma(x) \) is a base point of \( b \). Thus, the ramifications points of \( \delta \sigma \) are \( \{ \gamma^{-1}(x)/x \text{ is a base point of } b \} \). By Theorem 3.6 of [2] we know:

1. If \( \pi = 2g + 1 \), \( b = K + g_2^1 \) and \( b \) is base-point-free.
2. If \( \pi = 2g \), \( b = K + P \), with \( 2P \sim g_2^1 \) and \( b \) has one base point.
3. If \( \pi = 2g - 1 \), \( b = \sum_{1}^{g-2} g_2^1 + P + Q \), with \( 2P \sim 2Q \sim g_2^1 \), and \( b \) has two base points except when \( X \) is elliptic. But we have supposed \( \pi \geq 2 \) so in this case \( g > 1 \).

From this we see that the number of ramifications of \( \delta \sigma \) is 0, 2, 4 when \( \pi = 2g + 1, 2g, 2g - 1 \) respectively. Applying Hurwitz’s formula we obtain that the genus of \( \delta \sigma \) is \( \pi - g \).

Corollary 3.3 If \( \pi \geq 2 \) and \( g \geq 1 \) then \( Ch_{\pi}^{g} = Ch_{\pi}^{\pi - g} \).

References

[1] Arbarello, E.; Cornalba, M.; Griffiths, P. A.; Harris, J. Geometry of Algebraic Curves. Volume I. Grundlehren der mathematischen Wissenschaften 267. Springer–Verlag, 1985

[2] Fuentes, L.; Pedreira, M. Canonical geometrically ruled surfaces. Preprint. math.AG/0107114.

[3] Hartshorne, R. Algebraic Geometry. GTM, 52. Springer–Verlag, 1977.