Continuous-time mean-variance hedging under different loan and deposit rates

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Abstract
This paper investigates a continuous-time mean-variance hedging problem under different loan and deposit rates. The value function is shown to satisfy a fully nonlinear PDE and be of $C^{3,2}$ smooth by PDE method and verification theorem. We show that there are a borrowing and a saving boundary that divide the whole trading space into three regions: borrowing money region, no-trading region and saving money region. The optimal strategy is a mixture of the continuously trading strategy (as suggested by most continuous-time models) and discontinuously trading strategy (as suggested by models with transaction costs): one should put all her wealth in the stock in the middle no-trading region, and continuously trade the risky asset in the borrowing and saving money regions. Also one should never short sale the stock.

Keywords. Mean-variance hedging; fully nonlinear equation; free boundary problem; dual transformation; different loan and deposit rates;

2010 Mathematics Subject Classification. 35R35; 35Q93; 91G10; 91G30; 93E20.

1 Introduction

The famous work of Harry M. Markowitz [28] and his book of the same title [29] inaugurated a new era in modern finance. The mean-variance portfolio selection framework has

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become one of the most prominent topics in quantitative finance since its inception. Numerous extensions of Markowitz’s mean-variance portfolio selection model have been studied in the literature.

Zhou and Li [41] initiated the study of mean-variance portfolio selection in continuous time framework. They used an embedding technique and stochastic linear-quadratic (LQ) control method to solve the problem. Since then various realistic factors are taken into consideration in the continuous time mean-variance framework in the literature. For instance, Li, Zhou and Lim [23] considered a continuous-time mean-variance problem with no-shorting constraints. Hu and Zhou [18] extended to the case with random coefficients and cone constraints on the control variable. Lv, Wu and Yu [26] studied the model with random horizon in an incomplete market setting. Xiong and Zhou [37], and Xiong, Xu and Zheng [36] investigated partial information models. Zhou and Yin [42], Hu, Shi and Xu [16, 17] considered regime switching models with constraints. As a variant of mean-variance problem, Schweizer [33] introduced the mean-variance hedging (also called quadratic hedging) problem and subsequently extended to very general settings; see, e.g. Pham [32] and Gourieroux, Laurent and Pham [13]. We refer to Schweizer [34] for a guided tour.

Stochastic LQ control method was applied to study the problems in the aforementioned papers. This method is extremely powerful when dealing with mean-variance problems with trading constraints and random coefficients (see, e.g. [19, 23, 25, 40, 41, 42]), but less powerful when dealing with mean-variance problems with state constraints such as bankruptcy prohibition. The latter class of problems is often dealt by martingale approach (which usually requires the market to be complete) or partial differential equation (PDE) method (which requires the Markovian market setting). For instance, Bielecki, Jin, Pliska and Zhou [2] investigated a continuous-time mean-variance problem with bankruptcy prohibition. Using martingale approach, they turned the dynamic stochastic control problem into a static optimization problem that was eventually solved by optimization method. Li and Xu [22] studied a mean-variance problem with both trading and bankruptcy prohibition constraints by PDE method. Their idea is to transform the problem into an equivalent one with only bankruptcy prohibition constraint that has been solved in [2]. Xia [35] established a relationship between mean-variance problem and expected utility maximization with non-negative marginal utility in incomplete market with bankruptcy prohibition. Hou and Xu [15] considered the effect of intractable claims on the trading strategy by martingale approach.

All the optimal trading strategies obtained in the aforementioned papers are trading continuously in time, which are not consistent with reality most of the time. Following the idea of Dai and Yi [7], Dai, Xu and Zhou [5] considered a mean-variance problem with proportional transaction costs by PDE method. Inspired by the HJB equation, they first derived a related double-obstacle problem by intuitive argument. The solvability of the double-obstacle
problem was completed solved so that they can get a classical solution to the HJB equation. They showed that there are a selling and a buying boundary such that transactions only happen on these two boundaries. Such discontinuously trading strategy fit the real practice better than those continuously trading strategies suggested by the aforementioned other papers.

All the aforementioned papers assume that there is no difference between deposit and loan rates, namely the deposit and loan rates are the same all the time even if they are random. But, as is well-known, there always exists a gap between them, which is fairly large sometimes in practice. As loan rate is often higher than deposit rate, it discourages investors from borrowing money. There is a very limited number of works that have taken the gap between deposit and loan rates into consideration in the literature. For instance, Fleming and Zariphopoulou [9] considered optimal investment and consumption problems; Bergman [1], Korn [20], and Cvitanic and Karatzas [4] studied option pricing problems; Xu and Chen [38] investigated an optimal consumption-investment problem; Guan [14] studied a utility maximization problem by PDE method, but no verification theorem was provided in the paper so that whether the solution of the HJB equation is the value function of the problem is not known.

Fu, Lari-Lavassani and Li [11] is the only paper that we can find in the literature to study continuous-time mean-variance models under different deposit and loan rates. They constructed a piece-wise quadratic solution to the HJB equation, but did not show the solution is the value function of the original problem. In this paper, we will show the value function is of $C^{3,2}$ smooth, but their solution is not. Therefore, the mean-variance problem under different deposit and loan rates is still open. This paper aims to fill this gap.

We will investigate a continuous-time mean-variance hedging problem under different deposit and loan rates in the Black-Scholes market. It turns out that the effect that is caused by the gap of deposit and loan rates is similarly to that of the presence of transaction costs (such as [8, 7, 5]), that is, there are no-trading needed sometimes. We show there are a borrowing and a saving boundary that divide the whole trading space into three trading regions corresponding to the optimal strategies of borrowing money, no-trading and saving money. Economically speaking, we find that when an investor’s wealth is far from her target, she must borrow money to invest in the risky asset so as to maximize the chance to achieve her goal; by contrast, if her wealth is close to her target, she does not need to invest all her wealth in the risky asset and should save some in the money account to reduce her risk; while in the middle no-trading region, she should put all her wealth in the stock so that no trading is needed inside the region. Compared to the case of no gap between deposit and loan rates, the middle no-trading region is new. In the other two regions, the trading strategies are of the same form as the no gap case, except for that one should use the loan
rate in borrowing money region and the deposit rate in the saving money region. Therefore, our optimal strategy is a mixture of the continuously trading strategy (as suggested by most continuous-time models) and discontinuously trading strategy (as suggested by models with transaction costs): one does not need to trade in the middle no-trading region, and has to trade the risky asset continuously in time in the borrowing and saving regions.

Mathematically speaking, it is very important to notice that the different deposit and loan rates force the wealth dynamics to become piecewise linear, and no longer linear. As a consequence, the stochastic LQ control theory cannot be applied and new theory is called for to solve the problem. Indeed, one can apply stochastic LQ control method to solve a mean-variance problem only when the value function of the problem is of quadratic form, in which case one can determine its coefficients by solving the so-called Riccati equation. In this paper, we study a mean-variance hedging problem under different deposit and loan rates, whose value function is not quadratic form. Similar to Dai and Yi [7] and Guan [14], we solve the problem by PDE method. We first transform the HJB equation into a fully nonlinear parabolic PDE by intuitive argument. Using standard PDE tools including the truncation method, the Leray-Schauder fixed point theorem, the embedding theorem and the Schauder estimation, we get a solution to the fully nonlinear parabolic PDE, from which we eventually construct a $C^{3,2}$ solution to the original HJB equation. But different from [7] and [14], we will show the solution is indeed the value function to our mean-variance problem by a verification theorem. An optimal feedback strategy is also obtained during this process. The first order smoothness of the borrowing and saving boundaries are presented as well under a slightly stronger condition on the market parameters.

The reminder of this paper is organized as follows. In Section 2, we formulate a mean-variance hedging problem under different deposit and loan rates. In Section 3, we present our main results including the smoothness of the value function and the optimal control. Sections 4-6 are devoted to the proofs of the main results. We first derives a fully nonlinear parabolic PDE from the HJB equation by intuitive argument in Section 4; then show the parabolic PDE has a classical solution by PDE method in Section 5; and in Section 6, we prove the main results presented in Section 3. Some concluding remarks are given in Section 7.

## 2 Model Formulation

We call a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ the financial market. And assume that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is generated by a standard one-dimensional Brownian motion $\{W_t, t \geq 0\}$ defined in the probability space, argumented with all $\mathbb{P}$ null sets.

The market consists of a risk-free money account and a continuously traded risky asset
The deposit rate and loan rate of the money account are \( r_1 \) and \( r_2 \), respectively. The stock price process \( S_1^t > 0 \) follows a geometric Brownian motion:

\[
dS_1^t = S_1^t \left( \mu dt + \sigma dW_t \right),
\]

where \( \mu \) is the appreciation rate, and \( \sigma \) is the volatility rate of the stock. Economically speaking, one should pay a higher interest rate to borrow money than to save money. So we assume that the market parameters \( r_1, r_2, \mu \) and \( \sigma \) are constants and satisfy \( \sigma > 0 \) and

\[
\mu > r_2 > r_1.
\]

(2.1)

Remind that \( r_1, r_2 \) and \( \mu \) are allowed to be negative.

Consider an agent (“She”) faced with an initial endowment \( x \) and an investment horizon \([t, T]\). Let \( X_s \) and \( \pi_s \) denote her total wealth and total stock value at time \( s \), respectively. When \( \pi_s < X_s \), the agent saves the extra money \( X_s - \pi_s \) in the money account earning the deposit rate \( r_1 \); while when \( \pi_s > X_s \), the agent borrows money \( \pi_s - X_s \) from the money account at the loan rate \( r_2 \). Assume that the trading of shares is self-financed and takes place continuously, and there are no transaction costs or taxes. Then the wealth process \( X_s \) of the agent satisfies the following stochastic differential equation (SDE):

\[
\begin{cases}
  dX_s = \left( (r_1 \chi_{\pi_s < X_s} + r_2 \chi_{\pi_s > X_s}) (X_s - \pi_s) + \mu \pi_s \right) ds + \sigma \pi_s dW_s, & t \leq s \leq T, \\
  X_t = x.
\end{cases}
\]

(2.2)

Here \( \chi_S \) is the indicator function for statement \( S \): it is equal to 1 if \( S \) is true, and 0 otherwise.

We call the process \( \pi_s \), a portfolio of the agent. Define the set of admissible portfolios as

\[
\Pi_t := L^2_F([t, T]; \mathbb{R}),
\]

where \( L^2_F([t, T]; \mathbb{R}) \) denotes the set of all \( \mathbb{R} \)-valued, \( F_s \)-progressively measurable stochastic processes \( f(\cdot) \) on \([t, T]\) with \( \mathbb{E} \int_t^T |f(s)|^2 ds < +\infty \). For any admissible portfolio \( \pi \in \Pi_t \), the SDE (2.2) admits a unique strong solution \( X_s \) on \([t, T]\).

Fix a constant target \( d > 0 \), the agent’s objective is to find an admissible portfolio \( \pi^* \in \Pi_t \) to solve the following mean-variance hedging problem

\[
V(x, t) = \inf_{\pi \in \Pi_t} \mathbb{E} \left[ (X_T - d)^2 \mid X_t = x \right]
\]

for each \((x, t) \in Q_T\), where

\[
Q_T = \{(x, t) \mid xe^{r_1(T-t)} < d, \; 0 \leq t < T \}.
\]

If such an admissible portfolio \( \pi^* \) exists, we call it the optimal portfolio for the problem (2.3). The agent’s target \( d \) shall be higher than the outcome of saving all her money in the money account, so we put the constraint \( xe^{r_1(T-t)} < d \), leading to the above admissible region \( Q_T \).

The aim of this paper is to find an optimal portfolio to solve the stochastic control problem (2.3).
Remark 2.1 When bankruptcy is prohibited in the market, we need to replace \( Q^T \) by a bounded domain
\[
\{(x,t) \mid 0 < xe^{r_1(T-t)} < d, \ 0 \leq t < T\}.
\]
Meanwhile, we have an extra boundary condition \( V(0,t) = d^2, \ 0 \leq t < T \). Our argument after minor adjustment still works for this case. We encourage the readers to give the details.

Remark 2.2 As is well known, when running through \( d > 0 \) in (2.3), one can get the efficient mean-variance frontier. Therefore, there is no essential difference between the standard mean-variance problem and mean-variance hedging problem.

3 Main Results

Using standard viscosity theory (see, e.g. Grandall and Lions [3], Yong and Zhou [39]), one can prove that the value function of (2.3) is a viscosity solution of the following HJB equation with boundary and terminal conditions:
\[
\begin{cases}
  V_t + \inf_{\pi} \left( \frac{1}{2} \sigma^2 \pi^2 V_{xx} + \left( r_1 \chi_{x<x} + r_2 \chi_{x>x} \right)(x-\pi) + \mu \pi \right) V_x = 0 \text{ in } Q^T, \\
  V(e^{-r_1(T-t)}d,t) = 0, \quad 0 \leq t < T, \\
  V(x,T) = (x-d)^2, \quad x < e^{-r_1(T-t)}d.
\end{cases}
\tag{3.1}
\]

This paper does not adopt the viscosity approach because viscosity solution usually does not lead to good smoothness of the value function. Instead, we will prove that this HJB equation admits a good classical solution \( V \) (see the precise definition in Theorem 3.1 below). This together with Itô’s lemma can guarantee that \( V \) is the value function of the problem (2.3) (see Verification Theorem 3.4 below).

Theorem 3.1 (Solvability of the HJB Equation) There exists a solution \( V \in C^{3,2}(Q^T \setminus \{x = e^{-r_1(T-t)}d\}) \cap C(Q^T) \) to the HJB equation (3.1) such that
\[
\begin{align*}
  V_x &< 0, \\
  V_{xx} &> 0
\end{align*}
\tag{3.2}
\]
\[
\begin{align*}
  \lim_{x \to e^{-r_1(T-t)}d^-} V_x &= 0, \\
  \lim_{x \to -\infty} V_x &= -\infty, \quad \forall \ t \in [0,T].
\end{align*}
\tag{3.4}
\]

Proof: We leave the proof to Section 6.1. \qed
Based on this result, we can derive the optimal strategy stated in the following section.

3.1 Optimal Portfolio

Let $V$ be given in Theorem 3.1 and divide the whole state space $Q^T = \{(x, t) \mid xe^{r_1(T-t)} < d, 0 \leq t < T\}$ into three regions:

- **Borrowing Region** $\mathfrak{B} := \{ (x, t) \in Q^T \mid -\frac{\mu - r_2}{\sigma^2} \frac{V_x}{V_{xx}} > x \}$,
- **No-Trading Region** $\mathfrak{N} := \{ (x, t) \in Q^T \mid -\frac{\mu - r_2}{\sigma^2} \frac{V_x}{V_{xx}} \leq x \leq -\frac{\mu - r_1}{\sigma^2} \frac{V_x}{V_{xx}} \}$,
- **Saving Region** $\mathfrak{S} := \{ (x, t) \in Q^T \mid -\frac{\mu - r_1}{\sigma^2} \frac{V_x}{V_{xx}} < x \}$.

The following result shows that there exist two free boundaries to separate them.

**Proposition 3.2 (Optimal Trading Regions)** We have

$$\mathfrak{B} = \{ (x, t) \mid x < B(t), t \in [0, T) \},$$

$$\mathfrak{N} = \{ (x, t) \mid B(t) \leq x \leq L(t), t \in [0, T) \},$$

$$\mathfrak{S} = \{ (x, t) \mid L(t) < x < e^{-r_1(T-t)}d, t \in [0, T) \},$$

where $B(\cdot)$ and $L(\cdot)$ are called the borrowing and saving boundaries, respectively, defined by

$$B(t) := V_x^{-1}(\cdot, t)(-e^{b(T-t)}), \quad L(t) := V_x^{-1}(\cdot, t)(-e^{l(T-t)}),$$

with $V_x^{-1}(\cdot, t)$ being the inverse with respect to (w.r.t.) the spatial argument $x$, and $b(\cdot)$ and $l(\cdot)$ are given by (5.3) and (5.4). Moreover, the borrowing and saving boundaries have the terminal values $B(T) = \frac{\mu - r_2}{\sigma^2 + \mu - r_2}d$ and $L(T) = \frac{\mu - r_1}{\sigma^2 + \mu - r_1}d$, and satisfy the estimate

$$0 < B(t) < L(t) < e^{-r_1(T-t)}d, \quad t \in [0, T].$$
Remark 3.3 In fact

\[ B(t) = \sup \left\{ x \mid -\frac{\mu - r_2}{\sigma^2} \frac{V_x}{V_{xx}} > x, (x, t) \in Q^T \right\}, \]

and

\[ L(t) = \inf \left\{ x \mid -\frac{\mu - r_1}{\sigma^2} \frac{V_x}{V_{xx}} < x, (x, t) \in Q^T \right\}. \]

Figure 2: Optimal trading regions

We also establish the first-order smoothness of the boundaries \( B(\cdot) \) and \( L(\cdot) \) under certain conditions (see Proposition 5.11).

Theorem 3.4 (Verification Theorem) The function \( V \) given in Theorem 3.1 is the same as the value function \( V \) defined by (2.3). Moreover, the optimal portfolio to the problem (2.3), given in the feedback form, is

\[
\pi(x,t) = \begin{cases} 
-\frac{\mu - r_2}{\sigma^2} \frac{V_x}{V_{xx}}, & (x,t) \in \mathcal{B}, \\
x, & (x,t) \in \mathcal{N}, \\
-\frac{\mu - r_1}{\sigma^2} \frac{V_x}{V_{xx}}, & (x,t) \in \mathcal{S}.
\end{cases}
\]

Proof: We leave the proof to Section 6.3.

We have the following financial insights. When one’s wealth is far from her target (i.e. \( x < B(t) \)), she must borrow money to invest in the stock so as to maximize the chance to
achieve her goal $d$. By contrast, if her wealth is close to her target (i.e. $x > L(t)$), she does not need to invest all her wealth in the stock and can save some in the money account to reduce her risk. In the middle range (i.e. $B(t) \leq x \leq L(t)$), she does not need to borrow or save money, and shall invest all her wealth in the stock. Therefore the optimal strategy is a mixture of the continuously trading strategy in the first two scenarios (as suggested by most continuous-time models) and discontinuously trading strategy in the last scenario (as suggested by models with transaction costs).

Also the optimal portfolio is positive in all scenarios, so it is never optimal to short sale the stock. As a consequence, the portfolio is still optimal if we restrict us to the control set with no-shorting constraint:

$$\left\{ \pi_s \in L^2_F([t,T];\mathbb{R}) \mid \pi_s \geq 0, \ s \in [t,T] \right\}$$

in the problem (2.3).

When $r_2 \to r_1$, the optimal feedback portfolio reduces to

$$\pi(x,t) = -\frac{\mu - r_1}{\sigma^2} \frac{V_x}{V_{xx}}.$$  

This recovers the classical optimal portfolio when there is no gap between loan and deposit rates. In this case, the trading happens all the time, and the no-trading region is a zero measure set.

The reminder part of this paper is devoted to the proofs of the main results stated above.

## 4 Related Fully Nonlinear PDEs

To study the PDE (3.1), we first transform it into a fully nonlinear PDE (4.7) satisfying the usual structural conditions by heuristic argument in this section. Many a priori estimates of the solution will be used in this process. In the next section, we will rigorously prove the existence and uniqueness of the solution to the PDE (4.7) and prove those prior estimates used, and finally we will construct a solution to the PDE (3.1) from the solution to the PDE (4.7) in Section 6.

Our argument in the rest part of this section is intuitive and it will lead to a more tractable PDE (4.7) which will sever as our starting point of theoretical analysis.

Our below argument will be based on the conjecture that

$$V_x < 0, \quad V_{xx} > 0, \quad (x,t) \in Q^T,$$  

and

$$\lim_{x \to e^{s(t-T)}d_-} V_x = 0, \quad \lim_{x \to -\infty} V_x = -\infty, \quad t \in [0,T].$$
This conjecture in fact will be eventually proved in Theorem 3.1.

In order to solve the optimization problem in the HJB equation (3.1), write

\[ H(\pi) := \frac{1}{2} \sigma^2 \pi^2 V_{xx} + \left( (r_1 \chi_{\pi < x} + r_2 \chi_{\pi > x}) (x - \pi) + \mu \pi \right) V_x, \]

and

\[ \pi^*_i := -a_i \frac{V_x}{V_{xx}}, \quad a_i := \frac{\mu - r_i}{\sigma^2}, \quad i = 1, 2. \]

As \( a_1 > a_2 > 0 \) by our assumption (2.1), it follows from (4.1) that \( \pi^*_1 > \pi^*_2 \). Consequently, there are only three possible scenarios given as follows.

As a consequence, it follows from (4.1) that \( \pi^*_1 > \pi^*_2 \). Consequently, there are only three possible scenarios given as follows.

This leads to

\[
\arg\min_{\pi} H(\pi) = \begin{cases} 
-a_1 \frac{V_x}{V_{xx}}, & -a_1 \frac{V_x}{V_{xx}} < x, \\
- a_2 \frac{V_x}{V_{xx}} \leq x \leq -a_1 \frac{V_x}{V_{xx}}, \\
- a_2 \frac{V_x}{V_{xx}}, & -a_2 \frac{V_x}{V_{xx}} > x.
\end{cases}
\]

Putting this into (3.1), we get

\[
\begin{cases}
-V_t + \frac{\sigma^2 a_1^2}{2} V_{xx}^2 + (\sigma^2 a_1 - \mu) x V_x = 0, & -a_1 \frac{V_x}{V_{xx}} < x, \\
-V_t - \frac{\sigma^2}{2} x^2 V_{xx} - \mu x V_x = 0, & -a_2 \frac{V_x}{V_{xx}} \leq x \leq -a_1 \frac{V_x}{V_{xx}}, \\
-V_t + \frac{\sigma^2 a_2^2}{2} V_{xx}^2 + (\sigma^2 a_2 - \mu) x V_x = 0, & -a_2 \frac{V_x}{V_{xx}} > x, \\
V(e^{-r_1(T-t)}d, t) = 0, & 0 \leq t < T, \\
V(x, T) = (x-d)^2, & x < d.
\end{cases}
\]

This is a free boundary problem. But it does not satisfy the general structural conditions of nonlinear parabolic equation, so it is difficult to directly apply the existing conclusions to study it.
By the conjecture (4.1), \( V \) should be a convex function, so we can apply the dual transformation (see Pham [31]) to simplify (4.3). To this end, define
\[
v(y, t) := \inf_{x < e^{-(r-1)t}d} (V(x, t) + xy), \quad y > 0, \quad 0 \leq t \leq T.
\]
Then by the conjecture (4.2), for each fixed \( t \in [0, T] \), the optimal \( x \) corresponding to \( y \) is
\[
x = x(y, t) := V_x^{-1}(\cdot, t)(-y), \quad y > 0,
\]
where \( V_x^{-1}(\cdot, t) \) is the inverse of \( V_x(\cdot, t) \). This gives the following correspondence between \( v(y, t) \) and \( V(x, t) \),
\[
v(y, t) = V(x(y, t), t) + x(y, t)y,
\]
\[
v_y(y, t) = V_x(x(y, t), t)x_y(y, t) + yx_y(y, t) + x(y, t) = x(y, t),
\]
\[
v_{yy}(y, t) = x_y(y, t) = \frac{-1}{V_{xx}(x(y, t), t)},
\]
\[
v_t(y, t) = V_t(x(y, t), t) + V_x(x(y, t), t)x_t(y, t) + yx_t(y, t) = V_t(x(y, t), t).
\]
It then follows from (4.3) that
\[
\begin{align*}
-v_t - \frac{1}{2}\sigma^2a_1^2y^2v_{yy} - (\sigma^2a_1 - \mu)yv_y &= 0, \quad -\frac{v_y}{v_{yy}} > a_1y, \\
v_t + \frac{\sigma^2}{2}a_2y^2v_{yy} + \mu yv_y &= 0, \quad a_2y \leq -\frac{v_y}{v_{yy}} \leq a_1y, \\
-v_t - \frac{1}{2}\sigma^2a_2^2y^2v_{yy} - (\sigma^2a_2 - \mu)yv_y &= 0, \quad -\frac{v_y}{v_{yy}} < a_2y, \\
v(y, T) &= -\frac{1}{4}y^2 + dy, \quad y > 0.
\end{align*}
\]
Now we introduce
\[
u := -v_y.
\]
After differentiating (4.4) w.r.t. \( y \), we obtain an equation for \( u \):
\[
\begin{align*}
-u_t - \frac{1}{2}\sigma^2a_1^2y^2u_{yy} + (-\sigma^2a_1^2 - \sigma^2a_1 + \mu)uy_y + (\mu - \sigma^2a_1)u &= 0, \quad -\frac{u}{u_y} > a_1y, \\
u_t - \frac{\sigma^2}{2}(\frac{u}{u_y})^2u_{yy} + \sigma^2u + \mu uy_y + \mu u &= 0, \quad a_2y \leq -\frac{u}{u_y} \leq a_1y, \\
u_t - \frac{1}{2}\sigma^2a_2^2y^2u_{yy} + (-\sigma^2a_2^2 - \sigma^2a_2 + \mu)uy_y + (\mu - \sigma^2a_2)u &= 0, \quad -\frac{u}{u_y} < a_2y, \\
u(y, T) &= \frac{1}{2}y - d, \quad y > 0.
\end{align*}
\]
(4.5)
Making a transformation \( u(y, t) = w(z, s) \) for \( s = T - t \), \( z = \ln y \), we deduce
\[
u_t = -w_s, \quad u_y = w_z\frac{1}{y}, \quad u_{yy} = (w_{zz} - w_z)\frac{1}{y^2},
\]
so that (4.5) becomes

\[
\begin{aligned}
& w_s - \frac{1}{2}\sigma^2a_1^2w_{zz} + (\mu - \frac{1}{2}\sigma^2a_1^2 - \sigma^2a_1)w_z + (\mu - \sigma^2a_1)w = 0, \quad -\frac{w}{w_z} > a_1, \\
& w_s - \frac{\sigma^2}{2}(\frac{w}{w_z})^2(w_{zz} - w_z) + \sigma^2w + \mu w_z + \mu w = 0, \quad a_2 \leq -\frac{w}{w_z} \leq a_1, \\
& w_s - \frac{1}{2}\sigma^2a_2^2w_{zz} + (\mu - \frac{1}{2}\sigma^2a_2^2 - \sigma^2a_2)w_z + (\mu - \sigma^2a_2)w = 0, \quad -\frac{w}{w_z} < a_2, \\
& w(z, 0) = \frac{1}{2}e^z - d, \quad z \in \mathbb{R}.
\end{aligned}
\]

For convenience, we define a function

\[ A(\xi) := \min\left\{ \max\{a_2, -\xi\}, a_1 \right\} = \begin{cases} 
  a_1, & -\xi > a_1, \\
  -\xi, & a_2 \leq -\xi \leq a_1, \\
  a_2, & -\xi < a_2.
\end{cases} \]

It is clearly a Lipschitz-continuous bounded decreasing function. Using this notation, we can write (4.6) in a compact form as an initial value problem

\[
\begin{aligned}
& w_s - \mathcal{T}w = 0 \quad \text{in} \quad Q_T := \mathbb{R} \times (0, T] \\
& w(z, 0) = \frac{1}{2}e^z - d, \quad z \in \mathbb{R},
\end{aligned}
\]

where

\[ \mathcal{T}w := \frac{1}{2}\sigma^2A^2(\frac{w}{w_z})w_{zz} - \left( \mu - \frac{1}{2}\sigma^2A^2(\frac{w}{w_z}) - \sigma^2A(\frac{w}{w_z}) \right)w_z - \left( \mu - \sigma^2A(\frac{w}{w_z}) \right)w. \]

By definition we have $0 < a_2 \leq A \leq a_1$, so (4.7) is a fully nonlinear PDE of parabolic type, which satisfies the usual structural conditions. We will study its solvability and properties in the next section.

5 Solvability of (4.7)

We have got a fully nonlinear PDE (4.7) by intuitive argument in the previous section. From now on we do rigorous analysis and focus on the solvability and properties of the PDE (4.7) in this section.

Denote, from now on,

\[ \theta_1 := \sigma^2a_1^2 - 2r_1, \quad \theta_2 := \sigma^2a_2^2 - 2r_2, \]

and

\[ k := \max\left\{ 2\sigma^2a_1 + 4\sigma^2a_1^2, \theta_1 \right\}, \quad \kappa := 2\mu + \sigma^2(3a_1 + 1)(a_1 + 1). \]

Our main theoretical result is
Theorem 5.1 There exists a solution \( w \in C^{2+\alpha,1+\alpha/2}(\mathbb{R} \times [0,T]) \) (for some \( \alpha \in (0,1) \)) to the PDE (4.7) such that
\[
\frac{1}{2}e^{\theta s}e^{-z} - e^{-r_1 s}d \leq w \leq \frac{1}{2}e^{\theta_2 s}e^{z} - e^{-r_2 s}d, \tag{5.1}
\]
and
\[
\frac{1}{2}e^{-\kappa s}e^{-z} \leq w_z \leq \frac{1}{2}e^{k s}e^{z}. \tag{5.2}
\]

PROOF: The proof, which is cumbersome, needs some results of Sobolev space and a priori estimation method of parabolic equations, so we put it in Appendix A. \(\square\)

In the rest part of this paper we fix a solution \( w \) as in Theorem 5.1. Based on it, we will construct solutions to (4.5), (4.4), (4.3) and (3.1) in the following sections. In particular, Verification Theorem 3.4 will ensure such \( w \) is indeed unique.

Remark 5.2 The exact values of \( \theta_1, \theta_2, k \) and \( \kappa \) in Theorem 5.1 are not important. We just need to make sure that \( w \) and \( w_z \) are growth exponentially in \( z \), which will suffice to ensure such solution \( w \) to (4.7) is unique.

5.1 The Free Boundaries of (4.7)

In order to study the properties of (4.7), define
\[
\mathcal{B} := \{(z,s) \in Q_T \mid -\frac{w}{w_z} < a_2\},
\]
\[
\mathcal{N} := \{(z,s) \in Q_T \mid a_2 \leq -\frac{w}{w_z} \leq a_1\},
\]
\[
\mathcal{S} := \{(z,s) \in Q_T \mid -\frac{w}{w_z} > a_1\},
\]
and the boundaries
\[
b(s) := \sup \left\{ z \in \mathbb{R} \left| -\frac{w}{w_z}(z,s) \geq a_2 \right. \right\} = \sup \left\{ z \in \mathbb{R} \mid (w + a_2 w_z)(z,s) \leq 0 \right\}, \quad s \in (0,T], \tag{5.3}
\]
\[
l(s) := \inf \left\{ z \in \mathbb{R} \mid -\frac{w}{w_z}(z,s) \leq a_1 \right\} = \inf \left\{ z \in \mathbb{R} \mid (w + a_1 w_z)(z,s) \geq 0 \right\}, \quad s \in (0,T], \tag{5.4}
\]
where we used \( w_z > 0 \) to get the second expressions in above. They will be used to study the properties of the optimal portfolio for our original problem (2.3). Because \( a_1 > a_2 \) and \( w_z > 0 \), we see \( b(s) > l(s) \) for all \( s \in (0,T] \).

We have the following estimates for the two boundaries.
Lemma 5.3 For any $s \in (0, T]$,

$$b(s) < \ln(2d) - (r_1 + \theta_2)s. \quad (5.5)$$

PROOF: If $z \geq \ln(2d) - (r_1 + \theta_2)s$, then by $a_2 > 0$ and the lower bounds in (5.1) and (5.2) we have

$$w(z, s) + a_2 w_z > \frac{1}{2} e^{\theta z} e^z - e^{-r_1 s} \geq 0$$

implying (5.5). \qed

Lemma 5.4 For any $s \in (0, T]$,

$$l(s) \geq \ln(2d) - \ln(a_1 + e^{(\theta_2 - k)s}) - (r_1 + k)s. \quad (5.6)$$

PROOF: If $z < \ln(2d) - \ln(a_1 + e^{(\theta_1 - k)s}) - (r_2 + k)s$, by the upper bounds in (5.1) and (5.2) we have

$$w + a_1 w_z \leq \frac{1}{2} e^{\theta_1 s} e^z - e^{-r_2 s} d + a_1 \frac{1}{2} e^{k s} e^z = \frac{1}{2} e^{k s} e^z (e^{(\theta_1 - k)s} - 2e^{-(r_2 + k)s} e^z d + a_1) < 0,$$ 

which implies (5.6). \qed

Define

$$I = w + a_2 w_z$$

and

$$f(s) := \sup \{ z \in \mathbb{R} \mid I(z, s) < 0 \}, \quad s \in (0, T].$$

By definition $f(s) \leq b(s)$.

Lemma 5.5 If $f_*(s_0 -) < f^*(s_0)$ for some $s_0 \in (0, T]$, then

$$I(z, s_0) = 0, \quad \forall z \in (f_*(s_0 -), f^*(s_0)), \quad (5.7)$$

where $f_*(s_0 -) := \liminf_{s \to s_0 -} f(s)$ and $f^*(s_0) := \limsup_{s \to s_0} f(s)$.

PROOF: By the continuity of $I$ and the definition of $f(s)$, we have

$$I(z, s_0) \geq 0, \quad z \geq f_*(s_0 -). \quad (5.8)$$

If (5.7) were not true, then there would exit $z_0 \in (f_*(s_0 -), f^*(s_0))$ such that $I(z_0, s_0) > 0$. Owing to the continuity, we would have

$$I(z_0, s) > 0, \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon) \quad (5.9)$$
for sufficiently small $\varepsilon > 0$. Since $z_0 > f_s(s_0 - \varepsilon)$, we could suppose $z_0 > f(s_0 - \varepsilon)$ so that

$$I(z, s_0 - \varepsilon) \geq 0, \quad z > z_0. \quad (5.10)$$

Now we would prove $I > 0$ in $\mathcal{D} := (z_0, +\infty) \times (s_0 - \varepsilon, s_0 + \varepsilon)$. Indeed, suppose $\psi$ is the unique solution to

\[
\begin{align*}
\frac{\partial}{\partial s} \psi_s - \frac{1}{2} \sigma^2 a_2^2 \psi_{zz} + (\mu - \frac{1}{2} \sigma^2 a_2^2 - \sigma^2 a_2) \psi_z + (\mu - \sigma^2 a_2) & \psi = 0 \quad \text{in} \quad \mathcal{D}, \\
(\psi + a_2 \psi_z)(z_0, s) = I(z_0, s), & \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon), \\
\psi(z, s_0 - \varepsilon) = w(z, s_0 - \varepsilon), & \quad z > z_0.
\end{align*}
\]

under the exponential growth conditions on $\psi$ and $\psi_z$. Then the maximum principle gives $\psi \leq 0$. Differentiating the equation in (5.11) w.r.t. $z$, we have

$$\psi_{zs} - \frac{1}{2} \sigma^2 a_2^2 \psi_{zzz} + (\mu - \frac{1}{2} \sigma^2 a_2^2 - \sigma^2 a_2) \psi_{zz} + (\mu - \sigma^2 a_2) \psi_z = 0 \quad \text{in} \quad \mathcal{D},$$

So $\Psi = \psi + a_2 \psi_z$ satisfies

\[
\begin{align*}
\frac{\partial}{\partial s} \Psi_s - \frac{1}{2} \sigma^2 a_2^2 \Psi_{zz} + (\mu - \frac{1}{2} \sigma^2 a_2^2 - \sigma^2 a_2) & \Psi_z + (\mu - \sigma^2 a_2) \Psi = 0 \quad \text{in} \quad \mathcal{D}, \\
\Psi(z_0, s) = I(z_0, s), & \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon), \\
\Psi(z, s_0 - \varepsilon) = I(z, s_0 - \varepsilon), & \quad z > z_0.
\end{align*}
\]

Using (5.9) and (5.10), by the strong maximum principle, we have $\Psi > 0$ in $\mathcal{D}$. Together with $\psi \leq 0$ and $a_2 > 0$, we see $\psi_z > 0$ and $-\frac{\psi}{\psi_z} < a_2$. So $A\left(\frac{\psi}{\psi_z}\right) = a_2$ in $\mathcal{D}$, thus (5.11) can be rewritten as

\[
\begin{align*}
\frac{\partial}{\partial s} \psi_s - \frac{1}{2} \sigma^2 A^2(\psi) \psi_{zz} + (\mu - \frac{1}{2} \sigma^2 A^2(\psi) - \sigma^2 A(\psi)) \psi_z + (\mu - \sigma^2 A(\psi)) & \psi = 0 \quad \text{in} \quad \mathcal{D}, \\
(\psi + a_2 \psi_z)(z_0, s) = (w + a_2 w_z)(z_0, s), & \quad s \in (s_0 - \varepsilon, s_0 + \varepsilon) \\
\psi(z, s_0 - \varepsilon) = w(z, s_0 - \varepsilon), & \quad z > z_0.
\end{align*}
\]

(5.12)

Notice $\psi = w$ also satisfies this system, so by the uniqueness of its solution, we would conclude that $\psi = w$ in $\mathcal{D}$. Consequently, $I = \Psi > 0$ in $\mathcal{D}$. But, by the definition of $f(s_0)$, we would have $f(s) \leq z_0$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ so that $d^*(s_0) = \limsup_{s \to s_0} f(s) \leq z_0$, contradicting to $z_0 \in (f_*(s_0 - \varepsilon), f^*(s_0))$.

□
Lemma 5.6 Given $s \in (0, T]$, we have
\[ I(z, s) \leq 0, \quad \forall z \leq f(s). \tag{5.13} \]

Proof: Denote $\mathcal{C} := \{(z, s) \mid z \leq f(s), \ s \in (0, T]\}.$ If (5.13) were not true, i.e. $I$ would take positive values in $\mathcal{C}$. Since $\limsup_{z \to -\infty} I(z, s) < 0$ for any $s \in (0, T]$ by (5.1) and (5.2), there would exist $(z_0, s_0) \in \bar{\mathcal{C}}$ such that $I(z_0, s_0) = \max_{(z, s) \in \mathcal{C}} I(z, s) > 0$. Note that $(z_0, s_0) \in \mathcal{C}$ implies $z_0 \leq f^*(s_0) = \limsup_{s \to s_0-} f(s)$. By Lemma 5.5 we would have $z_0 < f^*(s_0) = \liminf_{s \to s_0-} f(s)$. Therefore, $I > 0$ in $\mathcal{D} := (z_0 - \varepsilon, z_0 + \varepsilon) \times (s_0 - \varepsilon, s_0) \subset \mathcal{C}$ for sufficiently small $\varepsilon > 0$. Then $A(w/w_z) = a_2$ in $\mathcal{D}$, so $I$ would satisfy a linear equation in $\mathcal{D}$. However, as $(z_0, s_0)$ is the maximum point of $I$ in $\mathcal{D}$, it is impossible by the maximum principle. □

By Lemma 5.6, we see $I(z, s_0) \geq 0$ for any $z > z_0$ if $I(z_0, s_0) > 0$. We continue to prove the following stronger conclusion.

Lemma 5.7 Given $s_0 \in (0, T]$, we have $I(z, s_0) > 0$ for any $z > z_0$ if $I(z_0, s_0) > 0$.

Proof: By the continuity of $I$, there exists $0 < \varepsilon < s_0$ such that
\[ I(z_0, s) > 0, \quad s \in (s_0 - \varepsilon, s_0]. \]

By Lemma 5.6 we further have
\[ I(z, s) \geq 0, \quad (z, s) \in (z_0, +\infty) \times (s_0 - \varepsilon, s_0] \]
and $I$ satisfies a linear equation in $(z_0, +\infty) \times (s_0 - \varepsilon, s_0]$. By the strong maximum principle we conclude $I > 0$ in $(z_0, +\infty) \times (s_0 - \varepsilon, s_0]$. □

By Lemma 5.7 and the definition (5.3), we conclude

Lemma 5.8 We have
\[ B = \{(z, s) \mid z > b(s), \ s \in (0, T]\} \tag{5.14} \]
with $b(0+) = \ln(2d) - \ln(1 + a_2)$.

Similarly, we can prove

Lemma 5.9 We have
\[ S = \{(z, s) \mid z < l(s), \ s \in (0, T]\} \tag{5.15} \]
with $l(0+) = \ln(2d) - \ln(1 + a_1)$.  

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Recall that $l(s) < b(s)$, so the above two lemmas imply

**Lemma 5.10** We have

$$
\mathcal{N} = \{(z, s) \mid l(s) \leq z \leq b(s), \ s \in (0, T]\}.
$$

Next, we prove that the boundaries $b(\cdot)$ and $l(\cdot)$ are smooth when the coefficients meet certain conditions.

**Proposition 5.11** If the coefficients satisfy the following conditions

\begin{align}
\sigma^2 a_2^2 + r_1 - 2r_2 &\geq 0, \tag{5.16} \\
1 + \frac{2r_1}{\mu - r_1} - 2a_1 + a_2 &\geq 0, \tag{5.17}
\end{align}

then the boundaries $b(\cdot), l(\cdot) \in C^1((0, T])$.

To prove this conclusion, we need

**Lemma 5.12** Under the condition (5.16), we have

$$
w_s + r_1 w \geq 0. \tag{5.18}
$$

**Proof:** The condition (5.16) is equivalent to $\theta_2 = \sigma^2 a_2^2 + 2\sigma^2 a_2 - 2\mu \geq -r_1$. Denote $\varphi = e^{r_1 s}w$, by the first inequality in (5.1) we have

$$
\varphi(z, s) \geq \frac{1}{2}e^{(\theta_2 + r_1)s}e^z - d \geq \frac{1}{2}e^z - d = \varphi(z, 0), \quad (z, s) \in Q_T.
$$

For any $\Delta s \in (0, T)$, let $\varphi(z, s) := \varphi(z, s + \Delta s)$, then by above and the equation in (4.7), we have

$$
\begin{cases}
\varphi_s - r_1 \varphi - T\varphi = 0 & \text{in } Q_{T-\Delta s}, \\
\varphi(z, 0) = \varphi(z, \Delta s) \geq \varphi(z, 0), \quad z \in \mathbb{R}.
\end{cases}
$$

By the comparison principle we have $\varphi \geq \varphi$ in $Q_{T-\Delta s}$, which implies $\varphi_s \geq 0$, so $w_s + r_1 w \geq 0$. \hfill \Box

**Lemma 5.13** Under the conditions (5.16) and (5.17),

$$
\partial_z \left(\frac{w}{w_z}\right) > 0 \quad \text{in } \mathcal{N}. \tag{5.19}
$$
PROOF: Recall that \( w < 0, w_z > 0 \) and \( a_2 \leq A(\frac{w}{w_z}) = \frac{-w}{w_z} \leq a_1 \) in \( \mathcal{N} \), so
\[
A^2(\frac{w}{w_z})w_{zz} = (\frac{w}{w_z})^2 w_{zz} = w\left(1 - \partial_z\left(\frac{w}{w_z}\right)\right) \text{ in } \mathcal{N}.
\]
By the equation in (4.7) and Lemma 5.12 we have
\[
\frac{\sigma^2}{2}w\left(1 - \partial_z\left(\frac{w}{w_z}\right)\right) = w_s + \left(\mu - \frac{1}{2}\sigma^2 A^2\left(\frac{w}{w_z}\right) - \sigma^2 A\left(\frac{w}{w_z}\right)\right)w_z + \left(\mu - \sigma^2 A\left(\frac{w}{w_z}\right)\right)w
\]
\[
= w_s - \frac{\mu}{A(\frac{w}{w_z})}w + \frac{1}{2}\sigma^2 A\left(\frac{w}{w_z}\right)w + \sigma^2 w + \left(\mu - \sigma^2 A\left(\frac{w}{w_z}\right)\right)w
\]
\[
= w_s - \frac{\mu}{A(\frac{w}{w_z})}w - \frac{1}{2}\sigma^2 A\left(\frac{w}{w_z}\right)w + (\sigma^2 + \mu)w
\]
\[
> -r_1 w - \frac{\mu}{a_1}w - \frac{1}{2}\sigma^2 a_2 w + (\sigma^2 + \mu)w
\]
\[
= \sigma^2 w\left(a_1 - \frac{\mu}{\sigma^2 a_1} - \frac{1}{2}a_2 + 1\right) \text{ in } \mathcal{N}.
\]
The inequality is strict because \( A(\frac{w}{w_z}) \) cannot equal \( a_1 \) and \( a_2 \) simultaneously. It follows
\[
\partial_z\left(\frac{w}{w_z}\right) > 1 - 2\left(a_1 - \frac{\mu}{\sigma^2 a_1} - \frac{1}{2}a_2 + 1\right) = 1 - 2a_1 + a_2 + \frac{2r_1}{\mu - r_1} \geq 0 \text{ in } \mathcal{N}.
\]
This completes the proof. \( \square \)

Now, we prove Proposition 5.11. Let \( J = -w/w_z, \) from the definition of \( b(\cdot) \) and \( l(\cdot) \) we have \( J(b(s), s) = a_2, J(l(s), s) = a_1, \) for all \( s \in (0, T] \). When the conditions (5.16) and (5.17) hold, the above result shows \( J_z(b(s), s) \) and \( J_z(l(s), s) < 0 \). So it follows from the implicit function existence theorem that \( b(\cdot), l(\cdot) \in C^1((0, T]) \).

6 Back to the HJB Equation (3.1) and Problem (2.3)

We are now ready to construct a classical solution to the PDE (3.1) from the function \( w \) given in Theorem 5.1 and deduce the optimal portfolio to the problem (2.3).

First, we rewritten the PDEs (4.5) and (4.4) of \( u \) and \( v \) in compact forms as follows
\[
\begin{cases}
-u_t - J u = 0 & \text{in } (0, +\infty) \times [0, T), \\
u(y, T) = \frac{1}{2}y - d, & y > 0,
\end{cases}
\]
(6.1)
and
\[
\begin{cases}
-v_t - H v = 0 & \text{in } (0, +\infty) \times [0, T), \\
v(y, T) = -\frac{1}{4}y^2 + dy, & y > 0,
\end{cases}
\]
(6.2)
where
\[ \mathcal{J}u := \frac{1}{2} \sigma^2 A^2 \left( \frac{u}{yu_y} \right) y^2 u_{yy} - \left( \mu - \sigma^2 A^2 \left( \frac{u}{yu_y} \right) \right) y u_y - \left( \mu - \sigma^2 A \left( \frac{u}{yu_y} \right) \right) u, \]
and
\[ \mathcal{H}v := \frac{1}{2} \sigma^2 A^2 \left( \frac{v_y}{yv_{yy}} \right) y^2 v_{yy} - \left( \mu - \sigma^2 A \left( \frac{v_y}{yv_{yy}} \right) \right) v y v_y. \]

**Lemma 6.1** Let \( w \) be given in Theorem 5.1 and set \( u(y, t) = w(ln y, T - t) \). Then \( u \in C^{2+\alpha,1+\frac{\alpha}{2}} ((0, +\infty) \times [0, T]) \) is a solution to the PDE (6.1) such that
\[
\begin{align*}
\frac{1}{2} e^{\theta_2(T-t)} y - e^{-r_1(T-t)} d & \leq u \leq \frac{1}{2} e^{\theta_1(T-t)} y - e^{-r_2(T-t)} d, \\
\frac{1}{2} e^{-\kappa(T-t)} \leq y u_y \leq \frac{1}{2} e^{k(T-t)},
\end{align*}
\]
in \((0, +\infty) \times [0, T]\).

This result can be easily verified. Moreover, we have

**Lemma 6.2** For any \( t \in [0, T] \),
\[
\lim_{y \to 0^+} u = -e^{-r_1(T-t)} d, \quad \lim_{y \to +\infty} u = +\infty, \quad \lim_{y \to 0^+} y u = 0, \quad \lim_{y \to 0^+} y^2 u_y = 0.
\]

**PROOF:** The second and third limits can be derived by (6.3), the fourth limit is due to (6.4). It is left to prove the first limit. Thanks to the estimate (5.6), there exists \( z_0 \in (-\infty, \inf_{s \in [0, T]} l(s)) \). Then \( A(w/w_z) = a_1 \) in \( \mathcal{D} := (-\infty, z_0] \times [0, T] \), and thus,
\[
w_y = \frac{1}{2} \sigma^2 a_1^2 w_{zz} + \left( \mu - \frac{1}{2} \sigma^2 a_1^2 - \sigma^2 a_1 \right) w_z + (\mu - \sigma^2 a_1) w = 0 \quad \text{in} \quad \mathcal{D}.
\]

Let \( M := \max \left\{ \frac{1}{2} e^{z_0}, \max_{s \in [0, T]} w(z_0, s) + e^{-r_1 s} d \right\} \) and denote \( \Psi(z, s) := M e^{\theta_1 z} e^{-z_0} - e^{-r_1 s} d. \)

Then
\[
\Psi_s - \frac{1}{2} \sigma^2 a_1^2 \Psi_{zz} + \left( -\frac{1}{2} \sigma^2 a_1^2 - \sigma^2 a_1 + \mu \right) \Psi_z + \left( \mu - \sigma^2 a_1 \right) \Psi
\]
\[
= M e^{\theta_1 z} e^{-z_0} \left( |\theta_1| - \frac{1}{2} \sigma^2 a_1^2 + \left( -\frac{1}{2} \sigma^2 a_1^2 - \sigma^2 a_1 + \mu \right) + \left( \mu - \sigma^2 a_1 \right) \right)
\]
\[
+ e^{-r_1 s} d(r_1 - (\mu - \sigma^2 a_1)) 
\geq 0,
\]
by recalling the definitions of \( \theta_1 \) and \( a_1 \). Moreover,
\[
\begin{cases}
\Psi(z, 0) = M e^{z_0} - d \geq \frac{1}{2} e^z - d = w(z, 0), \quad z \leq z_0, \\
\Psi(z_0, s) \geq M - e^{-r_1 s} d \geq w(z_0, s), \quad s \in [0, T],
\end{cases}
\]
by the comparison principle, we get \( \Psi \geq w \) in \( D \). Together with the first inequality in (5.1), we have \( \lim_{z \to -\infty} w = -e^{-r_1s}d \), which implies \( \lim_{y \to 0^+} u = -e^{-r_1(T-t)}d \). \hfill \Box 

By the above results, we see that that \( -u \) is one-to-one mapping \((0, +\infty) \times [0, T)\) to \( Q^T \).

**Lemma 6.3** Let \( u \) be given in Lemma 6.1. Define
\[
v(y, t) := -\int_0^y u(\xi, t)d\xi, \quad (y, t) \in (0, +\infty) \times [0, T].
\]
Then \( v \in C^{3,2}((0, +\infty) \times [0, T]) \) is a solution to the PDE (6.2) such that
\[
-\frac{1}{2}e^{\theta_2(T-t)} y + e^{-r_2(T-t)}d \leq v_y \leq -\frac{1}{2}e^{\theta_2(T-t)} y + e^{-r_1(T-t)}d, \quad (6.6)
\]
\[
-\frac{1}{2}e^{k(T-t)} \leq v_{yy} \leq -\frac{1}{2}e^{-\kappa(T-t)}, \quad (6.7)
\]
in \((0, +\infty) \times [0, T]\). Moreover, for any \( t \in [0, T] \),
\[
\lim_{y \to 0^+} v_y = -e^{-r_1(T-t)}d, \quad \lim_{y \to +\infty} v_y = -\infty, \quad \lim_{y \to 0^+} yv_y = 0 \quad \lim_{y \to 0^+} y^2v_{yy} = 0. \quad (6.8)
\]

**PROOF:** Clearly \( v_y = -u \), so (6.6), (6.7) and (6.8) are the direct consequences of (6.3), (6.4) and (6.5) respectively. Since \( u \in C^{2+\alpha, 1+\frac{\alpha}{2}}((0, +\infty) \times [0, T]) \), we have \( v, v_y \in C^{2+\alpha, 1+\frac{\alpha}{2}}((0, +\infty) \times [0, T]) \). Moreover, it is easy to check that \( \partial_y(-v_t - \mathcal{H}v) = u_t +Ju = 0 \) and \( -v_t - \mathcal{H}v(0, t) = 0 \), so
\[
(-v_t - \mathcal{H}v)(y, t) = (-v_t - \mathcal{H}v)(0, t) + \int_0^y \partial_y(-v_t - \mathcal{H}v)(\xi, t)d\xi = 0.
\]
Therefore, \( v \) is a solution to the PDE (6.2). As a consequence,
\[
-v_{tt} - \frac{1}{2}\sigma^2A^2\left(\frac{v_y}{yv_{yy}}\right)y^2v_{yyy} + \left(\mu - \sigma^2A\left(\frac{v_y}{yv_{yy}}\right)\right)yv_y = \partial_t(-v_t - \mathcal{H}v) = 0.
\]
Using Schauder interior estimation (see [21]), we get \( v_t \in C^{2+\alpha, 1+\frac{\alpha}{2}}((0, +\infty) \times [0, T]) \), therefore, we have \( v \in C^{3,2}((0, +\infty) \times [0, T]). \) \hfill \Box

**6.1 Proof of Theorem 3.1**

Now we are ready to prove Theorem 3.1. Let \( v \) be given in Lemma 6.3. Define
\[
V(x, t) := \sup_{y > 0}(v(y, t) - xy), \quad x < e^{-r_1(T-t)}d, \quad t \in [0, T]. \quad (6.9)
\]
We come to prove that the above \( V \) satisfies the requirements of Theorem 3.1.
For \( t \in [0, T] \), the estimates (6.7) and (6.8) imply \( v_y(\cdot, t) \) is strictly decreasing and maps \((0, \infty)\) to \((-\infty, e^{-r_1(T-t)}d)\), so
\[
J(x, t) := \operatorname{argmax}_{y > 0} (v(y, t) - xy) = (v_y(\cdot, t))^{-1}(x) > 0,
\]
and
\[
V(x, t) = v(J(x, t), t) - xJ(x, t), \quad x < e^{r_1(T-t)}d, \quad t \in [0, T]. \tag{6.10}
\]
Also the function \( J(x, t) \in C(Q^T) \) and is strictly increasing w.r.t. \( x \). Therefore,
\[
\begin{align*}
V_x(x, t) &= v_y(J(x, t), t)J_x(x, t) - xJ_x(x, t) - J(x, t) = -J(x, t) < 0, \\
V_{xx}(x, t) &= -J_x(x, t) = -\partial_x[(v_y(\cdot, t))^{-1}(x)] = \frac{-1}{v_{yy}(J(x, t), t)} > 0, \\
V_t(x, t) &= v_y(J(x, t), t)J_t(x, t) + v_t(J(x, t), t) - xJ_t(x, t) = v_t(J(x, t), t).
\end{align*}
\]
As \( v \in C^{3,2}((0, +\infty) \times [0, T]) \), we get \( V \in C^{3,2}(Q^T \setminus \{ x = e^{-r_1(T-t)}d \}) \). Since \( v \) is a solution to the PDE (6.2), which is equivalent to (4.4), one can check that \( V \) satisfies the PDE in (4.3). This together with \( V_x < 0 \) and \( V_{xx} > 0 \) shown above implies the PDE in (3.1).

From (6.8) we know for any \( t \in [0, T] \),
\[
\lim_{x \to e^{-r_1(T-t)}d-} J(x, t) = 0, \quad \lim_{x \to -\infty} J(x, t) = +\infty. \tag{6.11}
\]
So (3.4) holds. Moreover, (6.11) and (6.10) imply \( V(e^{-r_1(T-t)}d-, t) = v(0+, t) = 0 \), so the boundary condition in (3.1) holds.

Now, we verify the terminal condition. Thanks to (6.6) and \( v(0, t) = 0 \), we have
\[
-\frac{1}{4} e^{\theta_1(T-t)}y^2 + e^{-r_2(T-t)}dy \leq v \leq -\frac{1}{4} e^{\theta_2(T-t)}y^2 + e^{-r_1(T-t)}dy,
\]
and consequently,
\[
\begin{align*}
V(x, t) &= \sup_{y > 0} (v(y, t) - xy) \\
&\geq \sup_{y > 0} \left( -\frac{1}{4} e^{\theta_1(T-t)}y^2 + e^{-r_2(T-t)}dy - xy \right) \\
&= e^{-\theta_1(T-t)}(e^{-r_2(T-t)}d - x)^2, \tag{6.12}
\end{align*}
\]
and
\[
\begin{align*}
V(x, t) &\leq \sup_{y > 0} \left( -\frac{1}{4} e^{\theta_2(T-t)}y^2 + e^{-r_2(T-t)}dy - xy \right) \\
&= e^{-\theta_2(T-t)}(e^{-r_1(T-t)}d - x)^2. \tag{6.13}
\end{align*}
\]
Letting \( t \to T \) in the above two inequalities, it follows that \( V \) satisfies the terminal condition in (3.1). This completes the proof of Theorem 3.1.
6.2 Proof of Proposition 3.2

This is the consequence of Lemma 5.8 and Lemma 5.9. Thanks to (3.2) and (3.3), we see \((0, t) \in \mathcal{B}\), so \(B(t) > 0\).

6.3 Proof of Verification Theorem 3.4

In this section, we proves Verification Theorem 3.4.

Suppose \(V\) is the solution of (3.1) given in Theorem 3.1. Fix \((x, t) \in Q^T\), for any admissible portfolio \(\pi \in \Pi_t\), let \(X\) be the strong solution of (2.2). We set \(\tau_n = \inf\{s \geq t \mid |V_x(X_s, s)| + |\pi_s| \geq n\}\). Then \(s \mapsto \int_s^{\tau_n \wedge T} V_x(X_u, u)\sigma u dW_u\) is a martingale, whose mean is 0. Therefore, by Itô’s formula and the HJB equation (3.1),

\[
\mathbb{E}[V(X_{T \wedge \tau_n}, T \wedge \tau_n) \mid X_t = x] = V(x, t) + \mathbb{E}\left[\int_t^{T \wedge \tau_n} \left(V_t + \frac{1}{2}\sigma^2 \pi_s^2 V_{xx} + \left((r_1 \chi_{\pi_s < X_s} + r_2 \chi_{\pi_s > X_s})(X_s - \pi_s) + \mu \pi_s\right)V_x\right)(X_s, s) ds + \int_t^{T \wedge \tau_n} V_x(X_s, s)\sigma \pi_s dW_s \mid X_t = x\right] \geq V(x, t). \tag{6.14}
\]

Using the estimates (6.12) and (6.13), we have

\[
0 \leq V(X_{T \wedge \tau_n}, T \wedge \tau_n) \leq C\left(1 + \sup_{s \in [t, T]} |X_s|^2\right).
\]

By the standard estimate for SDE, the right hand side is integrable, so we can apply the dominated convergence theorem to \(\mathbb{E}[V(X_{T \wedge \tau_n}, T \wedge \tau_n) \mid X_t = x]\), and obtain

\[
V(x, t) \leq \mathbb{E}[\lim_{n \to \infty} V(X_{T \wedge \tau_n}, T \wedge \tau_n) \mid X_t = x] = \mathbb{E}[V(X_T, T) \mid X_t = x] = \mathbb{E}[(X_T - d)^2 \mid X_t = x]. \tag{6.15}
\]

Therefore, we have

\[
V(x, t) \leq \inf_{\pi \in \Pi_t} \mathbb{E}[(X_T - d)^2 \mid X_t = x].
\]

To show the reverse inequality, define a feedback control

\[
\pi^*(x, t) = \begin{cases} 
-a_1 \frac{V_x}{V_{xx}}, & -a_1 \frac{V_x}{V_{xx}} < x, \\
x, & -a_2 \frac{V_x}{V_{xx}} \leq x \leq -a_1 \frac{V_x}{V_{xx}}, \\
-a_2 \frac{V_x}{V_{xx}}, & -a_2 \frac{V_x}{V_{xx}} > x.
\end{cases}
\]

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Clearly (6.6) and (6.7) imply $|yv_{yy}| \leq C(1 + |v_y|)$ for some constant $C$, which is equivalent to $|\nabla_{xx}V| \leq C(1 + |x|)$, so $\pi^*(x, t)$ is linear growth in $x$. Moreover, $\pi^*(x, t)$ is locally Lipschitz continuous by the smoothness of $V_x$ and $V_{xx} > 0$. By Theorem 3.4, p.56, Mao [27], we conclude that there exists a unique strong solution $X^*$ to the following SDE:

$$
\begin{cases}
\begin{align*}
\frac{dX_s^*}{ds} &= \left[ (r_1\pi^*(X_s^*, s) < X_s^* + r_2\pi^*(X_s^*, s) > X_s^* ) (X_s^* - \pi^*(X_s^*, s)) \\
&\quad + \mu\pi^*(X_s^*, s) \right] ds + \sigma\pi^*(X_s^*, s) dW_s, \quad s \in [t, T], \\
X_t^* &= x.
\end{align*}
\end{cases}
$$

(6.16)

Furthermore, as $\pi^*(x, t)$ is linear growth in $x$, by Lemma 3.2, p.51, Mao [27], we see

$$
\mathbb{E} \left[ \sup_{s \in [t, T]} |X_s^*|^2 \right] < \infty,
$$

which further implies $\pi_s^* := \pi^*(X_s^*, s)$ is an admissible control in $\Pi_t$. Repeat the proceeding argument with this control, then the inequalities in (6.14) and (6.15) become equations, giving

$$
V(x, t) = \mathbb{E}[(X_T - d)^2 | X_t = x].
$$

Therefore, $V$ is the value function of the problem (2.3) and $\pi_s^*$ is an optimal control.

7 Concluding Remarks

Clearly Verification Theorem 3.4 implies the function $V$ given in Theorem 3.1 is unique. This uniqueness can be proved by pure PDE method as well. As a consequence, the functions $w$ in Theorem 5.1, $u$ in Lemma 6.1 and $v$ in Lemma 6.3 are also unique.

This paper used PDE method to solve the mean-variance hedging problem. This approach does not work if the system is not Markovian. So general control theory for piecewise linear quadratic problems is called for. Of course, it is of great importance to develop such theory, and also far beyonds the scope of this paper. But we hope our method can inspire the readers to develop such theory.
Appendix A

In this section, we prove Theorem 5.1.

Firstly, for fixed $0 < \varepsilon < 1$, define a continuous function

$$\Gamma_\varepsilon(\xi, \eta) := A\left(\frac{\xi}{\eta + \varepsilon}\right), \quad (\xi, \eta) \in (-\infty, +\infty) \times [0, +\infty).$$

Note that

$$\partial_\xi \Gamma_\varepsilon(\xi, \eta) = A'\left(\frac{\xi}{\eta + \varepsilon}\right)\frac{1}{\eta + \varepsilon} = \begin{cases} -\frac{1}{\eta + \varepsilon} \in [-\frac{1}{\varepsilon}, 0), a_2 < -\frac{\xi}{\eta + \varepsilon} < a_1 & -\frac{\xi}{\eta + \varepsilon} > a_1 \text{ or } -\frac{\xi}{\eta + \varepsilon} < a_2, \\ 0, & \end{cases}$$

$$\partial_\eta \Gamma_\varepsilon(\xi, \eta) = A'\left(\frac{\xi}{\eta + \varepsilon}\right)\frac{-\xi}{(\eta + \varepsilon)^2} = \begin{cases} \frac{\xi}{\eta + \varepsilon} \in [-\frac{\alpha}{\varepsilon}, 0), a_2 < -\frac{\xi}{\eta + \varepsilon} < a_1 & -\frac{\xi}{\eta + \varepsilon} > a_1 \text{ or } -\frac{\xi}{\eta + \varepsilon} < a_2, \\ 0, & \end{cases}$$

so the function $\Gamma_\varepsilon(\cdot, \cdot)$ is Lipschitz continuous in $(-\infty, +\infty) \times [0, +\infty)$. Moreover, for each fixed $c > 0$, $\partial_\xi \Gamma_\varepsilon(\xi, \eta)$ and $\partial_\eta \Gamma_\varepsilon(\xi, \eta)$ are uniformly bounded for all $(\xi, \eta, \varepsilon)(-\infty, +\infty) \times [c, +\infty) \times [0, 1]$.

Now, consider an approximation equation in a bounded domain

$$\begin{cases}
w_s^\varepsilon N - \frac{1}{2}\sigma^2 A^2(\frac{w_s^\varepsilon N}{|w_z^\varepsilon N| + \varepsilon}) w_{zz}^\varepsilon N + \left(\mu - \frac{1}{2}\sigma^2 A^2(\frac{w_s^\varepsilon N}{|w_z^\varepsilon N| + \varepsilon}) - \sigma^2 A(\frac{w_s^\varepsilon N}{|w_z^\varepsilon N| + \varepsilon})\right) w_z^\varepsilon N \\
\quad + \left(\mu - \sigma^2 A(\frac{w_s^\varepsilon N}{|w_z^\varepsilon N| + \varepsilon})\right) w_{z}^\varepsilon N = 0 \quad \text{in} \quad Q_T^N := (-N, N) \times [0, T],
\end{cases}$$

$$(w_{z}^\varepsilon N - w_{z}^\varepsilon N)(-N, s) = -e^{-r_2 s} d, \quad w_{z}^\varepsilon N(N, s) = \frac{1}{2}e^{\theta_1 s} e^N, \quad s \in [0, T],$$

$$w_{z}^\varepsilon N(z, 0) = \frac{1}{2}e^{z} - d, \quad -N < z < N,$$

(A.1)

The Leray-Schauder fixed point theorem (see [10, 12]) and embedding theorem (see [10]) imply the existence of $C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{Q}_T^N)$ (for some $\alpha \in (0, 1)$) solution to the problem (A.1). Moreover, the Schauder estimation (see [21]) implies $w_{z}^\varepsilon N \in C^{2+\alpha, 1+\alpha}(\bar{Q}_T^N)$.

In the below proof, we will frequently use the following facts without claim: $0 < a_2 \leq A(\xi) \leq a_1, |A'(\xi)| \leq 1, |A'(\xi)| \xi \leq a_1$ and $|A'(\xi)\xi|^2 \leq a_1^2$.

We now establish the estimates

$$\frac{1}{2}e^{\theta_2 s} e^z - e^{-r_1 s} d \leq w_{z}^\varepsilon N \leq \frac{1}{2}e^{\theta_1 s} e^z - e^{-r_2 s} d.$$  \hspace{1cm} (A.2)

Denote $\psi(z, s) = \frac{1}{2}e^{\theta_2 s} e^z - e^{-r_1 s} d$ and $A(\cdot) = A(\frac{w_s^\varepsilon N}{|w_z^\varepsilon N| + \varepsilon})$. Using the definitions of $\theta_2, a_1$
Applying the comparison principle for linear equation, the first inequality in (A.2) is established.

$$\psi_z - \frac{1}{2} \sigma^2 A^2(\cdot) \psi_{zz} + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \psi_z + \left( \mu - \sigma^2 A(\cdot) \right) \psi = \frac{1}{2} e^{\theta_z} e^z \left( \theta_z - \frac{1}{2} \sigma^2 A^2(\cdot) + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \right)$$

Moreover,

$$\Psi_s - \frac{1}{2} \sigma^2 A^2(\cdot) \Psi_{zz} + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \Psi_z + \left( \mu - \sigma^2 A(\cdot) \right) \Psi = \frac{1}{2} e^{\theta_s} e^z \left( \theta_s - \frac{1}{2} \sigma^2 A^2(\cdot) + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \right)$$

Notice $\theta_1 > \theta_2$, so

$$\left\{ \begin{array}{ll} \psi(z, 0) = \frac{1}{2} e^z - d = w^{\varepsilon, N}(z, 0), & -N < z < N, \\ (\psi - \psi_z)(-N, s) = -e^{-r_s} d \leq -e^{-r_s} d = (w^{\varepsilon, N} - w^{\varepsilon, N}_z)(-N, s), & s \in [0, T], \\ \psi_z(N, s) = \frac{1}{2} e^{\theta_z} e^N \leq \frac{1}{2} e^{\theta_1} e^N = w^{\varepsilon, N}_z(N, s), & s \in [0, T]. \end{array} \right.$$
where \( \theta_3 = \min\{\mu - \sigma^2 a_1(a_1 + 3), r_1\} \). Differentiating the equation in (A.1) w.r.t. \( z \) we have

\[
\begin{align*}
\partial_z w_{z}^{\varepsilon,N} - \frac{\sigma^2}{2} \partial_z (A^2(\cdot) \partial_z w_{z}^{\varepsilon,N}) + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \partial_z w_{z}^{\varepsilon,N} + \left( \mu - \sigma^2 A(\cdot) \right) w_{z}^{\varepsilon,N} \\
- \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) - \frac{w_{z}^{\varepsilon,N}}{(|w_{z}^{\varepsilon,N}| + \varepsilon)^2} \text{sgn}(w_{z}^{\varepsilon,N}) w_{z}^{\varepsilon,N} \right) (A(\cdot) + 1) w_{z}^{\varepsilon,N} \\
- \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) - \frac{w_{z}^{\varepsilon,N}}{(|w_{z}^{\varepsilon,N}| + \varepsilon)^2} \text{sgn}(w_{z}^{\varepsilon,N}) w_{z}^{\varepsilon,N} \right) w_{z}^{\varepsilon,N} = 0.
\end{align*}
\]

After reorganizing, we get an equation for \( w_{z}^{\varepsilon,N} \) in the divergence form:

\[
\begin{align*}
\partial_z w_{z}^{\varepsilon,N} - \frac{\sigma^2}{2} \partial_z (A^2(\cdot) \partial_z w_{z}^{\varepsilon,N}) + \left( - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) + \mu \right) \partial_z w_{z}^{\varepsilon,N} \\
+ \left( \mu - \sigma^2 A(\cdot) \right) w_{z}^{\varepsilon,N} - \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( A(\cdot) + 1 \right) w_{z}^{\varepsilon,N} \\
+ \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \text{sgn}(w_{z}^{\varepsilon,N}) \partial_z w_{z}^{\varepsilon,N} \\
+ \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \text{sgn}(w_{z}^{\varepsilon,N}) \partial_z w_{z}^{\varepsilon,N} - \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) w_{z}^{\varepsilon,N} = 0. \tag{A.4}
\end{align*}
\]

It is not hard to see all the coefficients in (A.4) are bounded. Denote \( \psi(z, s) = -e^{-\theta_3 s} d \), then

\[
\begin{align*}
\partial_z \psi - \frac{\sigma^2}{2} \partial_z (A^2(\cdot) \partial_z \psi) + \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdot) - \sigma^2 A(\cdot) \right) \partial_z \psi \\
+ \left( \mu - \sigma^2 A(\cdot) \right) \psi - \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( A(\cdot) + 1 \right) \psi \\
+ \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \text{sgn}(w_{z}^{\varepsilon,N}) \partial_z \psi \\
+ \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \text{sgn}(w_{z}^{\varepsilon,N}) \partial_z \psi - \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \psi \\
= e^{-\theta_3 s} d \left( \theta_3 - \mu + \sigma^2 A(\cdot) + \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \left( A(\cdot) + 1 \right) + \sigma^2 A'(\cdot) \left( \frac{w_{z}^{\varepsilon,N}}{|w_{z}^{\varepsilon,N}| + \varepsilon} \right) \right) \\
\leq e^{-\theta_3 s} d (\theta_3 - \mu + \sigma^2 a_1 + \sigma^2 a_1 (a_1 + 1) + \sigma^2 a_1) \leq 0,
\end{align*}
\]

thanks to the definition of \( \theta_3 \). Moreover,

\[
\begin{align*}
w_{z}^{\varepsilon,N}(z, 0) &= \frac{1}{2} e^z \geq 0 \geq \psi(z, 0), \\
w_{z}^{\varepsilon,N}(-N, s) &= w_{z}^{\varepsilon,N}(-N, s) + e^{-r_2 s} d > -e^{-r_1 s} d \geq \psi(-N, s), \quad \text{(by (A.2))} \\
w_{z}^{\varepsilon,N}(N, s) &= \frac{1}{2} e^{\theta_3 s} e^N \geq 0 \geq \psi(N, s).
\end{align*}
\]
Using the comparison principle for divergence form (see \cite{10} or \cite{30}), we obtain $w_{z,N}^\varepsilon \geq \psi$; giving (A.3).

We next to prove

\[
 w_{z,N}^\varepsilon \leq \frac{1}{2} e^{ks} e^z. \tag{A.5}
\]

Denote $g_{z,N}^\varepsilon(z,s) = e^{-z} w_{z,N}^\varepsilon(z,s)$. According to (A.4), we have

\[
 \partial_s g_{z,N}^\varepsilon - \frac{\sigma^2}{2} \partial_z \left( A^2(\cdot) g_{z,N}^\varepsilon \right) - \sigma^2 A^2(\cdot) g_{z,N}^\varepsilon - \frac{\sigma^2}{2} A^2(\cdot) g_{z,N}^\varepsilon \\
 - \sigma^2 A(\cdot) A'(\cdot) g_{z,N}^\varepsilon \left( \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \right) g - \left( \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \right) \text{sgn}(w_{z,N}^\varepsilon) \left( g_{z,N}^\varepsilon + g_{z,N}^\varepsilon \right) \\
 - \left( \mu - \frac{1}{2} \sigma^2 A(\cdot) - \sigma^2 A(\cdot) \right) \left( g_{z,N}^\varepsilon + g_{z,N}^\varepsilon \right) \\
 + \left( \mu - \sigma^2 A(\cdot) \right) g_{z,N}^\varepsilon - \sigma^2 A'(\cdot) \left( A(\cdot) + 1 \right) g_{z,N}^\varepsilon \\
 + \sigma^2 A'(\cdot) \left( \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \right)^2 \text{sgn}(w_{z,N}^\varepsilon) \left( g_{z,N}^\varepsilon + g_{z,N}^\varepsilon \right) - \sigma^2 A'(\cdot) \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} g_{z,N}^\varepsilon = 0. \tag{A.6}
\]

On the other hand, denote $\Psi(z,s) = \frac{1}{2} e^{ks}$, then

\[
 \partial_s \Psi - \frac{\sigma^2}{2} \partial_z \left( A^2(\cdot) \Psi \right) - \sigma^2 A^2(\cdot) \Psi \Psi - \frac{\sigma^2}{2} A^2(\cdot) \Psi \\
 - \sigma^2 A(\cdot) A'(\cdot) \left( \frac{w_{z,N}^\varepsilon}{w_{z,N}^\varepsilon} \right) \Psi - \left( \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \right) \text{sgn}(w_{z,N}^\varepsilon) \left( \Psi + \Psi \right) \\
 + \left( - \frac{1}{2} \sigma^2 A(\cdot) - \sigma^2 A(\cdot) + \mu \right) \left( \Psi + \Psi \right) \\
 + \left( \mu - \sigma^2 A(\cdot) \right) \Psi - \sigma^2 A'(\cdot) \left( A(\cdot) + 1 \right) \Psi \\
 + \sigma^2 A'(\cdot) \left( \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \right)^2 \text{sgn}(w_{z,N}^\varepsilon) \left( \Psi + \Psi \right) - \sigma^2 A'(\cdot) \frac{w_{z,N}^\varepsilon}{|w_{z,N}^\varepsilon| + \varepsilon} \Psi \\
 \geq \frac{1}{2} e^{ks} \left( k - \frac{1}{2} \sigma^2 a_1^2 - \sigma^2 a_1 - \frac{1}{2} \sigma^2 a_1^2 - \sigma^2 a_1 (a_1 + 1) - \sigma^2 a_1^2 - \sigma^2 a_1 \right) \geq 0,
\]

thanks to the definition of $k$. Notice $k \geq \theta_1$, so

\[
 \begin{cases}
 g_{z,N}(z,0) = \frac{1}{2} = \Psi(z,0), \\
 g_{z,N}(-N,s) = e^{N w_{z,N}^\varepsilon + e^{-rs} d}(-N,N) \leq \frac{1}{2} e^{k s} \leq \frac{1}{2} e^{ks} = \Psi(-N,s), \quad \text{by (A.2)} \\
 g_{z,N}(N,s) = \frac{1}{2} e^{k s} \leq \frac{1}{2} e^{ks} = \Psi(N,s).
\end{cases}
\]
Using the comparison principle for divergence form, we obtain \( g^{\varepsilon,N} \leq \Psi \), proving (A.5).

Thanks to (A.2), (A.3) and (A.5), for each \( a < b \), when \( N > \max \{|a|, |b|\} \), taking the \( C^{\alpha,2} \) interior estimate (see [21] or [24]) to the equation in (A.1) and (A.4) respectively, we obtain

\[
|w^{\varepsilon,N}|_{C^{\alpha,2}([a,b] \times [0,T])}, \quad |w^{\varepsilon,N}_z|_{C^{\alpha,2}([a,b] \times [0,T])} \leq C.
\]

where \( C \) is independent of \( \varepsilon \) and \( N \). Since \( \Gamma_\varepsilon(\cdot,\cdot) \) is Lipschitz continuous in \( (-\infty, +\infty) \times [0, +\infty) \), we have

\[
\left| A\left( \frac{w^{\varepsilon,N}}{|w^{\varepsilon,N}_z| + \varepsilon} \right) \right|_{C^{\alpha,2}([a,b] \times [0,T])} \leq C_\varepsilon
\]

i.e. the coefficients in the equation of (A.1) belong to \( C^{\alpha,2}([a,b] \times [0,T]) \), so we can take the Schauder interior estimate to the equation in (A.1) to get

\[
|w^{\varepsilon,N}|_{C^{2+\alpha,1+\alpha}([a,b] \times [0,T])} \leq C_\varepsilon.
\]

where the above two \( C_\varepsilon \)s are independent of \( N \). Therefore, there exists \( w^{\varepsilon} \in C^{2+\alpha,1+\alpha}(\overline{Q_T}) \) such that, for any region \( Q = (a, b) \times (0, T) \subset Q_T \), there exists a subsequence of \( w^{\varepsilon,N} \), which we still denote by \( w^{\varepsilon,N} \), such that \( w^{\varepsilon,N} \to w^{\varepsilon} \) in \( C^{2,1}(\overline{Q}) \) when \( N \to \infty \). So \( w^{\varepsilon} \) satisfies the initial problem

\[
\begin{aligned}
&\begin{cases}
  w^{\varepsilon}_s - \frac{1}{2} \sigma^2 A^2 \left( \frac{w^{\varepsilon}}{|w^{\varepsilon}_z| + \varepsilon} \right) w^{\varepsilon}_{zz} + \left( \mu - \frac{1}{2} \sigma^2 A^2 \left( \frac{w^{\varepsilon}}{|w^{\varepsilon}_z| + \varepsilon} \right) - \sigma^2 A \left( \frac{w^{\varepsilon}}{|w^{\varepsilon}_z| + \varepsilon} \right) \right) w^{\varepsilon}_z \\
  + \left( \mu - \sigma^2 A \left( \frac{w^{\varepsilon}}{|w^{\varepsilon}_z| + \varepsilon} \right) \right) w^{\varepsilon} = 0 & \text{in } Q_T,
\end{cases}

&w^{\varepsilon}(z, 0) = \frac{1}{2} e^{\varepsilon^2} - d
\end{aligned}
\]

with the exponential growth condition on \( w^{\varepsilon} \) and \( w^{\varepsilon}_z \) by the estimates (A.2), (A.3) and (A.5).

We now prove

\[
w^{\varepsilon}_z \geq \frac{1}{2} e^{-\kappa \varepsilon} e^{\varepsilon^2}.
\]

(A.10)
Denote \( g^\varepsilon(z,s) = e^{-z w^\varepsilon_s(z,s)} \) and \( A(\cdots) = A\left(\frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon}\right) \). Letting \( N \to \infty \) in (A.6), we obtain

\[
\partial_s g^\varepsilon - \frac{\sigma^2}{2} \partial_z \left( A^2(\cdots) g^\varepsilon \right) - \sigma^2 A^2(\cdots) g^\varepsilon - \frac{\sigma^2}{2} A^2(\cdots) g^\varepsilon
- \sigma^2 A(\cdots) A'(\cdots) \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} g^\varepsilon - \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \right) \operatorname{sgn}(w^\varepsilon) (g^\varepsilon + g^\varepsilon) \right)
+ \left( \mu - \frac{1}{2} \sigma^2 A^2(\cdots) - \sigma^2 A(\cdots) \right) (g^\varepsilon + g^\varepsilon)
+ \left( \mu - \sigma^2 A(\cdots) \right) g^\varepsilon - \sigma^2 A'(\cdots) \left( A(\cdots) + 1 \right) g^\varepsilon
+ \sigma^2 A'(\cdots) \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \left( A(\cdots) + 1 \right) \operatorname{sgn}(w^\varepsilon) (g^\varepsilon + g^\varepsilon)
+ \sigma^2 A'(\cdots) \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \right)^2 \operatorname{sgn}(w^\varepsilon) (g^\varepsilon + g^\varepsilon) - \sigma^2 A'(\cdots) \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} g^\varepsilon = 0.
\]

On the other hand, denote \( \Psi(z,s) = \frac{1}{2} e^{-\varepsilon s} \), we have

\[
\partial_s \Psi - \frac{\sigma^2}{2} \partial_z \left( A^2(\cdots) \Psi \right) - \sigma^2 A^2(\cdots) \Psi - \frac{\sigma^2}{2} A^2(\cdots) \Psi
- \sigma^2 A(\cdots) A'(\cdots) \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \Psi - \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \right) \operatorname{sgn}(w^\varepsilon) (\Psi + \Psi) \right)
+ \left( - \frac{1}{2} \sigma^2 A^2(\cdots) - \sigma^2 A(\cdots) + \mu \right) (\Psi + \Psi)
+ \left( \mu - \sigma^2 A(\cdots) \right) \Psi - \sigma^2 A'(\cdots) \left( A(\cdots) + 1 \right) \Psi
+ \sigma^2 A'(\cdots) \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \left( A(\cdots) + 1 \right) \operatorname{sgn}(w^\varepsilon) (\Psi + \Psi)
+ \sigma^2 A'(\cdots) \left( \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \right)^2 \operatorname{sgn}(w^\varepsilon) (\Psi + \Psi) - \sigma^2 A'(\cdots) \frac{w^\varepsilon}{|w^\varepsilon| + \varepsilon} \Psi
\leq \frac{1}{2} e^{-\varepsilon s} \left( - \kappa + \sigma^2 a_1 (1 + a_1) + \mu + \mu + \sigma^2 (a_1 + 1) + \sigma^2 a_1 (a_1 + 1) + \sigma^2 a_1 + \sigma^2 a_1 \right) = 0,
\]

thanks to the definition of \( \kappa \). Moreover, \( g(z,0) = \frac{1}{2} = \Psi(z,0) \). By the comparison principle we have \( g \geq \Psi \), hence, (A.10) is proved.

Thanks to (A.5) and (A.10), \( w^\varepsilon \) has positive lower and upper bounds which are independent of \( \varepsilon \) in any bounded region, noting that the bounds of \( |\partial_\eta \Gamma_\varepsilon(\xi,\eta)| \) and \( |\partial_\eta \Gamma_\varepsilon(\xi,\eta)| \) are independent of \( \varepsilon \) when \( \eta \) has a positive lower bound, so the constants \( C_\varepsilon s \) in the estimates (A.7) and (A.8) are independent of \( \varepsilon \). Let \( \varepsilon \to 0 \) in (A.9), we obtain a limit \( w \) that satisfies (4.7). Moreover, (5.1) and (5.2) are the direct consequences of (A.2), (A.5), (A.10).
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