PLANES IN SCHATTEN-3

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Abstract. We prove that any two-dimensional real subspace of Schatten-3 can be linearly isometrically embedded into $L_3$. This resolves the case $p = 3$ of Hanner’s inequality for Schatten classes, conjectured by Ball, Carlen and Lieb. We conjecture that similar isometric embedding is possible for any $p \geq 1$.

1. Introduction

For $p \geq 1$, let $S_p$ denote the Schatten-$p$ trace-class; the space of compact operators on complex separable Hilbert space $H$ for which the sequence of singular values $(\sigma_i(A))_{i=1}^{\infty}$ belongs to $\ell_p$, with the norm

\[ \|A\|_{S_p} = \left( \sum_{i=1}^{\infty} \sigma_i(A)^p \right)^{1/p} = \left( \text{tr}(A^*A)^{p/2} \right)^{1/p}. \] (1)

Our main result is the following.

Theorem 1.0.1. For any $A, B \in S_3$ the real space $\text{span}_R \{A, B\}$ (with $S_3$ norm) can be linearly isometrically embedded into $L_3$.

In other words, for any $A, B \in S_3$ (or $n \times n$ matrices $A$ and $B$) there exits $f, g \in L_3(0,1)$ such that for any $\alpha, \beta \in \mathbb{R}$ one has

\[ \|\alpha A + \beta B\|_{S_3} = \|\alpha f + \beta g\|_{L_3}. \]

As a consequence, to prove any inequality that only depends on $S_3$-norms of real linear combinations of two matrices, it is enough to consider diagonal matrices. This implies, for instance, that $S_3$ has the same modulus of uniform convexity and smoothness as $\ell_3$, fact first proven by Ball, Carlen and Lieb in 1994 [2].

Theorem 1.0.1 also implies Hanner’s inequality for $S_3$. Namely, for any pair of $n \times n$ matrices $A$ and $B$, the following inequality holds for $p = 3$:

\[ \|A + B\|_{S_p}^p + \|A - B\|_{S_p}^p \leq (\|A\|_{S_p} + \|B\|_{S_p})^p \left( \|A\|_{S_p} - \|B\|_{S_p} \right)^p. \] (2)

This inequality was first considered by Hanner in 1955 [7] for $L_p$. Hanner proved that for $p \geq 2$ and $f, g \in L_p(0,1)$ one has

\[ \|f + g\|_{L_p}^p + \|f - g\|_{L_p}^p \leq (\|f\|_{L_p} + \|g\|_{L_p})^p \left( \|f\|_{L_p} - \|g\|_{L_p} \right)^p. \] (3)

Hanner also proved that the reverse inequality holds for $1 \leq p \leq 2$. These inequalities were then used to estimate the modulus of uniform convexity of $L_p$.

The non-commutative version (2) was first discussed by Ball, Carlen and Lieb in [2]; the inequality (2) was proved for $p \geq 4$ and the reverse inequality, i.e. for any pair of $n \times n$ matrices $A$
and $B$

$$\|A + B\|_{S_p}^p + \|A - B\|_{S_p}^p \geq (\|A\|_{S_p} + \|B\|_{S_p})^p + \|A\|_{S_p} - \|B\|_{S_p})^p$$

was then proved by duality (see [2, Lemma 6]) for $1 \leq p \leq 4/3$. While the authors managed to find the modulus of uniform convexity of $S_p$ for every $1 < p < \infty$ using other means, it was conjectured that the inequalities (2) and (4) hold in full ranges $p \geq 2$ and $1 \leq p \leq 2$, respectively. Despite serious effort, these ranges of $p$ have not been improved until now. In addition to the case $p = 3$, the aforementioned duality argument then also implies (4) for $p = 3/2$. Understanding the non-commutative Hanner’s inequalities (2) and (4) was the original motivation for the main result Theorem 1.0.1.

We make the following obvious conjecture.

**Conjecture 1.0.2.** For any $p \geq 1$ and $A, B \in S_p$ the real space $\text{span}_\mathbb{R}\{A, B\}$ can be linearly isometrically embedded into $L_p$.

If true, this would imply the non-commutative Hanner’s inequalities (2) and (4) for full ranges of $p$. More generally, to prove any $S_p$-inequality depending only on norms of elements in a plane, it would be enough to consider commuting/diagonal operators. See section 3 for some discussion and further partial results towards this conjecture.

We also note that similar isometric embedding is not possible in general for 3-dimensional subspaces, for any $p \geq 1$.

**Theorem 1.0.3.** For any $1 \leq p < \infty$, $p \neq 2$ there exists a 3-dimensional subspace of $S_p$ which is not linearly isometric to any subspace of $L_p$. In fact one may consider the space of real symmetric $2 \times 2$ matrices.

The main result Theorem 1.0.1 is closely related to the following new result of independent interest.

**Theorem 1.0.4.** Let $f : (a, b) \rightarrow \mathbb{R}$ be a function with non-negative 4th derivative (in distributional sense). Then for any pair of Hermitian $n \times n$ matrices $A$ and $B$ the function

$$t \mapsto \text{tr} f(A + tB)$$

also has non-negative 4th derivative.

An analogous result was known earlier for second derivatives (convex functions), and first derivatives (increasing functions) with the additional assumption that $B$ is positive semidefinite (see for instance [14, Propositions 1 and 2]). We also prove similar result for third derivatives, and conjecture that an analogous statement holds true for derivatives of any order, under the additional assumption that $B$ is positive semidefinite for odd orders.

**Theorem 1.0.5.** Let $f : (a, b) \rightarrow \mathbb{R}$ be a function with non-negative 3rd derivative (in distributional sense). Then for any $n \times n$ matrices $A, B$ with $B \geq 0$ the function

$$t \mapsto \text{tr} f(A + tB)$$

also has non-negative 3rd derivative.

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1See however [12], [2] and [5] for partial results with special assumptions on $A$ and $B$. 
Conjecture 1.0.6. Let \( k \) be a positive integer and \( f : (a, b) \to \mathbb{R} \) a function with non-negative \( k \)’th derivative (in distributional sense). Let \( A, B \) be \( n \times n \) matrices, and if \( k \) is odd, additionally assume that \( B \) is positive semidefinite. Then the function
\[
t \mapsto \text{tr} f(A + tB)
\]
has non-negative \( k \)’th derivative.

In section 3 we show that if true, conjecture 1.0.6 would give a new proof (and be a vast generalization of) Stahl’s Theorem [18] (previously known as BMV conjecture [3]).

Theorem 1.0.7 (Stahl). Let \( A, B \) be Hermitian \( n \times n \) matrices with \( B \) positive semidefinite. Then
\[
t \mapsto \text{tr} \exp(A - tB)
\]
is a Laplace transform of a positive measure on \([0, \infty)\).

Section 2 contains the proofs of the main results, together with the necessary preliminaries. Section 3 contains discussion and partial results towards the aforementioned conjectures.

2. PROOFS OF THE MAIN RESULTS

2.1. Preliminaries. We make the usual conventions \( \ell_p := L_p(\mathbb{N}, \mathbb{R}) \) and \( L_p := L_p((0, 1), \mathbb{R}) \), with counting and Lebesgue measures, respectively. We say that a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) is \( \ell_p \)-norm, if \((\mathbb{R}^n, \| \cdot \|)\) is isometric to a subspace of \( L_p \). This is equivalent to the existence of a symmetric measure \( \mu \) on \( S^{n-1} \) (given any inner product \( \langle \cdot, \cdot \rangle \)) such that
\[
\|v\|^p = \int_{S^{n-1}} |\langle v, u \rangle|^p d\mu(u)
\]
for any \( v \in \mathbb{R}^n \) (see [13]). If \( p \) is not an even integer, such measure, whenever it exists, is unique [9, Theorem 2]. The issue of recovering the measure from the norm is discussed by Koldobsky in [10]; we shall need a simple corollary of the work therein.

Lemma 2.1.1. Let \( \| \cdot \| \) be a norm in \( \mathbb{R}^2 \) such that for any \( v, w \in \mathbb{R}^2 \) the 4th distributional derivative of
\[
t \mapsto \|tv + w\|^3
\]
is a finite positive Borel measure \( \mu \), for which \( \mu(|x|^3) < \infty \). Then \( \| \cdot \| \) is linearly isometrically embeddable into \( L_3 \).

See section 3 for version of Lemma 2.1.1 for general \( p \geq 1 \).

Proof of Lemma 2.1.1. It follows directly Theorem 5 of [10] and Remark 1 thereafter that for any \( v, w \in \mathbb{R}^2 \) there exists two functions \( h_v, h_w : \mathbb{R} \to \mathbb{R} \), with \( u_v \) polynomial, \( h_v(0) = 0 = h_w(0) \); and \( L_p \) norm \( \| \cdot \|_{3,v,w} \) on \( \mathbb{R}^2 \) such that
\[
\|xv + yw\|^3 = \|xv + yw\|^3_{3,v,w} + h_v(x) + h_w(y).
\]
Considering \( y = 0 \), one sees that \( h_v \) can only be polynomial if \( h_v = 0 \); similarly \( u_w(y) = \alpha |y|^3 \) for some \( \alpha \in \mathbb{R} \). But this means that
\[
\| \cdot \|^3 = \int_{S^1} |\langle \cdot, u \rangle|^3 d\mu(u),
\]
where $\mu$ is positive measure, except possibly at one point (depending on $v, w$ and the pairing). Changing $v, w$ and applying the uniqueness result \cite{[9]} Theorem 2, we see that $\mu$ is positive everywhere.

We shall need only very basic properties Schatten classes. Together with the norm \cite{[11]} they form Banach spaces. For $1 \leq p < \infty$ finite rank operators are dense in $S_p$ \cite{[12]}. While $\ell_p$ is isometric a subspace of $S_p$, $S_p$ is not isomorphic to any subspace of $L_p$ \cite{[15]}

We shall need basic properties of divided differences. Define

$$[x_0]f = f(x_0),$$

and recursively for any positive integer $k$

$$[x_0, x_1, \ldots, x_k]f = \frac{[x_0, x_1, \ldots, x_{k-1}]f - [x_1, x_2, \ldots, x_k]f}{x_0 - x_k},$$

when points $x_0, x_1, \ldots, x_k$ are pairwise distinct. If $f$ is $C^k$, the divided differences of order $k$ has continuous extension to all tuples of $(k+1)$ points. This extension satisfies (6) whenever $x_0 \neq x_k$ and

$$[x_0, x_0, \ldots, x_0]f = \frac{f^{(k)}(x_0)}{k!},$$

where $x_0$ appears $(k+1)$ times. For this and lot more, see for instance \cite{[6]}

2.2. Proofs of Theorems \ref{thm01} and \ref{thm03}. Our goal is to check the conditions of Lemma \ref{lem11}

**Lemma 2.2.1.** Let $A$ and $B$ be $n \times n$ Hermitian matrices such that $B$ is invertible. Then

$$(\text{tr} |A + tB|^3)^{(4)} = O(1/|t|^5)$$

when $|t| \to \infty$.

**Proof.** Recall that since $A$ and $B$ are Hermitian, by the result of Rellich \cite{[16]} the eigenvalues of $B + \varepsilon A$ are $n$ analytic functions with fixed sign for small enough $\varepsilon$. Consequently for large enough $t$ one has

$$\text{tr} |A + tB|^3 = \sum_{i=1}^n |t|^3 |\lambda_i(B)|^3 (1 + O(1/t)),$$

where the multipliers are analytic outside compact set. Differentiating this 4 times kills the polynomial part and each summand is $O(1/|t|^5)$ as required. \hfill \Box

**Theorem 2.2.2.** For any Hermitian $n \times n$ matrices $A$ and $B$, with $B$ invertible, the function

$$T := t \mapsto \text{tr} |A + tB|^3$$

is analytic outside finitely many points where $A + tB$ is singular. Outside these points the function has non-negative 4th derivative. At the singular points the function behaves like

$$C|t - c|^3 + D|t - c|^3(t - c) + f(t),$$

where $C \geq 0$ and $f$ is $C^4$ near $c$. Consequently the 4th derivative has non-negative multiple of $\delta$ measure at the singular points is therefore altogether non-negative measure.
**Lemma 2.2.3.** Let \( A, B \) be Hermitian \( n \times n \) matrices and \( f \) be analytic near the eigenvalues of \( A \). Write \( F(t) = \text{tr}(f(A + tB)) \). Then

\[
\frac{F^{(k)}(0)}{k!} = \frac{1}{k} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}]_f B_{i_1, i_2} B_{i_2, i_3} \cdots B_{i_k, i_1}.
\]

Here \( B_{i,j} \) is the matrix of \( B \) in the eigenbasis of \( A \), and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( A \).

**Proof.** The second claim follows straightforwardly from the analyticity of the eigenvalues. Indeed, assume that say 0 is a singular point. Now any eigenvalue can be written analytically as \( \lambda_i(t) \). If \( \lambda_i(0) \neq 0 \), \(|\lambda_i(t)|^3\) is smooth near 0. If on the other hand \( \lambda_i(0) = 0 \), we can write \( \lambda_i(t) = t^k \nu_i(t) \) for some analytic \( \nu_i \) with \( \nu_i(0) \neq 0 \) and \( k \) positive integer (recall that \( B \) is invertible). Now

\[
|\lambda_i(t)|^3 = |t|^{3k} |\nu_i(t)|^3.
\]

Note however that the second term is analytic near 0. If \( k > 1 \) the whole function is \( C^4 \) near 0. If \( k = 1 \), we may expand \(|\nu_i|^3\) at 0 to get required expansion for a single eigenvalue. Repeating this for all the eigenvalues yields the claim.

Let us then move to the heart of the matter. First note that since \( \det (At + B) \) is non-zero polynomial, it can only have finitely many zeroes, and there can consequently be only finitely many points where \( T \) is not analytic.

Next we need an identity for the smooth part.

**Lemma 2.2.3. (cf. [8] Theorem 2.3.1.)** Let \( A, B \) be Hermitian \( n \times n \) matrices and \( f \) be analytic near the eigenvalues of \( A \). Write \( F(t) = \text{tr}(f(A + tB)) \). Then

\[
(\text{tr}(A + tB)^m)^{(k)}(0) = \sum_{j_0, j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \text{tr}(A^{j_0} B A^{j_1} B \cdots B A^{j_k})
\]

\[
= \sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} (j_k + 1) \text{tr}(B A^{j_1} B \cdots B A^{j_k})
\]

\[
= \frac{m}{k} \sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \text{tr}(A^{j_1} B \cdots B A^{j_k} B)
\]

\[
= \frac{m}{k} \sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \lambda_{i_1}^{j_1} \lambda_{i_2}^{j_2} \cdots \lambda_{i_k}^{j_k} B_{i_1, i_2} B_{i_2, i_3} \cdots B_{i_k, i_1}.
\]

But since

\[
\sum_{j_1, \ldots, j_k \geq 0, \sum j_i = m-k} \lambda_{i_1}^{j_1} \lambda_{i_2}^{j_2} \cdots \lambda_{i_k}^{j_k} = [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}]_{x=m-1},
\]

we are done.

Let us now focus on the main expression \((7)\) with \( k = 4 \) and \( f = | \cdot |^3 \). Write \( g = f'/3 = (\cdot)| \cdot | \)

We have the following simple identities.
Lemma 2.2.4. Let \( a_1, a_2, a_3, a_4 < b_1, b_2, b_3, b_4 \). Then the following identities hold:

\[
(8) \quad [a_1, a_2, a_3, a_4]_g = 0
\]

\[
(9) \quad [a_1, a_2, a_3, b_1]_g = \frac{2b_1^2}{(b_1 - a_1)(b_1 - a_2)(b_1 - a_3)}
\]

\[
(10) \quad [a_1, a_2, b_1, b_2]_g = \frac{2(b_1 + b_2)a_1a_2 - 2(a_1 + a_2)b_1b_2}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)}
\]

\[
= \frac{-2a_1b_1}{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)} + \frac{-2a_2b_2}{(b_2 - a_2)(b_2 - a_1)(b_1 - a_2)}
\]

\[
(11) \quad [a_1, b_1, b_2, b_3]_g = \frac{2a_1^2}{(b_1 - a_1)(b_2 - a_1)(b_3 - a_1)}
\]

\[
(12) \quad [b_1, b_2, b_3, b_4]_g = 0.
\]

Proof. Straightforward to check from the definitions. \( \square \)

Let us then complete the proof of Theorem 2.2.2. Assume that the matrix \( A \) has \( n_1 \) negative (\( \lambda_1, \lambda_2, \ldots, \lambda_{n_1} \)) and \( n_2 \) positive (\( \lambda_{n_1+1}, \ldots, \lambda_{n} \)) eigenvalues \( (n_1 + n_2 = n) \). To prove \( 7 \) is non-negative, we first group it in several parts. Sum consists of \( n^4 \) summands; separate them in patterns depending on whether \( i_1, i_2, i_3 \) and \( i_4 \) are at most or greater than \( n_1 \). If say \( i_1, i_3 \leq n_1 < i_2, i_4 \), assign this term pattern \( (-, +, -, +) \). Write then \( S_{-, +, -, +} \) for the sum of all the terms with pattern \( (-, +, -, +) \). Note that since \( [x_0, x_1, \ldots, x_k]_f = [x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(k)}]_f \) for any permutation \( \sigma \), many of these sums are equal, and they can be in fact split into following groups:

(a) \( S_{-, -, -, -} \)

(b) \( S_{+, -, -, -}, S_{-, -, -, -}, S_{-, +, -, -}, S_{-, -, -} \)

(c) \( S_{+, -, -, -}, S_{+, -, -}, S_{-, -, +}, S_{+, -} \)

(d) \( S_{+, -, -, -}, S_{+, -} \)

(e) \( S_{-, +, +}, S_{-, +, +}, S_{+, +, +} \)

(f) \( S_{+, +, +} \).

Note also that by \( 8 \) and \( 12 \), \( S_{-, -, -} = 0 = S_{+, +, +} \). Moreover, by \( 9 \), \( 10 \) and \( 11 \), we have

\[
S_{+, -, -} = \sum_{i_1 = n_1+1}^{n} \sum_{i_2, i_3, i_4 = 1}^{n_1} \frac{2\lambda_{i_1}^2 B_{i_1, i_2} B_{i_2, i_3} B_{i_3, i_4} B_{i_4, i_1}}{\lambda_{i_1} - \lambda_{i_2} \lambda_{i_2} - \lambda_{i_3} \lambda_{i_1} - \lambda_{i_4}}
\]

\[
S_{+, +, +} = \sum_{i_1, i_2 = n_1+1}^{n} \sum_{i_3, i_4 = 1}^{n_1} -2\lambda_{i_1}^2 \lambda_{i_2} B_{i_1, i_2} B_{i_2, i_3} B_{i_3, i_4} B_{i_4, i_1}
\]

\[
\cdot \frac{\lambda_{i_1} - \lambda_{i_2} \lambda_{i_2} - \lambda_{i_3} \lambda_{i_1} - \lambda_{i_4}}{\lambda_{i_2} - \lambda_{i_3} \lambda_{i_1} - \lambda_{i_4}}
\]

\[
+ \sum_{i_1, i_2 = n_1+1}^{n} \sum_{i_3, i_4 = 1}^{n_1} -2\lambda_{i_2} \lambda_{i_3} B_{i_1, i_2} B_{i_2, i_3} B_{i_3, i_4} B_{i_4, i_1}
\]

\[
\cdot \frac{\lambda_{i_2} - \lambda_{i_3} \lambda_{i_1} - \lambda_{i_4}}{\lambda_{i_2} - \lambda_{i_3} \lambda_{i_1} - \lambda_{i_4}}
\]
Our goal is to prove that

\[ 7 = 4S_{+,--} + 4S_{++,--} + 2S_{--,+-} + 4S_{--,++} \geq 0. \]

We shall in fact prove that

\[ S_{+,--} + S_{++,--} + S_{--,++} \geq 0 \]

and

\[ S_{+,++} \geq 0. \]

The inequalities (13) and (14) correspond to terms where the (cyclic) sign sequence changes sign 2 and 4 times, respectively. To prove (13), first note that (simply relabel the indices, factor, and use the Hermitian property \(B_{j,i} = \overline{B_{i,j}}\))

\[
S_{+,--} = \sum_{i > n_1 \geq j} \frac{2\lambda_i^2}{\lambda_i - \lambda_j} \left| \sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_i - \lambda_l} \right|^2
\]

\[
S_{--,++} = \sum_{i > n_1 \geq j} \frac{2\lambda_j^2}{\lambda_i - \lambda_j} \left| \sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_l - \lambda_j} \right|^2.
\]

In a similar vein,

\[
S_{++,--} = \sum_{i > n_1 \geq j} \frac{-2\lambda_i\lambda_j}{\lambda_i - \lambda_j} \left( \sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_l - \lambda_j} \right) \left( \sum_{l=1}^{n_1} \frac{\overline{B_{i,l}}\overline{B_{l,j}}}{\lambda_l - \lambda_j} \right)
\]

\[
+ \sum_{i > n_1 \geq j} \frac{-2\lambda_i\lambda_j}{\lambda_i - \lambda_j} \left( \sum_{l=1}^{n_1} \frac{B_{i,l}B_{l,j}}{\lambda_l - \lambda_j} \right) \left( \sum_{l=1}^{n_1} \frac{\overline{B_{i,l}}\overline{B_{l,j}}}{\lambda_l - \lambda_j} \right).
\]

But this just means that

\[ S_{+,--} + S_{++,--} + S_{--,++} = 2 \sum_{i > n_1 \geq j} \frac{1}{\lambda_i - \lambda_j} \left| \sum_{l=1}^{n_1} \frac{\lambda_lB_{i,l}B_{l,j}}{\lambda_l - \lambda_j} \right| + \sum_{l=1}^{n_1} \frac{\lambda_jB_{i,l}B_{l,j}}{\lambda_j - \lambda_l} \right| ^2 \geq 0. \]
Quite similarly
\begin{equation}
S_{+,+,+} = \sum_{i_1, i_2 = 1}^{n_1} (-\lambda_{i_1} - \lambda_{i_2}) \sum_{l=1}^{n} \frac{\lambda_l B_{i_1, i_2} B_{i_1, i_2}}{(\lambda_{i_1} - \lambda_l)(\lambda_{i_2} - \lambda_l)}
\end{equation}

\begin{equation}
+ \sum_{i_1, i_2 = n_1 + 1}^{n} (\lambda_{i_1} + \lambda_{i_2}) \sum_{l=1}^{n_1} \frac{\lambda_l B_{i_1, i_2} B_{i_1, i_2}}{(\lambda_{i_1} - \lambda_l)(\lambda_{i_2} - \lambda_l)} \geq 0.
\end{equation}

The proof is thus complete. \hfill \Box

**Proof of Theorem 2.2.5.** By Theorem 2.2.2 and Lemmas 2.1.1 and 2.2.1 for any Hermitian $n \times n$ matrices $A, B$ with $B$ invertible there exists a finite measure on $S^3$ such that

$$\int_{S^3} |\langle v, w \rangle|^3 d\mu(w) = \text{tr} |v_1 A + v_2 B|^3 = \|v_1 A + v_2 B\|^3_3.$$ 

By simple approximation we see that we can drop the invertibility assumption; indeed on LHS one simply takes a limit point of measures of the approximant, which exists by compactness. For non-symmetric matrices, use the standard isometry trick

$$A \mapsto \frac{1}{27/3} \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

to note that any plane is isometric to a Hermitian plane. Finally for $A, B \in S_3$, approximate with finite rank operators and apply the matrix case. \hfill \Box

**Corollary 2.2.5.** Hanner’s inequality \cite{3} holds when $p = 3$ and (in reverse when) $p = 3/2$.

**Proof.** Take linear isometric embedding $T : \text{span}_R\{A, B\} \mapsto L_3$. Now by the isometry property and linearity

$$\|A + B\|^3_{S_3} + \|A - B\|^3_{S_3} \leq (\|A\|_{S_3} + \|B\|_{S_3})^3 + (\|A\|_{S_3} - \|B\|_{S_3})^3$$

$$\iff \|TA + TB\|^3_{S_3} + \|TA - TB\|^3_{S_3} \leq (\|TA\|_{S_3} + \|TB\|_{S_3})^3 + (\|TA\|_{S_3} - \|TB\|_{S_3})^3.$$ 

But the second inequality follows from Hanner’s original result for $L_p$ \cite{7}. The case $p = 3/2$ follows from duality (see \cite[Lemma 6]{2})). \hfill \Box

**Proof of Theorem 1.0.3.** Assume first that $p$ is not an even integer. Towards a contradiction, assume that there exists a measure $\mu$

$$\left\|\begin{bmatrix} z + x \\ y \\ z - x \end{bmatrix}\right\|_{S_p} = |z + \sqrt{x^2 + y^2}|^p + |z - \sqrt{x^2 + y^2}|^p = \int_{S^2} |xt_1 + yt_2 + zt_3|^p d\mu(t_1, t_2, t_3).$$

Setting $(x, y) = (r \cos(\theta), r \sin(\theta))$ we see that

$$|z + r|^p + |z - r|^p = \int_{S^2} |r(\cos(\theta)t_1 + \sin(\theta)t_2) + zt_3|^p d\mu(t_1, t_2, t_3).$$

Note however that by the uniqueness (for 2-dimensional $L_p$-spaces), the pushforward $f_*(\mu)$ with

$$f : (t_1, t_2, t_3) \mapsto \frac{(\cos(\theta)t_1 + \sin(\theta)t_2, t_3)}{\|\cos(\theta)t_1 + \sin(\theta)t_2, t_3\|_2}$$

should map the support of $\mu$ to $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, which clearly cannot be true for every $\theta$.
Consider then an even integer \( p \geq 4 \). Again, it is enough to refute the existence of measure \( \mu \) on \( S^2 \) such that (17) holds. Expanding both sides around \( r = 0 \) and averaging over \( \theta \in [0, 2\pi] \) we see that

\[
\begin{align*}
\mu(t_3^p) &= 2 \\
\mu(t_3^{p-2}(1-t_3^2)) &= 4 \\
\mu(t_3^{p-4}(1-t_3^2)^2) &= \frac{16}{3}.
\end{align*}
\]

But now \( \mu((2t_3^2-1)^2t_3^{p-4}) = -\frac{2}{3} \), which is impossible. \( \square \)

### 2.3. Proofs of Theorems 1.0.4 and 1.0.5

**Proof of Theorem 1.0.4.** By [4, Corollary 8], functions with non-negative 4th derivative are (on every compact interval) pointwise limits of functions of the form

\[
t(t) = p(t) + \sum w_i |t - c_i|^3,
\]

where \( p \) is a polynomial of degree at most 3 and \( w_i \geq 0 \). But Theorem 2.2.2 readily implies that for any such function the resulting trace function has non-negative 4th derivative, at least if \( B \) is invertible. Taking limit along invertible approximants of \( B \) and approximants of the form (18) yields the claim. \( \square \)

To prove Theorem 1.0.5 we modify proof of Theorem 2.2.2.

**Theorem 2.3.1.** For any Hermitian \( n \times n \) matrices \( A \) and \( B \), with \( B \) positive definite, the function

\[
T := t \mapsto \text{tr} |A + tB|(A + tB)
\]

is analytic outside finitely many points where \( A + tB \) is singular. Outside these points the function has non-negative 3rd derivative. At the singular points the function behaves like

\[
C|t - c|(t - c) + D|t - c|^2 + f(t),
\]

where \( C \geq 0 \) and \( f \) is \( C^3 \) near \( c \). Consequently the 3rd derivative has non-negative multiple of \( \delta \) measure at the singular points is therefore altogether non-negative measure.

**Proof.** The non-smooth part can be handled as in the proof of Theorem 2.2.2. Thanks to Lemma 2.2.3 we only need to understand

\[
\sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{i_3=1}^n [\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}]|B_{i_1,i_2}B_{i_2,i_3}B_{i_3,i_1}|
\]

**Lemma 2.3.2.** Let \( a_1, a_2, a_3 < 0 < b_1, b_2, b_3 \). Then we have the following identities:

\[
\begin{align*}
[a_1, a_2, a_3] &= 0 \\
[a_1, a_2, b_1] &= \frac{2b_1}{(b_1 - a_1)(b_1 - a_2)} \\
[a_1, b_1, b_2] &= \frac{-2a_1}{(b_1 - a_1)(b_2 - a_1)} \\
[b_1, b_2, b_3] &= 0
\end{align*}
\]

**Proof.** Straightforward calculation. \( \square \)
Assume again that the first \( n_1 \) eigenvalues of \( A \) are negative and the rest \( n_2 \) are positive. By cyclicity it suffices to understand

\[
S_{-,+,+} = \sum_{i_1=1}^{n_1} \sum_{i_2=i_1+1}^{n} \sum_{i_3=i_1+1}^{n} \left( -\frac{2\lambda_{i_3}}{\lambda_{i_2} - \lambda_{i_1}} \right) B_{i_1,i_2} B_{i_2,i_3} B_{i_3,i_1}
\]

and

\[
S_{+,--,} = \sum_{i_1=n_1+1}^{n} \sum_{i_2=1}^{n_1} \sum_{i_3=1}^{n} \left( \frac{2\lambda_{i_1}}{\lambda_{i_2} - \lambda_{i_1}} \right) B_{i_1,i_2} B_{i_2,i_3} B_{i_3,i_1}.
\]

But these are both non-negative as we can write

\[
S_{-,+,+} = 2 \sum_{i \leq n_1} -\lambda_i \langle B v_i, v_i \rangle
\]

\[
S_{+,--,} = 2 \sum_{i > n_1} \lambda_i \langle B u_i, u_i \rangle,
\]

where

\[
v_i = \sum_{j > n_1} \frac{B_{j,i}}{\lambda_j - \lambda_i} e_j
\]

\[
u_i = \sum_{j \leq n_1} \frac{B_{j,i}}{\lambda_j - \lambda_i} e_j,
\]

and \( e_i \) is the eigenvector of \( A \) corresponding to eigenvalue \( \lambda_i \).

**Proof of Theorem 1.0.5.** Make obvious modifications to the proof of Theorem 1.0.4.

**□**

### 3. Discussion

It would be extremely interesting to know whether conjecture 1.0.2 is true in general. In addition to the case \( p = 3 \), it is also clearly true for \( p = 2 \) (\( S_2 \) is Hilbert space) and \( p = \infty \) (\( L_\infty \) is universal).

The case \( p = 1 \) is also well known: every 2-dimensional normed space is isometric to a subspace of \( L_1 \), as observed by Lindenstrauss [11, Corollary 2].

In the following discussion we shall assume that \( A \) and \( B \) are Hermitian \( n \times n \) matrices.

**3.1. \( p \) is not an integer.** To prove conjecture 1.0.2 one might try to modify the proof of Theorem 1.0.1. Lemma 2.1.1 has generalization as follows. Define

\[
c_p = 2^{p+1} \pi^{1/2} \Gamma((p+1)/2)/\Gamma(-p/2).
\]

**Lemma 3.1.1.** Let \( p \geq 1 \) be not an even integer and let \( \| \cdot \| \) be a norm in \( \mathbb{R}^2 \) such that for any \( v, w \in \mathbb{R}^2 \) the \( (p+1) \):st distributional derivative of

\[
t \mapsto (1/c_p) \| tv + w \|^p
\]

is a finite positive Borel measure \( \mu \), for which \( \mu(\|x\|^p) < \infty \). Then \( \| \cdot \| \) is linearly isometrically embeddable into \( L_p \).

Here by distributional derivative of \( f \) order \( \alpha \) we mean the distribution \( (\hat{\cdot}, |\cdot|^\alpha f) \) (\( \hat{\cdot} \) stands for the Fourier transform). Exact value of the constant \( c_p \) here is of course irrelevant. Only the sign matters, which (that of \( \Gamma(-p/2) \)) happens to be \( +/- \) depending on if \( \lfloor p/2 \rfloor \) is odd/even. The proof of Lemma 3.1.1 is omitted, since we don’t know how to generalize the remaining steps.
3.2. $p$ is an odd integer. Lemma 3.1.1 is particularly convenient if $p$ is odd integer as then $(p+1)$st fractional derivative corresponds to the usual (distributional) derivative, up to a constant. For odd integers $p > 3$ one has expressions similar to (17); Lemma 2.2.1 goes also through without noticeable change. The main difficulty then is to find expressions similar to (15) and (16). We have partial work, which suggests that the Conjecture 1.0.2 is true for $p = 5$.

If one restricts attention to $2 \times 2$ matrices, the case of an odd integer can be checked without too much trouble. Indeed, one checks that

$$(\text{tr} | A + tB|^p)_{(p+1)}(0) = \sum_{2i+j=2p-1} (p-1)! \frac{1}{j! i! (i+1)!} \left( \frac{B_{1,1} \lambda_2 - B_{2,2} \lambda_1}{\lambda_2 - \lambda_1} \right)^j \left( \frac{-\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)^2} \right)^i$$

whenever $\lambda_1 < 0 < \lambda_2$. Here all the $(p+1)/2$ terms are clearly non-negative, and, akin to the (general $n$) case of $p = 3$, they correspond to different numbers of sign changes for the patterns introduced in the proof Theorem 2.2.2. Sadly for $p > 3$ and general $n$ such decomposition to $(p+1)/2$ non-negative parts is not possible.

3.3. $p$ is an even integer. These cases are somewhat special. While now there’s no more uniqueness in the representation (5), one can simply expand $\|A + tB\|_p^p$ as a polynomial in $t$. Equating the coefficients (in $t$) of

$$\int_{0}^{2\pi} |\cos(\theta) + \sin(\theta)t|^p d\mu(\theta) = \text{tr} | A + tB|^p,$$

one ends up with expressions of the Fourier coefficients of $\mu$, $\hat{\mu}(0), \hat{\mu}(2), \hat{\mu}(4), \ldots, \hat{\mu}(p)$ in terms of traces of non-commutative polynomials of $A$ and $B$. Since we assume that $\mu$ is symmetric, we may instead think about $\mu$ on $[0, \pi]$ in which case one obtains expressions for, $\hat{\mu}(0), \hat{\mu}(1), \hat{\mu}(2), \ldots, \hat{\mu}(p/2)$.

For $p = 2$, this yields

$$\hat{\mu}(0) = \text{tr}(A^2) + \text{tr}(B^2)$$

$$\hat{\mu}(1) = \text{tr}(A^2) - \text{tr}(B^2) + 2i \text{tr}(AB).$$

The necessary and sufficient condition for the existence of $\mu$ ($\hat{\mu}(0) \geq |\hat{\mu}(1)|$) simplifies to

$$\text{tr}(AB)^2 \leq \text{tr}(A)^2 \text{tr}(B)^2,$$

Cauchy–Schwarz inequality for the inner product $\langle A, B \rangle = \text{tr}(AB)$.

For $p = 4$ the situation is more complicated. One has

$$\hat{\mu}(0) = \text{tr}(A^4) + \frac{2}{3} (2 \text{tr}(A^2B^2) + \text{tr}(ABAB)) + \text{tr}(B^4)$$

$$\hat{\mu}(1) = \text{tr}(A^4) - \text{tr}(B^4) + 2i (\text{tr}(A^3B) + \text{tr}(AB^3))$$

$$\hat{\mu}(2) = \text{tr}(A^4) + \text{tr}(B^4) - 2 \left(2 \text{tr}(A^2B^2) + \text{tr}(ABAB)\right) + 4i (\text{tr}(A^3B) - \text{tr}(AB^3)).$$

Existence of the measure $\mu$ is now equivalent to (see for instance [1] Theorem 1.3.6]

$$\begin{bmatrix} \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) \\ \hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(1) \\ \hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) \end{bmatrix} \geq 0,$$
i.e. \( \mu(p) \geq 0 \) for any trigonometric polynomial \( p \) of degree 2 non-negative on \( S^1 \). By \cite[Theorem 1.1.7]{1} it is enough to check such polynomials with roots on the unit circle, namely that

\[
0 \leq \mu(\theta) \mapsto (e^{i\theta} - e^{-i\theta})(e^{i\theta} - e^{-i\theta})(e^{-i\theta} - e^{-i\theta})(e^{-i\theta} - e^{-i\theta}) = (4 + e^{i(\theta_1 - \theta_2)} + e^{i(\theta_2 - \theta_1)})\mu(0) - 2(e^{i\theta_1} + e^{-i\theta_2})\mu(1) - 2(e^{i\theta_1} + e^{i\theta_2})\mu(1) + e^{-i(\theta_1 + \theta_2)}(\mu(2) + e^{i(\theta_1 + \theta_2)}\mu(2))
\]

for any \( \theta_1, \theta_2 \in \mathbb{R} \).

But since this can be expressed as

\[
(2 - e^{i\theta_1} - e^{-i\theta_1})(2 - e^{i\theta_2} - e^{-i\theta_2}) \text{tr}(A^4) + 4i(e^{i\theta_1} + e^{i\theta_2} - e^{i(\theta_1 + \theta_2)} - e^{-i\theta_1} + e^{-i\theta_2} + e^{-i(\theta_1 + \theta_2)}) \text{tr}(A^3B) + \left( 8 + 2(e^{i(\theta_1 - \theta_2)} + e^{i(\theta_2 - \theta_1)}) - 6(e^{i(\theta_1 + \theta_2)} - e^{-i(\theta_1 + \theta_2)}) \right) \frac{2\text{tr}(A^2B^2) + \text{tr}(ABAB)}{3} + 4i(e^{i\theta_1} + e^{i\theta_2} + e^{i(\theta_1 + \theta_2)} - e^{-i\theta_1} - e^{-i\theta_2} - e^{-i(\theta_1 + \theta_2)}) \text{tr}(AB^3) + (2 + e^{i\theta_1} + e^{-i\theta_1})(2 + e^{i\theta_2} + e^{-i\theta_2}) \text{tr}(A^4)
\]

\[
= \|(e^{i\theta_1} - 1)(e^{i\theta_2} - 1)A^2 - i(e^{i(\theta_1 + \theta_2)} - 1)(AB + BA) - (e^{i\theta_1} + 1)(e^{i\theta_2} + 1)B^2\|^2 S_2 + \frac{|e^{i\theta_1} - e^{i\theta_2}|^2}{3}\|AB - BA\|^2 S_2 \geq 0,
\]

we obtain

**Theorem 3.3.1.** For any \( A, B \in S_4 \) the real space \( \text{span}_\mathbb{R}\{A, B\} \) can be linearly isometrically embedded into \( L_4 \).

Similar but more complicated argument can be carried out for \( p = 6 \), but it seems that such sum-of-squares expression is impossible for \( p = 8 \).

### 3.4. Stahl’s Theorem

We give a quick proof of Stahl’s result \cite[Theorem 1.0.4]{1} given Conjecture \cite[1.0.6]{1}

**Proof of Theorem 1.0.7** By the result of Bernstein on completely monotone functions (see \cite[Theorem 1.4]{17}) Theorem is equivalent to the function

\[
t \mapsto \text{tr}(\exp(A + tB))
\]

having non-negative \( k \)’th derivative for any \( k \geq 0 \). But since exponential function has non-negative \( k \)’th derivative for any \( k \), Conjecture \cite[1.0.6]{1} immediately implies the claim. \( \Box \)

### 3.5. Further questions

**Question 3.5.1.** Let \( X \) be a subspace of \( S_p \). We say that \( X \) is of \( p \)-width \( k \), if it is finitely representable\(^2\) in \( \oplus_{i=1}^\infty S_p^k \). Does there exist \( 1 \leq p < \infty, p \neq 2, \) and \( m, k \in \mathbb{N} \) with \( m \geq 3 \) such that any \( m \)-dimensional subspace of \( S_p \) has \( p \)-width at most \( k \)?

\(^2\)\( X \) is said to finitely representable in \( Z \) if for any finite dimensional subspace \( Y \) of \( X \) and \( \varepsilon > 0 \) there exists a subspace \( Z_Y \subset Z \) and isomorphism \( T : Y \to Z_Y \) such that \( \|T\|\|T^{-1}\| \leq 1 + \varepsilon \).
Question 3.5.2. Let $A, B \in S_p$ (not necessarily Hermitian). Does there exist $f, g \in L_1((0,1), \mathbb{C})$ such that

$$\|\alpha A + \beta B\|_{S_p} = \|\alpha f + \beta g\|_p$$

for any $\alpha, \beta \in \mathbb{C}$? In other words, is $\text{span}_C\{A, B\}$ complex linearly isometric to a subspace of complex $L_p$?

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References

[1] M. Bakonyi and H. J. Woerdeman. Matrix completions, moments, and sums of hermitian squares. In *Matrix Completions, Moments, and Sums of Hermitian Squares*. Princeton University Press, 2011.
[2] K. Ball, E. A. Carlen, and E. H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. *Inventiones mathematicae*, 115(1):463–482, 1994.
[3] D. Bessis, P. Moussa, and M. Villani. Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *Journal of Mathematical Physics*, 16(11):2318–2325, 1975.
[4] P. Bullen. A criterion for $n$-convexity. *Pacific Journal of Mathematics*, 36(1):81–98, 1971.
[5] V. M. Chayes. Matrix rearrangement inequalities revisited. *arXiv preprint arXiv:2009.04032*, 2020.
[6] C. de Boor. Divided differences. *arXiv preprint math/0502036*, 2005.
[7] O. Hanner. On the uniform convexity of $L_p$ and $l_p$. *Arkiv för Matematik*, 3(3):239–244, 1956.
[8] F. Hiai. Matrix analysis: matrix monotone functions, matrix means, and majorization. *Interdisciplinary Information Sciences*, 16(2):139–248, 2010.
[9] A. L. Koldobsky. Convolution equations in certain Banach spaces. *Proceedings of the American Mathematical Society*, 111(3):755–765, 1991.
[10] A. L. Koldobsky. Generalized Lévy representation of norms and isometric embeddings into $L_p$-spaces. In *Annales de l'IHP Probabilités et statistiques*, volume 28, pages 335–353, 1992.
[11] J. Lindenstrauss. On the extension of operators with a finite-dimensional range. *Illinois Journal of Mathematics*, 8(3):488–499, 1964.
[12] C. A. McCarthy. $C_p$. *Israel Math. J.*, 5:249–271, 1967.
[13] A. Neyman. Representation of $p$-norms and isometric embedding in $p$-spaces. *Israel Journal of Mathematics*, 48(2):129–138, 1984.
[14] D. Petz. A survey of certain trace inequalities. *Banach Center Publications*, 30(1):287–298, 1994.
[15] G. Pisier. Some results on Banach spaces without local unconditional structure. *Compositio Mathematica*, 37(1):3–19, 1978.
[16] F. Rellich and J. Berkowitz. *Perturbation theory of eigenvalue problems*. CRC Press, 1969.
[17] R. L. Schilling, R. Song, and Z. Vondracek. Bernstein functions. In *Bernstein functions*. de Gruyter, 2012.
[18] H. R. Stahl. Proof of the BMV conjecture. *Acta mathematica*, 211(2):255–290, 2013.

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