Abstract: This paper investigates the robustness of two well-known applications of second-order consensus dynamics, namely mechanical and power networks. For uniform subsystem parameters, we derive expressions for the $\mathcal{H}_\infty$ norms of mechanical and power networks, from external disturbances to body displacements and to generator phase angles, respectively. The closed-form expressions are in terms of the physical parameters (damping coefficients and inertias) of the dynamics and in terms of the spectrum of the grounded Laplacian matrix associated with the network. We then analyze the dependence of the $\mathcal{H}_\infty$ norm of each network on both the network structure and the physical parameters. For a fixed network topology, we find that each system norm can be minimized by choosing the damping coefficient within a specified range. Theoretical contributions are verified via two illustrative examples for mechanical and power networks, in which we show that the network structure, number of the reference nodes and their location in the network can have considerable effects on the system $\mathcal{H}_\infty$ norm.

Keywords: Network robustness, Mechanical networks, Power networks, Second order consensus, Grounded Laplacian matrix.

1. INTRODUCTION

Connecting system-theoretic notions to graph theory has attracted much attention in the past decade due to growing interest in networked control systems. Algebraic and spectral graph theory have become key tools in the study of systems on graphs, and have been used to study stability Olfati-Saber et al. (2007), robustness Bamieh et al. (2012); Patterson and Bamieh (2010); Pirani et al. (2017a), controllability Obhovsky et al. (2014); Pequito et al. (2016); Rahmani et al. (2009) and observability Sundaram and Hadjicostis (2011). A foundational system-theoretic concept is robustness to uncertainties or external disturbances. Among the robustness metrics proposed in systems and control theory, the $\mathcal{H}_\infty$ norm plays a substantial role in robust control synthesis and uncertainty modelling. Recently, extensive research has been done characterizing the $\mathcal{H}_2$ norm in terms of graph-theoretic notions Bamieh et al. (2012); Fitch and Leonard (2013); Patterson et al. (2015). Graph-theoretic characterizations of the $\mathcal{H}_\infty$ norm have received less attention however. In this paper we investigate some graph-theoretic approaches to the $\mathcal{H}_\infty$ norm of mechanical and power networks, which are two well-known examples of second-order consensus dynamics.

Second-order consensus dynamics have been heavily investigated in recent years Ren and R. Beard (2007), due to their diverse applications. Stability, performance and robustness of such systems have also been analyzed Lin et al. (2008a,b); Ren and Beard (2008). However, in order to investigate the dependence of system robustness on the physical model parameters, it is desirable to understand the precise relationship between the robustness metric and those parameters. The model of mechanical networks presented in this paper has diverse applications Acosta et al. (2005); Sandberg and Murray (2008, 2009). One of these applications is cooperative adaptive cruise control in vehicle platooning problems, in which vehicles are controlled to track reference inter-vehicular distances as well as a desired velocity, targeting safety issues and fuel consumption considerations Barooah et al. (2009); Hao and Barooah (2013); Jovanovic and Bamieh (2005); Peters et al. (2014); Sebastian et al. (2015). The other example of a second-order consensus systems considered in this paper is a network of power generators. During past years, there have been great efforts on revisiting power network dynamics from control-theoretic and cyber-physical perspectives Dorfler and Bullo (2010); Schiffer et al. (2014); Simpson-Porco et al. (2013). In this direction, some research has been done to analyze the effect of the network structure as well as physical control parameters on certain robustness and performance metrics Poolla et al. (2016); Tegling et al. (2015a,b); Teixeira et al. (2015).

The contributions of this paper are as follows:

- For homogeneous damping and mass/inertia constants, we derive closed-form expressions for the $\mathcal{H}_\infty$ norms of mechanical and power networks. The norms are in terms of physical parameters of the subsystems and in terms of the spectrum of the grounded Laplacian matrix.
- We investigate the dependence of these $\mathcal{H}_\infty$ norms on physical parameters (in particular, on the homoge-
neous damping coefficient $c$ and mass/inertia $m_i$), and on the spectrum of the grounded Laplacian matrix.
We demonstrate this analysis via two examples, one for each type of network.

The paper is organized as follows. In Section 2 some notations and definitions used in this paper are introduced. Section 3 presents the models for mechanical and power networks, and characterizes the spectral properties of the associated dynamic matrices in terms of the grounded Laplacian matrix. In Section 4, we derive expressions for the $H_{\infty}$ norm for mechanical and power networks and describe its dependence on the physical parameters of the system. The effect of network structure on the $H_{\infty}$ norm and characterizes the spectral properties of the networks, and expresses its dependence on the physical parameters of the system.

2. NOTATIONS AND DEFINITIONS

We denote an undirected graph (network) by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is a set of nodes (or vertices) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. The neighbors of node $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The unweighted adjacency matrix of the graph is given by a symmetric and binary $n \times n$ matrix $A$, where element $A_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and zero otherwise. The degree of node $v_i$ is denoted by $d_i = \sum_{j=1}^{n} A_{ij}$. For a given set of nodes $X \subset \mathcal{V}$, the edge-boundary (or just boundary) of the set is given by $\partial X = \{(v_i, v_j) \in \mathcal{E} \mid v_i \in X, v_j \in \mathcal{V} \setminus X\}$. The Laplacian matrix of the graph is given by $L \triangleq D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$. For a given subset $S \subset \mathcal{V}$ of nodes (which we term grounded nodes), the grounded Laplacian induced by $S$ is denoted by $L_{gS}$ or simply $L_g$, and is obtained by removing the rows and columns of $L$ corresponding to the nodes in $S$. For the case where the underlying network is connected and there exists at least one grounded node, the grounded Laplacian matrix $L_g$ is a positive-definite matrix (Pirani and Sundaram 2016). For a given set $I$, the number of members (cardinality) of the set is denoted by $|I|$.

3. NETWORK MODELS

In this section, we discuss two applications of second-order consensus dynamics: mechanical mass-spring-damper networks, and networks of power generators.

3.1 Mechanical Network

Consider a simple system comprised of two bodies connected to each other via a spring and a viscous damper, as shown in Fig. 1(a). Suppose that the position of one of the bodies (called the reference or leader) is fixed and the other acts as the follower. This is the simplest form of a network control system of masses, springs, and dampers, which will be addressed in this paper. The dynamics of the reference body is $\dot{x}_1(t) = 0$. From Newton's law, the dynamics of the follower body are

$$m_2 \ddot{x}_2 = k(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) + w_2(t),$$

where $m_2, c, k$ are mass, damping, and stiffness constants, respectively. The follower is subject to an external forcing disturbance, which we denote by $w_2(t)$.

Now we extend the above system to a network of connected masses with arbitrary topology. Consider a connected network of $n$ masses $V = \{v_1, v_2, \ldots, v_n\}$. Each mass $v_i \in V$ is either a follower $v_i \in F$ or a reference mass $v_i \in R$. The position and the velocity of each mass $v_i$ is denoted by scalars $x_i$ and $\dot{x}_i(t)$, respectively. The equation of motion of each follower is

$$m_i \ddot{x}_i = \sum_{j \in \mathcal{N}_i} k_{ij}(x_j - x_i) + c_{ij}(\dot{x}_j - \dot{x}_i) + w_i(t), \quad v_i \in F,$$

where $k_{ij}, c_{ij} > 0$ are spring stiffness and damping constant between $m_i$ and $m_j$ and $w_i(t)$ models the disturbance and uncertainties in dynamics of the $i$-th body. The dynamics of the reference body is

$$\dot{x}_1 = 0, \quad v_1 \in R.$$  

Here we consider homogeneous mass, stiffness and damping constants, i.e. $m_i = m$, $k_i = k$ and $c_i = c$, $\forall i = 1, 2, \ldots, n$. The objective of this study is to investigate the robustness of the networked control system to external disturbances. Thus, we only consider the dynamics of the followers, since they are affected by the disturbances. In vector notation, the dynamics of the followers are

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0_{|\mathcal{F}|} & I_{|\mathcal{F}|} \\ -k/m L_g & -c/m L_g \end{bmatrix} \begin{bmatrix} x \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0_{|\mathcal{F}|} \\ 1/m I_{|\mathcal{F}|} \end{bmatrix} w(t),$$

$$y = x,$$

where $L_g$ is the grounded Laplacian matrix formed by removing rows and columns corresponding to the reference bodies and $y(t)$ is the output of interest, in this case the positions of the follower bodies.

3.2 Power Network

Consider a power network of $n$ buses $V = \{v_1, v_2, \ldots, v_n\}$ and a set of edges $\mathcal{E}$, as shown in Fig. 1(b). Here we assume a Kron-reduced transmission network model, where all buses are modeled as generators and branch resistances are neglected. At each node $i = 1, 2, \ldots, n$, there is a generator with inertia $m_i$, damping $c_i$, and voltage phase angle $\theta_i$. The dynamics of the $i$-th generator is described by the swing equation

$$m_i \ddot{\theta}_i + c_i \dot{\theta}_i = P_{m,i} - P_{c,i} + w_i(t),$$

where $P_{m,i}$ is the constant mechanical power input from turbine and $w_i(t)$ models disturbances arising from changing generation or changing load. The term $P_{c,i}$ is the
real electrical power injected from \(i\)-th generator to the network, and is given by

\[
P_{e,i} = \sum_{j \in \mathcal{N}_i} V_i V_j B_{ij} \sin(\theta_i - \theta_j),
\]

where \(V_i\) is the nodal voltage magnitude and \(-B_{ij} < 0\) is the susceptance associated with edge \((v_i, v_j) \in \mathcal{E}\). We further approximate this using the so-called DC Power Flow, where \(V_i \approx V_j \approx 1\) and \(|\theta_i - \theta_j| \ll 1\), leading to the linear model

\[
P_{e,i} \approx \sum_{j \in \mathcal{N}_i} B_{ij} (\theta_i - \theta_j).
\]

Substituting (7) into (5) yields the LTI system

\[
m_i \ddot{\theta}_i + c_i \dot{\theta}_i \approx - \sum_{j \in \mathcal{N}_i} B_{ij} (\theta_i - \theta_j) + P_{m,i} + w_i(t),
\]

We consider a reference (slack) generator, \(i \in \mathcal{R}\), which has a constant phase

\[
\dot{\theta}_i = 0, \quad v_i \in \mathcal{R}.
\]

Similar to the case of mechanical network, here we assume that \(m_i = m, B_{ij} = b\) and \(c_i = c\) for all \(i = 1, 2, ..., |\mathcal{F}|\). Tegling et al. (2015b). After shifting the equilibrium point of (8) to the origin, the term proportional to \(P_{m,i}\) may be removed and the dynamics of the followers can be written in vector form as

\[
\begin{bmatrix}
\dot{\theta} \\
\ddot{\theta}
\end{bmatrix} = \begin{bmatrix}
0_{|\mathcal{F}|} \\
\frac{b}{m} I_{|\mathcal{F}|} \\
\frac{c}{m} I_{|\mathcal{F}|}
\end{bmatrix} \begin{bmatrix}
\theta \\
\dot{\theta}
\end{bmatrix} + \begin{bmatrix}
0_{|\mathcal{F}|} \\
I_{|\mathcal{F}|}
\end{bmatrix} w(t),
\]

where we have taken \(y = \theta\) as the output of interest; this is the phase difference between each bus angle and the (zero)\(^2\) angle of the reference bus. Other kinds of performance outputs (including frequency) are considered in Pirani et al. (2017b).

In Section 4 we derive expressions for the \(H_\infty\) norms of the systems (4) and (10), respectively. Since the smallest eigenvalue of \(L_g\) plays a crucial role in the robustness of (4) and (10) to external disturbances, the following theorem provides some graph-theoretic bounds on \(\lambda_i\).\(^3\)

**Theorem 1.** (Pirani and Sundaram (2016)). Consider a connected network \(\mathcal{G} = (\mathcal{V}, \mathcal{E})\) with a set of reference nodes \(\mathcal{R} \subset \mathcal{V}\). Let \(L_g\) be the grounded Laplacian matrix for \(\mathcal{G}\). Let \(\beta_i = |\mathcal{N}_i \cap \mathcal{R}|\) be the number of reference nodes in follower \(v_i\)'s neighborhood. Then

\[
\max \left\{ \frac{\|\mathcal{R}\|u_{\text{min}}}{|\mathcal{F}|}, \min_{i \in \mathcal{F}} \beta_i \right\} \leq \lambda_1 \leq \frac{\|\mathcal{R}\|}{|\mathcal{R}|} \leq \max_{i \in \mathcal{F}} \beta_i \leq |\mathcal{R}|,
\]

where \(u_{\text{min}}\) is the smallest component of the eigenvector \(u\) corresponding to \(\lambda_1\). Here we normalized the elements of \(x\) such that \(\|x\|_\infty = 1\).

### 4. \(H_\infty\) PERFORMANCE OF MECHANICAL AND POWER NETWORKS

We now present closed-form expressions for the \(H_\infty\) system norm of mechanical and power networks, from external disturbances to the mass displacements and the phase

\(^2\) Without loss of generality we assume that the states of reference nodes are zero, i.e., \(x = 0\) and \(\theta = 0\).

\(^3\) For simplicity, \(\lambda_i\) refers to \(\lambda_i(L_g)\), unless indicated.

angles, respectively. We then discuss how each system norm depends on the physical parameters of the subsystems.

**Theorem 2.** Consider the mechanical network described by (4) and the power network described by (10).

(i) **Mechanical Network:** For the mechanical network (4), the \(H_\infty\) norm from disturbances to position errors is

\[
||G||_{H_\infty}^{\text{Mech}} = \begin{cases}
\frac{2m}{c\lambda_1^2 \sqrt{4mk - c^2\lambda_1}} & \text{if } \frac{\lambda_1 c^2}{2km} \leq 1,
\frac{1}{k\lambda_1} & \text{otherwise}.
\end{cases}
\]

(ii) **Power Network:** For the power network (10), the \(H_\infty\) norm from disturbances to phase angle errors is

\[
||G||_{H_\infty}^{\text{Power}} = \begin{cases}
\frac{2m}{c\sqrt{4km\lambda_1 - c^2}} & \text{if } \frac{c^2}{2km\lambda_1} \leq 1,
\frac{1}{b\lambda_1} & \text{otherwise}.
\end{cases}
\]

**Proof.** We prove the result for the mechanical network; the proof for the power network is similar. Computing the transfer function of (4) gives

\[
G(s) = \left( ms^2 I + (cs + k)L_g \right)^{-1} = U \left( ms^2 I + (cs + k)\Lambda \right)^{-1} U^T,
\]

where \(U = [u_1, u_2, ..., u_{|\mathcal{F}|}]\) is a matrix formed by eigenvectors of \(L_g\) and diag\((G_1(s))\) is a diagonal matrix with diagonal elements \(G_i(s) = \frac{1}{ms^2 + c\lambda_i + s + k\lambda_i}\). Then we have

\[
|G_i(j\omega)|^2 = \frac{1}{(m\omega^2 - k\lambda_i)^2 + c^2\lambda_i^2 \omega^2}.
\]

Maximizing \(|G_i(j\omega)|^2\) with respect to \(\omega\) is equivalent to minimizing \(f(\omega)\). By setting \(\frac{df(\omega)}{d\omega} = 0\), we get \(\omega_1 = 0\) and \(\omega_2 = \left( \frac{k\lambda_i^2}{m} - c^2\lambda_i^2 \right)^{\frac{1}{2}}\) as critical points. Here \(\omega_2\) is the global minimizer of \(f(\omega)\), unless \(\frac{\lambda_1 c^2}{2km} > 1\). Hence, we have

\[
\omega = \begin{cases}
\left( \frac{k\lambda_i}{m} - \frac{c^2\lambda_i^2}{2m^2} \right)^{\frac{1}{2}} & \text{if } \frac{\lambda_1 c^2}{2km} \leq 1,
0 & \text{otherwise}.
\end{cases}
\]

Substituting the values in (16) into \(|G_i(j\omega)|\) yields

\[
\sup_{\omega} |G_i(j\omega)| = \begin{cases}
\frac{2m}{c\lambda_i^2 \sqrt{4mk - c^2\lambda_1}} & \text{if } \frac{\lambda_1 c^2}{2km} \leq 1,
\frac{1}{k\lambda_i} & \text{otherwise}.
\end{cases}
\]

Both terms in (17) are monotonic decreasing functions of \(\lambda_i\), which is clear from the second term, \(\frac{1}{k\lambda_i}\). For the first term, by differentiating with respect to \(\lambda_i\) we find that

\[
\arg\min_{\lambda_i} \left( \frac{2m}{c\lambda_i^2 \sqrt{4mk - c^2\lambda_1}} \right) = \frac{2km}{c^2},
\]

which is out of the range of \(\lambda_i \leq \frac{2km}{c^2}\). Hence, for system \(H_\infty\) norm we have

\[
||G||_{H_\infty} = \sup_{\omega} \max \left| |G_i(j\omega)| \right|.
\]
which yields the result. □

According to Theorem 2, the $\mathcal{H}_\infty$ norms for both mechanical and power networks are monotonic decreasing functions of $\lambda_1$. Hence, increasing the smallest eigenvalue of the grounded Laplacian matrix (by changing the network topology, and managing the number and location of the reference nodes) will always decrease the $\mathcal{H}_\infty$ norm and consequently provide a more robust network control system. We will return to the effect of network structure on the norm in Section 5. However, due to the fact that in most cases the network structure and the number and location of reference nodes are fixed, it is more convenient to use local feedback controllers to increase the robustness of the overall system. To this end, in the following proposition we analyze the behaviour of the damping coefficient $c$ and inertia $m$ on system $\mathcal{H}_\infty$ norms for both mechanical and power networks.

**Proposition 3.** Consider the mechanical network described by (4) and the power network described by (10). The following statements hold:

(i) **Mechanical Network:** The $\mathcal{H}_\infty$ norm of a mechanical network (12) is a non-increasing function of the damping coefficient $c$ and is bounded from below as

\[ ||G||_{\infty}^{\text{Mech}} \geq \frac{1}{k\lambda_1}, \]

with equality sign for all $c \geq \sqrt{\frac{2mk}{\lambda_1}}$. Moreover, the norm is a non-decreasing function of the mass $m$, and is bounded from below by (18), which holds with equality sign for all $m \leq \frac{c^2}{2\lambda_1}$.

(ii) **Power Network:** The $\mathcal{H}_\infty$ norm of the power network (13) is a non-increasing function of the damping coefficient $c$, and it is bounded from below as

\[ ||G||_{\infty}^{\text{Power}} \geq \frac{1}{b\lambda_1}, \]

with equality sign for all $c \geq \sqrt{\frac{2bm}{\lambda_1}}$. Moreover, the norm is a non-decreasing function of the inertia $m$, and is bounded from below by (19), which holds with equality sign for all $m \leq \frac{c^2}{2bm\lambda_1}$.

**Proof.** The proof for both the mechanical and power networks is similar; here we consider the power network case. First note that $||G||_{\infty}^{\text{Power}}$ is a continuous function of $c$ and $m$ (it can be verified by checking the values in boundaries between two sub-functions in (13)). Let $f(c)$ denote $||G||_{\infty}^{\text{Power}}$ as a function of $c$. For the first sub-function in (13) we have $f(c) = \frac{2m}{c\sqrt{4bm\lambda_1} - c^2}$. Differentiating with respect to $c$ yields

\[ f'(c) = \frac{2m}{(4bm\lambda_1 - c^2)^{\frac{3}{2}}} - \frac{2m}{c^2(4bm\lambda_1 - c^2)^{\frac{3}{2}}}. \]

By solving $f'(c) = 0$, we reach to the critical point $\tilde{c} = \sqrt{2bm\lambda_1}$, which is exactly the boundary of the range of this sub-function in (13). For $c < \tilde{c}$ we have $f'(c) < 0$ and for $c > \tilde{c}$, based on the second sub-function in (13) we have $\frac{dG}{dc} = 0$ (since it is independent of $c$). Hence the minimum value of $f(c)$ is attained for all $c \geq \sqrt{2bm\lambda_1}$.

Now consider $||G||_{\infty}^{\text{Power}}$ as a function of inertia, and denote this function by $g(m)$. Differentiating $g(m)$ for the first sub-function in (13) with respect to $m$ yields

\[ g'(m) = \frac{2}{c(4bm\lambda_1 - c^2)^{\frac{3}{2}}} - \frac{4mb\lambda_1}{c(4bm\lambda_1 - c^2)^{\frac{3}{2}}}, \]

where $\bar{m} = \frac{c^2}{2bm\lambda_1}$ is its critical point, which is the boundary of the range of this sub-function in (13). Considering the fact that for all $m < \bar{m}$ (based on the second sub-function), we have $g(m) = \frac{1}{b\lambda_1}$ and for $m > \bar{m}$, we have $g'(m) > 0$, the result is obtained. □

Fig. 2 shows the behavior of the power network $\mathcal{H}_\infty$ norm as a function of both damping and inertia, for several values of $\lambda_1$. We make two comments. First, while in general the $\mathcal{H}_\infty$ norm is non-differentiable, both expressions from Theorem 2 are continuously differentiable functions of $c$ and $m$. Second, given a fixed network structure and one of the values of $c$ or $m$, the other parameter can be adjusted to maximize system robustness. For example, if $m$ is fixed, one should choose $c = \sqrt{\frac{2m\lambda_1}{\lambda}}$ in mechanical and $c = \sqrt{\frac{2mb\lambda_1}{\lambda}}$ in power network to yield the minimum system $\mathcal{H}_\infty$ norm.

![Fig. 2](image-url)
Substituting this value into (12), and noting that here the reference set \( R \) forms a complete graph and two of them are connected to the broom tree. Fig. 4 shows a network called a nodes Pirani et al. (2016).

According to the above definition, a grounding central centrality of each vertex \( v \in V \) is given by \( I(G) = \arg \max_{v \in V} \lambda_1(L_{gs}) \).

Definition 1. Consider a graph \( G(V, E) \). The grounding centrality of each vertex \( v \in V \), denoted by \( I(s) \), is \( I(s) = \lambda_1(L_{gs}) \). The set of grounding central vertices in the graph \( G \) is given by \( IC(G) = \arg \max_{v \in V} \lambda_1(L_{gs}) \).

According to the above definition, a grounding central vertex \( v \in IC(G) \) is a vertex that maximizes \( \lambda_1 \), if chosen as a reference, over all possible choices of single reference nodes Pirani et al. (2016).

Fig. 4 shows a network called a broom tree, \( B_{n,\Delta} \), comprised of a star \( S_\Delta \) with \( \Delta \) leaf vertices and a path of length \( n - \Delta - 1 \) attached to the center of the star Stevanovic and Ilic (2010).

Consider Fig. 4 representing a power network with \( m = \frac{20}{\pi f} \) and \( c = \frac{10}{\pi f} \), based on Sauer and Pai (1999), with \( f = 60Hz \) and \( b = 1 \). Fig. 5 plots the value of the \( H_\infty \) norm as a function of the node which is chosen as a reference (the slack bus choice). As shown in this figure, the reference node which minimizes the system \( H_\infty \) norm is node number 8. This node does not have the highest degree (“degree central” node), nor is it the closest node to the other nodes (the “closeness central” node). In the graph shown in Fig. 4, node 7 is both the degree and closeness central node. By increasing \( \Delta \) and consequently \( n \) in the broom tree, the grounding central node becomes farther from the degree and closeness central nodes in the network.

6. SUMMARY AND CONCLUSIONS

In this paper we investigated the robustness of mechanical and power network dynamical systems in the sense of their system \( H_\infty \) norm. For homogeneous subsystem parameters, we derived-closed form expressions for the \( H_\infty \) norm in terms of physical properties of each dynamics (mass/inertia, coupling weights and damping coefficients) and in terms of the network structure, encapsulated in the smallest eigenvalue of the grounded Laplacian matrix. An interesting and important avenue for extending the results presented in this paper is to derive the system norms when the physical subsystem parameters are heterogeneous. Moreover, it would be informative to compare the \( H_\infty \) norm to the \( H_2 \) norm in terms of their dependency on physical parameters and network structure.

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REFERENCES

Acosta, J.A., Ortega, R., Astolfi, A., and Mahindrakar, A.D. (2005). Interconnection and damping assignment passivity-based control of mechanical systems with underactuation degree one. IEEE Transactions on Automatic Control, 50(12), 1936–1955.

Bamieh, B., Jovanovic, M.R., Mittra, P., and Patterson, S. (2012). Coherence in large-scale networks: Dimension-dependent limitations of local feedback. IEEE Transactions on Automatic Control, 57, 2235–2249.

Barooah, P., Mehta, P.G., and Hespanha, J.P. (2009). Mistuning-based control design to improve closed-loop stability margin of vehicular platoons. IEEE Transactions on Automatic Control, 54(9), 2100–2113.

Dorfler, F. and Bullo, F. (2010). Synchronization and transient stability in power networks and non-uniform kuramoto oscillators. SIAM Journal on Control and Optimization, 50, 1616–1642.

Fig. 4. Broom tree with \( \Delta = 6, n = 13 \).
Fig. 5. $H_{\infty}$ norm (13) versus the node label in Broom tree $B_{13.6}$.

Fitch, K.E. and Leonard, N.E. (2013). Information centrality and optimal leader selection in noisy networks. *IEEE Conference on Decision and Control*, 7510–7515.

Hao, H. and Barooah, P. (2013). Stability and robustness of large platoons of vehicles with double-integrator models and nearest neighbor interaction. *International Journal of Robust and Nonlinear Control*, 23(18), 2097–2122.

Jovanovic, M.R. and Bamieh, B. (2005). On the ill-posedness of certain vehicular platoon control problems. *IEEE Transactions on Automatic Control*, 50(9), 1307–1321.

Lin, P., Jia, Y., and Li, L. (2008a). Consensus over networks with dynamic channels. *American Control Conference*, 57, 2637–2642.

Lin, P., Jia, Y., and Li, L. (2008b). Distributed robust $H_{\infty}$ consensus control in directed networks of agents with time-delay. *Systems and Control Letters*, 57, 643–653.

Olfati-Saber, R., Fax, J.A., and Murray, R.M. (2007). Consensus and cooperation in networked multi-agent systems. *IEEE Transactions on Automatic Control*, 95, 215–233.

Olshevsky, A. (2014). Minimal controllability problems. *IEEE Transactions on Control of Network Systems*, 1, 249–258.

Patterson, S. and Bamieh, B. (2010). Leader selection for optimal network coherence. *49th IEEE Conference on Decision and Control*, 2692–2697.

Patterson, S., McGlohon, N., and Dyagilev, K. (2015). Efficient, optimal $k$-leader selection for coherent, one-dimensional formations. *European Control Conference*.

Pequito, S., K. Kar, and Aguiar, A.P. (2016). A framework for structural input/output and control configuration selection in large-scale systems. *IEEE Transactions on Automatic Control*, 61, 303–318.

Peters, A.A., Middleton, R.H., and Mason, O. (2014). Leader tracking in homogeneous vehicle platoons with broadcast delays. *Automatica*, 50(1), 64–74.

Pirani, M., Hashemi, E., Simpson-Porco, J.W., Fidan, B., and Khajepour, A. (2017a). A graph theoretic approach to the robustness of k-nearest neighbor vehicle platoons. *IEEE Transactions Intelligent Transportation Systems* (to appear).

Pirani, M., Shahrivar, E.M., Fidan, B., and Sundaram, S. (2016). Robustness of leader - follower networked dynamical systems. *arXiv:1604.08651v1*.

Pirani, M., Simpson-Porco, J.W., and Fidan, B. (2017b). System-theoretic performance metrics for low-inertia stability of power networks. *arXiv preprint arXiv:1703.02646*.

Pirani, M. and Sundaram, S. (2016). On the smallest eigenvalue of grounded Laplacian matrices. *IEEE Transactions on Automatic Control*, 61(2), 509–514.

Poolla, B.K., Bolognani, S., and Dorfler, F. (2016). Optimal placement of virtual inertia in power grids. *American Control Conference*.

Rahmani, A., Ji, M., Mesbahi, M., and Egerstedt, M. (2009). Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48, 162–186.

Ren, W. and Beard, R.W. (2008). Distributed consensus in multi-vehicle cooperative control. *Springer-Verlag, London*.

Ren, W. and Beard, E.A. (2007). Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine*, 27, 71–82.

Sandberg, H. and Murray, R.M. (2008). Model reduction of interconnected linear systems using structured gramians. *IFAC Proceedings Volumes*, 41(2), 8725–8730.

Sandberg, H. and Murray, R.M. (2009). Model reduction of interconnected linear systems. *Optimal Control Applications and Methods*, 30(3), 225–245.

Sauer, P. and Pai, M.A. (1999). Power system dynamics and stability. *Prentice Hall*.

Schiffer, J., Ortega, R., Astolfi, A., Raisch, J., and Sezi, T. (2014). Conditions for stability of droop-controlled inverter-based microgrids. *Automatica*, 50, 2457–2469.

Sebastian, V.D.H., Johansson, K.H., and Dimarogonas, D.V. (2015). Fuel-optimal centralized coordination of truck platooning based on shortest paths. *American Control Conference (ACC)*, 3740–3745.

Simpson-Porco, J.W., Dorfler, F., and Bullo, F. (2013). Synchronization and power sharing for droop-controlled inverters in islanded microgrids. *Automatica*, 49, 2603–2611.

Stevanovic, D. and Ilic, A. (2010). Distance spectral radius of trees with fixed maximum degree. *Electronic Journal of Linear Algebra*, 20, 168–179.

Sundaram, S. and Hadjicostis, C.N. (2011). Distributed function calculation via linear iterative strategies in the presence of malicious agents. *IEEE Transactions on Automatic Control*, 56(7), 1495–1508.

Tegling, E., Bamieh, B., and Gayme, D.F. (2015a). The price of synchrouny: Evaluating the resistive losses in synchronizing power networks. *IEEE Transactions on Control of Network Systems*, 2, 254–266.

Tegling, E., Gayme, D.F., and Sandberg, H. (2015b). Performance metrics for droop-controlled microgrids with variable voltage dynamics. *54th IEEE Conference on Decision and Control (CDC)*, 7502–7509.

Teixeira, A., Paridari, K., Sandberg, H., and Johansson, K.H. (2015). Voltage control for interconnected microgrids under adversarial actions. *IEEE 20th Conference on Emerging Technologies & Factory Automation (ETFA)*.