We find electric string solutions in Yang-Mills-Higgs theory.

Cosmic strings have attracted a lot of attention for nearly half a century since their discovery \[1, 2\]. Such strings carry magnetic flux and are analogous to vortices in superconductors. In contrast, strings that carry electric flux are expected to lead to confinement in QCD and classical solutions corresponding to electric strings are not known. Even if electric strings were to exist as classical solutions in a non-Abelian gauge theory, because gauge excitations (“gluons”) are massless and carry non-Abelian charge one might expect that rapid production of gluons by the Schwinger process \[3, 4\] would dissipate such strings. In view of these expectations it is surprising that electric string solutions do exist in certain non-Abelian gauge theories and are protected against the Schwinger process.

An essential element in constructing an electric string solution \(^1\) is that a non-Abelian electric field does not uniquely specify a gauge equivalent class of gauge fields. As Brown and Weisberger (BW) showed \[7\], there is a one parameter family of gauge inequivalent gauge fields that all result in the same electric field. Unlike a uniform non-Abelian electric field produced in analogy with Maxwell theory, BW gauge fields have been shown to be immune from decay due to Schwinger pair production \[8\]. This suggests the question: can there be string solutions containing BW gauge fields? In pure non-Abelian gauge theory, BW gauge fields do not solve the classical equations of motion. Instead they require external current and charge densities. Such external sources may arise due to quantum effects – after all the classical equations are expected to get modified due to the backreaction of quantum fluctuations – or they may be due to other fields in the system. Here we consider \(SU(2)\) gauge theory with a scalar field in the fundamental representation. The same solution can be embedded in models with larger gauge group that have an \(SU(2)\) subgroup \[9\]. The Lagrangian under consideration is,

\[
L = -\frac{1}{4} W_{\mu \nu}^a W^{a \mu \nu} + |D_\mu \Phi|^2 - V(\Phi) \tag{1}
\]

where

\[
W_{\mu \nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + ge^{abc} W_\mu^b W_\nu^c \tag{2}
\]

\[
D_\mu \Phi = \partial_\mu \Phi - \frac{g}{2} W_\mu^a \sigma^a \Phi \tag{3}
\]

\(V(\Phi) = m^2|\Phi|^2 + \lambda|\Phi|^4 \tag{4}\)

where \(\sigma^a\) are the Pauli spin matrices. The Lagrangian actually has an \(SU(2) \times U(1)\) symmetry but only the \(SU(2)\) is gauged, while the \(U(1)\) is global. (This corresponds to the \(\sin^2 \theta_w = 0\) limit of the electroweak model.)

The equations of motion are

\[
(\partial_\nu + ge^{abc} W_\nu^b) W^{\mu \nu c} = i \frac{g}{2} (\Phi^† \sigma^a D^\mu \Phi - h.c.) \tag{5}
\]

\[
D^\mu D_\mu \Phi + m^2 \Phi + 2\lambda|\Phi|^2 \Phi = 0. \tag{6}
\]

To find an electric field solution to the equations of motion, we first write the BW gauge field in temporal section \[8\],

\[
W_0^a = 0, \quad W_\mu^± = -\frac{\epsilon}{g} e^{\pm i\Omega f(r) \partial_\mu z}, \quad W_3^\mu = 0, \tag{7}
\]

where \(W_\mu^\pm = W_\mu^1 \pm iW_\mu^2\), and we have introduced a cylindrical profile function, \(f(r)\), with \(f(0) = 1\). This gauge field needs sources as given by the right-hand side of \(6\). Requiring that \(\Phi\) provides the currents that will produce the gauge field in \(7\) allows us to construct \(\Phi\), which we have to make sure also satisfies \(6\). Some algebra shows that the required form of \(\Phi\) is,

\[
\Phi = \frac{\epsilon}{g} \frac{\Omega}{\omega} f(r) \left( z_1 e^{i \omega t} + z_2 e^{-i \omega t} \right) \tag{8}
\]

where \(W_3^\pm = W_3^1 \pm iW_3^2\) and \(|z_1|^2 + |z_2|^2 = 1\). Without loss of generality we take \(\Omega > 0\) while \(\omega\) can be positive or negative. The \(\Phi\) field can then provide suitable sources for the gauge field as well as satisfy its classical equation of motion provided we take

\[
\Omega = m \left[ 4(1 - 4\lambda/g^2)^2 - 1 \right]^{-1/2} \tag{9}
\]

\[
\omega = 2(1 - 4\lambda/g^2) \Omega \tag{10}
\]

with restrictions that are necessary for \(\Omega\) to be real, namely, for \(m^2 > 0\), we need \(0 < \lambda < g^2/8\) (\(\omega > 0\)), or \(\lambda > 3g^2/8\) (\(\omega < 0\)), and for \(m^2 < 0\), \(g^2/8 < \lambda < g^2/4\) (\(\omega > 0\)) and \(g^2/4 < \lambda < 3g^2/8\) (\(\omega < 0\)). These constraints are summarized in Fig. \[1\]

The profile function satisfies the equation,

\[
f'' + \frac{f'}{r} + \Omega^2 \left( 1 - \frac{e^2}{2\omega \Omega^2 f^2} \right) f = 0. \tag{11}
\]

with boundary conditions \(f(0) = 1, f'(0) = 0\).
For $\epsilon^2 \ll 2|\omega|\Omega$ the solution is closely approximated by the zeroth order Bessel function, $J_0(\Omega r)$. For $\epsilon^2/2\omega\Omega > 1$, there is no well-behaved solution. We define a rescaled profile $F(r) \equiv \epsilon f(r)/\sqrt{2|\omega|\Omega}$ and a rescaled coordinate $R = \Omega r$. A numerical solution for $F(R)$ vs. $R$ for $\omega > 0$ is shown in Fig. 2. The asymptotic form of the solution is approximately described by the asymptotic form of the Bessel function,

$$F(R) \sim F(0) \sqrt{\frac{2}{\pi R}} \cos \left( R - \frac{\pi}{4} \right), \quad R \to \infty.$$  \hspace{1cm} (12)

The field strength for the solution is

$$W_{\mu\nu}^1 = -\frac{\epsilon}{g} e^{\pm i\Omega} \left[ \Omega f(r)(\partial_{\mu}r \partial_{\nu}z - \partial_{\nu}r \partial_{\mu}z) + f'(r)(\partial_{\mu}r \partial_{\nu}z - \partial_{\nu}r \partial_{\mu}z) \right]$$  \hspace{1cm} (13)

and $W_{\mu\nu}^3 = 0$. An SU(2) gauge rotation by

$$U = e^{i\sigma_2^\Theta/4} e^{i\sigma^+/4} e^{-i\sigma^+\Omega R/2}$$  \hspace{1cm} (14)

brings the field strength to the form,

$$W'_{\mu\nu} = U^{-1} W_{\mu\nu} U$$  \hspace{1cm} (15)

Then the electric field is in the third SU(2) direction and the spatial $z$-direction, while the magnetic field is in the second SU(2) direction and in the spatial azimuthal direction. The structure of the solution consists of a tube of electric field along $z$, wrapped by magnetic field along the azimuthal direction $\phi$, which is then within a sheet of electric field in the $-z$ direction, wrapped in magnetic field in the $-\phi$ direction, and so on. The distinction between electric and magnetic fields is frame-dependent and so we calculate the Lorentz invariant $-W_{\mu\nu}W^{\mu\nu} \propto F^2 - F'^2$ in Fig. 3 confirming the alternating sequence of electric and magnetic fields.

The energy density, $\mathcal{E}$ of the solution can be calculated from the expression,

$$\mathcal{E} = \frac{1}{2} (W_{0i}^a)^2 + \frac{1}{4} (W_{ij}^a)^2 + |D_t \Phi|^2 + |D_t \Pi|^2 + V(\Phi)$$  \hspace{1cm} (19)

In terms of rescaled variables, and restricting to $\omega > 0$,

$$\mathcal{E}' = \frac{g^2}{2|\omega|\Omega \chi} \mathcal{E} = \left( \frac{1}{2} + \frac{1}{\kappa} \right) F'^2 + \left( \frac{1}{2} + 2\kappa - \frac{1}{\kappa} \right) F^2 + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{\kappa} \right) F^4$$  \hspace{1cm} (20)

where (see (10)) $1 < \kappa \equiv |\omega|/\Omega < 2$. In Fig. 4 we show an example of $\mathcal{E}'$ vs. $R$. For $\omega < 0$, the only difference is that the coefficient of the $F^4$ term in (20) is $(3/2+1/\kappa)/2$ and the plot of $\mathcal{E}'$ vs. $R$ is very similar.

The slow fall off $\propto 1/\sqrt{R}$ of the gauge fields in (12) implies that $\mathcal{E} \propto 1/R$ and that the energy per unit length, $\mu$, diverges linearly with radial distance. Hence the string is not localized as in a Nielsen-Olesen string but is more
like a global string that has a logarithmically divergent energy per unit length, or like a global monopole with linearly divergent energy \[.\] We evaluate \(\mu\) numerically by integrating over \(R \in [0, R_c]\) where \(R_c\) is a radial cutoff to get,

\[
\mu(R_c) = 2\pi \int_0^{R_c} dR R \mathcal{E}' \\
\approx \left[ \left( \frac{1}{2} + \frac{1}{\kappa} \right) 0.54 + \left( \frac{1}{2} + 2\kappa - \frac{1}{\kappa} \right) 0.54 \right] R_c \\
+ \frac{1}{2} \left( \frac{1}{2} + \frac{1}{\kappa} \right) 0.09 \ln(R_c), \quad (\omega > 0)
\] (21)

While we have focused on the \(m^2 > 0\) case, similar solutions also exist for \(m^2 < 0\), i.e. in symmetry breaking potentials, in the allowed regions in Fig. 1. In that case, however, we still have \(|\Phi| \to 0\) in the asymptotic region. Since the true vacuum has \(|\Phi| \neq 0\), the solution has divergent energy per unit length and the divergence goes as \(R_c^2\) instead of \(R_c\).

Our electric string solution is with a scalar field in the fundamental representation. This leads to the question whether scalar fields in other representations can also provide suitable sources for the gauge fields in (7). We have examined the case of a scalar field in the adjoint representation, \(\phi^a\), and can show that a solution does not exist. Briefly, the gauge field equation is \(D_\mu W^{\mu \nu a} = j^{\mu a}\) where the current \(j^{\mu a}\) is now due to the adjoint scalar and has a form such that \(\phi^a j^{\mu a}_\mu = 0\) for every value of the index \(\mu\). This constraint then requires \(\phi^a D_\nu W^{\mu \nu a} = 0\) for every \(\mu\), where the gauge field is given in (7). The constraint is strong enough that it fixes the form of \(\phi^a\). Then we can calculate \(j^{\mu a}_\mu\) using \(\phi^a\) and we see that it does not satisfy the gauge field equation of motion: \(j^{\mu a}_\mu = D_\nu W^{\mu \nu a}\).

Another interesting question is if electric string solutions exist in the electroweak model. The difference between the model in (1) and the bosonic sector of the electroweak model is that the global \(U(1)\) symmetry of (1) is now gauged with coupling \(g'\) and \(\Phi\) has non-vanishing \(U(1)\) gauge charge (known as “hypercharge”). Then the \(U(1)\) hypercharge current is,

\[
j^{\mu}_Y = i \frac{g'}{2} \left( 0^+ D_\mu \Phi - h.c. \right) \tag{22}\]

Using (7) and (8) we find,

\[
j^{\mu}_Y = - \frac{g' \epsilon^2}{g^2} \Omega f^2 (|z_1|^2 - |z_2|^2) \partial_\mu t \\
- \frac{g' \epsilon^3}{2g^2} \frac{\Omega f^3}{\omega} \frac{1}{z_1 z_2^*} e^{i(\Omega + 2\omega)t} + c.c. \partial_\mu z, \tag{23}\]

which shows that in general there is a hypercharge charge density as well as a three current. If \(j^{\mu}_Y\) is non-vanishing then necessarily the hypercharge gauge field, \(Y^\mu\), is non-vanishing because it satisfies,

\[
\partial_\nu Y^{\mu \nu} = j^{\mu}_Y. \tag{24}\]

Our solution does not include a hypercharge component and so the question is if we can choose parameters such that \(j^{\mu}_Y = 0\). The \(\mu = 0\) component can be made to vanish by choosing \(|z_1|^2 = |z_2|^2\). The \(\mu = 3\) component is time dependent except if \(\Omega + 2\omega = 0\), which corresponds to \(8\lambda/g^2 = 5/2\) as follows from Eq. (10). With this choice we can make \(j^{\mu}_Y\) vanish by choosing \(z_1 \zeta_2^2 + \zeta_1^2 z_2 = 0\). A simple choice that meets all the requirements is \(z_1 = 1/\sqrt{2}\) and \(z_2 = i/\sqrt{2}\). Thus, even with the hypercharge gauge field, we find an electric string solution for \(8\lambda/g^2 = 5/2\), with \(W^a_\mu\) given by (7), \(Y_\mu = 0\), and

\[
\Phi = \frac{\epsilon}{g} f(r) \left( e^{-i\Omega t/2} + e^{+i\Omega t/2} \right) \tag{25}\]

Note that \(8\lambda/g^2 = 5/2\) is permissible only in the \(m^2 < 0\) region of Fig. 1. Thus the potential is necessarily of the symmetry breaking type. As we have pointed out above, this means that the potential energy density goes to a constant asymptotically since \(f \to 0\) as \(r \to \infty\). An alternate possibility is that there exist solutions with nonvanishing hypercharge gauge field \(Y_\mu\) and then we do not need to require that \(j^{\mu}_Y\) vanish. We have not been able to construct such solutions.

An important feature of the gauge field in the solution is that it is stationary. In other words, consider perturbations of only the gauge field,

\[
W^\pm_\mu = A^\pm_\mu + e^{\pm i\Omega t} Q^a_\mu, \quad W^3_\mu = Q^3_\mu, \quad \Phi = \Phi_0 \tag{26}\]

where \(A^a_\mu, \Phi_0\) denote the electric string solution. Then the action for \(Q^a_\mu\) does not contain any terms that are explicitly time-dependent and hence the solution is protected from decay to Schwinger pair production of gauge

\[\text{2 It is interesting that the experimentally measured value of} \quad 8\lambda/g^2 = m^2_{Higgs}/m^2_{W} \text{ is } 2.43 \approx 5/2 \text{ [22]}.\]
field excitations as shown in Ref. [8]. The inclusion of the scalar field, $\Phi$, does not make any difference to the analysis in Ref. [8] because the quadratic order interaction between $\Phi$ and $Q_\mu^a$ is simply $g^2|\Phi|^2(Q_\mu^a)^2/4$ and $|\Phi|^2$ is time-independent. (The linear order terms vanish because the solution obeys the classical equations of motion.) Fluctuations of the scalar field $\Phi$ can indeed get excited by the time dependence of the solution, and this means that Schwinger pair production of $\Phi$ excitations will occur. (This is similar to the Schwinger pair production of quarks on QCD strings.) In the limit of large mass parameter $m$, the Schwinger pair production of $\Phi$ will be suppressed.

A Maxwell electric field for example, with $W_3^\mu = -Et$ and all other components zero, is a classical solution of the SU(2) pure gauge theory. However, the Schwinger process for gluons in this background is non-vanishing at all momentum scales [4] and the Maxwell electric field will decay and evolve into another configuration. Since a BW electric field is stable to the Schwinger process, it is likely that it is the final state. In other words, if initially we start with a Maxwell electric field, it may evolve into a BW electric field.

In Ref. [13] the classical stability of a homogeneous electric field of the BW type was analyzed. Several unstable modes were found for the homogeneous configuration. For the electric string, these unstable modes will be suppressed due to the $|\Phi|^2(Q_\mu^a)^2$ term, since this term provides an effective mass to the gauge excitations wherever $|\Phi|^2$ is non-zero. The solutions with $m^2 < 0$ will almost certainly be unstable since $\Phi$ approaches the unstable vacuum, $|\Phi| = 0$, in the asymptotic region. We plan to carry out a detailed stability analysis in a future publication.

We close with a speculative remark. The solution we have found may be of interest in the context of pure non-Abelian gauge theories. In that case, the field $\Phi$ must arise as an effective degree of freedom due to quantum backreaction in the classical equations of motion. In this connection, it has been conjectured that magnetic monopoles at strong coupling transform as a scalar degree of freedom in the fundamental representation of a dual symmetry group [14]. While there are no classical monopole solutions in pure non-Abelian gauge theory, one can still write configurations that resemble the gauge fields of magnetic monopoles,

$$W_i^a = \frac{(1 - k(r))}{gr} \epsilon^{aij} \hat{x}^j$$

(27)

where $k(r)$ is a suitable profile function. Can the back-reaction due to such monopole configurations effectively behave like our scalar field $\Phi$?

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[1] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973).
[2] A. Vilenkin and E. P. S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, 2000), ISBN 978-0-521-65476-0.
[3] J. S. Schwinger, Phys. Rev. 82, 664 (1951).
[4] C. Cardona and T. Vachaspati, Phys. Rev. D 104, 045009 (2021), 2105.08782.
[5] R. Jackiw, L. Jacobs, and C. Rebbi, Phys. Rev. D 20, 474 (1979).
[6] K. Huang and R. Tipton, Phys. Rev. D 23, 3050 (1981).
[7] L. S. Brown and W. I. Weisberger, Nucl. Phys. B 157, 285 (1979), [Erratum: Nucl.Phys.B 172, 544 (1980)].
[8] T. Vachaspati, Phys. Rev. D 105, 105011 (2022), 2204.01902.
[9] T. Vachaspati and M. Barriola, Phys. Rev. Lett. 69, 1867 (1992).
[10] M. Barriola, T. Vachaspati, and M. Bucher, Phys. Rev. D 50, 2819 (1994), hep-th/9306120.
[11] J. Preskill, Phys. Rev. D 46, 4218 (1992), hep-ph/9206216.
[12] R. L. Workman and Others (Particle Data Group), PTEP 2022, 083C01 (2022).
[13] J. Pereira and T. Vachaspati, Phys. Rev. D 106, 096019 (2022), 2207.05102.
[14] P. Goddard, J. Nuyts, and D. I. Olive, Nucl. Phys. B 125, 1 (1977).