Colouring of generalized signed planar graphs

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Abstract

Assume $G$ is a graph. We view $G$ as a symmetric digraph, in which each edge $uv$ of $G$ is replaced by a pair of opposite arcs $e = (u,v)$ and $e^{-1} = (v,u)$. Assume $S$ is an inverse closed subset of permutations of positive integers. We say $G$ is $S$-k-colourable if for any mapping $\sigma : E(G) \to S$ with $\sigma(x,y) = (\sigma(y,x))^{-1}$, there is a mapping $f : V(G) \to [k] = \{1,2,\ldots,k\}$ such that for each arc $e = (x,y)$, $\sigma_{e}(f(x)) \neq f(y)$. The concept of $S$-k-colouring is a common generalization of many colouring concepts, including $k$-colouring, signed $k$-colouring defined by Mácajová, Raspaud and Škoviera, signed $k$-colouring defined by Kang and Steffen, correspondence $k$-colouring defined by Dvořák and Postle, and group colouring defined by Jaeger, Linial, Payan and Tarsi. We are interested in the problem as for which subset $S$ of $S_4$, every planar graph is $S$-colourable. Such a subset $S$ is called good. The famous four colour theorem is equivalent to say that $S = \{id\}$ is good. There are two conjectures on signed graph colouring, one is equivalent to $S = \{id, (12)\}$ be good and the other is equivalent to $S = \{id, (12)\}$ be good. We say two subsets $S$ and $S'$ of $S_k$ are conjugate if there is a permutation $\pi \in S_k$ such that $S' = \{\pi \sigma \pi^{-1} : \sigma \in S\}$. This paper proves that if $S$ is a good subset of $S_4$ containing $id$, then $S$ is conjugate to a subset of $\{id, (12), (34), (12)(34)\}$, however, it remains an open problem if there is any good subset $S$ which contains $id$ and has cardinality $|S| \geq 2$. We also prove that $S = \{(12), (13), (23), (123), (132)\}$ is not good.

Keywords: signed graph colouring, generalized signed graph, correspondence colouring, group colouring.

1 Introduction

A signed graph is a pair $(G, \sigma)$, where $G$ is a graph and $\sigma : E(G) \to \{+1, -1\}$ assigns to each edge $e$ a sign $\sigma_e$. 

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In the 1980’s, Zaslavsky studied vertex colouring of signed graphs [11, 12, 13]. He defined a colouring of a signed graph \((G, \sigma)\) as a mapping \(f : V(G) \to \{\pm k, \pm (k-1), \ldots, \pm 1, 0\}\) such that for any edge \(e = xy\) of \(G\), \(f(x) \neq \sigma_e f(y)\). In 2016, Mácajová, Raspaud and Škoviera [7] modified the definition as follows:

**Definition 1** Assume \((G, \sigma)\) is a signed graph and \(k\) is a positive integer. If \(k = 2q\) is even (respectively, \(k = 2q + 1\) is odd), then a \(k\)-colouring of \((G, \sigma)\) is a mapping \(f : V(G) \to \{\pm q, \pm q, \ldots, \pm 1\}\) (respectively, \(f : V(G) \to \{\pm q, \pm q, \ldots, \pm 1, 0\}\)) such that for any edge \(e = xy\) of \(G\), \(f(x) \neq \sigma_e f(y)\). We say a graph \(G\) is signed \(k\)-colourable if for any signature \(\sigma\) of \(G\), \((G, \sigma)\) is \(k\)-colourable.

Recently, Kang and Steffen introduced another type of colouring of signed graphs [3, 4].

**Definition 2** A \(k\)-colouring of a signed graph \((G, \sigma)\) is a mapping \(c : V(G) \to Z_k\) such that for each edge \(e = uv\), \(c(u) \neq \sigma_e c(v)\). Here \(c(u)\) is the inverse of \(c(v)\) in \(Z_k\), i.e., it is equal to \(-c(v)\) (mod \(k\)).

To distinguish these two \(k\)-colourability of signed graphs, we say a graph \(G\) is signed MRS-\(k\)-colourable or signed KS-\(k\)-colourable, if for any signature \(\sigma\) of \(G\), \((G, \sigma)\) is \(k\)-colourable according to Definition 1 or Definition 2, respectively.

Let

\[\chi_\pm(G, \sigma) = \min \{k : (G, \sigma) \text{ is MRS-}k\text{-colourable}\}\]

and

\[\chi(G, \sigma) = \min \{k : (G, \sigma) \text{ is KS-}k\text{-colourable}\}.

It is known [4] that for any signed graph \((G, \sigma)\),

\[\chi_\pm(G, \sigma) - 1 \leq \chi(G, \sigma) \leq \chi_\pm(G, \sigma) + 1\]

and these bounds are sharp.

The sign \(\sigma_e\) of an edge \(e\) is used to impose restriction on the colour pairs that can be assigned to the end vertices of \(e\). There are two possible signs for the edges, and hence there are two kinds of restrictions for the pairs of colours that can be assigned to the end vertices of an edge.

In a \(k\)-colouring of a graph, we usually use the set \([k] = \{1, 2, \ldots, k\}\) as the colour set. We may change the colour set in the \(k\)-colouring of a signed graph back to the set \([k]\). For both MRS-\(k\)-colouring and KS-\(k\)-colouring of a signed graph, there is an easy way to express the constraints of colour pairs that can be assigned to the end vertices of \(e\).

We denote by \(id\) the identity permutation on the set \([k]\), and let \(\pi = (12)(34)\ldots((2q-1)(2q))\) where \(q = \lfloor k/2 \rfloor\) and \(\pi' = (12)(34)\ldots((2q-1)(2q))\) where \(q = \lceil k/2 \rceil - 1\). Then an MRS-\(k\)-colouring (respectively, a KS-\(k\)-colouring) of a signed graph \((G, \sigma)\) is a mapping \(f : V(G) \to [k]\) such that if \(e = xy\) is a positive edge, then \(f(x) \neq f(y)\), and if \(e = xy\) is a negative edge, then \(\pi(f(x)) \neq f(y)\) (respectively, \(\pi'(f(x)) \neq f(y)\)).
Thus if \( k \) is odd, then MRS-\( k \)-colourability and KS-\( k \)-colourability are equivalent. However, if \( k \) is even, then the two \( k \)-colourabilities are distinct.

In this paper, we consider generalized signed graphs, in which edges of \( G \) may have more signs, and each sign imposes a restriction on pairs of colours that can be assigned to the end vertices of edges of that sign.

For convenience, we view an undirected graph \( G \) as a symmetric digraph, in which each edge \( uv \) of \( G \) is replaced by two opposite arcs \( e = (u, v) \) and \( e^{-1} = (v, u) \). We denote by \( E(G) \) the set of arcs of \( G \). A set \( S \) of permutations of positive integers is inverse closed if \( \pi^{-1} \in S \) for every \( \pi \in S \).

**Definition 3** Assume \( S \) is an inverse closed subset of permutations of positive integers. An \( S \)-signature of \( G \) is a mapping \( \sigma : E(G) \to S \) such that for every arc \( e \), \( \sigma_{e^{-1}} = \sigma_e^{-1} \). The pair \((G, \sigma)\) is called an \( S \)-signed graph.

**Definition 4** Assume \( S \) is an inverse closed subset of permutations of positive integers and \((G, \sigma)\) is an \( S \)-signed graph. A \( k \)-colouring of \((G, \sigma)\) is a mapping \( f : V(G) \to [k] = \{1, 2, \ldots, k\} \) such that for each arc \( e = (x, y) \) of \( G \), \( \sigma_e(f(x)) \neq f(y) \). We say \( G \) is \( S \)-\( k \)-colourable if \((G, \sigma)\) is \( k \)-colourable for every \( S \)-signature \( \sigma \) of \( G \).

In the signed graph colouring of a graph defined by Máčajová, Raspaud and Škoviera \cite{ma07}, or in the one defined by Kang and Steffen \cite{ka13,ka14}, two permutations are used: \( \sigma_e = id \) (if \( e \) is a positive edge) and \( \sigma_e = \pi \) for the definition of Máčajová, Raspaud and Škoviera, and \( \sigma_e = \pi' \) for the definition of Kang and Steffen. Since \( \pi^{-1} = \pi \) and \( \pi'^{-1} = \pi' \), the orientation of the edge is irrelevant and hence omitted.

It follows from the definition that if \( S = \{id\} \), then \( S \)-\( k \)-colourable is equivalent to \( k \)-colourable. If \( S = \{id, (12)(34) \ldots ((2q - 1)(2q))\} \) when \( q = [k/2] \) or \( q = [k/2] - 1 \), then \( G \) is \( S \)-\( k \)-colourable is equivalent to signed MRS-\( k \)-colourable or signed KS-\( k \)-colourable, respectively.

If \( S \) is the set of all permutations, then it turns out that \( S \)-\( k \)-colourable is equivalent to correspondence \( k \)-colourable (also called DP \( k \)-colourable), a concept introduced recently by Dvořák and Postle \cite{dv17}.

**Definition 5** Assume \( S \) is the set of all permutations of positive integers. We say a graph \( G \) is correspondence \( k \)-colourable if \( G \) is \( S \)-colourable. The DP-chromatic number \( \chi_{DP}(G) \) of \( G \) is the minimum positive integer \( k \) such that \( G \) is correspondence \( k \)-colourable.

The definition above of correspondence \( k \)-colourable and DP-chromatic number of a graph is differently formulated than the definition in \cite{dv17}, but is equivalent.

Dvořák and Postle introduced the concept of correspondence colouring as a generalization of list colouring. Let \( ch(G) \) be the choice number of \( G \). It is known \cite{dv17} that for any graph \( G \), \( ch(G) \leq \chi_{DP}(G) \).

Colouring of planar graphs has been an important source of challenging problems. Two challenging open problems in the colouring of signed graph are the following generalizations of the four colour theorem.
Conjecture 6 \[7\] Every planar graph is signed MRS-4-colourable.

Conjecture 6 is due to Kang and Steffen \[9\].

Conjecture 7 \[9\] Every planar graph is signed KS-4-colourable.

Conjecture 6 is equivalent to say that for \(S = \{id, (12)(34)\}\), every planar graph is \(S\)-colourable; and Conjecture 7 is equivalent to say that for \(S = \{id, (12)\}\), every planar graph is \(S\)-colourable.

By Four Colour Theorem, if \(S = \{id\}\), then every planar graph is \(S\)-colourable. The existence of non-4-choosable planar graphs shows that not every planar graph is \(S_4\)-colourable. Now an interesting problem is for which subsets \(S\) of \(S_4\), every planar graph is \(S\)-colourable.

Definition 8 Assume \(S\) is an inverse closed non-empty subset of \(S_4\). We say \(S\) is good if every planar graph is \(S\)-colourable, and \(S\) is bad otherwise.

Question 9 Which inverse closed subsets \(S\) of \(S_4\) are good?

Observe that if \(S, S'\) are inverse closed subsets of \(S_k\) and \(S \subseteq S'\), then any \(S'\)-colourable graph is \(S\)-colourable. Thus for \(S, S' \subseteq S_4\), if \(S \subseteq S'\), and \(S'\) is good, then \(S\) is good. In the poset consisting non-empty inverse closed subsets of \(S_4\), where is the boundary that separates good subsets of \(S_4\) from the bad ones? Intuitively, we are asking how tight is the four colour theorem. If all inverse closed subsets \(S\) with \(id \in S\) and with \(|S| \geq 2\) are bad, then the four colour theorem is very tight. There is no room to strengthen the four colour theorem in the sense of \(S\)-colourability defined in this paper. If this is not the case, then how much one can hope for a strengthening of the four colour theorem? Conjectures 6 and 7 claim some kind of strengthening of the four colour theorem.

We say two subsets \(S\) and \(S'\) are conjugate if there is a permutation \(\pi \in S_4\) such that \(S' = \{\pi \sigma \pi^{-1} : \sigma \in S\}\). It is obvious that if \(S\) and \(S'\) are conjugate, then \(S\) is good if and only if \(S'\) is good. In this paper, we prove that if \(id \in S\) and \(S\) is not conjugate to a subset of \(\{id, (12), (34), (12)(34)\}\), then \(S\) is bad. We also prove that \(S = \{(12), (13), (23), (123), (132)\}\) is bad. These results show that good sets are small. So if there is any room to strengthen the four colour theorem, the room is quite small. From another point of view, the results in this paper can be regarded as strengthening of the result that there are planar graphs that are not correspondence 4-colourable; or strengthening of the result that there are planar graphs that are not 4-choosable.

The results in this paper are also related to group colouring of graphs, a concept introduced by Jaeger, Linial, Panyan and Tarsi \[2\] in 1992.

Definition 10 Assume \(G\) is a graph and \(\Gamma\) is an additive Abelian group. We say \(G\) is \(\Gamma\)-colourable if for any mapping \(\sigma : E(G) \rightarrow \Gamma\) for which \(\sigma(e^{-1}) = \sigma(e)^{-1}\) for every directed edge \(e\), there is a mapping \(f : V(G) \rightarrow \Gamma\) such that for each directed edge \(e = (x, y)\) of \(G\), \(f(y) - f(x) \neq \sigma(e)\).
Group colouring of graphs is the dual concept of group connectedness, and has been studied extensively in the literature (see [6] for a survey).

Assume $\Gamma$ is an additive Abelian group with $|\Gamma| = k$. Let $\theta : \Gamma \to \{1, 2, \ldots, k\}$ be a one-to-one correspondence. Let $\pi : \Gamma \to S_k$ be the mapping which assigns to each element $a$ of $\Gamma$ a permutation $\pi_a$ defined as $\pi_a(j) = \theta(a + \theta^{-1}(j))$. Then $\pi$ is an isomorphism from $\Gamma$ to a subgroup $S = \pi(\Gamma)$ of $S_k$. It is straightforward to verify that $\Gamma$-colourable is equivalent to $S$-colourable. Therefore for any Abelian group $\Gamma$ with $|\Gamma| = k$, there is a subset $S$ of $S_k$ such that a graph $G$ is $\Gamma$-colourable if and only if $G$ is $S$-colourable.

For example, if $S$ is the subgroup of $S_k$ generated by $(12...k)$, then $S$-colourable is equivalent to $Z_k$-colourable. If $S = \{id,(12)(34),(13)(24),(14)(23)\} \subseteq S_4$, then $S$-colourable is equivalent to $Z_2 \times Z_2$-colourable.

Observe that for a subgroup $S$ of $S_k$, whether a graph $G$ is $S$-colourable depends on the permutations contained in $S$, not just on the group structure of $S$. For example, for $S = \{id,(12)(34),(13)(24),(14)(23)\}$ and $S' = \{id,(12),(34),(12)(34)\}$, both $S$ and $S'$ are subgroups of $S_4$ isomorphic to $Z_2 \times Z_2$. However, $S$-colourability is equivalent to $Z_2 \times Z_2$-colourability (as a group colouring), but $S'$-colourability is not. There are signed planar graphs that are $S'$-colourable but not $S$-colourable.

It is known [3, 8] that there are planar graphs that are not $Z_4$-colourable. The results in this paper is slightly stronger. For an inverse closed subset $S$ of $\Gamma$, we say a graph $G$ is $S$-$\Gamma$-colourable if for any mapping $\tau : E(G) \to S$ such that $\tau(e^{-1}) = \tau(e)^{-1}$ for each directed edge $e$ of $G$, there is a mapping $f : V(G) \to \Gamma$ such that for each directed edge $e = (x, y)$, $f(y) - f(x) = \tau(e)$. It is proved in this paper that for $S = \{0, 1, 3\} \subseteq Z_4$, there is a planar graph which is not $S$-$Z_4$-colourable.

On the other hand, Conjecture [17] is equivalent to say that for $\Gamma \in \{Z_4, Z_2 \times Z_2\}$, for any inverse closed subset $S$ of $\Gamma$ with $|S| \leq 2$, every planar graph is $S$-$\Gamma$-colourable.

Remark. To define an $S$-signature on $G$, we usually fix an orientation $D$ of $G$, and define $\sigma_e$ for each arc $e$ of $D$. If $e = (u, v)$, we write $\sigma_{(u,v)}$ for $\sigma_e$. If $\pi \in S$ and $\pi = \pi^{-1}$, then the orientation of the edge $e$ for which $\sigma_e = \pi$ is irrelevant. In this case, for convenience, we omit the orientation of the edge, and for $e = uv$, we write $\sigma_{uv}$ for $\sigma_e$.

2 The main results

Definition 11 A graph is uniquely $k$-colourable if there is a unique partition of $V(G)$ into $k$ independent sets.

Assume $G$ is uniquely $k$-colourable, and $V_1, V_2, \ldots, V_k$ is the unique partition of $V(G)$ into $k$ independent sets. There are $k!$ ways of assigning the $k$ colours $\{1, 2, \ldots, k\}$ to the independent sets. So there are actually $k!$ $k$-colourings of $G$. If $G$ is a uniquely 4-colourable planar graph, then there are exactly 24 4-colourings of $G$.

For a plane graph $G$, we denote by $F(G)$ the set of faces of $G$. 5
Lemma 12 There exists a uniquely 4-colourable plane triangulation $G'$, a set $\mathcal{F}$ of 24 faces of $G'$ and a one-to-one correspondence $\phi$ between $\mathcal{F}$ and the 24 4-colourings of $G'$ such that for each $F \in \mathcal{F}$, $\phi(F)(V(F)) = \{1, 2, 3\}$, where $\phi_F$ is the 4-colouring of $G'$ corresponding to $F$ and $V(F)$ is the set of vertices incident to $F$.

Proof. Build a plane triangulation which is uniquely 4-colourable and which has 24 faces. Such a graph can be constructed by starting from a triangle $T = uvw$, and repeat the following: choose a face $F$ (which is a triangle), add a vertex $x$ in the interior of $F$ and connect $x$ to each of the three vertices of $F$. Each iteration of this procedure increases the number of faces of $G$ by 2. We stop when there are 24 faces.

Let $\phi$ be an arbitrary one-to-one correspondence between the 24 4-colourings of $G$ and the 24 faces of $G$. For each face $F$ of $G$, we denote by $\phi_F$ the corresponding 4-colouring of $G$.

Let $\mathcal{F}'$ be the set of faces $F$ for which $\phi_F(V(F)) \neq \{1, 2, 3\}$.

For each $F \in \mathcal{F}'$, add a vertex $z_F$ in the interior of $F$, connect $z_F$ to each of the three vertices of $F$. The colouring $\phi_F$ is uniquely extended to $z_F$. Hence the resulting plane triangulation $G'$ is still uniquely 4-colourable. The face $F$ of $G$ is partitioned into three faces of $G'$. One of the three faces is coloured by $\{1, 2, 3\}$. We denote this face by $F'$ and use this face of $G'$ instead of the face $F$ of $G$ to be associated with the colouring $\phi_F$ (and we denote this colouring by $\phi_F$, after this operation).

Let

$$\mathcal{F} = \{F' : F \in \mathcal{F}'\} \cup \mathcal{F}(G) - \mathcal{F}'$$

The one-to-one correspondence between $\mathcal{F}$ and the 24 4-colourings of $G'$ defined above satisfies the requirements of the lemma. \hfill \blacksquare

Theorem 13 Assume $S$ is a subset of $S_4$ which contains $id$. If $S$ is not conjugate to a subset of $\{id, (12), (34), (12)(34)\}$, then $S$ is bad.

Proof. Assume $S$ is an inverse closed subset of $S_4$ which contains $id$ and is not conjugate to a subset of $\{id, (12), (34), (12)(34)\}$. It is straightforward to verify that $S$ contains a subset $S'$ which is conjugate to one of the following sets:

$$\{id, (123), (132)\}, \{id, (1234), (1432)\}, \{(id, (12), (13)\}, \{id, (12)(34), (13)\}, \{id, (12)(34), (13)(24)\}.$$ 

So it suffices to show that each of the subset listed above is bad.

By Lemma 12 there is a uniquely 4-colourable plane triangulation $G'$, a set $\mathcal{F}$ of 24 faces of $G'$ and a one-to-one correspondence $\phi$ between $\mathcal{F}$ and the 24 4-colourings of $G'$ so that for each $F \in \mathcal{F}$, $\phi_F(V(F)) = \{1, 2, 3\}$.

For each face $F \in \mathcal{F}$, for $i \in \{1, 2, 3\}$, let $v_{F,i}$ be the vertex with $\phi_F(v_{F,i}) = i$.

- add a triangle $T_F = a_F b_F c_F$ in the interior of $F$;
- connect $a_F$ to $v_{F,1}$ and $v_{F,2}$; connect $b_F$ to $v_{F,1}$ and $v_{F,3}$; connect $c_F$ to $v_{F,2}$ and $v_{F,3}$.
We denote the resulting plane triangulation by $G$.

1: If $S = \{id, (12), (13)\}$ or $S = \{id, (12)(34), (13)\}$, then $S$ is bad.

Figure 1: $S = \{id, (12), (13)\}$ and $S = \{id, (12)(34), (13)\}$ are bad, where dotted edge $e$ has either $\sigma_e = (12)$ or $\sigma_e = (12)(34)$ and dashed edges $e$ have $\sigma_e = (13)$

For each $F \in \mathcal{F}$, let $\sigma_{v_F,1}b_F = (12)$ or $\sigma_{v_F,1}b_F = (12)(34)$, $\sigma_{a_F}b_F = \sigma_{a_F}c_F = (13)$. For all the other edges $e$ of $G$, $\sigma_e = id$. See Figure 1.

Now we show that $G$ is not $\sigma$-colourable.

Assume $\psi$ is an $\sigma$-colouring of $G$. Then the restriction of $\psi$ to $G'$ is a proper 4-colouring of $G'$. As $G'$ is uniquely 4-colourable, the restriction of $\psi$ to $G'$ equals $\phi_F$ for some $F \in \mathcal{F}$. Consider the triangle $T_F$. Vertex $a_F$ is adjacent to vertices of colours 1 and 2 by edges $e$ with $\sigma_e = id$. Hence $\psi(a_F) \neq 1, 2$. So $\psi(a_F) \in \{3, 4\}$. Similarly, $\psi(c_F) \in \{1, 4\}$. Since $\sigma_{v_F,1}b_F(1) = 2$, we conclude that $\psi(b_F) \neq 2$. Since $\sigma_{v_F,3}b_F(3) = 3$, $\psi(b_F) \neq 3$. Hence $\psi(b_F) \in \{1, 4\}$.

If $\psi(c_F) = 4$, then since $\sigma_{c_F}b_F(4) = 3$, we conclude that $\psi(b_F) \neq 3$, hence $\psi(b_F) = 1$. Similarly, we must have $\psi(a_F) = 3$. As $\sigma_{b_F}a_F(1) = 3$, this is a contradiction.

If $\psi(c_F) = 1$, then the same argument shows that $\psi(a_F) = 4$ and $\psi(b_F) = 4$. But $\sigma_{a_F}b_F(4) = 4$, a contradiction.

2: If $S = \{id, (12)(34), (23)(14)\}$, then $S$ is bad.

Figure 2: $S = \{id, (12)(34), (23)(14)\}$ is bad, where dotted edge $e$ has $\sigma_e = (12)(34)$ and dashed edges $e$ have $\sigma_e = (23)(14)$

For each $F \in \mathcal{F}$, let $\sigma_{v_F,1}a_F = \sigma_{a_F}b_F = \sigma_{a_F}c_F = (23)(14)$, $\sigma_{v_F,2}c_F = (12)(34)$. For all the other edges $e$ of $G$, $\sigma_e = id$. See Figure 2.

Now we show that $G$ is not $\sigma$-colourable.
Assume \( \psi \) is an \( \sigma \)-colouring of \( G \). Similarly, the restriction of \( \psi \) to \( G' \) equals \( \phi_F \) for some \( F \in \mathcal{F} \). Similar argument as in the first case shows that \( \psi(a_F) \in \{1, 3\}, \psi(b_F) \in \{2, 4\} \) and \( \psi(c_F) \in \{2, 4\} \).

If \( \psi(c_F) = 4 \), then since \( \sigma_{a_Fb_F}(4) = 4 \), we conclude that \( \psi(b_F) \neq 4 \), hence \( \psi(b_F) = 2 \). Similarly, we must have \( \psi(a_F) = 3 \). As \( \sigma_{b_Fa_F}(2) = 3 \), this is a contradiction.

If \( \psi(c_F) = 2 \), then the same argument shows that \( \psi(a_F) = 1 \) and \( \psi(b_F) = 4 \). But \( \sigma_{a_Fb_F}(1) = 4 \), a contradiction.

3: If \( S = \{id, (123), (132)\} \), then \( S \) is bad.

Orient the edges of \( T_F \) as \( (b_F, a_F), (c_F, b_F), (a_F, c_F) \). The orientation of the other edges are arbitrary (and irrelevant).

Let \( D \) be the orientation of \( G \) defined above. See Figure 3.

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.3	extwidth]{figure3}
\caption{\( S = \{id, (123), (132)\} \) is bad, where arrowed edges \( e \) have \( \sigma_e = (123) \)}
\end{figure} \]

Let \( \sigma : E(D) \to S \) be defined as \( \sigma_e = id \) for all edges \( e \), except that for each \( F \in \mathcal{F} \), for the three edges \( e \) of \( T_F \), \( \sigma_e = (123) \).

Now we show that \( G \) is not \( \sigma \)-colourable.

Assume \( \psi \) is an \( \sigma \)-colouring of \( G \). Similarly, the restriction of \( \psi \) to \( G' \) equals \( \phi_F \) for some \( F \in \mathcal{F} \). The same argument as above shows that \( \psi(a_F) \in \{3, 4\}, \psi(b_F) \in \{2, 4\} \) and \( \psi(c_F) \in \{1, 4\} \).

If \( \psi(a_F) = 4 \), then since \( \sigma_{b_Fa_F}(4) = 4 \), we conclude that \( \psi(b_F) \neq 4 \), hence \( \psi(b_F) = 2 \). Similarly, we must have \( \psi(c_F) = 1 \). As \( \sigma_{c_Fb_F}(1) = 2 \), this is a contradiction.

Assume \( \psi(a_F) = 3 \). Then the same argument shows that \( \psi(c_F) = \psi(b_F) = 4 \). As \( \sigma_{c_Fb_F}(4) = 4 \), this is a contradiction.

4: If \( S = \{id, (1234), (1432)\} \), then \( S \) is bad.

Orient the added edges as \( (c_F, a_F), (c_F, b_F), (v_{F,3}, c_F) \) and \( (v_{F,2}, a_F) \). The orientation of the other edges are arbitrary (and irrelevant).

Let \( G \) be the resulting planar graph and \( D \) be the orientation of \( G \) defined above. See Figure 4.

Let \( \sigma : E(D) \to S \) be defined as \( \sigma_e = id \) for all edges \( e \), except that for each \( F \in \mathcal{F} \), for \( e \in \{(v_{F,2}, a_F), (c_F, a_F), (c_F, b_F), (v_{F,3}, c_F)\} \), \( \sigma_e = (1234) \).

Now we show that \( G \) is not \( \sigma \)-colourable.
Similarly, we must have \( \psi \) is not a \( k \)-colourable. Therefore, there is an edge \( e \) such that \( \psi(a_F) \in \{2, 4\} \), \( \psi(b_F) \in \{2, 4\} \) and \( \psi(c_F) \in \{1, 3\} \).

If \( \psi(c_F) = 1 \), then since \( \sigma_{(c_F, a_F)}(1) = 2 \), we conclude that \( \psi(a_F) \neq 2 \), hence \( \psi(a_F) = 4 \). Similarly, we must have \( \psi(b_F) = 4 \). But \( \sigma_{a_Fb_F}(4) = 4 \), this is a contradiction.

If \( \psi(c_F) = 3 \), then since \( \sigma_{(c_F, a_F)}(3) = 4 \), we conclude that \( \psi(a_F) \neq 4 \), hence \( \psi(a_F) = 2 \). Similarly, we must have \( \psi(b_F) = 2 \). Again this is a contradiction.

This completes the proof of Theorem 13. \( \square \)

Corollary 14 below follows from the fact that \( S = \{id, (1234), (1432)\} \) is bad.

**Corollary 14** For \( S = \{0, 1, 3\} \), there is a planar graph \( G \) which is not \( S-Z_4 \)-colourable.

**Lemma 15** Assume \( G \) is a graph, \( k \) is an integer, \( S \) is an inverse closed subset of \( S_k \) and \( \sigma \) is an \( S \)-signature on \( G \). Let \( S' \) be a subset of \( S_k \) and

\[
S'' = S'^{-1}SS' = \{\tau \sigma \tau' : \tau \in S'^{-1}, \sigma \in S, \tau' \in S'\}.
\]

Let \( \tau : V(G) \to S' \) be a mapping which assigns to each vertex \( v \) a permutation \( \tau_v \). Let \( \sigma' : E(G) \to S'' \) be defined as follows:

\[
\forall e = (v, u) \in E(G), \sigma'_e = \tau_u^{-1} \sigma_v \tau_u.
\]

Then \( \sigma' \) is an \( S'' \)-signature on \( G \) and \( (G, \sigma) \) is \( k \)-colourable if and only if \( (G, \sigma') \) is \( k \)-colourable.

**Proof.** Assume \( f \) is a \( k \)-colouring of \( (G, \sigma) \). Then \( g : V(G) \to [k] \) defined as \( g(v) = \tau_v^{-1}(f(v)) \) is a \( k \)-colouring of \( (G, \sigma') \). Indeed, if \( g \) is not a \( k \)-colouring of \( (G, \sigma') \), then there is an edge \( e = (v, u) \) such that \( g(u) = \sigma'_e(g(v)) \). This implies that \( \tau_u(g(u)) = \sigma_v(\tau_v(g(v))) \), i.e., \( f(u) = \sigma_v(f(v)) \). So \( f \) is not a \( k \)-colouring of \( (G, \sigma) \). Conversely, if \( f \) is not a \( k \)-colouring of \( (G, \sigma') \), then there is an edge \( e = (v, u) \) such that \( f(u) = \sigma_v(f(v)) \). This implies that \( \tau_v^{-1}(f(u)) = \tau_u^{-1} \sigma_v \tau_v(\tau_v^{-1}(f(v))) \), i.e., \( g(u) = \sigma'_e(g(v)) \) and \( g \) is not a \( k \)-colouring of \( (G, \sigma') \). \( \square \)

**Corollary 16** The set \( S = \{(12), (13), (23), (123), (132)\} \) is bad.
**Proof.** Let $G$ be the plane triangulation and $\sigma$ be the mapping in the proof of the third case of Theorem 13, where $\sigma_e \in \{\text{id},(123)\}$ for all edges $e$. Let $G',\mathcal{F},D$ be the graph, the set and the orientation of $G$ as in the proof of the third case of Theorem 13. Let $c$ be a 4-colouring of $G'$. Let $\tau:V(G)\to S_4$ be defined as follows:

- If $v \in V(T_F)$ for some $F \in \mathcal{F}$, then $\tau_v = \text{id}$.
- Otherwise, if $c(v) = 1$, then $\tau_v = (12)$.
- If $c(v) = 2$, then $\tau_v = (13)$.
- If $c(v) = 3$, then $\tau_v = (23)$.
- If $c(v) = 4$, then $\tau_v = (123)$.

For $e = (v,u) \in E(G)$, let $\sigma'_e = \tau_v^{-1}\sigma_e \tau_v$. By Lemma 15, $G$ is not $\sigma'$-colourable. On the other hand, it is easy to verify that for any arc $e$ of $G$, $\sigma'_e \in \{(12),(13),(23),(123),(132)\}$. So $S = \{(12),(13),(23),(123),(132)\}$ is bad.

3 Some open problems

The results in this paper suggest that most subsets $S$ of $S_4$ are bad. In some sense, this means that the four colour theorem is very tight. Still, maybe some strengthening of the four colour theorem is possible.

Two inverse closed subsets $S$ and $S'$ are conjugate if there is a permutation $\pi$ on $[4]$ such that $\pi(S) = S'$. For example, $S = \{(\text{id},(123),(132))\}$ is conjugate to $S' = \{(\text{id},(234),(243))\}$. It is obvious that if $S$ and $S'$ are conjugate then either both $S, S'$ are good or both are bad.

We have found some inverse closed subsets $S$ of $S_4$ with $|S| = 3$ and with $\text{id} \in S$ such that $S$ is bad. Up to isomorphisms, inverse closed subsets $S$ of $S_4$ with $|S| = 2$ are $S = \{\text{id},(12),(34)\}$, $S = \{\text{id},(12)\}$, $S = \{(12),(13)\}$ and $S = \{(12),(34)\}$. Conjecture 6 asserts that $S = \{\text{id},(12),(34)\}$ is good and Conjecture 7 asserts that $S = \{\text{id},(12)\}$ is good. For $S = \{(12),(13)\}$, since $\pi(1) \neq 1$ for any $\pi \in S$, we know that every graph is $S$-colourable. So $S = \{(12),(13)\}$ is good. We conjecture that $S = \{(12),(34)\}$ is also good. Thus we have the following conjecture which contains both Conjecture 6 and Conjecture 7.

**Conjecture 17** If $S$ is an inverse closed 2-element subset of $S_4$, then $S$ is good.

It is easy to verify that Conjecture 17 is equivalent to say that for $\Gamma \in \{Z_4,Z_2 \times Z_2\}$, for any inverse closed 2-subset $S$ of $\Gamma$, every planar graph is $S$-$\Gamma$-colourable.

The following conjecture is stronger than Conjecture 17.

**Conjecture 18** $S = \{\text{id},(12),(34),(12),(34)\}$ is good.
**Definition 19** A signed graph is called balanced if every cycle has an even number of negative edges.

**Conjecture 20** If $G$ is a signed planar graph, then $V(G)$ can be partitioned into two sets $V_1, V_2$ such that for each $i = 1, 2$, $G[V_i]$ is a balanced bipartite graph.

**Theorem 21** Conjecture 20 implies Conjecture 17.

**Proof.** Assume $S = \{id, (12)(34)\}$ or $S = \{id, (12)\}$ or $S = \{(12), (34)\}$, and $\sigma : V(G) \rightarrow S$. If $S = \{(12), (34)\}$, then we view edges $e$ with $\sigma_e = (12)$ as positive edges and edges $e$ with $\sigma_e = (34)$ as negative edges. In the other two cases, we view edges $e$ with $\sigma_e = id$ as positive edges, and the other edges are negative. Assume $G$ is a planar graph and $V(G) = V_1 \cup V_2$ is a partition as described in Conjecture 20.

We colour vertices in $V_1$ by colours 1 and 2. For each component of $G[V_1]$, we choose vertex $v$ from the component and order the vertices of the component by the distance from $v$, i.e., start from $v$, followed by the neighbours of $v$, then the neighbours of the neighbours of $v$, etc. We colour the vertices of $G[V_1]$ by colours 1, 2 greedily along this order. When we colour a vertex $u$, it follows from the fact that $G[V_1]$ is a balanced bipartite graph that all the coloured negative neighbours of $u$ (neighbours $v'$ for which $uv'$ is a negative edge) are coloured by the same colour, and the coloured positive neighbours of $u$ are coloured by the other colour. So $u$ can be properly coloured. So each component of $G[V_1]$, and hence $G[V_1]$ can be properly coloured by 1 and 2.

If $S = \{id, (12)(34)\}$ or $S = \{(12), (34)\}$, then $G[V_2]$ is coloured the same way by colours 3 and 4. If $S = \{id, (12)\}$, then we simply colour $G[V_2]$ properly as a graph. The resulting colouring is an $\sigma$-colouring of $G$. ■

**Lemma 22** Assume $S = \{id, (12)(34), (12), (34)\}$. If the vertex set of a graph $G$ can be partitioned into two sets $V_1, V_2$ such that $G[V_1], G[V_2]$ are forests, then $G$ is $S$-colourable.

**Proof.** We order the vertices of $V_1$ as $v_1, v_2, \ldots, v_p$ so that each vertex has at most one backward neighbour. Then we colour vertices of $V_1$ greedily by colours 1 and 2. When we colour $v_i$, $v_i$ has at most one coloured neighbour $v_j$ (for some $j < i$). Let $e = v_jv_i$. We only need to avoid the colour $\sigma_e(c(v_j))$. There is a legal colour in $\{1, 2\}$ for $v_i$.

Similarly, we can colour the vertices in $V_2$ by colours 3 and 4 so that for any edge $e = uv$ connecting two vertices in $V_2$, $c(u) \neq \sigma_e(c(v))$. For edges $e = uv$ connecting one vertex $u \in V_1$ and one vertex $v \in V_2$, as $\sigma_e \in \{id, (12)(34), (12), (34)\}$, we conclude that $c(u) \neq \sigma_e(c(v))$. So $c$ is a $\sigma$-colouring of $G$. ■

**Corollary 23** Assume $S = \{id, (12)(34), (12), (34)\}$. If $G$ is a plane graph whose dual graph has a Hamiltonian cycle, then $G$ is $S$-colourable.
Proof. It is known [10] that for such a plane graph $G$, $V(G)$ can be partitioned into two sets $V_1, V_2$ such that $G[V_1], G[V_2]$ are forests.

For the planar graph $G$ given in [5] that is not $Z_2 \times Z_2$-colourable, the dual of $G$ has a Hamiltonian cycle. Hence $G$ is $S$-colourable for $S = \{\text{id}, (12)(34), (12), (34)\}$. However, $G$ is not $S'$-colourable for $S' = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$, although $S$ and $S'$ are both subgroups of $S_4$ isomorphic to $Z_2 \times Z_2$.

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