The column group and its link invariants

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Abstract

The column group is a subgroup of the symmetric group on the elements of a finite blackboard birack generated by the column permutations in the birack matrix. We use subgroups of the column group associated to birack homomorphisms to define an enhancement of the integral birack counting invariant and give examples which show that the enhanced invariant is stronger than the unenhanced invariant.

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1 Introduction

Introduced in [8], a birack is a solution to the set-theoretic Yang-Baxter equation satisfying certain invertibility conditions. A blackboard birack is a type of strong birack with axioms corresponding to oriented blackboard-framed Reidemeister moves [17]. Special cases include quandles [10, 13], racks [7], strong biquandles [6] and semiquandles [9].

In [17] an invariant of knots and links is defined from any finite blackboard birack $X$ by counting labelings of a knot or link diagram with elements of $X$ over a set of framings of the knot or link determined by a quantity known as the birack rank of $X$.

In this paper we describe an enhancement of the birack counting invariant defined using a group determined by the columns of the birack operation matrices known as the column group. The paper is organized as follows. In section 2 we review the basics of blackboard biracks. In section 3 we define the column group and make a few useful observations. In section 4 we use the column group to enhance the blackboard birack counting invariants and give some examples and computations. In section 5 we collect questions for future research.

2 Blackboard biracks

In this section we review the basics of blackboard biracks. See [17] for more.

A blackboard framed link is an equivalence class of link diagrams under the blackboard framed Reidemeister moves:

Including a choice of orientation for each component gives us blackboard framed oriented links. Each component $C_k$ has a writhe $w_k$ equal to the sum of the crossing signs over the set of crossings where
both strands belong to $C_k$:

\[ \begin{align*}
\text{+1} & \quad \text{+1} \\
\text{−1} & \quad \text{−1}
\end{align*} \]

The writhe of each component is an invariant of oriented blackboard framed isotopy. Assuming a fixed choice of ordering of the components of a link $L = C_1 \cup \cdots \cup C_c$, we obtain a \textit{writhe vector} $w = (w_1, \ldots, w_c) \in \mathbb{Z}^c$. Setting these writhe vectors equal to to each other yields \textit{ambient isotopy} of unframed knots and links; this operation is equivalent to replacing the blackboard framed type I moves with the usual single-kink versions.

\textbf{Definition 1} Let $X$ be a set. A map $B = (B_1(x, y), B_2(x, y)) : X \times X \to X \times X$ is \textit{strongly invertible} if it satisfies the following conditions:

(i) $B$ is \textit{invertible}, i.e. there exists a map $B^{-1} : X \times X \to X \times X$ satisfying $BB^{-1} = \text{Id} = B^{-1}B$,

(ii) $B$ is \textit{sideways invertible}, i.e. there exists a unique invertible map $S : X \times X \to X \times X$ satisfying for all $x, y \in X$

\[ S(B_1(x, y), x) = (B_2(x, y), y), \]

and

(iii) $B$ is \textit{diagonally invertible}, i.e. the restrictions of the components of $S$ to the diagonal $\Delta = \{(x, x) \mid x \in X\}$ are bijections.

\textbf{Definition 2} Let $X$ be a set and $B : X \times X \to X \times X$ a strongly invertible map. The bijection $\pi : X \to X$ defined by

\[ \pi(x) = S_2|_\Delta^{-1} \circ S_1|_\Delta \]

is called the \textit{kink map} of the pair $(X, B)$. The exponent of $\pi$, i.e. the minimal integer $N \geq 1$ satisfying $\pi^N = \text{Id}$ (or $\infty$ if no such $N$ exists), is called the \textit{birack rank} of \textit{birack characteristic} of $(X, B)$.

\textbf{Definition 3} A \textit{blackboard birack} is a set $X$ with a strongly invertible map $B : X \times X \to X \times X$ which satisfies the \textit{set-theoretic Yang-Baxter equation}:

\[ (B \times \text{Id})(\text{Id} \times B)(B \times \text{Id}) = (\text{Id} \times B)(B \times \text{Id})(\text{Id} \times B) \]

where $\text{Id} : X \to X$ is the identity map.

\textbf{Example 1} Let $X$ be a module over the ring $\hat{\Lambda} = \mathbb{Z}[t^{\pm 1}, s, r^{\pm 1}]/(s^2 - (1 - tr)s)$. Then it is easy to check (see [17]) that

\[ B(x, y) = (ty + sx, rx) \]

defines a blackboard birack structure with $\pi(x) = (tr + s)$. We call this a \textit{(t, s, r)-birack}.

\textbf{Example 2} Let $X$ be a set. Two bijections $\tau, \sigma : X \to X$ define a blackboard birack structure on $X$ by

\[ B(x, y) = (\tau(y), \sigma(x)) \]

if and only if $\tau \sigma = \sigma \tau$. We call this a \textit{constant action birack}; such a birack has kink map $\pi = \sigma \tau^{-1}$. 

2
Example 3 Many previously studied algebraic structures in knot theory are special cases of blackboard biracks:

- A blackboard birack in which $\pi(x) = \text{Id}$ is a strong biquandle \([6]\)
- A blackboard birack in which $B_2(x, y) = x$ for all $x, y \in X$ is a rack \([7]\)
- A blackboard birack in which $\pi(x) = \text{Id}$ and $B_2(x, y) = x$ for all $x, y \in X$ is a quandle \([10, 13]\).

Given a finite set $X = \{x_1, \ldots, x_n\}$ we can define a blackboard birack structure on $X$ by specifying the operation tables of the components of $B$ with a matrix $M = [M_{B_1}, M_{B_2}]$ with two $n \times n$ blocks $M_{B_1}$ and $M_{B_2}$ whose $(i, j)$ entries respectively are $k$ and $l$ where $B_1(x_j, x_i) = x_k$ and $B_2(x_i, x_j) = x_l$.\(^4\) Given such matrices, we check cases to determine whether the blackboard birack axioms are satisfied (or rather, have a computer do so for us). It is not hard to see, for example, that such matrices must have columns which are permutations.

Example 4 Let $X = \{1, 2, 3, 4\}$ with $\tau = (12)$ and $\sigma = (34)$. Then $\tau \sigma = \sigma \tau$ so we have a constant action birack with nontrivial kink map $\pi = (12)(34)$. The birack matrix is given by

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 \end{bmatrix}.$$\(^3\)

Definition 4 Let $(X, B)$ and $(X', B')$ be blackboard biracks. As with other algebraic structures, we have the following useful notions:

- A map $f : X \to X'$ satisfying $B(f(x), f(y)) = (f(B_1(x, y)), f(B_2(x, y)))$ is a homomorphism of biracks,
- A subset $Y \subset X$ such that the restriction $B_Y = B|_{Y \times Y}$ defines a blackboard birack is a subbirack of $X$.

A blackboard birack can be used to label the semiarcs in an oriented blackboard-framed link diagram as indicated. Strong invertibility guarantees that such labelings are preserved under blackboard framed Reidemeister moves I and II in the sense that every labeling satisfying the pictured labeling condition at every crossing before a move corresponds to a unique such labeling after the move, while the Yang-Baxter condition does the same for type III moves. See \([17]\) for more.

If $(X, B)$ is a blackboard birack, then the set of labelings of a blackboard framed oriented link diagram $L$ such that the crossing conditions pictured above are satisfied at every crossing is an invariant of blackboard-framed isotopy known as the basic counting invariant, denoted $\text{Hom}(BBR(L), (X, B))$. If $(X, B)$ has finite birack rank $N$, then two blackboard-framed isotopic diagrams with congruent

\(^{3}\)Notice the reversed order of the inputs in $B_1$; this is for compatibility with notation in previous work.

\(^{4}\)Given such matrices, we check cases to determine whether the blackboard birack axioms are satisfied (or rather, have a computer do so for us). It is not hard to see, for example, that such matrices must have columns which are permutations.
writhe vectors modulo $N$ are related by the blackboard-framed oriented Reidemeister moves together with the $N$-phone cord move:

\[
\pi_N(x) = x \quad \pi^2(x) \quad \pi(x) \quad x
\]

Two such diagrams then have the same basic counting invariant with respect to ($X, B$). Thus, the quantity

\[
\Phi^Z_{(X, B)}(K) = \sum_{w \in (\mathbb{Z}/N)^c} |\text{Hom}(BBR(K, w), X, B)|
\]

is an invariant of ambient isotopy of knots and links, called the integral blackboard birack counting invariant.

An enhancement of $\Phi^Z_{(X, B)}(K)$ is a generally stronger invariant which associates to each labeling a signature which is invariant under birack-labeled Reidemeister moves. One standard example is the image enhanced blackboard birack counting invariant

\[
\Phi^{\text{Im}}_{(X, B)}(K) = \sum_{w \in (\mathbb{Z}/N)^c} \left( \sum_{f \in \text{Hom}(BBR(K, w), X, B)} t^{\text{Im}(f)} \right)
\]

where $\text{Im}(f)$ is the image of the labeling $f$ regarded as a homomorphism from the fundamental blackboard birack of $K$ to $(X, B)$, i.e. the smallest subbirack of $(X, B)$ containing all of the labels appearing in $f$. Another standard example is the writhe enhanced blackboard birack counting invariant given by

\[
\Phi^{\text{W}}_{(X, B)}(K) = \sum_{w \in (\mathbb{Z}/N)^c} |\text{Hom}(BBR(K, w), X, B)| q^w
\]

where $q^{w_1, \ldots, w_c} = \prod_{k=1}^c q^{w_k}$. This enhancement keeps track of which writhe vectors contribute which colorings, and for certain racks determines the linking number mod $N$ for links with two components [16].

Other examples of enhancements are known in special cases, such as quandle/biquandle/rack 2-cocycle enhancements [2, 3, 16], quandle/rack/biquandle polynomials [14, 15, 4], and various enhancements which use extra structure of the labeling objects, e.g. symplectic quandle enhancements [18] and Coxeter rack enhancements [19].

3 The column group

We can now define the column group of a blackboard birack.

Let $(X, B)$ be a blackboard birack with $n$ elements specified by a birack matrix $M$. As we have noted, the columns of $M$ determine permutations in the symmetric group $S_n$ — for each $x_j \in X$, define $\tau_j, \sigma_j : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by $\tau_j(i) = k$ and $\sigma_j(i) = l$ where $x_k = B_1(x_j, x_l)$ and $x_l = B_2(x_i, x_j)$. That is, $\tau_j$ and $\sigma_j$ are the permutations determined by the $j$th column of $M_{B_1}$ and $M_{B_2}$ respectively. We will call $\tau_j$ and $\sigma_j$ the upper and lower column permutations of the element $x_j$ respectively. Note that if $(X, B)$ is a rack then $\sigma_j = \text{Id}$ for all $j$, and if $(X, B)$ is a quandle, then $\tau_j$ has $j$ as a fixed point for each $j$. 







Definition 5 The column group $CG(X)$ of a finite blackboard birack $X$ with $n$ elements is the subgroup of $S_n$ generated by the elements $\tau_j, \sigma_j \in S_n$ corresponding to the columns of the birack matrix $M_{(X,B)}$. More generally, if $S \subseteq X$ is a subbirack then the column subgroup $CG(S \subseteq X)$ is the subgroup of $CG(X)$ generated by the permutations corresponding to the columns of the elements of $S$.

Example 5 Consider the $(t,s,r)$-birack $X = \mathbb{Z}_3$ with $t = 1$, $r = 2$ and $s = 2$. $X$ has birack matrix below with the listed upper and lower column permutations.

| $M_{(X,B)}$ | $\tau_1$ | $\tau_2$ | $\tau_3$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|-------------|----------|----------|----------|------------|------------|------------|
| $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ | (132) | (123) | (23) | (23) | (23) |

Thus, $CG(X)$ is the dihedral group of six elements; the subbirack $S = \{1\}$ has column subgroup $CG(S \subseteq X) \cong \mathbb{Z}_2$.

Remark 6 If $(X,B)$ is quandle or rack, then the operation $\triangleright$ defined by $x \triangleright y = B_1(y,x)$ is self-distributive. In this case, the column group $CG(X)$ is a subgroup of the automorphism group $\text{Aut}(X)$ of $(X,B)$, sometimes called the inner automorphism group of $X$. The column group is also related to the operator group defined in [7]. In the more general setting of blackboard biracks, however, the columns need not be automorphisms, so for simplicity we prefer the term “column group.”

Proposition 1 Let $(X,B)$ and $(X',B')$ be finite blackboard biracks. If there exists an isomorphism of biracks $\phi : X \rightarrow X'$, then $CG(X)$ is isomorphic to $CG(X')$.

Proof. We will show that $CG(X)$ and $CG(X')$ have presentations which differ only by relabeling.

Let us denote $B_1(x,y) = y \triangleright_1 x$, $B_2(x,y) = x \triangleright_2 y$. Then $\phi$ a birack isomorphism says

$$\phi(x \triangleright_1 y) = \phi(x) \triangleright_1 \phi(y) \quad \text{and} \quad \phi(x \triangleright_2 y) = \phi(x) \triangleright_2 \phi(y).$$

Then if $X = \{x_1, \ldots, x_n\}$ we have $X' = \{\phi(x_1), \ldots, \phi(x_n)\}$. Let us abbreviate $\phi(x_i)$ as $\phi(i)$.

Then the column groups $CG(X)$ and $CG(X')$ are generated by

$$\{\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_n\} \quad \text{and} \quad \{\tau_{\phi(1)}, \ldots, \tau_{\phi(n)}, \sigma_{\phi(1)}, \ldots, \sigma_{\phi(n)}\}$$

respectively. Note also that the finiteness of $X$ implies for any $\tau_i$ and $\sigma_i$ we have $\tau_i^{-1} = \tau_i^l$ and $\sigma_i^{-1} = \sigma_i^m$ for some $l,m > 0$.

Then

$$\phi(x_i \triangleright_1 x_j) = \phi(x_i) \triangleright_1 \phi(x_j) \iff \phi(\tau_j(i)) = \tau_{\phi(j)}(\phi(i))$$

and

$$\phi(x_i \triangleright_2 x_j) = \phi(x_i) \triangleright_2 (\phi(x_j)) \iff \phi(\sigma_j(i)) = \sigma_{\phi(j)}(\phi(i))$$

Let $\delta_i, \epsilon_i \in \{0,1\}$ and for all $i,j \in \{1, \ldots, n\}$ set $x_i \triangleright^0_j x_j = x_i \triangleright^0_j x_j = x_i$. It follows that for any relation $\tau_{i_1}^{\delta_1} \sigma_{i_1}^{\epsilon_1} \cdots \tau_{i_k}^{\delta_k} \sigma_{i_k}^{\epsilon_k} = \text{Id}$ satisfied in $CG(X)$, we have for all $x_j \in X$

$$\phi(x_j) = \phi((\tau_{i_1}^{\delta_1} \sigma_{i_1}^{\epsilon_1} \cdots \tau_{i_k}^{\delta_k} \sigma_{i_k}^{\epsilon_k}(x_j)) = \phi((\cdots(x_j \triangleright_{i_k}^0 x_{i_k}) \triangleright_{i_{k-1}}^0 x_{i_{k-1}}) \cdots \triangleright_{i_1}^0 x_{i_1}) = (\cdots((\phi(x_j) \triangleright_{i_k}^0 \phi(x_{i_k})) \triangleright_{i_{k-1}}^0 \phi(x_{i_{k-1}})) \cdots \triangleright_{i_1}^0 \phi(x_{i_1})) = \tau_{\phi(i_1)}^{\delta_1} \sigma_{\phi(i_1)}^{\epsilon_1} \cdots \tau_{\phi(i_k)}^{\delta_k} \sigma_{\phi(i_k)}^{\epsilon_k}(\phi(x_j))$$


and the relation \( \phi_{(i_1)}^{\delta_1} \circ \phi_{(i_2)}^{\delta_2} \circ \cdots \circ \phi_{(i_k)}^{\delta_k} = \text{Id} \) is satisfied in \( CG(X') \).

Replacing \( \phi \) with \( \phi^{-1} \) shows that every relation satisfied in \( CG(X') \) arises in this way. Thus, \( CG(X) \) and \( CG(X') \) have presentations which differ only by relabeling, and \( CG(X) \cong CG(X') \). \( \square \)

Note that, like quandle, biquandle and rack polynomials, the column subgroup of a subbirack carries information about how the subbirack is embedded in the overall birack. In particular, the column subgroup \( CG(S \subset X) \) is not in general isomorphic to the column group \( CG(S) \) considered as a stand-alone birack; isomorphic subbiracks \( S \subset X \) and \( T \subset X \) embedded differently in \( X \) generally have non-isomorphic column groups \( CG(S \subset X) \not\cong CG(T \subset X) \), as the next example illustrates.

**Example 7** The two 2-element subbiracks \( \{1,2\} \) and \( \{3,4\} \) of the birack with birack matrix

\[
M_{(X,B)} = \begin{bmatrix}
1 & 1 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4
\end{bmatrix}
\]

are both isomorphic to the trivial quandle of two elements, but \( CG(\{1,2\} \subset X) = 1 \) while \( CG(\{3,4\} \subset X) \cong \mathbb{Z}_2 \).

## 4 Enhancing the counting invariant

We will now use the column group to define an enhancement of the blackboard birack counting invariants.

**Definition 6** Let \( L = L_1 \cup \cdots \cup L_c \) be an oriented link of \( c \) components and \( (X, B) \) a finite blackboard birack with birack rank \( N \). The **column group enhanced birack multiset invariant** is the multiset of column subgroups

\[
\phi^{CG,M}_{(X,B)}(L) = \{CG(\text{Im}(f) \subset X) \mid f \in \text{Hom}(BBR(L,w),(X,B)), w \in (\mathbb{Z}_N)^c\}
\]

and the **column group enhanced birack polynomial invariant** is

\[
\phi^{CG}_{(X,B)}(L) = \sum_{w \in (\mathbb{Z}_N)^c} \left( \sum_{f \in \text{Hom}(BBR(L,w),(X,B))} u^{\text{CG}(\text{Im}(f) \subset X)} \right).
\]

That is, \( \phi^{CG,M}_{(X,B)}(L) \) is the multiset of column subgroups of the image subbiracks of labelings of a diagrams of \( L \) by \( (X, B) \) over a complete period of framings of \( L \) modulo \( N \). In the polynomial version \( \phi^{CG}_{(X,B)}(L) \) we trade some information (isomorphism type of a column subgroup is replaced with its cardinality) to get a more easily comparable invariant. In both cases, including column subgroup information enables the new invariants to distinguish between different labelings, resulting in a more sensitive invariant than simply counting labelings. Note that we can recover the integral birack counting invariant \( \Phi^Z_{(X,B)}(L) \) by specializing \( u = 1 \) in \( \phi^{CG}_{(X,B)}(L) \) or by taking the cardinality of \( \phi^{CG,M}_{(X,B)}(L) \).

**Example 8** The trefoil knot \( 3_1 \) has nine colorings by the \((t,s,r)\)-birack \( X = \mathbb{Z}_3 = \{1,2,3\} \) with \( t = 2 \), \( s = 1 \), \( r = 1 \) — in fact, these are the well-known Fox 3-colorings of the trefoil. Three of these labelings have singleton image subbiracks and six are surjective. Each element of \( X \) has \( \tau_i \) a transposition, so the column subgroups of the constant labelings are copies of \( \mathbb{Z}_2 \), while the surjective labelings have column subgroup generated by all three transpositions, i.e. isomorphic to \( S_3 \). Thus, the integral birack counting invariant value \( \Phi^Z_{(X,B)}(3_1) = |\text{Hom}(BBR(3_1),(X,B))| = 9 \) with column group enhancements becomes \( \phi^{CG,M}_{(X,B)}(L) = \{3 \times \mathbb{Z}_2, 6 \times S_3\} \) or \( \phi^{CG}_{(X,B)}(L)(3_1) = 3u^2 + 6u^6 \).
Example 9 Let us compute the column group enhanced birack counting invariant of the Hopf link with respect to the birack $X$ with birack matrix

\[
M_{(X,B)} = \begin{bmatrix}
2 & 2 & 2 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 3 & 3
\end{bmatrix}.
\]

The birack rank of $(X, B)$ is 2, so we need to consider diagrams of the Hopf link with both even and odd writhes on each component. The labeling rule can be expressed as follows: semiarcs labeled 1 switch to 2 and 2 switch to 1 when crossing under any arc, semiarcs labeled 3 or 4 retain their label when crossing under any arc; semiarcs labeled 3 switch to 4 and 4 switch to 3 when crossing over a 3 or 4, and all other overcrossings retain their labels when crossing over. Note that the labelings of the diagrams with writhe vectors $(0,1)$ and $(1,0)$ are the same due to the symmetry of the link. The reader can easily verify that the valid labelings are the ones listed in the table.

| x | y | z | w | u | v | s | t |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| 3 | 3 | 4 | 4 | 3 | 3 | 4 | 4 |
| 4 | 4 | 3 | 3 | 4 | 4 | 3 | 3 |

The integral birack counting invariant is thus $\Phi^{Z^2}_{(X,B)}(L) = 16$. The column group enhancement information distinguishes some of the labelings – the image subbiracks of labelings include \{1, 2\}, \{3, 4\} and \{1, 2, 3, 4\} with corresponding column groups $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ respectively. Thus, the
column group enhanced rack counting invariant is

\[ \phi_{CG}(L) = 4u^2 + 12u^4 \]

or in multiset form

\[ \phi_{CG,M}(L) = \{4 \times \mathbb{Z}_2, 12 \times \mathbb{Z}_2 \oplus \mathbb{Z}_2\} \].

As we have seen, the integral counting invariant can be obtained as a specialization of the column group enhanced invariant. Our last two examples show that the column group enhanced counting invariants are strictly stronger than the unenhanced counting invariants.

**Example 10** Consider the knots 5_1 and 6_1. Several other enhancements of counting invariants detect the difference between these two knots despite having the same integral counting invariant value, including generalized quandle polynomial enhancements and rack shadow enhancements [15, 5]. As expected, there is a blackboard birack \((X, B)\) whose column group enhancement distinguishes the knots 5_1 and 6_1 while we have \(\Phi_{CG,B}(5_1) = 30 = \Phi_{CG,B}(6_1)\).

\[
M_{(X,B)} = \begin{bmatrix}
1 & 3 & 5 & 2 & 4 & 1 & 5 & 2 & 1 & 4 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 2 & 4 & 1 & 3 & 4 & 3 & 2 & 1 & 5 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
4 & 1 & 3 & 5 & 2 & 1 & 5 & 4 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 5 & 2 & 4 & 1 & 3 & 2 & 1 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
2 & 4 & 1 & 3 & 5 & 5 & 4 & 3 & 2 & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
8 & 10 & 7 & 9 & 6 & 6 & 10 & 9 & 8 & 7 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 9 & 6 & 8 & 10 & 8 & 7 & 6 & 10 & 9 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
6 & 8 & 10 & 7 & 9 & 10 & 9 & 8 & 7 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
10 & 7 & 9 & 6 & 8 & 7 & 6 & 10 & 9 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
9 & 6 & 8 & 10 & 7 & 9 & 8 & 7 & 6 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10
\end{bmatrix}
\]

\[ \Phi_{CG}(X,B)(5_1) = 5u^2 + 5u^4 + 20u^{10} \]

\[ \Phi_{CG}(X,B)(6_1) = 5u^2 + 5u^4 + 20u^{20} \]

**Example 11** Let \((X, B)\) be the 27-element conjugation quandle on the conjugation classes of (13)(56) and (15643) in \(S_6\), i.e.

\[ X = \{ x \in S_6 \mid x = y^{-1}(13)(56)y \text{ or } x = y^{-1}(15643)y \text{ for some } y \in S_6 \} \]

with birack operation

\[ B(x,y) = (x^{-1}yx, x). \]

Our python computations say that both the trefoil 3_1 and the figure eight 4_1 have quandle counting invariant value \(|\text{Hom}(3_1, T)| = |\text{Hom}(4_1, T)| = 147\); however, the column group enhancement reveals distinct values. We list only the operation matrix for \(B_1\) since \(M_{B_2}\) is the trivial operation, i.e \(B_2(x, y) = x\).

\[ \phi_{CG}(X,B)(3_1) = 12u^5 + 15u^2 + 60u^6 + 60u^{60} \]

\[ \phi_{CG}(X,B)(4_1) = 12u^5 + 15u^2 + 120u^{60} . \]
In this section we collect questions for future research.

What kinds of groups can arise as column groups of a finite rack or quandle? That is, given a finite group, can one construct a blackboard birack with the specified column group? What is the relationship between the column group and birack polynomials?

A constant action rack always has a cyclic column group, generated by the single column permutation appearing in the birack matrix; what can one say about the column groups of specific types of blackboard biracks such as conjugation quandles, symplectic quandles, Coxeter racks, or $(t,s,r)$-biracks?

In [10], a construction is given which expresses any quandle in terms of a quandle structure on right cosets of the automorphism group of the original quandle; is a similar construction possible starting with the column group? What is the correct generalization of the column group to infinite biracks?

To maximize sensitivity of the column group enhanced invariant, we want blackboard biracks with as many subbiracks with distinct column subgroups as possible. On the other hand, biracks $(X,B)$ with larger cardinalities require more computation time. Finding fast algorithms for computing the sets of birack labelings of a diagram will improve the practical utility of column group enhanced invariants.

Our python code for computing the invariants defined in this paper is available at the second listed author's website, www.esotericka.org. Portions of this paper also appear in the first listed author's senior thesis.

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