DYNAMICS OF A LOTKA-VOLterra
COMPETITION-DIFFUSION MODEL WITH STAGE
STRUCTURE AND SPATIAL HETEROGENEITY

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Abstract. This paper is concerned with a Lotka-Volterra competition-diffusion model with stage structure and spatial heterogeneity. By analyzing the sign of the principal eigenvalue corresponding to each semi-trivial solution, we obtain the linear stability and global attractivity of the semi-trivial solution. In addition, an attracting region was obtained by means of the method of upper and lower solutions.

1. Introduction. The dynamical models in the form of reaction-diffusion equations have been extensively investigated in various natural sciences [33]. In order to reflect the real dynamical behaviors of models that depend on the past history of systems, it is reasonable to incorporate time delays into the systems [11, 13, 20, 34]. Especially in mathematical biology, many models of population dynamics can be described by delayed reaction–diffusion equations [2, 4, 5, 8, 9, 10, 12, 14, 19, 35, 36]. Lam and Ni [18] investigated the interactions between diffusion and heterogeneity of the environment in the following classical diffusive Lotka-Volterra type model:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d_1 \Delta u(x,t) + u(x,t)(m(x) - u(x,t) - cv(x,t)), (x,t) \in \Omega \times \mathbb{R}^+, \\
\frac{\partial v(x,t)}{\partial t} &= d_2 \Delta v(x,t) + v(x,t)(m(x) - bu(x,t) - v(x,t)), (x,t) \in \Omega \times \mathbb{R}^+,
\end{align*}
\]

(1)

where the function \(m(x)\) represents their common (spatially inhomogeneous) intrinsic growth rate or carrying capacity, the habitat \(\Omega\) is a bounded region in \(\mathbb{R}^n\) \((n \geq 1)\) with smooth boundary \(\partial \Omega\). Lam and Ni [18] established the uniqueness and the global asymptotic stability of coexisting steady states under various circumstances,
and also obtained a complete understanding of the change in dynamics when one of the interspecific competition coefficients is small.

In the natural world, there are many species that go through several stages during their lifetime, such as a single-species growth model with stage structure consisting of immature and mature stages is developed using a discrete time delay. For example, Aiello and Freedman [1] introduced the following population model consisting of immature and mature stages is developed using a discrete time delay. During their lifetime, such as a single-species growth model with stage structure.

\[
\begin{align*}
\frac{du_i(t)}{dt} &= \alpha u_m(t) - \gamma u_i(t) - \alpha e^{-\gamma \tau} u_m(t - \tau), \\
\frac{du_m(t)}{dt} &= \alpha e^{-\gamma \tau} u_m(t - \tau) - \beta u_m^2(t), \\
\end{align*}
\]

in which \(\alpha, \beta, \gamma, \tau\) are all positive, \(u_i(t)\) and \(u_m(t)\) are the densities of immature (juvenile) and mature (adult) members, respectively. The \(\alpha u_m\) term in the first equation of (2) is the birth rate, assumed proportional to the number of adults and \(-\gamma u_i\) is juvenile mortality, the \(e^{-\gamma \tau}\) term corrects for juvenile mortality. In the second equation of (2), the \(\beta u_m^2\) term represents adult mortality. The reader will see that mature and immature mortality are treated differently: quadratic for mature and linear for immature. Aiello and Freedman [1] investigated the existence of a globally asymptotically stable positive equilibrium.

Predator-prey population with stage structure is very important in the models of multi-species populations interactions and has been studied widely (see, for example, [21, 22, 28, 30]). In classical models of Lotka-Volterra type, such as system (1), it is assumed that all individuals have largely similar capabilities to hunt or reproduce [7]. However, for a number of animals, it seems reasonable to assume that the predator population feed on the mature prey because immature prey population are concealed in the mountain caves and are raised by their parents, and that the rate of predators attack at immature preys can be ignored (see, for example, [23]). Therefore, it is practical to introduce the stage structure into the competitive model. Furthermore, Liu et al. [23] considered the following two-species competitive population with stage structure:

\[
\begin{align*}
\frac{du_i(t)}{dt} &= \alpha u_m(t) - \gamma u_i(t) - \alpha e^{-\gamma \tau_1} u_m(t - \tau_1), \\
\frac{du_m(t)}{dt} &= \alpha e^{-\gamma \tau_1} u_m(t - \tau_1) - \beta u_m^2(t) - au_mv_m, \\
\frac{dv_i(t)}{dt} &= rv_i(t) - \gamma v_i(t) - re^{-\gamma \tau_2} v_m(t - \tau_2), \\
\frac{dv_m(t)}{dt} &= re^{-\gamma \tau_2} v_m(t - \tau_2) - bv_m^2(t) - cu_m v_m. \\
\end{align*}
\]

To find out how the stage structure affects the global behaviors of the competitive system (3), let delay \(\tau_2\) in (3) equal to zero, which means species 2 has only one stage, then we reduce system (3) to the following system

\[
\begin{align*}
\frac{du_i(t)}{dt} &= \alpha u_m(t) - \gamma u_i(t) - \alpha e^{-\gamma \tau} u_m(t - \tau), \\
\frac{du_m(t)}{dt} &= \alpha e^{-\gamma \tau} u_m(t - \tau) - \beta u_m^2(t) - au_m v, \\
\frac{dv(t)}{dt} &= v(r - cu_m - bv),
\end{align*}
\]
Motivated by the works of Lam and Ni [18], and Liu et al. [23], in this paper, we are concerned with the following Lotka-Volterra competition type model with stage structure in heterogeneous environments:

\[
\begin{align*}
\frac{\partial u_i(x, t)}{\partial t} &= d_1 \Delta u_i(x, t) + \alpha(x)u_m(x, t) - \gamma u_i(x, t) - \alpha(x)e^{-\gamma \tau} u_m(x, t - \tau), \\
\frac{\partial u_m(x, t)}{\partial t} &= d_1 \Delta u_m(x, t) + \alpha(x)e^{-\gamma \tau} u_m(x, t - \tau) - \beta u_m^2(x, t) - au_m(x, t)v(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= d_2 \Delta v(x, t) + v(x, t)[r(x) - cu_m(x, t) - bv(x, t)],
\end{align*}
\]

for all \( x \in \Omega \) and \( t > 0 \), where \( u_i(x, t) \) and \( u_m(x, t) \) are the densities of immature and mature members of species 1 at time \( t \) and location \( x \), respectively; \( v(x, t) \) is the population density of species 2 at time \( t \) and location \( x \); \( d_i > 0 \) (\( i = 1, 2 \)) is the diffusion coefficient, \( \tau \) is the time delay; \( \Omega \) is a connected bounded open domain in \( \mathbb{R}^n \) (\( n \geq 1 \)) with smooth boundary \( \partial \Omega \), \( \alpha(x) \) is the birth rate of species 1 at location \( x \), \( \gamma \) is the death rate of the immature of species 1 and \( \beta \) is the mature death and overcrowding rate of species 1, as in the logistic equation; \( r(x) \) represents the spatially inhomogeneous carrying capacities for species 2 or intrinsic growth rates of species 2 at location \( x \) in \( \Omega \), which reflects the situation of the resources and thus may vary from point to point. Throughout this paper, we shall assume \( \alpha(x) \) and \( r(x) \) satisfy the following hypothesis:

\( \text{(M)}: \) \( \alpha(x), r(x) \in C^r(\overline{\Omega}) \) with \( \epsilon \in (0, 1) \) are nonconstant, \( \alpha(x) > 0 \) and \( r(x) > 0 \) on \( \overline{\Omega} \); \( \bar{\alpha} = \frac{1}{|\Omega|} \int_\Omega \alpha(x) dx, \bar{r} = \frac{1}{|\Omega|} \int_\Omega r(x) dx. \)

Note that \( u_i \) don’t appear the second and third equations of system (5), then we just need to investigate the following system:

\[
\begin{align*}
\frac{\partial u_m(x, t)}{\partial t} &= d_1 \Delta u_m(x, t) + \alpha(x)e^{-\gamma \tau} u_m(x, t - \tau) - \beta u_m^2(x, t) - au_m(x, t)v(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= d_2 \Delta v(x, t) + v(x, t)[r(x) - cu_m(x, t) - bv(x, t)].
\end{align*}
\]

For simplicity, let \( \bar{u} = \beta u_m, \bar{v} = bv, \bar{a} = a/b, \bar{c} = c/\beta, \) and drop the bars, then system (6) can be rewritten as

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= d_1 \Delta u(x, t) + \alpha(x)e^{-\gamma \tau} u(x, t - \tau) - u^2(x, t) - au(x, t)v(x, t), \\
\frac{\partial v(x, t)}{\partial t} &= d_2 \Delta v(x, t) + v(x, t)[r(x) - cu(x, t) - v(x, t)].
\end{align*}
\]

In this paper, we consider system (7) with the following initial conditions and Neumann boundary conditions:

\[
\begin{align*}
u(x, s) = \phi_1(x, s), \quad v(x, 0) = \phi_2(x), \quad x \in \Omega, s \in [-\tau, 0], \\
\partial_n u(x, t) = \partial_n v(x, t) = 0, \quad x \in \partial \Omega, t > 0.
\end{align*}
\]

The zero Neumann (no-flux) boundary condition means that no individual crosses the boundary of the habitat, \( \partial_n = \nu \cdot \nabla \), where \( \nu \) denotes the outward unit normal vector on \( \partial \Omega \). Moreover, the functions \( \phi_1(\cdot, s) \) and \( \phi_2(\cdot) \) are non-negative and not.
identically zero. By analyzing the principal eigenvalue, we shall show that the semi-trivial solution \((\theta_{d,0}, \alpha, 0, 0)\) (respectively, \((0, \theta_{d, r})\)) is globally asymptotically stable provided that \(\tau < \tau^*\) (respectively, \(\tau > \tau^*\)). This result resembles that of the homogeneous system without diffusion, which has been investigated by Liu et al. [23], who have confirmed the negative effect of stage structure on the permanence and suggested that for a competitive community stage structure is also one of the important reasons that cause permanence and extinction. Moreover, this result is also consistent with result of the classical Lotka-Volterra competition model with just diffusion and heterogeneity of the environment, which has been considered in [15, 16, 18]. Note that system (7) possesses a mixed quasi-monotone property, then by the method of upper and lower solution, we can establish an attracting region of the solution, which implies that the system is uniformly persistent and permanent.

The rest of the paper is organized as follows. In Section 2, some concepts are introduced for later use. Sections 3 is devoted to the stability of the steady state solution of system (7) by analyzing the sign of the principal eigenvalues of the linearized system of (7) at this steady state solution. In Section 4, using the upper and lower method, we investigate the asymptotic behavior of the solution of system (7).

2. Preliminaries. To begin our discussion, we first recall the solution of the following equation:

\[
\begin{aligned}
\begin{cases}
d\Delta u + u(m(x) - u) &= 0 \quad \text{in } \Omega, \\
\partial_{\nu} u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\tag{9}
\]

where \(m(x) \in C^\epsilon(\overline{\Omega})\) with \(\epsilon \in (0, 1)\) is non-constant and satisfies \(m > 0\) in a set of positive measure in \(\Omega\). From the proof of existence and uniqueness results of (9) (see [3]), we know that if the principal eigenvalue \(\mu_1(d, m) < 0\), the unique solution of (9) is positive, denoted by \(\theta_{d,m}\). The principal eigenvalue will be given in Definition 2.1. Dividing the equation of \(\theta_{d,m}\) in the first equation of (9) by \(\theta_{d,m}\) and integrating over \(\Omega\), we obtain that

\[
\int_\Omega [m(x) - \theta_{d,m}(x)] \, dx = -d \int_\Omega \frac{|\nabla \theta_{d,m}|^2}{\theta_{d,m}^2} < 0, 
\tag{10}
\]

which implies that \(\int_\Omega m(x) \, dx < \int_\Omega \theta_{d,m}(x) \, dx\).

To characterize the principal eigenvalue (9), we need to introduce the following eigenvalue problem with indefinite weight

\[
\begin{aligned}
\begin{cases}
\Delta \varphi + \lambda h(x) \varphi &= 0 \quad \text{in } \Omega, \\
\partial_{\nu} \varphi &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\tag{11}
\]

where \(h\) is not a constant and could change sign in \(\Omega\). We say that \(\lambda_1(h)\) is a principal eigenvalue if (11) has a positive solution (Note that 0 is always a principal eigenvalue). Proposition 2.1 in [15] says that the principal eigenvalue \(\lambda_1(h)\) is nonzero if and only if \(h\) changes sign and \(\int_\Omega h \neq 0\). Next, we collect some facts concerning the elliptic eigenvalue problem with an indefinite weight.
Definition 2.1. Given a positive constant $d$ and a function $h \in L^\infty(\Omega)$, we define $\mu_k(d,k)$ to be the $k$th eigenvalue (counting multiplicities) of
\[
\begin{cases}
    d\Delta \psi + h(x)\psi + \mu \psi = 0 & \text{in } \Omega, \\
    \partial_\nu \psi = 0 & \text{on } \partial \Omega.
\end{cases}
\] (12)

In particular, we call $\mu_1(d,h)$ the first eigenvalue of (12), and have the following variational characterization
\[
\mu_1(d,h) = \inf_{\psi \in H_1(\Omega) \setminus \{0\}} \frac{\int_\Omega (d|\nabla \psi|^2 - h(x)\psi^2) \, dx}{\int_\Omega \psi^2 \, dx}.
\] (13)
The following proposition collects some important properties of $\mu_1(d,h)$ in connection with $\lambda_1(h)$ [15]. The more detailed proof can be found, for example, on page 95 of [3].

Proposition 1. The first eigenvalue $\mu_1(d,h)$ of (12) depends smoothly on $d > 0$ and continuously on $h \in L^\infty(\Omega)$. Moreover, it has the following properties:

(i): $\int_\Omega h \geq 0$ and $h \not\equiv 0 \Rightarrow \mu_1(d,h) < 0$ for all $d > 0$.

(ii): If $\int_\Omega h < 0$ and $h$ changes sign in $\Omega$, then $\mu_1(d,h) < 0$ for all $d < \frac{1}{\lambda_1(h)}$.

(iii): $\mu_1(d,h)$ is strictly increasing and concave in $d > 0$. Furthermore,
\[
limit_{d \rightarrow 0} \mu_1(d,h) = \min(-h) \text{ and } \lim_{d \rightarrow \infty} \mu_1(d,h) = -\bar{h},
\]
where $\bar{h}$ is the average of $h$.

(iv): If $h_1(x) \leq h_2(x)$ in $\Omega$, then $\mu_1(d,h_1) \geq \mu_1(d,h_2)$ with equality holds if and only if $h_1 = h_2$ a.e. in $\Omega$. Assume in addition that $h$ is nonconstant, then $\mu_1(d_1,h) < \mu_1(d_2,h)$ if $d_1 < d_2$.

Next, let us consider the following eigenvalue problem
\[
\begin{cases}
    d\Delta \psi + h(x)\psi + \alpha(x)\psi e^{-\gamma \tau - \lambda \tau} = \lambda \psi & \text{in } \Omega, \\
    \partial_\nu \psi = 0 & \text{on } \partial \Omega.
\end{cases}
\] (14)

For convenience, let $\bar{\lambda}(d,\tau,h)$ be the principal eigenvalue of (14). In fact, we need to study the sign of the principal eigenvalue $\bar{\lambda}(d,\tau,h)$ in order to investigate the stability of the solution of the model. Thanks to the proof of Theorem 2.2 in [31], we have the following result.

Lemma 2.2. $\bar{\lambda}(d,\tau,h)$ has the same sign as the principal eigenvalue of
\[
\begin{cases}
    d\Delta \psi + h(x)\psi + \alpha(x)e^{-\gamma \tau} \psi = \lambda \psi & \text{in } \Omega, \\
    \partial_\nu \psi = 0 & \text{on } \partial \Omega.
\end{cases}
\] (15)

That is, $\text{sgn}(\bar{\lambda}(d,\tau,h)) = -\text{sgn}(\mu_1(d,h + \alpha(x)e^{-\gamma \tau})))$.

3. Global dynamics of a single species with age structure. This section is devoted to the global dynamics of the following single-species model with age structure. It will play an important role in the investigation of global dynamics of system (7),
\[
\begin{cases}
    \frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \alpha(x)e^{-\gamma \tau} u(x,t-\tau) - u^2(x,t) - b(x)u(x,t), & x \in \Omega, \\
    \partial_\nu u(x,t) = 0, & x \in \partial \Omega,
\end{cases}
\] (16)
where $\tau > 0$, $\alpha(x)$, $b(x) \in C^0(\Omega)$ with $\varepsilon \in (0, 1)$ satisfy $\alpha(x)e^{-\varepsilon \tau} - b(x) > 0$ in a set of positive measure in $\Omega$. If $\mu_1(d, \alpha(x)e^{-\varepsilon \tau} - b(x)) < 0$, it follows from the discussion about (9) that system (16) has exactly one positive steady state solution, denoted by $\theta_{d,\alpha,b}$. In what follows, we shall employ the approach proposed by Huang [17] to show that every solution $u$ of (16) converges to $\theta_{d,\alpha,b}$ as $t \to \infty$ regardless of initial values $u(\cdot, \theta)$, $\theta \in [-\tau, 0]$.

The phase space for system (16) is $C = C([-\tau, 0], C(\Omega))$. The linear operator $d\Delta$ generates an analytic and compact semi-group $\{T(t)\}_{t \geq 0}$ on $C(\Omega)$. Moreover, the maximum principle implies that $T(t)$ is strongly positive for $t > 0$, that is, $T(t)\phi \gg 0$ for all $t > 0$ and $\phi > 0$. Define $F: C \to C(\Omega)$ by

$$
[F(\phi)](x) = \alpha(x)e^{-\tau} \phi(x, -\tau) - \phi^2(x, 0) - b(x)\phi(x, 0).
$$

It is obvious that $F$ is locally Lipschitz continuous. Therefore, from [24], for each $\phi \in C$, there exists a maximum $t_\phi > 0$ such that the initial value problem of the evolution equation

$$
\begin{align*}
\begin{cases}
 u(t) = T(t)\phi(0) + \int_{t_\phi}^t T(t-s)F(u_s)ds, & t \in (0, t_\phi), \\
u_0 = \phi, & t > 0,
\end{cases}
\end{align*}
$$

has a unique solution $u^\phi(t)$ which exists on $[-\tau, t_\phi)$. Here $u_s \in C$ is defined by $u_s(\theta) = u(s + \theta)$ for $\theta \in [-\tau, 0]$. In what follows, we present a series of lemmas to investigate the properties of the solution $u^\phi(t)$.

**Definition 3.1.** The solution $u^\phi(t)$ of (17) is called a mild solution of (16); $u \in C([t_1 - \tau, t_2], C_0(\Omega))$ with $t_2 > t_1$ is called a classical solution of (17) for $t \in [t_1, t_2]$ if $\partial u/\partial t$, $\Delta u$ are continuous and $u$ satisfies (17) for $(t, x) \in (t_1, t_2) \times \Omega$, where $C_0(\Omega) = \{\phi \in C(\Omega): \partial_c \phi(\partial \Omega) = 0\}$.

It is easy to see that $u^\phi(t)$ is Hölder continuous for $t > 0$. Hence if $t_\phi > \tau$ then $[F(u^\phi_t)](x)$ is Hölder continuous for $(t, x) \in (\tau, t_\phi) \times \Omega$. By a regularity theorem, we see that a mild solution of (16) is a classical solution for $t \in [\tau, t_\phi]$ if $t_\phi > \tau$. We now establish a uniform $C^1(\Omega)$ estimate for a solution of (16) with initial values.

**Lemma 3.2.** Let $\delta > 0$, $t_1 > \delta + \tau$ and $\kappa > 0$ be fixed, then there are constants $M = M(\delta, t_1, \kappa)$ and $M_1 = M_1(\delta, t_1, \kappa)$ such that if a mild solution $u(t) = u^\phi(t)$ exists on $[-\tau, t_1]$ and satisfies $\|u\|_{[-\tau, t_1]} = M$ for $t \in [-\tau, t_1]$, then

(i): $\|\Delta u(t)\|_{C^1(\Omega)} \leq M \|u\|_{[-\tau, t_1]}$ for $t \in [\delta + \tau, t_1]$;

(ii): $\|\partial_t u(t)\|_{C^1(\Omega)} \leq M_1 \|u\|_{[-\tau, t_1]}$, $\|u(t)\|_{C^1(\Omega)} \leq M_1 \|u\|_{[-\tau, t_1]}$.

The proof of Lemma 3.2 is similar to that of Lemma 2.3 and Corollary 2.4 of [17] and hence is omitted. Moreover, the following result is a direct consequence of Lemma 3.2.

**Corollary 1.** Let $\delta > 0$ be fixed. Then for each $\kappa > 0$, there is $\xi(\kappa) > 0$ such that if $u: [-\tau, \infty)$ is a mild solution of (17) satisfying $\|u(t)\|_{C(\Omega)} \leq \kappa$ for $t \geq 0$, then $\|\partial_t u(t)\|_{C^1(\Omega)} \leq \xi(\kappa)$ and $\|u(t)\|_{C^1(\Omega)} \leq \xi(\kappa)$ for all $t \geq \tau + \delta$.

Let $u^\phi(x, t)$ be the solution of (16) with the initial value $\phi \in C$ satisfying $\phi(\cdot, t) \geq 0$. Next, we shall prove that $\lim_{t \to \infty} \|u^\phi(\cdot, t) - \theta_{d,\alpha,b}\| = 0$. From the viewpoint of biology, only the nonnegative solution of (16) is of our interest. Moreover, if an initial function $\phi$ vanishes at $\theta = 0$, that is, $\phi(\cdot, 0) = 0$, then it is easy to observe
that $u^\phi(\cdot, t) = 0$ for all $t \geq 0$. Hence, from now on, let $C^+ = \{ \phi \in C : \phi(0) > 0 \}$ and investigate the properties of the solution $u^\phi$ of system (16) with initial value $\phi \in C^+$.

**Lemma 3.3.** For every $\phi \in C^+$, we have $u^\phi(t) \geq 0$ for all $t \in (0, t_\phi)$, and $\sup\{u^\phi(t, x) : t \in [0, t_\phi), x \in \Omega \} \leq \bar{m}(\phi)$, where $\bar{m}(\phi) = \max\{\|\phi\|_C, \sup\{\alpha(x) e^{-\gamma \tau} - b(x)\}\}$.

**Proof.** The proof of the statement $u^\phi(\cdot) \geq 0$ is similar to Lemma 3.1 of [17] and hence is omitted here. In what follows, we only prove the second conclusion of the lemma. From [27], we know that $u^\phi$ is continuous on $\Omega \times (0, t^*)$, and an open ball $S^* \subseteq \Omega$ exists for all $(x, s) \in \bar{\Omega} \times (0, t^*)$. From [37], we have $u^\phi(x, t) - \Delta u^\phi(x, s) < 0$ and the smoothness of $\partial \Omega$, there exist an $\varepsilon > 0$ and an open ball $S^* \subseteq \Omega$, such that $S^* \cap \partial \Omega = x$ and $\frac{\partial}{\partial \nu} u^\phi(x, t) - \Delta u^\phi(x, t) \leq 0$ for $(x, t) \in S^* \times [s_1 - \varepsilon, s_1 + \varepsilon]$. In view of Lemma 2.6(iii) [37], we have $\frac{\partial}{\partial \nu} u^\phi(x, t) \geq 0$.

Since for each $\phi \in C^+$, the solution $u^\phi(t)$ of (16) is nonnegative, bounded and exists for all $t \geq 0$, we can define a semigroup $\{S(t)\}_{t \geq 0}$ on $C^+$ as

$$S(t)\phi = u^\phi(t), \quad t \geq 0.$$ 

It follows from Corollary 1 and Lemma 3.3 that $S(t)$ is compact for all $t > \tau$. For each $\phi \in C^+$, the omega-limit set $\omega_\phi$ of $\phi$ is compact, where

$$\omega_\phi = \bigcap_{t \geq 0} CL \bigcup_{s \geq t} S(s)\phi.$$ 

The properties of the omega limit set $\omega_\phi$ can be found in [17].

**Proposition 2.** For each $\varphi \in \omega_\phi$, let

$$W_\varphi(x, \theta) = \frac{\varphi(x, \theta)}{\theta_d, a, b(x)}, \quad (x, \theta) \in \bar{\Omega} \times [-\tau, 0].$$

Then $W_\varphi$ is continuous on $\bar{\Omega} \times [-\tau, 0]$. Furthermore, $\sup_{\varphi \in \omega_\phi}\{\|W_\varphi\|_C\} < \infty$.

**Proposition 3.** For each fixed $\phi \in C^+$,

$$\beta_* = \inf_{\varphi \in \omega_\phi} \{\inf\{W_\varphi(x, \theta) : (x, \theta) \in \bar{\Omega} \times [-\tau, 0]\}\} > 0.$$
The proof of Proposition 3 can be found in [17] and hence is omitted. From Proposition 2 we have
\[ \beta^* = \sup_{\varphi \in \omega_\phi} \{ \sup \{ W_\varphi(x, \theta) : (x, \theta) \in \overline{\Omega} \times [-\tau, 0] \} \} < \infty. \]

To conclude that for each \( \phi \in C^+ \), \( u^\phi(\cdot, t) \) converges to \( \theta_{d,\alpha,b} \) as \( t \to \infty \), it suffices to prove that \( \omega_\phi = \{ \theta_{d,\alpha,b} \} \). This is equivalent to proving that
\[ \sigma \triangleq \sup_{\varphi \in \omega_\phi} \{ |W_\varphi(x, \theta) - 1| : (x, \theta) \in \overline{\Omega} \times [-\tau, 0] \} = 0. \] (18)

Let \( \chi > 0 \) be a constant and define
\[ W(x, t) = \frac{u(x, t)}{\theta_{d,\alpha,b}(x)}, \quad V(x, t) = u(x, t) - \chi \theta_{d,\alpha,b}(x), \]
for all \( (x, t) \in \overline{\Omega} \times [T - \tau, \infty) \), where \( u(\cdot, t) : [T - \tau, \infty) \to C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}) \) is a classical solution of (16) for \( t \geq T \).

Suppose (18) is not true, that is, \( \sigma > 0 \), then either \( \sigma = \beta^* - 1 \) or \( \sigma = 1 - \beta^* \). Firstly, suppose that \( \sigma = \beta^* - 1 \) then it follows from the compactness of \( \omega_\phi \) at an interior point \( (x_1, t_1) \) of \( \Omega \times [-\tau, 0] \) that one of the following two cases is true.

(i): There are \( \varphi^* \in \omega_\phi \) and \( (x_1, t_1) \in \Omega \times [-\tau, 0] \) such that \( \beta^* = \varphi^*(x_1, t_1) \);
(ii): \( W(x, t) < \beta^* \) for \( (x, t) \in \Omega \times \mathbb{R} \), and there are \( \varphi^* \in \omega_\phi \) and \( (x_1, t_1) \in \partial \Omega \times [-\tau, 0] \) such that \( \beta^* = \varphi^*(x_1, t_1) \).

For the case (i), let \( u(\cdot, t) = u^\varphi(\cdot, t) \). Then \( u(\cdot, t) \) exists for all \( t \in \mathbb{R} \) and \( u_t \in \omega_\phi \) because \( \omega_\phi \) is invariant. Then \( W(x, t) = u(x, t)/\theta_{d,\alpha,b}(x) \) attains its maximum \( \beta^* \) at an interior point \( (x_1, t_1) \) of \( \Omega \times \mathbb{R} \), that is,
\[ \frac{\partial}{\partial t} W(x_1, t_1) = 0, \quad \frac{\partial}{\partial x} W(x_1, t_1) = 0, \quad \Delta W(x_1, t_1) \leq 0. \] (19)

Since \( u(x, t) = W(x, t) \theta_{d,\alpha,b}(x) \), we have
\[ 0 = \theta_{d,\alpha,b}(x_1) \frac{\partial}{\partial t} W(x_1, t_1) \]
\[ = \frac{\partial}{\partial t} [W(x_1, t_1) \theta_{d,\alpha,b}(x_1)] = \frac{\partial}{\partial t} u(x_1, t_1) \]
\[ = d \Delta u(x_1, t_1) + \alpha(x) e^{-\gamma \tau} u(x_1, t_1 - \tau) - u^2(x_1, t_1) - b(x) u(x_1, t_1). \] (20)

Moreover, (19) implies that
\[ \Delta u(x_1, t_1) = \theta_{d,\alpha,b}(x_1) \Delta W(x_1, t_1) + W(x_1, t_1) \Delta \theta_{d,\alpha,b}(x_1). \] (21)

Note that \( \theta_{d,\alpha,b} \) is a steady state solution of (16). Substituting (21) into (20) yields
\[ 0 = \theta_{d,\alpha,b}(x_1) \frac{\partial}{\partial t} W(x_1, t_1) \]
\[ = d \theta_{d,\alpha,b}(x_1) \Delta W(x_1, t_1) \]
\[ - W(x_1, t_1) \left[ \alpha(x_1) e^{-\gamma \tau} \theta_{d,\alpha,b}(x_1) - \theta_{d,\alpha,b}^2(x_1) - b(x) \theta_{d,\alpha}(x_1) \right] \]
\[ + \alpha(x_1) e^{-\gamma \tau} u(x_1, t_1 - \tau) - u^2(x_1, t_1) - b(x) u(x_1, t_1) \]
\[ = d \theta_{d,\alpha,b}(x_1) \Delta W(x_1, t_1) - \alpha(x_1) [W(x_1, t_1) - 1] e^{-\gamma \tau} \theta_{d,\alpha,b}(x_1) \]
\[ + \alpha(x_1) e^{-\gamma \tau} [u(x_1, t_1 - \tau) - \theta_{d,\alpha,b}(x_1)] + u(x_1, t_1) [\theta_{d,\alpha,b}(x_1) - u(x_1, t_1)] \]
Lemma 3.4. Assume \( \mu_1(d, \alpha(x)e^{-\gamma t} - b(x)) < 0 \), let the initial function \( \phi \) be continuous on \( \Omega \times [-\tau, 0] \) and satisfy that \( \phi \geq 0 \) and \( \phi \neq 0 \). Then problem (16) with the initial value \( \phi \) has a unique positive solution \( u^\phi(\cdot, t) \), which satisfies
\[
u(\cdot, t) \rightarrow \theta_{d, \alpha, b} as t \rightarrow \infty.
\]

4. Stability of nonnegative steady states. In this section, we consider the local stability of nonnegative steady states, and also the global stability under some conditions.
4.1. Local stability. From the existence and uniqueness results of (9), we know that system (7) has a trivial steady state (0, 0) and two semi-trivial steady states \((\theta_{d_1, \alpha, 0}, 0)\) and \((0, \theta_{d_2, \tau})\). Moreover, if a steady state \((u, v)\) satisfying \(u \geq 0\) and \(v \geq 0\) is neither a trivial nor a semi-trivial steady state, then by the maximum principle we must have \(u > 0\) and \(v > 0\) in \(\Omega\), and we call \((u, v)\) a coexistence steady state. Now let us investigate the local stability of the semi-trivial steady state \((\theta_{d_1, \alpha, 0}, 0)\). Linearizing the steady state problem of (7) at \((u^*, v^*)\), and letting \((u, v) = (\Phi e^{\lambda t}, \Psi e^{\lambda t})\) for \((\Phi, \Psi) \in \mathbb{R}^2 \setminus (0, 0)\), we have

\[
\begin{align*}
&d_1 \Delta \Phi + (\alpha(x) e^{-\lambda t - \gamma \tau} - 2u^* - a v^*) \Phi - a \nu \Phi = \lambda \Phi, \quad x \in \Omega, \\
d_2 \Delta \Psi + (r(x) - c \theta_{d_1, \alpha, 0}) \Psi - c \theta_{d_1, \alpha, 0} \Psi = \lambda \Psi, \quad x \in \Omega, \\
\partial_\nu \Phi(x) = \partial_\nu \Psi(x) = 0, \quad x \in \partial \Omega.
\end{align*}
\]

(27)

In the following, we call a steady state \((u^*, v^*)\) of system (7) linearly stable (respectively, linearly unstable) if the principal eigenvalue \(\lambda_1\) of (27) is negative (respectively, positive). It is well known that a steady state of (7) is linearly stable (respectively, linearly unstable), then it is asymptotically stable (respectively, unstable) [29]. To describe the local asymptotic stability of \((\theta_{d_1, \alpha, 0}, 0)\), it suffices to consider (27) with \((u^*, v^*) = (\theta_{d_1, \alpha, 0}, 0)\), that is,

\[
\begin{align*}
&d_1 \Delta \Phi + \alpha(x) e^{-\lambda t \prod - \gamma \tau} - 2 \theta_{d_1, \alpha, 0} \Phi - a \nu \Phi = \lambda \Phi, \quad x \in \Omega, \\
d_2 \Delta \Psi + (r(x) - c \theta_{d_1, \alpha, 0}) \Psi = \lambda \Psi, \quad x \in \Omega, \\
\partial_\nu \Phi(x) = \partial_\nu \Psi(x) = 0, \quad x \in \partial \Omega.
\end{align*}
\]

(28)

Let \(\lambda\) be an eigenvalue of (28) with an associated eigenfunction \((\Phi, \Psi)\). If \(\Psi \neq 0\), then \(\lambda\) belongs to the spectrum of the self-adjoint operator \(d_2 \Delta + (r(x) - c \theta_{d_1, \alpha, 0})\) must be real and satisfy \(\lambda \leq -\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0})\). If \(\Psi \equiv 0\), then \(\Phi \neq 0\), and \(\lambda\) belongs to the spectrum of the infinitesimal generator \(A_{\lambda, \tau}\), which is given by

\[
A_{\lambda, \tau} \eta(\xi) = \begin{cases} 
\psi(x) \\
(d_1 \Delta - 2 \theta_{d_1, \alpha, 0}) \psi + \alpha(x) e^{-\gamma \tau} \psi(-\tau) \\
\eta(\xi) \end{cases} \quad \text{if } \xi \in (-\tau, 0), \\
\eta(\xi) \quad \text{if } \xi = 0.
\]

By Lemma 2.2, \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0})\) has the same sign as the principal eigenvalue of

\[
d_1 \Delta + \alpha(x) e^{-\gamma \tau} - 2 \theta_{d_1, \alpha, 0}.
\]

Note that \(-\mu_1(d_1, \alpha(x) e^{-\gamma \tau} - 2 \theta_{d_1, \alpha, 0}) < -\mu_1(d_1, \alpha(x) e^{-\gamma \tau} - \theta_{d_1, \alpha, 0}) = 0\), then we have

\[
\text{Re} \lambda \leq \max\{-\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0}), \hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0})\}.
\]

In summary, we obtain the following result.

**Theorem 4.1.** The linear stability of \((\theta_{d_1, \alpha, 0}, 0)\) is determined by \(\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0})\).

**Remark 1.** The maximum can be attainable under some suitable conditions. In fact, if \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0}) > -\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0})\) then \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0})\) is an eigenvalue of (28) with an associated eigenfunction \((\psi_1, 0)\), where \(\psi_1\) is the first eigenfunction corresponding to \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0})\). Hence, in this case, \((\theta_{d_1, \alpha, 0}, 0)\) is linearly stable. If \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0}) < -\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0})\) and \(\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0}) < 0\), the maximum can be attainable, but the semi-trivial steady state solution \((\theta_{d_1, \alpha, 0}, 0)\) is unstable. If \(\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0}) > 0\), the semi-trivial steady state solution \((\theta_{d_1, \alpha, 0}, 0)\) is linearly stable, however, we cannot make sure the maximum between \(\hat{\lambda}(d_1, \tau, -2 \theta_{d_1, \alpha, 0})\) and \(-\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, 0})\).
Now let us investigate the local stability of the semi-trivial steady state \((0, \theta_{d_2,r})\). Substituting \((0, \theta_{d_2,r})\) into the eigenvalue problem (27), we have
\[
\begin{cases}
d_1 \Delta \phi + \alpha(x) e^{-\lambda \tau - \gamma \tau} \phi - a \theta_{d_2,r} \phi = \lambda \phi, & x \in \Omega, \\
d_2 \Delta \psi + (r(x) - 2 \theta_{d_2,r}) \psi - c \theta_{d_2,r} \psi = \lambda \psi, & x \in \Omega, \\
\partial_n \phi(x) = \partial_n \psi(x) = 0, & x \in \partial \Omega.
\end{cases}
\]
(29)

Let \(\lambda\) be an eigenvalue of (29) with an associated eigenfunction \((\phi, \psi)\). If \(\Phi \neq 0\), then \(\lambda\) belongs to the spectrum \(\sigma(A_r)\) of the infinitesimal generator \(A_r\), which is given by
\[
(A_r \psi)(\xi) = \begin{cases} 
\psi & \text{if } \xi \in [-\tau, 0), \\
(d_1 \Delta - a \theta_{d_2,r}) \psi + \alpha(x) e^{-\gamma \tau} \psi(-\tau) & \text{if } \xi = 0.
\end{cases}
\]

Let \(s(A_r) = \sup \{\text{Re} \lambda : \lambda \in \sigma(A_r)\}\) be the spectral bound of \(A_r\). By the Krein-Rutman Theorem (see [29]) and Lemma 2.2, \(s(A_r)\) is the principal eigenvalue of \(A_r\). That is, \(\bar{\lambda}(d_1, \tau, -a \theta_{d_2,r}) = s(A_r)\). So we have \(\text{Re} \lambda \leq \bar{\lambda}(d_1, \tau, -a \theta_{d_2,r})\). If \(\Phi \equiv 0\), then \(\Psi \neq 0\), and \(\lambda\) belonging to the spectrum of \(d_2 \Delta + r(x) - 2 \theta_{d_2,r}\) must be real and satisfies \(\lambda < -\mu_1(d_2, r(x) - 2 \theta_{d_2,r})\). Note that \(\mu_1(d_2, r(x) - 2 \theta_{d_2,r}) > \mu(d_2, r(x) - \theta_{d_2,r}) = 0\), then we have
\[
\text{Re} \lambda \leq \max\{-\mu_1(d_2, r(x) - 2 \theta_{d_2,r}), \bar{\lambda}(d_1, \tau, -a \theta_{d_2,r})\}.
\]

Besides, we can obtain the maximum between \(-\mu_1(d_2, r(x) - 2 \theta_{d_2,r})\) and \(\bar{\lambda}(d_1, \tau, -a \theta_{d_2,r})\). In summary, we obtain the following result.

**Theorem 4.2.** The linear stability of \((0, \theta_{d_2,r})\) is determined by the sign of \(\bar{\lambda}(d_1, \tau, -a \theta_{d_2,r})\).

**Remark 2.** Using a similar argument, we see that the linear stability of \((0, 0)\) of system (7) is determined by the sign of \(\lambda(d_1, \tau, -a \theta_{d_2,r})\). In fact, \((0, 0)\) is always linearly unstable for any \(d_1, d_2 > 0\).

4.2. Global stability. Next, let us discuss the global stability of the steady state solutions.

**Theorem 4.3.** Assume that the principal eigenvalue satisfies that \(\mu_1(d_1, a e^{-\gamma \tau} \alpha - a \theta_{d_2,r}) < 0\) and \(\mu_1(d_2, r(x) - c \theta_{d_1, \tau} \alpha + a \theta_{d_2,r}) > 0\), then for any \(\phi \in C_\tau \times \mathbb{Y}\) satisfying \(\phi \neq 0\), the solution \(U^\phi(t, x)\) of system (7) satisfies that
\[
\lim_{t \to \infty} U^\phi(\cdot, t) = (\theta_{d_1, \tau} \alpha, 0, 0)
\]
uniformly in \(\Omega\).

**Proof.** For any \(\phi \in C_\tau \times \mathbb{Y}\) satisfying \(\phi \neq 0\), \(U^\phi(t, x) = (u(x, t), v(x, t))\) satisfies \(u(x, t) \geq 0\) and \(v(x, t) \geq 0\) for all \(t \geq 0\) and \(x \in \Omega\). Note that
\[
\begin{align*}
\lim_{\varepsilon \to 0} \mu_1(d_1, a e^{-\gamma \tau} + \alpha(x) \varepsilon - a \theta_{d_2,r}) &= \mu_1(d_1, a e^{-\gamma \tau} - a \theta_{d_2,r}) < 0, \\
\lim_{\varepsilon \to 0} \mu_1(d_2, r(x) - c \theta_{d_1, \tau} \alpha + a \theta_{d_2,r} - \varepsilon) &= \mu_1(d_2, r(x) - c \theta_{d_1, \tau} \alpha + a \theta_{d_2,r}) > 0,
\end{align*}
\]
then there exists some \(\varepsilon > 0\) such that
\[
\mu_1(d_1, a e^{-\gamma \tau} - a \theta_{d_2,r}) < 0, \quad \mu_1(d_2, r(x) - c \theta_{d_1, \tau} \alpha + a \theta_{d_2,r} - \varepsilon) > 0.
\]
(30)

Furthermore, it follows from the first equation of system (7) that
\[
\frac{\partial u(x, t)}{\partial t} \leq d_1 \Delta u(x, t) + \alpha(x) e^{-\gamma \tau} u(x, t - \tau) - u^2(x, t)
\]
for $x \in \Omega$ and $t > 0$. It follows from $\mu_1(d_1, \alpha e^{-\tau}) < 0$ and Lemma 3.4 that the unique positive steady state $\theta_{d_1, \alpha, 0}$ of (16) with $d = d_1$ and $b(x) = 0$ is globally attractive. A standard comparison theorem provides $t_0 = t_0(\phi) > 0$ such that

$$u(x, t) \leq \theta_{d_1, \alpha, 0} + \varepsilon$$

for all $x \in \Omega$ and $t \geq t_0$. Now, let us investigate the second equation of system (7), which satisfies

$$\frac{\partial v(x, t)}{\partial t} \leq d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t)].$$

(31)

Thus, for any initial value $\phi$, there exists $t_1 = t_1(\phi) > t_0$ such that $0 \leq v(x, t) \leq \theta_{d_2, \varepsilon}(x) + \varepsilon$ for $t > t_1$, which implies that

$$\frac{\partial u(x, t)}{\partial t} \geq d_1 \Delta u(x, t) + \alpha(x) e^{-\tau} u(x, t - \tau) - u^2(x, t) - au(x, t)[\theta_{d_2, \varepsilon}(x) + \varepsilon]$$

for all $x \in \Omega$ and $t \geq t_1$. It follows from (30) and Lemma 3.4 that the unique positive steady state $\theta_{d_1, \alpha, a(\theta_{d_2, \varepsilon} + \varepsilon)}$ of system (16) with $d = d_1$ and $b(x) = a(\theta_{d_2, \varepsilon}(x) + \varepsilon)$ is globally attractive. Again, a standard comparison theorem provides $t_2 = t_2(\phi) > 0$ such that

$$\theta_{d_1, \alpha, a(\theta_{d_2, \varepsilon} + \varepsilon)} - \varepsilon \leq u(\cdot, t) \leq \theta_{d_1, \alpha, 0} + \varepsilon \text{ in } \Omega$$

for all $t \geq t_2$. This, together with the second equation of system (7), implies that

$$\frac{\partial v(x, t)}{\partial t} \leq d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t) - c(\theta_{d_1, \alpha, a(\theta_{d_2, \varepsilon} + \varepsilon)}(x) - \varepsilon)].$$

(32)

In view of (30), the standard comparison theorem, and Corollary 2.2 in [6], we know that $\lim_{t \to \infty} v(x, t) = 0$ uniformly for $x \in \Omega$, and hence that $0 \leq v(x, t) \leq \varepsilon$ for $t > t_3$ with sufficiently large $t_3 = t_3(\phi) > t_2$. Therefore, the first equation system (7) satisfies

$$\frac{\partial u(x, t)}{\partial t} \geq d_1 \Delta u(x, t) + \alpha(x) e^{-\tau} u(x, t - \tau) - u^2(x, t) - \varepsilon au(x, t)$$

for all $x \in \Omega$ and $t \geq t_3$. In view of (30) and Proposition 1(iv), we have $\mu_1(d_1, \alpha e^{-\tau} - a\varepsilon) < 0$, which together with Lemma 3.4 implies that the positive steady state $\theta_{d_1, \alpha, a\varepsilon}$ of system (16) with $d = d_1$ and $b(x) = a\varepsilon$ is globally attractive. Again, a standard comparison theorem provides $t_4 = t_4(\phi) > t_3$ such that

$$\theta_{d_1, \alpha, a\varepsilon} - \varepsilon \leq u(\cdot, t) \leq \theta_{d_1, \alpha, 0} + \varepsilon \text{ in } \Omega$$

for all $t > t_4$. Therefore, the solution $U^\phi(t, x)$ of system (7) satisfies that $\lim_{t \to \infty} U^\phi(t, x) = (\theta_{d_1, \alpha, 0}, 0)$ uniformly for $x \in \Omega$. This completes the proof.

**Theorem 4.4.** Assume that the principal eigenvalue satisfies $\bar{\lambda}(d_1, \tau, -a\theta_{d_2, r - c\theta_{d_1, \alpha, 0}}) < 0$ and $\mu_1(d_2, r - c\theta_{d_1, \alpha, 0}) < 0$, then for any $\phi \in C_\tau \times \bar{\gamma}$ satisfying $\phi \neq 0$, the solution $U^\phi(t, x)$ of the system satisfies that

$$\lim_{t \to \infty} U^\phi(\cdot, t) = (0, \theta_{d_2, r})$$

uniformly in $\Omega$.

**Proof.** For any $\phi \in C_\tau \times \bar{\gamma}$ satisfying $\phi \neq 0$, $U^\phi(t, x) = (u(x, t), v(x, t))$ satisfies $u(x, t) \geq 0$ and $v(x, t) \geq 0$ for all $t \geq 0$ and $x \in \Omega$. Note that

$$\lim_{\varepsilon \to 0} \bar{\lambda}(d_1, \tau, -a(\theta_{d_2, r - c(\theta_{d_1, \alpha, 0} + \varepsilon)}) - \varepsilon)) = \bar{\lambda}(d_1, \tau, -a\theta_{d_2, r - c\theta_{d_1, \alpha, 0}}) < 0,$$

$$\lim_{\varepsilon \to 0} \mu_1(d_2, r - c(\theta_{d_1, \alpha, 0} + \varepsilon)) = \mu_1(d_2, r - c\theta_{d_1, \alpha, 0}) < 0,$$
then we can choose some $\varepsilon > 0$ such that
\[
\lambda(d_1, \tau, -a(\theta_{d_2, r - c(\theta_{d_1,0} + \varepsilon)}) < 0, \quad \mu_1(d_2, r - c(\theta_{d_1,0} + \varepsilon)) < 0. \tag{33}
\]
Obviously, $\mu_1(d_2, r(x)) < 0$, which implies that the unique positive steady state $\theta_{d_2, r}$ of the following system is globally attractive (see [6] for more details):
\[
\begin{cases}
\frac{\partial v(x, t)}{\partial t} = d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t)] & \text{in } \Omega, \\
\partial_v v(x, t) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Thus, it follows from (31) and a standard comparison theorem that there exists $t_0 = t_0(\phi) > 0$ such that
\[
v(x, t) \leq \theta_{d_2, r} + \varepsilon \quad \text{for } x \in \overline{\Omega} \text{ and } t \geq t_0.
\]
As stated in the proof of Theorem 4.3, there exists $t_1 = t_1(\phi) > t_0$ such that $0 \leq u(x, t) \leq \theta_{d_1,0} + \varepsilon$ for any $t > t_1$. Therefore, we have
\[
\frac{\partial v(x, t)}{\partial t} \geq d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t) - c(\theta_{d_1,0}(x) + \varepsilon)]. \tag{34}
\]
It follows from (33) and Lemma 3.4 that the positive steady state $\theta_{d_2, r - c(\theta_{d_1,0} + \varepsilon)}$ of the following system is globally attractive:
\[
\begin{cases}
\frac{\partial v(x, t)}{\partial t} = d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t) - c(\theta_{d_1,0}(x) + \varepsilon)] & \text{in } \Omega, \\
\partial_v v(x, t) = 0 & \text{on } \partial \Omega
\end{cases}
\]
Again, a standard comparison theorem provides $t_2 = t_2(\phi) > t_1$ such that
\[
\theta_{d_2, r - c(\theta_{d_1,0} + \varepsilon)} - \varepsilon \leq v(\cdot, t) \leq \theta_{d_2, r} + \varepsilon \quad \text{in } \overline{\Omega}
\]
for all $t > t_2$. Consequently, $u(x, t)$ satisfies
\[
\frac{\partial u(x, t)}{\partial t} \leq d_1 \Delta u(x, t) + a(x) e^{-\gamma \tau} u(x, t - \tau) - a \left[ \theta_{d_2, r - c(\theta_{d_1,0} + \varepsilon)}(x) - \varepsilon \right] u(x, t) - u^2(x, t). \tag{35}
\]
It follows from (33), Corollary 4.8 in [32], and a standard comparison theorem that
\[
\lim_{t \to \infty} u(\cdot, t) = 0 \quad \text{uniformly in } \overline{\Omega}.
\]
Thus, there exists $t_3 = t_3(\phi) > t_2$ such that $0 \leq u(\cdot, t) \leq \varepsilon$ in $\overline{\Omega}$ for $t \geq t_3$. Therefore, $v(\cdot, t)$ with $t > t_3$ satisfies
\[
\frac{\partial v(x, t)}{\partial t} \geq d_2 \Delta v(x, t) + v(x, t)[r(x) - c \varepsilon - v(x, t)]. \tag{36}
\]
From Proposition 1(iv), $\mu_1(d_2, r - c(\theta_{d_1,0} + \varepsilon)) < 0$ implies that $\mu_1(d_2, r - c \varepsilon) < 0$. This together with Lemma 3.4 that the positive steady state $\theta_{d_2, r - c \varepsilon}$ of the following system is globally attractive:
\[
\begin{cases}
\frac{\partial v(x, t)}{\partial t} = d_2 \Delta v(x, t) + v(x, t)[r(x) - v(x, t) - c \varepsilon] & \text{in } \Omega, \\
\partial_v v(x, t) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Again, a standard comparison theorem provides $t_4 = t_4(\phi) > t_3$ such that
\[
\theta_{d_2, r - c \varepsilon} \leq v(\cdot, t) \leq \theta_{d_2, r} \quad \text{in } \overline{\Omega}.
for \( t \geq t_4 \). Thus, the solution \( U^\theta(t,x) \) satisfies \( \lim_{t \to \infty} U^\theta(\cdot,t) = (0,\theta_{d_2,r}) \) uniformly in \( \Omega \). This completes the proof. \( \square \)

To state the global asymptotical stability, we first define the following two numbers:

\[
\tau_* = \frac{1}{\gamma} \ln \frac{c \inf_{\Omega} \alpha(x)}{(1 + ac) \sup_{\Omega} r(x)}, \quad \tau^* = \frac{1}{\gamma} \ln \frac{(1 + ac) \sup_{\Omega} \alpha(x)}{a \inf_{\Omega} r(x)}.
\]

By Theorem 4.1, it suffices to show that \( \mu_1(d_2, r(x) - c\theta_{d_1,\alpha,0}) > 0 \) to make sure that \((\theta_{d_1,\alpha,0}, 0)\) is locally stable. Theorem 4.3 says that the semi-trivial steady state \((\theta_{d_1,\alpha,0}, 0)\) is globally attractive when \( \mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r}) < 0 \) and \( \mu_1(d_2, r(x) - c\theta_{d_1,\alpha,0}) > 0 \). It follows from Proposition 1(iv) and \( \mu_1(d_2, r(x) - c\theta_{d_1,\alpha,0}) > 0 \) that \( \mu_1(d_2, r(x) - c\theta_{d_1,\alpha,0}) > 0 \). That is, if \((\theta_{d_1,\alpha,0}, 0)\) is globally attractive then it is locally asymptotically stable. In view of Lemma 2.3(i)(b) in [15], we have

\[
\begin{align*}
    r(x) - c\theta_{d_1,\alpha,0} &\leq \sup_{\Omega} r(x) - c \inf_{\Omega}(\alpha(x)e^{-\gamma r} - a\theta_{d_2,r}) \\
    &\leq \sup_{\Omega} r(x) - c \inf_{\Omega}(\alpha(x)e^{-\gamma r} + ac \sup_{\Omega} \theta_{d_2,r}) \\
    &\leq (1 + ac) \sup_{\Omega} r(x) - c \inf_{\Omega}(\alpha(x)e^{-\gamma r}).
\end{align*}
\]

Therefore, \( r(x) - c\theta_{d_1,\alpha,0} < 0 \) when \( \tau < \tau_* \). In view of Proposition 1(i), if

\[
\int_{\Omega} [ae^{-\gamma r} - a\theta_{d_2,r}] \, dx \geq 0,
\]

i.e., \( e^{\gamma r} < \bar{\alpha}/(ar) \), then \( \mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r}) < 0 \) for all \( d_1 > 0 \). If \( ac \leq 1 \) then

\[
\frac{c \inf_{\Omega} \alpha(x)}{(1 + ac) \sup_{\Omega} r(x)} \leq \frac{c \bar{\alpha}}{r} \leq \frac{\bar{\alpha}}{ar}.
\]

That is to say, when \( \tau < \tau_* \) and \( ac \leq 1 \), Theorem 4.3 holds and hence \((\theta_{d_1,\alpha,0}, 0)\) is globally asymptotically stable. Theorem 4.4 says that the semi-trivial steady state solution \((0, \theta_{d_2,r})\) is globally attractive when \( \lambda(d_1, \tau, -a\theta_{d_2,r} - c\theta_{d_1,\alpha,0}) < 0 \) and \( \mu_1(d_2, r(x) - c\theta_{d_1,\alpha,0}) < 0 \). Moreover, \((0, \theta_{d_2,r})\) is linearly stable when \( \lambda(d_1, \tau, -a\theta_{d_2,r}) < 0 \). From Lemma 2.2, we know that

\[
\begin{align*}
    \text{sgn}(\lambda(d_1, \tau, -a\theta_{d_2,r} - c\theta_{d_1,\alpha,0})) &= -\text{sgn}(\mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r} - c\theta_{d_1,\alpha,0})), \\
    \text{sgn}(\lambda(d_1, \tau, -a\theta_{d_2,r})) &= -\text{sgn}(\mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r})).
\end{align*}
\]

Note that \( \mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r} - c\theta_{d_1,\alpha,0}) < \mu_1(d_1, ae^{-\gamma r} - a\theta_{d_2,r}) \). Thus, if \((0, \theta_{d_2,r})\) is globally attractive then it is also linearly stable. Using a similar argument as above, we see that the semi-trivial solution \((0, \theta_{d_2,r})\) is globally asymptotically stable when \( \tau \geq \tau^* \) and \( ac \leq 1 \). In summary, we obtain the following results.

**Theorem 4.5.** Assume that (M) holds and \( ac \leq 1 \), then the following statements hold for system (7):

(i): If \( \tau \in (0, \tau_*) \), then for all \( d_1, d_2 > 0 \), \((\theta_{d_1,\alpha,0}, 0)\) is globally asymptotically stable.

(ii): If \( \tau \in (\tau^*, \infty) \), then for all \( d_1, d_2 > 0 \), \((0, \theta_{d_2,r})\) is globally asymptotically stable.
Remark 3. In the range \((\tau_*, \tau^*)\) where the strengths of the carrying capacities of the two competing species are “comparable”, the roles of diffusion rates \(d_1\) and \(d_2\) become more important. For example, for some \(d_1\) and \(d_2\) satisfying
\[
0 < d_1 < \frac{1}{\lambda_1(\alpha(x)e^{-\gamma \tau} - a \theta_{d_2,r})}, \quad d_2 > \frac{1}{\lambda_1(r(x) - c \theta_{d_1, \alpha, a \theta_{d_2,r}})},
\]
if each of \(r(x) - c \theta_{d_1, \alpha, a \theta_{d_2,r}}\) and \(\alpha(x) e^{-\gamma \tau} - a \theta_{d_2,r}\) changes sign in \(\Omega\), and
\[
\int_{\Omega} [r(x) - c \theta_{d_1, \alpha, a \theta_{d_2,r}}] \, dx < 0, \quad \int_{\Omega} [\alpha(x) e^{-\gamma \tau} - a \theta_{d_2,r}] \, dx < 0,
\]
then \(\mu_1(d_2, r(x) - c \theta_{d_1, \alpha, a \theta_{d_2,r}}) > 0\) and \(\mu_1(d_1, \alpha(x) e^{-\gamma \tau} - a \theta_{d_2,r}) < 0\) hold. In this case, \((\theta_{d_1, \alpha, 0}, 0)\) is also globally asymptotically stable. That is, when \(\tau \in (\tau_*, \tau^*)\), the semi-trivial steady states may be globally asymptotically stable for suitable \(d_1\) and \(d_2\). And the other \(d_1\) and \(d_2\) may contribute to the existence of coexistence steady states.

Remark 4. The quantity \(\zeta \triangleq \gamma \tau\) is said to be the degree of the stage structure for the species. Theorem 4.5 implies that in the stage-structured competitive system, stage structure brings negative effects on permanence of one species as well as contribution its extinction. And such effect can be estimated by \(e^\zeta\). In view of this point, the effect of stage-structure on the Lotka-Volterra competition model with diffusion and heterogenous of the resources is similar to its on the Lotka-Volterra competition model without diffusion and heterogenous; see, for example [23]. More precisely, the increase of species degree of stage structure can not only drive it into elimination but also ensure its competitor permanent.

5. Asymptotic behavior and global attractor. The purpose of this section is to investigate the asymptotic behavior of the solution of system (7). We first rewrite the system (7) as
\[
\begin{align*}
\partial u / \partial t - d_1 \Delta u &= f_1(x, u, u, v) \quad \text{in } \Omega, \\
\partial v / \partial t - d_2 \Delta v &= f_2(x, u, u, v) \quad \text{in } \Omega, \\
\partial u / \partial n &= \partial v / \partial n = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \(f_1(x, u, u, v) = \alpha(x) e^{-\gamma \tau} u(t - \tau) - u^2 - auv\), \(f_2(x, u, u, v) = v(r(x) - cu - v)\). Note that \(f_1(\cdot, u, u, v)\) is monotone non-decreasing in \(u\) and monotone non-increasing in \(v\), \(f_2(\cdot, u, u, v)\) is monotone non-increasing in \(u\). It follows that
\[
f(x, u, u, v) = (f_1(x, u, u, v), f_2(x, u, u, v))
\]
possesses a mixed quasi-monotone property in some subset \(\Xi\) of \(\mathbb{R}^2\) (see [26] for more details). Moreover, this property leads to the following definition.

Definition 5.1. A pair of vector functions \(\vec{U} = (\vec{u}, \vec{v})\) and \(\hat{U} = (\hat{u}, \hat{v})\) in \(C([0, T] \times \Omega) \cap C([0, T] \times \overline{\Omega})\) are called coupled upper and lower solutions of (37) if \(\hat{U} > \vec{U}\) in \([0, T] \times \overline{\Omega}\), and if
\[
\begin{align*}
\partial \vec{u} / \partial t - d_1 \Delta \vec{u} &\geq f_1(x, \vec{u}, \vec{u}, \vec{v}) \quad \text{in } \Omega, \\
\partial \vec{v} / \partial t - d_2 \Delta \vec{v} &\geq f_2(x, \vec{u}, \vec{u}, \vec{v}) \quad \text{in } \Omega,
\end{align*}
\]
and
\[
\begin{align*}
\partial \hat{u} / \partial t - d_1 \Delta \hat{u} &\leq f_1(x, \hat{u}, \hat{u}, \hat{v}) \quad \text{in } \Omega, \\
\partial \hat{v} / \partial t - d_2 \Delta \hat{v} &\leq f_2(x, \hat{u}, \hat{u}, \hat{v}) \quad \text{in } \Omega,
\end{align*}
\]
hold with initial conditions \(\eta_i(x, t)\) satisfying that \(\hat{U} \leq \eta_i(x, t) \leq \vec{U}\) for \((t, x) \in [-\tau, 0] \times \Omega\), and \(\partial \vec{U}(x) = \partial \vec{U}(x) = 0\) for \(x \in \partial \Omega\).
For a given pair of coupled upper and lower solutions \( \hat{U}, \tilde{U} \), we set \( \langle \hat{U}, \tilde{U} \rangle \triangleq \{ U \in C([0, T] \times \Omega) : \hat{U} \leq U \leq \tilde{U} \} \). Thus, the function \( f(v, u, v, \tau) \) is a \( C^r \) function and possesses a mixed quasi-monotone property in \( \langle \hat{U}, \tilde{U} \rangle \). Moreover, we have the following existence-comparison theorem from [25].

**Theorem 5.2.** Let \( \hat{U} \) and \( \tilde{U} \) be a pair of coupled upper and lower solutions of (37), then system (37) has a unique solution \( U^\star \) in \( \langle \hat{U}, \tilde{U} \rangle \). Moreover, there exist sequences \( \{ \hat{U}^{(m)} \} \) and \( \{ \tilde{U}^{(m)} \} \) converging monotonically from above and below, respectively, to \( U^\star \) as \( m \to \infty \).

To investigate the asymptotic behavior of the solution to (37), where the reaction function is mixed quasi-monotonic, we first consider the following elliptic system corresponding to (37):

\[
\begin{aligned}
-d_1 \Delta u &= \alpha(x)e^{-\gamma \tau}u - u^2 - auv \quad \text{in} \ \Omega, \\
-d_2 \Delta v &= v(r(x) - cu - v) \quad \text{in} \ \Omega, \\
\partial_\nu u &= \partial_\nu v = 0 \quad \text{on} \ \partial \Omega.
\end{aligned}
\]  

(38)

The definition of upper and lower solutions of system (38) can be found in Definition 4.1 in [26], and it says that any pair of coupled upper and lower solutions \( \hat{U}_s \) and \( \tilde{U}_s \) of (38) are also upper and lower solutions of (37). Hence by Theorem 5.2, the sector \( \langle \hat{U}_s, \tilde{U}_s \rangle \) is an invariant set of system (37). Unlike the usual scalar boundary problem, the upper and lower solutions are coupled. Our construction the lower solutions of system (38) is based on the positive steady state of the following problem

\[
\begin{aligned}
d_1 \Delta u + (m_i(x) - u)u &= 0, \quad \text{in} \ \Omega, \\
\partial_\nu u &= 0, \quad \text{on} \ \partial \Omega,
\end{aligned}
\]  

(39)

where \( m_i(x) \) is given continuous nonnegative functions on \( \Omega \). Recall from the problem (9), we know that system (39) has a unique positive solution \( \theta_{d_i,m_i} \), if \( \mu(d_i, m_i) < 0 \). Next, we seek a pair of the form \( \hat{U} = (\theta_{d_1,0,0}, \theta_{d_2,r}) \) and \( \tilde{U} = (\theta_{d_1,m_1}, \theta_{d_2,m_2}) \) satisfying

\[
\begin{aligned}
-d_1 \Delta \theta_{d_1,0,0} &\geq \alpha(x)\theta_{d_1,0,0}e^{-\gamma \tau} - \theta_{d_1,0,0} \left[ \theta_{d_1,0,0} + a\theta_{d_2,m_2} \right] \quad \text{in} \ \Omega, \\
-d_2 \Delta \theta_{d_2,r} &\geq \theta_{d_2,r} \left| r(x) - c\theta_{d_1,0,0} - \theta_{d_2,r} \right| \quad \text{in} \ \Omega,
\end{aligned}
\]  

(40)

and

\[
\begin{aligned}
-d_1 \Delta \theta_{d_1,m_1} &\leq \alpha(x)e^{-\gamma \tau} \theta_{d_1,m_1} - \theta_{d_1,m_1} \left( \theta_{d_1,m_1} + a\theta_{d_2,r} \right) \quad \text{in} \ \Omega, \\
-d_2 \Delta \theta_{d_2,m_2} &\leq \theta_{d_2,m_2} \left| r(x) - c\theta_{d_1,0,0} - \theta_{d_2,m_2} \right| \quad \text{in} \ \Omega.
\end{aligned}
\]  

(41)

Hence, choose

\[
m_1(x) = \alpha(x)e^{-\gamma \tau} - a\theta_{d_2,r}, \quad m_2(x) = r(x) - c\theta_{d_1,0,0}.
\]  

(42)

Then \( 0 < \theta_{d_1,m_1} < \theta_{d_1,0,0}, 0 < \theta_{d_2,m_2} < \theta_{d_2,r} \) and (40) and (41) hold. From Definition 4.1 in [26], we see that \( \hat{U} = (\theta_{d_1,0,0}, \theta_{d_2,r}) \) and \( \tilde{U} = (\theta_{d_1,m_1}, \theta_{d_2,m_2}) \) are a pair of coupled upper and lower solutions of system (38). Note that \( f(x, u, r, v) \) is a \( C^r \) function and possess a mixed quasi-monotone property in \( \langle \hat{U}, \tilde{U} \rangle \). Therefore, there exists \( K_1 > 0 \) such that

\[
|f_1(x, u_1, u_1, v_1) - f_1(x, u_2, u_2, v_2)| \leq K_1 |(u_1 - u_2) + |u_1 - u_2| + |v_1 - v_2|),
\]

and

\[
|f_2(x, u_1, u_1, v_1) - f_2(x, u_2, u_2, v_2)| \leq K_2 |(u_1 - u_2) + |v_1 - v_2|)
\]
for all \((u_1, v_1), (u_2, v_2) \in \langle \hat{U}, \bar{U} \rangle\). Construct the two sequences \((\bar{u}^{(n)}, \bar{v}^{(n)})\) and \((\underline{u}^{(n)}, \underline{v}^{(n)})\) via the following linear iteration process

\[-d_1 \Delta \bar{u}^{(n)} + K_1 \bar{u}^{(n)} = K_1 \bar{u}^{(n-1)} + f_1(x, \bar{u}^{(n-1)}, \bar{v}^{(n-1)}, \underline{v}^{(n-1)}),\]

\[-d_2 \Delta \bar{v}^{(n)} + K_2 \bar{v}^{(n)} = K_2 \bar{v}^{(n-1)} + f_2(x, \bar{u}^{(n-1)}, \bar{v}^{(n-1)}),\]

\[-d_1 \Delta \underline{u}^{(n)} + K_1 \underline{u}^{(n)} = K_1 \underline{u}^{(n-1)} + f_1(x, \underline{u}^{(n-1)}, \bar{v}^{(n-1)}, \underline{v}^{(n-1)}),\]

\[-d_2 \Delta \underline{v}^{(n)} + K_2 \underline{v}^{(n)} = K_2 \underline{v}^{(n-1)} + f_2(x, \underline{u}^{(n-1)}, \bar{v}^{(n-1)}),\]

and \((\bar{u}^{(0)}, \bar{v}^{(0)}) = (\theta_{d_1, \alpha, 0, \theta_{d_2, r}}), (\underline{u}^{(0)}, \underline{v}^{(0)}) = (\theta_{d_1, m_1, \theta_{d_2, m_2}})\). It is not difficult to know that,

\[
\begin{align*}
(\theta_{d_1, m_1}, \theta_{d_2, m_2}) & \leq \left( \bar{u}^{(n)}, \bar{v}^{(n)} \right) \leq \left( \underline{u}^{(n+1)}, \underline{v}^{(n+1)} \right) \\
& \leq \left( \bar{u}^{(n+1)}, \bar{v}^{(n+1)} \right) \leq \left( \bar{u}^{(n)}, \bar{v}^{(n)} \right) \leq \left( \theta_{d_1, \alpha, 0, \theta_{d_2, r}} \right) \\
\end{align*}
\]

for \(n \geq 1\). Furthermore,

\[
\begin{align*}
\lim_{n \to \infty} (\bar{u}^{(n)}, \bar{v}^{(n)}) & = (\bar{u}, \bar{v}), \quad \lim_{n \to \infty} (\underline{u}^{(n)}, \underline{v}^{(n)}) = (\underline{u}, \underline{v})
\end{align*}
\]

exist and satisfy

\[
\begin{align*}
-d_1 \Delta \bar{u} & = \alpha(x)e^{-\gamma \tau} \bar{u} - \bar{u}^2 - a \bar{u} \bar{v} \quad \text{in } \Omega, \\
-d_2 \Delta \bar{v} & = \bar{v}(r(x) - c \bar{u} - \bar{v}) \quad \text{in } \Omega,
\end{align*}
\]

and

\[
\begin{align*}
-d_1 \Delta \underline{u} & = \alpha(x)e^{-\gamma \tau} \underline{u} - \underline{u}^2 - a \underline{u} \underline{v} \quad \text{in } \Omega, \\
-d_2 \Delta \underline{v} & = \underline{v}(r(x) - c \underline{u} - \underline{v}) \quad \text{in } \Omega,
\end{align*}
\]

with \(\partial_\Omega \bar{u} = \partial_\Omega \bar{v} = \partial_\Omega \underline{u} = \partial_\Omega \underline{v} = 0\). The limits \((\bar{u}, \bar{v})\) and \((\underline{u}, \underline{v})\) are called quasi-solutions of system (38) in \(\langle U, \bar{U} \rangle\), and are ordered but in general are not true solutions. By Theorem 4.1 in [26], for any initial function \(\eta = (\eta_1, \eta_2) \in C([-\tau, 0] \times \Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})\) satisfying \(\theta_{d_1, m_1} \leq \eta_1(\cdot, s) \leq \theta_{d_1, \alpha, 0}\) for all \(s \in [-\tau, 0]\) and \(\theta_{d_2, m_2} \leq \eta_2 \leq \theta_{d_2, r}\), the corresponding solution \((u, v)\) of (37) satisfies

\[
u \leq u(\cdot, t) \leq \bar{u} \quad \text{and} \quad \underline{u} \leq v(\cdot, t) \leq \bar{v} \quad \text{as } t \to \infty \text{ in } \bar{\Omega}.
\]

This implies that the sector \(\langle \bar{U}, \hat{U} \rangle\) between the two quasisolutions \(\bar{U} = (\bar{u}, \bar{v})\) and \(\hat{U} = (\underline{u}, \underline{v})\) is an attractor for system (37) with initial function \(\eta = (\eta_1, \eta_2) \in C([-\tau, 0] \times \Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})\) satisfying \(\theta_{d_1, m_1} \leq \eta_1(\cdot, s) \leq \theta_{d_1, \alpha, 0}\) for all \(s \in [-\tau, 0]\) and \(\theta_{d_2, m_2} \leq \eta_2 \leq \theta_{d_2, r}\). In particular, if \(\underline{u} = \bar{u} = u^*\) and \(\underline{v} = \bar{v} = v^*\) then \((u^*, v^*)\) is the unique solution of system (38) in \(\langle \hat{U}, \bar{U} \rangle\) and \(u(\cdot, t) \to u^*\) and \(v(\cdot, t) \to v^*\) as \(t \to \infty\). Let \((u(\cdot, t), v(\cdot, t))\) be the solution of (37) corresponding to an arbitrary initial function \(\eta \in C([-\tau, 0] \times \Omega, \mathbb{R}) \times C(\Omega, \mathbb{R})\), if there exists \(t^* \geq 0\) such that \(\theta_{d_1, m_1} < u(\cdot, t) < \theta_{d_1, \alpha, 0}\) and \(\theta_{d_2, m_2} < v(\cdot, t) < \theta_{d_2, r}\) for all \(t \in [t^* - \tau, t^*]\), then Theorem 4.2 in [26] says that \((u(\cdot, t), v(\cdot, t))\) satisfies (45).

Note that

\[
\lim_{\varepsilon \to 0} \mu_1(d_1, \alpha(x)e^{-\gamma \tau} - a(\theta_{d_2, r} + \varepsilon)) = \mu_1(d_1, m_1) < 0,
\]

\[
\lim_{\varepsilon \to 0} \mu_1(d_2, r(x) - c(\theta_{d_1, \alpha, 0} + \varepsilon)) = \mu_1(d_2, m_2) < 0,
\]

then there exists some \(\varepsilon > 0\) such that

\[
\mu_1(d_1, \alpha(x)e^{-\gamma \tau} - a(\theta_{d_2, r} + \varepsilon)) < 0, \quad \mu_1(d_2, r(x) - c(\theta_{d_1, \alpha, 0} + \varepsilon)) < 0.
\]
For any nonzero initial function \( \eta \in C([-\tau, 0] \times \Omega, \mathbb{R}) \), note that the solution \((u, v)\) of system (37) satisfies that
\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &\leq \alpha(x) e^{-\gamma t} u(t - \tau) - u^2 & x \in \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &\leq v(r(x) - v) & x \in \Omega.
\end{align*}
\]
By standard comparison theorem there exists \( t_1(\eta) > 0 \) such that \( 0 \leq u(x, t) \leq \theta_{d_1, a, \bar{c}} + \varepsilon \) and \( 0 \leq v(x, t) \leq \theta_{d_2, r} + \varepsilon \) for all \( t \geq t_1(\eta) \) and \( x \in \Omega \). In addition, for \( t \geq t_1(\eta) \), and the same initial condition \( \eta \), the solution \((u, v)\) of system (37) satisfies that
\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &\geq \alpha(x) e^{-\gamma t} u(t - \tau) - u^2 - au(\theta_{d_2, r} + \varepsilon) & x \in \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &\geq v(r(x) - c(\theta_{d_1, a, \bar{c}} + \varepsilon) - v) & x \in \Omega.
\end{align*}
\]
Since (46) hold, by standard comparison theorem and Lemma 3.4, there exists \( t_2(\eta) > t_1(\eta) \) such that
\[
u(x, t) \geq \theta_{d_1, a, \bar{c}}(\theta_{d_2, r} + \varepsilon) - \varepsilon, \quad v(x, t) \geq \theta_{d_2, r} - c(\theta_{d_1, a, \bar{c}} + \varepsilon) - \varepsilon
\]
for all \( t > t_2(\eta) \) and \( x \in \Omega \). Since the above inequality holds for all sufficiently small \( \varepsilon > 0 \), thus, we have that the solution \((u(\cdot, t), v(\cdot, t))\) of (37) satisfies \( \theta_{d_1, m_1} \leq u(\cdot, t) \leq \theta_{d_1, a, \bar{c}} \) and \( \theta_{d_2, m_2} \leq v(\cdot, t) \leq \theta_{d_2, r} \) as \( t \to \infty \), and hence we also obtain the conclusion (45). Thus, we obtain the following results.

**Theorem 5.3.** Suppose that \( \mu_1(d_1, m_1) < 0 \) and \( \mu_1(d_2, m_2) < 0 \), where \( m_1 \) and \( m_2 \) are given in (42). Let \((\bar{u}, \bar{v})\) and \((\underline{u}, \underline{v})\) be the positive quasi-solutions which satisfy (43) and (44), and \((u, v)\) be a solution of (37) with nonzero initial function \( \eta = (\eta_1, \eta_2) \in C([-\tau, 0] \times \Omega, \mathbb{R}) \times C(\Omega, \mathbb{R}) \). Then \((u, v)\) possesses the property (45).

Theorem 5.3 implies that when \( \mu_1(d_1, m_1) < 0 \) and \( \mu_1(d_2, m_2) < 0 \) the solution of system (37) enters an attracting region \((\bar{U}, \bar{U})\) for some nonnegative initial functions. That is to say, the trivial solution \((0, 0)\) and the two semi-trivial solutions \((\theta_{d_1, a, \bar{c}}, 0)\) and \((0, \theta_{d_2, r})\) are all unstable. This property implies that system (37) is uniformly persistent and permanent.

**Remark 5.** From the construction of upper and lower solutions, we need \( \mu_1(d_1, \alpha(x) e^{-\gamma t} - a \theta_{d_2, r}) < 0 \) and \( \mu_1(d_2, r(x) - c \theta_{d_1, a, \bar{c}}) < 0 \). If
\[
\int_\Omega \left[ \alpha(x) e^{-\gamma t} - a \theta_{d_2, r} \right] dx \geq 0, \quad \int_\Omega \left[ r(x) - c \theta_{d_1, a, \bar{c}} \right] dx \geq 0,
\]
then by Proposition 1(i), \( \mu_1(d_1, \alpha(x) e^{-\gamma t} - a \theta_{d_2, r}) < 0 \) and \( \mu_1(d_2, r(x) - c \theta_{d_1, a, \bar{c}}) < 0 \) for all \( d_1, d_2 > 0 \). It is easy to see that the two inequalities of (47) are equivalent to
\[
e^{-\gamma t} \leq \frac{\bar{\alpha}}{a r} \quad \text{and} \quad e^{-\gamma t} \geq \frac{\bar{c} \bar{\alpha}}{r}
\]
respectively. If \( ac \leq 1 \), note that
\[
\frac{c \inf_\Omega \alpha(x)}{(1 + ac) \sup_\Omega r(x)} < \frac{c \inf_\Omega \alpha(x)}{\sup_\Omega r(x)} \leq \frac{\bar{c} \bar{\alpha}}{r} \leq \frac{(1 + ac) \sup_\Omega \alpha(x)}{a \inf_\Omega r(x)}.
\]
Thus, when \( \tau \in (\tau_*, \tau^*) \), it is possible that the solution of system (37) enters an attracting region, moreover, if \( \lim_{t \to \infty} u(\cdot, t) = \bar{u} = \underline{u} = u^* \) and \( \lim_{t \to \infty} v(\cdot, t) = \bar{v} = \underline{v} = v^* \), then system (7) may have a co-existence steady state solution. This observation is consistent with the result the weak competition Lotka-Volterra system which has been discussed in [16].
Finally, we take $\alpha(x) = 3 + \sin x, r(x) = 1 + \sin x, a = 0.5, c = 1.9, d_1 = 1, d_2 = 2, \gamma = 1$ and give some numerical simulations to illustrate our result. We can figure out $\tau_* = 0.3795$, $\tau^* = 2.7408$, $\ln \frac{\alpha}{a \bar{r}} \approx 1.4991$ and $\ln \frac{c \alpha}{\bar{r}} \approx 1.4403$. We consider the solution of model (7) with the following initial condition
\[ u(x, t) = \frac{1}{2} \sin^2 x \quad \text{and} \quad v(x, t) = \frac{1}{2} \sin^2 x \quad \text{for all} \quad -\tau < t \leq 0. \] 

Figures 1 and 2 show the global stability of the semi-trivial solution of $(\theta_{d_1, a, 0}, 0)$ and $(0, \theta_{d_2, r})$, respectively. Figure 3 shows that the existence of a positive spatially nonhomogeneous steady-state solution and illustrates that this steady-state solution is stable.

**Figure 1.** Solutions of model (7) with $\tau = 0.3 < \tau_*$ tend to the semi-trivial steady state solution $(\theta_{d_1, a, 0}, 0)$.

**Figure 2.** Solutions of model (7) with $\tau = 3 > \tau^*$ tend to the semi-trivial steady state solution $(0, \theta_{d_2, r})$. 
Figure 3. Solutions of model (7) with $\tau = 1.485 \in (\ln \bar{c}/\bar{F}, \ln \bar{\alpha}/\bar{r})$ tend to a positive steady state.

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