ABSTRACT. Let $k$ be a totally real field, and let $A/k$ be an absolutely irreducible, polarized Abelian variety of odd, prime dimension whose endomorphism rings is non-trivial and is defined over $k$. Then the only strictly compatible families of abstract, absolutely irreducible representations of $\text{Gal}(\overline{k}/k)$ coming from $A$ are tensor products of Tate twists of symmetric powers of two-dimensional $\lambda$-adic representations plus field automorphisms. The main ingredients of the proofs are the work of Borel and Tits on the ‘abstract’ homomorphisms of almost simple algebraic groups, plus the work of Shimura on the fields of moduli of Abelian varieties.

1. Introduction

Let $A$ be an absolutely irreducible Abelian threefold over a number field $k$. Suppose that the endomorphisms of $A$ are all defined over $k$. Let $l$ be a prime, and write $T_l(A)$ for the $l$-adic Tate module of $A$. Let $L/\mathbb{Q}_l$ be a finite extension, set $G_k := \text{Gal}(\overline{k}/k)$, and let $\rho : G_k \to GL_3(L)$ be an abstract (i.e. not necessarily continuous) group homomorphism which is a quotient of the $l$-adic representation $G_k \to \text{Aut}(T_l(A))$ by an abstract homomorphism $\text{Aut}(T_l(A)) \to GL_3(L)$. A special case of our main results (Theorems 2 and 3) says that if $l$ is sufficiently large and if $k$ is totally real, then $\rho$ is the Tate twist of the symmetric-square of an $l$-adic representation $G_k \to GL_2(L)$ plus a field automorphism $L \to L$. To put this result in perspective, we first recall how $l$-adic representations arise in the Langlands program.

Denote by $\mathbb{A}_k$ the adeles of a number field $k$. The Langlands program predicts that algebraic cuspidal automorphic forms on $GL_n(\mathbb{A}_k)$ correspond to motives, or, more concretely, compatible families of $n$-dimension $l$-adic representations coming from étale cohomology of algebraic varieties defined over $k$. For holomorphic eigenforms on $GL_2(\mathbb{A}_\mathbb{Q})$ this conjecture follows from the work of Eichler-Shimura, Deligne, and Deligne-Serre, by relating the eigenforms to the cohomology of certain Kuga fiber varieties. This has been extended to holomorphic $GL_2$-eigenforms over totally real fields, by Blasius-Rogawski \[5\] and Taylor \[27\]. When the automorphic form corresponds to a selfdual cuspidal representation, Clozel establishes the existence of these $l$-adic representations under mild conditions \[8, p. 150\]. Beyond that little is known for non-selfdual representations for $n \geq 3$.

Clozel raises the question of whether we can collect numerical evidence to support the Langlands correspondence for non-selfdual representations. Since then several authors have carried out numerical studies in the smallest open case $n = 3$ (\[1\], \[3\], \[11\], \[12\]). Such investigations have two parts. On the automorphic side, there is a well-known connection between cuspidal automorphic forms on $GL_3(\mathbb{A}_\mathbb{Q})$ and cuspidal cohomology of arithmetic
subgroup of $SL_3(\mathbb{Z})$ (cf. [17] for an exposition). Thus automorphic forms on $GL(3)$ can be systematically enumerated, at least for small levels, via Ash’s modular symbol algorithm on $GL(3)$ [2]. On the Galois side, however, there is no known algorithm that generates all irreducible 3-dimensional $l$-adic representations. Examples in the literature come from either ingenious but ad hoc geometric constructions ([11], [12]), or, in the case of mod-$l$ representations, number fields with special Galois groups ([4], [3]).

A familiar source of compatible families of $l$-adic representations comes from the Tate modules of Abelian varieties. It is natural to try to extract from them non-selfdual representations. To be specific, denote by $\mathcal{F}_\lambda$ the completion of a number field $\mathcal{F}$ at a finite place $\lambda$ of residual characteristic $l$. Suppose an abstract homomorphism $\rho_\lambda : G_k \to GL_N(\mathcal{F}_\lambda)$ comes from $A/k$, i.e. there exists an abstract homomorphism $\iota_\lambda : \text{Aut}(T_l(A)) \to GL_N(\mathcal{F}_\lambda)$ such that, along with the canonical inclusion $j_\lambda : GL_N(\mathcal{F}_\lambda) \hookrightarrow GL_N(\overline{\mathcal{F}_\lambda})$,

\begin{equation}
\begin{array}{cccc}
G_k & \rho_{A,l} & \text{Aut}(T_l(A)) & \subset \text{Aut}(V_l(A)) \\
& j_\lambda & & j_\lambda \\
GL_N(\mathcal{F}_\lambda) & & GL_N(\overline{\mathcal{F}_\lambda})
\end{array}
\end{equation}

Suppose $A/k$ is an Abelian threefold such that $\text{End}(A/k) \otimes \mathbb{Q}$ contains a quadratic field. Then $V_l(A)$ splits into a direct sum of two conjugate, odd-dimensional Galois representations. Furthermore, if $\text{End}(A/\mathbb{F}) \otimes \mathbb{Q}$ is exactly a quadratic field, by the Mumford-Tate conjecture [19] we expect the Lie algebra of the image of these conjugate representations to contain $\mathfrak{sl}_3(\mathbb{Z}_l)$. The Jacobian of a generic Picard quartic $y^3 = x^4 + ax^3 + bx^2 + cx + d$ over $k$ would satisfy the required conditions, thereby yielding interesting irreducible, non-selfdual 3-dimensional $l$-adic representations, provided that $k$ contains a primitive third-root of unity. One important consequence of the result stated in the first paragraph is that such examples cannot exist if $k$ is not totally real, even if we do not require that the representation be continuous.

The proofs of our main results have two parts. As a preliminary step, we utilize Shimura’s work on the fields of moduli of Abelian varieties with PEL structures to limit the possibilities for the endomorphism algebras of odd-dimensional Abelian varieties over a totally real field.

**Theorem 1.** Let $A$ be an absolutely simple, polarized Abelian variety of odd dimension $d$ defined over a totally real number field $k$. Suppose that the endomorphisms of $A$ are all defined over $k$. Then $E := \text{End}(A/k) \otimes \mathbb{Q}$ is a totally real number field of degree dividing $d$.

With $A/k$ as in Theorem [1], set $[E : \mathbb{Q}] = g$ and $\delta = d/g$. Since the action of $E$ on $V_l(A)$ commutes that of $G_k$, we have a direct sum decomposition

\begin{equation}
\rho_{A,l} \cong \tau_{l,1} \oplus \cdots \oplus \tau_{l,g},
\end{equation}

1we will explain shortly will we want to work with not-necessarily continuous maps.
where each \( r_{l,m} \) is a representation \( G_k \to GL_{2l}(E_{l,m}) \) for some completion \( E_{l,m} \) of \( E \) at a place above \( l \). Denote by
\[
\pi_{l,m} : \text{Aut}(T_l(A)) \to GL_{2l}(E_{l,m})
\]
the corresponding quotient. If \( \rho_\lambda \) is irreducible, it must factor through one of the \( r_{l,m} \). If \( \iota_\lambda \) is further assumed to be \( l \)-adic analytic, standard Lie theory then implies that \( \iota_\lambda \) is the composition of some \( \pi_{l,m} \) with a rational representation \( GL_{2l}(E_{l,m}) \to GL_N(T_\lambda) \). However, if we start out with a \( \lambda \)-adic representations that arise in arithmetic geometry, we only know that \( \rho_\lambda \) is continuous. So the additional hypothesis that \( \rho_\lambda \) comes from an Abelian variety does not automatically imply that \( \iota_\lambda \) is continuous, let alone being \( \lambda \)-adic analytic.

To address this issue, we invoke a deep result of Borel and Tits [6] which asserts that, over an infinite field \( K \), the ‘abstract’ homomorphisms of an almost simple matrix group \( G \) over \( K \) into \( PGL_n(K) \) are tensor products of the rational projective representations of \( G \) plus field automorphisms; additional work is required to handle ordinary representations. Before we state our main results we introduce some notation.

Let \( V \) be a two-dimension vector space over a field \( K \). Let \( \varphi : K \to \overline{K} \) be a field homomorphism. For any integer \( n \geq 1 \), denote by \( \sigma_\varphi^{(n)} \circ \varphi \) the \( n \)-th symmetric power of \( \varphi \) twisted standard representation \( GL_2(K) \to GL_2(\varphi(K)) \subset GL_2(\overline{K}) \). This is a \((n+1)\)-dimensional representation.

**Theorem 2.** With the notation as in Theorem 4, suppose further that there exists an abstract, absolutely irreducible representation \( \rho_\lambda : G_k \to GL_N(F_\lambda) \) coming from \( A \). Suppose \( l = \text{char}(\lambda) \) is sufficiently large.

(a) If \( N < 2d \) and \( \rho_\lambda \) is non-Abelian then \( [E : Q] > 1 \).

(b) Suppose \( d = [E : Q] \). Then there exist
- a choice \( r_{l,m} \) among the factors in the decomposition \( (\exists) \),
- positive integers \( r(\lambda) \) and \( n_1(\lambda), \ldots, n_r(\lambda) \) with \( \sum_{i=1}^{r(\lambda)}(n_i(\lambda) + 1) = N \),
- a field homomorphism \( \varphi_i(\lambda) : E_{l,m} \to \overline{E_{l,m}} \) for each \( 1 \leq i \leq r(\lambda) \), and
- a homomorphism of multiplicative groups \( f_\lambda : E_{l,m}^{\times} \to E_{l,m}^{\times} \),

such that, in the notation of diagram (1),
\[
\iota_\lambda = \left( \bigotimes_{i=1}^{\pi(r(\lambda))} \sigma_\varphi^{(n_i(\lambda))} \circ \varphi_i(\lambda) \right) \otimes (f_\lambda \circ \text{det}) \circ r_{l,m}.
\]

**Remark 1.** (a) Set \( d = N = 3 \) and we recover the result stated in the first paragraph.

(b) Our technique applies mutatis mutandis when \( [E : Q] < d \), once we replace the symmetric power representations of \( GL_2 \) by the Weyl modules associated to \( GL_n \) [10, §15.5]. The same goes for Theorem 2 below. In addition, Theorems 4 and 3 remain true if we replace \( \lambda \)-adic representations by mod-\( \lambda \) representations.

In practice, continuous \( \lambda \)-adic representations often arise in a compatible family. Our next result says that if \( \{\rho_\lambda\}_\lambda \) is a compatible family coming from an Abelian variety \( A/k \) as in Theorem 1, then the parameters \( r(\lambda), n_i(\lambda), f_\lambda \) and \( \varphi_i(\lambda) \) in Theorem 2 are essentially

\(^2\text{i.e. a morphism of algebraic groups.}\)
independent of \( \lambda \). To set up the notation we first we review the definition of compatible families of representations.

**Definition.** A strictly compatible family of \( \lambda \)-adic rational representations of \( G_k \) consists of

- a number field \( F/\mathbb{Q} \),
- a collection \( \{ \rho_\lambda : G_k \to GL_n(F_\lambda) \} \) of continuous Galois representations for each finite place \( \lambda \) of \( F \),
- a finite set \( S \) of finite places of \( \mathcal{O}_k \),

such that, with \( \Phi_\lambda : F \to F_\lambda \) denoting the canonical embedding,

- for any prime \( p \not\in S \cup \{ \lambda \} \), the coefficients of the characteristic polynomial \( \text{char}(\rho_\lambda, p) \) of \( \rho_\lambda(F_{\text{Frob}_p}) \) lies in \( \Phi_\lambda(F) \), and
- for any two distinct finite places \( \lambda, q \) and for any place \( p \not\in S \cup \{ \lambda, q \} \),

\[
\Phi_\lambda^{-1}(\text{char}(\rho_\lambda, p)) = \Phi_q^{-1}(\text{char}(\rho_q, p)).
\]

**Remark 2.** The phrase ‘rational representations’ refers to the condition that the image of each \( \rho_\lambda \) lies in \( \Phi_\lambda(F) \) for a fixed number field \( F \), rather than the whole \( F_\lambda \). In particular, this is different from the terminology in the theory of algebraic groups (cf. footnote 2).

**Theorem 3.** Let \( A/k \) be as in Theorem 2 with \( [E : \mathbb{Q}] = d \). Suppose \( \{ \rho_\lambda \} \) is a strictly compatible family of absolutely irreducible rational representations coming from \( A/k \). Following the notation in Theorem 2, for all but finitely many pairs \( (\lambda, q) \) of finite places of \( k \),

(i) \( r(\lambda) = r(q) \),
(ii) \( n_i(\lambda) = n_i(q) \) for all \( i \), up to reordering,
(iii) \( f_\lambda|_\mathbb{Z} = f_q|_\mathbb{Z} \), and
(iv) there exists a field isomorphism \( \xi_{\lambda, q} : \Phi_\lambda(E) \to \Phi_q(E) \) such that

\[
\xi_{\lambda, q}(\varphi_i(\lambda)|_{\Phi_\lambda(E)}) = (\varphi_i(q)|_{\Phi_q(E)}) \quad \text{for every } 1 \leq i \leq r(\lambda).
\]

**Remark 3.** We clarify the meaning of conditions (iii) and (iv). In the notation of Theorem 2 (and dropping the \( \lambda \) from \( r(\lambda) \) and \( n_i(\lambda) \)),

\[
i_\lambda = \left( \bigotimes_{i=1}^r \sigma_{\varphi_i(n_i)} \circ \varphi_i(\lambda) \right) \otimes (f_\lambda \circ \text{det}) \circ \tau_{l,m}.
\]

Since \( \Phi_\lambda(E) \subset E_\lambda \) is dense in \( E_\lambda \), for the purpose of computing \( i_\lambda \), condition (iv) completely determines \( \varphi_\lambda \). Similarly, for \( A/k \) as in Theorem 1, \( \text{det} \tau_{l,m} \) takes values in \( \mathbb{Z}_l - \{0\} \) and \( \mathbb{Z} \) is dense in \( \mathbb{Z}_l \), so condition (iii) completely determines \( f_\lambda \).

Since \( G_k \) is compact, if \( \rho_\lambda : G_k \to GL_N(F_\lambda) \) is continuous then its image is conjugate to a subgroup of \( GL_N(\mathcal{O}_\lambda) \), where \( \mathcal{O}_\lambda \) denotes the ring of integers of \( F_\lambda \) (cf. [18, Remark 1 on p. 1]). To say that \( \rho_\lambda \) comes from \( A/k \) is to say that the \( \mathcal{O}_\lambda[G_k] \)-module for \( \rho_\lambda \), after tensoring with \( \mathbb{Q} \), is a quotient of \( V_\iota(A) \). This suggests that we seek an integral form of our results, i.e. without taking the tensor product. This calls for an integral form of the work of Borel-Tits, i.e. a characterization of abstract homomorphisms of linear groups such as
Proposition 2 (Shimura \[23, \text{Prop. 1.6}\]). Let \(\mathbb{Q}_p\) denote \(p\)-adic numbers. Then \(\mathbb{Q}_p\) contains the number field generated by \(\mathbb{Q}\) equipped with the involution \(\cdot\). This is a very interesting question by itself, but little is known about the full results.

Theorem 4. For any integers \(n, N > 1\) there exists a constant \(c(n, N) > 0\) such that, for any prime \(l > c(n, N)\) and any two rings of integers \(\mathcal{O}\) and \(\mathcal{O}'\) of finite extensions of \(\mathbb{Q}_l\),

(a) every abstract homomorphism \(\varphi : \text{SL}_n(\mathcal{O}') \to \text{GL}_N(\mathcal{O})\) is continuous and \(\ker \varphi\) is contained in the subgroup of scalar matrices; and

(b) if \(N < n\), then there is no non-trivial abstract homomorphism \(\varphi : \text{SL}_n(\mathcal{O}') \to \text{GL}_N(\mathcal{O})\).

Part (a) remains true if we replace \(\text{SL}_n\) by \(\text{GL}_n, \text{Sp}_{2n}, \text{PSL}_n, \text{PGL}_n, \text{or} \text{PSp}_{2n}\). The same goes for Part (b), if for \(\text{GL}_n\) and \(\text{PGL}_n\) we require that \(\varphi\) be non-Abelian; and for \(\text{Sp}_{2n}\) and \(\text{PSp}_{2n}\), that \(N < 2n\).

2. Moduli of Abelian varieties

In this section we recall results of Shimura on Abelian varieties with prescribed endomorphism structures. Let \(\mathcal{A}\) be an Abelian variety over \(\mathbb{C}\). Let \(E\) be a semisimple algebra over a field \(k\). Let \(\theta : E \to \text{End}_\mathbb{C}(\mathcal{A}) \otimes \mathbb{Z} \mathbb{Q}\) be an isomorphism, and let \(\mathcal{C}\) be a polarization of \(\mathcal{A}\). Two such triples \((\mathcal{A}, \mathcal{C}, \theta)\) are said to be isomorphic to each other if there exists an isomorphism \(\mathcal{A} \xrightarrow{\lambda} \mathcal{A}'\) such that \(\lambda(X') \in \mathcal{C}\) for every divisor \(X' \in \mathcal{C}'\), and that \(\lambda \theta(\alpha) = \theta'(\alpha) \lambda\) for every \(\alpha \in E\).

Proposition 1 (Shimura \[23, \text{Prop. 5}\]). With \(\mathcal{P} = (\mathcal{A}, \mathcal{C}, \theta)\) as before, suppose further that \(\mathcal{A}\) is defined over a number field. Then there exists a unique subfield \(k/\mathbb{Q}\), called the field of moduli of \((\mathcal{A}, \mathcal{C}, \theta)\), which is characterized by the following property: For every automorphism \(\sigma\) of \(\mathcal{C}\), the triple \(\mathcal{P}^\sigma\) is isomorphic to \(\mathcal{P}\) if and only if \(\sigma|_k = \text{id}\).

Next, we recall Albert's work on the classification of endomorphism algebras of simple complex Abelian varieties. Let \(\mathcal{A}\) be a simple, polarized Abelian variety over the complex numbers. Then \(E := \text{End}(\mathcal{A}) \otimes \mathbb{Q}\) is a finite dimensional division algebra over \(\mathbb{Q}\) equipped
with the Rosati involution with respect to the polarization. Denote by $K$ the center of $E$, and by $K_0$ the fixed field of the Rosati involution restricted to $K$. Set

$$[E : K] = \delta^2, [K : Q] = e, [K_0 = Q] = e_0.$$ 

It is known that $\delta^2 e$ divides $2 \dim A$, and if $e = 2 \dim A$, then $K$ is a CM field [24, §2].

$E$ is said to be a division algebra of the first kind (resp. second kind) if the Rosati involution acts trivially on $K$ (resp. non-trivially). By the work of Albert, $E$ falls into one of four types, exactly three of which correspond to division algebras of the first kind. Moreover, we have the following restrictions on $e, e_0$ and $\delta$ ([14, Prop. 5.5.7]):

| End($A$) $\otimes Q$ | $\delta$ | $e_0$ | restriction |
|----------------------|--------|------|------------|
| totally real number field | 1      | $e$  | $e|\dim A$ |
| totally indefinite quaternion algebra | 2      | $e$  | $2e|\dim A$ |
| totally definite quaternion algebra | 2      | $e$  | $2e|\dim A$ |
| division algebra of the second kind | $\delta$ | $e/2$ | $e_0\delta^2|\dim A$ |

**Table 1.**

3. **Endomorphism algebras over totally real fields**

Let $(A, C, \theta)$ be an absolutely simple, polarized Abelian variety of type $\{E, \Phi, \rho\}$ over $k$ and of odd dimension $d$. Cf. Table 1 and we see that $E = \text{End}_{F}(A) \otimes Q$ is either a totally real number field of degree dividing $d$ or a division algebra of the second kind. To prove Theorem 1 we need to eliminate this second possibility when $k$ is totally real.

Suppose $E$ is a division algebra of the second kind. Following the notation in Table I, we have [22, §2]

$$E \otimes Q \mathbb{R} \simeq M_{\delta}(C) \oplus \cdots \oplus M_{\delta}(C),$$

where $M_{\delta}(C)$ denotes the algebra of $\delta \times \delta$ complex matrices. For $i = 1, \ldots, e_0$, denote by $\chi_i : E \otimes Q \mathbb{R} \rightarrow M_{\delta}(C)$ the projection onto the $i$-th factor. According to [22, §2], the inequivalent absolutely irreducible representations of $E$ are given by

$$\chi_1, \overline{\chi}_1, \ldots, \chi_g, \overline{\chi}_g,$$

where $\overline{\chi}_i$ denotes the complex conjugate of $\chi_i$. Moreover, for each $i$ there exist integers $r_i, s_i \geq 0$ satisfying

$$r_i + s_i = 2\delta(\dim A)/[E : Q]$$

$$= (\dim A)/(\delta e_0)$$

since $[E : Q] = [E : K][K : K_0][K_0 : Q] = \delta^2 \cdot 2 \cdot e_0$,

such that with

$$\Phi_i := (\chi_i \otimes I_{r_i}) \oplus (\overline{\chi}_i \otimes I_{s_i})$$

($I_j$ denotes the $j \times j$ identity matrix), the type $\{E, \Phi, \rho\}$ of $A/k$ satisfies [22, §2],

$$\Phi \simeq \Phi_1 \oplus \cdots \oplus \Phi_g.$$
Apply this to the case where \( \dim A \) is odd, it then follows from (4) that \( r_i + s_i \) is odd. In particular, \( r_i \neq s_i \) for every \( i \).

In light of the isomorphism (3), we can find \( \epsilon \in E \) such that

(i) \( \text{trace}(\chi_1(\epsilon)) \) is complex, and

(ii) the imaginary part of \( \chi_1(\epsilon) \) is large, but \( |\text{trace}(\chi_i(\epsilon))| \) is small for every \( i > 1 \).

Then

\[
\text{trace}(\Phi(\epsilon)) = \sum_i \text{trace}(\Phi_i(\epsilon)) \quad \text{by (6)}
\]

\[
= \sum_i [r_i \cdot \text{trace}(\chi_i(\epsilon)) + s_i \cdot \text{trace}(\chi_i(\epsilon))] \quad \text{by (3)}.
\]

Since \( r_1 \neq s_1 \), condition (i) implies that the term \( i = 1 \) in (7) is not real. Condition (ii) then implies that \( \text{trace}(\Phi(\epsilon)) \) is not real. By Proposition 3, the field of moduli of \((A, C, \theta)\) cannot be totally real. On the other hand, by Proposition 1, this field of moduli is contained in \( k \), which is totally real by hypothesis. This is a contradiction, and Theorem 1 follows.

4. ‘Abstract’ homomorphisms of \( \ell \)-adic groups

Let \( G \) be a linear algebraic group (i.e. a closed subgroup of \( GL_n \)) defined over a field \( K \). Fix an algebraic closure \( \overline{K} \) of \( K \). The group \( G \) is said to be almost simple (resp. absolutely almost simple) if it has no proper, normal subgroup over \( K \) (resp. over \( \overline{K} \)) of dimension \( > 0 \). It is said to be isotropic over \( K \) if it contains a non-zero, split \( K \)-torus. Both \( SL_n(K) \) and \( Sp_{2n}(K) \) are isotropic and are absolutely almost simple over \( \overline{K} \).

Let \( \varphi : K \rightarrow K' \) be a non-trivial field homomorphism. Denote by \( G_\varphi \) the base change of \( G \) to \( K' \) via \( \varphi \), i.e. the pullback of \( G \rightarrow \text{Spec } K \) via \( \varphi \):

\[
\begin{array}{ccc}
G_\varphi & \longrightarrow & G \\
\downarrow & & \downarrow \\
\text{Spec } K' & \longrightarrow & \text{Spec } K.
\end{array}
\]

If \( \rho : G \rightarrow PGL_n(K') \) is a rational representation, i.e. a morphism of algebraic groups, then we get a natural map \( \rho_\varphi : G_\varphi \rightarrow PGL_n(K') \).

The following special case of [8, Thm. 10.3] contains all the ingredients from the work of Borel-Tits for our subsequent applications.

**Theorem 5** (Borel-Tits). Let \( K \) be a field of characteristics zero, and let \( K' \) be an algebraically closed field. Let \( G \) be a connected linear algebraic group over \( K \) which is absolutely almost simple and is isotropic over \( K \). Let \( \rho : G(K) \rightarrow PGL_n(K') \) be an irreducible projective representation \((n \geq 2)\). Then there exists finitely many distinct, non-trivial field homomorphisms \( \varphi_i : K \rightarrow K' \), and for each \( i \), a non-trivial, irreducible rational projective representation \( \pi(i) \) of \( G \), such that

\[
\rho \simeq \otimes_i (\pi(i)_{\varphi_i} \circ \varphi_i).
\]

In particular, \( \text{char}(K') = 0 \). The pairs consisting of \( \varphi_i \) and the equivalence class of \( \pi_i \) are unique up to reordering.

The following fact is elementary.
Lemma 1. Let \(1 \rightarrow C \rightarrow G \rightarrow \tilde{G} \rightarrow 1\) be a central extension of groups. Given any group \(H\), any two lifts of a homomorphism \(H \rightarrow \tilde{G}\) differs by the multiple of a homomorphism \(H \rightarrow C\).

For the rest of this section, \(K\) denotes a field of characteristic zero. Let \(i \neq j\) be integer between 1 and \(n\). For any elements \(\alpha \in K^\times\) and \(\beta \in K\), define two matrices

\[
A_i(\alpha) = \text{the matrix obtained from the } 2 \times 2 \text{ identity matrix by replacing the } ii\text{-th entry by } \alpha,
\]

\[
B_{ij}(\beta) = \text{the matrix obtained from the } 2 \times 2 \text{ identity matrix by adding } \beta \text{ to the } ij\text{-th entry}.
\]

For \(i \neq j\) we have the identity

\[
(8) \quad A_i(\alpha)B_{ij}(\beta)A_i(\alpha)^{-1} = B_{ij}(\alpha^{\text{sgn}(j-i)}\beta).
\]

Proposition 3. Every abstract, absolutely irreducible representation \(GL_2(K) \rightarrow GL_d(K)\) is equivalent to

\[
[\otimes_{s=1}^r (\sigma_{\varphi_s}^{(n_s)} \circ \varphi_s)] \otimes (f \circ \det),
\]

where \(V\) is the underlying representation space associated to natural representation of \(GL_2(K)\); \(\det : GL_2(K) \rightarrow K^\times\) is the determinant map; and \(f : K^\times \rightarrow K^\times\) is a homomorphism of multiplicative groups.

Proof. Since \(\rho\) is absolutely irreducible, so are \(\rho|_{SL_2(K)}\) and

\[
\tilde{\rho}_S := \text{the projective representation } SL_2(K) \rightarrow PGL_d(K) \text{ associated to } \rho|_{SL_2(K)}.
\]

Define

\[
\sigma_{\varphi_s}^{(n_s)} := \text{the projective representation associated to } \sigma_{\varphi_s}^{(n_s)}.
\]

The absolutely irreducible rational projective representations of \(SL_2(K)\) are precisely the restrictions to \(SL_2(K)\) of the \(\sigma_{\varphi_s}^{(n_s)}\), so by Theorem III

\[
\tilde{\rho}_S \simeq \otimes_{s=1}^r [(\sigma_{\varphi_s}^{(n_s)} \circ \varphi_s)|_{SL_2(K)}],
\]

where \(n_s \geq 1\) and the \(\varphi_s : K \rightarrow K\) are field homomorphisms. Since \(SL_2(K)\) has no non-trivial Abelian quotients, by Lemma II there is a unique lift of \(\tilde{\rho}_S\) to \(\rho|_{SL_2(K)}\), whence

\[
(9) \quad \rho|_{SL_2(K)} \simeq \otimes_{s=1}^r [(\sigma_{\varphi_s}^{(n_s)} \circ \varphi_s)|_{SL_2(K)}].
\]

To determine \(\rho\) it then remains to study its action on the matrices \(A_i(\alpha)\) with \(\alpha \in K^\times\).

The identity (8) gives

\[
\sigma_\varphi^{(n)} B_{ij}(\varphi(\alpha^{\text{sgn}(j-i)}\beta)) = [\sigma_\varphi^{(n)} A_i(\varphi(\alpha))] [\sigma_\varphi^{(n)} B_{ij}(\varphi(\beta))] [\sigma_\varphi^{(n)} A_i(\varphi(\alpha))]^{-1},
\]

whence

\[
[\otimes_{s=1}^r \sigma_{\varphi_s}^{(n_s)} B_{ij}(\varphi_s(\alpha^{\text{sgn}(j-i)}\beta))] \equiv \left[\otimes_{s=1}^r \sigma_{\varphi_s}^{(n_s)} A_i(\varphi_s(\alpha))\right] \left[\otimes_{s=1}^r \sigma_{\varphi_s}^{(n_s)} B_{ij}(\varphi_s(\beta))\right] \left[\otimes_{s=1}^r \sigma_{\varphi_s}^{(n_s)} A_i(\varphi_s(\alpha))\right]^{-1}.
\]
In light of (9), this relation holds if we replace each $\otimes_{s=1}^{r} \sigma_{\varphi_s}^{(n_s)} B_{ij}(\varphi_s(-))$ by $\rho(B_{ij}(-))$. It follows that

$$\left[ \otimes_{s=1}^{r} \sigma_{\varphi_s}^{(n_s)} A_i(\varphi_s(\alpha)) \right] \left[ \rho(A_i(\alpha)) \right]^{-1}$$

commutes with every $B_{ij}(\varphi_s(\gamma))$. As $\gamma$ runs through all elements of $K$, the matrices $B_{ij}(\varphi_s(\gamma))$ generate $SL_2(\varphi_s(K))$ [4, Thm. 4.6]. Since $\rho|_{SL_2(K)}$ is absolutely irreducible, Schur’s Lemma implies that

$$\rho(A_i(\alpha)) = \left[ \otimes_{s=1}^{r} \sigma_{\varphi_s}^{(n_s)} A_i(\varphi_s(\alpha)) \right] f(\alpha)$$

for some element $\alpha \in \overline{K}^\times$. Since $\rho$ is a homomorphism, $f : K^\times \rightarrow \overline{K}^\times$ is in fact a homomorphism of multiplicative groups. Thus

$$\rho = \left[ \otimes_{s=1}^{r} \sigma_{\varphi_s}^{(n_s)} \circ \varphi_s \right] \otimes (f \circ \det)$$

when restricted to $SL_2(K)$ and to the subgroup of $GL_2(K)$ generated by $A_i(\alpha)$ for all the $\alpha \in K^\times$. These two subgroups together generate $GL_2(K)$, so we are done.

For every $s \in SL_2(K)$, the trace of $\sigma^{(n)}(s)$ is a $\mathbb{Z}$-polynomial of degree $n$ in trace($s$). Write

$$T_n(\text{trace}(s))$$

for this polynomial.

Lemma 2. Suppose there exists an infinite subset $\Sigma \subset SL_2(K)$ of elements with pairwise distinct traces, such that

$$\prod_{i=1}^{r} T_{n_i}(\text{trace}(s)) = \prod_{j=1}^{t} T_{m_j}(\text{trace}(s))$$

for every $s \in \Sigma$. Then $r = s$ and, up to reordering, $n_i = m_i$ for all $i$.

Proof. Denote by $\alpha$ and $1/\alpha$ the eigenvalues in $\overline{K}$ of $s \in \Sigma$. Write $\zeta_m$ for a fixed primitive $m$-th root of unity. Then

$$T_n(\text{trace}(s)) = \frac{\alpha^{n+1} - (1/\alpha)^{n+1}}{\alpha - (1/\alpha)} = \prod_{u=1}^{n}(\alpha - \zeta_{n+1}^u(1/\alpha)).$$

The equality (10) then becomes

$$\prod_{i=1}^{r} \prod_{u=1}^{n_i} (\alpha - \zeta_{1+n_i}^u(1/\alpha)) = \prod_{j=1}^{t} \prod_{v=1}^{m_j} (\alpha - \zeta_{1+m_j}^v(1/\alpha)).$$

Since this holds for infinitely many distinct $\alpha \in \overline{K}$, this becomes an equality of monic Laurent polynomials in the variable $\alpha$ over the field $L = K(\{\zeta_{1+n_i}, \zeta_{1+m_j})_{i,j}$. Since the ring of (one variable) Laurent polynomials over $L$ has unique factorization, we are done.
5. Odd-dimensional $l$-adic representations

Proof of Theorem 3. Suppose $E$ is a totally real field. We consider two cases.

Case: $[E : Q] = 1$, so $\text{End}_F(A) = \mathbb{Z}$.

Since $d$ is odd, Serre [20, 2.2.7 - 2.2.8] shows that the image of $\rho_{A,l}$ contains $Sp_{2d}(\mathbb{Z}_l)$ for all but finitely many $l$. Following the notation in [14], the restriction of the abstract homomorphism $\iota_{\lambda}$ to $Sp_{2d}(\mathbb{Q}_l)$ then gives rise to a non-trivial homomorphism $\iota'_{\lambda} : Sp_{2d}(\mathbb{Z}_l) \to GL_d(F_{\lambda})$ for some finite extension $F_{\lambda}/\mathbb{Q}_l$. Since $Sp_{2d}(\mathbb{Z}_l)$ is compact, the argument in [18] Remark 1 on p. 1] shows that $\iota'_{\lambda}$ is equivalent to a homomorphism with image in $GL_d(O_{F_{\lambda}})$. By Theorem 4(b), this is impossible for $l$ sufficiently large.

Case: $[E : Q] = d > 1$.

Suppose $\rho_{\lambda} : G_k \to GL_d(F_{\lambda})$ is an abstract, absolutely irreducible representation coming from $A/k$. Following the notation in [14] - (2), that means $\rho_{\lambda}$ is the composition of some $\iota_{l,m} : G_k \to GL_2(E_{l,m})$ with an abstract homomorphism $\psi : GL_2(E_{l,m}) \to G_d(O_{Q_i})$. Since $\rho_{\lambda}$ is absolutely irreducible, so does $\psi$. Apply Proposition 3 and we are done.

Proof of Theorem 3. Our goal is to understand the dependency of the parameters $r(\lambda), n_i(\lambda), f_\lambda$ and $\varphi_i(\lambda)$ on $\lambda$. Faltings shows that $\iota_{l,m}$ is semisimple, and that $E_{l,m}$ is the commutant of $\iota_{l,m}(G_k)$ in the $\text{End}(T_i(A))$. The argument in ([18, Thm. 4.5.4]) then shows that the Lie algebra of $\iota_{l,m}(G_k)$ is equal to $\{ u \in gl_2(E_{l,m}) : \text{trace}(u) \in Q_i \}$. Since $\iota_{l,m}$ is equivalent to a representation whose image lies in $GL_2(O_{E_{l,m}})$ (cf. [18, Remark 1 on p. 1]), it follows that

\begin{equation}
\text{for every } l, \ i, \ \iota_{l,m}(G_k) \text{ is (conjugate to) an open subgroup in } \{ g \in GL_2(O_{E_{l,m}}) : \det g \in \mathbb{Z}_l^\times \}.
\end{equation}

In particular, there exists an infinite subset $\{ g_i \} \subset G_k$ such that, for a fixed $l$, $\iota_{l,m}(g_i) \in SL_2(Z_l) \subset SL_2(O_{E_{l,m}})$ and with pairwise distinct traces. Since $\{ \iota_{l,m} \}_l$ is a strictly compatible family, that means $\tau_i := \text{trace}(\iota_{l,m}(g_i))$ lies in $\mathbb{Q}$ and is independent of $l$.

Let $\lambda, q$ be two distinct finite places of $F$ of residual characteristic $l$ and $q$ ($l = q$ is allowed). Following the notation in Lemma 2, we have

\begin{align*}
\Phi_{\lambda}^{-1} \left( \prod_{i=1}^{r(\lambda)} T_{n_i(\lambda)}(\tau_i) \right) &= \Phi_{\lambda}^{-1} \left( \text{trace}(\rho_{\lambda}(g_i)) \right) \\
&= \Phi_{q}^{-1} \left( \text{trace}(\rho_{q}(g_i)) \right) \quad \{ \rho_{\lambda}\}_l \text{ is a strictly compatible family} \\
&= \Phi_{q}^{-1} \left( \prod_{j=1}^{r(q)} T_{n_j(q)}(\tau_i) \right) \quad \text{and the Frobenius are dense in } G_k
\end{align*}

Since this holds for all $\tau_i$, by Lemma 2 that means $r(\lambda) = r(q)$ and, up to reordering, $n_i(\lambda) = n_i(q)$ for all $i$.

It remains to study the multiplicative homomorphism $f_\lambda : E_i^\times \to \mathbb{Q}_l^\times$. Since $\det \iota_{l,m}$ is the $l$-adic cyclotomic character [14], it is surjective onto $\mathbb{Z}_l^\times$ for all but finitely many $l$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_l$, we can find an infinite subset $\{ h_i \} \subset G_k$ and a prime $l$ such\footnote{In fact these traces lie in $\mathbb{Z}$, although we do not need it.}.

\begin{itemize}
  \item [\textbf{Proof of Theorem 3.}] Our goal is to understand the dependency of the parameters $r(\lambda), n_i(\lambda), f_\lambda$ and $\varphi_i(\lambda)$ on $\lambda$. Faltings shows that $\iota_{l,m}$ is semisimple, and that $E_{l,m}$ is the commutant of $\iota_{l,m}(G_k)$ in the $\text{End}(T_i(A))$. The argument in ([18, Thm. 4.5.4]) then shows that the Lie algebra of $\iota_{l,m}(G_k)$ is equal to $\{ u \in gl_2(E_{l,m}) : \text{trace}(u) \in Q_i \}$. Since $\iota_{l,m}$ is equivalent to a representation whose image lies in $GL_2(O_{E_{l,m}})$ (cf. [18, Remark 1 on p. 1]), it follows that
  \begin{equation}
  \text{for every } l, \ i, \ \iota_{l,m}(G_k) \text{ is (conjugate to) an open subgroup in } \{ g \in GL_2(O_{E_{l,m}}) : \det g \in \mathbb{Z}_l^\times \}.
  \end{equation}

In particular, there exists an infinite subset $\{ g_i \} \subset G_k$ such that, for a fixed $l$, $\iota_{l,m}(g_i) \in SL_2(Z_l) \subset SL_2(O_{E_{l,m}})$ and with pairwise distinct traces. Since $\{ \iota_{l,m} \}_l$ is a strictly compatible family, that means $\tau_i := \text{trace}(\iota_{l,m}(g_i))$ lies in $\mathbb{Q}$ and is independent of $l$.

Let $\lambda, q$ be two distinct finite places of $F$ of residual characteristic $l$ and $q$ ($l = q$ is allowed). Following the notation in Lemma 2, we have

\begin{align*}
\Phi_{\lambda}^{-1} \left( \prod_{i=1}^{r(\lambda)} T_{n_i(\lambda)}(\tau_i) \right) &= \Phi_{\lambda}^{-1} \left( \text{trace}(\rho_{\lambda}(g_i)) \right) \\
&= \Phi_{q}^{-1} \left( \text{trace}(\rho_{q}(g_i)) \right) \quad \{ \rho_{\lambda}\}_l \text{ is a strictly compatible family} \\
&= \Phi_{q}^{-1} \left( \prod_{j=1}^{r(q)} T_{n_j(q)}(\tau_i) \right) \quad \text{and the Frobenius are dense in } G_k
\end{align*}

Since this holds for all $\tau_i$, by Lemma 2 that means $r(\lambda) = r(q)$ and, up to reordering, $n_i(\lambda) = n_i(q)$ for all $i$.

It remains to study the multiplicative homomorphism $f_\lambda : E_i^\times \to \mathbb{Q}_l^\times$. Since $\det \iota_{l,m}$ is the $l$-adic cyclotomic character [14], it is surjective onto $\mathbb{Z}_l^\times$ for all but finitely many $l$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_l$, we can find an infinite subset $\{ h_i \} \subset G_k$ and a prime $l$ such\footnote{In fact these traces lie in $\mathbb{Z}$, although we do not need it.}.
Suppose there exists an injective map $g$ of some subgroup of $P$. Fix integers $m,n$ such that, for any prime $l > c_2(m,n,N)$, no subgroup of $GL_N(C)$ is isomorphic to $SL_n(Z/l^m), GL_n(Z/l^m), Sp_{2n}(Z/l^m), PSL_n(Z/l^m), PGL_n(Z/l^m)$, or $PSp_{2n}(Z/l^m)$.

Proof. Suppose there exists an injective map $\phi : SL_n(Z/l^m) \rightarrow GL_N(C)$. We can assume that $\phi$ is irreducible. Denote by $S$ the kernel of the reduction map $SL_n(Z/l^m) \rightarrow SL_n(Z/l)$. Every irreducible representation of a nilpotent group $P$ is induced from a degree one character of some subgroup of $P$ [1], Thm. 52.1, so the degree of every irreducible representation of the $l$-group $S$ is either 1 or is divisible by $l$. But $\deg(\phi|_S) = N$, so for $l > N$ every irreducible representation of $S$ appearing in $\phi|_S$ has degree 1. Thus $\phi(S)$ is Abelian. But $S$ is non-Abelian if $m > 2$, so $m \leq 2$. Lemma 3 covers the case $m = 1$, so it remains to consider the case $m = 2$. We first introduce some notation.

Let $H$ be a normal subgroup of a finite group $G$. For any character $\psi$ and any element $g \in G$, denote by $\psi^g$ the conjugate of $\psi$ by $g$:

$$\psi^g(h) := \psi(g^{-1}hg) \quad \text{for all } h \in H.$$ (12)

We now resume the proof of the Lemma. Take $m = 2$, so $S$ is Abelian. Clifford’s theory [1, §49] gives a decomposition

$$\phi|_S \simeq (\psi^{s_1} \oplus \cdots \oplus \psi^{s_r})^e,$$ (13)
where $e, r \geq 1$ are integers, $s_i \in SL_n(\mathbb{Z}/l^2)$, and $\psi$ is an irreducible representation of $S$. Moreover, $SL_n(\mathbb{Z}/l^2)$ permutes the $\psi^{g_i}$ transitively via (13). We claim that this action on the $\psi^{g_i}$, when restricted to $S$, is trivial. To see this, note that for any $h, s \in S$,

$$(\psi^{g_i})^s(h) = \psi(s^{-1}g_i^{-1}hg_is) = \psi(g_i^{-1}\sigma^{-1}h\sigma g_i) \text{ for some } \sigma \in S, \text{ since } S \text{ is normal}$$

$$= \psi(g_i^{-1}hg_i) \text{ since } S \text{ is Abelian}$$

$$= \psi^{g_i}(h).$$

This verifies the claim. As a result, $SL_n(\mathbb{Z}/l^2)/S \simeq SL_n(\mathbb{Z}/l)$ permutes the $\psi^{g_i}$ transitively. The left side of (13) has degree $N$, so $r \leq N$. Thus $SL_n(\mathbb{Z}/l)$ permutes transitively a set of size $r \leq N$. We have two cases:

- $r = 1$:
  Then $\phi|_{S}$ is the direct sum of copies of a fixed irreducible representation $\psi$ of $S$. Since $S$ is Abelian, that means $\deg \psi = 1$, whence $\phi|_{S}$ has cyclic image. This is impossible since $S$ is non-cyclic and $\phi$ is injective.

- $1 < r \leq N$:
  Then $SL_n(\mathbb{Z}/l)$ contains a proper subgroup of index $\leq N$. That means $SL_n(\mathbb{Z}/l)$ has a non-trivial permutation representation, and hence a non-trivial irreducible representation, of degree $\leq N$. This is impossible if $l$ is large enough, by Lemma 3 (note that $SL_n(\mathbb{Z}/l)$ has no non-trivial degree one character if $(n, l) \neq (2, 2)$).

This completes the proof of the Lemma for $SL_n$. The same argument applies to $PSL_n, PSp_{2n}$ and $PSp_{2m}$. Finally, if $GL_N(\mathbb{C})$ contains a subgroup isomorphic to $GL_n(\mathbb{Z}/l^m)$ or $PGL_n(\mathbb{Z}/l^m)$, then it contains one isomorphic to $SL_n(\mathbb{Z}/l^m)$ or $PSL_n(\mathbb{Z}/l^m)$, so we are done.

\[ \square \]

**Proposition 4.** Let $A$ be a local ring with maximal ideal $\lambda$. Suppose $\text{char}(A/\lambda) > 3$. Then a subgroup $H \subset GL_N(A)$ is normal if and only if there exists an ideal $a \subset A$ such that the image of $H$ in $GL_N(A/a)$ consists of the scalar matrices. The same holds for $SL_N, PSp_{2n}$, $PGL_N$, $PSL_N$ and $PSp_{2n}$.

**Proof.** For $GL_N(A)$ and $SL_N(A)$ see [13, p. 84 and p. 245]. For $Sp_{2d}(A)$ see [13, p. 210].

**Proof of Theorem 4.** Again we give the argument for $SL_n$. We can assume that $l > 3$. Denote by $\lambda$ and $\lambda'$ the maximal ideal of $O$ and $O'$, respectively.

(a) We can assume that $\varphi$ is not trivial. For $i > 0$, denote by $G_i$ the kernel of the reduction map $GL_N(O) \rightarrow GL_N(O/\lambda^i)$. Every quotient $G_i/G_{i+1}$ is a finite Abelian $l$-group, whence the same holds for $\varphi^{-1}(G_i)/\varphi^{-1}(G_{i+1})$.

**Lemma 5.** Every $\varphi^{-1}(G_i)$ has finite index in $SL_n(O')$.

**Proof.** Every $G_i$ is normal in $GL_N(O)$, so every $\varphi^{-1}(G_i)$ is normal in $SL_n(O')$. In light of Proposition 4, it suffices to show that none of the $\varphi^{-1}(G_i)$ is contained in the center of $SL_n(O')$. Suppose otherwise. Then $SL_n(O')/\varphi^{-1}(G_i)$ is infinite and injects into $GL_N(O)/G_i \simeq GL_N(O/\lambda^i)$, which is finite. This is a contradiction.

\[ \square \]
The $G_i$ form a basis of open neighborhoods of the identity element in $GL_N(O)$. Lemma 4 then implies that $\phi^{-1}(U)$ has finite index in $SL_n(O')$ for every sufficiently small open set $U \subseteq GL_N(O)$ containing the identity; and hence for all open sets $U \subseteq GL_N(O)$. That means $\phi$ is continuous.

Finally, suppose $\ker \phi$ is not contained in the subgroup of scalar matrices. In light of Proposition 4, for some $m \geq 1$ we then get an injective map of $SL_n(O'/\lambda^m)$ or $PSL_n(O'/\lambda^m)$ into $GL_N(O)$. Restrict this map to the subgroup $SL_n(Z/l^n)$ or $PSL_n(Z/l^n)$ and note that we have a (discontinuous) inclusion $GL_N(O) \subset GL_N(C)$, we get a contradiction, by Lemma 4. This completes the proof of Part (a).

(b) We claim that if such $\phi$ exists, then we can find an injective map of abstract groups

$$SL_n(F_l) \hookrightarrow GL_N(\overline{F_l}) \quad \text{or} \quad PSL_n(F_l) \hookrightarrow GL_N(\overline{F_l}).$$

By the work of Steinberg [26] it then follows that $N \geq n$.

For every integer $i \geq 1$, denote by $G_i$ the kernel of the reduction map $GL_N(O) \rightarrow GL_N(O/\lambda^i)$. Every quotient $G_i/G_{i+1}$ is a finite Abelian $l$-group, whence the same holds for the $\phi^{-1}(G_i)/\phi^{-1}(G_{i+1})$. We claim that

$$\phi^{-1}(G_1) \neq SL_n(O').$$

Suppose otherwise. Since $\ker(\phi)$ is a proper subgroup of $SL_n(O')$ and since $\cap G_i = \{I\}$, if (14) is false then there exists a smallest integer $j \geq 1$ such that $\phi^{-1}(G_{j+1}) \subseteq \phi^{-1}(G_j)$; necessarily $\phi^{-1}(G_j) = SL_n(O')$. On the other hand, every $\phi^{-1}(G_i)$ is normal in $SL_n(O')$. Since $l > 3$, Proposition 4 implies that

$$\phi^{-1}(G_{j+1}) = \ker(SL_n(O') \xrightarrow{\text{mod } \lambda^m} SL_n(O'/\lambda^m))$$

or

$$\phi^{-1}(G_{j+1}) = \ker(SL_n(O') \xrightarrow{\text{mod } \lambda^m} PSL_n(O'/\lambda^m))$$

for some $m \geq 1$. Then $\phi^{-1}(G_j)/\phi^{-1}(G_{j+1}) = SL_n(O')/\phi^{-1}(G_{j+1}) \simeq SL_n(O'/\lambda^m)$ or $PSL_n(O'/\lambda^m)$, which is not a finite Abelian $l$-group. This is a contradiction, so (14) must hold, and we get an injective map

$$SL_n(O')/\phi^{-1}(G_1) \hookrightarrow GL_N(O_L)/G_1 \simeq GL_N(O_L/\lambda).$$

By Proposition 4, we get an injective map

$$SL_n(O'/\lambda^m) \hookrightarrow GL_N(O_L/\lambda) \quad \text{or} \quad PSL_n(O'/\lambda^m) \hookrightarrow GL_N(O_L/\lambda).$$

for some $m \geq 1$. Every element of $GL_N(O_L/\lambda)$ either is semisimple or contains a non-trivial Jordan block (over $\overline{F}$). In the first case the element has order prime to $l$. In the second case, if $l > N$ then every non-semisimple element has order divisible by $l$ but not by $l^2$. On the other hand, since $l > 2$, both $SL_n(O'/\lambda^m)$ and $PSL_n(O'/\lambda^m)$ contain elements of order $l^m$ (coming from transvections). Since the maps in (13) are injective, that means $m = 1$. This completes the proof of Part (b).
Acknowledgement

My interest in odd-dimensional Galois representations was inspired by a lecture of Ash on the analog of Serre’s conjecture for $GL(3)$ modular representations. I am indebted to Humphreys for his help with the work of Borel-Tits, and to Ribet for his help concerning the image of $\lambda$-adic representations. I would like to thank Dettweiler for pointing out a mistake in an earlier draft. I also benefited from a useful conversation with Rosen.

References

[1] G. Allison, A. Ash and E. Avner, Galois representations, Hecke operators, and the mod-$p$ cohomology of $GL(3, \mathbb{Z})$ with twisted coefficients. Experiment. Math. 7 (1998) 361-390.
[2] A. Ash, D. Grayson and P. Green, Computations of cuspidal cohomology of congruence subgroups of $SL(3, \mathbb{Z})$. J. Number Theory 19 (1984) 412-436.
[3] A. Ash, R. Pinch and R. Taylor, An $A_4$ extension of $\mathbb{Q}$ attached to a nonselfdual automorphic form on $GL(3)$. Math. Ann. 291 (1991) 753-766.
[4] E. Artin, Geometric algebra. Interscience Publ., 1957.
[5] D. Blassius and J. Rogawski, Galois representations for Hilbert modular forms. Bull. Amer. Math. Soc. (N.S.) 21 (1989) 65-69.
[6] A. Borel and J. Tits, Homomorphismes ‘abstraits’ de groupes algébriques simples. Ann. Math. (2) 97 (1973) 499-571.
[7] W. C. Chi, $l$-adic and $\lambda$-adic representations associated to abelian varieties defined over number fields. Amer. J. Math. 114 (1992) 315-353.
[8] L. Clozel, Motifs et formes automorphes: applications du principe de fonctorialité, in: Automorphic forms, Shimura varieties, and $L$-functions, Vol. I (Ann Arbor, MI, 1988) 77-159. Academic Press, 1990.
[9] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras. Interscience Publ., 1962.
[10] W. Fulton and J. Harris, Representation theory. Springer-Verlag, 1991.
[11] B. van Geemen and J. Top, A non-selfdual automorphic representation of $GL_3$ and a Galois representation. Invent. Math. 117 (1992) 391-401.
[12] B. van Geemen and J. Top, Selfdual and non-selfdual 3-dimensional Galois representations. Compositio Math. 97 (1995) 51-70.
[13] V. Landazuri and G. M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups. J. Algebra 32 (1974) 418-443.
[14] H. Lange and C. Birkenhake, Complex Abelian varieties. Springer-Verlag, 1992.
[15] B. R. McDonald, Geometric algebra over local rings. M. Dekker, 1976.
[16] K. Ribet, Galois action on division points of Abelian varieties with real multiplications. Amer. J. Math. 98 (1976) 751-804.
[17] J. Schwermer, Cohomology of arithmetic groups, automorphic forms and $L$-functions, in: Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 1–29. LNM 1447, Springer-Verlag, 1990.
[18] J. P. Serre, Abelian $l$-adic representations and elliptic curves. Benjamin, New York, 1968.
[19] J. P. Serre, Représentations $l$-adiques, in: Algebraic number theory (Kyoto Internat. Sympos., Res. Inst. Math. Sci., Univ. Kyoto, Kyoto, 1976), 177-193.
[20] J. P. Serre, Algèbre et géométrie. Ann. Collège France 86 (1985/86) 95-100.
[21] G. Shimura, On the theory of automorphic functions. Ann. Math. (2) 70 (1959) 101-144.
[22] G. Shimura, On analytic families of polarized abelian varieties and automorphic functions. Ann. Math. (2) 78 (1963) 149-192.
[23] G. Shimura, On the field of definition for a field of automorphic functions. Ann. Math. (2) 80 (1964) 160-189.
[24] G. Shimura, Abelian varieties with complex multiplications and modular functions. Princeton Univ. Press, 1998.
[25] W. A. Simpson and J. S. Frame, The character tables for $SL(3, q), SU(3, q^2), PSL(3, q), PSU(3, q^2)$, \textit{Canad. J. Math.} \textbf{25} (1973) 486-494.

[26] R. Steinberg, Representations of algebraic groups. \textit{Nagoya Math. J.} \textbf{22} (1963) 33-56.

[27] R. Taylor, On Galois representations associated to Hilbert modular forms. \textit{Invent. Math.} \textbf{98} (1989) 265-280.

[28] H. Weyl, \textit{Classical groups}, 2nd. ed. Princeton University Press, 1946.

Department of Mathematics & Statistics, University of Massachusetts. Amherst, MA 01003-4515 USA

\textit{E-mail address: siman@math.umass.edu}