We show that in certain compactifications of \( \mathcal{M} \)-theory on eight-manifolds to three-dimensional Minkowski space-time the four-form field strength can have a non-vanishing expectation value, while an \( N = 2 \) supersymmetry is preserved. For these compactifications a warp factor for the metric has to be taken into account. This warp factor is non-trivial in three space-time dimensions due to Chern-Simons corrections to the five-brane Bianchi identity. While the original metric on the internal space is not Kähler, it can be conformally transformed to a metric that is Kähler and Ricci flat, so that the internal manifold has \( SU(4) \) holonomy.
1. Introduction

The duality symmetries between different string theories can be naturally understood from M-theory \[1\] [2] or its twelve-dimensional generalization, that has been called F-theory \[3\] [4]. M-theory contains membranes and fivebranes, which turn out to be dual in eleven dimensions. Membrane-fivebrane duality predicts the existence of a space-time correction to the eleven-dimensional supermembrane action \[5\]. Taking this anomaly into account, it is our goal in this paper to examine the conditions under which the ground state of M-theory can be supersymmetric and of the form \(M^3 \times K^8\), where \(M^3\) is three-dimensional Minkowski space-time and \(K^8\) is the internal eight-manifold. Compactifications of M-theory \[6\] and F-theory \[3\] on eight-manifolds are fascinating, since they may lead us to a way to understand the dynamics of \(N = 1\) supersymmetric field theories and string theories in \(D = 4\), supersymmetry breaking and to the solution of the cosmological constant problem along the lines proposed by Witten \[7\]. Our computation shows the existence of new vacua of M-theory having \(N = 2\) supersymmetry for which the four-form field strength can have a non-vanishing expectation value, while the three-dimensional cosmological constant vanishes.

2. M-Theory on Eight-Manifolds

The bosonic part of the action of the eleven-dimensional supergravity limit of M-theory is given by \[9\]:

\[
S_{11} = \frac{1}{2} \int d^{11}x \sqrt{-\hat{g}} \left[ \hat{R} - \frac{1}{2} \hat{F}_4 \wedge \ast \hat{F}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \right],
\]

(2.1)

where \(\hat{g}_{MN}\) is the space-time metric (the hat denotes eleven-dimensional quantities) and \(\hat{C}_3\) is a three-form with field strength \(\hat{F}_4 = d\hat{C}_3\). We have set the gravitational constant equal to one. The complete action is invariant under local supersymmetry transformations

\[
\delta \hat{e}^A_M = i \hat{\eta} \hat{\Gamma}^A \psi_M,
\]

\[
\delta \hat{C}_{MNP} = 3i \hat{\eta} \hat{\Gamma}_{[MNP]} \psi_P,
\]

\[
\delta \psi_M = \hat{\nabla}_M \hat{\eta} - \frac{1}{288} \left( \hat{\Gamma}^P_{MQRS} - 8 \hat{\delta}^P_M \hat{\Gamma}^{QRS} \right) \hat{F}_{PQRS} \hat{\eta},
\]

(2.2)

\[1\] For a field theory example see \[8\].
where $\hat{e}^A_M$ is the vielbein, $\psi_M$ is the gravitino, $\eta$ is an eleven-dimensional anticommuting Majorana spinor and $\hat{\nabla}_M$ denotes the covariant derivative involving the Christoffel connection as usual. Further notations and conventions will be given in the appendix. The field strength obeys the Bianchi identity:

$$d\hat{F}_4 = 0,$$

(2.3)

or in components $\partial_M \hat{F}_{PQRS} = 0$. This equation is metric independent. The field equation for $\hat{F}_4$ is:

$$d * \hat{F}_4 = -\frac{1}{2} \hat{F}_4^2,$$

(2.4)

or in components after dualizing

$$\hat{E}^{-1} \partial_M (\hat{E} \hat{F}^{MNPQ}) - \frac{1}{1152} \hat{e}^{NPQRSTU} \hat{F}_{RSTU} \hat{F}_{VWXYZ} = 0,$$

(2.5)

where $\hat{E} = \det \hat{e}^A_M$. The fivebrane soliton appears as a solution to the eleven-dimensional field equations and it couples to the dual seven-form field strength $\hat{F}_7 = *\hat{F}_4$. Equation (2.4) then becomes the Bianchi identity for the eleven-dimensional fivebrane.

This equation has in general gravitational Chern-Simons corrections associated to the sigma-model anomaly on the six-dimensional fivebrane worldvolume [5]. The corrected fivebrane Bianchi identity takes the form

$$d * \hat{F}_4 = -\frac{1}{2} \hat{F}_4^2 + (2\pi)^4 \beta X_8,$$

(2.6)

where $\beta$ is related to the fivebrane tension by $T_6 = 1/(2\pi)^3 \beta$. Henceforth we set $\beta = 1$. Since the gauge-fixed theory of the fivebrane is described by a chiral anti-self-dual tensor multiplet, the eight-form anomaly polynomial is expressed in terms of the Riemann tensor [10]

$$X_8 = \frac{1}{(2\pi)^4} \left( -\frac{1}{768} (\tr \hat{R}^2)^2 + \frac{1}{192} \tr \hat{R}^4 \right).$$

(2.7)

The anomaly leads to an additional term in the action (2.1)

$$\delta S_{11} = \frac{1}{2} \int \hat{C}_3 \wedge \left( -\frac{1}{768} (\tr \hat{R}^2)^2 + \frac{1}{192} \tr \hat{R}^4 \right).$$

(2.8)
The existence of this interaction can be verified by computing the one-point function of the two-form $B_{MN}$ in the type IIA string theory compactified on an eight-manifold [11]. The result of this calculation has no dilaton dependence, since this would spoil gauge invariance. It can therefore be extrapolated to eleven dimensions and it gives the previous answer.

A supersymmetric configuration is one that obeys for some Majorana spinor $\eta$ the conditions

$$
\delta_\eta \hat{e}^A_M = 0,
\delta_\eta \hat{C}_{MNP} = 0,
\delta_\eta \psi_M = 0. \tag{2.9}
$$

Since in the background the spinor $\psi_M$ vanishes, the first two of the above equations are satisfied, and only the gravitino equation remains to be solved

$$
\hat{\nabla}_M \eta - \frac{1}{288} \left( \hat{\Gamma}^P_{MQR} + 8 \delta^P_M \hat{\Gamma}^{QRS} \right) \hat{F}_{PQRS} \eta = 0. \tag{2.10}
$$

The most general ansatz for the metric that is consistent with maximal symmetry is\(^2\)

$$
\hat{g}_{MN}(x, y) = \Delta(y)^{-1} g_{MN}(x, y), \tag{2.11}
$$

where

$$
g_{MN}(x, y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}. \tag{2.12}
$$

Here $x$ are the three-dimensional external coordinates labeled by the indices $\mu, \nu, \ldots$ and $y$ the ones of the Euclidean eight-manifold labeled by $m, n, \ldots$. $\Delta(y)$ is a scalar function called the “warp factor”. We first would like to rewrite (2.10) in terms of $g_{MN}$. We can relate covariant derivatives with respect to conformally transformed metrics by using the formula:

$$
\hat{\nabla}_M \eta = \nabla_M \eta + \frac{1}{2} \Omega^{-1} \Gamma^N_M (\nabla_N \Omega) \eta, \tag{2.13}
$$

\(^2\) Compactifications of eleven-dimensional supergravity to $D = 4$ anti-de Sitter space with a warp factor but without the anomaly (2.8) have been considered before in [12]. In these theories the four-form field strength is proportional to the cosmological constant of the external space and vanishes therefore for compactifications to Minkowski space. Compactifications of type II superstring theories with a warp factor have also been discussed by [13]. For the heterotic string the warp factor is necessary in order to obtain solutions with torsion [14].
where \( \hat{g}_{MN} = \Omega^2 g_{MN} \). This gives the relation
\[
\hat{\nabla}_M \eta = \nabla_M \eta - \frac{1}{4} \Delta^{-1} \Gamma_M^N (\nabla_N \Delta) \eta.
\] (2.14)

Furthermore, \( \hat{\Gamma}_M \) matrices are related to \( \Gamma_M \) matrices as
\[
\hat{\Gamma}_M = \Delta^{-1/2} \Gamma_M \quad \text{and} \quad \hat{\Gamma}^M = \Delta^{1/2} \Gamma^M,
\] (2.15)

while \( \hat{F}_{MNPQ} \) will be kept fixed under the transformation (2.11). We then obtain for (2.10) in terms of \( g_{MN} \) the result:
\[
\nabla_M \eta - \frac{1}{4} \Gamma_M^N \partial_N (\log \Delta) \eta - \frac{1}{288} \Delta^{3/2} \left( \Gamma_M^{PQRS} - 8 \delta^P_M \Gamma^{QRS} \right) F_{PQRS} \eta = 0.
\] (2.16)

We make a decomposition of the gamma matrices that is appropriate to the \( 11 = 3 + 8 \) split, by taking
\[
\Gamma_\mu = \gamma_\mu \otimes \gamma_9,
\]
\[
\Gamma_m = 1 \otimes \gamma_m,
\] (2.17)

where \( \gamma_\mu \) and \( \gamma_m \) are the gamma matrices of \( M^3 \) and \( K^8 \) respectively and \( \gamma_9 \) is the eight-dimensional chirality operator, that satisfies \( \gamma_9^2 = 1 \) and anti-commutes with all the \( \gamma_m \)'s.

We decompose the eleven-dimensional spinor \( \eta \) as a sum of terms of the form
\[
\eta = \epsilon \otimes \xi,
\] (2.18)

where \( \epsilon \) is a three-dimensional anticommuting spinor, while \( \xi \) is a commuting eight-dimensional Majorana-Weyl spinor. Spinors of the form (2.18) that solve \( \delta_\eta \alpha = 0 \) for every field \( \alpha \), give unbroken supersymmetries. We shall be interested in compactifications having \( N = 2 \) supersymmetry in three dimensions for which two spinors on \( K^8 \) of the same chirality can be found. We can combine these real spinors into a complex spinor of a well defined chirality. Without loss of generality we will take the chirality to be positive. Compactifications for which spinors of the previous form can be found will, in general, have \( \int X^8 \neq 0 \).

In [15] it was shown that demanding the existence of a nowhere-vanishing eight-dimensional Majorana-Weyl spinor in the \( 8_c \) representation of \( SO(8) \) gives a relation between the Euler number \( \chi \) of the eight-manifold and the Pontryagin numbers, \( p_1 \) and \( p_2 \).
\[ p_1^2 - 4p_2 + 8\chi = 0. \]  

(2.19)

The Pontryagin numbers are obtained by integrating the first and second Pontryagin forms \[ P_1 = -\frac{1}{2} \text{tr} R^2 \quad \text{and} \quad P_2 = -\frac{1}{4} \text{tr} R^4 + \frac{1}{8} (\text{tr} R^2)^2, \]  

(2.20)

over \( K^8 \). Replacing the spinor field in the \( 8_c \) representation by a spinor in the \( 8_s \) representation of \( SO(8) \) corresponds to a change of sign in (2.19)

\[ p_1^2 - 4p_2 - 8\chi = 0. \]

(2.21)

Therefore if one asks for an \( 8_c \) and an \( 8_s \) nowhere-vanishing spinor field, one concludes that the Euler number of \( K^8 \) has to vanish \([13]\). However, it is also true that for every manifold having \(-8\chi = p_1^2 - 4p_2\) we can find another one which has \( 8\chi = p_1^2 - 4p_2 \), obtained by reversing the orientation of the original manifold. This corresponds to interchanging positive and negative chirality spinors.

Comparing (2.7) with (2.20) we observe that the anomaly polynomial \( X_8 \) is proportional to \( P_1^2 - 4P_2 \) and is therefore related to the Euler number of \( K^8 \)

\[ \int_{K^8} X_8 = -\frac{1}{4!(2\pi)^4} \chi, \]

(2.22)

which is a topological invariant. Finding nowhere-vanishing spinors of both chiralities as a solution of (2.16) thus implies that the integral of the anomaly polynomial (2.22) vanishes. Compactifications of eleven-dimensional supergravity on eight-manifolds of this type have been considered in \([16]\). For these compactifications no warp factor has been taken into account and the internal manifold is of the form \( K^2 \times K^6 \), where \( K^2 \) is a two-dimensional sphere or torus and \( K^6 \) is a six-dimensional Calabi-Yau manifold. They yield non-vanishing expectation values for the four-form field strength if the external space is anti-de Sitter and have an \( N = 4 \) supersymmetry in three dimensions. However, we shall see that the situation is rather different if the anomaly is taken into account. In this case we will find solutions that preserve an \( N = 2 \) supersymmetry if the external space is three-dimensional Minkowski space, while the four-form field strength gets a non-vanishing expectation value.
In compactifications with maximally symmetric three-dimensional space-time the non-vanishing components of $F_4$ are

$$F_{mnpq} \text{ arbitrary,}$$

$$F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} f_m,$$  \hspace{1cm} (2.23)

where $f_m$ is an arbitrary function that we will determine later on, as well as the explicit form of $F_{mnpq}$. $\epsilon_{\mu\nu\rho}$ is the completely anti-symmetric Levi-Civita tensor of $M^3$.

Consider now the $\mu$-component of the gravitino transformation law. Using (2.16), (2.17) and (2.23) we obtain

$$\delta \psi_\mu = \nabla_\mu \eta - \frac{1}{288} \Delta^{3/2} (\gamma_\mu \otimes \gamma_9 \gamma^{mnpq}) F_{mnpq} \eta + \frac{1}{6} \Delta^{3/2} (\gamma_\mu \otimes \gamma_m) f_m \eta - \frac{1}{4} \partial_n (\log \Delta) (\gamma_\mu \otimes \gamma_9 \gamma^n) \eta.$$  \hspace{1cm} (2.24)

The simplest way to satisfy the condition $\delta \psi_\mu = 0$ is to consider compactifications of $M$-theory to three-dimensional Minkowski space, so that we can find a spinor that satisfies:

$$\nabla_\mu \epsilon = 0.$$  \hspace{1cm} (2.25)

Since we assume the three-dimensional space to be maximally symmetric, the above condition implies that the external space is Minkowski.

Using (2.25) we get that (2.24) can be satisfied if we set

$$F_{mnpq} \gamma^{mnpq} \xi = 0,$$  \hspace{1cm} (2.26)

$$f_n = \partial_n \Delta^{-3/2}.$$  \hspace{1cm} (2.27)

The second equation gives the explicit solution for one of the non-vanishing components of $F_4$

$$F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} \partial_m \Delta^{-3/2},$$  \hspace{1cm} (2.28)

in terms of the warp factor.
Next we consider the $m$-component of the gravitino transformation law. Using the properties of the gamma-matrices of the appendix and equations (2.16), (2.23), (2.26) and (2.27) we obtain:

$$\delta \psi_m = \nabla_m \xi + \frac{1}{24} \Delta^{3/2} \gamma^{npq} F_{mnpq} \xi + \frac{1}{4} \partial_m (\log \Delta) \xi - \frac{3}{8} \partial_n (\log \Delta) \gamma^m \xi.$$  \quad (2.29)

It is now convenient to introduce transformed quantities:

$$\tilde{g}_{mn} = \Delta^{-3/2} g_{mn},$$
$$\tilde{\xi} = \Delta^{1/4} \xi,$$  \quad (2.30)

in terms of which the condition (2.10) takes the simple form

$$\tilde{\nabla}_m \tilde{\xi} + \frac{1}{24} \Delta^{-3/4} F_m \tilde{\xi} = 0,$$  \quad (2.31)

where we have introduced the notation $F_m = \tilde{\gamma}^{npq} F_{mnpq}$. The relation (2.31) guarantees the existence of a covariantly constant spinor

$$\tilde{\nabla}_m (\tilde{\xi}^\dagger \tilde{\xi}) = 0,$$  \quad (2.32)

and its norm can be chosen to be one

$$\tilde{\xi}^\dagger \tilde{\xi} = 1.$$  \quad (2.33)

For the components of $\tilde{\xi} = \tilde{\xi}_1 + i \tilde{\xi}_2$ we can choose

$$\tilde{\xi}_i^T \tilde{\xi}_i = \frac{1}{2} \quad \text{for} \quad i = 1, 2$$
$$\tilde{\xi}_i^T \tilde{\xi}_j = 0 \quad \text{for} \quad i \neq j.$$  \quad (2.34)

Now we would like to show that $K^8$ is a complex manifold. In terms of $\tilde{\xi}$, we can construct an almost complex structure

$$\tilde{J}_m^\dagger = i \tilde{\xi}^\dagger \gamma_m^\dagger \tilde{\xi},$$  \quad (2.35)

which is covariantly constant

$$\tilde{\nabla}_p \tilde{J}_m^\dagger = 0.$$  \quad (2.36)
This can be easily seen taking into account that \( \tilde{\xi}^1 \tilde{\gamma}^{a_1 \ldots a_n} \tilde{\xi} \) vanishes if \( n \) is odd, since the spinor involved is Weyl and \( \gamma_9 \) can be pulled through the expression. The tensor (2.33) has the property:

\[
\tilde{J}^n_m \tilde{J}^p_n = -\delta^p_m.
\] (2.37)

To do this computation it is convenient to use some formulas appearing in [17] [18] which are expressed in terms of a fourth-rank antisymmetric tensor

\[
\Omega^m_{i npq} = \xi^T_i \tilde{\gamma}^{mnpq} \xi^i
\] for \( i = 1, 2 \).
(2.38)

With the above normalization for the spinors it follows from [17] [18]

\[
\Omega^m_{i npq} \Omega^n_{i mnpq} = 84.
\] (2.39)

Furthermore using the Fierz rearrangement we can show that this tensor satisfies:

\[
\Omega^m_{i npq} \Omega^n_{j mnpq} = -12 \quad \text{for} \quad i \neq j.
\] (2.40)

\( \tilde{J}^n_m \) is a complex structure since the Nijenhuis tensor

\[
N^m_{mn} = \tilde{J}^q_m \tilde{J}^p_{[n;q]} - \tilde{J}^q_n \tilde{J}^p_{[m;q]},
\] (2.41)

vanishes. Equation (2.37) together with (2.41) imply that \( K^8 \) is a complex manifold.

We are then allowed to introduce complex coordinates as well as holomorphic and anti-holomorphic indices, which we will denote with \( a, b, \ldots \) and \( \bar{a}, \bar{b}, \ldots \) respectively. The metric \( \tilde{g}_{mn} \) is of type (1, 1) and it is related to the complex structure as follows

\[
\tilde{J}^a_{\bar{b}} = i \tilde{g}_{a\bar{b}}.
\] (2.42)

Since \( \tilde{J}^a_{\bar{b}} \) is covariantly constant, according to (2.36), it follows that \( K^8 \) is Kähler and \( \tilde{J}^a_{\bar{b}} \) is the Kähler form. From equation (2.42) it follows that \( \tilde{\gamma}^a_{\bar{a}} \) and \( \tilde{\gamma}^a_{\bar{a}} \) act as annihilation operators

\[
\tilde{\gamma}^a_{\bar{a}} \tilde{\xi} = \tilde{\gamma}^a_{\bar{a}} \tilde{\xi} = 0.
\] (2.43)

Next we would like to obtain the explicit form of the solution for the four-form field strength. Multiplying (2.31) with \( \tilde{\gamma}^a_{\bar{a}} \) and using (2.43) we obtain the condition

\[
F^a_{mnpq} \tilde{\gamma}^{a_{mnpq}} \tilde{\xi} = 0,
\] (2.44)
which is more restrictive than (2.26) and will allow us to obtain the solution for \( F_{mn pq} \).

All the components of the above expression must vanish separately. From the equation

\[
F_{abcd} \tilde{\gamma}^{abcd} \tilde{\xi} = 0, \tag{2.45}
\]

we obtain the solution

\[
F_{abcd} = 0. \tag{2.46}
\]

This can be easily seen by using (2.43) and the identity

\[
F_{abcd} = \frac{1}{384} F_{ef gh} \tilde{\xi}^\dagger \{ \tilde{\gamma}_{abcd}, \tilde{\gamma}^{ef gh} \} \tilde{\xi}, \tag{2.47}
\]

which follows from properties of gamma matrices of the appendix. By complex conjugation of (2.46) we get

\[
F_{\bar{a}\bar{b}\bar{c}\bar{d}} = 0. \tag{2.48}
\]

Similarly one gets from the equation

\[
F_{\bar{a}\bar{b}\bar{c}\bar{d}} \tilde{\gamma}^{\bar{a}\bar{b}\bar{c}\bar{d}} \tilde{\xi} = 0, \tag{2.49}
\]

the result

\[
F_{\bar{a}\bar{b}\bar{c}\bar{d}} = 0. \tag{2.50}
\]

By complex conjugation it follows

\[
F_{abcd} = 0. \tag{2.51}
\]

The vanishing of the remaining components of (2.44) can be written in the form

\[
F_{\bar{a}\bar{b}\bar{c}\bar{d}} \bar{J}^{\bar{c}\bar{d}} = 0. \tag{2.52}
\]

This expression reminds the Donaldson-Uhlenbeck-Yau equation [19] appearing in the heterotic string [14]. However, in this case the field strength is a four-index object instead of a two-index object and the “gauge group” is abelian instead of \( SU(N) \).

It is satisfying to see that equations (2.46), (2.48), (2.50), (2.51) and (2.52) represent a solution of the field equation (2.6). In fact, since we have derived these results from
supersymmetry it is natural to think that they will solve the field equation for $F_4$, which takes the form:

$$\hat{E}^{-1} \partial_m \left( \hat{E} F^{mnpq} \right) = \frac{1}{16} \epsilon^{npqrstu} \Delta^{-1} \partial_r \Delta \hat{F}_{stu}.$$  \hfill (2.53)

Here we have used that (2.8) is conformally invariant, so that the contribution of $X_8$ to the field equation (2.6) vanishes for this component of $F_4$. Choosing a basis in which the metric is diagonal and using the explicit form of the four-form field strength this equation can be transformed to

$$\partial_{[a} F_{b\bar{c}d\bar{e}]} = 0, \hfill (2.54)$$

which is nothing but the Bianchi identity (2.3). To summarize, the only non-vanishing components of $F_4$ are $F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} f^m$ and $F_{a\bar{b}c\bar{d}}$.

Taking this result into account and (2.52) it is easy to see that from expression (2.31) we get

$$\tilde{\nabla}_m \tilde{\xi} = 0, \hfill (2.55)$$

so that $K^8$ is Ricci flat. From this equation it follows $\tilde{R}_{mn} = 0$. Since we already showed that $K^8$ is Kähler, we conclude that the holonomy group is $SU(4)$ and that the internal manifold is a Calabi-Yau four-fold. These manifolds have vanishing first Chern class. The original metric appearing in (2.30) is not Kähler but conformal to the Kähler metric. These metrics are called “conformally Calabi-Yau” [14]. It is useful to recall at this point some properties of Calabi-Yau four-folds. Since the holonomy group is $SU(4)$ there are two covariantly constant spinors for a given chirality that come from the decomposition $8_c \rightarrow 6 \oplus 1 \oplus 1$ under the reduction of $SO(8)$ to $SU(4)$. The two singlets of this decomposition correspond to the two real covariantly constant spinors for a given chirality that we found.

We still need to find the explicit form of the solution to equation (2.52) on a Kähler space. Roughly as in [14] or [20], $F_{a\bar{b}c\bar{d}}$ can be written in terms of the harmonic four-forms $\omega^i_{(4)}$ on $K^8$

$$F = \sum_{i=1}^{h_{11}} \nu_i^{(4)} \omega^i_{(4)}. \hfill (2.56)$$
Here the $\nu^i$’s are constants, $h_{11}$ are the Hodge numbers and $F$ is a shorthand notation for the above component of $F_4$. There are several constraints on the $\nu^i$’s. The quantization of the magnetic charge implies that these constants should be integers.

A second constraint can be obtained from the Bianchi identity as follows. Inserting the solution (2.27) into the Bianchi identity (2.6) gives an equation for the warp factor

$$d \ast d \log \Delta = \frac{1}{3} F \wedge F - \frac{2}{3} (2\pi)^4 X_8. \tag{2.57}$$

Therefore, integrating the Bianchi identity over the eight-manifold we obtain a relation between the characteristic class represented by $\int F \wedge F$ and the Euler number

$$\int_{K^8} F \wedge F + \frac{1}{12} \chi = 0, \tag{2.58}$$

where we have used Stoke’s theorem and we have imposed the condition that $\ast F_4$ should be globally defined. For complex manifolds of real dimension eight there is a relation between the Euler number and the 4th Chern class $\chi = c_4(M)$ so that (2.58) can be written in the form

$$\int_{K^8} F \wedge F + \frac{1}{12} c_4(M) = 0. \tag{2.59}$$

Inserting (2.56) into (2.59) we obtain conditions on the constants $\nu^{(4)}_i$. Furthermore, the constraint (2.59) provides a topological restriction on the possible compactifications of $M$-theory to three dimensions.

At this point we have determined a complete solution to the supersymmetry transformations. Equation (2.36) and (2.55) state that the internal manifold is a Calabi-Yau four-fold, while the original metric (2.30) is non-Kähler but conformal to a Kähler metric. One of the non-vanishing components of $F_4$ has to satisfy (2.52) and can be expressed in terms of harmonic four-forms (2.56). The coefficients of this expansion should obey the constraint (2.59). Equation (2.57) is an equation for the warp factor. Finally, the warp factor determines the remaining non-vanishing component of $F_4$ (2.28).

3. Conclusion and Outlook

We have shown the existence of new vacua of $M$-theory compactified on an eight-manifold that preserve an $N = 2$ supersymmetry in $D = 3$. For these compactifications a
warp factor for the metric has been taken into account, which is non-trivial in three space-
time dimensions. Due to this fact, the four-form field strength acquires a non-vanishing
expectation value for compactifications to three-dimensional Minkowski space-time. This is
surprising and in constrast to the situation appearing in conventional compactifications of
eleven-dimensional supergravity, where the expectation value of the four-form field strength
has to vanish, if supersymmetry is unbroken [21].

While the original metric on the internal space is not Kähler, it can be conformally
transformed to a metric that is Kähler and Ricci flat, so that the internal manifold has
SU(4) holonomy.

This, of course, implies the existence of new vacua for the type IIA string theory in two
dimensions. A crucial ingredient to get this result was the existence of the anomaly in the
eleven-dimensional supermembrane action, which appears as a consequence of membrane-
fivebrane duality. Such an anomaly is not present in the type IIB string theory, and
a similar computation for compactifications on four- and six-manifolds gives a constant
warp factor, and vanishing expectation values for the field strength for compactifications
to Minkowski space [13] [22].

We have considered manifolds for which the holonomy group is SU(4). This leads,
as we have explained, to two covariantly constant spinors of a well defined chirality. One
could also consider compactifications on manifolds that admit only one covariantly con-
stant spinor. Examples of these manifolds are 8-manifolds with Spin(7) holonomy. The
corresponding spinor arises from the decomposition $8_c \to 7 \oplus 1$ [15]. Compactifications on
these manifolds have attracted recently some attention in connection to $\mathcal{F}$-Theory, and it
would be interesting to carry out a similar computation as the one presented herein.

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Appendix

Our notation and conventions are as follows

- The different types of indices that we use are:

\[
\begin{align*}
M, N, \ldots & \quad \text{are eleven-dimensional world indices}, \\
A, B, \ldots & \quad \text{are eleven-dimensional tangent space indices}, \\
m, n, \ldots & \quad \text{are real indices of the euclidean submanifold}, \\
a, b, \ldots & \quad \text{and } \bar{a}, \bar{b}, \ldots \quad \text{are complex indices of the euclidean submanifold}, \\
\mu, \nu, \ldots & \quad \text{are three-dimensional lorentzian indices},
\end{align*}
\]

We denote by \((x^0, x^1, x^2)\) the coordinates of the external space, while \((x^3, \ldots, x^{10})\) are the coordinates of the eight-manifold.

- \(\epsilon^{MNPQRSTUWX} \) denotes a tensor, rather than a tensor density, with

\[
\epsilon^{012\ldots10} = \hat{E},
\]

and analogously for the Levi-Civita tensors of \(M^3\) and \(K^8\) respectively.

- \(n\)-forms are defined with a \(1/n!\). For example:

\[
F = \frac{1}{4!} F_{mnpq} dx^m \wedge dx^n \wedge dx^p \wedge dx^q.\]

- The gamma-matrices \(\hat{\Gamma}^M\) are hermitian, for \(M = 1, \ldots, 10\) while \(\hat{\Gamma}_0\) is antihermitian. They satisfy:

\[
\{\hat{\Gamma}_M, \hat{\Gamma}_N\} = 2 \hat{g}_{MN},
\]

where \(\hat{g}_{MN}\) has the signature \((-,+\ldots,+\)\). \(\hat{\Gamma}_{M_1\ldots M_n}\) is the antisymmetrized product of gamma matrices:

\[
\hat{\Gamma}_{M_1\ldots M_n} = \hat{\Gamma}_{[M_1 \ldots \hat{\Gamma}_{M_n]}},
\]

where the square bracket implies a sum over \(n!\) terms with a \(1/n!\) prefactor.

- A representation of the \(d = 3\) gamma matrices is

\[
\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
they satisfy $\epsilon_{\mu\nu\rho} \gamma^{\nu\rho} = 2 \gamma_\mu$.

\[ [\nabla_m, \nabla_n] \xi = \frac{1}{4} R_{mnpq} \gamma^{pq} \xi, \]  

(3.7)

\[ [\gamma_m, \gamma^r] = 2 \gamma_m^r, \]
\[ \{\gamma_m, \gamma^r\} = 2 \delta_m^r, \]
\[ [\gamma_{mn}, \gamma^r] = 2 \gamma_{mn}^r, \]
\[ [\gamma_{mn}, \gamma^r] = -4 \delta^r_{[m} \gamma_{n]}, \]
\[ [\gamma_{mnp}, \gamma^r] = 2 \gamma_{mnp}^r, \]
\[ \{\gamma_{mnp}, \gamma^r\} = 6 \delta^r_{[m} \gamma_{np]}, \]
\[ \{\gamma_{mnpq}, \gamma^r\} = 2 \gamma_{mnpq}^r, \]
\[ [\gamma_{mnpq}, \gamma^r] = -8 \delta^r_{[m} \gamma_{npq]}, \]

(3.8)

\[ \text{The chirality operator is defined by} \]
\[ \gamma_9 = \frac{1}{8!} \epsilon_{mnprstu} \gamma^{mnprstu}. \]  

(3.9)

\[ \text{We use the Fierz identity} \]
\[ \chi \bar{\psi} \gamma_9 = \frac{1}{2[d/2]} \sum_{n=0}^{d} \frac{1}{n!} \Gamma^{c_n...c_1} \bar{\psi} \Gamma_{c_1...c_n} \chi. \]  

(3.10)

\[ \text{Our definition of Hodge } \ast \text{ is:} \]
\[ \ast(dx^{m_1} \wedge \ldots \wedge dx^{m_p}) = \frac{1}{(d-p)!} \epsilon^{m_1...m_p}_{m_{p+1}...m_d} dx^{m_{p+1}} \ldots \wedge dx^{m_d}. \]  

(3.11)
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