Generalized Poincaré Half-Planes

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Abstract

In this note, we give some generalisations of the classical Poincaré upper half-plane, which is the most popular model of hyperbolic plane geometry. For this, we replace the circular arcs by elliptical arcs with center on the $x$–axis, and foci on the $x$–axis or on the lines perpendicular to the $x$–axis at the center, in the upper half-plane. Thus, we obtain a class of generalized upper half-planes with infinite number of members.

Furthermore we show that every generalized Poincaré upper half-plane geometry is a neutral geometry satisfying the hyperbolic axiom. That is, it satisfies also all axioms of the Euclidean plane geometry except the parallelism.

Key Words: Metric, Hyperbolic geometry, Hyperbolic distance, Poincaré half-plane, Absolute geometry, Non-Euclidean geometries

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1 Introduction

The concept of upper half-plane is used to mean the Cartesian half-plane which consists of all points with positive ordinate. If half-lines which are perpendicular to the $x$–axis, and half-circles with center on the $x$–axis are defined as lines in the upper half-plane, one gets a model for the hyperbolic
plane. (For the other models of the hyperbolic plane geometries see [1], [4], [7], [8].) In this model, the hyperbolic length of an arbitrary curve $\gamma$ is defined by

$$\int_{\gamma} \frac{\sqrt{dx^2 + dy^2}}{y}$$

which reduces also a distance function and a metric known as Poincaré metric. The Euclidean half-plane together with Poincaré metric is generally known as the Poincaré half plane. (During the recent years many metric models have also been developed see [2], [3], [5], [6].)

In this note we define and examine some generalisations of the Poincaré Half-Plane.

2 Basic Concepts and Definitions

**Definition 2.1** Let $\mathcal{P}$ be the set of all Cartesian points on the upper half plane, that is

$$\mathcal{P} = \{(x, y) \mid x, y \in \mathbb{R}, y > 0\},$$

and let

$$\mathcal{L}_p = \{L_p \mid p \in \mathbb{R}\}$$

such that $L_p = \{(p, y) \in \mathcal{P}\}$;

that is, $L_p$ represent the Euclidean half line with the equation $x = p, y > 0$.

Let

$$\mathcal{L}_{kac} = \{L_{kac} \mid a, c, k \in \mathbb{R}, a > 0, k > 0, k \text{ constant}\}$$

such that

$$L_{kac} = \{(x, y) \in \mathcal{P} \mid (x - c)^2 + k^2 y^2 = a^2\}.$$ 

That is, $L_{kac}$ is an Euclidean half-ellipse with center $(c, 0)$ on the $x-$axis. Where positive real number $k$ is a given constant. $a$ represents the length of the semimajor axis or semiminor axis of the ellipse according as $k > 1$ or $0 < k < 1$. $a$ is always measured along the $x-$axis. If $b$ is the length of the other axis than $a = bk$.

Now, define

$$\mathcal{L} = \mathcal{L}_p \cup \mathcal{L}_{kac}.$$ 

Elements of $\mathcal{L}$ are called hyperbolic lines, shortly $h-$lines. Now consider the system

$$\mathbb{H}_k = \{\mathcal{P}, \mathcal{L}\}$$
which is called a **generalized Poincaré half plane**. Notice that, every positive real number $k$ determines a generalized Poincaré half plane. Thus, now we have a family of generalized Poincaré half-planes with infinite number of members. If $k = 1$ then $\mathbb{H}_1$ is the classical Poincaré half plane.

Notice that if $k > 1$ then $k$ is ratio of the length of semimajor axis to the length of semiminor axis of the ellipse. In this case the $x$–axis is the major axis.

If $0 < k < 1$ then $k$ is ratio of the length of semiminor axis to the length of semimajor axis of the ellipse. In this case the the major axis is perpendicular to the $x$–axis at $(c, 0)$.

**Proposition 2.1** There exist a unique $h$–line passing through two distinct points of $\mathcal{P}$ in $\mathbb{H}_k$.

**Proof:** Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2) \in \mathcal{P}$. If $x_1 = x_2$ then clearly $P_1P_2$ is the $h$–line $L_{x_1}$ with equation $x = x_1$, $y > 0$ by definition of $L_p$. And obviously, there is no $h$–line of type $L_{kac}$ passing through $P_1$ and $P_2$.

If $x_1 \neq x_2$ and $P_1, P_2 \in L_{kac}$ then

\[
(x_1 - c)^2 + k^2 y_1^2 = a^2
\]
\[
(x_2 - c)^2 + k^2 y_2^2 = a^2.
\]
Solving this system of equations for $a$ and $c$ one obtains

$$c = \frac{(x_1 - x_2)^2 + k^2(y_1^2 - y_2^2)}{2(x_1 - x_2)}$$

and

$$a = \left[ (x_1 - c)^2 + k^2y_1^2 \right]^{1/2}.$$

Since $k$ is constant there exist a unique pair $a$ and $c$ and consequently one obtains a unique $h$–line $L_{kac}$. Obviously there is no $h$–line of type $L_p$ having on such a pair of points.

**Corollary 2.1** Two hyperbolic lines meet at most one point in $H_k$.

Notice that although the $x$–axis is not in $H_k$, we can use the points on it in the definitions and calculations as follows:

**Definition 2.2** Every pair of two $h$–lines which are Euclidean half-lines are defined as parallel $h$–lines. Also, two different $h$–lines are called parallel iff their Euclidean extensions meet on the $x$–axis.

Thus, $L_p \parallel L_q$ for all $p \neq q$,

$L_p \parallel L_{kac} \iff L_p \cap L_{kac} \in \{(c - a, 0), (c + a, 0)\}$

$L_{kac} \parallel L_{kac'} \iff L_{kac} \cap L_{kac'} \in \{(c - a, 0), (c + a, 0)\}$.

**Proposition 2.2** $H_k$ satisfies the hyperbolic property, that is each $h$–line $L$ and each point $P \notin L$ there exist exactly two hyperbolic lines through $P$ and parallel to $L$.

**Proof:** Proof can be easily given using the definitions and proposition 2.1.

If one defines that two hyperbolic lines are parallel when they are disjoint, then, clearly, there exist infinitely many different hyperbolic lines through $P$ that are parallel to $L$.

### 3 Further Properties of $H_k$

As it is well known an incidence geometry is geometry $I$, consist of a set $P$, whose elements are called points, together with a collection $L$ of non-empty subsets of $P$, called lines, such that:
A1) Every two distinct points in $\mathcal{P}$ lies on a unique line,
A2) There exist three points in $\mathcal{P}$, which do not lie all on a line.

An incidence geometry is a metric geometry if
A3) There exists a distance function

$$d : \mathcal{P} \times \mathcal{P} \to \mathbb{R} \ni \text{ for all } P, Q \in \mathcal{P} \text{ such that}$$

i) $d(P, Q) \geq 0$; ii) $d(P, Q) = 0$ iff $P = Q$ and iii) $d(P, Q) = d(Q, P)$; and
A4) There exists a one-to-one, onto function $f : l \to \mathbb{R}$, $\forall l \in \mathcal{L}$ such that $|f(P) - f(Q)| = d(P, Q)$ for each pair of points $P$ and $Q$ on $l$ *(Ruler postulate)*.

Clearly, every $\mathbb{H}_k$ has an infinite number of points and lines and satisfies axioms of the incidence geometry.

Now the question is that whether the every $\mathbb{H}_k$ is a metric geometry or not. If it is a metric geometry what are its distance function and its ruler $f$.

It is known that if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points in the Poincaré plane, the distance function is given by

$$d(P, Q) = \begin{cases} 
\left| \ln \left( \frac{x_1 - c + r}{y_1} / \frac{x_2 - c + r}{y_2} \right) \right| & \text{if } x_1 \neq x_2 \\
\left| \ln \left( \frac{y_2}{y_1} \right) \right| & \text{if } x_1 = x_2.
\end{cases}$$

and the ruler $f$ is given by

$$f(x, y) = \begin{cases} 
\left| \ln \left( \frac{x - c + r}{y} \right) \right| & \text{if } (x, y) \in \mathcal{C} \\
\left| \ln \left( \frac{y}{y} \right) \right| & \text{if } (x, y) \in L_p.
\end{cases}$$

where $\mathcal{C}$ stands for the semi-circle

$$(x - c)^2 + y^2 = r^2, \ y > 0.$$ 

Now, we will use the above $d$ and $f$ to give a reasonable ruler and a distance function for $\mathbb{H}_k$. For this, consider the transformation

$$g : \mathcal{P} \to \mathcal{P} \ni g(x, y) = (x, ky)$$

which maps $L_{kac}$ to the above semicircle $\mathcal{C}$ and the line $L_p$ to itself one-to-one onto (see Fig.3).
Let us define distance function for $P_1 = (x_1, y_1)$ and $Q = (x_2, y_2)$ in $H_k$ as
\[
 d_k(P, Q) := d(g(P), g(Q)) = d((x_1, ky_1), (x_2, ky_2))
\]
\[
 = \begin{cases} 
 \left| \ln \left( \frac{x_1 - c + a}{ky_1} \right) / \left( \frac{x_2 - c + a}{ky_2} \right) \right| & \text{if } x_1 \neq x_2 \\
 \left| \ln \left( \frac{ky_2}{ky_1} \right) \right| & \text{if } x_1 = x_2
\end{cases}
\]
\[
 = \begin{cases} 
 \left| \ln \left( \frac{(x_1 - c + a)y_2}{(x_2 - c + a)y_1} \right) \right| & \text{if } x_1 \neq x_2 \\
 \left| \ln \left( \frac{y_2}{y_1} \right) \right| & \text{if } x_1 = x_2
\end{cases}
\]
\[
 = d(P, Q).
\]

And define the ruler $f_k$ as $f_k = f \circ g$, that is
\[
 f_k(x, y) = (f \circ g)(x, y) = f(g(x, y)) = f(x, ky)
\]
\[
 = \begin{cases} 
 \ln \left( \frac{x - c + a}{ky} \right) & \text{if } (x, y) \in L_{kac} \\
 \ln(ky) & \text{if } (x, y) \in L_p.
\end{cases}
\]

**Proposition 3.1** Every $(H_k, d, f_k)$ is a metric geometry.

**Proof:** Since $d_k = d$, the axioms i), ii) and iii) are satisfied.

To show that
\[
 f_k : L_{kac} \to \mathbb{R} \ni f_k(x, y) = \ln \left( \frac{x - c + a}{ky} \right)
\]
is one-to-one onto one must show that for every $t \in \mathbb{R}$ there is only one pair of $(x, y)$ which satisfies

$$(x - c)^2 + k^2 y^2 = a^2, \ y > 0$$

for $f(x, y) = t$.

If $f(x, y) = t$ then

$$f(x, y) = \ln \frac{x - c + a}{ky} = t \Rightarrow \frac{(x - c + a)}{ky} = e^t.$$

Thus,

$$e^{-t} = \frac{ky}{x - c + a} = \frac{ky}{x - c + a} \cdot \frac{x - c - a}{x - c - a} = \frac{ky(x - c - a)}{(x - c)^2 - a^2}$$

and

$$\frac{ky(x - c - a)}{-k^2 y^2} = \frac{x - c - a}{-ky}$$

and

$$e^t + e^{-t} = \frac{x - c + a}{ky} - \frac{x - c - a}{ky} = \frac{2a}{ky} \Rightarrow y = \frac{a}{k \cosh t},$$

and

$$e^t - e^{-t} = \frac{x - c + a}{ky} + \frac{x - c - a}{ky} = \frac{2(x - c)}{ky}.$$

Thus

$$\tanh t = \frac{2(x - c)}{ky} \cdot \frac{ky}{2a} = \frac{x - c}{a} \Rightarrow x = c + a \tanh t.$$

That is, the only possible solution to $f_k(x, y) = t$ is $x = c + a \tanh t$ and $y = a / k \cosh t$.

Similarly to show that

$$f_k : L_p \to \mathbb{R} \ni f_k(x, y) = \ln(ky)$$

is one-to-one onto, let $t \in \mathbb{R} \ni f_k(x, y) = t$. Then

$$\ln(ky) = t \Rightarrow ky = e^t \Rightarrow y = e^t / k$$

and

$$x \in L_p \Rightarrow x = p.$$

Consequently only solution is $(p, e^t / k)$ and $f_k$ is 1-1 onto.
Finaly, if \( x_1 \neq x_2 \) then \( P, Q \in L_{kac} \) and

\[
|f_k(P) - f_k(Q)| = \left| \ln \frac{x_1 - c + a}{ky_1} - \ln \frac{x_2 - c + a}{ky_2} \right| = \ln \left( \frac{x_1 - c + a}{y_1} \div \frac{x_2 - c + a}{y_2} \right) = \ln \left( \frac{x_1 - c + a}{y_1} \div \frac{x_2 - c + a}{y_2} \right) = d(P, Q)
\]

Thus \( f_k \) is a ruler for \( L_{kac} \).

If if \( x_1 = x_2 \) then \( P, Q \in L_p \) and

\[
|f_k(P) - f_k(Q)| = \left| \ln(ky_1) - \ln(ky_2) \right| = \left| \ln \frac{ky_1}{ky_2} \right| = \left| \ln \frac{y_1}{y_2} \right| = d(P, Q)
\]

and \( f_k \) is a ruler for \( L_p \).

Every distance on an incidence geometry doesn’t give a metric geometry. The ruler postulate is very strong condition to place an incidence geometry, which allows us to investigate further properties. If a metric geometry satisfies the plane separation axiom (PSA) below, then it is called Pasch Geometry.

**PSA.** For every line \( l \) in \( L \), there are two subsets \( H_1 \) and \( H_2 \) of \( \mathcal{P} \) (called half planes determined by \( l \)) such that

i) \( H_1 \cup H_2 = \mathcal{P} - l \) (\( \mathcal{P} \) with \( l \) removed)

ii) \( H_1 \) and \( H_2 \) are disjoint and each is convex,

iii) If \( A \in H_1 \) and \( B \in H_2 \), then \( AB \cap l \neq \emptyset \).

**Proposition 3.2** Every \((\mathcal{H}_k, d, f_k)\) satisfies plane separation axiom.

**Proof:** Let \( l \) be a \( h \)-line in \( \mathcal{H}_k \). Let \( H_1 \) and \( H_2 \) be the half planes determined by \( l \) such that

\[
H_1 = \left\{ (x, y) \in \mathcal{P} \mid x > p \right\} \quad \text{if} \quad l = L_p
\]

and

\[
H_1 = \left\{ (x, y) \in \mathcal{P} \mid (x - c)^2 + k^2y^2 > a^2 \right\} \quad \text{if} \quad l = L_{kac}
\]

\[
H_2 = \left\{ (x, y) \in \mathcal{P} \mid (x - c)^2 + k^2y^2 < a^2 \right\} \quad \text{if} \quad l = L_{p}
\]

It can be easily seen that \( H_1 \) and \( H_2 \) are disjoint and i) and iii) are satisfied. Furthermore convexity is a result of the fact that two \( h \)-lines meet at most one point in \( \mathcal{H}_k \) (see Fig 4).
It is very well known that a metric geometry satisfies PSA iff it satisfies Pasch Axiom: *A line which intersects one side of a triangle must intersect one of the other two sides.*

*Is $H_k$ a protractor geometry?*

Clearly, one can use Euclidean angle measure in a generalized Poincaré plane since it is a subset of the Euclidean plane and since its lines are defined in terms of Euclidean lines and ellipses. Here, the basic idea is to replace the elliptical rays that make up the angle by Euclidean rays that are tangents to the given elliptical rays. Thus without going into details we can give the following:

**Proposition 3.3** Every $(\mathbb{H}_k, d, f_k)$ with Euclidean angle measure is a protractor geometry.

A *neutral* (or absolute) geometry is a protractor geometry which satisfies Side-Angle-Side axiom (SAS). It is not difficult to deduce the result that *every $\mathbb{H}_k$ is a neutral geometry* if its angle measure is defined a similar way to that of the original Poincaré plane.

*Open Questions*

In this paper it has been shown that generalized upper half-plane with the Poincaré distance function gives a metric geometry $\mathbb{H}_k$. Is it possible to find a distance function distinct from that of the Poincaré distance for $\mathbb{H}_k$, using the hyperbolic distance of the elliptical arcs?

For this, one has to use the elliptic integral $\int \sqrt{k^2 + \cot^2 t} \, dt$, $k \neq 1$.

also, there are some problems that are worth studying. Think, what they are!

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