ON THE WELL-POSEDNESS OF THE INVISCID MULTI-LAYER QUASI-GEOSTROPHIC EQUATIONS

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Abstract. The inviscid multi-layer quasi-geostrophic equations are considered over an arbitrary bounded domain. The no-flux but non-homogeneous boundary conditions are imposed to accommodate the free fluctuations of the top and layer interfaces. Using the barotropic and baroclinic modes in the vertical direction, the elliptic system governing the streamfunctions and the potential vorticity is decomposed into a sequence of scalar elliptic boundary value problems, where the regularity theories from the two-dimensional case can be applied. With the initial potential vorticity being essentially bounded, the multi-layer quasi-equations are then shown to be globally well-posed, and the initial and boundary conditions are satisfied in the classical sense.

1. Introduction. At the mid-to-high latitudes, where the Coriolis parameter is away from zero, large-scale geophysical flows, namely the ocean and atmosphere, evolve around the so-called geostrophic balance, where the Coriolis force approximately counteract the horizontal pressure gradient. To the leading order, the flow is governed by the quasi-geostrophic (QG) equations. The QG equations take the form of a transport equation,

$$\frac{\partial}{\partial t} q + u \cdot \nabla q = F,$$

where $q$ represents the QG potential vorticity (PV), $u$ the horizontal velocity field, and $F$ on the right-hand side is a placeholder for other terms in the dynamics, such as the external forcing, the diffusion, etc. In the QG, the velocity $u$ can be derived from the QGPV $q$, and therefore, the QG, together with the suitable initial and boundary conditions, can be viewed as a closed system about a single quantity, the QGPV $q$. This simple and yet sophisticated model provides a unified framework for studying both the ocean and atmosphere ([21, 25, 17]).

Depending on the assumption on the vertical density profile, the QG equation(s) can take the form of a single scalar two-dimensional equation (the barotropic case with a uniform density profile), a system of two-dimensional scalar equations (the multi-layer case with a non-uniform discrete density profile), or a three-dimensional...
scalar equation (the 3D case with a non-uniform but continuous density profile). The QG equations form a hierarchy of models, with increasing complexity, for the large-scale geophysical flows. As a reference regarding the complexity, the barotropic QG is on the same level as the two-dimensional incompressible Euler equations; it is only a step forward from the latter with the inclusion of a free surface on the top. While the three-dimensional QG equation is posed on a three-dimensional spatial domain, the velocity vector at every point is horizontal, and therefore two-dimensional. Thus, the three-dimensional QG is simpler than the three-dimensional incompressible Euler equations, and hopefully, more amenable as well.

Several authors have studied the three-dimensional QG equation under idealized settings, in the unbounded half space, or a rectangular box. An early work is by Dutton ([13, 14]), who considered the three-dimensional QG model in a rectangular box with periodic boundary conditions on the sides, and homogeneous Neumann boundary conditions on the top and bottom. The uniqueness of a classical solution, if it exists, and the global existence of a generalized solution were established. Bourgeois and Beale ([6]) studied the equation in a similar setting, and the existence of a global strong solution was proved. Desjardins and Grenieer ([11]) also considered the equation in a similar setting, but included in their model the Ekman pumping effect which effectively add diffusion to the flow. The existence of a global weak solution is given. Puel and Vasseur ([22]) considered the inviscid QG in the upper half space, with the non-penetration boundary condition at the bottom of the fluid. The global existence of a weak solution was proven. In these works, the issue of uniqueness of the solutions was left open. In a recent work, Novack and Vasseur ([20]) considered the three-dimensional QG in the same spatial setting as in [22], but with an added diffusion term in the boundary at \( z = 0 \) due to the Ekman pumping effect. The existence and uniqueness of a global strong solution is proven. Novack ([19]) studies the existence of a weak solution to the inviscid 3D QG equation, with initial data in the Lebesgue spaces. The present work focuses on simpler models, but on more general settings, namely purely inviscid models on arbitrary bounded domains with physically relevant boundary conditions.

The well-posedness of the barotropic QG equation with a free top surface is the subject of a previous work ([8]). The goal of the present work is to address the issue of well-posedness for the multi-layer QG equations. Within the multi-layer QG equations, each layer behaves like a barotropic QG, and the layers interact with each other through pressure. Because of these interactions, the well-posedness of the barotropic QG does not directly transfer over to the multi-layer case. The layer interactions make the problem more interesting and more challenging at the same time.

A major challenge in the previous work ([8]) is the non-homogeneous boundary conditions on the streamfunction, which is imposed to accommodate the free fluctuations of the top surface. There, the challenge is dealt with by the superposition rule and an estimate on the constant non-zero value of the streamfunction. In this work, not only is the top surface left free, but also are the interior interfaces between layers. It turns out that the interior interfaces behave like the top surface, and can be treated as such. Therefore, the same type of boundary conditions are imposed on the interior interfaces, and they are treated in exactly the same way as in [8].

For both the ocean and the atmosphere, the density of the fluid is non-uniform, which is the basis for the multi-layer or three-dimensional models. Not only so,
the rate at which density varies against the height (or depth for the ocean) is also non-uniform. For example, in the ocean, the density of the water increases rapidly downward for the first couple of hundred meters, and then stay almost flat for the next thousands of meters ([21, 23]). Because of this non-uniform changing rate, the vertical interaction between layers takes the form of a second-order derivative with a non-uniform coefficient, in the continuous case,

\[
\frac{\partial}{\partial z} \left( \frac{1}{S(z)} \frac{\partial \psi}{\partial z} \right),
\]

where \( \psi \) stands for the streamfunction, and \( S(z) > 0 \) is determined by the vertical density profile. Under the usual homogeneous Neumann boundary conditions for \( \psi \), this operator is self-adjoint. In the multi-layer case, the non-uniformity in the changing rate gives rise to a non-symmetric matrix with non-positive eigenvalues, and the layer interactions are represented by a matrix-vector product,

\[ L\psi, \]

where \( \psi \) is a vector-valued function representing the streamfunction across the layers, and \( L \) is non-symmetric coefficient matrix. Because of its non-symmetry, even though \( L \) has only non-negative eigenvalues, the inner product

\[
(L\psi, \psi),
\]

which appears in the analysis of the elliptic boundary value problem governing the streamfunction \( \psi \) and the QGPV \( q \), is not negative definite. This lack of definiteness is not fatal for the analysis, as it can be remedied by a decomposition in the eigenmodes of the associated Sturm-Liouville problem in the vertical direction. The major hurdle, as it turns out, is related to another inner product involving a time derivative. Due to the non-symmetry of \( L \), the inner product

\[
\left( \frac{\partial}{\partial t} (L\psi), \psi \right)
\]

is no longer exactly integrable in time. To circumvent this difficulty imposed by the physical reality, we assume, in this study, that the density profile is linear with respect to the height, and the coefficient matrix \( L \) in the multi-layer case is actually symmetric. Of course, as pointed out above, this assumption runs against the physical reality. But this assumption does not significantly compromise the mathematical generality of the problem, because layer interactions are still included in the model. We also note that the corresponding differential operator in the continuous three-dimensional case is actually self-adjoint, which is the analogue of the symmetry of the discrete operator, and thus the current work can still serve as a stepping stone to the three-dimensional problem.

The rest of the paper is organized as follows. Section 2 presents the initial-boundary value problem for the multi-layer QGs in its complete form. Section 3 deals with an elliptic boundary value problem associated with the multi-layer QG. In Section 4, a weak formulation and some a priori results are obtained. Section 5 is devoted to the uniqueness of the weak solution, and Section 6 to the existence of this solution. The paper concludes in Section 7.
2. The initial and boundary conditions. We consider a 3-layer QG system,

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \mathbf{u}_1 \cdot \nabla \right) \left( \zeta_1 + \beta y + F_1^2(-\psi_1 + \psi_2) \right) &= f_1, \\
\left( \frac{\partial}{\partial t} + \mathbf{u}_2 \cdot \nabla \right) \left( \zeta_2 + \beta y + F_2^2(\psi_1 - 2\psi_2 + \psi_3) \right) &= f_2, \\
\left( \frac{\partial}{\partial t} + \mathbf{u}_3 \cdot \nabla \right) \left( \zeta_3 + \beta y + F_3^2(\psi_2 - \psi_3) \right) &= f_3.
\end{align*}
\]  

(2)

In the above, for each \( i = 1, 2, 3 \),

- \( \psi_i \) is pressure perturbation,
- \( \mathbf{u}_i = \nabla \perp \psi_i \) is horizontal velocity,
- \( \zeta_i = \nabla \times \mathbf{u}_i \) is relative vorticity,
- \( F_i \equiv \frac{L}{\sqrt{g'D_i/f_0}} \) is the Froude number.

In the specification for the Froude number, \( L \) represents the typical horizontal length scale of the flow, \( D_i \) the average layer depth, and \( g' \) is the reduced gravity within the flow.

This study focuses on the effect of the nonlinearity within each layer, as well as the interaction between the layers. For this reason, the diffusion terms have been omitted.

In reality, the Froude number \( F_i = O(1) \). We therefore take \( F_i = 1 \) in the equations (2). This choice has the added benefit that the coefficient matrix for the zeroth order terms is now symmetric, the significance of which has been discussed in the Introduction. The beta terms are mathematically insignificant, and therefore they will be neglected from now on. Thus we consider the following model,

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \mathbf{u}_1 \cdot \nabla \right) \left( \zeta_1 + \psi_2 \right) &= f_1, \\
\left( \frac{\partial}{\partial t} + \mathbf{u}_2 \cdot \nabla \right) \left( \zeta_2 + (\psi_1 - 2\psi_2 + \psi_3) \right) &= f_2, \\
\left( \frac{\partial}{\partial t} + \mathbf{u}_3 \cdot \nabla \right) \left( \zeta_3 + \psi_2 - \psi_3 \right) &= f_3.
\end{align*}
\]  

(3)

The variables \( \psi_i, \mathbf{u}_i, \) and \( \zeta_i \) are defined as before.

For this inviscid system, the no-flux boundary conditions are imposed on the velocity field, and in terms of the streamfunctions, these conditions can be written as

\[
\psi_i = \text{constant} \quad \text{for each } 1 \leq i \leq 3 \text{ on } \partial \Omega.
\]  

(4)

In order to uniquely determine the streamfunctions, a mass conservation constraint is imposed on each layer,

\[
\int_{\Omega} \psi_i(x, t) dx = 0.
\]  

(5)

Finally, the initial conditions are imposed on the streamfunctions as well,

\[
\psi_i(x, 0) = \psi_i^0(x), \quad \text{for each } 1 \leq i \leq 3 \text{ and } \forall x \in \Omega.
\]  

(6)
To present the analysis in a concise fashion, it is advisable to introduce some vector notations and rewrite the system in a vector format. We let

\[ \mathbf{q} = (q_1, q_2, q_3)^T, \]
\[ \psi = (\psi_1, \psi_2, \psi_3)^T, \]
\[ \varphi = (\phi_1, \phi_2, \phi_3)^T, \]
\[ \mathbf{f} = (f_1, f_2, f_3)^T, \]
\[ U = (u_1, u_2, u_3)^T. \]

The first four in the above are column vectors, while the last one stands for a \(3 \times 2\) tensor, because each \(u_i\) represents a vector in the horizontal direction. We designate the coefficients matrix for the zeroth order terms by

\[ L = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \]

Then the multilayer QG equations (3) can be succinctly written in the form of a transport equation,

\[ \frac{\partial}{\partial t} \mathbf{q} + U \cdot \nabla \mathbf{q} = \mathbf{f}, \quad (7) \]

with

\[ \mathbf{q} = \Delta \psi + L\psi, \quad (8a) \]
\[ U = \nabla^\perp \psi. \quad (8b) \]

The boundary conditions (4) and (5) and the initial conditions can also be recast in the vector variables,

\[ \psi(x, t) = l(t), \quad \forall x \in \partial M, \quad (9a) \]
\[ \int_M \psi(x, t) dx = 0, \quad (9b) \]

and

\[ \psi(x, 0) = \psi_0(x), \quad \forall x \in M. \quad (10) \]

3. A non-standard elliptic boundary value problem. When the potential vorticity \(q\) is known, the streamfunction \(\psi\) can be recovered by solving an elliptic boundary value problem,

\[ \begin{cases} \Delta \psi + L\psi = q, & x \in M, \\ \psi(x) = l, & x \in \partial M, \\ \int_M \psi(x) dx = 0. \end{cases} \quad (11) \]

The boundary conditions are of a non-standard type. The scalar version of (11) has been dealt with in REF, with the aid of the Green’s function for the Helmholtz equation. Our strategy for the system (11) is to transform and decouple it into a sequence of scalar elliptic boundary value problems. We note that the coefficient matrix \(L\) is symmetric, and therefore can be diagonalized. It has a set of non-positive eigenvalues \(\{\lambda_1, \lambda_2, \lambda_3\} = \{0, -1, -3\}\), and a corresponding set of distinct...
orthogonal eigenvectors,

\[
\begin{align*}
\mathbf{v}_0 &= \begin{pmatrix}
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}}
\end{pmatrix}, \\
\mathbf{v}_1 &= \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
-\frac{1}{\sqrt{2}}
\end{pmatrix}, \\
\mathbf{v}_2 &= \begin{pmatrix}
-\frac{1}{\sqrt{6}} \\
\frac{2}{\sqrt{6}} \\
-\frac{1}{\sqrt{6}}
\end{pmatrix},
\end{align*}
\]

corresponding to the barotropic mode, and the first and the second baroclinic modes in the vertical direction, respectively. Using these eigen-modes as a basis, we can transform the BVP (11) into a decoupled system. Specifically, We let

\[
P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3],
\]

and

\[
\psi = P\tilde{\psi}.
\]

For each \(i = 1, 2, 3\), the \(\tilde{\psi}_i\) solves the boundary value problem

\[
\begin{cases}
\Delta \tilde{\psi}_i + \lambda_i \tilde{\psi}_i = \tilde{q}_i, & x \in \mathcal{M}, \\
\tilde{\psi}_i = \tilde{l}_i, & x \in \partial\mathcal{M}, \\
\int_{\mathcal{M}} \tilde{\psi}_i(x) dx = 0.
\end{cases}
\tag{12}
\]

In the above, \(\tilde{q}_i\) and \(\tilde{l}_i\) are obtained, respectively, from

\[
\tilde{q} = P^{-1}q, \quad \tilde{l} = P^{-1}l.
\]

For \(i = 2, 3\), the non-standard scalar elliptic BVP with a zero-order term has been considered in [8]. There, with the Green’s function for the Helmholtz equation, it is shown that the constant boundary value \(|l_i|\) on the boundary can be bounded in terms of \(|q_i|_{\infty}\), and the solution \(\psi_i\) belongs to \(W^{2,p}(\mathcal{M})\) for any \(p > 1\), and is Hölder and quasi-Lipschitz continuous. The case with \(i = 0\) in (12) can be handled in a similar fashion, with the Green’s function for the Laplace operator, and the same regularity results can be obtained. These regularity results can then be transferred to the solution \(\psi\) of (11) via the transformation \(P^{-1}\).

Below, we shall formally state the regularity results for the elliptic boundary value problem (11). But, in order to do so, we need to first give the precise definitions of some relevant function spaces.

We denote by \(Q_T\) the spatial-temporal domain,

\[
Q_T = \mathcal{M} \times (0, T).
\]

We denote by \(L^\infty(\mathcal{M})\), or \(L^\infty(\mathcal{Q}_T)\) when time is also involved, the space of functions that are essentially bounded. We denote by \(C^{0, \gamma}(\mathcal{M})\), with \(\gamma > 0\), the space of Hölder-continuous functions on \(\mathcal{M}\), and similarly, \(C^{0, \gamma}(\mathcal{Q}_T)\) on \(\mathcal{Q}_T\). \(C^{0, \gamma}(\mathcal{M})\) and \(C^{0, \gamma}(\mathcal{Q}_T)\) are both Banach spaces under the usual Hölder norms.

**Lemma 3.1.** Let \(\partial\mathcal{M} \in C^2\), \(q \in L^\infty(\mathcal{M})\). Then the elliptic boundary value problem (11) has a unique solution \(\psi \in W^{2,p}(\mathcal{M})\) for every \(p > 1\), with the following estimate,

\[
\|\nabla^2 \psi\|_{L^p(\mathcal{M})} \leq C_p\|q\|_{L^\infty(\mathcal{M})}, \quad \forall p > 1.
\tag{13}
\]
In addition, the first derivatives of \( \psi \) are Hölder continuous and quasi-Lipschitz continuous,

\[
\| \nabla \psi \|_{C^{0,\gamma}(\mathcal{M})} \leq \frac{C}{1 - \gamma} \| q \|_{L^\infty(\mathcal{M})}, \quad \forall 0 < \gamma < 1, \quad (14)
\]

\[
| \nabla \psi(\xi) - \nabla \psi(\eta) | \leq C(\delta) \| q \|_{L^\infty(\mathcal{M})}, \quad \forall \xi, \eta \in \mathcal{M}. \quad (15)
\]

In the above,

\[
\chi(\delta) = \begin{cases} 
(1 - \ln \delta)\delta & \text{if } \delta < 1, \\
1 & \text{if } \delta \geq 1.
\end{cases}
\]

We denote by \( V \) the space of solutions to the elliptic boundary value problem (11) with \( q \in L^\infty(\mathcal{M}) \), i.e.,

\[
V := \{ \psi \mid \psi \text{ solves (11) for some } q \in L^\infty(\mathcal{M}) \}.
\]

The space \( V \) is equipped with the norm

\[
\| \psi \|_V := \| \Delta \psi + L\psi \|_{L^\infty(\mathcal{M})}.
\]

By the continuity of the inverse elliptic operator \((\Delta + L)^{-1}\), \( V \) is a Banach space.

In the analysis, we will also encounter functions that are differentiable with continuous first derivatives. The space of these functions will be denoted as \( C^1(\mathcal{M}) \), equipped with the usual \( C^1 \) norm.

When time is involved, we use \( L^\infty(0,T; V) \) to designate the space of functions that are essentially bounded with respect to the \( \| \cdot \|_V \) norm, and \( L^\infty(0,T; C^1(\mathcal{M})) \) for functions that are essentially bounded under the \( \| \cdot \|_{C^1(\mathcal{M})} \) norm.

In the sequel, we will need the following regularity result, which can be easily derived from the classical \( L^p \) theory for elliptic equations with Dirichlet boundary conditions (15).

**Lemma 3.2.** Let \( g \in L^p(\mathcal{M}) \) with \( p > 1 \), and let \( \psi \) be a solution of

\[
\Delta \psi + L\psi = \sum_{i=1}^2 c_i \frac{\partial}{\partial E_i} g, \quad \mathcal{M}, \quad (16a)
\]

\[
\psi = l, \quad \partial \mathcal{M}, \quad (16b)
\]

\[
\int_{\mathcal{M}} \psi dx = 0. \quad (16c)
\]

Then, \( \psi \) has one generalized derivative, and

\[
\| \psi \|_{W^{1,p}(\mathcal{M})} \leq Cp\| g \|_{L^p(\mathcal{M})}. \quad (17)
\]

Here \( C \) is a constant depending on \( \mathcal{M} \) and \( c_i \)’s only.

4. **Weak formulation and a priori estimates.** We assume that \( \psi \) is a classical solution of (7)–(8) subjecting to the constraints (9)–(10). We let \( \varphi \in C^\infty(Q_T) \) with \( \varphi|_{\partial \mathcal{M}} = \varphi|_{t=T} = 0 \). We take the inner product of (7) with \( \varphi \) and integrate by parts to obtain

\[
- \int_{\mathcal{M}} (\Delta \psi_0 + L\psi_0) \cdot \varphi(x,0) dx - \int_0^T \int_{\mathcal{M}} (\Delta \psi + L\psi) \cdot \frac{\partial \varphi}{\partial t} dx dt
\]

\[
- \int_0^T \int_{\mathcal{M}} (\Delta \psi + L\psi) \cdot U \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathcal{M}} f \cdot \varphi dx dt. \quad (18)
\]
Thus, every classical solution of the multilayer QG equation also solves the integral equation \( (18) \), but the converse is not true, for the QGPV \( q = \Delta \psi + L \psi \) may not be differentiable either in space \( x \) or in time \( t \). Solutions of \( (18) \) are called weak solutions of the multilayer QG.

We establish the well-posedness of the multilayer QG \((7)-(10)\) by working with its weak formulation first, whose precise statement is given here.

**Statement of the problem:**

Let \( \psi_0 \in V \), and \( f \in L^\infty(Q_T) \). Find \( \psi \in L^\infty(0,T; V) \) such that \( (18) \) holds for every \( \varphi \in C^\infty(Q_T) \) with \( \varphi|_{\partial \mathcal{M}} = \varphi|_{t=T} = 0 \). \hfill (19)

We first obtain a few *a priori* estimates on the solution(s) of \( (19) \). We choose \( \varphi(x,t) = g(t)\gamma(x) \) in \( (18) \) with \( g \in C^\infty([0,T]) \), \( g(T) = 0 \), and \( \gamma \in C_c^\infty(\mathcal{M}) \). Substituting this \( \varphi \) into \( (18) \), we have

\[
- g(0) \int_M (\Delta \psi_0 + L \psi_0) \cdot \gamma dx - \int_0^T g'(t) \int_M (\Delta \psi + L \psi) \cdot \gamma(x) dx dt \\
- \int_0^T g(t) \int_M (\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma dx dt = \int_0^T g(t) \int_M f \cdot \gamma dx dt. \hfill (20)
\]

If we take \( g(0) = 0 \) as well, then \( (20) \) becomes

\[
- \int_0^T g'(t) \int_M (\Delta \psi + L \psi) \cdot \gamma(x) dx dt = \int_0^T g(t) \int_M ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) dx dt. \hfill (21)
\]

This shows that

\[
\frac{d}{dt} \int_M (\Delta \psi + L \psi) \cdot \gamma(x) dx dt = \int_M ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) dx \quad \text{in } D'(0,T). \hfill (22)
\]

Thanks to the fact that \( C_c^\infty(\mathcal{M}) \) is dense in \( H_0^1(\mathcal{M}) \), the above also holds for every \( \varphi \in H_0^1(\mathcal{M}) \). Thus, we conclude that \( \Delta \psi + L \psi \) is weakly continuous in time in the following sense,

\[
\int_M (\Delta \psi + L \psi) \cdot \gamma dx \text{ is continuous in time for every } \gamma \in H_0^1(\mathcal{M}).
\]

Integrating by parts in \( (20) \), we find

\[
- g(0) \int_M (\nabla \psi_0 \cdot \nabla \gamma + L \psi_0 \cdot \gamma) dx + \int_0^T g'(t) \int_M (\nabla \psi \cdot \nabla \gamma + L \psi \cdot \gamma) dx dt = \\
\int_0^T g(t) \int_M ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) dx dt. \hfill (23)
\]

Again, taking \( g(0) = 0 \) yields

\[
\int_0^T g'(t) \int_M (\nabla \psi \cdot \nabla \gamma + L \psi \cdot \gamma) dx dt = \int_0^T g(t) \int_M ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) dx dt. \hfill (24)
\]
Since $C_c^\infty(\mathcal{M})$ is dense in the space $H^1_0(\mathcal{M})$ under the usual $H^1$-norm, the above holds for every $\gamma \in H^1_0(\mathcal{M})$. Thus,

$$
\frac{d}{dt} \int_\mathcal{M} (\nabla \psi \cdot \nabla + L \psi \cdot \gamma) \, dx = - \int_\mathcal{M} ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) \, dx \quad \text{in } D'(0, T).
$$

This implies that $\psi$ is weakly continuous in time for the $H^1$-norm,

$$
\int_\mathcal{M} (\nabla \psi \cdot \nabla + L \psi \cdot \gamma) \, dx \quad \text{is continuous in time for every } \gamma \in H^1_0(\mathcal{M}).
$$

To investigate the initial value of $\psi$, we take $g \in C_c^\infty([0,T])$ with $g(0) \neq 0$ and $g(T) = 0$. We multiply (22) by $g(t)$ and integrate by parts in $t$ to obtain

$$
g(0) \int_\mathcal{M} (\Delta \psi(x, 0) + L \psi(x, 0)) \cdot \gamma \, dx - \int_0^T g'(t) \int_\mathcal{M} (\Delta \psi + L \psi) \cdot \gamma(x) \, dx \, dt $$

$$
- \int_0^T g(t) \int_\mathcal{M} (\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma \, dx \, dt = \int_0^T g(t) \int_\mathcal{M} f \cdot \gamma \, dx \, dt. \quad (26)
$$

Comparing (26) with (20), we find that

$$
\int_\mathcal{M} (\Delta \psi(x, 0) + L \psi(x, 0)) \cdot \gamma \, dx = \int_\mathcal{M} (\Delta \psi_0 + L \psi_0) \cdot \gamma \, dx, \quad \forall \gamma \in H^1_0(\mathcal{M}). \quad (27)
$$

Multiplying (25) by the same $g(t)$ and integrating by parts in time, we obtain

$$
- g(0) \int_\mathcal{M} (\nabla \psi(x, 0) \cdot \nabla + \psi(x, 0) \cdot \gamma) \, dx + \int_0^T g'(t) \int_\mathcal{M} (\nabla \psi \cdot \nabla + L \psi \cdot \gamma) \, dx \, dt $$

$$
- \int_0^T g(t) \int_\mathcal{M} ((\Delta \psi + L \psi) \nabla^\perp \psi \cdot \nabla \gamma + f \cdot \gamma) \, dx \, dt. \quad (28)
$$

Comparing this equation with (23), we easily see that

$$
\int_\mathcal{M} (\nabla \psi(x, 0) \cdot \nabla \gamma + L \psi(x, 0) \cdot \gamma) \, dx = \int_\mathcal{M} (\nabla \psi_0 \cdot \nabla \gamma + L \psi_0 \cdot \gamma) \, dx, \quad \forall \gamma \in H^1_0(\mathcal{M}). \quad (29)
$$

We formally summarize these results in the following lemma.

**Lemma 4.1.** The solution $\psi$ to the weak formulation (18), if it exists, is weakly continuous in the following sense,

$$
\int_\mathcal{M} (\Delta \psi + L \psi) \cdot \gamma \, dx \quad \text{is continuous in time for every } \gamma \in H^1_0(\mathcal{M}), \quad (30a)
$$

$$
\int_\mathcal{M} (\nabla \psi \cdot \nabla + L \psi \cdot \gamma) \, dx \quad \text{is continuous in time for every } \gamma \in H^1_0(\mathcal{M}). \quad (30b)
$$

The initial condition is satisfied in the sense that

$$
\int_\mathcal{M} (\Delta \psi(x, 0) + L \psi(x, 0)) \cdot \gamma \, dx = \int_\mathcal{M} (\Delta \psi_0 + L \psi_0) \cdot \gamma \, dx, \quad \forall \gamma \in H^1_0(\mathcal{M}). \quad (31a)
$$

$$
\int_\mathcal{M} (\nabla \psi(x, 0) \cdot \nabla + L \psi(x, 0) \cdot \gamma) \, dx = \int_\mathcal{M} (\nabla \psi_0 \cdot \nabla + L \psi_0 \cdot \gamma) \, dx, \quad \forall \gamma \in H^1_0(\mathcal{M}). \quad (31b)
$$
By virtue of Lemma 3.1, any solutions of (19) automatically have second weak derivatives in space. In fact, it also has second temporal-spatial cross derivatives, according to the following lemma.

**Lemma 4.2.** Let \( \psi(x,t) \) be a generalized solution of (7)–(9) in the sense of (18). Then there exists generalized derivatives \( \partial^2 \psi/\partial x \partial t \) and, for any \( p \geq 1 \),

\[
\sup_{0 < t < T} \left\| \frac{\partial^2 \psi}{\partial x \partial t} \right\|_{L^p(M)} \leq C_p \sup_{0 < t < T} \left( \| F \|_{L^p(M)} + \| \psi \|_{L^\infty(0,T;V)} \cdot \| \nabla \psi \|_{L^p(M)} \right).
\]

**Proof.** From (22) one derives that, for a.e. \( t \in (0,T) \),

\[
(\Delta + L) \frac{\partial}{\partial t} \psi = \nabla \times F - \nabla \cdot (\nabla^\perp \psi \cdot (\Delta \psi + L \psi)) \in H^{-1}(M).
\]

Then, by Lemma 3.2,

\[
\left\| \frac{\partial \psi}{\partial t} \right\|_{W^{1,p}(M)} \leq C_p \left( \| F \|_{L^p(M)} + \| \psi \|_{L^\infty(0,T;V)} \cdot \| \nabla \psi \|_{L^p(M)} \right).
\]

Taking the supreme norm in time \( t \) on the right-hand side, and then on the left-hand side, we obtain

\[
\sup_{0 < t < T} \left\| \frac{\partial \psi}{\partial t} \right\|_{W^{1,p}(M)} \leq C_p \sup_{0 < t < T} \left( \| F \|_{L^p(M)} + \| \psi \|_{L^\infty(0,T;V)} \cdot \| \nabla \psi \|_{L^p(M)} \right).
\]

\[\square\]

5. **Uniqueness.** In this section, we establish the uniqueness of the weak solution of (7)–(10), if it exists.

**Theorem 5.1.** The solution to the weak problem (19), if it exists, must be unique.

The uniqueness proof largely follows the arguments laid out by Yudovich ([26]). What are new to the present problem include the presence of multiple vertical layers and the non-homogeneous boundary conditions that are needed to accommodate the free fluctuations of the layer interfaces.

**Proof.** We let \( \psi^1 \) and \( \psi^2 \) be two solutions to the weak problem for the same initial data \( \psi_0 \), and \( t \) be a fixed point in \([0,T]\). Then, for an arbitrary \( \varphi \in C^\infty(\overline{Q}_t) \) with \( \varphi|_{\partial M} = \varphi(\cdot,t) = 0 \), \( \psi^1 \) and \( \psi^2 \) satisfy the following equations, respectively,

\[
\int_M (\Delta \psi^0 + L \psi_0) \cdot \varphi(x,0) dx - \int_0^t \int_M (\Delta \psi^1 + L \psi_1) \cdot \frac{\partial \varphi}{\partial t} dx dt \\
- \int_0^T \int_M (\Delta \psi^1 + L \psi_1) \cdot \nabla^\perp \psi^1 \cdot \nabla \varphi dx dt = \int_0^T \int_M f \cdot \varphi dx dt, \quad (34)
\]

\[
\int_M (\Delta \psi_0 + L \psi_0) \cdot \varphi(x,0) dx - \int_0^t \int_M (\Delta \psi^2 + L \psi_2) \cdot \frac{\partial \varphi}{\partial t} dx dt \\
- \int_0^T \int_M (\Delta \psi^2 + L \psi_2) \cdot \nabla^\perp \psi^2 \cdot \nabla \varphi dx dt = \int_0^T \int_M f \cdot \varphi dx dt. \quad (35)
\]
Subtracting these two equations, and denoting \( h = \psi^1 - \psi^2 \), we obtain

\[
- \int_0^t \int_M (\Delta h + Lh) \cdot \frac{\partial \varphi}{\partial t} \, dx \, dt - \int_0^T \int_M (\Delta h + Lh) \cdot \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_M (\Delta \psi^2 + L\psi^2) \cdot \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt = 0. \tag{36}
\]

An integration by parts in space in the first term leads to

\[
\int_0^t \int_M \left( \nabla h \cdot \nabla \partial_t \varphi - Lh \cdot \partial_t \varphi \right) \, dx \, dt - \int_0^T \int_M \left( \Delta h + Lh \right) \cdot \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_M \left( \Delta \psi^2 + L\psi^2 \right) \cdot \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt = 0. \tag{37}
\]

Both \( \varphi^1 \) and \( \varphi^2 \) assume space-independent values on the boundary \( \partial M \), and so does the difference \( h \) between them. Thus, after a shifting in the vertical direction, \( h \) will vanish on the boundary. We denote this shifted function by \( h^\# \in L^\infty(0, t; H^1_0(M)) \). The functions \( h \) and \( h^\# \) are related via

\[
h(x, \tau) = h^\#(x, \tau) + l(\tau), \quad 0 \leq \tau \leq t \tag{38}
\]

for some function \( l(\tau) \). Both \( \psi^1 \) and \( \psi^2 \) have a zero average over \( M \), and so does their difference \( h \). Integrating \( (38) \) over \( M \) we establish a simple relation between \( l \) and \( h^\# \),

\[
l(\tau) = \frac{1}{|M|} \int_M h^\#(x, \tau) \, dx. \tag{39}
\]

Replacing \( h \) by \( h^\# + l \) in the first and third integrals of \( (37) \) yields

\[
\int_0^t \int_M \left( \Delta h^\# + Lh^\# \right) \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_M \left( \Delta h^\# + Lh^\# + LL \right) \cdot \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_M \left( \Delta \psi^2 + L\psi^2 \right) \cdot \nabla h^\# \cdot \nabla \varphi \, dx \, dt = 0. \tag{40}
\]

We integrate by parts in time \( t \) in the first integral, and use the facts that

\[
\begin{align*}
\left. h^\#(\cdot, \tau) \right|_{\tau=0} &\in H^1_0(M), \quad \text{a.e. } \tau, \\
\int_M (\Delta h + Lh) |_{\tau=0} \cdot \varphi(\cdot, 0) \, dx &= 0, \\
\int_M (\nabla h(\cdot, 0) \cdot \nabla \varphi - Lh(\cdot, 0) \cdot \varphi) \, dx &= 0,
\end{align*}
\]

and the regularity results from Lemma 4.2, we obtain

\[
- \int_0^t \int_M \left( \partial_t \nabla h^\# \cdot \nabla \varphi - L \partial_t h^\# \cdot \varphi \right) \, dx \, dt + \int_0^t \int_M \left( \nabla h^\# + Lh^\# + LL \right) \nabla \psi^1 \cdot \nabla \varphi \, dx \, dt + \int_0^T \int_M \left( \Delta \psi^2 + L\psi^2 \right) \nabla h^\# \cdot \nabla \varphi \, dx \, dt = 0. \tag{41}
\]
We note that each of the integrals is linear and continuous with respect to \( \varphi \) in the norm of \( L^2(0, T; H^1_0(M)) \). Thus, we can let \( \varphi \) tend to \( h^\# \) in \( L^2(0, T; H^1_0(M)) \), pass to the limit in (41), and noticing the fact that \( \nabla h^\# \cdot \nabla^\perp h^\# = 0 \), we obtain

\[
- \int_0^t \int_M (\partial_t \nabla h^\# \cdot \nabla h^\# - L \partial_t h^\# \cdot h^\#) dx dt + \int_0^t \int_M L \partial_t l \cdot h^\# dx dt \\
- \int_0^T \int_M (\Delta h^\# + L h^\# + Ll) \cdot \nabla^\perp \psi^1 \cdot \nabla h^\# dx dt = 0.
\]

(42)

Noticing that \( l \) is independent of \( x \) and the no-flux boundary conditions on \( \psi^1 \), we find

\[
\int_0^t \int_M Ll \cdot \nabla^\perp \psi^1 \cdot \nabla h^\# dx dt \\
= \int_0^t \nabla l \cdot \int_M \nabla^\perp (\psi^1 h^\#) dx dt \\
= \int_0^t Ll \cdot \int_{\partial M} \nabla^\perp \psi^1 \cdot n \cdot h^\# ds dt = 0.
\]

Using the fact that the coefficient matrix \( L \) is symmetric, we can write the second term on the left-hand side of (42) as

\[
\int_0^t \int_M L \partial_t h^\# \cdot h^\# dx dt \\
= \int_0^t \frac{1}{2} \frac{d}{dt} \int_M L h^\# \cdot h^\# dx dt \\
= \frac{1}{2} \int_M Lh^\#(x, t) \cdot h^\#(x, t) dx - \frac{1}{2} \int_M Lh^\#(x, 0) \cdot h^\#(x, 0) dx \\
= \frac{1}{2} \int_M Lh^\#(x, t) \cdot h^\#(x, t) dx.
\]

Regarding the third term on the left-hand side of (42), we again use the symmetry of the linear operator \( L \) and the formula (39) to obtain

\[
\int_0^t \int_M \partial_t l \cdot h^\# dx dt = \frac{-1}{2|\mathcal{M}|} \left( L \int_M h^\#(x, t) dx \right) \cdot \left( \int_M h^\#(x, t) dx \right)
\]
Inserting these identities into (42), we have

\[-\frac{1}{2}\|h^\#(\cdot, t)\|^2_{\mathcal{H}^1_0(M)}d\tau + \frac{1}{2}\int_M Lh^\#(x, t) \cdot h^\#(x, t)dx - \frac{1}{2}\int_M h^\#(x, t)dx \cdot \left(\int_M h^\#(x, t)dx\right)\]

\[= \int_0^t \int_M (\Delta h^\# + Lh^\#) \nabla \psi^1 \cdot \nabla h^\# dx d\tau. \quad (43)\]

\[\frac{1}{2}\|h^\#(\cdot, t)\|^2_{\mathcal{H}^1_0(M)}d\tau - \frac{1}{2}\int_M Lh^\#(x, t) \cdot h^\#(x, t)dx + \frac{1}{2}\int_M h^\#(x, t)dx \cdot \left(\int_M h^\#(x, t)dx\right)\]

\[= -\int_0^t \int_M (\Delta h^\# + Lh^\#) \nabla \psi^1 \cdot \nabla h^\# dx d\tau. \quad (44)\]

We expand \(h^\#\) in the orthonormal eigenvectors \(v_0, v_1\) and \(v_2\) of \(L\),

\(h^\#(x, t) = c_0(x, t)v_0 + c_1(x, t)v_1 + c_2(x, t)v_2\).

Substituting this expansion in the second term on the left-hand side of (44) and using the orthogonality of the eigenfunctions, one finds that

\[-\frac{1}{2}\int_M Lh^\#(x, t) \cdot h^\#(x, t)dx = -\frac{1}{2}(c_1\lambda_1v_1 + c_2\lambda_2v_2, c_0v_0 + c_1v_1 + c_2v_2)\]

\[= -\frac{1}{2}\int_M (\lambda_1c_1^2 + \lambda_2c_2^2)dx.\]

Similarly, substituting this expansion for \(h^\#\) into the third term on the left-hand side of the equation, and using the fact that the eigenvalues of \(L\) are non-positive, one finds that

\[\frac{1}{2}\int_M \left(\int_M h^\#(x, t)dx\right) \cdot \left(\int_M h^\#(x, t)dx\right)\]

\[= \frac{1}{2}\int_M \left(\lambda_1 \left(\int_M c_1(x, t)dx\right)^2 + \lambda_2 \left(\int_M c_2(x, t)dx\right)^2\right)\]

\[\geq \frac{1}{2}\int_M (\lambda_1c_1^2 + \lambda_2c_2^2)dx.\]

Thus the combination of the second and third terms on the left-hand side of (44) is positive, and the following inequality results,

\[\frac{1}{2}\|h^\#(\cdot, t)\|^2_{\mathcal{H}^1_0(M)} \leq -\int_0^t \int_M (\Delta h^\# + Lh^\#) \cdot \nabla \psi^1 \cdot \nabla h^\# dx d\tau. \quad (45)\]

Unlike in the barotropic case (see [8]), the inner product involving the zero order term on the right-hand side does not vanish, thanks to the vertical layer interactions. We proceed by obtaining an estimate on this term. Using the fact that the linear operator \(L\) is a constant coefficient matrix and \(\nabla \psi^1\) is Hölder continuous on \(M\), thanks to Lemma 3.1, we obtain that

\[\left|\int_0^t \int_M Lh^\# \cdot \nabla \psi^1 \cdot \nabla h^\# dx d\tau\right| \leq C(M)|U|^1_{\infty} \int_0^t \|\nabla h^\#\|^2_{L^2(M)} d\tau. \quad (46)\]
For the other term on the right-hand side of (45), we first notice that $\nabla^\perp \psi^1 \cdot \nabla h^# = 0$ on the boundary. Then, by an integration by parts, one obtains

$$\int_0^t \int_M \partial_t h^# \cdot \nabla \psi^1 \cdot \nabla h^# \, dx \, d\tau = - \int_0^t \int_M \nabla h^# \cdot \nabla \cdot (\nabla^\perp \psi^1 \cdot \nabla h^#) \, dx \, d\tau$$

To further investigate the integral on the right-hand side, we introduce index $i, j = 0$ on the boundary. Then, by an integration by parts, one obtains

$$\text{For the other term on the right-hand side of (45), we first notice that $\nabla^\perp \psi^1 \cdot \nabla h^# = 0$ on the boundary. Then, by an integration by parts, one obtains}$$

$$\int_0^t \int_M \partial_t h^# \cdot \nabla \psi^1 \cdot \nabla h^# \, dx \, d\tau$$

$$= - \int_0^t \int_M \sum_{l=1}^3 \partial_l h^#_i \partial_l (u_{ij} \partial_j h^#_i) \, dx \, d\tau$$

$$= - \int_0^t \int_M \sum_{l=1}^3 \partial_l h^#_i (\partial_i u_{ij} \partial_j h^#_i + u_{ij} \partial_j \partial_i h^#_i) \, dx \, d\tau$$

$$= - \int_0^t \int_M \sum_{l=1}^3 \partial_l h^#_i \partial_l u_{ij} \partial_j h^#_i + \frac{1}{2} u_{ij} \partial_j (\partial_i h^#_i)^2 \, dx \, d\tau$$

$$= - \int_0^t \int_M \sum_{l=1}^3 \partial_l h^#_i \partial_l u_{ij} \partial_j h^#_i \, dx \, d\tau.$$
handled with the technique employed in [26, 8]. We denote
\[ M_1 \equiv \sup_{0 < t < T} \| \psi^1(\cdot, t) \|_V, \]
\[ M_2 \equiv \sup_{0 < t < T} \| h(\cdot, t) \|_V. \]

It is then inferred from Lemma 3.1 that
\[ |U^1|_\infty \leq C(M) M_1, \]
\[ \sup_{0 < t < T} \| \psi^1(\cdot, t) \|_{W^{2, 2}(M)} \leq C_2 \epsilon M_1, \]
\[ \sup_{0 < t < T} \| \nabla h^\#(\cdot, t) \|_{L^{\infty}(M)} \leq C M_2. \]

We let \( \epsilon > 0 \) be arbitrary, and using the H"older’s inequality, we derive an estimate for the first integral on the right-hand side of (48),
\[
C \int_0^t \int_M |\nabla h^\#|^2 \cdot |\nabla^2 \psi^1| \, dx \, d\tau \leq C \left( \int_0^t \int_M |\nabla h^\#|^{2-\epsilon} \cdot |\nabla^2 \psi^1| \, dx \, d\tau \right)^{\frac{\epsilon}{2-\epsilon}} \left( \int_0^T |\nabla^2 \psi^1|^2 \, dx \right)^{\frac{2-\epsilon}{2}} \left( \int_0^t \int_M |\nabla h^\#| \, dx \right)^{\frac{2-\epsilon}{2}} \, d\tau.
\]

Applying this estimate in (48) yields
\[
\| h^\#(\cdot, t) \|_{H^2_0(M)}^2 \leq C(M) M_1 \left( \frac{M_2}{\epsilon} \int_0^t \| h^\#(\cdot, \tau) \|_{H^2_0(M)}^{2-\epsilon} \, d\tau + \int_0^t \| h^\#(\cdot, \tau) \|_{H^2_0(M)}^2 \, d\tau \right). \tag{49}
\]

We denote
\[ \sigma(\cdot, t) \equiv \| h^\#(\cdot, t) \|_{H^2_0(M)}. \]

Then (49) can be written as
\[
\sigma^2(t) \leq C(M, M_1) \left( \frac{M_2}{\epsilon} \int_0^t \sigma^{2-\epsilon}(\tau) \, d\tau + \int_0^t \sigma^2(\tau) \, d\tau \right). \tag{50}
\]

An estimate on \( \sigma \) can be obtained by the Gronwall inequality. Indeed, denoting the right-hand side as \( F(t) \), taking its derivative, one has
\[
\frac{d}{dt} F(t) \leq \frac{C M_2^2}{\epsilon} \sigma^{1-\frac{\epsilon}{2}}(t) + CF(t).
\]

An integration of this inequality yields
\[
F(t) \leq e^{C_1 t} M_2^2 \left( \frac{C_1 t}{2} \right)^{\frac{\epsilon}{2}}.
\]
Thus,

$$
\|h^\#(\cdot, t)\|_{L^2(M)}^2 \leq F(t) \leq e^{2Ct}M^2 \left( \frac{Ct}{2} \right)^{\frac{2}{\epsilon}}.
$$

(51)

We take

$$
t^* = \frac{1}{C}.
$$

Then, for

$$
0 \leq t \leq t^*,
$$

$$
\|h^\#(\cdot, t)\|_{L^2(M)}^2 \leq e^{2Ct^*}M^2 \left( \frac{1}{2} \right)^{\frac{2}{\epsilon}}.
$$

(52)

This estimate holds for arbitrary

$$
\epsilon > 0.
$$

Thus,

$$
h^\#(\cdot, t) \|_{L^2(M)}^2 = 0
$$

for a.e. 

$$
0 \leq t \leq T.
$$

Thus it has been proven that the solution to the weak problem (19), if it exists, must be unique.

6. Existence of a solution to the weak problem. Yudovich ([26]) establishes the existence of a weak solution to the two-dimensional Euler equation through an iterative scheme and the Schauder fixed-point theorem. The existence of a solution to the linearized problem is achieved via a regularization technique. Here, we largely follow the footsteps this work, but we treat the linearized problem with a flow map constructed out of a continuous velocity filed.

Given

$$
\psi \in L^\infty(0, T; C^1(M)),
$$

we compute the updated

$$
\psi^* \in L^\infty(0, T; V)
$$

via

$$
- \int_M (\Delta \psi_0 + L\psi_0) \cdot \varphi(\cdot, 0) dx - \int_0^T \int_M (\Delta \psi^* + L\psi^*) \cdot \frac{\partial \varphi}{\partial t} dx dt
$$

$$
- \int_0^T \int_M (\Delta \psi^* + L\psi^*) \cdot \nabla \psi \cdot \nabla \varphi dx dt = \int_0^T \int_M f \cdot \varphi dx dt,
$$

(53)

for every

$$
\varphi \in C^\infty(Q_T)
$$

with

$$
\varphi|_M = \varphi(\cdot, T) = 0.
$$

If a solution

$$
\psi^*
$$

exists, then

$$
\text{the weak problem (53) defines a mapping}
$$

$$
\mathcal{S} : L^\infty(0, T; C^1(M)) \longrightarrow L^\infty(0, T; V) \subset L^\infty(0, T; C^1(M)).
$$

The plan then is to show that this map has a fixed point. Since

$$
\mathcal{S}
$$

maps

$$
L^\infty(0, T; C^1(M))
$$

into

$$
L^\infty(0, T; V),
$$

this fixed point is a solution of the original weak problem (19).

**Lemma 6.1.** Let

$$
T > 0 \text{ and } f \in L^\infty(Q_T).
$$

Then for each

$$
\psi \in L^\infty(0, T; C^1(M)),
$$

the weak problem (53) has at least one solution

$$
\psi^* \in L^\infty(0, T; V).
$$

The basic idea for the proof of Lemma 6.1 is to show that its solution is the weak solution of the transport equation

$$
\begin{cases}
\frac{\partial}{\partial t} q' + u \cdot \nabla q' = f, \\
q'(x, 0) = q_0(x).
\end{cases}
$$

(54)
The weak solution of (54) can be constructed with a flow map \( \Phi_t(a) \) determined from the velocity field \( u \),

\[
\begin{align*}
\frac{d}{dt} \Phi_t(a) &= u(\Phi_t(a), t), \quad t > 0, \\
\Phi_0(a) &= a.
\end{align*}
\] (55)

We now show that, given an continuous velocity field within the domain \( \mathcal{M} \), there exists at least one flow map \( \Phi_t(a) \) satisfying (55).

**Lemma 6.2.** Assume that \( \partial \mathcal{M} \in C^2 \), and let \( T > 0 \) be arbitrary and \( u \in L^\infty(0, T; C(\mathcal{M})) \) with \( u \cdot n = 0 \) on \( \partial \mathcal{M} \). Then the initial value problem (55) has at least one solution \( \Phi_t(a) \) that is valid over \([0, T]\).

**Proof.** For each interior point, a solution can be constructed by the Peano method. The solution can be extended by the same method as long as it has not reached the boundary \( \partial \mathcal{M} \). Hence the proof is complete once it is shown that, starting from the a point on the boundary, there exists at least one solution \( \Phi_t(a) \) that remains on \( \partial \mathcal{M} \) for all time.

The boundary \( \partial \mathcal{M} \) is \( C^2 \) smooth. Then, locally, it can be parameterized by a single parameter \( \tau \in I \). We let \( b(\tau) \) be a vector-valued function representing the boundary. By assumption, \( b(\tau) \in C^2(I) \). If this parameterization of \( \partial \mathcal{M} \) is only local, then one can cover the entire \( \partial \mathcal{M} \) with a finite number of patches, each of which is parameterized by a single parameter. We now show that, starting from any point \( a \in b(I) \), there exists at least one solution \( \Phi_t(a) \) that remains in \( b(I) \) either for all time or until it exits from one of the end points of \( b(I) \). The velocity field on the boundary can also be expressed using the parameter \( \tau \),

\[
u = u(\tau, t).
\]

By assumption, \( u(\tau, t) \) is parallel to the tangential vector on the boundary, and the following relation holds,

\[
u(\tau, t) = \sigma(\tau, t)b'(\tau),
\]

for some scalar function \( \sigma(\tau, t) \) that is continuous in \( \tau \), and bounded in \( t \). We look for a solution of (55) in the form of

\[
\Phi_t(\tau_0) = b(\tau(\tau_0, t)).
\]

We let \( \tau(\tau_0, t) \) be such that

\[
\begin{align*}
\tau(\tau_0, 0) &= \tau_0, \\
\frac{d}{dt} \tau(\tau_0, t) &= \sigma(\tau(\tau_0, t), t).
\end{align*}
\] (56)

Then it is easy to check that \( \Phi_t(\tau_0) \) solves the initial-value problem (55), for

\[
\begin{align*}
\Phi_0(\tau_0) &= b(\tau_0), \\
\frac{d}{dt} \Phi_t(\tau_0) &= b'(\tau(\tau_0, t)) \cdot \sigma(\tau(\tau_0, t), t) = u(\tau(\tau_0, t), t).
\end{align*}
\]

The existence of a solution for the duration of \( \tau(\tau_0, t) \in I \) is just another application of the Peano method.

We now prove the existence of a solution to the linearized equation (53).
Proof of Lemma 6.1. We let \( \Phi_t(\cdot) \) be a global solution of the initial value problem (55). We define a new QGPV \( q' \) from the given initial state \( q_0 \) via

\[
q'(x, t) = q_0(\Phi_t(x)) + \int_0^t f(\Phi_{s-t}(x), s)ds, \quad \forall x \in \mathcal{M}, t > 0,
\]

or equivalently,

\[
q'(\Phi_t(a), t) = q_0(a) + \int_0^t f(\Phi_s(a), s)ds, \quad \forall a \in \mathcal{M}, t > 0.
\]

It is clear that, given \( f \in L^\infty(Q_T) \), \( q' \in L^\infty(Q_T) \). We now verify that \( q'(x, t) \) defined in (57) solves the transport equation (54) in the weak sense. We denote \( u = \nabla^\perp \psi \), and let \( \varphi \in C^\infty(Q_T) \) with \( \varphi|_{\partial \mathcal{M}} = \varphi(\cdot, T) = 0 \). Using the fact that the map \( x = \Phi_t(a) \) is area preserving, we derive that

\[
\int_0^T \int_{\mathcal{M}} q'(x, t) \left( \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) dxdt = \int_0^T \int_{\mathcal{M}} \left( q_0(a) + \int_0^t f(\Phi_s(a), s)ds \right) \frac{d}{dt} \psi(\Phi_t(a), t)dadt
\]

\[
= -\int_0^T \int_{\mathcal{M}} q_0(a) \varphi(a, 0)da - \int_0^T \int_{\mathcal{M}} f(\Phi_t(a), t)\varphi(\Phi_t(a), t)dadt
\]

\[
= -\int_{\mathcal{M}} q_0(x)\varphi(x, 0)dx - \int_0^T \int_{\mathcal{M}} f(x, t)\varphi(x, t)dxdt.
\]

Hence, we have shown that

\[
-\int_{\mathcal{M}} q_0(x)\varphi(x, 0)dx - \int_0^T \int_{\mathcal{M}} q'(x, t) \left( \frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi \right) dxdt = \int_0^T \int_{\mathcal{M}} f(x, t)\varphi(x, t)dxdt.
\]

We let \( \psi' \) be a solution to the boundary value problem (11) corresponding to the QGPV \( q' \). Then \( \psi' \in L^\infty(0, T; V) \) and it is a solution to the weak problem (53). \( \square \)

Lemma 6.3. The solution \( \psi' \) to the weak problem has the following estimates,

\[
|\psi'|_{L^\infty(0, T; V)} \leq |\psi_0|_V + |f|_{L^1(0, T; L^\infty(\mathcal{M}))},
\]

\[
\max_{0 \leq t \leq T} \left| \frac{\partial \psi'}{\partial t} \right|_{W^{1, p}(\mathcal{M})} \leq C \max_{0 \leq t \leq T} \left( |F|_{L^p} + |\psi'|_{L^\infty(Q_T)} |\nabla \psi|_{L^p(\mathcal{M})} \right). \quad (60)
\]

Proof. We start from the equation (57),

\[
|q'(\cdot, t)|_{L^\infty(\mathcal{M})} \leq |q_0|_{L^\infty(\mathcal{M})} + \int_0^t |f(\cdot, s)|_{L^\infty(\mathcal{M})}ds,
\]

\[
|q'|_{L^\infty(Q_T)} \leq |q_0|_{L^\infty(\mathcal{M})} + \int_0^T |f(\cdot, s)|_{L^\infty(\mathcal{M})}ds
\]

\[
= |q_0|_{L^\infty(\mathcal{M})} + |f|_{L^1(0, T; L^\infty(\mathcal{M}))}.
\]

The inequality (59) follows.

The inequality (60) can be established in a similar way as in Lemma 4.2. \( \square \)
Lemma 6.4. The mapping
\[ \psi \mapsto \psi' \text{ in } L^\infty(0, T; C^1(\mathcal{M})) \]
is compact.

Proof. We let \( \{ \psi_r \} \) be a bounded sequence in \( L^\infty(0, T; C^1(\mathcal{M})) \), and let \( \psi'_r \) be the corresponding sequence of solutions of (53). Then, by (59) and (60), we obtain, for \( \forall p > 1 \),
\[ \max_t \| \partial^2 \psi'_r \|_{L^p(\mathcal{M})} \leq C, \]
where \( C \) is a constant independent of the index \( r \). Thus \( \partial \psi'_r / \partial x \) has one generalized derivative in both \( t \) and \( x \), and
\[ \left\| \partial \psi'_r / \partial x \right\|_{W^{1, p}(Q_T)} \leq C, \quad \forall p > 1. \]
For \( p > 3 \), and by the Sobolev embedding theorem, \( \partial \psi'_r / \partial x \) is Hölder continuous in the temporal-spatial domain \( Q_T \), and
\[ \left\| \partial \psi'_r / \partial x \right\|_{C^{0, \lambda}(Q_T)} \leq C, \]
for some \( 0 < \lambda < 1 \). This shows that \( \partial \psi'_r / \partial x \) are equi-continuous in \( Q_T \), and so is \( \psi'_r \). By the Arzelá-Ascoli theorem, there exists a subsequence, still denoted by the index \( r \), such that
\[ \psi'_r \longrightarrow \psi, \]
\[ \partial \psi'_r / \partial x \longrightarrow \varphi. \]
Due to the completeness of the Banach space \( L^\infty(0, T; C^1(\mathcal{M})) \), we have that \( \psi \in L^\infty(0, T; C^1(\mathcal{M})) \) and \( \varphi = \partial \psi / \partial x \).

Theorem 6.5. There exists a solution to the weak problem (19) in \( L^\infty(0, T; V) \).

Proof. The mapping
\[ S : \psi \mapsto \psi' \]
is compact in \( L^\infty(0, T; C^1(\mathcal{M})) \). By Schauder’s fixed point theorem, it has a fixed point \( \psi \) in the same function space. Since \( S \) maps from \( L^\infty(0, T; C^1(\mathcal{M})) \) into \( L^\infty(0, T; V) \), the fixed point \( \psi \) belongs to \( L^\infty(0, T; V) \) as well.

Theorem 6.6. Let \( f = \nabla \times F \) be bounded and \( F \) be strongly continuous in time \( t \). Then the initial and boundary conditions (7)–(10) are satisfied in the classical sense, and \( \Delta \psi, \partial^2 \psi / \partial x \partial t \) are strongly continuous with respect to \( t \) on \([0, T]\) in \( L^p(\mathcal{M}) \) for any \( p > 1 \).

Proof. We first show that the QGPV \( q(\cdot, t) \) is continuous in \( t \) for any \( L^p(\mathcal{M}) \) norm with \( p > 1 \). This improves over (30a) of Lemma 4.1. Starting from (22) and by a well-known result ([24], Lemma 1.1 of Section 3.1), we derive that, for some \( 0 \leq \tau_1 < \tau_2 \leq T \),
\[ (q(\cdot, \tau_2), \varphi) - (q(\cdot, \tau_1), \varphi) = \int_{\tau_1}^{\tau_2} \int_{\mathcal{M}} (q \nabla \cdot \psi + f \cdot \varphi) \, dx \, dt. \]
We note that $\psi \in L^\infty(0, T; V)$, $q$ is bounded in $L^\infty(Q_T)$, and $\nabla^\perp \psi$ is uniformly bounded in $Q_T$. Thus, as $\tau_2 \rightarrow \tau_1$, the right-hand side vanishes, and one has
\begin{equation}
q(\cdot, \tau_2) \rightarrow q(\cdot, \tau_1) \quad \text{in any } L^p(M).
\end{equation}

Similar to (59), one can derive that, for $\forall p > 1$,
\begin{equation}
|q(\cdot, \tau_2)|_{L^p(M)} \leq |q(\cdot, \tau_1)|_{L^p(M)} + \int_{\tau_1}^{\tau_2} |f(\cdot, t)|_{L^p(M)} dt.
\end{equation}

From this estimate we conclude that
\begin{equation}
\lim_{\tau_2 \rightarrow \tau_1} |q(\cdot, \tau_2)|_{L^p(M)} \leq |q(\cdot, \tau_1)|_{L^p(M)}.
\end{equation}

By the Radon-Riesz theorem, $q(\cdot, \tau_2)$ converges to $q(\cdot, \tau_1)$, as $\tau_2 \rightarrow \tau_1$, in the strong norm of $L^p(M)$. Hence, $q(\cdot, t)$ is continuous in $t$ in any $L^p(M)$,
\begin{equation}
q \in C([0, T]; L^p(M)),
\end{equation}

which implies that the initial condition (10) is satisfied in a stronger norm,
\begin{equation}
q(\cdot, 0) = q_0(\cdot) \quad L^p(M), \, \forall \, p > 1.
\end{equation}

Concerning the continuity of $\partial^2/\partial x \partial t$, we derive from (7) that
\begin{equation}
(\Delta + L) \frac{\partial}{\partial t} \psi = \nabla \times F - \nabla \cdot (\nabla^\perp \psi (\Delta \psi + L \psi)) \quad \text{in the distribution sense.}
\end{equation}

Thus, formally, one has
\begin{equation}
\nabla \frac{\partial}{\partial t} \psi = \nabla (\Delta + L)^{-1} \nabla \times F - \nabla (\Delta + L)^{-1} \nabla \cdot (\nabla^\perp \psi (\Delta \psi + L \psi)),
\end{equation}

where $(\Delta + L)^{-1}$ is the solution operator of the elliptic boundary value problem (11). We note that $q \equiv \Delta \psi + L \psi$ is continuous in $t$ in any $L^p(M)$ with $p > 1$, and $\nabla^\perp \psi$ is uniformly bounded in $Q_T$. Thus, thanks to the continuity of the differential operator $\nabla (\Delta - I)^{-1} \nabla (\cdot)$, and the continuity of $F$ with respect to $t$, $\nabla \frac{\partial \psi}{\partial t}$ is continuous in $t$ in any $L^p(M)$ with $p > 1$.

By Lemma 3.1,
\begin{equation}
|\psi(\cdot, \tau_2) - \psi(\cdot, \tau_1)|_{W^{2, p}(M)} \leq Cp|q(\cdot, \tau_2) - q(\cdot, \tau_1)|_{L^p(M)}.
\end{equation}

Thus, as $\tau_2 \rightarrow \tau_1$,
\begin{equation}
\psi(\cdot, \tau_2) \rightarrow \psi(\cdot, \tau_1) \quad \text{in } W^{2, p}(M),
\end{equation}

This shows that the initial condition (10) holds in $W^{2, p}(M)$,
\begin{equation}
\psi(\cdot, 0) = \psi_0(\cdot) \quad \text{in } W^{2, p}(M).
\end{equation}

We also note that $\psi \in L^\infty(0, T; V)$ implies that
\begin{equation}
\frac{\partial \psi}{\partial x} \in L^\infty(0, T; W^{1, p}(M)).
\end{equation}

From Lemma 4.2, we have
\begin{equation}
\frac{\partial^2 \psi}{\partial t \partial x} \in L^\infty(0, T; L^p(M)) \subset L^p(Q_T).
\end{equation}

Combining (64) and (65), we derive that
\begin{equation}
\frac{\partial \psi}{\partial x} \in W^{1, p}(Q_T), \quad \forall \, p > 1.
\end{equation}
We take a $p > 3$. Then, by the Sobolev imbedding theorem,
\[
\frac{\partial \psi}{\partial x} \in C^{0,\lambda}(Q_T) \quad \text{for some } 0 < \lambda < 1.
\] (67)
Thus, the streamfunction $\psi$ is continuous in the spatial-temporal domain, and the initial and boundary conditions are satisfied in the classical sense.

Finally, (62), together with (67), implies that
\[
\Delta \psi \in C([0,T]; L^p(M)).
\]

We note that $q$ assumes its initial value $q_0$ in the $L^p$-norm ($\forall p > 1$), which is an improvement over (31a).

7. **Concluding remarks.** As far as model complexity is concerned, the suite of QG models sit between the purely planar two-dimensional Euler/Navier-Stokes equations and the fully three-dimensional Euler/Navier-Stokes equations. Even though the three-dimensional QG equation is posed over a three-dimensional spatial domain, the velocity vector at every point in the space is assumed to be horizontal, i.e. two-dimensional. The theory on the two-dimensional planar flow is rather complete, see, among many other references, [26, 3, 16]. On the other hand, the theory about the three-dimensional Euler or NSEs is rather incomplete. For example, it remains an open question whether the three-dimensional NSEs is globally well-posed ([24, 10]. The situation with the inviscid model (3D Euler) is generally worse ([4, 9, 5, 7]). It is then natural to ask how the QG equations fare as far as the well-posedness is concerned. The global well-posedness of the single-layer barotropic QG has been established in a previous study ([8]). The current work deals with the multi-layer QG equations. The multi-layer QG can be viewed as a stack of single-layer barotropic equations. The layers interact with each other through pressure. The layer interactions add some vertical variations to the model, and move it one step closer to the full three-dimensional fluid model. The added vertical variations make the well-posedness issue more interesting and more challenging at the same time. Notwithstanding the technical challenges, the inviscid multi-layer QG model is shown to be globally well-posed. In what follows, we briefly review the challenge and the general approach of the present work.

The presence of multiple layers and the interactions among them are dealt with by a combination of two techniques: mode decomposition and straightforward estimation. When the positive (or negative) definiteness is needed, a decomposition in the barotropic and baroclinic modes, which are the eigenmodes of the associated Sturm-Liouville problem, is employed. In other places, i.e. within the nonlinear terms, a straightforward estimation is carried out on the streamfunctions across all layers. The last technique works, of course, thanks to the finiteness in the number of layers and the consequential boundedness in the thickness of the layers.

Concerning the general approach of the analysis, in establishing the global well-posedness of the barotropic QG equation, Chen [8] follows [18], and employs the Picard iterations to prove the existence and uniqueness of the flow map and the convergence of the iterative scheme. The downside of this approach is that it has a higher regularity requirement on the right-hand side forcing, which needs to be uniformly continuous in space. Yudovich [26] establishes the convergence of the iterative scheme by the Schauder fixed-point theorem, and the right-hand side forcing
is only assumed to be essentially bounded. In order to remove the somewhat stringent requirement on the forcing in [8], the current work adopts the approach of [26], but with one modification. The existence of a solution to the linearized problem is established by a flow map corresponding to a continuous velocity field, instead of the regularization technique employed by Yudovich. The flow map of the merely continuous velocity field is not unique, and therefore the solution to the linearized problem is not unique either ([12, 1, 2]). Fortunately, the uniqueness is not required by the Schauder fixed-point theorem. The uniqueness of the weak solution to the nonlinear QG equation has been established by the a priori estimates.

The next target for the current project is naturally the three-dimensional QG equation. The equation represents a giant step from the barotropic or the multi-layer QG equations. It is a three-dimensional model with an infinite number of degrees of freedom in the vertical direction. The techniques of the current and the previous work ([8]), adept at treating a single layer, or a finite number of layers, will probably be inadequate for the three-dimensional model. It is expected that new techniques will have to be introduced or invented. Progresses in this regard will be reported elsewhere.

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