Ballistic aggregation: a solvable model of irreversible many particles dynamics

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The adhesive dynamics of a one-dimensional aggregating gas of point particles is rigorously described. The infinite hierarchy of kinetic equations for the distributions of clusters of nearest neighbours is shown to be equivalent to a system of two coupled equations for a large class of initial conditions. The solution to these nonlinear equations is found by a direct construction of the relevant probability distributions in the limit of a continuous initial mass distribution. We show that those limiting distributions are identical to those of the statistics of shocks in the Burgers turbulence. The analysis relies on a mapping on a Brownian motion problem with parabolic constraints.

1. INTRODUCTION

Rigorous solutions of the many-body dynamics are extremely rare. The present paper yields up such a precious case, reviewing parts of the existing works and completing the research which has been carried out in recent years.

The system under study is a one-dimensional gas of point particles forming aggregates through perfectly inelastic, adhesive collisions. The motion between collisions being free, the process is called ballistic aggregation. This dynamics is deterministic, randomness occurs only through the distribution of initial datas. In the context of statistical mechanics, the model was introduced and studied numerically in [1] and further investigated analytically in [2]. However in the context of fluid dynamics, it has been recognized much earlier (see e.g. [3], [4]) that the evolution of shocks in the inviscid limit of the (decaying) one-dimensional Burgers equation obeys the laws of ballistic aggregation. Moreover when the initial particle velocities are Gaussian and independent (equivalently when the initial Burgers field is a white noise) both models are isomorphic to a problem of Brownian motion under parabolic constraints. The latter problem has eventually received a solution in closed analytical form [5], [6] which enables to predict the exact distribution of shocks in the decaying Burgers turbulence, and correspondingly the distribution of masses and velocities of the aggregating particles. These close connections between problems of different origins is an attractive feature of ballistic aggregation. The purpose of this work is to establish precisely the above mentioned relations and to give an exact and fairly complete description of ballistic aggregation.

There are essentially two approaches. The first one consists in writing the dynamical hierarchy of equations coupling the many-particle distribution functions using the tools of kinetic theory, and hopefully solving these equations. The second route is by explicitly constructing the dynamics from suitably chosen initial conditions: the construction should of course yield a solution of the kinetic equations. Both approaches are discussed here and both will make the link with the dynamics of Burgers shock waves manifest.

After recalling the way in which one can describe the state of an infinite gas (Section II), we present in Section III the derivation of the infinite hierarchy of kinetic equations coupling the time evolution of distributions of many-particle clusters of nearest neighbours. The derived hierarchy, after integration over momenta of the aggregates, turns out to be identical to the hierarchy found in the study of the Burgers turbulence [6].

In Section IV an original result is derived. The infinite dynamical hierarchy is shown to be compatible with the factorization of the many-body distributions into products of two-particle conditional distributions and the one-particle density. Such a factorization is propagated by the ballistic aggregation. This yields an exact closure of the hierarchy and permits to reduce it to a system of two coupled equations (Section V) which allow self-similar solutions (Section VI).

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The Section VII is devoted to an explicit construction of the particle distributions. The construction, which has already been presented in III, uses a simple initial condition which enables to reformulate the problem within the theory of Brownian motion. To establish the relation with shock wave dynamics at this level, one must envisage the delicate question of the continuum limit. Indeed the Burgers velocity field takes values in the continuum, whereas masses of aggregates, resulting of sums of elementary initial masses, have discrete values. A strict isomorphism will only be obtained in the continuum limit of the aggregation process letting the distribution of initial masses tend to a continuous and uniform mass density. In terms of Brownian motion this leads to the subtle problem of controlling the cumulated effect of Brownian excursions in infinitely many small time intervals. This problem is addressed and extensively discussed in Section VII, in the continuum limit the particle distributions become identical to those of the shocks in Burgers dynamics so that all the results of immediately apply. All distributions of order three and more factorize in the way described in Section IV, so giving a complete statistical description of the aggregation process. In Section VIII the exact predictions for the mass and velocity distributions are compared to existing bounds as well as to the findings of previously formulated approximate theories. In particular significant discrepancies with mean field theory appear in the domains of large and small masses. The asymptotic form of the two-particle correlation function already been presented in [3], uses a simple initial condition which enables to reformulate the problem within the theory of Brownian motion. To establish the relation with shock wave dynamics at this level, one must envisage the cumulated effect of Brownian excursions in infinitely many small time intervals. This problem is addressed and

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II. STATES OF THE AGGREGATING GAS

At any time $t > 0$, the system is composed of point particles moving in $R^1$. The state of a given particle is entirely characterized by specifying its position $X$, its momentum $P$, and its mass $M$.

Consider a closed space interval $[L_1, L_2]$, where $L_1 < L_2$. We denote by

$$\mu_k(1, 2, \ldots, k; t|L_1, L_2)$$

the probability density for finding precisely $k$ particles within the interval $[L_1, L_2]$ at time $t$, occupying the sequence of states

$$j \equiv (X_j, P_j, M_j), \quad j = 1, 2, \ldots, k$$

ordered in space according to the inequalities

$$L_1 < X_1 < X_2 < \ldots < X_k < L_2$$

The set of densities $\mu_k(1, 2, \ldots, k; t|L_1, L_2)$, $k = 0, 1, 2, \ldots$, defined for arbitrary intervals $[L_1, L_2]$, provides a complete statistical description of the state of the infinite volume of the gas. Summing up the probability weights for all possible events within $[L_1, L_2]$ one gets the normalization condition

$$\mu_0(t|L_1, L_2) + \sum_{k=1}^{\infty} \int d1 \int d2 \cdots \int dk \theta(X_1 - L_1) \theta(L_2 - X_k) \prod_{j=2}^{k} \theta(X_j - X_{j-1}) \mu_k(1, 2, \ldots, k; t|L_1, L_2) = 1$$

where

$$\int dj = \int_{-\infty}^{+\infty} dX_j \int_{-\infty}^{+\infty} dP_j \int_{0}^{+\infty} dM_j$$

and $\theta(X)$ is the unit Heaviside step function.

Knowing the probability densities $\mu_k(1, 2, \ldots, k; t|L_1, L_2)$ one can evaluate the densities $\mu_k(1, 2, \ldots, k; t)$ of the nearest neighbours configurations by letting the extremities of the interval $[L_1, L_2]$ approach the positions of the first and the last particle, respectively

$$\mu_k(1, 2, \ldots, k; t) \equiv \lim_{L_1 \to X_1} \lim_{L_2 \to X_k} \mu_k(1, 2, \ldots, k; t|L_1, L_2)$$
In particular, $\mu_1(1; t)$ represents simply the number density of particles at time $t$ in the one-particle state $1 \equiv (X_1, P_1, M_1)$.

An alternative way of describing the state of an infinite system consists in determining the complete set of reduced distributions $\rho_k(1, 2, \ldots, k; t)$, representing the number density of ordered $k$-particle states $(1, 2, \ldots, k)$. The difference with respect to the densities $\mu_k(1, 2, \ldots, k; t)$ comes from the fact that the $k$ particles need not represent the set of nearest neighbours. Although in evaluating $\rho_k(1, 2, \ldots, k; t)$ the inequalities

$$X_1 < X_2 < \ldots < X_k$$

are still supposed to hold, one has to consider all configurations of the system compatible with the condition that the states $(1, 2, \ldots, k)$ are occupied at time $t$, with an arbitrary number of particles between pairs $(j, j+1)$, $j = 1, \ldots, k-1$. Clearly, there is then no difference at the level of the one-particle densities, and, as it has already been mentioned, the equality

$$\rho_1(1; t) = \mu_1(1; t)$$

holds.

For the two-particle reduced density $\rho_2(1, 2; t)$, the corresponding formula reads

$$\rho_2(1, 2; t) = \mu_2(1, 2; t) + \sum_{r=1}^{\infty} \int d1' \int d2' \ldots \int dr' \mu(X_1' - X_1) \theta(X_2 - X_r') \times \prod_{j=1}^{r-1} \theta(X_{r-1}' - X_j') \mu_{2+r}(1, 1', 2', \ldots, r', 2; t)$$

The possibility of the presence of $r$ particles ($r = 0, 1, 2, \ldots$) between the two particles at $X_1$ and at $X_2$, has been taken into account in (8). By definition (8) the density $\mu_{2+r}(1, 1', 2', \ldots, r', 2; t)$ vanishes outside the region $X_1 < X_1' < \ldots < X_r' < X_2$, reflecting the linear ordering of the particles.

In a similar way (i.e. by considering all possible intermediate states) one can express $\rho_k(1, 2, \ldots, k; t)$ in terms of the densities of nearest neighbours $\mu_{k+r}(1, 2, \ldots, k+r; t)$, $r = 0, 1, 2, \ldots$, for arbitrary $k = 3, 4, \ldots$. The general relations between both types of description of the states of an infinite system are discussed in Ruelle’s book [5].

### III. EVOLUTION OF THE DISTRIBUTION OF NEAREST NEIGHBOURS

Our main object here is the study of the dynamics of aggregation. The point masses forming the system move freely between collisions. When two of them collide, they instantaneously merge forming a new particle whose mass is equal to the sum of their masses. As the momentum is also conserved, the aggregate continues the motion along the center of mass trajectory of the pair it has been formed from. Hence, the microscopic dynamics consists of periods of a free motion separated by perfectly inelastic (sticky or adhesive) binary collisions.

Suppose that at some moment two particles occupy the states $a = (X_a, P_a, M_a)$ and $b = (X_b, P_b, M_b)$, with $X_a < X_b$. The notation $(a + b)$ will be used to denote the state corresponding to the center of mass motion of such an ordered pair. Hence

$$(a + b) \equiv \left( \frac{M_a X_a + M_b X_b}{M_a + M_b}, P_a + P_b, M_a + M_b \right)$$

We shall restrict our analysis to spatially homogeneous systems. The one-particle density does not depend in this case on the position variable

$$\mu_1(1; t) = \mu_1(P_1, M_1; t)$$

Therefore, the changes of $\mu_1(1; t)$ in the course of time are exclusively due to collisions. Clearly, in one dimension only the nearest neighbours can collide. The rate of collisions between adjacent particles $j$ and $(j+1)$ is proportional to the density of the nearest neighbours $\mu_2(j, j+1; t)$. It can be written as

$$C(j, j+1) \mu_2(j, j+1; t)$$

where the collision factor
\[ C(j, j + 1) \equiv \left( \frac{P_j}{M_j} - \frac{P_{j+1}}{M_{j+1}} \right) \theta \left( \frac{P_j}{M_j} - \frac{P_{j+1}}{M_{j+1}} \right) \delta(X_{j+1} - X_j - 0+) \]  \hspace{1cm} (13)

chooses (with the help of the Dirac \( \delta \)-distribution) only the precollisional configurations. As the motion between collisions is free, the rate \( \frac{\partial}{\partial t} \mu_1 \) is proportional to the relative velocity of the colliding pair (the \( \theta \)-factor in \( C(j, j + 1) \) assures the mutual approach of the particles).

In order to evaluate the time derivative of the one-particle density \( \mu_1 \), let us consider first the events which lead to the creation of the state 1 by adhesive collisions. Their rate of occurrence is given by

\[ \int d1' \int d2' C(1', 2') \delta[1 - (1' + 2')] \mu_1(1', 2'; t) \]  \hspace{1cm} (14)

Here the distribution

\[ \delta[1 - (1' + 2')] = \delta \left[ X_1 - \frac{M'_1 X'_1 + M'_2 X'_2}{M'_1 + M'_2} \right] \delta[P_1 - (P'_1 + P'_2)] \delta[M_1 - (M'_1 + M'_2)] \]  \hspace{1cm} (15)

picks out only those collisions which create the aggregate in the state 1.

Consider now the events leading to the destruction of state 1. When the precollisional state of one of the particles is 1, the collision removes this state from the system. The rate of such annihilating events is given by

\[ \int d1' C(1', 1) \mu_2(1', 1; t) + \int d1' C(1', 1) \mu_2(1', 1; t) \]  \hspace{1cm} (16)

Combining the gain term \( \frac{\partial}{\partial t} \mu_1 \) and the loss terms \( \frac{\partial}{\partial t} \mu_2 \), we arrive at the equation

\[ \frac{\partial}{\partial t} \mu_1(1; t) = \int d1' \int d2' C(1', 2') \delta[1 - (1' + 2')] \mu_2(1', 2'; t) \]  \hspace{1cm} (17)

In a similar way one can derive the evolution equation for the two-particle distribution of the nearest neighbours. It reads

\[ \left[ \frac{\partial}{\partial t} + L_{12} + C(1, 2) \right] \mu_2(1, 2; t) = \]  \hspace{1cm} (18)

where

\[ L_{12} = \frac{P_1}{M_1} \frac{\partial}{\partial X_1} + \frac{P_2}{M_2} \frac{\partial}{\partial X_2} \]  \hspace{1cm} (19)

is the generator of free streaming. On the left hand side there appears the loss term \( C(1, 2) \mu_2(1, 2; t) \) describing the destroying effect of a possible merging of the pair \((1, 2)\). The right hand side takes into account the processes of creation of aggregates in the states 1 or 2 (the first two terms). Finally, the last two terms represent the destruction of the two-particle state \((1, 2)\) by collisions with the left nearest neighbour of particle 1, and with the right nearest neighbour of particle 2, respectively. Equation \( \frac{\partial}{\partial t} \mu_1(1, 2; t) \) relates the rate of change of \( \mu_2(1, 2; t) \) to the three-particle density \( \mu_3(1, 2, 3; t) \).

Denoting by \( L_{1,...,k} \) the \( k \)-particle generator of free streaming

\[ L_{1,...,k} = \sum_{j=1}^{k} \frac{P_j}{M_j} \frac{\partial}{\partial X_j} \]  \hspace{1cm} (20)

we write the general \( k \)-th equation of the infinite hierarchy \((k = 1, 2, \ldots)\) in the form
The fact that \( \mu_k(1, \ldots, j, j + 1, \ldots, k; t) \) is the density of a sequence of \( k \) nearest neighbours imposes a restrictive condition on its support. Indeed, tracing backward in time the free trajectory of particle \( j \) to the moment \( t = 0 \), we find the position of the center of mass of the part of the initial system it has been formed from. So, if we denote by \( \rho \) the constant mass density at \( t = 0 \), the point

\[
X_j^{\text{right}}(t) = X_j - \frac{P_j}{M_j} t + \frac{M_j}{2\rho} \equiv X_j + Y_j^-
\]

has the meaning of the right extremity of the initial mass region which produced the mass \( M_j \) through adhesive collisions. This right extremity must coincide with the left extremity

\[
X_j^{\text{left}}(t) = X_{j+1} - \frac{P_{j+1}}{M_{j+1}} t + \frac{M_{j+1}}{2\rho} \equiv X_{j+1} - Y_{j+1}^+
\]

of the region which produced the mass \( M_{j+1} \). Otherwise, the pair \((j, j + 1)\) would not represent nearest neighbours. In (22) and (23) we have introduced the notation

\[
Y_j^\pm = \frac{P_j}{M_j} t \pm \frac{M_j}{2\rho}.
\]

We thus conclude that the density \( \mu_k(1, \ldots, k; t) \) contains necessarily the singular factor

\[
\prod_{j=1}^{k-1} \delta \left[ X_{j+1} - X_j - Y_{j+1}^+ + Y_j^- \right]
\]

This conclusion implies important consequences for collisional configurations where \( X_{j+1} = X_j + 0^+ \). The distribution (23) imposes then the condition

\[
Y_{j+1}^+ = Y_j^-
\]

Hence, in the hierarchy equations (21) one can replace the collision terms \( C(j, j + 1) \) (see (13)) by

\[
\frac{M_j + M_{j+1}}{2\rho t} \delta(X_{j+1} - X_j - 0^+)
\]

With the use of (24), one finds, after integrating over all momenta, the infinite hierarchy for the distribution of masses and distances between the particles which has already been derived in the study of the Burgers model of turbulence [8]. Our independent reasoning based on the microscopic laws of the aggregation process confirms the existence of a one-to-one correspondence between the two problems.

It is quite remarkable that the infinite set of equations (21) can be rigorously reduced to a system of two coupled nonlinear equations under a simple assumption, which can be shown to be satisfied for a large class of initial conditions. This important fact, derived in the next section, has not been noticed in [9]. The authors restricted their analysis therein to the first equation of the hierarchy, supplemented with an integral relation obtained by integrating equation (18) over both momenta and masses. In the search for self-similar solutions, they assumed a specific form of two-particle correlations, which eventually led them to an erroneous conclusion that the masses were exponentially distributed. The exponential distribution has been eighteen years later conjectured again in [1], and shown to follow from the hierarchy equations (21) within the weak mean-field approximation [3]. It turned out that a rigorous analysis invalidated these approximate results (see [4] and section IX).
IV. REDUCTION OF THE HIERARCHY

A fundamental role in further considerations will be played by the conditional probability density

$$\mu(2|1; t) = \frac{\mu_2(1, 2; t)}{\mu_1(1; t)}$$  \hspace{1cm} (28)

for finding the right nearest neighbour of a particle supposed to occupy the state 1, in the state 2 at time t. The normalization

$$\int d2 \mu(2|1; t) = 1$$ \hspace{1cm} (29)

expresses the fact that in a homogeneous system the right nearest neighbour does exist in some state with certainty.

The singular factor

$$\delta \left[ X_2 - X_1 - Y_2^+ + Y_1^- \right]$$\hspace{1cm} (30)

present in \(\mu_2(1, 2; t)\) (see (22)), and thus also in \(\mu(2|1; t)\), involves the combination \(Y_1^- = tP_1/M_1 - M_1/2\rho\) of mass and momentum of particle 1. This combination appears in formula (22) for the position of point \(X_j^{\text{right}}(t)\), separating from the right hand side the masses which contributed to the formation of particle \(j\) from the rest of the system.

The reduction of the dynamical hierarchy (21) presented in this section is possible if \(\mu(2|1; t)\) depends on \(P_1\) and \(M_1\) exclusively via the effective variable \(Y_1^-\). The following structure of the conditional probability density will be thus assumed

$$\mu(2|1; t) = \delta \left[ X_2 - X_1 - Y_2^+ + Y_1^- \right] \bar{\mu}(M_2, P_2, Y_1^-; t)$$  \hspace{1cm} (31)

In fact, it will be shown by construction in Section VII that for a large class of initial conditions the above hypothesis is verified.

The presence of factor (25) in \(\mu_k(1, \ldots, k; t)\) clearly indicates the existence of correlations between the states of the nearest neighbours. Let us suppose that only two-particle correlations exist, so that

$$\mu_k(1, 2, \ldots, k; t) = \mu_1(1; t) \prod_{j=1}^{j=k-1} \mu(j+1|j; t)$$  \hspace{1cm} (32)

We shall prove now that the above factorization is compatible with the dynamics of the system, and that the factorized densities (32) yield a solution to the hierarchy equations (21), provided \(\mu_1(1; t)\) and \(\mu(2|1; t)\) satisfy the system of coupled equations

\[
\frac{\partial}{\partial t} \mu_1(1; t) = \int d1' \int d2' C(1', 2') \delta[1 - (1' + 2')]\mu_1(1'; t)\mu(2'|1'; t) \\
- \int d1'C(1', 1)\mu_1(1'; t)\mu(1'|1'; t) - \int d1'C(1', 1)\mu_1(1; t)\mu(1'|1; t) + L_{12} + C(1, 2) \mu(2|1; t)
\]

\[
\left[ \frac{\partial}{\partial t} + L_{12} + C(1, 2) \right] \mu(2|1; t) = \int d1' \int d2' C(1', 2') \delta[2 - (1' + 2')]\mu(1'|1; t)\mu(2'|1'; t) + \left\{ \int d1'C(1', 1)\mu(1'|1; t) - \int d1'C(2, 1)\mu(1'|2; t) \right\} \mu(2|1; t)
\]

Equation (33) is just the first equation of the hierarchy (17) in which \(\mu_2(1, 2; t)\) has been replaced by \(\mu_1(1; t)\mu(2|1; t)\).

Let us consider the general equation (21) for \(k \geq 2\). Upon inserting the factorized form (32) of the densities \(\mu_k\) and \(\mu_{k+1}\) we get (with the use of the first equation (33)) the following relation

$$\mu_1(1; t) \left[ \frac{\partial}{\partial t} + L_{1\ldots k} + \sum_{j=1}^{k-1} C(j, j+1) \right] \prod_{r=1}^{k-1} \mu(r+1|r; t) =$$
So, taking into account the supposed structure (31) of condition al densities, we can replace

$$\int d1' \int d2'C(1', 2')\delta[1 - (1' + 2')]\mu_2(1', 2'; t) | \mu(2)2'; t) - \mu(2|2'; t)| \prod_{j=2}^{j=k-1} \mu(j + 1|j; t)$$

$$+ \mu_1(1; t) \sum_{j=2}^{k} \int d1' \int d2'C(1', 2')\delta[j - (1' + 2')] \prod_{r=1}^{j-2} \mu(r + 1|r; t)$$

$$\times \mu(1'|j - 1; t) \mu(2'|1'; t) \mu(j + 1|2'; t) \prod_{s=j+1}^{k-1} \mu(s + 1|s; t)$$

$$+ \mu_1(1; t) \left[ \int d1'C(1, 1') \mu(1'1; t) - \int d1'C(k, 1') \mu(1'k; t) \right] \prod_{j=1}^{k-1} \mu(j + 1|j; t).$$

The first term on the right hand side vanishes. Indeed, equation (26) implies the equality

$$- \frac{P_1}{M_1} t + \frac{M_1}{2\rho} = - \frac{P_1' + P_2' t + M_1' + M_2'}{2\rho} = - \frac{P_1'}{M_1'} t + \frac{M_2'}{2\rho}$$

So, taking into account the supposed structure (31) of conditional densities, we can replace \( \mu(2|2'; t) \) in (37) by \( \mu(2|1; t) \). Using exactly the same reasoning one can show that in the remaining gain terms in (35), involving the creation of particles \( j = 2, 3, \ldots, k \) through aggregation of \( 1' \) and \( 2' \), the density \( \mu(j + 1|2'; t) \) equals \( \mu(j + 1|j; t) \). This permits to rewrite the hierarchy equation (35) in the following form

$$\left[ \frac{\partial}{\partial t} + L_{1...k} + \sum_{j=1}^{k-1} C(j, j + 1) \right] \prod_{j=1}^{k-1} \mu(j + 1|j; t) =$$

$$\sum_{j=2}^{k} \int d1' \int d2'C(1', 2')\delta[j - (1' + 2')] \prod_{r=1}^{j-2} \mu(r + 1|r; t) \mu(1'|j - 1; t) \mu(2'|1'; t) \prod_{s=j}^{k-1} \mu(s + 1|s; t)$$

$$+ \left[ \int d1'C(1, 1') \mu(1'1; t) - \int d1'C(k, 1') \mu(1'k; t) \right] \prod_{j=1}^{k-1} \mu(j + 1|j; t).$$

The last term in (37) can be conveniently rewritten as

$$\sum_{j=2}^{k} \left[ \int d1'C(j - 1, 1') \mu(1'|j - 1; t) - \int d1'C(j, 1') \mu(1'|j; t) \right] \prod_{r=1}^{k-1} \mu(r + 1|r; t)$$

A straightforward calculation shows then that equation (35) is satisfied (for any \( k \geq 1 \)) if the conditional probability density \( \mu \) is a solution of equation (34).

We have thus proved that solving the system of coupled equations (33) and (34), one obtains the solution of the infinite hierarchy (24) in the factorized form (32). This remarkable reduction of the dynamical hierarchy to a closed system of two equations does not involve any approximation, provided \( \mu(2|1; t) \) has the structure (31). The adhesive collisions do correlate the motion of the nearest neighbours (the mean field approach is ruled out), but no three- or more-particle correlations are created in the course of time. The same kind of situation has been already discovered in the case of ballistic annihilation (10).

V. EVOLUTION OF REDUCED DENSITIES

The factorized form (32) of distributions \( \mu_k \) implies the analogous factorization of the reduced densities

$$\rho(1, \ldots, k; t) = \rho_1(1; t) \prod_{j=1}^{j=k-1} \rho(j + 1|j; t)$$

where \( \rho(2|1; t) \) is the conditional density.
\[
\rho(2|1; t) \equiv \rho(X_{21}, P_2, M_2|0, P_1, M_1; t) = \frac{\rho_2(1, 2; t)}{\rho_1(1; t)} \tag{40}
\]

From the closed system of equations (33), (34) for the nearest neighbour distributions one can derive the corresponding set of dynamical equations for the densities \(\rho_1 = \mu_1\) and \(\rho(2|1; t)\). First of all let us notice that the relation (3) implies the equality

\[
\rho_2(X_1, P_1, M_1, X_1+, P_2, M_2; t) = \mu_2(X_1, P_1, M_1, X_1+, P_2, M_2; t) \tag{41}
\]

Hence, equation (33) preserves its form

\[
\frac{\partial}{\partial t} \rho_1(1; t) = \int d1' \int d2' C(1', 2') \delta[1 - (1' + 2')] \rho_1(1'; t) \rho(2'|1'; t)
- \int d1'C(1', 1) \rho_1(1'; t) \rho(1|1'; t) - \int d1'C(1, 1') \rho_1(1; t) \rho(1'|1; t) \tag{42}
\]

In order to derive the closed evolution equation for \(\rho(2|1; t)\) one has to use the formula

\[
\rho(2|1; t) = \mu(2|1; t) + \sum_{r=1}^{\infty} \int d1' \int d2' \ldots \int dr' \theta(X_1' - X_1) \theta(X_2 - X_1')
\times \prod_{j=1}^{r-1} \theta(X_{j+1}' - X_j') \mu(1'|1; t) \mu(2'|1'; t) \cdots \mu(r'|r' - 1; t) \mu(2|r'; t) \tag{43}
\]

(see (3) and (34)). Using then repeatedly equation (34) one arrives at the equation

\[
\left[ \frac{\partial}{\partial t} + L_{12} + C(1, 2) \right] \rho(2|1; t) = \int d1' \int d2' C(1', 2') \delta[2 - (1' + 2')] \rho(1'|1; t) \rho(2'|1'; t)
+ \int d1' C(1', 1') \rho(1'|1; t) \rho(2|1; t) - \rho(2|1; t)]
\tag{44}
- \int d1' \{ C(1', 2) \rho(1'|1; t) \rho(2|1'; t) + C(2, 1') \rho(1'|2; t) \rho(2|1; t) \}
\]

Equations (33) and (34) form a closed system which suffices to determine the reduced densities of any order owing to the relation (33).

VI. SELF-SIMILAR SOLUTIONS

It seems interesting to check whether the aggregation dynamics is compatible with self-similar solutions. In other words, whether the evolution of the merging masses can be entirely reduced to rescaling of densities of a given shape (this idea has been followed in the study of the Burgers model of turbulence (3)). In order to investigate this question we insert into equations (33), (34) the assumed scaling formulas

\[
\mu_1(P_1, M_1; t) = t^{\gamma_1} \mu_1 \left( P_1 t^{1-2\alpha}, M_1 t^{-\alpha}; 1 \right) \tag{45}
\]

\[
\mu(X_{21}, P_2, M_2|0, P_1, M_1; t) = t^{\gamma_2} \mu(X_{21} t^{-\alpha}, P_2 t^{1-2\alpha}, M_2 t^{-\alpha}|0, P_1 t^{1-2\alpha}, M_1 t^{-\alpha}; 1) \tag{46}
\]

where \(X_{21} = X_2 - X_1\) and where we have used the fact that the three variables

\[
X_{21}, \frac{P}{M} t, \frac{M}{2P} \tag{47}
\]

have the same dimension and must thus scale in the same way.

As the mass is conserved, the integral

\[
\int_{-\infty}^{+\infty} dP \int_{0}^{\infty} dMM t^{\gamma_1} \nu(P t^{1-2\alpha}, M t^{\alpha}) \tag{48}
\]
representing the total mass density does not depend on time. This implies the relation
\[ \gamma_1 = 1 - 4\alpha. \]  
(49)

On the other hand, the normalization condition (24) leads to
\[ \gamma_2 = 1 - 4\alpha. \]  
(50)

It can be checked by a straightforward calculation that the distributions \( \mu_1(1; t) \) and \( \mu(2|1; t) \) of the form (13) and (10), respectively, with the exponents \((\alpha, \gamma_1, \gamma_2)\) satisfying equations (13) and (24), lead to a consistent closed system of equations for the self-similar distributions \( \mu_1(P'; M'; 1) \) and \( \mu(2'|1'; 1) \) when put into the reduced hierarchy (13), (24). It follows that the dynamics does not suffice to fix completely the values of the exponents. They can thus depend on the nature of the initial condition of the system.

VII. A STATISTICAL MECHANICAL MODEL

In this section, we give a microscopic description of the dynamics of ballistic aggregation from the viewpoint of statistical mechanics by recalling the model introduced in [1] and [3].

One considers the aggregation process developing from a simple initial condition. At \( t = 0 \), all the particles have the same mass \( m \) and are located on the sites \( X_j = ja \) of an infinite regular lattice with lattice constant \( a \) (\( j = 0, \pm1, \pm2, \ldots \)). The mass density \( \rho_j = m/a \). It is assumed that the initial momenta are uncorrelated, with a distribution corresponding to thermal equilibrium at inverse temperature \( \beta \):
\[ \varphi_m(p) = \left( \frac{\beta}{2\pi m} \right)^{1/2} \exp \left( -\frac{\beta p^2}{2m} \right) \]  
(51)

As it has already been explained in Section 11, the dynamics of the sticky gas has the remarkable property that any mass aggregate is found on the trajectory of the center of mass of the initial cluster it has been formed from. In particular, the state of an aggregate \( M_j = n_jm \) at time \( t > 0 \) determines uniquely the set of \( n_j \) consecutive initial masses which constitute it: they are located at \( t = 0 \) within the interval \([X_j - Y_j^+ + a/2, X_j - Y_j^- - a/2]\) (see also Eqs. (22), (23))). This property permits to write explicitly the kinematical constraints that select the subset of initial phase space configurations leading to the occurrence of a specific sequence of aggregate states \( (X_j, P_j, M_j), j = 1, \ldots, k \), within \([L_1, L_2]\) at time \( t > 0 \). The distributions \( \mu_k(1, \ldots, k; t|L_1, L_2) \) are then obtained by averaging these constraints over the initial state.

The origin of the constraints is two-fold. One class of them ensures that the \( n_j \) initial particles located in the interval \([X_j - Y_j^+ + a/2, X_j - Y_j^- - a/2]\) do merge and form the \( j \)'th aggregate before time \( t \). A necessary and sufficient condition here is that the trajectories of the centers of mass of all the pairs of subclusters of consecutive initial particles, which form a partition of the initial \( n_j \)-particle cluster, cross before time \( t \). After averaging, this leads to functions denoted by \( I_{n_j}(M_j, P_j; t) \) which are the probability densities for the formation of masses \( M_j = n_jm \), with momenta \( P_j \) from the corresponding sets of \( n_j \) initial neighbouring masses.

The second class of constraints guarantees that no particles other than the \( k \) specified aggregates are found within \([L_1, L_2]\) at time \( t \). The particles initially located in \((-\infty, ja]\) will be found to the left of \( L_1 \) if the positions of all the centers of mass of clusters of consecutive initial masses located in \([(j - i)a, ja], i = 1, 2, \ldots \) stay smaller than \( L_1 \) up to time \( t \). After averaging, this will be expressed by a function \( J_m(Y; t) \), where \( J_m(-L_1 + (j + 1/2)a; t) \) (resp. \( J_m(L_2 - (j + 1/2)a; t) \) is the probability for finding the particles initially located in \((-\infty, ja]\) (resp. \([(j + 1)a, \infty)\) in \((-\infty, L_1)\) (resp. in \([L_2, \infty])\). In particular, in terms of function \( J_m \) defined in this way the probability \( \mu_{0,m}(t|L_1, L_2) \) of finding the interval \([L_1, L_2]\) void of particles is given by
\[ \mu_{0,m}(t|L_1, L_2) = \sum_j J_m(-L_1 + (j + 1/2)a; t) J_m(L_2 - (j + 1/2)a; t) \]  
(52)

The probability density \( \mu_{1,m} \) \(^1\) reads \[ \]

\(^1\)We add the index \( m \) to keep in mind the discrete nature of the initial state.
\[ \mu_{1,m}(1; t) = \sum_{n_1=1}^{\infty} \delta(M_1-n_1m) \sum_j \delta \left( X_1 - Y_1^+ - (j+1/2)a \right) \times I_m(M_1, P_1; t) J_m \left( -Y_1^+; t \right) J_m \left( Y_1^-; t \right) \]

where the summation involving \( \delta \)-functions is a manifestation of the discreteness of the initial masses and the lattice positions. The function \( I_m \) in (53) ensures that the particles initially located in \([X_1 - Y_1^+ + a/2, X_1 - Y_1^- - a/2]\) have met to form the aggregate, while the functions \( J_m \) ensure that all the particles initially located on the left of \( X_1 - Y_1^+ \) (resp. on the right of \( X_1 - Y_1^+ \)) are at time \( t \) on the left (resp. on the right) of \( X_1 \).

The probability density \( \mu_{2,m} \) is given by

\[ \mu_{2,m}(1, 2; t) = \theta(X_2 - X_1) \sum_{n_1=1}^{\infty} \delta(M_1-n_1m) \sum_{n_2=1}^{\infty} \delta(M_2-n_2m) \times \sum_j \delta \left( X_1 - Y_1^+ - (j+1/2)a \right) \delta \left( X_2 - X_1 + Y_1^- - Y_2^+ \right) \times I_m(M_1, P_1; t) I_m(M_2, P_2; t) J_m \left( -Y_1^+; t \right) J_m \left( Y_2^-; t \right). \]  

(54)

In this expression the \( n_i \) and \( j \) summations involving \( \delta \)-functions reflect again the discreteness of the initial conditions. The additional \( \delta \)-function obtained here by construction reflects the fact that the aggregates are nearest neighbours and imposes precisely the condition (51) under which the hierarchy could be reduced. Once again, functions \( I_m \) stand for the formation of aggregates 1 and 2, while functions \( J_m \) ensure that particles initially located on the left of \( X_1 - Y_1^+ \) (resp. on the right of \( X_2 - Y_2^+ \)) stay at time \( t \) on the left of \( X_1 \) (resp. on the right of \( X_2 \)).

The higher order distributions \( \mu_{k,m} \) can be constructed along the same lines. They verify the factorization (52)

\[ \mu_{k,m}(1, \ldots, k; t) = \mu_{1,m}(1; t) \prod_{j=1}^{k-1} \mu_m(j+1|j; t) \]  

(55)

with the conditional probability \( \mu_m(2|1; t) \), derived from (53) and (54), of the form

\[ \mu_m(2|1; t) = \frac{\mu_{2,m}(1, 2; t)}{\mu_{1,m}(1; t)} \]

\[ = \theta(X_2 - X_1) \sum_{n_2=1}^{\infty} \delta(M_2-n_2m) \delta \left( X_2 - X_1 + Y_1^- - Y_2^+ \right) \frac{J_m(Y_2^-; t)}{J_m(Y_1^-; t)} \]

(56)

Note that this conditional probability has the structure (51).

To have the model in an explicit form, it remains to give the formulœ that express functions \( I_m \) and \( J_m \) in terms of the constraints: the details of the calculation can be found in [3]. The important point is that, due to the uncorrelated Gaussian initial velocity distribution, the whole aggregation dynamics can be mapped on the following equivalent Brownian motion problem:

Let \( P(\tau) \) be a Brownian path starting from the origin, \( P(0) = 0 \). Then, for \( M = nm \),

\[ I_m(M, P; t) = E_{(0,0)} \{ P(rm) \geq rm[(M - rm)/2\rho t + P/M], r = 1, \ldots, n-1 \} \mid P(nm) = P \}

(57)

is the conditional measure of such paths constrained to be above the parabolic barrier \( rm[(M - rm)/2\rho t + P/M] \) at discrete "times" \( \tau_r = rm, \ r = 1, \ldots, n-1 \) and to end in \( P \) at "time" \( nm \), see Fig.1.
FIG. 1. Brownian interpretation of the function $I_m(M,P;t)$. The Brownian motion starts at $P(0) = 0$, ends at $P(M) = P$ while overpassing the points $P(rm) \geq rm(2\rho Q^+ - rm)/2\rho t$, $(r = 1, \ldots, n - 1)$ with $Q^+ = Pt/M + M/2\rho$.

Likewise

$$J_m(Y;t) = E_{(0,0)}\{P(rm) \geq rm(2\rho Y - rm)/2\rho t, r = 1, 2, \ldots\}$$

(58)
is the measure of the paths that remain above the barrier $rm(Y - rm)/2\rho t$ for all discrete times $\tau = rm, r = 1, \ldots$; see Fig. 2. We recall that the dynamics is deterministic, and randomness enters only through the initial velocity distribution: $P(\tau)$ is a process in momentum space, as a function of the mass of aggregates.

FIG. 2. Brownian interpretation of the function $J_m(Y;t)$. The Brownian motion starts at $P(0) = 0$ and overpasses the points $P(rm) \geq rm(2\rho Y - rm)/2\rho t$, $(r = 1, 2, \ldots)$.

A simple and immediate consequence are the scaling properties of functions $I_m$ and $J_m$ with respect to time. Owing to the fact that the scaled Brownian motion $(\rho t)^{\alpha/2}P(\tau(\rho t)^{-\alpha})$ is equivalent in probability to $P(\tau)$, the functions $I_m$ and $J_m$ obey the scaling relations with $\alpha = 2/3$

$$I_m(M,P;t) = (\rho t)^{-1/3}I_{m'}(M',P';1) \equiv (\rho t)^{-1/3}I_{m'}(M',P')$$

$$J_m(Y;t) = J_{m'}(Y';1) \equiv J_{m'}(Y')$$

(59)

with

$$m' = m(\rho t)^{-2/3}, \quad P' = P(\rho t)^{-1/3}, \quad M' = nm' = M(\rho t)^{-2/3}, \quad Y' = (\rho t)^{-2/3}\rho Y$$

(60)
The construction (53)-(58) yields in principle a solution to the dynamical hierarchy that has the factorization property: $\mu_{1,m}(1;t)$ and $\mu_{m}(2;1;t)$ have to verify the coupled equations (53)-(54), adapted to the case of discrete masses. We shall not proceed to this verification now, but rather simplify first the discussion by taking the continuum limit of our statistical mechanical model.
The continuum limit amounts to let \( m \to 0 \) and \( a \to 0 \) while keeping the initial mass density \( \rho = m/a \) and time fixed. Because of (59) one can alternatively fix the scaled momentum \( P' \) and the scaled mass \( M' \) and look for the large time asymptotics \( t \to \infty \). In the first view, one looks for the formation of masses of order 1 at a given time arising from an initial infinitesimal dust. In the second view one looks for the formation of masses of size \( \sim mt^{2/3} \), \( t \to \infty \), while keeping the discrete initial condition. These two views are equivalent. The interesting point is that in the continuum limit the Brownian expressions (57) and (58) can be computed in a closed analytical form, thus providing an explicit complete statistical description of the aggregation process. Moreover this description exactly coincides with that of the statistics of shocks in the inviscid limit of the Burgers equation with white noise initial data.

To conclude one can check that the factorization property (55) persists for a wider class of initial conditions, for instance allowing any uncorrelated initial velocity distribution, and an arbitrary (non random) choice of initial positions and masses.

**VIII. THE CONTINUUM LIMIT**

In this section, we determine the continuum limits \( (m \to 0 \text{ with } \rho = m/a \text{ fixed}) \) of the one- and two-point probability densities

\[
\begin{align*}
\lim_{m \to 0} \mu_{1,m}(1; t) & = \mu_1(1; t) \\
\lim_{m \to 0} \mu_{2,m}(1, 2; t) & = \mu_2(1, 2; t).
\end{align*}
\]

Owing to the scaling relations (59) it is sufficient to calculate functions \( \mu_1(1; t) \) and \( \mu_2(1, 2; t) \) for \( t = 1 \) and \( \rho = 1 \). From now on we drop the time parameter from the notation setting simply \( \mu_{1,m}(1; t) = \mu_{1,m}(1) \), \( \mu_{2,m}(1, 2; t) = \mu_{2,m}(1, 2) \), and so on. Later in this section we will comment on the way to recover the time variable in the continuum limit of these densities.

Let us first introduce the transition probability density kernel

\[
K_{m,v}(M_1, P_1, M_2, P_2) = E_{(M_1, P_1)}\{P(r m) \geq f_v(r m), r = n_1 + 1, \ldots, n_2 - 1 | P(M_2) = P_2\}
\]

for the Brownian motion \( P(\tau) \) to start at \( P(M_1) = P_1 \), and end at \( P(M_2) = P_2 \) with \( M_1 = n_1 m \) and \( M_2 = n_2 m \), while overpassing the points \( f_v(r m) \), \( r = n_1 + 1, \ldots, n_2 - 1 \), where \( f_v(\tau) \) is the parabola

\[
f_v(\tau) = \nu \tau - \frac{\tau^2}{2},
\]

According to definitions (57) and (58), the functions \( I_m \) and \( J_m \) appearing in the one- and two-point densities can be expressed in terms of the transition kernel (62) as

\[
I_m(M, P) = K_{m,1}(0, 0, M, P)
\]

and

\[
J_m(Y) = \lim_{M \to \infty} \int_{f_y(M)}^\infty dP K_{m,y}(0, 0, M, P).
\]

In (62), Brownian paths are allowed to make excursions through holes of width \( m \) separating the discrete points \( rm \). In the continuum limit \( m \to 0 \), the weight of such excursions becomes vanishingly small and the paths become constrained to overpass the continuous barrier \( f_v(\tau) \), \( M_1 \leq \tau \leq M_2 \). Thus for \( P_1 > f_v(M_1) \) and \( P_2 > f_v(M_2) \)

\[
K_{m,v}(M_1, P_1, M_2, P_2) = K_v(M_1, P_1, M_2, P_2) + R_{m,v}(M_1, P_1, M_2, P_2)
\]

where \( \lim_{m \to 0} R_{m,v}(M_1, P_1, M_2, P_2) = 0 \), and

\[
K_v(M_1, P_1, M_2, P_2) = E_{(M_1, P_1)}\{P(\tau) > f_v(\tau), M_1 \leq \tau \leq M_2 | P(M_2) = P_2\}
\]

\[2\text{In (59) the density was set equal to 1/2.}\]
is the transition kernel for a Brownian motion with a continuous parabolic barrier. It satisfies the diffusion equation

$$\frac{\partial}{\partial M_2}K_\nu(M_1, P_1, M_2, P_2) = \frac{1}{2\beta} \frac{\partial^2}{\partial P_2^2}K_\nu(M_1, P_1, M_2, P_2)$$

(68)

with $K_\nu(M, P_1, M, P_2) = \delta(P_1 - P_2)$, and the Dirichlet conditions on the barrier: $K_\nu(M_1, P_1, M_2, P_2) = 0$ when $P_1 = f_\nu(M_1)$ or $P_2 = f_\nu(M_2)$.

Our object now is to express the distributions of aggregates in the continuum limit in terms of the kernel (67), which can be explicitly computed (Section II of [7]). However, in the continuum limit of $I_m$ and $J_m$ the starting point $(0, 0)$ of the Brownian motion lies on the parabola (63) where $K_\nu(0, 0, M, P) = 0$. Thus the determination of the densities $\mu_1(1)$ and $\mu_2(1, 2)$ requires the evaluation of the leading term in the asymptotic expansion of $K_{m,\nu}(0, 0, M, P)$ as $m \to 0$. This is not an elementary task since it involves the control of the cumulated effect of Brownian excursions in small intervals $(r - 1)m, rm, r = 1, 2, \ldots$ in the neighbourhood of the origin. The result is given in the following proposition.

**Proposition**

$$\lim_{m \to 0} \frac{K_{m,\nu}(0, 0, M, P)}{\sqrt{m}} = \frac{1}{\sqrt{2\beta}} \frac{\partial}{\partial P_0}K_\nu(M_0, P_0, M, P) \bigg|_{(M_0, P_0)=(0,0)} , \ M > 0, \ P \geq f_\nu(M)$$

(69)

With this result the continuum limits $m \to 0$ of functions $I_m$ and $J_m$, as expressed in (64) and (65), with fixed initial mass density $\rho = m/a$, read

$$\lim_{m \to 0} \frac{I_m(M, P)}{m} = \frac{1}{2\beta} \frac{\partial^2}{\partial P_1^2}\partial P_2 K_{Y^+}(0, P_1, M, P_2) \bigg|_{P_1=0, P_2=P} \equiv I(M, P)$$

(70)

and

$$\lim_{m \to 0} \frac{J_m(Y)}{\sqrt{m}} = \lim_{M \to \infty} \frac{1}{\sqrt{2\beta}} \int_{f_\nu(M)}^{\infty} dP \frac{\partial}{\partial P_1}K_Y(0, P_1, M, P) \bigg|_{P_1=0} \equiv J(Y).$$

(71)

In obtaining (70) we have taken into account that in (64), in addition to $(0,0)$, the paths have a second contact point with the parabola at $(M, P)$.

Then, using equations (53), (54), the continuum limits of the one- and two-point probability densities can be readily found. One multiplies and divides (53) by $m^2$ and takes (70) and (71) into account. Then, $m \to 0$ playing the role of an infinitesimal, the discrete sums go to the corresponding integrals, which can be evaluated owing to the $\delta$-functions, yielding eventually the formula

$$\mu_1(1) = I(M_1, P_1)J(-Y_1^+)J(Y_1^-)$$

(72)

and, in the same way

$$\mu_2(1, 2) = \theta(X_2 - X_1)\delta(X_2 - X_1 + Y_1^- - Y_2^+) I(P_1, M_1)I(P_2, M_2)J(-Y_1^+)J(Y_2^-).$$

(73)

The time and the initial density dependence can always be reintroduced in the continuum limit. Indeed, through Eqs.(59) and (70), (71) we get the scaling properties of functions $I$ and $J$

$$J(Y; t) = (pt)^{-1/3}J(Y'), \quad I(M, P; t) = (pt)^{-1}I(M', P')$$

(74)

with the scaling functions $J(Y)$ and $I(M, P)$ defined above. This implies the scaling behavior of the densities in the continuum limit of the form

$$\lim_{m \to 0} \mu_{k, m}(1, \ldots, k; t) = \mu_k(1, \ldots, k; t) = \rho^k(pt)^{-5k/3}\mu_k(1', \ldots, k')$$

(75)

with

$$j' \equiv (X'_j, P'_j, M'_j) = \left( \frac{\rho X_j}{(pt)^{2/3}}, \frac{P_j}{(pt)^{1/3}}, \frac{M_j}{(pt)^{2/3}} \right).$$

(76)
The scaling functions $\mu_1(1)$ and $\mu_2(1,2)$ are given by (72) and (73) and, for higher orders, the scaling functions are given by
\[\mu_k(1, \ldots, k) = \mu_1(1)^{k-1} \prod_{j=1}^{k-1} \frac{\mu_2(j,j+1)}{\mu_1(j)}.\] (77)

From Eq.(52), we find that the probability density to find an interval $[0, x)$ void of aggregates scales in the continuum limit as
\[\mu_0(x; t) = \mu_0(x')\] (78)
with $x' = \rho x(\rho t)^{-2/3}$ and
\[\mu_0(x) = \int dy J(y) J(x-y).\] (79)

Since the functions $I(M, P)$ and $J(Y)$ have been explicitly computed in [7], the results (72,73) and (77) give a complete solution of the ballistic aggregation model in a closed analytical form. We postpone the discussion of this solution to the next section and devote the rest of the present section to the proof of (69).

**Proof of the proposition**

The strategy to prove Eq.(69) is to bound the transition kernel (62) while linearizing the constraints imposed on the first $k = M_0/m$ points, $0 < M_0 < M$.

We have
\[(\nu - \frac{M_0}{2}) \tau \leq f_\nu(\tau) \leq \nu \tau, \quad 0 \leq \tau \leq M_0\] (80)

Let us introduce the kernel
\[K^{l}_{m,\nu}(0,0,M_0,P_0) = E_{(0,0)}\{P(rm) \geq \nu rm, r = 1, \ldots, k-1 | P(M_0) = P_0\}\] (81)
for paths starting at $(0,0)$, ending at $(M_0, P_0)$, while overcoming the linearly distributed sequence of discrete points $\nu rm, r = 1, \ldots, k-1$. Clearly (see Fig. 3)
\[K^{l}_{m,\nu}(0,0,M_0,P_0) \leq K_{m,\nu}(0,0,M_0,P_0) \leq K^{l}_{m,\nu-M_0/2}(0,0,M_0,P_0).\] (82)
Using the Markov property of the Brownian motion, one infers from (82) the bounds for the kernel $K_{m,\nu}(0,0,M,P)$ for $M > M_0 > 0$

$$K_{m,\nu}^{(\leq)}(0,0,M,P) \leq K_{m,\nu}(0,0,M,P) \leq K_{m,\nu}^{(\geq)}(0,0,M,P)$$

(83)

with

$$K_{m,\nu}^{(\leq)}(0,0,M,P) = \int dP_0 K_{m,\nu}^l(0,0,M_0,P_0)K_{m,\nu}(M_0,P_0,M,P)$$

(84)

and

$$K_{m,\nu}^{(\geq)}(0,0,M,P) = \int dP_0 K_{m,\nu}^l(M_0/2,0,0,M_0,P_0)K_{m,\nu}(M_0,P_0,M,P).$$

(85)

We shall calculate these bounds by first letting $m \to 0$ and then $M_0 \to 0$. They will be shown to coincide in this limit.

It is convenient to relate the transition kernel $K_{m,\nu}$ (53) for the Brownian process with discrete linear barrier $P(rm) \geq \nu rm$, $r = 1, \ldots, k - 1$ to that of a dimensionless Brownian process $q(\tau)$ with covariance equal to 1, the paths constrained to be positive at integer times, and with the corresponding transition kernel

$$G_k(q) = E_{(0,0)}\{q(n) \geq 0, n = 1, \ldots k - 1\}.$$ 

(86)

For this later process, we define the average $\langle f(q) \rangle_k = \int dq G_k(q) f(q)$. The moments $\langle q^j \rangle_k, j = 0, 1, 2, \ldots$ will be for us of particular interest.

The relation between the two processes is given by removing a linear drift

$$q(\tau) = \sqrt{\frac{\beta}{2m}} (P(\tau) - \nu \tau)$$

(87)

which leads to the relation

$$K_{m,\nu}^l(0,0,M_0,P_0) = \exp \left( -\frac{\beta \nu^2 M_0}{2} \right) \sqrt{\frac{\beta}{2m}} G_k \left( \sqrt{\frac{\beta}{2m}} (P_0 - \nu M_0) \right), \quad k = \frac{M_0}{m}.$$ 

(88)

When this is inserted into (84), one obtains

$$K_{m,\nu}^{(\leq)}(0,0,M,P) = \exp \left( -\frac{\beta \nu^2 M_0}{2} \right) \int dq G_k(q) e^{-\nu (2m \beta)^{1/2} q} K_{m,\nu} \left( M_0, \nu M_0 + \sqrt{\frac{2m}{\beta}} q, M, P \right).$$

(89)

Then, we take the following steps. First, for $M_0 > 0$ we can replace in (89) the kernel $K_{m,\nu}$ for the discrete parabolic barrier by its continuous limit (60). Next, we expand the integrand up to second order in the variable $\sqrt{m} q$. This gives

$$K_{m,\nu}^{(\leq)}(0,0,M,P) = e^{-\beta \nu^2 M_0/2} \left[ \langle 1 \rangle_k K_{\nu}(M_0, \nu M_0, M, P) \right.$$  

$$+ \sqrt{m} \langle q \rangle_k \left( -\nu \sqrt{2 \beta} K_{\nu}(M_0, \nu M_0, M, P) + \frac{\beta}{\beta \partial P_0} K_{\nu}(M_0, P_0, M, P)|_{P_0 = \nu M_0} \right) \right]$$

$$+ \langle q^2 \rangle_k \mathcal{O}(m) + \langle 1 \rangle_k o(m).$$

(90)

Recalling that $k = M_0/m$, we have to evaluate the first moments $\langle q^j \rangle_k$ of the distribution $G_k(q)$ for large $k$. This is done in appendix B with the help of the Sparre-Andersen theorem together with a Tauberian theorem. One gets

$$\langle 1 \rangle_k \sim \frac{M_0}{\pi m}, \quad \langle q \rangle_k \sim \frac{1}{2} \quad \langle q^2 \rangle_k \sim \sqrt{\frac{m}{\pi M_0}}, \quad m \to 0.$$ 

(91)

Hence
Finally, since $K_{\nu}(M_0, P_0, M, P)$ is differentiable with respect to $P_0$ and vanishes at $P_0 = 0$, one has $K_{\nu}(M_0, \nu M_0, M, P) = \mathcal{O}(M_0)$, $M_0 \to 0$, so that

$$
\lim_{M_0 \to 0} \lim_{m \to 0} \frac{K_{m,\nu}^{(\langle)}(0, 0, M, P)}{\sqrt{m}} = \frac{1}{\sqrt{2\pi\beta}} \frac{\partial}{\partial P_0} K_{\nu}(M_0, P_0, M, P) \bigg|_{(M_0, P_0) = (0, 0)} + \mathcal{O}(\sqrt{M_0}) \quad (93)
$$

The upper bound $K_{m,\nu}^{(\rangle)}(0, 0, M, P)$ is treated in the same way and attains the same limit, thus leading to the result Eq. (69) of the proposition.

The analysis performed in [3] was based on the lower bound $K_{m,\nu}(0, 0, M, P)$ is differentiable with respect to $P_0$ and vanishes at $P_0 = 0$. The result obtained therein differs from the exact limit (69) by a factor $1/\sqrt{\pi}$. In fact, the limit (69) involves contributions of infinitely many intervals, which can be summed up by applying the Sparre-Andersen theorem.

**IX. EXPLICIT SOLUTION**

The function $I(M, P)$ (70), explicitly computed in [7], has the form

$$
I(M, P) = 2b^3 \exp \left( -b^3 \left( \frac{P^2}{M} + \frac{M^3}{12} \right) \right) \mathcal{I}(M), \quad (95)
$$

where we set $b = (\beta/2)^{1/3}$ and

$$
\mathcal{I}(M) = \sum_{k \geq 1} e^{-b \omega_k M}. \quad (96)
$$

The function $J(Y)$ (71) reads

$$
J(Y) = \sqrt{b} e^{-b^{3/2} Y^{3/2}} \mathcal{J}(Y), \quad (97)
$$

with

$$
\mathcal{J}(Y) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} dw \frac{e^{bYw}}{\text{Ai}(w)}. \quad (98)
$$

In (94) and (98), $\text{Ai}(w)$ is the Airy function [11], solving the differential equation

$$
f''(w) - wf(w) = 0, \quad (99)
$$

$\text{Ai}(w)$ is analytic in the complex $w$ plane, and has an infinite countable set of zeros $-\omega_k$ on the negative real axis, $0 < \omega_1 < \omega_2 < \cdots$. The asymptotic behavior of these functions has been derived in [11]

$$
\mathcal{I}(\mu) \sim \frac{1}{\sqrt{4\pi b^3 \mu}}, \quad \mu \to 0; \quad \mathcal{I}(\mu) \sim \exp(-\omega_1 b \mu), \quad \mu \to \infty \quad (100)
$$

and

$$
\mathcal{J}(u) \sim \frac{e^{-b \omega_1}}{\text{Ai}(-\omega_1)}, \quad u \to \infty; \quad \mathcal{J}(u) \sim -2bu \exp \left( \frac{b^3 u^3}{3} \right), \quad u \to -\infty. \quad (101)
$$
In fact, these results, which have been announced in [4], were derived in details in the related framework of the Burgers turbulence [7]. The solutions of the one-dimensional Burgers equation with a white-noise initial condition, develop, in the limit of vanishing viscosity, a train of shock waves. In Burgers theory, a shock located at $X$ is characterized by two parameters $\mu$ and $\eta$, and can be identified with a particle of mass $M = \mu$ at point $X$ with momentum $P = -\eta$. Then it turns out that dynamics of shocks is completely equivalent to ballistic aggregation subject to mass and momentum conservation [9], [5]. Moreover, the white noise covariance is $D/2 = 1/(2\beta)$ and the mass density $\rho$ corresponds to a length scale in the Burgers equation. In this section, we recast the results of [7] in the language of ballistic aggregation of interest here.

We consider first the probability density (79) of finding an interval $[0, x)$ void of particles

$$\mu_0(x) = \int_{-\infty}^{\infty} dy J(y)J(x - y)$$

$$= \sqrt{\frac{\pi}{bx}} \exp\left(-\frac{b^3x^3}{12}\right) \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dw_1 \int_{-i\infty}^{+i\infty} dw_2 \frac{\exp\left(\frac{b^2}{4}(w_1 + w_2) + \frac{(w_1 - w_2)^2}{4\omega}\right)}{\text{Ai}(w_1)\text{Ai}(w_2)}$$

which is plotted on Fig. 4. We have $\lim_{x \to 0} \mu_0(x) = 1$, and asymptotically for $x \to \infty$

$$\mu_0(x) = \sqrt{\frac{\pi}{bx}} \exp\left(-\frac{b^3x^3}{12} - b\omega_1x\right) \left(1 + O\left(\frac{1}{x}\right)\right).$$

---

FIG. 4. The probability density $\mu_0(x)$ to find no aggregates in an interval $[0, x]$, for the parameters $t = 1$, $\rho = 1$ and $\beta = 2$ ($a = 1$ in Eq.(102)).

The one point probability density (72) reads

$$\mu_0(x) = \sqrt{\frac{\pi}{bx}} \exp\left(-\frac{b^3x^3}{12} - b\omega_1x\right) \left(1 + O\left(\frac{1}{x}\right)\right).$$

---

In Burgers language, one speaks of the shock strength $\mu/t$ and shock wavelength $\nu = \mu/2 - t\eta/\mu$

In the present paper, the Brownian motion $P(\tau)$ and the parabolic constraints $f_\nu(\tau)$ have the sign opposite to the corresponding objects in [7], namely $-\psi(y)$ and $-s_\nu(y)$. By invariance of Brownian motion under space reflexion, the functions $I$ and $J$ are the same in both papers, provided that $M$ and $P$ correspond to $\mu$ and $-\eta$ in the notation of [7].
\[ \mu_1(P, M) = I(M, P) J \left( \frac{P}{M} - \frac{M}{2} \right) J \left( \frac{P}{M} - \frac{M}{2} \right) \]

\[ = 2b^4 J \left( \frac{P}{M} - \frac{M}{2} \right) I(M, P) J \left( -\frac{P}{M} - \frac{M}{2} \right) \]

(104)

where \( I \) and \( J \) are given by Eqs.(102,103). Integration over the momentum space yields the mass density distribution

\[ \mu_1(M) = 2b^3 M I(M) \mathcal{H}(M) \]

(105)

with

\[ \mathcal{H}(M) = \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} du \frac{e^{-bMw}}{\Lambda^2(w)} \]

(106)

whose asymptotic behavior reads

\[ \mathcal{H}(\mu) \sim 1, \quad \mu \rightarrow 0; \quad \mathcal{H}(\mu) \sim \sqrt{\pi} b^3 \mu^3 \exp \left( \frac{b^3 \mu^3}{12} \right), \quad \mu \rightarrow \infty. \]

(107)

The shape of the mass density distribution (105) is plotted on Fig. 5.

The mass density \( \mu_1(M) \) for the parameters \( t = 1, \rho = 1 \) and \( \beta = 2 \) (\( a = 1 \) in Eq.(105)).

Its asymptotic behavior can be computed from (100) and (107) yielding the formulae

\[ \mu_1(M) = \sqrt{\frac{b^3}{\pi M}} + \mathcal{O}(M^{1/2}), \quad M \rightarrow 0 \]

(108)

and

\[ \mu_1(M) \sim 2\sqrt{\pi b^{9/2} M^{5/2}} \exp \left( -\frac{b^3 M^3}{12} - \omega_1 bM \right), \quad M \rightarrow \infty. \]

(109)

The exact scaling function obtained here is quite different from a simple exponential \( \exp(-M) \) suggested on the basis of numerical simulations in [1]. For example, one notices that small masses \( (M \ll t^{2/3}) \) are much more likely to be present in the system than suggested in [1], while large masses \( (M \gg t^{2/3}) \) have a much smaller chance to be present.
Let us remark that the exponential form has been also found in \cite{9} and \cite{2} by solving the dynamical equations in a mean-field-like approximation scheme. In the framework of Burgers turbulence, more refined numerical simulations were performed by Kida \cite{6} where the small mass behavior of the scaling function compatible with Eq.\eqref{108} has been found. Moreover, our result \eqref{108} is compatible with the rigorous upper and lower bounds derived in \cite{12}. The large mass behavior Eq.\eqref{109} fits into rigorous bounds of the type $\exp(-CM^3)$ found in \cite{13} and \cite{3}.

The density of particles of velocity $V = P/M$, given by

$$
\mu_1(V) = \int_0^\infty dM M I(M) J(V - M/2) J(-V - M/2)
$$

is plotted on Fig.\ref{fig:6}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{The velocity probability density $\mu_1(V)$ for the parameters $t = 1$, $\rho = 1$ and $\beta = 2$ ($a = 1$ in Eq.\eqref{110}).}
\end{figure}

From \eqref{110} and the asymptotic behavior \eqref{100}, \eqref{101} of $I$ and $J$, one can derive the bound

$$
\mu_1(V) \leq C|V| \exp\left(-\frac{b^3|V|^3}{3} - b|V|\omega_1\right), \quad |V| \to \infty.
$$

This shows that the large velocity behavior cannot be Gaussian and thus invalidates the numerically based hypothesis of Kida \cite{6}.

The density of nearest neighbours \eqref{73} has the form

$$
\mu_2(1, 2) = 4b^7\theta(X)\delta(X + Y_1^- - Y_2^+)
\times \exp\left(-\frac{b^3}{3}((Y_1^-)^3 + (Y_2^+)^3)\right) I(M_1) I(M_2) J(-Y_1^+) J(Y_2^-)
$$

with $X = X_2 - X_1$. Expression \eqref{112} permits to compute the collision frequency between the particles defined by

$$
\nu_2(M_1, M_2; t) = \int_{-\infty}^\infty dP_1 \int_{-\infty}^\infty dP_2 \frac{P_1}{M_1} - \frac{P_2}{M_2} \mu_2(0, P_1, M_1, 0+, P_2, M_2; t).
$$

First, we notice that relation \eqref{74} implies the scaling

$$
\nu_2(M_1, M_2; t) = \frac{\rho^2}{(\rho t)^3} \nu_2(M'_1, M'_2)
$$

19
with

\[ \nu_2(M_1, M_2) = \int_{-\infty}^{\infty} dP_1 \int_{-\infty}^{\infty} dP_2 \left[ \frac{P_1}{M_1} - \frac{P_2}{M_2} \right] \mu_2(1, 2) \]  

(115)

This last integral, upon inserting formula (112), gives

\[ \nu_2(M_1, M_2) = 2^{b^6} M_1 M_2 (M_1 + M_2) \mathcal{I}(M_1) \mathcal{I}(M_2) \mathcal{H}(M_1 + M_2) \]  

(116)

with \( \mathcal{H} \) given by Eq. (106). Clearly, the collision frequency does not factorize into a product of a function of \( M_1 \) and a function of \( M_2 \). This fact invalidates the weak mean-field hypothesis whose consequences where analysed in [3].

The two-point distribution function

The scaling form \( \rho_2(1, 2) = \rho_1 \rho_2(2|1) \) of the two-point reduced number density of aggregates enables to study the long distance correlations in the system. The calculation requires the summation over all possible configurations of aggregated masses in between the particles 1 and 2 (formula (13)). In principle, this summation can be be performed as follows. Because of translation invariance, the conditional probability \( \mu(2|1) \)

\[ \mu(X_2, P_2, M_2|X_1, P_1, M_1) = (P_1, M_1|T(X_2 - X_1)|P_2, M_2) \]  

(117)

can be considered as the kernel of an integral operator \( T(X) \) acting in the mass and momentum space. Then the series (13) can be summed up by applying the Laplace transformation. Setting

\[ \tilde{T}(s) = \int_0^{\infty} dX e^{-X s} T(X) \]

\[ \tilde{\rho}_2(s, P_1, M_1, P_2, M_2) = \int_0^{\infty} dX e^{-X s} \rho_2(0, P_1, M_1, X, P_2, M_2) \]

(118)

one finds by the convolution theorem

\[ \tilde{\rho}_2(s, P_1, M_1, P_2, M_2) = \rho_1(P_1, M_1) \sum_{r=1}^{\infty} (P_1, M_1|\tilde{T}^r(s)|P_2, M_2) \]

\[ = \rho_1(P_1, M_1)(P_1, M_1|\tilde{T}(s)(I - \tilde{T}(s))^{-1}|P_2, M_2) \]  

(119)

Inverting the operator \( I - \tilde{T}(s) \) and then performing the inverse Laplace transformation are not easy operations here. So, we shall proceed in a different way. In the context of the Burgers equation, \( \rho_2(1, 2) \) corresponds to the joint density to find two shocks at distance \( |X_2 - X_1| \). This density, in the Brownian interpretation, is the measure of paths that have contact points with two parabolas: one centered at \( X_1 \) another at \( X_2 \). Summing over all such paths amounts to sum over intermediate sequences of neighbouring aggregates in the particle language. We refer to [3] for details. One finds the following form of the two-point density

\[ \rho_2(1, 2) = J (-Y_1^+ ) I (M_1, P_1 ) [ \delta ( X + Y_1^- - Y_2^+ ) + \theta ( X + Y_1^- - Y_2^+ ) H ( X, Y_1^-, Y_2^+ ) ] I (M_2, P_2 ) J (Y_2^-) \]

(120)

The \( \delta \)-function term corresponds to the case when 1 and 2 are the nearest neighbours, whereas the function \( H (X, Y_1, Y_2) \) embodies precisely the sum over all configurations of intermediate particles. The function \( H \) will not be reproduced here. The important fact is that its large distance asymptotic behaviour can be calculated leading to the following cluster property of the two-point function [3]

\[ \rho_2(M_1, P_1, M_2, P_2, X) - \rho_1(M_1, P_1) \rho_1(P_2, M_2) \sim -b^{11/2} \frac{32 \sqrt{\pi}}{X^{3/2}} \exp \left( -\frac{b^3 X^3}{12} - b \omega_1 X \right) \]

\[ \times \exp \left( -b \omega_1 (Y_1^- - Y_2^+) \right) J (-Y_1^+) I (M_1) I (M_2) J (Y_2^-), \quad X \to \infty. \]  

(121)

We see that although the particles in the initial state are not correlated and the motion between collisions is free, the aggregation process induces dynamic correlations in the course of time. However, these correlations have a very short range since they stay dominated by the rapidly decaying cubic exponential factor \( \exp(-b^3 X^3/12) \).
X. RIGOROUS SOLUTION OF THE DYNAMIC HIERARCHY

We have shown in Section [V] how the infinite hierarchy (22) describing the aggregation process could be reduced to a system of two coupled equations (24), (25), satisfied by the one-particle density \( \mu_1(1; t) \), and the conditional probability density \( \mu(2;1; t) \). It is the right moment now to show that the constructed densities (72), (73) provide a rigorous solution to the aggregation dynamics.

In (72) and (73) the densities \( \mu_1(1; t) \) and \( \mu(2;1; t) \) are expressed in terms of the probability weights \( I(M, P) \) and \( J(Y) \) (to recover the time variable \( t \) the scaling relations (74) have to be used). In order to evaluate the rate of change of the state of the aggregating gas it is thus sufficient to calculate the time derivative of functions \( I(M, P; t) \), \( J(Y^+; t) \) and \( J(Y^-; t) \), where \( Y^\pm = tP/M \pm M/2\rho \). A straightforward calculation can be performed starting from the defining formulae (53), (54), whose explicit form can be found in [3]. Technically it is simple but lengthy, so we shall not reproduce it here. Taking then the continuum limit one finds the following system of equations

\[
\frac{\partial}{\partial t} I(M, P; t) = \int dM_1 \int dP_1 \int dM_2 \int dP_2 \left( \frac{P_1}{M_1} - \frac{P_2}{M_2} \right) \delta(M - M_1 - M_2) \delta(P - P_1 - P_2) \times \delta \left( \frac{P_1}{M_1} - \frac{P_2}{M_2} - \frac{M}{2\rho} \right) I(M_1, P_1; t) I(M_2, P_2; t) \tag{122}
\]

\[
\frac{\partial}{\partial t} J \left( \frac{P}{M} = \frac{M}{2\rho}; t \right) = - \int dM_1 \int dP_1 \left( \frac{P}{M} - \frac{P_1}{M_1} \right) \delta \left( \frac{P}{M} - \frac{P_1}{M_1} \right) t - \frac{M + M_1}{2\rho} \right) \times I(P_1, M_1; t) J \left( \frac{P_1}{M_1} - \frac{M_1}{2\rho}; t \right) \tag{123}
\]

A somehow laborious but straightforward analysis permits to check that the above system of equations, when combined with the relations (72), (73), implies the fundamental dynamic equations (24), (25). As one could expect from the interpretation of functions \( I \) and \( J \) given in Section [VI], the time derivative of \( I(M, P; t) \) is positive and produces gain terms in the kinetic equation (33), whereas the negative time derivatives of factors \( J(Y^+; t) \) and \( J(Y^-; t) \) generate loss terms.

XI. CONCLUDING COMMENTS

The hierarchy equations (22) describe the aggregation dynamics in a one-dimensional gas for arbitrary initial conditions. We considered here in detail the case where statistical correlations existed only between the states of the nearest neighbours. Under the precise condition (33), the evolution preserved this property, and the state of the system remained entirely characterized by the number density of aggregates and by the conditional distribution of nearest neighbours. This permitted to reduce the hierarchy to two coupled equations.

Rather than solving the obtained system of equations we determined the relevant distributions on the basis of their definitions, and only a posteriori we could verify that they satisfied the dynamical laws. So, finding a systematic way of solving the system (33), (34) remains an open problem.

When discussing the continuum limit we started from a very simple discrete initial condition, where no correlations were present at all. In the course of time, the system turned out to build up correlations by its internal dynamics, but only between the pairs of nearest neighbours. It is thus tempting to conjecture that the factorized form (22) of the many particle densities represents the asymptotic long time structure of the system for general initial conditions, as only the nearest neighbours get correlated by the process of aggregation. Let us recall that the correlations are rapidly decaying with the distance (see (121)).

A comment concerning the continuum limit seems also quite important. It has been noted at the end of Section [VI] that introduction of properly scaled variables permitted to interpret the continuum limit as a long time limit. One can thus expect that the dynamical properties derived here are quite universal, and do not depend on our particularly simple choice of the initial distribution. In particular, it would be interesting to clarify how large is the class of initial states which leads for long times to the mass density distribution plotted in Fig. (3).

Let us finally notice, that the dynamical scaling of the mass distribution predicted on the basis of intuitive arguments in [1] agrees with the rigorous scaling relations (53). The prediction of the Brownian motion exponent \( \alpha = 2/3 \) in [1] seems independent of the initial condition, whereas we derived it starting from a particular state. It would be thus interesting to clarify to what extent this type of scaling is universal. Indeed, our analysis showed that the aggregation dynamics was in principle compatible with other values of \( \alpha \), not necessarily equal to 2/3.
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APPENDIX A:

The moments of the distribution \( G_k \) (86) can be obtained from its generating function \( \gamma(s, \xi) \)

\[
\sum_{k=1}^{\infty} s^k \langle q^j \rangle_k = (-i)^j \frac{\partial^j \gamma(s, \xi)}{\partial \xi^j} \bigg|_{\xi=0} \tag{A1}
\]

with

\[
\gamma(s, \xi) = 1 + \sum_{k=1}^{\infty} s^k \int_0^\infty dq \, e^{i\xi q} G_k(q) \tag{A2}
\]

This generating function can be computed by applying the Sparre-Andersen theorem to the process with independent increments governed by the Gaussian distribution \( (\pi)^{-1/2} \exp(-q^2) \) (section XVIII.3 of [14])

\[
\gamma(s, \xi) = \exp \left[ \sum_{k=1}^{\infty} \frac{s^k}{k} \int_0^\infty dq \, e^{i\xi q} e^{-q^2/k} \right]. \tag{A3}
\]

Using (A3) we find

\[
\sum_{k=1}^{\infty} s^k \langle 1 \rangle_k = \gamma(s, 0) = \exp \left( \sum_{k=1}^{\infty} \frac{s^k}{2k} \right) = \frac{1}{\sqrt{1-s}} \tag{A4}
\]

and thus

\[
\langle 1 \rangle_k = \frac{\Gamma(k+1/2)}{\sqrt{\pi} \Gamma(k+1)} \sim \frac{1}{\sqrt{\pi k}} \quad k \to \infty. \tag{A5}
\]

For the first and second moments we find

\[
\sum_{k=1}^{\infty} s^k \langle q \rangle_k = -i \frac{\partial \gamma(s, \xi)}{\partial \xi} \bigg|_{\xi=0} = \frac{1}{\sqrt{1-s}} \sum_{k=1}^{\infty} \frac{s^k}{2k} \tag{A6}
\]

and

\[
\sum_{k=1}^{\infty} s^k \langle q^2 \rangle_k = - \frac{\partial^2 \gamma(s, \xi)}{\partial \xi^2} \bigg|_{\xi=0} = \frac{1}{4\sqrt{1-s}} \left[ \frac{1}{1-s} + \left( \sum_{k=1}^{\infty} \frac{s^k}{\sqrt{\pi k}} \right)^2 \right]. \tag{A7}
\]

To proceed, we need the following Tauberian theorem (Theorem 5, section XIII.5 in [14]):

Let \( c_k \geq 0 \) be a monotonic sequence and suppose that the series \( \sum_{k=1}^{\infty} c_k s^k \) converges for \( 0 \leq s < 1 \). Then, if \( L \) varies slowly at infinity and \( \alpha \geq 0 \), each of the two relations

\[
\sum_{k=1}^{\infty} c_k s^k \sim \frac{1}{(1-s)^\alpha} L \left( \frac{1}{1-s} \right), \quad s \to 1 \tag{A8}
\]

\( ^5 \)\( L \) varies slowly at infinity if \( \lim_{t \to \infty} L(tx)/L(t) = 1 \) for all \( x \), a condition which is verified in the subsequent applications where \( L \) is a constant.
implies the other.

From (A4), we have

\[ \sum_{k=1}^{\infty} s^k \langle q \rangle_k \sim \frac{1}{2(1-s)} \quad s \to 1 \]  \hspace{1cm} (A10)

and thus

\[ \langle q \rangle_k \sim \frac{1}{2}, \quad k \to \infty. \]  \hspace{1cm} (A11)

For the second moment we find

\[ \sum_{k=1}^{\infty} s^k \langle q^2 \rangle_k \sim \frac{1}{2(1-s)^{3/2}} \quad s \to 1 \]  \hspace{1cm} (A12)

leading to

\[ \langle q^2 \rangle_k \sim \frac{\sqrt{k}}{\sqrt{\pi}}, \quad k \to \infty. \]  \hspace{1cm} (A13)

To control the remainder in (90) by a limited Taylor expansion up to second order in \( q \) we use the fact that \( G_k(q) \) is non negative and that the second \( q \)-derivative of \( e^{-\nu \sqrt{2m\beta q} K_{m,\nu}(M_0, \nu M_0 + \sqrt{2m\beta q}, M, P)} \) has a \( q \)-integrable bound for \( M > 0 \) uniform when \( m \) and \( M_0 \) are in a neighbourhood of zero (irrespective of the sign of \( \nu \)). This is indeed the case because one can check that \( K_{\nu}(M_0, P_0, M, P) \) as well as its first and second \( P_0 \)-derivative obey Gaussian bounds of the form \( C_1 \exp(-C_2 P_0^2) \) with \( C_1 \) and \( C_2 \) independent of \( m \) and \( M_0, M_0 < M \)

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