Limiting state of plate with hole

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Abstract. The problem of determining the limit state of a plate with hole is considered. Solutions of two other problems are proposed to solve this problem, namely, determination of stress intensity factor in elements of structure with cracks and calculation of stress-strain state of a plate of finite width with hole. The first objective is based on the fracture mechanics for the crack plate using the software complex KOMPAS-3D (3D design system). And the second objective is based on the theoretical method of elasticity theory, applied to a plate of finite width with hole. Obtained results of both objectives were compared with known solutions.

1. Introduction

When designing structural elements, their strength calculation is performed. Structural elements must reliably perform certain functions during the defined service life. The main criterion for the operability of machine parts is their strength [1]. The structural elements may contain technological stress concentrators due to operational necessity. For such parts, the right choice of strength criteria will be an important point in the strength calculation. When determining the limit state of structural elements, the presence/absence of a crack is important for selecting the strength criterion. Then, in the presence of a crack, the methods of fracture mechanics are applied, and in the presence of a non-sharp-edged defect, classical and new criteria for the strength of a solid material are used.

The development and implementation of the principle of safe damage is possible only with the use of fracture mechanics methods. According to the conventional provisions, the stress state of bodies with cracks is fully characterized by the stress intensity factor (SIF) Almost all currently known criteria for brittle and quasi-brittle fracture, which describe the growth of fatigue cracks, are based on the preliminary determination of the SIF.

Various analytical and numerical methods are proposed for calculating stress intensity factors [2-3]. However, during practical calculations, especially for bodies of complex shapes, structural elements with damage, there is a need for a more convenient method for determining SIF, in order to obtain more accurate results.

The numerical method for determining SIF in structural elements with cracks using the finite element method makes it possible to significantly simplify the calculation method and expand the research.

2. Materials and methods

Strength and fracture criteria for continuum medium mechanics. The stress state of real parts even with the simplest schemes of force application is always complex. The results of the strength
calculation largely depend on the accepted strength criterion, on the basis of which it is possible, based on the data on material behavior under the simplest loads, to predict when a dangerous state will occur under the action of any complex load system.

The following four classical strength criteria are most widely known in engineering practice: the highest normal stress, linear strain, and shear stress, as well as the energy criterion [4].

**Analysis of material limit state criteria from crack mechanics positions.** The presence of sharp-pointed stress concentrators in the real body fundamentally complicates its strength calculation. In such cases, the classical approaches of continuum medium mechanics lead to incorrect results. In the classical theories of strength, for calculating the limiting state of a material, the special stress-strain state of the material near the apex of a sharp-pointed defect – a crack in the process of body deformation. This is due to the fact that the radius of the apex’s curvature of such a concentrator is comparable with the parameters of the material’s structure itself.

The basic idea of fracture mechanics of materials is based on the following. It is assumed that the transition of the deformable body element from a solid state to a fracture state is accompanied by an intermediate state. The most important feature of areas of deformable solid body, in which a pre-fracture state (pre-fracture areas) occurs, is that the material is always deformed beyond the elastic limit and that the most intense plastic flow occurs in them.

Therefore, assessing the strength of a solid, it is necessary to take into account its local physical and mechanical properties. Taking into account the states of material pre-fracture within the framework of continuum medium mechanics requires the introduction of new (in comparison with classical) computational models and concepts. Since the main characteristics that control the material behavior at the apex of the crack are stress, strain, and energy, then all the criteria of fracture mechanics similarly to the classical theories of strength, are divided into energy, force, and strain.

3. Characteristics and description of the proposed criterion for static fracture of components.

Consider a loaded plate with hole. It is known from experimental data that at the moment of fracture of a plate with a hole, the limiting stress state is not reached over the entire weakened section. Upon reaching a certain load near the hole in the dangerous section, a zone of the material limiting state appears. In this zone, a discontinuity of the material will be observed, which is equivalent to the formation of a crack. With a further increase in the load, the limit state zone (crack) will increase until the crack reaches the critical length. Then there will be almost instantaneous plate fracture.

The following conditions are accepted as criteria for static fracture of brittle materials:

\[ \sigma_{eq} \geq \sigma_B \cup \sigma_1 \geq S_K, \]  

where \( \sigma_{eq} \) is determined according to the Pisarenko-Lebedev strength theory; \( \sigma_B \) – is the ultimate strength; \( \sigma_1 \) – is the primary stress; \( \cup \) – is the logical summation sign; \( S_K \) – is the true resistance to separation.

Irwin condition is used as a fracture condition

\[ K_1 = K_c, \]  

Figure 1. Plate with central circular hole
where \( K_1 \) and \( K_c \) – are the stress intensity factors (SIF) of normal separation and its critical value in the case of the plane stress condition.

Thus, when using formulae (1) and (2) together, it is possible to determine the limit state of parts with stress concentrators, that differ from cracks, and their fracture. To apply conditions (1) and (2) to concrete details, it is necessary to know the stress state in the weakened section and the formula for determining SIF near a concentrator with a crack.

Hence, we write down the criterion for static fracture of a part with a non-crack-shaped concentrator:

\[
\begin{align*}
\sigma_{eq} & \geq \sigma_B \cup \sigma_1 \geq S_k \\
K_1 &= K_c.
\end{align*}
\]

(3)

4. Determination of the stress intensity factor in structural elements with cracks

Let us consider an infinite elastic body under the conditions of the plane problem of elasticity theory. The body has a narrow internal crack with a length of 2a (figure 1).

Let the crack propagate under the pressure effect \( P(x) \), which can vary along the crack.

We assume that there is symmetry about the x axis, so the problem is reduced to determining displacements and stresses in the elastic half-plane \( y \geq 0 \).

Boundary conditions in this case have the form, if \( y = 0 \):

\[
\begin{align*}
\tau_{xy} &= 0, \quad -\infty < x < \infty; \\
\sigma_y &= -p(x), \quad |x| \leq a; \\
v &= 0, \quad |x| \geq a,
\end{align*}
\]

where \( v \) – is the projection of the displacement vector onto the y axis.

Solving the mixed problem of elasticity theory, we find, if \( p(x) = p = \text{const} \):

\[
\begin{align*}
\sigma_y(x, 0) &= -p, \quad \text{if } |x| < a, \\
\sigma_y(x, 0) &= p \left[ \frac{|x|}{(x^2 - a^2)^{1/2}} - 1 \right], \quad \text{if } |x| > a.
\end{align*}
\]

(4)

The stress curve \( \sigma_y \), defined by the formula (4), is shown in figure 1. The stress \( \sigma_y \) approaches infinity at the crack apex, that is, it has a singularity at the tip.

To establish the nature of the normal stress singularity at the crack tip, we find the asymptotic image for \( \sigma_y \), if \( x \to a + 0 \). Letting in formula (4) \( x \) tend to \( a + 0 \) and discarding the constant summand, we find

\[
\sigma_y(x, 0) = \frac{pa^{1/2}}{2^{1/2}(x - a)^{1/2}}, \quad x \to a + 0.
\]

(5)

If we introduce the notation

\[
K_1 = p(a)^{1/2},
\]

(6)

then formula (5) is:

\[
\sigma_y(x, 0) = \frac{K_1}{(2\pi)^{1/2}(x - a)^{1/2}}, \quad x \to a + 0.
\]

(7)
where $K_1$ = SIF of normal separation.

Using formula (6), we determine SIF for a plane with a single crack, when uniform pressure $p$ is applied to the crack boundaries, or when the plane is subjected to uniform stretching along the normal to the crack line. In this case, SIF depends on the pressure and the crack size. Formula (7) is maintained when the crack is in an elastic body of finite sizes, while the pressure $p(x)$ may not be constant. However, in such cases, the expression for $K_1$ will differ from formula (6). For a finite body it also depends on its size.

The introduction of the concept of "SIF" has proven to be very useful in linear fracture mechanics. The stress intensity factor can be determined from both known stresses and from displacements. From formula (7) we find

$$K_1 = \lim \left[ 2\pi^{1/2}(x - a)^{1/2}\sigma_y(x, 0) \right].$$

(8)

Formula (8) allows you to determine SIF of normal separation by known stresses. The $K_1$ factor can also be found from the known displacements $v$:

$$K_1 = \frac{(2\pi)^{1/2}\mu}{1 + \chi} \lim_{x \to a} \left[ \frac{v(x, +0) - v(x, -0)}{(a - x)^{1/2}} \right];$$

(9)

$$K_1 = \frac{2\mu}{1 + \chi} \lim \left( (2\pi)^{1/2}(a - x)^{1/2} \frac{\partial}{\partial x} [v(x, +0) - v(x, -0)] \right),$$

(10)

where $\mu$ = is the shear modulus; $\chi = 3 - 4\nu$ – for plane strain;

$\chi = \frac{3 - \nu}{1 + \nu}$ – for the generalized plane stress state;

$\nu$ = Poisson's ratio.

For the determination of SIF of transverse $K_2$ and longitudinal shift $K_3$ there are formulas similar to the formulas (8)…(10) [5].

### 5. Numerical method for determining the stress intensity factor

Within the framework of linear fracture mechanics, which considers a model of an ideally elastic body and depicts a crack as a zero-thickness section, the surfaces of which are stress-free, the problem under consideration is reduced to the boundary value problem of elasticity theory.

The $K_1$ factor is the main basic characteristic of the stress-strain state of the material in the vicinity of a crack, which is extremely important in fracture mechanics.

Although real materials do not show perfectly elastic behavior during fracture, nevertheless, the models of linear fracture mechanics continue to be correct for them, provided that the zone of plastic processes is limited (implementing a quasi-brittle fracture scheme). This allows us to assume that the size of the pre-fracture zone and the state of the material in it as a whole is controlled by SIF [5].

For the problems under consideration, it is almost impossible to obtain an accurate analytical solution, therefore, so the numerical finite element method (FEM) in the form of the displacement method, implemented in the KOMPAS-3D software package, was used to calculate the $K_{1}$ factor.

For a generalized plane stress state, formula (6) is converted to a form that is convenient for use in the case of FEM applying:

$$K_1 = \frac{(2\pi)^{1/2}(1 + \nu)\mu}{4} \frac{v(x, +\frac{\delta}{2}) - v(x, -\frac{\delta}{2})}{(a - x)^{1/2}},$$

(11)

where $v(x, +\frac{\delta}{2})$ and $v(x, -\frac{\delta}{2})$ – displacement at points near the crack apex in the $y$-axis direction.
In some software complexes (KOMPAS-3D - 3D design system), SIF is calculated from several points at the crack apex.

The computational task was also to study the optimal number of points and their placement at the crack apex of the strained body during the determination of the $K_1$ factor and its comparison with the known solutions.

Two types of bodies with cracks are considered to solve the problem and to test the proposed method of determining SIF $K_1$:

- rectangular plate with a central crack (figure 4, a);
- strip with two symmetrical edge cracks under axial tension with normal stress $\sigma = 100$ MPa (figure 4, b).

Body thickness $t = 3$ mm, width $W = 150$ mm. For all variants, the crack thickness $\delta = 0.1$ mm is acceptable.

Body material – mild steel with characteristics:

\[ E = 2,1 \cdot 10^5 \text{ MPa}; \]
\[ \nu = 0.28; \]
\[ \mu = 7,9 \cdot 10^4 \text{ MPa}. \]

Analysis of stress-strain state of plate and strip is performed using FEM and software complex.

The plate and strip were divided into finite elements with automatic grid clustering around the crack apex and had such sizes that allowed us to study the strain behavior of the material bodies at the specified points at the crack tip.

The $K_1$ factors for normal separation cracks were determined by the formula (11) and compared with the solution [6] for the plate, obtained by the joint method of decomposition of complex stress potentials and boundary collocation

\[ K_1 = \sigma(\pi a)^{1/2} F_1(a, \beta), \]

and for the strip obtained by the volumetric force method: $K_1 = \sigma(\pi a)^{1/2} F_1(a);$ \n
\[ F(\alpha) = 1,122 - 0,154 \alpha + 0,807 \alpha^2 - 1,894 \alpha^3 + 2,494 \alpha^4; \]
\[ a = 2a/W; \beta = 2H/W, \]

where $F_1(a, \beta)$, $F(\alpha)$ – functions that depend on the geometry of bodies and are determined according to the work [6].
The results of studying the influence of the number of points and their placement at the crack apex of the plate on the SIF value showed, that the optimal sufficient option is to determine the displacements along the $y$ axis at two points located at a distance $\delta$ from the crack tip (figure 3).

The obtained results of comparison of the $K_{1}$ value indicate a greater accuracy in determining SIF values at two points at the crack apex. The discrepancy does not exceed 4% [6].

### 6. Comparison of calculated and experimental data

Using experimental data [7], we show the application of the fracture criterion (3) for the plate of final width $(2W = 140 \text{ mm}, H = 200 \text{ mm}, t = 1.83 \text{ mm})$, where $t$ – plate thickness, weakened by a central circular hole of radius $R = 14 \text{ mm}$, and subjected to uniform stretching (figure 1). The D16AT plate material has the following mechanical characteristics: true resistance to separation $S_{k} = 700 \text{ MPa}$, ultimate strength $\sigma_{B} = 450 \text{ MPa}$, fracture toughness $K_{c} = 37 \text{ MN/m}^{3/2}$. To determine the stress state in a weakened section, we apply the formula 2.24 obtained for a plate of finite sizes with a hole in [7] for figure 1:

$$\sigma_{y}(x, 0) = \sigma_{0}\left(1 + \frac{1}{2} \frac{R^{2}}{x^{2}} + \frac{3}{2} \frac{1 + \varepsilon}{1 - \varepsilon^{2}} \frac{R^{4}}{x^{4}}\right),$$

where $\varepsilon = R/w$, $R \leq x \leq W$.

To determine SIF near a concentrator with a crack - formula (12).

The application of criterion (3) and the work data [7] showed that the largest discrepancy between experimental and theoretical data obtained using the criterion $\sigma_{1} \geq S_{k}$ does not exceed 7%, and obtained using the criterion $\sigma_{eq} \geq \sigma_{B}$ – no more than 9%.

### 7. Conclusions

A numerical method for determining the SIF for structural elements with cracks is proposed. Several options for the placement of points were considered when determining the displacement field on the crack surface. Comparative analysis and numerical calculations show that the optimal case is when displacements are determined at two points (figure 3, points 1 and 2), located from the crack apex at a distance of its thickness $\delta$. In this case, the error usually does not exceed 5%. The proposed method of calculating SIF (8) can be widely used during the study of the limiting state of elements with cracks.

As can be seen from the example considered above, the error of the proposed static fracture criterion is relatively small and is quite acceptable for engineering calculations.

### References

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