Satellite Positioning with Large Constellations

Urs Niesen and Olivier Lévêque

Abstract

Modern global navigation satellite system receivers can access signals from several satellite constellations (including GPS, GLONASS, Galileo, BeiDou). Once these constellations are all fully operational, a typical receiver can expect to have on the order of 40–50 satellites in view. Motivated by that observation, this paper presents an asymptotic analysis of positioning algorithms in the large-constellation regime. We determine the exact asymptotic behavior for both pseudo-range and carrier-phase positioning. One interesting insight from our analysis is that the standard carrier-phase positioning approach based on resolving the carrier-phase integer ambiguities fails for large satellite constellations. Instead, we adopt a Bayesian approach, in which the ambiguities are treated as noise terms and not explicitly estimated.

I. INTRODUCTION

A. Motivation and Summary of Results

In order to determine its position and clock bias, a global navigation satellite system (GNSS) user needs to have at least four satellites in view. A fully operational GNSS constellation guarantees that this condition is always satisfied (in open sky), and traditional positioning algorithms were designed and analyzed with a number of visible satellites on that order in mind.

However, modern GNSS receivers can access signals from several different GNSS constellations including GPS (the US system), GLONASS (Russian), Galileo (European), and BeiDou (Chinese). Once these are fully deployed (which is already the case for GPS and GLONASS, and is expected by the end of this decade for Galileo and BeiDou), each of these constellations will have around 30 operational satellites. Thus, combined, there will be around 120 operational GNSS satellites [1]. A typical receiver in open-sky condition will then have access to on the order of 40–50 visible satellites.

This large number of visible satellites motivates an asymptotic analysis of the performance of GNSS positioning algorithms. We consider two different types of positioning approaches. The first approach uses only pseudo-range measurements. The second approach uses in addition carrier-phase measurements. We provide an asymptotic analysis of both these positioning approaches in the large-constellation regime.

When only pseudo-range measurements are available, the maximum likelihood (ML) estimate of the position is equal to the least-squares (LS) estimate. Its positioning performance depends on the satellite geometry and is summarized by the so-called dilution of precision (DOP). In order to analyze the performance behavior asymptotically, we introduce a simple stochastic model for the distribution of satellites across the sky. Using this model, Theorem 1 below shows that the DOP decreases as the inverse of the square root of the number of visible satellites and provides the exact scaling constant in front of the square-root term.

When carrier-phase measurements are also available, a more accurate positioning is possible. Unfortunately, the carrier-phase measurements are corrupted by an unknown integer ambiguity. The standard approach is to explicitly estimate these ambiguities as nuisance parameters. State-of-the-art estimation methods for resolving the ambiguities efficiently are LAMBDA [2] and modified LAMBDA [3]. For moderate number of visible satellites, these methods allow to resolve all ambiguities, leading to significantly improved performance compared to pseudo-range only positioning. However, for large number of visible satellites, insisting on resolving the integer ambiguities of all carrier-phase measurements leads to...
resolution errors for at least some of them, which deteriorates the positioning performance. A different treatment of these integer ambiguities is therefore required.

In the present paper, we instead adopt a Bayesian approach, treating the integer ambiguities as noise. This leads to an interesting expression for the maximum-likelihood estimate of the position (Equation (9) below), involving the minimum mean-squared error (MMSE) estimates of the integer ambiguities. We characterize the asymptotic behaviour of this ML estimate in Theorem 2 below. The standard deviation of the ML estimate is shown to decrease as the inverse of the square root of the number of satellites. Furthermore, the usefulness of the carrier-phase measurements is characterized by the ratio between the carrier wavelength and the carrier-phase noise standard deviation. More precisely, it is shown that the (rescaled) variance of the ML estimate can be expressed as a function that solely depends on this ratio.

B. Related Work

There has been significant recent interest in multi-constellation positioning, with performance evaluations both through simulations [1], [4] and experiments [5]–[7]. These results indicate that the many satellites available in these multi-constellation systems can lead to lower and less variable DOP [1], shorter convergence time of positioning algorithms [5], and better service availability in urban scenarios [4].

The traditional method used for carrier-phase positioning is to resolve integer ambiguities prior to estimating the baseline coordinates. Multiple methods have been developed along these lines, including the works of Teunissen [2] and Chang et al. [3] (already mentioned above), Hatch [8], Remondi [9], Al-Haifi et al. [10], and Hassibi and Boyd [11]. Recent interest in positioning with multiple satellite constellations has spurred the development of integer ambiguity resolution algorithms that scale more favorably with the number of satellites [12].

Parallel to this line of works, various authors have proposed following a Bayesian approach in order to estimate the baseline coordinates, treating integer ambiguities as noise and not imposing their resolution prior to the coordinates’ estimation [13]–[15]. Building on these approaches, de Lacy et al. have proposed in [16] to use Monte-Carlo simulations in order to refine the search space for the integer ambiguities. Also using a Bayesian framework, Garcia et al. have proposed in [17] an adaptive method to refine the estimation of the baseline coordinates when new measurements become available.

C. Organization

The remainder of this paper is organized as follows. Section II formally introduces the problem setting. Section III presents the main results. Section IV contains concluding remarks. All proofs are deferred to the appendix.

II. PROBLEM SETTING

We consider the GNSS positioning problem with \( S \) satellites, where \( S \) is assumed large. For each satellite \( s \in \{1, 2, \ldots, S\} \), we obtain two measurements, a pseudo-range measurement and a carrier-phase measurement.

After appropriate linearization around an approximate position solution and subtraction of known terms, the pseudo-range \( y_s \) satisfies the (approximately) linear measurement equation

\[
y_s \triangleq -u_s^T x + b + \sigma z_s.
\]

See, e.g., [18, Chapter 6.1.1] for a detailed derivation. Here, \( x \in \mathbb{R}^3 \) is the receiver position (technically the position approximation error), \( b \in \mathbb{R} \) is the receiver clock bias (again, technically the clock bias approximation error), \( u_s \) is the unit vector from the receiver to satellite \( s \) (computed from the approximate position solution), and \( \sigma z_s \) is receiver noise. This receiver noise is assumed here to be a Gaussian random variable with mean zero and variance \( \sigma^2 \), and to be independent and identically distributed (i.i.d.) across
satellites. Observe that here and in the following we use sans-serif font (i.e., $y_s$, $z_s$) to indicate random quantities.

It will be convenient to define the $S$-dimensional vector of pseudo-range measurements

$$
y \triangleq (y_s)_{s=1}^S \nonumber$$

and similar for $z$. Further, define the $S \times 4$ design matrix

$$G \triangleq \begin{pmatrix} -u_1^T & 1 \\
- u_2^T & 1 \\
\vdots & \vdots \\
- u_S^T & 1 \end{pmatrix}. \quad (2)$$

and the $4 \times 1$ vector of unknown parameters

$$w \triangleq \begin{pmatrix} x \\ b \end{pmatrix} \nonumber$$

With these definitions, the pseudo-range measurement equation (1) can be rewritten in vector form as

$$y = Gw + \sigma z. \nonumber$$

Similarly, the carrier phase $\tilde{y}_s$ can be linearized to satisfy the approximate measurement equation

$$\tilde{y}_s \triangleq -u_s^T x + b + \lambda m_s + \tilde{\sigma} z_s; \quad (3)$$

see, e.g., [18, Chapter 7]. Here $\lambda$ is the carrier wavelength (around 0.19 m for the GPS L1 signal), $m_s \in \mathbb{Z}$ is the unknown integer ambiguity, and $\tilde{\sigma} z_s$ is receiver noise. This receiver noise is assumed to be Gaussian with mean zero and variance $\tilde{\sigma}^2$, i.i.d. across satellites and independent of $z$.

We can again define the $S$-dimensional vector of carrier-phase measurements

$$\tilde{y} \triangleq (\tilde{y}_s)_{s=1}^S, \nonumber$$

and similar for $m$ and $\tilde{z}$. The carrier-phase measurement equation (3) then becomes

$$\tilde{y} = Gw + \lambda m + \tilde{\sigma} \tilde{z} \nonumber$$

with $G$ as defined in (2).

The carrier-phase measurements are much more precise than the pseudo-ranges. Typically, $\tilde{\sigma}$ is around a factor 100 smaller than $\sigma$ (see, e.g., [18, Chapter 5.5]). However, the carrier phases have the disadvantage that they contain an unknown integer ambiguity. Dealing with these integer ambiguities is one of the key challenges in carrier-phase positioning.

The above measurement model captures only first-order effects. In particular, atmospheric and ephemeris errors are neglected. Thus, this model is appropriate assuming that those errors have been corrected, for example using differential corrections.

The measurement equations (1) and (3) are stated for known, deterministic satellite positions (captured by the unit vectors $u_s$). To enable analytical evaluations of the positioning performance, we require a model for these unit vectors. We assume in the following that each $u_s$ is independently and uniformly distributed over the (say northern) hemisphere. This somewhat stylized model allows for analytical tractability and does again capture the first-order behavior.

With this assumption, the unit vector $u_s$ is now a random variable. As a consequence, the design matrix $G$ defined in (2) is now also a random matrix. The problem considered throughout the remainder of this paper is thus to estimate the receiver position $x$ and clock bias $b$ from the pseudo-ranges $y$, the carrier phases $\tilde{y}$, and the satellite unit vectors $u_1, \ldots, u_S$ (or, equivalently, the design matrix $G$). In particular, we will be interested in the estimation performance as the number of satellites $S$ increases.
III. MAIN RESULTS

We start with an analysis of pseudo-range only positioning in Section III-A. This will lay the foundation for our discussion of carrier-phase positioning in Section III-B.

A. Pseudo-Range Positioning

The ML estimator of the parameter vector \( \mathbf{w} \) (consisting of the receiver position \( \mathbf{x} \) and clock bias \( b \)) given the pseudo-range measurement vector \( \mathbf{y} \) and the design matrix \( \mathbf{G} \) is

\[
(G^T G)^{-1} G^T y
\]

This estimator is easily seen to be Gaussian with mean \( \mathbf{w} \) and covariance matrix

\[
\sigma^2 (G^T G)^{-1}
\]

(see, e.g., [18, Chapter 6.1]).

The quality of the estimate (4) depends on the satellite geometry through the design matrix \( \mathbf{G} \). This dependence is often summarized into a scalar quantity called the (geometric) dilution of precision, defined as

\[
\text{DOP}(\mathbf{G}) \triangleq \sqrt{\text{tr}((G^T G)^{-1})}
\]

(see again [18, Chapter 6.1]). Here \( \text{tr}(\cdot) \) denotes the trace. Observe that \( \text{DOP}(\mathbf{G}) \) is a random variable due to the random nature of \( \mathbf{G} \).

The DOP is lowest (and hence estimation performance best) if the satellites are well distributed across the hemisphere. For small number of visible satellites, the DOP can vary quite significantly. However, as our first theorem shows, this variability reduces as the number of satellites increases.

**Theorem 1.** The scaled pseudo-range positioning covariance matrix \( S \cdot \sigma^2 (G^T G)^{-1} \) converges in probability to \( \sigma^2 Q \) with

\[
Q \triangleq \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 12 & 6 \\
0 & 0 & 6 & 4
\end{pmatrix}
\]

as \( S \to \infty \). The corresponding scaled dilution of precision \( \sqrt{S} \cdot \text{DOP}(\mathbf{G}) \) converges in probability to \( \sqrt{22} \approx 4.69 \) as \( S \to \infty \).

The proof of Theorem 1 is reported in Appendix A.

Theorem 1 shows that, as the number of satellites grows, the dilution of precision decreases as \( \sqrt{22/S} \) with a stochastic variability (due to the random satellite geometry) that is much smaller than \( 1/\sqrt{S} \). This shows that increasing constellation size imparts two benefits. First, it improves positioning performance with root-mean-squared (RMS) error decreasing as the square root of the constellation size. Second, it reduces the variability of the positioning performance because we most often have well distributed satellites and consequently good satellite geometry.

The expression for the asymptotic covariance matrix \( Q \) in Theorem 1 also shows that the vertical positioning RMS is asymptotically twice as large as the horizontal positioning RMS (per dimension). This is in line with empirical observations from smaller satellite constellations [19]. Further, asymptotically only the vertical position and the clock bias estimation errors are correlated.

Theorem 1 only provides asymptotic information about the behavior of \( \text{DOP}(\mathbf{G}) \) as \( S \) increases. However, as Fig. 1 indicates, the limiting behavior is already apparent for \( S = 20 \) satellites.
Fig. 1. Dilution of precision as a function of number of satellites. The figure shows the expected value of $\sqrt{S} \cdot \text{DOP}(G)$ (solid black line) plus/minus one standard deviation (dotted black lines) as a function of the number of satellites $S$. Also shown is the limiting value $\sqrt{22}$ from Theorem 1 (dashed gray line).

B. Carrier-Phase Positioning

As mentioned earlier, dealing with the integer ambiguities present in the carrier-phase measurements is one of the key challenges in successfully using them for positioning. To see the potential value of the carrier-phase measurements, assume for the moment that we knew the integer ambiguities $m$ exactly. A short computation shows that the ML estimate of the parameter vector $w$ is then given by

$$
(G^T G)^{-1} G^T \left( \frac{\sigma^{-2}}{\sigma^{-2} + \tilde{\sigma}^{-2}} y + \frac{\tilde{\sigma}^{-2}}{\sigma^{-2} + \tilde{\sigma}^{-2}} (\tilde{y} - \lambda m) \right).
$$

(6)

This estimate constructs a convex combination between the pseudo-range measurements $y$ and the ambiguity-corrected carrier-phase measurements $\tilde{y} - \lambda m$. Since $\tilde{\sigma} \ll \sigma$, the ambiguity-corrected carrier-phase measurements have much higher weight in the convex combination than the pseudo-range measurements. This estimator is again Gaussian with mean $w$ (and therefore unbiased) and with covariance matrix

$$
\frac{1}{\sigma^{-2} + \tilde{\sigma}^{-2}} (G^T G)^{-1}.
$$

Comparing this to (5), we see that carrier-phase positioning with known ambiguities is much more precise than pseudo-range only positioning. From Theorem 1, we also see that this covariance matrix scaled by $S$ converges in probability to

$$
\frac{1}{\sigma^{-2} + \tilde{\sigma}^{-2}} Q
$$

(7)

as the number of satellites $S$ increases.

Of course, in reality we do not have direct access to the integer ambiguities and instead need to estimate or resolve them from the measurements. The standard approach for resolving the integer ambiguities consists of the following four steps:

1) Find the float estimate $\hat{m}$ of $m$ given the pseudo-range and carrier-phase measurements. This is the ML estimate ignoring the integer constraints.

2) Compute the fixed estimate $\hat{\hat{m}}$ of $m$ by finding the closest (in the least-squares sense, taking into account the covariance of the float estimates) integer vector to $\hat{m}$.

3) Validate the integer solution $\hat{\hat{m}}$ by applying a statistical test that guarantees that the probability $P(\hat{\hat{m}} = m)$ is above some threshold close to one.

As long as the pseudo-range only estimate of the clock bias is not very precise, it is difficult to disentangle the effect of the integer ambiguities and the clock bias on the carrier-phase measurements. To alleviate this problem, the standard procedure is to resolve the differenced ambiguities, i.e., $m_s - m_1$. However, for the purposes of this paper, we can focus on the undifferenced ambiguities.
4) Assuming validation was successful, estimate the receiver position and clock bias from the pseudo-range and carrier-phase measurements, treating the ambiguities as known and equal $\tilde{m}$.

As we will see next, this standard approach is unfortunately not appropriate for the regime of large satellite constellations. In fact, as $S$ increases, correct ambiguity resolution fails with probability approaching one. To see this, assume for the moment that a genie provides the correct value of the position $x$ and the clock bias $b$ to the receiver. Clearly, this knowledge can only increase the probability of successful integer ambiguity resolution. The ML (or, equivalently, the least-squares) integer estimate $\hat{m}_s$ of $m_s$ is then given by

$$\hat{m}_s \triangleq \left\lfloor \frac{(\tilde{y}_s + u_s^T x - b)}{\lambda} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes rounding to the closest integer. Now note that

$$\hat{m}_s = \left\lfloor m_s + \tilde{\sigma} z_s / \lambda \right\rfloor,$$

and therefore

$$\mathbb{P}(\hat{m}_s = m_s) = 1 - 2 \Phi\left(-\frac{\lambda}{2\tilde{\sigma}}\right)$$

with $\Phi(\cdot)$ denoting the standard Gaussian cumulative distribution function. Since the $\hat{m}_s$ are independent, this implies that

$$\mathbb{P}(\tilde{m} = m) = \left(1 - 2 \Phi\left(-\frac{\lambda}{2\tilde{\sigma}}\right)\right)^S \rightarrow 0 \quad \text{as } S \rightarrow \infty$$

provided that $\lambda/\tilde{\sigma}$ is finite. Thus, even with the aid of the genie providing $x$ and $b$, ambiguity resolution fails with probability one as the number of satellites increases. Clearly, the same conclusion holds without the aid of the genie.

From this discussion, we see that for a large number of satellites, we are unable to correctly resolve all the ambiguities. A different approach is therefore required. In order to avoid having to resolve the ambiguities, we will instead treat them as a noise term and estimate the position and clock bias directly from the pseudo-range and carrier-phase measurements. To this end, we place a prior distribution on the integer ambiguities. Specifically, we assume in the following that the ambiguities $m_1, \ldots, m_S$ are i.i.d. uniformly distributed over the set $\{-M, -M + 1, \ldots, M - 1, M\}$ with $M$ a fixed positive integer. We will mainly be interested in scenarios where $M$ is large enough to ensure that the resulting prior distribution on the ambiguities contains little information. Specifically, this is the case when $M \gg \sigma / \lambda$.

The probability density function of the combined carrier-phase noise (ambiguity plus receiver noise) is shown in Fig. 2. The combined noise has a mixture Gaussian density. For $\lambda/\tilde{\sigma} = 8$, the peaks of

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Fig. 2. Probability density function $p_{\tilde{y}_s|m_1,\ldots,m_S}$ of the combined carrier phase noise for different ratios of carrier wavelength to receiver noise standard deviation $\lambda/\tilde{\sigma}$. This combined noise includes the integer ambiguity, which is uniformly distributed on $\{-M, -M + 1, \ldots, M - 1, M\}$. In the figure, $M = 3$. 
the mixture components are clearly distinguishable. On the other hand, for \( \lambda/\hat{\sigma} = 2 \), they are virtually indistinguishable, and the combined carrier phase noise distribution is almost uniform over the interval \([-\lambda M, \lambda M]\).

Treating \( m \) as noise, the ML estimator \( \hat{w} \) of \( w \) is shown in Appendix B to be a solution of the equation

\[
\hat{w} = (G^T G)^{-1} G^T \left( \frac{\sigma^{-2}}{\sigma^{-2} + \hat{\sigma}^{-2}} y + \frac{\hat{\sigma}^{-2}}{\sigma^{-2} + \hat{\sigma}^{-2}} (y - \lambda \mathbb{E}_w(m | \hat{y}, G)) \right). \tag{9}
\]

Here \( \mathbb{E}_w(\cdot) \) denotes the expectation under the hypothesis that the true parameter vector \( w \) takes the value \( \hat{w} \). The conditional expectation in (9) has a simple closed-form expression given by

\[
\mathbb{E}_w(m | \hat{y}, G) = \frac{\sum_{m=-M}^{M} m \exp(-\frac{1}{2\hat{\sigma}^2}(\hat{y}_s - g_s^T \hat{w} - \lambda m)^2)}{\sum_{m=-M}^{M} \exp(-\frac{1}{2\hat{\sigma}^2}(\hat{y}_s - g_s^T \hat{w} - \lambda m)^2)} \tag{10}
\]

and can be efficiently evaluated.

Note that the value of the expectation \( \mathbb{E}_w(m | \hat{y}, G) \) depends itself on the value of \( \hat{w} \). As a consequence, the ML estimator \( \hat{w} \) is implicitly defined as a solution of (9). In general, there will be more than one solution \( \hat{w} \) satisfying (9) (see Fig. 4 below). To ensure uniqueness, we let \( \hat{w} \) be the solution of (9) that is closest (in Euclidean norm) to the pseudo-range only estimator \( (G^T G)^{-1} G^T y \).

Comparing (9) with (6), we see that the vector \( \mathbb{E}_w(m | \hat{y}, G) \) can be interpreted as an estimate of the integer ambiguities \( m \). In fact, it is the minimum mean-squared error (MMSE) estimator of the ambiguities. We emphasize that this MMSE estimator takes the integer nature of the ambiguities into account, as can be seen from (10).

The next theorem characterizes the asymptotic behavior of the estimator \( \hat{w} \) in (9) as the number of satellites \( S \) increases.

**Theorem 2.** For every fixed \( M > 0 \), the estimator \( \hat{w} \) is consistent, i.e., \( \hat{w} \) converges to \( w \) in probability as \( S \to \infty \). Further, the scaled estimation error \( \sqrt{S} \cdot (\hat{w} - w) \) converges in distribution to a Gaussian vector with mean zero and covariance matrix

\[
\frac{1}{\sigma^{-2} + h_M(\lambda/\hat{\sigma}) \cdot \hat{\sigma}^{-2}} Q
\]

as \( S \to \infty \), where the matrix \( Q \) was defined in Theorem 1 and where the function \( h_M(\cdot) \) is defined by (24) in Appendix B.

The proof of Theorem 2 is reported in Appendix B.
Observe the similarity of the asymptotic covariance expression for carrier-phase positioning in Theorem 2 with the ones for pseudo-range positioning in Theorem 1 and for carrier-phase positioning with known ambiguities in (7). Each of these expressions consist of the same matrix $Q$ pre-multiplied by a scalar factor. In order to illuminate the connection between these scalar factors, we plot in Fig. 3 the function $h_M(\cdot)$ appearing in Theorem 2 for $M = 20$. The figure shows that $h_M(\lambda/\tilde{\sigma})$ increases from 0 to 1 as a function of ratio $\lambda/\tilde{\sigma}$, which can be interpreted as a (square-root) signal-to-noise ratio of the carrier-phase signal.

When this ratio is small, $h(\lambda/\tilde{\sigma})$ is close to 0, and we (approximately) recover the result of Theorem 1. In this case, the asymptotic covariance of the ML estimator is given by $\sigma^2 Q$, which shows that the carrier-phase signal does not help. On the other hand, when $\lambda/\tilde{\sigma}$ increases, $h(\lambda/\tilde{\sigma})$ approaches the value of 1 (reaching this value approximately at $\lambda/\tilde{\sigma} = 8$). In this case, the asymptotic covariance of the ML estimator becomes $\frac{1}{\sigma^2 + \tilde{\sigma}^2} Q$, which by (7) is the same as if we knew the integer ambiguities exactly.

Recall that typically $\tilde{\sigma}$ is significantly smaller than $\sigma$. As a consequence, Theorem 2 implies that carrier-phase measurements can yield substantial performance gains even if $h_M(\lambda/\tilde{\sigma})$ is fairly small. In particular, we see from Fig. 3 that carrier-phase measurements are useful even if $\lambda/\tilde{\sigma}$ is below the value of 4. This contradicts the folklore rule of thumb that carrier-phase measurement noise with a standard deviation larger than one quarter of the wavelength renders those measurements unusable (see e.g. [20], [21]). Put differently, while it is difficult to resolve the integer ambiguities when $\lambda/\tilde{\sigma} \leq 4$, the Bayesian approach adopted here (which does not explicitly resolve the ambiguities) shows that the carrier-phase measurements can still be beneficial in this regime.

Our discussion so far has focused on the asymptotic behavior of carrier-phase positioning. We next evaluate the performance of the Bayesian carrier-phase positioning approach adopted in this paper for finite number $S$ of satellites. Recall that the ML estimator $\hat{w}$ of $w$ is a solution of (9), which results

\footnote{For fixed and finite $M$, the function $h_M(a)$ is not increasing on the entire domain $a \in [0, \infty)$. In fact, $h_M(0) = 1$ for all $M$, which can be seen by noting that for $\lambda = 0$ we always have $\lambda m_1 = 0$. However, for large enough $M$, the function $h_M(a)$ becomes increasing on the domain of interest. In particular, $h_M(a)$ is increasing on $a \in [2, \infty)$ for any fixed $M \geq 16$.}
from setting the derivative of the log-likelihood to zero. As Fig. 4 shows, the log-likelihood function has usually a fairly large number of local minima, maxima, and saddle points. Each of those corresponds to a solution of (9).

For our asymptotic analysis, we chose as estimator the solution of (9) closest to the pseudo-range only estimator. While this choice is asymptotically optimal and works well for very large numbers of satellites \( S \gg 100 \), it unfortunately performs poorly for more realistic numbers of satellites (say \( S = 50 \)). In this regime it is beneficial to instead choose the value of \( \hat{w} \) that directly maximizes the log-likelihood given by (12) in Appendix B. This maximizer can be found by running a global multi-start optimization procedure in a neighbourhood of the pseudo-range only estimator.

The performance of this approach is depicted in Fig. 5. The figure also shows the performance of pseudo-range only positioning and of carrier-phase positioning using the standard ambiguity resolution approach. The ratio of carrier wavelength to carrier-phase noise standard deviation is set to \( \lambda/\tilde{\sigma} = 4 \) and the number of satellites is \( S = 50 \). Recall from the discussion above that ambiguity resolution is considered difficult or impossible in the regime \( \lambda/\tilde{\sigma} = 4 \). Indeed, from (8), the probability of correctly resolving all the ambiguities using the standard approach is less 10%. This small probability of correct ambiguity resolution results in the poor performance of the standard approach: As Fig. 5 shows, carrier-phase positioning using the standard ambiguity resolution approach performs worse than pseudo-range only positioning about 70% of the time. In contrast, the Bayesian carrier-phase positioning approach adopted in this paper results in a noticeable performance improvement compared to pseudo-range only positioning.

IV. CONCLUSION

Motivated by the ever increasing number of available satellites, we have studied the problem of satellite positioning with large constellations. We have derived the asymptotic behavior of both pseudo-range and carrier-phase positioning. For carrier-phase positioning, we have argued that standard ambiguity resolution fails for large number of satellites, and a Bayesian approach of treating those ambiguities as additional receiver noise is more appropriate.

The results presented here raise several questions for follow-up work. First, the maximization of the likelihood function for carrier-phase positioning using a multi-start global search procedure is computationally quite demanding. Devising algorithms to (approximately) solve this optimization problem computationally more efficiently is therefore of interest.
Second, while we have shown with an example that carrier-phase positioning benefits from the Bayesian treatment of the ambiguities even for finite number of satellites, the same example also indicates that the number of satellites needs to be quite large for the asymptotic performance predictions to be accurate. It would therefore be beneficial to have analytical performance guarantees valid for smaller number of satellites.

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APPENDIX

A. Proof of Theorem

The matrix $\mathbf{G}^T \mathbf{G}$ can be written as

$$
\mathbf{G}^T \mathbf{G} = \sum_{s=1}^{S} \begin{pmatrix} \mathbf{-u}_s \\ 1 \end{pmatrix} \begin{pmatrix} -\mathbf{u}_s^T \\ 1 \end{pmatrix}.
$$

Observe that the matrices

$$
\begin{pmatrix} -\mathbf{u}_s \\ 1 \end{pmatrix} \begin{pmatrix} -\mathbf{u}_s^T \\ 1 \end{pmatrix}
$$

are i.i.d. as a function of $s$ (since the unit vectors $\mathbf{u}_s$ are) and have finite expected value. Hence, the weak law of large numbers applies and shows that

$$
\frac{1}{S} \mathbf{G}^T \mathbf{G} \xrightarrow{S \to \infty} \mathbb{E} \left( \begin{pmatrix} -\mathbf{u}_1 \\ 1 \end{pmatrix} \begin{pmatrix} -\mathbf{u}_1^T \\ 1 \end{pmatrix} \right),
$$

where $\xrightarrow{P}$ denotes convergence in probability, and where $\mathbb{E}(\cdot)$ denotes expectation.

We next compute this expectation. We start with the diagonal terms. By symmetry, we have

$$
\mathbb{E}(\mathbf{u}_{11}^2) = \mathbb{E}(\mathbf{u}_{12}^2) = \mathbb{E}(\mathbf{u}_{13}^2).
$$

Further, since $\mathbf{u}_1$ is a unit vector, we have

$$
\mathbb{E}(\mathbf{u}_{11}^2) + \mathbb{E}(\mathbf{u}_{12}^2) + \mathbb{E}(\mathbf{u}_{13}^2) = 1.
$$

Hence,

$$
\mathbb{E}(\mathbf{u}_{11}^2) = \mathbb{E}(\mathbf{u}_{12}^2) = \mathbb{E}(\mathbf{u}_{13}^2) = 1/3.
$$

The cross terms $\mathbf{u}_{1i} \mathbf{u}_{1j}$ for $i \neq j$ are easily seen to be zero by symmetry, and similar for the cross terms $-1 \cdot \mathbf{u}_{11}$ and $-1 \cdot \mathbf{u}_{12}$. It remains the cross term $-1 \cdot \mathbf{u}_{13}$. A straightforward calculation, making use of the standard expression for the area of a spherical cap, shows that the marginal distribution of $\mathbf{u}_{3i}$ is uniform on $[0, 1]$. Hence,

$$
\mathbb{E}(-1 \cdot \mathbf{u}_{13}) = -1/2.
$$

Together with (11), this shows that

$$
\frac{1}{S} \mathbf{G}^T \mathbf{G} \xrightarrow{S \to \infty} \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}.
$$
Observe that this last matrix is invertible. Since the matrix inverse is continuous, the continuous mapping theorem implies that

\[
\left( \frac{1}{S} \mathbf{G}^T \mathbf{G} \right)^{-1} \xrightarrow{P_{S \to \infty}} \mathbf{P} = \mathbf{S} \to \infty \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 12 & 6 \\ 0 & 0 & 6 & 4 \end{pmatrix}
\]

\[
= \mathbf{Q}.
\]

A second application of the continuous mapping theorem further shows that

\[
\sqrt{S} \cdot \text{DOP} (\mathbf{G}) \xrightarrow{P_{S \to \infty}} \sqrt{\text{tr} (\mathbf{Q})} = \sqrt{22},
\]

as claimed.

B. Proof of Theorem 2

Treating the integer ambiguities \( \mathbf{m} \) as noise uniformly distributed on \( \{-M, \ldots, M\}^S \) and considering the rows \( \mathbf{g}_s^T \) of the matrix \( \mathbf{G} \) as i.i.d. observations, we obtain that the ML estimator \( \mathbf{w} \) should maximize the following likelihood function (given the observables \( \mathbf{y} = \mathbf{y}, \tilde{\mathbf{y}} = \tilde{\mathbf{y}} \) and \( \mathbf{G} = \mathbf{G} \)):

\[
p_{y, \tilde{y}, \mathbf{G}} (\mathbf{y}, \tilde{\mathbf{y}}, \mathbf{G}; \mathbf{w}) = p_{\mathbf{G}} (\mathbf{G}) \sum_{\mathbf{m} \in \{-M, \ldots, M\}^S} p_{\mathbf{m}} (\mathbf{m}) p_{y, \tilde{y}, \mathbf{G}, \mathbf{m}} (\mathbf{y}, \tilde{\mathbf{y}}, \mathbf{w} | \mathbf{G}, \mathbf{m})
\]

\[
= C \exp \left( -\frac{1}{2\sigma^2} \| \mathbf{Gw} - \mathbf{y} \|^2 \right) \sum_{\mathbf{m} \in \{-M, \ldots, M\}^S} \exp \left( -\frac{1}{2\sigma^2} \| \mathbf{Gw} + \lambda \mathbf{m} - \tilde{\mathbf{y}} \|^2 \right)
\]

\[
= C \prod_{s=1}^S \exp \left( -\frac{1}{2\sigma^2} (\mathbf{g}_s^T \mathbf{w} - y_s)^2 \right) \sum_{m_s = -M}^M \exp \left( -\frac{1}{2\sigma^2} (\mathbf{g}_s^T \mathbf{w} + \lambda m_s - \tilde{y}_s)^2 \right),
\]

where the normalization constant \( C \) is given by

\[
C \triangleq \frac{p_{\mathbf{G}} (\mathbf{G})}{(2\pi\sigma\tilde{\sigma})^S (2M + 1)^S}.
\]

The corresponding log-likelihood function \( L (\mathbf{y}, \tilde{\mathbf{y}}, \mathbf{G}; \mathbf{w}) \) is given by

\[
L (\mathbf{y}, \tilde{\mathbf{y}}, \mathbf{G}; \mathbf{w}) = \log (C) + \sum_{s=1}^S \ell (y_s, \tilde{y}_s, \mathbf{g}_s; \mathbf{w})
\]

with

\[
\ell (y, \tilde{y}, \mathbf{g}; \mathbf{w}) \triangleq -\frac{1}{2\sigma^2} (\mathbf{g}^T \mathbf{w} - y)^2 + \log \left( \sum_{m = -M}^M \exp \left( -\frac{1}{2\tilde{\sigma}^2} (\mathbf{g}^T \mathbf{w} + \lambda m - \tilde{y})^2 \right) \right).
\]

By definition, the ML estimator \( \hat{\mathbf{w}} \) satisfies

\[
\frac{\partial L (\mathbf{y}, \tilde{\mathbf{y}}, \mathbf{G}; \mathbf{w})}{\partial w_i} \bigg|_{\mathbf{w} = \hat{\mathbf{w}}} = \sum_{s=1}^S \frac{\partial \ell (y_s, \tilde{y}_s, \mathbf{g}_s; \mathbf{w})}{\partial w_i} \bigg|_{\mathbf{w} = \hat{\mathbf{w}}} = 0
\]
for all $i \in \{1, \ldots, 4\}$. The partial derivative of the summand in the log-likelihood is

\[
\frac{\partial \ell(y, \tilde{y}, g; w)}{\partial w_i} = \frac{g_i}{\sigma^2} (y - g^T w) + \frac{g_i}{\sigma^2} (\tilde{y} - g^T w) - \frac{g_i \lambda}{\sigma^2} \sum_{m=-M}^{M} f_{\tilde{y}-g^T w}(m) \sum_{m=-M}^{M} f_{\tilde{y}-g^T w}(m)
\]

\[
= \frac{g_i}{\sigma^2} (y - g^T w) + \frac{g_i}{\sigma^2} (\tilde{y} - g^T w) - \frac{g_i \lambda}{\sigma^2} \langle m \rangle_{\tilde{y}-g^T w},
\]

where

\[
f_v(m) \triangleq \exp \left( -\frac{1}{2\sigma^2} (\lambda m - v)^2 \right) \quad \text{for } v \in \mathbb{R}
\]

and

\[
\langle m^k \rangle_v \triangleq \frac{\sum_{m=-M}^{M} m^k f_v(m)}{\sum_{m=-M}^{M} f_v(m)} \quad \text{for } k \in \mathbb{N} \text{ and } v \in \mathbb{R}.
\]

Note that the above bracket notation is justified by the fact that

\[
\frac{f_v(m)}{\sum_{m=-M}^{M} f_v(m)}
\]

is a probability mass function on $m \in \{-M, \ldots, M\}$ for every $v \in \mathbb{R}$. Given the particular form of $f_v(m)$, one may also interpret the above bracket as the conditional expectation

\[
\langle m^k \rangle_v = \mathbb{E}(m_s^k \mid \lambda m_s + \hat{s} \tilde{z}_s = v).
\]

This, in turn, may be rewritten as

\[
\langle m^k \rangle_v = \mathbb{E}(m_s^k \mid \tilde{y}_s - g_s^T w = v).
\]

We will use several times in the following that

\[
\frac{\partial \langle m^k \rangle_v}{\partial v} = \frac{\lambda}{\sigma^2} (\langle m^{k+1} \rangle_v - \langle m^k \rangle_v \langle m \rangle_v),
\]

which can be verified after a short calculation.

Substituting (14) and (16) into (13), we obtain that the ML estimator satisfies the equation

\[
\sum_{s=1}^{S} \left( \frac{g_{si}}{\sigma^2} (y_s - g_s^T \hat{w}) + \frac{g_{si}}{\sigma^2} (\tilde{y}_s - g_s^T \hat{w}) - \frac{g_{si} \lambda}{\sigma^2} \mathbb{E}_{\hat{w}}(m_s \mid \tilde{y}_s - g_s^T \hat{w}) \right) = 0
\]

for all $i \in \{1, \ldots, 4\}$, where $\mathbb{E}_{\hat{w}}(\cdot)$ denotes expectation under the hypothesis that the true parameter vector $w$ equals $\hat{w}$. Observing that

\[
\mathbb{E}_{\hat{w}}(m_s | \tilde{y}_s - g_s^T \hat{w}) = \mathbb{E}_{\hat{w}}(m_s | \tilde{y}_s, g_s) = \mathbb{E}_{\hat{w}}(m_s | \tilde{y}, G)
\]

by the independence of the observations under the hypothesis that $w = \hat{w}$, (18) may be rewritten more compactly as

\[
(G^T G) (\sigma^{-2} + \hat{\sigma}^{-2}) \hat{w} = G^T (\sigma^{-2} y + \hat{\sigma}^{-2} \tilde{y} - \hat{\sigma}^{-2} \lambda \mathbb{E}_{\hat{w}}(m | \tilde{y}, G))
\]

leading finally to (9) in Section III-B.

As pointed out earlier, this last equation may have multiple solutions, and we choose $\hat{w}$ as the one closest to the pseudo-range only estimator $(G^T G)^{-1} G^T y$. As the latter estimator is consistent (i.e., it converges in probability towards the true parameter $w$ as $S \to \infty$) by Theorem [1], this implies that $\hat{w}$ is also a consistent estimator by [22, p. 453].

The asymptotic normality of the ML estimator follows from [23, Theorem 5.4]. For that theorem to apply, the following seven conditions must be satisfied.
Condition 1: The parameter space is open. Since the parameter space is the whole $\mathbb{R}^4$, this is clearly the case.

Condition 2: The support of $\exp(\ell(y, \tilde{y}, g; w))$ does not depend on $w$. Clearly, the set $\{(y, \tilde{y}, g) : \exp(\ell(y, \tilde{y}, g; w)) > 0\}$ does not depend on $w$, so that this condition is satisfied.

Condition 3: The mapping $w \mapsto \exp(\ell(y, \tilde{y}, g; w))$ is three times continuously differentiable for every $(y, \tilde{y}, g)$. This condition again clearly holds.

Condition 4: For every $i \in \{1, \ldots, 4\}$, the equality
\[
\mathbb{E}\left( \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} \right) = 0
\]
holds. Observe first that $y_s - g_s^T w = \sigma z_s$ and $\tilde{y}_s - g_s^T w = \lambda m_s + \bar{\sigma} \bar{z}_s$. From (14), we then obtain
\[
\frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} = \frac{g_{si}}{\sigma^2} (\sigma z_s) + \frac{g_{si}}{\sigma^2} (\lambda m_s + \bar{\sigma} \bar{z}_s) - \frac{g_{si} \lambda}{\sigma^2} (m)_{\lambda m_s + \bar{\sigma} \bar{z}_s},
\]
which is actually independent of $w$. Using (16), we further obtain
\[
\frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} = g_{si} \left( \frac{z_s}{\sigma} + \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right).
\]
Therefore,
\[
\mathbb{E}\left( \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} \right) = \mathbb{E}\left( g_{si} \left( \frac{z_s}{\sigma} + \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right) \right)
\]
\[
= \mathbb{E}(g_{si}) \mathbb{E}\left( \frac{z_s}{\sigma} + \frac{\bar{z}_s}{\bar{\sigma}} \right)
\]
\[
= 0,
\]
where we have used the towering property of conditional expectation, that $g_{si}$, $m_s$, $z_s$, $\bar{z}_s$ are independent, and that $z_s$, $\bar{z}_s$ are centered.

Condition 5: For every $w \in \mathbb{R}^4$, the $4 \times 4$ matrix
\[
I(w) \triangleq \left( \mathbb{E}\left( \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_j} \right) \right)_{i,j \in \{1, \ldots, 4\}}
\]
is positive-definite. We start by deriving an explicit expression for the matrix $I(w)$. Reusing (19), we obtain
\[
I_{ij}(w) = \mathbb{E}\left( \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_i} \frac{\partial \ell(y_s, \tilde{y}_s, g_s; w)}{\partial w_j} \right)
\]
\[
= \mathbb{E}\left( g_{si} \left( \frac{z_s}{\sigma} + \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right) \right)
\]
\[
\times g_{sj} \left( \frac{z_s}{\sigma} + \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right)
\]
\[
= \mathbb{E}(g_{si} g_{sj}) \left( \frac{1}{\sigma^2} + \mathbb{E}\left( \left( \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right)^2 \right) \right),
\]
again by independence of $g_s$, $z_s$, $\bar{z}_s$ and $m_s$, and using that $z_s$ is centered. We can rewrite this last equation in matrix form as
\[
I(w) = I
\]
\[
= \mathbb{E}(g_s^T g_s) \left( \frac{1}{\sigma^2} + \mathbb{E}\left( \left( \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right)^2 \right) \right)
\]
\[
= Q^{-1} \left( \frac{1}{\sigma^2} + \mathbb{E}\left( \left( \frac{\bar{z}_s}{\bar{\sigma}} + \lambda \left( m_s - \mathbb{E}(m_s | \lambda m_s + \bar{\sigma} \bar{z}_s) \right) \right)^2 \right) \right),
\]
where $Q$ is the Cholesky decomposition of $\mathbb{E}(g_s^T g_s)$. This is typically provided by the optimization routine.
where $Q$ is the matrix given in Theorem 1 and where the notation $I$ is used to indicate that the right-hand side does not depend on $w$. Since $Q$ is positive definite, and since the scalar factor multiplying this matrix is positive, the matrix $I(w)$ is positive-definite for every $w \in \mathbb{R}^4$ as required.

For future reference, we further simplify the expression for $I_{ij}(w)$. We can rewrite the second expectation in (20) as

$$
\mathbb{E} \left( \left( \frac{\tilde{z}_s}{\sigma} + \frac{\lambda}{\sigma^2} (m_s - \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)) \right)^2 \right)
$$

$$
= \frac{1}{\sigma^2} + \frac{2\lambda}{\sigma^3} \mathbb{E} \left( \tilde{z}_s (m_s - \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)) \right) + \frac{\lambda^2}{\sigma^4} \mathbb{E} \left( (m_s - \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s))^2 \right)
$$

$$
= \frac{1}{\sigma^2} - \frac{2\lambda}{\sigma^3} \mathbb{E} \left( \tilde{z}_s \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s) \right) + \frac{\lambda^2}{\sigma^4} \left[ \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2) \right],
$$

using again the independence of $\tilde{z}_s$ and $m_s$, and using the towering property of conditional expectation to conclude that

$$
\mathbb{E} \left( \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s) \right) = \mathbb{E} \left[ \mathbb{E}(m_s \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s) | \lambda m_s + \tilde{\sigma} \tilde{z}_s) \right]
$$

$$
= \mathbb{E} \left( \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2 \right).
$$

We next use the fact that for $\tilde{z}_s \sim \mathcal{N}(0, 1)$ and for any continuously differentiable function $F(\cdot)$ with polynomial growth, the integration by parts formula gives

$$
\mathbb{E}(\tilde{z}_s F'(\tilde{z}_s)) = \mathbb{E}(F'\tilde{z}_s).
$$

Here, we would like to compute

$$
\mathbb{E}(\tilde{z}_s F(\lambda m_s + \tilde{\sigma} \tilde{z}_s))
$$

with

$$
F(v) \triangleq \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s = v) = \langle m \rangle_v.
$$

Using (17), the derivative of $F(\cdot)$ is given by

$$
F'(v) = \frac{\partial}{\partial v} \langle m \rangle_v = \frac{\lambda}{\sigma^2} (\langle m^2 \rangle_v - \langle m \rangle_v^2).
$$

Hence,

$$
\mathbb{E}(\tilde{z}_s \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)) = \mathbb{E}(\tilde{z}_s F(\lambda m_s + \tilde{\sigma} \tilde{z}_s))
$$

$$
= \tilde{\sigma} \mathbb{E}(F'(\lambda m_s + \tilde{\sigma} \tilde{z}_s))
$$

$$
= \frac{\lambda}{\sigma} \mathbb{E} \left( \langle m^2 \rangle_{\lambda m_s + \tilde{\sigma} \tilde{z}_s} - \langle m \rangle_{\lambda m_s + \tilde{\sigma} \tilde{z}_s}^2 \right)
$$

$$
= \frac{\lambda}{\sigma} \mathbb{E} \left( \mathbb{E}(m_s^2 | \lambda m_s + \tilde{\sigma} \tilde{z}_s) - \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2 \right)
$$

$$
= \frac{\lambda}{\sigma} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2) \right).
$$

Substituting this into (21) leads to

$$
\mathbb{E} \left( \left( \frac{\tilde{z}_s}{\sigma} + \frac{\lambda}{\sigma^2} (m_s - \mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)) \right)^2 \right) = \frac{1}{\sigma^2} - \frac{\lambda^2}{\sigma^4} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2) \right)
$$

and finally to

$$
I_{ij}(w) = (Q^{-1})_{ij} \left( \frac{1}{\sigma^2} + \frac{1}{\sigma^2} - \frac{\lambda^2}{\sigma^4} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2) \right) \right). \tag{22}
$$
We therefore have that
\[ J(w) \Delta \left( \mathbb{E}\left(-\frac{\partial^2 \ell(y, \bar{y}, g_s; w)}{\partial w_i \partial w_j} \right) \right)_{i,j \in \{1, \ldots, 4\}} \]
is positive-definite. It turns out in our case that \( J(w) = I(w) \) for every \( w \in \mathbb{R}^4 \). Since \( I(w) \) was already shown to be positive definite when verifying Condition 5, this implies that Condition 6 holds. To prove this equality, we compute the second-order partial derivatives. Starting from (14) and using (17), we obtain
\[ \frac{\partial^2 \ell(y, \tilde{y}, g; w)}{\partial w_i \partial w_j} = g_i g_j \left( -\frac{1}{\sigma^2} - \frac{\lambda^2}{\bar{s}^2} \left( \langle m^2 \rangle \bar{y} - g^T w - \langle m \rangle \bar{y} - g^T w \right) \right). \] (23)

Using again that \( y - g^T w = \lambda m + \bar{\sigma} z \), we have
\[ -\frac{\partial^2 \ell(y, \tilde{y}, g_s; w)}{\partial w_i \partial w_j} = g_{si} g_{sj} \left( \frac{1}{\sigma^2} + \frac{\lambda^2}{\bar{s}^2} \left( \langle m_s^2 \rangle \lambda m + \bar{\sigma} z - \langle m \rangle \lambda m + \bar{\sigma} z \right) \right). \]

Taking expectation and comparing to (22) leads to
\[ J_{ij}(w) = (Q^{-1})_{ij} \left( \frac{1}{\sigma^2} + \frac{\lambda^2}{\bar{s}^2} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m + \bar{\sigma} z)^2) \right) \right) = I_{ij}(w), \]
proving the claim.

**Condition 7:** For every \( w \in \mathbb{R}^4 \), \( i, j, k \in \{1, \ldots, 4\} \), there exists \( D_{ijk}(y_s, \tilde{y}_s, g_s) \) satisfying
\[ \mathbb{E}|D_{ijk}(y_s, \tilde{y}_s, g_s)| < +\infty \]
such that
\[ \left| \frac{\partial^3 \ell(y, \tilde{y}, g; w)}{\partial w_i \partial w_j \partial w_k} \right|_{w=\tilde{w}} \leq D_{ijk}(y_s, \tilde{y}_s, g_s) \]
holds for every \( \|\tilde{w} - w\| \leq \delta \). Starting from (23) and using (17), the third partial derivatives can be calculated as
\[ -\frac{\partial^3 \ell(y, \tilde{y}, g_s; w)}{\partial w_i \partial w_j \partial w_k} = -g_i g_j g_k \left( \frac{3 \lambda^3}{\bar{s}^6} \left( \langle m^3 \rangle \bar{y} - g^T w - 3 \langle m^2 \rangle \bar{y} - g^T w \langle m \rangle \bar{y} - g^T w + 2 \langle m^3 \rangle \bar{y} - g^T w \right) \right). \]

We therefore have that
\[ \left| \frac{\partial^3 \ell(y, \tilde{y}, g_s; w)}{\partial w_i \partial w_j \partial w_k} \right|_{w=\tilde{w}} \leq |g_{si} g_{sj} g_{sk}| \frac{\lambda^3}{\bar{s}^6} 6 M^3 \leq \frac{\lambda^3}{\bar{s}^6} 6 M^3, \]
which is independent of \( \tilde{w} \) and clearly integrable.

Under Conditions 1–7, [23, Theorem 5.4] states that \( \sqrt{S}(\tilde{w} - w) \) converges in distribution to a centered Gaussian random vector with covariance matrix \( J(w)^{-1} I(w) J(w)^{-1} = J(w)^{-1} = I^{-1} \) as \( S \to \infty \).

What remains to be computed is a more explicit expression for the scalar factor
\[ \frac{1}{\sigma^2} + \frac{1}{\bar{s}^2} - \frac{\lambda^2}{\bar{s}^2} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m + \bar{\sigma} z)^2) \right). \]

multiplying the matrix \( Q^{-1} \) in the expression (22) for \( I \). To this end, recall from (16) that
\[ \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m + \bar{\sigma} z)^2) = \mathbb{E}(\langle m_s^2 \rangle \lambda m + \bar{\sigma} z - \langle m \rangle \lambda m + \bar{\sigma} z) \]

3Strictly speaking, [23, Theorem 5.4] is only stated for scalar-valued i.i.d. observations. The conclusion of the theorem remains however valid for vector-valued i.i.d. observations, provided that the likelihood function derived from the vector-valued observations \( \langle y_s, \tilde{y}_s, g_s \rangle \) verifies Conditions 1–7.
and from [15] that
\[
\langle m^k \rangle_v = \frac{\sum_{m=-M}^{M} m^k f_v(m)}{\sum_{m=-M}^{M} f_v(m)} = \frac{\sum_{m=-M}^{M} m^k \exp \left(-\frac{1}{2\sigma^2} (\lambda m - v)^2 \right)}{\sum_{m=-M}^{M} \exp \left(-\frac{1}{2\sigma^2} (\lambda m - v)^2 \right)}.
\]

Thus,
\[
\langle m^k \rangle_{\lambda m_s + \tilde{z}_s} = \frac{\sum_{m=-M}^{M} m^k \exp \left(-\frac{1}{2\tilde{\sigma}^2} (\lambda m - m_s - \tilde{z}_s)^2 \right)}{\sum_{m=-M}^{M} \exp \left(-\frac{1}{2\tilde{\sigma}^2} (\lambda m - m_s - \tilde{z}_s)^2 \right)} = H_k(\lambda/\tilde{\sigma}, m_s, \tilde{z}_s),
\]

which depends on the parameters \( \lambda \) and \( \tilde{\sigma} \) only through their ratio \( \lambda/\tilde{\sigma} \). Finally, we obtain
\[
\frac{1}{\sigma^2} + \frac{1}{\tilde{\sigma}^2} - \frac{\lambda^2}{\tilde{\sigma}^4} \left( \mathbb{E}(m_s^2) - \mathbb{E}(\mathbb{E}(m_s | \lambda m_s + \tilde{\sigma} \tilde{z}_s)^2) \right) = \sigma^{-2} + \tilde{\sigma}^{-2} h_M(\lambda/\tilde{\sigma})
\]
with
\[
h_M(\lambda/\tilde{\sigma}) \triangleq 1 - \frac{\lambda^2}{\tilde{\sigma}^2} \mathbb{E}(H_2(\lambda/\tilde{\sigma}, m_s, \tilde{z}_s) - H_1(\lambda/\tilde{\sigma}, m_s, \tilde{z}_s)^2),
\]
where we have explicitly indicated the dependence of \( h_M(\cdot) \) on the support of the prior governed by \( M \). Substituting this expression into (22) shows that the inverse of the scaled asymptotic covariance matrix is
\[
I = (\sigma^{-2} + \tilde{\sigma}^{-2} h_M(\lambda/\tilde{\sigma})) Q^{-1},
\]
which can be rearranged as in the statement of the theorem. This concludes the proof.

\[\Box\]

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