Identification of Viscoelastic Property of Pile-Soil Interactions with Fractional Derivative Model

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Abstract
The viscoelastic property of pile-soil interactions is modeled by fractional derivative method and is used to characterize the structural dynamics of pile and rheological behavior of a thawing frozen soil. A study of the thawing frozen soil impacts on pile structure’s dynamic characteristic is carried out. The frequency spectrum analysis and empirical modal decomposition are used to identify the pile structure dynamic properties. The rocking mode vibration in which pile behaviors like rigid body supported by viscoelastic soil is extracted to capture the viscoelastic property of pile-soil interactions. The rocking mode exhibiting time-varying frequency features is modeled with fractional derivative model and the fractional derivative order is identified. This study proposes a quick and reliable method to identify the viscoelastic property of pile-soil interactions.

Keywords
Pile-soil interactions, fractional derivative model, viscoelastic materials, system identification

1. Introduction
Structure systems may have excessive settlement and failure when soil foundations exhibit viscoelastic property and creeping deformation under certain environments. In structural dynamic analysis, to study the soil’s force-deformation, many complicated models related the strain and shearing stress have been proposed and applied. The viscoelastic material’s conventional constitutive equation that behaves linearly is a standard linear differential equation connecting the time derivatives of strain and stress. For most viscoelastic materials, it requires many terms in the equation which involved higher order derivatives of strain and stress. Kelvin and Maxwell are the two basic models, and their models only with one first-order derivative term. Kelvin model describes the behavior of the solid viscoelastic materials, and Maxwell model is fit for the fluid system [1]. The material properties from the Kelvin model are frequency independent, so it is insufficient to analysis the structural vibration and design. In the constitutive equation, more derivative terms can provide a better description for the frequency dependence of the dynamic behavior of the material. But add more derivative terms will enhance the calculation complicated and laborious. As such, there is limited research effort that considers the general linear model to study the structural dynamics response with viscoelastic materials. Even if it is convenience to the conventional models to analysis the structures, it cannot show the varies in the material properties once the frequency changes. Another method to describe the behavior of viscoelastic material is to use a complex modulus. This model can provide an explicit description of strain and stress under the purely harmonic motion. But there are two major flaws: the

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complex modulus model based on the sinusoidal testing hence it limits apply to the sinusoidal motion. On the other hand, the inconsistency associated with the analysis’s assumptions.

Other model that can identify the material’s frequency dependence is the fractional derivative model. The studies show that the stress relaxation of viscoelastic materials is proportional to time raised to fractional powers \(2,3\). Bagley and Torvik established a constitutive equation by using the fractional derivatives to indicate the behavior of viscoelastic materials with memory \(4–6\). The research shown that only few parameters are required, and easier mathematical manipulation is needed to express the frequency dependent mechanical properties for the typical viscoelastic materials \(7–9\). This constitutive equation has been treated as fractional derivative model. The modulus for a viscoelastic material can be represent as a function of fractional powers of the frequency.

The fractional derivative model’s advantages including with few parameters, this model can provide physical explanations for the viscoelastic material and can characterize the frequency dependence of material properties. Other researchers have examined different problem relate to the numerical implementation of these kinds of models. It has been shown that both the classic general model and the fractional derivative model has the capability to sufficient fit and close to the test data from vibration test of the viscoelastic material, and therefore determining the frequency dependent characteristics of the system. The fractional derivative model is accurate and compact, and it only needs several parameters to comprehensively quantifying complex physical phenomena. According to the theory of viscoelasticity, fractional derivative method is used to model the dynamical behavior of soil foundation. The model parameters’ impacts on the dynamical properties are evaluated by a parametric study with varying values of the fractional order of differential operator. Compared with classic viscoelastic models, the fractional derivative model gives rise to accurate estimation of the problem with fewer parameters. The relationship of the order of fractional derivative and the viscoelastic physical properties have been well established in varied applications. Compared with the Kelvin-Voigt model, the fractional model has an accurate prediction of long-term settlements of soil foundation. Bagley and Torvik \(6\) conducted research on three-dimension constitutive relation of fractional derivative, limitation of thermodynamics on model parameters, and finite element method. Yin et al. \(10\) firstly considered fractional derivative element as soft-matter element while building the constitutive model of soil rheology phenomena, representing the material between an ideal fluid and solid.

By using the theory of viscoelasticity and fractional calculus, Liu et al. \(11\) proposed a fractional Kelvin-Voigt model to calculate the time-dependent behavior and lateral dynamic impedance of a single pile. Yin et al. \(12\) studied geomaterials in a condition of triaxial test by using Fractional order constitutive model. Based on using the fractional order to represent the soil mechanical properties, a fractional creep model for soil was proposed by introducing a variable-order fractional operator \(13\). Lai et al. \(14\) reviewed fractional calculus modeling for geotechnical engineering to quantify geotechnical rheology phenomenon. It indicated that the fractional derivative models have shown the merits of hereditary phenomena with long memory, and to be one of the most effective and accurate approaches to describe the rheology phenomenon.

Seasonally frozen has significant viscoelastic impact on the soil-pile system. Many studies on the topic of the viscoelastic effects of thawing frozen soil and its impacts on pile structures’ vibration. Lackner et al. \(15\) investigated the viscoelastic mechanical behavior of frozen granular soil using micromechanics-based material model. Vaziri and Han \(16\) studied the vibration of piles in partially frozen soils. The dynamic response of piles is influenced by the presence of a frozen soil layer at a modest thickness. Sritharan et al. \(17\) and Suleiman et al. \(18\) and studied a column extension system under different frozen depths and temperatures. The thermal effects on the effective stiffness has been qualified successfully. Yang et al. \(19,20\) investigated the changes of bridge fundamental frequency of a steel-column bridge. Xiong and Yang \(21\), Yang et al. \(22\) studied on the seasonally frozen soil impacts on the dynamics behavior of a soil–pile–bridge system. All these researches are based on the linear vibrational frequencies to study the seasonally effects on the frozen soil. There is limited study to identify the viscoelastic behavior and the seasonally frozen soils effects on the pile structures. In this case, the viscoelastic effects of thawed frozen soil on the dynamic behaviors of a pile structure were studied. It is important to characterize pile foundations by using the vibration signal. The impact response measuring has been used wide for modal parameter identification. The pile structure vibration signals are very complicated because of the geometrical and material properties of the soil-pile system. Because of the gapping and nonlinear behavior at the soil-pile interface, a soil-pile system exists hysteresis. The pile’s defects and soil weakening or hardening between the pile and soil next to the soil surface will influence the soil-pile system’s response \(23–27\). Many bridge designs adopt pile foundation. The thawing frozen soil could cause deterioration and degradation of load-carry capability performance of bridges’ pile foundations.
Most recently, [28–31] investigated fractional derivative models for various soils under different states. [32] employs a general and direct method to determine the time-dependent stress–strain behavior of soil using a Prony series method based on test of soil. The finite-element models are developed to simulate and validate the time-dependent constitutive relationship of the soil under both static and dynamic loading conditions. It is noted that the results of Prony series method could be same as that of the simplest fractional derivative representations [33]. [34] presents a hybrid numerical-analytical methodology for pile head impedance functions under inelastic soil-pile interaction. Results from three-dimensional nonlinear finite element analyses of single piles under dynamic pile head loading are used as input to the derivation algorithm. A new mathematical procedure for polynomial representation of the impedance functions is introduced. [35] theoretically investigated the vertical impedance of a tapered pile embedded in the saturated viscoelastic half-space. The constitutive behavior of the soil is described by a fractional viscoelastic model. Parametric studies are performed to investigate the influences of tapered angle, fluid permeability and fractional order of soil constitutive model. [36] presented a mathematical formulation to investigate the dynamic response of a single pile embedded in a semi-infinite partially saturated soil subjected to lateral dynamic excitations. An approximate method is implemented to decompose the pile-soil system into an extended water-air-filled poroelastic half-space and a fictitious pile. The extended porous medium is governed by the three-phase elastodynamic theory. By virtue of the Green’s function for a set of interior horizontal patch loads in unsaturated soil and the combination with the compatibility condition, the interaction problem is reduced to a Fredholm integral equation. The pile head impedance function in the frequency domain is derived and resolved numerically by a discretized method. Its accuracy was validated through a comparison with the existing solutions corresponding to single-phase and fully saturated soils.

In this study, a fractional derivative approach is used to model viscoelastic properties of a partially embedded pile, and the thawing frozen soil impacts on a pile structure vibration are investigated. Free-decay vibration signal is used to determine the dynamic behaviors of the pile partially embedded in thawing frozen soil. The conventional spectrum results are calculated to show the vibration properties in a linear system. The empirical modal decomposition (EMD) is used to decompose the vibration signal to identify the dynamic parameters [37–39]. EMD is adopted to extract the frequency dependence rocking mode which characterizes the viscoelastic properties of pile-soil interaction. The rocking mode is modeled by using fractional derivative method. The simulation of the fractional derivative model shows that the wide range rocking modes with varied frequency dependence can be attained by properly selecting the model fractional orders. This paper proposes a quick and reliable method to identify the viscoelastic property of pile-soil interactions.

2. Modeling Soil Viscoelastic Property using Fractional Derivative Method

Viscoelastic material, which behaves between elastic solid and the Newtonian fluid, generally exhibit hysteresis, creep or stress relaxation properties [7–9]. As a kind of viscoelastic materials, soils share the similar viscoelastic property. Conventionally, viscoelastic constitutive models with multiple coefficients such as Maxwell model, Kelvin-Voight model have been used to characterize the viscoelastic property. The recent development is to use advanced fractional model in which one fractional parameter \( \alpha \) is used with certain conventional parameters (elastic modulus and damping coefficient). In the framework of fractional calculus, one fractional parameter \( \alpha \) can efficiently and effectively play the role as multiple coefficients in traditional models. Generally, the fractional derivative is defined by,

\[
D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (n-1 < \alpha < n)
\]  

(1)

in which \( D^\alpha_t f(t) = d^\alpha f(t)/dt^\alpha \) is a generalized derivative with arbitrary order. For viscoelastic material, the stress-strain relationship is a convolution,

\[
\sigma(t) = \frac{d}{dt} \int_0^t E(\tau)\varepsilon(t-\tau) d\tau
\]

(2)

Out of many optional models, the fractional Kelvin-Voight model [40] is given as,

\[
\sigma(t) = E_0 \varepsilon(t) + E_1 D^\alpha_t \varepsilon(t)
\]

(3)
To describe the constitutive relationship of viscoelastic soil, where \( E_0 \) denotes Young’s modulus and \( E_1 \) denotes the fractional viscoelastic damping coefficient. This model is constructed by replacing the Newtonian viscous component of the Kelvin-Voigt model with the fractional time derivative component. And when the fractional derivative order \( \alpha = 1 \), this model is identical to the classical Kelvin-Voigt constitutive relationship. Applying the fractional Kelvin-Voigt model for a lumped-mass-spring-damper system with a viscoelastic damper, the vibration equation is given as,

\[
m \frac{d^2x}{dt^2} + c \frac{d^\alpha x}{dt^\alpha} + kx = f
\]  

(4)

In Eq. 4, \( m \) and \( k \) stand for the mass and stiffness coefficient of the oscillator. \( c \frac{d^\alpha x}{dt^\alpha} \) is the complicated damper force with \( c \) as coefficient and \( \alpha \) as fractional derivative order.

Based on the above equations, if \( \alpha \) is identified, the viscoelastic property of the soil can be determined together with conventional parameters.

### 3. Test-setup and Measurements

The test pile is a steel jacketed reinforced concrete pile and the pile’s diameter is 16-inch (406 mm) and the length is 25 feet (7620 mm). The reinforcement of the cross-section is used 8 reinforcing bars (Number 6 ASTM A706). The shear reinforcement is used Number 3 reinforcing bars, and the pipe wall thickness is 0.375-inch (9.525 mm). The soil-pile system has 20 feet (6,096 mm) in the soil and 5 feet (1,524 mm) exposed above ground. The above parameters are selected based on the relatively popular applications of pile structures in Alaska. The test system consists of a test pile and a reaction pile as shown in Figure 1. The pile tops were a free end. A lateral load at the 5,000lbs (22.24 kN) generated by hydraulic actuator was applied in between the test pile and reaction pile, and the load was quickly released with a quick disconnect device. An accelerometer was installed on the top of test pile to monitor free vibration signal in horizontal direction. The soil type at the test site is the Fairbanks silt and the mechanical properties documented in a published research report [42].

Figure 2 is the measured vibration signal and Figure 3 shows the corresponding FFT spectrum [18–21]. In Figure 3, their exist multiple peaks below 1000 Hz and many high frequencies non-salient peaks such as the one centered at frequency \( f_{\text{high}} = 970 \) Hz. There is a salient peak centered at \( f_{\text{low}} = 14 \) Hz. The multiple high frequency peaks can be pre-assumed as the natural bending modes of the beam, if the soil-pile system is idealized as a linear beam system supported by Winkler foundation. Since \( f_{\text{low}} = 14 \) Hz is much smaller than beam bending modes, it can be considered as a rocking mode for which pile works as a rigid body whereas soil performances like spring and damper.

In terms of the vibration theory of beam partially on Winkler foundation [43–45], if the length of the pile above ground is increased, the bending modes frequency will be reduced; and if the length of the pile under ground is increased, the bending modes frequency will be increased. The rocking mode of pile can be approximated by lumped mass/inertia-rotation spring model. Similarly, if the length of the pile above ground is increased, the

![Figure 1 Soil-pile system.](image-url)
rocking mode frequency of pile will be reduced; and if the length of the pile under ground is increased, the rocking mode frequency of pile will be increased.

The dominant components ($f_{\text{low}} = 14$ and $f_{\text{high}} = 970$ Hz) have been extract from the vibration signal for future extraction and analysis. Then, EMD method will be used to extract the specific signals and identify the system parameters. After obtained rocking mode from EMD, we will quantify its frequency dependence feature, then we evaluate and identify the fractional derivative model by using varied fractional orders so as to attain the specific frequency dependence feature extracted from experiment and EMD.

![Figure 2 Vibration signal.](image)

![Figure 3 Spectrum density.](image)
4. Empirical Mode Decomposition and Parameter Identification

EMD method is used to characterize the pile-soil interactions. This method can decompose a non-linear and non-stationary signal into a set of zero-mean amplitude-modulation frequency-modulation components, $x(t)$, can be represented as,

$$x(t) = \sum_{j=1}^{n} a_j(t)\cos[\phi_j(t)]$$  \hspace{1cm} (5)

In Eq. 5, $a_j(t)$ and $\phi_j(t)$ represent the instantaneous amplitude and the instantaneous phase of the $j$th component, and $n$ is the number of components. The level of the nonlinearity can be estimated by using this method, and the method can identify the modal frequency and damping. First, this study used the EMD method to decompose the vibration signal which has been characterized in the Figure 2 and Figure 3 by conducting a sifting process iteratively. The decomposed zero-mean components named Intrinsic Mode Functions (IMFs). Then, the vibration signal shows as $n$ intrinsic modes $c_i(t)$ and a residue $r_n(t)$.

$$x(t) = \sum_{i=1}^{n} c_i(t) + r_n(t)$$ \hspace{1cm} (6)

Then apply the Hilbert transform to all IMFs, to derive Hilbert spectrum to quantify instant frequency of respective components.

$$H[c_j(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c_j(t)}{t-\tau} \, d\tau$$ \hspace{1cm} (7)

Figure 4 shows the decomposed components of IMFs, $c_0-c_{12}$. Figure 5 shows the decomposed signal $c_1$ and $c_{10}$. Figure 6 is the decomposed signal $c_{10}$ (rocking mode) and Hilbert spectrum showing instantaneous frequency.

![Figure 4](image-url)  
*Figure 4* Decomposed components of IMFs, $c_0-c_{12}$ ($c_0-c_{12}$ in the Figure).
Figure 5 Decomposed specific components of IMFs, $c_1$ and $c_{10}$ ($c_1$ and $c_{10}$ in the Figure).

Figure 6 Decomposed $c_{10}$ component and instantaneous frequency.

Figure 7 Modelled response (normalized) of rock mode with different fractional derivative orders.
It can be seen that the specific frequency of the rocking mode drops by 20% during the decay process. By using the predictor-corrector method [46], Eq. (4) is numerically solved for various fractional parameters. Figure 7 shows the simulated response of fractional derivative model of rocking mode with different fractional parameters. From Figure 7, we can see that varied rocking modes can be quantified by selecting specific fractional derivative orders. Particularly, based on least square method, the feature of the frequency drop by 20% can be captured by using fractional derivative order $\alpha = 1.2$. It is noted that the extracted fractional derivative order is different from that in most of conventional engineering applications where the fractional parameter fall in the region of $0 < \alpha \leq 1$ [47]. The EMD extracted rocking mode fall in the range of the waveforms shown in Figure 7.

5. Conclusions

This paper studied the thawing frozen soil impacts on the dynamic behavior of a soil-pile system based on the free-decay vibration signal. In addition to the pile bending modes of lateral vibrations, a specific low frequency vibration mode is identified by using frequency spectrum analysis and EMD method. It is specified as pile rocking mode in which pile works as a rigid body. The rocking mode exhibit frequency dependence characteristics in the time-frequency domain and it is dominated by the complex soil effects. A fractional derivative model is used to quantify the viscoelastic property of the thawing frozen soil. Based on the extracted rocking mode the fractional derivative order for the constitutive relationship of the viscoelastic property of soil is identified. Compared with the existing research, this paper proposed the new methodology to identify the viscoelastic property of pile-soil interactions by jointly using testing, fractional derivative modeling, and EMD. It quantifies the dynamics of pile in thawing frozen soil with rheological behavior.

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