Finite Time Formation of Multiple Groups of Nonholonomic Robots Using Sliding Mode Control

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Abstract—Formation control of multiple groups of agents finds application in large area navigation by generating different geometric patterns and shapes, and also in carrying large objects. In this paper, Centroid Based Transformation (CBT) [36], has been applied to decompose the combined dynamics of Nonholonomic wheeled mobile robots (WMRs) into three subsystems: intra and inter group shape dynamics, and the dynamics of the centroid. Separate controllers have been designed for each subsystem. The gains of the controllers are such chosen that the overall system becomes singularly perturbed system. Then sliding mode controllers are designed on the singularly perturbed system to drive the subsystems on sliding surfaces in finite time. Negative gradient of a potential based function has been added to the sliding surface to ensure collision avoidance among the robots. Simulation results have been provided to demonstrate the effectiveness of the proposed controller.

I. INTRODUCTION

The study on the collective behaviour of birds, animals, fishes, etc. has not only drawn the attention of biologists, but also of computer scientists and roboticists. Thus several methods of cooperative control [13] of multi-agent system have evolved, where a single robot is not sufficient to accomplish the given task, like navigation and foraging of unknown territory. These methods can broadly be categorized as the behaviour based approach ([1]-[3]), leader follower based approach [4]-[5], virtual structure based approach [6]-[9], artificial potential based navigation [10]-[12], graph theoretic method [14]-[15], formation shape control [16]-[21]. Among other works carried out on single group of robots, cluster space control [32], distance based formation [33], formation control of nonholonomic robots [4], kinematic control [27], and mobile robots subject to wheel slip [34], segregation of heterogeneous robots [35], are to name a few.

The problem associated with the formation control of multi-agent system is that it becomes difficult to accurately position the robot within the group, as the number of robots increases. To address this issue shape control and region based shape control have been proposed, such that the robots form a desired shape during movement. The desired shape can be union or intersection of different geometric shapes. Region based shape control have been extended to multiple groups of robots [22]-[24]. However, accurate positioning of robots in a large group, has remained yet an issue over the years, and has prevented the research on multiple groups of robots with correct positioning.

In an attempt to solve the positioning accuracy, we propose a hierarchical topology, here in this paper, which is based on the centroid based transformations [16]-[19] for single group of robots. In this architecture, the large group of robots have been partitioned into relatively small basic units containing three or four robots. Then the centroid of each unit have been connected to form larger module containing more robots. Extending the process will give a hierarchical architecture which is a composition of relatively smaller modules. As the construction of this topology involves connecting the centroid, it has been named Centroid Based Topology (CBT). The CBTs basically capture the constraint relationship among the robots. Another advantage of CBT is that it separates shape variables from the centroid and thus separates the formation shape controller and tracking controller design. As the centroid moves, the entire structure moves maintaining the shape specified by the shape variables. In this paper, we have introduced suitable CBTs for multiple groups of robots to get a modular architecture distinguishable in the form of intra group shape variables, inter group shape variables along with centroid. Based on this modular structure, sliding mode based controllers have been designed for each module. The gains of the controllers have been chosen such that the subsystems reach the sliding surface at different time featuring the stretched time scale properties of singularly perturbed system. This technique allows us to give importance to the part of the formation dynamics (intra group, inter group, and tracking of centroid), which has to converge earlier than the others, based on priority. Furthermore, potential function based sliding surfaces have been selected to design controllers to avoid inter robot collision.

The main contributions of this paper are as follows

• Introduce CBTs for multiple groups of robots that separates intra group, inter group shape variables and centroid, and the general form of CBT for multiple group has been proposed.
• Apply the aforesaid transformations on the collective dynamics of nonholonomic WMRs to derive the modular architecture in the form of intra and inter group shape dynamics along with the dynamics of centroid.
• Using the modular architecture proposed, separate formation controllers are designed based on sliding mode control for intra group, inter group shape dynamics, and the tracking of centroid.
• Three time scale convergence analysis is introduced

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to show that different dynamics reach different sliding surfaces at different [finite] time.

• to design controller for the collision avoidance among the robots.

II. PROBLEM FORMULATION

Given a set of $N$ robots with nonholonomic constraint [10] - [11], given by

$$p_i = A_i(\theta_i, \dot{\theta}_i)p_i + B_i(\theta_i)u_i + C_i(\dot{\theta}_i)$$

(1)

where

$$A_i(\theta_i, \dot{\theta}_i) = \begin{bmatrix} -\sin \theta_i \cos \theta_i \dot{\theta}_i & -\sin^2 \theta_i \dot{\theta}_i \\ \cos^2 \theta_i \dot{\theta}_i & \cos \theta_i \dot{\theta}_i \end{bmatrix}$$

$$B_i(\theta_i) = \begin{bmatrix} \frac{\cos \theta_i}{m} - \frac{dR \sin \theta_i}{Jr} & \frac{\cos \theta_i}{m} + \frac{dR \sin \theta_i}{Jr} \\ \frac{dR \cos \theta_i}{m} & \frac{dR \cos \theta_i}{m} \end{bmatrix}$$

$$C_i(\dot{\theta}_i) = \begin{bmatrix} -d\dot{\theta}_i \cos \theta_i \\ -d\dot{\theta}_i \sin \theta_i \end{bmatrix}$$

$$u_i = [\tau_{ri}, \tau_{ti}]^T$$

And

$$J\ddot{\theta} = \frac{R}{r}(\tau_r - \tau_i)$$

where, $m$ is the mass of robot, $I = I_m d^2$, $I$ is moment of inertia, $R$ is the distance between left and right wheels, $r$ is the radius of each wheel, $d$ is the distance from wheel axis to the center of mass, and $\theta$ is the orientation. $\tau_r$ and $\tau_i$ are control input of individual mobile robot. The positions of the robots are described by $p_i = [x_i, y_i]^T$, $i = 1, 2, \ldots, N$ in the inertial coordinate frame, and $u_i$ is the control input. Then, for a single group of robots, a linear transformation $\Phi \in \mathbb{R}^{2N \times 2N}$, can be defined, that produces the following matrix equation

$$[z_1^T, z_2^T, \ldots, z_{N-1}^T, z_N^T]^T = \Phi[p_1^T, p_2^T, \ldots, p_N^T]^T$$

(2)

where $z_i = [x_i, y_i]^T \in \mathbb{R}^{2 \times 1}$, $i = 1, 2, \ldots, (N - 1)$ are the shape defining vectors or shape variables in transformed coordinate, and these vectors define the geometric shape of formation of swarms. Clearly, the transformation $\Phi$ generates shape variables along with the centroid for a single group of robots.

For multiple groups of robots, these shape variables can be categorized into two parts. The shape variables which represent the shape of each subgroup, are **intra group shape variables**. However, the variables which describe the interconnection among the groups, each group being considered as a single agent, centered on the centroid of that group, are **inter group shape variables**. Suppose there are $m$ subgroups and each subgroup contains $\rho_i$ number of robots, where $i = 1, 2, \ldots, m$ ($\sum_{i=1}^m \rho_i = N$, $N$ being the total number of robots). Then the total number of intra group shape variables is $\rho = \sum_{i=1}^m (\rho_i - 1)$, and total number of inter group shape variables is $(m - 1)$. The intra group shape variables for each subgroup is defined as

$$Z_j = [z_{j1}^T, z_{j2}^T, \ldots, z_{j(\rho_i-1)}^T]^T$$

where, $Z_j \in \mathbb{R}^{1 \times \rho_i - 1}$, $i, j = 1, 2, \ldots, m$ and $z_{jk} \in \mathbb{R}^{2 \times 1}$, $k = 1, 2, \ldots, \rho_i - 1$. Therefore, the intra groups shape vectors are defined in a compact form as

$$Z_s = [Z_1^T, Z_2^T, \ldots, Z_m^T]^T$$

. The inter group shape variables considering the centroid of each group as an agent, is defined as

$$Z_e = [z_{r1}^T, z_{r2}^T, \ldots, z_{r(m-1)}^T]^T$$

where, $Z_e \in \mathbb{R}^{1 \times (m-1)}$ and $z_{ri} \in \mathbb{R}^{2 \times 1}$, $i = 1, 2, \ldots, (m - 1)$. The geometric center of mass, $z_c$ is defined by

$$z_c = \frac{1}{N} \sum_{i=1}^N p_i$$

Using the above definitions for multiple groups of robots, the intra group, inter group shape variable, and centroid can be written in compact form using a CBT $\Phi_M$ as

$$[Z_s^T, Z_e^T, z_c^T]^T = \Phi_M[p_1^T, p_2^T, \ldots, p_N^T]^T$$

The detailed description of the matrix $\Phi$ and $\Phi_M$ is given in Section III and IV.

Define, the desired intra group shape variables $Z_{sd}$, the inter group shape variables $Z_{rd}$, and the desired trajectory of the centroid $z_{cd}$.

Let $Z = [Z_s^T, Z_e^T, z_c^T]^T$ and $X = [p_1^T, p_2^T, \ldots, p_N^T]^T$ and $Z_d = [Z_{sd}^T, Z_{rd}^T, z_{cd}^T]^T$ and $X_d = [p_{1d}^T, p_{2d}^T, \ldots, p_{Nd}^T]^T$. The following equation gives the transformation from Cartesian to the transformed coordinate.

$$Z = \Phi_M X; \ Z_d = \Phi_M X_d$$

Therefore, the convergence of $Z \rightarrow Z_d$ as $t \rightarrow \infty$ leads to the convergence of $X \rightarrow X_d$ as $t \rightarrow \infty$. However, our objective is that the intra group shape variables converge faster than the inter group shape variables, and the convergence of inter group shape variables will be faster than trajectory tracking of centroid. Using CBT, the resultant modular architecture of formation and tracking control law, is shown in Fig. 2. Based on this, the formation control problem has been divided into the following sub-problems:

**Intra Group Formation Control:** Given a reference constant $Z_{sd}$, determine a control law such that intra group shape variables $Z_s(t)$ converges to the desired value as

$$\lim_{t \rightarrow \tau_1} Z_s(t) \rightarrow Z_{sd}$$

**Inter Group Formation Control:** Given a reference constant $Z_{rd}$ determine a control law such that inter group shape variable $Z_r(t)$ converges to the desired value as

$$\lim_{t \rightarrow \tau_2} Z_r(t) \rightarrow Z_{rd}$$

**Trajectory Tracking:** Given a reference time varying trajectory $z_{cd}(t)$ determine a control law such that the centroid $z_c(t)$ converges to the desired trajectory as

$$\lim_{t \rightarrow \infty} z_c(t) \rightarrow z_{cd}(t)$$

where, $0 < \tau_1 < \tau_2 < \infty$. 
III. GENERAL FORM OF CENTROID BASED TRANSFORMATION

In centroid based representation [16]-[19] of formation of a single group of robots, the centroid is being retained, as it contains all the positional information of the group of robots. All other vectors (shape variables) describe the connectivity relationship among the robots in the group. However, the general transformation matrix for \( N \) robots can be given as

\[
\Phi = \left[ \frac{\Phi_r}{\Phi_c} \right] \otimes I_2; \quad \Phi = \left[ \frac{\Phi_r}{\Phi_c} \right] \otimes I_2
\]

where, ‘\( \otimes \)’ denotes the Kronekar product of matrices. \( I_2 \) is the identity matrix of dimension 2. The dimension of the matrix \( \Phi_r \) is \((N - 1 \times N)\) The matrix \( \Phi_c \) is \((1 \times N)\) and it captures the information of the coefficient to generate the centroid vector. As the centroid is to be retained, the last row of the block matrix \( \Phi \) is given by,

\[
\Phi_c = \left[ \frac{1}{N} \ldots \frac{1}{N} \right] \in R^{1 \times N}
\]

IV. TRANSFORMATION MATRIX FOR MULTIPLE GROUPS OF ROBOTS

This section mainly describes how to generate centroid based transformation matrices for multiple groups of robots. Fig. 4 shows three different groups of robots in triangular formation and the centroid of each group when connected together, forms another bigger triangle. The goal is to generate intra group and inter group shape variables.

The importance of this modularity is that we don’t need know what is the Jacobi or any other CBT for the entire group of robots (say, \( N = 9 \)). Instead, we choose relatively small subgroups of robots (say, \( N = 3 \) or \( N = 4 \)) and apply transformation. Then considering the centroid of each subgroup as a single agent, inter group shape variables are generated. Therefore, from the shape variables (intra and inter group) and centroid the transformation matrix is generated which captures the constraint relationship among robots.

Let’s illustrate with an example for the formation of robots described by Fig. 5. We’ve \( N_g = 3 \) groups of robots with \( \mu_i = 3 \) robots in each group. Therefore, we’ll have \( 8 (= 2 \times 3 + 2) \) shape variables along with the centroid of the groups. We choose to construct the transformation matrix for multiple groups of robots with the help of Jacob transformation here in this example, although other transformations can be similarly used. The derivation of this example of transformation gives a heuristic understanding and some intuitive feeling for the solution of the stated problem. However, we first write the shape variable and then give the transformation matrix for multiple groups.

where,

\[
\begin{align*}
Z_1 & \Rightarrow \{ z_{11} = \frac{1}{\sqrt{3}}(p_2 - p_1) \\
Z_2 & \Rightarrow \{ z_{12} = p_3 - \frac{1}{2}(p_1 + p_2) \\
Z_3 & \Rightarrow \{ z_{13} = \frac{1}{\sqrt{3}}(p_7 - p_8) \\
Z_r & \Rightarrow \{ z_{r1} = \frac{1}{\sqrt{2}}(\mu_1 - \mu_2) \\
Z_{r2} & \Rightarrow \{ z_{r2} = \mu_3 - \frac{1}{2}(\mu_1 + \mu_2)
\end{align*}
\]

The transformation matrix, \( \Phi_{M_1} \), is given below.

\[
\begin{pmatrix}
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

We can also choose two different transformation for subgroup and group representation. In the following example transformation \( \Phi_3 \) is applied to get the shape variables for the subgroups, and transformation \( \Phi_2 \) is applied to represent the combination of the subgroups. The overall transformation matrix \( \Phi_{M_2} \) is as follows.

\[
\begin{pmatrix}
\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\
\end{pmatrix}
\]
and the associated shape variables are given below.

\[ \Phi_{M2} \Rightarrow \begin{cases} 
Z_1 = \begin{bmatrix} z_{11} = x_1 - \frac{1}{2} (p_1 + p_2 + p_3) \\
z_{12} = x_2 - \frac{1}{2} (p_1 + p_2 + p_3) 
\end{bmatrix} \\
Z_2 = \begin{bmatrix} z_{21} = x_3 - \frac{1}{2} (p_4 + p_5 + p_6) \\
z_{22} = x_4 - \frac{1}{2} (p_4 + p_5 + p_6) 
\end{bmatrix} \\
Z_3 = \begin{bmatrix} z_{31} = x_5 - \frac{1}{2} (p_7 + p_8 + p_9) \\
z_{32} = x_6 - \frac{1}{2} (p_7 + p_8 + p_9) 
\end{bmatrix} \\
Z_r = \begin{bmatrix} z_{r1} = \frac{1}{\sqrt{2}} (\mu_1 - \mu_2) \\
z_{r2} = \frac{1}{\sqrt{2}} (\mu_1 + \mu_2) 
\end{bmatrix} 
\]

where,

\[ \begin{bmatrix} \mu_1 = \frac{1}{3} (p_1 + p_2 + p_3) \\
\mu_2 = \frac{1}{3} (p_4 + p_5 + p_6) \\
\mu_3 = \frac{1}{3} (p_7 + p_8 + p_9) \end{bmatrix} \]

A. General Form of The Transformation Matrix for Multiple Groups

Therefore, the general form of the transformation matrix for multiple groups of robots can be written as follows

\[ \Phi_M = \left[ \begin{array}{c|c} \Phi_1 & \Phi_2 \\
\Phi_m & \Phi_r \\
\Phi_c & \Phi_e \end{array} \right] \otimes I_2; \quad \Phi_M = \left[ \begin{array}{c|c|c|c|c} \Phi_1 & \Phi_2 & \cdots & \Phi_m & \Phi_r \\
\Phi_m & \Phi_r & \cdots & \Phi_c & \Phi_e \end{array} \right] \]

Where, \( \Phi_1, \Phi_2, \ldots, \Phi_m \) are \((p_1 - 1 \times N), (p_2 - 1 \times N), \ldots, (p_m - 1 \times N)\) dimensional matrices and \( m \) is the total number of subgroups. The scalar \((p_i - 1), \ i = 1, 2, \ldots, m\) are the number of shape variables required to represent \( i^{th} \) subgroup. \( \Phi_r \) is \((m - 1 \times N)\) and \( \Phi_c \) is \((1 \times N)\). We write the transformation matrix in a more compact form given below.

\[ \Phi_M = \left[ \begin{array}{c|c} \Phi_m & \Phi_r \\
\Phi_r & \Phi_e \end{array} \right] \otimes I_2; \quad \Phi_M = \left[ \begin{array}{c|c} \Phi_m & \Phi_r \\
\Phi_r & \Phi_e \end{array} \right] \]

Suppose there are \( m \) groups of robots and in the \( i^{th} \) group, there are \( n_i \) number of robots. Then the dimension of the matrix \( \Phi_m \) is \(((p_1 + p_2 + \cdots + p_m) - m) \times N\). Again \( \Phi_m \) can be written in block diagonal form as \( \Phi_m = \text{diag}\{\Phi_i\} \), where, \( \Phi_i \) denotes the transformation matrix for the \( i^{th} \) group of robots containing \( p_i \) number of robots. The dimension of \( \Phi_i \) is \((p_i - 1) \times p_i\).

Suppose, we’ve 3 groups of robots with 3, 4 and 5 robots in each group (12 being the total number of robots). Then there will be 2, 3 and 4 intra group shape vectors, and the dimension of the matrices \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) will be \( 2 \times 12, \ 3 \times 12, \) and \( 4 \times 12 \) respectively. The total number of intra group shape variables would be 2 and the dimension of the matrix \( \Phi_r \) is \( 2 \times 12 \).

B. Shape Variable Generation Algorithm for Multiple Groups of Robots

The following steps describe the generation algorithm for the shape variables of multiple groups of robots.

**step 1**: calculate intra group shape variables for each subgroup, i.e., \( Z_1, \ldots, Z_m \).

**step 2**: calculate the centroid of each subgroup, i.e., \( \mu_1, \ldots, \mu_m \)

**step 3**: calculate inter group shape variables for the overall group assuming each group as an agent.

**step 4**: calculate the overall centroid.

**step 5**: combine all the vectors to get the transformation matrix.

V. FORMATION CONTROLLER DESIGN AND STABILITY ANALYSIS

A. Formation Dynamics

The entire formation of \( N \) WMRs can be viewed as a deformable body whose shape and movement can be described by vectors in transformed coordinate. Define the notation \( A_i = A_i(\theta_i, \dot{\theta}_i) \), \( B_i = B_i(\theta_i) \) and \( C_i = C_i(\ddot{\theta}_i) \) for \( i = 1, 2, \ldots, N \), where \( \theta_i \) and \( \dot{\theta}_i \) are the orientation and angular speed of the \( i^{th} \) WMR.

Let \( Z = [z_1, z_2, \ldots, z_{N-1}, z_e]^T \) and \( X = [p_1, p_2, \ldots, p_N]^T \). The following equation gives the transformation from Cartesian to generalized \( z \) coordinate.

\[ Z = \Phi_M X \]  

Using (2) on (1), the overall translational dynamics of \( i = 1, 2, \ldots, N \) robots in a compact form is given by

\[ \begin{bmatrix} \dot{\hat{p}}_1 \\
\vdots \\
\dot{\hat{p}}_N 
\end{bmatrix} = \begin{bmatrix} A_1 \hat{p}_1 + B_1 u_1 + C_1 \\
\vdots \\
A_N \hat{p}_N + B_N u_N + C_N \end{bmatrix} \]

\[ \begin{bmatrix} \hat{\ddot{p}}_1 \\
\vdots \\
\hat{\ddot{p}}_N 
\end{bmatrix} + \begin{bmatrix} B_1 \\
\vdots \\
B_N \end{bmatrix} u_N \]

\[ \begin{bmatrix} C_1 \\
\vdots \\
C_N \end{bmatrix} \]

Let

\[ A = \text{diag}\{A_1, A_2, \ldots, A_N\} \]  

\[ B = \text{diag}\{B_1, B_2, \ldots, B_N\} \]  

\[ C = \text{diag}\{C_1, C_2, \ldots, C_N\} \]

and

\[ U = [u_1, u_2, \ldots, u_N]^T \]

Then the dynamic equation of \( N \) WMRs can be written by replacing (5), (6), (7), and (8) in equation (4) as

\[ \ddot{X} = AX + BU + C \]
Using the transformation given in \((3), (9)\) can be written as
\[
\dot{Z} = P\dot{Z} + QU + R \tag{10}
\]
where \(P = \Phi_M A \Phi_M^{-1} \), \(Q = \Phi_M B U \); \(R = \Phi_M C \). With equation \((10)\), we opt for designing a controller, by time scale division, such that, the convergence of the subgroup formation will be conducted first, then inter group formation, and finally trajectory tracking. For that we are required to use an important results of singular perturbation theory, given in the next subsection.

**B. Three time scale behaviour of multiple groups of robots**

As given in our previous work [36], the matrix \(P\) and \(R\) of equation \((10)\) in subsection A, can be written in the following form
\[
P = \begin{bmatrix}
P_s \\
P_r \\
P_c
\end{bmatrix}; \quad R = \begin{bmatrix}
R_s \\
R_r \\
R_c
\end{bmatrix}
\]
Therefore, the collective dynamics of \((9)\) can be separately written in the form of intra group shape dynamics \((Z_s)\), as follows
\[
\dot{Z}_s = P_s Z + F_s + R_s \tag{11}
\]
where, \(Z_s = \Phi_m X; \quad F_s = \Phi_m B U\). The inter group shape dynamics \((Z_r)\) is written as,
\[
\dot{Z}_r = P_r \dot{Z} + F_r + R_r \tag{12}
\]
where, \(Z_r = \Phi_r X; \quad F_r = \Phi_r B U\). The dynamics of the centroid \((z_c)\) is expressed as,
\[
\dot{z}_c = P_c \dot{Z} + f_c + R_c \tag{13}
\]
where, \(z_c = \Phi_c X; \quad f_c = \Phi_c B U\).

We define the intra group shape error vector as \(Z_{sce} = Z_s - Z_{srd}\), the inter group shape error vector \(Z_{cre} = Z_r - Z_{rd}\), and the tracking error of centroid \(z_{cce} = z_c - z_{cd}\), where \(Z_{srd}, Z_{rd}, \) and \(z_{cd}\) are the desired intra group, desired inter group shape variables, and desired trajectory of the centroid respectively.

Define a set of three time instants \(t_s, t_r, \) and \(t_c\), such that \(t_s = \frac{1}{\epsilon_1 \epsilon_2}; \quad t_r = \frac{1}{\epsilon_1}; \quad t_s \leq t_r \leq t_c \leq t\) as \(t \to \infty\). \(t\) is total time of operation, \(t_s\) and \(t_r\) are stretched time scale (within which the subsystems must converge), and \(\epsilon_1, \epsilon_2\) are selected as controller gain parameters as shown below. The controller is to be designed such that \(Z_{sce} \to 0\), during the interval \([t_0, t_s]\), \(Z_{cre} \to 0\), during the interval \([t_0, t_r]\), \(z_{cce} \to 0\), during the interval \([t_0, t_c]\). Here, \(Z_{sce}\) is ultra fast variable, \(Z_{cre}\) is fast variable, and \(z_{cce}\) is slow variable based on the choice of controller gain parameters \(\epsilon_1, \epsilon_2\) \((\epsilon_1 \gg \epsilon_2\) to conform the notion of ultra fast and fast variables). To achieve the desired formation and tracking, the following controllers is proposed for \((11) - (13)\).

1) **Control law for centroid**: The controller that manages the centroid to track the given trajectory, is designed to be the last to converge to the desired value. The switching surface for the centroid dynamics is defined by
\[
s_c(t) = c z_{cce} + \dot{z}_{cce} \tag{14}
\]
The equivalent control law (setting \(\dot{s}_c = 0\) is given by
\[
u_{cce} = -c \dot{z}_{cce} - P_c \dot{Z} - R_c + \ddot{z}_{cd} \tag{15}
\]
The control law \((15)\) only takes the system trajectory towards the origin along the sliding surface \((14)\). But trajectories which don’t initiate on the sliding surface is required to reach the surface so that they can sliding along the surface towards the origin. The following control law satisfies the reachability condition.
\[
\Delta u_c = -\delta_s sgn(s_c) \tag{16}
\]
where, \(\delta_s\) is a positive scalar.
\[
sgn(s) = \begin{cases}
1 & \text{for } s > 0 \\
-1 & \text{for } s < 0
\end{cases}
\]

Then the sliding mode control law for centroid dynamics can be written as
\[
u_c = u_{cce} + \Delta u_c \tag{17}
\]

**Theorem 1** The control law \((17)\) will asymptotically stabilize the subsystem \((13)\) in finite time.

**Proof**: Define a Lyapunov function for the subsystem \((13)\) as
\[
V(s_c) = \frac{1}{2} s_c^T s_c
\]
It’s time derivative gives
\[
\dot{V}(s_c) = -\delta_s s_c sgn(s_c) \leq -\delta_s \| s_c \|
\]
As \(\dot{V}(s_c) + \delta_s V(s_c)^{1/2} \leq 0\) it follows from Theorem 1 that the sliding surface \(s_c\) is finite time stable and there exist a finite time \(t_c \leq \frac{2}{\delta_s} V(s_c(0))^{1/2}\) such that \(z_{cce} \to 0\) for all \(t \geq t_c\).  

2) **Control law for inter group shape dynamics**: To serve the purpose of different time scale convergence, the control law of \((12)\) is chosen to be \(F_r = \frac{P_r}{\epsilon_1}\). Thereby, the system \((12)\) becomes singularly perturbed system as
\[
\epsilon_1 \dot{Z}_r = \epsilon_1 P_r \dot{Z} + F_r + \epsilon_1 R_r \tag{19}
\]
The sliding surface for the dynamics of \((19)\) is chosen to be
\[
s_r(t) = r Z_{cre} + \dot{Z}_{cre} \tag{20}
\]
The equivalent control law (setting \(\dot{s}_r = 0\) is given by
\[
u_{req} = -r \dot{Z}_{cre} - P_r \dot{Z} - R_r + \ddot{Z}_{rd} \tag{21}
\]
The reachability control for the system \((19)\) is
\[
\Delta u_r = -\delta_r sgn(s_r) \tag{22}
\]
where, \(\delta_r\) is a positive scalar. Then the sliding mode control law for inter group shape dynamics is written as
\[
u_r = u_{req} + \Delta u_r \tag{23}
\]

**Theorem 2** The control law \((23)\) will asymptotically stabilize the subsystem \((12)\) in finite time.
Proof: Define a Lyapunov function for the subsystem (12) as

\[ V(s_r) = \frac{1}{2} s_r^T s_r \]

It's time derivative gives

\[ \dot{V}(s_r) = -\frac{\delta_r}{\epsilon_1} s_r sgn(s_c) \leq \frac{-\delta_r}{\epsilon_1} | s_r | \]

As \( \dot{V}(s_r) + \frac{\delta_r}{\epsilon_1} V(s_r) \leq 0 \) it follows from Theorem 2 that the sliding surface \( s_r \) is finite time stable and there exist a finite time \( t_s \leq 2\frac{\epsilon_1}{\delta_r} V(s_r(0)) \) such that \( z_{rc} \to 0 \) for all \( t \geq t_s \).

3) Control law for intra group shape dynamics: To serve the purpose of different time scale convergence, the control law of (11) is chosen to be \( F_x = \frac{F_x}{\epsilon_1 \epsilon_2} \). Thereby, the system (11) becomes singularly perturbed system as

\[ \epsilon_1 \epsilon_2 \dot{Z}_r = \epsilon_1 \epsilon_2 P_r \dot{Z} + F_{r1} + \epsilon_1 \epsilon_2 R_r \]

The equivalent control law setting \( \dot{s}_a = 0 \) is given by

\[ u_{s_{eq}} = -s_Z s_a - P_s \dot{Z} - R_s + \dot{Z}_{ad} \]

The reachability control for the system (25) is

\[ \triangle u_s = -\delta_s \text{sgn}(s_a) \]

where, \( \delta_s \) is a positive scalar. Then the sliding mode control law for inter group shape dynamics is written as

\[ u_s = u_{s_{eq}} + \triangle u_s \]

Theorem 3 The control law (29) will asymptotically stabilize the subsystem (11) in finite time.

Proof: Define a Lyapunov function for the subsystem (11) as

\[ V(s_a) = \frac{1}{2} s_a^T s_a \]

It's time derivative gives

\[ \dot{V}(s_a) = -\frac{\delta_a}{\epsilon_1 \epsilon_2} s_a sgn(s_a) \leq -\frac{\delta_a}{\epsilon_1 \epsilon_2} | s_a | \]

As \( \dot{V}(s_a) + \frac{\delta_a}{\epsilon_1 \epsilon_2} V(s_a) \leq 0 \) it follows from Theorem 3 that the sliding surface \( s_a \) is finite time stable and there exist a finite time \( t_s \leq 2\frac{\epsilon_1 \epsilon_2}{\delta_a} V(s_a(0)) \) such that \( z_{ac} \to 0 \) for all \( t \geq t_s \).

Remark: As the settling times \( t_c, t_r, \) and \( t_s \) depend on the initial values \( V(s_r(0)), V(s_a(0)), \) and \( V(s_a(0)) \) respectively and on the parameters \( \delta_c, \delta_r, \) and \( \delta_a \) respectively, they can be selected such that \( t_a > t_r > t_c \).

VI. Collision Avoidance

The controllers of (17), (23), and (29) don’t guarantee collision avoidance among the robots. Therefore, the barrier-like function of (19) is chosen as a potential function for collision avoidance. The modified form of the function for the robots \( i, j \in \mathbb{N}, i, j = 1, 2, ..., N \) is given by

\[ V_{ij}(p_i, p_j) = \left( \min \left\{ 0, \| q_i - q_j \|^2 - R^2 \right\} \right)^2 \]

where \( R \) is the radius of sensing and \( q_i, q_j \) represents the position of \( i \)-th and \( j \)-th robot respectively. \( r \) denotes the permissible distance from the robot \( i \) to avoid collision. Then the control input for the collision avoidance of \( i \)-th robot is the summation of all potential defined by (31) of the robots \( j \) inside the permissible distance \( r \):

\[ \nabla f_i = -\sum_{j=1,j\neq i}^n \frac{\partial V_{ij}(p_i, p_j)^T}{\partial p_i} \]

To comply with the solutions of \( X = -\nabla F \) under the transformation \( Z = \Phi M X \), define a vector of control input in the transformed domain as

\[ F_{pot} = \Phi M \nabla F \]

The vector \( F_{pot} \) of (34) is partitioned as \( F_{pot} = [F_{pot1}, F_{pot2}, F_{pot3}]^T \), where, \( F_{pot1} \in \mathbb{R}^{2p \times 1} \), \( F_{pot2} \in \mathbb{R}^{2(m-1) \times 1} \), and \( F_{pot3} \in \mathbb{R}^{2 \times 1} \). Then the general sliding surface (10) for the intra and inter group and centroid dynamics given by

\[ s_i(t) = i Z_{ie} + \dot{Z}_{ie} + F_{pot} \]

where, \( i = c, r, s \). The equivalent control is then

\[ u_{s_{eq}} = -i \dot{Z}_{ie} - P_i \dot{Z} - R_s + \dot{Z}_{id} + \frac{\partial F_{pot}}{\partial t} \]

where, \( i = c, r, s \). If the potential term of (36) is bounded, i.e., \( \| \frac{\partial F_{pot}}{\partial t} \| \leq F_{pot} \) for some known \( F_{pot} \), \( i = c, r, s \), then \( \delta_i > F_{pot} \) for the equivalent control

\[ \triangle u_i = -\delta_i \text{sgn}(s_i) \]

where, \( i = c, r, s \).
VII. SIMULATION RESULTS

Fig. 2. Formation control using transformation $\Phi_M$

Fig. 3. Potential force based Formation control using transformation $\Phi_M$

The controllers developed in section V-VI has been simulated on three groups of robots with three robots in each group. The controller gain parameters are chosen as $s = r = c = 1$, $\delta_s = \delta_r = \delta_c = 1$ and $\epsilon_1 = 0.1$, $\epsilon_2 = 0.1$. All the figures in this section shows the trajectories of the robots moving in formation. The positions of the robots are marked by '⊿'. Potential force parameters are taken from [19]. The desired trajectory of the centroid of the formation is kept as $z_c = [t; 30 \sin(0.1t)]$. The rest of the desired vectors are framed as illustrated by Fig. 1. For the sake of simplicity, as shown in Fig. 1, we’ve kept the geometric shape of formation as simple as possible. The desired shape of individual group is an equilateral triangle and also desired shape of the bigger triangle that tangles all the groups, is equilateral triangle. From the Fig. 1, we can see that each side of the small triangle is $b = 7m$ and the sides of the big triangle is $a = 20m$. We’ve used the transformation $\Phi_M$, given in Section IV, here, in this simulation. The shape variables in the transformed domain, are given as $Z_{sd} = [(-4.9497, 0), (0, 6.0622), (-4.9497, 0), (0, 6.0622), (-4.9497, 0), (0, 6.0622)]^T$, $Z_{rd} = [(-14.1421, 0), (0, 17.3205)]^T$. The initial values of position of 9 robots is respectively as follows $(x_1, y_1) = (-4, 7)$, $(x_2, y_2) = (-3, 8)$, $(x_3, y_3) = (-6, 12)$, $(x_4, y_4) = (-1, -5)$, $(x_5, y_5) = (0, 6)$, $(x_6, y_6) = (1, -8)$, $(x_7, y_7) = (3, -12)$, $(x_8, y_8) = (-7, -16)$, $(x_9, y_9) = (4, 16)$. In Fig. 2, it is shown that the robots converge to the desired formation as can be seen in Fig. 1, from the initial conditions given above. The convergence of robots to the desired formation with collision avoidance, has been depicted in Fig. 3.

The Fig. 4 and 5, show the convergence time of the states in the transformed domain separately (without applying potential force). All the intra group shape variables $Z_1 \ldots Z_6$ converge faster than inter group shape variables $Z_7$ and $Z_8$. It can also be seen from the figures, that the convergence of the centroid is the slowest of all. It’s evident from Fig. 4 that the intra group shape variables converge to desired value at $t = 0.1sec$. The inter group shape variables converge at time $t = 1sec$ and the trajectory of centroid converges to the desired value at $t = 10sec$. Thus convergence of intra group shape variables are 10 times faster than the convergence of inter group shape variables. Again, convergence of the trajectory of centroid is 10 times faster than the convergence of inter group shape variables.

VIII. CONCLUSION

In the paper we propose, an intuitive and simple way of solving a complicated formation control problem. For that a centroid based transformation is given for multiple groups of robots such that a modular architecture results,
in the form of intra group, inter group shape variables, and centroid. Thus separate controller has been designed for each module of formation. The gains of the feedback controller are so selected that the error dynamics becomes singularly perturbed system and three time scale behaviour of the overall system is achieved. Thus the control laws ensure that the convergence of intra group shape variables is faster than the inter group shape variables, and the convergence of inter group shape variables is faster than the tracking of the centroid. For collision avoidance, negative gradient of potential function has also been appended the proposed feedback controller. Numerical simulation results including time varying formations has also been given to show the performance of the proposed formation controllers.

REFERENCES

[1] C. Reynolds Flocks, herds, and schools: A distributed behavioural model. Computer Graphics, vol. 21, no. 4, pp. 25-34, July 1987.
[2] T. Balch and R. C. Arkin, Behaviour-based formation control for multi robot systems. IEEE Transactions on Robotics and Automation, vol. 14, no. 6, pp. 926-939, December 1998.
[3] J. H. Reif and H. Wang, Social potential fields: A distributed behavioral control for autonomous robots. Robotics and Autonomous Systems, vol. 27, no. 3, pp. 171-194, May 1999.
[4] J. P. Desai, V. Kumar, and J. P. Ostrowski, Modeling and control of formations of nonholonomic mobile robots, IEEE Transactions on Robotics and Automation, 17(6), pp. 905 - 908, 2001.
[5] H. Tanner, G. Pappas, and V. Kumar, Leader-to-formation stability. IEEE Transactions on Robotics and Automation, 20, 443-455, 2004.
[6] N. E. Leonard and E. Fiorelli, Virtual leaders, artificial potentials and coordinated control of groups. Conference on Decision and Control, Florida, OR, USA, December 2001, pp. 2968-2973.
[7] M. A. Lewis and K. H. Tan, High Precision Formation Control of Mobile Robots using Virtual Structures, Autonomous Robots, 1997, Vol. 4, pp. 387-403
[8] M. Egerstedt and X. Hu, Formation constrained multi-agent control, IEEE Transactions on Robotics and Automation, 2001, Vol. 17, No. 6, pp. 947-951.
[9] W. Ren and R. W. Beard, Formation feedback control for multiple spacecraft via virtual structures, 2004, IEEE Proceedings - Control Theory and Applications, Vol. 151, No. 3, pp. 357-368.
[10] V. Gazi, Swarms aggregation using artificial potentials and sliding mode control. IEEE Transactions on Robotics, 2005, Vol. 21, No. 4, pp. 1208-1214.
[11] A. R. Pereira and L. Hsu, Adaptive formation control using artificial potentials for Euler-Lagrange agents, In Proc. of the 17th IFAC world congress, 2008, pp. 10788-10793.
[12] M.M. Zavlanos and G.J. Pappas, Potential fields for maintaining connectivity of Mobile networks, IEEE Transactions on Robotics, 2007, Vol. 23, No. 4, pp. 812-816.
[13] R. Olfati-Saber, J. A. Fax, and R. M. Murray, Consensus and cooperation in networked multi-agent systems. Proceedings of the IEEE, vol. 95, no. 1, pp. 215-233, January 2007.
[14] R. Olfati-Saber, and R. M. Murray, Graph Rigidity and Distributed Formation Stabilization of Multi-Vehicle Systems, 2002, Las Vegas, Nevada, USA.
[15] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, Flocking in fixed and switching networks. IEEE Transactions on Automatic Control, vol. 52, pp. 863-868, May 2007.
[16] V. Aquilanti and S. Cavalli, Coordinates for molecular dynamics: Orthogonal local systems, Journal of Chemical Physics, 1986, Vol. 85, pp. 1355-1361.
[17] F. Zhang, Geometric Cooperative Control of Particle Formations. IEEE Transactions of Automatic Control, vol. 55, no. 3, pp. 800-804, March 2010.
[18] H. Yang and F. Zhang, Robust Control of Horizontal Formation Dynamics for Autonomous Underwater Vehicles. International Conference on Robotics and Automation, Shanghai, China, May 2011, pp. 3364-3369.
[19] S. Mastellone, J. S. Mejia, D. M. Stipanovic, and M. W. Spong, Formation control and coordinated tracking via asymptotic decoupling for lagrangian multi-agent systems. Automatica, vol. 47, no. 11, pp. 2355-2363, November 2011.
[20] C. Belta and V. Kumar, Abstraction and control for groups of robots. IEEE Transactions on Robotics, vol. 20, no. 5, pp. 865-875, October 2004.
[21] C. C. Cheah, S. P. Hou, and J. J. E. Slotine, Region based shape control for a swarm of robots. Automatica, vol. 45, no. 10, pp. 2406-2411, October 2009.
[22] S.P. Hou, and C.C. Cheah, Dynamic compound shape control of robot swarm, IET Control Theory and Applications, 2012, Vol. 6, Issue 3, pp. 454-460.
[23] R. Haghhighi, C.C. Cheah, Multi-group coordination control for robot swarms, Automatica, 2012, Vol. 48, pp. 25262534.
[24] X. Yan, J. Chen, D. Sun, Multilevel-based topology design and shape control of robot swarms, Automatica, 2012, Vol. 48, issue 12, pp. 31223127.
[25] R. Fierro and F. L. Lewis, Control of a nonholonomic mobile robot: Backstepping kinematics into dynamics. Journal of Robotic Systems, vol. 14, no. 3, pp. 149-163, September 1997.
[26] Y. Yamamoto and X. Yun, Coordinating locomotion and manipulation of a mobile manipulator. in Recent Trends in Mobile Robots, Y. F. Zheng, Ed., World Scientific, 1993, pp. 157-181.
[27] A. Ailon and I. Zohar, Control Strategies for Driving a Group of Nonholonomic Kinematic Mobile Robots in Formation Along a Time-Parametrized Path. IEEE/ASME Transactions on Mechatronics, Vol. 17, No. 2, 2012, pp. 326-336.
[28] Khalil, H. (1995). Nonlinear Systems, 2nd Ed., Prentice-Hall
[29] P. Kokotovic, H. Khalil, J. O'Reilly, Singular perturbation methods in control: analysis and design. London: Academic Press, 1987.
[30] S. E. Roncerio, Three-Time-Scale Nonlinear Control of an Autonomous Helicopter on a Platform, PhD. Thesis, Automation, Robotics and Telematic Engineering, Universidad de Sevilla, July, 2011.
[31] V. R. Saksena, J. O'Reilly and P. V. Kokotovic, Singular Perturbations and Scale methods in Control Theory: Survey 1976-1983, Automatica, 1984, Vol. 20, No. 3, pp. 273-293.
[32] I. Mas and C. Kittes, Obstacle Avoidance Policies for Cluster Space Control of Nonholonomic Multirobot Systems, IEEE/ASME Transactions on Mechatronics, Dec. 2012, Vol. 17, No. 6, pp. 1068 - 1079.
[33] K. K. Oh and H.S. Ahn, Distance-based Formation Control Using Euclidean Distance Dynamics Matrix: Three-agent Case, American Control Conference, O’Farrell Street, San Francisco, CA, USA, June 29 - July 01, 2011, pp. 4810-4815.
[34] Y. Tian and N. Sarkar, Formation Control of Mobile Robots subject to Wheel Slip, IEEE International Conference on Robotics and Automation, 2012, pp. 4553-4558.
[35] Manish Kumar, Devendra P. Garg, Vijay Kumar: Segregation of Heterogeneous Units in a Swarm of Robotic Agents. IEEE Transactions on Automatic Control 55(3): 743-748 (2010)
[36] S. Sarkar and J. N. Kar, Formation Control of Multiple Groups of swarms. accepted for publication in Int. conf. on Decision and Control, Florence, Italy, 2013.