Simply-connected minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

Laurent Hauswirth and Harold Rosenberg developed in [4] the theory of minimal surfaces with finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. They showed that the total curvature of one such a surface must be a non-negative integer multiple of $-2\pi$. The first examples appearing in this context are vertical geodesic planes and Scherk minimal graphs over ideal polygonal domains. Other non simply-connected examples have been constructed recently in [6, 11, 14].

In the present paper, we show that the only complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ of total curvature $-2\pi$ are Scherk minimal graphs over ideal quadrilaterals. We also construct properly embedded simply-connected minimal surfaces with total curvature $-4k\pi$, for any integer $k \geq 1$, which are not Scherk minimal graphs over ideal polygonal domains.

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1 Introduction

In the classical theory of minimal surfaces in $\mathbb{R}^3$, the ones better known are those with finite total curvature. We recall that the total curvature of a surface $M$ is defined as $C(M) = \int_M K$, where $K$ denotes the Gauss curvature of $M$. If a minimal surface $M$ of $\mathbb{R}^3$ has finite total

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curvature (i.e. $|C(M)| < +\infty$) then either $M$ is a plane or it must be $C(M) = -4\pi k$, for some integer $k \geq 1$, and the equality only holds for $M$ being the catenoid or Enneper's surface (see [13, Theorems 9.2 and 9.4]).

In the last decade, the geometry of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ has been actively studied, and many examples have been constructed (see for instance [1, 3, 9, 10, 12, 15]). Hauswirth and Rosenberg started in [4] the study of complete minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$. The only known examples at that moment were the Scherk minimal graphs over ideal polygonal domains with an even number of edges, with boundary values $\pm \infty$ disposed alternately. Morabito and the authors constructed in [11, 14] non simply-connected properly embedded minimal surfaces with finite total curvature and genus zero. Quite recently, in a joint work with Martín and Mazzeo, the second author [6] has constructed properly embedded minimal surfaces with finite total curvature and positive genus.

The classification of minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ arises very naturally. The first result of classification appearing in this theory was that the only complete minimal surfaces with vanishing total curvature are the vertical geodesic planes (see [5, Corollary 5]). Quite recently, Hauswirth, Nelli, Sa Earp and Toubiana have proved in [7] that a complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane, must be one of the horizontal catenoids constructed in [11, 14]. In this paper, we show that the Scherk minimal graphs over ideal quadrilaterals (i.e. ideal polygonal domains bounded by four ideal geodesics) are the only complete minimal surfaces of total curvature $-2\pi$.

It was expected that each end of a minimal surface with finite curvature in $\mathbb{H}^2 \times \mathbb{R}$ were asymptotic to either a vertical geodesic plane or a Scherk graph over an ideal polygonal domain. We construct new simply-connected examples, that we call twisted Scherk examples, that highlight this is not the case. They all have total curvature an integer multiple of $-4\pi$, so we cannot expect a classification result for Scherk graphs over ideal polygonal domains bounded by $4k + 2$ edges as the only simply-connected complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with total curvature $-4k\pi$.

2 Preliminaires

We consider the Poincaré disk model of $\mathbb{H}^2$; i.e. $\mathbb{H}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$, with the hyperbolic metric $g_{-1} = \frac{4}{(1-|z|^2)^2} |dz|^2$. We denote by $\partial_\infty \mathbb{H}^2$ the infinite boundary of $\mathbb{H}^2$ (i.e. $\partial_\infty \mathbb{H}^2 = \{z \in \mathbb{C} \mid |z| = 1\}$) and by 0 the origin of $\mathbb{H}^2$. Also $t$ will denote the coordinate in $\mathbb{R}$.

Let $M$ be a complete orientable minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$. We define the total curvature of $M$ as $C(M) = \int_M K$, where $K \leq 0$ denotes the Gaussian curvature of $M$. We say that $M$ has finite total curvature when $|C(M)| < +\infty$. 

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In this section we summarize the geometric properties of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature given by Hauswirth and Rosenberg in [4].

We call the height function of $M$ the horizontal projection $h : M \to \mathbb{R}$, and we denote by $F$ the vertical projection of $M$ over $\mathbb{H}^2$. It is well-known that $h$ is a real harmonic function on $M$ and that $F$ is an harmonic map from $M$ to $\mathbb{H}^2$. Given a conformal parameter $w$ on $M$, Sa Earp and Toubiana [15] proved that $(h_w)^2 = -Q$, where $Q$ is the Hopf differential associated to $F$. Then the zeroes of $Q$ are of even order and, up to a sign (which corresponds to a reflection symmetry with respect to $\mathbb{H}^2 \times \mathbb{R}$),

$$h = \Re \left( -2i \int \sqrt{Q} \right),$$

see equation (3) in [4].

We fix a unit normal vector field $N$ on $M$. We now state the main theorem in [4].

**Theorem 1.** [4] Let $M$ be a complete, orientable, minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature. Then:

1. $M$ is conformally a closed Riemann surface $\mathbb{M}$ punctured in a finite number of points $p_1, \ldots, p_n$, called ends of $\Sigma$.

2. $Q$ is holomorphic on $M$ and extends meromorphically to its ends $p_i$. If we parameterize conformally a neighborhood of $p_i$ in $M$ by $\Omega = \mathbb{C} \setminus D_0$, where $D_0$ is the open unit disk in $\mathbb{C}$ centered at the origin, then

$$Q(z) = z^{2m_i}(dz)^2,$$

for some integer $m_i \geq -1$.

3. $N_3 = \langle N, \partial_t \rangle$ converges uniformly to zero on each end $p_i$.

4. The total curvature of $M$ is given by

$$\int_M K = 2\pi \left( 2 - 2g - 2n - \sum_{i=1}^{n} m_i \right).$$

**Remark 2.** Suppose $p_i$ is an end of $M$ for which $m_i = -1$. If we want to close periods in equation (1), then we have to choose $Q(z) = -z^{-2}(dz)^2$, $z \in \Omega$.

**Assertion 3.** In the second item of Theorem 1, $m_i$ cannot equal $-1$. 
Proof. Suppose $M$ (in the setting of Theorem 1) has an end $p_i$ for which $m_1 = -1$. We know that a neighborhood $E$ of $p_i$ can be conformally parameterized on $\Omega = \{ z \in \mathbb{C} \mid |z| \geq 1 \}$, where $Q(z) = -z^{-2}dz^2$ (see Remark 2). From (1) we then get $h(z) = 2 \Re \left( \int_M \frac{dz}{z} \right) = 2 \ln |z|$. Therefore, $E$ is a vertical annulus whose intersection with each horizontal slice $\mathbb{H}^2 \times \{ t \}$, $t \geq 0$, is a compact curve.

The boundary of $E$ (which corresponds to $\{|z| = 1\}$) consists of a horizontal compact curve $\Gamma$ at height zero. Consider $R > 0$ big enough so that the disc $D \subset \mathbb{H}^2$ of radius $R$ centered at the origin contains $\Gamma$ in its interior. And let $C$ be the complete vertical rotational catenoid constructed by Nelli and Rosenberg in [12] whose neck is $\partial D$. Since $E$ intersects each horizontal slice in a compact curve, we deduce using the Maximum Principle with vertically translated copies of $C$ that $E$ must be contained in $D \times \mathbb{R}$. But this is not possible: If we translate $C$ vertically up a distance $\pi$, we reach a contradiction by applying the Maximum Principle with the family of shrunk catenoids going from $C$ to the 2-sheeted covering of the punctured slice $(\mathbb{H}^2 - \{0\}) \times \{\pi\}$.

We finish this section by describing the asymptotic behavior of a complete, orientable, minimal surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature.

Lemma 4. [4] Let $M$ be a minimal surface in the hypothesis of Theorem 1, and $p_i$ an end of $M$. If $m_i \geq 0$ is the integer associated to $p_i$, as defined in Theorem 1, then $p_i$ corresponds to $m_i + 1$ geodesics $\gamma_1, \ldots, \gamma_{m_i+1} \subset \mathbb{H}^2 \times \{+\infty\}$, $m_i + 1$ geodesics $\Gamma_1, \ldots, \Gamma_{m_i+1} \subset \mathbb{H}^2 \times \{-\infty\}$, and $2(m_i + 1)$ vertical straight lines (possibly some of them coincide) in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, each one joining an endpoint of some $\gamma_j$ to an endpoint of some $\Gamma_j$.

3 Minimal examples with finite total curvature

Given any two points $p, q \in \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, we will denote by $\overline{pq}$ the geodesic arc joining $p, q$.

We consider an even number of different points $p_1, \cdots, p_{2k} \in \partial_\infty \mathbb{H}^2$ (cyclically ordered), with $k \geq 2$, and we call $A_i = \overline{p_{2i-1}p_{2i}}, B_i = \overline{p_{2i}p_{2i+1}}$, for any $1 \leq i \leq k$, where we consider the cyclic notation $p_{2k+1} = p_1$. Let $\Omega$ be the ideal polygonal domain bounded by $A_1, B_1, \cdots, A_k, B_k$. We call Scherk minimal graph over $\Omega$ to a minimal graph over $\Omega$ with boundary values $+\infty$ over the $A_i$ edges and $-\infty$ over the $B_i$ edges (in [1, 12] it is proved that it exists and it is unique up to a vertical translation). In [1, 4] it is proved that such a graph has total curvature $2\pi(1-k)$. Scherk graphs over ideal polygonal domains, together with the vertical geodesic planes, where the first known examples of minimal surfaces with finite total curvature.

In [11, 14] other non-simply-connected examples where presented, called minimal k-noids. We briefly explain their construction: Consider an even number of points $p_1, \cdots, p_{2k}$ (cyclically ordered) such that $p_{2i-1} \in \mathbb{H}^2$ and $p_{2i} \in \partial_\infty \mathbb{H}^2$. We call $A_i = \overline{p_{2i-1}p_{2i}}$ and $B_i = \overline{p_{2i}p_{2i+1}}$.
Consider the minimal graph $\Sigma$ over the polygonal domain bounded by $A_1, B_1, \cdots, A_k, B_k$ with boundary values $+\infty$ over the $A_i$ edges and $-\infty$ over the $B_i$ edges (it exists and is unique up to a vertical translation, by [1, 9]), which has total curvature $2\pi(1-k)$ (see [1]). The conjugate minimal surface $\Sigma^*$ of $\Sigma$ is a minimal graph contained in $\mathbb{H}^2 \times \{ t \geq 0 \}$, whose boundary consists of $k$ geodesic curvature lines in $\mathbb{H}^2 \times \{ 0 \}$. (The conjugation for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ was introduced by Daniel [2] and by Hauswirth, Sa Earp and Toubiana [5].) If we reflect $\Sigma^*$ with respect to $\mathbb{H}^2 \times \{ 0 \}$, we get a properly embedded minimal surface of genus zero, $k$ ends asymptotic to vertical geodesic planes and total curvature $4\pi(1-k)$. For $k=2$, the obtained examples are usually called horizontal catenoids, and have been recently classified by Hauswirth, Nelli, Sa Earp and Toubiana as the only complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane.

Using a gluing method, the second author has recently constructed in a joint work with Martín and Mazzeo a wide range of properly embedded minimal surfaces with finite total curvature and two ends, each one asymptotic to a vertical geodesic plane.

We wondered if Scherk minimal graphs were, together with the vertical geodesic planes, the only complete, embedded, simply-connected examples of finite total curvature and finite topology (with possibly positive genus).

We explain the simple construction of other different complete, embedded, simply-connected examples, that we will call twisted Scherk examples.

### 3.1 Twisted Scherk examples

Let us first construct an example with total curvature $-4\pi$. Let $p_1, p_2$ be two points in $\partial_\infty \mathbb{H}^2$. Up to an isometry of $\mathbb{H}^2$, we can assume $p_1 = 1$ and $p_2 = e^{i\theta}$, for some fixed $\theta \in (0, \pi/2]$ (see Figure 1). We call $A_1 = \overline{0p_1}$, $B_1 = \overline{p_1p_2}$ and $C_1 = \overline{0p_2}$. Let $\Delta$ be the geodesic triangle bounded by $A_1 \cup B_1 \cup C_1$. By the triangle inequality at infinity (see [1, Lemma 3]), we get that $\Delta$ satisfies the Jenkins-Serrin condition for the existence of a minimal graph $u$ over $\Delta$ with boundary values $+\infty$ on $A_1$, $-\infty$ on $B_1$ and $0$ on $C_1$ (see [1, Theorem 3] and [9, Theorem 3.3]).

Now let us see that the graph surface $\Sigma(u)$ of $u$ has finite total curvature: For any positive integer $n$, we denote $r = 1 - 1/(n+1)$ and $p_{1,n} = r$, $p_{2,n} = re^{i\theta}$. By Theorem 3 in [12], there exists a minimal graph $u_r(n)$ over the geodesic triangle of vertices $0, p_{1,n}, p_{2,n}$ taking boundary values $+n$ on $\overline{0p_{1,n}}$, $-n$ on $\overline{p_{1,n}p_{2,n}}$ and $0$ on $\overline{0p_{2,n}}$. By the Gauss-Bonnet formula, the graph surface of $u_r(n)$ has total curvature $\pi$. Since $u_r(n)$ converges uniformly on compact sets of $\Delta$ to $u$ as $n \to \infty$, the total curvature of $\Sigma(u)$ is at most $\pi$, and then finite.

By rotating $\Sigma(u)$ an angle $\pi$ about the horizontal geodesic ray $\overline{0p_2}$ contained in its boundary, we obtain a minimal graph whose boundary consists of the vertical geodesic $\{ 0 \} \times \mathbb{R}$. We extend such a graph by rotation of angle $\pi$ about its boundary, and we get a properly embedded simply-connected minimal surface $\Sigma_1$.

Since $\Sigma_1$ consists of four copies of $\Sigma(u)$, then it has finite total curvature. Then equation (2)
applies. In our case, $g = 0$, $n = 1$ and $m_1 = 2$ ($m_1 = 2$ follows from the fact that the intersection of $M$ with a horizontal slice $\mathbb{H}^2 \times \{t\}$, for $t > 0$ large enough, consists of three divergent curves, see Figure 1). Thus $\int_{\Sigma_1} K = -4\pi$.

Now, let us consider $k \geq 2$. Let $\Omega$ be a polygonal domain whose vertices are $0$ and $2k - 1$ different ideal points $p_1, \ldots, p_{2k - 1} \in \partial_\infty \mathbb{H}^2$. Assume that $\Omega$ satisfies the Jenkins-Serrin condition of Theorem 3 in [1] or Theorem 3.3 in [9]. The example below proves that there exist such domains. We call $\Sigma$ the minimal graph over $\Omega$ with boundary values $+\infty$ on $0\tilde{p}_1$ and on $p_{2i}p_{2i+1}$, for $1 \leq i \leq k - 1$; and $-\infty$ on $p_{2i-1}p_{2i}$, for $1 \leq i \leq k - 1$, and zero on $0p_{2k-1}$. By rotating $\Sigma$ an angle $\pi$ about the vertical geodesic line $\{0\} \times \mathbb{R}$ in its boundary, we obtain a properly embedded simply-connected minimal surface $\Sigma_k$. Arguing similarly as for $\Sigma_1$, we can prove that $\int_{\Sigma_k} K = -4k\pi$. Then we have proved the following theorem.

**Theorem 5.** For any integer $k \geq 1$, there exists a properly embedded simply-connected minimal surface $\Sigma_k$ of finite total curvature $-4k\pi$ which is not a minimal (vertical) graph.

Now let us construct a polygonal domain $\Omega$ in the above setting. For any $\theta \in (0, \frac{\pi}{2k})$, let $\Omega_\theta$ be the polygonal domain with vertices $0, \tilde{p}_1 = 1$, and

$$p_n = e^{i(n-1)\theta}, \quad 2 \leq n \leq k + 1.$$

We mark by $+\infty$ the edge $0\tilde{p}_1$ and those of the form $p_{2i}p_{2i+1}$; by $-\infty$ the edges of the form $p_{2i-1}p_{2i}$; and by 0 the edge $0p_{2k-1}$. It is clear that $\Omega_\theta$ does not satisfy the Jenkins-Serrin condition (see Theorem 3 in [1] or Theorem 3.3 in [9]), as we can consider the inscribed polygonal domain.
Theorem 6. If $M$ is a complete minimal surface of total curvature $-2\pi$ in $\mathbb{H}^2 \times \mathbb{R}$, then $M$ is the Scherk minimal graph over an ideal quadrilateral.

Proof. Since the total curvature of $M$ is $-2\pi$, we have by equation (2) in Theorem 1 that

$$-2\pi = 2\pi \left( 2 - 2g - 2n - \sum_{i=1}^{n} m_i \right).$$

We already know that $m_i \geq 0$, by Assertion 3. And $n \geq 1$, since a complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ cannot be compact. So the only possibility is $g = 0$, $n = 1$ (hence the complete minimal surface $M$ is simply-connected) and $m_1 = 1$. 

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Figure 2: Left: The fundamental piece of a twisted Scherk example $\Sigma_2$ with total curvature $-8\pi$. Right: Vertical projection of $\Sigma_2$. 

$\mathcal{P} \subset \Omega$ with vertices $0, \bar{p}_1, p_2, p_3$ and any choice of disjoints horocycles $H_1, H_2, H_3$ at $\bar{p}_1, p_2, p_3$ respectively, for which $\text{dist}_{\mathbb{H}^2}(0, H_1) + \text{dist}_{\mathbb{H}^2}(H_2, H_3) = \text{dist}_{\mathbb{H}^2}(0, H_3) + \text{dist}_{\mathbb{H}^2}(H_1, H_2)$.

To solve this problem, we consider a small perturbation of $\bar{p}_1$: Let $\Omega_{\theta,\beta}$ be the polygonal domain with vertices $p_1 = e^{-i\beta}$, for $\beta \in (0, \frac{\pi}{2} - k\theta]$ small, and $p_n$ defined as above, for $2 \leq n \leq k+1$. This domain $\Omega_{\theta,\beta}$ satisfies the Jenkins-Serrin condition if we label by $+\infty$ the edge $0, p_1$ and those of the form $p_{2i}p_{2i+1}$; by $-\infty$ the edges of the form $p_{2i-1}p_{2i}$; and by 0 the edge $0p_{k+1}$.

Let $R$ be the reflection with respect to the geodesic containing $0p_{k+1}$. Then $\Omega = \Omega_{\theta,\beta} \cup R(\Omega_{\theta,\beta})$ is in the desired conditions. See Figure 2.
As \( m_1 = 1 \), we know by Lemma 4 that there are four points \( p_1, p_2, p_3, p_4 \in \partial_{\infty} \mathbb{H}^2 \), with \( p_i \neq p_{i+1} \) for any \( i \), such that the end of \( M \) corresponds to

\[
(p_1p_2 \times \{+\infty\}) \cup (p_2p_3 \times \{-\infty\}) \cup (p_3p_4 \times \{+\infty\}) \cup (p_4p_1 \times \{-\infty\}),
\]

together with the complete vertical geodesics \( \{p_i\} \times \mathbb{R} \) in the ideal cylinder \( \partial_{\infty} \mathbb{H}^2 \times \mathbb{R} \) joining their endpoints.

Let us now prove that the four points \( p_i \) are all different. By the maximum principle using vertical geodesic planes, we know that at least three of them are different as \( M \) cannot be a vertical plane. Suppose \( p_1 = p_3 \) (the case \( p_2 = p_4 \) follows similarly). Also using the maximum principle with vertical geodesic planes, we get that the vertical projection \( \pi(M) \) of \( M \) is contained in the ideal geodesic triangle of vertices \( p_1, p_2, p_4 \). Even more, \( \pi(M) \) is contained in a domain \( T \subset \mathbb{H}^2 \) bounded by \( \overline{p_1p_2}, \overline{p_1p_4} \) and a strictly concave (with respect to \( T \)) curve \( \alpha \). We observe that the points in \( M \) projecting onto \( \alpha \) have horizontal normal vector. Suppose that the vertical projection of the limit normal vector of \( M \) (that we also call \( N \)) along \( \overline{p_1p_2} \times \{+\infty\} \) points to \( T \). We observe that the horizontal curves in \( M \) with endpoint in \( \{p_2\} \times \mathbb{R} \) arrive orthogonally to \( \partial_{\infty} \mathbb{H}^2 \times \mathbb{R} \). In particular, \( N \) is constant along the vertical asymptotic line \( \{p_2\} \times \mathbb{R} \). On one hand that implies, looking at the behavior of \( N \) along the asymptotic boundary of \( M \) (corresponding to the end) that the vertical projection of \( N \) along \( \overline{p_1p_2} \times \{-\infty\} \) also points to \( T \), and its projection along \( \overline{p_1p_4} \times \{\pm\infty\} \) goes out from \( T \). On the other hand, if we follow the projection of \( N \) along \( \alpha \), we obtain that it points to \( T \) along \( \overline{p_1p_4} \times \{\pm\infty\} \), a contradiction.

We now claim that \( p_1, p_2, p_3, p_4 \) are cyclically ordered. We define the solid cylinder \( C_{r,T} = \{(z,t) : |z| \leq r, |t| \leq T\} \), for \( r < 1 \) close to one and \( T \) large, and consider \( M_{r,T} = M \cap C_{r,T} \), which is a compact minimal surface bounded by two horizontal compact curves contained in \( \{t = T\} \) close to \( \overline{p_1p_2} \times \{T\} \) and \( \overline{p_3p_4} \times \{T\} \), two curves on \( \{t = -T\} \) close to \( \overline{p_2p_3} \times \{-T\} \) and \( \overline{p_4p_1} \times \{-T\} \), and four curves on \( \{|z| = r\} \) close to vertical lines. By the flux formula with respect to the Killing vector field \( \partial_t \) (see [8, Proposition 3]), we have

\[
\int_{\partial M_{r,T}} \langle \nu, \partial_t \rangle = 0,
\]

(3)

where \( \nu \) is the outward-pointing unit conormal to \( M_{r,T} \) along \( \partial M_{r,T} \). We get from (3), taking limits as \( r \to 1 \) and \( T \to +\infty \), that \( |\overline{p_1p_2}| + |\overline{p_3p_4}| = |\overline{p_2p_3}| + |\overline{p_4p_1}| \), where \( |\bullet| \) denotes (as in [1]) the hyperbolic length of the curve \( \bullet \) outside some disjoint horocycles at the ideal points \( p_i \), identifying \( \mathbb{H}^2 \) with the corresponding horizontal slice. By the triangle inequality at infinity [1, Lemma 3] we get that \( p_1, p_2, p_3, p_4 \) must be cyclically ordered.

We call \( \Omega \) the ideal quadrilateral with vertices \( p_1, p_2, p_3, p_4 \). By the maximum principle using vertical geodesic planes, we get that \( \pi(M) \subset \Omega \). On the other hand, the geometry of
the end of $M$ says that a neighborhood of $\partial \Omega$ is contained in $\pi(M)$. Since $M$ is complete and simply-connected, we conclude $\pi(M) = \Omega$.

Now let us show that the normal vector of $M$ is never horizontal. Suppose there exists a point $P \in M$ such that $N_3(P) = 0$. Let $\Gamma \times \mathbb{R}$ be the vertical geodesic plane tangent to $M$ at $P$. Since $M$ and $\Gamma \times \mathbb{R}$ have first contact order at $P$, their intersection consists of $k$ curves meeting at equals angles at $P$, with $k \geq 2$. Thus, there are at least four branches of $M \cap (\Gamma \times \mathbb{R})$ leaving $P$ (see Figure 3, left). Since $M$ is simply-connected, we deduce using the maximum principle with vertical planes that there cannot exist a compact cycle in $M \cap (\Gamma \times \mathbb{R})$. Hence $\Gamma$ cannot intersect two edges of $\Omega$, so it must have some $p_i$ as an endpoint. Denote by $\gamma = \gamma(t)$, $t \in \mathbb{R}$, the arc-length parameterized geodesic of $\mathbb{H}^2$ orthogonal to $\Gamma$ such that $\gamma(0) = \pi(P)$; and by $\Gamma_t$ the geodesic of $\mathbb{H}^2$ passing through $\gamma(t)$ orthogonally (in particular, $\Gamma_0 = \Gamma$). For $\varepsilon > 0$ small, $\Gamma_\varepsilon$ intersects two edges of $\Omega$, say $p_1p_2$ and $p_2p_3$, and the number of intersection curves between the vertical plane $\Gamma_\varepsilon \times \mathbb{R}$ and $M$ is at least two (see Figure 3, right). But only one branch of the intersection curves can arrive to $p_1p_2 \times \{+\infty\}$ (resp. $p_2p_3 \times \{-\infty\}$), the other branch should be a compact loop, a contradiction.

We have prove then that, for any point $q \in \Omega$, the intersection of $\{q\} \times \mathbb{R}$ with $M$ is transverse. So the number of intersection points does not depend on $q$. For $q$ near an edge of $\Omega$ this number is one. We conclude that $M$ is a graph over $\Omega$. \qed
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