EFFECTIVE ACTION OF $N = 2$ SUPERSYMMETRIC FIELD THEORIES IN HARMONIC SUPERSPACE APPROACH

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Abstract
The paper is a brief review of the works devoted to problem of effective action in $N = 2$ super Yang-Mills theories in harmonic superspace approach. The formulation of $N = 2$ superfield models in harmonic superspace is discussed, background field method for $N = 2$ super Yang-Mills theory is constructed and general structure of effective action in harmonic superspace is investigated. It is shown how the holomorphic and non-holomorphic contributions to effective action can be calculated within the harmonic superspace approach.

1 Introduction

$N = 2$ supersymmetric field theories possess remarkable properties both at the classical and quantum levels. Applications of $N = 2$ supersymmetry in a whole are very enormous and range from superstring theory to supergauge field models and so called special geometry (see [1] for a modern review). In particular, $N = 2$ super Yang-Mills theories are finite beyond one-loop approximation [2-7]. A modern interest to quantum aspects of $N = 2$ supersymmetric field theory was inspired by the work of Seiberg and Witten [8] where non-pertubative contribution to low-energy effective action was exactly found. The Seiberg-Witten approach has essentially been based on the structure of perturbative low-energy effective action proposed by Seiberg [9].

The paper under consideration is devoted to discussing a systematic method of investigation of effective action in $N = 2$ supersymmetric field theories within the harmonic superspace approach. This approach has been developed by A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky, E. Sokatchev [10, 11] and it is very well adapted to be applied for studying quantum aspects of $N = 2$ supersymmetric field theories. From point of view of quantum field theory the harmonic superspace approach possesses by two essential dignities:

(i) The $N = 2$ supersymmetric field theories are formulated in this approach in terms of $N = 2$ superfields defined on suitable superspace.

(ii) The theories are defined by actions depending on unconstrained superfields.

The first feature guarantees the manifest $N = 2$ supersymmetry on all stages of consideration. The second one allows to use the standard notions of propagators and vertices and hence to apply a standard quantum field theory technique.

The aspects of effective action problem for $N = 2$, supersymmetric field theories have been investigated in recent papers [12]. However all calculations were given in these papers using a formulation of $N = 2$ theories in terms of $N = 1$ superfields where $N = 2$ supersymmetry is not manifest. As well known although such a formulation is correct and sometimes convenient it does not allow to control the calculations on a base of explicit symmetry and can lead to different kinds of miraculous cancellations an actual reason of which is not seen. From this point of view developing a completely $N = 2$ supersymmetric approach to effective action problem looks like useful and important enough.

This paper is a brief review of such an approach which has been recently developed in the refs.[13, 14] and allows to preserve manifest $N = 2$ supersymmetry at all steps of effective action calculation.

2 Brief Review of Harmonic Superspace Approach

The aim of harmonic superspace approach is to formulate $N = 2$ supersymmetric field theories in terms of $N = 2$ unconstrained superfields. A basic idea is to introduce a superspace preserving $N = 2$ supersymmetry but having lesser number of anticommuting coordinates then in case of general $N = 2$ superspace.
A first step leading to suitable superspace is based on introducing the new bosonic coordinates $u^\pm_i, u^-_i; i = 1, 2$ forming a matrix belonging to $SU(2)$-group. These coordinates $u^\pm_i$ are called the harmonics. Using the decomposition

$$\theta^\pm_\alpha = u^\pm_i \theta^i_\alpha, \quad \bar{\theta}^\pm_\alpha = u^\pm_i \bar{\theta}^i_\alpha, \quad \tag{1}$$

where $\theta^i_\alpha, \theta^i_\alpha$ are the anticommuting coordinates of general $N = 2$ superspace we introduce the new space-time coordinates

$$x^m_A = x^m - i(\theta^i \sigma^m \bar{\theta}^i + \theta^i \sigma^m \bar{\theta}^i)u^+_i u^-_j \quad \tag{2}$$

and get a superspace with the coordinates

$$(x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\alpha, u^\pm_i). \quad \tag{3}$$

A second step begins with decomposition of spinor derivatives

$$D^\pm_\alpha = u^\pm_i \bar{D}^i_\alpha, \quad \bar{D}^\pm_\alpha = u^\pm_i \bar{D}^i_\alpha \quad \tag{4}$$

where $D^i_\alpha, \bar{D}^i_\alpha$ are the supercovariant spinor derivatives in general $N = 2$ superspace with the coordinates $(x^m, \theta^\alpha, \bar{\theta}^\alpha)$. One can show that the derivatives $D^\pm_\alpha, \bar{D}^\pm_\alpha$ have the very simple form

$$D^\pm_\alpha = \frac{\partial}{\partial \theta^\pm_\alpha}, \quad \bar{D}^\pm_\alpha = \frac{\partial}{\partial \bar{\theta}^\pm_\alpha} \quad \tag{5}$$

Taking these derivatives one can consider the superfields $\Phi(x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\alpha, u^\pm_i)$ satisfying the constraints

$$D^\pm_\alpha \Phi = 0, \quad \bar{D}^\pm_\alpha \Phi = 0 \quad \tag{6}$$

The constraints (6) show that we have a possibility to work with superfields $\Phi = \Phi(x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\alpha, u^\pm_i)$ which do not depend on $\theta^\pm_\alpha, \bar{\theta}^\pm_\alpha$, but preserve manifest $N = 2$ supersymmetry. Let us introduce a superspace with the coordinates $(x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\alpha)$. It is remarkable that these coordinates transform through each other under $N = 2$ supersymmetry transformations with full $N = 2$ transformation parameters. It alloys to treat the above superspace as an independent object. This superspace is called analytic subspace and it plays in $N = 2$ supersymmetry the same role as chiral subspace in $N = 1$ supersymmetry. The superfields defined on analytic subspace are called analytic. It is evident that any analytic superfields contains the same number of anticommuting coordinates as general $N = 1$ superfield. It leads to reducing a number of independent components in compare with general $N = 2$ superfields. However all components depend now on extra bosonic coordinates $u^\pm_i$. Therefore any analytic superfield contains infinite number of component fields from point of view of conventional supersymmetric field theory. It is worth to point out that a possibility to construct the analytic subspace is main dignity of harmonic superspace approach.

A third step is a construction of the action describing a dynamics of the superfields. Let us start with matter superfields which are called the hypermultiplets. A simplest hypermultiplet is described by analytic superfield $q^+(x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\alpha, u^\pm_i)$ with $U(1)$-charge +1. The action of the free theory looks as follows

$$S[q^+, q^+] = - \int d^4 \zeta(-4) du q^+ \bar{D}^+ q^+ \quad \tag{7}$$

where $d\zeta^{(4)} = d^4 x_A d^2 \theta^+ d^2 \bar{\theta}^+$. Here $\bar{q}^+$ means a special conjugation. Integral over harmonics $du$ was defined in the papers [10, 11] and

$$D^+ = u^+_i \frac{\partial}{\partial \theta^+_i} - 2i(\theta^+_i \sigma^m \bar{\theta}^+_i) \frac{\partial}{\partial x^m_A} + \theta^+_i \frac{\partial}{\partial \theta^{-}_i} + \bar{\theta}^+_i \frac{\partial}{\partial \bar{\theta}^{-}_i} \quad \tag{8}$$

is a specific operator acting on harmonics.

Another hypermultiplet is described by real analytic superfield $\omega(x_A, \theta^+; \bar{\theta}^+, u^\pm)$ and has the action

$$S[\omega] = \int d\zeta(-4) du (D^+ \omega)(D^+ \omega) \quad \tag{9}$$

Both $q^+$ and $\omega$ are the unconstrained superfields.
The equations of motion for hypermultiplets look like as follows
\[
D^{++} q^+ = 0 \\
(D^{++})^2 \omega = 0
\] (10)

If we expand the \(q^+\) and \(\omega\) in a power series in harmonics and substitute into equations of motion we will see that only a few first terms of expansion are not equal to zero on-shell. Hence, almost all \(u\)-dependence of \(q^+\) and \(\omega\) becomes to be unessential on-shell and the problem of extra bosonic coordinates dissapears.

To construct interacting theory one introduces the covariant derivatives
\[
\nabla^{++} = D^{++} + iV^{++}
\] (11)

where \(V^{++} = V^{++a} T^a\), \(V^{++a}\) is analytic superfields with \(U(1)\)-charge +2 and \(T^a\) are the generators of internal symmetry. Namely this superfield \(V^{++a}\) is an unconstrained prepotential for \(N = 2\) super Yang-Mills theory.

The final step is action for \(V^{++}\). It can be written in the form [15]
\[
S_{SYM}[V^{++}] = \frac{1}{g^2} \int d^4x d^8 \theta \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \int du_1...du_n \text{tr} V^{++}(z, u_1)...V^{++}(z, u_n) \frac{u_1^+ u_2^+}{(u_1^+ u_1^+)(u_2^+ u_2^+)}
\] (12)

Here \(z = (x^m, \theta^i, \bar{\theta}^i)\), \((u_1^+ u_2^+) = u_1^+ u_2^+\) and \(g\) is a coupling. This action is invariant under the gauge transformations [10, 11]
\[
\delta V^{++} = -D^{++} \Lambda + ...
\] (13)

where \(\Lambda\) is analytic superfield parameter. Taking into account this gauge transformation one can impose specific gauge fixing condition where practically whole \(u\)-dependence becomes to be unessential.

As result we obtain a formulation of super Yang-Mills theory coupled to a matter in a manifest \(N = 2\) supersymmetric form in terms of unconstrained superfields.

3 Background Field Method for \(N = 2\) Super Yang-Mills Theories

Background field method is a some specific construction of effective action in gauge field theories allowing to preserve classical gauge invariance in quantum theory. The matter is, to quantize a gauge theory we impose the gauge fixing conditions and destroy the classical gauge invariance of the theory. As a result, the effective action generally speaking is not invariant under initial gauge transformations. Only S-matrix is gauge independent object. However, there is a way to construct effective action which will be invariant under the same gauge transformations as the initial classical action. This way is based on splitting the fields into two pieces, the classical fields and the quantum ones and imposing the gauge fixing conditions only on quantum fields. In concrete theories one can find the suitable gauge fixing functions allowing to preserve classical gauge invariance with respect to above classical fields which are the functional arguments of the effective action.

Let us split the superfield \(V^{++}\) into background \(V^{++}\) and quantum \(v^{++}\) pisces
\[
V^{++} \rightarrow V^{++} + g v^{++}
\] (14)

where \(V^{++}\) is background field and \(v^{++}\) is quantum one. Then the initial gauge transformations can be realized in two different ways
(i) background transformations
\[
\delta V^{++} = -D^{++} \Lambda - i[V^{++}, \Lambda] = -\nabla^{++} \Lambda \\
\delta v^{++} = i[\Lambda, v^{++}]
\] (15)

(ii) quantum transformations
\[
\delta V^{++} = 0 \\
\delta v^{++} = -\frac{1}{g} \nabla^{++} \Lambda - i[v^{++}, \Lambda]
\] (16)
It is worth to point out here that the form of background - quantum splitting and corresponding background and quantum transformations are absolutely analogous to the conventional Yang-Mills theory but not to \( N = 1 \) super Yang-Mills theory. Our aim is to construct effective action as a functional of background field, invariant under background gauge transformations.

After substituting the background -quantum splitting into the action we obtain

\[
S_{SYM}[V^{++} + gv^{++}] = S_{SYM}[V^{++}] +
\]

\[
+ \frac{1}{4g} tr \int d\zeta^{-1} du v^{++} D^+ D^+ v^{++} W(\lambda) + \Delta S_{SYM}[V^{++}, v^{++}]
\]

\[
\Delta S_{SYM}[V^{++}, v^{++}] = -tr \int d^2 z \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \int du_1 ... du_n \frac{v^{++}_1(z, u_1) v^{++}_2(z, u_2) ... v^{++}_n(z, u_n)}{(u_1^{(1)} u_1^{(2)}) (u_2^{(1)} u_2^{(2)}) ... (u_n^{(1)} u_n^{(2)})}
\]

\[
W(\lambda) = e^{i\Omega} W e^{-i\Omega}
\]

\[
v^{++}_1 = e^{-i\Omega} v^{++}_1 e^{i\Omega}
\]

\[
(17)
\]

The superfield \( \Omega \) is called the bridge and it has been introduced in ref. [10]. The superfields \( W \) and \( \bar{W} \) are the \( N = 2 \) strengths introduced in ref. [17]. In the case under consideration the \( \Omega \) corresponds to background superfield \( V^{++} \). The action \( \Delta S_{SYM} \) depends on \( V^{++} \) via dependence \( v^{++}_1 \) on \( V^{++} \). We want to point out that each term in \( \Delta S_{SYM} \) is manifestly invariant under background gauge transformations.

To quantize a theory within background field method we should fix only quantum gauge transformations. One introduces the gauge fixing functions in the form

\[
\phi
\]

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\Delta S_{SYM}[V^{++}, v^{++}] = -tr \int d^2 z \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \int du_1 ... du_n \frac{v^{++}_1(z, u_1) v^{++}_2(z, u_2) ... v^{++}_n(z, u_n)}{(u_1^{(1)} u_1^{(2)}) (u_2^{(1)} u_2^{(2)}) ... (u_n^{(1)} u_n^{(2)})}
\]

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W(\lambda) = e^{i\Omega} W e^{-i\Omega}
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\[
\mathcal{F}^{(4)}_{(\tau)} = D^{++} v^{++}_1 = e^{-i\Omega} (\nabla^{++} v^{++} + i g [v^{++}, c]) e^{i\Omega} = e^{-i\Omega} \mathcal{F}^{(4)} e^{i\Omega}
\]

and applies Faddeev-Popov procedure. It leads to the following form for effective action \( \Gamma_{SYM}[V^{++}] \)

\[
e^{i\Gamma_{SYM}[V^{++}]} = e^{iS_{SYM}[V^{++}]} \int D^{++} D c^{(4)} [\Delta S_{SYM}[v^{++}, V^{++}]+S_{FP}] \delta(\mathcal{F}^{(4)} - f^{(4)})
\]

(19)

Here \( S_{FP} \) is Faddeev-Popov ghost action

\[
S_{FP}[b, c, v^{++}, V^{++}] = -tr \int d\zeta^{-1} du \nabla^{++} b(\nabla^{++} c + ig[v^{++}, c])
\]

(20)

\( b \) and \( c \) are the Faddeev-Popov ghosts and \( f^{(4)} \) is an arbitrary external analytic superfield.

The final step is averaging over all superfields \( f^{(4)} \). To do that we multiply both parts of the expression for \( e^{i\Gamma_{SYM}} \) with the unit

\[
1 = \Delta[V^{++}] \int D f^{(4)} \exp \left\{ \frac{i}{2\alpha} \int d^2 z du_1 du_2 f^{(4)}(z, u_1) \frac{(u_1^{(1)} u_1^{(2)}) (u_2^{(1)} u_2^{(2)})}{(u_1^{(1)} u_2^{(1)}) (u_1^{(2)} u_2^{(2)})} f^{(4)}(z, u_2) \right\}
\]

(21)

where \( \alpha \) is an arbitrary parameter and

\[
\Delta[V^{++}] = Det^{-\frac{1}{2}}(\nabla^{++})^2 Det^{-\frac{1}{2}}(\nabla^{--})
\]

\[
\nabla^{--} = -\frac{1}{2} (D^2)^2 (\nabla^{--})^2 = \nabla + ...
\]

(22)

with \( \nabla \) be the the standard dalambertian. The functional determinant \( Det^{-\frac{1}{2}}(\nabla^{++})^2 \) can be presented by path integral in the form

\[
Det^{-\frac{1}{2}}(\nabla^{++})^2 = \int D\phi e^{iS_{NK}[\phi, V^{++}]}
\]

\[
S_{NK}[\phi, V^{++}] = -\frac{1}{2} \int d\zeta^{-1} du \nabla^{++} \phi)(\nabla^{++} \phi)
\]

(23)

with bosonic real analytic superfield \( \phi \). This superfield \( \phi \) has sense of so called Nilsen-Kallosh ghost.

After doing averaging over \( f^{(4)} \) and putting \( \alpha = -1 \) we obtain the final form for effective action

\[
e^{i\Gamma_{SYM}[V^{++}]} = e^{iS_{SYM}[V^{++}]} \int D^{++} D c^{(4)} [\Delta S_{SYM}[v^{++}, V^{++}]+S_{int}] e^{iS_{total}}
\]

\[
S_{total}[v^{++}, b, c, \phi, V^{++}] = S_{2}[v^{++}, b, c, \phi, V^{++}] + S_{int}[v^{++}, b, c, V^{++}]
\]

(24)
\( S_2 \) plays a role of action of free theory
\[
S_2[v^{++}, b, c, \phi, V^{++}] = -\frac{1}{2} \int d\zeta^{(-4)} d\zeta \nabla^{+} v^{++} - \\
- \int d\zeta^{(-4)} d\zeta \nabla^{+} (v^{++} + c) - \frac{1}{2} \int d\zeta^{(-4)} d\zeta \nabla^{+} \phi (\nabla^{+} \phi)
\]
\[
S_{int}[v^{++}, b, c, V^{++}] = - \int d^{3}z d\zeta_{1}...d\zeta_{n} \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \frac{v^{++}_{(z)}(z, u_{1})v^{++}_{(z)}(z, u_{2})...v^{++}_{(z)}(z, u_{n})}{(u_{1}^{+} u_{2}^{-}) (u_{2}^{+} u_{3}^{-}) ... (u_{n}^{+} u_{1}^{-})}
\]
\[
-ig \int d\zeta^{(-4)} d\zeta (\nabla^{+} b)[v^{++}, c] \tag{25}
\]

The path integral defining \( \Gamma_{SYM}[V^{++}] \) has the form standart for quantum field theory. It contains the free action and interaction. The free action defines the propagators for super Yang-Mills field and ghosts and the interaction defines the vertices. Obtained form for effective action opens the possibilities to develop the manifestly \( N = 2 \) supersymmetric and gauge invariant procedures for calculating the effective action \( \Gamma_{SYM}[V^{++}] \).

### 4 General Structure of Effective Action

As in conventional field theory one can suggest that the effective action \( \Gamma[V^{++}] \) for \( N = 2 \) super Yang-Mills theory with matter is described in terms of effective Lagrangians
\[
\Gamma[V^{++}] = \int d^{4}x d^{4}\theta d^{4}\bar{\theta} \mathcal{L}_{eff} + (\int d^{4}x d^{4}\theta d^{4}\bar{\theta} \mathcal{L}_{eff}^{(c)} + c.c.) \tag{26}
\]

where \( \mathcal{L}_{eff} \) can be called general effective Lagrangian and \( \mathcal{L}_{eff}^{(c)} \) can be called chiral effective Lagrangian.

We will assume that the theory under consideration is formulated within background field method and hence the effective action \( \Gamma[V^{++}] \) is gauge invariant under initial classical gauge transformations. In this case the effective action should be constructed only from strengths and their covariant derivatives. Therefore the effective Lagrangians can be written as follows

\[
\mathcal{L}_{eff} = \mathcal{H}(W, \bar{W}) + \text{term depending on covariant derivatives of } W \text{ and } \bar{W}
\]

\[
\mathcal{L}_{eff}^{(c)} = \mathcal{F}(W) + \text{term depending on covariant derivatives of } W \text{ and preserving chirality}
\]

The term \( \mathcal{F}(W) \) in chiral effective Lagrangian depending only on \( W \) is called holomorphic effective action. Namely this term is leading in low-energy approximation and describes a vacuum structure of the theory. The term
\[
\int d^{4}x d^{4}\theta d^{4}\bar{\theta} \mathcal{H}(W, \bar{W})
\]
defines first non-leading correction to low-energy effective action and describes an effective dynamics.

The structure of effective action in \( N = 2 \) case turned out to be analogous to structure effective action depending on chiral and antichiral superfields in \( N = 1 \) case. This low-energy effective action \( \Gamma[\phi, \bar{\phi}] \) is described by three objects: Kahlerian effective potential \( K(\phi, \bar{\phi}) \), chiral effective potential \( V^{(c)}_{eff}(\phi) \) and so called effective action of auxiliary fields [18,19,16]. The chiral effective potential depends only on \( \phi \) and can be called a holomorphic effective potential. We see that \( V^{(c)}_{eff}(\phi) \) in \( N = 1 \) case is analogous to holomorphic effective action \( \mathcal{F}(W) \) in \( N = 2 \) case. The first non-leading correction \( \mathcal{H}(W, \bar{W}) \) in \( N = 2 \) case is analogous to Kahlerian effective potential \( K(\phi, \bar{\phi}) \) in \( N = 1 \) case.

### 5 One-Loop Holomorphic and Non-Holomorphic Contributions to Low-Energy Effective Action for Hypermultiplets

We define the effective action \( \Gamma[V^{++}] \) corresponding to \( q \)-hypermultiplet coupled to external super Yang-Mills field by path integral
\[
e^{i\Gamma[V^{++}]} = \int Dq^{+} Dq^{+} e^{iS[q^{+}, q^{+}, 4V^{++}]}
\]
where
\[ S[\bar{q}^+, q^+, V^{++}] = -\int d\zeta (-4) du \bar{q}^+ \nabla^{++} q^+ \] (30)

One can show [13] that effective action for \(\omega\)-hypermultiplet \(\Gamma_\omega[V^{++}]\) can be written as follows
\[ \Gamma_\omega[V^{++}] = 2\Gamma[V^{++}] \] (31)

Besides, the one-loop ghosts contribution to effective action within background field method can also be expressed in terms of \(\Gamma[V^{++}]\) therefore the \(q\)-hypermultiplet in external gauge superfield is basic model for investigating one-loop effective action.

Formal calculating the above path integral leads to
\[ \Gamma[V^{++}] = iT \text{Tr} \ln (\nabla^{++}) \] (32)

Further we will consider only case of abelian theory.

Let us write again the classical action
\[ S[\bar{q}^+, q^+, V^{++}] = -\int d\zeta (-4) du \bar{q}^+ (D^{++} + iV^{++}) q^+ \] (33)

It is convenient to decompose the \(V^{++}\) into two pieces in the form
\[ V^{++} = V_0^{++} + V_1^{++} \] (34)

where \(V_0^{++}\) possesses the constant strength and can be expressed as follows [13,20]
\[ V_0^{++} = -(\theta^+)^2 \bar{w}_0 - (\bar{\theta}^+)^2 W_0 \]
\[ w_0 = \text{const}, \quad \bar{w}_0 = \text{const}. \] (35)

If \(V_0^{++} = 0\) we have \(W_0 = \bar{W}_0 = 0\) and we get massless hypermultiplet. If \(V_0^{++} \neq 0\) then the equation of motion
\[ (D^{++} + iV_0^{++}) q^+ = 0 \] (36)

leads to the equation
\[ (\Box + m^2) q^+ = 0 \] (37)

where \(m^2 = \bar{W}_0 W_0\). Therefore this case corresponds massive hypermultiplet. The structure of low-energy effective action depends crucially on \(m = 0\) or 0.

To evaluate the functional trace \(Tr ln(V^{++})\) one introduces the Green function \(G^{(1,1)}(1, 2)\) of the operator \(\nabla^{++}\) by the equation
\[ \nabla^{++} G^{(1,1)}(1, 2) = \delta^{3,1}_A(1, 2) \] (38)

where \(\delta^{3,1}_A(1, 2)\) is so called analytic \(\delta\)-function investigated in the refs. [10,11]. Let us define the analytic kernel \(Q^{(3,1)}(1, 2)\) as follows
\[ G_0^{(1,1)}(1, 2) = \int d\zeta (-4) du_3 G^{(1,1)}(1, 3) Q^{(3,1)}(3, 2) \] (39)

where \(G_0^{(1,1)}\) is Green function of free hypermultiplet. It can be taken corresponding to or massless or massive theory. The above equation allows to express the kernel in the form
\[ Q^{(3,1)}(1, 2) = s^{(3,1)}_A(1, 2) + iV^{++}(1) G_0^{(1,1)}(1, 2) \] (40)

The effective action can be rewritten in terms of \(Q^{(3,1)}\) up to unessential constant
\[ \Gamma[V^{++}] = iT \text{Tr} ln Q^{(3,1)} \] (41)
Taking into account the structure of $Q^{(3,1)} \sim 1 + iV^{++}G_0$ we see that $\Gamma[V^{++}]$ is well defined within perturbation theory

$$\Gamma[V^{++}] = \sum_{n=1}^{\infty} \Gamma_n[V^{++}] = i^2 \sum_{n=1}^{\infty} G_0 + \frac{1}{2} i^3 \sum_{n=1}^{\infty} G_0 + \cdots$$

Here the wave line corresponds to external field $V^{++}$, the term $\Gamma_n[V^{++}]$ is given by supergraph with $n$ external lines.

Let us discuss briefly the results of supergraphs calculations leading to holomorphic and non-holomorphic contributions to effective action [13].

(i) Massless theory. Taking into account the structure of free Green function $G_0^{(1,1)}$ one can show that $\Gamma_n[V^{++}] = 0$ at odd $n$ and holomorphic contribution is absent at all. The first non-holomorphic contribution is $\Gamma_4[V^{++}]$ and corresponds to four-leg supergraph. The straightforward calculation [13] lead to

$$\Gamma_4[V^{++}] = \frac{1}{(16\pi)^2 \Lambda^4} \int d^4x d^8\theta W^2 W^2$$

where $\Lambda$ is infrared cutoff. This result has a simple physical interpretation. Let us keep non-vanishing only electromagnetic field component of $W$, that is $F_{\mu\nu}$. Then

$$\Gamma_4[V^{++}] = \frac{1}{(16\pi)^2 \Lambda^4} \int d^4x ((F_{\mu\nu}F^{\mu\nu})^2 + (\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})^2)$$

where $\tilde{F}_{\mu\nu}$ is dual to $F_{\mu\nu}$. This type of nonlinear corrections to Maxwell Lagrangian was originally discovered by Heisenberg and Euler (see e.g. [21]). Therefore our $\Gamma_4[V^{++}]$ can be interpreted as the $N=2$ supersymmetric generation of Heisenberg-Euler effective Lagrangian.

(ii) Massive theory. In this case $V^{++} \neq 0$. We can act here by two ways.

1. The free propagator $G_0^{(1,1)}$ corresponds to massless theory and we consider $V^{++} = V_0^{++} + V_1^{++}$ for external lines in the supergraphs.

2. The free propagator $G_0^{(1,1)}$ corresponds to massive theory and we consider only $V_1^{++}$ for external lines.

Both these ways lead to the same results. Holomorphic effective action is obtained in the form

$$F(W) = \frac{1}{64\pi^2} W^2 \ln \frac{W^2}{\mu^2}$$

where $\mu$ is renormalization scale. Imposing the renormalization condition

$$F(W)|_{W^2=M^2} = 0$$

with some scale $M$ one gets finally

$$F(W) = -\frac{1}{64\pi^2} W^2 \ln \frac{W^2}{M^2}$$

This function coincides on its structure with holomorphic effective action obtained in ref. [9]. Of course, we have another coefficient since we considere another theory.

In massive case we also will have non-holomorphic contribution which is not written here.
6 Summary and Open Problems

Let us summarize the results
(i) The background field method for $N = 2$ super Yang-Mills theory has developed within harmonic superspace approach. Ghost structure of the theory is established. Path integral for manifestly $N = 2$ supersymmetric and gauge invariant effective action is presented. This path integral has standard quantum field theory form where quadratic part of total action defines the propagators and all other parts defines the vertices.
(ii) The general approach to effective action of $N = 2$ abelian superfield coupled to hypermultiplet has developed. This approach is based on formulation of $N = 2$ theories in harmonic superspace and guarantees manifest $N = 2$ supersymmetry on all stages of calculations. The approach under consideration allows to calculate straightforwardly both holomorphic and non-holomorphic contributions to low-energy effective action.

In conclusion we should like to point out some open problems associated with effective action of $N = 2$ supersymmetric field theories in harmonic superspace approach.
(i) Developing a proper-time technique providing a most efficient way of gauge-invariant calculations of effective action.
(ii) Calculation of terms in effective action depending on the derivatives of $W$ and $\bar{W}$.
(iii) Higher-loop contributions to effective action.
(iii) Quantum aspects of $N = 2$ supergravity in harmonic superspace approach.

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