Information-Theoretic Bounds for Multiround Function Computation in Collocated Networks

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Abstract—We study the limits of communication efficiency for function computation in collocated networks within the framework of multi-terminal block source coding theory. With the goal of computing a desired function of sources at a sink, nodes interact with each other through a sequence of error-free, network-wide broadcasts of finite-rate messages. For any function of independent sources, we derive a computable characterization of the rate region - in terms of single-letter information measures. We show that when computing symmetric functions of binary sources, the sink will inevitably learn certain additional information which is not demanded in computing the function. This conceptual understanding leads to new improved bounds for the minimum sum-rate. The new bounds are shown to be orderwise better than those based on cut-sets as the network scales. The scaling law of the minimum sum-rate is explored for different classes of symmetric functions and source parameters.

I. Introduction

Both wired and wireless data networks such as the Internet and the mobile ad hoc and wireless mesh networks have been designed with the goal of efficient data transfer as opposed to data processing. As a result, computation takes place only after all the relevant data is moved. Two-way interaction would be utilized to primarily improve the reliability of data-reproduction than data processing efficiency. However, to maximize the data processing efficiency, it may be necessary for nodes to interact bidirectionally in multiple rounds to perform distributed computations in the network. In this paper we attempt to formalize this intuition through a distributed function computation problem where data processing efficiency is measured in terms of the total number of bits exchanged per sample computed. Our objective is to study the fundamental limits of multiround function computation efficiency within a distributed source coding framework, involving block-coding asymptotics and vanishing probability of function-computation error, for “collocated” networks where broadcasted messages can be heard by all nodes. We derive an information-theoretic characterization of the set of feasible coding-rates and explore the benefit of multiround communication.

This problem was studied in [1] within a communication complexity framework where computation is required to be error-free. For collocated networks and random planar multihop networks, the scaling law of the maximum rate of computation with respect to a growing size of the network was derived for divisible functions and two subclasses of symmetric functions namely type-sensitive and type-threshold functions. This work was extended in [2] to multihop networks having a finite maximum degree. In [2] it was also shown that for any network, if a nonzero per-sample error probability was allowed, the computation of a type-sensitive function could be converted to that of a type-threshold function. In [3] a min-cut bound was developed for acyclic network topology and was shown to be tight for tree networks and divisible functions.

In [4], a function computation problem in a collocated network was posed within a distributed block source coding framework, under the assumption that conditioned on the desired function, the observations of source nodes are independent. An information-theoretic lower bound for the sum-rate-distortion function was derived. It was shown that if the desired function and the observation noises are Gaussian, the lower bound is tight and there is no advantage to be gained, in terms of sum-rate, by broadcasting messages, in comparison to sending messages through point-to-point links from source nodes to the sink where the function is desired to be computed. Multiround (interactive) function computation in a two-terminal network was studied in [5], [6] within a distributed block source coding framework.

The impact of transmission noise on function computation was considered in [7]–[9] but without a block coding rate, i.e., only one source sample is available at each node. A joint source-channel function computation problem over noninteracting multiple-access channels was studied in [10]. Our focus is on the block source coding aspects of function computation and we assume that message exchanges are error-free.

The present work studies a multiround function computation problem in a collocated network within a multi-terminal source coding framework described in Sec. II. Sensors observe discrete memoryless stationary sources taking values in finite alphabets. The goal is to compute a samplewise function at a sink with a probability which tends to one as the block-length tends to infinity. We derive a computable characterization of the rate region and the minimum sum-rate in terms of in-
formations (Sec. III). For computation of symmetric functions of binary sources, the sink is shown to inevitably obtain certain additional information, which is not demanded in computing the function (Sec. IV-A). This key observation is formalized under the vanishing block-error probability criterion (Lemma 2) and also the zero-error criterion (Lemma 3). This conceptual understanding leads to improved bounds for the minimum sum-rate (Sec. IV-B). These bounds are shown to be orderwise better than cut-set bounds as the size of the network grows. The scaling law of the minimum sum-rate is evaluated in different cases in Sec. IV-C.

II. Multiround Computation in Collocated Networks

Consider a network consisting of \( m \) source nodes numbered \( 1, \ldots, m \), and one (un-numbered) sink (node). Each source node observes a discrete memoryless stationary source taking values in a finite alphabet. The sink has no source samples. For each \( j \in [1, m] \), let \( X_j := (X_j(1), \ldots, X_j(n)) \) denote the \( n \) source samples which are available at node-\( j \). To isolate the impact of the structure of the desired function on the efficiency of computation, we assume sources are independent, i.e., for \( i = 1, \ldots, n \), \((X_i(1), X_i(2), \ldots, X_i(n)) \sim \text{iid} \). Let \( f : X_1 \times \cdots \times X_m \rightarrow Z \) be the function of interest at the sink and \( Z(i) := f(X_1(i), \ldots, X_m(i)) \). The tuple \( Z := (Z(1), \ldots, Z(n)) \), which denotes \( n \) samples of the samplewise function of all the sources, is desired to be computed at the sink.

The communication takes place over \( r \) rounds. In each round, source nodes broadcast messages according to the schedule \( 1, \ldots, m \). Each message depends on the source samples and all the previous messages which are available to the broadcasting node. Nodes are collocated, meaning that every broadcasted message is recovered without error at every node. After \( mr \) message broadcasts over \( r \) rounds, the sink computes the samplewise function based on all the messages.

Definition 1: An \( r \)-round distributed block source code for function computation in a collocated network with parameters \((r, n, |\mathcal{M}_1|, \ldots, |\mathcal{M}_m|)\) is the tuple \((e_1, \ldots, e_r, g)\) of \( t := mr \) block encoding functions \( e_1, \ldots, e_r \) and a block decoding function \( g \), of block-length \( n \), where for every \( j \in [1, r] \), \( k = (j \mod m) \) and

\[
e_j : (X_k)^n \times \bigotimes_{i=1}^{j-1} \mathcal{M}_i \rightarrow \mathcal{M}_j, \quad g : \bigotimes_{j=1}^{r} \mathcal{M}_j \rightarrow Z.
\]

The output of \( e_j \), denoted by \( M_j \), is called the \( j \)-th message, \( r \) is the number of rounds, and \( t \) is the total number of messages. The output of \( g \) is denoted by \( Z \). For each \( j \), \((1/n) \log_2 |\mathcal{M}_j|\) is the \( j \)-th block-coding rate (in bits per sample).

Remarks: (i) Each message \( M_j \) could be a null message (\( |\mathcal{M}_j| = 1 \)). By incorporating null messages, the multiround coding scheme described above subsumes all orders of messages transfers from \( m \) source nodes, and an \( r \)-round coding scheme subsumes an \( r' \)-round coding scheme if \( r' < r \). (ii) Since the information available to the sink is also available to all source nodes, there is no advantage in terms of sum-rate to allow the sink to send any message.

Definition 2: A rate tuple \( \mathbf{R} = (R_1, \ldots, R_t) \) is admissible for \( r \)-round function computation if, \( \forall \epsilon > 0 \), \( \exists \tilde{m}(\epsilon, t) \) such that \( \forall n > \tilde{m}(\epsilon, t) \), there exists an \( r \)-round distributed block source code with parameters \((r, n, |\mathcal{M}_1|, \ldots, |\mathcal{M}_m|)\) satisfying

\[
\forall j \in [1, r], \quad \frac{1}{n} \log_2 |\mathcal{M}_j| \leq R_j + \epsilon, \quad \mathbb{P}(Z \neq \tilde{Z}) \leq \epsilon.
\]

The set of all admissible rate tuples, denoted by \( \mathcal{R}_r \), is called the operational rate region for \( r \)-round function computation. The minimum sum-rate \( R_{\text{sum},r} \) is given by \( \min_{\mathcal{R}_r} \left( \sum_{j=1}^{r} R_j \right) \). Note that since each message could be a null message, if \( r' < r \), \( R_{\text{sum},r'} \geq R_{\text{sum},r} \) holds. The goal of this work is to obtain a single-letter characterization of the rate region (a computable characterization independent of block-length \( n \)) to study the scaling behavior of \( R_{\text{sum},r} \), and to investigate the benefit of multiround function computation.

III. Rate Region

The rate region for \( r \)-round function computation for \( m \) independent sources can be characterized by Theorem 1 in terms of single-letter mutual information quantities involving auxiliary random variables satisfying Markov chain and conditional entropy constraints.

Theorem 1:

\[
\mathcal{R}_r = \{ \mathbf{R} : \exists U', \text{s.t. } \forall j \in [1, r], \text{ and } k = (j \mod m), \quad R_j \geq I(X_k; U_j | U'^{-1}), U_j - (U'^{-1}, X_k) - (X_k^{j-1}, X_k^{j+1}), \quad H(f(X_k^r) | U') = 0 \}, \quad (3.1)
\]

where \( U' \) are auxiliary random variables taking value in finite alphabets. Cardinality bounds on the alphabets of the auxiliary random variables can be derived using the Carathéodory theorem but are omitted.

The proof of achievability follows from standard random conditional coding arguments and is briefly outlined as follows. For the \( j \)-th message, \( j = 1, \ldots, t \), node-\( k = (j \mod m) \) quantizes \( X_k \) into \( U_j \) with \( U'^{-1} \) as side information, which is available at every node, so that every node can reproduce \( U_j \). After all the message transfers, the sink produces \( Z \) based on \( U' \). The constraints in (3.1) ensure that \( \mathbb{P}(Z = Z) \rightarrow 1 \) as \( n \rightarrow \infty \).

The (weak) converse, given in Appendix A, is proved using standard information inequalities, suitably defining auxiliary random variables, and using time-sharing arguments. Specifically, \( U_1 := (Q, U_1(Q)) \), \( Q \sim \text{Uniform}[1, n] \) independent of \( X_m^r \), for all \( q \in [1, n], U_1(q) = (M_1, X_m^1, \ldots, X_m^q - 1) \), and for all \( j \in [2, t], U_j := M_j \).

By adding all the rate inequalities in (3.1) and enforcing all the constraints, we have the following characterization of the minimum sum-rate.

Corollary 1:

\[
R_{\text{sum},r} = \min_U I(X_m^r; U'), \quad (3.2)
\]

where \( U' \) are subject to all the Markov chain and conditional entropy constraints in (3.1).
The Markov chain and conditional entropy constraints of (3.1) imply a key structural property which $U'$ need to satisfy. This property is described below in Lemma 1. This lemma provides a bridge between certain fundamental concepts which have been studied in the communication complexity literature [11] and distributed source coding theory. In order to state the lemma, we need to introduce some terminology used in the communication complexity literature [11]. A subset $\mathcal{A} \subseteq \bigotimes_{i=1}^{m} X_i$ is called a rectangle if for every $i \in [1,m]$, there exists $S_i \subseteq X_i$ such that $\mathcal{A} = \bigotimes_{i=1}^{m} S_i$. A set $\mathcal{A}$ is called $f$-monochromatic if the function $f$ is constant on $\mathcal{A}$. The support-set of a probability mass function $p$ is the set over which it is strictly positive and is denoted by supp($p$).

**Lemma 1:** Let $U'$ be any set of auxiliary random variables satisfying the Markov chain and conditional entropy constraints in (3.1). If supp($p_{X'}$) = $\bigotimes_{i=1}^{m} X_i$, then for any realization $u'$ of $U'$, $\mathcal{A}(u') := \{x^m|p_{X'|U'}(x^m,u') > 0\}$ is an $f$-monochromatic rectangle in $\bigotimes_{i=1}^{m} X_i$.

**Proof:** The Markov chains in (3.1) induce the following factorization of the joint probability.

$$p_{X'|U'}(x^m,u') = p_{X}(x^m)p_{U|X}(u|X_1)x_1p_{U|X_1,U_1}(u_2|x_2,u_1)\ldots$$

where $\phi_i$ is the product of all the factors having conditioning on $x_i$. For each $i \in [1,m]$, let $S_i(u') := \{x_i | \phi_i(x_i,u') > 0\}$.

Since $\forall x_i$, $p_{X_i}(x_i) > 0$, we have $\mathcal{A}(u') = \bigotimes_{i=1}^{m} S_i(u')$. Since $H(f(X^m)|U') = 0$ holds, $\mathcal{A}(u')$ is $f$-monochromatic.

**IV. Computing Symmetric Functions of Binary Sources**

In this section, we focus on the problem of computing symmetric functions of $m$ nontrivial Bernoulli sources: $\forall i \in [1,m]$, $X_i = \{0,1\}$, $p_{X_i}(1) = \pi_i$, where $\pi_i \in (0,1)$. Symmetric functions are invariant to any permutation of their arguments. A symmetric function $f(X^m)$ of binary sources is completely determined by the (integer) sum of the sources $S := \sum_{i=1}^{m} X_i$. In other words, $\exists f' : [0,m] \rightarrow \mathcal{Z}$, such that $f'(s) = f(X^m)$.

**Definition 3:** Given a function $f' : [0,m] \rightarrow \mathcal{Z}$, an interval $[a,b] \subseteq [0,m]$ is a maximal $f'$-monochromatic interval if (i) it is $f'$-monochromatic and (ii) it is not a proper subset of an $f'$-monochromatic interval.

The collection of all the maximal $f'$-monochromatic intervals can be constructed as follows. First, consider all the inverse images $f'(s)^{-1} : \mathcal{Z} \rightarrow [0,m]$. Each inverse image can be written as a disjoint union of nonadjacent intervals. The collection of all such intervals from all inverse images, denoted by $\{[a_i,b_i]\}_{i=1}^{\infty}$, forms the collection of all the maximal $f'$-monochromatic intervals. Note that they also form a partition of $[0,m]$. Without loss of generality, we assume that these intervals are ordered so that $a_1 = 0, b_{\max} = m$ and $\forall v \in [2, v_{\max}], a_v = b_{v-1} + 1$.

A. Sink learns more than the result of function computation

Note that if $f(S) = z$ then $S \in f^{-1}(z)$ which is, in general, a disjoint union of several maximal $f'$-monochromatic intervals. Thus, if the sink can successfully compute the function $f'(S)$, one may expect that the sink can only estimate the value of $S$ as belonging to the union of several intervals. Somewhat surprisingly, however, it turns out that due to the structure of the multiround code, the sink will, in fact, be able to identify a single maximal monochromatic interval to which $S$ belongs as opposed to the union of several intervals. More surprisingly, the sink will be able to correctly identify the source-values at certain nodes. Lemma 2 formalizes this unexpected property and plays a central role in proving Theorem 2(i).

**Lemma 2:** Let $f(x^m)$ be a symmetric function of binary variables and $\{(a_i,b_i)\}_{i=1}^{\infty}$ the collection of all the maximal $f'$-monochromatic intervals associated with $f$. Let $X^m$ be $m$ independent nontrivial Bernoulli random variables and $U'$ auxiliary random variables which satisfy the Markov chain and conditional entropy constraints in (3.1). Then for any $u' \in \text{supp}(p_{U'})$, the following conditions hold.

(i) There exists $v(u') \in [1,v_{\max}]$ such that

$$\mathbb{P}(S \in [a_{v(u')}, b_{v(u')}])|U' = u'\rangle = 1.$$

(ii) There exist $K_i(u') \subseteq [1,m]$ and $K_0(u') \subseteq [1,m]$ such that $K_i(u') \cap K_0(u') = \{\}$, $|K_i(u')| \geq a_{v(u')}$, $|K_0(u')| \geq m - b_{v(u')}$, and

$$\mathbb{P}(\forall i \in K_i(u'), \forall u' \in K_0(u'), X_i = 1, X_v = 0) |U' = u'\rangle = 1.$$

**Proof:** Due to Lemma 1, $\mathcal{A}(u') = \{x^m|p_{X'|U'}(x^m,u') > 0\}$ is an $f$-monochromatic rectangle, which can be expressed as $\bigotimes_{i=1}^{m} S_i(u')$, where $S_i(u')$ is either $[0] = [0]$ or $[1] = [1]$. Let $K_i(u') := \{i | S_i(u') = [0]\}$ and $K_0(u') := \{i | S_i(u') = [1]\}$. Let $\alpha(u') := |K_i(u')|$ and $\beta(u') := m - |K_0(u')|$. It can be shown that the projection of $\mathcal{A}(u')$ under the linear transformation $s = \sum_{i=1}^{m} x_i$ given by $\mathcal{A}(u') := \{(s_{i=1}^{m} x_i) | p_{X'|U'}(x^m,u') > 0\}$ is an $f'$-monochromatic interval $[\alpha(u'), \beta(u')]$. Since $\forall (a_i,b_i)_{i=1}^{\infty}$ is the collection of all the maximal $f'$-monochromatic intervals, $\exists v(u') \in [1,v_{\max}]$ such that $\mathcal{A}(u') \subseteq [a_{v(u')}, b_{v(u')}].$ Therefore (i) holds. Since $\mathcal{A}(u') = [a_{v(u')}, \beta(u')] \subseteq [a_{v(u')}, b_{v(u')}],$ we have $\alpha(u') \geq a_{v(u')}$ and $\beta(u') \leq b_{v(u')}$. Therefore (ii) holds.

Although learning that $f'(S) = z$ is equivalent to learning that $S \in f'^{-1}(z)$, which is generally a union of several intervals, Lemma 2(i) shows that the structure of block source coding for function computation in collocated networks is such that the sink will inevitably learn the exact interval $[a_{v(u')}, b_{v(u')}]$ in which $S$ resides even though this information is not demanded in computing $f'(S)$. Similarly, Lemma 2(ii) shows that although learning that $S \in [a_{v(u')}, b_{v(u')}]$ is equivalent to learning that there exist $a_{v(u')}$ nodes observing ones and $(m - b_{v(u')})$ nodes observing zeros, the sink will inevitably learn the identities of these nodes.

Lemma 3 describes a property of the single-letter characterization of the rate region. It does not, as such, have a direct operational significance. Hence the conclusions of the previous paragraph can be only accepted as intuitive interpretations. If, however, the block-error probability criterion $\mathbb{P}(\hat{Z} \neq Z) \leq \epsilon$ in Definition 2 is replaced by the zero-error criterion $\mathbb{P}(\hat{Z} \neq Z) = 0$ as in [1], we obtain Lemma 3 which holds for every sample realization and provides an operational significance to the results suggested by Lemma 2.
Lemma 3: Let \( f(x^m) \) be a symmetric function of binary variables and \([a_v, b_v]\) the collection of all the maximal \( f^*\)-monochromatic intervals associated with \( f \). Let \( X^m \) be \( m \) independent nontrivial Bernoulli sources. For any \( r\)-round, block-length \( n \) code\(^4\) for computing \( f \) in a collocated network, if \( \mathbb{P}(\hat{Z} \neq Z) = 0 \), then given all the messages \( M^i \), for every sample \( i \in [1, n] \), the following conditions hold. (i) There exists \( v(M^i, i) \in [1, v_{\text{max}}] \) such that \( S(i) := (\sum_{j=1}^m X_j(i)) \in \{v(M^i, b_{v(M^i)}), \ldots, v(M^i, a_{v(M^i)})\} \). (ii) There exist \( \mathcal{K}_i(M^i, i) \subseteq [1, m] \) and \( \mathcal{K}_0(M^i, i) \subseteq [1, m] \) such that: \( \mathcal{K}_i(M^i, i) \cap \mathcal{K}_0(M^i, i) = \{1, \mathcal{K}_i(M^i, i) \geq a_{v(M^i)}, \mathcal{K}_0(M^i, i) \geq m - b_{v(M^i)}, \text{and } \forall j \in \mathcal{K}_i(M^i, i) \implies \mathcal{K}_j(M^j, i) \) is a singleton.\(^5\) The proof of Lemma 3 given in Appendix B is similar in structure to those of Lemmas 1 and 2.

Example: (Parity function) Let \( f(x^m) = \left( \bigoplus_{i=1}^m x_i \right) \) be the Boolean XOR function of \( m \) binary variables. Then \( f^r-1(0) = \bigcup_{i=1}^m \{2i\} \) and \( f^r-1(1) = \bigcup_{i=1}^m \{2i - 1\} \). Thus for all \( v \in [1, (m+1)] \), \( a_v = b_v = v - 1 \), and all the \( f^*\)-monochromatic intervals are singletons. For every sample \( i \in [1, n] \), if \( f \) is computed with zero error, Lemma 3(i) shows that the sink ends up knowing \( S(i) \) exactly, because every interval is now a singleton. In addition, Lemma 3(ii) shows that the sink will also identify \( S(i) \) source nodes which observe ones and \( (m - S(i)) \) source nodes which observe zeros. Therefore the sink essentially needs all the raw data from all the source nodes in order to compute the parity function in a collocated network.

B. Bounds for minimum sum-rate

Returning to the block-error probability criterion, Lemma 2 leads to the following bounds for \( R_{\text{sum}, r} \) when \( X^m \) ~ iid Bernoulli(\( p \)), that is, \( \forall i \in [1, m], \pi_i = p \).

Theorem 2: Let \( f(x^m) \) be a symmetric function of binary variables and \([a_v, b_v]\) the collection of all the maximal \( f^*\)-monochromatic intervals associated with \( f \). If \( X^m \) ~ iid Bernoulli(\( p \)), \( p \in (0, 1) \), then for all \( r \in \mathbb{Z}^+ \), (i) \( R_{\text{sum}, r} \geq mh(p) - \sum_{v=1, a_v \neq b_v}^{v_{\text{max}}} (b_v - a_v) \frac{P(S \leq b_v)(S > b_v)}{(b_v - a_v)} \frac{P(S \geq a_v, b_v)}{P(S \geq a_v, b_v)} \), (ii) \( R_{\text{sum}, r} \geq mh(p) \max_{v=1, v_{\text{max}}} \left[ \frac{P(S \leq b_v)(S > b_v)}{P(S \geq a_v, b_v)} \right] \), (iii) \( R_{\text{sum}, r} \leq h(p) \sum_{v=1}^{v_{\text{max}}} \frac{a_v + m - b_v}{1 - p} \frac{P(S \geq a_v, b_v)}{P(S \geq a_v, b_v)} \), where \( h(\cdot) \) is the binary entropy function.

Remark: The minimum sum-rate for “data downloading” where all source samples are to be reproduced at the sink is \( H(X^m) = mh(p) \). Theorem 2(ii) explicitly bounds the efficiency of multiround broadcasting relative to data downloading. Since (ii) is proved by relaxing the lower bound in (i), the right side of (ii) is not greater than that of (i).

C. Scaling law of minimum sum-rate

Consider a sequence of problems, where in the \( m \)-th problem, \( m \in \mathbb{Z}^+ \), \( m \) source nodes observe Bernoulli(\( p_m \)) source samples \( X^m \) which are iid both across samples and across nodes and \( f_m \) is the desired function. Let \( R_{\text{sum}, r, m} \) be the minimum sum-rate of the \( m \)-th problem. The scaling law of \( R_{\text{sum}, r, m} \) with respect to \( m \) is explored in the following cases.

Case 1: We need to use the following fact.

Fact 1: For any \( \epsilon \in (0, 1/2) \), if \( m \geq \max_{1/2, \epsilon} \left[ \frac{P(S \leq b_v)(S > b_v)}{P(S \leq a_v, b_v)} \right] / \epsilon(1 - \epsilon) \), then \( \exists v \in [1, v_{\text{max}}] \) such that \( P(S \in [a_v, b_v]) > 1 - 2\epsilon \).

Proof: For any \( \epsilon \in (0, 1/2) \), if \( m \geq \max_{1/2, \epsilon} \left[ \frac{P(S \leq b_v)(S > b_v)}{P(S \leq a_v, b_v)} \right] / \epsilon(1 - \epsilon) \), then \( \forall v, P(S \leq b_v) \geq (1 - \epsilon)m(1 - \epsilon) \), which in turn implies that \( \epsilon(1 - \epsilon)m \geq \min[v(S \leq b_v) - 1 - \epsilon] \geq \epsilon(1 - \epsilon)m \). Then due to Theorem 2(ii), \( R_{\text{sum}, r, m} \geq \epsilon(1 - \epsilon)m(1 - \epsilon) \), which implies that \( R_{\text{sum}, r, m} = \Theta(m(1 - \epsilon)) \) and data downloading is orderwise optimal. Conversely, if \( R_{\text{sum}, r, m} = \Theta(m(1 - \epsilon)) \), then due to Fact 1, \( m \geq \max_{1/2, \epsilon} \left[ \frac{P(S \leq b_v)(S > b_v)}{P(S \leq a_v, b_v)} \right] / \epsilon(1 - \epsilon) \). Therefore there exists a vanishing sequence \( \epsilon_m \) such that \( m \geq \max_{1/2, \epsilon_m} \left[ \frac{P(S \leq b_v)(S > b_v)}{P(S \leq a_v, b_v)} \right] / \epsilon_m(1 - \epsilon_m) \) holds for the \( m \)-th problem. Due to Fact 1, there exists a sequence of maximal \( f_m^* \)-monochromatic intervals \( \left[ a_v, b_v \right] \) such that \( P(S \leq b_v) \geq m(1 - \epsilon_m) \). In other words, multiround computation of symmetric functions of iid binary sources in collocated networks is orderwise more efficient than data downloading only if each sample of \( f_m \) is determined with a probability which tends to one as \( m \to \infty \).

Case 2: \((p_m = 1/2)\) For any symmetric function of iid Bernoulli(1/2) sources, let \( \rho = m - \sum_{v=1}^{v_{\text{max}}} (b_v - a_v) \frac{P(S \leq a_v, b_v)}{P(S \geq a_v, b_v)} \). Theorem 2(ii) and (iii) imply that \( \rho \leq R_{\text{sum}, r, m} < R_{\text{sum}, 1, m} \leq 2p \). This shows that multiround computation can at most halve the minimum sum-rate of one-round computation. Since \( \rho \) can be easily computed using the binomial distribution, \( R_{\text{sum}, r, m} \) can be easily evaluated within a factor of 2 for all \( r \in \mathbb{Z}^+ \).

Case 3: \((p_m = 1/2)\) A sequence of symmetric functions \( f_m \) of binary variables is type-sensitive if \( \exists \gamma \in (0, 1) \) and \( \bar{m} \in \mathbb{Z}^+ \) such that \( \forall m > \bar{m} \), for every \( f_m^* \)-monochromatic interval \( [a_v, b_v] \), \( (b_v - a_v) \gamma \bar{m} \) (defined in [1], adapted to our notation). For example, the sum, mode, and parity functions are type-sensitive. For iid Bernoulli(1/2) sources, it can be shown that \( R_{\text{sum}, r, m} = \Theta(m) \) by applying Theorem 2(ii). Remark: For the zero-error criterion, the minimum worst-case sum-rate is also \( \Theta(m) \) [1].

Case 4: \((p_m = 1/2)\) A sequence of symmetric functions \( f_m \) of binary variables is type-threshold, if there exist \( \theta_0, \theta_1 \in \mathbb{N} \) such that \( (\theta_1, m - \theta_0) \) is \( f_m^* \)-monochromatic for every \( m \geq \theta_0 + \theta_1 \) (defined in [1], adapted to our notation).

\(^4\)The results of Lemma 3 hold for not only the multiround block coding strategy described in Definition 1 but also for the class of collision-free coding strategies defined in [1].

\(^5\)We cannot, however, let nodes send nothing and set the output of the sink to be the determined function value because then, for each \( m \) the probability of block error will tend to one with increasing block-length violating Definition 2.
For example, the minimum and maximum functions are type-
threshold. (i) If $p_m = p$, then $\Pr(S \in [\theta_1, m - \theta_0]) \to 1$ 
exponentially fast as $m \to \infty$. By applying Theorem 2(i) and (iii), we have $R_{\text{sum},m} = \Theta(1)$, which is orderwise less 
than $H(mn(p_m)) = \Theta(m)$. (ii) If $p_m = 1/m$ and $f_m(x^m) = 
\max_{i=1}^m x_i$, then $a_1 = b_1 = 0$, $a_2 = 1$, $b_2 = m$, and 
$\lim_{m \to \infty} \Pr(S \leq 0) = 1 - \frac{1}{e^2}$, due to Theorem 2(ii), $R_{\text{sum},m} = \Theta(mn(p_m)) = \Theta(\log m)$. 
Remark: For the zero-error criterion, the minimum worst-case sum-rate is $\Theta(\log m)$ [1].

D. Comparison to cut-set bounds

How do the bounds given in Sec. IV-B behave in comparison to 
bounds based on cut-sets? We will show that in some cases 
they are orderwise tighter than cut-set bounds and in some 
cases they coincide with them.

For any subset $S \subseteq [1,m]$, let $S^c := [1,m] \setminus S$. We 
can formulate a two-terminal interactive function computation 
problem with alternating message transfers [6] by regarding the 
set of source nodes in $S$ as supernode-$S$ and the other 
source nodes and the sink as supernode-$S^c$. The sources $[X_i]_{i \in S}$ 
and $[X_i]_{i \in S^c}$ are available to supernode-$S$ and supernode-$S^c$ 
respectively and the function $f(X^m)$ is to be computed at 
supernode-$S^c$. Let $R_{S,S^c}$ denote the directed sum-rate 
region of the two-terminal problem, which is the set of tuples 
$(R_{S,S^c}, R_{S^c,S})$ such that $R_{S,S^c}$ and $R_{S^c,S}$ are admissible 
directed sum-rates from $S$ to $S^c$ and from $S^c$ to $S$ 
respectively, for two-terminal interactive function computation with 
terminal alternating messages where $t \leq 2mr$ is the minimum number of 
messages needed in the two-terminal problem to simulate the 
multiround code.

For any multi-round code for a collocated network, for every 
i \in [1,m], let $r_i$ denote the sum-rate of the messages broadcasted 
by node-$i$. This code can be mapped into a two-terminal interaction code for the two-terminal problem described above, 
which generates the same computation result. The directed 
sum-rate tuples is $(R_{S,S^c}, R_{S^c,S}) = (\sum_{i \in S} r_i, \sum_{i \in S^c} r_i)$, which 
should belong to the directed sum-rate region of this 
problem. This leads to the following cut-set bound.

**Theorem 3: (cut-set bound)** For all $r \in \mathbb{Z}^+$,

$$R_{\text{sum},r} \geq R_{\text{cut}} := \min_{\forall S \subseteq [1,m], \sum_{i \in S} r_i \geq 0} \sum_{i=1}^m r_i. \quad (4.3)$$

One could also consider a different type of cut-set bound:

$$R_{\text{sum},r} \geq R'_{\text{cut}} := \max_{S \subseteq [1,m]} \sum_{i=1}^m r_i \min_{\forall S \subseteq [1,m], \sum_{i \in S} r_i \geq 0} \sum_{i=1}^m r_i$$

where $R'_{\text{cut}}$ is called the bi-directional minimum sum-rate 
of the two-terminal problem given by the cut-set $S$. Note that 
$R_{\text{cut}} \leq R'_{\text{cut}}$. In fact, $R'_{\text{cut}}$ can be orderwise looser than $R_{\text{cut}}$. For example, for the problem in Prop. 1, $R_{\text{cut}} \geq m$ and $R'_{\text{cut}} = 1$.

**Proposition 1:** If $X^m \sim \text{iid Bernoulli}(1/2)$ and $f_m(x^m) = \left( \bigoplus_{i=1}^m x_i \right)$, then $R_{\text{cut}} \geq m$.

**Proof:** For any $i \in [1,m]$, if $S = \{i\}$, by applying the 
cut-set bound for the two-terminal interactive function 
computation problem [6, Corollary 1(ii)], we have $\sum_{i \in j \in S^c} r_j \in R_{S,S^c}, r_i \geq H(f_m(X^m) | [X_i]_{i \in S^c}) = H(X_i) = 1$. Adding the $m$ 
inequalities $r_i \geq 1$ for all $i \in [1,m]$. We have $R_{\text{cut}} \geq m$.

Since $m$ is an admissible sum-rate for the problem stated in 
Prop. 1, the cut-set bound is tight. Note that Theorem 2(i) also 
gives the same bound $R_{\text{sum},m} \geq m$. However, in the following 
case, the cut-set bound is orderwise loose.

**Proposition 2:** If $X^m \sim \text{iid Bernoulli}(1/2)$ and $f_m(x^m) = \min_{i=1}^m x_i$, then $R_{\text{cut}} \leq 3m(2^m/2)$.

**Proof:** It is sufficient to show that $\forall i \in [1,m], r_i = 3/(2^m)$ is feasible for the minimization problem in [4.3], which requires showing $\forall S \subseteq [1,m], \sum_{i \in S} r_i, \sum_{i \in S^c} r_i = (3|S|/(2^m), 3|S^c|/(2^m)) \in R_{S,S^c}$. Let $Y_S := (\min_{i \in S} X_i(k))_{k=1}^n \sim \text{id Bernoulli}(1/(2^S))$ and $Y_{S^c} := (\min_{i \in S^c} X_i(k))_{k=1}^n \sim \text{id Bernoulli}(1/(2^{S^c}))$. The computation of the two-terminal problem can be performed by the following two schemes. (i) (One-message scheme) Supernode-$S$ sends $Y_S$ to supernode-$S^c$ at the rate $H(Y_S)$. Therefore $(H(Y_S), 0) \in R_{S,S^c}$, which implies that $R^1 := H(Y_S, \infty) \times [0, \infty) \subseteq R_{S,S^c}$. (ii) (Two-message scheme) Supernode-$S^c$ sends $Y_{S^c}$ to supernode-$S$ at the rate $H(Y_{S^c})$. Then supernode-$S$ computes the 
samplewise minimum of $Y_S$ and $Y_{S^c}$, and sends it back to 
supernode-$S^c$ with $Y_{S^c}$ as side information available to 
both supernodes, at the rate $H(\min(Y_S, Y_{S^c}))$. Therefore $(H(\min(Y_S, Y_{S^c})), H(Y_{S^c})) \in R_{S,S^c}$, which implies that $R^2 := H(\min(Y_S, Y_{S^c}), 0), \infty) \times H(Y_{S^c}, \infty) \subseteq R_{S,S^c}$. By evaluating the entropies, it can be shown that, if $|S| \geq m/2$, then $\sum_{i \in S} r_i, \sum_{i \in S^c} r_i \in R^1$, otherwise $\sum_{i \in S} r_i, \sum_{i \in S^c} r_i \in R^2$.

The detailed steps only for ourselves: (will be deleted in the final draft) If $|S| \geq m/2$, then

$$H(Y_S) = h\left(\frac{1}{2^{|S|}}\right) \leq \frac{\log_2(2^{|S|})}{2^{|S|}} \leq \frac{\log_2(2^{|S|})}{2^{|S|}} \leq \frac{3|S|}{2m/2}.$$ 

where the first inequality is because $h(p) \leq p \log_2(e/p)$ 
and the second inequality is because $e < 4^{|S|}$. Therefore 
$\sum_{i \in S} r_i, \sum_{i \in S^c} r_i = (3|S|/(2^m/2), 3|S^c|/(2^m/2)) \in R^1$. If $|S| < m/2$, then $H(Y_S) \leq 3|S^c|/(2^m/2)$. If $1 \leq |S| < m/2, then

$$H(\min(Y_S, Y_{S^c})) = \frac{1}{2^{|S|}} \leq \frac{3|S|}{2m/2}.$$ 

Otherwise ($|S| = 0$), $H(\min(Y_S, Y_{S^c})) = 0 = 3|S^c|/(2^m/2)$. Therefore $\sum_{i \in S} r_i, \sum_{i \in S^c} r_i = (3|S^c|/(2^m/2), 3|S^c|/(2^m/2)) \in R^2$.

Since the problem considered in Prop. 2 is a special case of 
Case 4(i), due to Theorem 2 we have $R_{\text{sum},r,m} = \Theta(1)$. 
Therefore the exponentially vanishing cut-set bound given by 
Theorem 3 is orderwise loose.

V. CONCLUDING REMARKS

We studied function computation in collocated networks using 
a distributed block source coding framework. We showed that in computing symmetric functions of binary sources, 
the sink will inevitably obtain certain additional information 
which is not part of the problem requirement. Leveraging this 
conceptual understanding we developed bounds for the
minimum sum-rate and showed that they can be better than cut-set bounds by orders of magnitude. Directions for future work include characterizing the scaling law of the minimum sum-rate for large source alphabets and general multihop networks.

Appendix A

Converse proof of Theorem 1

Suppose a rate tuple $(R_1, \ldots, R_t)$ is admissible for $r$-round function computation. By Definition 2 $\forall \epsilon > 0, \exists \delta(\epsilon, t)$, such that $\forall n > \delta(\epsilon, t)$, there exists an $r$-round distributed source code satisfying $\forall j \in [1, t], (1/n) \log_2 |M_j| < R_j + \epsilon$ and $P(Z \neq \tilde{Z}) < \epsilon$. Define auxiliary random variables as follows: $\forall k \in [1, n], U_1(k) := \{M_1, X^m(k-\bar{t})\}$ and for $\forall i \in [2, t], U_i := M_i$.

Information inequalities: For the first rate, we have

\[
n(R_1 + \epsilon) \geq H(M_1) \\
\geq I(X_1; M_1|X^m_2) \\
= H(X_1|M_1, X^m_2) - H(X_1|M_1) \\
= \sum_{k=1}^{n} (H(X_1(k)) - H(X_1(k)|X_1(k-\bar{t}), M_1, X^m_2)) \\
\geq \sum_{k=1}^{n} (H(X_1(k)) - H(X_1(k)|M_1, X^m(k-\bar{t}))) \\
= \sum_{k=1}^{n} I(X_1(k); U_1(k)). \quad (A.1)
\]

For the $i$-th rate, $i \in [2, t]$, let $j = (i \mod m)$.

\[
n(R_i + \epsilon) \geq H(M_i) \\
\geq I(X_i; M_i|M^{i-1}, X^{i-1}, X^m_{j+1}) \\
= H(X_i|M^{i-1}, X^{i-1}, X^m_{j+1}) - H(X_i|M^i, X^{i-1}, X^m_{j+1}) \\
= \sum_{k=1}^{m} (H(X_i(k)|X(k-\bar{t}), M^{i-1}, X^{i-1}, X^m_{j+1}) \cdot H(X_i(k)|X(k-\bar{t}), M^{i-1}, X^{i-1}, X^m_{j+1})) \\
= \sum_{k=1}^{n} (H(X_i(k))X^m(k-\bar{t}), M^{i-1}) - H(X_i(k)|X(k-\bar{t}), M^i, X^{i-1}, X^m_{j+1})) \\
\geq \sum_{k=1}^{n} (H(X_i(k))X^m(k-\bar{t}), M^{i-1}) - H(X_i(k)|X^m(k-\bar{t}), M^i)) \\
= \sum_{k=1}^{n} I(X_i(k); U_i|U_1(k), U^2_{j-1}). \quad (A.2)
\]

Due to the condition $P(Z \neq \tilde{Z}) \leq \epsilon$ and the Fano’s inequality [12], we have

\[
h_2(\epsilon) + \epsilon \log |Z|^n - 1) \\
\geq H(Z|M') \\
= \sum_{k=1}^{n} H(Z(k)|Z(k-\bar{t}), M') \\
\geq \sum_{k=1}^{n} H(Z(k)|Z(k-\bar{t}), M', X^m(k-\bar{t})) \\
= \sum_{k=1}^{n} H(f(X^m(k-\bar{t})))U_1(k), U^2_{j-1}). \quad (A.3)
\]

Step (b) is because $Z(k-\bar{t})$ is a function of $X^m(k-\bar{t})$.

Timesharing: We introduce a timesharing random variable $Q$ taking values in $[1, n]$ equally likely, which is independent of all the other random variables. For each $i \in [1, m]$, define $X_i := X_i(Q)$, and $U_i := (U_1(Q), Q)$. (A.1) becomes

\[
R_1 + \epsilon \geq \frac{1}{n} \sum_{k=1}^{n} I(X_1(k); U_1(k)) \\
= I(X_1(Q); U_1(Q)|Q) \\
\geq I(X_1(Q); U_1(Q), Q) \\
= I(X_1; U_1), \quad (A.4)
\]

where step (c) is because $I(X_1(Q); Q) = 0$, which is in turn implied by: (1), $Q$ is independent of all the other random variables, and (2), the distribution of $X_1(Q) \sim p_{X_1}$ does not depend on $Q$. Similarly, (A.2) and (A.3) become

\[
\forall i \in [2, t], R_i + \epsilon \geq I(X_i; U_i|U^{i-1}), \quad (A.5)
\]

\[
\frac{1}{n} h_2(\epsilon) + \epsilon \log |Z| \geq H(f(X^m)|U^i), \quad (A.6)
\]

Concerning the Markov chains, one can verify that $U_1(k) - X_1(k) - X^m_2(k)$ and $U_i - (U_1(k), U^2_{j-1}, X_i(k)) - (X^{i-1}, X^m_{j+1})$ form Markov chains for each $i \in [2, t]$, $k \in [1, n]$ and $j = (i \mod m)$, which imply that $U_i - (U^{i-1}, X_i) - (X^{i-1}, X^m_{j+1})$ forms a Markov chain for each $i \in [1, t]$ and $j = (i \mod m)$.

Cardinality bounds: The cardinalities of $\mathcal{U}$ can be bounded as for the rate region of the two-terminal interaction problem [6]. But they are omitted here.

Taking limits: As in [6], we consider a sequence $\{\epsilon_i\}$ which goes to zero as $l$ tends to infinity. Due to the continuity of conditional mutual information and conditional entropy measures, all the $\epsilon_i$‘s in (A.4) and (A.6) vanish and thus $(R_1, \ldots, R_t) \in \mathcal{R}_r$.

Appendix B

Proof of Lemma 3

For any $r$-round block-length $n$ code $(\epsilon_i’, g)$ for computing $f$ without any error in a collocated network, for every realization $M'$ of messages $M'$ and for every sample $i \in [1, n]$, let $\mathcal{A}(M', i) := \{X^m(i)|P_{X^m(i)|M'}(X^m(i), M') > 0\}$, i.e., the set of all possible $i$-th source samples that are
consistent with messages $m'$. We first show that $\mathcal{A}(m',i)$ is an $f$-monochromatic rectangle in $\bigotimes_{i=1}^{m} X_i$, which is similar to the statement of Lemma 1. Due to Definition 1 $\mathcal{A}(m',i) = (x^{(i)} m) \exists x^{(1) i}, \ldots, x^{(i - 1) i}, x^{(i + 1) i}, \ldots, x^{(n) i}$ such that $\forall j \in [1, i], k = (j \bmod m), e(x^{(i)} k) = m_j$, where $x_i$ stands for $(x_1(1), \ldots, x_n(i))$. For every $k \in [1, m]$, let $\mathcal{S}_k(m',i) := (x^{(i)} m) x_1(1), \ldots, x_1(i - 1), x_1(i + 1), \ldots, x_n(i)$ such that $\forall \rho \in [0, r - 1], j = k + \rho m, e(x^{(i)} k) = m_j$. Since $\mathcal{S}_k(m',i)$ contains all the constraints in $\mathcal{A}(m',i)$ related to source node $k$, we have $\mathcal{A}(m',i) = \bigotimes_{i=1}^{m} \mathcal{S}_k(m',i)$. Therefore $\mathcal{A}(m',i)$ is a rectangle in $\bigotimes_{i=1}^{m} X_i$. Since the code computes $f$ without any error for any inputs, $\mathcal{A}(m',i)$ is $f$-monochromatic.

The rest steps are parallel to the proof of Lemma 2. For all possible messages $m'$, $\mathcal{A}(m',i)$ is nonempty and $\mathcal{S}_k(m',i)$ is either $[0]$ or $[1]$ or $[0, 1]$. Let $\mathcal{K}_1(m',i) := \{i \mid \mathcal{S}_1(m',i) = [1]\}$ and $\mathcal{K}_0(m',i) := \{i \mid \mathcal{S}_0(m',i) = [0]\}$. Let $\alpha(m',i) := |\mathcal{K}_1(m',i)|$ and $\beta(m',i) := m - |\mathcal{K}_0(m',i)|$. It can be shown that the projection of $\mathcal{A}(m',i)$ under the linear transformation $s = (\sum_{i=1}^{m} x_i)$ given by $\mathcal{A}'(m',i) := \{\{\sum_{i=1}^{m} x_i(i)\} | \exists x^{(i)} m, \forall Y(m',i) > 0\}$ is an $f'$-monochromatic interval $[\alpha(m',i), \beta(m',i)]$. Since $[a_1, b_2] \subset \mathcal{A}'(m',i) \subseteq [a_0, b_2]$. Therefore (i) holds. Since $\mathcal{A}(m',i) = [\alpha(m',i), \beta(m',i)] \subseteq [a_0, b_2]$, we have $\alpha(m',i) \geq a_0$ and $\beta(m',i) \leq b_2$. Therefore (ii) holds.

**Appendix C**

**Proof of Theorem 2:**

**Proof of Theorem 2 (ii):**

**Lemma 4:** If $k \in [0, 1]$ is a random vector and $Y = \sum_{i=1}^{k} Y_i$, then $H(Y) \leq \mathbb{E}(H(Y_i))$.

**Proof:**

$H(Y) \leq \sum_{i=1}^{k} H(Y_i) = \sum_{i=1}^{k} \mathbb{E}(H(Y_i)) = \mathbb{E}(\sum_{i=1}^{k} H(Y_i)) = \mathbb{E}(H(Y_i))$.

Step (ii) is due to the concavity of $h(x)$ and the Jensen’s inequality.

Define an auxiliary random variable $V$ by $V := \eta$ if and only if $Y \in [a_1, b_1]$. Then $\mathbb{P}(V = \eta) = \delta_{\eta}$. Due to Corollary 1 we have

$R_{\text{sum}} = \min_{U} I(X^m; U') = \min_{U} [H(X^m) - H(X^m|U', V)]$

$= mh(p) - \max_{U} \sum_{i=1}^{m} H(X^m|U', V) = \mathbb{E}(h(Y)) - Y = \mathbb{E}(h(Y)) = \mathbb{E}(h(Y))$.

where $U'$ are subject to all the Markov chain and conditional entropy constraints in (3.1). Therefore we have $H(X^m|U', V) = \mathbb{E}(h(Y)) - Y$.

In order to prove Theorem 2 (ii), it is sufficient to show that $R_{\text{sum}} \geq mh(p) - p\mathbb{E}(h(Y))$ holds for every $Y \in [1, \mathbb{V}_{\text{max}} - 1]$. For any $Y \in [1, \mathbb{V}_{\text{max}} - 1]$, let $b := b_r$, $p_0 := \mathbb{E}(S \leq b)$ and $p_1 := 1 - p_0$. If $b = 0$ and $b \neq m - 1$, apply Lemma 5 to Theorem 2 (ii) by combining all the intervals greater than $b$ into $[b + 1, m]$ and all the intervals not greater than $b$ into $[0, b]$, we have

$R_{\text{sum}} \geq mh(p) - p\mathbb{E}(h(Y))$.

where $U'$ are subject to all the Markov chain and conditional entropy constraints in (3.1). Therefore we have $H(X^m|U', V) = \mathbb{E}(h(Y)) - Y$.

In order to prove Theorem 2 (ii), it is sufficient to show that $R_{\text{sum}} \geq mh(p) - p\mathbb{E}(h(Y))$ holds for every $Y \in [1, \mathbb{V}_{\text{max}} - 1]$. For any $Y \in [1, \mathbb{V}_{\text{max}} - 1]$, let $b := b_r$, $p_0 := \mathbb{E}(S \leq b)$ and $p_1 := 1 - p_0$. If $b = 0$ and $b \neq m - 1$, apply Lemma 5 to Theorem 2 (ii) by combining all the intervals greater than $b$ into $[b + 1, m]$ and all the intervals not greater than $b$ into $[0, b]$, we have

$R_{\text{sum}} \geq mh(p) - p\mathbb{E}(h(Y))$.

where $U'$ are subject to all the Markov chain and conditional entropy constraints in (3.1). Therefore we have $H(X^m|U', V) = \mathbb{E}(h(Y)) - Y$.
where

\[ \lambda_1 = \frac{b p_0}{m (1 - p_0 p_1)} \quad \lambda_2 = \frac{(m - b - 1) p_1}{m (1 - p_0 p_1)}, \]

\[ a_1 = \frac{\mathbb{E}(S|S \leq b)}{b}, \quad a_2 = \frac{\mathbb{E}(S|S \geq b + 1) - b - 1}{m - b - 1}, \]

and \( \lambda_1, \lambda_2 \) are arbitrary real numbers. (C.14) is guaranteed by the Jensen’s inequality if \( \lambda_1 a_1 + \lambda_2 a_2 + \lambda_4 = p \), \( \sum_{i=1}^4 \lambda_i = 1 \) and \( \lambda_i \geq 0 \), \( i = 1, \ldots, 4 \). The first two conditions imply that

\[ \lambda_3 = \frac{p p_0 (1 - p)}{1 - p_0 p_1} \left( \frac{m - b}{m - mp - p_1} \right), \quad \lambda_4 = \frac{p p_1}{1 - p_0 p_1} \left( \frac{b + 1 - m p - p_0}{m - b} \right). \]

We need to verify that \( \lambda_3 \geq 0 \) and \( \lambda_4 \geq 0 \). In order to get \( \lambda_4 \geq 0 \), it is sufficient to verify that

\[ p_0 = \mathbb{P}(S \leq b) \leq \frac{b + 1}{m p}. \quad (C.12) \]

When \( b \geq mp - 1 \), (C.12) holds immediately. When \( b < mp - 1 \), we verify (C.12), as follows.

The probability mass function \( p_Y(s) \) is nondecreasing when \( s \leq \mu_S \), where \( \mu_S = [mp + p] \) is the mode of the binomial random variable \( S \). Therefore \( \mathbb{P}(S \leq b)_{b=0}^{[mp]} \) is a convex sequence [13], which implies that for all integers \( 0 \leq b \leq [mp] \), the point \((b, \mathbb{P}(S \leq b))\) is below or on the line segment joining the point \((0, \mathbb{P}(S \leq 0)) = (0, (1 - p)^m)\) and the point \(([mp], \mathbb{P}(S \leq [mp])). \)

Since

\[ mp(1 - p)^m \leq mp(1 - p)^{m-1} \leq \sum_{k=0}^{m} (m \cdot k)^p (1 - p)^{m-1} = 1, \]

we have \( (1 - p)^m \leq 1/mp \). Also, \( \mathbb{P}(S \leq [mp]) = 1 < ([mp] + 1)/mp \). Therefore the line segment joining \((0, (1 - p)^m)\) and \((([mp], \mathbb{P}(S \leq [mp]))\) is below the line segment joining \((0, 1/mp)\) and \((([mp], ([mp] + 1)/mp))\), which is the graph of the function \((b + 1)/mp\) when \( 0 \leq b \leq [mp] \). Therefore we have shown that (C.12) holds for \( b \leq [mp] \) and completed the proof for \( \lambda_4 \geq 0 \). Similarly we have \( \lambda_3 \geq 0 \).

Proof of Theorem 2 (iii): Let \( r = 1 \) and for each \( i \in [1, m] \), define

\[ U_i := \begin{cases} 0, & \text{if } \exists v, s.t. N_i(U^{i-1}) \geq a_v, N_0(U^{i-1}) \geq m - b_v, \text{ and } X_i, \\ \text{otherwise}, & \end{cases} \]

where \( N_i(U^{i-1}) \) is the number of times the symbol \( x \) occurs in the sequence \( U^{i-1} \). Define a random variable \( V \) by \( V := v \) if and only if \( S \in [a_v, b_v] \). Define a random variable \( K \) by \( K := \min \{ i | \in [1, m], N_i(U^i) \geq a_v, N_0(U^i) \geq m - b_v \} \). In other words, we define \( U_1 = X_1, U_2 = X_2 \), and so on, until after \( K \)-steps, \( U^k = X^k \) contains at least \( a_v \) ones and at least \( m - b_v \) zeros, so that for arbitrary values of the remaining sources \( x_{k+1}^m \), \( S \) definitely belongs to \([a_v, b_v]\), which means that the desired function is determined. After the \( K \)-th step, no information is sent, because \( U_{k+1}^{m-k} = 0 \). One can verify that \( U^i \) satisfy the Markov chains and the conditional entropy equality in \( \text{(C.11)} \). Intuitively speaking, the above definition of \( U^i \) corresponds to the following one-round coding scheme: For each sample, the source nodes keep sending the original data until there exists \( v \in [1, v_{\max}] \) such that \( a_v \) ones and \( m - b_v \) zeros have appeared. Once it happens, the sum of sources definitely falls into an \( f \)-monochromatic interval \([a_v, b_v]\) so that the desired function is determined. Thus the computation for this sample is stopped. \( K \) is the stopping time of sending data.

For any source sequence \( x^m \), the corresponding values \( k \) and \( u^m \) satisfy \( u^k = x^k \) and \( u^m_{k+1} = 0 \). Therefore

\[ p_{Y^{(k)}}(u^k) = p_{Y^{(k)}}(u^k) p_{Y^{(k+1)}}(0|u^k) = p_{X^k}(x^k). \]

Then we have

\[ R_{nm,1} \leq I(X^m; U^m) \]

\[ = H(U^m) \]

\[ = \log \frac{1}{p_{Y^{(k)}}(U^m)} \]

\[ + \log \frac{1}{p_{X^k}(X^k)} \]

\[ = \mathbb{B}(H(X^k|K)) \]

\[ = \mathbb{B}(K)(p) \]

\[ = h(p) \sum_{v=1}^{v_{\max}} p_v(v) \mathbb{E}(K|V = v). \quad (C.13) \]

We need to bound \( \mathbb{E}(K|V = v) \) with respect to the joint distribution of \((K, V)\), which is given by: if \( k \in [1, m], v \in [1, v_{\max}], p_{K|V}(k) = p_{Y^{(k)}}(v) \mathbb{P}(X^k) \) contain \( a_v \) ones and \( (m - b_v) \) zeros, but \( X^k \) do not do it; otherwise \( p_{K|V}(k) = 0 \).

For each \( v \), we can define another random variable \( K^v \) as the number of iid bernoulli(p) trials to get \( a_v \) ones and \( (m - b_v) \) zeros. In other words, \( p_{K|V}(k) = \mathbb{P}(X^k) \) contain \( a_v \) ones and \( (m - b_v) \) zeros, but \( Y^{k-1} \) do not do it. \( K_i \sim \text{iid bernoulli}(p) \) for \( i \in \mathbb{N} \). Note that unlike \( K \), which does not exceed \( m \), \( K^v \) could be arbitrarily large.

Since \( \forall v \in [1, v_{\max}], \forall k \in [1, m], p_{K|V}(k) = p_{K^v}(k) \), the conditional distributions of \((K|V = v) \) and \((K^v|K^v \leq m) \) are the same. Therefore

\[ \mathbb{E}(K|V = v) = \mathbb{E}(K^v|K^v \leq m) \leq \mathbb{E}(K^v). \quad (C.14) \]

The last step is because for any random variable \( X \) and \( \forall a \in \mathbb{R}, \mathbb{E}(X|X \leq a) \leq \mathbb{E}(X). \)

Then, define two independent random variables \( W_{r,1} \) and \( W_{0,0} \) as follows: \( W_{1,1} \) is the number of iid bernoulli(p) trials to get \( a_v \) ones. \( W_{0,0} \) is the number of iid bernoulli(p) trials to get \((m - b_v) \) zeros. They are negative binomial distributed random variables \( \mathbb{E}(W_{1,1}) = a_v/p \) and \( \mathbb{E}(W_{0,0}) = (m - b_v)/(1 - p). \)

Since \( Y^{k+1} \) contains \( a_v \) ones and \( Y^{W_{1,1}+W_{0,0}} \) contains \( (m - b_v) \) zeros, then \( Y^{W_{1,1}+W_{0,0}} \) contains at least \( a_v \) ones and \( (m - b_v) \) zeros, we have \( K^v \leq W_{1,1} + W_{0,0}, \) which implies that

\[ \mathbb{E}(K^v) \leq \mathbb{E}(W_{1,1}) + \mathbb{E}(W_{0,0}) = a_v/p + (m - b_v)/(1 - p). \quad (C.15) \]

Combining \( \text{(C.13)}, \text{(C.14)} \) and \( \text{(C.15)} \) leads to the statement of Theorem 2 (iii).
REFERENCES

[1] A. Giridhar and P. Kumar, “Computing and communicating functions over sensor networks,” IEEE Journal on Selected Areas of Communication, vol. 23, no. 4, pp. 755–764, Apr 2005.

[2] S. Subramanian, P. Gupta, and S. Shakkottai, “Scaling bounds for function computation over large networks,” Proc. IEEE Intl. Symp. Info. Theory (ISIT), Jul. 2007.

[3] R. Appuswami, M. Franceschetti, N. Karamchandani, and K. Zeger, “Network coding for computing,” Proc. Allerton Conference, 2008.

[4] V. Prabhakaran, K. Ramchandran, and D. Tse, “On the role of interaction between sensors in the CEO problem,” Proc. Allerton Conference, 2004.

[5] A. Orlitsky and J. R. Roche, “Coding for computing,” IEEE Trans. Info. Theory, vol. IT–47, no. 3, pp. 903–917, Mar 2001.

[6] N. Ma and P. Ishwar, “Two-terminal distributed source coding with alternating messages for function computation,” Proc. IEEE International Symposium on Information Theory (ISIT), 2008.

[7] R. Gallager, “Finding parity in a simple broadcast network,” IEEE Tran. Info. Theory, vol. IT–34, pp. 176–180, March 1988.

[8] L. Ying, R. Srikant, and G. Dullerud, “Distributed symmetric function computation in noisy wireless sensor networks,” IEEE Tran. Info. Theory, vol. IT–53, pp. 4826–4833, Dec. 2007.

[9] O. Ayaso, D. Shah, and M. Dahleh, “Information theoretic bounds on distributed computation,” submitted to IEEE Transaction on Information Theory, 2008.

[10] B. Nazer and M. Gastpar, “Computation over multiple-access channels,” IEEE Tran. Info. Theory, vol. IT–53, no. 10, pp. 3498–3516, Oct 2007.

[11] E. Kushilevitz and N. Nisan, Communication Complexity. Cambridge University Press, 1997.

[12] T. M. Cover and J. A. Thomas, Elements of Information Theory. New York: Wiley, 1991.

[13] J. E. Pečarić, F. Proschan, and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications. Academic Press Inc, 1992.