Modified Bekenstein-Hawking system in $f(R)$ gravity

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The present work deals with four alternative formulation of Bekenstein system on event horizon in $f(R)$ gravity. While thermodynamical laws holds in universe bounded by apparent horizon, these laws break down on event horizon. With alternative formulation of thermodynamical parameters (temperature and entropy), thermodynamical laws hold on event horizon in Einstein Gravity. With this motivation, we extend the idea of generalised Hawking temperature and modified Bekenstein entropy in homogeneous and isotropic model of universe on event horizon and examine whether thermodynamical laws hold in $f(R)$ gravity. Specifically, we examine and compare validity of generalised second law of thermodynamics (GSLT) and thermodynamical equilibrium (TE) in four alternative modified Bekenstein scenarios. As Dark energy is a possible dominant candidate for matter in the universe and Holographic Dark Energy (HDE) can give effective description of $f(R)$ gravity, so matter in the universe is taken as in the form interacting HDE. In order to understand the complicated expressions, finally the above laws are examined from graphical representation using three Planck data sets and it is found that generalised/modified Hawking temperature has a crucial role in making perfect thermodynamical system.

Keywords: Modified theories of gravity; generalised second law of thermodynamics; thermodynamic

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equilibrium, modified Bekenstien entropy and generalised/modified Hawking temperature, event horizon.

I. INTRODUCTION

One of the interesting topics in general relativity (GR) is the relation between thermodynamics and gravity. This relation helps in understanding several aspects of GR [1]. The relations were first observed by Hawking and Bekenstein in 1970 in the context of black hole thermodynamics and it was realised black hole can be considered as thermodynamical system with a temperature and entropy [2]. The temperature and entropy are proportional to the surface gravity and horizon area respectively [2, 3]. Further, the first law of thermodynamics relates these temperature and entropy [4]. As entropy is a thermodynamical quantity and horizon is a geometrical quantity, people started predicting relation between black hole thermodynamics and Einstein field equations. Indeed, Jacobson in 1995 derived the Einstein field equations from the first law of thermodynamics. Padmanabhan, from the other side was able to derive first law of thermodynamics for a general static spherically symmetric space-time [5, 6].

Subsequently these ideas are generalised in cosmology, treating universe as a thermodynamical system. Generally apparent horizon \((R_A)\) is taken as boundary of the universe. The thermodynamics in de-Sitter’s space time was first investigated by Gibbons and Hawking [7]. Furthermore, the first law of thermodynamics and the Friedmann equations are shown to be equivalent [8]. On the other hand from the present observational data it is found that the current expansion of the universe is accelerating [9–16]. The event horizon is distinct from apparent horizon and its existence is assured in the accelerating universe. Wang et al. [17] in 2006 argued that the event horizon is larger than the apparent horizon and the universe bounded by the event horizon is not a Bekenstein system in the study of laws of thermodynamics when the universe is having accelerated expansion. Further, they concluded that the outside apparent horizon thermodynamical laws break down and thermodynamic description does not follow Bekenstein’s definition.

It is imperative to verify thermodynamical laws for modified gravity if thermodynamic interpretation of gravity near horizon is a generic feature. \(f(R)\) gravity is one of the prominent modified gravity which can explain naturally the present acceleration of the universe without any dark energy (DE). In \(f(R)\) gravity the action term is arbitrary function \(f(R)\) of the Ricci scalar \(R\). Some other classes of modified gravities are \(f(G)\), \(f(R,G)\), and \(f(T)\). These theories are considered as gravitational alternatives for DE and extensively explored for different purposes in the literature [18–38].

In recent years, lot of work has been done in \(f(R)\) gravity in the context of gravitational thermo-
dynamics [39–48]. Further, it has been extended in other gravity theories including $f(G)$ theory [8], Scalar tensor gravity [47–49], Lovelock theory [50] and braneworld scenarios [51]. All these works on universal thermodynamics mostly deal with apparent horizon but in case of event horizon due to its complicated nature there are few works related to it. It is imperative to investigate the validity of thermodynamical laws for event horizon as it separates out from apparent horizon in accelerating universe. In this regard it has been proved that generalised second law of thermodynamics holds in any gravity theory under some conditions in event horizon [52–57]. The main difficulty of studying the thermodynamics of the universe is to define the entropy and temperature on the horizons. Generally the entropy and hence temperature is taken from black hole thermodynamics but in modified gravity theories some corrections term may be needed.

In literature various forms of entropy and temperature have been proposed to study thermodynamics in expanding universe [58–60]. For instance by generalising the temperature or modifying the entropy in Einstein gravity, validity of thermodynamical laws and TE have been shown in various gravity theories [61–66]. In other words, generalised Hawking temperature or a modified Bekenstein entropy on event horizon helps in making perfect thermodynamical system in different gravity theories. Therefore, it is natural to ask, whether with these alternative definition of thermodynamical parameters (temperature and entropy), thermodynamical laws hold on event horizon in $f(R)$ gravity. With this motivation, here we extend the idea of generalised Hawking temperature and modified Bekenstein system on event horizon in $f(R)$ gravity. In this regard, we study and compare four different types of modified entropies/temperatures and examine the validity of thermodynamical laws with respect to these modified entropies and temperatures.

Dark energy (DE) is a possible dominant candidate for matter in the universe. One of the DE candidate which received lot of attention in recent years is Holographic Dark Energy (HDE) as it can alleviate coincidence problem [67]. The HDE model is based on the holographic principle which states that the number of degrees of freedom of a physical system is given by the area of the boundary [68, 69]. This model makes an attempt to apply holographic principle of quantum gravity to DE problem and one can obtain HDE density as $\rho_D = 3c^2 M_p^2 L^{-2}$, where $M_p^2 = 8\pi G$ is the reduced Planck mass, $L$ is the IR (infra red) cut-off (size of the region) and $c$ is numerical constant [70]. A comprehensive review of IR cut-off and various cosmological implications of HDE in accelerating universe can be obtained in Refs.[67, 71–73]. Further, HDE can give effective description of $f(R)$ gravity [74]. So in the present work matter in the universe is taken as in the form of interacting HDE. In order to understand the complicated expressions, finally, we have examined the validity of GSLT and TE from graphical representation using three Planck data sets.
The outline of the paper is as follows. The section II presents basic equations in $f(R)$ gravity. In section III we study the basic concepts of gravitational thermodynamics. Section IV deals with thermodynamical analysis in a universe dominated by HDE and finally in last section we discuss the summary of the work and possible conclusions.

II. BASIC EQUATIONS OF $f(R)$-GRAVITY

The modified Einstein-Hilbert action in $f(R)$ gravity is written as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_m,$$

with $S_m$ as the matter action, $R$ is Ricci scalar and $f(R)$ is the arbitrary real function of $R$. Taking variation of the action (1) with respect to the metric $g_{\mu\nu}$, we get

$$R_{\mu\nu} f_R - \frac{1}{2} g_{\mu\nu} f_R - \nabla_\mu \nabla_\nu f + g_{\mu\nu} \nabla^2 f_R = T^m_{\mu\nu},$$

where $f_R$ denotes the derivative of $f$ with respect to $R$ and $T^m_{\mu\nu} = \text{diag}(-\rho, p, p, p)$ is the energy momentum tensor for the matter field. The matter field is taken in the form of a perfect fluid with $\rho = \rho_m + \rho_d$, $p = p_d$. Here $\rho_d$, $p_d$ respectively denote the energy density and thermodynamic pressure of HDE and $\rho_m$ denotes the energy density of matter (dark matter +Baryonic)

Observations support flat, homogeneous and isotropic FRW metric given by

$$ds^2 = h_{ij}(x^i)dx^i dx^j + R^2_h d\Omega^2_2,$$

where $R_h = ar$ is the area radius ($a$ is the scale factor), $h_{ij} = \text{diag}(-1, a^2(t))$ and $d\Omega^2_2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on the unit 2-sphere. Here $i, j$ can take values 0 and 1, such that $x^0 = t, x^1 = r$

For a viable $f(R)$ gravity theory, if we take $f(R) = R + F(R)$, then the modified Friedmann equations in the above space time can be written as [42],

$$H^2 = \frac{8\pi G}{3} \rho_t,$$

$$\dot{H} = -4\pi G(\rho_t + p_t),$$
where $\rho_t = \rho + \rho_e$ and $p_t = p + p_e$. Here $\rho_e$ and $p_e$ represents the effective energy density and effective pressure due to the curvature contribution and they are given by

$$\rho_e = -\frac{1}{2} \left( F - RF_1 + 6H\dot{F}_1 + 6F_1H^2 \right), \quad (6)$$

$$\rho_e + p_e = \left( \dot{F}_1 - H\dot{F}_1 + 2\dot{H}F_1 \right), \quad (7)$$

where $F_1 = \frac{dF}{dR}$, $R = 6(\dot{H} + 2H^2)$ and from here onwards we are choosing $8\pi = 1$, and $G = 1$.

The energy conservation equations are given by

$$\dot{\rho} + 3H(\rho + p) = 0, \quad \dot{\rho}_t + 3H(\rho_t + p_t) = 0. \quad (8)$$

So the effective pressure and energy density also satisfies the conservation equation

$$\dot{\rho}_e + 3H(\rho_e + p_e) = 0. \quad (9)$$

The above equations will be used in deriving calculating time derivatives of modified entropies in four alternative Bekenstein-Hawking formulations given in Sec.IV.

### III. BASIC CONCEPTS OF GRAVITATIONAL THERMODYNAMICS

In this section we shall study some basic features of thermodynamics. The radius of event horizon $R_E$ is given by

$$R_E = a(t) \int_t^\infty \frac{dt'}{a(t')}, \quad (10)$$

and

$$\dot{R}_E = HR_E - 1. \quad (11)$$

On the other hand radius of apparent horizon in a flat universe is written as

$$R_A = \frac{1}{H}. \quad (12)$$
We know that the total entropy of an isolated macroscopic physical system can not decrease (GSLT), i.e. $\dot{S}_T \geq 0$, where $S_T$ is total entropy which is the sum of horizon entropy ($S_h$) and the entropy of the fluid bounded by the horizon ($S_f$). Also such a system evolves towards TE, a state having maximum entropy. For universe filled with a fluid and bounded by event horizon, the validity of GSLT and TE can be verified using the inequalities given below [75, 76]

$$\dot{S}_T \geq 0 \quad \text{(for GSLT)} \quad (13)$$

and

$$\ddot{S}_T < 0 \quad \text{(for TE)} \quad (14)$$

$S_f$ can be calculated from Gibb’s equation [52–57, 61, 62, 77]

$$T_f dS_f = dE_f + pdV_E, \quad (15)$$

where $E_f(= \rho V_E)$ is the energy flow across the horizon, $V_E(= \frac{4}{3} \pi R_E^3)$ is the volume of the fluid and $T_f$ is the temperature of the fluid. In the present work as we consider only event horizon, so we assumed the temperature of the fluid is same as the temperature of the horizon, i.e. $T_f = T_E$. Hence the time derivative of fluid entropy is given by

$$\dot{S}_f = \frac{1}{2T_f} R_E^2(\rho + p)(\dot{R}_E - HR_E) \quad (16)$$

Using above equation and entropy of the horizon, total time derivatives in various alternative Bekenstein-Hawking formulation can be calculated. In what follows we list four different temperatures and entropies for examining GSLT and TE in modified Bekenstein formulation in f(R) gravity.

### IV. MODIFIED BEKENSTEIN-HAWKING FORMULATION

In this section we list the following four alternative Bekenstein formulation for possible modification of entropy and temperature in $f(R)$ gravity. The modifications are carried out in a manner so that Clausius relation holds on the event horizon.

**Case I:** $S_E = \frac{R_{E}^2}{8}$ and $T_E = \frac{4\alpha R_{E}}{R_A}$
Case II: \( S_E = \frac{Af_R}{4} \) and \( T_E = 4d_1H \)

Case III: \( S_E = \frac{R_E^2}{8} \) and \( T_E = \frac{A R_E}{R_A^2} \)

Case IV: \( S_E = \frac{A E}{4} - \frac{1}{16} \int \left( \frac{R_E^2 R_A}{1-\epsilon} \right) \left( \rho_e + p_e \right) dR_E \) and \( T_E = 4 \left( \frac{R_E}{R_A} \right)^2 \left( \frac{1 - \frac{R_A}{R_E}}{R_E} \right) \).

Each case gives an alternate formulation of Bekenstein system and in each case \( T_E \) and \( S_E \) denote the temperature and entropy of the event horizon respectively. In what follows we give the motivation and background of each case:

Case I: It is well known that \( T_E = \frac{1}{2\pi R_E} \) and \( S_E = \frac{A}{4} \) are called Hawking temperature and Bekenstein entropy \([2, 3]\). Recently by generalising Hawking temperature, Chakraborty in ref \([60]\) have shown validity of GSLT and TE on event horizon in Einstein’s gravity.

The generalised Hawking temperature is given by

\[
T_E = \frac{4\alpha R_E}{R_A^2},
\]

where \( \alpha = \frac{v_A}{R_E} \), \( v_A = \dot{R}_A \), \( v_E = \dot{R}_E \) and for this value of \( \alpha \) first law of thermodynamics (FLT) holds on the event horizon \([60]\). In this case we do not change Bekenstein entropy given by \( S_E = \frac{R_E^2}{8} \), but we change the Hawking temperature to generalised Hawking temperature.

Case II: The temperature \( T = -\frac{H}{2\pi}(1 + \frac{H}{2\pi}) \) is known as Hayward-Kodama temperature \([78]\). To avoid the negative value of the temperature, it is redefined and is written as \( T \approx \frac{H}{2\pi} \) (often called Cai-Kim temperature \([8]\)). In general Cai-Kim temperature is written as \( T = \frac{d_1 H}{2\pi} \), where \( d_1 \) is a real constant and it shows deviations from Gibbons-Hawking temperature. For de-Sitter space, we have \( d_1 = 1 \). So in this case the event horizon temperature is taken as Cai-Kim temperature i.e.

\[
T_E = \frac{d_1 H}{2\pi} = 4d_1H
\]

The Bekenstein entropy in this case is modified because of \( f(R) \) gravity as \([79, 80]\)

\[
S_E = \frac{Af_R}{4} = \frac{AF_2}{4},
\]

with \( F_2 = f_R \) and \( A = \frac{R_E^2}{2} \) is the area of the horizon.

Case III: Similar to the apparent horizon the surface gravity on event horizon can be defined as \([81]\)

\[
\kappa_E = -\frac{1}{2} \frac{\delta}{\delta R} |_{R=R_E} = \frac{R_E}{R_A},
\]

so the Hawking temperature can be modified \([58]\) as \( T_E^m = \frac{\|\kappa_E\|}{2\pi} = \frac{4R_E}{R_A^2} \).
and the above temperature is known as modified Hawking temperature. The validity of first law of thermodynamics on the event horizon has been proved using this temperature in standard gravity in Ref. [58]

**Case IV**: In this case, entropy is evaluated from the validity of unified first law of thermodynamics which was introduced by Hayward [82–84]. For event horizon if \(d\xi_E^\pm = dt \mp adr\) is the one form orthogonal to the surface of the event, then the tangent vector \(\xi_E\) is given by [63]

\[
\xi_E = \frac{\partial}{\partial t} - \frac{1}{a} \frac{\partial}{\partial r}.
\]

(20)

Now projecting the unified first law along \(\xi_E\), one can write first law of thermodynamics for event horizon as [50, 85, 86]

\[
\langle dE, \xi_E \rangle = \kappa_E \langle dA, \xi_E \rangle + \langle WdV, \xi_E \rangle,
\]

(21)

and consequently the modified entropy on the event horizon can be evaluated as

\[
S_E = \frac{A_E}{4} - \frac{1}{16} \int \left( \frac{R_A^2 R_E}{1 - \epsilon} \right) \left(\frac{HR_E + 1}{HR_E - 1}\right) \left(\rho_e + p_e\right) dR_E
\]

(22)

where surface gravity \(\kappa_E\) is defined as \(\kappa_E = -\left(\frac{R_E}{R_A}\right) \left(\frac{1 - \epsilon}{R_E}\right)\) and \(\epsilon = \frac{\dot{R}_A}{2HR_A}\). The entropy expression shows the entropy of the event horizon actually differs from Bekenstein entropy by a correction term. Using the above from of surface gravity, one gets extended Hawking temperature as [62]

\[
T_E = \frac{||\kappa_E||}{2\pi} = 4 \left(\frac{R_E}{R_A}\right)^2 \left(\frac{1 - \dot{R}_A}{2HR_A}\right).
\]

(23)

Further, using this form of modified entropy and extended Hawking temperature validity of GSLT and TE have been examined in various gravity theories in ref [61]. For brevity, total variation of entropies in each case have been given in appendix. In what follows in following subsection we shall perform thermodynamic analysis considering the universe filled with HDE and make a comparative study of above four cases in the context of GSLT and TE. It may be noted that recently this type of analysis have been extensively done in various gravity theories [61, 62]

**V. THERMODYNAMIC ANALYSIS FOR UNIVERSE FILLED WITH HOLOGRAPHIC DARK ENERGY**

In this section we shall perform thermodynamical analysis and compare the above four cases for GSLT and equilibrium thermodynamics. We consider the universe filled with holographic dark energy
interacting with dark matter (DM) in the form of dust. So its total energy density is \( \rho = \rho_m + \rho_d \), where \( \rho_m \) and \( \rho_d \) are the energy densities of DM and DE respectively. As we consider the interaction between DM and DE, so \( \rho_m \) and \( \rho_d \) satisfies the following conservation equations

\[
\dot{\rho}_m + 3H \rho_m = Q, \quad (24)
\]

\[
\dot{\rho}_d + 3H \rho_d (1 + \omega_d) = -Q, \quad (25)
\]

where \( \omega_d \) is variable equation of state parameter of DE and \( Q = 3Hb^2\rho \) is interaction term, with \( b^2 \) as coupling parameter. Also \( \omega_d \) satisfies following equation \([81, 87]\)

\[
\omega_d = -\frac{1}{3} - \frac{2\sqrt{\Omega_d}}{3c} - \frac{b^2}{\Omega_d}, \quad (26)
\]

where \( c \) is a dimensionless constant and the density parameter is given by

\[
\Omega'_d = \Omega_d \left[ (1 - \Omega_d) \left( 1 + \frac{2\sqrt{\Omega_d}}{c} \right) - 3b^2 \right]. \quad (27)
\]

where \( t = \frac{\partial}{\partial x}, x = \ln a \). The velocities of the apparent \( (v_A) \) and event \( (v_E) \) horizons can be written as

\[
v_A = \frac{3}{2} \left[ (1 - b^2) - \frac{\Omega_d}{3} \left( 1 + \frac{2\sqrt{\Omega_d}}{c} \right) \right], \quad (28)
\]

and

\[
v_E = \left( \frac{c}{\sqrt{\Omega_d} - 1} \right). \quad (29)
\]

For graphical representation three data sets have been used from Table I \([64, 88–91]\). From observation, it is found that Planck data are more accurate than Wilkinson Microwave Anisotropy Probe (WMAP)-9 data. This accuracy can be increased more if we take External Astronomical data sets (EADS) and lensing data into account. Common EADS include the Baryonic Acoustic Oscillation (BAO) measurements from 6dFGS+SDSS+DR7(R)+BOSS DR9, Estimation of Hubble constant from Hubble Space Telescope (HST) and supernova data sets SNLS3 together with Union 2.1.

Using these data sets graphs of GSLT \( (\dot{S}_T) \) and TE \( (\ddot{S}_T) \) have been plotted each case in FIGS.1–6, considering \( H = 1, R_E = \frac{c}{H \sqrt{\Omega_d}} \) and \( d_1 = 1 \). (See appendix for expressions of total variation of entropies in each case)
TABLE I: Planck Data Sets

| Sl. No. | Data Sets                        | c     | \( \Omega_d \) |
|---------|----------------------------------|-------|----------------|
| 1       | Planck+CMB+SNLS3+lensing         | 0.603 | 0.699          |
| 2       | Planck+CMB+Union 2.1+lensing     | 0.645 | 0.679          |
| 3       | Planck+CMB+BAO+HST+lensing       | 0.495 | 0.745          |

TABLE II: Graphical Analysis of GSLT and TE when the universe is dominated by holographic dark energy

| Data Set | Case | GSLT                  | TE                  |
|----------|------|-----------------------|---------------------|
| 1        | I    | Holds for \( b^2 > 0.118 \) | Holds for \( b^2 < 0.228 \) |
| 1        | II   | Always holds          | Holds for \( 0.24 < b^2 < 0.359 \) |
| 1        | III  | Always holds          | Holds for \( b^2 > 0.448 \) |
| 1        | IV   | Never hold            | Holds for \( b^2 > 0.4 \) |
| 2        | I    | Holds for \( b^2 > 0.2 \) | Holds for \( b^2 < 0.3 \) or \( b^2 > 0.53 \) |
| 2        | II   | Always holds          | Never hold          |
| 2        | III  | Always holds          | Holds for \( b^2 < 0.2 \) or \( b^2 > 0.41 \) |
| 2        | IV   | Never hold            | Holds for \( b^2 > 0.4 \) |
| 3        | I    | Always holds          | Holds for \( b^2 < 0.244 \) or \( b^2 > 0.6 \) |
| 3        | II   | Always holds          | Never hold          |
| 3        | III  | Always holds          | Holds for \( b^2 > 0.526 \) |
| 3        | IV   | Never hold            | Never hold          |

VI. CONCLUSION

This paper deals with the study of thermodynamic analysis for the universe bounded by event horizon in \( f(R) \) gravity theory. In flat FRW model event horizon can only exist in the accelerating phase of the universe. As from recent observation the universe is going through an accelerated phase of expansion, so it is pertinent to consider the universe bounded by event horizon. Also as dark energy is a possible dominant candidate for the matter in the universe and HDE can give effective description of \( f(R) \) gravity. So we have for convenience chosen HDE as the dominant source of energy. We studied validity of GSLT and TE in four alternative Bekenstein formulation. It may be noted that recently, this type of alternate Bekenstein formulations have been extensively investigated in various gravity
FIG. 1: The time derivative of the total entropy is plotted against $b^2$ with $c = 0.603$ and $\Omega_d = 0.699$.

FIG. 2: The second order time derivative of total entropy is plotted against $b^2$ with $c = 0.603$ and $\Omega_d = 0.699$.

theories. Another physical motivation to study thermodynamical laws is that if two cosmological models satisfy equally observational constraints but one respects thermodynamical laws and other does not, then later one can be ruled out. So any sensible physical system must satisfy GSLT and TE. Furthermore, in order to understand the complicated expressions, the validity of GSLT and TE are examined graphically. To have a comparative study of above four different modified Bekenstein system, we have examined numerically our theoretical results with three Planck data sets presented in Table I.

The Table-II shows the region where GSLT and TE holds when the universe is filled with HDE. Our main results can be summarized as follows from the figures FIG.1 – 6:

- It is found that GSLT holds for the Case II and Case III in all three data sets but Case IV fails in all three data sets.

- However, in Case I GSLT holds under some restriction of $b^2$ in data sets 1 and 2 but is not satisfied in data set 3.

- On the other hand TE holds in all four cases for first data set but in second data set except Case II, all other three cases are satisfied under some restriction of $b^2$.

- In case of third data sets TE holds only in Case I and Case III under some restriction of $b^2$ but Case II and Case IV fails.
Therefore, from the above comparative study we can conclude that Case I \textit{i.e.}, Bekenstein entropy with generalised Hawking temperature and Case III \textit{i.e.} Bekenstein entropy with modified Hawking temperature are better compare to other two cases. It may be noted that in contrast to GR, when Bekenstein entropy is used then GSLT holds good and TE holds with some restriction but GSLT is violated when entropy is modified \cite{60}. Moreover, by changing the temperature it is shown that GSLT and TE holds good. So in f(R) gravity the generalised/modified Hawking temperature has a crucial role in formation of perfect thermodynamical system. It remains to be seen whether the generalised/modified Hawking temperature will have same role in other gravity theories. We leave it for our future work.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig3.png}
\caption{The time derivative of the total entropy is plotted against $b^2$ with $c = 0.645$ and $\Omega_d = 0.679$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig4.png}
\caption{The second order time derivative of total entropy is plotted against $b^2$ with $c = 0.645$ and $\Omega_d = 0.679$.}
\end{figure}

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FIG. 5: The time derivative of the total entropy is plotted against $b^2$ with $c = 0.495$ and $\Omega_d = 0.745$.

FIG. 6: The second order time derivative of total entropy is plotted against $b^2$ with $c = 0.495$ and $\Omega_d = 0.745$.

Appendix

In this appendix we will give the expressions of $\dot{S}_T$ and $\ddot{S}_T$ for four cases of modified Bekenstein-Hawking formulation. First we note that in each case the time variation of $(\rho_e + p_e)$ is given by

$$\frac{\partial}{\partial t}(\rho_e + p_e) = \left(\ddot{F}_1 + \dot{H}\dot{F}_1 - H\ddot{F}_1 + 2\dot{H}F_1\right)$$

(30)

So using Eq.(16), Eq.(30) and definition of $T_E$ and $S_E$ for each case we get the following expressions:

Case I:

Here the first time derivative of total entropy is given by

$$\dot{S}_T = \frac{1}{4} \left( R_E v_E - \frac{v_E(\rho + p)}{2v_A H^3} \right).$$

(31)

So the second time derivative of total entropy is given by

$$\ddot{S}_T = \frac{(R_E f_E + v_E^2)}{4} - \frac{1}{8} \left[ \frac{(\rho + p)(v_A H^3 f_E - v_E f_A H^3 - 3v_A v_E H^2 \dot{H})}{(v_A H^3)^2} \right. 
+ \left. \frac{v_A v_E H^3 \partial(\rho + p)}{(v_A H^3)^2} \right].$$

(32)
where \( f_E = \dot{v}_E \).

**Case II:**

In this case first time derivative of total entropy is given by

\[
\dot{S}_T = \frac{1}{8} \left[ (2R_E v_E F_2 + R_E^2 \dot{F}_2) - \frac{R_E^2 (\rho + p)}{d_1 H} \right].
\]  

(33)

The second time derivative of total entropy is

\[
\ddot{S}_T = \frac{1}{8} \left[ (2v_E^2 F_2 + 2R_E f_E F_2 + 4R_E v_E \ddot{F}_2 + R_E^2 \dddot{F}_2) + \frac{1}{d_1 H^2} \left( R_E (\rho + p) (2Hv_E - \dot{H} R_E) + HR_E^2 \frac{\partial (\rho + p)}{\partial t} \right) \right].
\]  

(34)

**Case III:**

The first time derivative of total entropy is given by

\[
\dot{S}_T = \frac{1}{4} \left( R_E v_E - \frac{R_E (\rho + p)}{2H^2} \right),
\]  

(35)

and second time derivative of total entropy is

\[
\ddot{S}_T = \frac{(R_E f_E + v_E^2)}{4} - \frac{1}{8H^3} \left[ (Hv_E - 2\dot{H} R_E)(\rho + p) + H R_E \frac{\partial (\rho + p)}{\partial t} \right].
\]  

(36)

**Case IV:**

The first derivative of total entropy is given by

\[
\dot{S}_T = \frac{R_E v_E}{4} - \frac{R_A^2 R_E}{2(2 - v_A)} \left[ (\frac{v_E + 2}{4})(\dddot{F}_1 - H \dddot{F}_1 + 2\dot{H} F_1) + v_A H^2 \right],
\]  

(37)

and the second derivative of total entropy is

\[
\ddot{S}_T = \frac{R_E f_E}{4} \left[ 1 - \frac{R_A^2 v_E}{2(2 - v_A)} \right] + \frac{v_E^2}{4} - \frac{R_A^2 R_E}{8(2 - v_A)} \left[ (v_E + 2) \left\{ \left( \frac{2v_A}{R_A} + \frac{v_E}{R_E} + \frac{f_A}{2 - v_A} \right) \right. \right.
\]
\[
\left. (\dddot{F}_1 - H \dddot{F}_1 + 2\dot{H} F_1) + (\dddot{F}_1 + H \dddot{F}_1 - H \dddot{F}_1 + 2\dot{H} F_1) \right\} + 4v_A H^2 \left[ \frac{v_E}{R_E} + \frac{2f_A}{v_A(2 - v_A)} \right].
\]  

(38)

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