Data augmentation with Möbius transformations

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Abstract

Data augmentation has led to substantial improvements in the performance and generalization of deep models, and remain a highly adaptable method to evolving model architectures and varying amounts of data—in particular, extremely scarce amounts of available training data. In this paper, we present a novel method of applying Möbius transformations to augment input images during training. Möbius transformations are bijective conformal maps that generalize image translation to operate over complex inversion in pixel space. As a result, Möbius transformations can operate on the sample level and preserve data labels. We show that the inclusion of Möbius transformations during training enables improved generalization over prior sample-level data augmentation techniques such as cutout and standard crop-and-flip transformations, most notably in low data regimes.

1. Introduction

Data augmentation has significantly improved the generalization of deep neural networks on a variety of image tasks, including image classification (Perez & Wang, 2017; DeVries & Taylor, 2017), object detection (Zhong et al., 2017; Guo & Gould, 2015), and instance segmentation (Wang et al., 2018). Prior work has shown that data augmentation on its own can perform better than, or on par with, highly regularized models using other regularization techniques such as dropout (Hernández-García & König, 2018). This effectiveness is especially prominent in low data regimes, where models often fail to capture the full variance of the data in the training set (Zhang et al., 2019).

Many data augmentation techniques rely on priors that are present in the natural world. Standard operations, such as translation, crop, and rotation, in addition to more recent methods, such as cutout (DeVries & Taylor, 2017), have improved generalization by encouraging equivariance to the transformation. For example, an image of a horse remains a horse in its vertical reflection or with its body partially occluded. As a result, these transformations are able to preserve the original label on the augmented sample, enabling straightforward and easy incorporation into the growing number of data augmentation algorithms for both fully supervised and semi-supervised learning. In a nutshell, these sample-level methods have been not only effective, but also interpretable, easy to implement, and flexible to incorporate.

Following the success of these methods, we focus this paper on augmentations that exploit natural patterns to preserve labels after transformation and that operate on the sample level. These transformations easily complement other methods, thus leveraged in a wide variety of data augmentation algorithms. In contrast, multi-sample augmentations, which have had comparably strong empirical results (Zhang et al., 2017), unfortunately connect less clearly to natural priors.
that would support equivariance to the augmentation. While performant on their own, these methods have had less success with integration into data augmentation algorithms and policies (Cubuk et al., 2019a;b; Xie et al., 2019), except for those tailored to them (Berthelot et al., 2019). In this paper, we propose a novel data augmentation technique, inspired by biological patterns, using bijective conformal maps known as Möbius transformations. Möbius transformations perform complex inversion in pixel space, extending standard translation to include divisibility. These transformations enable perspective projection—or transforming perceived distance of objects in an image—and are found naturally in the anatomy and biology of humans and other animals.

We apply Möbius transformations as a form of data augmentation during training to improve the generalization of deep models. We show empirically that the inclusion of Möbius transformations improves performance on the CIFAR-10 and CIFAR-100 benchmarks over prior sample-level data augmentation techniques, such as cutout (DeVries & Taylor, 2017) and standard crop-and-flip baselines. Möbius transformations are especially useful in low data settings particularly when applied in small quantities. Finally, we show that Möbius can complement other transformations, including cutout and crop-and-flip.

Our key contributions can be summarized as follows:

- **Method**: We introduce Möbius transformation as a method for data augmentation on images. This method works well at a small amount of inclusion, i.e. 20% of training set or only 80 images per class in the reduced CIFAR-10 dataset.

- **Performance**: Empirically, the inclusion of Möbius data augmentation improves neural network generalization over prior methods that use sample-level augmentation techniques, such as cutout (DeVries & Taylor, 2017) and standard crop-and-flip baselines.

- **Low data**: Möbius transformations are especially effective in low data settings, where the data quantity is on the order of hundreds of samples per class.

- **Complementary use**: Möbius data augmentation is designed to complement other methods, e.g. cutout. Implemented together, they can outperform either alone.

## 2. Möbius transformations

Möbius transformations are bijective conformal mappings that operate over complex inversion and preserve local angles. They are also known as bilinear or linear fractional transformations. We discuss their biological and perceptual underpinnings, and follow with a formal definition. Finally, we describe their application to data augmentation to improve generalization in convolutional neural networks.

### 2.1. Motivation

Möbius transformations have been studied in biology as 2D projections of specimens—such as humans, fungi, and fish—from their 3D configurations (Thompson et al., 1942; Petukhov, 1989; Lundh et al., 2011). Mathematically, most of these examples leverage Liouville’s theorem (Liouville, 1850), which states that smooth conformal mappings are Möbius transformations on a domain of $\mathbb{R}^n$ where $n \geq 2$. These biological patterns motivate our application of Möbius transformations to natural images, particularly those that include the relevant species.

Beyond biological underpinnings, Möbius transformations preserve the anharmonic ratio (Ahlffors, 1989; Needham, 1998), or the extent to which four collinear points on a projective line deviate from the harmonic ratio. This invariance is a property that Möbius transformations share with projective transformations, which are used widely in metrology (Criminisi et al., 2000). In the context of transforming natural images, such a transformation can be particularly useful for perspective projection. That is, an image can be transformed to an alternate perceived distance. This effect is visually apparent across examples in Figure 1.

### 2.2. Definition

Existing data augmentation techniques for image data belong to the class of affine mappings, i.e. the group of translation, scaling, and rotation, which can be generally described using a complex function $z \rightarrow az + b$, where the variable $z$ and the two parameters $a$ and $b$ are complex numbers. Möbius transformations represent the next level of abstraction by introducing division to the operation (Lie, 1893; Petukhov, 1989). The group of Möbius transformations can be described as all functions $f$ from $\mathbb{C} \rightarrow \mathbb{C}$ with the form

$$f(z) = \frac{az + b}{cz + d},$$  \hspace{1cm} (1)

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. As a result, the set of all Möbius transformations is a superset of several basic transformations, including translation, rotation, inversion, and and even number of reflections over lines.

One method for programatically specifying the different parameters $a, b, c, d$ in Eq. (1) is to use the fact that there exists a unique Möbius transformation sending any three points to any three other points in the extended complex plane (Needham, 1998, p. 150). That is, instead of specifying the four

\[\text{The anharmonic ratio, also denoted cross-ratio, stems from projective geometry and has been studied in biology with respect to Möbius transformations (Petukhov, 1989; Lundh et al., 2011).}\]
complex parameters \(a, b, c\) and \(d\) directly in (1), we can define three separate points \(z_1, z_2, z_3 \in \mathbb{C}\) in the image and then select three separate target points \(w_1, w_2, w_3 \in \mathbb{C}\), to which those initial points will be mapped in the resulting transformation. From these two sets of points, we can then compute the parameters of the transformation using the knowledge that anharmonic ratios—adding the points \(z_i\) and \(w_i\), where \(i = \{1, 2, 3\}\) completes the two quartets—are Möbius invariant (Needham, 1998, p. 154), resulting in the following equality:

\[
\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.
\]

(2)

We can rearrange this expression by solving for \(w\):

\[
\frac{w - w_1}{w - w_3} = \frac{(z - z_1)(z_2 - z_3)(w_2 - w_1)}{(z - z_3)(z_2 - z_1)(w_2 - w_3)} = \frac{A w_3 - w_1}{A - 1},
\]

where \(A = \frac{(z - z_1)(z_2 - z_3)(w_2 - w_1)}{(z - z_3)(z_2 - z_1)(w_2 - w_3)}\). This final expression for \(w\) is in the form of Eq. (1):

\[
f(z) = w = \frac{A w_3 - w_1}{A - 1} = \frac{az + b}{cz + d},
\]

from which we can compute the following values for \(a, b, c,\) and \(d\) using basic algebraic operations:

\[
a = w_1 w_2 z_1 - w_1 w_3 z_1 - w_1 w_2 z_2 + w_2 w_3 z_2 + w_1 w_3 z_3 - w_2 w_3 z_3,
\]

\[
b = w_1 w_3 z_1 z_2 - w_2 w_3 z_1 z_2 - w_1 w_2 z_1 z_3 + w_2 w_3 z_1 z_3 + w_1 w_3 z_2 z_3 - w_1 w_3 z_2 z_3,
\]

\[
c = w_2 z_1 - w_3 z_1 - w_1 z_2 + w_3 z_2 + w_1 z_3 - w_2 z_3,
\]

\[
d = w_1 z_1 z_2 - w_2 z_1 z_2 - w_1 z_1 z_3 + w_3 z_1 z_3 + w_2 z_2 z_3 - w_3 z_2 z_3.
\]

Alternatively, by solving Eq. (2) using linear algebra, i.e. evaluating a determinant from this construction using the Laplace expansion, one can elegantly express these algebraic expressions above as determinants:

\[
a = \begin{vmatrix} z_1 w_1 & w_1 & 1 \\ z_2 w_2 & w_2 & 1 \\ z_3 w_3 & w_3 & 1 \end{vmatrix}, \quad b = \begin{vmatrix} z_1 w_1 & z_1 & w_1 \\ z_2 w_2 & z_2 & w_2 \\ z_3 w_3 & z_3 & w_3 \end{vmatrix},
\]

\[
c = \begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{vmatrix}, \quad d = \begin{vmatrix} z_1 & z_1 & 1 \\ z_2 & z_2 & 1 \\ z_3 & z_3 & 1 \end{vmatrix}.
\]

This pointwise method is used in our work to construct valid image augmentations using Möbius transformations. Ultimately, this method can be leveraged to define parameters for specific types of Möbius transformations programmatically for needs within and beyond data augmentation.

### 2.2.1. Equivalent Framing: Circle Reflection

We introduce an equivalent formulation of Möbius transformations on images in \(\mathbb{R}^2\). The goal of this section is to lend intuition on constraints that we apply to Möbius data augmentation in Section 2.3 that follows.

Möbius mappings in the plane can also be defined as the set of transformations with an even number of reflections over circles and lines (i.e. circles with infinite radii) on the plane. A reflection, or inversion, in the unit circle is the complex transformation (Needham, 1998, pp. 124):

\[
z \in \mathbb{C} \rightarrow \frac{z}{|z|^2}.
\]

Thus, a Möbius transformation on an image is simply a reflection over the unit circle, with pixels inside of the circle projected outwards and pixels on the outside projected inwards. As such, Möbius transformations often reflect a different amount of pixels inwards as opposed to outwards, and this imbalance enables the scale distortions seen in Figure 1. Note that a circular shape can be left as an artifact after the transformation, e.g. if the reflection occurs at an edge without any pixels to project inwards (see Figure 2).
A large number of data augmentation techniques have recently emerged for effectively regularizing neural networks, including both sample-level augmentations, such as ours, as well as multi-sample augmentations that mix multiple images. We discuss these, as well as data augmentation algorithms that leverage multiple augmentations. Finally, we examine ways in which Möbius transformations have been applied to deep learning. To our knowledge, this is the first work using Möbius transformations for data augmentation in deep neural networks.

3.1. Sample-level augmentation

Möbius transformations generalize standard translation to include inversion as an operation under conformity, demonstrating outputs that appear to have gone through crop, rotation, and/or scaling, while preserving local angles from the original image. We recognize that the list of image transformations is extensive: crop, rotation, warp, skew, shear, random distortion, Gaussian blur, Gaussian noise, among many others. Additional sample-level data augmentation methods use occlusion such as cutout (DeVries & Taylor, 2017) and random erasing (Zhong et al., 2017), which apply random binary masks across image regions.

3.2. Multi-sample augmentation

Data augmentation on images also consists of operations applied to multiple input images. In such cases, original labels are often mixed. For example, MixUp (Zhang et al., 2017) performs a weighted average of two images (over pixels) and their corresponding labels in varying proportions to perform soft multi-label classification. Between-class
learning (Tokozume et al., 2018) and SamplePairing (Inoue, 2018) are similar techniques, though the latter differs in using a single label. Comparably, RICAP (Takahashi et al., 2019), VH-Mixup and VH-BC+ (Summers & Dinneen, 2019) form composites of several images into one. While these methods have performed well, we focus this paper on comparisons to sample-level augmentations that preserve original labels and that can be more readily incorporated into data augmentation algorithms and policies.

### 3.3. Algorithms and policies for data augmentation

Various strategies have emerged to incorporate multiple data augmentation techniques for improved performance. AutoAugment (Cubuk et al., 2019a), Adatransform (Tang et al., 2019), RandAugment (Cubuk et al., 2019b), and Population Based Augmentation (Ho et al., 2019) offer ways to select optimal transformations (and their intensities) during training. In semi-supervised learning, unsupervised data augmentation, or UDA (Xie et al., 2019), MixMatch (Berthelot et al., 2019), and FixMatch (Sohn et al., 2020) have shown to effectively incorporate unlabeled data by exploiting label preservation and consistency training. Tanda (Ratner et al., 2017) composes sequences of augmentation methods, such as crop followed by cutout then flip, that are tuned to a certain domain. DADA (Zhang et al., 2019) frames data augmentation as an adversarial learning problem and applies this method in low data settings. We do not test all of these augmentation schemes: our results suggest that Möbius transformations could add value as an effective addition to the search space of augmentations, e.g. in AutoAugment, or as a transformation that helps enforce consistency between original and augmented data, e.g. in UDA.

### 3.4. Möbius transformations in deep learning

Möbius transformations have been previously studied across a handful of topics in deep learning. Specifically, they have been used as building blocks in new activation functions (Özdemir et al., 2011) and as operations in hidden layers (Zammit-Mangion et al., 2019). Coupled with the theory of gyrovector spaces, Möbius transformations have inspired hyperbolic neural networks (Ganea et al., 2018). They also play an important component in deep fuzzy neural networks for approximating the Choquet integral (Islam et al., 2019). Finally, model activations and input-output relationships have been theoretically related to Möbius transformations (Mandic, 2000). While prior work has primarily leveraged them for architectural contributions, our work is the first to our knowledge to introduce Möbius transformations for data augmentation and their empirical success on image classification benchmarks.

### 4. Experiments

We experiment on CIFAR-10 and CIFAR-100 image classification benchmarks using standard data splits of 50k training and 10k test (Krizhevsky et al., 2009). CIFAR-10 has 10 classes with 10k images per class, while CIFAR-100 has 100 classes with 500 images per class, in their training sets. We additionally evaluate reduced CIFAR-10, a common low data regime for which only a subset of the training data—4k images with 400 images per class—is used. We do so in a fully supervised manner, without using unlabeled data. Thus, we explore three dataset settings: (1) CIFAR-10, (2) CIFAR-100, and (3) reduced CIFAR-10. The goal of these experiments is to assess the fundamental concept of

| Augmentation Method | Dataset     | Average # Images Per Class | Total # Training Images | Accuracy       |
|---------------------|-------------|----------------------------|-------------------------|----------------|
| Crop-and-flip       | CIFAR-10    | 5000                       | 50k                     | 96.47% ±0.04   |
| Cutout $l=16$       | CIFAR-10    | 5000                       | 50k                     | 97.13% ±0.03   |
| Möbius              | CIFAR-10    | 5000                       | 50k                     | 97.62% ±0.06   |
| Möbius + Cutout $l=16$ | CIFAR-10   | 5000                       | 50k                     | 97.19% ±0.07   |
| Crop-and-flip       | CIFAR-100   | 600                        | 50k                     | 81.91% ±0.20   |
| Cutout $l=8$        | CIFAR-100   | 600                        | 50k                     | 82.35% ±0.19   |
| Möbius              | CIFAR-100   | 600                        | 50k                     | 82.85% ±0.31   |
| Möbius + Cutout $l=8$ | CIFAR-100  | 600                        | 50k                     | 82.92% ±0.14   |
| Crop-and-flip       | CIFAR-10    | 400                        | 4k                      | 83.96% ±0.16   |
| Cutout $l=16$       | CIFAR-10    | 400                        | 4k                      | 85.20% ±0.22   |
| Möbius              | CIFAR-10    | 400                        | 4k                      | 86.07% ±0.24   |
| Möbius + Cutout $l=16$ | CIFAR-10   | 400                        | 4k                      | 85.70% ±0.26   |

Table 1. Results of main experiments across 5 runs, on several dataset settings: CIFAR-10, CIFAR-100, and reduced CIFAR-10. Möbius with cutout outperforms all other conditions, followed closely by Möbius alone in the CIFAR-100 and reduced CIFAR-10 cases, where the average number of images per class is on the order of hundreds. Next, cutout alone and finally standard crop-and-flip. On CIFAR-10, cutout exceeds Möbius alone, and both outperform the crop-and-flip baseline. We use the optimal lengths for cutout on each dataset (length 16 for CIFAR-10 and length 8 for CIFAR-100) based on prior work (DeVries & Taylor, 2017).
Varying Möbius representation in data augmentation

Figure 4. Results from increasing Möbius representation in data augmentation from 10% to 50% in 10% increments, across 5 runs. (a) On CIFAR-10, Möbius at only 10% with cutout demonstrates empirically best results. Möbius on its own performs best at 40%, though it still performs under cutout alone. (b) On CIFAR-100, Möbius reaches best performance at 20% on its own and at 10% with cutout. On both datasets, Möbius boosts the performance of cutout when applied together, particularly in small quantities of 10-30%.

including Möbius data augmentation during training in both regular and low data settings.

4.1. Evaluation of benchmarks

Following prior work on introducing novel data augmentation methods (DeVries & Taylor, 2017; Cubuk et al., 2019a), we use standard crop-and-flip transformations as the baseline across all experimental conditions. We design our experiments to both compare to, and complement, cutout (DeVries & Taylor, 2017), the previous state-of-the-art image transformation that operates on the sample level, preserves labels, and thus has been easy to incorporate into data augmentation policies. Cutout and standard crop-and-flip also remain the default augmentation choices in recent work (Cubuk et al., 2019a). Thus, we compare the following conditions: (1) baseline with only crop and flip (regular), (2) cutout, (3) Möbius, and (4) Möbius with cutout. Note that all conditions incorporate crop and flip transformations, following the original cutout paper (DeVries & Taylor, 2017). Because all augmentation techniques are sample-level and preserve labels, they are complementary and can be layered on each other. We further explore these effects by combining Möbius with cutout in our experiments.

Möbius is able to improve generalization when applied to only a small fraction of the data, contrasted with cutout and standard crop-and-flip that are applied to the full dataset. Across all experiments, we apply Möbius transformations to images with 20% probability during training. We perform additional experiments on modulating the representation of Möbius transformations in Section 4.2 where we discuss this insight further. Standard transformations and cutout are applied to all data.

We use the same model and retain the same hyperparameters across all experiments, which we largely draw from prior work on wide residual networks (Zagoruyko & Komodakis, 2016) and cutout (DeVries & Taylor, 2017). Specifically, our model is a wide residual network with a depth of 28, a widening factor of 10, and a dropout probability of 0.3 on all convolutional layers, which we denote WRN-28-10. The WRN-28-10 is trained using SGD with Nesterov momentum (0.9), weight decay (5e-4), and a learning rate (0.1) that follows a cosine annealing scheduler. On the full datasets, we evaluate training on 200 epochs and on the reduced, we use 2000 epochs. We measure average performance and standard deviation across 5 runs for every experimental condition. For hardware, we use 4 NVIDIA GeForce GTX 1070 GPUs on all experiments.

On cutout, we use the optimal length values for each dataset (DeVries & Taylor, 2017): a length of 16 on CIFAR-10 (cutout$_{16}$) and a length of 8 on CIFAR-100 (cutout$_{8}$). Möbius is the same across datasets with no fine-tuning.

As shown in Table 1, these experiments highlight several key observations:

- Empirically, Möbius outperforms cutout and standard crop-and-flip on CIFAR-100 and reduced CIFAR-10. On CIFAR-10, cutout performs better than Möbius, though Möbius with cutout outperforms both alone.

- Möbius’s performance gains are especially remarkable, because Möbius is only applied to 20% of the data and still achieves superior performance. This means that only 100 images per class in CIFAR-100 and 80 images per class in reduced CIFAR-10 receive Möbius
Table 2: Möbius data augmentation consistently improves the accuracy of image classification on animal superclasses (✓ on ↑), as opposed to non-animal superclasses. This empirical observation suggests that Möbius transformations, having been studied in animals, are particularly attuned to improving generalization in these classes. Note that this finding is interesting, though by no means definitive or theoretically potent.

| Animal Superclass                      | Crop-&-flip | Möbius ↑ | Non-Animal Superclass                  | Crop-&-flip | Möbius ↑ |
|----------------------------------------|-------------|----------|----------------------------------------|-------------|----------|
| Aquatic Mammals                        | 73.0%       | 74.4%    | Flowers                                | 86.0%       | 85.6%    |
| Fish                                   | 81.8%       | 82.4%    | Food Containers                        | 79.6%       | 80.6%    |
| Insects                                | 85.0%       | 85.8%    | Fruit and Vegetables                   | 86.2%       | 89.6%    |
| Large Omnivores and Herbivores         | 84.4%       | 84.8%    | Household Electrical Device            | 86.4%       | 87.0%    |
| Large Carnivores                       | 83.8%       | 84.4%    | Household Furniture                    | 86.4%       | 84.8%    |
| Non-insect Invertebrates               | 81.4%       | 82.0%    | Large Man-made Outdoor Things          | 89.8%       | 89.6%    |
| Medium-sized mammals                   | 82.6%       | 85.2%    | Large Natural Outdoor Scenes           | 85.2%       | 87.2%    |
| People                                 | 64.6%       | 66.8%    | Trees                                  | 75.8%       | 74.4%    |
| Small Mammals                          | 74.8%       | 79.0%    | Vehicles 1                             | 88.6%       | 90.8%    |
| Reptiles                               | 73.8%       | 77.2%    | Vehicles                               | 91.8%       | 90.2%    |

The results of this experiment suggest that a small amount of Möbius data augmentation can improve over cutout, the state-of-the-art performance on sample-level and label-preserving augmentation strategy. This effect is especially prominent in low data regimes, where there are fewer (on the order of hundreds) of samples per class.

4.2. Modulating the inclusion of Möbius

Given the inherent complexity of Möbius transformations, we additionally explore the effects of incorporating an increasing amount of Möbius transformations into the data augmentation process. We evaluate Möbius representations of 10% to 50%, at increments of 10% in between, on CIFAR-10 and CIFAR-100. The goal of this experiment is to examine the effects of modulating Möbius representation during the training process. Note that the experiments in Section 4.1 only focused on a stationary amount (20%) of Möbius.

We compare these increments of Möbius both with and without cutout. We then juxtapose these results with the baseline of cutout alone and that of standard crop-and-flip. We again report average performance and standard deviations across 5 runs on all experimental conditions. The results presented in Figure 4 emphasize the following findings:

- Möbius with cutout outperforms both Möbius alone and cutout alone on both full data settings, suggesting that the two techniques are complementary (Möbius without cutout performs best on reduced CIFAR-10). This is important, as we designed Möbius to combine easily with other augmentation methods.

The results of this experiment suggest that a small amount of Möbius data augmentation can improve over cutout, the state-of-the-art performance on sample-level and label-preserving augmentation strategy. This effect is especially prominent in low data regimes, where there are fewer (on the order of hundreds) of samples per class.

4.3. Random parameterization of Möbius

We run experiments comparing the performance of fully randomized parameterization against our defined parameterization on all dataset settings. Recall that our defined set is finite, while there exists infinite possible Möbius transformations in $\mathbb{R}^2$ under random parameterization. Figure 2 displays a visual comparison.

As shown in Table 3, we observe that random parameterizations of Möbius perform worse than our defined parameterizations, yet still outperform the baseline of crop-and-flip.
5. Conclusion

In this paper, we introduce Möbius data augmentation, a method that applies Möbius transformations to images during training to improve model generalization. Empirically, Möbius performs best when applied to a small subset of data in low data settings. Möbius transformations are complementary to other sample-level augmentations that preserve labels, such as cutout or standard affine transformations. In fact, across experiments on CIFAR-10 and CIFAR-100, we find that cutout and Möbius can be combined for superior performance over either alone. In future work, we plan to examine relaxing the constraints on possible Möbius transformations for a more general construction as well as integrating them into the many successful data augmentation policies for fully supervised and semi-supervised learning. Ultimately, this work presents the first foray into successfully employing Möbius transformations—the next level of mathematical abstraction from affine transformations—for data augmentation in neural networks and demonstrating the efficacy of this biologically motivated augmentation on image classification benchmarks.
Appendix A: Defined Möbius Parameters

For Möbius data augmentation, we parameterize eight different Möbius augmentations by specifying two sets of three points $z_i$ on the original image and $w_i$ as targets, where $i = \{1, 2, 3\}$. Visual examples are shown on different CIFAR-10 classes in Figure 3, which we denote: (1) clockwise twist, (2) clockwise half-twist, (3) spread, (4) spread twist, (5) counter clockwise twist, (6) counter clockwise half-twist, (7) inverse, and (8) inverse spread. Concretely, these parameters are presented below, where $\Re(p)$ and $\Im(p)$ denote the respective real and imaginary components of a point $p$, and height $x$ and width $y$ are dimensions of the original image.

1. Clockwise twist:
   \[
   \begin{align*}
   \Re(z) &= \{1 \cdot 0.5x, 0.6x\}, \\
   \Im(z) &= \{0.5y, 0.8y, 0.5y\}, \\
   \Re(w) &= \{0.5x, 0.5x + 0.3\sin(0.4\pi)y, 0.5x + 0.1\cos(0.1\pi)y\}, \\
   \Im(w) &= \{y - 1, 0.5y + 0.3\cos(0.4\pi)y, 0.5y - 0.1\sin(0.1\pi)x\}
   \end{align*}
   \]

2. Clockwise half-twist:
   \[
   \begin{align*}
   \Re(z) &= \{1 \cdot 0.5x, 0.6x\}, \\
   \Im(z) &= \{0.5y, 0.8y, 0.5y\}, \\
   \Re(w) &= \{0.5x, 0.5x + 0.4y, 0.5x\}, \\
   \Im(w) &= \{y - 1, 0.5y, 0.5y - 0.1x\}
   \end{align*}
   \]

3. Spread:
   \[
   \begin{align*}
   \Re(z) &= \{0.3x, 0.5x, 0.7x\}, \\
   \Im(z) &= \{0.5y, 0.7y, 0.5y\}, \\
   \Re(w) &= \{0.2x, 0.5x, 0.8x\}, \\
   \Im(w) &= \{0.5y, 0.8y, 0.5y\}
   \end{align*}
   \]

4. Spread twist:
   \[
   \begin{align*}
   \Re(z) &= \{0.3x, 0.6x, 0.7x\}, \\
   \Im(z) &= \{0.3y, 0.8y, 0.3y\}, \\
   \Re(w) &= \{0.2x, 0.6x, 0.8x\}, \\
   \Im(w) &= \{0.3y, 0.9y, 0.2y\}
   \end{align*}
   \]

5. Counter clockwise twist:
   \[
   \begin{align*}
   \Re(z) &= \{1 \cdot 0.5x, 0.6x\}, \\
   \Im(z) &= \{0.5y, 0.8y, 0.5y\}, \\
   \Re(w) &= \{0.5x, 0.5x + 0.4y, 0.5x\}, \\
   \Im(w) &= \{y - 1, 0.5y, 0.5y - 0.1x\}
   \end{align*}
   \]

6. Counter clockwise half-twist:
   \[
   \begin{align*}
   \Re(z) &= \{1 \cdot 0.5x, 0.6x\}, \\
   \Im(z) &= \{0.5y, 0.8y, 0.5y\}, \\
   \Re(w) &= \{0.5x, 0.5x + 0.3\sin(0.4\pi)y, 0.5x + 0.1\cos(0.1\pi)x\}, \\
   \Im(w) &= \{y - 1, 0.5y + 0.3\cos(0.4\pi)y, 0.5y - 0.1\sin(0.1\pi)x\}
   \end{align*}
   \]

\[
\text{where } \Re, \Im: \text{ dimensions of the original image.}
\]
7. Inverse:

\[ \Re(z) = \{1, 0.5x, x - 1\}, \]
\[ \Im(z) = \{0.5y, 0.9y, 0.5y\}, \]
\[ \Re(w) = \{x - 1, 0.5x, 1\}, \]
\[ \Im(w) = \{0.5y, 0.1y, 0.5y\} \]

8. Inverse spread:

\[ \Re(z) = \{0.1x, 0.5x, 0.9x\}, \]
\[ \Im(z) = \{0.5y, 0.8y, 0.5y\}, \]
\[ \Re(w) = \{x - 1, 0.5x, 1\}, \]
\[ \Im(w) = \{0.5y, 0.1y, 0.5y\} \]
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