Mass generation in the large N Gross-Neveu model: a constructive proof without intermediate field

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Research Article

Keywords: Gross-Neveu model, constructive field theory, cluster expansion, mass generation, bubble diagrams

Posted Date: June 22nd, 2021

DOI: https://doi.org/10.21203/rs.3.rs-536320/v1

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Mass generation in the large $N$ Gross-Neveu model: a constructive proof without intermediate field

May 10, 2021

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We give a new constructive proof of the infrared behavior of the Euclidean Gross-Neveu model in two dimensions with small coupling and large component number $N$. Our argument does not rely on the use of an intermediate (auxiliary bosonic) field. Instead bubble series are resummed by hand, and determinant bounds replaced by a control of local factorials relying on combinatorial arguments and Pauli’s principle.

The discrete symmetry-breaking is ensured by considering the model directly with a mass counterterm chosen in such a way as to cancel tadpole diagrams. Then the fermion two-point function is shown to decay (quasi-)exponentially as in [12].

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1 Introduction

The Gross-Neveu model is a theory of $N$ interacting charged fermions $(\psi_a, \bar{\psi}_a)$, $a = 1, \ldots, N$, with $\psi_a = \begin{pmatrix} \psi^a_0 \\ \psi^a_1 \end{pmatrix}$, $\bar{\psi}^a = \begin{pmatrix} \bar{\psi}^a_0 \\ \bar{\psi}^a_1 \end{pmatrix}$ two-component spinor fields on $\mathbb{R}^2$. We study the model in imaginary time, i.e. after a Wick rotation, so that the theory is rotation-invariant in the coordinate plane $(x^0, x^1)$. The field index $a$ is called flavour index (by reference to QCD). The covariance of the free fermion field is chosen to be $C(p) = \chi(|p|)$, where $\chi$ is a scale 1, rotation-invariant UV cut-off in momentum space (see later on in the text for details), and $\gamma = \gamma_\mu p^\mu$ (Einstein’s summation intended, with $\mu = 0, 1$), where $\gamma$-matrices are chosen as $\gamma_0 = i\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The interaction Lagrangian is

$$\mathcal{L}(\psi, \bar{\psi}; x) = -\frac{\lambda}{2N} (\bar{\psi} \cdot \psi)^2(x) = -\frac{\lambda}{2N} \sum_{a,b=1}^{N} (\bar{\psi}^a \psi^b \bar{\psi}^b \psi^a)(x),$$

(1.1)

with $\bar{\psi} \cdot \psi = \sum_{i=0,1} \sum_{a=1}^{N} \bar{\psi}^a_i \psi^a_i$, where $\lambda$ is a small enough, positive coupling constant, and $N$ is large enough (depending on $\lambda$).

The model is interesting per se from a theoretical point of view, as it is one of the simplest QFT models exhibiting chiral symmetry breaking. This was understood long ago by D. Gross and A. Neveu themselves [10]. Briefly said, the model can be rewritten in terms of a coupled fermion/boson model with an auxiliary, scalar, bosonic field $\sigma$, featuring a ”QED”-like (non-derivative) cubic vertex $\sqrt{\frac{N}{\lambda}} \sigma (\bar{\psi} \cdot \psi)$. Integrating out fermion fields yields a purely bosonic theory whose effective potential $V(\sigma)$ has the form of a ”Mexican hat”, i.e. is symmetric under the inversion $\sigma \to -\sigma$, and has global minimum at $\sigma = \pm \sigma^*$ with $\sqrt{\frac{N}{\lambda}} \sigma^* \approx e^{-\pi/\lambda}$, zero being a local maximum. As in the low-temperature analysis
of the Ising model, it may be proved that the model exhibits two pure phases, which can be selected by using suitable boundary conditions. These statements were proved at the level of mathematical rigor by C. Kopper, J. Magnen and V. Rivasseau more than twenty years ago [12], using a single-scale cluster expansion and a detailed analysis of the bosonic action functional \( \mathcal{L}(\sigma) = \text{Tr} \log(\tilde{\phi} + \sqrt{\frac{\lambda}{N}} \sigma) \) after an appropriate translation of the \( \sigma \)-field, \( \sigma \rightarrow \sigma - \sqrt{\frac{N}{X}} m_\psi \), where \( m_\psi \approx e^{-\pi/\lambda} \) is chosen such that \( \sqrt{\frac{N}{X}} m_\psi \approx \sigma^* \). Their method also allows a computation of the connected fermion two-point function, which is shown to be massive, i.e. exponentially decaying at large distances with a decay rate \( \approx m_\psi \).

The aim of this article is to prove discrete symmetry breaking and massiveness of the fermion field by an alternative method, avoiding the introduction of the auxiliary field \( \sigma \). Generally speaking, this participates in an effort to discuss rigorously QFT theories with discrete or continuous symmetry breaking, for which auxiliary fields are gauge fields. These fields have several lowest-energy configurations related by the symmetries, which makes them difficult to handle in combination with cluster expansions and multi-scale decompositions, in particular in the case of continuous symmetry breaking. So the present article may be seen as a first attempt in this direction.

In order to motivate our strategy, let us first discuss the lowest-order terms in the bosonic action functional \( \mathcal{L}(\sigma) \). The first-order term is calculated in terms of the tadpole diagram \( \mathcal{T} \).

\[
A(\mathcal{T}) = \frac{1}{(2\pi)^2} \int dp \left( \frac{\chi(|p|)}{\tilde{\phi} + m_\psi} \right) \delta_{0,0}.
\] (1.2)
and (in function of the transfer momentum $q$)

$$\mathcal{A}_q(\Upsilon) = -\frac{1}{(2\pi)^2} \text{Tr}_2 \int dp \frac{\chi(|p|)}{\not p + m_\psi} \frac{\chi(|p+q|)}{\not p + q + m_\psi}$$

for some UV cut-off function $\chi$. Note our particular spinor component convention; $\mathcal{A}(\Upsilon)$ has been chosen to be one of the two (equal) diagonal components $\frac{1}{(2\pi)^2} \int dp \chi(|p|) \not p + m_\psi$, $i = 0, 1$, while $\mathcal{A}(\Upsilon)$ is evaluated as a trace $\text{Tr}_2$ over spinor components, taking into account the $(-1)$ fermionic loop factor. In practice, these diagrams come with an extra prefactor $\lambda N$ due to the coupling constant, and with a flavor-counting factor $O(N)$ which is computed by direct inspection.

To first order, requiring that $\sqrt{N} m_\psi$ is a local extremum of $\mathcal{L}$ is equivalent to requiring the gap equation to hold, see [12], eq. (75),

$$\text{(gap equation)} \quad \frac{\lambda}{N} (2N - 1) \mathcal{A}(\Upsilon) = m_\psi$$

This equation expresses the exact compensation of the first-order term by the one coming from the free part of the action $\frac{1}{2} \sigma^2$.

Then the lowest-order term is the quadratic term $1 - \lambda \mathcal{A}_q(\Upsilon)$. This kernel (denoted $1 - \pi(q)$ in [12]) is shown there to be positive; its inverse $(1 - \pi(q))^{-1}$ is then interpreted as the covariance kernel of the subtracted field $\sigma - \sqrt{N} m_\psi$ (see eq. (84), (86) and Lemma 5). It turns out that the fermionic mass is also essentially the effective mass of the $\sigma$-field: the decrease rate of $(1 - \pi(q))^{-1}$ in coordinate space is $\approx m_\psi$ (see [12] eq. p. 122, and eq. (209)).

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Drawing from these well-established facts, our general line of conduct, in close connection to the above computations, is as follows. As in [12], $\lambda$ is assumed to be small enough, and $N$ larger than some power of $e^{\pi/\lambda} \approx m_\psi^{-1}$.

1. We choose a smooth UV cut-off function $\chi : \mathbb{R}_+ \to [0,1]$ such that $\chi\big|_{[0,1/2]} = 1$, $\chi\big|_{[2,\infty)} = 0$. Contrary to [12], we do not bother to take an analytic cut-off. The price to pay is that we only prove quasi-exponential decay of fermion two-point functions in the end (see Theorem 1.1 below).

2. We choose $m_\psi \approx e^{-\pi/\lambda}$ to be solution of the above gap equation (see Lemma 4.6).

3. We make the substitutions

$$C(p) = \frac{\chi(|p|)}{\not p} \leadsto C^*(p) = \frac{\chi(|p|)}{\not p + m_\psi},$$

a $2N \times 2N$ matrix with $2 \times 2$ flavor components $C^*_{ab}(p) = \delta_{a,b} C^*_{a,a}(p)$;
\[ \mathcal{L}_{\text{int}}(\psi, \bar{\psi}) \sim \mathcal{L}_{\text{int}}(\psi, \bar{\psi}) + \delta \mathcal{L}(\psi, \bar{\psi}) = \mathcal{L}_{\text{int}}(\psi, \bar{\psi}) - m_\psi (\bar{\psi} \cdot \psi). \] (1.6)

Except for terms interacting with the UV cut-off, the model is the same. However, the latter terms would have the effect of breaking the discrete \( \sigma \leftrightarrow -\sigma \) symmetry, should one introduce an auxiliary field \( \sigma \) as in [KMR]. It turns out that discrete symmetry-breaking in our model is a consequence of (1.5,1.6), which spares us the need to carefully choose boundary conditions.

The gap equation then ensures the exact cancellation of tadpole diagrams by the counterterm \( \delta \mathcal{L} \).

4. Schwinger functions with \( N_{\text{ext}} \) external points are completely expanded in terms of the free fermionic measure \( \langle \cdot \rangle_0 \):

\[
\langle \left( \prod_{i=1}^{N_{\text{ext}}/2} \psi^a_i(x_i) \right) \left( \prod_{j=1}^{N_{\text{ext}}/2} \bar{\psi}^{a'}_j(x'_j) \right) \rangle = \left( \prod_{i=1}^{N_{\text{ext}}/2} \psi^a_i(x_i) \right) \left( \prod_{j=1}^{N_{\text{ext}}/2} \bar{\psi}^{a'}_j(x'_j) \right) \sum_{n=0}^{+\infty} \frac{1}{n!} \prod_{i=1}^{n} \int dz_i (-\mathcal{L}(\psi, \bar{\psi}; z_i)) \right)_0 \quad (1.7)
\]

and rewritten as a sum of Feynman diagrams using Wick’s formula. Then bubble chains are resummed by hand and represented as dashed wiggling lines

\[
\begin{array}{c}
\bar{\psi}^a \\
\psi^a \\
\end{array}
\begin{array}{c}
\cdots
\end{array}
\begin{array}{c}
\bar{\psi}^b \\
\psi^b \\
\end{array}
\]

Fig. 1.3. Delocalized vertex

connecting two field pairs \( \bar{\psi}^a \psi^a, \bar{\psi}^b \psi^b \). Resumming the bubble series

\[
\begin{array}{c}
\bar{\psi}^a \\
\psi^a \\
\end{array}
+ \frac{\lambda}{N} 
\begin{array}{c}
\bar{\psi}^a \\
\psi^a \\
\end{array}
+ \left( \frac{\lambda}{N} \right)^2 
\begin{array}{c}
\bar{\psi}^a \\
\psi^a \\
\end{array}
+ \cdots
\]

Fig. 1.4. Bubble series.

at transfer momentum \( q \) yields the \( \Sigma \)-kernel,

\[
\Sigma(q) = \frac{\lambda/N}{1 - \lambda A_q(\Gamma)}, \quad (1.8)
\]
where $A_q(\Upsilon)$ is given exactly by (1.3). In particular, computations (see Lemma 4.3) show that

$$A_0(\Upsilon) \sim_{\lambda \to 0} 1/\lambda$$

(1.9)

at zero transfer momentum, calling for a detailed analysis of the denominator of (1.7) at small momenta. Letting

$$\pi(q) := \lambda A_q(\Upsilon),$$

(1.10)

$$\Sigma(q) = \lambda N (1 - \pi(q))^{-1}$$

is (up to normalization) the covariance kernel of the auxiliary field $\sigma$ in [12]. It is proved in Lemma 4.5 that

$$|\Sigma(x)| = \frac{1}{N} m_\psi^2 O((1 + m_\psi |x|)^{-\infty}), \quad |x| > 1/m_\psi$$

(1.11)

where "$O((1 + m_\psi |x|)^{-\infty})$" means: $\leq C_n (1 + m_\psi |x|)^{-n}$ for every $n \geq 0$, where $C_n$ is a constant independent from the parameters $\lambda$ and $N$. Thus the kernel $\Sigma(x)$ exhibits a quasi-exponential decay with rate $\approx m_\psi$, in general agreement with the estimates proved in [12]. The prefactor in (1.11) deteriorates for $|x|$ smaller, but the relevant quantity is the $L^1$-norm of $\Sigma$, for which we prove

$$||\Sigma||_{L^1} = \int dx |\Sigma(x)| = O(1/N).$$

(1.12)

Note the absence of logarithmic prefactor $\log(1/m_\psi) \approx 1/\lambda$ in (1.12), as opposed to (1.9).

At the end of this stage, we have a series of Feynman diagrams with delocalized vertices, thereafter called: partially resummed Feynman diagrams or simply (if no ambiguity can arise) Feynman diagrams. Feynman diagrams in our sense make up a set $\mathcal{FD}$ which is precisely defined in Definition 2.2. Identifying (by formal analogy) delocalized vertices $\Sigma(x,y)$ with a photon-photon propagator $\langle \sigma(x)\sigma(y) \rangle_0$, the set $\mathcal{FD}$ may be identified as a subset of the set of Feynman diagrams of QED with $N$ fermions.

We pause to comment on large orders of the theory. The power-counting for just renormalizable theories with a $1/N$-expansion is well-known; let us redo computations here. Consider a Feynman diagram with $V$ vertices, $I$ internal lines, $N_{\text{ext}}$ external lines, and $L$ loops. Graph topology yields $L = I - V + 1$ and $4V = 2I + N_{\text{ext}}$. Generally speaking, one expects

(i) that the effective volume of integration $\text{Vol}$ for a vertex is $O(m_\psi^{-2})$ because of the quasi-exponential decay rate $m_\psi$;

(ii) that $|C_{a,a}^*(x,y)| \lesssim \text{Vol}^{-1} \sum_{i,j=0,1} ||C_{a,i,a,j}^*||_{L^1} = O(m_\psi)$ in average (see Lemma 4.1).

Taking into account the coupling constant, the power-counting for an integrated vertex is $\approx \frac{1}{N} m_\psi^{-2}$. Finally, one must count $N$ per fermionic loop (sum over flavor indices).
Disregarding \( N_{\text{ext}} \) (which is fixed), the naive power-counting for a large Feynman diagram is therefore expected to be roughly

\[
m_I(\frac{\lambda}{N} m_\psi^{-2}) V N^L \approx m_\psi^2(\frac{\lambda}{N} m_\psi^{-2}) V N^V \approx \lambda^V
\]

(1.13)

which looks convergent. However, by substituting in (ii) an averaged quantity to \( C^*(x, y) \) one has overlooked the fact that the prefactor in front of \( C^*(x, y) \) (see (4.2)) is not \( m_\psi \) but \( \frac{1}{1 \vee |x - y|} \) for \( |x - y| < m_\psi^{-1} \), where: \( 1 \vee |x - y| := \max(1, |x - y|) \). The square of this function has logarithmic divergence when \( m_\psi \rightarrow 0 \). For small but fixed \( m_\psi \), this translates into the logarithmic behavior in \( O(1/\lambda) = O(\log(1/m_\psi)) \) of the bubble at zero transfer momentum (see (4.20)).

From the above it may be conjectured that logarithms appear only due to bubbles, and that they disappear by resumming bubble chains into the integrable kernel \( \Sigma \). This is precisely what we do. The main result of the article is the following.

**Theorem 1.1 (main result)** There exists \( \lambda_{\text{max}} > 0 \) such that the following holds. Assume \( \lambda < \lambda_{\text{max}} \) and \( N \gg m_\psi^{120} \), where \( m_\psi \approx e^{-\pi/\lambda} \) is the solution of the gap equation (4.48). Then the (connected) two-point function \( \langle \psi^a_i(x) \bar{\psi}^a_j(x') \rangle \) of the model defined by 1., 2., 3. above has quasi-exponential decay at large distances,

\[
\left| \langle \psi^a_i(x) \bar{\psi}^a_j(x') \rangle \right| = O((1 + m_\psi|x - x'|)^{-\infty})
\]

(1.14)

where "\( O((1 + m_\psi|x - x'|)^{-\infty}) \)" means: \( \leq C_n (1 + m_\psi|x - x'|)^{-n} \) for every \( n \geq 0 \), where \( C_n \) is a constant independent from the parameters \( \lambda \) and \( N \).

We may now resume our general presentation, and discuss the main elements of the proof of Theorem 1.1.

5. A well-known problem in constructive field theory is that of local factorials, which we set about to explain briefly. Let us partition coordinate space into \( \psi_{\Delta \in D} \Delta \), where \( \Delta \) are square boxes of size \( m_\psi^{-1} \), corresponding to the inverse of the decay rate of the kernels \( C^* \) and \( \Sigma \). Forgetting about signs (so that our remark also holds for bosonic theories), the sum of the absolute values of the amplitudes of individual Feynman diagrams with a fixed number \( n_\Delta^a \) of fields \( \psi^a \) per box \( \Delta \in D \) (and therefore, the same number of conjugate fields \( \bar{\psi}^a \)) involves the combinatorial factor \( \prod_{\Delta \in D} \prod_a O(n_\Delta^a !) \), as a consequence of Wick’s formula, a product over boxes \( \Delta \) of so-called local factorials.

There are several known approaches to cure this problem in the case of a fermionic theory, that can all go under the name of **determinant bounds**. (i) One is to note that the sum over all diagrams obtained by Wick’s rule from

\[
\langle \left( \prod_{i=1}^{N_{\text{ext}}/2} \psi^a_i(x_i) \right) \left( \prod_{j=1}^{N_{\text{ext}}/2} \bar{\psi}^a_j(x'_j) \right) \prod_{i=1}^{n} (-\mathcal{L})(\psi, \bar{\psi}; z_i) \rangle_0
\]
may be written as a determinant of the general form \( \prod_{a=1}^{N} \det \left( C^*(y_a^a, (y'_a)_j) \right) \), where \( y_a^a \), resp. \( (y'_a)_j \), are the locations of the fields \( \psi^a \), resp. \( \bar{\psi}^a \), and to use for example Gram determinant bounds, see e.g. [14, 15], and also e.g. [2, 3, 4, 8] for some applications to important condensed matter models. (ii) Another argument, which is due to Feldman and can be found in the appendix of Iagolnitzer-Magnen [11], consists in Taylor expanding to a certain fixed order \( k \) the fields \( \psi^a(y_a^a) \) located in a cell \( \delta \) around the center \( x_\delta \) of the cell (which is equivalent to Taylor expanding the covariance functions \( C^* \)). Expanding by multilinearity and keeping in mind that determinants of a matrix with two identical lines vanish, we see that all but a finite number of lines \( i \) with \( y_a^a \in \delta \) must feature a Taylor remainder of the form \( (y_a^a - x_\delta)^k \nabla_k C^*((1-t)x_\delta + ty_a^a, \cdot) \), with \( 0 \leq t \leq 1 \) and \( |y_a^a - x_\delta| \leq \text{diam}(\delta) \). Choosing carefully the order \( k \) (large enough) and the diameters of the cells (small enough) produces extra inverse local factorials to the desired power, compensating the above-mentioned local factorials.

The problem now is that the partial resummation (step 4.) destroys the determinantal structure of Schwinger functions (the \( \Sigma \)-kernel is a bosonic two-point function). However, we have been able to modify Feldman’s argument to make it compatible with the bubble resummation; see section 2. The core of the modified argument is the following. Determinants with two equal lines vanish as a consequence of the fact that they are alternating multilinear forms, namely, permuting fields, \( (\psi(y_a^a))_i \rightarrow (\psi(\sigma_i)_a) \), according to a permutation of indices \( \sigma \) leaves the determinant invariant up to a sign \( \varepsilon(\sigma) \) (signature of the permutation). Then summing over all permutations yields zero because \( \sum_\sigma (-1)^\sigma \varepsilon(\sigma) = 0 \). Each individual term in the determinant (arising from Leibniz’s expansion into a sum over all permutations) is a Feynman diagram, so permutations define a mapping on the set of Feynman diagrams. Now, we prove that a (large, even if partial) group of permutations leaves our set of partially resummed Feynman diagrams \( \mathcal{FD} \) invariant. This is the key to our generalized determinant bound.

6. The last point is to produce bounds for the sum of terms produced by the above expansions. Leaving aside the field translations and gradients produced by the Taylor expansion in 5., Feynman diagrams still have the same topological structure as Feynman diagrams of QED, namely, they present themselves as a set of single-flavored fermion loops connected by \( \Sigma \)-kernels. The general aim is to prove that all diagrams not reduced to a single propagator have extra \( 1/N \) factors. The first step is to choose a loop spanning tree and distinguish between connecting vertices (edges of the tree) and non-connecting vertices. The power-counting of connecting vertices is computed in such a way as to include the sum over the flavors of the loops. On the other hand, there is no flavor sum involved with non-connecting vertices, i.e. there is a \( 1/N \) factor involved with each of them. We conclude by proving that the number of non-connecting vertices increases linearly with the total number of vertices, and that the associated \( 1/N \)-factors ensure the convergence of the series of perturbations for \( N \) large enough.

Here is a very brief outline of the article. Our generalized determinant bound (see 5.) is presented in section 2. Constructive bounds (see 6.), in particular, a proof of Theorem 1.1,
are presented in section 3. Finally, diagram and kernel estimates used in the proofs, together with a fixed-point argument to solve the gap equation, are presented in Appendix.

The ultra-violet limit of the model is not discussed in the present article. It was understood perturbatively in the original work by Gross and Neveu, and later on proved at the level of mathematical rigor in [6, 7], that the model is asymptotically free at high energies. It would be interesting to couple our analysis with that of [6, 7], and also to be able to overcome the strong, \( \lambda \)-dependent condition on \( N \), in such a way as to 'construct' the model without any cut-off for \( N \) large enough, and an effective coupling constant \( \lambda \) small enough at momentum scale 1. We plan to do this in a near future.

Acknowledgements. Many warm thanks go to J. Magnen (even though he did not work actively on this project) for the long-standing collaboration which gave rise to this project.

Notations. We frequently use the notation \( f(\lambda, N) \lesssim g(\lambda, N) \) when there exists a constant \( c > 0 \) independent of \( \lambda, N \) (possibly depending on the cut-off function \( \chi \)) such that \( f(\lambda, N) \leq cg(\lambda, N) \); the inequality holds in a \( (\lambda, N) \) region \( (\lambda \text{ small enough}, N \text{ large enough}) \) which will be clear in the context. Similarly, \( (g(\lambda, N) \gtrsim f(\lambda, N)) \Leftrightarrow (f(\lambda, N) \lesssim g(\lambda, N)), \) and \((f(\lambda, N) \approx g(\lambda, N)) \Leftrightarrow ((f(\lambda, N) \lesssim g(\lambda, N)) \text{ and } (g(\lambda, N) \lesssim f(\lambda, N)))\).

2 The generalized determinant bound

The section (see step 5. in the Introduction) is divided into two parts. In §2.1, we discuss the set \( \mathcal{FD} \) of partially resummed Feynman graphs (which we simply call: set of Feynman graphs thereafter), and the action of a group of permutations \( \mathcal{G} \) on \( \mathcal{FD} \). In §2.2, we Taylor expand covariance kernels of diagrams and show that averaging w.r. to the action of \( \mathcal{G} \) kills leading order terms when the number of fields in a given cell of side \( m^{-1}_\psi \) exceeds a certain finite value, leaving small factors that compensate local factorials.

Definition 2.1 (coordinate space partition) A box \( \Delta \) is an open square of the form \((k_0m^{-1}_\psi, (k_0 + 1)m^{-1}_\psi) \times (k_1m^{-1}_\psi, (k_1 + 1)m^{-1}_\psi)\), with \( k_0, k_1 \in \mathbb{Z} \).

Boxes are cells of side \( m^{-1}_\psi \) (later on, in §2.2, we introduce smaller cells \( \delta \subset \Delta \); only cells \( \Delta \) of side \( m^{-1}_\psi \) are called boxes). All computations below are done assuming that all vertices are enclosed in a square volume \( \Omega \) of the type \([-nm^{-1}_\psi, nm^{-1}_\psi]^2\), with \( n \in \mathbb{N} \) arbitrarily large. We denote by \( \mathbb{D} \) the set of boxes \( \Delta \) included in \([-nm^{-1}_\psi, nm^{-1}_\psi]^2\). The proof of Theorem 1.1 is given assuming implicitly that \( n \to \infty \), i.e. in the thermodynamic limit.

2.1 Action of permutations on the set of Feynman graphs

We fix in the ensuing discussion:
- a finite set of external fields \( \Psi_{ext} = \{ (\psi^{\alpha_i}(x_i)), (\bar{\psi}^{\bar{\alpha}_j}(x'_j)) \} \);
- a finite set of disjoint cells \( \Delta = (\Delta_1, \ldots, \Delta_p) \);
– for each $\Delta \in \Delta$ and each flavor $a$, an integer number $n_a^\Delta$ (the number of vertices of type $(\bar{\psi}^a \psi^a)(x)$ with $x$ located in $\Delta$, or equivalently, the number of fields $\psi^a(x)$ with $x$ located in $\Delta$; see Definition below). By assumption $\sum_a n_a^\Delta > 0$.

Let $\mathbf{n} := (n_a^\Delta)_{\Delta \in \Delta, 1 \leq a \leq N}$ and $|\mathbf{n}| := \sum_{\Delta, a} n_a^\Delta$.

Below, we define an integer $m_a^\Delta$ which is $\geq \lceil n_a^\Delta / 5 \rceil = \min\{m \in \mathbb{N} \mid m \geq n_a^\Delta / 5\}$.

Delocalized vertices (see Fig. 3.1) are made up of two half-vertices $v, v'$ with locations $x, x'$ connected by the $\Sigma$-kernel $\Sigma(x, x')$. In the remainder of the section, we shall say vertex instead of half-vertex. Fermion propagators (full lines) are oriented (following usual convention in QFT) from $\bar{\psi}$ to $\psi$.

Following fermion lines, one obtains single-flavored, oriented fermion loops. Fermion loops are connected by $\Sigma$-kernels. The precise structure of such diagrams will now be described.

**Definition 2.2** Let $\mathcal{FD}_n$ be the set of Feynman graphs $\Gamma = (V, E)$ such that:
- $V = V_{\text{int}} \uplus V_{\text{ext}}$ (set of internal/external vertices), where $V_{\text{int}}$ is a set indexed by $1, \ldots, |\mathbf{n}|$;
- each vertex $v \in V_{\text{int}}$ has a flavor $a(v)$ and a cell localization $\Delta(v) \in \Delta$; letting $V_{a}^\Delta$ be the subset of internal vertices with flavor $a$ and localized in $\Delta$, $|V_{a}^\Delta| = n_a^\Delta$;
- $E = \left( \biguplus_{1 \leq a \leq n} E^a \right) \uplus E^{\text{bub}}$, where elements of $E^a$ are oriented edges (drawn as simple lines with an arrow giving the orientation), and elements of $E^{\text{bub}}$ non-oriented edges (drawn as dashed wiggling lines); a vertex $v \in V_{\text{int}}$ has two oriented edges with same flavor index $a$ attached to it (one incoming, the other outgoing), and one non-oriented;
- following oriented edges, one obtains closed loops, and open loops originated from and ending in external vertices;
- non-oriented edges connect (closed or open) loops;
- a closed loop $\gamma$ is connected to $\geq 3$ (closed or open) loops, i.e. there are $\geq 3$ vertices along $\gamma$.

The length $n(\gamma)$ of a loop $\gamma$ is the number of vertices along $\gamma$. If $n \geq 0$, we let

$$\mathcal{FD}_n := \bigcup_{|\mathbf{n}|=n} \mathcal{FD}_n$$

be the set of Feynman diagrams with $n$ vertices. Finally, $\mathcal{FD} := \bigcup_{n \geq 0} \mathcal{FD}_n$. The evaluation $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma; (x_v)_{v \in V})$ is the product $\prod_a \prod_{\ell = (x, x')} \mathcal{A}(x, x') = \prod_{\ell = (x, x')} \mathcal{B}(x, x')$.

A detailed example. Below, a Feynman diagram with 14 internal vertices numbered from 1 to 14 (see Definition 2.2), two external vertices (located at $x, x'$), and three loops (to which we have given here arbitrary labels 1, 2, 3 for convenience)
The last property of the Definition ("there are $\geq 2$ vertices along $\gamma$...") is not necessarily preserved by an arbitrary exchange of $\psi^a$-fields.

**Definition 2.3 (distance of vertices along a loop)** Let $(v_1, \ldots, v_{n(\gamma)})$ be the set of internal vertices along a loop $\gamma$ obtained by following its oriented edges (if the loop is open, then the ordering of this set is unambiguous, otherwise one chooses arbitrarily a first vertex $v_1$). Then

$$d(v_i, v_j) := \begin{cases} |j - i| & (\gamma \text{ open}) \\ \min(|j - i|, |j - i - n(\gamma)|, |j - i + n(\gamma)|) & (\gamma \text{ closed}) \end{cases}$$

(2.2)

(called: **distance of** $v_i$ **and** $v_j$ **along** $\gamma$) is the minimal number of oriented edges along $\gamma$ connecting $v_i$ to $v_j$ or $v_j$ to $v_i$.

If $v, v'$ are internal vertices which do not belong to the same loop, one defines $d(v, v') = +\infty$.

Let $\Gamma \in \mathcal{F}_n$. Fix $\Delta \in \Delta$ and $a = 1, \ldots, N$. Assume $n^a_\Delta \geq 1$. We define inductively a shortlist of $m^a_\Delta$ internal vertices $v_{i_1} < \ldots < v_{i_{m^a_\Delta}}$ extracted from the set of $n^a_\Delta$ internal vertices $v_1 < \ldots < v_{n^a_\Delta}$

- $i_1 := 1$;
- $i_2 := \min\{j = 2, \ldots, n^a_\Delta \mid d(v_{i_1}, v_j) \geq 3\}$;
- if $i_1 < i_2 < \ldots < i_k$ have been constructed, then

$$i_{k+1} := \min\{j = i_k + 1, \ldots, n^a_\Delta \mid \min_{1 \leq i \leq k} d(v_{i_1}, v_j) \geq 3\}.$$  

(2.3)

The algorithm stops when the set in (2.3) is empty, and $m^a_\Delta$ is the last value of $k$.

Then the **ordering of internal vertices** in $\Gamma$ is permuted by shuffling them in such a way that $v_{i_1}, \ldots, v_{i_{m^a_\Delta}}$ come first in the list. Explicitly, $v_i$ is now called $v_{\sigma^{-1}(i)}$, where $\sigma(1) =
\(i_1, \ldots, \sigma(m^\Delta_{\Delta}) = i_{m^\Delta_{\Delta}}\) and \(\{1, \ldots, n^\Delta_{\Delta}\} \setminus \{i_1, \ldots, i_{m^\Delta_{\Delta}}\} = \{\sigma(m^\Delta_{\Delta} + 1) < \cdots < \sigma(n^\Delta_{\Delta})\}\). (For more readability we skipped the \((\Delta, a)\)-indices, but \(v_i, i = 1, \ldots, n^\Delta_{\Delta}\) are rewritten \(v^\Delta_{\Delta,i}, i = 1, \ldots, n^\Delta_{\Delta}\) below when \((\Delta, a)\) are allowed to vary).

We must still prove that \(m^\Delta_{\Delta} \geq \lceil n^\Delta_{\Delta} / 5 \rceil\). The neighborhood \(\mathcal{N}(v)\) of an internal vertex located on an (open or closed) \(\gamma\) is the set of internal vertices \(v'\) along \(\gamma\) such that \(d(v, v') \leq 2\); its cardinal is \(|\mathcal{N}(v)| \leq \min(5, n(\gamma)) \leq 5\). Cutting out \(\mathcal{N}(v)\) from \(\gamma\) yields (depending on the case) a shortened open loop \(\gamma'\) or two loops \(\gamma'_1, \gamma'_2\).

We define inductively a sequence of non-empty sets of loops \(\gamma^1, \gamma^2, \ldots, \gamma^k, \ldots, \gamma^{m^\Delta_{\Delta}}\). Initially \((k = 1)\), \(\gamma^1\) is the set of loops containing at least one of the vertices in \(V^\Delta_{\Delta}\). Cutting out \(\mathcal{N}(v_{i_1,1})\) from \(\gamma^1\), we get a new set of loops \(\gamma^2\). Then \(i_2\) is the first index such that \(v_{i_2}\) belongs to \(\gamma^2\); one cuts out \(\mathcal{N}(v_{i_2})\) from \(\gamma^2\), and so on. The process may go on as long as \(\gamma^k\) is non-empty. Since the total number of vertices along \(\gamma^k\) is \(\geq n^\Delta_{\Delta} - 5(k - 1)\), \(m^\Delta_{\Delta} = \max\{k \mid \gamma^k \neq \emptyset\} \geq \lceil n^\Delta_{\Delta} / 5 \rceil\).

Consider now the following diagram

\[\text{Diagram}\]
where $v_{\Delta,i}^a, v_{\Delta,j}^a \in \{v_1, \ldots, v_{m^a_\Delta}\}$ belong to the shortlist. Exchanging the two $\psi^a$ attached to $v_{\Delta,i}^a$, resp. $v_{\Delta,j}^a$ (in red on the figures), one gets the new diagram

This graphical operation on Feynman diagrams is called $\Pi_{\Delta,a}(i,j)$. There are two different cases from a topological point of view; we let $\tilde{d}$ be the new distance function between vertices. We use the fact that closed loops have $\geq 3$ vertices along them. (i) If $v_{\Delta,i}^a, v_{\Delta,j}^a$ were originally on two distinct fermionic loops, now the two loops have merged, and $\tilde{d}(v_{\Delta,i}^a, v_{\Delta,j}^a) \geq \min(d'(v_{\Delta,i}^a, v) + 1, d'(v_{\Delta,j}^a, v') + 1) \geq 3$, where $d'$ is the distance function when the two edges $(v, v_{\Delta,i}^a)$ and $(v', v_{\Delta,j}^a)$ are cut out. (ii) If $v_{\Delta,i}^a$ and $v_{\Delta,j}^a$ were originally on the same loop, then the loop has split into two parts, and they are now on two different loops, so $\tilde{d}(v_{\Delta,i}^a, v_{\Delta,j}^a) = \infty$. Also, by construction $d(v_{\Delta,i}^a, v_{\Delta,j}^a) \geq 3$, hence $d(v_{\Delta,i}^a, v') \geq 2$, which means that the number of vertices along the new loop containing $v_{\Delta,i}^a$ (and similar for the one containing $v_{\Delta,j}^a$) is $\geq 3$. (In particular, the new diagram is a Feynman diagram in our language).

The above action of transposition extends by composition to an action of the group

$$\mathfrak{S}_n := \prod_{\Delta,a} \mathfrak{S}_{\Delta,a}$$

on $\mathcal{FD}^*_n$, where $\mathfrak{S}_{\Delta,a}$ is the group of permutations acting on the fields $\psi^a$ attached to $\{v_{\Delta,1}^a, \ldots, v_{\Delta,m}^a\}$, where $m$ is any integer $\leq m^a_\Delta$ (in §2.2 below we choose for simplicity $m = \lceil n^a_\delta/5 \rceil$ instead of $m^a_\delta$).

Note that $m^a_\Delta$ and the list of indices $i_1, \ldots, i_{m^a_\Delta}$ is left unchanged through the action of $\mathfrak{S}_n$.

Now comes our main observation. The sign attached to a permutation $\sigma$ is the signature $\varepsilon(\sigma)$. In particular, the following property (P) holds,

(P) Assume that all fields $\psi_{i_1}^a, \ldots, \psi_{i_{m^a_\Delta}}^a$ attached to $v_{i_1}, \ldots, v_{m^a_\Delta}$ are located at the same point $x$, and $m^a_\delta \geq 2$. Then the sum $\sum_{\sigma \in \mathfrak{S}_{\Delta,a}} \varepsilon(\sigma)A(\sigma(\Gamma); (x_v)_{v \in V})$ vanishes.
In the next subsection §2.2), we shall consider cells $\delta$ which are obtained by partitioning a box $\Delta \in \mathcal{D}$, and fields $\psi_{i_1}^a, \ldots, \psi_{i_{m_\Delta}}^a$ will actually be gradient fields $\nabla^{\kappa_1} \psi_{i_1}^a, \ldots, \nabla^{\kappa_{m_\Delta}} \psi_{i_{m_\Delta}}^a$. Then the same conclusion holds provided there exist $1 \leq j \neq k \leq i_{m_\Delta}$ such that multi-indices $\kappa_j, \kappa_k$ coincide.

A detailed example (continued). See Fig. 2.1. We assume that all 14 vertices have same flavor index $a$ and box location $\Delta$. The fist step consists in reshuffling indices as explained below (2.3); the list $\{i_1, \ldots, i_{m_\Delta}\}$ of vertices constructed in (2.3) is $\{1, 4, 5, 10, 13\}$ and has cardinal $5 \geq \lceil 14/5 \rceil = 3$. After reshuffling, we get

![Diagram](image)

The group $\mathfrak{S}$ of permutations of $\{1, 2, 3, 4, 5\}$ acts on this diagram by exchanging the $\psi$-fields attached to vertices labeled from 1 to 5, generating a set of diagrams; in particular, the action $\Pi_{\Delta,a}(1, 2)$ of the transposition $\sigma_{12}$ (exchange of 1 and 2) yields

![Diagram](image)

The effect has been to split the fermion loop labeled 1 into two loops along the edges $7 \rightarrow 1$ and $8 \rightarrow 2$ leading to 1 and 2 along the oriented loop.

The action of $\sigma_{13}$ yields instead

\[ \begin{array}{c}
\end{array} \]
The effect has been to merge the fermions loops labeled 1 and 2 along the edges 7 → 1 and 11 → 3.

Note that splitting/merging operations do not necessarily preserve diagram connectedness. On the other hand, it preserves box-connectedness: if the diagram $\Gamma$ obtained from $\Gamma$ by gluing together vertices located in the same box is connected, then the same holds for the image of $\Gamma$ by $\sigma \in S_n$, in other words, $\sigma(\Gamma)$ is box-connected.

2.2 Generalized bound

As explained in the Introduction, we adapt an argument from [11]. Fix $k \in \mathbb{N}$ large enough, but independently of all parameters $N, \lambda, \cdots$ (the value of $k$ will be fixed in the end of the argument). For a given box $\Delta$ and color $a = 1, \ldots, N$, propagators $C_a(x, \cdot) = \langle \psi_a(x) \bar{\psi}_a(\cdot) \rangle_0$ with $x \in \Delta$ ($\Delta \in \mathbb{D}$) can be replaced using a Taylor expansion by

$$\sum_{|\kappa| < k} \left\{ \frac{(x - x_{\delta})^\kappa}{k!} \nabla^\kappa \right\} C_a(x_{\delta}, \cdot) + \sum_{|\kappa| = k} (x - x_{\delta})^\kappa \int_0^1 dt \frac{(1 - t)^{k-1}}{(k-1)!} \nabla^\kappa \right\} C_a((1 - t)x_{\delta} + tx, \cdot), \tag{2.5}$$

where $x_{\delta}$ is the center of a sub-box $\delta \subset \Delta$ containing $x$. This is equivalent to displacing the field $\psi(x)$ to the location $x_{\delta}$. Terms on the first line (2.5) are called fully expanded terms, there are $2(1 + 2 + 2^2 + \cdots + 2^{k-1}) = 2(2^k - 1) < 2^{k+1}$ of them; terms on the second line (2.6) are called Taylor remainders.

Let $n_a^\delta$ be the number of flavor $a$ fields located in $\delta$, and (after reshuffling of the list of $\psi^a$ fields located in $\delta$, see (2.3)) $\psi_1, \ldots, \psi_{\lfloor n_a^\delta / 5 \rfloor}$ the $\lfloor n_a^\delta / 5 \rfloor$ first $\psi^a$-fields, forming what we call the shortlist. The key argument is the following: by Property (P) (see below (2.4)), contributions with two fields $\psi^a(x_{\delta})$ in the shortlist with same (flavor, spinor and gradient) indices and located at the same point vanish after averaging over the action of the
permutation group $\mathcal{S}_{\delta,a}$. Therefore, at most $2^{k+1}$ fields can be fully expanded; if $\lfloor n_\Delta^a/5 \rfloor \geq 2^{k+1}$, then fully expanded terms associated to remaining $\lfloor n_\Delta^a/5 \rfloor - 2^{k+1}$ fields give zero contribution, so remaining fields can be replaced by their Taylor remainders.

Choosing carefully the size of sub-boxes $\delta$ of a given box $\Delta$ will make Taylor remainders "combinatorially small". Namely, fixing $a$ and $\Delta$, we subdivide equally $\Delta$ into $\lfloor n_\Delta^a \Delta/((5 \times 2^{k+2}) \times 2^{k+2}) \rfloor$ boxes with size $(n_\Delta^a \Delta/((5 \times 2^{k+2}) \times 2^{k+2})) - 1/2$ provided $n_\Delta^a \Delta > 5 \times 2^{k+2}$ (otherwise we simply let $\delta = \Delta$ be the unique sub-box). Then, in Taylor remainders, $|x - x_\delta|^\kappa = O(m_\psi^{-k})$, times the small factor $(n_\Delta^a \Delta/((5 \times 2^{k+2}) \times 2^{k+2})) - 1/2$.

Collecting remaining factors, we are left with at most $O((m_\psi^{-1})^{1/5}) = O(m_\psi^{-4})$ per field $\psi$.

In the next section, one shall show that diagrams with $n$ vertices have extra $(1/N)^{n/3}$ factors, which compensate the factor $O(m_\psi^{-4})$ per annihilation field just found provided $N$ if larger than some inverse power of $m_\psi$.

3 Constructive bounds

We prove here Theorem 1.1. We separate the two main issues. We first solve the problem of summing over topologies of Feynman diagrams (see §3.1); it is a power-counting argument, characteristic of $1/N$ expansions met in all theories with $N$ flavors and $O(1/N)$-coupling constant, based in particular on the construction of a loop spanning tree. In the second part (see §3.2), we expound a classical, general constructive argument implying convergence of the sum over the box locations of vertices.

3.1 Loop spanning tree and $1/N$ expansion

Applying Taylor's formula (§2.2) has 'displaced' the fields, $\psi_t^a(x) \rightsquigarrow \nabla^k \psi_t^a(x_\delta)$, but not changed the topological structure of diagrams: we have a theory with delocalized vertices

![Fig. 3.1. A delocalized vertex.](image)
made up of two half-vertices \( v = \bar{\psi}^a \psi^a, v' = \bar{\psi}^b \psi^b \) connected by a \( \Sigma \)-kernel; following half-vertices, one gets monochromatic fermionic loops. The first step is to choose a loop spanning tree, i.e. a tree connecting loops.

![Fig. 3.2. A loop spanning tree with \( L' = 5 \).](image)

Delocalized vertices which are edges of the loop spanning tree have been drawn in the above Figure as full wiggling lines; we call them connecting vertices. All other delocalized vertices (dashed on the Figure) are called non-connecting vertices. Loops are vertices of the tree, denoted \( \gamma_1, \ldots, \gamma_{L'} \); their color indices are \( a_1, \ldots, a_{L'} \). Extra delocalized vertices produce a graph connecting \( \gamma_1, \ldots, \gamma_{L'} \). N.B. In the sequel, "vertices" refer to delocalized vertices as on Fig. 3.1, not to loops.

The key point is that the number \( n_\gamma \) of half-vertices along a given loop \( \gamma \) is \( \geq 3 \): \( \gamma_v = 1 \) is excluded because tadpoles are compensated by the mass counterterm (as a consequence of the gap equation), and \( \gamma_v = 2 \) is excluded because bubble chains have been resummed.

A mild complication comes from the fact that kernels \( C^* (x) \) and \( \Sigma(x) \) have a much worse behavior when \( |x| < m_\psi^{-1} \) than the behavior expected from the large-distance estimates,

\[
|C^*_{aa}(x)| \leq m_\psi O((1 + m_\psi |x|)^{-\infty}), \quad |\Sigma(x)| \leq \frac{1}{N} m_\psi^2 O((1 + m_\psi |x|)^{-\infty}) \quad (|x| > m_\psi^{-1})
\]

(3.1)

where "\( O((1 + m_\psi |x|)^{-\infty}) \)" means \( \leq C_p \left((1 + m_\psi |x|)^{-p}\right) \) for every \( p \geq 0 \). Indeed, (letting \( ||f||_\infty := \sup_x |f(x)| \)),

\[
||C^*_{aa}||_\infty \lesssim 1, \quad ||\Sigma||_\infty \lesssim 1/N
\]

(3.2)

(see Lemmas 4.1 and 4.5 for all these), but bounds for \( |x| \) varying from 1 to \( m_\psi^{-1} \) have a characteristic inverse polynomial decrease in \( (m_\psi |x|)^{-1} \) or \( |x|^{-2} (\log(1/m_\psi |x|))^{-2} \). A multi-scale expansion – or, more simply perhaps, a detailed analysis of the convolution of \( C^* \) and \( \Sigma \)-kernels in a diagram – would do justice to these factors in a proper way. Here we shall be content with using the less-than-optimal bounds (3.2), in which a loss of one or two powers of \( m_\psi \) compared to (3.1) is manifest.

Let us redo the power-counting of (1.13). The result is generally expressed as a product of contributions coming from connecting vertices (explored inductively from the root of the loop spanning trees to its leaves by any search algorithm) and of contributions coming from non-connecting vertices.
(i) Count \( N \) for the choice of the color \( a_1 \) of the loop \( \gamma_1 \).

(ii) For a connecting vertex made up of two-half vertices \( v, v' \) located at \( x, x' \), count

- \( N \) for the choice of the color of the newly explored loop;
- \( |\Sigma(x, x')| \lesssim |\Sigma|_\infty \lesssim 1/N \);
- \( O((m_\psi^{-2})^2) = O(m_\psi^{-4}) \) for the integral over \( x \) and \( x' \);
- \( \left( \sqrt{\sup_{y, y'} |C_{aa}(y, y')|} \right)^4 \lesssim 1 \) for the four contracted fermions \( \psi^a, \bar{\psi}^a, \psi^b, \bar{\psi}^b \) (the square-root is there to avoid double-counting, since \( C^* \)-kernels are shared by two consecutive vertices);

all together, \( O(m_\psi^{-4}) \);

(iii) For a non-connecting vertex, there is no sum over colors, so the power-counting is

\[ O\left( \frac{m_\psi^{-4}}{N} \right). \]

Let \( n \) be the total number of vertices of the graph, \( n' \) the number of connecting vertices, and \( n'' = n - n' \) the number of non-connecting vertices. Because (as emphasized above) there are at least 3 half-vertices along a given loop, \( L' \leq \frac{2}{3} n \). Now, \( n' = L' - 1 \) by construction, so

\[ n' \leq \frac{2}{3} n, \quad n'' \geq \frac{1}{3} n \quad (3.3) \]

and (assuming \( N \geq m_\psi^{-4} \)) the total weight of the above graph is at most of order

\[ N \times (m_\psi^{-4})^{n'} \times \left( \frac{m_\psi^{-4}}{N} \right)^{n''} \leq N \left( \frac{m_\psi^{-4}}{N} \right)^{3n/3} = N \left( \frac{m_\psi^{-12}}{N} \right)^{n/3}. \quad (3.4) \]

Thus, if e.g.

\[ N \gg m_\psi^{-72}, \quad (3.5) \]

we get the small factor \( O(N^{-1/6}) \) per vertex, times \( O((m_\psi^{-12+72/2})^{1/3}) = O(m_\psi^8) \) per vertex, which we turn into \( O(m_\psi^4) \) per annihilation field \( \psi \), compensating the \( O(m_\psi^{-4}) \) per field \( \psi \) found in C.1.2 above. The small factor \( O(N^{-1/6}) \) per vertex makes it possible to sum Feynman diagrams of large order.

### 3.2 Sum over cell locations

We prove in this section Theorem 1.1. We fix a flavor \( a \); two spin indices \( i, j \); two boxes \( \Delta_{ext}, \Delta'_{ext} \in \mathbb{D}; x \in \Delta_{ext}, x' \in \Delta'_{ext} \); and compute \( \langle \psi^a_i(x) \bar{\psi}^a_j(x') \rangle \) using the expansion (1.7). We resum by hand bubble chains as indicated in the Introduction, and apply the Taylor expansion of §2.2. Expanding produces a sum over permutations of an (unordered) set of \( n \) vertices, with \( n = 0, 1, \ldots \). We consider only box-connected contributions (see last paragraph of §2.1). Thus we must find a way to generate all box-connected diagrams (with vertices located in the same box glued together, and up to permutations of edges connecting the same pair of boxes \( (\Delta, \Delta'), \Delta, \Delta' \in \mathbb{D}, \) where possibly \( \Delta = \Delta' \) containing boxes \( \Delta_{ext}, \Delta'_{ext}; \) and bound the sum of all these terms. The combinatorial factor resulting from the sum
over permutations of fields located in the same box has been taken into account in §2.2, and powers of $N$ carefully unraveled in §3.1; as a result, we are left with $O(N^{-1/6})$ per vertex, times the product of the decay factors for each kernel (see below) to help us sum over box-connected diagrams.

This is a standard argument in statistical mechanics and constructive field theory; see e.g. [13], Corollary 5.3. Let $q \geq 1$. The following algorithm generates all box-connected diagrams containing $\Delta_{\text{ext}}$ (and possibly $\Delta'_{\text{ext}}$) and a total number of $q$ covariance kernels (counting both $C^*$ and $\Sigma$ kernels). Start from $\Delta_1 := \Delta_{\text{ext}}$. Sum at step 1 over all possible boxes $\Delta_1' \in \mathbb{D}$ (including $\Delta_1$), and add a link between $\Delta_1$ and $\Delta_1'$, i.e. a pairing $\langle \psi^a(x_1)\bar{\psi}^a(x_1') \rangle_0$ or $\langle \bar{\psi}^a(x_1)\psi^a(x_1') \rangle_0$ between fields located at $x_1 \in \Delta_1, x_1' \in \Delta_1'$, or a pairing $\Sigma(x_1 - x_1')$. In both cases, the decay factor $O((1 + m_\psi |x|^{-\infty})$ (see (3.1)) contributes $O\left((1 + d(\Delta_1, \Delta_1'))^{-p}\right)$ for any $p$, where (letting $x_\Delta, x_{\Delta'}$ be the center of boxes $\Delta, \Delta'$

$$d(\Delta, \Delta') := m_\psi |x_\Delta - x_{\Delta'}|$$

is a scaled distance between two boxes. Summing over all possible cases yields a factor $O(C_p)$, with

$$C_p := \sum_{\Delta_1'} \left(1 + d(\Delta_1, \Delta_1')\right)^{-p}.$$  

By ordering boxes by their distance to $\Delta_1$, one obtains

$$C_p \lesssim \sum_{i \geq 1} i^{1-p} < \infty$$

provided $p \geq 3$. Continue in a second step by picking a second link between $\Delta_1$ and a box $\Delta_1''$, and so on, until all pairings between fields in $\Delta_1$ and fields either in $\Delta_1$ or in any other box have been exhausted, each time producing a new multiplicative factor $O(C_p)$ for the sum over all possibilities. Now one must look for all possible pairings between a field located in $\Delta_2$ defined as $\Delta_i'$, where $i := \min\{i' \geq 1 \mid \Delta_i' \neq \Delta_1\}$, and a field located in $\Delta_i' \neq \Delta_1$, and so on. After $q$ steps, the procedure stops.

The outcome is as follows. First, the sum over all possibilities has produced a factor $(O(C_p))^q$. Then (since a delocalized vertex as on Fig. 3.1 contributes three kernels: two $C^*$ and one $\Sigma$) we have a small factor $O(N^{-1/6})^{q/3}$ (see after (3.5)) left from the procedure in §3.1. Summing over $q$ yields $\sum_{q \geq 0} O(C_p)^q O(N^{-1/6})^{q/3} = O(1)$ for $N$ large enough.

We want better than a $O(1)$ bound – a bound that decreases quasi-exponentially as $d(\Delta_{\text{ext}}, \Delta'_{\text{ext}}) \to \infty$. For that we modify our counting scheme. Instead of explicitly summing at each step over all possibilities, we rewrite the current term $\left(1 + d(\Delta, \Delta')\right)^{-p}$ as 1 if $\Delta = \Delta'$, and $(2d(\Delta, \Delta'))^{-p/2}$ times $O\left((1 + d(\Delta, \Delta'))^{-p/2}\right)$ otherwise. We set aside the product of the prefactors $(2d(\Delta, \Delta'))^{-p/2}$ for the links between different boxes. At the end of the algorithm, each term contains one or several paths $\Delta_{\text{ext}} \equiv \Delta_1 \to \Delta_2 \to \cdots \to \Delta_m \to \Delta_{m+1} \equiv \Delta_{\text{ext}}$ of length $m \geq 1$ connecting $\Delta_1$ to $\Delta_1'$ through links $(\Delta_i, \Delta_{i+1}), i = 1, \ldots, m$ such that $\Delta_i \neq \Delta_{i+1}$. Choose one such path, and consider the product $\prod_{i=1}^m (2d(\Delta_i, \Delta_{i+1}))^{-p/2}$. Remark that $d(\Delta_i, \Delta_{i+1}) \geq 1$ and that $(2x)(2x') \geq 2(x + x')$ if $x, x' \geq 1$, from which
(by induction) \( \prod_{i=1}^{m}(2x_i) \geq 2 \sum_{i=1}^{m} x_i \) if \( x_1, \ldots, x_m \geq 1 \). Hence \( \prod_{i=1}^{m}(2d(\Delta_i, \Delta_{i+1}))^{-p/2} \leq \left( \frac{2 \sum_{i=1}^{m}d(\Delta_i, \Delta_{i+1})}{p/2} \right) \leq (2d(\Delta_{ext}, \Delta'_{ext}))^{-p/2} \). Then (if \( p \geq 6 \)) we finally obtain for the sum of all possibilities with \( q \) fixed \( (2d(\Delta_{ext}, \Delta'_{ext}))^{-p/2} O(C_p/2)^q \). Summing over \( q \) yields as above \( O(1) \times (2d(\Delta_{ext}, \Delta'_{ext}))^{-p/2} \).

4 Appendix

4.1 Diagram and kernel estimates

Lemma 4.1 \((L^1\text{-bound for the covariance kernel})\) Let

\[
||C_{a,i;a,j}^*||_{L^1} := \int dx |C_{a,i;a,j}^*(x)|
\]

be the \( L^1 \)-norm of the \((a,i;a,j)\) component of the covariance matrix. Then

\[
|C_{a,i;a,j}^*(x)| \leq (1 \lor |x|)^{-1} O((1 + m_\psi|x|)^{-\infty})
\]

from which

\[
||C_{a,i;a,j}^*||_{L^1} = O(m_\psi^{-1}).
\]

The notation “\( |f(x)| = O((1 + m_\psi|x|)^{-\infty}) \)” means that, for every \( \tilde{n} \geq 0 \), there exists a constant \( C_n \) (independent of \( m_\psi \)) such that \( |f(x)| \leq C_n(1 + m_\psi|x|)^{-\tilde{n}} \).

We split the proof into several points.

1. Let

\[
G_\chi(x) := \frac{1}{(2\pi)^2} \int \frac{dp}{p^2 + m_\psi^2} \chi(|p|) e^{ip_\mu x_\mu}, \quad G_{1-\chi}(x) := \frac{1}{(2\pi)^2} \int \frac{dp}{p^2 + m_\psi^2} (1 - \chi(|p|)) e^{ip_\mu x_\mu}
\]

and

\[
G(x) := G_\chi(x) + G_{1-\chi}(x).
\]

Using polar coordinates \( p = (\rho, \theta) \), we obtain

\[
G_{1-\chi}(x) = \frac{1}{2\pi} \int_{0}^{+\infty} \frac{\rho d\rho}{\rho^2 + m^2} (1 - \chi)(\rho) J_0(\rho|x|),
\]

where \( J_0(\rho|x|) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\rho|x|\cos \theta} d\theta \) is the Bessel function of order 0, and (differentiating and using \( J'_0 = -J_1 \)),

\[
\partial_{x_\mu} G_{1-\chi}(x) = -\frac{1}{2\pi |x|} \int_{0}^{+\infty} \frac{\rho^2 d\rho}{\rho^2 + m^2} (1 - \chi)(\rho) J_1(\rho|x|).
\]

Hankel’s asymptotic expansion (see e.g. [1], §9.2.5) yields for any pair of positive integers \( \nu, n \)

\[
J_\nu(\rho|x|) \sim_{\rho|x| \to \infty} \sqrt{\frac{2}{\pi \rho |x|}} \left( P_\nu((\rho|x|)^{-1}) \cos(\rho|x| - \frac{\pi}{4}) + (\rho|x|)^{-1} Q_\nu((\rho|x|)^{-1}) \sin(\rho|x| - \frac{\pi}{4}) + O((\rho|x|)^{-n-1}) \right)
\]

(4.8)
where $P_\nu$, resp. $Q_\nu$ is a polynomial of degree $n$, resp. $n - 1$. Assume $|x| \geq 1$. Since (as follows from repeated integrations by parts) $\int_0^{+\infty} \frac{dp}{p^2 + m_\psi^2} \frac{\rho^{\nu/2}}{(\rho |x|)^{n'}} (1 - \chi)(\rho) e^{\pm ip|x|} = O((1 + |x|)^{-n})$ ($\nu = 0, 1$, $n' = 0, 1, \ldots, n$), we get a quasi-exponential decrease,

$$|G_{1-\chi}(x)|, |\partial_{x_\mu} G_{1-\chi}(x)| = O((1 + |x|)^{-n}), \quad |x| \geq 1. \quad (4.9)$$

This holds for any $n \geq 1$; for sake of brevity we write this as

$$|G_{1-\chi}(x)|, |\partial_{x_\mu} G_{1-\chi}(x)| = O((1 + |x|)^{-\infty}), \quad |x| \geq 1. \quad (4.10)$$

2. Similarly, $G(x) = \frac{1}{2\pi} \int_0^{+\infty} \frac{dp \rho}{p^2 + m_\psi^2} J_0(\rho |x|) = K_0(m_\psi |x|)$ ([9], formula 6.532 (4)). The functions $K_0(y), -K_0'(y) = K_1(y)$ are decreasing functions of their argument on $\mathbb{R}_+$. When $m_\psi |x|$ is large (see [1], §9.6 and 9.7), and $\nu = 0, 1$, $K_{\nu}(m_\psi |x|) \sim \sqrt{\frac{\pi}{2m_\psi |x|}} e^{-m_\psi |x|}(1 + O(\frac{1}{m_\psi |x|})).$ For a small argument instead, one gets $K_0(m_\psi |x|) \sim -\log(m_\psi |x|)$ and $K_1(m_\psi |x|) \sim \frac{1}{m_\psi |x|}$. 

3. The covariance kernel is $C_{aa}(x) = \frac{1}{(2\pi)^7} \int \frac{dp \mu}{\gamma_\mu p^2 + m_\psi^2} \chi(|p|) e^{ip^\mu x_\mu}$. Rewriting $\frac{1}{\gamma_\mu p^2 + m_\psi^2}$ as \(\frac{m_\psi}{\mu^2 + m_\psi^2} - \frac{\gamma_\mu}{p^2 + m_\psi^2}\), we get

$$C_{aa}(x) = m_\psi (G - G_{1-\chi})(x) + i\gamma_\mu \partial_{x_\mu} (G - G_{1-\chi})(x) \quad (4.11)$$

It is a $2 \times 2$ matrix, sum of a scalar and of a traceless contribution. When $m_\psi |x| \gg 1$, $m_\psi G(x), \partial_{x_\mu} G(x) = O(m_\psi e^{-m_\psi |x|})$ by 2. When $1 \leq |x| < \frac{1}{m_\psi}$, the small argument behavior yields instead $m_\psi G(x) = O(m_\psi \log(1/m_\psi |x|))$ and $\partial_{x_\mu} G(x) = O\left(\frac{1}{|x|}\right)$. Note that

$$\int_{|x| < \frac{1}{m_\psi}} m_\psi \log(1/m_\psi |x|) \, dx = -\frac{2\pi}{m_\psi} \int_{m_\psi}^1 \log z \, dz = \frac{2\pi}{m_\psi} \left(\frac{m_\psi^2}{2} \log m_\psi - \frac{1}{2} + \frac{1}{4}\right) = \frac{\pi}{2} m_\psi^{-1} + O(m_\psi \log(1/m_\psi)),$$ 

and $\int_{|x| > \frac{1}{m_\psi}} m_\psi e^{-m_\psi |x|} \, dx \approx m_\psi^{-1}$. 

4. Consider finally the case when $|x| \leq 1$. Then

$$|m_\psi G_\chi(x)| \leq m_\psi \int \frac{dp}{p^2 + m_\psi^2} \chi(|p|) = O(m_\psi \log(1/m_\psi)) \quad (4.12)$$

and (by parity)

$$|\partial_{x_\mu} G_\chi(x)| = \left| \int dp \chi(|p|) \frac{p^\mu}{p^2 + m_\psi^2} (e^{ip^\mu x} - 1) \right| \leq |x| \int dp \chi(|p|) = O(|x|) \quad (4.13)$$

Putting everything together, we see that there are three quite distinct contributions to the covariance kernel $C_{aa,i\neq j}(x)$:

- A negligible bounded contribution with quasi-exponential decay rate of order 1 coming from the scale $1$ UV cut-off (see 1.), a $O((1 + |x|)^{-\infty})$, with $L^1$ norm $O(1)$;
- (if $i = j$ only) a scalar contribution coming from the Bessel function $m_\psi K_0(m_\psi |x|)$, which is (i) maximal, of order $O(m_\psi \log(1/m_\psi))$ when $|x| \leq 1$; (ii) decreases as $O(m_\psi \log(1/m_\psi |x|))$ as $|x|$ increases from 1 to $O(1/m_\psi)$, and finally (iii) decreases with exponential decay rate $m_\psi$ as $O(m_\psi e^{-m_\psi |x|})$ as $|x|$ increases from $O(1/m_\psi)$ to infinity. Regimes (i) and (ii) may
be combined into \( O(m_\psi \log(1/m_\psi (1 \vee |x|))) \), where \( 1 \vee |x| := \max(1, |x|) \). As proved in 3., the \( L^1 \) norm is \( O(m_\psi^{-1}) \):

- a traceless contribution coming mainly from the Bessel \( K_1 \)-function, which is (i) \( O(|x|) \) when \( |x| \leq 1 \); (ii) \( O(1/|x|) \) when \( 1 \leq |x| \leq 1/m_\psi \) (comparing with the scalar contribution in the same \( x \)-region, we note that \( 1/|x| \geq m_\psi \log(1/m_\psi |x|) \) since \( y \mapsto y \log(1/y) \) is bounded when \( 0 < y \leq 1 \); (iii) \( O(m_\psi e^{-m_\psi |x|}) \) (as for the scalar contribution) when \( |x| \gtrsim 1/m_\psi \). The \( L^1 \)-norm is of order \( \int_{|x| \leq 1} dx |x| + \int_{1 \leq |x| \leq 1/m_\psi} \frac{dx}{|x|} + \int_{|x| > 1/m_\psi} dx m_\psi e^{-m_\psi |x|} = O(m_\psi^{-1}). \)

**Tadpole diagram.**

Recall

\[
\mathcal{A}(\mathcal{T}) = \frac{1}{(2\pi)^2} \int dp \left( \frac{\chi(|p|)}{p + m_\psi} \right)_{0,0}. \tag{4.14}
\]

Note that the actual contribution to fermionic two-point functions of tadpole diagrams is

\[
\frac{\lambda}{2N} (2N - 1) \mathcal{A}(\mathcal{T}) \ (\frac{\lambda}{2N} \text{ for the coupling constant, and } 2N - 1 \text{ for the sum over flavor and spinor indices different from } (a,i), \text{ see Fig. 1}). \text{ This accounts for the prefactor in the gap equation, see Lemma 4.6.}
\]

**Lemma 4.2 (tadpole diagram)** There exists a constant \( c_1 > 0 \) of order 1, depending on the UV cut-off, but independent of \( m_\psi \), such that

\[
\mathcal{A}(\mathcal{T}) = \frac{m_\psi}{2\pi} \left( \log(c_1/m_\psi) + O(m_\psi^2) \right). \tag{4.15}
\]

**Proof.** The identity \( \frac{1}{p^2 + m_\psi} = \frac{-m_\psi}{p^2 + m_\psi^2} \) and the symmetry \( p \leftrightarrow (-p) \) yield

\[
\mathcal{A}(\mathcal{T}) = \frac{m_\psi}{(2\pi)^2} \int dp \frac{\chi(|p|)}{p^2 + m_\psi^2}. \tag{4.16}
\]

Substituting to the smooth UV cut-off function \( \chi \) a sharp cut-off at \( |p| = 1 \), we get (with \( u = p^2 \))

\[
\mathcal{A}(\mathcal{T}) = \frac{1}{(2\pi)^2} m_\psi \int_{|p| < 1} \frac{dp}{p^2 + m_\psi^2} + \text{err.} = \frac{m_\psi}{4\pi} [\log(u + m_\psi^2)]_{u=0}^{u=1} + \text{err.} = \frac{m_\psi}{2\pi} \left( \log(1/m_\psi) + O(m_\psi^2) \right) + \text{err.}. \tag{4.17}
\]

The error term \( \frac{m_\psi}{2\pi} \int dp \frac{\chi(|p|)}{p^2 + m_\psi^2} \) is due to the UV cut-off and equal to \( \frac{m_\psi}{2\pi} \left( \log(c_1) + O(m_\psi^2) \right) \), where \( \log(c_1) := \frac{1}{2\pi} \int dp \frac{\chi(|p|)}{p^2} \) is finite and does not depend on \( m_\psi \). Thus

\[
\mathcal{A}(\mathcal{T}) = \frac{m_\psi}{2\pi} \left( \log(c_1/m_\psi) + O(m_\psi^2) \right). \tag{4.18}
\]
Bubble diagram estimates.

Recall
\[
\frac{\pi(q)}{\lambda} = A_q(\Upsilon) = -\frac{1}{(2\pi)^2} \text{Tr}_2 \int dp \frac{\chi(|p|)}{p + m} \frac{\chi(|p + q|)}{p + f + m}
\] (4.19)

We note to begin with that
\[
\left| \frac{\pi(q)}{\lambda} \right| = \left| \int dx \text{Tr}_2(|C_{aa}(x)|^2) e^{iq \cdot x} \right| \leq \frac{\pi(0)}{\lambda}.
\] (4.20)

**Lemma 4.3 (bubble diagram)** The following estimates hold:

1. There exists a constant \(c_2 > 0\) of order 1, depending on the UV cut-off but independent of \(m_\psi\), such that
\[
\frac{\pi(0)}{\lambda} = \frac{1}{\lambda} - c_2 + O(m_\psi^2)
\] (4.21)

2. If \(|q| \gg m_\psi\),
\[
\left| \frac{\pi(q)}{\lambda} \right| \leq \frac{1}{\pi} \log(|q|/\lambda) + O(1)
\] (4.22)

and, if \(|\kappa| \geq 1\),
\[
|\nabla_q^\kappa \left( \frac{\pi(q)}{\lambda} \right) | \lesssim |q|^{-|\kappa|}.
\] (4.23)

3. If \(|q| \lesssim m_\psi\), and \(|\kappa| \geq 1\), then
\[
|\nabla_q^\kappa \left( \frac{\pi(q)}{\lambda} \right) | \lesssim m_\psi^{-|\kappa|}.
\] (4.24)

**Proof.**

1.
\[
\frac{\pi(0)}{\lambda} = -\frac{1}{(2\pi)^2} \text{Tr}_2 \int dp \frac{\chi^2(|p|)}{(p + m_\psi)^2} = -\frac{2}{(2\pi)^2} \int dp \chi^2(|p|) \frac{m_\psi^2 - p^2}{(p^2 + m_\psi^2)^2}
\]
\[
= \frac{1}{2\pi} \left[ \log(p^2 + m_\psi^2) + \frac{2m_\psi^2}{p^2 + m_\psi^2} \right] + \text{err}.
\]
\[
= \frac{1}{\pi} \log(1/m_\psi) - \frac{1}{\pi} + O(m_\psi^2) + \text{err}
\]
\[
= \frac{1}{\lambda} - \frac{1}{\pi} + O(m_\psi^2) + \text{err}.
\] (4.25)

which is eq. (107) in [12].

Error terms (indicated by \(\text{err.}\)) come from the UV cut-off and are of order \(O(1)\); in order for the \(\Sigma\)-kernel to be massive, it is important that we check that \(\pi(0) < 1\), so we need to be more precise. (1) The \(\frac{1}{\pi} \log(1/m_\psi)\) term comes from the integration of \(\frac{2}{(2\pi)^2} \int dp \frac{\chi^2(|p|)}{p^2 + m_\psi^2}\). Since \(\chi(|p|) \leq 1\), there exists some constant \(c_2' > 0\) of order 1, independent of \(m_\psi\), such that this integral is equal to \(-c_2' + O(m_\psi^2)\), plus \(\frac{2}{(2\pi)^2} \int dp \frac{\chi(|p|)}{p^2 + m_\psi^2}\).
2. Let us first deal with the case integral in momentum space of $\text{Tr}(A)$. Thus we get (using the same decomposition and general strategy as for $c_2 := c_2 + \frac{1}{\pi}$, we obtain (4.21).

(ii) $|p| > 2|q|$: then the leading term is obtained by replacing $\frac{1}{p + q + m_\psi}$ by $\frac{1}{p + m_\psi}$, i.e. by letting $q = 0$. One obtains then the same integral as in 1., but restricted to a smaller domain in $p$, yielding $\frac{1}{2\pi} \left[ \log(p^2 + m_\psi^2) + \frac{2m_\psi^2}{p^2 + m_\psi^2} \right] |q| + \text{err} = \frac{1}{\pi} \left[ \log(1/|q|) - \frac{m_\psi^2}{2|q|^2} \right] + O(1) = \frac{1}{\pi} \log(1/|q|) + O(1).

(iii) the complementary region $\Omega := \text{supp}(\chi) \setminus (|p| > 2|q|) \cup \{|p| < |q|/2\} \cup \{|p + q| < |q|/2\}$ is characterized by $|p| \approx |p + q| \approx |q|$. Thus the integral restricted to $\Omega$ is bounded by pure scaling as $|q|^2 \times (1/|q|)^2 = O(1)$.

Error terms have in factor

(i) $\frac{1}{p + q + m_\psi} - \frac{1}{p + m_\psi} = O\left(\frac{|q|}{|p|^2}\right)$, yielding an integral $\int_{|p| > 2|q|} dp O\left(\frac{|q|}{|p|^2}\right) = O(1)$.

(ii) $\frac{1}{p + q + m_\psi} - \frac{1}{q + m_\psi} = O\left(\frac{|q|}{|q|^2}\right)$, yielding $\int_{|p| < |q|/2} dp O\left(\frac{|q|}{|q|^2}\right) = O(1)$, and $\frac{1}{q + m_\psi} - \frac{1}{q} = O\left(\frac{m_\psi}{|q|^2}\right)$, yielding $\int_{|p| < |q|/2} dp O\left(\frac{|q|}{|q|^2}\right) = O\left(\frac{m_\psi}{|q|^2}\right) = O(1)$ too.

We now consider the case $|\kappa| \geq 1$. The action of the operator $\partial_{q_1} \cdots \partial_{q_{|\kappa|}}$ on $\frac{1}{p + q + m_\psi}$ is given by iterating (4.29), yielding up to combinatorial coefficients $\frac{1}{p + q + m_\psi}$ to the power $(|\kappa| + 1)$, with $\gamma_{i_1}, \ldots, \gamma_{i_{|\kappa|}}$ sandwiched in-between. We must now bound the integral in momentum space of $\text{Tr}(A_1 \cdots A_n)$ with $n = 2 + 2|\kappa|$ and $A_1 = \frac{1}{\psi + m_\psi}$; $A_{2k} = \frac{1}{\psi + m_\psi}$ for $k = 1, 2, \ldots, 1 + |\kappa|$, and $A_q$ a $\gamma$-matrix if $q = 2k + 1$, $k = 1, \ldots, |\kappa|$. For $2 \times 2$ matrices, all classical norms (in particular the matrix norm $\| \cdot \|$ and the Hilbert-Schmidt norm) are equivalent, so $|\text{Tr}(A_1 \cdots A_n)| \lesssim \|A_1 \cdots A_n\| \leq \prod_{k=1}^n ||A_k|| \lesssim \prod_{k=1}^n \left(\text{Tr}(A_k A_k^\dagger)\right)^{1/2}$, with $A_1 A_1^\dagger = \frac{1}{p^2 + m_\psi^2}$ and $A_{2k} A_{2k}^\dagger = \frac{1}{(p + q)^2 + m_\psi^2}$. Thus we get (using the same decomposition and general strategy for $\kappa = 0$)
\[ \int_{|p|>2|q|} \chi(|p|) \left| \nabla_{\bar{q}}^\kappa \left( \frac{1}{\bar{p}+\bar{f}+m_\psi} \right) \right| \leq \int_{2|q|}^{1} dp \frac{1}{\sqrt{p^2+m_\psi}} \left( \frac{1}{\sqrt{(p+q)^2+m_\psi}} \right)^{|\kappa|+1} \]
\[ \leq \int_{|q|}^{1} dp \rho^{-|\kappa|-1} = O(|q|^{-|\kappa|}). \quad (4.26) \]

(ii) Assume e.g. \(|p| < \frac{|q|}{2}\) and use as above \(\frac{1}{\bar{p}+m_\psi} = \frac{-\bar{p}+m_\psi}{\bar{p}^2+m_\psi}\) and \(\frac{1}{\bar{p}+\bar{q}+m_\psi} = \frac{1}{\bar{q}+O\left(\frac{|q|}{|\kappa|^2}\right)}\). The integral of the leading term vanishes once again thanks to the symmetry \(p \leftrightarrow (-p)\) and to the fact that a product of an odd number of matrices is traceless. Error terms feature \(\int_{|p|<|q|/2} dp \rho^{-1-|\kappa|} dp \rho = O(|q|^{-|\kappa|})\).

(iii) By pure scaling, one gets for the integral \(|q|^2 \times (1/|q|)^{|\kappa|+2} = O(|q|^{-|\kappa|})\).

3. Assume now \(|q| \lesssim m_\psi\). The beginning of the proof is the same, yielding

\[ |\nabla_{\bar{q}}^\kappa \left( \frac{\pi(q)}{\lambda} \right) | \leq \int_{0}^{1} dp \frac{1}{\sqrt{p^2+m_\psi}} \left( \frac{1}{\sqrt{(p+q)^2+m_\psi}} \right)^{|\kappa|+1} \]
\[ \leq \left( \int_{0}^{1} dp \left( \frac{1}{\sqrt{p^2+m_\psi}} \right)^{|\kappa|+2} \right)^{1/(|\kappa|+2)} \left[ \left( \int_{0}^{1} dp \left( \frac{1}{\sqrt{(p+q)^2+m_\psi}} \right)^{|\kappa|+1} \right)^{1/(|\kappa|+2)} \right] |\kappa|+1 \]

\[ (4.27) \]

by the generalized Cauchy-Schwarz inequality. All \(|\kappa|+2\) integrals in the last expression are bounded by \(O(1) \int_{m_\psi}^{1} \rho^{-1-|\kappa|} dp + O(1) m^{-2-|\kappa|} \int_{0}^{m_\psi} \rho dp = O(m_\psi^{-|\kappa|})\).

The following Lemma can be skipped; it is only used in the preliminary informal discussion of \(\Sigma\)-kernel estimates.

**Lemma 4.4** Let \(|\nabla_q|^2 := \partial_{\bar{q}}^2 + \partial_q^2\), then

\[ -|\nabla_q|^2 \left( \frac{\pi(q)}{\lambda} \right) \bigg|_{q=0} = \frac{1}{12\pi m_\psi^2} + O(1). \quad (4.28) \]

**Remark.** From (4.20), we already know that \(-|\nabla_q|^2 \left( \frac{\pi(q)}{\lambda} \right) \bigg|_{q=0} \geq 0\).

**Proof.** We use the following identities,

\[ \partial_{\bar{q}} \frac{1}{\bar{p}+\bar{f}+m_\psi} = \frac{1}{\bar{p}+\bar{f}+m_\psi}, \quad \partial_q \frac{1}{\bar{p}+\bar{f}+m_\psi} = -\frac{1}{\bar{p}+\bar{f}+m_\psi} \gamma_i \frac{1}{\bar{p}+\bar{f}+m_\psi} \]

\[ (4.29) \]

\[ \gamma_i^2 = -I, \quad \{\gamma_j, \gamma_k\} := \gamma_j \gamma_k + \gamma_k \gamma_j = \text{Tr}_2(\gamma_j \gamma_k) I = -2\delta_{jk} I \]
\[ \text{Tr}_2 \left( \gamma_j \gamma_i \gamma_k \gamma_i \right) = \text{Tr}_2 \left( \{ \gamma_i, \gamma_j \} \gamma_i \gamma_k + \gamma_j \gamma_k \right) = -2\delta_{jk}(1 - 2\delta_{ij}) \]

Then

\[
- \sum_i \partial_{q_i}^2 \left( \frac{\pi(q)}{\lambda} \right) \bigg|_{q=0} = \frac{1}{(2\pi)^2} \text{Tr}_2 \sum_i \partial_{q_i} \left( \int dp \frac{\chi(|p|)}{p + m_\psi} \partial_{q_i} \frac{\chi(|p + q|)}{p + q + m_\psi} \right) \bigg|_{q=0} = \frac{1}{(2\pi)^2} \text{Tr}_2 \sum_i \int_{|p|<1} dp \frac{1}{(p + m_\psi)^2} \gamma_i \frac{1}{p + m_\psi} + \text{err.} = \frac{1}{(2\pi)^2} \int_{|p|<1} dp \frac{1}{(p^2 + m_\psi^2)^{\frac{3}{2}}} \text{Tr}_2 \sum_i \left( (m_\psi^2 - p^2 - 2m_\psi \pi) \gamma_i (p + m_\psi) \gamma_i (p + m_\psi) \right)
\]

where \( \text{err.} \) denotes \( O(1) \) error terms coming from the UV cut-off. Now, \( \gamma \)-identities yield

\[
\sum_i (m_\psi^2 - p^2) \text{Tr}_2 (\gamma_i (p + m_\psi) \gamma_i (p + m_\psi)) = 2(p^2 - m_\psi^2) \sum_{i,j,k} p_j p_k \delta_{jk}(1 - 2\delta_{ij}) = 0
\]

\[
\sum_i (m_\psi^2 - p^2) \text{Tr}_2 (\gamma_i m_\psi \gamma_i m_\psi) = 2m_\psi^2 (p^2 - m_\psi^2)
\]

\[
-2m_\psi \sum_i \text{Tr}_2 (\gamma_i (p + m_\psi) \gamma_i (p + m_\psi)) = -4m_\psi^2 \sum_{i,j,k} p_j p_k \delta_{jk}(1 - 2\delta_{ij}) - 4m_\psi^2 p^2 = 4m_\psi^2 p^2
\]

Other terms contain an odd number of \( \gamma \)-matrices and therefore vanish. Hence (letting \( u = p^2 \))

\[
- \sum_i \partial_{q_i}^2 \left( \frac{\pi(q)}{\lambda} \right) \bigg|_{q=0} = \frac{1}{(2\pi)^2} \int_{|p|<1} dp \frac{6m_\psi^2 u - 8m_\psi^4}{(u + m_\psi^2)^4} + \text{err.} = \frac{1}{4\pi} \int_0^1 du \frac{6m_\psi^2 (u + m_\psi^2) - 8m_\psi^4}{(u + m_\psi^2)^4} + \text{err.} = \frac{1}{4\pi} \left[ -\frac{3m_\psi^2}{u(m_\psi^2)^2} - \frac{8m_\psi^4}{u(m_\psi^2)^3} \right]_0 + \text{err.} = \frac{1}{12\pi m_\psi^2} + \text{err.}
\]

with error terms \( O(1) \) coming from the UV cut-off.

### \( \Sigma \)-kernel estimates.

Recall

\[
\Sigma(q) = \frac{\lambda/N}{1 - \pi(q)},
\]

(4.32)
Note that $\Sigma(q) = \frac{1}{N} \left( 1 - \frac{1}{N} (N\mathcal{A}_q(Y))^{-1} \right)$ is the sum of bubble chains (see Fig. 4), the $N$-prefactor in front of $\mathcal{A}_q(Y)$ accounting for the sum over flavor indices.

We start with a somewhat informal discussion, whose outcome (4.34, 4.36) shows the connection to the auxiliary field approach in [12], but is not needed for our estimates. From Lemma 4.3 1. and Lemma 4.4, to leading order as $|q| \to 0$ and $\lambda \to 0$,

$$1 - \pi(q) \sim 1 - \left( \pi(0) + \frac{1}{4} |\nabla_q|^2 \pi(q) \right)_{q=0} |q|^2 + O(|q|^4)$$

so that

$$\Sigma(q) \sim \frac{1}{c_2 N} \frac{m^2_\sigma}{q^2 + m^2_\sigma} \quad (4.34)$$

where

$$m_\sigma := \sqrt{48\pi c_2 m_\psi} \quad (4.35)$$

could be interpreted as the mass of an equivalent intermediate field as in [12]. Fourier inversion yields asymptotically

$$\Sigma(x) \sim \frac{m^2_\sigma}{c_2 N} G(m_\sigma; x)$$

$$\sim \frac{m^2_\sigma}{c_2 N} \sqrt{\frac{\pi}{2m_\sigma|x|}} e^{-m_\sigma|x|} \quad (4.36)$$

as $m_\sigma|x| \to \infty$, i.e. for $|x| \gg 1/m_\psi$, where $G(m_\sigma; x) := \frac{1}{(2\pi)^N} \int \frac{dq}{q^2 + m^2_\sigma} e^{iq\cdot x}$ is the same function as the function $G$ studied in the proof of Lemma 4.1 1.

The above computations are simply based on the second-order Taylor expansion of the $\pi$-kernel around 0, so they should not be taken at face value, and in any case, they give no indication on the behavior of $\Sigma(x)$ for $|x| \lesssim 1/m_\psi$, nor on the $L^1$-norm of $\Sigma$. This is the purpose of our next Lemma.

**Lemma 4.5 (bounds for the $\Sigma$-kernel)** The following bounds hold:

1. Assume $|q| \gg m_\psi$, then $\left| \frac{1 - \pi(q)}{\lambda} \right| \gtrsim \log (|q|/m_\psi)$ and

$$|\Sigma(q)| \lesssim \frac{1}{N} \frac{1}{\log (|q|/m_\psi)}, \quad |\nabla^\kappa \Sigma(q)| \lesssim \frac{1}{N} \frac{1}{|q|^\kappa \log^2 (|q|/m_\psi)} \quad (4.37)$$

2. If $|q| \lesssim m_\psi$, then, for all $\kappa \geq 0$, $|\nabla^\kappa \Sigma(q)| \lesssim \frac{1}{N} m_\psi^{-\kappa}$.

3. $||\Sigma||_{L^1} := \int dx \ |\Sigma(x)| \lesssim \frac{1}{N}$.

4. $|\Sigma(x)| \lesssim \begin{cases} \frac{m^2_\psi}{N} (m_\psi|x|)^{-2} \log (1/m_\psi|x|))^{-2}, & |x| \leq 1/2m_\psi \\ \frac{1}{N} m^2_\psi O((1 + m_\psi|x|)^{-\infty}), & |x| > 1/2m_\psi \end{cases}$. Furthermore, $|\Sigma(x)| \lesssim \frac{1}{N}$ if $|x| < 1$. Thus

$$||\Sigma||_{\infty} := \sup_x |\Sigma(x)| \lesssim \frac{1}{N}. \quad (4.38)$$
Proof.

1. From Lemma 4.6, \( \frac{1}{\pi} \log(1/m_{\psi}) = \frac{1}{\lambda} + O(1) \), whence (using Lemma 4.3 2.) \( \frac{|\pi[q]|}{\lambda} \leq \frac{1}{\lambda} - \frac{1}{\pi} \log(|q|/m_{\psi}) + O(1) \), implying \( \frac{1-\pi(q)/\lambda}{\lambda} \geq \frac{1}{\pi} \log(|q|/m_{\psi}) \), and then \( |\Sigma(q)| \lesssim \frac{1}{\log(|q|/m_{\psi})} \). Taking gradients,

\[
|\nabla \Sigma(q)| = \frac{1}{N} \frac{|1-\pi(q)/\lambda|}{\lambda^2} |\nabla \frac{\pi(q)}{\lambda}| \lesssim \frac{1/N}{|q| \log^2(|q|/m_{\psi})}
\]

(4.39) and then (by induction on \( |\kappa| \), using (4.23)) \( |\nabla^\kappa \Sigma(q)| \lesssim \frac{1/N}{|q|^{(\kappa)} \log^2(|q|/m_{\psi})} \).

2. If \( |q| \lesssim m_{\psi} \), \( \frac{|1-\pi(q)/\lambda|}{\lambda} \geq \frac{1-\pi(0)/\lambda}{\lambda} = c_2 + O(m_{\psi}^2) \) (see (4.20) and Lemma 4.3) and then, \( \Sigma(q) \lesssim \frac{1}{N} \). The extension to arbitrary \( \kappa \) is based on Lemma 4.3 3. and an induction on \( |\kappa| \) as in 1.

3. \((L^1\text{-bound})\) We let \( j_\phi := \left\lfloor \frac{\pi/\lambda}{\ln(2)} \right\rfloor \) be the gap energy scale, so that

\[
2^{-j_\phi} \approx m_{\psi},
\]

(4.40) and cut \( \Sigma(q) = \sum_{j_{\phi}=0}^{j_{\phi}} \Sigma^j(q) \) into scales, i.e. introduce a smooth multi-scale partition of unity in momentum space, \( 1 = \sum_{j_{\phi}=0}^{j_{\phi}} \chi^j(|q|) \), where \( \chi^j(|q|) = f(2^j|q|) \)

(1 \leq j \leq j_{\phi}-1), \( \chi^0(|q|) = \sum_{j_{\phi}=-\infty}^{0} f(2^j|q|), \chi^{j_{\phi}}(|q|) = \sum_{j_{\phi}}^{+\infty} f(2^j|q|) \) for some smooth function \( f : \mathbb{R}_+ \rightarrow [0, 1] \) with support \( \subset [1/3, 3] \), and let \( \Sigma^j(q) := \chi^j(|q|) \Sigma(q) \), so that \( \text{supp}(\Sigma^j) \subset \left\{ \begin{array}{ll} [1/3, O(1)], & j = 0 \\ [2^{-j}[1/3, 3], & 1 \leq j < j_{\phi} \end{array} \right. \) . Integrations by parts give \( \Sigma^j(x) = [0, O(m_{\psi})] \),

\[
\int_{|x| \leq 2^j} \frac{\partial}{\partial x} \nabla^\kappa \Sigma^j(q) e^{iq \cdot x},
\]

we use \( \kappa = 1 \) for \( |x| \lesssim 2^j \), and \( \kappa \) arbitrarily large for \( |x| \gtrsim 2^j \). Let \( n \geq 1 \). Then, if \( j < j_{\phi} \),

\[
|\Sigma^j(x)| = O(|x|^{-n}) O(\text{Vol}(\text{supp}(\chi^j))) \max_{|\kappa|=n} \sup_q |\nabla^\kappa \Sigma^j(q)| \lesssim \frac{1/N}{(j_{\phi} - j)^2} 2^{-2j} (2^{-j}|x|)^{-n}, \quad |x| \gtrsim 2^j
\]

(4.41) and

\[
|\Sigma^j(x)| \lesssim \frac{1/N}{(j_{\phi} - j)^2} 2^{-2j} (2^{-j}|x|)^{-1}, \quad |x| \lesssim 2^j
\]

(4.42) Note that \( \int_{|x| < 2^j} dx (2^{-j}|x|)^{-1} = O(2^{2j}) \). Summing w.r. to \( j \) and integrating w.r. to \( x \),

\[
\sum_{j < j_{\phi}} \int dx |\Sigma^j(x)| \lesssim \sum_{j < j_{\phi}} \frac{1/N}{(j_{\phi} - j)^2} = O(1/N).
\]

(4.43) The contribution of the infra-red scale \( j_{\phi} \) is simpler; by 2.,

\[
|\Sigma^{j_{\phi}}(x)| = O(\frac{1}{N}) 2^{-2j_{\phi}} (1 + 2^{-j_{\phi}} |x|)^{-n} = O(\frac{1}{N}) m_{\psi}^2 (1 + m_{\psi}|x|)^{-n}
\]

(4.44)
as shown by simple integration for $|x| < 2^{j\phi}$ and by successive integrations by parts as in (4.41) for $|x| > 2^{j\phi}$. Hence $\int dx |\Sigma^j(x)| = O(1/N)$.

4. (bounds in coordinate space). Using (4.41), (4.42), (4.44), assuming $|x| \approx 2^k$ for some $0 \leq k \leq j\phi$, and summing over $j$, one proves immediately that

$$|\Sigma(x)| \lesssim \frac{1}{N} \left\{ \sum_{j < k} \frac{1}{(j\phi - j)^2} 2^{-2j} (2^{k-j} - n) + \sum_{j = k}^{j\phi} \frac{1}{(j\phi - j)^2} 2^{-(j-k)2-2k} \right\}$$

$$\lesssim \frac{1}{N} \frac{2^{-2k}}{(j\phi - k)^2} \approx \frac{m^2}{N} (m\psi|x|)^{-2} (\log(1/m\psi|x|))^{-2}, \quad 1 \leq |x| \leq 1/2m\psi$$

(4.45)

Integrating, one finds $\int_{1/|x| < 1/m\psi} dx |\Sigma(x)| = O(1/N)$ (Bertrand integral). If $|x| \geq 1/2m\phi$, then (by the same method) one sees that $|\Sigma(x)|$ is bounded like $|\Sigma^j(x)|$ by

$$|\Sigma(x)| = \frac{1}{N} m^2 \psi O((1 + m\psi|x|)^{-\infty}), \quad |x| > 1/2m\psi$$

(4.46)

also contributing $O(1/N)$ to the $L^1$-norm. Finally, there is no singularity at $x = 0$: using the easy inequality $|\Sigma(q)| \lesssim 1/N$ (Lemma 4.5 1., 2.) and Fourier inversion, one gets simply

$$|\Sigma(x)| \lesssim 1/N, \quad |x| < 1$$

(4.47)

with again a contribution $O(1/N)$ to the $L^1$-norm.

\[\square\]

4.2 Fixed-point theorem for $m\psi$

We prove here that the gap equation has a unique solution in a neighborhood of $c_1 e^{-\pi/\lambda}$ for $\lambda$ small enough and $N$ correspondingly large enough.

**Lemma 4.6 (gap equation)** There exists an open neighborhood $N = [0, \lambda_{\text{max}}) \times [0, c_2)$ ($\lambda_{\text{max}} > 0, c_2 > 1$) of $(0,1)$ in $\mathbb{R}_+ \times \mathbb{R}_+$ such that the following holds. If $\lambda < \lambda_{\text{max}}$, and $N \geq \frac{1}{2N}^\phi$, the gap equation

$$\frac{\lambda}{N}(2N - 1)\mathcal{A}(T) = m\psi$$

(4.48)

has a unique solution $m\psi = m\psi(\lambda)$ in $(c_2^{-1} c_1 e^{-\pi/\lambda}, c_2 c_1 e^{-\pi/\lambda})$, which is of the form

$$m\psi = (1 - \frac{\pi}{2N\lambda} + O(\lambda^2)) c_1 e^{-\pi/\lambda}.$$ 

(4.49)

**Proof.** Let $t := \frac{1}{2N\lambda}$. By assumption $0 \leq t \leq 1$. We fix $t$ and prove that the gap equation $\frac{\lambda}{N}(2N - 1)\mathcal{A}(T) = m\psi$ has a unique solution in $m\psi$ for $(\lambda, m\psi)$ in some small neighborhood of the origin.
Let $\gamma := m_\psi e^{\pi/\lambda}$. In terms of the variables $(\gamma; \lambda)$, the gap equation may be rewritten as $F(\gamma; \lambda) = 0$, where

$$F(\gamma; \lambda) := \frac{1}{2} \left( \frac{2N - 1}{N} \frac{A(\mathcal{T})}{m_\psi} - \frac{1}{\lambda} \right) = (1 - \lambda^2 t) \frac{A(\mathcal{T})}{m_\psi} - \frac{1}{2\lambda} \quad (4.50)$$

Recall from Lemma 4.2

$$\frac{A(\mathcal{T})}{m_\psi} = \frac{1}{2\pi} \left( \log\left( \frac{c_1}{m_\psi} \right) + O(m_\psi^2) \right) = \frac{1}{2\lambda} + \frac{1}{2\pi} \log\left( \frac{c_1}{\gamma} \right) + O(m_\psi^2) \quad (4.51)$$

We write $f(\lambda) = O(\lambda^\infty)$ if $f$ is a function defined in a neighbourhood of 0 in $\mathbb{R}_+$ such that $f(\lambda) = O_{\lambda \to 0}(\lambda^n)$ for every $n \geq 0$; in particular, $e^{-\pi/\lambda} = O(\lambda^\infty)$. Let $\gamma = c_1$, then

$$F(c_1; \lambda) = (1 - \lambda^2 t) \left( \frac{1}{2\lambda} + O(m_\psi^2) \right) - \frac{1}{2\lambda} = \frac{\lambda}{2} t + O(\lambda^\infty) \quad (4.52)$$

vanishes when $\lambda \to 0$. Differentiating (4.50) w.r. to $\lambda$, we get

$$\frac{\partial F}{\partial \lambda}(c_1; \lambda) = -t + O(\lambda^\infty) \to_{\lambda \to 0} -t/2. \quad (4.53)$$

To obtain $\frac{\partial F}{\partial \gamma}$, we note that

$$\frac{\partial (A(\mathcal{T})/m_\psi)}{\partial \gamma} = \frac{m_\psi}{\gamma} \frac{\partial (A(\mathcal{T})/m_\psi)}{\partial m_\psi} = -\frac{1}{\pi\gamma} \int dp \chi(|p|) \frac{m_\psi^2}{(p^2 + m_\psi^2)^2} = \frac{1}{2\pi\gamma} \left[ \frac{m_\psi^2}{p^2 + m_\psi^2} \right]_0^1 + \text{err.} = -\frac{1}{2\pi\gamma} + \text{err.} \quad (4.54)$$

Error terms come from the UV cut-off and are $O(m_\psi^2)$. (The result is coherent with the one obtained by naively differentiating (4.51)). Hence $\frac{\partial F}{\partial \gamma} = \left( 1 - \lambda^2 t \right) \left( -\frac{1}{2\pi\gamma} + \text{err.} \right) \to_{\gamma \to c_1, \lambda \to 0} -\frac{1}{2\pi c_1} \neq 0$.

All hypotheses are met for an application of the fixed-point theorem found in [5] in an open neighborhood $\Omega$ of $(c_1; 0)$ in $\mathbb{R} \times \mathbb{R}_+$. Hence we find for $\Omega$ small enough a unique differentiable function $\gamma = \gamma(\lambda)$ such that the gap equation holds if and only if $m_\psi = \gamma(\lambda)e^{-\pi/\lambda}$. The implicit function theorem yields furthermore

$$\frac{\partial \gamma}{\partial \lambda} = -\frac{\partial F/\partial \lambda}{\partial F/\partial \gamma} \to_{\lambda \to 0} -\pi c_1 t \quad (4.55)$$

equivalently, $\gamma = (1 - \pi \lambda t + O(\lambda^2)) c_1$, or $m_\psi = (1 - \pi \lambda t + O(\lambda^2)) c_1 e^{-\pi/\lambda}$. \qed
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