O(2,2) TRANSFORMATIONS AND THE STRING GEROCH GROUP

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ABSTRACT

The 1–loop string background equations with axion and dilaton fields are shown to be integrable in four dimensions in the presence of two commuting Killing symmetries and $\delta c = 0$. Then, in analogy with reduced gravity, there is an infinite group that acts on the space of solutions and generates non–trivial string backgrounds from flat space. The usual $O(2,2)$ and $S$–duality transformations are just special cases of the string Geroch group, which is infinitesimally identified with the $O(2,2)$ current algebra. We also find an additional $Z_2$ symmetry interchanging the field content of the dimensionally reduced string equations. The method for constructing multi–soliton solutions on a given string background is briefly discussed.

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1 Introduction

In realistic string models the target space is described by $M_4 \times K$, where $M_4$ is a 4–dim space with signature $- + + +$ and $K$ is some internal space, which is usually represented by a conformal field theory, so that the total central charge of the string theory is at its critical value. The 4–dim Minkowski space provides a particular choice for $M_4$, but it is well known by now that there are many other choices with non–trivial dilaton and antisymmetric tensor (or axion) fields that are compatible with local scale invariance quantum mechanically [1]. A variety of cosmological and black hole type solutions have been constructed explicitly using conformal field theory techniques, duality symmetries, and $O(d,d)$ transformations in general; the list of references [2–10] is indicative, but by no means complete. Although the solution generating techniques we have available at the moment are quite interesting for further study, they appear to be rather restrictive in many ways. It would be interesting to find new (and perhaps complementary) methods for constructing general classes of string vacua and study the transition between them in a systematic way.

The string background equations with axion and dilaton fields provide a natural generalization of the vacuum Einstein equations in $M_4$. Finding their most general solution is a hopeless problem. The best we can hope for our solution generating techniques will be the construction of non–trivial string backgrounds with a certain degree of symmetry, which effectively reduces the dimensionality of the corresponding string equations. In the mini–superspace approach the string background equations are reduced to a 1–dim non–linear system, which is of interest for some cosmological applications, but quite restrictive otherwise. In this paper we adopt a midi–superspace approach, which preserves the field theory aspects of the problem, while making the construction of a large class of solutions still possible.

We consider 4–dim backgrounds with two commuting Killing symmetries and show that the dimensionally reduced 1–loop string equations are in fact integrable. We also find an infinite dimensional symmetry on the space of solutions, which is infinitesimally described by the $SL(2,R) \times SL(2,R) \simeq O(2,2)$ current algebra. Then, finite group elements (i.e. elements further away from the vicinity of the identity) can be used to generate (at least formally) non–trivial string backgrounds from flat space or from other known solutions with two Killing symmetries. We call this new infinite dimensional symmetry the string Geroch group, because for constant axion and dilaton fields it becomes the infinite group of reduced vacuum Einstein gravity generated by the $SL(2,R)$ current algebra [11–16]. In the string case, the usual $O(2,2)$ and $S$–duality transformations can be explicitly identified with certain modes of the $O(2,2)$ Geroch group. As we will see later, we have to go beyond the zero modes to achieve this in the axion formalism.

It should be noted that the explicit construction of new solutions with the aid of the Geroch group is not an easy project to complete, because in the general case we have to solve the associated Riemann-Hilbert problem, which is described by a singular integral equation. The class of solutions which are more accessible to explicit calculations,
however, is that of the (multi)–soliton solutions on any given string background. It should be emphasized that soliton solutions correspond to special elements of the full string Geroch group and cannot be obtained by the usual $O(2,2)$ or $S$–duality transformations. Many non–trivial solutions we already know in reduced vacuum Einstein gravity admit a solitonic interpretation on the background of much simpler solutions; for example, the exterior of a rotating black hole can be described as a double stationary soliton on flat space [14]. Hence, it is natural to expect that straightforward generalization of the Belinski-Sakharov method will produce an infinite (but discrete) set of new 4–dim string solutions descending from known ones. Multi–soliton excitations in dimensionally reduced string theory will be characterized by two positive integers $(n,m)$, referring to the soliton numbers associated with the two $SL(2,R)/U(1)$ $\sigma$–models that appear in the effective 2–dim description of the string background equations.

The dimensional reduction will be performed in the case that both Killing symmetries of $M_4$ are space–like. This is merely done for practical reasons, fixing the notation and sign conventions throughout the calculations. We note, however, that the whole discussion can be easily generalized to the case of one space–like and one time–like Killing symmetry, as well as to 4–dim spaces with Euclidean or $(2,2)$ signature. A few remarks will be made about these other cases, but the generalization will not be discussed in detail.

The remaining part of this paper is organized as follows. In section 2 we consider the string background equations with zero cosmological constant (i.e. $\delta c = 0$) and perform the dimensional reduction in the presence of two commuting Killing symmetries. In section 3 we describe the infinitesimal action of the string Geroch group and identify its generators with the $O(2,2)$ current algebra. The embedding of the $O(2,2)$ and $S$–duality transformations will be discussed in detail. In section 4 we find that every solution with two space–like Killing symmetries possess a “mirror image” that is obtained by interchanging the field content of the two $SL(2,R)/U(1)$ $\sigma$–models that appear in the formalism. Any two solutions of this kind admit different space–time interpretations, in general. In section 5 we review the general method for constructing (multi)–soliton solutions and describe their form around simple string background. Meron–like solutions are also briefly discussed. Finally, in section 6 we present the conclusions and directions for future work.

## 2 Reduced String Background Equations

String propagation in a non–trivial background is described by a generalized 2–dim non–linear $\sigma$–model, which in the conformal gauge, $e^{2\sigma}dzd\bar{z}$, assumes the form

$$ S = \frac{1}{4\pi\alpha'} \int dzd\bar{z} \left( (O_{\mu\nu}^{(\sigma)}(X) + B_{\mu\nu}(X)) \partial X^\nu \bar{\partial} X^\nu - 4\alpha' (\partial \bar{\partial} \sigma) \Phi \right). $$

(2.1)

Here $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ are the target space metric, the antisymmetric tensor and the dilaton fields, respectively. It is well known that these fields have to satisfy certain
consistency conditions so that the quantum theory possesses local scale invariance. In particular, at the 1–loop level in the coupling constant $\alpha'$ (inverse string tension), the vanishing conditions for the beta functions $\beta^{\Phi}$, $\beta_{\mu \nu}^{G}$ and $\beta_{\mu \nu}^{B}$ are [1]

$$4(\nabla \Phi)^2 - 4\nabla^2 \Phi - R^{(4)}[G^{(\sigma)}] + \frac{1}{12}H^2 = 0,$$  

(2.2)

$$R^{(4)}_{\mu \nu}[G^{(\sigma)}] - \frac{1}{4}H_{\mu}^{\ \lambda \sigma}H_{\nu \lambda \sigma} + 2\nabla_{\mu} \nabla_{\nu} \Phi = 0,$$  

(2.3)

$$\nabla_{\lambda}H_{\mu \nu \lambda} - 2(\nabla_{\lambda} \Phi)H_{\mu \nu \lambda} = 0,$$  

(2.4)

respectively, where $H_{\mu \nu \lambda} = 3\nabla_{[\mu}B_{\nu \lambda]}$ is the field strength of the antisymmetric tensor field. The right–hand side of eq. (2.2) is set equal to zero because we assume that the central charge deficit $\delta c$ (cosmological constant) is zero to first order in $\alpha'$. In other words we make the assumption that string propagation takes place in $M_4 \times K$ with $c(M_4) = 4 + \delta c$ and $c(K) = 22 - \delta c$, where $\delta c = \mathcal{O}(\alpha')$. The details of the internal space $K$ will not be needed for the purposes of the present work. We also recall for completeness that eq. (2.2) is a consequence of eqs. (2.3) and (2.4), using the Bianchi identities.

It is convenient to rewrite the 4–dim string background equations in a form that will later lead to a separation of variables upon dimensional reduction. After a conformal rescaling of the metric,

$$G_{\mu \nu} = e^{-2\Phi}G_{\mu \nu}^{(\sigma)},$$  

(2.5)

we pass from the $\sigma$–model frame to the Einstein frame, in which the string equations (2.2)–(2.4) are equivalently described by the classical equations of motion of the effective action

$$S_{\text{eff}} = \int_{M_4} d^4X \sqrt{-\text{det} G} \left( R^{(4)}[G] - 2(\nabla \Phi)^2 - \frac{1}{12}e^{-4\Phi}H^2 \right).$$  

(2.6)

In four dimensions we can also trade $B_{\mu \nu}(X)$ for a scalar field $b(X)$ by duality. The axion field can be consistently defined in the Einstein frame as follows,

$$\partial_{\mu}b = \frac{e^{-4\Phi}}{6} \sqrt{-\text{det} G} \epsilon_{\mu \nu \rho \sigma}H_{\nu \rho \sigma},$$  

(2.7)

where $\epsilon_{0123} = 1$. In $M_4$ with signature $-+++$ we may further define the complex conjugate fields

$$S_{\pm}(X) = b \pm ie^{-2\Phi}.$$  

(2.8)

The axion formalism should be introduced directly to the classical equations of motion and not to the effective action (2.6), since otherwise sign discrepancies will arise. Then, in the Einstein frame, the 1–loop effective action assumes the form

$$S_{\text{eff}} = \int_{M_4} d^4X \sqrt{-\text{det} G} \left( R^{(4)}[G] + 2G^{\mu \nu} \frac{\partial_{\mu}S_{\pm} \partial_{\nu}S_{\mp}}{(S_{+} - S_{-})^2} \right),$$  

(2.9)

which describes a 4–dim $SL(2,\mathbb{R})/U(1)$ non–linear $\sigma$–model coupled to gravity. In spaces with Minkowski signature, like $M_4$, this $\sigma$–model is Euclidean since its target space has
signature ++. On the contrary, in 4-dim spaces with Euclidean or \((2,2)\) signature the \(\sigma\)–model is Lorentzian and the relevant \(\sigma\)–model variables are \(S_\pm = b \pm e^{-2\Phi}\).

It will be quite useful for our purposes to adopt an alternative description of the \(SL(2,R)/U(1)\) \(\sigma\)–model by introducing the symmetric \(2 \times 2\) matrix

\[
\lambda(X) = e^{2\Phi} \begin{pmatrix}
1 & b \\
b & b^2 + e^{-4\Phi}
\end{pmatrix}
\]  

with \(\det \lambda = 1\), so that the 4–dim effective string action is equivalently written as

\[
S_{\text{eff}} = \int_{M_4} d^4X \sqrt{-\det G} \left( R^{(4)}[G] - \frac{1}{4} \text{Tr}(J\mu J^\mu) \right),
\]  

where

\[
J_\mu(X) = \lambda^{-1} \partial_\mu \lambda.
\]  

We mention for completeness that for the Lorentzian \(SL(2,R)/U(1)\) \(\sigma\)–model the matrix element \(b^2 + e^{-4\Phi}\) has to be replaced by \(b^2 - e^{-4\Phi}\), \(b\) also being the axion field, so that \(\det \lambda = -1\) in that case.

The string background equations, which follow from the action (2.12), will be dimensionally reduced in the presence of two commuting space–like Killing symmetries. We assume, therefore, that the components of \(G_{\mu\nu}\) depend only on \(X^0\) and \(X^1\). We also make the extra assumption of orthogonal transitivity, which physically means that \(M_4\) possess a reflection symmetry under \((X^2, X^3) \rightarrow (-X^2, -X^3)\). This implies that the metric has the block diagonal form

\[
ds^2 = h_{ij}(X^0, X^1)dX^i dX^j + g_{AB}(X^0, X^1)dX^A dX^B,
\]  

with \(G_{iA} = 0\). Here \(i, j\) take the values 0 or 1, while \(A, B\) take 2 or 3. Also, the 2–dim metric \(g\) has Euclidean signature. Since the \(h\)–part of the metric can always be brought into a conformally flat form, we may rewrite the line element (2.13) as

\[
ds^2 = -f(\eta, \xi)d\eta d\xi + g_{AB}(\eta, \xi)dX^A dX^B,
\]  

where \(f(\eta, \xi)\) is the conformal factor and \(\eta, \xi\) are the light–cone coordinates

\[
\eta = \frac{1}{2}(X^0 - X^1), \quad \xi = \frac{1}{2}(X^0 + X^1).
\]  

As for the axion and dilaton fields, we also assume that they depend only on \(\eta\) and \(\xi\). Using eq. (2.7) we immediately see that

\[
B_{23} = -B_{32} = B(\eta, \xi),
\]  

while all other components of the antisymmetric tensor field vanish in this case. Then, in the light–cone variables, the relations (2.7) can be written as

\[
\partial_\xi b = \frac{e^{-4\Phi}}{\sqrt{\det g}} \partial_\xi B, \quad \partial_\eta b = -\frac{e^{-4\Phi}}{\sqrt{\det g}} \partial_\eta B.
\]
It will turn out that the sector of the string background equations described by this ansatz is an integrable 2–dim system, generalizing a similar situation encountered in vacuum Einstein gravity [12–16].

The string background equations can be easily cast into the form

\[ R^{(4)}_{\mu\nu} [G] = \frac{1}{4} \text{Tr}(J_\mu J_\nu), \]

\[ \nabla_\mu J^\mu = 0, \]

after eliminating the Ricci scalar curvature term by contraction. For the special class of metrics (2.12) we are considering here, the dimensional reduction of the string equations yields

\[ R^{(4)}_{iA} [G] = 0, \]

\[ R^{(4)}_{AB} [G] = 0, \]

\[ R^{(4)}_{ij} [G] = \frac{1}{4} \text{Tr}(J_i J_j), \]

\[ \nabla_i J^i + \frac{\partial_i \sqrt{\text{det} g}}{\sqrt{\text{det} g}} J^i = 0. \]

It can be verified that eq. (2.20) is just an identity and has no field content. All the information about the dynamics of the field variables is contained in eqs. (2.21)–(2.23), which we now analyse case by case in the light–cone variables.

Equation (2.21) does not depend on the axion–dilaton system and hence it is identical to the same equation of reduced vacuum Einstein gravity. Explicit calculation shows that it is equivalent to the following continuity equation for the \(2 \times 2\) metric \(g\),

\[ \partial_\eta (\sqrt{\text{det} g} g^{-1} \partial_\xi g) + \partial_\xi (\sqrt{\text{det} g} g^{-1} \partial_\eta g) = 0 \]

and

\[ \partial_\eta \partial_\xi (\sqrt{\text{det} g}) = 0. \]

Equation (2.23) yields a similar equation for the \(2 \times 2\) symmetric matrix \(\lambda\), namely

\[ \partial_\eta (\sqrt{\text{det} g} \lambda^{-1} \partial_\xi \lambda) + \partial_\xi (\sqrt{\text{det} g} \lambda^{-1} \partial_\eta \lambda) = 0. \]

At this point there appears to be a slight difference between eqs. (2.24) and (2.26) in that \(\text{det} g \neq \text{det} \lambda = 1\). We note, however, that if we define \(\lambda' = \sqrt{\text{det} g} \lambda\) so that \(\text{det} \lambda' = \text{det} g\), the continuity equation for \(\lambda'\) is identical to that of \(\lambda\) thanks to the Laplacian condition (2.25). As a result, the field variables \(g, \lambda\) satisfy the classical equations of motion of two \(SL(2, R)/U(1)\) non–linear \(\sigma\)–models in two dimensions, modified by the presence of the \(\sqrt{\text{det} g}\) factor in their currents. Modified \(\sigma\)–models of this type are usually called Ernst models. They first arose in Geroch’s treatment of reduced gravity [12–14] and were subsequently studied in detail by Ernst and collaborators, in the vacuum and electrovacuum cases [15]. In \(M_4\) with two space–like Killing symmetries, both \(\sigma\)–models
have Euclidean signature. In section 4 we will use this fact to produce new solutions by interchanging their field content.

The 2–dim wave equation (2.25) implies that \( \sqrt{\det g} \) can be written as a sum of two arbitrary functions, one depending on \( \eta \) and the other on \( \xi \) only. Let \( \alpha(\eta, \xi) \) and \( \beta(\eta, \xi) \) be a pair of conjugate solutions such that

\[
\partial_\xi \alpha = \partial_\xi \beta, \quad \partial_\eta \alpha = -\partial_\eta \beta. \tag{2.27}
\]

Then, without loss of generality, we may choose

\[
\alpha = \xi + \eta, \quad \beta = \xi - \eta, \tag{2.28}
\]

since the form of the line element (2.14) remains invariant under the transformation. The differential equations we are going to derive for the conformal factor \( f(\eta, \xi) \) simplify in the frame \( \sqrt{\det g} = \alpha = \xi + \eta. \tag{2.29} \)

Its essential properties remain the same, however, while the more general picture can be easily described by a simple transformation.

The conditions on the conformal factor \( f \) follow from eq. (2.22), which describes its dependence on \( g \) and \( \lambda \). We first find

\[
R^{(4)}_{ij}[G] = R^{(2)}_{ij}[h] - \nabla_i \nabla_j (\log \sqrt{\det g}) - \frac{1}{4} \text{Tr} \left( (g^{-1} \partial_i g)(g^{-1} \partial_j g) \right). \tag{2.30}
\]

Then, in the light–cone metric (2.14), since

\[
R^{(2)}_{\eta \eta} = 0 = R^{(2)}_{\xi \xi}, \tag{2.31}
\]

we obtain the following first–order conditions

\[
\partial_\xi (\log f) = -\frac{1}{\alpha} + \frac{\alpha}{4} \text{Tr} \left( (g^{-1} \partial_i g)^2 + (\lambda^{-1} \partial_i \lambda)^2 \right), \tag{2.32}
\]

\[
\partial_\eta (\log f) = -\frac{1}{\alpha} + \frac{\alpha}{4} \text{Tr} \left( (g^{-1} \partial_i g)^2 + (\lambda^{-1} \partial_i \lambda)^2 \right), \tag{2.33}
\]

provided that eq. (2.29) is satisfied. In a more general frame, the first term on the right–hand side has to be replaced by \( \partial [\log (\partial [\log \sqrt{\det g}]) \right) \) and the overall coefficient of the trace term by \( 1/4 \partial (\log \sqrt{\det g}) \); the derivatives are taken with respect to \( \xi \) or \( \eta \) when referring to the generalization of eq. (2.32) or (2.33), respectively.

This simple system of equations for \( \log f \) can be easily integrated, once a pair of solutions \( (g, \lambda) \) for the two \( SL(2, R)/U(1) \) Ernst \( \sigma \)–models is known. This system is also compatible with the other equations following from the dimensional reduction. It can be verified that their compatibility is fully encoded in the \( (\eta \xi) \)–component of eq. (2.22), which is the last one to examine. For this we also have to make use of the expression

\[
R^{(2)}_{\eta \xi} = -\partial_\eta \partial_\xi (\log f) \tag{2.34}
\]
for the 2–dim curvature. Hence, we find that in this case the problem of generating new solutions to the string background equations reduces to the solution of two $SL(2, R)/U(1)$ Ernst $\sigma$–models. The two $\sigma$–models are essentially decoupled because $\sqrt{\text{det} \, g}$ satisfies the wave equation (2.25). This decoupling is manifest in the special coordinate system (2.28), which eliminates the determinant degree of freedom. Our formalism also reproduces the classical equations of reduced gravity for constant matrix $\lambda$.

One might think that it would be possible to choose the condition $\sqrt{\text{det} \, g} = 1$ and still be able to describe a large class of solutions to the string background equations. There is a uniqueness theorem, however, which was proved for reduced gravity a long time ago [11] and can be easily extended to the string case. In particular, for $\sqrt{\text{det} \, g} = 1$, the only physical solution is the flat Minkowski space with trivial dilaton and axion fields. The point is that although there are infinitely many solutions of the ordinary $SL(2, R)/U(1)$ $\sigma$–model (to which the Ernst equation reduces for $\sqrt{\text{det} \, g} = 1$), the additional conditions on the conformal factor $f(\eta, \xi)$ reduce the number of physical possibilities to the trivial one. This will be used later to clarify the meaning of the string Geroch group.

### 3 On the String Geroch Group

In $M_4$ with orthogonal transitivity and two commuting space–like Killing symmetries, the string background equations are described by the 2–dim system of Ernst equations (2.24)–(2.26), plus a simple system of first–order differential equations for the conformal factor. There are six degrees of freedom in the problem, but because of the decoupling that essentially occurs in their dynamics the target space integrability of string theory boils down to the integrability of the $SL(2, R)/U(1)$ Ernst $\sigma$–model. This is a well studied model in reduced gravity and exhibits infinitely many symmetries, in analogy with ordinary 2–dim non–linear $\sigma$–models. We will briefly review some of its integrability aspects and then identify the known $O(2, 2)$ and $S$–duality transformations of string theory in the context of the corresponding string Geroch group, which is much larger.

Consider the Ernst $\sigma$–model

$$\partial_\xi (\alpha g^{-1} \partial_\eta g) + \partial_\eta (\alpha g^{-1} \partial_\xi g) = 0,$$

(3.1)

where $\alpha = \sqrt{\text{det} \, g}$, but without necessarily making the same choice of coordinates as in (2.28). It is convenient for our purposes to adopt a complexified description of the model and impose the reality conditions at the end. For this we first introduce a twist potential matrix $\psi(\eta, \xi)$ such that

$$\partial_\xi \psi = \alpha \epsilon g^{-1} \partial_\xi g, \quad \partial_\eta \psi = -\alpha \epsilon g^{-1} \partial_\eta g,$$

(3.2)

where $\epsilon$ is the usual antisymmetric matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
The existence of $\psi$ is guaranteed by the integrability condition of the system (3.2), which is nothing else but the $\sigma$–model equation (3.1). We then introduce the so–called Ernst potential matrix \cite{15}

$$E = g + i\psi$$

and reformulate the problem in terms of $E$.

It can be verified that the linearization system of the theory (3.1) is given by

$$\partial_\xi F(l) = \frac{l}{1 - 2l(\beta + \alpha)} (\partial_\xi E) i\epsilon F(l),$$

$$\partial_\eta F(l) = \frac{l}{1 - 2l(\beta - \alpha)} (\partial_\eta E) i\epsilon F(l),$$

where $F(l) = F(\eta, \xi; l)$ is a $2 \times 2$ matrix depending on a spectral parameter $l$ with $F(l = 0) = 1$. Here $\alpha, \beta$ is the pair of conjugate solutions (2.27) of the 2–dim wave equation for $\sqrt{\det g}$. This linearization is sufficient to establish the integrability of the $SL(2, R)/U(1)$ Ernst $\sigma$–model. We also note some properties of $g$ and $\psi$, which are useful for the calculations,

$$g \epsilon g = \alpha^2 \epsilon, \quad \psi - \psi^t = 2\beta \epsilon,$$

where $\psi^t$ denotes the transpose of $\psi$.

The model exhibits a hidden symmetry of (non–local) transformations, in close analogy with 2–dim principal chiral models \cite{17}. Following \cite{16}, we define

$$\delta_T E = -\frac{1}{l} \left( F(l) TF(l)^{-1} - T \right) (i\epsilon),$$

where $l$ is taken to be real and $T$ is an element of the $SL(2, R)$ algebra,

$$T = \epsilon_+ T_+ + \epsilon_- T_0 + \epsilon_0 T_0,$$

with arbitrary infinitesimal real parameters $\epsilon_\pm, \epsilon_0$. It follows that

$$\delta_T g = -\frac{1}{l} \mathrm{Re} \left( F(l) TF(l)^{-1} i\epsilon \right),$$

is a symmetric $2 \times 2$ matrix, which defines the infinitesimal action of the Geroch group on the space of solutions of the model. An important property of this transformation is

$$\delta_T \alpha = 0 = \delta_T \beta,$$

and hence $\sqrt{\det g}$ remains invariant.

Introducing the loop expansion of the variation

$$\delta_T = \sum_{n=0}^{\infty} l^n \delta_T^{(n)},$$

it has been proven in the literature \cite{15–17} (see also \cite{12, 13} for alternative derivations) that

$$[\delta_T^{(n)}, \delta_T^{(m)}] = \delta_T^{(n+m)}.$$
which is an $SL(2, R)$ current algebra. The key formula in deriving this result is

$$
\delta_T F(t) = \frac{t}{t-l} \left( F(l)T F(l)^{-1} - F(t)T F(t)^{-1} \right) F(t).
$$

(3.14)

So far we have actually described only half of the modes since both $m, n$ are $\geq 0$. The negative modes can be appended as well by a slight modification of the formalism, but it is well known that the latter give rise to trivial (gauge) transformations on $g$.

In the string case, where we have two such Ernst $\sigma$–models, the corresponding string Geroch group is generated by the $SL(2, R) \times SL(2, R) \simeq O(2, 2)$ current algebra. However, only its non–negative modes will lead to non–trivial transformations of $g$ and $\lambda$, leaving the string equations invariant. The variation of the conformal factor $f$ follows immediately from the transformation of $g$ and/or $\lambda$, so that it still satisfies eqs. (2.32), (2.33) (or their generalization discussed earlier).

In the following we need some more explicit expressions for the variation of $g$ (and $\lambda$). One may use for this the linearized system of equations (3.5), (3.6) writing $F(l)$ as a path–ordered exponential of gauge connections depending on $E$. The results will be valid off shell as well if we use only one of the two equations, say eq. (3.5), in analogy with a similar derivation in 2–dim principal chiral models [17]. With this in mind, expanding the path–ordered exponential form of $F(l)$ in powers of $l$ around $l = 0$, we obtain the following result

$$
\delta_T^{(0)} g = [g \epsilon , T] \epsilon
$$

(3.15)

to zeroth order. To first order we have

$$
\delta_T^{(1)} g = [g \epsilon , T \psi \epsilon - 2 \psi \epsilon T + \psi^t \epsilon T] \epsilon,
$$

(3.16)

using the properties (3.7), and so on. Unlike the variation $\delta_T^{(0)} g$, which is local, $\delta_T^{(1)} g$ is non–local in $g$ since the right–hand side involves the twist potential matrix $\psi$. This will be very important later on. These transformations apply equally well to the $g$ and $\lambda$ sectors of the dimensionally reduced string background equations. We note for completeness that since $g$ is a symmetric matrix, $ge$ is traceless; in these variables the variations (3.15) and (3.16) look closer to the similar variations of ordinary $\sigma$–models.

To justify our claim that the usual $O(2, 2)$ and $S$–duality transformations are just special cases of the string Geroch group, we have to formulate them in the Einstein frame (2.5), on which our previous discussion was based.

Recall that any element of the $O(2, 2)$ group can be represented in the form

$$
D = \begin{pmatrix}
D_1 & D_2 \\
D_3 & D_4
\end{pmatrix}, \quad D \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} D^t = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

(3.17)

where $D_1, \cdots, D_4$ are $2 \times 2$ matrices otherwise arbitrary. Expanding $D$ to first order in an infinitesimal parameter $\epsilon$, we obtain

$$
D_1 = 1 + \epsilon d_1, \quad D_2 = \epsilon d_2, \quad D_3 = \epsilon d_3, \quad D_4 = 1 + \epsilon d_4,
$$

(3.18)
where the Lie algebra elements $d_1, \ldots, d_4$ satisfy the $O(2, 2)$ constraints
\[ d_2 = -d_1^t, \quad d_3 = -d_3^t, \quad d_4 = -d_1^t. \] (3.19)

The decomposition of $O(2, 2)$ into two commuting $SL(2, R)$ algebras can be described in this representation by choosing the basis elements
\[ d_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad d_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (3.20)

for one of them and
\[ d_+ = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad d_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \] (3.21)

for the other $SL(2, R)$ factor, so that $[d_+, d_-] = d_0$ and $[d_0, d_±] = ±2d_±$.

It is known that 4–dim string backgrounds with two commuting Killing fields exhibit an $O(2, 2)$ symmetry, which leaves the 1–loop beta function equations invariant [8, 9]. For the special class of block diagonal metrics (2.14), we consider the $\sigma$–model variables
\[ g_{AB}^{(σ)}(η, ξ) = g_{AB} + ε_{AB}B, \quad f^{(σ)}(η, ξ) = e^{2Φ}f, \] (3.22)

with $B$ defined as in eq. (2.16) and $ε_{23} = 1$. Then, the invariant action of an arbitrary $O(2, 2)$ group element $D$, (3.17), is described by the transformation
\[ G^{(σ)} \rightarrow ˜G^{(σ)} = (D_1G^{(σ)} + D_2)(D_3G^{(σ)} + D_4)^{-1}, \] (3.23)

\[ f^{(σ)} \rightarrow ˜f^{(σ)} = f^{(σ)}, \] (3.24)

\[ e^{2Φ} \rightarrow ˜e^{2Φ} = e^{2Φ} \frac{\det ˜g^{(σ)}}{\det g^{(σ)}}. \] (3.25)

Note that eq. (3.25) implies that in the Einstein frame
\[ \sqrt{\det ˜g} = \sqrt{\det g}, \] (3.26)

which is a common property with the Geroch group (cf. eq. (3.11)). This motivates us to examine whether the remaining $O(2, 2)$ transformations are also part of the string Geroch group.

The infinitesimal form of the action (3.23) is
\[ δG^{(σ)} = ε(d_1G^{(σ)} + G^{(σ)}d_1^t + d_2 - G^{(σ)}d_3G^{(σ)}), \] (3.27)
where  is a collective infinitesimal parameter (not to be confused with the antisymmetric matrix (3.3)). Taking the symmetric and antisymmetric parts of this variation we obtain in the  frame the expressions for  and  respectively. The results of the calculation will be described separately for the two  subalgebras to which  is decomposed.

(i). For the  subalgebra (3.20) we obtain

\[
\begin{align*}
\delta g^{(\sigma)}_{22} &= 2(\epsilon_- g_{23}^{(\sigma)} + \epsilon_0 g_{22}^{(\sigma)}), \\
\delta g^{(\sigma)}_{33} &= 2(\epsilon_+ g_{23}^{(\sigma)} - \epsilon_0 g_{33}^{(\sigma)}), \\
\delta g^{(\sigma)}_{23} &= \epsilon_+ g_{22}^{(\sigma)} + \epsilon_- g_{33}^{(\sigma)},
\end{align*}
\]

where  are infinitesimal parameters corresponding to the three  generators and

\[
\delta B = 0.
\]

In this case we have  and so eq. (3.25) implies

\[
\delta \Phi = 0.
\]

We also conclude from eq. (2.17) that  and so \(\delta \lambda = 0\).

Since  the variation of the metric matrix  is independent of the frame and eqs. (3.28)–(3.30) can be summarized as follows,

\[
\delta g = \left[ g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \epsilon_0 & \epsilon_- \\ \epsilon_+ & -\epsilon_0 \end{pmatrix} \right] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

They coincide with the variation (3.15) in the fundamental representation of  . Therefore, in the Einstein frame the first  subalgebra of the usual  string symmetry can be identified with the zero modes of the first  current subalgebra of the string Geroch group.

(ii). For the other  subalgebra the situation is more complicated, requiring the use of non–local transformations. Using the basis elements (3.21), we find that the corresponding variation (3.27) with infinitesimal parameters  yields

\[
\delta g^{(\sigma)}_{AB} = 2(\epsilon_0 + \epsilon_+) g^{(\sigma)}_{AB},
\]

\[
\delta B = -\epsilon_- + 2\epsilon_0 B + \epsilon_+(B^2 - \det g^{(\sigma)}).
\]

We can easily calculate the variation of  and then use eq. (3.25) to find the corresponding variation of the dilaton field; it reads

\[
\delta \Phi = \epsilon_0 + \epsilon_+ B.
\]

Going to the Einstein frame we find

\[
\delta g_{AB} = 0.
\]
and so the second $SL(2, R)$ subalgebra of the $O(2, 2)$ string symmetry acts only on the $\lambda$–sector, contrary to case (i), where the situation was the other way around.

We note that $\delta \Phi$ does not depend on $\epsilon_-$, while $\delta B$ depends trivially on it (constant $B$–shift). All the non–trivial dependence of the variations is on $\epsilon_0$ and $\epsilon_+$, which we now examine separately. We set first $\epsilon_+ = 0 = \epsilon_-$ and find that the axion–dilaton system transforms (up to an overall constant) as

$$
\delta b = -2\epsilon_0 b, \quad \delta \Phi = \epsilon_0, \quad (3.38)
$$

which for the matrix $\lambda$ implies the result

$$
\delta \lambda = \begin{bmatrix}
\lambda & 0 & 1 \\
-1 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
0 & 0 & \epsilon_0 \\
0 & -\epsilon_0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}. \quad (3.39)
$$

Comparison with eq. (3.15) shows that in this case the variation of $\lambda$ can be identified with $\delta^{(0)} \lambda$, namely the zero mode of the diagonal $U(1)$ subalgebra of the (other) $SL(2, R)$ current algebra. For $\epsilon_0 = 0 = \epsilon_-$, however, the transformation of $\lambda$ is non–local; we have to go beyond the zero modes to describe it in a similar way.

It follows from eqs. (3.35) and (3.36) that

$$
\delta (\partial_\xi \Phi) = \epsilon_+ e^{4\Phi} \sqrt{\det g} \partial_\xi b, \quad \delta (\partial_\eta \Phi) = -\epsilon_+ e^{4\Phi} \sqrt{\det g} \partial_\eta b \quad (3.40)
$$

and

$$
\delta (e^{2\Phi} \partial_\xi b) = -2\epsilon_+ \partial_\xi (e^{2\Phi} \sqrt{\det g}), \quad \delta (e^{2\Phi} \partial_\eta b) = 2\epsilon_+ \partial_\eta (e^{2\Phi} \sqrt{\det g}). \quad (3.41)
$$

We claim that this non–local variation of the matrix $\lambda$ can be written in the form (3.16),

$$
\delta \lambda = - \begin{bmatrix}
\lambda \epsilon & 0 & 0 \\
0 & \epsilon_+ & 0
\end{bmatrix}
\psi_\lambda \epsilon + \psi_\lambda^t \epsilon (-2\psi_\lambda) \epsilon
\begin{pmatrix}
0 & 0 & \epsilon_+ \\
0 & -\epsilon_+ & 0
\end{pmatrix} \epsilon, \quad (3.42)
$$

where $\psi_\lambda$ is the twist potential matrix of $\lambda$.

$$
\partial_\xi \psi_\lambda = \sqrt{\det g} \epsilon (\sqrt{\det g} \lambda)^{-1} \partial_\xi (\sqrt{\det g} \lambda) \quad (3.43)
$$

$$
\partial_\eta \psi_\lambda = -\sqrt{\det g} \epsilon (\sqrt{\det g} \lambda)^{-1} \partial_\eta (\sqrt{\det g} \lambda) \quad (3.44)
$$

and $\epsilon$ is the antisymmetric matrix (3.3) as before (not to be confused with the infinitesimal parameter $\epsilon_+$ of the variation). In other words, this transformation is identified with the $-\delta^{(-1)}_\lambda$ mode of the second $SL(2, R)$ current algebra of the string Geroch group. The $SL(2, R)$ algebra of case (ii) will then be completely described in terms of the generators $\delta^{(-1)}_+, \delta^{(0)}_0$ and $\delta^{(1)}_+$. We recall that generators with negative modes correspond to trivial transformations, which explains why the $\epsilon_-$ part of the variation can be naturally identified with $\delta^{(-1)}_+ \lambda$ to close the $SL(2, R)$ algebra.

*We scale $\lambda$ to $\sqrt{\det g} \lambda$ in order to apply the same formalism as was used for the metric sector of the theory. This modification, however, does not affect eq. (3.42) because $\delta \sqrt{\det g} = 0$ and so the determinant factor cancels from both sides.
The proof of eq. (3.42) is straightforward once the right guess was made. It is sufficient to check the insertion

$$\delta e^{2\Phi} = 2\epsilon_+\psi^{11}_\lambda e^{2\Phi}, \quad \delta (be^{2\Phi}) = 2\epsilon_+\psi^{21}_\lambda e^{2\Phi},$$  \hspace{1cm} (3.45)

where (11) and (21) denote the corresponding elements of the $2 \times 2$ matrix $\psi_\lambda$. On the other hand, the defining relations (3.43) and (3.44) imply

$$\partial_\xi \psi^{11}_\lambda = \sqrt{\det g} e^{4\Phi} \partial_\xi b, \quad \partial_\eta \psi^{11}_\lambda = -\sqrt{\det g} e^{4\Phi} \partial_\eta b$$  \hspace{1cm} (3.46)

and

$$\partial_\xi \psi^{21}_\lambda = \sqrt{\det g} (e^{4\Phi} b \partial_\xi b - 2\partial_\xi \Phi) - \partial_\xi \sqrt{\det g}, \quad \partial_\eta \psi^{21}_\lambda = -\sqrt{\det g} (e^{4\Phi} b \partial_\eta b - 2\partial_\eta \Phi) + \partial_\eta \sqrt{\det g}.$$  \hspace{1cm} (3.47)

Then, using eqs. (3.46)–(3.48) it can be verified that the variation (3.45) coincides with (3.40) and (3.41) after differentiating $\psi_\lambda$ with respect to $\eta$ and $\xi$.

Summarizing the results so far, we have found that the usual $O(2, 2)$ string symmetries are part of a much larger group of transformations generated by the $O(2, 2)$ current algebra. The embedding of $O(2, 2)$, however, is not the obvious one and requires going beyond the zero modes in the axion formalism of the problem. In particular, the first $SL(2, R)$ subalgebra corresponds to $\delta^{(0)}_+, \delta^{(0)}_0$ and $\delta^{(0)}_-$, while the second one corresponds to $\delta^{(-1)}_+ \delta^{(0)}_0$ and $\delta^{(1)}_-$ in the mode expansion of the two $SL(2, R)$ current algebras into which the string Geroch group is decomposed. In both cases, the variation of the conformal factor $f$ follows immediately from eq. (3.24) after transforming it to the Einstein frame.

It is natural to query at this point the meaning of the global symmetry generated by the zero modes of the second $SL(2, R)$ current algebra. Since it only acts on the $\lambda$–sector of the theory, its infinitesimal form will be

$$\delta^{(0)} \lambda = \left[ \lambda \epsilon, \left( \begin{array}{cc} \epsilon_0 & \epsilon_- \\ \epsilon_+ & -\epsilon_0 \end{array} \right) \right] \epsilon,$$  \hspace{1cm} (3.49)

in terms of an arbitrary element of the $SL(2, R)$ algebra. The transformation (3.49) can be equivalently stated as

$$\delta \Phi = \epsilon_0 + \epsilon_- b$$  \hspace{1cm} (3.50)

and

$$\delta b = \epsilon_+ - 2\epsilon_0 b - \epsilon_-(b^2 - e^{-4\Phi}).$$  \hspace{1cm} (3.51)

Then, in terms of the $S_\pm$ variables (2.8) that provide an alternative $\sigma$–model description of the $\lambda$–sector, this transformation reads

$$\delta S_\pm = \epsilon_+ - 2\epsilon_0 S_\pm - \epsilon_- S_\pm^2.$$  \hspace{1cm} (3.52)

It is straightforward to verify that this is the infinitesimal form of the global transformation

$$S_\pm \rightarrow \frac{AS_\pm + B}{CS_\pm + D}, \quad AD - BC = 1$$  \hspace{1cm} (3.53)
that leaves the $S$–part of the effective action (2.9) invariant. The relevant infinitesimal expansion is $A = 1 - \epsilon_0$, $B = \epsilon_+$, $C = \epsilon_-$ and $D = 1 + \epsilon_0$, which is an alternative description of the Lie algebra elements. The $SL(2, R)$ symmetry (3.53) has been encountered before in 4–dim string theory, using different methods, and it is known as $S$–duality [10]. We have just demonstrated that it is also part of the string Geroch group.

There are many more $SL(2, R)$ subalgebras that can be extracted from the present formalism. More precisely, the generators $\delta_+^{(n)}$, $\delta_0^{(0)}$, $\delta_-^{(-n)}$ satisfy the commutation relations

$$\left[\delta_+^{(n)}, \delta_-^{(-n)}\right] = \delta_0^{(0)}, \quad \left[\delta_0^{(0)}, \delta_\pm^{(\pm n)}\right] = \pm 2\delta_\pm^{(n)}$$

(3.54)

for any integer $n$, but they are realized non–locally for $n \neq 0$. Transformations of this type can be applied equally well to the $g$ or $\lambda$ sectors of string theory, but their physical interpretation for arbitrary $n$ lies beyond the scope of the present work. More generally, the full string Geroch group can be used to continuously generate infinitely many solutions descending from any given string background. It is not clear if its action on the space of solutions is transitive. If this were the case, any solution could be obtained (at least formally) from the trivial one in an appropriately chosen coordinate system. In section 5 we will be mostly concerned with the construction of multi–soliton solutions, while the more general problem will be left open to future work.

4 A Discrete Symmetry of 4–dim Strings

The 1–loop string background equations in $M_4$ with two commuting space–like Killing fields and orthogonal transitivity exhibit an additional $Z_2$ symmetry. This manifests itself by interchanging the field content of the two $SL(2, R)/U(1)$ Ernst $\sigma$–models that describe the $g$ and $\lambda$ sectors of the theory. We consider the transformation

$$\begin{pmatrix} g_{22} & g_{23} \\ g_{23} & g_{33} \end{pmatrix} \leftrightarrow \sqrt{\det g} e^{2\Phi} \begin{pmatrix} 1 & b \\ b & b^2 + e^{-4\Phi} \end{pmatrix},$$

(4.1)

which is legitimate in the Einstein frame because it leaves the reduced string equations invariant, without changing the conformal factor. Indeed, since

$$\text{Tr} \left( (g^{-1}\partial g)^2 + (\lambda^{-1}\partial \lambda)^2 \right) = \text{Tr} \left( (g^{-1}\partial g)^2 + \left( (\sqrt{\det g} \lambda)^{-1}\partial (\sqrt{\det g} \lambda) \right)^2 \right) - 2 \left( \partial (\log \sqrt{\det g}) \right)^2,$$

(4.2)

it can be readily verified that the discrete transformation (4.1) has no effect on $f$ (cf. eqs. (2.32) and (2.33) or their generalization). Also, $\sqrt{\det g}$ remains invariant.

In $M_4$ with signature $- + ++$, both $\sigma$–models are Euclidean. Therefore, it makes perfect sense to perform the field interchange and construct the “mirror image” of any
solution there is available. The space–time interpretation of the solutions, however, will be quite different from their “mirror images”. We also note that this transformation is not an element of the usual \( O(2, 2) \) group, in general, since otherwise the dilaton field would have to stay invariant as well (cf. eq. (3.24) in the Einstein frame). If we were considering strings in \( M_4 \) with one space–like and one time–like Killing symmetry, this field interchange would not be physically correct; in this case the \( \sigma \)–model \( g \) would be Lorentzian, while the \( \sigma \)–model \( \lambda \) would stay Euclidean. Similarly, if we were considering string propagation in 4–dim Euclidean space with two Killing symmetries, the \( \sigma \)–model \( g \) would be Euclidean and the \( \lambda \) Lorentzian. The only other case that the field interchange is consistent with the target space character of the \( \sigma \)–models is string propagation in 4–dim spaces with signature \( (2, 2) \); in this case both \( \sigma \)–models are Lorentzian, provided that there are one space–like and one time–like Killing symmetries in the theory.

An interesting example of 4–dim strings with two space–like Killing symmetries in \( M_4 \) has been constructed in the literature using conformal field theories techniques [6, 7]. In particular, the Lorentzian \((SL(2, R) \times SU(2))/(U(1) \times U(1))\) coset model provides a 4–dim solution with non–trivial dilaton and axion fields and with \( \delta c = 0 \) to first order in \( \alpha' \), which describes a closed, inhomogeneous, expanding and recollapsing universe. The metric [7]
\[
ds_{(\sigma)}^2 = -(dX^0)^2 + (dX^1)^2 + g_{22}^{(\sigma)}(dX^2)^2 + g_{33}^{(\sigma)}(dX^3)^2,
\]
where
\[
g_{22}^{(\sigma)} = \frac{2(1 - \sin \theta)(\sin X^0 \sin X^1)^2}{(1 - \cos 2X^0 \cos 2X^1) + \sin \theta(\cos 2X^0 - \cos 2X^1)},
\]
\[
g_{33}^{(\sigma)} = \frac{2(1 + \sin \theta)(\cos X^0 \cos X^1)^2}{(1 - \cos 2X^0 \cos 2X^1) + \sin \theta(\cos 2X^0 - \cos 2X^1)}
\]
describes the classical geometry of this model in the \( \sigma \)–model frame of string theory. The coordinates \( X^0, X^1 \) take values in the close interval \([0, \pi/2]\) and \( \theta \) is a free parameter that enters in defining the gauged WZW model. Also, the dilaton field is
\[
\Phi = -\frac{1}{2} \log \left( (1 - \cos 2X^0 \cos 2X^1) + \sin \theta(\cos 2X^0 - \cos 2X^1) \right),
\]
while the only non–zero component of the antisymmetric tensor field is \( B_{23} \equiv B \),
\[
B = -\frac{1}{2} \frac{(\cos 2X^0 - \cos 2X^1) + \sin \theta(1 - \cos 2X^0 \cos 2X^1)}{(1 - \cos 2X^0 \cos 2X^1) + \sin \theta(\cos 2X^0 - \cos 2X^1)}
\]
The fields depend only on \( X^0, X^1 \) and the metric \( g^{(\sigma)}_{AB} \) is diagonal.

As an application, we are going to use the discrete transformation (4.1) to generate a new solution, which is the “mirror image” of this cosmological string model. We first find the axion field
\[
b = \cos \theta(\cos 2X^0 - \cos 2X^1)
\]
and
\[
\sqrt{\det g} = \frac{1}{2} \cos \theta \sin 2X^0 \sin 2X^1
\]
in the Einstein frame.† The solution that is obtained by performing the field interchange (4.1) in the Einstein frame has zero axion field because the original metric (4.3) is diagonal. Then, the resulting metric–dilaton system can be rotated back to the σ–model frame, where it assumes the following form: the dilaton is still logarithmic,

$$\tilde{\Phi} = \frac{1}{2} \left( \log(\tan X^0) + \log(\tan X^1) \right) + \text{const.},$$

(4.10)

but the metric $g_{AB}^{(\sigma)}$ develops off–diagonal elements that are proportional to the original axion field (4.8). We have explicitly

$$d\tilde{s}^2 = f_1^{(\sigma)} \left( -(dX^0)^2 + (dX^1)^2 \right) + f_2^{(\sigma)} \gamma_{AB}^{(\sigma)} dX^A dX^B,$$

(4.11)

where

$$f_1^{(\sigma)} = \tan X^0 \tan X^1 \left( (1 - \cos 2X^0 \cos 2X^1) + \sin \theta (\cos 2X^0 - \cos 2X^1) \right),$$

(4.12)

$$f_2^{(\sigma)} = \frac{2 \cos \theta (\sin X^0 \sin X^1)^2}{(1 - \cos 2X^0 \cos 2X^1) + \sin \theta (\cos 2X^0 - \cos 2X^1)}$$

(4.13)

and

$$\gamma_{22}^{(\sigma)} = 1, \quad \gamma_{23}^{(\sigma)} = \cos \theta (\cos 2X^0 - \cos 2X^1),$$

(4.14)

$$\gamma_{33}^{(\sigma)} = (\cos 2X^0 - \cos 2X^1)^2 + (1 - \cos 2X^0 \cos 2X^1)^2 \nonumber$$

$$+ 2 \sin \theta (\cos 2X^0 - \cos 2X^1)(1 - \cos 2X^0 \cos 2X^1),$$

(4.15)

up to an overall normalization factor $(1 - \sin \theta) / \cos \theta$ coming from the constant term of the dilaton field (4.10). Therefore, only $g_{22}^{(\sigma)}$ remains invariant under the field interchange (4.1) in this case.

The solution we have just obtained has a singularity structure that is different from the original model. Clearly, the space–time points where a curvature singularity occurs should coincide with the points that the dilaton field blows up. In the cosmological solution (4.3)–(4.5) the only curvature singularities are at $X^0 = 0 = X^1$ (initial singularity) or $X^0 = \pi/2 = X^1$ (final singularity). In the “mirror image” solution the dilaton field (4.10) blows up separately at each boundary point of $X^0, X^1$, namely at $X^0 = 0$ or $X^1 = 0$ or $X^0 = \pi/2$ or $X^1 = \pi/2$. At these points $\sqrt{\det g}$ also vanishes, as can be readily verified from eq. (4.9). If we had chosen to work with the special coordinate system (2.28), (2.29) in the Einstein frame, with $\sqrt{\det g}$ playing the role of $X^0$, the singularities would have appeared at $X^0 \equiv \alpha = 0$. It is not clear to us whether the “mirror image” of the Lorentzian ($SL(2, R) \times SU(2))/(U(1) \times U(1))$ coset model admits a conformal field theory description as well.

It would also be interesting to study the intertwining of the string Geroch group with the additional discrete symmetry that interchanges the field content of the two Ernst σ–models. This might be necessary for finding the right algorithm to generate (at least formally) all 4–dim string backgrounds with two commuting space–like Killing symmetries. Much work remains to be done in this direction.

†√det $g$ satisfies the 2–dim wave equation as it should, and the conjugate solution is $\beta = -\frac{1}{2} \cos \theta \cos 2X^0 \cos 2X^1$. 

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5 Multi–soliton Solutions

In this section we describe a special class of solutions, called multi–solitons, using the integrability properties of the dimensionally reduced string background equations. String multi–solitons are characterized by two positive integers \((n, m)\) and descend as a family from any given string background. Their construction requires the use of the spectral parameter that appears in the zero curvature formulation of the reduced string equations. Hence, they correspond to special elements of the full string Geroch group lying beyond the range of the usual \(O(2,2)\) or \(S–duality\) transformations. Soliton calculations in string theory can be carried out in exact analogy with reduced vacuum Einstein gravity by applying the Belinski–Sakharov method to either of the two \(SL(2,R)/U(1)\) Ernst \(\sigma–model\)s. The pair \((n, m)\) will therefore refer to the soliton numbers corresponding to the \(g\) and \(\lambda\) sectors of the theory. Once this construction is done in the Einstein frame, the conformal factor \(f\) can be calculated by a simple integration.

We briefly review the construction of solitons in the Ernst \(\sigma–model\), since the method is not widely known to string theorists. It is convenient for this purpose to adopt an alternative formulation of the linearization system (3.5), (3.6) without making use of the twist potential matrix, but introducing derivatives with respect to the spectral parameter \(l\). Following Belinski and Sakharov [14], we introduce the differential operators

\[
D_1 = \partial_\xi - 2(\partial_\xi \alpha) \frac{l}{l - \alpha} \partial_\xi, \quad D_2 = \partial_\eta + 2(\partial_\eta \alpha) \frac{l}{l + \alpha} \partial_\eta, \tag{5.1}
\]

where \(\alpha \equiv \sqrt{\det g}\) as usual. Their commutativity, \([D_1, D_2] = 0\), is a consequence of the wave equation \(\partial_\eta \partial_\xi (\sqrt{\det g}) = 0\). Then, the following system of equations

\[
D_1 \Psi = \frac{A}{l - \alpha} \Psi, \quad D_2 \Psi = \frac{B}{l + \alpha} \Psi, \tag{5.2}
\]

where \(\Psi(\eta, \xi; l)\) is a complex matrix function \((l\) can be complex) and

\[
A = -\alpha \partial_\xi gg^{-1}, \quad B = \alpha \partial_\eta gg^{-1}, \tag{5.3}
\]

linearize the Ernst \(\sigma–model\) as eq. (3.1) provides their compatibility condition. The matrix \(\Psi\) is a generalization of \(g\) with a spectral parameter so that

\[
g(\eta, \xi) = \Psi(\eta, \xi; l = 0). \tag{5.4}
\]

Suppose now that a solution \(g_0(\eta, \xi)\) is already known and \(\Psi_0(\eta, \xi; l)\) is the corresponding solution of the linear system (5.2). Any other solution descending from it has

\[
\Psi(l) = \chi(l) \Psi_0(l), \tag{5.5}
\]

where \(\chi(\eta, \xi; l)\) is a matrix satisfying the equations

\[
D_1 \chi = \frac{1}{l - \alpha} (A\chi - \chi A_0), \quad D_2 \chi = \frac{1}{l + \alpha} (B\chi - \chi B_0). \tag{5.6}
\]

\(^\dagger\)Soliton methods have been applied before in de Sitter space–time to construct multi–string solutions [18].
A₀ and B₀ are the currents (5.3) of the “seed” matrix g₀. We also have
\[ g(\eta, \xi) = \chi(l = 0)g₀ \]  
(5.7)
and therefore finding \( \chi \) will produce new solutions of the model. Note that \( \chi \) has to satisfy the following additional conditions:

(a) \( \chi(\bar{l}) = \chi(l) \), so that \( \chi \) is real on the real axis of the complex \( l \)-plane.

(b) \( g = \chi(\alpha^2/l)g₀\chi'(l) \), with \( \chi(l = \infty) = 1 \), so that \( g \) is symmetric.

(c) \( \det \chi(l = 0) = 1 \), so that \( \sqrt{\det g} = \sqrt{\det g₀} = \alpha \). If this condition is not satisfied, we simply have to rescale \( g \) by \( \alpha/\sqrt{\det g} \) so that the determinant will remain invariant in the coordinate system where the calculations are performed.

Soliton solutions correspond to the special ansatz for \( \chi \),
\[ \chi(\eta, \xi; l) = 1 + \sum_{k=1}^{n} \left( \frac{R_k(\eta, \xi)}{l - p_k(\eta, \xi)} + \frac{\bar{R}_k(\eta, \xi)}{l - \bar{p}_k(\eta, \xi)} \right) \],  
(5.8)
up to an overall normalization, which will be determined from the condition (c); \( R_k, \bar{R}_k \) are the residue matrices and \( p_k, \bar{p}_k \) the locations of the poles in the complex \( l \)-plane. They can all be determined explicitly by substituting the expression (5.8) in eq. (5.6) and comparing the pole structure of the left and right–hand sides [14]. Setting \( l = p_k \), we first determine the location of the poles,
\[ p_k = c_k - \beta - \sqrt{(c_k - \beta)^2 - \alpha^2} \],  
(5.9)
where \( c_k \) are arbitrary complex constants and \( \beta \) is the conjugate variable of \( \alpha \) (cf. eq. (2.27)). Then,
\[ q_k = c_k - \beta + \sqrt{(c_k - \beta)^2 - \alpha^2} = \frac{\alpha^2}{p_k} \]  
(5.10)
determine the location of the poles of the inverse matrix \( \chi^{-1}(l) \). Furthermore, \( R_k(\eta, \xi) \) turn out to be degenerate \( 2 \times 2 \) matrices whose elements depend on the \( 2 \)-vectors
\[ V^{(k)}_A(\eta, \xi) = c^{(k)}_B \Psi^{-1}_0(\eta, \xi; l = p_k)_{BA} \],  
(5.11)
where \( c^{(k)}_B \) are arbitrary constant vectors.

For example, for \( n = 1 \), explicit calculation yields
\[ R_1(\eta, \xi)_{AB} = \frac{1}{\Delta} \left( \frac{V^{(1)}_D V^{(1)}_E (g₀)_{DE} V^{(1)}_C (g₀)_{CA} - \bar{V}^{(1)}_D \bar{V}^{(1)}_E (g₀)_{DE} V^{(1)}_C (g₀)_{CA}}{q₁ - \bar{p}₁} \right) V^{(1)}_B, \]  
(5.12)
where
\[ \Delta = \frac{|V^{(1)}_A V^{(1)}_B (g₀)_{AB}|^2}{|q₁ - p₁|^2} - \frac{|\bar{V}^{(1)}_A \bar{V}^{(1)}_B (g₀)_{AB}|^2}{|\bar{q}₁ - p₁|^2}. \]  
(5.13)
The new solution has \( \det g = \alpha^6/|p₁|^4 \), which should be rescaled according to the condition (c) above. The final result for the (physical) metric \( g \) reads
\[ g = \frac{|p₁|^2}{\alpha^2} \left( 1 - \frac{R₁}{p₁} - \frac{\bar{R}₁}{\bar{p}₁} \right) g₀ \]  
(5.14)
and describes the double–soliton solution on a given background $g_0$. It is not uniquely fixed, however, as it clearly depends on three complex parameters $c_1, c^{(1)}_B$. Even in this simple case, $\chi(0)$ is highly non–trivial, depending on $g_0 = \Psi_0(l = 0)$ and $\Psi_0(l = p_1)$.

The only non–trivial step in the calculation is finding $\Psi_0(\eta, \xi; l)$ by solving the system (5.2) for $g = g_0$. $\Psi_0$ can be determined recursively by introducing a power series expansion

$$\Psi_0(\eta, \xi; l) = \Psi_0^{(0)}(\eta, \xi) + \sum_{k=1}^{N} l^k \Psi_0^{(k)}(\eta, \xi),$$

(5.15)

where $\Psi_0^{(0)} = g_0$. This series typically terminates at a finite integer $N$, depending on $g_0$. Then, the system of differential equations (5.2) yields the following recursive relations,

$$\partial_\xi \Psi_0^{(k)} = (\alpha \partial_\xi + 2(k + 1)(\partial_\xi \alpha) + A_0) \Psi_0^{(k+1)},$$

(5.16)

$$\partial_\eta \Psi_0^{(k)} = - (\alpha \partial_\eta + 2(k + 1)(\partial_\eta \alpha) - B_0) \Psi_0^{(k+1)},$$

(5.17)

for all $k \geq 0$. It is clear that for finite $N$, the leading coefficient $\Psi_0^{(N)}$ is a constant $2 \times 2$ matrix. The result from the iteration of the recursive relations (5.16), (5.17) cannot be brought into a closed form for general $g_0$. Therefore, $\Psi_0(\eta, \xi; l)$ has to be determined separately in each case.

Once this is done, any multi–soliton solution can be constructed explicitly by straightforward generalization of the $n = 1$ case. For $n > 1$, the new solutions of the Ernst $\sigma$–model will depend on $3n$ free complex parameters. Their form, however, becomes considerably more complicated for arbitrary $n$. This solution generating technique can be applied to either of the two $\sigma$–models sectors of string theory in the Einstein frame. As a result, an $(n, m)$ family of multi–soliton solutions can be constructed explicitly on any given string background, provided that $\Psi_0(\eta, \xi; l)$ has been determined in each sector. The conformal factor of the soliton solutions, as has already been pointed out, can be found by simple integration, in analogy with reduced gravity. The dressing formula (5.7) has the advantage of providing us with explicit formulas for the soliton solutions. The difficult part is finding which particular element of the Geroch group corresponds to $\chi(0)$, if one wishes to do so.

We note that the ansatz (5.8) describes double, quadruple, etc., soliton solutions for $n = 1, 2, \ldots$, etc. The reason we are not considering single, triple, etc., soliton solutions is that their space–time interpretation is problematic. For the single soliton solution, for example, both $c_1$ and $p_1$ in eq. (5.9) have to be real. In this case we are in the region of space–time with $(c_1 - \beta)^2 \geq \alpha^2$; $p_1$ becomes complex in the complementary region $(c_1 - \beta)^2 < \alpha^2$ where the continuation of the single–soliton solution is a double–soliton with the special property $|p_1|^2 = \alpha^2$. It is well known in this case that the solution (5.14) will remain unperturbed, i.e. $g = g_0$, because the poles of $\chi(l)$ are situated on the circle $|l|^2 = \alpha^2$ in the complex $l$–plane [14]. The point now is that although the single soliton solution can be defined continuously everywhere, its first derivatives will be discontinuous at the points $(c_1 - \beta)^2 = \alpha^2$, which is considered to be problematic.
Many non–trivial solutions of reduced gravity have been constructed explicitly by applying the soliton technique to various backgrounds [14]. These results can also be generalized to 4–dim string theory with non–trivial dilaton and axion fields. We may consider as an application the (1, 0) soliton solution on the cosmological background (4.3)–(4.7). Going to the Einstein frame, where our methods are applicable, we find the following simple result for the \( g \)–sector of the model:

\[
\Psi_0(X^0, X^1; l) = g_0(X^0, X^1) - \frac{l}{\cos \theta} \begin{pmatrix} 1 - \sin \theta & 0 \\ 0 & 1 + \sin \theta \end{pmatrix}; \quad (5.18)
\]

\( \alpha \) and \( \beta \) have been determined in section 4 and therefore all the quantities (5.9)–(5.13) can be explicitly found. Substituting \( p_1, R_1 \) and their complex conjugates in eq. (5.14), while keeping the \( \lambda \)–sector untouched, we obtain the metric of the (1, 0) soliton in the same coordinate system. One may similarly work out the (0, 1) soliton solution on the “mirror image” background (4.10)–(4.15). The final expressions will be omitted for simplicity. We point out, nevertheless, that the matrix \( R_1 \) (and hence \( g \) of the (1, 0) soliton) has non–zero off–diagonal elements. This also implies that the (0, 1) soliton solution on the “mirror image” background carries a non–trivial axion field.

More general solitons can be constructed on this cosmological background by finding \( \Psi_0(\eta, \xi; l) \) of the \( \lambda \)–sector as well, but the final result is much more complicated. A detailed analysis of the corresponding \( (n, m) \) soliton solutions and their space–time interpretation will be presented elsewhere. It is also conceivable that the Nappi–Witten universe can be itself interpreted as a soliton solution around a much simpler string background, but this possibility will not be explored here.

It is very difficult to solve the system of equations (5.6) outside the soliton sector. However, some other classes of solutions have been constructed in the literature using different methods. It is worth mentioning that meron–like solutions also exist for the Ernst \( \sigma \)–model [19], in analogy with ordinary non–linear \( \sigma \)–models [20]. This last class is parametrized by two arbitrary functions \( f_1(\eta) \) and \( f_2(\xi) \), which essentially describe the general solution of the 2–dim wave equation \( \partial_\eta \partial_\xi (\sqrt{\det g}) = 0 \). The meron–like solutions of the Ernst \( \sigma \)–model are of the form

\[
g = \frac{1}{\sqrt{f_1 f_2}} \left( C_1 \log \frac{f_1}{f_2} + C_2 \right) \begin{pmatrix} C_3(f_1 f_2 + 1) & f_1 f_2 - 1 \\ f_1 f_2 - 1 & (f_1 f_2 + 1)/C_3 \end{pmatrix}, \quad (5.19)
\]

where \( C_1, C_2 \) and \( C_3 \) are arbitrary constants.

It would be interesting to find under what circumstances these special classes of solutions admit a conformal field theory interpretation in string theory.
6 Conclusions and Discussion

In this paper we have investigated the target space integrability of 4–dim string theory with dilaton and axion fields in the presence of two commuting Killing symmetries. The 1–loop string background equations simplify considerably in the Einstein frame, reducing to two $SL(2,R)/U(1)$ Ernst $\sigma$–models, plus a linear system of first–order differential equations for the conformal factor. The non–local symmetries of the two $\sigma$–models combine to the string Geroch group, which acts on the space of solutions generalizing the usual $O(2,2)$ and $S$–duality transformations. This infinite dimensional symmetry can be used (at least formally) as a solution generating technique for new non–trivial string backgrounds. If $\delta c \neq 0$ to first order in $\alpha'$, this infinite dimensional structure will not survive. It will be interesting to see what happens in this case.

It might be possible to obtain all the solutions from the trivial one by exponentiating the infinitesimal action of the underlying $O(2,2)$ current algebra. This expectation does not contradict the uniqueness theorem of section 2. One might naively think that the Geroch group cannot generate non–trivial solutions from flat space because its action keeps $\sqrt{\det g}$ invariant. This would certainly be true if we had chosen a coordinate system with $\det g = 1$ in Minkowski space. We note, however, that the action of the string Geroch group does not commute, in general, with coordinate transformations. Therefore, starting from Minkowski space with an appropriately chosen coordinate system, so that $\det g$ is not constant, non–trivial solutions can emerge. A typical example that illustrates this point is the use of polar coordinates in 4–dim Minkowski space. Indeed, if we define new coordinates $(X^1, X^2) \rightarrow (r, \varphi),$

$$X^1 = r \sin \varphi, \quad X^2 = r \cos \varphi,$$

the metric will become

$$ds^2 = -(dX^0)^2 + dr^2 + r^2 d\varphi^2 + (dX^3)^2,$$

which has non–trivial determinant. The real difficulty lies in making use of the Geroch group for the explicit construction of new solutions from old ones. The identification of group elements is not straightforward even for the multi–solitons which are much simpler to describe in the Belinski-Sakharov formalism. More work is certainly required in this direction.

Another related issue is the problem of boundary conditions. Infinite symmetries in integrable systems do not necessarily respect the boundary conditions. Their use for generating new solutions normally introduces various free parameters, as in the multi–soliton case. Some of them can be uniquely determined by imposing the right boundary conditions, but some arbitrariness may remain. This is physically interesting because if non–trivial solutions are going to emerge from flat space (or other simple backgrounds) their characteristic parameters will have to emerge as well. It would be very desirable to prove uniqueness theorems for various 4–dim string backgrounds, in analogy with reduced gravity.
The present results can be generalized to include electromagnetism (or any number of additional $U(1)$ fields) in the string background equations. It is well known that the reduced electrovacuum Einstein equations exhibit a hidden $SL(3, R)$ symmetry [12, 15] generalizing the Geroch group of the pure vacuum case. In the presence of $n$ $U(1)$ fields the corresponding group is $SL(2+n, R)$. The appropriate generalization to string theory should be described by the $O(2, 2+n)$ current algebra, but the details have to be worked out. Generalizing the results to the supersymmetric case is also an interesting problem, which could clarify the role of the infinitely many string symmetries. In fact, our results might have a more natural interpretation in terms of hidden symmetries in supergravity theories [21].

We have demonstrated that the non–local symmetries of $\sigma$–models can be used in string theory to enlarge the known $O(2, 2)$ and $S$–duality transformations to an infinite dimensional group. It would be interesting to formulate the action of the full string Geroch group in the $\sigma$–model frame using the antisymmetric tensor field instead of the axion. Non–local symmetries can become local and vice versa in these variables, because $b$ and $B$ are non–locally related to each other. Whether this symmetry survives at higher orders in $\alpha'$ is an important open question. It might be that only $N = 4$ superconformal theories with $\delta c = 0$ to all orders in $\alpha'$ [6] allow for this possibility. Also, the presence of local symmetries in the axion–dilaton formalism of string theory has not been addressed in this paper. The construction of local conservation laws in target space might help in clarifying further the integrability aspects of 4–dim string backgrounds. Furthermore, the world–sheet interpretation of the infinitely many target space symmetries is lacking for the moment and it should be considered separately.

Finally, the group of duality transformations $O(2, 2; Z)$ might have a natural generalization in the context of the string Geroch group. If this is indeed the case, our present understanding of unbroken string symmetries will improve considerably.

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