How viscous is a superfluid neutron star core?

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We discuss the effects of superfluidity on the shear viscosity in a neutron star core. Our study combines existing theoretical results for the viscosity coefficients with data for the various superfluid energy gaps into a consistent description. In particular, we provide a simple model for the electron viscosity which is relevant both when the protons form a normal fluid and when they become superconducting. This model explains in a clear way why proton superconductivity leads to a significant strengthening of the shear viscosity. We present our results in a form which permits the use of data for any given modern equation of state (our final formulas are explicitly dependent on the proton fraction). We discuss a simple description of the relevant superfluid pairing gaps, and construct a number of models (spanning the range of current uncertainty) which are then used to discuss the superfluid suppression of shear viscosity. We conclude by a discussion of a number of challenges that must be met if we are to make further progress in this area of research.

I. INTRODUCTION

Neutron stars are often thought of as exciting cosmic laboratories. This is natural since their description depends on much complex physics [1]. With a mass of about one and a half times that of the Sun compressed inside a radius of ten kilometers or so, they are compact enough to require a fully general relativistic description. With central densities several times the nuclear saturation density it may be energetically favourable for exotic phases of matter, like kaon condensates, hyperons and/or deconfined quarks, to be present in the core. With temperatures much below the Fermi temperature for the various constituents, neutron stars are cold on the nuclear scale which means that the presence of both solid (the outer layers form a kilometer sized nuclear lattice) and superfluid regions is expected. Neutron star dynamics is also intriguing. The obvious example of this is the glitches, which are taken as evidence of at least two weakly coupled interior components, observed in a number of radio pulsars [2]. In addition, the evolution of the sample of the neutron star population which is spinning rapidly may be affected by both hydrodynamical and/or radiation-driven instabilities [3]. This is particularly interesting from the point of view of gravitational-wave observations. With a generation of large-scale gravitational-wave interferometers now reaching design sensitivity, it is worth emphasising that a signal from an (unstable?) neutron star pulsation mode could provide a unique, and perhaps the only, probe of the bulk motion of matter inside such stars. The information that could be gleaned from such data would be truly unprecedented [4, 5].

As a useful illustration of a problem which plays a key role in neutron star dynamics, let us consider the coupling between the elastic crust of nuclei and the fluid core. The relevant coupling timescale impacts on, for example, the damping rate of oscillations driven unstable by gravitational-radiation emission (eg. the r-modes [3]), the relaxation following a pulsar glitch [6, 7, 8], and possible neutron star free precession [9, 10, 11]. In its simplest form the problem is analogous to the classic “spin-up problem” in fluid dynamics, where one studies the rotation rate of a viscous fluid following a change in rotation of the container. It is well-known that the fluid velocity changes due to the formation of a so-called Ekman layer at the interface. The role of this viscous boundary layer is to ensure that the fluid motion satisfies a no-slip condition at the interface. Its presence results in a coupling to the bulk of the fluid on a timescale

\[ t_E \sim t_\nu \left( \frac{\delta_E}{R} \right) \quad \text{with} \quad \delta_E = \left( \frac{\eta \rho \Omega}{\rho \Omega} \right)^{1/2} \]  

(1)

where \( \delta_E \) is the width of the Ekman layer [65], \( R \) is the radius of the container, \( \Omega \) is the rotation rate, \( \rho \) is the mass density at the interface and \( \eta \) is the shear viscosity coefficient. For a neutron star, where \( \delta_E/R \sim 10^{-6} \) it is clear that the resulting timescale is much shorter than that of viscous diffusion, i.e.

\[ t_\nu \sim \frac{R^2 \rho}{\eta} \]  

(2)

Analogously, in the case of an oscillation in the fluid the Ekman layer provides an efficient damping mechanism, see [8] for a recent review.

However, it is a serious oversimplification to model the core-crust interface as a viscous fluid/solid wall transition. In particular, the fact that the crust is expected to be permeated by superfluid neutrons should also be considered.
Similarly, the multi-fluid nature of the expected superfluid neutron/superconducting proton mixture in the outer core must be accounted for. This involves many difficult issues associated with the presence of rotational vortices in the neutron superfluid, and magnetic fluxtubes in the proton fluid (if it forms a type-II superconductor). The interactions involving the vortices, leading to the non-dissipative entrainment effect \cite{12} as well as the mutual friction \cite{13}, and the potential pinning of vortices to the nuclear lattice \cite{2,3} are not yet understood in detail. It is clear that one faces a problem which stretches our understanding of neutron star physics considerably. In a series of papers, we aim to make progress by considering each of the relevant issues in turn.

The purpose of the present paper is to discuss the shear viscosity in a superfluid neutron star. In doing this, we are taking a small step towards a fully consistent description of dissipative neutron star cores. As a first step in this development we revisit the standard shear viscosity (ignoring for the moment the multi-fluid dynamics aspects), and ask how well understood the relevant viscosity coefficients are. This leads to a set of useful formulas that can readily be used in future work. In addition, our discussion identifies a number of challenging research problems that require further attention.

II. REVISING THE SHEAR VISCOSITY

We begin by making the following observation: In their by now classic paper on the effect of viscosity on damping of oscillations in superfluid neutron stars, Cutler and Lindblom \cite{14} state: “It is interesting to note that contrary to our experience with other superfluids like He$_4$, neutron star matter becomes more viscous in the superfluid state than it was in the normal state.” This, at first sight, counter-intuitive result provides ample motivation for our investigation.

Estimates of the effect that shear viscosity has on the fluid motion in neutron stars have so far almost exclusively been based on the work of Flowers and Itoh \cite{15,16}, who calculated the relevant transport coefficients at supranuclear densities. Based on the results of these papers, Cutler and Lindblom \cite{14} discussed the viscous damping of oscillations in the neutron star fluid. Their analysis was based on, first of all, a fit to the result in \cite{16} for the total shear viscosity densities. Based on the results of these papers, Cutler and Lindblom \cite{14} discussed the viscous damping of oscillations in the neutron star fluid. Their analysis was based on, first of all, a fit to the result in \cite{16} for the total shear viscosity in normal fluid neutron stars, which can be written

\[ \eta_n \approx 2 \times 10^{20} \rho_{15}^{9/4} T_S^{-2} \text{ g/cm s} \]  

where $\rho_{15} = \rho/10^{15}$ g/cm$^3$ and $T_S = T/10^8$ K. As stated by Flowers and Itoh (and as is also evident from Figure 2 in \cite{16}), the dominant contribution to $\eta_n$ is due to neutron-neutron scattering. This result makes sense since the neutrons make up the bulk of the fluid at the relevant densities (and more exotic massive particles like hyperons are not being considered). However, the neutrons in the outer core are expected to become superfluid as soon as the star cools below (say) $10^9$ K, i.e. soon after its birth in a supernova core collapse. Below the neutron superfluid transition temperature the neutron-neutron scattering is suppressed and the dominant contribution to the shear viscosity is made by scattering processes involving the relativistic electrons. Cutler and Lindblom \cite{14} argue that the relevant estimate (for electron-electron scattering) in a superfluid neutron star is

\[ \eta_{ee}^{CL} \approx 6 \times 10^{20} \rho_{15}^{2} T_S^{-2} \text{ g/cm s} \]  

This estimate for $\eta_{ee}$ has since been used in a number of contexts. The puzzling fact that $\eta_{ee}^{CL} > \eta_n$ throughout a typical neutron star does not seem to have caused much concern. In addition to possible confusion about this result, the estimate \cite{14} is not quite satisfactory. One reason is that it is associated with a particular supranuclear equation of state (derived by Baym, Bethe and Pethick in 1971 \cite{17}). In practise, one would like to have an estimate which explicitly includes the dependency on the electron (=proton) fraction. This is important since one may then be able to better understand the effect of varying the supranuclear equation of state (and the proton fraction) within the current range of uncertainty.

To derive the relevant estimate, we take as our starting point the general formula (cf. \cite{17,18,19})

\[ \eta_e = \frac{n_e p_e}{5m_e^3} \tau \]  

where $n_e$ is the electron number density, $p_e = \hbar k_F = \hbar (3\pi^2 n_e)^{1/3}$ is the corresponding Fermi momentum, $m_e^* = p_e/c$ is the effective electron mass and $\tau$ is the timescale for momentum transfer due to the various scattering processes in which the electrons are involved. In a normal fluid neutron star core, scattering off of the (nonrelativistic) protons provides the main channel for momentum exchange. However, if the protons become superconducting then electron-electron scattering takes over as the dominant contribution to $\tau$ \cite{3,16}.
In order to model the resultant shear viscosity, we use the timescale for electron-proton scattering derived by Easson and Pethick [18]:

\[
\tau_{ep} = \frac{4}{\pi^2\alpha^2} \left( \frac{\epsilon_{Fp}}{k_B T} \right)^2 \frac{k_{Ft}}{ck_{Fp}^2}
\]

where \(\alpha = 1/137\) is the fine-structure constant, \(\epsilon_{Fp}\) is the proton Fermi energy and \(k_{Ft}\) is the Thomas-Fermi screening wave-vector (to be discussed later). Note that we are assuming that the matter is charge neutral, i.e. we take \(n_e = n_p\).

We can extract an analogous formula for electron-electron scattering from Eq. (91) in [15]. This equation can be written

\[
\frac{1}{\eta_{ee}} = \frac{15 \pi^4 \alpha^2}{2 \hbar k_{Fp}^2 \frac{p_e c}{m_e c}} \left( \frac{2k_{Fp}}{k_{Ft}} \right)^2 \left[ \frac{5}{2} + 3 \left( \frac{m_e c}{p_e} \right)^2 \left( \frac{m_e c}{p_e} \right)^4 \right]
\]

Noting that

\[
\frac{m_e c}{p_e} \approx 8 \times 10^{-4} \left( \frac{n_e}{1 \text{ fm}^{-3}} \right)^{-1/3} \approx 4.5 \times 10^{-3} \left( \frac{x_p}{10^{-2}} \right)^{-1/3} \rho_{15}^{-1/3}
\]

where \(x_p\) is the proton (electron) fraction, we see that we only need to retain the leading order contribution (in \(m_e c/p_e\)). By comparing (5) and (7) we then find that

\[
\tau_{ee} = \frac{4}{10\pi^2\alpha^2} \left( \frac{\epsilon_{Fe}}{k_B T} \right)^2 \frac{k_{Ft}}{ck_{Fp}^2}
\]

where \(\epsilon_{Fe} = \hbar c k_{Fp}^2\) for relativistic electrons.

In order to combine the two results for \(\tau_{ep}\) and \(\tau_{ee}\), we need to note that individual scattering processes add like “parallel resistors”, cf. [15], which means that we have

\[
\tau = \left[ \frac{1}{\tau_{ee}} + \frac{1}{\tau_{ep}} \right]^{-1}
\]

This shows that the most important contribution to the shear viscosity comes from the most frequent scattering process (as one would intuitively expect). Using

\[
\frac{\tau_{ee}}{\tau_{ep}} = \frac{1}{10} \frac{\epsilon_{Fe}}{\epsilon_{Fp}} = \frac{1}{10} \frac{2m_e^* c}{\hbar k_{Fp}^2}
\]

where \(\epsilon_{Fp} = \hbar^2 k_{Fp}^2/2m_e^*\) and \(m_e^*\) is the effective proton mass, we find that \(\tau_{ee} \gg \tau_{ep}\) under normal circumstances. Thus the electron-proton scattering provides the dominant contribution to the electron shear viscosity, in accordance with the discussion of Flowers and Itoh [16].

We now want to account for the likely possibility that the protons are superconducting. In that case the electron-proton scattering will be suppressed, essentially because there will be fewer states available for the protons to scatter into. In order to allow for the transition to proton superconductivity, we introduce a suppression factor \(R_p\) such that

\[
\tau_{ep} \rightarrow \tau_{ep} \frac{R_p}{R_p}
\]

Far below the critical transition temperature at which the protons become superconducting we should have \(R_p \rightarrow 0\), and we see from (10) that the electron-electron scattering then dominates the shear viscosity [6].

Combining the above results we obtain the following final formula

\[
\tau = \frac{4}{10\pi^2\alpha^2} \left( \frac{\epsilon_{Fe}}{k_B T} \right)^2 \frac{k_{Ft}}{ck_{Fp}^2} \left[ 1 + \frac{R_p}{10} \left( \frac{\epsilon_{Fe}}{\epsilon_{Fp}} \right)^2 \right]^{-1}
\]

which should be used together with (5). It should be noted that we have neglected the contribution from electron-neutron scattering. This should always be a valid approximation.
Let us now turn to the screening factor $k_{ft}$. Electron scattering is screened by both electrons and protons, and the relevant screening wave vector is given by

$$k_{ft}^2 = \frac{4\alpha}{\pi} k_F^2 \left( 1 + \frac{m_p^* c}{\hbar k_F} \right)$$ (14)

The first term in the bracket is due to the electron screening, while the second is due to the protons. It is straightforward to show that the latter tends to dominate under the conditions that prevail in a neutron star core.

Having introduced the key ingredients, we can discuss two limiting cases. In the case of normal protons we take $R_p = 1$ in (13). This leads to

$$\eta_{ep} \approx 1.8 \times 10^{18} \left( \frac{x_p}{0.01} \right)^{13/6} \rho_{15}^{13/6} T_8^{-2} \text{ g/cm s}$$ (15)

Here, and in the following, we have taken (for simplicity) $m_p^* = m_p$ in evaluating $k_{ft}$. Our estimate (15) agrees with the result derived by Easson and Pethick. In the opposite limit the protons are (strongly) superconducting, which means that $R_p = 0$, and we arrive at the estimate

$$\eta_{ee} \approx 4.4 \times 10^{19} \left( \frac{x_p}{0.01} \right)^{3/2} \rho_{15}^{3/2} T_8^{-2} \text{ g/cm s}$$ (16)

This result should be compared to the formula used by Cutler and Lindblom in deriving (14). One can easily show that the two results are compatible. This means that the key reason for the emergence of the dominant electron shear viscosity is the superconductivity of the protons.

The results illustrated in Figure 1 (and others discussed throughout the paper) were obtained using one of the phenomenological PAL equations of state (with compression modulus $K = 240$ MeV) [22]. In order to obtain a suitably simple model we have followed Kaminker et al [23] who provide the following fit for the two nucleon number densities:

$$n_x = a \rho_{14}^b / (1 + c \rho_{14} + d \rho_{14}^2)$$ (17)

where

$$a = 0.1675, \quad b = 1.8185, \quad c = 2.0288, \quad d = 0.02444 \quad \text{for } x = n$$

and

$$a = 0.0006823, \quad b = 2.6727, \quad c = 0.1946, \quad d = 0.01604 \quad \text{for } x = p$$

The total number density $n_b$ and the proton fraction $x_p$ follow immediately from $n_b = n_n + n_p$ and $x_p = n_p/n_b$, respectively.

In one of the few alternative calculations of the neutron shear viscosity (that we are aware of), Mornas [24] derives the following expression [cf. her Eq. (4.9b)]

$$\eta_n \approx 2.4 \times 10^{19} \rho_{15}^{3/5} T_8^{-2} \text{ g/cm s}$$ (18)

It is notable that the scaling with the density is very different from that in Eq. (13), possibly because of the density dependence of the effective neutron mass. As can be seen from Figure 1, Eq. (18) leads to a considerably weaker viscosity at high density. It should also be noted that Mornas’s formula leads to results that are very close to the electron-electron scattering results for the crust region determined by Flowers and Itoh [17]. We have no rational explanation for why this should be the case (in fact, it must be a coincidence), but it is a potentially useful observation.

### III. MODELLING THE EFFECTS OF SUPERFLUIDITY

The formulas given in the previous section provide a useful step towards modelling the dynamics of realistic neutron stars since they allow us to determine a consistent electron viscosity for various supranuclear equations of state and core temperatures. Of course, in order to use these formulas we need not only the total number density and the proton
fraction, we also need to i) determine whether the protons and/or neutrons are superfluid, and ii) if so, quantify the associated suppression factors. This forces us to venture into the thorny area of nucleon superfluidity (see [25] for a recent review).

In this section, we summarise the current understanding of the relevant superfluid energy gaps and discuss the suppression factors we require to complete our model. From a survey of the vast superfluid gap-literature [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], it is clear that the determination of the relevant parameters constitute a serious challenge. In many ways this is not too surprising. After all, significant uncertainties concerning the supranuclear equation of state remain. Given the fact that the superfluid energy gaps are sensitive to the internal constitution of the star, we have to accept a range of possibilities at the present time.

It is natural to adopt a strategy similar to that used in the (closely related) discussion of neutron star cooling, see for example the work by Kaminker and colleagues [23, 41, 42, 43] and the recent discussion of a “minimal cooling paradigm” by Page et al [44]. That is, to consider the plausible range of superfluid parameters suggested by the theoretical studies and ask whether observations provide useful constraints on the models. Since our focus is on the shear viscosity in a neutron star core, the relevant observations concern pulsar glitches and potential future data for core fluid oscillations. In particular, it is known that gravitational waves from internal dynamics could provide an excellent probe of the core physics [5].

In principle, superfluidity in the singlet state appears if the formation of Cooper pairs from particles with opposite momenta and spins leads to a lowering of the ground state energy. A key parameter in any discussion of nucleon superfluidity is the energy gap $\Delta(k)$, which corresponds to the energy needed to create a quasiparticle of momentum $k$ in the superfluid [32]. The gap on the Fermi surface $\Delta(k_F)$ can be interpreted as half the energy required to break the Cooper pairs. In the following we will focus on estimates of this quantity since it allows us to approximate both the critical temperature at which the matter becomes superfluid and the reduction factors required in our analysis of shear viscosity.
The strong interaction provides several channels in which pairing is possible. The indications are that Cooper pairs with zero angular momentum in a spin-singlet state $^1S_0$ should form at low momenta. However, since the effective interaction is expected to become repulsive near the nuclear saturation density, the corresponding energy gap will disappear as the density increases $^{34}$. As a result, because of the comparatively low proton number density one would expect proton $^1S_0$ pairing at densities too high for neutron pairing in that phase $^{27, 28}$. All existing models suggest that the dominant proton pairing takes place well inside the core-crust transition. At the same time, the energetic advantage of neutron pairing in the anisotropic $^3P_2$ state at higher densities may be attributed to the tensor force $^{29}$.

Given the practical difficulties associated with numerical many-body calculations, most studies have been based on the “standard” BCS theory, including more and more refined pairing interactions by adding successive corrections to the bare nucleon-nucleon potential $^{31}$. The next order beyond the bare interaction in this scheme corresponds to various medium effects. These are often accounted for by renormalising the single-particle energies, e.g. by lowering the effective nucleon mass. It is by now well-known that this leads to a smaller energy gap. The key factor determining the effective mass is the number of interaction partners of opposite type that a given nucleon has, i.e. the number of neutrons per proton and vice versa. The greater the number, the greater will be the effect of the medium in reducing the effective mass. Hence, the effective proton mass tends to be smaller than that for neutrons, and one may expect the maximum gap for the $^1S_0$ proton superfluid to be smaller than that for neutrons. Similarly, a major difference between results for pure neutron matter and nuclear matter is due to the induced lowering of the effective neutron mass by the relatively small proton contaminant $^{24}$.

While the results for the bare interaction have converged towards a predicted maximum $^1S_0$ neutron gap of about 3 MeV at $k_F \approx 0.85 \text{ fm}^{-1}$ $^{34, 35, 37, 39}$ (model $A$ in Table I), the higher order calculations still lead to a range of possibilities. Particularly important is the fact that pairing polarises the medium $^{26}$. After accounting for the polarisation effects one finds that the maximum gap has been quenched to nearly 1 MeV at $k_F \approx 3 \text{ MeV at } 0.7 - 0.8 \text{ fm}^{-1}$ $^{32, 34, 37}$ (models $a-d$ in Table I). A nice explanation for this effect is provided by Lombardo and Schulze $^{23}$ who show that low density polarisation effects suppress the BCS gap by roughly a factor $(4e)^{-1/3} \approx 0.45$.

The $^1S_0$ proton and $^3P_2$ neutron gaps provide further challenges. In the former case one must take into account the background neutron influence (e.g. the polarisation terms), while in the latter one must solve the anisotropic gap equations (in principle ten coupled equations) in a consistent way. In the case of the proton gap the result is particularly sensitive to the so-called “symmetry energy”. This is expected since this parameter governs the proton abundance, cf. $^{22, 31}$. The available results suggest a maximum gap for proton $^1S_0$ pairing of approximately 1 MeV at $k_F \approx 0.4 - 0.5 \text{ fm}^{-1}$ (models $e$ and $g$ in Table I). This is roughly a factor of 3 smaller than the maximum neutron gap. The difference can be understood from the fact that the effective proton mass is smaller than the effective neutron mass. (In a pure proton medium the proton gap should, because of the charge symmetry of the nuclear interaction, be identical to the neutron gap in a pure neutron fluid $^{24}$.) One would expect polarisation effects to further influence the proton gap, leading to a suppression (at least?) similar to that found for the $^1S_0$ neutron channel (a factor of a few) $^{32}$. Thus the maximum proton gap may be reduced to perhaps 0.2-0.3 MeV (a level similar to model $f$ in Table I).

In order to study the anisotropic $^3P_2$ neutron gap in detail one must extend BCS theory, and replace the single gap equation for the $^1S_0$ case by ten coupled equations. In fact, the added attraction from the tensor coupling is essential for the existence of superfluidity in this state $^{35}$. As in the case of the isotropic gaps, one finds that the $^3P_2$ gap is reduced significantly (by a factor of two or so) by the lower neutron effective mass in nuclear matter. The available results for the $^3P_2$ gap illustrate the extent to which this problem remains to be understood: The calculation of Elgarøy et al. $^{35}$ (models $k$ and $l$ in Table I) suggests a much smaller gap than that predicted in the work by Baldo and colleagues $^{40}$ (models $h-j$ in Table I). The difficulties associated with superfluid pairing in the $^3P_2$ channel are exacerbates by the fact that relativistic effects come into play at the relevant densities. While the $^1S_0$ results remain largely unaffected by the inclusion of relativistic effects (model $f$ in Table I), the associated change in the single-particle energies reduces the $^3P_2$ gap by about a factor of two $^{38}$ (model $m$ in Table I). Finally, in contrast to the case for the $^1S_0$ gaps, it has been suggested that polarisation effects (which have yet to be accounted for) may increase the $^3P_2$ neutron gap.

In order to consider various gap models in our analysis, without complicating things excessively, we take a lead from Kaminker and colleagues $^{24, 41, 42, 43}$ and represent the energy gap (at the Fermi surface) by the phenomenological formula

$$\Delta(k_F) = \Delta_0 \frac{(k_F - k_1)^2}{(k_F - k_1)^2 + k_2 (k_F - k_3)^2 + k_4} \left(\frac{k_F}{k_0}\right)^{(3/2)}$$  \hspace{1cm} (19)$$

where $k_F$ is the Fermi momentum of the relevant nucleon. This expression makes sense because we know from the weak-coupling formula of BCS theory that $\Delta \sim k_F^3$ at low densities (leaving out an exponential factor which further suppresses the gap as $k_F \to 0$). The results in the literature also indicate that the bare interaction gap function is
TABLE I: Detailed parameters for our various gap models constructed from (19). The models are based on calculations in the given references and represent the current range of possibilities. Model A is for the bare interaction and is relevant in a pure neutron (proton) medium. Models a – d are for the $^1S_0$ neutron pairing, while models e – g correspond to the $^1S_0$ proton results and models h – m are for the $^3P_2$ neutron channel.

| model | $\Delta_0$ (MeV) | $k_1$ (fm$^{-1}$) | $k_2$ (fm$^{-1}$) | $k_3$ (fm$^{-1}$) | $k_4$ (fm$^{-1}$) | Reference/Comments |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|------------------|
| A     | 9.3             | 0.02            | 0.6             | 1.55            | 0.32            | [34, 36, 37, 39] (bare interaction) |
| a     | 68              | 0.1             | 4               | 1.7             | 4               | [33]             |
| b     | 4               | 0.4             | 1.5             | 1.65            | 0.05            | [37]             |
| c     | 22              | 0.3             | 0.09            | 1.05            | 4               | [29] (Reid potential w. m$_*$) |
| d     | 2.9             | 0.3             | 0.017           | 1.3             | 0.07            | [29] (Reid potential w. polarisation) |
| e     | 61              | 0               | 6               | 1.1             | 0.6             | [38] (relativistic) |
| f     | 55              | 0.15            | 4               | 1.27            | 4               | [27] (OPEG potential w. m$_*$ = 0.6) |
| g     | 2.27            | 0.1             | 0.07            | 1.05            | 0.25            | [31]             |
| h     | 4.8             | 1.07            | 1.8             | 3.2             | 2               | [40] (Bonn B potential, free spectrum) |
| i     | 10.2            | 1.09            | 3               | 3.45            | 2.5             | [40] (Argonne $V_{14}$ potential, free spectrum) |
| j     | 2.2             | 1.05            | 1               | 2.82            | 0.6             | [40] (Paris potential, free spectrum) |
| k     | 0.425           | 1.1             | 0.5             | 2.7             | 0.5             | [35] (non-relativistic) |
| l     | 0.068           | 1.28            | 0.1             | 2.37            | 0.02            | [35] (non-relativistic) |
| m     | 2.9             | 1.21            | 0.5             | 1.62            | 0.5             | [38] (relativistic) |

FIG. 2: The various energy gaps for the $^1S_0$ pairing, corresponding to the data in Table I, are shown as functions of $k_F$. The left panel corresponds to the neutrons, while the right panel shows the proton results.

roughly symmetric [34, 36, 37, 39]. This suggests that $\Delta \sim (k_F - k_3)^2$, where the gap vanishes for $k_F > k_3$, for higher densities. The additional parameters $k_2$ and $k_4$ permit us to adapt the shape of $\Delta$ according to various results which incorporate, for example, polarisation effects.

In Table I we provide data for a set of gap models based on the literature [67]. The models have been selected to span the current range of uncertainty. The given parameters lead to good representations of the original results, although it should be noted that we have not worried too much about the numerical precision of each individual gap. The pairing gaps obtained from the data in Table I are shown in Figures 2 and 3.

Having surveyed the current thinking about the nucleon superfluid energy gaps, and modelled the results in a useful way, we want to use the results to derive, first of all, the associated transition temperatures and then turn our attention to the suppression of shear viscosity. As discussed in detail by Yakovlev et al [46] (see also [30, 31] for useful approximations), the critical temperatures can be approximated by

$$\frac{k_B T_c}{\Delta(k_F)} \approx \begin{cases} 
0.5669 & \text{for } ^1S_0 \\
0.8416 & \text{for } ^3P_2 (m_J = 0) \\
0.4926 & \text{for } ^3P_2 (|m_J| = 2) 
\end{cases}$$

(20)
FIG. 3: The various energy gaps for the $^3P_2$ neutron pairing, corresponding to the data in Table I, are shown as functions of $k_F$. The two panels correspond to strong and weak neutron superfluidity in the core, respectively.

FIG. 4: Critical temperatures associated with the various gap models listed in Table I. For clarity we only show the $^3P_2$ neutron results for the $m_J = 0$ pairing. The results for the $|m_J| = 2$ case would be suppressed by a factor of roughly 0.6, cf. (20). As in Figure 1, the grey region represents the neutron star crust, in which there are no free protons and hence no corresponding $T_c$. The inset in the right panel helps identify the two sets of gap-models we combine to describe "strong" and "weak" superfluidity.

Here the effect of the anisotropy of the $^3P_2$ channel, for which $m_J$ represents the projection of the angular momentum of the Cooper pair on the quantisation axis, is apparent. (Note that, even though the $|m_J| = 1$ case has not been studied in detail it is expected to be qualitatively similar to the $m_J = 0$ case.)

If we combine these approximations with our various gap models (and the data for the PAL equation of state from (17)) we obtain the results shown in Figure 4. The two panels distinguish between cases that can be referred to as representing "strong" and "weak" superfluidity. We also distinguish the crust and core regions. To make this distinction we have assumed that the transition takes place at 0.6 of the nuclear saturation density $\rho_0 \approx 2.8 \times 10^{14}$ g/cm$^3$, i.e. at $\rho_c \approx 1.7 \times 10^{14}$ g/cm$^3$. It should be noted that there are no free protons in the crust, and hence no proton gap in that region (we are not accounting for the possibility that nuclei may exhibit pairing). It is also worth commenting on the fact that the models for the $^1S_0$ neutron gap in the crust do not account for interactions with the crust nuclei in detail. There are many difficulties associated with this analysis which need to be addressed in future research. Anyway, from Figure 4 we see that superfluid neutrons are expected to be present in the crust for
temperatures above a few times $10^9$ K. The same is true for the proton superfluid in the core. This means that, all but newly born (less that a month old?) neutron stars should contain both superfluid neutrons and superconducting protons in the $^1S_0$ phase. However, it is not clear whether one should expect the corresponding regions in the star to overlap. Figure 4 certainly gives examples of $^1S_0$ neutron superfluids which are entirely confined to the crust.

At this point it is relevant to mention possible constraints on the neutron star superfluidity provided by glitch data. By assuming that the glitches are associated with the neutron superfluid in the inner crust, one can put constraints on the corresponding moment of inertia required to explain the observations. This leads to a requirement that at least $1.5 - 2.5\%$ of the star’s moment of inertia is in the superfluid, see for example \cite{8,47}. Comparing to the models given in Table I, this constraint could mean that model $d$ (and possibly also model $e$) is ruled out. Of course, this conclusion has many caveats and one should avoid making definitive statements given the lack of truly quantitative glitch models.

As we have already discussed, the results for the $^3P_2$ neutron gap are associated with a great deal of uncertainty. From Figure 4 we see that the critical temperature may, in fact, be as low as $10^8$ K. This extreme case would mean that the core of an accreting neutron star in a low-mass X-ray binary, for which nuclear burning in the crust is expected to equilibrate the temperature around a few times $10^8$ K, may not contain superfluid neutrons at all. It should, of course, be emphasised that most of the models have significantly higher critical temperatures for the $^3P_2$ superfluidity. It would certainly be surprising, given the present results, if a neutron star with core temperature at the level of $10^8$ K did not contain a region where superconducting protons coexist with superfluid neutrons in the $^3P_2$ state.

Next, introducing the variable $\tau = T/T_c$, we learn from Ref. \cite{47} that the temperature dependency of the gap functions can be approximated by

$$y = \frac{\Delta(T)}{k_BT} \approx \begin{cases} \sqrt{1 - \tau} (1.456 - 0.157/\sqrt{\tau} + 1.764/\tau) & \text{for } ^1S_0 \\ \sqrt{1 - \tau} (0.7893 + 1.188/\tau) & \text{for } ^3P_2 (m_J = 0) \\ \sqrt{1 - \tau^4} (2.030 - 0.4903\tau + 0.1727\tau^4) / \tau & \text{for } ^3P_2 (|m_J| = 2) \end{cases}$$

(21)

Combining these results with the critical temperatures from \cite{20}, we can estimate the pairing gaps at finite temperature. This is a key ingredient in constructing the superfluid suppression factor $R_p$, which represents the extent to which the superconducting protons no longer scatter the electrons. To estimate this factor we use eq. (45) from \cite{20}, which can be written

$$R_p \approx \left\{ 0.7694 + \sqrt{(0.2306)^2 + (0.07207y)^2} + (27y^2 + 0.1476y^4) \exp \left[ -\sqrt{(4.273)^2 + y^2} \right] \right\} \exp \left( 1.187 - \sqrt{(1.187)^2 + y^2} \right)$$

(22)

The form of this fit to the numerical results was chosen to lead to the limits, $R_p \to 1$ as $y \to 0$ and $R_p \sim 0.2362y \exp(-y)$ as $y \to \infty$. For temperatures above $T_c$ one should simply use $R_p = 1$. It is perhaps worth pointing out that the exact form of this suppression factor is only relevant very close to the transition temperature, and that an accurate model may only be important for a small subset of neutron star models.

We need an analogous suppression factor for neutron superfluidity. In this case we have carried out a fit to the expressions for $R_{n1}$ and $R_{n2}$ given by eqn. (45) in \cite{45}. Thus we find

$$R_n \approx \left[ 0.9543 - \sqrt{(0.04569)^2 + (0.6971y)^2} \right]^3 \exp \left( 0.1148 - \sqrt{(0.1148)^2 + 4y^2} \right)$$

(23)

obviously with $y$ evaluated for the relevant pairing gap according to \cite{20}. Given this suppression factor, we use

$$\eta_n \rightarrow R_n\eta_n$$

(24)

for the shear viscosity due to neutron-neutron scattering.

If we now combine all the results we have discussed, we can calculate the electron shear viscosity coefficient as a function of the density for various neutron star temperatures. In Figures 5 and 6 we show the result for strong and weak superfluidity, respectively. The former model combines gaps $a$, $e$ and $h$ from Table I while the latter follows if we use gaps $d$, $f$ and $l$. The two Figures show the shear viscosity as a function of the density for four different temperatures $T = 10^8$ K, $5 \times 10^8$ K, $10^9$ K and $5 \times 10^9$ K, also indicated by horizontal dotted lines in Figure 4. The illustrated data shows (very nicely) how the neutron viscosity becomes strongly suppressed at lower temperatures. One can also clearly distinguish, eg. from the transition that takes place at a density of about $7 \times 10^{14}$ g/cm$^3$ in the last three panels of Figure 5 the region where the protons are superconducting and the (stronger) electron-electron viscosity dominates. Taken together, the two figures provide a useful insight into the complexity of the neutron star interior and how the viscosity changes as the star matures.
FIG. 5: Viscosity coefficients for the “strong” superfluidity case discussed in the text. The dashed line represents the neutron-neutron scattering viscosity, while the solid line shows the electron viscosity. The grey region shows the extent of the neutron star crust. Note that, in the final panel the neutron viscosity is suppressed to levels below those shown.

FIG. 6: Viscosity coefficients for the “weak” superfluidity case discussed in the text. The dashed line represents the neutron-neutron scattering viscosity, while the solid line shows the electron viscosity. The grey region shows the extent of the neutron star crust.

These figures represent the final results of this paper. Having reached the stage where we can generate this data, we are now able to turn our attention to the implications for neutron star dynamics. In particular, we should be able to study the impact of nucleon superfluidity on neutron star oscillations in more detail than has been possible so far. We can also return to well-known problems like the “spin-up” problem and re-assess the viscosity due to the core-crust interaction and the associated Ekman layer.
**IV. FUTURE CHALLENGES**

Hoping to stimulate further work on the many relevant issues, we have discussed the effects of superfluidity on the shear viscosity in a neutron star core. As should be clear from our analysis, an understanding of this problem requires input from several different areas of research, most notably nuclear physics and relativistic astrophysics. Although the results we have collated may not be original, we are not aware of any previous study which combines the data into a consistent description of the shear viscosity in a superfluid neutron/superconducting proton mixture. This is in sharp contrast to the many discussions of the role of superfluidity in neutron star cooling, see for example [23, 41, 12, 14, 14]. We believe that this paper provides a useful update on the core viscosities which should be valuable for future modelling of neutron star oscillations and associated instabilities [8].

We have provided a simple model for the electron viscosity which is relevant both when the protons form a normal fluid and when they become superconducting. This result explains (as in [8]) why it is natural that proton superconductivity leads to a significant strengthening of the shear viscosity. It also clarifies some confusion associated with the description of Cutler and Lindblom [14]. In particular, it is clear that the superfluidity of the neutrons is not the key factor which leads to electron-electron scattering becoming the main shear viscosity agent. Rather, it is the fact that the onset of superconductivity suppresses the electron-proton scattering. This is an important point. As can be seen from Figure 1, the electron-electron shear viscosity is not too different from the result for neutrons scattering off of each other. This means that, in the temperature range where shear viscosity dominates, the damping of neutron star oscillations will be quite similar (modulo multi-fluid effects) in the extreme cases when i) the neutrons and protons are both normal, and ii) when the neutrons and protons are superfluid/superconducting, respectively. The contrast with the case when the neutrons are superfluid and the protons normal (and viscosity is dominated by \( \eta_{ep} \)) is clear from Figure 1. This is an interesting observation because it shows that proton superconductivity (or rather absence thereof) could have a significant effect on the dynamics of a neutron star core.

Our results are in a form which permits the use of data for any given modern equation of state. In contrast to the often used expression obtained by Cutler and Lindblom [14], our final formulas are explicitly dependent on the proton fraction (as well as the total mass density). This is a crucial difference because of the simple fact that the result in [14] was obtained for the now seriously outdated equation of state derived by Baym, Bethe and Pethick [17] in 1971. We can obtain an indication of the likely “error bars” associated with the electron shear viscosity from [10] which suggests that \( \eta_e \sim x_p^{3/2} \). As the proton fraction varies by up to perhaps a factor of two for different equations of state we can expect the viscosity coefficient to be uncertain at the level of (at least) a factor of three.

Our discussion also highlights a number of challenges that must be met if we are to make further progress. First of all, it is clear that an understanding of the various superfluid gaps is crucial. We believe that the results summarised in Table 1, i.e. the “phenomenological” models we have constructed from a range of results in the relevant literature (in line with the philosophy adopted in [23, 41, 42, 43]), provide a useful survey of the theoretical range of possibilities. These models are also readily included in quantitative studies like that discussed in this paper. Of course, many uncertainties remain concerning the superfluid parameters. Most important are the suppression factors for the various scattering rates which are needed to evaluate the viscosity coefficients (\( R_p \) and \( R_n \) in our analysis). Any detailed discussion of these coefficients is certainly valuable. Especially since it allows us to make the analysis of temperatures near the critical temperature \( T_c \) quantitative rather than qualitative (as in the present discussion).

In addition to these problems, there are many related challenges. Although it is commonly acknowledged that superfluid components play a crucial role in neutron star dynamics (e.g. in pulsar glitches [2, 23, 48]), our understanding of the multi-fluid aspects and the relevant dissipation mechanisms can be improved considerably. That this area provides exciting possibilities is nicely illustrated by the demonstration that a two-stream instability may operate from Figure 1, the electron-electron shear viscosity is not too different from the result for neutrons scattering off of each other. This means that, in the temperature range where shear viscosity dominates, the damping of neutron star oscillations will be quite similar (modulo multi-fluid effects) in the extreme cases when i) the neutrons and protons are both normal, and ii) when the neutrons and protons are superfluid/superconducting, respectively. The contrast with the case when the neutrons are superfluid and the protons normal (and viscosity is dominated by \( \eta_{ep} \)) is clear from Figure 1. This is an interesting observation because it shows that proton superconductivity (or rather absence thereof) could have a significant effect on the dynamics of a neutron star core.

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variety of “shapes”, ranging from droplets to rods to plates, as the core-crust phase-transition is approached \(^5\). An effective description of these exotic phases may differ significantly from the standard picture. In addition to this, one should account for the presence of superfluid neutrons throughout the bulk of the crust. The relevant parameters for this superfluid phase must be considered as poorly known since the various gap-calculations do not (yet) account for the presence of the nuclei. This constitutes a difficult theory problem, but the result could be crucial for a number of scenarios ranging from the pulsar glitches to gravitational-wave driven mode-instabilities. An indication of how different the parameters of the crust superfluidity may be compared to those in the core fluid is provided by the work of Carter, Chamel and Haensel \(^5\). Their analysis of the entrainment near the base of the crust shows that the results differ greatly from those typically used to model the outer core. The essential difference is in the effective mass of the neutrons. As in the other cases, the dynamical repercussions of this result remain to be investigated.

Finally, let us turn to the physics of the neutron star inner core. It is generally expected that, at densities above a few times nuclear saturation density, it will be energetically favourable for exotic phases of matter to be present. The two most commonly considered cases concern hyperons and deconfined quarks. In the first of these cases it has been pointed out that hyperons provide an efficient refrigerant, since direct URCA reactions will be in operation. These reactions would cool the star much faster than indicated by the observational data. The only way to avoid contradictions with the observations is to appeal to hyperon superfluidity \(^6\), which would quench the reactions and slow down the cooling rate. It is also worth pointing out that the fast nuclear reactions in a hyperon core are thought to lead to rapid damping of oscillations due to the resultant bulk viscosity \(^5\). This effect would be significantly suppressed by superfluidity \(^6\). In the context of the present paper it is relevant to make two comments. First of all, it is known that an increase of the hyperon population tends to lead to a significant depletion of the electron number density. If we suppose that all the other components (neutrons, protons and hyperons) are superfluid, then the electrons may provide the only channel for shear viscosity. If their number density is depressed to say \(n_e \sim 10^{-3}\) then we would estimate that the shear viscosity is something like five orders of magnitude weaker than that illustrated in Figure \(^1\). The dynamical effects of this could be important. Secondly, it is worth pointing out that neither the many possible vortex-vortex and electron-vortex interactions, nor the multi-fluid aspects of superfluid hyperon mixtures have yet been addressed.

The situation is quite similar for the deconfined quark core. The mixture of s, d, and u quarks is thought to be able to exhibit a variety of “colour superconducting” pairings \(^5\). Whether this leads to multi-fluid dynamics is not clear, although it seems likely for at least some of the possible phases. There have not yet been many attempts to estimate the relevant transport coefficients. Yet, it is clear that such studies are of great importance. Especially since they may help unveil the true ground state of matter. In this context, it is worth mentioning an interesting result. Madsen \(^6\) has combined estimated viscosity coefficients for strange stars with the radiation reaction results for the r-mode instability to demonstrate that, if the fastest spinning pulsar were a pure colour-flavour locked quark star then it ought to be unstable (see \(^6\) for an alternative). One interpretation of this result is that this particular form of quark pairing is not present in the bulk of a compact star. There are of course many caveats to this statement, but it is nevertheless relevant. In particular, it demonstrates the need to put current and future models of exotic neutron star physics in an astrophysical context and ask whether observations provide useful constraints on the theory. It is particularly important to consider this at the present time when gravitational-wave observations of compact object dynamics seem a definite possibility.

Taking the situation at face value, neutron star (astro)physics is a vibrant area of research which provides a number of exciting challenges for the future. In order to meet these challenges we need a closer dialogue between researchers in different fields, like nuclear physics and relativistic astrophysics. We hope that this paper will help stimulate such discussion.

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[1] J.M. Lattimer and M. Prakash, Science \(^3\) 536 (2004)
[2] A.G. Lyne, S.L. Shemar and F. Graham Smith, MNRAS \(^3\) 534 (2000)
[3] N. Andersson, Class. Quantum Grav. \(^2\) R105 (2003)
[4] K.D. Kokkotas, T.A. Apostolatos and N. Andersson, MNRAS \(^2\) 307 (2001)
[5] N. Andersson and G.L. Comer, Phys. Rev. Lett. \(^7\) 241101 (2001)
It is worth pointing out that one would not expect an Ekman layer to form on the crust-side of the phase transition. The main reason for this is that the elastic shear stresses will overwhelm those associated with the viscosity. Hence, all quantities in the equation should be evaluated for the core fluid at the base of the crust.

In a draft version of this paper we followed Gnedin and Yakovlev [20] who argued that the protons should not contribute to the screening once they become superconducting. We are grateful to the authors of [20] for pointing out to us that this
result was erroneous, and for helpful discussions concerning the screening in general.

[67] It should be noted that $\Delta(k_F)$ is only well defined in the isotropic case. For triplet state pairing, the gap varies over the Fermi surface and one must be careful to choose a “characteristic” gap.