Acyclic Edge Coloring of Planar Graphs

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Abstract

An acyclic edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The acyclic chromatic index of a graph is the minimum number k such that there is an acyclic edge coloring using k colors and is denoted by \( a'(G) \). It was conjectured by Alon, Sudakov and Zaks (and much earlier by Fiamcik) that \( a'(G) \leq \Delta + 2 \), where \( \Delta = \Delta(G) \) denotes the maximum degree of the graph. We prove that if \( G \) is a planar graph with maximum degree \( \Delta \), then \( a'(G) \leq \Delta + 12 \).

Keywords: Acyclic edge coloring, acyclic edge chromatic number, planar graphs.

1 Introduction

All graphs considered in this paper are finite and simple. A proper edge coloring of \( G = (V, E) \) is a map \( c: E \to C \) (where \( C \) is the set of available colors) with \( c(e) \neq c(f) \) for any adjacent edges \( e, f \). The minimum number of colors needed to properly color the edges of \( G \), is called the chromatic index of \( G \) and is denoted by \( \chi'(G) \). A proper edge coloring \( c \) is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in \( G \). The acyclic edge chromatic number (also called acyclic chromatic index), denoted by \( a'(G) \), is the minimum number of colors required to acyclically edge color \( G \). The concept of acyclic coloring of a graph was introduced by Grünbaum [17]. The acyclic chromatic index and its vertex analogue can be used to bound other parameters like oriented chromatic number and star chromatic number of a graph, both of which have many practical applications, for example, in wavelength routing in optical networks (see [4], [19]). Let \( \Delta = \Delta(G) \) denote the maximum degree of a vertex in graph \( G \). By Vizing’s theorem, we have \( \Delta \leq \chi'(G) \leq \Delta + 1 \) (see [9] for proof). Since any acyclic edge coloring is also proper, we have \( a'(G) \geq \chi'(G) \geq \Delta \).

It has been conjectured by Alon, Sudakov and Zaks [2] (and much earlier by Fiamcik [10]) that \( a'(G) \leq \Delta + 2 \) for any \( G \). Using probabilistic arguments Alon, McDiarmid and Reed [11] proved that \( a'(G) \leq 60\Delta \). The best known result up to now for arbitrary graph, is by Molloy and Reed [20] who showed that \( a'(G) \leq 16\Delta \). Muthu, Narayanan and Subramanian [21] proved that \( a'(G) \leq 4.52\Delta \) for graphs \( G \) of girth at least 220 (Girth is the length of a shortest cycle in a graph).

Though the best known upper bound for general case is far from the conjectured \( \Delta + 2 \), the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov and Zaks [2] proved that there exists a constant \( k \) such that \( a'(G) \leq \Delta + 2 \) for any graph \( G \) whose girth is at least \( k\Delta \log \Delta \). They also proved that \( a'(G) \leq \Delta + 2 \) for almost all \( \Delta \)-regular graphs. This result was improved by Nešetřil and Wormald [23] who showed that for a random \( \Delta \)-regular graph \( a'(G) \leq \Delta + 1 \). Muthu, Narayanan and Subramanian proved the conjecture for grid-like graphs [22]. In fact they gave a better bound of \( \Delta + 1 \) for these class of graphs. From Burnstein’s [8] result it follows that the conjecture is true for subcubic graphs. Skulrattankulchai [25] gave a polynomial time algorithm to color a subcubic graph using \( \Delta + 2 = 5 \) colors. Fiamcik [12], [11] proved that every subcubic graph, except for \( K_4 \) and \( K_{3,3} \), is acyclically edge colorable using 4 colors.

Determining \( a'(G) \) is a hard problem both from theoretical and algorithmic points of view. Even for the simple and highly structured class of complete graphs, the value of \( a'(G) \) is still not determined exactly. It has also been shown by Alon and Zaks [3] that determining whether \( a'(G) \leq 3 \) is NP-complete for an arbitrary graph \( G \). The vertex version of this problem has also been extensively studied (see [17], [8], [7]). A generalization of the acyclic edge chromatic number has

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been studied: The \( r \)-acyclic edge chromatic number \( a'_c(G) \) is the minimum number of colors required to color the edges of the graph \( G \) such that every cycle \( C \) of \( G \) has at least \( \min\{|C|, r\} \) colors (see [15], [16]).

**Our Result:** The acyclic chromatic index of planar graphs has been studied previously. Fiedorowicz, Hauszczak and Narayanan [14] gave an upper bound of \( 2\Delta + 29 \) for planar graphs. Independently Hou, Wu, GuiZhen Liu and Bin Liu [18] gave an upper bound of \( \max(2\Delta - 2, \Delta + 22) \), which is the best known result up to now for planar graphs. Note that for \( \Delta \geq 24 \), it is equal to \( 2\Delta - 2 \). In this paper, we prove the following theorem,

**Theorem 1.** If \( G \) is a planar graph, then \( a'(G) \leq \Delta + 12 \).

The acyclic chromatic index of special classes of planar graphs based on girth and absence of short cycles as also been studied. In [14], an upper bound of \( \Delta + 6 \) for triangle free planar graphs has been proved. In [18] an upper bound of \( \Delta + 2 \) for planar graphs of girth at least five has been proved. Fiedorowicz and Borowiecki [13] proved an upper bound of \( \Delta + 1 \) for planar graphs of girth at least six and an upper bound of \( \Delta + 15 \) for planar graphs without four cycles.

Our proof is constructive and yields an efficient polynomial time algorithm. We have presented the proof in a non-algorithmic way. But it is easy to extract the underlying algorithm from it.

## 2 Preliminaries

Let \( G = (V, E) \) be a simple, finite and connected graph of \( n \) vertices and \( m \) edges. Let \( x \in V \). Then \( N_G(x) \) will denote the neighbours of \( x \) in \( G \). For an edge \( e \in E \), \( G - e \) will denote the graph obtained by deletion of the edge \( e \). For \( x, y \in V \), when \( e = (x, y) = xy \), we may use \( G - \{xy\} \) instead of \( G - e \). Let \( c : E \rightarrow \{1, 2, \ldots, k\} \) be an acyclic edge coloring of \( G \). For an edge \( e \in E \), \( c(e) \) will denote the color given to \( e \) with respect to the coloring \( c \). For \( x, y \in V \), when \( e = (x, y) = xy \) we may use \( c(x, y) \) instead of \( c(e) \). For \( S \subseteq V \), we denote the induced subgraph on \( S \) by \( G[S] \).

Many of the definitions, facts and lemmas that we develop in this section are already present in our earlier papers [6], [5]. We include them here for the sake of completeness. The proofs of the lemmas will be omitted whenever it is available in [6], [5].

**Partial Coloring:** Let \( H \) be a subgraph of \( G \). Then an edge coloring \( c' \) of \( H \) is also a partial coloring of \( G \). Note that \( H \) can be \( G \) itself. Thus a coloring \( c \) of \( G \) itself can be considered a partial coloring. A partial coloring \( c \) of \( G \) is said to be a proper partial coloring if \( c \) is proper. A proper partial coloring \( c \) is called acyclic if there are no bichromatic cycles in the graph. Sometimes we also use the word valid coloring instead of acyclic coloring. Note that with respect to a partial coloring \( c \), \( c(e) \) may not be defined for an edge \( e \). So, whenever we use \( c(e) \), we are considering an edge \( e \) for which \( c(e) \) is defined, though we may not always explicitly mention it.

Let \( e \) be a partial coloring of \( G \). We denote the set of colors in the partial coloring \( c \) by \( C = \{1, 2, \ldots, k\} \). For any vertex \( u \in V(G) \), we define \( F_u(c) = \{c(u, z) | z \in N_G(u)\} \). For an edge \( ab \in E \), we define \( S_{ab}(c) = F_b - \{c(a, b)\} \). Note that \( S_{ab}(c) \) need not be the same as \( S_{ba}(c) \). We will abbreviate the notation to \( F_u \) and \( S_{ab} \) when the coloring \( c \) is understood from the context.

To prove the main result, we plan to use contradiction. Let \( G \) be the minimum counter example with respect to the number of edges for the statement in the theorems that we plan to prove. Let \( G = (V, E) \) be a graph on \( m \) edges where \( m \geq 1 \). We will remove an edge \( e = (x, y) \) from \( G \) and get a graph \( G' = (V, E') \). By the minimality of \( G \), the graph \( G' \) will have an acyclic edge coloring \( c : E' \rightarrow \{1, 2, \ldots, t\} \), where \( t \) is the claimed upper bound for \( a'(G) \). Our intention will be to extend the coloring \( c \) of \( G' \) to \( G \) by assigning an appropriate color for the edge \( e \) thereby contradicting the assumption that \( G \) is a minimum counter example.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial coloring of \( G - e \) to \( G \).

**Maximal bichromatic Path:** An \((\alpha, \beta)\)-maximal bichromatic path with respect to a partial coloring \( c \) of \( G \) is a maximal path consisting of edges that are colored using the colors \( \alpha \) and \( \beta \) alternatingly. An \((\alpha, \beta, a, b)\)-maximal bichromatic path is an \((\alpha, \beta)\)-maximal bichromatic path which starts at the vertex \( a \) with an edge colored \( \alpha \) and ends at \( b \). We emphasize that the edge of the \((\alpha, \beta, a, b)\)-maximal bichromatic path incident on vertex \( a \) is colored \( \alpha \) and the edge incident on vertex \( b \) can be colored either \( \alpha \) or \( \beta \). Thus the notations \((\alpha, \beta, a, b)\) and \((\alpha, \beta, b, a)\) have different meanings. Also, note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of proper edge coloring:
Fact 1. Given a pair of colors $\alpha$ and $\beta$ of a proper coloring $c$ of $G$, there can be at most one maximal $(\alpha, \beta)$-bichromatic path containing a particular vertex $v$, with respect to $c$.

A color $\alpha \neq c(e)$ is a candidate for an edge $e$ in $G$ with respect to a partial coloring $c$ of $G$ if none of the adjacent edges of $e$ are colored $\alpha$. A candidate color $\alpha$ is valid for an edge $e$ if assigning the color $\alpha$ to $e$ does not result in any bichromatic cycle in $G$.

Let $c = (a, b)$ be an edge in $G$. Note that any color $\beta \notin F_a \cup F_b$ is a candidate color for the edge $ab$ in $G$ with respect to the partial coloring $c$ of $G$. A sufficient condition for a candidate color being valid is captured in the Lemma below (See Appendix for proof):

Lemma 1. [6] A candidate color for an edge $e = ab$ is valid if $(F_a \cap F_b) - \{c(a, b)\} = (S_{ab} \cap S_{ba}) = \emptyset$.

Now even if $S_{ab} \cap S_{ba} \neq \emptyset$, a candidate color $\beta$ may be valid. But if $\beta$ is not valid, then what may be the reason? It is clear that color $\beta$ is not valid if and only if there exists $\alpha \neq \beta$ such that a $(\alpha, \beta)$-bichromatic cycle gets formed if we assign color $\beta$ to the edge $e$. In other words, if and only if, with respect to coloring $c$ of $G$ there existed a $(\alpha, \beta, a, b)$ maximal bichromatic path with $\alpha$ being the color given to the first and last edge of this path. Such paths play an important role in our proofs. We call them critical paths. It is formally defined below:

Critical Path: Let $ab \in E$ and $c$ be a partial coloring of $G$. Then a $(\alpha, \beta, a, b)$ maximal bichromatic path which starts out from the vertex $a$ via an edge colored $\alpha$ and ends at the vertex $b$ via an edge colored $\alpha$ is called an $(\alpha, \beta, a, b)$ critical path. Note that any critical path will be of odd length. Moreover the smallest length possible is three.

An obvious strategy to extend a valid partial coloring $c$ of $G$ would be to try to assign one of the candidate colors to an uncolored edge $e$. The condition that a candidate color being not valid for the edge $e$ is captured in the following fact.

Fact 2. Let $c$ be a partial coloring of $G$. A candidate color $\beta$ is not valid for the edge $e = (a, b)$ if and only if $\exists \alpha \in S_{ab} \cap S_{ba}$ such that there is a $(\alpha, \beta, a, b)$ critical path in $G$ with respect to the coloring $c$.

Color Exchange: Let $c$ be a partial coloring of $G$. Let $u, i, j \in V(G)$ and $ui, uj \in E(G)$. We define Color Exchange with respect to the edge $ui$ and $uj$, as the modification of the current partial coloring $c$ by exchanging the colors of the edges $ui$ and $uj$ to get a partial coloring $c'$, i.e., $c'(u,i) = c(u,j), c'(u,j) = c(u,i)$ and $c'(e) = c(e)$ for all other edges $e$ in $G$. The color exchange with respect to the edges $ui$ and $uj$ is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges $ui$ and $uj$ is valid if and only if the coloring obtained after the exchange is acyclic. The following fact is obvious:

Fact 3. Let $c'$ be the partial coloring obtained from a valid partial coloring $c$ by the color exchange with respect to the edges $ui$ and $uj$. Then the partial coloring $c'$ will be proper if and only if $c(u, i) \notin S_{uj}$ and $c(u, j) \notin S_{ui}$.

The color exchange is useful in breaking some critical paths as is clear from the following lemma (See Appendix for proof):

Lemma 2. [6], [5] Let $u, i, j, a, b \in V(G)$, $ui, uj, ab \in E$. Also let $\{\lambda, \xi\} \in C$ such that $\{\lambda, \xi\} \cap \{c(u, i), c(u, j)\} \neq \emptyset$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Suppose there exists an $(\lambda, \xi, ab)$-critical path that contains vertex $u$, with respect to a valid partial coloring $c$ of $G$. Let $c'$ be the partial coloring obtained from $c$ by the color exchange with respect to the edges $ui$ and $uj$. If $c'$ is proper, then there will not be any $(\lambda, \xi, ab)$-critical path in $G$ with respect to the partial coloring $c'$.

Multisets and Multiset Operations: Recall that a multiset is a generalized set where a member can appear multiple times in the set. If an element $x$ appears $t$ times in the multiset $S$, then we say that multiplicity of $x$ in $S$ is $t$. In notation $\text{mult}_S(x) = t$. The cardinality of a finite multiset $S$, denoted by $\| S \|$ is defined as $\| S \| = \sum_{x \in S} \text{mult}_S(x)$. Let $S_1$ and $S_2$ be two multisets. The reader may note that there are various possible ways to define union of $S_1$ and $S_2$. For the purpose of this paper we will define one such union notion- which we call as the join of $S_1$ and $S_2$, denoted as $S_1 \uplus S_2$. The multiset $S_1 \uplus S_2$ will have all the members of $S_1$ as well as $S_2$. For a member $x \in S_1 \uplus S_2$, $\text{mult}_{S_1 \uplus S_2}(x) = \text{mult}_{S_1}(x) + \text{mult}_{S_2}(x)$. Clearly $\| S_1 \uplus S_2 \| = \| S_1 \| + \| S_2 \|$. We also need a specially defined notion of the multiset difference of $S_1$ and $S_2$, denoted by $S_1 \setminus S_2$. It is the multiset of elements of $S_1$ which are not in $S_2$, i.e., $x \in S_1 \setminus S_2$ iff $x \in S_1$ but $x \notin S_2$ and $\text{mult}_{S_1 \setminus S_2}(x) = \text{mult}_{S_1}(x)$. 

3
3 Proof of Theorem 1

Proof. A well-known strategy that is used in proving coloring theorems in the context of planar graphs is to make use of induction combined with the fact that there are some unavoidable configurations in any planar graph. Typically the existence of these unavoidable configurations are proved using the so called charging and discharging argument (See [24], for a comprehensive exposition). Loosely speaking, for the purpose of this paper, a configuration is a set \( \{v\} \cup N(v) \), where \( v \) is some vertex in \( G \), along with some information regarding the degrees of the vertices in \( \{v\} \cup N(v) \). For example, the following lemma illustrates how certain unavoidable configurations appear in a planar graph.

**Lemma 3.** [26] Let \( G \) be a simple planar graph with \( \delta \geq 2 \), where \( \delta \) is the minimum degree of graph \( G \). Then there exists a vertex \( v \) in \( G \) with exactly \( d(v) = k \) neighbours \( v_1, v_2, \ldots, v_k \) with \( d(v_1) \leq d(v_2) \leq \ldots \leq d(v_k) \) such that at least one of the following is true:

\[
\begin{align*}
(A1) & \quad k = 2, \\
(A2) & \quad k = 3 \text{ and } d(v_1) \leq 11, \\
(A3) & \quad k = 4 \text{ and } d(v_1) \leq 7, d(v_2) \leq 11, \\
(A4) & \quad k = 5 \text{ and } d(v_1) \leq 6, d(v_2) \leq 7, d(v_3) \leq 11.
\end{align*}
\]

Let graph \( G \) be a minimum counter example with respect to the number of edges for the statement in Theorem 1. From Lemma 3 we know that there exists a vertex \( v \in G \) such that \( |N(v)| = \alpha \) and by Lemma 1 all the candidate colors are valid, a contradiction to the assumption that \( G \) is a counter example. Thus we have \( \Delta(G') = \Delta(G) = \Delta \). To prove the theorem for \( G \), we may assume that \( G \) is 2-connected since if there are cut vertices in \( G \), the acyclic edge coloring of the blocks \( B_1, B_2 \ldots B_k \) of \( G \) can easily be extended to \( G \). Thus we have, \( \delta(G) \geq 2 \). We present the proof in two parts based on which configuration the vertex \( v \) belongs to. The first part deals with the case where \( \Delta \geq 2 \) and the second part deals with the case when \( \Delta = 1 \) where \( \delta(v) \geq 3 \). Thus \( v \) belongs to configuration A2, A3 or A4.

3.1 There exists a vertex \( v \) that belongs to configuration A2, A3 or A4

**Claim 1.** For any valid coloring \( \alpha' \) of \( G' \), \( |F_v \cap F_{v_1}| \geq 2 \).

**Proof.** Suppose not. The case \( |F_v \cap F_{v_1}| = 0 \) is trivial. The reader can verify from close examination of configurations A2-A4 that \( |F_v \cap F_{v_1}| \) will be maximum for configuration A2 and therefore \( |F_v \cap F_{v_1}| = |F_v| + |F_{v_1}| - |F_v \cap F_{v_1}| \leq 11 \) and hence there are at least \( \Delta + 1 \) candidate colors for the edge \( vv_1 \). Let \( F_v \cap F_{v_1} = \{v\} \) and let \( u \in N(v) \) be a vertex such that \( \alpha'(v, u) = \alpha \). Now if none of the \( \Delta + 1 \) candidate colors are valid for the edge \( vv_1 \), then by Fact 2 for each \( \gamma \in C - (F_v \cup F_{v_1}) \), there exists a \((\alpha, \gamma, vv_1)\) critical path. Since \( \alpha'(v, u) = \alpha \), we have all the critical paths passing through the vertex \( u \) and hence \( S_{vu} \subseteq C - (F_v \cup F_{v_1}) \). This implies that \( |S_{vu}| \geq |C - (F_v \cup F_{v_1})| \geq (\Delta + 12) - 11 = \Delta + 1 \), a contradiction since \( |S_{vu}| \leq \Delta - 1 \). Thus we have a valid color for the edge \( vv_1 \), a contradiction to the assumption that \( G \) is a counter example. Thus \( |F_v \cap F_{v_1}| \geq 2 \).

Let \( S_v \) be a multiset defined as \( S_v = S_{vv_2} \cup S_{vv_3} \cup \ldots \cup S_{vv_k} \). In view of Claim 1 and Lemma 2 \( 2 \leq |F_v \cap F_{v_1}| \leq 4 \). We consider each case separately.
case 1: $|F_v \cap F_{v_1}| = 2$

Let $F_v \cap F_{v_1} = \{1, 2\}$ and let $v', v'' \in N_G(v)$ and $u', u'' \in N_G(v_2)$ be such that $c'(v, v') = c'(v_1, u') = 1$ and $c'(v, v'') = c'(v_1, u'') = 2$. It is easy to see that $|F_v \cup F_{v_1}| \leq 10$. Thus there are at least $\Delta + 2$ candidate colors for the edge $vv_1$. If any of the candidate colors are valid for the edge $vv_1$, we are done. Thus none of the candidate colors are valid for the edge $vv_1$. This implies that there exist either a $(1, \theta, vv_1)$ or $(2, \theta, vv_1)$ critical path for each candidate color $\theta$.

Claim 2. With respect to the coloring $c'$, the multiset $S_v$ contains at least $|F_{v_1}| - 1$ colors from $F_{v_1}$.

Proof. Suppose not. Then there are at least two colors in $F_{v_1}$ which are not in $S_v$. Let $\nu$ and $\mu$ be any two such colors. Now assign colors $\nu$ and $\mu$ to the edges $vv'$ and $vv''$ respectively to get a coloring $c''$. Now since $\nu, \mu \notin S_v$, we have $\nu \notin S_{vv'}$ and $\mu \notin S_{vv''}$.

Note that this cannot be a $(\nu, \mu)$ bichromatic cycle since $\mu \notin S_{vv''}$. Therefore it has to be a $(\nu, \lambda)$ or $(\mu, \lambda)$ bichromatic cycle where $\lambda \in F_v(c'') - \{\nu, \mu\}$. Let $u$ be a vertex such that $c''(v, u) = \lambda$. This means that there was already a $(\lambda, \nu, vv')$ or $(\lambda, \mu, vv'')$ critical path with respect to the coloring $c''$. This implies that $\nu \in S_{vu}$ or $\mu \in S_{vu}$, implying that $\nu \in S_v$ or $\mu \in S_v$, a contradiction. Thus the coloring $c''$ is acyclic also.

Note that $|F_v \cup F_{v_1}| \leq 10$ (The maximum value of $|F_v \cup F_{v_1}|$ is attained when the graph has configuration A2). Therefore there are at least $\Delta + 2$ candidate colors for the edge $vv_1$. If any of the candidate colors are valid for the edge $vv_1$, then we are done as this is a contradiction to the assumption that $G$ is a counterexample. Thus none of the candidate colors are valid for the edge $vv_1$ and therefore there exist either a $(\nu, \theta, vv_1)$ or $(\mu, \theta, vv_1)$ critical path for each candidate color $\theta$. Let $C_v$ and $C_{v_1}$ respectively be the set of bichromatic cycles which are forming critical paths with colors $\nu$ and $\mu$. Then clearly $C_v \subseteq S_{vv_1}$ and $C_{v_1} \subseteq S_{vv_2}$ since $c'(v_1, u_1) = \nu$ and $c'(v_1, u_2) = \mu$. Now we exchange the colors of the edges $vv'$ and $vv''$ to get a modified coloring $c$. Note that $c$ is proper since $\nu \notin S_{vv'}$ and $\nu \notin S_{vv''}$. By Lemma 2 all $(\nu, \beta, vv_1)$ critical paths where $\beta \in C_v$ and all $(\mu, \gamma, vv_1)$ critical paths where $\gamma \in C_{v_1}$ are broken. Now if none of the colors in $C_v$ are valid for edge $vv_1$, then it means that for each $\beta \in C_v$, there exists a $(\mu, \beta, vv_1)$ critical path with respect to coloring $c$, implying that $C_v \subseteq S_{vv_2}$. Since the recoloring involved no candidate colors, we still have $C_{v_1} \subseteq S_{vv_2}$. Thus we have $(C_v \cup C_{v_1}) \subseteq S_{vv_2}$. But $|C_v \cup C_{v_1}| \geq \Delta + 2$ which implies that $|S_{vv_2}| \geq \Delta + 2$, a contradiction since $|S_{vv_2}| \leq \Delta - 1$.

Claim 3. With respect to the coloring $c'$, there exists at least two colors $\alpha$ and $\beta$ in $C - F_{v_1}$ with multiplicity at most one in $S_v$.

Proof. In view of Claim 2 we have $\sum_{x \in C - F_v} \text{mult}_{S_v}(x) = |S_v| - |(F_v) - 1|$. Thus if $|S_v| - |(F_v) - 1| \leq 2(|C - F_{v_1}|) - 3$, then there exist at least two colors $\alpha$ and $\beta$ in $C - F_{v_1}$ with multiplicity at most one in $S_v$. Thus it is enough to prove $|S_v| \leq 2|C| - |F_{v_1}| - 4 \leq 2\Delta + 24 - |F_{v_1}| - 4 = 2\Delta + 20 - |F_{v_1}|$. Now we can easily verify that $|S_v| + |F_{v_1}| \leq 2\Delta + 20$ for configurations A2 - A4 as follows:

- For A2, $|S_v| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + |F_{v_1}| = (\Delta - 1) + (\Delta - 1) + 10 = 2\Delta + 8$.
- For A3, $|S_v| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + |F_{v_1}| = 10 + (\Delta - 1) + (\Delta - 1) + 6 = 2\Delta + 14$.
- For A4, $|S_v| + |F_{v_1}| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + (d(v_5) - 1) + |F_{v_1}| = 6 + 10 + (\Delta - 1) + (\Delta - 1) + 5 = 2\Delta + 19$.

The colors $\alpha$ and $\beta$ of Claim 3 are crucial to the proof. Now we make another claim regarding $\alpha$ and $\beta$.

Claim 4. With respect to the coloring $c'$, $\alpha$ and $\beta$ in $F_v$.

Proof. Without loss of generality, let $\alpha \notin F_v$. Then recalling that $\alpha \notin F_{v_1}$, $\alpha$ is a candidate for the edge $vv_1$. If it is not valid, then there exists either a $(1, \alpha, vv_1)$ or $(2, \alpha, vv_1)$ critical path with respect to $c'$. Since the multiplicity of $\alpha$ in $S_v$ is at most one, we have the color $\alpha$ in exactly one of $S_{vv'}$ or $S_{vv''}$. Without loss of generality let $\alpha \in S_{vv''}$. Hence there exists either a $(2, \alpha, vv_1)$ critical path with respect to $c'$.

Now recolor the edge $vv'$ with color $\alpha$ to get a coloring $c$. It is obvious that the recoloring $c$ is proper since $\alpha \notin F_v(c')$ and $\alpha \notin S_{vv'}(c')$. It is also valid since if a bichromatic cycle gets formed due to this recoloring, it has to be a $(\alpha, \gamma)$ bichromatic cycle for some $\gamma \in F_v(c) - c(v, v')$. Let $a \in N_G(v)$ be such that $c(v, a) = \gamma$. Then the $(\alpha, \gamma)$ bichromatic
cycle should contain the edge \( va \) and therefore \( \gamma \in S_{va} \) with respect to \( c \). But we know that \( v'' \) is the only vertex in \( N_{G'}(v) \) such that \( \alpha \in S_{vv'} \). Therefore \( u = v'' \). This implies that \( \gamma = 2 \) and there existed a \((2, \alpha, vv')\) critical path with respect to the coloring \( c' \). This is a contradiction to the \text{Fact 1} since there already existed a \((2, \alpha, vv')\) critical path with respect to the coloring \( c' \). Thus the recoloring \( c \) is valid. Now with respect to the coloring \( c' \), \( |F_v \cap F_{v_v}| = 1 \), a contradiction to \text{Claim 4}.

\[ \square \]

Note that \( \alpha, \beta \notin \{1, 2\} \) since \( \alpha, \beta \notin F_{v_v} \). In view of \text{Claim 4} we have \( \{1, 2, \alpha, \beta\} \subseteq F_v \) and thus \( |F_v| \geq 4 \), which implies that \( d(v) \geq 5 \). Thus the vertex \( v \) belongs to configuration A4. Therefore \( d(v) = 5 \) and \( F_v = \{1, 2, \alpha, \beta\} \). There are at least \( \Delta + 12 - (5 + 4 - 2) = \Delta + 5 \) candidate colors for the edge \( vv_1 \). Also recall that \( d(v_2) \leq 7 \), \( c'(v, v') = c'(v_1, u') = 1 \) and \( c'(v, v'') = c'(v_1, u'') = 2 \).

\textbf{Claim 5. With respect to the coloring \( c' \), \( v_2 \notin \{v', v''\} \).}

\textbf{Proof.} Suppose not. Then without loss of generality let \( v_2 = v' \) and \( c'(v, v_2) = 1 \). Now if none of the \( \Delta + 5 \) candidate colors are valid for the edge \( vv_1 \), then they all form critical paths that contain either the edge \( vv' \) or the edge \( vv'' \). Now \( |S_{vv'}| \cup |S_{vv''}| \leq 6 + \Delta - 1 = \Delta + 5 \). Since each of the \( \Delta + 5 \) candidate colors has to be present in either in \( S_{vv'} \) or \( S_{vv''} \), we infer that \( S_{vv'} \cup S_{vv''} \) is exactly the set of candidate colors, i.e., \( |S_{vv'}| + |S_{vv''}| = \Delta + 5 \). This requires that \( |S_{vv'}| = 6 \), \( |S_{vv''}| = \Delta - 1 \) and \( S_{vv'} \cap S_{vv''} = \emptyset \). Since for each \( \gamma \in S_{vv'} \), we have \( (2, \gamma, vv_1) \) critical path containing \( u'' \), we can infer that \( S_{vv''} \subseteq S_{v_1 u''} \) (Recall that \( c'(v_1, u'') = 2 \)). But since \( |S_{v_1 u''}| \leq \Delta - 1 \), we infer \( S_{vv''} = S_{v_1 u''} \). Thus we have \( S_{v_1 u} \cap S_{vv'} = S_{vv''} \cap S_{vv''} = \emptyset \).

Now we exchange the colors of the edges \( vv' \) and \( vv'' \) to get a coloring \( c' \) i.e., \( c(v, v') = 2 \) and \( c(v, v'') = 1 \). The coloring \( c' \) is proper since \( 2 \notin S_{vv'}(c') \) and \( 1 \notin S_{vv''}(c') \) (Recall that \( S_{vv'}(c') \) and \( S_{vv''}(c') \) contain only candidate colors). The coloring is also valid: If a bichromatic cycle gets formed it has to be a \((1, \eta)\) or \((2, \eta)\) bichromatic cycle where \( \eta \notin F_v \). Clearly it cannot be a \((1, 2)\) bichromatic cycle since \( 1 \notin S_{vv'}(c) \) and therefore \( \eta = \alpha \) or \( \beta \) (Recall that \( F_v = \{1, 2, \alpha, \beta\} \)). This implies that either \( \alpha \) or \( \beta \) belongs to \( S_{vv'} \cup S_{vv''} \). But we know that \( S_{vv'} \cup S_{vv''} \) is exactly the set of candidate colors for the edge \( vv_1 \), a contradiction since \( \alpha, \beta \in F_v \) cannot be candidate colors for the edge \( vv_1 \). Therefore the coloring \( c \) is acyclic. By \text{Lemma 2} all the existing critical paths are broken. Now consider a color \( \gamma \in S_{vv'} \). If it is still not valid then there has to be a \((2, \gamma, vv_1)\) critical path since \( c(v, v') = 2 \) and \( \gamma \notin S_{vv''}(c) \). This implies that \( \gamma \in S_{v_1 u''}(c) \), a contradiction since \( S_{v_1 u''}(c) \cap S_{vv'}(c) = \emptyset \). Thus we have a valid color for the edge \( vv_1 \), a contradiction to the assumption that \( G \) is a counter example. Thus \( v_2 \notin \{v', v''\} \).

\[ \square \]

From \text{Claim 5} we infer that \( c'(v, v_2) \notin F_v \cup F_{v_v} \) since \( F_v \cup F_{v_v} = \{c'(v, v'), c(v, v''\}) = \{1, 2\} \). Therefore we have \( c'(v, v_2) \in \{\alpha, \beta\} \) since \( F_v = \{1, 2, \alpha, \beta\} \). Without loss of generality let \( c(v, v_2) = \alpha \). We know that the color \( \beta \) can be in at most one of \( S_{vv'} \) and \( S_{vv''} \) by \text{Claim 3}.

Now let \( v' \) be such that \( \beta \notin S_{vv'} \). Note that \( C - (S_{vv'} \cup F_v \cup F_{v_v}) = \emptyset \) since \( |S_{vv'} \cup F_v \cup F_{v_v}| \leq \Delta - 1 + 4 + 5 - 2 = \Delta + 6 \). Assign a color \( \theta \in C - (S_{vv'} \cup F_v \cup F_{v_v}) \) to the edge \( vv' \) to get a coloring \( c' \).

If it is valid, then let \( c = c'' \).

If the recoloring is not valid then there has to be a bichromatic cycle created due to the recoloring. Now the bichromatic cycle should involve one of the colors \( 2, \alpha, \beta \) along with \( \theta \). Since the bichromatic cycle contains a color from \( S_{vv'} \) and \( \beta \notin S_{vv'} \), it cannot be a \((\theta, \beta)\) bichromatic cycle. Now with respect to the coloring \( c' \), color \( \theta \) was not valid for the edge \( vv_1 \) implying that there existed either a \((1, \theta, vv_1)\) or a \((2, \theta, vv_1)\) critical path. But \((1, \theta, vv_1)\) critical path was not possible since \( \theta \notin S_{vv'} \) by the choice of \( \theta \). Thus there existed a \((2, \theta, vv_1)\) critical path with respect to \( c' \). Thus by \text{Fact 1} there cannot be a \((2, \theta, vv')\) critical path with respect to \( c' \) and hence there cannot be a \((2, \theta)\) bichromatic cycle in \( c'' \) formed due to the recoloring. Thus if there is a bichromatic cycle formed, then it has to be a \((\alpha, \gamma)\) bichromatic cycle (Note that the \((2, \gamma)\) and \((\beta, \gamma)\) bichromatic cycles are argued out as before). But \( \gamma \notin S_{vv_2} \), a contradiction. Thus the coloring \( c \) is acyclic.

With respect to the coloring \( c \) we have \( |F_v \cap F_{v_v}| = 1 \), a contradiction to \text{Claim 4}.
case 2: $|F_v \cap F_{v_1}| = 3$

Note that in this case $|F_v| \geq 3$ and therefore $d(v) \geq 4$. Thus $v$ belongs to either configuration A3 or A4. Let $S'_v$ be a multiset defined by $S'_v = S_v \setminus (F_v \cup F_{v_1})$. Let $v', v'', v''' \in N_G(v)$ be such that \{c(v, v'), c(v, v''), c(v, v''')\} = F_v \cap F_{v_1}$. Also let $c(v, v') = 1, c(v, v'') = 2$ and $c(v, v''') = 3$.

Claim 6. With respect to $c'$, $||S'_v|| \leq 2\Delta + 11$.\[ \]

Proof. When $d(v) = 4$, it is clear that $||S'_v|| \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) \leq 10 + \Delta - 1 + \Delta = 2\Delta + 1 + \Delta$. On the other hand when $d(v) = 5$, try to recolor one of the edges $vv', vv''$ using a color in $C - (F_v \cup F_{v_1})$. There are $\Delta + 6$ colors in $C - (F_v \cup F_{v_1})$ and if any of these colors is valid for one of $vv', vv''$ or $vv'''$, the situation reduces to case 1 i.e., $|F_v \cap F_{v_1}| = 2$. Otherwise there has to be a bichromatic cycle formed during each recoloring. Since such a bichromatic cycle has to be $(\gamma_1, \gamma_2)$ bichromatic cycle where $\gamma_1$ is the color used in the recoloring and $\gamma_2 \in F_v - \{\gamma_1\}$, we infer that $S_{vv'}, S_{vv''}$ and $S_{vv'''}$ contain at least one color from $F_v$. Thus we have $||S'_v|| \leq ||S_v|| - 3 \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) \leq 6 + 10 + \Delta - 1 + \Delta = 2\Delta + 11$.\]

Claim 7. With respect to $c'$, there exists at least one color $\alpha \in C - (F_v \cup F_{v_1})$ with multiplicity at most one in $S'_v$.

Proof. Since $v$ belongs to either configuration A3 or configuration A4, we have $|F_v \cup F_{v_1}| \leq 9 - 3 = 6$. Thus $|C - (F_v \cup F_{v_1})| \leq \Delta + 6$. By Claim 6 we have $||S'_v|| \leq 2\Delta + 11$ and from this it is easy to see that there exists at least one color $\alpha \in C - (F_v \cup F_{v_1})$ with multiplicity at most one in $S'_v$.

Note that $\alpha \in C - (F_v \cup F_{v_1})$, where $\alpha$ is the color from Claim 7 is a candidate color for the edge $vv_1$. If it is not valid then there has to be a $(\theta, \alpha, v_1)$ critical path, where $\theta \in \{1, 2, 3\}$. By Claim 7, $\alpha$ can be present in at most one of $S_{vv'}, S_{vv''}$ and $S_{vv'''}$. Without loss of generality let $\alpha \in S_{vv'}$. Thus there exists a $(2, \alpha, v_1)$ critical path with respect to the coloring $c'$. Recolor the edge $vv'$ using the color $\alpha$ to get a coloring $c$. Clearly the recoloring is proper since $\alpha \notin S_{vv''}$ and $\alpha \notin F_v$. The recoloring is valid since if a bichromatic cycle gets formed then it has to contain the color $\alpha$ as well as a color $\gamma \in F_v(v) - \{\alpha\}$. If $\gamma = c(v, w)$, then $\alpha \in S_{wv}$, for the $(\alpha, \gamma)$ bichromatic cycle to get formed. But $v''$ is the only vertex in $N_G(v)$ such that $\alpha \in S_{vv''}$. Thus $w = v''$, $\gamma = 2$ and it has to be a $(\alpha, 2)$ bichromatic cycle. This means that there exist a $(2, \alpha, vv')$ critical path with respect to the coloring $c'$, a contradiction by Fact 1 since there already existed a $(2, \alpha, vv_1)$ critical path with respect to the coloring $c'$. Thus the coloring $c$ is acyclic. This reduces the situation to case 1.

case 3: $|F_v \cap F_{v_1}| = 4$

Note that in this case $|F_v| \geq 4$ and since $d(v) \leq 5$, we have $d(v) = 5$. In other words, $v$ belongs to configuration A4. Let $S'_v$ be a multiset defined by $S'_v = S_v \setminus (F_v \cup F_{v_1})$. Also let $c(v, v_2) = 1, c(v, v_3) = 2, c(v, v_4) = 3$ and $c(v, v_5) = 4$.

Now try to recolor an edge incident on $v$ with a candidate color from $C - (F_v \cup F_{v_1})$. If the recoloring is valid then the situation reduces to case 2. Otherwise there has to be a bichromatic cycle created due to recoloring with one of the colors from $F_v$. This implies that $F_v \cap S'_v \neq \emptyset$. Thus we have $||S'_v|| \leq ||S_v|| - 1 \leq (d(v_2) - 1) + (d(v_3) - 1) + (d(v_4) - 1) + (d(v_5) - 1) \leq 6 + 10 + \Delta - 1 + \Delta - 1 - 1 = 2\Delta + 13$. Note that $\alpha \in C - (F_v \cup F_{v_1}) \geq \Delta + 12 - (4 + 5 - 4) = \Delta + 7$ candidate colors and $||S'_v|| \leq 2\Delta + 13$, it is easy to see that there exists at least one candidate color $\alpha$ with multiplicity at most one in $S'_v$.

Note that $\alpha \in C - (F_v \cup F_{v_1})$ is a candidate color for the edge $vv_1$. If it is not valid then there has to be a $(\theta, \alpha, v_1)$ critical path, where $\theta \in \{1, 2, 3, 4\}$. Note that we can present in at most one of $S_{vv_2}, S_{vv_3}, S_{vv_4}$ and $S_{vv_5}$. Without loss of generality let $\alpha \in S_{vv_2}$. Thus there exists a $(2, \alpha, vv_1)$ critical path with respect to the coloring $c'$. Recolor the edge $vv_2$ using the color $\alpha$ to get a coloring $c$. Clearly the recoloring is proper since $\alpha \notin S_{vv_3}$ and $\alpha \notin F_v$. The recoloring is valid since if a bichromatic cycle gets formed then it has to contain the color $\alpha$ as well as a color $\gamma \in F_v(c) - \{\alpha\}$. If $\gamma = c(v, w)$, then $\alpha \in S_{wv}$, for the $(\alpha, \gamma)$ bichromatic cycle to get formed. But $v_3$ is the only vertex in $N_G(v)$ such that $\alpha \in S_{vv_3}$. Thus $w = v_3$, $\gamma = 2$ and it has to be a $(\alpha, 2)$ bichromatic cycle. This means that there exist a $(2, \alpha, vv_2)$ critical path with respect to the coloring $c'$, a contradiction by Fact 1 since there already existed a $(2, \alpha, vv_1)$ critical path with respect to the coloring $c'$. Thus the coloring $c$ is acyclic. This reduces the situation to case 2.
3.2 There exists no vertex $v$ that belongs to configuration $A_2, A_3$ or $A_4$

Then clearly by Lemma 3 we can assume that there is a vertex $v$ that belongs to configuration $A_1$, i.e., $d(v) = 2$. Now delete all the degree 2 vertices from $G$ to get a graph $H$. Now since the graph $H$ is also planar, there exists a vertex $v'$ in $H$ such that $v'$ belongs to one of the configurations $A_1 \to A_4$, say $A'$. The vertex $v'$ was not already in configuration $A'$ in $G$. This means that the degree of at least one of the vertices of the configuration $A'$ i.e., $\{v'\} \cup N_H(v')$, got decreased by the removal of 2-degree vertices. Let $P = \{x \in \{v'\} \cup N_H(v') : d_H(x) < d_G(x)\}$. Let $u$ be the minimum degree vertex in $P$ in the graph $H$. Now it is easy to see that $d_H(u) \leq 11$ since $v'$ did not belong to $A'$ in $G$.

Let $N'(u) = \{x | x \in N_G(u)$ and $d_G(u) = 2\}$. Let $N''(u) = N_G(u) - N'(u)$. It is obvious that $N''(u) = N_H(u)$.

Since $u \in P$ and $d_H(u) \leq 11$, we have $|N'(u)| \geq 1$ and $|N''(u)| \leq 11$. In $G$ let $u' \in N'(u)$ be a two degree neighbour of $u$ such that $N(u') = \{u, u''\}$. Now by induction $G - \{uu'\}$ is acyclically edge colorable using $\Delta + 12$ colors. Let $c'$ be such a coloring. With respect to a partial coloring $c'$ let $F'_u(c') = \{c'(u, x) | x \in N'(u)\}$ and $F''_u(c') = \{c'(u, x) | x \in N''(u)\}$. Now if $c(u', u'') \notin F_u$ we are done since $|F_u \cup F''_u| \leq \Delta$ and thus there are at least 12 candidate colors which are also valid by Lemma 10.

We know that $|F''_u| \leq 11$. If $c'(u', u'') \in F'_u$, then let $c = c'$. Else if $c'(u', u'') \in F''_u$, then recolor edge $u'u''$ using a color from $C - (S_{u', u''} \cup F''_u)$ to get a coloring $c$ (Note that $|C - (S_{u', u''} \cup F''_u)| \geq \Delta + 12 - (\Delta - 1) = 2$ and since $u'$ is a pendant vertex in $G - \{uu'\}$ the recoloring is valid). Now if $c(u', u'') \notin F_u$ the proof is already discussed. Thus $c(u', u'') \notin F'_u$.

With respect to coloring $c$, let $a \in N'(v)$ be such that $c(v, a) = c(u', u'') = 1$. If none of the candidate colors in $C - (F_u \cup F''_u)$ are valid for the edge $uu'$ then by Fact 2 for each $v \in C - (F_u \cup F''_u)$, there exists a $(1, \gamma, uu')$ critical path. Since $c(v, a) = 1$, we have all the critical paths passing through the vertex $a$ and hence $S_{va} \subseteq C - (F_u \cup F''_u)$. This implies that $|S_{va}| \geq |C - (F_u \cup F''_u)| \geq \Delta + 12 - (1 + \Delta - 1) = 13$, a contradiction since $|S_{va}| = 1$. Thus we have a valid color for the edge $uu'$, a contradiction to the assumption that $G$ is a counter example.

$\square$

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Lemma 1. A candidate color for an edge $e = uv$, is valid if $(F_u \cap F_v) - \{c(u, v)\} = (S_{uv} \cap S_{vu}) = \emptyset$.

Proof. Any cycle containing the edge $uv$ will also contain an edge incident on $u$ (other than $uv$) as well as an edge incident on $v$ (other than $uv$). Clearly these two edges are colored differently since $(S_{uv} \cap S_{vu}) = \emptyset$. Thus the cycle will have at least 3 colors and therefore any of the candidate colors for the edge $uv$ is valid.

Lemma 2. Let $u, i, j, a, b \in V(G)$, $ui, uj, ab \in E$. Also let $\{\lambda, \xi\} \subseteq C$ such that $\{\lambda, \xi\} \cap \{c(u, i), c(u, j)\} \neq \emptyset$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Suppose there exists an $(\lambda, \xi, ab)$-critical path that contains vertex $u$, with respect to a valid partial coloring $c$ of $G$. Let $c'$ be the partial coloring obtained from $c$ by the color exchange with respect to the edges $ui$ and $uj$. If $c'$ is proper, then there will not be any $(\lambda, \xi, ab)$-critical path in $G$ with respect to the partial coloring $c'$.
Proof. Firstly, $\{\lambda, \xi\} \neq \{c(u, i), c(u, j)\}$. This is because, if there is a $(\lambda, \xi, ab)$-critical path that contains vertex $u$, with respect to a valid partial coloring $c$ of $G$, then it has to contain the edge $ui$ and $uj$. Since $i \notin \{a, b\}$, vertex $i$ is an internal vertex of the critical path which implies that both the colors $\lambda$ and $\xi$ (that is $c(u, i)$ and $c(u, j)$) are present at vertex $i$. That means $c(u, j) \in S_{ui}$ and this contradicts Fact 3 since we are assuming that the color exchange is proper. Thus $\{\lambda, \xi\} \neq \{c(u, i), c(u, j)\}$.

Now let $P$ be the $(\lambda, \xi, ab)$ critical path with respect to the coloring $c$. Without loss of generality assume that $\gamma = c(u, i) \in \{\lambda, \xi\}$. Since vertex $u$ is contained in path $P$, by the maximality of the path $P$, it should contain the edge $ui$ since $c(u, i) = \gamma \in \{\lambda, \xi\}$. Let us assume without loss of generality that path $P$ starts at vertex $a$ and reaches vertex $i$ before it reaches vertex $u$. Now after the color exchange with respect to the edges $ui$ and $uj$, i.e., with respect to the coloring $c'$, there will not be any edge adjacent to vertex $i$ that is colored $\gamma$. So if any $(\lambda, \xi)$ maximal bichromatic path starts at vertex $a$, then it has to end at vertex $i$. Since $i \neq b$, by Fact 1 we infer that the $(\lambda, \xi, ab)$ critical path does not exist.

□