Oscillatory decay of a two-component Bose-Einstein condensate

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We study the decay of a two-component Bose-Einstein condensate with negative effective interaction energy. With a decreasing atom number due to losses, the atom-atom interaction becomes less important and the system undergoes a transition from a bistable Josephson regime to the monostable Rabi regime, displaying oscillations in phase and number. We study the equations of motion and derive an analytical expression for the oscillation amplitude. A quantum trajectory simulation reveals that the classical description fails for low emission rates, as expected from analytical considerations. Observation of the proposed effect will provide evidence for negative effective interaction.

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The experimental realization of Bose-Einstein condensates offers the possibility of studying quantum matter waves. A paradigmatic effect in such systems is the coherent exchange of particles between weakly connected states, first predicted for superconductors. This so-called Josephson effect is expected to be displayed also by a Bose-Einstein condensate (BEC) in a split trap. It has been argued that an “internal” Josephson effect may also occur in condensates whose atoms can reside in two different hyperfine states coupled by Raman transitions. Depending on the relative strength of the interaction between atoms and the hopping between states, the system may lie in the Rabi regime, where atoms behave as independent, or in the Josephson regime, where interactions are sufficiently strong to create a new collective behavior. The coherent exchange of atoms in the Rabi regime has been realized between two hyperfine states of $^{87}\text{Rb}$ atoms.

It is possible to have two-component condensates in which it is energetically favorable for atoms to be in the same state. This “ferromagnetic” coupling results from an appropriate combination of the three relevant scattering lengths, and does not necessarily require a negative scattering length. On the other hand, atom losses by one- and three-body collisions cause the condensate to decay. In this article, we show that the decay of a two-component condensate with ferromagnetic coupling may display spontaneous oscillations due to a fundamental dependence of the energy on the relative particle number.

We consider Bose condensed atoms with two internal hyperfine levels $|A\rangle$ and $|B\rangle$. The atoms interact via $AA$, $BB$, and $AB$ elastic collisions. Additionally, a laser field induces Raman transitions $|A\rangle \leftrightarrow |B\rangle$, generating an internal Josephson effect. If the orbital wave functions of states $|A\rangle$ and $|B\rangle$ may be regarded as rigid, the system can be adequately described by the two-mode Hamiltonian

\begin{equation}
H = -\frac{\hbar \omega_R}{2} (a^\dagger b + ab^\dagger) + \frac{E_c}{8} (a^\dagger a - b^\dagger b)^2,
\end{equation}

where the Rabi frequency $\omega_R$ depends on the detailed laser tuning, and where the “charging” energy $E_c = (u_{AA} + u_{BB} - 2u_{AB})/4$ encompasses the various interactions. The effective interaction constants are given by $u_{ij} = (4\pi \hbar^2 \sigma_{ij}/m) \int dx |\varphi_i(x)|^2 |\varphi_j(x)|^2$, where the mode functions $\varphi_i(x)$, $i = A, B$, are normalized to unity. We have assumed $u_{AA} = u_{BB}$, so that a contribution linear in the atom number difference vanishes. In the present work we focus on the case where the effective interaction energy for the internal Josephson dynamics is negative ($E_c < 0$). This is also the case for a double-well condensate with a negative scattering length.

First we analyze the structure of the classical phase space. For that in the Heisenberg equations of motion, we replace the operators $a, b$ by $\sqrt{n_A} e^{-i\phi} \sqrt{n_B} e^{-i\theta}$. For the variables $z \equiv (n_A - n_B)/N \equiv 2\nu/N$ and $\phi \equiv (\phi_A - \phi_B)$ this yields

\begin{equation}
\dot{\phi} = \frac{z}{\sqrt{1 - z^2}} \cos \phi - \lambda z,
\end{equation}

\begin{equation}
\dot{z} = -\sqrt{1 - z^2} \sin \phi.
\end{equation}

Hereafter, $\lambda \equiv -N E_c/2\hbar \omega_R > 0$ gives the effective interaction strength, and the time is measured in units of $1/\omega_R$. Equations (2),(3) can also be obtained from the classical nonrigid pendulum Hamiltonian

\begin{equation}
H(z, \phi, \lambda) = -\sqrt{1 - z^2} \cos \phi - \frac{\lambda}{2} z^2,
\end{equation}

where $(z, \phi)$ are canonically conjugate coordinates.

The pendulum Hamiltonian (4) obeys the symmetry $H(z, \phi, \lambda) = -H(z, \phi + \pi, -\lambda)$ and, thus, a sign change of the effective interaction $\lambda$ corresponds to a phase shift $\phi \rightarrow \phi + \pi$ plus time reversal. This means that the $\pi$-states which are found for positive $E_c$ (4), at maxima of the Hamiltonian, correspond to minima at $\phi = 0$ for negative $E_c$. 

The classical fixed points of the pendulum Hamiltonian can easily be found by linearizing around its extrema. For 0 ≤ λ < 1 one finds the stable solution
\[ z_0 = 0, \quad \phi_0 = 0, \quad \Omega_0 = \sqrt{1-\lambda}, \]  
(5)
where \( \Omega_0 \) is the frequency of small oscillations around the potential minimum. For \( \lambda > 1 \) its potential curvature becomes negative. Thus at \( \lambda = 1 \) there is a bifurcation whereby \( z_0 \) becomes unstable and splits into two symmetry-related fixed points
\[ z_\pm = \pm \sqrt{1 - 1/\lambda^2}, \quad \phi_\pm = 0, \quad \Omega_\pm = \sqrt{\lambda^2 - 1}. \]  
(6)
As \( \lambda \) is proportional to the atom number, a decaying condensate will necessarily cross the bifurcation point \( \lambda = 1 \), a process which we study below.

![Figure 1](image_url)

**FIG. 1.** Classical time evolution of the relative number imbalance \( z(t) \) (dashed) for \( \gamma = 0.01 \) and \( \lambda_0 = 2 \). The full line denotes the equilibrium value for the corresponding total atom number. The inset shows the effective potential \( \mathcal{H} \) at \( \phi = 0 \) for the interaction strengths \( \lambda = 1.5 \) (dashed), 1 (full line), and 0.5 (dotted).

The bifurcation process can be appreciated in the inset of Fig. 1, where \( \mathcal{H}(z, 0, \lambda) \) is represented for several values of \( \lambda \). In the conventional case of \( E_c > 0 \), it has been noted that the point where \( \Lambda \equiv E_c N/2\hbar \omega_R = 1 \) marks the limit between the Rabi (\( \Lambda < 1 \)) and the Josephson (\( \Lambda > 1 \)) regimes. Thus we may view the transition from 3 to 1 as the negative \( E_c \) version of the crossover from an interaction dominated to a hopping dominated regime. Once \( \lambda < 1 \), the dynamics is controlled by hopping and becomes insensitive to the sign of the interaction.

The master equation that results from the inclusion of random losses is such that the classical equations of motion 2, 3 remain formally the same, but the effective interaction constant acquires a time dependence,
\[ \lambda(t) = \lambda_0 e^{-\gamma t}, \]  
(7)
due to the exponential decrease of the total atom number. We integrate the classical equations of motion 2, 3 with the initial condition \( z(0) = z_+(\lambda_0) \), \( \phi(0) = 0 \), and \( \lambda_0 = 2 \), i.e., we start from a minimum of the pendulum Hamiltonian \( \mathcal{H} \) in the bistable regime, where most of the atoms are in the state \( |A \rangle \). Figure 1 shows the corresponding time evolution of the number imbalance. As the total number decreases due to losses, the two minima of the effective potential merge towards a single minimum at \( z = 0 \) (see inset). The system evolves adiabatically until \( \lambda \) approaches and crosses the critical value of 1. We will see that a departure from adiabaticity necessarily occurs when \( \lambda \) gets sufficiently close to 1.

After crossing the bistability threshold the system finds itself distant from the new equilibrium point and begins to oscillate spontaneously around it. These oscillations initiate in the crossover region between the Josephson and the Rabi regimes.

The oscillation amplitude can be estimated analytically if one assumes (i) that outside the transition region \( \lambda \approx 1 \) the effective potential changes sufficiently slowly and (ii) that the crossover region is traversed fast enough for the process to be described as a sudden change in the potential. For \( \lambda > 1 \) the condition for adiabaticity reads
\[ 1 > \left| \frac{d}{dt} \frac{1}{\Omega_\pm^2} \right| = \frac{\gamma \lambda^2}{\Omega_\pm^3}, \]  
(8)
while later, for \( \lambda < 1 \), the condition is
\[ 1 > \left| \frac{d}{dt} \frac{1}{\Omega_0} \right| = \frac{\gamma \lambda}{2\Omega_0^3}. \]  
(9)
For \( \lambda \approx 1 \) the r.h.s. of the inequalities (8) and (9) become \( \gamma / [2(\lambda - 1)]^{3/2} \) and \( \gamma / [2(1 - \lambda)]^{3/2} \), respectively. This indicates that, sufficiently close to the bifurcation point \( \lambda = 1 \), adiabaticity breaks down.

Initially, the system follows the potential minimum \( z_+(\lambda(t)) \) adiabatically until the inequality (8) is violated for the first time. This occurs for
\[ z = z_+(1) \big|_{\Omega_0^2(1)} = \gamma \lambda^3; \]  
(10)
where \( \lambda \approx 1 \) has been assumed. As \( \lambda \) crosses the transition region, we may assume that there is a sudden change of the effective potential in the sense that \( z \) and \( \phi \) are effectively constant. Then the adiabatic condition, now reading as in (8), is recovered for \( \lambda \) sufficiently below 1.

The oscillating dynamics can be conveniently studied by noting that, for a harmonic oscillator, the action \( J = \pi E_c/\Omega_0 \) is an adiabatic invariant. The energy \( E = \frac{1}{2} (\phi^2 + \Omega_0^2 z^2) \) is estimated from Eq. (10) leading to \( J = 2\pi \Omega_0 z_+^2 = \pi \gamma \). Thus the instantaneous amplitude reads \( \sqrt{\gamma / \Omega_0(\tau)} \) and the imbalance oscillates like
\[ z(\tau) \approx \sqrt{\gamma / \Omega_0(\tau)} \cos[\Omega_0(\tau)\tau], \]  
(11)
where \( \tau \) is the time elapsed after the oscillation sets in, and \( \Omega_0(\tau) = (1 - e^{-\gamma \tau})^{1/2} \) is the instantaneous oscillation frequency. For sufficiently small \( \gamma \), \( \Omega_0(\tau) = (\gamma \tau)^{1/2} \)
which yields, for the time $\tau_k$ at which the $k$-th maximum of the oscillation happens, the condition $2\pi k = \Omega_0(\tau_k) \tau_k \approx \gamma^{1/2} k^{3/2}$. The corresponding amplitudes therefore read

$$z_k \approx (\gamma^2/2\pi k)^{1/6}. \quad (12)$$

We have numerically checked the values of $z_k$ for $1 \leq k \leq 5$ in the range $10^{-5} < \gamma < 10^{-1}$, and have found good agreement with this analytical prediction.

The classical treatment above requires a smooth potential, such that its curvature is basically constant on the scale of the width $\Delta z$ of the wave packet. While approaching the transition point $\lambda = 1$, the potential becomes flatter and, thus, the stationary wave packet spreads. At the same time the distance $|z_+ - z_-|$ between the two potential minima decreases. As long as

$$\Delta z < |z_+ - z_-|, \quad (13)$$

the classical description should be adequate. When inequality (13) is violated, a quantum description is necessary. For that we take the variables $z$ and $\phi$ to obey the commutation relation $[n,e^{i\phi}] = i e^{i\phi}$, which for small phase uncertainty amounts to $[z,\phi] = 2i/N$. For the Hamiltonian (8) in the harmonic approximation, this yields the number r.m.s. $(\Delta z)^2 = 1/N\Omega_+^2$.

When $\lambda \to 1^+$, the adiabaticity condition (8) reads $[2(\lambda - 1)]^{3/2} > \gamma$, while (13) amounts to $[2(\lambda - 1)]^{3/2} > 1/4N$. We expect the classical description of the passage through the bifurcation to be correct if, as $\lambda$ decreases in the bistable region, adiabaticity breaks down before classicality does (18). This is the case if $N\gamma > 1$, i.e., if losses are sufficiently intense that more than one atom is lost in a typical Rabi cycle. On the contrary, we expect the classical predictions to fail for $N\gamma \lesssim 1$. A similar conclusion is reached for the regime $\lambda < 1$ if we replace (13) by the condition that the width $(N\Omega_0)^{-1/2}$ of the ground state wave function is smaller than the scale $\sqrt{1-\lambda}$ below which the effective potential behaves as harmonic. Below we see that these expectations are confirmed by the numerical simulation.

To analyze the quantum dynamics of a single experiment, we employ the quantum jump method (19) where the time evolution over a sufficiently small time interval $\Delta t$ is simulated as follows. If an atom is lost from condensate $A$ ($B$), the wave function changes as $|\psi(t + \Delta t)\rangle \rightarrow Q|\psi(t)\rangle$, $Q = a, b$. These losses occur with the probabilities $p_{AB}(t,\Delta t) = \gamma \Delta t|\psi(t)\rangle Q|\psi(t)\rangle$. If no loss takes place, the system propagates form $t$ to $t + \Delta t$ with the effective Hamiltonian $H_{\text{eff}} = H - i\hbar(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b})/2$. Since the propagation with $H_{\text{eff}}$ as well as the atom losses are non-unitary, the wave function has to be normalized after each time step. Note that in the case of two identical loss rates the total atom number commutes also with the effective Hamiltonian and terms in $H_{\text{eff}}$ which depend only on the total atom number can still be neglected.

We start our simulation in the bistable region with $\lambda_0 = 2$. Thus the initial state is the “local ground state” at the potential minimum $z_+$ and satisfies condition (13). Here we make the very reasonable assumption that the parity symmetry is effectively broken due to the large value of $N_0$, i.e., that any Schrödinger cat state has evolved into a mixture due to tiny perturbations from the environment.

Figure 2(a) depicts the quantum mechanical time evolution for the same parameters as used in Fig. 1. Both plots agree quite well, which indicates that the system behaves classically. However, for a smaller value of the decay rate [Fig. 2(b)] we find substantial differences: (i) the oscillation starts well before the transition to the monostable regime and (ii) its amplitude is larger than in the corresponding classical case (see Fig. 3). This is so because the finite width of the wave packet allows to cross the point $z = 0$ when there is still a finite barrier.

![Graph](attachment:image.png)
i.e., roughly at the value for which the classical description was predicted to fail. Then the amplitude is comparable to the phase uncertainty and therefore difficult to observe. A decay rate $N_0\gamma \approx 10$ seems optimal for the experimental observation, since then there are still sufficiently many oscillations before the condensate has decayed and the amplitude is still larger than the uncertainty.

Let us assume that the rigid wave functions $\varphi_{A,B}(x)$ are given by the Thomas-Fermi (TF) approximation for $N_0 = 1000$ and $\lambda_0 = 2$. Each data point results from an average over 20 simulation runs. The full line displays the classical result \(^\text{(14)}\) for $k = 1$.

In summary, we have shown that a two-component condensate with negative interaction energy ($E_c < 0$) will exhibit spontaneous oscillations in phase and number during its decay due to atom losses. These oscillations have their root in the fundamental dependence of the Josephson coupling energy on the relative boson number. As the atom number decreases, interaction between atoms becomes less important than hopping between states. Thus the system evolves from a Josephson regime, which is bistable due to the negative sign of $E_c$, to a monostable Rabi regime dominated by hopping. A classical description of the oscillatory decay dynamics gives a good qualitative account of the decay process for high enough loss rates. The observation of this effect will provide unambiguous evidence of the Josephson effect with negative $E_c$, by identifying the existence of a bifurcation that is manifested through the spontaneous appearance of Josephson-Rabi oscillations. By accurately monitoring the decay process, it will be possible to quantitatively check our current understanding of the internal Josephson dynamics of two-component condensates.

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![FIG. 3. Amplitude of the quantum mechanical oscillations (circles) and its quantum uncertainty (boxes) for $N_0 = 1000$ and $\lambda_0 = 2$. Each data point results from an average over 20 simulation runs. The full line displays the classical result \(^\text{(14)}\) for $k = 1$.](image)

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