MONOTONICITY AND RIGIDITY OF SOLUTIONS TO SOME ELLIPTIC SYSTEMS WITH UNIFORM LIMITS

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Abstract. In this paper we prove the validity of Gibbons’ conjecture for a coupled competing Gross-Pitaevskii system. We also provide sharp a priori bounds, regularity results and additional Liouville-type theorems.

1. Introduction

This paper concerns the study of the qualitative properties, with particular emphasis to symmetry and monotonicity, of non-negative solutions to the elliptic system

\[
\begin{align*}
-\Delta u &= u - u^3 - \Lambda uv^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - \Lambda u^2 v \quad \text{in } \mathbb{R}^N \\
u, v &\geq 0 \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

(P)

As discussed in [1, 2, 17, 31] (see also the references therein), the problem under investigation with \( \Lambda > 1 \) arises in the study of domain walls and interface layers of two-components Bose-Einstein condensates in the segregation regime. Motivated by the physical interpretation, in [2] it is proved that (P) in dimension \( N = 1 \) has a solution satisfying the conditions

\[
\begin{align*}
u(t) &\to 1 & v(t) &\to 0 & \text{as } t \to +\infty \\
u(t) &\to 0 & v(t) &\to 1 & \text{as } t \to -\infty,
\end{align*}
\]

that is, an heteroclinic connection between the equilibria \((0, 1)\) and \((1, 0)\). This solution satisfies the monotonicity condition

\[
u' > 0 \quad \text{and} \quad v' < 0 \quad \text{in } \mathbb{R},
\]

is spectrally and nonlinearly stable (see [2]), and is unique (modulo translation) within the class of solutions with a monotone component (see [1]). Furthermore, in [1] precise asymptotic estimates in the limit \( \Lambda \to +\infty \) are provided.

By the study of the 1-dimensional problem, and by the results in [1], it emerges a deep connection between problem (P) and

\[
\begin{align*}
\Delta U &= UV^2 \quad \text{in } \mathbb{R}^N \\
\Delta V &= U^2 V \quad \text{in } \mathbb{R}^N \\
U, V &\geq 0 \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

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Firstly, also (1.3) in \( \mathbb{R} \) has a unique stable solution (modulo translations and scalings) which satisfies the same monotonicity property as in (1.2), and has limits

\[
U(t) \to +\infty \quad V(t) \to 0 \quad \text{as } t \to +\infty \\
U(t) \to 0 \quad V(t) \to +\infty \quad \text{as } t \to -\infty,
\]

see [10, 11]. Secondly, it is shown in [1, 38] that such 1-dimensional solution appears as limit in a suitable blow-up analysis near the regular part of the interface of solutions to (P) in the strong competition regime \( \Lambda \to +\infty \).

Problem (1.3) has been intensively studied in the last years also in higher dimension, starting from the seminal paper [10]. We refer to [11, 36, 37] for existence, and to [11, 20, 23, 32, 35, 40, 39] for classification results. In particular, in [23] the authors proved that in any dimension \( N \geq 2 \), a solution to (1.3) with at most polynomial growth at infinity and satisfying

\[
U(x', x_N) \to +\infty \quad V(x', x_N) \to 0 \quad \text{as } x_N \to +\infty \\
U(x', x_N) \to 0 \quad V(x', x_N) \to +\infty \quad \text{as } x_N \to -\infty,
\]

the limit being uniform in \( x' \in \mathbb{R}^{N-1} \), depends only on \( x_N \). This gives an affirmative answer to a conjecture raised in [10], which is the natural counterpart of the famous Gibbons' conjecture for the Allen-Cahn equation, for which we refer to [6, 9, 21].

Motivated by the 1-dimensional analysis carried on in [1, 2], and inspired by the results in [23], in this paper we address the following issue: is it true that, in any dimension \( N \geq 1 \), any solution to (P) satisfying the condition (1.1) as \( x_N \to \pm \infty \), uniformly in the other variables, depends only on \( x_N \)? The first of our main results is the positive answer to this question.

**Theorem 1.1.** Let \( N \geq 1 \), \( \Lambda > 1 \), and let \((u, v) \in L_{\text{loc}}^3(\mathbb{R}^N) \times L_{\text{loc}}^3(\mathbb{R}^N)\) be a distributional solution of (P), satisfying the assumption

\[
(h_\infty) \quad u(x', x_N) \to 1 \quad v(x', x_N) \to 0 \quad \text{as } x_N \to +\infty \\
u(x', x_N) \to 0 \quad v(x', x_N) \to 1 \quad \text{as } x_N \to -\infty,
\]

uniformly with respect to \( x' \in \mathbb{R}^{N-1} \). Then \((u, v)\) is smooth, depends only on \( x_N \), and

\[
\partial_N u > 0, \quad \partial_N v < 0 \quad \text{in } \mathbb{R}^N.
\]

Here and in the rest of the paper, we use the common notation \( x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \) for points of \( \mathbb{R}^N \), and we denote by \( \partial_N \) the partial derivative with respect to \( x_N \). In general, in what follows we always write \( \partial_{\nu}w \) to denote the directional derivative of a function \( w \) in a direction \( \nu \in S^{N-1} \).

As byproduct of Theorem 1.1, we obtain the uniqueness (modulo translations) of the positive 1D heteroclinic connections between \((0,1)\) and \((1,0)\), without any additional assumption on \((u,v)\). For minimal solutions (in a suitable sense), this result was conjectured in [2, Section 5], and proved in [1] as a consequence of [1, Theorem 1.3], where it is showed the uniqueness of positive 1D heteroclinic connections with one monotone component. Here we can remove the monotonicity assumption and extend the uniqueness in any dimension.

**Corollary 1.2.** For \( N \geq 1 \) and \( \Lambda > 1 \), there exists exactly one distributional solution \((u,v) \in L_{\text{loc}}^3(\mathbb{R}^N) \times L_{\text{loc}}^3(\mathbb{R}^N)\) of (P) satisfying \((h_\infty)\), modulo translations.

The first step in the proof of Theorem 1.1 is the following universal estimate regarding any solution to (P) (not necessarily positive), with arbitrary \( \Lambda > 0 \), which we think can be of independent interest.
Theorem 1.3. Assume $N \geq 1$ and let $(u,v) \in L^3_{\text{loc}}(\mathbb{R}^N) \times L^3_{\text{loc}}(\mathbb{R}^N)$ be a distributional solution of
\begin{align*}
-\Delta u &= u - u^3 - \Lambda u v^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - \Lambda u^2 v \quad \text{in } \mathbb{R}^N.
\end{align*}
It results that:

(i) If $\Lambda > 0$, then $u$ and $v$ are smooth and satisfy
$$|u| \leq 1, \quad |v| \leq 1 \quad \text{in } \mathbb{R}^N.$$ 

(ii) If $\Lambda \geq 1$, then
$$u^2 + v^2 \leq 1 \quad \text{in } \mathbb{R}^N.$$ 

(iii) If $\Lambda \in (0,1)$, then
$$u^2 + v^2 \leq \frac{2}{\Lambda + 1} \quad \text{in } \mathbb{R}^N.$$ 

Remark 1.4. 1) The previous estimates are sharp. To see this, it is sufficient to observe that the constant solutions $(\pm 1, 0)$ or $(0, \pm 1)$ realize the equality in items (i) and (ii), and $(\frac{1}{\sqrt{\Lambda+1}}, \frac{1}{\sqrt{-\Lambda}})$ realizes the equality in item (iii).

2) Regarding item (i), we note that, whenever we consider a nonconstant solution, we immediately gain the strict inequalities $|u|, |v| < 1$ by the strong maximum principle.

3) By elliptic estimates, we can also obtain universal bounds (depending only on the dimension $N$) for all the derivatives of any solution to $(P)$.

We also observe that item (ii) permits to weaken the assumptions of Theorem 1.1 in the following way:

Corollary 1.5. Let $N \geq 1$, $\Lambda > 1$, and let $(u,v) \in L^3_{\text{loc}}(\mathbb{R}^N) \times L^3_{\text{loc}}(\mathbb{R}^N)$ be a distributional solution of $(P)$, satisfying the assumption
\begin{equation}
\lim_{x_N \to \pm \infty} \left( u(x', x_N) - v(x', x_N) \right) = \pm 1,
\end{equation}
uniformly with respect to $x' \in \mathbb{R}^{N-1}$. Then $(u,v)$ is smooth, satisfies $(h_\infty)$, depends only on $x_N$, and
$$\partial_{x_N} u > 0, \quad \partial_{x_N} v < 0 \quad \text{in } \mathbb{R}^N.$$ 

Clearly, this result gives also a stronger version of Corollary 1.2, where assumption $(h_\infty)$ is replaced by (1.5).

Let us focus now on the case $\Lambda \in (0,1)$. The existence of heteroclinic connection between $(0,1)$ and $(1,0)$ was not proved in this setting. We can show that this is natural, as specified by the following Liouville-type theorem.

Theorem 1.6. Assume $\Lambda \in (0,1)$, $N \geq 1$, and let $(u,v)$ be a solution of $(P)$:
\begin{align*}
-\Delta u &= u - u^3 - \Lambda u v^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - \Lambda u^2 v \quad \text{in } \mathbb{R}^N \\
u, v &> 0 \quad \text{in } \mathbb{R}^N.
\end{align*}
Then $u = v = \frac{1}{\sqrt{1+\Lambda}}$.

The case $\Lambda = 1$ is not covered by Theorems 1.1 and 1.6. Also in such setting the existence of non-constant positive solutions is an open problem, and we can show that once again no such solution can exist, at least in dimension $N \leq 2$. 
Theorem 1.7. Assume $N \leq 2$, and let $(u,v)$ be a solution of $(P)$ with $\Lambda = 1$:

$$
\begin{align*}
-\Delta u &= u - u^3 - uv^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - u^2v \quad \text{in } \mathbb{R}^N \\
u, v &> 0 \quad \text{in } \mathbb{R}^N.
\end{align*}
$$

Then $(u,v)$ is constant, necessarily satisfying $u^2 + v^2 = 1$.

The case $\Lambda = 1$ in higher dimension is open.

Further considerations are devoted to the special case $\Lambda = 3$. In such a situation, system $(P)$ together with the limit condition $(h_\infty)$ in $\mathbb{R}$ has the explicit solutions

$$
\begin{align*}
u(t) &= \frac{1 + \tanh \left( \frac{t}{\sqrt{2}} \right)}{2}, \quad v(t) = \frac{1 - \tanh \left( \frac{t}{\sqrt{2}} \right)}{2},
\end{align*}
$$

as already observed in [17, 31]. We can actually provide a complete classification of the heteroclinic connections between $(0, 1)$ and $(1, 0)$ in arbitrary dimension, without any sign-assumption on $(u,v)$.

Theorem 1.8. Assume $N \geq 1$, and let $(u,v)$ be a solution of

$$(1.6) \begin{align*}
-\Delta u &= u - u^3 - 3uv^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - 3u^2v \quad \text{in } \mathbb{R}^N,
\end{align*}$$

such that

$$
\begin{align*}
h_\infty(u(x',x_N)) \to 1 & \quad v(x',x_N) \to 0 \quad \text{as } x_N \to +\infty \\
u(x',x_N) \to 0 & \quad v(x',x_N) \to 1 \quad \text{as } x_N \to -\infty,
\end{align*}
$$

uniformly with respect to $x' \in \mathbb{R}^{N-1}$. Then,

$$
\begin{align*}
u(x) &= \frac{1 + \tanh \left( \frac{x_N + \alpha}{\sqrt{2}} \right)}{2}, \quad \nu(x) = \frac{1 - \tanh \left( \frac{x_N + \alpha}{\sqrt{2}} \right)}{2},
\end{align*}
$$

for some $\alpha \in \mathbb{R}$, i.e., $u$ and $v$ are 1-dimensional and monotone and the solution is unique and explicit, up to translations.

Restricting our attention to positive solutions, we can provide additional results.

Theorem 1.9. Assume $N \geq 1$, and let $(u,v)$ be a solution of

$$(1.7) \begin{align*}
-\Delta u &= u - u^3 - 3uv^2 \quad \text{in } \mathbb{R}^N \\
-\Delta v &= v - v^3 - 3u^2v \quad \text{in } \mathbb{R}^N \\
u, v &> 0 \quad \text{in } \mathbb{R}^N.
\end{align*}$$

Then:

(i) $\partial_N u > 0$ on $\mathbb{R}^N$ if and only if $\partial_N v < 0$ on $\mathbb{R}^N$.

(ii) If $N \leq 3$ and $\partial_N u > 0$ on $\mathbb{R}^N$, then

$$
\begin{align*}
u(x) &= \frac{1 + \tanh \left( \frac{x_N + \alpha}{\sqrt{2}} \right)}{2}, \quad \nu(x) = \frac{1 - \tanh \left( \frac{x_N + \alpha}{\sqrt{2}} \right)}{2},
\end{align*}
$$

for some unit vector $a$ such that $a_N > 0$ and some $\alpha \in \mathbb{R}$.

(iii) If $N \leq 8$, $\partial_N u > 0$ on $\mathbb{R}^N$ and

$$
\begin{align*}
limit_{x_N \to -\infty} u(x',x_N) &= 1, \quad \lim_{x_N \to -\infty} u(x',x_N) = 0
\end{align*}
$$

point-wise for every $x' \in \mathbb{R}^{N-1}$, then the same conclusion of point (ii) holds.
For $N > 8$, problem (1.7) possesses a solution $(u, v)$ satisfying $\partial_N u > 0$, $\partial_N v < 0$ on $\mathbb{R}^N$ and
\[
\lim_{x_N \to +\infty} u(x', x_N) = 1, \quad \lim_{x_N \to +\infty} u(x', x_N) = 0
\]
\[
\lim_{x_N \to -\infty} v(x', x_N) = 0, \quad \lim_{x_N \to -\infty} v(x', x_N) = 1
\]
point-wisely for every $x' \in \mathbb{R}^{N-1}$, and which is not 1-dimensional.

Remark 1.10. 1) Theorems 1.8 and 1.9 are, respectively, positive answers to Gibbons’ and De Giorgi’s conjecture for system (1.6). The proofs rely on a complete characterization of solutions to (1.6) and (1.7) in terms of pairs of solutions to the Allen-Cahn equation, see Propositions 6.1 and 6.2 below. It will then be evident that further conclusions could be obtained combining our method and the main results in [25]. We do not write down explicit statements only for the sake of brevity.

2) Item (i) (whence all the other conclusions follow) is false without the sign condition $u, v > 0$ in $\mathbb{R}^N$. A counterexample is provided by the 1-dimensional solution
\[
u(x) = \frac{\tanh \left( \frac{x_1}{\sqrt{2}} \right) + \tanh \left( \frac{x_2 + x_3}{\sqrt{2}} \right)}{2}, \quad v(x) = \frac{\tanh \left( \frac{x_1}{\sqrt{2}} \right) - \tanh \left( \frac{x_2 + x_3}{\sqrt{2}} \right)}{2},\]
with $\alpha > 0$. That this is a solution follows by the forthcoming Proposition 6.1 (or it can be checked by direct computations), and it is immediate to observe that $u$ changes sign with $\partial_N u$ positive everywhere, while $v$ is negative with $\partial_N v$ sign changing.

If $\Lambda \neq 3$, the explicit classification above is out of reach, but it is natural to wonder whether or not the above results hold for generic $\Lambda > 1$. This is left as an open problem. In this direction, it may be useful to observe that we can still obtain further information about the solutions using the comparison with $\Lambda = 3$.

Theorem 1.11. Assume $N \geq 1$, $\Lambda > 1$, and let $(u, v)$ be a solution of (P). We have:

(i) If $\Lambda > 3$, then
\[u + v < 1 \quad \text{in} \quad \mathbb{R}^N.\]

(ii) If $\Lambda < 3$, then
\[u + v > 1 \quad \text{in} \quad \mathbb{R}^N.\]

The rest of the paper is devoted to the proof of the previous results. Before proceeding, we complete the introduction with further references related to our study.

Symmetry of solutions to elliptic systems of gradient type

(1.8)
\[\Delta(u, v) = \nabla W(u, v) \quad \text{in} \quad \mathbb{R}^N\]

with multi-well potential has attracted increasing attention in the last years. Beyond the aforementioned results regarding problem (1.3), we refer in particular to [15, 27], where the authors proved, under some assumptions on the potential $W$, symmetry for monotone or stable solutions in dimension $N = 2$ or $N = 3$, and to [4], where the authors investigated rigidity properties of minimal solutions to suitable symmetric systems; see also [14, 15, 16, 26, 28] for related results in low dimension, regarding more general operators and possibly unbounded solutions. Notice that system (P) falls within the general case (1.8), with

(1.9)
\[W(u, v) = \frac{(u^2 - 1)^2}{4} + \frac{(v^2 - 1)^2}{4} + \frac{\Lambda}{2} u^2 v^2.\]

Then, it is not difficult to check that the main results in [15, 27] apply to give 1-dimensional symmetry of any bounded solution to (P) satisfying $\partial_N u > 0$ and $\partial_N v < 0$ in $\mathbb{R}^N$ with $N \leq 3$. Theorem 1.1 here is the first symmetry result applying to (P) and holding in any dimension.
Since in some of the quoted papers the authors could deal with quite general potentials \( W \), it seems natural to wonder if we can relax our assumptions as well. The answer to this question is essentially negative: on one side, from the proof of Theorem 1.1 it will be evident that we could replace \( W \) in (1.9) with

\[
W(u,v) = f(u) + g(v) + \frac{\Lambda}{2} u^2 v^2,
\]

with \( f(s) \) and \( g(s) \) behaving like \( (s^2-1)^2 \), in particular, our method does not rely on the symmetry of the system with respect to the involution \((u,v) \mapsto (v,u)\). But on the other hand, there is no hope to work in a completely general setting, since the results in [3] provide counterexample to Gibbons-type results such as Theorem 1.1 for multi-well potential systems, under particular assumptions on \( W \) (clearly such assumptions are not satisfied in the setting studied here, see Remark 3.5 for more details).

**Structure of the paper.** Sections 2-4 are devoted to the proof of Theorem 1.1. As we have already anticipated, the first step consist in the derivation of the universal estimates collected in Theorem 1.3, and is the object of Section 2.

As second step, we aim at applying the moving planes method to deduce that \( u \) and \( v \) have the desired monotonicity in \( x_N \). Due to the competitive nature of the problem (A is positive), several complications arise when trying to apply the moving plane argument (in particular, it is not easy to show that the moving plane method can start). Following the scheme used in [23], in Section 3 we first study the “monotonicity as \( x_N \to +\infty \)” with an ad-hoc argument, and then we carry on the moving planes to obtain the monotonicity in the whole space. Although the general strategy is similar to the one in [23], most of the intermediate proofs are completely different.

Afterwards, we turn to the monotonicity in all the directions of the upper semi-hphere

\[
\mathbb{S}^{N-1}_+ := \{ \nu \in \mathbb{S}^{N-1} : (\nu, e_N) > 0 \},
\]

adapting the argument firstly introduced in [21]. This is the object of Section 4, and gives 1-dimensional symmetry completing the proof of Theorem 1.1.

Afterwards, we focus on the case \( \Lambda \in (0,1) \), proving Theorems 1.6 and 1.7 in Section 5. The former result is obtained using in a decisive way the estimate in item (iii) of Theorem 1.3, while the latter one is essentially a consequence of the Liouville theorem for superharmonic functions in \( \mathbb{R}^2 \).

Finally, we analyze the special case \( \Lambda = 3 \) in Section 6, proving Theorems 1.8, 1.9 and 1.11. The main step in the proof is the complete characterization of solutions to (1.6) in terms of pairs of solutions to the Allen-Cahn equation, whence we will easily deduce our thesis.

## 2. Universal bounds

In this section we prove Theorem 1.3. In the next proof and in the rest of the paper, we denote by \( \mathds{1}_A \) the characteristic function of the set \( A \).

**Proof.** Let us start with item \((i)\). Since \((u,v) \in L^3_{\text{loc}}(\mathbb{R}^N) \times L^3_{\text{loc}}(\mathbb{R}^N)\) then both \( u - 1 \) and \( \Delta(u-1) \) belong to \( L^1_{\text{loc}}(\mathbb{R}^N) \). Therefore we can apply Kato’s inequality [12, 30] to \( u - 1 \) to obtain

\[
\Delta(u - 1)^+ \geq \Delta u \mathds{1}_{\{u-1>0\}} = (u(u^2 - 1) + \Lambda u^2) \mathds{1}_{\{u-1>0\}} \geq [(u-1)^+]^2
\]

in the sense of distribution on \( \mathbb{R}^N \). The latter and Lemma 2 of [12] imply that \((u - 1)^+ \leq 0\) on \( \mathbb{R}^N \), i.e., \( u \leq 1 \) a.e. on \( \mathbb{R}^N \). The same argument applied to \( v - 1 \) also yields that \( v \leq 1 \) a.e. on \( \mathbb{R}^N \). Finally, we observe that also \((-u,-v)\) is a solution of (1.4) and so, the above argument implies that \(-u \leq 1\) and \(-v \leq 1\) a.e. on \( \mathbb{R}^N \). By summarizing, we proved that

\[
|u| \leq 1, \quad |v| \leq 1 \quad \text{a.e. in } \mathbb{R}^N,
\]
and thus $u$ and $v$ are smooth by standard elliptic estimates.

To prove item $(ii)$ we use the smoothness of $u$ and $v$ and, once again, Kato’s inequality applied to the function $u^2 + v^2 - 1$:

$$
\Delta (u^2 + v^2 - 1)^+ \geq (2u\Delta u + 2v\Delta v)\mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

$$
= 2[(u^4 + v^4 + 2\Lambda u^2 v^2) - (u^2 + v^2)]\mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

$$
\geq 2[(u^2 + v^2)^2 - (u^2 + v^2)]\mathbb{I}_{\{u^2 + v^2 > 1\}} \geq [(u^2 + v^2 - 1)^+]^2
$$

since $\Lambda \geq 1$. Therefore, $(u^2 + v^2 - 1)^+ \leq 0$, which gives the desired conclusion.

To prove item $(iii)$ we observe, as before, that

$$
\Delta \left( u^2 + v^2 - \frac{2}{\Lambda + 1} \right)^+ \geq 2[(u^4 + 4v^4 + 2\Lambda u^2 v^2) - (u^2 + v^2)]\mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

and that, for $\Lambda \in (0, 1)$, the following algebraic inequality holds true:

$$(u^4 + v^4 + 2\Lambda u^2 v^2) \geq \frac{\Lambda + 1}{2}(u^2 + v^2)^2.
$$

Hence

$$
\Delta \left( u^2 + v^2 - \frac{2}{\Lambda + 1} \right)^+ \geq 2 \left[ \frac{\Lambda + 1}{2}(u^2 + v^2)^2 - (u^2 + v^2) \right] \mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

$$
\geq (\Lambda + 1)(u^2 + v^2) \left[ (u^2 + v^2) - \frac{2}{\Lambda + 1} \right] \mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

$$
\geq \left( u^2 + v^2 - \frac{2}{\Lambda + 1} \right)^+ \mathbb{I}_{\{u^2 + v^2 > 1\}}
$$

and we conclude as above. \qed

3. Monotonicity with respect to $x_N$

The purpose of this section consists in showing that any $(u, v)$ fulfilling the assumptions of Theorem 1.1 has the desired monotonicity with respect to $x_N$.

**Proposition 3.1.** Under the assumptions of Theorem 1.1, we have that

$$
\partial_N u > 0 \quad \text{and} \quad \partial_N v < 0 \quad \text{in} \ \mathbb{R}^N.
$$

The proof is based upon an application of the moving planes method, suitably adapted in order to deal with a non-cooperative system. For $\lambda \in \mathbb{R}$, we set

$$
u (x, x_N) := u(x', 2\lambda - x_N) \quad \text{and} \quad \Sigma_\lambda := \{ x_N > \lambda \}.
$$

We aim at proving that

$$
u (x, x_N) \leq u(x) \quad \text{and} \quad v (x, x_N) \geq v(x) \quad \forall x \in \Sigma_\lambda, \ \forall \lambda \in \mathbb{R},
$$

This and the strong maximum principle give the thesis of Proposition 3.1.

In order to prove that (3.1) is satisfied, we show that

$$
\Theta := \{ \lambda \in \mathbb{R} : u_\theta \leq u \text{ and } v_\theta \geq v \text{ in } \Sigma_\theta \text{ for every } \theta \geq \lambda \} = \mathbb{R}.
$$

This can be done in two steps: at first we prove that $\Theta \neq \emptyset$, so that by definition it is an unbounded interval of type $(\lambda, +\infty)$ (or eventually $(\lambda, +\infty)$). In a second time, we show that necessarily $\lambda = -\infty$.

For the first step, it is useful to recall the following statement:
Lemma 3.2 ([22, Lemma 2.1]). Let \( \vartheta > 0 \) and \( \gamma > 0 \) such that \( \vartheta < 2^{-\gamma} \). Moreover let \( R_0 > 0 \), \( C > 0 \) and
\[
\mathcal{L} : (R_0, +\infty) \rightarrow \mathbb{R}
\]
a non-negative and non-decreasing function such that
\[
\begin{cases}
\mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) + g(R) & \forall R > R_0, \\
\mathcal{L}(R) \leq CR^\gamma & \forall R > R_0,
\end{cases}
\]
where \( g : (R_0, +\infty) \rightarrow \mathbb{R}^+ \) is such that
\[
\lim_{R \to +\infty} g(R) = 0.
\]
Then
\[
\mathcal{L}(R) = 0.
\]
The lemma will be used in the next proof.

Lemma 3.3. There exists \( \lambda \in \mathbb{R} \) sufficiently large such that
\[
u \geq u_\lambda \quad \text{and} \quad v \leq v_\lambda \quad \text{in} \ \Sigma_\lambda
\]
for any \( \lambda \geq \bar{\lambda} \).

Proof. The pair \((u_\lambda, v_\lambda)\) solves
\[
(P_\lambda)
\]
\[
\begin{cases}
-\Delta u_\lambda = u_\lambda - u_\lambda^3 - \Lambda u_\lambda v_\lambda^2 & \text{in} \ \mathbb{R}^N, \\
-\Delta v_\lambda = v_\lambda - v_\lambda^3 - \Lambda u_\lambda^2 v_\lambda & \text{in} \ \mathbb{R}^N, \\
0 < u_\lambda, v_\lambda < 1 & \text{in} \ \mathbb{R}^N.
\end{cases}
\]

Let \( \varphi_R \) be a standard \( C^1 \) cut off function such that \( \varphi_R = 1 \) in \( B_R \), \( \varphi_R = 0 \) outside \( B_{2R} \), with \( |\nabla \varphi_R| \leq 2/R \). We consider the test functions
\[
(u_\lambda - u)^+ \varphi_R^2 \mathbf{1}_{\{x_N \geq \lambda\}}, \quad (v - v_\lambda)^+ \varphi_R^2 \mathbf{1}_{\{x_N \geq \lambda\}}.
\]

Let us set \( g(t) = t - t^3 \). By \((P)\) and \((P_\lambda)\), using the test functions above and subtracting, we deduce that
\[
\int_{\Sigma_\lambda} |\nabla (u_\lambda - u)^+|^2 \varphi_R^2 = -2 \int_{\Sigma_\lambda} \varphi_R (u_\lambda - u)^+ \nabla (u_\lambda - u)^+ \cdot \nabla \varphi_R
\]
\[
+ \int_{\Sigma_\lambda} (g(u_\lambda) - g(u))(u_\lambda - u)^+ \varphi_R^2
\]
\[
- \int_{\Sigma_\lambda} (\Lambda u_\lambda v_\lambda^2 - \Lambda u^2 v_\lambda)(u_\lambda - u)^+ \varphi_R^2.
\]

Now we make the first assumption on \( \bar{\lambda} \) and we suppose that \( \lambda \geq \bar{\lambda} \), with \( \bar{\lambda} \) large enough in order that
\[
u \geq \sqrt{2/3} \quad \text{in} \ \Sigma_\lambda.
\]
In this way, for any \( x \) such that \( u_\lambda(x) \geq u(x) \) we have
\[
g(u_\lambda(x)) - g(u(x)) = g'(\xi(x))(u_\lambda(x) - u(x)) \leq -(u_\lambda(x) - u(x))
\]
for some \( \xi(x) \in [u(x), u_\lambda(x)] \). Furthermore,
\[
-(\Lambda u_\lambda v_\lambda^2 - \Lambda u^2 v_\lambda)(u_\lambda - u)^+ = -\Lambda v_\lambda^2 ((u_\lambda - u)^+)^2 + \Lambda u(v^2 - v_\lambda^2)(u_\lambda - u)^+,
\]
so that, setting
\[
\mathcal{C}_R := \Sigma_\lambda \cap B_R,
\]
we deduce that for any $\vartheta \in (0, 1)$
\[
\int_{C_{2R}} |\nabla (u_\lambda - u)^+|^2 \leq \vartheta \int_{C_{2R}} |\nabla (u_\lambda - u)^+|^2 \\
+ \frac{1}{\vartheta R^2} \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 - \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 \\
+ \int_{C_{2R}} \Lambda u(v + v_\lambda)(v - v_\lambda)^+(u_\lambda - u)^+ \varphi_R.
\]
To estimate the last term on the right hand side, we observe that $(v + v_\lambda) \leq 2v$ in the support of $(v - v_\lambda)^+$; then, recalling also that $0 < u, u_\lambda < 1$ in $\mathbb{R}^N$, we obtain
\[
\int_{C_{2R}} \Lambda u(v + v_\lambda)(v - v_\lambda)^+(u_\lambda - u)^+ \varphi_R^2 \leq \Lambda \|v\|_{L^\infty(\Sigma_\lambda)} \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 \\
+ \Lambda \|v\|_{L^\infty(\Sigma_\lambda)} \int_{C_{2R}} ((v - v_\lambda)^+)^2 \varphi_R^2.
\]
As a consequence
\[
\int_{C_{2R}} |\nabla (u_\lambda - u)^+|^2 \leq \vartheta \int_{C_{2R}} |\nabla (u_\lambda - u)^+|^2 \\
+ \frac{1}{\vartheta R^2} \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 - \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 \\
+ \Lambda \|v\|_{L^\infty(\Sigma_\lambda)} \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 \\
+ \Lambda \|v\|_{L^\infty(\Sigma_\lambda)} \int_{C_{2R}} ((v - v_\lambda)^+)^2 \varphi_R^2.
\]
(3.3)
We proceed further with similar estimates on $(v - v_\lambda)^+$. As above, by $(P)$ and $(P_\lambda)$
\[
\int_{\Sigma_\lambda} |\nabla (v - v_\lambda)^+|^2 \varphi_R^2 = -2 \int_{\Sigma_\lambda} \varphi_R(v - v_\lambda)^+ \nabla (v - v_\lambda)^+ \cdot \nabla \varphi_R \\
+ \int_{\Sigma_\lambda} (g(v) - g(v_\lambda))(v - v_\lambda)^+ \varphi_R^2 \\
+ \int_{\Sigma_\lambda} (\Lambda u^2 \varphi_R - \Lambda u^2 v)(v - v_\lambda)^+ \varphi_R^2 \\
= -2 \int_{\Sigma_\lambda} \varphi_R(v - v_\lambda)^+ \nabla (v - v_\lambda)^+ \cdot \nabla \varphi_R \\
+ \int_{\Sigma_\lambda} (g(v) - g(v_\lambda))(v - v_\lambda)^+ \varphi_R^2 - \Lambda u^2((v - v_\lambda)^+)^2 \varphi_R^2 \\
+ \Lambda \int_{\Sigma_\lambda} v_\lambda(u^2 \varphi_R^2)(v - v_\lambda)^+ \varphi_R^2.
\]
Let now $\varepsilon > 0$ be such that $1 + \varepsilon < \Lambda$ (recall that $\Lambda > 1$). By assumption $(h_\infty)$, if necessary enlarging $\lambda$, we can suppose that for any $\lambda > \lambda$
\[
u^2 > \frac{1 + \varepsilon}{\Lambda} \quad \text{in } \Sigma_\lambda, \text{ and } \quad 2\Lambda \|v\|_{L^\infty(\Sigma_\lambda)} < \min\{1, \varepsilon\}.
\]
Using the first of these conditions, and the fact that \( g'(t) \leq 1 \) for every \( t \in \mathbb{R} \), and that \( v_\lambda \leq v \) in the support of \( (v - v_\lambda)^+ \), we infer that

\[
\int_{C_R} |\nabla (v - v_\lambda)^+|^2 \leq \partial \int_{C_{2R}} |\nabla (v - v_\lambda)^+|^2 + \frac{1}{\vartheta R^2} \int_{C_{2R}} ((v - v_\lambda)^+)^2 \varphi_R^2 - \varepsilon \int_{C_{2R}} ((v - v_\lambda)^+)^2 \varphi_R^2 + 2\Lambda \int_{C_{2R}} v(u_\lambda - u)^+(v - v_\lambda)^+ \varphi_R^2
\]

(3.5)

Now we set

\[
L_\lambda(R) := \int_{C_R} |\nabla (u_\lambda - u)^+|^2 + \int_{C_R} |\nabla (v - v_\lambda)^+|^2,
\]

observing that \( L_\lambda(R) \leq CR^N \) since \( |\nabla u| \in L^\infty(\mathbb{R}^N) \) and \( |\nabla v| \in L^\infty(\mathbb{R}^N) \). Having in mind Lemma 3.2, we fix \( \vartheta := 2^{-(N+1)} \). Adding (3.3) and (3.5), we deduce that for any \( \varepsilon > 0 \) fixed as above, \( \lambda > \bar{\lambda} \) fixed and \( R > 1 \), it results that

\[
L_\lambda(R) \leq \partial L(2R) + \left( 2\Lambda ||v||_{L^\infty(\Sigma_\lambda)} + \frac{1}{\vartheta R^2} - 1 \right) \int_{C_{2R}} ((u_\lambda - u)^+)^2 \varphi_R^2 + \left( 2\Lambda ||v||_{L^\infty(\Sigma_\lambda)} + \frac{1}{\vartheta R^2} - \varepsilon \right) \int_{C_{2R}} ((v - v_\lambda)^+)^2 \varphi_R^2.
\]

Recalling (3.4), this implies that for sufficiently large \( R \) we have

\[
L_\lambda(R) \leq \partial L_\lambda(2R).
\]

Therefore we are in position to apply Lemma 3.2, and we conclude that

\[
L_\lambda(R) = 0.
\]

This holds for any \( \lambda > \bar{\lambda} \), and hence, recalling that \( u = u_\lambda \) and \( v = v_\lambda \) on \( \partial \Sigma_\lambda \), the proof is complete. \( \square \)

As already observed, Lemma 3.3 implies that the quantity \( \bar{\lambda} := \inf \Theta \) (with \( \Theta \) defined in (3.2)) is either a real number, or \( -\infty \). We can actually rule out the former possibility, thus completing the proof of Proposition 3.1.

**Lemma 3.4.** It results that \( \bar{\lambda} = -\infty \).

**Proof.** The proof of this fact is similar to the one of step 2 of Proposition 5.1 in [23]. We report the details for the sake of completeness. Assume by contradiction that \( \bar{\lambda} > -\infty \) is a real number. In this case by continuity \( \Theta = [\bar{\lambda}, +\infty) \), and by definition of inf there exist sequences \( (\lambda_i) \subset (-\infty, \lambda) \) and \( (x^i) \subset \Sigma_\lambda \), such that \( \lambda_i \to \bar{\lambda} \) as \( i \to \infty \), and at least one between

(3.6a) \[ u_\lambda(x^i) > u(x^i) \]

(3.6b) \[ v_\lambda(x^i) < v(x^i) \]
holds true for every $i$.

Assume that (3.6a) holds true. We claim that the sequence $(x^i_N) \subset \mathbb{R}$ is bounded. If not, as $x^i_N > \lambda_i$ and $\lambda_i$ is bounded, up to a subsequence $x^i_N \to +\infty$ as $i \to \infty$. It follows that $2\lambda_i - x^i_N \to -\infty$, and in light of assumption ($h_\infty$) we obtain
\[ \lim_{i \to \infty} u_{\lambda_i}(x^i) = \lim_{i \to \infty} u((x^i)'', 2\lambda_i - x^i_N) = 0 \quad \text{and} \quad \lim_{i \to \infty} u(x^i) = 1, \]
in contradiction with (3.6a) for $i$ sufficiently large. Hence the claim is proved and, up to a subsequence, $x^i_N \to x^\infty_N$ as $i \to \infty$.

Let us set
\[ u^i(x) := u((x^i)' + x', x_N) \quad \text{and} \quad v^i(x) := v((x^i)' + x', x_N). \]
Since $(u, v)$ is bounded, by standard gradient estimates $|\nabla u|, |\nabla v| \in L^\infty(\mathbb{R}^N)$. Thus \{(u^i, v^i)\} is uniformly bounded and equi-Lipschitz-continuous, and by elliptic estimates up to a subsequence $(u^i, v^i)$ converges via a diagonal process in $C^2_{\text{loc}}(\mathbb{R}^N)$ to a limit $(u^\infty, v^\infty)$, still solution of $(P)$ in $\mathbb{R}^N$.

We wish to show that $x^\infty_N = \tilde{\lambda}$. From the absurd assumption, equation (3.6a), we obtain
\[ u^\infty_{\lambda}(0', x^\infty_N) = u^\infty(0', 2\tilde{\lambda} - x^\infty_N) = \lim_{i \to \infty} u((x^i)'', 2\lambda_i - x^i_N) \]
\[ = \lim_{i \to \infty} u_{\lambda_i}(x^i) = \lim_{i \to \infty} u(x^i) = u^\infty(0', x^\infty_N). \]

On the other hand, we observe that $(x^i)' + x', x_N) \in \Sigma_{\tilde{\lambda}}$ whenever $(x', x_N) \in \Sigma_{\lambda}$, and by definition $u_{\lambda} \leq u$ in $\Sigma_{\lambda}$. Consequently, by the convergence of $u^i$ to $u^\infty$ we deduce that
\[ u^\infty_{\lambda}(x', x_N) = \lim_{i \to \infty} u^i(x', 2\tilde{\lambda} - x_N) = \lim_{i \to \infty} u((x^i)' + x', 2\tilde{\lambda} - x_N) \]
\[ \leq \lim_{i \to \infty} u((x^i)' + x', x_N) = \lim_{i \to \infty} u^i(x', x_N) = u^\infty(x', x_N) \]
for every $(x', x_N) \in \Sigma_{\tilde{\lambda}}$. Analogously, as $v_{\lambda} \geq v$ in $\Sigma_{\lambda}$, we have $v^\infty_{\lambda} \geq v^\infty$ in $\Sigma_{\lambda}$.

Now,
\[ \begin{cases} \begin{aligned} -\Delta(u^\infty - u^\infty_{\lambda}) + c(x)(u^\infty - u^\infty_{\lambda}) &= \Lambda((u^\infty_{\lambda})^2 - (v^\infty)^2)u^\infty_{\lambda} \\ u^\infty - u^\infty_{\lambda} &\geq 0 \\ u^\infty - u^\infty_{\lambda} &= 0 \end{aligned} \end{cases} \quad \text{in} \quad \Sigma_{\lambda} \]
\[ \text{on} \quad \partial \Sigma_{\lambda}, \]
with $c \in L^\infty(\Sigma_{\lambda})$ defined by
\[ c(x) := \Lambda u^2(x) - c_0(x), \quad c_0(x) := \begin{cases} \frac{g(u_{\lambda_i}(x)) - g(u(x))}{u_{\lambda_i}'(x) - u(x)} & \text{if} \ u_{\lambda_i}(x) \neq u(x) \\ g'(u(x)) & \text{if} \ u_{\lambda_i}(x) = u(x) \end{cases} \]
(recall that $g(t) = t - t^3$). Hence, the strong maximum principle together with assumption ($h_\infty$) implies that necessarily $u^\infty - u^\infty_{\lambda_i} > 0$ in $\Sigma_{\lambda_i}$, and a comparison with (3.7) reveals that $x^\infty_N = \tilde{\lambda}$, as desired.

At this point we are ready to reach a contradiction. On one side, by the absurd assumption (3.6a)
\[ 0 < u_{\lambda_i}(x^i) - u(x^i) = u^i(0', 2\lambda_i - x^i_N) - u^i(0', x_N) = 2\partial_N u^i(0', \xi^i(\lambda_i - x^i_N)) \quad \forall i, \]
for some $\xi^i \in (2\lambda_i - x^i_N, x^i_N)$. As $\lambda_i < x^i_N$ for every $i$ this implies $\partial_N u^i(x', \xi^i_N) < 0$ for every $i$, and passing to the limit we infer that
\[ \partial_N u^\infty(0', \tilde{\lambda}) \leq 0, \]
where we used the fact that $\lambda_i \leq \xi^i \leq x^i_N$ with $\lambda_i, x^i_N \to \tilde{\lambda}$. 
On the other side, thanks to (3.8) and the fact that \( u^\infty - u_\lambda^\infty > 0 \) in \( \Sigma_\lambda \), the Hopf Lemma implies that
\[
-2\partial_N u^\infty(0', \tilde{\lambda}) = \partial_{-\epsilon_N}(u^\infty(0', \tilde{\lambda}) - u_\lambda^\infty(0', \tilde{\lambda})) < 0,
\]
in contradiction with (3.9).

The above argument establishes that (3.6a) cannot occur. With minor changes, we can show that also (3.6b) cannot be verified, and in conclusion \( \tilde{\lambda} \) cannot be finite. \( \square \)

**Proof of Proposition 3.1.** By (3.2), we directly deduce that \( \partial_N u \geq 0 \) and \( \partial_N v \leq 0 \) in \( \mathbb{R}^N \). Since
\[
\begin{cases}
-\Delta (\partial_N u) + (3u^2 + \Lambda v^2 - 1)\partial_N u = -2\Lambda uv\partial_N v \geq 0 & \text{in } \mathbb{R}^N \\
-\Delta (\partial_N v) + (3v^2 + \Lambda u^2 - 1)\partial_N v = -2\Lambda uv\partial_N u \leq 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]
the strict inequality follows by the strong maximum principle and \( (h_\infty) \).

**Remark 3.5.** In this section we established the monotonicity of any solution to \((P)\) satisfying assumption \( (h_\infty) \). In particular, this result holds in dimension \( N = 1 \), and combined with Theorem 1.3 in [1] implies the uniqueness of the positive heteroclinic connection between \((0, 1)\) and \((1, 0)\). This permits to verify that, in the setting considered here, the assumptions in [3] are not satisfied. Indeed, one of the crucial assumption in [3] is the non-uniqueness (modulo translations) of the 1D minimal heteroclinic connection between two global minima of the potential, in this case \((0, 1)\) and \((1, 0)\). But any such connection is positive by minimality (as shown in [2, page 582]), and hence unique by the above discussion. In addition, we can also observe that to apply [3, Theorem 1.1] one need the odd-symmetry of one component of the connection, which cannot be verified by positive solutions.

4. 1-DIMENSIONAL SYMMETRY

In this section we pass from the monotonicity in \( x_N \) to the monotonicity in all the directions of the upper hemi-sphere \( \mathbb{S}^{N-1}_+ := \{ \nu \in \mathbb{S}^{N-1} : \langle \nu, e_N \rangle > 0 \} \). We follow the idea introduced in [21] (see also [24]), which was already successively adapted in [23] to deal with competitive systems.

In any case, with respect to [23], the adaptation here presents several differences.

**Proposition 4.1.** For every \( \nu \in \mathbb{S}^{N-1}_+ \), we have
\[
\partial_\nu u > 0 \quad \text{and} \quad \partial_\nu v < 0 \quad \text{in } \mathbb{R}^N.
\]

We divide the proof in several steps.

**Lemma 4.2.** Let \( \sigma > 0 \) be arbitrarily chosen. There exists an open neighborhood \( \mathcal{O}_{e_N} \) of \( e_N \) in \( \mathbb{S}^{N-1} \) such that
\[
\frac{\partial u}{\partial \nu}(x) > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) < 0 \quad \forall x \in \overline{S}_\sigma, \forall \nu \in \mathcal{O}_{e_N},
\]
where \( S_\sigma := \mathbb{R}^{N-1} \times (-\sigma, \sigma) \).

**Proof.** Let \( \sigma > 0 \) be arbitrarily chosen. At first, we claim that there exists \( \epsilon = \epsilon(\sigma) > 0 \) such that
\[
(4.1) \quad \partial_N u(x) \geq \epsilon \quad \text{and} \quad \partial_N v(x) \leq -\epsilon \quad \forall x \in \overline{S}_\sigma.
\]
By contradiction, assume that there exists \((x^i) \subset S_\sigma\) such that at least one between
\[
(4.2a) \quad \lim_{i \to +\infty} \partial_N u(x^i) = 0
\]
\[
(4.2b) \quad \lim_{i \to +\infty} \partial_N v(x^i) = 0
\]
holds true for every \( i \). Assume e.g. that (4.2a) holds. We define
\[
u^i(x) := \nu(x + x^i) \quad \text{and} \quad u^i(x) := u(x + x^i).
\]
The sequence $\{(u^i, v^i)\}$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}^N)$, and hence by elliptic regularity $(u^i, v^i) \to (u^\infty, v^\infty)$ in $C^2_{loc}(\mathbb{R}^N)$ up to a subsequence and via a diagonal process, where $(u^\infty, v^\infty)$ is still a solution to $(P)$. By the convergence, we have

$$\partial_N u^\infty \geq 0 \quad \text{and} \quad \partial v^\infty_N \leq 0 \quad \text{in} \ \mathbb{R}^N,$$

and $\partial_N u^\infty(0) = 0$. Furthermore,

$$-\Delta (\partial_N u^\infty) + (3(v^\infty)^2 + \Lambda(v^\infty)^2 - 1) (\partial_N u^\infty) = -2\Lambda u^\infty v^\infty (\partial_N v^\infty) \geq 0 \quad \text{in} \ \mathbb{R}^N.$$

The strong maximum principle implies that either $\partial_N u^\infty > 0$ or $\partial_N u^\infty \equiv 0$. The former one is in contradiction with the fact that $\partial_N u^\infty(0) = 0$, the latter one is in contradiction with assumption $(h_\infty)$, which is also satisfied by the limit profile $(u^\infty, v^\infty)$ since $(x_N^\infty)$ is bounded. Thus, $(4.2a)$ cannot occur. A similar argument shows that also $(4.2b)$ does not hold, and completes the proof of claim $(4.1)$.

Now we claim that

$$(4.3) \quad \text{The map} \ \nu \mapsto (\partial_{\nu} u, \partial_{\nu} v) \text{ is in } C^{0.1}(\mathbb{S}^{N-1}, (C^0(\mathbb{R}^N))^2).$$

This is a simple consequence of the global Lipschitz continuity of $(u, v)$, which implies that

$$\left| \frac{\partial u}{\partial \nu_1}(x) - \frac{\partial u}{\partial \nu_2}(x) \right| + \left| \frac{\partial v}{\partial \nu_1}(x) - \frac{\partial v}{\partial \nu_2}(x) \right| \leq 2C|\nu_1 - \nu_2|$$

for every $x \in \mathbb{R}^N$.

Combining $(4.1)$ and $(4.3)$, the thesis follows. \hfill $\square$

**Lemma 4.3.** $u$ is strictly increasing and $v$ is strictly decreasing with respect to all the unit vectors of an open neighborhood of $e_N$ in $\mathbb{S}^{N-1}$.

**Proof.** First, we write down the equations satisfied by the directional derivatives $\partial_{\nu} u = u_{\nu}$ and $\partial_{\nu} v = v_{\nu}$:

$$(4.4) \begin{cases} -\Delta u_{\nu} = c_1(x) u_{\nu} - 2\Lambda uu_{\nu} & \text{in} \ \mathbb{R}^N, \\ -\Delta v_{\nu} = c_2(x) v_{\nu} - 2\Lambda uu_{\nu} & \text{in} \ \mathbb{R}^N, \end{cases}$$

where

$$c_1(x) := 1 - \Lambda v^2(x) - 3u^2(x), \quad \text{and} \quad c_2(x) := 1 - \Lambda u^2(x) - 3v^2(x).$$

We choose $\sigma > 0$ sufficiently large, in such a way that for a positive small $\varepsilon$

$$(4.5) \begin{align*} \sup_{S^\varepsilon_1} c_1 & \leq -\varepsilon, \\ \sup_{S^\varepsilon_2} c_2 & \leq -\varepsilon, \\ \Lambda \sup_{\{x_N > \sigma\}} v & < \frac{\varepsilon}{2}, \\ \Lambda \sup_{\{x_N < -\sigma\}} u & < \frac{\varepsilon}{2}, \end{align*}$$

with $S_\sigma$ as in Lemma 4.2. The existence of such $\sigma$ is guaranteed by $(h_\infty)$ and by the fact that $\Lambda > 1$.

Let also $\mathcal{O}_{e_N}$ be the neighborhood of $e_N$ given by Lemma 4.2. We test the first equation in $(4.4)$ with $u_{\nu} \varphi^2_R$ in $\Sigma_\sigma = \{x_N > \sigma\}$, where $\varphi_R$ is chosen exactly as in Lemma 3.3: writing $C_R := \Sigma_\sigma \cap B_R$, and using the fact that $u_{\nu} \geq 0$ on $\{x_N = \sigma\}$, we easily obtain

$$\int_{C_R} |\nabla u_{\nu}|^2 \leq -2 \int_{C_R} u_{\nu} \varphi_R \nabla u_{\nu} \cdot \nabla \varphi_R + \int_{C_R} c_1(u_{\nu} \varphi_R)^2 + 2\Lambda \int_{C_R} \varphi^2_R uu_{\nu} v^+_{\nu}$$

$$\leq \vartheta \int_{C_R} |\nabla u_{\nu}|^2 + \left( \frac{1}{\vartheta R^2} + \sup_{\Sigma_\sigma} c_1 \right) \int_{C_R} (u_{\nu} \varphi_R)^2$$

$$+ \Lambda \sup_{\Sigma_\sigma} v \int_{C_R} \varphi^2_R \left[ (u_{\nu}^-)^2 + (v^+_{\nu})^2 \right],$$
where $0 < \vartheta < 2^{-N}$. In a similar way, we also deduce that
\[
\int_{\mathcal{C}_R} |\nabla u_\nu^+|^2 \leq \vartheta \int_{\mathcal{C}_{2R}} |\nabla u_\nu^+|^2 + \left( \frac{1}{\vartheta R^2} + \sup_{\Sigma_\sigma} c_2 \right) \int_{\mathcal{C}_{2R}} (u_\nu^+ \varphi_R)^2
+ \Lambda \sup_{\Sigma_\sigma} \varphi_R^2 \left[ (u_\nu^-)^2 + (u_\nu^+)^2 \right].
\]
Summing up the terms in the above inequalities, we infer that
\[
\int_{\mathcal{C}_R} |\nabla u_\nu^-|^2 + |\nabla u_\nu^+|^2 \leq \vartheta \int_{\mathcal{C}_{2R}} |\nabla u_\nu^-|^2 + |\nabla u_\nu^+|^2 + \left( \frac{1}{\vartheta R^2} + \sup_{\Sigma_\sigma} c_1 + \Lambda \sup_{\Sigma_\sigma} u \right) \int_{\mathcal{C}_{2R}} (u_\nu^- \varphi_R)^2
+ \left( \frac{1}{\vartheta R^2} + \sup_{\Sigma_\sigma} c_2 + \Lambda \sup_{\Sigma_\sigma} \varphi_R \right) \int_{\mathcal{C}_{2R}} (u_\nu^+ \varphi_R)^2
\leq \vartheta \int_{\mathcal{C}_{2R}} |\nabla u_\nu^-|^2 + |\nabla u_\nu^+|^2
\]
for $R$ sufficiently large, where we used estimate (4.5) and assumption $(h_\infty)$. As a consequence, Lemma 3.2 is applicable, and implies that $u_\nu \geq 0$ and $v_\nu \leq 0$ in $\Sigma_\sigma = \{ x_N > \sigma \}$. Arguing exactly in the same way, we can show that the same conditions are satisfied in $\{ x_N < -\sigma \}$, and finally by Lemma 4.2 we deduce that $u_\nu \geq 0$ and $v_\nu \leq 0$ in $\mathbb{R}^N$ for every $\nu \in \mathcal{O}_{e_N}$, with both $u_\nu \neq 0$ and $v_\nu \neq 0$ by $(h_\infty)$, whence the thesis follows.

**Proof of Proposition 4.1.** Here we can essentially apply the same argument used in step 4 of Proposition 6.1 in [23]. We report the details for the sake of completeness. Let $\Omega$ be the set of the directions $\nu \in S^{N-1}_+$ for which there exists an open neighborhood $\mathcal{O}_\nu \subset S^{N-1}_+$ of $\nu$ such that
\[
\partial_\mu u > 0 \quad \text{and} \quad \partial_\mu v < 0 \quad \text{in} \ \mathbb{R}^N, \ \forall \mu \in \mathcal{O}_\nu.
\]
The set $\Omega$ is open by definition, and contains $e_N$ by Lemma 4.3. Since $S^{N-1}_+$ is arc-connected, if we can show that $\partial \Omega \cap S^{N-1}_+ = \emptyset$, then we conclude that $\Omega = S^{N-1}_+$, as desired. Thus, let us suppose by contradiction that $\nu \in \partial \Omega \cap S^{N-1}_+$ (notice in particular that $\langle e_N, \nu \rangle > 0$). By definition, there exists $(\nu_\alpha) \subset \Omega$ such that $\nu_\alpha \to \nu$. As
\[
\partial_{\nu_\alpha} u > 0 \quad \text{and} \quad \partial_{\nu_\alpha} v < 0 \quad \text{in} \ \mathbb{R}^N, \ \forall \alpha,
\]
by continuity
\[
\partial_{\nu} u \geq 0 \quad \text{and} \quad \partial_{\nu} v \leq 0 \quad \text{in} \ \mathbb{R}^N.
\]
By the strong maximum principle, recalling that $(u_\nu, v_\nu)$ solves (4.4), either $u_\nu \equiv 0$ or $u_\nu > 0$ in $\mathbb{R}^N$, and analogously either $v_\nu \equiv 0$ or $v_\nu < 0$ in $\mathbb{R}^N$. But the alternatives $u_\nu \equiv 0$ and $v_\nu \equiv 0$ are in contradiction with assumption $(h_\infty)$, since $\nu$ is not orthogonal to $e_N$, and hence
\[
\partial_{\nu} u > 0 \quad \text{and} \quad \partial_{\nu} v < 0 \quad \text{in} \ \mathbb{R}^N.
\]
Having established (4.6), it is possible to adapt the same proof of Lemmas 4.2 and 4.3, with $\nu$ instead of $e_N$, to deduce that $u_\nu > 0$ and $v_\nu < 0$ in $\mathbb{R}^N$ in all the direction of an open neighborhood $\mathcal{O}_\nu$ of $\nu$ in $S^{N-1}_+$. Thus, we have that $\nu \in \Omega \cap \partial \Omega$, in contradiction with the openness of $\Omega$. This shows that $\partial \Omega \cap S^{N-1}_+ = \emptyset$, which, as already observed, implies $\Omega = S^{N-1}_+$.

We are ready to proceed with the:

**Conclusion of the proof of Theorem 1.1.** By Proposition 4.1, we immediately obtain both $\partial_\tau u \equiv 0$ and $\partial_\tau v \equiv 0$ for every $\tau \in S^{N-1}$ orthogonal to $e_N$.

**Proof of Corollary 1.2.** By Theorem 1.1, any solution to $(P)$-$(h_\infty)$ is 1-dimensional and has monotone components. Therefore, the thesis follows by the uniqueness (modulo translations) of the 1D monotone heteroclinic connections proved in [1].
**Proof of Corollary 1.5.** If \((u - v) \to 1\) as \(x_N \to +\infty\), then by \(0 < u, v < 1\) we immediately deduce that \(u \to 1\) and \(v \to 0\). The same discussion applies for the limit as \(x_N \to -\infty\), and hence the corollary follows by Theorem 1.1. \(\square\)

5. **The case \(\Lambda \in (0, 1]\)**

We discuss separately the cases \(\Lambda \in (0, 1]\) and \(\Lambda = 1\), starting from the former one.

**Proof of Theorem 1.6.** By item (iii) of Theorem 1.3 we know that \(-v^2 \geq u^2 - \frac{2}{1+\Lambda}\) on \(\mathbb{R}^N\), and thus \(u\) solves

\[
\begin{aligned}
-\Delta u &\geq u - u^3 + \Lambda u \left(\frac{1}{1+\Lambda} - u^2\right) = (1 - \Lambda)u \left(\frac{1}{1+\Lambda} - u^2\right) \\
\text{in } &\mathbb{R}^N
\end{aligned}
\]

Given \(\gamma \in \left(0, \frac{1}{\sqrt{1+\Lambda}}\right)\), there is \(\delta = \delta(\Lambda, \gamma) > 0\) such that

\[
(1 - \Lambda)t \left(\frac{1}{1+\Lambda} - t^2\right) \geq \delta t \quad \forall t \in [0, \gamma]
\]

here we have used, in a crucial way, that \(1 - \Lambda > 0\). Therefore, there is \(R = R(\delta) > 0\) such that the principal eigenvalue \(\lambda_1 = \lambda_1(B_R)\) of \(-\Delta\) in the open ball \(B_R\) under zero Dirichlet boundary conditions satisfies \(\lambda_1 < \delta\). Let \(\phi_1\) be the eigenfunction of \(-\Delta\) in \(B_R\) such that \(\max_{B_R} \phi_1 = 1\); then, for every \(\varepsilon \in (0, \min\{\gamma, \min_{\overline{B_R}} u\})\), the function \(w = \varepsilon \phi_1\) satisfies

\[
\begin{aligned}
0 < w &\leq \varepsilon < u \\
-\Delta w &\leq (1 - \Lambda)w \left(\frac{1}{1+\Lambda} - w^2\right) \\
w &\geq 0 \quad \text{in } B_R \\
w &\geq 0 \quad \text{in } \partial B_R
\end{aligned}
\]

since \(\lambda_1 w \leq \delta w \leq (1 - \Lambda)w \left(\frac{1}{1+\Lambda} - w^2\right)\) thanks to (5.2). Hence, in view of (5.1), we can use either the sliding method (cf. e.g. [8, Lemma 3.1]) or the sweeping method (cf. e.g. [34]) to infer that \(u \geq \varepsilon > 0\) on the whole of space \(\mathbb{R}^N\). With this information in our hands we can apply Kato’s inequality to (5.1) to get that

\[
\Delta \left(\frac{1}{\sqrt{1+\Lambda}} - u\right) + (1 - \Lambda)\varepsilon \left[\left(\frac{1}{\sqrt{1+\Lambda}} - u\right)\right]^2
\]

which, in turn, provides \(\left(\frac{1}{\sqrt{1+\Lambda}} - u\right) \leq 0\) on \(\mathbb{R}^N\) and so \(u^2 \geq \frac{1}{1+\Lambda}\) on \(\mathbb{R}^N\). Since the system is symmetric in \(u\) and \(v\), the previous argument also gives that \(v^2 \geq \frac{1}{1+\Lambda}\) on \(\mathbb{R}^N\). Comparing these informations with \(u^2 + v^2 \leq \frac{2}{1+\Lambda}\) on \(\mathbb{R}^N\), we immediately infer that \(u^2 = v^2 = \frac{1}{1+\Lambda}\) on \(\mathbb{R}^N\), concluding the proof. \(\square\)

**Proof of Theorem 1.7.** By item (ii) of Theorem 1.3 we know that \(u^2 + v^2 \leq 1\) on \(\mathbb{R}^N\), and hence, being \(\Lambda = 1\), both \(u\) and \(v\) are positive super-harmonic functions in \(\mathbb{R}^N\). Since \(N \leq 2\), they must be constant. \(\square\)

6. **The (special) case \(\Lambda = 3\)**

In this section we first study the special system (1.6). Afterwards, we obtain the improved estimates of Theorem 1.11 regarding the case \(\Lambda \neq 3\).

The first step in our analysis is represented by the following statement.
Proposition 6.1. Assume $N \geq 1$. A pair $(u, v)$ (not necessarily non-negative) is a solution of (1.6) if and only if there exist $w_1$ and $w_2$, solutions to the Allen-Cahn equation $-\Delta w = w - w^3$ in $\mathbb{R}^N$, such that

\begin{align}
  u &= \frac{w_1 + w_2}{2} \quad \text{in} \quad \mathbb{R}^N \\
  v &= \frac{w_1 - w_2}{2} \quad \text{in} \quad \mathbb{R}^N.
\end{align}

Proof. Since $\Lambda = 3$, by adding the two equations in (6.1) we immediately see that $u + v$ solves the Allen-Cahn equation

$$\Delta (u + v) = (u + v)^3 - (u + v) \quad \text{on} \quad \mathbb{R}^N.$$ 

In a similar way we see that $u - v$ solves the Allen-Cahn equation

$$\Delta (u - v) = (u - v)^3 - (u - v) \quad \text{on} \quad \mathbb{R}^N.$$ 

The desired conclusion the follows by setting $w_1 = u + v$ and $w_2 = u - v$.

Conversely, it is a straightforward computation to see that the pair $(\frac{w_1 + w_2}{2}, \frac{w_1 - w_2}{2})$ is a solution of (6.1) if $w_1$ and $w_2$ solve the Allen-Cahn equation $-\Delta w = w - w^3$ in $\mathbb{R}^N$. \hfill \Box

This permits us to prove Theorem 1.8.

Proof of Theorem 1.8. We observe that $w_1 = u + v \to 1$ as $x_N \to \pm \infty$, while $w_2 = u - v \to 1$ as $x_N \to +\infty$ and $w_2 = u - v \to -1$ as $x_N \to -\infty$. Theorem 1.1 of [19] implies that $w_1 \equiv 1$ on $\mathbb{R}^N$ and $w_2 = \tanh (\frac{x_1 + \alpha}{\sqrt{2}})$ for some $\alpha \in \mathbb{R}$. The conclusion then follows from Proposition 6.1. \hfill \Box

Now we focus on positive solutions.

Proposition 6.2. Assume $N \geq 1$. A pair $(u, v)$ is a solution of (1.7) if and only if

$$u = \frac{1 + w_2}{2}, \quad v = \frac{1 - w_2}{2} = 1 - u$$

where $w_2$ is a solution of the Allen-Cahn equation $-\Delta w = w - w^3$ in $\mathbb{R}^N$, with $w_2 \neq \pm 1$.

Proof. We observe that $w_1 = u + v > 0$ in $\mathbb{R}^N$ and so, Theorem 1.1 of [19] implies that $w_1 \equiv 1$ on $\mathbb{R}^N$. The conclusion then follows from Proposition 6.1, by also recalling that any non-constant entire solution $w$ of the Allen-Cahn equation satisfies the universal bound $|w| < 1$ everywhere on $\mathbb{R}^N$ ([18]). \hfill \Box

As a consequence:

Proof of Theorem 1.9. Item (i) is an immediate consequence of Proposition 6.2, where we proved that $v = 1 - u$.

For item (ii), we observe that $w_2 = u - v = 2u - 1$ in $\mathbb{R}^N$, again by Proposition 6.2. Therefore, $\partial_N w_2 > 0$ on $\mathbb{R}^N$, and so the known results regarding De Giorgi’s conjecture for the Allen-Cahn equation [7, 29, 5] tell us that

$$w_2(x) = \tanh \left( \frac{a \cdot x + \alpha}{\sqrt{2}} \right),$$

for some unit vector $a$ such that $a_N > 0$ and some $\alpha \in \mathbb{R}$. In the same way, by [33] we can prove item (iii).

To prove the last claim of the Theorem we recall (see [13]) that in dimension $N > 8$ there exists an entire solution $w_2$ of the Allen-Cahn equation satisfying $\partial_N w_2 > 0$ on $\mathbb{R}^N$ and

$$\lim_{x_N \to +\infty} w_2 (x', x_N) = 1, \quad \lim_{x_N \to -\infty} w_2 (x', x_N) = -1$$
point-wisely for every \( x' \in \mathbb{R}^{N-1} \), and which is not 1-dimensional. Thus, an application of Proposition 6.1 with \( w_1 = 1 \) and the above solution \( w_2 \) yields that the pair \((\frac{1+w_2}{2}, \frac{1-w_2}{2})\) is a solution of (1.7) with all the desired properties. \( \square \)

Finally, we give the proof of Theorem 1.11.

**Proof of Theorem 1.11.** We observe that, for \( \Lambda > 0 \),

\[
(6.2) \quad \Delta(u + v) = (u + v)^3 - (u + v) + (\Lambda - 3)uv(u + v) \quad \text{in} \quad \mathbb{R}^N.
\]

For \( \Lambda > 3 \), from (6.2) and Kato’s inequality we get

\[
\Delta(u + v - 1) \geq [(u + v)^3 - (u + v)] \mathbbm{1}_{\{u + v - 1 > 0\}} \geq [(u + v - 1)^+]^3 \quad \text{in} \quad \mathbb{R}^N,
\]

and so \( u + v \leq 1 \) on \( \mathbb{R}^N \). Now, if \( u(x_0) + v(x_0) = 1 \), \( x_0 \in \mathbb{R}^N \), then

\[
0 \geq (\Delta(u + v))(x_0) = 1 - 1 + (\Lambda - 3)u(x_0)v(x_0)(1) > 0
\]

a contradiction.

For \( \Lambda \in (0, 3) \), from (6.2) we deduce that

\[
-\Delta(u + v) \geq (u + v) - (u + v)^3 = (u + v)(1 - (u + v)^2) \quad \text{in} \quad \mathbb{R}^N
\]

and, by proceeding as in the proof of Theorem 1.6 we first see that \( u + v \geq \varepsilon \) on \( \mathbb{R}^N \), for some \( \varepsilon > 0 \), and then, by Kato’s inequality, we infer that

\[
\Delta(1 - (u + v))^+ \geq \varepsilon [(1 - (u + v))^+]^2 \quad \text{in} \quad \mathbb{R}^N
\]

which provides \( u + v \geq 1 \). The strict inequality then follows as before. \( \square \)

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