Coherence Scaling of Noisy Second-Order Scale-Free Consensus Networks

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Abstract—A striking discovery in the field of network science is that the majority of real networked systems have some universal structural properties. In generally, they are simultaneously sparse, scale-free, small-world, and loopy. In this paper, we investigate the second-order consensus of dynamic networks with such universal structures subject to white noise at vertices. We focus on the network coherence $H_{SO}$ characterized in terms of the $H_2$-norm of the vertex systems, which measures the mean deviation of vertex states from their average value. We first study numerically the coherence of some representative real-world networks. We find that their coherence $H_{SO}$ scales sublinearly with the vertex number $N$. We then study analytically $H_{SO}$ for a class of iteratively growing networks—pseudo fractal scale-free webs (PSFWs), and obtain an exact solution to $H_{SO}$, which also increases sublinearly in $N$, with an exponent much smaller than 1. To explain the reasons for this sublinear behavior, we finally study $H_{SO}$ for Sierpinski gaskets, for which $H_{SO}$ grows superlinearly in $N$, with a power exponent much larger than 1. Sierpinski gaskets have the same number of vertices and edges as the PSFWs, but do not display the scale-free and small-world properties. We thus conclude that the scale-free and small-world, and loopy topologies are jointly responsible for the observed sublinear scaling of $H_{SO}$.

Index Terms—Distributed average consensus, multi-agent systems, Gaussian white noise, network coherence, scale-free network, small-world network.

I. INTRODUCTION

A fundamental problem in interdisciplinary areas ranging from control systems to computer science and physics, consensus has been attracting extensive attention [1–6]. It can be applied to various practical scenarios, such as load balancing [7, 8], multi-agent rendezvous [9], UAV flocking [10], and sensor networks [11–13]. For multi-agent systems, consensus means that the agents reach an agreement on certain quantities or values, such as load, direction, and pace. However, when the system operates in uncertain environments with noisy disturbances imposed on agents, the system will never reach consensus, with the state of each agent fluctuating around their average. In this case, what we are concerned with is the performance of the system in resisting noise.

The essence of various dynamical networks is the interaction among elements, which can be described by the powerful analytic tool—graphs, where vertices represent the elements and edges represent their relationships [14]. With this network representation, the interactions of vertex systems are organized into a complex topological structure as a network, characterized by various measurements including degree distribution, average shortest distance, and distribution of cycles or loops of different lengths. These structural properties have striking consequences on the behaviors and performance of dynamical processes running on the networked systems [15]. In the scenario of networks of agents, the consensus problem has been intensively studied, establishing nontrivial effects of network topological properties on various aspects of the problem, for example, convergence rate [16–19] and robustness to time delay [19–21], which are determined jointly by the second smallest eigenvalue and the largest eigenvalue of the Laplacian matrix associated with the graph.

In addition to convergence rate and time delay, many other interesting quantities about consensus dynamics are also governed by the eigenvalues of the graph Laplacian matrix $L$. For example, for first- and second-order noisy networks without leaders, their network coherence defined in terms of the system’s $H_2$-norm (i.e., the average of deviations of agent states from the current average value) is determined by the sum of the reciprocal of the square of each nonzero eigenvalue of $L$ [22–26]. For the first-order consensus problem, the network coherence has been studied for graphs with different structures, including paths [27], stars [27], cycles [27], Vicsek fractal trees and T-fractal trees [25, 28], tori and lattices [24], Farey graphs [29, 30], Koch networks [31], hierarchical graphs [19], Sierpinski graphs [19], and some real-world networks [32]. These works revealed nontrivial impacts of the network topology on the behavior of first-order network coherence.

Compared with the first-order setting, the network coherence for second-order consensus problem is relatively rarely studied, in spite of the fact that it can well describe many practical applications, for example, formation control [33] and clock synchronization [34]. It has been analyzed only for several special graphs, such as tori [24], classic fractals [19, 25, 28], Koch networks [31], and hierarchical graphs [19]. However, these networks cannot well mimic most real networked systems, which exhibit universal topological properties [15]: power-law degree distribution [35], small-world behavior [36],

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and pattern with cycles at various scales [37, 38], where a cycle is a path plus an edge connecting its two ending nodes. It has been shown that the aggregation of these properties has a critical effect on the first-order network coherence [32]. To date, their effects on second-order network coherence are still largely unknown, which are expected to be quite different from those for the first-order case, since the intrinsic mechanisms governing their dynamics differ significantly.

To fill this gap, in this paper, we study the second-order coherence of noisy consensus networks with the aforementioned universal properties observed in many real-life networks. The main work and contribution of this paper are summarized as follows.

First, we consider the coherence of scale-free small-world networks with cycles at distinct scales. We show that, for these networks, the second-order coherence scales sublinearly with the number of nodes, which is in sharp contrast to their corresponding first-order coherence that converges to a constant independent of the network size [32].

Then, we address the second-order coherence of a family of deterministically iterative networks, called pseudofractal scale-free webs (PSFWs) [39–41], which display some structural properties similar to those of the real networks studied. By exploiting the self-similarity of the graphs, we establish some recursion relations for the characteristic polynomials of the Laplacian matrices of the PSFWs and their subgraphs at consecutive iterations, based on which we further find exact expressions for the second-order coherence and its leading scaling, which also behaves sublinearly with the network size.

Finally, we show that the sublinear scaling for second-order coherence observed for both real and model networks lies in the composition of scale-free behavior, small-world effect, and the cycles of various scales in the considered networks. For this purpose, we study the second-order coherence of Sierpinski gaskets [39, 40], which have the same numbers of nodes and edges as the PSFWs but are homogeneous and large-scale, an architecture quite different from that of PSFWs. We obtain explicit formulas for the second-order coherence and its dominating behavior, which grows superlinearly with the number of nodes.

Our results presented in this paper provide insights into understanding the noisy second-order consensus dynamics and have far-reaching implications for the structural design of communication networks.

II. PRELIMINARIES

In this section, we introduce some basic concepts about a graph, its Laplacian matrix, related distances associated with the eigenvalues and the eigenvectors of the Laplacian matrix, as well as the first-order and second-order noisy consensus problems to be studied.

A. Graph, Laplacian Matrix and Related Distances

We use $G = (V,E)$ to denote an undirected connected graph with $N = |V|$ vertices and $M = |E|$ edges, where $V$ is the node set, $E$ is the edge set, and $|\cdot|$ denotes the cardinality of a set.

The adjacency matrix $A$ of a graph $G$ is an $N \times N$ symmetric matrix, representing the adjacent relations of its vertices. The entry $a_{ij}$ of $A$ at row $i$ and column $j$ is defined as follows: $a_{ij} = a_{ji} = 1$ if the vertex pair $(i,j) \in E$, and $a_{ij} = 0$ otherwise. Let $\Gamma_i$ be the set of the neighbors for vertex $i$. Then, the degree of vertex $i$ in graph $G$ is defined as $d_i = \sum_{j=1}^{N} a_{ij} = \sum_{j \in \Gamma_i} a_{ij}$. The average degree of $G$ is $\bar{d} = \frac{1}{N} \sum_{i=1}^{N} d_i = 2M/N$. If $\bar{d}$ is a constant, independent of $N$, we call $G$ a sparse graph. For a graph $G$, its degree matrix $D$ is a diagonal matrix, with the $i$th diagonal entry equal to $d_i, i = 1, 2, \ldots, N$.

Let $P(d)$ be the degree distribution of graph $G$. If $P(d) \sim d^{-\gamma}$, we call $G$ a scale-free network [35]. In a scale-free network, there exist some large-degree nodes, with the maximum-degree vertices called hub vertices, each having degree $d_{\text{max}} = N^{1/(\gamma - 1)}$. It has been shown [15] that many real networks are scale-free.

Another important matrix related to a graph $G$ is the Laplacian matrix $L$ defined by $L = D - A$ [42]. It is an $N \times N$ positive semi-definite matrix with a unique zero eigenvalue and $N - 1$ positive eigenvalues if the graph is connected. Let $\lambda_i, i = 1, 2, \ldots, N$, be the $N$ eigenvalues of $L$ arranged in ascending order, namely, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. Let $u_k, k = 1, 2, \ldots, N$, denote the corresponding mutually orthogonal unit eigenvectors, with the $i$th component being $u_{ki}, x = 1, 2, \ldots, N$. Using eigenvalues $\lambda_k$ and their corresponding eigenvectors $u_k, k = 1, 2, \ldots, N$, one can define various distances for a graph, such as resistance distance and biharmonic distance. The resistance distance $\Omega_{ij}$ between two nodes $i$ and $j$ is [43]

$$\Omega_{ij} = \sum_{k=2}^{N} \frac{1}{\lambda_k} (u_{ki} - u_{kj})^2, \quad (1)$$

while the biharmonic distance $\Theta_{ij}$ between $i$ and $j$ is defined as [44]

$$\Theta_{ij} = \sum_{k=2}^{N} \frac{1}{\lambda_k^2} (u_{ki} - u_{kj})^2. \quad (2)$$

The sum of resistance distances over all the $N(N-1)/2$ pairs of vertices in graph $G$ is called its Kirchhoff index [43], denoted by $R(G)$, which can be represented by all positive eigenvalues of the Laplacian matrix $L$ as [45]

$$R(G) = \sum_{i,j \in V, i < j} \Omega_{ij} = \frac{1}{2} \sum_{i,j \in V} \Omega_{ij} = N \sum_{i=2}^{N} \frac{1}{\lambda_i}. \quad (3)$$

The sum of biharmonic distances over all the $N(N-1)/2$ pairs of vertices in graph $G$ is called its biharmonic index [26], denoted by $B(G)$. Similarly to $R(G)$, $B(G)$ can be represented in terms of the $N - 1$ non-zero eigenvalues of $L$:

$$B(G) = \sum_{i,j \in V, i < j} \Theta_{ij} = N \sum_{i=2}^{N} \frac{1}{\lambda_i}. \quad (4)$$

It was shown [46] that $B(G)$ can be expressed in terms of $R(G)$ and the resistance distances of some vertex pairs.
B. Noisy First-Order Consensus Dynamics

A graph \( G \) can be considered as a multiagent system, where a vertex corresponds to an agent and an edge is associated with available information flow between two agents. In the first-order consensus network, each agent has a single state. We express the states of the system at time \( t \) by an \( N \)-dimensional real vector \( x(t) \in \mathbb{R}^N \), where the \( i \)th element \( x_i(t) \) represents the state of vertex \( i \). Every agent adjusts its state according to its local information. In the presence of noise, each agent is subject to stochastic disturbances. For simplicity, we suppose that every agent is independently influenced by Gaussian white noise with identical intensity, which is a stationary and ergodic random process with zero mean and the following fundamental property: two values of noise at any pair of times are statistically independent. Then, the evolution of the system state can be written in a matrix form as

\[
\dot{x}(t) = -Lx(t) + w(t),
\]

where \( w(t) = (w_1(t), w_2(t), ..., w_N(t)) \in \mathbb{R}^N \) is a Gaussian signal with zero-mean and unit variance.

Due to the impact of noise, the agents will never reach agreement and their states fluctuate around the average value of the current states for all agents. The variance of these fluctuations can be captured by network coherence, characterized by the \( H_2 \)-norm of the system [27]. Without loss of generality, we assume that the initial condition \( \frac{1}{N} \sum_{i=1}^{N} x_i(0) = 0 \). The concept of network coherence represents the extent of the fluctuations [24, 25, 28].

**Definition II.1.** For a graph \( G \), the first-order network coherence \( H_{FO} \) is defined as the mean steady-state variance of the deviation from the average of the current agent states:

\[
H_{FO} := \frac{1}{N} \lim_{t \to \infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{j=1}^{N} x_j(t) - \frac{1}{N} \sum_{i=1}^{N} x_i(t) \right\}. \tag{4}
\]

It was shown [22, 24, 27, 28, 47] that \( H_{FO} \) is purely determined by the Kirchhoff index or the \( N - 1 \) nonzero eigenvalues of \( L \), as given by

\[
H_{FO} = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i} = \frac{R(G)}{2N^2}. \tag{5}
\]

\( H_{FO} \) measures the performance of the system robustness to noise. Low \( H_{FO} \) corresponds to good robustness, indicating that every agent keeps close to the average of the current states.

C. Noisy Second-Order Consensus Dynamics

In the second-order consensus problem, at time \( t \), each agent \( i \) has two scalar-valued states: \( x_{1,i}(t) \) and \( x_{2,i}(t) \). Thus, the states of all agents can be represented by two vectors: position vector \( x_1(t) \) and velocity vector \( x_2(t) \), where \( x_2(t) \) is the first-order derivative of \( x_1(t) \) with respect to time \( t \). Different from the first-order case, every agent in the second-order consensus dynamics updates its state by changing the value of \( x_2(t) \), on the basis of its own states and the states of its neighbors. The noisy second-order consensus system can be described by:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
-L & -L
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
I
\end{bmatrix} w(t), \tag{6}
\]

where vector \( w(t) \in \mathbb{R}^N \) represents the uncorrelated Gaussian white noise process with zero-mean and unit variance; \( 0 \) and \( I \) are the \( N \times N \) zero matrix and identity matrix, respectively. We note that only the variable \( x_2(t) \) is subject to disturbances.

The network coherence of the above second-order dynamics only reflects the derivation of state \( x_1(t) \) from the average value of the current states of all agents.

**Definition II.2.** For a graph \( G \), the second-order network coherence \( H_{SO} \) is defined as the mean steady-state variance of the deviation of state \( x_1(t) \) from the current average:

\[
H_{SO} := \frac{1}{N} \lim_{t \to \infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{i=1}^{N} \left( x_{1,i}(t) - \frac{1}{N} \sum_{j=1}^{N} x_{1,j}(t) \right) \right\}. \tag{7}
\]

Similarly to \( H_{FO} \), \( H_{SO} \) is completely determined by the nonzero eigenvalues of the Laplacian matrix [24]. Specifically, \( H_{SO} \) is determined by the biharmonic index of the network [26]:

\[
H_{SO} = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i^2} = \frac{B(G)}{2N^2}. \tag{8}
\]

A low \( H_{SO} \) means that the network structure is robust to random disturbances to the second-order consensus system.

D. Related Work

The notion of network coherence was introduced by Bamieh et al. for both noisy first- and second- order consensus dynamics [24]. There are many works focusing on the first-order network coherence. Young et al. [27] derived analytical formulas for first-order network coherence of cycles, paths, and star graphs. Patterson and Bamieh gave exact expressions for the first-order network coherence of some fractal trees [25], as well as tori and lattices [24] of different fractal dimensions. Some co-authors of the present paper presented explicit solutions to the first-order network coherence in Farey graphs [30], Koch graphs [31], self-similar hierarchical graphs [19] and Sierpiński graphs [19]. These works unveiled some non-trivial effects of network architecture on first-order network coherence. In a recent paper [32], the upper and lower bounds of the first-order network were provided for an arbitrary graph, where the lower bound can be approximately reached in most real-world networks.

Relatively to the first-order case, related works about second-order network coherence \( H_{SO} \) are much less, with the exception of few particular graphs, such as tori [24], fractal trees [25], Koch networks [31], hierarchical graphs [19] and Sierpiński graphs [19]. Very recently, Yi, Zhang, and Patterson [26] established a connection between the biharmonic distance of a graph and its second-order network coherence. They provided exact solutions to second-order network coherence of complete graphs, star graphs, cycles, and paths. However, these studied graphs can not well mimic real-world networks,
most of which are sparse, displaying simultaneously scale free, small-world and loopy structures. Thus far, it has not been unexplored how the second-order coherence behaves in networks with these general properties. Particularly, there is no exact result about second-order coherence in scale-free small-world and loopy networks.

In the sequel, we study the second-order network coherence $H_{SO}$ for scale-free, small-world and loopy networks. First we experimentally study various real-world networks with scale-free small-world structure and loops of different length. Then, we derive an exact expression for $H_{SO}$ of a family of scale-free small-world and loopy networks, the PSFWs to be detailed below. We show that their $H_{SO}$ behaves sublinear with the vertex number $N$. Finally, we obtain an explicit expression for $H_{SO}$ of loopy Sierpiński gaskets, which are neither scale-free nor small-world, but have the same number of vertices and edges as those of PSFWs. We found that the $H_{SO}$ of Sierpiński gaskets scales superlinearly with $N$. We argue that the observed sublinear scaling lies in the aggregation of scale-free, small-world, and loopy properties of the studied networks.

III. COHERENCE OF SOME REAL NETWORKS

In this section, we evaluate the second-order coherence for 26 real-world networks chosen from the Koblenz Network Collection [48], which are scale-free, small-world, and loopy. All these realistic networks are typical and representative, including different types of networks such as information networks, social networks, metabolic networks, and technological networks. Table I summarizes the information of the 26 networks, listed in increasing order of their number of vertices.

Using formula (8), we determine the second-order coherence $H_{SO}$ for the largest connected component of each studied network, as shown in Fig. 1. From this figure, we can observe that, for all networks of different sizes, their second-order coherence $H_{SO}$ is approximately a sublinear function of their vertex number $N$, that is, $H_{SO} \sim (N')^{\alpha}$, with $0 < \alpha < 1$. This is in sharp contrast to the first-order coherence $H_{FO}$, which tends to small constants much less than 1, and is independent of $N'$ [32].

IV. NETWORK COHERENCE IN PSEUDOFRACTAL SCALE-FREE NETWORKS

In this section, we study analytically the coherence for a family of deterministic scale-free model networks, called pseudofractal scale-free webs [39–41], which display some remarkable properties as observed in many real networks. It is thus expected that the behavior of the network coherence is similar to that for real networks.

A. Network Construction and Properties

The pseudofractal scale-free webs (PSFWs) are generated in an iterative way. We denote by $G_n$, the pseudofractal scale-free network after $n(n \geq 0)$ iterations. For $n = 0$, $G_0$ is a triangle consisting of three vertices and three edges. When $n \geq 1$, $G_n$ is obtained from $G_{n-1}$ as follows. Every existing edge in $G_{n-1}$ introduces a new vertex connected to both ends of the edge. Figure 2 illustrates the construction process for the first three generations.

Let $N_n$ and $E_n$ denote, respectively, the number of vertices and the number of edges in $G_n$. It is easy to verify that $N_n = 3^{n+1} + 3$ and $E_n = 3^{n+1}$. In network $G_n$, the three vertices generated at $n = 0$ have the largest degree $2^{n+1}$, which are called hub vertices and are denoted by $A_n$, $B_n$, and $C_n$, respectively.

The PSFWs are self-similar, which suggests another construction method highlighting their self-similar structure. This approach creates the networks as follows. Given the $n$th generation network $G_n$, the $(n+1)$th generation $G_{n+1}$ is obtained by joining three copies of $G_n$ at their hubs, see Fig. 3. Let $G_n^{(\theta)}$, $\theta = 1, 2, 3$, represent the three replicas of $G_n$, and let $A_n^{(1)}$, $B_n^{(1)}$, and $C_n^{(1)}$, $\theta = 1, 2, 3$, represent the three hub vertices of $G_n^{(\theta)}$, respectively. Then, $G_{n+1}$ can be generated by merging $G_n^{(\theta)}$, $\theta = 1, 2, 3$, with $A_n^{(1)}$ and $B_n^{(3)}$ being identified as $A_{n+1}$, $B_n^{(2)}$, and $C_n^{(1)}$ being identified as $B_{n+1}$, and $A_n^{(2)}$ and $C_n^{(3)}$ being identified as $C_{n+1}$.

The PSFWs exhibit some typical properties of realistic networks. They are scale-free, with the degree distribution $P(d)$ obeying a power law form $P(d) \sim d^{-0.6+3ln3/ln2}$ [49]. They are also small-world, with the average distance scaling logarithmically with $N_n$. Moreover, they are highly clustered, with the average clustering coefficient converging to $0.29$. Finally, they have many cycles of different lengths, the distribution of which is studied in [37].
TABLE I  
Statistics of 26 realistic networks. For a network with $N$ vertices and $M$ edges, we represent the number of vertices and edges in its largest connected component by $N'$ and $M'$, respectively. $d'$ represents the average degree of the largest connected component, equalling $2M'/N'$. $\gamma$ denotes the power-law exponent. $I$ is the average shortest path distance.

| Network                      | $N$ | $M$ | $N'$ | $M'$ | $d'$ | $\gamma$ | $I$ |
|------------------------------|-----|-----|------|------|------|---------|-----|
| Karate                       | 34  | 78  | 34   | 78   | 4.588| 2.161   | 2.408|
| Windsurfers                  | 43  | 336 | 43   | 336  | 15.628| 4.001  | 1.671|
| Dolphins                     | 62  | 159 | 62   | 159  | 5.129| 5.001   | 3.357|
| Lesmis                       | 77  | 254 | 77   | 254  | 6.597| 1.521   | 2.641|
| Adjnoun                      | 112 | 425 | 112  | 425  | 7.589| 3.621   | 2.536|
| ElectronicCircuit(S208)      | 122 | 189 | 122  | 189  | 3.098| 4.161   | 4.928|
| ElectronicCircuit(S420)      | 252 | 399 | 252  | 399  | 3.167| 4.021   | 5.806|
| C. elegans                   | 297 | 2,148| 297 | 2,148| 14.465| 2.101  | 3.077|
| HamsterFull                  | 2,426| 16,631 | 2,000 | 16,098 | 16,098 | 2.421 | 3.588|
| WordAdjacency(JAP)           | 2,704| 7,998| 2,698| 7,995| 5.927| 2.101   | 3.077|
| FacebookNIPS                 | 4,039| 88,234| 4,039| 88,234| 43.691| 2.501  | 3.093|
| GrQc                         | 5,242| 14,484| 5,138| 14,422| 6.456| 2.121   | 6.049|
| Reactome                     | 6,327| 146,160| 5,973| 145,778| 48.812| 1.741  | 4.214|
| RouteViews                   | 6,474| 12,572| 6,474| 12,572| 3.884| 2.141   | 3.705|
| WordAdjacency(SPA)           | 7,381| 44,207| 7,377| 44,205| 11.985| 2.201  | 2.778|
| HighEnergy                   | 7,610| 15,751| 5,835| 13,815| 4.735| 3.441   | 7.026|
| HepTh                        | 9,875| 25,973| 8,638| 24,806| 5.743| 5.481   | 5.945|
| Biecatalog                   | 10,312| 333,983| 10,312| 333,983| 64.776| 2.081  | 2.582|
| PrettyGoodPrivacy             | 10,680| 24,316| 10,680| 24,316| 4.261| 2.461   | 7.486|
| HepPh                        | 12,006| 118,489| 11,204| 117,619| 20.996| 2.081  | 4.673|
| AstroPh                      | 18,772| 198,050| 17,903| 196,972| 22.004| 2.861  | 4.194|
| CAIDA                        | 26,475| 53,381| 26,475| 53,381| 4.033| 2.101   | 3.876|
| EnronEmail                   | 36,692| 183,831| 33,696| 180,811| 10.732| 1.981  | 4.025|
| CondensedMatter              | 39,577| 175,692| 36,458| 171,735| 9.421| 3.681   | 5.499|
| Brightkite                   | 58,228| 214,078| 56,739| 212,945| 7.353| 2.501   | 4.917|

Fig. 3. Second construction of pseudo fractal scale-free webs, highlighting the self-similar property.

B. Exact Solutions and Scalings for Network Coherence

Let $L_n$ denote the Laplacian matrix of network $G_n$, with a unique zero eigenvalue $\lambda_1(n)$ and $N_n - 1$ nonzero eigenvalues $\lambda_2(n), \lambda_3(n), \ldots, \lambda_{N_n}(n)$. Let $HFO(n)$ and $HSO(n)$ represent, respectively, the first-order network coherence and second-order network coherence of $G_n$. To determine $HFO(n)$ and $HSO(n)$, we define two quantities $S_n$ and $T_n$ by $S_n = \sum_{i=2}^{N_n} \lambda_i(n)$ and $T_n = \sum_{i=2}^{N_n} \lambda_i^2(n)$. Then, $HFO(n) = S_n/(2N_n)$ and $HSO(n) = T_n/(2N_n^2)$. We next find $S_n$ and $T_n$.

1) Recursive Relations for Related Polynomials and Quantities: To determine $S_n$ and $T_n$, we introduce some quantities. Let $P_n(\lambda)$ denote the characteristic polynomial of matrix $L_n$, i.e.,

$$P_n(\lambda) = \det(L_n - \lambda I_n),$$

where $I_n$ is the $N_n \times N_n$ identity matrix. Let $Q_n$ be an $(N_n - 1) \times (N_n - 1)$ submatrix of $(L_n - \lambda I_n)$, obtained by removing from $(L_n - \lambda I_n)$ the row and column corresponding to a hub vertex in $G_n$. Let $R_n$ represent a submatrix of $(L_n - \lambda I_n)$ with an order $(N_n - 2) \times (N_n - 2)$, obtained from $(L_n - \lambda I_n)$ by removing from it two rows and columns corresponding to two hub vertices in $G_n$. Let $X_n$ represent a submatrix of $(L_n - \lambda I_n)$ with an order $(N_n - 1) \times (N_n - 1)$, obtained from $(L_n - \lambda I_n)$ by removing from it one row corresponding to a hub vertex and one column corresponding to another hub vertex. Moreover, let $Q_n(\lambda), R_n(\lambda), X_n(\lambda)$ denote, respectively, the determinants of $Q_n, R_n$ and $X_n$. As will be shown later, $S_n$ and $T_n$ can be expressed in terms of the coefficients of some related polynomials.

Lemma IV.1. For any nonnegative integer $n$,

$$P_{n+1}(\lambda) = 2Q_n(\lambda)^2 + 6P_n(\lambda)Q_n(\lambda)R_n(\lambda) +$$

$$9\lambda Q_n(\lambda)^2 R_n(\lambda) + 3\lambda P_n(\lambda)R_n(\lambda)^2 +$$

$$6\lambda^2 Q_n(\lambda)R_n(\lambda)^2 + \lambda^3 R_n(\lambda)^3 + 2X_n(\lambda)^3, \tag{10}$$

$$Q_{n+1}(\lambda) = 3Q_n(\lambda)^2 R_n(\lambda) + P_n(\lambda)R_n(\lambda)^2 +$$

$$4\lambda Q_n(\lambda)R_n(\lambda)^2 + \lambda^2 R_n(\lambda)^3, \tag{11}$$

$$R_{n+1}(\lambda) = 2R_n(\lambda)^2 Q_n(\lambda) + \lambda R_n(\lambda)^3, \tag{12}$$

$$X_{n+1}(\lambda) = 2Q_n(\lambda)R_n(\lambda)X_n(\lambda) + \lambda R_n(\lambda)^2 X_n(\lambda) -$$

$$R_n(\lambda)X_n(\lambda)^2. \tag{13}$$

Proof: By definition, $P_{n+1}$ can be expressed as

$$P_{n+1}(\lambda) = \begin{vmatrix}
2^{n+2} - \lambda & -1 & -1 & s_n & s_n & 0 \\
-1 & 2^{n+1} - \lambda & -1 & 0 & t_n & 0 \\
0 & -1 & 2^{n+1} - \lambda & 0 & 0 & t_n \\
0 & 0 & 0 & R_n & 0 & 0 \\
s_n & s_n & 0 & t_n & R_n & 0 \\
0 & 0 & 0 & t_n & R_n & 0
\end{vmatrix}, \tag{14}$$
where \(2^{n+2}\) denotes the degree of the hub vertices \(A_{n+1}, B_{n+1},\) and \(C_{n+1}\) in network \(G_{n+1}^1; s_n(\ell_n)\) is a vector of order \(N_n - 2\) with \(2^{n+1} - 1\) nonzero entries \(-1/2\) and \(N_n - 2^{n+1} - 1\) zero entries, in which each \(-1\) describes the connection between the hub vertex \(A_{n+1}(B_{n+1})\) and vertices belonging to \(G_{n+1}^1, G_{n+1}^2\) or \(G_{n+1}^3;\) the superscript \(T\) of a vector represents its transpose.

In a similar way, we obtain

\[
Q_{n+1}(\lambda) = \begin{bmatrix}
2^{n+2} - \lambda & -1 & t_n & 0 & s_n \\
-1 & 2^{n+2} - \lambda & 0 & t_n & t_n \\
0 & t_n & R_n & 0 & 0 \\
0 & t_n^T & 0 & R_n & 0 \\
S_n^T & t_n^T & 0 & 0 & R_n
\end{bmatrix},
\]  

(15)

\[
R_{n+1}(\lambda) = \begin{bmatrix}
2^{n+2} - \lambda & 0 & t_n & t_n \\
0 & R_n & 0 & 0 \\
0 & t_n & R_n & 0 \\
0 & t_n^T & 0 & R_n
\end{bmatrix},
\]  

(16)

\[
X_{n+1}(\lambda) = \begin{bmatrix}
-1 & -1 & t_n & 0 & s_n \\
-1 & 2^{n+2} - \lambda & 0 & t_n & t_n \\
S_n^T & t_n^T & 0 & R_n & 0 \\
S_n^T & t_n^T & 0 & R_n & 0 \\
S_n^T & t_n^T & 0 & R_n & 0 \\
S_n^T & t_n^T & 0 & R_n & 0
\end{bmatrix},
\]  

(17)

In the sequel, we will show how to derive the recursive relations for \(P_{n+1}(\lambda), Q_{n+1}(\lambda), R_{n+1}(\lambda),\) and \(X_{n+1}(\lambda)\) by the Laplace theorem, we have

\[
P_{n+1}(\lambda) = \begin{bmatrix}
2^{n+1} - \lambda & -1 & 0 & s_n & 0 & 0 \\
-1 & 2^{n+1} - \lambda & -1 & t_n & 0 & s_n \\
0 & S_n^T & t_n^T & 0 & R_n & 0 \\
0 & S_n^T & t_n^T & 0 & R_n & 0 \\
0 & S_n^T & t_n^T & 0 & R_n & 0 \\
0 & S_n^T & t_n^T & 0 & R_n & 0
\end{bmatrix},
\]  

(18)

According to the properties of determinants, it is straightforward to obtain (10) from (18) by using the approach in [50]. Similarly, we can derive (11), (12), and (13).

2) Analytical Solutions for Intermediary Quantities: Having derived the recursive relations for the above four characteristic polynomial \(P_n(\lambda), Q_n(\lambda), R_n(\lambda),\) and \(X_n(\lambda),\) we now determine the coefficients of \(P_n(\lambda),\) Define \(p_n^{(i)}(0 \leq i \leq 2)\) as the coefficient of the term \(\lambda^i\) corresponding to \(S_n^T(\lambda)\). Then, \(p_n^{(0)}\) denotes the constant item, \(p_n^{(1)}\) and \(p_n^{(2)}\) are the coefficients of the terms with degree 1 and 2, respectively. According to Vieta’s formulas, we obtain

\[
S_n = \sum_{i=2}^{N_n} \frac{1}{\lambda_i(n)} = -\frac{p_n^{(1)}}{p_n^{(0)}},
\]  

(19)

\[
T_n = \sum_{i=2}^{N_n} \frac{1}{\lambda_i(n)} = \left(\sum_{i=2}^{N_n} \frac{1}{\lambda_i(n)}\right)^2 - 2 \sum_{2 \leq i < j \leq N_n} \frac{1}{\lambda_i(n)\lambda_j(n)} = S_n^2 - 2p_n^{(2)} \frac{p_n^{(0)}}{p_n^{(0)}},
\]  

(20)

Thus, the problem of determining \(S_n\) and \(T_n\) is reduced to determining \(p_n^{(0)}, p_n^{(1)},\) and \(p_n^{(2)}\). In order to find these three coefficients, we introduce some additional quantities. Let \(q_n^{(i)}, r_n^{(i)}, x_n^{(i)}, 0 \leq i \leq 3,\) be the coefficients of term \(\lambda^i\) corresponding to \(Q_n(\lambda), R_n(\lambda),\) and \(X_n(\lambda),\) respectively.

Lemma IV.2. For any nonnegative integer \(n,\)

\[
p_n^{(0)} = -2^{\frac{1}{4}}(-7t^{3+1}+n-2n)3^{\frac{1}{2}}(5t^{3+1}+2n+2)(1+3n),
\]  

(21)

\[
q_n^{(0)} = 2^{-\frac{3}{2}} + \frac{3^{1+n}}{t^{3+n}} - \frac{3^{1+n} + 2}{3^{1+n} + 2},
\]  

(22)

\[
r_n^{(0)} = 2^{\frac{1}{2}} + \frac{1^{1+n}}{t^{3+n}} + \frac{2}{3^{1+n} + 2},
\]  

(23)

\[
x_n^{(0)} = -2^{\frac{1}{4}}(-7t^{3+1}+n-2n)3^{\frac{1}{2}}(7t^{3+1}+6n)
\]  

(24)

\[
p_n^{(1)} = \frac{1}{7}2^{\frac{1}{2}}(-11t^{3+1}+n-2n)3^{\frac{1}{2}}(1+3t^{3+1}+2n)\times
\]  

\[
\left(25 \times 2^n - 7 \times 3^n + 8 \times 3^{1+2n} + 25 \times 3^{1+3n} + 5 \times 6^{1+n} - 35 \times 18^n\right),
\]  

(25)

\[
q_n^{(1)} = \frac{1}{7}2^{\frac{1}{4}}(-11t^{3+1}+n-2n)3^{\frac{1}{2}}(-3t^{3+1}+2n)\times
\]  

\[
\left(-11 \times 2^{2+n} + 7 \times 3^n - 25 \times 3^{1+2n}\right),
\]  

(26)

\[
r_n^{(1)} = \frac{1}{7}2^{\frac{1}{2}}(-7t^{3+1}+n+2n)3^{\frac{1}{2}}(-7t^{3+1}+n-6n)\times
\]  

\[
(3 \times 2^{2+n} - 25 \times 3^{1+2n} + 7 \times 3^n (-1 + 2^{2+n}),
\]  

(27)

\[
x_n^{(1)} = \frac{1}{7}2^{\frac{1}{2}}(-11t^{3+1}+n-2n)3^{\frac{1}{2}}(-3t^{3+1}+2n)\times
\]  

\[
(21^{2+n} - 7 \times 3^n + 25 \times 3^{1+2n} - 7 \times 6^{1+n}),
\]  

(28)

\[
p_n^{(2)} = \frac{1}{5635}2^{\frac{1}{2}}(-27t^{3+1}+n-2n)3^{\frac{1}{2}}(-7t^{3+1}+6n)\times
\]  

\[
\left(-41 \times 2^{7+2}n^3 + 9775 \times 2^{1+3n} + 3^{2+n} + 129283 \times 3^{1+4n} - 71875 \times 3^{1+5n} + 9039 \times 4^{2+n} - 20125 \times 6^{1+n} + 93541 \times 9^n + 79373 \times 4^{2+n} + 417163 \times 9^{1+n} + 100625 \times 2^{1+n} 9^{1+2n} - 64975 \times 5^{1+n}\right),
\]  

(29)

\[
q_n^{(2)} = \frac{1}{5635}2^{\frac{1}{2}}(-3-3t^{3+1}+2n)3^{\frac{1}{2}}(-11t^{3+1}+6n)\times
\]  

\[
(8855 \times 2^{3+n}9^{1+n} - 1127 \times 2^{7+2}n^3 3^{1+n} + 71875 \times 3^{1+4n} - 18819 \times 4^{2+n} - 93541 \times 9^n - 61985 \times 4^{2+n}9^n + 31625 \times 3^{1+2n}9^{1+n}\)
that in [50], it is not difficult to derive the following recursive
proof:

\[ p_n^{(0)} = 6p_n q_n r_n + 9[q_n r_n]^2 r_n + 6[r_n q_n]^2 q_n + 6[x_n]^2 x_n, \]
\[ q_n^{(0)} = 3[q_n r_n]^2 r_n, \]
\[ r_n^{(0)} = 2q_n r_n^2, \]
\[ x_n^{(0)} = 2n x_n + r_n x_n - r_n x_n^2, \]
\[ p_n^{(1)} = 3p_n r_n^2 + 9[q_n r_n]^2 r_n + 6[q_n r_n]^2 q_n + 6[p_n r_n]^2 p_n + 18q_n r_n q_n r_n + 6(q_n^2 q_n)^2 + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6[r_n q_n]^2 q_n + 6[q_n^2 q_n]^2 q_n + 6[x_n]^2 x_n, \]
\[ q_n^{(1)} = p_n r_n^2 + 4q_n r_n^2 r_n + 6n r_n q_n + \frac{1}{2} q_n^2 q_n^3, \]
\[ r_n^{(1)} = [r_n]^3 + 2[r_n]^2 q_n + 4q_n r_n r_n, \]
\[ x_n^{(1)} = [r_n(x_n)^2 q_n + 2q_n(x_n)^2 q_n r_n + 6(p_n)^2 0_n + q_n q_n r_n + 2[q_n r_n]^2 r_n + 6p_n p_n r_n + 12q_n r_n p_n r_n + 6(q_n + q_n)^2 q_n + 18q_n r_n q_n r_n + 6(q_n^2 q_n)^2 + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6(r_n q_n)^2 q_n + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6[r_n q_n]^2 q_n + 6[q_n^2 q_n]^2 q_n + 6[x_n]^2 x_n, \]
\[ p_n^{(2)} = [r_n]^3 + 3[r_n]^2 q_n + 6[r_n q_n]^2 q_n + \frac{1}{2} q_n^2 q_n^3, \]
\[ q_n^{(2)} = [r_n]^3 + 3[r_n]^2 q_n + 4[r_n q_n]^2 q_n + 3[r_n]^3 q_n + \frac{1}{2} q_n^2 q_n^3, \]
\[ r_n^{(2)} = 3[r_n]^2 q_n + 4[q_n r_n]^2 r_n + 2q_n r_n r_n + 4p_n r_n p_n r_n + 12q_n r_n p_n r_n + 6(q_n + q_n)^2 q_n + 18q_n r_n q_n r_n + 6(q_n^2 q_n)^2 + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6(r_n q_n)^2 q_n + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6[r_n q_n]^2 q_n + 6[q_n^2 q_n]^2 q_n + 6[x_n]^2 x_n, \]
\[ x_n^{(2)} = 2r_n x_n + 3r_n q_n + 4r_n q_n r_n + 2q_n r_n + 4p_n r_n p_n r_n + 12q_n r_n p_n r_n + 6(q_n + q_n)^2 q_n + 18q_n r_n q_n r_n + 6(q_n^2 q_n)^2 + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + \frac{3}{2} q_n^2 q_n^3, \]
\[ q_{n+1}^{(3)} = 3[r_n]^2 q_n + 2r_n p_n q_n + 8p_n q_n^2 r_n + 3[p_n]^2 q_n + 4q_n r_n^2 q_n + 3[r_n]^2 q_n + 4[r_n q_n]^2 r_n + 2q_n r_n r_n + 4p_n r_n p_n r_n + 12q_n r_n p_n r_n + 6(q_n + q_n)^2 q_n + 18q_n r_n q_n r_n + 6(q_n^2 q_n)^2 + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6(r_n q_n)^2 q_n + 6q_n r_n q_n r_n + 9[q_n r_n]^2 r_n + 6[r_n q_n]^2 q_n + 6[q_n^2 q_n]^2 q_n + 6[x_n]^2 x_n. \]

The above equations are solved to give

\[ q_n = \frac{r_{n+1}}{r_n^2}, \]
\[ r_n = \frac{3}{2} n, \]
\[ k_n = 2 \left( \frac{3}{2} \right)^n. \]
With this exact result of \( k_n \), (50) is recast as
\[
\ln r_n^{(0)} = 3 \ln r_{n-1}^{(0)} + \ln k_n^{(0)}.
\]
(53)

Considering \( \ln r_0^{(0)} = \ln 2 \) and the expression for \( k_n \) given in (52), (53) is solved inductively to yield
\[
\ln r_n^{(0)} = 3^n \ln r_0^{(0)} + \sum_{i=0}^{n-1} 3^i \ln \left[ 2 \left( \frac{3}{2} \right)^{i+1} \right],
\]
(54)

which implies
\[
r_n^{(0)} = 2^{\frac{3}{4} + \frac{3^i + n}{4} + \frac{3^i}{4}} 3^{\frac{3}{4} + \frac{3^i + n}{4} + \frac{3^i}{4}},
\]
as shown in (23).

After deriving \( r_n^{(0)} \), we continue to calculate \( q_n^{(0)} \). Plugging (23) into (48) gives
\[
q_n^{(0)} = 2 \left( 2^{\frac{3}{4} + \frac{3^i + n}{4} - \frac{3^i}{4}} 3^{\frac{3}{4} + \frac{3^i + n}{4} + \frac{3^i}{4}} \right) - 1.
\]
(55)

In this way, we obtain (22). Finally, we determine \( x_n^{(0)} \). By inserting (34) into (36), one obtains
\[
x_n^{(0)} = \frac{2 x_{n+1}^{(0)}}{3 q_{n+1}^{(0)}} - \frac{1}{3} \left( \frac{x_n^{(0)}}{q_n^{(0)}} \right)^2.
\]
(56)

In order to obtain the exact expression for \( x_n^{(0)} \), we introduce another quantity \( y_n \) defined by
\[
y_n = \frac{x_n^{(0)}}{q_n^{(0)}}.
\]
(57)

Then, (55) can be rewritten as
\[
y_{n+1} = \frac{2}{3} y_n - \frac{1}{3} y_n^2,
\]
which, under the initial condition \( y_0 = x_0^{(0)}/q_0^{(0)} = -1 \), is solved to yield
\[
y_n = -1.
\]
(58)

Combine (58), (56) and \( r_n^{(0)} \), we obtain
\[
x_n^{(0)} = -2^{\frac{3}{4} + \frac{3^i + n}{4} + \frac{3^i}{4}} 3^{\frac{3}{4} + \frac{3^i + n}{4} + \frac{3^i}{4} + \frac{3^i}{4}},
\]
as provided by (24).

3) Explicit Expression and Behavior for Network Coherence: With the above-obtained quantities, we can get accurate solutions for both the first-order and the second-order network coherence of network \( G_n \), from which we can further reveal their asymptotical behaviors.

**Theorem IV.3.** For the PSFW \( G_n \) with \( n \geq 1 \), the first-order coherence \( H_{FO}(n) \) and the second-order coherence \( H_{SO}(n) \) of the system with dynamics in (5) and (8) are
\[
H_{FO}(n) = \frac{1}{28 (1 + 3^n)^2 3^{2^n}} \times (2^{2n} - 7 \times 3^n + 8 \times 3^{1 + 2n} + 25 \times 3^{1 + 3n} + 5 \times 6^{1 + n} - 35 \times 18^n),
\]
(59)
\[
H_{SO}(n) = \frac{90160 (1 + 3^n)^3}{(1 + 3^n)^3} \times (69538 \times 3^{2 + 5n} + 360249 \times 4^n + 35 \times 2^{2 + 5n} 3^{1 + n} (-575 + 1539 \times 2^n) + 322 \times 2^n (1135 - 3225 \times 2^{1 + n} + 847 \times 2^{1 + 2n}) + 3^{1 + 4n} (51262 - 60375 \times 2^{2 + n} + 140875 \times 4^n) + 2 \times 9^n (55223 - 94875 \times 2^{1 + n} + 480487 \times 4^n).)
\]
(60)

Moreover,
\[
\lim_{n \to \infty} H_{FO}(n) = \frac{25}{84},
\]
(61)
\[
\lim_{n \to \infty} H_{SO}(n) = \frac{25}{432} (N_n)^{(\log_3 4) - 1}.
\]
(62)

**Proof:** Since \( H_{FO}(n) = S_n/(2N_n) \) and \( H_{SO}(n) = T_n/(2N_n) \), combining the above-obtained related quantities and using (19) and (20), we obtain (59) and (60), which lead to (61) and (62) for sufficiently large \( n \).

Notice that (59) and (61) were previously obtained in [32] by using a different technique. However, (60) and (62) are novel. Theorem IV.3 shows that in large networks \( G_n \), the first-order coherence \( H_{FO}(n) \) does not depend on \( n \), thus not on \( N_n \), while the second-order coherence \( H_{SO}(n) \) increases sublinearly with \( N_n \).

V. NETWORK COHERENCE IN SIERPIŃSKI GASKETS

In the previous section, we obtained an explicit formula of network coherence \( H_{FO}(n) \) for PSFW \( G_n \), and showed that \( H_{FO}(n) \) is a sublinear function of \( N_n \). To unveil the effect of scale-free and small-world topologies on the scaling of network coherence, in this section we derive an analytical expression for network coherence in Sierpiński gaskets with the same numbers of vertices and edges as those of PSFWs. We will show that the leading scalings of both first-order and second-order network coherence for \( S_n \) are significantly different from those associated with \( G_n \).

The Sierpiński gaskets are also iteratively constructed. Let \( S_n (n \geq 0) \) denote the networks after \( n \) iterations. Then, the Sierpiński gaskets are generated as follows. When \( n = 0 \), \( S_0 \) is an equilateral triangle with three vertices and three edges. For \( n \geq 1 \), the three edges of the equilateral triangle \( S_0 \) are bisected and the central triangle is removed, yielding \( S_1 \) containing three copies of the original triangle. For \( n > 0 \), \( S_n \) is generated from \( S_{n-1} \) by performing the bisection and removal procedure for each upward pointing triangle in \( S_{n-1} \). Fig. 4 illustrates the first three generations of Sierpiński gaskets.

Both the number of vertices and the number of edges in \( S_n \) are the same as those of \( G_n \). That is, there are \( N_n = \frac{3^{n+1}}{2} + 3 \) vertices and \( E_n = 3^{n+1} \) edges in \( S_n \). Many other properties of \( S_n \) and \( G_n \) are quite different from each other. For example, Sierpiński gaskets are neither scale-free nor small-world. They are homogeneous, with the degrees of the three outmost

![Fig. 4. The first three generations of Sierpiński gaskets.](image-url)
vertices equal to 2, while the the degrees of other vertices being 4.

Despite the difference between $S_n$ and $G_n$, there are some similarity between them. For instance, both graphs have cycles of various lengths. Moreover, Sierpiński gaskets are likewise self-similar, as can be seen from the following alternative construction method. We denote the three outmost vertices in $S_n$ with degree 2 by $A_n$, $B_n$, and $C_n$, respectively. Then $S_{n+1}$ is obtained from $S_n$ by joining three copies of $S_n$ at their outmost vertices, as shown in Fig. 5. Let $S_n^{(θ)}$, $θ = 1, 2, 3$, represent the three replicas of $S_n$, with outmost vertices $A_n^{(θ)}$, $B_n^{(θ)}$, and $C_n^{(θ)}$. Then, $S_{n+1}$ is created by coalescing $S_n^{(θ)}$, $θ = 1, 2, 3$, with $A_n^{(1)}, B_n^{(2)}$, and $C_n^{(3)}$ being the three outmost vertices of $S_{n+1}$.

Let $H_{FO}(n)$ and $H_{SO}(n)$ denote, respectively, the first-order network coherence and second-order network coherence for $S_n$. By using a similar method and procedure, we can obtain exact solutions for $H_{FO}(n)$ and $H_{SO}(n)$ and the leading scalings for $S_n$, as summarized in the following theorem.

**Theorem V.1.** For the Sierpiński gasket $S_n$ with $n ≥ 1$, the first-order coherence $H_{FO}(n)$ and the second-order coherence $H_{SO}(n)$ of the system with dynamics in (5) and (8) are

$$H_{FO}(n) = \frac{1}{20(3^{2n}+1+3^n)2(4 \times 3^n+2 \times 3^{2n}+2+3^{1+3^n}+13 \times 3^{1+3^n}+5^n+4 \times 5^{1+n}+14 \times 45^n)}.$$

$$H_{SO}(n) = \frac{1}{400(1+3^n)3^3g^{2n}(86 \times 3^{1+4^n}-2 \times 3^{2+5n}+75 \times 3^{1+2n}5^n+568 \times 3^{1+3n}n^5+32 \times 3^{2+n}5^{1+n}+119 \times 9^n+28 \times 5n9^{1+2n}+64 \times 15^{1+n}+8 \times 9^{1+2n}25^n+24 \times 25^{1+n}+320 \times 27^n+1237 \times 225^n+394 \times 675^n)}$$

Moreover,

$$\lim_{n \to \infty} H_{FO}(n) = \frac{7}{90}(N_n^{(log_35)^{-1}})$$

and

$$\lim_{n \to \infty} H_{SO}(n) = \frac{1}{450}(N_n^{(log_325)^{-1}}).$$

Theorem V.1 shows that the behaviors of both first-order coherence $H_{FO}(n)$ and second-order coherence $H_{SO}(n)$ in Sierpiński gaskets significantly differ from those of PSFWs. For large Sierpiński gaskets $S_n$, the first-order coherence $H_{FO}(n)$ increases sublinearly with $N_n$, while the second-order coherence $H_{SO}(n)$ behaves superlinearly with $N_n$.

In Fig. 6, we report a direct comparison of approximate and exact results about first-order coherence $H_{FO}(n)$ and second-order coherence $H_{SO}(n)$ in PSFWs $G_n$ and Sierpiński gaskets $S_n$ for various $n$. For moderately large $n$, the exact and approximate results agree with each other.

**VI. RESULT ANALYSIS**

In the previous sections, we have investigated the noisy second-order consensus dynamics of some real-life scale-free networks and a class of scale-free model networks. For all these studied scale-free networks, their second-order network coherence scales sublinearly with the number of vertices, $N$. Note that for a network, its second-order coherence is completely determined by the sum of the square reciprocal of every non-zero eigenvalue of its Laplacian matrix. Since for scale-free networks, the eigenvalues and their distributions are closely related to the network structures [51], the sublinear scaling for coherence observed for the considered scale-free networks lies in their intrinsic structural characteristics, particularly the scale-free small-world topology and cycles of different lengths.

As shown in [26], the second-order coherence of a network is determined by the average of biharmonic distances $Θ_{ij}$ over all pairs of vertices. By formulas (1) and (2), for any pair of vertices $i$ and $j$, both resistance distance $Ω_{ij}$ and biharmonic distance $Θ_{ij}$ are combinations of $\frac{1}{N_k}(u_{ki} - u_{kj})^2$, $k = 1, 2, \ldots, N - 1$. For the resistance distance, the weight of each term is 1, while for the biharmonic distance, the weight varies, with a larger term $\frac{1}{N_k}(u_{ki} - u_{kj})^2$ corresponding to a larger weight $\frac{1}{N_k}$. Thus, for most graphs, $Θ_{ij}$ is greater than $Ω_{ij}$. In a scale-free network, the existence of large-degree vertices connected to many other vertices is accompanied by the small-world property, characterized by at most a logarithmic growth average path length [15]. Moreover, for a scale-
free small-world network, its average resistance distance is even smaller, converging to a constant [32]. In contrast to the constant average resistance distance, the average of biharmonic distances is dependent on $N$, scaling sublinearly with $N$. Next, we show that this sublinear scaling is an aggregation of scale-free, small-world and loopy structures, since neither power-law small-world behavior nor cycles alone can ensure a sublinear coherence, but it leads to a linear or superlinear scaling.

It was reported that for the scale-free small-world Koch network [31], its second-order network coherence $H_{SO}$ behaves linearly with the number of vertices, $N$, which is also observed for the small-world hierarchical graphs [19] with an exponential degree distribution. Both Koch networks and hierarchical graphs are highly clustered, but have only small cycles such as triangles, lacking cycles of various lengths. Thus, the existence of cycles of different lengths is necessary for sublinear scaling of $H_{SO}$ in a network. However, cycles do not suffice to guarantee a sublinear scaling of $H_{SO}$. For example, in Sierpiński gaskets, there are cycles of various lengths, but their $H_{SO}$ is a superlinear function of $N$ as given by formula (66). This, in turn, indicates that scale-free small-world topology is only necessary for a sublinear scaling of $H_{SO}$ in a sparse network.

Note that although the small-world is an accompanying phenomenon of the power-law behavior [15], small-world and loopy structures cannot lead to the sublinear scaling of second-order network coherence. For example, using the result in [30] and the technique in this current paper, we can determine the analytical expression of second-order coherence for the Farey graphs, which scales linearly with the number of nodes, being quite different from the sublinear scaling observed for the PSFWs. By construction, Farey graphs are subgraphs of PSFWs, both of which are small-world and highly clustered, with many cycles at different scales. The reason for the scaling distinction of the second-order coherence between Farey graphs and PSFWs lies in, at least partially, the scale-free topology of PSFWs that the Farey graphs do not possess.

VII. CONCLUSION

A large variety of real-world networks are sparse and loopy, and exhibit simultaneously scale-free and small-world features. These structural properties have a substantial influence on different dynamics running on such networks. In this paper, we presented an extensive study on second-order consensus in noisy networks with these properties, focusing on its robustness measured by network coherence that is characterized by the average steady-state variance of the system. We first studied numerically the network coherence for some representative real scale-free networks, which grows sublinearly with the vertex number $N$. We then determined exactly the coherence for a class of deterministic scale-free networks, PSFWs, which is also a sublinear function of $N$. Moreover, we studied analytically the coherence for Sierpiński gaskets with the same numbers of vertices and edges as the PSFWs, the leading scaling of which scales superlinearly in $N$. We concluded that the scale-free, small-world, and loopy structures are responsible for the observed sublinear scaling of coherence for the studied networks.

It should be mentioned that we only addressed second-order noisy consensus on undirected binary networks, concentrating on scale-free, small-world, and loopy properties on the effects of network coherence. Future work should include the following directions. First, it would be of interest to consider second-order noisy consensus on directed [52, 53] and weighted [54, 55] communication graphs, with an aim to explore the influences of one-way action or distribution of edge weights on network coherence. Another direction is to examine second-order noisy linear consensus networks in the presence of time-delay [56, 57]. Moreover, of particular interest is to consider the case that both scalar-valued states of each agent are subject to disturbances. Finally, our method and process for computing the network coherence are only applicable to deterministically growing self-similar networks, it is of great significance to modify or extend them to more general networks.

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