Local Geometric Phase and Quantum State Tomography in a Superconducting Qubit

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We investigate quantum state reconstruction of a superconducting qubit threaded by an Aharonov-Bohm flux, with particular attention to the local geometric phase. A state reconstruction scheme is introduced with a proper account of the local geometric phase generated by Faraday’s law of induction. Our scheme is based on measurement of three complementary quantities, that is, the extra charge and two local currents. Incorporating time-reversal symmetry and the Faraday’s law, we show that the full density matrix can be reconstructed without ambiguity in the choice of gauge. This procedure clearly demonstrates that the quantum Faraday effect plays an essential role in the dynamics of a quantum system that involves Aharonov-Bohm flux.

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Aharonov-Bohm (AB) effect is regarded as a purely topological phenomenon that arises even in the absence of electromagnetic force, as far as the magnetic field is localized inside a loop and vanishes in the region of the electronic path. The situation is different if an AB loop involves a time-dependent flux. Faraday’s law of induction plays a central role in quantum state evolution in the presence of time-dependent magnetic flux, even in the adiabatic limit. The geometric nature of the Faraday-induced phase has been investigated in Ref. 3, which is essential in quantum state dynamics of generic AB loops. It has been predicted that this phase is observable in a flux-switching experiment with double-dot AB loop. In addition, estimating the state of a qubit is an essential ingredient for quantum information processing. The most developed candidate for a solid-state realization is the superconducting qubits. AB flux is an essential control parameter in various types of superconducting qubits. In particular, in a single Cooper pair box (SCB) with two parallel-coupled Josephson junctions, the AB flux penetrating the loop between the two junctions is used to control the effective coupling strength of the two charge states. Also, flux switching may be useful for estimating the qubit state in a SCB. However, as pointed out previously by the author, it is not possible to specify the state evolution of a qubit involving a change in the AB flux, without proper consideration of the Faraday’s law of induction. It is rather puzzling that this effect has been widely ignored.

In this Letter, we investigate how quantum state tomography (QST) can be performed for a flux-tunable superconducting qubit, with particular attention to the Faraday-induced local geometric phase. We introduce a QST scheme with detection of the local charge and the two local currents flowing through each junction. The Faraday-induced local phase plays an essential role in this procedure. Starting from a system with a time-reversal symmetry (that is, with a vanishing external magnetic field), the density matrix can be fully reconstructed without ambiguity, and the phase evolution of its off-diagonal elements is completely determined by Faraday’s law of induction. Notably, this is in strong contrast with the arbitrary (gauge-dependent) local phase in the conventional description of the AB effect. In addition, this procedure provides particular insight for characterizing an equilibrium state.

A superconducting Cooper pair box - We consider a superconducting Cooper pair box (SCB) with two Josephson junctions threaded by a magnetic flux (Fig. 1). This is one of the simplest quantum systems involving the AB phase. The SCB has been extensively studied in the context of quantum information processing, and is one of the best candidates for investigating the Faraday-induced local phase. A theoretical scheme with a SCB describes a state reconstruction with charge detection followed by voltage/flux switching. However, the Faraday effect has not been considered in Ref. 7. In fact, without considering the Faraday effect, the evolution of the qubit state cannot be properly described. As we will show here, characterization of the local phase, which is directly related to Faraday’s law of induction, is essential for a QST. Despite recent progress in realizing the QST with superconducting qubits, the essential role of the Faraday-induced local phase has never been addressed.

A SCB (Fig. 1) is described by the Hamiltonian

\[ H = \sum_{n=0}^{1} E_n |n\rangle \langle n| - \frac{1}{2} \left( \tilde{E}_f |1\rangle \langle 0| + \tilde{E}_f^* |0\rangle \langle 1| \right). \]  

The qubit energy level \( E_n \) (\( n = 0, 1 \)) is \( E_n = E_c (n - n_g)^2 \), where \( E_c \) is the charging energy of a Cooper pair. This level is tunable via the gate-dependent parameter \( n_g \). Josephson energy, \( E_J \), is assumed to be identical for the two junctions, which gives the effective Josephson coupling

\[ \tilde{E}_f = 2E_J e^{i(\varphi_a - \varphi_b)/2} \cos (\varphi/2), \]

where \( \varphi_a (\varphi_b) \) is the local phase shift across the junction \( a(b) \) (Fig. 1). The magnitude of \( \tilde{E}_f \) is controlled by the AB phase \( \varphi = \varphi_a + \varphi_b \), whereas the choice of the two
local phases $\varphi_a$ and $\varphi_b$ is arbitrary. The phase factor $e^{i(\varphi_a-\varphi_b)/2}$ is widely ignored for convenience, which is fine for describing any phenomena with a time-independent flux. However, this phase factor plays a major role in our context. It is useful to rewrite the Hamiltonian in a Bloch-sphere representation (assigning the pseudospin states $|\uparrow\rangle = |0\rangle$, $|\downarrow\rangle = |1\rangle$) as

$$H = -\frac{1}{2} \mathbf{B} \cdot \vec{\sigma},$$  \hspace{1cm} (2a)$$

where

$$\mathbf{B} = \left( \text{Re}(\vec{E}_f), \text{Im}(\vec{E}_f), E_c(1 - 2n_y) \right),$$  \hspace{1cm} (2b)$$

and $\vec{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$. As one can find from Eqs. (1), the Hamiltonian depends on the phase factor $e^{i(\varphi_a-\varphi_b)/2}$. Therefore, variation in the local phases affects the quantum state dynamics, and one can already expect that the local phase variation should be related to some physical process.

**Faraday-induced local phase** - Any choice of $\varphi_a$ and $\varphi_b$ (with the constraint $\varphi_a + \varphi_b = \varphi$) is fine unless a change of the flux is involved. However, when the flux varies in time, even in the adiabatic limit, the Faraday’s law of induction plays a crucial role in the state evolution. This should be considered for the choice of gauge. A change in the flux induces changes in the local phases $\varphi_a$ and $\varphi_b$, as well as the AB phase $\varphi$, namely $\delta \varphi = \delta \varphi_a + \delta \varphi_b$. For the SCB under consideration, we adopt the representation with single-valued time-independent energy levels $E_n (n = 0, 1)$ for each qubit state. This is equivalent to the choice of time-independent scalar potential. Then, we find

$$\delta \varphi_{a(b)} = \frac{2e}{\hbar c} \int_{a(b)} \delta \mathbf{A} \cdot d\mathbf{r},$$  \hspace{1cm} (3)$$

where $\int_{a(b)}$ is the integral along the path $a(b)$. The important point here is that the change in the vector potential $\delta \mathbf{A}$ is not gauge dependent but is proportional to the Faraday-induced momentum kick $\langle \delta \mathbf{p} \rangle$ as

$$\delta \mathbf{A} = -c \int \mathbf{E}_t \, dt = -\frac{c}{e} \delta \mathbf{p},$$  \hspace{1cm} (4)$$

where $\mathbf{E}_t$ denotes the time-dependent contribution of the electric field generated by the Faraday’s induction. Therefore, $\delta \varphi_a(\delta \varphi_b)$ is a gauge-invariant physical quantity, whereas $\varphi_a(\varphi_b)$ itself can be chosen arbitrarily to describe the AB effect.

Basically, the Faraday-induced local phase is determined by the geometry of the system, and can be measured via flux switching and the charge response of the qubit, as also described in Ref. 3 for a double-dot AB loop. This is because the quantum dynamics of the qubit upon flux switching is uniquely determined by Faraday’s law of induction.

**Quantum state reconstruction procedure** - Once (variation of) the local geometric phase is well defined due to the law of Faraday induction, it is possible to carry out tomography of an arbitrary quantum state. Three independent quantities should be measured for qubit state reconstruction; namely, the three components of the pseudospin average $\langle \langle \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 \rangle \rangle$ in the Bloch-sphere representation. This is easily understood because any single-qubit density matrix can be represented by

$$\rho = \frac{1}{2} \sum_{k=0}^{3} \langle \hat{\sigma}_k \rangle \hat{\sigma}_k,$$  \hspace{1cm} (5)$$

where $\hat{\sigma}_0 = 1$ denotes the unit matrix. It is also possible to perform a QST by measuring only one variable $\langle \langle \hat{\sigma}_3 \rangle \rangle$ for instance) followed by appropriate single-bit operations. It can be done for a SCB with a charge detection combined with pseudo-spin rotations, where the pseudo-spin rotation is performed by voltage and flux switching. Here we introduce an alternative, instructive rather than practical, approach. Our scheme does not require the voltage or flux switching necessary for single-bit operations. Instead, a series of direct measurements can be made for the three physical variables. Note that the conclusion we draw here does not depend on the kind of state-reconstruction scheme. In any case, the Faraday-induced phase shift should be included, which is indeed the essential factor determining the off-diagonal components of the density matrix.

Our scheme is based on direct measurement of the three complementary quantities. For a SCB, the three complementary variables correspond to the excess charge in the box and the two local currents across each junction $a$ or $b$. The charge $\hat{q}$

$$\hat{q} = 2e \frac{\partial H}{\partial E_t} = 2e|1\rangle\langle 1|,$$  \hspace{1cm} (6)$$

gives the $z$-component of the pseudospin due to the relation

$$\hat{\sigma}_3 = 1 - \hat{q}/e.$$  \hspace{1cm} (7)$$

The local current $\hat{I}_\alpha$ flowing through junction $\alpha$ ($= a$ or $b$) is given by

$$\hat{I}_\alpha = -\frac{2e}{\hbar} \frac{\partial H}{\partial \varphi_\alpha} = -I_0 \left( \hat{\sigma}_1 \sin \varphi_\alpha + \hat{\sigma}_2 \cos \varphi_\alpha \right),$$  \hspace{1cm} (8)$$

where $I_0 = eE_f/q$ is the current amplitude, and the sign $(-)$ in the equation is for the case $\alpha = a(b)$. This relation is derived from the fact that the local current density $\hat{j}(r)$ of an arbitrary quantum system is obtained from the functional derivative of the Hamiltonian with respect to the vector potential $\mathbf{A}$, as

$$\hat{j}(r) = -\frac{c}{e} \frac{\delta H}{\delta \mathbf{A}(r)}.$$  \hspace{1cm} (9)$$
From Eq. 8, we find
\[ \hat{\sigma}_1 = -\frac{1}{I_0 \sin \varphi} \left( \hat{I}_a \cos \varphi_b + \hat{I}_b \cos \varphi_a \right), \quad (10a) \]
\[ \hat{\sigma}_2 = \frac{1}{I_0 \sin \varphi} \left( \hat{I}_a \sin \varphi_b - \hat{I}_b \sin \varphi_a \right). \quad (10b) \]

The average values of the three quantities, \( \langle \hat{q} \rangle \), \( \langle \hat{I}_a \rangle \), and \( \langle \hat{I}_b \rangle \) provide full information of the average values of the three complementary variables, \( \langle \hat{\sigma}_1 \rangle \), \( \langle \hat{\sigma}_2 \rangle \), and \( \langle \hat{\sigma}_3 \rangle \). Therefore, a complete reconstruction for a given qubit state is possible from Eq. 5.

**Time-reversal symmetry (TRS) and the local phase**

While \( \langle \hat{\sigma}_3 \rangle \) is uniquely determined by measuring excess charge, \( \langle \hat{\sigma}_1 \rangle \) and \( \langle \hat{\sigma}_2 \rangle \) depend on the choice of gauge \( \varphi_a, \varphi_b \). In the Bloch-sphere representation, this is equivalent to the choice of the \( x - y \) axes. However, further constraints on the gauge can be provided by imposing the symmetry of the system. For example, let us start a quantum state reconstruction procedure from the case with TRS. The TRS is achieved when the external magnetic field is zero in all regions of the system. The time-reversed Hamiltonian \( \hat{H} \) is related to the original Hamiltonian of Eqs. 12 as
\[ \hat{H} = \hat{H}(\varphi_a, \varphi_b) = H(-\varphi_a, -\varphi_b). \quad (11) \]

The TRS condition, \( \hat{H} = \hat{H}_0 \), is satisfied by imposing the constraint \( \varphi_a = \varphi_b = 0 \). That is, for a system with TRS, the local phase is uniquely determined, in contrast to an arbitrary choice (with \( \varphi = 0 \)) for describing an AB loop. In fact, this constraint of the local phase is equivalent to the theorem: the non-degenerate eigenfunction of a time-reversal invariant Hamiltonian should be real (more generally, a real function times phase factor independent of position)

Starting from the case with perfect TRS, a complete state reconstruction procedure can be provided as follows: (i) A quantum state \( \rho \) (density matrix) is prepared in the absence of the external magnetic field \( \langle \varphi_a = \varphi_b = 0 \rangle \). (ii) The density matrix \( \rho \) is reconstructed, without ambiguity in the local phase, by measuring \( \langle \hat{q} \rangle \), \( \langle \hat{I}_a \rangle \), and \( \langle \hat{I}_b \rangle \), as described in Eqs. 5, 10. This is possible because the arbitrariness of the local phase is removed by TRS. (iii) Magnetic field is turned on and varied, which modifies the local phases by \( \delta \varphi_a \) and \( \delta \varphi_b \). Note that these local phases are uniquely defined by the Faraday-induced momentum kick (Eq. 9). (iv) The procedure of (i) and (ii) is repeated to provide a full reconstruction of state \( \rho \) as a function of the applied external magnetic field \( \langle \varphi_a \text{ and } \varphi_b \rangle \) are uniquely given. (v) In this way, an arbitrary state can be reconstructed for an arbitrary distribution of the external magnetic field.

**Equilibrium state reconstruction and persistent current**

While the QST described above is valid for any state and is not limited to an equilibrium, it is worth investigating the equilibrium state in relation to the persistent current. The standard definition of the persistent current of an AB loop is given by the derivative of the Hamiltonian with respect to the AB phase,
\[ \dot{I} = -\frac{2e}{\hbar} \frac{\partial H}{\partial \varphi}. \quad (12) \]

which leads to the relation
\[ \dot{I} = -\frac{I_0}{2} (\sin \varphi_a + \sin \varphi_b) \dot{\sigma}_1 + \frac{I_0}{2} (\cos \varphi_a - \cos \varphi_b) \dot{\sigma}_2 = \frac{1}{2} (\dot{I}_a + \dot{I}_b), \quad (13) \]
in our SCB. This is an interesting expression in that the standard definition of the persistent current is equivalent to the average of the two local currents. In fact, the persistent current of the definition in Eq. 12 is meaningful only for an equilibrium state, whereas the local currents \( \dot{I}_a \) and \( \dot{I}_b \) have their direct physical meaning for any quantum state. This also implies that the conventional description of the circulating persistent current is a limiting case of our local-current based scheme. In an equilibrium state, the two local currents should be balanced, \( \langle \dot{I}_a \rangle = \langle \dot{I}_b \rangle \). Naturally, it results in the obvious relation \( \langle I \rangle = \langle \dot{I}_a \rangle = \langle \dot{I}_b \rangle \).

Further, by imposing this equilibrium condition \( \langle \dot{I}_a \rangle = \langle \dot{I}_b \rangle \), we find
\[ \langle \dot{\sigma}_2 \rangle = \langle \dot{\sigma}_1 \rangle \tan \frac{\varphi_a - \varphi_b}{2}. \quad (14) \]

It can be shown by a straightforward evaluation that the qubit state is an incoherent mixture of the two eigenstates \( |+\rangle \) and \( |-\rangle \) under the condition of Eq. 14. That is, the density matrix is reduced to the form
\[ \rho = \rho_{eq} = a_+|+\rangle\langle+| + a_-|\rangle\langle-|, \quad (15) \]
where \( a_\pm \) satisfies \( a_+ + a_- = 1 \) with \( 0 \leq a_\pm \leq 1 \). This result is equivalent to the basic postulate of equilibrium quantum statistical mechanics that the interference between different eigenstates vanishes. Interestingly, this property is not postulated here but derived from equilibration of the local current, and can also be understood in terms of the stationary nature of the eigenstates.

**Conclusion**

In conclusion, the local geometric phase induced by Faraday’s law of induction plays a central role in quantum state tomography of a superconducting qubit in the presence of an external magnetic flux. A state reconstruction scheme has been proposed for a superconducting Cooper pair box which involves a change in the flux. Together with the constraint of time-reversal symmetry and the Faraday-induced local phase, any quantum state can be reconstructed from measuring three complementary quantities, without ambiguity in the gauge dependence. It is also important to note that our conclusion is not limited to the specific case of a superconducting qubit but can be widely applied to any
quantum system that involves a change in magnetic flux, which calls for further study.

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[11] To be precise, \( \varphi_a = -\varphi_b = \pi \) is an equally possible choice satisfying both TRS and \( \varphi = 0 \). However, this is equivalent to the case with negative \( E_J \) and \( \varphi_a = \varphi_b = 0 \).
[12] Note that this is not in contradiction with the freedom of gauge choice (that gives an arbitrary choice of the local phase) in the standard description of the Aharonov-Bohm effect with time-independent magnetic flux.
[13] See e.g., J. J. Sakurai, Modern Quantum Mechanics, revised ed., p.276 (Addison-Wesley, 1994).
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FIG. 1. (Color online) Schematic of a single Cooper pair box with two Josephson junctions $(a,b)$ threaded by a magnetic flux $\Phi$. The qubit state is controlled by the gate voltage and the flux.