Large–$N$ limit of the generalized 2D
Yang–Mills theory on cylinder

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Abstract

Using the collective field theory approach of large–$N$ generalized two-dimensional Yang–Mills theory on cylinder, it is shown that the classical equation of motion of collective field is a generalized Hopf equation. Then, using the Itzykson–Zuber integral at the large–$N$ limit, it is found that the classical Young tableau density, which satisfies the saddle–point equation and determines the large–$N$ limit of free energy, is the inverse of the solution of this generalized Hopf equation, at a certain point.

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1 Introduction

The 2D Yang–Mills theory (YM$_2$) is a theoretical tool for understanding one of the most important theories of particle physics, i.e., QCD$_4$. It is known that the YM$_2$ theory is a solvable model, and in the recent years there have been much efforts to analyze the different aspects of this theory. The lattice formulation of YM$_2$ has been known for a long time [1], and many of the physical quantities of this model, e.g. the partition function and the expectation values of the Wilson loops, have been calculated in this context [2,3]. The continuum (path integral) approach of YM$_2$ has also been studied in [4] and, using this approaches, besides the above mentioned quantities, the Green functions of field strengths have also been calculated [5].

It is known that the YM$_2$ theory is defined by the Lagrangian $\text{tr}(\mathcal{F}^2)$ on a Riemann surface. In an equivalent formulation, one can use $i\text{tr}(BF) + \text{tr}(B^2)$ as the Lagrangian of this model, where $B$ is an auxiliary pseudo-scalar field in the adjoint representation of the gauge group. Path integration over the field $B$ leaves an effective Lagrangian of the form $\text{tr}(\mathcal{F}^2)$.

Now the YM$_2$ theory is essentially characterized by two important properties: invariance under area-preserving diffeomorphisms and the lack of propagating degrees of freedom. These properties are not unique to the $i\text{tr}(BF) + \text{tr}(B^2)$ Lagrangian, but rather are shared by a wide class of theories, called the generalized 2D Yang–Mills theories (gYM$_2$’s). These theories are defined by replacing the $\text{tr}(B^2)$ term by an arbitrary class function $f(B)$ [6]. Several properties of gYM$_2$ theories have been studied in recent years, for example the partition function [7,8], and the Green functions on arbitrary Riemann surface [9].

One of the important features of YM$_2$, and also gYM$_2$’s, is its behaviour in the case of large gauge groups, i.e., the large–$N$ behaviour of $SU(N)$ (or $U(N)$) gauge theories. Study of the large–$N$ limit of these theories is motivated on one hand by an attempt to find a string representation of QCD in four dimension [10]. It was shown that the coefficients of $1/N$ expansion of the partition function of $SU(N)$ gauge theories are determined by a sum over maps from a two-dimensional surface onto the two-dimensional target space. These kinds of calculations have been done in [11] and [12] for YM$_2$ and in [8] for gYM$_2$.

On the other hand, the study of the large–$N$ limits is useful in exploring more general properties of large–$N$ QCD. To do this, one must calculate, for example, the large–$N$ behaviour of the free energy of these theories. This is done by replacing the sum over irreducible representations of $SU(N)$ (or $U(N)$), appearing in the expressions of partition function, by a path integral over continuous Young tableaus, and calculating the area-dependence of the free energy from the saddle-point configuration. In [13], the logarithmic behaviour of the free energy of $U(N)$ YM$_2$ on a sphere with area $A < A_c = \pi^2$ has been obtained, and in [14] the authors have considered the case $A > A_c$ and proved the existence of a third–order phase
transition in YM$_2$. A fact that has been known earlier in the context of lattice formulation \cite{15}. In the case of gYM$_2$ models, the same transition has been shown for $f(B) = \text{tr}(B^4)$ \text{and for } $f(B) = \text{tr}(B^6)$ and $f(B) = \text{tr}(B^2) + g\text{tr}(B^4)$ \text{in } \cite{17}, all on the sphere.

Such kinds of investigations are much more involved in the cases of surfaces with boundaries. This is because in these cases, the characters of the group elements, which specify the boundary conditions, enter in the expressions of the partition functions and this makes the saddle–point equations too complicated. In \cite{18} (see also \cite{21}), the authors considered the YM$_2$ theory on cylinder and investigated its large–N behaviour. If we denote the two circles forming the boundaries of the cylinder by $C_1$ and $C_2$, then the boundary conditions are specified by fixed holonomy matrices $U_{C_1} = \text{Pexp}_{C_1} A_\mu(x)dx^\mu$ and $U_{C_2} = \text{Pexp}_{C_2} A_\mu(x)dx^\mu$. In the large–N limit, in which the eigenvalues of these matrices become continuous, the eigenvalue densities of $U_{C_1}$ and $U_{C_2}$ are denoted by $\sigma_1(\theta)$ and $\sigma_2(\theta)$, respectively, where $\theta \in [0, 2\pi]$. Then it was shown that the free energy of YM$_2$ on cylinder, minus some known functions, satisfies a Hamilton–Jacobi equation with a Hamiltonian describing a fluid of a certain negative pressure \cite{18}. The time coordinate of this system is the area of the cylinder between one end and a loop ($0 \leq t \leq A$), and its position coordinate is $\theta$, and there are two boundary conditions $\sigma(\theta)|_{t=0} = \sigma_1(\theta)$ and $\sigma(\theta)|_{t=A} = \sigma_2(\theta)$. It is found that the classical equation of motion of this fluid is the Hopf (or Burgers) equation. Further, it was shown that the Young tableau density $\rho_c$, satisfying the saddle–point equation (and therefore specifying the representation which has the dominant contribution in the partition function at large–N), satisfies $\pi \rho_c(-\pi \sigma_0(\theta)) = \theta$. $\sigma_0(\theta)$ is $\sigma(\theta, t)$ at a time (area) $t$ at which the fluid is at rest. When $U_{C_1} = U_{C_2}$, $\sigma_0(\theta)$ is the solution of Hopf equation at $t = A/2$. In the case of a disc, $\sigma_2(\theta) = \delta(\theta)$, the authors have calculated the critical area $A_c$ by using the results of the Itzykson–Zuber integral \cite{19} at large–N limit.

Studying the same problem for gYM$_2$ has begun in \cite{20}, in the context of master field formalism. In this paper we study this problem, gYM$_2$ on cylinder, using the above described technique. The plan of the paper is as following. In section 2, by calculating the classical Hamilton–Jacobi equation, we obtain the corresponding Hamiltonian for the eigenvalue density for almost general $f(B)$ and find the classical equations of motion. It is found that these equations are the generalized Hopf equation. In section 3, we show that the Young tableau density $\rho_c$ is the inverse function of the solution of the generalized Hopf equation at some certain time (area).
2 The collective field theory and the generalized Hopf equation

As it is shown in [7,8,9], the partition function of a gYM on a cylinder is

\[ Z = \sum_R \chi_R(U_1)\chi_R(U_2)e^{-AC(R)}, \]  

(1)

where \( U_1 \) and \( U_2 \) are Wilson loops corresponding to the boundaries of the cylinder, the summation is over all irreducible representations of the gauge group, \( \chi_R \) is the trace of the representation \( R \) of the group element, and \( C \) is a certain Casimir of the group, characterizing the particular gYM\(_2\) theory we are working with. For the gauge group \( U(N) \), the group we are working with, the representation \( R \) is labeled by \( N \) integers \( l_1 \) to \( l_N \), satisfying

\[ l_i < l_j, \quad i < j. \]  

(2)

The group element \( U \) (an \( N \times N \) unitary matrix) has \( N \) eigenvalues \( e^{i\theta_1} \) to \( e^{i\theta_N} \). The character \( \chi_R(U) \) is then

\[ \chi_R(U) = \frac{\det \{ e^{i\theta_j} \}}{J \{ e^{i\theta_k} \}}, \]  

(3)

where

\[ J \{ e^{i\theta_k} \} = \prod_{j<k} (e^{i\theta_j} - e^{i\theta_k}). \]  

(4)

The Casimir \( C \) is a function of \( l_i \)'s. In its simplest form, \( C \) has an expression

\[ C(R) = \sum_{i=1}^N c(l_i), \]  

(5)

where \( c \) is an arbitrary function. Here, we restrict ourselves to this form.

In the Large-\( N \) limit, it is convenient to define a set of scaled parameters \( y_i \), instead of \( l_i \)'s:

\[ y_i := \frac{l_i}{N} - \frac{1}{2}. \]  

(6)

In the same limit, also two density functions \( \rho \) and \( \sigma \), corresponding to the distribution of \( y_i \)'s and \( \theta_i \)'s, respectively, are defined:

\[ \sigma(\theta) := \frac{1}{N} \sum_{j=1}^N \delta(\theta - \theta_j) \]  

(7)

\[ \rho(y) := \frac{1}{N} \sum_{j=1}^N \delta(y - y_j). \]  

(8)
Inserting (3), (4), and (5) in (1), using (6), and making some obvious redefinition of the function \( c \), one arrives at

\[
Z = K \sum_R \frac{\det \{ e^{iNy_1 \theta_1} \} \det \{ e^{iNy_2 \theta_2} \}}{D\{ \theta_1 \} D\{ \theta_2 \}} e^{-NA \sum_k g(y_k)}. \tag{9}
\]

Here we have defined

\[
D\{ \theta_k \} := \prod_{j<k} \sin \frac{\theta_j - \theta_k}{2}, \tag{10}
\]

and \( K \) is an unimportant constant.

To proceed, we use the change of variable

\[
\tau_k := i \theta_k, \tag{11}
\]

and rewrite (9) as

\[
Z = \tilde{K} \sum_R \frac{\det \{ e^{Ny_1 \tau_k} \} \det \{ e^{Ny_2 \tau_k} \}}{D\{ \tau_1 \} D\{ \tau_2 \}} e^{-NA \sum_k g(y_k)}, \tag{12}
\]

where

\[
D\{ \tau_k \} := \prod_{j<k} \sinh \frac{\tau_j - \tau_k}{2}. \tag{13}
\]

We can then use, along the line of [18],

\[
Z = \tilde{K} e^{N^2 F}, \tag{14}
\]

and differentiate it to obtain

\[
- \frac{\partial F}{\partial A} = \frac{1}{D^1 Z} \frac{1}{N} \sum_k g \left( \frac{\partial}{N \partial \tau_k} \right) (D^1 Z), \tag{15}
\]

where \( D^1 \) is \( D\{ \tau_k \} \). The function \( g \) is assumed to be a polynomial. So, to calculate the right–hand side of (15), let’s first calculate it for a monomial. We have

\[
\frac{1}{ND^1 Z} \sum_k \left( \frac{\partial}{N \partial \tau_k} \right)^n (D^1 Z) = \frac{1}{ND^1 Z} \sum_{k,m} \binom{n}{m} \left( \frac{\partial}{N \partial \tau_k} \right)^m (D^1) = \\
\times \left( \frac{\partial}{N \partial \tau_k} \right)^{(n-m)} Z = \\
\times \left[ (N \frac{\partial F}{\partial \tau_k})^{n-m} + O \left( \frac{1}{N^2} \right) \right]. \tag{16}
\]
In the large-$N$ limit, one can of course omit the $O(1/N^2)$ term (which contains higher derivatives of $F$). In this limit, one must also note the limiting behaviours

\[
\frac{1}{N} \sum b_k \to \int d\tau \tilde{\sigma}(\tau)b(\tau),
\]

and

\[
N \frac{\partial b}{\partial \tau_k} \to \frac{\partial}{\partial \tau} \left[ \frac{\delta b}{\delta \tilde{\sigma}(\tau)} \right] |_{\tau = \tau_k}.
\]

Here we have used a density function $\tilde{\sigma}(\tau)$ instead of $\sigma(\theta)$, corresponding to the change-of-variable (11).

Returning to (16), we define

\[
D_k := \frac{1}{N} \frac{\partial}{\partial \tau_k} \ln |D^1| = \frac{1}{2N} \sum_{j \neq k} \coth \frac{\tau_k - \tau_j}{2}.
\]

This remains finite, as $N$ tends to infinity. One then has

\[
\frac{1}{D^1} \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} \right)^m D^1 = \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} + D_k \right)^m = \sum_l \binom{m}{l} D_k^{m-l} \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} + D_k \right)_s^l.
\]

Here, the subscript $s$ denotes that part of expression which contains only the derivatives of $D_k$, not $D_k$ itself. The first equality simply comes from $(D^1)^{-1} N^{-1} (\partial/\partial \tau_k) D^1 = N^{-1} (\partial/\partial \tau_k) + D_k$. To obtain the second equality, one may consider a term with $l$ factors of $D_k$. There are $m!/l!(m-l)!$ ways to choose $l$ factors of $D_k$ from $m$ factors present. The other $D_k$’s, either are differentiated or are not present.

It may seem that this $s$ part vanishes at the large-$N$ limit, since it contains factors of $1/N$. This is, however, not the case, since there are singular terms in $D_k$ at $j \sim k$. To calculate the non–vanishing part of this expression, let us define a generating function:

\[
q_k(u) := \sum \frac{u^m}{m!} \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} + D_k \right)^m = e^{u \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} + D_k \right)}.
\]

This is easily seen to be

\[
q_k(u) = \frac{1}{D^1} e^{\frac{u}{N} \frac{\partial}{\partial \tau_k}} D^1 = \frac{D^1 \left( \tau_k + \frac{u}{N} \right)}{D^1(\tau_k)}.
\]
\[ \prod_{j \neq k} \frac{\sinh \left( \frac{\tau_k - \tau_j + u/N}{2} \right)}{\sinh \left( \frac{\tau_k - \tau_j}{2} \right)}. \tag{22} \]

The s–part of this expression is contained in terms with small values for \( \tau_k - \tau_j \). To obtain this, we use
\[ \tau_k - \tau_j \approx \frac{k - j}{N \tilde{\sigma}(\tau_k)}. \tag{23} \]

let \( j \) run from \(-\infty\) to \( \infty \) (but \( j \neq k \)), and keep only the leading terms in \( \tau_k - \tau_j \). It is easily seen that if one uses this prescription for \( D_k \) itself, \( D_k \) vanishes. So, using the above–mentioned prescription in (22) gives exactly \( q_{ks}(u) \), the s–part of \( q_k(u) \). That is,
\[ q_{ks}(u) = \prod_{j \neq k} \left\{ 1 - \left[ \frac{u \tilde{\sigma}(\tau_k)}{k - j} \right]^2 \right\}, \tag{24} \]

So,
\[ q_{ks}(u) = \frac{\sin[\pi u \tilde{\sigma}(\tau_k) \pi u \tilde{\sigma}(\tau_k)]}{\pi u \tilde{\sigma}(\tau_k)}. \tag{25} \]

Having found this, we return to (20) and arrive at
\[ \frac{1}{D^1} \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} \right)^m D^1 = \sum_l \binom{m}{l} D_k^{m-l} a_l[\pi \tilde{\sigma}(\tau_k)]^l, \tag{26} \]

where the coefficients \( a_l \) are defined through
\[ \frac{\sin x}{x} =: \sum_{l=0}^{\infty} \frac{a_l}{l!} x^l = \sum_{l=0}^{\infty} \frac{\cos(\pi l/2)}{l + 1} x^l. \tag{27} \]

Inserting (24) in (16), one obtains
\[
\begin{align*}
\frac{1}{ND^1 Z} \sum_k \left( \frac{1}{N} \frac{\partial}{\partial \tau_k} \right)^n (D^1 Z) &= \frac{1}{N} \sum_{k,m,l} \binom{n}{m} \binom{m}{l} a_l[\pi \tilde{\sigma}(\tau_k)]^l D_k^{m-l} \left( N \frac{\partial F}{\partial \tau_k} \right)^{n-m} \\
&= \frac{1}{N} \sum_{k,l} a_l[\pi \tilde{\sigma}(\tau_k)]^l \left( \frac{n-1}{n} \right) \sum_m \binom{n-1}{m-1} D_k^{m-l} \left( N \frac{\partial F}{\partial \tau_k} \right)^{n-m} \\
&= \frac{1}{N} \sum_{k,l} a_l[\pi \tilde{\sigma}(\tau_k)]^l \left( \frac{n-1}{n} \right) \left( N \frac{\partial F}{\partial \tau_k} + D_k \right)^{n-l} \\
\end{align*}
\]
\[
\sum_l a_l \int d\tau \bar{a}(\tau)[\pi\bar{a}(\tau)]^l \left\{ \frac{\partial}{\partial \bar{a}} \left[ \frac{\delta S}{\delta \bar{a}(\tau)} \right] \right\}^{n-l}.
\]

(28)

In the last step, we have defined a function \( S \) through

\[
\begin{align*}
\left\{ \begin{array}{l}
N \frac{\partial}{\partial \nu_k} S := N \frac{\partial}{\partial \nu_k} F + D_k \\
\frac{\partial S}{\partial A} := \frac{\partial F}{\partial A}
\end{array} \right. 
\]

(29)

Combining (13) with (28), we arrive at

\[
- \frac{\partial S}{\partial A} = \sum_{n,l} \binom{n}{l} a_l g_n \int d\tau \bar{a}(\tau)[\pi\bar{a}(\tau)]^l \left\{ \frac{\partial}{\partial \bar{a}} \left[ \frac{\delta S}{\delta \bar{a}(\tau)} \right] \right\}^{n-l},
\]

(30)

where \( g_n \)'s are the coefficients of the Taylor–series expansion of \( g \). Considering \( A \) as a time variable, (30) can be regarded as the Hamilton–Jacobi equation corresponding to the Hamiltonian

\[
H = \sum_{l,n} \binom{n}{l} g_n a_l \int d\tau \bar{a}(\tau)[\pi\bar{a}(\tau)]^l \left[ \frac{\partial \bar{\Pi}(\tau)}{\partial \tau} \right]^{n-l},
\]

(31)

where \( \bar{\Pi} \) is the momentum conjugate to \( \bar{a} \).

The summations in (31) are easily carried out to yield

\[
H = \frac{1}{2\pi i} \int d\tau \left\{ G \left[ \pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right] - G \left[ -\pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right] \right\},
\]

(32)

where \( G \) is an integral of \( g \):

\[
\frac{dG}{dx} = g(x).
\]

(33)

From (32), one can obtain the equations of motion for \( \bar{a} \) and \( \bar{\Pi} \):

\[
\dot{\bar{a}} = \frac{\delta H}{\delta \bar{\Pi}} = -\frac{1}{2\pi i} \frac{\partial}{\partial \tau} \left[ g \left( \pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right) - g \left( -\pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right) \right],
\]

(34)

and

\[
\dot{\bar{\Pi}} = -\frac{\delta H}{\delta \bar{a}} = \frac{i\pi}{2\pi i} \left[ g \left( \pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right) + g \left( -\pi\bar{a}(\tau) + \frac{\partial \bar{\Pi}}{\partial \tau} \right) \right].
\]

(35)

Defining

\[
\bar{v} := \frac{\partial \bar{\Pi}}{\partial \tau},
\]

(36)
as a velocity field, in correspondence with what defined in [18,19], one can combine (34) and (35) into a generalized Hopf equation:

\[
\dot{\tilde{v}} \pm i\pi \dot{\tilde{\sigma}} = -\frac{\partial}{\partial \tau} [g(\tilde{v} \pm i\pi \tilde{\sigma})]
\] (37)

In the case of YM2, where \( g(y_k) = \frac{1}{2} y_k^2 \), this equation reduces to Hopf equation found in [18]. In this case, when one of the boundaries shrinks, so that one has a disc instead of a cylinder, that is \( \sigma_2(\theta) = \delta(\theta) \), the Itzykson–Zuber integral can be used to obtain a solution for the Hopf equation and from that the critical area of the disc has been obtained [18]. In our problem, \( g_{YM2} \), we do not know such an integral representation.

### 3 The dominant representation

It is shown in [19] that the character \( \Xi_R(U) \), can be written as

\[
\chi_R(U) = e^{N^2 \Xi[\rho, \tilde{\sigma}]},
\] (38)

where, for large \( N \),

\[
\Xi[\rho, \tilde{\sigma}] = \Sigma[\rho, \tilde{\sigma}] + \frac{1}{2} \int dy y^2 \rho(y) + B[\tilde{\sigma}].
\] (39)

Here \( B \) is some functional of \( \tilde{\sigma} \), and \( \Sigma \) satisfies a Hamilton–Jacobi equation

\[
\frac{\partial \Sigma}{\partial t} = \frac{1}{2} \int dx \mu(x) \left[ \left( \frac{\partial \Sigma}{\partial \mu} \right)^2 - \frac{1}{3} \pi^2 \mu^2(x) \right],
\] (40)

in which the variable \( \mu \) satisfies the boundary conditions

\[
\mu(x, t = 0) = \tilde{\sigma}(x),
\] (41)

and

\[
\mu(x, t = 1) = \rho(x),
\] (42)

Defining \( V \) as the derivative of the momentum conjugate to \( \mu \), as in the previous section, it is seen that the Hamilton–Jacobi equation (10) is equivalent to the following evolution equation for \( \mu \) and \( V \).

\[
\frac{\partial \Phi}{\partial t} - \Phi \frac{\partial \Phi}{\partial x} = 0,
\] (43)

where

\[
\Phi := V + i\pi \mu.
\] (44)

and \( V = \frac{\partial}{\partial y} \frac{\delta \Sigma}{\delta \mu} \).

Using (39) in (11) and (12), we see that in the large–\( N \) limit, the dominant representation satisfies the following saddle–point equation
\[ \sum_i \frac{\partial}{\partial y} \frac{\delta \Sigma_i}{\delta \rho(y)} = Ag'(y), \]  
(45)

or

\[ \sum_i \frac{\partial}{\partial y} \frac{\delta \Sigma_i}{\delta \rho(y)} = Ag'(y) - \sum_i y, \]  
(46)

where the summation is over boundaries. Now recall (37). Rewriting it as

\[ \dot{\tilde{f}} + \frac{\partial}{\partial \tau} g(\tilde{f}) = 0, \]  
(47)

where

\[ \tilde{f} := \tilde{v} - i\pi \tilde{\sigma}, \]  
(48)

one can write an implicit solution to (47) as

\[ f(\tau, b) = f(\tau - (b - a)g[f(\tau, b)], a), \]  
(49)

where \( a \) and \( b \) are two particular values of the time variable (here, actually the area). The same thing can be done to solve (43). In fact, one can define two functions \( H^+ \) and \( H^- \) as

\[ H^+(x) := x - (t - T)\Phi(x, T), \]  
(50)

and

\[ H^-(x) := x + (t - T)\Phi(x, t), \]  
(51)

and see that \( \Phi \) is a solution to (43) if these two functions are inverses of each other, i.e. \( H^-[H^+(x)] = x \).

As an ansatz for \( H^\pm \) (with \( t = 1 \) and \( T = 0 \)), we take

\[ H^- := A_i g' + i\pi \rho, \]  
(52)

and

\[ H^+ := \tilde{v}_i - i\pi \tilde{\sigma}_i = \tilde{f}_i \]  
(53)

where

\[ \tilde{f}(x) = \tilde{f}_0 \{x - A_i g'(\tilde{f}_i(x))\}. \]  
(54)

Here \( A_i \) is the area between a curve for which \( \tilde{f} \) is \( \tilde{f}_0 \) and the \( i \)-th boundary. The meaning of (52) and (53) is that we are seeking a solution to (43) with boundary conditions

\[ V_i(x, 1) = A_i g'(x) - x, \]  
(55)
and

$$\mu(x, 0) = \tilde{\sigma}(x).$$  \hfill (56)  

For such a solution, (46) is obviously satisfied.

It is now easily seen that

$$H^{-}_{i}[H^{+}_{i}(x)] = A_{i}g'[\tilde{f}_{i}(x)] + i\pi\rho(\tilde{f}_{0}\{x - A_{i}g'[\tilde{f}_{i}(x)]\}).$$  \hfill (57)

If $\tilde{f}_{0}(x)$ is the inverse of $i\pi\rho$,

$$i\pi\rho[\tilde{f}_{0}(x)] = x,$$  \hfill (58)

then we have

$$H^{-}_{i}[H^{+}_{i}(x)] = x,$$  \hfill (59)

So (16) is satisfied if $\rho$ satisfies (58). But $\tilde{f}_{0}$ is generally a complex function, whereas $\rho$ should be real. It is now better to return to the earlier variables $\theta$ and $\sigma$. We have

$$i\tilde{\sigma}(i\theta) = \sigma(\theta).$$  \hfill (60)

So (58) can be written as

$$\pi\rho[-if_{0}(x)] = x,$$  \hfill (61)

where

$$f := v - i\pi\sigma.$$  \hfill (62)

The argument of $\rho$ in (61) is real if $v_{0}$ is zero. So, if there exits a loop on the surface, for which the velocity field is zero, then there exists a dominant representation $\rho$, satisfying

$$\pi\rho[-\pi\sigma_{0}(\theta)] = \theta.$$  \hfill (63)

This is the same as that obtained in [18]. Note, however, that the equation governing the evolution of $\sigma$, (37), is different from the corresponding equation in [18].

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