SUPERMEMBRANES WITH FEWER SUPERSYMMETRIES

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ABSTRACT

The usual supermembrane solution of \( D = 11 \) supergravity interpolates between \( \mathbb{R}^{11} \) and \( \text{AdS}_4 \times \text{round } S^7 \), has symmetry \( P_3 \times SO(8) \) and preserves \( 1/2 \) of the spacetime supersymmetries for either orientation of the round \( S^7 \). Here we show that more general supermembrane solutions may be obtained by replacing the round \( S^7 \) by any seven-dimensional Einstein space \( M^7 \). These have symmetry \( P_3 \times G \), where \( G \) is the isometry group of \( M^7 \). For example, \( G = SO(5) \times SO(3) \) for the squashed \( S^7 \). For one orientation of \( M^7 \), they preserve \( N/16 \) spacetime supersymmetries where \( 1 \leq N \leq 8 \) is the number of Killing spinors on \( M^7 \); for the opposite orientation they preserve no supersymmetries since then \( M^7 \) has no Killing spinors. For example \( N = 1 \) for the left-squashed \( S^7 \) owing to its \( G_2 \) Weyl holonomy, whereas \( N = 0 \) for the right-squashed \( S^7 \). All these solutions saturate the same Bogomol’nyi bound between the mass and charge. Similar replacements of \( S^{D-p-2} \) by Einstein spaces \( M^{D-p-2} \) yield new super \( p \)-brane solutions in other spacetime dimensions \( D \leq 11 \). In particular, simultaneous dimensional reduction of the above \( D = 11 \) supermembranes on \( S^1 \) leads to a new class of \( D = 10 \) elementary string solutions which also have fewer supersymmetries.

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1 Introduction

Recent developments in non-perturbative string theory have emphasized the need to incorporate supersymmetric extended objects of more than two worldvolume dimensions [1]: the super \(p\)-branes. They appear as solitons of the fundamental string theory and may even need to be treated as fundamental in their own right. At the same time, there has been a revival in the fortunes of \(D = 11\) dimensions: \(D = 11\) supergravity [2, 3] emerges as the strong coupling limit of the Type \(IIA\) string [4] and there are hints that the \(D = 11\) supermembrane [5, 6] on \(S^1\) and the \(D = 10\) Type \(IIA\) string theories may actually be equivalent when all the solitonic states are taken into account [7].

Certainly, the Green-Schwarz action of the latter follows by simultaneous worldvolume/spacetime \(S^1\) dimensional reduction of the former [5]. Moreover, there is a web of interconnections between the \(D = 11\) supermembrane and the other four superstring theories [8, 9, 10, 11] and it has even been suggested that the \(D = 11\) supermembrane might be the fundamental theory underlying all the others. This therefore seems an appropriate time to reevaluate the status of the \(D = 11\) supermembrane. This object was first described by a manifestly spacetime supersymmetric Green-Schwarz action [7] whose kappa-symmetry forces the supermultiplet of background fields (\(g_{MN}, \Psi_M, A_{MNP}\)) to obey the field equations of \(D = 11\) supergravity. Only later was the supermembrane recognized as a solution of the \(D = 11\) supergravity field equations [8], whose zero modes correspond to the physical degrees of freedom of the Green-Schwarz action. This contrasts with some of the other super \(p\)-branes whose existence was first established by solving the supergravity field equations and for which, in some cases, the Green-Schwarz action is still unknown.

The usual supermembrane solution of \(D = 11\) supergravity interpolates between \(R^{11}\) and \(AdS_4 \times \text{round } S^7\) [7], has symmetry \(P_3 \times SO(8)\), preserves 1/2 of the spacetime supersymmetries and admits 8 bosonic plus 8 fermionic zero modes [8] (for either orientation of the round \(S^7\).) Here we show that more general supermembrane solutions may be obtained by replacing the round \(S^7\) by any seven-dimensional Einstein space \(M^7\). These have symmetry \(P_3 \times G\), where \(G\) is the isometry group of \(M^7\). For example, \(G = SO(5) \times SO(3)\) for the squashed \(S^7\). For one orientation of \(M^7\), they preserve \(N/16\) spacetime supersymmetries where \(1 \leq N \leq 8\) is the number of Killing spinors on \(M^7\); for the opposite orientation they preserve no supersymmetries since then \(M^7\) has no Killing spinors. For example \(N = 1\) for the left-squashed \(S^7\) owing to the \(G_2\) Weyl holonomy, whereas \(N = 0\) for the right-squashed \(S^7\) [8]. All these solutions saturate the same Bogomol’nyi bound between the mass and charge. Similar replacements of \(S^{D-p-2}\) by Einstein spaces \(M^{D-p-2}\) yield new super \(p\)-brane solutions in other spacetime dimensions \(D \leq 11\).

We begin in section [2] by showing in general how new \(p\)-brane solutions of supergravity in various spacetime dimensions \(D\) may be obtained by replacing the unit \(S^{D-p-2}\) which appears in
the transverse space line element by general Einstein spaces $M^{D-p-2}$ with the same cosmological constant. As an important special case we find new elementary membrane ($p = 2$) solutions of $D = 11$ supergravity. In section 3 we derive a Bogomol’nyi bound between the mass per unit area of the membrane and the central charges appearing in the $D = 11$ supersymmetry algebra. All the elementary membrane solutions, both old and new, saturate the bound. The zero modes are analyzed in section 4. In contrast to the round $S^7$, however, these new membrane solutions exhibit no translation zero modes or their fermionic partners. The non-supersymmetric orientation yields just $N$ fermionic zero modes while the supersymmetric orientation yields no zero modes at all. The new elementary membrane solutions exhibit $\delta$-function singularities at their location and hence require the introduction of membrane sources. As discussed in section 5, the same ansatz that worked for the usual membrane continues to work for the new membranes. Another interesting special case provides the focus of section 6, where simultaneous dimensional reduction of the above $D = 11$ supermembranes on $S^1$ leads to a new class of $D = 10$ elementary string solutions which also have fewer supersymmetries. Having replaced the round $S^{D-p-2}$ by Einstein spaces $M^{D-p-2}$ with the same positive cosmological constant, the question arises as to whether we can find new super $p$-brane solutions when $M^{D-p-2}$ is Ricci flat. As discussed in section 7, the answer is yes, and we give examples such as the seven-dimensional Joyce manifolds which also have $G_2$ holonomy and hence yield $N = 1$. In this case, one finds $N$ bosonic and $N$ fermionic zero-modes. These Ricci-flat manifolds do give rise to $p$-branes with rather unusual asymptotic behavior, however. We finish with some concluding remarks in section 8.

## 2 New $p$-brane solutions

Our principle focus in this paper is to find new supermembrane solutions of $D = 11$ supergravity. However, we shall begin by considering the more general case of a $p$-brane in $D$ dimensions. As in the previously discussed elementary and solitonic $p$-brane solutions [16, 19], the relevant fields that participate in the solutions are the metric, a dilaton and an $n$’th rank antisymmetric tensor. The relevant part of the bosonic Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\kappa^2}(eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{2n!}ee^{-a\phi}F_n^2),$$

(2.1)

where $e = \sqrt{-g}$ is the determinant of the vielbein. The constant $a$ takes different values for supergravities in diverse dimensions and for different values of $n$. It is conveniently parametrized as

$$a^2 = \Delta - \frac{2(n-1)(D-n-1)}{D-2}.$$

(2.2)
If $F_n$ represents the field strength of the antisymmetric tensor that couples to the $p$-brane worldvolume via the Wess-Zumino term, then always $\Delta = 4$ [19]. However, for the purpose of finding $p$-brane solutions, $F_n$ might sometimes represent some different linear combination of the original $n$’th rank antisymmetric tensors in the supergravity theory. Under these circumstances, one can have $\Delta < 4$ [19, 20].

We shall take the ansatz for the canonical metric for the $D$ dimensional spacetime to be

$$ds^2 = e^{2A}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}(dr^2 + r^2 ds^2) ,$$

(2.3)

where $x^\mu (\mu = 0, \ldots, d-1)$ are the coordinates of the $(d-1)$-brane world volume. The remaining coordinates of the $D$ dimensional spacetime are $r$ and the coordinates on a $(D-d-1)$ dimensional space $M^{D-d-1}$ whose metric is $ds^2$. This metric is taken to be an Einstein metric with $\bar{R}_{\alpha\beta} = \Lambda \delta_{\alpha\beta}$ (here $\alpha, \beta, \cdots$ denote tangent-space indices on $M^{D-d-1}$.) The functions $A$ and $B$ depend only on $r$. In the elementary and solitonic solutions that have been discussed previously, $M^{D-d-1}$ has been taken to be the usual unit $(D-d-1)$ sphere. Here we investigate the properties of $p$-brane solutions with other choices of $M^{D-d-1}$.

A convenient choice of vielbein for the metric (2.3) is $e^\mu = e^A dx^\mu$, $e^r = e^B dr$ and $e^\alpha = re^B \bar{e}^\alpha$, where $\bar{e}^\alpha$ is the vielbein for the Einstein metric $d\bar{s}^2$. The corresponding spin connection is

$$\omega^\mu = -e^{-B} A' e^\mu , \quad \omega^{r\alpha} = -e^{-B} (B' + \frac{1}{r}) e^\alpha , \quad \omega^{\alpha\beta} = \bar{\omega}^{\alpha\beta} ,$$

(2.4)

where $\bar{\omega}^{\alpha\beta}$ is the spin connection for the vielbein $\bar{e}^\alpha$, and $A' = \frac{\partial A}{\partial r}$, etc. The Ricci tensor can be easily obtained; its tangent-space components are

$$R_{\mu\nu} = -e^{-2B} \left( \bar{d} A' B' + A'' + dA'^2 + \frac{\bar{d} + 1}{r} A' \right) \eta_{\mu\nu} ,$$
$$R_{rr} = -e^{-2B} \left( (\bar{d} + 1) B'' - dA'B' + dA'' - dA'^2 + \frac{\bar{d} + 1}{r} B' \right) ,$$
$$R_{\alpha\beta} = -e^{-2B} \left( B'' + \bar{d} B'^2 + dA'B' + \frac{2\bar{d} + 1}{r} B' + \frac{d}{r} A' + \frac{\bar{d} - \Lambda}{r^2} \right) \delta_{\alpha\beta} .$$

(2.5)

There are two discrete possibilities for the metric on the Einstein space $M^{D-d-1}$, namely $\Lambda$ zero or non-zero. The case when $\Lambda = 0$ shall be discussed in section (7). In this and the next two sections we shall discuss only the case with $\Lambda$ positive. Without loss of generality, we can choose the same cosmological constant $\Lambda = \bar{d}$ as that of the unit $(\bar{d} + 1)$ sphere, the Ricci tensor (2.5) for the metric $ds^2$ is identical to that for the case when $M$ is the unit sphere. Thus the bosonic equations giving rise to the $p$-brane solutions will be identical to those for the case when $M$ is the unit sphere.

For the elementary $p$-brane solutions, the ansatz for the antisymmetric tensor is given in terms of its potential, and it takes the form

$$A_{\mu_1 \cdots \mu_{n-1}} = \epsilon_{\mu_1 \cdots \mu_{n-1} \nu} e^C ,$$

(2.6)
and hence

\[ F_{\mu_1 \ldots \mu_{n-1}} = \epsilon_{\mu_1 \ldots \mu_{n-1}} (e^C)' , \quad (2.7) \]

where \( C \) is a function of \( r \) only, and the indices on \( A \) and \( F \) are world indices. Here and throughout this paper \( \epsilon_{M \ldots N} \) and \( \epsilon^{M \ldots N} \) have purely numerical components \( \pm 1 \) or \( 0 \). The dimension of the world volume is given by \( d = n - 1 = p + 1 \) for the elementary \( p \)-brane solutions.

The elementary \( p \)-brane solution to the equations of motion following from (2.1) is given by

\[
\begin{align*}
e^A &= \left(1 + \frac{k}{r^d}\right)^{\frac{2}{D(D-2)}}, & e^C &= \frac{2}{\sqrt{\Delta}} \left(1 + \frac{k}{r^d}\right)^{-1}, \\
B &= -\frac{d}{dA}, & \phi &= \frac{a(D-2)}{d} A ,
\end{align*}
\]

(2.8)

where \( k \) is an arbitrary integration constant and we have set the asymptotic value of the dilaton equal to zero. Note that the \( p \)-brane solution (2.8) is consistent with the case where the dilaton is absent in the supergravity theory, \( i.e. a = 0 = \phi \). Note also that when \( M^{D-d-1} \) is the usual unit sphere, the metric (2.3) is asymptotic to \( D \)-dimensional Minkowski spacetime as \( r \) approaches infinity. On the other hand, if \( M^{D-d-1} \) is any other Einstein space, then the metric (2.3) is no longer asymptotic to Minkowski spacetime, although it is still asymptotically flat. In fact, the metric on the \((D-d)\)-dimensional transverse space is asymptotic to a Ricci-flat metric on a generalized cone:

\[ ds^2_{D-d} = dr^2 + r^2 d\bar{s}^2 . \quad (2.9) \]

This metric has a conical singularity at the apex \( r = 0 \), except when \( d\bar{s}^2 \) is the usual round sphere metric. In \cite{[23]} the metric (2.3) was rounded off at its apex to a ‘bolt of the second kind’, giving a complete Ricci-flat metric. In our paper, the singularity of \( r = 0 \) is eliminated. One can see this by first going to the \( \sigma \)-model metric \cite{[16]}

\[ ds^2(\sigma - model) = e^{-a\phi/\bar{d}} ds^2(\text{canonical}) \quad (2.10) \]

where the canonical metric is given in (2.3), and then noting that the \( \sigma \)-model metric there approaches \((AdS)_{d+1} \times M^{\bar{d}+1} \) for \( \bar{d} \neq 2 \) or \((\text{Mink})_{d+1} \times M^3 \) for \( \bar{d} = 2 \) \cite{[17]}. Note that the translational invariance of the \( 8 \)-dimensional metric (2.3) is absent except in the case of the round \( S^7 \). This will prove important when we come to analyse the zero-modes in section (4). Of course, the complete metric still has the Poincaré symmetry in the world volume, since the only modification is in the structure of the transverse space.

A particular case which is important to us in the present paper is the elementary membrane solution of \( D = 11 \) supergravity. In this case, there is no dilaton, and a single fourth-rank antisymmetric tensor. In fact, it fits into the general pattern of solutions given above, with \( d = 3, \bar{d} = 6, \)
\[ \Delta = 4 \] and hence \( a = 0 \). From (2.8) we therefore have

\[ e^A = \left( 1 + \frac{k}{r^6} \right)^{1/3}, \quad e^C = \left( 1 + \frac{k}{r^6} \right)^{-1} \quad B = -\frac{1}{2} A, \quad (2.11) \]

Note that the replacement of the round \( S^7 \) by the Einstein space \( M^7 \) has not changed the singularity structure discussed in [17]: by introducing the Schwarzschild-like coordinate \( \rho \) given by

\[ r^6 = \rho^6 - k^6 \quad (2.12) \]

one may see that there is an horizon at \( \rho = k \) and that the metric may be analytically continued through the horizon up to a curvature singularity at \( \rho = 0 \).

### 3 Supersymmetry and Bogomol'nyi bound

In the previous section, we obtained new \( p \)-brane solutions by replacing the usual round sphere that foliate the transverse space with a more general Einstein space. In particular, we obtained new elementary membrane solutions in \( D = 11 \) supergravity. In this section we shall investigate the supersymmetry of these solutions.

The gravitino transformation rule in \( D = 11 \) supergravity with bosonic background is given by

\[ \delta \psi_A = D_A \epsilon - \frac{1}{288} (\Gamma_A^{BCDE} - 8 \delta^B_A \Gamma^{CDE}) F_{BCDE} \epsilon \equiv \hat{D}_A \epsilon, \quad (3.1) \]

where \( \lambda, \nu, \ldots \) are tangent space indices. Decomposing these indices in terms of \( \lambda = (\mu, r, \alpha) \), we first make a \( 3 + 1 + 7 \) split of gamma matrices:

\[ \Gamma^\mu = \gamma^\mu \otimes \sigma_3 \otimes 1, \quad \Gamma^r = 1 \otimes \sigma_1 \otimes 1, \quad \Gamma^\alpha = 1 \otimes \sigma_2 \otimes \gamma^\alpha. \quad (3.2) \]

Substituting the solutions from the previous section, the gravitino transformation rule (3.1) becomes

\[ \delta \psi_\mu = e^{-A} \partial_\mu \epsilon - \frac{1}{2} e^{-B} A' \gamma^\mu \otimes \sigma_1 (\sigma_3 - 1) \otimes 1 \epsilon, \]
\[ \delta \psi_r = e^{-B} \epsilon' - \frac{1}{2} e^{-B} A' 1 \otimes \sigma_3 \otimes 1 \epsilon, \]
\[ \delta \psi_\alpha = \frac{1}{r} e^{-B} \left( 1 \otimes 1 \otimes \bar{D}_\alpha \epsilon - \frac{i}{2} 1 \otimes \sigma_3 \otimes \gamma_\alpha \epsilon \right) + \frac{i}{4} e^{-B} A' 1 \otimes (\sigma_3 - 1) \otimes \gamma_\alpha \epsilon. \quad (3.3) \]

It is easy to see that these variations all vanish provided that

\[ \epsilon = e^{\frac{1}{2} A} \epsilon_+^0 \otimes \eta_-, \quad 1 \otimes \sigma_3 \epsilon_+^0 = \epsilon_+^0, \quad (3.4) \]

where \( \eta_- \) is a Killing spinor on the Einstein space \( M^7 \), satisfying

\[ \bar{D}_\alpha \eta_- - \frac{i}{2} \gamma_\alpha \eta_- = 0. \quad (3.5) \]
Thus if there are $N$ Killing spinors $\eta_-$ satisfying (3.5) on the Einstein space $M^7$, then $N/16$ of the $D = 11$ spacetime supersymmetry is preserved. For example, if $M^7$ is the usual round seven sphere, for which there are eight Killing spinors satisfying (3.5), we recover the usual half breaking of supersymmetry, which was found in [8]. Other examples are provided by the squashed seven sphere, which has $N = 1$ [18]; the $M^{pqrs}$ spaces, which have $N = 2$ [22]; and the $N^{pqrs}$ spaces, which have $N = 1$ or $N = 3$ [23]. These examples give rise to new supermembrane solutions in $D = 11$ supergravity which preserve $1/16$, $1/8$ and $3/16$ of the original $D = 11$ supersymmetry. If, on the other hand we consider the same spaces with their orientations reversed then, with the exception of the round $S^7$, all supersymmetries are broken. This is the skew-whiffing theorem [3]; the round $S^7$ admits eight solutions of $\bar{D}_a \eta_- - \frac{1}{2} \gamma_\alpha \eta_- = 0$ and eight solutions of $\bar{D}_a \eta_+ + \frac{1}{2} \gamma_\alpha \eta_+ = 0$, but any other non-Ricci-flat $M^7$ can admit solutions of either the former or the latter equation (depending on orientation) but not both. It is worth remarking that except for the round $S^7$, when the membrane is absent, i.e. $k = 0$, the metric (2.3) and (2.8) is not Minkowskian, and already breaks $(1 - N/16)$ of the $D = 11$ supersymmetry. The remaining supersymmetry is either broken or unbroken by the presence of the membrane, depending on the orientation of $M^7$. We shall see in section 4 that this leads to the the unusual counting of zero modes of the solution.

It is instructive to examine the relation between the mass per unit area, $m$, and the Page charge [24], $Z$, which gives rise to a central charge in the supersymmetry algebra:

$$\{Q, Q\} = \Gamma^A P_A + \Gamma^{AB} U_{AB} + \Gamma^{ABCD} V_{ABCD} .$$

(3.6)

The elementary solutions will contribute to the $U$-type central charge while the solitonic solutions, which we have not considered so far, will contribute to the $V$-type central charge. To calculate the central charges, we need to compute the supercharges and then their anticommutator.

The supersymmetric variation of $D = 11$ supergravity takes the form $\delta I = \int \bar{D}_\alpha \psi J^\alpha$, where the Noether current is given by $J^\alpha = \Gamma^{MNP} \bar{D}_N \psi_P$, modulo the bosonic field equations. Thus the supercharges per unit membrane area are given by

$$Q_\epsilon = \int_{\partial \Sigma} \epsilon \Gamma^{ABC} \bar{\psi}_C d\Sigma_{AB} ;$$

(3.7)

where $\Sigma$ is an eight dimensional space-like surface. When $r \to \infty$, we take $\epsilon$ to be such that $\epsilon \to \epsilon_0 \otimes \eta$, where $\epsilon_0$ is a constant four-component spinor on the space of $(x^\mu, r)$ and $\eta$ is a Killing spinor on the Einstein space $M^7$, satisfying (3.5). The commutator of the resulting conserved supercharges is given by

$$[Q_{\epsilon_1}, Q_{\epsilon_2}] = \delta_{\epsilon_1} Q_{\epsilon_2} = \int_{\partial \Sigma} N^{AB} d\Sigma_{AB} ;$$

(3.8)

where $N^{AB} = \bar{\epsilon}_1 \Gamma^{ABC} \delta \bar{\psi}_C$. Using (3.1), we obtain

$$N^{AB} = \bar{\epsilon}_1 \Gamma^{ABC} D_C \epsilon_2 + \frac{1}{8} \bar{\epsilon}_1 \Gamma^{C_1 C_2} \epsilon_2 F^{AB} c_1 c_2 + \frac{1}{96} \bar{\epsilon}_1 \Gamma^{AB C_1 \ldots C_4} \epsilon_2 F_{C_1 \ldots C_4} .$$

(3.9)
Substituting elementary membrane solution (2.11) and gamma matrix decomposition (3.2), the last term of (3.9) vanishes in (3.8) since only the \(d\Sigma_0\) component of the area element contributes. Thus we obtain the Bogomol’nyi matrix \(\mathcal{M}\):

\[
\int_{M \text{ at } r \to \infty} N^{0r} r^7 d\Omega_7 = -2/3 \epsilon_1^+ \mathcal{M} \epsilon_2 \omega_7 ,
\]

where \(\omega_7\) is the volume of \(M\), and

\[
\mathcal{M} = m \mathbf{1} + Z \sigma_3 ,
\]

where we have suppressed the common-factor gamma matrices in the world volume, \(m\) is the mass per unit membrane area corresponding to the first term in (3.9) and \(Z\) is the central charge corresponding to the second term in (3.9). In fact, \(Z\) is precisely equal to the Page charge defined by \(\int *F/\omega_7\), with \(F = 6k/r^7 dr \wedge dx^0 \wedge dx^1 \wedge dx^2\) asymptotically at large \(r\). With the new solutions obtained in this paper, it is easy to see that \(m = k = Z\). The matrix \(\mathcal{M}\) has two eigenvalues \(\{m - Z, m + Z\} = \{0, 2k\}\). The zero eigenvalue implies that 1/16 of the \(D = 11\) supersymmetry is preserved for each Killing spinor. Thus if there are \(N\) Killing spinors satisfying (3.5) on the Einstein space \(M^7\), then \(N/16\) of the spacetime supersymmetry is preserved, which is the same conclusion we obtained at the beginning of this section by looking explicitly at the supersymmetry transformation rules.

For an arbitrary configuration, it can be shown that there is a Bogomol’nyi bound \(m \geq |Z|\). See for example [27]. Thus we see that this bound is saturated for any of the elementary supermembrane solutions obtained in this paper. It is interesting to note that the mass-charge ratio is independent of the choice of the Einstein space \(M^7\). Since the choice of the Einstein space determines how much of the \(D = 11\) dimensional supersymmetry is broken, it follows that for these types of solutions, the mass-charge ratio is independent of how much of the supersymmetry is broken. In this respect, the solutions presented in this paper are qualitatively different from previous examples of p-brane solutions breaking more than half the supersymmetries, such as the extreme black fourbrane and sixbrane [14] of the \(N = 1, D = 11\) theory (which break 3/4 and 7/8, respectively), the octonionic string soliton [13] of the \(N = 1, D = 10\) theory (which breaks 15/16), the double instanton string soliton [17] of the \(N = 1, D = 10\) theory (which breaks 3/4), the elementary particle solution [26] of the \(N = 2, D = 9\) theory which breaks 3/4, the dyonic string [30] of the \(N = 2, D = 6\) theory which breaks 3/4, the extreme \(a = 0\) and \(a = 1/\sqrt{3}\) black holes of the \(N = 4, D = 4\) theory which both break 3/4 [28, 29], and the string solitons of the \(N = 4, D = 4\) theory which break 3/4 and 7/8 [30]. In the case of the double instanton string and the octonionic string, for example, the mass per unit length actually diverges, in contrast to the elementary strings discussed in section (3). In supergravities with \(N\)-extended supersymmetry, there will be several central charges \(Z_1\),
$Z_2...Z_{N/2}$ and the mass is bounded by $Z_{\text{max}}$. The number of supersymmetries preserved by the solution will then be related to the number of $Z$s that are equal to $M$. In these examples, the Bogomol’nyi bounds are different from the solutions breaking just $1/2$ of the supersymmetries because one changes the value of $\Delta$. In this paper, we obtain fewer supersymmetries by keeping $\Delta$ fixed but replacing $S^{D-p-2}$ by $M^{D-p-2}$. In fact as we saw earlier, $M^{D-p-2}$ has already partially broken the $D = 11$ supersymmetry. The presence of the $p$-brane itself breaks none or all of the remaining supersymmetry depending on the orientation.

The new super $p$-brane solutions discussed in this section arise as a consequence of the existence of Einstein spaces $M^{D-p-2}$ other than round spheres that admit Killing spinors. Such spaces seem to be known only for the dimensions $\geq 5$. Thus in $D$ dimensions, there could only be new solutions with world volume dimension $\leq D - 6$.

## 4 Zero Modes

Each broken supersymmetry transformation of a $p$-brane solution gives rise to a corresponding fermionic Goldstone zero-mode. There will also be bosonic zero modes associated with the breaking of local bosonic gauge symmetries. If supersymmetry remains partially unbroken by the solution, then the fermionic and bosonic zero modes will form supermultiplets of the unbroken symmetry. These multiplet furnish a linear realization of the unbroken supersymmetry. The full supersymmetry of the $D$ dimensional theory is still realised of course, albeit non-linearly.

In the case of the round $S^{D-p-2}$, these zero-modes will include the $(D - d)$ translational zero modes corresponding to the breaking of the $D$-dimensional translational symmetry down to the $d$-dimensional world volume translations. However, our new solutions are dramatically different in this respect: the spacetime even in the absence of the $p$-brane (i.e. when $k = 0$) has translational invariance only in the worldvolume directions and not in the transverse directions. Consequently there are no Goldstone zero-modes corresponding to the breaking of the $(D - d)$-dimensional translations. In fact, as we shall see, in the case where there is some residual supersymmetry, there are no zero-modes at all! This presumably means that the corresponding Green-Schwarz action is that of a $p$-brane in $p + 1$ dimensions which has no continuous physical degrees of freedom.

We shall first consider the fermionic zero modes. Note that unlike the case of the round $S^7$, even when the membrane solutions are absent, i.e. $k = 0$, there are only $N < 8$ components of supersymmetry, where $N$ is the number of Killing spinors of $M^7$. The gravitino supersymmetry transformation in the backgrounds of the new elementary supermembrane solutions are given by (3.3). As we shall see now, the supersymmetry of the solution depends on the orientation of $M^7$,
which we denote by $M_7^-$. For $M_7^-$, where the Killing spinors satisfy $ar{D}_\alpha \eta_- - \frac{1}{2} i \gamma_\alpha \eta_- = 0$, the variations (3.3) vanish for spinors $\epsilon$ satisfying (3.4) and (3.5), which includes a chirality condition and Killing-spinor condition. They correspond to the $N$ unbroken supersymmetry generators. The fermionic zero modes would correspond to the spinors $\epsilon$ for which the variations (3.3) do not vanish. However, these “zero modes” are not normalizable since the first term of $\delta \psi_\alpha$ in (3.3) does not vanish fast enough as $r \to \infty$. Thus in this case, there are no normalizable fermionic zero modes. In fact this is understandable since the membrane solution preserves all $N$ components of supersymmetry. It follows by supersymmetry that there are no bosonic zero modes either in this case.

For the case of $M_7^+$, for which the Killing spinors satisfy $\bar{D}_\alpha \eta_+ + \frac{1}{2} i \gamma_\alpha \eta_+ = 0$, there is no spinor $\epsilon$ such that the gravitino supersymmetry variations (3.3) vanish. Thus in this case, the membrane solution breaks all $N$ components of supersymmetry which is present when the membrane is absent. Accordingly there are $N$ normalizable fermionic zero modes, given by $\epsilon = e^{-\frac{1}{2} A} e_0 \otimes \eta_+$ and $1 \otimes \sigma_3 e^0_\perp = -e^0_\perp$, where $\eta_+$ is any of the $N$ Killing spinors on $M_7^+$. In this non-supersymmetric membrane solution, a separate counting of the bosonic zero modes is necessary. Since skew-whiffing seems not to change the bosonic symmetries, we would expect that there are no bosonic zero modes in this case either.

5 Including the sources

The $D = 11$ supermembrane solution (2.11) does not solve the field equations everywhere because of the appearance of $\delta$-function singularities at $r = 0$. Instead of

$$\Box_8 e^{-C} = 0$$

(5.1)

for example, we have

$$\Box_8 e^{-C} = -6k\Omega_7 \delta^8(r)$$

(5.2)

In order that (2.11) be solutions everywhere it is necessary that the pure supergravity equations be augmented by source terms. This source is, of course, just the supermembrane itself. Here, however, we encounter the problem that the covariant Green-Schwarz action for the new membranes is unknown. We know the correct action describing just the usual 8 translation zero modes and their superpartners [5]. In the case of the new supermembranes with no zero-modes, the action is presumably just a truncation of this. In the non-supersymmetric case, there will be extra terms. Nevertheless, the same ansatz that was used in the case of the round $S^7$ [5] continues to solve the field equations. It is not difficult to verify that the correct source terms for the antisymmetric tensor and Einstein equations, as well as the membrane equations, are given by choosing the static
gauge choice
\[ X^\mu = \xi^\mu, \quad \mu = 0, 1, 2. \] (5.3)

where \( X^\mu (\mu = 0, 1, 2) \) are the first three spacetime coordinates, and \( \xi^i (i = 0, 1, 2) \) are the worldvolume coordinates, and then setting all the other zero mode variables to zero. The inclusion of the source now requires, as usual [8], that the constant \( k \) be given by
\[ k = \frac{k^2 T}{3 \Omega_7}, \] (5.4)

where \( T \) is the membrane tension.

6 New elementary string solutions

An important special case of new \( D < 11 \) solutions is provided by the elementary strings obtained by simultaneous dimensional reduction of the new \( D = 11 \) supermembranes. The logic proceeds in just the same way as for the round \( S^7 \) case. Let us denote all \( D = 11 \) variables by a carat, and then make the ten-one split
\[ \hat{x}^M = (x^M, x^2), \quad M = 0, 1, 3, ... 10. \] (6.1)

Then the solutions for the \( D = 10 \) canonical metric \( g_{MN} \), 2-form \( B_{MN} \) and dilaton \( \phi \) follow from
\[ \hat{g}_{MN} = e^{-\phi/6} g_{MN}, \]
\[ \hat{g}_{22} = e^{4\phi/3}, \]
\[ \hat{A}_{012} = B_{01}. \] (6.2)

From (2.11), we see that they are given by (2.8) in the special case \( d = 2, D = 10 \) and, like the elementary solution of [23], will solve the heterotic and Type IIB superstring equations as well as Type IIA. Moreover, like the elementary solution of [23], they will have finite mass per unit length, in contrast to the octonionic string soliton [13] and the double instanton string soliton [15]. The physical interpretation of such (non-critical?) strings remains obscure.

7 Further solutions

Both the elementary and solitonic \( p \)-brane solutions considered above have an interesting generalization in which the Einstein space \( M^{D-p-2} \) has zero rather than positive cosmological constant \( \Lambda \). The ansatz for the antisymmetric tensor is the same as that for the previously discussed solutions, given by (2.6) and (2.7). It is convenient to express the metric ansatz (2.3) in terms of
\[ ds^2 = e^{2A} dx^\mu dx^n \eta_{\mu\nu} + e^{2B} (dr^2 + ds^2) \]

where \( ds^2 \) is the metric for a Ricci-flat Einstein space \( M^{D-p-2} \). The spin connection and Ricci tensors are given by \( (2.4) \) and \( (2.7) \) respectively with all the explicit \( r \) dependent terms dropped. The elementary \( p \)-brane solution to the equations of motion following from \( (2.1) \) is given by

\[
\begin{align*}
    e^A &= (1 - \lambda r)^{-1/c} , \\
    e^C &= \frac{2}{\Delta} \sqrt{\frac{2cd}{D-2}} (1 - \lambda r)^{-\Delta(D-2)/(2cd)} , \\
    B &= -\frac{d}{dA} A , \\
    \phi &= \frac{a(D-2)}{d} A ,
\end{align*}
\]

where \( c = 3d - \Delta(D-2)/\delta \). In particular in \( D = 11 \), the solutions are given by

\[
\begin{align*}
    ds^2 &= (1 - \lambda r)^{-2/3} dx^\mu dx^n \eta_{\mu\nu} + (1 - \lambda r)^{1/3} (dr^2 + ds^2) , \\
    (e^C) &= \frac{1}{1 - \lambda r} .
\end{align*}
\]

The analysis of the supersymmetry of these membrane solutions in Ricci-flat manifolds is similar to that for the supermembrane solutions discussed in the previous sections. The amount of supersymmetry that is preserved depends on the number of Killing spinors (which on a Ricci-flat space are the same as covariantly constant spinors) in the Ricci flat manifold \( M \). For example, in \( D = 11 \), if \( M \) admits \( N \) such spinors, then \( N/16 \) of the \( D = 11 \) supersymmetry is preserved. An interesting example for \( M \) is provided by Joyce manifolds, which are compact Ricci-flat seven manifolds with \( G_2 \) holonomy \([31]\). (In fact they are three-torus bundles over \( K3 \).) Accordingly, they admit one covariantly constant spinor, and thus \( 1/16 \) of the original supersymmetry is preserved. Other examples are \( T^7 \), which has \( N = 8 \); \( K3 \times T^3 \), which has \( N = 4 \); and \( Y \times S^1 \) where \( Y \) is a Ricci-flat Calabi-Yau space, which has \( N = 2 \).

The discussion of the zero modes is different from that for the solutions discussed in the previous sections, however, since the skew-whiffing theorem ceases to apply in the Ricci-flat case \([3]\). Nevertheless, we shall follow the same logic that we used to discuss the zero modes for the membrane solutions with \( M^7 \) non-Ricci-flat in section 4, and compare the symmetries with and without a membrane. For Ricci-flat \( M^7 \), the supersymmetry transformation rule for the gravitino is given by \( (3.3) \) except that the first term in \( \delta \psi_\alpha \) turns into \( e^{-B} \mathbb{1} \otimes \mathbb{1} \otimes \bar{D}_\alpha \xi \). Thus when the membrane is absent, \( i.e. \lambda = 0 \), there are \( 2N \) components of the 11-dimensional supersymmetry that are preserved, corresponding to \( N \) Killing spinors \( \epsilon^0_+ \eta \) and \( N \) Killing spinors \( \epsilon^0_- \eta \), where \( \bar{D}_\alpha \eta = 0 \). It can be seen from \( (3.3) \) that the variations are all proportional to \( \lambda \) except for the first term in \( \delta \psi_\alpha \), which also vanishes for the above \( 2N \) Killing spinors. However, when \( \lambda \neq 0 \), the variations \( (3.3) \) vanish only for the \( N \) Killing spinors \( \epsilon^0_+ \eta \). Thus the presence of the membrane breaks further half of the supersymmetry, giving rise to \( N \) fermionic zero modes. The supersymmetry of the membrane then dictates that the solution must have \( N \) bosonic zero modes.
8 Conclusions

We have seen that the usual super $p$-brane solutions of supergravity are merely special cases of more general solutions which can have fewer supersymmetries. The construction of a Green-Schwarz action for these new $p$-branes remains an interesting unsolved problem. Curiously, the superfivebrane in $D = 11$, which is conjectured to be dual to the supermembrane, seems not to admit new non-Ricci-flat versions with fewer supersymmetries since $S^4$ is the only Einstein $M^4$ known to admit Killing spinors. The situation is different in the case where $M^4$ is Ricci flat, since then $K3$ provides a supersymmetric example. It also remains to be seen how these new objects fit into the already bewildering mesh of interconnections between $p$-branes and strings.

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