A new analytic solution for 2nd-order Fermi acceleration

Philipp Mertsch

Rudolf Peierls Centre for Theoretical Physics, University of Oxford,
1 Keble Road, Oxford OX1 3NP, UK
E-mail: p.mertsch1@physics.ox.ac.uk

Abstract. A new analytic solution for 2nd-order Fermi acceleration is presented. In particular, we consider time-dependent rates for stochastic acceleration, diffusive and convective escape as well as adiabatic losses. The power law index $q$ of the turbulence spectrum is unconstrained and can therefore account for Kolmogorov ($q = 5/3$) and Kraichnan ($q = 3/2$) turbulence, Bohm diffusion ($q = 1$) as well as the hard-sphere approximation ($q = 2$). This considerably improves beyond solutions known to date and will prove a useful tool for more realistic modelling of 2nd-order Fermi acceleration in a variety of astrophysical environments.

Keywords: particle acceleration, cosmic ray theory
1 Introduction

Stochastic acceleration of relativistic particles by plasma wave turbulence – a 2nd-order Fermi process [1] – is dominating the production of non-thermal particle distributions in a variety of astrophysical environments. Radio galaxies [2–7], clusters of galaxies [8–11], gamma-ray bursts [12, 13], extra-galactic large-scale jets [14, 15], blazars [16, 17], solar flares [18–20], the interstellar medium [21, 22], the galactic centre [23, 24], supernova remnants [25–27] and even the recently discovered “Fermi bubbles” [28] have been suggested as sites of stochastic acceleration.

The dynamics of the phase-space density $f(p, x, t)$ of relativistic particles interacting with a turbulent magnetised plasma is governed by a Fokker-Planck equation. If the time for pitch-angle scattering is much smaller than the timescales of interest, i.e. acceleration, escape and loss times, it suffices to consider the isotropic part of the phase-space density, $f(p, x, t)$ where $p = \sqrt{\mathbf{p}^2}$. Here we consider only relativistic energies such that energy $E$ and momentum $p$ are essentially the same, $E = pc$. Furthermore, we constrain ourselves to a distribution function $f(p, t)$ independent of position, e.g. the spatial average if the rates only change slowly over the acceleration region. In the special case of transport coefficients constant in time (see discussion in Sec. 2), this constraint can be relaxed as the spatial and the momentum parts of the problem decouple, see e.g. Ch. 14 of Ref. [29].

Stochastic acceleration is a biased diffusion in momentum space and results in a broadening and systematic shift of the injection spectrum to higher momenta. In quasi-linear theory, particles interact resonantly with turbulent plasma waves of a range of wave-lengths similar to the particles gyroradius $r_g$, and the resulting momentum diffusion coefficient $D_{pp}$ depends on the spectrum of turbulence. In particular, if we consider resonant interaction with a power law spectrum $W(k) \propto k^{-q}$ of MHD waves of velocity $v_A = \beta_A c$, the diffusion coefficient takes the form [30–32]

$$D_{pp} = \frac{\zeta \beta_A^2 p^2 c}{r_g^{2-q} A^{q-1}}, \quad (1.1)$$
where $\zeta = (\delta B)^2/B^2$ is the energy in turbulence $(\delta B)^2 = \int_{k_1}^{k_2} dk' W(k')$, compared to the energy of the background magnetic field $B$ and $\lambda_2 = 2\pi/k_1$ is the longest wave-length of the MHD modes. The momentum dependence of the diffusion coefficient reflects the power law behaviour of the turbulence spectrum, $D_{pp} \propto p^q$. In the Kolmogorov and Kraichnan phenomenologies the spectral index is $q = 5/3$ and $3/2$, respectively [33]. A well known and particularly straightforward approximation is the so-called “hard-sphere” limit in which $D_{pp} \propto p^2$. If one however assumes the scattering mean-free path to be equal to the gyroradius one finds $D_{pp} \propto p$ (Bohm limit).

In eq. 1.1 we have omitted a numerical factor which depends on $q$ as well as on the magnetic and cross helicity, e.g. in the case of slab-Alfvén turbulence [34]. For $1.5 \lesssim q \lesssim 2.5$ and the simplest case of right-/left-handed waves propagating parallel and antiparallel with the same power, this factor is $\mathcal{O}(1)$. If Alfvén waves however only propagate into one direction, the momentum diffusion coefficient vanishes. Furthermore, also allowing for obliquely propagating modes one needs to consider that Alfvénic turbulence is inherently anisotropic [35, 36] which reduces the efficiency of second-order acceleration. In contrast, fast-mode waves or turbulence at super-Alfvénic scales (which cascades hydrodynamically) are isotropic [36] and probably dominate 2nd-order Fermi acceleration [37, 38].

Despite the relevance of stochastic acceleration for a variety of astrophysical environments, however, only solutions for a somewhat limited range of $q$ and assumptions about the acceleration, escape and loss rates have been presented. The first solution of the Fokker-Planck equation including Bremsstrahlung losses but only a constant escape rate was given for the hard-sphere limit ($q = 2$) [39] using Mellin transforms with respect to particle energy. Still for $q = 2$, this was later extended [40] to include time-dependent escape but no Bremsstrahlung. A comprehensive review of known time-dependent solutions (and a discussion of the importance of boundary conditions and its relevance for the steady state spectrum) was presented in Ref. [41], although only for time-independent rates and limited values of $q$. Finally, the transport equation has been solved for arbitrary $q$ [29] and extensively discussed [42, 43], however, again only for time-independent rates. We conclude that for time-dependent rates, a solution only exists for $q = 2$ whereas for general $q$ the loss and gain rates must be assumed to be time-independent.

For certain classes of environments, however, this assumption is difficult to justify. For example for very young ($\lesssim 100$ yr) supernova remnants, the acceleration, escape and adiabatic loss rates are expected to vary on timescales less than or equal to the acceleration or loss times [26, 44]. Another example are blazar jets with variability on timescales down to minutes [45]. What is therefore needed is the time-dependent solution of the Fokker-Planck equation for arbitrary $q$ and allowing for time-dependent energy gain and loss rates. In this paper, we present a time-dependent solution of the Fokker-Planck equation for general $q$ considering time-dependent stochastic acceleration, escape and adiabatic losses or gains. We employ a combination of integral transforms to reduce the transport equation to the heat equation in $(\gamma + 1)$-dimensional spherical coordinates (where $\gamma$ is a function of $q$). Our result improves beyond solutions known to date and constitutes an important contribution to particle transport theory.

In Section 2 of this paper we derive the Green’s function of the Fokker-Planck equation with time-dependent stochastic acceleration, diffusive or convective escape and adiabatic loss/gain terms for arbitrary $q$, using a combination of integral transforms. We apply this newly found solution to four specific (toy) models of time-dependencies in Section 3. In particular, assuming all rates to be constant the solution of Ref. [42] is recovered which
constitutes a non-trivial test of our calculation. We conclude in Section 4 with some remarks on boundary conditions and the existence of the steady state solution.

2 The Green’s function

We start from the transport equation for the isotropic and spatially averaged phase space density \( f(p,t) \) in flux conservation form \([32, 46]\),

\[
\frac{\partial f(p,t)}{\partial t} = - \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 \left( -D_{pp}(p,t) \frac{\partial f(p,t)}{\partial p} + A(p,t)f(p,t) \right) \right) - \frac{f(p,t)}{\tau(p,t)} + \frac{S(p,t)}{4\pi p^2}. \tag{2.1}
\]

The terms on the right hand side describe biased diffusion in momentum space with a diffusion coefficient \( D_{pp}(p,t) \), additional energy gain/loss processes with a rate \( A(p,t) \), global escape with a rate \( 1/\tau(p,t) \) and injection with a rate \( S(p,t)/(4\pi p^2) \).

For adiabatic losses/gains the rate is proportional to \( p \), so we define \( a(t) \) by

\[
A(p,t) = mc \left( \frac{p}{mc} \right) a(t), \tag{2.2}
\]

where \( m \) is the mass of the particle. We note that this form can in principle also account for bremsstrahlung losses and gains by 1st-order Fermi acceleration at shocks. However, with this particular momentum dependence, cooling by synchrotron radiation or inverse Compton scattering cannot be accounted for because the momentum dependence of the loss rate \( A(p,t) \) is more complex than the usually assumed \( p^2 \) which is only valid in the Thomson regime.

While in certain limits, e.g. in the steady-state case (see e.g. \([29]\)), analytical solutions might be possible, the fully general case is only amenable to numerical approaches. If the diffusion in momentum space is due to resonant interactions with MHD waves, the momentum dependence of the diffusion coefficient \( D_{pp} \) reflects the spectrum of the turbulence cascade. In particular, assuming the spectral energy density \( W(k) \) to be \( \propto k^{-q} \), we have \( D_{pp} \propto p^q \), and we define the acceleration rate \( k(t) \) by

\[
D_{pp}(p,t) = k(t)(mc)^2 \left( \frac{p}{mc} \right)^q, \tag{2.3}
\]

where \( q = 1 \) for Bohm diffusion, \( q = 3/2 \) for Kraichnan turbulence, \( q = 5/3 \) for Kolmogorov turbulence and \( q = 2 \) in the hard-sphere approximation.

2.1 Diffusive escape

Assuming that the interactions with the same turbulent MHD waves dominate spatial diffusion and therefore the diffusive escape from the acceleration region, fixes the energy dependence of the escape rate. In particular, the time for diffusive escape from a region of spatial extent \( L \) is \( \tau \sim L^2/D_{xx} \) where the spatial diffusion coefficient \( D_{xx} \) is related to the momentum diffusion coefficient \( D_{pp} \) by \( D_{xx}D_{pp} = \xi v_A^2 p^2 \) with \( v_A \) the Alfvén velocity. Here, \( \xi \) is a factor that depends on \( q \) as well as the magnetic and cross helicity of the magnetic turbulence. In this case, \( \tau \propto p^{q-2} \) and we define \( \tau_d(t) \) through the relation

\[
\tau(p,t) = \tau_d(t) \left( \frac{p}{mc} \right)^{q-2}. \tag{2.4}
\]
We are now looking for the Green’s function to the transport equation 2.1, that is $f(p, t)$ for mono-energetic, impulsive injection $S(p, t) = \delta(p - p_0)\delta(t - t_0)/(4\pi p^2)$. Introducing the dimensionless momentum variable $x \equiv p/(mc)$, the transport equation reads,

$$
\frac{\partial f}{\partial t} + 3 a(t)f + (a(t) - (2 + q)k(t)x^q - x) \frac{\partial f}{\partial x} - k(t)x^q \frac{\partial^2 f}{\partial x^2} + \frac{f}{\tau_d(t)} x^{2-q} = \frac{\delta(x-x_0)\delta(t-t_0)}{(mc)^3 4\pi x_0^2},
$$

(2.5)

with $x_0 \equiv p_0/(mc)$.

We make the substitutions,

$$
\rho(x, t) = 2x^{(2-q)/2} \sqrt{g(t)}\psi(t) \quad \text{where} \quad g(t) = \exp \left[-(2-q) \int_{t_0}^{t} dt' a(t') \right],
$$

(2.6)

$$
\eta = \varphi(t) \quad \text{and} \quad f = \hat{f} \exp \left[ y \alpha(t) - \int_{t_0}^{t} dt' \lambda(t') \alpha(t') - 3 \int_{t_0}^{t} dt' a(t') \right],
$$

(2.7)

where we choose

$$
\frac{d\alpha}{dt} = (2-q)^2 \alpha(t)^2 k(t)g(t) - \frac{1}{\tau_d(t)g(t)},
$$

(2.8)

$$
\psi(t) = \exp \left[ 2(2-q)^2 \int_{t_0}^{t} dt' \alpha(t')k(t')g(t') \right],
$$

(2.9)

$$
\varphi(t) = (2-q)^2 \int_{t_0}^{t} dt' k(t')g(t')\psi(t').
$$

(2.10)

Eq. 2.5 then transforms to

$$
\frac{\partial \hat{f}}{\partial \eta} = \frac{\partial^2 \hat{f}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \hat{f}}{\partial \rho} + \exp \left[-x\alpha + \int_{t_0}^{t} dt' \lambda \alpha + 3 \int_{t_0}^{t} dt' a(t') \right] (\varphi(t))^{-1} \delta(x-x_0)\delta(t-t_0),
$$

(2.11)

that is the heat equation with spherical symmetry in $(\gamma+1)$-dimensional spherical coordinates where $\gamma = (4+q)/(2-q)$. Equation 2.8 is a special case of the Riccati equation and solutions $\alpha(t)$ for explicit $k(t), g(t)$ and $\tau_d(t)$ are known and have been compiled in Refs. [47–49].

The bounded Green’s function, i.e. the solution to

$$
\frac{\partial \hat{f}}{\partial \eta} = \frac{\partial^2 \hat{f}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \hat{f}}{\partial \rho} + \delta(\rho - \rho_0)\delta(\eta - \eta_0)
$$

(2.12)

that remains finite for all $\rho$ and $\eta > 0$, is

$$
\hat{f}(\rho, \rho_0, \eta, \eta_0) = \frac{\rho_0}{2(\eta - \eta_0)} \exp \left[ -\frac{\rho^2 + \rho_0^2}{4(\eta - \eta_0)} \right] I_{\gamma-1}^{\gamma-1} \left( \frac{\rho \rho_0}{2(\eta - \eta_0)} \right) \left( \frac{\rho}{\rho_0} \right)^{(1-\gamma)/2},
$$

(2.13)

with $I_{(\gamma-1)/2}$ the modified Bessel function of the first kind. Resubstituting for $\rho = \rho(x, t)$ and $\eta = \eta(t)$ one finds for $q \neq 2,

$$
f = \frac{1}{(mc)^3 4\pi x_0^2} \exp \left[ \frac{3}{2} \int_{t_0}^{t} dt' a(t') \right] \exp \left[ x^{2-q} g \alpha - x_0^{2-q} \alpha_0 \right] \frac{(x x_0)^{2-q}}{\varphi} \sqrt{\psi}
$$

$$
\times \exp \left[ -\frac{x^{2-q} g \psi + x_0^{2-q} \psi_0}{\varphi} \right] I_{\frac{\gamma+q}{2-q}}^{\frac{\gamma+q}{2-q}} \left( \frac{2(x x_0)^{2-q}}{\varphi^2} \sqrt{\psi} \right) \left( \frac{x}{x_0} \right)^{-3/2}.
$$

(2.14)
which, together with eqs. 2.8 - 2.10, constitutes the main result of this paper.

For the hard-sphere approximation, one needs to carefully take the limit \( q \to 2 \). After some tedious algebra including a number of non-trivial cancellations, one arrives at the known result [26, 40],

\[
\begin{align*}
    f &= \frac{1}{(mc)^3 4\pi x_0^2} \frac{1}{x} \frac{1}{4\pi} \exp\left[-\frac{\int_{t_0}^{t} dt'}{\tau_4}\right] \exp\left[-\frac{\left(\ln x - \ln x_0 - \int_{t_0}^{t} dt' a(t') - 3 \int_{t_0}^{t} dt' k\right)^2}{4 \int_{t_0}^{t} dt' k}\right].
\end{align*}
\] (2.15)

We stress that this result has been derived in a way completely independent from the one in Refs. [26, 40] and therefore constitutes a valuable test of our calculation.

### 2.2 Convective escape

If we assume an energy-independent escape time as is for example the case if convection out of the acceleration zone is dominating over diffusive escape,

\[
\tau_{\text{esc}}(p, t) \equiv \tau_c(t),
\] (2.16)

the Green’s function takes a form similar to eq. 2.14,

\[
\begin{align*}
    f &= \frac{1}{(mc)^3 4\pi x_0^2} \frac{2 - q}{x} \exp\left[-\frac{\int_{t_0}^{t} dt' a(t')}{\tau_c}\right] \exp\left[-\frac{3}{2} \int_{t_0}^{t} dt' k\right] \frac{(x x_0)^{\frac{2-q}{q}} \sqrt{g}}{(q-2)^2 \int_{t_0}^{t} dt' k g}
    
    &\times \exp\left[-\frac{x^2-q}{g} + \frac{x_0^2-q}{g}\right] I_{\frac{q+1}{2-q}} \left(\frac{2 (x x_0)^{\frac{2-q}{q}} \sqrt{g}}{(q-2)^2 \int_{t_0}^{t} dt' k g} \right) \left(\frac{x}{x_0}\right)^{-\frac{3}{2}}.
\end{align*}
\] (2.17)

For \( q = 2 \), the result is again eq. 2.15 since in the hard-sphere approximation the escape time for diffusive and convective escape are both energy-independent, \( \tau_c \sim \tau_d \).

### 3 Examples

#### 3.1 Constant acceleration and escape rates, but no adiabatic losses

We assume that the acceleration and escape rates are constant, \( k(t) \equiv k_0, \tau_a(t) \equiv \tau_{d0} \), and that there are no adiabatic losses, \( a(t) \equiv 0 \). The Riccati equation 2.8 is solved by

\[
\alpha = -\frac{1}{\varepsilon \sqrt{k_0 \tau_{d0}}},
\] (3.1)

and we find for \( \psi(t) \) and \( \varphi(t) \)

\[
\begin{align*}
    \psi(t) &= \exp\left[-2(2 - q) \frac{k_0}{\tau_{d0}} (t - t_0)\right],
    
    \varphi(t) &= \frac{1}{2} (2 - q) \sqrt{k_0 \tau_{d0}} (1 - \psi).
\end{align*}
\] (3.2)
(3.3)

For \( q \neq 2 \), the particle density \( n(x, x_0, t, t_0) = 4\pi x^2 f(x, x_0, t, t_0) \) for impulsive injection reads

\[
\begin{align*}
    n(x, x_0, t, t_0) &= \frac{2 - q}{x} \sqrt{\frac{x}{x_0}} \frac{2 (x x_0)^{(2-q)/2} \sqrt{\psi}}{x_0 (2 - q) \sqrt{k_0 \tau_{d0}} (1 - \psi)} \exp\left[-\frac{(x^2-q+x_0^2-q)(1+\psi)}{(2 - q) \sqrt{k_0 \tau_{d0}} (1 - \psi)}\right]
    
    &\times I_{\frac{q+1}{2-q}} \left(\frac{4 (x x_0)^{(2-q)/2} \sqrt{\psi}}{(2 - q) \sqrt{k_0 \tau_{d0}} (1 - \psi)}\right),
\end{align*}
\] (3.4)
and for \( q = 2 \) this reduces to

\[
n(x, x_0, t, t_0) = \frac{1}{p} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{k_0(t-t_0)}} \exp \left[ -\frac{(\ln x - \ln x_0 - 3k_0(t-t_0))^2}{4k_0(t-t_0)} \right]. \tag{3.5}
\]

Equations 3.4 and 3.5 are identical to eqs. 46 and 49 of Ref. [42] (setting their \( a \equiv 0 \)), respectively (see also [29]). Their result was however derived assuming constant acceleration rate and escape time. Reproducing this result as a special case of our more general solution therefore constitutes a non-trivial test of our calculation.

In the left column of Fig. 1 (compare with Fig. 2 of Ref. [42]) we show the particle spectrum for impulsive injection, \( n(x, 1, t, 0) \), for Kraichnan turbulence, Kolmogorov turbulence and the hard-sphere approximation. We fixed \( \tau_{d0} = 1 \) and all timescales (rates) are in units of \( 1/k_0 \) \((k_0)\). Diffusion and advection in momentum space lead to a broadening of the spectrum and monotonous increase of the mean energy with time. Since in the hard-sphere approximation the acceleration timescale \( p^2/D_{pp} \) is the same for all momenta, the spectrum (in the logarithmic momentum variable, \( \log x \)) is even with respect to the mean logarithmic energy, \( \log x_0 + 3k_0(t-t_0) \). For Kraichnan and Kolmogorov turbulence the behaviour is qualitatively different as the acceleration time is now increasing with energy. At higher energies, the spectrum becomes gradually softer and asymmetric. For a fixed escape time, the spectrum rolls over at much smaller energies than in the hard-sphere case.

In the right column of Fig. 1 we show the spectra for steady injection,

\[
n(x, x_0, t, t_0) = \int_{t_0}^t dt' f(x, x_0, t, t'), \tag{3.6}
\]

including the steady state spectrum, i.e. the limit \( t \to \infty \). In general, it is not clear whether this integral in fact converges for \( t \to \infty \) but for the case of constant rates, injection and escape exactly balance each other. We note that for the hard-sphere approximation, acceleration and escape rates have the same momentum behaviour and consequently there is no preferred momentum scale. This leads to a power law steady state spectrum whereas for Kraichnan and Kolmogorov phenomenology the spectrum exhibits a long exponential roll-over with a characteristic momentum defined by the equality of acceleration and escape times.

### 3.2 Constant acceleration rate

We use the relation between the space and momentum diffusion coefficients, \( D_{xx}D_{pp} = \xi v_A^2 p^2 \), to express the escape time \( \tau(t,p) = \tau_A(t)x^{-2} \) in terms of the acceleration rate \( k(t) \),

\[
\frac{1}{\tau_A(t)} = \frac{\xi v_A^2(t)}{k(t)L^2} = \frac{1}{k(t)\rho_m} \left( \frac{B(t)}{L(t)} \right)^2, \tag{3.7}
\]

where \( L(t) \) is the size of the acceleration region and the Alfvén velocity \( v_A(t) = B(t)/\sqrt{\rho_m} \), with \( B \) the background magnetic field and \( \rho_m \) the thermal gas mass density. Specifying the adiabatic loss/gain rate, \( a(t) \), to the case of an expanding/contracting flux tube [26],

\[
a(t) = \frac{1}{3} \left( \frac{d \ln L(t)}{dt} - \frac{d \ln B(t)}{dt} \right) \Rightarrow g = \exp \left[ -3 \int_{t_0}^t dt' a(t') \right] = \frac{L_0}{L(t)} \frac{B(t)}{L_0}, \tag{3.8}
\]

with \( L_0 = L(t_0) \) and \( B_0 = B(t_0) \), we can write

\[
\frac{1}{\tau_A(t)} = \frac{1}{k(t)\rho_m} \left( \frac{B_0}{L_0} \right)^2 g^2(t). \tag{3.9}
\]
The Riccati equation 2.8 now reads,

\[
\frac{d\alpha}{dt} = \alpha^2 (q - 2)^2 k g - \frac{\xi}{\rho_m} \left( \frac{B_0}{L_0} \right)^2 \frac{g}{k},
\]

(3.10)
Figure 2. Particle spectrum $n(x, 1, t, 0)$ for impulsive (left panel) and steady injection (right panel) for Kraichnan turbulence and assuming $a = a_0 \exp [-\lambda t]$ with $a_0 = -1$, $\lambda = 0.5$ and $(\xi B_0^2 / L_0^2 \rho_m) = 0.1$. The solid lines are for fixed times $t = 0.01, 0.03, 0.1, 0.3, 1, 3$ and the dashed line denotes the steady state spectrum. All timescales (rates) are in units of $1/k_0$ ($k_0$).

For a constant acceleration rate, $k(t) \equiv k_0$, this is solved by

$$\alpha(t) \equiv \alpha_0 = \frac{1}{q-2} \sqrt{\frac{\xi}{\rho_m} \frac{B_0}{L_0} k_0}, \quad (3.11)$$

leading to

$$\psi(t) = \exp \left[ 2(2-q)^2 \alpha_0 k_0 \int_{t_0}^{t} dt' g(t') \right], \quad (3.12)$$

$$\varphi(t) = \frac{1}{2\alpha_0} (\psi - 1). \quad (3.13)$$

In Fig. 2 we show the particle spectrum $n(x, 1, t, 0)$ for impulsive and steady injection and Kraichnan turbulence. We have chosen $L(t)$ and $B(t)$ such that

$$a(t) = a_0 \exp [-\lambda t], \quad (3.14)$$

with $a_0 = -1$, $\lambda = 0.5$ and $(\xi B_0^2 / L_0^2 \rho_m) = 0.1$. Again, all times (rates) are understood to be in units of $1/k_0$ ($k_0$). For early times ($t \ll 1$), the acceleration and adiabatic loss rate are similar, $|a(t)| \approx k_0$ and escape can be neglected. For intermediate times ($t \sim 1$), the loss rate starts declining which leads to a rather hard spectrum. Finally, for late times ($t \gg 1$), escape becomes important such that the spectrum does not extend to any higher energies. The late onset of escape also reflects in very hard spectra for steady injection and a steady state spectrum extending over several orders of magnitude.

### 3.3 Constant adiabatic loss rate and exponentially decreasing acceleration rate

We assume that the adiabatic loss rate $a(t) < 0$ is constant, $a(t) \equiv a_0$. This is a fair assumption for environments with a blast wave where the shock radius and speed are $r(t) \propto t^{2/5}$ and $v(t) \propto t^{-3/5}$ such that the adiabatic loss rate $a(t) \propto (r^2 v) \propto t^{4/5}$ has a weak time dependence. We find for $g(t)$,

$$g = g_0 \exp [\lambda t] \quad \text{with} \quad \lambda = -(2-q) a_0 > 0 \quad \text{and} \quad g_0 = -\lambda t_0. \quad (3.15)$$
For even later times, escape becomes dominant since its rate is \( \propto \) the acceleration rate has started decreasing while the adiabatic loss rate stays constant. Similarly as in the cases considered before. For intermediate times \((0 \lesssim t \lesssim 1)\), the Green’s function becomes heavily suppressed. This behaviour is even more prominent in the spectrum for steady injection, see right panel of Fig. 3. We show the particle spectrum for Kraichnan turbulence. We assumed \( a = a_0 = \text{const.} \) and \( k = k_0 \exp[-\lambda t] \) with \( a_0 = -0.6 \) and \((\xi B_0^2/L_0^2\rho_m) = 0.2\). The solid lines are for fixed times \( t = 0.01, 0.03, 0.1, 0.3, 1, 3 \). All timescales (rates) are in units of \(1/k_0 \) (\(k_0\)).

We assume further that \( k(t) \) is of the form \( k_0 \exp[-\kappa t] \) and that \( \kappa = \lambda \). (In fact, this can be easily extended to \( \kappa \neq \lambda \) but for clarity we here constrain ourselves to the case \( \kappa = \lambda \).) The Riccati equation 3.10 now reads

\[
\frac{d\alpha}{dt} = \alpha^2 (q - 2)^2 k_0 g_0 - \frac{\xi}{\rho_m} \left( \frac{B_0}{L_0} \right)^2 \frac{g_0}{k_0} \exp[2\lambda t].
\]

and substituting \( y = \exp[2\lambda t] \), \( w = -1/\alpha \) leads to the standard form

\[
\frac{dw}{dy} = \mathcal{A}^2 w^2 + \mathcal{B} \frac{1}{y} \quad \text{with} \quad \mathcal{A} \equiv -\frac{1}{\rho_m} \left( \frac{B_0}{L_0} \right)^2 \frac{g_0}{k_0} \frac{1}{2\lambda} \text{ and } \mathcal{B} \equiv (q - 2)^2 \frac{k_0 g_0}{2\lambda}.
\]

The solution for \( w(y) \) is [49]

\[
w = -1 \frac{1}{\mathcal{A} u} \frac{du}{dy} \quad \text{where} \quad u = \sqrt{y} J_1 \left(2\sqrt{\mathcal{A}B\sqrt{y}}\right),
\]

with \( J_1 \) the Bessel function of the first kind. For \( \alpha(t) \) and \( \psi(t) \) we find

\[
\alpha(t) = \sqrt{\frac{\mathcal{A}}{B}} \frac{J_1 \left(2\sqrt{\mathcal{A}B\sqrt{y_0}}\right)}{J_0 \left(2\sqrt{\mathcal{A}B\sqrt{y}}\right)} \quad \text{and} \quad \psi(t) = \left( \frac{J_1 \left(2\sqrt{\mathcal{A}B\sqrt{y_0}}\right)}{J_0 \left(2\sqrt{\mathcal{A}B\sqrt{y}}\right)} \right)^2,
\]

where \( y_0 = y(t_0) \). \( \varphi(t) \) is again given by eq. 2.10.

In the left panel of Fig. 3 we show the particle spectrum \( n(x, 1, t, 0) \) for impulsive injection for Kraichnan turbulence and for \( \lambda = 0.3 \) and \((\xi B_0^2/L_0^2\rho_m) = 0.2\). All timescales (rates) are in units of \(1/k_0 \) (\(k_0\)). For early times \((t \lesssim 0.1)\), the Green’s function behaves similarly as in the cases considered before. For intermediate times \((0.1 \lesssim t \lesssim 1)\), however, the acceleration rate has started decreasing while the adiabatic loss rate stays constant. For even later times, escape becomes dominant since its rate is \( \propto g^2 \propto \exp[2\lambda t] \) and the Green’s function becomes heavily suppressed. This behaviour is even more prominent in the spectrum for steady injection, see right panel of Fig. 3. We show the particle spectrum for...
intermediate and late times \((t \gtrsim 1)\) only since for early times \((t \lesssim 1)\) it looks very similar to the previously considered examples. Around \(t \approx 5\) the spectrum starts becoming noticeably asymmetric because of the dominance of adiabatic losses over acceleration. The escape rate \(1/\tau_d \propto g^2(t) \propto \exp[2\lambda t]\) keeps increasing and particles injected early have already escaped. However, particles which are injected late \((t \gtrsim 10)\) do not get accelerated anymore and hardly have time to lose energy before escaping. The resulting spectrum therefore converges against the \(\delta\)-like injection though with decreasing amplitude.

### 3.4 No escape

Assuming a vanishing escape rate, \(\tau_d \to \infty\), the Ricatti equation can be directly integrated,

\[
\frac{d\alpha}{dt} = (q - 2)^2 \alpha^2 k g \quad \Rightarrow \quad \alpha = \alpha_0 \left(1 - (q - 2)^2 \alpha_0 \int_{t_0}^{t} dt' k g\right)^{-1}.
\]

(3.20)

Furthermore, \(\psi(t)\) and \(\varphi(t)\) are

\[
\psi(t) = \left(\frac{\alpha(t)}{\alpha_0}\right)^2 \quad \text{and} \quad \varphi(t) = \left((q - 2)^2 \int_{t_0}^{t} dt' k g\right)^{-1} - \alpha_0^{-1}.
\]

(3.21)

For illustration, we consider both \(k(t)\) and \(a(t)\) to be exponentially declining, i.e. \(a(t) = a_0 \exp[-t/\lambda]\) and \(k(t) = k_0 \exp[-t/\kappa]\). In Fig. 4 we show the particle spectrum \(n(x, x_0, t, t_0)\) for impulsive and steady injection for Kraichnan turbulence and choose \(a_0 = -0.5, \lambda = 2, \kappa = 1\). All timescales (rates) are in units of \(1/k_0\) \((k_0)\). At early times, the Green’s function (Fig. 4, left panel) again behaves very similar to the cases considered above. For late times, however, it stalls as both \(a(t)\) and \(k(t)\) approach zero. For momenta \(x > 1\) and steady injection, the steady state is reached rather quickly as only particles injected early enough can reach higher momenta. Close to the injection momentum however, the spectrum still changes even at late times due to the injection of new particles. As \(t \to \infty\), the spectrum reaches its steady state at all momenta except the injection momentum \(x_0\).
4 Summary and conclusion

We have presented a new solution to the Fokker-Planck equation for stochastic particle acceleration by plasma wave turbulence. In extension to previously known solutions we allow the rates for stochastic acceleration, both diffusive and convective escape as well as adiabatic losses to be time-dependent. Furthermore, we do not need to constrain ourselves to specific values of the turbulence spectral index \( q \) and can therefore apply this result to the phenomenologically interesting cases of Kolmogorov \( (q = 5/3) \) and Kraichnan \( (q = 3/2) \) turbulence. We have investigated four examples to illustrate the qualitatively different behaviour of spectra due to the time-dependent rates and extended range in \( q \). In the first example we constrained ourselves, however, to constant rates and neglected adiabatic losses in order to compare our solution to previously presented results which constitutes a non-trivial test of our calculation. We have also considered the case of constant acceleration and exponentially decreasing adiabatic loss rate which leads to rather hard spectra. In contrast, with exponentially decreasing acceleration and constant adiabatic losses, the spectra become very soft and no steady-state solution exists. Finally, for infinite escape time we have presented an example with exponentially decreasing acceleration and adiabatic loss rates which leads to rather soft spectra in particular close to the injection energy.

A few words about boundary conditions are in order. It was pointed out [41] that the initial value problem (IVP), eq. 2.1, with boundary conditions at \( x = 0 \) and \( x \to \infty \) is singular (see Ref. [41] for a definition and discussion of a singular IVP). Therefore, the status of the boundary conditions is not clear but must be determined more carefully. Furthermore, the spectral theory of second order differential equations proves helpful in investigating the conditions under which a steady state solution exists. Unfortunately, it is not possible to analyse the problem at hand in this framework, since the variable coefficients of the partial differential equation prevent the formulation of an equivalent boundary-value problem (BVP) of Sturm-Liouville type. However, the transformed IVP, i.e. the heat equation type eq. 2.11, must satisfy boundary conditions for \( \hat{f} \) at \( \rho \to 0, \infty \) similar to those that the original IVP, eq. 2.5, satisfies for \( f \) at \( x \to 0, \infty \) since both are connected by non-singular transformations. We find that \( \rho \to 0, \infty \) are both limit points (for a definition see Ref. [41]) such that the appropriate boundary conditions at \( \rho \to 0, \infty \) are simply \( ||\hat{f}|| < \infty \) which justifies the choice of the bounded solution in deriving eq. 2.11. Finally, we note that the discussion about the existence of the steady state solution cannot be applied here either due to the variable coefficients. As we have seen in the above examples, the existence of the steady state is largely determined by the time-dependence and asymptotic behaviour of the acceleration and loss rates, \( k(t) \), \( a(t) \) and \( \tau_d(t) \) and therefore needs to be discussed on a case-by-case basis.

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