Coherent states and quantum numbers for twist-deformed oscillator model

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Abstract
The coherent states for twist-deformed oscillator model provided in article [1] are constructed. Besides, it is demonstrated that the energy spectrum of considered model is labeled by two quantum numbers - by so-called main and azimuthal quantum numbers respectively.
The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in \cite{2}. Recently, there were also found formal arguments based mainly on Quantum Gravity \cite{3}, \cite{4} and String Theory models \cite{5}, \cite{6}, indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. \cite{7}, \cite{8}) as well as with field theoretical models (see e.g. \cite{9}, \cite{10}), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic \cite{11} and nonrelativistic \cite{12} symmetries, one can distinguish three basic types of space-time noncommutativity:

1) The canonical (soft) deformation\
\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}, \]
with constant and antisymmetric tensor $\theta_{\mu\nu}$. The explicit form of corresponding Poincare Hopf algebra has been provided in \cite{13}, \cite{14}, while its nonrelativistic limit has been proposed in \cite{15}.

2) The Lie-algebraic case\
\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho} x_\rho, \]
with particularly chosen constant coefficients $\theta_{\mu\nu}^{\rho}$. Particular kind of such space-time modification has been obtained as representations of $\kappa$-Poincare \cite{16}, \cite{17} and $\kappa$-Galilei \cite{18} Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in \cite{19}, \cite{20} and \cite{15}, respectively.

3) The quadratic deformation\
\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau, \]
with constant coefficients $\theta_{\mu\nu}^{\rho\tau}$. Its Hopf-algebraic realization was proposed in \cite{21}, \cite{22} and \cite{20} in the case of relativistic symmetry, and in \cite{23} for its nonrelativistic counterpart.

Besides, it has been demonstrated in \cite{24}, that in the case of so-called N-enlarged Newton-Hooke Hopf algebras \( U_0^{(N)}(NH_\pm) \), the twist deformation provides the new space-time noncommutativity of the form\(^2\)
\[ [t, x_i] = 0, \quad [x_i, x_j] = i f_{\kappa,\pm}(t) \theta_{ij}(x), \]
with time-dependent functions
\[ f_{\kappa,\pm}(t) = \kappa f \left( \sinh \left( \frac{t}{\tau} \right), \cosh \left( \frac{t}{\tau} \right) \right), \quad f_{\kappa,\pm}(t) = \kappa f \left( \sin \left( \frac{t}{\tau} \right), \cos \left( \frac{t}{\tau} \right) \right), \]
\(^1x_0 = ct.\)
\(^2\) The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. \cite{13}, \cite{14}), for quantum N-enlarged Newton-Hooke Hopf algebras.
\(\theta_{ij}(x) \sim \theta_{ij} = \text{const} \) or \(\theta_{ij}(x) \sim \theta_{ij}^k x_k\) and \(\tau\) as well as \(\kappa\) denoting the cosmological constant and deformation parameter respectively. It should be also noted that different relations between all mentioned above quantum spaces (1), (2), (3) and (4)) have been summarized in article [24].

Let us now turn to the quantum oscillator model defined on the twist-deformed phase space [1]

\[
[ t, \bar{x}_i ] = 0 \quad , \quad [ \bar{x}_1, \bar{x}_2 ] = i f_\kappa(t) \quad , \quad [ \bar{x}_i, \bar{p}_j ] = i\hbar \delta_{ij} \quad , \quad [ \bar{p}_i, \bar{p}_j ] = 0 . \tag{5}
\]

It’s dynamic is given by the following Hamiltonian function with constant mass \(m\) and frequency \(\omega\)

\[
\hat{H}(\bar{p}, \bar{x}) = \frac{1}{2m} (\bar{p}_1^2 + \bar{p}_2^2) + \frac{1}{2} m\omega^2 (\bar{x}_1^2 + \bar{x}_2^2) , \tag{6}
\]

In order to analyze the above system we represent the noncommutative variables \((\bar{x}_i, \bar{p}_i)\) on classical phase space \((x_i, p_i)\) as follows (see e.g. [25], [26])

\[
\bar{x}_1 = \hat{x}_1 - \frac{f_\kappa(t)}{2\hbar} \hat{p}_2 \quad , \quad \bar{x}_2 = \hat{x}_2 + \frac{f_\kappa(t)}{2\hbar} \hat{p}_1 , \tag{7}
\]

where

\[
[ \hat{x}_i, \hat{x}_j ] = 0 = [ \hat{p}_i, \hat{p}_j ] \quad , \quad [ \hat{x}_i, \hat{p}_j ] = i\hbar \delta_{ij} . \tag{8}
\]

Then, the Hamiltonian (6) takes the form

\[
H_f(t) = \frac{(\hat{p}_1^2 + \hat{p}_2^2)}{2M_f(t)} + \frac{1}{2} M_f(t)\Omega_f^2(t) (\hat{x}_1^2 + \hat{x}_2^2) - \frac{f_\kappa(t)}{2\hbar} m\omega^2 \hat{L} , \tag{9}
\]

with symbol

\[
\hat{L} = \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1 \tag{10}
\]

denoting angular momentum of particle. Besides, the coefficients \(M_f(t)\) and \(\Omega_f(t)\) present in the above formula denote the time-dependent functions given by

\[
M_f(t) = \frac{m}{1 + \frac{m^2\omega^2 f_\kappa^2(t)}{4\hbar^2}} , \quad \Omega_f(t) = \omega \sqrt{1 + \frac{m^2\omega^2 f_\kappa^2(t)}{4\hbar^2}} , \tag{11}
\]

respectively, such that

\[
M_f(t)\Omega_f^2(t) = m\omega^2 = \text{const} . \tag{12}
\]

Further, we introduce a set of time-dependent creation \((a_A^\dagger(t))\) and annihilation \((a_A(t))\) operators

\[
\hat{a}_\pm(t) = \frac{1}{2\sqrt{\hbar}} \left[ \frac{\hat{p}_1 \pm i\hat{p}_2}{\sqrt{M_f(t)\Omega_f(t)}} - i\sqrt{M_f(t)\Omega_f(t)}(\hat{x}_1 \pm i\hat{x}_2) \right] , \tag{13}
\]

\[3\]See type 4) of quantum space-time.
\[4\]It should be noted that for \(f_\kappa(t) = \theta\) we get the canonically deformed oscillator model provided in [20].
satisfying the standard commutation relations
\[
[\hat{a}_A, \hat{a}_B] = 0, \quad [\hat{a}_A^\dagger, \hat{a}_B^\dagger] = 0, \quad [\hat{a}_A, \hat{a}_B^\dagger] = \delta_{AB}; \quad A, B = \pm.
\] (14)

Then, one can easily check that in terms of the operators (13) the Hamiltonian function (9) looks as follows
\[
\hat{H}_f(t) = \Omega_+(t) \left( \hat{N}_+(t) + \frac{1}{2} \right) + \Omega_-(t) \left( \hat{N}_-(t) + \frac{1}{2} \right),
\] (15)

with
\[
\Omega_\pm(t) = \Omega_f(t) \mp \frac{f_\kappa(t)m\omega}{2\hbar},
\] (16)

and number operators in ± direction given by
\[
\hat{N}_\pm(t) = \hat{a}_\pm^\dagger(t)\hat{a}_\pm(t),
\] (17)

respectively. Moreover, we see that the energy eigenvectors can be generated in a standard way as follows
\[
|n_+, n_-, t> = \frac{1}{\sqrt{n_+!}\sqrt{n_-!}} \left( \hat{a}_+^\dagger(t) \right)^{n_+} \left( \hat{a}_-^\dagger(t) \right)^{n_-} |0> .
\] (18)

while the corresponding (parameterized by \(n_+\) and \(n_-\)) eigenvalues are
\[
E_{n_+, n_-}(t) = \Omega_+(t) \left( n_+ + \frac{1}{2} \right) + \Omega_-(t) \left( n_- + \frac{1}{2} \right), \quad n_+, n_- = 0, 1, 2, \ldots .
\] (19)

Besides, using operator representation (13) one finds
\[
(\Delta \hat{x}_i)^2|n_+, n_-, t> = \frac{\hbar^2}{4}(1 + n_+ + n_-)^2 ,
\] (20)

where symbol \((\Delta \hat{a})|\varphi>\) denotes the uncertainty of observable \(\hat{a}\) in quantum state \(|\varphi>\).

The above result means that momentum-position uncertainty relations for eigenstates (18) become saturated only for \(n_+ = n_- = 0\), i.e. only for vacuum vector \(|0>\). Apart from that it is easy to see that the momentum operator (10) can be written as follows
\[
\hat{L} = \hbar \left( \hat{a}_-^\dagger(t)\hat{a}_+(t) - \hat{a}_+^\dagger(t)\hat{a}_-(t) \right).
\] (21)

while it’s action on quantum states (18) is given by
\[
\hat{L}|n_+, n_-, t> = \hbar(n_- - n_+)|n_+, n_-, t> .
\] (22)

Consequently, the energy spectrum (19) can be written in terms of eigenvalues (22) as follows
\[
E_{n_+, n_-}(t) = \hbar\Omega_f(t)(n_+ + n_- + 1) + \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2}(n_- - n_+) .
\] (23)
Let us now solve two problems. First of them concerns the construction of so-called coherent states for considered model, i.e. the quantum vectors which saturate the momentum-position Heisenberg uncertainty relations. The second problem applies to the proper interpretation of quantum numbers \( n = n_+ + n_- \) and \( l = n_- - n_+ \) labeling the energy spectrum (23).

Hence, let us consider the quantum states of the form
\[
|c_+, c_-, t> = \sum_{n_+, n_-} \frac{c_+^{n_+} e^{-\frac{1}{2}|c_+|^2}}{\sqrt{n_+!}} \frac{c_-^{n_-} e^{-\frac{1}{2}|c_-|^2}}{\sqrt{n_-!}} |n_+, n_-, t>,
\]
which play the role of the eigenfunctions for annihilation operators (13)
\[
\hat{a}_\pm(t)|c_+, c_-, t> = c_\pm |c_+, c_-, t> .
\]

By direct calculation one may check that
\[
(\Delta p_i)^2|c_+, c_-, t> = \frac{\hbar M_f(t) \Omega_f(t)}{2}, \quad (\Delta x_i)^2|c_+, c_-, t> = \frac{1}{2} \frac{\hbar}{M_f(t) \Omega_f(t)} , \quad i = 1, 2,
\]
what leads to the saturated momentum-position Heisenberg uncertainty relations
\[
(\Delta p_i)^2|c_+, c_-, t> (\Delta x_i)^2|c_+, c_-, t> = \frac{\hbar^2}{4}, \quad i = 1, 2.
\]

Consequently, we see that the vectors (24) are (in fact) nothing else than the coherent states for twist-deformed oscillator model, satisfying
\[
<\hat{H}_f>|c_+, c_-, t> = E|0, 0, t>(t) + \frac{\Omega_f(t)}{\hbar} (\Delta L)^2|c_+, c_-, t> + \frac{M_f(t) \Omega_f^2(t) f_\kappa(t)}{2 \hbar} <L>|c_+, c_-, t>,
\]
with
\[
<\hat{L}>|c_+, c_-, t> = \hbar (|c_-|^2 - |c_+|^2) , \quad (\Delta L)^2|c_+, c_-, t> = \hbar^2 (|c_-|^2 + |c_+|^2).
\]

In the case of second problem one should to solve the eigenvalue equation for Hamiltonian (9) written in terms of polar coordinates
\[
\hat{H}_f(t)\psi(r, \varphi, t) = E(t)\psi(r, \varphi, t),
\]
where
\[
\hat{H}_f(t) = -\frac{\hbar^2}{2M_f(t)} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{\hbar^2} \frac{\hat{L}^2}{r^2} \right) + \frac{M_f(t) \Omega_f^2(t)}{2} - \frac{f_\kappa(t) M_f(t) \Omega_f^2(t)}{2 \hbar} \hat{L},
\]

5
\[
\hat{L} = -i\hbar \frac{\partial}{\partial \varphi} , \quad [\hat{H}, \hat{L}] = 0 .
\] (33)

To this aim, it is convenient to take the corresponding eigenfunctions in the form
\[
\psi(r, \varphi, t) = \phi(\varphi) R(r, t) ,
\] (34)

with its azimuthal part \( \phi(\varphi) \) satisfying
\[
\hat{L}\phi_l(\varphi) = \hbar l \phi_l(\varphi) , \quad \phi_l(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i l \varphi} , \quad l = 0, \pm 1, \pm 2, \ldots .
\] (35)

Then, the proper equation for radial function \( R(r, t) \) looks as follows
\[
\left( -\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{l^2}{\rho^2} + \frac{\rho^2}{4} - \mathcal{E}_l(t) \right) R_l(\rho(t)) = 0 ,
\] (36)

where \( \rho(t) = r \sqrt{2M_f(t)\Omega_f(t)/\hbar} \) plays the role of dimensionless variable. Its physical solution can be written as
\[
R_l^{(n)}(\rho(t)) = w_l^{(n)}(\rho(t))e^{\rho^2(t)/4} ,
\] (37)

with \( w_l^{(n)}(\rho(t)) \) denoting the polynomial of degree \( n \). Then, the equation (36) reduces to the following one
\[
-\frac{\partial^2 w_l^{(n)}(\rho(t))}{\partial \rho^2} + \frac{\rho^2 - 1}{\rho} \frac{\partial w_l^{(n)}(\rho(t))}{\partial \rho} + \frac{l^2}{\rho^2} w_l^{(n)}(\rho(t)) - (\mathcal{E}_l(t) - 1)w_l^{(n)}(\rho(t)) = 0 ,
\] (38)

for which the solution (this time) is given by
\[
w_l^{(n)}(\rho(t)) = a_l^{(n)} \left( 1 + \sum_{k=1}^{(n-l)/2} \left[ \prod_{s=1}^{k} \frac{n + 2 - (2s + |l|)}{l^2 - (2s + |l|)^2} \right] \rho^{2k}(t) \right) \rho^{|l|}(t) ,
\] (39)

only when
\[
\mathcal{E}_l(t) \rightarrow \mathcal{E}_l^{(n)}(t) = n + 1 , \quad l \in \{-n, -n + 2, \ldots, n - 2, n\} , \quad n = 0, 1, 2, 3, \ldots ,
\] (40)
or (equivalently)
\[
\mathcal{E}_l^{(n)}(t) = \hbar \Omega_f(t)(n + 1) + \frac{f_s(t) M_f(t) \Omega_f^2(t)}{2} l .
\] (41)

Consequently, after substitution \( n = n_+ + n_- \) and \( l = n_- - n_+ \) into eigenvalues (41) we get (in fact) the energy spectrum (23) labeled by \( n_+ \) and \( n_- \) parameters. For this reason as well as due to the formulas (35), (37) and (41) the quantities \( n \) and \( l \) may be called the “main” and “azimuthal” quantum numbers respectively.

\footnote{The symbol \( a_l^{(n)} \) denotes the normalization factor.}
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References

[1] M. Daszkiewicz, C.J. Walczyk, Acta Phys. Pol. B 40, 293 (2009); arXiv: 0812.1264 [hep-th]
[2] H.S. Snyder, Phys. Rev. 72, 68 (1947)
[3] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39 (1994); Comm. Math. Phys. 172, 187 (1995); hep-th/0303037
[4] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); hep-th/9612084
[5] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802, 003 (1998)
[6] N. Seiberg and E. Witten, JHEP 9909, 032 (1999); hep-th/9908142
[7] A. Deriglazov, JHEP 0303, 021 (2003); S. Ghosh, Phys. Lett. B 648, 262 (2007)
[8] V.P. Nair, A.P. Polychronakos, Phys. Lett. B 505, 267 (2001); M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001)
[9] P. Kosinski, J. Lukierski, P. Maslanka, Phys. Rev. D 62, 025004 (2000)
[10] M. Chaichian, P. Preˇ snajder and A. Tureanu, Phys. Rev. Lett. 94, 151602 (2005); G. Fiore, J. Wess, Phys. Rev. D 75, 105022 (2007)
[11] S. Zakrzewski, Comm. Math. Phys. 185, 285 (1997); q-alg/9602001
[12] Y. Brihaye, E. Kowalczyk, P. Maslanka, Mod. Phys. Lett. A 16, 321 (2001); math/0006167
[13] R. Oeckl, J. Math. Phys. 40, 3588 (1999)
[14] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004)
[15] M. Daszkiewicz, Mod. Phys. Lett. A 23, 7 (2008), IFT-UWR-LV-420; arXiv: 0801.1206 [hep-th]
[16] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331 (1991)
[17] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 293, 344 (1992)
[18] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)
[19] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, J. Phys. A 27, 2389 (1994)

[20] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006); hep-th/0508083

[21] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Comm. Math. Phys. 150, 495 (1992)

[22] P. Aschieri, L. Castellani, A.M. Scarfone, Eur. Phys. J. C 7, 159 (1999); q-alg/9709032

[23] M. Daszkiewicz, Mod. Phys. Lett. A 23, 1757 (2008), IFT-UWR-LV-724; arXiv: 0807.0133 [hep-th]

[24] M. Daszkiewicz, Mod. Phys. Lett. A27 (2012) 1250083; arXiv: 1205.0319 [hep-th]

[25] J. Lukierski, P. Stichel, W.J. Zakrzewski, Ann. Phys. 260, 224 (1997); hep-th/9612017

[26] A. Kijanka, P. Kosinski, Phys. Rev. D 70, 127702 (2004); hep-th/0407246