Quantum Mechanics on a Torus

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Abstract

We present here a canonical description for quantizing classical maps on a torus. We prove theorems analogous to classical theorems on mixing and ergodicity in terms of a quantum Koopman space $L^2(\mathfrak{A}_\hbar, \tau_\hbar)$ obtained as the completion of the algebra of observables $\mathfrak{A}_\hbar$ in the norm induced by the following inner product $(A, B) = \tau_\hbar(A^\dagger B)$, where $\tau_\hbar$ is a linear functional on the algebra analogous to the classical “integral over phase space.” We also derive explicit formulas connecting this formulation to the $\theta$-torus decomposition of Bargmann space introduced in ref.\[1]
I. Introduction

Quantum mechanics on a toroidal phase space has reared its head in many varied fields: from string theory (ref. 2) to quantum computation (ref. 3), the quantized torus seems to present a simple and workable framework for understanding the quantum world when phase space is compact. In fact, two seemingly different approaches to the “quantum torus” are widely used in the literature.

In physics papers, an ad hoc “quantization” is used in which a prescription for a unitary $N \times N$ matrix (with $N = \text{inverse Planck’s constant}$) is given which yields the classical dynamics in the limit as $N \to \infty$ ($\hbar \to 0$) (see ref. 4). A more common approach in mathematical physics, and one which we adopt here as our starting point, is to quantize the classical algebra generated by $\exp(2\pi i x)$ and $\exp(2\pi i p)$, the generators of the periodic functions on the plane, and hence functions on the torus. In refs. 1 and 5 a connection between these two approaches was given explicitly.

In this latter, algebraic approach, classical observables (functions) over a toroidal phase space are replaced by non-commuting operators in a canonical fashion:

$$\exp(2\pi i x) \to U$$
$$\exp(2\pi i p) \to V$$

where $U$ and $V$ satisfy the canonical commutation relation

$$UV = e^{4\pi^2 i \hbar} VU.$$ 

The Hilbert space on which these operators act is taken to be the usual $L^2(\mathbb{R}, dx)$ or, equivalently, Bargmann space $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ (see below). Corresponding to any classical function over the torus $f$ we specify a quantum operator $Q_\hbar(f)$. That is, $Q_\hbar$ also gives an ordering prescription for the quantum operator.

We say $\{U, V, U^{-1}, V^{-1}\}$ (operators on $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$) generate the quantum algebra of observables $\mathfrak{A}_\hbar$. The only appropriate “quantum mechanical questions” must be operators which depends on $U$ and $V$. For example, it is appropriate to ask: What is $(U + U^\dagger)/2$ (an observable)? Whereas, the question: What is $\exp(-\hat{x}^2)$? is not allowed. Different closures on series generated by these operators and their inverses will yield different algebras:
the smallest are the $\mathbb{C}^*$-algebras which are norm closed, while von Neumann algebras are only weakly closed. The quantization of classical maps can proceed in a such a fashion for all values of Planck’s constant. In this framework, a natural quantum version of chaos can be defined. The procedure is, simply: formulate the classical definition algebraically, and substitute the quantum noncommutative algebra for the classical algebra.

The dynamical component of the quantization involves finding the (highly non-unique) unitary quantum propagator $F$ such that

$$
\lim_{\hbar \to 0} F^\dagger Q_\hbar (f) F = Q_\hbar (f \circ T)
$$

in a suitable topology. Here $T$ represents the classical map on the torus. What this means is we require the propagator to return the classical dynamics in the limit $\hbar \to 0$. As is common in the literature, we write the quantum evolution as an action of the group $\mathbb{Z}$ on $\mathfrak{A}_\hbar$. That is, for any $A \in \mathfrak{A}_\hbar$,

$$
\alpha_n (A) = F^{-n}AF^n.
$$

Finally, for discussions of quantum ergodicity, we define a “quantum ensemble average” $\tau_\hbar$ as a linear functional on $\mathfrak{A}_\hbar$. The defining property of this functional is the same as the classical “phase space average”:

classical : $\int_{T^2} e^{2\pi i (mx+np)} dxdp = \delta_{m_0} \delta_{n_0},$

quantum : $\tau_\hbar (U^m V^n) = \delta_{m_0} \delta_{n_0}.$

We summarize this discussion with the following general definition:

**Definition 1** A quantum map is a quadruple $(\mathfrak{A}_\hbar, \mathbb{Z}, \alpha_n, \tau_\hbar)$ (or $(\mathfrak{A}_\hbar, \mathbb{Z}, F, \tau_\hbar)$)

such that

(i) $\mathfrak{A}_\hbar$ is an algebra of bounded operators.

(ii) $\alpha_n : \mathbb{Z} \to \text{Aut} (\mathfrak{A}_\hbar)$ is an action of $\mathbb{Z}$ on $\mathfrak{A}_\hbar$.

(iii) $\tau_\hbar$ is a $\mathbb{Z}$-invariant faithful tracial state on $\mathfrak{A}_\hbar$:

$$
\tau_\hbar (\alpha_n (A)) = \tau_\hbar (A).
$$

Recall that a classical map may be defined as the triplet $(M, \mu, T_n)$, with $M$ a smooth manifold, $\mu$ a positive measure on $M$, and $T_n$ a one-parameter group of measure-preserving diffeomorphisms, with $n \in \mathbb{Z}$. This leads to a natural definition of what it means to a have a “quantization of a classical dynamics”: 3
Definition 2. The quantum map \((\mathcal{A}_\hbar, \mathbb{Z}, \alpha_n, \tau_\hbar)\) is furthermore the quantization of a classical map \((M, \mu, T_n)\) if, for any \(f \in \mathcal{A}_0\) (the classical algebra of functions on \(M\)),

\[
\begin{align*}
\mathcal{A}_\hbar &= Q_\hbar(\mathcal{A}_0), \\
\alpha_n(Q_\hbar(f)) &\to Q_\hbar(f \circ T_n) \quad \text{as} \quad \hbar \to 0, \\
\tau_\hbar(Q_\hbar(f)) &\to \int_M f \, d\mu \quad \text{as} \quad \hbar \to 0.
\end{align*}
\]

An important point is that Definition 1 for a quantum map can be “weakened” by relaxing conditions (ii) and (iii) to hold in the limit \(\hbar \to 0\).

II. Classical Maps and Classical Ergodicity

Much work has recently been done in the field of classical chaos. In particular, a hierarchy of just how “chaotic” a dynamics can be has been established:

Anosov Dynamics \(\Rightarrow\) Mixing \(\Rightarrow\) Ergodic.

A system is defined to be Anosov if the phase space is compact and the flow is globally hyperbolic. A canonical example of such a system is the dynamics of a particle constrained to a compact surface of negative curvature with genus \(g \geq 2\) (see ref. \(^4\)). In such a system two particles which are nearby almost always fall away from each other exponentially fast. This, plus the fact the dynamics is constrained to a compact phase space induces the strong form of chaos known as the Anosov property. It is remarkable just how “random” this motion can get. Anosov systems satisfy the Bernoulli property, in which two points initially close to each other will eventually be as different as two different infinite series of coin tosses. (See ref. \(^8\) for a full discussion, and ref. \(^9\) for a proof.) A beautiful example of a map which satisfies this Bernoulli property is the baker’s map, whose quantization is studied in refs. \(^4\) and \(^10\).

A weaker notion of chaos is known as the mixing property. (It was shown by Sinai in ref. \(^11\) that all Anosov systems are mixing.) A dynamical system is mixing if all areas of initial particles eventually spread uniformly throughout phase space. More precisely, a map \((M, \mu, T_n)\) is mixing if for every \(f, g \in L^2(M, \mu)\)

\[
\lim_{n \to \infty} \int_M f(T_n(x)) g(x) \, d\mu(x) = \int_M f(x) \, d\mu(x) \int_M g(x) \, d\mu(x) . \tag{3}
\]
A basic result of ergodic theory is that mixing implies ergodicity.

**Definition 3** A system is ergodic if any invariant measurable function $f \in L^2(M, \mu)$ is constant almost everywhere.

A more common (and physically clear), but equivalent definition of ergodic is “time average = phase space average”. More precisely, a map $(M, \mu, T_n)$ is ergodic if for any $f \in L^2(M, \mu)$,

$$s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} f \circ T^m = \int f(x) \, d\mu(x).$$

Ergodicity is weaker than mixing in that a measurable set can actually remain intact in time while individual points still visit the entire phase space. That is, a “packet” can survive. A good example of a system which is ergodic but not mixing is the Kronecker map (see ref. 1).

III. Quantum Ergodicity

We can associate with the quadruple $(\mathfrak{A}_h, \mathbb{Z}, \alpha_n, \tau_h)$ a quantum Koopman space as a Hilbert space of operators derived from $\mathfrak{A}_h$.

**Definition 4** The quantum Koopman space is the Hilbert space $L^2(\mathfrak{A}_h, \tau_h)$ obtained as the completion of $\mathfrak{A}_h$ in the norm induced by the following inner product

$$(A, B) = \tau_h(A^\dagger B),$$

for $A, B \in \mathfrak{A}_h$.

What we shall now demonstrate that this “quantum Hilbert space” $L^2(\mathfrak{A}_h, \tau_h)$ has associated with it many of the theorems in ergodicity that are classically associated with $L^2(M, \mu)$. For clarity, we give here explicit proofs for the case $G = \mathbb{Z}$ of quantum maps, although the results can easily be generalized to the case of any amenable group.

First, we observe that the quantum evolution is unitary on $L^2(\mathfrak{A}_h, \tau_h)$. Specifically, the automorphism $\alpha$ which implements the quantum dynamics
on $A_h$ is unitary on $L^2(\mathfrak{A}_h, \tau_h)$. This follows almost immediately from eqn. 2. We see

$$\langle A, \alpha (B) \rangle = \tau_h (A^\dagger \alpha (B)) = \tau_h (\alpha_{-1} (A^\dagger \alpha (B))) = \tau_h (\alpha_{-1} (A^\dagger) B)$$

Thus $\alpha_{-1} = \alpha^\dagger$.

We can think of $\alpha$ as the “quantum Liouville operator” analogous to the unitary Liouville operator of classical dynamics.

We now define “quantum ergodicity” in analogy with the classical definition 3.

**Definition 5** A quantum map is called ergodic if the only $A \in L^2(\mathfrak{A}_h, \tau_h)$ invariant under $\mathbb{Z}$ are scalar multiples of the identity $I$.

In what sense is this a statement of “time average = phase space average”? We first demonstrate the existence of the “quantum time average,” analogous to the von Neuman theorem of classical dynamical systems (see ref. 13).

**Theorem 6** For any $A \in L^2(\mathfrak{A}_h, \tau_h)$, the strong limit

$$s = \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) = A_*$$

exists.

Proof. We divide the proof into three steps.

Step1. Let $A \in L^2(\mathfrak{A}_h, \tau_h)$ have the form

$$A = B - \alpha_m (B)$$

where $B \in \mathfrak{A}_h$. Observe that

$$\frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) = \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (B - \alpha_m (B))$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (B) - \alpha_{m+n} (B)$$

6
\[\begin{align*}
&= \frac{1}{N} \left( \sum_{n=0}^{N-1} \alpha_n (B) - \sum_{n=m}^{N+m-1} \alpha_n (B) \right) \\
&= \frac{1}{N} \left( \sum_{n=-\infty}^{\infty} \chi_{[0,N-1]} (n) \alpha_n (B) - \sum_{n=-\infty}^{\infty} \chi_{[m,m+N-1]} (n) \alpha_n (B) \right) \\
&= \frac{1}{N} \left( \sum_{n=-\infty}^{\infty} \left( \chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n) \right) \alpha_n (B) \right)
\end{align*}\]

where \(\chi_{[a,b]}\) is the characteristic function

\[\chi_{[a,b]} (n) = \begin{cases} 1, & a \leq n \leq b \\ 0, & \text{otherwise} \end{cases}\]

Then

\[\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) \right\|_{L^2} \leq \frac{1}{N} \sum_{n=-\infty}^{\infty} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| \left\| \alpha_n (B) \right\|_{L^2}
\]

\[= \left\| B \right\|_{L^2} \frac{1}{N} \sum_{n=-\infty}^{\infty} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right|\]

Now choose \(N > m\). Then

\[\sum_{n=-\infty}^{\infty} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| = \sum_{n=-\infty}^{m-1} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| + \sum_{n=m}^{N-1} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| + \sum_{n=N}^{m+N-1} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| + \sum_{n=N+m}^{\infty} \left| (\chi_{[0,N-1]} (n) - \chi_{[m,m+N-1]} (n)) \right| = 2m.
\]

so that

\[\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) \right\|_{L^2} \leq \left\| B \right\|_{L^2} \frac{2m}{N} \to 0 \text{ as } N \to \infty.\]
Step 2. We denote the closed subspace in $L^2(\mathfrak{M}_h, \tau_h)$ generated by operators of the form $B - \alpha_m(B)$ as $\mathcal{H}$. Then for any $A \in \mathcal{H}$, and any $\epsilon$, there exists an $A_0 = B - \alpha_m(B)$ such that

$$\|A - A_0\|_{L^2} \leq \epsilon/2.$$ 

Thus

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A) \right\|_{L^2} \leq \left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A_0) \right\|_{L^2} + \left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A - A_0) \right\|_{L^2} \leq \left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A_0) \right\|_{L^2} + \|A - A_0\|_{L^2}.$$ 

By the results of step 1, we can choose an $N$ such that

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A_0) \right\|_{L^2} \leq \epsilon/2$$

so that for all $A \in \mathcal{H}$,

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A) \right\|_{L^2} \to 0 \text{ as } N \to \infty.$$

Step 3. By the Riesz lemma (see [§]), any $A \in L^2(\mathfrak{M}_h, \tau_h)$ can be written $A = A^\parallel + A^\perp$, where $A^\parallel \in \mathcal{H}$ and $A^\perp$ is in the orthogonal complement $\mathcal{H}^\perp$. Observe that for any $B \in L^2(\mathfrak{M}_h, \tau_h)$,

$$0 = (A^\parallel, B - \alpha_m(B)) = (\alpha_m(A^\perp), \alpha_m(B) - B) = (\alpha_m(A^\perp), \alpha_m(B)) - (\alpha_m(A^\perp), B) = (A^\perp - \alpha_m(A^\perp), B).$$

Thus, for any $m$

$$(A^\perp, B) = (\alpha_m(A^\perp), B).$$

from which it follows that $A^\perp = \alpha_m(A^\perp)$. Thus

$$\frac{1}{N} \sum_{n=0}^{N-1} \alpha_n(A^\perp) = A^\perp$$

and the theorem is proved. $\blacksquare$

Another important result is that the time average is in fact invariant under the dynamics.
Lemma 7 For any \( n \in \mathbb{Z} \), \( \alpha_n (A_*) = A_* \).

Proof. Observe that

\[
\alpha_n (A_*) = s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} \alpha_{m+n} (A) \\
= s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=n}^{M+n-1} \alpha_m (A) \\
= \left( s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} \alpha_m (A) \right) - \left( s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{n-1} \alpha_m (A) \right) \\
+ \left( s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=M}^{M+n-1} \alpha_m (A) \right) \\
= A_* - \left( s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{n-1} \alpha_m (A) \right) + \left( s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=M}^{M+n-1} \alpha_m (A) \right). 
\]

Now consider the middle term. We have

\[
\left\| s - \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{n-1} \alpha_m (A) \right\|_{L^2} \leq \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{n-1} \| \alpha_m (A) \|_{L^2} \\
= \lim_{M \to \infty} \frac{n}{M} \| A \|_{L^2} = 0.
\]

by the unitarity of \( \alpha \). The same argument holds for the last term, and the lemma is thus proved. \( \blacksquare \)

We are now in a position to derive the more conceptual statement of quantum ergodicity, whereby the time average equals the phase space average.

Theorem 8 A quantum map \((\mathcal{A}_h, \alpha, \mathbb{Z}, \tau_h)\) is quantum ergodic if and only if

\[
s - \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) = \tau_h (A) I \tag{4}
\]

for all \( A \in \mathcal{A}_h \).
Proof. For an ergodic quantum map, we see from Definition 5 and Lemma 7 that
\[ s - \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) = c_A I. \]

We next apply \( \tau_h \) to both sides to get
\[ s - \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tau_h (\alpha_n (A)) = c_A. \]

By the invariance of \( \tau_h \), we see immediately that \( c_A = \tau_h (A) \). Conversely, if 4 holds, then suppose \( \alpha_n (A) = A \) for some \( A \in \mathcal{A}_h \). Then
\[ s - \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) = A = \tau_h (A) I, \]
i.e. \( A \) is a multiple of the identity operator. \( \blacksquare \)

We can also formulate an analogous definition for quantum mixing.

**Definition 9** A quantum map \((\mathcal{A}_h, \alpha, Z, \tau_h)\) is called mixing if for all \( A, B \in \mathcal{A}_h \)
\[ \lim_{n \to \infty} \tau_h (\alpha_n (A) B) = \tau_h (A) \tau_h (B). \]

(5)

Comparison of this with 3 demonstrates that 5 is indeed a quantum mechanical version of mixing. As in the classical case, quantum mixing is a stronger statement than quantum ergodicity.

**Theorem 10** For a quantum map \((\mathcal{A}_h, \alpha, Z, \tau_h)\), quantum mixing implies quantum ergodicity.

Proof. Consider the case where \( \tau_h (A) = 0 \). Observe that
\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) \right\|^2 = \tau_h \left( \left( \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) \right)^\dagger \left( \frac{1}{M} \sum_{m=0}^{M-1} \alpha_m (A) \right) \right)
\]
\[
= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \tau_h (\alpha_n (A^\dagger) \alpha_m (A))
\]

10
\[
\begin{align*}
&= \frac{1}{NM} \sum_{n=0}^{P-1} \sum_{m=0}^{Q-1} \tau_h (\alpha_n (A^\dagger) \alpha_m (A)) \\
&\quad + \frac{1}{NM} \sum_{n=P}^{N-1} \sum_{m=Q}^{M-1} \tau_h (\alpha_n (A^\dagger) \alpha_m (A)) \\
&\leq \frac{PQ}{NM} \|A\|^2 + \frac{1}{NM} \sum_{n=P}^{N-1} \sum_{m=Q}^{M-1} \tau_h (\alpha_n (A^\dagger) \alpha_m (A)) \\
&\leq \frac{PQ}{NM} \|A\|^2 + \sup_{P \geq N, Q \geq M} \left| \tau_h (\alpha_P (A^\dagger) \alpha_Q (A)) \right|,
\end{align*}
\]

for some fixed \( P \leq N \) and \( Q \leq M \). By the definition of mixing, we can choose a \( P \) and \( Q \) such that the second set of terms factorizes, i.e. is less than \( \epsilon/2 \) for any \( \epsilon \). For this \( P \) and \( Q \), we can now find an \( M \) and \( N \) such that the first term is less than \( \epsilon/2 \). Thus, for any \( \epsilon \), and sufficiently large \( N \),

\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} \alpha_n (A) \right\|^2 \leq \epsilon.
\]

This proves that mixing implies ergodicity for the case \( \tau_h (A) = 0 \). For the general case, we simply let \( A_0 = A - \tau_h (A) \) in (6). This completes the proof of the theorem.

What we see here is that many of the connections the different formulations of ergodicity in classical mechanics remain true quantum mechanically. This provides an impetus to believe that quantum mechanics does preserve notions of chaotic dynamics, despite the uncertainty principle. What's more, it also gives hope that a physically sound and well-accepted hierarchy of "quantum chaos" is within reach.

### IV. The Quantum Torus

#### A. Bargmann Space

The classical dynamics on a toroidal phase space are fundamentally phase space dynamics, that is, the momentum coordinate is not necessarily the symplectic dual of the position coordinate. This leads to the use of a Bargmann representation of Hilbert space. (See for a comprehensive review of Bargmann
space.) Bargmann space $\mathcal{H}^2(\mathbb{C}, d\mu_h)$ is the Hilbert space of entire functions on the complex plane which are square integrable with respect to the measure $d\mu_h(z) = (\pi h)^{-1} \exp \left(-|z|^2/h\right)$. In fact, as we shall presently review, this is equivalent to $L^2(\mathbb{R}, dx)$ via the Bargmann transform. The idea behind Bargmann space is to project a general state $|\psi\rangle$ onto a coherent state $|z\rangle$ then multiply out the non-analytic part to obtain an entire function on the complex plane. Thus, we write

$$\psi(z) = e^{|z|^2/2h} \langle z | \psi \rangle.$$ 

In fact, there is an inner product isomorphism between $\mathcal{H}^2(\mathbb{C}, d\mu_h)$ and $L^2(\mathbb{R}, dx)$, implemented by the Bargmann transform. For $x \in \mathbb{R}$, and $z \in \mathbb{C}$, and using an appropriate normalization, this can be found easily

$$\psi(x) = \langle x | \psi \rangle = \frac{1}{\pi h} \int_{\mathbb{C}} \langle x | z \rangle \langle z | \psi \rangle d^2z$$

$$= (\pi h)^{-1/4} \int_{\mathbb{C}} \exp \left( i\sqrt{2xz - x^2/2 - z^2/2} / h \right) \psi(z) d\mu_h(z).$$

Hence,

$$B : \mathcal{H}^2(\mathbb{C}, d\mu_h) \rightarrow L^2(\mathbb{R}, dx),$$

$$B(z, x) = \frac{e^{i\sqrt{2xz - x^2/2 - z^2/2}/h}}{(\pi h)^{1/4}},$$

and $B^{-1}(z, x) = B(z, x)$. We now review some of the important properties of Bargmann space. First of all, the polynomials in $z$ (obviously entire) span a countably dense subset, and in fact, $1, z, z^2, z^3, ...$ are orthogonal functions. We can see this easily by using spherical coordinates with $z = re^{i\phi}$:

$$(z^n, z^m) = \int_{\mathbb{C}} z^n z^m d\mu_h(z) = 2^{-n} \pi^{1/2} h^{-n/2} (2n - 1)!! \delta_{nm}.$$ 

Bargmann representation also carries a unitary projective representation of the group of translations on the plane. Let $a$ and $b$ be fixed complex numbers and let $\psi \in \mathcal{H}^2(\mathbb{C}, d\mu_h)$. Then

$$U(a) \psi(z) = \exp \left( \frac{1}{h} (\bar{a}z - |a|^2/2) \right) \phi(z - a),$$

(8)
so that
\[ U(a)U(b) = e^{i\text{Im}(ab)/\hbar}U(a+b). \] (9)

Also important is the fact that Bargmann space has a reproducing kernel \( \exp(\pi z/\hbar) \) satisfying the equation
\[ \int_{\mathbb{C}} e^{\pi z/\hbar} \psi(w) d\mu_h(w) = \psi(z). \] (10)

Note that the fact that it is a projective representation of the group of translations rather than a representation is a manifestation of the uncertainty principle, where we interpret the real part of \( z \) as the canonical position coordinate and the imaginary part as the canonical momentum. With this in mind, we next define the particular quantization ordering we shall use throughout. We define \( z = (x - ip)/\sqrt{2} \) with \( x \) and \( p \in \mathbb{R} \). Corresponding to a classical symbol \( \sigma \) (a function over phase space) we define the quantum observable as the Toeplitz operator \( T_\hbar(\sigma) \) acting on \( \mathcal{H}^2(\mathbb{C}, d\mu_h) \) so that
\[ T_\hbar(\sigma) \psi(z) = \int_{\mathbb{C}} e^{\pi z/\hbar} \sigma(w) \psi(w) d\mu_h(w). \] (11)

Observe some of the remarkable properties of this quantization. First of all, acting on a symbol which is entire over the plane, it is just multiplication by the symbol. In particular,
\[ \hat{z} \psi(z) \equiv T_\hbar(z) \psi(z) = z \psi(z). \]

The quantization of anti-entire functions also satisfies the property
\[ \hat{z} \psi(z) = T_\hbar(\bar{z}) \psi(z) = \int_{\mathbb{C}} e^{\pi \bar{w}/\hbar} \bar{w} \psi(w) d\mu_h(w) \]
\[ = \hbar \frac{d}{dz} \int_{\mathbb{C}} e^{\pi \bar{w}/\hbar} \psi(w) d\mu_h(w) \]
\[ = \hbar \frac{d \psi(z)}{dz}. \]

With this, we see that acting on the ground state \( \Omega = 1 \), the operators \( \hat{z} \) and \( \hat{\bar{z}} \) ladder up and down, respectively, the basis state \( \{ z^n \}_{n \in \mathbb{Z}} \). We thus define the operators
\[ A^\dagger = \hat{z}, \]
\[ A = \hat{\bar{z}}, \] (12)
and notice

\[ [A, A^\dagger] = \hbar. \]

We can now see that Toeplitz quantization is canonical. Specifically,

\[ [\hat{x}, \hat{p}] = i [A, A^\dagger] = i\hbar. \]

Finally, we note that Toeplitz quantization gives an anti-Wick prescription for operator ordering. For \( \phi \in \mathcal{H}^2(\mathbb{C}, d\mu_\hbar) \),

\[
T_\hbar (z^mw^n) \psi (z) = \left( \hbar \frac{d}{dz} \right)^n \int_{\mathbb{C}} e^{z\overline{w}/\hbar} w^m \psi (w) \, d\mu_\hbar (w) = A^n (A^\dagger)^m \psi (z),
\]

where we have used the fact that \( \exp (z \overline{w}/\hbar) \) is the reproducing kernel.

We next begin an explicit construction of the quantum torus in this representation. Let us define the operators analogous to the function \( \exp (2\pi i x) \) and \( \exp (2\pi i p) \) which generate the classical algebra of observables on the torus. We let

\[
U = U \left( -i\hbar\pi \sqrt{2} \right) \quad \quad \quad V = V \left( \hbar\pi \sqrt{2} \right). \quad \quad (13)
\]

Notice from eqn. 9 that they obey the commutation relations \( UV = e^{i\lambda} VU \) with \( \lambda = 4\pi^2\hbar \). We can see directly that eqn. (13) is the quantization of the generators of the classical algebra. Observe

\[
e^{2\pi i z} \psi (z) = e^{\sqrt{2}\pi i (A + A^\dagger)} \psi (A^\dagger) e^{-\sqrt{2}\pi i (A + A^\dagger)} e^{\sqrt{2}\pi i (A + A^\dagger)} \cdot 1
\]

\[
= \psi (A^\dagger + i\hbar\pi \sqrt{2}) e^{\sqrt{2}\pi i A^\dagger - \pi^2 h} \cdot 1
\]

\[
= e^{-\pi^2 h + \sqrt{2}\pi i z} \psi \left( z + i\hbar\pi \sqrt{2} \right)
\]

\[
= U \psi (z).
\]

Likewise,

\[
e^{2\pi i p} \psi (z) = V \psi (z).
\]

We call \( U \) and \( V \) the generators of the quantum algebra \( \mathfrak{A}_\hbar \) on the torus. More precisely, we have the following definition.

**Definition 11** The (von Neuman) algebra \( \mathfrak{A}_\hbar \) is the weak closure of the set of bounded operators generated by \( \{U, V\} \).
What is this algebra? In fact, the algebra is different for different values of Planck’s constant. Consider now the case of Planck’s constant irrational. (The rational case provides for another beautiful level of structure we study in more detail later (for $h = 1/N$).) For instance, does there exist a bounded operator on $L^2(\mathbb{R})$ which is not in $\mathfrak{A}_h$? The answer to this is yes, as can be seen from the fact that

\[ [U, e^{i\widehat{x}/\hbar}] = [V, e^{i\widehat{p}/\hbar}] = [U, e^{i\widehat{p}/\hbar}] = [V, e^{i\widehat{p}/\hbar}] = 0 \]

In other words, the algebra is contained in the commutant of the set of bounded operators generated by $\exp (i\widehat{x}/\hbar)$ and $\exp (i\widehat{p}/\hbar)$: $\mathfrak{A}_h \subset \{ e^{i\widehat{x}/\hbar}, e^{i\widehat{p}/\hbar} \}' \equiv \mathfrak{A}$. In fact, we can show that $\mathfrak{A}_h = \mathfrak{A}$, and in the process learn quite a bit about the properties of this algebra. We follow here a suggestion of Coleman (ref.15) to look at the Wigner representation. It should be noted that there is a vast mathematical literature and that the following calculation is by no means original. We present here an argument for physicists which can be thought of as densely filling the actual proof. For a full and rigorous presentation, the reader is referred to ref.16

A von Neumann algebra $\mathfrak{a}$ is a set of bounded operators equal to its bicommutant: $\mathfrak{a} = \mathfrak{a}''$. A beautiful theorem of von Neumann is that such an algebra is weakly closed. In our case, $\mathfrak{A} = \mathfrak{A}''$, since it is always true that $\mathfrak{a}' = \mathfrak{a}'''$ (see ref.17). Thus by construction, $\mathfrak{A}$ is a von Neumann algebra.

Recall the Wigner representation $a(p,q)$ of an operator $W$ is given by

\[ a(p,q) = \frac{1}{2\pi\hbar} \int e^{-ipx/\hbar} \langle q + x/2 | A | q - x/2 \rangle dx, \]

for a $\psi \in L^2(\mathbb{R})$. The Wigner representation of the operator $W$ has the property that

\[ \int a(p,q) dp = \langle q | A | q \rangle, \]

and

\[ \int a(p,q) dq = \langle p | A | p \rangle. \]

Now, an operator in the commutant $\{ e^{i\widehat{x}/\hbar}, e^{i\widehat{p}/\hbar} \}'$ satisfies

\[ e^{i\widehat{x}/\hbar} A e^{-i\widehat{x}/\hbar} = A \]

\[ e^{i\widehat{p}/\hbar} A e^{-i\widehat{p}/\hbar} = A. \]
What does this give for the Wigner functions? With a simple calculation, we see

\[ a(p, q) = a(p + 1, q) = a(p, q + 1). \]

That is, the Wigner representation of an operator \( A \in \mathfrak{A} \) must be periodic with period one.

For \( h \) irrational, the weak closure of the set generated by \( \{U, V, e^{i\hat{x}/\hbar}, e^{i\hat{p}/\hbar}\} \) comprise all bounded operators on the Hilbert space \( \mathfrak{B}(\mathcal{H}) \). To see this, we must simply show that the commutant of this set consists only of the identity operator. (Note that \( \mathfrak{B}(\mathcal{H}) = \mathfrak{B}(\mathcal{H})'' = \{I\}' \). We follow here an argument due to Faddeev.

We already know what the conditions are for an operator to commute with \( \exp(i\hat{x}/\hbar) \) and \( \exp(i\hat{p}/\hbar) \), that is, its Wigner functions must be periodic with period one. In exactly the same way, we see that for an operator to commute with \( U \) and \( V \), we must have

\[ a(p + h, q) = a(p, q + h) = a(p, q). \]

Now the only function which is periodic with two periods irrationally related is the constant function: \( a(p, q) = a_0 \), corresponding to the operator \( A = a_0I \).

We thus see that any operator in \( A \in \mathfrak{A}(\mathcal{H}) \) can be written as the weak limit of finite polynomials of the form

\[ A_i = \sum_{a, b, c, d} \alpha_{a, b, c, d} U^a V^b \left( e^{i\hat{x}/\hbar} \right)^c \left( e^{i\hat{p}/\hbar} \right)^d \]

(14)

where the sum is taken to be finite. Now suppose further that \( A_i \in \mathfrak{A} \). Then for all \( M \),

\[ A_i = \frac{1}{M} \sum_{m=0}^{M-1} \left( e^{i\hat{x}/\hbar} \right)^m A_i \left( e^{i\hat{p}/\hbar} \right)^{-m}. \]

Substituting this into (14), we see

\[ \sum_{a, b, c, d} \alpha_{a, b, c, d} U^a V^b \left( e^{i\hat{x}/\hbar} \right)^c \left( e^{i\hat{p}/\hbar} \right)^d \left( 1 - \frac{1}{M} \sum_{m=0}^{M-1} e^{2\pi imd/\hbar} \right) = 0. \]

As \( M \to \infty \), the only way this can hold for \( h \) irrational is if \( d = 0 \). In a similar fashion we see \( c = 0 \). Thus for \( A \in \mathfrak{A} \), \( A \) must be the weak limit of a finite polynomial operator of the form

\[ A_i = \sum_{a, b} \alpha_{a, b} U^a V^b \]

16
which is precisely the von Neumann algebra $\mathfrak{A}_h$ defined above. Thus we have demonstrated $\mathfrak{A}_h = \mathfrak{A}$.

C. Quantum Ergodicity on the Torus

With this in mind, we can now give precisely what we mean by “quantum mixing” and “quantum ergodicity” on the torus. We can define a “quantum phase average” as a trace over the algebra $\mathfrak{A}_h$. Let $\phi \in H^2 (\mathbb{C}, d\mu_h)$ be an arbitrary vector of norm one. For $A \in \mathfrak{A}_h$, we define

$$\tau_h (A) = \int_{T^2} (U (l) \phi, AU (l) \phi) \, d^2 l .$$  \hspace{1cm} (15)

As shown in ref. this functional has the desired property that $\tau_h (U^m V^n) = \delta_{m0} \delta_{n0}$. Another fact about $\tau_h$ is that its value on a Toeplitz operator is equal to the integral of the symbol of the operator:

$$\tau_h (T_h (f)) = \int_{T^2} f (x, p) \, dx dp$$  \hspace{1cm} (16)

where $\tau (f)$ is the phase-space integral of $f$ over the torus.

**Definition 12** We define quantum ergodicity of a quantum toral map $(\mathfrak{A}_h, \alpha, \mathbb{Z}, \tau_h)$ to be the property

$$s - \lim_{M \to \infty} \langle A \rangle_M = \tau_h (A)$$  \hspace{1cm} (17)

for any $A \in \mathfrak{A}_h$, where

$$\langle A \rangle_M = \frac{1}{M} \sum_{m=0}^{M-1} F^m A F^{-m} .$$

**Definition 13** Likewise, we define a quantum toral map to be “quantum mixing”, if for any $A, B \in \mathfrak{A}_h$,

$$\lim_{M \to \infty} \tau_h \left( F^M A F^{-M} B \right) = \tau_h (A) \tau_h (B) .$$  \hspace{1cm} (18)
V. The $\theta$-torus

It was shown in ref. [1] that a remarkable set of properties can be associated with quantum dynamics on a torus if we let Planck’s constant satisfy the integrality condition

$$h = 1/N.$$  

We review these results briefly here and then present further work. The algebra $\mathcal{A}_h$ has a natural (and non-trivial) center generated by

$$X = U^N,$$
$$Y = V^N.$$  (19)

That is, for $h = 1/N$, we easily see that

$$[X, Y] = [X, U] = [X, V] = [Y, U] = [Y, V] = 0.$$  

This insight was used to construct an $N$-dimensional “quantum torus” in the following manner. The simultaneous eigenvalue problem:

$$X \phi(z) = e^{2\pi i \theta_1} \phi(z),$$
$$Y \phi(z) = e^{2\pi i \theta_2} \phi(z),$$

where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, was used to decompose Bargmann space into the direct sum space

$$\kappa : \mathcal{H}^2(\mathbb{C}, d\mu_h) \to \int_{T^2} \mathcal{H}_h(\theta) d\theta$$

where an element $\phi_n^{(\theta)}$ is a simultaneous generalized eigenvectors of $X$ and $Y$, and $\mathcal{H}_h(\theta)$ denote the $N$-dimensional space of simultaneous eigenvectors with fixed $\theta$. The isomorphism in fact preserves the inner product, given by

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}^2(\mathbb{C}, d\mu_h)} = \int_{T^2} \int_D \kappa \phi_1(z, \theta) \kappa \phi_2(z, \theta) d\mu_h(z) d\theta.$$  (20)

The space $\mathcal{H}_h(\theta)$ thus has an inner product defined as an integral over the fundamental domain $D = [0, 1] \times [0, 1]$. A natural set of orthonormal position state basis vectors exist for this given by the following functions:

$$\phi_m^{(\theta)}(z) = C_m(\theta) e^{-N\pi z^2 + 2\sqrt{2\pi}(\theta_1 + m)z} \left(-i\sqrt{2}Nz + i(\theta_1 + i\theta_2 + m), iN\right)$$
where
\[ C_m(\theta) := (2/N)^{1/4} e^{-\pi(\theta_1 + m)^2/N - 2\pi i \theta_2 m/N}, \]
and
\[ \theta(\omega, \tau) = \sum_{k \in \mathbb{Z}} e^{i\pi k^2 \tau + 2\pi ik \omega}. \]
is a Jacobi \( \theta \)-function (see, for example, [18]). These functions satisfy
\[ \int_D \phi_n^{(\theta)}(z) \phi_m^{(\theta)}(z) d\mu(z) = \delta_{mn}. \]

We can see now why these functions are called “position-state eigenvectors”. Under the isomorphism \( \kappa \), \( U \) and \( V \) are block diagonal, that is, they leave a vector at a particular value of \( \theta \) at the same value, and furthermore, these \( \phi_m^{(\theta)} \) are also eigenvalues of \( U^n \) for \( n \in [0, N - 1] \):
\[ \kappa U \kappa^{-1} \phi_m(\theta, z) = e^{2\pi i(\theta_1 + m)/N} \phi_m(\theta, z), \]
\[ \kappa V \kappa^{-1} \phi_m(\theta, z) = e^{2\pi i \theta_2 / N} \phi_{m+1}(\theta, z). \]

We are now in a position to calculate explicitly the form for the Hilbert space of operators \( L^2(\mathfrak{A}_\hbar, \tau_\hbar) \) for the quantized torus discussed in the introduction. We have the following result.

**Theorem 14** For the quantized torus, \( L^2(\mathfrak{A}_\hbar, \tau_\hbar) \simeq \mathcal{M}_{N \times N} \otimes L^2(\mathbb{T}^2) \).

Proof. Recall that any \( A \in \mathfrak{A}_\hbar \) can be written as the weak closure of the finite sums
\[ \sum_{n,m} a_{n,m} U^n V^m = \sum_{c,d} \beta_{c,d} X^c Y^d \sum_{a,b=0}^{N-1} \alpha_{a,b} U^a V^b, \]
where we have let \( n = a + Nc, m = b + Nd \). Observe also that
\[
\tau_\hbar(A^\dagger A) = \int_{\mathbb{T}^2} (U(l) \phi, A^\dagger A U(l) \phi)_{\mathcal{H}^2} d^2l
\]
\[
= \int_{\mathbb{T}^2} (U(l) \phi, \left( \sum_{g,h \in \mathbb{Z}} \beta_{g,h}^* X^{-g} Y^{-h} \sum_{e,f=0}^{N-1} \alpha_{e,f}^* U^{-e} V^{-f} \right) \sum_{c,d \in \mathbb{Z}} \beta_{c,d} X^c Y^d \sum_{a,b=0}^{N-1} \alpha_{a,b} U^a V^b U(l) \phi)_{\mathcal{H}^2} d^2l
\]
\[
= \int_{\mathbb{T}^2} \left( U(l) \phi, \sum_{c,d,g,h \in \mathbb{Z}} \beta_{c,d}^* \beta_{g,h} X^{-g} Y^{d-h} \sum_{a,b,e,f=0}^{N-1} \alpha_{a,b} \alpha_{e,f}^* U^{-e} V^{-f} U(l) \phi \right)_{\mathcal{H}^2} d^2l.\]
We can write this as

\[
\tau_h (A^\dagger A) = \int_{T^2} \int_{T^2} (\kappa U (l) \phi (\theta), \kappa \sum_{a,b,e,f=0}^{N-1} \alpha_{a,b} \alpha_{e,f}^* U^{a-e} V^{b-f} \kappa^{-1} \kappa U (l) \phi (\theta)) \frac{d^2 \theta d^2 l}{p}
\]

\[
= \int_{T^2} \int_{T^2} (\kappa U (l) \phi (\theta), \kappa \sum_{a,b,e,f=0}^{N-1} \alpha_{a,b} \alpha_{e,f}^* U^{a-e} V^{b-f} \kappa^{-1} \kappa U (l) \phi (\theta)) \frac{d^2 l}{p}
\]

\[
\times \sum_{c,d,g,h \in \mathbb{Z}} \beta_{c,d} \beta_{g,h}^* e^{2\pi i (c-g) / N} e^{2\pi i (d-h) / N} d^2 \theta
\]

\[
= \int_{T^2} (U (l) \phi, \sum_{a,b,e,f=0}^{N-1} \alpha_{a,b} \alpha_{e,f}^* U^{a-e} V^{b-f} U (l) \phi) \frac{d^2 l}{H^2}
\]

\[
\times \int_{T^2} \sum_{c,d,g,h \in \mathbb{Z}} \beta_{c,d} \beta_{g,h}^* e^{2\pi i (c-g) / N} e^{2\pi i (d-h) / N} d^2 \theta.
\]

The set of operators \(\{U^j V^k\}\) in the last line of the above expression form an \(N^2\)-dimensional vector space isomorphic to \(\mathfrak{M}_{N \times N}\). From the second set of integrals it is clear that the function \(f (\theta) = \sum_{c,d,g,h \in \mathbb{Z}} \beta_{c,d} e^{2\pi i \theta_1} e^{2\pi i \theta_2} \) span a dense subspace of \(L^2 (T^2)\). The usual continuity argument completes the proof of the theorem. ■

A. The Quantum Torus from \(L^2 (\mathbb{R})\)

What we have demonstrated is a transformation of Bargmann space into a “\(\theta\)-torus” space. As there is an explicit isomorphism between \(H^2 (\mathbb{C}, d\mu_h)\) and \(L^2 (\mathbb{R}, dx)\), we can similarly construct the transformation between \(L^2 (\mathbb{R}, dx)\) and \(\int_{T^2} H_h (\theta) d\theta\). Applying the Bargmann transformation to the basis functions \(\phi_m^{(\theta)} \in H(\theta)\), we find

\[
\Phi_m^{(\theta)} (x) = B^{-1} \phi_m^{(\theta)} (x) = \frac{e^{2\pi i \theta_2 m / N}}{N^{1/2}} \sum_{k \in \mathbb{Z}} e^{2\pi i \theta_2 k} \delta \left( x - \frac{m + \theta_1 + NK}{N} \right).
\]

We have thus found explicitly the \(\delta\)-comb wavefunctions described informally in the physics literature (see, for example, [19]). For later convenience, we
express this now in Dirac notation as follows:

$$\Phi_m(\theta) = e^{2\pi i \theta m / N} \sum_{k \in \mathbb{Z}} e^{2\pi i \theta k} \left| \frac{m + \theta_1}{N} + k \right>_x.$$  \hspace{1cm} (23)

We can, of course, just as easily work in momentum representation. In fact, for $h = 1/N$, a rather interesting calculational identity can be found.

**Lemma 15** For $h = 1/N$,

$$\Phi_m(\theta) = e^{-2\pi i \theta_1 \theta_2 / N} \sum_{n=0}^{N-1} F_{mn} \tilde{\Phi}_n(\theta),$$

where $\{\tilde{\Phi}_n(\theta)\}_{0 \leq n \leq N-1}$ are the momentum-state wave functions on the torus,

$$\tilde{\Phi}_n(\theta) = e^{-2\pi i n \theta_1 / N} \sqrt{N} \sum_k e^{-2\pi i \theta_1} \left| \frac{\theta_2 + n}{N} + k \right>_p,$$  \hspace{1cm} (24)

and $F_{mn}$ is the discrete Fourier transform,

$$F_{mn} = e^{-2\pi i mn / N}.$$

**Remark 1** We see in particular that for the subsets $\theta_1 = 0$ or $\theta_2 = 0$, changing coordinates from momentum representation to position representation is simply a discrete Fourier transform.

**Proof.** The proof is a direct calculation. We have

$$\int_{\mathbb{R}} |p\rangle \langle p| \Phi_m(\theta) dp = \frac{e^{2\pi i \theta_2 m / N}}{N^{1/2}} \sum_{k \in \mathbb{Z}} e^{2\pi i \theta k} \int_{\mathbb{R}} |p\rangle \langle p| \frac{m + \theta_1}{N} + k \rangle_x dp$$

$$= e^{2\pi i \theta_2 m / N} \sum_{k \in \mathbb{Z}} e^{2\pi i \theta k} \int_{\mathbb{R}} |p\rangle \exp \left\{ -2\pi N ip \left( \frac{m + \theta_1}{N} + k \right) \right\} dp$$

$$= e^{2\pi i \theta_2 m / N} \int_{\mathbb{R}} |p\rangle e^{-2\pi i p (m + \theta_1)} \sum_{k \in \mathbb{Z}} e^{2\pi i k (\theta_2 - Np)} dp$$

$$= \frac{e^{2\pi i \theta_2 m / N}}{N} \int_{\mathbb{R}} |p\rangle e^{-2\pi i p (m + \theta_1)} \sum_{k \in \mathbb{Z}} \delta \left( p - \frac{\theta_2 + k}{N} \right) dp$$

$$= \frac{e^{2\pi i \theta_2 m / N}}{N} \sum_{k \in \mathbb{Z}} \left| \frac{\theta_2 + k}{N} \right>_p \exp \left\{ -2\pi i \left( \frac{\theta_2 + k}{N} \right) (m + \theta_1) \right\}.$$
We now let \( k \to n + kN \), with \( n \in \{0, \ldots, N - 1\} \), and \( k \in \mathbb{Z} \), to find

\[
\int_{\mathbb{R}} |p\rangle \langle p| \Phi_m^{(\theta)} dp = \frac{e^{2\pi i \theta / N}}{N} \sum_{n} \sum_{k} \left| \frac{\theta_2 + n}{N} + k \right|_p \\
\times \exp \left\{ -2\pi i \left( \frac{\theta_2 + n + Nk}{N} \right) (m + \theta_1) \right\} \\
= \frac{e^{-2\pi i \theta_1 / N}}{N} \sum_{n} e^{-2\pi i mn / N} e^{-2\pi i \theta_1 / N} \sum_{k} e^{-2\pi i k} \left| \frac{\theta_2 + n}{N} + k \right|_p \\
= e^{-2\pi i \theta_1 / N} \sum_{n} \mathcal{F}_{mn} \Phi_n^{(\theta)},
\]

as claimed. \( \blacksquare \)

Analogous to eqn. 20, we can also find an explicit expression for the inner product over the \( N \)-dimensional Hilbert space at each point on the \( \theta \)-torus as an integral over the fundamental domain \( [0, 1] \) of the real line. The inner product defined in eqn. 20 can be written as

\[
(\Psi_1(\theta), \Psi_2(\theta)) = \int_{0}^{1} \overline{\Psi_1(x, \theta)}(K \Psi_2)(x, \theta) dx,
\]

where

\[
K(x, y) = g \left( \frac{x - y}{2\hbar} \right),
\]

and

\[
g(r) = \frac{1}{2\pi \hbar} e^{-\hbar^2 + \sin r \frac{r}{r}}.
\]

**Proof.** The proof is again a direct calculation. We have

\[
\int_{D} \overline{\psi_1(z, \theta)} \psi_2(z, \theta) d\mu_h(z) = \int_{D} \overline{B \Psi_1(z, \theta)} \Psi_2(z, \theta) d\mu_h(z) \\
= \frac{1}{2(\pi \hbar)^{3/2}} \int_{\mathbb{R}^2} \overline{\Psi_1(x, \theta)} \Psi_2(y, \theta) e^{-(x^2 + y^2)/2\hbar} \\
\times \int_{D} e^{-(u^2 - x(u - iv) + y(u + iv))/\hbar} dudv dx dy \\
= \frac{1}{2(\pi \hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_{0}^{1} dx \int_{-\infty}^{\infty} dy \overline{\Psi_1(x + k, \theta)} \Psi_2(y, \theta) e^{-(x+k)^2 + y^2)/2\hbar} \\
\times \int_{D} e^{-(u^2 - (x+k)(u - iv) + y(u + iv))/\hbar} dudv.
\]
We next use the fact that both $\Psi_1$ and $\Psi_2$ satisfy $X\Psi = e^{2\pi i \theta_1}\Psi$ and $Y\Psi = e^{2\pi i \theta_2}\Psi$. Substituting in, we find

$$\frac{1}{2(\pi \hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_0^1 dx \int_{-\infty}^{\infty} dy \int_D dudv e^{-(u^2-(x+k)(u-k)+y(u+v))/\hbar} \Psi_1(x,\theta) \Psi_2(y,\theta) e^{-(x+k)^2+y^2)/2\hbar}$$

$$= \frac{1}{2(\pi \hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_0^1 dx \int_{-\infty}^{\infty} dy \Psi_1(x,\theta) \Psi_2(y-k,\theta) e^{-(x+k)^2+y^2)/2\hbar}$$

$$\times \int_D e^{-(u^2-(x+k)(u-k)+y(u+v))/\hbar} dudv$$

$$= \frac{1}{2(\pi \hbar)^{3/2}} \int_0^1 dx \Psi_1(x,\theta) \int_{-\infty}^{\infty} dy \Psi_2(y,\theta) e^{-(x^2+y^2+i(x-y))/2\hbar} \sin \left( \frac{x-y}{2\hbar} \right)$$

$$\times \int_{-\infty}^{\infty} e^{-(u^2-(x+k)^2+y^2)/\hbar} du$$

$$= \int_0^1 \Psi_1(x,\theta) (K\Psi_2)(x,\theta) dx,$$

and the claim follows. □

We see that the kernel $K(x,y)$ is a type of quantum diffraction in keeping with the uncertainty principle. In fact it can be readily seen that as $\hbar \to 0$, $K(x,y) \to \delta(x-y)$. We demonstrate this numerically for $\hbar = 1/10$ and $\hbar = 1/100$.

$$|g(r/2\hbar)|^2 \text{ for } \hbar = 1/10.$$

Observe also that with respect to this inner product, the basis elements
\{ \Phi_m^{(\theta)} \} \text{ are orthonormal:}

**Corollary 16** \( (\Phi_m^{(\theta)}, \Phi_n^{(\theta)}) = \delta_{mn} \).

Proof. The proof is a straightforward calculation, but also follows immediately from the Bargmann transformation on the inner product eqn. [20].

\[ \square \]
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