The Simplification of Spinor Connection and Classical Approximation

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The standard spinor connection in curved space-time is represented in a compact form. In this form the calculation is complicated, and its physical effects are concealed. In this paper, we split spinor connection into two vectors $\Upsilon_\mu$ and $\Omega_\mu$, where $\Upsilon_\mu$ is only related to geometrical calculations, but $\Omega_\mu$ leads to dynamical effects, which couples with the spin of a spinor. The representation depends only on metric but is independent of Dirac matrices, so it is valid for both Weyl spinors and Dirac spinor. In the new form, we can clearly define classical concepts for a spinor and then derive its complete classical dynamics. By detailed calculation we find the classical approximation is just Newtonian second law. The dynamical connection $\Omega_\mu$ couples with the spin of a particle with a tiny energy in weak field, which provides location and navigation functions for a spinor. This term may be also important to form magnetic field of a celestial star. From the results, we find the spinor has marvelous structure and wonderful property, and the interaction between spinor and gravity is subtle. This study may be also helpful to clarify the relations between relativity, quantum mechanics and classical mechanics.

Keywords: spinor connection, spinor structure, spin, gravitomagnetic field, principle of equivalence

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I. INTRODUCTION

The classical theory of motion for a spinor in a gravitational field is firstly studied by Mathisson[1], and then developed by Papapetrou[2] and Dixon[3]. A detailed derivation can be found in [4]. Where by the commutator of the usual covariant derivative of the spinor $[\nabla_\alpha, \nabla_\beta]$, we get an extra approximate acceleration of the spinor as follows

$$a_\alpha(x^\mu) = -\frac{\hbar}{4m}R_{\alpha\beta\gamma\delta}(x^\mu)u^\beta(x^\mu)S^{\gamma\delta}(x^\mu),$$

(1.1)
where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature, $u^\alpha$ 4-vector speed and $S^{\gamma\delta}$ the half commutator of the Dirac matrices.

(1.1) leads to the violation of Einstein’s equivalence principle. This problem was discussed by many authors\cite{4–11}. In \cite{5}, the exact Cini-Touschek transformation and the ultra-relativistic limit of the fermion theory were derived, but the Foldy-Wouthuysen transformation is not uniquely defined. The following calculations also show that, the usual covariant derivative $\nabla_{\mu}$ includes cross terms, which is not parallel to the speed $u^\mu$ of the spinor.

For a classical spin such as gyroscope, the frame dragging effect was predicted by Lense and Thirring \cite{12, 13}, and the non-relativistic formula for the effect was derived by L. Schiff \cite{14–16}. It has also been shown that the gravitomagnetic interaction plays a part in both shaping the lunar orbit\cite{17}, and in contributing to the periastron precession of binary and especially double pulsars\cite{18}. For applications to the analysis of gravitational phenomena, a general metric tensor field expansion for the gravitational potentials in a broad class of theories was developed\cite{19–22}. This parameterized post-Newtonian framework yields a gravitomagnetic contribution to the equation of motion\cite{23}. The spin precession was studied in \cite{24}.

In this paper, by projecting the spinor connection onto the tetrad or Pauli matrices, and splitting it into geometrical and dynamical parts, we get two 4-d vectors ($\Upsilon_{\mu}, \Omega_{\mu}$) from the connection. These vectors of connection are only determined by metric but independent of Dirac matrices, and the classical approximation is parallel to 4-vector speed of particle. In this representation of connection, we can clearly define classical concepts such as coordinate, speed, momentum for a spinor, and then derive the classical mechanics in detail. $\Upsilon_{\mu}$ only corresponds to the geometrical calculations, but $\Omega_{\mu}$ leads to tiny dynamical effects. $\Omega_{\mu}$ couples with the spin $s^\mu$ of a spinor, which provides location and navigation functions for a spinor with little energy. So this form of connection is helpful to understand the subtle interaction between spinor and gravity.

II. SIMPLIFICATION OF THE SPINOR CONNECTION

At first we introduce some notations and conventions. We take $\hbar = c = 1$ as units, the Minkowski metric is given by $\eta_{ab} = \text{diag}(1, -1, -1, -1)$, the Pauli and Dirac matrices in Minkowski space-time is as follows

$$\sigma^a \equiv \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

(2.1)

$$\tilde{\sigma}^a \equiv (\sigma^0, -\sigma), \quad \tilde{\sigma} = (\sigma^1, \sigma^2, \sigma^3).$$

(2.2)
\[ \gamma^a \equiv \begin{pmatrix} 0 & \tilde{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \] (2.3)

The element of space-time is described by
\[ dx = \tilde{\gamma}_\mu dx^\mu = \gamma_a \delta X^a, \quad \tilde{\gamma}^\mu = h^\mu_a \gamma_a, \quad \tilde{\gamma}_\mu = l^a_\mu \gamma_a, \] (2.4)
in which \( \gamma_a \) and \( \tilde{\gamma}_\mu \) act as tetrad frames satisfying the following \( C\ell(1,3) \) Clifford algebra,
\[ \gamma_a \gamma_b + \gamma_b \gamma_a = 2 \eta_{ab}, \quad \tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu = 2 g_{\mu\nu}. \] (2.5)

In this paper, we use the indices \((a, b \in \{0, 1, 2, 3\})\) for the Minkowski space-time, Greek characters \((\mu, \nu \in \{0, 1, 2, 3\})\) for the curved space-time, and \((j, k, l \in \{1, 2, 3\})\) for the simultaneous hypersurface or space.

In the flat space-time, the Dirac equation for free bispinor \( \phi \) is equivalent to
\[ \gamma^a i \partial_a \phi = m \phi. \] (2.6)

In chiral representation we get dynamics of Weyl spinors,
\[ \begin{cases} \sigma^a i \partial_a \psi = m \bar{\psi}, \\ \tilde{\sigma}^a i \partial_a \bar{\psi} = m \psi, \end{cases} \quad \phi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \quad \tilde{\gamma}^\mu = \begin{pmatrix} 0 & \tilde{\varrho}^\mu \\ \tilde{\varrho}^\mu & 0 \end{pmatrix}. \] (2.7)

which is more convenient for calculation than (2.6) in some cases.

In curved space-time we have Pauli and Dirac matrices as follows
\[ \begin{cases} \varrho^\mu = h^\mu_a \sigma^a, & \varrho_\mu = l^a_\mu \sigma^a, \\ \tilde{\varrho}^\mu = h^\mu_a \tilde{\sigma}^a, & \tilde{\varrho}_\mu = l^a_\mu \tilde{\sigma}^a, \end{cases} \quad \tilde{\gamma}^\mu = \begin{pmatrix} 0 & \tilde{\varrho}^\mu \\ \tilde{\varrho}^\mu & 0 \end{pmatrix}. \] (2.8)

The spinor equation (2.7) becomes
\[ \begin{cases} \varrho^\mu i \nabla_\mu \psi = m \bar{\psi}, \\ \tilde{\varrho}^\mu i \nabla_\mu \bar{\psi} = m \psi, \end{cases} \] (2.9)

where \( \nabla_\mu = \partial_\mu + \Gamma_\mu \), \( \tilde{\nabla}_\mu = \partial_\mu + \tilde{\Gamma}_\mu \) are the covariant derivatives of \( \psi \) and \( \bar{\psi} \), \( \Gamma_\mu \) and \( \tilde{\Gamma}_\mu \) are spinor affine connections[25–29],
\[ \Gamma_\mu = \frac{1}{4} \tilde{\varrho}_\nu \varrho^\nu_\mu, \quad \tilde{\Gamma}_\mu = \frac{1}{4} \tilde{\varrho}_\nu \tilde{\varrho}^\nu_\mu, \] (2.10)
in which \( \varrho^\mu_\nu = \partial_\nu \varrho^\mu + \Gamma^\mu_\alpha \varrho^\alpha \). For Dirac bispinor \( \phi \), by (2.10) it is easy to check
\[ \nabla_\mu \phi = (\partial_\mu + \Gamma_\mu) \phi, \quad \tilde{\nabla}_\mu \phi = (\partial_\mu + \tilde{\Gamma}_\mu) \phi. \] (2.11)
The Lagrangian corresponding to (2.9) is given by

\[ \mathcal{L}_m = \Re \left( \psi^+ \psi \left( \bar{\psi}^+ \bar{\psi} + \bar{\psi}^+ \bar{\psi} \right) - m (\bar{\psi}^+ \psi + \psi^+ \bar{\psi}) \right), \]

\[ \mathcal{L}_m = \Re \left( \psi^+ \psi \left( \bar{\psi}^+ \bar{\psi} + \bar{\psi}^+ \bar{\psi} \right) + \psi^+ \Omega \psi + \bar{\psi}^+ \bar{\Omega} \bar{\psi} - m (\bar{\psi}^+ \psi + \psi^+ \bar{\psi}) \right), \]  

(2.12)

in which \( \Omega \) and \( \bar{\Omega} \) are two Hermitian matrices defined by

\[ \begin{align*}
\Omega &\equiv i \left[ \psi^+ \Omega \psi - (\partial_{\mu} \Delta_{\alpha}) \bar{\Omega} \bar{\psi} \right], \\
\bar{\Omega} &\equiv i \left[ \bar{\psi}^+ \bar{\Omega} \bar{\psi} - (\partial_{\mu} \bar{\Delta}_{\alpha}) \psi \right].
\end{align*} \]

(2.13)

For any diagonal metric, it is easy to check \( \Omega = \bar{\Omega} = 0 \). By variation of (2.12) with respect to \( \psi^+ \) and \( \bar{\psi}^+ \), we get dynamics equivalent to (2.9) as follows

\[ \begin{align*}
\partial_{\mu} \psi^+ &\equiv \partial_{\mu} \psi + \Gamma_{\mu}^{\alpha} \psi^+ = (l_{\mu}^a \partial_{\alpha} h_a^0 + \partial_{\mu} \ln \sqrt{g}) \psi^+. 
\end{align*} \]

(2.14)

Projecting \( \partial_{\mu} \psi^+ \) onto the basis \( \psi^+ \), i.e., we define \( k_{\mu} \) as follows

\[ \partial_{\mu} \psi^+ = \partial_{\mu} h_{\alpha}^a \sigma^a \equiv k_{\mu} \psi^+ = k_{\mu} h_{\alpha}^a \sigma^a, \]

(2.15)

then we have \( \partial_{\mu} h_{\alpha}^a = k_{\mu} h_{\alpha}^a \) or \( k_{\mu} = l_{\mu}^a \partial_{\alpha} h_a^0 \), and

\[ \psi^+ = \psi + \Gamma_{\mu}^{\alpha} \psi^+ = (l_{\mu}^a \partial_{\alpha} h_a^0 + \partial_{\mu} \ln \sqrt{g}) \psi^+. \]

(2.16)

So we can define the geometrical part of connection by

\[ \Upsilon = \Upsilon_{\mu} \psi^+ \equiv \frac{1}{2} \psi^+ \Upsilon_{\mu}, \quad \Upsilon_{\mu} \equiv \frac{1}{2} \left( l_{\mu}^a \partial_{\alpha} h_a^0 + \partial_{\mu} \ln \sqrt{g} \right) = \frac{1}{2} h_{\alpha}^0 \partial_{\mu} l_{\alpha}^a - \partial_{\nu} l_{\mu}^a. \]

(2.17)

For any 3-d vectors \( \vec{A} \) and \( \vec{B} \), we have

\[ (\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}. \]

(2.18)

Denoting

\[ \psi^a = h_{\alpha}^0 + \tilde{\psi}^a \cdot \sigma, \quad \bar{\psi}^a = h_{\alpha}^0 - \tilde{\psi}^a \cdot \sigma, \quad \partial_{\alpha} \psi = \partial_{\alpha} l_{\alpha}^0 - \partial_{\nu} \tilde{l}_{\alpha}^b \cdot \sigma, \]

(2.19)

where \( \tilde{h}_{\alpha} \) and \( \tilde{l}_{\alpha} \) are the original \( h_{\alpha}^0 \) and \( l_{\alpha}^a \), we get

\[ \Omega = -\frac{1}{4} \left( (\tilde{h}_{\alpha}^0 \times \tilde{h}_{\beta}^0) \cdot \partial_{\alpha} \tilde{l}_{\beta} - \partial_{\alpha} l_{\alpha}^0 (\tilde{h}_{\alpha}^0 \times \tilde{h}_{\beta}^0) \cdot \sigma + (h_{\alpha}^0 \tilde{h}_{\beta}^0 - h_{\beta}^0 \tilde{h}_{\alpha}^0) \times \partial_{\alpha} l_{\beta}^0 \cdot \sigma - \frac{1}{4} \Omega_{\mu} \psi^+ \right), \]

(2.20)

Let \( \Omega = \omega_{\sigma} \psi^+ = \Omega_{\mu} \psi^+ \), then we have

\[ \Omega_{\mu} = \frac{1}{4} \left( (\tilde{h}_{\alpha}^0 \times \tilde{h}_{\beta}^0) \cdot (l_{\alpha}^0 \partial_{\alpha} \tilde{l}_{\beta} - l_{\alpha}^0 \partial_{\alpha} \tilde{l}_{\beta}^b) + \tilde{l}_{\mu}^b \cdot [(h_{\alpha}^0 \tilde{h}_{\beta}^0 - h_{\beta}^0 \tilde{h}_{\alpha}^0) \times \partial_{\alpha} \tilde{l}_{\beta}^b] \right), \]

(2.21)
where $\vec{\omega} = (\omega^1, \omega^2, \omega^3)$. In [30] we get $(\Omega^a, \omega^a)$ expressed by $\partial_\mu g_{\mu\nu}$ as follows,

$$\omega^d = \frac{1}{8} \epsilon^{dabc} h^\alpha_a S^\mu_{bc} \partial_\alpha g_{\mu\nu}, \quad \Omega^a = \frac{1}{8} \epsilon^{dabc} h^\beta_a S^\mu_{bc} \partial_\beta g_{\mu\nu}. \quad (2.22)$$

(2.21) or (2.22) defines the dynamical part of the spinor connection.

Similarly we have

$$\tilde{\Upsilon} = \Upsilon^\mu \tilde{\nu} = \frac{1}{2} \tilde{\nu}^\mu, \quad \tilde{\Omega} = \tilde{\Omega}^\mu \tilde{\nu} = -\omega^a \tilde{\sigma}^a, \quad \tilde{\omega}_\mu = -\Omega_\mu. \quad (2.23)$$

By (2.17) and (2.23), the dynamical equation (2.14) becomes

$$\begin{cases}
\varrho^\mu [i(\partial_\mu + \Upsilon_\mu) + \Omega_\mu] \psi = m \tilde{\psi}, \\
\bar{\varrho}^\mu [i(\partial_\mu + \Upsilon_\mu) - \Omega_\mu] \bar{\psi} = m \psi.
\end{cases} \quad (2.24)$$

Correspondingly, the Dirac equation (2.6) in the curved space-time becomes

$$\tilde{\gamma}^\mu [i(\partial_\mu + \Upsilon_\mu) + \Omega_\mu \gamma_4] \phi = m \phi. \quad (2.25)$$

In order to characterize the rotational degrees of freedom, the decomposition of spinor connection in Clifford algebra was derived by J. M. Nester as follows[31, 32],

$$\tilde{\gamma}^\mu \nabla_\mu \phi = \tilde{\gamma}^\mu \partial_\mu \phi - \frac{1}{2} \tilde{\gamma}^\mu \tilde{q}_\mu \phi + \frac{1}{2 \cdot 3!} \tilde{q}_{\mu\nu\omega} \tilde{\gamma}^\mu \gamma^\nu \gamma^\omega \phi. \quad (2.26)$$

Simplifying the grade-3 Clifford algebra by $\gamma^{abc} = \epsilon^{abcd} \gamma^d_{0123}$ and combining like terms, we find the two splits (2.25) and (2.26) are equivalent. However, (2.24) and (2.25) are more convenient for discussion and practical calculation as shown below.

The vector connection $\Upsilon_\mu$ and $\Omega_\mu$ are only determined by metric and get rid of the influence of coefficient matrices. The following discussion shows that, $\Upsilon_\mu$ and $\Omega_\mu$ have different physical meanings. $\partial_\mu + \Upsilon_\mu$ as a whole operator is similar to the covariant derivatives $\nabla_\mu$ for tensors, it only has geometrical effect. But $\Omega_\mu$ couples with the spin of a particle, and leads to dynamical effects.

We calculate Dirac equation in diagonal metric. In general case, the metric is given by

$$g_{\mu\nu} = \text{diag}(N_0^2, -N_1^2, -N_2^2, -N_3^2), \quad \sqrt{g} = N_0 N_1 N_2 N_3, \quad (2.27)$$

where $N_\mu = N_\mu(x^\alpha)$. Then we have $\Omega_\mu = 0$, and

$$\tilde{\gamma}^\mu = \left( \begin{array}{c} \frac{\gamma^0}{N_0} \\ \frac{\gamma^1}{N_1} \\ \frac{\gamma^2}{N_2} \\ \frac{\gamma^3}{N_3} \end{array} \right), \quad \Upsilon_k = \frac{1}{2} \partial_k \ln \left( \frac{\sqrt{g}}{N_k} \right), \quad (2.28)$$

where $k = 0, 1, 2, 3$. 
For Dirac equation in Schwarzschild metric,

\[ g_{\mu\nu} = \text{diag}(B(r), -A(r), -r^2, -r^2 \sin^2 \theta), \]  

we have

\[ \tilde{\gamma}^\mu = \left( \gamma^0 / \sqrt{B}, \gamma^1 / \sqrt{A}, \frac{\gamma^2}{r}, \gamma^3 / (r \sin \theta) \right), \quad \Upsilon_\mu = \left( 1, \frac{1}{r} + \frac{B'}{4B}, \frac{1}{2} \cot \theta, 0 \right). \]  

The Dirac equation for free spinor is given by

\[ i \left[ \gamma^0 \partial_t + \frac{\gamma^1}{\sqrt{B}} \left( \partial_r + \frac{1}{r} + \frac{B'}{4B} \right) + \frac{\gamma^2}{r} \left( \partial_\theta + \frac{1}{2} \cot \theta \right) + \frac{\gamma^3}{r \sin \theta} \partial_\phi \right] \phi = m\phi. \]  

Set \( A = B = 1 \), we get Dirac equation in spherical coordinate system

\[ i \left[ \gamma^0 \partial_t + \frac{\gamma^1}{\sqrt{A}} \left( \partial_r + \frac{1}{r} \right) + \frac{\gamma^2}{r} \left( \partial_\theta + \frac{1}{2} \cot \theta \right) + \frac{\gamma^3}{r \sin \theta} \partial_\phi \right] \phi = m\phi. \]  

### III. THE CLASSICAL APPROXIMATION OF DIRAC EQUATION

In this section, we derive the classical mechanics of a spinor moving in gravity, and disclose the physical meaning of connection \( \Upsilon_\mu \) and \( \Omega_\mu \). We introduce the local Gaussian normal coordinate system (GCS) with metric \( \text{diag}(1, -\tilde{g}_{jk}) \), because only in such coordinate system we can define simultaneity and then clearly establish the Hamiltonian formalism and calculate the Noether charges. In GCS, we have

\[ h^0_0 = l^0_0 = 1, \quad \vec{h}^0_0 = \vec{l}^0_0 = 0. \]  

Then by (2.17) we get

\[ \Upsilon_\mu = \frac{1}{2} \left( \partial_t \ln \sqrt{\tilde{g}}, \vec{h}^0_j \partial_j \vec{h}^0 + \partial_k \ln \sqrt{\tilde{g}} \right). \]  

In GCS, to lift and lower the index of a vector means \( \Upsilon^0 = \Upsilon_0, \Upsilon^k = -\tilde{g}^{kl} \Upsilon_l \).

More generally, we consider Dirac equation with electromagnetic potential \( eA^\mu \), then (2.25) can be rewritten in Hamiltonian formalism

\[ i(\partial_t + \Upsilon_t)\phi = H\phi, \]  

where the Hamiltonian is defined by

\[ H = -\alpha^k \vec{p}_k + eA_0 + m\gamma_0 - \Omega_\mu \tilde{s}^\mu, \quad \alpha^\mu \equiv \gamma_0 \tilde{\gamma}^\mu = \text{diag}(\bar{e}^\mu, \bar{e}^\mu), \]
in which \(\alpha^\mu\) is current operator, and \(\hat{p}_\mu\) and \(\hat{s}^\mu\) are respectively momentum and spin operators defined by

\[
\hat{p}_\mu = i(\partial_\mu + \Gamma_\mu) - eA_\mu, \quad \hat{s}^\mu \equiv \alpha^\mu\gamma_4 = \text{diag}(\sigma^\mu, -\sigma^\mu). \tag{3.5}
\]

It is easy to check \(\vec{s} = \text{diag}(\vec{\sigma}, \vec{\sigma})\) is the usual spin for any representation of Dirac bispinor.

Similarly to the case in flat space-time\([33, 34]\), we define classical concepts such as coordinate \(\vec{X}\) and speed \(\vec{v}\) of the spinor as follows,

\[
\vec{X}(t) = \int_{S^3} \vec{x}q^0\sqrt{g}d^3x, \quad \vec{v} = \frac{d}{dt}\vec{X}, \tag{3.6}
\]

where \(S^3\) stands for the total simultaneous hypersurface, \(q^\mu\) is current

\[
q^\mu = \phi^+\alpha^\mu\phi = \psi^+\rho^\mu\psi + \bar{\psi}^+\bar{\rho}^\mu\bar{\psi}. \tag{3.7}
\]

By definition (3.6) and current conservation law \(q^\mu_{,\mu} = 0\), it is easy to check

\[
\vec{v} = \int_{S^3} \vec{x}\partial_t(q^0\sqrt{g})d^3x = \int_{S^3} \vec{x}q^0_t\sqrt{g}d^3x = -\int_{S^3} \vec{x}q^k_0\sqrt{g}d^3x = \int_{S^3} \bar{q}\sqrt{g}d^3x. \tag{3.8}
\]

With normalizing condition \(\int_{S^3} q^0\sqrt{g}d^3x = 1\), we have point-particle model,

\[
q^\mu \to u^\mu\sqrt{1 - \bar{g}_{kl}v^kv^l}\delta^3(\vec{x} - \vec{X}), \quad u^\mu = \frac{dX^\mu}{d\tau} = (1, \vec{v})/\sqrt{1 - \bar{g}_{kl}v^kv^l}, \tag{3.9}
\]

where the Dirac-\(\delta\) means \(\int_{S^3} \delta^3(\vec{x} - \vec{X})\sqrt{g}d^3x = 1\) and \(\delta^3(\vec{x} - \vec{X}) = 0\) if \(\vec{x} \neq \vec{X}\), \(\tau\) is proper time \(d\tau = \sqrt{1 - \bar{g}_{kl}v^kv^l}dt\).

For any Hermitian operator \(\hat{P}\), by (3.3) we have following generalized Ehrenfest theorem,

\[
\frac{dP}{dt} = \frac{d}{dt}\int_{S^3} \sqrt{g}\phi^+\hat{P}\phi d^3x = \Re \int_{S^3} \sqrt{g}\left(\phi^+(\partial_\mu\hat{P})\phi + i(\partial_\mu\phi)^+\hat{P}\phi - i\phi^+\hat{P}(i\partial_\mu\phi) + \phi^+\hat{P}\phi\partial_\mu\ln \sqrt{g}\right) d^3x,
\]

\[
= \Re \int_{S^3} \sqrt{g}\left(\phi^+(\partial_\mu\hat{P})\phi + i[H, \phi]^+\hat{P}\phi - i\phi^+\hat{P}[H, \phi]\right) d^3x,
\]

\[
= \Re \int_{S^3} \sqrt{g}\phi^+\left(\partial_\mu\hat{P} + (\partial_k\alpha^k + \alpha^k\partial_k\ln \sqrt{g} - 2\alpha^k\gamma_k)\hat{P} + i[H, \hat{P}]\right)\phi d^3x,
\]

\[
= \Re \int_{S^3} \sqrt{g}\phi^+\left(\partial_\mu\hat{P} + i[H, \hat{P}]\right)\phi d^3x, \tag{3.10}
\]

where any Hermitian operator \(\hat{P}\) means \(P = \int_{S^3} \sqrt{g}\phi^+\hat{P}\phi d^3x\) is real for any \(\phi\). (3.10) clearly shows the connection \(\Gamma^\mu\) has only geometrical effect, which cancels the derivatives of \(\sqrt{g}\). Obviously, we cannot get (3.10) from conventional definition of spinor connection (\(\Gamma_\mu, \Gamma_{\mu}\)).

Define 4-dimensional momentum of the spinor by

\[
p^\mu = \Re \int_{S^3} \phi^+\hat{p}^\mu\phi\sqrt{g}d^3x. \tag{3.11}
\]
For a spinor at energy eigenstate, we have classical approximation \( p^\mu = mu^\mu \), where \( m \) defines the classical mass of the spinor. Let \( \hat{P} = \hat{p}_\mu \), we get classical approximation as \( q^\mu \to v^\mu \delta^3(\vec{x} - \vec{X}) \),

\[
\frac{d}{dt}p_\mu = R \int_{S^3} (e(\partial_\mu A_\nu - \partial_\nu A_\mu)q^\nu + \phi^+ \partial_\mu (\Omega_\nu S^\nu)\phi - \phi^+(\partial_\mu \alpha^\nu)\hat{p}_\nu \phi) \sqrt{g} d^3x. 
\]

\[
\to [e(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu + s^a \partial_\mu \omega_a] \sqrt{1 - g_{kl}v^k v^l} - K_\mu, 
\]

\[
\Rightarrow s^a = R \int_{S^3} \phi^+ \hat{s}^a \phi \sqrt{g} d^3x / \sqrt{1 - g_{kl}v^k v^l} = L^a_b \hat{s}^b, 
\]

\[
K_\mu = R \int_{S^3} \phi^+ (\partial_\mu \alpha^\nu)\hat{p}_\nu \phi \sqrt{g} d^3x, 
\]

in which \( \hat{s}^b = \int_{R^3} \phi^+ \hat{s}^b \phi d^3X \) is proper spin of the spinor. \( \hat{s}^b \) equals to \( \pm \frac{1}{2} \hbar \) in one direction but vanishes in other directions. \( L^a_b \) is the local Lorentz transformation between local tetrad and the central coordinate system of the spinor[34].

Now we prove the following classical approximation of \( K_\mu \),

\[
K_\mu \to g_{\mu \nu} \Gamma^\nu_{\alpha \beta} p^\alpha u^\beta \sqrt{1 - g_{kl}v^k v^l} - p^\nu \frac{dg_{\mu \nu}}{dt} 
\]

\[
= \left( \frac{1}{2} m (\partial_\alpha g_{\mu \beta} + \partial_\beta g_{\mu \alpha} - \partial_\mu g_{\alpha \beta})u^\alpha u^\beta - m u^\nu u^\mu \partial_\alpha g_{\mu \alpha} \right) \sqrt{1 - g_{kl}v^k v^l} 
\]

\[
= - \frac{1}{2} (\partial_\mu g_{\alpha \beta}) mu^\alpha u^\beta \sqrt{1 - g_{kl}v^k v^l}. 
\]

in which we used \( \frac{d}{dt} = u^\alpha \partial_\alpha \). (3.16) can be proved by using Theorem 4 in [30] as follows. In this case we have \( \alpha^\nu = h^\nu_a \tilde{\alpha}^a \), where \( \tilde{\alpha}^a \) is matrix in Minkowski space-time. By Theorem 4 we have

\[
\frac{\partial h^\nu_a}{\partial g_{\alpha \beta}} = - \frac{1}{4} (h^\rho_a h^{\nu \beta} + h^\rho_a g^{\rho \nu} - \frac{1}{2} S^{\alpha \beta}_{ab} h^{\nu}_{na} \eta^{nb} \tilde{\alpha}^a p_\nu), 
\]

\[
S^{\alpha \beta}_{ab} = \frac{1}{2} (h^\rho_a h^\beta_{b} + h^\beta_a h^\rho_{b}) \text{sgn}(a - b). 
\]

Then we get

\[
(\partial_\mu \alpha^\nu)\hat{p}_\nu = \partial_\mu g_{\alpha \beta} \frac{\partial h^\nu_a}{\partial g_{\alpha \beta}} \tilde{\alpha}^a p_\nu = \partial_\mu g_{\alpha \beta} \left( - \frac{1}{4} (\alpha^\rho p^\beta + \alpha^\beta p^\rho) - \frac{1}{2} S^{\alpha \beta}_{ab} h^{\nu}_{na} \eta^{nb} \tilde{\alpha}^a p_\nu \right) 
\]

\[
= - \frac{1}{4} \partial_\mu g_{\alpha \beta} \left( (\alpha^\rho p^\beta + \alpha^\beta p^\rho) + 2 S^{\alpha \beta}_{ab} h^{\nu}_{na} \eta^{nb} \tilde{\alpha}^a p_\nu \right). 
\]

For classical approximation we have

\[
\phi^+ \tilde{\alpha}^a \phi \to \tilde{v}^a \delta^3(\vec{x} - \vec{X}), \ h^\nu_{na} \eta^{nb} p_\nu \phi \to m \hat{u}^b \phi, \ S^{\alpha \beta}_{ab} = - S^{\alpha \beta}_{ba}. 
\]

Substituting (3.19) and (3.20) into (3.14), we get the right hand term of (3.16). The proof is finished.

Substituting (3.15) into (3.12) and noticing \( dt = \sqrt{1 - g_{kl}v^k v^l} dt \), we get Newtonian second law for the spinor

\[
\frac{d}{dt}p^\mu + \Gamma^\mu_{\alpha \beta} p^\alpha u^\beta = g^{\alpha \mu} (e(\partial_\alpha A_\beta - \partial_\beta A_\alpha)u^\beta + s^a \partial_\alpha \omega_a). 
\]

(3.21)
Although we derive (3.21) in GCS, it obviously holds in all coordinate system due to the covariant form. Clearly, the additional acceleration in (3.21) is a little different from (1.1). If the spin-gravity coupling potential \( s_\mu \Omega^\mu \) can be ignored, (3.21) satisfies ‘mass shell constraint’ \( \frac{d}{dt}(p^\mu p_\mu) = 0 \). In this case, the classical mass of the spinor is a constant and the free spinor moves along geodesic.

In (3.21) we get a spin-gravity coupling potential

\[
\Psi \equiv \omega_s s^a = \Omega_\alpha s^\alpha. \tag{3.22}
\]

This potential provides an explanation for the relevance between magnetic field and rotation of a celestial body. For a static star without rotation, the magnetic field is also very weak, because in this case we have \( \Omega_\mu = 0 \) and the spins of all particles have not a dominant direction, and their magnetic fields are canceled each other. In a rotational star, we have \( \Omega_\mu \neq 0 \), and the spins are automatically arranged in order to generate macro magnetic field. This macro magnetic field is in turn enhanced by the orbital magnetic moment of particles.

For many body problem, dynamics of the system should be juxta posed (3.3) due to the superposition of Lagrangian,

\[
i(\partial_t + \Upsilon_t)\phi_n = H_n \phi_n, \quad H_n = -\alpha^k \hat{\rho}_k + eA_0 + m_n \gamma_0 - \Omega_\mu \hat{s}_\mu. \tag{3.23}
\]

The coordinate, speed and momentum of \( n \)-th spinor are defined by\([33, 34]\),

\[
\vec{X}_n(t) = \int_{S^3} \vec{x}_0 \sqrt{g} d^3x, \quad \vec{v}_n = \frac{d}{dt} \vec{X}_n, \quad p^\mu_n = \Re \int_{S^3} \phi^*_n \hat{p}^\mu \phi_n \sqrt{g} d^3x. \tag{3.24}
\]

The classical approximation condition for point-particle model reads,

\[
q^\mu_n \rightarrow u^\mu_n \sqrt{1 - \bar{g}_{kl} v^k_n v^l_n} \delta^3(\vec{x} - \vec{X}_n), \quad u^\mu_n = \frac{dX^\mu_n}{d\tau} = (1, \vec{v}_n) / \sqrt{1 - \bar{g}_{kl} v^k_n v^l_n}. \tag{3.25}
\]

Repeating the derivation from (3.12) to (3.20), we get classical dynamics for each spinor,

\[
\frac{d}{d\tau} p^\mu_n + \Gamma^\mu_{\alpha\beta} p^\alpha_n u^\beta_n = g^{\alpha\mu} \left( \epsilon_\alpha (\partial_\alpha A_\beta - \partial_\beta A_\alpha) u^\beta_n + s^\alpha_\mu \partial_\alpha \omega_n \right), \quad (\forall n). \tag{3.26}
\]

**IV. DISCUSSION AND CONCLUSION**

To split the spinor connection into \( \Upsilon_\mu \) and \( \Omega_\mu \) not only makes calculation simple, but also highlights their different physical meanings. \( \Upsilon_\mu \) corresponds to geometrical calculations, but \( \Omega_\mu \) has complex form and leads to dynamical effects. \( \Omega_\mu \) couples with the spin \( s^\mu \) of a spinor, which provides location and navigation functions for a spinor. In this representation, the connection only
depends on metric but is independent of Dirac or Pauli matrices, and their classical approximation is parallel to the speed of spinor.

The new vector $\Omega_\mu$ provides an explanation for the origin of magnetic field of celestial body. In weak gravity, the spin-gravity coupling energy is a higher order infinitesimal, but in a neutral star, this term may become dominant. In a diagonal metric we have $\Omega_\mu = 0$, and a static planet is usually of very weak magnetic field. In (3.21), the gravitomagnetic force is caused by Christoffel symbols $\Gamma^\mu_{\alpha\beta}$. In harmonic coordinate system, the main part of gravitomagnetism has a similar structure of Maxwell equation system which was derived in [20, 21, 36]. The gravitomagnetic potential is equal to $\vec{A} = (g^{01}, g^{02}, g^{03})$, and field intensity $\vec{B} = \nabla \times \vec{A}$. The gravitomagnetic field only interacts with speed $\vec{v}$ of a particle, but is independent of spin $s^\mu$. This feature is different from electromagnetic field.

By (3.5) we find the spin is actually a true 4-d vector, which is different from angular momentum, the latter is an axial vector. Besides, $\Omega_\mu$ is also irrelevant with gravitomagnetic field. So this study may be helpful to understand the marvelous structure and wonderful property of a spinor, as well as subtle interaction between spinor and space-time.

In conventional classical approximation we usually use inadequate limitations such as $\hbar \to 0$, $c \to \infty$. They are constants act as units of physical variables. We can only make approximation such as $v \ll c$ or (3.9) if the mean radius of a spinor is much less than moving scale. Most paradoxes and puzzles in physics are caused by such ambiguous statements or overlapping concepts of different logical systems. A detailed discussion for such problems in Minkowski space-time is given in [34, 37]. One of purposes of this paper is to show the consistence of general relativity, quantum mechanics and classical mechanics.

It is a good choice to take Pauli or Dirac matrices as tetrad, and then the expression of equations and meanings of parameters become simpler and clearer as shown above. In fact, all current fundamental physical theories can be simply unified in this elegant language as follows:

A1. The space-time is described by

$$dx = \tilde{\gamma}_\mu dx^\mu = \gamma_\alpha \delta X^\alpha,$$

in which $\gamma_\alpha$ and $\tilde{\gamma}_\mu$ satisfy the $C\ell(1,3)$ Clifford algebra (2.5).

A2. The dynamics for a definite physical system is given by

$$\partial \Psi = F(\Psi), \quad \partial \equiv \tilde{\gamma}_\mu \partial_\mu,$$

in which $\Psi = (\psi_1, \psi_2, \cdots, \psi_n)^T$, and $F(\Psi)$ consists of some tensorial products of $\Psi$, so that the total equation is covariant.
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