FLOPS AND S-DUALITY CONJECTURE

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To the memory of Kentaro Nagao

Abstract. We prove the transformation formula of Donaldson-Thomas (DT) invariants counting two dimensional torsion sheaves on Calabi-Yau 3-folds under flops. The error term is described by the Dedekind eta function and the Jacobi theta function, and our result gives evidence of a 3-fold version of Vafa-Witten’s S-duality conjecture. As an application, we prove a blow-up formula of DT type invariants on the total spaces of canonical line bundles on smooth projective surfaces. It gives an analogue of the similar blow-up formula in the original S-duality conjecture by Yoshioka, Li-Qin and Göttsche.

1. Introduction

1.1. Background and motivation. The purpose of this paper is to give evidence of a 3-fold version of Vafa-Witten’s S-duality conjecture [VW94]. The original S-duality conjecture predicts the (at least almost) modularity of the generating series of Euler characteristics of moduli spaces of stable torsion free sheaves on algebraic surfaces. This is still an open problem, but there exist several evidence, and we refer to [Gö99] for the developments so far. One of the important evidence is that the above generating series transform under a blow-up at a point of a surface by a multiplication of a certain modular form. Such a blow-up formula was predicted by Vafa-Witten [VW94], and proved by Yoshioka [Yos96], Li-Qin [LQ99] and Göttsche [Gö99].

Instead of stable torsion free sheaves on algebraic surfaces, we study semistable pure two dimensional torsion sheaves on 3-folds and the generating series of their counting invariants. Let $X$ be a smooth projective Calabi-Yau 3-fold, i.e. $K_X = 0$ and $H^1(O_X) = 0$. For an ample divisor $\omega$ on $X$ and a cohomology class $v \in H^*(X, \mathbb{Q})$, we have the (generalized) Donaldson-Thomas (DT) invariant (cf. [Tho00], [JS12], [KS])

$$DT_{\omega}(v) \in \mathbb{Q}$$

(1)

which virtually counts $\omega$-semistable sheaves $E \in \text{Coh}(X)$ satisfying $v(E) = v$. Here $v(E)$ is the Mukai vector\footnote{Taking the Mukai vector, rather than the usual Chern character, is crucial for our purpose. See Remark 3.19} of $E$:

$$v(E) := \text{ch}(E) \cdot \sqrt{\text{td}_X} \in H^*(X, \mathbb{Q}).$$

We are interested in the DT invariants (1) for the classes $v$ of the form

$$v = (0, P, -\beta, -n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X).$$
Below we regard $\beta, n$ as elements of $H_2(X, \mathbb{Q})$ via Poincaré duality. By fixing $P$, we consider the generating series
\begin{equation}
\sum_{\beta \in H_2(X), n \in \mathbb{Q}} \DT_{\omega}(0, P, -\beta, -n) q^n t^\beta.
\end{equation}
The sheaves which contribute to the series (2) are supported on two-dimensional subschemes in $X$. The S-duality conjecture for our 3-fold $X$ is stated that the series (2) satisfies (almost) modular transformation properties of Jacobi forms. Such an expected modularity of the series (2) plays an important role in [DM] to derive the Ooguri-Strominger-Vafa (OSV) conjecture [OSV04] in string theory. The OSV conjecture, which is not yet formulated in a mathematically precise way, may be stated as a certain approximation between the series (2) for $|P| \gg 0$ and the generating series of Gromov-Witten invariants on $X$. The study of the S-duality conjecture for the series (2) is an important subject toward a mathematical approach of the OSV conjecture.

In general, computing the generating series (2) in concrete examples may be more difficult than those which appear in the S-duality conjecture for smooth surfaces, due to the possible singularities in the supports of two-dimensional sheaves on 3-folds. Nevertheless, we can ask how the generating series (2) transforms under a birational transformation, and whether the error term is described in terms of Jacobi forms or not. This is an important process to check the validity of the S-duality conjecture for the series (2), as well as the blow-up formula in [Yos96], [LQ99], [G99] toward the S-duality conjecture for surfaces.

1.2. Main result. Our main result shows that a variant of the generating series (2) transforms under a flop by a multiplication of a certain meromorphic Jacobi form. Let $X$ be a smooth projective Calabi-Yau 3-fold which fits into a flop diagram (cf. Definition 2.1)
\begin{equation}
(C \subset X) \xrightarrow{\phi} (X^{\dagger} \supset C^{\dagger}) \xrightarrow{f} (p \in Y).
\end{equation}
Here $C, C^{\dagger}$ are exceptional locus of $f, f^{\dagger}$ and $f(C) = f(C^{\dagger}) = p$. We assume that $C$ is an irreducible rational curve and let $l$ be the scheme theoretical length of $f^{-1}(p)$ at the generic point of $C$. For a fixed divisor class $P \in H^2(X)$ and an ample divisor $\omega$ on $Y$, we consider the generating series
\begin{equation}
\DT_{f^*\omega}(P) := \sum_{\beta \in H_2(X), n \in \mathbb{Q}} \DT_{f^*\omega}(0, P, -\beta, -n) q^n t^\beta.
\end{equation}
The following theorem, which will be proved in Subsection 3.7 is the main result in this paper:

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2 We refer to [EZ85] for a basic of Jacobi forms.
3 The invariant $\DT_{f^*\omega}(v)$ counts $f^*\omega$-slope semistable sheaves $E$ on $X$ with $v(E) = v$, which is well-defined although $f^*\omega$ is not ample.
Theorem 1.1. There exist $n_j \in \mathbb{Z}_{\geq 1}$ for $1 \leq j \leq l$ such that we have the following formula:

\[
\text{DT}_{f^* \omega}(\phi_* P) = \phi_* \text{DT}_{g^* \omega}(P) \cdot \prod_{j=1}^{l} \left\{ (q^{P_C - 1} \eta(q)^{-1} \vartheta_{1,1}(q, (-1)^j \phi_* P C^j))^j \right\}^{j n_j P_C}.
\]

Here $\phi_*$ is the variable change $(n, \beta) \mapsto (n, \phi_* \beta)$, $\eta(q)$ is the Dedekind eta function and $\vartheta_{a,b}(q,t)$ is the Jacobi theta function, given as follows:

\[
\eta(q) = q^{\frac{1}{24}} \prod_{k \geq 1} (1 - q^k), \quad \vartheta_{a,b}(q,t) = \sum_{k \in \mathbb{Z}} q^{\frac{1}{2}(k+a)^2} (-1)^b t^{k+a^2}.
\]

Recall that $\eta(q)$ is a modular form of weight $1/2$, $\vartheta_{1,1}(q,t)$ is a Jacobi form of weight $1/2$ and index $1/2$. The result of Theorem 1.1 shows that the series (4) transforms under a flop by a multiplication of a meromorphic Jacobi form of weight $0$ and index $\sum_{j=1}^{l} j^3 n_j (P \cdot C)/2$. The integer $n_j$ also coincides with the genus zero Gopakumar-Vafa invariant with class $j[\mathcal{C}]$ in the sense of [Kat08].

We will also prove some variants of Theorem 1.1: the Euler characteristic version and a fixed supported version. The former version is a flop formula for DT type invariants without Behrend functions [Beh09]. Such invariants are not deformation invariant, and the computation of the error term of the flop formula is more subtle. We will give that version only in the case of $l = 1$, while we don’t need the Calabi-Yau condition. The latter version compares DT type invariants on $X$ and $X^\dagger$, which count sheaves supported on a fixed divisor $S \subset X$ and its strict transform $S^\dagger \subset X^\dagger$ respectively. The above variants give interesting applications. For instance, we give a blow-up formula of DT type invariants on canonical line bundles on surfaces in Section 4. In the next paper [Todc], we also apply the result in this paper to show the modularity of the generating series of Hilbert schemes of points with $A_n$-type singularities.

1.3. Blow-up formula. As an application of a variant of Theorem 1.1, we prove a blow-up formula for DT type invariants without Behrend functions on canonical line bundles on surfaces, which gives an analogue of the similar blow-up formula in the original S-duality conjecture [Yos96], [LQ99], [G99]. Let $S$ be a smooth projective surface and $\pi: \omega_S \to S$ the total space of the canonical line bundle of $S$, which is a non-compact Calabi-Yau 3-fold. For an ample divisor $L$ on $S$ and an element

\[(r, l, s) \in H^0(S) \oplus H^2(S) \oplus H^4(S)\]

the invariant

\[
\text{DT}^\chi_L(r, l, s) \in \mathbb{Q}
\]

is defined to be (roughly speaking) the naive Euler characteristic of the moduli space of $L$-semistable sheaves $E \in \text{Coh}(\omega_S)$ supported on the zero section of $\pi$ such that the Chern character of $\pi_* E$ coincides with $(r, l, s)$. Let $g: S^\dagger \to S$ be a blow-up at a point in $S$ with exceptional curve $C^\dagger \subset S^\dagger$. The following result will be proved in Theorem 4.3.
Theorem 1.2. For fixed \( r \in \mathbb{Z}_{\geq 1} \) and \( l \in H^2(S) \), we have the following formula:

\[
\sum_{s,a} DT^x_{g^*L}(r, g^*l - aC^\dagger, -s)q^{\frac{r}{2} + \frac{s}{2} + a + \frac{t}{2}} = \sum_s DT^x_{L}(r, l, -s)q^s \cdot \eta(q)^{-r} \partial_{1,0}(q, t)^r.
\]

It is easy to see that (cf. Example 4.4 for the rank two case) the error term of the formula in Theorem 1.2 coincides with the error term in the blow-up formula in the original S-duality conjecture \([Yos96], [LQ99], [G99]\). The idea of the proof of Theorem 1.2 is as follows: we find a suitable flop diagram (3) such that \( X, X^\dagger \) are compactifications of \( \omega_S, \omega_S^\dagger \) respectively. Moreover the strict transform of the zero section \( S \subset \omega_S \subset X \) coincides with the zero section \( S^\dagger \subset \omega_S^\dagger \subset X^\dagger \). Then a variant of Theorem 1.1 compares the invariants (6) on \( S \) and \( S^\dagger \), which proves the result.

1.4. Sketch of the proof of Theorem 1.1. Here we give a rough sketch of the proof of Theorem 1.1. Let \( f: X \to Y \) be a 3-fold flopping contraction as in Theorem 1.1. By Bridgeland \([Bri02]\), there exist hearts of perverse t-structures \( p\text{Per}^{\leq 2}(X/Y) \subset D^b\text{Coh}^{\leq 2}(X) \) for \( p = 0, -1 \). Here \( \text{Coh}^{\leq 2}(X) \) is the category of torsion sheaves on \( X \). We introduce the \( f^*\omega \)-slope stability on \( p\text{Per}^{\leq 2}(X/Y) \) (cf. Definition 2.10), and construct another heart

\[
pA_{f^*\omega}^\mu \subset D^b\text{Coh}^{\leq 2}(X)
\]

by a tilting of \( p\text{Per}^{\leq 2}(X/Y) \) determined by the \( f^*\omega \)-slope stability on it and a choice of \( \mu \in \mathbb{Q} \). Moreover we construct the abelian subcategory (cf. Subsection 2.4)

\[
pB_{f^*\omega}^\mu \subset pA_{f^*\omega}^\mu
\]

which consists of objects in \( pA_{f^*\omega}^\mu \) whose Chern characters satisfy a certain linear equation. It has the following properties: first for a given element \( v \in H^*(X) \), the set of objects in \( pB_{f^*\omega}^\mu \) with Mukai vector \( v \) is bounded (cf. Proposition 2.17). Hence we are able to define the completion of the stack theoretic Hall algebra of \( pB_{f^*\omega}^\mu \), denoted by \( \hat{H}(pB_{f^*\omega}^\mu) \). Next we see that the category \( pB_{f^*\omega}^\mu \) contains any \( f^*\omega \)-slope semistable two dimensional sheaf on \( X \) with slope \( \mu \) (cf. Lemma 2.14). Hence the moduli spaces of such objects, which define the series (4), determine an element of \( \hat{H}(pB_{f^*\omega}^\mu) \).

For a flop diagram (3), there is an equivalence \( \Phi \) between \( D^b\text{Coh}(X) \) and \( D^b\text{Coh}(X^\dagger) \) by Bridgeland \([Bri07]\). The equivalence \( \Phi \) restricts to an equivalence between \( 0B_{f^*\omega}^\mu \) and \( -1B_{f^*\omega}^\mu \). Noting this fact, we construct the generating series

\[
p\hat{DT}_{f^*\omega}(P) = \sum_{\beta, n} p\hat{DT}_{f^*\omega}(0, P, -\beta, -n)q^n t^\beta
\]

(7)
whose coefficients satisfy
\begin{equation}
0 \hat{DT}_{f^*\omega}(v) = -1 \hat{DT}_{f^*\omega}(\Phi_*v).
\end{equation}

Here $\Phi_*$ is the isomorphism between $H^*(X)$ and $H^*(X^\dagger)$ induced by $\Phi$, which is described in Lemma 2.9.

In Proposition 3.7, we describe the relationship between moduli spaces which define (4) and (7) in terms of the algebra $\hat{H}(\mathcal{PB}_{f^*\omega}^\mu)$. Together with the integration map on the Lie algebra of virtual indecomposable objects in $\hat{H}(\mathcal{PB}_{f^*\omega})$ by \cite{JS12}, it enables us to describe the relationship between the series \begin{equation}
(8)
\end{equation}
with \begin{equation}
\Phi_*
\end{equation}
described in Lemma 2.9. In Proposition 3.7, we describe the relationship between moduli spaces which define (4) and (7) in terms of the algebra $\hat{H}(\mathcal{PB}_{f^*\omega}^\mu)$. Together with the integration map on the Lie algebra of virtual indecomposable objects in $\hat{H}(\mathcal{PB}_{f^*\omega})$ by \cite{JS12}, it enables us to describe the relationship between the series \begin{equation}
(8)
\end{equation}
with \begin{equation}
\Phi_*
\end{equation} described in Lemma 2.9. In proving the Euler characteristic version of Theorem 1.1, we give a direct classification of parabolic stable pairs when $l = 1$ (cf. Lemma 3.20).

1.5. Related works. A flop formula for DT type curve counting invariants was obtained in the papers \cite{HL12}, \cite{NN11}, \cite{Tod13b}, \cite{Cal}. Among them, the papers \cite{Tod13b}, \cite{Cal} (also see \cite{Bri11}) use similar Hall algebra methods, but we need to work with the relevant abelian category $\mathcal{PB}_{f^*\omega}^\mu$ which did not appear in the above papers. On the other hand, there are few mathematical literatures in which DT invariants of the form $\text{DT}_{\omega}(0, P, \beta, n)$ are studied. In \cite{GSa}, the modularity of these invariants is discussed for nodal K3 fibrations using degeneration formula. In \cite{Tod13a}, \cite{GST}, some relationship between the invariants $\text{DT}_{\omega}(0, P, \beta, n)$ and DT type curve counting invariants are studied. In \cite{GSb}, the invariant $\text{DT}_{\omega}(0, P, \beta, n)$ on local $\mathbb{P}^2$ is studied for small $P$. In physics literatures, a few of the D4 brane counting which corresponds to the invariants of the form $\text{DT}_{\omega}(0, P, \beta, n)$ are computed \cite{GSY}, \cite{GY}. Also the flop formula of D4D2D0 bound states on the resolved conifold is studied in \cite{Nis}, \cite{NY} using Kontsevich-Soibelman’s wall-crossing formula \cite{KS}. The result of Theorem 1.1 is interpreted as a mathematical justification and a generalization of the arguments in the physics articles \cite{Nis}, \cite{NY}.

1.6. Acknowledgment. This paper is dedicated to the memory of Kentaro Nagao, who made significant contributions to the Donaldson-Thomas theory and its relationship to flops, non-commutative algebras \cite{NN11}, \cite{Nag13}, during his short life. This work is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan. This work is also supported by Grant-in Aid for Scientific Research grant (22684002) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

1.7. Notation and convention. In this paper, all the varieties are defined over $\mathbb{C}$. For a $d$-dimensional variety $X$, we denote by $H^*(X, \mathbb{Q})$ the even part of the singular cohomologies of $X$, and write its element as $(a_0, a_1, \cdots, a_d)$ for $a_i \in H^{2i}(X, \mathbb{Q})$. We sometimes abbreviate $\mathbb{Q}$ and just write $H^{2i}(X, \mathbb{Q})$. 


\[ H^2_1(X, \mathbb{Q}) \text{ as } H^2(X), H^2_2(X). \] For subschemes \( Z_1, Z_2 \subset X \), the intersection \( Z_1 \cap Z_2 \subset X \) always means the scheme theoretical intersection. For a triangulated category \( \mathcal{D} \) and a set of objects \( S \) in \( \mathcal{D} \), we denote by \( \langle S \rangle_{\text{ex}} \) the smallest extension closed subcategory in \( \mathcal{D} \) which contains \( S \). For a variety \( X \) and a sheaf of (possibly non-commutative) algebras \( A \) on \( X \), we denote by \( \text{Coh}(A) \) the abelian category of coherent right \( A \)-modules on \( X \), and \( D^b \text{Coh}(A) \) its bounded derived category. We write \( \text{Coh}(\mathcal{O}_X) \) as \( \text{Coh}(X) \) as usual. For \( i \in \mathbb{Z} \), we denote by \( \text{Coh}^{\leq i}(A) \subset \text{Coh}(A) \) the subcategory of objects \( E \in \text{Coh}(A) \) whose support \( \text{Supp}(E) \) as an \( \mathcal{O}_X \)-module satisfies \( \dim \text{Supp}(E) \leq i \). For \( E \in D^b \text{Coh}(A) \), we denote by \( H^i(E) \in \text{Coh}(A) \) its \( i \)-th cohomology.

2. Flops, perverse t-structures and their tilting

2.1. 3-fold flops. Let us recall the notion of flopping contractions and their flops.

**Definition 2.1.** A projective birational morphism \( f: X \to Y \) is called a flopping contraction if \( f \) is isomorphic in codimension one, \( Y \) has only Gorenstein singularities, and the relative Picard number of \( f \) equals to one. A flop of a flopping contraction \( f: X \to Y \) is a non-isomorphic birational morphism \( \phi: X / \text{axis} \to X^\dagger \) which fits into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^\dagger \\
\downarrow f & & \downarrow f^\dagger \\
Y & & Y \\
\end{array}
\]

such that \( f^\dagger \) is also a flopping contraction.

It is known that a flop is unique if it exists, and any birational map between minimal models are decomposed into a finite number of flops [Kaw08]. We say that \( f: X \to Y \) is a 3-fold flopping contraction if \( f \) is a flopping contraction and \( X \) is a smooth 3-fold. In this case, the exceptional locus \( C \) of \( f \) is a tree of smooth rational curves \( C = C_1 \cup \cdots \cup C_N, \ C_i \cong \mathbb{P}^1. \)

If \( f^\dagger: X^\dagger \to Y \) is a flop of \( f \), the exceptional locus \( C^\dagger \) of \( f^\dagger \) is also a tree of smooth rational curves \( C_i^\dagger \) with \( 1 \leq i \leq N \). A projective line on a smooth 3-fold is called \((a, b)\)-curve if its normal bundle is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \). It is well-known (cf. [Rei]) that each \( C_i \) is \((a, b)\)-curve for either \( (a, b) = (-1, -1) \), \((0, -2)\), or \((1, -3)\).

**Example 2.2.** Let \( Y \subset \mathbb{C}^4 \) be the 3-fold singularity, given by

\[
Y = \{ xy + z^2 - w^{2n} = 0 : (x, y, z, w) \in \mathbb{C}^4 \}.
\]

Then there is a flop diagram [14], where \( f, f^\dagger \) are blow-ups at the ideals

\[
I = (x, z - w^n) \subset \mathcal{O}_Y, \quad I^\dagger = (x, z + w^n) \subset \mathcal{O}_Y
\]

respectively. Both of the exceptional locus \( C, C^\dagger \) of \( f \), \( f^\dagger \) are \((-1, -1)\)-curves if \( n = 1 \), and \((0, -2)\)-curves otherwise. By [Rei], the birational map...
\( \phi \) is obtained as a Pagoda diagram

(12) \[ X \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-1}} X_{n-1} \xleftarrow{f_n} X_n \xleftarrow{f_{n+1}} X_{n+1} \xleftarrow{f_{n+2}} \cdots \xleftarrow{f_n} X_n \xrightarrow{f_1} X. \]

Here \( f_i, f_i^\dagger \) for \( 1 \leq i \leq n-1 \) are blow-ups at \((0,-2)\)-curves, and \( f_n, f_n^\dagger \) are blow-ups at \((-1,-1)\)-curves.

Let \( f : X \to Y \) be a 3-fold flopping contraction whose exceptional locus is an irreducible rational curve \( C \subset X \). We denote by \( l \) the length of \( \mathcal{O}_{f^{-1}(p)} \) at the generic point of \( C \), where \( p = f(C) \) and \( f^{-1}(p) \) is the scheme theoretic fiber of \( f \) at \( p \). Then we have

\[
l \in \{1, 2, 3, 4, 5, 6 \}
\]

and \( l = 1 \) if and only if \( C \) is not a \((1,-3)\)-curve (cf. [KM92]). Moreover if \( l = 1 \), then \( \hat{\mathcal{O}}_{Y,p} \) is isomorphic to the completion of the singularity \((11)\) for some \( n \in \mathbb{Z}_{\geq 1} \) at the origin (cf. \([Rci]\)). In this case, the integer \( n \) is called the width of \( C \).

In general, we consider the following completion:

(13) \[ \hat{f}: \hat{X} := X \times_Y \text{Spec} \hat{\mathcal{O}}_{Y,p} \to \hat{Y} := \text{Spec} \hat{\mathcal{O}}_{Y,p}. \]

By \([BKL01]\), there exists a flat deformation

(14) \[ \hat{X} \xrightarrow{h} \hat{Y} \]

where \( \Delta \) is a Zariski open neighborhood of \( 0 \in \mathbb{A}^1 \) such that \( h_0: \hat{X}_0 \to \hat{Y}_0 \) is isomorphic to \( \hat{f} \), and \( h_t: \hat{X}_t \to \hat{Y}_t \) for \( t \in \Delta \setminus \{0\} \) is a flopping contraction whose exceptional locus is a disjoint union of \((-1,-1)\)-curves. More precisely, the exceptional locus of \( h_t \) consists of disjoint \((-1,-1)\)-curves \( C_{j,k} \subset \hat{X}_t \) for \( 1 \leq j \leq l \) and \( 1 \leq k \leq n_j \) for some \( n_j \in \mathbb{Z}_{\geq 1} \) such that \( C_{j,k} \) is class \( j[C] \), i.e. for any line bundle \( L \) on \( \hat{X} \), we have

\[
\deg(L|_{C_{j,k}}) = j \deg(L|_C)
\]

where we regard \( C \) as a curve on the central fiber of \( \hat{X} \to \Delta \). The integer \( n_j \) is given as follows: there is a subscheme \( C_j \subset f^{-1}(p) \) with class \( j[C] \) such that \( n_j \) is the multiplicity of the Hilbert scheme of subschemes in \( X \) at \( C_j \).

If \( l = 1 \), the number \( n_1 \) equals to the width \( n \).

2.2. Perverse t-structures. Let \( f : X \to Y \) be a 3-fold flopping contraction. In this situation, Bridgeland \([Bri02]\) associates the subcategories \( p\text{Per}(X/Y) \subset D^b\text{Coh}(X) \) for \( p = 0, -1 \) as follows:

**Definition 2.3.** We define \( p\text{Per}(X/Y) \subset D^b\text{Coh}(X) \) for \( p = 0, -1 \) to be

\[
p\text{Per}(X/Y) := \left\{ E \in D^b\text{Coh}(X) : \begin{array}{c} \text{R}f_* E \in \text{Coh}(Y) \\ \text{Hom}^{-p}(E, C) = \text{Hom}^{<p}(C, E) = 0 \end{array} \right\}.
\]

Here \( C := \{ F \in \text{Coh}(X) : \text{R}f_* F = 0 \} \).
We also define $p\Per_{\leq i}(X/Y)$ to be

$$p\Per_{\leq i}(X/Y) := \{E \in p\Per(X/Y) : \mathbb{R}f_*E \in \Coh_{\leq i}(Y)\}.$$  

It is proved in [Bro02] that $p\Per(X/Y)$ are the hearts of bounded t-structures on the category $D^b\Coh(X)$. Similarly, the subcategories $p\Per_{\leq i}(X/Y)$ are shown to be the hearts of bounded t-structure on $D^b\Coh_{\leq i}(X)$, hence they are abelian categories. By convention, we write $p\Per_{\leq 0}(X/Y)$ as $p\Per_0(X/Y)$. We set

$$\delta_H, t_H : \Coh(X) \to \Per(X/Y)$$

such that $\delta_H$ gives a small deformation of $\Per_0(X/Y)$ with $1 \leq \delta < 1$. Hence the result essentially follows from [Bri07, Proposition 5.2 (iii)].

Proof. The result essentially follows from [Tod08a] Proposition 5.2 (iii). For simplicity, suppose that $p = 0$. For $B + iH \in H^2(X, \mathbb{C})$ and $F \in D^b\Coh_0(X/Y)$, we set

$$\mu_H(F) := \frac{\text{ch}_2(F)}{\text{ch}_1(F) \cdot H}.$$  

Here $\mu_H(F)$ is set to be $\infty$ if the denominator is zero. The above slope function defines the $\mu_H$-stability on $\Coh_0(X/Y)$ in the usual way.

Lemma 2.4. We have the following descriptions of $p\mathcal{F}$:

$$0\mathcal{F} = \{E \in \Coh_0(X/Y) : E \text{ is } \mu_H\text{-semistable with } \mu_H(E) < 0\}_{\text{ex}}$$

$$-1\mathcal{F} = \{E \in \Coh_0(X/Y) : E \text{ is } \mu_H\text{-semistable with } \mu_H(E) \leq 0\}_{\text{ex}}.$$  

Proof. The result essentially follows from [Tod08a] Proposition 5.2 (iii). For simplicity, suppose that $p = 0$. For $B + iH \in H^2(X, \mathbb{C})$ and $F \in D^b\Coh_0(X/Y)$, we set

$$Z_{B,H}(F) := -\text{ch}_3(F) + (B + iH)\text{ch}_2(F).$$  

Then by [Tod08a] Proposition 5.2 (iii), the pair

$$\sigma_0 := (Z_{-\delta H, 0}, 0\Per_0(X/Y)),$$  

$$0 < \delta \ll 1$$

determines a Bridgeland stability condition [Bri07] on $D^b\Coh_0(X/Y)$. Note that $Z_{-\delta H, 0}(F)$ for any non-zero $F \in 0\Per_0(X/Y)$ is contained in $\mathbb{R}_{\leq 0}$. Moreover the pair

$$\sigma_t := (Z_{-\delta H,tH}, \Coh_0(X/Y)), $$  

$$0 < t \ll 1$$

gives a small deformation of $\sigma_0$ in the space of Bridgeland stability conditions on $D^b\Coh_0(X/Y)$. Let $\mathcal{P}_t(\phi) \subset D^b\Coh_0(X/Y)$ be the full subcategory for each $\phi \in \mathbb{R}$, which gives the slicing in the sense of [Bri07] corresponding to $\sigma_t$. Let us take $E \in \mathcal{P}_t(\phi)$ with $1/2 < \phi \leq 1$. We have $E \in \Coh_0(X/Y)$, and there is an exact sequence

$$0 \to T \to E \to F \to 0$$

for $T, F \in 0\mathcal{F}$. Since $F[1] \in 0\Per_0(X/Y) \cap (\Coh_0(X/Y)[1])$, it follows that $\arg Z_{-\delta H,tH}(F) \in (0, \pi/2)$. Hence the $Z_{-\delta H,tH}$-semistability of
$E$ yields $F = 0$, i.e. $E \in \mathcal{O}_X$. This implies that, in the notation of [Bri07], we have $\mathcal{P}_1(1/2, 1) \subset \mathcal{O}_X$, and a similar argument also shows that $\mathcal{P}_1(0, 1/2) \subset \mathcal{O}_X$. Since both of $(\mathcal{O}_X, \mathcal{O}_Y)$ and $(\mathcal{P}_1(1/2, 1), \mathcal{P}_1(0, 1/2))$ are torsion pairs of $\text{Coh}_0(X/Y)$, they must coincide. Note that an object $E \in \text{Coh}_0(X/Y)$ is contained in $\mathcal{P}_i(\phi)$ for $0 < \phi < 1/2$ and $0 < t \ll 1$ if and only if $E$ is $\mu_H$-semistable with $\mu_H(E) < -\delta$. Since we can take $\delta > 0$ arbitrary close to zero, we obtain the result.

Let $\phi: X \to X^\dagger$ be the flop of $f$. By [Bri02], there is an equivalence

$$\Phi: \mathcal{D}^b(\text{Coh}(X)) \xrightarrow{\sim} \mathcal{D}^b(\text{Coh}(X^\dagger)) \tag{18}$$

which takes $\mathcal{O}_{X} \in \mathcal{O}_{X}$ to $\mathcal{O}_{X^\dagger}$. Furthermore the equivalence $\Phi$ is given by the Fourier-Mukai functor with kernel $\mathcal{O}_{X \times Y}$ (cf. [Che02]), hence $\Phi$ also takes $\mathcal{D}^b(\text{Coh}_{\leq i}(X))$ to $\mathcal{D}^b(\text{Coh}_{\leq i}(X^\dagger))$ and $\mathcal{O}_{X}$ to $\mathcal{O}_{X^\dagger}$.

**Lemma 2.5.** We have $\Phi(\mathcal{O}_X) \cong \mathcal{O}_{X^\dagger}$.

**Proof.** The object $\mathcal{O}_X \in \mathcal{O}_{X}$ is a local projective object in $\mathcal{O}_{X}$ by [dIH04] Lemma 3.2.4], hence $\Phi(\mathcal{O}_X) \in \mathcal{O}_{X^\dagger}$ is a local projective object. By [dIH04] Proposition 3.2.6], the object $\Phi(\mathcal{O}_X)$ must be a line bundle, hence it must be isomorphic to $\mathcal{O}_{X^\dagger}$.

The abelian categories $\mathcal{P}(\mathcal{O}_X)$ are also related to sheaves of non-commutative algebras. By [dIH04], there are vector bundles $\mathcal{E}$ on $X$ which admit equivalences

$$\mathcal{P}\Phi := Rf_* R\text{Hom}(\mathcal{P}(\mathcal{E}), *) : \mathcal{D}^b(\text{Coh}(X)) \xrightarrow{\sim} \mathcal{D}^b(\text{Coh}(\mathcal{P}(\mathcal{E}))) \tag{19}$$

where $\mathcal{P}(\mathcal{E}) := f_* \text{End}(\mathcal{E})$ are sheaves of non-commutative algebras on $Y$. The equivalence $\Phi$ restricts to equivalences between $\mathcal{P}(\mathcal{O}_X)$ and $\text{Coh}(\mathcal{P}(\mathcal{E})).$

### 2.3. Induced morphism on cohomologies

For an object $E \in \mathcal{D}^b(\text{Coh}(X)$, its Mukai vector is defined by

$$v(E) := \text{ch}(E) \cdot \sqrt{\text{td}_X} \in H^*(X).$$

The $H^{2i}(X)$-component of $v(E)$ is denoted by $v_i(E)$. Let $\Phi$ be the derived equivalence [IS]. By the Grothendieck duality, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^b(\text{Coh}(X)) & \xrightarrow{\Phi} & \mathcal{D}^b(\text{Coh}(X^\dagger)) \\
v \downarrow & & \downarrow v \\
H^*(X, \mathbb{Q}) & \xrightarrow{\Phi^*} & H^*(X^\dagger, \mathbb{Q}).
\end{array} \tag{20}
$$

Here $\Phi_*$ is defined by the correspondence $v(\mathcal{O}_{X \times Y \times X^\dagger})$. The map $\Phi_*$ is an isomorphism, which does not preserve the grading, but takes $H^{2i}(X)$ to $H^{2i}(X^\dagger)$ for any $i \in \mathbb{Z}$. By taking the subquotients, we obtain the graded isomorphism

$$\phi_* : H^*(X) \xrightarrow{\sim} H^*(X^\dagger).$$

The isomorphism $\phi_*$ is given by the correspondence $[X \times_Y X^\dagger]$. 

We set \( \Gamma \) to be
\[
\Gamma := \text{Im} \left( \psi(X): D^b \text{Coh}_{\leq 2}(X) \to H^{\geq 2}(X, \mathbb{Q}) \right).
\]
We identify \( H^4(X), H^6(X) \) with \( H_2(X), H_0(X) \cong \mathbb{Q} \) via Poincaré duality respectively. We write an element of \( \Gamma \) as \((P, \beta, n)\) for \( P \in H^2(X), \beta \in H_2(X) \) and \( n \in \mathbb{Q} \). Let \( H_2(X/Y) \) be the kernel of
\[
f_*: H_2(X) \to H_2(Y).
\]
We use the following description of the action of \( \Phi_* \) on \( \Gamma \):

**Lemma 2.6.** There exist linear maps
\[
\psi_1: H^2(X) \to H_2(X/Y), \quad \psi_0: H^2(X) \to \mathbb{Q}
\]
such that we have
\[
\Phi_*(P, \beta, n) = (\phi_*P, \phi_*\beta + \psi_1(P), n + \psi_0(P))
\]
for any \((P, \beta, n) \in \Gamma\).

**Proof.** The result follows from \cite{Toda08a}, Proposition 5.2. \(\square\)

For \( \beta \in H_2(X) \), we write \( \beta > 0 \) if \( \beta \) is a numerical class of a non-zero effective one cycle on \( X \). We have the following lemma:

**Lemma 2.7.** For any divisor class \( P \in H^2(X) \) and curve class \( \beta \in H_2(X) \), we have \( P \cdot \beta = \phi_*P \cdot \phi_*\beta \). In particular, \( \beta \in H_2(X/Y) \) satisfies \( \beta > 0 \) if and only if \( \phi_*\beta < 0 \) in \( H_2(X^1/Y) \).

**Proof.** Let us take \( E \in \text{Coh}_{\leq 2}(X) \) and \( F \in \text{Coh}_{\leq 1}(X) \). Since \( \Phi \) is an equivalence, we have \( \chi(E, F) = \chi(\Phi(E), \Phi(F)) \). By the Riemann-Roch theorem, the LHS coincides with \( -\chi_1(E) \cdot \chi_2(F) \). By Lemma 2.6 the RHS coincides with \( -\phi_*\chi_1(E) \cdot \phi_*\chi_2(F) \). Hence the result follows. \(\square\)

In the following lemma, we give some more precise descriptions of \( \psi_0, \psi_1 \):

**Lemma 2.8.** There exists \( a \in \mathbb{Q} \) such that we have \( \psi_1(P) = a(P \cdot C_1)C_1^\dagger \) and
\[
(22) \quad \psi_0(P) = \frac{1}{24} \left( c_2(X) - \phi_*^{-1}c_2(X^\dagger) \right) P
\]
for any divisor class \( P \).

**Proof.** Obviously \( \psi_1(f^*D) = 0 \) for any Cartier divisor \( D \) on \( Y \). Hence we can write \( \psi_1(P) = a(P \cdot C_1)C_1^\dagger \) and \( \psi_0(P) = b(P \cdot C_1) \) for some \( a, b \in \mathbb{Q} \). Here we have used that every \( C_1 \) is numerically proportional to \( C_1 \) by the definition of a flopping contraction. In order to obtain \((22)\), it is enough to prove this for \( P = [S] \) for any irreducible divisor \( S \subset X \). Since \( \Phi(O_X) = O_S \) by Lemma 2.5, we have \( \chi(O_X, O_S) = \chi(O_X, \Phi(O_S)) \). Applying the Riemann-Roch theorem and the commutative diagram \((20)\), we obtain
\[
\frac{P^3}{6} + \frac{c_2(X)}{6} P = \frac{P^3}{6} + \frac{c_2(X)}{24} P + \frac{c_2(X^\dagger)}{24} \phi_* P + \psi_0(P).
\]
By Lemma 2.7, we obtain \((22)\). \(\square\)

If the exceptional locus of \( f \) is an irreducible rational curve, the linear maps \( \psi_0, \psi_1 \) are described using the integers \( n_1, \ldots, n_l \) in Subsection 2.1.
Proposition 2.9. Suppose that the exceptional locus \( C \) of \( f \) is an irreducible rational curve. Then we have

\[
\psi_0(P) = \frac{1}{12} \sum_{j=1}^{l} jn_j(P \cdot C), \quad \psi_1(P) = -\frac{1}{2} \sum_{j=1}^{l} j^2n_j(P \cdot C)C^1.
\]

Proof. It is enough to prove the claim for one divisor class \( P \) with \( P \cdot C \neq 0 \). We first prove the claim in the case that \( C \) is a \((-1,-1)\)-curve and there is a smooth surface \( S \subset X \) such that \( S \cap C \) is a one point, e.g. a suitable compactification of a flopping contraction in Example 2.2 with \( n = 1 \). In this case, \( l = n_1 = 1 \), and the flop \( \phi \) is obtained as

\[
X \xrightarrow{g} Z \xrightarrow{\phi} X^\dagger
\]

where \( g, g^\dagger \) are blow-ups at \( C, C^\dagger \) respectively. Let \( E \) be the exceptional locus of \( g \), which is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( t_1, t_2 \) be the lines in \( E \) which are contracted to points by \( g, g^\dagger \) respectively. By the blow-up formula of Chern classes (cf. [Ful]), we have

\[
c_2(Z) = g^*c_2(X) + l_2 - t_1 = g^\dagger t^\dagger c_2(X^\dagger) + l_1 - l_2
\]

which shows \( c_2(X) - \phi_s^{-1}c_2(X^\dagger) = -2C \). By (22), we obtain the desired formula for \( \psi_0 \) in this case. On the other hand, since the equivalence \( \Phi \) coincides with \( Rg^\dagger \circ g^* \) in this case, we have \( \Phi(O_S) \cong O_{S^1} \), where \( S^1 \) is the strict transform of \( S \). The commutative diagram (20) shows that

\[
\psi_1(S) = \frac{1}{2} \phi_s(S^2) - \frac{1}{2}S^12
\]

By the base point free theorem for \( f : X \to Y \), there is a divisor \( S' \) on \( X \) which is linearly equivalent to \( S \) such that \( S' \cap C \) is a one point which is different from \( S \cap C \). The intersection \( S^1 \cap S^\dagger \) contains \( C^1 \) as a connected component, which is reduced. Hence we have \( S^12 - \phi_s(S^2) = C \), and we obtain the desired formula for \( \psi_1 \) in this case.

Next we prove the general case. Let \( S \subset X \) be an irreducible divisor which is sufficiently ample and does not contain \( C \). We have

\[
\Phi(O_S) \cong R_{p_{X^1*}}(O_{S \times_X X^1})
\]

which is a sheaf since \( S \times_Y X^\dagger \to X^\dagger \) is finite onto its image. Here \( p_{X^1} \) is the projection from \( X \times_Y X^1 \) to \( X^1 \). Let \( F \) be the sheaf obtained as a cokernel of the natural injection \( O_{S^1} \to p_{X^1*}O_{S \times_X X^1} \). Note that \( F \) is supported on \( C^\dagger \), and it depends only on the pair \((S,X)\) restricted to the completion \([13]\). Using (21), (22) and noting \( ch_3(F) = \chi(F) \), we obtain

\[
(23) \quad \psi_0(S) = \frac{1}{12} S^{13} - \frac{1}{12} S^3 + \frac{1}{2} \chi(F)
\]

\[
\psi_1(S) = \frac{1}{2} \phi_s(S^2) - \frac{1}{2} S^{12} + [F].
\]

Let \( S' \subset X \) be another divisor which is linearly equivalent to \( S \) and does not contain \( C \). We can take \( S' \) so that \( S \cap S' \cap C = \emptyset \). Then we have \( S^{12} - \phi_sS^2 = cC^1 \), where \( c \) is the length of \( S^1 \cap S^\dagger \) at the generic point of \( C \), and \( S^{13} - S^3 = -c(S \cdot C) \) by Lemma \([27]\). Hence \( \psi_i(S) \) depends only on the data \((S,S',X)\) restricted to the completion \( \hat{X} \), denoted by \((\hat{S},\hat{S}',\hat{X})\). In particular, the
result holds for any 3-fold flopping contraction which contracts a \((-1,-1)\)-curve to a point.

Let \( h : \tilde{X} \to \tilde{Y} \) be a deformation as in \([14]\), and take its flop \( h^\dagger : \tilde{X}^\dagger \to \tilde{Y} \). Since \( H^2(\mathcal{O}_{\tilde{X}}) = 0 \), by shrinking \( \Delta \) if necessary, the divisors \( \tilde{S}, \tilde{S}' \) deform to \( h \)-ample divisors \( \tilde{S}, \tilde{S}' \subset \tilde{X} \) which are flat over \( \Delta \). Let \( \mathcal{F} \) be the sheaf on \( \tilde{X}^\dagger \) obtained as a cokernel of the injection \( \mathcal{O}_{\tilde{X}^\dagger} \to p^*\mathcal{O}_{\tilde{S} \times_{\tilde{Y}} \tilde{X}^\dagger} \), where \( p^* \) is the projection from \( \tilde{X} \times_{\tilde{Y}} \tilde{X}^\dagger \) to \( \tilde{X}^\dagger \). The sheaf \( \mathcal{F} \) is a flat deformation of \( F \), and for \( t \in \Delta \setminus \{0\} \) the restriction \( \mathcal{F}_t = \mathcal{F}|_{X_t} \) decomposes into the direct sum of \( \mathcal{F}_{t,k,j} \) where \( \mathcal{F}_{t,k,j} \) is supported on \( C_{t,k,j} \). Also let \( \tilde{S}, \tilde{S}' \subset \tilde{X}^\dagger \) be the strict transforms of \( S, S' \) respectively. The intersection \( \tilde{S} \cap \tilde{S}' \) is a flat deformation of \( S \cap S' \). For \( t \in \Delta \setminus \{0\} \), the fundamental cycle of \( (\tilde{S} \cap \tilde{S}')_t = \tilde{S}_t \cap \tilde{S}'_t \) is written as \( \sum_{j,k} c_{j,k} C_{j,k}^\dagger \) for some \( c_{j,k} \in \mathbb{Z}_{\geq 0} \). By \([23]\), the result for the \((-1,-1)\)-flopping contractions shows that

\[
- \frac{1}{12} c_{j,k} (\hat{S}_t \cdot C_{j,k}) + \frac{1}{2 \chi(F_{t,j,k})} = - \frac{1}{12} (\hat{S}_t \cdot C_{j,k}) \\
- \frac{1}{2} c_{j,k} C_{j,k}^\dagger + [F_{t,j,k}] = - \frac{1}{2} (\hat{S}_t \cdot C_{j,k}) C_{j,k}^\dagger.
\]

Note that \( c = \sum_{j,k} j c_{j,k}, [F] = \sum_{j,k} [F_{t,j,k}], [C_{j,k}] = \hat{j}[C] \) and \( [C_{j,k}^\dagger] = \hat{j}[C^\dagger] \). By taking the sum for all \( k,j \), we obtain the desired formula for \( \psi_0, \psi_1 \). \( \square \)

2.4. Tilting via slope stability conditions. Let \( X \) be a smooth projective 3-fold and \( f : X \to Y \) a flopping contraction. Let \( \mathcal{L}_Y \) be an ample line bundle on \( Y \) with first Chern class \( \omega \). We consider the \( f^*\omega \)-slope stability conditions on \( \text{Coh}_{\leq 2}(X) \) and \( \text{Per}_{\leq 2}(X/Y) \) based on the slope stability conditions for torsion sheaves in \([11, 17]\) Definition 1.6.8 which use \( \mu \) for the notation of slope. Namely for \( E \in D^b \text{Coh}_{\leq 2}(X) \), its Hilbert polynomial is written as

\[
\chi(E \otimes f^*\mathcal{L}_Y^\oplus m) = \alpha_2, f^*\omega(E)m^2/2 + \alpha_1, f^*\omega(E)m + \alpha_0, f^*\omega(E)
\]

with \( \alpha_i, f^*\omega(E) \in \mathbb{Q} \). Moreover \( \alpha_2, f^*\omega(E) \) is a positive integer if \( E \) is a two dimensional sheaf outside a codimension two subset in \( X \). We set

\[
\mu_{f^*\omega}(E) := \frac{\alpha_1, f^*\omega(E)}{\alpha_2, f^*\omega(E)} \in \mathbb{Q} \cup \{\infty\}.
\]

Here we set \( \mu_{f^*\omega}(E) = \infty \) if the denominator is zero. By the Riemann-Roch theorem, \( \mu_{f^*\omega}(E) \) is written as

\[
(24) \quad \mu_{f^*\omega}(E) = \frac{(\text{ch}_2(E) + c_1(X) \text{ch}_1(E)/2)}{\text{ch}_1(E) f^*\omega^2}.
\]

The slope function \( \mu_{f^*\omega} \) satisfies the weak see-saw property on \( \text{Coh}_{\leq 2}(X) \) and \( \text{Per}_{\leq 2}(X/Y) \), i.e. if there is an exact sequence \( 0 \to F \to E \to G \to 0 \) in \( \text{Coh}_{\leq 2}(X) \) or \( \text{Per}_{\leq 2}(X/Y) \), we have either

\[
\mu_{f^*\omega}(F) \geq \mu_{f^*\omega}(E) \geq \mu_{f^*\omega}(G) \text{ or } \mu_{f^*\omega}(F) \leq \mu_{f^*\omega}(E) \leq \mu_{f^*\omega}(G).
\]

Hence the slope function \( \mu_{f^*\omega} \) defines weak stability conditions on \( \text{Coh}_{\leq 2}(X) \) and \( \text{Per}_{\leq 2}(X/Y) \):
Definition 2.10. An object \( E \in \text{Coh}_{\leq 2}(X) \) (resp. \( \text{Per}_{\leq 2}(X/Y) \)) is \( \mu_{f^* \omega} \)-\( (\text{semi})\)stable if for any exact sequence \( 0 \to F \to E \to G \to 0 \) in \( \text{Coh}_{\leq 2}(X) \) (resp. \( \text{Per}_{\leq 2}(X/Y) \)), we have the inequality
\[
\mu_{f^* \omega}(F) < (\leq) \mu_{f^* \omega}(G).
\]

Remark 2.11. If \( E \in \text{Coh}_{\leq 2}(X) \) is scheme theoretically supported on a smooth surface \( S \subset X \) with \( f^* \omega|_S \) ample, then \( E \) is \( \mu_{f^* \omega} \)-\( (\text{semi})\)stable if and only if it is a torsion free \( f^* \omega|_S \)-slope (semi)stable sheaf on \( S \) in the classical sense.

Remark 2.12. In general \( f^* \omega \) is not ample on a support of a two dimensional sheaf, so we need to take a little care in dealing with some properties of \( \mu_{f^* \omega} \)-stability. The existence of Harder-Narasimhan filtrations follows from a standard argument (say, using the same argument of [Tod13c, Lemma 3.6]). The boundedness of \( \mu_{f^* \omega} \)-semistable objects will follow from Lemma 2.13 and Proposition 2.17 below.

For \( \mu \in \mathbb{Q} \), let \( (\text{PT}_{f^* \omega}^\mu, \text{PF}_{f^* \omega}^\mu) \) be the pair of subcategories in \( \text{Per}_{\leq 2}(X/Y) \) given as follows:
\[
\text{PT}_{f^* \omega}^\mu := \langle E \in \text{Per}_{\leq 2}(X/Y) : E \text{ is } \mu_{f^* \omega} \text{-semistable with } \mu_{f^* \omega}(E) > \mu \rangle_{\text{ex}}
\]
\[
\text{PF}_{f^* \omega}^\mu := \langle E \in \text{Per}_{\leq 2}(X/Y) : E \text{ is } \mu_{f^* \omega} \text{-semistable with } \mu_{f^* \omega}(E) \leq \mu \rangle_{\text{ex}}.
\]

By the existence of Harder-Narasimhan filtrations in \( \text{Per}_{\leq 2}(X/Y) \), the pair of subcategories \( (\text{PT}_{f^* \omega}^\mu, \text{PF}_{f^* \omega}^\mu) \) forms a torsion pair on \( \text{Per}_{\leq 2}(X/Y) \). The associated (shifted) tilting is given by
\[
\text{PA}_{f^* \omega}^\mu := \langle \text{PF}_{f^* \omega}^\mu, \text{PT}_{f^* \omega}^\mu[-1] \rangle_{\text{ex}} \subset D^b \text{Coh}_{\leq 2}(X).
\]

By a general theory of tilting, \( \text{PA}_{f^* \omega}^\mu \) is the heart of a bounded t-structure on \( D^b \text{Coh}_{\leq 2}(X) \). By the construction, any object \( E \in \text{PA}_{f^* \omega}^\mu \) satisfies the inequality
\[
\alpha_{1, f^* \omega}(E) - \mu \cdot \alpha_{2, f^* \omega}(E) \leq 0.
\]

Hence the category
\[
\text{PB}_{f^* \omega}^\mu := \{ E \in \text{PA}_{f^* \omega}^\mu : \alpha_{1, f^* \omega}(E) - \mu \cdot \alpha_{2, f^* \omega}(E) = 0 \}
\]
is an abelian subcategory of \( \text{PA}_{f^* \omega}^\mu \). Note that \( \text{PB}_{f^* \omega}^\mu \) is written as
\[
\text{PB}_{f^* \omega}^\mu = \left\{ F, \text{Per}_0(X/Y)[-1] : F \in \text{Per}_{\leq 2}(X/Y) \text{ is } \mu_{f^* \omega} \text{-semistable with } \mu_{f^* \omega}(F) = \mu \right\}_{\text{ex}}.
\]

Let \( \Phi \) be the equivalence given by [18]. We have the following lemma:

Lemma 2.13. The equivalence \( \Phi \) restricts to the equivalence \( 0 \text{B}_{f^* \omega}^\mu \cong -1 \text{B}_{f^* \omega}^\mu \).

Proof. By Lemma 2.13 and 2.14, we have \( \mu_{f^* \omega}(E) = \mu_{f^* \omega}(\Phi(E)) \) for any object \( E \in \text{Per}_{\leq 2}(X/Y) \). Therefore the result is obvious. \( \square \)
2.5. Some properties of $^p\mathcal{B}^\mu_{f^*}$. This subsection is devoted to showing some properties of the abelian category $^p\mathcal{B}^\mu_{f^*}$. We first prove that it contains any $\mu_{f^*}$-semistable sheaf with slope $\mu$. We define the category $\mathcal{C}^\mu_{f^*}$ to be

$$\mathcal{C}^\mu_{f^*} := \{ E \in \text{Coh} \leq 2(X) : E \text{ is } \mu_{f^*} \text{-semistable with } \mu_{f^*}(E) = \mu \}.$$ 

We have the following lemma:

**Lemma 2.14.** We have $\mathcal{C}^\mu_{f^*} \subseteq ^p\mathcal{B}^\mu_{f^*}$.

**Proof.** Let us take an object $E \in \mathcal{C}^\mu_{f^*}$. Since $^p\text{Per}(X/Y)$ is a tilting of $\text{Coh}(X)$ by [dBO4], we have the distinguished triangle

$$\mathcal{H}^0_p(E) \rightarrow E \rightarrow \mathcal{H}^1_p(E)[-1]$$

where $\mathcal{H}^i_p(E) \in ^p\text{Per} \leq 2(X/Y)$ is the $i$-th cohomology of $E$ with respect to the $t$-structure on $\text{D}^b\text{Coh} \leq 2(X)$ with heart $^p\text{Per} \leq 2(X/Y)$. Applying $\mathbf{R}f_*$, we obtain the distinguished triangle in $\text{D}^b\text{Coh} \leq 2(Y)$

$$\mathbf{R}f_*\mathcal{H}^0_p(E) \rightarrow \mathbf{R}f_*E \rightarrow \mathbf{R}f_*\mathcal{H}^1_p(E)[-1].$$

Since $\mathbf{R}f_*$ takes $^p\text{Per} \leq 2(X/Y)$ to $\text{Coh} \leq 2(Y)$, we have $\mathbf{R}f_*\mathcal{H}^i_p(E) \cong R^1f_*E$, and it is a zero dimensional sheaf. Hence $\mathcal{H}^i_p(E) \in ^p\text{Per}_0(X/Y)$, and it remains to show that $\mathcal{H}^0_p(E)$ is a $\mu_{f^*}$-semistable object in $^p\text{Per}_2(X/Y)$. It is enough to check that $\text{Hom}(F, \mathcal{H}^0_p(E)) = 0$ for any $F \in ^p\text{Per}_1(X/Y)$. This follows from [20] and the distinguished triangle

$$\mathcal{H}^{-1}(F)[1] \rightarrow F \rightarrow \mathcal{H}^0(F)$$

with $\mathcal{H}^i(F) \in \text{Coh} \leq 1(X)$, together with the fact that $E$ is a pure two dimensional sheaf.

We next show that any object in $^p\mathcal{B}^\mu_{f^*}$ admits a certain filtration, which plays an important role in the proof of Theorem 1.1. Let $(^p\mathcal{F}, ^p\mathcal{F})$ be the torsion pair of Coh$_0(X/Y)$ as in (15). We have the following proposition:

**Proposition 2.15.** For any $E \in ^p\mathcal{B}^\mu_{f^*}$, there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E$$

such that $F_i = E_i/E_{i-1}$ satisfy $F_1 \in ^p\mathcal{F}$, $F_2 \in \mathcal{C}^\mu_{f^*}$ and $F_3 \in ^p\mathcal{T}[-1]$.

**Proof.** By (16), we may assume that $E \notin ^0\text{Per}_0(X/Y)[-1]$. We have the exact sequence in $^p\mathcal{B}^\mu_{f^*}$

$$0 \rightarrow F \rightarrow E \rightarrow T[-1] \rightarrow 0$$

such that $F \in ^p\text{Per} \leq 2(X/Y)$ is $\mu_{f^*}$-semistable with $\mu_{f^*}(F) = \mu$ and $T \in ^p\text{Per}_0(X/Y)$. By (16), we also have the exact sequence in $^p\mathcal{B}^\mu_{f^*}$

$$0 \rightarrow F' \rightarrow T[-1] \rightarrow T'[\leq 1] \rightarrow 0$$

with $F' \in ^p\mathcal{F}$ and $T'[\leq 1] \in ^p\mathcal{T}[-1]$. Combining the above two exact sequences, we have the subobject $E_2 \subset E$ in $^p\mathcal{B}^\mu_{f^*}$ with $E/E_2 \in ^p\mathcal{T}[-1]$ which fits into the exact sequence

$$0 \rightarrow F \rightarrow E_2 \rightarrow F' \rightarrow 0$$
in $p\mathcal{B}^{\mu}_{f,\omega}$. Note that the $\hat{\mu}_{f,\omega}$-semistability of $F$ implies that $F \in \text{Coh}_{\leq 2}(X)$, hence $E_2 \in \text{Coh}_{\leq 2}(X)$. We set $E_1 \subseteq E_2$ to be the maximal one dimensional subsheaf of $E_2$. Note that $E_2/E_1$ is a pure two dimensional sheaf, hence it is an object in $C_{f,\omega}^{\mu}$. It is enough to show that $E_1 \in p\mathcal{F}$. Applying $p\Phi$ in (19) to (28), we obtain the distinguished triangle in $D^b\text{Coh}(p\mathcal{A}_Y)$

$$p\Phi(F) \rightarrow p\Phi(E_2) \rightarrow p\Phi(F').$$

Here $p\Phi(F) \in \text{Coh}_{\leq 2}(p\mathcal{A}_Y)$ and $p\Phi(F') \in \text{Coh}_0(p\mathcal{A}_Y)[-1]$. In particular, we have $H^0(p\Phi(E_2)) = p\Phi(F)$, which is pure two dimensional by the $\hat{\mu}_{f,\omega}$-stability of $F$. Because there is an injection $H^0 p\Phi(E_1) \rightarrow H^0 p\Phi(E_2)$ in $\text{Coh}(p\mathcal{A}_Y)$, we have $H^0 p\Phi(E_1) = 0$, hence $p\Phi(E_1) \in \text{Coh}_0(p\mathcal{A}_Y)[-1]$. This implies that $E_1 \in p\mathcal{F}$. □

The filtration in the above proposition may be interpreted as a Harder-Narasimhan filtration with respect to a certain weak stability condition on $p\mathcal{B}^{\mu}_{f,\omega}$ in [Lod10]. Indeed, we have the following lemma:

**Lemma 2.16.** If we set $C_1 = p\mathcal{F}$, $C_2 = C_{f,\omega}^{\mu}$ and $C_3 = p\mathcal{T}[-1]$, we have $\text{Hom}(C_i, C_j) = 0$ for $i < j$. In particular, the filtration (22) is unique up to an isomorphism.

*Proof.* The result is obvious from the definition of $C_i$. □

Finally we show the boundedness of the set of objects in $p\mathcal{B}^{\mu}_{f,\omega}$ with a fixed Mukai vector. We define $p\Gamma^{\mu}_{f,\omega}$ to be

$$p\Gamma^{\mu}_{f,\omega} := \text{Im} \left( v(s) : p\mathcal{B}^{\mu}_{f,\omega} \rightarrow \Gamma \right).$$

**Proposition 2.17.** For any $v \in p\Gamma^{\mu}_{f,\omega}$, the set of objects $E \in p\mathcal{B}^{\mu}_{f,\omega}$ with $v(E) = v$ is bounded.

*Proof.* We write $v = (\rho, \beta, n)$. If $P = 0$, we have $E \in p\text{Per}_0(X/Y)[-1]$ and the result follows since $p\text{Per}_0(X/Y)$ is the extension closure of $\mathcal{O}_x$ for $x \in X \setminus \text{Ex}(f)$ and a finite number of sheaves up to shift supported on $\text{Ex}(f)$ (cf. [B04]). Hence we may assume that $P$ is a non-zero class of an effective divisor in $X$. For $E \in p\mathcal{B}^{\mu}_{f,\omega}$ with $v(E) = v$, there exists an exact sequence in $p\mathcal{B}^{\mu}_{f,\omega}$

$$0 \rightarrow F \rightarrow E \rightarrow T[-1] \rightarrow 0$$

such that $F$ is a $\hat{\mu}_{f,\omega}$-semistable object in $p\text{Per}_{\leq 2}(X/Y)$, and $T$ is an object in $p\text{Per}_0(X/Y)$. Applying the equivalence $p\Phi$ in (19) and forgetting the $p\mathcal{A}_Y$-module structures, we obtain the distinguished triangle in $D^b\text{Coh}_{\leq 2}(Y)$

$$p\Phi(F) \rightarrow p\Phi(E) \rightarrow p\Phi(T)[-1].$$

(29)

Here $p\Phi(T)$ is a zero dimensional sheaf on $Y$, and $p\Phi(F)$ is a sheaf on $Y$ which is pure two dimensional by the $\hat{\mu}_{f,\omega}$-semistability of $F$.

For $M \in \text{Coh}_{\leq 2}(Y)$, its Hilbert polynomial is written as

$$\chi(M \otimes \mathcal{L}_Y^m) = \alpha_{2,\omega}(M)m^2/2 + \alpha_{1,\omega}(M)m + \alpha_{0,\omega}(M)$$
for $\alpha_{i, \omega}(M) \in \mathbb{Q}$. It defines the $\mu_\omega$-stability on $\text{Coh}_{\leq 2}(Y)$ by setting $\mu_\omega(M) = \frac{\alpha_1(M)}{\alpha_2(M)}$ as in Subsection 2.4. Let $\mu_{\omega}^{\max}(M)$ be the maximal $\mu_\omega$-slope among the Harder-Narasimhan factors of $M \in \text{Coh}_{\leq 2}(Y)$ with respect to the $\mu_\omega$-stability. For a fixed $v$, we claim that the set

$$\{ \mu_{\omega}^{\max}(p^{\Phi}(F)) : E \in \mathcal{B}_{f^*}^0, v(E) = v \} \subset \mathbb{Q}$$

is bounded above. For simplicity, we prove the claim only for the case of $p = 0$. The case of $p = -1$ is similarly proved. We need to recall a construction of the vector bundle $0E$ on $X$ which gives an equivalence (19).

Let $\mathcal{L}_X$ be a globally generated ample line bundle on $X$. By replacing $\mathcal{L}_Y$ with $\mathcal{L}_Y^k$ for $k \gg 0$, we may assume that there is a surjection of sheaves $(\mathcal{L}_Y^k)^{\oplus m} \to R^1 f_* \mathcal{L}_X$ for some $m > 0$. Taking the adjunction, we obtain the exact sequence of vector bundles

$$0 \to \mathcal{L}_X^k \to 0 E' \to f^*(\mathcal{L}_Y^k)^{\oplus m} \to 0.$$

Then $0E$ is given by $\mathcal{O}_X \oplus 0 E'$. By the above construction of $0E$, an upper bound of $\mu_{\omega}^{\max}(p^{\Phi}(F))$ is obtained if we give upper bounds of $\mu_{\omega}^{\max}(R f_* F)$ and $\mu_{\omega}^{\max}(f_*(E \otimes \mathcal{L}_X))$, where $f_*(E \otimes \mathcal{L}_X) := \mathcal{H}^0 \mathcal{O}_X$. Because $F \in 0 \text{Per}_{\leq 2}(X/Y)$ is $\mu_{f^* \omega}$-semistable, $R f_* F$ is a $\mu_{\omega}$-semistable sheaf on $Y$. Hence $\mu_{\omega}^{\max}(R f_* F) = \mu_{\omega}(R f_* F) = \mu_{f^* \omega}(E)$ which is constant. As for $f_*(E \otimes \mathcal{L}_X)$, let $F' \in \text{Coh}_{\leq 2}(Y)$ be the $\mu_{\omega}$-semistable factor of $f_* (E \otimes \mathcal{L}_X)$ such that $\mu_{\omega}^{\max}(f_*(E \otimes \mathcal{L}_X)) = \mu_{\omega}(F')$. Since $R f_* F$ is $\mu_{\omega}$-semistable, we have the inequality

$$\mu_{\omega}(F' \otimes f_* \mathcal{L}_X) \leq \mu_{\omega}(R f_* F)$$

which implies that

$$\mu_{\omega}(F') \leq \mu_{f^* \omega}(E) + \frac{[F'] f_* c_1(\mathcal{L}_X) \omega}{[F'] \omega^2}.$$

Here $[F'] \in H^2(Y)$ is the fundamental class of $F'$. Since $[F']$ has only a finite number of possibilities, it follows that $\mu_{\omega}(F')$ is bounded above.

By the upper boundedness of (30), the result of Langer [Lan04, Theorem 4.4], [Lan09, Theorem 3.8] shows that $\alpha_{0, \omega}(p^{\Phi}(F))$ is bounded above. Because $\alpha_{0, \omega}(p^{\Phi}(T))$ is non-negative and

$$\alpha_{0, \omega}(p^{\Phi}(F)) - \alpha_{0, \omega}(p^{\Phi}(T)) = \alpha_{0, \omega}(p^{\Phi}(E))$$

is constant, the set

$$\{(\alpha_{0, \omega}(p^{\Phi}(F)), \alpha_{0, \omega}(p^{\Phi}(T))) : E \in \mathcal{B}_{f^*}^0, v(E) = v \}$$

is a finite set. Again by [Lan04, Theorem 4.4], [Lan09, Theorem 3.8], and noting that $p^{\Phi}(T)$ is a zero dimensional sheaf with bounded length, the set of objects

$$\{ p^{\Phi}(F), p^{\Phi}(T) : E \in \mathcal{B}_{f^*}^0, v(E) = v \}$$

is bounded as $\mathcal{O}_Y$-modules. Now for $M \in \text{Coh}_{\leq 2}(Y)$, the set of $p^{-}\text{Ay}$-module structures on $M$ is contained in the set of morphisms $p^{-}\text{Ay} \to \text{End}(M)$, which is finite dimensional. Therefore the set of objects (31) is bounded also as $p^{-}\text{Ay}$-modules. By the distinguished triangle (27), the set of objects $p^{\Phi}(E)$ for $E \in \mathcal{B}_{f^{-}}$ with $v(E) = v$ is also bounded as $p^{-}\text{Ay}$-modules, hence so is the set of such $E$ as $p^{\Phi}$ gives an equivalence (19). \qed
The following corollary is immediate from the above proposition:

**Corollary 2.18.** For any $0 \neq v \in \mathcal{P}_{f_{*\omega}}$, there is only a finite number of ways to decompose $v$ into $v_1 + \cdots + v_l$ for some $l \geq 1$ and $0 \neq v_i \in \mathcal{P}_{f_{*\omega}}$.

### 3. Flop formula of Donaldson-Thomas type invariants

This section is devoted to proving Theorem 1.1. In this section, we always assume that $f: X \to Y$ is a 3-fold flopping contraction with $X$, $Y$ projective, and $\omega$ is an ample divisor on $Y$. We use the notation of the flop diagram (9).

#### 3.1. Hall algebras

Let $\mathcal{M}$ be the moduli stack of objects $E \in \mathcal{D}^b \text{Coh}(X)$ with $\text{Ext}^0(E, E) = 0$, which is an algebraic stack locally of finite type (cf. [Lie06]). The same arguments as in [Tod08b] easily imply that we have the open substack $\text{Obj}(\mathcal{P'}_{f_{*\omega}}) \subset \mathcal{M}$ which parametrizes all the objects $E \in \mathcal{P}_{f_{*\omega}}$. It decomposes into the connected components

$$\text{Obj}(\mathcal{P}'_{f_{*\omega}}) = \coprod_{v \in \mathcal{P}'_{f_{*\omega}}} \text{Obj}_v(\mathcal{P}'_{f_{*\omega}})$$

where $\text{Obj}_v(\mathcal{P}'_{f_{*\omega}})$ parametrizes objects $E \in \mathcal{P}_{f_{*\omega}}$ with $v(E) = v$. By Lemma 2.17, $\text{Obj}_v(\mathcal{P}'_{f_{*\omega}})$ is an algebraic stack of finite type.

Recall that the stack theoretic Hall algebra $H(\mathcal{P}'_{f_{*\omega}})$ of $\mathcal{P}'_{f_{*\omega}}$ is $\mathbb{Q}$-spanned by the isomorphism classes of the symbols (cf. [Joy08])

$$[\mathcal{X} \xrightarrow{\rho} \text{Obj}(\mathcal{P}'_{f_{*\omega}})]$$

where $\mathcal{X}$ is an algebraic stack of finite type with affine geometric stabilizers and $\rho$ is a 1-morphism. The relation is generated by

$$[\mathcal{X}_1 \xrightarrow{\rho_1} \text{Obj}(\mathcal{P}'_{f_{*\omega}})] * [\mathcal{X}_2 \xrightarrow{\rho_2} \text{Obj}(\mathcal{P}'_{f_{*\omega}})] = [\mathcal{X}_3 \xrightarrow{\rho_3} \text{Obj}(\mathcal{P}'_{f_{*\omega}})]$$

where $(\mathcal{X}_3, \rho_3 = p_3 \circ (\rho_1', \rho_2'))$ is given by the following Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{X}_3 & \xrightarrow{(\rho_1', \rho_2')} & \mathcal{E}x(\mathcal{P}'_{f_{*\omega}}) & \xrightarrow{p_3} & \text{Obj}(\mathcal{P}'_{f_{*\omega}}) \\
\downarrow & & & & \\
\mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(\rho_1, \rho_2)} & \text{Obj}(\mathcal{P}'_{f_{*\omega}})^2.
\end{array}
$$
Assumption 3.1.

The unit element is given by

\[ 1 = [\text{Spec } \mathbb{C} \to \text{Obj}(\mathcal{P}B^\mu_{f^*})] \in H(\mathcal{P}B^\mu_{f^*}) \]

which corresponds to \(0 \in \mathcal{B}^\mu_{f^*}\). The algebra \(H(\mathcal{P}B^\mu_{f^*})\) is \(\mathcal{P}\Gamma^\mu_{f^*}\)-graded: it decomposes as

\[ H(\mathcal{P}B^\mu_{f^*}) = \bigoplus_{v \in \mathcal{P}\Gamma^\mu_{f^*}} H_v(\mathcal{P}B^\mu_{f^*}) \]

such that \(H_v(\mathcal{P}B^\mu_{f^*}) \ast H_w(\mathcal{P}B^\mu_{f^*}) \subset H_{v+w}(\mathcal{P}B^\mu_{f^*})\). The component \(H_v(\mathcal{P}B^\mu_{f^*})\) is \(\mathbb{Q}\)-spanned by the elements of the form \([X \xrightarrow{\rho} \text{Obj}_v(\mathcal{P}B^\mu_{f^*})]\).

We consider the following completion of \(H(\mathcal{P}B^\mu_{f^*})\)

\[ \hat{H}(\mathcal{P}B^\mu_{f^*}) := \prod_{v \in \mathcal{P}\Gamma^\mu_{f^*}} H_v(\mathcal{P}B^\mu_{f^*}). \]

By Corollary 2.18 the \(\ast\)-product on \(H(\mathcal{P}B^\mu_{f^*})\) extends to the \(\ast\)-product on \(\hat{H}(\mathcal{P}B^\mu_{f^*})\). Moreover for any \(\gamma \in \hat{H}(\mathcal{P}B^\mu_{f^*})\) with \(H_0(\mathcal{P}B^\mu_{f^*})\)-component zero, the following elements are well-defined:

\[ \exp(\gamma), \; \log(1 + \gamma), \; (1 + \gamma)^{-1} \in \hat{H}(\mathcal{P}B^\mu_{f^*}). \]

For a subcategory \(\mathcal{C} \subset \mathcal{P}B^\mu_{f^*}\), suppose that there are constructible subsets \(\text{Obj}_v(\mathcal{C}) \subset \text{Obj}_v(\mathcal{P}B^\mu_{f^*})\) whose closed points correspond to objects \(E \in \mathcal{C}\) with \(v(E) = v\). By the relation (32), we are able to define the following element:

\[ \delta_\mathcal{C} := \sum_{v \in \mathcal{P}\Gamma^\mu_{f^*}} [\text{Obj}_v(\mathcal{C}) \subset \text{Obj}_v(\mathcal{P}B^\mu_{f^*})] \in \hat{H}(\mathcal{P}B^\mu_{f^*}). \]

By [Joy07, Section 5.2], there is a Lie subalgebra \(\hat{H}^{\text{Lie}}(\mathcal{P}B^\mu_{f^*})\) of \(\hat{H}(\mathcal{P}B^\mu_{f^*})\), consisting of elements called virtual indecomposable objects. It contains elements of the form \([X \xrightarrow{\rho} \text{Obj}_v(\mathcal{P}B^\mu_{f^*})]\) with \(X\) a \(\mathbb{C}^\ast\)-gerb over an algebraic space. Moreover, it also contains the elements of the form

\[ \epsilon_\mathcal{C} := \log \delta_\mathcal{C} \in \hat{H}^{\text{Lie}}(\mathcal{P}B^\mu_{f^*}). \]

3.2. Integration map. Let \(\chi\) be the pairing on \(\mathcal{P}\Gamma^\mu_{f^*}\), given by

\[ \chi(v_1, v_2) = D_2 \beta_1 - D_1 \beta_2 - \frac{1}{2} c_1(X) D_1 D_2 \]

where we write \(v_i = (D_i, \beta_i, n_i)\). Note that \(\chi\) is anti-symmetric if either \(D_1\) or \(D_2\) or \(c_1(X)\) is zero. Let us take \(E_i \in \mathcal{P}B^\mu_{f^*}\) with \(v(E_i) = v_i\). If \(D_1\) or \(D_2\) or \(c_1(X)\) is zero, we have the equality

\[ \chi(v_1, v_2) = \dim \text{Hom}(E_1, E_2) - \dim \text{Ext}^1(E_1, E_2) \]

\[ + \dim \text{Ext}^1(E_2, E_1) - \dim \text{Hom}(E_2, E_1) \]

by the Riemann-Roch theorem and the Serre duality.

Let \(\nu\) be a locally constructible function on \(\mathcal{M}\), satisfying the following assumption:

**Assumption 3.1.** The function \(\nu\) is one of the following:
• \( \nu \) is Behrend function \cite{Beh09} on the algebraic stack \( \mathcal{M} \) (cf. \cite{JS12}), denoted by \( \chi_B \). In this case, we always assume that \( X \) is a Calabi-Yau 3-fold, i.e. \( K_X = 0 \) and \( H^1(\mathcal{O}_X) = 0 \).

• \( \nu \equiv 1 \). In this case, \( X \) is an arbitrary smooth projective 3-fold.

We define \( \tilde{C}(\mathcal{P}_f^{\mu}) \) to be

\[
\tilde{C}(\mathcal{P}_f^{\mu}) := \prod_{v \in \mathcal{P}_f^{\mu}} \mathbb{Q} \cdot c_v
\]

with bracket \([c_{v_1}, c_{v_2}]_\nu\) given by

\[
[c_{v_1}, c_{v_2}]_\nu := (-1)^{\epsilon(v_1, v_2)} \chi(v_1, v_2) c_{v_1 + v_2}.
\]

Here \( \epsilon(v_1, v_2) = \chi(v_1, v_2) \) if \( \nu = \chi_B \), and \( \epsilon(v_1, v_2) = 0 \) if \( \nu \equiv 1 \). Let \( \mathcal{P}_0 \subset \mathcal{P}_f^{\mu} \) be the subset of \((D, \beta, n)\) with \( D = 0 \). Note that \( \mathcal{P}_0 \) is the set of Mukai vectors of objects in \( \mathcal{P}_{\text{Per}}(X/Y)[-1] \), and it is independent of \( \mu \) and \( \omega \). For three elements \( c_{v_1}, c_{v_2}, c_{v_3} \) in \( \mathcal{P}_f^{\mu} \), the bracket \((37)\) satisfies the Leibniz rule if two of \( v_i \) are contained in \( \mathcal{P}_0 \), or \( c_1(X) = 0 \). By Lemma 2.18, the bracket \((37)\) defines the well-defined bracket on \( \tilde{C}(\mathcal{P}_f^{\mu}) \).

The same arguments of \cite{JS12} Theorem 5.12, \cite{Joy07} Theorem 6.12 show that there is a \( \mathbb{Q} \)-linear homomorphism called integration map

\[
\Pi^\nu : \tilde{H}^{\text{Lie}}(\mathcal{P}_f^{\mu}) \to \tilde{C}(\mathcal{P}_f^{\mu})
\]

such that if \( \mathcal{X} \) is a \( \mathbb{C}^* \)-gerb over an algebraic space \( \mathcal{X}' \), we have

\[
\Pi^\nu([\mathcal{X} \xrightarrow{\nu} \text{Obj}_{\mathcal{X}}(\mathcal{P}_f^{\mu})]) = \left( \sum_{k \in \mathbb{Z}} k \cdot \chi(\nu^{-1}(k)) \right) c_v
\]

where \( \chi(*) \) on the RHS is taken for the constructible subsets in \( \mathcal{X}' \). Let us take \( \gamma_i \in \tilde{H}^{\text{Lie}}(\mathcal{P}_f^{\mu}) \cap \text{H}_v(\mathcal{P}_f^{\mu}) \) with \( i = 1, 2 \), and suppose that one of \( v_i \) is contained in \( \mathcal{P}_0 \), or \( c_1(X) = 0 \). Moreover if \( \nu = \chi_B \), suppose that \( \gamma_i \) is written as

\[
\gamma_i = \sum_j a_{i,j} [\mathcal{X}_{i,j} \xrightarrow{\rho_{i,j}} \text{Obj}_{v_i}(\mathcal{P}_f^{\mu})]
\]

with \( a_{i,j} \in \mathbb{Q} \) such that for any closed point \( x \in \mathcal{X}_{i,j} \), the object in \( \mathcal{P}_f^{\mu} \) corresponding the point \( \rho_{i,j}(x) \) is a sheaf. Then together with \((38)\) and Assumption 3.1, the arguments of \cite{JS12} Theorem 5.12 for \( \nu = \chi_B \) and \cite{Joy07} Theorem 6.12 for \( \nu \equiv 1 \) show that

\[
\Pi^\nu[\gamma_1, \gamma_2]_\nu = [\Pi^\nu(\gamma_1), \Pi^\nu(\gamma_2)]_\nu.
\]

**Remark 3.2.** By \cite{JS12} Theorem 5.5, the condition that \( \rho_{i,j}(x) \) corresponds to a sheaf implies that the stack \( \text{Obj}_{\mathcal{X}}(\mathcal{P}_f^{\mu}) \) is analytic locally at \( \rho_{i,j}(x) \) written as a critical locus of a holomorphic Chern Simons function. This property is required to show \((39)\) in the proof of \cite{JS12} Theorem 5.12.
3.3. Donaldson-Thomas type invariants. Let $\mathcal{C}_{f^\omega}^\mu \subset \mathcal{B}_{f^\omega}^\mu$ be the subcategory in Lemma 2.14 and consider the element [43] for $\mathcal{C} = \mathcal{C}_{f^\omega}^\mu$. We define the DT type invariant which depends on a choice of $\nu$ in Assumption 3.1 as follows:

**Definition 3.3.** For $v \in \mathcal{G}_{f^\omega}^\mu$, we define $\text{DT}_{f^\omega}^\nu(v) \in \mathbb{Q}$ by the following formula:

$$
\Pi^\nu(\epsilon_{f^\omega} ) = \sum_{v \in \mathcal{G}_{f^\omega}^\mu} (-1)^{\epsilon(\nu)} \text{DT}_{f^\omega}^\nu(v) \cdot c_v.
$$

Here $\epsilon(\nu) = 1$ if $\nu = \chi_B$ and $\epsilon(\nu) = 0$ if $\nu \equiv 1$.

**Remark 3.4.** It is obvious that $\text{DT}_{f^\omega}^\nu(v)$ does not depend on a choice of $p$. Alternatively, the invariant $\text{DT}_{f^\omega}^\nu(v)$ can be defined by applying the integration map to the element $\epsilon_{f^\omega}^\nu$ in the Hall algebra of $\text{Coh}_{\leq 2}(X)$, as in [JS12]. When $\nu = \chi_B$, the invariant $\text{DT}_{f^\omega}^\nu(v)$ is the generalized DT invariant $\text{DT}_{f^\omega}^\nu(v)$ in Theorem 1.4.

**Remark 3.5.** Let $\mathcal{M}_{f^\omega}^\nu(v)$ be the moduli stack of $\mu_{f^\omega}$-semistable sheaves $E \in \text{Coh}_{\leq 2}(X)$ with $v(E) = v$. If any $[E] \in \mathcal{M}_{f^\omega}^\nu(v)$ is $\mu_{f^\omega}$-stable then $\mathcal{M}_{f^\omega}^\nu(v)$ is the $\mathbb{C}^*$-gerb over a projective scheme $\mathcal{M}_{f^\omega}^\nu(v)$ and $\text{DT}_{f^\omega}^\nu(\omega)$ coincides with the $\nu$-weighted Euler characteristic of $\mathcal{M}_{f^\omega}^\nu(v)$ up to sign. For instance, this is the case if $v = (P,\beta,n)$ and $P$ does not decompose into $P_1 + P_2$ for effective divisor classes $P_i$.

For $n \in \mathbb{Z}$ and $\beta \in H_2(X/Y)$ with $\beta \geq 0$, there is another invariant

$$
N_{n,\beta}^\nu \in \mathbb{Q}
$$

defined to be the DT type invariant counting $\mu_{H}$-semistable sheaves $F \in \text{Coh}_0(X/Y)$ satisfying

$$
(\text{ch}_2(F),\text{ch}_3(F)) = ([F],\chi(F)) = (\beta,n).
$$

Here the $\mu_H$-stability on $\text{Coh}_0(X/Y)$ is defined in Subsection 2.2 and $\nu$ is either $\nu = \chi_B$ or $\nu \equiv 1$ on the moduli stack of objects in $\text{Coh}_0(X/Y)$ as in Assumption 3.1. We refer to [JS12, Subsection 6.4], [Tod10, Proposition-Definition 5.7], [Tod11, Section 2.3] for the detail. In our situation, the invariant (41) for $n \leq 0$ and $\beta > 0$ can be defined in the following way. For $\mu' \in \mathbb{Q}_{\leq 0}$, let $\mathcal{C}_{H}^{\mu'} \subset -1F$ be the subcategory consisting of $\mu_H$-semistable sheaves $F$ with $\mu_H(F) = \mu'$ which makes sense by Lemma 2.3. Then $N_{n,\beta}^{\nu}$ is defined by

$$
\Pi'^\nu(\epsilon_{\mathcal{C}_{H}^{\mu'}} ) = \sum_{n/H-\beta = \mu'} (-1)^{\epsilon(\nu)} N_{n,\beta}^{\nu} \cdot c_{(0,\beta,n)}.
$$

The invariant (41) is known to be independent of $H$ (cf. [JS12, Theorem 6.16]). By convention, we set $N_{n,\beta}^{\nu} = N_{-n,-\beta}^{\nu}$ for $\beta \leq 0$. The invariant (41) satisfies the following relation [Tod13b, Lemma 5.1]

$$
N_{n,\beta}^{\nu} = N_{-n,-\beta}^{\nu} = N_{n,\beta}^{\nu}.
$$

We have the following lemma:
Lemma 3.6. We have the following identities:

\begin{equation}
\Pi^\nu(c_0\mathcal{F}) = \sum_{n<0,\beta>0, f_\beta=0} (-1)^{\ell(\nu)} N^\nu_{n,\beta} \cdot c_{(0,\beta,n)}
\end{equation}

\begin{equation}
\Pi^\nu(\epsilon_{-1}\mathcal{F}) = \sum_{n\leq0,\beta>0, f_\beta=0} (-1)^{\ell(\nu)} N^\nu_{n,\beta} \cdot c_{(0,\beta,n)}.
\end{equation}

Proof. We prove the second identity. By Lemma 2.4, the existence of Harder-Narasimhan filtrations with respect to the $\mu_H$-stability yields

\[ \delta_{-1}\mathcal{F} = \prod_{\mu' \leq 0} \delta_{\mathcal{C}_{\mu'}}. \]

In the RHS, we take the product with the decreasing order of $\mu'$. We refer to [Joy08, Theorem 5.11] for the detail of the above product formula. We take the logarithm of both sides and apply the integration map. Since $\chi(v_1, v_2) = 0$ for $v_i \in -1\Gamma_0$, the property of the integration map (39) shows that

\[ \Pi^\nu(\epsilon_{-1}\mathcal{F}) = \sum_{\mu' \leq 0} \Pi^\nu(\epsilon_{\mathcal{C}_{\mu'}}). \]

By (42), we obtain the desired identity. \hfill \square

3.4. Flop formula. In this subsection, we describe the flop transformation formula of the invariants $DT^\nu_{f^*\omega}(v)$ in terms of the invariants $N^\nu_{n,\beta}$.

Proposition 3.7. We have the following identity in $\hat{H}(\mathbb{P}B^\mu_{f^*\omega})$:

\begin{equation}
\delta_{\mathbb{P}B^\mu_{f^*\omega}} = \delta_{\mathcal{F}} \ast \delta_{\mathcal{C}_{f^*\omega}} \ast \delta_{\mathcal{T}[-1]}.
\end{equation}

Proof. The equality (45) follows from Proposition 2.15, Lemma 2.16 and the same argument of [Joy08, Theorem 5.11]. \hfill \square

Lemma 3.8. We have the following identity in $\hat{H}(\mathbb{P}B^\mu_{f^*\omega})$:

\begin{equation}
\delta_{\mathcal{F}} \ast \delta_{\mathcal{C}_{f^*\omega}} \ast \delta_{\mathcal{T}[-1]}^{-1} = \delta_{\mathbb{P}B^\mu_{f^*\omega}} \ast \delta_{\mathcal{P}_{\text{Per}0}(X/Y)[-1]}^{-1}.
\end{equation}

Proof. The equality (46) follows from (45) and $\delta_{\mathcal{F}} \ast \delta_{\mathcal{T}[-1]} = \delta_{\mathcal{P}_{\text{Per}0}(X/Y)[-1]}$, where the last equality follows from (16). \hfill \square

For $v \in \mathbb{P}B^\mu_{f^*\omega}$, we define the invariant $p\hat{DT}^\nu_{f^*\omega}(v) \in \mathbb{Q}$ by

\begin{equation}
\Pi^\nu \log \left( \delta_{\mathbb{P}B^\mu_{f^*\omega}} \ast \delta_{\mathcal{P}_{\text{Per}0}(X/Y)[-1]}^{-1} \right) = \sum_{v \in \mathbb{P}B^\mu_{f^*\omega}} p\hat{DT}^\nu_{f^*\omega}(v) \cdot c_v.
\end{equation}

For a fixed divisor class $P \in H^2(X)$, we set

\[ DT^\nu_{f^*\omega}(P) := \sum_{(P-\beta,-n) \in \mathbb{P}B^\mu_{f^*\omega}} DT^\nu_{f^*\omega}(P,-\beta,-n) q^n t^\beta. \]

\[ p\hat{DT}^\nu_{f^*\omega}(P) := \sum_{(P-\beta,-n) \in \mathbb{P}B^\mu_{f^*\omega}} p\hat{DT}^\nu_{f^*\omega}(P,-\beta,-n) q^n t^\beta. \]
For a curve class $\beta$, we set $\langle P, \beta \rangle \in \mathbb{Z}$ to be
\[
\langle P, \beta \rangle := \left\{ \begin{array}{cl}
-1 & P \cdot \beta \quad \nu = \chi_B \\
1 & P \cdot \beta \quad \nu \equiv 1.
\end{array} \right.
\] (48)

**Proposition 3.9.** We have the following formulas:
\[
0^{\nu} DT^\mu_{f^* \omega}(P) = \prod_{n>0, \beta>0} f^*\omega_n \cdot \langle P, \beta \rangle \cdot DT^\mu_{f^* \omega}(P)
\]
\[
-1^{\nu} DT^\mu_{f^* \omega}(P) = \prod_{n>0, \beta>0} f^*\omega_n \cdot \langle P, \beta \rangle \cdot DT^\mu_{f^* \omega}(P).
\]

**Proof.** By the arguments so far, we have the following identities:
\[
\sum_{v \in \Gamma^\nu_{f^* \omega}} p^{\nu} DT^\mu_{f^* \omega}(v) \cdot c_v = \Pi^{\nu} \log \left( \delta_{P\omega_{f^* \omega}} \ast \delta_{P\omega_{f^* \omega}^{-1}} \right)
\]
\[
= \Pi^{\nu} \log \left( \delta_{\epsilon_{\mathcal{F}}} \ast \delta_{\epsilon_{\mathcal{C}}} \ast \delta_{\epsilon_{\mathcal{F}}}^{-1} \right)
\]
\[
= \Pi^{\nu} \log \left( \exp (\epsilon_{\mathcal{F}}) \ast \exp (\epsilon_{\mathcal{C}}) \ast \exp (-\epsilon_{\mathcal{F}}) \right)
\]
\[
= \sum_{m \geq 0} \frac{1}{m!} \text{Ad}_{\Pi^{\nu} \epsilon_{\mathcal{F}}} (\Pi^{\nu} \epsilon_{\mathcal{C}}).
\]

Here the first equality is (47), the second equality is (46), the third equality is (34), and the last equality follows from (39) and the Baker-Campbell-Hausdorff formula. We note that we are able to apply (39) since the objects in $p_{f^* \omega}$ and $C^\nu_{f^* \omega}$ are sheaves. For instance, suppose that $p = 0$ and $\nu = \chi_B$.

Using (43), the desired formula follows from the above equality. The other cases are similarly discussed. \qed

We define the following generating series:
\[
DT^\nu_{f^* \omega}(P) := \sum_{\mu \in \mathbb{Q}} DT^\nu_{f^* \omega}(P).
\] (49)

Let $\psi_0$ and $\psi_1$ be the linear functions given in Lemma 2.6. The following theorem is the main result of this subsection:
Theorem 3.10. We have the following formula:
\[
\phi_* \text{DT}^\mu_{f^* \omega}(P) = q^{\psi_0(P)} t^{\psi_1(P)} \text{DT}^\nu_{f^* \omega}(\phi_* P) \\
\cdot \prod_{n > 0, \beta > 0, \beta^! = 0} \exp \left( N^\nu_{n, \beta} q^n t^\beta \right)^{(P, \beta)} \cdot \prod_{n \geq 0, \beta > 0, \beta^! = 0} \exp \left( N^\nu_{n, \beta} q^n t^{-\beta} \right)^{(P, \beta^!)}.
\]

Here \( \beta^! \) in the RHS are elements of \( H_2(X/Y) \), and \( \phi_* \) is the variable change \((n, \beta) \mapsto (n, \phi_* \beta)\).

Proof. By Lemma 2.13 there is an isomorphism of algebras
\[
\Phi_*^H : \wedge(\mathcal{B}^\mu_{f^* \omega}) \cong \wedge(-\mathcal{B}^\mu_{f^* \omega}).
\]

By the commutative diagram (20), we have the commutative diagram
\[
\begin{array}{ccc}
\wedge \text{Lie}(\mathcal{B}^\mu_{f^* \omega}) & \xrightarrow{\Phi_*^H} & \wedge \text{Lie}(-\mathcal{B}^\mu_{f^* \omega}) \\
\Pi^\nu \downarrow & & \downarrow \Pi^\nu \\
\wedge \text{Lie}(\mathcal{C}^\mu_{f^* \omega}) & \xrightarrow{\Phi_*^C} & \wedge \text{Lie}(-\mathcal{C}^\mu_{f^* \omega}).
\end{array}
\]

Here \( \Phi_*^C \) sends \( c_v \) to \( c_{\phi_* v} \). Moreover we have
\[
\Phi_*^H \left( \delta_{0, \text{Per}_0(X/Y)} \right) = \delta_{-1 \text{Parab}(X/Y)[-1]}, \quad \Phi_*^H \left( \delta_{0, \text{Per}_0(X/Y)} \right) = \delta_{-1 \text{Parab}(X/Y)[-1]}.
\]

Therefore we obtain the equality
\[
\Phi_*^C \Pi^\nu \left( \delta_{0, \text{Parab}(X/Y)} \right) = \Pi^\nu \left( \delta_{-1 \text{Parab}(X/Y)} \right).
\]

Using Lemma 2.13 the above equality implies
\[
(50) \quad \phi_* \text{DT}^\mu_{f^* \omega}(P) = q^{\psi_0(P)} t^{\psi_1(P)} \cdot -\text{DT}^\mu_{f^* \omega}(\phi_* P).
\]

The desired formula follows from the above equality and Proposition 3.10 together noting that
\[
\phi_* \prod_{n > 0, \beta > 0, \beta^! = 0} \exp \left( N^\nu_{n, \beta} q^n t^\beta \right)^{(P, \beta)} = \prod_{n > 0, \beta > 0, \beta^! = 0} \exp \left( N^\nu_{n, \beta} q^n t^{-\beta} \right)^{(\phi_* P, \beta^!)}.
\]

Here we have set \( \beta^! = -\phi_* \beta \in H_2(X/Y) \), which is effective by Lemma 2.7. We have also used the fact \( N^\nu_{n, \beta} = N^\nu_{n, \beta^!} \) by [Toda] Theorem 5.6, and \( \langle P, \beta \rangle = -\langle \phi_* P, \beta^! \rangle \) by Lemma 2.7.

\[\square\]

3.5. Parabolic stable pairs. The error term of the formula in Theorem 3.10 is described in terms of invariants counting parabolic stable pairs, introduced in [Toda]. Let \( H \) be a \( f \)-ample effective divisor in \( X \), which intersects with \( \text{Ex}(f) \) transversally. Recall that there is the \( \mu_H \)-stability on \( \text{Coh}_0(X/Y) \) given in Subsection 2.2.

Definition 3.11. A \( H \)-parabolic stable pair consists of \((F, s)\), where \( F \in \text{Coh}_0(X/Y) \) and \( s \in F \otimes \mathcal{O}_H \), satisfying the following conditions:

- \( F \) is a one dimensional \( \mu_H \)-semistable sheaf.
• For any surjection \( \pi : F \to F' \) with \( \mu_H(F) = \mu_H(F') \), we have \( (\pi \otimes O_H)(s) \neq 0 \).

For \( \beta \in H_2(X/Y) \) and \( n \in \mathbb{Z} \), let \( M^{par}_H(\beta, n) \) be the moduli space of parabolic stable pairs \((F, s)\) of above parabolic stable pairs (cf. [Toda, Theorem 1.2]).

**Definition 3.13.** For \( \beta \in H_2(X/Y) \) and \( n \in \mathbb{Z} \), we define the invariant \( \text{Par}^{\nu}_H(\beta, n) \in \mathbb{Z} \) to be

\[
\text{Par}^{\nu}_H(\beta, n) := \sum_{k \in \mathbb{Z}} k \cdot \chi(\nu^{-1}(k)).
\]

We note that, contrary to the invariants \( N^\nu_{n, \beta} \), the invariants \( \text{Par}^{\nu}_H(\beta, n) \) are always integers. However they are related as follows: for \( \mu \in \mathbb{Q} \), let us set

\[
\text{Par}^{\nu,\mu}_H(q, t) := 1 + \sum_{n/\beta-H=\mu} \text{Par}^{\nu}_H(\beta, n)q^n t^\beta.
\]

Then we have the following identity (cf. [Toda Theorem 4.12])

\[
(51) \quad \text{Par}^{\nu,\mu}_H(q, t) = \prod_{n/\beta-H=\mu\atop f_\beta=0,\beta>0} \exp \left( N^\nu_{n, \beta} q^n t^\beta \right)^{(H, \beta)}.
\]

For a divisor class \( P \) on \( X \), it is either \( f \)-ample, \( f \)-trivial, or \( f \)-anti ample by the definition of a flopping contraction. We set \( s = 1 \) in the first case, \( s = 0 \) in the second case and \( s = -1 \) in the last case. By the \( f \)-relative base point free theorem, there is a \( f \)-ample effective divisor \( H \) on \( X \) which intersects with \( \text{Ex}(f) \) transversally such that \( P \cdot \beta = sH \cdot \beta \) for any \( \beta \in H_2(X/Y) \). This implies that

\[
\{ \text{Par}^{\nu,\mu}_H(q, t) \}^s = \prod_{n/\beta-H=\mu\atop f_\beta=0,\beta>0} \exp \left( N^\nu_{n, \beta} q^n t^\beta \right)^{(P, \beta)}.
\]

Therefore we obtain the following corollary:

**Corollary 3.14.** In the same situation of Theorem 3.12, there is a \( f^\dagger \)-ample effective divisor \( H^\dagger \) on \( X^\dagger \) which intersects with \( \text{Ex}(f^\dagger) \) transversally, and \( s \in \{1, 0, -1\} \) such that the following identity holds:

\[
\phi_* \text{DT}^{\nu^*}_{f^\dagger \omega}(P) = q^{\psi_0(P) t^{\psi_1(P)}} \text{DT}^{\nu^*}_{f^\dagger \omega}(\phi_* P) \cdot \left\{ \prod_{\mu \in \mathbb{Q}_{>0}} \text{Par}^{\nu,\mu}_H(q, t) \prod_{\mu \in \mathbb{Q}_{>0}} \text{Par}^{\nu,\mu}_H(q, t^{-1}) \right\}^s.
\]
Here $s$ is determined by $\phi_s P \cdot \beta^1 = s H^1 \cdot \beta^1$ for any $\beta^1 \in H_2(X/Y)$.

3.6. Computation of the error term. From this subsection until the end of this section, we assume that $f$ contracts only an irreducible rational curve $C \subset X$ to a point $p \in Y$. We compute the error term of the formula in Corollary 3.14 under the above assumption. Let $H$ be an effective $f$-ample divisor on $X$ which intersects with $C$ transversally. As in Subsection 2.1 we denote by $l$ the scheme theoretic length of $f^{-1}(p)$ at the generic point of $C$, and $n_j$ for $1 \leq j \leq l$ the associated integers. By convention, we put $n_j = 0$ for $j > l$.

**Lemma 3.15.** If $\nu = \chi_B$, we have the following formula for $m \geq 1$:

$$N^\nu_{n,m}[C] = \sum_{k \geq 1,k \mid (n,m)} \frac{n_{m/k}}{k^2}.$$  

**Proof.** The invariant $N^\nu_{n,m}[C]$ for $m > 0$ depends only on the formal completion ([13]) as any object which contributes to $N^\nu_{n,m}[C]$ is supported on $C$. The moduli space of Joyce-Song [JS12] stable pairs $(O_{\hat{X}} \to F(aH))$ for $a > 0$ on $\hat{X}$ is a projective scheme if $[F] = m[C]$, since the latter condition implies that $F$ is supported on $C$. Hence the same argument of [JS12] shows that $N^\nu_{n,m}[C]$ is invariant under the deformation ([14]) of $\hat{X}$. Let $C_{j,k} \subset \hat{X}$ be $(-1,-1)$-curves as in Subsection 2.1. The invariance of $N^\nu_{n,\beta}$ under the deformation ([14]) yields

$$N^\nu_{n,m}[C] = \sum_{1 \leq j \leq l, 1 \leq k \leq n_j} \sum_{m_j,k=m} N^\nu_{n,\sum_{1 \leq j \leq l, 1 \leq k \leq n_j} m_{j,k}[C_{j,k}]}.$$  

The invariant $\sum_{1 \leq j \leq l, 1 \leq k \leq n_j} m_{j,k}[C_{j,k}]$ in the RHS is non-zero only if $m_{j,k} \neq 0$ for only one $(j,k)$ since $C_{j,k}$ are disjoint, and we have $j|m$, $m_{j,k} = \frac{m}{j}$ for such $(j,k)$. Since $C_{j,k}$ is a $(-1,-1)$-curve, the computation in [JS12, Example 6.2] shows that $N^\nu_{m,m_{j,k}[C_{j,k}]}$ is non-zero only if $m_{j,k} | n$, and it equals to $1/m_{j,k}^2 = j^2/m^2$ in this case. Therefore we obtain

$$N^\nu_{n,m}[C] = \sum_{j \geq 1, j \mid m, (m/j) | n} n_j \frac{j^2}{m^2}.$$  

By putting $k = m/j$, we obtain the desired identity. \qed

**Proposition 3.16.** If $\nu = \chi_B$, we have the following formula:

$$\text{Par}_H^\nu(q,t) = \prod_{1 \leq j \leq l} (1 - (-1)^j H^C q^n j^C)_{j \mid (H,C)=\mu}.$$  

**Proof.** By Toda [Proposition 4.5], the invariants $N^\nu_{n,\beta}$ satisfy the following multiple cover formula

$$N^\nu_{n,\beta} = \sum_{k \in \mathbb{Z} \geq 1} \frac{1}{k^2} N^\nu_{1,\beta/jk}$$  

(52)
if and only if we have the following formula:

\[(53) \quad \text{Par}_H^{\nu \cdot \mu}(q, t) = \prod_{j \geq 1, n/j(H-C) = \mu} (1 - (-1)^j(H-C)q^n t^jC)^{N_{1,j[H]}(H-C)} . \]

By Lemma 3.15, \(N_{1,j[H]}(H-C) = n_j\) (which also holds from [Kat08]), and the identity (52) holds for any \(\beta = j[H]\) with \(j > 0\). Therefore we obtain the desired formula.

3.7. **Proof of Theorem 1.1** The following theorem, which proves Theorem 1.1, is the main result in this section:

**Theorem 3.17.** Suppose that \(\nu = \chi_B\). Then we have the following formula:

\[
\text{DT}^{\nu}_{\gamma} (\phi_\ast P) = \phi_\ast \text{DT}^{\nu}_{\gamma} (P) \\
\cdot \prod_{j=1}^l \left\{ i^{P-C} \eta(q)^{-1} \vartheta_{1,1}(q, ((-1)^j \phi_\ast P)t)^{j_nj} \right\}^{n_j \cdot (H^l-C^l)} .
\]

**Proof.** By Proposition 2.9 Corollary 3.14 and Proposition 3.16 we have

\[
\phi_\ast \text{DT}^{\nu}_{\gamma} (P) = q^{-\sum_j j n_j (P-C)/2} t^{-\sum_j j n_j (P-C)C^l} \text{DT}^{\nu}_{\gamma} (\phi_\ast P) \\
\cdot \prod_{j=1}^l \left\{ \prod_{n \geq 0} (1 - (-1)^j (H^l-C^l)q^n t^jC^l) \prod_{n \geq 0} (1 - (-1)^j (H^l-C^l)q^n t^jC^l) \right\}^{n_j \cdot (H^l-C^l)} .
\]

The result of Theorem 1.1 follows from the above identity together with \(\phi_\ast P \cdot C^l = sH^l \cdot C^l = -P \cdot C\) by Lemma 2.7, and the Jacobi triple product

\[(54) \quad \vartheta_{1,1}(q, t) = \frac{1}{1 - q^{-1} \eta(q)} \prod_{n \geq 1} (1 - q^n t)(1 - q^n t^{-1}). \]

\[\square\]

**Remark 3.18.** If \(l = 1\), then \(n_1 = n\) where \(n\) is the width of \(C\), and we obtain the following formula:

\[
\text{DT}^{\nu}_{\gamma} (\phi_\ast P) = \phi_\ast \text{DT}^{\nu}_{\gamma} (P) \cdot \left\{ i^{P-C} \eta(q)^{-1} \vartheta_{1,1}(q, ((-1)^j \phi_\ast P)t)^{j_nj} \right\}^{n \cdot (H-C)} .
\]

**Remark 3.19.** If we take the generating series \((49)\) w.r.t. the Chern characters (not Mukai vectors), then the error term in Theorem 3.17 is further multiplied by some monomial in \(q, t\), which is not a Jacobi form. This indicates that taking the generating series w.r.t. the Mukai vectors is crucial in order to obtain the modularity of the generating series.

3.8. **Euler characteristic version.** We next treat the case \(\nu \equiv 1\). In this case, the invariant \(N_{n, \beta}\) is not deformation invariant and its computation is more subtle. We focus on the case of \(l = 1\), and let \(n\) be the width of \(C\). We take a smooth divisor \(\tilde{H}\) in \(\tilde{X}\) such that \(\tilde{H} \cap C\) is a one point, which always exist. Also we set the polynomial \(f_n(x)\) to be

\[(55) \quad f_n(x) := 1 + x + \cdots + x^n .
\]

We have the following lemma:
Lemma 3.20. In the above situation, we have $\text{Par}^{s,\mu}_{i}(q,t) = 1$ unless $\mu \in \mathbb{Z}$.

If $\mu \in \mathbb{Z}$, we have

$$\text{Par}^{s,\mu}_{i}(q,t) = f_{n}(q^{\mu}t^{C_{1}}).$$

Proof. Let $F \in \text{Coh}_{0}(X/Y)$ be a $\mu_{i}$-semistable sheaf with $[F] = m[C]$ for $m \geq 1$, and $F_{1}, \ldots, F_{e}$ the $\mu_{i}$-stable factors of $F$. Then each $F_{i}$ is $O_{f^{-1}(i)} = O_{C}$-module, hence isomorphic to $O_{C}(k_{i})$ for some $k_{i} \in \mathbb{Z}$. Since $\mu_{i}(F) = \mu_{i}(F_{i}) = k_{i} + 1$, the integer $k_{i}$ is independent of $i$, and $\mu_{i}(F)$ is an integer. This implies that $\text{Par}^{s,\mu}_{i}(q,t) = 1$ unless $\mu \in \mathbb{Z}$.

Suppose that $\mu \in \mathbb{Z}$. It is enough to show that $\text{Par}^{s}_{i}(mC, m\mu)$ equals to one for $1 \leq m \leq n$, and zero otherwise. Applying $\otimes O_{\hat{X}}(-\mu\hat{H})$, we have $\text{Par}^{s}_{i}(mC, m\mu) = \text{Par}^{s}_{i}(mC, 0)$, so we may assume that $\mu = 0$. Let $(F,s)$ be a $\hat{H}$-parabolic stable pair on $\hat{X}$ with $([F], \chi(F)) = (m[C], 0)$. Then by the above argument, we have $F \in \langle O_{C}(-1) \rangle_{\text{ex}}$. We first show that $F$ must be indecomposable. Indeed suppose that $F$ decomposes as $F_{1} \oplus F_{2}$ with $F_{1} \neq 0$, and write $s = (s_{1}, s_{2})$ with $s_{i} \in F_{i} \otimes O_{\hat{H}}$. We choose surjections $\pi_{i}: F_{i} \rightarrow O_{C}(-1)$, and set $s'_{i} = \pi_{i}(s_{i}) \in O_{C}(-1) \otimes \hat{H} \cong C$. By the parabolic stability, we have $(s'_{1}, s'_{2}) \in C^{2} \setminus \{0\}$. Then the composition

$$\pi: F_{1} \oplus F_{2} \rightarrow O_{C}(-1) \oplus_{2} (s_{2}^{\mu} - s_{1}^{\mu}) \rightarrow O_{C}(-1)$$

satisfies $(\pi \otimes O_{\hat{H}})(s) = 0$, which contradicts to the parabolic stability of $(F,s)$. Hence $F$ is indecomposable.

Next we classify indecomposable objects in $\langle O_{C}(-1) \rangle_{\text{ex}}$. The derived category $D^{b}\text{Coh}(\hat{X})$ is known to be equivalent to the derived category of finitely generated right modules over the completion of the path algebra $A$ of the quiver of the form (cf. [DW, Example 3.10])

$$y_{2} \xleftarrow{a_{1}} \circ \circ \xrightarrow{a_{2}} y_{1}$$

with relations given by $y_{1}a_{i} = a_{i}y_{2}$, $y_{2}b_{i} = b_{i}y_{1}$, $2y_{1}^{n} = a_{i}b_{i} - a_{i}b_{i}$ and $2y_{2}^{n} = b_{i}a_{i} - b_{i}a_{i}$. Under the above derived equivalence, the category $\langle O_{C}(-1) \rangle_{\text{ex}}$ is equivalent to the subcategory of mod $A$ generated by a simple object corresponding to one of the vertex in $[50]$, say $\circ$. Hence $\langle O_{C}(-1) \rangle_{\text{ex}}$ is equivalent to mod $(\mathbb{C}[y_{1}] / (y_{1}^{m})$. The indecomposable objects in the latter category consist of $\mathbb{C}[y_{1}] / (y_{1}^{m})$ with $1 \leq m \leq n$, hence $\text{Par}^{s}_{i}(mC, 0) = 0$ for $m > n$.

Suppose that $1 \leq m \leq n$, and $F \in \langle O_{C}(-1) \rangle_{\text{ex}}$ corresponds to $\mathbb{C}[y_{1}] / (y_{1}^{m})$. Let $\pi': F \rightarrow F'$ be the quotient corresponding to the surjection $\mathbb{C}[y_{1}] / (y_{1}^{m}) \rightarrow \mathbb{C}$. Then $\pi'$ factors through any quotient $F \rightarrow F''$ in $\langle O_{C}(-1) \rangle_{\text{ex}}$, hence $(F,s)$ is a parabolic stable pair if and only if $0 \neq (\pi' \otimes O_{\hat{H}})(s) \in C$. Since $F \otimes O_{\hat{H}} \cong \mathbb{C}^{m}$, the set of such parabolic stable pairs is identified with $\mathbb{C}^{*} \times \mathbb{C}^{m-1}$. On the other hand, $\text{Aut}(F)$ is isomorphic to $\text{Aut}(\mathbb{C}[y_{1}] / (y_{1}^{m}))$ in mod$(\mathbb{C}[y_{1}] / (y_{1}^{m}))$, and the latter group consists of $1 \mapsto a_{0} + a_{1}y_{1} + \cdots + a_{m-1}y_{1}^{m-1}$ with $a_{i} \in \mathbb{C}$, $a_{0} \neq 0$. Hence $\text{Aut}(F)$ is isomorphic to $\mathbb{C}^{*} \times \mathbb{C}^{m-1}$, and it is easy to see that the action of $\text{Aut}(F) = \mathbb{C}^{*} \times \mathbb{C}^{m-1}$ on the above set $\mathbb{C}^{*} \times \mathbb{C}^{m-1}$ of parabolic stable pairs is transitive. This implies that the moduli
We have the following result:

**Theorem 3.21.** Suppose that $\nu \equiv 1$ and $l = 1$. Let $n$ be the width of $C$, and $f_n(x)$ the polynomial \((53)\). Then we have the following identity:

$$\text{DT}^\nu_{f_1^*\omega}(\phi_* P) = \phi_* \text{DT}^\nu_{f_* \omega}(P)$$

\((57)\)

**Proof.** We take a smooth divisor $\hat{H}^1$ in $\hat{X}^1$ such that $\hat{H}^1 \cap C^1$ is a one point. By Proposition \[\text{2.19}\] Theorem \[\text{3.10}\] and \((51)\), we have

$$\phi_* \text{DT}^\nu_{f_* \omega}(P) = \text{DT}^\nu_{f_1^*\omega}(\phi_* P)$$

$$\cdot \left\{ q_{\hat{H}^1} t^{C^1} \prod_{m \in \mathbb{Z} \geq 0} f_n(q^m t^{C^1}) \prod_{m \in \mathbb{Z} \geq 0} f_n(q^m t^{-C^1}) \right\}^{P.C.}.$$

Applying Lemma \[\text{2.7}\] and Lemma \[\text{3.20}\] we obtain the desired result. \[\square\]

**Remark 3.22.** Noting that $f_n(x) = \prod_{j=1}^n (1 - \xi^j x)$ for $\xi = e^{\frac{2\pi i}{n}}$, \((54)\), and $\prod_{j=1}^n (-i\xi^{-\frac{1}{2}}) = (-1)^n$, the formula \((57)\) is also written as

$$\text{DT}^\nu_{f_1^*\omega}(\phi_* P) = \phi_* \text{DT}^\nu_{f_* \omega} \cdot \prod_{j=1}^n \left\{ -\eta(q)^{-1} \gamma_{1,1}(q_i, \xi^j t^{C^1}) \right\}^{P.C.}.$$

### 3.9. A fixed supported version.

We can similarly prove a variant of Theorem \[\text{[19]}\] Theorem \[\text{3.21}\] for DT type invariants counting semistable sheaves supported on a fixed effective divisor $S \subset X$. We denote by $\text{Coh}_S(X)$ the subcategory of $\text{Coh}_{\leq 2}(X)$ consisting of sheaves supported on $S$. We set $\text{C}^\mu_{f^* \omega}$ to be

$$\text{C}^\mu_{f^* \omega} := \text{C}^\mu_{f^* \omega} \cap \text{Coh}_S(X).$$

For $v \in \Pi^\mu_{f^* \omega}$, we define $\text{DT}^\nu_{f^* \omega}(v) \in \mathbb{Q}$ as follows:

$$\Pi^\nu \left( \text{C}^\mu_{f^* \omega} \right) = \sum_{v \in \Pi^\nu_{f^* \omega}} (-1)^{e(v)} \text{DT}^\nu_{f^* \omega}(v) \cdot c_v.$$

Similarly to $\text{DT}^\nu_{f^* \omega}(P)$, we set

$$\text{DT}^\nu_{f^* \omega}(P) := \sum_{n, \beta} \text{DT}^\nu_{f^* \omega}(P, -\beta, -n) q^n t^\beta$$

for $P \in H^2(X)$. Note that if $S$ is an irreducible divisor then $\text{DT}^\nu_{f^* \omega}(P)$ is non-zero only if $P$ is a positive multiple of the cohomology class of $S$. Let $S^1 \subset X^1$ be the strict transform of $S$, and $f_n(x)$ the polynomial as in \((55)\). We have the following result:
Theorem 3.23. (i) If $\nu = \chi_B$, we have the following formula:

$$DT_{f^*\omega}^{\nu,S}(\phi_*P) = \phi_* DT_{f^*\omega}^{\nu,S}(P)$$

$$\cdot \prod_{j=1}^{l} \left\{ i^j_{P,C} \right\}_{j=1}^{l} \left( q_{1,1}(-1)^{j} \right) j_{\nu,P,C}^{n_j}.$$

(ii) If $\nu \equiv 1$ and $l = 1$, let $n$ be the width of $C$, and set $\xi = e^{2\pi i}$. We have the following formula:

$$DT_{f^*\omega}^{\nu,S}(\phi_*P) = \phi_* DT_{f^*\omega}^{\nu,S}(P)$$

(59)

$$\cdot \left\{ q_{1,1}^{P,C} \right\}_{m \in \mathbb{Z}_{>0}} \prod_{m \in \mathbb{Z}_{>0}} f_n(q^m t^{C_1}) \prod_{m \in \mathbb{Z}_{\geq 2}} f_n(q^m t^{-C_1}) \right\}^{P,C}.$$

Proof. Let $B_{f^*\omega}^{\nu,S} \subset B_{f^*\omega}^{\nu}$ be the subcategory consisting of objects $E \in B_{f^*\omega}^{\nu,S}$ such that $E|_{X\setminus\text{Ex}(f)}$ is supported on $S$. It is easy to check that $B_{f^*\omega}^{\nu,S}$ is an abelian subcategory of $B_{f^*\omega}^{\nu}$. We claim that the following equality holds in $\hat{H}(B_{f^*\omega}^{\nu})$:

$$\delta_{B_{f^*\omega}^{\nu,S}} = \delta_{f^*\omega} \ast \delta_{f^*\omega} \ast \delta_{f^*\omega}[1].$$

Similarly to Proposition 3.7, it is enough to show the following: for any $E \in B_{f^*\omega}^{\nu,S}$ if we take a filtration (27), then $F_2 \in C_{f^*\omega}^{\nu,S}$ holds. The condition $E \in B_{f^*\omega}^{\nu,S}$ implies that $F_2|_{X\setminus\text{Ex}(f)}$ is supported on $S$. On the other hand, the sheaf $F_2$ is $f^*\omega$-semistable, hence it is pure two dimensional. Therefore $F_2$ must be supported on $S$, which implies $F_2 \in C_{f^*\omega}^{\nu,S}$. Hence the equality (60) holds.

Let $\Phi$ be the derived equivalence (18). By Lemma 2.13 it is obvious that $\Phi$ takes $B_{f^*\omega}^{\nu,S}$ to $-B_{f^*\omega}^{\nu,S}$. Hence we have the equality

$$\Phi^* \delta_{B_{f^*\omega}^{\nu,S}} = \delta_{-B_{f^*\omega}^{\nu,S}}.$$

Therefore the same argument of Theorem 3.10 is applied and we obtain the following formula:

$$\phi_* DT_{f^*\omega}^{\nu,S}(P) = q \psi(P) \psi(P) DT_{f^*\omega}^{\nu,S}(\phi_*P)$$

$$\cdot \prod_{n>0, \beta>0} f_{n}^{P,\beta} \prod_{n>0, \beta>0} \exp \left( N_{n,\beta}^{\nu} q^{n-\beta} \right) \prod_{n>0, \beta=0} f_{n}^{P,\beta} \prod_{n>0, \beta=0} \exp \left( N_{n,\beta}^{\nu} q^{n-\beta} \right).$$

Then the computations of the error term in the previous subsections give the desired result. 

4. Blow-up formula for DT type invariants on canonical line bundles on surfaces

Let $S$ be a smooth projective surface, and

$$\pi: \omega S \to S$$
the total space of the canonical line bundle of $S$, which is a non-compact Calabi-Yau 3-fold. The purpose of this section is to compare DT type invariants on $\omega_S$ under a blow-up $g: S^\dagger \to S$ at a point $p \in S$. In what follows, we regard $S, S^\dagger$ as subvarieties of $\omega_S, \omega_{S^\dagger}$ by the zero sections.

4.1. DT type invariants on canonical line bundles. Let $\text{Coh}_c(\omega_S)$ be the abelian category of coherent sheaves on $\omega_S$ with compact supports. We have the following inclusions of abelian subcategories

$$\text{Coh}(S) \subset \text{Coh}_S(\omega_S) \subset \text{Coh}_c(\omega_S).$$

For $E \in \text{Coh}_c(\omega_S)$, we set

$$\text{ch}(\pi_* E) = (r, l, s) \in H^0(S) \oplus H^2(S) \oplus H^4(S).$$

Let $L$ be an ample divisor on $S$. We define the slope $\mu_L(E)$ to be

$$\mu_L(E) := \frac{l \cdot L}{r} \in \mathbb{Q} \cup \{\infty\}.$$ 

Here we set $\mu_L(E) = \infty$ if $r = 0$. The above slope function defines the $\mu_L$-stability on $\text{Coh}_c(\omega_S)$, which restricts to the usual $L$-slope stability on $\text{Coh}(S)$. Let $\text{Coh}_c(\omega_S)$ be the stack of all the objects in $\text{Coh}_c(\omega_S)$. The stack $\text{Coh}_c(\omega_S)$ is an open substack of the algebraic stack of all the objects in $\text{Coh}(X)$ for any projective compactification $\omega_S \subset X$. Hence $\text{Coh}_c(\omega_S)$ is an algebraic stack locally of finite type. Similarly to Subsections 3.1, 3.2, we can define the Hall algebra $H(\text{Coh}_c(\omega_S))$ of $\text{Coh}_c(\omega_S)$, the Lie subalgebra $H^{\text{Lie}}(\text{Coh}_c(\omega_S))$ of virtual indecomposable objects, and the unweighted integration map

(61) \begin{align*}
\Pi^{\text{v}1}: H^{\text{Lie}}(\text{Coh}_c(\omega_S)) &\to \bigoplus_{(r, l, s) \in H^*(S)} \mathbb{Q} \cdot c_{(r, l, s)}.
\end{align*}

For $\mu \in \mathbb{Q}$, let $C^t_\mu \subset \text{Coh}_c(\omega_S)$ be the substack corresponding to $\mu_L$-semistable sheaves $E \in \text{Coh}_S(\omega_S)$ with slope $\mu$. The map (61) extends to appropriate completions of both sides, and the invariant $\text{DT}^t_{\mu}(r, l, s) \in \mathbb{Q}$ is defined by

$$\Pi^{\text{v}1}(\epsilon_{C^t_\mu}) = \sum_{l \cdot L / r = \mu} \text{DT}^t_{\mu}(r, l, s) \cdot c_{(r, l, s)}.$$ 

Remark 4.1. Let $\mathcal{M}_L(r, l, s)$ be the moduli stack of $\mu_L$-semistable sheaves $E \in \text{Coh}_S(\omega_S)$ with $\text{ch}(\pi_* E) = (r, l, s)$. If any $[E] \in \mathcal{M}_L(r, l, s)$ is $\mu_L$-stable, then $\mathcal{M}_L(r, l, s)$ is a $\mathbb{C}^*$-gerb over a projective scheme $M_L(r, l, s)$, and $\text{DT}^t_{\mu}(r, l, s)$ coincides with the naive Euler characteristic of $M_L(r, l, s)$.

For $(r, l, s) \in H^*(S^\dagger)$, we can similarly define the invariant $\text{DT}^t_{S^\dagger L}(r, l, s) \in \mathbb{Q}$ by replacing $(S, L)$ by $(S^\dagger, g^* L)$ in the above construction.

4.2. Blow-up and 3-fold flop. The following lemma relates a blow-up of a surface with a 3-fold flop:

Lemma 4.2. There exist smooth projective 3-folds $X, X^\dagger$ and a flop diagram satisfying the following conditions:

- Both of the exceptional locus $C = \text{Ex}(f), C^\dagger = \text{Ex}(f^\dagger)$ are irreducible $(-1, -1)$-curves.
• There are closed embeddings

\[ i : S \hookrightarrow X, \quad i^! : S^! \hookrightarrow X^! \]

such that \( S \cap C \) is a one point, the strict transform of \( S \) in \( X^! \) coincides with \( S^! \), and \( C^! \subset S^! \) coincides with the exceptional locus of \( g : S^! \to S \). Moreover \( f^!(S^!) = S, \ f^!|_{S^!} = g \) and \( kL \) for \( k \gg 0 \) extends to an ample divisor \( L' \) on \( Y \).

• There are open neighborhoods

\[ (62) \]
\[ S \subset X_0, \quad S^! \subset X_0^! \]

which are isomorphic to \( \omega_S, \omega_{S^!} \) respectively, such that the embeddings (62) are identified with the zero sections.

Proof. Let \( C^! \subset S^! \) be the exceptional locus of \( g \), and \( m_p \subset \mathcal{O}_S \) be the ideal sheaf of \( p \). We set \( U^! = \omega_{S^!} \) and consider the following commutative diagram

\[ \begin{array}{ccc}
U^! = Spec_{\mathcal{O}_{S^!}} & \left( \bigoplus_{k \geq 0} \omega_{S^!}^{-k} \right) & \xrightarrow{h^!} V = Spec_{\mathcal{O}_S} \left( \bigoplus_{k \geq 0} m_p^k \otimes \omega_S^{-k} \right) \\
\pi^! & & \pi_0 \\
S^! & \xrightarrow{g} & S.
\end{array} \]

Here the morphism \( h^! \) is induced by \( g_\ast \omega_{S^!}^{-k} \cong m_p^k \otimes \omega_S^{-k} \), and it is a birational morphism. We embed \( S^! \) into \( U^! \) by the zero section of \( \pi^! \). It is easy to check that the curve \( C^! \subset S^! \) is a \((-1, -1)\)-curve in \( U^! \), which coincides with the exceptional locus of \( h^! \), and \( h^!(C^!) \) is an ordinary double point in \( V \). Hence \( h^! \) is a 3-fold flopping contraction. Let \( h : U \to V \) be the flop of \( h^! \), and \( C \subset U \) the exceptional locus of \( h \). Note that \( U \) and \( U^! \) are related by the diagram \( U \leftarrow W \to U^!, \) where the left morphism is a blow-up at \( C \), and the right morphism is a blow-up at \( C^! \). Hence the strict transform of \( S^! \) in \( U \) coincides with \( S \), which intersects with \( C \) at \( p \).

We show that \( U \) contains an open neighborhood of \( S \) which is isomorphic to \( \omega_S \). Let us consider the divisor \( \pi^{-1}(C^!) \) in \( U^! \), and its strict transform \( D \subset U \). Note that \( S \cap D = \emptyset \), and \( D \cap C = \{ q \} \) with \( p \neq q \). We claim that \( U \setminus D \) is isomorphic to \( \omega_S \), such that \( S \subset U \setminus D \) is identified with the zero section. Note that \( U \setminus D \) is set theoretically written as \( \pi^{-1}(S \setminus \{ p \}) \sqcup (C \setminus \{ q \}) \), where \( \pi : \omega_S \to S \) is the projection. We consider the map from \( U \setminus D \) to \( S \) by sending \( x \in \pi^{-1}(S \setminus \{ p \}) \) to \( \pi(x) \) and \( x \in C \setminus \{ q \} \) to \( p \). It is easy to check that this map is a Zariski locally trivial \( \mathbb{A}^1 \)-fibration, hence \( U \setminus D \) is a total space of some line bundle \( \mathcal{L} \) on \( S \). Since \( \mathcal{L} \) is isomorphic to \( \omega_S \) outside \( p \), it must be isomorphic to \( \omega_S \). Hence \( U \setminus D \) is isomorphic to \( \omega_S \).

We consider the \( \mathbb{P}^1 \)-bundle over \( S^! \) given by

\[ \Phi^! : X^! = \mathbb{P}(\mathcal{O}_{S^!} \oplus \omega_{S^!}) \to S^!. \]

Note that \( X^! \) is a projective compactification of \( U^! \). Let \( E := X^! \setminus U^! \) be the boundary divisor. By the base point free theorem, the divisor \( k\Phi^! \cdot g^*L + E \) is globally generated for \( k \gg 0 \). The resulting morphism \( f^! : X^! \to Y \) is a birational morphism whose exceptional locus coincides with \( C^! \). Hence \( f^! \) is a 3-fold flopping contraction and \( Y \) is a projective compactification.
of $V$. By the construction, there is an ample divisor $L'$ on $Y$ such that $k\pi^!g^*L + E = f^!L'$, which implies $L'|_S = kL|_S$. By taking the flop of $f^!$, we obtain a desired flop diagram. □

4.3. Blow-up formula. We compare the DT type invariants $\text{DT}^v_L(*)$ and $\text{DT}^v_{g^*L}(*)$ in Subsection 4.1 using Theorem 3.23 and Lemma 4.2. Let $\omega_S \subset X$ be a compactification as in Lemma 4.2. By the Grothendieck Riemann-Roch theorem, an object $E \in \text{Coh}_S(\omega_S)$ satisfies $\text{ch}(\pi^*E) = (r, l, s)$ if and only if $v(E) = (rS, i_*(l - \frac{r}{2}K_S), \frac{rK_S^2}{12} - \frac{KSl}{2} + \frac{r\chi(O_S)}{2} + s)$ in $H^{22}(X)$. Hence $E \in \text{Coh}_S(\omega_S)$ is $\mu_L$-(semi)stable if and only if $E \in \text{Coh}_S(X)$ and it is $\hat{\mu}_{f^*L}$-(semi)stable, where $L'$ is an ample divisor on $Y$ as in Lemma 4.2. We have the following identity for $\nu = 1$: $\text{DT}^v_L(r, l, s) = \text{DT}^\nu_{f^*L'}(rS, i_*(l - \frac{r}{2}K_S), \frac{rK_S^2}{12} - \frac{KSl}{2} + \frac{r\chi(O_S)}{2} + s)$. Here we have used the notation in Subsection 3.9. Similarly we have the identity for $\nu = 1$: $\text{DT}^v_{g^*L}(r, l, s) = \text{DT}^\nu_{f^*L'}(rS^!, i^!(l - \frac{r}{2}K_{S!}), \frac{rK_{S!}^2}{12} - \frac{K_{S!}l}{2} + \frac{r\chi(O_{S!})}{2} + s)$. We have the following result:

**Theorem 4.3.** For fixed $r \in \mathbb{Z}_{\geq 1}$ and $l \in H^2(S)$, we have the following formula:

$$\sum_{s,a} \text{DT}_{g^*L}^v(r, g^*l - aC^!, -s)q^{\frac{r}{2}l + \frac{s}{2}a + \frac{s}{2}} = \sum_s \text{DT}_L^\nu(r, l, -s)q^s \cdot \eta(q)^{-r} \varphi_{1,0}(q, t)^v.$$  

**Proof.** Let $\phi: X \dashrightarrow X^!$ be a flop as in Lemma 4.2. Note that $\phi_*: H^4(X) \to H^4(X^!)$ takes elements of the form $i_*l$ for $l \in H^2(S)$ to $i^!_*g^*l$. Noting that $H^2(S^!) = g^*H^2(S) \oplus \mathbb{Q} \cdot [C^!]$ and applying Theorem 3.23 for $\nu = 1$, we
obtain
\[
\sum_{l,s} \mathcal{DT}_L^S(r, l, -s) q^{-\frac{rK^2}{12} + \frac{K_{Sl}}{2} - \frac{r\chi(O_S)}{2} + s\theta^\ast(-l + \frac{rK^2}{2})}
\]
\[
= \phi_s \left( \sum_{l,s} \mathcal{DT}_{f_r L}^{\nu^S} \left( rS, l, \left( l - \frac{rK_S}{2} \right), \frac{rK_S^2}{12} - \frac{K_{Sl}}{2} + \frac{r\chi(O_S)}{2} - s \right) \right)
\]
\[
= \sum_{l,s,a} \mathcal{DT}_{f_r L}^{\nu^S} \left( rS, l, \left( g^l - aC^\dagger - \frac{rK_{Sl}}{2} \right), \frac{rK_{Sl}^2}{12} - \frac{K_{Sl}}{2} \left( g^l - aC^\dagger \right) \right)
\]
\[
\cdot q^{-\frac{rK^2}{12} + \frac{K_{Sl}}{2} - \frac{r\chi(O_S)}{2} + s\theta^\ast(-l + \frac{rK^2}{2})}
\]
\[
\cdot \left\{ \eta(q)^{-1} \vartheta_{1,1}(q, -tC^\dagger) \right\}^{-r}
\]
\[
= \sum_{l,s,a} \mathcal{DT}_{g_r L}^S(r, g^l - aC^\dagger, -s) q^{-\frac{rK^2}{12} + \frac{K_{Sl}}{2} - \frac{r\chi(O_S)}{2} + \frac{rK_{Sl}}{2} + \frac{s}{2} + \frac{1}{2}}
\]
\[
\cdot \theta^\ast(-l + \frac{rK_{Sl}}{2}) + (a + \frac{r}{2})C^\dagger \cdot \eta(q)^{\vartheta_{1,0}(q, tC^\dagger)} \cdot \eta(q)^{r}. \]

Therefore we obtain the desired result.

\[\square\]

**Example 4.4.** Suppose that \(\nu = 1\) and \(r = 2\). We take \(a \in \{0, 1\}\) and set \(\tilde{l} = g^l - aC^\dagger\). Then the result of Theorem 4.3 shows that
\[
\sum_{s} \mathcal{DT}_{g_r L}(2, \tilde{l}, -s) q^{\frac{s}{2} + \frac{rK_{Sl}}{2}} = q^{\frac{1}{2}} \frac{\vartheta_a(q)}{\eta(q)^2} \sum_{s} \mathcal{DT}_{L}^S(2, l, -s) q^{s + \frac{rK_{Sl}}{2}}.
\]

Here \(\vartheta_a(q) = \sum_{k \in \mathbb{Z}} q^{(k + \frac{r}{2})^2}\). The above formula coincides with the blow-up formula obtained by Li-Qin \cite{LQ99}.

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