A LOWER BOUND FOR THE HAUSDORFF DIMENSION OF THE SET OF WEIGHTED SIMULTANEOUSLY APPROXIMABLE POINTS OVER MANIFOLDS

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Abstract. Given a weight vector \( \tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n \) with each \( \tau_i \) bounded by certain constraints, we obtain a lower bound for the Hausdorff dimension of the set \( W_n(\tau) \cap M \), where \( M \) is a twice continuously differentiable manifold. From this we produce a lower bound for \( W_n(\Psi) \cap M \) where \( \Psi \) is a general approximation function with certain limits. The proof is based on a technique developed by Beresnevich et al. in [4], but we use an alternative mass transference style theorem proven by Wang, Wu and Xu [22] to obtain our lower bound.

1. Introduction

Diophantine approximation over \( \mathbb{R} \) is largely complete with regards to the Lebesgue measure and Hausdorff measure when considering monotonic approximation functions. In higher dimensions there are two branches of Diophantine approximation studied in depth; dual and simultaneous. In the simultaneous setting we consider the set of numbers

\[
S_n(\psi) := \{ x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |qx_i - p_i| < \psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z}^n \times \mathbb{N} \}
\]

where \( \psi : \mathbb{N} \to \mathbb{R}_+ \) such that \( \psi(q) \to 0 \) as \( q \to \infty \) is an approximation function. Many measure results for this set have already been established. For example, where \( \psi \) is of the form \( \psi(q) = q^{-\tau} \) with \( \tau \leq 1/n \) then Dirichlet’s theorem states that there are infinitely many \( (p, q) \in \mathbb{Z}^n \times \mathbb{N} \) for any \( x \in \mathbb{R}^n \) i.e. \( S_n(\tau) = \mathbb{R}^n \). Further, where \( \tau > 1/n \) we can deduce via the Borel-Cantelli Lemma [12] [13] that such set is of Lebesgue measure zero. In \( n \) dimensions, where the function \( \psi(q) = q^{-1/n} \) is the best possible approximation function i.e. \( x \notin S_n(\tau) \) for any \( \tau > 1/n \) then \( x \) is called an extremal point. In the case that there does exist \( \tau \) such that \( x \in S_n(\tau) \) for some \( \tau > 1/n \) then we say \( x \) is very well approximable.

Using Hausdorff measure and Hausdorff dimension we can be more precise in distinguishing between two sets of zero Lebesgue measure. For a set \( X \subset \mathbb{R}^n \) and \( \rho > 0 \), define a \( \rho \)-cover of \( X \) as a sequence of balls \( \{B_i\} \) such
that $X \subset \bigcup_i B_i$, and for all balls $r(B_i) \leq \rho$, where $r(.)$ is the radius. Define a dimension function $f : \mathbb{R}_+ \to \mathbb{R}_+$ as an increasing function with $f(r) \to 0$ as $r \to 0$. Let
\[
\mathcal{H}_f^\rho(X) = \inf \left\{ \sum_i f(r(B_i)) : \{B_i\} \text{ is a } \rho - \text{ cover of } X \right\},
\]
where the infimum is take over all $\rho$-covers of $X$. Then define the $f$-Hausdorff measure by
\[
\mathcal{H}_f^\rho(X) = \lim_{\rho \to 0} \mathcal{H}_f^\rho(X).
\]
We will usually take the dimension function $f(x) = x^s$ for some $s \in \mathbb{R}_+$, and we will denote $\mathcal{H}_f$ as $\mathcal{H}^s$. With this notation we define the Hausdorff dimension as
\[
\dim X = \inf \{ s > 0 : \mathcal{H}^s(X) = \infty \} = \sup \{ s > 0 : \mathcal{H}^s(X) < \infty \}.
\]

The following theorem is a foundational result proved independently by Jarnik [16] and Besicovitch [11] on the Hausdorff dimension of the set of simultaneously approximable points.

**Theorem.** Let $\tau \geq \frac{1}{n}$ then
\[
\dim \mathcal{S}_n(\tau) = \frac{n + 1}{\tau + 1}.
\]

The result gives a tool to differentiate the size of simultaneous $\tau$-approximable sets where $\tau > 1/n$. This form of simultaneous approximation can be thought of as the set of all $x \in \mathbb{R}^n$ that lie in infinitely many $n$-dimensional hypercubes with side length $2\psi(q)$ and centres $p/q = (\frac{p_1}{q}, \ldots, \frac{p_n}{q}) \in \mathbb{Q}^n$, where each $p_i \in \mathbb{Z}^n$. Weighted simultaneously approximable numbers can be considered in a very similar way, with the $n$-dimensional hypercubes being replaced by $n$-dimensional hyperrectangles. This set is defined as
\[
\mathcal{W}_n(\Psi) := \{ x \in \mathbb{R}^n : |qx_i - p_i| < \psi_i(q) 1 \leq i \leq n \text{ for i.m } (p,q) \in \mathbb{Z}^n \times \mathbb{N} \}
\]
where $\Psi : \mathbb{N} \to \mathbb{R}_+, \text{ with } \Psi(q) = (\psi_1(q), \ldots, \psi_n(q))$ and each $\psi_i : \mathbb{N} \to \mathbb{R}_+$ for $1 \leq i \leq n$. There are many measure theoretic results that have been established for $\mathcal{W}_n(\Psi)$. A Dirichlet style theorem for weighted simultaneous approximation can be deduced from Minkowski’s linear forms theorem (Section 1.4.1 of [5]) to show that $\mathcal{W}_n(\tau) = \mathbb{R}^n$ for all vectors $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_n^n$ satisfying $\sum_{i=1}^n \tau_i \leq 1$. Further, there is also a Khintchine style result [17] that states for $\sum_{i=1}^n \tau_i > 1$ the set has zero Lebesgue measure. The Hausdorff dimension result for such sets was determined by Rynne [19].
Theorem 1. Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ with $\tau_1 \geq \cdots \geq \tau_n$ and suppose that $\sum_{i=1}^n \tau_i \geq 1$. Then

$$\dim W_n(\tau) = \min_{1 \leq j \leq n} \left\{ n + 1 + \sum_{i=j}^n (\tau_j - \tau_i) \right\}.$$ 

It is worth noting that in [19] Rynne proved much more than the above theorem. He obtained a dimension result for $W_n(\tau)$ when the set of points were approximated by some infinite subset $Q \subset \mathbb{N}$, rather than all $\mathbb{N}$. In Rynne's paper [19], the $\tau$ approximation function can be replaced with a monotonic decreasing $\Psi$ approximation function providing that the limit

$$\tau_i^* = \lim_{q \to \infty} -\frac{\log \psi_i(q)}{\log q}$$

exists for $1 \leq i \leq n$. We can do this by considering the inequality

$$q^{-\tau_i^* - \epsilon} \leq \psi_i(q) \leq q^{-\tau_i^* + \epsilon}$$

for sufficiently large $q \in \mathbb{N}$. Observing that, for $\tau^* = (\tau_1^*, \ldots, \tau_n^*)$, the corresponding sets then satisfy

$$W_n(\tau^* + \epsilon) \subset W_n(\Psi) \subset W_n(\tau^* - \epsilon)$$

where $\epsilon = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n$. Thus,

$$\dim W_n(\tau^* - \epsilon) \leq \dim W_n(\Psi) \leq \dim W_n(\tau^* + \epsilon)$$

and as $\epsilon$ is arbitrary we have that $\dim W_n(\Psi) = \dim W_n(\tau^*)$.

We now consider Diophantine approximation on manifolds. When considering manifolds we look at them locally on some open $U \subset \mathbb{R}^d$ and use the following Monge parametrisation without loss of generality

$$\mathcal{M} := \left\{ (x, f(x)) : x \in U \subset \mathbb{R}^d \right\},$$

where $U$ is some open subset in $\mathbb{R}^d$ for $d$ the dimension of the manifold, and $f$ is a function of the form $f : U \to \mathbb{R}^m$ with $m$ being the codimension of the manifold. As the manifold is of this form we can consider the approximation of the coordinates $x$ and $f(x)$ separately. When doing this we will refer to $x$ as the independent variables, and the codomain of $f$ as the dependent variables. When considering the set of simultaneously approximable points on manifolds the approximation functions on both the independent and dependent variables are the same.

Much progress has been made in establishing measure theoretic results for the set $S_n(\psi) \cap \mathcal{M}$, we highlight some of these results below. The first research in this field involved extremal manifolds. Spindzhuk established many of the foundational results in this area which he referred to as dependent variables in [20]. A differentiable manifold is called extremal if almost all points, with respect to the induced Lebesgue measure of the manifold,
are extremal, whereby we mean that the Dirichlet approximation of the space cannot be improved. It was first conjectured [21] and later proven by Kleinbock and Margulis [18] that any non-degenerate submanifold of $\mathbb{R}^n$ is extremal. One of the first advances with respect to the Hausdorff dimension of the set $\mathcal{S}_2(\psi) \cap \mathcal{M}$ was by Beresnevich and Velani in [3], where they determined the dimension of the set of simultaneously approximable points on sufficiently curved planar curves in $\mathbb{R}^2$. There is also a related paper [8] which uses a similar technique to find the Hausdorff dimension of $\mathcal{W}_2(\tau) \cap \mathcal{M}$ for $\tau = (\tau_1, \tau_2)$ bounded below and above by 0 and 1 respectively. Both papers give an equality for the dimension rather than just a lower bound as presented in this paper. The following is Theorem 4 from [8]. We denote the set of $n$ times continuously differentiable functions by $C^{(n)}$.

**Theorem 2.** *(Beresnevich et al. [8])* Let $f$ be a $C^{(3)}$ function over an interval $I_0$, and let $C_f := \{(x, f(x)) : x \in I_0\}$. Let $\tau = (\tau_1, \tau_2)$ where $\tau_1$ and $\tau_2$ are positive numbers such that $0 < \min\{\tau_1, \tau_2\} < 1$ and $\tau_1 + \tau_2 \geq 1$. Assume that

$$
\dim \{x \in I_0 : f''(x) = 0\} \leq \frac{2 - \min\{\tau_1, \tau_2\}}{1 + \max\{\tau_1, \tau_2\}}.
$$

Then

$$
\dim \mathcal{W}_2(\tau) \cap C_f = \frac{2 - \min\{\tau_1, \tau_2\}}{1 + \max\{\tau_1, \tau_2\}}.
$$

Theorem 4 from [3] is the case where $\tau_1 = \tau_2$. The common approach in both papers is ubiquity, as established in [2], to determine the lower bound. The upper bound is found though a combination of Huxley’s estimate [15], which gives an upper estimate on the number of rational points within a specified neighbourhood of the curve, and the property given by (1). This result has been further improved by Beresnevich and Zorin [10] who showed that the dimension result holds for weakly non-degenerate manifolds (see Theorem 4 of [10]). Beresnevich et al. [4] proved the following result.

**Theorem 3.** *(Beresnevich et al. [4])* Let $\mathcal{M}$ be any twice continuously differentiable submanifold of $\mathbb{R}^n$ of codimension $m$ and let

$$
\frac{1}{n} \leq \tau < \frac{1}{m}.
$$

Then

$$
\dim \mathcal{S}_n(\tau) \cap \mathcal{M} \geq s := \frac{n + 1}{\tau + 1} - m.
$$

Furthermore,

$$
\mathcal{H}^s(\mathcal{S}_n(\tau) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}).
$$

To prove this result they used Mass Transference Principle, a new tool developed in [7] to solve such problems. In this paper we adapt the arguments
given in [4] to establish the following theorem, a weighted simultaneous version on Theorem [3]. We now give the following theorem.

**Theorem 4.** Let $\mathcal{M} := \{(x, f(x)) : x \in \mathcal{U} \subset \mathbb{R}^d\}$ where $f : \mathcal{U} \to \mathbb{R}^m$ with $f \in C^{(2)}$. Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ with

$$\tau_1 \geq \tau_2 \geq \cdots \geq \tau_d \geq \tau_{d+1} \quad \text{and} \quad \frac{1}{m} > \tau_{d+1} \geq \tau_{d+2} \geq \cdots \geq \tau_n \geq \frac{1}{n}.$$  

Then

$$\dim (W_n(\tau) \cap \mathcal{M}) \geq \min_{1 \leq j \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^n (\tau_j - \tau_i)}{\tau_j + 1} - m \right\}.$$  

It is worth making a few remarks. Firstly, the minimum is taken over only the first $d \tau_i$, that is the approximation functions over the independent variables $x \in \mathbb{R}^d$. Secondly we only have a lower bound here rather than equality as in theorem [2]. This lower bound agrees with both theorem [2] and theorem [3] so in the case of theorem [2] this is the best lower bound. We also see that this lower bound can be viewed as some function dependent on $\tau$ less the codimension of the manifold which is similar to theorem [3].

We would like to generalise theorem [4] to a more general approximation function. To achieve this we must apply some constraints on our approximation function. Given a decreasing approximation function $\Psi = (\psi_1, \ldots, \psi_n)$ define the upper order $v(\Psi) = (v_1, \ldots, v_n)$ of the function $\Psi$, where

$$v_i := \limsup_{q \to \infty} -\log \frac{\psi_i(q)}{\log q}, \quad 1 \leq i \leq n.$$  

Given such a function, we can state the following Corollary.

**Corollary 5.** Let $\mathcal{M} := \{(x, f(x)) : x \in \mathcal{U} \subset \mathbb{R}^d\}$ where $f : \mathcal{U} \to \mathbb{R}^m$ with $f \in C^{(2)}$. For any approximation function $\Psi = (\psi_1, \ldots, \psi_n)$ such that (2) are positive finite, and

$$v_1 \geq v_2 \geq \cdots \geq v_d \geq v_{d+1} \quad \text{and} \quad \frac{1}{m} > v_{d+1} \geq v_{d+2} \geq \cdots \geq v_n \geq \frac{1}{n},$$  

we have that

$$\dim (W_n(\Psi) \cap \mathcal{M}) \geq \min_{1 \leq j \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^n (v_j - v_i)}{v_j + 1} \right\}.$$  

**Proof.** By properties of the approximation function given by (2) we have that, for any $\epsilon > 0$ there exists a $q_0 \in \mathbb{N}$ such that for all $q > q_0$

$$\psi_i(q) \geq q^{-v_i - \epsilon}, \quad \text{for each} \quad 1 \leq i \leq n.$$  

Using this property, for $\epsilon = (\epsilon, \ldots, \epsilon) \in \mathbb{R}_+^n$, we obtain that

$$W_n(v(\Psi) + \epsilon) \subset W_n(\Psi).$$
so by Theorem 4 and letting $\varepsilon \to 0$ we have that
\[
\dim (W_n(\Psi) \cap \mathcal{M}) \geq \dim (W_n(v(\Psi)) \cap \mathcal{M}) \geq \min_{1 \leq i \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^{n} (v_j - v_i)}{v_j + 1} \right\}
\]
as required. \hfill \Box

Remark 6. Note that this proof is similar to the proof of Corollary 1 from [19]. However we can use the weaker condition of the lim sup rather than lim as we only need the lower bound rather than equality.

We will now review several key theorems required in the proof of Theorem 4.

2. Mass Transference style theorems

The Mass Transference Principle (MTP) was developed by Beresnevich and Velani in [7]. Before stating the MTP we need to introduce some notation. We will use $B = B(x, r)$ to denote a ball $B \subset \mathbb{R}^n$ with centre $x \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_+$. Then we denote $B_s$ to be the ball with centre $x$ and radius $r^{s/n}$, that is $B_s = B(x, r^{s/n})$.

**Theorem 7.** (The Mass Transference Principle) Let $U$ be an open subset of $\mathbb{R}^k$. Let $\{B_i\}_{i \in \mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ centred in $U$ with radius $r_i \to 0$ as $i \to \infty$. Let $s > 0$ and suppose that for any ball $B$ in $U$
\[
H^k \left( B \cap \lim_{{i \to \infty}} \sup B_i^s \right) = H^k(B).
\]
Then, for any ball $B$ in $U$
\[
H^s \left( B \cap \lim_{{i \to \infty}} \sup B_i^k \right) = H^s(B).
\]

Remark 8. In this theorem we take the dimension function $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be $f(r) = r^s$. However in [7] it is shown that we can choose any dimension function providing it is an increasing function with $f(r) \to 0$ as $r \to 0$.

The MTP is a key tool in establishing Theorem 3. The general method for establishing such results is to formulate a Dirichlet style theorem in the desired setting, and then increasing the radius by the power of $s/n$ to form the original lim sup set. Geometrically we can consider this as moving from a lim sup set of balls with radius $r$ to a lim sup set of balls with radius $r^{s/n}$. With the set $W_n(\Psi)$ we have the problem that the set is described by a lim sup set of hyperrectangles rather than balls, so we cannot use the MTP. The following theorem by Wang, Wu and Xu [22] is similar to the MTP with the difference that we move from a lim sup set of balls to a lim sup set of hyperrectangles. In this case we use very similar notation to that of above, with the exception that we introduce a weight vector. Define a weight vector...
to be a vector of the form $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$ with $a_1 \geq \cdots \geq a_n$.

Then, given a ball $B = B(x, r) \subset \mathbb{R}^n$ define $B^\mathbf{a} = B(x, (r^{a_1}, r^{a_2}, \ldots, r^{a_n})) = \{y \in \mathbb{R}^n : |x_i - y_i| < r^{a_i}\}$, i.e. a hyperrectangle with side lengths $2r^{a_i}$ and centre $x$.

**Theorem 9.** (Wang, Wu, Xu [22]) Let $(x_j)_{j \in \mathbb{N}}$ be a sequence of points in $\mathbb{R}^n$ and $(r_j)_{j \in \mathbb{N}}$ be a sequence of positive real numbers such that $r_j \to 0$ as $j \to \infty$, then let $B_j = B(x_j, r_j)$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. Suppose that

$$\mu_n \left( \limsup_{j \to \infty} B_j \right) = 1$$

where $\mu_n$ is the $n$ dimensional Lebesgue measure. Then,

$$\dim \left( \limsup_{j \to \infty} B^\mathbf{a}_j \right) \geq \min_{1 \leq k \leq n} \left\{ n + \sum_{i=k}^{n} \frac{(a_k - a_i)}{a_k} \right\}.$$ 

We see that to use Theorem 9 we need two key ingredients. Firstly we need a lim sup set of balls with full Lebesgue measure. Secondly, we need to construct a weight vector $\mathbf{a}$ that we can use to transform our set of full Lebesgue measure to our desired lim sup set of hyperrectangles. We will now illustrate how this theorem can be used to attain the lower bound of Theorem 1 as shown in [1].

**Proof.** (Lower bound of theorem 1)

Let

$$N := \{(p, q) \in \mathbb{Z}^n \times \mathbb{N} : 0 \leq p_i \leq q \text{ for } 1 \leq i \leq n\},$$

and also let

$$B_{(p, q)} = B \left( \left( \frac{p_1}{q}, \ldots, \frac{p_n}{q} \right), q^{-1-1/n} \right)$$

for some $(p, q) \in N$. By Dirichlet’s theorem we have that

$$\mu_n \left( \limsup_{(p, q) \in N} B_{(p, q)} \right) = 1.$$ 

If we take $\mathbf{a} = (a_1, \ldots, a_n)$ to be the weight vector with coefficients

$$a_i = \frac{n(1 + \tau_i)}{1 + n} \quad \text{for } 1 \leq i \leq n,$$

then

$$B^\mathbf{a}_{(p, q)} = \left\{ \mathbf{x} \in \mathbb{R}^n : \left| x_i - \frac{p_i}{q} \right| < q^{-1-\tau_i} \quad \text{for } 1 \leq i \leq n \right\}.$$ 

So we have that

$$\dim \mathcal{W}_n(\tau) \geq \dim \left( \limsup_{(p, q) \in N} B^\mathbf{a}_{(p, q)} \right).$$
Using Theorem 9 we get that
\[
\dim W_n(\tau) \geq \min_{1 \leq j \leq n} \left\{ \frac{n + \sum_{i=j}^{n} \left( \frac{n(1+\tau_j)}{1+n} - \frac{n(1+\tau_i)}{1+n} \right)}{n(1+\tau_j)} \right\}
\]
\[
\geq \min_{1 \leq j \leq n} \left\{ \frac{n + 1 + \sum_{i=j}^{n} (\tau_j - \tau_i)}{1 + \tau_j} \right\}
\]
as required. \[\square\]

The last measure theoretic result we will be using to prove 4 is a lemma from [9], which essentially states that the Lebesgue measure of a lim sup set remains the same when the balls are altered by some fixed constant.

**Lemma 10.** Let \( \{B_i\} \) be a sequence of balls in \( \mathbb{R}^k \) with \( \mu(B_i) \to 0 \) as \( i \to \infty \) (here \( \mu \) is the \( k \)-dimensional Lebesgue measure of the set). Let \( \{U_i\} \) be a sequence of Lebesgue measurable sets such that \( U_i \subset B_i \) for all \( i \). Assume that for some \( c > 0 \), \( \mu(U_i) \geq c \mu(B_i) \) for all \( i \). Then the sets
\[
\lim_{i \to \infty} \sup U_i \quad \text{and} \quad \lim_{i \to \infty} \sup B_i
\]
have the same Lebesgue measure.

We can use Lemma 10 to change the radius of the balls used in our construction of the lim sup set by a constant and still ensure we have full Lebesgue measure.

### 3. Dirichlet Style theorem on Manifolds

In order to apply Theorem 9 we need to construct a set of full Lebesgue measure which is also a lim sup set of balls. We will achieve this by varying the approximation function only over the dependent variables so we can form a lim sup set from the balls centred at certain rational points in the independent variable space. The theorem below constructs such a set.

**Theorem 11.** Let \( \mathcal{M} := \{ (x, f(x)) : x \in U \subset \mathbb{R}^d \} \) where \( f : U \to \mathbb{R}^m \) with \( f \in C^{(2)} \). Let \( \tau = (\tau_1, \ldots, \tau_m) \in \mathbb{R}^m_+ \) with
\[
\frac{1}{m} > \tau_1 \geq \tau_2 \geq \cdots \geq \tau_m \geq \frac{1}{n}
\]
and let \( \tilde{\tau} = \frac{1}{m} \sum_{i=1}^{m} \tau_i \). Then for any \( x \in U \) there is an infinite subset \( Q \subset \mathbb{N} \) such that for any \( Q \in Q \) there exists \( (p_1, \ldots, p_n) \in \mathbb{Z}^n \times \mathbb{N} \) with \( 1 \leq q \leq Q \) and \( (\frac{p_1}{q}, \ldots, \frac{p_n}{q}) \in \mathcal{U} \) such that
\[
|x_i - \frac{p_i}{q}| < \frac{4^m/d}{q(Q^{1+\tau_m})^{1/d}} \quad 1 \leq i \leq d
\]
and

$$\left| f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - \frac{p_{d+j}}{q} \right| < \frac{q^{-\tau_j - 1}}{2} \quad 1 \leq j \leq m. \tag{4}$$

Further, for any \( x \in U \setminus \mathbb{Q}^d \) there exists infinitely many \((p_1, \ldots, p_n, q) \in \mathbb{Z}^n \times \mathbb{N} \) with \((\frac{p_1}{q}, \ldots, \frac{p_d}{q}) \in U\) satisfying (11) and

$$\left| x_i - \frac{p_i}{q} \right| < 4^{m/d}q^{-1-(1-\tilde{\tau}m)/d} \quad \text{for} \ 1 \leq i \leq d. \tag{5}$$

Before proving Theorem 11 we will state several properties of our manifold \( \mathcal{M} \) that we will be using. Given that \( \mathcal{M} \) is constructed by a twice continuously differentiable function \( f \) we can choose a suitable \( U \) such that, without loss of generality, the following two constants exist. We define

$$C = \max_{1 \leq i, k \leq d} \sup_{1 \leq j \leq m} \left| \frac{\partial^2 f_j}{\partial x_i \partial x_k}(x) \right| < \infty, \tag{6}$$

and

$$D = \max_{1 \leq i \leq d} \sup_{1 \leq j \leq m} \left| \frac{\partial f_j}{\partial x_i}(x) \right| < \infty. \tag{7}$$

A brief outline of the proof is as follows; firstly we alter the system of inequalities to a suitable form so Minkowski’s Theorem for systems of linear forms can be applied. We then use Taylor’s approximation Theorem to return the system of inequalities to the initial form and show that the dependent variable inequalities can be displayed in terms of the independent variable approximation. We finish by concluding that there are infinitely many different integer solutions via a proof by contradiction.

**Proof.** Define

$$g_j := f_j - \sum_{i=1}^{d} x_i \frac{\partial f_j}{\partial x_i} \quad \text{for} \ 1 \leq j \leq m,$$

and consider the system of inequalities

$$\left| qg_j(x) + \sum_{i=1}^{d} p_i \frac{\partial f_j(x)}{\partial x_i} - p_{d+j} \right| < \frac{Q^{-\tau_j}}{4} \quad \text{for} \ 1 \leq j \leq m, \tag{8}$$

$$|qx_i - p_i| < \frac{4^{m/d}}{Q^{(1-\tau m)/d}} \quad \text{for} \ 1 \leq i \leq d, \tag{9}$$

$$|q| \leq Q. \tag{10}$$
Taking the product of the right hand side of the above inequalities and taking the det of the matrix

\[
A = \begin{pmatrix}
g_1 & \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_d} & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
g_2 & \frac{\partial f_m}{\partial x_1} & \ldots & \frac{\partial f_m}{\partial x_d} & 0 & \ldots & -1 \\
x_1 & -1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
x_d & 0 & \ldots & -1 & 0 & \ldots & 0 \\
1 & 0 & \ldots & \ldots & \ldots & 0
\end{pmatrix},
\]
then we see that by Minkowski’s Theorem for systems of linear forms, there exists a non-zero integer solution to the inequalities (8)-(10). Now we must show that this system of inequalities are equivalent to the inequalities (3)-(4). Firstly, fix some \( x \in U \) and as \( U \) is open there exists a ball \( B(x,r) \) for some \( r > 0 \) which is contained in \( U \). Define

\[
Q := \left\{ Q \in \mathbb{N} : (4^{-m}Q^{1-\tau m})^{-1/d} < \min\left\{ 1, r, \left( \frac{1}{2Cd^2} \right)^{1/2} \right\} \right\},
\]
where \( C \) is defined by (6). As \( \tau_i < \frac{1}{m} \) for all \( 1 \leq i \leq m \) we have that \( \tau m < 1 \), so \( Q \) is an infinite subset of \( \mathbb{N} \). We will show that for any \( Q \in Q \) the solution \((p_1, \ldots, p_n, q)\) to the system of inequalities (8)-(10) is a solution to (3)-(4). Suppose \( q = 0 \), then by the definition of the set \( Q \) we have that

\[
(4^{-m}Q^{1-\tau m})^{-1/d} < 1.
\]
By the set of inequalities (3) we have that \( |p_i| < 1 \), hence \( p_i = 0 \) for \( 1 \leq i \leq d \). Further, from (3) we can see that

\[
|p_{d+j}| < \frac{Q^{-\tau_0}}{4} < 1,
\]
for \( 1 \leq j \leq m \). This would conclude that our solution \((p_1, \ldots, p_n, q) = 0\) which contradicts Minkowski’s theorem of systems of linear forms, thus \( |q| \geq 1 \). Now dividing (9) by \( q \) gives us that \( \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) \in B(x,r) \subset U \). We also note that (3) is satisfied upon dividing (3) by \( q \).

Lastly we need to prove (4). By Taylor’s theorem

\[
f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) = f_j(x) + \sum_{i=1}^{d} \frac{\partial f_j}{\partial x_i}(x) \left( \frac{p_i}{q} - x_i \right) + R_j(x, \hat{x}),
\]
for some \( \hat{x} \) on the line connecting \( x \) and \( \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) \), and

\[
R_j(x, \hat{x}) = \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f_j}{\partial x_i \partial x_k}(x) \left( \frac{p_i}{q} - x_i \right) \left( \frac{p_k}{q} - x_k \right).
\]
We may rewrite (8) using Taylor’s theorem and our definition of $g_j$ as

$$q g_j(x) + \sum_{i=1}^d p_i \frac{\partial f_j}{\partial x_i}(x) - p_{d+j} = q f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - p_{d+j} - q R_j(x, \hat{x}).$$

Using the triangle inequality and the assumption that

$$(11) \quad |q R_j(x, \hat{x})| < \frac{q^{-\tau_i}}{4},$$

we can obtain that

$$\left| q f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - p_{d+j} \right| < \frac{q^{-\tau_i}}{4} + \frac{q^{-\tau_i}}{4}.$$

Noting the monotonicity of the approximation function and dividing by $q$ we obtain

$$\left| f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - p_{d+j} \right| < \frac{q^{-\tau_i-1}}{2},$$

thus (II) is satisfied. To complete the first part of the theorem it remains to show that (III) is satisfied for all $Q \in \mathbb{Q}$. Using the definition of $R_j(x, \hat{x})$ we see that

$$|q R_j(x, \hat{x})| = \left| \frac{q}{2} \sum_{i=1}^d \sum_{k=1}^d \frac{\partial^2 f_j}{\partial x_i \partial x_k}(\hat{x}) \left( \frac{p_i}{q} - x_i \right) \left( \frac{p_k}{q} - x_k \right) \right|,$$

$$\leq C q d^2 \left( \frac{4^{m/d}}{q Q^{(1 - \tau_m)/d}} \right)^2,$$

for $1 \leq j \leq m$. Hence we must show that

$$\frac{C q d^2}{2} \left( \frac{4^{m/d}}{q Q^{(1 - \tau_m)/d}} \right)^2 < \frac{q^{-\tau_i}}{4},$$

for $1 \leq j \leq m$. Rearranging the equation and dividing through by $q$ we obtain the inequality

$$\left( \frac{4^{m}}{Q^{(1 - \tau_m)}} \right)^{1/d} < \left( \frac{q^{1-\tau_i}}{2C d^2} \right)^{1/2}.$$

Considering that $\tau_i < \frac{1}{m}$, we have that $\inf_{q \in \mathbb{N}} q^{1-\tau_i} = 1$. Thus by the definition of the set $Q$ the above inequality is satisfied by all $Q \in \mathbb{Q}$, so (III) is true for all $1 \leq j \leq m$.

We now prove the second part of the theorem, that is that there is infinitely many vector solutions. Suppose that there are only finitely many such $q$ and let $A$ be the corresponding set. As $x \in U \setminus \mathbb{Q}^d$ there exists some $1 \leq i \leq d$ where $x_i \notin \mathbb{Q}$. Hence there exists some $\delta > 0$ such that

$$\delta \leq \min_{q \in A, p_i \in \mathbb{Z}} |qx_i - p_i|.$$
By (3) we now have that

$$\delta \leq |qx_i - p_i| \leq \frac{4^{m/d}}{Q(1-\tilde{\tau}m)/d} \quad \text{for } 1 \leq i \leq d.$$  

However, as $Q$ is an infinite set and $Q^{(1-\tilde{\tau}m)/d} \to \infty$ as $Q \to \infty$ we have a contradiction so there are infinitely many different $q$. Lastly, as $q \leq Q$, we can replace $Q$ by $q$ in (3) to obtain (5) as desired. \hfill \Box

4. Proof of Theorem 4

We are now in a position to prove Theorem 4. To do so we construct a lim sup set of balls satisfying the conditions on Theorem 11. The lim sup set will thus have full Lebesgue measure. Next we choose a suitable weight vector $a$ that we use to transform our lim sup set of balls to a lim sup set of hyperrectangles with a known lower bound for its Hausdorff dimension. The proof is completed by showing that the constructed lim sup set is at least contained within our set $W_n(\tau) \cap M$, thus our lower bound is a lower bound for $W_n(\tau) \cap M$.

**Proof.** Take the set

$$N(f, \tau) := \left\{ (p, q) \in \mathbb{Z}^n \times \mathbb{N} : \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) \in \mathcal{U} \right\}$$

and

$$\left| f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - \frac{p_{d+j}}{q} \right| < q^{-\tau_{d+j} - 1} 1 \leq j \leq m \}.$$  

In view of Theorem 11 we have that for almost all $x \in \mathcal{U}$ there are infinitely many different vectors $(p, q) \in N(f, \tau)$ satisfying

$$\left| x_i - \frac{p_i}{q} \right| < 4^{m/d} q^{-1-(1-\tilde{\tau}m)/d}, \quad 1 \leq i \leq d,$$

where $\tilde{\tau} = \frac{1}{m} \sum_{i=d+1}^{n} \tau_i$. By Lemma 10 we can choose a constant $k > 0$ such that for almost every $x \in \mathcal{U}$ there are infinitely many different vectors $(p, q) \in N(f, \tau)$ satisfying

$$\left| x_i - \frac{p_i}{q} \right| < k q^{-1-(1-\tilde{\tau}m)/d}, \quad 1 \leq i \leq d.$$  

Now take the ball

$$B(p, q) := \left\{ x \in \mathcal{U} : \left| x_i - \frac{p_i}{q} \right| < k q^{-1-(1-\tilde{\tau}m)/d} \quad \text{for} \quad 1 \leq i \leq d \right\},$$

and considering Theorem 11 and Lemma 10 we have that

$$\mu_d \left( \limsup_{(p, q) \in N(f, \tau)} B(p, q) \right) = 1$$
where $\mu_d$ is the induced $d$ dimensional Lebesgue measure over $M$. Let $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ be a weight vector with each

\[(12) \quad a_i = \frac{d(1 + \tau_i)}{d + 1 - \tau_i}, \quad 1 \leq i \leq d.
\]

Then $B^a_{(p,q)}$ is the hyperrectangle with the following properties:

\[B^a_{(p,q)} = \left\{ x \in U^d : \left| x_i - \frac{p_i}{q} \right| < k^{a_i} q^{-1 - \tau_i}, \quad 1 \leq i \leq d \right\}.
\]

By Theorem 3 we have that

\[\dim \left( \lim \sup_{(p,q) \in N(f,\tau)} B^a_{(p,q)} \right) \geq \min_{1 \leq j \leq d} \left\{ \frac{d + \sum_{i=j}^d (a_j - a_i)}{a_j} \right\}.
\]

We can then replace our value for $a_i$ by (12),

\[\dim \left( \lim \sup_{(p,q) \in N(f,\tau)} B^a_{(p,q)} \right) \geq \min_{1 \leq j \leq d} \left\{ \frac{d + \sum_{i=j}^d (a_j - a_i)}{a_j} \right\}.
\]

Using the definition of $\bar{\tau}$ and that $d = n - m$ we may rewrite this as

\[\dim \left( \lim \sup_{(p,q) \in N(f,\tau)} B^a_{(p,q)} \right) \geq \min_{1 \leq j \leq d} \left\{ \frac{n - m + 1 - \sum_{i=d+1}^n \tau_i + \sum_{i=j}^d (\tau_j - \tau_i)}{1 + \tau_j} \right\},
\]

\[\geq \min_{1 \leq j \leq d} \left\{ \frac{n - m + 1 - \sum_{i=d+1}^n (\tau_j - \tau_i) - m \tau_j + \sum_{i=j}^d (\tau_j - \tau_i)}{1 + \tau_j} \right\},
\]

\[\geq \min_{1 \leq j \leq d} \left\{ \frac{n + 1 - m (1 + \tau_j)}{1 + \tau_j} \right\},
\]

\[\geq \min_{1 \leq j \leq d} \left\{ \frac{n + \sum_{i=j}^d (\tau_j - \tau_i)}{1 + \tau_j} - m \right\},
\]

as required. We now finish by showing that

\[\dim(W_n(\tau) \cap M) \geq \dim \left( \lim \sup_{(p,q) \in N(f,\tau)} B^a_{(p,q)} \right).
\]

Note that for any $y \in S_a(\tau) \cap M$ it must have infinitely many solutions $(p_1, \ldots, p_2, q) \in \mathbb{Z}^n \times \mathbb{N}$ to the following system of inequalities

\[(13) \quad |qx_i - p_i| < q^{-\tau_i}, \quad 1 \leq i \leq d,
\]

\[(14) \quad |qf(x) - p_{d+j}| < q^{-\tau_{d+j}}, \quad 1 \leq j \leq m,
\]
where \( y = (x, f(x)) \) for some \( x = (x_1, \ldots, x_d) \in \mathcal{U} \). Let the set of \( x \) satisfying (13)-(14) be denoted by \( \pi_d(W_n(\tau) \cap \mathcal{M}) \). This set is the orthogonal projection of \( W_n(\tau) \cap \mathcal{M} \) onto \( \mathbb{R}^d \). By Proposition 3.3 of [14], as the projection is bi-Lipschitz it is sufficient to prove that

\[
\dim \pi_d(W_n(\tau) \cap \mathcal{M}) \geq \min_{1 \leq j \leq d} \left\{ \frac{n + 1 + \sum_{i=j}^n (\tau_i - \tau_j)}{1 + \tau_j} - m \right\}.
\]

Let \( x \in B_{(p,q)}^a \) for some \( (p, q) \in N(f, \tau) \). On using the triangle inequality, the mean-value theorem, and (7) we have that for any \( 1 \leq j \leq m, \)

\[
\left| f_j(x) - \frac{p_{d+j}}{q} \right| \leq \left| f_j(x) - f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) \right| + \left| f_j \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) - \frac{p_{d+j}}{q} \right|,
\]

\[
\leq \left| \left( \frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_d}{\partial x_d} \right) \cdot \left( x - \left( \frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) \right) \right| + \frac{q^{-1-\tau_{d+j}}}{2},
\]

\[
\leq D \sum_{i=1}^d \left| x_i - \frac{p_i}{q} \right| + \frac{q^{-1-\tau_{d+j}}}{2},
\]

\[
\leq D d \max_{1 \leq i \leq d} \left| x_i - \frac{p_i}{q} \right| + \frac{q^{-1-\tau_{d+j}}}{2},
\]

\[
\leq D d k^a q^{-1-\tau_d} + \frac{q^{-1-\tau_{d+j}}}{2}.
\]

We can choose \( k \) sufficiently small, and note that \( \tau_d \geq \tau_{d+j} \) for all \( 1 \leq j \leq m, \) so that we have

\[
\left| f_j(x) - \frac{p_{d+j}}{q} \right| \leq q^{-1-\tau_j}, \quad 1 \leq j \leq m.
\]

We have that

\[
B_{(p,q)}^a \subset \left\{ x \in \mathcal{U} : \left| x_i - \frac{p_i}{q} \right| < q^{-1-\tau_i}, \quad 1 \leq i \leq d \right\}
\]

for \( i.m \) \( (p, q) \in N(f, \tau) \). Hence for any \( x \in \limsup_{(p,q)\in N(f,\tau)} B_{(p,q)}^a \), (13)-(14) are satisfied for infinitely many \( (p_1, \ldots, p_2, q) \in \mathbb{Z}^n \times \mathbb{N} \), thus

\[
\dim \pi_d(S_n(\tau) \cap \mathcal{M}) \geq \dim \left( \limsup_{(p,q)\in N(f,\tau)} B_{(p,q)}^a \right),
\]

as required. \( \square \)

5. Conclusion

Using the arguments above and, principally applying the MTP of [8], we have established a lower bound for \( W_n(\tau) \cap \mathcal{M} \) which coincides with that of \( S_n(\psi) \cap \mathcal{M} \) from Theorem 3. The natural question is can equality be determined. That is, can an upper bound be found which agrees with our calculated lower bound? Thus achieving a complete analogue of Theorem 4. In trying to attain an upper bound, it is necessary to find an estimate for the
number of rational points within a \(\tau\)-neighbourhood of the manifold. Until recently, one of the only results in this direction was due to Huxley [15]. However, Beresnevich et. al. [6] have recently established a \(n\)-dimensional version of Huxley’s estimate for ‘large’ approximation functions. While these set of approximations are out of the range we can consider in this setting it might be possible to adapt these results and prove a suitable upper bound corresponding to Theorem [4].

Note that, Corollary [5] whilst being relatively general, does not cover all approximating functions. For example, the approximation function \(\Psi(q) = (|\sin q|q^{-\tau_1}, \ldots, |\sin q|q^{-\tau_n})\) has infinite upper order. It would be of interest to extend the class of approximating functions somehow.

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