The Hardy–Rellich inequality for polyharmonic operators

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The Hardy–Rellich inequality given here generalizes a Hardy inequality of Davies, from the case of the Dirichlet Laplacian of a region \( \Omega \subseteq \mathbb{R}^N \) to that of the higher-order polyharmonic operators with Dirichlet boundary conditions. The inequality yields some immediate spectral information for the polyharmonic operators and also bounds on the trace of the associated semigroups and resolvents.

1. Introduction

The Hardy inequality originated in 1920 in [7] as an integral inequality for functions defined on the real half-line. Its original representation can be easily reformulated, for \( 1 < p < \infty \), as

\[
\int_0^\infty \frac{|f(x)|^p}{x^p} \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty |f'(x)|^p \, dx
\]

(1.1)

for all \( f \in C_c^\infty((0,\infty)) \). Since its appearance, various generalizations of particular aspects of the inequality have been made. In [10], for example, there is a detailed treatment of weighted Hardy-type inequalities in an \( L^p \) setting.

The Rellich inequality appeared first in [13] as a generalization of inequality (1.1) to two derivatives. The simplest form of such an inequality is

\[
\int_0^\infty \frac{|f(x)|^p}{x^{2p}} \, dx \leq \left( \frac{p^2}{(p-1)(2p-1)} \right)^p \int_0^\infty |f''(x)|^p \, dx
\]

(1.2)

for all \( f \in C_c^\infty((0,\infty)) \).

In this paper we study a generalization for all derivatives, within the \( L^2 \) setting. The variable \( x \) in the denominator of inequalities (1.1) and (1.2) is replaced by a pseudodistance \( a_m(x) \), where \( m \) is the number of derivatives in the dominating integrand. We formulate our result as a Hardy–Rellich operator inequality, and use it as a tool in the spectral analysis of polyharmonic operators. The Rellich inequalities found in [6] concern a distinct but related class of operator inequalities.

In order to state our result properly, we need the following definitions, in which \( \Omega \) denotes an open subset of \( \mathbb{R}^N \):

**Definition 1.1.** Let \( Q_m \) be the closure of the quadratic form defined on \( C_c^\infty(\Omega) \subseteq L^2(\Omega) \) by

\[
Q_m(f) = \langle (-\Delta)^m f, f \rangle.
\]
The domain of the closure is the Sobolev space $W_0^{m,2}(\Omega)$. The polyharmonic operator $(-\Delta)^m|_{\text{DIR}}$ is defined as the non-negative self-adjoint operator associated with $Q_m$. See [5] for details. Where the implied region is not contextually evident, the operator is denoted by $H_{\Omega,m}$.

The boundary conditions classically associated with the operators $(-\Delta)^m|_{\text{DIR}}$ and $(-\Delta|_{\text{DIR}})^m$ are different. The inequality,

$$H_{\Omega,m} \geq H_{\Omega,1}^m,$$  \hspace{1cm} (1.3)

valid in the quadratic form sense, may be verified by considering the domains of the corresponding quadratic forms.

In the case $m = 2$, the biharmonic operator $\Delta^2|_{\text{DIR}}$ is central to the theory of vibrating of elastic shells with clamped boundary. See [14, ch. 6.2] for a brief account of the theory. This is a counterpart to the theory of vibrating membranes, in which the Dirichlet Laplacian ($m = 1$) is the relevant operator.

**DEFINITION 1.2.** Let $\omega \in S^{N-1}$ and define $d_\omega : \Omega \to (0, +\infty]$ by

$$d_\omega(x) := \min\{|s| : x + s\omega \not\in \Omega\}. \hspace{1cm} (1.4)$$

Define the pseudodistances $a_m : \Omega \to (0, +\infty]$ for $1 \leq m \in \mathbb{R}$ by

$$a_m(x) = \left[ \int_{S^{N-1}} d_\omega(x)^{-2m} d^{N-1}\omega \right]^{-1/2m}, \hspace{1cm} (1.5)$$

where $d^{N-1}\omega$ is the normalized surface measure on the unit spherical shell $S^{N-1}$.

In theorem 2.3 we shall prove the inequality

$$(-\Delta)^m|_{\text{DIR}} \geq \frac{(N + 2m - 2)(N + 2m - 4)\ldots N(2m - 1)(2m - 3)\ldots 1}{4^m a_m^{2m}}, \hspace{1cm} (1.6)$$

in the quadratic form sense, thus comparing the polyharmonic operator $H_{\Omega,m} = (-\Delta)^m|_{\text{DIR}}$, acting in $L^2(\Omega)$, with a multiplication operator which is large near the boundary of the region. This generalizes an inequality of Davies [2] for the case $m = 1$ of the Dirichlet Laplacian.

As a special case of (1.6), if $\Omega$ is convex we see that

$$(-\Delta)^m|_{\text{DIR}} \geq \frac{(2m - 1)^2(2m - 3)^2\ldots 1^2}{4^m d^{2m}}, \hspace{1cm} (1.7)$$

in the quadratic form sense, where $d : \Omega \to (0, +\infty]$ is defined by

$$d(x) := \min\{|y - x| : y \not\in \Omega\}. \hspace{1cm} (1.8)$$

More generally, for regular regions, where the pseudodistances $a_m$ are comparable with the distance $d$, a similar inequality is valid. The constants in inequalities (1.6) and (1.7) are shown to be optimal.

Using the Hardy–Rellich inequality (1.6) it is possible to find an upper bound on the trace of the semigroup $e^{-H_{\Omega,m}^t}$. Following a technique of Davies [3] of decomposing a region with finite inradius into dyadic cubes we find a similar lower
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bound. More explicitly,

\[
b_{m,N} t^{-N/2m} \int_{\Omega} \exp[-c_{m,N} td^{-2m}] \leq \text{tr}[e^{-H_{\Omega,m}t}] \\
\leq b'_{m,N} t^{-N/2m} \int_{\Omega} \exp[-c'_{m,N} t a^{-2m}], \tag{1.9}
\]

where \( b_{m,N}, c_{m,N}, b'_{m,N} \) and \( c'_{m,N} \) are positive constants. For regular regions \( \Omega \) this yields an immediate equivalent condition for \( \text{tr}[e^{-H_{\Omega,m}t}] \) to be finite, and, as a corollary, a condition for \( \text{tr}[H_{\Omega,m}^\gamma] \) to be finite. This generalizes work of Davies [3] for the corresponding semigroups and resolvents of the Dirichlet Laplacian.

2. The Hardy–Rellich operator inequality

Our starting point is a one-dimensional version of the Hardy–Rellich inequality in the \( L^2 \) setting. For \( m = 1 \) and \( m = 2 \), the following lemma respectively resembles inequalities (1.1) and (1.2), where we set \( p = 2 \).

**Lemma 2.1.** Let \( \Omega \) be an open (not necessarily connected) set in \( \mathbb{R} \). Define the distance \( d : \Omega \to (0, +\infty) \) as in equation (1.8). Then

\[
(2m-1)^2(2m-3)^2 \ldots 1^2 \int_{\Omega} \frac{|f(x)|^2}{d(x)^{2m}} \, dx \leq \int_{\Omega} |f^{(m)}(x)|^2 \, dx
\]

for all \( f \in C^\infty_c(\Omega) \).

**Proof.** We prove the statement only for open intervals \((a,b) \subseteq \mathbb{R} \). Suppose that the above statement is true for some \( m \). Then applying [5, lemma 5.3.1] with \( \alpha = -2m \),

\[
\frac{(2m+1)^2}{4} \int_{a}^{b} \frac{|f(x)|^2}{d(x)^{2(m+1)}} \, dx
\]

\[
= \frac{(1+2m)^2}{4} \left[ \int_{0}^{(b-a)/2} x^{-2m-2} |f(x+a)|^2 \, dx + \int_{0}^{(b-a)/2} x^{-2m-2} |f(b-x)|^2 \, dx \right]
\]

\[
\leq \int_{0}^{(b-a)/2} x^{-2m} |f'(x+a)|^2 \, dx + \int_{0}^{(b-a)/2} x^{-2m} |f'(b-a)|^2 \, dx
\]

\[
= \int_{a}^{b} \frac{|f'(x)|^2}{d(x)^{2m+1}} \, dx
\]

\[
\leq \frac{4^m}{(2m-1)^2(2m-3)^2 \ldots 1^2} \int_{a}^{b} |f^{(m+1)}(x)|^2 \, dx.
\]

The first step of induction is dealt with by [5, corollary 5.3.2]. \( \square \)

**Lemma 2.2.** Let \( \omega \in S^{N-1} \). Then

\[
\int_{S^{N-1}} \langle \xi, \omega \rangle^{2m} d^{N-1} \omega = \frac{(2m-1)(2m-3) \ldots 1}{(N + 2m - 2)(N + 2m - 4) \ldots N} |\xi|^{2m}. \tag{2.2}
\]
Proof. Since the above integral is rotationally invariant and homogeneous of degree 2m with respect to \( \xi \), we see that
\[
\int_{S^{N-1}} \langle \xi, \omega \rangle^{2m} d^{N-1}\omega = c|\xi|^{2m}.
\]
Setting \( \xi = (1,0,\ldots,0) \), we see that for \( N \geq 3 \)
\[
c = \int_{S^{N-1}} \langle \xi, \omega \rangle^{2m} d^{N-1}\omega
\begin{align*}
&= \frac{1}{\omega_{N-1}} \int_{-\pi}^{\pi} \cdots \int_{0}^{\pi} \cos^{2m} \theta_1 \sin^{N-2} \theta_1 \cdots \sin \theta_{N-2} \, d\theta_1 \cdots d\theta_{N-1} \\
&= \frac{(2m-1)(2m-3)\cdots 1}{(N+2m-2)(N+2m-4)\cdots N},
\end{align*}
\]
where \( \omega_{N-1} \) denotes the surface area of the unit spherical shell \( S^{N-1} \) regarded as a subset of \( \mathbb{R}^N \). The last step of this calculation requires elementary analysis and is therefore omitted. The cases \( N = 1, 2 \) are simple. □

We may now prove the Hardy–Rellich operator inequality:

**Theorem 2.3.** Let \( (-\Delta)^m \big|_{\text{DIR}} = H_{\Omega,m} \) be the polyharmonic operator of order 2m acting in \( L^2(\Omega) \), where \( \Omega \) is a region in \( \mathbb{R}^N \), and let \( a_m \) be the corresponding pseudodistance. Then, in the quadratic form sense,
\[
(-\Delta)^m \big|_{\text{DIR}} \geq \frac{(N+2m-2)(N+2m-4)\cdots N(2m-1)(2m-3)\cdots 1}{4^m a_m^{2m}}.
\tag{2.3}
\]

**Proof.** Let \( f \in C_c^\infty(\Omega) \). Let \( \omega \in S^{N-1} \) be fixed, and let \( \{u_1 = \omega, u_2, \ldots, u_N\} \) be an orthonormal basis of \( \mathbb{R}^N \). Let \( v = (v_1, \ldots, v_N) \) denote coordinates with respect to that basis and let \( P \) be the coordinate transition matrix \( x = vP \) from \( v \) coordinates to standard coordinates. Let \( \hat{v} = (v_2, \ldots, v_N) \) be fixed, and let \( \Omega_{\hat{v}} \) be the open (not necessarily connected) set
\[
\Omega_{\hat{v}} = \{v_1 \in \mathbb{R} : vP \in \Omega\},
\]
where \( v = (v_1, \hat{v}) \). Define \( g_{\hat{v}} : \Omega_{\hat{v}} \to \mathbb{R} \) and \( d_{\hat{v}} : \Omega_{\hat{v}} \to (0, +\infty] \) by
\[
\begin{align*}
g_{\hat{v}}(v_1) &:= f(vP) \\
d_{\hat{v}}(v_1) &:= d_\omega(vP).
\end{align*}
\]
Then \( g_{\hat{v}} \in C_c^\infty(\Omega_{\hat{v}}) \) and
\[
d_{\hat{v}}(v_1) = \min\{|y - v_1| : y \not\in \Omega_{\hat{v}}\}.
\]
Using lemma 2.1,
\[
\frac{(2m-1)^2(2m-3)^2\cdots 1^2}{4^m} \int_{\Omega_{\hat{v}}} \frac{|g_{\hat{v}}(v_1)|^2}{d_{\hat{v}}(v_1)^{2m}} \, dv_1 \leq \int_{\Omega_{\hat{v}}} |g_{\hat{v}}^{(m)}(v_1)|^2 \, dv_1,
\]
and hence
\[
\frac{(2m-1)^2(2m-3)^2 \ldots 1}{4^m} \int_{\Omega} |f(x)|^2 a_m(x)^m \, d^N x
\]
\[
= \frac{(2m-1)^2(2m-3)^2 \ldots 1}{4^m} \int_{S^{N-1}} \int_{\Omega} |f(x)|^2 d^N x \, d^{N-1} \omega
\]
\[
= \frac{(2m-1)^2(2m-3)^2 \ldots 1}{4^m} \int_{S^{N-1}} \int_{\Omega} \int_{\Omega_0} \frac{|g_0^{(m)}(v_1)|^2}{d_0^{(m)}} \, dv_1 \, d^{N-1} \hat{\omega} \, d^{N-1} \omega
\]
\[
\leq \int_{S^{N-1}} \int_{\Omega} |\partial^m f(x)|^2 d^N x \, d^{N-1} \omega
\]
\[
= \int_{S^{N-1}} \int_{\Omega} \langle \xi, \omega \rangle^2 \|\hat{f}(\xi)\|^2 d^N \xi \, d^{N-1} \omega
\]
\[
= \frac{(2m-1)(2m-3) \ldots 1}{(N+2m-2)(N+2m-4) \ldots N} \int_{\mathbb{R}^N} |\xi|^{2m} |\hat{f}(\xi)|^2 d^N \xi
\]
\[
= \frac{(2m-1)(2m-3) \ldots 1}{(N+2m-2)(N+2m-4) \ldots N} Q_m(f).
\]

\[\square\]

**Corollary 2.4.** Suppose that \( \Omega \) is a convex region in \( \mathbb{R}^N \). Then
\[
(-\Delta)^m |_{\text{DIR}} \geq \frac{(2m-1)^2(2m-3)^2 \ldots 1}{4^m d^{2m}}.
\]  

**Proof.** Let \( x \in \Omega \) and let \( y \in \partial \Omega \) be such that \( |y - x| = d(x) \). Suppose \( z \in \Omega \). Constructing the point
\[
p = y + \frac{\langle z - y, x - y \rangle}{\langle z - y, z - y \rangle} (z - y),
\]
we see that
\[
|p - x|^2 = |y - x|^2 - \frac{\langle z - y, x - y \rangle^2}{\langle z - y, z - y \rangle}
\]
so either
\[
\langle z - y, x - y \rangle = 0
\]
or
\[
|p - x| < d(x).
\]
In the second case, \( p \) will lie in \( \Omega \), and so by convexity the line segment joining \( z \) and \( p \) lies in \( \Omega \). See figure 1. Since \( y \notin \Omega \), it cannot lie on this segment so
\[
\langle z - y, p - y \rangle > 0.
\]
From the definition of \( p \), this implies that
\[
\langle z - y, x - y \rangle > 0.
\]
In both cases, \( z \) lies in the set \( \{ z \in \mathbb{R}^N : \langle z - y, x - y \rangle \geq 0 \} \).

Since \( \Omega \) is open, it must therefore be a subset of the open half
\[
H := \{ z \in \mathbb{R}^N : \langle z - y, x - y \rangle > 0 \}
\]

of \( \mathbb{R}^N \). From the definition (1.4) of \( d_\omega \) we see that
\[
d_\omega(x)|\langle y - x, \omega \rangle| \leq \min\{ |s| : x + s\omega \notin H \} |\langle y - x, \omega \rangle| d(x) = d(x)^2.
\]

See figure 2 for a diagrammatic representation of this last step. Hence
\[
|\langle y - x, \omega \rangle|^{2m} d(x)^{-4m} \leq d_\omega(x)^{-2m}.
\]

Therefore
\[
\frac{(2m-1)(2m-3)\ldots1}{(N+2m-2)(N+2m-4)\ldots N} d(x)^{-2m} = \frac{(2m-1)(2m-3)\ldots1}{(N+2m-2)(N+2m-4)\ldots N} |y - x|^{2m} d(x)^{-4m} = \int_{S^{N-1}} |\langle y - x, \omega \rangle|^{2m} d(x)^{-4m} d^{N-1}\omega 
\leq \int_{S^{N-1}} d_\omega(x)^{-2m} d^{N-1}\omega = a_m(x)^{-2m}.
\]

Using theorem 2.3 we see that for \( f \in C^\infty_c(\Omega) \),
\[
\frac{(2m-1)^2(2m-3)^2\ldots1}{4^m} \int_\Omega \frac{|f(x)|^2 dN x}{d(x)^{2m}} \leq \frac{(N+2m-2)(N+2m-4)\ldots N(2m-1)(2m-3)\ldots1}{4^m} \int_\Omega a_m(x)^{2m} dN x \leq Q_m(f).
\]
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NOTE 2.5. It is simple to deduce a crude lower bound

\[ \lambda_1 \geq \frac{(2m-1)^2(2m-3)^2\ldots1}{4^m \text{Inradius}(\Omega)^{2m}} \]  \hspace{1cm} (2.8)

on the first eigenvalue of \((-\Delta)^m|_{\text{DIR}}\) for convex regions with finite inradius

\[ \text{Inradius}(\Omega) := \sup_{x \in \Omega} d(x). \]  \hspace{1cm} (2.9)

Since the strength of the inequality (2.4) lies in the behaviour near the boundary of the ‘potential’ function on the right-hand side, the constant in the bound (2.8) (which loses information at the boundary) is not sharp.

NOTE 2.6. The constants in theorem 2.3 and corollary 2.4 are optimal. This can be seen by choosing \( \Omega = \{ (x_1, \ldots, x_N) : x_1 > 0 \} \) and by considering the sequence of functions \( f_n \in C_c^\infty(\Omega) \) defined by

\[ f_n(x) = x_1^{m-1/2} \phi_n(x_1) \psi(\hat{x}), \]

where \( \psi \in C_c^\infty(\mathbb{R}^{N-1}) \) and \( \phi_n \in C_c^\infty((0, \infty)) \) is chosen so that \( \phi_n = 1 \) on the interval \([2/n, 1], \phi_n = 0 \) on \( \mathbb{R} \setminus [1/n, 2], |D^j \phi_n| \leq cn^j \) on \([1/n, 2/n]\) and \( |D^j \phi_n| \leq c \) on \([1, 2], \) for \( j = 0, 1, \ldots, 2m. \)

Calculations now show that

\[ \int_\Omega \frac{|f_n(x)|^2}{d(x)^{2m}} d^N x \geq \int_{[2/n, 1] \times \mathbb{R}^{N-1}} x_1^{-1} |\psi(\hat{x})|^2 d^N x = \| \psi \|_2^2 \ln n/2 \]

and

\[ Q_m(f_n) \leq \frac{(2m-1)^2(2m-3)^2\ldots1}{4^m} \| \psi \|_2^2 \ln n/2 + c'. \]

The constant in corollary 2.4 is therefore optimal, and so the constant in theorem 2.3 must also be optimal.

3. Spectral implications of the inequality

In the course of proving corollary 2.4 we show, in inequality (2.7), that the pseudo-distance \( a_m \) is uniformly comparable to the boundary distance function \( d. \) This motivates the introduction of the following terminology:

DEFINITION 3.1. A region \( \Omega \) is said to be regular if there is a constant \( k < \infty \) such that

\[ d(x) \leq a_1(x) \leq kd(x) \]  \hspace{1cm} (3.1)

for all \( x \in \Omega. \) More generally, we shall say that a region \( \Omega \) is \( m \)-regular if there is a constant \( k_m < \infty \) such that

\[ d(x) \leq a_m(x) \leq k_m d(x) \]  \hspace{1cm} (3.2)

for all \( x \in \Omega. \)

DEFINITION 3.2. We shall say that \( \Omega \) satisfies a uniform external ball condition if there exist positive constants \( \alpha, \beta \) such that for any \( y \in \Omega \) and \( 0 < s \leq \beta \) there exists a ball \( B(a;r) \) with centre \( a \) satisfying \( |a - y| \leq s, \) and radius \( r \) satisfying \( r \geq \alpha s, \) which does not meet \( \Omega. \)
EXAMPLES 3.3. If any one of the following geometrical conditions is satisfied, then the region \( \Omega \subseteq \mathbb{R}^N \) is regular:

(i) \( \Omega \) satisfies a uniform external ball condition with \( \beta = \infty \).

(ii) \( \Omega \) has finite inradius and satisfies a uniform external ball condition.

(iii) There exists a positive constant \( c \) such that

\[
\{ y \notin \Omega : |y - a| < r \} \geq cr^N
\]

for all \( a \in \partial \Omega \) and all \( r > 0 \).

Proof. See \([4, \text{theorems 1.5.5 and 1.5.4}]\) and \([5, \text{theorem 5.3.6}]\). The common characteristic of these situations is that at any point \( x \in \Omega \), the directional distance \( d_\omega(x) \) to the boundary is uniformly comparable to the actual distance \( d(x) \) to the boundary over a uniform solid angle. \( \square \)

Lemma 3.4. Let \( x \in \Omega \) be fixed. Then \( a_m(x) \) is a decreasing function of \( m \). Hence if \( \Omega \) is regular then it is \( m \)-regular for all \( m \geq 1 \).

Proof. Let \( \| . \|_p \) be norms on the spaces \( L^p(S^{N-1},d^{N-1}\omega) \). Since the surface measure \( d^{N-1}\omega \) in definition 1.2 is normalized, Hölder’s inequality implies that for \( m \geq n \)

\[
\|d_\omega(x)^{-1}\|_{2n} \leq \|d_\omega(x)^{-1}\|_{2m}\|1\|_{2m/n/(m-n)} = \|d_\omega(x)^{-1}\|_{2m}.
\]

Hence from definition 1.2,

\[
a_m(x) = \|d_\omega(x)^{-1}\|_{2m}^{-1} \leq \|d_\omega(x)^{-1}\|_{2m}^{-1} = a_n(x). \quad \square
\]

Theorem 3.5. Suppose that \( \Omega \) is \( m \)-regular. Then \( 0 \notin \text{Spec}(H_{\Omega,m}) \) if and only if the inradius of \( \Omega \) is finite.

Proof. Suppose that \( \Omega \) has finite inradius. The Hardy–Rellich operator inequality (2.3) and \( m \)-regularity (3.2) imply that

\[
H_{\Omega,m} \geq \frac{(N + 2m - 2)(N + 2m - 4) \ldots N(2m - 1)(2m - 3) \ldots 1}{4^m k_m^{2m} \text{Inradius}(\Omega)^{2m}}. \tag{3.3}
\]

Conversely, suppose that \( d(x) \) is unbounded. For any \( r > 0 \) there exists a ball \( B_r \) with radius \( r \), contained in \( \Omega \). Using the Rayleigh–Ritz variational formula (see \([5, \S 4.5])

\[
0 \leq \min(\text{Spec}(H_{\Omega,m})) \leq \min(\text{Spec}(H_{B_r,m})) = r^{-2m} \min(\text{Spec}(H_{B_1,m})).
\]

Hence \( 0 \in \text{Spec}(H_{\Omega,m}) \). \( \square \)

In the special case where \( \Omega \) is regular, one should note that the above result can be proved by using the original Hardy inequality \( (m = 1) \) and inequality (1.3). Such an approach is not, however, valid for a proof of the following theorem. Here, we extend the domain of the distance functions \( d, a_m \) to the whole of \( \mathbb{R}^N \) by setting \( d(x) = a_m(x) = 0 \) for \( x \notin \Omega \).
Theorem 3.6. Suppose that $\Omega$ is $m$-regular. Then $H_{\Omega,m}^{-1}$ is compact if and only if $d(x) \to 0$ as $|x| \to \infty$. If $\Omega$ is bounded, then $H_{\Omega,m}^{-1}$ is compact without any assumption about $m$-regularity.

Proof. Using the Hardy–Rellich inequality (2.3) we see that

$$H_{\Omega,m} \geq \frac{1}{2} H_{\Omega,m} + \frac{(N + 2m - 2)(N + 2m - 4) \ldots N(2m - 1)(2m - 3) \ldots 1}{2^m a_{2m}^m}$$

$$\geq \frac{1}{2} H_{K^N,m} + \frac{(N + 2m - 2)(N + 2m - 4) \ldots N(2m - 1)(2m - 3) \ldots 1}{2^{2m+1} k_{2m}^2 d^{2m}}$$

as quadratic forms in $L^2(\mathbb{R}^N)$. The last operator in the above inequality has compact resolvent because it is a Schrödinger operator whose potential

$$V = \frac{(N + 2m - 2)(N + 2m - 4) \ldots N(2m - 1)(2m - 3) \ldots 1}{2^{2m+1} k_{2m}^2 d^{2m}}$$

satisfies $V(x) \to \infty$ as $|x| \to \infty$. (The compactness of the resolvent of such a Schrödinger operator is proved in [9, theorem 12.5.5], although with the unnecessary restriction that $N < 2m$. Simple modification of the proof of [11, theorem XIII.67] yields the result without any such restriction.) It now follows that $H_{\Omega,m}^{-1}$ is compact.

Conversely, suppose $d(x)$ does not converge to zero as $x \to \infty$, $x \in \Omega$. Then there exist $r > 0$ and a sequence of disjoint balls $B_i \subseteq \Omega$, each with radius $r$. Let $\phi_i$ be the ground state of the operator $H_{B_i,m}$. Then

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}, \quad (3.4)$$

$$\langle H_{\Omega,m}^{1/2} \phi_i, H_{\Omega,m}^{1/2} \phi_j \rangle = c \delta_{ij}, \quad (3.5)$$

where $c$ is independent of $i, j$. Using the Rayleigh–Ritz formula of [5, §4.5] we see that $H_{\Omega,m}^{-1}$ cannot be compact. □

4. Lower bound on the trace of the polyharmonic semigroup

In the remaining sections we build upon the methods of Davies [3] to obtain lower and upper bounds on the trace of the semigroup $e^{-H_{\Omega,m} t}$ and the resolvent $H_{\Omega,m}^{-1}$. The proof of the lower bound in theorem 4.4 requires the following sequence of lemmas:

Lemma 4.1. Let $\lambda_{m,n}$ denote the $n$th eigenvalue of the polyharmonic operator

$$(-\Delta)^m|_{\text{DIR}}$$

acting in $L^2((0,1))$. Then

$$[n \pi]^{2m} \leq \lambda_{m,n} \leq [(m + n - 1) \pi]^{2m}. \quad (4.1)$$

Proof. The left-hand inequality is a consequence of inequality (1.3). We prove the other inequality as follows. Let $f_r \in W_0^{m,2}((0,1))$ be defined by

$$f_r(x) = \sin^{m-1} \pi x \sin r \pi x.$$

Then $f_r \in M_{r+m-1}$, where

$$M_s = \text{lin}\{1, \sin \pi x, \cos \pi x, \ldots, \sin s \pi x, \cos s \pi x\}.$$
Let $L_n \subseteq W_{0,m}^2([0,1])$ be defined by

$$L_n = \text{lin}\{f_r : 1 \leq r \leq n\}.$$  

Then $L_n \subseteq M_{n+m-1}$, and by the Rayleigh–Ritz formula [5],

$$\lambda_{m,n} \leq \sup\{Q_m(f) : f \in L_n, \|f\|_2 = 1\} = \sup\{\|D^m f\|_2^2 : f \in L_n, \|f\|_2 = 1\} \leq \sup\{\|D^m f\|_2^2 : f \in M_{n+m-1}, \|f\|_2 = 1\}. \quad (4.2)$$

Suppose that $f \in M_s$ and $\|f\|_2 = 1$. Then

$$f(x) = \alpha_0 + \sum_{r=1}^{s} (\alpha_r \sqrt{2} \cos \pi r x + \beta_r \sqrt{2} \sin \pi r x),$$

where

$$\sum_{r=0}^{s} (\alpha_r^2 + \beta_r^2) = 1.$$  

Now

$$\|D^m f\|_2^2 = \sum_{r=1}^{s} (\alpha_r^2 (r\pi)^{2m} + \beta_r^2 (r\pi)^{2m}) \leq (s\pi)^{2m}.$$  

Hence by inequality (4.2),

$$\lambda_{m,n} \leq [(m+n-1)\pi]^{2m}.$$

\[\Box\]

**Lemma 4.2.** The operator

$$H' = \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} \right)^m$$

acting in $L^2(C)$ with Dirichlet boundary conditions, where $C = (0, \delta)^N$, is uniformly elliptic, homogeneous of order $2m$, and has compact resolvent. The eigenvalues of $H'$ are given by

$$\mu_n = \delta^{-2m} \sum_{i=1}^{N} \lambda_{m,n_i}, \quad (4.4)$$

where $n = (n_1, \ldots, n_N)$ is a non-negative multi-index and $\lambda_{m,n_i}$ are the eigenvalues of the one-dimensional polyharmonic operator in lemma 4.1.

**Proof.** The proof of this lemma is elementary and therefore omitted. \[\Box\]

To find the lower bound on $\text{tr}[e^{-H'_{a,m} t}]$ we use the technique of decomposing $\Omega$ into dyadic cubes by introducing Dirichlet boundary conditions along various internal partitioning surfaces. The cubes $C \subseteq \Omega$ we use are of the form

$$C = \left\{ x \in \mathbb{R}^n : \frac{a_i}{2^n} < x_i < \frac{a_i + 1}{2^n} \right\} \quad (4.5)$$
for some \( n \in \mathbb{Z} \) and some \( a \in \mathbb{Z}^N \). Ordering the dyadic cubes (4.5) by inclusion, let \( \{ C_r : r \in \mathbb{N} \} \) be an enumeration of the maximal cubes contained in \( \Omega \), provided at least one exists. Let \( \delta_r \) be the side length of \( C_r \) and let \( \Omega' = \bigcup_{r=1}^{\infty} C_r \).

**Lemma 4.3.** The cubes \( C_r \) are disjoint. Suppose that the inradius of \( \Omega \) is finite. Then \( \Omega' = \Omega \), and moreover for \( x \in C_r \) we have \( d(x) \leq 2N^{1/2}\delta_r \).

*Proof.* The inclusion \( \Omega' \subseteq \overline{\Omega} \) is obvious. Conversely, suppose that \( x \in \Omega \). Then \( B(x; d(x)) \subseteq \overline{\Omega} \) and so \( x \) will lie in some closed dyadic cube \( C \subseteq \overline{\Omega} \) with diameter at least \( d(x)/2 \). The edge length of such a cube will be at least \( d(x)/(2N^{1/2}) \).

Since \( d(x) \) is bounded, the point \( x \) will lie in a maximal cube \( C_r \) with edge length \( \delta_r \geq d(x)/(2N^{1/2}) \). Hence \( \Omega \subseteq \overline{\Omega'} \) and \( d(x) \leq 2N^{1/2}\delta_r \). \( \square \)

We shall always assume that \( \Omega \) has finite inradius, for otherwise using theorem 3.6, we may deduce that \( \text{tr}[e^{-H_{\infty},m^t}] = \infty \).

**Theorem 4.4.** For \( 0 < t < \infty \)

\[
b_{m,N} t^{-N/2m} \int_\Omega \exp[-c_{m,N} d^{-2m} t] \leq \text{tr}[e^{-H_{\infty},m^t}],
\]

where

\[
b_{m,N} = N^{-N(m-1)/2m} (2\pi)^{-N} \Gamma(1 + 1/2m) 2^{N/2m}
\]

and

\[
c_{m,N} = (4mN\pi)^{2m}/2.
\]

*Proof.* Let \( \Omega = (0,1) \subseteq \mathbb{R} \). Using the notation of lemma 4.1 and the spectral mapping theorem, we see that the trace of the semigroup \( e^{-H_{[0,1]}t} \) is \( \sum_{n=1}^{\infty} e^{-\lambda_{m,n} t} \), and moreover that

\[
\sum_{n=1}^{\infty} e^{-\lambda_{m,n} t} \geq \sum_{n=0}^{\infty} e^{-[(m+n)\pi]^{2m} t}
\]

\[
\geq \sum_{n=0}^{\infty} e^{-2^{m-1}(m^{2m} + n^{2m})\pi^{2m} t}
\]

\[
\geq e^{-(2m\pi)^{2m} t/2} \int_0^\infty e^{(2m\pi)^{2m} t/2} dx
\]

\[
= [(2\pi)^{2m} t/2]^{-1/2m} \Gamma(1 + 1/2m) e^{-(2m\pi)^{2m} t/2}
\]

\[
= b_{m,1} t^{-1/2m} \exp[-c_{m,1} 2^{-2m} t].
\]

Let \( H' \) denote the operator (4.3) acting in \( L^2(C_r) \) with Dirichlet boundary conditions. The symbol

\[
a(x, \xi) = \sum_{i=1}^{N} \xi_i^{2m}
\]

of \( H' \) satisfies

\[
N^{-(m-1)} |\xi|^{2m} \leq a(x, \xi) \leq |\xi|^{2m}
\]
and therefore

$$H_{C_r,m} \leq N^{m-1} H'.$$  \hfill (4.10)

Using lemma 4.2 and equation (4.7),

$$\text{tr}[e^{-H_{C_r,m} t}] \geq \text{tr}[e^{-N^{m-1} H' t}]$$

$$= \sum_{n \in N^N} \exp \left( -N^{m-1} \delta_r^{-2m} t \sum_{i=1}^N \lambda_{m,n_i} \right)$$

$$= \sum_{n \in N^N} \prod_{i=1}^N e^{-N^{m-1} \delta_r^{-2m} t \lambda_{m,n_i}}$$

$$= \left[ \sum_{n=1}^\infty e^{-N^{m-1} \delta_r^{-2m} t \lambda_{m,n}} \right]^N$$

$$\geq b_{m,1}^N [N^{m-1} \delta_r^{-2m} t]^{-N/2m} \exp[-c_{m,1} 2^{-2m} N^m \delta_r^{-2m} t]$$

$$= b_{m,N} \delta_r^N t^{-N/2m} \exp[-c_{m,N} (2N^{1/2} \delta_r)^{-2m} t].$$

Using the results of lemma 4.3,

$$\text{tr}[e^{-H_{\Omega,m} t}] \geq \text{tr}[e^{-H_{\Omega',m} t}]$$

$$\geq \sum_{r=1}^\infty \text{tr}[e^{-H_{C_r,m} t}]$$

$$\geq \sum_{r=1}^\infty b_{m,N} \delta_r^N t^{-N/2m} \exp[-c_{m,N} (2N^{1/2} \delta_r)^{-2m} t]$$

$$\geq b_{m,N} t^{-N/2m} \sum_{r=1}^\infty \int_{C_r} \exp[-c_{m,N} t d(x)^{-2m}] d^n x$$

$$= b_{m,N} t^{-N/2m} \int_{\Omega} \exp[-c_{m,N} t d^{-2m}].$$

\hfill \square

5. Upper bound on the trace of the polyharmonic semigroup

In order to prove an upper bound on the trace, we shall need to assume that the region $\Omega$ satisfies the following condition.

**Condition 5.1.** Let $\Omega$ be a region such that the kernel $K_{\Omega}(t,x,y)$ of $e^{-H_{\Omega,m} t}$ exists, is jointly continuous and satisfies

$$|K_{\Omega}(t,x,y)| \leq c t^{-N/2m}. \hfill (5.1)$$

for some $c = c_{\Omega}$, and for all $t > 0$ and $x, y \in \Omega$. Let $b'_{m,N} = 2^{N/2m} c$.

Two special cases in which this condition is satisfied are given in the following two examples:
EXAMPLE 5.2. For all \( N \) the Laplacian \((-\Delta)|_{\text{DIR}}\) acting in \( L^2(\Omega) \) has a heat kernel \( K(t, x, y) \) which satisfies
\[
0 \leq K(t, x, y) \leq (4\pi t)^{-N/2}
\]
for all \( x, y \in \Omega \) and \( t > 0 \).

Proof. See [4, example 2.1.8]. \( \square \)

EXAMPLE 5.3. Suppose that \( \Omega \subseteq \mathbb{R}^N \) and \( N < 2m \). Then \((-\Delta)^m|_{\text{DIR}}\) acting in \( L^2(\Omega) \) has a heat kernel which satisfies
\[
|K(t, x, y)| \leq ct^{-N/2m}
\]
for all \( x, y \in \Omega \) and all \( t > 0 \).

Proof. By the spectral mapping theorem we see that
\[
\|H_{\Omega, m}^{1/2}e^{-H_{\Omega, m}t}\|_{2, 2} \leq ct^{-1/2}.
\]
For \( f \in L^2(\Omega) \) and \( t > 0 \), let \( f_t = e^{-H_{\Omega, m}t}f \in W_0^{m, 2}(\Omega) \). Using a standard Sobolev embedding theorem,
\[
\|f_t\|_\infty \leq c\|H_{\Omega, m}^{1/2}f_t\|_2^{N/2m}\|f_t\|_2^{1-N/2m}
\leq c\|H_{\Omega, m}^{1/2}e^{-H_{\Omega, m}t}\|_{2, 2}^{N/2m}\|f\|_2^{N/2m}\|f\|_2^{1-N/2m}
\leq ct^{-N/4m}\|f\|_2.
\]
Hence
\[
\|e^{-H_{\Omega, m}t}\|_{\infty, 2} \leq ct^{-N/4m}.
\]
By duality,
\[
\|e^{-H_{\Omega, m}t}\|_{\infty, 1} \leq \|e^{-H_{\Omega, m}t}\|_{\infty, 2}\|e^{-H_{\Omega, m}t}\|_{2, 1} \leq ct^{-N/2m}. \tag{5.2}
\]
Define \( \phi_t : \Omega \to L^2(\Omega) \cap L^\infty(\Omega) \) by the property
\[
\langle f, \phi_t(x) \rangle = (e^{-H_{\Omega, m}t}f)(x)
\]
for all \( f \in L^1(\Omega) \cap L^2(\Omega) \). Since \( e^{-H_{\Omega, m}t}f \) is a smooth function, the map \( x \mapsto \phi_t(x) \) from \( \Omega \to L^2(\Omega) \) is smooth in the weak Hilbert space sense and hence, by [1, corollary 1.42], it is smooth. Define
\[
K(t, x, y) = [\phi_t(x)](y).
\]
Then by the definition of \( \phi_t \), we see that \( K(t, \ldots) \) is an integral kernel of \( e^{-H_{\Omega, m}t} \). Using the identity
\[
[\phi_{s+t}(x)](y) = \langle \phi_s(x), \phi_t(y) \rangle
\]
for all \( t, s > 0 \) we see that \( K \) is smooth in \( x \) and \( y \). Moreover, by (5.2) we see that
\[
|K(t, x, y)| \leq ct^{-N/2m}.
\]
\( \square \)
THEOREM 5.4. Suppose that $\Omega$ satisfies condition 5.1. Then
\[ \text{tr}[e^{-H_{\Omega,m}t}] \leq b'_{m,N} t^{-N/2m} \int_{\Omega} \exp[-c'_{N,m}a_m^{-2m}t], \quad (5.3) \]
where $b'_{m,N}$ is determined by condition 5.1, and
\[ c'_{m,N} = 2^{-2m-1}(N + 2m - 2)(N + 2m - 4)\ldots N(2m - 1)(2m - 3)\ldots 1. \]

Proof. The Hardy–Rellich inequality (2.3) shows that
\[ H_{\Omega,m} \geq \frac{1}{2} H_{\Omega,m} + \frac{(N + 2m - 2)(N + 2m - 4)\ldots N(2m - 1)(2m - 3)\ldots 1}{2.4m^2}. \]
Using the Golden–Thompson inequality [8], integration of the kernel along the diagonal [12, pp. 65 and 66], and condition 5.1 we see that
\[ \text{tr}[e^{-H_{\Omega,m}t}] \leq \text{tr}[\exp[-H_{\Omega,m}t/2 - c'_{N,m}a_m^{-2m}t]] \]
\[ \leq \text{tr}[e^{-H_{\Omega,m}t/4} \exp[-c'_{N,m}a_m^{-2m}t]e^{-H_{\Omega,m}t/4}] \]
\[ = \int_{\Omega} K_{\Omega}(t/2, x, x) \exp[-c'_{N,m}a_m(x)^{-2m}t] d^Nx \]
\[ \leq b'_{m,N} t^{-N/2m} \int_{\Omega} \exp[-c'_{N,m}a_m(x)^{-2m}t] d^Nx. \]

\[ \square \]

6. Equivalent conditions for finite trace

We can now use the lower and upper bounds of theorems 4.4 and 5.4 to give conditions for finite trace of $e^{-H_{\Omega,m}t}$ and $H_{\Omega,m}^{-\gamma}$ in terms of integrals involving the distance function $d$.

THEOREM 6.1. Suppose that $\Omega$ is $m$-regular and satisfies condition 5.1. Then
\[ \text{tr}[e^{-H_{\Omega,m}t}] < \infty \]
for all $t \in (0, \infty)$ if and only if
\[ \int_{\Omega} e^{-td^{-2m}} < \infty \]
for all $t \in (0, \infty)$.

Proof. Since $\Omega$ is $m$-regular, inequality (1.9) becomes
\[ bt^{-N/2m} \int_{\Omega} \exp[-ctd^{-2m}] \leq \text{tr}[e^{-H_{\Omega,m}t}] \leq b't^{-N/2m} \int_{\Omega} \exp[-c'tk_m d^{-2m}]. \]

\[ \square \]

COROLLARY 6.2. Suppose that $\Omega$ is $m$-regular and satisfies condition 5.1, and that $\gamma > N/2m$. Then
\[ \text{tr}[H_{\Omega,m}^{-\gamma}] < \infty \]
if and only if
\[ \int_{\Omega} d^{2m\gamma-N} < \infty. \]
The Hardy–Rellich inequality for polyharmonic operators

Proof. Using Fubini’s theorem for traces we see that

\[
\int_0^\infty \text{tr} [e^{-H_{\Omega,m}t}] \gamma^{-1} dt = \text{tr} \left[ \int_0^\infty e^{-H_{\Omega,m}t} \gamma^{-1} dt \right] = \Gamma(\gamma) \text{tr}[H_{\Omega,m}^{-\gamma}].
\]

Integration of inequality (5.3) gives

\[
\int_0^\infty \text{tr} [e^{-H_{\Omega,m}t}] \gamma^{-1} dt \leq \int_\Omega \int_0^\infty b't^{-N/2m+\gamma-1} \exp[-c'ta_m(x)^{-2m}] dt \, d^N x
\]

\[
= b'c^{-\gamma+N/2m} \Gamma(\gamma - N/2m) \int_\Omega a_m^{2m(\gamma-N)}
\]

and similarly by integrating inequality (4.6) we see that

\[
bce^{-\gamma+N/2m} \Gamma(\gamma - N/2m) \int_\Omega d^{2m(\gamma-N)} \leq \int_0^\infty \text{tr} [e^{-H_{\Omega,m}t}] \gamma^{-1} dt.
\]

The result follows as in theorem 6.1 because \( \Omega \) is \( m \)-regular. \( \square \)

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