Fast Linking Numbers for Topology Verification of Loopy Structures

ANTE QU and DOUG L. JAMES, Stanford University, USA

It is increasingly common to model, simulate, and process complex materials based on loopy structures, such as in yarn-level cloth garments, which possess topological constraints between inter-looping curves. While the input model may satisfy specific topological linkages between pairs of closed loops, subsequent processing may violate those topological conditions. In this paper, we explore a family of methods for efficiently computing and verifying linking numbers between closed curves, and apply these to applications in geometry processing, animation, and simulation, so as to verify that topological invariants are preserved during and after processing of the input models. Our method has three stages: (1) we identify potentially interacting loop–loop pairs, then (2) carefully discretize each loop’s spline curves into line segments so as to enable (3) efficient linking number evaluation using accelerated kernels based on either counting projected segment–segment crossings, or by evaluating the Gauss linking integral using direct or fast summation methods (Barnes–Hut or fast multipole methods). We evaluate CPU and GPU implementations of these methods on a suite of test problems, including yarn-level cloth and chainmail, that involve significant processing: physics-based relaxation and animation, user-modeled deformations, curve compression and reparameterization. We show that topology errors can be efficiently identified to enable more robust processing of loopy structures.

CCS Concepts: • Computing methodologies → Animation; Shape modeling; • Theory of computation → Computational geometry.

Additional Key Words and Phrases: Topology, checksum, linking number, Gauss linking integral, Barnes–Hut, fast multipole method, yarn-level cloth

1 INTRODUCTION

It is increasingly common to model, simulate, and process complex materials based on loopy structures, such as in knitted yarn-level cloth garments, which possess topological constraints between inter-looping curves. In contrast to a solid or continuum model, the integrity of a loopy material depends on the preservation of its loop–loop topology. A failure to preserve these links, or an illegal creation of new links between unlinked loops, can result in an incorrect representation of the loopy material.

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Unfortunately, common processing operations can easily destroy the topological structure of loop materials. For example, time-stepping dynamics with large time steps or contact solver errors that cause pull-through events, deformation processing that allows loops to separate or interpenetrate, or even applying compression and reparameterization on curves, can all cause topology errors. Furthermore, once the topology of a loopy material has been ruined, the results can be disastrous (an unraveling garment) or misleading (incorrect yarn-level pattern design).

In this paper, we explore a family of methods for efficiently computing and verifying linking numbers between closed curves. Intuitively, if two loops are unlinked (i.e., they can be pulled apart), the linking number is zero; otherwise, the linking number is a nonzero signed integer corresponding to how many times they loop through one another, with a sign to disambiguate the looping direction (see Figure 2). In mathematical terms, the linking number is a homotopically invariant that describes the “linkage” of two oriented closed curves in 3D space. While a closed curve can be either a “knot” or a simple “loop”, we will use the term “loop” to refer to all closed curves. Given two disjoint, oriented loops \( γ_1 \) and \( γ_2 \) where each maps \( S^1 \rightarrow \mathbb{R}^3 \), the linking number is an integer-valued function, \( \lambda(γ_1, γ_2) \), that counts the number of times each curve winds around the other [Rolfsen 1976]. Curve deformations that result in one part of a curve crossing the other loop at a single point must change the linking number by \( ±1 \). By evaluating (or knowing) the linking numbers before a computation, then evaluating them afterwards, we can detect certain topological changes. A linking number checksum can therefore be useful as a sanity check for topology preservation—we cannot rule out intermediate topology changes when two states have the same linking number, different linking numbers imply a topology change. We efficiently and systematically explore this idea and apply it to several applications of loop materials in computer graphics where it is useful to verify the preservation of topological invariants during and after processing (see Figure 1 for a preview of our results).

One way to compute the linking number is to compute crossings in a link diagram. Suppose our curves are discretized into polylines. The link diagram is a 2D regular projection of this set of 3D curves that also stores the above–belowness at every intersection; that is, if the projection plane is the XY plane, then the Z coordinate of each curve indicates which curve is above or below. A projection is deemed regular if no set of 3 or more points, or 2 or more vertices, coincide at the same projected point [Rolfsen 1976].

To compute the linking number, start with \( \lambda(γ_1, γ_2) = 0 \), and at every detected intersection, compare the orientation of the curves with Figure 3, and increment the linking number by \( +1/2 \) if it is positive, or \( -1/2 \) if it is negative.

Gauss’s Integral Form. Another mathematically equivalent method to compute the linking number, which doesn’t require a projection plane or geometric overlap tests, is to compute the linking integral \( \lambda \), first introduced by Gauss [Rolfsen 1976]:

\[
\lambda(γ_1, γ_2) = \frac{1}{4\pi} \int_{γ_1} \int_{γ_2} \frac{r_1 - r_2}{|r_1 - r_2|^3} \cdot (dr_1 \times dr_2).
\] (1)
This integral form can be interpreted as an application of Ampère’s law from magnetostatics for a current loop and a test loop\(^1\).

Direct Summation Methods. If the loops \(y_1, y_2\) are discretized into line segments indexed by \(j\) and \(i\), and we denote the midpoints of \(y_j\) by \(r_j\) and \(y_2\) by \(r_i\), and the line segment length vectors by \(s_j\) and \(s_i\) respectively, then taking a midpoint quadrature, we have

\[
λ(y_1, y_2) ≈ \frac{1}{4\pi} \sum_{j,i} \frac{r_j - r_i}{|r_j - r_i|^3} \cdot (s_j × s_i).
\]

This midpoint approximation is only accurate when the segments are short compared to the distance between each pair. Fortunately, an exact expression for polyline loops is known of the form

\[
λ(y_1, y_2) = \sum_{j,i} λ_{ji}
\]

where \(λ_{ji}\) is the contribution from a pair of line segments [Arai 2013; Berger 2009]. If we denote the vertices (endpoints of the segments) of each loop by \(l_j, k_i\), the contribution from the segment pair \((j, i)\) is given in [Arai 2013] (which uses the signed solid angle formula [Van Oosterom and Strackee 1983]):

\[
λ_{ji} = \frac{1}{2\pi} \left( \begin{align*}
\text{atan} & \left( \frac{a \cdot (b × c)}{|a||b||c| + (a \cdot b)|c| + (c \cdot a)|b| + (b \cdot c)|a|} \right) \\
& + \text{atan} \left( \frac{c \cdot (d × a)}{|c||d||a| + (c \cdot d)|a| + (a \cdot c)|d| + (d \cdot a)|c|} \right) \end{align*} \right).
\]

where

\[
a = l_j - k_i, \quad b = l_j - k_{i+1}, \quad c = l_{j+1} - k_i, \quad d = l_{j+1} - k_{i+1}, \quad \text{atan}(y/x) = \text{atan2}(y, x).
\]

This computation requires two trigonometric operations per segment pair. More recently, Bertolazzi et al. [2019] used the arctan addition formula to eliminate all trigonometric operations.

Fast Summation Methods. If we discretize the loops into \(N\) samples, Gauss’s double integral can be approximated using fast summation methods, such as the Barnes–Hut algorithm which uses a tree to compute \(N\)-body Gaussian summations in \(O(N \log N)\) time. Alternately one can use the Fast Multiple Method (FMM); some FMM libraries provide off-the-shelf implementations of the Biot–Savart integral, which is the inner integral of the linking integral.

2.2 Related Work

Many works in the past have tackled the topic of robust collision processing. Epsilon geometry and robust predicates [Salesin et al. 1989; Shewchuk 1997] have been used to precisely answer “inside–outside” questions, such as where a point lies with respect to a flat face (“vertex–face”), or a straight line with respect to another (“edge–edge”). These determinant-based geometric tests (such as [Feito and Torres 1997]) are also useful for detecting inversions in finite elements and preventing them from collapsing [Irving et al. 2004; Selle et al. 2008]. Some papers [Harmon et al. 2011; Kim et al. 2019; Selle et al. 2008; Volino and Magnenat-Thalmann 2006; Zhang et al. 2007] propose methods to repair interpenetrations between objects by minimizing a computable quantity, such as an intersection contour, a space–time intersection volume, an altitude spring, or another form of energy. For models with a large number of elements, a few works use hierarchical bounding volumes to compute energy-based collision verification certificates [Barbić and James 2010; Zheng and James 2012], or to validate geometry in hair [Kaufman et al. 2014], holey shapes [Bernstein and Wojtan 2013], or cloth [Baraff et al. 2003].

Many of the above methods either perform a check or repair a violation by answering discrete collision-detection questions about a point vs. a flat face or a straight edge vs. another straight edge. To generalize these questions to arbitrary surfaces (and soups and clouds), a few papers [Barill et al. 2018; Jacobson et al. 2013] compute generalized winding numbers to answer the inside–outside question. The generalized winding number can be expressed as a Gaussian integral, and Barill et al. [2018] uses Barnes–Hut style algorithms to speed up the computation.

Continuous collision detection (CCD) methods [Bash 1999; Bridson et al. 2002; Brochu et al. 2012; Harmon et al. 2011; Wang and Cao 2021] can verify that a deformation trajectory is collision free. Such methods could be used to detect curve–curve crossings and thus topology changes. In contrast, our approach of computing a topology verification checksum is a discrete before/after test: we verify the topology at a fixed state rather than the geometry along the entire deformation path. While our method cannot guarantee a deformation is collision free, it works even if the deformation path is ambiguous (e.g., in verifying model reparameterizations, which we will show in §4.3.2), or, when the path is known, it can be performed as sparsely or frequently as the user requires.

For simulations and deformations that are more suitable for 1D rod elements rather than volumetric elements, it can be difficult to apply discrete inside–outside checks. For example, in yarn-level simulations of cloth, yarns are often modeled as cylindrical spline or rod segments [Kaldor et al. 2008, 2010; Leaf et al. 2018; Wu et al. 2020]. While the inside–outside question applied to the cylindrical volumes can determine interpenetration when detected, oftentimes an illegal or large simulation step can cause a “pull-through” where the end state has no interpenetration but rather a topological change in the curve configuration. Pull-throughs can cause changes to the appearance and texture of the resulting relaxed pattern, yet remain undetected unless spotted by a human eye. Many of these simulations [Leaf et al. 2018; Yuksel et al. 2012] prevent pull-throughs by using large penalty contact forces and limiting step sizes using a known guaranteed displacement limit. Some stitch meshing papers [Guo et al. 2020; Narayanan et al. 2018, 2019; Wu et al. 2018, 2019] avoid closed loops as a result of making the model knittable or machine knittable. In contrast, many of the models from [Yuksel et al. 2012] consist of yarns that form closed loops. We propose a method to automatically detect pull-throughs in closed-loop yarn models after they occur, so that the simulation can take more aggressive steps and backtrack or exit when the violation is detected. Other

\(^1\)Magnetostatics Interpretation: If we imagine one loop has current flowing through it, the Gauss linking integral essentially applies Ampère's law on the other loop, using the magnetic field generated by the current loop (given to us by the Biot–Savart law), to compute the total current enclosed. Similar to the winding number computation in [Barill et al. 2018], we note that the first term of the integrand is the gradient of the Laplace Green’s function \(G(r_1, r_2) = -1/(4\pi |r_1 - r_2|)\); in magnetostatics, the vector potential \(A\) of a monopole current source at \(r_1\) with current \(I\) and length vector \(s_1\) is \(\mu_0 I s_1 G(r_1, r_2)\), and the magnetic field at \(r_2\) due to \(r_1\) is \(B = \nabla_r \times A = -\mu_0 I s_1 \times \nabla_r G(r_1, r_2)\). Applying Ampère’s law gives us the linking integral. This interpretation will later allow us to apply computational tools for multipole expansions.
modeling applications in graphics that are more suitable for 1D elements include chains and chainmail [Bergou et al. 2008; Mazhar et al. 2016; Tasora et al. 2016], threads [Bergou et al. 2010], knots [Bergou et al. 2008; Harmon et al. 2011; Spellmann and Teschner 2008; Yu et al. 2020], hair [Chang et al. 2002; Kaufman et al. 2014; Selle et al. 2008], and necklaces [Guibas et al. 2002].

To detect pull-through violations of edge–edge contacts, one can locally verify the sign of the volume of the tetrahedron (computed as a determinant) spanned by the two edges. To generalize this local notion to arbitrary closed curves, one can compute the linking number between the two curves. Interestingly, Gauss’s integral form of the linking number uses, in the integrand, this exact determinant, applied on differential vectors, but scaled so that it is the degree, or signed area of the image, of the Gauss map of the link.

In graphics, some works [Dey et al. 2013, 2009] compute the linking number, by projecting and computing crossings, to determine if a loop on an object’s surface is a handle versus a tunnel, which is useful for topological repair and surface reparameterization. Another work [Edelsbrunner and Zomorodian 2001] computes the linking numbers of simplicial complexes within DNA filtrations, by using Seifert surface crossings (which are generated from the filtration process), to detect nontrivial tangleing in biomolecules. In the computational DNA topology community, many papers [Clauvelin et al. 2012; Fuller 1978; Klenin and Langowski 2000; Krajina et al. 2018; Sierzega et al. 2020] use topological invariants such as the linking number, twist, and writhe to understand DNA ribbons; in particular, Berger [2009] summarizes how to compute these quantities, and [Arai 2013; Bertolazzi et al. 2019] present more robust exact expressions for line segment curves using the signed solid angle formula [Van Oosterom and Strackee 1983]. Moore [2006] also discusses how to subdivide spline curves for computational topology. A few works in animation and robotics [Ho and Komura 2009; Ho et al. 2010; Pokorny et al. 2013; Zarubin et al. 2012] also use the linking integral as a topological cost in policy optimization, e.g., for grasping. However, none of these works propose using acceleration structures to speed up linking-number computation for general closed curves.

The linking integrand uses the Green’s function for the Laplace problem in a Biot–Savart integral, and there are numerous Fast Multipole libraries for the Laplace problem, including FMM3D [Cheng et al. 1999] and FMMTL [Cecka and Layton 2015]. FMMTL provides an implementation of the Biot–Savart integral and has recently been used to simulate ferrofluids in graphics [Huang et al. 2019]. Another library [Burtscher and Pingali 2011] implements a very efficient, parallel Barnes–Hut algorithm for the N-Body Laplace problem on the GPU.

### 3.1 Method Overview

#### 3.1.1 Potential Link Search (PLS) (§3.2)

This stage exploits the fact that any pair of loops with disjoint bounding volumes have no linkage. It takes a sequence of loops and produces a list of potentially linked loop pairs by using a bounding volume hierarchy (BVH).

#### 3.1.2 Discretization (§3.3)

This stage discretizes the input curves, if they are not in line segment form, into a sequence of line segments for each loop, taking care to ensure that the process is link homotopic (i.e., if we continuously deform the original curves into the final line segments, no curves cross each other) and thus preserves linking numbers.

#### 3.1.3 Linking Number Computation (§3.4)

This stage computes the linking matrix for the potentially linked loops, and can be performed with many different linking-number methods.

### Fast Verification

Verification of the linking matrix can be done exhaustively by computing and comparing the full linking matrix computed using Algorithm 1. However, in some applications it is sufficient to know that any linkage failed, e.g., so that a simulation can be restarted with higher accuracy. In such cases, an “early exit” strategy can be used to quickly report topological failure by exiting when any linking number differs from the input.

### Bounding Volumes

All bounding volumes in our method are convex. The convexity provides guarantees in §3.2 and §3.3.
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1. Potential Link Search

When the number of loops, \( L \), is large, computing the linking integral between all loop pairs can be expensive, as there are \( \frac{L(L-1)}{2} \) pairs. For this stage of the method, we input a list of loops \( \{\gamma_i\} \), and output a list of potentially linked loop pairs, \( P = \{(i, j)\} \) (where \( i < j \) are loop indices), which gets passed to the next stages (§3.3 and §3.4). All other loop pairs must have 0 linkage.

We exploit the fact that if two loops have a nonzero linkage, then their bounding volumes, which are convex, must overlap. Therefore, we precompute an axis-aligned bounding box (AABB) for each loop, and build an AABB-tree of loops. Edelsbrunner and Zomorodian [2001] also use bounding boxes to cull the list of potentially linked loops. For any two loops with overlapping bounding volumes, we add the pair to a list of potentially linked loops. See Algorithm 2 for pseudocode, and Figure 4 for an illustration.

ALGORITHM 2: Potential Link Search

```
Input : \( \Lambda = \{\gamma_i\} \), a list of looped curves.
Output: \( P = \{(i, j)\} \), a list of pairs of loop indices that potentially link.
// This function returns a list of pairs of loops that have overlapping bounding boxes.

Function PotentialLinkSearch(\( \Lambda \)):
    Boxes ← {};
    for \( i ← 0 \) to \( |\Lambda| - 1 \) do
        Boxes\[i\] ← ComputeBoundingBoxOfLoop(\( \gamma_i \));
    end
    Tree ← BuildBVH(Boxes);
    \( I \) ← GetIntersectingBoxes(Tree, Tree);
    \( P \) ← \{(i, j) : (i, j) ∈ I \land (i < j)\};
    return \( P \);
```

3.3 Discretization

Given a set of input loops discretized into polynomial splines or line segments and a set of overlapping loop pairs, our discretization step computes a set of corresponding loops discretized into homotopically equivalent line segments, so that we can use the method in the next stage (Section 3.4) to compute the exact linkage. See Figure 5 for an illustration. While there are many ways to refine spline segments into line segments [De Casteljau 1959; Micchelli and Frautzsch 1989], many applications simply refine them to a fixed tolerance. Our method is spatially adaptive and refines the spline further only when curves from different loops pass closely by each other. The discretization routine takes advantage of the fact that if two convex volumes don’t overlap, then two curves bounded by these volumes can be deformed into two line segments within these volumes without crossing each other. Example pseudocode is provided in Algorithm 3, and the routine is as follows.

Suppose every spline segment is parameterized in \( t \) as \( p(t) \). We take multiple passes. At the beginning, all spline segments are unprocessed. In each pass, we start with a list of unprocessed spline segments.
ALGORITHM 3: Discretization

Input: \( \Lambda = \{ \gamma_l \} \), a list of looped curves, each of which can be a cubic spline or a polyline;
\( P = \{(i_k, j_k)\} \), a list of pairs of loops that potentially link.
Output: \( \{ \Gamma_l \} \), a list of polylines corresponding to the curve list.

This function homotopically refines input curves into polylines.

Function Discretize(\( A, P \)):

1. Initialize the sets of segments:
   \( L \leftarrow |A|; \quad \text{UnprocessedSegments} \leftarrow \{ \{ \} \}; \quad \text{L sets of segments.} \)

2. Define auxiliary parameters:
   \( \bar{\varepsilon} \leftarrow \text{the average coordinate magnitude;} \quad \varepsilon \leftarrow \text{machine epsilon;} \)

3. Ensure all segments have finite, nonzero length:

4. Parallel process for \( i \leftarrow 0 \) to \( L - 1 \) do
   - \( \Gamma_i \leftarrow \{ \}; \quad \text{UnprocessedSegments}[i] \leftarrow \{ \}; \)
   - For each Curve segment \( s \in \gamma_l \) do
     - Compute bounding box:
       \( s.\text{ComputeBoundingBox}(); \)
     - If \( \text{DiameterOfBoundingBox}(s) < \varepsilon \bar{\varepsilon} \) then
       - \text{return Error("Segment has zero-length segments.");}
     - Otherwise
       - \text{UnprocessedSegments}[i].InsertSegment(s); \)
   - \text{Trees} \leftarrow \{ \}; \quad \text{One tree for each curve.}
   - While (At least one UnprocessedSegments[i] is not empty) do
     - Parallel process for \( i \leftarrow 0 \) to \( L - 1 \) do
       - \text{(i) Build an AABB-tree for each curve.}
       - \text{Trees}[i] \leftarrow \text{BuildBVH(UnprocessedSegments[i]);}
     - Parallel process for \( i \leftarrow 0 \) to \( L - 1 \) do
       - \text{CurrentSegments} \leftarrow \text{UnprocessedSegments}[i];
       - \text{UnprocessedSegments}[i] \leftarrow \{ \};
       - For each \( (i, j) \in P \) do
         - \text{(ii) Traverse BVHs, and (iii–v) subdivide the input curves if necessary.}
       - For each \( (s, \{ \} \in \text{GetIntersectingBoxes}(\text{Trees}[j], \text{Trees}[j])); \)
         - For each \( r, t \in \text{SplitSegmentIntoHalves}(s); \)
           - \text{r. ComputeBoundingBox();}
           - \text{t. ComputeBoundingBox();}
           - If \( \text{DiameterOfBoundingBox}(r \text{ or } t) < \varepsilon \bar{\varepsilon} \) then
             - \text{return Error("Curves \( i \) and \( j \) intersect.");}
           - \text{UnprocessedSegments}[i].InsertSegment(r, t);}
       - \text{CurrentSegments} \leftarrow \{ \};
     - End
   - End
   - \text{(vi-vii) Collect remaining segments into results.}
   - \( \Gamma_i \leftarrow \Gamma_i \cup \text{CurrentSegments}; \)
   - End
   - End
   - End
   - End
   - End
   - End

3.4 Linking Number Computation

Given a set of loops discretized into line segments, and a set of loop pairs to check, we wish to compute the respective linkages and output a linking matrix. We present several classes of methods here based on different prior work, and our results and analysis section (§4) will discuss the relative advantages of each method.

When running this stage on the CPU, we parallelize across the list of potentially linked loop pairs and have each thread compute its own linking number individually. We only use the parallelized version of these methods if we have very few or only one loop pair, which can arise when there are very few loops. For the GPU-accelerated methods, we just iterate through the list of loop pairs and launch the GPU kernels sequentially.

3.4.1 Method 1: Count Crossings (\( CC_{ana} \) and CC). The first approach is, given a pair of loops, find a regular projection and use it to count crossings. We modify the implementation provided with [Dey et al. 2013] for computing the linking number, and evaluate two optimized methods (\( CC_{ana} \) and CC). Their projection code takes three steps to heuristically find a regular projection: first, it uses approximate nearest neighbors (ANN) to estimate a projection direction away from all line-segment directions among the two loops,
defining an initial frame whose z axis will be the direction of projection. Next, it corrects the frame by rotating about the z axis into the direction such that all the projected segments would be oriented, at minimum, as far away as possible from vertical (y) or horizontal (x) directions. Thirdly, it rotates the frame about the y axis so that all the projected segments would be, at minimum, the farthest from being degenerate (aligned with the z axis). Each of these steps take \(O(N \log N)\) time to sort the projected segment angles and perform the maximin. Afterwards, they project the segments onto the XY plane, and then perform an \(O(N_k N_l)\) (where \(N_k\) and \(N_l\) are the numbers of segments in each input loop) brute-force intersection check between all segment pairs. The intersection check uses 2D exact line segment queries from CGAL [Brönnimann et al. 2021] and exits with an error when degeneracies are detected.

While this procedure is sufficient for the loops generated from the surface mesh analysis in [Dey et al. 2013], our linking number computation must support larger inputs. We accelerated the intersection search by building a BVH for one of the two loops when their average loop size exceeds 750 segments, and evaluating segments from the other loop against this tree in parallel. We refer to this modified method as CC\textsubscript{ann}. The ANN call also introduces a large overhead, so, after experimentation, we simply replace the initial frame with a randomized frame. This greatly speeds up performance for many examples, and we report the performance for this method (which we call CC) as well as robustness tests in the results §4.1.2. See Algorithm 4 for pseudocode.

ALGORITHM 4: ComputeLink: Counting Crossings

**Input:** \(\{l_i\}, \{k_j\}\), the two polyline loops, each represented as a list of vertices. Each vertex is a column vector.

**Output:** \(\lambda\), the linking number

// This function computes the linking number between two closed polylines by counting crossings and uses a BVH.

**Function** ComputeLink(\(\{l_i\}, \{k_j\}\)):

\[
\lambda \leftarrow 0; \\
N_k \leftarrow |\{k_j\}|; \\
// Find a regular projection frame. \\
F \leftarrow (k\hat{y}, k\hat{z}) \leftarrow GenerateRegularProjectionFrame(\{l_i\}, \{k_j\}); \\
// Rotate into the frame. \\
\{l_i\} \leftarrow \{FTl_i\}; \quad \{k_j\} \leftarrow \{FTk_j\}; \\
// Build a tree from the projected segments of \(\{l_i\}\). \\
tree \leftarrow BuildBVH(\{(l_{i_x}, l_{i_y}), (l_{j_x}, l_{j_y})\})); \\
// Sum the orientations of all crossings \\
parallel for i = 0 to N_k do \\
\quad ProjectedSegment \leftarrow ((k_{l_{i_x}}, k_{l_{i_y}}), (k_{l_{i_x}+1}, k_{l_{i_y}})); \\
\quad foreach j \in tree.Query(ProjectedSegment) do \\
\quad\quad SegmentI, SegmentJ \leftarrow (k_{l_{i_x}}, l_{j_{x}}), (l_{j_{y}}, k_{l_{i_y}}); \\
\quad\quad \lambda \leftarrow \lambda + 0.5 \text{Orientation}(SegmentI, SegmentJ); \\
end \\
return \lambda;
\]

3.4.2 Method 2: Direct Summation (DS). This approach simply uses the exact double-summation formula for Gauss’s integral from [Arai 2013] (which uses the signed solid angle formula [Van Oosterom and Strackee 1983]), produced above as (4). That is, for loops \(y_1, y_2\), consisting of \(N_l, N_k\) line segments enumerated by \(j, i\) respectively,

\[
\lambda(y_1, y_2) = \sum_{i}^{N_l} \sum_{j}^{N_k} \lambda_{ji}. \tag{5}
\]

See Algorithm 5 for a simple implementation. Unfortunately this approach computes \(2N_l N_k\) arcantachs for loops of sizes \(N_l\) and \(N_k\), which is expensive for large loops.

When this is a single-threaded computation, the approach from [Bertolazzi et al. 2019] removes all arcantachs by using angle summations (counting negative x-axis crossings) instead. We use a modified version of their approach, and for robustness, we add a single arctan at the end to compute the remainder after the angle summation. We only compute one triple scalar product per segment pair because \(a \cdot (b \times c) = c \cdot (b \times a)\). See Appendix B.1 for details on how we multithread this on the CPU and the GPU.

ALGORITHM 5: ComputeLink: Direct Summation.

**Input:** \(\{l_i\}, \{k_j\}\), the two polyline loops, each represented as a list of vertices. Each vertex is a column vector.

**Output:** \(\lambda\), the linking number

// This function computes the linking number between two closed polylines using the [Arai 2013] expression.

**Function** ComputeLink(\(\{l_i\}, \{k_j\}\)):

\[
\lambda \leftarrow 0; \\
N_l \leftarrow |\{l_i\}|; \quad N_k \leftarrow |\{k_j\}|; \\
// Find a regular projection frame. \\
F \leftarrow (k\hat{y}, k\hat{z}) \leftarrow GenerateRegularProjectionFrame(\{l_i\}, \{k_j\}); \\
// Rotate into the frame. \\
\{l_i\} \leftarrow \{FTl_i\}; \quad \{k_j\} \leftarrow \{FTk_j\}; \\
// Build a tree from the projected segments of \(\{l_i\}\). \\
tree \leftarrow BuildBVH(\{(l_{i_x}, l_{i_y}), (l_{j_x}, l_{j_y})\})); \\
// Sum the orientations of all crossings \\
\parallel for i = 0 to N_k do \\
\quad ProjectedSegment \leftarrow ((k_{l_{i_x}}, k_{l_{i_y}}), (k_{l_{i_x}+1}, k_{l_{i_y}})); \\
\quad foreach j \in tree.Query(ProjectedSegment) do \\
\quad\quad SegmentI, SegmentJ \leftarrow (k_{l_{i_x}}, l_{j_{x}}), (l_{j_{y}}, k_{l_{i_y}}); \\
\quad\quad \lambda \leftarrow \lambda + 0.5 \text{Orientation}(SegmentI, SegmentJ); \\
\quad end \\
\return \lambda; \\
\]

3.4.3 Method 3: Fast Multipole Method (FMM). We can also use external FMM libraries (provided by FMML [Cecka and Layton 2015] or FMM3D [Cheng et al. 1999]) to compute the linking integral using fast summation approximations. In particular, if we represent \(y_2\) as a source current, the Fast Multipole Method on the Biot–Savart integral will approximately evaluate, at each target point \(r\), the field

\[
f(r) = \frac{1}{4\pi} \int_{y_2} \frac{dr_2}{|r - r_2|^3}. \tag{6}
\]

The final linking number is then \(\lambda(y_1, y_2) = \int_{y_1} f(r_1) \cdot dr_1\). In this notation, loop \(y_2\) is the "source," and \(y_1\) is the "target."

Because many of these libraries do not directly support finite line-segment inputs, we use segment midpoints for the source and target points. After evaluation by the library, we compute a set of finite-segment corrections for close segment pairs poorly approximated by

ACM Trans. Graph., Vol. 40, No. 4, Article 106. Publication date: August 2021.
midpoint samples, and add the correction. While we already know the exact evaluation expressions for line-segment pairs and could have modified libraries to use them, our goal is to make no modifications to existing FMM libraries that operate on point samples, so that our results can be easily reproduced and improvements to FMM libraries on newer hardware can be easily incorporated. This is why we perform post-evaluation correction rather than directly modify the FMM kernels and tree building structures.

Given the aforementioned loop discretizations of \( y_1, y_2 \) in the direct evaluation section, denote the midpoint of each line segment as \( r_j \) or \( r_i \), respectively, and the displacement vector (difference of endpoints) as \( s_j, s_i \), respectively. FMM TL directly provides a Biot–Savart implementation, so we simply have to pass in the line-segment displacements \( s_i \) as the sources, with positions \( r_i \), and FMM TL computes the fields \( f_j = f(r_j) \) at positions \( r_f \).

The linking number can then be computed from the field \( f_j \) using

\[
\lambda(y_1, y_2) = \sum_j s_j \cdot f_j. \tag{7}
\]

To reduce the duplicate work of building the FMM trees, we batch evaluation points for each loop. That is, given source loop \( y_1 \), we concatenate the target evaluation points \( r_f \) for all target loops to compute \( F_j \) for all target loops, and then compute each linking-number-sum individually.

When a library does not provide a native Biot–Savart implementation, we can alternately pass in a list of three-vector sources \( \{s_i\} \) into a Laplace FMM. The gradient output gives us Eq. (6), except with an outer instead of a cross product. The output is a 3 x 3 tensor \( C_j = \sum_i s_i \otimes \nabla G(r_j, r_f) \) at each evaluation point \( r_f \), and we can simply grab the field \( f_j \) from the definition of cross product:

\[
f_j = (C_{j23} - C_{j32}, C_{j31} - C_{j13}, C_{j12} - C_{j21}). \tag{8}
\]

After the FMM library computation, we add a finite-segment correction, because we passed in our line segments as point sources and targets. We pick a distance ratio \( \beta \) and apply this correction to all segment pairs that are less than \( \beta(l_1 + l_2)/2 \) apart, where \( l_1, l_2 \) are their segment lengths. Specifically, before the FMM calls we (i) build a bounding volume for each segment, and (ii) dilate each bounding volume by \( \beta l/2 \), where \( l \) is the segment length. We then (iii) build a BVH of segment bounding volumes for each loop. For any loop pair, (iv) traverse their BVH trees, and (v) accumulate the difference between expression (4) and its approximation (2) for every pair of overlapping segment bounding volumes. We then (vi) add this correction to the original FMM result. In our implementation, we parallelize steps (i–iii) by loop and steps (iv–v) by loop pair, with a reduction at the end. We used a finite-segment correction because the correction is faster to compute in practice than using denser sample points along each segment for the FMM input; in our results in §4, the correction takes a small fraction (< 1/8) of the runtime.

### 3.4.4 Method 4: Barnes–Hut (BH)

Similar to [Barill et al. 2018], simple Barnes–Hut trees [Barnes and Hut 1986] can be sufficient for approximately evaluating the integral using fast summation without the overhead of the full FMM. For each loop pair, the FMM computes a field at every “target” point; however, we only need a single scalar overall. Since each loop can participate in many loop pairs, we can instead precompute a tree for each loop. Then for each loop pair, we traverse both trees from the top down using Barnes–Hut; this can save effort especially when the loops do not hug each other closely and multiple loop pairs use the same target loop.

The Barnes–Hut algorithm takes advantage of a far-field multipole expansion. Given two curves \( y_1, y_2 \), parameterize \( y_1 \) as \( r_1 = r_1(t) \) and \( y_2 = r_2(t) \), and denote \( dr_1/dt \) and \( dr_2/dt \) by \( r_1' \) and \( r_2' \). Suppose also that we are only integrating the portions of \( y_1 \) near \( r_1 \), and the portions of \( y_2 \) near \( r_2 \). Also, let \( \nabla \) denote differentiation with respect to \( r_2 \) (note that \( \nabla_r G \) is just \( -\nabla_r G \otimes \) denote a tensor (outer) product, and - denote an inner product over all dimensions. Taylor expanding (1) with respect to \( r_1 \) and \( r_2 \), we have

\[
\lambda = -\int ds \, \left( r_1' \times r_2' \right) \cdot \nabla G(r_1, r_2). \tag{9}
\]

\[
\lambda = -\int ds \, \left[ (r_1' \times r_2') \cdot \nabla G(r_1, r_2) + \left( (r_1' \times r_2') \otimes \left( -(r_1 - r_1') + (r_2 - r_2') \right) \right) \cdot \nabla^2 G(r_1, r_2) + \frac{1}{2} \left( (r_1' \times r_2') \otimes \left( -(r_1 - r_1') + (r_2 - r_2') \right) \otimes \left( -(r_1 - r_1') + (r_2 - r_2') \right) \right) \right] \cdot \nabla^2 G(r_1, r_2) + O(\left| r_1 - r_2 \right|^5) \right]. \tag{10}
\]

Expanding (10) and applying some algebra, every term can be split into a product of two separate integrals, one for the \( r_1 \) factors and one for the \( r_2 \) factors. These integrals, known as moments, can be precomputed when constructing the tree. For each node on both curves, we precompute the following moments:

\[
c_{M} = \int r' \, ds, \tag{11}
\]

\[
c_{D} = \int (r' - \hat{r})^T \, ds, \tag{12}
\]

\[
c_{Q} = \int r' \otimes (r - \hat{r}) \otimes (r - \hat{r}) \, ds. \tag{13}
\]

Having just the \( c_{M} \) moment at each node is sufficient to give us the first term in (10). Adding the \( c_{D} \) and \( c_{Q} \) moments gives us the second and third terms respectively. The derivatives of \( G(\hat{r}_1, \hat{r}_2) \), how we compute these moments, and how we parallelize the Barnes–Hut on the CPU and GPU (using the Barnes–Hut N-Body implementation from [Burtscher and Pingali 2011]) are in Appendix B.2.

We set a \( \beta \) tolerance parameter, similar to most Barnes–Hut algorithms, to distinguish between the near field and the far field: two nodes at locations \( p \) and \( q \) are in the far field of each other if \( |p - q| > \beta \) times the sum of the bounding-box radii of the two nodes. Starting from both tree roots, we compare their bounding boxes against each other. If the nodes are in the far field, we use the precomputed moments and evaluate the far-field expansion; otherwise, we traverse down the tree of the larger node and check its children. If both nodes are leaves, we use the direct arctangent expression (4). See Algorithm 6 for a summary.

In our implementation, we first run Barnes–Hut with the second-order expansion in (10) (using all three terms) with \( \beta_{\text{init}} = 2 \). Unlike other problems that use Barnes–Hut, the linking number certificate requires the same absolute error tolerance, regardless of the magnitude of the result. In large examples, some regions can induce a strong magnetic field from one loop on many dense curves from the
We perform all our experiments on a single 18-core Intel® Xeon® CPU E5-2697 v4 @ 2.30GHz with 64GB of RAM, and the GPU versions are evaluated on an NVIDIA GeForce GTX 1080. In Table 1, we evaluate the methods on inputs with varying numbers of loops, loop sizes, and loop shapes, and report runtimes and accuracies of the linkage values. For CPU Barnes–Hut, we used $\beta_{\text{init}} = 2$, $\beta_{\text{max}} = 10$, and $E_{\text{target}} = 0.2$. We chose $\beta_{\text{init}}$ so that the absolute errors were well below $10^{-2}$ for most examples. For Barnes–Hut on the GPU, since we only evaluated up to dipole moments with no error estimation, we used $\beta = 4$ if $N_1N_2 < 10^5$ and $\beta = 16$ otherwise. We also included the runtime and accuracies achieved using direct summation, the FMFMTL library [Cecka and Layton 2015; Cheng et al. 1999], and the modified count-crossings method from [Dey et al. 2013] accelerated with parallelism and a BVH. The number of discretized line segments for each input is listed in Table 2. In most cases, Discretization did not significantly increase the segment count (The exception is the chevron $3 \times 3$, which uses a special procedure defined in §4.3.5 where each curve participates in two virtual loops.).

In general, the Counting Crossings (CC) and the two Barnes–Hut (BH) implementations give the fastest runtimes, with the CC, BH, and BHG all taking under 2 seconds for all examples shown in the video (first 11 rows of Table 1). As a result, we believe either a CC or a BH method is performant enough for most applications.

The GPU implementations perform best when they have very few, or one, pairs of large loops to evaluate, and this can be seen in the ribbon, $\lambda = 10^5$ torus, and low-frequency (low $\nu$) woundball. In particular, the smaller double-helix ribbons ($N_1 = 200000$) examples illustrate that the GPU Direct Summation (DS) is a 370x speedup over the CPU DS, and both DS implementations have very little error (mostly under $10^{-5}$ for the single-precision GPU implementation). The Barnes–Hut CPU implementation benefits from traversing two trees, a quadrupole expansion, as well as error estimation, allowing it to perform almost as well as the BH GPU on a wide range of examples that involve more than one loop pair. However, both Barnes–Hut implementations suffer when the segment lengths are relatively long (compared to the distance between curves), which is true in the higher-$\nu$ woundball (a higher $\nu$ has longer segments in order to traverse the ball more times), because their limited far-field expansion requires a relatively high $\beta$. This effect is worse for the BH GPU on the high-$\nu$ woundball, which is slower than the BH CPU here because our GPU version only implements monopole and dipole terms and thus requires an even higher $\beta$ to achieve the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Large-Input Stress Tests: Synthetic examples were generated for (Top Left) DNA-like double-helix ribbons, (Top Right) thick square links, (Bottom Left) tori, and (Bottom Right) woundballs, of variable sizes. The "thick square links" example has $L^2/a$ links each with $\lambda = 1$, and the torus has a single $\lambda$ equal to the product of the number of toroidal periods in the green loop and poloidal periods in the blue loop. We also generated a $\lambda = 0$ torus where both loops simply alternate directions every period, and it also has 2 more larger copies of the blue loop. The woundball has two similar concentric spherical curves, one winding a ball half the size of the other, and a parameter $\nu$ indicating the number of times each curve crosses the North pole.}
\end{figure}

\section{4 RESULTS AND ANALYSIS}

We now compare the many aforementioned methods on several benchmarks and applications. Please see the supplemental video for all animation results.

\subsection{4.1 Experiments and Verification}

We perform all our experiments on a single 18-core Intel® Xeon® CPU E5-2697 v4 @ 2.30GHz with 64GB of RAM, and the GPU versions are evaluated on an NVIDIA GeForce GTX 1080. In Table 1, we evaluate the methods on inputs with varying numbers of loops, loop sizes, and loop shapes, and report runtimes and accuracies of the linkage values. For CPU Barnes–Hut, we used $\beta_{\text{init}} = 2$, $\beta_{\text{max}} = 10$, and $E_{\text{target}} = 0.2$. We chose $\beta_{\text{init}}$ so that the absolute errors were well below $10^{-2}$ for most examples. For Barnes–Hut on the GPU, since we only evaluated up to dipole moments with no error estimation, we used $\beta = 4$ if $N_1N_2 < 10^5$ and $\beta = 16$ otherwise. We also included the runtime and accuracies achieved using direct summation, the FMFMTL library [Cecka and Layton 2015; Cheng et al. 1999], and the modified count-crossings method from [Dey et al. 2013] accelerated with parallelism and a BVH. The number of discretized line segments for each input is listed in Table 2. In most cases, Discretization did not significantly increase the segment count (The exception is the chevron $3 \times 3$, which uses a special procedure defined in §4.3.5 where each curve participates in two virtual loops.).

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\begin{algorithm}
\caption{ComputeLink: Barnes–Hut}
\begin{algorithmic}
\State \textbf{Input}: $\text{tree1, tree2}$, the root nodes of the two trees, with $r$ the node’s center, $R$ the bounding box radius, and $E_M$, $C_D$, $C_Q$ the moments, precomputed at each node.
\State \textbf{Output}: $\lambda$, the linking number
\State // This function computes the linking number between two closed polylines using the Barnes–Hut algorithm with a quadrupole far-field expansion.
\Function{ComputeLink}{tree1, tree2}:
\If{($(\text{tree1}.r-\text{tree2}.r) > \beta(\text{tree1}.R+\text{tree2}.R)$)}
\State return FarFieldEvaluation(tree1, tree2); // Use (10)
\ElsIf{(tree1 and tree2 have no children)}
\State return DirectEvaluation(tree1.Segment, tree2.Segment);
\ElsIf{(tree1.R > tree2.R and tree1 has children)}
\State return $\sum_c \text{ComputeLink}(\text{tree1}.child[c], \text{tree2})$;
\Else
\State return $\sum_c \text{ComputeLink}(\text{tree1}, \text{tree2}.child[c])$;
\EndIf
\EndFunction
\end{algorithmic}
\end{algorithm}
Table 1. Linkage Computation for Various Inputs: In this table, \(N\) is the total number of input curve segments, \(L\) is the number of input loops, and \(P\) is the computed number of potentially linked loop pairs. The first 11 models correspond to animated results shown in the supplemental video, and the rest are stress tests depicted in Fig. 6. The sweater and glove models originate from [Yuksel et al. 2012], with “Sweater” referring to “sweater flame ribbing.” “PLS Time” is the Potential Link Search Time and “Dscr Time” is the Discretization time; DS is direct summation; CC and CC for counting crossings with ANN and randomized direction initialization, respectively; BH for Barnes–Hut, and FMM for Fast Multipole Method (using FMMTL with batched target points; FMM3D is slower in almost all cases). DSG and BHG run the DS and BH, respectively, on the GPU. A “N/A” indicates the run took longer than a half hour or did not fit in memory. \(\lambda\) indicates the linkage between curves in a model, and \(\nu\) is a winding number parameter explained in Fig. 6.

| Model                  | \(N\)  | \(L\)  | \(P\)  | PLS Time | Dscr. Time |
|------------------------|-------|--------|--------|----------|------------|
| Alien Sweater (Initial)| 139506| 146    | 1335   | 0.013    | 0.059      |
| Alien Sweater (Final)  | 139506| 146    | 2426   | 0.017    | 0.050      |
| Sheep Sweater          | 729628| 549    | 4951   | 0.024    | 0.152      |
| Sweater                | 271902| 253    | 2985   | 0.015    | 0.120      |
| Glove                  | 58537 | 70     | 1022   | 0.012    | 0.046      |
| Knit Tube (Initial)    | 18228 | 39     | 233    | 0.012    | 0.032      |
| Knit Tube (Final)      | 18228 | 39     | 180    | 0.012    | 0.082      |
| Chainmail (Initial)    | 211680| 14112 | 18781  | 0.028    | 0.070      |
| Chainmail (Final)      | 211680| 14112 | 61417  | 0.056    | 0.086      |
| Chevron 3 × 3          | 11741 | 32     | 58     | 0.013    | 0.049      |
| Rubber Bands           | 51200 | 1024   | 20520  | 0.018    | 0.035      |
| Double Helix Ribbon \(\lambda = 10\) | 200000 | 2     | 1      | 0.018    | 0.076      |
| Double Helix Ribbon \(\lambda = 10^3\) | 200000 | 2    | 1      | 0.016    | 0.112      |
| Double Helix Ribbon \(\lambda = 10^4\) | 2E7    | 2     | 1      | 0.465    | 6.816      |
| Thick Square Link      | 500000| 1250   | 41E5   | 0.054    | 0.146      |
| Thick Square Link      | 4E6   | 5000   | 6.4E6  | 0.531    | 1.874      |
| Torus \(\lambda = 10^6\) | 2E7    | 2     | 1      | 0.456    | 7.027      |
| Torus \(\lambda = 0\)  | 4E7   | 4      | 6      | 0.519    | 7.698      |
| Woundball \(\nu = 10^3\) | 1E6   | 2     | 1      | 0.048    | 0.387      |
| Woundball \(\nu = 10^4\) | 2E6   | 2     | 1      | 0.056    | 0.756      |

Table 2. Discretization Results: This table shows, for the first 10 models in Table 1, how many line segments each model was discretized into. All other models were input as line segments separated enough that Discretization did not change the segment count. Most of these 10 models ended with only a little more than one segment per control point, with the exception of the twisted knit tube and the chevron pattern.

| Model                  | \(N\)  | \(N_D\) |
|------------------------|-------|--------|
| Alien Sweater (Initial)| 139506| 158708 |
| Alien Sweater (Final)  | 139506| 143228 |
| Sheep Sweater          | 729628| 801447 |
| Sweater                | 271902| 296813 |
| Glove                  | 58537 | 69598  |
| Knit Tube (Initial)    | 18228 | 18941  |
| Knit Tube (Final)      | 18228 | 27625  |
| Chainmail (Initial)    | 211680| 211680 |
| Chainmail (Final)      | 211680| 212552 |
| Chevron 3 × 3          | 11741 | 32804  |

same accuracy. In contrast, the FMM excels on the high-\(\nu\) woundball, giving the fastest performance, because it uses arbitrary-order expansion terms as needed. Both BH and FMM suffer from hard-to-predict absolute errors, especially in examples with high \(\lambda\). In general, CC appears to perform the best in the most scenarios, and because it tabulates an integer, it has 0 error in the runs we observed.

4.1.1 Barnes–Hut \(\lambda\) behavior: We illustrate how varying \(\beta\) can impact accuracy in Figure 7. The error appears to behave as \(\beta^{-3}\) for the first-order (dipole) expansion, whereas for the second-order (quadrupole) expansion the error appears to behave as \(\beta^{-4}\).

4.1.2 Numerical conditioning and robustness: For direct summation (and lowest-level evaluations of the tree algorithms), we use the expression from Arai [2013] (reproduced in §2.1 as (4)), because it is well defined over the widest range of input. In particular, it is only singular when the two line segments intersect or collinearly overlap, or if either has zero length. Our method already detects and flags these conditions in the Discretization step. Furthermore, the signed solid angle formula is widely used due to its simplicity and stability [Van Oosterom and Strackee 1983].
All methods compute using double precision when implemented on the CPU, while the GPU implementations use single precision.

To learn more about the robustness of the projection in the count-crossings (CC) method, we tested it, using random initial frames before the rotation step, 50,000 times on the compressed chainmail (final), and 5,000 times on the alien sweater (initial), and, based on their $P$ values in Table 1, this sums up to 3.08 billion projections. If our implementation encounters a degeneracy, the method would fail and exit. For each loop pair, we recomputed the linking number with a new projection every trial, and it completed with the correct result every time. We also tested it with very large inputs such as the dense tori and woundballs (Fig. 6), which span a large range of angles, and the linking number was successfully computed, albeit the crossings computation was very slow for the unlinked $\lambda = 0$ torus. This robustness is likely because degeneracies are unlikely to appear with this input size in double precision. That said, we do not guarantee with our implementation that it is able to find a non-degenerate, regular projection on every input. If this is a concern, the Gauss summation implementations such as the Barnes–Hut algorithm, which do not require a projection, can also be used on most inputs with comparable speed.

### 4.2 Performance

For all results shown in the video (first 11 rows of Table 1), PLS and Discretization took under 200 ms combined. For the third stage, we plot the runtimes versus discretized model size in Figure 8. Both the count-crossings and Barnes–Hut methods significantly outperform direct summation, and also outperform FMM in most cases. We also tested four classes of large input as stress tests; see Fig. 6 for an illustration.

#### 4.2.1 Usual-case theoretical estimates

Similar to many tree-based methods, the performance depends on the distribution of input, and certain inputs can severely degrade performance. For inputs with relatively “short” segments (i.e. segments are short compared to the spacing between curves) and uniformly sized loops, we expect PLS to take $O(L \log L)$ time and Discretization to take $O(N \log N)$ time, where $L$ is the number of loops and $N$ is the number of input segments. The Barnes–Hut, Count-Crossings, and FMM computations are all expected to take $O(PN_L \log N_L)$ time, where $P$ is the number of potentially linked pairs and $N_L$ approximates the number of segments per loop (FMM is not linear in $N_L$ because we compute the finite-segment correction, although for all the inputs we reported in Table 1, the finite-segment correction took under 1/6 of the FMM runtime). For chainmail, the total simplifies to $O(C \log C)$ where $C$ is the number of rings, and for most stitch-mesh knits patterns where each row interacts with a small number of other rows, this simplifies to $O(R \log R + N \log N_L)$ where $R$ is the number of rows.

However, if segments are extremely long compared to the distance between curves, which can happen after a simulation has destabilized, every algorithm can slow down to $O(N^2)$ performance to finish. This is where the early-exit approach can be a huge gain; furthermore, if our method runs periodically during a simulation, topology violations can be detected long before this occurs, as we will show in the Knit Tube Simulation (Figure 13).

### 4.3 Applications

In this section we demonstrate the versatility of our method across several applications. In most scenarios (except Example 4.3.1) we first compute the linking matrix of the initial model, and then compare it against the linking matrices of the deformed models.

#### 4.3.1 Yarn model analysis

Many yarn-level models in graphics and animation are represented as a set of closed curves; a major example is the models generated from StitchMesh [Yuksel et al. 2012]. Implicit (and/or implicit constraint direction) yarn simulations solve smaller systems when the model uses closed curves, and the model stays intact without the requirement of pins. Our method can aid in the
Fig. 9. **Yarn Model Analysis**: We examine the "sweater flame ribbing" yarn model from Stitch Mesh [Wu and Yuksel 2017; Yuksel et al. 2012], which was provided in relaxed form. This model contains small yarn loops to join rows with mismatched wale direction (Bottom Left, reproduced from Figure 8 in [Yuksel et al. 2012]). Even with these loops, the entire stitch mesh should have zero pairwise linkage between its yarn rows. We verify the relaxed yarn model using our algorithm, and mark linkages in red (Top). From this, we zoom in on one of the linkages, and find that the small green loop (Bottom Right), which should topologically match the small green loops in the bottom left figure, has only 3, rather than 4, curve segments passing through it, indicating there was yarn pull-through during relaxation.

topology validation of these models, by computing a linking matrix certificate before and after a deformation, whether the deformation is a simulation, relaxation, reparameterization, or compression. In addition, we can examine the integrity of input models from prior work. In our examples we analyze four models from [Wu and Yuksel 2017; Yuksel et al. 2012]: the “alien sweater,” “sweater flame ribbing,” “glove,” and the “sheep sweater,” which are provided after a first-order implicit yarn-level relaxation. Stitches between rows introduce zero pairwise linkage, and our method verifies whether the relaxed models have valid stitches. We found that the “glove” and the “alien sweater” have no pairwise linkages, indicating no obvious issues, while the “sheep sweater” and “sweater flame ribbing” contain linkage violations. See Figure 9 for a violation in the “sweater flame ribbing” model.

4.3.2 **Reparameterizing and compressing yarn-level models**: Our methods can be used to verify the topology of yarn-level cloth models following common processing operations. Spline reparameterization can be used to reduce the number of spline control points prior to simulation, or afterwards for model export. Furthermore, it is common to compress control point coordinates using quantization for storage and transmission. Unfortunately such operations can introduce topological errors, which limits the amount of reduction that can occur. Our method can be used to verify these reparameterization and compression processes. Since large modifications of the model may introduce multiple linking errors that could cancel each other out, we leverage our methods’ speed to analyze a sweep of the reparameterization/compression process to better detect errors. See Figure 10 for results.

4.3.3 **Embedded deformation of yarn-level models**: It is computationally appealing to animate and pose yarn-level models using cheaper deformers than expensive yarn-level physics, but also retain yarn topology for visual and physical merits. To this end, we animated the yarn-level glove model using embedded deformation based on a tetrahedral mesh simulated using Houdini 18.5 Vellum. For the modest deformations shown in Figure 11 the yarn-level model retains the correct topology throughout the animation, and therefore can be used for subsequent yarn-level processing.

---

**Fig. 10. Compression and Reparameterization of Detailed Yarn Models**: (Left) We compress a sweater model down to 16-bit (Top Left) and 12-bit (Bottom Left) precision by quantizing coordinates. Our algorithm verifies that the 16-bit compression is valid, but highlights the loops that contain topology violations in the 12-bit compression. (Right) If we aggressively resample the Catmull–Rom splines of the sheep sweater model to reduce the number of control points from 729628 (Top Right) to 547264 (Bottom Right), our method highlights violations in the latter.

**Fig. 11. Embedded Deformation of Yarn-Level Models**: A coarse single-layer tetrahedral mesh (simulated by Houdini’s Vellum solver) is used to place a yarn-level glove model on a surface using embedded deformation. We verify that no topology changes are introduced by this deformation.
4.3.4 Implicit yarn-level simulation: Simulation of yarn, as in [Kaldor et al. 2008], is challenging because of the large number of thin and stiff yarns in contact, and the strict requirement to preserve topology (by avoiding yarn–yarn “pull through”), both of which can necessitate small time steps. For large models, these relaxations or simulations can take hours or days, and there may not be an expert user in the loop to visually monitor the simulations everywhere. While sometimes penalty-based contact forces will cause the simulation to blow up, alerting the user to restart, at other times pull-through failures can be silent and lead to incorrect results.

Our method is useful for validating yarn models during yarn-level relaxation, when large forces can cause topological changes, as well as large-step, preview implicit simulations of yarn-level models. We use implicit integration [Baraff and Witkin 1998; Bergou et al. 2010; Kaldor et al. 2008; Leaf et al. 2018] to step through the stiff forces, which enables larger timesteps at the risk of introducing topology errors; our PCG implementation uses Eigen [Guennebaud et al. 2010] and falls back to the SYM-ILDL package [Greif et al. 2015] when the energy gets too nonconvex [Kim 2020]. In Figure 12, we simulate the alien sweater model from [Yuksel et al. 2012] using larger implicit steps (1/2400 s) for a preview simulation as well as smaller implicit–explicit step sizes (1/6600 s). Our method detects and marks the yarn loops that have engaged in pull through violations in the former, and validates that the latter maintains its topology. In Figure 13, we perform a common stress test where we grab a knitted cylindrical tube by the ends and wring (twist) the ends in opposing directions until the simulation breaks. Even in this simulation with an obvious failure, our method is still useful because it detects the first linkage violations over a thousand steps before the simulation destabilizes. In all of these simulations, the runtime of our method is small compared to the runtime of the simulation: we only run our method once every animation frame, whereas the simulation takes dozens of substeps per frame.

4.3.5 Open-curve verification: chevron $3 \times 3$ stitch pattern relaxation: Our method can be applied not only on closed loopy structures, but also on finite, open curves in specific circumstances, to detect topology violations between different curves. In particular, this notion is well defined when the curves form a braid, that is, when the curves connect fixed points on two rigid ends, and the rigid ends remain separated. This can be useful for verifying the relaxation of a yarn stitch pattern, if the top and bottom rows are either cast on or bound off, and the ends of each row are pinned to two rigid ends, turning the stitch pattern into a braid.

When the curves form a braid, we can connect them using virtual connections to form virtual closed loops (see Figure 14). We propose a method to automatically form virtual connections which we include in Appendix C. To test this, we use a stitch pattern that repeats a chevron three times horizontally and vertically ($3 \times 3$). We apply our proposed method to the chevron $3 \times 3$ to verify the relaxation of the stitch pattern (see Figure 15). In the video, we also detect an early pull through in another relaxation with larger steps, hundreds of steps before it becomes visually apparent.
4.3.6 Chainmail simulation: In Figure 1, we simulated a fitted Kusari 4-in-1 chainmail garment with 14112 curved rings and 18752 links. Inspired by the 3D printing example from [Mazhar et al. 2016; Tasora et al. 2016], we simulated packing the garment into a small box for 3D printing using the Bullet rigid-body solver in Houdini 18.5. Without exceptionally small timesteps, the solver destroys links and creates spurious others as shown in Figure 1 and supplemental animations. By using varying numbers of substeps, we could illustrate linkage violations in Figure 1 (b) and (c), and certify that the result in (d) maintains its topological integrity, which would be otherwise difficult to ascertain until after expensive 3D printing.

4.3.7 Rubber band simulation: Errors in collision and contact handling can result in previously unlinked loops becoming entangled, and thus necessitate more accurate but expensive simulation parameters. We demonstrate this scenario by the simulation of 1024 unlinked rubber bands dropped onto the ground in Figure 16. Our methods can be used to monitor and detect topological errors introduced by low-quality simulation settings (too few substeps), and inform practitioners when to increase simulation fidelity.

4.3.8 Ribbons and DNA: Some applications in computational biology require the computation of linking numbers (and related quantities such as writhe) for extremely long DNA strands, which can change their linking numbers as a result of topoisomerase, thereby producing interesting effects such as super-coiling [Clavelin et al. 2012]. For computational purposes, the strands are essentially ribbons with two twisted double-helix curves along each edge. While existing approaches typically only consider direct evaluation methods for relatively small models, e.g., [Sierzega et al. 2020], we demonstrate performance on very long and closely separated closed ribbons (akin to circular DNA) depicted in the top left of Fig. 6. Results shown in Table 1 illustrate excellent performance by several of the algorithms; most notably, the GPU Barnes–Hut approach finishes 200,000-segment ribbons in under 30 milliseconds and, with sufficiently high β, finishes 20-million-segment ribbons in 5.3 seconds or faster; combined with the first two stages, our method takes up to 157 ms to compute the linking matrix for the former ribbon and 12.6 seconds for the larger ribbon.

5 CONCLUSIONS
We have explored practical algorithms and implementations for computing linking numbers for efficient monitoring and verification of the topology of loopy structures in computer graphics and animation. Linking-number certificates provide necessary (but not sufficient) conditions to ensure the absence of topology errors in simulation or processing that would go undetected previously. We hope that the community can benefit from these improved correctness monitoring techniques especially given the many immediate uses in graphics applications, e.g., yarn-level cloth simulation, that have previously relied on linkage correctness but lacked practical tools for verification.

By comparing multiple exact and approximate linking-number numerical methods, and CPU and GPU implementations, we discovered that several methods give accurate and fast results for many
types of problems, but alas there are no clear winners. Direct summation (DS) using exact formulae, although popular outside graphics for its simplicity and accuracy, was uncompetitive with the other methods for speed except for very small examples; GPU acceleration of the double summation was competitive in some cases. Our CPU parallel implementation of counting crossings (CC) was perhaps the simplest and fastest method for many problems; it relies on finding a suitable regular projection which is often cited as a weakness and potential source of numerical problems, but we were unable to generate incorrect results with it in double precision. CPU Barnes–Hut was a strong performer and competitive with CC in most cases, and GPU-accelerated Barnes–Hut was the fastest on several of the very largest examples, most notably for DNA ribbons. In some cases the GPU method struggles with accuracy due to having only dipole terms and single precision, and our GPU hardware and implementation were unable to store Barnes–Hut trees for large numbers of small loops, e.g., chainmail, for which CC or DS did best. The CPU-based Fast Multipole Method (FMM) was able to achieve high-accuracy fast summations due to the increased number of multipole terms, e.g., relative to Barnes–Hut, but it did not translate to beating CPU Barnes–Hut in general.

5.1 Limitations and Future Work

There are several limitations of our approaches, and opportunities for future work. While changes in the linking number between two curves imply a change in topology, the same linking number does not rule out intermediate topological events, e.g., two knit yarn loops with \( \lambda \) zero could “pull through” twice during a simulation and still have zero linkage at the end. Furthermore, collision violations can occur without a topology change: for example, given a key ring with three rigid keys, it is physically impossible for two adjacent keys to swap positions, yet the topology would still be maintained after a swap. Therefore, our approach can flag linking number changes, but a passing certificate does not preclude other errors.

We use AABB-trees to find overlapping loop AABBs in Potential Link Search (PLS) and overlapping spline-segment AABBs in Discretization, but tighter bounds than AABBs might reduce the number of linking-number calculations and discretized segments enough to provide further speedups.

We have considered linking numbers between closed loops and virtually closed loops for fixed patterns, but many loopy structures involve open loops, e.g., knittable yarn models [Guo et al. 2020; Narayanan et al. 2018, 2019; Wu et al. 2018, 2019], periodic yarn-level cloth patterns [Leaf et al. 2018], and hair, and it would be useful to generalize these ideas for open and “nearly closed” loops, e.g., to reason about “pull through” between adjacent yarn rows.

Topology checks could be closely integrated with simulations and geometry processing to encourage topologically correct results and enable roll-back/restart modes. Our proposed method with the “early exit” strategy could be better optimized to aggressively verify links before finishing PLS and Discretization on the entire model.

Finally, our GPU implementations and comparisons could be improved by specialized implementations for linking-number computations, all-at-once loop–loop processing instead of serial, better memory management for many-loop scenarios, e.g., chainmail, and also more recent hardware.

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THEOREM A.1. If all input loops have only polynomial spline segments, and no two loops intersect, then Algorithm 3 terminates in finite time if it uses tight AABBs as the bounding volumes in its overlap check.

PROOF. Write each input spline segment using a parameter $t$ that has a domain of $(0, 1)$. Since there is a finite number of input spline segments and each segment is polynomial over $t$ with a finite domain, the derivative along the curve with respect to $t$ has a global maximum, $M$. Since the loops don’t intersect, let $d$ be the minimum separation between two loops. After pass $P$ of the algorithm, the parameter $t$ of any “under-refined” spline segment now spans a domain of length $2^{-P}$. Because polynomial spline segments are continuous, their arc lengths must be at most $2^{-P} M$. Their bounding AABBs must therefore have diameters that are also at most $2^{-P} M$. If the minimum separation between any two loops is $d$, then after $P > \log_2(2M/d)$ passes, no two bounding boxes of different loops can overlap, and so all splines segments must have been discretized.

THEOREM A.2. If Algorithm 3 terminates, then it produces a line segment discretization that is homotopically equivalent to the input loops.

PROOF. We can deform the input following the steps of the algorithm. At each pass, let us deform one loop, $\xi$, at a time. All the spline segments of this loop, $\xi$, that are “processed” during the pass can be deformed continuously to their line segments (which share the same endpoints) by parameterizing the line segment to have the same $t$ domain and performing a point-wise straight-line deformation at each $t$. Because our bounding boxes are convex, this deformation sweeps a surface that is entirely within the bounding box. The box overlaps with no boxes from other loops, so loop $\xi$ does not cross any other loop during the deformation. Notice, however, that we do not guarantee that a loop does not cross itself during its deformation; this is allowed in link homotopy and does not affect the linking number. After all the passes, every loop has been deformed into its output polyline, and we are done.

B IMPLEMENTATION DETAILS OF LINKING NUMBER COMPUTATION ROUTINES

B.1 Direct Summation

We reproduce the direct sum expression (4) here for reference. For loops $\gamma_1, \gamma_2$ consisting of $N_1, N_2$ line segments respectively, with $i, j$ enumerating the segments respectively, we want to compute

$$\lambda(\gamma_1, \gamma_2) = \sum_{i} \sum_{j} \lambda_{ji}. \quad (16)$$

$$\lambda_{ji} = \frac{1}{2\pi} \arctan \left( \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{a}| |\mathbf{b}| |\mathbf{c}| + (\mathbf{a} \cdot \mathbf{b}) |\mathbf{c}| + (\mathbf{b} \cdot \mathbf{c}) |\mathbf{a}|} \right) + \arctan \left( \frac{|\mathbf{c}| |\mathbf{d}| |\mathbf{a}| + (\mathbf{c} \cdot \mathbf{d}) |\mathbf{a}| + (\mathbf{a} \cdot \mathbf{c}) |\mathbf{d}| + (\mathbf{d} \cdot \mathbf{a}) |\mathbf{c}|}{|\mathbf{c}| |\mathbf{d}|}\right). \quad (17)$$

In our implementation, both on the CPU and for large examples on the GPU, we multithread the outer loop of $i$, and perform angle summations (using the arctan addition formula) in the inner loop.
with a single arctangent at the end of each loop, and perform a sum reduction on the total. This results in \( N_k \) arctangents. When performing the inner loop, we split the \( j \) loop into batches, and each threadblock fetches the entire \( I_j \) batch in parallel into shared memory before each thread iterates through the batch. See Alg. 7 for more details. To maximize GPU usage, for small examples \((N_n, N_k < 1.4 \times 10^6)\), we parallelize both for-loops, and within each iteration we perform one arctangent.

**B.2 Barnes–Hut: Derivatives of \( g(\mathbf{r}_1, \mathbf{r}_2) \), Moment Computation, and Parallel Implementation**

For reference, here is the Barnes–Hut far-field expansion (Eq. (10)) again, expanded about \((\mathbf{r}_1, \mathbf{r}_2)\):

\[
\lambda = \int ds \, \left( (\mathbf{r}_1' \times \mathbf{r}_2') \cdot \nabla G(\mathbf{r}_1, \mathbf{r}_2) \right),
\]

\[
\lambda = \int ds \, \left( (\mathbf{r}_1' \times \mathbf{r}_2') \cdot \nabla G(\mathbf{r}_1, \mathbf{r}_2) + \left( (\mathbf{r}_1' \times \mathbf{r}_2') \otimes (-(\mathbf{r}_1 - \mathbf{r}_1) + (\mathbf{r}_2 - \mathbf{r}_2)) \right) \cdot \nabla^2 G(\mathbf{r}_1, \mathbf{r}_2) \right. 
+ \frac{1}{2} \left( (\mathbf{r}_1' \times \mathbf{r}_2') \otimes (-(\mathbf{r}_1 - \mathbf{r}_1) + (\mathbf{r}_2 - \mathbf{r}_2)) \otimes (-(\mathbf{r}_1 - \mathbf{r}_1) + (\mathbf{r}_2 - \mathbf{r}_2)) \right) 
\cdot \left. \nabla^3 G(\mathbf{r}_1, \mathbf{r}_2) + O(|\mathbf{r}|^{-5}) \right). 
\]

Now here are the derivatives of \( g(\mathbf{r}_1, \mathbf{r}_2) \). Let \( \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \). Then

\[
\nabla G(\mathbf{r}_1, \mathbf{r}_2) = \frac{r}{4\pi|\mathbf{r}|^3},
\]

\[
\nabla^2 G(\mathbf{r}_1, \mathbf{r}_2) = \frac{3|\mathbf{r}|}{4\pi|\mathbf{r}|^5} (\mathbf{r} \otimes \mathbf{r} - 3\mathbf{r} \otimes \mathbf{r}),
\]

\[
\nabla^3 G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{3}{4\pi|\mathbf{r}|^5} \sum_i (\mathbf{r} \otimes \mathbf{c}_i \otimes \mathbf{c}_i + \mathbf{c}_i \otimes \mathbf{r} \otimes \mathbf{c}_i + \mathbf{c}_i \otimes \mathbf{c}_i \otimes \mathbf{r})
\]

\[
+ \frac{15}{4\pi|\mathbf{r}|^7} (\mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r}),
\]

where \( \mathbf{c}_i \) is the \( i \)-th basis element and repeated indices are summed.

Note that these are identical to the basis functions in Appendix A of [Barill et al. 2018], except the third derivative there (Eq. 24 in [Barill et al. 2018]) has a misprint. These expressions can be expanded and rearranged so that for each term, the \( s \) and \( t \) integrals can be separated; this allows us to use the multipole moments directly.

Now let’s discuss how to build the moment tree. For reference, here are the moments (from Eqs. (11), (12), and (13)) again:

\[
c_M = \int \mathbf{r}' \, ds,
\]

\[
c_D = \int \mathbf{r}' (\mathbf{r} - \mathbf{r}')^T \, ds,
\]

\[
c_Q = \int \mathbf{r}' \otimes (\mathbf{r} - \mathbf{r}) \otimes (\mathbf{r} - \mathbf{r}) \, ds.
\]

For a line-segment element, we set \( \mathbf{r} \) to the midpoint, and, evaluating the integrals over a line segment, we get that \( c_M \) is the displacement between the endpoints, \( c_D = 0 \), and \( c_Q = \frac{1}{12} c_M \otimes c_M \otimes c_M \).

When we combine bounding boxes into their parents, we need to sum and shift moments. Let \( \mathbf{r}_c \) be the displacement \( \mathbf{r}_c - \mathbf{r}_p \) (from the parent node, indexed by \( p \), to the child node, \( c \)), and simply use

\[
c_{M_p} = \sum_c c_{M_c},
\]

\[
c_{D_p} = \sum_c c_{D_c} + c_{M_c} \mathbf{r}_c^T,
\]

\[
c_{Q_p} = \sum_c c_{Q_c} + c_{D_c} + c_{M_c} \mathbf{r}_c^T + c_{M_{c'}^T} + c_{M_{c}^T}.
\]

To compute the parent moments \( c_{M_p}, c_{D_p}, \text{ and } c_{Q_p} \).
On the CPU, we parallelized the two-tree method by gathering the list of child node pairs to evaluate after each recursive pass, and proceeding to children only after a pass is finished (breadth-first). We first run breadth-first passes single-threaded until the list has over 1000 node pairs. We then multithread the evaluation with breadth-first passes when it has between 1000 and 500,000 node pairs to evaluate. Beyond the 500,000-node-pair upper limit, memory allocation overhead (for the node-pair list) dominates, and we simply call the depth-first recursive algorithm in parallel for the 500,000+ node pairs.

On the GPU, we use only a first-order far-field expansion, dropping the quadrupole terms. Although we use the parallel, two-trees algorithm on the CPU, we parallelize one one-tree algorithm when evaluating on the GPU by basing it on the GPU Barnes–Hut N-Body implementation from [Burtscher and Pingali 2011]. In particular, we use Kernels 1 through 5. We first build a tree (Kernels 1 through 4) for each loop, then for each loop pair, we evaluate Kernel 5 using the source tree from one loop against target points from the other loop. We modified Kernel 3 to gather and compute the monopole and dipole moments of each node and also propagate the max segment length, $l_M$, from its children. Because segments are only inserted at their midpoints, we replace the $\beta$ far-field condition in Alg. 6 by checking whether $|r_{\text{tree},i} - tree.r| > \beta (\text{tree.R + tree.lM/2 + l_i/2})$, where $l_i$ is the length of the segment at $r_i$. We modified Kernel 5 to evaluate a first-order (dipole) far-field expansion when the $\beta$ far-field condition is met by all threads; otherwise the arctan expression. However, one key difference from [Burtscher and Pingali 2011] is that because the source and target trees are different and our input is already spatially coherent, Kernel 5 is actually slower if it evaluates the points in the sorted order from Kernel 4 than if it evaluates the points in the input order. So we keep Kernel 4 for sorting the individual source trees, but we evaluate Kernel 5 on the target evaluation points in the input order instead of the sorted order.

### C FORMING VIRTUAL CLOSED LOOPS IN BRAIDS

This section describes the formation of virtual closed loops for Open-Curve Verification §4.3.5. The key is that every curve in the braid participates in two overlapping virtual closed loops, and every virtual closed loop contains two curves from the braid. Then in the Potential Link Search (PLS, §3.2), we exclude loop pairs that share a common curve.

Let $a_1, \cdots, a_L$ denote the open curves, and let one rigid end be the “left” end and the other be the “right” end. Furthermore, define a small rigid end volume attached to each end, such that no curve passes through it. Then follow this procedure, also illustrated in Figure 14:

1. To ease bookkeeping, parameterize the curves so that they are oriented left to right.
2. Form $L$ total closed loops, $y_i$. For each $i \in [1, L - 1]$, form $y_i$ from $a_i$ and $a_{i+1}$ by doing the following:
   (a) Start with a copy of $a_i$.
   (b) Append a virtual connection, entirely within the right end volume, from the right end of $a_i$ to the right end of $a_{i+1}$, carefully making sure it intersects no other virtual curve,
   (c) Append $a_{i+1}$ in the reverse direction, and then
   (d) Append a virtual connection, entirely within the left end volume, from the left end of $a_{i+1}$ to the left end of $a_i$, taking the same care to ensure it intersects no other curve.
3. Also form $y_L$ using $a_1$ and $a_L$ the same way as the last step. At this point, every curve $\alpha$ should be part of two closed loops $y$, and every loop $y$ should contain two curves from the model.
4. Run our method from §3 using the input loops ($y_i$), with the difference that, at the end of the PLS stage, we exclude pairs $(i, j) \in P$, where $y_i$ and $y_j$ share an $\alpha$ as a component (which is when $i + 1 = j$ or $(i = 1$ and $j = L)$), from $P$. The rest of the method proceeds as usual, computing a sparse linking matrix as a certificate.
5. When this procedure is used again on a deformed model, the virtual connections at each end must be appended with the same exact paths, up to a rigid transformation (this is possible because of the rigid ends), ensuring that the virtual connections don’t pull through each other.

We also require that $L \geq 4$.

The certificate meaningfully verifies the topology between open curves in the model, as long as no real curve enters the rigid end volumes and the end volumes don’t mutually interpenetrate (ensuring no real or virtual curve moves through a virtual connection). To see how, suppose a single pull through occurs, between curves $a_i$ and $a_j$, $i < j$. Then this certificate must change, for these reasons:

- Call indices “cyclically adjacent” if they are either adjacent (such as $i$ and $i + 1$) or the first and last (such as $1$ and $L$). If $(i, j)$ are not cyclically adjacent, then the linkage between $y_i$ and $y_j$ will be computed (because they don’t share a curve), and found to be different, since the former contains $a_i$, and the latter contains $a_j$.
- Otherwise if $(i, j)$ are cyclically adjacent, then the linkage between $y_{i-1}$ (modulo $L$) and $y_j$ will be calculated (because $L \geq 4$, they don’t share a curve) and found to be different, because the former contains $a_i$, and the latter contains $a_j$.

Conversely, if a linking number changes, then there must have been at least one topology violation among the real curves, because no virtual connection participated in any pull through.

Furthermore, this procedure can be applied independently on many interacting braids as long as each braid has at least 4 curves.