REPRESENTING THE DELIGNE–HINICH–GETZLER ∞-GROUPOID

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ABSTRACT. The goal of the present paper is to introduce a smaller, but equivalent version of the Deligne–Hinich–Getzler ∞-groupoid associated to a homotopy Lie algebra. In the case of differential graded Lie algebras, we represent it by a universal cosimplicial object.

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1. INTRODUCTION

The fundamental principle of deformation theory, due to Deligne, Grothendieck and many others and recently formalized and proved in the context of ∞-categories by Pridham and Lurie, states that “Every deformation problem in characteristic 0 is encoded in the space of Maurer–Cartan elements of a differential graded Lie algebra.”

Therefore, one is naturally led to the study of Maurer–Cartan elements of differential graded Lie algebras and, more generally, homotopy Lie algebras.

In order to encode the Maurer–Cartan elements, gauge equivalences between them and higher relations between gauge equivalences, Hinich [Hin97] introduced the Deligne–Hinich–Getzler ∞-groupoid. It is a Kan complex associated to any complete Lie algebra modeling the space of its Maurer–Cartan elements. Since it is a very big object, Getzler introduced in [Get09] a smaller but weakly equivalent Kan complex γ• which, however, is more difficult to manipulate. In this paper, we introduce another simplicial set associated to any Lie algebra with the following nice properties, which we prove (not in this order):

1. it is weakly equivalent to the Deligne–Hinich–Getzler ∞-groupoid,
2. it is a Kan complex,
3. it is contained in the Getzler ∞-groupoid γ•, and
4. if we restrict to the category of complete dg Lie algebras, there is an explicit cosimplicial dg Lie algebra m•, representing this object.

The cosimplicial dg Lie algebra m• was already introduced in the works of Buijs–Murillo–Félix–Tanré [BFMT15] in the context of rational homotopy theory. We show here that it plays a key role in deformation theory.

Results coming from operad theory play a crucial role throughout the paper, especially in the second part. In particular, we use the explicit formulæ for the ∞-morphisms induced by the Homotopy Transfer Theorem given in [LV12] and various theorems proven in [RN17].

Little after the apperition of the present article, Buijs–Murillo–Félix–Tanré gave an alternative proof of Corollary 5.3 in [BFMT17]. Their proof doesn’t rely on general operadic results, but rather on explicit combinatorial computations.

2010 Mathematics Subject Classification. Primary 17B55; Secondary 18G55, 55U10.
Key words and phrases. Deformation theory, Deligne groupoid, differential graded Lie algebras, Maurer–Cartan elements.

The author was supported by grants from Région Ile-de-France, and the grant ANR-14-CE25-0008-01 project SAT.
The author was made aware by Marco Manetti in a private conversation that many of the results of this article are already present in the unpublished PhD thesis [Ban14] of his student Ruggero Bandiera. We acknowledge this, but we consider that the present article remains interesting in that the methods used to prove the results are different. In particular, in view of Bandiera’s results, Sections 3 and 4 can be interpreted as an alternative construction of the Getzler $\infty$-groupoid $\gamma_*$ with new proofs of its properties.

Structure of the paper. In Section 2 we give a short review of the Deligne groupoid, the Deligne–Hinich–Getzler $\infty$-groupoid, and the main theorems in this context. In Section 3 we state and prove our main theorem, giving a new simplicial set encoding the Maurer–Cartan space of $L_\infty$-algebras. Next, in Section 4 we study some properties of this object. In particular, we prove that it is a Kan complex, and that it is “small” in a precise sense. Finally, we focus on the special case of dg Lie algebras in Section 5, showing that our Kan complex is represented by a cosimplicial dg Lie algebra in this.

Notation and conventions. We work over a fixed field $\mathbb{K}$ of characteristic 0. We abbreviate “differential graded” by dg, and sometimes omit it completely. All algebras are differential graded unless stated otherwise.

Since we work with differential forms, we adopt the cohomological convention. Therefore, we work over cochain complexes, and Maurer–Cartan elements of dg Lie and $L_\infty$-algebras (i.e. homotopy Lie algebras) are in degree 1, not $-1$. All cochain complexes are $\mathbb{Z}$-graded.

For the definitions and the necessary basic notions about (filtered) $L_\infty$-algebras and (filtered) $\infty$-morphisms we refer the reader to the article [DRT15].

Acknowledgments. I thank Ezra Getzler, Marco Manetti, Chris Rogers and Jim Stasheff for their useful comments, both editorial and mathematical. I am naturally also extremely grateful to my advisor Bruno Vallette for all the help, support and continuous discussion. I am thankful to Yangon, Don Mueang, and Narita international Airports for the almost pleasant working atmosphere they provided, as the bulk of this paper was typed in between flights.

2. The Deligne–Hinich–Getzler $\infty$-groupoid

An object of fundamental interest in deformation theory is the Deligne groupoid $\text{Del}(g)$ associated to a complete dg Lie algebra $g$. There is a higher generalization of the Deligne groupoid in the form of the Deligne–Hinich–Getzler $\infty$-groupoid. It is a simplicial set with nice properties and which 1-truncation gives back the Deligne groupoid. It was introduced in [Hin97] and then studied in depth and further generalized in [Get09].

2.1. The Deligne groupoid. Let $g$ be a dg Lie algebra. Then we can associate a groupoid $\text{Del}(g)$ to $g$, called the Deligne groupoid, as follows. The objects of the Deligne groupoid are the Maurer–Cartan elements of $g$, i.e. the degree 1 elements $\alpha \in g^1$ satisfying the Maurer–Cartan equation

$$d\alpha + \frac{1}{2} [\alpha, \alpha] = 0.$$ 

Definition 2.1. The set of Maurer–Cartan elements of $g$ is denoted by $\text{MC}(g)$. 
We have the set of objects of $\text{Del}(g)$, we still need to define its morphisms. To an element $\lambda \in g^0$, one can associate a “vector field” by sending $\alpha \in g^1$ to

$$d\lambda + [\lambda, \alpha] \in g^1.$$  

It is tangent to the Maurer–Cartan locus, in the sense that if $\alpha(t)$ is the flow of $\lambda$, that is

$$\dot{\alpha}(t) = d\lambda + [\lambda, \alpha(t)]$$

with $\alpha(0) \in \text{MC}(g)$, then $\alpha(t) \in \text{MC}(g)$ for all $t$. We say that two Maurer–Cartan elements $\alpha_0, \alpha_1 \in \text{MC}(g)$ are gauge equivalent if there exists such a flow $\alpha(t)$ such that $\alpha(i) = \alpha_i$ for $i = 0, 1$. The Deligne groupoid is the groupoid associated to this equivalence relation, this means that the morphisms are

$$\text{Del}(g)(\alpha_0, \alpha_1) := \{ \lambda \in g^0 \mid \text{the flow of } \lambda \text{ starting at } \alpha_0 \text{ gives } \alpha_1 \text{ at time } 1 \}.$$  

See for example [GM88].

The assignment of the Deligne groupoid to a dg Lie algebra is functorial and has a good homotopical behavior: it sends filtered quasi-isomorphisms to equivalences, as can be seen by the Goldman-Millson theorem ([CM88], [Yek12]).

### 2.2. Generalization: the Deligne–Hinich–Getzler $\infty$-groupoid.

Let $g$ be a complete $L_\infty$-algebra. The Maurer–Cartan equation can be generalized to

$$dx + \sum_{n \geq 2} \frac{1}{n!} \ell_n(x, \ldots, x) = 0$$

for $x \in g^1$. Again, we denote by $\text{MC}(g)$ the set of all elements satisfying this equation.

**Remark 2.2.** Notice that the condition that $g$ be complete is necessary to make it so that the left-hand side of the Maurer–Cartan equation is well defined.

#### 2.2.1. The Deligne–Hinich–Getzler $\infty$-groupoid.

**Definition 2.3.** The Sullivan algebra is the simplicial dg commutative algebra

$$\Omega_n := \mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/\left( \sum_{i=0}^n t_i = 1, \sum_{i=0}^n dt_i = 0 \right)$$

endowed with the unique differential satisfying $d(t_i) = dt_i$.

This object was introduced by Sullivan in the context of rational homotopy theory [Sul77]. At level $n$, it is the algebra of polynomial differential forms on the standard geometric $n$-simplex. Now let $g$ be a nilpotent dg Lie algebra. Then tensoring $g$ with $\Omega_n$ gives us back a nilpotent dg Lie algebra, of which we can consider the Maurer–Cartan elements.

**Definition 2.4.** The Deligne–Hinich–Getzler $\infty$-groupoid (DHG $\infty$-groupoid) is the simplicial set

$$\text{MC}_\bullet(g) := \text{MC}(g \otimes \Omega_\bullet).$$

This association is natural in $g$, and thus defines a functor

$$\text{MC}_\bullet : \{ \text{nilpotent } L_\infty\text{-algebras} \} \rightarrow \text{sSet}.$$  

We will rather consider the following slight generalization: Let $g$ be a complete $L_\infty$-algebra, then

$$g \cong \lim_{\leftarrow n} g/F_n^{L_\infty} g$$

is the limit of a sequence of nilpotent $L_\infty$-algebras. Thus we can define

$$\text{MC}_\bullet(g) := \lim_{\leftarrow n} \text{MC}_\bullet(g/F_n^{L_\infty} g).$$

Notice that the elements in $\text{MC}_\bullet(g)$ in this case are not polynomials with coefficients in $g$ anymore, but rather power series with some “vanishing at infinity” conditions. We state all the following results in this setting.

**Theorem 2.5.** Let either:

- [Get09] Prop. 4.7: $g, h$ be nilpotent $L_\infty$-algebras and $\Phi : g \rightarrow h$ be a surjective strict morphism of $L_\infty$-algebras, or
- [Rog16] Thm. 2: $g, h$ be filtered $L_\infty$-algebras and $\Phi : g \rightarrow h$ be a filtered $\infty$-morphism that induces a surjection at every level of the filtrations.
Then
\[ MC_\bullet(\Phi) : MC_\bullet(\mathfrak{g}) \rightarrow MC_\bullet(\mathfrak{h}) \]
is a fibration of simplicial sets. In particular, for any $L_\infty$-algebra $\mathfrak{g}$, the simplicial set $MC_\bullet(\mathfrak{g})$ is a Kan complex.

This result was originally proven by Hinich [Hin97, Th. 2.2.3] for strict surjections between nilpotent dg Lie algebras concentrated in positive degrees, and then generalized by Getzler and by Rogers to the version stated above.

Generalizing the Goldman–Millson theorem, Dolgushev and Rogers proved in [DR15] that the Deligne–Hinich–Getzler $\infty$-groupoid behaves well with respect to homotopy theory: it sends filtered quasi-isomorphisms of filtered $L_\infty$-algebras to weak equivalences.

2.2.2. Basic forms, Dupont’s contraction and Getzler’s functor $\gamma_\bullet$. The Sullivan algebra has a subcomplex $C_\bullet$ linearly spanned by the so-called basic forms
\[ \omega_I := k! \sum_{j=1}^{k} (-1)^j t_{i_0} \cdots \hat{t}_{i_j} \cdots t_{i_k} \in \Omega_n \]
for $I = \{i_0 < i_1 < \cdots < i_k\} \subseteq \{0, \ldots, n\}$. This is in fact the (co)cellular complex for the standard geometric $n$-simplex $\Delta^n$. In order to prove a simplicial version of the de Rham theorem, Dupont [Dup76] introduced a homotopy retract
\[ h_\bullet \subseteq \Omega_\bullet \xrightarrow{p_\bullet} C_\bullet \]
where all the maps are simplicial. Homotopy retract means that
\[ p_\bullet h_\bullet = 1, \quad \text{and} \quad 1 - i_\bullet p_\bullet = dh_\bullet + h_\bullet d. \]
Moreover, the maps satisfy the side conditions
\[ h_\bullet i_\bullet = 0, \quad p_\bullet h_\bullet = 0, \quad \text{and} \quad h_\bullet^2 = 0. \]
A retraction satisfying the side conditions is called a contraction.

This contraction will be a fundamental ingredient in the rest of the present paper. As the DHG $\infty$-groupoid is always a big object, Getzler defined the following subobject.

**Definition 2.6.** The sub-simplicial set $\gamma_\bullet(\mathfrak{g})$ of $MC_\bullet(\mathfrak{g})$ is given by
\[ \gamma_n(\mathfrak{g}) := \{ \alpha \in MC_n(\mathfrak{g}) \mid h_\alpha = 0 \}. \]

**Theorem 2.7 ([Get09]).** The simplicial set $\gamma_n(\mathfrak{g})$ is a Kan complex, and it is weakly equivalent to the DHG $\infty$-groupoid $MC_\bullet(\mathfrak{g})$.

A part of the definition of $h_\bullet$ and $p_\bullet$, which we will need in what follows is the (formal) integration of a form in the Sullivan algebra over a simplex, which is given by:
\[ \int_{\Delta^n} t_{i_0}^{a_0} \cdots t_{i_n}^{a_n} dt_1 \cdots dt_n := \frac{a_1! \cdots a_n!}{(a_1 + \cdots + a_n + n)!}. \]
It corresponds to the usual integration when working over $K = \mathbb{R}$.

**Remark 2.8.** We have
\[ \int_{\Delta^p} \omega_I = 1 \]
for $p + 1 = |I|$, where $\Delta^p$ is the subsimplex of $\Delta^n$ with vertices indexed by $I$.

**Definition 2.9.** A form $\alpha \in \gamma_n(\mathfrak{g})$ is said to be thin if
\[ \int_{\Delta^n} \alpha = 0. \]

**Theorem 2.10 ([Get09]).** For every horn in $\gamma_n(\mathfrak{g})$, there exists a unique thin simplex filling it.

**Remark 2.11.** The existence of a set of thin simplices such that every horn has a unique thin filler is what is meant by Getzler when he speaks of an $\infty$-groupoid. We use the term simply to mean Kan complex (e.g. when speaking of the DHG $\infty$-groupoid).
3. MAIN THEOREM

In this section, we give a reminder on the Homotopy Transfer Theorem for commutative and for $L_\infty$-algebras, before going on to state and prove the main theorem of the present article.

3.1. Reminder on the Homotopy Transfer Theorem. Let $V, W$ be cochain complexes, and suppose that we have a retraction

$$h : V \rightarrow W,$$

that is, we have

$$ip - 1 = dh + hd$$

and $pi = 1$. Furthermore, as we are working in characteristic $0$, we can always suppose that $h^2 = 0$, $hi = 0$, and $ph = 0$.

Then we can transfer algebraic structures from $V$ to $W$. More precisely, the specific cases of interest to us are the following ones.

**Theorem 3.1** (Homotopy Transfer Theorem for commutative algebras). Suppose $V$ is a commutative algebra. There is a $C_\infty$-algebra structure on $W$ such that $p$ and $i$ extend to $\infty$-quasi isomorphisms $p_\infty$ and $i_\infty$ of $C_\infty$-algebras between $V$ and $W$ endowed with the respective structures.

**Theorem 3.2** (Homotopy Transfer Theorem for $L_\infty$-algebras). Suppose $V$ is an $L_\infty$-algebra. There is an $L_\infty$-algebra structure on $W$ such that $p$ and $i$ extend to $\infty$-quasi isomorphisms $p_\infty$ and $i_\infty$ of $L_\infty$-algebras between $V$ and $W$ endowed with the respective structures.

For details on this theorem, see e.g. [LV12, Sect. 10.3] (where it is proven in the general context of algebras over operads). See also [LV12, Sect. 10.3.5–6] for the explicit formulæ for the $\infty$-morphisms $p_\infty$ and $i_\infty$.

3.2. Statement of the main theorem. Let $g$ be a complete $L_\infty$-algebra. Then the Dupont contraction induces a contraction

$$1 \otimes h : g \otimes \Omega_\bullet \rightarrow (1 \otimes \mu_\bullet) : g \otimes C_\bullet,$$

of $g \otimes \Omega_\bullet$ to $g \otimes C_\bullet$. Applying the Homotopy Transfer Theorem to this contraction, we obtain a simplicial $L_\infty$-algebra structure on $g \otimes C_\bullet$. We also know that we can extend the maps $1 \otimes p_\bullet$ and $1 \otimes i_\bullet$ to simplicial $\infty$-morphisms of simplicial $L_\infty$-algebras $(1 \otimes p_\bullet)_\infty$ and $(1 \otimes i_\bullet)_\infty$. We denote $P_\bullet$ and $I_\bullet$ the induced maps on Maurer–Cartan elements. We will also use the notation

$$(1 \otimes r_\bullet)_\infty := (1 \otimes i_\bullet)_\infty (1 \otimes p_\bullet)_\infty,$$

and we dub $R_\bullet$ the map induced by $(1 \otimes r_\bullet)_\infty$ on Maurer–Cartan elements.

**Theorem 3.3.** Let $g$ be a complete $L_\infty$-algebra. The maps $P_\bullet$ and $I_\bullet$ are inverse one to the other in homotopy, and thus provide a weak equivalence

$$MC_\bullet(g) \simeq MC(g \otimes C_\bullet)$$

of simplicial sets which is natural in $g$.

**Remark 3.4.** The simplicial $L_\infty$-algebra $g \otimes C_\bullet$ has the advantage of being quite smaller than $g \otimes \Omega_\bullet$, since $C_n$ is finite dimensional for each $n$. The price to pay is that the algebraic structure is much more convoluted.

3.3. Proof of the main theorem. The rest of this section is dedicated to the proof of this result. We begin with the following lemma.

**Lemma 3.5.** We have

$$P_\bullet I_\bullet = \text{id}_{MC(g \otimes C_\bullet)}.$$

**Proof.** This is because $(1 \otimes p_\bullet)_\infty (1 \otimes i_\bullet)_\infty$ is the identity, see e.g. [DSV16 Theorem 5], and the functoriality of the Maurer–Cartan functor $MC$. \(\square\)
Therefore, it is enough to prove that the map
\[ R_\ast = 1 \otimes p_\ast : MC_\ast(g) \to MC_\ast(g) \]
is a weak equivalence. The idea is to use the same methods as in [DR15]. The situation is however slightly different, as the map \( R_\ast \) is not of the form \( 1 \otimes 1 \otimes \) and thus Theorem 2.2 of loc. cit. cannot be directly applied. The first, easy step is to understand what happens at the level of \( \pi_0 \).

**Lemma 3.6.** The map
\[ \pi_0(R_\ast) : \pi_0MC_\ast(g) \to \pi_0MC_\ast(g) \]
is a bijection.

**Proof.** We have \( \Omega_0 = C_0 = \mathbb{K} \), and the maps \( i_0 \) and \( p_0 \) both are the identity of \( \mathbb{K} \). Therefore, the map \( R_0 \) is the identity of \( MC_0(g) \), and thus obviously induces a bijection on \( \pi_0 \). \hfill \Box

For the higher homotopy groups, we start with a simplified version of [DR15 Prop. 2.4], which gives in some sense the base for an inductive argument. If the \( \mathcal{L}_\infty \)-algebra \( g \) is abelian, i.e. all of its brackets vanish, then so do the brackets at all levels of \( g \otimes \Omega_\ast \). In this case, the Maurer–Cartan elements are exactly the cocycles of the underlying cochain complex, and therefore \( MC_\ast(g) \) is a simplicial vector space.

**Lemma 3.7.** If the \( \mathcal{L}_\infty \)-algebra \( g \) is abelian, then \( R_\ast \) is a weak equivalence of simplicial vector spaces.

**Proof.** Recall that the Moore complex of a simplicial vector space \( V_\ast \) is defined by
\[ \mathcal{M}(V_\ast)_n := s^nV_n \]
endowed with the differential
\[ \partial := \sum_{i=0}^{n} (-1)^i d_i, \]
where the maps \( d_i \) are the face maps of the simplicial set \( V_\ast \). It is a standard result that
\[ \pi_0(V_\ast) = H_0(\mathcal{M}(V_\ast)) \]
for all \( i \geq 1 \) and \( v \in V_0 \), and that a map of simplicial vector spaces is a weak equivalence if and only if it induces a quasi-isomorphism between the respective Moore complexes [GJ09 Cor. 2.5, Sect. III.2]. In our case,
\[ V_\ast := MC_\ast(g) = Z^1(g \otimes \Omega_\ast) \]
is the simplicial vector space of 1-cocycles of \( g \otimes \Omega_\ast \). As in [DR15], it can be proven that the map
\[ \mathcal{M}(1 \otimes p_\ast) : \mathcal{M}(Z^1(g \otimes \Omega_\ast)) \to \mathcal{M}(Z^1(g \otimes C_\ast)) \]
is a quasi-isomorphism. But as the bracket vanishes, this is exactly \( P_\ast \). Now
\[ \mathcal{M}(1 \otimes p_\ast) \mathcal{M}(1 \otimes i_\ast) = 1_{\mathcal{M}(Z^1(g \otimes \Omega_\ast))}, \]
which implies that \( \mathcal{M}(1 \otimes i_\ast) \) also is a quasi-isomorphism. This implies that \( R_\ast \) is a weak equivalence, concluding the proof. \hfill \Box

Now we basically follow the structure of [DR15 Sect. 4]. We define a filtration of \( g \otimes \Omega_\ast \) by
\[ F_k(g \otimes \Omega_\ast) := (F_k^Z \otimes g) \otimes \Omega_\ast. \]
We denote by
\[ (g \otimes \Omega_\ast)^{(k)} := g \otimes \Omega_\ast_{/F_k^Z \otimes (g \otimes \Omega_\ast) = (g^{(k)}) \otimes \Omega_\ast}. \]
The composite \( (1 \otimes i_\ast)(1 \otimes p_\ast) \) induces an endomorphism \( (1 \otimes i_\ast)^{(k)}(1 \otimes p_\ast)^{(k)} \) of \( (g \otimes \Omega_\ast)^{(k)} \). All the \( \infty \)-morphisms coming into play obviously respect this filtration, and moreover \( 1 \otimes h_\ast \) passes to the quotients, so that we have
\[ 1_{(g \otimes \Omega_\ast)^{(k)}} - (1 \otimes i_\ast)^{(k)}(1 \otimes p_\ast)^{(k)} = d(1 \otimes h_\ast)^{(k)} + (1 \otimes h_\ast)^{(k)}d \]
for all \( k \), which shows that \( (1 \otimes r_\ast)^{(k)} \) is a filtered \( \infty \)-quasi-isomorphism. The next step is to reduce the study of the homotopy groups with arbitrary basepoint to the study of the homotopy groups with basepoint \( 0 \in MC_0(g) \).
Lemma 3.8. Let $\alpha \in \MC(g)$, and let $g^\alpha$ be the $L_\infty$-algebra obtained by twisting $g$ by $\alpha$, that is the $L_\infty$-algebra with the same underlying graded vector space, but with differential
\[
d^\alpha(x) := dx + \sum_{n \geq 2} \frac{1}{(n-1)!} \ell_n(\alpha, \ldots, \alpha, x)
\]
and brackets
\[
\ell^\alpha(x_1, \ldots, x_m) := \sum_{n \geq m} \frac{1}{(n-m)!} \ell_n(\alpha, \ldots, \alpha, x_1, \ldots, x_m).
\]
Let
\[
\text{Shift}_\alpha : \MC_*(g^\alpha) \to \MC_*(g)
\]
be the isomorphism of simplicial sets induced by the map given by
\[
\beta \in g \mapsto \alpha + \beta \in g^\alpha.
\]
Then the following diagram commutes
\[
\begin{array}{ccc}
\MC_*(g^\alpha) & \xrightarrow{\text{Shift}_\alpha} & \MC_*(g) \\
\downarrow R_* & & \downarrow R_* \\
\MC_*(g^\alpha) & \xrightarrow{\text{Shift}_\alpha} & \MC_*(g)
\end{array}
\]
where
\[
R_*^k(\beta) := \sum_{k \geq 1} (1 \otimes r_*)_k^\alpha (\beta \otimes k)
\]
and
\[
(1 \otimes r_*)_k^\alpha (\beta_1 \otimes \cdots \otimes \beta_k) := \sum_{j \geq 0} \frac{1}{j!} (1 \otimes r_*)_{k+j} (\alpha \otimes j \otimes \beta_1 \otimes \cdots \otimes \beta_k)
\]
is the twist of $(1 \otimes f_*)_\infty$ by the Maurer–Cartan element $\alpha$. Here, we identified $\alpha \in g$ with $\alpha \otimes 1 \in g \otimes \Omega_*$. 

Proof. The proof in [DR16, Lemma 4.3] goes through mutatis mutandis. \hfill \Box

Now we proceed by induction to show that $F^{(k)}$ is a weak equivalence from $\MC_*(g^{(k)})$ to itself for all $k \geq 2$. As the $L_\infty$-algebra $(g \otimes \Omega_*)^{(2)}$ is abelian, the start of the induction is given by Lemma 3.7.

Lemma 3.9. Let $m \geq 2$. Suppose that
\[
R_*^{(k)} : \MC(g^{(k)}) \to \MC(g^{(k)})
\]
is a weak equivalence for all $2 \leq k \leq m$. Then $R_*^{(m+1)}$ is also a weak equivalence.

Proof. The zeroth homotopy set $\pi_0$ has already been taken care of in Lemma 3.6. Thanks to Lemma 3.8, it is enough to prove that $R_*^{(m+1)}$ induces isomorphisms of homotopy groups $\pi_i$ based at 0, for all $i \geq 1$. Consider the following commutative diagram
\[
\begin{array}{cccccc}
0 & \xrightarrow{F_m(g \otimes \Omega_*)} & (g \otimes \Omega_*)^{(m+1)} & \xrightarrow{(g \otimes \Omega_*)^{(m)}} & 0 \\
\downarrow & & \downarrow & & \\
0 & \xrightarrow{F_m(g \otimes \Omega_*)} & (g \otimes \Omega_*)^{(m+1)} & \xrightarrow{(g \otimes \Omega_*)^{(m)}} & 0
\end{array}
\]
where the leftmost vertical arrow is given by the linear term $(1 \otimes i_*)_{(m+1)}(1 \otimes r_*)_{(m+1)}$ of $(1 \otimes r_*)_{(m+1)}$ since all higher terms vanish, as can be seen by the explicit formulas for the $\infty$-quasi isomorphisms induced by the Homotopy Transfer Theorem given in [LY12, Sect. 10.3.5–6]. Therefore, it is a weak equivalence as the $L_\infty$-algebras in question are abelian. The first term in each row is the fibre of the next map, which is surjective. By Theorem 2.5, we know that applying the $\MC_*$ functor makes the horizontal maps on the right into fibrations of simplicial sets, while the objects we obtain on the left are easily seen to be the fibres. Taking the long sequence in homotopy and using the five-lemma, we see that all we are left to do is to prove that $R_*^{(m+1)}$ induces an isomorphism on $\pi_1$. Notice that it is necessary to prove this, as the long sequence is exact everywhere except on the level of $\pi_0$.

The long exact sequence of homotopy groups (truncated on both sides) reads
\[
\begin{align*}
\pi_2 \MC_*(g^{(m)}) & \xrightarrow{\partial} \pi_1 \MC_*(F_{m+1}^{\Lie} g) \\
\pi_1 \MC_*(g^{(m+1)}) & \xrightarrow{\partial} \pi_1 \MC_*(g^{(m)}) \\
\pi_0 \MC_*(F_{m+1}^{\Lie} g) & \xrightarrow{\partial} \pi_0 \MC_*(F_{m+1}^{\Lie} g)
\end{align*}
\]
where in the higher homotopy groups we left the basepoint implicit (as it is always 0). The map
\[ \partial : \pi_1 MC_\bullet(\mathfrak{g}^{(m)}) \to \pi_0 MC_\bullet \left( \frac{F_{\mathfrak{g}}^{Lie}}{F_{m+1}^{Lie} \mathfrak{g}} \right) = H^1(\frac{F_{m+1}^{Lie} \mathfrak{g}}{F_{m}^{Lie} \mathfrak{g}}) \]
is seen to be the obstruction to lifting an element of \( \pi_1 MC_\bullet(\mathfrak{g}^{(m)}) \) to an element of \( \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \) (e.g. [Gj09 Lemma 7.3]).

The map \( \pi_1(\mathcal{R}_\bullet^{(m+1)}) \) is surjective: Let \( y \in \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \) and denote by \( \overline{y} \) its image in \( \pi_1 MC_\bullet(\mathfrak{g}^{(m)}) \). By the induction hypothesis, there exists a unique \( \overline{x} \in \pi_1 MC_\bullet(\mathfrak{g}^{(m)}) \) which is mapped to \( \overline{y} \) under \( F^{(m)} \).

As \( \overline{y} \) is the image of \( y \), we have \( \partial(\overline{y}) = 0 \), and this implies that \( \partial(\overline{x}) = 0 \), too. Therefore, there exists \( x \in \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \) mapping to \( \overline{x} \). Denote by \( y' \) the image of \( x \) under \( R^{(m+1)}_\bullet \). Then \( y' y^{-1} \) is in the kernel of the map
\[ \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \to \pi_1 MC_\bullet(\mathfrak{g}^{(m)}). \]

By exactness of the long sequence, and the fact that \( R^\bullet \) induces an automorphism of \( \pi_1 MC_\bullet \left( \frac{F_{\mathfrak{g}}^{Lie}}{F_{m+1}^{Lie} \mathfrak{g}} \right) \), there exists an element \( z \in \pi_1 MC_\bullet \left( \frac{F_{\mathfrak{g}}^{Lie}}{F_{m+1}^{Lie} \mathfrak{g}} \right) \) mapping to \( y' y^{-1} \) under the composite
\[ \pi_1 MC_\bullet \left( \frac{F_{\mathfrak{g}}^{Lie}}{F_{m+1}^{Lie} \mathfrak{g}} \right) \xrightarrow{F} \pi_1 MC_\bullet \left( \frac{F_{m+1}^{Lie} \mathfrak{g}}{F_{m}^{Lie} \mathfrak{g}} \right) \to \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}). \]

Let \( x' \) be the image of \( z \) in \( \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \), then \( (x')^{-1} x \) maps to \( y \) under \( R^{(m+1)}_\bullet \). This proves the surjectivity of the map \( \pi_1(\mathcal{R}^{(m+1)}_\bullet) \).

The map \( \pi_1(\mathcal{R}^{(m+1)}_\bullet) \) is injective: Assume \( x, x' \in \pi_1 MC_\bullet(\mathfrak{g}^{(m+1)}) \) map to the same element under \( R^{(m+1)}_\bullet \). Then \( (x')^{-1} x \) maps to the neutral element 0 under \( R^{(m+1)}_\bullet \). As it can be seen in the diagram below, it follows that there is a \( z \in \pi_2 MC_\bullet(\mathfrak{g}^{(m)}) \) mapping to \( x'x^{-1} \) which must be such that its image \( w \) is the image of some \( \tilde{w} \in \pi_2 MC_\bullet(\mathfrak{g}^{(m)}) \) under the map \( \partial \). But by the induction hypothesis and the exactness of the long sequence, this implies that \( z \) is in the kernel of the next map, and thus that \( x'x^{-1} \) is the identity element. Therefore, the map \( \pi_1(\mathcal{F}^{(m+1)}) \) is injective.

This concludes the proof of the lemma. □

Finally, we can conclude the proof of Theorem 3.3.

Proof of Theorem 3.3: Lemma 3.9 together with all we have said before, shows that \( F^{(m)} \) is a weak equivalence for all \( m \geq 2 \). Therefore, we have the following commutative diagram:

\[ \cdots \]

\[ \downarrow \]

\[ \downarrow \]

\[ \text{MC}_\bullet(\mathfrak{g}^{(4)}) \xrightarrow{\sim} \text{MC}_\bullet(\mathfrak{g}^{(4)}) \]

\[ \downarrow \]

\[ \downarrow \]

\[ \text{MC}_\bullet(\mathfrak{g}^{(3)}) \xrightarrow{\sim} \text{MC}_\bullet(\mathfrak{g}^{(3)}) \]

\[ \downarrow \]

\[ \downarrow \]

\[ \text{MC}_\bullet(\mathfrak{g}^{(2)}) \xrightarrow{\sim} \text{MC}_\bullet(\mathfrak{g}^{(2)}) \]

where all objects are Kan complexes, all horizontal arrows are weak equivalences, and all vertical arrows are Kan fibrations by Theorem 2.3. It follows that the collection of horizontal arrows defines a weak equivalence between fibrant objects in the model category of tower of simplicial sets, see [Gj09 Sect. VI.1]. The functor from towers of simplicial sets to simplicial sets given by taking the limit is right adjoint to the constant tower functor, which trivially preserves cofibrations and weak equivalences. Thus, the constant tower functor is a left Quillen functor, and it follows that the limit functor is a right Quillen functor. In particular, it preserves weak equivalences between fibrant objects. Applying this to the diagram above proves that \( R^\bullet \) is a weak equivalence. □
4. Properties and comparison

The main theorem (3.3) shows that the simplicial set $MC(g \otimes C_\bullet)$ is a new model for the DHG $\infty$-groupoid. This section is dedicated to the study of some properties of this object. We start by showing that it is a Kan complex, then we give some conditions on the differential forms representing its simplices. We show how we can use it to rectify cells of the DHG $\infty$-groupoid, which provides an alternative, simpler proof of [DR15]. Finally we compare it with Getzler’s functor $\gamma_\bullet$, proving that our model is contained in Getzler’s. We conjecture that the inclusion is strict in general.

4.1. Properties of $MC_\bullet(g \otimes C_\bullet)$. The following proposition is the analogue to Theorem 2.5 for our model.

**Proposition 4.1.** Let $g, h$ be two complete $L_\infty$-algebras, and suppose that $\Phi : g \rightarrow h$ is an $\infty$-morphism of $L_\infty$-algebras inducing a fibration of simplicial sets under the functor $MC_\bullet$. Then the induced morphism

$$MC(\phi \otimes id_{C_\bullet}) : MC(g \otimes C_\bullet) \rightarrow MC(h \otimes C_\bullet)$$

is also a fibration of simplicial sets. In particular, for any complete $L_\infty$-algebra $g$, the simplicial set $MC(g \otimes C_\bullet)$ is a Kan complex.

**Proof.** By assumption, the morphism $MC_\bullet(\phi) : MC_\bullet(g) \rightarrow MC_\bullet(h)$ is a fibration of simplicial set, and by Lemma 3.5 the following diagram exhibits $MC(\phi \otimes id_{C_\bullet})$ as a retract of $MC_\bullet(\phi)$.

$$\begin{array}{ccc}
MC(g \otimes C_\bullet) & \xrightarrow{I_\bullet} & MC_\bullet(g) \\
MC(\phi \otimes id_{C_\bullet}) & \downarrow & \downarrow MC(\phi) \\
MC(g \otimes C_\bullet) & \rightarrow & MC(\phi \otimes id_{C_\bullet}) \\
I_\bullet & \downarrow & \downarrow P_\bullet \\
MC(g \otimes C_\bullet) & \rightarrow & MC(g \otimes C_\bullet)
\end{array}$$

As the class of fibrations is closed under retracts, this concludes the proof. □

We consider the composite $R_\bullet = I_\bullet P_\bullet$, which is not the identity.

**Definition 4.2.** We call the morphism

$$R_\bullet : MC_\bullet(g) \rightarrow MC_\bullet(g)$$

the rectification map.

The following is a (more general) alternative version of [DR15, Lemma B.2], as well as a motivation for the name “rectification map” for $R_\bullet$.

**Proposition 4.3.** We consider an element $\alpha := \alpha_1(t_0, \ldots, t_n) + \cdots \in MC_n(g)$, where the dots indicate terms in $g^{1-k} \otimes \Omega^n_k$ with $1 \leq k \leq n$. Then $\beta := R_\bullet(\alpha) \in MC_n(g)$ is of the form

$$\beta = \beta_1(t_0, \ldots, t_n) + \cdots + \xi \otimes \omega_0 \ldots \omega_n,$$

where the dots indicate terms in $g^{1-k} \otimes \Omega^n_k$ with $1 \leq k \leq n - 1$, where $\xi$ is an element of $g^{1-n}$, and where $\alpha_1$ and $\beta_1$ agree on the vertices of $\Delta^n$. In particular, if $\alpha \in MC_1(g)$, then $\beta = F(\alpha) \in MC_1(g)$ is of the form

$$\beta = \beta_1(l) + \lambda dt$$

for some $\lambda \in g^0$, and satisfies

$$\beta_1(0) = \alpha_1(0) \quad \text{and} \quad \beta_1(1) = \alpha_1(1).$$

**Remark 4.4.** As $R_\bullet$ is a projector, this proposition in fact gives information on the form of all the elements of $MC(g \otimes C_\bullet)$. 

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Proof. First notice that the map $R_\bullet$ commutes with the face maps and is the identity on 0-simplices, thus evaluation of the part of $\beta$ in $g^1 \otimes \Omega^0_n$ at the vertices gives the same result as evaluation at the vertices of $\alpha$. Next, we notice that $\beta$ is in the image of $I_\bullet$. We use the explicit formula for $(1 \otimes i_n)_\infty$ of [LV12, Sect. 10.3.5]: the operator acting on arity $k$ is composed by the following two parts:

- the differential $d$ in the image of $I_\bullet$.
- the differential $d$ in the image of $I_\bullet$.

But the $1 \otimes h$ at the root lowers the degree of the part of the form in $\Omega_n$ by 1, and thus we cannot get something in $g^{1-n} \otimes \Omega^0_n$ from these terms. The only surviving term is therefore the one coming from $(1 \otimes i_n)(P(\alpha))$, given by $\xi \otimes \omega_{0\ldots n}$ for some $\xi \in g^{1-n}$.

4.2. Comparison with Getzler’s $\infty$-groupoid $\gamma_\bullet$. Finally, we compare the simplicial set $MC(g \otimes C_\bullet)$ with Getzler’s Kan complex $\gamma_\bullet(g)$. We start with an easy result that follows directly from our approach, before exposing Bandieras’ result that these two simplicial sets are actually isomorphic.

Lemma 4.5. We have

$$I_\bullet MC(g \otimes C_\bullet) \subseteq \gamma_\bullet(g).$$

Proof. We have $h_\bullet i_\bullet = 0$. Therefore, by the explicit formula formula for $(i_\bullet)_\infty$ given in [LV12, Sect. 10.3.5], we have $h_\bullet(i_\bullet(\beta)) = 0$ for any $\beta \in g \otimes \Omega_\bullet$ in the image of $I_\bullet$. Thus

$$h_\bullet(MC(g \otimes C_\bullet)) = h_\bullet i_\bullet P_\bullet(MC_\bullet(g)) = 0,$$

which proves the claim. □

In his thesis [Ban14], Bandiera proves the following.

Theorem 4.6 ([Ban14, Thm. 2.3.3 and Prop 5.2.7]). The map

$$(P_\bullet, 1 \otimes h_\bullet) : MC_\bullet(g) \longrightarrow MC(g \otimes C_\bullet) \times (\text{Im}(1 \otimes h_\bullet) \cap (g \otimes C_\bullet)^1)$$

is bijective. In particular, its restriction to $\gamma_\bullet(g)$ is $\ker(1 \otimes h_\bullet) \cap MC_\bullet(g)$ gives an isomorphism of simplicial sets $P_\bullet : \gamma_\bullet(g) \longrightarrow MC(g \otimes C_\bullet)$.

Remark 4.7. Thanks to our approach, we immediately have an inverse for the map $P_\bullet$: it is of course the map $I_\bullet$. As a consequence of Bandiera’s result and of Proposition 4.3, we can partially characterize the differential forms appearing in $\gamma_\bullet(g)$, as follows.

Lemma 4.8. For each $n \geq 1$, the thin elements contained in $\gamma_n(g)$ have no term in $g^{1-n} \otimes \Omega^0_n$.

Proof. By Proposition 4.3 and Theorem 4.6, we know that if $\alpha \in \gamma_n(g)$, then $\alpha$ is of the form

$$\alpha = \cdots + \xi \otimes \omega_{0\ldots n}$$

for some $\xi \in g^{1-n}$, where the dots indicate terms in $g^{1-k} \otimes \Omega^0_k$ for $0 \leq k \leq n-1$, which will give zero after integration. Integrating, we get

$$\int_{\Delta^n} \alpha = \xi \otimes \int_{\Delta^n} \omega_{0\ldots n} = \xi \otimes 1.$$

Therefore, $\alpha$ is thin if, and only if $\xi = 0$. □

5. The case of Lie algebras

In this section, we focus on the case where $g$ is actually a dg Lie algebra. In this situation, we are able to represent the functor $MC(g \otimes C_\bullet)$ by a cosimplicial dg Lie algebra. The main tools used here are results from the article [RN17].

5.1. Reminder on the complete cobar construction. What we explain here is a special case of [LV12, Ch. 11.1-3], namely where we take $\mathcal{P} = \text{Lie}$ and only consider the canonical twisting morphism $\pi : BLie \rightarrow Lie$, where $BLie$ is the bar construction of the operad $\text{Lie}$ encoding Lie algebras. In fact, we consider a slight variation on the material presented there, as we remove the conilpotency condition on coalgebras but additionally add the requirement that algebras be complete. See also [RN17, Sect. 6.2].

Let $X$ be a dg $BLie$-coalgebra. The complete cobar construction of $X$ is the complete dg Lie algebra

$$\widehat{\Omega}_c X := \left(\widehat{Lie}(X), d := d_1 + d_2\right),$$

where the differential $d$ is composed by the following two parts:

1. The differential $-d_1$ is the unique derivation extending the differential $d_X$ of $X$. 
2. Additional terms involving the complete Lie algebra structure.
(2) The differential $-d_2$ is the unique derivation extending the composite

$$X \xrightarrow{\Delta_X} \widehat{BLie}(X) \xrightarrow{\rho_{01X}} \widehat{Lie}(X).$$

Notice that as $X$ is not assumed to be conilpotent, the decomposition map $\Delta_X$ really lands in the product

$$\widehat{BLie}(X) := \prod_{n \geq 0} (BLie(n) \otimes X^\otimes n)^{\wedge_n}$$

and not the direct sum. Thus it is necessary to consider the free complete Lie algebra over $X$. Also, there is a passage from invariants to coinvariants that is left implicit here, as the decomposition map lands in invariants, but the elements of the complete free Lie algebra $\widehat{Lie}(X)$ are coinvariants. This introduces factors of the form $\frac{1}{n!}$ when computing explicit formulæ for $d_2$.

The complete cobar construction $\hat{\Omega}_n$ defines a functor from dg $BLie$-coalgebras to complete dg Lie algebras.

5.2. Representing $MC(g \otimes C_\bullet)$. Using the Dupont contraction, the Homotopy Transfer Theorem gives the structure of a simplicial $C_\infty$-algebra to $C_\bullet$. As the underlying cochain complex $C_n$ is finite dimensional for each $n$, it follows that its dual is a cosimplicial $B(\mathcal{F} \otimes Lie)$-algebra. Therefore, the desuspension $sC_\bullet^\vee$ is a cosimplicial $BLie$-algebra, and we can take its canonical cobar construction.

**Definition 5.1.** We denote this cosimplicial dg Lie algebra by $mc_\bullet := \Omega_\pi(sC_\bullet^\vee)$.

**Theorem 5.2.** Let $g$ be a complete dg Lie algebra. There is a canonical isomorphism

$$MC(g \otimes C_\bullet) \cong \hom_{dgLie}(mc_\bullet, g).$$

It is natural in $g$.

**Proof.** By [RN17 Th. 5.1], the $C_\infty$-algebra structure we have on $g \otimes C_\bullet$ is the same as the structure that we obtain on the tensor product of the dg Lie algebra $g$ with the simplicial $C_\infty$-algebra $C_\bullet$ by using [RN17 Th. 3.4] with $\mathcal{P} = \mathcal{Q} = Lie$ and $\Psi = id_{Lie}$. Therefore, we can apply [RN17 Cor. 6.6], which gives the desired isomorphism.

With this form for $MC(g \otimes C_\bullet)$, Theorem 5.3 reads as follows.

**Corollary 5.3.** Let $g$ be a complete dg Lie algebra. There is a weak equivalence of simplicial sets

$$MC_\bullet(g) \simeq \hom_{dgLie}(mc_\bullet, g),$$

natural in $g$.

We can completely characterize the first levels of the cosimplicial dg Lie algebra $mc_\bullet$. Recall from [LS10] the Lawrence–Sullivan algebra: it is the unique free complete dg Lie algebra generated by two Maurer–Cartan elements in degree 1 and a single element in degree 0 such that the element in degree 0 is a gauge between the two generators.

**Proposition 5.4.** The first two levels of the cosimplicial dg Lie algebra $mc_\bullet$ are as follows.

1. The dg Lie algebra $mc_0$ is isomorphic to the free dg Lie algebra with a single Maurer–Cartan element as the only generator.

2. The dg Lie algebra $mc_1$ is isomorphic to the Lawrence–Sullivan algebra.

**Proof.** For $\Omega_1$, we have $\Omega_0 \cong \mathbb{K} \cong C_0$, both $p_0$ and $i_0$ are the identity, and $h_0 = 0$. It follows that, as a complete graded free Lie algebra, $mc_0$ is given by

$$mc_0 = \widehat{Lie}(s\mathbb{K}).$$

We denote the generator by $\alpha := s1^\vee$. It has degree 1. Let $g$ be any complete dg Lie algebra, then a morphism

$$\phi : mc_0 \longrightarrow g$$

is equivalent to the Maurer–Cartan element

$$\phi(\alpha) \otimes 1 \in MC(g \otimes C_\bullet) \cong MC(g).$$

Conversely, through $P_0$ every Maurer–Cartan element of $g$ induces a morphism $mc_0 \rightarrow g$. As this is true for any dg Lie algebra $g$, it follows that $\alpha$ is a Maurer–Cartan element.
To prove (2), we start by noticing that
\[
C_1 := \mathbb{K} \omega_0 \oplus \mathbb{K} \omega_1 \oplus \mathbb{K} \omega_{01}
\]
with \(\omega_0, \omega_1\) of degree 0 and \(\omega_{01}\) of degree 1. Denoting by \(\alpha_i := s \omega_i^\vee\) and by \(\lambda := s \omega_0^\vee\), we have
\[
mc_1 = \tilde{L}(\alpha_0, \alpha_1, \lambda)
\]
as a graded Lie algebra. Let \(\mathfrak{g}\) be any dg Lie algebra, then a morphism
\[
\phi : mc_1 \longrightarrow \mathfrak{g}
\]
is equivalent to a Maurer–Cartan element
\[
\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01} \in MC(\mathfrak{g} \otimes C_1),
\]
see [RN17 Sect. 6.3–4]. Applying \(I_1\), as in the proof of Proposition 4.3 we obtain
\[
I_1(\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01}) = a(t_0, t_1) + \phi(\lambda) \otimes \omega_{01} \in MC_1(\mathfrak{g})
\]
with \(a(1, 0) = \phi(\alpha_0)\) and \(a(0, 1) = \phi(\alpha_1)\). The Maurer–Cartan equation for \(a(t_0, t_1) + \phi(\lambda) \otimes \omega_{01}\) then shows that \(\phi(\lambda)\) is a gauge from \(\phi(\alpha_0)\) to \(\phi(\alpha_1)\). Conversely, if we are given the data of two Maurer–Cartan elements of \(\mathfrak{g}\) and a gauge equivalence between them, then this data gives us a Maurer–Cartan element of \(\mathfrak{g} \otimes \Omega_1\). Applying \(P_1\), then gives back a non-trivial morphism \(mc_1 \rightarrow \mathfrak{g}\). As this is true for any \(\mathfrak{g}\), it follows that \(mc_1\) is isomorphic to the Lawrence–Sullivan algebra. \(\square\)

Remark 5.5. Alternatively, one could write down explicitly the differentials for both \(mc_0\) (which is straightforward) and \(mc_1\) (with the help of [CG08 Prop. 19]). An explicit description of \(mc_*\) is made difficult by the fact that one needs to know all the \(C_\infty\)-algebra structure on \(C_*\), in order to write down a formula for the differential.

5.3. Relations to rational homotopy theory. The cosimplicial dg Lie algebra \(mc_*\) has already made its appearance in the literature not long ago, in the paper [BFMT15], in the context of rational homotopy theory, where it plays the role of a Lie model for the geometric \(n\)-simplex. With the goal of simplifying comparison and interaction between our work and theirs, we provide here a short review and a dictionary between our vocabulary and the notations used in [BFMT15].

| Notation of this paper | Notation of [BFMT15] |
|------------------------|---------------------|
| \(mc_*\)               | \(\Sigma_*\) or \(\Sigma_{\Delta}^*\) |
| \(\Omega_*\)           | \(A_{PL}(\Delta^*)\) |
| \(B_*\)                | Quillen functor \(\mathcal{C}\) |
| \(\text{hom}_{dgLie}(mc_*, -)\) | \((-\) ) |
| \(\text{hom}_{dgCom}(-, \Omega_*\) | \((-\) )_{S} |

The following theorem has non-empty intersection with our results. We say a dg Lie algebra is of finite type if it is finite dimensional in every degree and if its degrees are bounded either above or below.

Theorem 5.6 ([BFMT15 Th. 8.1]). Let \(\mathfrak{g}\) be a dg Lie algebra of finite type with \(H^n(\mathfrak{g}, d) = 0\) for all \(n > 0\). Then there is a homotopy equivalence of simplicial sets
\[
\text{hom}_{dgLie}(mc_*, \mathfrak{g}) \simeq \text{hom}_{dgCom}(B_\cdot(\mathfrak{g}^\vee), \Omega_\cdot) .
\]
We can easily recover an analogous result, which works on complete dg Lie algebras of finite type, but without restrictions on the cohomology, using our main theorem and some results of [RN17].

Proposition 5.7. Let \(\mathfrak{g}\) be a complete dg Lie algebra of finite type. Then there is a weak equivalence of simplicial sets
\[
\text{hom}_{dgLie}(mc_*, \mathfrak{g}) \simeq \text{hom}_{dgCom}(B_\cdot(\mathfrak{g}^\vee), \Omega_\cdot) .
\]

Proof. The proof is given by the sequence of equivalences
\[
\text{hom}_{dgCom}(B_\cdot(\mathfrak{g}^\vee), \Omega_\cdot) \cong \text{hom}_{dgCom}(\Omega_\cdot(s^{-1}\mathfrak{g}^\vee), \Omega_\cdot) \\
\cong MC(\mathfrak{g} \otimes \Omega_\cdot) \\
\cong \text{hom}_{dgLie}(mc_*, \mathfrak{g}) .
\]

In the first line we used the natural isomorphism
\[
B_\cdot (s^{-1}\mathfrak{g}^\vee) \cong \Omega_\cdot(s^{-1}\mathfrak{g}^\vee) .
\]
Notice that the assumptions on $g$ make it so that $g^\vee$ is a Lie$^\vee$-coalgebra. In the second line we used a straightforward generalization [RN17, Cor. 6.6] for $Q = P = \text{Com}$ and $\Psi$ the identity morphism of $\text{Com}$. Finally, in the third line we used our Corollary [5,3].

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