THE GEOMETRY OF MIRROR SYMMETRY

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1. Introduction

Mirror symmetry was discovered in the late 1980s by physicists studying superconformal field theories (SCFTs). One way to produce SCFTs is from closed string theory; in the Riemannian (rather than Lorentzian) theory the string’s worldline gives a map of a Riemannian 2-manifold into the target with an action which is conformally invariant, so the 2-manifold can be thought of as a Riemann surface with a complex structure. Making sense of the infinities in the quantum theory (supersymmetry and anomaly cancellation) forces the target to be 10 dimensional – Minkowski space times by a 6-manifold \( X \) – and \( X \) to be (to first order) Ricci flat and so to have holonomy in \( SU(3) \). That is \( X \) is a Calabi-Yau 3-fold \((X, \Omega, \omega)\). So SCFTs come from \( \sigma \)-models (mapping Riemann surfaces into Calabi-Yau 3-folds) but, it turns out, in two different ways – the A-model and the B-model. Deformations of the SCFT and either \( \sigma \)-model are isomorphic, so over an open set the two coincide. So it was natural to conjecture that pretty much all of the relevant SCFTs came from geometry – from an A or B \( \sigma \)-model. In particular, the A-model of a Calabi-Yau \( X \) should therefore give the same SCFT as the B-model on another Calabi-Yau \( \bar{X} \). It turns out then that the A-model on \( \bar{X} \) should also be isomorphic to the B-model on \( X \); thus mirror symmetry should give an involution on the set of Calabi-Yau manifolds. (The full picture is slightly more complicated, to do with large complex structure limits, described below, multiple mirrors and flops.) By studying the SCFTs, Greene and Plesser predicted the mirror of the simplest Calabi-Yau 3-fold, the quintic in \( \mathbb{P}^4 \), and mirror symmetry was born.

Topological observables, i.e. certain path integrals over the space of all maps, can be calculated by the semi-classical approximation as integrals over the space of classical minima – (anti)holomorphic curves in the Calabi-Yau (these minimize volume in a fixed homology class). From the zero homology class we get the constant maps – points in \( X \) – and so integrals over \( X \). In some cases, by Poincaré duality, these can be thought of as intersections of cycles; we think of the string worldsheet lying at a point of intersection. When the worldsheet has a nontrivial homology class it allows more general ‘intersections’ where the cycles need not intersect but are connected by a holomorphic curve, giving a perturbation of the usual intersection product on cohomology called quantum cohomology. Namely, there is a contribution \(!a.\beta_b.\beta_c.\beta_\epsilon^\omega!\) to the quantum triple product \( a.b.c \) of three four-cycles \( a, b, c \in H^{1,1} \cong H^2 \cong H_4 \) from each holomorphic curve \( \beta \) (of genus 0, in the 0-loop approximation to the physics) in \( X \) of area \( \int_\beta \omega \) (where \( \omega \) is the Kähler form). The A-model correlation functions can be determined from this data; the B-model computation involves no such quantum correction and can be computed purely in terms of integrals over cycles (“periods”) and their derivatives, discussed in (4) below. So it is in some sense easier and, in a historic tour-de-force, was calculated in [COGP] for the Greene-Plesser mirror of the quintic. Comparing with the A-model computation on the quintic gave remarkable predictions about the number of holomorphic rational curves on the quintic. These were way beyond mathematical capabilities at the time, and sparked enormous mathematical interest. The predictions (and more) have now been proved to be true by Givental and Lian-Liu-Yau, while mirror symmetry has begun to be understood geometrically. But in some sense the mathematical reason for the relationship between the Yukawa couplings and the quantum cohomology of the mirror is still a little mysterious; it is the hardest part of mirror symmetry to see in the geometry, yet for the physics it was the easiest and first prediction.

We survey, non-chronologically, some of the geometry of mirror symmetry as it is now understood, mainly in dimension \( n = 3 \). For the many topics omitted the reader should consult the references.
2. The geometric set-up

A Calabi-Yau 3-fold \((X, \Omega, \omega)\) is a Kähler manifold \((X, \omega)\) with a holomorphic trivialization \(\Omega\) of its canonical bundle

\[
K_X = \Lambda^2 T^* X
\]

(i.e. a nowhere vanishing holomorphic volume form, locally \(dz_1 \wedge dz_2 \wedge dz_3\)), and \(b_1(X) = 0\). It follows that the Hodge numbers \(h^{0,2}, h^{1,1}\) vanish, and so \(H^2(X, \mathbb{C}) = H^{1,1}\) and \(H^3(X, \mathbb{R}) \cong H^{2,1} \oplus H^{3,0}\). By Yau’s theorem the Kähler metric can be changed within its \(H^2(X, \mathbb{R})\) cohomology class to a unique Ricci-flat Kähler metric; equivalently \(\Omega\) is parallel, so the induced metric on \(K_X\) within its canonical bundle is sometimes extended by adding a nowhere vanishing holomorphic volume form, \(\Omega\), up to scaling by \(C^*\) on the (conjectural) mirror \(\check{X}\). Kähler deformations are unobstructed, forming an open set \(\mathcal{K}_X\) in \(H^2(X, \mathbb{R})\).

Its closure \(\overline{\mathcal{K}_X}\) is sometimes extended by adding the Kähler cones of all birational models of \(X\) to give Kawamata’s moveable cone. This is because work of Aspinwall, Greene, Morrison and Witten suggested that all birational models of \(X\) are indistinguishable to string theory and so are all mirrors of \(X\), corresponding to a different choice of \((1,1)\)-form \(\omega\) which is a Kähler form on one model only. \(\mathcal{K}_X\) is also complexified by including in the A-model data any “B-field” \(B \in H^2(X, \mathbb{R}/\mathbb{Z})\), and divided by holomorphic automorphisms of \(X\) to give a moduli space of complex dimension \(h^{1,1}(X)\). Deformations of complex structure are also unobstructed by the non-trivial Bogomolov-Tian-Todorov theorem, so form a smooth space with tangent space

\[
H^1(T \check{X}) \cong H^1(\Lambda^2 T^* \check{X}) = H^{2,1}(\check{X}).
\]

(Given a deformation of complex structure, this isomorphism takes the \(H^{2,1}\)-component of the derivative of the \((3,0)\)-form \(\Omega\).) So for the moduli spaces to match up, we get the first and simplest prediction of mirror symmetry, that

\[
(1) \quad h^{1,1}(X) = h^{2,1}(\check{X}) \quad \text{and} \quad h^{2,1}(X) = h^{1,1}(\check{X}).
\]

This is where mirror symmetry gets its name, the above relation making the Hodge diamonds of \(X\) and \(\check{X}\) mirror images of each other.

As the complexified Kähler cone is a tube domain it has natural partial complex compactifications (due to Looijenga, and suggested in the context of mirror symmetry by Morrison [M]). The simplest case is where we ignore the moveable cone and automorphisms and assume there is an integral basis \(e_1, \ldots, e_n\) of both \(\mathcal{K}_X\) and \(H^2(X, \mathbb{Z})/\text{torsion}\). The complexified Kähler moduli space is then

\[
\mathcal{K}_X^C := H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) + i\mathcal{K}_X = \{ B + i\omega \},
\]

with natural coordinates \(x_i, y_i \geq 0\) pulled back from the first and second factors respectively, induced by the \(e_i\). \(x_i\) is multiply-valued with integer periods, so

\[
(2) \quad z_i = \exp(2\pi i(x_i + iy_i))
\]

is a well defined holomorphic coordinate, giving an isomorphism to the product of \(n\) punctured unit disks in \(\mathbb{C}^n\):

\[
\mathcal{K}_X^C \cong (\mathbb{C}^*)^n = \{(z_i) : 0 < |z_i| \leq 1\} \subset (\mathbb{C}^*)^n.
\]

The compactification \(\Delta^n\) comes from adding in the origins in the disks, which we reach by going to infinity (in various directions) in \(\mathcal{K}_X^C\). We call the point \((0, \ldots, 0) \in \Delta^n\) the large Kähler limit point (LKLP) in this case. Moving along the ray generated by \(\sum k_i e_i \in \mathcal{K}_X\), \(k_i \geq 0\), complexifies in the holomorphic structure (2) to give the analytic curve

\[
(3) \quad z_i^{k_i} = z_j^{k_j}, \quad \forall i, j,
\]

in \(\mathcal{K}_X^C\). For \(k_i \in \mathbb{Q}\) \(\forall i\) this extends to a complete curve in the compactification. Without loss of generality we can assume the \(k_i\) are integers with no common factor, then the link of the curve winds around the LKLP \((0, \ldots, 0) \in \Delta^n\) with winding number

\[
(k_1, \ldots, k_n) \in \pi_1(H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) + i\mathcal{K}_X) = H^2(X, \mathbb{Z}) = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n.
\]

This is because multiplying the ray \(\mathbb{R}\sum k_i e_i \in \mathcal{K}_X\) by \(i\) gives the direction \(\mathbb{R}\sum k_i e_i \in \mathcal{K}_X\) of \(H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})\) of B-fields, with the given winding number. For \(k_i\) not rational we get an analytic mess; the direction in the space of B-fields does not close up to give a circle.

There is no obvious mirror to these rays since we consider \(\Omega\) only up to scale. So mirror symmetry predicts an isomorphism between \(\mathcal{K}_X^C\) and the moduli space \(\mathcal{M}_X\) of complex structures on \(\check{X}\),


and a distinguished limit in \( \mathcal{M}_X \), the large complex structure limit point (LCLP), the mirror of the LKLP \((0, \ldots, 0) \in \Delta^n \) above. Morrison has given a rigorous definition of LCLPs and the canonical coordinates on \( \mathcal{M}_X \) dual to the \( z_i \) on \( K^C_X \); see Section 7.1 below. The holomorphic curves in \((\Delta)^n \) described above, corresponding to rational rays of \( \Lambda^h \) described above, give degenerations of (the complex structure on) \( \check{X} \) to the LCLP whose monodromy we will discuss in Section 5.

LCLPs play a vital role in mirror symmetry; in fact mirror symmetry is really a statement about LCLPs and families of Calabi-Yau manifolds near LCLPs. Most predictions only really hold near or at the LCLP, and complex structure moduli space only looks like \( \Delta^n \) near the LCLP. For instance, manifolds can have many LCLPs and accordingly many mirrors. This also explains one obvious paradox – that rigid Calabi-Yau manifolds, those with no complex structure deformations, \( h^{2,1} = 0 \), and so no LCLP, can have no mirror, since a \( \check{K} \) (or symplectic) manifold has \( h^2 = h^{1,1} \neq 0 \).

The first predicted refinement of (1) is, as discussed in the introduction, that the variation of Hodge structure (VHS) on \( \check{X} \) should be describable in terms of Gromov-Witten invariants of \( X \). Here VHS is governed by how the ray \( \mathbb{C} \cdot \Omega_t = H^3(X, \mathbb{C}) \) sits inside \( H^3(X_t, \mathbb{C}) \) as the complex structure on \( X_t \) varies, parametrized by \( t \in \mathcal{M}_X \). By Poincaré duality, it is sufficient to know how \( \Omega_t \) pairs with \( H^3(X) \), i.e. to compute the period integrals

\[
\int_{A_i} \Omega_t, \quad i = 1, \ldots, 2k = 2h^{2,1} + 2,
\]

where \( A_i \) form a basis of \( H^3(X, \mathbb{Z}) \). (In fact we can choose the \( A_i \) to be a symplectic basis, \( A_i A_j = \delta_{i+k,j} \), and then knowledge of only the periods of the first \( k \) \( A_i \) suffices, locally in moduli space.) These periods determine \( \Omega_t \) and so the Yukawa coupling

\[
H^1(T \check{X}_t) \cong \mathbb{C} \ni H^3(A^3 T \check{X}_t) \xrightarrow{\partial \delta^2} H^3(K_X) \cong \mathbb{C}.
\]

On \( X \) we get the cubic form on \( H^2(X) \) described roughly in the introduction in terms of numbers of rational curves in \( X \). These numbers are in fact independent of the almost complex structure on \( X \) so long as it is compatible with the symplectic form \( \omega \), so give the symplectic invariants of Gromov and Witten. The cubic form depends on \( \omega = \omega_t \) as it moves in \( K_X \) (or in \( K^C_X \), replacing \( \omega \) by \( -i(B + i \omega_t) \)). Under the predicted local isomorphism \( K^C_X \cong \mathcal{M}_X \) near the LKLP and LCLP, the equality of these cubic forms gives the predictions of numbers of rational curves in \( X \) mentioned in the introduction. This has been carried out, and the predictions checked rigorously, in quite some generality, for instance for mirror pairs produced by Batyrev’s toric methods.

There is, of course, a flat connection, the Gauss-Manin connection on the bundle over \( \mathcal{M}_X \) with fiber \( H^3(X_t, \mathbb{C}) \) over \( t \in \mathcal{M}_X \), given by the local system \( H^3(X_t, \mathbb{Z}) \subset H^3(X_t, \mathbb{C}) \). Mirror to this, Dubrovin has shown how to put a flat connection on the bundles with fibers \( H^2(X_t) \) and \( H^{\text{ev}}(X_t) \) using Gromov-Witten invariants.

### 3. Homological mirror symmetry

Building on work of Witten, in 1994 Kontsevich [K] proposed a remarkable conjecture that purported to explain mirror symmetry, all the more surprising because it appeared to have little to do with what was thought to be mirror symmetry at the time. The conjecture is now reasonably well understood, while the link to Gromov-Witten invariants and Yukawa couplings is more mysterious, though it is known how both data should be encoded in the conjecture.

Kontsevich proposed that mirror symmetry is a (non-canonical) equivalence of triangulated categories between the derived Fukaya category \( D^F(X) \) of \((X, \omega) \) and the bounded derived category of coherent sheaves \( D^b(\check{X}) \) on its mirror \( \check{X} \). This second category consists of chain complexes of holomorphic bundles, with quasi-isomorphisms (maps of chain complexes inducing isomorphisms on cohomology) formally inverted, i.e. decreed to be isomorphisms. For zero B-field the first category should be constructed using Lagrangian submanifolds \( L \subset X \) carrying flat unitary connections \( A \). That is, \( L \) is middle (three) dimensional, and

\[
\omega|_L \cong 0, \quad F_A = 0.
\]

For \( B \neq 0 \) this needs modifying to \( F_A + 2\pi i B \cdot \text{id} = 0 \) (so in particular we require that \( L \) satisfies \( [B]|_L = 0 \in H^2(L, \mathbb{R}/\mathbb{Z}) \)). There are also various technical conditions like the choice of a relative spin structure, the Maslov class of \( L \) must vanish (i.e.
the map \((\Omega_L^*/\text{vol}_L): L \to \mathbb{C}^*\) has winding number zero) and we pick a grading on \(L\) (a choice of logarithm of this map). Morphisms are defined by Floer cohomology \(HF^*\) of Lagrangian submanifolds; roughly speaking this assigns a vector space to each intersection point (the \(\text{Homs}\) between the fibers of the two unitary bundles carried by the Lagrangians at this point), made into a chain complex by a certain counting of holomorphic disks between intersection points. Deep work of Fukaya-Oh-Ohta-Ono shows that this gives the structure of an \(A^\infty\)-category which can then be “derived” into a triangulated category in a formal way by taking “twisted cochains”. The construction is still very technical and hard to calculate with, but the key points are that we get a category depending only on the symplectic structure, that certain “unobstructed” Lagrangian submanifolds give objects of this category, and that \(\text{Hamiltonian isotopy}\) unobstructed Lagrangian submanifolds give isomorphic objects.

Since the introduction of \(D\)-branes there is a physical interpretation of this conjecture in terms of open string theory: the objects of the two categories are boundary conditions for open strings, and morphisms correspond to strings beginning on one object and ending on the other. So, for instance, intersections of Lagrangians give morphisms corresponding to constant strings at the intersection point, while the Floer differential gives instanton tunneling corrections.

One paradox this formulation immediately sheds light on is automorphisms on both sides of mirror symmetry. While symplectomorphisms of \((X, \omega)\) are abundant, there are few holomorphic automorphisms of a Calabi-Yau \(\tilde{X}\). The former induce automorphisms of \(D^F(X)\); Kontsevich’s suggestion is that mirror to this there should be an autoequivalence of \(D^b(\tilde{X})\); this need not be induced by an automorphism of \(\tilde{X}\). Motivated by this, groups of autoequivalences of derived categories of sheaves of Calabi-Yau manifolds have now been found that were predicted by mirror symmetry; a few are mentioned below. Thus homological mirror symmetry suggests a SCFT is equivalent to a triangulated category, and the ambiguities in geometrizing a SCFT (finding a Calabi-Yau of which it is a \(\sigma\)-model) are seen in the category – not all automorphisms come from an automorphism of a Calabi-Yau, Calabi-Yau manifolds \(\tilde{X}\) with equivalent derived categories give multiple mirrors to \(X\), and not all appropriate categories need even come from a Calabi-Yau. Supporting this suggestion, Bondal-Orlov and Bridgeland have shown that indeed birational Calabi-Yau manifolds \(\tilde{X}\) have equivalent derived categories.

Finally, Kontsevich explained how deformation theory of the categories should involve derived morphisms on the product from the diagonal (thought of as a Lagrangian in the A-model, its structure sheaf as a coherent sheaf in the B-model) to itself, giving quantum cohomology in the A-model and Hodge structure in the B-model. For instance the holomorphic disks used to compute the Floer cohomology of the diagonal on the product \(X \times X\) give holomorphic rational curves on \(X\). So one should be able to see some parts of “classical” mirror symmetry.

Below, as we describe more of the geometry of mirror symmetry that has emerged since Kontsevich’s conjecture, we will mention at each stage how his conjecture fits in with it.

4. The SYZ Conjecture

To recover more geometry from Kontsevich’s conjecture, there are some objects of \(D^b(\tilde{X})\) that obviously reflect the geometry of \(\tilde{X}\) – the structure sheaves \(O_p\) of points \(p \in \tilde{X}\). Calculating their self-\(\text{Homs}\),

\[
\text{Ext}^\bullet(O_p, O_p) \cong \Lambda^\bullet T_p \tilde{X} \cong \Lambda^\bullet \mathbb{C}^3 \cong H^\bullet(T^3, \mathbb{C}),
\]

shows that if they are mirror to Lagrangians \(L\) in \(X\) (with flat connections \(A\) on them) then we must have

\[
HF^*((L, A), (L, A)) \cong H^\bullet(T^3, \mathbb{C}),
\]

as graded vector spaces. Since the left hand side is, modulo instanton corrections, \(H^\bullet(L, \mathbb{C})^{\mathbb{Z}, r}\), where \(r\) is the rank of the bundle carried by \(L\), this suggests that the mirror should be \(L \cong T^3\) with a flat \(U(1)\) connection \(A\) over it. There are then reasons why the Floer cohomology of such an object should not be quantum corrected, and so be isomorphic to \(\text{Ext}^\bullet(O_p, O_p)\).

For any Lagrangian \(L\), the symplectic form gives an isomorphism between \(T^*L\) and its normal bundle \(N_L\); thus Lagrangian tori have trivial normal bundles, and locally one can fiber \(X\) by them. Thus one might hope that \(X\) is fibered by Lagrangian
tori, and the mirror $\tilde{X}$ is (at least over the locus of smooth tori) the dual fibration. This is because the set of flat $U(1)$ connections on a torus is naturally the dual torus.

This is the kind of philosophy that led to the SYZ conjecture [SYZ], though Strominger, Yau and Zaslow were working with physical D-branes, and not Kontsevich’s conjecture. Therefore their D-branes are not the “topological D-branes” of Kontsevich, but those minimizing some action. That is, instead of holomorphic bundles in the B-model, we deal with bundles with a compatible connection satisfying an elliptic PDE (like the Hermitian-Yang-Mills equations (HYM), or some perturbation thereof); instead of Lagrangian submanifolds up to Hamiltonian isotopy in the A-model, we consider special Lagrangians (sLags) \(5\). The SYZ conjecture is that a Calabi-Yau \(X\) should admit a sLag torus fibration, and that the mirror \(\tilde{X}\) should admit a fibration which is dual, in some sense.

A sLag is a Lagrangian submanifold of a Calabi-Yau manifold \(X\) satisfying the further equation that the unit norm complex function (phase)

\[
\frac{\Omega_L}{\text{vol}_L} = e^{i\theta} = \text{constant}.
\]

(5) So sLags have Maslov class zero, in particular.) This equation uses the complex structure on \(X\) as well as the symplectic structure, and the resulting Ricci-flat metric of Yau, to define a metric on \(L\) and so its Riemannian volume form \(\text{vol}_L\). SLags are calibrated by \(\text{Re}(e^{-i\theta}\Omega)\) and so minimize volume in their homology class. This is similar to the HYM equations on the mirror \(\tilde{X}\), which are defined on holomorphic bundles on the complex manifold \(\tilde{X}\) via a Kähler form \(\omega\), and minimize the Yang-Mills action. The Donaldson-Uhlenbeck-Yau theorem says that for holomorphic bundles that are polystable (defined using \(\omega\), this is true for the generic bundle), there is a unique compatible HYM connection. Thus modulo stability, HYM connections are in one-to-one correspondence with holomorphic bundles. Thomas and Yau conjecture, and prove in some special cases, a similar correspondence for (special) Lagrangians: that modulo a stability condition (which can be formulated precisely), sLags are in one-to-one correspondence with Lagrangian submanifolds up to Hamiltonian isotopy. That is, there should be a unique sLag in the Hamiltonian isotopy class of a Lagrangian if and only if it is stable. Currently only the uniqueness part of this conjecture has been worked out, but morally at least, we do not lose much by considering only Lagrangian torus fibrations.

The SYZ conjecture is thought to hold only near the LCLPs and LKLPs of \(X\) and \(\tilde{X}\); away from these the sLag fibers may start to cross. Due to work of Joyce, the discriminant locus of the fibration on \(X\) is expected to be a codimension one ribbon graph in a base \(S^3\) near the limit points, while the discriminant locus of the dual fibration \(\tilde{X}\) may be different – i.e. the smooth parts of the fibration and its dual are compactified in different ways. In the limit of moving to the limit points, however, both discriminant loci shrink onto the same codimension two graph. In this limit the fibers shrink to zero size, so that \(X\) (with its Ricci-flat metric) tends in the Gromov-Hausdorff sense to its base \(S^3\) (with a singular metric). This formal picture has been made precise in two dimensions, for \(K3\)-surfaces, by Gross and Wilson. The limiting picture suggests that if we are only interested in topological or Lagrangian torus fibrations then we might hope for codimension two discriminant loci, and such fibrations might make sense well away from limit points. Work of Gross and Ruan carries this out in examples such as the quintic and its mirror, and makes sense of dualizing the fibration by dualizing monodromy around the discriminant locus and specifying a canonical compactification over the discriminant locus. This gives the correct topology for toric varieties and their mirrors, and flips Hodge numbers \(1\), for instance. Approaching the LCLP in a different way (in the example \(3\) this corresponds to altering the rational numbers \(k_i\) can give a different graph and different fibration on \(X\); the dual fibration can then be a topologically different manifold, giving a different birational model of the mirror \(\tilde{X}\).

We focus only on Lagrangian fibrations, as they are better behaved and understood. We can expect them to be \(C^\infty\) fibrations with codimension two discriminant loci, for instance. Below we see how to put a complex structure on the smooth part of the fibration, but extending this over the compactification is much harder and will involve “instanton corrections” coming from holomorphic disks. Fukaya
5. Lagrangian Torus Fibrations

If \((X^{2n}, \omega) \to B^n\) is a smooth Lagrangian fibration with compact fibers, then the fibration is naturally an affine bundle of torus groups (i.e., a bundle of groups once we pick a Lagrangian zero-section – an identity in each fiber), and the base \(B\) inherits a natural integral affine structure: it looks like a vector space \(V\) with an integral structure \(V \cong \Lambda \otimes \mathbb{R}\) up to translation by elements of \(V\). This is the classical theory of action-angle variables. \(T^*_bB\) acts on the fiber \(X_b = \pi^{-1}(b)\): by pullback and contraction with the symplectic form, \(\sigma \in T^*_bB\) gives a vector field \(\sigma\) acting on \(X_b\), and the time-one flow along \(\sigma\) gives the action. By compactness and smoothness of \(X_b\) the kernel is a full rank lattice \(\Lambda_b \subset T^*_bB\), giving the isomorphism

\[
X_b \cong T^*_bB/\Lambda_b.
\]

We define the integral affine structure on \(B\) by specifying the integral affine functions \(f\) (up to translation) to be those whose time-one flow along \(df\) is the identity (i.e., on the universal cover the time-one flow is to a section of the bundle of lattices \(\Lambda\)).

The situation that interests us is where \(B\) is a 3-manifold \(\overline{B}\) (usually \(S^3\)) minus a graph; then the monodromy around the graph preserves the integral affine structure:

\[
\pi_1(B) \to \mathbb{R}^3 \times GL(3, \mathbb{Z}).
\]

A great deal of mirror symmetry can be seen from just this knowledge of the smooth locus of the fibration; in particular Gross [G] has shown how mild assumptions about the compactification (with singular fibers over \(\overline{B}\)) are enough to determine much of the topology of \(X\). The dual fibration \(\tilde{\pi}\) should have the monodromy dual to (6), and he shows how this implies the switching of the Hodge numbers (1) by the Leray spectral sequence; the rough idea being the obvious isomorphism

\[
R^i \pi_* \mathbb{R} \cong \Lambda^i T^*B \cong \Lambda^{3-i} T^*B \cong R^{3-i} \tilde{\pi}_* \mathbb{R}
\]

induced by a trivialization of \(\Lambda^3 T^*B\). That is, intuitively, the flipping of Betti numbers arises by representing cycles by those with linear intersection with the fibers, and replacing this linear space by its annihilator in the dual torus. This also agrees with the equivalence taking Lagrangians to coherent sheaves described in the next section.

The dual fibration \(\tilde{\pi}\) has a natural complex structure; here the affine structure is essential, as in general a tangent bundle \(TB\) has a natural almost complex structure along its zero-section. Since, up to translation, locally \(B \cong V\) is a vector space, \(TB = V \times V \cong V \otimes \mathbb{R} \mathbb{C}\) has a natural complex structure which descends to

\[
\tilde{\pi}: \tilde{X} = TB/\Lambda^* \to B.
\]

Gross suggests that the \(B\)-field on \(X\) should lie in the piece

\[
H^1(R^1 \pi_* \mathbb{R}/\mathbb{Z}) = H^1(TB/\Lambda^*)
\]

of the Leray spectral sequence for \(H^2(X, \mathbb{R}/\mathbb{Z})\). So it gives a \(C\)ech cocycle \(e\) on overlaps of an open cover of \(B\) with values in the dual bundle of groups \(TB/\Lambda^*\). Using this to twist (7) and reglue it via transition functions translated by \(e\), we get a new complex manifold \((e\) is locally constant so translation by \(e\) is holomorphic which we think of as mirror to \(X\) with complexified form \(B + i\omega\). In this way Gross manages to match up complexified symplectic deformations of \(X\) with complex structures on \(\tilde{X}\).

6. The Two-Torus

Mirror symmetry is non-trivial even for the simplest Calabi-Yau – the two-torus. We write it as an SYZ fibration \(T^2 \to B = S^1\), and write \(B = \mathbb{R}/a\mathbb{Z}\) with its standard integral affine structure induced by \(Z \subset \mathbb{R}\). This trivializes \(T^*B = B \times \mathbb{R}\) and the lattice \(\Lambda\) in it as \(B \times Z \subset B \times \mathbb{R}\). So as a symplectic manifold,

\[
T^2 = T^*S^1 / \Lambda = \frac{[0,a] \times [0,1]}{(0, p) \sim (a, p), (q, 0) \sim (q, 1)},
\]

with symplectic coordinates \((q, p)\) in which the symplectic form is \(\omega = dp \wedge dq\) (so \(\int_{T^2} \omega = a\)). Again the \(B\)-field, \(b \in H^1(R^1 \pi_* \mathbb{R}/\mathbb{Z}) = H^2(T^2, \mathbb{R}/\mathbb{Z})\), is in \(H^1\) of the locally constant sections of the dual fibration.

In our trivialization \(B \cong \mathbb{R}/a\mathbb{Z}\), \(\Lambda^* 
 times \(TB\) is also standard: \(B \times Z \subset B \times \mathbb{R}\), so the mirror has the same description (8) in which the complex structure
is standard: \( J\partial_p = \partial_q \). That is, \( p + iq \) gives a local holomorphic coordinate.

For nonzero B-field \( b \neq 0 \), twisting the dual fibration by \( b \) gives

\[
T^2 = \frac{T^*S^1}{\Lambda} = \frac{[0, a] \times [0, 1]}{(0, p) \sim (a, b + p), (q, 0) \sim (q, 1) }
\]

again with holomorphic structure given by \( p + iq \) and SYZ fibration \( \pi \) being projection onto \( q \). So as a complex manifold the mirror is \( \mathbb{C} \) divided by the lattice

\[
\Lambda = \langle 1, b + ia \rangle.
\]

Changing \( b \) to \( b + 1 \) does not alter this lattice, so the construction is well defined for \( b \in \mathbb{R}/\mathbb{Z} \cong H^1(R^1\pi_* \mathbb{R}/\mathbb{Z}) \), and we have the standard description of an elliptic curve via its period point \( \tau = b + ia \) in the upper half plane (as \( a > 0 \)). Mirror symmetry has indeed swapped the complexified symplectic parameter \( b + ia = \int_{T^2}(b + i\omega) \) for the complex structure modulus \( \tau = b + ia \). \( SL(2, \mathbb{Z}) \) acts on both sides (in the standard way on \( \tau \), and as symplectomorphisms modulo those isotopic to the identity on the \( A \)-side) permuting the choices of SYZ fibration. We note that in this case, the fibrations are special Lagrangian in the flat metric, with no singular fibers.

Polishchuk and Zaslow have worked out in detail how Kontsevich’s conjecture works in this case. The general picture for any torus fibration is an extension of the fiberwise duality that led to SYZ. Namely, Lagrangian multisectiions \( L \) of the fibration, of degree \( r \) over the base, give \( r \) points on each fiber, and so \( r \) flat \( U(1) \) connections on the dual fiber. The resulting \( U(1)^r \) connections can be glued together and twisted by the flat connection on \( L \), to give a rank \( r \) vector bundle with connection on the mirror. Arinkin and Polishchuk show that in general the Lagrangian condition implies the integrability condition \( F^{0,2} = 0 \) of the resulting connection, giving a holomorphic structure on the bundle. Leung-Yau-Zaslow show that the special Lagrangian condition gives a perturbation of the HYM equations on the connection. Branching of sections has been dealt with by Fukaya, and requires instanton corrections from holomorphic disks. Other Lagrangians with linear intersection with the fibers can be dealt with similarly. \( T^2 \) is simpler because all Lagrangians with vanishing Maslov class can be isotoped into straight lines (i.e. sLags in the flat metric) with no branching. The upshot is that the slope of the sLag over the base corresponds to the slope \( \left( \int_{T^2} c_1/r \right) \in [-\infty, \infty] \) of the mirror sheaf.

### The large complex structure limit.

The LKLP for \( T^2 \) is clearly lim \( a \to \infty \). On the mirror then, the LCLP is at \( \tau = b + ia \to b + i\infty \), the nodal torus compactifying the moduli of elliptic curves. Metrically, however, in the (Ricci) flat metric, things look different; if we rescale to have fixed diameter, the torus collapses to the base of its SYZ fibration, and all of its fibers contract. This is an important general feature of the difference between complex and metric descriptions of LCLPs; see the description of the quintic in the next section.

We note that as in the compactifications of Section 2, the monodromy around this LCLP is given by rotating the B-field: \( b \mapsto b + 1 \). This gives back the same elliptic curve, but after a monodromy diffeomorphism \( T \) which we see from (9) to be

\[
T: q \mapsto q, \quad p \mapsto p + q/a.
\]

On \( H^1(T^2) = \mathbb{Z}[fiber] \oplus \mathbb{Z}[section] \) this acts as

\[
T_* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

This is called a Dehn twist. Picking the zero-section \( O = \{ p = 0 \} \) in the mirror (9) when \( b = 0 \), this is taken to the section

\[
T(O) = \{ p = q/a \},
\]

and \( T \) is in fact translation by this section \( T(O) \) on \( T^2 \), using the group structure on the fibers (now we have chosen a zero-section). Again, Gross [G] has shown that this is a general feature of LCLPs.

If we pick a Kähler structure on this family of complex tori, \( T \) turns out to be a symplectomorphism. Importantly, its mirror is not a holomorphic automorphism, but an equivalence of the derived category of coherent sheaves. As above, the section \( T(O) \) corresponds to a slope-one line bundle \( L \) on the mirror, and the monodromy action corresponds to

\[
\otimes L: D^b \to D^b
\]

on the derived category. Again, this is a more general feature of these LCLPs, with \( L \) such that \( c_1(L) \) equals the symplectic form which generated the ray along which the original LKLP was reached. In general the SYZ fiber is the invariant cycle under
of points are invariant under $\otimes L$. On the cohomology of $T^2$, cupping with $ch(L) = e^{c_1(L)} = 1 + c_1(L)$ has the same action (10) on $H^{ev} = \mathbb{Z}(c_1(L)) \otimes \mathbb{Z}(1)$.

Notice we have used the choices of fibration and zero-section to product the equivalence of triangulated categories and to equate the monodromy actions. Kontsevich’s conjectural equivalence is not canonical, but is fixed by a choice of fibration and zero-section. In turn a fibration should be fixed by a choice of LCLP or LKLP from the resulting collapse on the fibers and so on the Fukaya category. On the fibers and so on the Fukaya category, $\mathcal{O}_X$ and $LCLP$ monodromy on $\mathcal{O}_{\mathcal{X}}$ mirror to the zero-section, and an $\mathcal{O}_X$ the quintic is any crepant ($\mathcal{Z}$ of unity. Dividing by the diagonal of hyperplanes $\{0\}$ resolved by fifth roots of unity. Dividing by the diagonal $\mathbb{Z}/5$ projective stabilizer we get a free $(\mathbb{Z}/5)^3$-action; the mirror of the quintic is any crepant ($K = \mathcal{O}$) resolution of the quotient:

$$Q_\lambda = \mathcal{O}_\lambda / (\mathbb{Z}/5)^3.$$  

Different resolutions have isomorphic $H^2$ but different Kähler cones therein. The union of these Kähler cones is the moveable cone, whose complexification is locally isomorphic to the complex structure moduli space of $Q$. $h^{1,1}(\mathcal{Q}_\lambda) = 101$ for any crepant resolution, and $h^{2,1}(\mathcal{Q}_\lambda) = 1$ corresponds locally to the one complex structure deformation (12). In fact for $\alpha^5 = 1$, multiplying $x_0$ by $\alpha$ shows that $\mathcal{Q}_\lambda \cong \mathcal{Q}_\alpha \lambda$, and $\lambda^5$ parametrizes the complex structure moduli.

The large complex structure limit point is at $\lambda = \infty$, i.e. it is a resolution of the quotient of the union of hyperplanes

$$Q_\infty = \left\{ \prod_{i=0}^{4} x_i = 0 \right\} = \{x_0 = 0\} \cup \ldots \cup \{x_4 = 0\}.$$

This is a union of toric varieties, each with a $T^3$-action inherited from the toric $T^4$-action on $\mathbb{P}^4$. Much more generally, Batyrev’s construction considers the anticanonical divisors (and even more generally, complete intersections) in toric varieties fibered over the boundary of the moment polytope, and takes as mirror the anticanonical divisor of the.

The simplest Calabi-Yau 3-fold is given by the zeros $Q$ of a homogeneous quintic polynomial on $\mathbb{P}^4$, i.e. an anticanonical divisor of $\mathbb{P}^4$. By adjunction this has trivial canonical bundle, and so is Calabi-Yau. By the Lefschetz hyperplane theorem, it has $h^{1,1} = 1$, so computing its Euler number to be $e = -200$, we find $h^{2,1} = 101$ is its number of complex deformations. Alternatively this can be seen by showing all such deformations are themselves quintics, then dividing the 126-dimensional space of quintic polynomials by the 25-dimensional $GL(5, \mathbb{C})$. Thus its mirror has 1 complex structure deformation and 101 Kähler classes.

Greene and Plesser prescribed the following mirror. Take the special one-dimensional family of Fermat quintics

$$Q_\lambda = \left\{ \sum_{i=0}^{4} x_i^5 - \lambda \prod_{i=0}^{4} x_i = 0 \right\} \subset \mathbb{P}^4,$$

with the action of $\{\alpha_0, \ldots, \alpha_4 \} \in (\mathbb{Z}/5)^5$: $\prod_{i=0}^{4} \alpha_i = 1 \equiv (\mathbb{Z}/5)^3$ given by rescaling the $x_i$ by fifth roots of unity. Dividing by the diagonal $\mathbb{Z}/5$ projective stabilizer we get a free $(\mathbb{Z}/5)^3$-action; the mirror of the quintic is any crepant ($K = \mathcal{O}$) resolution of the quotient:

$$Q_\lambda = \mathcal{O}_\lambda / (\mathbb{Z}/5)^3.$$
toric variety associated to the dual polytope. Most of the geometry is visible in this quintic example, however.

(13) is the analogue of the nodal torus of the last section, and we emphasize again that metrically it looks nothing like this; the Ricci-flat metric collapses the $T^3$ toric fibers to the base $S^3$ (with a singular metric). General LCLPs look rather similar, with such “as bad as possible” normal crossing singularities. Smoothing a local model (in singular metric). General LCLPs look rather similar, with such “as bad as possible” normal crossing singularities. Smoothing a local model (in singular metric). General LCLPs look rather similar, with such “as bad as possible” normal crossing singularities. Smoothing a local model (in singular metric).

$$T^3 = \left\{ |x_1| = \delta_1, |x_2| = \delta_2, |x_3| = \delta_3, x_4 = \frac{\epsilon}{x_1 x_2 x_3} \right\}.$$ \hspace{1cm} (14)

These are even Lagrangian in the standard symplectic form on the local model, and fiber the smoothing over the base $\{ (\delta_1, \delta_2, \delta_3) \}$. It turns out that metrically, these tori which vanish into the normal crossings singularity at the LCLP actually form a large part of the smooth Calabi-Yau. This enlightens the apparent paradox between the SYZ conjecture and the Batyrev construction, i.e. why a vertex of the original moment polytope (corresponding to the deepest type of singularity $(0, 0, 0) \in \{ \prod_{i=1}^4 x_i = 0 \}$) can be replaced by the dual 3-dimensional face in the dual polytope. This was first suggested by Leung and Vafa.

Gross and Siebert [GS] exploit this to extend SYZ and Batyrev’s construction to non-toric LCLP Calabi-Yau manifolds; it is only the local toric nature of the normal crossing singularities of the LCLP that they use. It seems possible that their construction will give the mirrors of all Calabi-Yau manifolds with LCLPs. Much of mirror symmetry should soon be reduced to graphs (the discriminant locus of a Lagrangian torus fibration) in spheres, and further graphs over which D-branes (such as holomorphic curves) fiber, as in recent conjectures of Kontsevich and Soibelman and Fukaya [F]. Before long it may be possible to write down a triangulated category in terms of such data. The full geometric story (involving Joyce’s description of sLag fibrations, for instance) is still some way off, however; we cannot even write down an explicit Ricci-flat metric on a compact Calabi-Yau.

7.1. Monodromy around the LCLP. As well as the SYZ torus fiber (14) we can also see a Lagrangian zero-section on the quintic and its mirror as a component of the real locus of (12) for $\lambda > 5$. Remarkably, like the torus (14), this cycle was already described and used in [COGP], long before the relevance of torus fibrations was suspected.

Gross and Ruan are able to describe the quintic and its mirror (at least topologically or symplectically) very explicitly as a simple torus fibration over this $S^3$ with a natural integral affine structure and codimension two graph discriminant locus. See for example [GHJ].

Under monodromy about $\lambda = \infty$, the zero-section is moved to another section $T(O)$, and $T$ is given by translation by $T(O)$ using the group structure on the fibers. This is the analogue of the Dehn twist (10), and one can choose a basis of $H_3(Q^\vee)$ (with first element the invariant cycle, the $T^3$-fiber, second element a cycle fibered over a curve in $S^3$, third fibered over a surface, and last the zero-section itself) such that

$$T_* = \begin{pmatrix} 1 & 1 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ \hspace{1cm} (15)

Like the Dehn twist (10), it turns out that $T_*$ is maximally unipotent; that is, in $n$-dimensions, we have

$$(T_* - 1)^{n+1} = 0 \text{ but } (T_* - 1)^n \neq 0.$$ \hspace{1cm} Again, this is a general feature of LCLPs as formulated by Morrison [M] as part of the definition.

This should be compared with the Lefschetz operator $L = \cup \omega$ on the cohomology of the mirror, which also satisfies $L^n \neq 0$, $L^{n+1} = 0$ (or, more relevantly, $\exp(L)$, which satisfies $(e^L - 1)^n \neq 0$, $(e^L - 1)^{n+1} = 0$). Their similarity was noticed by the Griffiths school working on VHS in the late 1960s. Now we know that for Calabi-Yau manifolds at a LCLP dual to a LKLP along a ray $\omega = c_1(L)$ on the mirror, they should be considered mirror operators (up to some factors of the Todd class of the underlying Calabi-Yau, to do with the relationship between the Chern character $e^c$ of the line bundle $L (11)$ and the Riemann-Roch formula).

Both, by linear algebra of the nilpotent operator $N = \log T_* = \sum_{k=1}^n (T_* - 1)^k$, induce a natural...
filtration $W_\bullet: 0 \leq W_0 \leq \ldots \leq W_{2n} = H$ on the cohomology on which they operate (which is $H = H^n$ for $N = \log T_\bullet$ and $H = H^{ev}$ for $N = L = \cup \omega$):

\begin{equation}
0 \leq \text{im}(N^n) \leq \text{im}(N^{n-1}) \cap \ker(N) \leq \ldots
\leq \ker(N^{n-1}) + \text{im}(N) \leq \ker(N^n) \leq H.
\end{equation}

We refer to the references for the construction of this monodromy weight filtration. It plays a key role in studying degenerations of varieties and Hodge structures, in this case as we approach the LCLP.

It is a beautiful result of Gross that this filtration coincides with the Leray filtration on $H^n$ induced by the fibration. That is, under Poincaré duality, the weight filtration on cycles is by the minimal dimension (over all homologous cycles) of the image in the base over which the cycle is fibered. So the first graded piece is spanned by the invariant cycle, the $T^3$-fiber, supported over a point, and the last by the zero-section; cf. (15). (Similarly on the mirror, the filtration for the Lefschetz operator $\cup e^\omega$ has first piece spanned by the cohomology class of a point, which is invariant under the monodromy action $\otimes L$ (11), etc.)

Letting $\gamma_0$ be the class of a fiber and $\gamma_1$ span $W_2/W_0$ (which is one dimensional) over the integers, then $T_* \gamma_1 = \gamma_1 + \gamma_0$. It follows that

\begin{equation}
q = \exp \left( 2\pi i \frac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega} \right)
\end{equation}

is invariant under monodromy. This is the higher dimensional analogue of the coordinate $\exp(2\pi i\tau)$ on the moduli space of elliptic curves, where $\tau$ is the period point. It is this coordinate $q$ that is mirror to the coordinate

\begin{equation}
\int_{\text{line}} \omega
\end{equation}

on the Kähler moduli space on the mirror quintic, which allows one to compute the correspondence between VHS and Gromov-Witten invariants mentioned in the introduction.

More generally, following Morrison [M], one can make a rigorous definition of a LCLP using features we have seen above, extended to the case of $h^{2,1} > 0$; see for instance [CK]. Roughly, the upshot is that $M_X$ (of dimension $s = h^{2,1}(X)$) should be compactified with $s$ divisors $(D_i)_{i=1}^s$ (parameterizing singular varieties) forming a normal crossings divisor meeting at the LCLP, with monodromies $T_i$ about them. There should be a unique (up to multiples) integral cycle $\gamma_0$ (our torus fiber, of course) invariant under all $T_i$, and cycles $(\gamma_i)_{i=1}^s$ such that

\begin{equation}
\tau = \frac{\int_{\gamma_0} \Omega}{\int_{\gamma_i} \Omega}
\end{equation}

is logarithmic at $D_i$; i.e. $\tau_i = \frac{1}{2\pi i} \log(z_i)$, where $z_i$ is a local parameter for $D_i = \{ z_i = 0 \}$.

So $z_i = \exp(2\pi i \tau_i)$ form local coordinates for moduli space, mirror to the polydisk coordinates (2) on $\mathcal{M}_X$. The direction we approached the LKLP in that section corresponds to the holomorphic curve $z_i = z_j^k$ (3) we take through the LCLP ($z_i = 0 \forall i$), and the monodromy $\sum N_i T_i$ varies accordingly, but the corresponding weight filtration $W_\bullet$ remains constant if $k_i \neq 0 \forall i$, by a theorem of Cattani and Kaplan.

Morrison then requires that the $(\gamma_i)_{i=0}^s$ should form an integral basis for $W_2 = W_3$ (with $\gamma_0$ a basis of $W_0 = W_1$). Finally, part definition and part conjecture, we should be able to choose that they satisfy $\log T_i(\gamma_j) = \delta_{ij} \gamma_0$.

Of course, as has been emphasized, Morrison’s definition of a LCLP is really where the mathematics and geometry of mirror symmetry begin, and should have been the starting point of this article. But that would have required a lot of knowledge of abstract VHS that are best understood, in this context, through the new geometry of Lagrangian torus fibrations that mirror symmetry has inspired.

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