Zero Mach Number Limit of the Compressible Primitive Equations: Ill-prepared Initial Data

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Abstract

In the work, we consider the zero Mach number limit of compressible primitive equations in the domain $\mathbb{R}^2 \times 2T$ or $\mathbb{T}^2 \times 2T$. We identify the limit equations to be the primitive equations with the incompressible condition. The convergence behaviors are studied in both $\mathbb{R}^2 \times 2T$ and $\mathbb{T}^2 \times 2T$, respectively. This paper takes into account the high oscillating acoustic waves and is an extension of our previous work in [29].

Mathematics Subject Classification: 35B25, 35B40, 76N10.

Keywords: Low Mach number limit, Compressible primitive equations, incompressible primitive equations.

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1 Introduction

1.1 Zero Mach number limit

As an example of multi-scale analysis, the zero Mach number limit problem for compressible flows has been an important classical problem in the study of hydrodynamic equations. Pioneered by Klainerman and Majda in [19, 20], it is shown that the solutions of inviscid compressible Euler equations converge to that of inviscid incompressible Euler equations for the isentropic flows in $\mathbb{R}^d$ and $\mathbb{T}^d$, $d \in \mathbb{Z}^+$. The initial data are well-prepared and almost incompressible. The result is further studied in bounded domains for non-isentropic flows by Schochet in [38, 39]. In [43], the author establishes the zero Mach number limit with general (ill-prepared) initial data in $\mathbb{R}^d$. As pointed out in [33], in comparison to the case when the initial data are well-prepared, the time derivatives are no longer uniformly bounded with respect to the Mach number when the system is complemented with ill-prepared initial data. This leads to high frequency acoustic waves with large amplitude.

We have studied the zero Mach number limit of compressible primitive equations with well-prepared initial data in [29]. In this work, we are considering such a singular limit problem with ill-prepared initial data.

To present the general idea of singular limit problems with ill-prepared initial data, consider the following equation of the unknown function $G$,

$$\partial_t G + \frac{1}{\varepsilon} L(\partial_x) G = \mathcal{N}(G), \quad (1.1)$$

where $\varepsilon \in (0, \infty)$ is the singular parameter, $L(\partial_x)$ is an anti-symmetric differential operator with constant coefficients, and $\mathcal{N}(G)$ is the nonlinearity of $G$. Then one can perform the $H^s$ estimate for some large $s \in \mathbb{Z}^+$, on system (1.1). Since $L(\partial_x)$ is anti-symmetric, $\|G\|_{H^s}$ is uniformly bounded with respect to $\varepsilon$, at least locally in time. If one considers the projection of
in the kernel of operator \( L(\partial_x) \), which is called the non-singular part of the equation, it follows that
\[
\partial_t P_{\ker(L(\partial_x))} G = P_{\ker(L(\partial_x))} \mathcal{N}(G)
\]
is uniformly bounded in \( H^{s'} \) space for some \( s' \in \mathbb{Z}^+ \cup \{0\} \) with respect to \( \varepsilon \). On the other hand, \( G - P_{\ker(L(\partial_x))} G \) satisfies
\[
\partial_t (G - P_{\ker(L(\partial_x))} G) + \frac{1}{\varepsilon} L(\partial_x)(G - P_{\ker(L(\partial_x))} G) = \mathcal{N}(G) - P_{\ker(L(\partial_x))} \mathcal{N}(G),
\]
which has wave packet solutions with fast oscillations as \( \varepsilon \to 0^+ \). Consequently, \( G - P_{\ker(L(\partial_x))} G \) is oscillating at high frequency. Moreover, as \( \varepsilon \to 0^+ \), the \( H^s \) norm preserving property of the anti-symmetric operator \( L(\partial_x) \) implies that the amplitude of the oscillations does not vanish in general. Instead, the limit of the oscillatory part is driven by certain PDEs. To resolve the singular limit problem of (1.1) as \( \varepsilon \to 0^+ \), one has to study the interactions of the non-singular part and the oscillatory part in the non-linearity \( \mathcal{N}(G) \) in order to identify the limit equations. Such a method of studying the singular limit problem of (1.1) is developed by Schochet in [37] for hyperbolic PDEs with applications to the incompressible limit problem of Euler equations, nonlinear wave equations and the theory of weakly non-linear geometric optics. Later, this method is further developed for some parabolic equations by Gallagher in [16]. We remark here that, if equation (1.1) is complemented with well-prepared initial data, the amplitude of oscillations is small, and consequently, the interaction between the non-singular part and the oscillatory part is much weaker.

In the study of zero Mach number limit of hydrodynamic equations for isentropic flows, if one writes the equations in the form of (1.1), the corresponding kernel of \( L(\partial_x) \) consists the solenoidal velocity field. The equations corresponding to the fast oscillation equations (1.2) are referred to as the acoustic wave equations. Using these terminologies, the theorem developed by Ukai in [43] is basically showing that the acoustic waves decay to zero as \( \varepsilon \to 0^+ \) in \( \mathbb{R}^d \). Such a fact is the consequence of the Strichartz estimates for linear wave equations, as pointed out by Desjardins and Grenier in [10] (see, e.g., [17, 25] about the Strichartz estimate). In \( \mathbb{T}^d, \) Lions and Masmoudi study the resonance of the high frequency oscillations and show that in the sense of distribution, the solutions to the compressible Navier–Stokes equations converge to that to the incompressible Navier–Stokes equations as the
Mach number goes to zero in [27]. Later in [31], Masmoudi studies the incompressible, inviscid limit, with low Mach number and large Reynolds number, of compressible Navier–Stokes equations and identifies the equations in both $\mathbb{R}^d$ and $\mathbb{T}^d$. We refer readers, for further developments, to [6–8, 40].

Additionally, we would like to mention that when considering the non-isentropic flows, the corresponding operator $L(\partial_x)$ in (1.1) has coefficients depending on space and time variables. This causes non-trivial difficulties to study the resonances between the oscillations (see, e.g., [1–3, 33, 34]). Moreover, when taking into account the stratification effect of gravity, the compressible Navier–Stokes equations with gravity may converge to the Oberbeck-Boussinesq equations or the anelastic equations, depending on the strength of the gravity effect. We refer readers, for more discussions of related topics, to [4, 9, 11–15, 32, 35, 36, 44]. Also, for more multi-scale analysis, we refer readers to [21–24, 30].

1.2 The compressible primitive equations

As mentioned before, we aim at studying the low Mach number limit of the compressible primitive equations. As discussed in our previous work [29], this is part of the justification of the PE diagram (see Figure 1 in [29]). We refer readers, for more background of the compressible primitive equations, to [29].

Let $\varepsilon \in (0, \infty)$ denote the Mach number, and let $\rho_\varepsilon \in \mathbb{R}, v_\varepsilon \in \mathbb{R}^2, w_\varepsilon \in \mathbb{R}$ represent the density, the horizontal and the vertical velocities, respectively. Then the compressible primitive equations can be written as, after rescaling the original CPE, similar to that of the compressible Navier–Stokes equations (see, e.g., [14]):

\[
\begin{align*}
\partial_t \rho_\varepsilon + \text{div}_h(\rho_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon) &= 0 \quad \text{in } \Omega_h \times (0, 1), \\
\partial_t (\rho_\varepsilon v_\varepsilon) + \text{div}_h(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon v_\varepsilon) &+ \frac{1}{\varepsilon^2} \nabla_h P(\rho_\varepsilon) = \text{div}_h S(v_\varepsilon) + \partial_{zz} v_\varepsilon \quad \text{in } \Omega_h \times (0, 1), \\
\partial_z P(\rho_\varepsilon) &= 0 \quad \text{in } \Omega_h \times (0, 1). 
\end{align*}
\]

(CPE)

where $P(\rho_\varepsilon) = \rho_\varepsilon^\gamma$, $S(v_\varepsilon) = \mu(\nabla_h v_\varepsilon + \nabla_h v_\varepsilon^\top) + (\lambda - \mu)\text{div}_h v_\varepsilon I_2$ represent the pressure for isentropic flows and the viscous stress tensor for Newtonian flows, respectively. Here, we assume that $\mu, \lambda > 0$ and $\gamma > 1$. We consider $\Omega_h = \mathbb{R}^2$ or $\mathbb{T}^2$ in this paper, where $\mathbb{T}^2$ represents the periodic domain with period 1 in both directions in $\mathbb{R}^2$. (CPE) is complemented with the
stress-free and non-permeable boundary conditions:

\[ \partial_z v_\varepsilon \big|_{z=0,1} = 0, \quad w_\varepsilon \big|_{z=0,1} = 0. \]  

(BC-CPE)

Hereafter, we have and will use \( \nabla_h, \text{div}_h \) and \( \Delta_h \) to represent the horizontal gradient, the horizontal divergence, and the horizontal Laplace operator, respectively; that is,

\[ \nabla_h := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad \text{div}_h := \nabla_h \cdot, \quad \Delta_h := \text{div}_h \nabla_h \cdot. \]

Notice, (CPE) with (BC-CPE) is invariant with respect to the following symmetry:

\[ v_\varepsilon \text{ and } w_\varepsilon \text{ are even and odd, respectively, in the } z \text{-variable.} \]  

(SYM)

Owing to such symmetry, in order to study the limit system of (CPE) as \( \varepsilon \to 0^+ \), it suffices to consider the following system:

\[
\begin{aligned}
\partial_t \rho_\varepsilon + \text{div}_h(\rho_\varepsilon v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon) &= 0 \quad \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_t(\rho_\varepsilon v_\varepsilon) + \text{div}_h(\rho_\varepsilon v_\varepsilon \otimes v_\varepsilon) + \partial_z(\rho_\varepsilon w_\varepsilon v_\varepsilon) + \frac{1}{\varepsilon^2} \nabla_h P(\rho_\varepsilon) &= \text{div}_h S(v_\varepsilon) + \partial_{zz} v_\varepsilon \quad \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_z P(\rho_\varepsilon) &= 0 \quad \text{in } \Omega_h \times 2\mathbb{T},
\end{aligned}
\]

subject to the periodic boundary condition and symmetry (SYM). Here \( 2\mathbb{T} \) is the periodic domain with period 2 in \( \mathbb{R} \).

We remark that the restrictions of solutions to (1.3) in \( \Omega_h \times [0,1] \) solve (CPE) with (BC-CPE), provided that the solutions exist and are regular enough.

We can rewrite (1.3) as, provided that \( \rho_\varepsilon > 0 \),

\[
\begin{aligned}
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \frac{1}{\varepsilon^2} \nabla_h \left( \frac{\gamma}{\gamma - 1} \rho_\varepsilon^{\gamma - 1} \right) &= \frac{\mu}{\rho_\varepsilon} \Delta_h v_\varepsilon \\
+ \frac{\lambda}{\rho_\varepsilon} \nabla_h \text{div}_h v_\varepsilon + \frac{1}{\rho_\varepsilon} \partial_{zz} v_\varepsilon.
\end{aligned}
\]

Also from the continuity equation (1.3), one can derive

\[
\partial_t \rho_\varepsilon^{\gamma - 1} + v_\varepsilon \cdot \nabla_h \rho_\varepsilon^{\gamma - 1} + (\gamma - 1) \rho_\varepsilon^{\gamma - 1} \text{div}_h v_\varepsilon + (\gamma - 1) \rho_\varepsilon^{\gamma - 1} \partial_z w_\varepsilon = 0.
\]

We consider \( \rho_\varepsilon^{\gamma - 1} \) with perturbations around the constant state given by \( \rho_\varepsilon^{\gamma - 1} = \bar{\rho}^{\gamma - 1} \in (0, \infty) \). Then we define the perturbation variable \( \xi_\varepsilon \) by

\[
\xi_\varepsilon := \frac{1}{\varepsilon} \log \left( \frac{\gamma}{\gamma - 1} \frac{\rho_\varepsilon^{\gamma - 1}}{\bar{\rho}^{\gamma - 1}} \right).
\]
with \( c^2 := \frac{\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} \). Let \( \alpha := \frac{1}{\gamma-1} \), \( c_1 := \left( \frac{\gamma}{(\gamma-1)c^2} \right)^{\alpha} \). With such notations, \( \rho_\varepsilon = c_1^{-1} \exp(\varepsilon \alpha \xi_\varepsilon) \). Without loss of generality, we take \( \bar{\rho} \equiv 1 \). Hence (1.3) can be written as

\[
\begin{aligned}
\frac{\partial \xi_\varepsilon}{\partial t} + v_\varepsilon \cdot \nabla h v_\varepsilon + \frac{\gamma - 1}{\varepsilon} (\text{div}_h v_\varepsilon + \partial_z w_\varepsilon) &= 0 \quad \text{in } \Omega_h \times 2\mathbb{T}, \\
\frac{\partial v_\varepsilon}{\partial t} + v_\varepsilon \cdot \nabla h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + c_2^2 \varepsilon \xi_\varepsilon &\nabla_h \xi_\varepsilon \quad \varepsilon \in (0, 1), \quad \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_z \xi_\varepsilon &= 0 \quad \text{in } \Omega_h \times 2\mathbb{T},
\end{aligned}
\]

(1.4)

complemented with periodic initial data with symmetry \((\text{SYM})\). In fact, (1.4) is already in the form of (1.1). In this work, we study the asymptotic limit of (1.4) as \( \varepsilon \to 0^+ \). In fact, we will demonstrate that the existence time of strong solutions to (1.4) has a uniform lower bound, independent of \( \varepsilon \in (0, \varepsilon_0) \) for some small \( \varepsilon_0 \in (0, 1) \). In addition, in the sense of distribution, the limit equations of (1.4), as \( \varepsilon \to 0^+ \), are the primitive equations (2.9) with the incompressible condition, below. Also, the associated acoustic wave equations for the three dimensional system (1.4) are only two dimensional (see (4.20), below). Consequently, we are able to adopt the strategy of studying the acoustic wave equations for the compressible Navier–Stokes equations in \( \mathbb{R}^2 \) and \( \mathbb{T}^2 \) to investigate the oscillatory part of the equations. This is done in section 4 below.

The rest of this work is organized as follows. In the next section, we will introduce some notations as well as some function spaces. Also, the main theorems in this work are stated in both \( \Omega_h = \mathbb{R}^2 \) and \( \Omega_h = \mathbb{T}^2 \). Next, in section 3 we establish the uniform local well-posedness of strong solutions to system (1.4). This is done with uniform \textit{a priori} estimates in section 3.1, a local existence theorem in section 3.2 and a continuity argument in section 3.3. In section 4 we first identity the primitive equations, i.e., (2.9), below, as the limit of system (1.4) as \( \varepsilon \to 0^+ \) in the sense of distribution. Then in section 4.1 we argue that the acoustic waves decay to zero in the case when \( \Omega_h = \mathbb{R}^2 \); in section 4.2 we study the oscillation equations and identify the limit equations of oscillations in the case when \( \Omega_h = \mathbb{T}^2 \). Consequently, we conclude the compactness in both cases as stated in the main theorems, below.
2 Preliminaries and main theorems

In this work, we denote by
\[ \overline{f}(x, y) := \int_0^1 f(x, y, z') dz', \quad \text{and} \quad \tilde{f}(x, y, z) := f(x, y, z) - \overline{f}(x, y), \]
as the average and the fluctuation of any function \( f = f(x, y, z) \) over the \( z \)-variable. We use \( \partial_h \) to denote the horizontal derivatives, i.e., \( \partial_h \in \{ \partial_x, \partial_y \} \).

\( \partial_t \) and \( \partial_z \) denote the time derivative and the vertical derivative, respectively. For any function \( f, \) \( \partial_y f \) is sometimes denoted as \( f_y \). For \( g \in \{ t, x, y, z, h \} \). Similar notations are also adopted for higher order derivatives.

As in [26], we introduce the following function spaces. Denote by
\[
C_\sigma^\infty := \{ u \in C_0^\infty(\Omega_h \times 2T; \mathbb{R}^2) \mid \int_0^1 \text{div}_h u \, dz = 0 \},
\]
\[
C_\tau^\infty := \{ u \in C_0^\infty(\Omega_h \times 2T; \mathbb{R}^2) \mid \nabla_h \psi, \text{ for } \psi \in C^\infty(\Omega_h; \mathbb{R}^2) \}.
\]

Then with respect to the \( L^2 \)-inner product, \( C_\sigma^\infty \perp C_\tau^\infty \) and \( C_0^\infty = C_\sigma^\infty \cup C_\tau^\infty \). We denote the closures in the \( L^2 \) norm of \( C_\sigma^\infty, C_\tau^\infty \) as \( L_\sigma^2(\Omega_h \times 2T; \mathbb{R}^2), L_\tau^2(\Omega_h \times 2T; \mathbb{R}^2) \), respectively. \( \mathcal{D}' = \mathcal{D}_\sigma' \cup \mathcal{D}_\tau', H^s = H_\sigma^s \cup H_\tau^s \) are the spaces of test functions and Sobolev functions in \( \Omega_h \times 2T \), defined similarly for \( s \in \mathbb{Z}^+ \). We define the projection operators \( \mathcal{P}_\sigma, \mathcal{P}_\tau \) in the following. Let \( u \in C_0^\infty(\Omega_h \times 2T; \mathbb{R}^2) \). Consider the elliptic problem
\[
\Delta_h \psi_u = \int_0^1 \text{div}_h u \, dz,
\]
\[
\begin{aligned}
\lim_{|(x, y)| \to \infty} \psi_u &= 0 \quad \text{in the case when } \Omega_h = \mathbb{R}^2, \\
\int_{\Omega_h} \psi_u \, dx dy &= 0 \quad \text{in the case when } \Omega_h = T^2.
\end{aligned}
\]

Let
\[
\mathcal{P}_\sigma u := \nabla_h \psi_u, \quad \mathcal{P}_\tau u := u - \nabla_h \psi_u.
\]

Then \( \mathcal{P}_\sigma, \mathcal{P}_\tau \) are the projection operators from \( C_0^\infty \) to \( C_\sigma^\infty, C_\tau^\infty \), respectively. Also, by a density argument, \( \mathcal{P}_\sigma, \mathcal{P}_\tau \) can act on functions in \( L^2, H^s, s \in (0, \infty) \). In particular, the standard elliptic estimates yield, provided that \( u \in H^s(\Omega_h \times 2T; \mathbb{R}^2), s \in (0, \infty) \),
\[
|\psi_u|_{H^{s+1}} \lesssim |\nabla u|_{H^s} \lesssim \| u \|_{H^s}.
\]

Thus \( \mathcal{P}_\sigma u \in H_\sigma^s, \mathcal{P}_\tau u \in H_\tau^s \), \( u = \mathcal{P}_\sigma u + \mathcal{P}_\tau u \) and
\[
\| \mathcal{P}_\sigma u \|_{H^s} \lesssim \| u \|_{H^s}, \quad \| \mathcal{P}_\tau u \|_{H^s} \lesssim \| u \|_{H^s}. \quad (2.3)
\]
System (1.4) is complemented with initial data \((\xi_0, v_0) \in H^2(\Omega_h \times 2T; \mathbb{R}) \times H^2(\Omega_h \times 2T; \mathbb{R}^2)\) with the compatibility conditions:
\[
\partial_z \xi_0 = 0; \quad v_0 \text{ is even in the } z\text{-variable.} \tag{2.4}
\]
Also, in the case when \(\Omega_h = \mathbb{R}^2\), the far field condition
\[
\lim_{|(x,y)| \to \infty} |v_\varepsilon| = 0, \tag{2.5}
\]
is also imposed. We denote the constant \(M \in (0, \infty)\) satisfying
\[
\frac{1}{2} \|v_0\|_{H^2}^2 + \frac{c^2}{\gamma - 1} \|\xi_0\|_{H^2}^2 < M, \tag{2.6}
\]
to be the bound of initial data \((\xi_0, v_0)\). Now we describe our main theorems in this work. The first theorem is stating that the \(H^2\) norms of the solutions to (1.4) are uniformly bounded with respect to \(\varepsilon\), provided that \(\varepsilon\) is small enough.

**Theorem 2.1.** With initial data \((\xi_0, v_0) \in H^2(\Omega_h \times 2T; \mathbb{R}) \times H^2(\Omega_h \times 2T; \mathbb{R}^2)\) satisfying the compatibility conditions in (2.4), let \(M\) be the upper bound of the initial data, i.e., (2.6). Then there are positive constants \(\varepsilon_1 \in (0, 1), T \in (0, \infty)\) depending only on \(M\), such that for any \(\varepsilon \in (0, \varepsilon_1)\), there is a unique strong solution \((\xi_\varepsilon, v_\varepsilon)\) to system (1.4) in the time interval \([0, T]\). \((\xi_\varepsilon, v_\varepsilon)\) satisfies
\[
\xi_\varepsilon \in L^\infty(0, T; H^2(\Omega_h \times 2T)), \quad \partial_t \xi_\varepsilon \in L^\infty(0, T; H^1(\Omega_h \times 2T)),
\]
\[
v_\varepsilon \in L^\infty(0, T; H^2(\Omega_h \times 2T)) \cap L^2(0, T; H^3(\Omega_h \times 2T)),
\]
\[
\partial_t v_\varepsilon \in L^\infty(0, T; L^2(\Omega_h \times 2T)) \cap L^2(0, T; H^1(\Omega_h \times 2T)). \tag{2.7}
\]
In addition, there exist positive constants \(M_0, M_1, M_2 \in (0, \infty)\) independent of \(\varepsilon\) such that
\[
\|\xi_\varepsilon\|_{L^\infty(0, T; H^2(\Omega_h \times 2T))} < M_0, \quad \|v_\varepsilon\|_{L^\infty(0, T; H^2(\Omega_h \times 2T))} < M_1,
\]
\[
\|\nabla v_\varepsilon\|_{L^2(0, T; H^2(\Omega_h \times 2T))} < M_2. \tag{2.8}
\]
Next, as \(\varepsilon \to 0^+\), formally, we expect the solutions \(v_\varepsilon\) obtained in Theorem 2.1 will converge to a solution to the following primitive equations:
\[
\begin{aligned}
div_h v_p + \partial_z w_p &= 0 & \text{in } \Omega_h \times 2T, \\
\partial_t v_p + v_p \cdot \nabla_h v_p + w_p \partial_z v_p + \nabla_h P &= c_1(\mu \Delta_h v_p + \lambda \nabla_h \div_h v_p + \partial_{zz} v_p) & \text{in } \Omega_h \times 2T, \\
\partial_z P &= 0 & \text{in } \Omega_h \times 2T,
\end{aligned} \tag{2.9}
\]
complemented with initial data \( v_p|_{t=0} = v_{p,0} = P_\sigma v_0 \in H_0^2(\Omega_h \times 2T) \). The convergence behaviors are different in the case when \( \Omega_h = \mathbb{R}^2 \) and the case when \( \Omega_h = T^2 \). We summarize the results in the following:

**Theorem 2.2.** Under the same assumptions as in Theorem 2.1, as \( \varepsilon \to 0^+ \), one has

\[
\begin{align*}
v_\varepsilon \rightharpoonup^* v_p & \quad \text{weak-* in } L^\infty(0,T;H^2(\Omega_h \times 2T)), \\
v_\varepsilon \rightharpoonup v_p & \quad \text{weakly in } L^2(0,T;H^3(\Omega_h \times 2T)),
\end{align*}
\]

(2.10)

where \( v_p \in L^\infty(0,T;H^2(\Omega_h \times 2T)) \cap L^2(0,T;H^3(\Omega_h \times 2T)) \) is the unique strong solution to (2.9) with initial data \( v_p|_{t=0} = v_{p,0} = P_\sigma v_0 \in H_0^2(\Omega_h \times 2T) \). Moreover, the following strong convergence holds,

\[
P_\sigma v_\varepsilon \to v_p \quad \text{in } C([0,T];H^1_{\sigma,\text{loc}}(\Omega_h \times 2T)) \cap L^2(0,T;H^2_{\sigma,\text{loc}}(\Omega_h \times 2T)).
\]

(2.11)

In addition,

- **in the case when \( \Omega_h = \mathbb{R}^2 \), as \( \varepsilon \to 0^+ \),**

\[
\xi_\varepsilon \to 0, \quad P_\tau v_\varepsilon \to 0 \quad \text{in } L^2(0,T;W^{1,6}(\mathbb{R}^2)),
\]

(2.12)

and therefore,

\[
\|v_\varepsilon - v_p\|_{L^2(0,T;L^6(\mathbb{R}^2 \times 2T))} + \|\xi_\varepsilon\|_{L^2(0,T;L^6(\mathbb{R}^2 \times 2T))} \to 0;
\]

(2.13)

- **in the case when \( \Omega_h = T^2 \), as \( \varepsilon \to 0^+ \), only the weak convergences in (2.10) hold, and there exists a function \( V^\circ \in L^\infty(0,T;H^2(T^2)) \cap C(0,T;H^1(T^2)) \) and a constant \( g^\circ \), satisfying equation (4.71), below in section 4.2, such that, as \( \varepsilon \to 0^+ \),

\[
\left\| \begin{pmatrix} \xi_\varepsilon - g^\circ \\ v_\varepsilon - v_p \end{pmatrix} - \mathcal{L}(t_\varepsilon) V^\circ \right\|_{L^\infty(0,T;H^1(T^2 \times 2T))} \to 0
\]

(2.14)

where \( \mathcal{L} \) is the solution operator to equation (4.27), defined in (4.26), below in section 4.

Theorem 2.1 is the direct consequence of Proposition 3 below. Theorem 2.2 is the consequence of Propositions 4, 5, 6 below.
3 Uniform stability

In this section, we will establish the uniform local existence of solutions to \((1.4)\) with respect to \(\varepsilon \in (0, \varepsilon_0)\) for some \(\varepsilon_0 \in (0, \infty)\) small enough. This is done via a series of \(a \ priori\) estimates, a local well-posedness theorem and a continuity argument. To simplify the presentation, in this section, we shorten the notations by dropping the subscript \(\varepsilon\). That is, we denote \(\xi = \xi_\varepsilon, v = v_\varepsilon, w = w_\varepsilon\).

3.1 \(a \ priori\) estimates

We first establish some \(a \ priori\) estimates, which are independent of \(\varepsilon\). Indeed, for \(\varepsilon\) small enough, the \(a \ priori\) estimates obtained in this subsection allow us to establish a uniform existence time in subsection 3.3. We remind readers system \((1.4)\):

\[
\begin{aligned}
\partial_t \xi + v \cdot \nabla_h \xi + \frac{\gamma - 1}{\varepsilon} (\text{div}_h v + \partial_\xi w) &= 0 \quad \text{in } \Omega_h \times 2T, \\
\partial_t v + v \cdot \nabla_h v + w \partial_\xi v + \frac{\varepsilon^2 c \varepsilon_\xi}{\varepsilon} \nabla_h \xi &= c_1 e^{-\varepsilon \alpha_\xi} (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_\xi w) \quad \text{in } \Omega_h \times 2T, \\
\partial_\xi v &= 0 \\
\partial_\xi \xi &= 0
\end{aligned}
\]

We will show the following:

**Proposition 1.** Let \((\xi, v)\) be a local strong solution to \((1.4)\) in the time interval \([0, T]\), \(T \in (0, \infty)\), and \((\xi, v)\big|_{t=0} = (\xi_0, v_0) \in H^2(\Omega_h \times 2T)\) with the compatibility conditions in \((2.3)\). Then the following inequality holds: for any \(\delta \in (0, 1)\) with corresponding \(C_\delta \simeq \delta^{-1}\),

\[
\sup_{0 \leq t \leq T} \left\{ \frac{1}{2} \| v(t) \|_{H^2}^2 + \frac{\varepsilon^2}{2(\gamma - 1)} e^{-\varepsilon} \| \xi(t) \|_{H^2} \| \xi(t) \|_{H^2}^2 \right\} \\
+ c_1 \int_0^T \left( e^{-\varepsilon \alpha} \| \xi(t) \|_{H^2} \| \nabla_h v(t) \|_{H^2}^2 + \lambda \| \text{div}_h v(t) \|_{H^2}^2 \right) \| \partial_\xi v(t) \|_{H^2}^2 \right) \right) dt \\
+ \| \partial_\xi v(t) \|_{H^2}^2 \right) dt \\
+ \delta \int_0^T \| \nabla v(t) \|_{H^2}^2 dt + \varepsilon \int_0^T \left( \mathcal{H}_1 \| \xi(t) \|_{H^2}, \| v(t) \|_{H^2} \right) \times \| \nabla v(t) \|_{H^2}^2 \right) dt \\
+ C_\delta \int_0^T \mathcal{H}_2 \| \xi(t) \|_{H^2}, \| v(t) \|_{H^2} \right) dt,
\]

\(\mathcal{H}_1, \mathcal{H}_2\) are some suitable embeddings into \(L^2\).
where $\mathcal{H}_1(\cdot), \mathcal{H}_2(\cdot)$ are smooth and bounded functions of the arguments. Also, $\mathcal{H}_1(0) = 0, \mathcal{H}_2(0) = 0$. Moreover, with the same notations, below, we have the inequalities:

\[
\begin{align*}
\| \partial_t \xi \|_{H^1} &\leq \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \\
\| \partial_t \mathcal{P}_\sigma v \|_{L^2} &\leq \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \\
\| \partial_t \mathcal{P}_\sigma v \|_{H^1} &\leq \mathcal{H}_1(\| \xi \|_{H^2}, \| v \|_{H^2}) \| \nabla v \|_{H^2} \\
&\quad + \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \\
\| \partial_t \xi \|_{H^1} &\leq \varepsilon^{-1} \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \\
\| \partial_t v \|_{L^2} &\leq (1 + \varepsilon^{-1}) \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \\
\| \partial_t v \|_{H^1} &\leq \mathcal{H}_1(\| \xi \|_{H^2}, \| v \|_{H^2}) \| \nabla v \|_{H^2} \\
&\quad + (1 + \varepsilon^{-1}) \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}).
\end{align*}
\]

In order to establish such a priori estimates, we first represent the vertical velocity $w$ in terms of $(\xi, v)$. In fact, after averaging (1.4) in the $z$-variable, one has

\[
\partial_t \xi + \nabla_h \xi + \frac{\gamma - 1}{\varepsilon} \text{div}_h \mathbf{v} = 0,
\]

and consequently, after comparing the above equation with (1.4), it follows that

\[
\tilde{v} \cdot \nabla_h \xi + \frac{\gamma - 1}{\varepsilon} \text{div}_h \tilde{v} + \partial_z w = 0.
\]

Then the following representations of the vertical velocity and its derivatives hold (recall that $\alpha = 1/(\gamma - 1)$):

\[
\begin{align*}
w &\quad = - \int_0^z \left( \varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \text{div}_h \tilde{v} \right) \, dz, \\
w_z &\quad = -\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi - \text{div}_h \tilde{v}, \\
w_h &\quad = - \int_0^z \left( \varepsilon \alpha (\tilde{v} \cdot \nabla_h \xi)_h + \text{div}_h \tilde{v}_h \right) \, dz, \\
w_{zz} &\quad = -\varepsilon \alpha \tilde{v}_z \cdot \nabla_h \xi - \text{div}_h \tilde{v}_z, \\
w_{hh} &\quad = -\varepsilon \alpha \tilde{v}_h \cdot \nabla_h \xi - \varepsilon \alpha \tilde{v} \cdot \nabla_h \xi_h - \text{div}_h \tilde{v}_h.
\end{align*}
\]

In the following, we separate the proof of Proposition 1 in three parts: estimates on the horizontal derivatives; estimates on the vertical derivatives; and estimates on the time derivatives.
Estimates on the horizontal derivatives

Denote by $\partial_h \in \{\partial_x, \partial_y\}$. Then after applying $\partial_h^2 = \epsilon\Delta_h$, it follows

$$
\begin{aligned}
&\partial_t \xi_{hh} + \frac{\gamma - 1}{2} (\text{div}_h v_{hh} + \partial_z w_{hh}) = \mathcal{G} \quad \text{in } \Omega_h \times 2\mathbb{T}, \\
&\partial_t v_{hh} + \frac{c_2^2 \epsilon \xi}{\epsilon} \nabla_h \xi_{hh} - c_1 e^{-\alpha \xi}(\mu \Delta_h v_{hh} \\
&\quad + \lambda \nabla_h \text{div}_h v_{hh} + \partial_z w_{hh}) = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 \quad \text{in } \Omega_h \times 2\mathbb{T},
\end{aligned}
$$

where we have denoted by

$$
\mathcal{G} := -(v \cdot \nabla_h \xi)_{hh},
$$
$$
\mathcal{F}_1 := -(v \cdot \nabla_h v)_{hh} - (w \partial_z v)_{hh},
$$
$$
\mathcal{F}_2 := -c_2 e^{\epsilon \xi}(\xi_{hh} + \epsilon \xi_{hh}^2) \nabla_h \xi - 2c_2 e^{\epsilon \xi} \xi_h \nabla_h \xi_h,
$$
$$
\mathcal{F}_3 := -2\epsilon c_1 \alpha e^{-\alpha \xi} \xi_h (\mu \Delta_h v_h + \lambda \nabla_h \text{div}_h v_h + \partial_z w_h),
$$
$$
\mathcal{F}_4 := -\epsilon c_1 \alpha e^{-\alpha \xi} (\xi_{hh} - \epsilon \alpha \xi_{hh}^2)(\mu \Delta_h v_h + \lambda \nabla_h \text{div}_h v_h + \partial_z w_h).
$$

After taking the $L^2$-inner product of (3.13) with $v_{hh}$ in $\Omega_h \times 2\mathbb{T}$, it holds

$$
\frac{d}{dt} \left\{ \frac{1}{2} \int |v_{hh}|^2 \, d\bar{x} \right\} + \int \frac{c_2^2 e^{\epsilon \xi}}{\epsilon} \nabla_h \xi_{hh} \cdot v_{hh} \, d\bar{x} + c_1 \int \left( e^{-\alpha \xi} (\mu |\nabla_h v_{hh}|^2 \\
&\quad + \lambda |\text{div}_h v_{hh}|^2 + |v_{hhz}|^2) \right) \, d\bar{x} = \int \mathcal{F}_1 \cdot v_{hh} \, d\bar{x} + \int \mathcal{F}_2 \cdot v_{hh} \, d\bar{x} \\
&\quad + \int \mathcal{F}_3 \cdot v_{hh} \, d\bar{x} + \int \mathcal{F}_4 \cdot v_{hh} \, d\bar{x} + \epsilon c_1 \alpha \int \left( e^{-\alpha \xi} (\mu (\nabla_h \xi \cdot \nabla_h v_{hh}) \cdot v_{hh} \\
&\quad + \lambda (v_{hh} \cdot \nabla_h \xi) \text{div}_h v_{hh}) \right) \, d\bar{x} =: I_1 + I_2 + I_3 + I_4 + I_5.
$$

On the other hand, after applying integration by parts and substituting (3.13) 1, one has

$$
\int \frac{c_2^2 e^{\epsilon \xi}}{\epsilon} \nabla_h \xi_{hh} \cdot v_{hh} \, d\bar{x} = - \int \frac{c_2^2 e^{\epsilon \xi}}{\epsilon} \xi_{hh} \text{div}_h v_{hh} \, d\bar{x} \\
- \int c_2^2 e^{\epsilon \xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi) \, d\bar{x} = \int \frac{c_2^2 e^{\epsilon \xi}}{\gamma - 1} \xi_{hh} (\partial_h \xi_{hh} - \mathcal{G} - \frac{\gamma - 1}{\epsilon} \partial_z w_{hh}) \, d\bar{x} \\
- \int c_2^2 e^{\epsilon \xi} \xi_{hh} (v_{hh} \cdot \nabla_h \xi) \, d\bar{x} = \frac{d}{dt} \left\{ \frac{c_2^2}{2(\gamma - 1)} \int e^{\epsilon \xi} |\xi_{hh}|^2 \, d\bar{x} \right\} \\
- \frac{c_2^2}{2(\gamma - 1)} \int e^{\epsilon \xi} \xi_{hh} |\xi_{hh}|^2 \, d\bar{x} - \frac{c_2^2}{\gamma - 1} \int e^{\epsilon \xi} \xi_{hh} \mathcal{G} \, d\bar{x}.
$$
Then it follows, after substituting (3.9) and (3.10), that

$$
1 \sim \epsilon dt + \text{div} \int I_1 \cdot G d\bar{x},
$$

one has

$$\frac{d}{dt} \left\{ \frac{1}{2} \| v_{hh} \|^2_{L^2} + \frac{\epsilon^2}{2(\gamma - 1)} \| \epsilon^{\xi/2} \xi_{hh} \|^2_{L^2} \right\} + c_1(\mu \| e^{-\alpha \xi/2} \nabla v_{hh} \|^2_{L^2} + \| e^{-\alpha \xi/2} v_{hh} \|^2_{L^2}) = \sum_{i=1}^8 I_i. \tag{3.14}
$$

Now we will estimate the right-hand side of (3.14). After applying integration by parts, $I_1$ can be written as

$$I_1 = \int \left( \frac{1}{2} \text{div} v |v_{hh}|^2 - (2v \cdot \nabla v_{hh} + v_{hh} \cdot \nabla v) \cdot v_{hh} \right) d\bar{x}
+ \int \frac{1}{2} |w_z| v_{hh}^2 d\bar{x} + \int \left( w_h \partial_z v \cdot v_{hh} + w_h \partial_z v \cdot v_{hh} 
+ 2w_h \partial_z v \cdot v_{hh} \right) d\bar{x} = I'_1 + I''_1 + I'''_1.
$$

Then it follows, after substituting (3.9) and (3.10), that

$$I'_1 \lesssim \int |\nabla v| |\nabla^2 v|^2 d\bar{x} \lesssim \| \nabla v \|_{L^2} \| \nabla^2 v \|_{L^3} \| \nabla^2 v \|_{L^6}
\lesssim \| \nabla v \|_{L^2} (\| \nabla^2 v \|_{L^2}^{1/2} \| \nabla^2 v \|_{L^2}^{1/2} \| \nabla^2 v \|_{L^2}) (\| \nabla^2 v \|_{L^2} + \| \nabla^3 v \|_{L^2}^2 + C \delta (1 + \| \nabla v \|_{L^2}^4) \| \nabla^2 v \|_{L^2}^2),
$$

$$I''_1 = -\frac{1}{2} \int \left( \frac{\epsilon}{\gamma - 1} \nabla v \xi + \text{div} v \nabla \xi \right) |v_{hh}|^2 d\bar{x} = \frac{1}{2} \int \left( \frac{\epsilon}{\gamma - 1} \xi \text{div} v \nabla \xi 
- \text{div} v \xi |v_{hh}|^2 \right) d\bar{x} + \int \frac{\epsilon}{\gamma - 1} \xi (v \cdot \nabla v_{hh}) \cdot v_{hh} d\bar{x}
\lesssim \epsilon \int |\xi| (|\nabla v| |\nabla^2 v|^2 + |v| |\nabla^2 v| |\nabla^3 v|) d\bar{x} + \int |\nabla v| |\nabla^2 v|^2 d\bar{x}
\lesssim \epsilon \| \xi \|_{L^\infty} \| \nabla^2 v \|_{L^3} (\| \nabla v \|_{L^2} \| \nabla^2 v \|_{L^6} + \| v \|_{L^6} \| \nabla^3 v \|_{L^2})$$
Next, to estimate $I_3$, $I_4$, $I_5$, applying the Hölder, Sobolev embedding, and Young inequalities. Hence we have

\[
I_1 \lesssim (\varepsilon \| v \|_{H^2}^2 + \varepsilon^2 \| v \|_{H^2}^2)^2 + (\varepsilon \| v \|_{H^2}^2) + \| v \|_{H^2}^2 + \| v \|_{H^2}^3
\]

Next, to estimate $I_3$, $I_4$, $I_5$, applying the Hölder, Sobolev embedding, and Young inequalities yields,

\[
I_3 = -2\varepsilon c_\alpha \int e^{-\varepsilon\alpha}\xi_h \left( (\mu \Delta_h v_h + \lambda \nabla_h \text{div}_h v_h) \cdot v_{hh} - \partial_z v_h \cdot v_{hhz} \right) d\tilde{x}
\]
\[ I_4 = -\varepsilon c_1 \alpha \int \left( e^{-\varepsilon \alpha \xi} (\xi h - \varepsilon \alpha \xi_h^3) \cdot (\mu \Delta_h v + \lambda \nabla_h div_h v) \cdot v_{hh} - \partial_z v \cdot v_{hhz} \right) d\vec{x} \lesssim \varepsilon \int_0^1 \left( e^{\varepsilon \alpha \xi} \| \xi \|_{L^\infty} (|\xi h|_{L^2} + \varepsilon |\xi_h^3|_{L^2}) \left( |\nabla^2 v|_{L^1} + |\partial_z v|_{L^2} \right) \right) d\vec{x} \lesssim \varepsilon \varepsilon c_1 \alpha \int_0^1 \left( e^{\varepsilon \alpha \xi} \| \xi \|_{L^\infty} (\| \xi \|_{H^2} + \varepsilon \| \xi \|_{H^2}^2) \right) \left( |\nabla^2 v|_{L^2} + |\partial_z v|_{H^2} \right) d\vec{x} \lesssim \varepsilon \left( \| \xi \|_{H^2} + \varepsilon \| \xi \|_{H^2}^2 \right) e^{2 \varepsilon \alpha} \| \xi \|_{H^2} \left( \| \nabla^2 v \|_{L^2} + \varepsilon \| v \|_{H^2} \right), \]

In order to estimate \( I_6, I_7 \), after substituting (1.4) and applying integration by parts, it holds

\[ I_6 = c_2^2 \alpha \int e^{\varepsilon \xi} |\xi h|^2 (-\varepsilon v \cdot \nabla_h \xi - (\gamma - 1) div_h v) d\vec{x}, \]

\[ I_7 = -c_2^2 \alpha \int e^{\varepsilon \xi} \xi_h (v_{hh} \cdot \nabla_h \xi + 2v_h \cdot \nabla_h \xi_h) d\vec{x} \]

Therefore, one has

\[ I_6 + I_7 + I_8 = -c_2^2 \alpha \int e^{\varepsilon \xi} \xi_h (v_{hh} \cdot \nabla_h \xi + 2v_h \cdot \nabla_h \xi_h) d\vec{x} \]

\[ + \frac{(2 - \gamma)c_2^2 \alpha}{2} \int e^{\varepsilon \xi} |\xi h|^2 \cdot div_h v d\vec{x} + c^2 \int e^{\varepsilon \xi} \xi_h (v_{hh} \cdot \nabla_h \xi) d\vec{x} \]

\[ \lesssim \varepsilon \| \xi \|_{L^\infty} \int_0^1 \left( |\xi h|_{L^2} \left( |v_{hh}|_{L^4} + |\nabla_h \xi|_{L^4} \right) + |\nabla v|_{L^\infty} |\nabla h \xi|_{L^2} \right) d\vec{x} \]
\[ \lesssim e^\epsilon \| \xi \|_{L^\infty} \int_0^1 \left( |\xi|^2_{H^2} \left( |\nabla_h^2 v|_{L^2} + |v|_{H^2} \right) \right) \, dx \lesssim \delta \| \nabla_h^2 v \|_{L^2} \]
\[ + C \delta e^{2\epsilon} \| \xi \|_{H^2} \| \xi \|_{H^2} \left( |\xi|^2_{H^2} + \| v \|_{H^2} \right), \]
where we have applied the Hölder, Sobolev embedding and Young inequalities. Similarly, \( I_2 \) can be estimated as following:
\[ I_2 \lesssim e^\epsilon \| \xi \|_{L^\infty} \int_0^1 \left( (|\nabla_h \xi h\|_{L^2} + \epsilon |\xi h|_{L^4}) |\nabla_h \xi|_{L^4} |v h|_{L^4} \right) \, dz \]
\[ \lesssim e^\epsilon \| \xi \|_{L^\infty} \int_0^1 \left( (|\xi|^2_{H^2} + |\xi|^3_{H^2}) (|\nabla_h v h|_{L^2} |v h|_{L^2}^3 + |v h|_{L^2}) \right) \, dz \]
\[ \lesssim \delta \| \nabla v h \|_{L^2}^2 + C \delta \left( \| \xi \|_{H^2}^{8/3} + \| \xi \|_{H^2}^4 \right) e^{4\epsilon/3} \| \xi \|_{H^2} \]
\[ \times \| v \|_{H^2} + (\| \xi \|_{H^2}^2 + \| \xi \|_{H^2}^3) e^{\epsilon} \| \xi \|_{H^2} \| v \|_{H^2}. \]
Therefore, (3.14) implies, after summing up the estimates above with \( \partial_h \in \{ \partial_x, \partial_y \} \),
\[ \frac{d}{dt} \left\{ \frac{1}{2} \| \nabla_h^2 v \|_{L^2}^2 + \frac{c^2}{2(\gamma - 1)} \| e^{\epsilon \xi /2} \nabla_h^2 \xi \|_{L^2}^2 \right\} \]
\[ + c_1 (\mu) e^{-e \alpha \epsilon /2} \| \nabla_h^2 v \|_{L^2}^2 + \lambda \| e^{-e \alpha \epsilon /2} \nabla_h^2 d_{v h} v \|_{L^2}^2 \]
\[ + \| e^{-e \alpha \epsilon /2} \nabla_h^2 v \|_{L^2}^2 \leq \delta \| \nabla \nabla_h^2 v \|_{L^2}^2 \]
\[ + e \mathcal{H}_1 (\| \xi \|_{H^2}, \| v \|_{H^2}) \| \nabla \nabla_h^2 v \|_{L^2}^2 + C \delta \mathcal{H}_2 (\| \xi \|_{H^2}, \| v \|_{H^2}), \]
where
\[ \mathcal{H}_1 = \mathcal{H}_1 (\| \xi \|_{H^2}, \| v \|_{H^2}), \mathcal{H}_2 = \mathcal{H}_2 (\| \xi \|_{H^2}, \| v \|_{H^2}), \]
are two regular functions of the arguments and \( \mathcal{H}_1(0) = \mathcal{H}_2(0) = 0 \). We will adopt the same notations for functions with such properties in this work.
After taking the \( L^2 \)-inner product of (1.3) with \( v, v_h \), respectively, repeating similar arguments as above yields the following estimate:
\[ \frac{d}{dt} \left\{ \frac{1}{2} \| v \|_{L^2}^2 + \frac{1}{2} \| \nabla_h v \|_{L^2}^2 + \frac{c^2}{2(\gamma - 1)} \| e^{\epsilon \xi /2} \xi \|_{L^2}^2 \right\} \]
\[ + c_1 (\mu) e^{-e \alpha \epsilon /2} \| \nabla_h v \|_{L^2}^2 + \mu \| e^{-e \alpha \epsilon /2} \nabla_h v \|_{L^2}^2 + \lambda \| e^{-e \alpha \epsilon /2} \nabla h d_{v h} v \|_{L^2}^2 \]
\[ + \| e^{-e \alpha \epsilon /2} \nabla h d_{v h} v \|_{L^2}^2 \leq \delta \| \nabla v \|_{H^2}^2 + C \delta \mathcal{H}_2 (\| \xi \|_{H^2}, \| v \|_{H^2}). \]
Estimates on the vertical derivatives

After applying $\partial_z$ to (1.4), it holds
\[
\begin{align*}
\partial_t v_z - c_1 e^{-\varepsilon \alpha \xi} (\mu \Delta_h v_z + \lambda \nabla_h \text{div}_h v_z + \partial_z v_z) &= -v_z \cdot \nabla_h v \\
&\quad - v \cdot \nabla_h v_z - w_z \partial_z v - w \partial_z v_z.
\end{align*}
\] (3.19)

Again, applying $\partial_z$ to (3.19) yields,
\[
\begin{align*}
\partial_t v_{zz} - c_1 e^{-\varepsilon \alpha \xi} (\mu \Delta_h v_{zz} + \lambda \nabla_h \text{div}_h v_{zz} + \partial_z v_{zz}) &= -v \cdot \nabla_h v_{zz} \\
&\quad - 2v_z \nabla_h v_z - v_z \cdot \nabla_h v - w_z \partial_z v - 2w_z \partial_z v_z.
\end{align*}
\] (3.20)

Next, we take the $L^2$-inner produce of (3.20) with $v_{zz}$. After applying integration by parts, one has
\[
\begin{align*}
\frac{d}{dt} \left\{ \frac{1}{2} \left\| v_{zz} \right\|_{L^2}^2 \right\} + c_1 \left( \mu \left\| e^{-\varepsilon \alpha \xi} \nabla_h v_{zz} \right\|_{L^2}^2 + \lambda \left\| e^{-\varepsilon \alpha \xi} \text{div}_h v_{zz} \right\|_{L^2}^2 \right) \\
&\quad + \left\| e^{-\varepsilon \alpha \xi} v_{zz} \right\|_{L^2}^2 = I_9 + I_{10} + I_{11},
\end{align*}
\] (3.21)

where
\[
\begin{align*}
I_9 &:= - \int (v \cdot \nabla_h v_{zz} + 2v_z \nabla_h v_z + v_{zz} \cdot \nabla_h v) \cdot v_{zz} \, d\bar{x}, \\
I_{10} &:= - \int (w_z \partial_z v + \frac{3}{2} w_z \partial_z v_z) \cdot v_{zz} \, d\bar{x}, \\
I_{11} &:= c_1 \varepsilon \alpha \int e^{-\varepsilon \alpha \xi} (\mu (\nabla_h \xi \cdot \nabla_h v_{zz}) + \lambda (v_{zz} \cdot \nabla_h \xi) \text{div}_h v_{zz}) \, d\bar{x}.
\end{align*}
\]

After applying the Hölder, Sobolev embedding and Young inequalities, it holds,
\[
\begin{align*}
I_9 &\lesssim \left( \left\| v \right\|_{L^6} \left\| \nabla_h v_{zz} \right\|_{L^2} + \left\| v_z \right\|_{L^6} \left\| \nabla_h v_z \right\|_{L^2} + \left\| v_{zz} \right\|_{L^6} \left\| \nabla_h v \right\|_{L^6} \right) \\
&\quad \times \left\| v_{zz} \right\|_{L^3} \lesssim \left( \left\| v \right\|_{H^1} \left\| \nabla_h v_{zz} \right\|_{L^2} + \left\| v \right\|_{H^2}^2 \right) \left( \left\| v_{zz} \right\|_{L^2} \left\| \nabla v_{zz} \right\|_{L^2} \right) \\
&\quad + \left\| v_{zz} \right\|_{L^2} \lesssim \delta \left\| \nabla v_{zz} \right\|_{L^2}^2 + C_\delta \left( \left\| v \right\|_{H^2} + \left\| v \right\|_{H^2} \right) \left\| v \right\|_{H^2}^2, \\
I_{10} &= \varepsilon \alpha \int \left( \left( \nabla_h v_z \right) (\partial_z v \cdot v_{zz}) + \frac{3}{2} \left( \nabla_h \xi \right) (\partial_z v \cdot v_{zz}) \right) \, d\bar{x} \\
&\quad + \int \left( \text{div}_h \nabla_h v_z \right) (\partial_z v \cdot v_{zz}) + \frac{3}{2} \text{div}_h \nabla_h v_z \, d\bar{x} \\
&\lesssim \varepsilon \alpha \left( \left\| v \right\|_{L^4} \left\| \nabla_h \xi \right\|_{L^4} \left\| v_z \right\|_{L^3} + \left\| v \right\|_{L^6} \left\| \nabla_h \xi \right\|_{L^6} \left\| v_{zz} \right\|_{L^3} \right) \left\| v_{zz} \right\|_{L^6} \\
&\quad + \left( \left\| \nabla_h v_z \right\|_{L^2} \left\| v_z \right\|_{L^3} + \left\| \nabla_h v \right\|_{L^3} \left\| v_{zz} \right\|_{L^2} \right) \left\| v_{zz} \right\|_{L^6}.
\end{align*}
\]
\[
\lesssim (\varepsilon \alpha \| \xi \|_{H^2} + 1) \| v \|_{H^2}^2 (\| \nabla v_{zz} \|_{L^2} + \| v \|_{H^2}) \\
\lesssim \delta \| \nabla v_{zz} \|_{L^2}^2 + C_\delta (\varepsilon^2 \alpha^2 \| \xi \|_{H^2}^2 + 1) (\| v \|_{H^2}^2 + \| v \|_{H^2}) \| v \|_{H^2}^2,
\]

\[
I_{11} \lesssim \varepsilon \alpha e^{\varepsilon \alpha \| \xi \|_{L^\infty}} \| \nabla h \xi \|_{L^6} \| v_{zz} \|_{L^3} \| \nabla h v_{zz} \|_{L^2} \lesssim \varepsilon \alpha e^{\varepsilon \alpha \| \xi \|_{H^2}} \| \xi \|_{H^2} \\
\times (\| v_{zz} \|_{L^2}^{1/2} \| \nabla v_{zz} \|_{L^2}^{1/2} + \| v_{zz} \|_{L^2}^2) \| \nabla h v_{zz} \|_{L^2} \lesssim \delta \| \nabla v_{zz} \|_{L^2}^2 \\
+ C_\delta (\varepsilon^4 \| \xi \|_{H^2}^4 + \varepsilon^2 \| \xi \|_{H^2}^2) e^{4\varepsilon \alpha \| \xi \|_{H^2}} \| v \|_{H^2}^2,
\]

where we have substituted (3.9) and (3.11) into I_{10}. Therefore, we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \| v_{zz} \|_{L^2}^2 \right\} + c_1 (\mu \| e^{-\varepsilon \alpha \xi^2/2} \nabla h v_{zz} \|_{L^2}^2 + \lambda \| e^{-\varepsilon \alpha \xi^2/2} \text{div}_h v_{zz} \|_{L^2}^2)
\]

\[
\quad + \| e^{-\varepsilon \alpha \xi^2/2} v_{zz} \|_{L^2}^2 \right\} \lesssim \delta \| \nabla v_{zz} \|_{L^2}^2 + C_\delta H_2 (\| \xi \|_{H^2}, \| v \|_{H^2}).
\]

Next, we establish the estimate of v_{hz}. Apply \partial_h to (3.19). It follows

\[
\partial_z v_{hz} - c_1 e^{-\varepsilon \alpha \xi} (\mu \Delta_h v_{hz} + \lambda \nabla_h \text{div}_h v_{hz} + \partial_z v_{hz}) = -v_{hz} \cdot \nabla_h v \\
- v_{zz} \cdot \nabla_h v_{hz} - v_h \cdot \nabla_h v_z - v \cdot \nabla_h v_{hz} - w_{hh} \partial_z v - w_{zz} \partial_z v_{hz} \\
- w \partial_z v_{hz} - w_h \partial_z v_z - c_1 \varepsilon \alpha e^{-\varepsilon \alpha \xi} \xi_{hh} (\mu \Delta_h v_z + \lambda \nabla_h \text{div}_h v_z \\
+ \partial_{zz} v_z).
\]

Take the \(L^2\)-inner product of (3.23) with \(v_{hz}\) and apply integration by parts in the resultant equation. It follows,

\[
\frac{d}{dt} \left\{ \frac{1}{2} \| v_{hz} \|_{L^2}^2 \right\} + c_1 (\mu \| e^{-\varepsilon \alpha \xi^2/2} \nabla h v_{hz} \|_{L^2}^2 + \lambda \| e^{-\varepsilon \alpha \xi^2/2} \text{div}_h v_{hz} \|_{L^2}^2)
\]

\[
\quad + \| e^{-\varepsilon \alpha \xi^2/2} v_{hz} \|_{L^2}^2 \right\} = I_{12} + I_{13} + I_{14} + I_{15} + I_{16},
\]

where

\[
I_{12} := - \int (v_{hz} \cdot \nabla_h v + v_z \cdot \nabla_h v_h + v_h \cdot \nabla_h v_z - \frac{1}{2} (\text{div}_h v) v_{hz}) \cdot v_{hz} \, d\vec{x},
\]

\[
I_{13} := - \int (w_{hz} \partial_z v + w_z \partial_z v_h - \frac{1}{2} (w_z) v_{hz}) \cdot v_{hz} \, d\vec{x},
\]

\[
I_{14} := - \int w_h (\partial_z v_z \cdot v_{hz}) \, d\vec{x},
\]

\[
I_{15} := -c_1 \varepsilon \alpha \int e^{-\varepsilon \alpha \xi} \xi_{hh} (\mu \Delta_h v_z + \lambda \nabla_h \text{div}_h v_z + \partial_{zz} v_z) \cdot v_{hz} \, d\vec{x},
\]

\[
I_{16} := c_1 \varepsilon \alpha \int e^{-\varepsilon \alpha \xi} (\mu (\nabla_h \xi \cdot \nabla_h v_{hz}) \cdot v_{hz} + \lambda (v_{hz} \cdot \nabla_h \xi) \text{div}_h v_{hz}) \, d\vec{x}.
\]
After substituting (3.12), (3.9) in $I_{13}$ and (3.10) in $I_{14}$, applying the Hölder, Sobolev embedding, Minkowski and Young inequalities yields,

$$I_{12} \lesssim \left( \| \nabla h v_z \|_{L^2} \| \nabla_h v \|_{L^3} + \| v_z \|_{L^3} \| \nabla_h v h \|_{L^2} \right) \| v h_z \|_{L^6}
\lesssim \| v \|_{H^2}^2 \left( \| \nabla h v_z \|_{L^2} + \| v h_z \|_{L^2} \right) \lesssim \delta \| \nabla v h_z \|_{L^2}^2
+ C_{\delta} \| v \|_{H^2}^4 + \| v \|_{H^2}^3,$$

$$I_{13} = \varepsilon \alpha \int \left( \langle \tilde{v}_h \cdot \nabla h \xi + \tilde{v} \cdot \nabla h \xi h \rangle (\partial_z v \cdot v h_z) + \frac{1}{2} (\tilde{v} \cdot \nabla h \xi) (v h_z \cdot v h_z) \right) d\vec{x}
+ \int \left( \text{div} h \tilde{v}_h (\partial_z v \cdot v h_z) + \frac{1}{2} \text{div} h \tilde{v} (v h_z \cdot v h_z) \right) d\vec{x}
\lesssim \varepsilon \alpha \left( \| v h \|_{L^6} \| \nabla h \xi \|_{L^2} + \| v \|_{L^6} \| \xi h \|_{L^2} \right) \| \partial_z v \|_{L^6}
+ \| v \|_{L^6} \| \nabla h \xi \|_{L^6} \| v h_z \|_{L^2} \right) \| v h_z \|_{L^6} + \left( \| \nabla h v h \|_{L^2} \| \partial_z v \|_{L^3} \right)
\times \left( \| \nabla v h_z \|_{L^2} + \| v h_z \|_{L^2} \right) \lesssim \delta \| \nabla v h_z \|_{L^2}^2
+ C_{\delta} (\varepsilon^2 \| \xi \|_{H^2}^2 \| v \|_{H^2}^2 + \| v \|_{H^2}^2 + 1) \| v \|_{H^2}^2,$$

$$I_{14} = \int \left[ \int_0^z \left( \varepsilon (\tilde{v}_h \cdot \nabla h \xi + \tilde{v} \cdot \nabla h \xi h) + \text{div} h \tilde{v}_h \right) d\vec{x} \times (\partial_z v \cdot v h_z) \right] d\vec{x}
\lesssim \int_0^1 \left( \varepsilon (|v h|_{L^4} |\nabla h \xi|_{L^4} + |v|_{L^\infty} |\nabla h \xi h|_{L^2}) + |\nabla h v h|_{L^2} \right) d\vec{z}
\times \int_0^1 \| \partial_z v_z \|_{L^4} |v h_z|_{L^4} d\vec{z} \lesssim \int_0^1 \left( \varepsilon |v|_{H^2} |\xi|_{H^2} + |v|_{H^2} \right) d\vec{z}
\times \int_0^1 \left( \left( |\partial_z v_z|_{L^2}^{1/2} |\nabla h \partial_z v_z|_{L^2}^{1/2} + |\partial_z v_z|_{L^2} \right) \left( |v h_z|_{L^2}^{1/2} |\nabla h v h_z|_{L^2}^{1/2}
+ |v h_z|_{L^2} \right) \right) d\vec{z} \lesssim \delta \| \nabla v h_z \|_{L^2}^2 + C_{\delta} (\varepsilon^2 \| \xi \|_{H^2}^2 \| v \|_{H^2}^2
+ \| v \|_{H^2}^2 + 1) \| v \|_{H^2}^2,$$

$$I_{15} \lesssim \varepsilon \varepsilon^\alpha \| \xi \|_{L^\infty} \| \nabla^3 v \|_{L^2} \| v h_z \|_{L^4} \| \xi h \|_{L^6} \lesssim \delta \| \nabla^3 v \|_{L^2}
+ C_{\delta} \varepsilon^4 \varepsilon^4 \| \xi \|_{H^2} \left( \| \xi \|_{H^2}^4 + \| \xi \|_{H^2}^2 \right) \| v \|_{H^2}^2,$$

$$I_{16} \lesssim \varepsilon \varepsilon^\alpha \| \xi \|_{L^\infty} \| \nabla h \xi \|_{L^5} \| \nabla v h_z \|_{L^2} \| v h_z \|_{L^3} \approx \varepsilon \varepsilon^\alpha \| \xi \|_{H^2} \| \xi \|_{H^2}
\| \nabla v h_z \|_{L^2} \left( |v h_z|_{L^2}^{1/2} \| \nabla v h_z \|_{L^2}^{1/2} + |v h_z|_{L^2} \right) \lesssim \delta \| \nabla v h_z \|_{L^2}
+ C_{\delta} (\varepsilon^4 \varepsilon^4 \alpha \| \xi \|_{H^2}^4 + 1) \| v \|_{H^2}^2.$$

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Therefore, summing up the estimates above with $\partial \xi \in \{\partial_x, \partial_y\}$ leads to,

$$
\frac{d}{dt} \left\{ \frac{1}{2} \left\| \nabla_h v_z \right\|_{L^2}^2 \right\} + c_1 (\mu) \left\| e^{-\varepsilon \alpha \xi / 2} \nabla_h^2 v_z \right\|_{L^2}^2 \\
+ \lambda \left\| e^{-\varepsilon \alpha \xi / 2} \nabla_h \text{div}_h v_z \right\|_{L^2}^2 + \left\| e^{-\varepsilon \alpha \xi / 2} \nabla_h v_z \right\|_{L^2}^2 (3.25)
$$

Similarly, after taking the $L^2$-inner product of (3.19) with $v_z$ and performing estimates as above, one has,

$$
\frac{d}{dt} \left\{ \frac{1}{2} \left\| v_z \right\|_{L^2}^2 \right\} + c_1 (\mu) \left\| e^{-\varepsilon \alpha \xi / 2} \nabla_h v_z \right\|_{L^2}^2 + \lambda \left\| e^{-\varepsilon \alpha \xi / 2} \text{div}_h v_z \right\|_{L^2}^2 (3.26)
$$

Consequently, after integrating in the time variable, (3.16), (3.18), (3.22), (3.26) yield (3.21).

**Estimates on the time derivatives**

Now we establish the final pieces of Proposition 1. After multiplying (1.4) with $\varepsilon e^{\varepsilon \xi}$ and averaging the resulting equation in the $z$-variable, one has

$$
\partial \xi e^{\varepsilon \xi} + e^{\varepsilon \xi} \nabla_h \text{div}_h \xi + (\gamma - 1) e^{\varepsilon \xi} \text{div}_h \bar{v} = 0. (3.27)
$$

Then applying the triangle inequality and the Sobolev embedding inequality in (3.27) leads to

$$
\left\| \partial \xi e^{\varepsilon \xi} \right\|_{L^2} \lesssim \varepsilon e^{\varepsilon} \left\| \xi \right\|_{L^\infty} \left\| v \right\|_{L^\infty} \left\| \nabla_h \xi \right\|_{L^2} + e^{\varepsilon} \left\| \xi \right\|_{L^\infty} \left\| \nabla_h v \right\|_{L^2} (3.28)
\lesssim e^{\varepsilon} \left\| \xi \right\|_{H^2} \left\| v \right\|_{H^2} \left\| \xi \right\|_{H^2} + e^{\varepsilon} \left\| \xi \right\|_{H^2} \left\| v \right\|_{H^2}
\lesssim \mathcal{H}_2 (\left\| \xi \right\|_{H^2}, \left\| v \right\|_{H^2}).
$$

Meanwhile, we have, after applying $\partial \xi$ to (3.27),

$$
\partial_t (e^{\varepsilon \xi})_h + e^2 e^{\varepsilon \xi} \nabla_h \xi \cdot \nabla_h \xi + e^{\varepsilon \xi} \text{div}_h \xi + e^{\varepsilon \xi} \text{div}_h \bar{v} + (\gamma - 1) e^{\varepsilon \xi} \text{div}_h \bar{v} = 0. (3.29)
$$

Thus it holds,

$$
\left\| \partial_t (e^{\varepsilon \xi})_h \right\|_{L^2} \lesssim e^{\varepsilon} \left\| \xi \right\|_{L^\infty} (e^2 \left\| \nabla_h \xi \right\|_{L^4}^2 \left\| v \right\|_{L^\infty} + \varepsilon \left\| \nabla_h \xi \right\|_{L^4}) \times \left\| \nabla_h v \right\|_{L^4} + \varepsilon \left\| \xi \right\|_{L^\infty} \left\| \nabla_h \xi_h \right\|_{L^2} + \left\| \nabla_h v_h \right\|_{L^2} (3.30)
\lesssim e^{\varepsilon} \left\| \xi \right\|_{H^2} (e^2 \left\| \xi \right\|_{H^2}^2 \left\| v \right\|_{H^2} + \left\| v \right\|_{H^2}) \lesssim \mathcal{H}_2 (\left\| \xi \right\|_{H^2}, \left\| v \right\|_{H^2}).
$$
Consequently, (3.28) and (3.30) imply (3.2).

On the other hand, after applying the projection operator $P_\sigma$ (defined in (2.2)) to (1.4), we have the following:

$$\partial_t P_\sigma v = -P_\sigma(v \cdot \nabla_h v) - P_\sigma(w \partial_z v) + c_1 P_\sigma(e^{-\varepsilon \alpha \xi} (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v)).$$

In order to estimate the $L^2$ and $H^1$ norms of $\partial_t P_\sigma v$, we apply the Hölder and Sobolev embedding inequalities as follows:

$$\|v \cdot \nabla_h v\|_{L^2} \lesssim \|v\|_{L^3} \|\nabla_h v\|_{L^6} \lesssim \|v\|_{H^2},$$

$$\|e^{-\varepsilon \alpha \xi} (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v)\|_{L^2} \lesssim e^{\varepsilon \alpha} \|\xi\|_{L^\infty} \times \|\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v\|_{L^2} \lesssim e^{\varepsilon \alpha} \|\xi\|_{H^2} \|v\|_{H^2},$$

$$\|\partial(v \cdot \nabla_h v)\|_{L^2} \lesssim \|\partial v \cdot \nabla_h v\|_{L^2} + \|v \cdot \nabla_h \partial v\|_{L^2} \lesssim \|v\|_{H^2}^2,$$ (3.32)

$$\|\partial(e^{-\varepsilon \alpha \xi} (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v))\|_{L^2} \lesssim e \|e^{-\varepsilon \alpha \xi} \partial \xi (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v)\|_{L^2} + \|e^{-\varepsilon \alpha \xi} \partial (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v)\|_{L^2} \lesssim e^{\varepsilon \alpha} \|\xi\|_{H^2} (\|\xi\|_{H^2} + 1) \|v\|_{H^3}.$$}

Here $\partial \in \{\partial_x, \partial_y, \partial_z\}$ denotes the spatial derivatives. To estimate the $L^2$ and $H^1$ norms of $w \partial_z v$, we first substitute the identities in (3.8), (3.9), (3.10) to $w, \partial_z w, \partial_h w$, respectively, and write down the following:

$$w \partial_z v = -\int_0^z (\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \text{div}_h \tilde{v}) \, dz \partial_z v,$$

$$\partial_h (w \partial_z v) = \partial_h w \partial_z v + w \partial_z v_h$$

$$= -\int_0^z (\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \varepsilon \alpha \tilde{v} \cdot \nabla_h \xi_h + \text{div}_h \tilde{v}_h) \, dz \partial_z v$$

$$- \int_0^z (\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \text{div}_h \tilde{v}) \, dz \partial_z v_h,$$

$$\partial_z (w \partial_z v) = \partial_z w \partial_z v + w \partial_z v_z$$

$$= -(\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \text{div}_h \tilde{v}) \partial_z v - \int_0^z (\varepsilon \alpha \tilde{v} \cdot \nabla_h \xi + \text{div}_h \tilde{v}) \, dz \partial_z v_z.$$

Therefore, after applying the Hölder, Minkowski, and Sobolev embedding
inequalities, the following estimates hold:

\[
\| w \partial_z v \|_{L^2}^2 = \int_0^1 |w \partial_z v|_{L^2}^2 \, dz \lesssim \int_0^1 \left( \int_0^1 (\varepsilon^2 \| v \|_{H^2} \| \xi \|_{H^2}^2 + \| v \|_{H^2}^2) \, dz \right) \lesssim \int_0^1 (\varepsilon^2 \| v \|_{H^2} \| \xi \|_{H^2}^2 + \| v \|_{H^2}^2) \, dz,
\]

\[
\| \partial_v (w \partial_z v) \|_{L^2}^2 = \int_0^1 |\partial_v (w \partial_z v)|_{L^2}^2 \, dz \lesssim \int_0^1 \left( \int_0^1 (\varepsilon^2 \| v \|_{H^2} \| \xi \|_{H^2}^2 + \| v \|_{H^2}^2) \, dz \right) \lesssim \int_0^1 (\varepsilon^2 \| v \|_{H^2} \| \xi \|_{H^2}^2 + \| v \|_{H^2}^2) \, dz.
\]

(3.3)

Combining the above estimates, together with (2.3), we have shown (3.3) and (3.4).

In addition, notice that \( \partial_t \xi = \varepsilon^{-1} e^{-\varepsilon \xi} \partial_t e^{\xi} \). From (3.2), one has

\[
\| \partial_t \xi \|_{L^2} \lesssim \varepsilon^{-1} \| e^{\xi} \|_{H^2} \| \partial_t e^{\xi} \|_{L^2} \lesssim \varepsilon^{-1} \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}),
\]

\[
\| \partial_t \xi \|_{H^1} \lesssim \| \partial_t \xi \|_{L^2} + \varepsilon^{-1} \| e^{\xi} \|_{H^2} (\| \xi \|_{L^2} + \| \partial_t e^{\xi} \|_{L^2} + \| \partial_t (e^{\xi})_h \|_{L^2})
\]

\[
\lesssim \| \partial_t \xi \|_{L^2} + \varepsilon^{-1} \| e^{\xi} \|_{H^2} (\| \xi \|_{H^2} + \| \partial_t e^{\xi} \|_{H^1} + \| \partial_t (e^{\xi})_h \|_{L^2})
\]

\[
\lesssim \varepsilon^{-1} \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}).
\]

(3.5)

On the other hand, (1.4), (3.2), and (3.3) yield

\[
\| \partial_t v \|_{L^2} \lesssim \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}) + \varepsilon^{-1} \| e^{\xi} \|_{L^2}
\]

\[
\lesssim (1 + \varepsilon^{-1}) \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}),
\]

(3.6)
\[ \| \partial_t v \|_{H^1} \lesssim \mathcal{H}_1(\| \xi \|_{H^2}, \| v \|_{H^2}) \| \nabla v \|_{H^2} \\
+ \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}) + \varepsilon^{-1} \| e^{\varepsilon x} \nabla e \xi \|_{H^1} \\
\lesssim \mathcal{H}_1(\| \xi \|_{H^2}, \| v \|_{H^2}) \| \nabla v \|_{H^2} \\
+ (1 + \varepsilon^{-1}) \mathcal{H}_2(\| \xi \|_{H^2}, \| v \|_{H^2}), \tag{3.57} \]

where in the last inequality we have used the fact that
\[ \| \partial_h(e^{\varepsilon x} \nabla e \xi) \|_{L^2} \lesssim \varepsilon \| e^{\varepsilon x} e \xi h \nabla e \xi \|_{L^2} + \| e^{\varepsilon x} \nabla e \xi h \|_{L^2} \]
\[ \lesssim \varepsilon \| \xi \|_{H^2} \| e \xi h \|_{L^2} \| \xi h \|_{L^6} + \| \xi \|_{H^2} \]
\[ \lesssim \varepsilon \| \xi \|_{H^2} \| \xi \|_{H^2} + \| \xi \|_{H^2}. \]

This finishes the proof of Proposition 1.

### 3.2 Local-in-time a priori estimates and local well-posedness

In this subsection, we aim at establishing the following proposition:

**Proposition 2.** In the case when either \( \Omega_h = \mathbb{T}^2 \) or \( \Omega_h = \mathbb{R}^2 \), consider initial data \((\xi_0, v_0) \in H^2(\Omega_h \times 2T)\), satisfying the compatibility conditions in (2.4). Let \( M_0, M_1 \) be two positive constants satisfying
\[ \| \xi_0 \|_{H^2}^2 \leq M_0, \| v_0 \|_{H^2}^2 \leq M_1. \tag{3.34} \]

Then for some positive constant \( \varepsilon_0 \in (0, 1) \) small enough, any \( \varepsilon \in (0, \varepsilon_0) \), there exists \( T_\varepsilon \in (0, \infty) \) such that there exists a unique strong solution \((\xi, v)\) to (1.4) in the time interval \([0, T_\varepsilon]\) with
\[ \xi \in L^\infty(0, T_\varepsilon; H^2(\Omega_h \times 2T)), \partial_t \xi \in L^\infty(0, T_\varepsilon; H^1(\Omega_h \times 2T)), \]
\[ v \in L^\infty(0, T_\varepsilon; H^2(\Omega_h \times 2T)) \cap L^2(0, T_\varepsilon; H^3(\Omega_h \times 2T)), \tag{3.35} \]
\[ \partial_t v \in L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2T)) \cap L^2(0, T_\varepsilon; H^1(\Omega_h \times 2T)). \]

Moreover, there exist positive constants \( C_0, C_1 \) independent of \( \varepsilon \), and \( C_2 = C_2(\varepsilon, C_0, M_0, M_1) \) such that
\[ \sup_{0 \leq t \leq T_\varepsilon} \| \xi(t) \|_{H^2}^2 \leq C_0 M_0, \sup_{0 \leq t \leq T_\varepsilon} \| v(t) \|_{H^2}^2 + \int_0^{T_\varepsilon} \| \nabla v(t) \|_{H^2}^2 \, dt \leq C_1 M_1, \]
\[ \sup_{0 \leq t \leq T_\varepsilon} \{ \| \xi(t) \|_{H^1}^2 + \| \partial_t v(t) \|_{L^2}^2 \} + \int_0^{T_\varepsilon} \| \partial_t v(t) \|_{H^1}^2 \, dt \leq C_2. \tag{3.36} \]

Here \( C_0 \in (1, \infty) \), \( C_1, C_2 \) are determined by (3.57), (3.16), and (3.58), below, and \( T_\varepsilon \) depends on \( M_0, M_1 \) and \( \varepsilon \).
Proposition 2 can be shown by applying the Banach fixed point theorem. In the following, without going into too much details, we will only sketch the proper steps to construct this local strong solution.

**Sketch of constructing strong solutions.** Let $\xi', v'$ be regular enough functions. Consider the following linear system associated with (1.4):

$$
\begin{aligned}
&\partial_t \xi + \overline{v} \cdot \nabla_h \xi + \frac{\gamma - 1}{\varepsilon} \text{div}_h \overline{v} = 0 \quad \text{in } \Omega_h \times 2T, \\
&\partial_t v + v' \cdot \nabla_h v + w' \partial_z v + \frac{\varepsilon^2}{\varepsilon} \nabla_h \xi' = c_1 e^{-\varepsilon \alpha \xi'} \left( \mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v \right) \quad \text{in } \Omega_h \times 2T, \\
&\partial_z \xi = 0 \quad \text{in } \Omega_h \times 2T,
\end{aligned}
$$

(3.37)

where $w'$ is given by

$$
w' := - \int_0^z (\varepsilon \alpha \overline{v} \cdot \nabla_h \xi' + \text{div}_h \overline{v}') \, dz.
$$

(3.38)

Then for $(\xi', v')$ with $(\xi', v')|_{t=0} = (\xi_0, v_0)$ and satisfying the same regularity and bounds as in (3.36), there is a unique solution to system (3.37) with initial data $(\xi, v)|_{t=0} = (\xi_0, v_0)$, after applying the standard existence theory for linear hyperbolic and parabolic equations. Moreover, similar *a priori* estimates as in our previous work [28] show that the solution $(\xi, v)$ satisfies the same regularity and bounds of norms in (3.35) and (3.36). We define the following function framework in order to apply the Banach fixed point theorem.

Consider the function space

$$
\mathcal{Y}_{T_\varepsilon} := \{ (\xi, v) | \xi \in L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2T)), \\
v \in L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2T)) \cap L^2(0, T_\varepsilon; H^1(\Omega_h \times 2T)), \\
\partial_z \xi = 0 \}
$$

(3.39)

and the following subset of $\mathcal{Y}_{T_\varepsilon}$

$$
\mathcal{X}_{T_\varepsilon} := \{ (\xi, v) | (\xi, v) \in \mathcal{Y}_{T_\varepsilon} \text{ and the bounds in (3.36) hold} \}.
$$

(3.40)

For any $(\xi, v) \in \mathcal{Y}_{T_\varepsilon}$, define the norm

$$
\left\| (\xi, v) \right\|_{\mathcal{Y}_{T_\varepsilon}}^2 := \left\| \xi \right\|_{L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2T))}^2 + \left\| v \right\|_{L^\infty(0, T_\varepsilon; L^2(\Omega_h \times 2T))}^2 + \left\| \nabla v \right\|_{L^2(0, T_\varepsilon; L^2(\Omega_h \times 2T))}^2.
$$

(3.41)
Then \((\mathcal{Y}_{T_{\varepsilon}}, \|\cdot\|_{\mathcal{Y}_{T_{\varepsilon}}})\) is a complete metric space. In addition, we define the following map: with same initial data \((\xi', v')\big|_{t=0} = (\xi, v)\big|_{t=0} = (\xi_0, v_0),\)

\[
\mathcal{T} : (\xi', v') \leadsto (\xi, v) \text{ mapping } (\xi', v') \in \mathcal{X}_{T_{\varepsilon}} \subset \mathcal{Y}_{T_{\varepsilon}}
\]
to the solution to system (3.37) (3.42) with trivial extension outside the set \(\mathcal{X}_{T_{\varepsilon}}\).

Then we show that, For \(\varepsilon_0 \in (0, 1)\) small enough and any \(\varepsilon \in (0, \varepsilon_0)\), there exists \(T_{\varepsilon} \in (0, \infty)\), such that \(\mathcal{T} : \mathcal{Y}_{T_{\varepsilon}} \mapsto \mathcal{Y}_{T_{\varepsilon}}\) is a contraction mapping in \(\mathcal{X}_{T_{\varepsilon}}\). Notice, once this is proved, then the Banach fixed point theorem implies that we have a unique strong solution to (1.4) with \(\varepsilon, T_{\varepsilon}\) as described above.

In the rest of this subsection, we focus on showing that for \(\varepsilon \in (0, \varepsilon_0)\) with \(\varepsilon_0\) small enough, and some \(T_{\varepsilon} \in (0, \infty)\) depending on \(\varepsilon\), we have:

1. \(\mathcal{T}\) maps \(\mathcal{X}_{T_{\varepsilon}}\) into \(\mathcal{X}_{T_{\varepsilon}}\); i.e., \((\xi, v) = T(\xi', v') \in \mathcal{X}_{T_{\varepsilon}}\) for any \((\xi', v') \in \mathcal{X}_{T_{\varepsilon}}\);

2. \(\mathcal{T}\) is a contraction mapping in \(\mathcal{X}_{T_{\varepsilon}}\) with respect to the topology of \(\mathcal{Y}_{T_{\varepsilon}}\);

i.e., for any \((\xi_i, v_i) \in \mathcal{X}_{T_{\varepsilon}} \subset \mathcal{Y}_{T_{\varepsilon}}\) and \((\xi, v_i) = T(\xi_i', v_i')\), \(i = 1, 2\), one has

\[
\| (\xi_1 - \xi_2, v_1 - v_2) \|_{\mathcal{Y}_{T_{\varepsilon}}} \leq q \| (\xi_1' - \xi_2', v_1' - v_2') \|_{\mathcal{Y}_{T_{\varepsilon}}},
\]

for some \(q \in (0, 1)\).

We will only show the corresponding \textit{a priori} estimates to show (1) and (2).

**Proof of (1): Estimates of \(\xi\).** First, we shall derive the estimates of \(\xi\) from (3.37). We shall only show the highest order estimates. After applying \(\partial_{hh}\) to (3.37), we have

\[
\partial_t \xi_{hh} + \nabla_h \cdot \nabla_h \xi_{hh} = -2 \nabla_h \cdot \nabla_h \xi - \nabla_h \cdot \nabla_h \xi - \frac{\gamma - 1}{\varepsilon} \text{div}_h \nabla_h \xi_{hh}.
\]

Then after taking the \(L^2\)-inner product of (3.44) with \(2\xi_{hh}\) in \(\Omega_h\), it follows

\[
\frac{d}{dt} \| \xi_{hh} \|_{L^2}^2 = \int_{\Omega_h} \text{div}_h \nabla' \| \xi_{hh} \|_2^2 \ dx \ dy - 4 \int_{\Omega_h} (\nabla_h \cdot \nabla_h \xi_{hh}) \xi_{hh} \ dx \ dy
\]

\[
- 2 \int_{\Omega_h} (\nabla_h \cdot \nabla_h \xi_{hh}) \ dx \ dy - \frac{2(\gamma - 1)}{\varepsilon} \int_{\Omega_h} \text{div}_h \nabla_{hh} \xi_{hh} \ dx \ dy
\]

\[
\leq C \| \nabla' \|_{H^2} \| \xi_{hh} \|_{H^2}^2 + C \varepsilon^{-1} \| \nabla' \|_{H^2} \| \xi_{hh} \|_{L^2}.
\]

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Similar arguments also hold for $|\xi_h|_{L^2}$ and $|\xi|_{L^2}$, and therefore we have
\[
\frac{d}{dt}|\xi|_{H^2}^2 \leq C \|\nabla'\|_{H^2}|\xi|_{H^2}^2 + C\varepsilon^{-1} \|\nabla'\|_{H^2}|\xi|_{H^2}^2
\]
\[
\leq 2C \|\nabla'\|_{H^2}|\xi|_{H^2}^2 + C\varepsilon^{-2} \|\nabla'\|_{H^2}^2.
\]
Then applying the Grönwall inequality and the Hölder inequality yields
\[
\sup_{0 \leq t \leq T_\varepsilon} \|\xi(t)\|_{H^2}^2 \leq e^{\int_{0}^{T_\varepsilon} 2C \|\nabla'\|_{H^2} dt} \left(\|\xi_0\|_{H^2}^2 \right)
\]
\[
+ C\varepsilon^{-2} \int_{0}^{T_\varepsilon} \|\nabla'\|_{H^2} dt \leq e^{2C T_\varepsilon^{1/2}(\int_{0}^{T_\varepsilon} \|\nabla'\|_{H^2}^2 dt)^{1/2}} \left(\|\xi_0\|_{H^2}^2 \right)
\]
\[
+ C\varepsilon^{-2} T_\varepsilon^{1/2} \left(\int_{0}^{T_\varepsilon} \|\nabla'\|_{H^2}^2 dt\right)^{1/2}
\]
\[
\leq e^{2CT_\varepsilon^{1/2}(C_1 M_1)^{1/2}} (M_0 + C\varepsilon^{-2} T_\varepsilon^{1/2}(C_1 M_1)^{1/2}) \leq C_0 M_0,
\]
where we have chosen $T_\varepsilon$ sufficiently small in the last inequality and $C_0 \in (1, \infty)$.

On the other hand, from (3.37), we have
\[
|\partial_t \xi|_{H^1}^2 \leq C|\nabla' \cdot \nabla h \xi|_{H^1}^2 + \varepsilon^{-2} C \|\nabla h \nabla' \xi|_{H^1}^2 \leq C(C_0 M_0 + \varepsilon^{-2}) C_1 M_1 < \infty.
\]

**Estimates of $v$.** Next we shall present the estimates of $v$. Similarly, we will only sketch the highest order estimates. After applying $\partial_{hh}$ to (3.37), one obtains
\[
\partial_t v_{hh} + v' \cdot \nabla v_{hh} + w' \partial_z v_{hh} - c_1 e^{-\varepsilon \alpha} (\mu \Delta_h v_{hh} + \lambda \nabla_h \text{div}_h v_{hh} + \partial_z v_{hh})
\]
\[
= -\left(\frac{2e^{\varepsilon \xi'}}{\varepsilon} \nabla_h \xi'_h\right)_{hh} - v' \partial_z v - 2u' \cdot \nabla_h v - w' \partial_z v - 2w' \partial_z v_{hh}
\]
\[
- \varepsilon c_1 \alpha e^{-\varepsilon \alpha} (\xi'_h - \alpha (\xi'_h)^2) (\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_z v)
\]
\[
- 2\varepsilon c_1 \alpha e^{-\varepsilon \alpha} \xi'_h (\mu \Delta_h v_{hh} + \lambda \nabla_h \text{div}_h v_{hh} + \partial_z v_{hh}).
\]

Then after taking the $L^2$-inner product of (3.47) with $2v_{hh}$ in $\Omega_h \times 2T$, it follows, after applying integration by parts,
\[
\frac{d}{dt} \|v_{hh}\|_{L^2}^2 + 2c_1 \int e^{-\varepsilon \alpha} (\mu |\nabla_h v_{hh}|^2 + \lambda |\text{div}_h v_{hh}|^2 + |\partial_z v_{hh}|^2) \, dx
\]
\[
= \int \left(\frac{2e^{\varepsilon \xi'}}{\varepsilon} \nabla_h \xi'_h\right) \cdot v_{hh} \, dx + \int \left(\text{div}_h v' v_{hh}^2 - 2(v' \cdot \nabla_h v) v_{hh}\right)
\]

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\[ -4(v_h' \cdot \nabla_h v_h) \cdot v_{hh} - \text{div}_h \tilde{v}'|v_{hh}|^2 \right) \, dx - 2 \int (w'_h \partial_z v_h \cdot v_{hh} \\
- w'_h \partial_z v \cdot v_{hh} \right) \, dx - \varepsilon \alpha \int (\tilde{v}' \cdot \nabla_h \xi')|v_{hh}|^2 \, dx \\
- 2 \varepsilon c_1 \alpha \int e^{-\varepsilon \alpha \xi'}(\xi'_h - \varepsilon \alpha (\xi'_h)^2)(\mu \Delta_h v + \lambda \nabla_h \text{div}_h v + \partial_{zz} v) \cdot v_{hh} \, dx \\
- 4 \varepsilon c_1 \alpha \int e^{-\varepsilon \alpha \xi'}(\xi'_h + \lambda \nabla_h \text{div}_h v + \partial_{zz} v) \cdot v_{hh} \, dx \\
+ 2 \varepsilon c_1 \alpha \int e^{-\varepsilon \alpha \xi'}(\mu (\nabla_h \xi' \cdot \nabla_h v_{hh}) \cdot v_{hh} + \lambda (v_{hh} \cdot \nabla_h \xi')(\text{div}_h v_{hh})) \, dx \\
=: L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 ,
\]

where we have substituted the identity
\[ w'_z = - \varepsilon \alpha \tilde{v}' \cdot \nabla_h \xi' - \text{div}_h \tilde{v}' , \]

which is obtained by taking \( \partial_z \) to (3.38).

Similarly as in the last section, from (3.38), we have
\[ u'_h = - \int_0^z \left( \varepsilon \alpha \tilde{v}'_h \cdot \nabla_h \xi' + \varepsilon \alpha \tilde{v}' \cdot \nabla_h \xi'_h + \text{div}_h \tilde{v}'_h \right) \, dz . \tag{3.48} \]

Then following similar arguments as before, one can derive, for any \( \delta \in (0, 1) \) and fixed \( \varepsilon \in (0, 1) \),
\[ L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 \leq \left( \delta + \varepsilon \mathcal{H}_1(\| \xi' \|_{H^2}, \| v' \|_{H^2}) \right) \\
\times \| \nabla v \|_{H^2}^2 + \delta \| \nabla v' \|_{H^2}^2 \| v \|_{H^2}^2 + \mathcal{H}_2(\varepsilon, C_{\delta}, \| \xi' \|_{H^2}, \| v' \|_{H^2}) \| v \|_{H^2}^2 \\
+ \mathcal{H}_3(\varepsilon, C_{\delta}, \| \xi' \|_{H^2}, \| v' \|_{H^2}) , \tag{3.49} \]

where \( C_{\delta} \approx \delta^{-1} \). We remind readers that we have been using \( \{ \mathcal{H}_i \}_{i=1,2,3} \) to denote regular functions of the arguments with property \( \mathcal{H}_i(0) = 0 \). Details are listed below, for readers’ reference:

\[ L_1 \lesssim \| v_h \|_{L^2}(\varepsilon^{-1} \varepsilon \| \xi' \|_{H^2} + \| v' \|_{H^2} + \| \xi' \|_{H^2}^2) \]
\[ \lesssim \delta \| \nabla v \|_{H^2}^2 + C_{\delta} \| v \|_{H^2}^2 \| \xi' \|_{H^2}^2 + \| v' \|_{H^2}^2 \| v \|_{H^2}^2 \| \xi' \|_{H^2}^4 , \]
\[ L_2 \lesssim \| \nabla v' \|_{L^6} \| v_h \|_{L^3} \| v_{hh} \|_{L^2} + \| v'_{hh} \|_{L^2} \| \nabla_h v \|_{L^3} \| v_{hh} \|_{L^6} \]
\[ \lesssim \| v' \|_{H^2} \| v \|_{H^2}^{3/2} \| \nabla v \|_{H^2}^{1/2} + \| v \|_{H^2} \| \nabla v \|_{H^2} \]
\[ \lesssim \delta \| \nabla v \|_{H^2}^2 + C_{\delta}(\| v' \|_{H^2} + 1) \| v \|_{H^2}^2 , \]

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Then we have arrived at the estimate
\[
L_3 \lesssim \int_0^1 \left( \varepsilon |v'_h|_{L^4} \xi'_h |L^4_\varepsilon + \varepsilon |v'|_{L^\infty} |\xi'|_{H^2} \right) dz \cdot \int_0^1 \left( |\partial_z v_h|_{L^4} |v_{hh}|_{L^4} \right) dz
+ |\partial_z v|_{L^\infty} |v_{hh}|_{L^2} dz + \int_0^1 |v'_h|_{L^4} dz \cdot \int_0^1 \left( |\partial_z v_h|_{L^2} |v_{hh}|_{L^4} \right) dz
+ |\partial_z v|_{L^4} |v_{hh}|_{L^2} dz \lesssim \int_0^1 \varepsilon |v'|_{H^2} |\xi'|_{H^2} dz \cdot \int_0^1 \left( |\nabla v|_{H^2} (|\nabla v|_{H^1} + |\nabla v|_{H^2}) \right) dz + \int_0^1 |v'|_{H^2}^{1/2} |\nabla v|_{H^2}^{1/2} dz \cdot \int_0^1 \left( |\nabla v|_{H^2}^{3/2} |\nabla v|_{H^2}^{1/2} \right) dz
+ |\nabla v|_{H^2} \left( |\nabla v|_{H^2} + \delta \right) |\nabla v|_{H^2}^2
\]
+ \delta \|\nabla v\|_{H^2}^2 v \|_{H^2}^2 + \varepsilon \|\xi\|_{H^2} \|v\|_{H^2}^2 + C_\delta (\|v\|_{H^2}^2 + 1) \|v\|_{H^2}^2,

L_4 \lesssim \varepsilon \|v\|_{H^2} \|\nabla_h \xi\|_{L^6} \|v_{hh}\|_{L^2} \|v_{hh}\|_{L^3} \lesssim \varepsilon \|v\|_{H^2}^2 \times \|\xi\|_{H^2} \|v\|_{H^2}^2 \frac{3}{2} \|\nabla v\|_{H^2}^{1/2} \lesssim \delta \|\nabla v\|_{H^2}^2
+ C_\delta \varepsilon^{4/3} \|v\|_{H^2}^2 \|\xi\|_{H^2} \|v\|_{H^2}^2,

L_5 \lesssim \varepsilon \varepsilon^{4/3} \|\xi\|_{H^2} \left( \|\xi\|_{H^2} + \varepsilon \|\xi\|_{H^2}^2 \right) \|\nabla v\|_{L^2} \|v_{hh}\|_{L^6} \lesssim \varepsilon \varepsilon^{4/3} \|\xi\|_{H^2} \times \left( \|\xi\|_{H^2} + \varepsilon \|\xi\|_{H^2}^2 \right) \times \left( \|v\|_{H^2}^{1/2} \|\nabla v\|_{H^2}^{3/2} \lesssim \delta \|\nabla v\|_{H^2}^2
+ C_\delta \varepsilon^{4/3} \|\xi\|_{H^2} \|v\|_{H^2}^2 \right),

L_6 + L_7 \lesssim \varepsilon \varepsilon^{4/3} \|\xi\|_{H^2} \|\xi\|_{H^2} \|\nabla v\|_{H^2}^{3/2} \|v\|_{H^2}^{1/2} \lesssim \delta \|\nabla v\|_{H^2}^2
+ \varepsilon^{4/3} \|\xi\|_{H^2} \|v\|_{H^2}^2 \|v\|_{H^2}^2.

Similar estimates also hold for \|v_{hh}\|_{L^2}, \|v_{zz}\|_{L^2}, \|v_h\|_{L^2}, \|v_z\|_{L^2}, \|v\|_{L^2}.

Then we have arrived at the estimate
\[
\frac{d}{dt} \|v\|_{H^2}^2 + c_1 \min \{ \mu, \lambda, 1 \} \|\nabla v\|_{H^2}^2 \leq (\delta + \varepsilon H_1(\|\xi\|_{H^2}, \|v\|_{H^2}^2)) \times \|\nabla v\|_{H^2}^2 + \delta \|\nabla v\|_{H^2}^2 \|v\|_{H^2}^2 + H_2(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v\|_{H^2}^2) \|v\|_{H^2}^2
+ H_3(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v\|_{H^2}^2),
\]
where we have chosen \(\varepsilon \in (0, \varepsilon_0)\), with \(\varepsilon_0\) small enough such that
\[
\varepsilon \alpha \|\xi\|_{H^2} \leq \varepsilon \alpha (C_0 M_0)^{1/2} \leq \log 2, \text{ and thus } 1/2 \leq e^{-\varepsilon \alpha \xi} < 2.
\]
Next, we choose $\varepsilon_0, \delta$ small enough such that
\[
\delta + \varepsilon_0 \mathcal{H}_1(\|\xi\|_{H^2}, \|v'\|_{H^2}) \leq \delta + \varepsilon_0 \mathcal{H}_1((C_0M_0)^{1/2}, (C_1M_1)^{1/2}) \leq \frac{c_1 \min\{\mu, \lambda, 1\}}{2}.
\] (3.52)

Then for $\varepsilon \in (0, \varepsilon_0)$, after applying the Grönwall inequality to (3.50), we have
\[
\sup_{0 \leq t \leq T_\varepsilon} \|v(t)\|_{H^2}^2 + \frac{c_1 \min\{\mu, \lambda, 1\}}{2} \int_0^{T_\varepsilon} \|\nabla v\|_{H^2}^2 \, dt \leq e^{\delta} \int_0^{T_\varepsilon} \|v'(t)\|_{H^2}^2 + T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v'\|_{H^2}) \times \left(\|v_0\|_{H^2}^2 + T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v'\|_{H^2})\right). \tag{3.53}
\]

Now we let $\delta$ small enough such that
\[
\delta \int_0^{T_\varepsilon} \|\nabla v'\|_{H^2}^2 \, dt \leq \delta C_1 M_1 \leq \log 2. \tag{3.54}
\]

Then for fixed $\varepsilon_0, \delta$ satisfying (3.52) and (3.54), and $\varepsilon \in (0, \varepsilon_0)$, let $T_\varepsilon$ small enough, depending on $\varepsilon, \delta$, such that
\[
T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v'\|_{H^2}) \leq T_\varepsilon \mathcal{H}_2(\varepsilon, C_\delta, (C_0M_0)^{1/2}, (C_1M_1)^{1/2}) < \log 2,
\]
\[
T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, \|\xi\|_{H^2}, \|v'\|_{H^2}) \leq T_\varepsilon \mathcal{H}_3(\varepsilon, C_\delta, (C_0M_0)^{1/2}, (C_1M_1)^{1/2}) < \frac{M_1}{2}. \tag{3.55}
\]

Then (3.53) yields
\[
\sup_{0 \leq t \leq T_\varepsilon} \|v(t)\|_{H^2}^2 + \frac{c_1 \min\{\mu, \lambda, 1\}}{2} \int_0^{T_\varepsilon} \|\nabla v\|_{H^2}^2 \, dt \leq 4(M_1 + M_1/2)
\]
\[
= 6M_1 \leq \min\left\{\frac{1}{3}, \frac{c_1 \min\{\mu, \lambda, 1\}}{6}\right\} C_1 M_1. \tag{3.56}
\]

Here we have required $C_1$ to be sufficiently large such that
\[
6 \leq \min\left\{\frac{1}{3}, \frac{c_1 \min\{\mu, \lambda, 1\}}{6}\right\} C_1. \tag{3.57}
\]

On the other hand, from (3.57), we have
\[
\|\partial_t v\|_{L^2}^2 \lesssim (\|v'\|_{H^2}^2 + e^{2\varepsilon_0} \|\xi\|_{H^2}^2 + \varepsilon^2 \|v'\|_{H^1}^2 \|\xi'\|_{H^2}^2) \|v\|_{H^2}^2
\]
\[
+ \varepsilon^{-2} e^{2\varepsilon} \|\xi'\|_{H^2} \|\xi'\|_{H^1}^2,
\]
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\[ \| \partial_t v \|_{H^2}^2 \lesssim (\| v' \|_{H^2}^2 + \varepsilon^{2\alpha} \| \xi' \|_{H^2}^2 + \varepsilon^2 \| v' \|_{H^1}^2 \| \xi' \|_{H^2}^2) \| v \|_{H^2}^2 \\
+ ((\| \xi' \|_{H^2}^2 + 1)\varepsilon^{2\alpha} \| \xi' \|_{H^2}^2 + \varepsilon^2 \| \xi' \|_{H^2}^2 \| v' \|_{H^2}^2 + \| v' \|_{H^2}^2) \| \nabla v \|_{H^2}^2 \\
+ \varepsilon^{-2} \varepsilon^{2\gamma} \| \xi' \|_{H^2}^2 \| \xi' \|_{H^2}^2 + \varepsilon^{2\gamma} \| \xi' \|_{H^2}^2, \]

after applying similar arguments as in (3.32) and (3.33). Then we have the following

\[ \| v_t \|_{L^2}^2 \leq \mathcal{H}_1(\varepsilon, \| \xi' \|_{H^2}, \| v' \|_{H^2}) \| v \|_{H^2}^2 + \mathcal{H}_2(\varepsilon, \| \xi' \|_{H^2}) \]
\[ \leq \mathcal{H}_1(\varepsilon, (C_0 M_0)^{1/2}, (C_1 M_1)^{1/2}) 6 M_1 + \mathcal{H}_2(\varepsilon, (C_0 M_0)^{1/2}) < \infty, \]
\[ \int_0^T \| v_t \|_{H^2}^2 \leq \int_0^T \left( \mathcal{H}_1(\varepsilon, (C_0 M_0)^{1/2}, (C_1 M_1)^{1/2}) 6 M_1 \\
+ \mathcal{H}_2(\varepsilon, (C_0 M_0)^{1/2}, (C_1 M_1)^{1/2}) \| v \|_{H^2}^2 + \mathcal{H}_3(\varepsilon, (C_0 M_0)^{1/2}) \right) dt < \infty. \]

(3.58)

This finishes the proof of (1).

**Proof of (2):** Denote by \((\xi_{12}, v_{12}) := (\xi_1 - \xi_2, v_1 - v_2)\) and \((\xi'_{12}, v'_{12}) := (\xi'_1 - \xi'_2, v'_1 - v'_2)\). Then \((\xi_{12}, v_{12})\) satisfies the following system:

\[ \left\{ \begin{array}{l}
\partial_t \xi_{12} = -\nabla_h \xi_{12} - \nabla_h \xi_2 - \frac{\gamma - 1}{\varepsilon} \text{div}_h v'_{12} \\
\partial_t v_{12} - c_1 (\mu \Delta_h v_{12} + \lambda \nabla_h \text{div} v_{12}) \quad \text{in } \Omega_h \times 2T, \\
\end{array} \right. \]
\[ = -v'_1 \cdot \nabla_h v_{12} - v'_{12} \cdot \nabla_h v_2 - w'_1 \partial_z v_{12} - (w'_1 - w'_2) \partial_z v_2 \\
- \frac{c^2 \varepsilon^{2\xi}}{\varepsilon} \nabla_h \xi_{12} - \frac{c^2 \varepsilon^{2\xi'_2}}{\varepsilon} \nabla_h \xi'_2 + c_1 (e^{-\varepsilon \alpha \xi'_1} - 1) \\
(\mu \Delta_h v_{12} + \lambda \nabla_h \text{div} v_{12} + \partial_z v_{12}) \\
+ c_1 (e^{-\varepsilon \alpha \xi'_1} - e^{-\varepsilon \alpha \xi'_2}) (\mu \Delta_h v_{12} + \lambda \nabla_h \text{div} v_{12} + \partial_z v_{12}) \quad \text{in } \Omega_h \times 2T, \]

(3.59)

with \((\xi_{12}, v_{12})|_{t=0} = 0\). Then after taking the \(L^2\)-inner product of (3.59) with \(2 \xi_{12}\) in \(\Omega_h\), one has, for any \(\delta \in (0, 1)\) with corresponding \(C_\delta \simeq \gamma^{-1}\),

\[ \frac{d}{dt} \| \xi_{12} \|_{L^2}^2 = \int_{\Omega_h} \text{div}_h \nabla v'_{1} \| \xi_{12} \|_{L^2}^2 dxdy - 2 \int_{\Omega_h} (\nabla h \xi_{12}) \text{div} v_{12} dxdy \\
- \frac{2(\gamma - 1)}{\varepsilon} \int_{\Omega_h} \text{div}_h \nabla v_{12} \xi_{12} dxdy \lesssim \| \nabla h \xi'_{1} \|_{L^\infty} \| \xi_{12} \|_{L^2}^2 \\
+ \| \nabla h \xi_{12} \|_{L^4} \| \xi_{12} \|_{L^4} + \| \nabla h \nabla \xi_{12} \|_{L^4} \| \xi_{12} \|_{L^4} \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 \\
+ C_\delta (1 + \| \nabla v_1 \|_{H^2} + \| \xi_{12} \|_{H^3} \| \xi_{12} \|_{L^2} + C_\delta \| v_{12} \|_{L^2}^2, \]
where we have applied the Hölder, Sobolev embedding and Young inequalities. Therefore, applying the Grönwall inequality in the above inequality yields,

\[
\sup_{0 \leq t \leq T} \| \xi_{12} \|_{L^2}^2 \leq e^{C_4 L^2 T_e (1 + \| \nabla v'_1 \|_{H^2} + \| \xi_2 \|_{H^2})} dt \\
\times \int_0^{T_e} \left( \delta \| \nabla v'_{12} \|_{L^2}^2 + C_\delta \| v'_{12} \|_{L^2}^2 \right) dt \\
\leq e^{C_4 (1 + C_\delta M_0) T_e + C_\delta (C_\delta M_1)^{\frac{4}{3} + \frac{1}{2}}} \left( \delta \| \nabla v'_{12} \|_{L^2(0,T_e;L^2(\Omega_h \times 2T))}^2 \\
+ C_\delta T_e \| v'_{12} \|_{L^\infty(0,T_e;L^2(\Omega_h \times 2T))}^2 \right). 
\]

(3.60)

On the other hand, after taking the \( L^2 \)-inner product of (3.59) \(_2\) with \( 2v_{12} \) in \( \Omega_h \times 2T \) and applying integration by parts, one has

\[
\frac{d}{dt} \| v_{12} \|_{L^2}^2 + c_1 (\mu \| \Delta_h v_{12} \|_{L^2} + \lambda \| \text{div}_h v_{12} \|_{L^2} + \| \partial_z v_{12} \|_{L^2}^2) \\
= \int (\text{div}_h v'_1 - \varepsilon \alpha \psi'_{12} \cdot \nabla_h \xi'_{12}) |v_{12}|^2 d\bar{x} - \int (v'_{12} \cdot \nabla_h v_2) \cdot v_{12} d\bar{x} \\
+ \int \int_0^z (\varepsilon \alpha \psi'_{12} \cdot \nabla_h \xi'_{12} + \text{div}_h \psi'_{12}) d\bar{x} \times (\partial_z v_2 \cdot v_{12}) d\bar{x} \\
- \int \left( \int_0^z (\varepsilon \alpha (\text{div}_h \psi'_2) \cdot \nabla_h (\partial_z v_2 \cdot v_{12} \right) \\
+ \int \left( \frac{C^2 \varepsilon^2 \xi'_{12}}{\varepsilon} \text{div}_h v_{12} + c_2 \varepsilon \xi'_{12} \cdot \nabla_h \xi'_{12} \right) d\bar{x} \\
- \int \frac{2 (e^{\varepsilon \xi'_{12}} - e^{-\varepsilon \xi'_{12}})}{\varepsilon} \nabla_h \xi'_{12} \cdot v_{12} d\bar{x} - c_1 \int \left( e^{-\varepsilon \alpha \xi'_{12}} - 1 \right) (\mu |\nabla_h v_{12}|^2 \\
+ \lambda |\text{div}_h v_{12}|^2 + |\partial_z v_{12}|^2) d\bar{x} + c_1 \varepsilon \alpha \int \left( e^{-\varepsilon \alpha \xi'_{12}} (\mu (\nabla_h \xi'_{12} \cdot \nabla_h v_{12}) \cdot v_{12} \\
+ \lambda (v_{12} \cdot \nabla_h \xi'_{12} (\text{div}_h v_{12})) d\bar{x} + c_1 \int \left( e^{-\varepsilon \alpha \xi'_{12}} - e^{-\varepsilon \alpha \xi'_{12}} \right) \\
\times (\mu \Delta_h v_{12} + \lambda \nabla_h \text{div}_h v_2 + \partial_z v_2) \cdot v_{12}) d\bar{x} =: L_8 + L_9 + L_{10} + L_{11} \\
+ L_{12} + L_{13} + L_{14} + L_{15} + L_{16},
\]

where we have substituted \( \psi'_i \), \( i = 1, 2 \). Consequently, after applying similar arguments as before, we have the following estimates of the
terms on the right-hand side: for any \( \delta \in (0, 1) \) and some constant \( C_\delta \approx \delta^{-1} \),

\[
L_8 \lesssim (\| \nabla h v'_1 \|_{L^3} + \| v'_1 \|_{L^6} \| \nabla h \xi'_1 \|_{L^6}) \| v_{12} \|_{L^2} \| v_{12} \|_{L^6} \\
\lesssim (\| v'_1 \|_{H^2} + \| v'_1 \|_{H^1} \| \xi'_1 \|_{H^2}) \| v_{12} \|_{L^2} (\| \nabla v_{12} \|_{L^2} + \| v_{12} \|_{L^2}) \\
\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, C_0 M_0, C_1 M_1) \| v_{12} \|_{L^2}^2, \\
L_9 \lesssim \| v'_1 \|_{L^2} \| \nabla h v_2 \|_{L^6} \| v_{12} \|_{L^3} \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + C_\delta \| v_{12} \|_{L^2}^2 \\
+ C_\delta \| v_2 \|_{H^2} \| v'_{12} \|_{L^2}^2, \\
L_{10} \lesssim \int_0^1 (\varepsilon \| v'_{12} \|_{L^2} \| \nabla h \xi'_1 \|_{L^4} + \| \nabla h v'_{12} \|_{L^2}) \, dz \times \int_0^1 |\partial_z v_2|_{L^4} |v_{12}|_{L^4} \, dz \\
\lesssim (\varepsilon \| v'_{12} \|_{L^2}^{1/2} \| v'_1 \|_{H^1} \| \xi'_1 \|_{H^2} + \| \nabla h v'_{12} \|_{L^2}) \\
\times (\| \partial_z v_2 \|_{H^1} \| v_{12} \|_{L^2}^{1/2} \| v_{12} \|_{H^1}^{1/2} \| v_{12} \|_{H^1}^{1/2}) \lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \delta \| \nabla v'_{12} \|_{L^2}^2 \\
+ \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) (\| v_{12} \|_{L^2}^2 + \| v'_{12} \|_{L^2}^2), \\
L_{11} \lesssim \varepsilon \int_0^1 \| \nabla h v'_{12} \|_{L^8} \| \xi'_1 \|_{L^2} \| v_{12} \|_{L^4} \, dz \times \int_0^1 |\partial_z v_2|_{L^4} |v_{12}|_{L^4} \, dz \\
+ \varepsilon \int_0^1 |\xi'_{12} \|_{L^2} \| v'_{12} \|_{L^8} \, dz \times \int_0^1 (|\partial_z v_2|_{L^4} \| \nabla h v_{12} \|_{L^2} \\
+ |\nabla h \partial_z v_2|_{L^4} \| v_{12} \|_{L^4} \, dz) \lesssim \varepsilon \| v'_2 \|_{H^1} \| \xi'_1 \|_{L^2} \\
\times \left( (\| v_2 \|_{H^1} \| v_{12} \|_{H^1}^{1/2} \| v_{12} \|_{H^1}^{1/2}) \| \partial_z v_2 \|_{H^1} \| \xi'_{12} \|_{L^2} \\
+ \| \nabla^2 v_2 \|_{L^2} \| \nabla^2 v_{12} \|_{L^2} \| v_{12} \|_{H^1}^{1/2} \| v_{12} \|_{H^1}^{1/2}) \lesssim \delta \| \nabla v_{12} \|_{L^2} \right) \\
+ \| v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) \| v_{12} \|_{H^2} \| \xi'_{12} \|_{L^2} \right), \\
L_{12} \lesssim \| \xi'_{12} \|_{L^2} (\varepsilon^{-1} e^{\varepsilon} \| \xi'_1 \|_{H^2} \| \nabla h v_{12} \|_{L^2} + e^{\varepsilon} \| \xi'_1 \|_{H^2} \| v_{12} \|_{L^2} \| \nabla h \xi'_1 \|_{L^6} \\
\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) (\| \xi'_{12} \|_{L^2}^2 + \| v_{12} \|_{L^2}^2) , \\
L_{13} \lesssim \varepsilon \| \xi'_1 \|_{H^2} + \| \xi'_2 \|_{H^2} \| \xi'_{12} \|_{L^2} \| \nabla h \xi'_2 \|_{L^6} \| v_{12} \|_{L^3} \\
\lesssim \delta \| \nabla v_{12} \|_{L^2}^2 + \mathcal{H}(C_\delta, \varepsilon, C_0 M_0, C_1 M_1) (\| \xi'_{12} \|_{L^2}^2 + \| v_{12} \|_{L^2}^2), \\
L_{14} \lesssim \varepsilon e^{2\alpha} \| \xi'_{12} \|_{H^2} \| \xi'_1 \|_{H^2} \| \nabla v_{12} \|_{L^2} \| v_{12} \|_{L^2} \| \nabla v_{12} \|_{L^2}^2 \lesssim \varepsilon e^{2\alpha(C_0 M_0)^{1/2}} (C_0 M_0)^{1/2} \\
\times \| \nabla v_{12} \|_{L^2}^2, \\
L_{15} \lesssim \varepsilon e^{2\alpha} \| \xi'_{12} \|_{H^2} \| \nabla h \xi'_1 \|_{L^6} \| \nabla v_{12} \|_{L^2} \| v_{12} \|_{L^3} \| \nabla v_{12} \|_{L^2}^2 \\
+ \mathcal{H}(C_\delta, \varepsilon, C_0 M_0) \| v_{12} \|_{L^2}^2, \\
L_{16} \lesssim \varepsilon e^{2\alpha} (\| \xi'_{12} \|_{H^2} + \| \xi'_{12} \|_{H^2}) \| \xi'_{12} \|_{L^2} \| \nabla^2 v_{12} \|_{L^3} \| v_{12} \|_{L^6} \right)
ε \leq \delta \| \nabla v_{12} \|^2_{L^2} + \delta \| v_{12} \|^2_{L^2} + \mathcal{H}(C_0, \varepsilon, C_0 M_0) (\| \nabla v_2 \|_{H^2} + 1) \| \xi_1 \|^2_{L^2},

where \mathcal{H}(\cdot), as before, is a regular function of the arguments. Consequently, one has

\begin{align*}
\frac{d}{dt} \| v_{12} \|^2_{L^2} &+ c_1 (\| \nabla_h v_{12} \|^2_{L^2} + \| \nabla v_{12} \|^2_{L^2} + \| \partial_z v_{12} \|^2_{L^2}) \\
\leq (\delta + \varepsilon \varepsilon^{o(C_0 M_0)^1/2} (C_0 M_0)^{1/2}) \| \nabla v_{12} \|^2_{L^2} + \| \nabla v_{12} \|^2_{L^2} \\
&+ \mathcal{H}(C_0, \varepsilon, C_0 M_0, C_1 M_1) (\| v_{12} \|^2_{L^2} + \| v_{12} \|^2_{L^2} + \| \xi_1 \|^2_{L^2}) \\
&+ \mathcal{H}(C_0, \varepsilon, C_0 M_0, C_1 M_1) \| \nabla v_2 \|_{H^2} \| \xi_1 \|^2_{L^2}.
\end{align*}

Notice that \( C_0, C_1 \) are independent of \( \varepsilon \). For \( \varepsilon_0, \delta \in (0,1) \) small enough and \( \varepsilon \in (0, \varepsilon_0) \), applying the Grönwall inequality in the above inequality yields

\begin{align*}
\sup_{0 \leq t \leq T} \| v_{12} \|^2_{L^2} &+ \frac{c_1 \min \{ \mu, \lambda, 1 \}}{2} \int_0^T \| \nabla v_{12} \|^2_{L^2} dt \\
\leq e^{T \mathcal{H}(C_0, \varepsilon, C_0 M_0, C_1 M_1)} (\delta \| \nabla v_{12} \|^2_{L^2(\Omega \times 2T)}) \\
&+ T \| v_{12} \|^2_{L^\infty(\Omega \times 2T)} + T \| \xi_1 \|^2_{L^\infty(\Omega \times 2T)} \\
&+ T^{1/2} \| \nabla v_2 \|_{L^2(\Omega \times 2T)} \mathcal{H}(C_0, \varepsilon, C_0 M_0, C_1 M_1) \\
&\times \| \xi_1 \|^2_{L^\infty(\Omega \times 2T)}.
\end{align*}

(3.61)

Then for fixed \( \varepsilon \in (0, \varepsilon_0) \), by first choosing \( \delta \) small enough, and then \( T \varepsilon \) small enough, (3.60) and (3.61) yield inequality (3.43) with \( q = \frac{1}{2} \). This finishes the proof of (2) and completes the proof of Proposition 2. \( \square \)

### 3.3 Uniform stability

In this section, we will show that the existence time \( T \varepsilon \) of the strong solutions constructed in Proposition 2 is uniform in \( \varepsilon \) provided \( \varepsilon \in (0, \varepsilon_1) \) for some \( \varepsilon_1 \in (0, \varepsilon_0) \). In order to show this, it suffices to show that there is a positive constant \( \varepsilon_1 \) such that the \( H^2 \)-norms of \( (\xi, v) \) remain bounded in a time interval \( (0, T) \) with \( T \in (0, \infty) \) independent of \( \varepsilon \), provided \( \varepsilon \in (0, \varepsilon_1) \) with \( \varepsilon_1 \in (0, 1) \) small enough. We perform a continuity argument in the following.

Recall that we are given initial data \( (\xi_0, v_0) \in H^2(\Omega \times 2T) \), and a positive constant \( M \in (0, \infty) \) satisfying

\begin{equation}
\frac{1}{2} \| v_0 \|^2_{H^2} + \frac{\varepsilon^2}{\gamma - 1} \| \xi_0 \|^2_{H^2} < M.
\end{equation}

(2.6)
Denote by
\[ M_0 := \frac{8(\gamma - 1)}{c^2} M, \quad M_1 := 4M. \] (3.62)

Then it is obvious that
\[ \| \xi_0 \|_{H^2} < M_0, \quad \| v_0 \|_{H^2} < M_1. \]

From Proposition 2, there is a strong solution \((\xi, v)\) satisfying (3.35) and (3.36) in the time interval \([0, T_\varepsilon]\), for some \(T_\varepsilon \in (0, \infty)\). Then for any \(t \in [0, T_\varepsilon]\), we have
\[ \| \xi(t) \|_{H^2} \leq C_0 M_0 = \frac{8(\gamma - 1)}{c^2} C_0 M, \quad \| v(t) \|_{H^2} \leq C_1 M_1 = 4C_1 M, \] (3.63)
where \(C_0, C_1\) are given in Proposition 2. We remind readers that \(T_\varepsilon\) depends on \(M_0, M_1\) and \(\varepsilon\). On the other hand, consider \(\varepsilon_1\) satisfying
\[ \sup_{0 \leq t \leq T_\varepsilon} \max\{1, \alpha\} \varepsilon_1 \| \xi(t) \|_{H^2} \leq \max\{1, \alpha\} \varepsilon_1 \times (C_0 M_0)^{1/2} \leq \log 2, \]
\[ \sup_{0 \leq t \leq T_\varepsilon} \varepsilon_1 \mathcal{H}_1(\| \xi(t) \|_{H^2}, \| v(t) \|_{H^2}) \leq \varepsilon_1 \mathcal{H}_1(C_0 M_0, C_1 M_1) \leq \frac{\min\{\mu, \lambda, 1\} c_1}{8}, \]
where \(\mathcal{H}_1\) is as in the right-hand side of (3.1). Then the a priori estimate in (3.1) implies that, for \(\varepsilon \in (0, \varepsilon_1)\) and \(t \in [0, T_\varepsilon]\),
\[ \frac{1}{2} \| v(t) \|_{H^2}^2 + \frac{c^2}{4(\gamma - 1)} \| \xi(t) \|_{H^2}^2 + \frac{\min\{\mu, \lambda, 1\} c_1}{2} \int_0^t \| \nabla v(s) \|_{H^2}^2 ds \]
\[ \leq \frac{1}{2} \| v_0 \|_{H^2}^2 + \frac{c^2}{\gamma - 1} \| \xi_0 \|_{H^2}^2 + \frac{\varepsilon_0}{8} \int_0^t \| \nabla v(s) \|_{H^2}^2 ds \]
\[ + \frac{\min\{\mu, \lambda, 1\} c_1}{8} \int_0^t \| \nabla v(s) \|_{H^2}^2 ds + C_\delta \int_0^t \mathcal{H}_2(C_0 M_0, C_1 M_1) ds. \]

Hence after substituting (2.6) and choosing
\[ \delta = \delta' := \frac{\min\{\mu, \lambda, 1\} c_1}{8}, \quad \text{and} \quad T := \frac{M}{C_{\delta'} \mathcal{H}_2(C_0 M_0, C_1 M_1)}, \] (3.64)

it follows
\[ \frac{1}{2} \| v(t) \|_{H^2}^2 + \frac{c^2}{4(\gamma - 1)} \| \xi(t) \|_{H^2}^2 + \frac{\min\{\mu, \lambda, 1\} c_1}{4} \int_0^t \| \nabla v(s) \|_{H^2}^2 ds \]
\[ \leq M + C_{\delta'} \mathcal{H}_2(C_0 M_0, C_1 M_1) t < 2M, \] (3.65)
for 
\[ t \in [0, \min\{T_\varepsilon, T\}] \].

Notice that \( T \) is independent of \( \varepsilon \) once \( \varepsilon \in (0, \varepsilon_1) \). Also from (3.65), we have

\[
\sup_{0 \leq t \leq \min\{T_\varepsilon, T\}} \| \xi(t) \|_{L^2}^2 < \frac{8(\gamma - 1)}{c^2} M = M_0, \\
\sup_{0 \leq t \leq \min\{T_\varepsilon, T\}} \| v(t) \|_{L^2}^2 < 4M = M_1, \\
\int_0^{\min\{T_\varepsilon, T\}} \| \nabla v(t) \|_{L^2}^2 dt \leq \frac{8M}{c_1 \min\{\mu, \lambda, 1\}} =: M_2.
\]

Therefore, supposed that \( T_\varepsilon < T \), we will have

\[
\| \xi(T_\varepsilon) \|_{L^2}^2 < M_0, \quad \| v(T_\varepsilon) \|_{L^2}^2 < M_1.
\]

Then by setting \((\xi(T_\varepsilon), v(T_\varepsilon))\) as the new initial data, applying Proposition 2 again, estimate (3.63) holds in \([0, 2T_\varepsilon]\). Thus the arguments between (3.63) and (3.65) hold with \( T_\varepsilon \) replaced by \( 2T_\varepsilon \), without needing to choose the smallness of \( \varepsilon_1 \) and \( T \). Then estimate (3.65) holds in the time interval \([0, \min\{2T_\varepsilon, T\}]\). Repeat such arguments \( n \) times, \( n \in \mathbb{Z}^+ \), until \( nT_\varepsilon \geq T \). This extends the existence time of the local strong solution \((\xi, v)\) to (1.4) to \( T > 0 \), which is independent of \( \varepsilon \) provided \( \varepsilon \in (0, \varepsilon_1) \) with \( \varepsilon_1 \) given as above. Consequently, we conclude that:

**Proposition 3.** Consider \((\xi_0, v_0) \in H^2(\Omega_h \times 2T)\) with the compatibility conditions in (2.4), bounded as in (2.6). Then there is a positive constant \( \varepsilon_1 \in (0, 1) \) such that for any \( \varepsilon \in (0, \varepsilon_1) \), there is a unique strong solution to (1.4) in the time interval \([0, T]\) satisfying the regularity (3.35). Here \( T \in (0, \infty) \) is a positive time which is independent of \( \varepsilon \). Moreover, there are positive constants \( M_0, M_1, M_2 \in (0, \infty) \), given in (3.62) and (3.66) and independent on \( \varepsilon \), such that

\[
\| \xi \|_{L^\infty(0,T;H^2(\Omega_h \times 2T))}^2 < M_0, \quad \| v \|_{L^\infty(0,T;H^2(\Omega_h \times 2T))}^2 < M_1, \\
\| \nabla v \|_{L^2(0,T;H^2(\Omega_h \times 2T))}^2 < M_2.
\]

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4 The limit equations

In this section, we will identify the limit equations of (1.4) as $\varepsilon \to 0^+$ in the distribution sense. Recall that,

\[
\begin{aligned}
\begin{cases}
\partial_t \xi_\varepsilon + v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} (\text{div}_h v_\varepsilon + \partial_z w_\varepsilon) = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon + \frac{\varepsilon^2 e^{\varepsilon \xi_\varepsilon}}{\varepsilon} \nabla_h \xi_\varepsilon = c_1 e^{-\varepsilon \alpha \xi_\varepsilon} (\mu \Delta_h v_\varepsilon + \lambda \nabla_h \text{div}_h v_\varepsilon + \partial_{zz} v_\varepsilon) & \text{in } \Omega_h \times 2\mathbb{T}, \\
\partial_z \xi_\varepsilon = 0 & \text{in } \Omega_h \times 2\mathbb{T}, \\
w_\varepsilon = 0 & \text{on } \Omega_h \times Z,
\end{cases}
\end{aligned}
\]

(4.1)

where the initial data are taken as $(\xi_\varepsilon, v_\varepsilon)|_{t=0} = (\xi_0, v_0) \in H^2(\Omega_h \times 2\mathbb{T})$ satisfying the compatibility conditions in (2.4), bounded as in (2.6). We first list the uniform estimates obtained in the previous section. In fact, from (3.67) and (3.2)–(3.4), the following estimates hold uniformly in $\varepsilon \in (0, \varepsilon_1)$ for some constant $C, T \in (0, \infty)$ independent of $\varepsilon$ and depending only on the initial data,

\[
\begin{aligned}
&\sup_{0 \leq t \leq T} \left\{ \| v_\varepsilon(t) \|_{H^2} + \| \xi_\varepsilon(t) \|_{H^2} + \| \partial_t (e^{\varepsilon \xi_\varepsilon}) (t) \|_{H^1} + \| \partial_t P_\sigma v_\varepsilon(t) \|_{L^2} \right\} \\
&+ \| \nabla v_\varepsilon \|_{L^2(0,T;H^2)} + \| \partial_t P_\sigma v_\varepsilon \|_{L^2(0,T;H^1)} < C.
\end{aligned}
\]

(4.1)

Then as $\varepsilon \to 0^+$, there exist $\xi_p, v_p, \zeta_p, v_{p,\sigma}$ with

\[
\begin{aligned}
\xi_\varepsilon \rightharpoonup^* \xi_p & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^2), \\
v_\varepsilon & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \\
\zeta_p & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^1), \\
v_{p,\sigma} & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^2) \cap L^2(0,T;H^3),
\end{aligned}
\]

such that,

\[
\begin{aligned}
\xi_\varepsilon \rightharpoonup^* \xi_p & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^2), \\
v_\varepsilon \text{ (and } P_\sigma v_\varepsilon) \rightharpoonup v_p \text{ (respectively } v_{p,\sigma}) & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^2) \text{ (respectively } L^\infty(0,T;H^2)), \\
v_\varepsilon \text{ (and } P_\sigma v_\varepsilon) \to v_p \text{ (respectively } v_{p,\sigma}) & \quad \text{weakly in } L^2(0,T;H^3) \text{ (respectively } L^2(0,T;H^3)),
\end{aligned}
\]

(4.3)

\[
\begin{aligned}
\partial_t (e^{\varepsilon \xi_\varepsilon}) \rightharpoonup^* \zeta_p & \quad \text{weak-}^* \text{ in } L^\infty(0,T;H^1), \\
\partial_t P_\sigma v_\varepsilon \rightharpoonup \partial_t v_{p,\sigma} & \quad \text{weak-}^* \text{ in } L^\infty(0,T;L^2), \\
\partial_t P_\sigma v_\varepsilon \to \partial_t v_{p,\sigma} & \quad \text{weakly in } L^2(0,T;H^1).
\end{aligned}
\]
Also applying the Aubin compactness lemma (see, e.g., [42, Theorem 2.1] and [5, 41]) yields, as \( \varepsilon \to 0^+ \),
\[
\mathcal{P}_\sigma v_\varepsilon \to v_{p,\sigma} \quad \text{in} \quad L^2(0,T;H_{\sigma,\text{loc}}^2) \cap C([0,T];H_{\sigma,\text{loc}}^1). \tag{4.4}
\]

Moreover, from (4.1), we have
\[
\varepsilon \| \xi_\varepsilon \|_{H^2} < C \| v_\varepsilon \|_{L^\infty(0,T;H^2)} < \varepsilon C \quad \text{as} \quad \varepsilon \to 0^+. \tag{4.5}
\]

Thus we have, as \( \varepsilon \to 0^+ \),
\[
e^{\varepsilon \xi_\varepsilon} \to 1 \quad \text{in} \quad L^\infty(0,T;H^2),
\]
and \( \partial_t (e^{\varepsilon \xi_\varepsilon}) \to 0 = \zeta_p \) in the sense of distribution. \( \tag{4.6} \)

On the other hand, recall that the vertical velocity is represented by
\[
w_\varepsilon = - \int_0^z \left( \frac{\varepsilon}{\gamma - 1} v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \text{div}_h \tilde{v}_\varepsilon \right) dz. \tag{3.8}
\]

Therefore, a direct calculation yields that \( w_\varepsilon, \partial_z w_\varepsilon \in L^\infty(0,T;H^1) \) and we have the following uniform bounds:
\[
\| w_\varepsilon \|_{L^\infty(0,T;H^1)} + \| \partial_z w_\varepsilon \|_{L^\infty(0,T;H^1)} \\
\lesssim \varepsilon \| v_\varepsilon \|_{L^\infty(0,T;H^2)} \| \xi_\varepsilon \|_{L^\infty(0,T;H^2)} < C. \tag{4.7}
\]

Therefore, there exists
\[
w_p \in L^\infty(0,T;H^1) \quad \text{with} \quad \partial_z w_p \in L^\infty(0,T;H^1) \tag{4.8}
\]
such that,
\[
w_\varepsilon \quad \text{(and} \quad \partial_z w_\varepsilon) \quad \overset{\ast}{\to} \quad w_p \quad \text{(respectively} \quad \partial_z w_p) \quad \text{weak-}^* \quad \text{in} \quad L^\infty(0,T;H^1), \tag{4.9}
\]
as \( \varepsilon \to 0^+ \). Notice, after applying the trace theorem, \( \{ w_\varepsilon \}_{\varepsilon \in (0,\varepsilon_1)} \subset L^\infty(0,T;H^{1/2}(\Omega_h)) \) and is uniformly bounded. Then it follows from (3.8) that \( w_\varepsilon \big|_{z \in \mathcal{Z}} = 0 \) and hence \( w_p \big|_{z \in \mathcal{Z}} = 0 \). Moreover, after multiplying (1.4) with \( e^{\varepsilon \xi_\varepsilon} \), one has
\[
\partial_t e^{\varepsilon \xi_\varepsilon} + e^{\varepsilon \xi_\varepsilon} v_\varepsilon \cdot \nabla_h \xi_\varepsilon + (\gamma - 1) e^{\varepsilon \xi_\varepsilon} (\text{div}_h v_\varepsilon + \partial_z w_\varepsilon) = 0. \tag{4.10}
\]

Then, (4.1) and (4.6) imply that as \( \varepsilon \to 0^+ \), (4.10) converges to
\[
\text{div}_h v_p + \partial_z w_p = 0 \quad \text{in the sense of distribution.}
\]
Consequently, we have shown that

\[ \text{div}_h v_p + \partial_z w_p = 0, \quad \text{and} \quad w_p \bigg|_{z \in \mathbb{Z}} = 0. \]  

(4.11)

Next, we will identify the limit equation of the momentum equation (1.4). First, (4.2) and (4.11) imply that \( v_p \in L^\infty(0,T; H^2_\sigma) \cap L^2(0,T; H^3_\sigma). \)

We first show that \( v_{p,\sigma} \equiv v_p. \) Let \( u \in C^\infty(0,T; C^\infty(\Omega_h \times 2\mathbb{T}, \mathbb{R}^2)). \) Then we have, from (4.3),

\[
\int_0^T \int u \cdot v_p \, d\vec{x} \, dt = \lim_{\varepsilon \to 0^+} \int_0^T \int u \cdot v_\varepsilon \, d\vec{x} \, dt = \lim_{\varepsilon \to 0^+} \int_0^T \int u \cdot v_{p,\sigma} \, d\vec{x} \, dt = \int_0^T \int u \cdot v_p \, d\vec{x} \, dt,
\]

which shows that \( v_{p,\sigma} \equiv v_p. \) Then (4.4) can be written as

\[ P_\sigma v_\varepsilon \rightarrow v_p \quad \text{in} \quad L^2(0,T; H^2_{\sigma,\text{loc}}) \cap C([0,T]; H^1_{\sigma,\text{loc}}), \]  

(4.12)
as \( \varepsilon \to 0^+. \) In particular, \( v_p(t = 0) = P_\sigma v_0. \) Moreover, from (4.3), since \( v_{p,\sigma} \equiv v_p, \) we have the following: as \( \varepsilon \to 0^+, \)

\[
P_\tau v_\varepsilon = v_\varepsilon - P_\sigma v_\varepsilon \rightharpoonup 0 \quad \text{weak-* in} \quad L^\infty(0,T; H^2_\tau),
\]

\[ P_\tau v_\varepsilon = v_\varepsilon - P_\sigma v_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(0,T; H^3_\tau). \]  

(4.13)

Also, denote \( P_\tau v_\varepsilon = \nabla_h \psi_\varepsilon \) with \( \psi_\varepsilon \) defined by the following elliptic problem (see, e.g., (2.1)),

\[
\Delta_h \psi_\varepsilon = \int_0^1 \text{div}_h v_\varepsilon \, dz \quad \text{in} \quad \Omega_h,
\]

\[
\left\{ \begin{array}{ll}
\lim_{\|x,y\| \to \infty} \psi_\varepsilon = 0 & \quad \text{in the case when} \quad \Omega_h = \mathbb{R}^2, \\
\int_{\Omega_h} \psi_\varepsilon \, dxdy = 0 & \quad \text{in the case when} \quad \Omega_h = \mathbb{T}^2.
\end{array} \right.
\]

(4.14)

Then \( P_\tau \partial_t v_\varepsilon = \nabla_h \partial_t \psi_\varepsilon \) and we have the identity:

\[
P_\tau v_\varepsilon \cdot \nabla_h P_\tau v_\varepsilon + w_\varepsilon \partial_z P_\tau v_\varepsilon = \nabla_h \psi_\varepsilon \cdot \nabla_h \nabla_h \psi_\varepsilon + w_\varepsilon \partial_z \nabla_h \psi_\varepsilon
\]

\[
= \frac{1}{2} \nabla_h |\nabla_h \psi_\varepsilon|^2.
\]

Therefore, one has

\[
v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon = P_\sigma v_\varepsilon \cdot \nabla_h v_\varepsilon + P_\tau v_\varepsilon \cdot \nabla_h P_\sigma v_\varepsilon + w_\varepsilon \partial_z P_\sigma v_\varepsilon
\]

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Proposition 4. Let \((\xi_0, v_0) \in H^2(\Omega_h \times 2T)\) satisfying the compatibility conditions in (2.4) and \(\varepsilon \in (0, \varepsilon_1)\) with \(\varepsilon_1\) given in Proposition 3. Then

\[
\partial_t v_p + v \cdot \nabla_h v_p + w_p \partial_z v_p + \nabla_h P = c_1(\mu \Delta_h v_p + \lambda \nabla_h \div_h v_p + \partial_zz v_p),
\]

(4.18)
in the sense of distribution, where \(w_p\) is determined by (4.11), and \(\partial_z P = 0\).

To conclude, we have shown the following:

**Proposition 4.** Let \((\xi_0, v_0) \in H^2(\Omega_h \times 2T)\) satisfying the compatibility conditions in (2.4) and \(\varepsilon \in (0, \varepsilon_1)\) with \(\varepsilon_1\) given in Proposition 3. Then
the unique strong solutions \( \{(\xi_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_1)} \) obtained in Proposition 3 in the time interval \([0, T], T \in (0, \infty), \) and \( w_\varepsilon \), given as in (3.3), converge to \((\xi_0, v_0)\) and \( w_0 \), with the convergences in (4.3), (4.6), (1.9), (4.12) and (1.13), as \( \varepsilon \to 0^+ \). \( \xi_0, v_0, w_0 \) satisfy the regularity (4.2) and (4.8). Also, \((v_\varepsilon, w_\varepsilon)\) satisfies the primitive equations (2.9) with the initial data \( v_\varepsilon|_{t=0} = v_{0,0} = \mathcal{P}_\sigma v_0 \in H^2_\sigma(\Omega \times 2T) \).

In the rest of this section, we will discuss the converging behaviors of \( \xi_\varepsilon \) and \( \mathcal{P}_\tau v_\varepsilon \) as \( \varepsilon \to 0^+ \) in some strong sense. We will consider such a problem in two cases: \( \Omega_\varepsilon = \mathbb{R}^2 \) and \( \Omega_\varepsilon = T^2 \). In fact, we will investigate the following acoustic wave equations associated with (1.4):

\[
\begin{aligned}
\partial_t \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} \text{div}_h \mathcal{P}_\tau v_\varepsilon &= -v_\varepsilon \cdot \nabla_h \xi_\varepsilon \\
- \frac{\gamma - 1}{\varepsilon} \left( \text{div}_h \mathcal{P}_\sigma v_\varepsilon + \partial_z w_\varepsilon \right) &= 0 \\
\partial_t \mathcal{P}_\tau v_\varepsilon + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon &= \frac{c^2 (1 - e^{e \xi_\varepsilon})}{\varepsilon} \nabla_h \xi_\varepsilon - \partial_t \mathcal{P}_\sigma v_\varepsilon \\
\partial_t v_\varepsilon \nabla_h v_\varepsilon - w_\varepsilon \partial_z v_\varepsilon &= c_1 e^{-e \xi_\varepsilon} (\mu \Delta_h v_\varepsilon) \\
+ \lambda \nabla_h \text{div}_h v_\varepsilon + \partial_z \varepsilon v_\varepsilon &= 0.
\end{aligned}
\]

To study the acoustic wave, notice that, \( \mathcal{P}_\tau v_\varepsilon = \nabla_h \psi_\varepsilon \) is a function independent of the \( z \)-variable, where \( \psi_\varepsilon \) is defined in (4.14). After averaging (4.19), in the \( z \)-variable and applying projection operator \( \mathcal{P}_\tau \) to (4.19), system (4.19) is reduced to,

\[
\begin{aligned}
\partial_t \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} \text{div}_h \nabla_h \psi_\varepsilon &= G_1 \quad \text{in } \Omega_h, \\
\partial_t \nabla_h \psi_\varepsilon + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon &= G_2 + G_3 \quad \text{in } \Omega_h,
\end{aligned}
\]

where we have used \( \nabla_h \psi_\varepsilon \) (defined in (4.14)) to represent \( \mathcal{P}_\tau v_\varepsilon \), and

\[
\begin{aligned}
G_1 &= -v_\varepsilon \cdot \nabla_h \xi_\varepsilon, \\
G_2 &= \frac{c^2 (1 - e^{e \xi_\varepsilon})}{\varepsilon} \nabla_h \xi_\varepsilon + \mathcal{P}_\tau \left( c_1 e^{-e \xi_\varepsilon} (\mu \Delta_h v_\varepsilon + \lambda \nabla_h \text{div}_h v_\varepsilon + \partial_z \varepsilon v_\varepsilon) \right), \\
G_3 &= -\mathcal{P}_\tau (v_\varepsilon \cdot \nabla_h v_\varepsilon + w_\varepsilon \partial_z v_\varepsilon).
\end{aligned}
\]

Therefore, the acoustic wave system is only two dimensional. From (3.3), (3.33), (4.1), we have

\[
|G_1|_{L^\infty(0,T;H^1(\Omega_h))} + |G_2|_{L^\infty(0,T;L^2(\Omega_h))} + |G_2|_{L^2(0,T;H^1(\Omega_h))}
+ |G_3|_{L^\infty(0,T;L^2(\Omega_h))} + |G_3|_{L^2(0,T;H^1(\Omega_h))} < C.
\]
Now denote by the linear operator $L$ defined on $(\mathcal{D}'(\Omega_h))^3$ as

$$L(U) := \left( \frac{(\gamma - 1) \text{div}_h u}{c^2 \nabla_h q} \right), \quad \text{where \ } U = \left( \begin{array}{c} q \\ u \end{array} \right),$$

(4.22)

where $q \in \mathcal{D}'(\Omega_h)$, $u \in (\mathcal{D}'(\Omega_h))^2$. Then (4.20) can be written as

$$\partial_t U_\varepsilon + \frac{1}{\varepsilon} L U_\varepsilon = G \quad \text{in } \Omega_h,$$

(4.23)

where

$$G := \left( \begin{array}{c} G_1 \\ G_2 + G_3 \end{array} \right), \quad U_\varepsilon := \left( \begin{array}{c} \xi_\varepsilon \\ \nabla h \psi_h \\ \mathcal{P}_\tau v_\varepsilon \end{array} \right).$$

Also, the initial data are given as

$$U_\varepsilon(t = 0) = U_0 = \left( \begin{array}{c} \xi_0 \\ \mathcal{P}_\tau v_0 \end{array} \right).$$

(4.24)

Also, we have

$$\xi_\varepsilon \in L^\infty(0, T; H^2(\Omega_h)), \quad \mathcal{P}_\tau v_\varepsilon \in L^\infty(0, T; H^2(\Omega_h)) \cap L^2(0, T; H^3(\Omega_h)),$$

$$G \in L^\infty(0, T; L^2(\Omega_h)) \cap L^2(0, T; H^1(\Omega_h)), \quad U_0 \in H^2(\Omega_h),$$

and from (2.30), (4.1) and (4.21),

$$|\xi_\varepsilon|_{L^\infty(0, T; H^2(\Omega_h))} + |\mathcal{P}_\tau v_\varepsilon|_{L^\infty(0, T; H^2(\Omega_h)) \cap L^2(0, T; H^3(\Omega_h))} + |G|_{L^\infty(0, T; L^2(\Omega_h)) \cap L^2(0, T; H^1(\Omega_h))} + |U_0|_{H^2(\Omega_h)} < C.$$  

(4.25)

Define the associated solution operator of $L$ by

$$\mathcal{L} : t \mapsto \mathcal{L}(t) = \exp(-Lt).$$

(4.26)

That is, given $V_0 \in (\mathcal{D}')^3$, $\mathcal{L}(t)V_0$ satisfies the linear equation:

$$\partial_t (\mathcal{L}(t)V_0) + L(\mathcal{L}(t)V_0) = 0.$$  

(4.27)

Notice that $\mathcal{L}$ is linear and $\mathcal{L}(t_1 + t_2) = \mathcal{L}(t_1)\mathcal{L}(t_2)$ for $t_1, t_2 \in (0, \infty)$. Then the solution $U_\varepsilon$ of (4.23) can be represented as, using the Duhamel principle,

$$U_\varepsilon(t) = \mathcal{L}(\frac{t}{\varepsilon})U_0 + \int_0^t \mathcal{L}(\frac{t-s}{\varepsilon})G(s) \, ds.$$  

(4.28)
4.1 The case when $\Omega_h = \mathbb{R}^2$: dispersion of the acoustic waves

In this section, we shall establish the strong convergence of $U_\varepsilon$ in the case when $\Omega_h = \mathbb{R}^2$. To be precise, we will show the following:

**Proposition 5.** Under the same assumptions as in Proposition 4 in the case when $\Omega_h = \mathbb{R}^2$, we have

$$\xi_\varepsilon \rightarrow 0, \quad P_\tau v_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+, \quad \text{in } L^2(0, T; W^{1,6}(\mathbb{R}^2)).$$

In particular, we have, as $\varepsilon \rightarrow 0^+$,

$$\|v_\varepsilon - v_p\|_{L^2(0, T; L^6(\mathbb{R}^2))} + \|\xi_\varepsilon\|_{L^2(0, T; L^6(\mathbb{R}^2))} \rightarrow 0.$$ 

In order to show Proposition 5, we will use the dispersion estimates of operator $L$ in the whole space $\mathbb{R}^2$. In particular, we will employ the Strichartz inequalities as in [10], to derive the appropriate decay as $\varepsilon \rightarrow 0^+$. Indeed, let $\phi_0 \in (D'(\mathbb{R}^2))^3$. The following inequality is from [10, equation (3.4)]:

$$|L(t)\phi_0|_{L^q(\mathbb{R}^2; L^p(\mathbb{R}^2))} \leq C_{p,q} |\phi_0|_{H^\eta(\mathbb{R}^2)},$$

(4.29)

provided that the right-hand side is finite, where $p, q > 2$ and $\eta \in (0, \infty)$ satisfying

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p} \quad \text{and} \quad \eta q = 3. \quad (4.30)$$

We will make use of (4.29) to derive the strong dispersion. To begin with, the following dispersion inequalities are analogies to inequalities (3.6) and (3.7) in [10].

**Lemma 1.** For $\phi_0$ and $\phi(s) \in (D'(\mathbb{R}^2))^3$, $p, q, \eta$ satisfying (4.30) and $s \geq 0$, we have

$$|L(t)\phi_0|_{L^q(0, T; W^{s,p}(\mathbb{R}^2))} \leq C_{p,q}^{1/q} |\phi_0|_{H^\eta+s(\mathbb{R}^2)},$$

(4.31)

and

$$\left|\int_0^t L\left(\frac{t-s}{\varepsilon}\right)\phi(s) \, ds\right|_{L^2(0, T; W^{s,p}(\mathbb{R}^2))} \leq C(1 + T)\varepsilon^{1/q} |\phi|_{L^2(0, T; H^{\eta+s}(\mathbb{R}^2))},$$

(4.32)

provided the right-hand sides are finite.

**Proof.** Indeed (4.31) is a direct consequence of rescaling the temporal variable and replacing $\phi_0$ with $(I - \Delta_h)^{s/2}\phi_0$ in (4.29). In order to show (4.32), notice that after applying the Minkowski and Hölder inequalities, one has

$$\left|\int_0^t L\left(\frac{t-s}{\varepsilon}\right)\phi(s) \, ds\right|^2_{W^{s,p}(\mathbb{R}^2)} \leq Ct \int_0^t \left|L\left(\frac{t-s}{\varepsilon}\right)\phi(s)\right|^2_{W^{s,p}(\mathbb{R}^2)} \, ds.$$
Then by employing the Fubini theorem, it holds
\[
\int_0^T \int_0^t L\left(\frac{t-s}{\varepsilon}\right) \phi(s) \, ds \, W^{s,p}(\mathbb{R}^2) \, dt \\
\leq C T \int_0^T \int_0^t |L\left(\frac{t-s}{\varepsilon}\right) \phi(s)|^2 \, W^{s,p}(\mathbb{R}^2) \, dt \, ds \\
\leq CT \int_0^T (T-s)^{1-2/q} \left( \int_s^T |L\left(\frac{t-s}{\varepsilon}\right) \phi(s)|^q \, W^{s,p}(\mathbb{R}^2) \, dt \right)^{2/q} \, ds \\
\leq \varepsilon^{2/q} C T^{2-2/q} \int_0^T |\phi(s)|^2 \, H^{\gamma+\varepsilon}(\mathbb{R}^2) \, ds = \varepsilon^{2/q} C T^{2-2/q} |\phi|^q \, L^2(0,T;H^{\gamma+\varepsilon}(\mathbb{R}^2)).
\]
This finishes the proof of (4.32).

\textbf{Proof of Proposition 5.} Consider \( \eta = \frac{1}{2}, s = \frac{1}{2}, q = 6, p = 6 \) in (4.31) and (4.32), which satisfy (4.30). Then from (4.28), after applying the triangle and Hölder inequalities,

\[
|U_\varepsilon| \, L^2(0,T;W^{\frac{1}{2},\frac{1}{6}}(\mathbb{R}^2)) \lesssim T^\frac{1}{2} |L\left(\frac{t}{\varepsilon}\right) U_0| \, L^6(0,T;W^{\frac{1}{6},\frac{1}{6}}(\mathbb{R}^2)) \\
+ \left| \int_0^T L\left(\frac{t-s}{\varepsilon}\right) G(s) \, ds \right| \, L^2(0,T;W^{\frac{1}{2},\frac{1}{6}}(\mathbb{R}^2)) \lesssim T^\frac{3}{2} \varepsilon^{\frac{1}{6}} |U_0| \, H^1(\mathbb{R}^2) \\
+ (1+T) \varepsilon^{\frac{1}{6}} |G| \, L^2(0,T;H^1(\mathbb{R}^2)) \lesssim (1+T) \varepsilon^{\frac{1}{6}},
\]

where the second and the third inequalities follow from (4.31), (4.32) and (4.25). This yields (2.12) and finishes the proof of Proposition 5.

\section{The case when \( \Omega_h = \mathbb{T}^2: \) the fast oscillation}

We will establish the convergence behavior of \( U_\varepsilon \) in the case when \( \Omega_h = \mathbb{T}^2 \) in this subsection. Indeed, we shall investigate the fast oscillations of the acoustic waves as \( \varepsilon \to 0^+ \). This is motivated by [37] (see also [16]).

\textbf{The oscillation equations and the convergence of oscillations}

To begin with, (4.20) can be written as

\[
\partial_t \mathcal{P}_\tau v_\varepsilon + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon - c_1 (\mu \Delta_h \mathcal{P}_\tau v_\varepsilon + \lambda \nabla_h \text{div} \mathcal{P}_\tau v_\varepsilon) \\
= \frac{c^2 (1-\varepsilon \xi_\varepsilon)}{\varepsilon} \nabla_h \xi_\varepsilon - \mathcal{P}_\tau (v_\varepsilon \cdot \nabla_h v_\varepsilon) - \mathcal{P}_\tau (w_\varepsilon \partial_z v_\varepsilon) \\
+ \mathcal{P}_\tau (c_1 (e^{-\varepsilon \alpha \xi_\varepsilon} - 1)(\mu \Delta_h v_\varepsilon + \lambda \nabla_h \text{div} v_\varepsilon + \partial_z v_\varepsilon)).
\]
Moreover, for any \( u \in (\mathcal{D}'(\mathbb{T}^2 \times 2\mathbb{T}))^2 \), consider \( \tau(x, y) = \int_0^1 u \, dz \in (\mathcal{D}'(\mathbb{T}))^2 \) as a function on \( \mathbb{T}^2 \times 2\mathbb{T} \). One has \( \mathcal{P}_\tau u = \mathcal{P}_\tau \tau u \). Then one has, after applying integration by parts and substituting (3.9),

\[
\mathcal{P}_\tau(w_\varepsilon \partial_z v_\varepsilon) = \mathcal{P}_\tau(- \int_0^1 \partial_z w_\varepsilon v_\varepsilon \, dz)
\]

\[
\begin{aligned}
&= \mathcal{P}_\tau(\int_0^1 ((\varepsilon \alpha \tilde{\nu}_\varepsilon \cdot \nabla_h \xi_\varepsilon + \text{div}_h \tilde{\nu}_\varepsilon) v_\varepsilon) \, dz) \\
&= \mathcal{P}_\tau(\int_0^1 ((\varepsilon \alpha \tilde{P}_\sigma \nu_\varepsilon \cdot \nabla_h \xi_\varepsilon) \mathcal{P}_\sigma v_\varepsilon + \text{div}_h \tilde{P}_\sigma \nu_\varepsilon \mathcal{P}_\sigma v_\varepsilon) \, dz)
\end{aligned}
\]

\[
\mathcal{P}_\tau((-\varepsilon \alpha \xi_\varepsilon - 1)\partial_z v_\varepsilon) = 0,
\]

where we have used the facts that \( \int_0^1 \tilde{P}_\sigma v_\varepsilon \, dz = 0 \), \( \tilde{\nu}_\varepsilon = \tilde{P}_\sigma \nu_\varepsilon \) and that \( \mathcal{P}_\tau v_\varepsilon, \xi_\varepsilon \) are independent of the \( z \)-variable.

On the other hand, notice that

\[
v_\varepsilon \cdot \nabla_h \xi_\varepsilon = \mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon = \text{div}_h (\xi_\varepsilon \mathcal{P}_\sigma v_\varepsilon) - \xi_\varepsilon \text{div}_h \mathcal{P}_\sigma v_\varepsilon
\]

\[
+ \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon.
\]

(4.20) can be written as

\[
\partial_t \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} \text{div}_h \mathcal{P}_\tau v_\varepsilon = -\text{div}_h (\xi_\varepsilon \int_0^1 \mathcal{P}_\sigma v_\varepsilon \, dz) - \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon.
\]

Additionally, integrating (4.35) in \( \mathbb{T}^2 \) yields

\[
\frac{d}{dt} \int_{\mathbb{T}^2} \xi_\varepsilon \, dx \, dy = - \int_{\mathbb{T}^2} (\mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon) \, dx \, dy.
\]

Consequently, after combining equations (4.33), (4.34), (4.35), (4.36), we have the following system of oscillations:

\[
\begin{cases}
\partial_t \xi_\varepsilon + \frac{\gamma - 1}{\varepsilon} \text{div}_h \mathcal{P}_\tau v_\varepsilon = -\text{div}_h (\xi_\varepsilon \int_0^1 \mathcal{P}_\sigma v_\varepsilon \, dz) \\
- \mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon + \int_{\mathbb{T}^2} (\mathcal{P}_\tau v_\varepsilon \cdot \nabla_h \xi_\varepsilon) \, dx \, dy
\end{cases}
\quad \text{in } \mathbb{T}^2,
\]

\[
\begin{aligned}
\partial_t \mathcal{P}_\tau v_\varepsilon + \frac{c^2}{\varepsilon} \nabla_h \xi_\varepsilon^\circ - c_1 (\mu \Delta_h \mathcal{P}_\sigma v_\varepsilon + \lambda \nabla_h \text{div}_h \mathcal{P}_\tau v_\varepsilon) \\
= -\mathcal{P}_\tau (v_\varepsilon \cdot \nabla_h v_\varepsilon + (\varepsilon \alpha \tilde{P}_\sigma \nu_\varepsilon \cdot \nabla_h \xi_\varepsilon) \mathcal{P}_\sigma v_\varepsilon)
\end{aligned}
\]

\[
\begin{aligned}
+ \mathcal{P}_\tau (\text{div}_h \tilde{P}_\sigma \nu_\varepsilon \mathcal{P}_\sigma v_\varepsilon) + \frac{c^2}{\varepsilon} (1 - e^{\varepsilon \xi_\varepsilon}) \nabla_h \xi_\varepsilon^\circ \\
+ \mathcal{P}_\tau (c_1 (e^{-\varepsilon \alpha \xi_\varepsilon} - 1)(\mu \Delta_h v_\varepsilon + \lambda \nabla_h \text{div}_h v_\varepsilon))
\end{aligned}
\quad \text{in } \mathbb{T}^2,
\]

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where
\[ \xi^o_\varepsilon := \xi - \int_{T^2} \xi_x \, dx \, dy. \]  
(4.38)

In the following, denote by
\[ U^o_\varepsilon := \left( \xi^o_\varepsilon \right)_{\mathcal{P} \tau V_\varepsilon} = \left( \xi - \int_{T^2} \xi_x \, dx \, dy \right) \], \quad \varepsilon := \int_{T^2} \xi_x \, dx \, dy, \quad \text{and}
\[ V^o_\varepsilon := \mathcal{L}(\frac{-t}{\varepsilon}) U^o_\varepsilon = \left( \mathcal{L}_1(\frac{-t}{\varepsilon}) U^o_\varepsilon \right) \left( \mathcal{L}_2(\frac{-t}{\varepsilon}) U^o_\varepsilon \right) \] (so \( U^o_\varepsilon = \mathcal{L}(\frac{t}{\varepsilon}) V^o_\varepsilon \)),
\[ (4.39) \]

where \( \mathcal{L}_1 : (\mathcal{D}'(T^2))^3 \mapsto \mathcal{D}'(T^2) \), \( \mathcal{L}_2 : (\mathcal{D}'(T^2))^3 \mapsto (\mathcal{D}'(T^2))^2 \) will be referred to as the first and the second components of \( \mathcal{L} \), respectively. Then \( V^o_\varepsilon \) satisfies
\[ \partial_t V^o_\varepsilon + Q_{\varepsilon,1}(P_\sigma V_\varepsilon, V^o_\varepsilon) + Q_{\varepsilon,2}(V^o_\varepsilon, V^o_\varepsilon) + Q_{\varepsilon,3}(g_\varepsilon, V^o_\varepsilon) - \mathcal{A}_\varepsilon(D) V^o_\varepsilon \]
\[ = \mathcal{L}(\frac{-t}{\varepsilon}) \left( K + L_1 + L_2 \right), \]
(4.40)

where the linear and bi-linear operators are defined as follows: for \( V_1, V_2 \in (\mathcal{D}'(T^2))^3, u \in (\mathcal{D}'(T^2 \times 2T))^2, g \in \mathbb{R}, \)
\[ \mathcal{A}_\varepsilon(D) V_1 := \mathcal{L}(\frac{-t}{\varepsilon}) \left( c_1(\mu \Delta_h \mathcal{L}_2(\frac{-t}{\varepsilon}) V_1 + \lambda \nabla_h \text{div}_h \mathcal{L}_2(\frac{-t}{\varepsilon}) V_1) \right), \]
\[ Q_{\varepsilon,1}(P_\sigma u, V_1) := \mathcal{L}(\frac{-t}{\varepsilon}) \left( \text{div}_h(\mathcal{L}_1(\frac{-t}{\varepsilon}) V_1 \int_0^t P_\sigma u \, dz) \right), \]
\[ Q_{\varepsilon,2}(V_1, V_2) := \mathcal{L}(\frac{-t}{\varepsilon}) \left( \mathcal{L}_2(\frac{-t}{\varepsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{-t}{\varepsilon}) V_2 + f_{T_2} \mathcal{L}_2(\frac{-t}{\varepsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{-t}{\varepsilon}) V_2 \right), \]
\[ Q_{\varepsilon,3}(g, V_1) := \mathcal{L}(\frac{-t}{\varepsilon}) \left( c_2^2 g \nabla_h \mathcal{L}_1(\frac{-t}{\varepsilon}) V_1 \right). \]
(4.41)
Here,

\[ K := \mathcal{P}_\tau(-\mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h \mathcal{P}_\sigma v_\varepsilon - div_h \mathcal{P}_\sigma v_\varepsilon \mathcal{P}_\sigma v_\varepsilon), \]

\[ L_1 := -\varepsilon \alpha \mathcal{P}_\tau((\mathcal{P}_\sigma v_\varepsilon \cdot \nabla_h \xi_\varepsilon) \mathcal{P}_\sigma v_\varepsilon) + \mathcal{P}_\tau(\varepsilon^2 (1 + \varepsilon \xi_\varepsilon - \varepsilon^2 \xi_\varepsilon) \nabla_h \xi_\varepsilon^0), \quad (4.42) \]

\[ L_2 := \mathcal{P}_\tau(c_1 (e^{-\varepsilon \alpha \xi_\varepsilon} - 1)(\mu \Delta_h v_\varepsilon + \lambda \nabla_h \nabla_h h_\varepsilon)). \]

Also \( g_\varepsilon(t) = \int_{T^2} \xi_\varepsilon \, dx \, dy \) satisfies, from (4.36),

\[
\frac{d}{dt} g_\varepsilon = -\int_{T^2} ((U_\varepsilon^o)_2 \cdot \nabla_h (U_\varepsilon^o)_1) \, dx \, dy \\
= -\int_{T^2} L_2 \left( \frac{t}{\varepsilon} \right) V_\varepsilon^o \cdot \nabla_h L_1 \left( \frac{t}{\varepsilon} \right) V_\varepsilon^o \, dx \, dy, \tag{4.43}
\]

with \( g_\varepsilon(t = 0) = \int_{T^2} \xi_0 \, dx \, dy \).

Now we will address the strong convergence of \( V_\varepsilon^o, g_\varepsilon \) as \( \varepsilon \to 0^+ \). Notice that \( L \) defined in (4.22) is anti-symmetric with respect to \( A := \text{diag}(c_2, \gamma - 1, \gamma - 1) \) (i.e., \( \int V^\top ALU \, d\tilde{x} = -\int U^\top ALV \, d\tilde{x} \)), linear and commutative with \( \partial_h \in \{ \partial_x, \partial_y \} \).

Consequently the standard \( H^s \) estimate of the solutions to equation (4.27) implies that the operator \( L \) preserves the \( H^s \) norm, i.e.,

\[ |L(t)V|_{H^s} \simeq |V|_{H^s} \text{ for } t \in (0, \infty), s \in \mathbb{N}, V \in (\mathcal{D}'(T^2))^3. \tag{4.44} \]

Therefore one has, as the consequence of (4.25) and (4.39),

\[ |V_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} + |U_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} + |(U_\varepsilon^o)_2|_{L^2(0,T;H^3(T^2))} < C. \tag{4.45} \]

Then it is easy to see from (4.43) that

\[ \sup_{0 \leq t \leq T} \{ |g_\varepsilon| + |\frac{d}{dt} g_\varepsilon| \} \leq (1 + T)|U_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} < C. \tag{4.46} \]

Therefore the Arzelà-Ascoli theorem implies that as \( \varepsilon \to 0^+ \),

\[ g_\varepsilon \to g^o \text{ uniformly for some } g^o \in C([0, T]), \tag{4.47} \]

and \( |g^o| < C \). On the other hand, after applying the Hölder and Sobolev embedding inequalities, (4.1), (4.41), (4.45), and (4.46) imply the following
estimates:
\[
|\mathcal{A}(D)V_\varepsilon^o|_{L^\infty(0,T;L^2(T^2)) \cap L^2(0,T;H^1(T^2))} \lesssim |V_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} + |(U_\varepsilon^o)|_{L^2(0,T;H^3(T^2))} < C,
\]
\[
|Q_{\varepsilon,1}(P_\sigma v_\varepsilon, V_\varepsilon^o)|_{L^\infty(0,T;H^1(T^2))} \lesssim |V_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} \times \|P_\sigma v_\varepsilon\|_{L^\infty(0,T;H^2(T^2))} < C,
\]
\[
|Q_{\varepsilon,2}(V_\varepsilon^o, V_\varepsilon^o)|_{L^\infty(0,T;H^1(T^2))} \lesssim |V_\varepsilon^o|^2_{L^\infty(0,T;H^2(T^2))} < C,
\]
\[
|Q_{\varepsilon,3}(g_\varepsilon, V_\varepsilon^o)|_{L^\infty(0,T;H^1(T^2))} \lesssim \sup_{0 \leq t \leq T} |g_\varepsilon| \times |V_\varepsilon^o|_{L^\infty(0,T;H^2(T^2))} < C,
\]
and
\[
|K|_{L^\infty(0,T;H^1(T^2))} \lesssim \|P_\sigma v_\varepsilon\|^2_{L^\infty(0,T;H^2(T^2))} < C,
\]
\[
|L_1|_{L^\infty(0,T;H^1(T^2))} \lesssim \varepsilon \|P_\sigma v_\varepsilon\|^2_{L^\infty(0,T;H^2(T^2))} \|\xi_\varepsilon\|_{L^\infty(0,T;H^2(T^2))} + \varepsilon \varepsilon \|\xi_\varepsilon\|^2_{L^\infty(0,T;H^2(T^2))} + 1 < C,
\]
\[
|L_2|_{L^\infty(0,T;L^2(T^2)) \cap L^2(0,T;H^1(T^2))} \lesssim \varepsilon \varepsilon \|\xi_\varepsilon\|_{L^\infty(0,T;H^2(T^2))} \times (\|\xi_\varepsilon\|^2_{L^\infty(0,T;H^2(T^2))} + 1) \|v_\varepsilon\|_{L^\infty(0,T;H^2(T^2)) \cap L^2(0,T;H^3(T^2))} < C.
\]

Therefore, equation (4.40) implies
\[
|\partial_t V_\varepsilon^o|_{L^\infty(0,T;L^2(T^2)) \cap L^2(0,T;H^1(T^2))} < C.
\]

Then the Aubin compactness lemma (see, i.e., [42, Theorem 2.1] and [43]) implies that, there is
\[
V^o \in L^\infty(0,T;H^2(T^2)) \cap C(0,T;H^1(T^2)) \quad \text{with} \quad \partial_t V^o \in L^\infty(0,T;L^2(T^2)) \cap L^2(0,T;H^1(T^2)),
\]

such that as \(\varepsilon \to 0^+\),
\[
V_\varepsilon^o \rightharpoonup V^o \quad \text{weak-* in} \ L^\infty(0,T;H^2(T^2)),
\]
\[
V_\varepsilon^o \to V^o \quad \text{in} \ L^\infty(0,T;H^1(T^2)) \cap C(0,T;H^1(T^2)),
\]
\[
\partial_t V_\varepsilon^o \rightharpoonup \partial_t V^o \quad \text{weak-* in} \ L^\infty(0,T;L^2(T^2)),
\]
\[
\partial_t V_\varepsilon^o \rightharpoonup \partial_t V^o \quad \text{weakly in} \ L^2(0,T;H^1(T^2)).
\]
Also
\[ V^\circ_\varepsilon(t = 0) = V^\circ(t = 0) = \left( \frac{\xi_0 - \int_{2T} \xi_0 \, dx \, dy}{\mathcal{P}_\varepsilon v_0} \right), \] (4.52)

Consequently, as \( \varepsilon \to 0^+ \), since \( \mathcal{L} \) preserves the \( H^1 \) norm,
\[ \mathcal{U}^\circ_\varepsilon - \mathcal{L} \left( \frac{t}{\varepsilon} \right) V^\circ \to \mathcal{L} \left( \frac{t}{\varepsilon} \right) (V^\circ_\varepsilon - V^\circ) \to 0 \quad \text{in} \quad L^\infty(0,T; H^1(\mathbb{T}^2)). \] (4.53)

In conclusion, as \( \varepsilon \to 0^+ \), (4.12), (4.47) and (4.53) imply that
\[ \| v_\varepsilon - v_p - \mathcal{L}_2 \left( \frac{t}{\varepsilon} \right) V^\circ \|_{L^\infty(0,T; H^1(\mathbb{T}^2 \times 2T))} \]
\[ + \| \xi_\varepsilon - g^\circ \|_{L^\infty(0,T; H^1(\mathbb{T}^2 \times 2T))} \]
\[ \leq \| g^\circ - g^\circ \|_{L^\infty(0,T)} + \| \mathcal{U}^\circ_\varepsilon - \mathcal{L} \left( \frac{t}{\varepsilon} \right) V^\circ \|_{L^\infty(0,T; H^1(\mathbb{T}^2))} \]
\[ + \| \mathcal{P}_\sigma v_\varepsilon - v_p \|_{L^\infty(0,T; H^1(\mathbb{T}^2 \times 2T))} \to 0. \] (4.54)

Next, we will identify the limit equations of (4.40) and (4.43); that is, the equation satisfied by \( V^\circ \) and \( g^\circ \).

**The limit equations of oscillations**

In order to identify the limit equations of (4.40) and (4.43) as \( \varepsilon \to 0^+ \), we first introduce the Fourier representation of the operators defined in (4.41). Notice that (4.27) implies \( \int_{2T^2} V^\circ_\varepsilon \, dx \, dy = \int_{2T^2} \mathcal{U}^\circ_\varepsilon \, dx \, dy = 0 \). It suffices to study in the space consisting of functions in \( \mathcal{D}'(\mathbb{T}^2)^3 \) with zero average.

Notice that, on the other hand, \( \mathcal{U}^\circ_\varepsilon \) is inside the orthogonal complement of the kernel of operator \( L \) with respect to the \( L^2 \)-inner product, and thanks to (4.27), so is \( V^\circ_\varepsilon \). In fact, \((\ker L)^\perp = \{ (q,u) \in \mathcal{D}'(\mathbb{T}^2) \times (\mathcal{D}'(\mathbb{T}^2))^2 \mid \int_{2T^2} q \, dx \, dy = 0, u = \nabla_h \psi, \psi \in \mathcal{D}'(\mathbb{T}^2) \}\). Next, we introduce, as in [6] and [31], the following basis of \((\ker L)^\perp\): for \( k \in 2\pi \mathbb{Z}^2 \setminus \{(0,0)\} \),
\[ V^+_k(\vec{x}_h) := \frac{1}{\sqrt{\gamma - 1 + c^2 |k|}} \begin{pmatrix} \sqrt{\gamma - 1} |k| \\ -c \text{sgn}(k) |k| \end{pmatrix} e^{ik \cdot \vec{x}_h}, \]
\[ V^-_k(\vec{x}_h) := \frac{1}{\sqrt{\gamma - 1 + c^2 |k|}} \begin{pmatrix} \sqrt{\gamma - 1} |k| \\ +c \text{sgn}(k) |k| \end{pmatrix} e^{ik \cdot \vec{x}_h}, \] (4.55)
and the corresponding conjugates
\[ V^+_{k^*}(\vec{x}_h) := \frac{\sqrt{\gamma - 1 + c^2}}{2c\sqrt{\gamma - 1} |k|} \begin{pmatrix} c |k| \\ -\sqrt{\gamma - 1} \text{sgn}(k) |k| \end{pmatrix} e^{ik \cdot \vec{x}_h}, \]
\[ V^-_{k^*}(\vec{x}_h) := \frac{\sqrt{\gamma - 1 + c^2}}{2c\sqrt{\gamma - 1} |k|} \begin{pmatrix} c |k| \\ +\sqrt{\gamma - 1} \text{sgn}(k) |k| \end{pmatrix} e^{ik \cdot \vec{x}_h}. \]
Here $sg(k)$ is the generalized sign function for vector $k \in 2\pi T^2 \setminus \{(0,0)\}$, defined as,

$$sg(k) := \begin{cases} 
1 & \text{if and only if } k_1 > 0 \text{ or } k_1 = 0, k_2 > 0, \\
-1 & \text{otherwise.}
\end{cases}$$

Notice that $V^\pm_k = \overline{V^\mp_k}$, where, hereafter, $\overline{\cdot}$ represents the complex conjugate.

Then by defining $\zeta = c\sqrt{\gamma - 1}$, $\{V_k^\pm\}_{k \in 2\pi T^2 \setminus \{(0,0)\}}$ are the eigenfunctions of $L$ with the eigenvalues $\{\pm i\zeta \sg(k)k\}_{k \in 2\pi T^2 \setminus \{(0,0)\}}$, i.e.,

$$LV_k^+ = -i\zeta \sg(k)k |V_k^+|, \quad \text{and} \quad LV_k^- = i\zeta \sg(k)k |V_k^-|, \quad k \in 2\pi T^2 \setminus \{(0,0)\}.$$ 

Also any $V \in (\ker L)^\perp$ can be represented in terms of Fourier series as

$$V = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} (a_k^+ V_k^+ + a_k^- V_k^-)$$

with the Fourier coefficients $\{a_k^\pm\}_{k \in 2\pi T^2 \setminus \{(0,0)\}}$ determined by

$$a_k^\pm = \langle V, V_{k,^\pm} \rangle_{C} \deq \int_{T^2} V \cdot \overline{V_{k,^\pm}} \, dxdy. \quad (4.57)$$

If $V$ is real-valued, $a_k^- = \overline{a_k^+}$. In addition, one has

$$\mathcal{L}(s)V = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} (a_k^+ V_k^+ e^{i\zeta \sg(k)k} s + a_k^- V_k^- e^{-i\zeta \sg(k)k} s). \quad (4.58)$$

Denote by $P_{(\ker L)^\perp} : (\mathcal{D}'(T^2))^3 \to (\ker L)^\perp \subset (\mathcal{D}'(T^2))^3$, being the projection operator on $(\ker L)^\perp$ with respect to the $L^2$-inner product. Then for any $q \in D'(T^2)$ and $u \in (D'(T^2 \times 2T))^2$, it satisfies,

$$P_{(\ker L)^\perp}\left( \int_0^q u \, dz \right) = \left( q - \int_{T^2} q \, dxdy \overline{\mathcal{P}_\tau u} \right). \quad (4.59)$$

Moreover, as in (4.56) and (4.57), one has

$$P_{(\ker L)^\perp}\left( \int_0^q u \, dz \right) = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} (b_k^+ V_k^+ + b_k^- V_k^-), \quad (4.60)$$

with $\{b_k^\pm\}_{k \in 2\pi T^2 \setminus \{(0,0)\}}$ determined by

$$b_k^\pm = \langle \left( \int_0^q u \, dz \right), V_{k,^\pm} \rangle_{C} \deq \int_{T^2} \left( \int_0^q u \, dz \right) \cdot \overline{V_{k,^\pm}} \, dxdy.$$
We will now investigate the action of the operators in (4.41) on $V^\pm_k$. Without further mentioning, we will use the representations (4.59) and (4.60), below. To shorten the notations, the multi-indices $k, l, m, \ldots$ in the following calculations are always subject to the set $2\pi T^2 \setminus \{(0,0)\}$. For instance,

$$\sum_k = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}}, \text{ etc.}$$

**Calculation of $A^\varepsilon(D)$**. To calculate $A^\varepsilon(D)V^\pm_k$, notice that,

$$c_1(\mu \Delta h L_2(t^\varepsilon) V^\pm_k + \lambda \nabla h \nabla h L_2(t^\varepsilon) V^\pm_k)$$

$$= \pm c_1(\mu + \lambda) \frac{c \text{sg}(k)|k|e^{\pm i\text{sg}(k)|k|^\frac{1}{4}}e^{\pm i\text{sg}(k)|k|t^\varepsilon}_e^{ik_x h}.$$ 

Denote by

$$\begin{pmatrix} 0 \\ c_1(\mu \Delta h L_2(t^\varepsilon) V^\pm_k + \lambda \nabla h \nabla h L_2(t^\varepsilon) V^\pm_k) \end{pmatrix} =: A_k = \alpha_k V^\pm_k + \beta_k V^\mp_k.$$ 

Then

$$\alpha_k = \int_{T^2} A_k \cdot V^\pm_k dx dy = \frac{\sqrt{\gamma - 1 + c^2}}{2c\sqrt{\gamma - 1}|k|^\frac{1}{4}}$$

$$\times \left( \pm c_1(\mu + \lambda) \frac{c \text{sg}(k)|k|e^{\pm i\text{sg}(k)|k|^\frac{1}{4}}e^{\pm i\text{sg}(k)|k|t^\varepsilon}_e^{ik_x h} \right) \cdot \left( \frac{\frac{e^{|k|^\frac{1}{4}}}{\sqrt{\gamma - 1} \text{sg}(k)k}}{\mp} \right)$$

$$= -\frac{c_1(\mu + \lambda)}{2} |k|^2 e^{\pm i\text{sg}(k)|k|^\frac{1}{4}}, \text{ and similarly}$$

$$\beta_k = \int_{T^2} A_k \cdot V^\mp_k dx dy = \frac{c_1(\mu + \lambda)}{2} |k|^2 e^{\pm i\text{sg}(k)|k|^\frac{1}{4}}.$$ 

Therefore

$$A^\varepsilon(D)V^\pm_k = L\left(-\frac{t^\varepsilon}{\varepsilon}\right) A_k = -\frac{c_1(\mu + \lambda)}{2} |k|^2 V^\pm_k$$

$$+ \frac{c_1(\mu + \lambda)}{2} |k|^2 V^\mp_k e^{\pm 2 i\text{sg}(k)|k|^\frac{1}{4}}.$$

**Calculation of $Q_{\sigma,1}$**. To calculate $Q_{\sigma,1}(P_\sigma u, V^\pm_k)$, we first represent $P_\sigma u$ as, after expanding it as Fourier series in the horizontal direction,

$$P_\sigma u = \sum_k u_k(z, t)e^{ik_x h}, \quad \int_0^1 P_\sigma u dz = \sum_k \bar{u}_k e^{ik_x h}, \quad \text{(4.61)}$$
Here, recall that \( \overline{u}_k \cdot \mathbf{k} = 0 \). Then the relations (4.59) and (4.60) allow us to write down the following:

\[
\left( \text{div}_h (L_1 (\frac{t}{\varepsilon}) V_k^\pm \int_0^1 \mathcal{P}_u u dz) \right. \\
\left. + \mathcal{P}_\tau (\mathcal{P}_\sigma u \cdot \nabla_h L_2 (\frac{t}{\varepsilon}) V_k^\pm \mathcal{P}_\sigma u) \right)
\]

\[
= \mathcal{P}_{(\ker L)^\perp} \left( \int_0^1 \mathcal{P}_u u dz \cdot \nabla_h L_2 (\frac{t}{\varepsilon}) V_k^\pm + \mathcal{L}_2 (\frac{t}{\varepsilon}) V_k^\pm \cdot \nabla_h \int_0^1 \mathcal{P}_u u dz \right)
\]

\[
= : \sum_m (\xi_m V_m^\pm + \zeta_m V_m^\mp).
\]

Denote by

\[
B_k := \left( \int_0^1 \mathcal{P}_u u dz \cdot \nabla_h L_2 (\frac{t}{\varepsilon}) V_k^\pm + \mathcal{L}_2 (\frac{t}{\varepsilon}) V_k^\pm \cdot \nabla_h \int_0^1 \mathcal{P}_u u dz \right)
\]

\[
= \sum_1 \left( \frac{i \sqrt{\gamma - 1} \cdot \mathbf{k} + \mathbf{l}}{\sqrt{\gamma - 1 + c^2} |\mathbf{k}|} \right) e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|} e^{i (\mathbf{k} \cdot \mathbf{1}) \cdot \hat{x}_h}.
\]

Then

\[
\xi_m = \int_{\mathbb{T}^2} B_k \cdot V_m^\pm \ dx dy = \int_{\mathbb{T}^2} B_k \cdot V_m^\pm \ dx dy \frac{\gamma - 1 + c^2}{2 \sqrt{\gamma - 1}} e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|} \sqrt{\gamma - 1 + c^2} |\mathbf{k}| \]

\[
\times \left( \frac{i \sqrt{\gamma - 1} \cdot \mathbf{k} + \mathbf{l}}{\sqrt{\gamma - 1 + c^2} |\mathbf{k}|} \right) \left( \frac{e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|}}{\sqrt{\gamma - 1 + c^2} |\mathbf{k}|} \right)
\]

\[
= \frac{i}{2} (\mathbf{k} \cdot \hat{u}_1) (1 + (\mathbf{m} \cdot \mathbf{k} + \mathbf{k} \cdot \mathbf{l}) \frac{\text{sg}(|\mathbf{m}|) \text{sg}(|\mathbf{k}|)}{|\mathbf{m}| |\mathbf{k}|}) e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|} , \quad \text{and similarly}
\]

\[
\zeta_m = \int_{\mathbb{T}^2} B_k \cdot V_m^\mp \ dx dy \frac{\gamma - 1 + c^2}{2 \sqrt{\gamma - 1}} e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|} \sqrt{\gamma - 1 + c^2} |\mathbf{k}| \]

\[
\times \left( \frac{i \sqrt{\gamma - 1} \cdot \mathbf{k} + \mathbf{l}}{\sqrt{\gamma - 1 + c^2} |\mathbf{k}|} \right) \left( \frac{e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|}}{\sqrt{\gamma - 1 + c^2} |\mathbf{k}|} \right)
\]

\[
= \frac{i}{2} (\mathbf{k} \cdot \hat{u}_1) (1 - (\mathbf{m} \cdot \mathbf{k} + \mathbf{k} \cdot \mathbf{l}) \frac{\text{sg}(|\mathbf{m}|) \text{sg}(|\mathbf{k}|)}{|\mathbf{m}| |\mathbf{k}|}) e^{\pm i \text{sg}(|\mathbf{k}|) |\mathbf{k}|} ,
\]
where \( l = m - k \) and we have used \((4.62)\). Therefore

\[
\mathcal{Q}_{\xi,1}(P_{\sigma}u, V_k^\pm) = \mathcal{L}(-\frac{t}{\epsilon}) \sum_m (\zeta_m V_m^\pm + \zeta_m V_m^\mp)
\]

\[
= \frac{i}{2} \sum_{m-l=k} ((k \cdot \hat{u}_1)(1 + \frac{sg(m) sg(k)}{|m| |l|} k \cdot (m + l))) V_m^\pm
\]

\[
\times e^{\pm i(\frac{sg(k) |k| - sg(m) |m|) t}{\epsilon}}
\]

\[
+ ((k \cdot \hat{u}_1)(1 - \frac{sg(m) sg(k)}{|m| |l|} k \cdot (m + l))) V_m^\mp
\]

\[
\times e^{\pm i(\frac{sg(k) |k| + sg(m) |m|) t}{\epsilon}}.
\]

**Calculation of \( \mathcal{Q}_{\xi,2} \).** To calculate \( \mathcal{Q}_{\xi,2}(V_k^\pm, V_1^\pm) \) and \( \mathcal{Q}_{\xi,2}(V_k^\pm, V_1^\mp) \), as before, notice that, for \( V_1, V_2 \in (D'((\mathbb{T}^2))^3, \) the relation \((4.59)\) implies the following identity:

\[
\left( \mathcal{L}_2(\frac{t}{\epsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2 - f_{T_2} \mathcal{L}_2(\frac{t}{\epsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2 \right)
\]

\[
\mathcal{P}_{\perp} \mathcal{L}_2(\frac{t}{\epsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2 + c^2 \mathcal{L}_1(\frac{t}{\epsilon}) V_1 \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2
\]

\[
= \mathcal{P}_{\text{ker } L} \left( \mathcal{L}_2(\frac{t}{\epsilon}) V_1 \cdot \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2 + c^2 \mathcal{L}_1(\frac{t}{\epsilon}) V_1 \nabla_h \mathcal{L}_1(\frac{t}{\epsilon}) V_2 \right).
\]

Thus following the same arguments as in the calculation of \( \mathcal{Q}_{\xi,1} \) yields:

\[
\mathcal{Q}_{\xi,2}(V_k^\pm, V_1^\pm) = -\frac{ic}{2\sqrt{\gamma - 1 + c^2}} \left( \frac{sg(k) k \cdot l}{|k|} + \frac{sg(m) sg(k) sg(l)(1 \cdot k)(1 \cdot m)}{|m| |k| |l|} \right)
\]

\[
+ \frac{(\gamma - 1) sg(m) m \cdot m}{|m|} V_m^\pm e^{\frac{\pm i(\frac{sg(k) |k| + sg(l) |l| - sg(m) |m|)}{\epsilon}}}
\]

\[
\mathcal{Q}_{\xi,2}(V_k^\pm, V_1^\mp) = -\frac{ic}{2\sqrt{\gamma - 1 + c^2}} \left( \frac{sg(k) k \cdot l}{|k|} + \frac{sg(m) sg(k) sg(l)(1 \cdot k)(1 \cdot m)}{|m| |k| |l|} \right)
\]

\[
- \frac{(\gamma - 1) sg(m) m \cdot m}{|m|} V_m^\mp e^{\frac{\pm i(\frac{sg(k) |k| + sg(l) |l| + sg(m) |m|)}{\epsilon}}},
\]

\[
Q_{\xi,2}(V_k^\pm, V_1^\pm) = \frac{ic}{2\sqrt{\gamma - 1 + c^2}} \left( \frac{sg(k) k \cdot l}{|k|} + \frac{sg(m) sg(k) sg(l)(1 \cdot k)(1 \cdot m)}{|m| |k| |l|} \right)
\]

\[
+ \frac{(\gamma - 1) sg(m) m \cdot m}{|m|} V_m^\pm e^{\frac{\pm i(\frac{sg(k) |k| + sg(l) |l| - sg(m) |m|)}{\epsilon}}},
\]

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where \( m = k + l \).

**Calculation of \( Q_{\varepsilon,3} \).** This will be similar to the calculation of \( A_{\varepsilon}(D) \). The result is

\[
Q_{\varepsilon,3}(g, V_k^\pm) = \mp \frac{ic}{2} \sqrt{\gamma - 1} \text{sg}(k)|V_k^\pm| e^{\pm i|\text{sg}(k)|k - \text{sg}(l)|l| + \text{sg}(m)|m|} t, \quad (4.65)
\]

\[
\pm \frac{ic}{2} \sqrt{\gamma - 1} \text{sg}(k)|V_k^\pm| e^{\pm 2i|\text{sg}(k)|k} t.
\]

The limits of the operators \( A_{\varepsilon}(D) \), \( Q_{\varepsilon,1} \), \( Q_{\varepsilon,2} \), \( Q_{\varepsilon,3} \) as \( \varepsilon \to 0^+ \). For fixed \( k, l \), in the sense of distribution, we have the following: as \( \varepsilon \to 0^+ \),

\[
A_{\varepsilon}(D)V_k^\pm \to -\frac{(\mu + \lambda)c_1}{2}|k|^2 V_k^\pm = \frac{(\mu + \lambda)c_1}{2} \Delta_h V_k^\pm =: A(D)V_k^\pm, \quad (4.63)
\]

\[
Q_{\varepsilon,1}(P_\sigma u, V_k^\pm) \to i \sum_{m - 1 = k \atop \text{sg}(k)|k| = \text{sg}(m)|m|} \frac{(\hat{u}_1 \cdot k)(m \cdot k)}{|k|^2} V_m^\pm =: Q_1(P_\sigma u, V_k^\pm), \quad (4.64)
\]

\[
Q_{\varepsilon,2}(V_k^\pm, V_1^\pm) \to \mp \frac{ic(\gamma + 1)}{2\sqrt{\gamma - 1} + c^2} \text{sg}(l)|l| V_m^\pm =: Q_2(V_k^\pm, V_1^\pm), \quad (4.65)
\]

where \( m = k + l, \text{sg}(k)|k| + \text{sg}(l)|l| = \text{sg}(m)|m| \), \( k \) and \( l \) are co-linear,

\[
Q_{\varepsilon,3}(g, V_k^\pm) \to \pm \frac{ic(\gamma - 1)}{2} \text{sg}(k)|k| V_k^\pm =: Q_3(g, V_k^\pm), \quad (4.67)
\]

where we have used the facts that

\[
\{m = k + l, \text{sg}(k)|k| + \text{sg}(l)|l| = \text{sg}(m)|m|\} = \{k, l, m \text{ are co-linear, and } l \cdot k = \text{sg}(l)\text{sg}(k)|l||k|, l \cdot m = \text{sg}(l)\text{sg}(m)|l||m|\},
\]

and \( \{m = k + l, \text{sg}(k)|k| + \text{sg}(l)|l| = -\text{sg}(m)|m|\} = \emptyset, \)

\[
\{m = k + l, \text{sg}(k)|k| - \text{sg}(l)|l| = \pm \text{sg}(m)|m|\} = \emptyset.
\]

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The limit equations. Now we have prepared enough to identify the limit equations of equations (4.40) and (4.43) as $\varepsilon \to 0^+$.

We rewrite equation (4.40) in the following fashion:

$$
\partial_t V_\varepsilon^0 + Q_{\varepsilon,1}(v_p, V_\varepsilon^0) + Q_{\varepsilon,2}(V_\varepsilon^0, V_\varepsilon^0) + Q_{\varepsilon,3}(g^0, V_\varepsilon^0) - A_\varepsilon(D)V_\varepsilon^0
\quad = - Q_{\varepsilon,1}(P_\sigma v_\varepsilon - v_p, V_\varepsilon^0) - Q_{\varepsilon,1}(v_p, V_\varepsilon^0 - V_\varepsilon^0)
\quad - Q_{\varepsilon,2}(V_\varepsilon^0 - V_\varepsilon^0, V_\varepsilon^0) - Q_{\varepsilon,2}(V_\varepsilon^0, V_\varepsilon^0 - V_\varepsilon^0)
\quad - Q_{\varepsilon,3}(g_\varepsilon - g^0, V_\varepsilon^0) - Q_{\varepsilon,3}(g^0, V_\varepsilon^0 - V_\varepsilon^0)
\quad + A_\varepsilon(D)(V_\varepsilon^0 - V_\varepsilon^0) + \mathcal{L}(\frac{t}{\varepsilon})(K + L_1 + L_2).
$$

(4.68)

Then, from the regularity in (4.52) and (4.50), the weak convergence of $\partial_t V_\varepsilon^0$ in (4.51), and the convergence of operators in (4.63), (4.64), (4.65), (4.66) and (4.67), as $\varepsilon \to 0^+$, the left-hand side of (4.68) converges in the sense of distribution to

$$
\partial_t V^0 + Q_1(v_p, V^0) + Q_2(V^0, V^0) + Q_3(g^0, V^0) - A(D)V^0.
$$

Indeed, one can replace $v_p, V^0$ with their finite dimensional Fourier truncations on the left-hand side of (4.68), and similar estimates as in (4.43) of the operators for such truncations imply that the actions of the operators on the remainders are bounded by certain norms of the remainders uniformly in $\varepsilon$. Then by letting $\varepsilon$ go to zero, since the truncations approximate the identity operator, one can show the convergence of the actions of the operators.

We claim that the right-hand side of (4.68) converges in the sense of distribution to 0. Indeed, from the definition of the operators in (4.41) and the norm preserving property (4.44), applying the Hölder and Sobolev embedding inequalities implies that, as $\varepsilon \to 0^+$,

$$
|Q_{\varepsilon,1}(P_\sigma v_\varepsilon - v_p, V_\varepsilon^0)|_{L^\infty(0,T;L^2(T^2))} \lesssim \mathcal{L}(\frac{t}{\varepsilon})|V_\varepsilon^0|_{L^\infty(0,T;L^\infty(T^2))}
\times \left\| \nabla h(P_\sigma v_\varepsilon - v_p) \right\|_{L^\infty(0,T;L^2(T^2))} + \left\| \nabla h \mathcal{L}(\frac{t}{\varepsilon})V_\varepsilon^0 \right\|_{L^\infty(0,T;L^1(T^2))}
\times \left\| P_\sigma v_\varepsilon - v_p \right\|_{L^\infty(0,T;L^4(T^2))} \lesssim |V_\varepsilon^0|_{L^\infty(0,T;H^2(T^2))}
\times \left\| P_\sigma v_\varepsilon - v_p \right\|_{L^\infty(0,T;H^1(T^2))} \to 0,
\quad |Q_{\varepsilon,1}(v_p, V_\varepsilon^0 - V_\varepsilon^0)|_{L^\infty(0,T;L^2(T^2))} \lesssim |V_\varepsilon^0 - V_\varepsilon^0|_{L^\infty(0,T;H^1(T^2))}
\times \left\| v_p \right\|_{L^\infty(0,T;H^2(T^2))} \to 0,
\quad |Q_{\varepsilon,2}(V_\varepsilon^0 - V_\varepsilon^0)|_{L^\infty(0,T;L^2(T^2))} \lesssim \mathcal{L}(\frac{t}{\varepsilon})(V_\varepsilon^0 - V_\varepsilon^0)|_{L^\infty(0,T;L^1(T^2))}
$$

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Consider any $\psi \in (D'(\mathbb{T}^2))^3$. By making use of the above relation, we have

$$
\int_{\mathbb{T}^2} \mathcal{A}_\varepsilon(D)(V^\alpha_\varepsilon - V^o) \cdot \psi \, dx \, dy
= - \int_{\mathbb{T}^2} \left( c_1 (\mu \Delta_h \mathcal{L}_2(t_\varepsilon)(V^\alpha_\varepsilon - V^o) + \lambda \nabla_h \text{div}_h \mathcal{L}_2(t_\varepsilon)(V^\alpha_\varepsilon - V^o)) \right) \cdot \mathcal{L}(t_\varepsilon) \psi \, dx \, dy
= c_1 \int_{\mathbb{T}^2} \left( \mu \nabla_h \mathcal{L}_2(t_\varepsilon)(V^\alpha_\varepsilon - V^o) \cdot \nabla_h \mathcal{L}_2(t_\varepsilon) \psi
+ \lambda (\text{div}_h \mathcal{L}_2(t_\varepsilon)(V^\alpha_\varepsilon - V^o)) \times (\text{div}_h \mathcal{L}_2(t_\varepsilon) \psi) \right) \, dx \, dy
\lesssim |\nabla_h \psi|_{H^1(\mathbb{T}^2)}
\times |V^\alpha_\varepsilon - V^o|_{L^4(\mathbb{T}^2)} \rightarrow 0 \quad \text{in} \ L^\infty(0,T), \quad \text{as} \ \varepsilon \rightarrow 0^+,
$$

where we have applied (4.12), (4.17), (4.19), (4.50), (4.51). On the other hand, the norm preserving property (4.44) implies that, for $t \in (0, \infty)$ and any $V_1, V_2 \in (D'(\mathbb{T}^2))^3$,

$$
0 = |V_1 + \mathcal{L}(t)V_2|^2_{L^2(\mathbb{T}^2)} - |\mathcal{L}(-t)V_1 + V_2|^2_{L^2(\mathbb{T}^2)}
= 2 \int_{\mathbb{T}^2} V_1 \cdot \mathcal{L}(t)V_2 \, dx \, dy + 2 \int_{\mathbb{T}^2} \mathcal{L}(-t)V_1 \cdot V_2 \, dx \, dy.
$$

Consider any $\psi \in (D'(\mathbb{T}^2))^3$. That is, we will show in the following, as $\varepsilon \rightarrow 0^+$, that

$$
\mathcal{F} := \mathcal{L}(t_\varepsilon) \left( \frac{0}{K + L_1 + L_2} \right) \rightarrow 0, \quad \text{in the sense of distribution.} \quad (4.69)
$$
On the one hand, the estimates in (4.49) and the norm-preserving property (4.44) yield that,

\[ \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \begin{array}{c} 0 \\ L_1 + L_2 \end{array} \right) \to 0, \quad \text{in } L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2)), \]

as \( \epsilon \to 0^+ \). On the other hand,

\[ K = \mathcal{P}_\tau (-v_p \cdot \nabla h v_p - (\text{div}_h \tilde{v}_p) v_p) \\
+ \mathcal{P}_\tau (-\mathcal{P}_\sigma v_\varepsilon \cdot \nabla h (\mathcal{P}_\sigma v_\varepsilon - v_p) - (\text{div}_h \mathcal{P}_\sigma v_\varepsilon) (\mathcal{P}_\sigma v_\varepsilon - v_p) \\
- (\mathcal{P}_\sigma v_\varepsilon - v_p) \cdot \nabla h v_p - (\text{div}_h (\mathcal{P}_\sigma v_\varepsilon - \tilde{v}_p)) v_p) =: K' + K'', \]

where, as the consequence of (4.12) and the Hölder and Sobolev embedding inequalities, we have, as \( \epsilon \to 0^+, \)

\[ K'' \to 0 \quad \text{in } L^\infty(0, T; L^2(\Omega_h)) \cap L^2(0, T; H^1(\Omega_h)). \]

Therefore, one only has to investigate,

\[ \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \begin{array}{c} 0 \\ K' \end{array} \right) = \sum_{k \in 2\pi \mathbb{T}^2 \setminus \{(0,0)\}} (\hat{K}'_k V^+_k e^{-i\varepsilon \text{sg}(k)|k|\frac{t}{\epsilon}} + \hat{K}'_k V^-_k e^{i\varepsilon \text{sg}(k)|k|\frac{t}{\epsilon}}), \]

(4.70)

where we have substituted the representation (4.58), and it is represented, using the relation (4.59) and (4.60),

\[ \left( \begin{array}{c} 0 \\ K' \end{array} \right) = \sum_{k \in 2\pi \mathbb{T}^2 \setminus \{(0,0)\}} (\hat{K}'_k V^+_k + \hat{K}'_k V^-_k). \]

Together with the norm preserving property (4.44) and the fact that

\[ |K'|_{L^\infty(0,T;H^1(\mathbb{T}^2))} \leq \|v_p\|^2_{L^\infty(0,T;H^2(\mathbb{T}^2 \times 2\mathbb{T}))} < \infty, \]

(4.70) yields, as \( \epsilon \to 0^+, \)

\[ \mathcal{L} \left( -\frac{t}{\epsilon} \right) \left( \begin{array}{c} 0 \\ K' \end{array} \right) \rightharpoonup 0 \quad \text{weak-* in } L^\infty(0,T;H^1(\mathbb{T}^2)). \]

This finishes the proof of (4.69). Therefore, we have identified the limit equation of (4.40):

\[ \partial_t V^o + Q_1(v_p, V^o) + Q_2(V^o, V^o) + Q_3(g^o, V^o) - \mathcal{A}(D)V^o = 0. \]

(4.71)
To identity the limit equation of (4.43), notice that equation (4.43) can be written as,
\[
\frac{d}{dt} g_\varepsilon = - \int_{T^2} \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V^0 \cdot \nabla_h \mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V^0 \, dxdy \\
+ \int_{T^2} \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) (V^o - V^o_\varepsilon) \cdot \nabla_h \mathcal{L}_1\left(\frac{t}{\varepsilon}\right) V^0 \, dxdy \\
+ \int_{T^2} \mathcal{L}_2\left(\frac{t}{\varepsilon}\right) V^o_\varepsilon \cdot \nabla_h \mathcal{L}_1\left(\frac{t}{\varepsilon}\right) (V^o - V^o_\varepsilon) \, dxdy := M_1 + M_2 + M_3.
\]
Due to the norm preserving property of $L$, we have $M_2 + M_3 \to 0$ as $\varepsilon \to 0^+$ from (4.45), (4.50) and (4.51). To investigate $M_1$, since $V^o$ is real-valued, we denote, with $\hat{V}^\pm_k = \hat{V}^\pm_k c$
\[
V^o = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} (\hat{V}^+_k V^+_k + \hat{V}^-_k V^-_k).
\]
(4.72)
Then after applying (4.58), direct calculations show that
\[
M_1 = \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} \left( -i \hat{V}^+_k \hat{V}^-_k \frac{c\sqrt{\gamma - 1} \text{sgn}(k)|k|}{\gamma - 1 + c^2} \\
+ i \hat{V}^-_k \hat{V}^-_k \frac{c\sqrt{\gamma - 1} \text{sgn}(k)|k|}{\gamma - 1 + c^2} \\
- 2i \hat{V}^+_k \hat{V}^-_k \frac{c\sqrt{\gamma - 1} \text{sgn}(k)|k|}{\gamma - 1 + c^2} e^{2i\text{sgn}(k)|k|} \right)
\]
\[
\times \frac{c\sqrt{\gamma - 1}}{\gamma - 1 + c^2} \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} \left( -i \hat{V}^+_k \hat{V}^-_k \text{sgn}(k)|k| + i \hat{V}^-_k \hat{V}^-_k \text{sgn}(k)|k| \right)
\]
\[
= i \frac{c\sqrt{\gamma - 1}}{\gamma - 1 + c^2} \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} \left( -|\hat{V}^+_k|^2 + |\hat{V}^-_k|^2 \right) \text{sgn}(k)|k| 
\]
in $L^\infty(0, T)$.
Moreover, direct integrating (4.43) in the temporal variable yields
\[
g_\varepsilon(t) - \int_{T^2} \xi_0 \, dxdy = \int_0^t M_1 \, dt + \int_0^t (M_2 + M_3) \, dt.
\]
After taking $\varepsilon \to 0$, we have for any $t \in [0, T]$,
\[
g^o(t) - \int_{T^2} \xi_0 \, dxdy \\
= i \frac{c\sqrt{\gamma - 1}}{\gamma - 1 + c^2} \sum_{k \in 2\pi T^2 \setminus \{(0,0)\}} \left( -|\hat{V}^+_k|^2 + |\hat{V}^-_k|^2 \right) \text{sgn}(k)|k|,
\]
57
and since \( g^0 \) is real-valued, this implies

\[
g^0 \equiv \int_{\mathbb{T}^2} \xi_0 \, dx \, dy. \tag{4.73}
\]

We summarize the discussion in this subsection in the following:

**Proposition 6.** Under the same assumptions as in Proposition 4 in the case when \( \Omega_h = \mathbb{T}^2 \), there are a function \( V^0 \) satisfying the regularity in (4.50) and a constant \( g^0 \) given by (4.73), such that \( (V^0, g^0) \) satisfies equation (4.71). The convergence in (4.54) holds as \( \varepsilon \to 0^+ \).

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