We unify $\kappa$-Poincaré algebra and $\kappa$-Minkowski spacetime by embedding them into quantum phase space. The quantum phase space has Hopf algebroid structure to which we apply the twist in order to get $\kappa$-deformed Hopf algebroid structure and $\kappa$-deformed Heisenberg algebra. We explicitly construct $\kappa$-Poincaré-Hopf algebra and $\kappa$-Minkowski spacetime from twist. It is outlined how this construction can be extended to $\kappa$-deformed super algebra including exterior derivative and forms. Our results are relevant for constructing physical theories on noncommutative spacetime by twisting Hopf algebroid phase space structure.

**Keywords:** noncommutative space, $\kappa$-Minkowski spacetime, $\kappa$-Poincaré algebra, $\kappa$-deformed phase space, Hopf algebra, Hopf algebroid, twist.
I. INTRODUCTION

The noncommutative (NC) spacetime is a natural setting for investigating the properties of physical theories and the structure of spacetime at very small distances [1, 2]. There are arguments based on quantum gravity [3, 4], and string theory models [5, 6], which suggest that spacetime at Planck length is quantum, i.e. it should be noncommutative. One of the main motivations for dealing with NC spaces is related to the fact that general theory of relativity together with Heisenberg uncertainty principle leads to coordinate uncertainty $\Delta x_\mu \Delta x_\nu > l_{\text{Planck}}^2$ which can naturally be realized via noncommuting coordinates [3]. In this setting the spacetime becomes “fuzzy” and the notion of smooth spacetime geometry and its symmetry are generalized using Hopf algebraic methods.

$\kappa$-Minkowski spacetime [7]-[23] is a Lie algebraic deformation of Minkowski spacetime, where $\kappa$ is the deformation parameter usually interpreted as Planck mass or the quantum gravity scale. The full symmetry of special relativity, i.e. Minkowski spacetime is algebraically described by Poincaré-Hopf algebra $\mathcal{U}(\mathcal{P})$ generated by rotations, boosts and translations. Analogously, the symmetries of $\kappa$-Minkowski spacetime are encoded in the $\kappa$-Poincaré-Hopf algebra. Generalized Poncaré algebras related to $\kappa$-Minkowski spacetime were considered in [21].

Some of the main features of physical theories on $\kappa$-Minkowski spacetime are modification of particle statistics [24]-[26], deformed Maxwell’s equations [27, 28], Aharonov-Bohm problem [29] and quantum gravity effects [30–33]. The construction of QFT’s on NC spaces is of immense importance and is still under investigation [34]-[38]. $\kappa$-Minkowski spacetime is also related to doubly-special and deformed relativity theories [39–43].

It is known that the deformations of the symmetry group can be realized through the application of the Drinfeld twist on that symmetry group [44, 45]. The main virtue of the twist formulation is that the deformed (twisted) symmetry algebra is the same as the original undeformed one and the only thing that changes is the coalgebra structure which then leads to the same free field structure as the corresponding commutative field theory.

One of the ideas presented by the group of Wess et al. [46, 47] is that the symmetries of general relativity, i.e. the diffeomophsms, are considered as the fundamental objects and are deformed using twist [48, 49]. The generalization of the diffeormorphism symmetry is formulated in the language of Hopf algebras, a setting suitable for studying quantization of Lie groups and algebras. Physical applications of this approach are investigated in [50], and especially for black holes in [51]. Our main motivation is to generalize the ideas of the group of Wess et al. to the notion of the Hopf algebroid [52–54] and to construct both QFT and gravity in Hopf algebroid setting, which is more general and it seams more natural since it deals with the
whole phase space [23].

So far there have been attempts in the literature to obtain $\kappa$-Poincaré-Hopf algebra from Drinfeld twist, but none of them succeeded to accomplish this completely. The Abelian twists [12, 14, 25] and Jordanian twists [19] compatible with $\kappa$-Minkowski spacetime were constructed, but the problem with these twists is that they cannot be expressed in terms of the Poincaré generators and the coalgebra runs out into $\mathcal{U}(i\mathfrak{gl}(4)) \otimes \mathcal{U}(i\mathfrak{gl}(4))$.

In this letter we will show that the key for resolving these problems is to analyze the whole quantum phase space $\mathcal{H}$ and its Hopf algebroid structure. Here, we use the Abelian twist, satisfying cocycle condition. This twist is not an element of $\kappa$-Poincaré-Hopf algebra, but an element of $\mathcal{H} \otimes \mathcal{H}$. By applying the twist to the Hopf algebroid structure of quantum phase space $\mathcal{H}$ we obtain the Hopf algebroid structure of $\kappa$-deformed Heisenberg algebra $\hat{\mathcal{H}}$. Moreover, this twist also provides the correct Hopf algebra structure of $\kappa$-Poincaré algebra when applied to the generators of rotation, boost and momenta.

In section II, we present the $\kappa$-Poincaré-Hopf algebra in the bicrossproduct basis and show how the generators of $\kappa$-Poincaré-Hopf algebra and $\kappa$-Minkowski spacetime can be viewed as elements of both quantum phase space $\mathcal{H}$ and $\kappa$-deformed Heisenberg algebra $\hat{\mathcal{H}}$. In section III, we first give the Hopf algebroid structure of quantum phase space $\mathcal{H}$ and then using the twist $\mathcal{F}$ construct the Hopf algebroid structure of $\kappa$-deformed Heisenberg algebra $\hat{\mathcal{H}}$. At the end of section III, we show that the twist $\mathcal{F}$ provides the correct Hopf algebra structure of $\kappa$-Poincaré algebra. In section IV, we illustrate the construction of exterior derivative and NC one-forms compatible with $\kappa$-Poincaré algebra and comment some of the problems which could be resolved by using the extended twist within super-Heisenberg algebra.

II. $\kappa$-POINCARÉ-HOPF ALGEBRA IN BICROSSPRODUCT BASIS

For $\kappa$-Poincaré-Hopf algebra $\mathcal{U}(\mathcal{P}_\kappa)$ generated by Lorentz generators $M_{\mu\nu}$ and momentum generators $p_\mu$, the coproducts $\Delta$ in bicrossproduct basis [10] are

$$\begin{align*}
\Delta p_0 &= p_0 \otimes 1 + 1 \otimes p_0, \\
\Delta p_i &= p_i \otimes 1 + e^{\alpha p_0} \otimes p_i, \\
\Delta M_{i0} &= M_{i0} \otimes 1 + e^{\alpha p_0} \otimes M_{i0} - a_0 p_j \otimes M_{ij}, \\
\Delta M_{ij} &= M_{ij} \otimes 1 + 1 \otimes M_{ij}
\end{align*}$$

(1)

where $a_0 \propto \frac{1}{\kappa}$ is the deformation parameter and $\kappa$ can be interpreted as Planck mass or the quantum gravity scale. Equations in (1) describe the coalgebra structure of the $\kappa$-Poincaré-Hopf algebra and together with

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4 Greek indices ($\mu, \nu,...$) are going from 0 to 3, and latin indices ($i, j,...$) from 1 to 3. Summation over repeated indices is assumed. We use $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.  

antipode $S: \mathcal{U}(\mathcal{P}_\kappa) \mapsto \mathcal{U}(\mathcal{P}_\kappa)$ and counit $\epsilon: \mathcal{U}(\mathcal{P}_\kappa) \mapsto \mathbb{C}$

$$
\epsilon(p_\mu) = \epsilon(M_{\mu\nu}) = 0
$$

$$
S(p_0) = -p_0 \quad S(p_i) = -p_i e^{-a_0 p_0}
$$

$$
S(M_{ij}) = -M_{ij} \quad S(M_{i0}) = -e^{-a_0 p_0} \left( M_{i0} + a_0 M_{ij} p_j \right)
$$

(2)

make the $\kappa$-Poincaré-Hopf algebra. The $\kappa$-Poincaré-Hopf algebra $\mathcal{U}(\mathcal{P}_\kappa)$ is generated by Lorentz generators $M_{\mu\nu}$ and momentum generators $p_\mu$, where $M_{\mu\nu}$ generate undeformed Lorentz algebra,

$$
[M_{\mu\nu}, M_{\lambda\rho}] = -i \left( \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda} \right)
$$

(3)

and $p_\mu$ satisfies

$$
[p_\mu, p_\nu] = 0.
$$

(4)

Additionally, the commutation relations $[M_{\mu\nu}, p_\lambda]$ are

$$
[M_{ij}, p_k] = i \left( \delta_{ik} p_j - \delta_{jk} p_i \right), \quad [M_{ij}, p_0] = 0
$$

$$
[M_{i0}, p_k] = \delta_{ik} \left( \frac{1 - e^{2a_0 p_0}}{2ia_0} - \frac{i a_0}{2 p_i^2} \right) + ia_0 p_i p_k
$$

$$
[M_{i0}, p_0] = ip_i + ia_0 p_i p_0
$$

(5)

Note that in the limit $a_0 \to 0$ we have $\Delta \to \Delta_0$, $S \to S_0$ and the $\kappa$-deformed Poincaré algebra (3-5) reduces to the undeformed Poincaré-Hopf algebra $\mathcal{U}(\mathcal{P})$.

In $\kappa$-Minkowski space with deformed coordinates $\{\hat{x}_\mu\}$ we have:

$$
[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = ia_0 \hat{x}_i.
$$

(6)

Aiming at covariant notation we define $a_\mu = (a_0, \vec{0})$ and for eq.(6) get

$$
[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)
$$

(7)

NC coordinates $\hat{x}$ and momentum $p$ generate the $\kappa$-deformed Heisenberg algebra $\hat{\mathcal{H}}$.

We introduce the action $\triangleright: \hat{\mathcal{H}}(\hat{x}, p) \mapsto \hat{\mathcal{A}}(\hat{x})$, where $\hat{\mathcal{H}}(\hat{x}, p)$ is the unital algebra generated by $\hat{x}_\mu$ and $p_\mu$ and $\hat{\mathcal{A}}(\hat{x})$ is a unital subalgebra of $\hat{\mathcal{H}}(\hat{x}, p)$ generated by $\hat{x}_\mu$. This action is defined by the following requirements

$$
\hat{x}_\mu \triangleright \hat{g}(\hat{x}) = \hat{x}_\mu \hat{g}(\hat{x}), \quad p_\mu \triangleright 1 = 0, \quad M_{\mu\nu} \triangleright 1 = 0
$$

$$
p_\mu \triangleright \hat{x}_\nu = -i \eta_{\mu\nu}, \quad M_{\mu\nu} \triangleright \hat{x}_\lambda = -i \left( \eta_{\nu\lambda} \hat{x}_\mu - \eta_{\mu\lambda} \hat{x}_\nu \right).
$$

(8)

5 For more details see [20].
Namely, using coproducts (1), action (8) and \( G \hat{x}_\mu = m(\Delta G(\triangleright \otimes 1)(\hat{x}_\mu \otimes 1)), \ \forall G \in \mathcal{U}(\mathcal{P}_\kappa) \), one can extract the commutation relations \([M_\mu, \hat{x}_\lambda]\) and \([p_\mu, \hat{x}_\nu]\):

\[
[p_0, \hat{x}_\mu] = -i\eta_{0\mu}, \quad [p_k, \hat{x}_\mu] = -i\eta_{k\mu} + ia_\mu p_k, \tag{9}
\]

\[
[M_{\mu\nu}, \hat{x}_\lambda] = -i(\eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu + a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}). \tag{10}
\]

The Lorentz generators \( M_\mu \) can be expressed in terms of \( \hat{x}_\mu \) and \( p_\mu \), that is as an element in \( \hat{H} \):

\[
M_{\mu0} = \hat{x}_\mu \left( \frac{e^{2a_0 p_0}}{2a_0} - \frac{a_0}{2} p_k^2 \right) - \hat{x}_0 p_i \tag{11}
\]

\[
M_{ij} = \hat{x}_i p_j - \hat{x}_j p_i
\]

Eqs.(11) are completely compatible with eqs.(1-10). In this way we have embedded the \( \kappa \)-Poincaré algebra (3-5) and \( \kappa \)-Minkowski space (6) into \( \kappa \)-deformed Heisenberg algebra \( \hat{H} \).

The realization for \( M_{\mu\nu} \) and \( \hat{x}_\mu \) in terms of \( x_\mu \) and \( p_\mu \), that is as an element in quantum phase space \( \mathcal{H} \) (see (13)), corresponding to bicrossproduct basis is :

\[
\hat{x}_i = x_i, \quad \hat{x}_0 = x_0 - a_0 x_k p_k,
\]

\[
M_{\mu0} = x_i \left( \frac{Z^2 - 1}{2a_0} - \frac{a_0}{2} p_k^2 \right) - (x_0 - a_0 x_k p_k) p_i, \quad M_{ij} = x_i p_j - x_j p_i, \tag{12}
\]

where \( Z = e^A \) and \( A = a_0 p_0 \) (for more details see [14] and [16]). In this way we have embedded the \( \kappa \)-Poincaré algebra (3-5) and \( \kappa \)-Minkowski space (6) into quantum phase space \( \mathcal{H} \) (which is defined in the next section, see (13)).

### III. QUANTUM PHASE SPACE AND TWIST

#### A. Quantum phase space and Hopf algebroid structure

The quantum phase space, i.e. Heisenberg algebra \( \mathcal{H} \) is defined by

\[
[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0 \tag{13}
\]

\[
[p_\mu, x_\nu] = -i\eta_{\mu\nu}
\]

We define the action \( \triangleright : \mathcal{H}(x, p) \mapsto \mathcal{A}(x) \), where \( \mathcal{A} \) is a subalgebra, \( \mathcal{A}(x) \subset \mathcal{H}(x, p) \), generated by \( x_\mu \).

Heisenberg algebra \( \mathcal{H}(x, p) \) can be written as \( \mathcal{H} = \mathcal{A} \ T \), where \( \mathcal{T} \) is a subalgebra, \( \mathcal{T}(p) \subset \mathcal{H}(x, p) \), generated by \( p_\mu \). For any element \( f(x) \in \mathcal{A}(x) \) we have

\[
x_\mu \triangleright f(x) = x_\mu f(x), \quad p_\mu \triangleright f(x) = -i \frac{\partial f}{\partial x^\mu}, \tag{14}
\]

\(^6\) where \( m \) is a multiplication map \( m(A \otimes B) = AB \)
It is known that for Heisenberg algebra $\mathcal{H}$ there is no Hopf algebra structure. However, there exists Hopf algebroid structure defined by undeformed coproduct $\Delta'_0$, counit $\epsilon'_0$ and antipode $S'_0$

$$\Delta'_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu, \quad \Delta'_0 x_\mu = x_\mu \otimes 1,$$

$$\epsilon'_0 (h) = h \triangleright 1, \quad \forall h \in \mathcal{H},$$

$$S'_0 (p_\mu) = -p_\mu, \quad S'_0 (x_\mu) = x_\mu$$

where we have generated an equivalence class in $\mathcal{A} \otimes \mathcal{A}$ by the relations $(\mathcal{R}_0)_\mu \equiv x_\mu \otimes 1 - 1 \otimes x_\mu$ (for details see [23]). The $S'_0$ is antimultiplicative map, $S'_0 : \mathcal{H} \mapsto \mathcal{H}$ and a generalization of the antipode $S_0$ in Hopf algebra (see [52] and [53]). The $\epsilon'_0 : \mathcal{H} \mapsto \mathcal{A}$ is a generalization of a counit $\epsilon_0$ in Hopf algebra. If the coproduct $\Delta'_0$, counit $\epsilon'_0$ and antipode $S'_0$ are applied only to the generators of the Poincaré algebra, they act in the same way as $\Delta_0$, $\epsilon_0$ and $S_0$ and they satisfy the axioms of a Hopf algebra. However, note that: $S'_0 (x_\mu p_\nu) = -p_\nu x_\mu$ and in $\mathfrak{gl}(4)$-Hopf algebra $S_0 (x_\mu p_\nu) = -x_\mu p_\nu$ so for $\mu = \nu$ we have that $S'_0 (x_\mu p_\nu) \neq S_0 (x_\mu p_\nu)$. The full treatment of Hopf algebroid structure of quantum phase space and $\kappa$-deformed phase space will be presented elsewhere.

B. $\kappa$-deformed phase space from twist

Let us introduce the twist operator $\mathcal{F}$ (see eq.(61) in [12] and also see [14])

$$\mathcal{F} = \exp (-iA \otimes x_k p_k)$$

where $A = a_0 p_0$, which is Abelian and satisfies the cocycle condition [25]. We apply it to the Hopf algebroid structure of quantum phase space $\mathcal{H}$

$$\Delta' h = \mathcal{F} \Delta'_0 h \mathcal{F}^{-1}$$

$$\epsilon' (h) = m \left[ \mathcal{F}^{-1} (\epsilon \otimes 1)(\epsilon'_0 (h) \otimes 1) \right] = h \triangleright 1$$

$$S' (h) = \chi S'_0 (h) \chi^{-1} \quad \forall h \in \mathcal{H}$$

where $\chi^{-1} = m \left[ \left( S'_0 \otimes 1 \right) \mathcal{F}^{-1} \right] = e^{-iA x_k p_k}$ and $\chi = e^{iA x_k p_k}$ (note that in the Hopf algebroid $\chi \neq m \left[ \left( 1 \otimes S'_0 \right) \mathcal{F} \right]$). The results for the generators $x_\mu$ and $p_\mu$ are

$$\Delta' p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu \equiv \Delta p_\mu \quad \Delta' p_i = p_i \otimes 1 + e^A \otimes p_i \equiv \Delta p_i$$

$$\Delta' x_0 = x_0 \otimes 1 + 1 \otimes a_0 x_k p_k \quad \Delta' x_i = x_i \otimes 1$$

$$\epsilon' (x_\mu) = \hat{x}_\mu \quad \epsilon' (p_\mu) = 0$$

$$S' (p_0) = -p_0 \equiv S (p_0) \quad S' (p_i) = -p_i e^{-A} \equiv S (p_i)$$

$$S' (x_0) = x_0 - a_0 x_k p_k \quad S' (x_i) = x_i e^A$$
where we have generated an equivalence class in \((\mathcal{A} \otimes \mathcal{A})_F\) by the relations \(R_{\mu} = F(R_0)_{\mu} F^{-1}\) (for details see [23]). We point out that the coproduct \(\Delta'\) and antipode \(S'\) reduce to \(\Delta\) and \(S\) when applied to the generators of \(\kappa\)-Poincaré algebra (see subsection C). With the coproduct \(\Delta'\), antipode \(S'\) and counit \(\epsilon'\) given in (18), we have defined the \(\kappa\)-deformed Hopf algebroid structure of quantum phase space \(\mathcal{H}\).

The NC coordinates \(\hat{x}\) are given by
\[
\hat{x}_i = m\left(F^{-1}(\hat{x}_i \otimes 1)(x_i \otimes 1)\right) = x_i
\]
\[
\hat{x}_0 = m\left(F^{-1}(\hat{x}_0 \otimes 1)(x_0 \otimes 1)\right) = x_0 - a_0 x_k p_k
\]
and satisfy the \(\kappa\)-Minkowski algebra (6). Hence, from equations (18) we have
\[
\Delta' \hat{x}_\mu = \hat{x}_\mu \otimes 1 \quad \epsilon' (\hat{x}_\mu) = \hat{x}_\mu \quad 1 = \hat{x}_\mu
\]
\[
S'(\hat{x}_i) = \hat{x}_i e^{A} \quad S'(\hat{x}_0) = \hat{x}_0 + a_0 p_k \hat{x}_k
\]
which defines the Hopf algebroid structure of \(\kappa\)-deformed Heisenberg algebra \(\hat{H}\).

C. \(\kappa\)-Poincaré-Hopf algebra from twist

The realization of Lorentz generators \(M_{\mu \nu}\) is given in terms of \(\mathcal{H}\) in (12). Then using the eq.(15) for coproduct and homomorphism of \(\Delta'\) we can calculate \(\Delta'_0 M_{ij} = \Delta'_0 x_i \Delta'_0 \left(\frac{Z^2 - 1}{2a_0} - \frac{a_0}{2} p_k^2\right) - \Delta'_0 x_0 \Delta'_0 p_i\) and \(\Delta'_0 M_{ij} = \Delta'_0 x_i \Delta'_0 p_j - \Delta'_0 x_j \Delta'_0 p_i = M_{ij} \otimes 1 + 1 \otimes M_{ij} \equiv \Delta_0 M_{ij}\). Note that \(\Delta'_0 M_{ij} \neq M_{ij} \otimes 1 + 1 \otimes M_{ij}\).

Furthermore, applying the twist \(F\) we get the deformed coproduct
\[
\Delta' M_{ij} = F \Delta'_0 M_{ij} F^{-1} = M_{ij} \otimes 1 + 1 \otimes M_{ij} \equiv \Delta M_{ij}
\]
\[
\Delta' M_{i0} = F \Delta'_0 M_{i0} F^{-1} = \Delta' x_i \Delta' \left(\frac{Z^2 - 1}{2a_0} - \frac{a_0}{2} p_k^2\right) - \Delta' x_0 \Delta' p_i
\]
\[
= (x_i \otimes 1) \left(\frac{Z^2 \otimes Z - 1 \otimes 1}{2a_0} - \frac{a_0}{2} \Delta' (p_k^2)\right) - (x_0 \otimes 1) \Delta' p_i
\]
\[
= M_{i0} \otimes 1 + e^{a_0 p_0} \otimes M_{i0} - a_0 p_j \otimes M_{ij} \equiv \Delta M_{i0}
\]
where in the last line we have used \(x_i \otimes 1 = Z^{-1} \otimes x_i\) which is derived from \(R_0\), [23]. Similarly, using (18) and antihomomorphism of the antipode \(S'_0\) we get
\[
S'(M_{ij}) = \chi' S'_0 (M_{ij}) \chi^{-1} = S'(p_j) S'(x_j) - S'(p_i) S'(x_j) = -M_{ij} \equiv S(M_{ij})
\]
\[
S'(M_{i0}) = \chi' S'_0 (M_{i0}) \chi^{-1} = S' \left(\frac{Z^2 - 1}{2a_0} - \frac{a_0}{2} p_k^2\right) S'(x_i) - S'(p_i) S'(x_0)
\]
\[
= -e^{-a_0 p_0} \left(M_{i0} + a_0 M_{ij} p_j\right) \equiv S(M_{i0})
\]
Hence, the twist \(F\) in (16) applied to the Hopf algebroid structure gives the correct Hopf algebra structure of \(\kappa\)-Poincaré algebra generated by \(M_{\mu \nu}\) and \(p_\mu\), eqs. (1-5).
In this letter we have presented the $\kappa$-Poincaré-Hopf algebra in the bicrossproduct basis from twist (16). This can be generalized to the twist corresponding to any realization [54]. The flip operator and universal $R$-matrix were analyzed in [23].

IV. EXTERIOR DERIVATIVE, NC FORMS AND $\kappa$-DEFORMED SUPER ALGEBRA

We define the exterior derivative $\hat{d}$ and the NC one form $\hat{\xi}_\mu$ in a usual way [11, 15, 16, 22, 55]

$$\hat{d}^2 = 0, \quad \hat{\xi}_\mu = [\hat{d}, \hat{x}_\mu]$$

(23)

with the properties

$$\{\hat{\xi}_\mu, \hat{\xi}_\nu\} = 0, \quad \{\hat{d}, \hat{\xi}_\mu\} = 0$$

(24)

and the commutator between NC forms and NC coordinates $[\hat{\xi}_\mu, \hat{x}_\nu]$ has to be constructed in order to be consistent with (super)Jacobi identities and in doing so, the most general expression reads

$$[\hat{\xi}_\mu, \hat{x}_\nu] = K^\lambda_{\mu\nu}(p)\hat{\xi}_\lambda$$

(25)

The equation for $\kappa$-Minkowski space (7) together with (23) implies the consistency condition

$$[\hat{\xi}_\mu, \hat{x}_\nu] - [\hat{\xi}_\nu, \hat{x}_\mu] = i(a_\mu \hat{\xi}_\nu - a_\nu \hat{\xi}_\mu)$$

(26)

In order to have a differential calculus compatible with $\kappa$-Poincaré algebra we require

$$[M_{\mu\nu}, \hat{d}] = [p_\mu, \hat{d}] = [p_\mu, \hat{\xi}_\nu] = 0$$

$$[M_{\mu\nu}, \hat{\xi}_\lambda] = -i(\eta_{\lambda\nu}\hat{\xi}_\mu - \eta_{\lambda\mu}\hat{\xi}_\nu)$$

(27)

Hence, we have unified the algebra generated by $\{\hat{x}, p, M_{\mu\nu}, \hat{\xi}, \hat{d}\}$ in a super-algebra.

Sitarz in [11] constructed differential algebra $\{\hat{x}, \hat{d}, \phi\}$ which can be exactly realized in the bicrossproduct basis (12) as:

$$\hat{\xi}_0 = \xi_0(2 - \frac{a_0^2}{2} p_1^2 Z^{-1} - \cosh(A)) - a_0 \xi_k p_k$$

$$\hat{\xi}_k = \xi_k + a_0 \xi_0 p_k Z^{-1},$$

$$\phi = - \hat{d}_x = \xi_0\left(\frac{i}{a_0} \sinh(A) - \frac{ia_0}{2} p_1^2 Z^{-1}\right)$$

(28)

However, we point out that there does not exist realization for the generators $M_{\mu\nu}$ compatible with $[M_{\mu\nu}, \hat{d}] = 0$ and Lorentz algebra (3).
In the papers [22, 55] it is shown that the above super algebra has no realization if \( \hat{x}_i = x_i \) and \( \hat{x}_0 = x_0 - a_0 x_k p_k \).

The twist \( \mathcal{F} \) defined in (16) leads to \( \hat{\xi}_\mu = m \left( \mathcal{F}^{-1}(\triangleright \otimes 1)(\xi_\mu \otimes 1) \right) = \xi_\mu \) and this is in contradiction with the consistency condition (26). The analysis of these problems is presented in [55], using super Heisenberg algebra and the extended twist \( \mathcal{F}_{ext} \). There we have constructed a bicovariant calculus compatible with \( \kappa \)-deformed \( \mathfrak{gl}(4) \)-Hopf algebra, for which the consistency condition is satisfied. The physical consequences of this construction is still a work in progress and will be presented elsewhere.

In this letter we have analyzed the whole phase space and its Hopf algebroid structure. The idea is to construct quantum field theory and gravity in the Hopf algebroid setting generalizing the ideas presented by the group of Wess et al. In [56, 57] NC fluid was analyzed. Using the realization formalism for Snyder space, fluid equations of motion and their perturbative solutions were derived. It is of interest to generalize this approach using realization of \( \kappa \)-deformed phase space. This letter should be considered as a starting point for a further investigation along this lines (work in progress).

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