UNIQUE CONTINUATION FROM A CRACK’S TIP
UNDER NEUMANN BOUNDARY CONDITIONS

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Abstract. We derive local asymptotics of solutions to second order elliptic equations at the edge of a \((N - 1)\)-dimensional crack, with homogeneous Neumann boundary conditions prescribed on both sides of the crack. A combination of blow-up analysis and monotonicity arguments provides a classification of all possible asymptotic homogeneities of solutions at the crack’s tip, together with a a strong unique continuation principle.

Keywords. Crack singularities; monotonicity formula; unique continuation; blow-up analysis.

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1. Introduction

In this paper we establish a strong unique continuation principle and analyse the asymptotic behaviour of solutions, from the edge of a flat crack \(\Gamma\), for the following elliptic problem with homogeneous Neumann boundary conditions on both sides of the crack

\[
\begin{aligned}
\begin{cases}
-\Delta u = f u, & \text{in } B_R \setminus \Gamma, \\
\partial^+ u = \partial^- u = 0, & \text{on } \Gamma,
\end{cases}
\end{aligned}
\]

where

\(B_R = \{ x \in \mathbb{R}^N : |x| < R \} \subset \mathbb{R}^N, \quad N \geq 2,\)

\(\Gamma\) is a closed subset of \(\mathbb{R}^{N-1} \times \{0\}\) with \(C^1,1\)-boundary, and the potential \(f\) satisfies either assumption (H1) or assumption (H2) below. The boundary operators \(\partial^+\) and \(\partial^-\) in (1) are defined as

\[
\begin{aligned}
\partial^+ u := -\frac{\partial}{\partial x_N} \left( u |_{B_R^+} \right) \quad \text{and} \quad \partial^- u := \frac{\partial}{\partial x_N} \left( u |_{B_R^-} \right),
\end{aligned}
\]

where we are denoting, for all \(r > 0,\)

\[
\begin{aligned}
B_r^+ := \{(x', x_N, x_N) \in B_r : x_N > 0\}, \quad B_r^- := \{(x', x_{N-1}, x_N) \in B_r : x_N < 0\},
\end{aligned}
\]

being the total variable \(x \in \mathbb{R}^N\) written as \(x = (x', x_N) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}^\times\).

The interest in elliptic problems in domains with cracks is motivated by elasticity theory, see e.g. [24] [11]. In particular, in crack problems, the coefficients of the asymptotic expansion of solutions near the crack’s tip are related to the so called stress intensity factor, see [11]. We refer to [9] [10] [15] and references therein for the study of the behaviour of solutions at the edge of a cut.

We recall that a family of functions \(F = \{f_i\}_{i \in I},\) with \(f_i : A \to \mathbb{R}, \ A \subseteq \mathbb{R}^N,\) satisfies the strong unique continuation property if no function in \(F,\) besides possibly the trivial null function, has a zero of infinite order at any point \(x_0 \in A.\) The first significant contribution to the study of strong unique continuation for second order elliptic equations was given by Carleman in [8] for bounded potentials in dimension 2, by means of weighted a priori inequalities. The so-called Carleman estimates are still today one of the main techniques used in this research field and have been adapted by many authors to generalize Carleman’s results and prove unique continuation for more general classes of elliptic equations; among the numerous contributions in this area we mention [4] [23] [28] [32] and in particular [23], where strong unique continuation is established under sharp scale invariant assumptions on the potentials. Garofalo and Lin developed in [21] an alternative
approach to the study of unique continuation, based on local doubling inequalities, which are in turn deduced by the monotonicity of an Almgren type frequency function, see [3]. In the present paper we follow this latter approach and study the Almgren frequency function $N$ around the point 0 lying on the edge of the crack, defined as the ratio between the local energy function

$$E(r) := \frac{1}{r^N} \int_{B_r \setminus \Gamma} (|\nabla u|^2 - fu^2) \, dx$$

and the local mass or height

$$H(r) := \frac{1}{r^{N-1}} \int_{\partial B_r} u^2 \, d\sigma,$$

i.e.

$$N(r) := \frac{E(r)}{H(r)}.$$  

The boundedness of the frequency function $N$ will imply a strong unique continuation principle from the edge of $\Gamma$. Furthermore, the monotonicity properties of the quotient $N$ will allow us to obtain energy estimates, which will be combined with a blow-up analysis for scaled solutions to prove that any $u \in H^1(B_R \setminus \Gamma)$ weakly solving (1) behaves, asymptotically at the edge of the crack $\Gamma$, as a homogeneous function with half-integer degree of homogeneity. We mention that an analogous procedure for classifying all possible asymptotic homogeneity degrees of solutions by monotonicity formula and blow-up analysis was introduced in [18, 19, 20] for equations with singular potentials and adapted to domains with corners in [17].

The derivation of a monotonicity formula around a boundary point presents some additional difficulties with respect to the interior case, due to the role that the regularity and the geometry of the domain may play.

Among papers dealing with unique continuation from the boundary under homogeneous Dirichlet conditions we cite [1, 2, 17, 26]. Instead, for Neumann problems, we refer to [1] and [30] for the homogeneous case and to [14] for unique continuation from the vertex of a cone under non-homogeneous Neumann conditions. We also mention that unique continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems was established in [16].

The high non-smoothness of the domain $B_R \setminus \Gamma$ at points on the edge of the crack causes two kinds of difficulties in the proof of the Pohozaev type identity which is needed to estimate the derivative of the Almgren frequency function, see Proposition 5.10. A first difficulty is a lack of regularity that can prevent us from integrating Rellich-Nečas identities of type (74). A second issue is related to the interference with the geometry of the crack, which manifests in the form of extra terms, produced by integration by parts, which could be problematic to estimate.

In [12], where homogeneous Dirichlet conditions on the crack are considered, this latter difficulty is overcome by assuming a local star-shapedness condition for the cracked domain, which forces the extra terms produced by integration by parts to have a sign favourable to the desired estimates. The problem produced by lack of regularity is instead solved in [12] by approximating $B_r \setminus \Gamma$ with a sequence of smooth domains $\Omega_{n,r} \subset B_r$ and constructing approximating problems in $\Omega_{n,r}$, whose solutions $u_n$ converge in $H^1(B_r)$ to the solution of the original cracked problem for $r \in (0, R)$ small enough. Each function $u_n$ is sufficiently regular to satisfy a Pohozaev type identity, in which it is possible to pass to the limit as $n \to \infty$, obtaining the inequality needed to estimate the derivative of the Almgren frequency function.

In the present paper we use a similar approximation technique, which however entails additional difficulties and requires substantial modifications due to the Neumann boundary conditions. In particular, the existence of an extension operator for Sobolev functions on $\Omega_{n}$, uniform with respect to $n$, which is obvious under Dirichlet boundary conditions, turns out to be more delicate in the Neumann case, see Proposition 4.2. Furthermore the different boundary conditions produce remainder terms with different signs, requiring a modified profile for the approximating domains, see Section 4.

Unlike [12], we do not require any geometric star-shapedness condition on the crack $\Gamma$, limiting ourselves to a $C^{1,1}$-regularity assumption, see (5) below. The removal of the star-shapedness condition assumed in [12] requires a more sophisticated monotonicity formula, developed for the
to the main results of this paper, we introduce now our assumptions on the crack $\Gamma$ and the potential $f$. We suppose that $\Gamma$ is a closed set of the form

$$\Gamma := \{(x_1,0) : x_1 \in [0, +\infty)\} \text{ if } N = 2$$

and

$$\Gamma := \{(x',x_{N-1},0) \in \mathbb{R}^N : g(x') \leq x_{N-1}\} \text{ if } N \geq 3,$$  

where

$$g : \mathbb{R}^{N-2} \to \mathbb{R}, \quad g \in C^{1,1}(\mathbb{R}^{N-2}),$$

and

$$g(0) = 0, \quad \nabla g(0) = 0.$$  

Assumption (6) is not restrictive, being a free consequence of an appropriate choice of the Cartesian coordinate system. We are going to study the behaviour of solutions to (1) near $0$, which belongs to the edge of the crack $\Gamma$ defined in (3)–(4).

For every closed set $K \subseteq \mathbb{R}^N \times \{0\}$ and $r > 0$, we define the functional space $H^1_{0,\partial B_r}(B_r \setminus K)$ as the closure in $H^1(B_r \setminus K)$ of the set

$$\{v \in H^1(B_r \setminus K) : v = 0 \text{ in a neighbourhood of } \partial B_r\}.$$  

A weak solution to (1) is a function $u \in H^1(B_R \setminus \Gamma)$ such that

$$\int_{B_R \setminus \Gamma} (\nabla u \cdot \nabla \phi - fu\phi) \, dy = 0,$$

for all $\phi \in H^1_{0,\partial B_R}(B_R \setminus \Gamma)$.

The following unique continuation principle for solutions to (1) is our main result.

**Theorem 1.1.** Let $u$ be a weak solution to (1) with $\Gamma$ as in (3)–(4) and $f$ satisfying either (H1) or (H2). If $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for all $k \in \mathbb{N}$, then $u \equiv 0$ in $B_R$.

In Theorem 7.5 we provide a classification of blow-up limits in terms of the eigenvalues of the following problem

$$-\Delta_{S^{N-1}} \psi = \mu \psi, \quad \text{on } S^{N-1} \setminus \Sigma,$$

$$\frac{\partial^+ \psi}{\partial v^+} = \frac{\partial^- \psi}{\partial v^-} = 0, \quad \text{on } \Sigma,$$

on the unit $(N-1)$-dimensional sphere $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ with a cut on the half-equator

$$\Sigma := \{(x',x_{N-1},0) \in S^{N-1} : x_{N-1} \geq 0\},$$

where, letting $e_N := (0,\ldots,1)$,

$$S^{N-1}_+ := \{(x',x_{N-1},x_N) \in S^{N-1} : x_N > 0\}, \quad S^{N-1}_- := \{(x',x_{N-1},x_N) \in S^{N-1} : x_N < 0\},$$

the boundary operators $\frac{\partial^+}{\partial v^+}$ and $\frac{\partial^-}{\partial v^-}$ are defined as

$$\frac{\partial^+ \psi}{\partial v^+} := -\nabla_{S^{N-1}_+} (\psi|_{S^{N-1}_+}) \cdot e_N \quad \text{and} \quad \frac{\partial^- \psi}{\partial v^-} := \nabla_{S^{N-1}_-} (\psi|_{S^{N-1}_-}) \cdot e_N,$$
see Section 6 for the weak formulation of (7). In Section 6 we prove that the set of the eigenvalues of (7) is \( \{ \mu_k : k \in \mathbb{N} \} \) where
\[
\mu_k = \frac{k(k + 2N - 4)}{4}, \quad k \in \mathbb{N}.
\]
As a consequence of the classification of blow-up limits, we obtain the following unique continuation result from the edge with respect to crack points.

**Theorem 1.2.** Let \( u \) be a weak solution to (1) with \( \Gamma \) as in (3)–(4) and \( f \) satisfying either (H1) or (H2). Let us also assume that \( u \) vanishes at 0 at any order with respect to crack points, namely that either \( \text{Tr}_\Gamma^+ u(z) = O(|z|^6) \) as \( |z| \to 0^+ \), \( z \in \Gamma \), for all \( k \in \mathbb{N} \) or \( \text{Tr}_\Gamma^- u(z) = O(|z|^6) \) as \( |z| \to 0^- \), \( z \in \Gamma \), for all \( k \in \mathbb{N} \), where \( \text{Tr}_\Gamma^+ u \), respectively \( \text{Tr}_\Gamma^- u \), denotes the trace of \( u|_{B_R^+} \), respectively \( u|_{B_R^-} \), on \( \Gamma \). Then \( u \equiv 0 \) in \( B_R \).

If \( N \geq 3 \), a combination of the blow-up analysis with an expansion in Fourier series with respect to a orthonormal basis made of eigenfunctions of (7), allows us to classify the possible asymptotic homogeneity degrees of solutions at 0.

**Theorem 1.3.** Let \( N \geq 3 \) and let \( u \in H^1(B_R \setminus \Gamma) \), \( u \not\equiv 0 \), be a non-trivial weak solution to (1), with \( \Gamma \) defined in (3)–(4) and \( f \) satisfying either assumption (H1) or assumption (H2). Then there exist \( k_0 \in \mathbb{N} \) and an eigenfunction \( Y \) of problem (7), associated to the eigenvalue \( \mu_{k_0} \), such that, letting
\[
\Phi(x) := |x|^\frac{4N-4}{2N} Y \left( \frac{x}{|x|} \right),
\]
we have that
\[
\lambda^{-\frac{4N-4}{2N}} u(\lambda) \to \Phi \quad \text{and} \quad \lambda^{1-\frac{4N}{2N}} \left( \nabla_{B_R \setminus \Gamma} u \right)(\lambda) \to \nabla_{R^N \setminus \tilde{\Gamma}} \Phi \quad \text{in} \quad L^2(B_1)
\]
as \( \lambda \to 0^+ \), where
\[
\tilde{\Gamma} := \{ x = (x', x_{N-1}, 0) \in \mathbb{R}^N : x_{N-1} \geq 0 \}
\]
and \( \nabla_{B_R \setminus \Gamma} \) and \( \nabla_{R^N \setminus \tilde{\Gamma}} \) denote the distributional gradients in \( B_R \setminus \Gamma \) and \( \mathbb{R}^N \setminus \tilde{\Gamma} \) respectively.

A more precise version of Theorem 1.3 relating \( k_0 \) to the limit of a frequency function and characterizing the eigenfunction \( Y \), will be proved in Section 8; see Theorem 8.3.

The paper is organized as follows. In Section 2 an equivalent problem in a domain with a straightened crack is constructed. Sections 3 contains some trace and embedding inequalities for the space \( H^1(B_i \setminus \tilde{\Gamma}) \). Section 4 is devoted to the construction of the approximating problems. In Section 5 we develop the monotonicity argument, which is first used to prove Theorem 1.1 and later, in Section 4, to perform a blow-up analysis and prove Theorem 1.2, taking into account the structure of the spherical eigenvalue problem (7) studied in Section 6. Finally, Theorem 1.3 is proved in Section 8.

## 2. An equivalent problem with straightened crack

In this section we straighten the boundary of the crack in a neighbourhood of 0. If \( N \geq 3 \) we use the local diffeomorphism \( F \) defined in (13) Section 2, see also (11); for the sake of clarity and completeness we summarize its properties in Propositions 2.1 and 2.2 below, referring to [13] Section 2 for their proofs. If \( N = 2 \), the crack is a segment and we simply take \( F = \text{Id} \), where \( \text{Id} \) is the identity function on \( \mathbb{R}^2 \).

**Proposition 2.1.** (13) Section 2] Let \( N \geq 3 \) and \( \Gamma \) be defined in (4) with \( g \) satisfying (5) and (6). There exist \( F = (F_1, \ldots, F_N) \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N) \) and \( r_1 > 0 \) such that \( F|_{B_{r_1}} : B_{r_1} \to F(B_{r_1}) \) is a diffeomorphism of class \( C^{1,1} \),
\[
F(y', 0, 0) = (y', g(y')) , \quad \text{for any} \ y' \in \mathbb{R}^{N-1}, \quad \text{and} \quad F(\tilde{\Gamma} \cap B_{r_1}) = \Gamma \cap F(B_{r_1}),
\]
with $\bar{\beta}$ as in \([5]\). Furthermore, letting $J_F(y)$ be the Jacobian matrix of $F$ at $y = (y', y_{N-1}, y_N) \in B_{r_1}$ and
\begin{equation}
A(y) := |\det J_F(y)|(J_F(y))^{-1}((J_F(y))^{-1})^T,
\end{equation}
the following holds:
i) $J_F$ depends only on the variable $y'' = (y', y_{N-1})$ and
\begin{equation}
J_F(y) = J_F(y'') = \text{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+,
\end{equation}
where $\text{Id}_N$ denotes the identity $N \times N$ matrix and $O(|y''|)$ denotes a matrix with all entries being $O(|y''|)$ as $|y''| \to 0^+$;
ii) $\det J_F(y) = \det J_F(y', y_{N-1}) = 1 + O(|y'|^2) + O(y_{N-1})$ as $|y'| \to 0^+$ and $y_{N-1} \to 0$;
iii) $\frac{\partial F_i}{\partial y_n} = \frac{\partial F_i}{\partial y_y} = 0$ for any $i = 1, \ldots, N - 1$ and $\frac{\partial F_n}{\partial y_n} = 1$;
iv) the matrix-valued function $A$ can be written as
\begin{equation}
A(y) = A(y', y_{N-1}) = \begin{pmatrix}
D(y', y_{N-1}) & 0 \\
0 & \det J_F(y', y_{N-1})
\end{pmatrix},
\end{equation}
with
\begin{equation}
D(y', y_{N-1}) = \begin{pmatrix}
\text{Id}_{N-2} + O(|y'|^2) + O(y_{N-1}) & O(y_{N-1}) \\
O(y_{N-1}) & 1 + O(|y'|^2) + O(y_{N-1})
\end{pmatrix},
\end{equation}
where $\text{Id}_{N-2}$ denotes the identity $(N - 2) \times (N - 2)$ matrix and $O(y_{N-1})$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_{N-1})$ as $y_{N-1} \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$.
v) $A$ is symmetric with coefficients of class $C^{0,1}$ and
\begin{equation}
\|A(y)\|_{C^0(R^N, R^N)} \leq 2 \quad \text{for all } y \in B_{r_1},
\end{equation}
\begin{equation}
\frac{1}{2}|z|^2 \leq A(y)z \cdot z \leq 2|z|^2 \quad \text{for all } z \in \mathbb{R}^N \text{ and } y \in B_{r_1}.
\end{equation}
We observe that
\begin{equation}
A = \text{Id}_2 \quad \text{if } N = 2.
\end{equation}
Moreover \([12]-[13]\) easily imply that
\begin{equation}
A(y) = A(y'') = \text{Id}_N + O(|y''|) \quad \text{as } |y''| \to 0^+.
\end{equation}
Under the same assumptions and with the same notation of Proposition \([2.1]\) we define
\begin{equation}
\mu(y) := \frac{A(y)y \cdot y}{|y|^2} \quad \text{and} \quad \beta(y) := \frac{A(y)y}{\mu(y)} \quad \text{for any } y \in B_{r_1} \setminus \{0\}.
\end{equation}

**Proposition 2.2.** \([13]\) Section 2] Under the same assumptions as Proposition \([2.1]\) let $\mu$ and $\beta$ be as in \([15]\). Then, possibly choosing $r_1$ smaller from the beginning,
\begin{equation}
\frac{1}{2} \leq \mu(y) \leq 2 \quad \text{for any } y \in B_{r_1} \setminus \{0\},
\end{equation}
\begin{equation}
\mu(y) = 1 + O(|y|) \quad \text{as } |y| \to 0^+,
\end{equation}
\begin{equation}
\nabla \mu(y) = O(1) \quad \text{as } |y| \to 0^+.
\end{equation}
Moreover $\beta$ is well-defined and
\begin{equation}
\beta(y) = y + O(|y|^2) = O(|y|) \quad \text{as } |y| \to 0^+,
\end{equation}
\begin{equation}
J_\beta(y) = A(y) + O(|y|) = \text{Id}_N + O(|y|) \quad \text{as } |y| \to 0^+,
\end{equation}
\begin{equation}
\text{div}(\beta)(y) = N + O(|y|) \quad \text{as } |y| \to 0^+.
\end{equation}
We also define $dA(y)zz$, for every $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ and $y \in B_{r_1}$, as the vector of $\mathbb{R}^N$ with $i$-th component, for $i = 1, \ldots, N$, given by

\[(dA(y)zz)_i = \sum_{h,k=1}^N \frac{\partial a_{kh}}{\partial y_i} z_h z_k,\]

where we have defined the matrix $A = (a_{k,h})_{k,h=1,\ldots,N}$ in (10).

**Remark 2.3.** For any measurable function $f : F(B_{r_1}) \to \mathbb{R}$ we set

\[\tilde{f} : B_{r_1} \to \mathbb{R}, \quad \tilde{f} := |\det J_F| (f \circ F).\]

Then, in view of i) and ii) in Proposition 2.1, the function $\tilde{f}$ satisfies assumptions $\text{(H1)}$ or $\text{(H2)}$ on $B_{r_1}$ if and only if $f$ satisfies assumptions $\text{(H1)}$ or $\text{(H2)}$ on $F(B_{r_1})$.

It is easy to see that, if $u$ is a solution to (11), then the function $U := u \circ F$ belongs to $H^1(B_{r_1} \setminus \hat{\Gamma})$ and is a weak solution of the problem

\[
\begin{cases}
- \text{div}(A\nabla U) = \tilde{f}u, & \text{in } B_{r_1} \setminus \hat{\Gamma}; \\
A\nabla^+ U \cdot \nu^+ = A\nabla^- U \cdot \nu^- = 0, & \text{on } \hat{\Gamma},
\end{cases}
\]

where

\[\nabla^+ U = \nabla \left( U \big|_{B_{r_1}^+} \right), \quad \nabla^- U = \nabla \left( U \big|_{B_{r_1}^-} \right), \quad \text{and } \nu^- = -\nu^+ = (0, \ldots, 1).\]

By saying that $U$ a weak solution to (20) we mean that $U \in H^1(B_{r_1} \setminus \hat{\Gamma})$ and

\[
\int_{B_{r_1} \setminus \hat{\Gamma}} (A\nabla U \cdot \nabla \phi - \tilde{f} U \phi) \, dy = 0
\]

for all $\phi \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \hat{\Gamma})$.

3. **Traces and Embeddings for the Space $H^1(B_{r_1} \setminus \hat{\Gamma})$**

In this section, we present some trace and embedding inequalities for the space $H^1(B_{r_1} \setminus \hat{\Gamma})$ which will be used throughout the paper.

We define the even reflection operators

\[
\mathcal{R}^+(v)(y', y_{N-1}, y_N) = v(y', y_{N-1}, |y_N|),
\]

(27)

\[
\mathcal{R}^-(v)(y', y_{N-1}, y_N) = v(y', x_{N-1}, -|y_N|),
\]

(28)

and observe that, for all $r > 0$, $\mathcal{R}^+ : H^1(B_r \setminus \hat{\Gamma}) \to H^1(B_r)$ and $\mathcal{R}^- : H^1(B_r \setminus \hat{\Gamma}) \to H^1(B_r)$. We have that $\mathcal{R}^+(v), \mathcal{R}^-(v) \in L^p(B_r)$ for some $p \in [1, \infty)$ if and only if $v \in L^p(B_r)$; in such a case we have that

\[
\|\mathcal{R}^+(v)\|_{L^p(B_r)} = 2\|v\|_{L^p(B_r)}, \quad \|\mathcal{R}^-(v)\|_{L^p(B_r)} = 2\|v\|_{L^p(B_r)},
\]

and

\[
\|v\|_{L^p(B_r)} = \frac{1}{2} \left( \|\mathcal{R}^+(v)\|_{L^p(B_r)}^p + \|\mathcal{R}^-(v)\|_{L^p(B_r)}^p \right).
\]

Furthermore, for every $v \in H^1(B_r \setminus \hat{\Gamma})$,

\[
\int_{B_r \setminus \hat{\Gamma}} |\nabla v|^2 \, dy = \frac{1}{2} \left( \int_{B_r} |\nabla \mathcal{R}^+(v)|^2 \, dy + \int_{B_r} |\nabla \mathcal{R}^-(v)|^2 \, dy \right).
\]

**Proposition 3.1.** For any $r > 0$ there exists a linear continuous trace operator

\[\gamma_r : H^1(B_r \setminus \hat{\Gamma}) \to L^2(\partial B_r).\]

Furthermore $\gamma_r$ is compact.
Proof. Since $B_r^+$ and $B_r^-$ are Lipschitz domains, there exist two linear, continuous and compact trace operators $\gamma_r^+: H^1(B_r^+) \to L^2(\partial B_r^+ \cap \partial B_r)$ and $\gamma_r^-: H^1(B_r^-) \to L^2(\partial B_r^- \cap \partial B_r)$. By setting
\[
\gamma_r(v)(y) := \begin{cases} 
\gamma_r^+(v)(y), & \text{if } y_N > 0, \\
\gamma_r^-(v)(y), & \text{if } y_N < 0,
\end{cases}
\]
we complete the proof.

Letting $\gamma_r$ be the trace operator introduced in Proposition 3.1, we observe that
\[
\int_{\partial B_r} |\gamma_r(v)|^2 dS = \frac{1}{2} \left( \int_{\partial B_r} |\gamma_r(R^+(v))|^2 dS + \int_{\partial B_r} |\gamma_r(R^-(v))|^2 dS \right)
\]
for every $v \in H^1(B_r \setminus \tilde{\Gamma})$. With a slight abuse of notation we will often write $v$ instead of $\gamma_r(v)$ on $\partial B_r$.

Proposition 3.2. If $N \geq 3$ and $r > 0$, then, for any $v \in H^1(B_r \setminus \tilde{\Gamma})$,
\[
\left( \frac{N-2}{2} \right)^2 \int_{B_r} \frac{v^2}{|x|^4} dx \leq \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 dx + \frac{N-2}{2r} \int_{\partial B_r} v^2 dS.
\]

Proof. By scaling, Theorem 1.1] proves the claim for $R^+(v)$ and $R^-(v)$. Then we conclude by (30), (31), and (32).

Proposition 3.3. Let $N \geq 2$ and $q \geq 1$ be such that $1 \leq q \leq 2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $1 \leq q < \infty$ if $N = 2$. Then
\[
H^1(B_r \setminus \tilde{\Gamma}) \subset L^q(B_r)
\]
and there exists $S_{N,q} > 0$ (depending only on $N$ and $q$) such that
\[
\|v\|_{L^q(B_r)} \leq S_{N,q} r^{\frac{N(2-q)+2q}{q}} \left( \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right),
\]
for all $r > 0$ and $v \in H^1(B_r \setminus \tilde{\Gamma})$.

Proof. Since
\[
\left( \int_{B_r} |\nabla v|^2 dx + \int_{\partial B_r} v^2 dS \right)^{\frac{1}{2}}
\]
is an equivalent norm on $H^1(B_r)$, from a scaling argument and Sobolev embedding Theorems it follows that, for all $q \in [1,2^*]$ if $N \geq 3$ and $q \in [1,\infty]$ if $N = 2$, there exists $S_{N,q} > 0$ such that, for all $r > 0$ and $v \in H^1(B_r)$,
\[
\|v\|_{L^q(B_r)} \leq S_{N,q} r^{\frac{N(2-q)+2q}{q}} \left( \int_{B_r} |\nabla v|^2 dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right).
\]
Using (29), (30), (31) and (32) we complete the proof.

Proposition 3.4. For any $r > 0$, $h \in L^{\frac{N+4\epsilon}{N-2+2\epsilon}}(B_r)$ with $\epsilon > 0$, and $v \in H^1(B_r \setminus \tilde{\Gamma})$, there holds
\[
\int_{B_r} |h| v^2 \leq \eta_h(r) \left( \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right),
\]
where
\[
\eta_h(r) = S_{N,q} \|h\|_{L^{\frac{N+4\epsilon}{N-2+2\epsilon}}(B_r)} r^{\frac{4\epsilon}{N-4\epsilon}} \quad \text{and} \quad q_e := \frac{2N+4\epsilon}{N-2+2\epsilon}.
\]

Proof. For any $v \in H^1(B_r \setminus \tilde{\Gamma})$
\[
\int_{B_r} |h| v^2 dx \leq \|h\|_{L^{\frac{N+4\epsilon}{N-2+2\epsilon}}(B_r)} \left( \int_{B_r} |v|^{q_e} dx \right)^{2/q_e} \leq S_{N,q_e} \|h\|_{L^{\frac{N+4\epsilon}{N-2+2\epsilon}}(B_r)} r^{\frac{4\epsilon}{N-4\epsilon}} \left( \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 dx + \frac{1}{r} \int_{\partial B_r} v^2 dS \right).
\]
thanks to Hölder inequality and (34).

\[ \square \]

**Remark 3.5.** If \( f \) satisfies (12), then \( f \in L^{2+\epsilon}(B_R) \), so that Proposition 3.4 applies to potentials satisfying either (11) or (12).

**Remark 3.6.** By (35), (19) and (15), for any \( r \in (0, r_1) \), \( h \in L^{2+\epsilon}(B_r) \), and \( v \in H^1(B_r \setminus \tilde{\Gamma}) \), we have that

\[
\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dy \leq 2 \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - hv^2) \, dy + 2\eta_h(r) \left( \int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dy + \frac{2}{r} \int_{\partial B_r} \mu \nu^2 \, dS \right)
\]

and therefore, if \( \eta_h(r) < \frac{1}{2} \),

\begin{equation}
\int_{B_r \setminus \tilde{\Gamma}} |\nabla v|^2 \, dy \leq \frac{2}{1 - 2\eta_h(r)} \int_{B_r \setminus \tilde{\Gamma}} (A \nabla v \cdot \nabla v - hv^2) \, dy + \frac{4\eta_h(r)}{(1 - 2\eta_h(r))r} \int_{\partial B_r} \mu \nu^2 \, dS.
\end{equation}

### 4. Approximating Problems

In this section we construct a sequence of problems in smooth sets approximating the straightened cracked domain. We define, for any \( n \in \mathbb{N} \setminus \{0\} \),

\[ g_n : \mathbb{R} \rightarrow \mathbb{R}, \quad g_n(t) := nt^4 \]

and, for any \( r \in (0, r_1) \),

\[ \Omega_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} < g_n(y_N)\} \]

and

\[ \Gamma_{n,r} := \{(y', y_{N-1}, y_N) \in B_r : y_{N-1} = g_n(y_N)\} = \partial \Omega_{n,r} \cap B_r. \]

It is clear that, for every \( y \in B_r \setminus \tilde{\Gamma} \), there exists a \( n \in \mathbb{N} \setminus \{0\} \) such that \( y \in \Omega_{n,r} \) for all \( n \geq n \). Moreover \( \Omega_{n,r} \cap \tilde{\Gamma} = \emptyset \) for any \( r \in (0, r_1) \) and \( n \in \mathbb{N} \setminus \{0\} \). We also note that \( \Omega_{n,r} \) is a Lipschitz domain and \( \Gamma_{n,r} \) is a \( C^2 \)-smooth portion of its boundary.

**Proposition 4.1.** Let \( \nu(y) \) be the outward normal vector to \( \partial \Omega_{n,r_1} \) in \( y \). Then

\begin{align}
\nu(y) \cdot y &\leq 0 \quad \text{for all } y \in \Gamma_{n,r_1}, \\
A(y) \nu(y) \cdot y &\leq 0 \quad \text{for all } y \in \Gamma_{n,r_1}.
\end{align}

**Proof.** As a first step we notice that

\begin{equation}
g_n(t) - \frac{1}{3} g_n'(t) = nt^4 - \frac{4}{3} nt^3 = -\frac{1}{3} nt^4 \leq 0, \quad g_n(t) - \frac{1}{3} g_n'(t) \leq 0
\end{equation}

and that

\[ \nu(y) = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \quad \text{for all } y \in \Gamma_{n,r_1}. \]

Then, for all \( y \in \Gamma_{n,r_1} \),

\[ \nu(y) \cdot y = \frac{(0, 1, -g_n'(y_N))}{\sqrt{1 + (g_n'(y_N))^2}} \cdot (y', g_n(y_N), y_N) = \frac{g_n(y_N) - y_N g_n'(y_N)}{\sqrt{1 + (g_n'(y_N))^2}} \leq 0 \]

due to (40). We have then proved (35) (and 39 in the case \( N = 2 \) in view of (10)).

If \( N \geq 3 \), possibly choosing \( r_1 \) smaller in Proposition 2.1, for all \( y \in \Gamma_{n,r_1} \) we have that

\[ \sqrt{1 + (g_n'(y_N))^2} A(y) \nu(y) = g_n(y_N) (1 + O(|y'|) + O(y_{N-1}')) - \det JF(y) y_N g_n'(y_N) \leq \frac{3}{2} g_n(y_N) - \frac{1}{2} y_N g_n'(y_N) = \frac{3}{2} g_n(y_N) - \frac{1}{3} y_N g_n'(y_N), \]

thanks to ii) in Proposition 2.1 (12) and (13). Then, by (10) we finally obtain (39) also for \( N \geq 3 \).
Let
\[
\mathbb{R}_+^N := \{ y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N > 0 \} \quad \text{and} \quad \mathbb{R}_-^N := \{ y = (y', y_{N-1}, y_N) \in \mathbb{R}^N : y_N < 0 \}.
\]
For any \( r \in (0, r_1) \) and \( n \in \mathbb{N} \setminus \{0\} \) let
\[
\Omega_{n,r}^+ := \Omega_{n,r} \cap B_r^+, \quad \Omega_{n,r}^- := \Omega_{n,r} \cap B_r^-,
\]
\( S_{n,r} := \partial \Omega_{n,r} \cap \partial B_r \).
For all \( n \in \mathbb{N} \setminus \{0\} \) we also define
\[
K_{n,r_1}^+ := \{ y = (y', y_{N-1}, y_N) \in \mathbb{R}_+^N : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1 \},
\]
\[
K_{n,r_1}^- := \{ y = (y', y_{N-1}, y_N) \in \mathbb{R}_-^N : \text{either } y_{N-1} < g_n(y_N) \text{ or } |y| > r_1 \}.
\]
Since \( \Omega_{n,r} \) is a Lipschitz domain, for any \( r \in (0, r_1) \) and \( n \in \mathbb{N} \setminus \{0\} \) there exists a trace operator
\[
\gamma_{n,r} : H^1(\Omega_{n,r}) \to L^2(\partial \Omega_{n,r}).
\]
We define
\[
H^1_{0,S_{n,r}}(\Omega_{n,r}) := \{ u \in H^1(\Omega_{n,r}) : \gamma_{n,r}(u) = 0 \text{ on } S_{n,r} \}.
\]
The following proposition provides an extension operator from \( H^1_{0,S_{n,r}}(\Omega_{n,r}) \) to \( H^1(B_{r_1} \setminus \tilde{\Gamma}) \) with an operator norm bounded uniformly with respect to \( n \).

**Proposition 4.2.** For any \( r \in (0, r_1) \) and \( n \in \mathbb{N} \setminus \{0\} \) there exists an extension operator
\[
\xi_{n,r}^0 : H^1_{0,S_{n,r}}(\Omega_{n,r}) \to H^1(B_{r_1} \setminus \tilde{\Gamma})
\]
such that, for any \( \phi \in H^1_{0,S_{n,r}}(\Omega_{n,r}) \),
\[
(44) \quad \xi_{n,r}^0(\phi) |_{\Omega_{n,r}} = \phi, \quad \xi_{n,r}^0(\phi) = 0 \text{ on } \Omega_{n,r} \setminus \Omega_{n,r}, \quad \xi_{n,r}^0(\phi) \in H^1_{0,\partial B_{r_1}}(B_{r_1} \setminus \tilde{\Gamma}),
\]
and
\[
(45) \quad \|\xi_{n,r}^0(\phi)\|_{H^1(B_{r_1} \setminus \tilde{\Gamma})} \leq c_0 \|\phi\|_{H^1(\Omega_{n,r})} = c_0 \left( \int_{\Omega_{n,r}} (\phi^2 + |\nabla \phi|^2) \, dy \right)^{1/2},
\]
where \( c_0 > 0 \) is independent of \( n, r \), and \( \phi \).

**Proof.** It is well known that, since \( K_{n,r_1}^+ \) and \( K_{n,r_1}^- \) are uniformly Lipschitz domains, there exist continuous extension operators \( \xi_{n,r}^+ : H^1(K_{n,r_1}^+) \to H^1(\mathbb{R}_+^N) \) and \( \xi_{n,r}^- : H^1(K_{n,r_1}^-) \to H^1(\mathbb{R}_-^N) \), see \([29, 7, 27]\). Furthermore, since the Lipschitz constants of the parameterization of \( \partial K_{n,r_1}^+ \) and \( \partial K_{n,r_1}^- \) are uniformly bounded with respect to \( n \), there exists a constant \( C > 0 \), which does not depend on \( n \), such that
\[
(46) \quad \|\xi_{n,r}^+(v)\|_{H^1(\mathbb{R}_+^N)} \leq C \|v\|_{H^1(K_{n,r_1}^+)} \quad \text{and} \quad \|\xi_{n,r}^-(w)\|_{H^1(\mathbb{R}_-^N)} \leq C \|w\|_{H^1(K_{n,r_1}^-)}
\]
for all \( v \in H^1(K_{n,r_1}^+) \) and \( w \in H^1(K_{n,r_1}^-) \).

If \( \phi \in H^1_{0,S_{n,r}}(\Omega_{n,r}) \) then the trivial extension \( \tilde{\phi}_+ \) of \( \phi |_{\Omega_{n,r}} \) to \( K_{n,r_1}^+ \) belongs to \( H^1(K_{n,r_1}^+) \) and the trivial extension \( \tilde{\phi}_- \) of \( \phi |_{\Omega_{n,r}} \) to \( K_{n,r_1}^- \) belongs to \( H^1(K_{n,r_1}^-) \). Then we define
\[
(47) \quad \xi_{n,r}^0(\phi)(y) := \begin{cases} \xi_{n,r}^+(\tilde{\phi}_+(y)), & \text{if } y \in B_{r_1}^+, \\ \xi_{n,r}^-(\tilde{\phi}_-(y)), & \text{if } y \in B_{r_1}^-. \end{cases}
\]
which belongs to \( H^1(B_{r_1} \setminus \tilde{\Gamma}) \) and satisfies \([45]\) in view of \([46]\). Furthermore \([44]\) follows directly from the definition of \( \xi_{n,r}^0 \). □

The following proposition establishes a Poincaré type inequality for \( H^1_{0,S_{n,r}}(\Omega_{n,r}) \)-functions, with a constant independent of \( n \).
Proposition 4.3. For any \( r \in (0, r_1] \), \( n \in \mathbb{N} \setminus \{0\} \), and \( \phi \in H^1_{0, S_{n,r}}(\Omega_{n,r}) \)

\[
\int_{\Omega_{n,r}} \phi^2 dy \leq \frac{r_1^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 dy
\]

and

\[
\|\phi\|_{H^1_{0, S_{n,r}}(\Omega_{n,r})} := \left( \int_{\Omega_{n,r}} |\nabla \phi|^2 dy \right)^{\frac{1}{2}}
\]

is an equivalent norm on \( H^1_{0, S_{n,r}}(\Omega_{n,r}) \).

Proof. For any \( \phi \in C^\infty(\overline{\Omega_{n,r}}) \) such that \( \phi = 0 \) in a neighbourhood of \( S_{n,r} \) we have that

\[
\text{div}(\phi^2 y) = 2\phi \nabla \phi \cdot y + N\phi^2
\]

so that

\[
N \int_{\Omega_{n,r}} \phi^2 dy = -2 \int_{\Omega_{n,r}} \phi \nabla \cdot (\phi y) dy + \int_{\Gamma_{n,r}} \phi^2 y \cdot \nu dS \leq \int_{\Omega_{n,r}} \phi^2 dy + r^2 \int_{\Omega_{n,r}} |\nabla \phi|^2 dy,
\]

since \( y \cdot \nu \leq 0 \) on \( \Gamma_{n,r} \) by \( [35] \). Then we may conclude that

\[
\int_{\Omega_{n,r}} \phi^2 dy \leq \frac{r_1^2}{N-1} \int_{\Omega_{n,r}} |\nabla \phi|^2 dy
\]

for all \( \phi \in C^\infty(\overline{\Omega_{n,r}}) \) such that \( \phi = 0 \) in a neighbourhood of \( S_{n,r} \). Since \( \Omega_{n,r} \) is a Lipschitz domain, \( [17] \) holds for any \( \phi \in H^1_{0, S_{n,r}}(\Omega_{n,r}) \) by \( [6] \) Theorem 3.1. The second claim is now obvious. \( \square \)

From now on we consider on \( H^1_{0, S_{n,r}}(\Omega_{n,r}) \) the norm \( \| \cdot \|_{H^1_{0, S_{n,r}}} \) defined in \( [13] \).

Proposition 4.4. Let \( r \in (0, r_1] \), \( n \in \mathbb{N} \setminus \{0\} \), \( h \in L^{\frac{N}{N-1}}(B_r) \) with \( \epsilon > 0 \), and \( q_r \) be as in \( [30] \). Then, for any \( \phi \in H^1_{0, S_{n,r}}(\Omega_{n,r}) \),

\[
\int_{\Omega_{n,r}} |h| \phi^2 dy \leq c_\phi^N \left( \frac{N-1+r^2}{N-1} \right) S_{N,q_r} r_1^{\frac{4q_r}{N-1-q_r}} \|h\|_{L^{\frac{N}{N-1}}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 dy.
\]

Proof. We have, for every \( \phi \in H^1_{0, S_{n,r}}(\Omega_{n,r}) \),

\[
\int_{\Omega_{n,r}} |h|^2 \phi^2 dy \leq \int_{B_r} |h| \phi_{n,r}^\epsilon(\phi) |\nabla \phi|^2 dy \leq \|h\|_{L^{\frac{N}{N-1}}(B_r)} \left( \int_{B_{r_1}} \phi_{n,r}^\epsilon(\phi) |\nabla \phi|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq \frac{c_\phi^N}{N-1+r^2} S_{N,q_r} r_1^{\frac{4q_r}{N-1-q_r}} \|h\|_{L^{\frac{N}{N-1}}(B_r)} \int_{\Omega_{n,r}} |\nabla \phi|^2 dy,
\]

thanks to H"{o}lder’s inequality, \( [34] \), Proposition 4.2 and Proposition 4.3. \( \square \)

Hereafter we fix a potential \( f \) satisfying either \( [H_1] \) or \( [H_2] \) and define \( \tilde{f} := |\det J_F|(f \circ F) \) as in Remark 2.3. Thanks to Remark 2.3 we have that \( \tilde{f} \) satisfies either \( [H_1] \) or \( [H_2] \) as well. If \( f \) (and consequently \( \tilde{f} \)) satisfies \( [H_2] \), we define

\[
f_n(y) = \begin{cases} n, & \text{if } \tilde{f}(y) > n, \\ \tilde{f}(y), & \text{if } |\tilde{f}(y)| \leq n, \\ -n, & \text{if } \tilde{f}(y) < -n, \end{cases}
\]

so that

\[
f_n \in L^\infty(B_{r_1}) \quad \text{and} \quad |f_n| \leq |\tilde{f}| \text{ a.e. in } B_{r_1} \quad \text{for all } n \in \mathbb{N} \setminus \{0\}
\]

and

\[
f_n \rightarrow \tilde{f} \text{ a.e. in } B_{r_1}.
\]
If \( f \) satisfies (H1), we just let
\[
(53) \quad f_n := \tilde{f} \quad \text{for any } n \in \mathbb{N}.
\]
We observe that
\[
(54) \quad f_n \to \tilde{f} \quad \text{in } L^{\frac{N}{N+2}}(B_{r_1}) \quad \text{as } n \to \infty
\]
as a consequence of (51), (52) and the Dominated Convergence Theorem if assumption (H2) holds
and \( f_n \) is defined in (50), in view of Remark 3.5 on the other hand (54) is obvious if assumption (H1)
holds and \( f_n \) is defined in (52).

Since under both assumptions (H1) and (H2) we have that \( \tilde{f} \in L^{\frac{N}{N+2}}(B_{r_1}) \) (see Remark 3.5),
by the absolute continuity of the Lebesgue integral we can choose \( r_0 \in (0, \min\{1, r_1\}) \) such that
\[
(55) \quad \eta_f(r_0) < \frac{1}{2} \quad \text{and} \quad \frac{c_0 N - 1 + r_1^{N-r}}{N-1} S_{N,q, r_1^{N-r}} \| \tilde{f} \|_{L^{\frac{N}{N+2}}(B_{r_1})} < \frac{1}{4},
\]
where \( q_c \) and \( \eta_f \) are defined in (56).

Let \( U = u \circ F \), where \( u \) is a fixed weak solution to (H1) and \( F \) is the diffeomorphism introduced
in Section 2, so that \( U \) weakly solves (20). For any \( n \in \mathbb{N} \setminus \{0\} \), we consider the following sequence
of approximating problems, with potentials \( f_n \) defined in (50)–(53):
\[
(56) \quad \begin{cases}
- \text{div}(A \nabla U_n) = f_n U_n, & \text{in } \Omega_{n,r_0}, \\
A \nabla U_n \cdot \nu = 0, & \text{on } \Gamma_{n,r_0}, \\
\gamma_{n,r_0}(U_n) = \gamma_{n,r_0}(U), & \text{on } S_{n,r_0},
\end{cases}
\]
with \( r_0 \) as in (55). A weak solution to problem (56) is a function \( U_n \in H^1(\Omega_{n,r_0}) \) such that
\( U_n - U \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \) and
\[
\int_{\Omega_{n,r_0}} (A \nabla U_n \cdot \nabla \phi - f_n U_n \phi) \, dy = 0
\]
for all \( \phi \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \). If \( U_n \) weakly solves (56), then \( W_n := U - U_n \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \) and
\[
(57) \quad \int_{\Omega_{n,r_0}} (A \nabla W_n \cdot \nabla \phi - f_n W_n \phi) \, dy = \int_{\Omega_{n,r_0}} (A \nabla U \cdot \nabla \phi - f_n U \phi) \, dy
\]
for any \( \phi \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \).

For every \( n \in \mathbb{N} \setminus \{0\} \), let us consider the bilinear form
\[
(58) \quad B_n : H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \times H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad B_n(v, \phi) := \int_{\Omega_{n,r_0}} (A \nabla v \cdot \nabla \phi - f_n v \phi) \, dy,
\]
and the functional
\[
(59) \quad L_n : H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}) \to \mathbb{R}, \quad L_n(\phi) := \int_{\Omega_{n,r_0}} (A \nabla U \cdot \nabla \phi - f_n U \phi) \, dy.
\]

**Proposition 4.5.** The bilinear form \( B_n \) defined in (58) is continuous and coercive; more precisely
\[
(60) \quad B_n(\phi, \phi) \geq \frac{1}{4} \| \phi \|_{H^1_{0, S_{n,r_0}}(\Omega_{n,r_0})}^2 \quad \text{for all } \phi \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}).
\]
Furthermore the functional \( L_n \) defined in (59) belongs to \( (H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}))^* \) and there exists a constant \( \ell > 0 \) independent of \( n \) such that
\[
(61) \quad |L_n(\phi)| \leq \ell \| \phi \|_{H^1_{0, S_{n,r_0}}(\Omega_{n,r_0})} \quad \text{for all } \phi \in H^1_{0, S_{n,r_0}}(\Omega_{n,r_0}).
\]
**Proof.** The continuity of $B_\ell$ and (60) easily follow from (15), (51), (49) and (55). Thanks to H"older’s inequality, (51), (14), (35), (49) and (55)

\[ |L_n(\phi)| \leq 2 \|\nabla U\|_{L^2(\Omega_n,r_0)} \|\phi\|_{H^1_{0,S_n,r_0}(\Omega_n,r_0)} + \left( \int_{B_{r_0}} |f|^2 dx \right)^\frac{1}{2} \left( \int_{\Omega_n,r_0} |f|\phi^2 dx \right)^\frac{1}{2} \]

\[
\leq \left( 2 \|\nabla U\|_{L^2(B_{r_0}\setminus\tilde{\Gamma})} + \frac{1}{2} \sqrt{\eta f(r_0)} \left( \int_{B_{r_0}\setminus\tilde{\Gamma}} |\nabla U|^2 dx + \frac{1}{r_0} \int_{\partial B_{r_0}} U^2 dS \right)^\frac{1}{2} \right) \|\phi\|_{H^1_{0,S_n,r_0}(\Omega_n,r_0)},
\]

thus implying (61). \[\square\]

**Corollary 4.6.** Let $u$ be a weak solution to (11) and $U = u \circ F$. Let either (H1) hold and $\{f_n\}$ be as in (55), or (H2) hold and $\{f_n\}$ be as in (50). Let $r_0$ be as in (55) and $\ell$ be as in Proposition 4.5. Then, for any $n \in \mathbb{N} \setminus \{0\}$, there exists a solution $W_n \in H^1_{0,S_n,r_0}(\Omega_n,r_0)$ of (57) such that

\[\|W_n\|_{H^1_{0,S_n,r_0}(\Omega_n,r_0)} \leq 4\ell.\]

**Proof.** The existence of a solution $W_n$ of (57) follows from the Lax-Milgram Theorem, taking into account Proposition 4.5. Estimate (62) follows from (60) and (61) with $\phi = W_n$. \[\square\]

We are now in position to prove the main result of this section.

**Theorem 4.7.** Suppose that $f$ satisfies either (H1) or (H2), $u$ is a weak solution of (11), and $U = u \circ F$ with $F$ as in Section 3. Let $\{f_n\}_{n \in \mathbb{N}}$ satisfies (53) under hypothesis (H1) or (50) under hypothesis (H2). Let $r_0 \in (0,r_1)$ be as in (55). Then there exists $\{U_n\}_{n \in \mathbb{N} \setminus \{0\}} \subset H^1(B_{r_0} \setminus \Gamma)$ such that $U_n$ weakly solves (56) for any $n \in \mathbb{N} \setminus \{0\}$ and $U_n \to U$ in $H^1(B_{r_0} \setminus \Gamma)$ as $n \to \infty$. Furthermore $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0,r_0)$ and $n \in \mathbb{N} \setminus \{0\}$.

**Proof.** Let $r_0 \in (0,r_1)$ be as in (55). For any $n \in \mathbb{N} \setminus \{0\}$, let $W_n \in H^1_{0,S_n,r_0}(\Omega_n,r_0)$ be the solution to (57) given by Corollary 4.6. Then $U - W_n$ weakly solves problem (56) and we define $U_n := U - \xi_{n,r_0}^0(W_n)$, with $\xi_{n,r_0}^0$ being the extension operator introduced in Proposition 4.2. We observe that $U_n \in H^1(B_{r_0} \setminus \tilde{\Gamma})$. To prove that $U_n$ converges to $U$ in $H^1(B_{r_0} \setminus \tilde{\Gamma})$ as $n \to \infty$, we notice that

\[\|U - U_n\|^2_{H^1(B_{r_0}\setminus\tilde{\Gamma})} \leq c_0^2 \|W_n\|^2_{H^1(\Omega_n,r_0)} \leq 4 c_0^2 \frac{N - 1 + r_0^2}{N - 1} \int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2) dy,
\]

by Proposition 4.2, (47), and (69). Therefore it is enough to prove that

\[\lim_{n \to \infty} \int_{\Omega_{n,r_0}} (A\nabla W_n \cdot \nabla W_n - f_n W_n^2) dy = 0.
\]

Let

\[O_n := (B_{r_1} \setminus \tilde{\Gamma}) \setminus \Omega_{n,r_1}.
\]

\[O_n := (B_{r_1} \setminus \tilde{\Gamma}) \setminus \Omega_{n,r_1}.
\]
for any $n \in \mathbb{N} \setminus \{0\}$. Since $W_n \in H^1_{\Gamma, r_0}(\Omega_n, \rho_0)$ solves \((57)\) and $U$ is a solution to \((26)\), by Hölder’s inequality, \((13)\) and Proposition \((4.2)\) we have that

\[
\left| \int_{\Omega_n, r_0} (A \nabla W_n \cdot \nabla W_n - f_n W_n^2) \, dy \right| = \left| \int_{\Omega_{n, r_1}} (A \nabla U \cdot \nabla (\xi_{n, r_0}^0(W_n)) - f_n U \xi_{n, r_0}^0(W_n)) \, dy \right|
\]

\[
= \left| \int_{B_{r_1}} (A \nabla U \cdot \nabla (\xi_{n, r_0}^0(W_n)) - f_n U \xi_{n, r_0}^0(W_n)) \, dy \right| \]

\[
- \int_{\Omega_n} (A \nabla U \cdot \nabla (\xi_{n, r_0}^0(W_n)) - f_n U \xi_{n, r_0}^0(W_n)) \, dy \right| \]

\[
= \left| \int_{B_{r_1}} (A \nabla U \cdot \nabla (\xi_{n, r_0}^0(W_n)) - f_n U \xi_{n, r_0}^0(W_n)) \, dy \right| \]

\[
+ \int_{B_{r_1}} (\tilde{f} - f_n) U \xi_{n, r_0}^0(W_n)) \, dy \right| \]

\[
- \int_{\Omega_n} (A \nabla U \cdot \nabla (\xi_{n, r_0}^0(W_n)) - f_n U \xi_{n, r_0}^0(W_n)) \, dy \right| \]

\[
\leq 2 \left\| \nabla U \right\|_{L^2(\Omega_n)} \left\| \nabla \xi_{n, r_0}^0(W_n) \right\|_{L^2(\Omega_{n, r_1} \setminus \bar{\Gamma})} + \left\| f_n \right\|_{L^\infty(\Omega_n)} \left\| \xi_{n, r_0}^0(W_n) \right\|_{L^\infty(\Omega_{n, r_1} \setminus \bar{\Gamma})} \]

\[
+ \left\| \tilde{f} - f_n \right\|_{L^\infty(\Omega_{n, r_1} \setminus \bar{\Gamma})} \left\| \nabla \xi_{n, r_0}^0(W_n) \right\|_{L^\infty(\Omega_{n, r_1} \setminus \bar{\Gamma})} \]

\[
\leq 4c_0 \frac{(N - 1 + r_0^2)}{\sqrt{N - 1}} \left( 2 \left\| \nabla U \right\|_{L^2(\Omega_n)} + \sqrt{S_{N, dr_1}} \left\| \tilde{f} \right\|_{L^\infty(\Omega_n)} \left\| \nabla \xi_{n, r_0}^0(W_n) \right\|_{L^\infty(\Omega_{n, r_1} \setminus \bar{\Gamma})} \right),
\]

where $|O_n|$ is the $N$-dimensional Lebesgue measure of $O_n$. Then, since $\nabla U \in L^2(B_{r_1} \setminus \bar{\Gamma})$, $U \in L^{\infty}(B_{r_1})$ by Proposition \((5.3)\) and $\tilde{f} \in L^{\frac{2N}{N - 1}}(B_{r_1})$, \((63)\) follows by the absolute continuity of the integral and convergence \((54)\).

We observe that $f_n U_n \in L^2(\Omega_{n, r_0})$. Indeed, under assumption \((11)\), by Remark \((2.3)\), we have that $\tilde{f} \in W^{1, \frac{2N}{N - 1}}(B_{r_1} \setminus \bar{\Gamma})$ and then, by Sobolev embeddings and Hölder’s inequality, easily obtain that $f_n U_n = \tilde{f} U_n \in L^2(\Omega_{n, r_0})$. Under assumption \((12)\), $f_n$ is defined in \((50)\) and $f_n \in L^\infty(B_{r_1})$, hence $f_n U_n \in L^2(\Omega_{n, r_0})$.

Since $\Gamma_{n, r_0}$ is $C^\infty$-smooth and $f_n U_n \in L^2(\Omega_{n, r_0})$, by classical elliptic regularity theory, see e.g. [22] Theorem 2.2.2.5], we deduce that $U_n \in H^1(\Omega_{n, r})$ for any $r \in (0, r_0)$. The proof is thereby complete.

\[
5. \ The \ Almgren \ Type \ Frequency \ Function
\]

Let $u \in H^1(B_{r_1} \setminus \bar{\Gamma})$ be a non-trivial weak solution to \((1)\) and $U = u \circ F \in H^1(B_{r_1} \setminus \bar{\Gamma})$ be the corresponding solution to \((26)\). Let $r_0 \in (0, \min\{1, r_1\})$ be as in \((55)\). For any $r \in (0, r_0]$, we define

\[
H(r) := \frac{1}{r^{N - 1}} \int_{\partial B_r} \mu U^2 \, dS,
\]

where $\mu$ is the function introduced in \((18)\), and

\[
E(r) := \frac{1}{r^{N - 1}} \int_{B_r \setminus \bar{\Gamma}} (A \nabla U \cdot \nabla U - \tilde{f} U^2) \, dy.
\]

Proposition 5.1. If $r \in (0, r_0]$ then $H(r) > 0$. 

Remark 5.4. Provided by Theorem 4.7.

Proof. For any \( r \in (0, r_0) \) such that \( H(r) = 0 \). By (19), it follows that \( U \) weakly solves (20) with the extra condition \( U = 0 \) on \( \partial B_r \). Then by (57) we obtain that \( U = 0 \) on \( B_r \). By classical uniqueness continuation principles for elliptic equations, see e.g. [21], we conclude that \( u = 0 \) on \( BR \), which is a contradiction.

Proposition 5.2. We have that \( H \in W^{1,1}_{\text{loc}}((0, r_0]) \) and

\[
H'(r) = \frac{1}{r^{N-1}} \left( 2 \int_{\partial B_r} \mu U \frac{\partial U}{\partial v} \, dS + \int_{\partial B_r} U^2 \nabla \mu \cdot \nu \, dS \right)
\]

for \( r \to 0^+ \), in a distributional sense and for a.e. \( r \in (0, r_0) \).

Remark 5.3. To explain in what sense the term \( \frac{\partial U}{\partial v} \) in (67) is meant, we observe that, if \( \nabla U \) is the distributional gradient of \( U \) in \( B_{r_1} \setminus \tilde{\Gamma} \), then \( \nabla U \in L^2(B_{r_1}, \mathbb{R}^N) \) and \( \frac{\partial U}{\partial v} := \nabla U \cdot \frac{\nu}{|\nu|} \in L^2(B_{r_1}) \). By the Coarea Formula it follows that \( \nabla U \in L^2(\partial B_{r_1}, \mathbb{R}^N) \) and \( \frac{\partial U}{\partial v} \in L^2(\partial B_r) \) for a.e. \( r \in (0, r_1) \).

Proof. For any \( \phi \in C^\infty_0((0, r_0]) \) we define \( v(y) := \phi(|y|) \). Then we have

\[
\int_0^r H(r) \phi(r) \, dr = \int_0^r \frac{1}{r^{N-1}} \left( \int_{\partial B_r} \mu U^2 \, dS \right) \phi(r) \, dr
\]

\[
= \int_{B_{r_0}} \frac{1}{|y|^N} \mu(y) U^2(y) \nabla v(y) \cdot y \, dy + \int_{B_{r_0}} \frac{1}{|y|^N} \mu(y) U^2(y) \nabla v(y) \cdot y \, dy
\]

\[
= -\int_{B_{r_0} \setminus \tilde{\Gamma}} \frac{1}{|y|^N} (\mu(y) U^2(y) \nabla v(y) \cdot y + \nabla v U(y) \cdot y + v U(\nabla \mu)(y) \cdot y) \, dy
\]

which proves (67) thanks to (21). Since \( r^{-N+1} \) is bounded in any compact subset of \( (0, r_0] \), then, by (19), (21) and the Coarea Formula, \( H \) and \( H' \) are locally integrable so that \( H \in W^{1,1}_{\text{loc}}((0, r_0]) \).

Now we turn our attention to \( E \). Henceforth we let \( \{f_n\} \) be as in (53), if \( f \) satisfies (11), or as in (50), if \( f \) satisfies (12), and we consider the sequence \( \{U_n\} \) converging to \( U \) in \( H^1(B_{r_0} \setminus \tilde{\Gamma}) \) provided by Theorem 4.7.

Remark 5.4. By Proposition 3.3 and (36), \( U_n \to U \) in \( L^p(B_r \setminus \tilde{\Gamma}) \). Then, since \( f_n \to \tilde{f} \) in \( L^{N+\epsilon}(B_{r_0}) \) by (54), from Hölder’s inequality it easily follows that

\[
\lim_{n \to \infty} \int_{B_{r_0}} |\tilde{f} U^2 - f_n U_n^2| \, dy = 0.
\]

Moreover, if \( f \) satisfies (11), \( \nabla \tilde{f} \in L^{N+\epsilon}(B_{r_0}, \mathbb{R}^N) \) and hence

\[
\lim_{n \to \infty} \int_{B_{r_0} \setminus \tilde{\Gamma}} |(\nabla \tilde{f} \cdot \beta) (U^2 - U_n^2)| \, dx = 0,
\]

since the vector field \( \beta \) defined in (11S) is bounded in view (22).

Lemma 5.5. If \( F_n \to F \) in \( L^1(B_{r_0}) \), then there exists a subsequence \( \{F_{n_k}\}_{k \in \mathbb{N}} \) such that, for a.e. \( r \in (0, r_0) \),

\[
\lim_{k \to \infty} \int_{B_r} |F - F_{n_k}| \, dS = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_{S_{n_k,r}} F_{n_k} \, dS = \int_{\partial B_r} F \, dS,
\]

where the notation \( S_{n,r} \) has been introduced in (11).

Proof. Let \( h_n(r) := \int_{\partial B_r} |F - F_n| \, dS \). Since, by assumption and the Coarea Formula,

\[
\lim_{n \to \infty} \int_{B_{r_0}} |F - F_n| \, dy = \lim_{n \to \infty} \int_0^{r_0} h_n(r) \, dr = 0,
\]
we have that $h_n \to 0$ in $L^1(0, r_0)$. Hence there exists a subsequence $\{h_{n_k}\}_{k \in \mathbb{N}}$ converging to 0 a.e. in $(0, r_0)$. Therefore $F_{n_k} \to F$ in $L^1(\partial B_r)$ for a.e. $r \in (0, r_0)$. It follows that, for a.e. $r \in (0, r_0)$,
\[
\int_{S_{n_k,r}} F_{n_k} \, dS - \int_{\partial B_r} F \, dS = \int_{\partial B_r} \chi_{S_{n_k,r}} (F_{n_k} - F) \, dS + \int_{\partial B_r} (\chi_{S_{n_k,r}} - 1) F \, dS \to 0
\]
as $k \to \infty$, thus yielding the conclusion.

\[\square\]

**Proposition 5.6.** We have that $E \in W^{1,1}_{\text{loc}}((0, r_0))$,

\begin{equation}
E(r) = \frac{1}{r^{N-1}} \int_{\partial B_r} U A \nabla U \cdot \nu \, dS \geq \frac{r}{2} H'(r) + r H(r)O(1) \quad \text{as } r \to 0^+
\end{equation}

and

\begin{equation}
E'(r) = (2 - N) \frac{1}{r^{N-1}} \int_{B_r \setminus \Gamma} (A \nabla U \cdot \nabla U - fU^2) \, dy + \frac{1}{r^{N-2}} \int_{\partial B_r} (A \nabla U \cdot \nabla U - fU^2) \, dS
\end{equation}
in the sense of distributions and for a.e. $r \in (0, r_0)$.

**Proof.** The fact that $E \in W^{1,1}_{\text{loc}}((0, r_0))$ and (71) follow from the Coarea Formula and (55). To prove (70) we consider the sequence $\{U_n\}$ introduced in Theorem 4.7. For every $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$,
\[
\frac{1}{r^{N-2}} \int_{\Omega_{n,r}} (A \nabla U_n \cdot \nabla U_n - f_n U_n^2) \, dy = \frac{1}{r^{N-2}} \int_{S_{n,r}} U_n A \nabla U_n \cdot \nu \, dS
\]
since $U_n$ solve (50) and $U_n \in H^2(\Omega_{n,r})$ by Theorem 4.7. Thanks to Remark 5.5 the Dominated Convergence Theorem, and Lemma 5.5 we can pass to the limit, up to a subsequence, as $n \to \infty$ in the above identity for a.e. $r \in (0, r_0)$, thus proving the first equality in (70). To prove the second equality in (70) we define
\[
\zeta(y) := \mu(y)(\beta(y) - y) = \frac{A(y)y}{|y|} - \frac{A(y)g \cdot y}{|y|^3} \cdot y.
\]

Then, since $\zeta(y) \cdot y = 0$ and $\zeta \cdot (0, \ldots, 0, 1) = 0$ on $\tilde{\Gamma}$, we have that
\[
\int_{\partial B_r} UA \nabla U \cdot \nu \, dS - \int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu} \, dS = \frac{1}{2} \int_{\partial B_r} \zeta \cdot \nabla(U^2) \, dS
\]
\[= - \frac{1}{2} \int_{\partial B_r} \text{div}(\zeta) U^2 \, dS = r^{N-1} H(r)O(1)
\]
as $r \to 0$, where we have used in the last equality the estimate
\[
\text{div}(\zeta)(y) = \left( \frac{\nabla \mu(y)}{|y|} - \frac{\mu(y)y}{|y|^3} \right)(\beta(y) - y) + \frac{\mu(y)}{|y|} (\text{div}(\beta)(y) - N) = O(1)
\]
which follows from Proposition 2.2. Then we conclude by (67).  

\[\square\]

The approximation procedure developed above also allows us to derive the following integration by parts formula.

**Proposition 5.7.** There exists a set $\mathcal{M} \subset [0, r_0]$ having null 1-dimensional Lebesgue measure such that, for all $r \in (0, r_0] \setminus \mathcal{M}$, $A \nabla U \cdot \nu \in L^2(\partial B_r)$ and
\[
\int_{B_r \setminus \tilde{\Gamma}} A \nabla U \cdot \nabla \phi \, dx = \int_{B_r} \tilde{f} U \phi \, dx + \int_{\partial B_r} (A \nabla U \cdot \nu) \phi \, dS
\]
for every $\phi \in H^1(B_{r_0} \setminus \hat{\Gamma})$, where $A \nabla U \cdot \nu$ on $\partial B_r$ is meant in the sense of Remark 5.3.

**Proof.** Since $U_n \to U$ in $H^1(B_{r_0} \setminus \hat{\Gamma})$ in view of Theorem 4.7 by Lemma 5.5 there exist a subsequence $\{U_{n_k}\}$ and a set $\mathcal{M} \subset [0, r_0]$ having null 1-dimensional Lebesgue measure such that
$A\nabla U \cdot \nu \in L^2(\partial B_r)$ and $A\nabla U_n \cdot \nu \to A\nabla U \cdot \nu$ in $L^2(\partial B_r)$ for all $r \in (0, r_0) \setminus \mathcal{M}$. Since $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$ by Theorem 5.1 from (30) it follows that

$$
\int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla \phi - f_n U_n \phi) \, dy = \int_{\partial \Omega_{n,r}} \phi A\nabla U_n \cdot \nu \, dS.
$$

Arguing as in the proof of Proposition 5.6 we can pass to the limit along $r$ which can be rewritten as

$$(\text{Pohozaev type inequality}) \text{ Under either assumption (H1) or assumption (H2), for any } r \in (0, r_0) \text{ we have that}$$

$$r \int_{\partial B_r} A\nabla U \cdot \nabla U \, dS \geq 2r \int_{\partial B_r} \frac{|A\nabla U \cdot \nu|^2}{\mu} \, dS + \int_{B_r} (A\nabla U \cdot \nabla U) \, \text{div}(\beta) \, dy$$

$$+ 2 \int_{B_r \setminus \tilde{f}} (A\nabla U \cdot \nabla U) \, \text{div}(\beta) \, dy + \int_{B_r \setminus \tilde{f}} (\tilde{f} \, \text{div}(\beta) + \nabla \tilde{f} \cdot \beta) U^2 \, dy$$

which can be rewritten as

$$r \int_{\partial B_r} A\nabla U \cdot \nabla U \, dS \geq 2r \int_{\partial B_r} \frac{|A\nabla U \cdot \nu|^2}{\mu} \, dS$$

$$+ \int_{B_r \setminus \tilde{f}} (A\nabla U \cdot \nabla U) \, \text{div}(\beta) \, dy + \int_{B_r \setminus \tilde{f}} (\tilde{f} \, \text{div}(\beta) + \nabla \tilde{f} \cdot \beta) U^2 \, dy$$

if $f$ satisfies (H1).

Proof. By Theorem 1.7 we have that $U_n \in H^2(\Omega_{n,r})$ for any $r \in (0, r_0)$ and $n \in \mathbb{N} \setminus \{0\}$. Then, since $A$ is symmetric by Proposition 2.1 we may write the following Rellich-Nečas identity in a distributional sense in $\Omega_{n,r}$:

$$\text{div}((A\nabla U_n \cdot \nabla U_n)\beta - 2(\beta \cdot \nabla U_n)A\nabla U_n) = (A\nabla U_n \cdot \nabla U_n) \text{div}(\beta)$$

$$- 2(\beta \cdot \nabla U_n) \text{div}(A\nabla U_n) + (dA\nabla U_n \cdot \nabla U_n) \cdot \beta - 2J_\beta(A\nabla U_n) \cdot \nabla U_n.$$

Since $U_n \in H^2(\Omega_{n,r})$ and the components of $A$ and $\beta$ are Lipschitz continuous by Propositions 2.1 and 2.2 then $(A\nabla U_n \cdot \nabla U_n)\beta - 2(\beta \cdot \nabla U_n)A\nabla U_n) \in W^{1,\infty}(\Omega_{n,r})$. Therefore we can integrate both sides of (74) on the Lipschitz domain $\Omega_{n,r}$ and apply the Divergence Theorem to obtain, in view of (15) and (56),

$$r \int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla U_n - 2(\beta \cdot \nabla U_n)A\nabla U_n) \, dS$$

$$= \int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla U_n) \, \text{div}(\beta) \, dy + 2 \int_{\Omega_{n,r}} \frac{A\nabla U_n \cdot y}{\mu} f_n U_n \, dy$$

$$+ \int_{\Omega_{n,r}} (dA\nabla U_n \cdot \nabla U_n) \cdot \beta \, dy - 2 \int_{\Omega_{n,r}} J_\beta(A\nabla U_n) \cdot \nabla U_n \, dy.$$

From Proposition 1.1 (15), and (19) it follows that, for all $n \in \mathbb{N} \setminus \{0\}$ and $r \in (0, r_0)$,

$$\int_{\Omega_{n,r}} (A\nabla U_n \cdot \nabla U_n) \frac{A y \cdot \nu}{\mu} \, dS \leq 0.$$

We recall from Theorem 1.7 that $U_n \to U$ strongly in $H^1(B_{r_0} \setminus \tilde{f})$, while Propositions 2.1 and 2.2 imply that

$$\mu \in L^\infty(B_{r_0}, \mathbb{R}), \quad \beta \in L^\infty(B_{r_0}, \mathbb{R}^N), \quad \text{div} \beta \in L^\infty(B_{r_0}, \mathbb{R}),$$

$$A \in L^\infty(B_{r_0}, \mathbb{R}^{N^2}), \quad \left\{ \frac{\partial h_i}{\partial y_h} \right\}_{i,j,h=1,...,N} \in L^\infty(B_{r_0}, \mathbb{R}^n).$$
Furthermore, under assumption \( \text{(H1)} \), we have that, by Sobolev embeddings (see Proposition 3.3), if \( N \geq 3 \), then \( f_n = \tilde{f} \in L^N(B_{r_0}) \) and \( U_n \to U \) strongly in \( L^2(B_{r_0}) \), whereas, if \( N = 2 \), then \( f_n = \tilde{f} \in L^{2(1+\varepsilon)/(1-\varepsilon)}(B_{r_0}) \) and \( U_n \to U \) strongly in \( L^{(1+\varepsilon)/\varepsilon}(B_{r_0}) \); then, since \( \nabla U_n \to \nabla U \) in \( L^2(B_{r_0}) \), Hölder’s inequality ensures that

\[
(f_n U_n A \nabla U_n \cdot y) \to \tilde{f} U A \nabla U \cdot y \quad \text{in} \quad L^1(B_{r_0}).
\]

Under assumption \( \text{(H2)} \), we have that Hardy’s inequality (see Proposition 3.2), Proposition 3.1 and (51) yield that

\[
\int_{B_{r_0}} |f_n y(U_n - U)|^2 \, dy \leq \text{const} r_0^{4\mu} \int_{B_{r_0}} |y|^2 |U_n - U|^2 \, dy \to 0 \quad \text{as} \quad n \to \infty
\]

which, thanks to Proposition 3.2 again and the Dominated Convergence Theorem, easily implies that

\[
f_n y U_n \to \tilde{f} y U \quad \text{in} \quad L^2(B_{r_0}),
\]

thus proving (78) also under assumption \( \text{(H2)} \).

Then, thanks to the Dominated Convergence Theorem, (25), (78) and Lemma 5.5, we can pass to the limit in (75) as \( n \to \infty \), up to a subsequence, and, taking into account (74), we obtain inequality (72).

If assumption \( \text{(H1)} \) holds then by (18), (53) and Proposition 2.2 we have that

\[
2 \int_{\Omega_{n,r}} \frac{A \nabla U_n \cdot y}{\mu} f_n U_n \, dy = 2 \int_{\Omega_{n,r}} (\beta \cdot \nabla U_n) \tilde{f} U_n \, dy
\]

\[
= - \int_{\Omega_{n,r}} (\tilde{f} \nabla (\beta) + \nabla \tilde{f} \cdot \beta) U_n^2 \, dy + r \int_{\Gamma_{n,r}} \tilde{f} U_n^2 \, dS + \int_{\Gamma_{n,r}} \tilde{f} U_n^2 \beta \cdot \nu \, dS.
\]

We define

\[
O_{+}^{n,r} := O_n \cap B_{+}^{r} , \quad O_{-}^{n,r} := O_n \cap B_{-}^{r} ,
\]

\[
\Gamma_{+}^{n,r} := \Gamma_{n,r} \cap \partial B_{+}^{r} , \quad \Gamma_{-}^{n,r} := \Gamma_{n,r} \cap \partial B_{-}^{r} ,
\]

where \( O_n \) is defined in (63). Taking into account that \( \beta \cdot \nu = \frac{A\nu}{\mu} \cdot \nu = 0 \) on \( \partial O_{+}^{n,r} \cap \partial \mathbb{R}^{N} \) since \( \nu = -(0, \ldots, 1) \) and (12) holds, the Divergence Theorem yields that

\[
\int_{\Gamma_{n,r}^{+}} \tilde{f} U_n^2 \beta \cdot \nu \, dS = - r \int_{\partial O_{+}^{n,r} \cap \partial B_{r}} \tilde{f} U_n^2 \beta \cdot \nu \, dS
\]

\[
+ \int_{O_{-}^{n,r}} \left( \tilde{f} U_n^2 \text{div} \beta + U_n^2 \nabla \tilde{f} \cdot \beta + 2 \tilde{f} U_n \nabla U_n \cdot \beta \right) \, dy.
\]

By (63), (74), and Lemma 5.5 there exists a subsequence \( \{ \tilde{f} U_n^2 \beta \cdot \nu \}_{k \in \mathbb{N}} \) converging in \( L^1(\partial B_r) \) and hence equi-integrable in \( \partial B_r \) for a.e. \( r \in (0, r_0) \), hence

\[
\lim_{k \to \infty} \int_{\partial O_{+}^{n,r} \cap \partial B_{r}} \tilde{f} U_n^2 \beta \cdot \nu \, dS = 0 \quad \text{for a.e.} \quad r \in (0, r_0).
\]

Since \( \nabla U_n \to \nabla U \) in \( L^2(B_{r_0}^{+}, \mathbb{R}^{N}) \), \( U_n \to U \) in \( L^q(B_{r_0}^{+}) \) and \( \tilde{f} \in L^{N+2\varepsilon/\varepsilon}(B_{r_0}^{+}) \) by \( \text{(H1)} \) and classical Sobolev embeddings, from (77) and Hölder’s inequality we deduce that

\[
\tilde{f} U_n \nabla U_n \cdot \beta \to \tilde{f} U \nabla U \cdot \beta \quad \text{in} \quad L^1(B_{r_0}^{+}),
\]

so that \( \{ \tilde{f} U_n \nabla U_n \cdot \beta \}_{n \in \mathbb{N}} \) is equi-integrable in \( B_{r_0}^{+} \). Therefore

\[
\lim_{n \to \infty} \int_{O_{+}^{n,r}} \tilde{f} U_n \nabla U_n \cdot \beta \, dy = 0 \quad \text{for all} \quad r \in (0, r_0).
\]

Moreover, also \( \{ \text{div} \beta \tilde{f} U_n^2 + U_n^2 \nabla \tilde{f} \cdot \beta \}_{n \in \mathbb{N}} \) is equi-integrable thanks to (18) and (61). It follows that

\[
\lim_{n \to \infty} \int_{O_{+}^{n,r}} (\text{div} \beta \tilde{f} U_n^2 + \nabla \tilde{f} \cdot \beta U_n^2) \, dy = 0 \quad \text{for all} \quad r \in (0, r_0).
\]
Then from (81) we conclude that

$$\lim_{k \to \infty} \int_{\Gamma_{h_k}^r} \tilde{f}U_{nk}^2 \beta \cdot \nu dS = 0.$$  

In a similar way we obtain that  

$$\lim_{k \to \infty} \int_{\Gamma_{h_k}^r} \tilde{f}U_{nk}^2 \beta \cdot \nu dS = 0$$  

so that

$$\lim_{k \to \infty} \int_{\Gamma_{h_k}^r} \tilde{f}U_{nk}^2 \beta \cdot \nu dS = 0.$$  

Therefore (73) follows by passing to the limit in (73) and (79) as $n \to \infty$ along a subsequence, taking into account Proposition 4.4, the Dominated Convergence Theorem, (25), Remark 5.4 and Lemma 5.5. □

$$\textbf{Proposition 5.9.}$$ For a.e. $r \in (0, r_0)$

$$E'(r) \geq 2r^{2-N} \int_{\partial B_r} \frac{|A \nabla U \cdot \nu|^2}{\mu} dS + r^{1-N} \int_{B_r \setminus \hat{F}} (\nu U^2 + 2 - N)A \nabla U \cdot \nabla U dy$$

$$+ r^{1-N} \int_{B_r \setminus \hat{F}} (\tilde{f}(\nu + N - 2) + \nabla \tilde{f} \cdot \nu) U^2 dy$$

$$+ r^{1-N} \int_{B_r \setminus \hat{F}} (dA \nabla U \cdot \nabla U) \cdot \nu dS$$

if (H1) holds, and

$$E'(r) \geq 2r^{2-N} \int_{\partial B_r} \frac{|A \nabla U \cdot \nu|^2}{\mu} dS - r^{2-N} \int_{\partial B_r} \tilde{f}U^2 dS + (N - 2)r^{1-N} \int_{B_r} \tilde{f}U^2 dy$$

$$+ r^{1-N} \int_{B_r \setminus \hat{F}} (A \nabla U \cdot \nu)(\nu U^2 + 2 - N - \tilde{f}U^2) dy$$

$$+ r^{1-N} \int_{B_r \setminus \hat{F}} (dA \nabla U \cdot \nabla U) \cdot \nu dS$$

if (H2) holds.

Proof. Estimates (81)–(82) are direct consequences of (71), (72), and (73). □

We now introduce the Almgren frequency function, defined as

$$N : (0, r_0) \to \mathbb{R}, \quad N(r) := \frac{E(r)}{H(r)}.$$  

The above definition of $N$ is well posed thanks to Proposition 5.1.

$$\textbf{Proposition 5.10.}$$ If either assumption (H1) or assumption (H2) hold, then $N \in W^{1,1}_{\text{loc}}((0, r_0])$ and, for any $r \in (0, r_0)$,

$$N(r) \geq -2\eta_f(r).$$

Furthermore, for a.e. $r \in (0, r_0)$,

$$N'(r) \geq \mathcal{V}(r) + W(r)$$

where

$$\mathcal{V}(r) = \frac{2r \left( \int_{\partial B_r} \frac{|A \nabla U \cdot \nu|^2}{\mu} dS \right) \left( \int_{\partial B_r} \mu U^2 dS \right) - \left( \int_{\partial B_r} U A \nabla U \cdot \nu dS \right)^2}{\left( \int_{\partial B_r} \mu U^2 dS \right)^2} \geq 0$$

and

$$W(r) = O \left( r^{-1+\frac{2N}{1-\alpha}} \right) (1 + N(r)) \quad \text{as } r \to 0^+.$$

Proof. Since \(1/H, E \in W^{1,1}_{\text{loc}}((0,r_0))\), then \(N \in W^{1,1}_{\text{loc}}((0,r_0))\). Furthermore \((87)\) directly implies \((84)\).

By \((74)\), for a.e. \(r \in (0,r_0)\)

\[
N'(r) = \frac{E'(r)H(r) - E(r)H'(r)}{H^2(r)} = \frac{E'(r)H(r) - \frac{2}{r}E^2(r)}{H^2(r)} + \frac{E(r)O(1)}{H(r)}
\]

as \(r \to 0^+\). By Proposition \(2.1\), Proposition \(2.2\), \(30\) and \(34\)

\[
\int_{B_r} (A\nabla U \cdot \nabla U)(\text{div}(\beta) + 2 - N) - 2J_\beta(A\nabla U) \cdot \nabla U + (dA\nabla U \nabla U) : \beta \, \mu \, \text{d}S
\]

\[
\leq O(r) \left( \int_{B_r} |\nabla U|^2 \, \text{d}S \right) + O\left(r^{\frac{2-N}{2}}\right) \left( \int_{\partial B_r} \mu \, U^2 \, \text{d}S \right)
\]

as \(r \to 0^+\) and, by \((24)\), the same holds for \(\int_{B_r}(\text{div} \beta - N + 2) \tilde{f} U^2 \, \text{d}S\). In the same way from \((22)\) it follows that, if \((11)\) holds,

\[
\int_{B_r} \nabla \cdot \beta U^2 \, \mu \, \text{d}S \leq O\left(r^{\frac{4-N}{2}}\right) \left( \int_{B_r} \nabla^2 U \cdot \nabla^2 U \, \mu \, \text{d}S \right) + O\left(r^{\frac{2-N}{2}}\right) \left( \int_{\partial B_r} \mu \, U^2 \, \text{d}S \right)
\]

as \(r \to 0^+\).

Under assumption \((12)\), by Remark \(2.3\), \(20\), \(14\), \(36\), \(35\), \(37\) and Hölder’s inequality,

\[
\int_{B_r} \frac{A\nabla U \cdot \gamma \tilde{f} U \, \mu \, \text{d}S}{\text{d}S} = O(r) \left( \int_{B_r} \nabla^2 U \cdot \nabla^2 U \, \mu \, \text{d}S \right)^{\frac{1}{2}}
\]

\[
\leq O\left(r^{\frac{N-1}{2}}\right) \left( \int_{B_r} \frac{A\nabla U \cdot \nabla U - \tilde{f} U^2}{\text{d}S} \right) \mu \, \text{d}S \]

\[
\leq O\left(r^{\frac{N-1}{2}}\right) \left( \int_{B_r} \frac{A\nabla U \cdot \nabla U - \tilde{f} U^2}{\text{d}S} \right) \mu \, \text{d}S \]

Under assumptions \((12)\), thanks to Remark \(2.3\) and \(19\),

\[
\int_{\partial B_r} \tilde{f} U^2 \, \text{d}S = O\left(r^{\frac{N-2}{2}}\right) \left( \int_{\partial B_r} \mu \, U^2 \, \text{d}S \right)
\]

Collecting the above estimates, we conclude that \((55)\), \((58)\) and \((57)\) follow from \((51)\) or \((52)\) under hypotheses \((11)\) or \((12)\) respectively. From the Cauchy–Schwarz inequality we also deduce that \(\mathcal{V} \geq 0\) a.e. in \((0,r_0)\).

We now prove that \(\mathcal{N}\) is bounded.
Proposition 5.11. There exists a constant $C > 0$ such that, for every $r \in (0, r_0)$,
(89) \[ N(r) \leq C. \]

Proof. By Proposition 5.10, there exists a constant $\kappa > 0$ such that, for a.e. $r \in (0, r_0)$,
\[ (N + 1)'(r) \geq W(r) \geq -\kappa r^{-1+\frac{1}{\kappa N}} (N(r) + 1). \]
Since $N + 1 > 0$ by (84) and the choice of $r_0$ in (55), it follows that
\[ \log(N + 1)'(r) \geq -\kappa r^{-1+\frac{1}{\kappa N}}. \]
An integration over $(r, r_0)$ yields
\[ N(r) \leq -1 + \exp\left(\frac{N + 2 \epsilon}{4\epsilon} r_0^{-\frac{1}{\kappa N}}\right) (N(r_0) + 1) \]
and the proof is thereby complete. \qed

Proposition 5.12. There exists the limit
(90) \[ \gamma := \lim_{r \to 0^+} N(r). \]
Furthermore $\gamma$ is finite and $\gamma \geq 0$.

Proof. From Proposition 5.10 and (89) there exists a constant $\kappa > 0$ such that
\[ N'(r) \geq W(r) \geq -\kappa r^{-1+\frac{1}{\kappa N}} (N(r) + 1) \geq -\kappa(C + 1)r^{-1+\frac{1}{\kappa N}} \quad \text{for a.e. } r \in (0, r_0). \]
Then
\[ \frac{d}{dr} \left( N(r) + \frac{\kappa(C + 1)(N + 2 \epsilon)}{4\epsilon} r^{-\frac{1}{\kappa N}} \right) \geq 0 \]
for a.e. $r \in (0, r_0)$. We conclude that $\lim_{r \to 0^+} N(r)$ exists; moreover such a limit is finite thanks to (89) and (84). Furthermore from (86) and (84) we deduce that $\gamma \geq 0$. \qed

Proposition 5.13. There exists a constant $\alpha > 0$ such that, for every $r \in (0, r_0)$,
(91) \[ H(r) \leq \alpha r^{2\gamma}. \]
Furthermore for every $\sigma > 0$ there exist $\alpha_0 > 0$ and $r_\sigma \in (0, r_0)$ such that, for every $r \in (0, r_\sigma)$,
(92) \[ H(r) \geq \alpha_0 r^{2\gamma + \sigma}. \]

Proof. By (70), (89), (86), (87), and (89) there exists a constant $\kappa > 0$ such that, for a.e. $r \in (0, r_0)$,
\[ \frac{H'(r)}{H(r)} = \frac{2}{r} N'(r) + O(1) \]
\[ = \frac{2}{r} \int_0^r N'(t) \, dt + \frac{2\gamma}{r} + O(1) \geq \frac{2}{r} \int_0^r W(t) \, dt + O(1) + \frac{2\gamma}{r} \geq -\kappa r^{-1+\frac{1}{\kappa N}} + \frac{2\gamma}{r}. \]
It follows that, integrating between $r$ and $r_0$,
\[ H(r_0) \geq H(r) \left( \frac{r_0}{r} \right)^{2\gamma} \exp\left( -\kappa \frac{N + 2 \epsilon}{4\epsilon} r_0^{-\frac{1}{\kappa N}} \right) \]
for all $r \in (0, r_0]$, so that (91) is proved. To prove (92) we notice that, by (90) and (70), for every $\sigma > 0$ there exists $r_\sigma \in (0, r_0)$ s.t.
\[ \frac{H'(r)}{H(r)} \leq \frac{2\gamma + \sigma}{r} \quad \text{for all } r \in (0, r_\sigma] \]
and an integration between $r \in (0, r_\sigma)$ and $r_\sigma$ yields
\[ H(r_\sigma) \leq \left( \frac{r_\sigma}{r} \right)^{2\gamma + \sigma} H(r) \]
thus proving (92). \qed

Proposition 5.14. The limit $\lim_{r \to 0^+} r^{-2\gamma} H(r)$ exists and is finite.
Proof. By (10) we only need to prove that the limit exists. For any \( r \in (0, r_0) \) we have that
\[
\frac{d}{dr} \frac{H(r)}{r^{2\gamma}} = \frac{2r^{2\gamma-2}E(r) - 2r^{2\gamma}H(r) + r^{2\gamma}H(r)O(1)}{r^{2\gamma}} = 2r^{-2\gamma-1}H(r) \left( \int_0^r N'(t) \, dt + rO(1) \right),
\]
by (70) and Proposition 5.10. Thanks to Proposition 5.10, integrating between \( r \) and \( r_0 \) we obtain
\[
\frac{H(r_0)}{r_0^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{r_0} 2s^{-2\gamma-1}H(s) \left( \int_0^s (N'(t) - W(t)) \, dt \right) \, ds
\]
\[
+ \int_r^{r_0} 2s^{-2\gamma-1}H(s) \left( sO(1) + \int_0^s W(t) \, dt \right) \, ds.
\]
We note that there exists a constant \( \kappa > 0 \) such that
\[
\left| 2s^{-2\gamma-1}H(s) \left( sO(1) + \int_0^s W(t) \, dt \right) \right| \leq \kappa s^{-1+\frac{4k}{2\gamma}}
\]
by Proposition 5.10, 59, and (11). Since \( s^{-1+\frac{4k}{2\gamma}} \in L^1(0, r_0) \), then
\[
\lim_{r \to 0^+} \int_r^{r_0} 2s^{-2\gamma-1}H(s) \left( sO(1) + \int_0^s W(t) \, dt \right) \, ds
\]
exists and is finite. Moreover, since \( N' - W \geq 0 \) by Proposition 5.10
\[
\lim_{r \to 0^+} \int_r^{r_0} 2s^{-2\gamma-1}H(s) \left( \int_0^s (N'(t) - W(t)) \, dt \right) \, ds
\]
exists, being possibly infinite. Then the right hand side of (93) admits a limit as \( r \to 0^+ \) and the conclusion follows. \( \square \)

From the properties of the height function \( H \) derived above, in particular from estimate (92), we deduce the unique continuation property stated in Theorem 1.1.

Proof of Theorem 1.1 Let \( u \) be a weak solution to (1) such that \( u(x) = O(|x|^k) \) as \( |x| \to 0^+ \) for all \( k \in \mathbb{N} \). To prove that \( u \equiv 0 \) in \( B_R \), we argue by contradiction and assume that \( u \not\equiv 0 \). Then we can define a frequency function for \( U = u \circ F \) as in (65), (66) and (83). Choosing \( k \in \mathbb{N} \) such that \( k > \gamma + \frac{2}{\gamma} \), we would obtain that \( H(r) = O(r^{2k}) = o(r^{2\gamma+\frac{2}{\gamma}}) \) as \( r \to 0 \), contradicting estimate (92). \( \square \)

## 6. Neumann Eigenvalues on \( \mathbb{S}^{N-1} \setminus \Sigma \)

In this section we study the spectrum of (7). We recall that \( \mu \in \mathbb{R} \) is an eigenvalue of (7) if there exists \( \psi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma) \setminus \{0\} \) such that
\[
\int_{\mathbb{S}^{N-1} \setminus \Sigma} \nabla_{\mathbb{S}^{N-1} \setminus \Sigma} \psi \cdot \nabla_{\mathbb{S}^{N-1} \setminus \Sigma} \phi \, dS = \mu \int_{\mathbb{S}^{N-1} \setminus \Sigma} \psi \phi \, dS \quad \text{for any} \ \phi \in H^1(\mathbb{S}^{N-1} \setminus \Sigma).
\]
A Rellich-Kondrakov type theorem is needed to apply the classical Spectral Theorem to problem (7).

### Proposition 6.1.

The embedding \( H^1(\mathbb{S}^{N-1} \setminus \Sigma) \hookrightarrow L^2(\mathbb{S}^{N-1}) \) is compact.

Proof. Let \( \{\phi_n\}_{n \in \mathbb{N}} \) be a bounded sequence in \( H^1(\mathbb{S}^{N-1} \setminus \Sigma) \). We observe that \( \mathbb{S}^{N-1} \) and \( \mathbb{S}^{N-1} \setminus \Sigma \) are smooth compact manifolds with boundary and that the sequences of restrictions \( \{\phi_n|_{\mathbb{S}^{N-1}}\}_{n \in \mathbb{N}} \) and \( \{\phi_n|_{\mathbb{S}^{N-1} \setminus \Sigma}\}_{n \in \mathbb{N}} \) are bounded in \( H^1(\mathbb{S}^{N-1}) \) and \( H^1(\mathbb{S}^{N-1} \setminus \Sigma) \) respectively. Then we can extract a subsequence \( \{\phi_{n_k}\}_{k \in \mathbb{N}} \) such that \( \{\phi_{n_k}|_{\mathbb{S}^{N-1}}\}_{n \in \mathbb{N}} \) converges in \( L^2(\mathbb{S}^{N-1}) \) by the classical Rellich-Kondrakov Theorem on compact manifolds with boundary, see [5]. Proceeding in the same way for \( \{\phi_{n_k}|_{\mathbb{S}^{N-1} \setminus \Sigma}\}_{n \in \mathbb{N}} \) in \( H^1(\mathbb{S}^{N-1} \setminus \Sigma) \), we conclude that there exists a subsequence \( \{\phi_{n_{k_l}}\}_{l \in \mathbb{N}} \) which converges both in \( L^2(\mathbb{S}^{N-1}) \) and in \( L^2(\mathbb{S}^{N-1} \setminus \Sigma) \), hence in \( L^2(\mathbb{S}^{N-1}) \). \( \square \)

### Proposition 6.2.
(i) The point spectrum of (7) is a diverging and increasing sequence of non-negative eigenvalues \( \{\mu_k\}_{k \in \mathbb{N}} \) of finite multiplicity and the eigenvalue \( \mu_0 = 0 \) is simple. Letting \( N_k \) be the multiplicity of \( \mu_k \) and \( V_k \) be the eigenspace associated to \( \mu_k \), there exists an orthonormal basis of \( L^2(S^{N-1}) \) consisting of eigenfunctions \( \{Y_{k,i}\}_{k \in \mathbb{N}, i = 1, \ldots, N_k} \) such that \( \{Y_{k,i}\}_{i = 1, \ldots, N_k} \) is a basis of \( V_k \) for any \( k \in \mathbb{N} \).

(ii) For any \( k \in \mathbb{N} \)

\[
\mu_k = \frac{k(k + 2N - 4)}{4}.
\]

Moreover any eigenfunction of (7) belongs to \( L^\infty(S^{N-1}) \).

Proof. The proof of (i) follows from the classical Spectral Theorem for compact self-adjoint operators, taking into account Proposition 6.1. We prove now (ii). If \( \mu \) is an eigenvalue of (7) and \( \Psi \) an associated eigenfunction, let \( \sigma := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu} \)

\[
W(r\theta) := r^\sigma \Psi(\theta), \text{ for any } r \in [0, \infty), \theta \in S^{N-1} \setminus \Sigma.
\]

Since \( \Psi \) is an eigenfunction of (7) then \( W \) is harmonic on \( B_1 \setminus \tilde{\Gamma} \) and \( \frac{\partial W}{\partial r^+} = \frac{\partial W}{\partial r^-} = 0 \) on \( \tilde{\Gamma} \).

Therefore from [10] that there exists \( k \in \mathbb{N} \) such that \( \sigma = \frac{k}{2} \) and so \( \mu = \frac{k(k + 2N - 4)}{4} \).

Moreover from [10] it also follows that \( W \in L^\infty(B_1) \) hence \( \Psi \in L^\infty(S^{N-1}) \).

Viceversa, if we let \( k \in \mathbb{N} \) and define \( W \) in cylindrical coordinates as

\[
W(x', r \cos(t), r \sin(t)) := r^{\frac{k}{2}} \cos \left( \frac{kt}{2} \right) \text{ for any } x' \in \mathbb{R}^{N-2}, r \in [0, \infty), \text{ and } t \in [0, 2\pi],
\]

then \( W \) is harmonic on \( B_1 \setminus \tilde{\Gamma} \) and \( \frac{\partial W}{\partial r^+} = \frac{\partial W}{\partial r^-} = 0 \) on \( \tilde{\Gamma} \). Since \( W \) is homogeneous of degree \( k/2 \), then

\[
W(r\theta) = r^\frac{k}{2} \Psi(\theta), \text{ for any } r \in [0, \infty), \text{ and } \theta \in S^{N-1} \setminus \Sigma,
\]

where \( \Psi = W_{|_{S^{N-1}}} \). Then from

\[
r^{\frac{k-4}{4}} \left( \frac{k(k - 2)}{4} \Psi(\theta) + \frac{k(N - 1)}{2} \Psi(\theta) + \Delta_{S^{N-1}} \Psi(\theta) \right) = 0, \text{ } r \in [0, \infty), \theta \in S^{N-1} \setminus \Sigma,
\]

we deduce that \( \Psi \) solves (7) with \( \mu = \frac{k(k + 2N - 4)}{4} \).

\[\qed\]

Remark 6.3. The traces of eigenfunctions of problem (7) on both sides of \( \Sigma \) (i.e. the traces of restrictions to \( S^{N-1}_+ \) and \( S^{N-1}_- \)) cannot vanish identically.

Indeed, if an eigenfunction \( \Psi \) associated to the eigenvalue \( \mu_k \) is such that the trace of \( \Psi_{|_{S^{N-1}}} \) on \( \Sigma \) vanishes, then the function \( W(x) = |x|^{k/2} \Psi(|x|/|x|) \) would be a harmonic function in \( \mathbb{R}^N \setminus \tilde{\Gamma} \) satisfying both Dirichlet and Neumann homogeneous boundary conditions on the upper side of the crack, thus violating classic unique continuation principles.

7. The blow-up analysis

Throughout this section we let \( u \in H^1(B_R \setminus \Gamma) \) be a non-trivial weak solution to (11) with \( f \) satisfying either (11) or (12), \( U = u \circ F \in H^1(B_{r_1} \setminus \tilde{\Gamma}) \) be the corresponding solution to (24), \( r_0 \) be as in (25) and \( r_1 \) be as in Proposition 2.1. For all \( \lambda \in (0, r_0) \), let

\[
W^\lambda(y) := \frac{U(\lambda y)}{\sqrt{H(\lambda)}} \text{ for any } y \in B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}.
\]

For any \( \lambda \in (0, r_0) \) it is easy to verify that \( W^\lambda \in H^1(B_{\lambda^{-1}r_1} \setminus \tilde{\Gamma}) \) and \( W^\lambda \) satisfies

\[
\int_{B_{\lambda^{-1}r_1}} A(\lambda y) \nabla W^\lambda(y) \cdot \nabla \phi(y) dy - \lambda^2 \int_{B_{\lambda^{-1}r_1}} f(\lambda y) W^\lambda(y) \phi(y) dy = 0
\]
for any \( \phi \in H^1_{0,\partial B_{\lambda^{-1}r}}(B_{\lambda^{-1}r} \setminus \hat{\Gamma}) \). In other words \( W^\lambda \) is a weak solution of

\[
\begin{aligned}
- \text{div}(A(\lambda)\nabla W^\lambda) &= \lambda^2 f(\lambda)W^\lambda, & & \text{in } B_{\lambda^{-1}r} \setminus \hat{\Gamma}, \\
A(\lambda)\nabla^+ W^\lambda \cdot \nu^+ &= A(\lambda)\nabla^- W^\lambda \cdot \nu^-, & & \text{on } \Gamma,
\end{aligned}
\]

(98)

for any \( \lambda \in (0, r_0) \). Since \( B_1 \subset B_{\lambda^{-1}r} \) for all \( \lambda \in (0, r_0) \), it follows that, for any \( \lambda \in (0, r_0) \),

\[
\int_{B_1 \setminus \hat{\Gamma}} A(\lambda)\nabla W^\lambda(y) \cdot \nabla \phi(y) \, dy - \lambda^2 \int_{B_1} f(\lambda y)W^\lambda(y)\phi(y) \, dy = 0,
\]

(99)

for any \( \phi \in H^1_{0,\partial B_1}(B_1 \setminus \hat{\Gamma}) \). Furthermore by a change of variables, (96) and (55),

\[
\int_{\mathbb{S}^{N-1}} \mu(\lambda \theta)|W^\lambda(\theta)|^2 \, dS = 1 \quad \text{for every } \lambda \in (0, r_0). \tag{100}
\]

**Proposition 7.1.** Let \( W^\lambda \) be as in (96). Then \( \{W^\lambda\}_{\lambda \in (0, r_0)} \) is bounded in \( H^1(B_1 \setminus \hat{\Gamma}) \).

**Proof.** We have

\[
\int_{B_1 \setminus \hat{\Gamma}} |\nabla W^\lambda|^2 \, dy = \frac{\lambda^{N-2}}{H(\lambda)} \int_{B_1 \setminus \hat{\Gamma}} |\nabla U(y)|^2 \, dy \leq \frac{2}{1 - 2\eta_f(\lambda)} N(\lambda) + \frac{4\eta_f(\lambda)}{1 - 2\eta_f(\lambda)},
\]

by (97). Then thanks to (96), (60), (61), (62), (63), and (100) we conclude. \( \square \)

The following proposition is a doubling type result.

**Proposition 7.2.** There exists a constant \( C_1 > 0 \) such that for any \( \lambda \in (0, \frac{r_0}{2}) \) and \( T \in [1, 2] \)

\[
\frac{1}{C_1} H(T\lambda) \leq H(\lambda) \leq C_1 H(T\lambda), \tag{101}
\]

\[
\int_{B_T} |W^\lambda(y)|^2 \, dy \leq 2^N C_1 \int_{B_1} |W^{T\lambda}(y)|^2 \, dy, \tag{102}
\]

and

\[
\int_{B_T \setminus \hat{\Gamma}} |\nabla W^\lambda(y)|^2 \, dy \leq 2^{N-2} C_1 \int_{B_1 \setminus \hat{\Gamma}} |\nabla W^{T\lambda}(y)|^2 \, dy. \tag{103}
\]

**Proof.** From (89), (84), (70), and (55) we deduce that there exist two constants \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \) such that, for any \( r \in (0, r_0) \),

\[
-\frac{2}{r} < -\frac{2\eta_f(r)}{H(r)} < \frac{H'(r)}{H(r)} \leq \frac{2N}{r} < \frac{\kappa_1}{r} \leq \frac{\kappa_2}{r}.
\]

Then (101) follows from an integration in \( (\lambda, T\lambda) \) of the above inequality. Furthermore from (101) we obtain that, for any \( \lambda \in (0, \frac{r_0}{2}) \) and \( T \in [1, 2] \),

\[
\int_{B_T} |W^\lambda(y)|^2 \, dy = \frac{\lambda^{-N}}{H(\lambda)} \int_{B_{rT}} |U(y)|^2 \, dy \leq \frac{C_1 2^N}{(T\lambda)^N H(T\lambda)} \int_{B_{T\lambda}} |U(y)|^2 \, dy = C_1 2^N \int_{B_1} |W^{T\lambda}(y)|^2 \, dy.
\]

In the same way (103) follows from (101). \( \square \)

**Proposition 7.3.** Let \( \mathcal{M} \) be as in Proposition 7.7 and \( W^\lambda \) be defined in (96). Then there exist \( M > 0 \) and \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \) there exists \( T_\lambda \in [1, 2] \) such that \( \lambda T_\lambda \notin \mathcal{M} \) and

\[
\int_{\partial B_{r_0\lambda}} |\nabla W^\lambda|^2 \, dS \leq M \int_{B_{r_0\lambda} \setminus \hat{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy. \tag{104}
\]
Proof. Since \( \{W^\lambda\}_{\lambda \in (0, r_0/2)} \) is bounded in \( H^1(B_2 \setminus \hat{\Gamma}) \) by Proposition 3.1, 102 and 103, then
\[
\limsup_{\lambda \to 0^+} \int_{B_2 \setminus \hat{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy < +\infty.
\]
By the Coarea formula, for any \( \lambda \in (0, \frac{r_0}{2}) \) the function
\[
g_\lambda(r) := \int_{B_2 \setminus \hat{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy
\]
is absolutely continuous in \([1, 2]\) with weak derivative
\[
g'_\lambda(r) = \int_{\partial B_r} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dS \quad \text{for a.e. } r \in [1, 2],
\]
where the integral \( \int_{\partial B_r} |\nabla W^\lambda|^2 \, dS \) is meant in the sense of Remark 5.3. To prove the statement we argue by contradiction. If the conclusion does not hold, for any \( M > 0 \) there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset (0, r_0/2) \) such that \( \lim_{n \to \infty} \lambda_n = 0 \) and
\[
\int_{\partial B_r} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2) \, dS > M \int_{B_2 \setminus \hat{\Gamma}} (|\nabla W^{\lambda_n}|^2 + |W^{\lambda_n}|^2) \, dy
\]
for any \( n \in \mathbb{N} \) and \( r \in [1, 2] \setminus \frac{1}{\lambda_n} \mathcal{M} \), and hence for a.e. \( r \in [1, 2] \). Hence
\[
g'_{\lambda_n}(r) > Mg_{\lambda_n}(r) \quad \text{for any } n \in \mathbb{N} \text{ and a.e. } r \in [1, 2].
\]
An integration in \([1, 2]\) yields
\[
\limsup_{n \to \infty} g_{\lambda_n}(1) \leq e^{-M} \limsup_{n \to \infty} g_{\lambda_n}(2)
\]
hence
\[
\liminf_{\lambda \to 0^+} g_{\lambda}(1) \leq e^{-M} \limsup_{\lambda \to 0^+} g_{\lambda}(2).
\]
In view of (105), letting \( M \to \infty \) we conclude that
\[
\liminf_{\lambda \to 0^+} \int_{B_1 \setminus \hat{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy = 0.
\]
Then there exists a sequence \( \{\rho_n\}_{n \in \mathbb{N}} \) such that \( W^{\rho_n} \to 0 \) strongly in \( H^1(B_1 \setminus \hat{\Gamma}) \) as \( n \to \infty \). Due to the continuity of the trace operator \( \gamma_1 \) defined in Proposition 3.1 and (20), this is in contradiction with (100). \( \square \)

**Proposition 7.4.** There exists \( \overline{M} > 0 \) such that
\[
\left(106\right) \quad \int_{S^{N-1}} |\nabla W^\lambda|^2 \, dS \leq \overline{M} \quad \text{for all } \lambda \in \left(0, \min\left\{ \frac{r_0}{2}, \lambda_0\right\} \right).
\]

**Proof.** Since
\[
\int_{S^{N-1}} |\nabla W^\lambda|^2 \, dS = \frac{\lambda^2 T_\lambda^{3-N}}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla U(\lambda y)|^2 \, dS = T_\lambda^{3-N} \frac{H(\lambda)}{H(\lambda T_\lambda)} \int_{\partial B_{T_\lambda}} |\nabla W^\lambda|^2 \, dS,
\]
then, by (101), (102), (103), (104), and the fact that \( 1 \leq T_\lambda \leq 2 \), for any \( \lambda \in \left(0, \min\left\{ \frac{r_0}{2}, \lambda_0\right\} \right) \) we have that
\[
\int_{S^{N-1}} |\nabla W^\lambda|^2 \, dS \leq 2C_1 M \int_{B_{T_\lambda} \setminus \hat{\Gamma}} (|\nabla W^\lambda|^2 + |W^\lambda|^2) \, dy \leq 2^{N+1} C_2^2 M \int_{B_1 \setminus \hat{\Gamma}} (|\nabla W^{T_\lambda}|^2 + |W^{T_\lambda}|^2) \, dy.
\]
Therefore we conclude thanks to Proposition 7.3. \( \square \)

Thanks to the estimates established above, we can now prove a first blow-up result.
Proposition 7.5. Let $u \in H^1(B_1 \setminus \Gamma)$, $u \not\equiv 0$, be a non-trivial weak solution to \((11)\), with $\Gamma$ defined in \((10)\) and $f$ satisfying either \((111)\) or \((112)\), and let $U = u \circ F$ be the corresponding solution to \((20)\). Let $\gamma$ be as in \((90)\). Then

\begin{equation}
\text{(107)}
\frac{U(\lambda_n y)}{\sqrt{H(\lambda_n)}} \to \left| y \right|^\gamma \Psi \left( \frac{y}{\left| y \right|} \right) \text{ strongly in } H^1(B_1 \setminus \tilde{\Gamma}).
\end{equation}

Proof. Let $W^\lambda$ be as in \((90)\) for any $\lambda \in (0, \min \{ \frac{m}{r}, \lambda_0 \})$ and let us consider a sequence $\{\lambda_n\} \in \mathbb{N}$ such that $\lim_{n \to \infty} \lambda_n = 0$. For any sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ and any $\lambda \in (0, \min \{ \frac{m}{r}, \lambda_0 \})$, it follows that $W^\lambda$ is bounded in $H^1(B_1 \setminus \tilde{\Gamma})$. Therefore there exists a subsequence $\{W^\lambda_{\lambda_n T_{\lambda_n k}}\} \subset H^1(B_1 \setminus \tilde{\Gamma})$ and a function $W \in H^1(B_1 \setminus \tilde{\Gamma})$ such that $W^\lambda_{\lambda_n T_{\lambda_n k}} \to W$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$. By compactness of the trace operator $\gamma_1$ (see Propositions 3.3, 20, and 110), it follows that

\begin{equation}
\int_{\partial B_1} W^2 \, dS = 1
\end{equation}

and so $W \not\equiv 0$ on $B_1 \setminus \tilde{\Gamma}$.

By Hölder’s inequality and \((59)\), we have that, for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$,

\begin{equation}
\text{(110)}
\left| \lambda^2 \int_{B_1} \tilde{f}(\lambda y) W^\lambda(y) \phi(y) \, dy \right| \\
\leq \lambda^2 \eta \tilde{f}(\lambda)(1) \left( \int_{B_1 \setminus \tilde{\Gamma}} |\nabla W^\lambda|^2 \, dy + \int_{\partial B_1} |W^\lambda|^2 \, dS \right)^{\frac{1}{2}} \left( \int_{B_1} |\nabla \phi|^2 \, dy + \int_{\partial B_1} \phi^2 \, dS \right)^{\frac{1}{2}}.
\end{equation}

By \((56)\) and a change of variables we have that

\begin{equation}
\text{(111)}
\lambda^2 \eta \tilde{f}(\lambda)(1) = S_{N,q} \lambda^2 \left( \int_{B_1} |\tilde{f}(\lambda y)|^{\frac{2}{r}} \, dy \right)^{\frac{r}{N+2}} \\
= S_{N,q} \lambda^\frac{r}{N+2} \|\tilde{f}\|_{L^{\frac{N+2}{r}}(B_1)} \to 0 \quad \text{as } \lambda \to 0^+.
\end{equation}

From \((110)\), \((111)\), the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \tilde{\Gamma})$ (established in Proposition 7.1) and of the traces (following from Proposition 3.1), we deduce that

\begin{equation}
\text{(112)}
\lim_{k \to \infty} \lambda^2_{\lambda_n T_{\lambda_n k}} \int_{B_1} \tilde{f}(\lambda_n T_{\lambda_n} y) W^\lambda_{\lambda_n T_{\lambda_n k}}(y) \phi(y) \, dy = 0,
\end{equation}

for every $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$.

Let $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$. We can test \((99)\) with $\phi$ to obtain

\begin{equation}
\text{(113)}
\int_{B_1 \setminus \tilde{\Gamma}} A(\lambda_n T_{\lambda_n y}) \nabla W^\lambda_{\lambda_n T_{\lambda_n k}}(y) \cdot \nabla \phi(y) \, dy \\
= (\lambda_n T_{\lambda_n k})^2 \int_{B_1} \tilde{f}(\lambda_n T_{\lambda_n} y) W^\lambda_{\lambda_n T_{\lambda_n k}}(y) \phi(y) \, dy,
\end{equation}

for any $k \in \mathbb{N}$. Since $W^\lambda_{\lambda_n T_{\lambda_n k}} \to W$ weakly in $H^1(B_1 \setminus \tilde{\Gamma})$, by \((17)\) we have that

\begin{equation}
\text{(114)}
\lim_{k \to \infty} \int_{B_1 \setminus \tilde{\Gamma}} A(\lambda_n T_{\lambda_n y}) \nabla W^\lambda_{\lambda_n T_{\lambda_n k}}(y) \cdot \nabla \phi(y) \, dy = \int_{B_1 \setminus \tilde{\Gamma}} \nabla W \cdot \nabla \phi \, dy.
\end{equation}

Therefore, for any $\phi \in H^1_{0,\partial B_1}(B_1 \setminus \tilde{\Gamma})$ we can pass to the limit as $k \to \infty$ in \((113)\) thus obtaining, in view of \((114)\) and \((112)\),

\begin{equation}
\int_{B_1 \setminus \tilde{\Gamma}} \nabla W \cdot \nabla \phi \, dy = 0,
\end{equation}

where $W$ is the weak limit of $W^\lambda_{\lambda_n T_{\lambda_n k}}$ as $k \to \infty$.}
i.e. $W$ is a weak solution of

$$
\begin{aligned}
\begin{cases}
-\Delta W = 0, & \text{on } B_1 \setminus \tilde{\Gamma}, \\
\frac{\partial W}{\partial v^+} = \frac{\partial W}{\partial v^-} = 0, & \text{on } \tilde{\Gamma}.
\end{cases}
\end{aligned}
$$

(115)

We note that, by classical elliptic regularity theory, $W$ is smooth in $B_1 \setminus \tilde{\Gamma}$.

In view of Proposition 7.4 and (14) there exists a function up to a subsequence. By the weak convergence

$$
\int_{B_1 \setminus \tilde{\Gamma}} A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} (y) \cdot \nabla \phi(y) \, dy 
$$

$$
- (\lambda_n T_{\lambda_n})^2 \int_{B_1} \tilde{f}(\lambda_n T_{\lambda_n} y) W^{\lambda_n T_{\lambda_n}} (y) \phi(y) \, dy
$$

$$
= \int_{\partial B_1} (A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} (y) \cdot \nu) \phi(y) \, dS.
$$

(116)

Thanks to Proposition 7.4 and (14) there exists a function $h \in L^2(\partial B_1)$ such that

$$
(A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} (y) \cdot \nu) \to h \text{ weakly in } L^2(\partial B_1),
$$

up to a subsequence. By the weak convergence $W^{\lambda_n T_{\lambda_n}} \to W$ in $H^1(B_1 \setminus \tilde{\Gamma})$, (17), (112) and (117), passing to the limit as $k \to \infty$ in (116), we obtain that

$$
\int_{B_1 \setminus \tilde{\Gamma}} \nabla W \cdot \nabla \phi \, dy = \int_{\partial B_1} h \phi \, dS
$$

(118)

for any $\phi \in H^1(B_1 \setminus \tilde{\Gamma})$. From the compactness of the trace operator $\gamma_1$ (see Proposition 36) and (117) it follows that

$$
\lim_{k \to \infty} \int_{\partial B_1} (A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} (y) \cdot \nu) W^{\lambda_n T_{\lambda_n}} (y) \, dS = \int_{\partial B_1} h W \, dS.
$$

Therefore, recalling estimates (110), (111), and the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \tilde{\Gamma})$, choosing $\phi = W^{\lambda_n T_{\lambda_n}}$ in (116) and passing to the limit as $k \to \infty$, we obtain that

$$
\lim_{k \to \infty} \int_{B_1 \setminus \tilde{\Gamma}} A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} \cdot \nabla W^{\lambda_n T_{\lambda_n}} \, dy = \int_{\partial B_1} h W \, dS.
$$

(119)

From (118) and (119) it follows that

$$
\lim_{k \to \infty} \int_{B_1 \setminus \tilde{\Gamma}} A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} \cdot \nabla W^{\lambda_n T_{\lambda_n}} \, dy = \int_{B_1 \setminus \tilde{\Gamma}} |\nabla W|^2 \, dy
$$

and so, thanks to (17),

$$
W^{\lambda_n T_{\lambda_n}} \to W \text{ strongly in } H^1(B_1 \setminus \tilde{\Gamma}).
$$

(120)

For any $k \in \mathbb{N}$ and $r \in (0, 1)$ let us define

$$
E_k(r) := r^{2-N} \int_{B_1 \setminus \tilde{\Gamma}} (A(\lambda_n T_{\lambda_n} y) \nabla W^{\lambda_n T_{\lambda_n}} \cdot \nabla W^{\lambda_n T_{\lambda_n}}
$$

$$
- (\lambda_n T_{\lambda_n})^2 \tilde{f}(\lambda_n T_{\lambda_n} y) |W^{\lambda_n T_{\lambda_n}}|^2 \, dy,
$$

$$
H_k(r) := r^{1-N} \int_{\partial B_r} \mu(\lambda_n T_{\lambda_n} y) |W^{\lambda_n T_{\lambda_n}}|^2 \, dS, \text{ and } N_k(r) := \frac{E_k(r)}{H_k(r)}.
$$

By a change of variables it is easy to verify that, for any $r \in (0, 1)$,

$$
N_k(r) = \frac{E_k(r)}{H_k(r)} = \frac{E(\lambda_n T_{\lambda_n} r)}{H(\lambda_n T_{\lambda_n} r)} = N(\lambda_n T_{\lambda_n} r).
$$

(121)

For any $r \in (0, 1)$, we also define

$$
H_W(r) := r^{1-N} \int_{\partial B_r} |W|^2 \, dS, \quad E_W(r) := r^{2-N} \int_{B_1 \setminus \tilde{\Gamma}} |\nabla W|^2 \, dy \text{ and } N_W(r) := \frac{E_W(r)}{H_W(r)}.
$$

(122)
The definition of $\mathcal{N}_W$ is well posed. Indeed, if $H_W(r) = 0$ for some $r \in (0, 1)$, then we may test the equation (115) on $B_r$ with $W$ and conclude that $W = 0$ in $B_r$. Thanks to classical unique continuation principles for harmonic functions, this would imply that $W = 0$ in $B_1$, thus contradicting (119).

Thanks to (120), (110)–(111) together with the boundedness of $\{W^\lambda\}$ in $H^1(B_1 \setminus \Gamma)$, (17), (20), and Proposition 5.12, passing to the limit as $k \to \infty$ in (121) we obtain that

$$\mathcal{N}_W(r) = \lim_{k \to \infty} N_k(r) = \lim_{k \to \infty} \mathcal{N}(\lambda_{n_k}T_{\lambda_{n_k}}r) = \gamma \quad \text{for any} \ r \in (0, 1).$$

Then $\mathcal{N}_W$ is constant in $(0, 1)$. Following the proof of Proposition 5.10 in the case $f \equiv 0$ and $g \equiv 0$ (where $g$ is the function defined in (5)–(6)), so that $A = \text{Id}_N$ and $\mu = 1$, we obtain that

$$0 = \mathcal{N}_W'(r) \geq \frac{2r \left( \int_{\partial B_r} \left| \frac{\partial W}{\partial r} \right|^2 \ dS \right) \left( \int_{\partial B_r} W^2 \ dS \right) - \left( \int_{\partial B_r} W \frac{\partial W}{\partial r} \ dS \right)^2}{\left( \int_{\partial B_r} W^2 \ dS \right)^2} \geq 0$$

for a.e. $r \in (0, 1)$. It follows that $\left( \int_{\partial B_r} \left| \frac{\partial W}{\partial r} \right|^2 \ dS \right) \left( \int_{\partial B_r} W^2 \ dS \right) = \left( \int_{\partial B_r} W \frac{\partial W}{\partial r} \ dS \right)^2$ for a.e. $r \in (0, 1)$, i.e. equality holds in the Cauchy-Schwarz inequality for the vectors $W$ and $\frac{\partial W}{\partial r}$ in $L^2(\partial B_r)$ for a.e. $r \in (0, 1)$. It follows that there exists a function $\zeta(r)$ such that

$$\frac{\partial W}{\partial r}(r\theta) = \zeta(r)W(r\theta) \quad \text{for any} \ \theta \in S^{N-1} \setminus \Sigma \text{ and a.e.} \ r \in (0, 1].$$

Multiplying by $W(r\theta)$ and integrating on $S^{N-1}$ we obtain

$$\int_{S^{N-1}} \frac{\partial W}{\partial r}(r\theta)W(r\theta) \ dS = \zeta(r) \int_{S^{N-1}} W^2(r\theta) \ dS,$$

so that $\zeta(r) = \frac{\mathcal{N}_W(r)}{2W(r\theta)} = \frac{\gamma}{r}$ by Proposition 5.2 and (122). Integrating (128) between $r \in (0, 1)$ and 1 we obtain that

$$W(r\theta) = r^\gamma W(\theta) = r^\gamma \Psi(\theta) \quad \text{for any} \ \theta \in S^{N-1} \setminus \Sigma \text{ and any} \ r \in (0, 1],$$

where $\Psi = W|_{S^{N-1} \setminus \Sigma}$. Then $\Psi \in H^1(S^{N-1} \setminus \Sigma)$; furthermore, substituting $W(r\theta) = r^\gamma \Psi(\theta)$ in (119) we find out that $\Psi$ is an eigenfunction of (4) with $(\gamma + N - 2) \gamma$ as an associated eigenvalue. Hence by Proposition 6.2 there exists $k_0 \in \mathbb{N}$ such that $(\gamma + N - 2) \gamma = \frac{k_0(k_0 + 2N - 4)}{4}$. Recalling from Proposition 5.12 that $\gamma \geq 0$, we then obtain (117).

To conclude the proof it is enough to show that $W^{\lambda_{n_k}} \to W$ strongly in $H^1(B_1 \setminus \Gamma)$ (possibly along a subsequence). Since $\{W^{\lambda_{n_k}}\}_{k \in \mathbb{N}}$ is bounded in $H^1(B_1 \setminus \Gamma)$ by Proposition 7.1 there exists a function $\tilde{W} \in H^1(B_1 \setminus \Gamma)$ and $T \in [1, 2]$ such that $W^{\lambda_{n_k}} \rightharpoonup \tilde{W}$ weakly in $H^1(B_1 \setminus \Gamma)$ and $T_{\lambda_{n_k}} \to T$, up to a subsequence.

Moreover, since $\{W^{\lambda_{n_k}}T_{\lambda_{n_k}}\}_{k \in \mathbb{N}}$ and $\{|W^{\lambda_{n_k}}T_{\lambda_{n_k}}|\}_{k \in \mathbb{N}}$ converge strongly in $L^2(B_1)$ by (120), they are dominated by a measurable $L^2(B_1)$-function, up to a subsequence. Similarly, thanks to (101), we can suppose that, up to a subsequence, the limit

$$\zeta := \lim_{k \to \infty} \frac{H(\lambda_{n_k}T_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and it is finite and strictly positive. Then for any $\phi \in C_c^\infty(B_1)$ we have that

$$\lim_{k \to \infty} \int_{B_1} W^{\lambda_{n_k}}(y)\phi(y) \ dy = \lim_{k \to \infty} T_{\lambda_{n_k}}^N \int_{B_T^{-1}} W^{\lambda_{n_k}}(T_{\lambda_{n_k}}y)\phi(T_{\lambda_{n_k}}y) \ dy$$

$$= \lim_{k \to \infty} T_{\lambda_{n_k}}^N \sqrt{\zeta} \int_{B_T^{-1}} W(y)\phi(Ty) \ dy = \sqrt{\zeta} \int_{B_1} W(y/T)\phi(y) \ dy,$$
thanks to the Dominated Convergence Theorem. By density the same holds for any \( \phi \in L^2(B_1) \). It follows that \( W^{\lambda_{nk}} \rightharpoonup \sqrt{\gamma}(W/T) \) weakly in \( L^2(B_1) \). Hence, by uniqueness of the weak limit, we have that \( \tilde{W}(\cdot) = \sqrt{\gamma}(W/T) \) and \( W^{\lambda_{nk}} \rightharpoonup \sqrt{\gamma}(W/T) \) weakly in \( H^1(B_1 \setminus \tilde{\Gamma}) \). Furthermore

\[
\lim_{k \to \infty} \int_{B_1 \setminus \tilde{\Gamma}} |\nabla W^{\lambda_{nk}}(y)|^2 \, dy = \lim_{k \to \infty} T_{\lambda_{nk}}^N \int_{B_{T \lambda_{nk}} \setminus \Gamma} |\nabla W^{\lambda_{nk}}(T_{\lambda_{nk}} y)|^2 \, dy \\
= \lim_{k \to \infty} T_{\lambda_{nk}}^{N-2} \frac{H(\lambda_{nk}, T_{\lambda_{nk}})}{H(\lambda_{nk})} \int_{B_{T \lambda_{nk}} \setminus \Gamma} |\nabla W_{T \lambda_{nk}}^{\lambda_{nk}}(y)|^2 \, dy \\
= T^{N-2} \zeta \int_{B_{T \Gamma} \setminus \Gamma} |\nabla W(y)|^2 \, dy = \int_{B_1 \setminus \tilde{\Gamma}} |\sqrt{\zeta}(W(T/T))|^2 \, dy.
\]

Then we can conclude that \( W^{\lambda_{nk}} \rightharpoonup \tilde{W} = \sqrt{\gamma}(W/T) \) strongly in \( H^1(B_1 \setminus \tilde{\Gamma}) \). Moreover, by compactness of the trace operator \( \gamma_1 \) (see Proposition 3.1, 29, and 100), we deduce that \( \int_{\partial B_1} \tilde{W}^2 \, dS = 1 \). Then, since \( W(r\theta) = r^\frac{k}{nk} \Psi(\theta) \), we deduce that

\[
\tilde{W}(r\theta) = \sqrt{\gamma}(W \left( \frac{r}{T} \theta \right)) = \left( \frac{\zeta}{T^{nk}} \right)^{\frac{k}{nk}} r^\frac{k}{nk} \Psi(\theta) = \left( \frac{\zeta}{T^{nk}} \right)^{\frac{k}{nk}} W(r\theta)
\]

and

\[
1 = \int_{\partial B_1} \tilde{W}^2 \, dS = \frac{\zeta}{T^{nk}} \int_{\partial B_1} W^2 \, dS = \frac{\zeta}{T^{nk}}.
\]

thanks to (100). Therefore \( W = \tilde{W} \) and the proof is complete. \( \square \)

We are now in position of prove Theorem 1.2.

**Proof of Theorem 1.2.** Let us assume that \( \text{Tr}_{\Gamma}^+ u(z) = O(|z|^k) \) as \( |z| \to 0^+ \), \( z \in \Gamma \), for all \( k \in \mathbb{N} \) (a similar argument works under the assumption \( \text{Tr}_{\Gamma}^+ u(z) = O(|z|^{\delta}) \)). Letting \( U = u \circ F \), by the properties of the diffeomorphism \( F \) described in Proposition 2.1, we have that \( \text{Tr}_{\tilde{\Gamma}}^+ U(z) = O(|z|^k) \) as \( |z| \to 0^+ \), so that, for all \( k \in \mathbb{N} \),

\[
\|\lambda^{-k} \text{Tr}_{\tilde{\Gamma}}^+ U(\lambda \cdot)\|_{L^2(B_1 \cap \Gamma)} \to 0 \quad \text{as} \quad \lambda \to 0^+.
\]

On the other hand, if, by contradiction, \( u \neq 0 \), by Proposition 7.3 and classical trace theorems there exist \( k_0 \in \mathbb{N} \), a sequence \( \lambda_n \to 0^+ \), and an eigenfunction \( \Psi \) of problem (7) such that

\[
\lim_{n \to \infty} \frac{\|\text{Tr}_{\tilde{\Gamma}}^+ U(\lambda_n \cdot)\|_{L^2(B_1 \cap \Gamma)}}{H(\lambda_n)^{\frac{k_0}{nk}}} = \|\text{Tr}_{\tilde{\Gamma}}^+ \left( |y|^\gamma \Psi \left( \frac{y}{|y|} \right) \right)\|_{L^2(B_1 \cap \Gamma)} \neq 0,
\]

where the above limit is nonzero thanks to Remark 6.3. Combining (124) and (125) we obtain that

\[
\lim_{n \to \infty} \sqrt{\frac{H(\lambda_n)}{\lambda_n^{k_0}}} = 0 \quad \text{for all} \quad k \in \mathbb{N},
\]

thus contradicting estimate (122). \( \square \)

8. ASYMPTOTICS OF THE HEIGHT FUNCTION \( H(\lambda) \) AS \( \lambda \to 0^+ \), WHEN \( N \geq 3 \)

In dimension \( N \geq 3 \), we can further specify the behaviour of \( U(\lambda \cdot) \) as \( \lambda \to 0^+ \), deriving the asymptotics of the function \( H(\lambda) \) appearing as a normalization factor in the blowed-up family (10). Let \( \{Y_{k,i}\}_{k \in \mathbb{N}, i = 1, \ldots, N_k} \) be the basis of \( L^2(S^{N-1}) \) given by Proposition 6.2. Let \( N \geq 3 \), \( u \in H^1(B_R \setminus \Gamma) \) be a weak solution to (11), with \( \Gamma \) defined in (3)–(4) and \( f \) satisfying either (111) or (112), and let \( U = u \circ F \) be the corresponding solution to (29). For any \( \lambda \in (0, r_0) \), \( k \in \mathbb{N} \) and \( i = 1, \ldots, N_k \) we define

\[
\varphi_{k,i}(\lambda) := \int_{S^{N-1}} U(\lambda \theta) Y_{k,i}(\theta) \, dS
\]

(126)
A direct calculation shows that, in the sense of distributions on \((0, r)\), we deduce that, for every \(r > 0\), we obtain that, for every \(r > 0\), thanks to (95). Integrating the right-hand side of the equation above by parts, since (129) holds, we may consider its derivative in the sense of distributions. A direct calculation shows that

\[
\mathcal{Y}_{k,i}(\lambda) := - \int_{B_\lambda \setminus \Gamma} (A - \text{Id}_N) \nabla U \cdot \frac{\nabla g_{N-1} Y_{k,i}(y/|y|)}{|y|} \, dy \\
+ \int_{B_\lambda} \tilde{f}(y)U(y)Y_{k,i}(y/|y|) \, dy + \int_{\partial B_\lambda}(A - \text{Id}_N) \nabla U \cdot \frac{y}{|y|} Y_{k,i}(y/|y|) \, dS.
\]

**Proposition 8.1.** Let \(k_0\) be as in Proposition 7.5. Then, for any \(i = 1, \ldots, N_{k_0}\) and \(r \in (0, r_0]\),

\[
\varphi_{k_0,i}(\lambda) = \frac{\lambda^k}{2(N + k_0 - 2)} \int_0^r s^{N+1-k-\frac{k}{N}} \mathcal{Y}_{k_0,i}(s) \, ds \\
+ \frac{k_0 r^{-N+2-k}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k}{N}-1} \mathcal{Y}_{k_0,i}(s) \, ds + o(\lambda^{k_0}) \quad \text{as } \lambda \to 0^+.
\]

**Proof.** For any \(k \in \mathbb{N}\) and any \(i = 1, \ldots, N_k\) we consider the distribution \(\zeta_{k,i}\) on \((0, r_0)\) defined as

\[
\mathcal{D}'((0, r_0)) (\zeta_{k,i}, \omega)_{\mathcal{D}(0,r_0)} := \int_0^{r_0} \omega(\lambda) \left( \int_{S^{N-1}} \tilde{f}(\lambda \theta) U(\lambda \theta) Y_{m,k}(\theta) \, dS_\theta \right) \, d\lambda \\
- \int_{B_{r_0} \setminus \Gamma} (A - \text{Id}_N) \nabla U \cdot \nabla \left( \left| y \right|^{1-N} \omega(|y|) Y_{m,k}(y/|y|) \right) \, dy,
\]

for any \(\omega \in \mathcal{D}(0, r_0)\).

Since \(\mathcal{Y}_{k,i} \in L^1_{\text{loc}}(0, r_0)\) by (127), we may consider its derivative in the sense of distributions. In the sense of distributions on \((0, r_0)\). From the definition of \(\zeta_{k,i}\), (26), and the fact that \(Y_{k,i}\) is a solution of (94) we deduce that

\[
-\varphi_{k,i}'(\lambda) = \frac{N-1}{\lambda} \varphi_{k,i}(\lambda) + \frac{\mu_k}{\lambda^2} \varphi_{k,i}(\lambda) = \zeta_{k,i}(\lambda)
\]

in the sense of distribution in \((0, r_0)\); the above equation can be rewritten as

\[
-\left(\lambda^{N-1+k} (\lambda^{1/2} \varphi_{k,i}(\lambda))'\right)' = \lambda^{N-1+k/2} \zeta_{k,i}(\lambda),
\]

thanks to (95). Integrating the right-hand side of the equation above by parts, since (129) holds, we obtain that, for every \(r \in (0, r_0)\), \(k \in \mathbb{N}\) and \(i = 1, \ldots, N_k\) there exists a constant \(c_{k,i}(r)\) such that

\[
(\lambda^{1/2} \varphi_{k,i}(\lambda))' = -\lambda^{N-1+k} (\lambda^{1/2} \varphi_{k,i}(\lambda))' = \lambda^{N-1+k} \left( c_{k,i}(r) + \int_0^r s^{k/2-1} \mathcal{Y}_{k,i}(s) \, ds \right)
\]

in the sense of distribution on \((0, r_0)\). Then \(\varphi_{k,i}(\lambda) \in W^{1,1}_{\text{loc}}(0, r_0)\) and a further integration yields

\[
\varphi_{k,i}(\lambda) = \lambda^{\frac{k}{2}} \left( r^{\frac{k}{2}} \varphi_{k,i}(r) + \int_0^r s^{N-1+k/2} \mathcal{Y}_{k,i}(s) \, ds \right) \\
+ \frac{k}{2} \lambda^{\frac{k}{2}} \left( \int_0^r s^{N-1+k} \left( c_{k,i}(r) + \int_0^r t^{k/2-1} \mathcal{Y}_{k,i}(t) \, dt \right) \, ds \right) \\
= \lambda^{\frac{k}{2}} \left( r^{\frac{k}{2}} \varphi_{k,i}(r) + \frac{2N+k-4}{2(N+k-2)} \int_0^r s^{N-1+k/2} \mathcal{Y}_{k,i}(s) \, ds \right) \\
- \lambda^{\frac{k}{2}} \frac{k c_{k,i}(r) r^{N+2-k}}{2(N+k-2)} + \frac{k \lambda^{N-2+k/2}}{2(N+k-2)} \left( c_{k,i}(r) + \int_0^r t^{k/2-1} \mathcal{Y}_{k,i}(t) \, dt \right).
\]

Now we claim that, if \(k_0\) is as in Proposition 7.5, then

\[
\text{the function } s \to s^{N+1-k/2} \mathcal{Y}_{k_0,i}(s) \text{ belongs to } L^1(0, r_0).
\]
To this end we will estimate each terms in (124). Thanks to (17), Hölder’s inequality, a change of variables and Proposition (7.1) we have that

$$\left| \int_{B_s \setminus \Gamma} (A - \text{Id}_N) \nabla U \cdot \frac{\nabla s_{N-1} Y_{k_0,i}(y/|y|)}{|y|} \, dy \right| \leq \text{const} \int_{B_s \setminus \Gamma} |y| |\nabla U| \left| \frac{\nabla s_{N-1} Y_{k_0,i}(y/|y|)}{|y|} \right| \, dy$$

$$\leq \text{const} \left( \int_{B_s \setminus \Gamma} |\nabla U|^2 \, dy \right)^{\frac{1}{2}} \left( \int_{B_s \setminus \Gamma} \left| \frac{\nabla s_{N-1} Y_{k_0,i}(y/|y|)}{|y|} \right|^2 \, dy \right)^{\frac{1}{2}}$$

$$\leq \text{const} \, s^{\frac{N-2}{2}} \sqrt{H(s)} \left( \int_{B_s \setminus \Gamma} |\nabla W|^2 \, dy \right)^{\frac{1}{2}} \leq \text{const} \, s^{N-1} \sqrt{H(s)}.$$
We argue by contradiction assuming that there exists $r \in (0, r_0)$ such that (134) does not hold. Then by (130) and (133)
\begin{equation}
\varphi_{k_0,i}(\lambda) \sim \frac{k_0 \lambda^{-N+2-b_k}}{2(N+k_0-2)} \left( c_{k_0,i}(r) + \int_{\lambda}^{r} t^{\frac{b_k}{N+k_0-2} - 1} Y_{k_0,i}(t) \, dt \right)
\end{equation}
as $\lambda \to 0^+$. From Hölder’s inequality, a change of variables, and (33)
\begin{equation}
\int_{0}^{r_0} \lambda^{N-3} |\varphi_{k_0,i}(\lambda)|^2 \, d\lambda \leq \int_{0}^{r_0} \lambda^{N-3} \left( \int_{S^{N-1}} |U(\lambda \theta)|^2 \, dS \right) \, d\lambda = \int_{B_{r_0}} \frac{|U|^2}{|y|^2} \, dy < +\infty
\end{equation}
thus contradicting (135). Hence (134) is proved. Furthermore from (132) and (134)
\begin{equation}
\left| \lambda^{-N+2-b_k} \left( c_{k_0,i}(r) + \int_{\lambda}^{r} t^{\frac{b_k}{N+k_0-2} - 1} Y_{k_0,i}(t) \, dt \right) \right|
\leq \lambda^{-N+2-b_k} \int_{0}^{\lambda} t^{\frac{b_k}{N+k_0-2} - 1} Y_{k_0,i}(t) \, dt
\leq \lambda^{\frac{b_k}{N+k_0-2}} \int_{0}^{\lambda} t^{\frac{b_k}{N+k_0-2} - 1} Y_{k_0,i}(t) \, dt = O \left( \lambda^{\frac{b_k}{N+k_0-2}} \right)
as \lambda \to 0^+.
\end{equation}
Then the conclusion follows from (130), (134), and (136).

\begin{prop}
Let $\gamma$ be as in (90). Then
\[ \lim_{r \to 0^+} r^{2\gamma} H(r) > 0. \]
\end{prop}
\begin{proof}
For any $\lambda \in (0, r_0)$ the function $U(\lambda \cdot)$ belongs to $L^2(S^{N-1})$. Then we can expand it in Fourier series respect to the basis \( \{Y_{k,i}\}_{k \in \mathbb{N}, i=1,\ldots,N_k} \) introduced in Proposition 6.2
\[ U(\lambda \cdot) = \sum_{k=0}^{\infty} \sum_{i=1}^{N_k} \varphi_{k,i}(\lambda) Y_{k,i} \quad \text{in} \quad L^2(S^{N-1}), \]
where we have defined $\varphi_{k,i}(\lambda)$ in (125) for any $k \in \mathbb{N}$ and any $i = 1, \ldots, N_k$. From (20), a change of variables and the Parseval identity
\begin{equation}
H(\lambda) = (1 + O(\lambda)) \int_{S^{N-1}} U^2(\lambda \theta) \, dS = (1 + O(\lambda)) \sum_{k=0}^{\infty} \sum_{i=1}^{N_k} |\varphi_{k,i}(\lambda)|^2.
\end{equation}
We argue by contradiction assuming that $\lim_{r \to 0^+} r^{2\gamma} H(r) = 0$. Then by (137), letting $k_0$ be as in (107),
\begin{equation}
\lim_{\lambda \to 0^+} \lambda^{-\frac{b_k}{N+k_0-2}} \varphi_{k_0,i}(\lambda) = 0 \quad \text{for any} \quad i = 1, \ldots, N_{k_0}.
\end{equation}
From (128) it follows that
\begin{equation}
r^{-\frac{b_k}{N+k_0-2}} \varphi_{k_0,i}(r) + \frac{2N+k_0-4}{2(N+k_0-2)} \int_{0}^{r} s^{-N+1+b_k} Y_{k_0,i}(s) \, ds
+ \frac{k_0 r^{-N+2+b_k}}{2(N+k_0-2)} \int_{0}^{r} s^{\frac{b_k}{N+k_0-2} - 1} Y_{k_0,i}(s) \, ds = 0
\end{equation}
for any $r \in (0, r_0)$ and any $i = 1, \ldots, N_{k_0}$.

In view of (90), (126), (134), and (136), (139) implies that
\begin{equation}
\sqrt{H(\lambda)} \int_{S^{N-1}} W^\lambda Y_{k_0,i} \, dS = \varphi_{k_0,i}(\lambda) = O \left( \lambda^{\frac{b_k}{N+k_0-2}} \right) \quad \text{as} \quad \lambda \to 0^+
\end{equation}
for all $i = 1, \ldots, N_{k_0}$. From (122) with $\sigma = \frac{4k}{N+2\epsilon}$ we have that $\sqrt{H(\lambda)} \geq \sqrt{1 - \frac{4k}{N+2\epsilon} \lambda^{\frac{b_k}{N+k_0-2}}}$ in a neighbourhood of 0, so that (140) implies that
\begin{equation}
\int_{S^{N-1}} W^\lambda Y_{k_0,i} \, dS = O \left( \lambda^{\frac{b_k}{N+k_0-2}} \right) = o(1) \quad \text{as} \quad \lambda \to 0^+
\end{equation}
for all $i = 1, \ldots, N_{k_0}$.

On the other hand, by Proposition 3.4 and continuity of the trace map $\gamma_1$ (see Proposition 3.1), for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and $\Psi \in \text{span}\{Y_{k_0,i} : m = 1, \ldots, N_{k_0}\}$ such that

\[
\|\Psi\|_{L^2(\mathbb{S}^{N-1})} = 1 \quad \text{and} \quad W^{\lambda_{n_k}} \to \Psi \quad \text{in} \quad L^2(\mathbb{S}^{N-1}).
\]

From (141) and (142) it follows that

\[
0 = \lim_{k \to \infty} \int_{\mathbb{S}^{N-1}} W^{\lambda_{n_k}} \Psi \, dS = \|\Psi\|^2_{L^2(\mathbb{S}^{N-1})} = 1,
\]

thus reaching a contradiction. $\square$

We are now ready to prove the following result, which is a more complete version of Theorem 1.3.

**Theorem 8.3.** Let $N \geq 3$ and let $u \in H^1(B_R \setminus \Gamma)$ be a non-trivial weak solution to (11), with $\Gamma$ defined in (3) and $f$ satisfying either assumption (111) or assumption (112). Then there exists $k_0 \in \mathbb{N}$ such that, letting $N$ be as in Section 4.

\[
\lim_{r \to 0^+} N(r) = \frac{k_0}{2}.
\]

Moreover if $N_{k_0}$ is the multiplicity of the eigenvalue $\mu_{k_0}$ of problem (4) and $\{Y_{k_0,i}\}_{i=1, \ldots, N_{k_0}}$ is a $L^2(\mathbb{S}^{N-1})$-orthonormal basis of the eigenspace associated to $\mu_{k_0}$, then

\[
\lambda^\frac{k_0}{2} u(\lambda \cdot) \to \Phi \quad \text{and} \quad \lambda^{1-\frac{k_0}{2}} (\nabla_{\mathbb{R}^N \setminus \Gamma} u)(\lambda \cdot) \to \nabla_{\mathbb{S}^{N-1}} \Phi \quad \text{in} \quad L^2(B_1) \quad \text{as} \quad \lambda \to 0^+,
\]

where

\[
\Phi = \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \left( \frac{y}{|y|} \right)
\]

$(\alpha_1, \ldots, \alpha_{N_{k_0}}) \neq (0, \ldots, 0)$ and, for all $i \in \{1, 2, \ldots, N_{k_0}\}$,

\[
\alpha_i = r^{-k_0/2} \int_{\mathbb{S}^{N-1}} u(F(r\theta)) Y_{k_0,i}(\theta) \, dS + \frac{1}{2 - N - k_0} \int_0^r \left( 2 - N - \frac{k_0}{s^{N-2+k_0}} - \frac{k_0 s^{k_0-1}}{2^{2N-2+k_0}} \right) Y_{k_0,i}(s) \, ds
\]

for any $r \in (0, r_0)$ for some $r_0 > 0$, where we have defined $Y_{k_0,i}$ in (127) and $F$ is the diffeomorphism introduced in Proposition 2.7.

**Proof.** (143) directly comes from (104). Let $U = u \circ F$ and $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \to \infty} \lambda_n = 0^+$. By Proposition 7.5 and Proposition 8.2 there exist a subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ and constants $a_1, \ldots, a_{N_{k_0}}$ such that $(a_1, \ldots, a_{N_{k_0}}) \neq (0, \ldots, 0)$ and

\[
\lambda_{n_k} U(\lambda_{n_k}y) \to |y|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \left( \frac{y}{|y|} \right) \quad \text{in} \quad H^1(B_1 \setminus \Gamma) \quad \text{as} \quad k \to \infty.
\]

Now we show that the coefficients $a_1, \ldots, a_{N_{k_0}}$ do not depend on $\{\lambda_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$. Thanks to the continuity of the trace operator $\gamma_1$ introduced in Proposition 3.1

\[
\lambda_{n_k} U(\lambda_{n_k}y) \to \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \quad \text{in} \quad L^2(\mathbb{S}^{N-1}) \quad \text{as} \quad k \to \infty
\]

and therefore, letting $\varphi_{k_0,i}$ be as in (129) for any $i = 1, \ldots, N_{k_0}$,

\[
\lim_{k \to \infty} \lambda^\frac{k_0}{2} \varphi_{k_0,i}(\lambda_{n_k}) = \lim_{k \to \infty} \int_{\mathbb{S}^{N-1}} \lambda_{n_k}^{k_0/2} U(\lambda_{n_k} \theta) Y_{k_0,i}(\theta) \, dS = \sum_{j=1}^{N_{k_0}} \alpha_j \int_{\mathbb{S}^{N-1}} Y_{k_0,j} Y_{k_0,i} \, dS = \alpha_i.
\]
On the other hand by (128)
\[
\lim_{k \to \infty} \lambda^{\frac{k_0}{2}} \varphi_{k_0,i}(\lambda_{n_k}) = r \lambda^{\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_0^r s^{-N+\frac{k_0}{2}} \varphi_{k_0,i}(s) \, ds + \frac{k_0 r^{-N+2-k_0}}{2(N + k_0 - 2)} \int_0^r s^{\frac{k_0}{2}-1} \varphi_{k_0,i}(s) \, ds,
\]
for all \(i = 1, \ldots, N_{k_0}\) and \(r \in (0, r_0)\), where we have defined \(\varphi_{k_0,i}\) in (127). We deduce that
\[
\alpha_i = r^{\frac{k_0}{2}} \varphi_{k_0,i}(r) + \frac{2N + k_0 - 4}{2(N + k_0 - 2)} \int_0^r s^{-N+\frac{k_0}{2}} \varphi_{k_0,i}(s) \, ds
\]
and so \(\alpha_i\) does not depend on \(\{\lambda_n\}_{n \in \mathbb{N}}\) nor on its subsequence \(\{\lambda_{n_k}\}_{k \in \mathbb{N}}\) thus implying that
\[
\lambda^{\frac{k_0}{2}} U(\lambda y) \to \|y\|^{\frac{k_0}{2}} \sum_{i=1}^{N_{k_0}} \alpha_i Y_{k_0,i} \left( \frac{y}{|y|} \right) \quad \text{in } H^1(B_1 \setminus \Gamma) \quad \text{as } \lambda \to 0^+.
\]
To prove (144) we note that
\[
\lambda^{\frac{k_0}{2}} u(\lambda x) = \lambda^{\frac{k_0}{2}} U(\lambda G_\lambda(x)), \quad \nabla \left( \lambda^{\frac{k_0}{2}} u(\lambda x) \right) = \nabla \left( \lambda^{\frac{k_0}{2}} U(\lambda x) \right) (G_\lambda(x)) J_{G_\lambda}(x),
\]
where \(G_\lambda(x) = \frac{1}{\lambda} F^{-1}(\lambda x)\) and \(F\) is the diffeomorphism introduced in Proposition 2.1. We also have by Proposition 2.1 that
\[
G_\lambda(x) = x + O(\lambda) \quad \text{and} \quad J_G(x) = \text{Id}_N + O(\lambda)
\]
as \(\lambda \to 0^+\) uniformly respect to \(x \in B_1\). Then from (147) we deduce (144) and (135) follows from (146) and (127). 

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