A Fast Randomized Geometric Algorithm for Computing Riemann-Roch Spaces

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Abstract

We propose a probabilistic Las Vegas variant of Brill-Noether’s algorithm for computing a basis of the Riemann-Roch space \( L(D) \) associated to a divisor \( D \) on a projective plane curve \( C \) over a sufficiently large perfect field \( k \). Our main result shows that this algorithm requires at most \( O(\max(\deg(C)^2, \deg(D+)^\omega)) \) arithmetic operations in \( k \), where \( \omega \) is a feasible exponent for matrix multiplication and \( D+ \) is the smallest effective divisor such that \( D+ \geq D \). This improves the best known upper bounds on the complexity of computing Riemann-Roch spaces. Our algorithm may fail, but we show that provided that a few mild assumptions are satisfied, the failure probability is bounded by \( O(\max(\deg(C)^4, \deg(D+)^2)/|E|) \), where \( E \) is a finite subset of \( k \) in which we pick elements uniformly at random. We provide a freely available C++/NTL implementation of the proposed algorithm and we present experimental data. In particular, our implementation enjoys a speed-up larger than 9 on several examples compared to the reference implementation in the Magma computer algebra system. As a by-product, our algorithm also yields a method for computing the group law on the Jacobian of a smooth plane curve of genus \( g \) within \( O(g^\omega) \) operations in \( k \), which slightly improves in this context the best known complexity \( O(g^{\omega+\epsilon}) \) of Khuri-Makdisi’s algorithm.

1 Introduction

The Riemann-Roch theorem is a fundamental result in algebraic geometry. In its classical version for smooth projective curves, it provides information on the dimension of the linear space of functions with some prescribed zeros and poles. The computation of such Riemann-Roch spaces is a subroutine used in several areas of computer science and computational mathematics. One of its most prominent applications is the construction of algebraic-geometric error-correcting codes [9]: Such codes are precisely (subspaces of) Riemann-Roch spaces. Another direct application is the computation of the group law on the Jacobian of a curve: representing a point in the Jacobian of a genus-\( g \) curve \( C \) as \( D - gO \), where \( D \) is an effective divisor of degree \( g \) and \( O \) is a fixed rational point (or more generally, a fixed divisor of degree 1), the sum of the classes of \( D_1 - gO \) and \( D_2 - gO \) can be computed by finding a function \( f \) in the Riemann-Roch space \( L(D_1+D_2-gO) \). Indeed, by setting \( D_3 = D_1 + D_2 - gO + (f) \), the divisor \( D_3 - gO \) is linearly equivalent to \( (D_1 - gO) + (D_2 - gO) \).

State of the art and related works. In this paper, we focus on the classical geometric approach attributed to Brill and Noether for computing Riemann-Roch spaces. The general algorithmic setting for this approach is described by Goppa in its landmark paper [9, §4]. Given a divisor \( D \) on a plane projective curve \( C \subset \mathbb{P}^2 \), this method proceeds by finding first a common denominator to all the functions in the Riemann-Roch space \( L(D) \). This is done by computing a form \( h \) on the curve such that the associated principal effective divisor \( (h) \) satisfies \( (h) \geq D \). Then the residual divisor \( (h) - D \) is computed. From this, a basis of the Riemann-Roch space is found by computing the kernel of a linear map. The correctness of this method is ensured by the residue theorem of Brill and Noether, which works even in the presence of ordinary singularities by using the technique of adjoint curves, see [20, § 42][7, Sec. 8.1].
its original version [9, §4], Goppa’s algorithm works only for finite fields, and some parts of the algorithm use exhaustive search. During the 90s, several versions of Goppa’s algorithm have been proposed, incorporating tools of modern computer algebra. In particular, Huang and Ferardi provide in [13] a deterministic algorithm for computing Riemann-Roch spaces of plane curves $C$ all singularities of which are ordinary within $O(\deg(C)^\omega \deg(D_+)^2)$ arithmetic operations in the base field, where $D_+$ is the smallest effective divisor such that $D_+ \geq D$. In fact, writing $D_- = D_+ - D$, we can assume without loss of generality that $\deg(D_+) \geq \deg(D_-)$, since $L(D) = L(D_+ - D_-)$ is reduced to zero if $\deg(D) < 0$. Consequently, $\deg(D_+)$ is a relevant measure of the size of the divisor $D$. Haché [10] proposes the first implementation of Brill-Noether’s approach in a computer algebra system, using local desingularizations to handle singularities encountered during the algorithm. For lines of research closely related to this topic, we refer to [16, 11] and references therein.

A few years later, a breakthrough is achieved by Hess [12]: He provides an arithmetic approach to the Riemann-Roch problem, using fast algorithms for algebraic function fields. Hess’ algorithm is now considered as a reference method for computing Riemann-Roch spaces, and it is proved to be polynomial in the input size [12, Remark 6.2].

An important special case of the computation of Riemann-Roch spaces is the computation of the group law on Jacobians of curves. In this special case, $\deg(D_+) \leq g$, where $g$ is the genus of the curve. Volchock [23] describes an algorithm with complexity $O(\max(\deg(C), g)^7)$ in this context. The best known complexity for computing the group law on Jacobians of general curves is currently achieved by Khuri-Makdisi in [15], where he gives an algorithm which requires $O(g^{\omega+\epsilon})$ operations in the base field, where $\omega$ is a feasible exponent for matrix multiplication and $\epsilon$ is any fixed positive number.

**Main results.** We propose a probabilistic Las Vegas algorithm for computing Riemann-Roch spaces on plane projective curves $C \subset \mathbb{P}^2$ defined over sufficiently large perfect fields. Our main result is that its complexity is bounded by $O(\max(\deg(C)^{2\omega}, \deg(D_+)^2))$ and that, provided that some mild assumptions are satisfied, its failure probability is bounded above by $O(\max(\deg(C)^4, \deg(D_+)^2)/|E|)$, where $E$ is a finite subset of the base field $k$ in which we can draw elements uniformly at random. One of the assumptions we need is that the cardinality of $k$ is greater than $\left(\frac{\deg(D_+) + 1}{2}\right)^{\deg(C)^{\omega+\epsilon}}$. This is a very mild requirement, since if $k$ is a small finite field, then this inequality can be enforced by replacing $k$ by an algebraic extension of degree $\log\left(\frac{\deg(D_+) + 1}{2}\right) + 1)/\log(|k|))$. If $C$ is singular, then we require some mild assumptions on the input divisor. Roughly speaking, we ask that no singular point gets in the way during the algorithm. In particular, this means that the input divisor must not involve any singular point. Also, we shall make some technical assumptions which ensure the existence of functions on the curve satisfying some properties but which do not vanish at any singular point. We emphasize that these assumptions are satisfied in most cases. We expect that it may be possible to remove these assumptions without harming the global complexity by computing a local desingularization on the fly if we encounter a singular point, by doing for instance as in [11, Section 3]. However, the complexity analysis in this case would be beyond the scope of this paper, so we keep this question open for future works.

Up to our knowledge, the complexity that we obtain is the best bound for the general problem of computing Riemann-Roch spaces. Even in the special case of the group law on the Jacobian of plane smooth curves where $\deg(D_+) = O(g)$ and $\deg(C) = O(\sqrt{g})$ by the genus-degree formula, the complexity becomes $O(g^\omega)$ which improves slightly the best known complexity bound $O(g^{\omega+\epsilon})$ of Khuri-Makdisi’s algorithm. Moreover, the algorithm that we propose requires very few assumptions, and its efficiency relies on classical building blocks in modern computer algebra: fast arithmetic of univariate polynomials and fast linear algebra. Consequently, our algorithm can be easily made practical by using existing implementations of these building blocks. We have made a C++/NTL implementation of our algorithm which is freely distributed under LGPL-2.1+ license and which is available at https://gitleab.inria.fr/pspaenele/rrspace. We also provide experimental results which seems to indicate that our prototype software is competitive with the reference implementation in the Magma computer algebra system [1].
Organization of the paper. Section 2 provides an overview of the main algorithm. Section 3 focuses on the data structures used to represent effective divisors. Algorithms to perform additions and subtractions of divisors with this representation are described in Section 4. Then Section 5 gives the details of the subroutines used in the main algorithm, and their correctness is proved. Section 6 focuses on the complexity of the subroutines and of the main algorithm. Then Section 7 is devoted to the analysis of the failure probability. Finally, Section 8 presents some experimental results obtained with our NTL/C++ implementation.

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2 Overview of the algorithm

This section is devoted to the description of the general setting of the Brill-Noether’s method and of the algorithm that we propose, without giving yet all the details on the data structures that we use to represent mathematical objects.

Throughout this paper, $k$ is a perfect field and $C \subset \mathbb{P}^2$ is an absolutely irreducible projective curve defined over $k$. Also, we let $\bar{k}$ denote an algebraic closure of $k$. By divisor, we always mean a Weil divisor on the curve $C$, i.e. a formal sum with integer coefficients of closed points of $C$. For simplicity, we will fix an affine open subset $C^0$ of $C$, so that $k[C^0] = k[X,Y]/q(X,Y)$ for some polynomial $q$. We shall also require that all the divisors that we consider are defined over $k$, i.e. that they are invariant under the natural action of the Galois group $\text{Gal}(\bar{k}/k)$ for any extension $K$ of $k$. In this setting, effective divisors on $C^0$ which do not involve any singular point can be thought of as nonzero ideals $I$ in $k[C^0]$ such that $I + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^0]$. For two divisors $D,D'$ on $C$, we write $D \leq D'$ if the valuation of $D$ at any place of $k(C)$ is at most the valuation of $D'$. If $g \in k[C^0]$ is a nonzero regular function on $C^0$ or if $g \in k[C]$ is a nonzero form on $C$, then we let $(g)$ denote the associated effective principal divisor, as defined in [7, Sec. 8.1]. If $f \in k(C)$ is a nonzero function on $C$, i.e. a quotient $f = g/h$ of two nonzero forms $g,h \in k[C]$ of the same degree, then we let $(f)$ denote the associated degree-0 principal divisor. Finally, for a divisor $D$ we let $L(D) = \{ f \in k(C) \setminus \{0\} \mid (f) \geq -D \} \cup \{0\}$ denote the Riemann-Roch space associated to $D$.

Assumptions on the input divisor. If the curve $C$ is singular, then we need three mild assumptions on the input divisor $D$ to ensure that our algorithm does not always fail. First, the support of the input divisor $D$ must not contain any singular point of the curve. To describe the second assumption – which is more technical – we need some insight on the data structure that we will use: The input divisor $D$ will be given as a pair of effective divisors $(D_+, D_-)$ such that $D = D_+ - D_-$. Set

$$d = \begin{cases} \lfloor \deg(D_+)/\deg(C) + (\deg(C) - 1)/2 \rfloor & \text{if } \lfloor \deg(C) + 1 \rfloor \leq \deg(D_+) \\ \lfloor (\sqrt{1 + 8 \deg(D_+)} - 1)/2 \rfloor & \text{otherwise}. \end{cases}$$

We will see in the sequel that this value of $d$ is in fact the smallest integer which ensures the existence of a nonzero form $h \in k[C]$ of degree $d$ such that $(h) \geq D_+$. Our second assumption is that there exists such a form $h$ of degree $d$ which does not vanish at any singular point of $C$. The third assumption is that there exists a basis $\{f_i/h_i \mid f_i,h_i \in k[C]\}_{i \in [1,L(D)]}$ of $L(D)$ which is such that the forms $h_i$ do not vanish at any singular point of the curve. We expect these three conditions to be satisfied by a generic linear projection to $\mathbb{P}^2$ of a nonsingular projective model of the curve. However, proving this statement would be beyond the scope of this paper.

Algorithm 1 gives a bird’s eye view of our algorithm for computing Riemann-Roch spaces. We now describe briefly what is done at each step of the algorithm. The routine INTERPOLATE takes as input an effective divisor $D_+$, and it returns a form $h$ such that $(h) \geq D_+$.\end{document}
can be written as 

g/h

to the vector space spanned by the output of Algorithm 1. To this end, we must prove that 

\[ \lambda g/h = (f) \]

We first prove that there exists a basis of 

\[ L(h) \]

Proof. Let 

\[ D, D' \]

be two linearly equivalent effective divisors on \( C \), and \( g, g' \in k[C] \) be two forms such that 

\[ D + (g) = D' + (g') \]

Let \( h \in k[C] \) be a form such that \( (h) = D + A \) for some effective divisor \( A \). If \( g' \) does not vanish at any singular point of the curve, then there exists a form \( h' \in k[C] \) of the same degree as \( h \) such that \( (h') = D' + A \).

Proof. The roadmap of the proof is similar to that of the residue theorem in [7, Sec. 8.1]. First, we notice that we have the inequality 

\[ (gh) = D' + A + (g') \geq (g') \]

since \( D' \) and \( A \) are both effective. Next, we use the fact that \( g' \) does not vanish at any singular point of the curve and that the localization of the coordinate ring at a nonsingular point is a discrete valuation ring (see e.g. [7, Sec. 3.2, Thm. 1]). This ensures that the inequality \( (gh) \geq (g') \) implies that Noether’s conditions are satisfied in the AF+BG theorem with \( F = \lambda g/h \) and \( H = gh \), with the notation \( F, G, H \) as in [7, Sec. 5.5]. Applying Noether’s AF+BG theorem yields a form \( h' \in k[C] \) of the same degree as \( h \) such that \( gh = g'h' \) in \( k[C] \) and hence \( (h') = (gh) - (g') = D' + A \) as expected.

We can now prove the general correctness of the main algorithm, assuming that all the subroutines behave correctly.

**Theorem 2.** If all the subroutines \( \text{Interpolate}, \text{CompPrincDiv}, \text{SubtractDivisors}, \text{AddDivisors}, \text{NumeralorBasis} \) are correct, then Algorithm 1 is correct: It returns a basis of the space \( L(D) \).

Proof. We first prove that there exists a basis of \( L(D) \) such that any basis element \( f \) belongs to the vector space spanned by the output of Algorithm 1. To this end, we must prove that \( f \) can be written as \( g/h \) where \( h \) is the output of the subroutine \( \text{Interpolate} \) and \( g \) belongs to the vector space spanned by the output of the subroutine \( \text{NumeralorBasis} \). Using the third assumption on the input divisor (described at the beginning of this section), we can choose a basis of \( L(D) \) such the denominators do not vanish at singular points of \( C \). Proposition 1 with \( D = D_+, D' = D_+ + (f) \) and \( h \), together with the fact that the denominator of \( f \) does not vanish at any singular point, implies that there exists a form \( g \in k[C] \) such that \( (g/h) = (f) \), where \( g \) has the same degree as \( h \). Therefore, \( f = \lambda g/h \) for some nonzero \( \lambda \in k \). It remains to
prove that $g$ belongs to the vector space spanned by the output of \texttt{NumeratorBasis}. Since
$f \in L(D)$, we must have $(g) = (f) + (h) \geq (h) - D = (D_h - D_+) + D_- = D_{\text{res}} + D_- = D_{\text{num}}$. But \texttt{NumeratorBasis} returns precisely a basis of the space of forms $\alpha$ of the same degree as $h$ such that $(\alpha) \geq D_{\text{num}}$.

Conversely, let $f$ be a function returned by Algorithm 1. Then $f \cdot h$ belongs to $B$, and hence
$(f \cdot h) = (f) + (h) \geq D_{\text{num}} = (h) - D$. This implies that $(f) \geq - D$ and hence $f \in L(D)$.

### 3 Data structures

#### Data structure for the curve $C$. Naming $X,Y,Z$ the homogeneous coordinates of $\mathbb{P}^2$, let
$\mathcal{C}^0 \subset \mathbb{A}^n$ be the affine curve obtained by intersecting $\mathcal{C}$ with the open subset $\{Z \neq 0\} \subset \mathbb{P}^2$. It is described by a bivariate polynomial $q(X,Y) \in k[X,Y]$. Closed points of $\mathcal{C}^0$ correspond to maximal ideals in $k[\mathcal{C}^0] = k[X,Y]/q(X,Y)$. Let $k[\mathcal{C}]$ denote the homogeneous coordinate ring $k[X,Y,Z]/(Z^{\deg(q)}q(X/Z,Y/Z))$. We assume that $q$ is monic in $Y$ (i.e. it is monic when seen as a polynomial in $k(X)[Y]$), and that the degree in $Y$ of $q$ equals its total degree. These two conditions on $q$ imply that $C$ is in projective Noether position with respect to the projection on the line $Y = 0$, i.e. that the canonical map $k[X,Z] \to k[\mathcal{C}]$ is injective and that it defines an integral ring extension. Note that this implies that the map $k[X] \to k[\mathcal{C}]$ is also an integral ring extension. We refer to [8, Sec. 3.1] for more details on the projective Noether position. We emphasize that the projective Noether position is achieved in generic coordinates. Consequently, it can be enforced by a harmless linear change of coordinates.

#### Data structure for forms. We will represent forms on $C$ – namely elements in $k[\mathcal{C}] = k[X,Y,Z]/(Z^{\deg(q)}q(X/Z,Y/Z))$ – by their affine counterpart in the affine chart $Z = 1$. Consequently, we shall represent a form $g \in k[\mathcal{C}]$ as an element in $k[X,Y]/q(X,Y)$, given by a representative $\tilde{g} \in k[X,Y]$ such that $\deg_{Y}(\tilde{g}) < \deg(C)$, using the fact that $q$ is monic in $Y$. This representation is not faithful since it does not encode what happens on the line $Z = 0$ at infinity. In order to encode the behaviour on this line and obtain a faithful representation, it is enough to adjoin to $\tilde{g}$ the degree $d$ of the form $g$, since $g$ is the class of $Z^{d}\tilde{g}(X/Z,Y/Z)$ in $k[\mathcal{C}]$. In the sequel of this paper, we do not mention further this issue and we often identify $g$ with $\tilde{g}$ by slight abuse of notation when the context is clear.

#### Data structure for divisors. For representing divisors, we use a data structure strongly inspired by the Mumford representation for divisors on hyperelliptic curves and by representations of algebraic sets by primitive elements as in [3]. Our data structures for divisors require some mild assumptions on the divisor that we represent.

We recall that we only consider divisors which do not involve singular points, in particular because our representation will contain local analytic data of the curve at any point in the support of the input divisor. Another restriction is that we assume that none of the points in supports of the divisors that we represent lie at infinity. In fact, this is not a strong restriction since all points can be brought to an affine chart via a projective change of coordinate. If one does not wish to change the coordinate system, a solution is to maintain three representations, one for each of the three canonical affine charts covering $\mathbb{P}^2$. Another solution (which is often used in practice) is to use an additional data structure to represent the multiplicities of a divisor at the places at infinity: Since there are only finitely-many such places at infinity, this has a negligible impact on the complexity.

We shall represent a divisor $D$ as a pair of effective divisors $(D_{+},D_{-})$ such that $D = D_{+} - D_{-}$. One crucial point for the representation of effective divisors is that the 0-dimensional algebraic set corresponding to the support of an effective divisor $D$ can be described by a finite étale algebra which is a quotient of $k[\mathcal{C}]^{0}$ by a nonzero ideal. This étale algebra is isomorphic to a quotient of a univariate polynomial ring if it admits a primitive element. Using primitive elements to represent 0-dimensional algebraic sets is a classical technique in computer algebra, see e.g. [3, Sec. 2][8, Sec. 3.2].
Lemma 3. Let $R$ be a finite étale $k$-algebra, i.e. a finite product of finite extensions of $k$. Let $z \in R$ be an element, and let $m_z$ denote the multiplication by $z$ in $R$, seen as a $k$-linear endomorphism. The following statements are equivalent:

1. The element $z$ generates $R$ as a $k$-algebra;
2. The elements $1, z, z^2, \ldots, z^{\dim_k(R)-1}$ are linearly independent over $k$;
3. The characteristic polynomial of $m_z$ equals its minimal polynomial;
4. The characteristic polynomial of $m_z$ is squarefree.

If $z$ satisfies these four properties, then $z$ is called a primitive element for $R$.

Proof. (2) $\Rightarrow$ (1): By definition, the element $z$ generates $R$ as a $k$-algebra if and only if its powers generates $R$ as a $k$-vector space. (1) $\Rightarrow$ (2): Let $n_0$ be the smallest positive integer such that $1, z, z^2, \ldots, z^{n_0}$ are linearly dependent. The integer $n_0$ must be finite since $\dim_k(R)$ is finite. Write $z^{n_0} = \sum_{i=0}^{n_0-1} a_i z^i$ for some $a_0, \ldots, a_{n_0} \in k$. By multiplying this relation by $z^{n-n_0}$ and by induction on $n$, we obtain that for any $n \geq n_0$, $z^n$ belongs to the vector space generated by $1, z, \ldots, z^{n_0-1}$. This implies that the algebra generated by $z$ has dimension $n_0$ as a $k$-vector space. By (1), we obtain that $n_0 = \dim_k(R)$.

(2) $\Rightarrow$ (3): By (2), the minimal polynomial of $m_z$ has degree at least $\dim_k(R)$, and hence it equals its characteristic polynomial. (3) $\Rightarrow$ (2): The degree of the characteristic polynomial is $\dim_k(R)$. (3) $\Rightarrow$ (4): Let $\xi$ be the squarefree part of the characteristic polynomial of $m_z$. By (3), $\xi(z)$ is nilpotent in $R$. But the only nilpotent element in an étale algebra in 0, so $\xi$ must be a multiple of the minimal polynomial of $m_z$. Hence, by (3), $\xi$ is the characteristic polynomial of $m_z$. (4) $\Rightarrow$ (3): This is a consequence of the facts that the characteristic polynomial is the minimal polynomial and the minimal polynomial have the same set of roots, and the minimal polynomial divides the characteristic polynomial. \qed

We are now ready to define the data structure that we will use to represent effective divisors on the curve. An effective divisor $D$ will be represented as:

- A scalar $\lambda \in k$;
- Three univariate polynomials $\chi, u, v \in k[S]$, such that $\chi$ is monic, $\chi$ has degree $\deg(D)$ and $u, v$ have degree at most $\deg(D) - 1$.

such that

(Div-H1) $q(u(S), v(S)) \equiv 0 \mod \chi(S);
(Div-H2) \lambda u(S) + v(S) = S;
(Div-H3) \text{GCD}(\frac{\partial}{\partial X}(u(S), v(S))) - \lambda \frac{\partial}{\partial Y}(u(S), v(S)), \chi(S)) = 1.$

We call the data structure above a primitive element representation of an effective divisor. An important ingredient of the primitive element representation is that (Div-H3) enables us to use Hensel’s lemma to encode the multiplicities. More precisely, (Div-H3) implies that $\lambda X + Y$ is a uniformizing element for all the discrete valuation rings associated to each of the points in the support of the divisor.

Not all effective divisors can be described by this primitive element representation. A first obstruction appears if the effective divisor $D$ that we want to represent involves a singular point of the curve: In this case, (Div-H3) cannot be satisfied. Another serious problem arises if the quotient of $[k[C^0]]$ corresponding to the divisor does not admit any primitive element of the form $AX + Y$ which satisfies all the wanted properties. Fortunately, Proposition 4 below shows that such a primitive element exists as soon as $k$ contains more than $(\deg(D)+1)$ elements.

Before going any further, we summarize here the data structures for the input and the output of Algorithm 1 and the properties that they must satisfy.

Input data:

- A bivariate polynomial $q \in k[X,Y]$. This polynomial encodes the curve $C$.
- A divisor $D = D_+ - D_-$ given by two tuples $(\lambda_+, \chi_+, u_+, v_+)$ and $(\lambda_-, \chi_-, u_-, v_-)$ with $\lambda_\pm \in k, \chi_\pm, u_\pm, v_\pm \in k[S]$.

The input data must satisfy the following constraints:
1. The bivariate polynomial $q \in k[X,Y]$ is absolutely irreducible, and its base field $k$ is perfect;
2. The degree of $q$ is at least 2;
3. The total degree of $q$ equals its degree with respect to $Y$;
4. The inequalities $\deg(u_{\pm}) < \deg(x_{\pm})$ and $\deg(v_{\pm}) < \deg(x_{\pm})$ hold;
5. Both tuples $(\lambda_{+}, \chi_{+}, u_{+}, v_{+})$ and $(\lambda_{-}, \chi_{-}, u_{-}, v_{-})$ satisfy (Div-H1) to (Div-H3);

Output data:
- A bivariate polynomial $h \in k[X,Y]$;
- A finite set of bivariate polynomials $B \subset k[X,Y]$.

The output data satisfies that the set $\{b/h \mid b \in B\}$ is a basis of the Riemann-Roch space associated to $D$ on $C$.

The rest of this section is devoted to some technical proofs about the primitive element representation. The statements below will be used for proving the correctness of the subalgorithms, but they may be skipped without harming the general understanding of this paper.

The following proposition shows that a primitive representation of an effective divisor which does not involve any singular point exists provided that the base field is large enough.

**Proposition 4.** Let $J$ be a nonzero ideal of $k[C^0] = k[X,Y]/q(X,Y)$ such that $J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k^e$. Assume that the cardinality of $k$ is larger than $(\dim_{k[C^0]}(J)/2 + 1)$. Then there exist $\lambda \in k$ and polynomials $\chi, u, v \in k[S]$ satisfying (Div-H1) to (Div-H3) such that the map $k[C^0]/J \to k[S]/\chi(S)$ sending $X$ and $Y$ to the classes of $u$ and $v$ is an isomorphism of $k$-algebras.

Before proving Proposition 4, we need some technical lemmas. First, the following lemma generalizes slightly the classical fact that ideals in the coordinate rings of smooth curves admit a unique factorization. Here, we do not assume that $C^0$ is nonsingular, but the factorization property holds only for ideals of regular functions which do not vanish at any singular point.

**Lemma 5.** Let $I$ be a nonzero ideal of $k[C^0]$ such that $I + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k^e$. Then there exists a unique factorization $I = \prod_{i=1}^{\ell} m_i^{\alpha_i}$ as a product of maximal ideals of $k[C^0]$.

**Proof.** First, we prove the existence of such a factorization. Let $m \subset k[C^0]$ be an ideal containing $I$. If $m$ is a nonsingular closed point of $C^0$, then the local ring $k[C^0]_m$ is a discrete valuation ring by [7, Sec. 3.2, Thm. 1]. Let $\text{val} m(I) \in \mathbb{Z}_{\geq 0}$ denote the integer such that $I = m^{\text{val} m(I)}$ in this local ring. Let $J$ be the ideal $J = \prod_{m \ni I} m^{\text{val} m(I)}$. By Noether’s theorem [7, Sec. 5.5], the equality $I = J$ holds if and only if it holds in all the local rings $k[C^0]_m$ where $m \supset I$. Since the maximal ideals $m$ are nonsingular closed points of $C^0$, this equality holds true because the corresponding local rings are discrete valuation rings, hence the equality of ideals is equivalent to the equality of their $m$-valuation.

We now prove the unicity of this factorization: By contradiction, assume that $I$ has two distinct factorization $\prod_{1 \leq i \leq t} m_i^{\alpha_i}$ and $\prod_{1 \leq i \leq \ell} m_i^{\alpha_i}$ of the ideal. Without loss of generality, assume that $m_1$ does not occur in the second factorization, or that it appears with a different multiplicity. This would lead to a contradiction since it would lead to distinct valuations of the same ideal in the local ring at $m_1$. □

An ideal $I \subset k[X,Y]$ in a polynomial ring is called 0-dimensional if the dimension of $k[X,Y]/I$ as a $k$-vector space is finite. The following lemma identifies values of $\lambda$ for which $\lambda X + Y$ is not a primitive element for a 0-dimensional algebraic set.

**Lemma 6.** Let $I \subset k[X,Y]$ be a radical 0-dimensional ideal, with associated variety $V = \{\alpha \}_{1 \leq \alpha \leq \dim_{k}(k[X,Y]/I)} \subset \overline{k}$ and let $u, v \in k[X,Y]$ be elements such that for any distinct points $\alpha, \beta \in V$, $u(\alpha) \neq u(\beta)$ or $v(\alpha) \neq v(\beta)$. Then the set of $\lambda \in k$ such that $\lambda u + v$ is not a primitive element for $k[X,Y]/I$ is contained in the set of roots of a nonzero univariate polynomial with coefficients in $\overline{k}$ of degree $(\dim_{k}(k[X,Y]/I))/2$.
Proof. Writing $\alpha_i = (\alpha_{i,x}, \alpha_{i,y})$, let $K \subset \mathbb{F}$ be a field extension of $k$ where $I$ factors as a product of degree-1 maximal ideals $\mathfrak{m}_i = (X - \alpha_{i,x}, Y - \alpha_{i,y})$. This provides an isomorphism of $K$-algebras between $K[X,Y]/I$ and $K^{\dim_k(k[X,Y]/I)}$ sending polynomials to their evaluations at $\alpha_{1,i}, \ldots, \alpha_{\dim_k(k[X,Y]/I)}$. Using this isomorphism, and letting $e_i$ denote the $i$-th canonical vector in $K^{\dim_k(k[X,Y]/I)}$, we observe that $e_i$ is an eigenvector of the endomorphism of multiplication by $\lambda u_i + v_i$, with associated eigenvalue $\lambda u_i + v_i$. Next, $\lambda u_i + v_i$ is a primitive element for $k[X,Y]/I$ if all these eigenvalues are distinct by Lemma 3. This is the case if and only if the discriminant of the characteristic polynomial is nonzero. Since the discriminant is the product of the squared differences of the roots, it equals

$$
\prod_{1 \leq i < j \leq \dim_k(k[X,Y]/I)} (\lambda(u(\alpha_i) - u(\alpha_j)) + v(\alpha_i) - v(\alpha_j))^2.
$$

This discriminant is a polynomial in $k[\lambda]$ of degree $2^{\dim_k(k[X,Y]/I)}$. It is nonzero because of the assumption that for any distinct $\alpha_i, \alpha_j$, either $u(\alpha_i) \neq u(\alpha_j)$ or $v(\alpha_i) \neq v(\alpha_j)$. The polynomial obtained by taking its square root satisfies the wanted properties.

In the following lemma, the notation $\text{red}(R)$ stands for the quotient of a ring $R$ by its Jacobson radical (i.e. the intersection of its maximal ideals). If $I$ is an ideal of $R$, we use the notation $\sqrt{I}$ to denote the radical of $I$. The ring $\text{red}(k[C^n])$ can be thought of as the coordinate ring of the $n$-dimensional algebraic set corresponding to the points in the support of the effective divisor associated to $J$.

Lemma 7. Let $J$ be a nonzero ideal of $k[C^n]$ that $J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^n]$. Then there exists a nonzero univariate polynomial $\Delta$ with coefficients in $k$ of degree at most $(\dim_k(k[C^n]/J) + 1)$, such that for any $\lambda$ which is not a root of $\Delta$, the element $\lambda X + Y$ is primitive for $\text{red}(k[C^n]/J)$ and $\partial q/\partial X - \lambda \partial q/\partial Y$ is invertible in $\text{red}(k[C^n]/J)$.

In particular, if $k$ has cardinality larger than $(\dim_k(k[C^n]/J) + 1)$, then there exists a value of $\lambda$ in $k$ which is not a root of $\Delta$.

Proof. First, notice that $k[C^n]/J$ is isomorphic to $k[X,Y]/(J + \langle q \rangle)$, by using the classical fact that ideals of a quotient ring $R/I$ correspond to ideals of $R$ containing $I$. Since $q$ is irreducible, $J + \langle q \rangle \subset k[X,Y]$ is a zero-dimensional ideal, and hence $\text{red}(k[X,Y]/(J + \langle q \rangle)) = k[X,Y]/\sqrt{J + \langle q \rangle}$. Notice that $\dim_k(k[X,Y]/\sqrt{J + \langle q \rangle}) \leq \dim_k(k[C^n]/J)$. Next, using the fact that two distinct points in the variety have distinct coordinates, Lemma 6 provides a nonzero polynomial $\Delta_0$ of degree at most $(\dim_k(k[C^n]/J) + 1)$ such that $\lambda X + Y$ is not a primitive element for $\text{red}(k[C^n]/J)$ only if $\lambda$ is a root of $\Delta_0$.

Next, we notice that since $\sqrt{J + \langle q \rangle} \subset k[X,Y]$ is a radical 0-dimensional ideal, it can be decomposed as a product $\prod_{1 \leq i \leq \ell} \mathfrak{m}_i$ of maximal ideals. Consequently, $\partial q/\partial X - \lambda \partial q/\partial Y$ must not vanish in any of these maximal ideals. Equivalently, $\partial q/\partial X - \lambda \partial q/\partial Y$ must not vanish in any of the residue fields $\kappa_i = \mathfrak{m}_i/m_i$. Notice that the norm $N_{\kappa_i/k}(\partial q/\partial X - \lambda \partial q/\partial Y)$ is a polynomial $\Delta$ in $\lambda$ with coefficients in $k$. It is nonzero since $J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^n]$ and hence either $\partial q/\partial X$ or $\partial q/\partial Y$ is nonzero in $\kappa_i$. Therefore $\Delta$ is either constant (if $\partial q/\partial Y$ vanishes in $\kappa_i$), or it has degree $[\kappa_i : k]$. Finally, the proof is concluded by noticing that $\sum_{1 \leq i \leq \ell} [\kappa_i : k] = \dim_k(k[X,Y]/\sqrt{J + \langle q \rangle}) \leq \dim_k(k[C^n]/J)$, so that the product $\Delta_0 \cdot \prod_{1 \leq i \leq \ell} \Delta_i$ has degree at most $(\dim_k(k[C^n]/J) + 1)$ and satisfies all the desired properties.

We now have all the tools that we need to prove Proposition 4.

Proof of Proposition 4. First, we assume that $J = \mathfrak{m}^2$ is a power of a maximal ideal in $k[C^n]$ such that $\mathfrak{m} + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^n]$. Then $\text{red}(k[C^n]/J) = k[C^n]/\mathfrak{m}$. Let $\lambda \in k$ be an element which is not a root of the polynomial $\Delta$ provided by Lemma 7. Such an element exists since the cardinality of $k$ is larger than the degree of $\Delta$. Therefore, $\lambda X + Y$ is a primitive element for
k[C^0]/m and hence there exists univariate polynomials \( \tilde{u}, \tilde{v} \in k[S] \) such that \( X = \tilde{u}(\lambda X + Y) \) and \( Y = \tilde{v}(\lambda X + Y) \) in \( k[C^0]/m \). Let \( \tilde{\chi}(S) \) be the minimal polynomial of \( \lambda X + Y \) in \( k[C^0]/m \), which is irreducible since \( k[C^0]/m \) is a field. Note that the map \( k[C^0]/m \rightarrow k[S]/\tilde{\chi}(S) \) sending the classes of \( X, Y \) to \( \tilde{u}, \tilde{v} \) is an isomorphism of \( k \)-algebras. Next, set \( \chi(S) = \tilde{\chi}(S)^n \) and consider the bivariate system

\[
\begin{align*}
q(X, Y) &= 0 \\
\lambda X + Y - S &= 0.
\end{align*}
\]

(1)

By construction, this system has solution \((\tilde{u}, \tilde{v})\) over \( k[S]/\tilde{\chi}(S) \). The Jacobian of this system is \( \partial q/\partial X(\lambda X + Y) - \lambda \partial q/\partial Y(\lambda X + Y) \), which is invertible in \( \text{red}(k[C^0]/J) \) by Lemma 7, and therefore \( \partial q/\partial X(\tilde{u}(S), \tilde{v}(S)) - \lambda \partial q/\partial Y(\tilde{u}(S), \tilde{v}(S)) \) is invertible in \( k[S]/\tilde{\chi}(S) \). By Hensel’s lemma, there exist polynomials \( u, v \in k[S]_{<\deg(\chi)} \), which are solutions of (1) over \( k[S]/\chi(S) \); Indeed, for \( i > 1 \), if \((\tilde{u}, \tilde{v})\) is a solution of (1) over \( k[S]/\tilde{\chi}(S)^i \), then a Taylor expansion of the system at order 1 shows that

\[
\left[ \frac{\partial u}{\partial \lambda} \right] - \left[ \frac{\partial q/\partial X}{\lambda} \cdot \frac{\partial q/\partial Y}{1} \right]^{-1} \left[ \frac{q(\tilde{u}, \tilde{v})}{\lambda \tilde{u} + \tilde{v} - S} \right]
\]

is a solution of Eq. (1) over \( k[S]/\tilde{\chi}(S)^{2i} \).

The map \( k[C^0]/J \rightarrow k[S]/\chi(S) \) is well-defined because \( m \) maps to 0 modulo \( \tilde{\chi} \) and hence \( J = m^\alpha \) maps to 0 modulo \( \chi = \tilde{\chi}^n \). It is an isomorphism because \( k[C^0]/J \) and \( k[S]/\chi(S) \) have the same dimension as vector spaces over \( k \) and the map \( S \mapsto \lambda X + Y \) is the right inverse to the map \( (X, Y) \mapsto (u(S), v(S)) \). It remains to prove that (Div-H3) is satisfied by \( \chi, u, v \), which is a direct consequence of the fact that by Lemma 7, \( \frac{\partial u}{\partial \lambda} - \lambda \frac{\partial v}{\partial \lambda} \) does not belong to \( m \) and hence \( \frac{\partial u}{\partial \lambda} (u(S), v(S)) - \lambda \frac{\partial v}{\partial \lambda} (u(S), v(S)) \) is invertible modulo \( \chi(S) \).

Next, we consider the general case where \( J \) is a nonzero ideal in \( k[C^0] \) such that \( J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^0] \). Lemma 5 implies that \( J \) can be written as a product \( J = \prod_{i=1}^{\ell} m_i^{\alpha_i} \) of powers of maximal ideals. For each \( i \), using the previous argument, we can construct univariate polynomials \( \chi_i, u_i, v_i \in k[S] \) satisfying (Div-H1) to (Div-H3) such that the maps \( k[C^0]/m_i^{\alpha_i} \rightarrow k[S]/\chi_i(S) \) sending \( X, Y \) to \( u_i(S), v_i(S) \) are isomorphisms of \( k \)-algebras. Setting \( \chi(S) = \prod_{i=1}^{\ell} \chi_i(S) \) and using the CRT, let \( u, v \in k[S]_{<\deg(\chi)} \) be such that for all \( i \), we have \( u(S) \equiv u_i(S) \mod \chi_i(S) \) and \( v(S) \equiv v_i(S) \mod \chi_i(S) \). Then the fact that the CRT is a ring morphism allows us to conclude that the map \( k[C^0]/J \rightarrow k[S]/\chi(S) \) is an isomorphism and that \( \chi, u, v \) satisfy (Div-H1) to (Div-H3).

The next lemma shows that any data satisfying (Div-H1) to (Div-H3) actually encodes a well-defined effective divisor with no singular point in its support.

Lemma 8. Let \( (\lambda, \chi, u, v) \) be such that (Div-H1) to (Div-H3) are satisfied, and let \( I = (X - u(S), V - v(S), \chi(S)) \cap k[X, Y] \). Then \( k[X, Y]/I \) is isomorphic as a \( k \)-algebra to \( k[C^0]/J \) where \( J \) is a nonzero ideal in \( k[C^0] \). Moreover, \( J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^0] \), \( \lambda X + Y \) is a primitive element for \( \text{red}(k[X, Y]/I) \), and its minimal polynomial is the squarefree part of \( \chi_0 \).

Proof. Nonzero ideals of \( k[C^0] \) correspond to ideals of \( k[X, Y] \) containing properly the principal ideal \( \langle q(X, Y) \rangle \). First, notice that (Div-H1) implies that \( q(X, Y) \in I \). Also, by (Div-H2), we get that \( \chi(\lambda X + Y) \in I \). Notice that \( \chi(\lambda X + Y) \) factors as a product of polynomials of degree 1 over the algebraic closure of \( k \). Since \( q \) is supposed to be absolutely irreducible and to have degree at least 2, this implies that \( \chi(\lambda X + Y) \) does not belong to the principal ideal \( \langle q(X, Y) \rangle \). Consequently, \( I \) contains properly \( \langle q(X, Y) \rangle \) and this proves the isomorphism between \( k[X, Y]/I \) and \( k[C^0]/J \). Using the isomorphism between \( k[C^0]/J \) and \( k[S]/\chi(S) \) described in Proposition 4, we obtain that \( \text{red}(k[X, Y]/I) \) is isomorphic to \( \text{red}(k[S]/\chi(S)) \), which in turn isomorphic to \( k[S]/\tilde{\chi}(S) \), where \( \tilde{\chi}(S) \) is the squarefree part of \( \chi(S) \). Then (Div-H3) implies that \( \partial q/\partial X - \lambda \partial q/\partial Y \) is invertible in \( k[C^0]/J \), and hence \( k[C^0] = J + \langle \partial q/\partial X, \partial q/\partial Y \rangle \). Therefore, \( J + \langle \partial q/\partial X, \partial q/\partial Y \rangle = k[C^0] \). Finally, the proof is concluded by noticing that \( S \) is a primitive element for \( k[S]/\tilde{\chi}(S) \) with minimal polynomial \( \tilde{\chi}(S) \).
The following lemma explicits the link between the primitive element representation and the ideal vanishing on the 0-dimensional algebraic set that it represents.

**Lemma 9.** Let \((\lambda, \chi, u, v)\) be data satisfying (Div-H2). Set \(I = \langle \chi(S), X - u(S), Y - v(S) \rangle \subset k[X, Y, S]\) and \(J = \langle \chi(X + Y), X - u(\lambda X + Y), Y - v(\lambda X + Y) \rangle\). Then \(I \cap k[X, Y] = J\).

**Proof.** By (Div-H2) and by using the fact that \(X - u(S), Y - v(S) \in I\), we deduce that \(S - (\lambda X + Y) \in I\). This implies that \(I \cap k[X, Y] = \{ f(X, Y, \lambda X + Y) \mid f \in I \}\).

The primitive element representation of an effective divisor is not unique: Two tuples \((\lambda_1, \chi_1, u_1, v_1)\) and \((\lambda_2, \chi_2, u_2, v_2)\) may encode the same effective divisor. The cases where this happens are detailed in the following proposition.

**Proposition 10.** Let \((\lambda_1, \chi_1, u_1, v_1), (\lambda_2, \chi_2, u_2, v_2) \in k \times k[S]^3\) be data which satisfy (Div-H2). Let \(I_1, I_2 \subset k[X, Y, S]\) be the associated ideals \(I_1 = \langle \chi_1(S), X - u_1(S), Y - v_1(S) \rangle, I_2 = \langle \chi_2(S), X - u_2(S), Y - v_2(S) \rangle\).

Then \(I_1 \cap k[X, Y] = I_2 \cap k[X, Y]\) if and only if \(\chi_1\) is the characteristic polynomial of \(\lambda_1 u_2 + v_2 \in k[S]/\chi_2(S)\), \(u_1(\lambda_1u_2(S) + v_2(S)) \equiv u_2(S) \mod \chi_2(S)\) and \(v_1(\lambda_1u_2(S) + v_2(S)) \equiv v_2(S) \mod \chi_2(S)\).

**Proof.** We first prove the “if” part of the statement. First, we notice that \(k[X, Y]/(I_1 \cap k[X, Y])\) and \(k[X, Y]/(I_2 \cap k[X, Y])\) are \(k\)-vector space of the same finite dimension, since \(\deg(\chi_1)\) must equal \(\deg(\chi_2)\). Therefore it is enough to show one inclusion to prove the equality. Let \(f(X, Y) \in I_1 \cap k[X, Y]\). Using the equalities modulo \(\chi_2\), we obtain that \(f(X, Y) \equiv f(u_1(\lambda_1u_2(S) + v_2(S)), v_1(\lambda_1u_2(S) + v_2(S))) \mod I_2\), which is divisible by \(\chi_1(\lambda_1u_2(S) + v_2(S))\) because \(f\) is in \(I_1\) and by using Cayley-Hamilton theorem. Finally, we use the fact that \(\chi_1(S)\) is the characteristic polynomial of \(\lambda_1 u_2(S) + v_2(S)\) and hence \(\chi_2\) divides \(\chi_1(S)\), which finishes to prove that \(f \in I_2\).

Conversely, assume that \(I_1 \cap k[X, Y] = I_2 \cap k[X, Y]\). By composing the isomorphisms

\[
\begin{align*}
\frac{k[S]}{\chi_1(S)} & \rightarrow \frac{k[X, Y]}{(I_1 \cap k[X, Y])} & \frac{k[X, Y]}{(I_2 \cap k[X, Y])} & \rightarrow \frac{k[S]}{\chi_2(S)} \\
S & \rightarrow \lambda_1X + Y & X & \rightarrow u_2(S) \\
Y & \rightarrow v_2(S)
\end{align*}
\]

we obtain that the map \(\frac{k[S]}{\chi_1(S)} \rightarrow \frac{k[S]}{\chi_2(S)}\) which sends \(S\) to \(\lambda_1u_2(S) + v_2(S)\) is an isomorphism. This proves that \(\chi_1\) is the characteristic polynomial of \(\lambda_1u_2(S) + v_2(S)\) in \(k[S]/\chi_2(S)\). To prove the two congruence relations, we observe that for all \(f \in k[X, Y]\), \(f(u_1, v_1) \equiv 0 \mod \chi_1\) if and only if \(f(u_2, v_2) \equiv 0 \mod \chi_2\). In particular, the polynomial \(P(X, Y) = u_1(\lambda_1X + Y) - X\) satisfy \(P(u_1, v_1) \equiv 0 \mod \chi_1\), and hence \(P(u_2, v_2) \equiv 0 \mod \chi_2\). The proof of the last congruence relation is similar.

**4 Divisor arithmetic**

The first step to perform arithmetic operations on divisors given by primitive element representations is to agree on a common primitive element. In order to achieve this, the routine CHANGEPRIMELT (Algorithm 2) performs the necessary change of primitive element by using linear algebra. We will prove in Propositions 20 and 26 that the complexity of this step is the same as the complexity of the subroutine NUMERATORBASIS in the main algorithm. Therefore, decreasing the complexity of CHANGEPRIMELT would not change the global complexity and hence we make no effort to optimize it, although we are aware that it might be possible to obtain a better complexity for this step by using a method similar to [8, Algo. 5].

Throughout this paper, for \(d > 0\) we let \(k[S]_{<d}\) denote the vector space of univariate polynomials with coefficients in \(k\) of degree less than \(d\).

**Proposition 11.** Algorithm 2 (CHANGEPRIMELT) is correct: If it does not fail, then \((\tilde{\lambda}, \tilde{\chi}, \tilde{u}, \tilde{v})\) satisfies properties (Div-H1) to (Div-H3) and it represents the same effective divisor as \((\lambda, \chi, u, v)\).
which is again proved directly by using the isomorphism \( \Psi \) on equation 10, this amounts to show that \( \lambda, \chi, u, v \) is a primitive element representation of \( D \) or "fail".

**Proof.** First, we prove that \( (\widetilde{\chi}, \widetilde{u}, \widetilde{v}) \) satisfies Properties (Div-H1) to (Div-H3). We notice that the map \( \tilde{\psi} \) in Algorithm 2 can be extended to an isomorphism \( \Psi \) of \( k \)-algebras between \( k[S]/\tilde{\chi}(S) \) and \( k[S]/\chi(S) \). Property (Div-H1) follows from the fact that in \( k[S]/\chi(S) \), we have \( q(\tilde{u}, \tilde{v}) = q(\tilde{\psi}^{-1}(u), \tilde{\psi}^{-1}(v)) = \tilde{\psi}^{-1}(q(u, v)) = 0 \). Property (Div-H2) follows from the equalities \( S = \Psi^{-1}(\tilde{\chi}(S)) = \tilde{\psi}^{-1}(\tilde{\chi}(S)) = \tilde{\psi}^{-1}(\tilde{\chi}(S), \tilde{\psi}(S)) = \tilde{\chi}(\tilde{\chi}(S)) + \tilde{\psi}(S) = \tilde{\chi}(S) \), which holds in \( k[S]/\chi(S) \) if \( \tilde{\chi} \) satisfies Property (Div-H2). The fact that equality \( \tilde{\chi}(S) \) holds in \( k[S]/\chi(S) \) if \( \tilde{\chi} \) satisfies Property (Div-H2). Finally, we must prove that both representations encode the same divisor. By Proposition 10, this amounts to show that

\[
\begin{aligned}
\tilde{\chi} \text{ is the characteristic polynomial of } \tilde{\chi}(S) + v(S) \\
\tilde{u}(\tilde{\chi}(S) + v(S)) &\equiv u(S) \mod \chi(S) \quad \text{and} \\
\tilde{v}(\tilde{\chi}(S) + v(S)) &\equiv v(S) \mod \chi(S),
\end{aligned}
\]

which is again proved directly by using the isomorphism \( \Psi^{-1} \).

**Algorithm 2:** Changing the primitive element in the representation.

**Algorithm 3:** A step of Newton-Hensel’s lifting.
Proposition 12. Algorithm 3 (HenselLiftingStep) is correct: \((\lambda, \chi^2, \tilde{u}, \tilde{v})\) satisfies (Div-H1) to (Div-H3).

Proof. This is a special case of the Newton-Hensel’s lifting. Using Taylor expansion,
\[
\left[ \frac{q(X,Y)}{\lambda X + Y - S} \right] = \left[ \frac{q(u(S), v(S))}{\lambda u(S) + v(S) - S} \right] + \left[ \frac{\partial q}{\partial X}(u(S), v(S)) + \frac{\partial q}{\partial Y}(u(S), v(S)) \right] \left[ X - u(S) \right] + \varepsilon(X,Y,S),
\]
where \(\varepsilon\) is such that \(\varepsilon(\tilde{u}(S), \tilde{v}(S), S) \equiv 0 \mod \chi(S)^2\) for any polynomials \(\tilde{u}, \tilde{v} \in k[S]\) such that \(\tilde{u} \equiv u \mod \chi\) and \(\tilde{v} \equiv v \mod \chi\). Next, notice that the denominators in the definitions of \(\tilde{u}\) and \(\tilde{v}\) are invertible modulo \(\chi(S)^2\) because they are invertible modulo \(\chi(S)\). The proof of (Div-H1) and (Div-H2) follows from a direct computation by plugging the values of \(\tilde{u}\) and \(\tilde{v}\) in the Taylor expansion, and by noticing that \(\tilde{u} \equiv u \mod \chi\) and \(\tilde{v} \equiv v \mod \chi\), so that \(\varepsilon(\tilde{u}(S), \tilde{v}(S), S) \equiv 0 \mod \chi(S)^2\). Finally, (Div-H3) is a direct consequence of the fact that \(\frac{\partial q}{\partial X}(u(S), v(S)) - \lambda \frac{\partial q}{\partial Y}(u(S), v(S))\) is invertible modulo \(\chi(S)\).

\]

Function ADDDivisors;

Data: A polynomial \(q \in k[X,Y]\) and two effective divisors \(D_1, D_2\) given by primitive element representations \((\lambda_1, \chi_1, u_1, v_1)\) and \((\lambda_2, \chi_2, u_2, v_2)\).

Result: A primitive element representation of the divisor \(D_1 + D_2\) or “fail”.

\[
\lambda \leftarrow \text{Random}(k);
(\tilde{\chi}_1, \tilde{u}_1, \tilde{v}_1) \leftarrow \text{CHANGEPRIMELT}(\lambda, \lambda_1, \chi_1, u_1, v_1);
(\tilde{\chi}_2, \tilde{u}_2, \tilde{v}_2) \leftarrow \text{CHANGEPRIMELT}(\lambda, \lambda_2, \chi_2, u_2, v_2);
\]

if \(\tilde{u}_1 \not\equiv \tilde{u}_2 \mod \text{GCD}(\tilde{\chi}_1, \tilde{\chi}_2)\) then
  1. Return “fail”
\end
\]

\[
\tilde{\lambda} \leftarrow \tilde{\chi}_1 \cdot \tilde{\chi}_2;
\tilde{\lambda} \leftarrow \text{LCM}(\tilde{\chi}_1, \tilde{\chi}_2);
\tilde{u}_{12} \leftarrow \text{XCRT}((\tilde{\chi}_1, \tilde{\chi}_2, (\tilde{u}_1, \tilde{u}_2)) \in k[S]_{\leq \deg(\tilde{\lambda})};
\tilde{v}_{12} \leftarrow \text{XCRT}((\tilde{\chi}_1, \tilde{\chi}_2, (\tilde{v}_1, \tilde{v}_2)) \in k[S]_{\leq \deg(\tilde{\lambda})};
(\tilde{u}, \tilde{v}) \leftarrow \text{HenselLiftingStep}(q, \tilde{\lambda}, \tilde{\chi}, \tilde{u}_{12}, \tilde{v}_{12});
\tilde{u} \leftarrow \tilde{u} \mod \tilde{\lambda};
\tilde{v} \leftarrow \tilde{v} \mod \tilde{\lambda};
\text{Return } (\tilde{\lambda}, \tilde{\chi}, \tilde{u}, \tilde{v}).
\]

Algorithm 4: Computing the sum of two effective divisors.

Algorithm 4 uses a variant of the CRT, which we call the Extended Chinese Remainder Theorem and which we abbreviate as XCRT. Given four univariate polynomials \(u_1, u_2, \chi_1, \chi_2 \in k[S]\) such that \(u_1 \equiv u_2 \mod \text{GCD}(\chi_1, \chi_2)\), it returns a polynomial \(u \in k[S]\) of degree less than \(\deg(\text{LCM}(\chi_1, \chi_2))\) such that \(u \equiv u_1 \mod \chi_1\) and \(u \equiv u_2 \mod \chi_2\). The main difference with the classical CRT is that we do not require \(\chi_1\) and \(\chi_2\) to be coprime. A minimal solution to the XCRT problem is given by
\[
\text{XCRT}((\chi_1, \chi_2), (u_1, u_2)) = (u_1 \alpha_1 (\chi_1/g) + u_2 \alpha_2 (\chi_2/g)) \mod \text{LCM}(\chi_1, \chi_2),
\]
where \(g = \text{GCD}(\chi_1, \chi_2)\) and \(\alpha_1, \alpha_2 \in k[S]\) are Bézout coefficients for \(\chi_1, \chi_2\), i.e. they satisfy \(\alpha_1 \chi_1 + \alpha_2 \chi_2 = g\). Notice that the XCRT is in fact a \(k\)-algebra isomorphism between \(k[S]/\text{LCM}(\chi_1(S), \chi_2(S))\) and the subalgebra of \(k[S]/\chi_1(S) \times k[S]/\chi_2(S)\) formed by pairs \((u_1, u_2)\) such that \(u_1 \equiv u_2 \mod \text{GCD}(\chi_1, \chi_2)\).

Proposition 13. Algorithm 4 (ADDDivisors) is correct: If it does not fail, then it returns a primitive element representation of the effective divisor \(D_1 + D_2\).
Proof. Let $I_1$, $I_2$, $J$ denote the three following ideals of $k[C^0]$:  

$$I_1 = \langle \chi_1(\lambda_1 X + Y), \ X - u_1(\lambda_1 X + Y), \ Y - v_1(\lambda_1 X + Y) \rangle;$$

$$I_2 = \langle \chi_2(\lambda_2 X + Y), \ X - u_2(\lambda_2 X + Y), \ Y - v_2(\lambda_2 X + Y) \rangle;$$

$$J = \langle \hat{\chi}(\hat{\lambda} X + Y), \ X - \hat{u}_1(\hat{\lambda} X + Y), \ Y - \hat{v}_2(\hat{\lambda} X + Y) \rangle.$$ 

Proving that Algorithm 4 is correct amounts to showing that $I_1 \cdot I_2 = J$, and that (Div-H1) to (Div-H3) are satisfied by $\hat{\lambda}, \hat{\chi}, \hat{u}, \hat{v}$. First, let $I'_1, I'_2 \subset k[C^0]$ be the ideals

$$I'_1 = \langle \hat{\chi}_1(\hat{\lambda} X + Y), X - \hat{u}_1(\hat{\lambda} X + Y), Y - \hat{v}_1(\hat{\lambda} X + Y) \rangle;$$

$$I'_2 = \langle \hat{\chi}_2(\hat{\lambda} X + Y), X - \hat{u}_2(\hat{\lambda} X + Y), Y - \hat{v}_2(\hat{\lambda} X + Y) \rangle.$$ 

By Proposition 11 and Lemma 9, the equalities $I_1 = I'_1$ and $I_2 = I'_2$ hold.

We start by proving that $\hat{\lambda}, \hat{\chi}, \hat{u}, \hat{v}$ satisfy (Div-H1) to (Div-H3). For (Div-H1) and (Div-H2), Proposition 11 ensures that $q(\tilde{u}_i(S), \tilde{v}_i(S)) \equiv 0 \mod \tilde{\chi}_i(S)$ for $i \in \{1, 2\}$. Then, using the fact that the XCRT is a morphism, we obtain that $q(\tilde{u}_{12}(S), \tilde{v}_{12}(S)) \equiv 0 \mod LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))$ and $\tilde{\lambda}_{12}(S) + \tilde{v}_{12}(S) \equiv S \mod LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))$. Next, Proposition 12 shows that $q(\tilde{u}(S), \tilde{v}(S)) \equiv 0 \mod LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))^2$ and $\tilde{\lambda} u(S) + \tilde{v}(S) \equiv S \mod LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))^2$. Since $\tilde{\chi} = \tilde{\chi}_1 \cdot \tilde{\chi}_2$ divides $LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))^2$, we get that $q(\tilde{u}(S), \tilde{v}(S)) \equiv 0 \mod \tilde{\chi}$ and $\lambda \tilde{u}(S) + \tilde{v}(S) = S$. For (Div-H3), we observe that the fact that the XCRT is a ring morphism implies that $\frac{\partial}{\partial X}(\tilde{u}_{12}(S), \tilde{v}_{12}(S)) - \lambda \frac{\partial}{\partial X}(\tilde{u}_{12}(S), \tilde{v}_{12}(S))$ is invertible in $k[S] / LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))$. Consequently, $\frac{\partial}{\partial X}(\tilde{u}(S), \tilde{v}(S)) - \lambda \frac{\partial}{\partial X}(\tilde{u}(S), \tilde{v}(S))$ is invertible in $\tilde{\chi} / LCM(\tilde{\chi}_1(S), \tilde{\chi}_2(S))$, and hence it is also invertible in $k[S] / \tilde{\chi}$.

We prove now that $I'_1 \cdot I'_2 = J$. Using the factorization as a product of maximal ideals given by Lemma 5, it is sufficient to prove that a power $m^k \subset k[C^0]$ of a maximal ideal contains $I'_1 \cdot I'_2$ if and only if it contains $J$. Notice that the powers of maximal ideals which contain $I'_1$ are of the form $\langle \chi_m(\hat{\lambda} X + Y)^e, X - u_m(\hat{\lambda} X + Y), Y - v_m(\hat{\lambda} X + Y) \rangle$, where $\chi_m$ is a prime factor of $\hat{\chi}$ and $u_m(S) \equiv \tilde{u}_1(S) \mod \chi_m(S)^e$, $v_m(S) \equiv \tilde{v}_1(S) \mod \chi_m(S)^e$. We conclude by noticing that $\tilde{\chi} = \tilde{\chi}_1 \cdot \tilde{\chi}_2$ and by using the properties of the XCRT and of the Hensel’s lifting:

$$\tilde{u}(S) \equiv \tilde{u}_1(S) \mod \tilde{\chi}_1(S);$$

$$\tilde{u}(S) \equiv \tilde{u}_2(S) \mod \tilde{\chi}_2(S);$$

$$\tilde{v}(S) \equiv \tilde{v}_1(S) \mod \tilde{\chi}_1(S);$$

$$\tilde{v}(S) \equiv \tilde{v}_2(S) \mod \tilde{\chi}_2(S).$$

Algorithm 5 (SubtractDivisors) provides a method for subtracting effective divisors given by primitive element representations. We emphasize that the divisor returned is the subtraction $D_1 - D_2$ only if the result is also effective, i.e. if $D_1 \geq D_2$. If this is not the case, then it returns the positive part of the subtraction.

**Proposition 14.** Algorithm 5 (SubtractDivisors) is correct: If it does not fail, then it returns a primitive element representation of the effective divisor $[D_1 - D_2]_+$, where the notation $[D]_+$ denotes the positive part of the divisor $D$, i.e. the smallest effective divisor $D'$ such that $D' \geq D$.

Proof. Let $I_1$, $I_2$, $J$ denote the three following ideals of $k[C^0]$, using the notation in Algorithm 5:

$$I_1 = \langle \chi_1(\lambda_1 X + Y), \ X - u_1(\lambda_1 X + Y), \ Y - v_1(\lambda_1 X + Y) \rangle;$$

$$I_2 = \langle \chi_2(\lambda_2 X + Y), \ X - u_2(\lambda_2 X + Y), \ Y - v_2(\lambda_2 X + Y) \rangle;$$

$$J = \langle \hat{\chi}(\hat{\lambda} X + Y), \ X - \hat{u}_1(\hat{\lambda} X + Y), \ Y - \hat{v}_2(\hat{\lambda} X + Y) \rangle.$$ 

The effective divisor $[D_1 - D_2]_+$ corresponds to the colon ideal $I_1 : I_2 = \{ f \in k[C^0] \mid f \cdot I_2 \subset I_1 \}$. Consequently, we must prove that $(\lambda, \tilde{\chi}, \tilde{u}, \tilde{v})$ satisfies (Div-H1) to (Div-H3) and that $J =$
The proof is concluded by noticing that for any prime factor $\Phi$ of $m$,
As in the proof of Proposition 13, the maximal ideals of $\Phi$ which divide $m$
the ideal $I_q$ using Noether's theorem together with the fact that $D_1$ does not involve any singular point of the curve by (Div-H3), the equality $I_1 : I_2 = J$ holds if and only if the powers of maximal ideals $m^\ell \subset k[C^0]$ which contain $I_1 : I_2$ are exactly those which contain $J$. Equivalently, this means that if $m^\ell_1$ is the largest power of $m$ which contains $I_1$ and if $m^\ell_2$ is the largest power of $m$ which contains $I_2$, then $m^{\max(\ell_1 - \ell_2, 0)}$ is the largest power of $m$ which contains $J$. As in the proof of Proposition 13, the maximal ideals $m \subset k[C^0]$ which contain $I_1$ have the form $\langle \chi_m(\lambda X + Y), X - u_m(\lambda X + Y), Y - v_m(\lambda X + Y) \rangle$, where $u_m \equiv \hat{u}_1 \mod \chi_m, v_m \equiv \hat{v}_1 \mod \chi_m$. The proof is concluded by noticing that for any prime factor $\Phi$ of $\hat{X}_1$, if $\Phi^{f_2}$ is the largest power of $\Phi$ which divides $\hat{X}_1$ and $\Phi^{f_2}$ is the largest power of $\Phi$ which divides $\hat{X}_2$, then the largest power $\Phi$ which divides $\hat{X} = \hat{X}_1 / \text{GCD}(\hat{X}_1, \hat{X}_2)$ is $\Phi^{\max(f_1 - f_2, 0)}$.

5 Description and correction of the subroutines

5.1 Interpolation

This section focuses on the following interpolation problem: Given an effective divisor $D$ on a plane projective curve whose support contains only nonsingular points of $C^0$, find a element $h \in k[C^0]$ such that its associated principal divisor $(h)$ satisfies $(h) \geq D$.

Proposition 15. Algorithm 6 (INTERPOLATE) is correct: The kernel of $\varphi$ has positive dimension, and its nonzero elements $h$ satisfy $(h) \geq D$.

Proof. The fact that the kernel $\varphi$ has positive dimension follows from a dimension count, which is postponed to Lemma 16. We now prove the second part of the proposition. First, notice that $\deg Y(h) < \deg Y(q)$ for any nonzero $h$ in the kernel of $\varphi$, hence nonzero elements in the kernel cannot be multiple of $q$, which implies that $\langle 0 \rangle \subset (h) \subset k[C^0]$. Next, by Lemmas 8 and 9, the ideal $I_D = \{ f \in k[C^0] \mid (f) \geq D \}$ equals $\langle \chi(\lambda X + Y), X - u(\lambda X + Y), Y - v(\lambda X + Y) \rangle$. 

\begin{algorithm}
\textbf{Function} SUBTRACTDIVISORS;
\textbf{Data:} Two effective divisors given by primitive element representations:

\begin{align*}
D_1 &= (\lambda_1, \chi_1, u_1, v_1), \\
D_2 &= (\lambda_2, \chi_2, u_2, v_2).
\end{align*}

\textbf{Result:} A primitive element representation of the divisor $[D_1 - D_2]_+$ or “fail”.

\begin{align*}
\lambda &\leftarrow \text{RANDOM}(k); \\
(\hat{\chi}_1, \hat{u}_1, \hat{v}_1) &\leftarrow \text{CHANGEPRIME}(\lambda, \chi_1, u_1, v_1); \\
(\hat{\chi}_2, \hat{u}_2, \hat{v}_2) &\leftarrow \text{CHANGEPRIME}(\lambda, \chi_2, u_2, v_2);
\end{align*}

\textbf{if} \ $\hat{u}_1 \neq \hat{u}_2 \mod \text{GCD}(\hat{\chi}_1, \hat{\chi}_2)$ \textbf{then}

\begin{itemize}
\item Return “fail”
\end{itemize}

\begin{algorithm}
\begin{align*}
\hat{\chi} &\leftarrow \hat{\chi}_1 / \text{GCD}(\hat{\chi}_1, \hat{\chi}_2); \\
\hat{u}(S) &\leftarrow \hat{u}_1(S) \mod \hat{\chi}(S); \\
\hat{v}(S) &\leftarrow \hat{v}_1(S) \mod \hat{\chi}(S);
\end{align*}

Return $(\hat{\lambda}, \hat{\chi}, \hat{u}, \hat{v})$.

\end{algorithm}

Algorithm 5: Computing the subtraction of effective divisors.
The proof is concluded by noticing that the polynomials in $I_D$ are exactly those which satisfy $f(u(S), v(S)) \equiv 0 \mod \chi(S)$, using the isomorphism in Proposition 4. By construction, $h(u(S), v(S)) \equiv 0 \mod \chi(S)$ for any $h \in \ker \varphi$. □

The following lemma ensures that Algorithm 6 actually returns a nonzero element, i.e. that the kernel of $\varphi$ has positive dimension.

**Lemma 16.** With the notation in Algorithm 6,

$$\deg(\chi) < \dim_k \{ f \in k[X, Y] \mid \deg(f) \leq d, \deg_Y(f) < \delta \} \leq 3 \deg(\chi).$$

Consequently, $\varphi$ is not injective.

**Proof.** First, a direct dimension count gives

$$\dim_k \{ f \in k[X, Y] \mid \deg(f) \leq d, \deg_Y(f) < \delta \} = \begin{cases} \delta(d - (\delta - 3)/2) & \text{if } d \geq \delta \\ \binom{d + 2}{2} & \text{otherwise.} \end{cases}$$

On one hand, if $\left(\frac{\delta + 1}{2}\right) \leq \deg(\chi)$, then

$$d = \frac{\deg(\chi)}{\delta + (\delta - 1)/2} \geq \left(\frac{\delta + 1}{2}\right) / \delta + (\delta - 1)/2 \geq \delta,$$

and hence

$$\delta(d - (\delta - 3)/2) > \delta \left(\frac{\deg(\chi)}{\delta} + (\delta - 1)/2 - 1 - (\delta - 3)/2\right) = \frac{\delta(\deg(\chi)/\delta + (\delta - 1)/2 - 1 - (\delta - 3)/2)}{\deg(\chi) + \delta} \leq \frac{\delta(\deg(\chi)/\delta + (\delta - 1)/2 - (\delta - 3)/2)}{2 \deg(\chi)} \leq \frac{\delta(d - (\delta - 3)/2)}{2 \deg(\chi)}.$$

On the other hand, if $\left(\frac{\delta + 1}{2}\right) > \deg(\chi)$, then

$$d = \left[\frac{\sqrt{1 + 8 \deg(\chi)} - 1}{2}\right] < \left[\frac{\sqrt{1 + 4 \deg(\chi) + 1}}{2}\right] = \left[\frac{\sqrt{2 \delta} + 1}{2}\right] = \delta.$$
Since \((r^2 - 2) - \deg(\chi) > 0\) for any \(x > (\sqrt{1 + 8 \deg(\chi)} - 3)/2\), we get that \(\deg(\chi) < (d+2)/2\) as expected. Finally, the last inequality follows from \(\left(\frac{1}{2}(\sqrt{1 + 8 \deg(\chi)} - 1)/2\right) + 2 \leq \deg(\chi) + (1 + \sqrt{1 + 8 \deg(\chi)})/2\), and direct computations show that \((1 + \sqrt{1 + 8 \deg(\chi)})/2 \leq 2 \deg(\chi)\). 

5.2 Computing the principal divisor associated to a regular function on the curve

The section is devoted to the following problem: Given a polynomial \(h \in k[C^0]\), compute a primitive element representation of the principal effective divisor \((h)\) associated to \(h\). Let us mention that the principal effective divisor \((h)\) associated to \(h\) may involve some singular points. In this case, Algorithm CompPrincDiv will fail. Also, \(h\) may vanish at infinity. Ignoring these zeros may lead to functions having unauthorized poles at infinity in the basis returned by Algorithm 1. As we already mentioned in Section 3, handling what happens at infinity is not a problem: This issues can be solved for instance by adjoining to a divisor some data describing the multiplicities at the places at infinity. Notice also that it is easy to detect if \(h\) has zeros at infinity: This happens if and only if the degree of the resultant is strictly less than the Bézout bound \(\deg(h)\deg(C)\), thanks to the fact that we assumed that the polynomial \(q\) is monic in \(Y\). For simplicity, we will not discuss further this issue in the sequel of this paper.

**Function** CompPrincDiv;

**Data:** A squarefree bivariate \(q \in k[X,Y]\) such that \(\deg(q) = \deg_Y(q)\), and a bivariate polynomial \(h \in k[X,Y]\).

**Result:** A primitive element representation \((\lambda, \chi(S), u(S), v(S))\) of the principal effective divisor \((h)\) or “fail”.

\[
\lambda \leftarrow \text{Random}(k);
\]

if \(\lambda = 0\) or \(q((S - Y)/\lambda, Y) \in k[Y]\) is not monic in \(Y\) then

1. Return “fail”

\[
\chi(S) \leftarrow \text{Resultant}_Y(q((S - Y)/\lambda, Y), h((S - Y)/\lambda, Y));
\]

\[
a_0(S) + Ya_1(S) \leftarrow \text{FirstSubRes}_Y(q((S - Y)/\lambda, Y), h((S - Y)/\lambda, Y));
\]

if \(\text{GCD}(a_1(S), \chi(S)) \neq 1\) then

1. Return “fail”

\[
v(S) \leftarrow -a_0(S) \cdot a_1(S)^{-1} \bmod \chi(S);
\]

\[
u(S) \leftarrow (S - v(S))/\lambda;
\]

if \(\text{GCD}(\frac{\partial}{\partial Y}(u(S), v(S)) - \lambda \frac{\partial}{\partial Y}(u(S), v(S)), \chi(S)) \neq 1\) then

1. Return “fail”

Return \((\lambda, \chi(S), u(S), v(S))\).

**Algorithm 7:** Computing a primitive element representation of \((h)\).

**Proposition 17.** Algorithm 7 (CompPrincDiv) is correct: If it does not fail, then it returns a primitive element representation of the principal divisor \((h)\) associated to \(h\).

Before proving Proposition 17, we need the following technical lemma.

**Lemma 18.** With the notation in Algorithm 7, let \(s \in \mathbb{F} \setminus \{0\}\) be such that \(\frac{\partial}{\partial Y}(q((s - Y)/\lambda, Y))\) is invertible in \(\mathbb{F}[Y]/q((s - Y)/\lambda, Y)\). Let \(y_1, \ldots, y_r \in \mathbb{F}\) be the roots of the polynomial \(q((s - Y)/\lambda, Y)\), and for \(i \in [1, r]\), let \(m_i\) denote the valuation of \(h\) in the discrete valuation ring \((\mathbb{F}[X,Y]/q)(X-(s-y_i)/\lambda, Y-y_i)\). Then \(s\) is a root of multiplicity \(\sum_{i=1}^{r} m_i\) in Resultant\(_Y(q((S - Y)/\lambda, Y), h((S - Y)/\lambda, Y))\).
Proof. Set \( \tilde{q}(S,Y) = q((S-Y)/\lambda,Y) \in k[S,Y] \). Since \( \partial \tilde{q}/\partial Y(s,y_i) \) is nonzero for all \( y_i \), Hensel's lemma gives us the existence of a series expansion at \( S = s \) given by a power series \( \tilde{y}_i(S) \in \mathcal{T}[[S-s]] \) such that \( \tilde{q}(S,\tilde{y}_i(S)) = 0 \) in the ring \( \mathcal{T}[[S-s]] \). Here, \( \mathcal{T}[[S-s]] \) is the completion of the ring \( k[S] \) w.r.t. the polynomial \( S-s \), see [5, Sec. 7.1]. The fact that \( \frac{1}{\partial} q'(s,Y) \) is invertible in \( \mathcal{T}[Y]/\tilde{q}(s,Y) \) implies that each root is simple, and hence \( \deg_q(q((S-Y)/\lambda,Y) = \ell \). Therefore the polynomial \( q((S-Y)/\lambda,Y) \) splits as

\[
q((S-Y)/\lambda,Y) = \alpha \prod_{i=1}^{\ell} (Y - \tilde{y}_i(S)) \in \mathcal{T}[[S-s]][Y],
\]

where \( \alpha \in k \). Next, using the multiplicativity property of the resultant [14, Sec. 5.7], we get

\[
\text{Resultant}_Y(q((S-Y)/\lambda,Y), h((S-Y)/\lambda,Y)) = \alpha^{\deg_Y(h((S-Y)/\lambda,Y))} \prod_{i=1}^{\ell} h((S-\tilde{y}_i(S))/\lambda, \tilde{y}_i(S)).
\]

The proof is concluded by noticing that \( m_t \) precisely corresponds to the largest integer \( \gamma \) such that \( (S-s)\gamma \) divides \( h((S-\tilde{y}_i(S))/\lambda, \tilde{y}_i(S)) \) since \( S-s \) is a uniformizing element for all the discrete valuation rings \( (\mathcal{T}[X,Y]/(q)) \langle (x-(s-y_i)/\lambda,Y-y_i) \rangle \).

Proof of Proposition 17. In order to prove Proposition 17, we must prove that the output \((\lambda,\chi,u,v)\) satisfy (Div-H1) to (Div-H3) and that the two ideals \((h) \subset k[c^0] \) and \((\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y)) \subset k[c^0] \) are equal. (Div-H2) follows directly from the definitions of \( u(S) \) and \( v(S) \) in Algorithm 7. To prove (Div-H1), we shall prove that the equality holds modulo \((S-s)\gamma \) for any root \( s \in \mathcal{T} \) of \( \chi \) of multiplicity \( \gamma \). A classical property of the subresultants is that they belong to the ideal generated by the input polynomials. This implies that for any root \( s \in \mathcal{T} \) of \( \chi \) we have

\[
an_0(S) + Y a_1(S) \in \langle q((S-Y)/\lambda,Y), h((S-Y)/\lambda,Y) \rangle \subset \mathcal{T}[[S-s]][Y].
\]

If the algorithm does not fail, then \( a_1(S) \) is invertible modulo \( \chi(S) \). Consequently, it is also invertible in \( \mathcal{T}[[S-s]] \) and hence

\[
Y - (a_0(S) a_1(S) - 1) \in \langle q((S-Y)/\lambda,Y), h((S-Y)/\lambda,Y) \rangle \subset \mathcal{T}[[S-s]][Y].
\]

Since \( \text{Frac}(\mathcal{T}[[S-s]][Y]) \) is a principal ring, it implies that \( q((S-a_0(S) a_1(S) - 1)/\lambda,Y-a_0(S) a_1(S) - 1) = 0 \) in \( \mathcal{T}[[S-s]][Y] \). Considering this equation modulo \((S-s)\gamma \) and using the CRT over all the roots of \( \chi \) finishes the proof of (Div-H1). Finally, (Div-H3) is explicitly tested and hence it must be satisfied if the algorithm does not fail.

It remains to prove the equality of the ideals \((h) \subset k[c^0] \) and \((\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y)) \subset k[c^0] \). Using the isomorphism between \( k[X,Y]/(\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y)) \) and \( k[S]/\chi(S) \) (see Proposition 4), the elements in \( \chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y)) \) are precisely the classes of the bivariate polynomials \( \psi(X,Y) \in k[X,Y] \) such that \( \psi(u(S),v(S)) \equiv 0 \mod \chi(S) \). Using a proof identical to that of (Div-H1) we get that \( h(u(S),v(S)) \equiv 0 \mod \chi(S) \) which proves that \( (h) \subset \langle \chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y) \rangle \) if this inclusion holds in the local ring associated to any maximal ideal \( m \subset \mathcal{T}[c^0] \) which contains \((\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y)) \). Over \( \mathcal{T} \), these maximal ideals have the form \( \langle \lambda X + Y - s, X-u(s), Y-v(s) \rangle \), where \( s \in \mathcal{T} \) is a root of \( \chi \). The assumption \( \text{GCD}(\lambda X - s,Y) = 1 \) ensures that all these maximal ideals correspond to nonsingular points, and hence the associated local rings are discrete valuation rings. For \( s \in \mathcal{T} \) a root of \( \chi \), let \( y_1, \ldots, y_t \) be the roots of the univariate polynomial \( q(s-Y)/\lambda,Y) \in \mathcal{T}[Y] \). Let \( m_t \) denote the intersection multiplicity of \( h \) at the point \((s-y_t)/\lambda,y_t) \) of \( c^0 \). Using the fact that \( \partial \partial/(\lambda X + Y - s,Y) \) and Lemma 18, we get that \( m_t \) is the multiplicity of the root \( s \) in \( \chi \). Let \( k \) be the integer such that \( y_k = v(s) \). Then \( m_k \leq \alpha \), which shows that we have \((\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y) \subset (h) \) in the local ring at the point \((u(s),v(s)) \). Noether’s AF+BG theorem concludes the proof of the inclusion \( (\chi(\lambda X + Y),X-u(\lambda X + Y),Y-v(\lambda X + Y) \subset (h). \)
5.3 Computing the linear space of regular functions of bounded degree having prescribed zeros

The task accomplished by Algorithm NumeratorBasis is similar to what Algorithm Interpolate does: It computes a basis of the vector space of regular functions having prescribed zeros. The only difference with Algorithm Interpolate is that Algorithm NumeratorBasis returns a basis of this linear space.

\begin{algorithm}
\textbf{Function NumeratorBasis;}
\textbf{Data:} A positive integer \( \delta \), an effective divisor given by a primitive element representation \((\lambda, \chi(S), u(S), v(S))\), and a positive integer \( d \).
\textbf{Result:} A basis of the space of polynomials \( g \in k[X, Y] \) such that \( \deg(g) \leq d, \deg_Y(g) < \delta \) and the associated divisor satisfies \( (g) \geq D \).

Construct the matrix representing the linear map \\
\( \varphi : \{f \in k[X, Y] \mid \deg(f) \leq d, \deg_Y(f) \leq \delta\} \to k[S]_{<\deg(\chi)} \) defined as \\
\( \varphi(f(X, Y)) = f(u(S), v(S)) \mod \chi(S) \);
Compute and return a basis of the kernel of \( \varphi \).
\end{algorithm}

Proposition 19. Algorithm 8 (NumeratorBasis) is correct: the nonzero elements \( g \) in the kernel of \( \varphi \) are not divisible by \( q \) and they satisfy \( (g) \geq D \).

Proof. The proof is similar to that of Proposition 15.

6 Complexity

All complexity bounds count the number of arithmetic operations (additions, subtractions, multiplications, divisions) in \( k \), all at unit cost. We do not include in our complexity bounds the cost of generating random elements, nor the cost of monomial manipulations, nor multiplications by fixed integer constants. In particular, we do not include in our complexity bounds the cost of computing the partial derivatives of a polynomial. We use the classical \( O() \) and \( \tilde{O}() \) notation, see e.g. [24, Sec. 25.7]. The notation \( M(n) \) stands for the number of arithmetic operations required in \( k \) to compute the product of two univariate polynomials of degree \( n \) with coefficients in \( k \). By [4], \( M(n) = O(n \log n \log \log n) \). In the sequel, \( \omega \) is a feasible exponent for matrix multiplication, i.e. \( \omega \) is such that there is an algorithm for multiplying two \( N \times N \) matrices with entries in \( k \) within \( O(N^{\omega}) \) arithmetic operations in \( k \). The best known bound is \( \omega < 2.3729 \) [17]. In the following, we make the assumption that \( \omega > 2^1 \).

Proposition 20. Algorithm 2 (ChangePrimEl t) requires at most \( O(\deg(\chi)^{\omega}) \) arithmetic operations in \( k \).

Proof. In order to construct the matrix \( M \) in Algorithm 2, we must compute the remainders \( S^i \cdot (\lambda u(S) + v(S)) \mod \chi(S) \) for \( i \in \{0, \ldots, \deg(\chi) - 1\} \). Each of these computations costs \( O(M(\deg(\chi))) \) arithmetic operations, so the total cost of constructing the matrix \( M \) is bounded by \( O(M(\deg(\chi)) M(\deg(\chi))) \), which is bounded above by \( O(\deg(\chi)^2) \). Computing the characteristic polynomial of \( M \) can be done within \( O(\deg(\chi)^{\omega}) \) arithmetic operations [18]. We emphasize that in [18], it is assumed that the cardinality of \( k \) is at least \( 2 \deg(\chi)^2 \), so that the probability of failure is bounded by \( 1/2 \). In fact, using the same algorithm and the same proof as in [18], the assumption on the cardinality of \( k \) can be removed but the probability of failure will then only be bounded by \( \deg(\chi)^2/|E| \), where \( E \subset k \) is a finite subset in which we can draw elements uniformly at random. We will incorporate this probability of failure for the computation of

\footnote{If \( \omega = 2 \), then the \( O() \) in Theorem 27 should be replaced by \( \tilde{O}() \).}
the characteristic polynomial in our bound for the probability of failure of the main algorithm, see the proof of Theorem 36.

Constructing the matrix \( N \) is done by computing successively the remainders \((\hat{\lambda}\nu(S) + v(S))i \mod \chi(S)\) for \( i \in \{0, \ldots, \deg(\chi) - 1\} \) at a total cost of \( O(\deg(\chi) M(\deg(\chi))) \) which is again bounded by \( O(\deg(\chi)^2) \). Finally, inverting \( N \) and applying the inverse linear map can be done using \( O(\deg(\chi)^2) \) operations in \( k \) by using [2].

**Proposition 21.** Algorithm 3 (HenselLiftingStep) requires at most \( O(\deg(q)^2 M(\deg(\chi))) \) arithmetic operations in \( k \).

**Proof.** Algorithm 3 consists in evaluations of \( q \) and its partial derivatives at \((u(S), v(S))\), together with finitely many arithmetic operations in \( k[S]/\chi(S)^2 \). Each of the arithmetic operations modulo \( \chi^2 \) costs \( O(M(\deg(\chi))) \) arithmetic operations in \( k \). Evaluating \( q \) at \((u(S), v(S))\) modulo \( \chi(S)^2 \) can be done by computing the remainders \( u(S)^i v(S)^j \mod \chi(S)^2 \) for all \((i, j) \in \mathbb{Z}_{\geq 0} \) such that \( i + j \leq \deg(q) \), then by multiplying these evaluations by the corresponding coefficients in \( q \) and by summing them. Computing all the modular products can be done in \( O(\deg(q)^2 M(\deg(\chi))) \) operations in \( k \), by considering the pairs \((i, j)\) in increasing lexicographical ordering. Multiplying by the coefficients and summing then costs \( O(\deg(q)^2 \deg(\chi)) \) arithmetic operations in \( k \). Computing the evaluations of the partial derivatives of \( q \) is done similarly and it has a similar cost.

**Proposition 22.** Algorithm 4 (AddDivisors) requires at most \( O(\deg(q)^2 M(\nu) + \nu^2) \) arithmetic operations in \( k \), where \( \nu = \max(\deg(\chi_1), \deg(\chi_2)) \).

**Proof.** Algorithm 4 starts by two calls to the function ChangePrimElt, with respective costs \( O(\deg(\chi_1)^2) \) and \( O(\deg(\chi_2)^2) \) by Proposition 20. The polynomial GCD \((\hat{\chi}_1, \hat{\chi}_2)\) can be computed at cost \( O(\max(M(\deg(\hat{\chi}_1))) \log(\deg(\hat{\chi}_1)), M(\deg(\hat{\chi}_2)) \log(\deg(\hat{\chi}_2)))) = O(M(\nu) \log(\nu)) \) using the fast GCD algorithm [24, Coro. 11.9]. The product \( \hat{\chi} \) in Algorithm 4 and the LCM are then also computed at costs \( O(M(\nu)) \) and \( O(M(\nu) \log(\nu)) \). The XCRT can be computed at cost \( O(M(\nu) \log(\nu)) \) by using Equation (2) together with the fact that Bézout coefficients can be computed within quasi-linear complexity [24, Coro. 11.9]. Finally, the Hensel lifting step can be achieved at cost \( O(\deg(q)^2 M(\nu)) \) by Proposition 21.

**Proposition 23.** Algorithm 5 (SubtractDivisors) requires at most \( O(\nu^2) \) arithmetic operations in \( k \), where \( \nu = \max(\deg(\chi_1), \deg(\chi_2)) \).

**Proof.** Most of the steps of Algorithm 5 are similar to steps of Algorithm 4, except that Hensel lifting is not required here. The complexity analysis is similar and we refer to the proof of Proposition 22. The only step which does not appear in Algorithm 4 is the exact division of \( \hat{\chi}_1 \) by the GCD. The cost of this step does not hinder the global complexity since exact division of polynomials can be done in quasi-linear complexity [24, Thm. 9.1].

In practice, if \( k \) is sufficiently large, then choosing a global value for \( \lambda \) and using the same value for all the representations of divisors would succeed with large probability. In this case, we do not need to call the function ChangePrimElt within Algorithms AddDivisors and SubtractDivisors. This would decrease significantly the complexities of AddDivisors and SubtractDivisors. In any case, this would not change the global asymptotic complexity of Algorithm 1.

**Proposition 24.** Algorithm 6 (Interpolate) requires at most \( O(\deg(\chi)^2) \) arithmetic operations in \( k \) and it returns a polynomial of degree less than \( \deg(\chi)/\delta + \delta \).

**Proof.** The computation of the degree \( d \) does not cost any arithmetic operations in \( k \). The construction of the matrix representing the linear map \( \varphi \) can be done by computing all the modular products \( u(S)^i v(S)^j \mod \chi(S) \) for pairs \((i, j)\) such that \( i + j \leq d \) and \( j < \delta \). Lemma 16 states that the number of such pairs is bounded above by \( 3 \deg(\chi) \). By considering the pairs \((i, j)\) in increasing lexicographical ordering, computing all these modular products can be done
Proof. This is a direct consequence of Propositions 22, 23, 24, 25 and 26.

7 Lower bounds on the probability of success

In this section, we examine all possible sources of failures for the main algorithm. In fact, if the assumptions detailed in Section 2 are satisfied, then failure can only come from a bad
choice of an element chosen at random in $k$. More precisely, we show that these bad choices can be characterized algebraically and that they are included in the set of roots of polynomials. Bounding the degrees of these polynomials provides us with lower bounds on the probability of success if random elements in $k$ are picked uniformly at random in a finite subset $E \subset k$.

First, we investigate which values of $\lambda$ make Algorithm 2 (\textsc{ChangePrimEl t}) fail. These are the values of $\lambda$ such that there is a line of equation $\lambda X + Y + \gamma$ for some $\gamma \in K$ which goes either through two distinct points in the support of the input divisor, or which is tangent to $C^0$ at a point in the support of the divisor.

**Proposition 28.** Given an effective divisor $D = (\lambda, \chi, u, v)$, the set of $\tilde{\lambda} \in k$ such that Algorithm 2 (\textsc{ChangePrimEl t}) with input $D, \tilde{\lambda}$ fails is contained in the set of roots of a nonzero univariate polynomial with coefficients in $k$ of degree at most $(\deg(\lambda)+1)/2$.

**Proof.** There are two possible sources of failures for Algorithm 2: if the vector $(1, -\tilde{\lambda})$ is tangent to the curve $C^0$ at one of the points in the support of the effective divisor (first test) or if $\tilde{\lambda}X + Y$ is not a primitive element (second test).

The first test fails only if $\Delta_1(\tilde{\lambda}) = \text{Resultant}_S(\frac{\partial \lambda}{\partial X}(u(S), v(S)) - \bar{\lambda} \frac{\partial u}{\partial Y}(u(S), v(S)), \chi(S))$ vanishes. This yields a univariate polynomial $\Delta_1$ of degree at most $\deg(\chi(S))$ in $\tilde{\lambda}$ which vanishes only if the first test fails. This polynomial is nonzero because its evaluation at $\lambda$ is nonzero by (Div-H3).

With the notation of Algorithm 2, let $\Delta_2(\tilde{\lambda}) \in k[\tilde{\lambda}]$ be the determinant of the matrix which represents the linear map

\[
\begin{align*}
\tilde{k}(\lambda)[S]_{< \deg(\chi)} & \rightarrow \tilde{k}(\lambda)[S]_{< \deg(\chi)} \\
f(S) & \mapsto f(\lambda u(S) + v(S)),
\end{align*}
\]

written in the basis $1, S, S^2, \ldots, S^{\deg(\chi)-1}$ of $k(\lambda)[S]_{< \deg(\chi)}$. Entries on the $i$-th row of this matrix are polynomials of degree at most $i - 1$ in $\tilde{\lambda}$. Consequently, the determinant $\Delta_2(\tilde{\lambda})$ of this matrix is a univariate polynomial in $\tilde{\lambda}$ of degree at most $\deg(\chi) \cdot (\deg(\chi) - 1)/2$. If the evaluation of $\Delta$ at $\tilde{\lambda}$ is not zero, then the algorithm does not fail. Finally, we prove that $\Delta_1$ is a nonzero polynomial using (Div-H2): $\Delta_2(\lambda)$ is the determinant of the identity matrix, hence $\Delta_2(\lambda) = 1$. The product $\Delta_1 \cdot \Delta_2$ satisfies the desired properties. \qed

Before investigating Algorithm 4 (AddDivisors) and Algorithm 5 (SubtractDivisors), we need a technical lemma.

**Lemma 29.** Let $\phi : R \to S$ be a surjective morphism of finite étale $k$-algebras, and let $z$ be a primitive element for $R$. Then $\phi(z)$ is a primitive element for $S$.

**Proof.** Since $\phi$ is surjective, any element $y \in S$ equals $\phi(x)$ for some $x \in R$. Since $z$ is primitive, there exists a univariate polynomial $w(S) \in k[S]$ such that $x = w(z)$. Consequently, $y = \phi(w(z)) = w(\phi(z))$. Therefore, $\phi(z)$ is primitive for $S$. \qed

**Proposition 30.** For a given input $(q, D_1, D_2)$ of Algorithm 4 (AddDivisors), the set of $\tilde{\lambda}$ which makes Algorithm 4 fail is contained in the set of roots of a nonzero univariate polynomial with coefficients in $K$ and of degree bounded by $(\deg(\chi_1) + \deg(\chi_2)+1)/2$.

**Proof.** There are three possible sources of failure for Algorithm 4: The two invocations of \textsc{ChangePrimEl t}, and the conditional test. For $i \in \{1, 2\}$, consider $I_i = \langle U - \bar{u}_i(S), V - \bar{v}_i(S), \chi_i(S) \rangle \cap k[U, V]$. We prove now the following claim: If $\tilde{\lambda}U + V$ is a primitive element for $\text{red}(k[U, V]/(I_1 \cdot I_2))$ such that

\[
\begin{align*}
\text{GCD} \left( \frac{\partial q}{\partial X}(u_1(S), v_1(S)) - \tilde{\lambda} \frac{\partial q}{\partial Y}(u_1(S), v_1(S)), \chi_1(S) \right) & \neq 1, \\
\text{GCD} \left( \frac{\partial q}{\partial X}(u_2(S), v_2(S)) - \tilde{\lambda} \frac{\partial q}{\partial Y}(u_2(S), v_2(S)), \chi_2(S) \right) & \neq 1,
\end{align*}
\]

(3)}
then both calls to \texttt{CHANGEPRIMELT} succeed, and \( \tilde{u}_1 \equiv \tilde{u}_2 \mod \gcd(\tilde{\chi}_1, \tilde{\chi}_2) \). The fact that the calls to \texttt{CHANGEPRIMELT} succeed is a direct consequence of Lemma 29, using the canonical projections \( \text{red}(k[U,V]/(I_1 \cdot I_2)) \to \text{red}(k[U,V]/I_i) \) for \( i \in \{1, 2\} \) together with Proposition 28. Then \( \tilde{\lambda}U + V \) must be a primitive element for \( \text{red}(k[U,V]/I_1) \) and for \( \text{red}(k[U,V]/I_2) \) by Lemma 8. Let \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \) denote the minimal polynomials of \( \tilde{\lambda}U + V \) in \( \text{red}(k[U,V]/I_1) \) and \( \text{red}(k[U,V]/I_2) \). Also, set \( \xi = \text{LCM}(\tilde{\chi}_1, \tilde{\chi}_2) \). Consequently, \( \tilde{\chi}_1(\tilde{\lambda}U + V) \cdot \tilde{\chi}_2(\tilde{\lambda}U + V) \in \sqrt{I_1 \cdot I_2} \) and \( \xi \) is the minimal polynomial of \( \tilde{\lambda}U + V \) in \( \text{red}(k[U,V]/(I_1 \cdot I_2)) = k[U,V]/(\sqrt{I_1} \cap \sqrt{I_2}) \). If \( \tilde{\lambda}U + V \) is a primitive element for \( \text{red}(k[U,V]/(I_1 \cdot I_2)) \), then the canonical map
\[
\text{red}(k[U,V]/(I_1 \cdot I_2)) \to \text{red}(k[U,V]/I_2) \times \text{red}(k[U,V]/I_1)
\]
becomes a map
\[
k[S]/\xi(S) \to k[S]/\tilde{\chi}_1(S) \times k[S]/\tilde{\chi}_2(S).
\]
This implies that there exists an element \( \tilde{u}_{12} \in k[S]/\xi(S) \) such that \( \tilde{u}_{12} \equiv \tilde{u}_1 \mod \tilde{\chi}_1 \) and \( \tilde{u}_{12} \equiv \tilde{u}_2 \mod \tilde{\chi}_2 \). As a consequence, \( \tilde{u}_1 \equiv \tilde{u}_2 \mod \gcd(\tilde{\chi}_1, \tilde{\chi}_2) \). Using Hensel’s lemma, the property (Div-H3) and the CRT, we obtain that the equation \( (\xi(u(S), S - \lambda u(S)) = 0 \) has a unique solution \( s \) in \( k[S]/\text{gcd}(\tilde{\chi}_1(S), \tilde{\chi}_2(S)) \) such that \( s \equiv \tilde{u}_1 \equiv \tilde{u}_2 \mod \gcd(\tilde{\chi}_1, \tilde{\chi}_2) \). By (Div-H1), both \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are solutions, and therefore \( \tilde{u}_1 \equiv \tilde{u}_2 \mod \gcd(\tilde{\chi}_1, \tilde{\chi}_2) \).

Finally, Lemma 7 for the ideal \( I_1 \cdot I_2 \) yields a polynomial \( \Delta \) of degree at most \( (\deg(\chi_1) + \deg(\chi_2) + 1) \) such that elements \( \tilde{\lambda} \in k \) which are not roots of \( \Delta \) satisfy the wanted properties. \( \square \)

**Proposition 31.** For a given input \( (q, D_1, D_2) \) of Algorithm 5 (\texttt{SUBTRACTDIVISORS}), the set of \( \tilde{\lambda} \) which makes Algorithm 5 fail is contained in the set of roots of a nonzero univariate polynomial with coefficients in \( \mathbb{F}_p \) and of degree bounded by \( (\deg(\chi_1) + \deg(\chi_2) + 1)^2 \).

**Proof.** The proof is similar to the first part of the proof of Proposition 30. With the same notation as in the proof of Proposition 30, Algorithm 5 fails only if \( \tilde{\lambda}U + V \) is not a primitive element for \( \text{red}(k[U,V]/(I_1 \cdot I_2)) \) or if the vector \( (1, -\tilde{\lambda}) \) is tangent to the curve at one of the points in the support of one of the divisors. Using a proof similar to that of Proposition 30, this happens only when \( \tilde{\lambda} \) is in the set of roots of the nonzero polynomial of degree at most \( (\deg(\chi_1) + \deg(\chi_2) + 1)^2 \) provided by Lemma 7 for the ideal \( I_1 \cdot I_2 \). \( \square \)

Next, we wish to bound the probability that the regular function \( h \) returned by Algorithm 6 (\texttt{INTERPOLATE}) vanishes at a singular point of the curve. To this end, we first bound the number of singular points of the curve; then we will show that the set of regular functions which vanish at a given singular point is contained in a linear subspace of codimension 1.

The number of singular points of the curve can be bounded by the following lemma.

**Lemma 32.** For any \( \lambda \in k \), there are at most \( \deg(C)^2(\deg(C) - 1) \) maximal ideals \( m \) in \( k[C^0] \) which contain \( \partial q/\partial Y \).

**Proof.** Let \( m \) be a maximal ideal in \( k[C^0] \). The partial derivative \( \partial q/\partial Y \) vanishes in the residue field \( k[C^0]/m \) only if the discriminant of \( q \) with respect to \( Y \) does so. Since \( q \) is monic when regarded as a polynomial in \( k[X][Y] \), this discriminant is a univariate polynomial in \( k[X] \) of degree bounded by \( \deg(C) \cdot (\deg(C) - 1) \). Since we assumed that \( C \) has degree at least 2 and that it is absolutely irreducible, this discriminant is not identically zero. For a given \( x \in \mathbb{F}_p \), there are at most \( \deg(C) \) points in \( C^0(\mathbb{F}_p) \) with abscissa \( x \), and hence there are at most \( \deg(C)^2 \cdot (\deg(C) - 1) \) points in \( C^0(\mathbb{F}_p) \) whose \( x \)-coordinate is a root of the discriminant. \( \square \)

Before stating the next proposition, we recall the second assumption that we have made on the input divisor and which is described in Section 2. It ensures the existence of a form \( h \) of given degree such that \( (h) \geq D_+ \) and \( h \) does not vanish at any singular point. With the notation in the following proposition, this assumption precisely means that \( A \neq \ker(\varphi) \).
Proposition 33. Let $A \subset \ker(\varphi) \subset k[X, Y]$ be the subset of all the regular functions $h$ in the kernel of $\varphi$ in Algorithm 6 which are such that the support of $(h)$ contains a singular point of $C^0$. If $A \neq \ker(\varphi)$, then $A$ is contained in the join of at most $\deg(C^2)(\deg(C) - 1)$ hyperplanes in $\ker(\varphi)$. Said otherwise, there is a nonzero polynomial in $k[Z_1, \ldots, Z_{\dim(\ker(\varphi))}]$ of degree at most $\deg(C)^2(\deg(C) - 1)$ which vanishes at any value of $(\mu_1, \ldots, \mu_{\dim(\ker(\varphi))})$ for which the second test in Algorithm 7 fails.

Proof. If $(h)$ involves a singular point of $C^0$, then in particular $h$ must vanish at a point $(x, y) \in C^0(\mathbb{K})$ such that $\frac{\partial q}{\partial Y}(u(S), v(S))$ vanishes at $S = \lambda x + y$, which implies that $\frac{\partial q}{\partial Y}(x, y) = 0$. Said otherwise, $h$ must be contained in a maximal ideal $m$ which contains $\partial q/\partial Y$. The set of elements in the kernel of $\varphi$ which belong to $m$ forms a $k$-vector space. This vector space cannot equal $\ker(\varphi)$ because this would imply that $A = \ker(\varphi)$, which would contradict our hypotheses. Therefore, it is contained in a hyperplane. The proof is concluded by noticing that there are at most $\deg(C)^2(\deg(C) - 1)$ such maximal ideals, by Lemma 32.

To prove the last sentence, consider a hyperplane $H \subset \ker(\varphi)$ corresponding to a maximal ideal containing $\partial q/\partial Y$. Such a hyperplane can be described by a linear form $\psi$ in $k[Z_1, \ldots, Z_{\dim(\ker(\varphi))}]$ such that $\psi(\mu_1, \ldots, \mu_{\dim(\ker(\varphi))}) = 0$ if and only if $\sum_{i=1}^{\dim(\ker(\varphi))} \mu_i b_i \in H$, where $b_1, \ldots, b_{\dim(\ker(\varphi))}$ is a basis of $\ker(\varphi)$. The product of all these linear forms over all maximal ideals containing $\partial q/\partial Y$ yields a polynomial in $k[Z_1, \ldots, Z_{\dim(\ker(\varphi))}]$ satisfying the desired properties.

Proposition 34. The set of values of $\lambda$ which make the first test in Algorithm 7 fail is contained within the set of roots of a univariate polynomial with coefficients in $k$ of degree $\deg(C) + 1$.

Proof. Writing $\tilde{q}(S, Y) = q((S - Y)/\lambda, Y)$, the first test fails if $\lambda = 0$ or if $\tilde{q}$ is not monic in $Y$. In particular, this latter condition happens when the coefficient of the monomial $Y^{\deg(C)}$ in $\tilde{q}$ vanishes. Writing explicitly the change of variables, we obtain that this coefficient equals $\sum_{i=0}^{\deg(C)} (-1)^i \langle \mu_1 \cdots \mu_i h(x, y) \rangle_{\deg(C) - i}$, where $q_{i,j}$ stands for the coefficient of $X^i Y^j$ in $q$. Multiplying by $\lambda^{\deg(C) + 1}$ clears the denominator and adds the root $0$ to exclude the case $\lambda = 0$; this provides a polynomial satisfying the desired properties.

Proposition 35. Let $h \in k[C^0]$ be a regular function such that $\langle h, \partial q/\partial X, \partial q/\partial Y \rangle = k[C^0]$. Then the set of $\lambda$ which makes Algorithm 7 (COMPPRINCDIV) with input $q, h$ fail is contained in the set of roots of a nonzero univariate polynomial with coefficients in $\mathbb{K}$ and of degree bounded by $\left(\frac{\deg(C)}{2} \deg(h) + 1\right) + \deg(C) + 1$.

Proof. First, let $\Delta_1$ be the univariate polynomial constructed in Proposition 34. The first test in Algorithm 7 does not fail only if $\lambda$ is not a root of $\Delta_1$.

By Bézout theorem, the effective divisor $(h)$ has degree at most $\deg(C) \deg(h)$. Consequently, Lemma 7 yields a nonzero polynomial $\Delta_2$ of degree at most $\frac{\deg(C)}{2} \deg(h) + 1$ such that the set of $\lambda$ such that $\lambda X + Y$ is not a primitive element for $\text{red}(k[C^0]/(h))$ or such that the last test in Algorithm 7 fails is contained within the set of roots of $\Delta_2$.

We claim that the product $\Delta_1 \cdot \Delta_2$ satisfies the required properties. To prove this claim, it remains to show that if $\lambda$ is not a root of $\Delta_1 \cdot \Delta_2$, then $a_1(S)$ is invertible modulo $\chi(S)$. To this end, we notice that $a_1(S)$ is invertible modulo $\chi(S)$ if and only if $a_1(s)$ is nonzero for any root $s \in \mathbb{K}$ of $\chi(S)$. By [6, Cor. 5.1], this is equivalent to fact that the GCD of the polynomials $q((s - Y)/\lambda, Y)$, $h((s - Y)/\lambda, Y)$ has degree 1 for any root $s$ of $\chi(S)$. Next, we note that if $\lambda$ is not a root of $\Delta_2$, then any common root $y$ of $q((s - Y)/\lambda, Y)$ and $h((s - Y)/\lambda, Y)$ has multiplicity 1 in $q((s - Y)/\lambda, Y)$: Indeed, the vanishing of the derivative $\partial q/\partial Y$ of $q((s - Y)/\lambda, Y)$ at $Y = y$ would precisely mean that the vector $(1, -\lambda)$ is tangent to the curve at the intersection point, which is impossible by definition of $\Delta_2$. Consequently, the GCD of the polynomials $q((s - Y)/\lambda, Y)$, $h((s - Y)/\lambda, Y)$ must be squarefree. Finally, let $y_1, y_2 \in \mathbb{K}$ be two common roots of $q((s - Y)/\lambda, Y)$, $h((s - Y)/\lambda, Y)$. This means that $((s - y_1)/\lambda, y_1)$ and $((s - y_2)/\lambda, y_2)$ are two common zeros of $q(Y, X)$ and $h(X, Y)$. Since $\lambda$ is not a root of $\Delta_1$, $AX + Y$ is a primitive element for $\text{red}(k[C^0]/(h)) = k[X, Y]/\langle q, h \rangle$, which implies that $\lambda x + y$ takes distinct
values at all points \((x, y)\) in the variety associated to the system \(h(X, Y) = q(X, Y) = 0\) (the endomorphism of multiplication by \(\lambda X + Y\) must have distinct eigenvalues, see e.g. the proof of Lemma 6 for more details). In particular, this means that \(y_1 = y_2\). Consequently, the GCD of the polynomials \(q((s - Y)/\lambda, Y), h((s - Y)/\lambda, Y)\) is a squarefree polynomial with at most one root, hence it has degree at most 1. Since \(\text{Resultant}(q((s - Y)/\lambda, Y), h((s - Y)/\lambda, Y))\) vanishes and \(q((S - Y)/\lambda, Y)\) is monic as a polynomial in \(k[S][Y]\) because \(\lambda\) is not a root of \(\Delta_1\), this GCD must have degree at least 1. Therefore, this GCD has degree exactly 1, and hence \(a_1(S)\) is invertible modulo \(\chi(S)\).

Finally, we can derive our bound on the probability that the top-level algorithm fails by summing the probabilities that the subroutines fail.

**Theorem 36.** Let \(E \subset k\) be a finite set. Assume each call to the function \(\text{RANDOM}(k)\) is done by picking an element uniformly at random in \(E\). Then the probability that Algorithm 1 fails is bounded above by

\[
O(\max(\deg(C)^4, \deg(D_+)^2)/|E|).
\]

**Proof.** Propositions 30, 31, together with the fact that the number of roots in \(k\) of a univariate polynomial is bounded by its degree, directly imply that the probabilities of failure of Algorithms \(\text{ADD\ DIVISORS}\) and \(\text{SUBTRACT\ DIVISORS}\) are bounded by \(O(\max(\deg(D_1), \deg(D_2))^2/|E|)\), if the computation of the characteristic polynomial in Algorithm \(\text{CHANGE\ PRIME\ ELT}\) succeeds. Following [18] (see also the remark in the proof of Proposition 20), the probability that the computation of the characteristic polynomial in \(\text{CHANGE\ PRIME\ ELT}\) fails is bounded by \(\deg(\chi)^2/|E|\). Therefore, the probabilities that Algorithms \(\text{ADD\ DIVISORS}\) and \(\text{SUBTRACT\ DIVISORS}\) fail are still bounded by \(O(\max(\deg(D_1, D_2))^2/|E|)\) when we take into account the probability that the computations of the characteristic polynomials fail. Notice that our second technical assumption (described in Section 2) on the input divisor ensures that \(A \neq \ker(\varphi)\) in Proposition 33. Using Proposition 33, Schwartz-Zippel lemma [24, Lemma 6.44], and Proposition 35, we bound the probability that \(\text{COMP\ PRINC\ DIV}\) fails by \(O(\max(\deg(C)^3, \deg(C)^2 \deg(h)^2)/|E|)\).

The failure probabilities are summed up in Table 2. Next, notice that the probability of failure of Algorithm 1 is bounded by the sum of the probabilities of the subroutines. Finally, the proof is concluded by using the inequality \(\deg(h) < \deg(D_+)/\deg(C) + \deg(C)\) (Proposition 24) and the degree bounds in Table 1 for the divisors arising in Algorithm 1.

### 8 Experimental results

We have implemented Algorithm 1 in C++ for \(k = \mathbb{Z}/p\mathbb{Z}\), relying on the NTL library for all operations on univariate polynomials and for linear algebra. We have also implemented the group law on the Jacobian of a curve via Riemann-Roch space computations. Our software \texttt{rrspace} is freely available at \url{https://gitlab.inria.fr/pspaenle/rrspace} and it is distributed under the LGPL-2.1+ license.

All the experiments presented below have been conducted on a Intel(R) Core(TM) i5-6500 CPU@3.20GHz with 16GB RAM. The comparisons with the computer algebra system Magma have been done with its version V2.23-8. The experiments in this section have been

| Algorithm         | Failure probability                                | Statement                        |
|-------------------|---------------------------------------------------|----------------------------------|
| \texttt{CHANGE\ PRIME\ ELT} | \(\deg(D)^2/|E|\)                              | Prop. 28                          |
| \texttt{ADD\ DIVISORS}        | \(O(\max(\deg(D_1), \deg(D_2))^2/|E|)\)          | Prop. 30                          |
| \texttt{SUBTRACT\ DIVISORS}    | \(O(\max(\deg(D_1), \deg(D_2))^2/|E|)\)          | Prop. 31                          |
| \texttt{COMP\ PRINC\ DIV}     | \(O(\max(\deg(C)^3, \deg(C)^2 \deg(h)^2)/|E|)\) | Prop. 33, Prop. 35, Schwartz-Zippel lemma [19, Coro. 1] |

Table 2: Probabilities of failure.
done over the field $\mathbb{Z}/p\mathbb{Z}$ for $p = 65521$. We have also done experiments for other primes $p < 2^{28}$, and the choice of $p$ does not seem to have a significant impact on the timings.

Our first experimental data is generated as follows. We set $k = \mathbb{Z}/65521\mathbb{Z}$. For $i$ from 10 to 100, we consider a curve $C$ defined by a random bivariate polynomial of degree 10 over $k$, and we generate $i$ random irreducible $k$-defined effective divisors $D_1, \ldots, D_i$ of degree 10 on $C$. Then we set $D = D_1 + \cdots + D_i$ and we measure the time used for computing a basis of $L(D)$ by using either Magma (via its function `RiemannRochSpace`) or the software rrspace. The experimental results are displayed in Figure 1. For these parameters, we observe that rrspace has a speed-up of approximately 9.5 compared to Magma. Since we do not have access to the implementation of the function `RiemannRochSpace` in Magma, we cannot explain the small variations in the Magma timings which seem to depend on the parity of $i$. A linear regression on these data seems to indicate that the timings of Magma grow as $\Theta(\text{deg}(D)^{2.25})$ while the timings of rrspace grow as $\Theta(\text{deg}(D)^{2.65})$. In this setting, the theoretical asymptotic complexity bound proved in Theorem 27 would give $O(\text{deg}(D)^\omega)$.

Our second experimental data is generated as follows. Again, we set $k = \mathbb{Z}/65521\mathbb{Z}$. For $d$ from 8 to 25, we generate a random smooth curve $C$ defined by a random bivariate polynomial of degree $d$ over $k$, a random rational point $O$ on $C$, and two random $k$-defined irreducible effective divisors $D_1, D_2$ of degree $g = (d - 1)(d - 2)/2$ on $C$. Then we wish to compute an effective divisor $D_3$ of degree $g$ such that $(D_1 - gO) + (D_2 - gO)$ is linearly equivalent to $D_3 - gO$. This is motivated by the fact that every point in the Jacobian has a representative of the form $D - gO$, with $D$ effective of degree $g$, so this operation models the group law on the Jacobian of $C$. A solution to this problem is $D_3 = D_1 + D_2 - gO + (f)$, where $f \in L(D_1 + D_2 - gO)$. We measure the time required by rrspace and Magma to compute such a divisor $D_3$. In rrspace, we have implemented a function which takes $D_1, D_2$ and $O$ as input and which returns $D_3$. In Magma, we split this computation in two steps: first the computation of $L(D_1 + D_2 - gO)$, then the computation of $D_3$. Figure 2 displays the experimental results. The green curve records the total time used by Magma for doing the two steps, while the red curve displays only the part of the time which was used by Magma for the computation of the Riemann-Roch space.

![Figure 1: Comparison of the time required by rrspace and Magma to compute a basis of $L(D)$ on a fixed smooth curve of degree 10 over $\mathbb{Z}/65521\mathbb{Z}$, and where $D$ is the sum of random irreducible effective divisors of degree 10. Both axes are in logarithmic scale.](image-url)
The black curve displays the total time used by \texttt{rrspace}. Here, \texttt{rrspace} seems to have a speedup of a factor more than 10 compared to the Riemann-Roch computation in \texttt{Magma}. The runtime for \texttt{rrspace} seems to grow as $\Theta(d^{5.2})$, whereas the Riemann-Roch part of the \texttt{Magma} computation seems to grow as $\Theta(d^{6.1})$. The theoretical bound provided by Theorem 27 would give $O(d^{2\omega})$.

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