PONTRJAGIN FORMS AND INVARIANT OBJECTS RELATED TO THE Q-CURVATURE

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ABSTRACT. We clarify the conformal invariance of the Pontrjagin forms by giving them a manifestly conformally invariant construction; they are shown to be the Pontrjagin forms of the conformally invariant tractor connection. The Q-curvature is intimately related to the Pfaffian. Working on even-dimensional manifolds, we show how the $k$-form operators $Q_k$ of [10], which generalise the Q-curvature, retain a key aspect of the $Q$-curvature’s relation to the Pfaffian, by obstructing certain representations of natural operators on closed forms. In a closely related direction, we show that the $Q_k$ give rise to conformally invariant quadratic forms $\Theta_k$ on cohomology that interpolate, in a suitable sense, between the integrated metric pairing (at $k = n/2$) and the Pfaffian (at $k = 0$). Using a different construction, we show that the $Q_k$ operators yield a generalisation of the period map which maps conformal structures to Lagrangian subspaces of the direct sum $H^k \oplus H_k$ (where $H_k$ is the dual of the de Rham cohomology space $H^k$). We couple the $Q_k$ operators with the Pontrjagin forms to construct new natural densities that have many properties in common with the original Q-curvature; in particular these integrate to global conformal invariants. We also work out a relevant example, and show that the proof of the invariance of the (nonlinear) action functional whose critical metrics have constant Q-curvature extends to the action functionals for these new Q-like objects. Finally we set up eigenvalue problems that generalise to $Q_k$-operators the Q-curvature prescription problem.

1. Introduction

1.1. Overview. In recent years there has been much interest in Q-curvature on even-dimensional conformal manifolds [7, 16, 17, 18, 19, 27, 28, 30, 32, 33, 34, 35]. Q-curvature naturally appears in Polyakov-type formulas for the determinant or torsion quotient with respect to a differential operator or elliptic complex with good conformal behavior. It is shown in [32] that the integral of the Q-curvature is the action functional for the Fefferman-Graham obstruction tensor. This is in turn associated with the AdS/CFT correspondence, and in fact quantities having some properties in common with the Q-curvature appear in the volume asymptotics of conformally compact Poincare-Einstein spaces. In terms of elementary geometric analysis, the prescription problem for the Q-curvature provides a higher-dimensional generalisation of the 2-dimensional Gauss curvature prescription problem.

The above-mentioned Polyakov-type formulas have motivated study of the structure of natural densities $U$ having conformally invariant integral. Conformal indices in the sense of [12], or trace anomalies in Physics, are quantities built from spectral zeta functions $\zeta(s)$ (and specifically from their conformally invariant values $\zeta(0)$ at $s = 0$); these take the form $\int U$. The functional determinant or one-loop effective action is a spectral invariant that provides a regularisation of a divergent functional integral; a torsion quantity is a certain well-chosen linear combination of functional log-determinants $\zeta'(0)$. The local part of the conformal variation of these determinants are given by integrals of quantities $U$ as above against arbitrary functions (conformal factors). This makes it worthwhile
to understand the space of natural densities $U$ that can appear; see e.g. [21]. Though this is easy to do in, for example, dimension 4, by writing out all the possibilities, the situation is not well understood in higher dimensions, despite the relevance of this problem to string and brane theories; see [5] for some work in dimension 8. A conjecture of central importance is the proposal that each natural density $U$ with conformally invariant integral is the sum of a multiple of the Pfaffian, a natural exact divergence, and a natural locally conformally invariant density. Settling this is proving to be not at all straightforward, see [1] for some recent progress. The Q-curvature captures the conformal content carried by the Pfaffian, in the sense that its integral is a fixed multiple of the Euler characteristic in the locally conformally flat case, so this conjecture is equivalent to the analogous one in which the Pfaffian is replaced by the Q-curvature. A separate conjecture [11] asserts that the Q-curvature and suitable modifications are the only terms that can appear in a certain way in the local part of a Polyakov-type formula.

A central result here is a new construction of natural densities that are “Q-like” in the sense that each of these densities integrates (in the compact setting) to a conformal invariant and has conformal variation by a formally self-adjoint linear operator on the log of the conformal factor. There has been recent progress in the construction of such densities [28, 30]. The construction here in Theorem 5.2 takes a completely new direction in that the components of the construction are essentially non-local; they arise from coupling Pontrjagin forms with the local expression for a global (integrated) conformal pairing.

Our analysis of the structure of the deRham complex from the viewpoint of conformal structure in [10] has uncovered a series of linear differential operators $Q_k$ on lower degree cocyclic forms that admit linear conformal change laws – that is, which “act like Q-curvatures”. In fact, the classical Q-curvature $Q$ of [6] is essentially the lowest-valence member of this series ($Q = Q_01$). The $Q_k$ compress to maps between cohomologies and spaces of conformal harmonics and this generalises the fact that $\int Q_0$ is a numerical conformal invariant. One of the key results is Theorem 3.2 which proves that these $Q_k$-operators encode global information. The conformal change of $Q$ is by an exact divergence, yet $Q$ is not just the sum of a local conformal invariant and an exact divergence. In the conformally flat case at least, the Pfaffian is the obstruction. Theorem 3.2 shows the analogue is true for the $Q_k$ operators. The $Q_k$ changes conformally by an operator with a left composition factor of the differential form coderivative $\delta$, yet $Q_k$ is not just the sum of a conformally invariant operator and something with a left $\delta$ factor. This establishes that Q-curvatures and Q-operators encode deep information about conformal structures beyond what is available from the classification of locally conformally invariant tensors and operators. In particular, while they do give conformally invariant cohomology maps these do not arise from objects with local conformal invariance. This is an indication that the $k$-form Q operators for $k \neq 0$ are conformal analogues of the (scalar) Pfaffian.

The global conformal content of the $Q_k$-operators is captured by a global pairing in Theorem 4.1. This shows that in a suitable sense and under suitable conditions on the conformal structure (conditions which we expect to be generic), the $Q_k$ give rise to conformally invariant real-valued quadratic forms $\Theta_k$ on the cohomologies $H_k$. When $k = n/2$, this quadratic form is the integrated metric pairing, while for connected, conformally flat manifolds, $\Theta_0$ is a multiple of the Euler characteristic. (Relaxing the connectedness assumption, it multiplies the class in 0-cohomology contributed by a given connected component by that component’s Euler characteristic.) Thus one may view
the $Q_k$ operators as special local and natural expressions for these global quantities. In Section 7.1 we establish eigenvalue problems that generalise to $Q_k$-operators the $Q$-curvature prescription problem. For a class of 10-dimensional manifolds, we exhibit a solution to such a prescription problem in Section 6. These examples establish that the new objects constructed in Theorem 5.2 are not trivial in general.

It should be interesting to determine whether the $\Theta_k$ distinguish conformally flat structures on a given fixed manifold or whether, in this setting, they reflect only topological data. On the other hand in general we expect the $Q_k$ operators to be sensitive to conformal structure and in Proposition 4.2 we construct a map from the space of conformal structures on $M$ to the Grassmannian of Lagrangian subspaces of $H^k \oplus H_k$. This generalises a special case of this generalised period map developed by Eastwood and Singer [24]. For some applications of period maps to gauge theory, see [22], Section 4.3, and [38, 36].

Apart from those mentioned already, the main results are as follows. We show that the Pontrjagin classes of the standard conformal tractor bundle agree with the Pontrjagin classes for $TM$. This leads to a new and simple proof that the Pontrjagin classes of the tangent bundle obstruct conformally flat metrics. The argument uses no explicit information about forms representing the classes. We then show that in fact the Pontrjagin forms for the tractor bundle (not just their cohomology classes) agree with the usual Pontrjagin forms. This provides a transparent explanation for the conformal invariance of these forms (first proved by Chern and Simons [20]). It also suggests the point of view that the Pontrjagin forms are more naturally associated to the tractor connection than to the Levi-Civita connection, since the latter involves choices (in particular a metric from the conformal class) that are not necessary in order to write down these forms.

Theorem 5.3 shows that the Q-operator of lowest positive (in fact second) order is capable of producing Q-like objects even when coupled to an auxiliary bundle and connection, while Theorem 5.4 shows that the higher-order Q operators definitely do not perform this service. Theorem 7.1 establishes the conformal transformation invariance of the action functional associated to the problem of prescribing constancy for Q-like invariants.

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1.2. Basic constructions. We define a vector space of Q-like objects, the linear Q-space, as follows. Let $\mathcal{I}$ be the space of natural $(-n)$-densities $A$ on pseudo-Riemannian manifolds of even dimension $n$; that is, quantities built in a universal and polynomial way from a pseudo-Riemannian metric and its inverse, together with the associated covariant derivative $\nabla$ and Riemann curvature $R$. Our normalisation of conformal density weights is such that $(-n)$-densities are integrable given only a conformal structure; henceforth we shall simply use the term densities to refer to these. In a natural density, for each monomial summand, the number of $\nabla$ plus twice the number of $R$ occurring is $n$. Under a conformal change of metric $\tilde{g} = e^{2\omega} g$, where $\omega$ is a smooth function, we have

$$\tilde{A} = A + A^1 (d\omega, g, g^{-1}, \nabla, R) + \cdots + A^n (d\omega, g, g^{-1}, \nabla, R),$$
where \( A^s \) is a universal polynomial which is \( s \)-homogeneous in \( d\omega \). It is an elementary exercise to show that if \( A^s \) is identically zero (for all \( g \) and \( \omega \)), then so are the \( A^t \) for \( t > s \). Thus \( \mathcal{I}^s := \{ A \mid A^{s+1} = 0 \} \) is a filtration of the set of natural densities. In particular, \( \mathcal{I}^0 \) is the space of conformally invariant natural densities. It is also elementary to show that if \( A \in \mathcal{I}^s \), then

\[
\omega \mapsto A^s(d\omega, g, g^{-1}, \nabla, R)
\]

is a conformally invariant \( s \)-homogeneous differential operator from functions to densities. In particular, if \( s = 1 \), this is an invariant linear differential operator.

The linear Q-space is a certain subspace of \( \mathcal{I}^1 \), the densities with a linear conformal change law. The additional condition that we shall impose reflects the origin of the Q-curvature concept in the study of determinant and torsion quotient formulas [13]. Because the conformal variation of a Q-curvature should be the second conformal variation of a (possibly nonlocal) quantity, there should be a symmetry condition for this second variation. This works out to be as follows. For any \( A \in \mathcal{I} \), let \((bA)\omega := A^1(d\omega, g, g^{-1}, \nabla, R)\). Since \( bA \) is a linear differential operator from functions to densities, and these are naturally dual given a conformal structure, the formal adjoint \((bA)^*\) is also a linear differential operator from functions to densities. We define \( \mathcal{I}^{\text{FSA}} \) to be the subspace of \( A \in \mathcal{I} \) for which \( bA \) is formally self-adjoint. Note that \( d \) is a right composition factor for any \( bA \). Thus if \( A \in \mathcal{I}^{\text{FSA}} \), then (from the normal form of [23]) the real coefficient operator \( bA \) has a factorisation \( bA = \delta N d \) for some natural operator \( N \).

The linear Q-space is

\[
\mathcal{I}^Q := \mathcal{I}^1 \cap \mathcal{I}^{\text{FSA}}.
\]

By the remark around (1), for each \( Q \in \mathcal{I}^Q \), the operator \( bQ \) is conformally invariant; we also know it is formally self-adjoint and of the form \( \delta N d \). In particular,

\[
\int \tilde{Q} = \int (Q + \delta N d\omega) = \int Q,
\]

since \( \delta N d\omega \) is an exact divergence. This shows that \( \int Q \) is conformally invariant. Elements of \( \mathcal{I}^Q \) representing nonzero classes modulo \( \mathcal{I}^0 \) (i.e. elements which are not just local conformal invariants) are especially interesting.

The linear Q-space makes precise a notion of “Q-like”. In this announcement, we indicate how one may allow the \( Q_k \) to act on characteristic forms to obtain elements of the linear Q-space. Alternatively, instead of maximally reducing the valence in this way, we may use characteristic forms to obtain additional operators on cocyclic forms with linear conformal change laws.

2. Conformal forms

For a complex vector bundle \( V \) of rank \( N \), let us write \( AB \) to mean the composition of sections \( A, B \in \Gamma(\text{End}(V)) \). By construction the function

\[
s_k(A) = \text{Tr}(AA\cdots A)
\]

is an invariant which is pointwise polynomial (and homogeneous of degree \( k \)) in the \( N^2 \) components of \( \text{End}(V) \). In the following, we shall call such objects polynomial invariants. We recall that the \( s_k(A) \) polynomially generate all such invariants.

Since exterior multiplication is commutative on even forms, for a given polynomial invariant \( P(A) \) we can replace the section \( A \in \text{End}(V) \) with a 2-form \( F \) taking values in
End(\mathcal{V}) to obtain an invariant, and in general non-trivial, 2k-form that we denote \( P(F) \). That is, \( P \) determines a map

\[ P : \mathcal{E}^2(\text{End}(\mathcal{V})) \to \mathcal{E}^{2k}, \]

which is algebraic in the sections \( F \in \mathcal{E}^2(\text{End}(\mathcal{V})) \) since it arises from a bundle map. This construction is most familiar when \( F = F^D \) is the curvature of some connection \( D \) on \( \mathcal{V} \). In this case the following results are well known (see e.g. [37]).

**Lemma 2.1.** For each polynomial invariant \( P \) and connection \( D \), \( P(F^D) \) is a closed form. The de Rham cohomology class \( \left[ P(F^D) \right] \in H^*(M; \mathbb{C}) \) is independent of the choice of connection \( D \) on \( \mathcal{V} \), and isomorphic vector bundles define identical cohomology classes.

Recall that the \( \ell \)-th Chern character class of the complex vector bundle \( \mathcal{V} \) is the class in \( H^{2\ell}(M; \mathbb{C}) \) given by

\[ c_\ell(\mathcal{V}) = \frac{1}{\ell!} \left[ s_\ell \left( \frac{-1}{2\pi \sqrt{-1}} F^D \right) \right], \]

for any connection \( D \) on \( \mathcal{V} \). (It is slightly more convenient here to use these character classes rather than the Chern classes, whose definition involves the elementary symmetric polynomial \( \sigma_\ell \) instead of \( s_\ell \).) The \( k \)-th Pontrjagin character class of a real vector bundle \( \mathcal{V} \) is simply \( c_{2k}(\mathcal{V}_\mathbb{C}) \), and takes values in real cohomology. In this setting we will call the forms

\[ p_k^D = \frac{1}{(2k)!} s_{2k} \left( \frac{-1}{2\pi \sqrt{-1}} F^D \right) \]

*Pontrjagin forms.*

It is result of Chern and Simons [20] that the Pontrjagin forms for a Riemannian connection are conformally invariant. We will show here that in fact these forms arise from the conformally invariant tractor connection \( \nabla \) [3, 14] on the (real) tractor bundle \( \mathbb{T} \). Let us write \( \Omega \) for the curvature of the standard tractor connection. Since this connection preserves a metric on \( \mathbb{T} \), the invariant \( p \)-forms \( P(\Omega) \) vanish identically unless they have degree \( p = 4k \). (Contraction with several copies of the metric produces a self- or skew-adjoint endomorphism according to whether \( p \) is of the form \( 4k \) or \( 4k + 2 \).)

**Definition:** For each \( k \geq 1 \) we will call the closed 4k-form \( \tau_k = p_k^\mathbb{T} \) the \( k \)-th conformal Pontrjagin (character) form for a Riemannian or pseudo-Riemannian manifold and we will use the term conformal form for any invariant form generated by the conformal Pontrjagin (character) forms. We first observe that these recover the usual character classes.

**Theorem 2.2.** For \( k \geq 1 \) the class \( [\tau_k] \in H^{4k}(M) \) is \( \phi_k(TM) \), the \( k \)-th Pontrjagin character class of the tangent bundle.

**Proof:** From its definition [3, 14] the tractor bundle \( \mathbb{T} \) is isomorphic (as a vector bundle, without considering further structure) to the direct sum bundle \( \mathbb{R} \oplus TM \oplus \mathbb{R} \). The latter may be equipped with a connection which is a trivial extension of any connection on \( TM \), so the result follows from the definition and Lemma 2.1. □

Since the tractor curvature vanishes on manifolds that are (locally) conformally flat, it follows that we have an alternative proof of the following result of [20]:

**Corollary 2.3.** The Pontrjagin character classes

\[ \phi_k(TM) \quad k \geq 1 \]
vanish on manifolds \( M \) that admit a metric (of any signature) which is locally conformally flat.

Alternatively one obtains Theorem 2.2 from the stronger result that follows. Here \( R \) denotes the Riemann curvature tensor for Riemannian (or pseudo-Riemannian) structure on \( M \), and \( C \) the corresponding Weyl curvature.

**Proposition 2.4.** On a manifold with a metric \( g \) of any signature,

\[
\tau_k = \frac{1}{(2k)!} s_{2k} \left( \frac{-1}{2\pi \sqrt{-1}} C \right) = \frac{1}{(2k)!} s_{2k} \left( \frac{-1}{2\pi \sqrt{-1}} R \right),
\]

where in the middle expression we regard the Weyl curvature as an \( \text{End}(TM) \)-valued 2-form.

**Proof:** Following the notation of [30], in terms of the metric \( g \) the tractor curvature is given by

\[
\Omega_{ab}^D = Z^C e Z^d C_{ab}^c d - 2 X^C Z^d \nabla_{[a} P_{b]d} - 2 X^D Z^C d \nabla_{[a} P_{b]d}.
\]

The upper case tractor indices may be raised and lowered using the tractor metric, and the tractor projectors \( Z \) and \( X \) combine via the tractor metric according to

\[
Z^D d Z^D e = \delta^d_e \quad \text{(the Kronecker delta)}, \quad X^D d Z^D e = 0 \quad \text{and} \quad X^D d X^D = 0.
\]

It follows that all pairings vanish other than \( ZZ \) pairings, and the first equality in (4) is then immediate.

The equality relating \( s_{2k}(C) \) to \( s_{2k}(R) \) follows easily from a direct calculation; see e.g. [2]. \( \square \)

3. **The \( Q_k \) Operators**

In [10] the authors constructed, for \( k = 0, 1, \ldots, n/2 + 1 \), natural operators \( Q^g_k : \mathcal{E}^k \to \mathcal{E}_k \), with \( Q^0_{n/2} \) a nonzero constant, \( Q^0_{n/2+1} = 0 \) and otherwise with properties as follows. (Here and below, \( \mathcal{E}^k \) denotes the bundle of \( k \)-forms or, by way of notational abuse, the sections of this bundle; and \( \mathcal{E}_k \) is the tensor product of \( \mathcal{E}^k \) with the \( (2k-n) \)-densities.)

We shall sometimes suppress the dependence of \( Q^g_k \) on the metric \( g \) and write simply \( Q_k \).

**Proposition 3.1.** Up to a non-zero constant scale, \( Q^g_k \) has the form

\[
(d\delta)^{n/2-k} + \text{LOT}
\]

and \( Q_01 \) is the (Branson) \( Q \)-curvature. Upon restriction to the closed \( k \)-forms \( \mathcal{C}^k \), \( Q^g_k \) has the conformal transformation law

\[
Q^g_k u = Q^g_k u + \delta Q^g_{k+1} d(\omega u)
\]

where \( \hat{g} = e^{2\omega} g \) with \( \omega \) a smooth function.

It follows that \( \delta Q^g_{k+1} d =: L_k : \mathcal{E}^k \to \mathcal{E}_k \) is conformally invariant.

A question of some interest is as follows. If \( S \) is a natural Riemannian density with \( \int S \) conformally invariant on compact manifolds, then is it necessary that

\[
S = \text{const-Pff} + L + V,
\]

where \( L \) is a local conformal invariant and \( V \) is an exact divergence?

This question has implications for the \( Q \)-curvature (and other natural densities in the linear \( Q \)-space), since \( \int Q \) is conformally invariant. From our current viewpoint, \( Q \) arises as \( Q_01 \), where \( Q_0 \) is a natural differential operator as in Proposition 3.1. Rephrasing things to respect this viewpoint, we come to the following equivalent question. Let
$C_k := \mathcal{E}_k/R(\delta)$, and let $\pi_k : \mathcal{E}_k \to C_k$ be the quotient map. Suppose $S$ is a natural differential operator $\mathcal{E}^0 \to \mathcal{E}_0$ with the property that $\pi_0 \circ S|_{\mathcal{E}_0}$ is conformally invariant. Is it necessary that

$$S|_{\mathcal{E}_0} = [\text{const-Pff} + L + \delta U]|_{\mathcal{E}_0},$$

where $L : \mathcal{E}^0 \to \mathcal{E}_0$ is a natural differential operator with $L|_{\mathcal{E}_0}$ conformally invariant, and $U$ is a natural differential operator $U : \mathcal{E}^0 \to \mathcal{E}_1$? (Note that the composition just mentioned acts $\delta U : \mathcal{E}^0 \to \mathcal{E}_0$.) Of course in the last display we view $\text{const-Pff}$ as a multiplication operator.

More generally, there is an analogue of this question for operators on $k$-forms with $k = 0, 1, \ldots, n/2 - 1$. If $S : \mathcal{E}^k \to \mathcal{E}_k$ is a natural differential operator with $\pi_k \circ S|_{\mathcal{E}_k} : C^k \to C_k$ conformally invariant, is it necessary that

$$S|_{\mathcal{E}_k} = [O + L + \delta U]|_{\mathcal{E}_k},$$

where $L : \mathcal{E}^k \to \mathcal{E}_k$ is a natural differential operator with $L|_{\mathcal{E}_k}$ conformally invariant, $U$ is a natural differential operator $\mathcal{E}^k \to \mathcal{E}_{k+1}$, and $O$ means $\text{const-Pff}$ for $k = 0$, and 0 for all other $k$?

It turns out that we can answer this more general question in the negative if $k \geq 1$, with the $Q_k$ operators as counterexamples. Note that by (5), the invariance hypothesis on $\pi_k \circ Q_k|_{C^k}$ is satisfied. Let us fix $k \in \{1, \ldots, n/2 - 1\}$ and suppose, with a view to contradiction, that as an operator on $C^k$ we have $Q_k = L + \delta U$ with $L$ and $U$ as above. Now we consider the situation on a conformally flat manifold. Since

$$L : C^k \to \mathcal{E}_k$$

is conformally invariant, it follows that

$$Ld : \mathcal{E}^{k-1} \to \mathcal{E}_k$$

is conformally invariant. From the classification of invariant operators on locally conformally flat manifolds (see e.g. [25]) it follows that $Ld = 0$. Thus $\delta Q_k d = 0$ on conformally flat manifolds. This contradicts Proposition 3.1, which asserts that the leading term of $\delta Q_k d$ is a non-zero constant multiple of $(\delta d)^{n/2 + 1 - k}$. In summary we have the following:

**Theorem 3.2.** For $k \in \{1, \ldots, n/2 - 1\}$ the operators $Q_k : \mathcal{E}^k \to \mathcal{E}_k$ of Proposition 3.1 cannot be written in the form $L + \delta U$ where $L : \mathcal{E}^k \to \mathcal{E}_k$ is a natural differential operator with $L|_{\mathcal{E}_k}$ conformally invariant, and $U$ is a natural differential operator carrying $\mathcal{E}^k \to \mathcal{E}_{k+1}$.

Without speculating on any possible implications that the $Q_k$ might have for topology, this theorem says that from the viewpoint of conformal geometry, and modulo local conformally invariant operators and divergence type operators, the $Q_k$ are $k$-form analogues of the Pfaffian. That is, for forms of degree $k \geq 1$, they play at least one of the roles that the Pfaffian plays for $k = 0$.

4. Conformal forms and a period map

Suppose now that $M^n$ even is compact without boundary (but not necessarily oriented or connected) and equipped with a Riemannian signature conformal structure $[g]$. We construct a family of invariant bilinear forms.

Recall that a conformal structure $[g]$ is equivalent to a canonical non-degenerate symmetric bilinear form $g$ that takes values in densities of weight 2. We term this the conformal metric (see e.g. [10] for further discussion) and if $g$ is a metric in the conformal class then $g = \sigma^{-2}g$ for some non-vanishing weight 1 density $\sigma$. For $k$-forms $\xi, \eta$, $\mathcal{E}_k := \mathcal{E}_k/\mathcal{R}(\delta)$, and let $\pi_k : \mathcal{E}_k \to C_k$ be the quotient map.
let \(\xi, \eta\) be the local form inner product determined by \(g^{-1}\), and \(\langle \xi, \eta \rangle := \int \xi \cdot \eta\). Consider the quadratic form

\[
Q_k : C^k \times C^k \to \mathbb{R}, \quad Q(\xi, \eta) = \langle \xi, Q_k \eta \rangle.
\]

Since the operator \(Q_k\) is formally self-adjoint at any metric from \([g]\), the form \(Q_k\) is symmetric.

Let \(H^k_G := \{\xi \in C^k \mid \delta Q_k \xi = 0\}\). Because the \(Q_k\) acts as \((d\delta)^{n/2-k} + \text{LOT}\) on closed forms, the system of equations (at each given conformal scale) satisfied by elements of \(H^k_G\) is elliptically coercive, and thus \(H^k_G\) is finite dimensional. In more detail, each \(\xi \in H^k_G\) satisfies a fixed system of the form

\[
\left( \delta \left\{ (d\delta)^{n/2-k} + \text{LOT} \right\} \right) \xi = 0,
\]

and so also satisfies the Laplace-type equation

\[
0 = \left( (d\delta)^{n/2-k} \delta \mid d \right) \left( \delta \left\{ (d\delta)^{n/2-k} + \text{LOT} \right\} \right) \xi = \left\{ (\delta d + d\delta)^{n/2-k+1} + \text{LOT} \right\} \xi.
\]

Let \(\tilde{\Theta}_k\) be the restriction of \(Q_k\) to a quadratic form \(H^k_G \times H^k_G \to \mathbb{R}\). Let \(H^k_{G_2}\) be the image of \(H^k_G\) in the cohomology \(H^k\), under the cohomology class map \(\xi \mapsto [\xi]\). Note that is a conformally invariant subspace of \(H^k\). We claim that \(\tilde{\Theta}_k\) gives rise to a quadratic form \(\Theta_k\) on \(H^k_G\) via

\[
\Theta_k([\xi], [\eta]) = \tilde{\Theta}_k(\xi, \eta).
\]

To see that this is well-defined, let \(\xi, \xi' \in H^k_G\) with \(\xi' - \xi = df\). Since

\[
\tilde{\Theta}_k(df, \eta) = \langle df, Q_k \eta \rangle = \langle f, \delta Q_k \eta \rangle = 0,
\]

we have \(\tilde{\Theta}_k(\xi', \eta) = \tilde{\Theta}_k(\xi, \eta)\).

In [10], a conformal manifold \((M, [g])\) is called \((k-1)\)-regular if the map \(H^k_G \to H^k\) above is surjective; that is, if \(H^k_{G} = H^k\). (We expect this condition to hold for generic conformal manifolds \((M, [g])\).

**Theorem 4.1.** If \((M, [g])\) is \((k-1)\)-regular, then \(\Theta_k : H^k \times H^k \to \mathbb{R}\) is a conformally invariant quadratic form on cohomology, \(Q_{n/2}\) is the identity, every \((M, [g])\) is \((n/2-1)\)-regular, and \(\Theta_{n/2}\) is the integrated metric pairing. Every \((M, [g])\) is \((-1)\)-regular, and in the conformally flat case, up to a universal positive constant multiple, \(\Theta_0 = \text{diag}(\chi(M_1), \ldots, \chi(M_p))\), where \(M = M_1 \sqcup \cdots \sqcup M_k\) is the connected component decomposition of \(M\).

**Proof:** Everything is clear or established above, except the final statement. By [8], in the conformally flat case, each natural density with conformally invariant integral takes the form \(c \cdot \text{Pff} + \delta \rho\), where \(c\) is a constant and \(\rho\) is a natural one-form-(2 - n)-density. Applying this to \(Q = Q_{01}\) and integrating, we find that

\[
c \cdot \chi(M_j) = \int_{M_j} Q,
\]

The round sphere, where \(Q = (n - 1)!\), serves to normalise \(c\):

\[
2c = (n - 1)! \varpi_n,
\]
where \( \omega_n \) is the volume of the round sphere. Each connected component contributes one dimension of 0-cohomology (generated by the class of its characteristic function); this establishes that
\[
\Theta_0 = \frac{(n-1)! \omega_n}{2} \text{diag}(\chi(M_1), \cdots, \chi(M_p)). \quad \Box
\]

The theorem indicates that in some sense, the conformally invariant forms \( \Theta_k \) interpolate between the integrated metric pairing of middle-forms and the Euler characteristic.

In another direction, we get a map from the set of conformal structures that generalises the celebrated period map, but involves forms not of the middle degree. Recall that above we used \( \mathcal{E}_k \) to denote the tensor product of \( \Lambda^k TM \) with \((-n)\)-densities. Then we again use \( \delta \) to denote the adjoint of \( d \) afforded by the global pairing of \( \mathcal{E}_k \) with \( \mathcal{E}_k \) and we write \( H_k \) for the corresponding cohomology space. On a fixed conformal structure this change makes no difference at all. On the other hand this contravariant point of view has the advantage that the operators \( \delta \) and the cohomology spaces \( H_k \) are now just objects belonging to the smooth structure, and in particular are not affected by moving the conformal structure.

For \( 0 \leq k \leq n/2 \), let us equip \( H^k \oplus H_k \) with the obvious symplectic structure. We obtain the following generalisation of the period map, modelled on the construction of Eastwood and Singer in [24] for the case \( n = 4, k = 1 \).

**Proposition 4.2.** For each conformal structure \( [g] \) on \( M \), there is a well-defined map
\[
I[g] : H^k_{\mathcal{E}} \to H^k \oplus H_k
\]
given by \( \phi \mapsto (\phi, [Q_k \phi]) \), the range of which is a Lagrangian subspace. Thus we obtain a map
\[
\Phi : \{ \text{conformal structures on } M \} \to G_M
\]
where \( G_M \) is the Grassmannian of Lagrangian subspaces of the symplectic vector space \( H^k \oplus H_k \), given by \( \Phi([g]) = \mathcal{R}(I[g]) \).

**Proof:** By the remarks preceding the proposition, we may regard \( H^k \oplus H_k \) as a fixed target space, independent of the conformal structure. From the transformation law (5) it follows that the map \( \phi \mapsto ([\phi], [Q_k \phi]) \) depends only on conformal structure. (Or see section 2.0 of [10] where \( I[g] \) is also discussed). That \( \mathcal{R}(I[g]) \) is an isotropic subspace follows immediately from the result [10] that, in any scale \( g \), \( Q_k^g \) is formally self-adjoint. Then \( \dim(\mathcal{R}(I[g])) \) is the \( k \)th Betti number from Corollary 2.12 of [10] and so \( \mathcal{R}(I[g]) \) is Lagrangian (i.e. it is isotropic and has half the dimension of \( H^k \oplus H_k \)). \( \Box \)

### 5. Natural scalars in the linear Q-space

For a metric \( g \) from the conformal class and \( k \in \{0, 1, \cdots, n/2 + 1\} \), let us fix a pair \( (\xi, \eta) \) in \( \mathcal{C}^k \times \mathcal{C}^k \) and make the definitions
\[
\tilde{Q}^g_{\xi, \eta} := \xi Q_k^g \eta, \quad Q_k^g := \frac{1}{2}(\tilde{Q}^g_{\xi, \eta} + \tilde{Q}^g_{\eta, \xi}), \quad Q^g_{\xi} := Q_k^g \xi.
\]
(Recall that \( \xi \cdot \eta \) is the pointwise form inner product determined by the conformal metric \( g \).) This makes each quantity in the display a density. At the moment, we make no
naturality assumption on $\xi$ or $\eta$. Related to these constructions are the differential operators $\tilde{L}_{\xi,\eta}$, $L_{\xi,\eta}$, and $L_\xi$ carrying $\mathcal{E}^0$ to $\mathcal{E}_0$, defined by

$$L_{\xi,\eta} := \tilde{L}_{\xi,\eta}(f) = \xi L_k(f(\eta)), \quad L_{\xi,\eta} = \frac{1}{2}(\tilde{L}_{\xi,\eta} + \tilde{L}_{\eta,\xi}), \quad L_\xi = L_{\xi,\xi}.$$  

Note that by construction each operator in (8) is conformally invariant. Some properties of these are summarised in the following proposition.

**Proposition 5.1.** (i) At a fixed conformal scale $g$, $Q^g_{\xi,\eta}$ is a density. Under a conformal change of metric to $\hat{g} = e^{2\omega}g$ this has the conformal transformation

$$Q^g_{\xi,\eta} = Q^{\hat{g}}_{\xi,\eta} = Q^{\hat{g}}_{\xi,\eta} + L_{\xi,\eta} \omega.$$  

(ii) The operator $L_{\xi,\eta}$ has the form $\delta M_{\xi,\eta}^g$, and is formally self-adjoint.

(iii) On compact manifolds $\int Q_{\xi,\eta}$ is conformally invariant.

(iv) In the case $k = 0$ we have that $Q^1_{1,1}$ is the $Q$-curvature.

**Proof:** Part (i). Since $\tilde{Q}^g_{\xi}$ takes values in $\mathcal{E}_k$ and the form inner product carries a density weight of $-2k$, the scalar densities $\tilde{Q}^g_{\xi,\eta}$ have weight $-n$. Thus this also holds for $Q_{\xi,\eta}$. The transformation law is an immediate consequence of the transformation law (5) for the $Q^g_k$ operator. (In fact, the stronger statement obtained by attaching a tilde to each quantity in (9) also holds.)

Part (ii). Since $\eta$ is closed,

$$L_k(f_1 \eta) = \delta Q^k_{k+1}(\varepsilon(df_1)\eta)$$

for any function $f_1$. Thus $\tilde{L}_{\xi,\eta}$ has $d$ as a right composition factor. On the other hand for another function $f_2$, from the fact that $\xi$ is closed we have

$$\langle f_2, \delta Q^k_{k+1}(\varepsilon(df_1)\eta) \rangle = \langle \varepsilon(df_2)\xi, Q^k_{k+1}(\varepsilon(df_1)\eta) \rangle,$$

where recall $\langle \phi, \psi \rangle = \int \phi \psi$. Thus $\delta$ is a left composition factor of $\tilde{L}_{\xi,\eta}$ and overall we have that $\tilde{L}_{\xi,\eta} = \delta \tilde{M}_{\xi,\eta}^g d$ for some differential operator $\tilde{M}_{\xi,\eta}^g$. Continuing our integration by parts we see that

$$\langle d(f_2 \xi), Q^k_{k+1}(df_1) \rangle = \langle Q^k_{k+1}(df_2) \xi, df_1 \eta \rangle = \langle df_1 \eta, Q^k_{k+1}(df_2 \xi) \rangle$$

and so $\tilde{L}_{\eta,\xi}$ is the formal adjoint of $\tilde{L}_{\xi,\eta}$. Symmetrising over $\xi$ and $\eta$ the claimed results follow.

Part (iii) is immediate from parts (i) and (ii). Finally, by definition $Q^1_{1,1} = Q^0_{0,1}$ and so (iv) is immediate from the result $Q = Q^0_{0,1}$ in Proposition 3.1. \qed

An interesting possibility is to take $\xi, \eta$ in (7) to be invariant forms arising from a connection $D$, for example $\xi = \eta = s_\ell(F^D)$. Then (for example) $\int Q^g_{\xi}$ gives some conformal coupling between the geometric structure and the connection $D$.

In the case that $D$ is the conformal tractor connection this yields natural invariants $Q_\xi$. As a result, we have the following.

**Theorem 5.2.** For each pair of conformal $k$-forms $\tau, \kappa$, where $0 \leq 4k \leq n/2 - 1$, the scalar $Q^g_{\tau,\kappa}$ field is a natural invariant in the linear $Q$-space $\mathcal{I}^Q$. On compact manifolds $\int Q^g_{\tau,\kappa}$ is an invariant of the conformal structure.
5.1. **A coupled generalisation of** $Q_{n/2-1}$. The operator

$$Q_{n/2-1} : \mathcal{E}^{n/2-1} \to \mathcal{E}^{n/2-1}$$

has the explicit formula $Q_{n/2-1}^g = d\delta - 4P\sharp + 2J$. Here we view the Schouten tensor $P$ as an endomorphism of the tangent bundle and $\sharp$ indicates the usual ($\otimes$-derivation) action of such an endomorphism on tensors (in this case on forms). $J$ is the trace of $P$.

Acting on the closed forms $\mathcal{C}^{n/2-1}$, we have the conformal transformation law

$$Q_{n/2-1}^\hat{g}\kappa = Q_{n/2-1}^g\kappa + 2\delta d(\omega\kappa).$$

This transformation law is preserved if we couple to a connection. Suppose $D$ is a connection on some vector bundle $\mathcal{V}$. Then write $Q_{n/2-1}^{D,g}$ for the operator on the $\mathcal{V}$-valued $(n/2 - 1)$-forms $\mathcal{E}^{n/2-1}(\mathcal{V})$ given by the formula

$$d^D \delta^D - 4P\sharp + 2J.$$

Write $\mathcal{C}^k(\mathcal{V})$ for the space of $\mathcal{V}$-valued $k$-forms $\kappa$ satisfying the identity $d^D\kappa = 0$. By direct calculation one readily verifies the following result.

**Theorem 5.3.** For any vector bundle with connection $(\mathcal{V}, D)$

$$Q_{n/2-1}^{D,\hat{g}}\kappa = Q_{n/2-1}^{D,g}\kappa + 2\delta^D d^D(\omega\kappa),$$

for $\kappa \in \mathcal{C}^{n/2-1}(\mathcal{V})$ and $\hat{g} = e^{2\omega}g$.

Since the curvature $F$ of the connection $D$ and its exterior powers are annihilated by $d^D$, it is straightforward to use these to construct coupled quantities in the linear Q-space. Once again natural scalar fields are obtained when we specialise to the case in which $D$ is the tractor connection $\nabla$ on the standard tractor bundle $\mathcal{T}$ and its tensor powers. Writing $\Omega$ for the curvature of the tractor connection on $\mathcal{T}$, in dimension 6 we have

$$\Omega^A_{\; B} \Omega^B_{\; A},$$

for example. Here we have displayed abstract tractor indices but omitted form indices. Direct computation yields the following description of this invariant in terms of classical curvatures. Let $A_{abc} := 2\nabla_c P_{[b|a}$ be the Cotton tensor, let

$$W_{abcde} := \nabla_e C_{abcd} + 2g_{e[a}A_{b]cd} + 2g_{e[c}A_{d]ab},$$

and let

$$U_{abcd} := \nabla_a A_{bcd} - P_{a}^{\; c} C_{ebcd}.$$

Then [26, 31, 15]

$$I := |W|^2 - 16(C, U) + 16|A|^2$$

is conformally invariant in dimension 6. Gover and Peterson [30] note that $G := \Delta |C|^2 = \delta d|C|^2$ admits the linear and formally self-adjoint conformal change law

$$\hat{G} = G + 4\delta(\Delta |C|^2 d\omega).$$

Fefferman and Hirachi [28] note that

$$H := -C_{abcd} C^{abcde} P^d_{\; e} + |A|^2 + \frac{1}{4} |\Omega|^2 J$$
has a linear conformal change law, which can be shown by direct computation to be formally self adjoint; in fact,

$$\hat{H} = H + \left( \frac{1}{4} |C|^2 \Delta + 4 \nabla e P_{ab} C^{a c b d} \nabla_d + C^a_{c d e} C^{c b d e} \nabla_a \nabla_b \right) \omega.$$ 

The invariant (10) is

$$\Omega^A_B Q_2^\Omega \Omega^B_A = \frac{1}{4} I + \frac{1}{8} G - 2 \hat{H} - \frac{1}{4} C_{abcd} e f C^{c d e f} - C_{abcd} e^{g} f^{h} C^{g h}.$$ 

In dimension 10, examples include (skewing over the unexpressed tensor indices in each $\Omega \otimes \Omega$)

$$\Omega^A_B \Omega^B_C Q_4^\Omega C E \Omega^E A \quad \text{and} \quad \Omega^A_B \Omega^C_E Q_4^\Omega C \Omega^B_A.$$ 

One might expect that there is a result generalising Theorem 5.3 to the $Q_k$ for $k \leq n/2 - 2$. This is not the case.

**Theorem 5.4.** Suppose that

$$Q_{n/2-k}^{D,g} : \mathcal{C}^k(\mathcal{V}) \rightarrow \mathcal{E}_k(\mathcal{V}), \quad k \in \{0, 1, \cdots, n/2 - 2\}$$

is given by a universal polynomial formula, with natural coefficients, in the connection $D$ coupled with the Levi-Civita connection. Suppose further that for any vector bundle/connection pair $(\mathcal{V}, D)$ we have

$$(11) \quad Q_{n/2-k}^{D,\hat{g}} = Q_{n/2-k}^{D,g} + S^{a.D} d^D \omega$$

whenever $\hat{g} = e^{2\omega} g$. (In the last term of (11) we view $\omega$ as a multiplication operator.) Then the operator $S^{a.D} d^D$ vanishes on conformally flat structures.

**Proof:** It is straightforward to show that without loss of generality we may assume that $S^{a.D} d^D$ is also given by a universal polynomial formula, with natural coefficients, in the coupled connection. From formula (11) it follows that $S^{a.D} d^D$ is conformally invariant acting on $\omega \mathcal{C}^k(\mathcal{V})$ for each function $\omega$. Thus by linearity we obtain an operator $S^{a.D} d^D : \mathcal{E}^k(\mathcal{V}) \rightarrow \mathcal{E}_k(\mathcal{V})$ which is conformally invariant for any vector bundle/connection pair $(\mathcal{V}, D)$. But Proposition 1.1 of [29] shows that any such operator vanishes on conformally flat structures. \qed

### 5.2. A more complete picture

Although it is a slight digression we should point out that many of our constructions generalise without difficulty in an obvious way. For example Proposition 5.1 shows that $Q_k$ generates scalar fields in the linear Q-space via (7). However if we take $(\xi, \eta) \in \mathcal{C}^{k-\ell} \times \mathcal{C}^{k-\ell}$, for $\ell \leq k$, then the formula (7) may be used to instead give an operator $\tilde{Q}_{\xi,\eta}^{k,g} : \mathcal{E}^\ell \rightarrow \mathcal{E}_\ell$. In abstract index notation, we take (up to nonzero constant multiples) $\xi^{a_k \cdots \ell+1 \cdots a_k} Q_{a_1 \cdots a_k} b_1 \cdots b_k \eta^{b_{k+\ell-1} \cdots b_k}$ to obtain $(\tilde{Q}_{\xi,\eta})_{a_1 \cdots a_k} b_1 \cdots b_k$, and set $Q_{\xi,\eta} = \frac{1}{2} (\tilde{Q}_{\xi,\eta} + \tilde{Q}_{\eta,\xi})$. These maps have properties as follows.

**Proposition 5.5.** (i) $Q_{\xi,\eta}^{k,g}$ is a differential operator which, upon restriction to the closed forms $\mathcal{C}^\ell$, has the conformal transformation law

$$Q_{\xi,\eta}^{k,\hat{g}} = Q_{\xi,\eta}^{k,g} + \delta Q_{\xi,\eta}^{k+1,g} d\omega \quad \text{where} \quad \hat{g} = e^{2\omega} g;$$

and on the right hand side $\omega$ is viewed as a multiplication operator.

(ii) The operator $\delta Q_{\xi,\eta}^{k+1,g} d$ is formally self-adjoint.
Pontrjagin forms, Q-space

(iii) \( Q^k_{\xi,\eta} \) determines a conformally invariant operator

\[
Q^k_{\xi,\eta} : \mathcal{C}^\ell \rightarrow \mathcal{C}_\ell := \mathcal{E}_\ell / \delta \mathcal{E}_{\ell+1}.
\]

(iv) On compact manifolds \( Q^k_{\xi,\eta} \) induces a conformally invariant pairing between \( \mathcal{N}(\delta Q^k_{\xi,\eta} + 1, g \xi, \eta) \) and \( \mathcal{C}^\ell \) given by

\[
(\mu, \rho) \mapsto \int \mu \cdot Q^k_{\xi,\eta} \rho.
\]

In particular we get a pairing on \( \mathcal{C}^\ell \times \mathcal{C}^\ell \).

(v) \( \delta Q^k_{\xi,\eta} \) is conformally invariant on the null space of \( \delta Q^k_{\xi,\eta} + 1, g \xi, \eta \). Thus in particular it is conformally invariant on \( \mathcal{C}^\ell \).

**Proof:** Parts (i) and (ii) are proved by obvious adaptations of the earlier arguments. Parts (iii) and (iv) then follow immediately. It is shown in [10] that on \( \mathcal{E}^k \) we have

\[
\delta Q^k = \delta Q^k + c \omega \delta Q^k + 1, g \xi, \eta d, \text{ for some constant } c.
\]

Part (v) follows easily. □

The Proposition shows that the operators \( Q^k_{\xi,\eta} \) generalise in a natural way the operators \( Q^k_{\xi,\eta} \) of [10]. Many of the other results for the \( Q^k_{\xi,\eta} \) carry over to the \( Q^k_{\xi,\eta} \). In fact the story is still more general since we can compose the \( Q_k \) on the right by exterior multiplication with any closed form and on the left by interior multiplication with any closed form. In these matters we are making no attempt be complete in the current note.

By using conformal forms for the \( \xi, \eta \) in these constructions the operators in the proposition become natural operators. Of course since the conformal forms vanish on conformally flat structures the resulting operators all vanish (apart from the \( Q^k_{\xi,\eta} \)), and this includes the natural scalar fields in the linear Q-space that we constructed earlier as a special case.

6. An example

Consider the case of a 10-dimensional manifold \( M = \mathbb{C}P^2 \times N \), where \( N \) is a conformally flat 6-dimensional manifold of constant scalar curvature \( \nu \). The factor \( \mathbb{C}P^2 \) is supplied with the Fubini-Study metric. Under these circumstances the Pontrjagin 4-form \( \kappa \) of \( M \) is just (i.e. pulls back to under inclusion of a submanifold \( \mathbb{C}P^2 \cong \mathbb{C}P^2 \times \{ y \} \)) that of \( \mathbb{C}P^2 \), and this is an eigenform of \( Q_4 \) with eigenvalue a nonconstant affine function of \( \nu \). Thus, except for one value of \( \nu \), the quantity \( Q_4 \) is nonzero.

First we claim that on general oriented Riemannian 4-manifolds, the Pontrjagin 4-form of the Levi-Civita connection is

\[
(12) \quad p^{1LC}_1 = \frac{1}{96\pi^2} (|C_+|^2 - |C_-|^2) E,
\]

where \( E \) is the volume form. In fact, from (3) and Proposition 2.4, we have

\[
p^{1LC}_1 = -\frac{1}{8\pi^2} s_2(C).
\]

An elementary calculation gives

\[
4!g(s_2(C), E) = C^a_{bce} C^b_{aef} E^{cde} = 2 C^a_{bde} (\pm C)^b_{ac} = 2(C_+ + C_-)^a_{bce} (C_+ - C_-)^b_{aef} = -2(|C_+|^2 - |C_-|^2),
\]

so that

\[
s_2(C) = -\frac{1}{12} (|C_+|^2 - |C_-|^2) E.
\]
Equation (12) follows immediately.

Now consider the special case of $\mathbb{CP}^2$ with the Fubini-Study metric $g$. $\mathbb{CP}^2$ is orientable, its Euler characteristic is $\chi = 3$, and its signature is $\sigma = 1$. The usual metric normalisation has scalar curvature $24$ (and thus Schouten scalar $24$) and volume $\pi^2/2$. The Pfaffian in dimension 4 is

$$\text{Pf} = \frac{1}{32\pi^2}(|C|^2 - 8|S|^2 + 6J^2),$$

where $S$ is the trace-free Schouten tensor. Since $g$ is locally symmetric, $\nabla C$ vanishes and $|C|^2$ is constant. Since $g$ is Einstein, $S = 0$; this and the above data give $|C|^2 = 96$. The Hirzebruch polynomial (signature integrand) in dimension 4 is

$$L = \frac{1}{48\pi^2}(|C_+|^2 - |C_-|^2),$$

so $|C_+|^2 = 96$ and $|C_-|^2 = 0$, reflecting the fact that $\mathbb{CP}^2$ is half conformally flat, and indicating which half is flat. In particular,

$$\text{Pf}^\text{LC}(\mathbb{CP}^2, g) = E/\pi^2,$$

where $E$ is the volume form of $g$, and $\int_{\mathbb{CP}^2} \text{Pf}^\text{LC} = \frac{1}{2}$.

Now consider the 10-dimensional manifold $\mathbb{CP}^2 \times N$. Since the Riemann curvature tensor is additive over products it follows easily that the Pontrjagin forms of a direct product depend only on the Weyl tensors of the factors. Now

$$Q_4 \kappa = (d\delta - 4P_+ + 2J)\kappa = 2(J - 2P_+)\kappa,$$

since $\kappa$ is (parallel and hence) harmonic. The scalar curvature of $M$ is $24 + \nu$, so the Schouten scalar is $J = (24 + \nu)/18$. The pullback to $\mathbb{CP}^2$ of the Schouten tensor $P$ of $M$ is $(84 - \nu)g/144$, so $P_+\kappa = (84 - \nu)\kappa/36$. Thus $\kappa$ solves the eigenvalue problem

$$Q_4 \kappa = 2(\nu - 30)\kappa/9$$

(see Section 7.1). Since by construction $\nabla$ annihilates $Q_4 = \kappa Q_4 \kappa$, the function corresponding to this density under the metric $g$ is constant, and the above shows that this constant is nonzero as long as $\nu \neq 30$. In particular, $\int Q_\kappa$ is nonzero if $\nu \neq 30$.

7. INVARIANT NONLINEAR FUNCTIONALS AND PRESCRIPTION PROBLEMS

For the moment, suppose our conformal manifold is compact, but not necessarily of Riemannian signature. For each quantity $Q$ in the linear $Q$-space, we define a two point functional $\mathcal{K}(\widehat{g}, g) = \mathcal{K}_Q(\widehat{g}, g)$ on the conformal class $[g]$ by

$$\mathcal{K}(\widehat{g}, g) = \frac{1}{2} \int_M \omega(\widehat{g}, g)(Q^\widehat{g} + Q^\widehat{g}),$$

where $\omega(g, \widehat{g})$ is the unique function on $M$ satisfying $\widehat{g} = e^{2\omega(g, \widehat{g})} g$. As a $C^\infty(M)$-valued function on the conformal class $[g]$, $\omega$ is a cocycle:

$$\omega(g_2, g_1) = -\omega(g_1, g_2), \quad \omega(g_3, g_2) + \omega(g_2, g_1) = \omega(g_3, g_1)$$

for any metrics $g_1, g_2, g_3 \in [g]$. The real valued function $\mathcal{K}$ is also a cocycle in this sense: it is alternating because $\omega$ is, and with $\omega_{ij} := \omega(g_i, g_j)$ and $Q_{ij} := Q^{ij}$, we have

$$\omega_{21}(Q_1 + Q_2) + \omega_{32}(Q_2 + Q_3) = 2\omega_{31}Q_1 + \omega_{21}L\omega_{21} + \omega_{32}L\omega_{32} + 2\omega_{32}L\omega_{21},$$

using the conformal invariance of $L$ and the conformal transformation law for $Q$. This differs from

$$\omega_{31}(Q_1 + Q_3) = 2\omega_{31}Q_1 + (\omega_{21} + \omega_{32})L(\omega_{21} + \omega_{32})$$
by
\[ \omega_{22} L \omega_{21} - \omega_{21} L \omega_{22}, \]
which is a divergence since \( L \) is formally self-adjoint. This proves that
\[ \mathcal{K}(g_3, g_2) + \mathcal{K}(g_2, g_1) = \mathcal{K}(g_3, g_1). \]

Fixing a metric \( g_0 \), we wish to look for metrics that are critical for the functional \( \mathcal{K}(g, g_0) \) with respect to conformal variations of \( g \). Because uniform scaling of \( g \) results in the addition of constant multiples of the conformal invariant \( c := \int Q \) to the functional, we can either restrict ourselves to equal-volume perturbations (thus freezing out uniform scaling), or add a volume penalty to the functional, as in:
\[
\mathcal{M}(g, g_0) = -\frac{c}{n} \log \frac{\text{vol}(g)}{\text{vol}(g_0)} + \mathcal{K}(g, g_0) =: \mathcal{V}(g, g_0) + \mathcal{K}(g, g_0).
\]

Note that the functional \( \mathcal{V} \) depends on our \( Q \)-quantity through the constant \( c \). The functional \( \mathcal{M}(g, g_0) \) is invariant under uniform scaling: addition of a constant \( b \) to \( \omega(g, g_0) \) increases \( \mathcal{K} \) by \( bc \), and decreases \( \mathcal{V} \) by the same amount. In addition, the penalised functional \( \mathcal{M} \) is a cocycle on \([g]\). Thus the problem of finding critical metrics or of extremising \( \mathcal{M}(g, g_0) \) is independent of the choice of base metric \( g_0 \), since choosing base metric \( g_1 \) instead just adds the constant \( \mathcal{M}(g_0, g_1) \) to the functional.

**Remark:** In what follows, we shall integrate both densities and functions. The integral of a density, for example \( \int Q \), makes sense given only a conformal class. To integrate a function \( f \), we need to choose a metric \( g \) within the conformal class and use its (pseudo-)Riemannian measure; we shall denote this process by \( \int f \, dv_g \). Though \( Q \) is a density, the notion of a metric \( g \) of constant \( Q \) is well-defined. The choice \( g \) of scale induces an identification of all the density bundles \( \mathcal{E}[w] \); in particular there is a canonical function (0-density) corresponding to \( Q \) via the metric \( g \), to which we also give the name \( Q \).

Now restrict to the Riemannian signature setting, keeping the underlying manifold \( M \) compact. Taking an equal-volume conformal curve of metrics \( e^{2\beta_t} g \), with \( \beta_0 = 0 \), and differentiating to find the variation, we have
\[
\frac{d}{dt} \mathcal{M}(e^{2\beta_t} g, g_0) \bigg|_{t=0} = \frac{d}{dt} \mathcal{K}(e^{2\beta_t} g, g_0) \bigg|_{t=0} = \frac{d}{dt} \left( \mathcal{K}(g, g_0) + \mathcal{K}(e^{2\beta_t} g, g) \right) \bigg|_{t=0} = \int \beta Q^g dv_g,
\]
where \( \beta := (d/dt)\beta_t \big|_{t=0} \). As \( \{\beta_t\} \) runs through all equal-volume conformal perturbations, \( \beta \) runs through all smooth functions with \( \int \beta dv_g = 0 \); that is, all \( \beta \) which are \( g \)-orthogonal to the constants. In fact, the value of the constant at a critical metric \( g \) is determined by the conformal invariance of \( c = \int Q \) to be \( Q^g = c/\text{vol}(g) \). Note that because \( \mathcal{M} \) is invariant under uniform scaling, it was sufficient (in determining the critical points) to take equal-volume perturbations.

**Remark:** The function \( Q \) satisfies the exponential prescription equation
\[
P \omega + Q = \tilde{Q} e^n \omega,
\]
where \( P \) is the operator on functions obtained from the scale-induces density/function correspondence described in the last remark. The behavior of this prescription equation as a PDE problem varies with the order of \( L \). At the high-order end is the original
Q-curvature, whose $L$ has the form $\Delta^{n/2} + \text{LOT}$, and in particular is positively elliptic. (A special case is Gauss curvature prescription in dimension 2.) At the other extreme are $Q$-quantities with $L = 0$; i.e. local conformal invariants. For such a quantity, the prescription problem is just algebraic: Given $g$, we can find a conformally related $\tilde{g}$ with constant $\bar{Q}$ if and only if the sign (+, −, or 0) of $Q$ is constant; and then such a $\tilde{g}$ is unique up to uniform dilations.

Suppose that we have an $\mathcal{M}$-critical metric $g_0$, i.e. a metric with constant $Q_0$, and consider the corresponding one-metric functional

$$\mathcal{H}^0(g) := \mathcal{M}(g, g_0) = -\frac{c}{n} \log \frac{\text{vol}(g)}{\text{vol}(g_0)} + \mathcal{K}(g, g_0),$$

where $c$ is the conformal invariant $\int Q$.

**Theorem 7.1.** If $Q_0$ is constant, the functional $\mathcal{H}^0(g)$ is invariant under conformal diffeomorphisms $h$ in the identity component of the conformal group of $(M, [g_0])$, in the sense that $\mathcal{H}^0(h \cdot g) = \mathcal{H}^0(g)$, where on covariant tensors $h \cdot (h^{-1})^*$.

**Proof:** We need to know that $\mathcal{M}(h \cdot g, g_0) = \mathcal{M}(g, g_0)$. We know that

$$\mathcal{M}(h \cdot g, h \cdot g_0) = \mathcal{M}(g, g_0),$$

since this is true of the $\mathcal{V}$ functional (diffeomorphism does not change the volume), while for the $\mathcal{K}$ functional,

$$2\mathcal{K}(g, g_0) = \int \omega(Q_0 + Q) = \int h \cdot \{\omega(Q_0 + Q)\} = \int (h \cdot \omega)(Q^{h \cdot g_0} + Q^{h \cdot g}),$$

by naturality of $Q$. Since $h \cdot \omega(g, g_0) = \omega(h \cdot g, g_0)$, this is

$$2\mathcal{K}(g, g_0) = \int \omega(h \cdot g, h \cdot g_0)(Q^{h \cdot g_0} + Q^{h \cdot g}) = 2\mathcal{K}(h \cdot g, h \cdot g_0).$$

Using the cocycle condition and (14), we have

$$\mathcal{H}(h \cdot g) = \mathcal{M}(h \cdot g, h \cdot g_0) + \mathcal{M}(h \cdot g_0, g_0) = \mathcal{M}(g, g_0) + \mathcal{M}(h \cdot g_0, g_0) = \mathcal{H}(g) + \mathcal{M}(h \cdot g_0, g_0).$$

Thus what we need to show is that the very last term $\mathcal{M}(h \cdot g_0, g_0)$ vanishes. Since it vanishes with $\mathcal{V}$ in place of $\mathcal{M}$, what we need to show is the vanishing of

$$2\mathcal{K}(h \cdot g_0, g_0) = \int \omega(h \cdot g_0, g_0)(Q_0 + h \cdot Q_0)$$

$$= Q_0 \int \omega(h \cdot g_0, g_0)(dv_{g_0} + dv_{h \cdot g_0}),$$

the last simplification depending on the fact that as a function, $Q_0$ is constant (and so equals its pushout $h \cdot Q_0$ under $h$).

The corresponding infinitesimal statement comes from taking $h = h_t$ in a one-parameter group of conformal transformations, and computing $(d/dt)|_{t=0}$ of (15). This yields $2Q_0 \int \omega dv_{g_0}$, where $\omega$ is the infinitesimal conformal factor of the conformal vector field $T$ generating the $h_t$:

$$L_T g_0 = 2\omega g_0,$$
where \( L_T \) is the Lie derivative. However \( n \int \omega \, dv_{g_0} \) is the variation of the volume in this conformal diffeomorphism direction, and so it vanishes. This establishes invariance of \( \mathcal{H} \) under transformations from the identity component of the conformal group \( G \) of \((M, [g])\) (in which the one-parameter groups generate a dense set). \( \square \)

**Remark**: For a conformal transformation \( h \) in another connected component \( G_1 \) of the conformal group of \((M, [g_0])\), we get the same invariance statement provided \( G_1 \) contains an isometry of \( g_0 \). Indeed, \( \mathcal{H} \) is clearly invariant under isometries: the integrand of (15) vanishes identically for the isometry invariance problem.

Given a critical metric \( g_0 \), an interesting way to rewrite the functional \( \mathcal{H}(g) = \mathcal{M}(g, \kappa, \lambda) \) is as follows. Using the constancy of \( Q_0 \) and the conformal transformation law \( e^{n \omega} Q = Q_0 + L_0 \omega \) (where \( \omega := \omega(g, g_0) \)), we have

\[
\mathcal{H}(g) = -\frac{c}{n} \log \frac{\int e^{n \omega} dv_{g_0}}{\text{vol}(g_0)} + \left( Q_0 \int \omega \, dv_{g_0} + \frac{1}{2} \int \omega(L_0 \omega) dv_{g_0} \right).
\]

Since \( Q_0 = c / \text{vol}(g_0) \), the first term on the right side can be combined with the left side to give

\[
(16) \quad \mathcal{H}(g) = -\frac{Q_0 \text{vol}(g_0)}{n} \log \frac{\int e^{n(\omega - \bar{\omega})} dv_{g_0}}{\text{vol}(g_0)} + \frac{1}{2} \int \omega(L_0 \omega) dv_{g_0},
\]

where \( \bar{\omega} \) is the \( g_0 \)-average value of \( \omega \).

In the case of the original Q-curvature, for which \( L \) is the critical GJMS operator, and of the standard conformal class on the sphere \( S^n \), the quantity on the right hand side of (16) is the one asserted to be nonnegative by the celebrated Beckner-Moser-Trudinger inequality [4]. In this case \( Q = (n - 1)! \) (at the round metrics, which constitute one orbit within the conformal class under the conformal diffeomorphism group). What we have done above is to generalise this form and its conformal transformation invariance to arbitrary Q-quantities:

**Corollary 7.2.** If \( Q_0 \) is constant, the functional (16) is invariant under conformal diffeomorphisms \( h \) in the identity component of the conformal group of \((M, [g_0])\).

### 7.1. Generalised prescription problems

The prescription problem for the (classical) Q-curvature is generalised by the prescription/eigenvalue problem of finding a triple \((\tilde{g}, \kappa, \lambda) \in [g] \times \mathcal{H}^k_G \times \mathbb{R} \) with the property that \( Q_k \kappa = \lambda \kappa \). From the viewpoint of an arbitrary scale \( g \in [g] \), this is the differential equation \( L(\omega \kappa) + Q_k \kappa = e^{(n - 2k) \omega} \lambda \kappa \), where \( \omega = \omega(\tilde{g}, g) \). (We could also state a version of this problem where we demand only that \( \kappa \in \mathcal{C}^k \). So one may ask, for example, whether the problem may be solved with \( \kappa \) a Pontrjagin form.) Note that for any solution \((\tilde{g}, \kappa, \lambda) \) with \( \lambda \neq 0 \), the conformal harmonic \( \kappa \) is actually a harmonic at the scale \( \tilde{g} \), since \( 0 = \delta Q_k \kappa = \lambda \delta \kappa \). Note also that the example worked out in Section 6 is an example of this eigenvalue phenomenon. If we nominate and fix a conformal harmonic (or closed form) \( \kappa \), then the overall problem generalises constant Q-curvature prescription: here the general harmonic is a linear combination of indicator functions for the connected components of our compact manifold. In any case where we have a solution \((\tilde{g}, \kappa, \lambda) \), the conformally invariant quantity \( \langle \kappa, \tilde{Q} \kappa \rangle \) is the eigenvalue \( \lambda \) times \( \int \langle \kappa, \kappa \rangle \), where the latter is computed in the scale \( \tilde{g} \). In the usual (i.e. \( k = 0 \)) problem \( \int \langle \kappa, \kappa \rangle \) is a nonnegative linear combination of the component \( \tilde{g} \)-volumes, in which the coefficients are the squares of those on the indicator functions in the original expression for \( \kappa \). Put another way, we see that the information in \( \langle \kappa, \tilde{Q} \kappa \rangle \) for general form order generalises that provided in the 0-form case by the conformal invariants \( \int_M Q \)
for $M$, the connected components of $M$. The eigenvalue problem described above is a corresponding generalisation of the problem of prescribing constant $\hat{Q}$ on some chosen union of connected components.

Knowledge of the scale $\hat{g}$ in which the eigenvalue $\lambda$ is attained yields further information on $\lambda$ and $\kappa$ which depends explicitly on $\hat{g}$. To describe this information, note that in the equation $\langle \kappa, \hat{Q}\kappa \rangle = \lambda \langle \kappa, \kappa \rangle$, the left side uses the conformally invariant quadratic form $\tilde{\Theta}_k$ (of Sec. 4) on $H^k_G$, while the right side uses the (scale-dependent) $\hat{g}$-metric form. Let $B_k := \dim H^k_G$. Composing $\hat{\Theta}_k$ with the inverse of the $\hat{g}$-form gives an endomorphism $T^{\hat{g}}$ of $H^k_G$ with a full complement $\lambda_1 \leq \cdots \leq \lambda_{B_k}$ of real eigenvalues (by the symmetry of the conformally invariant quadratic form). Denoting by $\kappa_i$ the corresponding eigenvectors, the $(\lambda, \kappa)$ solving our problem must be one of the pairs $(\lambda_i, \kappa_i)$ from this list.

References

[1] S. Alexakis, On the decomposition of global conformal invariants, I, II in progress.
[2] A. Avez, Characteristic classes and Weyl Tensor: Applications to general relativity, Proc. Nat. Acad. Sci. 66 (1970) 265–268.
[3] T.N. Bailey, M.G. Eastwood, and A.R. Gover, Thomas’s structure bundle for conformal, projective and related structures. Rocky Mountain J. Math. 24 (1994) 1191–1217.
[4] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Annals of Math. 138 (1993) 213–242.
[5] N. Boulanger and J. Erdmenger, A classification of local Weyl invariants in $D = 8$, preprint. hep-th/0405228
[6] T. Branson, “The Functional Determinant”, Global Analysis Research Center Lecture Note Series, Number 4, Seoul National University (1993).
[7] T. Branson, S-Y.A. Chang, and P. Yang, Estimates and extremals for zeta function determinants on four-manifolds. Commun. Math. Phys. 149 (1992) 241–262.
[8] T. Branson, P. Gilkey, and J. Pohjanpelto, Invariants of conformally flat manifolds. Trans. Amer. Math. Soc. 347 (1995) 939–954.
[9] T. Branson and A.R. Gover, Electromagnetism, metric deformations, ellipticity and gauge operators on conformal 4-manifolds. Diff. Geom. and its Applications 17 (2002), 229–249.
[10] T. Branson, and A.R. Gover, Conformally invariant operators, differential forms, cohomology and a generalisation of $Q$-curvature. Commun. Partial Diff. Eq., to appear. math.DG/0309085
[11] T. Branson and B. Ørsted, Detour torsion, preprint. In preparation.
[12] T. Branson and B. Ørsted, Conformal indices of Riemannian manifolds. Compositio Math. 60 (1986) 261–293.
[13] T. Branson and B. Ørsted, Explicit functional determinants in four dimensions. Proc. Amer. Math. Soc. 113 (1991) 671–684.
[14] A. Čap and A.R. Gover, Tractor bundles for irreducible parabolic geometries. SMF Séminaires et congrès 4 (2000) 129–154, electronically available at http://smf.emath.fr/SansMenu/Publications/SeminairesCongres/
[15] A. Čap and A.R. Gover, Standard tractors and the conformal ambient metric construction, Annals of Global Analysis and Geometry, 24, 3 (2003) 231–259.
[16] S.-Y.A. Chang, M. Gursky, and P. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature. Annals of Math. 155 (2002) 709–787.
[17] S.-Y.A. Chang, J. Qing, and P. Yang, Compactification of a class of conformally flat 4-manifold. Invent. Math. 142 (2000) 65–93.
[18] S.-Y.A. Chang and P. Yang, Extremal metrics of zeta function determinants on 4-manifolds. Annals of Math. 142 (1995) 171–212.
[19] S.-Y.A. Chang and P. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry. Math. Res. Lett. 4 (1997) 91–102.
[20] S-S. Chern and J. Simons, Characteristic forms and geometric invariants, Ann. Math. 99 (1974) 48–69.
Pontrjagin forms, Q-space

[21] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, Phys. Lett. B 309 (1993) 279.
[22] S.K. Donaldson and P.B. Kronheimer, “The geometry of four-manifolds.” Oxford University Press, New York, 1990.
[23] M.G. Eastwood and A.R. Gover, Formal Adjoints and a Canonical Form for Linear Operators. Twistor Newsletter, 41 35–36.
[24] M.G. Eastwood and M. Singer, The Fröhlicher spectral sequence on a twistor space. J. Diff. Geom. 38 (1993) 653–669.
[25] M.G. Eastwood and J. Slovák, Semikolonomische Verma modules. J. Algebra 197 (1997) 424–448.
[26] C. Fefferman and C.R. Graham, Conformal invariants, in Élie Cartan et les Mathématiques d’Aujourd’hui, (Astérisque, hors série), (1985), 95–116.
[27] C. Fefferman and C.R. Graham, Q-curvature and Poincare metrics. Math. Res. Lett. 9 (2002) 139–151.
[28] C. Fefferman and K. Hirachi, Ambient metric construction of Q-curvature in conformal and CR geometries, Math. Res. Lett. 10 (2003) 819–832.
[29] A.R. Gover, Conformal de Rham Hodge theory and operators generalising the Q-curvature, Rend. Circ. Matem. Palermo, Ser. II, Suppl. 75 (2005) 109–137.
[30] A.R. Gover and L.J. Peterson, Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus. Commun. Math. Phys. 235 (2003) 339–378.
[31] C.R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, Rend. Circ. Mat, Palermo, Ser. II, Suppl. 63 (2000) 31–42.
[32] C.R. Graham and K. Hirachi, The ambient obstruction tensor and Q-curvature. In “AdS-CFT Correspondence: Einstein Metrics and their Conformal Boundaries,” IRMA Lectures in Mathematics and Theoretical Physics, vol. 8, European Mathematical Society (2005) 59–71.
[33] C.R. Graham and M. Zworski, Scattering matrix in conformal geometry. Invent. Math. 152 (2003) 89–118.
[34] M. Gursky, The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics. Annals of Math. 148 (1998) 315–337.
[35] M. Gursky and J. Viaclovsky, A fully nonlinear equation on four-manifolds with positive scalar curvature. J. Diff. Geom. 63 (2003) 131–154.
[36] M. Ishida and C. LeBrun, Curvature, connected sums, and Seiberg-Witten theory. Comm. Anal. Geom. 11 (2003) 809–836.
[37] S. Kobayashi and K. Nomizu, Foundations of differential geometry I,II, Interscience, New York, 1963 and 1969.
[38] C. LeBrun, Ricci curvature, minimal volumes, and Seiberg-Witten theory. Invent. Math. 145 (2001) 279–316.

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