Parametric amplification and up-conversion of optical image in coupled nonlinear optical processes

E V Makeev and A S Chirkin
M.V. Lomonosov Moscow State University, Moscow 119992, Russia
E-mail: makeev@newmail.ru, aschirkin@pisem.net

Abstract. We develop quantum theory of process of the parametric optical image amplification at high-frequency pumping which is accompanied by up-conversion one. Such interactions can be implemented by two coupled three-frequency optical processes. The theory developed takes into account the diffraction effect and the difference of group velocities of interacting waves. The expressions for mean photon numbers and noise figures of the interacting waves are obtained and the dependence of these parameters on the interaction length are examined.

1. Introduction
Couple of two nonlinear optical processes in a single crystal with second order nonlinearity can be implemented in both homogeneous crystals and inhomogeneous ones (see [1, 2] cited references there). In the homogeneous nonlinear optical crystal the phase matching conditions, necessary for efficient energy exchange between interacting waves, may be simultaneously realized for two three-frequency processes by means of noncollinear wave interactions. While in the inhomogeneous (periodically or aperiodically poled) nonlinear crystal the collinear quasi-phase matching conditions may be executed by adjusting the modulation period of nonlinear susceptibility.

Opportunity of implementation of coupled nonlinear optical processes in the single crystal increases the number of wavelengths generated simultaneously [1, 2]. Besides, such interactions possess attractive properties from viewpoint of quantum information and quantum imaging. Three partite entangled states can be observed as a result of coupled nonlinear optical interactions [3, 4, 5]. It should be also noted that in the periodically poled nonlinear crystals the polarization entangled states can be formed at the collinear geometry of wave interactions [6].

The coupled wave interactions in the periodically poled nonlinear crystals allow us to realize the process of parametric amplification at low frequency pumping [7]. In this process at the interaction of waves with multiple frequencies (phase-sensitive interaction) the radiation with the frequency higher than the pumping one can possess sub-Poissonian photon statistics [8]. The quantum theory of the parametric optical image amplification at low frequency pumping is developed in [9]. It is shown that in the case of the interaction of waves with nonmultiple frequencies, besides of the input amplified image with the frequency more than the pumping frequency, two new images, which have frequencies lower than the pumping frequency, appear.
The parametric image amplification in the phase-sensitive coupled wave interactions has been studied in [10]. Conditions have been found for which the signal-to-noise ratio for the amplified image and up-converted one may be the same as for the input image. This is the so-called noiseless amplification and up-conversion of the optical image. To the best of our knowledge, the parametric amplification and the up-conversion of optical image was observed for the first time in [11].

The aim of our paper is to study the amplification and the up-conversion of image in the coupled nonlinear optical processes when the interacting waves have nonmultiple frequencies and the input image has frequency lower than the pumping frequency. Results of the present paper supplements the results obtained in [9].

The paper is organized as follows. In Section 2 we outline scheme realizing the parametric amplification and up-conversion of the optical image. In Section 3 equations which describe coupled nonlinear optical interactions and their solution are given. Section 4 deals with analysis of quantum properties of the output images. Sections 5 concludes the paper.

2. Optical scheme

Scheme of simultaneous parametric amplification and up-conversion of optical image is shown in figure 1. This scheme is similar to the scheme used in [12] where, however, a conventional parametric image amplification at high frequency pumping has occured in the nonlinear crystal. The weak input optical image is located in the object plane $P_1$. This image is projected by the lens $L_1$ into the input $P_2$ of a nonlinear crystal such as, for example, periodically poled LiNbO$_3$ crystal, in which the collinear coupled nonlinear optical interactions can proceed. The amplified image and the images, generated at new frequencies, are projected by the lens $L_2$ from the output of the crystal (plane $P_3$) onto the image plane $P_4$. The lenses $L_1$ and $L_2$ have the same focal lengths $f$. The object plane and the image one, as well as the input and the output of the nonlinear crystal, are both at the distance $f$.

We will denote the field operators in the object plane and the image one as $A^\text{in}_j(\vec{\rho}, t)$ and $A^\text{out}_j(\vec{r}, t)$, respectively, and in the input and the output planes of the nonlinear crystal as $A^\text{in}_j(\vec{r}_1, t)$ and $A^\text{out}_j(\vec{r}_2, t)$, respectively. The index $j$ is connected with the frequency $\omega_j$. The operators $A^\text{in}_j$, $A^\text{out}_j$, $A^\text{in}_j$ and $A^\text{out}_j$, $A_j$ are related to the Fourier transformations (see figure 1 and [12, 13])

$$A^\text{in}_0(\vec{r}_1, t) = \frac{1}{\lambda f} \int_{-\infty}^{\infty} A^\text{in}(\vec{\rho}, t)e^{-i\frac{2\pi}{\lambda f}\vec{\rho}\cdot\vec{r}_1}d^2\rho,$$

$$A^\text{out}(\vec{r}, t) = \frac{1}{\lambda f} \int_{-\infty}^{\infty} A(\vec{r}_2, t)P(\vec{r}_2)e^{-i\frac{2\pi}{\lambda f}\vec{r}_2\cdot\vec{r}_2}d^2r_2,$$

which are performed by lens $L_1$ and lens $L_2$, respectively. $P(\vec{r}_2)$ is the pupil frame function that accounts for the finite area $S_\text{a}$ of the pupil, $d^2r = dx dy$.

The operators $A^\text{in}_0(\vec{r}, t)$, $A^\text{out}_j(\vec{r}, t)$, $A^\text{in}_j(\vec{r}, t)$ and $A^\text{out}_j(\vec{r}, t)$ should obey the commutation relations

$$[A(\vec{r}, t; z), A^\dagger(\vec{r}', t'; z)] = \delta(\vec{r} - \vec{r}')\delta(t - t'), \quad [A(\vec{r}, t; z), A(\vec{r}', t'; z)] = 0,$$

In order to satisfy (3) the operator $A^\text{out}_j(\vec{r}, t)/(2)$ should contain also term associated with the vacuum fluctuations outside of the pupil’s aperture. However, this term gives no contribution to any normal-ordered operational expressions which are related to measurable values, so that this term can be omitted [13].
where integration goes at the image plane over pixel area $S_p$ near with point with coordinate $\vec{r}'$ and over temporal observation interval $T_0$ near the time moment $t$.

The mean value $\langle \hat{n}(\vec{r}', t; z) \rangle = \langle A_{\text{out}}^{\text{out}}(\vec{r}', t; z) A_{\text{out}}^{\text{out}}(\vec{r}', t; z) \rangle$ determines the mean photon-flux density in photons in cross-section $z$ per cm$^2$ at point $z$ and per second at time $t$.

In order to find the connection between operators at the output $A_j(\vec{r}_1, t)$ and the input $A_{j0}(\vec{r}_1, t)$ of the nonlinear crystal it is necessary to consider nonlinear processes in the nonlinear crystal.

3. Coupled parabolic equations and their solution

We will consider wave interactions which consist of the following coupled processes

$$\omega_p \rightarrow \omega_1 + \omega_2, \quad \omega_p + \omega_1 \rightarrow \omega_3.$$  \hspace{1cm} (5)

Here $\omega_p$ is the frequency of intense pump wave, and $\omega_1, \omega_2$ and $\omega_3$ are the frequencies of generated waves with $\omega_p$ and $\omega_1$ being the shared frequencies of two processes. The first down-conversion process in (5) is a conventional parametric amplification at high-frequency pumping, and the second one is the up-conversion process. As it was mentioned above, both these processes (5) are possible simultaneously to implement in periodical poled nonlinear crystals in geometry of the collinear interaction by choosing modulation period of the second order nonlinear susceptibility (see, for instance, [14]).

Taking into account the diffraction phenomenon the processes (5) in the first order of the dispersion theory are described by the system of coupled equations [9]

$$\begin{cases} \frac{\partial A_1}{\partial z} + \frac{1}{u_1} \frac{\partial A_1}{\partial t} - \frac{i}{2k_1} \Delta_\perp A_1 = i\gamma_1 A_1^\dagger + i\gamma_2^* A_3, \\
\frac{\partial A_2}{\partial z} + \frac{1}{u_2} \frac{\partial A_2}{\partial t} - \frac{i}{2k_2} \Delta_\perp A_2 = i\gamma_1 A_1^\dagger, \\
\frac{\partial A_3}{\partial z} + \frac{1}{u_3} \frac{\partial A_3}{\partial t} - \frac{i}{2k_3} \Delta_\perp A_3 = i\gamma_2 A_1, \end{cases}$$  \hspace{1cm} (6)

where $u_j$ is the group velocity on the frequency $\omega_j$, $\Delta_\perp = \nabla^2 (x, y)$ is the transversal laplacian, $A_j^\dagger = A_j(r, t; z) \quad (A_j = A_j(r, t; z))$ is the creation (annihilation) operator of a photon with frequency $\omega_j \ (j = 1, 2, 3)$.

Real nonlinear coupling coefficients $\gamma_1$ and $\gamma_2$ are proportional to the second order nonlinearity and absolute value of the pump wave amplitude [14]. Eqs. (6) are described in the undepleted pump plane wave approximation of lossless nonlinear crystal and for the case of the pump phase $\varphi_p = 0$.

Notice, that nonlinear wave interactions in the periodical inhomogeneous nonlinear crystals can be described by equations for wave interactions in homogeneous nonlinear crystals if the number of the modulation periods $N \gg 1$ and the intensity of pump wave is less than the critical intensity of parametric trapping [15]. These conditions is assumed to be fulfilled.

In order to solve the system of Eqs. (6) we use the Fourier transformations

$$A_j(\vec{r}, t; z) = \int_{-\infty}^{\infty} a_j(\vec{q}, \Omega; z) e^{-i(\Omega t - \vec{q} \vec{r})} d^2q d\Omega,$$

$$a_j(\vec{q}, \Omega; z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} A_j(\vec{r}, t; z) e^{i(\Omega t - \vec{q} \vec{r})} d^2r dt.$$  \hspace{1cm} (7)
Then Eqs. (6) take the form
\[
\begin{align*}
\frac{da_1}{dz} &= i\mu_1 a_1 + i\gamma_1 a_2^\dagger + i\gamma_2 a_3, \\
\frac{da_2}{dz} &= -i\mu_2 a_2^\dagger - i\gamma_1 a_1, \\
\frac{da_3}{dz} &= i\mu_3 a_3 + i\gamma_2 a_1,
\end{align*}
\]
where \( a_j = a_j(\vec{q}, \Omega; z) \) and
\[
\mu_j = \frac{\Omega}{u_j} - \frac{k_j}{2} \phi_j^2, \quad \phi_j = \frac{q}{k_j}, \quad j = 1, 2, 3.
\]

It is convenient to present the solution of Eqs. (6) in the matrix form
\[
\begin{pmatrix} a_1 \\ a_2^\dagger \\ a_3 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\
Q_{21} & Q_{22} & Q_{23} \\
Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix}
\]
where \( Q_{jm} = Q_{jm}(q, \Omega) \) is the transfer function and \( a_{j0} \) is the value of the operators \( a_j \) at the input of nonlinear crystal. Functions \( Q_{jm} \) are studied in [16] in details.

Eqs. (6), (8) allow us to consider the various situations of the interaction of wave with nonmultiple frequencies in the dependence on boundary conditions. The parametric optical image amplification of optical image at low frequency pumping was studied in [9]. In this case the image had carrying frequency \( \omega_3 \) (\( \omega_3 > \omega_p \), see Eqs. (5)). Here we examine the situation when the optical image incoming on the nonlinear crystal has carrying frequency \( \omega_1 \) so that the input image is parametrically amplified and simultaneously converted upon frequency \( \omega_3 \). The input image is assumed to be coherent and monochromatic, that is \( A_1^\alpha(\rho, \Omega)|\alpha_1(\rho, \Omega)⟩ = a_0(\rho, \Omega)|\alpha_1(\rho, \Omega)⟩ \), where the operator eigenvalue \( a_0(\rho, \Omega) = a_0(\rho)\delta(\Omega) \).

4. Quantum properties of the output images
According to Eqs. (10) at the output of nonlinear crystal the operator, for example, \( a_1(\vec{q}, \Omega) \) reads
\[
a_1(\vec{q}, \Omega) = Q_{11}(q, \Omega)a_{10}(\vec{q}, \Omega) + Q_{12}(q, \Omega)v_2^1(\vec{q}, -\Omega) + Q_{33}(q, \Omega)v_3(\vec{q}, \Omega).
\]
where operators \( v_2 = a_{20} \) and \( v_3 = a_{30} \) refer to the vacuum states.

In order to calculate statistical moments of the measured photon number (4) we use the following assumptions: spatial frequency band of the parametric trapping is wider than the band of the input image. The latter is wider than the width of the function \( P(\vec{q}) \). This circumstance makes it possible in some cases to replace \( P(\vec{q}) \) by \( \delta \)-function. We consider also that observation time \( T_0 \) is significantly larger than \( 1/\Delta\Omega_{tr} \), where \( \Delta\Omega_{tr} \) is the frequency trapping band of the parametric amplification.

The analysis has shown (see also [9, 12]) that the expressions for the mean values and the variance of the photon numbers in the image plane \( P^4 \) (see figure 1) have the similar structures. Of course, the expressions for the mean photon numbers have more simple form. They read
\[
\langle N_1(\vec{r}) \rangle = |Q_{11}(\kappa_1)|^2 \langle N_0(-\vec{r}) \rangle + |Q_{12}(\kappa_1)|^2,
\]
\[
\langle N_2(\vec{r}) \rangle = |Q_{21}(\kappa_2)|^2 \langle N_0(-\vec{r}) \rangle + |Q_{22}(\kappa_2)|^2 + |Q_{23}(\kappa_2)|^2,
\]
\[
\langle N_3(\vec{r}) \rangle = |Q_{31}(\kappa_3)|^2 \langle N_0(-\vec{r}) \rangle + |Q_{32}(\kappa_3)|^2.
\]
where $\langle N_0(\vec{r}) \rangle = |\alpha_0(\vec{r})|^2$ is the mean photon number for the input image, $Q_{jm}(\kappa_j) = Q_{jm}(k_j, \Omega = 0)$, $\kappa_j = k_j r / f$. The first terms are associated with the presence of the input image and the other terms are due to the parametric amplification of the vacuum fluctuations. The latter produces background in the absence of the input image. The background values of the mean photon number $\langle N_{j,b}(\vec{r}) \rangle$ and variance $\sigma^2_{j,b}(\vec{r})$ ($j = 1, 2, 3$) can be measured before incoming the optical image on nonlinear crystal.

Thus the useful mean number of photon on the frequency of interest for us and the variance connected with the signal photons can be determined by the equations

$$\langle N_{j,u}(\vec{r}) \rangle = \langle N_j(\vec{r}) \rangle - \langle N_{j,b}(\vec{r}) \rangle,$$

$$\langle \sigma^2_{j,u}(\vec{r}) \rangle = \langle \sigma^2_j(\vec{r}) \rangle - \langle \sigma^2_{j,b}(\vec{r}) \rangle.$$  

Figure 2 (a) shows behavior of the mean photon numbers on the interacting frequencies. You can see that image amplification process at both signal and new generated frequencies takes place.

In this case the signal-to-noise ratio for output image is equal to

$$\left( \frac{S}{N} \right)^{\text{out}}_j = \frac{\langle N_{j,u}(\vec{r}) \rangle^2}{\sigma^2_{j,u}(\vec{r})},$$

One defines the noise figure as

$$NF_j = \left( \frac{S}{N} \right)^{\text{in}}_1 / \left( \frac{S}{N} \right)^{\text{out}}_j.$$  

For the coherent input image under consideration $(S/N)^{\text{in}}_1 = \langle N_0(\vec{r}) \rangle$.

Results of calculations $NF_j$ can be presented in the compact form

$$NF_j = |Q_{j1}(\kappa_j)|^2 \left\{ |Q_{j1}(\kappa_j)|^2 + \frac{1}{2} \left( 1 - (-1)^{j-1} \right) |Q_{j2}(\kappa_j)|^2 + |Q_{j3}(\kappa_j)|^2 \right\}.$$  

Dependencies $NF_j(\vec{r})$ (19) are shown in figure 2 (b) as a function of the interaction length. One can see that in the case of small interaction lengths the images on the generated frequencies $\omega_2$ and $\omega_3$ possess essentially larger level of fluctuations than the input image. With growing interaction length the fluctuations levels on frequencies $\omega_1$ and $\omega_3$ tend to identical value depending on ratio of the nonlinear coupling coefficients. Asymptotic value of the noise figure equals $\gamma_1^2 / (\gamma_1^2 - \gamma_2^2)$. Whereas the asymptotic fluctuation level on $\omega_2$ frequency is found twice more than one for the signal frequency $\omega_1$. Analysis has shown that there is always an interaction length at which the fluctuation level on the $\omega_3$ frequency is equal to one of the input image on the $\omega_1$ frequency.

5. Conclusion

Quantum theory of the parametric amplification and simultaneously up-conversion of the optical image is elaborated. Such a type of the nonlinear optical interaction can be realized by means of two coupled three-frequency processes. The theory developed takes into account diffraction phenomenon at the interacting waves and difference of their group velocities. Optical scheme includes a nonlinear optical crystal and two identical lenses. The mean photon numbers and the noise figures are calculated for the interacting waves. It is established that with increasing interaction length (increasing amplification) the noise figures tend to the asymptotic values depending on relation of the nonlinear coefficients $\gamma_1$ and $\gamma_2$. It is interesting to note that the
asymptotic noise figures for amplified image and the generated ones differ from unit a little when efficiency of parametric amplification prevails of up-conversion ($\gamma_1 \gg \gamma_2$).

From viewpoint of the amplification and conversion of an optical image the trapping band of spatial frequencies of nonlinear process is important parameter. It is determined by spatial dependence of the transfer functions $Q_{jm}$ and has been studied in [9, 16].

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Figure 1. Scheme of optical image amplification and up-conversion.
Figure 2. Mean photon number $\langle N_j \rangle$ (a), normalized on the input value, and noise figure $NF_j$ (b) as function of interaction length $z$ for frequencies $\omega_1$ (curves 1), $\omega_2$ (curves 2) and $\omega_3$ (curves 3) in the case $\gamma_1 = 9.0 \text{ cm}^{-1}$ and $\gamma_2 = 5.0 \text{ cm}^{-1}$. 

