ABS EQUATIONS ARISING FROM DISCRETE PAINLEVÉ SYSTEMS: 
ω-LATTICE FOR THE (A₂ + A₁)⁽¹⁾ CASE

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ABSTRACT. We demonstrate how to construct a lattice where quad-equations of ABS type appear from τ functions of Painlevé systems. In particular, we consider the q-Painlevé equations which have the affine Weyl group symmetry of type (A₂ + A₁)⁽¹⁾. Moreover, we discuss the hypergeometric solutions of the quad-equations on the resulting ω-lattice.

1. Introduction

Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which include discrete analogs of the six Painlevé equations: P₁, P₁I, ..., and P₆VI. The geometric classification of discrete Painlevé equations, based on types of rational surfaces connected to affine Weyl groups [29], is well known. Together with the Painlevé equations, they are now regarded as one of the most important classes of equations in the theory of integrable systems (see, e.g., [5]). Another important class is given by partial difference equations, which are discrete versions of integrable PDEs, such as Korteweg-de Vries equation (KdV equation). They are usually constructed as relations on vertices of a quadrilateral on a lattice and are often called lattice or quad-equations. In [1], integrable lattice equations were constructed and classified by a property known as consistency around the cube, where on every face of the cube copies of the same quad-equation hold (we call this a symmetric cube). The resulting list of equations are now collectively referred to as the ABS equations. Boll has extended these results, allowing different types of quad-equations to be fitted consistently on a cube (asymmetric cube) [2]. Subsequent to these development, the relation between discrete Painlevé equations and ABS equations became an object of intense interest in the theory of integrable systems, motivated by the fact that the six Painlevé equations can be obtained from integrable PDEs by similarity reduction. Here, we consider a discrete analog of similarity reduction called a periodic reduction.

Many types of periodic reductions from ABS equations to discrete Painlevé equations have been investigated [3, 6–8, 10, 19, 26, 27]. It is well known that some discrete Painlevé equations can be derived from ABS equations by periodic reductions with suitable choice of dependent variables. However, there are still problems for the periodic reductions approach: (i) after applying a periodic reduction to an ABS equation, we do not know that which of discrete Painlevé equations appear; (ii) discrete Painlevé equations obtained by periodic reductions often have insufficient number of parameters. To solve the above-mentioned problems, it is necessary to study the periodic reductions not only from the viewpoint of ABS equations but also from that of Painlevé systems.

The focus of this paper lies on the following system of q-difference equations [17, 29]:

\[ \frac{\bar{g}}{g} = \frac{j^2}{g f (t + f)}, \quad \frac{j}{x} = \frac{j^2}{a f (at + \bar{g})} \]  

(1.1)

where

\[ f = f(t), \quad g = g(t), \quad t \rightarrow qt, \quad a, t, q \in \mathbb{C}^*. \]  

(1.2)

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Equation (1.1) is known as a $q$-Painlevé III equation (denoted by $q$-$P_{III}$) since the continuum limit yields the third Painlevé equation $P_{III}$. It is known that $q$-$P_{III}$ (1.1) is iterated on an $A^{(1)}_2$-surface and has the (extended) affine Weyl group symmetry of type $(A_2 + A_1)^{(1)}$. In concrete terms, by constructing a lattice where quad-equations appear from the $q$-Painlevé equation, we show how to obtain the reverse, by investigating underlying birational representations of a transformation group isomorphic to the affine $W$-group symmetry of type $(A_2 + A_1)^{(1)}$. Furthermore, we construct the hypergeometric $	au$-functions for the discrete Painlevé equation. The theory of birational representations of affine Weyl groups provides us with an algebraic tool to study the Painlevé systems ([15, 16]) through the search for the explicit formulae of the hypergeometric and algebraic solutions. It is now known that theory of the discrete Painlevé equations is regarded as carrying the underlying fundamental mathematical structures. Concerning the discrete Painlevé equations, investigation of $	au$ functions started [15, 16] through the search for the explicit formulae of the hypergeometric and algebraic solutions. It is now known that theory of birational representations of affine Weyl groups provides us with an algebraic tool to study the Painlevé systems [21–25]. Moreover, a geometric framework of the two-dimensional Painlevé systems has been presented based on certain rational surfaces [11, 29]. Combining these results enables us to study the Painlevé systems effectively.

The purpose of this paper is to look into ABS equations from the viewpoint of Painlevé theory, in particular, as an example we treat the case of $A^{(1)}_2$-surface $q$-Painlevé systems, which has the affine Weyl group symmetry of type $(A_2 + A_1)^{(1)}$. In concrete terms, by constructing a lattice where quad-equations appear from the $q$-Painlevé systems, we shall show that the quad-equations are of ABS type.

This paper is organized as follows: in Section 2, we introduce the quad-equations of $A^{(1)}_2$-surface $q$-Painlevé systems, which have the affine Weyl group symmetry of type $(A_2 + A_1)^{(1)}$. Moreover, we show that $q$-Painlevé equations can be derived from a birational transformation group isomorphic to the affine Weyl group of type $(A_2 + A_1)^{(1)}$. In Section 3, we construct a lattice where quad-equations appear, and then derive various quad-equations of ABS type as the relations on the lattice. Furthermore, we construct the hypergeometric solutions of the quad-equations on the lattice.

1.1. The periodic reductions of $H^{3}_{3,0}$. In this section, we demonstrate that $q$-$P_{III}$ (1.1) can be obtained from and $H^{3}_{3,0}$ (1.3) via a periodic reductions. By letting

$$U_{l,m} = \lambda^l \Omega_{l,m},$$  \hspace{1cm} (1.7)

and applying the $(1, -2)$-reduction

$$\Omega_{l+1, m-2} = \Omega_{l,m},$$  \hspace{1cm} (1.8)
which implies the condition on the parameters

\[
\frac{\alpha}{\bar{\alpha}} = \frac{\beta}{\bar{\beta}}, \quad (1.9)
\]

\(H3_{l,m} (1.3)\) can be reduced to

\[
\frac{\ddot{\Omega}}{\Omega} = \frac{\ddot{\Omega} - \lambda \frac{\alpha}{\bar{\beta}} \ddot{\Omega}}{\lambda \left( \frac{\ddot{\Omega}}{\Omega} - \frac{\alpha}{\bar{\beta}} \right)}, \quad (1.10)
\]

where

\[\Omega = \Omega_{l,m}.\] (1.11)

Substituting

\[
f = \frac{\ddot{\Omega}}{\Omega}, \quad g = \frac{\ddot{\Omega}}{\Omega}, \quad t = -\frac{\alpha}{\bar{\beta}}, \quad a = \frac{\bar{\beta}}{\beta}, \quad q = \frac{\alpha}{\bar{\beta}},\]

in Equation (1.10), we obtain \(q\)-P\(_{\text{III}}\) (1.1). On the other hand, another type of periodic reduction from \(H3_{l,m} = 0\) to the \(q\)-P\(_{\text{III}}\) is reported in [10]. Let us consider three copies of \(H3_{l,m}\):

\[
\begin{align*}
\hat{u} & = \frac{\alpha \bar{u} - \beta \bar{u}}{\alpha^{-1}} , \\
\ddot{u} & = \frac{\beta \bar{u} - \gamma \bar{u}}{\beta^{-1}}, \\
\dddot{u} & = \frac{\gamma \bar{u} - \alpha \bar{u}}{\gamma^{-1}}, \\
\end{align*}
\]

(1.13a)

(1.13b)

(1.13c)

where

\[
\begin{align*}
\hat{u} & = u_{l,m,n}, \quad \alpha = \alpha_l, \quad \beta = \beta_m, \quad \gamma = \gamma_n, \\
\end{align*}
\]

\(\vdash: l \rightarrow l + 1, \quad \vdash: m \rightarrow m + 1, \quad \vdash: n \rightarrow n + 1, \quad l, m, n \in \mathbb{Z}.\) (1.14)

Setting

\[
u_{l,m,n} = \Lambda^n u_{l,m,n},\]

and applying the similarity constraint

\[
u_{l+1,m+1,n+1} = \nu_{l,m,n} \]

(1.15)

(1.16)

to System (1.13), which implies the condition on the parameters

\[
\frac{\alpha}{\bar{\alpha}} = \frac{\beta}{\bar{\beta}} = \frac{\gamma}{\bar{\gamma}}, \quad (1.17)
\]

we obtain the following system of quad-equations:

\[
\begin{align*}
\dddot{\omega} & = \frac{\alpha \dddot{\omega} - \beta \dddot{\omega}}{\alpha^{-1}} , \\
\dddot{\omega} & = \frac{\beta \dddot{\omega} - \gamma \dddot{\omega}}{\beta^{-1}}, \\
\dddot{\omega} & = \frac{\gamma \dddot{\omega} - \alpha \dddot{\omega}}{\gamma^{-1}}, \\
\end{align*}
\]

(1.18a)

(1.18b)

(1.18c)

where

\[
\omega = \omega_{l,m,n}.\] (1.19)
Substituting

\[ f = \frac{\bar{\omega}}{\omega}, \quad g = A \frac{\bar{\omega}}{\omega}, \quad t = q^\frac{\alpha}{\gamma}, \quad a = \frac{\gamma}{\beta}, \quad q = \frac{\bar{\alpha}}{\beta} = \frac{\bar{\gamma}}{\gamma} \]  

in Equations (1.18a) and (1.18c), we get \( q \)-P\(_{\text{II}} \) (1.1).

\section{Construction of Lattices from Affine Weyl Groups}

In this section, we describe new three-dimensional structures constructed by using the symmetry groups of discrete Painlevé equations. While the groups themselves are well known, the novel perspective we focus on is the construction of three-dimensional lattices based on \( \tau \) functions and \( q \)-Painlevé systems.

\subsection{The \( \tau \)-lattice}

We describe the action of the family of Bäcklund transformations of \( q \)-P\(_{\text{II}} \) (1.1) on six particular variables associated with this system [30]. Iterating these variables under the Weyl group actions, we obtain a system of \( \tau \) functions, which form a \( \tau \)-lattice.

The transformation group \( \tilde{W}(A_2 + A_1) \) has 7 generators \( s_0, s_1, s_2, \pi, w_0, w_1, r \). Below, we describe actions on parameters: \( a_i, i = 0, 1, 2, \) and on variables: \( \tau_i, \bar{\tau}_i, i = 0, 1, 2 \).

Actions on parameters are given by

- \( s_i : (a_0, a_1, a_2, c) \rightarrow (a_0^{-1}, a_0a_i, a_0a_{i+1}, c) \),
- \( \pi : (a_0, a_1, a_2, c) \rightarrow (a_1, a_2, a_0, c) \),
- \( w_0 : (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, c^{-1}) \),
- \( w_1 : (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, q^{-2}c^{-1}) \),
- \( r : (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, q^{-1}c^{-1}) \),

while its actions on variables are given by

- \( s_i(\tau_i) = \frac{u_i\tau_{i+1}\bar{\tau}_{i+1} + \bar{\tau}_{i+1}\tau_{i+1}}{u_i^{1/2}\bar{\tau}_{i+1}} \), \( s_j(\tau_j) = \tau_j \) (\( i \neq j \)),
- \( s_i(\bar{\tau}_i) = \frac{v_i\bar{\tau}_{i+1}\tau_{i+1} + \tau_{i+1}\bar{\tau}_{i+1}}{v_i^{1/2}\tau_{i+1}} \), \( s_j(\bar{\tau}_j) = \bar{\tau}_j \) (\( i \neq j \)),
- \( \pi(\tau_i) = \tau_{i+1} \), \( \pi(\bar{\tau}_i) = \bar{\tau}_{i+1} \),
- \( w_0(\tau_i) = a_i^{1/3}(\tau_i\tau_{i+1}\tau_{i+2} + u_i^{-1}\tau_{i+1}\tau_{i+2} + u_i^{1/2}\tau_{i+1}\bar{\tau}_{i+2}) \), \( w_0(\bar{\tau}_i) = \bar{\tau}_i \),
- \( w_1(\tau_i) = a_i^{1/3}(\tau_i\bar{\tau}_{i+1}\bar{\tau}_{i+2} + v_i^{-1}\tau_{i+1}\bar{\tau}_{i+2} + v_i\tau_{i+1}\bar{\tau}_{i+2}) \), \( w_1(\bar{\tau}_i) = \bar{\tau}_i \),
- \( r(\tau_i) = \bar{\tau}_i \), \( r(\bar{\tau}_i) = \tau_i \),

where

\[ u_i = q^{-1/3}c^{-2/3}a_i, \quad v_i = q^{1/3}c^{2/3}a_i, \quad q = a_0a_1a_2, \]  

and \( i, j \in \mathbb{Z}/3\mathbb{Z} \). For each element \( w \in \tilde{W}(A_2 + A_1) \) and function \( F = F(a_i, c, \tau_j, \bar{\tau}_j) \), we use the notation \( w.F \) to mean \( w.F = F(w.a_i, w.c, w.\tau_j, w.\bar{\tau}_j) \), that is, \( w \) acts on the arguments from the left.

The following proposition shows that \( \tilde{W}(A_2 + A_1) \) is isomorphic to the affine Weyl group of type \( (A_2 + A_1) \).

\begin{proposition} [30] \end{proposition}

The group of transformations \( \tilde{W}(A_2 + A_1) = \langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle \) forms the (extended) affine Weyl group of type \( (A_2 + A_1) \). Namely, the transformations satisfy the fundamental relations

\[ s_i^2 = (s_is_{i+1})^3 = \pi^3 = 1, \quad \pi s_i = s_{i+1} \pi, \quad (i \in \mathbb{Z}/3\mathbb{Z}), \]  

\[ w_0^2 = w_1^2 = r^2 = 1, \quad rw_0 = w_1 r, \]  

and the action of \( \tilde{W}(A_2) = \langle s_0, s_1, s_2, \pi \rangle \) and that of \( \tilde{W}(A_1) = \langle w_0, w_1, r \rangle \) commute. Note that \( q = a_0a_1a_2 \) and \( c \) are invariant under the action of \( \tilde{W}(A_2 + A_1) \) and \( W(A_2) \), respectively.
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To iterate each variable $\tau_i$, $\bar{\tau}_i$, we need the following translations $T_i$ ($i = 1, 2, 3, 4$), defined by

$$ T_1 = \pi s_2 s_1, \quad T_2 = \pi s_0 s_2, \quad T_3 = \pi s_1 s_0, \quad T_4 = r w_0. \quad (2.4) $$

The actions of these on the parameters are given by

$$ T_1 : (a_0, a_1, a_2, c) \rightarrow (qa_0, q^{-1}a_1, a_2, c), \quad (2.5) $$

$$ T_2 : (a_0, a_1, a_2, c) \rightarrow (a_0, qa_1, q^{-1}a_2, c), \quad (2.6) $$

$$ T_3 : (a_0, a_1, a_2, c) \rightarrow (q^{-1}a_0, a_1, qa_2, c), \quad (2.7) $$

$$ T_4 : (a_0, a_1, a_2, c) \rightarrow (a_0, a_1, a_2, qc). \quad (2.8) $$

Note that $T_i, i = 1, 2, 3, 4$, commute with each other and $T_1 T_2 T_3 = 1$. We define $\tau$ functions by

$$ \tau^{n,m}_N = T_n^{-1} T_2^{m} T_4^{N}(\tau_1), \quad (2.9) $$

where $n, m, N \in \mathbb{Z}$ and the $\tau$-lattice as in Figure 1. We note that

$$ \tau_0 = \tau_0^{-1,0}, \quad \tau_1 = \tau_0^{0,0}, \quad \tau_2 = \tau_0^{0,1}, \quad \tau_0 = \tau_1^{-1,0}, \quad \tau_1 = \tau_1^{0,0}, \quad \tau_2 = \tau_1^{0,1}. \quad (2.10) $$

**Remark 2.2.** By definition, action of $\tilde{W}(A_2 + A_1)^{(1)}$ gives the relations of points on $\tau$-lattice (bilinear equations) and any point of $\tau$-lattice (or $\tau$ function) is determined by six initial points: $\tau_0, \tau_1, i = 0, 1, 2$.

### 2.2. The discrete Painlevé lattice.

In this section, we construct a three-dimensional lattice that relates ratios of $\tau$ functions. The ratios, defined in (2.11), turn out to satisfy a rich set of relations, which give rise not only to $q$-$P_{III}$ (1.1), but also to the $q$-Painlevé IV equation (2.17) and the $q$-Painlevé II equation (2.23) [13, 14]. We derive these $q$-Painlevé equations as relations on the three-dimensional lattice.

The key starting point is the definition of the following ratios

$$ f_0 = q^{1/3} c^{2/3} \frac{\tau_1 \tau_2}{\tau_1 \tau_2}, \quad f_1 = q^{1/3} c^{2/3} \frac{\tau_0 \tau_2}{\tau_0 \tau_2}, \quad f_2 = q^{1/3} c^{2/3} \frac{\tau_0 \tau_1}{\tau_0 \tau_1} \quad (2.11) $$

![Figure 1. Configuration of $\tau$ functions on the $\tau$-lattice. $\tau$ functions are defined on the intersections of four lines.](image-url)
The action of $\overline{W}(\pm(A_2 + A_1))$ on the variables $f_i$ is given by

$$s_i(f_{i\pm1}) = f_{i\pm1} \pm a_if_i/a_i \pm f_i, \quad s_i(f_i) = f_i, \quad s_i(f_{i+1}) = f_{i+1} + a_if_i + f_i \quad \pi(f_i) = f_{i+1},$$

$$w_0(f_i) = \frac{a_if_{i+1}(a_i-1)f_{i+1} + f_{i+1}}{f_i(a_if_{i+1} + f_{i+1})},$$

$$w_1(f_i) = \frac{a_i+1(a_i-1)f_{i+1}}{a_if_{i+1} + a_if_{i+1} + f_i} \quad r(f_i) = f_i^{-1},$$

where $i \in \mathbb{Z}/3\mathbb{Z}$. We note that $f_i$ satisfies the condition

$$f_0f_1f_2 = qe^2. \quad (2.12)$$

Define $f$-functions by

$$f_{0,N}^{n,m} = T_1^nT_2^mT_4^N(f_0), \quad f_{1,N}^{n,m} = T_1^nT_2^mT_4^N(f_1), \quad f_{2,N}^{n,m} = T_1^nT_2^mT_4^N(f_2), \quad (2.13)$$

where $n, m, N \in \mathbb{Z}$. These form the edges of a lattice, which we refer to as the $f$-lattice, shown in Figure 2. This lattice is three-dimensional, with coordinate axes given by $n, m$, and $N$.

The relations in the $T_1$-direction on the lattice:

$$T_1(f_1) = \frac{qe^2 1 + a_0f_0}{f_0f_1} \quad T_1(f_0) = \frac{qe^2 1 + a_0d_2T_1(f_1)}{f_0T_1(f_1)} \quad T_1(f_{i+1}) = \frac{qe^2 1 + a_0d_2T_1(f_i)}{f_0T_1(f_i)} \quad (2.14)$$

lead to a system of first-order ordinary difference equations, which is equivalent to $q$-$\Pi_{\Pi}$ (1.1):

$$f_{1,N}^{n+1,m} = q^{2N+1}e^2 1 + q^a(a_0f_{0,N}^{n,m}) \quad f_{0,N}^{n+1,m} = q^{2N+1}e^2 1 + q^{a-m}a_0f_{1,N}^{n+1,m} \quad (2.15)$$

In a similar manner, in each of the $T_2$- and $T_3$-directions, we also obtain $q$-$\Pi_{\Pi}$ (1.1).

In contrast, the action of $T_4$ on the variables $f_i$ can be expressed as

$$T_4(f_0) = a_0a_1f_1 \quad (2.16a)$$

$$T_4(f_1) = a_1a_2f_2 \quad (2.16b)$$

$$T_4(f_2) = a_2a_0f_1 \quad (2.16c)$$
By considering the restricted edges. This is consistent with the observation that discrete Painlevé equations (which are known as a f-lattice where \( f_{0,N} = f_{1,N} \), \( f_{0,N-1} = f_{1,N-1} \), \( f_{0,N+1} = f_{1,N+1} \), \( f_{0,N} = f_{1,N} \)).

Therefore, any function associated with an edge on this lattice is determined by two initial parameters subspace: \( f_{0,N} = f_{0,N} \) satisfying \( f_{0,N} = f_{1,N} \). We introduce the half-translation \( \tilde{R}_1 = \pi^2 s_1 \) satisfying \( R_1^2 = T_1 \). Let

\[
f^M_N = R^M_1 T^N_{21}(f_0),
\]

where

\[
f^{2M-1}_N = f^{M,0}_{1,N} \quad f^{2M}_N = f^{M,0}_{0,N}.
\]

By considering the restricted f-lattice where \( f^M_N \) are defined (see Figure 3), System (2.15) becomes the following system:

\[
f^{n+1,0}_{1,N} = q^n a^n f^{n,0}_{1,N} + a^n a^n f^{n,0}_{0,N} = f^{n+1,0}_{1,N},
\]

\[
f^{n+1,0}_{0,N} = q^n a^n f^{n,0}_{1,N} + a^n a^n f^{n,0}_{0,N} = f^{n+1,0}_{0,N},
\]

which is equivalent to the following single equation:

\[
f^{M+1}_N = q^{N+1,0}_1 \left( 1 + R_{1}^M(a_0) f^{M}_N \right),
\]

In addition, by assuming \( a_2 = q^{1/2} \), transformation \( R_1 \) becomes the translational motion in the parameter subspace:

\[
R_1 : (a_0, a_1) \rightarrow (q^{1/2} a_0, q^{-1/2} a_1),
\]

then Equation (2.21) can be regarded as the single second-order ordinary difference equation:

\[
f^{M+1}_N = q^{N+1,0}_1 \left( 1 + a^n f^{M,0}_N \right),
\]

which is known as a q-Painlevé II equation [28].

**Remark 2.3.** By definition, action of \( W(A_2 + A_1)^{(1)} \) gives the relations on each edge of the f-lattice. Since the variables \( f_i \) satisfy Equation (2.12), only two are independent. Therefore, any function associated with an edge on this lattice is determined by two initial edges. This is consistent with the observation that discrete Painlevé equations (which are second order ordinary) are embedded in this lattice.
3. QUAD-EQUATIONS OF ABS TYPE FROM THE $\omega$-LATTICE

In the previous section, we showed how to construct a $T$-lattice by starting with 6 initial variables and how to obtain discrete Painlevé equations as relations on the $f$-lattice. In this section, we show how to construct a lattice by starting with 3 variables and applying the action of the affine Weyl group to find their iterates. In the resulting $\omega$-lattice, we discover higher dimensional integrable partial difference equations, commonly known as quad-equations (because they relate vertices of quadrilaterals), that were classified by Adler, Bobenko and Suris [1]. Furthermore, we obtain the hypergeometric solutions of the quad-equations appearing on the $\omega$-lattice.

3.1. The $\omega$-lattice. We extend the parameter space (defined by $a_i$, $i = 0, 1, 2, c$ in the previous section) and introduce new parameters $\kappa_j$, $j = 0, 1, 2$, satisfying the condition

$$k_0k_1k_2 = \lambda,$$  

where

$$\lambda = q^{1/2}c.$$  

The first step is to define the action of the affine Weyl group on the new parameters.

**Definition 3.1.** The action of $\tilde{W}((A_2 + A_1)^{(1)})$ on parameters $k_i$ is defined by

$$s_0 : (k_0, k_1, k_2) \rightarrow (k_0^{-1}, k_1k_0, k_2k_0),$$

$$s_1 : (k_0, k_1, k_2) \rightarrow (k_0k_1, k_1^{-1}, k_2k_1),$$

$$s_2 : (k_0, k_1, k_2) \rightarrow (k_0k_2, k_1k_2, k_2^{-1}),$$

$$\pi : (k_0, k_1, k_2) \rightarrow (k_1, k_2, k_0),$$

$$w_0 : (k_0, k_1, k_2) \rightarrow (a_0k_0^{-1}, a_1k_1^{-1}, a_2k_2^{-1}),$$

$$w_1 : (k_0, k_1, k_2) \rightarrow (a_0^{-1}k_0^{-1}, a_1^{-1}k_1^{-1}, a_2^{-1}k_2^{-1}),$$

$$r : (k_0, k_1, k_2) \rightarrow (k_0^{-1}, k_1^{-1}, k_2^{-1}).$$

Note that $\tilde{W}((A_2 + A_1)^{(1)})$ forms the (extended) affine Weyl group of type $(A_2 + A_1)^{(1)}$ on the level of the parameters $k_i$. From definition (2.4), it follows that the actions of translations $T_i$ $(i = 1, 2, 3, 4)$ on parameters $k_i$ $(i = 0, 1, 2)$ are given by the following:

$$T_1 : (k_0, k_1, k_2) \rightarrow (\lambda k_0, \lambda^{-1}k_1, k_2),$$

$$T_2 : (k_0, k_1, k_2) \rightarrow (k_0, \lambda k_1, \lambda^{-1}k_2),$$

$$T_3 : (k_0, k_1, k_2) \rightarrow (\lambda^{-1}k_0, k_1, \lambda k_2),$$

$$T_4 : (k_0, k_1, k_2) \rightarrow (a_0k_0, a_1k_1, a_2k_2).$$

Now we are in a position to define the 3 initial variables

$$\omega_0 = \frac{k_0^{1/3}}{k_1^{1/3}}, \quad \omega_1 = \frac{k_1^{1/3}}{k_2^{1/3}}, \quad \omega_2 = \frac{k_2^{1/3}}{k_0^{1/3}},$$

(3.7)

whose iterates (constructed below) will provide us with the $\omega$-lattice.

The action of $\tilde{W}((A_2 + A_1)^{(1)})$ on these variables $\omega_i$ is given by the following lemma, which follows from the above definitions.

**Lemma 3.2.** The action of $\tilde{W}((A_2 + A_1)^{(1)})$ on variables $\omega_i$ is given by

$$s_i(\omega_i) = \omega_i \frac{a_i + \lambda \omega_{i+1} + k_i \omega_{i+2}}{\lambda \omega_{i+1} + a_i \omega_{i+2}},$$

$$s_i(\omega_{i+1}) = k_i^{-1} \omega_{i+1}, \quad s_i(\omega_{i+2}) = k_i \omega_{i+2},$$

$$\pi(\omega_i) = \omega_{i+1}, \quad w_0(\omega_i) = \frac{a_{i+1}k_i \omega_{i+1} + a_{i+1}2\omega_i + k_i \lambda \omega_i}{a_{i+1}k_i 2\omega_i + a_{i+1}2k_i \lambda \omega_i + \lambda \omega_i},$$

$$w_1(\omega_i) = \frac{a_{i+1}k_i \omega_{i+1} + a_{i+1}2\omega_i + k_i \lambda \omega_i}{a_{i+1}k_i 2\omega_i + a_{i+1}2k_i \lambda \omega_i + \lambda \omega_i}, \quad r(\omega_i) = \omega_{i-1},$$

where $i \in \mathbb{Z}/3\mathbb{Z}$.
We define \( \omega \)-functions by

\[
\omega_{l_1, l_2, l_3, N} = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^N(\omega_0),
\]

where \( l_1, l_2, l_3, N \in \mathbb{Z} \) and the \( \omega \)-lattice as in Figure 4. We note that

\[
\omega_0 = \omega_{0,0,0,0}, \quad \omega_1 = \kappa_2^{-1} \omega_{1,0,0,0}, \quad \omega_2 = \kappa_1 \omega_{1,1,0,0}.
\]

**Remark 3.3.** Since for all \( w \in \tilde{W}(A_2 + A_1)^{(1)} \),

\[
w(\omega_i) \in \mathcal{L}, \quad (i = 0, 1, 2),
\]

where \( \mathcal{L} = \mathcal{K}(\omega_0, \omega_1, \omega_2) \) is the field of rational functions in \( \omega_i \) \((i = 0, 1, 2)\) with coefficient field \( \mathcal{K} = \mathbb{C}(\kappa, \kappa_1, \lambda) \), every point on the \( \omega \)-lattice is determined by 3 initial points. This implies that quad-equations appear as the relations on the \( \omega \)-lattice. Moreover, the relations on the \( f \)-lattice can be expressed by those on the \( \omega \)-lattice because of the following correspondence:

\[
f_0 = \kappa_1 \kappa_2 \frac{\omega_1}{\omega_2}, \quad \text{or} \quad f_{0, N}^{l_1, l_2, l_3} = \frac{\omega_{l_1+1, l_2, l_3, N}}{\omega_{l_1+1, l_2+1, l_3, N}},
\]

\[
f_1 = \kappa_2 \kappa_0 \frac{\omega_2}{\omega_0}, \quad \text{or} \quad f_{1, N}^{l_1, l_2, l_3} = q^N \frac{\omega_{l_1+1, l_2, l_3, N}}{\omega_{l_1, l_2, l_3, N}},
\]

\[
f_2 = \kappa_0 \kappa_1 \frac{\omega_0}{\omega_1}, \quad \text{or} \quad f_{2, N}^{l_1, l_2, l_3} = q^N \frac{\omega_{l_1, l_2+1, l_3, N}}{\omega_{l_1, l_2, l_3, N}}.
\]

We have constructed the \( \omega \)-lattice associated with \( \tilde{W}(A_2 + A_1)^{(1)} \). Henceforth, let us consider the quad-equations appearing on the \( \omega \)-lattice.

Figure 4. Configuration of \( \omega \)-functions on the \( \omega \)-lattice. Note that each \( \omega \)-function is defined on the intersection of four lines.
Lemma 3.4. The following equations hold on the $\omega$-lattice:

\begin{align}
\omega_{i+1,j+1,N} &= q^{1-i-1} a_0 \omega_{i+1,j+1,N} - \omega_{i,j+1,N+1} \quad (3.12a) \\
\omega_{i+1,j+1,N} &= q^{1-i+1} a_1 \omega_{i,j+1,N} - \omega_{i,j+1,N} \\
\omega_{i+j+1,N} &= q^2 L \left( q^{1-i+1} a_1 \omega_{i+1,j+1,N} - q^N \lambda \omega_{i,j+1,N} \right) \\
\omega_{i,j+1,N+1} &= \frac{q^{2-i+1} a_2 \omega_{i,j+1,N} - q^N \lambda \omega_{i,j+1,N}}{q^{2-i+1} a_2 \omega_{i,j+1,N} - \omega_{i,j+1,N+1}} \quad (3.12b) \\
\omega_{i+1,j+1,N} &= \frac{q^2 L \left( q^{1-i+1} a_1 \omega_{i,j+1,N} - q^N \lambda \omega_{i,j+1,N} \right)}{q^{2-i+1} a_2 \omega_{i,j+1,N} - \omega_{i,j+1,N+1}} \quad (3.12c)
\end{align}

The following correspondences show that this system is a periodic reduction of the $H_{3<0}$ system of equations (1.18):

\begin{align}
\omega &= \omega_0, \\
\alpha &= a_1^{-1}, \\
\beta &= a_2^{-1}, \\
\gamma &= q a_0^{-1}, \\
\rho &= T_1, \\
\lambda &= T_2, \\
\mu &= T_3. 
\end{align} 

Moreover, we also find $H^6$ type equations:

\begin{align}
\omega_{i+1,j+1,N+1} &= \frac{q^{2N+1} a_1^2 - 1}{q^{2N+1} a_1} \omega_{i,j+1,N} + q^N \lambda \omega_{i,j+1,N} \quad (3.14a) \\
\omega_{i,j+1,N+1} &= \frac{q^{2N+1} a_2^2 - 1}{q^{2N+1} a_2} \omega_{i,j+1,N} \quad (3.14b) \\
\omega_{i,j+1,N+1} &= \frac{a_2 \left( q^{2N+1} a_1^2 - 1 \right)}{q^{2N+1} a_2} \omega_{i,j+1,N} \quad (3.14c)
\end{align}

and an additional partial difference equation:

\begin{align}
\omega_{i,j+1,N+1} &= \frac{(q^{2N+1} a_1^2 - 1) \omega_{i,j+1,N} + q^N \lambda \omega_{i,j+1,N}}{q^{2N+1} a_1^2 - 1} \omega_{i,j+1,N} + q^N \lambda \omega_{i,j+1,N} \\
&+ \frac{q^{2N+1} a_2 \left( q^{2N+1} a_1^2 - 1 \right)}{q^{2N+1} a_2} \omega_{i,j+1,N} - \omega_{i,j+1,N+1} \omega_{i,j+1,N} \quad (3.15)
\end{align}

**Proof.** First, we prove System (3.12). From the action of translations $T_1$, $T_2$ and $T_3$, it holds that

\begin{align}
\omega_1 &= a_0 k_2 k_0 T_1(\omega_2) - \omega_0 \\
\omega_2 &= a_0 k_2 k_0 T_2(\omega_2) - \omega_0 \\
\omega_3 &= a_1 k_0 k_1 T_3(\omega_3) \quad (3.16) \\
\omega_4 &= a_2 k_1 k_0 T_4(\omega_4) - \omega_0 \\
\omega_5 &= a_2 k_1 k_0 T_5(\omega_5) \quad (3.17) \\
\omega_6 &= a_2 k_1 k_0 T_6(\omega_6) \quad (3.18)
\end{align}

Applying $T_1^i T_2^h T_3^{l+1} T_4^N, T_1^i T_2^h T_3^l T_4^N$ and $T_1^i T_2^{l+1} T_3^{l+1} T_4^N$ on Equations (3.16)–(3.18), we obtain Equations (3.12a)–(3.12c), respectively.

We now consider the derivation of System (3.14). From the action of the translation $T_4$, it easily verified that

\begin{align}
\frac{T_4(\omega_1)}{\omega_1} &= \frac{1}{k_2} - \frac{1}{k_2} \omega_1 = q \lambda^2 - 1 \\
\frac{T_4(\omega_2)}{\omega_2} &= \frac{a_2 k_1 k_2}{a_2 k_1 k_2} \omega_1 = q \lambda^2 - 1 \\
\frac{T_4(\omega_3)}{\omega_3} &= \frac{1}{k_1} - \frac{1}{k_1} \omega_3 = \frac{a_2 \left( q \lambda^2 - 1 \right)}{q \lambda^2} \\
\frac{T_4(\omega_4)}{\omega_4} &= \frac{a_1 k_0}{a_1 k_0} \omega_1 = q \lambda \\
\frac{T_4(\omega_5)}{\omega_5} &= \frac{1}{k_1} - \frac{1}{k_1} \omega_5 = \frac{a_1 \left( q \lambda^2 - 1 \right)}{q \lambda} \quad (3.19)
\end{align}

Applying $T_1^i T_2^h T_3^l T_4^N, T_1^{l+1} T_2^h T_3^l T_4^N$ and $T_1^{l+1} T_2^{l+1} T_3^{l+1} T_4^N$ on Equations (3.19)–(3.21), we obtain Equations (3.14a)–(3.14c), respectively.
Finally, we prove Equation (3.15). By eliminating $\omega_2$ from Equation (3.17) and

$$T_4(\omega_0) = \frac{\omega_0\omega_1 + a_1\kappa_0(\lambda_2\omega_0 + \kappa_2\omega_1)\omega_2}{a_1\kappa_0\kappa_1\omega_0}$$

(3.22)

the following relation can be derived:

$$T_4(\omega_0) = \frac{(a_1^2 - 1)\omega_1T_2(\omega_0) + a_1a_2(a_1\kappa_0\kappa_1T_2(\omega_0) - \omega_1)\omega_0}{a_1(a_1\omega_1 - \kappa_0\kappa_1T_2(\omega_0))}$$

(3.23)

Applying $T_4^bT_2^bT_3^bT_4^N$ on Equation (3.23), we obtain Equation (3.15). This completes the proof.

In [10], we showed the following proposition by using Lemma 3.4:

**Proposition 3.5 ([10]).** By using the quad-equations given in Lemma 3.4, it is shown that $\omega$-lattice can be obtained from an asymmetric 4D cube which has twelve $H_3$-type equations.

Proposition 3.5 shows that the above quad-equations are the only ones that relate four points on the $\omega$-variables. Therefore, we have shown the following theorem:

**Theorem 3.6.** All quad-equations appearing on the $\omega$-lattice are of ABS type.

3.2. Relations to discrete Schwarzian KdV equation. Other quad-equations also appear in the $\omega$-lattice. In this section, we show how to find the discrete Schwarzian KdV equation (or $Q_{12}(a_0)$ in ABS classification). We also consider a periodic reduction.

In a recent work [9], Hay et al. showed that by setting

$$z_{l_1,l_2,l_3,N} = T_4^{l_1}T_2^{l_2}T_3^{l_3}T_4^N(z),$$

(3.24)

where

$$z = e^{2\lambda_2(\omega_1^+/a_2)(1/3)\log a_2^{2/3}}T_3(\tau_0)$$

(3.25)

one can obtain the discrete Schwarzian KdV equation:

$$\frac{\left((z_{l_1,l_2,l_3,N} - z_{l_1+1,l_2,l_3,N})(z_{l_1,l_2+1,l_3,N} - z_{l_1+1,l_2+1,N})\right)}{\left((z_{l_1,l_2,l_3,N} - z_{l_1+1,l_2,l_3,N})(z_{l_1,l_2+1,l_3,N} - z_{l_1+1,l_2+1,N})\right)} = q^{2l_1-l_0-1}a_0^2,$$

(3.26)

Equation (3.26) can be found from the $\omega$-lattice because of the following relation:

$$z = T_4(\omega_0)\omega_0, \quad \text{or} \quad z_{l_1,l_2,l_3,N} = \omega_{l_1,l_2,l_3,N} + \omega_{l_1,l_2,l_3,N}.$$

(3.27)

Furthermore, we can also obtain the following equations:

$$\frac{(z_{l_1,l_2,l_3,N} - q^{2N+1}a_2^2z_{l_1,l_2+1,l_3,N})(z_{l_1,l_2+1,l_3,N} - q^{2N+1}a_2^2z_{l_1+1,l_2+1,l_3,N})}{(z_{l_1,l_2,l_3,N} - z_{l_1+1,l_2,l_3,N})(z_{l_1+1,l_2+1,l_3,N} - z_{l_1+2,l_2+1,l_3,N})} = q^{-2l_1+2l_2+2N+1}a_1^2a_2^2,$$

(3.28)

$$\frac{(z_{l_1,l_2,l_3,N} - q^{2N+1}a_2^2z_{l_1,l_2+1,l_3,N})(z_{l_1,l_2+1,l_3,N} - q^{2N+1}a_2^2z_{l_1+1,l_2+1,l_3,N})}{(z_{l_1,l_2,l_3,N} - z_{l_1+1,l_2,l_3,N})(z_{l_1+1,l_2+1,l_3,N} - z_{l_1+2,l_2+1,l_3,N})} = q^{2l_1-2l_2-2N-1}a_1^2a_2^2,$$

(3.29)

from the following relations:

$$\frac{(z - q^2T_2(z))(T_4(z) - q^2T_1T_2(z))}{(z - T_1(z))(T_2(z) - T_1T_2(z))} = q^2a_1^2,$$

(3.30)

$$\frac{(z - T_2(z))(T_3(z) - T_2T_3(z))}{(z - q^2T_2(z))(T_3(z) - q^2T_2T_3(z))} = q^{-1}a_1^2a_2^2.$$

(3.31)
Remark 3.7. The system of equations (3.26), (3.28) and (3.29) arises as a periodic reduction of three copies of $Q_{1,0,n}$:

\[
\begin{align*}
(u - \pi)(\bar{u} - \pi) &= \alpha, \\
(u - \bar{u})(\bar{u} - \pi) &= \beta, \\
(u - \pi)(\bar{u} - \pi) &= \gamma,
\end{align*}
\]

(3.32)

where we have used the notation (1.14), and the reduction is defined by

\[
u_{l,m,n} = q^n \hat{\lambda}^m z_{l,m,n},
\]

(3.35)

with the similarity constraint

\[
z_{l+1,m+1,n+1} = z_{l,m,n}.
\]

(3.36)

**Proof.** Equations (3.32)–(3.36) imply the following conditions on the parameters

\[
\frac{\alpha}{\hat{\beta}} = \frac{\beta}{\gamma} = \frac{\gamma}{q^2}.
\]

(3.37)

Therefore, we obtain the following quad-equations:

\[
\begin{align*}
(z - q^2 \bar{z}) (z - q^2 \bar{z}) &= q^4 \hat{\lambda}^2 \beta, \\
(z - \bar{z}) (z - \bar{z}) &= \frac{\alpha}{\gamma}, \\
(z - \bar{z}) (z - \bar{z}) &= \frac{\alpha}{\gamma}.
\end{align*}
\]

(3.38)

(3.39)

(3.40)

where $z = z_{l,m,n}$. This proves that the system of equations (3.26), (3.28) and (3.29) is equivalent to the system (3.38)–(3.40) with the following correspondence:

\[
\frac{\alpha}{\beta} = a_1^{-2}, \quad \frac{\beta}{\gamma} = a_2^{-2}, \quad \frac{\gamma}{\alpha} = q^2 a_0^{-2}, \quad \hat{\gamma} = T_1, \quad \hat{\gamma} = T_2, \quad \hat{\gamma} = T_3.
\]

(3.41)

\[
\square
\]

### 3.3. The restricted $\omega$-lattice.

In the case of the $f$-lattice, we showed that System (2.15) can be rewritten as the single equation (2.21) on the restricted lattice (projective reduction). The concept of projective reduction applies not only for the $f$-lattice but also for the $\omega$-lattice. Set

\[
\omega_{1,N} = R_1^l T_4^N (\omega_0),
\]

(3.42)

where

\[
\omega_{21-1,N} = \frac{q^N \lambda}{a_2^N k_2} \omega_{1,0,N} \quad \omega_{21,N} = \omega_{1,0,N}.
\]

(3.43)

Considering the restricted $\omega$-lattice where $\omega_{1,N}$ are defined (see Figure 5), Equations (3.12a) and (3.12b) can be rewritten as

\[
\begin{align*}
\omega_{1,0,0,N} &= \omega_{1,0,0,N} + q^N \lambda \omega_{1,0,0,N}, \\
\omega_{1,1,0,N} &= \omega_{1,0,0,N} + q^N \lambda \omega_{1,0,0,N},
\end{align*}
\]

(3.44)

(3.45)
respectively. The system of equations (3.44) and (3.45) is expressed by the single equation

\[
\omega_{l+3,N} = \omega_{l,N} \frac{q^{N}A \left( R_{1}^{1}T_{4}^{N}(\kappa_{2})\omega_{l+1,N} + q^{N}A R_{1}^{1}(a_{0})\omega_{l+2,N} \right)}{R_{1}^{1}T_{4}^{N}(\kappa_{2}) \left( q^{N}A\omega_{l+2,N} + R_{1}^{1}(a_{0})R_{1}^{1}T_{4}^{N}(\kappa_{2})\omega_{l+1,N} \right)}. \tag{3.46}
\]

In a similar manner, Equation (3.14a) becomes

\[
\frac{\omega_{l+2,N+1}}{\omega_{l,N}} - \frac{\omega_{l+1,N+1}}{\omega_{l+2,N}} = \frac{q^{2N+1}A^{2} - 1}{q^{N}A R_{1}^{1}(a_{1})}, \tag{3.47}
\]

and furthermore, Equations (3.14b) and (3.14c) can be expressed by

\[
\frac{\omega_{l+2,N+1}}{\omega_{l+1,N+1}} - \frac{1}{R_{1}^{1-1}T_{4}^{N}(a_{2}k_{2})} \frac{\omega_{l+1,N+1}}{\omega_{l+2,N}} = \frac{q^{2N+1}A^{2} - 1}{q^{N}A R_{1}^{1}T_{4}^{N}(a_{2}k_{2})}, \tag{3.48}
\]

on the restricted \( \omega \)-lattice.

**Remark 3.9.** Equation (3.46) is equivalent to the \((2, -1)\)-reduction of \(H_{3,0}^{2,0} (1.10)\).

**Proof.** Setting

\[
\Omega_{l,N} = R_{1}^{1}T_{4}^{N}(\Lambda_{2}^{3/2} \kappa_{1}) \omega_{l,N}, \tag{3.49}
\]

Equation (3.46) can be rewritten as

\[
\frac{\Omega_{l+3,N}}{\Omega_{l,N}} = \frac{\Omega_{l+1,N} + q^{N}A R_{1}^{1}(a_{0})\Omega_{l+2,N}}{q^{N}A \left( q^{N}A\Omega_{l+2,N} + R_{1}^{1}(a_{0})\Omega_{l+1,N} \right)}. \tag{3.50}
\]

which is equivalent to Equation (1.10) with the following correspondence:

\[
\frac{\alpha}{\beta} = a_{0}, \quad \gamma = T_{1}, \quad \gamma = R_{1}. \tag{3.51}
\]

This completes the proof. \(\square\)

We note that on the restricted \( \omega \)-lattice, Equations (3.26) and (3.28) can be also rewritten as

\[
\frac{(\zeta_{l,N} - \zeta_{l+2,N})(\zeta_{l+1,N} - \zeta_{l+3,N})}{(q^{N+1/2}A^{1/2} \zeta_{l,N} - \zeta_{l+3,N})(q^{N+1/2}A^{1/2} \zeta_{l+2,N} - \zeta_{l+3,N})} = q^{N+1/2}A^{-1} R_{1}^{1}(a_{0}^{2}), \tag{3.52}
\]

where

\[
\zeta_{l,N} = R_{1}^{1}T_{4}^{N}(a_{2}^{-1/2}k_{2}^{-1})R_{1}^{1}T_{4}^{N}(z) = R_{1}^{1}T_{4}^{N}(a_{2}^{-1/2}k_{2}^{-1}) \omega_{l,N} \omega_{l,N+1}. \tag{3.53}
\]

**Remark 3.10.** Equation (3.52) can be obtained by a periodic reduction of \(Q_{1,0}^{2,0}\):

\[
\frac{(U - \overline{U}) \sqrt{(U - \overline{U})}}{(U - \overline{U}) \sqrt{(U - \overline{U})}} = \frac{\alpha}{\beta}, \tag{3.54}
\]

where we have used the notation (1.4), and the reduction is defined by

\[
U_{l,N} = q^{-m/2}A^{-m} \zeta_{l,m}. \tag{3.55}
\]
with the similarity constraint
\[ \zeta_{l+1,m-2} = \zeta_{l,m}. \]  

**Proof.** Equation (3.54)–(3.56) imply the condition on the parameters
\[ \frac{\alpha}{\beta} = \frac{\beta}{\hat{\beta}} = q^2, \]  

Therefore, Equation (3.54) can be reduced to
\[ \frac{(\zeta - \hat{\zeta})(\zeta - \hat{\zeta})}{(q^{1/2} \zeta - \hat{\zeta})(q^{1/2} \hat{\zeta} - \hat{\zeta})} = \frac{\alpha}{q^{1/2} \lambda \beta}, \]  

where
\[ \zeta = \zeta_{l,m}. \]  

Then, the statement follows since Equation (3.52) is equivalent to Equation (3.58) with the following correspondence:
\[ \frac{\alpha}{\beta} = a_0^2, \quad \hat{\alpha} = T_1, \quad \hat{\beta} = R_1. \]  

\[ \square \]

### 3.4. Hypergeometric solutions of quad-equations on the \( \omega \)-lattice.

In the previous section, we derived various quad-equations of ABS type on the \( \omega \)-lattice. In this section, we give their hypergeometric solutions. We use the following conventions of \( q \)-analysis [4].

- **\( q \)-Shifted factorials:**
  \[ (a; q)_k = \prod_{j=0}^{k-1} (1 - aq^{j+1}), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^{j+1}). \]  

- **Basic hypergeometric series:**
  \[ \phi_s \left( a_1, \ldots, a_s; b_1, \ldots, b_r; q, \xi \right) = \sum_{n=0}^{\infty} \left( a_1, \ldots, a_s; b_1, \ldots, b_r; q, q \right)_n (-1)^n q^{n(n-1)/2} \xi^n, \]  

  where
  \[ (a_1, \ldots, a_s; q)_n = \prod_{i=1}^{s} (a_i; q)_n. \]  

- **Jacobi theta function:**
  \[ \Theta \left( a; q \right) = (a; q)_\infty (qa^{-1}; q)_\infty. \]  

- **Elliptic gamma function:**
  \[ \Gamma \left( a; p, q \right) = \frac{(pqa^{-1}; p, q)_\infty}{(a; p, q)_\infty}, \]  

  where
  \[ (a; p, q)_k = \prod_{i,j=0}^{k-1} (1 - p^i q^j a). \]  

Note that the following relations hold:
\[ \Theta \left( qa; q \right) = -a^{-1} \Theta \left( a; q \right), \]  
\[ \Gamma \left( qa; q, q \right) = \Theta \left( a; q \right) \Gamma \left( a; q, q \right). \]
In [18], Nakazono showed that the following functions provide solutions of all bilinear equations in the \( \tau \)-lattice. Define \( \chi^{n,m}_N \) \((n, m, N \in \mathbb{Z}) \) by

\[
\chi^{n,m}_N = (-1)^{(N+1)/2} q^{-(2m-n)N^2/3 + Nn} a_0^{-(-2N^2 + 3N)/3} a_2^{-N^2/3} \\
\times \left( \frac{\Theta(-q^{-m} a_0^{-1} q) \Theta(-q^{-m} a_2^{-1} q)}{\Theta(q^{-m} a_0^{-1} a^2^{-2} q)} \right)^N \Gamma(q^{-m+1/6} a_0^{-1/3} a_2^{-1/3} q^{1/6}, q^{1/6}) \\
\times \Gamma(q^{-m+1/6} a_0^{-1/3} a_2^{-1/3} q^{1/6}, q^{1/6}) \chi^{n-1,m}_{N-1}.
\]

(3.69)

Here \( \psi^{n,m}_N \) and its matrix element \( F_{n,m} \) are given by

\[
\psi^{n,m}_N = \begin{cases} 
F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\
F_{n-1,m} & F_{n,m} & \cdots & F_{n+N-2,m} \\
\vdots & \vdots & \ddots & \vdots \\
F_{n-N+1,m} & F_{n-N+2,m} & \cdots & F_{n,m} \\
1, & & & (N > 0), \\
0, & & & (N = 0), \\
& & & (N < 0), 
\end{cases}
\]

(3.70)

\[
F_{n,m} = \frac{A_{n,m}}{(a_2^{-2} q^{2m+2}; q^4)_\infty} \varphi \left( \begin{array}{c} 0 \\ a_2 q^{2m-2}; q^2 \\ a_2^2 a_0 q^{2n-2m} \end{array} \right) \\
+ \frac{B_{n,m}}{(a_2^{-2} q^{2m+2}; q^4)_\infty} \varphi \left( \begin{array}{c} 0 \\ a_2^{-2} q^{2m+2}; q^2 \\ a_0 q^{2n+2} \end{array} \right),
\]

(3.71)

where \( A_{n,m} \) and \( B_{n,m} \) are periodic functions of period one with respect to \( n \) and \( m \), that is,

\[
A_{n+1,m} = A_{n,m+1} = A_{n,m}, \quad B_{n+1,m} = B_{n,m+1} = B_{n,m}.
\]

(3.72)

The following proposition provides the crucial connection needed to solve the bilinear equations:

**Proposition 3.11 ([18]).** By setting

\[
c^{1/6} = 1,
\]

(3.73)

the functions \( \psi^{n,m}_N = \chi^{n,m}_N \) \((n, m, N \in \mathbb{Z}) \) satisfy all bilinear equations on the \( \tau \)-lattice and

\[
\tau^{n,m}_N = T_1^n T_2^m (\chi^{0,0}_N).
\]

(3.74)

In addition, under the assumption

\[
c^{1/6} = 1, \quad R_1(A_{n,m}) = B_{n,m},
\]

(3.75)

the functions \( \psi^{n,m}_N = \chi^{n,m}_N \) \((n, N \in \mathbb{Z}, m \in \{0, 1\}) \) also satisfy

\[
\tau^{n,0}_N = R_1^n (\chi^{0,0}_N), \quad \tau^{n,1}_N = R_1^{n-1} (\chi^{0,0}_N).
\]

(3.76)

From Proposition 3.11, we obtain the following theorem:

**Theorem 3.12.** The functions

\[
\omega_{1,2,3,6,N} = T_1^{l_1} T_2^{l_2} T_3^{l_3} T_4^{l_4} \left( \frac{k_2^{1/3}}{k_1^{1/3}} \right)^{l_1-l_2-l_3} \phi^{n-1,l_1,l_2}_{N,m} \\
= q^{-(l_1-2l_2+3l_3)/3} (a_1^{-l_1} a_2^{-l_2} a_3^{-l_3} a_4^{l_3})^{1/3} \frac{\chi^{n-1,l_1,l_2}_{N,m}}{\chi^{n,0}_N},
\]

(3.77)
where \( n, m \in \mathbb{Z} \) and \( N \in \mathbb{Z}_{\geq 0} \), give the hypergeometric solutions of all equations holding on the \( \omega \)-lattice, defined in Section 3, under the assumption (3.73). Furthermore, under the assumption (3.75), the functions

\[
\omega_{l,N} = R^{l} T_{N}^{l} \left( \frac{k^{1/3}}{k^{1/3}} \right) R^{l} \left( \frac{X^{-1,0}}{X^{0,1}} \right)
\]

\[
= \left\{ \begin{array}{l}
q^{(N/3)k/3} a_{0}^{N/3} a_{1}^{N/3} k_{1}^{1/3} k_{2}^{1/3} \frac{X^{k+1}}{X^{1+k}} \quad (l = 2k - 1), \\
q^{(N/3)k/3} a_{0}^{N/3} a_{2}^{N/3} k_{1}^{1/3} k_{2}^{1/3} \frac{X^{-1,0}}{X^{0,1}} \quad (l = 2k),
\end{array} \right.
\]

where \( l \in \mathbb{Z} \) and \( N \in \mathbb{Z}_{\geq 0} \), give the hypergeometric solutions of all equations that hold on the restricted \( \omega \)-lattice.

**Remark 3.13.** Theorem 3.12 provides hypergeometric solutions of Equations (3.12), (3.14), (3.15), (3.26), (3.28) and (3.29). Moreover, the results also provide hypergeometric solutions of Equations (3.46)–(3.48) and (3.52).

4. **Concluding remarks**

In this paper, we constructed an \( \omega \)-lattice associated with the affine Weyl group of type \((A_{2} + A_{1})^{(1)}\) and showed that all quad-equations obtained from this lattice are members of the classification derived in [1, 2]. Furthermore, we obtained hypergeometric solutions of these quad-equations by finding the corresponding solutions of the \( \omega \)-lattice.

More general \( \omega \)-lattices are possible. They share certain fundamental properties with the \( \omega \)-lattice constructed in Section 3. In particular, all \( f \)-functions arise as rational combinations of \( \omega \)-functions and all \( \omega \)-functions in each connected component of an \( \omega \)-lattice are determined by 3 initial variables in that component. We will explore the general constructions of \( \omega \)-lattices in subsequent works. An interesting future project is to construct various \( \omega \)-lattices associated with Painlevé systems of other surface types in Sakai’s classification [29].

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