RELATIVISTIC PERFECT FLUIDS NEAR KASNER SINGULARITIES

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Abstract. We establish the existence of a stable family of solutions to the Euler equations on Kasner backgrounds near the singularity with the full expected asymptotic data degrees of freedom and no symmetry or isotropy restrictions. Existence is achieved through transforming the Euler equations into the form of a symmetric hyperbolic Fuchsian system followed by an application of a new existence theory for the singular initial value problem. Stability is shown to follow from the existence theory for the (regular) global initial value problem for Fuchsian systems that was developed in \cite{16}. In fact, for each solution in the family, we prove the existence of an open set of nearby solutions with the same qualitative asymptotics and show that any such perturbed solution agrees again with another solution of the singular initial value problem. All our results hold in the regime where the speed of sound of the fluid is large in comparison to all Kasner exponents. This is interpreted as the regime of stable fluid asymptotics near Kasner big bang singularities.

1. Introduction

Perfect fluids on Kasner backgrounds. The main result of this paper concerns the dynamics of solutions of the compressible relativistic Euler equations in the vicinity of the in general highly anisotropic singularity of Kasner spacetimes. Relativistic compressible perfect fluids with linear equations of state

\[ P = c_s^2 \rho, \]  

where \( P \) is the fluid pressure, \( \rho \) is the fluid density and \( c_s \) the speed of sound, are common matter models in cosmology. In the standard model of cosmology \cite{33}, large portions of the history of the universe are taken to be dominated by perfect fluids with \( c_s = 0 \) (dust) while the early hot universe, which we are mostly concerned with in this paper here, is dominated by radiation fluids given by \( c_s^2 = 1/3 \).

One of the main goals in mathematical cosmology is to characterise the big bang asymptotics of general solutions to the coupled Einstein-matter equations, a formidable task. However, the matter does not matter hypothesis, part of the famous BKL conjecture \cite{8–10,20,32,51}, suggests that the study of the matter equations defined on some relevant class of fixed singular cosmological model spacetimes is a useful and significant first step for this long-term endeavour. This hypothesis states that the big bang asymptotics of the geometry for solutions of the Einstein-matter equations is governed by the Einstein-vacuum equations, and therefore, the evolution of the geometry effectively decouples from the dynamics of the matter. We take this as a justification to restrict our attention to the perfect fluid equations on fixed background spacetimes in this paper as a first step towards the long-term objective of understanding the dynamics of the coupled Einstein-Euler equations near a big-bang singularity. It is believed that the matter does not matter hypothesis is valid only if the matter is not “too extreme”. In the case of \( (1.1) \) it is expected that the condition \( c_s^2 < 1 \) is sufficient while the dynamics of the “extreme” case \( c_s^2 = 1 \), the stiff fluid (or scalar field) case, is expected to be significantly different \cite{2,6,8,44,45}. Interestingly, our results do continue to apply to fluids on Kasner-scalar field background spacetimes as we explain below.

The family of Kasner spacetimes holds a prominent place in the mathematical cosmology literature as one of the most important singular cosmological models. This is particularly true for the so-called Kasner-scalar field spacetimes – solutions of the Einstein-scalar field equations – which are expected to be limit points of the coupled Einstein-matter equations and, most importantly, to which the results of this article can be easily applied, see Remark 3.2. In the presentation here, we, however, focus on the Kasner vacuum spacetimes, often referred to as simply Kasner spacetimes. A Kasner-(vacuum) spacetime is a spatially homogeneous solution \((M,g)\) of Einstein’s vacuum equations for \( M = (0, \infty) \times \Sigma \) with \( \Sigma = T^3 \).

\[ ^{1,1} \text{In this paper, we choose physical units such that } c = 1 \text{ for the speed of light and } G = 1/(8\pi) \text{ for Newton’s gravity constant.} \]
and
\[ g = -d\tilde{t} \otimes d\tilde{t} + \tilde{t}^{2p_1} d\tilde{x} \otimes d\tilde{x} + \tilde{t}^{2p_2} d\tilde{y} \otimes d\tilde{y} + \tilde{t}^{2p_3} d\tilde{z} \otimes d\tilde{z}, \]
for Kasner exponents \( p_1, p_2 \) and \( p_3 \) in \( \mathbb{R} \) satisfying the Kasner relations
\[ \sum_{i=1}^{3} p_i = 1, \quad \sum_{i=1}^{3} p_i^2 = 1. \]

Except for the flat Kasner cases \( (p_1, p_2, p_3) = (1, 0, 0) \), \( (p_1, p_2, p_3) = (0, 1, 0) \) or \( (p_1, p_2, p_3) = (0, 0, 1) \), all Kasner metrics \( g \) have curvature singularities in the limit \( \tilde{t} \searrow 0 \).

In this paper, we consider the compressible relativistic Euler equations defined on Kasner spacetimes in the framework of [24, 52] in which the perfect fluid is represented by a single timelike vector field \( V^\alpha \) and the Euler equations take the form
\[ A^\delta_{\alpha\beta} \nabla_\delta V^\beta = 0, \quad A^\delta_{\alpha\beta} = -\frac{3c_s^2 + 1}{c_s^2} V^\delta V^\alpha \delta^\alpha + V^\delta g_{\alpha\beta} + 2g^\delta_{(\beta V^\alpha)}, \]
provided \( c_s^2 > 0 \). Despite its relevance for cosmology, the case \( c_s^2 = 0 \) is excluded from our results, note the inequality (1.6) from Theorem 1.1 below. It is worth noting that this is not due to our use of the formulation (1.4) of the Euler equations, which is undefined at \( c_s = 0 \).

In writing (1.4), we have made use of the Einstein summation convention where indices are lowered and raised with the given Kasner spacetime metric \( g \). The derivative \( \nabla \) in (1.4) is the covariant derivative associated with \( g \). The key property which characterises this form of the Euler equations is that (1.4) is explicitly symmetric hyperbolic, that is, the coefficients \( A^\delta_{\alpha\beta} \) in (1.4) are symmetric in \( \alpha \) and \( \beta \) and \( A^0_{0,\beta} \) is positive-definite (if the fluid is timelike), which turns out to be crucial for the analysis. According to [24, 52], the fluid pressure \( P \), density \( \rho \) and normalised fluid 4-vector field \( u^\alpha \) can be calculated from \( V^\alpha \) via the expressions\(^1^2\)
\[ P = \rho_0 c_s^2 (-V^\lambda V^\alpha)^{-\frac{c_s^2 + 1}{2c_s}}, \quad \rho = \rho_0 (-V^\lambda V^\alpha)^{-\frac{c_s^2 + 1}{2c_s}}, \quad u^\alpha = V^\alpha / \sqrt{-V^\lambda V^\lambda}, \]
for an arbitrary constant \( \rho_0 \) carrying the correct physical units of an energy density.

We are now in the position to state an informal version of the main result of this paper. The precise version, which is a combination of two results, is given in Theorems 3.1 and 3.6, and the proof of these theorems can be found in Section 3. We emphasise in this article we make no symmetry assumptions for the fluid, nor impose isotropy restrictions on the background spacetime. Furthermore, our results continue to hold for the more general family of Kasner-scalar field background spacetimes; see Remark 3.2 below and [2, 44, 45]. These are spacetimes of the same form as (1.2) that solve the coupled Einstein-scalar field equations as opposed to the Einstein-vacuum equations. For these spacetimes, the Kasner relations (1.3) take a more general form, see (3.15), that involves the scalar field strength parameter \( A \), and special cases include the isotropic Friedmann-Robertson-Walker spacetimes for \( A = \pm \sqrt{\frac{2}{3}} \) as well as the vacuum spacetimes for \( A = 0 \). It was a major breakthrough of [23, 44, 45] to establish that Kasner-scalar field spacetimes (in the sub-critical regime) are stable under generic nonlinear perturbations of the Einstein-scalar field equations. The fact that our results here extend to this family of Kasner-scalar field background spacetimes will therefore be crucial in any extension of our line of research here to include coupling to Einstein gravity. However, for the purposes of this paper, there is no loss of generality, as we argue in Remark 3.2, to restricting our attention to the special case of vacuum Kasner background spacetimes.

**Theorem 1.1 (Informal version).** Suppose \( p_1, p_2, p_3 \) are Kasner exponents satisfying
\[ 1 > c_s^2 > \max\{p_1, p_2, p_3\} \]
and \( W_0^0 > 0, W_1^1, W_2^2 \) and \( W_3^3 \) are given functions on \( \mathbb{T}^3 \) at \( \tilde{t} = 0 \) that together define the asymptotic data for the Euler equations. Then near the singularity at \( \tilde{t} = 0 \) of the Kasner vacuum solution with\(^1^2\) the signature of Lorentzian metrics is taken to be \( (-, +, +, +) \) in this paper. The fluid vector field \( V^\alpha \) is therefore timelike if and only if \( V^\alpha V_\alpha < 0 \).
exponents $p_1$, $p_2$ and $p_3$ given by (1.2) – (1.3), there exists a solution $V^\alpha$ of the Euler equations (1.4) with equation of state (1.1) that can be represented as
\[
V^\alpha = \tilde{\xi}^2 \left( W_0 e_0^\alpha + \tilde{\xi}^{2-p_1} W_1 e_1^\alpha + \tilde{\xi}^{2-p_2} W_2 e_2^\alpha + \tilde{\xi}^{2-p_3} W_3 e_3^\alpha \right)
\]  
(1.7)
for some uniformly bounded functions $W_0$, $W_1$, $W_2$, and $W_3$ on $(0, T_0] \times \mathbb{T}^3$ that agree in the limit $\tilde{\xi} \to 0$ with the asymptotic data $W_0^0 > 0$, $W_1^1$, $W_2^2$, and $W_3^3$, respectively, and $V^\alpha$ is everywhere timelike including in the limit $\tilde{\xi} \to 0$. Here $(e_0^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha)$ is the orthonormal frame of the Kasner spacetime given by
\[
e_0 = \partial_t, \quad e_1 = \tilde{\xi}^{-p_1} \partial_x, \quad e_2 = \tilde{\xi}^{-p_2} \partial_y, \quad e_3 = \tilde{\xi}^{-p_3} \partial_z.
\]  
(1.8)
Moreover this solution is unique within the class of solutions with the asymptotics (1.7), and it is non-linearly stable against sufficiently small perturbations and the behaviour of the resulting perturbed solution again has the asymptotics (1.7).

The main physical restriction in Theorem 1.1 is (1.6), which we refer to as the stability condition. This condition determines an additional restriction on the fluid sound speed $c_s$ and the Kasner exponents $p_1$, $p_2$ and $p_3$ beyond the non-negativity of $c_s$ and the Kasner relation (1.3). According to this condition, the square of the speed of sound must be smaller than 1; that is, the fluid is not stiff\(^{1,3}\), and greater than all Kasner exponents. Before we explain the meaning of the lower bound on the sound speed, let us assume for the moment that (1.6) holds. For this case, Theorem 1.1 guarantees the existence of perfect fluid solutions described by (1.7) near the big bang singularity at $\tilde{\xi} = 0$ represented in terms of the orthonormal frame (1.8) with respect to a (by assumption fixed) background Kasner spacetime. Physically, this orthonormal frame can be interpreted as a cosmological reference frame. The functions $W_0$, $W_1$, $W_2$ and $W_3$ in (1.7) distinguish the different fluid solutions and are all convergent at $\tilde{\xi} = 0$. The main fluid dynamics near $\tilde{\xi} = 0$ is therefore described by the powers of $\tilde{\xi}$ in (1.7), most notably the powers $\tilde{\xi}^{2-p_i}$ in front of the spatial fluid components. The fact that these powers are all positive and therefore bounded at $\tilde{\xi} = 0$ by virtue of (1.6) is a first indication of the importance of the stability condition. As a consequence of (1.5) and (1.7), we note that
\[
P \sim \tilde{\xi}^{-(1+c_s^2)}, \quad \rho \sim \tilde{\xi}^{-(1+c_s^2)},
\]  
(1.9)
and
\[
a^{\alpha} = \frac{e_0^\alpha + Q^i e_i^\alpha}{\sqrt{1 - (Q^i)^2}}, \quad Q^i = \tilde{\xi}^{2-p_i} W_i W_0^{-1}, \quad i = 1, 2, 3.
\]  
(1.10)
where
\[
Q^i := \tilde{\xi}^{2-p_i} W_i W_0^{-1}, \quad i = 1, 2, 3.
\]  
(1.11)
It follows from (1.10) that the quantities $Q^i$ can be interpreted as a measure for the peculiar motions of the fluids relative to the cosmological reference frame. Provided the stability condition holds, we deduce from Theorem 1.1 that $Q^i \to 0$ as $\tilde{\xi} \to 0$, and it follows that all the fluids are asymptotically co-moving; i.e., the peculiar motions die out at $\tilde{\xi} = 0$. This is consistent with one of the key ideas of the standard model of cosmology, namely that peculiar motions of matter should be small.

It is clear that the stability condition imposes a significant restriction on the fluid sound speed. For instance, it follows from (1.3) that
\[
\max\{p_1, p_2, p_3\} \geq 2/3,
\]
and hence the lower bound for $c_s^2$ in (1.6), notably, rules out radiation fluids $c_s^2 = 1/3$ – one of the most popular matter models for the early universe\(^{1,4}\). It is therefore natural to question what happens to the fluids when (1.6) is violated. In general this is an open question. But the following special case provides strong evidence that the fluid behaviour is significantly different when (1.6) is violated (without loss of generality we assume that $p_3 \geq p_2 \geq p_1$ in the following). For this special case, we assume that $W_1 = W_2 = 0$ and that $W_0$ and $W_3$ are functions of $\tilde{\xi}$ only; in particular, this means that the fluid is

\(^{1,3}\)The case of (irrotational) stiff fluids is handled in [23, 44, 45].

\(^{1,4}\)According to Remark 3.2, it is interesting to notice that in the special case of the FLRW Kasner–scalar field background, the radiation fluid case $c_s^2 = 1/3$ is exactly at the border of the stability region.
spatially homogeneous and only flows in the direction of the largest Kasner exponent. It is then possible to integrate the Euler equations to obtain solutions in the following implicit form\textsuperscript{1,5}

\[
\frac{|Q^3(\tilde{t})|}{|1 - (Q^3(\tilde{t}))^2|^{(1-c^2_s)/2}} = C\tilde{r}^{2-p_3}, \quad \tilde{t} > 0,
\]

(1.12)

for a constant $C \geq 0$ where $Q^3$ is as defined above by (1.11). Observe here how the factor $\tilde{r}^{2-p_3}$ whose behaviour we control by means of the stability condition appears, and that the fluid is co-moving; i.e., $Q^3 \equiv 0$ if and only if $C = 0$. Let us suppose that $C \neq 0$ so that the fluid has some peculiar motion. If the stability condition holds, i.e., if $p_3 < c^2_s < 1$, the right hand side converges to zero at $\tilde{t} = 0$ which means by virtue of (1.12) that $Q^3 \to 0$ in consistency with our result before. However, if the stability condition is violated and $c^2_s < p_3 < 1$, then the right-hand side diverges in this limit (since $C \neq 0$) from which it follows that $1 - (Q^3)^2 \to 0$. The peculiar motions of the fluids are therefore unstable and the fluid becomes “asymptotically null” (as suggested by the normalisation factor in (1.10)). This behaviour is not consistent with the standard model of cosmology and the before-mentioned matter does not matter hypothesis. The borderline case $c^2_s = p_3$ is characterised by $Q^3 = \text{const}$ in this special case. We remark that the critical dependence of the fluid dynamics on the signs of $c^2_s - p_3$ was first observed in [14–16] while [26] provides more general analyses of tilted fluids in spatially homogeneous settings.

Even though more work is clearly needed to fully support our claims beyond Theorem 1.1, in particular, coupling to Einstein gravity needs to be taken into account, we nevertheless assert that (1.6) is both sufficient and necessary to obtain a stable fluid flow at the big bang singularity $\tilde{t} = 0$ with the potential for interesting critical dynamics when $c^2_s = \max\{p_1, p_2, p_3\}$. If this turns out to be correct, then this would have important implications for physics as it might rule out some of the standard matter models used in cosmology to describe the early universe, and in particular, the radiation fluid $c^2_s = 1/3$.

\textit{Forward and backward Fuchsian methods.} The main results of this paper, in particular, Theorems 1.1, 3.1 and 3.6, are established through the use of complementary forward and backward Fuchsian techniques. The forward Fuchsian method is a well-established technique that has been used extensively to establish the existence of solutions to singular initial value problems for Fuchsian systems; for example, see [3–6, 11–14, 17–19, 22, 25, 27, 29–31, 36–39, 47]. Its purpose here is to prove the existence of solutions $V^\alpha$ of the form (1.7) with functions $W^0, W^1, W^2, W^3$ that converge at $\tilde{t} = 0$ to specific chosen limits $W^0 \ast, W^1 \ast, W^2 \ast, W^3 \ast$ the asymptotic data. This yields a family of fluid solutions parametrised by four functional degrees of freedom. The precise result is given in Theorem 3.1. This method is based on forward evolutions away from the singular time $\tilde{t} = 0$ launching from $\tilde{t} = 0$. The backward Fuchsian method on the other hand is a relatively new technique, first introduced in [35] and further developed in [16, 21], for establishing the global existence of solutions to initial value problems for Fuchsian systems of equations; see Section A for a brief summary of this method. In this work here, the purpose of the backward method is to construct perturbations of the solutions obtained by the forward method globally backwards in time towards the singular time $\tilde{t} = 0$ and to investigate the limits at $\tilde{t} = 0$. In fact we show that small perturbations of any solution of the forward problem given by asymptotic data $W^0, W^1, W^2, W^3$ converges to limits $\tilde{W}^0, \tilde{W}^1, \tilde{W}^2, \tilde{W}^3$ close to the asymptotic data. Our results establish that the forward and backward solutions of the Euler equations are, in a sense, in one-to-one correspondence – a question pioneered by Ringström [41–43]. However, the important problem of characterising this correspondence as an invertible continuous map with respect to a given topology remains open. We plan on returning to this question in future work.

In order to illustrate how these techniques are employed in this article, it suffices to consider the following model equation

\[
\partial_t u(t) = \frac{1}{t} bu(t) + F(t), \quad t > 0,
\]

(1.13)

where here, $b \in \mathbb{R}$ is a constant and $F$ is some sufficiently smooth function on $t > 0$. This ordinary differential equation is the prototype of a Fuchsian equation; a class of in general nonlinear partial

\textsuperscript{1,5}This implicit solution is derived without imposing the second Kasner constraint in (1.3). In particular, the same result is obtained when the background is a Kasner-scalar field spacetime with (3.15).
differential equations introduced in Section 2.2. For this simple problem, the solution is clearly

$$u(t) = t^b \left( t_0^{-b} u_0 + \int_{t_0}^t s^{-b} F(s) \, ds \right), \quad t > 0,$$

(1.14)

where \( t_0 > 0 \) and \( u_0 \in \mathbb{R} \) are arbitrary. Since \( u(t_0) = u_0 \), we can interpret (1.14) as the solution of the initial value problem for (1.13) with initial data \( u_0 \) imposed at the regular time \( t = t_0 > 0 \). We refer to this initial value problem as the backward problem with the idea that the initial data is imposed at a regular time \( t_0 > 0 \) and the Fuchsian equation is solved backwards in time all the way down to the singular time at \( t = 0 \). By a backward method, we then mean a method to construct solutions to the backward problem and analyse their asymptotics as \( t \searrow 0 \).

In contrast, the forward problem, which we also often refer to as the singular initial value problem, involves solving the Fuchsian equation forward in time starting from prescribed asymptotic data at \( t = 0 \). A forward method then refers to a method to construct solutions of the forward problem. In the case of (1.13), assuming that \( F(t) = O(t^{b-1+v}) \) for some \( v > 0 \), this corresponds to expressing the solution given by (1.14) as

$$u(t) = t^b \left( \tilde{u}_0 + \int_0^t s^{-b} F(s) \, ds \right), \quad t > 0,$$

(1.15)

where here, the constant \( \tilde{u}_0 \) defines the asymptotic data. The leading order term \( u_0(t) = t^b \tilde{u}_0 \) of the solution (1.15) is therefore parametrised by the asymptotic data while the remainder \( w(t) = t^b \int_0^t s^{-b} F(s) \, ds \) is of higher order at \( t = 0 \), i.e., \( w(t) = O(t^{b+v}) \).

In the case of the model equation (1.13), it is clear that both the backward and the forward problems are equivalent provided that \( F(t) = O(t^{b-1+v}) \), and consequently, they contain the same information about the class of all solutions. While it has been established that this equivalence between the forward and backward problems also holds for certain Fuchsian systems of partial differential equations [2,28,41], it is definitely not the case that these two complementary problems always contain the same information [21,40,41]. In the case of perfect fluids on Kasner spacetimes, however, they do as we prove in this article, and this leads to the detailed description of the singular dynamics given in Theorems 1.1, 3.1 and 3.6.

In Section 2, we develop a new forward method to solve the singular initial value problem for symmetric hyperbolic systems of Fuchsian equations, which yields a new existence result given in Theorem 2.2 that has two significant advantages over current results. The first advantage is that Theorem 2.2 allows the coefficients of the spatial derivatives in the Fuchsian equations to contain \( 1/t \) singular terms. Spatial derivatives are therefore not necessarily required to be negligible at \( t = 0 \). While it is expected that this will open up a significant range of new applications, we do not exploit this particular property here. The second advantage of our new existence result, which is exploited here, is that it is formulated using the same framework that is employed in [16] to solve backward problem for Fuchsian equations. The importance of this is that it makes it possible, for the first time, to efficiently study the forward and the backward problems together using the same methodology for large classes of nonlinear partial differential equations. The application to the Euler equations in this paper demonstrates the enormous potential that the forward and backward Fuchsian methods have when used in combination to characterise the asymptotics of solutions to nonlinear partial differential equations.

Outline of the paper. The technical arguments of this paper consist of two distinct parts that are contained in Section 2 and Section 3. In Section 2, we consider the forward problem for general symmetric hyperbolic Fuchsian systems, and we state and prove a new existence and uniqueness result for the forward problem. The precise statement of the existence and uniqueness result is given in Theorem 2.2. This forward method together with the backward method from [16, Theorem 3.8], see Theorem A.2 from the appendix for details, are then applied to the Euler equations on Kasner backgrounds in Section 3. In particular, we use Theorem 2.2 to prove an existence and uniqueness result in Section 3.1 for the singular initial value problem for the Euler equations. This yields solutions with the full expected asymptotic data degrees of freedom and leads to a detailed description of the asymptotics of a large class of fluid solutions. In Section 3.2, we complement this by establishing the nonlinear stability of these solutions via an application of Theorem A.2. Specifically, we show that for each of the solutions of the forward problem there is an open set of nearby solutions of the backward problem with the same qualitative asymptotics. Combining
this with the uniqueness result for the forward method, we finally establish that each perturbation of a solution of the forward problem of the Euler equations is again a solution of the forward problem.

2. Symmetric hyperbolic Fuchsian systems and the singular initial value problem

2.1. Preliminaries. The definitions and notation set out here agree, for the most part, with those of [16].

2.1.1. Spatial manifolds, coordinates, indexing and partial derivatives. Throughout this article, unless stated otherwise, $\Sigma$ will denote a closed $n$-dimensional manifold, lower case Latin indices, e.g. $i, j, k$, will run from 1 to $n$ and will index coordinate indices associated to a local coordinate system $x = (x^i)$ on $\Sigma$, and $t$ will denote a time coordinate on intervals of the form $[T_0, T_1)$ or $[T_0, T_1]$ for $T_1 \leq 0$. Note that this convention agrees with that of [16] where the time $t$ is taken to be negative. It worthwhile noting that this convention is opposite to that employed in the introduction where time is assumed to be positive. However, this cause no real difficulties since it is trivial to change between the two conventions using the transformation $t \mapsto -t$.

Partial derivatives with respect to the coordinates will be denoted by

$$\partial_t = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x^i}.$$ 

2.1.2. Vector Bundles. In the following, we will let $\pi : V \rightarrow \Sigma$ denote a rank $N$ vector bundle with fibres $V_x = \pi^{-1}(\{x\})$, $x \in \Sigma$, and use $\Gamma(V)$ to denote the set of all smooth sections of $V$. We will assume that $V$ is equipped with a time-independent connection\footnote{1)} $\nabla$, and a time-independent, compatible\footnote{2)} positive definite metric $h \in \Gamma(T^2_0(V))$. We will also denote the vector bundle of linear operators that act on the fibres of $V$ by $L(V) = \cup_{x \in \Sigma} \mathcal{L}(V_x) \cong V \otimes V^*$. The transpose of $A_x \in L(V_x)$, denoted by $A_x^t$, is then defined as the unique element of $L(V_x)$ satisfying

$$h(x)(A_x^tu, v_x) = h(u, A_xv_x), \quad \forall u_x, v_x \in V_x.$$

Furthermore, given two vector bundles $V$ and $W$ over $\Sigma$, we will use $L(V, W) = \cup_{x \in \Sigma} \mathcal{L}(V_x, W_x) \cong V \otimes W^*$ to denote the vector bundle of linear maps from the fibres of $V$ to the fibres of $W$.

Additionally, we employ the standard notation $V \oplus W$ for the direct sum, which has fibres $(V \oplus W)_x = V_x \times W_x$ and where addition and scalar multiplication are defined via $\lambda(v_1, w_2) + (v_2, w_2) = (\lambda v_1 + w_2, \lambda v_2 + w_2)$. Furthermore, if $V \subset V$ and $W \subset W$ are subsets satisfying $\pi(V) = \pi(W) = \Sigma$, then we define

$$V \oplus W = \bigcup_{x \in \Sigma} V_x \times W_x \subset V \oplus W$$

where $V_x = V_x \cap V$ and $W_x = W_x \cap W$. We will often abuse notation and write elements of $V \oplus W$ as $(v, w)$ where $v \in V, w \in W$ and $\pi(v) = \pi(w)$.

For any given vector bundle over $\Sigma$, e.g. $V, L(V), V \otimes V$, etc., we will generally use $\pi$ to denote the canonical projection onto $\Sigma$.

Here and below, unless stated otherwise, we will use upper case Latin indices, i.e., $I, J, K$, that run from 1 to $N$ to index elements of $V$ with respect to a local basis $\{e_I\}$ of $V$. By introducing a local basis $\{e_I\}$, we can represent $u \in \Gamma(V)$ and the inner-product $h$ locally as

$$u = u^Ie_I \quad \text{and} \quad h = h_{IJ}\theta^I \otimes \theta^J,$$

respectively, where $\{\theta^I\}$ is local basis of $V^*$ determined from the basis $\{e_I\}$ by duality. Moreover, assuming that the local coordinates $(x^i)$ and the local basis $\{e_I\}$ are defined on the same open region of $\Sigma$, we can represent the covariant derivative $\nabla u \in \Gamma(V \otimes T^*M)$ locally by

$$\nabla u = \nabla_Iu^Ie_I \otimes dx^i,$$

where

$$\nabla_Iu^I = \partial_Iu^I + \omega_{IJ}^Iu^J,$$

and the $\omega_{IJ}^I$ are the connection coefficients determined, as usual, by

$$\nabla_Ie_J = \omega_{IJ}^Ke_I.$$
We further assume that $\Sigma$ is equipped with a time-independent \(^2\) \(^3\) Riemannian metric \(^4\) \(g \in \Gamma(T^0_x(\Sigma))\) that is given locally in coordinates \((x^i)\) by

\[ g = g_{ij} dx^i \otimes dx^j. \]

Since the metric determines the Levi-Civita connection on the tensor bundle \(T^*_x(M)\) uniquely, we can, without confusion, use $\nabla$ to also denote the Levi-Civita connection. The connection on $V$ and the Levi-Civita connection on \(T^*_x(M)\) then determine a connection on the tensor product $V \otimes T^*_{x}(\Sigma)$ in a unique fashion, which we will again denote by $\nabla$. This connection is compatible with the positive definite inner-product induced on \(V \otimes T^*_{x}(\Sigma)\) by the inner-product $h$ on $V$ and the Riemannian metric $g$ on $\Sigma$. With this setup, the **covariant derivative of order** $s$ of a section $u \in \Gamma(V)$, denoted $\nabla^s u$, defines an element of $\Gamma(V \otimes T^0_x(\Sigma))$ that is given locally by

\[ \nabla^s u = \nabla_{i_1} \cdots \nabla_{i_s} u^i e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s}. \]

When $s = 2$, the components of $\nabla^2 u$ can be computed using the formula

\[ \nabla_j \nabla_i u^I = \partial_j \nabla_i u^I - \Gamma^k_{ji} \nabla_k u^I + \omega_{ij}^k \nabla_k u^I, \]

where $\nabla_i u^I$ is as defined above, and $\Gamma^k_{ij}$ are the Christoffel symbols of $g$. Similar formulas exist for the higher covariant derivatives.

### 2.1.3. Inner-products and operator inequalities

We define the **norm** of $v \in V_x$, $x \in \Sigma$, by

\[ |v|^2 = h(x)(v, v). \]

Using this norm, we then define, for $R > 0$, the **bundle of open balls of radius** $R$ in $V$ by

\[ B_R(V) = \{ v \in V \mid |v| < R \}. \]

Given elements $v, w \in V_x \otimes T^0_x(M_x)$, we can expand them in a local basis as

\[ v = v^j_{i_1 i_2 \cdots i_s} e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s} \quad \text{and} \quad w = w^j_{i_1 i_2 \cdots i_s} e_I \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_s}, \]

respectively. Using these expansions, we define the inner-product of $v$ and $w$ by

\[ (v|w) = g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_s j_s} h_{I J} v^j_{i_1 i_2 \cdots i_s} w^j_{j_1 j_2 \cdots j_s}, \]

while the norm of $v$ is defined via

\[ |v|^2 = (v|v). \]

For $A \in L(V_x)$, the **operator norm** of $A$, denoted $|A|_{op}$, is defined by

\[ |A|_{op} = \sup \{ |(v|Av)| \mid w \in B_1(V_x) \}. \]

We also define a related norm for elements $A \in L(V_x) \otimes T^*_x M$ by

\[ |A|_{op} = \sup \{ |(v|Aw)| \mid (v, w) \in B_1(V_x \otimes T^*_x M) \times B_1(V_x) \}. \]

From this definition, it not difficult to verify that

\[ |A|_{op} = \sup \{ |(v|Aw)| \mid (v, w) \in B_1(V_x \otimes T^0_{x+1}(T_x M)) \times B_1(V_x \otimes T^0_x(T_x M)) \}. \]

This definition can also be extended to elements of $A \in L(V_x) \otimes T^*_x M \otimes T_x M$ in a similar fashion; that is,

\[ |A|_{op} = \sup \{ |(v|Aw)| \mid (v, w) \in B_1(V_x \otimes T^*_x M \otimes T_x M) \times B_1(V_x) \}, \]

where again we have that

\[ |A|_{op} = \sup \{ |(v|Aw)| \mid (v, w) \in B_1(V_x \otimes T^1_{x+1}(T_x M)) \times B_1(V_x \otimes T^0_x(T_x M)) \}. \]

Finally, for $A, B \in L(V_x)$, we define

\[ A \leq B \]

if and only if

\[ (v|Av) \leq (v|Bv), \quad \forall v \in V_x. \]

\(^2\)\(^3\) $\partial g = 0$.

\(^4\)This reference metric $g$ is in general unrelated to the spacetime metric $g$. 
2.1.4. Constants, inequalities and order notation. We will use the letter $C$ to denote generic constants whose exact dependence on other quantities is not specified and whose value may change from line to line. For such constants, we will often employ the standard notation

$$ a \lesssim b $$

for inequalities of the form

$$ a \leq Cb. $$

On the other hand, when the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norm $\|u\|_{L^\infty}$, we use the notation

$$ C = C(\|u\|_{L^\infty}). $$

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments.

We now turn to defining a notion of the order notation that is applicable for maps from one vector bundle to another. This notation will be used frequently in the proof of various nonlinear estimates. The definition begins with four vector bundles $V$, $W$, $Y$ and $Z$ that sit over $\Sigma$, and maps

$$ f \in C^0([T_0, 0), C^\infty(W \oplus B_R(V), Z)) \quad \text{and} \quad g \in C^0([T_0, 0), C^\infty(B_R(V), Y)), $$

where $W \subset W$ is open and $\pi(W) = \Sigma$. We then say that

$$ f(t, w, v) = O(g(t, v)) $$

if there exists a $\tilde{R} \in (0, R)$ and a map

$$ \tilde{f} \in C^0([T_0, 0), C^\infty(W \oplus B_{\tilde{R}}(V), L(Y, Z))) $$

such that$^{2,5}$

$$ f(t, w, v) = \tilde{f}(t, w, v)g(t, v), $$

$$ |\tilde{f}(t, w, v)| \leq 1 \quad \text{and} \quad |\nabla_{w, v}^s \tilde{f}(t, w, v)| \lesssim 1 $$

for all $(t, w, v) \in [T_0, 0) \times W \oplus B_{\tilde{R}}(V)$ and $s \geq 1$. For situations where we want to bound $f(t, w, v)$ by $g(t, v)$ up to an undetermined constant of proportionality, we define

$$ f(t, w, v) = O(g(t, v)) $$

if there exists a $\tilde{R} \in (0, R)$ and a map

$$ \tilde{f} \in C^0([T_0, 0), C^\infty(W \oplus B_R(V), L(Y, Z))) $$

such that

$$ f(t, w, v) = \tilde{f}(t, w, v)g(t, v) $$

and

$$ |\nabla_{w, v}^s \tilde{f}(t, w, v)| \lesssim 1 $$

for all $(t, w, v) \in [T_0, 0) \times W \oplus B_{\tilde{R}}(V)$ and $s \geq 0$.

Finally, we use the notation $C^0_b([0, T), C^\infty(W, Z))$ to denote the subspace of $C^0([0, T), C^\infty(W, Z))$ consisting of all the maps $f \in C^0([0, T), C^\infty(W, Z))$ satisfying $f(t, w) = O(1).$

---

$^{2,5}$Here, we are using $\nabla_{w, v}$ to denote a covariant derivative operator on the product manifold $W \times V$. Since $\Sigma$ is compact, we know that such a covariant derivative always exists and it does not matter for our purposes which one is employed.
2.1.5. Sobolev spaces. The $W^{k,p}$, $k \in \mathbb{Z}_{\geq 0}$, norm of a section $u \in \Gamma(V)$ is defined by
\[
\|u\|_{W^{k,p}(\Sigma)} = \left\{ \begin{array}{ll}
\left( \sum_{\ell=0}^{k} \int_{\Sigma} |\nabla^{\ell} u|^p \nu_q \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\
\max \sup_{0 \leq \ell \leq k} |\nabla^{\ell} u(x)| & \text{if } p = \infty
\end{array} \right.,
\]
where $\nu_q \in \Omega^n(\Sigma)$ denotes the volume form of $g$. The Sobolev space $W^{k,p}(\Sigma, V)$ can then be defined as the completion of the space of smooth sections $\Gamma(V)$ in the norm $\| \cdot \|_{W^{k,p}(\Sigma)}$. When $V = \Sigma \times \mathbb{R}$ or the vector bundle is clear from context, we will often write $W^{k,p}(\Sigma)$ instead. Furthermore, when $p = 2$, we will employ the standard notation $H^k(\Sigma, V) = W^{k,2}(\Sigma, V)$, and we recall that $H^k(\Sigma, V)$ is a Hilbert space with the inner-product given by
\[
\langle u|v \rangle_{H^k(\Sigma)} = \sum_{\ell=0}^{k} \int_{\Sigma} \langle \nabla^{\ell} u | \nabla^{\ell} v \rangle,
\]
where the $L^2$ inner-product $\langle \cdot | \cdot \rangle$ is defined by
\[
\langle w|z \rangle = \int_{\Sigma} \langle w|z \rangle \nu_g.
\]
We also employ the notation
$C^0_b([T_0,0), H^k(V)) = C^0([T_0, 0), H^k(\Sigma, V)) \cap L^\infty([T_0, 0), H^k(\Sigma, V))$
for the set of continuous and bounded maps from $[T_0,0)$ to $H^k(\Sigma, V)$.

2.2. The class of symmetric hyperbolic Fuchsian systems. In this section, we consider partial differential equations of the form
\[
B^0(t, w_1, u) \partial_t u + B^1(t, w_1, u) \nabla_i u = \frac{1}{t} B(t, w_1, u) u + F(t, w_2, u) \quad \text{in } [T_0,0) \times \Sigma,
\]
where $T_0 < 0$, the coefficients satisfy the conditions below, and $w_1$ and $w_2$ should be interpreted as prescribed time-dependent sections of vector bundles $Z_1$ and $Z_2$, respectively, over $\Sigma$ while the solution $u$ takes values in a vector bundle $V$ over $\Sigma$. Since the assumptions below imply, in particular, that (2.1) is symmetric hyperbolic, we know from standard local-in-time existence and uniqueness theory for symmetric hyperbolic equations that given initial data at time $t_\ast \in [T_0,0)$, there exists a unique solution of (2.1) that is defined for a small interval of time about $t_\ast$ and agrees with the initial data at $t_\ast$. What is not at all clear from the local-in-time existence theory is if there are solutions that exist for all times in the interval $[T_0,0)$.

**Definition 2.1** (Symmetric hyperbolic Fuchsian systems). A partial differential equation of the form (2.1) is called a symmetric hyperbolic Fuchsian system for the constants $T_0 < 0$, $R > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $q \geq 0$, $0 < p \leq 1$, $\mu \in \mathbb{R}$, $\lambda > -1$, $\theta \geq 0$ and $\beta \geq 0$, open and bounded subsets $Z_1 \subset Z_1$ and $Z_2 \subset Z_2$ with $\pi(Z_1) = \pi(Z_2) = \Sigma$, and maps $B^0$, $B^1$, $B$, $F$, $F_0$, $B_0$, $B_1$ and $B_1$ provided the following conditions are satisfied:

1. The maps $(t, z_1, v) \mapsto B^0(t, z_1, |t|^{\mu} v)$ and $(t, z_1, v) \mapsto B(t, z_1, |t|^{\mu} v)$ are in
$C^0_b([T_0,0), C^\infty(Z_1 \oplus B_R(V), L(V))) \cap C^1([T_0,0), C^\infty(Z_1 \oplus B_R(V), L(V)))$
and $C^0_b([T_0,0), C^\infty(Z_1 \oplus B_R(V), L(V)))$, respectively, and
$\pi(B^0(t, z_1, |t|^{\mu} v)) = \pi(B(t, z_1, |t|^{\mu} v)) = \pi(z_1, v)$,
$\left( B^0(t, z_1, |t|^{\mu} v) \right)^{\gamma_1} = B^0(t, z_1, |t|^{\mu} v)$
and
$\frac{1}{\gamma_1} \text{id}_{V^{\gamma_1}} \leq B^0(t, z_1, |t|^{\mu} v) \leq \gamma_2 \text{id}_{V^{\gamma_2}}$
for all $(t, z_1, v) \in [T_0,0) \times Z_1 \oplus B_R(V)$. Moreover, there exists a map $\bar{B}^0 \in C^0_b([T_0,0), C^\infty(Z_1, L(V)))$ satisfying
$\pi(\bar{B}^0(t, z_1)) = \pi(z_1)$
for all \((t, z_1) \in [T_0, 0) \times Z_1\) and

\[
B^0(t, z_1, |t|^p v) - \tilde{B}^0(t, z_1) = O(v).
\]

(2) The map \(F\) can be expanded as

\[
F(t, z_2, \cdot) = |t|^\lambda \tilde{F}(t, z_2) + |t|^\lambda F_0(t, z_2, \cdot)
\]

where \(\tilde{F} \in C^0([T_0, 0), C^\infty(\mathbb{Z}_2, V))\), the map \((t, z_2, v) \mapsto F_0(t, z_2, |t|^p v)\) is in \(C^0_0([T_0, 0), C^\infty(\mathbb{Z}_2 \oplus B_R(V), V))\), and

\[
\pi(\tilde{F}(t, z_2)) = \pi(z_2) \quad \text{and} \quad \pi(F_0(t, z_2, |t|^p v)) = \pi((z_2, v))
\]

for all \((t, z_2, v) \in [T_0, 0) \times \mathbb{Z}_2 \oplus B_R(V)\) and

\[
F_0(t, z_2, |t|^p v) = O(v).
\]

(3) The map \(^2^6\) \(B\) locally defined as \(B_{ij}^I \theta^I \otimes e_I \otimes \partial_i\) can be expanded as

\[
B(t, z_1, \cdot) = |t|^{-(1-p)} B_0(t, z_1, \cdot) + |t|^{-1} B_1(t, z_1, \cdot)
\]

where the maps \((t, z_1, v) \mapsto B_0(t, z_1, |t|^p v)\) and \((t, z_1, v) \mapsto B_1(t, z_1, |t|^p v)\) are in \(C^0_0([T_0, 0), C^\infty(\mathbb{Z}_1 \oplus B_R(V), L(V) \otimes T\Sigma))\) and satisfy

\[
[\sigma(\pi(v))(B(t, z_1, |t|^p v))]_{tr} = \sigma(\pi(v))(B(t, z_1, |t|^p v))
\]

and

\[
\pi(B_0(t, z_1, v)) = \pi(B_1(t, z_1, v)) = \pi((z_1, v))
\]

for all \((t, z_1, v) \in [T_0, 0) \times \mathbb{Z}_1 \oplus B_R(V)\) and for all \(\sigma \in \mathcal{X}^*(\Sigma)\). Moreover, there exists a map \(\tilde{B}_1 \in C^0_0([T_0, 0), C^\infty(\mathbb{Z}_1, L(V) \otimes T\Sigma))\) such that

\[
\pi(\tilde{B}_1(t, z_1)) = \pi(z_1)
\]

for all \((t, z_1) \in [T_0, 0) \times \mathbb{Z}_1\) and

\[
B_1(t, z_1, |t|^p v) - \tilde{B}_1(t, z_1) = O(v).
\]

(4) The map \(\text{div} B(t, z_1, z_1', v, v')\) defined locally by \(^2^7\)

\[
\text{div} B(t, x, z_1, z_1', z_2, v, v') = \partial_i B^0(t, x, z_1, v) + D_{z_1} B^0(t, x, z_1, v) \cdot p|t|^{q-1} z_1'
\]

\[
+ D_{z_2} B^0(t, x, z_1, v) \cdot (B^0(t, x, z_1, v))^{-1} \left[ -B^1(t, x, z_1, v) \cdot v_i' - \frac{1}{t} B(t, x, z_1, v) v + F(t, x, z_2, v) \right] + \partial_i B^1(t, x, z_1, v)
\]

\[
+ D_{z_2} B^1(t, x, z_1, v) \cdot (z_1' - \omega_i(x) z_1) + D_{x_2} B^1(t, x, z_1, v) \cdot (v_i' - \omega_i(x) v)
\]

\[
+ \Gamma_{ij} (x) B^1(t, x, z_1, v) + \omega_i(x) B^1(t, x, z_1, v) - B^1(t, x, z_1, v) \omega_i(x),
\]

where \(v = (v_i'), v' = (v_i'''), z_1 = (z_1'''), z_1' = (z_1'''), z_1'' = (z_1'''), z_2 = (z_2''), \omega_i = (\omega_i'''), B^2 = (B^1)'\), \(x = \pi(v) = \pi(z_1) = \pi(z_2)\), and

\[
(t, z_1, z_1', z_2, v) \in [T_0, 0) \times \mathbb{Z}_1 \oplus B_R(Z_1) \oplus B_R(Z_1 \otimes T\Sigma) \oplus \mathbb{Z}_2 \oplus B_R(V) \oplus B_R(V \otimes T\Sigma),
\]

satisfies

\[
\text{div} B(t, z_1, z_1', z_2, |t|^p v, |t|^p v') = O(|t|^{-(1-p)} \theta + |t|^{-1} \beta)
\]

\(^2^6^6\)Here, we are using the notation \(\sigma(A)\) to denote the natural action of a differential 1-form \(\sigma \in \mathcal{X}^*(\Sigma)\) on an element of \(A \in \Gamma(L(V) \otimes T\Sigma)\), which if we express \(A\) and \(\sigma\) locally as

\[
A = A_{ij}^I \theta^I \otimes e_I \otimes \partial_i \quad \text{and} \quad \sigma = \sigma dx^i,
\]

respectively, is given by

\[
\sigma(A) = \sigma A_{ij}^I \theta^I \otimes e_I.
\]

\(^2^7^7\)The notation \(D_v\) denotes the linear operator associated to differentiating with respect to the variable slot occupied by \(\#\); e.g. \(D_v f(v, w) v' = \frac{df}{dt}_{t=0} f(v + tv', w)\).
for some constants \( q, \theta, \beta \geq 0 \). We note, by definition, that \( \text{div} B \) satisfies
\[
\text{div} B(t, w_1, p^{-1}|t|^{1-q} \partial_t w_1, \nabla w_1, w_2, u, \nabla u) = \partial_t \left( B^0(t, w_1, u) \right) + \nabla_i (B^i(t, w_1, u))
\] (2.8)
for any solution \( u \) of (2.1).

As we shall see in our application to the Euler equations in Section 3, the similarity between the structural assumptions given above by Definition 2.1 for the forward problem, i.e., going away from the singularity at \( t = 0 \), and those given in Appendix A.1 for the backward problem, i.e., going towards the singularity at \( t = 0 \), is a decisive advantage as it allows us to analyse both the forward and backward evolution problems under a similar set of conditions. We expect that this feature will be important in future application.

The main genuine difference between the assumptions from Definition 2.1 and those from Appendix A.1 is the upper bound in (2.14) versus the lower bound in (A.3) for the linear operator \( B \), which is one of the key structural differences between forward and backward problems as has been noted previously in [2, 41]. We further note that the parameter \( \lambda \) from Definition 2.1, which is not present in Appendix A.1, introduces control of the decay of the source term function in a manner that is crucial for the forward problem as obvious from (2.10) below. Making appropriate choices for the parameter \( \mu \) in Definition 2.1, which also appears in (2.10), is as critical. It is worth noting that some of the conditions in Definition 2.1 become less restrictive the larger we choose \( \mu \) due to the factor \( |t|^\mu \) that appears in the arguments of some of the maps there. However, \( \mu \) must be no larger than \( \lambda + 1 \) as a particular consequence of (2.10) given that \( \alpha \) and \( \rho \) are positive.

We close this subsection with the remark that it might be of general interest to allow \( \lambda, \rho \) and \( \mu \) to be sufficiently smooth functions on \( \Sigma \). Depending on the application, this might yield a more localised and finer control. In this paper, however, we fully restrict to the constant case for simplicity.

### 2.3. The singular initial value problem

We are now ready to state our new existence and uniqueness theorem for the forward problem, which is the singular initial value problem for Fuchsian systems. The constant \( C_{\text{Sob}} \) that appears in the following theorem is the constant from Sobolev’s inequality [48, Ch. 13, Prop. 2.4] for \( k \in \mathbb{Z}_{>n/2+1} \), that is,
\[
\max \left\{ \| f \|_{L^\infty(\Sigma)}, \| \nabla f \|_{L^\infty(\Sigma)} \right\} \leq C_{\text{Sob}} \| f \|_{H^k(\Sigma)}. \tag{2.9}
\]

**Theorem 2.2** (The singular initial value problem for symmetric hyperbolic Fuchsian systems). Suppose \( k \in \mathbb{Z}_{>n/2+3} \). Eq. (2.1) is a symmetric hyperbolic Fuchsian system for constants \( T_0 < 0, \mathcal{R} > 0, R > 0, \gamma_1 > 0, \gamma_2 > 0, q \geq 0, 0 < p \leq 1, \mu \in \mathbb{R}, \lambda > -1, \theta \geq 0 \) and \( \beta \geq 0 \), open bounded sets \( \mathcal{Z}_1 \subset \mathcal{Z}_1 \) and \( \mathcal{Z}_2 \subset \mathcal{Z}_2 \) with \( \pi(\mathcal{Z}_1) = \pi(\mathcal{Z}_2) = \Sigma \), and maps \( B^0, \tilde{B}^0, \tilde{B}, F, F_0, B_0, B_1 \), and \( \tilde{B}_1 \) as in Definition 2.1 with
\[
\lambda + 1 - (1 + \alpha) \rho = \mu \tag{2.10}
\]
for some \( \alpha > 0, w_1 \in C^0_\rho([T_0, 0), H^k(\Sigma, Z_1)] \) and \( \| f \|_{H^k(\Sigma)} \) and \( w_2 \in C^0_\rho([T_0, 0), H^{k-1}(\Sigma, Z_2)] \) satisfy
\[
\sup_{T_0 < t < 0} \max \left\{ \| \nabla w_1(t) \|_{H^{k-1}(\Sigma)}, \| |t|^{1-q} \partial_t w_1(t) \|_{H^{k-1}(\Sigma)} \right\} < \frac{\mathcal{R}}{C_{\text{Sob}}}, \tag{2.11}
\]
and
\[
w_a([T_0, 0) \times \Sigma) \subset \mathcal{Z}_a, \quad a = 1, 2.
\]

Let
\[
\mathcal{B} = \sup_{T_0 \leq t \leq 0} \left( \left\| \frac{\partial}{\partial t} \left( \tilde{B}^0(t) \tilde{B}^0(t)^{-1} \tilde{B}_1(t) \right) \right\|_{L^\infty} + \left\| \nabla \tilde{B}_1(t) \right\|_{L^\infty} \right). \tag{2.12}
\]
where \( \tilde{B}^0(t) = \tilde{B}^0(t, w_1(t)) \) and \( \tilde{B}_1(t) = \tilde{B}_1(t, w_1(t)) \), and in addition, suppose there exists a \( \eta > 0 \) satisfying
\[
2\eta - \gamma_1 \beta - k(k + 1) b \gamma_1 > 0 \tag{2.13}
\]
for which the inequality
\[
\mathcal{B}(t, z_1, \| t \|^\mu v) \leq (\mu - \eta) |B^0(t, z_1, \| t \|^\mu v) \leq (\mu - \eta) |B^0(t, z_1, \| t \|^\mu v) \tag{2.14}
\]
holds for all \( (t, z_1, v) \in [T_0, 0) \times \mathcal{Z}_1 \oplus B_R(V) \). Then there exist constants \( C, \delta > 0 \) such that if
\[
\int_{T_0}^0 |s|^{p-1} \left\| \tilde{F}(s, w_2(s)) \right\|_{H^k(\Sigma)} ds < \delta, \tag{2.15}
\]
then there exists a solution
\[ u \in C^0 \left( [T_0, 0), H^k(\Sigma, V) \right) \cap C^1 \left( \left[ T_0, 0 \right), H^{k-1}(\Sigma, V) \right) \subset C^1 \left( [T_0, 0) \times \Sigma, V \right) \]
(2.16)
of (2.1) that is bounded by
\[ \| t^{-(\lambda+1-p)}u(t) \|_{H^p(\Sigma)} \leq C \int_t^0 |s|^{p-1} \| \tilde{F}(s, w_2(s)) \|_{H^k(\Sigma)} \, ds, \quad T_0 < t < 0, \]
(2.17)
and
\[ \sup_{t \in [T_0, 0)} \max \left\{ \left\| t^{-p}u(t) \right\|_{L^\infty(\Sigma)}, \left\| t^{-p} \nabla u(t) \right\|_{L^\infty(\Sigma)} \right\} < R. \]
(2.18)
Moreover, the solution \( u \) is the unique solution of (2.1) within the class \( C^1 \left( [T_0, 0) \times \Sigma, V \right) \) satisfying (2.18).

Remark 2.3. If the map \( B_1 \), see (2.5), vanishes, then the regularity requirement in the statement of Theorem 2.2 can be lowered to \( k \in \mathbb{Z}_{\geq n/2+1} \). This is also true for the statements of Propositions 2.6 and 2.7 below. The origin of this improvement is that Lemma 3.5 from [16] is no longer needed to derive the estimate (2.37) in the proof of Proposition 2.6.

The proof of Theorem 2.2 is given in Section 2.4 below, but before we consider the proof, we first make a few observations. First, a direct application of Theorem 2.2 to the model problem (1.13) would only address the case with zero asymptotic data \( \bar{u} = 0 \), cf. (1.15). However, this is no loss of generality since Theorem 2.2 can instead be applied to the equation for the remainder \( w = u - \bar{u}_0 \) defined from the original unknown \( u \) and the leading-order term \( \bar{u}_0 \) parametrised by the asymptotic data, e.g. \( u_0(t) = t^\beta \bar{u}_* \in (1.15) \). In this way, we can obtain the solution (1.15) to the model Fuchsian equation (1.13) via an application of Theorem 2.2.

It is an important property of Fuchsian systems, as we demonstrate in the proof of Theorem 2.2 below, that a solution to the singular initial value problem can be approximated on \([T_0, 0)\) with arbitrary accuracy by a solution of the (regular) initial value problem on intervals \([T_0, t_*)\) with suitable (regular) initial data imaged at sufficiently small (regular) times \( t_* < 0 \). The central idea of the proof of Theorem 2.2 is to show that the sequence of solutions \( u_{t_*}(t) \) generated by solving the regular initial value problem of (2.1) with zero Cauchy data \( u_{t_*}(t_*) = 0 \) is a monotonic sequence of positive times \( t_* \) that converges to a solution of the singular initial value problem. This idea was put forward in [12] and then used in [3, 4] to establish the first existence proof of solutions of the singular initial value problem for quasilinear symmetric hyperbolic Fuchsian equations. Besides being a useful analytic technique, this approximation technique can be employed to construct numerical solutions of the singular initial value problem [12, 13, 15].

2.4. Proof of Theorem 2.2.

2.4.1. Transformation to a canonical Fuchsian system. The first step in the proof of Theorem 2.2 is to show that the Fuchsian system (2.1) can be transformed into canonical form, which we now define. The point of this transformation is that it allows us to deduce existence for the general class of Fuchsian systems (2.1) from an existence result for the simpler class of canonical Fuchsian systems.

Definition 2.4 (Canonical symmetric hyperbolic Fuchsian systems). A system of partial differential equations of the form (2.1) is called a canonical symmetric hyperbolic Fuchsian system for the constants \( T_0 < 0, R > 0, R > 0, \gamma_1 > 0, \gamma_2 > 0, \eta > 0, \lambda > 0, \theta > 0 \) and \( \beta \geq 0 \), open sets \( \mathcal{Z}_1 \subset Z_1 \) and \( \mathcal{Z}_2 \subset Z_2 \) with \( \pi(Z_1) = \pi(Z_2) = \Sigma \), and maps \( B^0, B^1, \bar{B}, \bar{F}, \bar{F}_0, \bar{F}_1, \bar{B}_0, \bar{B}_1 \) and \( \bar{B}_1 \) if these choices of constants, sets and maps along with \( p = 1 \) and \( \mu = 0 \) make the system symmetric hyperbolic according to Definition 2.1.

The following lemma guarantees that the Fuchsian system (2.1) can always be brought into canonical form provided the condition (2.28) below holds. Since the proof follows from a straightforward calculation, the details will be left to the reader.

Lemma 2.5. Suppose \( p \in \mathbb{R} \setminus \{0\} \), \( \lambda \in \mathbb{R} \) and \( T_0 < T_1 \leq 0 \). Then \( u \in C^1 \left( [T_0, T_1) \times \Sigma, V \right) \) is a solution of (2.1) if and only if
\[ u(t, \tau) = (-\tau)^{-\lambda/p}u(-\tau)^{1/p}, \quad \tau = -(t)^p, \]
(2.19)
is a solution of
\[ B^0(\tau, \bar{w}_1, \bar{u}) \partial_\tau \bar{u} + \bar{B}^i(\tau, \bar{w}_1, \bar{u}) \nabla_i \bar{u} = \frac{1}{\tau} \bar{B}(\tau, \bar{w}_1, \bar{u}) \bar{u} + \bar{F}(\tau, \bar{w}_2, \bar{u}), \]  
(2.20)
where
\[ \bar{w}_1(\tau, x) = w_1(-(-\tau)^{1/p}, x), \]  
(2.21)
\[ \bar{w}_2(\tau, x) = w_2(-(-\tau)^{1/p}, x), \]  
(2.22)
\[ \bar{B}^0(\tau, z_1, v) = B^0(-(-\tau)^{1/p}, z_1, (-\tau)^{\lambda/p_v}), \]  
(2.23)
\[ \bar{B}^i(\tau, z_1, v) = \frac{(-\tau)^{1/p} B^i(-(-\tau)^{1/p}, z_1, (-\tau)^{\lambda/p_v})}{p}, \]  
(2.24)
\[ \bar{B}(\tau, z_1, v) = \frac{B(-(-\tau)^{1/p}, z_1, (-\tau)^{\lambda/p_v}) - \lambda B^0(-(-\tau)^{1/p}, z_1, (-\tau)^{\lambda/p_v})}{p}, \]  
(2.25)
\[ \bar{F}(\tau, z_2, v) = \frac{(-\tau)^{1/p} - \lambda}{p} F(-(-\tau)^{1/p}, z_2, (-\tau)^{\lambda/p_v}), \]  
(2.26)
with \( \bar{u} \in C^1([-(-T_0)^p, -(-T_1)^p) \times \Sigma, V) \). Furthermore, the map \( \text{div} \bar{B} \) defined by (2.6) transforms as
\[ \text{div} \bar{B}(\tau, z_1, z_1', z_2, v', v') = \frac{(-\tau)^{1/p}}{p} \text{div} B(-(-\tau)^{1/p}, z_1, p \bar{z}_1 \bar{p}/p, z_1', z_2, (-\tau)^{\lambda/p_v}, (-\tau)^{\lambda/p_v'}), \]  
(2.27)
provided \( \bar{q} \) and \( p \) are the constants used to define the divergence map on the left side of (2.27) and \( q \) and \( p \) are the ones used to define the divergence map on the right side with \( \bar{q} = q/p \).

In particular, if (2.1) is a symmetric hyperbolic Fuchsian system for constants \( T_0 < 0, R > 0, R > 0, \gamma_1 > 0, \gamma_2 > 0, q \geq 0, 0 < p \leq 1, \mu \in \mathbb{R}, \lambda > -1, \theta > 0 \) and \( \beta \geq 0 \), open bounded sets \( Z_1 \subset \mathbb{R} \) and \( Z_2 \subset \mathbb{R} \) with \( \pi(Z_1) = \pi(Z_2) = \Sigma, \) and maps \( B^0, \bar{B}^0, B, \bar{F}, F_0, B_0, B_1, \) and \( \bar{B} \) as in Definition 2.1 such that
\[ \mu = \lambda + 1 - (1 + \alpha)p \]  
(2.28)
for some \( \alpha > 0 \), then (2.20) – (2.26) with
\[ p = p, \quad \lambda = \mu \]
is a canonical symmetric hyperbolic Fuchsian system for the choice of constants \( T_0 = -(-T_0)^p, \bar{q} = q/p, \) \( R = \mathbb{R}, \bar{R} = R, \bar{\gamma}_1 = \gamma_1, \bar{\gamma}_2 = \gamma_2, \lambda = \alpha, \theta = \theta/p \) and \( \beta = \beta/p \), the sets \( \bar{Z}_1 = Z_1, \bar{Z}_2 = Z_2, \) and maps
\[ \bar{B}^0(\tau, z_1) = \bar{B}^0(-|\tau|^{1/p}, z_1) \]
and
\[ \bar{F}(\tau, z_2) = \bar{F}(-|\tau|^{1/p}, z_2)/p, \quad F_0(\tau, z_2, v) = F_0(-|\tau|^{1/p}, z_2, (-\tau)^{\mu/p_v})/p. \]
Furthermore, \( \bar{w} \) and \( w \) are related via
\[ |\tau|^{1-q} \partial_\tau \bar{w}(\tau, x) = \frac{1}{p} (|\tau|^{1-q} \partial_\tau w)(-(-\tau)^{1/p}, x). \]

2.4.2. Existence and uniform estimates on time intervals \([T_0, t_\infty)\) with \( t_\infty < 0 \). The next step in the proof of Theorem 2.2 is to establish the existence of solutions to canonical Fuchsian systems on time intervals of the form \([T_0, t_\infty)\) where \( t_\infty \) can be chosen as small as we like. This existence result is established in Proposition 2.6 below. Crucially, the initial data smallness parameter \( \delta \) in Proposition 2.6 is independent of \( t_\infty \). This property will allow us to send \( t_\infty \) to zero in Proposition 2.7 below to obtain, in the limit, solutions of the singular initial value problem for canonical Fuchsian systems. We note that for this construction it essential that the other constants, e.g. \( \Delta, C \), appearing in Proposition 2.6 are also independent of \( t_\infty \).

Proposition 2.6. Suppose \( k \in \mathbb{Z}_{>n/2+3}, \) Eq. (2.1) is a canonical symmetric hyperbolic Fuchsian system for the constants \( T_0 < 0, R > 0, \gamma_1 > 0, \gamma_2 > 0, q \geq 0, \lambda > 0, \theta > 0 \) and \( \beta \geq 0 \), open bounded sets \( Z_1 \subset \mathbb{R} \) and \( Z_2 \subset \mathbb{R} \) with \( \pi(Z_1) = \pi(Z_2) = \Sigma, \) and maps \( B^0, \bar{B}^0, B, \bar{F}, F_0, F_1, B_0, B_1 \)
and $\bar{B}_1$ as in Definition 2.4, and $w_1 \in C_0^0([T_0, 0), H^k(\Sigma, Z_1))] \cap C^1([T_0, 0), H^{k-1}(\Sigma, Z_1))]$ and $w_2 \in C_0^0([T_0, 0), H^k(\Sigma, Z_2))]$ satisfy
\[
\sup_{T_0 < t < 0} \max \left\{ \| \nabla w_1(t) \|_{H^{k-1}(\Sigma)}, \| |t|^{1-q} \partial_t w_1(t) \|_{H^{k-1}(\Sigma)} \right\} < \frac{R}{C_{Sob}}
\]
and
\[
w_a([T_0, 0) \times \Sigma) \subset Z_a, \quad a = 1, 2.
\]
In addition, suppose there exists a constant $\eta > 0$ such that
\[
B(t, z_1, v) \leq -\eta B^0(t, z_1, v)
\]
for all $(t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)$ and that
\[
2\eta - \gamma_1 b - k(k + 1) b \gamma_1 > 0,
\]
where $b$ is defined in (2.12). Then there exist constants $0 < \Delta < R/C_{Sob}$ and $C > 0$, independent of $T_1 \in [T_0, 0)$ and $t_* \in (T_1, 0)$, such that every solution
\[
u \in C^0([T_1, t_*], H^k(\Sigma, V)) \cap C^1([T_1, t_*], H^{k-1}(\Sigma, V)) \subset C^1([T_1, t_*] \times \Sigma, V)
\]
of (2.1) that is bounded by
\[
\sup_{t \in [T_1, t_*]} \| u(t) \|_{H^k(\Sigma)} \leq \Delta
\]
also satisfies
\[
\| |t|^{-\nu} u(t) \|_{H^k(\Sigma)} \leq C \left\{ \| |t_*|^{-\nu} u(t_*) \|_{H^k(\Sigma)} + \int_{t_*}^{t} \| s \|^{1-\nu} \bar{F}(s, w_2(s)) \|_{H^k(\Sigma)} ds \right\}
\]
for all $t \in (T_1, t_*)$ and $\nu \in \mathbb{R}$. Moreover, there exists a $\delta > 0$ such that, for any $t_* \in (T_0, 0)$ and
\[
u_0 \in H^k(\Sigma)
\]
with
\[
\max \left\{ \| \nu_0 \|_{H^k(\Sigma)}, \int_{T_0}^{0} \| \bar{F}(s, w_2(s)) \|_{H^k(\Sigma)} ds \right\} \leq \delta,
\]
there exists a unique solution
\[
u \in C^0([T_0, t_*], H^k(\Sigma)) \cap C^1([T_0, t_*], H^{k-1}(\Sigma))
\]
of (2.1) that satisfies the initial condition
\[
u(t_*) = \nu_0
\]
and the estimates (2.31) and (2.32) for $T_1 = T_0$.

Proof. The first step of the proof is to derive the estimate (2.32). The derivation of this estimate is modelled on the derivation of related estimates from the existence part of the proof of Theorem 3.8 in [16]. To complete the proof, we then use the estimate (2.32) in conjunction with the continuation principle for symmetric hyperbolic equations to establish the existence of solutions to (2.1) on intervals of the form $[T_0, t_*]$ where the initial data is specified at a fixed time $t_*$ in the interval $(T_0, 0)$. Crucially, we show for sufficiently small initial data that the time $t_*$ can be chosen as close to zero as we like.

To begin the first step, we suppose $t_* \in (T_1, 0)$ and $\nu \in C^0([T_1, t_*], H^k(\Sigma, V)) \cap C^1([T_1, t_*], H^{k-1}(\Sigma, V))$ is a solution of (2.1) that satisfies the initial condition (2.34) and the bound (2.31) for some $\Delta < R/C_{Sob}$, which we note by the Sobolev inequality (2.9) implies that
\[
\sup_{t \in [T_1, t_*]} \max \left\{ \| \nabla u(t) \|_{L^\infty}, \| u(t) \|_{L^\infty} \right\} < R.
\]
Then by a slight variation of the arguments in [16], we see by taking $\ell$ spatial derivatives of (2.1) for $0 \leq \ell \leq k$ that $\nabla^\ell u$ satisfies
\[
B^0 \partial_t \nabla^\ell u + B^1 \nabla_1 \nabla^\ell u = \frac{1}{\ell} \left[ B \nabla^\ell u - [\nabla^\ell, B^0](B^0)^{-1} B u + [\nabla^\ell, B^0] u \right] + [\nabla^\ell, B^0](B^0)^{-1} B^1 \nabla_1 u
\]
\[- [\nabla^\ell, B^1] \nabla_1 u - B^1 \nabla^\ell u - [\nabla^\ell, B^0](B^0)^{-1} F + \nabla^\ell F.
\]
For this proof, we consider $w_1$ and $w_2$ as fixed and for brevity, we omit these from the argument list for all maps. As a consequence of the Moser and Sobolev inequalities, all constants in the following depend on $w_1$ and $w_2$ only via their uniform $H^k(\Sigma)$ bounds over $[T_0, 0)$. 


From the standard $L^2$-energy identity, we have
\begin{equation}
\frac{1}{2} \partial_t \langle \nabla^\ell u | B^0 \nabla^\ell u \rangle = -\frac{1}{|t|} \langle \nabla^\ell u | B \nabla^\ell u \rangle + \frac{1}{2} \langle \nabla^\ell u | \div B \nabla^\ell u \rangle + \langle \nabla^\ell u | G_\ell \rangle, \quad 0 \leq \ell \leq k, \tag{2.35}
\end{equation}
where $G_\ell = F$ and
\begin{equation}
G_\ell = |t|^{-1} \left( |\nabla^\ell, B| (B^0)^{-1} B u \right) - |\nabla^\ell, B| u + |\nabla^\ell, B^0| (B^0)^{-1} B^\ell \nabla_i u
- |\nabla^\ell, B^\ell| \nabla_i u - B^2 |\nabla^\ell, \nabla_i| u - |\nabla^\ell, B^0| (B^0)^{-1} F + \nabla^\ell F, \quad 1 \leq \ell \leq k.
\end{equation}

By a slight variation of the estimates from Propositions 3.4., 3.6. and 3.7. from [16] and the expansion (2.4) with $p = 1$ and $\mu = 0$, we then have
\begin{equation}
\langle u | G_\ell \rangle \leq -C \left( \| u \|^2_{L^2(\Sigma)} + \| u \|^2_{L^2(\Sigma)} \right)\tag{2.36}
\end{equation}
and
\begin{equation}
\langle \nabla^\ell u | G_\ell \rangle \leq -\frac{1}{|t|} \left[ (\beta_1 \| u \|^2_{H^s(\Sigma)} + C(\| u \|^2_{H^s(\Sigma)} \| u \|^2_{H^{s-1}(\Sigma)} + \| u \|_{H^s(\Sigma)} \| u \|^2_{H^{s-1}(\Sigma)}) \right]
- C(\| u \|^2_{H^s(\Sigma)} \| u \|^2_{H^s(\Sigma)}) \tag{2.37}
\end{equation}
for $1 \leq \ell \leq k$ where $b$ is defined in (2.12). We see also from (2.7) with $\mu = 0$ and $p = 1$ that
\begin{equation}
| \langle \nabla^\ell u | \div B \nabla^\ell u \rangle | \leq \theta \| \nabla^\ell u \|^2_{L^2} + |t|^{-1} \beta \| \nabla^\ell u \|^2_{L^2}. \tag{2.38}
\end{equation}

Before proceeding, we define the energy norm
\begin{equation}
\| u \|_s^2 = \sum_{\ell=0}^s \langle \nabla^\ell u | B^0 \nabla^\ell u \rangle,
\end{equation}
which we note is equivalent to the standard Sobolev norm $\| u \|_{H^s(\Sigma)}$ since
\begin{equation}
\frac{1}{\sqrt{2\ell!}} \| u \|_{H^s} \leq \| u \|_s \leq \sqrt{2\ell!} \| u \|_{H^s}
\end{equation}
holds by (2.3) with $\mu = 0$. We will employ this equivalence below without comment.

Now, by (2.29) and (2.35) – (2.38), we find that
\begin{equation}
\frac{1}{2} \partial_t \| \nabla^\ell u \|^2_0 \geq \frac{1}{|t|} \left[ 2\eta - \gamma_1 \beta_1 \right] \| \nabla^\ell u \|^2_0 - \gamma_1 \beta_2 \| \nabla^\ell u \|^2_0
- \frac{1}{|t|} \left[ (\beta_1 \| u \|^2_{H^s(\Sigma)} + C(\| u \|^2_{H^s(\Sigma)} \| u \|^2_{H^{s-1}(\Sigma)} + \| u \|_{H^s(\Sigma)} \| u \|^2_{H^{s-1}(\Sigma)}) \right]
- C(\| u \|^2_{H^s(\Sigma)} \| u \|_{H^s(\Sigma)}) \tag{2.39}
\end{equation}
for $0 \leq \ell \leq k$.

\begin{equation}
\partial_t \| u \|^2_0 \geq \frac{1}{|t|} \left[ 2\eta - \gamma_1 \beta_1 \right] \| u \|^2_0 - C(\| u \|^2_{H^s(\Sigma)} \| u \|_{H^s(\Sigma)}) \| u \|^2_{H^s(\Sigma)} \tag{2.40}
\end{equation}

Summing these estimates over $\ell$ from 0 to $k$ yields
\begin{equation}
\partial_t \| u \|^2_k \geq \frac{1}{|t|} \left[ 2\eta - \gamma_1 \beta_1 \right] \| u \|^2_k - C(\| u \|^2_{H^s(\Sigma)} \| u \|_{H^s(\Sigma)}) \| u \|^2_{H^s(\Sigma)} \tag{2.41}
\end{equation}
From this inequality and the bound (2.31), we see that
\begin{equation}
\partial_t \| u \|^2_0 \geq \frac{1}{|t|} \rho_0 \| u \|^2_0 - C(\Delta) \| u \|^2_k - C(\Delta) \| \nabla^\ell F \|_{H^s(\Sigma)} \| u \|^2_k \tag{2.42}
\end{equation}
where $\rho_0$ is defined by setting $k = 0$ in
\begin{equation}
\rho_k = 2\eta - \gamma_1 \beta_1 - k(k+1)\beta_1.
\end{equation}
Due to assumption (2.30), we note that $\rho_0 > 0$ and $\rho_k > 0$. 
Applying the Sobolev interpolation estimates\(^2\)\(^8\) [1, Thm. 5.2] to (2.40), we find, with the help of the bound (2.31), that
\[ \partial_t \| u \|^2_k \geq \frac{1}{| \nu |} \left[ 2 \eta_\gamma - \gamma_1 \beta_1 - k(k+1)b_\gamma_1 - C(\Delta) (\epsilon + \| u \|_k) - \epsilon \gamma_1 \beta_0 \right] \| u \|^2_k \]
\[ - \frac{1}{| \nu |} c(\Delta, \epsilon^{-1}) \| u \|^2_0 - C(\Delta) \| u \|^2_k - C(\Delta) \| \dot{F} \|_{H^1(\Sigma)} \| u \|_k \]
for any \( \epsilon > 0 \). Adding \( 1/\rho_0 c(\Delta, \epsilon^{-1}) \) times (2.39) to the above inequality yields
\[ \partial_t \left( \| u \|^2_k + \frac{1}{\rho_0} c(\Delta, \epsilon^{-1}) \| u \|^2_0 \right) \geq \frac{1}{| \nu |} \left[ 2 \eta_\gamma - \gamma_1 \beta_1 - k(k+1)b_\gamma_1 - C(\Delta) (\epsilon + \| u \|_k) - \epsilon \gamma_1 \beta_0 \right] \| u \|^2_k \]
\[- C(\Delta, \epsilon^{-1}) \| u \|^2_0 - C(\Delta, \epsilon^{-1}) \| \dot{F} \|_{H^1(\Sigma)} \| u \|_k, \]
(2.42)
which holds for all \( t \in [T_1, t_*] \) and \( \epsilon > 0 \) small enough. Since \( \rho_k > 0 \) by assumption, it follows, by choosing \( \epsilon > 0 \) sufficiently small and by shrinking \( \Delta \in (0, R/C_{\text{Sob}}) \) if necessary, that
\[ \tilde{\rho}_k := \rho_k - C(\Delta)(\epsilon + \Delta) > 0. \]
Assuming \( \Delta \) and \( \epsilon \) are chosen so that this holds, we shall no longer write the monotonic dependencies of the constants on our particular choices of \( \Delta \) and \( \epsilon \) as we now consider these choices as fixed.

Now, observe that (2.42) can be written as
\[ \partial_t \left( \| u \|^2_k + \frac{c}{\rho_0} \| u \|^2_0 - \int_T^{t_*} \frac{\tilde{\rho}_k}{s} \| u(s) \|^2_{H^1} ds \right) \geq -C \left( \| u \|^2_k + \| \dot{F} \|_{H^1(\Sigma)} \| u \|_k \right) , \]
which implies the differential energy inequality
\[ \partial_t E(t) \geq -CE(t) - C\| \dot{F} \|_{H^1(\Sigma)} \| u \|_k \sqrt{E(t)}. \]
But this implies
\[ \partial_t \sqrt{E(t)} \geq -C \sqrt{E(t)} - C\| \dot{F} \|_{H^1(\Sigma)}, \]
(2.43)
and so we conclude, by Grönwall’s inequality, that
\[ \sqrt{E(t)} \leq e^{C T_1} \left( \sqrt{E(t_*)} + C \int_{\nu}^{t_*} \| \dot{F}(s) \|_{H^1(\Sigma)} ds \right) , \]
where here and below \( \dot{F}(s) \) will be used to denote \( \dot{F}(s, \nu_2(s)) \). Since this inequality holds for all \( t \in (T_1, t_*] \), we have established the estimate (2.32) for \( \nu = 0 \). We further note that the differential energy inequality (2.43) implies
\[ \partial_t \left( |t|^{-\nu} \sqrt{E(t)} \right) \geq \frac{\nu}{| \nu |} |t|^{-\nu} \sqrt{E(t)} - C |t|^{-\nu} E(t) - C\| \dot{F} \|_{H^1(\Sigma)} \geq -C |t|^{-\nu} \sqrt{E(t)} - C\| \dot{F} \|_{H^1(\Sigma)}, \]
which shows, via another application of Grönwall’s inequality, that (2.32) holds for arbitrary \( \nu \in \mathbb{R} \).

To complete the proof, we now turn to establishing the small data existence result for the Cauchy problem of (2.1) and (2.34) on \([T_0, t_*]\) for arbitrary \( t_* \in (T_0, 0) \). Let us start by imposing the small data condition
\[ \| u_0 \|_{H^k(\Sigma)} \leq \delta < \frac{R}{5} \]
where
\[ R = \frac{R}{C_{\text{Sob}}}, \]
(2.44)
which we note by Sobolev’s inequality (2.9) implies that \( \max \{ \| u(t_*) \|_{L^\infty(\Sigma)}, \| \nabla u(t_*) \|_{L^\infty(\Sigma)} \} \leq R/5 \). As a consequence of the standard local-in-time existence and uniqueness result for the Cauchy problem of quasilinear symmetric hyperbolic systems, see e.g. [48, Ch. 16, Prop. 1.4.], there is therefore a \( T^* \in [T_0, t_*] \) such that this Cauchy problem has a unique solution
\[ u \in C^0((T^*, t_*], H^k(\Sigma, V)) \cap C^1((T^*, t_*], H^{k-1}(\Sigma, V)) \]
(2.45)
satisfying
\[ \|u(t)\|_{H^k(\Sigma)} < \frac{R}{2} \] (2.46)
for all \( t \in (T^*, t_*) \). In fact, we can assume that \( T^* \) is the maximal time for which (2.45) and (2.46) hold for all \( t \in (T^*, t_*) \). Suppose now that \( T^* \) was strictly larger than \( t_0 \), i.e. the solution \( u \) can not be extended to the whole interval \((T_0, t_*]\) as a solution with the properties (2.45) and (2.46). As a consequence of continuity of \( u \) on \((T^*, t_*)\) and the fact that \( \|u(t)\|_{H^k(\Sigma)} \leq R/5 \), it follows that either \( \|u(t)\|_{H^k(\Sigma)} < R/3 \) for all \( t \in (T^*, t_*) \) or, there exists a first \( T_* \in (T^*, t_*) \) such that \( \|u(T_*)\|_{H^k(\Sigma)} = R/3 \). In the first case, we set \( T_* = T^* \). In both cases, we have
\[ \|u(t)\|_{H^k(\Sigma)} \leq \frac{R}{3} \] (2.47)
for all \( t \in (T_*, t_*) \).

Now, shrinking \( R \) from its preliminary choice (2.44) so that \( R \leq \Delta \) holds now, we can therefore guarantee by (2.47) that (2.31) holds for \( T_1 = T_* \), and by (2.32) with \( \nu = \lambda \), that
\[ \|u(t)\|_{H^k(\Sigma)} \leq C(\delta + |T_0|^\lambda \int_{T_0}^{t_*} \|\tilde{F}(s)\|_{H^k(\Sigma)} ds) \]
for all \( t \in (T_*, t_*) \). Assuming now that
\[ \int_{T_0}^{t_*} \|\tilde{F}(s)\|_{H^k(\Sigma)} ds \leq \delta \]
and tightening the condition for \( \delta \) so that \( \delta \leq \frac{R}{\max(1+|T_0|^\lambda)} \), it follows that \( \|u(t)\|_{H^k(\Sigma)} \leq \frac{R}{4} \) for all \( t \in (T_*, t_*) \). By the definition of \( T_* \) and continuity, we conclude that \( T_* = T^* \) and that therefore \( \|u(t)\|_{H^k(\Sigma)} \leq \frac{R}{4} \) for all \( t \in (T^*, t_*) \). This implies that
\[ \max\{\|u(t)\|_{L^\infty(\Sigma)}, \|u(t)\|_{L^\infty(\Sigma)}\} \leq \frac{R}{4}, \quad T^* < t < t_* \]
and so we conclude from the maximality of \( T^* \) and the continuity principle for symmetric hyperbolic equations [48, Ch. 16, Prop. 1.5] that we must in fact have that \( T^* = T_0 \) and the solution extends continuously to the closed time interval \([T_0, t_*]\), which completes the proof. \( \square \)

2.4.3. Singular initial value problem existence and uniqueness for canonical Fuchsian systems. We now turn to establishing the existence and uniqueness of solutions to the singular initial value problem for canonical Fuchsian systems. Existence for the singular initial value problem is obtained from the solutions to the Cauchy problem on the intervals \([T_0, t_*]\) from Proposition 2.6 by letting \( t_* \not\to 0 \).

**Proposition 2.7.** Suppose \( k \in \mathbb{Z}_{>n/2+3} \), Eq. (2.1) is a canonical symmetric hyperbolic Fuchsian system for the constants \( T_0 < 0 \), \( R > 0 \), \( \gamma_1 > 0 \), \( \gamma_2 > 0 \), \( q > 0 \), \( \lambda > 0 \), \( \theta \geq 0 \) and \( \beta \geq 0 \), open bounded sets \( Z_1 \subset Z_1 \) and \( Z_2 \subset Z_2 \) with \( \pi(Z_1) = \pi(Z_2) = \Sigma \), and maps \( B^0, \tilde{B}, B, \tilde{B}, \tilde{F}, F_0, F_1, B_0, B_1, B_2 \) as in Definition 2.4, and \( w_1 \in C^0_b([T_0, 0), H^k(\Sigma, Z_1)] \cap C^1([T_0, 0), H^{k-1}(\Sigma, Z_1)] \) and \( w_2 \in C^0([T_0, 0), H^k(\Sigma, Z_2)] \) satisfy
\[ \sup_{t_0 < t < 0} \max \left\{ \|\nabla w_1(t)\|_{H^{k-1}(\Sigma)}, \|t^{1-\eta} \partial_t w_1(t)\|_{H^{k-1}(\Sigma)} \right\} < \frac{R}{C_{Sob}} \]
and
\[ w_a([T_0, 0] \times \Sigma) \subset Z_a, \quad a = 1, 2. \]
In addition, suppose there exists a constant \( \eta > 0 \) such that
\[ B(t, z_1, v) \leq -\eta B^0(t, z_1, v) \] (2.48)
for all \((t, z_1, v) \in (T_0, 0) \times Z_1 \times B_R(V) \) and that (2.30) holds, where \( b \) is defined in (2.12). Then there are constants \( C, \delta > 0 \) such that if
\[ \int_{T_0}^{0} \|\tilde{F}(s, w_2(s))\|_{H^k(\Sigma)} ds < \delta, \] (2.49)
there exists a classical solution
\[ u \in C^0_b([T_0, 0), H^k(\Sigma, V)] \cap C^1([T_0, 0), H^{k-1}(\Sigma, V)] \subset C^1([T_0, 0) \times \Sigma, V) \] (2.50)
of (2.1) that is bounded by
\[
\max\left\{ \|u\|_{L^\infty([T_0, 0] \times \Sigma)}, \|\nabla u\|_{L^\infty([T_0, 0] \times \Sigma)} \right\} < R
\] (2.51)
and
\[
\|t\|^{-\lambda} u(t)\|_{H^k(\Sigma)} \leq C \int_0^t \|\tilde{F}(s, w_2(s))\|_{H^k(\Sigma)} ds
\] (2.52)
for all \( t \in [T_0, 0) \). Moreover, the solution \( u \) is the unique solution within the class \( C^1([T_0, 0] \times \Sigma, V) \) satisfying the bound (2.51).

Proof.

Existence: Let \( \{t_n\}_{n=1}^\infty \subset [T_0, 0) \) be a monotonically increasing sequence satisfying \( \lim_{n \to \infty} t_n = 0 \). Then by Proposition 2.6, we know, for \( \delta > 0 \) small enough, that there exist solutions
\[
u_n \in C^0([T_0, t_n], H^k(\Sigma, V)) \cap C^1([T_0, t_n], H^{k-1}(\Sigma, V))
\]
of (2.1) satisfying the initial condition
\[
u_n(t_n) = 0.
\] (2.53)
Moreover, there exist constants \( \Delta, C \) that satisfy \( \Delta < R/C_{\text{Sob}} \) and are both independent of \( n \) such that the solutions \( \nu_n \) are bounded by
\[
\sup_{t \in [T_0, t_n]} \|\nu_n(t)\|_{H^k(\Sigma)} \leq \Delta
\] (2.54)
and
\[
\|t\|^{-\lambda} \nu_n(t)\|_{H^k(\Sigma)} \leq C \int_0^{t_n} \|\tilde{F}(s)\|_{H^k(\Sigma)} ds
\] (2.55)
for all \( t \in [T_0, t_n] \), where here, and below \( \tilde{F}(s) \) is used to denote \( \tilde{F}(s, w_2(s)) \). Throughout this proof, \( w_1 \) and \( w_2 \) are taken to be fixed and for brevity, we omit these from the argument list for all maps. As a consequence of the Moser and Sobolev inequalities, all constants in the following depend on \( w_1 \) and \( w_2 \) only via their uniform \( H^k(\Sigma) \) bounds over \([T_0, 0)\).

Now, using the evolution equation (2.1), which \( \nu_n \) satisfies, we can express \( \partial_t \nu_n \) in terms of \( \nu_n \) and its spatial derivative as follows
\[
\partial_t \nu_n = B^0(t, u_n)^{-1} (-tB'(t, u_n)\nabla u_n + B(t, u_n)u_n + tF(t, u_n)).
\] (2.56)
With the help of the calculus inequalities, in particular the Sobolev, Product and Moser calculus inequalities, see [48, Ch. 13, §2 & 3], it is then not difficult to verify from the formula (2.56) and the coefficient assumptions, see Definition (2.4), that (2.49) and (2.54) imply the uniform bound
\[
\int_{t_n}^0 \|\partial_t \nu_n(s)\|_{H^{k-1}(\Sigma)} ds \lesssim 1, \quad n \geq 1.
\] (2.57)
To proceed, we define, for \( s_1, s_2 \in \mathbb{Z}_{\geq 0}, \rho \in \mathbb{R} \) and \( 1 \leq p \leq \infty \), the spaces \( X^{s_1, s_2; p} \) as the closure of \( C^\infty([T_0, 0] \times \Sigma, V) \) in the norm
\[
\|v\|_{X^{s_1, s_2; p}} = \sum_{0 \leq i \leq s_1} \left\|\|t\|^{-\beta_i} \partial_t^i v(t)\|_{H^{s_2-1}(\Sigma)}\right\|_{L^p([T_0, 0])}.
\]
The spaces \( X^{s_1, s_2; p} \) are reflexive for \( 1 < p < \infty \), and we know from the Rellich-Kondrachov theorem [1, Thm. 6.3] that the inclusion\(^{2,9}\)
\[
X^{1, 0, 1; \tilde{\rho}} \subset X^{0, k-1; q}
\] (2.58)
is compact for \( 1 \leq q < \infty \) and \( \tilde{\rho} > \rho \).

Next, we extend the solutions \( \nu_n \) to the whole time interval \([T_0, 0] \) by defining
\[
\tilde{u}_n(t, x) = H_n(t)u_n(t, x)
\] (2.59)
where

\[ H_n(t) = \begin{cases} 
1 & \text{if } t \leq t_n \\
0 & \text{if } t > t_n 
\end{cases} \]

is the unit step function with jump at \( t_n \). Then, differentiating \( \tilde{u}_n \), we see, in the sense of distributions, that

\[ \partial_t \tilde{u}_n(t, x) = u_n(t, x) \partial_t H_n(t) + H_n(t) \partial_t u_n(t, x) = u_n(t, x) \delta_{t_n}(t) + H_n(t) \partial_t u_n(t, x), \]

which by (2.53) reduces to

\[ \partial_t \tilde{u}_n(t, x) = H_n(t) \partial_t u_n(t, x). \]  
(2.60)

By (2.54), (2.57), (2.59), and (2.60), we see that the sequence \( \tilde{u}_n \) is bounded by

\[ \| \tilde{u}_n \|_{X_{0,k,p}^{n}} \leq \| 1 \|_{L^p([T_0, 0])} \Delta \quad \text{and} \quad \| \tilde{u}_n \|_{X_{0,k,1}^{n}} \leq C \]  
(2.61)

for \( 1 < p < \infty \) and \( n \geq 1 \) where the constant \( C \) is independent of \( n \). Since \( X_{0,k,p} \) is reflexive for \( 1 < p < \infty \), we conclude from the first inequality in (2.61) and the sequential Banach–Alaoglu theorem\(^{2,10}\) the existence of a subsequence of \( \tilde{u}_n \), which we also denote by \( \tilde{u}_n \), that converges weakly to an element \( u \in X_{0,k,p} \) satisfying the estimate

\[ \| u \|_{X_{0,k,p}^{n}} \leq \| 1 \|_{L^p([T_0, 0])} \Delta \]  
for \( 1 < p < \infty \). Using the fact that \( \sup_{t_0 \leq t \leq t_1} |f(t)| = \lim_{p \to \infty} \left( \frac{1}{0} \int_{T_0}^0 |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \) holds for measurable functions \( f(t) \) on \([T_0, 0)\) satisfying \( |f|^{p_*} \in L^1([T_0, 0)) \) for some \( p_* \in \mathbb{R} \), we deduce from the above inequality that

\[ \| u \|_{X_{0,k,p}^{n}} \leq \Delta. \]

Moreover, by the compactness of the inclusion (2.58), we can, by selecting a further subsequence of \( \tilde{u}_n \) if necessary, assume that the sequence \( \tilde{u}_n \) converges strongly in \( X_{0,k-1,q}^{n} \) to \( u \) for any \( \epsilon > 0 \) and \( 1 \leq q < \infty \) due to the second inequality in (2.61). Since \( \Delta < R/C_{\text{Sob}} \), we note from the above bound and Sobolev’s inequality (2.9) that

\[ \max \left\{ \| u \|_{L^\infty([T_0, 0) \times \Sigma)}, \| \nabla u \|_{L^\infty([T_0, 0) \times \Sigma)} \right\} < R \quad \text{and} \quad \sup_{t \in [T_0, 0)} \| u(t) \|_{H^k(\Sigma)} < \frac{R}{C_{\text{Sob}}}. \]  
(2.62)

Since the sequence \( t_n \) is monotonically increasing and converges to 0 as \( n \to \infty \), it is clear from the definition (2.59) that for any test function \( \psi \in C_0^\infty([T_0, 0) \times \Sigma, V) \) there exists an \( N = N(\psi) \in \mathbb{Z}_{>0} \) such that

\[ \partial_t \bar{\psi} \bar{u}_n = \partial_t \psi u_n \quad \text{and} \quad \psi \bar{u}_n = \psi u_n, \quad n \geq N, \]  
(2.63)

and so we conclude from the weak convergence \( \bar{u}_n \rightharpoonup u \) in \( X_{0,k,p}^n \) that

\[ \langle \partial_t \psi | u \rangle = \lim_{n \to \infty} \langle \partial_t \psi | u_n \rangle = -\lim_{n \to \infty} \langle \psi | \partial_t u_n \rangle, \]

where \( \langle \cdot | \cdot \rangle \) is the \( L^2 \) inner-product on \([T_0, 0) \times \Sigma \). Using (2.56) and (2.63), we can write this as

\[ \langle \partial_t \psi | u \rangle = -\lim_{n \to \infty} \left( \psi \left| B^0(t, \bar{u}_n)^{-1} \left( -B^i(t, \bar{u}_n) \nabla_i \bar{u}_n + t^{-1} B(t, \bar{u}_n) \bar{u}_n + F(t, \bar{u}_n) \right) \right. \right). \]

It is not difficult to verify from the strong convergence \( \bar{u} \rightharpoonup u \) in \( X_{0,k-1,q} \) that, in the limit \( n \to \infty \), we can replace \( \bar{u}_n \) with \( u \) in the above expression to get

\[ \langle \partial_t \psi | u \rangle = -\left( \psi \left| B^0(t, u)^{-1} \left( -B^i(t, u) \nabla_i u + t^{-1} B(t, u) u + F(t, u) \right) \right. \right). \]

Since the test function \( \psi \) was chosen arbitrarily, we conclude that \( u \) defines a weak solution of the Fuchsian system (2.1) where

\[ \partial_t u = B^0(t, u)^{-1} \left( -B^i(t, u) \nabla_i u + t^{-1} B(t, u) u + F(t, u) \right). \]

By another application of the calculus inequalities (i.e. Sobolev, Product and Moser calculus inequalities, see [48, Ch. 13, §2 & 3]), we get, for each \( T_1 \in (T_0, 0) \), that \( \sup_{t \in [T_0, T_1)} \| \partial_t u(t) \|_{H^{k-1}(\Sigma)} \lesssim 1 \), and hence, that

\[ u \in L^\infty([T_0, 0), H^k(\Sigma)) \cap W^{1,\infty}_{\text{loc}}([T_0, 0), H^{k-1}(\Sigma)). \]  
(2.64)

\(^{2,10}\) See Theorem 3.17 in [46].
By standard local-in-time existence and uniqueness theory for symmetric hyperbolic equations, e.g. [48, Ch. 16, Prop. 1.4.1], it follows that the solution \( u \) enjoys the improved regularity

\[
\begin{align*}
   u & \in C^0([T_0, 0), H^k(\Sigma, V)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, V)) \subset C^1([T_0, 0) \times \Sigma, V),
\end{align*}
\]

and consequently, \( u \) defines a classical solution of \((2.1)\) on \([T_0, 0) \times \Sigma\).

To finish the existence part of the proof, let \( \eta \in C^\infty_0([T_0, 0)) \) be any non-negative test function. Then we observe from \((2.55)\) and \((2.63)\) that there exists an \( N = N(\eta) \in \mathbb{Z}_{>0} \) such that

\[
\| \eta(t)|t|^{-\lambda} \tilde{u}_n(t) \|_{H^k(\Sigma)} \lesssim \eta(t) \int_0^t \| \tilde{F}(s) \|_{H^k(\Sigma)} ds \leq \delta \eta(t)
\]

for all \( t \in [T_0, 0), n \geq N \) and \( 1 < p < \infty \). But this implies

\[
\| \eta \tilde{u}_n \|_{X^p_{0,k,p}} \lesssim \int_{T_0}^0 \eta(t)^p \left( \int_t^0 \| \tilde{F}(s) \|_{H^k(\Sigma)} ds \right)^p dt, \quad n \geq N,
\]

and so, we conclude via the weak convergence \( \eta \tilde{u}_n \rightharpoonup \eta u \in X^p_{0,k,p} \) that

\[
\| \eta u \|_{X^p_{0,k,p}} = \int_{T_0}^0 \eta(t)^p \| |t|^{-\lambda} u(t) \|_{H^k(\Sigma)} dt \lesssim \int_{T_0}^0 \eta(t)^p \left( \int_t^0 \| \tilde{F}(s) \|_{H^k(\Sigma)} ds \right)^p dt, \quad 1 < p < \infty.
\]

Letting \( p \searrow 1 \) in the above expression then gives

\[
\int_{T_0}^0 \eta(t) |||t|^{-\lambda} u(t)||_{H^k(\Sigma)} dt \lesssim \int_{T_0}^0 \eta(t) \int_t^0 \| \tilde{F}(s) \|_{H^k(\Sigma)} ds dt
\]

by the Dominated Convergence Theorem. However, since \( \eta \) was an arbitrarily chosen non-negative test function, we must have

\[
\| |t|^{-\lambda} u(t) \|_{H^k(\Sigma)} dt \leq C \int_{T_0}^0 \| \tilde{F}(s) \|_{H^k(\Sigma)} ds, \quad T_0 < t < 0,
\]

which completes the existence part of the proof.

Uniqueness: To establish uniqueness, suppose that \( \hat{u} \in C^1([T_0, 0) \times \Sigma, V) \) is another classical solution of \((2.1)\) that satisfies

\[
\max\left\{ \| \hat{u} \|_{L^\infty([T_0, 0) \times \Sigma)}, \| \nabla \hat{u} \|_{L^\infty([T_0, 0) \times \Sigma)} \right\} < R.
\]

Letting

\[ w = \hat{u} - u \]

denote the difference between the two solutions, a short calculation shows that \( w \) satisfies

\[
B^0(t, \hat{u}) \partial_t w + B^i(t, \hat{u}) \nabla_i w = \frac{1}{t} B(t, \hat{u}) w + G,
\]

where

\[
G = (B^0(t, u) - B^0(t, \hat{u})) \partial_t u + (B^i(t, u) - B^i(t, \hat{u})) \nabla_i u + \frac{1}{t} \left( B(t, \hat{u}) - B(t, u) \right) u + F(t, \hat{u}) - F(t, u)
\]

\[
= (B^0(t, u) - B^0(t, \hat{u})) B^0(t, u)^{-1} \left( -B^i(t, u) \nabla_i u + \frac{1}{t} B(t, u) u + F(t, u) \right)
\]

\[ + \left( B^i(t, u) - B^i(t, \hat{u}) \right) \nabla_i u + \frac{1}{t} \left( B(t, \hat{u}) - B(t, u) \right) u + F(t, \hat{u}) - F(t, u). \]

Observe here that \( G \) vanishes if \( w = 0 \), i.e., \( \hat{u} = u \).

Next, we derive from the evolution equation \((2.67)\) the \( L^2 \)-energy identity

\[
\frac{1}{2} \partial_t \| w \|^2 = -\frac{1}{|t|} \langle w | B(t, \hat{u}) w \rangle + \frac{1}{2} \langle w | \text{div} B(t, \hat{u}, \nabla \hat{u}) w \rangle + \langle w | G \rangle
\]

where

\[
\| w \|^2 = \langle w | B^0(t, \hat{u}) w \rangle
\]
is the energy norm. But we have, from the bounds (2.66), and the fact that coefficients $B^{0}$ and $B^{1}$ satisfy the assumptions of Definition 2.4, see in particular (2.2) and (2.7) with $\mu = 0$ and $p = 1$, that

$$\frac{1}{\gamma_1} \|w\|_{L^2(\Sigma)}^2 \leq \|w\|_{L^2(\Sigma)}^2 \leq \gamma_2 \|w\|_{L^2(\Sigma)}^2,$$

(2.69)
and

$$\langle w|B(t, \dot{u})w\rangle \leq -\eta \|w\|^2.$$

With the help of these inequalities and the energy identity (2.68), we obtain the energy estimate

$$\partial_t \|w\|^2 \geq \frac{2\eta - \gamma_1 \beta}{|t|} \|w\|^2 - \theta \|w\|^2 + \langle w|G\rangle.$$

(2.70)

Next, from the bounds (2.62), (2.65) and (2.66), it is not difficult to verify from the assumptions of Definition 2.4 on the coefficients $B^{0}$, $B^{i}$, $B$ and $F$, and H{"o}lder’s and Sobolev’s inequalities that we can estimate $G$ by

$$\|G\|_{L^2} \lesssim \frac{1}{|t|^{1-\gamma_2}} \|w\|_{L^2}$$

uniformly for all $t \in [T_0, 0)$. Observe that it is crucial for this argument here to exploit the positive decay for $u$ implied by (2.65) in the limit $t \nearrow 0$ and the fact that $\lambda$ is assumed to be positive. Using this, we get from (2.70) and the Cauchy-Schwartz inequality that

$$\partial_t \|w\|^2 \geq \frac{2\eta - \gamma_1 \beta}{|t|} \|w\|^2 - C \|w\|^2$$

for some constant $C > 0$. From this differential inequality, we then conclude, via an application of Gr{"o}nwall’s inequality, that

$$\|w(t)\|_{L^2(\Sigma)} \lesssim \left(\frac{|t_s|}{|t|}\right)^{2\eta - \gamma_1 \beta} \|w(t_s)\|_{L^2(\Sigma)}, \quad T_0 \leq t < t_*.$$

But by H{"o}lder’s inequality we have that $\|w(t_*)\|_{L^2(\Sigma)} \lesssim \|w(t_*)\|_{L^\infty(\Sigma)}$, and so we see from (2.57) and (2.69) that

$$\lim_{t_* \nearrow 0} \left(\frac{|t_s|}{|t|}\right)^{2\eta - \gamma_1 \beta} \|w(t_*)\|_{L^2(\Sigma)} = 0$$

since $2\eta - \gamma_1 \beta > 0$ by assumption. Thus we conclude that $w = 0$ in $[T_0, 0) \times \Sigma$, which establishes uniqueness. \hfill \Box

2.4.4. Completion of the proof of Theorem 2.2. We are now ready to complete the proof of Theorem 2.2. By Lemma 2.5, we know that the symmetric hyperbolic Fuchsian system (2.1), which satisfies the hypotheses of Theorem 2.2, can be transformed into a canonical symmetric hyperbolic Fuchsian system with $\lambda = \alpha > 0$. The existence and uniqueness of solutions to the singular initial value problem for this canonical Fuchsian system is guaranteed by Proposition 2.7. Transforming back from the canonical Fuchsian system to the starting one, see Lemma 2.5, it is then not difficult to verify from the properties of the canonical solution, as determined by Proposition 2.7, that the corresponding solution to the singular initial value problem for the original Fuchsian system satisfies all the properties as stated in Theorem 2.2.

3. Euler equations on Kasner backgrounds

In this section, we state and prove a precise version of Theorem 1.1. This precise version is separated into two parts. The first part concerns the existence and uniqueness of solution to the singular initial value problem for the Euler equations (i.e. the forward problem) and statement of the result is given below in Theorem 3.1. The second part, which is presented below in Theorem 3.6, expresses the nonlinear stability of solutions from Theorem 3.1 under perturbations of the initial data at $t = T_0$ sufficiently close to 0 (i.e. the backward problem).
In the following, we find it advantageous to change from the coordinates \((\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})\) in (1.2) to a new set of coordinates \((t, x, y, z)\) defined by
\[
\tilde{t} = \frac{4}{K^2 + 3} (t - \frac{K^2 + 3}{4}), \quad \tilde{x} = \left(\frac{K^2 + 3}{4}\right)^{\frac{1}{K^2 + 3}} x, \quad \tilde{y} = \left(\frac{K^2 + 3}{4}\right)^{\frac{1}{K^2 + 3}} y, \quad \tilde{z} = \left(\frac{K^2 + 3}{4}\right)^{\frac{1}{K^2 + 3}} z.
\]

(3.1)

A short calculation then shows that, in terms of these new coordinates, the Kasner metric (1.2) takes the form
\[
g = (-t)^{\frac{2K}{K^2 + 3}} (dt \otimes dt + dx \otimes dx) + (-t)^{1-K} dy \otimes dy + (-t)^{1+K} dz \otimes dz.
\]

(3.2)

As a consequence of (1.3), the Kasner exponents \(p_1, p_2\) and \(p_3\) can be expressed in terms of the single parameter \(K \in \mathbb{R}\) by
\[
p_1 = (K^2 - 1)/(K^2 + 3), \quad p_2 = 2(1 - K)/(K^2 + 3), \quad p_3 = 2(1 + K)/(K^2 + 3),
\]

(3.3)

which is often referred to as the asymptotic velocity. Notice that, in agreement with the conventions in Section 2, the time \(t\) is negative now. We will always assume that \(x, y\) and \(z\) are periodic over the domain \([0, 2\pi]\), and rather than using \(c_s^2\) to parameterise the square of the sound speed, we will instead use the constant
\[
\gamma = c_s^2 + 1
\]

(3.4)
as is common in the mathematical cosmology literature. With this definition, the equation of state (1.1) then reads
\[
P = (\gamma - 1)\rho.
\]

(3.5)

We will also interpret the fluid vector field \(V = V^\alpha \partial_\alpha\) as a time-dependent section of the trivial vector bundle
\[
\mathbb{V} = T^3 \times \mathbb{R}^4
\]
that sits over the spatial manifold \(\Sigma = \mathbb{T}^3\) by using the representation \(V = V^\alpha \partial_\alpha\) in terms of the coordinate frame \(\{\partial_t, \partial_x, \partial_y, \partial_z\}\) to identify \(V = V^\alpha \partial_\alpha\) with its components \(V^0, V^1, V^2, V^3\). Furthermore, on the spatial manifold \(\Sigma = \mathbb{T}^3\), we employ the flat metric \(dx \otimes dx + dy \otimes dy + dz \otimes dz\), which allows us to identify the associated covariant derivative, see Section 2, with the spatial partial derivatives \(\partial_x, \partial_y, \partial_z\), while on \(\mathbb{V}\), we use the metric \(h\) defined by
\[
b(V_1, V_2) = \delta_{\alpha\beta} V_1^\alpha V_2^\beta.
\]

(3.6)

3.1. Singular initial value problem.

3.1.1. Statement of the theorem. We are now in the position to formulate our first main result concerning relativistic fluids on Kasner spacetimes. This theorem establishes the existence of solutions to the relativistic Euler equations near the Kasner singularities with the asymptotics stated in Theorem 1.1.

**Theorem 3.1** (Singular initial value problem for fluids on Kasner spacetimes). **Suppose**
\[
K \in [0, 1), \quad 2 > \gamma > \frac{K^2 + 2K + 5}{K^2 + 3},
\]

(3.7)

and set
\[
\Gamma_1 = \frac{3\gamma - 2 - K^2(2 - \gamma)}{4}, \quad \Gamma_2 = \frac{3\gamma - 5 + 2K + K^2(\gamma - 1)}{4}, \quad \Gamma_3 = \frac{3\gamma - 5 - 2K + K^2(\gamma - 1)}{4}.
\]

(3.8)

Furthermore, suppose \(k \in \mathbb{Z}_{\geq 3}, \ell \in \mathbb{Z}_{\geq 1}\) satisfies \(\ell > \Gamma_1/q\) for \(q = \min\{1 - \Gamma_1, 2\Gamma_3\}\), and set \(\epsilon = \min\{\Gamma_1 + \Gamma_3, 1, (1 - \Gamma_1)/2\Gamma_3\} - \Gamma_1\).

Then for each choice of asymptotic data \(v_* = (v_*^0, \ldots, v_*^3)^T \in H^{k+\ell}(\Sigma)\) where \(v_*^0 > 0\), there exists, for \(T_0 < 0\) close enough to zero, a solution
\[
V \in C^2_0([T_0, 0), H^k(\Sigma)) \cap C^1([T_0, 0), H^{k-1}(\Sigma)) \subset C^1([T_0, 0) \times \Sigma, \mathbb{V})
\]

(3.9)
of the Euler equations (1.4) with equation of state (3.5) satisfying the decay estimate
\[
\begin{align*}
&\|(-t)^{-\Gamma_1} V^0(t) - v_*^0\|_{H^k(\Sigma)} + \|(-t)^{-2\Gamma_1} V^1(t) - v_*^1\|_{H^k(\Sigma)} + \|(-t)^{-2\Gamma_2} V^2(t) - v_*^2\|_{H^k(\Sigma)} + \|(-t)^{-2\Gamma_3} V^3(t) - v_*^3\|_{H^k(\Sigma)} 
\lesssim |t|^{\epsilon}
\end{align*}
\]

(3.10)
for all \(t \in [T_0, 0)\).
Furthermore, if \( \tilde{V} \in C^1(T_0, 0) \times \Sigma, V \) is any classical solution of the Euler equations (1.4) with equation of state (3.5) for which
\[
\sup_{t \in [T_0, 0)} \max \left\{ \| |t|^{-\mu}T^{-1}(V(t) - \tilde{V}(t))\|_{L^\infty(\Sigma)}, \| |t|^{-\mu}T^{-1}(\nabla V(t) - \nabla \tilde{V}(t))\|_{L^\infty(\Sigma)} \right\} \lesssim 1 \tag{3.11}
\]
for some \( \mu > \Gamma_1 \), where
\[
T := \text{diag}(|t|^{\Gamma_1}, |t|^{\Gamma_2}, |t|^{\Gamma_3}),
\]
then \( V = \tilde{V} \) in \([T_0, 0) \times \Sigma\).

The proof of this theorem is presented in Section 3.1.2; but before we proceed with the proof, we first make a number of remarks regarding this theorem. We begin with noting that the existence result contained in Theorem 3.1 does not rely on any symmetry or isotropy assumptions. This should be contrasted with previous results from the literature on relativistic fluids near Kasner singularities. For example, the existence results for the Einstein-Euler equations from [14] assumes \( T^2 \)-symmetry while those from [7,49,50] rely on an isotropy assumption.

We further observe that Theorem 3.1, by itself, does not characterise the asymptotics of \emph{generic} solutions. After all, this theorem is concerned with a family of solutions with the particular leading-order behaviour given by (3.10), albeit with the full expected number of free data functions. With that said, a significant result of this paper is that we are able to show in Theorem 3.6 that the solutions from Theorem 3.1 are stable under nonlinear perturbations.

Next, we claim that the condition \( K \in [0, 1) \) of Theorem 3.1 is not much of a restriction at all. To see why, given an arbitrary \( K \in \mathbb{R} \setminus \{\pm 1\} \), we can employ a diffeomorphism to map the Kasner metric (3.2) to another Kasner metric (3.2) with \( K = \tilde{K} \) where \( \tilde{K} \in [0, 1] \) as follows. First, if \( K < 0 \), we can use the diffeomorphism \((t, x, y, z) \mapsto (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})\) to map the Kasner metric (3.2) to another Kasner metric of the same form with \( \tilde{K} = -K > 0 \). Therefore, without loss of generality, we can restrict our attention to the parameter values \( K \geq 0 \). We further note that \( K = 1 \) is excluded by the second inequality in (3.7), and therefore, we can assume that \( K \neq 1 \). Now, if \( 1 < K \leq 3 \), we can apply the transformation (3.1) to bring the Kasner metric (3.2) into the form (1.2) with exponents given by (3.3). Then by applying the transformation \((t, x, y, z) \mapsto (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})\) and following it with the inverse transformation (3.1), we obtain a Kasner metric of the form (3.2) with the new value \( \tilde{K} = (3 - K)/(1 + K) \in (0, 1) \) up to some irrelevant additional constant factors for the spatial components. On the other hand if \( K > 3 \), we can first apply to the Kasner metric (3.2) the transformation (3.1). We then follow it with the transformation \((\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) \mapsto (\tilde{t}, \tilde{y}, \tilde{x}, \tilde{z})\) and finish with the inverse transformation of (3.1) to yield a new Kasner metric (3.2) with parameter value \( \tilde{K} = (K - 3)/(1 + K) \in (0, 1) \) for the Kasner metric in (3.2).

For the subsequent analysis, it is important to note that the assumption (3.3) from the statement of Theorem 3.1 implies that the Kasner exponents satisfy \( p_3 \geq p_2 \geq p_1 \) while it is clear from (3.3) and the assumption \( K \in [0, 1) \) that \( p_1 < 0 \). Due to (3.3), (3.4) and (3.8), we can express the constants \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) as
\[
\Gamma_1 = \frac{\gamma^2 - p_3}{1 - p_3}, \quad \Gamma_2 = \frac{\gamma^2 - p_2}{1 - p_2}, \quad \text{and} \quad \Gamma_3 = \frac{\gamma^2 - p_1}{1 - p_1},
\]
respectively. From these expressions and the inequalities \( p_3 \geq p_2 \geq p_1 \) and \( p_1 < 0 \), it is clear that the constants \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are ordered by \( \Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \). Now, while the restriction \( \gamma < 2 \) in (3.7) implies that \( \Gamma_1 < 1 \), we note that \( \Gamma_3 > 0 \) is a consequence of the lower bound in (3.7). Combining these observations, we deduce that the constants \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) satisfy

\[
1 > \Gamma_1 \geq \Gamma_2 \geq \Gamma_3 > 0
\]
as a consequence of (3.7).

The asymptotic data \( \nu = (\nu^0, ..., \nu^3)^\top \) in Theorem 3.1 determines via the decay estimate (3.10) the leading order behavior of the solution given by \( V_0 = (\nu^0(-t)^{\Gamma_1}, \nu^1(-t)^{2\Gamma_1}, \nu^2(-t)^{2\Gamma_1}, \nu^3(-t)^{2\Gamma_1})^\top \). As we shall see below, the proof of Theorem 3.1 is more complicated than simply applying Theorem 2.2 directly to the equation satisfied by the remainder \( V - V_0 \). The main reason for this is that the leading order term is, in general, not accurate enough and it must be improved in an iterative process until it is accurate enough to apply Theorem 2.2 to the remainder. It is interesting that the more the exponents \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) in (3.8) differ, the more steps are needed in the iterative process to improve the leading order term. As a consequence, the analysis becomes simpler as the the background spacetime becomes
more isotropic. This observation is particularly relevant in the context of Remark 3.2 below. Moreover, since we lose an order of differentiability in each step of the iterative process, the differentiability of the solutions is tied to the isotropy of the background Kasner spacetime with decreasing isotropy leading to an increase in the differentiability requirements as measured by the integer $\ell$ in Theorem 3.1.

Additionally, we note that in the coordinates of the metric (3.2), the restriction that $u^0$ is positive means that the fluid is time-oriented towards the big bang time $t = 0$. In cosmology, where the time orientation is usually chosen such that the big bang singularity represents the past, one mostly cares about fluids with the opposite time orientation. This is easy reconciled by noting that (1.4) is invariant under the transformation $V^0 \mapsto -V^0$. In particular, every past directed solution asserted by Theorem 3.1 can be transformed into a future directed one and vice versa, and so the choice of orientation is immaterial.

As another remark, we emphasize that $T_0$ is in general expected to depend on the choice of asymptotic data given that some fluid solutions are expected to break down earlier than others in virtue of shock formation etc.

We close the discussion of Theorem 3.1 with a remark about the quantities $\epsilon$ and $\ell$ that appear in its statement. First, we note that $\epsilon$ is positive as a consequence of the assumption $\ell > \Gamma_1/q$ from Theorem 3.1 and the inequality (3.14) satisfied by the constants $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. We further observe that the value of $\epsilon$ in Theorem 3.1 depends on the choice of $\ell$, which determines the the order of differentiability of the solutions. As the following calculations show, the largest possible value for $\epsilon$ is $\epsilon_0 = \min\{\Gamma_3, 1 - \Gamma_1\}$ and it can be achieved choosing $\ell \geq \Gamma_1/q + 1$:

**Case $\Gamma_1 + 2\Gamma_3 \leq 1$:** In this case, we have $q = 2\Gamma_3$. Since $\Gamma_1 + \Gamma_3 < \Gamma_1 + 2\Gamma_3 \leq 1$, we get $\epsilon = \min\{\Gamma_1 + \Gamma_3, 2\Gamma_3\} - \Gamma_1$. The largest value $\epsilon_0 = \Gamma_3$ is therefore obtained for $\ell$ satisfying

$$\ell \geq \frac{\Gamma_1 + \Gamma_3}{2\Gamma_3} = \frac{\Gamma_1 + 1}{2}.$$

**Case $\Gamma_1 + \Gamma_3 \leq 1 \leq \Gamma_1 + 2\Gamma_3$:** In this case, we have $q = 1 - \Gamma_1$ and $\epsilon = \min\{\Gamma_1 + \Gamma_3, \ell(1 - \Gamma_1)\} - \Gamma_1$. The largest value $\epsilon_0 = \Gamma_3$ is therefore obtained for $\ell$ satisfying

$$\ell \geq \frac{\Gamma_1 + \Gamma_3}{1 - \Gamma_1} = \frac{\Gamma_1}{q} + \frac{\Gamma_3}{1 - \Gamma_1}.$$

Hence, the condition $\ell \geq \frac{\Gamma_1}{q} + 1$ is sufficient.

**Case $1 \leq \Gamma_1 + \Gamma_3$:** In this case, we have $q = 1 - \Gamma_1$ and $\epsilon = \min\{1, \ell(1 - \Gamma_1)\} - \Gamma_1$. The largest value $\epsilon_0 = 1 - \Gamma_1$ is therefore obtained for $\ell$ satisfying

$$\ell \geq \frac{1}{1 - \Gamma_1} = \frac{\Gamma_1}{q} + 1.$$

**Remark 3.2.** For future applications, we make here the important observation that Theorem 3.1 and Theorem 3.6 continue to apply to the larger class of *Kasner-scalar field spacetimes* \[2, 44, 45\] within which (1.2) – (1.3) is a special case. As we have noted already in the introduction, the Kasner-scalar field spacetimes provide important singularity models for solutions to the Einstein equations coupled to matter fields some of which are known to have stable big-bang singularities \[44, 45\].

By definition, the Kasner-scalar field spacetimes are spatially homogeneous solutions of the Einstein-scalar field equations (minimally coupled, zero potential) where the scalar field is $\phi = A \log t + B$ for constants $A \in [-\sqrt{2/3}, \sqrt{2/3}]$ and $B \in \mathbb{R}$, and the metric is of the same form as (1.2), but with exponents

$$\sum_{i=1}^{3} p_i = 1, \quad \sum_{i=1}^{3} p_i^2 = 1 - A^2,$$

instead of (1.3). The parameter $B$ is non-dynamical and can be assumed to be zero without loss of generality. In the case $A = 0$, we obtain the *vacuum* Kasner solutions with the exponents satisfying (1.3). The solution given by the *maximal scalar field strength* $A = \pm \sqrt{2/3}$ is isotropic, i.e., $p_1 = p_2 = p_3 = 1/3$, and therefore agrees with a spatially flat Friedmann-Robertson-Walker model.

We claim that all of the results of this paper can be easily generalised to hold for the whole family of Kasner-scalar field spacetimes. The main reason for the validity of this generalisation is that the Kasner-scalar field spacetimes become less anisotropic the larger the parameter $A$, and consequently, the vacuum case $A = 0$ is the most technically challenging case. Because of this property, we assert that our results, which are written for $A = 0$, imply that analogous results continue to hold for $A \neq 0$. To see why this is,
consider an arbitrary Kasner–scalar field spacetime with $A \in [-\sqrt{2/3}, \sqrt{2/3}]$. Then it is straightforward to find a coordinate transformation that brings the metric (1.2) given by Kasner exponents satisfying (3.15) to the form
\begin{equation}
g = (-t)^{k-1} (\ - dt \otimes dt + dx \otimes dx) + (-t)^{1-L} dy \otimes dy + (-t)^{1+L} dz \otimes dz \end{equation}
where
\begin{equation}
p_1 = (K^2 - 1)/(K^2 + 3), \quad p_2 = 2(1 - L)/(K^2 + 3), \quad p_3 = 2(1 + L)/(K^2 + 3),
\end{equation}
which we observe is very similar to (3.2) and (3.3). We also notice that (3.15) is equivalent to
\begin{equation}
K^2 = L^2 + \frac{A^2}{8}(3 + K^2)^2.
\end{equation}
It follows in the vacuum case $A = 0$ that $L = \pm K$ and (3.16) reduces to (3.2) as expected. Given any $A$ with $0 < A^2 \leq 2/3$, the restriction $L^2 \geq 0$ implies the following restriction for $K^2$ in (3.18):
\begin{equation}
K^2 \in \left[\frac{4 - 3A^2 - 2\sqrt{2}A^2 - 3A^4}{A^2}, \frac{4 - 3A^2 + 2\sqrt{2}A^2 - 3A^4}{A^2}\right],
\end{equation}
which degenerates to no restriction in the case $A = 0$. Given all this, we now define
\begin{equation}
\Gamma_1 = \frac{3\gamma - 2 - K(2 - \gamma)}{4}, \quad \Gamma_2 = \frac{3\gamma - 5 + 2L + K(\gamma - 1)}{4}, \quad \Gamma_3 = \frac{3\gamma - 5 - 2L + K(\gamma - 1)}{4},
\end{equation}
which reduces to (3.8) for $A = 0$. Observe that (3.13) holds for these quantities $\Gamma_1, \Gamma_2, \Gamma_3$ in (3.20) for all $A$ with $0 \leq A^2 \leq 2/3$ as a consequence of (3.17).

Anticipating parts of the proofs of Theorem 3.1 and Theorem 3.6, it is a remarkable fact that the Euler equations take the same form (3.21)–(3.28), whether $A = 0$ or not, as long as the fluid exponents $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are defined by (3.20) as opposed to by (3.8). It is a further remarkable fact of the proofs that condition (3.7) never needs to be invoked directly; all arguments in the proofs rely on condition (3.14) instead (which is equivalent to (1.6) as a consequence of (3.13) irrespective of the value of $A$). We conclude from this that all the results about the dynamics of fluids given in Theorem 3.1 and Theorem 3.6 are valid for fluids on Kasner–scalar field spacetimes for any $A \in [-\sqrt{2/3}, \sqrt{2/3}]$ provided the results are expressed in terms of the fluid exponents in (3.20) and provided (3.14) holds (while (3.7) can be ignored). We remark that also the expression (1.7) then turns out to hold when the fluid vector $V^\alpha$ is expressed in terms of the orthonormal frame (1.8) irrespective of the value of $A$.

In order to illustrate this further, let us consider the isotropic case $A = \pm \sqrt{2/3}$ as an example. In this case, (3.19) implies that $K^2$ has to take the value 3 and (3.18) implies that $L = 0$. (3.20) then yields that $\Gamma_1 = \Gamma_2 = \Gamma_3 = (3\gamma - 4)/2$. Our discussion therefore implies that all our results about fluids in Theorem 3.1 (and similarly in Theorem 3.6) hold for arbitrary equation of state parameters $\gamma$ with $2 \geq \gamma > 4/3$. This is an interesting outcome since it shows that radiation fluids given by $\gamma = 4/3$ correspond to the borderline case of stability for $A = \pm \sqrt{2/3}$ (while they are in the unstable regime for $A = 0$).

3.1.2. Proof of Theorem 3.1. The proof of Theorem (3.1) involves four steps. The first step is, given a leading order term $U_\ast$, which can be thought of as an approximate solution, to transform the Euler equations (1.4) into a symmetric hyperbolic Fuchsian system in accord with Definition 2.1. The second step is to apply Theorem 2.2 to obtain the existence and uniqueness of a solution to the Fuchsian system, which yields a corresponding solution to the Euler equations. This is a conditional existence result because it relies on the leading order term $U_\ast$ being sufficiently accurate. This leads to the third step where sufficiently accurate leading order terms are constructed. With a sufficiently accurate leading order term in hand, we are then able, in the final step, to obtain solutions to the Euler equations from the conditional existence result.

Step 1: Transformation to Fuchsian form

We multiply (1.4) through with $(-t)^{-(K^2-1)/2-5T}V_\lambda V^\lambda$ and express that system as a partial differential equation for the unknown
\begin{equation}
U = T^{-1}V,
\end{equation}
where $T$ is given by (3.12). In this way, the Euler equations take the form
\begin{align*}
B^0(U)\partial_t U + B^1(U)\partial_\alpha U + B^2(t, U)\partial_\gamma u + B^3(t, U, u)\partial_\beta u = \frac{1}{t}B(U)U + G(t, U),
\end{align*}
(3.22)
where
\begin{align*}
B^0(v) &= \begin{pmatrix} P_0 v^0 & -Q_0 v^1 & -Q_0 v^2 & -Q_0 v^3 \\ -Q_0 v^1 & Q_1 v^0 & v^0 v^2 (3\gamma - 2) & v^0 v^2 v^3 (3\gamma - 2) \\ -Q_0 v^2 & v^0 v^2 (3\gamma - 2) & Q_2 v^0 & v^0 v^2 v^3 (3\gamma - 2) \\ -Q_0 v^3 & v^0 v^2 v^3 (3\gamma - 2) & Q_3 v^0 & 0 \end{pmatrix},
\end{align*}
(3.23)
\begin{align*}
B^1(v) &= \begin{pmatrix} Q_0 v^1 & -Q_1 v^0 & -v^0 v^2 (3\gamma - 2) & -v^0 v^2 v^3 (3\gamma - 2) \\ -Q_1 v^0 & P_1 v^1 & Q_1 v^2 & Q_1 v^3 \\ -v^0 v^2 (3\gamma - 2) & Q_1 v^2 & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^3 \\ -v^0 v^2 v^3 (3\gamma - 2) & Q_1 v^3 & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^3 \end{pmatrix},
\end{align*}
(3.24)
\begin{align*}
B^2(t, v) &= (-t)^{\Gamma_2 - \Gamma_1} \begin{pmatrix} -v^0 v^2 (3\gamma - 2) & Q_1 v^2 & v^0 v^2 v^3 (3\gamma - 2) & Q_2 v^0 \\ -Q_2 v^0 & P_2 v^2 & Q_2 v^3 & Q_3 v^2 \\ -v^0 v^2 v^3 (3\gamma - 2) & Q_2 v^3 & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^3 \\ Q_2 v^3 & Q_3 v^3 & Q_3 v^3 & Q_3 v^3 \end{pmatrix},
\end{align*}
(3.25)
\begin{align*}
B^3(t, v) &= (-t)^{\Gamma_3 - \Gamma_1} \begin{pmatrix} -v^0 v^3 (3\gamma - 2) & -v^0 v^2 v^3 (3\gamma - 2) & -Q_3 v^0 \\ -v^0 v^2 v^3 (3\gamma - 2) & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^1 \\ -Q_3 v^0 & Q_3 v^1 & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^2 \\ -Q_3 v^0 & Q_3 v^2 & v^1 v^2 v^3 (3\gamma - 2) & Q_3 v^3 \end{pmatrix},
\end{align*}
(3.26)
\begin{align*}
B(v) &= r v^0 \text{diag} (0, \Gamma_1, \Gamma_2, \Gamma_3)
\end{align*}
(3.27)
and
\begin{align*}
G(t, v) &= -r \frac{\Gamma_1 (v^1)^2 + \Gamma_2 (v^2)^2 + \Gamma_3 (v^3)^2}{t} (1, 0, 0, 0)^r.
\end{align*}
(3.28)

With $t < 0$ and $v \in V^3$. Here, $r$, $P_0$, ..., $P_3$ and $Q_0$, ..., $Q_3$ are all quadratic polynomials in the components\textsuperscript{3.2} $v \in \mathbb{R}^3$ with $(t, x, y, z)$-independent coefficients, which only depend on the parameter $\gamma$ and not, for example, on $K$. These polynomials are all chosen to be strictly positive whenever they are evaluated at $v = (v^0, 0, 0, 0)^T$ with $v^0 > 0$, and in particular,
\begin{align*}
& r(v^0, 0, 0, 0) = Q_1 (v^0), 0, 0, 0) = Q_2 (v^0, 0, 0, 0) = Q_3 (v^0, 0, 0, 0) = (\gamma - 1)(v^0)^2, \\
& P_0 (v^0, v^1, v^2, v^3) = (v^0)^2 + 3(\gamma - 1)((v^1)^2 + (v^2)^2 + (v^3)^2) \implies P_0 (v^0, 0, 0, 0) = (v^0)^2.
\end{align*}
(3.29)
(3.30)

To proceed with the transformation to Fuchsian form, we introduce a leading order term $U_\ast$, to be specified\textsuperscript{3.3}, and we formulate the Euler equations in terms of the remainder
\begin{align*}
u = U - U_\ast.
\end{align*}
(3.31)

We will also find it convenient at times to work with a rescaled version of $U_\ast$, denoted by $\hat{U}_\ast$, that is defined via
\begin{align*}
U_\ast = \hat{T}^{-1} \hat{U}_\ast
\end{align*}
(3.32)
where
\begin{align*}
\hat{T} = \hat{T}(t) := \text{diag} \left( 1, (-t)^{-\Gamma_1}, (-t)^{-\Gamma_2}, (-t)^{-\Gamma_3} \right)
\end{align*}
(3.33)
and
\begin{align*}
\hat{U}_\ast = (\hat{U}_0^0, \hat{U}_1^1, \hat{U}_2^2, \hat{U}_3^3)^T.
\end{align*}

Now, a straightforward calculation shows that the Euler equations (3.22) can be expressed in terms of the remainder (3.31) as
\begin{align*}
B^0(U_\ast, u)\partial_t u + B^1(U_\ast, u)\partial_\alpha u + B^2(t, U_\ast, u)\partial_\gamma u + B^3(t, U_\ast, u)\partial_\beta u = \frac{1}{t}B(U_\ast, u)U_\ast + F(t, W_\ast, u)
\end{align*}
(3.34)
\textsuperscript{3.1}Recall that $V$ is the trivial bundle $T^3 \times \mathbb{R}^4$.
\textsuperscript{3.2}Observe that strictly speaking $v$ is an element of the trivial bundle $V$. Thinking of $v$ as a vector in $\mathbb{R}^4$ is justified by our conventions above.
\textsuperscript{3.3}A sufficient list of precise assumptions for $U_\ast$ and the related quantity $\hat{U}_\ast$, defined in (3.32) is given in Proposition 3.4.
where

\[ W_* = (U_*, (-t)^{-\gamma} \partial_t U_*, \partial_x U_*, \partial_y U_*, \partial_z U_*), \quad q \geq 0, \]  

and by a slight abuse of notation, we have set

\[ B^0(\tilde{Z}, v) = B^0(\tilde{Z} + v), \]  

\[ B^1(\tilde{Z}, v) = B^1(\tilde{Z} + v), \]  

\[ B^2(t, \tilde{Z}, v) = B^2(t, \tilde{Z} + v), \]  

\[ B^3(t, \tilde{Z}, v) = B^3(t, \tilde{Z} + v), \]  

\[ B(\tilde{Z}, v) = B(\tilde{Z} + v), \]  

for \( t \in [T_0, 0) \) and \((\tilde{Z}, v) \in Z_1 \oplus \mathbb{V} \) with \( Z_1 = \Sigma \times \mathbb{R}^4 \). In the following, we label components of \( \tilde{Z} \in Z_1 \) by \((\tilde{Z}^0, \tilde{Z}^1, \tilde{Z}^2, \tilde{Z}^3)\). The map \( F \) can be expressed as

\[ F(t, Z, v) = (-t)^3 \tilde{F}(t, Z) + (-t)^3 F_0(t, Z, v) \]  

where \([T_0, 0) \times Z_2 \oplus \mathbb{V} \) and \( Z_2 \) is the trivial bundle \( Z_1 \oplus Z_1 \oplus Z_1 \oplus Z_1 \oplus Z_1 \) over \( \Sigma \). We will label the components of \( Z \in Z_2 \) by

\[ (Z_0, Z_1, Z_21, Z_{22}, Z_{23}). \]  

With this notation, \( \tilde{F} \) and \( F_0 \) are then given by

\[ \tilde{F}(t, Z) = (-t)^{-\lambda} \left( B^0(\tilde{T}(t)^{-1}Z_0)(-t)^{q-1}Z_1 + B^1(\tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_{21} + B^2(t, \tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_{22} + B^3(t, \tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_{23} + \frac{1}{t}(B(\tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_0 - G(\tilde{T}(t)^{-1}Z_0)) \right) \]  

and

\[ F_0(t, Z, v) = (-t)^{-\lambda} \left( -(B^0(\tilde{T}(t)^{-1}Z_0 + v) - B^0(\tilde{T}(t)^{-1}Z_0))(-t)^{q-1}Z_1 - (B^1(\tilde{T}(t)^{-1}Z_0)v - B^1(\tilde{T}(t)^{-1}Z_0))\tilde{T}(t)^{-1}Z_{21} - (B^2(t, \tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_{22}) - (B^3(t, \tilde{T}(t)^{-1}Z_0)\tilde{T}(t)^{-1}Z_{23}) + \frac{1}{t}(B(\tilde{T}(t)^{-1}Z_0 + v) - B(\tilde{T}(t)^{-1}Z_0))\tilde{T}(t)^{-1}Z_0 + G(\tilde{T}(t)^{-1}Z_0 + v) - G(\tilde{T}(t)^{-1}Z_0)) \right). \]  

The point of expressing the Euler equations this way is, as will be verified in the following lemma, that (3.34) is now in a form to which Theorem 2.2 applies. Before stating the lemma, we define a family of open and bounded sets \( \tilde{Z}_{r,R}, 0 < r < R, \) in \( Z_1 \) by

\[ \tilde{Z}_{r,R} = \{ \tilde{Z} \in Z_1 \mid \tilde{Z}^0 > r, |\tilde{Z}| < R \}. \]  

**Lemma 3.3.** Suppose \( K \) and \( \gamma \) satisfy (3.7), \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are defined by (3.8), \( q > 0, \mu > \Gamma_1, R > 0, 0 < r < R, \gamma_1 = 2/(r^2(\gamma - 1)), \gamma_2 = 2R^3, 0 < p \leq \min\{1 - \Gamma_1 + \Gamma_3, \mu, q, \Gamma_3\}, \beta = 0 \) and \(-1 < \lambda \leq \min\{\mu + q - 1, \mu + \Gamma_3 - 1\} \). Then for \( T_0 < 0 \) close enough to zero, there exists a constant \( \theta > 0 \) such that (3.34) is a symmetric hyperbolic Fuchsian system according to Definition 2.1 for the above choices of constants, the open bounded subsets sets \( Z_1 = \tilde{Z}_{r,R} \) and \( Z_2 = Z_1 \oplus B_R(Z_1) \oplus B_R(Z_1) \oplus B_R(Z_1) \) of \( Z_1 \) and \( Z_2 \), respectively, and the maps (3.36)-(3.40), (3.41)-(3.44), \( B^0(t, \tilde{Z}) = B^0(\tilde{Z}, 0), B^0_0(t, \tilde{Z}, v) = t^{1-q} B^0(t, \tilde{Z}, v) \) and \( B^0_0 \) = 0.

**Proof.** Since \( \mu > \Gamma_1 \), it is clear from the inequality (3.14), the definitions (3.23)-(3.27) and (3.36)-(3.40), and (3.29)-(3.30) that, for any \( 0 < r < R \) and \( R > 0 \), the map

\[ (t, \tilde{Z}, z) \mapsto B^0(t, \tilde{Z}, |t|^\mu z) \]
is in $C^1([T_0,0), C^\infty(Z_1 \oplus B_R(\mathcal{V}), L(\mathcal{V}))$ while the maps

$$
(t, \tilde{Z}, v) \mapsto t^{1-p} B^1(t, \tilde{Z}, |t|^p v), \\
(t, \tilde{Z}, v) \mapsto t^{1-p} B^2(t, \tilde{Z}, |t|^p v), \\
(t, \tilde{Z}, v) \mapsto t^{1-p} B^3(t, \tilde{Z}, |t|^p v), \\
(t, \tilde{Z}, z) \mapsto B(\tilde{Z}, |t|^p v),
$$

are in $C^0_b([T_0,0), C^\infty(Z_1 \oplus B_R(\mathcal{V}), L(\mathcal{V}))$ provided that

$$0 < p \leq 1 - \Gamma_1 + \Gamma_3.
$$

Next, we observe from the definition (3.45) of the set $Z_1 = \tilde{Z}_{r,R}$, the formulas (3.23), (3.29)-(3.30) and (3.36), and the fact that $\mu > 0$, that, by choosing $T_0 < 0$ close enough to 0, we can ensure that $B^0(t, \tilde{Z}, v)$ satisfies

$$\frac{1}{\gamma_1} \leq B^0(\tilde{Z}, |t|^p v) \leq \gamma_2 \mathbb{I}
$$

for all $(t, \tilde{Z}, v) \in [T_0,0) \times \tilde{Z}_{r,R} \oplus B_R(\mathcal{V})$ where $\mathbb{I}$ is the identity map on $\mathcal{V}$,

$$\gamma_1 = \frac{2}{r^3(\gamma - 1)} \quad \text{and} \quad \gamma_2 = 2R^3.
$$

Setting

$$\tilde{B}^0(t) = B^0(\tilde{Z}, 0),
$$

it is also not difficult to verify that $\tilde{B}^0 \in C^\infty(Z_1, L(\mathcal{V}))$ and that

$$B^0(\tilde{Z}, |t|^p v) - \tilde{B}^0(\tilde{Z}) = O(v)
$$

for all $(t, \tilde{Z}, v) \in [T_0,0) \times Z_1 \oplus B_R(\mathcal{V})$. Turning to the map $\tilde{F}$ defined by (3.43), it is clear from this definition along with the formulas (3.23)-(3.30), (3.33) and (3.42) that the map $(t, Z) \mapsto \tilde{F}(t, Z)$ is in $C^0([T_0,0), C^\infty(Z_2, \mathcal{V}))$. In anticipation of the discussion below, it is useful to note by (3.43) that $(t, Z) \mapsto \tilde{F}(t, Z)$ is in $C^0_b([T_0,0), C^\infty(Z, \mathbb{R}^3))$ for any

$$\lambda \leq \min\{q - 1, \Gamma_3 - 1\}. \quad (3.46)
$$

Now, considering the map $F_0$ defined by (3.44), we first observe from (3.14), (3.23)-(3.30), (3.33), (3.42) and the assumption $\mu > \Gamma_1 > 0$ that the maps

$$
(t, Z, v) \mapsto |t|^{-(\mu+q-1)} \left(B^0(\tilde{T}^{-1}(t)Z_0, |t|^p v) - B^0(\tilde{T}^{-1}(t)Z_0, 0)\right)(-t)^{\mu-1} Z_1,
$$

$$
(t, Z, v) \mapsto |t|^{-\mu} \left(B^1(\tilde{T}^{-1}(t)Z_0, |t|^p v) - B^1(\tilde{T}^{-1}(t)Z_0, 0)\right)\tilde{T}^{-1}(t)Z_21,
$$

$$
(t, Z, v) \mapsto |t|^{-(\mu-\Gamma_1+\Gamma_3)} \left(B^2(t, \tilde{T}^{-1}(t)Z_0, |t|^p v) - B^2(t, \tilde{T}^{-1}(t)Z_0, 0)\right)\tilde{T}^{-1}(t)Z_22,
$$

$$
(t, Z, v) \mapsto |t|^{-(\mu-\Gamma_1+\Gamma_3)} \left(B^3(t, \tilde{T}^{-1}(t)Z_0, |t|^p v) - B^3(t, \tilde{T}^{-1}(t)Z_0, 0)\right)\tilde{T}^{-1}(t)Z_23,
$$

$$
(t, Z, v) \mapsto |t|^{-(\mu+q-1)1} \left(B(\tilde{T}^{-1}(t)Z_0, |t|^p v) - B(\tilde{T}^{-1}(t)Z_0, 0)\right)\tilde{T}^{-1}(t)Z_0,
$$

are all in $C^0_b([T_0,0), C^\infty(Z_2 \oplus B_R(\mathcal{V}), \mathcal{V}))$. From this, (3.14), (3.44) and the assumption $\mu > \Gamma_1$, we see immediately that

$$
(t, Z, v) \mapsto F_0(t, Z, |t|^p v)
$$

will be in $C^0_b([T_0,0), C^\infty(Z \oplus B_R(\mathcal{V}), \mathcal{V}))$ provided that $\lambda$ satisfies

$$\lambda \leq \min\{\mu + q - 1, \mu + \Gamma_3 - 1\}.
$$

To complete the proof of the lemma, we see from (2.6) that the map div$B$ is given by

$$
\text{div}B(t, \tilde{Z}, \tilde{Z}', v, v') = D_{\tilde{Z}} B^0(\tilde{Z}, v) |t|^{q-1} \tilde{Z} + D_v B^0(\tilde{Z}, v) (B^0(\tilde{Z}, v))^{-1} \left[-B^1(t, \tilde{Z}, v) v' + \frac{1}{t} B(\tilde{Z}, v) + F(t, Z, v)\right] + D_{\tilde{Z}} B^1(t, \tilde{Z}, v) \tilde{Z}' + D_v B^1(t, \tilde{Z}, v) v'.
$$
From similar considerations as above, it is not difficult to verify from the above, in particular (3.46), that
\[
\text{div}B(t, \tilde{Z}, \tilde{Z}', |t|^{\mu}v, |t|^\mu v') = O(\theta |t|^{-(1-p)})
\]
for some \(\theta > 0\) provided that
\[
p \leq \min\{q, \mu, \Gamma_3\}.
\]

\[\Box\]

Step 2: Conditional existence and uniqueness

Given Lemma 3.3, we can now invoke Theorem 2.2, while observing Remark 2.3, to derive precise conditions (3.48)-(3.50) for the leading order term \(U_*\). The following proposition states that if these conditions are satisfied, then the singular initial value problem for the Euler equations has a solution. In Step 3 below, see Lemma 3.5, we then construct leading order terms for the Euler equations which we show in Step 4 to be consistent with these conditions.

**Proposition 3.4.** Let \(K\) and \(\gamma\) satisfy (3.7), \(R > 0\), \(\Gamma_1, \Gamma_2\) and \(\Gamma_3\) be as defined by (3.8), and \(k \in \mathbb{Z}_{\geq 3}\).

Suppose there exist constants \(\mu > \Gamma_1, q > 0, r_0 > 0\) and \(\lambda\) with
\[
\mu - 1 < \lambda \leq \min\{q, \Gamma_3\} + \mu - 1,
\]
such that
\[
\dot{U}_* = (\dot{U}_0^0, \dot{U}_1^0, \dot{U}_2^0, \dot{U}_3^0) \in C^0([T_0, 0), H^{k+1}(\Sigma, Z_1)) \cap C^1([T_0, 0), H^k(\Sigma, Z_1))
\]
satisfies
\[
\dot{U}_* \geq r_0 \quad \text{in} \quad [T_0, 0] \times \Sigma, \quad t^{-\gamma} \partial_t U_*(t) \in C^0([T_0, 0), H^k(\Sigma, Z_1))
\]
and
\[
\tilde{F}(t, W_*(t)) \in C^0([T_0, 0), H^k(\Sigma, V))
\]
where \(U_*, W_*\) and \(\tilde{F}\) are as defined by (3.32), (3.35) and (3.43), respectively. Then there exists a solution \(u \in C^0([T_0, 0), H^k(\Sigma, V)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, V)) \subset C^1([T_0, 0) \times \Sigma, V)\) of (3.34). Moreover, this solution satisfies
\[
\sup_{t \in [T_0, 0]} \max\left\{\|t|^{-\mu} u(t)\|_{L^\infty(\Sigma)}, \|t|^{-\mu} \nabla u(t)\|_{L^\infty(\Sigma)}\right\} < R
\]
and
\[
\|u(t)\|_{H^k(\Sigma)} \lesssim |t|^{\lambda+1}
\]
for all \(t \in [T_0, 0)\), and it is unique within the \(C^1([T_0, 0) \times \Sigma, V)\) class of solutions satisfying (3.52).

**Proof.** First we notice, since \(\Gamma_3\), by (3.14), and \(q\), by assumption, are both positive, that the inequality for \(\lambda\) in (3.47) is consistent. Moreover we know from (3.14) that \(\Gamma_1 > 0\) and hence, \(\mu > 0\). It therefore follows that \(\lambda\) satisfies the required inequality in Lemma 3.3. By (3.14) and (3.47), we also have \(\min\{1 - \Gamma_1 + \Gamma_3, \mu, q, \Gamma_3, \lambda + 1 - \mu\} > 0\). So, we can choose \(p\) to satisfy
\[
0 < p < \min\{1 - \Gamma_1 + \Gamma_3, \mu, q, \Gamma_3, \lambda + 1 - \mu\},
\]
which implies that \(p\) is consistent with the inequality in Lemma 3.3. Consequently, we conclude by Lemma 3.3 that the system (3.34) is symmetric hyperbolic Fuchsian for any choice of constants \(r \in (0, r_0)\) and \(R > r_0\) provided that \(T_0 < 0\) is chosen close enough to zero.

Next, we set
\[
\alpha = \frac{\lambda + 1 - p - \mu}{p}
\]
in order to satisfy (2.10), and we note that \(\alpha > 0\) due to our choice of \(p\) above. We see also by \(\mu > \Gamma_3\), (3.14), (3.23), (3.27), (3.29) and (3.30) that for every sufficiently small \(\eta > 0\), there exists a small \(T_0 < 0\) so that (2.14) holds. Because \(\beta\) and \(b\) (defined in (2.12)) are zero as a consequence of Lemma 3.3, condition (2.13) is also fulfilled. Finally we notice that the smallness condition (2.15) can be verified for any \(\delta > 0\) by choosing a sufficiently small \(|T_0|\), as a consequence of (3.50) for any \(p\) with (3.54).
Having verified that all the conditions for Theorem 2.2 are met, we conclude, with the help of Remark 2.3, that there exists a unique solution
\[ u \in C^0_b([T_0, 0), H^k(\Sigma, \mathbb{V})] \cap C^1([T_0, 0), H^{k-1}(\Sigma, \mathbb{V})) \subset C^1([T_0, 0) \times \Sigma, \mathbb{V}) \]
of (3.34) that is bounded by
\[ ||t|^{-(\lambda+1-p)}u(t)||_{H^k(\Sigma)} \leq C \int_0^t |s|^{p-1}||\tilde{F}(s, W^*(s))||_{H^k(\Sigma)} ds, \quad T_0 \leq t < 0, \] (3.55)
and
\[ \sup_{t \in [T_0, 0)} \max \left\{ ||t|^{-(\lambda+1-p)}u(t)||_{L^\infty(\Sigma)}, ||t|^{-(\lambda+1-p)}\nabla u(t)||_{L^\infty(\Sigma)} \right\} < R. \] (3.56)
Moreover, this is the unique solution of (3.34) within the class \( C^1([T_0, 0) \times \Sigma, \mathbb{V}) \) satisfying (3.56). Finally, given (3.50), we note from (3.55) that
\[ ||t|^{-(\lambda+1-p)}u(t)||_{H^k(\Sigma)} \lesssim |t|^p \]
for all \( t \in [T_0, 0) \). Multiplying both sides of this inequality by \(|t|^{\lambda+1-p}\) yields the estimate (3.53). This completes the proof of Proposition 3.4. \( \square \)

Step 3: Constructing accurate leading order terms

We now turn to constructing leading order terms \( U_* \) that satisfy the conditions (3.48)-(3.50). We do this by constructing a finite sequence \( U_n, n \geq 0 \), of solutions of the following approximate equations:
\[ B^0_0 \partial_t U_{n+1} + B^i_0 \partial_i U_{n+1} + B^0_n \hat{T}^{-1} \partial_t \hat{T} (U_{n+1} - U_n) - \frac{1}{t} B_n U_n - G_n = 0 \] (3.57)
where
\[ B^0_n = B^0(U_n), \quad B^1_n = B^1(U_n), \quad B^2_n = B^2(t, U_n), \]
\[ B^3_n = B^3(t, U_n), \quad B_n = B(U_n), \quad G_n = G(U_n), \] (3.58) (3.59)
and \( \hat{T} \) is defined above by (3.33). The leading order term \( U_* \) will then be identified with \( U_n \) for some sufficiently large \( n \). In order to explain how this is done, we note first that, by setting
\[ W_n = \hat{T} U_n, \] (3.60)
the approximate equations (3.57) can be expressed more compactly as
\[ \partial_t W_{n+1} + \hat{B}^i_n \partial_i W_n - \hat{G}_n = 0 \] (3.61)
where
\[ \hat{B}^i_n = \hat{T} (B^0_n)^{-1} B^i_n \hat{T}^{-1} \quad \text{and} \quad \hat{G}_n = \frac{1}{t} \hat{T} (B^0_n)^{-1} B_n U_n + \partial_t \hat{T} U_n + \hat{T} (B^0_n)^{-1} G_n. \] (3.62)
A straightforward but lengthy calculation shows that

\[ \dot{B}_n^1 = \frac{|t|^{-\Gamma_1}}{S_0 W_n^6} \left( \begin{array}{c} -S_0 W_n^0 \\
-2Q_1 |t|^{2T_1} W_n^3 \\
2 |t|^{2T_1} W_n^2 \\
0 \\
0 \\
0 \end{array} \right) \left( \begin{array}{c} |t|^2 W_n^9 \\
|t|^2 W_n^3 \\
|t|^2 W_n^9 \\
|t|^2 W_n^3 \\
|t|^2 W_n^9 \\
|t|^2 W_n^3 \end{array} \right) \right) \]

\[ \dot{B}_n^2 = \frac{|t|^{-\Gamma_1}}{S_0 W_n^6} \left( \begin{array}{c} 0 \\
0 \\
0 \\
0 \\
-2Q_1 |t|^{2T_1} W_n^3 \\
2 |t|^{2T_1} W_n^2 \end{array} \right) \left( \begin{array}{c} -S_0 W_n^0 \\
|t|^2 W_n^3 \\
|t|^2 W_n^3 \\
|t|^2 W_n^3 \\
|t|^2 W_n^3 \\
|t|^2 W_n^3 \end{array} \right) \right) \]

\[ \dot{B}_n^3 = \frac{|t|^{-\Gamma_1}}{S_0 W_n^6} \left( \begin{array}{c} 0 \\
0 \\
0 \\
0 \\
0 \end{array} \right) \left( \begin{array}{c} |t|^2 W_n^9 \\
|t|^2 W_n^3 \\
|t|^2 W_n^9 \\
|t|^2 W_n^3 \\
|t|^2 W_n^9 \\
|t|^2 W_n^3 \end{array} \right) \right) \]

and

\[ \dot{G}_n = \frac{\Gamma_1 (W_n^1)^2 |t|^{2T_1} + \Gamma_2 (W_n^2)^2 |t|^{2T_2} + \Gamma_3 (W_n^3)^2 |t|^{2T_3}}{t S_0} \left( \begin{array}{c} \gamma W_n^9 \\
\gamma W_n^3 \end{array} \right), \]

where \( Y_1, Y_2, Y_3, Q_1, Q_2, Q_3, P_1, P_2, P_3, S_0, S_1, S_2 \) and \( S_3 \) are quadratic polynomials in the components of \( U_n = \tilde{T}^{-1} W_n \) with \( (t, x, y, z) \)-independent coefficients that only depend on \( \gamma \). All these polynomials have been chosen to be strictly positive when evaluated at \( U_n = (U_n^0, 0, 0, 0) \) with \( U_n^0 > 0 \).

In the following lemma, we fix the starting value \( U_0 \) of the iteration sequence and solve the approximate equations (3.61) to obtain \( U_1, U_2, \ldots, U_{\ell} \) for any integer \( \ell \geq 0 \).

**Lemma 3.5.** Suppose \( K \) and \( \gamma \) satisfy (3.7), \( \ell, k \in \mathbb{Z}_{\geq 1}, q \leq \min\{1 - \Gamma_1, 2\Gamma_3\} \), \( v_* = (v_*^0, v_*^1, v_*^2, v_*^3) \in H^{k+\ell}\gamma (\Sigma, \mathbb{V}) \) with \( v_*^0 > 0 \) in \( \Sigma \), and let

\[ U_0 = (v_0^0, v_0^1, v_0^2, v_0^3) (-t)^{T_1}, v_0^1 (-t)^{T_2}, v_0^2 (-t)^{T_3}). \]

Then for \( T_0 < 0 \) close enough to zero, there exists solutions \( U_0, U_1, \ldots, U_{\ell} \) of (3.57) with the properties:

\[ U_n \in C^0_b([T_0, 0), H^{k+\ell-n}(\Sigma, \mathbb{V})), \]

\[ (-t)^{-q}\tilde{T} U_n - v_* \in C^0_b([T_0, 0), H^{k+\ell-n}(\Sigma, \mathbb{V})), \]

\[ (-t)^{-q}\tilde{T} (U_n - U_{n-1}) \in C^0_b([T_0, 0), H^{k+\ell-n}(\Sigma, \mathbb{V})). \]

**Proof.** We proceed by induction.

**Base case:** Given \( v_* = (v_*^0, v_*^1, v_*^2, v_*^3) \in H^{k+\ell}(\Sigma, \mathbb{V}) \), we note by (3.33) that

\[ U_0 = (v_0^0, v_0^1 (-t)^{T_1}, v_0^2 (-t)^{T_2}, v_0^3 (-t)^{T_3}) \]

satisfies

\[ \tilde{T} U_0 - v_* = 0. \]

---

3.4 Despite having the same labels, the polynomials here do not agree with the polynomials used to express (3.23) – (3.28).
From these expressions, it is then clear that $U_0$ satisfies all asserted properties (3.67), (3.68) and (3.70) for $n = 0$. Thanks to (3.71), we notice that the base case of (3.69) for $n = 1$ is equivalent to (3.68) for $n = 1$.

**Induction hypothesis:** We now assume that (3.67)-(3.70) hold for $n = 0, 1, \ldots, m - 1$ for some integer $m$ satisfying $1 \leq m \leq t - 1$. For use below, we observe by setting

$$w_n = W_n - v_* = \hat{T}U_n - v_*$$

(3.72) that we can write (3.61) as

$$\partial_tw_{n+1} = \hat{F}_n := -\hat{B}^t_i\partial_i(v_* + w_n) + \hat{G}_n,$$

(3.73) and moreover, that $w_{n+1} - w_n$ satisfies

$$\partial_t(w_{n+1} - w_n) = \hat{f}_n := -\hat{B}_i^t\partial_i(w_n - w_{n-1}) - (\hat{B}_i^t - \hat{B}_i^{n-1})\partial_i(v_* + w_n) - \hat{G}_n - \hat{G}_{n-1}.$$  

(3.74)

**Induction step:** With the help of the calculus inequalities, in particular, the Sobolev, Product and Moser calculus inequalities, see [48, Ch. 13, §2 & §3], we see from (3.14), (3.63)-(3.63), (3.73)-(3.74), and the induction hypothesis that $\hat{F}_{m-1}$ can be estimated by

$$\|\hat{F}_{m-1}\|_{H^{k+t-m}(\Sigma)} \leq \|\hat{B}_{m-1}\|_{H^{k+t-m}(\Sigma)} \|D(v_* + w_{m-1})\|_{H^{k+t-m}(\Sigma)} + \|\hat{G}_{m-1}\|_{H^{k+t-m}(\Sigma)}$$

$$\leq C(\|v_* + w_{m-1}\|_{H^{k+t-m}(\Sigma)}) (|t|^{-\Gamma_1} \|v_* + w_{m-1}\|_{H^{k+t-m(1)}(\Sigma)}) + |t|^{2\Gamma_3 - 1})$$

$$\lesssim |t|^{-\Gamma_1} + |t|^{2\Gamma_3 - 1},$$

(3.75) while a similar argument shows that $\hat{f}_{m-1}$ can be estimated by

$$\|\hat{f}_{m-1}\|_{H^{k+t-m}(\Sigma)} \leq \|\hat{B}_{m-1}\|_{H^{k+t-m}(\Sigma)} \|D(w_{m-1} - w_{m-2})\|_{H^{k+t-m}(\Sigma)}$$

$$+ \|\hat{B}_m - \hat{B}_m - \hat{B}_{m-2}\|_{H^{k+t-m}(\Sigma)} \|D(v_* + w_{m-1})\|_{H^{k+t-m}(\Sigma)}$$

$$+ \|\hat{G}_{m-1} - \hat{G}_{m-2}\|_{H^{k+t-m}(\Sigma)}$$

$$\lesssim (|t|^{-\Gamma_1} + |t|^{2\Gamma_3 - 1}) \|w_{m-1} - w_{m-2}\|_{H^{k+t-m(1)}(\Sigma)}$$

$$\lesssim |t|^{(m-1)q - \Gamma_1} + |t|^{(m-1)q + 2\Gamma_3 - 1}.$$  

(3.76)

Using (3.75), it follows, since $\Gamma_1 < 1$ and $2\Gamma_3 - 1 > -1$ by assumption, that we can solve (3.73) for $n = m - 1$ by setting

$$w_m(t) = -\int_0^t \hat{F}_{m-1}(s) \, ds$$

(3.77) where $w_m$ and $\partial_tw_m$ are bounded by

$$\|w_m(t)\|_{H^{k+t-m}(\Sigma)} \lesssim |t|^{-\Gamma_1} + |t|^{2\Gamma_3}$$

(3.78) and

$$\|\partial_tw_m(t)\|_{H^{k+t-m}(\Sigma)} \lesssim |t|^{-\Gamma_1} + |t|^{2\Gamma_3 - 1}$$

(3.79) for all $t \in [T_0, 0)$. But $q \leq \min\{1 - \Gamma_1, 2\Gamma_3\}$, and so, we see immediately from (3.33), (3.77), (3.78) and (3.79) that $U_m = \hat{T}^{-1}(w_m + v_*)$ satisfies (3.67), (3.68) and (3.70) for $m = n$. Similar arguments using (3.74), (3.76), and the fact that $\lim_{t \to 0}(w_m(t) - w_{m-1}(t)) = 0$ by (3.78) and the induction hypothesis shows that $w_m(t) - w_{m-1}(t)$ can be expressed as

$$w_m(t) - w_{m-1}(t) = -\int_t^0 \hat{f}_{m-1}(s) \, ds,$$

(3.80) and, in turn, bounded by

$$\|w_m(t) - w_{m-1}(t)\|_{H^{k+t-m}(\Sigma)} \lesssim |t|^{(m-1)q+1-\Gamma_1} + |t|^{(m-1)q+2\Gamma_3}.$$  

But as $q \leq \min\{1 - \Gamma_1, 2\Gamma_3\}$ and $\hat{T}(U_m - U_{m-1}) = w_m - w_{m-1}$, we see that (3.69) is satisfied for $n = m$, which completes the proof.
Step 4: Existence and uniqueness

Now, we are in the position to complete the proof of Theorem 3.1. Assume that \( k \in \mathbb{Z}_{\geq 3} \), \( K \) and \( \gamma \) satisfy (3.7), \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are defined by (3.8),

\[
q = \min \{ 1 - \Gamma_1, 2\Gamma_3 \},
\]

(3.81)

\( \mu > \Gamma_1 \), and \( v_* = (v_1^*, v_2^*, v_3^*) \in H^{k+\ell}(\Sigma, V) \) with \( v_3^0 > 0 \) in \( \Sigma \). Then for \( \ell \in \mathbb{Z}_{\geq 2} \) and \( T_0 < 0 \) chosen sufficiently small, let \( U_0, U_1, \ldots, U_{\ell-1} \) be the sequence of solutions to (3.57) from Lemma 3.5.

Setting

\[
U_* = U_{\ell-1}
\]

(3.82)

and letting, as above, \( \dot{U}_* \) and \( W_* \) be defined in terms of \( U_* \) by (3.32) and (3.35), respectively, we see by (3.43) and (3.58)–(3.59) that

\[
\tilde{F}(t, W_*) = (-t)^{-\lambda}F_{\ell-1}(t)\partial_t U_{\ell-1} + \frac{1}{t}B_{\ell-1}U_{\ell-1} - G_{\ell-1}
\]

(3.83)

Using the definitions (3.60) and (3.62) together with the evolution equation (3.61), it then is not difficult to verify via a short calculation that we can express \( \tilde{F}(t, W_*) \) as

\[
\tilde{F}(t, W_*) = (-t)^{-\lambda}B_{\ell-1}^{-1}F_{\ell-1}(t)(B_{\ell-1}^{-1}\partial_t(W_{\ell-1} - W_{\ell-2}) + (B_{\ell-1}^{-1} - B_{\ell-2})\partial_t W_{\ell-2} + \tilde{G}_{\ell-2} - \tilde{G}_{\ell-1})
\]

(3.84)

Using this expression, we can, with the help of (3.33), (3.23), (3.58), (3.63) – (3.66), and the calculus inequalities (i.e. Sobolev, Product and Moser calculus inequalities, see [48, Ch. 13, §2 & 3]), estimate \( \tilde{F}(t, W_*) \) by

\[
\|\tilde{F}(t, W_*(t))\|_{H^k(\Sigma)} \leq C \left( \|v_*\|_{H^k(\Sigma)}, \|W_{\ell-1}(t)\|_{H^{k+1}(\Sigma)}, \|W_{\ell-2}(t)\|_{H^{k+2}(\Sigma)} \right) \times (|t|^{-\Gamma_1 - \lambda} + |t|^{2\Gamma_3 - 1 - \lambda}) \left( \|W_{\ell-1}(t) - W_{\ell-2}(t)\|_{H^{k+1}(\Sigma)} \right).
\]

By Lemma 3.5 and the above estimate, we have

\[
\|\tilde{F}(t, W_*(t))\|_{H^k(\Sigma)} \lesssim |t|^{-\Gamma_1 - \lambda + (\ell - 1)q} + |t|^{2\Gamma_3 - 1 - \lambda + (\ell - 1)q}.
\]

(3.85)

Condition (3.50) is therefore satisfied provided

\[
\lambda \leq \min \{ 1 - \Gamma_1, 2\Gamma_3 \} - 1 + (\ell - 1)q.
\]

Given our choice of \( q \) above and the restriction for \( \lambda \) and \( \mu \) in Proposition 3.4, we find

\[
\mu - 1 < \lambda \leq \min \{ q + \mu, \Gamma_3 + \mu, \ell q \} - 1,
\]

(3.86)

and therefore

\[
\ell > \frac{\mu}{q}.
\]

(3.87)

Due to the uniform bound (3.50), the fact that \( U_* = U_{\ell-1} \) where \( U_{\ell-1} \) satisfies (3.68) and (3.70), and the definition \( \dot{U}_* = TU_* \), we are now in a position, after choosing \( T_0 < 0 \) closer to zero if necessary, to apply Proposition 3.4. For any choice of constants \( R > 0 \), this yields the existence of a solution

\[
u \in C^0([T_0, 0), H^k(\Sigma, V)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, V)) \subset C^1([T_0, 0) \times \Sigma, V)
\]

of (3.34) that is bounded by

\[
\|\nu(t)\|_{H^k(\Sigma)} \lesssim |t|^{\lambda + 1}, \quad T_0 \leq t < 0,
\]

(3.88)

and is unique within the \( C^1([T_0, 0) \times \Sigma, V) \) class of solutions satisfying (3.52).

By (3.12), (3.21), \( U = U_* + \nu = U_{\ell-1} + u \) and the fact that \( \lambda + 1 > \Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \) as a consequence of \( \mu > \Gamma_1 \), and (3.14) and (3.83), it follows that

\[
V = TU_{\ell-1} + Tu \in C^0([T_0, 0), H^k(\Sigma, V)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, V))
\]

determines a solution of the relativistic Euler equations (1.4). Now, according to (3.60) and (3.72), we have

\[
\dot{T}T^{-1}V - \nu = w_{\ell-1} + \dot{T}u,
\]

which we observe, in turn, implies, via (3.14) and (3.33) and the estimates (3.68) and (3.85), that

\[
\|\dot{T}T^{-1}V - \nu\|_{H^k(\Sigma)} \lesssim |t|^{\ell}
\]
where
\[ \epsilon = \min\{\lambda + 1 - \Gamma_1, q\} = \min\{\lambda + 1 - \Gamma_1, 1 - \Gamma_1, 2\Gamma_3\}, \] (3.86)
see (3.81).

Now, by choosing \( \ell \) sufficiently large, (3.84) and (3.83) allow us to choose \( \mu \) and \( \lambda \) arbitrarily large. This makes sense since the larger we choose \( \ell \), the more accurately we expect the leading order term \( U_{\ell-1} \) to describe the actual solution \( U \) and therefore the faster the remainder \( u \) to decay according to (3.85). However, whatever large values we choose for \( \ell, \mu \) and \( \lambda \), this does not allow us to obtain a better decay exponent \( \epsilon \) than \( \min\{1 - \Gamma_1, 2\Gamma_3\} \) in accordance with (3.86). Moreover, the larger we choose \( \ell \) the more orders of differentiability we lose, and the larger we choose \( \mu \) the weaker the uniqueness statement (3.52) becomes. In order to optimise differentiability and uniqueness, we obtain the result stated in Theorem 3.1 as follows. First, we select any \( \ell \) with
\[ \ell > \frac{\Gamma_1}{q} \] (3.87)
with \( q \) given by (3.81), and we make the specific choice
\[ \lambda = \min\{1, \Gamma_1 + \Gamma_3, \ell q\} - 1 =: \mu_0 - 1. \]

If we then select any \( \mu \) with
\[ \mu \in (\Gamma_1, \mu_0), \]
where we notice that \( \mu_0 > \Gamma_1 \) by (3.14) and (3.87), we see that all the conditions \( \mu > \Gamma_1 \), (3.83) and (3.84) are satisfied. Given (3.86) and our choice for \( \lambda \), we find that
\[ \epsilon = \min\{\ell q - 1, \Gamma_1 + \Gamma_3\}. \] (3.88)
The estimate holds for every \( \ell \) consistent with (3.87) and we have \( \epsilon > 0 \). The largest value for \( \epsilon \), which we can achieve by choosing \( \ell \) sufficiently large, is therefore \( \min\{1 - \Gamma_1, \Gamma_3\} \). In particular, this establishes (3.10).

Having established the existence of this solution, which we can write in the form \( U = U_{\ell-1} + u \), consider now any other classical solution \( \tilde{U} \) of (3.22) such that \( \tilde{U} \) and \( \tilde{U} \) satisfy (3.11), where we recall that \( V \) and \( \tilde{V} \) are defined in terms of \( U \) and \( \tilde{U} \) by \( U = T^{-1}V + \tilde{U} = T^{-1}\tilde{V} \), respectively. It follows that \( u = U - U_{\ell-1} \) and \( \tilde{u} = \tilde{U} - U_{\ell-1} \) are both solutions of the same symmetric hyperbolic Fuchsian system (3.34) with \( U_0 = U_{\ell-1} \). According to the uniqueness result of Proposition 3.4, it follows that \( \tilde{u} = u \) and hence \( U = \tilde{U} \) if the bound (3.52) holds for \( u \) for some \( \mu \in (\Gamma_1, \mu_0) \). Expressing \( \tilde{u} \) as \( \tilde{u} = u + (\tilde{U} - U) \), it then follows from (3.11), the definitions \( U = T^{-1}V \) and \( \tilde{U} = T^{-1}\tilde{V} \), and the bound (3.52) satisfied by \( u \) (recall that \( \lambda + 1 = \mu_0 \) that \( \tilde{u} \) satisfies the required bound for uniqueness. We therefore conclude that \( u = \tilde{u} \) and the proof of Theorem 3.1 is complete.

3.2. Stability. Having established the existence of a class of solutions of the Euler equations with the asymptotics given in Theorem 3.1, we now turn to establishing the nonlinear stability of these solutions. The precise sense in which these solutions are stable is given below in Theorem 3.6. The proof of this theorem is carried out in Section 3.2.1.

**Theorem 3.6 (Stability of fluids on Kasner spacetimes).** Suppose \( K \) and \( \gamma \) satisfy (3.7), \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are defined by (3.8), and \( k, \ell \) and \( \ell_0 \) are integers satisfying \( k \geq 3, \ell > \Gamma_1/q \) and
\[ \ell_0 \geq \ell + 2, \quad \ell_0 > \ell + \frac{3}{2} + \frac{\Gamma_1}{2\Gamma_3}, \] (3.89)
respectively, where \( q \) is defined in (3.81). Further, for given asymptotic data \( v_\ast = (v_\ast^0, \ldots, v_\ast^3)^T \in H^{k + \ell_0 + \ell}(\Sigma) \) with \( v_\ast^0 > 0 \), let \( V_\ast \) be the solution of the singular initial value problem of the Euler equations (1.4) with equation of state (3.5) defined on the Kasner spacetime given by (3.2) on a time interval \([\tilde{T}_0, 0)\) asserted by Theorem 3.1.

Then there exists \( \delta > 0 \) such that for all sufficiently small \( T_0 \in [\tilde{T}_0, 0) \) and for any \( V_0 \in H^{k + \ell_0}(\Sigma) \) with \( \|V_0\|_{H^k + \ell_0(\Sigma)} \leq \delta \), the Cauchy problem of the Euler equations (1.4) with equation of state (3.5) defined on the Kasner spacetime given by (3.2) with Cauchy data \( V_{C,0} = V_\ast(T_0) + \tilde{T}^{-1}(T_0)T(T_0)V_0 \) imposed at \( t = T_0 \) has a unique solution \( V_C(t) = T(t)U_C(t) \) with
\[ U_C \in C^0_0([T_0, 0), H^{k + \ell_0}(\Sigma)) \cap C^1([T_0, 0), H^{k + \ell_0 - 1}(\Sigma)), \]
where $T(t)$ and $\dot{T}(t)$ are defined in (3.12) and (3.33), respectively. Moreover:

1. The limits $\lim_{t \to 0} T_0^3 C(t)$ in $H^{k+\ell-1}(\Sigma)$, $\lim_{t \to 0} |t|^{-1} U_0^2$ in $H^{k+\ell}(\Sigma)$, $\lim_{t \to 0} |t|^{-1} U_0^3$ in $H^{k+\ell}(\Sigma)$ and $\lim_{t \to 0} |t|^{-1} U_0^3$ in $H^{k+\ell-2}(\Sigma)$, which we denote by $W_0 C(t)$, $W_0^1 C(t)$, $W_0^2 C(t)$, and $W_0^3 C(t)$, respectively, exist, and for any $\sigma > 0$, the estimates

$$\|(t) - T_1 C(t) - W_0 C(t))||_{H^{k+\ell-1}(\Sigma)} \lesssim |t|^{1-\Gamma_1 + \Gamma_2 + |t|^{2(\Gamma_3 - \sigma)}}$$

(3.90)

and

$$\|(t) - T_2 C(t) - W_0 C(t))||_{H^{k+\ell}(\Sigma)} + \|(t) - T_2 C(t) - W_0 C(t))||_{H^{k+\ell}(\Sigma)}$$

(3.91)

hold for all $t \in [T_0, 0]$.

2. The solution $V_C$ agrees with the solution $V_S$ of the singular initial value problem of the Euler equations (1.4) with equation of state (3.5) asserted by Theorem 3.1 for the asymptotic data $v_0 = (W_0^0 C(t), W_0^1 C(t), W_0^2 C(t), W_0^3 C(t))$. 

3. The asymptotic data $v_\ast$ and $\tilde{v}_\ast = (W_0^0 C(t), W_0^1 C(t), W_0^2 C(t), W_0^3 C(t))^{tr}$ are close in the sense

$$\|v_\ast - v_\ast||_{H^{k+\ell-1}(\Sigma)} \lesssim \|V_0||_{H^{k+\ell}(\Sigma)} + |T_0|$$

(3.92)

with $\epsilon$ defined in Theorem 3.1, and,

$$\|v_\ast - v_\ast||_{H^{k+\ell}(\Sigma)} \lesssim \|V_0||_{H^{k+\ell}(\Sigma)}$$

(3.93)

$$+ |T_0|^{1-\Gamma_1} + |T_0|^{2(\Gamma_3 - \sigma) - 2\sigma} + |T_0|^{2(\ell_0 - \ell - 1)(\Gamma_3 - \sigma) - (\Gamma_1 - \Gamma_3) - \sigma},$$

for each $i = 1, 2, 3$.

Before proceeding with the proof of this theorem, we briefly discuss its consequences and make a few remarks. First, given a solution of the singular initial value problem $V_S$ from Theorem 3.1 that is determined by the asymptotic data $(v_0, v_1, v_2, v_3)^{tr}$, Theorem 3.6 yields an open family of perturbations of $V_S$ that are obtained by solving the regular Cauchy problem from the initial time $t = T_0$ towards $t = 0$ with Cauchy data perturbed around the value $V_S(T_0)$ of the solution of the singular initial value problem at $t = T_0$. Furthermore, Theorem 3.6 guarantees, for a sufficiently small perturbation of $V_S(T_0)$, that the resulting solution of the Cauchy problem extends all the way to $t = 0$ (global existence) and that the components of the fluid vector field, when rescaled with the appropriate powers of $t$, converge as $t \to 0$ to the asymptotic data $(W_0^0 C(t), W_0^1 C(t), W_0^2 C(t), W_0^3 C(t))^{tr}$. It is important to note that the asymptotic data generated this way will, in general, be different from the asymptotic data $(v_\ast, v_\ast, v_\ast, v_\ast)^{tr}$ that determines the singular solution $V_S$. However, it is a consequence of Theorem 3.6 that the perturbed solution will agree with the solution to the singular initial value problem that is generated from the asymptotic data $(W_0^0 C(t), W_0^1 C(t), W_0^2 C(t), W_0^3 C(t))^{tr}$ via Theorem 3.1. This leads to the important conclusion that perturbations of solutions of the singular initial value problem will again be solutions of the singular initial value problem.

Next, we observe that in the unperturbed case, that is, $V_0 = 0$ in Theorem 3.6, we have $V_C = V_S$ and $(v_\ast, v_\ast, v_\ast, v_\ast)^{tr} = (W_0^0 C(t), W_0^1 C(t), W_0^2 C(t), W_0^3 C(t))^{tr}$. Comparing (3.10), (3.90) and (3.91) with the given value of $\epsilon$, it is interesting to note that Theorem 3.6 yields larger decay exponents compared to Theorem 3.1. The reason for this difference is due to the regularity requirements which are higher for Theorem 3.6.

Let us comment on part (3) of the theorem which establishes the closeness of the given asymptotic data $v_\ast$ (which determine $V_S$) and the perturbed asymptotic data $\tilde{v}_\ast$ obtained as the limit of the perturbed solution $V_C$ observing that all powers of $|T_0|$ in (3.92) and (3.93) are positive by virtue of the hypothesis. We expect however that all terms involving powers of $|T_0|$ on the right sides of these estimates can in fact be removed by performing a more detailed analysis especially of $V_S$ and the nonlinear terms in the Euler equations (see the corresponding remark after Proposition 3.7 and, especially, the suboptimal treatment of nonlinear terms in the step from (3.150) to (3.151) which we have performed here for brevity). As a consequence of this we cannot conclude here that the map $V_0 \mapsto \tilde{v}_\ast$ given by the theorem is continuous from the ball of radius $\delta$ in $H^{k+\ell}(\Sigma)$ to $H^{k+\ell}(\Sigma)$. Even though we think that establishing this notion of continuity would be possible with our techniques, we think that it would only be interesting if we could pair it with a corresponding notion of continuity for the forward map given by Theorem 3.1 to show that it is a homeomorphism in a $C^\infty$-topology as pioneered by Ringström [41–43]. Conclusions regarding the
\(C^\infty\)-setting can however not be drawn from the current versions of Theorems 2.2 and A.2, but we expect that it would not be difficult to modify the proofs of these theorems so that such conclusions could be drawn.

We conclude our discussion of Theorem 3.6 with two additional comments. First, Remark 3.2 also applies to Theorem 3.6, and consequently this stability result automatically extends to the more general case of fluids on Kasner-scalar field spacetimes. Second, the stability of Euler fluids near Kasner singularities in the sense of Theorem 3.6 was outstanding even in the Gowdy symmetric setting; see [16] and the related numerical investigations from [15] for details.

3.2.1. Proof of Theorem 3.6. The proof of Theorem 3.6 involves three main steps. We begin by fixing asymptotic data \(v_* = (v^0_*, \ldots, v^3_*)^T \in H^{k+\ell}(\Sigma)\) with \(v^0_* > 0\) and \(\ell\) as defined in Theorem 3.1. We then label the resulting solution to the singular value problem of the Euler equations that is generated by this asymptotic data via Theorem 3.1 by \(V_*\).

Step 1: Global existence and preliminary estimates of small data perturbations of \(V_*\).

Letting \(V_C\) denote the solution of the Cauchy problem for the Euler equations (1.4) with initial data at \(T_0 < 0\) determined by

\[
V_C(T_0) = V_{C,0} := V_S(T_0) + V_0, \tag{3.94}
\]

where \(V_0\) is to be considered as the perturbed initial data that measures the deviation from the solution \(V_S(T_0)\) at \(t = T_0 > 0\), we set

\[
U_C = T^{-1}V_C, \tag{3.95}
\]

where \(T = T(t)\) is defined by (3.12), and we recall that \(U_C\) satisfies the system (3.22) – (3.28). To proceed, we subtract off from \(U_C\) the first component of the asymptotic data \(v_*\) by setting

\[
u := U_C - v_x, \quad v_x := (v^0_*, 0, 0, 0)^T. \tag{3.96}
\]

A straightforward calculation then shows that the system (3.22) – (3.28) is given in terms of \(u\) by

\[
B^0(v_x + u)\partial_t u + B^1(v_x + u)\partial_x u + B^2(t, v_x + u)\partial_y u + B^3(t, v_x + u)\partial_z u = \frac{1}{t}B(v_x + u)u - B^1(t, v_x + u)\partial_x v_x - B^2(t, v_x + u)\partial_y v_x - B^3(t, v_x + u)\partial_z v_x + G(t, u), \tag{3.97}
\]

where in deriving this we have exploited the fact that \(v_x\), defined in (3.96), does not depend on \(t\), \(B(v_x + u)v_x = 0\) as a consequence of (3.27), and \(G(v_x + u) = G(u)\) thanks to (3.28).

Next, we choose \(V_0 \in H^k(\Sigma, V)\), and we observe that the Cauchy data (3.94) for the Euler equations translate into the Cauchy data

\[
u(T_0) = u_0 := T(T_0)^{-1}V_S(T_0) - v_x + \tilde{T}(T_0)^{-1}V_S. \tag{3.98}
\]

for the system (3.97). Recall that \(T(t)\) and \(\tilde{T}(t)\) are defined in (3.12) and (3.33), respectively. Notice here that it is a consequence of (3.14), (3.95), and (3.96), and Theorem 3.1, in particular (3.10) for \(V = V_S\), that (3.98) implies

\[
\|u_0\|_{H^k(\Sigma)} \lesssim \|T_0\|^k + \|V_0\|_{H^k(\Sigma)}, \tag{3.99}
\]

where the implicit constant depends monotonically on \(T_0\) and on the choice of asymptotic data \(v_*\) and therefore on \(V_S(T_0)\), but not on \(V_0\). This estimate will allow us to meet all smallness requirement for \(u_0\), which we will encounter in the proof of Theorem 3.6, by imposing appropriate smallness conditions on \(T_0\) together with corresponding bounds on \(V_0\).

To complete the first step of the proof, we now establish the existence of solutions to the Cauchy problem for the system (3.97) where the initial data at time \(t = T_0\) is specified according to (3.98) and must be chosen suitably small. The precise statement for the existence result is given in the following proposition. The proposition also yields a first preliminary estimate of the behaviour of the solutions at \(t = 0\). This preliminary estimate, which is not yet sharp, will be improved successively in Step 2 of the proof. We express all the results in terms of the quantity \(\tilde{U}_C\) instead of \(u\); cf. (3.96).

\[\text{3.5}^\text{The stronger regularity requirements for } v_* \text{ in Theorem 3.6 will only become relevant in Step 2 of this proof.}\]

\[\text{3.6}^\text{Observe carefully the variable } u \text{ defined here is in general different from the variable } u \text{ introduced in Section 3.1.2 and (3.34).}\]

\[\text{3.7}^\text{The stronger regularity requirements for } V_0 \text{ in Theorem 3.6 will only become relevant in Step 2 of this proof.}\]
Proposition 3.7. Suppose $K$ and $\gamma$ satisfy (3.7), and $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are defined by (3.8), $k \in \mathbb{Z}_{\geq 3}$, $l \in \mathbb{Z}_{\geq 1}$ with $l > \Gamma_1/q$ for $q = \min\{1 - \Gamma_1, 2\Gamma_3\}$ and $\sigma \in (0, \Gamma_3)$. Further, for given asymptotic data $v_* = (v_*^0, \ldots, v_*^3)^T \in H^{k+2}$ with $v_*^0 > 0$, let $V_0$ be the solution asserted by Theorem 3.1 of the singular initial value problem of the Euler equations (1.4) with equation of state (3.5) defined on the Kasner spacetime given by (3.2) on a time interval $[T_0, 0)$.

Then there exists $\delta > 0$ such that for all sufficiently small $T_0 \in [\bar{T}_0, 0)$ and any $V_0 \in H^k(\Sigma)$ with $\|V_0\|_{H^k(\Sigma)} \leq \delta$, the Cauchy problem of the Euler equations (1.4) with equation of state (3.5) defined on the Kasner spacetime given by (3.2) with Cauchy data $V_{C, 0} = V_S(T_0) + \hat{T}^{-1}(T_0)T(T_0)V_0$ imposed at $t = T_0$ has a unique solution $V_C(t) = T(t)U_C(t)$ with

$$U_C \in C^0_\delta([T_0, 0), H^k(\Sigma)) \cap C^1([T_0, 0), H^{k-1}(\Sigma)),$$

where $T(t)$ and $\hat{T}(t)$ are defined in (3.12) and (3.33), respectively, and $U_C$ is bounded by

$$\|U_C^1(t)\|_{H^{k-1}(\Sigma)} + \|U_C^2(t)\|_{H^{k-1}(\Sigma)} + \|U_C^3(t)\|_{H^{k-1}(\Sigma)} \lesssim |t|^{\Gamma_3 - \sigma}$$

for all $t \in [T_0, 0)$. Moreover, the limit $\lim_{t \to 0} U_C^0$, denoted $U_C^0(0)$, exists in $H^{k-1}(\Sigma)$ and satisfies

$$\|U_C^0(0) - v_*^0\|_{H^{k-1}(\Sigma)} \lesssim |T_0|^{\epsilon} + \|V_0\|_{H^k(\Sigma)},$$

with $\epsilon$ defined in Theorem 3.1, and the estimate

$$\|U_C^0(t) - U_C^0(0)\|_{H^{k-1}(\Sigma)} \lesssim |t|^{1 - \Gamma_1 + \Gamma_3} + |t|^{2(\Gamma_3 - \sigma)}$$

holds for all $t \in [T_0, 0)$. Finally, $U_C$ satisfies $U_C^0(0) > 0$ provided $[T_0]$ and $\|V_0\|_{H^k(\Sigma)}$ are chosen sufficiently small.

We emphasize here that the implicit smallness condition for $T_0$ does not depend on $V_0$ so long as $\|V_0\|_{H^k(\Sigma)} \leq \delta$. This proposition establishes the existence of a limit $U_C^0(0)$ for the time component of the fluid vector field and (3.102) states that this limit is close to the asymptotic data component $v_*^0$ provided $T_0$ is small. We expect that the first term on the right side of (3.102) can be avoided by using more detailed estimates of $V_S$; in fact, it follows from uniqueness for the Cauchy problem that $v_* = U_C^0(0)$ if $V_0 = 0$. In any case, it will be the purpose of Proposition 3.8 to establish corresponding statements for the spatial components of the fluid vector field.

Proof of Proposition 3.7. We will prove this proposition by first showing that the system (3.97) satisfies all of the coefficient assumptions from Appendix A.1. After doing so, the proof will then follow directly from an application of Theorem A.2 together with Remark A.3.(i). To this end, we express (3.97) in the form

$$B^0(v_x, u)\partial_t u + B^1(v_x, u)\partial_x u + B^2(t, v_x, u)\partial_y u + B^3(t, v_x, u)\partial_z u = \frac{1}{t}B(v_x, u)\bar{\partial}u + F(t, w_x, u)$$

(3.104)

where

$$w_x = (v_x, \partial_x v_x, \partial_y v_x, \partial_z v_x),$$

$$\bar{\partial} = \text{diag}(0, 1, 1, 1), \quad \partial^\perp = \text{diag}(1, 0, 0, 0),$$

and, with the same slight abuse of notation as in Section 3.1.2, we have set

$$B^0(\bar{Z}, v) = B^0(\bar{Z} + v),$$

$$B^1(\bar{Z}, v) = B^1(\bar{Z} + v),$$

$$B^2(t, \bar{Z}, v) = B^2(t, \bar{Z} + v),$$

$$B^3(t, \bar{Z}, v) = B^3(t, \bar{Z} + v),$$

$$B(\bar{Z}, v) = B(\bar{Z} + v) + r(\bar{Z} + v)(\bar{Z}^2 + v^2) \partial^\perp$$

(3.106)

where $r$ is the polynomial used in (3.23)–(3.28) and

$$F(t, Z, v) = -B^1(Z_0 + v)Z_{01} - B^2(t, Z_0 + v)Z_{02} - B^3(t, Z_0 + v)Z_{03} + G(t, v).$$

(3.107)

Here, $a \in \mathbb{R}$ is a constant to be fixed below, $v \in V$, $\bar{Z} \in Z_1$, $Z_1$ is the rank-1-subbundle of $V$ defined by elements of the form $\bar{Z} = (\bar{Z}^0, 0, 0, 0)^T$, and $Z \in Z_2$ where $Z_2$ is the trivial rank-4 bundle $Z_2 = Z_1 \oplus Z_1 \oplus Z_1 \oplus Z_1$, whose elements we can express in components $Z = (Z_0, Z_{01}, Z_{02}, Z_{03})^T$. $Z_1$
Now, for any $R > 0$ and $r \in (0, R)$, we further define $Z_1$ as the bounded open subset of $Z_1$ given by vectors $\tilde{Z}$ with components $(\tilde{Z}^0, 0, 0)$ with $r < \tilde{Z}^0 < R$. Correspondingly, the bounded open subset $Z_2$ of $Z_2$ is defined as $Z_1 \oplus B_R(Z_1) \oplus B_R(Z_1) \oplus B_R(\mathbb{Z}_1)$. Given these definitions and the expression (3.23) and (3.27), it is not difficult to verify, for any $R > 0$, that
\[
B^0 \in C^\infty (Z_1 \oplus B_R(V), L(\mathbb{V})) \quad \text{and} \quad \tilde{B} \in C^\infty (Z_1 \oplus B_R(V), L(\mathbb{V})).
\] (3.108)

Moreover, given
\[
\sigma \in (0, \Gamma_3),
\]
we can, by reducing the size of $R$ if necessary, ensure that the inequality
\[
\frac{1}{\gamma_1} \text{id}_{\gamma(v)} \leq B^0(\tilde{Z}, v) \leq \frac{1}{\kappa} \tilde{B}(\tilde{Z}, v) \leq \frac{1}{\gamma_2} \text{id}_{\gamma(v)}
\] (3.109)
holds for all $(\tilde{Z}, v) \in Z_1 \oplus B_R(V)$ by setting
\[
\gamma_1 = \frac{2}{r^3(\gamma - 1)}, \quad \kappa = \Gamma_3 - \sigma, \quad a = \Gamma_3/((\gamma - 1),
\]
and choosing $\gamma_2 > 0$ accordingly making use of (3.7) and (3.14).

We further observe, with the help of (3.23) and (3.27), that the matrices $P$ and $\tilde{B}$ commute, that is,
\[
\left[ P, B^0(\tilde{Z}, v) \right] = 0
\] (3.111)
for all $(t, \tilde{Z}, v) \in [T_0, 0] \times Z_1 \oplus B_R(V)$, and the matrix $B^0$ is symmetric and satisfies
\[
P B^0(\tilde{Z}, v) P^\perp = (P^\perp B^0(\tilde{Z}, v) P)^\dagger = O(P v).
\] (3.112)
It also follows from (3.23), (3.27), (3.29) and (3.110) that $\tilde{B}$ and $B^0$ can be expanded, respectively, as
\[
\tilde{B}(\tilde{Z}, v) = \tilde{B}(\tilde{Z}) + O(v), \quad \tilde{B}(\tilde{Z}) = (\gamma - 1)(\tilde{Z}^0)^3 \text{diag}(a, \Gamma_1, \Gamma_2, \Gamma_3),
\] (3.113)
and
\[
B^0(\tilde{Z}, v) = B^0(\tilde{Z}) + O(v), \quad B^0(\tilde{Z}) = (\tilde{Z}^0)^3 \text{diag}(1, \gamma - 1, \gamma - 1, \gamma - 1),
\] (3.114)
where the maps $\tilde{B}^0$ and $\tilde{B}$ lie in $C^0([T_0, 0], C^\infty(Z_1 + B_R(V), L(\mathbb{V})))$. We further note by (3.24) – (3.26) that the matrices $B^1, B^2$ and $B^3$ are all symmetric and satisfy
\[
B^1 \in C^\infty (Z_1 \oplus B_R(V), L(\mathbb{V})) \quad \text{and} \quad B^2, B^3 \in C^0([T_0, 0], C^\infty(Z_1 \oplus B_R(V), L(\mathbb{V}))).
\] (3.115)

Next, we turn our attention to the map $F$ defined above by (3.107). We decompose $F$ as
\[
F(t, Z, v) = F_1(t, Z, v) + \frac{1}{|t|^{1-p}} F_2(t, Z, v) + \frac{1}{|t|^p} F_3(t, Z, v)
\] (3.116)
where
\[
F_1(t, Z, v) = -(1-t)^{1-p} \left( B^1(\tilde{Z}(t, Z_0, 0)Z_0 + B^2(t, Z_0, 0)Z_0 + B^3(t, Z_0, 0)Z_0 \right),
\] (3.117)
\[
F_2(t, Z, v) = -(1-t)^{1-p} \left( (B^1(\tilde{Z}(t, Z_0, 0) - B^1(\tilde{Z}(t, Z_0, 0) - B^2(t, Z_0, 0)Z_0 + B^3(t, Z_0, 0)Z_0 \right)
\] (3.118)
and
\[
F_3(t, Z, v) = -t G(t, v).
\] (3.119)
Assuming for the moment that $p \in (0, 1]$, it is not difficult to verify from (3.24) – (3.26), (3.28) and (3.117) – (3.119) that
\[
\tilde{F} \in C^0([T_0, 0], C^\infty(Z_2, \mathbb{V})) \quad \text{and} \quad F_0, F_2 \in C^0([T_0, 0], C^\infty(Z_2 \oplus B_R(V), \mathbb{V})).
\] (3.120)
We further observe from (3.28) and (3.118)-(3.119) that
\[
F_0 = O(v), \quad P F_0 = 0 \quad \text{and} \quad F_2 = O \left( \frac{\lambda_{\gamma} v \otimes v}{R} \right)
\] (3.121)
for some constant $\lambda_3 = O(R)$.
Before proceeding, we note that because \( v_0^a \in H^{k+1}(\Sigma) \) satisfies \( v_0^a > 0 \) and is time-independent by assumption, we have by (3.96) and the Sobolev inequality that there exist constants \( 0 < r < R_0 \) such that

\[
    v_\times \in H^{k+1}(\Sigma, \mathbb{Z}_1) \quad \text{and} \quad \partial_r v_\times = 0
\]

for any \( R \geq R_0 \). This, in turn, implies with the help of the (3.105) and the Sobolev inequality that

\[
    w_\times \in H^k(\Sigma, \mathbb{Z}_2) \quad \text{and} \quad \partial_r w_\times = 0 \quad (3.122)
\]

for \( R \) chosen sufficiently large.

To complete the analysis of the coefficients of the Fuchsian equation (3.104), we need to examine the divergence map \( \text{div} B \) given locally by (2.6), where all the connection coefficients in (2.6) vanish since the connection we are using is flat and its coefficients vanish in the global frame \( \{ \partial_x, \partial_y, \partial_z \} \). Furthermore, since \( v_\times \) is independent of \( t \), we can take the map \( \text{div} B \) to be independent of the variable \( \dot{z}_1 \) since \( z_1 = v_\times \) and \( \partial_t v_\times = 0 \). This implies in particular that the constant \( q \) in (2.6) plays no role here. We recall also that the map \( \text{div} B \), when evaluated on solutions of (3.104), is given by \( \text{div} B = \partial_t B^0 + \partial_x B^1 + \partial_y B^2 + \partial_z B^3 \), see (2.8).

Now, it is easy to see that all the terms in \( \text{div} B \) other than

\[
    M := \frac{1}{t} D_t B^0(x, z_1, v) \cdot \left( B_0(x, z_1, v) \right)^{-1} \left[ B(x, z_1, v) \mathcal{P} v + tG(t, x, v) \right]
\]

are \( O(|t|^{-(1-p)} \theta) \) for some \( \theta > 0 \) provided \( p \) is chosen to satisfy

\[
    p = 1 - \Gamma_1 + \Gamma_3.
\]

To proceed with the analysis, we decompose the \( M \) as follows

\[
    M = \frac{1}{t} \left[ \left( D_{2x} B^0(x, z_1, v) \cdot \frac{\mathcal{P}(B_0(x, z_1, v))^{-1}}{\mathcal{P} v} + D_{2y} B^0(x, z_1, v) \cdot \frac{\mathcal{P}(B_0(x, z_1, v))^{-1}}{\mathcal{P} v} \right) B(x, z_1, v) \mathcal{P} v \right]
\]

are \( O(|t|^{-1} \mathcal{P} v) \) for some \( \theta > 0 \) provided \( p \) is chosen to satisfy

\[
    p = 1 - \Gamma_1 + \Gamma_3.
\]

From this expression, it is clear that there exist constants \( \beta_1, \beta_3 \) and \( \beta_5 \) of \( O(R) \) such that

\[
    P \mathcal{M} \mathcal{P} = O(|t|^{-1} \beta_1), \quad P \mathcal{M} \mathcal{P}^\perp = O \left( |t|^{-1} \frac{\beta_3}{\mathcal{P} v} \right), \quad P^\perp \mathcal{M} \mathcal{P} = O \left( |t|^{-1} \frac{\beta_5}{\mathcal{P} v} \right). \quad (3.123)
\]

In order to show that there also exists a constant \( \beta_7 = O(R^2) \) such that

\[
    P^\perp \mathcal{M} \mathcal{P}^\perp = O \left( |t|^{-1} \frac{\beta_7}{R^2} \mathcal{P} v \right), \quad (3.124)
\]

we need to have a closer look at \( P^\perp D_{2x} B^0(x, z_1, v) \mathcal{P}^\perp \) in the above formula for \( M \). Since it follows immediately from (3.23), (3.30) and (3.106) that \( P^\perp D_{2x} B^0(x, z_1, v) \mathcal{P} \) \( \mathcal{P} \) \( = O(\mathcal{P} v) \), we see that (3.124) does in fact hold for some constant \( \beta_7 = O(R^2) \).

From the above analysis, and in particular the results (3.106), (3.108) – (3.116), (3.120)-(3.121) and (3.123) – (3.124), it follows that the Fuchsian equation (3.104) satisfies all of the coefficient assumptions from Appendix A.1 for the following choice of constants:

(i) \( \gamma_1, \gamma_2, \kappa, p, \theta, \lambda_3, \beta_1, \beta_3, \beta_5 \) and \( \beta_7 \) as above,

(ii) and \( p = 0, \lambda_1 = \lambda_2 = 0, \) and \( \beta_3 = \beta_2 = \beta_4 = \beta_6 = 0. \)

Noting that the constant \( b \) defined by (A.4) vanishes, it is clear, by reducing the size of \( R \) if necessary, that we can ensure that all the constants \( \beta_i \) and \( \lambda_i \) are small enough so that \( \kappa \) satisfies the inequality (A.2). Additionally, we note from (3.14), (3.24) – (3.26), (3.117) and (3.122) that

\[
    \| \mathcal{F}(t, w_\times) \|_{H^k(\Sigma)} \lesssim |t|^{1-\Gamma_1+\Gamma_3-p},
\]

and, since \( \mathcal{F}(t, w_\times) \) is independent of the choice of \( V_0 \), the implicit constant here does not depend on \( V_0 \) (but on \( \mathcal{V}_0 \)). From this, we deduce, since \( 1 - \Gamma_1 + \Gamma_3 > 0 \), that

\[
    \int_{T_0}^0 |s|^{p-1} \| \mathcal{F}(s, w_\times) \|_{H^k(\Sigma)} ds \lesssim \int_{T_0}^0 (-s)^{\Gamma_3-\Gamma_1} ds \lesssim |T_0|^\Gamma_3-\Gamma_1+1. \quad (3.125)
\]
Thus, we can arrange that
\[
\int_{T_0}^0 |s|^{p-1} \| \hat{F}(s, w_\infty) \|_{H^k(\Sigma)} \, ds \leq \delta \tag{3.126}
\]
in a way that is independent of the choice of $V_0$ for any given $\delta > 0$ by reducing the size of $|T_0|$ as needed. This allows us to satisfy the smallness condition in Theorem A.2 for the term $\hat{F}$ from the expansion (3.116) for $F$.

Given all this, an application of Theorem A.2 from the Appendix A.2 yields, for $u_0$ satisfying $\|u_0\|_{H^k(\Sigma)} \leq \delta$ with $\delta > 0$ sufficiently small, the existence of a unique solution $u$ of the Cauchy problem of (3.97) with $u(T_0) = u_0$ and satisfying
\[
u \in C^0_b\left(\{T_0, 0\}, H^k(\Sigma)\right) \cap C^1\left(\{T_0, 0\}, H^{k-1}(\Sigma)\right) . \tag{3.127}
\]
Shrinking $R$ even further if necessary to make $\beta_1$ sufficiently small, we can make the quantity $\zeta$ in Theorem A.2 arbitrarily close to $\kappa = \kappa$, and the estimate
\[
\|u(t)\|_{H^{k-1}(\Sigma)} + \|u^0(t)\|_{H^{k-1}(\Sigma)} + |t|^{1-\Gamma_1+\Gamma_3} + |t|^{2(\Gamma_3-\sigma)}, \quad T_0 \leq t < 0, \tag{3.128}
\]
follows from the decay estimate (A.6) for $\|\mathcal{P}u\|_{H^{k-1}}$ from Theorem A.2 since $\lambda_1 = \alpha = b = 0$, $\kappa = \Gamma_3 - \sigma > 0$, and the projection matrix $\mathcal{P}$ is defined by (3.106). Theorem A.2 also yields the existence of the limit $\lim_{t \to 0} u^1$ in $H^{k-1}(\Sigma)$, which we denote as $u^0(0)$, and the estimate
\[
\|u^0(0) - u^0(0)\|_{H^{k-1}(\Sigma)} \leq |t|^{1-\Gamma_1+\Gamma_3} + |t|^{2(\Gamma_3-\sigma)}, \quad T_0 \leq t < 0, \tag{3.129}
\]
is also a direct consequence of this theorem and the improvement to the decay estimate for $\|u(t) - u(0)\|_{H^{k-1}}$ described in Remark A.3.(i). Finally, from the energy estimate from Theorem A.2, we have that
\[
\|u(t)\|_{H^{k-1}(\Sigma)} \leq \|u_0\|_{H^k(\Sigma)} + \int_{T_0}^0 |s|^{p-1} \| \hat{F}(s, w_\infty) \|_{H^k(\Sigma)} \, ds .
\]
In the light of (3.125), it is then clear that the inequality
\[
\|u^0(0)\|_{H^{k-1}(\Sigma)} \leq \|u_0\|_{H^k(\Sigma)} + |T_0|^{1-\Gamma_1+\Gamma_3} \tag{3.130}
\]
holds.

Finally, we notice that the existence of the solution $V_S$ is guaranteed by Theorem 3.1 under our hypothesis here. Since $U^0_C(0) = u^0 + v^0(0)$, the proof is completed by invoking (3.96), (3.98) and (3.99).

\section*{Step 2: Improved asymptotics and existence of limits for all components of $V_C$}

Setting
\[
W_C = \hat{T}UC \tag{3.131}
\]
where $\hat{T}$ is defined by (3.33), Proposition 3.7 implies that
\[
W^0_C, |t|^{-(\Gamma_3-\Gamma_1-\sigma)}W^1_C, |t|^{-(\Gamma_3-\Gamma_2-\sigma)}W^2_C, |t|^\sigma W^3_C \in C^0_b\left(\{T_0, 0\}, H^{k-1}(\Sigma)\right) , \tag{3.132}
\]
and that the limit $W^0_C(0) = \lim_{t \to 0} W^0_C$ exists in $H^{k-1}(\Sigma)$ and is given by
\[
W^0_C(0) = U^0_C(0) .
\]
The aim of this step of the proof is now to establish the existence of limits for all components of $W_C$ at the cost of some amount of differentiability. The precise statement is summarised in the following proposition.

\begin{proposition}
Suppose in addition to the hypothesis of Proposition 3.7 that the following hold: $\nu = (u^0, \ldots, v^0)^T \in H^{k+\kappa_0 + \ell}(\Sigma)$ with $u^0 > 0$, $V_0 \in H^{k+\kappa_0}(\Sigma)$ with $\|V_0\|_{H^{k+\kappa_0}(\Sigma)} \leq \delta$, and $0 < \sigma < \min\{\Gamma_3/4, \Gamma_2 - \Gamma_3\}$ for any integer $\kappa_0$ satisfying
\[
\ell_0 \geq \ell + 2, \quad \ell_0 > \ell + \frac{1}{2} + \frac{\Gamma_1}{2(\Gamma_3 - \sigma)}. \tag{3.133}
\]
Then the solution $U_C$ asserted by Proposition 3.7 has the property that
\[
U_C \in C^0_b\left(\{T_0, 0\}, H^{k+\kappa_0}(\Sigma)\right) \cap C^1\left(\{T_0, 0\}, H^{k+\kappa_0-1}(\Sigma)\right) ,
\]

\end{proposition}
and the limits $\lim_{t \to 0} U_0^k$ in $H^{k+\ell_0-1}(\Sigma)$, $\lim_{t \to 0} |t|^{-1} U_0^k$ in $H^{k+\ell}(\Sigma)$, $\lim_{t \to 0} |t|^{-2} U_0^k$ in $H^{k+\ell}(\Sigma)$ and $\lim_{t \to 0} |t|^{-3} U_0^k$ in $H^{k+\ell_0-2}(\Sigma)$, which we denote by $W_C^0(0)$, $W_C^1(0)$, $W_C^2(0)$, and $W_C^3(0)$, respectively, all exist. Moreover, the estimates

$$
\|W_0^0(t) - W_C^0(0)\|_{H^{k+\ell_0-1}(\Sigma)} \lesssim |t|^{1-\Gamma_1+\Gamma_3} + |t|^{2(\Gamma_3-\sigma)},
$$

$$
\|W_0^1(t) - W_C^1(0)\|_{H^{k+\ell}(\Sigma)} + \|W_0^2(t) - W_C^2(0)\|_{H^{k+\ell}(\Sigma)} \lesssim |t|^{1-\Gamma_1} + |t|^{2(\Gamma_3-\sigma)-2\sigma} + |t|^{2(\ell_0-\ell-1)(\Gamma_3-\sigma)-(\Gamma_1-\Gamma_3) - \sigma},
$$

and

$$
\|W_0^3(t) - W_C^3(0)\|_{H^{k+\ell_0-2}(\Sigma)} \lesssim |t|^{1-\Gamma_1} + |t|^{2(\Gamma_3-\sigma)-2\sigma},
$$

hold for all $t \in [T_0, 0]$.

Given another field $\tilde{V}_0 \in H^{k+\ell_0}(\Sigma)$ with $\|\tilde{V}_0\|_{H^{k+\ell_0}(\Sigma)} \leq \delta$, the limits $W_{C}^0(t)$ and $\tilde{W}_{C}^0(t)$ attained by the corresponding solutions $W_C(t)$ and $\tilde{W}_C(t)$, respectively, launched from the same initial time $T_0$, satisfy the estimate

$$
\|W_C^0(0) - \tilde{W}_C^0(0)\|_{H^{k+\ell}(\Sigma)} \leq \|V_0^0 - \tilde{V}_0^0\|_{H^{k+\ell}(\Sigma)} + C([T_0]^{1-\Gamma_1} + [T_0]^{2(\Gamma_3-\sigma)-2\sigma} + [T_0]^{2(\ell_0-\ell-1)(\Gamma_3-\sigma)-(\Gamma_1-\Gamma_3) - \sigma}),
$$

for each $i = 1, 2, 3$.

Before we prove Proposition 3.8, we observe that every $\ell_0$ that satisfies (3.89) can also be made to satisfy (3.133) by choosing $\sigma > 0$ sufficiently small. Further, we note that the estimate (3.90) is the same as (3.134), and the estimate (3.91) can be deduced from (3.135) and (3.136) if

$$
2(\ell_0 - \ell - 1)(\Gamma_3 - \sigma) - (\Gamma_1 - \Gamma_3) - \sigma \geq 2(\Gamma_3 - \sigma) - 2\sigma
$$

which we establish as follows. We observe that every $\ell_0$ that satisfies (3.89) also satisfies

$$
\ell_0 - \ell \geq \frac{3}{2} + \frac{\Gamma_1}{2(\Gamma_3 - \sigma)} - \frac{\sigma}{\Gamma_3 - \sigma}
$$

provided we choose $\sigma > 0$ sufficiently small. Given this, we have

$$
2(\ell_0 - \ell - 1)(\Gamma_3 - \sigma) - (\Gamma_1 - \Gamma_3) - \sigma - 2(\Gamma_3 - \sigma) + 2\sigma
$$

$$
= 2(\ell_0 - \ell - 3/2)(\Gamma_3 - \sigma) - (\Gamma_3 - \sigma) - \Gamma_1 + \Gamma_3 + \sigma
$$

$$
\geq (\Gamma_1 - 2\sigma) - (\Gamma_3 - \sigma) - \Gamma_1 + \Gamma_3 + \sigma = 0,
$$

and consequently, the estimate (3.91) follows from (3.135) and (3.136). Proposition 3.8 therefore can be used to complete the proof of Theorem 3.6 except for the last statement regarding the solution $\tilde{V}_S$ of the singular initial value problem, which we will address in Step 3 of this proof below.

**Proof.** By assumption $k \geq 5$ and $\ell_0 \geq 0$ (as a consequence of (3.133)), and so, we have by (3.132) that

$$
W_C^0, |t|^{\Gamma_1-\Gamma_3+\sigma} W_C^1, |t|^{\Gamma_2-\Gamma_3+\ell} W_C^2, |t|^\ell W_C^3 \in C^0([T_0, 0), H^{k+\ell_0-1}(\Sigma)).
$$

(3.138)

Additionally, we observe from the definitions (3.60) and (3.131) that $W_C$ satisfies (3.61) and (3.62) if we set $W_{n+1} = W_n = W_C$, $B_{n+1} = B_n = B_C$, and $G_{n+1} = G_n = G_C$ in all the expressions there. Doing so shows that $W_C$ satisfies

$$
\partial_t W_C = -B_C \partial_t W_C + G_C.
$$

(3.140)

Integrating this equation in time and expressing the result in components, we find that the components $W_C^j$, $j = 0, 1, 2, 3$, satisfy

$$
W_C^j(t) = W_C^j(T_0) + \int_{T_0}^t \left( -\sum_{i=1}^3 \sum_{k=0}^3 \tilde{B}_C^{ki} \partial_t W_C^k(s) + G_C^i(s) \right) ds
$$

(3.141)

for all $t \in [T_0, 0]$. Applying the $H^l(\Sigma)$ norm to this expression for any integer $l$ with $2 \leq l \leq k + \ell_0 - 2$, we obtain, with the help of the triangle inequality, the estimate

$$
\|W_C^j(t)\|_{H^l(\Sigma)} \leq \|W_C^j(T_0)\|_{H^l(\Sigma)} + \int_{T_0}^t \left( -\sum_{i=1}^3 \sum_{k=0}^3 \tilde{B}_C^{ki} \partial_t W_C^k(s) + G_C^i(s) \right) \|\|H^l(\Sigma)ds.
$$

(3.141)
In order to successively improve the estimates (3.138) for the components \(W^1_C\) and \(W^2_C\), we now set up the following inductive argument to show that

\[
|t|^{-\min\{2m(\Gamma_3 - \sigma) - (\Gamma_1 - \Gamma_3) - \sigma, 0\}} W^1_C, |t|^{-\min\{2m(\Gamma_3 - \sigma) - (\Gamma_2 - \Gamma_3) - \sigma, 0\}} W^2_C \in C^0_b([T_0, 0), H^{k+\ell_0-1-m}(\Sigma))
\]

for each integer \(m \in [0, k + \ell_0 - 4]\). The base case \(m = 0\) is given by (3.138). For the induction step \(m \mapsto m + 1\), let us assume that (3.142) holds for an arbitrary integer \(m \in [0, k + \ell_0 - 5]\) and that \(W^1_C\) and \(W^2_C\) satisfy (3.138). For convenience, let us define

\[
\eta_a := -\min\{2m(\Gamma_3 - \sigma) - (\Gamma_a - \Gamma_3), -\sigma\} = \max\{\Gamma_a - \Gamma_3 - 2m(\Gamma_3 - \sigma), \sigma\},
\]

for \(a = 1, 2\). From this, the condition for \(m\) above, the condition for \(\sigma\) in the proposition, and (3.14), we easily see that

\[
0 \leq \eta_a \leq \Gamma_a - \Gamma_3, \quad \eta_2 \leq \eta_1.
\]

A lengthy analysis involving (3.63)–(3.66) and (3.142) then shows, observing that \(k + \ell_0 - 1 - (m + 1) > 3/2\) as required to apply the calculus inequalities, in particular, the product and Moser estimates, we conclude with the help of the analogue of (A.5) for Proposition 3.7 (which was not listed there for brevity) and (3.126) both for \(k\) replaced by \(k + \ell_0\), and, the condition \(\|V_0\|_{H^{k+\ell_0}(\Sigma)} \leq \delta\), that

\[
\| - \sum_{i=1}^{3} \sum_{k=0}^{3} \tilde{B}^{1i}_{3,k} \partial_i W^2_C(s) + C^1_C(s) \|_{H^{k+\ell_0-1-(m+1)}(\Sigma)} \lesssim \|s|^{-\min\{2(\Gamma_3 - \sigma) - \eta_1 - \sigma, 1 - \Gamma_1\}} - 1,
\]

(3.145)

\[
\| - \sum_{i=1}^{3} \sum_{k=0}^{3} \tilde{B}^{2i}_{3,k} \partial_i W^2_C(s) + C^2_C(s) \|_{H^{k+\ell_0-1-(m+1)}(\Sigma)} \lesssim \|s|^{-\min\{2(\Gamma_3 - \sigma) - \eta_2 - \sigma, 1 - \Gamma_1\}} - 1,
\]

(3.146)

\[
\| - \sum_{i=1}^{3} \sum_{k=0}^{3} \tilde{B}^{3i}_{3,k} \partial_i W^2_C(s) + C^3_C(s) \|_{H^{k+\ell_0-1-(m+1)}(\Sigma)} \lesssim \|s|^{-\min\{2(\Gamma_3 - \sigma) - \eta_1 - \sigma, 1 - \Gamma_1\}} - 1,
\]

(3.147)

with implicit constants that are independent of \(V_0\) so long as \(\|V_0\|_{H^{k+\ell_0}(\Sigma)} \leq \delta\). From this (3.141) implies that

\[
|t|^{-\min\{2(\Gamma_3 - \sigma) - \eta_1 - \sigma, 0\}} W^1_C, |t|^{-\min\{2(\Gamma_3 - \sigma) - \eta_2 - \sigma, 0\}} W^2_C \in C^0_b([T_0, 0), H^{k+\ell_0-1-(m+1)}(\Sigma)),
\]

where in deriving this we have invoked (3.14) again. For each \(a = 1, 2\), we have

\[
\min\{2(\Gamma_3 - \sigma) - \eta_a - \sigma, 0\} = \min\{2(\Gamma_3 - \sigma) - \sigma + 2m(\Gamma_3 - \sigma) - (\Gamma_a - \Gamma_3), 2(\Gamma_3 - \sigma) - 2\sigma, 0\} = \min\{2(m + 1)(\Gamma_3 - \sigma) - (\Gamma_a - \Gamma_3) - \sigma, 0\}
\]

thanks to the condition for \(\sigma\) in the hypothesis. This establishes the induction step \(m \mapsto m + 1\) and therefore confirms that (3.142) holds for every integer \(m \in [0, k + \ell_0 - 4]\).

Now, consider an arbitrary monotonically increasing sequence \(\{t_n\}\) of negative numbers in \([T_0, 0]\) that converges to 0. Eq. (3.140) implies

\[
W^j_C(t_n) - W^j_C(t_\bar{n}) = \int_{t_\bar{n}}^{t_n} \left( - \sum_{i=1}^{3} \sum_{k=0}^{3} \tilde{B}^{ij}_{3,k} \partial_i W^k_C(s) + C^j_C(s) \right) ds,
\]

and hence, that

\[
\|W^j_C(t_n) - W^j_C(t_\bar{n})\|_{H^{k+\ell_0-1-(m+1)}(\Sigma)} \leq \int_{t_\bar{n}}^{t_n} \| - \sum_{i=1}^{3} \sum_{k=0}^{3} \tilde{B}^{ij}_{3,k} \partial_i W^k_C(s) + C^j_C(s)\|_{H^{k+\ell_0-1-(m+1)}(\Sigma)} ds
\]

(3.148)

given any integer \(m \in [0, k + \ell_0 - 5]\) provided \(t_\bar{n} \leq t_n\).

Let us now pick \(m = \ell_0 - \ell - 2\). It follows that \(m \in [0, k + \ell_0 - 5]\) because of the first condition for \(\ell_0\) in (3.133) and because \(m = \ell_0 - \ell - 2 \leq (k - 3) + \ell_0 - \ell - 2 \leq k + \ell_0 - 5\) given that \(k \geq 3\) and \(\ell \geq 0\) by assumption. With this choice of \(m\) we also have that \(2(\Gamma_3 - \sigma) - \eta_1 - \sigma > 0\) since, by (3.143), we have \(2(\Gamma_3 - \sigma) - \eta_1 - \sigma = \min\{2(m + 1)(\Gamma_3 - \sigma) - (\Gamma_1 - \Gamma_3) - \sigma, 2(\Gamma_3 - \sigma) - 2\sigma\}\), from which we get
that and obtain the solution guarantees that which establishes (3.133). Estimates (3.145), (3.146) together with (3.148) therefore imply that \{W^{\alpha}_{C}(t_n)\} is a Cauchy sequence in \(H^{k+\ell}(\Sigma)\) for \(\alpha = 1, 2\). Consequently, the sequence \{W^{\alpha}_{C}(t_n)\} has a limit in \(H^{k+\ell}(\Sigma)\), which we denote by \(W^{\alpha}_{C}(0)\) for \(\alpha = 1, 2\). We recall that \(t_0\) is the same as before for two Cauchy data perturbations \(V_0\) and \(\tilde{V}_0\) imposed at the same time \(T_0 \in \{\tilde{T}_0, 0\}\) satisfying \(\|V_0\|_{H^{k+\ell}(\Sigma)} \leq \delta\) and \(\|\tilde{V}_0\|_{H^{k+\ell}(\Sigma)} \leq \delta\). According to (3.21) and (3.131) and the definitions of \(V_0\) and \(\tilde{V}_0\), this estimate also holds in the limit \(t_n \not\to 0\), we find, with the help of (3.147), that

\[
\|W^{\alpha}_{C}(t) - W^{\alpha}_{C}(0)\|_{H^{k+\ell}(\Sigma)} \leq |t|^{\min\{2(\Gamma_3 - \sigma) - \sigma, 1 - \Gamma_1\}}, \quad \alpha = 1, 2,
\]

from which we deduce (3.136).

Finally, we establish (3.137) by considering, for fixed \(V_S\), two solutions \(W_C\) and \(\tilde{W}_C\) obtained under the same conditions as before for two Cauchy data perturbations \(V_0\) and \(\tilde{V}_0\) imposed at the same time \(T_0 \in \{\tilde{T}_0, 0\}\) satisfying \(\|V_0\|_{H^{k+\ell}(\Sigma)} \leq \delta\) and \(\|\tilde{V}_0\|_{H^{k+\ell}(\Sigma)} \leq \delta\). According to (3.21) and (3.131) and the definitions of \(V_0\) and \(\tilde{V}_0\), the difference \(W_C(T_0) - \tilde{W}_C(T_0)\) can be written with the help of (3.140) as

\[
W^{\alpha}_{C}(t) - \tilde{W}^{\alpha}_{C}(t) = W^{\alpha}_{C}(T_0) - \tilde{W}^{\alpha}_{C}(T_0) + \int_{T_0}^{t} \left( -\sum_{i=1}^{3} \sum_{k=0}^{3} \partial_i W^{\alpha}_{C}(s) + C^{\alpha}_{C}(s) \right) ds - \int_{T_0}^{t} \left( -\sum_{i=1}^{3} \sum_{k=0}^{3} \partial_i \tilde{W}^{\alpha}_{C}(s) + \tilde{C}^{\alpha}_{C}(s) \right) ds.
\]

(3.150)

And therefore, exploiting the \(m = \ell_0 - \ell - 2\) of (3.145) – (3.147), (3.143) and (3.144)

\[
\|W^{\alpha}_{C}(t) - \tilde{W}^{\alpha}_{C}(t)\|_{H^{k+\ell}(\Sigma)} \leq \|W^{\alpha}_{C}(T_0) - \tilde{W}^{\alpha}_{C}(T_0)\|_{H^{k+\ell}(\Sigma)} + C\|T_0\|_{H^{\min\{2(\Gamma_3 - \sigma) - \sigma, 1 - \Gamma_1\}}},
\]

for all \(t \in \{\tilde{T}_0, 0\}\) and for each \(\alpha = 1, 2, 3\), where the constant \(C > 0\) is in particular independent of \(t\), \(V_0\) and \(\tilde{V}_0\). Due to the convergence result above for this choice of \(m\), this estimate also holds in the limit \(t_n \not\to 0\). We now establish (3.137) by re-expressing the power of \(|T_0|\) as above and using (3.149). □

Step 3: Establish that \(V_C\) agrees with the solution \(\tilde{V}_S\) of the singular initial value problem

To complete the proof of Theorem 3.6, we need to verify the solution \(V_C\) from Proposition 3.8 that determines via (3.95) and (3.131) the limits \(W^{\alpha}_{C}(0)\), \(W^{\alpha}_{S}(0)\), \(W^{\alpha}_{C}(0)\) agrees with the solution \(V_S\) from Theorem 3.1 that is generated by the asymptotic data \(\tilde{v}_* = (\tilde{W}^0_{C}(0), \tilde{W}^1_{C}(0), \tilde{W}^2_{C}(0), \tilde{W}^3_{C}(0))^{\Gamma_T}\). Note that Proposition 3.8 guarantees that \(\tilde{v}_* \in H^{k+\ell}(\Sigma)\) and \(W^{0}_{C}(0) > 0\), which is sufficient to apply Theorem 3.1 and obtain the solution \(\tilde{V}_S\) satisfying (3.9).

Since \(V_C\) defines a classical solution of the Euler equations (1.4) on \(\{\tilde{T}_0, 0\} \times \Sigma\), we would be able to conclude from the uniqueness statement from Theorem 3.1 that \(V_C = \tilde{V}_S\), which is what we need to establish, if we can show that

\[
\sup_{t \in \{\tilde{T}_0, 0\}} \|t^{-\mu}(U_C - \tilde{U}_S)\|_{H^{k}(\Sigma)} \lesssim 1
\]

(3.152)
for some $\mu > \Gamma_1$ where $U_C = T^{-1}V_C$ and $\tilde{U}_S = T^{-1}\tilde{V}_S$. But, by Theorem 3.1, we have

$$\tilde{U}_S = \tilde{U}_{\ell - 1} + \tilde{u}$$

where $\tilde{u}$ is given by Proposition 3.4 and $\tilde{U}_{\ell - 1}$ is constructed in Lemma 3.5 from the asymptotic data $\tilde{v}_*$; see in particular, (3.31) and (3.82). Therefore, due to (3.53) and the fact that $\lambda + 1 > \Gamma_1$ in Proposition 3.4, it is sufficient to show that

$$\sup_{t \in [T_0, 0]} \| t^{-\mu} (\tilde{U}_{\ell - 1} - U_C) \|_{H^{k} (\Sigma)} \lesssim 1$$

for some $\mu > \Gamma_1$ in order to guarantee that (3.152) holds. Using the same notation as previously, see (3.72), we set

$$\tilde{U}_{\ell - 1} = \tilde{T}^{-1} \tilde{W}_{\ell - 1} = \tilde{T}^{-1} (\tilde{v}_* + \tilde{w}_{\ell - 1})$$

and define analogously

$$U_C = \tilde{T}^{-1} W_C = \tilde{T}^{-1} (\tilde{v}_* + w_C).$$

Then because of (3.14) and (3.33), we conclude that (3.152) will hold provided that

$$\sup_{t \in [T_0, 0]} \| t^{-\mu} (\tilde{w}_{\ell - 1} - w_C) \|_{H^{k} (\Sigma)} \lesssim 1$$

for some $\mu > \Gamma_1$.

Let us now consider the finite sequence of leading-order term fields $(\tilde{w}_m)$ for $m = 0, \ldots, \ell - 1$ constructed as in Lemma 3.5. This means that for each $m \in [0, \ell - 1]$, the function $\tilde{w}_m$ is given by (3.73) with $n + 1 = m$ and with $v_*$ replaced by $\tilde{v}_*$. Since $W_C = \tilde{v}_* + w_C$, it is clear from (3.139) that $w_C$ will satisfy (3.73) if we set $w_{n+1} = w_n = w_C$, $\tilde{B}^i_n = \tilde{B}^i_n = \tilde{B}^i_{C}$, $G_{n+1} = G_n = G_C$ and $v_* = \tilde{v}_*$. In full analogy to (3.74) and (3.80), this implies that

$$\partial_t (\tilde{w}_m - w_C) = f_{m-1} := -\tilde{B}^i_{m-1} \partial_i (\tilde{w}_{m-1} - w_C) - (\tilde{B}^i_{m-1} - \tilde{B}^i_{C}) \partial_i (\tilde{v}_* + w_C) + \tilde{G}_{m-1} - \tilde{G}_C$$

and

$$\tilde{w}_m(t) - w_C(t) = -\int_0^t \tilde{f}_{m-1}(s) \, ds. \tag{3.155}$$

We now employ induction to verify that the inequality

$$\| \tilde{w}_m - w_C \|_{H^{k+\ell - m} (\Sigma)} \lesssim |t|^{(m+1)q - 4\sigma} \tag{3.156}$$

holds for each integer $0 \leq m \leq \ell - 1$, where $q$ is defined by (3.81). For the base case $m = 0$, we observe that $\tilde{w}_0 = 0$, see (3.71), and the estimates (3.90) and (3.91), which we know hold by Step 2, can be combined to give

$$\| \tilde{w}_0 - w_C \|_{H^{k+\ell} (\Sigma)} \lesssim |t|^{-\sigma}.$$

For the induction step $m \mapsto m + 1$, suppose that (3.156) holds for some $0 \leq m \leq \ell - 2$. Using analogous arguments to the ones used to derive (3.76), we find that

$$\| \tilde{f}_m \|_{H^{k+\ell - (m+1)} (\Sigma)} \lesssim (|t|^{1 - \Gamma_1} + |t|^{2\Gamma_3 - 1}) \| \tilde{w}_m - w_C \|_{H^{k+\ell - m} (\Sigma)}$$

for all $t \in [T_0, 0)$, and hence, due to (3.156), that

$$\| \tilde{f}_m \|_{H^{k+\ell - (m+1)} (\Sigma)} \lesssim (|t|^{1 - \Gamma_1} + |t|^{2\Gamma_3}) |t|^{(m + 1)q - 4\sigma - 1} = |t|^{(m + 2)q - 4\sigma - 1},$$

where in deriving this expression, we have again made use of the definition of $q$ given by (3.81). Eq. (3.155), with $m$ replaced by $m + 1$, then gives

$$\| \tilde{w}_{m+1} - w_C \|_{H^{k+\ell - (m+1)} (\Sigma)} \lesssim |t|^{(m + 2)q - 4\sigma},$$

and we conclude that (3.156) holds for each integer $0 \leq m \leq \ell - 1$. Setting $m = \ell - 2$, we deduce that

$$\| \tilde{w}_{\ell - 1} - w_C \|_{H^{k+\ell} (\Sigma)} \lesssim |t|^{q - 4\sigma}.$$

But, since $\ell q > \Gamma_1$ and we are allowed to choose $\sigma$ as small as we like, we see that (3.153) holds, and so, we conclude that $V_C = V_S$, which completes the proof of Theorem 3.6. \qed
APPENDIX A. CAUCHY PROBLEM FOR FUChSIAN EQUATIONS

In [16], it was established that the initial value problem

\[ B^0(t, u)\partial_t u + B^i(t, u)\nabla_i u = \frac{1}{t}B(t, u)u + F(t, u) \quad \text{in } [T_0, 0) \times \Sigma, \]

\[ u = u_0 \quad \text{in } \{0\} \times \Sigma, \]

for Fuchsian equations admits solutions under a small initial data assumption provided that the coefficients satisfy certain structural conditions. Strictly speaking, the existence results from [16] do not apply to the initial value problem for Fuchsian equations of the form (2.1), that is,

\[ B^0(t, w_1, u)\partial_t u + B^i(t, w_1, u)\nabla_i u = \frac{1}{t}B(t, w_1, u)u + F(t, w_2, u) \quad \text{in } [T_0, 0) \times \Sigma, \]

\[ u = u_0 \quad \text{in } \{0\} \times \Sigma, \]

which are considered in this article due to the presence of the “background” fields \(w_1\) and \(w_2\). However, it is not difficult to verify that all of the results of [16] still apply. Since the modifications needed to adapt the results from [16] to the current setting are so straightforward, we will only discuss the essential changes below. Instead, we will content ourselves with stating the corresponding existence result for the initial value problem (A.1).–(A.2). However, before we state the existence theorem, we first list the coefficient assumptions that the Fuchsian equation (A.1) needs to satisfy.

A.1. COEFFICIENT ASSUMPTIONS. As in Definition 2.1, we assume here that \(Z_1\) and \(Z_2\) are open and bounded subsets of the vector bundles \(Z_1\) and \(Z_2\), respectively, where \(\pi(Z_1) = \pi(Z_2) = \Sigma\), and \(T_0 < 0\), \(R > 0\), \(\mathcal{R} > 0\) and \(p \in (0, 1]\) are fixed constants. We then assume the coefficients of the Fuchsian equation (A.1) satisfy the following set of assumptions, where here, we employ the same notation as in Section 2.1:

1. The section \(P \in \Gamma(L(V))\) satisfies

\[ P^2 = P, \quad P^\tau = P, \quad \partial_t P = 0 \quad \text{and} \quad \nabla P = 0. \]

2. There exist constants \(\kappa, \gamma_1, \gamma_2 > 0\) such that the maps

\[ B^0 \in C^0_0([T_0, 0), C^\infty(Z_1 \oplus B_R(V), L(V))] \cap C^1([T_0, 0), C^\infty(Z_1 \oplus B_R(V), L(V))] \]

and

\[ B \in C^0_0([T_0, 0), C^\infty(Z_1 \oplus B_R(V), L(V))] \]

satisfy

\[ \pi(B^0(t, z_1, v)) = \pi(B(t, z_1, v)) = \pi((z_1, v)), \]

and

\[ \frac{1}{\gamma_1}\text{id}_{V_{\pi(v)}} \leq B^0(t, z_1, v) \leq \frac{1}{\kappa}B(t, z_1, v) \leq \gamma_2\text{id}_{V_{\pi(v)}} \tag{A.3} \]

for all \((t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)\). Moreover,

\[ [P(\pi(v)), B(t, z_1, v)] = 0 \quad \text{and} \quad (B^0(t, z_1, v))^\tau = B^0(t, z_1, v) \]

for all \((t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)\),

\[ P(\pi(v))B^0(t, z_1, v)P(\pi(v)) = O(|t|^\frac{\pi}{2} + |P(\pi(v))v|) \]

and

\[ P(\pi(v))B^0(t, z_1, v)P(\pi(v)) = O(|t|^\frac{\pi}{2} + |P(\pi(v))v|), \]

where \(P = \mathbb{1} - P\), and there exist maps \(\tilde{B}^0, \tilde{B} \in C^0_0([T_0, 0), C^\infty(Z_1, L(V))]\) satisfying

\[ \pi(\tilde{B}^0(t, z_1)) = \pi(\tilde{B}(t, z_1)) = \pi(z_1) \quad \text{and} \quad [P(\pi(v)), \tilde{B}(t, z_1)] = 0 \]

for all \((t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)\) such that

\[ B^0(t, z_1, v) - \tilde{B}^0(t, z_1) = O(v) \quad \text{and} \quad B(t, z_1, v) - \tilde{B}(t, z_1) = O(v). \]
(3) The map $F \in C^0([T_0, 0), C^\infty(Z_2 \oplus B_R(V), V))$ can be expanded as

$$F(t, z_2, v) = |t|^{-(1-p)} \tilde{F}(t, z_2) + |t|^{-(1-p)} F_0(t, z_2, v) + |t|^{-(1-p)} F_1(t, z_2, v) + |t|^{-1} F_2(t, z_2, v)$$

where $\tilde{F} \in C^0([T_0, 0), C^\infty(Z_2, V))$, the maps $F_0, F_1, F_2 \in C^0_b([T_0, 0), C^\infty(Z_2 \oplus B_R(V), V))$ satisfy

$$\pi(\tilde{F}(t, z_2)) = \pi(z_2),$$
$$\pi(F_a(t, z_2, v)) = \pi((z_2, v)), \quad a = 0, 1, 2,$$

and

$$\mathbb{P} \pi(v) F_2(t, z_2, v) = 0$$

for all $(t, z_2, v) \in [T_0, 0) \times Z_2 \oplus B_R(V)$, and there exist constants $\lambda_\alpha \geq 0, \alpha = 1, 2, 3$, such that

$$F_0(t, z_2, v) = O(v),$$
$$\mathbb{P} \pi(v) F_1(t, z_2, v) = O(\lambda_1 v),$$

and

$$\mathbb{P} \pi(v) F_2(t, z_2, v) = O \left( \frac{\lambda_2}{R} \mathbb{P} \pi(v) v \otimes \mathbb{P} \pi(v) v \right).$$

(4) The map $B \in C^0([T_0, 0), C^\infty(Z_1 \oplus B_R(V), L(V) \otimes T^* \Sigma))$ satisfies

$$\pi(B(t, z_1, v)) = \pi((z_1, v))$$

and

$$[\sigma(\pi(v))(B(t, z_1, v))]^{1T} = \sigma(\pi(v))(B(t, z_1, v))$$

for all $(t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)$ and $\sigma \in \mathbb{R}^n(\Sigma)$. Moreover, $B$ can be expanded as

$$B(t, z_1, v) = |t|^{-(1-p)} B_0(t, z_1, v) + |t|^{-(1-p)} B_1(t, z_1, v) + |t|^{-1} B_2(t, z_1, v)$$

where $B_0, B_1, B_2 \in C^0_b([T_0, 0), C^\infty(Z_1 \oplus B_R(V), L(V) \otimes T^* \Sigma))$ satisfy

$$\pi(B_a(t, z_1, v)) = \pi((z_1, v))$$

for all $(t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)$, and there exist a constant $\alpha \geq 0$ and a map $\tilde{B}_2 \in C^0([T_0, 0), C^\infty(Z_1, L(V) \otimes T^* \Sigma))$ such that

$$\mathbb{P} \pi(v) B_1(t, z_1, v) \mathbb{P} \pi(v) = O(1),$$
$$\mathbb{P} \pi(v) B_1(t, z_1, v) \mathbb{P} \pi(v) = O(\alpha),$$

and

$$\mathbb{P} \pi(v) B_1(t, z_1, v) \mathbb{P} \pi(v) = O(\mathbb{P} \pi(v) v),$$
$$\mathbb{P} \pi(v) B_2(t, z_1, v) \mathbb{P} \pi(v) = O(\mathbb{P} \pi(v) v),$$
$$\mathbb{P} \pi(v) B_2(t, z_1, v) \mathbb{P} \pi(v) = O(\mathbb{P} \pi(v) v),$$
$$\mathbb{P} \pi(v) B_2(t, z_1, v) \mathbb{P} \pi(v) = O(\mathbb{P} \pi(v) v)$$

and

$$\mathbb{P} \pi(v) (B_2(t, z_1, v) - \tilde{B}_2(t, z_1)) \mathbb{P} \pi(v) = O(v).$$

(5) There exist constants $\eta, \theta \geq 0$, and $\beta_\alpha \geq 0, \alpha = 0, 1, \ldots, 7$, such that the map $\text{div} B$ defined locally by (2.6) satisfies

$$\mathbb{P} \pi(v) \text{div} B(\xi) \mathbb{P} \pi(v) = O \left( |t|^{-(1-p)} \eta + |t|^{-(1-p)} \theta + \frac{|t|^{-1} \beta_0}{R} + \frac{|t|^{-1} \beta_1}{R} \mathbb{P} \pi(v) v \right),$$

$$\mathbb{P} \pi(v) \text{div} B(\xi) \mathbb{P} \pi(v) = O \left( |t|^{-(1-p)} \theta + |t|^{-(1-p)} \beta_2 + \frac{|t|^{-1} \beta_3}{R} \mathbb{P} \pi(v) v \right),$$

and

$$\mathbb{P} \pi(v) \text{div} B(\xi) \mathbb{P} \pi(v) = O \left( |t|^{-(1-p)} \theta + |t|^{-(1-p)} \beta_4 + \frac{|t|^{-1} \beta_5}{R} \mathbb{P} \pi(v) v \right).$$
and

\[ \mathbb{P}^\perp(\pi(v)) \text{div} B(\xi) \mathbb{P}^\perp(\pi(v)) = O\left( \left| t \right|^{-\left(1-p\right)} + \frac{\left| t \right|^{-\left(1-p^2\right)}}{R} \mathbb{P}(\pi(v))v \right. + \frac{\left| t \right|^{-\left(1-p^2\right)}}{R^2} \mathbb{P}(\pi(v))v \otimes \mathbb{P}(\pi(v))v \)

where \( \xi = (t, z_1, \dot{z}_1, z_2, v, v') \) denotes an element of \([0, T_0) \times Z_1 \oplus B_R(Z_1) \oplus B_R(Z_2 \times T^* \Sigma) \oplus Z_2 \oplus B_R(V) \oplus B_R(V \otimes T^* \Sigma)\).

**Remark A.1.** By (A.3), we note that there exist constants \(0 < \tilde{\gamma}_1 \leq \gamma_1 \) and \( \tilde{\kappa} \geq \kappa > 0 \) such that

\[ \frac{1}{\tilde{\gamma}_1} \mathbb{P}(\pi(v)) \leq \mathbb{P}(\pi(v))B^0(t, z_1, v)\mathbb{P}(\pi(v)) \leq \frac{1}{\kappa} B(t, z_1, v)\mathbb{P}(\pi(v)) \leq \gamma_2 \mathbb{P}(\pi(v)) \]

for all \((t, z_1, v) \in [T_0, 0) \times Z_1 \oplus B_R(V)\).

**A.2. Existence and uniqueness.**

**Theorem A.2.** Suppose \(k \in \mathbb{Z}_{n/2+3}, \sigma > 0, w_0 \in H^k(\Sigma, V)\), assumptions (1)-(5) from Appendix A.1 are fulfilled, \(w_1 \in C^0_0([T_0, 0), H^k(\Sigma, Z_1)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, Z_1))\) and \(w_2 \in C^0_0([T_0, 0), H^k(\Sigma, Z_2))\) satisfy

\[ \sup_{T_0 < t < 0} \max \left\{ \left\| \nabla w_1(t) \right\|_{H^{k-1}(\Sigma)}, \left\| \left| t \right|^{-q} \partial_t w_1(t) \right\|_{H^{k-1}(\Sigma)} \right\} < \frac{\mathcal{R}}{C_{Sob}} \]

and

\[ w_0([T_0, 0) \times \Sigma) \subset Z_a, \quad a = 1, 2, \]

and the constants \(\gamma_1, \lambda_3, \beta_0, \beta_1, \beta_3, \beta_5, \tilde{\beta}_7\) from Appendix A.1 satisfy

\[ \kappa > \frac{1}{2} \gamma_1 \max \left\{ \sum_{a=0}^3 \beta_{2a+1} + 2 \lambda_3, \beta_1 + 2k(k+1) \right\} \]

where

\[ b = \sup_{T_0 \leq t < 0} \left( \left\| \mathbb{P}B(t) \nabla (\tilde{B}(t)^{-1} \tilde{B}_0(t)^{-1} \mathbb{P}B_2(t)) \mathbb{P}_{\text{op}} \right\|_{L^\infty(\Sigma)} + \left\| \mathbb{P}B(t) \nabla (\tilde{B}(t)^{-1} \tilde{B}_2(t)) \mathbb{P}_{\text{op}} \right\|_{L^\infty(\Sigma)} \right) \]

(A.4)

with \(\tilde{B}_0(t) = \tilde{B}_0(t, w_1(t))\) and \(\tilde{B}_1(t) = \tilde{B}_1(t, w_1(t))\).

Then there exists a \(\delta > 0\) such that if

\[ \max \left\{ \left\| u_0 \right\|_{H^k(\Sigma)}, \int_{T_0}^0 \left| s \right|^{p-1} \left\| \tilde{F}(s) \right\|_{H^k(\Sigma)} ds \right\} \leq \delta, \]

where \(\tilde{F}(t) = \tilde{F}(t, w_2(t))\), then there exists a unique solution

\[ u \in C^0_0([T_0, 0), H^k(\Sigma, V)) \cap C^1([T_0, 0), H^{k-1}(\Sigma, V)) \]

of the IVP (A.1)-(A.2) such that the limit \(\lim_{t \to 0} \mathbb{P}^\perp u(t)\), denoted \(\mathbb{P}^\perp u(0)\), exists in \(H^{k-1}(\Sigma, V)\).

Moreover, for \(T_0 \leq t < 0\), the solution \(u\) satisfies the energy estimate

\[ \left( \left\| u(t) \right\|_{H^k(\Sigma)}^2 - \int_{T_0}^t \frac{1}{2} \left\| Pu(\tau) \right\|_{H^k(\Sigma)}^2 d\tau \right)^\frac{1}{2} \leq \left( \left\| u_0 \right\|_{H^k(\Sigma)} + \int_{T_0}^t \left| s \right|^{p-1} \left\| \tilde{F}(s) \right\|_{H^k(\Sigma)} ds \right) \]

(A.5)

and the decay estimates

\[ \left\| \mathbb{P}u(t) \right\|_{H^{k-1}(\Sigma)} \lesssim \begin{cases} \left| t \right|^p \left( \lambda_1 + \alpha \right)\left| t \right|^\frac{\sigma}{2} & \text{if } \zeta > p \\ \left| t \right|^{\zeta-\sigma} \left( \lambda_1 + \alpha \right)\left| t \right|^\frac{\sigma}{2} & \text{if } \frac{p}{2} < \zeta \leq p \\ \left| t \right|^{\zeta-\sigma} & \text{if } 0 < \zeta \leq \frac{p}{2} \end{cases} \]

(A.6)

and

\[ \left\| \mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0) \right\|_{H^{k-1}(\Sigma)} \lesssim \begin{cases} \left| t \right|^{p/2} \left| t \right|^{\zeta-\sigma} & \text{if } \zeta > p/2 \\ \left| t \right|^{\zeta-\sigma} & \text{if } 0 \leq \zeta \leq p/2 \end{cases}, \]

(A.7)
where
\[ \zeta = \tilde{\kappa} - \frac{1}{2} \xi_1 (\beta_1 + (k - 1)k\tilde{b}) \]
and
\[ \tilde{b} = \sup_{T_0 < t < 0} \left( \| \mathbb{P} \tilde{B}(t) \nabla (\tilde{B}(t)^{-1} \mathbb{P} \tilde{B}'(t))\mathbb{P}\|_{L^\infty(\Sigma)} + \| \mathbb{P} \tilde{B}(t) \nabla (\tilde{B}(t)^{-1} \tilde{B}_2(t))\mathbb{P}\|_{L^\infty(\Sigma)} \right). \]

Proof. The proof follows from a straightforward adaptation of the proof of Theorem 3.8. from [16] together with the time-reparameterisation argument from Section 3.4. of that same article; see also Lemma 2.5 which shows how the Fuchsian system (A.1) transform under the time-reparameterization \( t = -(-t)^p \). The only important change to note is the following simple improvement to the proof of [16, Theorem 3.8.]:

Adapting the arguments in the proof of [16, Theorem 3.8.], it is not difficult to verify that the following variation of the differential energy estimate given on the line equation below equation (3.71) in [16] holds in our setting:
\[ \partial_t E_k \leq C(\delta, \delta^{-1})(E_k + \| \tilde{F} \|_{H^k} \sqrt{E_k}), \quad T_0 \leq t < T_0, \]
where \( \tilde{F} = \tilde{F}(t, w_2(t)) \) and the energy \( E_k \) is now defined by
\[ E_k(t) = \| u(t) \|_k^2 + \rho \| c(\delta, \delta^{-1})\| u(t) \|_0^2 - \int_{T_0}^t \frac{\partial_k}{\tau} \| P u(\tau) \|_0^2 \, d\tau. \]

The main advantage of this variation is that it allows us to write the above differential energy inequality as
\[ \partial_t \sqrt{E_k} \leq C(\delta, \delta^{-1})\left( \sqrt{E_k} + \| \tilde{F} \|_{H^k(\Sigma)} \right), \]
which, in turn, yields via Grönwall’s inequality the energy estimate
\[ \sqrt{E_k(t)} \leq C(\delta, \delta^{-1})T_0 \left( \sqrt{E_k(T_0)} + \int_{T_0}^t \| \tilde{F}(s) \|_{H^k(\Sigma)} \, ds \right). \tag{A.8} \]

Comparing this estimate with the estimate (3.73) from [16], we see that the advantage of the new energy estimate (A.8) is that it only requires \( \int_{T_0}^t \| \tilde{F}(s, w_2(s)) \|_{H^k(\Sigma)} \, ds \) to be small rather than \( \sup_{T_0 < t < 0} \| \tilde{F}(s, w_2(s)) \|_{H^k(\Sigma)} \) as in the proof of [16, Theorem 3.8.]. With the exception of this change, the rest of the proof mimics that of [16, Theorem 3.8.] with the obvious changes needed to account for the background fields \( w_1 \) and \( w_2 \).

Remark A.3.

(i) By [16, Remark 3.10.(a),(ii)], we know, taking into account the time-reparameterization argument from [16, §3.4.], that if the maps \( B_1 \) and \( B_2 \) satisfy
\[ \mathbb{P}^\perp B_1 = \mathbb{P}^\perp B_2 = 0, \]
then the decay estimate for \( \mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0) \) from Theorem A.2 improves to
\[ \| \mathbb{P}^\perp u(t) - \mathbb{P}^\perp u(0) \|_{H^{k-1}} \lesssim \begin{cases} |t|^p & \text{if } \zeta > p, \\ |t|^p + |t|^{2(\zeta-\sigma)} & \text{if } \zeta \leq p, \end{cases} \quad T_0 \leq t < 0. \]

(ii) If the map \( B_2 \) satisfies
\[ B_2 = 0, \]
then the regularity requirement in the statement of Theorem A.2 can be lowered to \( k \in \mathbb{Z}_{>n/2+1} \). This is because in this situation the use of [16, Lemma 3.5.] can be avoided in the proof of [16, Theorem 3.8.], which leads to the reduction in the required regularity.
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