Sonic crystal lenses that obey Lensmaker’s formula

Chao-Hsien Kuo and Zhen Ye
Wave Phenomena Laboratory, Department of Physics, National Central University, Chungli, Taiwan 32054, R. O. C.
(Dated: October 30, 2018)

This paper presents a theoretical study of the phenomenon of acoustic imaging by sonic crystals, which are made of two-dimensional regular arrays of rigid cylinders placed in parallel in air. The scattering of acoustic waves is computed using the standard multiple scattering theory, and the band structures are computed by the plane-wave expansion method. It is shown that properly arranged arrays not only can behave as acoustic lenses, but also the focusing effect can be well described by Lensmaker’s formula. Possible applications are also discussed.

PACS numbers: 43.20.+g, 43.90.+v

When propagating through periodic structures, waves reveal a particularly important property. That is, the propagation of waves will be modulated by the periodic structures. As a result, the dispersion of waves will no longer behave as in a free space, and the so called band structures appear.

The phenomenon of band structures was first investigated for electronic systems and was put on a solid foundation in the context of Bloch’s theorem.[1] Since the central physics behind electronic band structures lies in the wave nature of electrons, by analogy an immediate inquiry would naturally be whether such a phenomenon can be demonstrated in classical systems involving, for instance, acoustic and electromagnetic (EM) waves. Although this problem was early addressed by Brillouin [2], then by Yariv and Yeh [3], this seemingly straightforward question has not received serious attention, until late 1980s. In 1987, Yablonovitch [4] and John [5] proposed that such a band structure is indeed possible when EM waves travel in periodic dielectric media. Since then, the study of EM waves in periodic structures has been booming, eventually leading to the exploration of acoustic waves in periodic structures (e. g., Refs. [6, 7, 8, 9, 10, 11]). This has induced the establishment of the field of sonic crystals (SCs).

The exciting phenomenon of band structures in sonic crystals allows for many possible applications. It has been recognized that SCs could be used as sound shields and acoustic filters [1, 7, 8, 12, 13, 14, 15]. These applications mostly rely on the existence of complete sonic bandgaps in which acoustic waves are prohibited from transmission in all directions. Thus most of early studies have been focused on the formation of bandgaps.

Recently, interest in the low frequency region, in which the dispersion relation is more or less linear, has started. Since the wavelength in this region is very large compared to the lattice constant, the wave sees the media as if it were homogeneous. Consequently, it has been proposed that refractive devices could be developed to converge acoustic waves by SCs, that is the sonic crystal lenses. A necessary condition to be satisfied to construct a sonic crystal lens is that the acoustic impedance contrast between the SC and the medium should not be large; otherwise acoustic waves will be mostly reflected. Once this condition is satisfied, the converging lens can be either convex or concave depending on whether the sound speed in the SCs is smaller or greater than that in the medium.

Recently, it was experimentally demonstrated [10] that proper SCs can indeed make refractive devices such as Fabry-Perot interferometers and acoustically converging lenses. This interesting observation has stimulated further explorations of acoustic imaging by SCs [17, 18, 19], and has been also extended to the optical imaging by photonic crystal lenses [20, 21]. In light of these novel developments, in this paper we wish to explore further the properties of the sonic crystal made lenses.

This paper presents a first attempt to quantify the imaging effect of sonic crystals. Here, we carry out numerical simulations on the focusing of acoustic waves by SCs. To comply with the current experiments [10] so that experimental verifications could be readily done, we will consider the SCs made of arrays of rigid cylinders in air. It will become clear that the present approach can be readily extended to other SCs and also to photonic crystals. The results will demonstrate that SCs can make refractive lenses which nicely obey Lensmaker’s formula. It is also shown that the refraction dominates the focusing properties.

Before continuing, it is worth noting here that pursuing acoustic lenses can be dated back to 70s. For example, ultrasonic lenses made from bulk materials were proposed by Beaver et al. [22] and Szilard et al. [23]. However, these lenses are quite different from the sonic crystal lenses. The essential principles that govern the focusing phenomenon by sonic crystals are (1) the wave propagation are subdue by the crystal structure, leading to the refraction of waves commonly seen at the interface of two different materials; yet (2) the frequencies of travelling waves are still within the passing band, and therefore no propagation inhibition will incur, allowing efficient transmission through the crystal [10].
1, 2,...N to form regular arrays. An acoustic line source transmitting monochromatic waves is placed at \( \vec{r}_s \). The scattered wave from each cylinder is a linear response to the incident waves which are composed of the direct wave from the source and the multiply scattered waves from other cylinders, and subsequently contributes to the total waves, forming a self-consistent multiple scattering scheme. The final wave reaching a receiver located at \( \vec{r}_r \) is the sum of the direct wave from the source and the scattered waves from all the cylinders. Such a scattering process has been formulated exactly by Twersky [24], and was detailed in Refs [10, 11]. For regular arrays of cylinders, the band structures are computed by the plane-wave method to obtain the acoustic phase speed inside the arrays.

In the following computation, the rigid cylinders of radius \( a \) are arranged to form triangular arrays in air with lattice constant \( d \). The following parameters from the experiment [14] are used in the simulations: (1) the radius of the cylindrical rods is 2.0cm; (2) the lattice constant is 6.35cm; (3) The filling factor, defined as the area occupation of the cylinders per unit area \( f_a = (\pi/2\sqrt{3})(d/a)^2 \), is equal to 0.3598; (4) the sound speed in air is \( v_a = 351\text{m/s} \). Additionally, the acoustic transmission through the cylinder arrays is normalized as \( T = p/p_0 \), and the incident wave in this entire project is along the \( \Gamma X \) direction. We note here that the rigid cylinders can be any solid rods. As shown in Ref. [8], any material whose acoustic impedance with respect to the air exceeds roughly 10 can be used as the composition material for the rods.

We note that when waves propagate in structure media, it is usually necessary to distinguish the group velocity, roughly characterizing the energy flow, and the phase velocity. In the linear dispersion regions, the two velocities are nearly the same. We have also computed the phase velocity in other directions, yielding the consistent values. These results imply that the cylinder arrays may be regarded as an effective refractive medium, making it possible that SCs can be used as the refractive acoustic devices. This will be supported by the following simulation of the sonic crystal lenses.

A conceptual layout of the acoustic lensing system made from a sonic crystal is shown by Fig. 1. The geometrical factors, important to the following discussion, are clearly indicated in the figure. The source is place at a distance \( s_o \) from the lens. The image is at a distance of \( s_i \). The cylinders, represented by the black dots, are placed inside the lenticular area. Inside the area, the cylinders are arranged to form the triangular lattice with the parameters given above. Fig. 2 actually shows a double two-dimensional convex lens of a refractive sonic crystal whose surfaces have radii of curvature \( R_1 \) and \( R_2 \). We note that a three-dimensional lens can be fabricated by aligning two perpendicular two-dimensional lenses.

![FIG. 1](image1.png)

**FIG. 1:** The band structures computed by the plane wave expansion method for triangular lattice of the rigid cylinders in air.

Figure 1 presents the band structure result for the triangular array of rigid cylinders. We see that the dispersion in the low frequency region is linear. Our subsequent calculations will be restricted to this region only. We calculate the sound phase speed from the linear dispersive curve as \( v_{p,c} = \omega/K = 295\text{m/s} \). Then the refraction index is calculated as \( n = v_a/v_{p,c} = 1.1898 \). We note that in structure media, it is usually necessary to distinguish the group velocity, roughly characterizing the energy flow, and the phase velocity. In the linear dispersion regions, the two velocities are nearly the same. We have also computed the phase velocity in other directions, yielding the consistent values. These results imply that the cylinder arrays may be regarded as an effective refractive medium, making it possible that SCs can be used as the refractive acoustic devices. This will be supported by the following simulation of the sonic crystal lenses.

A conceptual layout of the acoustic lensing system made from a sonic crystal is shown by Fig. 1. The geometrical factors, important to the following discussion, are clearly indicated in the figure. The source is placed at a distance \( s_o \) from the lens. The image is at a distance \( s_i \). The cylinders, represented by the black dots, are placed inside the lenticular area. Inside the area, the cylinders are arranged to form the triangular lattice with the parameters given above. Fig. 2 actually shows a double two-dimensional convex lens of a refractive sonic crystal whose surfaces have radii of curvature \( R_1 \) and \( R_2 \). We note that a three-dimensional lens can be fabricated by aligning two perpendicular two-dimensional lenses.

![FIG. 2](image2.png)

**FIG. 2:** The conceptual layout of an acoustic convex lens, in line with the optical lens depicted in Fig. 5.16 of Hecht [23].

We first consider the acoustic transmission through the lenticular structure of SCs. The results are presented in Fig. 3. The scenario is illustrated by Fig. 3(a). The two dimensional spatial distribution of the normalized transmitted intensity is shown in Fig. 3(b). Here the \( x \) axis is a horizontal line towards right, and the \( y \) axis is placed vertically upward. We see that the focusing of the transmitted wave is evident. Since the exact location of the maximum intensity point is not clear from Fig. 3(b), we have computed the variation of the field intensity along the \( x \) axis with \( y = 0 \), and also along the \( y \) axis with \( x = 20\text{m} \). The results are shown in Figs. 3(c) and 3(d). Fig. 3(c) shows that there is a peak at \( x \approx 20\text{m} \). The intensity variation along the \( y \) axis shows a significant
peak at \( y = 0 \). Here we observe that the intensity is better confined along the \( y \) axis than along the \( x \) axis. Some relevant parameters are listed in the figure caption.

![Figure 3](image)

**FIG. 3:** (a) The line source denoted by \( S \) and the lenticular arrangement of rigid cylinders; (b) two dimensional spatial distribution of the transmitted intensity \( |T|^2 \) on the right hand side of the crystal; (c) the variation of the intensity along the \( x \) axis at \( y = 0 \) and (d) the variation of the intensity along the \( y \) axis at \( x = 20 \) m. Parameters: (1) the source is placed at 21 m away from the center of the crystal; (2) the origin is placed at the rightmost point of the crystal; (3) the frequency is taken as 1500 Hz, and the incident wave is along the \( \Gamma X \) direction; (4) the radii of the curvatures of the lenses are chosen to be equal, i.e. \( R_1 = R_2 = 4 \) m. The thickness of the lens is about 80 cm or 12 lattice constants, while the height amounts to 340 cm or 53 lattice constants; therefore it is a thin lens. The total number of scatterers is 546.

Fig. 4 clearly demonstrate the acoustic converging effects by SCs. Since the lenticular shape is thin, the focusing effect may be quantified. According to the refraction theory, it is possible to relate the object distance or length \( s_o \), the image distance \( s_i \), and the focal length \( f \) by the thin formula

\[
\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}.
\]  

(1)

If the medium can be regarded as a refractive medium, the focal length will be related to the radii of the curvatures of the convex surfaces by

\[
\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right).
\]  

(2)

Eqs. (1) and (2) lead to Lensmaker’s formula

\[
\frac{1}{s_o} + \frac{1}{s_i} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right),
\]  

(3)

where \( n \) is the refraction index.

We have verified that the acoustic imaging by the SCs discussed above does obey Lensmaker’s formula. First from the band structure, we obtain the phase speed and subsequently the refraction index as the ratio between the speeds in the air and in the SCs. Then the focal length can be calculated from Eq. (2) with known \( R_1 \) and \( R_2 \). This will give the theoretical value for the focal length. The focal length can also be obtained by simulation. We vary the location of the source and then find the image position. Taking these two values into Eq. (1), we can get a simulated focal length for each source location. If the device indeed behave as a lens, the simulated focal lengths at various source locations should be consistent. Furthermore, if the lens is refractive, the simulated values should also be consistent with the theoretical value. We found these claims are indeed valid for SCs.

![Figure 4](image)

**FIG. 4:** (a) The focal lengths versus the object lengths; (b) image distance versus the object length \( s_o \). Both radii of the curvatures are taken as 4 m. The theoretical results are shown by solid lines.

![Figure 5](image)

**FIG. 5:** The focal lengths versus the radius of curvature at frequency 1500 Hz. The crosses represent the simulation results, and the straight line refers to the theoretical estimates.

As an example, we consider the lenticular shape of arrays of cylinders described in the figure caption of Fig. 3.
The results are shown in Figure 1. Here, both results from the theoretical and simulations are presented. In (a), the focal lengths estimated from the simulation for various object distances are shown by the crosses. As the comparison, the theoretical focal length calculated from Eq. 2 is also shown. In (b), the image distance versus the object distance is plotted. Here we see that (1) from (a) the simulated focal lengths are consistent with each other, and also consistent with the theoretical value; (2) from (b) the relation between the image distance and the object distance can be well described by Eq. 1 and also agrees remarkably well with the theoretical curve.

We have further verified the above agreement by varying the radii of the curved surfaces of the convex surfaces. The comparison with the theoretical prediction is shown in Fig. 2. In the simulation, the thickness of the lens is kept unchanged at about 80 cm, when the radii are varied. We also take that both the radii of the curved surfaces are the same; so the theoretical focal length is \( f = \frac{R}{n-1} \). The simulated value for the focal length is obtained as follows. For a fixed radius of the curvature, we vary the object distance and obtain the corresponding image distance. Then the focal length can be obtained through Eq. 1 for each object location. The focal length will be averaged over all the locations. Then we change to other values for the radius of the curvature. In this way we obtain the simulated focal lengths as a functions of the radius. The theoretical results are obtained from Eq. 2 with the refraction index obtained from the band structures. Figure 3 shows that the estimated focal lengths are in excellent agreement with the theoretical prediction. Here we would also like to note that the agreement will deteriorate when the size of lenses is too small. In fact, when the size is too small, the diffraction effect will become dominant and therefore the focusing can no longer be described by the refraction process.

In summary, here we demonstrate that SCs not only can make acoustic focusing devices, but also the focusing behavior can be well described by Lensmaker’s formula. This finding will help design of actual acoustic refractive devices, such as focusing audio speaker systems or sonars for large scale oceanographical probing purposes. Underwater sonars with sonic crystal lenses may improve the detection of fish and other marine organisms. Although the present work has been focused upon the acoustic cases and on the rigid cylinders, it is easy to see that the ideas can be readily applied to other SCs, and as well as the photonic crystals.

This work received support from the National Central University, National Science Council of Republic of China.

* Electronic address: zhen@phy.ncu.edu.tw

[1] F. Bloch, Z. Physik 52 (1928).
[2] L. Brillouin, Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices, (McGraw Hill, New York, 1946; Dover Publications, New York, 1953); ibid, Wave Propagation and Group Velocity, (New York, Academic Press, 1960).
[3] A. Yariv and P. Yeh, Optical waves in crystals, (New York, Wiley, 1984).
[4] E. Yablonovitch, Phys. Rev. Lett. 58, 2059 (1987).
[5] S. John, Phys. Rev. Lett. 58, 2486 (1987).
[6] J. P. Dowling, J. Acoust. Soc. Am. 91, 2539 (1992).
[7] R. Martínez-Sala, J. Sancho, J. V. Sánchez-Pérez, J. Linares, and F. Meseguer, Nature (London) 378, 241 (1995).
[8] J. V. Sánchez-Pérez et al., Phys. Rev. Lett. 80, 5325 (1998).
[9] M. S. Kushwaha, Recent Res. Devel. Appl. Phys. 2, 743 (1999).
[10] Y.-Y. Chen and Z. Ye, Phys. Rev. Lett. 87, 184301 (2001).
[11] Y.-Y. Chen and Z. Ye, Phys. Rev. E 64, 036616 (2001).
[12] D. Caballero et al., Phys. Rev. E 60, R6316 (1999).
[13] W. M. Robertson and W. F. Rudy III, J. Acoust. Soc. Am., 69, 3080 (1992).
[14] L. Sanchis et al., J. Acoust. Soc. Am., 109, 2598 (2001).
[15] M. S. Kushwaha, Appl. Phys. Lett., 70, 3218 (1997).
[16] F. Cervera, et al., Phys. Rev. Lett., 88, 023902 (2002).
[17] L. Sanchis, A. Hakansson, F. Cervera and J. Sanchez-Dehesa, Phys. Rev. B 67, 035422 (2003).
[18] N. Garcia, M. Nieto-Vesperinas, E. V. Ponizovskaya, and M. Torres, Phys. Rev. E 67, 046606 (2003).
[19] B. Gupta and Z. Ye, Phys. Rev. E 67, 036603 (2003).
[20] P. Halevi, A. A. Krokhin, and J. Arriaga, Phys. Rev. Lett. 82, 719 (1999).
[21] B. Gupta and Z. Ye, Phys. Rev. B 67, 153109 (2003).
[22] W. L. Beaver, IEEE Trans. on Sonic and Ultrasonic SU-24, 235 (1977).
[23] T. Szilard and M. Kidger, Ultrasonics 73, 268 (1976).
[24] V. Twersky, J. Acoust. Soc. Am. 24, 42 (1951).
[25] E. Hecht, Optics, (Addison-Wiley, New York, 1998).
[26] H. Medwin and C. S. Clay, Fundamentals of acoustical oceanography, (Academic Press, New York, 1998).
[27] Z. Ye, J. Acoust. Soc. Am. 98, 2727 (1995); Z. Ye, T. Curran and D. Lemon, ICES J. Mar. Science, 53, 317 (1996); L. Ding and Z. Ye, J. Acoust. Soc. Am. 102, 1977 (1997).
[28] D. Chu and Z. Ye, J. Acoust. Soc. Am. 106, 1732 (1999).