INCOMPRESSIBLE INVISCID RESISTIVE MHD SURFACE WAVES IN 2D

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Abstract. We consider the dynamics of a layer of an incompressible electrically conducting fluid interacting with the magnetic field in a two-dimensional horizontally periodic setting. The upper boundary is in contact with the atmosphere, and the lower boundary is a rigid flat bottom. We prove the global well-posedness of the inviscid and resistive problem with surface tension around a non-horizontal uniform magnetic field; moreover, the solution decays to the equilibrium almost exponentially. One of the key observations here is an induced damping structure for the fluid vorticity due to the resistivity and transversal magnetic field.

1. Introduction

1.1. Formulation in Eulerian coordinates. We consider the motion of an incompressible electrically conducting fluid interacting with the magnetic field in a 2D moving domain

\[ \Omega(t) = \{ y \in \mathbb{T} \times \mathbb{R} \mid -1 < y_2 < h(t, y_1) \} . \]

We assume that the domain is horizontally periodic and \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) is the usual 1-torus. Note that the upper boundary of the domain is a free boundary that is the graph of the unknown function \( h : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R} \). The dynamics of the fluid is described by the velocity, the pressure and the magnetic field, which are given for each \( t \geq 0 \) by \( u(t, \cdot) : \Omega(t) \to \mathbb{R}^2 \), \( p(t, \cdot) : \Omega(t) \to \mathbb{R} \) and \( B(t, \cdot) : \Omega(t) \to \mathbb{R}^2 \), respectively. For each \( t > 0 \), \((u, p, B, h)\) is required to satisfy the following free boundary problem for the incompressible inviscid and resistive magnetohydrodynamic equations (MHD) \([12, 23]\):

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p & = B \cdot \nabla B \quad \text{in } \Omega(t) \\
\text{div } u & = 0 \quad \text{in } \Omega(t) \\
\partial_t B + u \cdot \nabla B - \kappa \Delta B & = B \cdot \nabla u \quad \text{in } \Omega(t) \\
\text{div } B & = 0 \quad \text{in } \Omega(t) \\
\partial_t h & = u_2 - u_1 \partial_1 h \quad \text{on } \{ y_2 = h(t, y_1) \} \\
p & = gh - \sigma H, \quad B = \bar{B} \quad \text{on } \{ y_2 = h(t, y_1) \} \\
u_2 & = 0, \quad B = \bar{B} \quad \text{on } \{ y_2 = -1 \}.
\end{aligned}
\]

Here \( \bar{B} \) is the constant magnetic field in the outside of the fluid. \( \kappa > 0 \) is the magnetic diffusion coefficient, \( g > 0 \) is the strength of gravity and \( \sigma > 0 \) is the surface tension coefficient. \( H \) is twice the mean curvature of the free boundary given by the formula

\[ H = \partial_1 \left( \frac{\partial_1 h}{\sqrt{1 + (\partial_1 h)^2}} \right) . \]

The fifth equation in (1.2) implies that the free boundary is advected with the fluid. Note that in (1.2) we have shifted the gravitational forcing to the free boundary and eliminated the constant

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atmospheric pressure $p_{atm}$, the magnetic pressure $|B|^2/2$ and the constant outside magnetic pressure $\bar{|B}|^2/2$, in the usual way by adjusting the actual pressure $\bar{p}$ according to

$$p = \bar{p} + gy_2 - p_{atm} + \frac{|B|^2}{2} - \frac{\bar{|B}|^2}{2}. \quad (1.4)$$

To complete the statement of the problem, one must specify the initial conditions. Suppose that the initial upper boundary is given by the graph of the function $h(0) = h_0 : \mathbb{T} \to \mathbb{R}$, which yields the initial domain $\Omega(0)$ on which the initial velocity $u(0) = u_0 : \Omega(0) \to \mathbb{R}^2$ and the initial magnetic field $B(0) = B_0 : \Omega(0) \to \mathbb{R}^2$ are specified. We will assume that $h_0 > -1$, which means that at the initial time the boundaries do not intersect with each other.

In the global well-posedness theory of the problem (1.2), we assume further the following zero-average condition

$$\int_\mathbb{T} h_0 dy_1 = 0 \quad \text{and} \quad \int_{\Omega(0)} (u_0)_1 dy = 0. \quad (1.5)$$

For sufficiently regular solutions, the condition (1.5) persists in time, that is,

$$\int_\mathbb{T} h(t) dy_1 = 0 \quad \text{and} \quad \int_{\Omega(t)} u_1(t) dy = 0 \quad \text{for} \quad t \geq 0. \quad (1.6)$$

Indeed, using the equations in (1.2) except the third equation, by the divergence theorem, one has, writing $\nu = (-\partial h,1)/\sqrt{1 + |\partial_1 h|^2}$ for the outward-pointing unit normal on $\{y_2 = h(t,y_1)\}$,

$$\frac{d}{dt} \int_\mathbb{T} h \, dy_1 = \int_\mathbb{T} \partial_t h \, dy_1 = \int_{\{y_2 = h(t,y_1)\}} u \cdot \nu \, ds = \int_{\Omega(t)} \text{div} \, u \, dy = 0 \quad (1.7)$$

and

$$\frac{d}{dt} \int_{\Omega(t)} u_1 \, dy = \int_{\Omega(t)} \partial_t u_1 \, dy + \int_{\mathbb{T}} u_1(t,h(t,y_1)) \partial_1 h \, dy_1 = \int_{\Omega(t)} (\partial_1 u_1 + u \cdot \nabla u_1) \, dy_1$$

$$= \int_{\Omega(t)} (-\partial_1 \nu + B \cdot \nabla B)_1 \, dy = -\int_{\{y_2 = h(t,y_1)\}} p \nu_1 \, ds$$

$$= \int_{\mathbb{T}} (gh - \sigma H) \partial_1 h \, dy_1 = \frac{1}{2} \int_{\mathbb{T}} \partial_1 \left( g |h|^2 + 2\sigma \sqrt{1 + |\partial_1 h|^2} \right) \, dy_1 = 0. \quad (1.8)$$

The condition (1.6) allows one to apply Poincaré’s inequalities for $h$ and $u_1$ for all $t \geq 0$, respectively. Moreover, we will work in a functional framework for which $h(t) \to 0$ and $u_1(t) \to 0$ as $t \to \infty$ in high order Sobolev norms; due to the conservation, one cannot expect this decay unless that (1.5) has been assumed.

1.2. Formulation in flattening coordinates. The movement of the free boundary and the subsequent change of the domain create numerous mathematical difficulties. To circumvent these, as usual, we will use a coordinate system in which the boundary and the domain stay fixed in time. To this end, we define the fixed domain

$$\Omega := \mathbb{T} \times (-1,0), \quad (1.9)$$

for which the coordinates are written as $x \in \Omega$. Set $\Sigma := \mathbb{T} \times \{0\}$ for the upper boundary, $\Sigma_{-1} := \mathbb{T} \times \{-1\}$ for the lower boundary, and so $\partial \Omega = \Sigma \cup \Sigma_{-1}$. We will think of $h$ as a function on $\mathbb{R}_+ \times \Sigma$, and flatten the fluid region via the mapping

$$\Omega \ni x \mapsto (x_1, \varphi(t,x_1,x_2) := x_2 + \eta(t,x_1,x_2)) =: \Phi(x,t) = (y_1, y_2) \in \Omega(t), \quad (1.10)$$

where $\eta = (1 + x_2) \mathcal{P} h$, and $\mathcal{P} h$ is the harmonic extension of $h$ onto $\{x_2 \leq 0\}$ with $\mathcal{P}$ defined by (A.1). Note that if $h$ is sufficiently small in an appropriate Sobolev space, then $\partial_2 \varphi = 1 + \partial_2 \eta > 0$ and hence the mapping $\Phi$ is a diffeomorphism. This allows one to transform the problem (1.2) to one in the fixed spatial domain $\Omega$ for each $t \geq 0$. For this, we define

$$v(t,x) = u(t, \Phi(t,x)), \quad q(t,x) = p(t, \Phi(t,x)) \quad \text{and} \quad b(t,x) = B(t, \Phi(t,x)) - \bar{B}. \quad (1.11)$$
Set
\[ \partial^i_t = \partial_t - \frac{\partial_i \varphi}{\partial x_i} \frac{\partial}{\partial x_i}, \quad i = 1, 2, \quad \partial^2_t = \frac{1}{\partial_2 \varphi} \partial_2. \]  
(1.12)

Then the free boundary problem (1.2) is equivalent to the following problem for \((v, q, b, h)\) in new coordinates:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial^2_t v + v \cdot \nabla^2 v + \nabla^2 q = (\bar{B} + b) \cdot \nabla^2 b & \text{in } \Omega \\
\nabla^2 v = 0 & \text{in } \Omega \\
\partial^2_t b + v \cdot \nabla^2 b - \kappa \Delta^2 b = (\bar{B} + b) \cdot \nabla^2 v & \text{in } \Omega \\
\nabla^2 \cdot b = 0 & \text{in } \Omega \\
\partial_t h = v \cdot N & \text{on } \Sigma \\
\partial_t h = v \cdot N & \text{on } \Sigma \\
q = gh - \sigma H, \quad b = 0 & \text{on } \Sigma \\
v_2 = 0, \quad b = 0 & \text{on } \Sigma_{-1}
\end{array} \right.
\end{aligned}
\]  
(1.13)

Here \((\nabla^2) = \partial_2^2 v, v = \partial_2^2 v_1, \Delta^2 = \partial_2^2 \partial_2^2 v, \) and \(N = (-\partial_1 \varphi, 1)^t \equiv (-\partial_1 \eta, 1)^t\) which reads as \(N = (-\partial_1 h, 1)^t\) on \(\Sigma\) and \(N = e_2\) on \(\Sigma_{-1}\).

Note that the initial condition (1.5) is transformed to
\[ \int_\Sigma h_0 \, dx_1 = 0 \quad \text{and} \quad \int_\Omega (v_0) \, dV_0 = 0, \]  
(1.14)

and (1.6) becomes
\[ \int_\Sigma h(t) \, dx_1 = 0 \quad \text{and} \quad \int_\Omega v_1(t) \, dV = 0 \quad \text{for } t \geq 0. \]  
(1.15)

Here \(dV_1\) stands for the volume element induced by the change of variable (1.10): \(dV_1 = \partial_2 \varphi \, dx\).

The problem (1.13) possesses the natural physical energy-dissipation law: for sufficiently regular solutions to the problem, one has
\[
\frac{1}{2} \frac{d}{dt} \left( \int_\Omega \left( |v|^2 + |b|^2 \right) \, dV_1 + \int_\Sigma \left( gh^2 + 2\sigma \left( \sqrt{1 + |\partial_1 h|^2} - 1 \right) \right) \, dx_1 \right) + \kappa \int_\Omega |\nabla^2 b|^2 \, dV_1 = 0.
\]  
(1.16)

The structure of this energy evolution equation is the basis of the energy method we will use to analyze (1.13).

1.3. Related works. Free boundary problems in fluid mechanics have been studied intensively in the mathematical community. There are a huge amount of mathematical works; it is impossible to provide a thorough survey of the literature here, and we mention only briefly some of them below. We may refer to the references cited in these works for more proper survey of the literature.

For the free-boundary incompressible Euler equations, the early works were focused on the irrotational fluids, which began with the local well-posedness for the small initial data \[27, 42, 43\] and was generalized to the general initial data by the breakthrough of Wu \[38, 39\], see also Lannes \[24\] and Ambrose and Masmoudi \[2, 3\]. For the general incompressible Euler equations without the irrotational assumption, the first local well-posedness was obtained by Lindblad \[25\] for the case without surface tension and by Coutand and Shkoller \[2\] for the case with (and without) surface tension, see also Christodoulou and Lindblad \[8\], Schweizer \[29\], Shatah and Zeng \[30\] and Zhang and Zhang \[43\]. We also refer to the local well-posedness in conormal Sobolev spaces as the inviscid limits of the free-boundary incompressible Navier-Stokes equations, see Masmoudi and Rousset \[26\] and the authors \[37\].

It is natural to consider the question whether there is the global well-posedness for free boundary problems or not. Despite the recent very interesting works of the formation of splash or splat singularities for free boundary problems, Castro, Córdoba, Fefferman, Gancedo and Gómez-Serrano \[6, 7\] (see also Coutand and Shkoller \[10, 11\]), which implies the development of singularity in finite time for both inviscid and viscous fluids with some general large initial data, it is still not clear whether the free-boundary incompressible Euler equations for the general small initial data admits a global unique solution or not, even in 2D. This is significantly different.
from the incompressible Euler equations in a fixed domain, where the global well-posedness has been established even for the general large initial data in 2D. To our best knowledge, there have been two types of mechanisms that lead to the global well-posedness of free boundary problems for the small initial data. One is the irrotational assumption for the free-boundary incompressible Euler equations in the horizontally infinite setting, for which certain dispersive effects can be used to establish the global well-posedness for the small initial data; we refer to Wu \[40\] [41], Germain, Masmoudi and Shatah \[14\], Ionescu and Puasateri \[20\] and Alazard and Delort \[1\] for the case without surface tension, and Germain, Masmoudi and Shatah \[15\] and Ionescu and Puasateri \[21\] for the case with surface tension. The other one is the dissipation effect presented on the fluid velocity, e.g., the viscosity; the global well-posedness for the free-boundary incompressible Navier-Stokes equations has been established; we refer to, for instance, Beale \[4\], Hataya \[19\] and Guo and Tice \[16\], \[17\], \[18\] for the case without surface tension, and Beale \[5\], Nishida, Teramoto and Yoshihara \[28\] and Tan and the first author \[34\] for the case with surface tension, see also the series of works of Solonnikov \[31\], \[32\] for the problem of an isolated mass of viscous fluid bounded entirely by a free boundary.

In this paper, we will illustrate the third type of mechanism that leads to the global well-posedness of incompressible inviscid free boundary problems for the small initial data, that is, the dissipation induced by the coupling with a diffusive magnetic field around a non-horizontal uniform magnetic field in 2D. Note that it is reasonable to expect the global well-posedness of the free-boundary viscous and resistive MHD, see Solonnikov and Frolova \[33\] for instance. In \[36\], the first author proved the global well-posedness of the free-interface viscous and non-resistive MHD around a non-horizontal uniform magnetic field. These results rely heavily on the dissipation and regularizing effects of the viscosity. It seems to be more subtle and difficult to prove the global well-posedness of the free-boundary inviscid and resistive MHD since the flow is transported by the velocity; indeed, even the global existence of classical solutions to the Cauchy problem in 2D is unknown to us. Our analysis depends on the finite depth of the fluid in our setting, which allows for the Poincaré-type inequality to hold.

2. Main results

2.1. Statement of the results. The aim of this paper is to show the global well-posedness of the following problem \[(1.13)\] around the equilibrium state \((0,0,0,0)\) when \(B_2 \neq 0\).

Before stating the main results, we first mention the issue of compatibility conditions for the initial data \((v_0,b_0,h_0)\) since the problem \[(1.13)\] is considered in a domain with boundary. We will work in a high-regularity context, essentially with regularity up to \(2N\) temporal derivatives for \(N \geq 8\) an integer. This requires one to use \((v_0,b_0,h_0)\) to construct the initial data \(\partial_t^j b(0)\) for \(j = 1,\ldots,2N+1\), \(\partial_t^j v(0)\) and \(\partial_t^j b(0)\) for \(j = 1,\ldots,2N\) and \(\partial_t^j q(0)\) for \(j = 0,\ldots,2N-1\), inductively. The construction is very similar to that of the incompressible viscous surface wave problem \[16\], \[34\], and thus omitted. These data must satisfy the following conditions

\[
\begin{aligned}
\nabla v^{0} \cdot v_0 &= 0 \text{ in } \Omega, \quad (v_0)_2 = 0 \text{ on } \Sigma \\
\nabla v^{0} \cdot b_0 &= 0 \text{ in } \Omega, \quad b_0 = 0 \text{ on } \partial \Omega, \quad \partial_t^j b(0) = 0, \quad j = 1,\ldots,2N-1, \text{ on } \partial \Omega,
\end{aligned}
\]

which in turn require \((v_0,b_0,h_0)\) to satisfy the necessary compatibility conditions; these are natural for the local well-posedness of \[(1.13)\] in the functional framework below.

Let \(H^m(\Omega)\) with \(m \geq 0\) and \(H^s(\Sigma)\) with \(s \in \mathbb{R}\) be the usual Sobolev spaces, whose norms are denoted by \(\| \cdot \|_m\) and \(\| \cdot \|_s\), respectively. The anisotropic Sobolev norm is defined as

\[
\| f \|_{m,\ell} := \sum_{j \leq \ell} \left\| \partial_t^j f \right\|_m.
\]

Note that \(\| \cdot \|_{m,0} = \| \cdot \|_m\). For a generic integer \(n \geq 3\), define the high-order energy as

\[
E_n := \sum_{j=0}^{n} \left\| \partial_t^j v \right\|_{n-j}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j b \right\|_{n-j+1}^2 + \left\| \partial_t^n b \right\|_0^2
\]
Remark 2.4. Theorem 2.2 implies in particular that for each \( t \) and \( E \) one may change coordinates to \( y \in \Omega(t) \) to produce a global-in-time, decaying solution to (1.2).

Remark 2.3. Since \( h \) is such that the mapping \( \Phi(t, \cdot) \), defined in (1.10), is a diffeomorphism for each \( t \geq 0 \), one may change coordinates to \( y \in \Omega(t) \) to produce a global-in-time, decaying solution to (1.2).

Remark 2.4. Theorem 2.2 implies in particular that \( \sqrt{\mathcal{E}_{N+4}(t)} \lesssim (1 + t)^{-\frac{N+4}{4}} \), which is integrable in time for \( N \geq 8 \). One may refine the nonlinear estimates so as to lower the index \( N \); we have forgone this for the simplifications. Since \( N \) may be taken to be arbitrarily large, this decay result can be regarded as an “almost exponential” decay rate.
Remark 2.5. Theorem 2.2 provides the first result of the global well-posedness of free boundary problems without viscosity for rotational flows. This is due to the strong coupling between the fluid and the diffusive magnetic field.

Remark 2.6. The global well-posedness theory here relies heavily on that the magnetic field $B$ is non-horizontal in 2D. Hence, it would be interesting to consider the case when $B$ is horizontal in 2D. However, it seems unlikely to generalize the results of global well-posedness with decay in 2D to the 3D case. For example, if $B = e_1$ and taking $B = B$ and $v_1 = 0$, then the problem in 3D will reduce to the free-boundary incompressible Euler equations in 2D; if $B = e_3$ and taking $B = B$, $h = 0$ and $v_3 = 0$, then the problem in 3D will reduce to the incompressible Euler equations in $\mathbb{T}^2$ for which the exponential growth of the vorticity gradient has been proved in Zlatoš [44], see also Kiselev and Šverák [22] for the double-exponential growth of the vorticity gradient in a disc.

Remark 2.7. It seems difficult to generalize the results here to the case without surface tension, i.e., $\sigma = 0$. In this case, it seems that only $H^{2N}$ regularity for $h$ is available, however, to deal with the magnetic diffusion term which involves the differential operator $\Delta^2$, an additional 1/2 order of regularity for $h$ is required; indeed, such a difficulty occurs already for the local well-posedness. This is different from the viscous fluid, where the viscosity has a regularizing effect of 1/2 order for $h$.

2.2. Strategy of the proof. Since for the local-in-time theory of (1.13), the Lorentz force term is of a lower order regularity compared to the magnetic diffusion term, the local well-posedness in our functional framework can be established by combining the construction of solutions to the free-boundary incompressible Euler equations with surface tension [9] with our idea of deriving the estimates for the Euler part and a bit work of treating the magnetic field terms. We thus omit the construction of local solutions and focus on the derivation of the a priori estimates, the main part of proving Theorems 2.2 which are recorded in Theorem 2.3.

The basic ingredient in our analysis is to use the energy-dissipation structure (1.16). As needed to work with the higher order energy functionals to control the nonlinear terms, one applies the temporal and horizontal spatial derivatives $\partial^\alpha$ for $\alpha \in \mathbb{N}^{1+1}$ with $|\alpha| \leq 2N$ that preserve the boundary conditions to (1.13) and then derives the tangential energy evolution

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} (|\partial^\alpha v|^2 + |\partial^\alpha b|^2) \, dv + \int_{\Sigma} \left( g |\partial^\alpha h|^2 + \sigma |\partial_1 \partial^\alpha h|^2 \right) \, dx_1 \right) + \kappa \int_{\Omega} |\nabla^\varphi \partial^\alpha b|^2 \, dv \ni$$

$$= - \int_{\Omega} \sigma \partial^\alpha H [\partial^\alpha, N] \cdot v \, dx + \int_{\Omega} \partial^\alpha q [\partial^\alpha, \nabla^\varphi] \cdot v \, dv + \sum_{R}. \tag{2.8}$$

Here $\sum_{R}$ denotes the nonlinear terms, which, after some delicate arguments, can be controlled well, say, by $\sqrt{\mathcal{E}_{N+1} \mathcal{E}_{2N}}$ as $\mathcal{E}_{2N}$ is small. When $\alpha_0 \leq 2N - 1$, due to the definition (2.9) with $n = 2N$ of $\mathcal{E}_{2N}$, the first two terms in the right hand side of (2.8) are also of $\sum_{R}$. It turns out that the most delicate argument is to treat them for the case $\alpha_0 = 2N$. Noting that $\nabla^\varphi \cdot v \partial_2 \varphi = N \cdot \partial_2 v + \partial_2 \varphi \partial_1 v_1 = 0$, one has

$$[\partial^\alpha_{2N}, \nabla^\varphi] \cdot v \partial_2 \varphi = [\partial^\alpha_{2N}, -\partial_1 \eta] \partial_2 v_1 + [\partial^\alpha_{2N}, \partial_2 \eta] \partial_1 v_1, \tag{2.9}$$

and hence the right hand side of (2.8) can be rewritten as

$$\int_{\Sigma} \sigma \partial^\alpha_{2N} H^{2N} \partial_2 \eta \partial_1 v_1 \, dx + \int_{\Omega} \partial^\alpha_{2N} q^{2N} \partial_2 \eta \partial_1 \eta \partial^\alpha_{2N} \partial_1 v_1 \, dx + \sum_{Q} + \sum_{R}. \tag{2.10}$$

Here $\sum_{Q}$ stands for the nonlinear terms related to $q$, which are easier to estimate after some delicate arguments than the second term in (2.10). The main difficulties now are that there is no any estimate for $\partial^\alpha_{2N} q$ so that it is difficult to bound the second term, and there is a loss of 1/2 order of regularity for $\partial^\alpha_{2N} h$ or $\partial^\alpha_{2N} v_1$ in controlling the first term. In fact, such a difficulty also occurs for the local well-posedness. For the free-boundary incompressible Euler equations with surface tension [9] [29], through a careful study of the vorticity equation, it can be shown that $\partial^\alpha_{2N-1} v \in H^{\frac{3}{2}}(\Omega)$ (i.e., a gain of 1/2 order of regularity) and hence the estimates can be closed. Such an idea may be employed to establish the local well-posedness of (1.13) in
the framework of the energy functional used in [1, 29], however, it is not suitable for the global well-posedness of (1.13) here since it would involve treating the linearized Lorentz force term as the forcing term for the Euler part, which is harmful for the global-in-time estimates of the vorticity.

Our observation to overcome these difficulties mentioned above is to integrate by parts over $t$ and $x_2$ in an appropriate order so that one has a crucial cancellation. More precisely, we integrate by parts in $x_2$ first and then in $t$ for the pressure term, using the boundary condition $q = gh - \sigma H$ on $\Sigma$, to obtain

\[
\int_\Sigma \sigma \partial_t^2 \nu H 2N_0 \partial_1 h \partial_i^2 \nu_1 - v_1 \, dx + \int_\Omega \partial_i^2 q 2N_\partial_1 \partial_i \eta \partial_i^2 \nu_1 - v_1 \, dx \\
= \int_\Sigma (\sigma \partial_t^2 H + \sigma \partial_t^2 q) 2N_0 \partial_1 h \partial_i^2 \nu_1 - v_1 \, dx_1 - \int_\Omega \partial_2 (\partial_i^2 q 2N_\partial_1 \partial_i \eta) \partial_i^2 \nu_1 - v_1 \, dx \\
= \int_\Sigma g \partial_i^2 \nu h 2N_0 \partial_1 h \partial_i^2 \nu_1 - v_1 \, dx_1 - \frac{d}{dt} \int_\Omega \partial_2 (\partial_i^2 q 2N_\partial_1 \partial_i \eta) \partial_i^2 \nu_1 - v_1 \, dx + \sum_R. \tag{2.11}
\]

Note now that by the definition of $E_{2N}$, the first term in the right hand side of (2.11) is of $\sum_R$. This can then lead to the tangential energy evolution estimates, and one concludes that

\[
\bar{E}_{2N}(t) + \int_0^t \bar{D}_{2N}(s) \, ds \lesssim E_{2N}(0) + E_{2N}^{3/2}(t) + \int_0^t \sqrt{E_{N+4}(s)} \bar{E}_{2N}(s) \, ds, \tag{2.12}
\]

where $\bar{E}_n$ and $\bar{D}_n$ represent the tangential energy and dissipation functionals, defined by (1.1) and (1.2), respectively. It should be pointed out that this observation provides also a new energy functional for the local well-posedness of the free-boundary incompressible Euler equations with surface tension.

Note that, as already seen from the estimate (2.12), to close the estimates, if the desired dissipation can not dominate the energy, then one needs to show that $\sqrt{E_{N+4}(t)}$ is integrable in time. Unfortunately, this is indeed the case here. Our strategy is to show that $E_{N+4}(t)$ decays sufficiently fast in time. To this end, employing a quite different but careful argument, we will be able to derive a related set of tangential energy evolution estimates

\[
\frac{d}{dt} (\bar{E}_n + B_n) + \bar{D}_n \lesssim \sqrt{E_{2N} D_n}, \quad n = N + 4, \ldots, 2N - 2, \tag{2.13}
\]

where $B_n$ are some nonlinear terms satisfying $|B_n| \lesssim \sqrt{E_{2N} E_n}$. Note that the restriction $n \leq 2N - 2$ in (2.13) is due to that certain controls on regularities in $D_n$ are weaker than those in $E_n$.

With the control of the tangential energy evolution estimates we then proceed to derive the full energy estimates by employing further the structures of the equations (1.13). The crucial starting point here is that one can derive the desired boundary estimates of $v_2$ or $v \cdot N$ on the upper boundary $\Sigma$. The boundary estimates in the energy can be obtained directly from the control of $\bar{E}_n$ by applying the normal trace estimates, while for the estimates in the dissipation it is crucial to derive the estimates of $\left\| \bar{B} \cdot \nabla \partial_i^2 v_1 \right\|_{0,n-j-1}, j = 0, \ldots, n - 1$, from the control of $\bar{D}_n$ for $\kappa > 0$ and then use the Poincaré-type inequality related to $\bar{B} \cdot \nabla$ for $\bar{B} \neq 0$ with $v_2 = 0$ on $\Sigma_1$. With these boundary estimates, we can then use the elliptic problems for the pressure $q$ and the boundary conditions to derive the estimates for $q$ and $h$. Note that the regularity of $q$ in the dissipation is weaker than that in the energy, and this then reflects also the regularity of the other components of the solution.

The remaining in completing the energy estimates is to derive the rest of the estimates for $v$ and $b$. The natural way of estimating the normal derivatives of $v$, as for the incompressible Euler equations, is to consider the equations for the vorticity $\text{curl}^2 v = \partial_i^2 v_2 - \partial_2^2 v_1$:

\[
\partial_i^2 \text{curl}^2 v + v \cdot \nabla^2 \text{curl}^2 v = \bar{B} \cdot \nabla^2 \text{curl}^2 b + \cdots. \tag{2.14}
\]

Here $\cdots$ means plus some nonlinear terms. Now the difficulty is that one can not treat the linear term $\bar{B} \cdot \nabla^2 \text{curl}^2 b$ on the right hand side of (2.14) just as a forcing term, and one can not use the equation of $\text{curl}^2 b$ to balance this linear term as done for the tangential energy.
evolution estimates due to the presence of the magnetic diffusion. Our key point here then lies in
the treatment of this linear term: by using the fourth, third and second equations in (1.13),
one finds

\[ \vec{B} \cdot \nabla \times \nabla^r \vec{B} = \vec{B} \times \Delta^r \vec{B} - \vec{B} \times \nabla \times (\frac{1}{\kappa} \vec{B} \cdot \nabla \times \nabla^r \vec{B}) + \cdots \]

\[ = \frac{\vec{B}^2}{\kappa} \nabla^r \vec{B} - \frac{1}{\kappa} \partial_t (\vec{B} \cdot \nabla \times \vec{B} \cdot \nabla^r \vec{B}) + \frac{1}{\kappa} \partial_t (\vec{B} \times \vec{B}) + \cdots. \] (2.15)

One then arrives at the following equation of \( \nabla \times \vec{B} \):

\[ \partial_t \nabla^r \vec{B} + \nabla^r \nabla^r \vec{B} + \frac{\vec{B}^2}{\kappa} \nabla^r \vec{B} = \frac{1}{\kappa} \partial_t (\vec{B} \cdot \nabla \times \vec{B} \cdot \nabla^r \vec{B}) + \frac{1}{\kappa} \partial_t (\vec{B} \times \vec{B}) + \cdots. \] (2.16)

One thus sees again the key roles of the positivity of the magnetic diffusion coefficient \( \kappa > 0 \) and
the non-vanishing of \( \vec{B}_2 \neq 0 \); it induces the damping term which provides the mechanism for the
global-in-time estimates of the vorticity curl \( \vec{B} \). Note that for the estimates in the energy one
can estimate the \( \partial_t \vec{B} \) terms in the right hand of (2.10) by the control of \( \mathcal{E}_N \); for the estimates in
the dissipation one has to estimate them by instead combing the dissipation estimates of \( \vec{B} \cdot \nabla \vec{B} \),
the Poincaré-type inequalities and the incompressibility condition, where the dimension of 2 is
used crucially. Basing on these estimates and the transport-damping structure of curl \( \vec{B} \) in
(2.16) and employing the Hodge-type elliptic estimates of \( \vec{B} \), we can derive the desired estimates
of \( \vec{B} \) in a recursive way in terms of the number of normal derivatives of \( \vec{B} \); the desired estimates
of \( \vec{B} \) are derived along by employing the elliptic estimates of \( \vec{B} \). The conclusion is that one can
then improve (2.12) and (2.13) to be

\[ \mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) ds \lesssim \mathcal{E}_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{N+4}}(s) \mathcal{E}_{2N}(s) ds \] (2.17)

and

\[ \frac{d}{dt} \mathcal{E}_n + \mathcal{D}_N \leq 0, \quad n = N + 4, \ldots, 2N - 2. \] (2.18)

Note that if \( \mathcal{E}_{N+4}(t) \) decays at a sufficiently fast rate, then the estimate (2.17) can be closed
to be (2.18). This will be achieved by using (2.18). One does not have that \( \mathcal{E}_n \lesssim \mathcal{D}_n \), which
rules out the exponential decay; also, \( \mathcal{D}_n \) cannot control \( \mathcal{E}_n \) with respect to not only the spatial
regularity but also the temporal regularity, which prevents one from using the spatial regularity
Sobolev interpolation argument as [17, 18] to bound \( \mathcal{E}_n \lesssim \mathcal{E}_{N+4}^{1-\theta} \mathcal{D}_n^\theta \), \( 0 < \theta < 1 \) so as to derive
the algebraic decay. Our key ingredient to get around this is to observe that \( \mathcal{E}_2 \lesssim \mathcal{D}_{\ell+1} \) and
employ a time weighted inductive argument. To begin with, we may rewrite (2.18) as

\[ \frac{d}{dt} \mathcal{E}_{2N-j-1} + \mathcal{D}_{2N-j-1} \leq 0, \quad j = 1, \ldots, N - 5. \] (2.19)

Multiplying (2.19) by \( (1 + t)^j \), one has

\[ \frac{d}{dt} \left( (1 + t)^j \mathcal{E}_{2N-j-1} \right) + (1 + t)^j \mathcal{D}_{2N-j} \leq (1 + t)^j \mathcal{E}_{2N-j-2} \lesssim (1 + t)^j \mathcal{D}_{2N-j-1}. \] (2.20)

Integrating (2.20) in time directly, by a suitable linear combination of the resulting inequalities,
one obtains

\[ \sum_{j=1}^{N-5} (1 + t)^j \mathcal{E}_{2N-j-1}(t) + \sum_{j=1}^{N-5} \int_0^t (1 + s)^j \mathcal{D}_{2N-j-1}(s) ds \lesssim \int_0^t \mathcal{D}_{2N-1}(s) ds. \] (2.21)

This together with (2.6) yields (2.7) and hence implies a decay of \( \mathcal{E}_{N+4} \) with the rate \( (1+t)^{-N+5} \).
Consequently, this scheme of the a priori estimates can be closed by requiring \( N \geq 8 \).
2.3. Notation. We now set the conventions for notations to be used later. The Einstein convention of summation over repeated indices will be used. Throughout the paper $C > 0$ denotes a generic constant that does not depend on the data, but can depend on the the parameters of the problem, $g, \kappa, \sigma, N$ and $\Omega$. We refer to such constants as “universal”. Such constants are allowed to change from line to line. We employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$. To avoid the constants in various time differential inequalities, we employ the following two conventions:

$$\partial_t A_1 + A_2 \lesssim A_3$$
and

$$\partial_t (A_1 + A_2) + A_3 \lesssim A_4$$

for an integer $N$, $T$ is given on the interval $[0, T]$. Section 4 provides the tangential energy evolution estimates. Section 5 gives the estimates of the pressure and the free boundary function. Section 6 yields the estimates of the velocity and the magnetic field. Section 7 concludes the global energy estimates, as recorded in estimates of the free-boundary inviscid surface waves.

Also, $\mathbb{N} = \{0, 1, 2, \ldots \}$ denotes for the collection of non-negative integers. When using space-time differential multi-indices, we write $\mathbb{N}^{1+d} = \{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \}$ to emphasize that the 0–index term is related to temporal derivatives. For just spatial derivatives, we write $\mathbb{N}^d$. For $\alpha \in \mathbb{N}^{1+d}$, $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. For any differential operator $\partial^\alpha$, one can define the standard commutator

$$[\partial^\alpha, f] g = \partial^\alpha (fg) - f \partial^\alpha g$$

and the symmetric commutator

$$[\partial^\alpha, f, g] = \partial^\alpha (fg) - f \partial^\alpha g - \partial^\alpha fg.$$

2.4. Organization of the paper. The rest of the paper is organized as follows. Some analytic tools are collected in Appendix A. Section 3 contains some preliminary results for the a priori estimates. Section 4 provides the tangential energy evolution estimates. Section 5 gives the estimates of the pressure and the free boundary function. Section 6 yields the estimates of the velocity and the magnetic field. Section 7 concludes the global energy estimates, as recorded in Theorem 1.2, and the proof of Theorem 1.3 is completed.

3. Preliminaries for the a priori estimates

In this section we give some preliminary results to be used in the derivation of the a priori estimates for solutions to (1.13). It will be assumed throughout Sections 3-6 that the solution is given on the interval $[0, T]$ and obey the a priori assumption

$$E_2N(t) \leq \delta, \quad \forall t \in [0, T]$$

for an integer $N \geq 4$ and a sufficiently small constant $\delta > 0$. This implies in particular that

$$\frac{1}{2^N} \leq \partial_2 \varphi(t, x) \leq \frac{3}{2}, \quad \forall (t, x) \in [0, T] \times \Omega.$$

We remark that (3.1) and (3.2) are always used; in particular, the smallness (3.1) is used in many nonlinear estimates so that the various polynomials of $E_2N$ are bounded by $C E_2N$.

In order to use the energy-dissipation structure (1.16) to derive the energy evolution estimates for temporal and horizontal spatial derivatives of the solution to (1.13), as for the free-boundary incompressible Euler equations, it is natural to utilize the geometric structure of the equations given in (1.13). For this, one applies the differential operators $\partial^\alpha$ for $\alpha \in \mathbb{N}^{1+1}$ that preserve the boundary conditions to (1.13) to find that

$$\begin{aligned}
\partial_t \tilde{\varphi} + v \cdot \nabla \tilde{\varphi} &+ \nabla \partial^\alpha v + \partial^\alpha q = (\tilde{B} + b) \cdot \nabla \tilde{\varphi} \partial^\alpha b + F^{1, \alpha}, \\
\nabla \tilde{\varphi} \cdot \partial^\alpha v &+ F^{2, \alpha}, \quad \text{in } \Omega, \\
\partial_t \partial^\alpha b &+ \partial^\alpha v \cdot N + F^{3, \alpha}, \quad \text{on } \Sigma, \\
\partial^\alpha q &+ g \partial^\alpha h - \sigma \partial^\alpha H, \quad \partial^\alpha b = 0, \quad \text{on } \Sigma, \\
\partial^\alpha v_2 &+ 0, \quad \partial^\alpha b = 0, \quad \text{on } \Sigma_{-1},
\end{aligned}$$

where

$$F^{1, \alpha} = [\partial^\alpha, (\tilde{B} + b) \cdot \nabla \tilde{\varphi}] b - [\partial^\alpha, \partial_t \tilde{\varphi} + v \cdot \nabla \tilde{\varphi}] v - [\partial^\alpha, \nabla \tilde{\varphi}] q.$$
\[ F^{2,\alpha} = -[\partial^\alpha, \nabla \varphi] \cdot v, \quad (3.5) \]
\[ F^{3,\alpha} = [\partial^\alpha, (B + b) \cdot \nabla \varphi] \cdot v - [\partial^\alpha, \partial^\varphi + v \cdot \nabla \varphi] \cdot b + \kappa [\partial^\alpha, \nabla \varphi] \cdot \nabla \varphi \cdot b, \quad (3.6) \]
\[ F^{4,\alpha} = \nabla \varphi \cdot \tilde{F}^{4,\alpha}, \quad \tilde{F}^{4,\alpha} = \kappa [\partial^\alpha, \nabla \varphi] \cdot b, \quad (3.7) \]
\[ F^{5,\alpha} = [\partial^\alpha, N] \cdot v. \quad (3.8) \]

Note that in (3.7), \( F^{4,\alpha} \) is written in a divergence form which will be helpful in the tangential energy evolution estimates. Furthermore, direct calculations show that for \(|\alpha| \geq 1,\)
\[ \partial^\alpha H = \partial_t \left( \frac{\partial_t \partial^\alpha h}{\sqrt{1 + |\partial_t h|^2}} + F^{6,\alpha} \right), \quad (3.9) \]
where
\[ F^{6,\alpha} = -\left[ \partial^{\alpha-\alpha'}, \frac{\partial_t h}{\sqrt{1 + |\partial_t h|^2}} \right] \partial_t \partial^{\alpha'} h \partial_t h + \left[ \partial^\alpha, \frac{1}{\sqrt{1 + |\partial_t h|^2}}, \partial_t h \right] \quad (3.10) \]
for any \( \alpha' \leq \alpha \) with \(|\alpha'| = 1.\)

Then it holds the following natural energy evolution associated to (3.3).

**Proposition 3.1.** Let \( \alpha \in \mathbb{N}^{1+1} \) be with \(|\alpha| \geq 1.\) For (3.3), it holds that

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} \left( |\partial^{\alpha} v|^2 + |\partial^{\alpha} b|^2 \right) \, d\nu_t + \int_{\Sigma} \left( q |\partial^{\alpha} h|^2 + \sigma \frac{|\partial_t \partial^{\alpha} h|^2}{1 + |\partial_t h|^2} \right) \, dx \right) + \kappa \int_{\Omega} |\nabla \varphi \partial^{\alpha} b|^2 \, d\nu_t
\]
\[ = \int_{\Omega} \left( F^{1,\alpha} \cdot \partial^\alpha v + \partial^\alpha q F^{2,\alpha} + (F^{3,\alpha} + F^{4,\alpha}) \cdot \partial^\alpha b \right) \, d\nu_t \]
\[ + \int_{\Sigma} \left( \left( g \partial^\alpha h - \sigma \partial^\alpha H \right) F^{5,\alpha} + \sigma \partial_t F^{6,\alpha} \partial_t \partial^\alpha h + \frac{\sigma}{2} \partial_t \left( \frac{1}{1 + |\partial_t h|^2} \right) |\partial_t \partial^\alpha h|^2 \right) \, dx_t. \quad (3.11) \]

**Proof.** Taking the inner product of the first equation in (3.3) with \( \partial^\alpha v \) and the third equation with \( \partial^\alpha b \), integrating by parts over \( \Omega \) by using \( \nabla \varphi \cdot v = 0, \partial_t h = v \cdot N \) on \( \Sigma, v_2 = 0 \) on \( \Sigma_1 \) and \( \partial^\alpha b = 0 \) on \( \partial \Omega \), and then adding the resulting equations together, one has
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |\partial^{\alpha} v|^2 + |\partial^{\alpha} b|^2 \right) \, d\nu_t + \int_{\Omega} \nabla \varphi \partial^{\alpha} q \cdot \partial^{\alpha} v \, d\nu_t + \kappa \int_{\Omega} |\nabla \varphi \partial^{\alpha} b|^2 \, d\nu_t \]
\[ = \int_{\Omega} \left( B + b \right) \cdot \nabla \varphi \left( \partial^\alpha b \cdot \partial^\alpha v \right) \, d\nu_t + \int_{\Omega} \left( F^{1,\alpha} \cdot \partial^\alpha v + (F^{3,\alpha} + F^{4,\alpha}) \cdot \partial^\alpha b \right) \, d\nu_t. \quad (3.12) \]

The integration by parts over \( \Omega \) shows that, by using \( \partial^\alpha b = 0 \) on \( \partial \Omega \) and \( \nabla \varphi \cdot b = 0, \)
\[ \int_{\Omega} \left( B + b \right) \cdot \nabla \varphi \left( \partial^\alpha b \cdot \partial^\alpha v \right) \, d\nu_t = \int_{\Omega} \nabla \varphi \cdot b \partial^\alpha b \cdot \partial^\alpha v \, d\nu_t = 0. \quad (3.13) \]

By the fifth, fourth and second equations in (3.3), one integrates by parts over \( \Omega \) again, using \( \partial^\alpha v_2 = 0 \) on \( \Sigma_1 \), to obtain
\[ \int_{\Omega} \nabla \varphi \partial^{\alpha} q \cdot \partial^{\alpha} v \, d\nu_t = \int_{\Sigma} \partial^\alpha q \partial^\alpha v \cdot N \, dx_t - \int_{\Omega} \partial^\alpha q \nabla \varphi \cdot \partial^{\alpha} v \, d\nu_t \]
\[ = \int_{\Omega} \left( g \partial^\alpha h - \sigma \partial^\alpha H \right) \left( \partial_t \partial^\alpha h - F^{5,\alpha} \right) \, dx_t - \int_{\Omega} \partial^\alpha q F^{2,\alpha} \, d\nu_t. \quad (3.14) \]

Integrating by parts in \( t \) leads to
\[ \int_{\Sigma} \partial^\alpha h \partial_t \partial^\alpha h \, dx_t = \frac{1}{2} \frac{d}{dt} \int_{\Sigma} \left| \partial^\alpha h \right|^2 \, dx_t. \quad (3.15) \]

By (3.14), one may write
\[ -\int_{\Sigma} \sigma \partial^\alpha H \partial_t \partial^\alpha h \, dx_t = -\int_{\Sigma} \sigma \partial_t \left( \frac{\partial_t \partial^\alpha h}{\sqrt{1 + |\partial_t h|^2}} + F^{6,\alpha} \right) \partial_t \partial^\alpha h \, dx_t. \quad (3.16) \]
Integrating by parts in both \( x_1 \) and \( t \) yields
\[
- \int_\Sigma \sigma \partial_t \left( \frac{\partial_t \varphi^h}{\sqrt{1 + |\partial_t h|^2}} \right) \partial_\alpha \varphi^h \, dx_1 = \int_\Sigma \sigma \partial_t \varphi^h \partial_\alpha \partial_t \varphi^h \, dx_1
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_\Sigma \sigma \frac{|\partial_t \varphi^h|^2}{\sqrt{1 + |\partial_t h|^2}} \, dx_1 - \frac{1}{2} \int_\Sigma \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\partial_t h|^2}} \right) |\partial_t \varphi^h|^2 \, dx_1. \tag{3.17}
\]
Consequently, in light of (3.13)–(3.17), (3.12) yields the identity (3.11).

Next, we shall estimate these nonlinear terms \( F^{i,\alpha} \) with \( |\alpha| \leq 2N \).

**Lemma 3.2.** For \( |\alpha| \leq 2N \), it holds that
\[
\left\| F^{1,\alpha} \right\|_0^2 + \left\| F^{2,\alpha} \right\|_0^2 + \left\| F^{3,\alpha} \right\|_0^2 + \left\| F^{4,\alpha} \right\|_0^2 + \left\| F^{5,\alpha} \right\|_0^2 + \left\| F^{6,\alpha} \right\|_{3/2}^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{3.18}
\]
For \( |\alpha| \leq 2N, \alpha_0 \leq 2N - 1 \), it holds that
\[
\left\| F^{5,\alpha} \right\|_{1/2}^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{3.19}
\]

**Proof.** To estimate \( F^{1,\alpha} \), one first notes that by the definition (1.12) of \( \partial_t^\varepsilon \),
\[
\partial_t^\varepsilon + \nabla^\varepsilon = \partial_t + v_1 \partial_1 + v_2 \partial_2, \quad \text{where} \quad v_z = \frac{1}{\partial_2 \varphi} (v \cdot N - \partial_t \varphi) \equiv \frac{1}{\partial_2 \varphi} (v \cdot N - \partial_t \eta), \tag{20.20}
\]
and hence,
\[
[\partial_\alpha^2, \partial_\varepsilon^2 + \nabla^\varepsilon] v = [\partial_\alpha^2, v_1] \partial_1 v + [\partial_\alpha^2, v_2] \partial_2 v. \tag{20.21}
\]
It then follows from the Leibniz rule and the Sobolev embeddings that
\[
\left\| [\partial_\alpha^2, v_z] \partial_2 v \right\|_0^2 \lesssim \sum_{0 < |\alpha'| \leq |\alpha|} \left\| \partial_\alpha^2 v_z \partial_\alpha^2 v_z \right\|_0^2
\]
\[
\lesssim \sum_{0 < |\alpha'| \leq |\alpha|} \left\| \partial_\alpha^2 v_z \right\|_{L^\infty}^2 \left\| \partial_\alpha^2 v_z \right\|_0^2 + \sum_{|\alpha'| > |\alpha|} \left\| \partial_\alpha^2 \partial_\alpha^2 v_z \right\|_0^2 \left\| \partial_\alpha^2 v_z \right\|_0^2
\]
\[
\lesssim \sum_{0 < |\alpha'| \leq |\alpha|} \left\| \partial_\alpha^2 v_z \right\|_2^2 \left\| \partial_\alpha^2 \partial_\alpha^2 v_z \right\|_0^2 + \sum_{|\alpha'| > |\alpha|} \left\| \partial_\alpha^2 \partial_\alpha^2 v_z \right\|_2^2 \left\| \partial_\alpha^2 v_z \right\|_0^2. \tag{20.22}
\]
To bound the \( H^0 \) norms in the right hand side of (20.22), since \( v_z \) involves \( v \) and \( \partial_\varepsilon \varphi \) for \( \beta' \in \mathbb{N}^{1+2} \) with \( |\beta'| = 1 \), one may check directly that the terms of the highest order derivatives involved are \( \partial_\alpha^2 \partial_\alpha^2 v_z \) for \( \alpha' \in \mathbb{N}^{1+1} \) with \( |\alpha'| = 1 \), \( \partial_\alpha v_z \) and \( \partial_\alpha \partial_\alpha^2 \varphi \) for \( \beta' \in \mathbb{N}^{1+2} \) with \( |\beta'| = 1 \). Noting the term \( |\partial_1^2 \partial_1 h|_{-1/2}^2 \) in \( \mathcal{E}_n \) so that when \( \partial_\alpha^2 \beta' = \partial_1^{2N+1} \), one gets by (1.10) and Lemma A.1 that
\[
\left\| \partial_1^{2N+1} \varphi \right\|_0^2 = \left\| \partial_1^{2N+1} \eta \right\|_0^2 \lesssim \left\| \partial_1^{2N+1} \varphi \right\|_0^2 \lesssim \left\| \partial_1^{2N+1} \varphi \right\|_0^2 \lesssim \left\| \partial_1^{2N+1} \varphi \right\|_0^2 \lesssim \mathcal{E}_{2N}. \tag{20.23}
\]
Then the \( H^0 \) norms in the right hand side of (20.22) are bounded by \( \mathcal{E}_{2N} \), due to (2.20) with \( n = 2N \) and Lemmas A.1 and A.2. On the other hand, by Lemmas A.1 and A.2 again along with the definition (2.5) of \( \mathcal{E}_n \), one notes that the extra 4 derivatives in \( \mathcal{E}_{N+4} \) has been chosen so as to be sufficient for those \( H^2 \) norms in the right hand side of (20.22) to be bounded by \( \mathcal{E}_{N+4} \). Hence \( \left\| \partial_\alpha^2 v_z \partial_2 v \right\|_0^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N} \), and estimating the other terms in \( F^{1,\alpha} \) in the same way, one may conclude that
\[
\left\| F^{1,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\]
Similarly, one has that \( \left\| F^{2,\alpha} \right\|_0^2 + \left\| F^{4,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N} \). To estimate \( F^{3,\alpha} \), one may argue again as \( F^{1,\alpha} \) to derive the desired estimates with one exceptional term \( \kappa [\partial_\alpha^2 \nabla \varphi] \cdot \nabla \chi b \), which involves new types of terms of the highest order derivatives: \( \partial_\alpha^2 \partial_\alpha \varphi \) for \( \alpha' \in \mathbb{N}^{1+1} \) with \( |\alpha'| = 1 \). By the definition of \( \mathcal{E}_n \), (1.10) and Lemma A.1 one obtains
\[
\left\| \partial_\alpha^2 \partial_\alpha \varphi \right\|_0^2 \lesssim \left\| \partial_\alpha^2 \varphi \right\|_2^2 \lesssim \left\| \partial_\alpha^2 \varphi \right\|_2^2 \lesssim \left\| \partial_\alpha^2 \varphi \right\|_2^2 \lesssim \mathcal{E}_{2N}. \tag{20.24}
\]
and since $\alpha_0 - \alpha'_0 \leq 2N - 1$,
\[ \left\| \partial^{\alpha - \alpha'} \partial_2 \nabla b \right\|^2_0 \leq \left\| \partial^{\alpha - \alpha'} b \right\|^2_2 \leq \mathcal{E}_{2N}. \] (3.25)

Hence, one can get $\left\| F^{5,\alpha} \right\|^2_0 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}$.

To estimate $F^{5,\alpha}$, since the terms of the highest order derivatives are $\partial^{\alpha - \alpha'} v_1$ for $\alpha' \in \mathbb{N}^{1+1}$ with $|\alpha'| = 1$ and $\partial^{\alpha} \partial_1 h$, one then separates the cases $\alpha_0 = 2N$ and $\alpha_0 \leq 2N - 1$. By Lemma A.2, the trace theory and the definitions of $\mathcal{E}_n$ and $\mathcal{E}_n$, one deduces that for $\alpha_0 = 2N$,
\[ |F^{5,\alpha}|^2_0 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N} \] and that for $\alpha_0 \leq 2N - 1$, $|F^{5,\alpha}|^2_{1/2} \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}$.

Finally, to estimate $F^{6,\alpha}$, since the terms of the highest order derivatives are $\partial^{\alpha - \alpha'} \partial_1 h$ for $\alpha' \in \mathbb{N}^{1+1}$ with $|\alpha'| = 1$, similarly, one obtains that $|F^{6,\alpha}|^2_{3/2} \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}$.

Consequently, the estimates (3.18)--(3.19) follow. □

We now present the specialized estimates of $F^{\alpha}$ when $|\alpha| \leq 2N - 2$.

Lemma 3.3. For $|\alpha| \leq 2N - 2$, it holds that
\[ \left\| F^{1,\alpha} \right\|^2_0 + \left\| \partial_1 F^{1,\alpha} \right\|^2_0 + \left\| F^{2,\alpha} \right\|^2_1 + \left\| \partial_1 F^{2,\alpha} \right\|^2_0 + \left\| \partial_1^2 F^{2,\alpha} \right\|^2_0 \]
\[ + \left\| F^{3,\alpha} \right\|^2_0 + \left\| F^{4,\alpha} \right\|^2_0 + \left\| F^{5,\alpha} \right\|^2_2 + \left\| F^{6,\alpha} \right\|^2_2 + \left\| \partial_1 F^{6,\alpha} \right\|^2_1 \lesssim \mathcal{D}_{N+4}\mathcal{E}_{2N} \] (3.26)

and
\[ \left\| F^{1,\alpha} \right\|^2_0 + \left\| F^{2,\alpha} \right\|^2_0 + \left\| \partial_1 F^{2,\alpha} \right\|^2_0 + \left\| F^{6,\alpha} \right\|^2_{1/2} \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \] (3.27)

Proof. Recall the definition (2.14) of $\mathcal{D}_n$. The proof proceeds similarly as Lemma 3.2. □

Though the form (1.13) is faithful to the geometry of the free boundary problem, yet it is more convenient to write (1.13) in a linear perturbed form to derive many of the estimates. The reason for this is that the differential operators become constant-coefficient, which is more convenient for many of the a priori estimates when utilizing the linear structure of the equations such as employing the elliptic regularities or deriving some special linear structures. The system can be rewritten as
\[
\begin{aligned}
\partial_1 v + \nabla q &= \bar{B} \cdot \nabla b + G^1 & \text{in } \Omega \\
\text{div } v &= G^2 & \text{in } \Omega \\
\partial_1 b - \kappa \Delta b &= \bar{B} \cdot \nabla v + G^3 & \text{in } \Omega \\
\text{div } b &= G^4 & \text{in } \Omega \\
\partial_1 h &= v_2 + G^5 & \text{on } \Sigma \\
q &= gh - \sigma \partial_1^2 h + G^6, & b = 0 & \text{on } \Sigma \\
v_2 = 0, & b = 0 & \text{on } \Sigma_{-1},
\end{aligned}
\] (3.28)

where
\[ G^1 = \partial_1 \eta \partial_1^2 v + \nabla \eta \partial_1^2 q - \bar{B} \cdot \nabla \eta \partial_1^2 b - v \cdot \nabla \partial_1^2 v + b \cdot \nabla \partial_1^2 b, \] (3.29)
\[ G^2 = \nabla \eta \cdot \partial_1^2 v, \] (3.30)
\[ G^3 = \partial_1 \eta \partial_1^2 b - \bar{B} \cdot \nabla \eta \partial_1^2 v + \kappa (\Delta \partial_1 v - \Delta) b - v \cdot \nabla \partial_1 b + b \cdot \nabla \partial_1 v, \] (3.31)
\[ G^4 = \nabla \eta \cdot \partial_1^2 b, \] (3.32)
\[ G^5 = -v_2 \partial_1 h, \] (3.33)
\[ G^6 = -\sigma \partial_1 ((1 + |\partial_1 h|^2)^{-1/2} - 1) \partial_1 h). \] (3.34)

Here one has used $\partial_1^2 v = -\partial_1 \eta \partial_1^2 v$ for $i = t, 1, 2$ due to (1.12).

The nonlinear terms $G^i$ are estimated as follows.

Lemma 3.4. It holds that
\[ \sum_{j=0}^{2N-1} \left\| \partial_1^j G^1 \right\|^2_{2N-j-1} + \sum_{j=0}^{2N-1} \left\| \partial_1^j G^2 \right\|^2_{2N-j-1} + \sum_{j=0}^{2N-1} \left\| \partial_1^j G^3 \right\|^2_{2N-j-1} + \sum_{j=0}^{2N-1} \left\| \partial_1^j G^4 \right\|^2_{2N-j} \]
\[ + \sum_{j=0}^{2N-1} \left| \partial_t^j G^5 \right|_{2N-j-1/2}^2 + \sum_{j=0}^{2N-1} \left| \partial_t^j G^6 \right|_{2N-j-1/2}^2 \lesssim \min \{ \mathcal{E}_{N+4}, \mathcal{D}_{N+4} \} \mathcal{E}_{2N}. \]  

(3.35)

**Proof.** The proof follows in the same way as for Lemma 3.2. \[ \square \]

The pressure \( q \) is governed by elliptic problems. Indeed, the first equation in (1.13) implies that, by \( \nabla \cdot v = 0 \),
\[ \Delta \varphi q = -\partial_t^2 v_j \partial_t^2 v_i + \partial_t^2 b_j \partial_t^2 b_i \quad \text{in} \Omega. \]  

(3.36)

Projecting the first equation in (1.13) along \( \mathbf{N} \) onto \( \Sigma \) and \( \Sigma^{-1} \) yields, recalling (3.20),
\[ \nabla \varphi q \cdot \mathbf{N} = -\partial_t v \cdot \mathbf{N} - v_i \partial_t v \cdot \mathbf{N} \quad \text{on} \Sigma \]  

(3.37)

and
\[ \partial_2 q = 0 \quad \text{on} \Sigma^{-1}. \]  

(3.38)

Here in (3.37) one has used the fact that on \( \Sigma \), using \( b = 0 \) on \( \Sigma \) and \( \nabla \varphi \cdot b \equiv \partial_1 b_1 + \partial_2^2 b \cdot \mathbf{N} = 0 \),
\[ (\bar{\mathcal{B}} + b) \cdot \nabla \varphi \cdot \mathbf{N} = \bar{\mathcal{B}} \cdot \nabla \varphi \cdot \mathbf{N} = \bar{\mathcal{B}} \cdot \mathbf{N} \partial_2^2 b \cdot \mathbf{N} = -\bar{\mathcal{B}} \cdot \mathbf{N} \partial_1 b_1 = 0, \]  

(3.39)

and in (3.38) one has used further the fact that \( \mathbf{N} = -e_2 \) and \( v_2 = 0 \) on \( \Sigma^{-1} \).

It is noted that for \( q \), there are two choices of boundary conditions on \( \Sigma \), i.e., the Neumann boundary condition (3.37) and the Dirichlet boundary condition
\[ q = gh - \sigma H \quad \text{on} \Sigma. \]  

(3.40)

Without surface tension, i.e., \( \sigma = 0 \), one can use the elliptic problem (3.36), (3.40) and (3.38) to establish the regularity estimates for \( q \). The subtlety lies in that \( \tilde{E}_{2N} \) provides the needed estimates for those boundary terms in the case without surface tension. When there is surface tension, however, \( \tilde{E}_{2N} \) does not provide enough estimates for the boundary term \( -\sigma H \) (which is of 1/2 regularity less). So for the desired regularity estimates of \( q \), one needs to use the elliptic problem (3.36)–(3.38). Then one sees also the need of estimating the time derivatives of the solution even for the local well-posedness. Though, as one cannot guarantee that \( q \) has zero average, one shall estimate \( \|q\|_0 \) by Poincaré’s inequality with the estimate of \( |q|_0 \) by using the boundary condition (3.40). It is noticed that there is an essential difficulty arising: when doing tangential energy evolution estimates with time derivatives up to 2N order, as one can only obtain the estimates of time derivatives of \( q \) up to 2N – 1 order, the energy estimates seems difficult to be closed since the 2N order time derivative of \( q \) is involved. We will explain this and our way to overcome it in more details in the next section.

To estimate the highest temporal derivative \( \partial_t^{n-1} q \), one needs to use again the geometric structure. Applying \( \partial_t^{n-1} \) to the problem (3.36)–(3.38), one obtains
\[
\begin{cases}
\Delta \varphi \partial_t^{n-1} q = P^{1,n-1} & \text{in} \Omega \\
\nabla \varphi \partial_t^{n-1} q \cdot \mathbf{N} = -\partial_t^n v \cdot \mathbf{N} + P^{2,n-1} & \text{on} \Sigma \\
\partial_2 \partial_t^{n-1} q = 0 & \text{on} \Sigma^{-1},
\end{cases}
\]  

(3.41)

where \( P^{1,n-1} = -[\partial_t^{n-1}, \Delta \varphi] q + \partial_t^{n-1} \left( -\partial_t^2 v_j \partial_t^2 v_i + \partial_t^2 b_j \partial_t^2 b_i \right) \),
\[ P^{2,n-1} = -[\partial_t^{n-1}, \mathbf{N} \cdot \nabla \varphi] q - [\partial_t^{n-1}, \mathbf{N}] \cdot \partial_t v - \partial_t^{n-1} (v_i \partial_t v \cdot \mathbf{N}), \]  

(3.42)

(3.43)

which can be estimated as follows.

**Lemma 3.5.** For \( n = N + 4, \ldots, 2N \), it holds that
\[ \| P^{1,n-1} \|_0^2 + \| P^{2,n-1} \|_{-1/2}^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}. \]  

(3.44)

**Proof.** The proof proceeds similarly as Lemma 3.2 with one exception of the estimate of the term \( |v_1 \partial_t \partial_t^{n-1} v \cdot \mathbf{N}|_{-1/2}^2 \) in estimating \( |P^{2,n-1}|_{-1/2}^2 \). For this term, one uses Lemmas A.3 and A.2 and the trace theory to bound
\[ |v_1 \partial_t \partial_t^{n-1} v \cdot \mathbf{N}|_{-1/2}^2 \lesssim |\partial_t \partial_t^{n-1} v|^2_{1/2} |v_1 \mathbf{N}|^2_{1/2} |\partial_t \partial_t^{n-1} v|^2_{1/2} |v_1 \mathbf{N}|^2_{1/2} \lesssim \mathcal{E}_{2N} \mathcal{E}_{N+4}. \]  

(3.45)
Then the estimate of $|P^{2,n-1}|^2_{1/2}$ can be concluded. □

Again, sometimes it is more convenient to write (3.36)–(3.38) in the linear perturbed form as

$$
\begin{cases}
\Delta q = Q^1 & \text{in } \Omega \\
\partial_2 q = -\partial_i v_2 + Q^2 & \text{on } \Sigma \\
\partial_2 q = 0 & \text{on } \Sigma_-, 
\end{cases}
$$

(3.46)

where

$$
Q^1 = -\partial_i^\nu v_j \partial_j^\nu v_i - \partial_i^\nu b_j \partial_j^\nu b_i - (\Delta^\varphi - \Delta) q,
$$

(3.47)

$$
Q^2 = \partial_i v_1 \partial_j h - v_1 \partial_i v \cdot \mathbf{N} + \nabla \eta \cdot \nabla^\varphi q.
$$

(3.48)

**Remark 3.6.** It is direct to check that

$$
\int_\Omega Q^1 \, dx = \int_\Sigma (-\partial_i v_2 + Q^2) \, dx_1,
$$

(3.49)

which is the compatibility condition of the solvability of (3.46).

Note that one can not use the problem (3.46) but rather (3.41) to estimate $\partial_t^{n-1} q$ since one can not control $\partial_t^n v_2$ but rather $\partial_t^n v \cdot \mathbf{N}$. $Q^1$ and $Q^2$ can be estimated as follows.

**Lemma 3.7.** It holds that

$$
\sum_{j=0}^{2N-2} \left\| \partial_t^j Q^1 \right\|_{2N-j-2}^2 + \sum_{j=0}^{2N-2} \left\| \partial_t^j Q^2 \right\|_{2N-j-3/2}^2 \lesssim \min \{ \mathcal{E}_{N+4}, \mathcal{D}_{N+4} \} \mathcal{E}_{2N}.
$$

(3.50)

**Proof.** The proof follows in the same way as Lemma 3.2. □

## 4. Tangential energy evolution

In this section we will derive the tangential energy evolution estimates of the solution to (1.13). For a generic integer $n \geq 3$, we define the tangential energy that involves the temporal and horizontal spatial derivatives by, employing the anisotropic Sobolev norm (2.2),

$$
\mathcal{E}_n := \sum_{j=0}^n \left\| \partial_t^j v \right\|^2_{0,n-j} + \sum_{j=0}^n \left\| \partial_t^j b \right\|^2_{0,n-j} + \sum_{j=0}^n \left\| \partial_t^j h \right\|^2_{n-j+1}
$$

(4.1)

and the corresponding dissipation by

$$
\mathcal{D}_n := \sum_{j=0}^n \left\| \partial_t^j b \right\|^2_{1,n-j}.
$$

(4.2)

### 4.1. Energy evolution at the $2N$ level

We first derive the following time-integrated tangential energy evolution estimate at the $2N$ level.

**Proposition 4.1.** It holds that

$$
\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0) + \mathcal{E}_{2N}^{3/2}(t) + \int_0^t \sqrt{\mathcal{E}_{N+4}(s)} \mathcal{E}_{2N}(s) \, ds.
$$

(4.3)

**Proof.** Let $\alpha \in \mathbb{N}^{1+1}$ be so that $1 \leq |\alpha| \leq 2N$. Recall the identity (3.11) of Proposition 3.1 and then estimate the right hand side term by term. One has directly

$$
\frac{1}{2} \int_\Sigma \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\partial_1 h|^2}} \right) \left| \partial_t \partial^\alpha h \right|^2 \, dx_1 \lesssim \sqrt{\mathcal{E}_{N+4}} \left| \partial^\alpha h \right|_{1/2}^2 \lesssim \sqrt{\mathcal{E}_{N+4}} \mathcal{E}_{2N}.
$$

(4.4)

It follows from (3.18) that

$$
\int_\Sigma (g \partial^\alpha h F_5^\alpha + \sigma \partial_1 F_6^\alpha \partial_1 \partial^\alpha h) \, dx_1 \lesssim \left| F_5^\alpha \right|_0 \left| \partial^\alpha h \right|_0 + \left| F_6^\alpha \right|_{3/2} \left| \partial_t \partial^\alpha h \right|_{-1/2} \lesssim \sqrt{\mathcal{E}_{N+4}} \mathcal{E}_{2N} \sqrt{\mathcal{E}_{2N}}.
$$

(4.5)
and
\[
\int_{\Omega} \left( F^{1,\alpha} \cdot \partial^\alpha v + F^{3,\alpha} \cdot \partial^\alpha b \right) d\mathcal{V}_t \lesssim \left\| F^{1,\alpha} \right\|_0 \left\| \partial^\alpha v \right\|_0 + \left\| F^{3,\alpha} \right\|_0 \left\| \partial^\alpha b \right\|_0
\]
\[
\lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}} \mathcal{E}_{2N}.
\]
(4.6)

To estimate the \( F^{4,\alpha} \) term, since \( \partial^\alpha b = 0 \) on \( \partial \Omega \), by using the expression (3.7) and then integrating by parts over \( \Omega \), one deduces that by (3.18),
\[
\int_{\Omega} F^{4,\alpha} \cdot \partial^\alpha b d\mathcal{V}_t = \int_{\Omega} \nabla \varphi \cdot \tilde{F}^{4,\alpha} \cdot \partial^\alpha b d\mathcal{V}_t = -\int_{\Omega} \tilde{F}^{4,\alpha} \cdot \nabla \varphi \partial^\alpha b d\mathcal{V}_t
\]
\[
\leq \left\| \tilde{F}^{4,\alpha} \right\|_0 \left\| \nabla \varphi \partial^\alpha b \right\|_0 \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}} \left\| \nabla \varphi \partial^\alpha b \right\|_0.
\]
(4.7)

Now we turn to estimate the most delicate two remaining terms,
\[
- \int_{\Sigma} \sigma \partial^\alpha H F^{5,\alpha} dx_1 \quad \text{and} \quad \int_{\Omega} \partial^\alpha q F^{2,\alpha} d\mathcal{V}_t.
\]
(4.8)

As explained in Section 2, one needs to consider the cases \( \alpha_0 \leq 2N - 1 \) and \( \alpha_0 = 2N \) separately. For the case \( \alpha_0 \leq 2N - 1 \), one has that by (3.19),
\[
- \int_{\Sigma} \sigma \partial^\alpha H F^{5,\alpha} dx_1 \lesssim \left\| \partial^\alpha H \right\|_{-1/2} \left| F^{5,\alpha} \right|_{1/2} \lesssim \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}}
\]
(4.9)

and by (3.18),
\[
\int_{\Omega} \partial^\alpha q F^{2,\alpha} d\mathcal{V}_t \lesssim \left\| \partial^\alpha q \right\|_0 \left\| F^{2,\alpha} \right\|_0 \lesssim \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}}.
\]
(4.10)

For the case \( \alpha_0 = 2N \), the main difficulty is that there is no any estimate of \( \partial^2 t^N q \) and so one needs to integrate by parts in \( t \) for the pressure term, and there is a 1/2 regularity loss of \( \partial^2 t^N h \) or \( \partial^2 t^{N-1} v_1 \) so that it is insufficient to control the surface tension term. The crucial observation here is that these two terms will enjoy some cancellation by performing some careful computations. We start with the integration by parts in \( t \) for the pressure term, and we will use a variant of the expression of \( F^{2,(2N,0)} \) defined by (3.5). Indeed, \( \nabla \varphi \cdot v = 0 \) yields
\[
\nabla \varphi \cdot v \partial_2 \varphi = \mathbf{N} \cdot \partial_2 v + \partial_2 \varphi \partial_1 v_1 = 0.
\]
(4.11)

Applying \( \partial^2 t^N \) to (4.11) and using the definition of \( F^{2,(2N,0)} \), one gets that
\[
- \partial_2 \varphi F^{2,(2N,0)} = \left[ \partial^2 t^N, -\partial_1 \eta \right] \partial_2 v_1 + \left[ \partial^2 t^N, \partial_2 \eta \right] \partial_1 v_1.
\]
(4.12)

Moreover, one needs to single out the highest \( 2N - 1 \) order time derivative terms of \( v_1 \) and the highest \( 2N \) order time derivative terms of \( \eta \) as
\[
- \partial_2 \varphi F^{2,(2N,0)} = \sum_{i=1}^{5} F_i^{2,(2N,0)},
\]
(4.13)

where
\[
F_1^{2,(2N,0)} = -2N \partial_1 \eta \partial_2^N \partial_2 v_1, \quad F_2^{2,(2N,0)} = 2N \partial_1 \partial_2 \eta \partial_2^N \partial_1 v_1, \quad F_3^{2,(2N,0)} = -\partial^2 t^N \partial_1 \eta \partial_2 v_1, \quad F_4^{2,(2N,0)} = \partial^2 t^N \partial_2 \eta \partial_1 v_1, \quad F_5^{2,(2N,0)} = \sum_{l=2}^{2N-1} C^l_{2N} \left( -\partial^l \partial_1 \eta \partial_2^N \partial_2 v_1 + \partial^l \partial_2 \eta \partial_2^N \partial_1 v_1 \right).
\]
(4.14)

Accordingly,
\[
\int_{\Omega} \partial^2 t^N q F^{2,(2N,0)} d\mathcal{V}_t = -\sum_{i=1}^{5} \int_{\Omega} \partial^2 t^N q F_i^{2,(2N,0)} dx.
\]
(4.17)
One integrates by parts in $t$ for the last four terms as
\[
- \sum_{i=2}^{5} \int_{\Omega} \partial_{t}^{N} q F_{i}^{2,(2N,0)} \, dx = - \sum_{i=2}^{5} \frac{d}{dt} \int_{\Omega} \partial_{t}^{N-1} q F_{i}^{2,(2N,0)} \, dx + \sum_{i=2}^{5} \int_{\Omega} \partial_{t}^{2N-1} q \partial_{t} F_{i}^{2,(2N,0)} \, dx.
\] 
(4.18)

One may directly bound, by estimating $\left\| \partial_{t} F_{5}^{2,(2N,0)} \right\|_{0} \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}}$ as Lemma 3.2
\[
\int_{\Omega} \partial_{t}^{2N-1} q \partial_{t} F_{5}^{2,(2N,0)} \, dx \leq \left\| \partial_{t}^{2N-1} q \right\|_{0} \left\| \partial_{t} F_{5}^{2,(2N,0)} \right\|_{0} \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{E}_{N+4} \mathcal{E}_{2N}}.
\] 
(4.19)

Upon an integration by parts in $x_{2}$ and estimating as in Lemma 3.2 one has
\[
\int_{\Omega} \partial_{t}^{2N-1} q \partial_{t} F_{4}^{2,(2N,0)} \, dx = \int_{\Omega} \partial_{t}^{2N-1} q \left( \partial_{t}^{2N+1} \eta \partial_{t} \partial_{1} v_{1} + \partial_{t}^{2N} \partial_{2} \partial_{1} \partial_{1} v_{1} \right) \, dx
\]
\[
= \int_{\Sigma} \partial_{t}^{2N+1} \eta \partial_{t} \partial_{1} v_{1} \, dx_{1} - \int_{\Omega} \left( \partial_{t}^{2N+1} \eta \partial_{2} \left( \partial_{t}^{2N-1} q \partial_{1} v_{1} \right) + \partial_{t}^{2N-1} q \partial_{2} \partial_{2} \partial_{1} v_{1} \right) \, dx
\]
\[
\lesssim \left| \partial_{t}^{2N+1} \eta \right|_{-\frac{1}{2}} \left| \partial_{t}^{2N-1} q \partial_{1} v_{1} \right|_{\frac{1}{2}} + \left\| \partial_{t}^{2N+1} \eta \right\|_{0} \left\| \partial_{2} \left( \partial_{t}^{2N-1} q \partial_{1} v_{1} \right) \right\|_{0}
\]
\[
+ \left\| \partial_{t}^{2N} \partial_{1} \eta \right\|_{0} \left\| \partial_{t}^{2N-1} q \partial_{2} \partial_{1} v_{1} \right\|_{0}
\]
\[
\lesssim \sqrt{\mathcal{E}_{2N} \mathcal{E}_{N+4} \mathcal{E}_{2N}}.
\] 
(4.20)

Similarly, by integrating by parts in $x_{1}$, one deduces
\[
\int_{\Omega} \partial_{t}^{2N-1} q \left( \partial_{t} F_{2}^{2,(2N,0)} + \partial_{t} F_{3}^{2,(2N,0)} \right) \, dx \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{E}_{N+4} \mathcal{E}_{2N}}.
\] 
(4.21)

It remains to deal with the most difficult term, the first term involving $F_{1}^{2,(2N,0)}$ in (4.17). One integrates by parts in $x_{2}$ first to get
\[
- \int_{\Omega} \partial_{t}^{2N} q F_{1}^{2,(2N,0)} \, dx = \int_{\Sigma} \partial_{t}^{2N} q 2N \partial_{t} \partial_{1} h \partial_{t}^{2N-1} v_{1} \, dx_{1} - \int_{\Omega} \partial_{2} \left( \partial_{t}^{2N} q 2N \partial_{t} \partial_{1} \eta \right) \partial_{t}^{2N-1} v_{1} \, dx.
\] 
(4.22)

Then integrating by parts in $t$ for the second term in the right hand side of (4.22) yields
\[
- \int_{\Omega} \partial_{2} \left( \partial_{t}^{2N} q 2N \partial_{t} \partial_{1} \eta \right) \partial_{t}^{2N-1} v_{1} \, dx
\]
\[
= - \frac{d}{dt} \int_{\Omega} \partial_{2} \left( \partial_{t}^{2N-1} q 2N \partial_{t} \partial_{1} \eta \right) \partial_{t}^{2N-1} v_{1} \, dx
\]
\[
+ \int_{\Omega} \partial_{2} \left( \partial_{t}^{2N-1} q 2N \partial_{t} \partial_{1} \eta \right) \partial_{t}^{2N} v_{1} + \partial_{2} \left( \partial_{t}^{2N-1} q 2N \partial_{2} \partial_{1} \eta \right) \partial_{t}^{2N-1} v_{1} \, dx
\]
\[
\leq - \frac{d}{dt} \int_{\Omega} \partial_{2} \left( \partial_{t}^{2N-1} q 2N \partial_{t} \partial_{1} \eta \right) \partial_{t}^{2N-1} v_{1} \, dx + C \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N} \mathcal{E}_{2N}}.
\] 
(4.23)

Note carefully that we integrate by parts in $x_{2}$ here first rather than in $t$ since there are no estimates for $\partial_{t}^{2N} v_{1}$ on the boundary. This also indicates the difficulty in controlling the first term in the right hand side of (4.22) since one can no longer integrate by parts in $t$. Recall here that there was also one term out of control, that is, the surface tension term in (4.18) when $\alpha = (2N,0)$. Our crucial observation is that there is a cancellation between them since $\partial_{t}^{2N} q = g \partial_{t}^{2N} h - \sigma \partial_{t}^{2N} H$ on $\Sigma$. Indeed, one obtains
\[
- \int_{\Sigma} \sigma \partial_{t}^{2N} H F_{5}^{5,(2N,0)} \, dx_{1} + \int_{\Sigma} \partial_{t}^{2N} q 2N \partial_{t} \partial_{1} h \partial_{t}^{2N-1} v_{1} \, dx_{1}
\]
\[
= - \int_{\Sigma} \sigma \partial_{t}^{2N} H \left( F_{5}^{5,(2N,0)} + 2N \partial_{t} \partial_{1} h \partial_{t}^{2N-1} v_{1} \right) \, dx_{1}
\]
\[
+ \int_{\Sigma} \left( \sigma \partial_{t}^{2N} H + \partial_{t}^{2N} q \right) 2N \partial_{t} \partial_{1} h \partial_{t}^{2N-1} v_{1} \, dx_{1}
\]
\[ - \int \sigma \partial_t^2 H \left( v_1 \partial_t \partial_t^2 h + \tilde{F}^{5,(2N,0)} \right) dy + \int \sigma \partial_t^2 h 2 N \partial_t h \partial_t^{2N-1} v_1 \, dx, \tag{4.24} \]

where

\[ \tilde{F}^{5,(2N,0)} = \sum_{\ell=2}^{n-1} \partial_t \partial_t^2 h \partial_t^{2N-\ell} v_1. \tag{4.25} \]

One has directly

\[ \int \sigma \partial_t^2 h 2 N \partial_t h \partial_t^{2N-1} v_1 \, dx \lesssim \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}. \tag{4.26} \]

Note that \( \| \tilde{F}^{5,(2N,0)} \|_1 \lesssim \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}} \) as Lemma 3.2, so integrating by parts in \( x_1 \) yields

\[ - \int \sigma \partial_t^2 H v_1 \partial_t \partial_t^2 h \, dx = - \int \sigma \partial_t \left( \frac{\partial_t \partial_t^2 h}{\sqrt{1 + |\partial_t h|^2}} + F^{6,(2N,0)} \right) v_1 \partial_t \partial_t^2 h \, dx. \tag{4.28} \]

Then (3.9) implies

\[ - \int \sigma \partial_t F^{6,(2N,0)} v_1 \partial_t \partial_t^2 h \, dx \lesssim \| F^{6,(2N,0)} \|_1 \| \partial_t \partial_t^2 h \|_1 \lesssim \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}. \tag{4.29} \]

Integrating by parts in \( x_1 \), one can deduce that

\[ - \int \sigma \partial_t \left( \frac{\partial_t \partial_t^2 h}{\sqrt{1 + |\partial_t h|^2}} \right) v_1 \partial_t \partial_t^2 h \, dx = \int \sigma \frac{\partial_t \partial_t^2 h}{\sqrt{1 + |\partial_t h|^2}} \partial_t (v_1 \partial_t \partial_t^2 h) \, dx \]

\[ = \int \sigma \left( \partial_t v_1 - \frac{1}{2} \partial_t \left( \frac{|v_1|}{\sqrt{1 + |\partial_t h|^2}} \right) \right) |\partial_t \partial_t^2 h|^2 \, dx \]

\[ \lesssim \sqrt{\mathcal{E}_{N+4}} \| \partial_t \partial_t^2 h \|_1^2 \lesssim \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}. \tag{4.30} \]

Hence, it follows from (4.20)–(4.30) and (4.24) that

\[ - \int \sigma \partial_t^2 H F^{5,(2N,0)} \, dx + \int \sigma \partial_t^{2N} q 2 N \partial_t h \partial_t^{2N-1} v_1 \, dx \lesssim \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}. \tag{4.31} \]

This, together with (4.17)–(4.23), implies that

\[ - \int \sigma \partial_t^2 H F^{5,(2N,0)} \, dx + \int \sigma \partial_t^{2N} q F^{3,(2N,0)} \, dx \leq - \frac{d}{dt} B_{2N} + C \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}. \tag{4.32} \]

where

\[ B_{2N} := \sum_{i=2}^{5} \int \partial_t^{2N-1} q F^{2,(2N,0)} \, dx + \int \partial_t \left( \partial_t^{2N-1} q 2 N \partial_t h \partial_t \partial_t^{2N-1} v_1 \right) \, dx. \tag{4.33} \]

As a consequence of the estimates (4.1)–(4.7), (4.9), (4.10) and (4.32), one deduces from (3.11) with summing over such \( \alpha \) and (1.16) that, by Cauchy’s and Poincaré’s inequalities and then integrating in time from 0 to \( t \),

\[ \mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0) + B_{2N}(0) - B_{2N}(t) + \int_0^t \sqrt{\mathcal{E}_{N+4}(s)} \mathcal{E}_{2N}(s) \, ds. \tag{4.34} \]
Note that \( \| F_i^{2,(2N,0)} \|_0 \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}} \), \( i = 2, \ldots, 5 \), as Lemma 3.2. Thus
\[
|B_{2N}| \leq \sum_{i=2}^{5} \left\| \partial_t^{N-1} q_i \right\|_0 \left\| F_i^{2,(2N,0)} \right\|_0 + \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}} \lesssim \mathcal{E}^{3/2}. \tag{4.35}
\]

Then the estimate (4.33) follows.

4.2. Energy evolution at the lower levels. Now we present the following time-differential tangential energy evolution estimate at the \( n = N+4, \ldots, 2N-2 \) levels.

**Proposition 4.2.** For \( n = N+4, \ldots, 2N-2 \), it holds that
\[
\frac{d}{dt} (\bar{\mathcal{E}}_n + B_n) + \bar{D}_n \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{D}_n}, \tag{4.36}
\]
where \( B_n \) is defined by (4.54) below and satisfies the estimate
\[
|B_n| \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{E}_n}. \tag{4.37}
\]

**Proof.** Let \( n \) denote \( N+4 \ldots, 2N-2 \) throughout the proof and \( \alpha \in \mathbb{N}^{1+1} \) be so that \( 1 \leq |\alpha| \leq n \). We will estimate the right hand side of (3.11) in a quite different way from the arguments that lead to the estimates (1.14–1.17), (4.9), (4.10) and (3.32) in the proof of Proposition 4.1.

First, integrating by parts in \( x_1 \) yields
\[
\frac{1}{2} \int_{\Sigma} \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\partial_t h|^2}} \right) |\partial_t \partial^\alpha h|^2 \, dx_1 = -\frac{1}{2} \int_{\Sigma} \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\partial_t h|^2}} \right) \partial_t |\partial^\alpha h|^2 \, dx_1 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}} \mathcal{D}_n \tag{4.38}
\]
It follows from (3.26) that
\[
\int_{\Sigma} (F^{3,\alpha} + F^{4,\alpha}) \cdot \partial^\alpha b \, d\mathcal{V}_t \leq \left( \left\| F^{3,\alpha} \right\|_0 + \left\| F^{4,\alpha} \right\|_0 \right) \left\| \partial^\alpha b \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N} \mathcal{D}_n} \tag{4.39}
\]
and
\[
\int_{\Sigma} g \partial^\alpha h^2 \, dx_1 \lesssim |\partial^\alpha h|_0 \left\| F^{5,\alpha} \right\|_0 \lesssim \sqrt{\mathcal{D}_n \mathcal{D}_{N+4} \mathcal{E}_{2N}} \tag{4.40}
\]
Using the formula (3.9) and then integrating by parts in \( x_1 \) two times yield, by (3.26),
\[
-\int_{\Sigma} \sigma \partial^\alpha h F^{5,\alpha} \, dx_1 = \sigma \int_{\Sigma} \left( \frac{\partial_t F^{5,\alpha}}{\sqrt{1 + |\partial_t h|^2}} + F^{6,\alpha} \right) \partial_t F^{5,\alpha} \, dx_1
\]
\[
= \sigma \int_{\Sigma} \left( -\partial^\alpha h \partial_t \left( \frac{\partial_t F^{5,\alpha}}{\sqrt{1 + |\partial_t h|^2}} \right) + F^{6,\alpha} \partial_t F^{5,\alpha} \right) \, dx_1
\]
\[
\lesssim |\partial^\alpha h|_0 \left\| F^{5,\alpha} \right\|_1^{2} + |F^{6,\alpha}| _0 |F^{5,\alpha}|_1 \lesssim \sqrt{\mathcal{D}_n \mathcal{D}_{N+4} \mathcal{E}_{2N}} \mathcal{D}_n \tag{4.41}
\]
Next, we consider the term involving \( F^{6,\alpha} \). If \( |\alpha| \leq n-1 \), then by (3.26), one has
\[
\int_{\Sigma} \sigma \partial_t F^{6,\alpha \alpha} \partial_t \partial^\alpha h \, dx_1 \lesssim |F^{6,\alpha}| _1 |\partial_t \partial^\alpha h|_0 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N} \mathcal{D}_n}. \tag{4.42}
\]
If \( |\alpha| = n, \alpha_1 \geq 1 \), then we integrate by parts in \( x_1 \) to obtain, by (3.26),
\[
\int_{\Sigma} \sigma \partial_t F^{6,\alpha} \partial_t \partial^\alpha h \, dx_1 = \int_{\Sigma} \sigma \partial_t^2 F^{6,\alpha} \partial_t \partial^\alpha (\partial_1 \partial^\alpha (\alpha_0, \alpha_1 - 1)) h \, dx_1
\]
\[
\lesssim |F^{6,\alpha}| _2 |\partial^{(\alpha_0 + 1, \alpha_1 - 1)} h| _0 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N} \mathcal{D}_n}. \tag{4.43}
\]
The remaining case is that \( \alpha_0 = n \), then integrating by parts in \( t \) and using (3.26) show that
\[
\int_{\Sigma} \sigma \partial_t F^{6,(n,0)} \partial_t \partial^\alpha h \, dx_1 = \frac{d}{dt} \int_{\Sigma} \sigma \partial_t F^{6,(n,0)} \partial_t^\alpha h \, dx_1 - \int_{\Sigma} \sigma \partial_t \partial_t F^{6,(n,0)} \partial_t^\alpha h \, dx_1
\]
\[ \frac{d}{dt} \left( \int_{\Omega} \sigma \partial_t F^{6,(n,0)} \partial_t^2 h \, dx \right) + C \left| \partial_t F^{6,(n,0)} \right|_1 \left| \partial_t^2 h \right|_0 \]

\[ \leq \frac{d}{dt} \left( \int_{\Sigma} \sigma \partial_t F^{6,(n,0)} \partial_t^2 h \, dx + C \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}} \sqrt{\mathcal{D}_n} \right). \quad (4.44) \]

We now consider the term involving \( F^{1,\alpha} \). If \(|\alpha| \leq n - 1\), then \( (3.20) \) implies

\[ \int_{\Omega} F^{1,\alpha} \cdot \partial^\alpha v \, dV_t \leq \left\| F^{1,\alpha} \right\|_0 \left\| \partial^\alpha v \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}} \sqrt{\mathcal{D}_n}. \quad (4.45) \]

If \(|\alpha| = n, \alpha_1 \geq 1\), then integrating by parts in \( x_1 \) and \( (3.26) \) give

\[ \int_{\Omega} F^{1,\alpha} \cdot \partial^\alpha v \, dV_t - \int_{\Omega} \partial_1 \left( F^{1,\alpha} \partial_2 \varphi \right) \cdot \partial_t^{\alpha_1-1} v \, dx \lesssim \left\| F^{1,\alpha} \right\|_1 \left\| \partial^{(\alpha_0,\alpha_1-1)} v \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}} \sqrt{\mathcal{D}_n}. \quad (4.46) \]

For the remaining case, \( \alpha_0 = n \), integrating by parts in \( t \) and using \( (3.26) \) show

\[ \int_{\Omega} F^{1,(n,0)} \cdot \partial_t^n v \, dV_t = \frac{d}{dt} \int_{\Omega} F^{1,(n,0)} \cdot \partial_t^{n-1} v \, dV_t + C \left( \left\| F^{1,(n,0)} \right\|_0 + \left\| \partial_t F^{1,(n,0)} \right\|_0 \right) \left\| \partial_t^{n-1} v \right\|_0 \]

\[ \leq \frac{d}{dt} \int_{\Omega} F^{1,(n,0)} \cdot \partial_t^{n-1} v \, dV_t \]

\[ \leq \frac{d}{dt} \int_{\Omega} F^{1,(n,0)} \cdot \partial_t^{n-1} v \, dV_t + C \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}} \sqrt{\mathcal{D}_n}. \quad (4.47) \]

Finally, we treat the term involving \( F^{2,\alpha} \). If \(|\alpha| \leq n - 1, \alpha_0 \leq n - 2\), then \( (3.20) \) implies

\[ \int_{\Omega} \partial^\alpha q F^{2,\alpha} \, dV_t \lesssim \left\| \partial^\alpha q \right\|_0 \left\| F^{2,\alpha} \right\|_0 \lesssim \sqrt{\mathcal{D}_n} \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}}. \quad (4.48) \]

If \( \alpha = (n-1,0) \), then we integrate by parts in \( t \) to have, by \( (3.26) \),

\[ \int_{\Omega} \partial_t^{n-1} q F^{2,(n-1,0)} \, dV_t = \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} q F^{2,(n-1,0)} \, dV_t - \int_{\Omega} \partial_t^{n-2} q \partial_t \left( F^{2,(n-1,0)} \partial_2 \varphi \right) \, dx \]

\[ \leq \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} q F^{2,(n-1,0)} \, dV_t + C \left( \left\| \partial_t^{n-2} q \right\|_0 \left( \left\| F^{2,(n-1,0)} \right\|_0 + \left\| \partial_t F^{2,(n-1,0)} \right\|_0 \right) \right) \]

\[ \leq \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} q F^{2,(n-1,0)} \, dV_t + C \sqrt{\mathcal{D}_n} \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}}. \quad (4.49) \]

If \(|\alpha| = n, \alpha_0 \leq n - 2\) and hence \( \alpha_1 \geq 2\), then integrating by parts in \( x_1 \) and \( (3.26) \) show

\[ \int_{\Omega} \partial^\alpha q F^{2,\alpha} \, dV_t = - \int_{\Omega} \partial^{\alpha_0,\alpha_1-1} q \partial_t F^{2,\alpha} \, dV_t \]

\[ \lesssim \left\| \partial^{\alpha_0,\alpha_1-1} q \right\|_0 \left\| F^{2,\alpha} \right\|_1 \lesssim \sqrt{\mathcal{D}_n} \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}}. \quad (4.50) \]

If \( \alpha = (n-1,1) \), then we integrate by parts in \( t \) to have, by \( (3.26) \),

\[ \int_{\Omega} \partial_t^{n-1} \partial_t q F^{2,(n-1,1)} \, dV_t = \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} \partial_t q F^{2,(n-1,1)} \, dV_t - \int_{\Omega} \partial_t^{n-2} q \partial_t^2 \partial_t \left( F^{2,(n-1,1)} \partial_2 \varphi \right) \, dx \]

\[ \leq \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} \partial_t q F^{2,(n-1,1)} \, dV_t + C \left( \left\| \partial_t^{n-2} q \right\|_1 \left( \left\| F^{2,(n-1,1)} \right\|_0 + \left\| \partial_t F^{2,(n-1,1)} \right\|_0 \right) \right) \]

\[ \leq \frac{d}{dt} \int_{\Omega} \partial_t^{n-2} \partial_t q F^{2,(n-1,1)} \, dV_t + C \sqrt{\mathcal{D}_n} \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}}. \quad (4.51) \]

The remaining case, \( \alpha_0 = n \), can be handled by the integration by parts in \( t \) two times and using \( (3.20) \) as

\[ \int_{\Omega} \partial_t^n q F^{2,(n,0)} \, dV_t = \frac{d}{dt} \int_{\Omega} \partial_t^{n-1} q F^{2,(n,0)} \, dV_t - \int_{\Omega} \partial_t^{n-1} q \partial_t \left( F^{2,(n,0)} \partial_2 \varphi \right) \, dx \]

\[ = \frac{d}{dt} \left( \int_{\Omega} \partial_t^{n-1} q F^{2,(n,0)} \, dV_t - \int_{\Omega} \partial_t^{n-2} q \partial_t \left( F^{2,(n,0)} \partial_2 \varphi \right) \, dx \right) + \int_{\Omega} \partial_t^{n-2} q \partial_t^2 \left( F^{2,(n,0)} \partial_2 \varphi \right) \, dx \]
\[ \frac{d}{dt} \left( E_n + B_n \right) + D_n \lesssim \sqrt{D_{N+4}E_{2N}} \sqrt{D_n}, \quad (4.53) \]

where \( B_n \) is defined by

\[ B_n := - \int_\Omega \sigma \partial_t F_0 \partial_t^1 h \, dx_1 - \int_\Omega \partial_1^1 F_0 \partial_1^1 v \, dx_1 - \int_\Omega \partial_1^1 \partial_1^2 q \, dx_1 - \int_\Omega \partial_1^2 \partial_1^3 q \, dx_1 + \int_\Omega \partial_1^2 \partial_1^3 q \partial_1^2 \varphi \, dx, \quad (4.54) \]

which can be estimated by \((4.32)\) as

\[ |B_n| \lesssim \left( F_0^2,0,0 \right)_1^1 \left\| \partial_1^1 h \right\|_0 + \left\| F_0^1,0,0 \right\|_0^0 \left\| \partial_1^1 v \right\|_0 + \left\| \partial_1^1 \partial_1^2 q \right\|_0 + \left\| F_0^2,0,0 \right\|_0 \]

\[ + \left\| \partial_1^2 \partial_1^3 q \right\|_0 \left( \left\| F_0^2,0,0 \right\|_0 + \left\| \partial_1^1 F_0^2,0,0 \right\|_0 \right) \]

\[ \lesssim \sqrt{E_{N+4}E_{2N}} \sqrt{E_n}. \quad (4.55) \]

Then the estimates \((4.36)\) and \((4.37)\) follow since \( n \geq N + 4 \).

5. Estimates of \( q \) and \( h \)

From this section on we shall derive the full energy estimates of the solutions to \((1.13)\) by using the structure of the equations \((1.13)\) combined with the tangential energy evolution estimates derived previously in Section 4. The crucial starting point is that one can derive the desired boundary estimates of the normal component of the velocity \( v \) on the upper boundary \( \Sigma \) in both the energy and the dissipation, from the assumed control of \( E_n \) and \( D_n \), respectively. With these boundary regularities one can then derive the estimates of the pressure \( q \) and the free boundary function \( h \) by employing the elliptic estimates and the boundary conditions.

5.1. Instantaneous energy. We begin with the estimates in the energy.

**Proposition 5.1.** For \( n = N + 4, \ldots, 2N \), it holds that

\[ \sum_{j=0}^{n-1} \left\| \partial_1^j v \right\|_{n-j}^2 + \sum_{j=0}^{n-1} \left\| \partial_1^j h \right\|_{n-j+3/2}^2 + \left\| \partial_1^j h \right\|_{1/2}^2 + \left\| \partial_1^{j+1} h \right\|_{-1/2}^2 \lesssim E_n + E_{N+4}E_{2N}. \quad (5.1) \]

**Proof.** Assume that \( N + 4 \leq n \leq 2N \). We first derive the boundary estimates of the normal component of \( v \) on \( \Sigma \). It follows from the Sobolev interpolation on \( \Sigma \), the normal trace estimates \((A.11)\) and \((A.12)\) and the second equation in \((3.3)\) that for \( j = 0, \ldots, n - 1 \), by the definition \((4.1)\) of \( E_n \) and \((3.3)\),

\[ \left\| \partial_1^j v \right\|_{n-j-1/2}^2 \lesssim \left\| \partial_1^j v \right\|_{n-j-1/2}^2 + \left\| \partial_1^{n-j} \partial_1^j v \right\|_{-1/2}^2 \]

\[ \lesssim \left\| \partial_1^j v \right\|_0^2 + \| \text{div} \partial_1^j v \|_0^2 + \left\| \partial_1^{n-j} \partial_1^j v \right\|_0^2 + \| \text{div} \partial_1^{n-j-1} \partial_1^j v \|_0^2 \]

\[ \lesssim E_n + \left\| \partial_1^j G \right\|_{n-j}^2 \lesssim E_n + E_{N+4}E_{2N}. \quad (5.2) \]
Note that the estimate (5.12) excludes the case \( j = n \), which can be handled as follows. Using the normal trace estimate (A.14), by the second equation in (3.3), one obtains that, by (4.1) and (3.18),

\[
|\partial_t v \cdot N|_{-1/2} \lesssim \|\partial_t v\|^2_0 + \|\nabla^2 \cdot \partial_t v\|^2_0 \\
\lesssim \bar{F}_n + \|\mathcal{F}^{(n,0)}\|_0^2 \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\tag{5.3}
\]

Next, we estimate \( \left\| \nabla \partial_t^j q \right\|_0^2 \), \( j = 0, \ldots, n - 1 \). It follows from taking the inner product of the first equation in (3.41) with \( \partial_t^{n-1} q := \partial_t^{n-1} q - \int_{\partial \Omega} \partial_t^{n-1} q \, dx \) and then integrating by parts over \( \Omega \), and using the boundary conditions in (3.41) and the trace theory that

\[
\int_{\Omega} \left| \nabla \partial_t^{n-1} q \right|^2 \, dv = \int_{\Omega} \left( -\partial_t^n v \cdot N + P^{2,n-1} \partial_t^{n-1} q \right) dx_1 + \int_{\Omega} P^{1,n-1} \partial_t^{n-1} q \, dv \\
\lesssim \left( \|\partial_t^n v \cdot N\|_{-1/2} + \|P^{2,n-1}\|_{-1/2} \right) \|\partial_t^{n-1} q\|_{1/2} + \|P^{1,n-1}\|_0 \|\partial_t^{n-1} q\|_0 \\
\lesssim \left( \|\partial_t^n v \cdot N\|_{-1/2} + \|P^{2,n-1}\|_{-1/2} + \|P^{1,n-1}\|_0 \right) \left\| \partial_t^{n-1} q \right\|_1.
\tag{5.4}
\]

By Poincaré’s and Cauchy’s inequalities, one then gets from (5.3), (5.3) and (3.41) that

\[
\left\| \nabla \partial_t^{n-1} q \right\|_0^2 \lesssim \|\partial_t^n v \cdot N\|^2_{-1/2} + \|P^{2,n-1}\|^2_{-1/2} + \|P^{1,n-1}\|^2_0 \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\tag{5.5}
\]

On the other hand, applying the time derivatives \( \partial_t^j \), \( j = 0, \ldots, n - 2 \), to (3.40) leads to

\[
\begin{aligned}
\Delta \partial_t^j q &= \partial_t^j Q^1 \\
\partial_t \partial_t^j q &= -\partial_t^{j+1} v_2 + \partial_t^j Q^2 \\
\partial_\nu \partial_t^j q &= 0
\end{aligned}
\quad \text{in } \Omega
\tag{5.6}
\]

on \( \Sigma \), and then integrating by parts over \( \Sigma \), using the elliptic estimates (A.21) of Lemma (A.10) with \( r = n - j \geq 2 \) and the problem (5.5), one has that for \( j = 0, \ldots, n - 2 \), by (5.5) and (5.2),

\[
\left\| \nabla \partial_t^j q \right\|^2_{n-j-1} \lesssim \left\| \partial_t^j Q^1 \right\|^2_{n-j-2} + \left\| \partial_t^{j+1} v_2 \right\|^2_{n-j-3/2} + \left\| \partial_t^j Q^2 \right\|^2_{n-j-3/2} \\
\lesssim \left\| \partial_t^{j+1} v_2 \right\|^2_{n-(j+1)-1/2} + \mathcal{E}_{N+4}\mathcal{E}_{2N} \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\tag{5.7}
\]

Now note that since \( \partial_t^j q \) may not have zero average, we shall estimate \( \left\| \partial_t^j q \right\|_0^2 \) by using Poincaré’s inequality with estimating instead \( \left\| \partial_t^j q \right\|_0^2 \) on \( \Sigma \). Indeed, it follows from the sixth equation in (3.28), (1.1) and (3.35) that for \( j = 0, \ldots, n - 1 \),

\[
\left\| \partial_t^j q \right\|_0^2 \lesssim \left\| \partial_t^j h \right\|^2_2 + \left\| \partial_t^j G^6 \right\|_0^2 \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\tag{5.8}
\]

This, together with the estimates (5.7) and (5.3), implies that for \( j = 0, \ldots, n - 1 \),

\[
\left\| \partial_t^j q \right\|_{n-j}^2 \lesssim \left\| \nabla \partial_t^j q \right\|^2_{n-j-1} + \left\| \partial_t^j q \right\|^2_{n-j} \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N},
\tag{5.9}
\]

where one has used the Poincaré’s inequality.

Finally, we improve the estimates of \( h \) by using the estimates of \( q \) derived in (5.29). Indeed, applying \( \partial_t^j \), \( j = 0, \ldots, n - 1 \), to the sixth equation in (3.28) shows

\[
-\sigma \partial_t^j \partial_t^j h + g \partial_t^j h = \partial_t^j q + \sigma \partial_t^j G^6 \quad \text{on } \Sigma.
\tag{5.10}
\]

This implies that for \( j = 0, \ldots, n - 1 \), by the trace theory, (3.35) and (5.9),

\[
\left\| \partial_t^j h \right\|^2_{n-j+3/2} \lesssim \left\| \partial_t^j q \right\|^2_{n-j-1/2} + \left\| \partial_t^j G^6 \right\|^2_{n-j-1/2} \\
\lesssim \left\| \partial_t^j q \right\|^2_{n-j} + \mathcal{E}_{N+4}\mathcal{E}_{2N} \lesssim \bar{F}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\tag{5.11}
\]
It follows from the fourth equation in (3.3), (3.4) and (3.13) that
\[ |\partial_t^{n+1}h|_{n-j-1/2}^2 \lesssim |\partial_t^j q|_{n-j}^2 + |\partial_t^q \cdot N|_{n-j}^2 + |F^{5,(0)}|_{n-j-1/2}^2 \lesssim \varepsilon_n + \mathcal{E}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.12)

Consequently, combining (5.9), (5.11) and (5.12) yields the estimate (5.1) by noting that $|\partial_t^q h|^2$ is already contained in the definition of $\varepsilon_n$. \hfill \square

5.2. Dissipation. Now we consider the estimates in the dissipation.

**Proposition 5.2.** For $n = N + 4, \ldots, 2N$, it holds that
\[ \sum_{j=0}^{n-2} \int \partial_t^j \|q\|^2_{n-j} + \sum_{j=0}^{n-2} \int \partial_t^j \|h\|^2_{n-j+1} + |\partial_t^{n-1}h|^2_{1} + |\partial_t^q h|^2_{0} \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.13)

**Proof.** Assume that $N + 4 \leq n \leq 2N$. Note that by the definition (3.2) $\bar{D}_n$ does not contain any estimates of $v$, so to derive the boundary estimates of the normal component of $v$ on $\Sigma$ in the dissipation we will need to employ a different argument from that of Proposition 5.1. Indeed, it follows from the second component of the third equation and the fourth equation in (3.28) that
\[ \bar{B} \cdot \nabla v_2 = \partial_t b_2 - \kappa \Delta b_2 - G_2^3 \]
\[ = \partial_t b_2 - \kappa \partial_t^2 b_2 + \kappa \partial_t \partial_t b_1 - \partial_2 G^4 - G_2. \]  \hspace{1cm} (5.14)

It then follows from (5.13), (5.2) and (3.33) that for $j = 0, \ldots, n-1$,
\[ \int \bar{B} \cdot \nabla \partial_t^j v_2 \|^2_{0,n-j-1} \lesssim \int \partial_t^{j+1} b_2 \|_{0,n-j-1}^2 + \int \partial_t^j b \|_{1,n-j}^2 + \int \partial_t^j G^3 \|_{n-j}^2 + \int \partial_t^j G^4 \|_{n-j}^2 \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.15)

Since $\bar{B}_2 \neq 0$ by the Poincare-type inequality (A.8) and noting that the Sobolev regularity on the boundary $\Sigma$ only involving $\partial_t^1$, one obtains that for $j = 0, \ldots, n-1$, by (5.15),
\[ \int \partial_t^j v_2 \|_{n-j-1}^2 \lesssim \int \partial_t^j \|_{0}^2 + \int \partial_t \partial_t^{j-1} v_2 \|_{0}^2 \lesssim \int \bar{B} \cdot \nabla \partial_t^j v_2 \|_{0,n-j-1}^2 \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.16)

Now we estimate the pressure $q$. For $j = 0, \ldots, n-2$, employing the elliptic estimates (A.21) of Lemma (A.10) with $r = n-j-1/2 \geq 3/2$ to the problem (5.6), one has that, by (3.5) and (5.16),
\[ \int \nabla \partial_t^j q \|_{n-j-3/2}^2 \lesssim \int \partial_t^{j+1} q \|_{n-j-5/2}^2 + \int \partial_t^j q \|_{n-j-2}^2 \lesssim \int \partial_t^{j+1} v_2 \|_{n-j-1} + \bar{D}_{n+4} \mathcal{E}_{2N} \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.17)

To estimate $\int \partial_t^j q \|_{10}^2$ on the upper boundary $\Sigma$, we estimate $h$ first. Indeed, applying $\partial_t^{j-1}, j = 1, \ldots, n$, to the fifth equation in (3.28), one obtains by (3.35) and (5.16) that for $j = 1, \ldots, n$,
\[ \int \partial_t^j h \|_{n-j}^2 \lesssim \int \partial_t^{j-1} v_2 \|_{n-j}^2 + \int \partial_t^{j-1} G \|_{n-j}^2 \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.18)

Hence by the sixth equation in (3.28), (5.18) and (3.35), one has that for $j = 1, \ldots, n-2$,
\[ \int \partial_t^j q \|_{10}^2 \lesssim \int \partial_t^j h \|_{12}^2 + \int \partial_t^j G \|_{0}^2 \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.19)

For $\|q\|_{0}$, a different argument is needed since we have not controlled $|h|_{0}^2$ yet. Note that by the trace theory, the estimate (5.17) with $j = 0$ implies in particular that
\[ \int \partial_t^j q \|_{n-2}^2 \lesssim \|\partial_t^j q\|_{n-3/2} \lesssim \bar{D}_n + \mathcal{D}_{n+4} \mathcal{E}_{2N}. \]  \hspace{1cm} (5.20)
Then it follows from the sixth equation in (3.28), (5.20) and (5.25) that
\[ |\partial_t h|^2_n \lesssim |\partial_t q|^2_n + |\partial_t G|^2_n \lesssim \mathcal{D}_n + \mathcal{D}_n + \mathcal{E}_{2N}. \] (5.21)
But since \( \int_{\Sigma} h \, dx_1 = 0 \) due to (1.15), so Poincare’s inequality yields
\[ |h|^2_{n+1} \lesssim |\partial_t h|^2_n \lesssim \mathcal{D}_n + \mathcal{D}_n + \mathcal{E}_{2N}, \] (5.22)
which in turn implies, by the sixth equation in (5.28) and (5.35) again,
\[ |q|^2_n \lesssim |h|^2_n + |G|^2_n \lesssim \mathcal{D}_n + \mathcal{D}_n + \mathcal{E}_{2N}. \] (5.23)

Now by Poincaré’s inequality, one deduces from the estimates (5.17), (5.19) and (5.23) that for \( j = 0, \ldots, n-2, \)
\[ \left\| \partial_t q \right\|^2_{n-j-1/2} \lesssim \left\| \nabla \partial_t q \right\|^2_{n-j-3/2} + \left| \nabla \partial_t q \right|^2_n \lesssim \mathcal{D}_n + \mathcal{D}_n + \mathcal{E}_{2N}. \] (5.24)
This in turn, by the trace theory and the equation (5.10), improves the estimates of \( \partial_t h \) that for \( j = 1, \ldots, n-2, \)
\[ \left\| \partial_t h \right\|^2_{n-j+1} \lesssim \left\| \partial_t q \right\|^2_{n-j-1} + \left| \partial_t G \right|^2_n \lesssim \mathcal{D}_n + \mathcal{D}_n + \mathcal{E}_{2N}. \] (5.25)
Consequently, combining (5.21), (5.22), (5.25) and (5.18) with \( j = n-1 \) and \( n \) yields the estimate (5.14).

6. Estimates of \( v \) and \( b \)

In this section we will complete the estimates for the velocity \( v \) and the magnetic field \( b \). By Propositions 4.1 and 4.2, it remains to estimate the normal derivatives of \( v \) and \( b \) in the energy and to derive the estimates for \( v \) and improve the normal derivatives estimates for \( b \) in the dissipation.

For the estimates of the normal derivatives of \( v \), as for the incompressible Euler equations, a natural way is to estimate instead the vorticity \( \text{curl}^e v = \partial_t^e v_2 - \partial_2^e v_1 \) to get rid of the pressure term \( \nabla^e q \) to avoid the loss of derivatives and then use the Hodge-type elliptic estimates. Applying \( \text{curl}^e \) to the first equation in (1.13), one finds that, by using \( \nabla^e \cdot v = 0, \)
\[ \partial_t^e \text{curl}^e v + v \cdot \nabla^e \text{curl}^e v = \bar{B} \cdot \nabla^e \text{curl}^e b + \text{curl}^e (b \cdot \nabla^e b). \] (6.1)
The difficulty is that there is a linear forcing term \( \bar{B} \cdot \nabla^e \text{curl}^e b \) on the right hand side of (6.1), and one can not use the equations of \( \text{curl}^e b \) to balance this term as done for the tangential energy evolution estimates in Section 4 due to the usual difficulties caused by the diffusion term \( \Delta^e b \) with the Dirichlet boundary condition. This is harmful for the global-in-time uniform estimate of \( \text{curl}^e v \). But, on the other hand, if there were without this term, then there would be no hope of deriving the global-in-time uniform estimates of the higher order derivatives for \( \text{curl}^e v \) just as for the incompressible Euler equations. The crucial observation here is that there is a new damping structure for the vorticity \( \text{curl}^e v \). Indeed, it follows from the fourth equation in (1.13) that
\[ \bar{B} \cdot \nabla^e \text{curl}^e b = \bar{B}_1 \partial_1^e \partial_2^e b_2 - \bar{B}_1 \partial_2^e \partial_1^e b_1 + \bar{B}_2 \partial_1^e \partial_1^e b_2 - \bar{B}_2 \partial_2^e \partial_2^e b_1 = \bar{B}_1 \Delta^e b_2 - \bar{B}_2 \Delta^e b_1 \equiv \bar{B} \times \Delta^e b. \] (6.2)
Next, applying \( \bar{B} \times \) to the third equation in (1.13) yields
\[ \kappa \bar{B} \times \Delta^e b = \partial_1^e (\bar{B} \times b) + v \cdot \nabla^e (\bar{B} \times b) - \bar{B} \cdot \nabla^e (\bar{B} \times v) - b \cdot \nabla^e (\bar{B} \times v). \] (6.3)
The second equation in (1.13) implies
\[ \bar{B} \cdot \nabla^e (\bar{B} \times v) = \bar{B}_1 \partial_1^e (\bar{B} \times v) + \bar{B}_2 \bar{B}_1 \partial_2^e v_2 - \bar{B}_2^2 \partial_2^e v_1 \]
\[ = \bar{B}_1 \partial_1^e (\bar{B} \times v) + \bar{B}_2 \text{curl}^e v - \bar{B}_2 \partial_2^e v_1 - \bar{B}_2 \bar{B}_1 \partial_2^e v_1 \]
\[ \equiv \nabla \times v + \frac{\nabla \times v}{\kappa} (\hat{B}_1 B \times v - \hat{B}_2 B \cdot v). \]

Hence, as a consequence of (6.2)–(6.4), (6.1) can be rewritten as

\[ \partial_t \nabla \times v + v \cdot \nabla \nabla \times v + \frac{\hat{B}_2^2}{\kappa} \nabla \times v = -\frac{1}{\kappa} \partial_t (\hat{B}_1 B \times v - \hat{B}_2 B \cdot v) + \frac{1}{\kappa} \partial_t (\hat{B} \times b) + \Phi, \]

where

\[ \Phi = \frac{1}{\kappa} \partial_t \nabla \times (\hat{B}_1 B \times v - \hat{B}_2 B \cdot v) - \frac{1}{\kappa} \partial_t \eta \nabla \times (\hat{B} \times b) \]

\[ + \frac{1}{\kappa} (v \cdot \nabla \times (\hat{B} \times b) - b \cdot \nabla \times (\hat{B} \times v)) + \nabla \times (b \cdot \nabla \times b). \]

This equation yields a transport-damping evolution structure for \( \nabla \times v \), and one then sees again the key roles of the positivity of the magnetic diffusion coefficient \( \kappa > 0 \) and the non-vanishing of \( \hat{B} \neq 0 \).

Applying \( \partial^{\alpha} \) for \( \alpha \in \mathbb{N}^{1+2} \) with \( |\alpha| \leq 2N - 1 \) to (6.5) gives that

\[ \partial_t^\alpha \nabla \times v + v \cdot \nabla \nabla \times v + \frac{\hat{B}_2^2}{\kappa} \partial^{\alpha} \nabla \times v \]

\[ = -\frac{1}{\kappa} \partial^{\alpha} \partial_t (\hat{B}_1 B \times v - \hat{B}_2 B \cdot v) + \frac{1}{\kappa} \partial^{\alpha} \partial_t (\hat{B} \times b) + \Phi^\alpha, \]

where

\[ \Phi^\alpha = -[\partial^{\alpha}, \partial^{\alpha}_t + v \cdot \nabla \times] \nabla \times v + \partial^{\alpha} \Phi, \]

which can be estimated as follows.

**Lemma 6.1.** For \( \alpha \in \mathbb{N}^{1+2} \) with \( |\alpha| \leq 2N - 1 \), it holds that

\[ \|\Phi^\alpha\|_0^2 \lesssim \min\{\mathcal{E}_{N+4}, \mathcal{D}_{N+4}\} \mathcal{E}_{2N}. \]

**Proof.** The proof follows in the same way as Lemma 3.2. \( \square \)

The difference between \( \nabla \times v \) and \( \nabla \times v \) can be estimated as follows.

**Lemma 6.2.** For \( \alpha \in \mathbb{N}^{1+2} \) with \( |\alpha| \leq 2N - 1 \), it holds that

\[ \|\partial^{\alpha} (\nabla \times v - \nabla \times v)\|_0^2 \lesssim \min\{\mathcal{E}_{N+4}, \mathcal{D}_{N+4}\} \mathcal{E}_{2N}. \]

**Proof.** Note that \( \nabla \times v - \nabla \times v = -\nabla \eta \times \partial^2_v v \), and the proof then follows in the same way as Lemma 3.2. \( \square \)

### 6.1. Bounded normal energy estimates

We first derive the normal energy estimates of \( v \) and \( b \) at the \( 2N \) level, with assuming the control of \( \mathcal{E}_{2N} \). Recall the definition (4.1) with \( n = 2N \) of \( \mathcal{E}_{2N} \).

**Proposition 6.3.** It holds that

\[ \sum_{j=0}^{2N-1} \left( \left\| \partial^j \right\|_{2N-j}^2 + \left\| \partial^j b(t) \right\|_{2N-j+1}^2 \right) \]

\[ \lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup_{0 \leq s \leq t} \mathcal{E}_{N+4}(s) \mathcal{E}_{2N}(s). \]

**Proof.** We fix first \( j = 0, \ldots, 2N - 1 \) and then \( \ell = 0, \ldots, 2N - j - 1 \). Let \( \alpha \in \mathbb{N}^{1+2} \) with \( |\alpha| \leq 2N - 1 \) be so that \( \alpha_0 = j \) and \( \alpha_2 \leq 2N - j - 1 - \ell \). Taking the inner product of (6.7) with \( \partial^{\alpha} \nabla \times v \), integrating by parts over \( \Omega \) by using \( \nabla \cdot v = 0 \), \( \partial_t h = v \cdot N \) on \( \Sigma \) and \( v_2 = 0 \) on \( \Sigma_{-1} \), one obtains

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \partial^{\alpha} \nabla \times v \right|^2 \, dv + \frac{\hat{B}_2^2}{\kappa} \int_{\Omega} \left| \partial^{\alpha} \nabla \times v \right|^2 \, dv_t \]

\[ \lesssim \left( \left\| \partial^{\alpha} \partial_t v \right\|_0 + \left\| \partial^{\alpha} \partial_t b \right\|_0 + \left\| \Phi^\alpha \right\|_0 \right) \left\| \partial^{\alpha} \nabla \times v \right\|_0. \]
It follows from Cauchy’s inequality, (6.13) and the anisotropic Sobolev norm (2.2) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial^\alpha \text{curl}^\alpha v|^2 \, d\mathcal{V}_t + \frac{\mathcal{B}^2}{\kappa} \int_{\Omega} |\partial^\alpha \text{curl}^\alpha v|^2 \, d\mathcal{V}_t \\
\lesssim \|\partial^\alpha \partial_t v\|_0^2 + \|\partial^\alpha \partial_t b\|_0^2 + \|\Phi^\alpha\|_0^2 \\
\lesssim \left\| \partial^\alpha_t v \right\|_{2N-j-1-\ell, \ell+1}^2 + \left\| \partial^\alpha_{t+1} b \right\|_{2N-j-1}^2 + \mathcal{C}_{N+4} \mathcal{E}_{2N}. \tag{6.13}
\]
So a Gronwall type argument on (6.13) yields
\[
\|\partial^\alpha \text{curl}^\alpha v(t)\|_0^2 \lesssim e^{-\frac{\mathcal{B}^2 s}{\kappa}} \|\partial^\alpha \text{curl}^\alpha v(0)\|_0^2 + \int_0^t e^{-\frac{\mathcal{B}^2 s}{\kappa} (t-s)} \left( \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s) \right) \, ds \\
\lesssim \|\partial^\alpha \text{curl}^\alpha v(0)\|_0^2 + \sup_{0 \leq s \leq t} \left\| \partial^\alpha_t v(s) \right\|_{2N-j-1-\ell, \ell+1}^2 \\
+ \sup_{0 \leq s \leq t} \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \sup_{0 \leq s \leq t} \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.14}
\]
This, together with (6.10), implies that, summing over such \(\alpha\),
\[
\left\| \partial^\alpha_t \text{curl}^\alpha v \right\|_{2N-j-1-\ell, \ell}^2 \lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \left\| \partial^\alpha_t v(s) \right\|_{2N-j-1-\ell, \ell+1}^2 \\
+ \sup_{0 \leq s \leq t} \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \sup_{0 \leq s \leq t} \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.15}
\]
On the other hand, it follows from the second equation in (5.28) and (5.33) that
\[
\left\| \partial^\alpha_t \text{div} v \right\|_{2N-j-1}^2 = \left\| \partial^\alpha_t G^\alpha \right\|_{2N-j-1}^2 \lesssim \mathcal{C}_{N+4} \mathcal{E}_{2N}. \tag{6.16}
\]
Hence, employing the Hodge-type elliptic estimates (A.16) of Lemma A.8 with \( r = 2N-j-\ell \geq 1 \), by (6.15), (6.16) and (6.2) with \( n = 2N \), one obtains
\[
\left\| \partial^\alpha_t v(t) \right\|_{2N-j-1-\ell, \ell}^2 \lesssim \left\| \partial^\alpha_t v(t) \right\|_{0, \ell}^2 + \left\| \partial^\alpha_t \text{curl}^\alpha v(t) \right\|_{2N-j-1-\ell, \ell}^2 \\
+ \left\| \partial^\alpha_t \text{div} v(t) \right\|_{2N-j-1-\ell, \ell}^2 + \left\| \partial^\alpha_t v_2(t) \right\|_{2N-j-1/2}^2 \\
\lesssim \mathcal{E}_{2N}(t) + \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \left\| \partial^\alpha_t v(s) \right\|_{2N-j-1-\ell, \ell+1}^2 \\
+ \sup_{0 \leq s \leq t} \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \sup_{0 \leq s \leq t} \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.17}
\]
Taking the supremum of (6.17) over \([0, t]\), by an induction argument on \( \ell = 0, \ldots, 2N-j-1 \), one gets that for \( j = 0, \ldots, 2N-1 \),
\[
\sup_{0 \leq s \leq t} \left\| \partial^\alpha_t v(s) \right\|_{2N-j}^2 \lesssim \mathcal{E}_{2N}(0) + \mathcal{E}_{2N}(t) + \sup_{0 \leq s \leq t} \left\| \partial^\alpha_t v(s) \right\|_{0, 2N-j}^2 \\
+ \sup_{0 \leq s \leq t} \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \sup_{0 \leq s \leq t} \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s) \tag{6.18}
\]
\[
\lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup_{0 \leq s \leq t} \left\| \partial^\alpha_{t+1} b(s) \right\|_{2N-j-1}^2 + \sup_{0 \leq s \leq t} \mathcal{C}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.19}
\]
Next, applying the time derivatives \( \partial^\alpha_t, j = 0, \ldots, 2N-1 \) to the third equation in (3.28), one finds
\[
\begin{cases}
-\kappa \Delta \partial^\alpha_t b = B \cdot \nabla \partial^\alpha_t v - \partial^\alpha_{t+1} b + \partial^\alpha_t G^\alpha & \text{in } \Omega \\
\partial^\alpha_t b = 0 & \text{on } \partial\Omega.
\end{cases} \tag{6.19}
\]
It follows from (6.9), (6.10) and (6.24) that, summing over such \( s \),
\[
\left\| \partial_t^j b \right\|_{2N-j+1}^2 \lesssim \left\| \partial_t^j v \right\|_{2N-j}^2 + \left\| \partial_t^{j+1} b \right\|_{2N-j-1}^2 + \left\| \partial_t^j G \right\|_{2N-j-1}^2 \\
\lesssim \left\| \partial_t^j v \right\|_{2N-j}^2 + \left\| \partial_t^{j+1} b \right\|_{2N-j-1}^2 + E_{N+4} E_{2N}. \tag{6.20}
\]

Now combining the estimates (6.18) and (6.20) yields that for \( j = 0, \ldots, 2N - 1 \),
\[
\sup_{0 \leq s \leq t} \left( \left\| \partial_t^j v(s) \right\|_{2N-j}^2 + \left\| \partial_t^j b(s) \right\|_{2N-j+1}^2 \right) \\
\lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup_{0 \leq s \leq t} \left\| \partial_t^{j+1} b(s) \right\|_{2N-j-1}^2 + \sup \mathcal{E}_{N+4}(s) \mathcal{E}_{2N}(s) \\
= \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup_{0 \leq s \leq t} \left\| \partial_t^{2N} b(s) \right\|_0^2 + \sup \mathcal{E}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.21}
\]

By an induction argument on \( j = 0, \ldots, 2N - 1 \), one gets from (6.21) that
\[
\sup_{0 \leq s \leq t} \sum_{j=0}^{2N-1} \left( \left\| \partial_t^j v(s) \right\|_{2N-j}^2 + \left\| \partial_t^j b(s) \right\|_{2N-j+1}^2 \right) \\
\lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup_{0 \leq s \leq t} \left\| \partial_t^{2N} b(s) \right\|_0^2 + \sup \mathcal{E}_{N+4}(s) \mathcal{E}_{2N}(s) \\
\lesssim \mathcal{E}_{2N}(0) + \sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \sup \mathcal{E}_{N+4}(s) \mathcal{E}_{2N}(s). \tag{6.22}
\]

This yields the estimate (6.11).

6.2. Normal energy-dissipation estimates. This subsection is devoted to derive the energy-dissipation estimates of \( v \) and \( b \) at the \( n = N + 4, \ldots, 2N \) levels. We start with the dissipation estimates of \( v \) and the normal dissipation estimates of \( b \), with assuming the control of \( D_n \). Recall the definition (4.2) of \( D_n \). It is important to note that \( \| \text{curl} v \|_{n-2}^2 \) will be also controlled in the energy along the way.

**Proposition 6.4.** For \( n = N + 4, \ldots, 2N \), it holds that
\[
\frac{d}{dt} \| \text{curl} v \|_{n-2}^2 + \| v \|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \| b \|_n^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j b \right\|_{n-j-1/2}^2 \\
\lesssim D_n + D_{N+4} E_{2N}. \tag{6.23}
\]

**Proof.** Assume that \( N + 4 \leq n \leq 2N \). Fix \( \ell = 0, \ldots, n - 2 \). Let \( \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq n - 2 \) be so that \( \alpha_2 \leq n - 2 - \ell \) and then recall from the estimate (6.13) in the proof of Proposition 6.3 the following
\[
\frac{d}{dt} \int_\Omega |\partial^\alpha \text{curl} v|^2 \, d\mathcal{V} + \frac{B_2^2}{\kappa} \int_\Omega |\partial^\alpha \text{curl} v|^2 \, d\mathcal{V} \lesssim \| \partial^\alpha \partial_t v \|_0^2 + \| \partial^\alpha \partial_t b \|_0^2 + \| \Phi \|_0^2. \tag{6.24}
\]

It follows from (6.9), (6.10) and (6.24) that, summing over such \( \alpha \),
\[
\frac{d}{dt} \| \text{curl} v \|_{n-2-\ell}^2 + \| v \|_{n-2-\ell}^2 \lesssim \| \partial_t v \|_{n-2-\ell}^2 + \| \partial_t b \|_{n-2}^2 + D_{N+4} E_{2N}. \tag{6.25}
\]

The situation here is different from that of Proposition 6.3 one needs to estimate the term \( \| \partial_t v \|_{n-2-\ell}^2 \) in the right hand side of (6.25) in the dissipation rather than the energy. But one has not controlled \( \| v \|_{0, \ell}^2 \) in the dissipation yet, which prevents one from using the Hodge-type elliptic estimates for the moment, and hence one must employ a different argument. For this, recall first the estimate (6.15) with \( j = 0 \):
\[
\| \bar{B} \cdot \nabla v_2 \|_{0, n-1}^2 \lesssim D_n + D_{N+4} E_{2N}. \tag{6.26}
\]
Since $B_2 \neq 0$ and $v_2 = 0$ on $\Sigma_{-1}$ by the Poincaré-type inequality (A.7) and (6.20), one obtains
\[ \|v_2\|_{0,n-1}^2 \lesssim \|B \cdot \nabla v_2\|_{0,n-1}^2 \lesssim \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.27)
It then follows from (6.20) and (6.47) that
\[ \|\partial_2 v_2\|_{0,n-2}^2 = \frac{1}{B_2^2} \|\bar{B} \cdot \nabla v_2 - \bar{B}_1 \partial_1 v_2\|_{0,n-2}^2 \lesssim \|\bar{B} \cdot \nabla v_2\|_{0,n-2}^2 + \|v_2\|_{0,n-1}^2 \lesssim \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.28)
On the other hand, by the second equation in (6.28) and (6.33), one has
\[ \|\text{div } v\|_{n-2}^2 = \|G^2\|_{n-2}^2 \lesssim \mathcal{D}_{N+4} E_{2N}. \] (6.29)
Hence, (6.29) and (6.28) imply
\[ \|\partial_1 v_1\|_{0,n-2}^2 = \|\text{div } v - \partial_2 v_2\|_{0,n-2}^2 \lesssim \|\text{div } v\|_{0,n-2}^2 + \|\partial_2 v_2\|_{0,n-2}^2 \lesssim \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.30)
Now taking $\ell = n - 2$ in (6.25), one has, by (6.30) and (6.27),
\[ \frac{d}{dt} \|\text{curl } v\|_{0,n-2}^2 + \|\text{curl } v\|_{n-2}^2 \lesssim \|\partial_1 v_1\|_{0,n-2}^2 + \|\partial_2 b\|_{n-2}^2 + \mathcal{D}_{N+4} E_{2N} \lesssim \|\partial_2 b\|_{n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.31)
By (6.27) again, one obtains
\[ \|\partial_2 v_1\|_{0,n-2}^2 = \|\text{curl } v + \partial_1 v_2\|_{0,n-2}^2 \lesssim \|\text{curl } v\|_{0,n-2}^2 + \|v_2\|_{0,n-1}^2 \lesssim \|\text{curl } v\|_{0,n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.32)
But since $\int v_1 D_1 = 0$ from (1.15), by Poincaré’s inequality, (6.30) and (6.32), one gets
\[ \|v_1\|_{1,n-2}^2 \lesssim \|\nabla v_1\|_{0,n-2}^2 + \int_{\Omega} v_1 dx = \|\nabla v_1\|_{0,n-2}^2 + \int_{\Omega} v_1 \partial_2 v_2 dx \lesssim \|\text{curl } v\|_{0,n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.33)
Hence, by (6.27), (6.28) and (6.33), one may improve (6.31) to be
\[ \frac{d}{dt} \|\text{curl } v\|_{n-2}^2 + \|v\|_{1,n-2}^2 \lesssim \|\partial_2 b\|_{n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.34)
Next, for $\ell = 0, \ldots, n - 3$, employing the Hodge-type elliptic estimates (A.16) of Lemma A.8 with $r = n - 1 - \ell \geq 2$, by (6.29) and (7.10) with $j = 0$, one obtains
\[ \|v\|_{n-1-\ell,\ell}^2 \lesssim \|v\|_{n,\ell}^2 + \|\text{curl } v\|_{n-2-\ell,\ell}^2 + \|\text{div } v\|_{n-2-\ell,\ell}^2 + |v_2|_{n-3/2}^2 \lesssim \|v\|_{n,\ell}^2 + \|\text{curl } v\|_{n-1-(\ell+1),\ell}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.35)
Hence, (6.29) and (6.35) imply that for $\ell = 0, \ldots, n - 3$,
\[ \frac{d}{dt} \|\text{curl } v\|_{n-2-\ell,\ell}^2 + \|v\|_{n-1-\ell,\ell}^2 \lesssim \|v\|_{n-1-(\ell+1),\ell+1}^2 + \|\partial_2 b\|_{n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.36)
By an induction argument on $\ell = 0, \ldots, n - 3$, one gets from (6.36) that
\[ \frac{d}{dt} \|\text{curl } v\|_{n-2}^2 + \|v\|_{n-1}^2 \lesssim \|v\|_{n-1}^2 + \|\partial_2 b\|_{n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.37)
This together with (6.31) yields
\[ \frac{d}{dt} \|\text{curl } v\|_{n-2}^2 + \|v\|_{n-1}^2 \lesssim \|\partial_2 b\|_{n-2}^2 + \bar{D}_n + \mathcal{D}_{N+4} E_{2N}. \] (6.38)
Next, applying (A.18) of Lemma A.9 with $r = n$ to the problem (6.19) for $j = 0$, one has, by (6.35),
\[ \|b\|_{n}^2 \lesssim \|v\|_{n-1}^2 + \|\partial_2 b\|_{n-2}^2 + \|G^2\|_{n-2}^2 \lesssim \|v\|_{n-1}^2 + \|\partial_2 b\|_{n-2}^2 + \mathcal{D}_{N+4} E_{2N}. \] (6.39)
Now the first equation in (6.42) implies that for \( j = 1, \ldots, n - 1 \), by (6.36) and (6.39),
\[
\begin{align*}
\left\| \partial_t^j v \right\|_{n-j-1/2}^2 & \lesssim \left\| \partial_t^{j-1} b \right\|_{n-j+1/2}^2 + \left\| \partial_t^{j-1} q \right\|_{n-j+1/2}^2 + \left\| \partial_t^{j-1} G \right\|_{n-j-1/2}^2 \\
& = \left\| \partial_t^{j-1} b \right\|_{n-(j-1)-1/2}^2 + \left\| \partial_t^{j-1} q \right\|_{n-(j-1)-1/2}^2 + \left\| \partial_t^{j-1} G \right\|_{n-(j-1)-3/2}^2 \\
& \lesssim \left\| \partial_t^{j-1} b \right\|_{n-(j-1)-1/2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\end{align*}
\]
Next, applying (A.18) of Lemma A.9 with \( r = n-j+1/2 \geq 3/2 \), \( j = 1, \ldots, n-1 \) to the problem (6.19), one has that for \( j = 1, \ldots, n-1 \), by (6.36),
\[
\left\| \partial_t^j b \right\|_{n-j+1/2}^2 \lesssim \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| \partial_t^{j+1} b \right\|_{n-j-3/2}^2 + \left\| \partial_t^{j+1} G \right\|_{n-j-3/2}^2 \\
\lesssim \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| \partial_t^{j+1} b \right\|_{n-(j+1)-1/2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
Combining (6.40) and (6.41), by a simple inductive argument on \( j = 1, \ldots, n-1 \), one gets
\[
\sum_{j=1}^{n-1} \left( \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| \partial_t^j b \right\|_{n-j+1/2}^2 \right) \lesssim \left\| \partial_t^0 b \right\|_{0}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^{j-1} b \right\|_{n-(j-1)-1/2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N} \\
\lesssim \sum_{j=0}^{n-2} \left\| \partial_t^j b \right\|_{n-j-1/2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
This together with the Sobolev interpolation implies that
\[
\sum_{j=1}^{n-1} \left( \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| \partial_t^j b \right\|_{n-j+1/2}^2 \right) \lesssim \left\| b \right\|_{n-1/2}^2 + \sum_{j=1}^{n-2} \left\| \partial_t^j b \right\|_{0}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N} \\
\lesssim \left\| b \right\|_{n-1/2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
Finally, collecting the estimates (6.38), (6.39) and (6.43) yields that
\[
\frac{d}{dt} \left\| \text{curl} v \right\|_{n-2}^2 + \left\| v \right\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| b \right\|_{n}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j b \right\|_{n-j+1/2}^2 \\
\lesssim \left\| b \right\|_{n-1/2}^2 + \left\| \partial_t b \right\|_{n-2}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
This together with the Sobolev interpolation implies that
\[
\frac{d}{dt} \left\| \text{curl} v \right\|_{n-2}^2 + \left\| v \right\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j v \right\|_{n-j-1/2}^2 + \left\| b \right\|_{n}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j b \right\|_{n-j+1/2}^2 \\
\lesssim \left\| b \right\|_{0}^2 + \left\| \partial_t b \right\|_{0}^2 + \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N} \\
\lesssim \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
This is the estimate (6.23). \(\square\)

Now, we shall use \( \mathcal{E}_n + \left\| \text{curl} v \right\|_{n-2}^2 \) to derive the normal energy estimates of \( v \) and \( b \).

**Proposition 6.5.** For \( n = N + 4, \ldots, 2N \), it holds that
\[
\left\| v \right\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j v \right\|_{n-j}^2 + \left\| b \right\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j b \right\|_{n-j+1}^2 \\
\lesssim \left\| \text{curl} v \right\|_{n-2}^2 + \mathcal{E}_n + \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\]

**Proof.** Assume that \( N + 4 \leq n \leq 2N \). The proof is almost similar to that of Proposition 6.4 but a bit simpler. First, it follows from the second equation in (6.28) and (6.35) that
\[
\left\| \text{div} v \right\|_{n-2}^2 = \left\| G^2 \right\|_{n-2}^2 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\]
By (A.10) of Lemma A.8 with \( r = n - 1 \), (6.10), (6.27) and (5.2) with \( j = 0 \), one obtains
\[
\|v\|_{n-1}^2 \lesssim \|v\|_0^2 + \|\text{curl} \, v\|_{n-2}^2 + \|\text{div} \, v\|_{n-2}^2 + |v_2|_{n-3/2}^2 \\
\lesssim \|\text{curl} \, v\|_{n-2}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.48)
Applying (A.18) of Lemma A.9 with \( r = n \) to the problem (6.19) for \( j = 0 \), one has, by (3.35),
\[
\|b\|_n^2 \lesssim \|v\|_{n-1}^2 + \|\partial_t b\|_{n-2}^2 + \|G^3\|_{n-2}^2 \lesssim \|v\|_{n-1}^2 + \|\partial_t b\|_{n-2}^2 + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.49)
Next, by the first equation in (3.28), (5.1) and (3.35), one has that for \( j = 1, \ldots, n-1 \),
\[
\left\| \partial^j v \right\|_{n-j}^2 \lesssim \left\| \partial^j b \right\|_{n-j+1}^2 + \left\| \partial^j q \right\|_{n-j+1}^2 + \left\| \partial^j G^1 \right\|_{n-j}^2 \\
= \left\| \partial_{j-1} b \right\|_{n-j}^2 + \left\| \partial_{j-1} q \right\|_{n-j}^2 + \left\| \partial_{j-1} G^1 \right\|_{n-j-1}^2 \\
\lesssim \left\| \partial_{j-1} b \right\|_{n-j}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.50)
Now applying (A.18) of Lemma A.9 with \( r = n - j + 1 \geq 2 \), \( j = 1, \ldots, n-1 \), to the problem (6.19), one has that for \( j = 1, \ldots, n-1 \), by (3.35),
\[
\left\| \partial^j b \right\|_{n-j+1}^2 \lesssim \left\| \partial^j v \right\|_{n-j}^2 + \left\| \partial^j b \right\|_{n-j}^2 + \left\| \partial^j G^3 \right\|_{n-j}^2 \\
\lesssim \left\| \partial^j v \right\|_{n-j}^2 + \left\| \partial^j b \right\|_{n-j}^2 + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.51)
Combining (6.50) and (6.51), by a simple inductive argument on \( j = 1, \ldots, n-1 \), one gets
\[
\sum_{j=1}^{n-1} \left( \left\| \partial^j v \right\|_{n-j}^2 + \left\| \partial^j b \right\|_{n-j}^2 \right) \lesssim \left\| \partial^j b \right\|_0^2 + \sum_{j=1}^{n-1} \left\| \partial^j b \right\|_{n-j}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N} \\
\lesssim \sum_{j=0}^{n-2} \left\| \partial^j b \right\|_{n-j}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.52)
This together with the Sobolev interpolation implies that
\[
\sum_{j=1}^{n-1} \left( \left\| \partial^j v \right\|_{n-j}^2 + \left\| \partial^j b \right\|_{n-j+1}^2 \right) \lesssim \left\| b \right\|_n^2 + \sum_{j=1}^{n-2} \left\| \partial^j b \right\|_{n-j}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N} \\
\lesssim \left\| b \right\|_n^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.53)
Finally, collecting the estimates (6.48)–(6.49) and (6.53) shows that
\[
\|v\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial^j v \right\|_{n-j}^2 + \|b\|_n^2 + \sum_{j=1}^{n-1} \left\| \partial^j b \right\|_{n-j}^2 \\
\lesssim \|\text{curl} \, v\|_{n-2}^2 + \|\partial_t b\|_{n-2}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.54)
This together with the Sobolev interpolation implies that
\[
\|v\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial^j v \right\|_{n-j}^2 + \|b\|_n^2 + \sum_{j=1}^{n-1} \left\| \partial^j b \right\|_{n-j}^2 \\
\lesssim \|\text{curl} \, v\|_{n-2}^2 + \|\partial_t b\|_0^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N} \\
\lesssim \|\text{curl} \, v\|_{n-2}^2 + \bar{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.55)
This yields the desired estimate (6.46). □

7. Global energy estimates

In this section we will derive the global-in-time full energy estimates by collecting the estimates derived previously in Sections 4–6.
7.1. Boundedness at the $2N$ level. We first show the the boundedness of $\mathcal{E}_{2N}$ in term of the initial data.

**Theorem 7.1.** Let $N \geq 8$ be an integer. There exists a universal constant $\delta > 0$ such that if
\[
\mathcal{E}_{2N}(t) + (1 + t)^{N-5}\mathcal{E}_{N+4}(t) \leq \delta, \quad \forall t \in [0, T],
\]
then
\[
\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0), \quad \forall t \in [0, T].
\]

**Proof.** By the assumption (7.1), Proposition 5.2 and 6.4, and recalling the definitions of $\mathcal{E}$ and $\bar{\mathcal{E}}$.

Theorem 7.2. Let $N \geq 8$ be an integer. There exists a universal constant $\delta > 0$ such that if
\[
\mathcal{E}_{2N}(t) + (1 + t)^{N-5}\mathcal{E}_{N+4}(t) \leq \delta, \quad \forall t \in [0, T],
\]
then
\[
\int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0), \quad \forall t \in [0, T]
\]
and
\[
\sum_{j=1}^{N-5} (1 + t)^j \mathcal{E}_{2N-j-1}(t) + \sum_{j=1}^{N-5} \int_0^t (1 + s)^j \mathcal{D}_{2N-j-1}(s) \, ds \lesssim \mathcal{E}_{2N}(0), \quad \forall t \in [0, T].
\]

**Proof.** By the assumption (7.1), Propositions 5.2 and 6.4 and recalling the definitions of $\mathcal{D}_n$ and $\mathcal{D}_n$,

\[
\frac{d}{dt} \| \text{curl}^2 v \|_{n-2}^2 + \mathcal{D}_n \lesssim \mathcal{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N} \lesssim \mathcal{D}_n + \delta \mathcal{D}_{N+4},
\]
which implies that, since $\delta$ is small and $n \geq N + 4$,

\[
\frac{d}{dt} \| \text{curl}^2 v \|_{n-2}^2 + \mathcal{D}_n \lesssim \mathcal{D}_n.
\]

We first derive the dissipation estimate (7.5). Taking $n = 2N$ in (7.8) and then integrating in time directly, by the estimate (7.2), one has in particular that
\[
\int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \| \text{curl}^2 v(0) \|_{2n-2}^2 + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0).
\]
This is the estimate (7.5).

We now show the decay estimates (7.6). It follows from Propositions 5.4 and 6.5 and the definitions of $\mathcal{E}_n$ and $\bar{\mathcal{E}}_n$ that for $n = N + 4, \ldots, 2N - 2$,
\[
\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n + \| \text{curl}^2 v \|_{n-2}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_n + \| \text{curl}^2 v \|_{n-2}^2 + \delta \mathcal{E}_{N+4},
\]
which implies that, since $\delta$ is small and $n \geq N + 4$,
\[
\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n + \| \text{curl}^2 v \|_{n-2}^2.
\]
On the other hand, Proposition 7.1 and 7.2 show that for \( n = N + 4, \ldots, 2N - 2 \),
\[
\frac{d}{dt} \left( \mathcal{E}_n + \mathcal{B}_n + \| \text{curl}^r v \|_{n-2}^2 \right) + \mathcal{D}_n \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{D}_n} \lesssim \delta \mathcal{D}_n, \tag{7.12}
\]
which implies that, since \( \delta \) is small,
\[
\frac{d}{dt} \left( \mathcal{E}_n + \mathcal{B}_n + \| \text{curl}^r v \|_{n-2}^2 \right) + \mathcal{D}_n \leq 0. \tag{7.13}
\]
By (7.37) and (7.11), it holds that
\[
\mathcal{E}_n \lesssim \mathcal{E}_n + \mathcal{B}_n + \| \text{curl}^r v \|_{n-2}^2 \lesssim \mathcal{E}_n. \tag{7.14}
\]
Hence, (7.13) implies that for \( n = N + 4, \ldots, 2N - 2 \),
\[
\frac{d}{dt} \mathcal{E}_n + \mathcal{D}_n \leq 0. \tag{7.15}
\]

Note that \( \mathcal{D}_n \) cannot control \( \mathcal{E}_n \), which can be seen by checking both the spatial and the temporal regularities in their definitions. This rules out not only the exponential decay of \( \mathcal{E}_n \) but also prevents one from using the spatial Sobolev interpolation as \( 17 \) \( 18 \) to bound \( \mathcal{E}_n \lesssim \mathcal{E}_{2N}^1 \mathcal{D}_n^\theta, 0 < \theta < 1 \) so as to derive the algebraic decay. Observe that \( \mathcal{E}_t \leq \mathcal{D}_{t+1} \) and then we will employ a time weighted inductive argument here. To begin with, we may rewrite (7.13) as that for \( j = 1, \ldots, N - 5 \),
\[
\frac{d}{dt} \mathcal{E}_{2N-j-1} + \mathcal{D}_{2N-j-1} \leq 0. \tag{7.16}
\]
Multiplying (7.16) by \((1 + t)^j\), one has, by using \( \mathcal{E}_{2N-j-1} \leq \mathcal{D}_{2N-j} \),
\[
\frac{d}{dt} ((1 + t)^j \mathcal{E}_{2N-j-1}) + (1 + t)^j \mathcal{D}_{2N-j} \leq j(1 + t)^{j-1} \mathcal{E}_{2N-j-1} \leq j(1 + t)^{j-1} \mathcal{D}_{2N-j-1}. \tag{7.17}
\]
Integrating (7.17) in time directly, by a suitable linear combination of the resulting inequalities, one obtains, by (7.3),
\[
\sum_{j=1}^{N-3} (1 + t)^j \mathcal{E}_{2N-j-1}(t) + \sum_{j=1}^{N-3} \int_0^t (1 + s)^j \mathcal{D}_{2N-j-1}(s) \, ds \lesssim \int_0^t \mathcal{D}_{2N-1}(s) \, ds \lesssim \mathcal{E}_{2N}(0). \tag{7.18}
\]
This is the estimate (7.3). \( \square \)

### 7.3. Proof of Theorem 2.2
Now the proof of Theorem 2.2 follows, in a standard way, by the local well-posedness theory, a continuity argument and the following a priori estimates.

**Theorem 7.3.** Let \( N \geq 8 \) be an integer. There exists a universal constant \( \delta > 0 \) such that if
\[
\mathcal{E}_{2N}(t) + (1 + t)^{N-5} \mathcal{E}_{N+4}(t) \leq \delta, \quad \forall t \in [0, T], \tag{7.19}
\]
then
\[
\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0), \quad \forall t \in [0, T] \tag{7.20}
\]
and
\[
\sum_{j=1}^{N-5} (1 + t)^j \mathcal{E}_{2N-j-1}(t) + \sum_{j=1}^{N-5} \int_0^t (1 + s)^j \mathcal{D}_{2N-j-1}(s) \, ds \lesssim \mathcal{E}_{2N}(0), \quad \forall t \in [0, T]. \tag{7.21}
\]

**Proof.** The conclusion follows directly from Theorems 7.1 and 7.2. \( \square \)

### Appendix A. Analytic tools

In this appendix we will collect the analytic tools which are used throughout the paper.
A.1. Harmonic extension. We define the Poisson integral in $\mathbb{T} \times (-\infty, 0)$ by
\[
\mathcal{P}f(x) = \sum_{\xi \in \mathbb{Z}} e^{2\pi i \xi x_1} e^{2\pi i |\xi| x_2} \hat{f}(\xi),
\] (A.1)
where for $\xi \in \mathbb{Z},$
\[
\hat{f}(\xi) = \int_{\mathbb{T}} f(x_1) e^{-2\pi i \xi x_1} dx_1.
\] (A.2)
It is well known that $\mathcal{P} : H^s(\Sigma) \to H^{s+1/2}(\mathbb{T} \times (-\infty, 0))$ is a bounded linear operator for $s > 0$. However, if restricted to the domain $\Omega$, one has the following improvement.

Lemma A.1. It holds that for all $s \in \mathbb{R},$
\[
\|\mathcal{P}f\|_s \lesssim |f|_{s-1/2}.
\] (A.3)
Proof. One may refer to Lemma A.9 of [16] or Proposition 3.1 of [26] for instance. \qed

A.2. Estimates in Sobolev spaces. The following standard estimates in Sobolev spaces are needed.

Lemma A.2. Let $U$ denote either $\Omega$ or $\Sigma$, and $d$ be the dimension.

1. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_1 > d/2$. Then
\[
\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.
\] (A.4)
2. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + d/2$. Then
\[
\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}.
\] (A.5)
The following $|\cdot|_{-1/2}$ product estimates are also useful.

Lemma A.3. Let $m > 3/2$. Then
\[
|fg|_{-1/2} \lesssim |f|_{-1/2} \|g\|_m.
\] (A.6)
Proof. It follows by the duality and the estimate (A.5) with $r = s_1 = 1/2$, $d = 1$ and $s_2 = m$. \qed

A.3. Poincaré-type inequality related to $B \cdot \nabla$. The following Poincaré-type inequality holds.

Lemma A.4. For any constant vector $B \in \mathbb{R}^2$ with $B_2 \neq 0$, it holds that for all $f$ with $f = 0$ on $\Sigma_{-1},$
\[
\|f\|_0^2 \leq \frac{1}{B_2^2} \|(B \cdot \nabla)f\|_0^2
\] (A.7)
and
\[
|f|_0 \leq \frac{1}{B_2} \|(B \cdot \nabla)f\|_0.
\] (A.8)
Proof. For any $f$ with $f = 0$ on $\Sigma_{-1}$, by the fundamental theory of calculus and the Cauchy-Schwarz inequality, since $B_2 \neq 0$, one deduces that for any $x = (x_1, x_2) \in \Omega,$
\[
f(x_1, x_2) = f\left(x_1 + \frac{1}{B_2} \left. \frac{1}{B_2} x_2 \right|_{B_1, -1} \right) + \int_{-1-x_2}^0 \frac{d}{ds} \left. (f(x_1 + sB_1, x_2 + sB_2)) \right|_{s_2} \, ds
\]
\[
= \int_{-1-x_2}^0 (B \cdot \nabla)f(x_1 + sB_1, x_2 + sB_2) \, ds
\]
\[
\leq \left(\frac{1 + x_2}{B_2}\right)^{\frac{1}{2}} \left(\int_{-1-x_2}^0 |(B \cdot \nabla)f(x_1 + sB_1, x_2 + sB_2)|^2 \, ds\right)^{\frac{1}{2}}.
\] (A.9)
By taking the square of (A.9) and then integrating over \( x_1 \in T \), using the Fubini theorem and the change of variables, one has
\[
\int_T f^2(x_1, x_2) \, dx_1 \leq \frac{1}{B_2} \int_T \int_{\frac{1}{B_2}}^0 \left| (B \cdot \nabla) f(x_1 + sB_1, x_2 + sB_2) \right|^2 \, ds \, dx_1
\]
\[
= \frac{1}{B_2} \int_{\frac{1}{B_2}}^0 \int_T \left| (B \cdot \nabla) f(x_1 + sB_1, x_2 + sB_2) \right|^2 \, dx_1 \, ds
\]
\[
= \frac{1}{B_2} \int_{\frac{1}{B_2}}^0 \int_T \left| (B \cdot \nabla) f(x_1, x_2 + sB_2) \right|^2 \, dx_1 \, ds
\]
\[
= \frac{1}{B_2} \int_{\frac{1}{B_2}}^0 \left| (B \cdot \nabla) f(x_1, s') \right|^2 \, dx_1 \, ds' \leq \frac{1}{B_2} \| B \cdot \nabla f \|_0^2. \tag{A.10}
\]
Integrating (A.10) in \( x_2 \) over \((-1, 0)\) implies (A.7), while taking \( x_2 = 0 \) yields (A.8).

A.4. Normal trace estimates. The following classical normal trace estimate for vector functions is valid.

Lemma A.5. It holds that
\[
|v_2|_{-1/2} \lesssim \|v\|_0 + \| \text{div} \, v \|_0. \tag{A.11}
\]

Proof. It is indeed a special case of Lemma A.7 with \( \varphi = x_2 \).

The following normal trace estimate for \( \partial_1 v_2 \) is also needed.

Lemma A.6. It holds that
\[
|\partial_1 v_2|_{-1/2} \lesssim \| \partial_1 v \|_0 + \| \text{div} \, v \|_0 \tag{A.12}
\]

Proof. Let \( \psi \in H^{1/2}(\Sigma) \), and let \( \tilde{\psi} \in H^1(\Omega) \) be a bounded extension which vanishes near \( \Sigma \). Then
\[
\int_{\Sigma} \psi \partial_1 v_2 \, dx_1 = \int_{\Omega} \text{div}(\tilde{\psi} \partial_1 v) \, dx
\]
\[
= \int_{\Omega} \left( \nabla \tilde{\psi} \cdot \partial_1 v + \tilde{\psi} \text{div} \, \partial_1 v \right) \, dx = \int_{\Omega} \left( \nabla \tilde{\psi} \cdot \partial_1 v - \partial_1 \tilde{\psi} \text{div} \, v \right) \, dx
\]
\[
\lesssim \| \tilde{\psi} \|_1 \left( \| \partial_1 v \|_0 + \| \text{div} \, v \|_0 \right) \lesssim |\psi|_{1/2} \left( \| \partial_1 v \|_0 + \| \text{div} \, v \|_0 \right). \tag{A.13}
\]

Then (A.12) follows from (A.13) immediately.

Finally, the following \( H^{-1/2} \) boundary estimate holds for functions satisfying \( v \in L^2 \) and \( \nabla \varphi \cdot v \in L^2 \).

Lemma A.7. Assume that \( \| \nabla \varphi \|_{L^\infty} \leq C \), then
\[
|v \cdot N|_{-1/2} \lesssim \|v\|_0 + \| \nabla \varphi \cdot v \|_0. \tag{A.14}
\]

Proof. One can adapt the proof of Lemma 3.3 [16]. Let \( \psi \in H^{1/2}(\Sigma) \), and let \( \tilde{\psi} \in H^1(\Omega) \) be a bounded extension which vanishes near \( \Sigma \). Then
\[
\int_{\Sigma} \psi v \cdot N \, dx_1 = \int_{\Omega} \nabla \varphi \cdot (\tilde{\psi} v) \, d\nu_1 = \int_{\Omega} \left( \nabla \varphi \tilde{\psi} \cdot v + \tilde{\psi} \nabla \varphi \cdot v \right) \, d\nu_1
\]
\[
\lesssim \| \tilde{\psi} \| \| \nabla \varphi \cdot v \|_0 + \| \nabla \tilde{\psi} \|_0 \|v\|_0 \lesssim |\psi|_{1/2} \left( \|v\|_0 + \| \nabla \varphi \cdot v \|_0 \right). \tag{A.15}
\]

Then (A.14) follows immediately from (A.15).
A.5. Elliptic estimates. The derivation of the high order energy estimates for the velocity $v$ is based on the following well-known Hodge-type elliptic estimates.

**Lemma A.8.** Let $r \geq 1$, then it holds that

$$
\|v\|_r \lesssim \|v\|_0 + \|\text{curl} \, v\|_{r-1} + \|\text{div} \, v\|_{r-1} + |v_2|_{r-1/2}.
$$

(A.16)

**Proof.** The estimate is well-known and follows from the identity $-\Delta v = \text{curl} \, \text{curl} \, v - \nabla \text{div} \, v$. One may refer to Section 5.9 of [35]. □

We use the standard elliptic theory with Dirichlet boundary conditions for the magnetic field $b$,

$$
\begin{cases}
-\kappa \Delta b = f & \text{in } \Omega \\
b = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(A.17)

**Lemma A.9.** Let $r \geq 1$. If $f \in H^{r-2}(\Omega)$, then

$$
\|b\|_r \lesssim \|f\|_{r-2}.
$$

(A.18)

We use the standard elliptic theory with Neumann boundary conditions for the pressure $q$,

$$
\begin{cases}
-\Delta q = f & \text{in } \Omega \\
\partial_2 q = g & \text{on } \Sigma \\
\partial_2 q = 0 & \text{on } \Sigma_{-1}.
\end{cases}
$$

(A.19)

**Lemma A.10.** Let $r \geq 1$. If $f \in H^{r-2}(\Omega)$ and $g \in H^{r-3/2}(\Sigma)$ satisfy the compatibility condition

$$
\int_{\Omega} f \, dx + \int_{\Sigma} g \, dx_1 = 0,
$$

(A.20)

then

$$
\|\nabla q\|_{r-1} \lesssim \|f\|_{r-2} + |g|_{r-3/2}.
$$

(A.21)

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