Embeddings of local fields in simple algebras and simplicial structures on the Bruhat-Tits building

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- $\mathbb{N} = \{1, 2, \ldots\}$  $\mathbb{N}_r := \{1, \ldots, r\}$.

- $(F, \nu)$ non-archimedean local field, $D|F$ a central skewfield, $d := \sqrt{[D : F]} < \infty$. $L|F$ max. unramified field in $D$, $[L : F] = d$

$$D \supseteq L \supseteq F$$

- Assume that $\pi_D$ normalizes $L$.

$$D = L \oplus L\pi_D \oplus L\pi_D^2 \ldots \oplus L\pi_D^{d-1}$$

- $A := M_m(D)$ the and $V := D^m$, right $D$ vector space, $m \in \mathbb{N}$ fixed.
Martin Grabitz and Paul Broussous have classified embeddings

\[ E^\times \subseteq \text{compact modulo center group} \subseteq M_m(D) \]

and introduced invariants. The question of E.W. Zink was: Is there a geometric way to find the invariants using euclidean Bruhat Tits buildings as geometrical object together with an affine map.
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**Definition 1** A *hereditary order* \( a \subseteq M_m(D) \) is a subring of \( M_m(D) \), s.t. there is a \( g \in GL_m(D) \) s.t. \( gag^{-1} \) is of the form

\[
\begin{pmatrix}
D \circ & D \circ \circ & \ldots & D \circ \circ \\
\vdots & \ldots & \ldots & \vdots \\
D \circ & \ldots & D \circ & D \circ \circ \\
D \circ & \ldots & D \circ & D \circ \\
\end{pmatrix}^{n_1, n_2, \ldots, n_r}
\]

where \( \sum n_i = m \).
Definition 4  An embedding is a pair \((E, \alpha)\) satisfying

1. \(E\) is a field extension of \(F\) in \(A\),

2. \(\alpha \in \text{Her}(A)\) is normalised by \(E^\times\).

\((E, \alpha) \sim (E', \alpha')\) if there is a \(g \in A^\times\), such that \(gE_Dg^{-1} = E'_D\) and \(gag^{-1} = \alpha'\).

An example for embeddings are pearl embeddings. (soon)
**Definition 6** Let \( f|d \) and \( r \leq m \). An *embedding datum* is a \( f \times r \)-matrix \( \lambda \) of non-negative integer entries s.t. in every column is non-zero, and the sum of all entries is \( m \). The *pearl embedding* of \( \lambda \) is the embedding \((E, a)\), s.t.

1. \([E : F] = f\) and \( E \) is in the image of
   \[
   x \in L \mapsto \text{diag}(M_1(x), M_2(x), \ldots, M_r(x)) \quad \text{where}
   \]
   \[
   M_j(x) = \text{diag}(\sigma^0(x)I_{\lambda_1,j}, \sigma^1(x)I_{\lambda_2,j}, \ldots, \sigma^{f-1}(x)I_{\lambda_f,j})
   \]

2. \( a \in \text{Her}(A) \) in standard form according to
   \[
   m = n_1 + \ldots + n_r \quad \text{where} \quad n_j := \sum_{i=1}^{f} \lambda_{i,j}.
   \]
Equivalent vectors

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 1 & 0
\end{pmatrix}^T.
\]

Definition 7

1. \( w = (w_1, \ldots, w_t) \sim w' = (w'_1, \ldots, w'_t) \) (real entries) if there is a \( k \), s.t.

\[
w = (w'_k, \ldots, w'_t, w'_1, \ldots, w'_{k-1}).
\]

We write \(< w >\) for the equivalence class.

2. For a \( t \times s \)-matrix \( M \) we put

\[
\text{row}(M) := (m_{1,1}, \ldots, m_{1,s}, m_{2,1}, \ldots, m_{2,s}, \ldots, m_{t,s}).
\]

3. \( M \sim N \) if \( \text{row}(M) \sim \text{row}(N) \).
**Theorem 1**  \[BG00, 2.3.3 \text{ and } 2.3.10\]

1. *Two pearl embeddings are equivalent if and only if the embedding datas are.*

2. *In any class of embeddings lies a pearl embedding.*

**Definition 8**  By the theorem to an embedding corresponds one class of embedding datas, called *embedding type* (notion from V. Secherre).
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A building of rank $m - 1$ is a poset $(\Omega, \leq)$ s.t.

- $\bar{S} := \{S' \in \Omega | S' \leq S\}$ is poset isom. to a simplex, $S \in \Omega$ (faces).
- Every face has not more then $m - 1$ vertices (= minimal elements).
- Every face lies in a face with $m - 1$ vertices (= maximal elements = chambers).
- $\Omega = \bigcup \mathcal{A}$, where $\mathcal{A}$ is a set of chamber subcomplexes of rank $m - 1$, apartments.
- There are poset isomorphisms between $\Sigma, \Sigma' \in 'A$. 

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Euclidean building

A building is called euclidean if every apartment is a isomorphic to a cell decomposition of an f.d. euclidean space with an infinite affine reflection group.

\(|S| := \{\sum_{\text{vertex of } S} \lambda_v v | \sum \lambda_v = 1 \lambda_v > 0\}\) geometric realisation g.r. of \(S\)

\(|\Omega| := \bigcup{|S| | S \in \Omega}\).
Lattice functions

With $\text{Latt}_{D^\circ}^{m,V}$ we denote the set of full $D^\circ$-lattices in $V$. The word full will be omitted. Definitions:

- A left continuous monoton decreasing (all w.r.t. $\subseteq$) function $r \in \mathbb{R} \rightarrow \Lambda(r) \in \text{Latt}_{D^\circ}^{m,V}$ is called $D^\circ$-lattice function of $V$, if $\forall r \in \mathbb{R} : \Lambda(r)\pi_D = \Lambda(r + \frac{1}{d})$.

- The set of $D^\circ$ lattice functions is denoted by $\text{Latt}_{D^\circ}^1 V$.

- $\Lambda_1 \sim \Lambda_2$ iff $\exists s \in \mathbb{R} : \forall r \in \mathbb{R} : \Lambda_1(r) = \Lambda_2(r + s)$.

- $\text{Latt}_{D^\circ} V := \text{Latt}_{D^\circ}^1 V / \sim$
**Affine Structure**

**Definition 10** A $D$-basis $(v_i)$ of $V$ is called splitting basis of a lattice function $[\Lambda]$, if

$$\forall r \in \mathbb{R} : \Lambda(r) = \bigoplus_{i=1}^m (\Lambda(r) \cap R_i).$$

**Affine structure:** For $[\Lambda]$ and $[\Lambda']$ we can find a splitting basis $(v_i)$, thus

$$\Lambda(r) = \bigoplus_{i=1}^m v_i D^{\circ \circ [r-\alpha_i]} + \text{ and } \Lambda'(r) = \bigoplus_{i=1}^m v_i D^{\circ \circ [r-\alpha'_i]} + .$$

For $\lambda \in [0, 1]$ one defines

$$\lambda[\Lambda] + (1 - \lambda)[\Lambda'] := [\Lambda''] \text{ with}$$

$$\Lambda''(r) := \bigoplus_{i=1}^m v_i D^{\circ \circ [r-\lambda \alpha_i - (1-\lambda)\alpha'_i]} + .$$
The g.r. of the eucl. building of $GL_m(D)$ we denote by $\mathcal{I}$.

**Theorem 5 ([BL02] section 1 (2.5))** $\mathcal{I} \cong \text{Latt}_{D^\circ} V$

$GL(D)^\times$-equivariant, affine.

**Apartments:** A frame $R = \{R_i| 1 \leq i \leq m\}$ is a set of $m$ linearely independent 1-dim. $D$-subspaces of $V$.

$$\text{Latt}_R V := \{[\Lambda]| \Lambda \text{ is splitt by } R\}.$$

Apartments $= \{\text{Latt}_R V| R \text{ frame}\}$.

**Faces:** They are given by the hereditary orders of $A$,

$$\text{Her}(A) := \{a| a \text{ is a hereditary order}\}$$

Def.: $a \leq a'$ if $a \supseteq a'$
A lattice function $[\Lambda]$ lies on the face $\alpha_\Lambda = \{ a \in A | a\Lambda(r) \subseteq \Lambda(r) \ \forall r \in \mathbb{R} \}$.

The range of a lattice function is a lattices chain. This lattice chain represents the face $\tilde{F}$ of the simplicial building s.t. $p \in |\tilde{F}|$.

Lattice chains are in 1-1 correspondence to hereditary orders.
Theorem 6 (P. Broussous, B. Lemaire)

1. The simplicial complex of $\mathcal{I}$ is isomorphic to $(\text{Her}(A), \supseteq)$.

2. The hereditary order of rank $k$ correspond to the faces of rank $k$, i.e. of dimension $k - 1$.

3. Maximal her. orders, correspond to the vertices and minimal her. orders to the chambers.
The affine map $j_E$
\[ A = M_m(D) \supseteq B = C_A(E) \supseteq E \supseteq F \]

- \(E|F\) is a unram. field extension of degree \([E:F]|d\) in \(A\).
- \(B\) is the centraliser of \(E\) in \(A\).
- It is \(\mathcal{I}_E\) the g.r. of the eucl. building of \(B\).
Existence and Uniqueness of $j_E$

**Theorem 8** [BL02, part of Thm 1.1.] There exists a unique application $j_E : \mathcal{I}_E^\times \to \mathcal{I}_E$ such that

1. $j_E$ is $B^\times$-equivariant.
2. $j_E$ is affine.

Moreover $j_E^{-1}$ can be characterised as the unique $B^\times$-equivariant affine map $\mathcal{I}_E \to \mathcal{I}$.
\( \mathbf{j}_E \) in terms of lattice functions 1

This is due to Broussous and Lemaire [BL02] II 3.1. We have \( E \cong \mathfrak{i}(E) \subseteq L \) (\( F \)-Algebrahomomorphism).

\[ E \otimes_F \mathfrak{i}(E) \cong \bigoplus_{k=0}^{[E:F]-1} \mathfrak{i}(E) \] with the decomposition

\[ 1 = \sum_{k=0}^{[E:F]-1} 1^k \]

So we get \( V = \bigoplus_k V^k \), \( V^k := 1^k V \), w.l.o.g. s.t. \( V^{k+1} = V^k \pi_D \)

and \( V^{[E:F]-1} \pi_D = V^0 \).

**Remark 3** The skewfield \( \Delta := C_D(\mathfrak{i}(E)) \) is central over \( \mathfrak{i}(E) \) of index \( \frac{d}{[E:F]} \).

1. \( B \cong \text{End}_\Delta(V^0) \).

2. \( B \cong M_m(\Delta) \).
\[ j_E \text{ in terms of lattice functions 2} \]

**Theorem 9** [BL02, II 3.1.] *In terms of lattice functions \( j_E \) has the form*

\[
j_E^{-1}([\Theta]) = [\Lambda],
\]

*with*

\[
\Lambda(s) := \bigoplus_{k=0}^{f-1} \Theta(s - \frac{k}{d})\pi_D^k, \quad s \in \mathbb{R}.
\]
Barycentric coordinates
Orientation

For the simplicial complexes of $\mathcal{I}, \mathcal{I}_E$ we write $(\Omega, \leq), (\Omega_E, \leq)$. For the lattices corresponding to a face $H$ or point $x$ we write $\text{lattices}(H), \text{lattices}(x)$. We define an orientation on $\Omega_E$.

**Definition 11** An edge $H = \{e, e'\} \in \Omega_E$ is said to be oriented towards $e'$ if there are $\Gamma \in \text{lattices}(e)$ and $\Gamma' \in \text{lattices}(e')$, such that $\dim_{\kappa_D}(\Gamma/\Gamma') = 1$. (write $e_1 \rightarrow e_2$) An oriented chamber is a tupel $(e_1, \ldots, e_m)$ of $m$ different vertices which lie in a common chamber s.t. $e_i \rightarrow e_{i+1}$ and $e_m \rightarrow e_1$. 
Oriented barycentric coordinates type

Definition 12 Assume \( x \in \mathcal{I}_E \). An equivalence class of a tuple \( \mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+ \) is called the local type of \( x \), if there is an oriented chamber \( (e_1, \ldots, e_m) \) of \( \Omega_E \) such that \( x = \sum_{i=1}^{m} \mu_i e_i \).

Proposition 1 For \( x \in \mathcal{I}_E \) there is only one local type.
The theorem
**Definition 13** $m', t \in \mathbb{N}$. Take

$$w \in \text{Row}(m', t) := \{w \in \mathbb{N}_0^{m'} | \sum_i w_i = t\}, \ \text{i.e.}$$

$$w = (0, \ldots, 0, w_{i_0}, 0, \ldots, 0, w_{i_1}, 0, \ldots, 0, w_{i_k}, 0, \ldots, 0)$$

with $w_{i_j} > 0$, and we can represent $\langle w \rangle$ by a $(k + 1)$-tupel of pairs

$$(w_{i_0}, i_1-i_0), (w_{i_1}, i_2-i_1), \ldots, (w_{i_{k-1}}, i_k-i_{k-1}), (w_{i_k}, i_0+m'-1-i_k)$$

In this way we can map $\langle w \rangle$ to a class of a vector of pairs, which we denote:

$$\text{pairs}(\langle w \rangle) := \langle (w_{i_0}, i_1-i_0), (w_{i_1}, i_2-i_1), \ldots, (w_{i_k}, i_0+m'-1-i_k)$$
There is a duality map $<>^c$: $\text{Row}(m', t) \rightarrow \text{Row}(t, m')$.

**Definition 14** Given $w$ as above and pairs($< w >$) = $(a_0, b_0), \ldots, (a_k, b_k)$ we define the *complement of* $< w >$, denoted by $< w >^c$ to be the class $< w' >$, such that pairs($< w' >$) = $(b_0, a_1), (b_1, a_2), (b_2, a_3), \ldots, (b_k, a_0)$.

**Theorem 10 (S.)** Given $\alpha \in \text{Her}(A)^{E^\times}$ and a matrix $\lambda$ s.t. $< \lambda >$ is the embedding type of $(\alpha, E)$ and assume $< \mu >$ to be the local type of $j_E(M_\alpha)$, where $M_\alpha$ is the barycentre of the face corresponding to $\alpha$. $< \text{row}(\lambda) >$ is obtained as follows

1. $r f \mu \in \mathbb{N}_0^m$ and
2. $< \text{row}(\lambda) > = <fr\mu >^c$. 
Example

For example take \( r = 2, \ [E : F] = 6, \ \dim_D V = 7, \)

\[
j_E(M_\alpha) = \frac{3}{12} b_0 + \frac{2}{12} b_1 + \frac{1}{12} b_2 + \frac{0}{12} b_3 + \frac{0}{12} b_4 + \frac{4}{12} b_5 + \frac{2}{12} b_6.
\]

\[
< 12\mu > = < 3, 2, 1, 0, 0, 4, 2 > \\
\equiv < (3, 1), (2, 1), (1, 3), (4, 1), (2, 1) > \\
< 12\mu >^c \equiv < (1, 2), (1, 1), (3, 4), (1, 2), (1, 3) > \\
\equiv < 1, 0, 1, 3, 0, 0, 0, 1, 0, 1, 0, 0 > .\text{Applying theorem 10 we get the embedding data}
\]

\[
\begin{pmatrix}
1 & 0 \\
1 & 3 \\
0 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]
Bibliography

[BG00] P. Broussous and M. Grabitz. Pure elements and intertwining classes of simple strata in local central simple algebras. *COMMUNICATION IN ALGEBRA*, 28(11):5405–5442, 2000.

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