The Cooper Pair Pump as a Quantized Current Source

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A new charge quantization in a phase-polarized Cooper Pair Pump (CPP) is proposed, based on the topological properties of its Hamiltonian ground state over a three-dimensional parameter space \( P \). The charge is quantized using a set of path in \( P \) covering the surface of a torus, and is a multiple of the integer Chern index \( c_1 \) of this surface. This quantization is asymptotic but the pumped charge converges rapidly to the quantized value with the increase in the path frequency. The topological nature of the current makes this CPP implementation an excellent candidate for a metrological current standard.

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Topological defects occur in a number of different settings. In superfluids or magnets, the ground state is described by an order parameter. Topological defects (vortices, skyrmions...) are singular points of the order parameter field \( \Psi \). A topological charge is assigned to the mapping from physical space to order parameter space. In quantum systems with few degrees of freedom, the energy levels and eigenstates may depend on a set of continuous classical parameters (or band indices). When the number of parameters is sufficient (three for a complex Hamiltonian), isolated degeneracies between two neighboring levels (or bands) can occur. The presence of a degeneracy has many physical consequences. For quantum wavefunctions, the degeneracies can then be viewed as a defect in their phase field \( \tilde{\Psi} \). Some classical examples are well known in polyatomic molecules \( \Psi_0 \), where the nuclear coordinates form a set of semiclassical parameters (within the Born-Oppenheimer approximation) for the electronic structure. Energy manifolds can have conical intersections at isolated values of the nuclear coordinates \( \Psi_1 \). When a topological charge is assigned to these degeneracies, the change of multiplicity of rotation-vibrations forms a function of nuclear coordinates easily understood in topological terms. In molecular magnets, isolated degeneracies have also been found for specific directions and values of the magnetic field \( \Psi_2 \). The integer quantum Hall effect has also been interpreted in term of Chern indices \( \Psi_3 \) which are the total topological charge included within a filled Landau band. In all these examples, a topological charge is associated to an isolated degeneracy between neighboring bands, where the defect in the wavefunctions phase field occur. Its physics can be described by a “monopole” placed at the degenerate point in parameter space. The flux of the magnetic field produced by the monopole through any surface enclosing the defect is proportional to the Chern index for this surface. For a the two dimensional electron gas, the Hall conductivity of a band is \( \sigma_{xy} = \frac{e^2}{\pi} \times c_1 \) where \( c_1 \) is the Chern index assigned to the band \( \Psi_4 \). The phase change of a wavefunction on a closed path is proportional to the integral of the potential vector, which is also the flux through the surface bounded by the path. This phase change, better known as Berry’s phase, takes a particular value \( (\pi) \) for some of the physically relevant paths enclosing degeneracies. In molecular magnets, this phase change leads to destructive interferences which quench quantum tunnelling at the degeneracy points.

In this letter, the relevance of this physics to superconducting circuits is illustrated around an example, the Cooper Pair Pump (CPP), which lowest two energy levels form two bands which depend on continuous parameters (gate charges or voltages and magnetic flux). The parameter space is in this case three dimensional and degeneracies (defects) occur at isolated points in parameter space. We demonstrate how a charge quantization, proportional to the defect topological charge, can be exploited to produce a current source with metrological accuracy.

The CPP-circuit represented in Fig. 1 includes a small inductance in serie with the Cooper pair pump. The CPP has three Josephson junctions in series which define two superconducting islands. As long as the device is dominated by the Coulomb charging energy \( E_C = \frac{(2e)^2}{4c} \) of the islands, the topological properties of the circuit do not

![FIG. 1: The Cooper Pair Pump.](image-url)
bias is set by the "classical" phase energy can be tuned through three independent parameters: the two lowest levels as a function of the gate polarization charge induced by two gate voltages $V_1$ and $V_2$. The electrostatic charges on each island are $n_1 - n_{g1}$ and $n_2 - n_{g2}$, where $n_{g1}$ and $n_{g2}$ are classical variables tuning the islands electrostatic energies. The inductance $L$ serves two purposes: the phase $\phi$ across the CPP is controlled by the magnetic flux $\Phi$ it threads, and it is also used as the input for the readout circuit measuring the circulating current through the pump. For a small inductance, the magnetic energy $\frac{1}{2L} \left( \dot{\phi} - 2\pi \frac{\Phi}{\Phi_0} \right)^2$ defines a sufficiently deep potential minimum: quantum fluctuations of $\phi$ across the CPP are quenched and the phase bias is set by the "classical" phase $\phi = 2\pi \frac{\Phi}{\Phi_0}$. The circuit energy can be tuned through three independent parameters $n_{g1}, n_{g2}$ and $\varphi$ defining the parameter space $P$.

Charging $\mathcal{H}_C$ and Josephson $\mathcal{H}_J$ Hamiltonian contribute to the CPP energy. It is convenient to write the charging energy in term of $\hat{n}_s = \hat{n}_1 + \hat{n}_2$, the total charge on the CPP and $\hat{n}_d = \hat{n}_1 - \hat{n}_2$, the charge imbalance between the two islands. Their canonical conjugates, the phase variables $\hat{\Theta}_s$ and $\hat{\Theta}_d$ enter in the Josephson Hamiltonian. In term of $\hat{n}_s$ and $\hat{n}_d$ the charging Hamiltonian is

$$\mathcal{H}_C = E_C \left( \left( \hat{n}_s - n_{gs} \right)^2 + \kappa_0 \left( \hat{n}_d - n_{gd} \right)^2 \right),$$

where $E_C = \frac{\left( 2e \right)^2}{4\kappa_0}$, $n_{gs} = n_{g1} + n_{g2}$, $n_{gd} = n_{g1} - n_{g2}$ are the corresponding polarization charges and $\kappa_0 = \frac{1}{\sqrt{2}}$ is a capacitance ratio. Each eigenstate $| \hat{n}_s, n_{gd} \rangle$ is the ground state of $\mathcal{H}_C$ inside an hexagonal area $h(n_s, n_{gd})$, centered at the point $(n_{gs} = n_{g1}, n_{gd} = n_{g2})$, in the $n_{gs}$-$n_{gd}$ plane (shown as the base plane in Fig. 3). Its boundaries are electrostatic degeneracy lines between "neighboring" charge states, and the vertices are points of triple degeneracies. The vertices $\{ T_1 \} = (n_{gs} = \frac{1}{4}, n_{gd} = 0)$ and $\{ T_2 \} = (n_{gs} = \frac{3}{4}, n_{gd} = 0)$ form the unit cell of this hexagonal lattice. With an appropriate gauge choice, the Josephson Hamiltonian $\mathcal{H}_J$ depends on the phase bias $\varphi$ as

$$\mathcal{H}_J = -2E_J \cos \hat{\Theta}_s \cos \hat{\Theta}_d - E_J \cos(2\hat{\Theta}_d + \varphi).$$

The total Hamiltonian is 2π-periodic in $\varphi$. Lattice translations in the hexagonal lattice $(n_{gs}-n_{gd})$ connect equivalent but physically different states. Because the parameter space $P$ is three dimensional, $\mathcal{H}_J$ lifts the electrostatic degeneracies except at isolated points where the ground level is still degenerate, in $P$ (a consequence of the von Neumann-Wigner theorem[10]). These points occur at the vertices $\{ T_1 \}$ and $\{ T_2 \}$ of the hexagonal lattice for $\varphi = \pi$, as illustrated in Fig. 3. In realistic implementation of the Cooper pair pump, it is favorable to keep the ratio between Josephson and charging energy $\beta = \frac{E_J}{E_C}$ small, an assumption made throughout this paper.

In the limit of zero Josephson coupling, the electrostatic states $|0,0\rangle, |1,1\rangle$ and $|1,-1\rangle$ are degenerate at the vertex $\{ T_1 \}$. For finite Josephson couplings, the ground state remains degenerate at this point when $\varphi = \pi$. The two degenerate ground states at $\{ T_1 \} = \{ T_1, \pi \}$ are

$$| g_{\pm} \rangle = \frac{1}{\sqrt{3}} \left[ |0,0\rangle + \sqrt{3} |1,1\rangle \pm \frac{\sqrt{3} \pm 1}{2} |1,-1\rangle \right]$$

with energy $E_g(\{ T_1 \}) = \frac{E_J}{2} + \frac{1}{2} E_C - \frac{1}{2} E_J$ while the first excited state $| \psi \rangle$ is well separated from the ground state ($E_\varphi = E_g + \frac{\delta E_J}{2} \psi$). For a small deviation $\delta \mathbf{R} = (\sigma = n_{gs} - \frac{1}{4}, \beta = n_{gd}, \psi = \varphi - \pi)$ from degeneracy $\{ T_1 \}$, a two-level approximation $(| g_{\pm} \rangle)$ is appropriate for small $\beta$. The eigenenergies $E_{\pm}(\delta \mathbf{R}) = E_g \pm \frac{\delta E_J}{2} \pm \frac{\delta}{2} E_C \sqrt{\sigma^2 + \frac{3}{4} + \frac{1}{16} \beta^2 \psi^2}$ display a characteristic conical intersection at $\{ T_1 \}$. After rescaling the deviation $\delta \mathbf{R}$ from $\{ T_1 \}$, as $b_x = \frac{\sqrt{2}}{2} E_J \sigma, b_y = -\frac{\sqrt{2}}{4} E_J \psi, b_z = \frac{\sqrt{2}}{4} \psi \delta$, the two-level Hamiltonian maps onto an isotropic spin-$\frac{1}{2}$ problem,

$$\tilde{\mathcal{H}}_{\pm}(\mathbf{R}) = E_g \mathbf{1} + \frac{1}{2} \sigma \cdot \mathbf{b}(\mathbf{R}).$$

This makes the conical nature of the intersection explicit ($E_{\pm}(\delta \mathbf{R}) = E_g \pm |b_3|/2$), and spinor eigenstates can be specified using the direction of $\mathbf{b}$ rather than $\delta \mathbf{R}$.

The topological nature of this degeneracy becomes clear when considering the adiabatic evolution of quantum states along different closed paths $\Gamma$ "around" this

FIG. 2: (Color) Plot of the energies (in units of the charging energy) of the two lowest states as a function of the gate charges $n_{gs}$ and $n_{gd}$ at $\varphi = \pi$ in the vicinity of the vertex $\{ T_1 \}$. This "diabolical point" is a conical intersection of the energy sheets between the two lowest bands. It represents a topological defect in the wavefunction phase fields and cannot be removed by any perturbations which merely shift its position in parameter space.
closed paths considered are sufficiently close to the de-  
spin-
 proactive eigenstates of the Hamiltonian, after a period  
γ  
evolution of a point  
depends on two phases,  
the parameter space  
direction of  
this cylinder has the topology of a torus, and its surface  
FIG. 3: Representation of the degeneracy point  \{T_1, \varphi = \pi\} in  
the parameter space \(P\). The cylindrical surface which encloses  
this point has the topology of a torus. The Chern index is  
defined as the flux through this surface normalized by 2\pi.  
Several paths on this cylinder are considered in the text.

point. One of them is represented in Fig. 3 as the time  
evolution of a point \(R(t)\) in \(P\). If  \(|g_{\pm}(R(t))\rangle\) are instan-
taneous eigenstates of the Hamiltonian, after a period  
\(T\) around any closed path  \(|g_{\pm}(T)\rangle = e^{-i(\eta_0 - \gamma_\pm)}|g_{\pm}(0)\rangle\)  
depends on two phases,  
\(\eta_\pm = \frac{1}{\pi} \int_0^T E_{g_{\pm}}(R(t)) dt\), the dyna-

mical phase and  \(\gamma_\pm = i \oint R \cdot A_{\pm}(R) \cdot dR\), the geometrical  
phase. It is expressed as an integral of the vector potential  
\(A_{\pm}(R) = \langle g_{\pm}(R) | \nabla g_{\pm}(R) \rangle\) over the path. If the  
closed paths considered are sufficiently close to the de-
generacy \(T_1\) where the two-level approximation is valid,  
Berry’s geometrical phase can also be computed using the  
spin-- representation as  \(\gamma_\pm = i \oint A_{\pm}(b) \cdot db\), where the  
vector potential is azimuthal in this textbook example:  
\(A_{\pm}(b) = \pm \cos \theta \sin \varphi \epsilon_\theta\) (the Euler angles \(\theta, \varphi\) specify  
the direction of \(b\)). Its line integral over a closed path \(\Gamma\) is  
\(\gamma_\pm(\Gamma) = \pm \frac{\Omega(\Gamma)}{2}\), where  \(\Omega(\Gamma)\) is the solid angle seen  
from the degeneracy. The vector potentials  \(A_{\pm}(b)\) can be  
recognized as the gauge field of Dirac monopoles of strength  
\(\mp 1\) placed at \(T_1\). They produce a radial magnetic field  
\(B_{\pm} = \pm \frac{\epsilon_{\varphi}}{2\pi R}\), which flux is \(\mp 2\pi\) when integrated over any  
surface enclosing this single degeneracy. When normal-
ized to 2\pi, this produces the Chern index  \(c_{\pm}^{(\pi)} = \mp 1\)  
of the surface with respect to the eigenstates  \(|g_{\pm}\rangle\) and  
one assigns to the degeneracy \(T_1\) topological charges  
\(q_{\pm}(T_1) = \mp 1\).

This topological picture is not restricted to the im-
mediate neighborhood of \(T_1\) where a two-level approx-
imation can be made. Let us consider the cylindrical  
surface represented in Fig. 3. Since the top and bottom  
planes \(\varphi = 0\) and \(\varphi = 2\pi\) are physically equivalent,  
this cylinder has the topology of a torus, and its surface  
is the constant radius \(|R| = \rho\) cylinder. The magnetic  
flux of  \(B_- = \nabla \times A_-\) through this closed surface de-

fines its Chern index  \(c_{\pm}^{(\pi)}\) of the ground state  \(|g_{\pm}(R)\rangle\).  
This index is non-zero only if the topology is non trivial,  
i.e. if the surface encloses degeneracies. Here the “cylin-
drical surface” encloses only the isolated degeneracy \(T_1\)  
and Gauss theorem guarantees that its index is identical  
to the one of the small sphere around \(T_1\), just consid-
ered. For a cylinder around \(T_2 = (\{T_2, \pi\})\) the Chern  
indices have opposite sign and  \(q_{\pm}(T_2) = \pm 1\). For a larger  
surface, the Chern index is the sum of the topological  
charges it encloses and when no degeneracies are present,  
it is zero.

In a CPP, pumping of charges through the circuit is  
achieved when moving the ground state  \(|g_{\pm}(R)\rangle\) adia-
batically along a one-dimensional path in \(P\). In the liter-
ature, the paths considered so far have been closed tra-
jectories around \(\{T_1\}\) or \(\{T_2\}\) at constant \(\varphi\) for which  
the charge transferred is not quantized. When the path  
chosen is to cover a surface enclosing one of the singular-

arities, the charge transferred becomes an integer multiple  
of the ground state Chern index. This quantization is not  
only observable but can be exploited to make a current  
standard of metrological accuracy, in a similar way as the  
quantum Hall effect is a standard of resistance based on  
the Chern indices of Landau bands. Let us consider a  
helical path  \(\Gamma_{\varphi_0}\) on the surface of the cylinder (Fig. 3)  
starting from the plane  \(\varphi = \varphi_0\) and ending in the plane  
\(\varphi = \varphi_0 + 2\pi\) while making an integer number  \(p\) of wind-
ings (in Fig. 3  \(\varphi_0 = 0\) and  \(p = 2\)). The pumped charge  
\(Q_{\text{pump}}\) on this path has a dynamical  \(Q_{\text{dyn}}\) and a geometrical  
contribution  \(Q_{\text{geo}}\). The expressions

\[
Q_{\text{dyn}} = 2e \frac{d\gamma}{d\varphi_0}(\Gamma_{\varphi_0}), \quad Q_{\text{geo}} = -2\pi \frac{d\gamma}{d\varphi_0}(\Gamma_{\varphi_0}), \quad (5)
\]

proved elsewhere[13], generalize earlier results[13, 14] to  
arbitrary three dimensional closed paths in \(P\). We first  
show that the geometrical contribution is quantized when  
averaged over the initial phase \(\varphi_0\) of the helix. Taking  
two helices  \(\Gamma_{\varphi_0}\) and  \(\Gamma_{\varphi_0 + \delta\varphi}\) shifted in  \(\varphi\) by an infinites-
imal increment \(\delta\varphi\), a closed path  \(\Sigma(\varphi_0)\) on the surface  
of the cylinder (shown in Fig. 3) can be constructed by  
adding two small opposite “vertical” segments connecting  
the paths  \(\Gamma_{\varphi_0}\) and  \(\Gamma_{\varphi_0 + \delta\varphi}\) which line integrals cancel.  
Using Eq. 3 the transferred charge is

\[
Q_{\text{geo}} = 2e \frac{\gamma_-(\Gamma_{\varphi_0}) + \gamma_-(\Gamma_{\varphi_0 + \delta\varphi}) - \gamma_-(\Sigma(\varphi_0))}{\delta\varphi} = 2e \frac{\gamma_-(\Sigma(\varphi_0))}{\delta\varphi}. \quad (6)
\]

Using Stokes theorem, the above numerator is also  
\(\oint_{\delta S} B_-(R) \cdot \mathbf{n} dS\), where  \(\delta S\) is the surface of the strip  
between the two helices (shown in Fig. 3 for  \(\varphi_0 = 0\)) and  
\(B_-(R)\) is the “magnetic” field produced by the monopole  
at the degeneracy. When the initial angle  \(\varphi_0\) is integrated  
from 0 to 2\pi, adding all the elementary strips  \(\delta S\) gener-
In order to quantify the error in the transferred charge, we introduce the function $Q_{\phi_0}^\text{dyn}(\Gamma_{\phi_0})$ as a function of the number of windings $p$ in the interval $[\phi_0; \phi_0 + 2\pi]$ for a charging energy of 10 meV, a Josephson coupling $E_J = 0.5 E_C$ and a period of $10^{-8}$ sec.

$Q_{\phi_0}^\text{dyn}(\Gamma_{\phi_0})$ quantizes by the Chern index $c$ of the number of windings $p$. If the number of windings $p$ is quantized by the Chern index $c$, it is not completely controlled, or the path may not close periodically and odd-dependence of $Q_{\phi_0}(\Gamma_{\phi_0})$ is a 2\pi-periodic function of $\phi_0$ since the Hamiltonian is 2\pi-periodic in $\varphi$. Its magnitude $\approx \frac{\tau}{\hbar} E_N E_J$ depends critically on the ratio $\beta = \frac{E_N}{E_J}$, however the periodic and odd-dependence of $Q_{\phi_0}^\text{dyn}$ with $\phi_0$ guarantees that its $\phi_0$-average is zero. In realistic situations, $\phi_0$ may not be completely controlled, or the path may not close perfectly because of flux-noise. It is therefore instructive to plot the evolution of the peak amplitude of $Q_{\phi_0}^\text{dyn}$ as a function of the number of winding $p$ in the $\varphi$-period. In Fig. 3, this peak amplitude looses four order of magnitudes when $p$ is increased by 10, using a ratio $\beta = 0.5$ is an order of magnitude larger than its optimal value. For smaller $\beta$, the decrease is even more dramatic. Hence, the CPP delivers a very accurate DC current

$$P_{\text{pump}} = Q_{\text{pump}}^\nu \nu^{-\infty} = 2e \nu,$$

where $\nu$ is the winding frequency.

We now consider the possible sources of errors in a practical implementation of this circuit as a current standard. Low frequency noise in the parameters will introduce some jitter in the path covered in $\Phi$: the surface generated by the path will not be a perfect cylinder. On the other hand, as long as the jitter amplitude is small compared to the radius, the surface generated still encloses the topological charge, and no significant error in quantization should result. The most detrimental source of error comes from inelastic transitions between the ground state and the first excited state in the adiabatic evolution, because their topological charges are opposite. This means that if the time spent in the first exited state is $\tau$ in a period, the relative error in the transferred charge will be $-4e \pi \frac{\tau}{\hbar}$. There are two ways this can happen. For each winding around $\Gamma_1$, three saddle points between the ground and first excited states have to be crossed (along the hexagonal lines shown in Fig. 3), where the probability of Landau-Zener tunnelling is largest. For the optimal radius ($\approx \frac{\pi}{4}$) this tunnelling probability is of order $P \approx e^{-\left(\frac{\tau}{\pi}\right)^2 \frac{E_N}{E_J} \frac{E_J}{\hbar} \nu}$. To stay below the part per million level requires that the winding frequency $\nu$ to stay below 100 MHz for a Josephson coupling of 0.05 $E_C$, limiting the pumped current amplitude 2$e\nu$ to 32 pA. The other source of inelastic transitions to the first excited state is the back-action of the measuring device on the CPP, which needs to be rigourously controlled.

The physics discussed in this paper is by no means limited to the CPP: as long as a circuit has three tunable parameters or more, degeneracies can occur. In this paper, we emphasized the topological quantization and its strong robustness to adiabatic parameters fluctuations, a key point for metrological applications. The realization of this metrological source is under way in our group: it is a beautiful scientific challenge for fundamental quantum electronics of real practical value. R. Leone is supported by a fellowship of the Rhône-Alpes Micro-Nano cluster. The authors are grateful to F. Faure for enlightening discussions.