A LAZY APPROACH TO ON-LINE BIPARTITE MATCHING

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Abstract. We present a new approach, called a lazy matching, to the problem of on-line matching on bipartite graphs. Originally, each arriving element is irrevocably matched with some already existing one. In this paper we allow matching a new element to a group of elements (possibly decreasing other groups) with additional restrictions that every two groups have to be disjoint and decreasing a group is allowed as long as it stays not empty. We present a deterministic optimal algorithm and prove that its competitive ratio equals $1 - \pi / \cosh(\frac{\sqrt{3}}{2}) \approx 0.588$. The lazy approach allows to break the barrier of $1/2$, which is the best competitive ratio that can be guaranteed by any deterministic algorithm in the classical on-line matching.

1. Introduction

Many problems of task-server assignment can be modeled as finding a matching in a bipartite graph. Vertices of one part correspond to servers, and the vertices of the other to tasks. An edge between a task and a server indicates that the server is capable of performing the task. In a simple setting, when one server can realize at most one task, the problem of maximization of the number of realized tasks reduces to finding a maximum matching. In real-life applications it is very common that not all tasks are known a priori and some decisions about assignments have to be taken with no knowledge about future tasks. A simple model for this situation is on-line bipartite matching. In this setting servers are known from the beginning and tasks are revealed one by one. The decision about assignment of each task has to be made just after its arrival and cannot be changed in the future. Suppose that there are $n$ servers, $n$ tasks are going to be revealed, and capabilities of servers are such that it is possible to realize all the tasks. (i.e. there exists a perfect matching in the tasks-servers graph). It is an easy exercise to show that even with these restrictions it is possible to present tasks in such a way, that the constructed assignment is at most $\lceil n/2 \rceil$. On the other hand, any greedy assigning strategy guarantees that at least half of the tasks will be assigned.

In the classical paper [11], authors take other approach in which the graph to be presented is fixed before the first task is presented. In particular the presented graph does not depend on the decisions of the assigning algorithm. It does not make any difference for the worst-case analysis of the algorithm, but it provides a framework for analyzing randomized ones. The authors presented a randomized algorithm which on the average...
constructs a matching of size at least about \((1 - 1/e)n\). They also argued that the result is asymptotically best possible for any randomized algorithms (the original paper [11] contained a mistake, which has been corrected in [8], see also [4] for a simple exposition). The approach of [11] has been applied to many variants of the original problem, with various practical applications (switch routing problem [1, 2], on-line auctions [13] and Adwords problem [5, 8, 15] etc.) Recently a lot of interest is put into a problem of on-line stochastic matching [3, 6, 10, 13, 14] where a competitive ratio can be even greater then \(1 - \frac{1}{e}\). A different approach (called \(b\)-matching) is presented in [9] where authors allow a server to realize up to \(b\) tasks at the same time. They showed an optimal deterministic algorithm with competitive ratio \(1 - \frac{1}{(1 + \frac{1}{b})^b}\) (which tends to \(1 - \frac{1}{e}\) with \(b \to \infty\)).

The approach proposed in this paper (called \(\alpha\)-lazy matching) is based on another relaxation of the original assumptions. In some application, e.g. when the cost of maintaining idle server and sever realizing some tasks is roughly the same, it might be profitable to start realization some task by many servers (at most \(\alpha\)). If it is sufficient that only one of these servers accomplishes the task, we can reassign all but one of them to different tasks if there is such need in the future. This enables to postpone the decision about which server is going to accomplish given task. Exploiting this possibility we present a deterministic algorithm which attains a worst-case fraction of assigned tasks (i.e. the competitive ratio) of \(1 - \frac{\alpha}{1 + \frac{\alpha}{\pi}} \prod_{i=1}^{\alpha} \frac{1 + \frac{\alpha}{\pi}}{1 + \frac{\alpha}{\pi}}\) (which tends to \(1 - \frac{\pi}{\cosh(\sqrt{3}\pi)} \approx 0.588\) when \(\alpha \to \infty\)). Moreover, we prove that the obtained ratio is best possible.

The lazy approach was first introduced by Felsner in [7] (by the name of adaptive) as a generalization of the on-line chain partitioning problem. He noted that the adaptive on-line chain partitioning of up-growing partial orders is equivalent to the on-line dimension of up-growing partial orders. A corrected proof of that observation can be found in [12].

1.1. Problem definition. For a positive integer \(\alpha\), an \(\alpha\)-lazy matching game is played in rounds between Scheduler and Builder. At the beginning of the game a set of vertices \(D\) is revealed. Then in each round:

1. Builder presents one vertex \(u\) and a set of its neighbors \(N(u) \subseteq D\).
2. Scheduler assigns to \(u\) a set \(m(u) \subseteq N(u)\) of size at most \(\alpha\).
3. For each previously presented vertex \(x\), the assigned set is updated as follows
   \[ m(x) := m(x) \setminus m(u). \]

The set of vertices presented by Builder during the game is denoted by \(U\). Note that at each moment of the game each vertex \(d \in D\) is assigned to at most one vertex \(u \in U\). After the game, the sets \((N(u))_{u \in U}\) interpreted as neighbourhoods of vertices from \(U\) provide a structure of bipartite graph denoted by \(G = (U, D, E)\). We say that \(\alpha\)-lazy matching game has size \(n\) if the size of the maximum matching of \(G\) is \(n\). If we drop the restriction on the limit size of \(m(u)\) then the game is called \(\infty\)-lazy matching or just lazy matching.

We say that the move of Scheduler (2) is illegal if a nonempty set \(m(x)\) becomes empty after (3), otherwise it is legal. By an analogy to the classical setting, we restrict Scheduler to make only legal moves (hence a task that has been started is going to be eventually realized by at least one server). The assumption is insignificant since the proof from section 2 works also for case with illegal moves and the optimal algorithm presented in section 3 makes only legal moves.
The goal of Scheduler is to maximize the number of vertices \( u \in U \) with non-empty set \( m(u) \) after all vertices from \( U \) has been presented. We refer to this number as the size of the matching constructed by Scheduler. The goal of Builder is to make this number as small as possible.

Vertices from \( D \) can be interpreted as servers, vertices from \( U \) as tasks and neighborhood \( N(u) \) as the servers capable of realizing task \( u \). Each new task \( u \) should be assigned to some set of servers \( m(u) \) each of them being capable of realizing it. The quality of Scheduler is measured by the number (or the fraction) of tasks which are being realized at the end of the game. Note that the game reduces to classical on-line bipartite matching for \( \alpha = 1 \).

Let \( A \) be an algorithm which assigns incoming tasks. By \( \text{val}_A(n) \) we denote the worst case (maximum) of matching constructed by \( A \) in all possible games of size \( n \). The value of the \( \alpha \)-lazy matching problem \( \text{val}_\alpha(n) \) is the minimum value of \( \text{val}_A(n) \) among all \( \alpha \)-assigning algorithms \( A \). Since no algorithm produces matching larger that \( n \) we additionally use a competitive ratio defined as \( \liminf_{n \to \infty} \frac{\text{val}(\alpha)}{n} \).

1.2. Main results. We solve the problem of \( \alpha \)-lazy on-line matching by presenting an optimal algorithm, called \( \alpha \)-BALANCED, with competitive ratio \( 1 - \frac{\alpha}{1+\alpha} \prod_{i=1}^{n-1} \frac{i+\alpha}{1+i+\alpha} \).

For \( \alpha = \infty \) we obtain ratio \( 1 - \prod_{i=1}^{\infty} \frac{i+2}{1+i+\frac{\pi}{2}} \), which turns out to have closed form \( 1 - \frac{\pi}{\cosh \frac{\sqrt{3}}{2}} \approx 0.588 \). The algorithm is deterministic and greedy, i.e., no task is rejected if there is a possibility to run it. For \( \alpha = 2, 3 \) we get the competitive ratio \( \frac{5}{9} \approx 0.556 \) and \( \frac{4}{7} \approx 0.571 \), respectively.

The proof of optimality of \( \alpha \)-BALANCED is split into two parts. The following schema of system of inequalities is crucial for our arguments:

\[
\begin{aligned}
(1 + \alpha)x_0 &\leq n, \\
(x_0 + \ldots + x_i)(1 + x_i) &\leq n - i, \quad i = 1, \ldots, k, \\
x_1 &\geq x_2 \geq \ldots \geq x_k \geq 0, \\
x_0 + \ldots + x_k &\geq 0.
\end{aligned}
\]

We are going to work with \( n \) and \( \alpha \) fixed. Then, the schema is parametrized by a positive integer \( k \). We say that a pair \((k, x)\) satisfy system (1) if \( x = (x_0, x_1, \ldots, x_k) \) is an integer vector satisfying instance of the schema for this particular \( k \). In Section 2 we prove that among all graphs with maximum matching of size \( n \) any \( \alpha \)-lazy algorithm constructs a matching of the size at most \( n - (x_0 + \ldots + x_k) \) for pair \((k, (x_0, \ldots, x_k))\) satisfying (1). Hence, to find the best upper bound one has to find such pair with maximal sum \( x_0 + \ldots + x_k \). On the other hand, to prove optimality of \( \alpha \)-BALANCED we show in Section 3 that if \( \alpha \)-BALANCED constructed a matching of size \( n' \) during a game of size \( n \), then there exists a pair \((k, (x_0, \ldots, x_k))\) satisfying (1) such that \( n' = n - (x_0 + \ldots + x_k) \). In Section 4 we determine the competitive ratio of \( \alpha \)-BALANCED.

2. Worst case scenario for a lazy algorithm

Inequalities (1) allow \( x_0 \) to be negative. We start with a technical proposition, which implies that in order to maximize the sum \( x_0 + \ldots + x_k \) it is enough to consider only solutions of (1) with \( x_0 = \lfloor \frac{n}{1+\alpha} \rfloor \).
Proposition 2.1. For any pair \((k, x)\) satisfying (1) there exists a pair \((k, x')\) satisfying (1) such that
- \(x_0' = \left\lfloor \frac{n}{1 + \alpha} \right\rfloor\),
- \(x_i' \leq x_i\) for \(0 < i \leq k\),
- \(x_0' + \ldots + x_k' \geq x_0 + \ldots + x_k\).

Proof. The case when \(x_1 = 0\) is trivial. We assume that \(x_1 > 0\). The claim is proved by induction on \(\left\lfloor \frac{n}{1 + \alpha} \right\rfloor - x_0\). The base, when \(x_0 = \left\lfloor \frac{n}{1 + \alpha} \right\rfloor\), is obvious. For the induction step let us assume that \(x_0 < \left\lfloor \frac{n}{1 + \alpha} \right\rfloor\). Let \(j\) be the greatest index for which \(x_j = x_1\). Consider the sequence \((x_0', \ldots, x_k')\) satisfying \(x_0' = x_0 + 1, x_j' = x_j - 1\) and \(x_i' = x_i\) for all positive \(i \neq j\). For \(i \geq j\) we have \(x_0' + \ldots + x_i' = x_0 + \ldots + x_i\). Therefore
\[
(x_0' + \ldots + x_i')(1 + x_i) \leq (x_0 + \ldots + x_i)(1 + x_i) \leq n - i.
\]
For positive \(i < j\) we get
\[
(x_0' + \ldots + x_i')(1 + x_i) = (x_0 + \ldots + x_i + 1)(1 + x_j) \leq (x_0 + \ldots + x_{j-1} + x_j)(1 + x_j) \leq n - j \leq n - i.
\]
Since \(x_0' \leq \left\lfloor \frac{n}{1 + \alpha} \right\rfloor\) sequence \((k, (x_0', \ldots, x_k'))\) satisfies (1). Finally, \(\left\lfloor \frac{n}{1 + \alpha} \right\rfloor - x_0' < \left\lfloor \frac{n}{1 + \alpha} \right\rfloor - x_0\), so by the induction hypothesis there exists a solution \((k, (x_0'', \ldots, x_k''))\) of (1) satisfying the claim. \hfill \(\square\)

Proposition 2.2. For any pair \((k, x)\) satisfying (1) there exists a strategy for Builder in the \(\alpha\)-lazy matching game of size \(n\) such that any Scheduler constructs matching of size at most \(n - (x_0 + \ldots + x_k)\).

Proof. All the following arguments does not depend on the legality of Scheduler’s moves.

By Proposition 2.1 it is enough to consider pairs with \(x_k > 0\). Without loss of generality we assume that \(x_k > 0\) and describe a strategy for Builder, that does not allow Scheduler to construct matching larger than \(n - (x_0 + \ldots + x_k)\). Regardless of the strategy of Scheduler, graph \(G = (U, D, E)\) presented by Builder, is going to have a specific structure. Both sets \(U\) and \(D\) will have cardinality \(n\). During the game Builder builds partitions of \(U\) and \(D\) satisfying:
- \(U = U_0 \cup U_1 \cup \ldots \cup U_k \cup R\),
- \(D = D_0 \cup D_1 \cup \ldots \cup D_k \cup S\),
- \(|U_0| = |D_0| = x_0\),
- \(|U_i| = |D_i| = 1 + x_i\) for \(i = 1, \ldots, k\),
- \(N(u_i) = D - (D_0 \cup \ldots \cup D_{i-1})\) for each \(u_i \in U_i\),
- \(N(r) = S\) for each \(r \in R\).

Observe that (1) guarantees that \(x_0 + \ldots + x_k \leq n - k\) and thus it is possible to partition sets \(U\) and \(D\) as described above. It is also straightforward to verify that any bipartite graph which can be partitioned in such a way contains a perfect matching.

The strategy of Builder is divided into \(k + 2\) phases. In the beginning of the \(i\)-th phase \((i \leq k)\) sets \(U_j\) and \(D_j\), for \(j < i\), are already fixed. During the \(i\)-th phase (for
We need to find one more vertex $y$ with neighborhoods defined by (2). Then, Builder chooses set $D_i \subseteq D - (D_0 \cup \ldots \cup D_{i-1})$ of size $x_i + 1$ (or $x_0$ when $i = 0$), and proceeds to the phase $i + 1$. The choice of $D_i$ is described below. In the beginning of the last phase, sets $R$ and $S$ are already determined. Builder presents the vertices from $R$ with neighborhoods defined by (3).

The strategy of Builder is to choose $D_i$ in such a way that the following condition is satisfied:

\[ (\star) \text{ Each phase } i \leq k \text{ there exist } i \text{ different vertices } y_1, \ldots, y_i \in U_0 \cup \ldots \cup U_i \text{ such that } D_j \cap m(y) = \emptyset \text{ for any } 0 \leq j \leq i \text{ and } y \in U \setminus \{y_1, \ldots, y_i\}. \]

Observe that for a fixed $X \subseteq D$ and a fixed $y \in U$ once the condition $X \cap m(y) = \emptyset$ is satisfied it will stay so to the end of the game. It is true also in the case of illegal moves. Therefore, we prove by induction that choosing such $D_i$ is possible in every phase by successively finding correct $y_i$’s. For the case $i = 0$ recall that $|m(y)| \leq \alpha$ for all $y \in U_0$. To satisfy $(\star)$ Builder picks any $D_0 \subseteq D - m(U_0)$ of size $x_0$ since by \footnote{Let $m(x) = \bigcup_{x \in X} m(x)$ for a set $X$.} we have $|D - m(U_0)| \geq n - \alpha x_0 \geq x_0$.

Suppose now that $(\star)$ holds after phase $i - 1 \geq 0$ with vertices $Y = \{y_1, \ldots, y_{i-1}\}$. We need to find one more vertex $y_i$ after phase $i$. Let $U' = U_0 \cup \ldots \cup U_i$ and $D' = D - (D_0 \cup \ldots \cup D_{i-1})$. Consider set $X$ of all vertices in $D'$ which are not matched to a vertex in $U' - Y$, i.e. $X = D' - m(U' - Y)$. Thus, every vertex $d \in D' - X$ belongs to $m(u)$ for some $u \in U' - Y$. Hence the average size of $m(u)$ for $u \in U' - Y$ is

\[
\frac{|D'| - |X|}{|U' - Y|} = \frac{n - i - |X| - (x_0 + \ldots + x_{i-1} - 1)}{x_0 + \ldots + x_i + 1} \geq \frac{(x_0 + \ldots + x_i)(1 + x_i) - |X| - (x_0 + \ldots + x_{i-1} - 1)}{x_0 + \ldots + x_i + 1} = x_i + 1 - \frac{|X|}{x_0 + \ldots + x_i + 1} > x_i - |X|.
\]

Therefore, there exists a vertex $y_i \in U' - Y$ such that $|m(y_i)| \geq 1 + x_i - |X|$ and thus $|X \cup m(y_i)|$ equals at least $1 + x_i$. It is enough to pick for $D_i$ any subset of $X \cup m(y_i)$ of size $x_i + 1$, to keep the property $(\star)$ satisfied after the $i$-th phase.

The condition $(\star)$ after the $k$-th phase gives $Y = \{y_1, \ldots, y_k\}$ such that $D - S$ and $m(y)$ are disjoint for any $y \in U - Y$. Therefore, if $m(u) \neq \emptyset$ for some $u \in U$ then either $u \in Y$ or $m(u) \subseteq S$. It means that the number of such $u$’s is at most $|Y| + |S| = k + n - (|D_0| + \ldots + |D_k|) = n - (x_0 + \ldots + x_k)$. Consequently, the size of the matching produced by Scheduler is at most $n - (x_0 + \ldots + x_k)$.

\[ \Box \]

3. The best matching algorithm

At any moment during a lazy matching game, we say that $d \in D$ is available for $u \in U$ if $d \in N(u)$ and $m(x) \neq \{d\}$ for any $x$ presented earlier. Also, $d$ is strongly available for $u$ if it is available for $u$ and $d$ does not belong to any $m(x)$. Vertex $e \in U$ is ready for $u$ if $m(e)$ contains an element which is available for $u$.

We present an algorithm for Scheduler called $\alpha$-BALANCED. Suppose that vertex $u$ has just been presented and let $U'$ be the set of vertices presented so far (including $u$).
Each set $m(x)$ for $x \in U - \{u\}$ is already known and the algorithm has to construct set $m(u)$. The construction is described below – $m(u)$ is increased, one element at a time, until a certain condition is satisfied. During the process some other sets $m(x)$ may be decreased.

### Algorithm 1: $\alpha$–BALANCED($u$)

1. let $m(u) := \emptyset$
2. pick up at most $\alpha$ strongly available elements for $u$ and put it into $m(u)$
3. **while** there exists a vertex $e \in U$ that is ready for $u$ and satisfy $|m(u)| + 2 \leq |m(e)|$
   - do
     - from the set of all such vertices pick $e$ with maximal size of $m(e)$
     - move one vertex available for $u$ from $m(e)$ to $m(u)$

Condition in line 2 guarantees that the size of $m(u)$ will be at most $\alpha$. Note that $\alpha$-BALANCED never leaves $m(u)$ empty if there exists an available element for $u$, so in this respect $\alpha$-BALANCED can be considered as greedy. For $\alpha = 1$ the above algorithm is just a simple greedy construction of a bipartite matching.

The following proposition describes the performance of $\alpha$-BALANCED.

**Proposition 3.1.** The size $k$ of matching produced by $\alpha$-BALANCED in a lazy matching game of size $n$ equals $n - (x_0 + x_1 + \ldots + x_k)$ for pair $(k, (x_0, \ldots, x_k))$ satisfying (1).

**Proof.** Consider an instance of lazy matching game of size $n$ in which Builder produced graph $G = (U, D, E)$, and algorithm $\alpha$-BALANCED constructed matching $m : U \to \mathcal{P}(D)$. Suppose that $N$ rounds has been played in the game. Presenting time of element $u \in U$ is the index of the round in which $u$ has been presented.

We denote by $m^t : U \to \mathcal{P}(D)$, the (partial) matching constructed up to round $t$. In particular, for $u \in U$ which is presented in round $t_0$, we have $m^t(u) = \emptyset$ for $t < t_0$ and then $(m^{t_0}(u), m^{t_0+1}(u), \ldots, m^N(u))$ is a weakly decreasing sequence of sets with $m^N(u) = m(u)$.

Let $X$ be a set of all vertices in $D$ such that $m(u) \cap X = \emptyset$ for each $u \in U$. The size of the matching produced be $\alpha$-BALANCED is equal to the size of $Y = \{u \in U : m(u) \neq \emptyset\}$.

For the proof of the proposition we need the following claims.

**Claim 3.2.** Suppose that $d \in m^{t_1}(x) \cap m^{t_2}(y)$ and $t_1 < t_2$, then $|m^{t_1}(x)| \geq |m^{t_1+1}(x)| \geq |m^{t_2}(y)|$.

**Proof.** It is sufficient to verify the claim for $t_2 = t_1 + 1$. The case when $x = y$ is obvious, since $m^t(x)$ cannot increase by the rules of the game. Suppose that $x \neq y$. It means that during round $t_2$ algorithm $\alpha$-BALANCED removed $d$ from $m(x)$ and inserted it into $m(y)$. This happens only when the condition from the line 3 of the algorithm is satisfied and $x$ is a vertex with maximum size of assigned set among vertices ready for $y$. Let $s$ be the size of the set assigned to $x$ at that moment (in terms of listing it is $|m(x)|$). Clearly $|m^{t_1}(x)| \geq s$, since $|m^{t_1}(x)|$ is the size of the set assigned to $x$ in the beginning of round $t_2$, and that set can only get smaller during the round.
Condition from the line 3 of α-BALANCED guarantees that the set that has just been increased has no more elements than the one that has been decreased. That property, and the fact that no vertex that was ready for \( y \) had assigned set greater than \( s \), gives \( s > |m^l(y)| \). The claim follows.

**Claim 3.3.** If \( t \) is the presenting time of \( u \) and \( N(u) \cap X \neq \emptyset \), then \( |m^t(u)| = \alpha \).

**Proof.** Let \( d \in N(u) \cap X \). By the definition of \( X \) element \( d \) is strongly available for \( u \) when \( u \) is presented. But \( d \) is not chosen in the line 2 of α-BALANCED. It means there were at least \( \alpha \) other strongly available elements for \( u \) which were added to \( m^t(u) \). Thus, indeed \( |m^t(u)| = \alpha \). \( \square \)

**Claim 3.4.** For any subset \( Y \subseteq Y \) we have

\[
(|Q| - |Y|)(\mu - 1) + |M| \leq |D - X|,
\]

where \( \mu = \min\{|m(y)| : y \in Y\}, M = \bigcup_{y \in Y} m(y) \) and \( Q \) is the set of all vertices \( q \in U \) for which \( N(q) \cap (M \cup X) \neq \emptyset \).

**Proof.** The claim is obvious for \( \mu = 1 \) since \( M \cap X = \emptyset \). We assume that \( \mu > 1 \). Let \( i = |Y|, s = |Q| \) and let \((q_1, \ldots, q_s)\) be the enumeration of \( Q \) for which the sequence of corresponding presenting times \((t_1, \ldots, t_s)\) is strictly decreasing. For each \( q_j \) we recursively define set \( Z_j := m^t(q_j) - (Z_1 \cup \ldots \cup Z_{j-1}) \subseteq D - X \). Observe that \(|Z_1| + \ldots + |Z_i| \leq |D - X|\) and to finish the proof it suffices to show that \(|Z_j| \geq \mu - 1\), for all \( j \in [s] \), and \( Z_j \supseteq m(q_j) \) for \( q_j \in Y \).

Consider index \( j \in [s] \) such that \( q_j \in Y \). Set \( m^t(q_j) \) can only get smaller, after vertex \( q_j \) has been presented, therefore we have \( m(q_j) \subseteq m^t(q_j) \) for all \( t \geq t_j \). In particular, for every \( j' \in [j-1] \), we have \( t_{j'} \geq t_j \), hence \( m(q_j) \cap Z_{j'} = \emptyset \). That gives \( Z_j \supseteq m(q_j) \).

Suppose now, that \( q_j \notin Y \) and let \( m^t(q_j) = Z_j \cup R_j \), where \( R_j \subseteq Z_1 \cup \ldots \cup Z_{j-1} \).

Assume also that \(|Z_j| < \alpha \) since otherwise \(|Z_j| = \alpha > \mu - 1\). We consider two cases:

**Case 1:** \( R_j = \emptyset \). By Claim 3.3 and the definition of set \( Q \), inequality \(|Z_j| < \alpha \) implies that \( N(q_j) \cap M \neq \emptyset \). It means that some element \( d \in M \) was available for \( q_j \) at the time when \( q_j \) was presented. Element \( d \) must belong to \( m^t(u) \) for some \( u \in U \) presented earlier (otherwise the algorithm would put it in \( m^t(q_j) \), but since after the game we have \( d \in M \), it would imply that \( d \in R_j \)). Now \( m^t(u) \cap m(y) \neq \emptyset \), for some \( y \in Y \), therefore, by Claim 3.2 \( |m^t(u)| \geq |m(y)| \geq \mu \). On the other hand the algorithm in round \( t_j \) did not choose element \( d \) to be assigned to \( q_j \), which means that at the end of the round the inequality in the line 3 of the algorithm was not satisfied. That means that \(|Z_j| = |m^t(q_j)| \geq |m^t(u)| - 1 \geq \mu - 1\).

**Case 2:** \( R_j \neq \emptyset \). Let \( t' \) be the smallest \( t > t_j \) for which \( R_j \cap m^t(q_j) = \emptyset \) (it is a straightforward consequence of the definitions that such \( t \) exists). Clearly \(|Z_j| \geq |m^t(q_j)|\). Consider any \( d \in m^{t'-1}(q_j) \cap R_j \) and note there is \( l < j \) such that \( d \in Z_l \subseteq m^t(q_j) \) with \( t' - 1 < t_l \). By Claim 3.2 it means that \(|m^{t'}(q_j)| \geq |m^t(q_j)|\), thus \(|Z_j| \geq |Z_l|\).

Straightforward induction (with Case 1 as basis) gives \(|Z_j| \geq \mu - 1\). \( \square \)

We are ready to prove the proposition. Fix any optimal (maximum) matching in graph \( G \) and let \( F \subseteq D \) be the set of all elements in \( D \) outside the matching. Consider an enumeration \((y_1, \ldots, y_k)\) of \( Y \) such that, for \( x_i = |m(y_i)| - 1 \), we have \( x_1 \geq x_2 \geq \ldots \geq x_k \geq 0 \). Let \( x' = |X - F| \), \( f' = |F - X| \) and \( x_0 = x' - f' \). Observe that
\(|F| = f' + |X| - x'\). It implies
\begin{equation}
|D - X| = |D| - |X| = n + |F| - |X| = n - x_0.
\end{equation}

To show that \((k, (x_0, x_1, \ldots, x_k))\) satisfy (1) we apply Claim 3.3 to subsets \(\{y_1, \ldots, y_i\}\), for \(i = 1, \ldots, k\). Fix \(i\) and let \(Y = \{y_1, \ldots, y_i\}\). Then \(|M| = x_1 + \ldots + x_i + i\) and \(\mu = x_i + 1\). Recall that in the chosen optimal matching each vertex in \(\alpha\) can be rewritten to \(\alpha\) in any \(-\)-lazy matching game of size \(n\). Suppose that, pair \((\alpha, n)\) it is enough to find a pair \((\alpha, n)\) satisfying (1) which maximizes \(|M| - i + x_0\). Hence, \(|M| - i + x_0|\alpha - 1| + |M| \leq |D - X| \equiv n - x_0,
which can be rewritten into
\[(x_0 + x_1 + \ldots + x_i) \cdot (1 + x_i) \leq n - i.
\]

Let \(Q'\) be the set of all vertices \(q \in U\) for which \(N(q) \cap X \neq \emptyset\). Next, we define \(s(q)\) as the set of all strong available elements assigned to \(m(q)\) in the line 2 of the algorithm. Observe that \((q_1)\) and \((q_2)\) are disjoint for distinct \(q_1, q_2 \in U\). By Claim 3.3 we get \(|s(q)| = \alpha\) for each \(q \in Q'\). Thus, \(\alpha |Q'| \leq n - x_0\) since \(\bigcup_{y \in Q'} s(y) \subseteq D - X\). Also, since each element in \(D - F\) has a unique match in \(U\), we have \(|Q'| \geq |X - F| = x' \geq x_0\). Therefore \((1 + \alpha) x_0 \leq n\).

To finish the proof recall that \(\bigcup_{y \in Y} m(y) = D - X\). Thus, \(x_1 + \ldots + x_k + k = n - x_0\) and consequently the size of the matching constructed by the algorithm equals \(k = n - (x_0 + x_1 + \ldots + x_k)\). Also, since \(k\) cannot be larger then \(n\) we have \(x_0 + \ldots + x_k \geq 0\). □

Combining Proposition 2.2 and Proposition 3.1 we get

**Theorem 3.5.** \(\alpha\)-BALANCED is an optimal strategy in \(\alpha\)-lazy matching game.

4. Competitiveness of \(\alpha\)-BALANCED algorithm

Let \(\text{bal}(\alpha, n)\) be the worst (minimum) size of matching constructed by \(\alpha\)-BALANCED in any \(\alpha\)-lazy matching game of size \(n\). Competitive ratio of \(\alpha\)-BALANCED is defined as \(\text{bal}(\alpha) = \lim \inf_{n \to \infty} \text{bal}(\alpha, n)/n\). Propositions 2.2 and 3.1 imply that in order to determine \(\text{bal}(\alpha, n)\) it is enough to find a pair \((k, (x_0, x_1, \ldots, x_k))\) satisfying (1) which maximizes \(\sum_{i=0}^{k} x_i\). Moreover, by Proposition 2.1 we can assume that in the maximizing solution we have \(x_0 = \lfloor \frac{n}{1+\alpha} \rfloor\). From now on we consider \(x_0\) in system (1) as fixed together with \(n\) and \(\alpha\). Suppose that, pair \((k, (x_1, \ldots, x_k))\) satisfies (1). Note that, for \(i \geq 1\), if \(x_i = x_{i+1}\) then the \((i + 1)\)-th inequality of system (1) implies the \(i\)-th inequality. That suggests another representation of the solutions. For a pair \((k, x)\) satisfying (1) with \(x = (x_1, \ldots, x_k)\), let \(Y(x) = (y_1, \ldots, y_m)\) be such that \(m = 1 + x_1\) and \(y_j = \{|i \geq 0 : 1 + x_i = j\}\). Then, for every \(i\) for which \(x_{i+1} \neq x_i\), inequality
\[(x_0 + x_1 + \ldots + x_i)(1 + x_i) \leq n - i
\]
can be rewritten to
\[(x_0 + (m - 1) \cdot y_m + \ldots + (t - 1) \cdot y_t) \cdot t \leq n - (y_m + \ldots + y_t),
\]
where \( t = x_i + 1 \). By the above discussion sequence \((y_1, \ldots, y_m)\) belongs to the image of \( Y \) whenever it satisfies the following system of inequalities

\[
 t \cdot x_0 + \sum_{i=t}^{m} (1 + (i - 1)t)y_i \leq n \quad \text{for} \quad t = 1, \ldots, m
\]

denoted by \( \Psi_{n,m}(x_0) \). Therefore in order to determine \( \text{bal}(\alpha, n) \) we can focus on finding a positive integer \( m \) and nonnegative integer vector \( y = (y_1, \ldots, y_m) \) which satisfies \( \Psi_{n,m}(x_0) \) and maximizes \( \sum_{i=1}^{m} (i - 1)y_i \). Then \( \text{bal}(\alpha, n) = n - x_0 - \sum_{i=1}^{m} (i - 1)y_i \).

4.1. Lower bound. In order to find an upper bound for \( \text{bal}(\alpha, n) \) we consider relaxed problem in which, for fixed positive integer \( m \), we are interested in nonnegative rational solution \( y \) that satisfies \( \Psi_{n,m}(x_0) \) and maximizes \( \sum_{i=1}^{m} (i - 1)y_i \).

Fix parameter \( m \). We say that an inequality from \( \Psi_{n,m}(x_0) \) is not saturated by some vector \( y \), if its left hand side is strictly smaller than the right hand side. Specific structure of inequalities \( \Psi_{n,m}(x_0) \) guarantees that the maximizing vector saturates all inequalities. Indeed, let \( y \) be a nonnegative rational vector which satisfies \( \Psi_{n,m}(x_0) \) and suppose that at least one of inequalities of \( \Psi_{n,m}(x_0) \) is not saturated by \( y \). Let \( j \) be the index of the first such inequality. In such a case we show how to change values of \( y \) to saturate all inequalities in \( \Psi_{n,m}(x_0) \) without decreasing target function \( \sum_{i=1}^{m} (i - 1)y_i \). Consider three cases, in each one we describe a transformation of the current solution \( y \) that does not invalidate inequalities but in some sense improves the saturation of the system:

- **(T1)** If \( j = 1 \), then we can increase \( y_1 \) by the amount that makes the first inequality saturated. This transformation does not violate the other inequalities and does not change the target function.

- **(T2)** If \( j = 2 \), then we can increase \( y_2 \) by the amount that makes the second inequality saturated and decrease \( y_1 \) by the half of that value. After this transformation both inequalities, first and second, are saturated. The shape of inequalities in \( \Psi_{n,m}(x_0) \) guarantees \( y_1 > 0 \). Moreover the other inequalities are not affected by the change and the value of the target function increases.

- **(T3)** When \( j > 2 \) we know that every inequality \( i \), for \( i < j \) is saturated. In particular, we have \( y_{j-1} > 0 \). We choose largest \( \epsilon > 0 \) such that \( y \) with \( y_j \) increased by \( \epsilon \) still satisfies the \( j \)-th inequality and \( y_{j-1} - \frac{\epsilon^2 - 1}{2} \) is still nonnegative. Let \( y' \) be vector \( y \) with increased \( y_j \) and decreased \( y_{j-1} \). It is straightforward to check that \( y' \) still satisfy \( \Psi_{n,m}(x_0) \) and achieve the same value of the target function as \( y \).

Note that for any initial vector \( y \), after using transformation \((T3)\) the next transformation we use is \((T1)\) and then \((T2)\) which will increase the target function. If we start from transformation \((T1)\) then either all inequalities will become saturated or after a few next steps we will use \((T2)\) increasing again the target function. Altogether it justifies that the target function is maximized when all the inequalities are saturated, so to determine the maximizing vector it is sufficient to solve the following system of equations

\[
 t \cdot x_0 + \sum_{i=t}^{m} y_i + t \sum_{i=t}^{m} (i - 1)y_i = n \quad \text{for} \quad t = 0, \ldots, m,
\]

where equation for \( t = 0 \), defining \( y_0 \), has been added for technical convenience.
Combining equations for \( t = 0 \) and \( t = 1 \) we get

\[
x_0 + \sum_{i=1}^{m} (i-1)y_i = y_0.
\]

Therefore for a pair \((m, y)\) where \(y = (y_0, y_1, \ldots, y_m)\) is the nonnegative rational vector satisfying (3) and maximizing \(\sum_{i=1}^{m} (i-1)y_i\), we get

\[
\text{bal}(\alpha, n) \geq n - y_0.
\]

System is already in triangular form and subtracting consecutive equations we obtain

\[
-x_0 + y_t + t(t-1)y_t - \sum_{i=t+1}^{m} (i-1)y_i = 0 \quad \text{for } t = 0, \ldots, m - 1.
\]

One more subtraction gives

\[
y_t = y_{t+1} \frac{(t+1)^2}{1 - t + t^2} \quad \text{for } t = 0, \ldots, m - 2.
\]

Together with initial values

\[
y_m = \frac{n - m \cdot x_0}{1 + m(m-1)} \quad y_{m-1} = \frac{x_0 + (m-1)y_m}{1 + (m-1)(m-2)}
\]

we get

\[
y_0 = y_{m-1} \prod_{i=0}^{m-2} \frac{(i+1)^2}{1 - i + i^2} = (x_0 + (m-1)y_m) \prod_{i=0}^{m-2} \frac{(i+1)^2}{1 + i + i^2}
\]

\[
= (x_0 + (m-1)y_m)(m-1) \prod_{i=1}^{m-2} \frac{i + i^2}{1 + i + i^2} = \left( \frac{(m-1)(n + x_0)}{m^2 - m + 1} \right) \prod_{i=1}^{m-2} \frac{i + i^2}{1 + i + i^2} = \frac{(m-1)n + x_0}{m} \prod_{i=1}^{m-1} \frac{i + i^2}{1 + i + i^2}.
\]

Let \(F(z, m) = \frac{(m-1)n + x_0}{m} \prod_{i=1}^{m-1} \frac{i + i^2}{1 + i + i^2} \). Then, for fixed \(m\), we get \(y_0 = n \cdot F(x_0/n, m)\).

Note also that vector \(y\) defined by the equations is positive, whenever \(y_m\) is positive. This restricts \(m\) to satisfy \(n - m \cdot x_0 > 0\). In general, for positive \(x_0\), we have

\[
(7) \quad \text{bal}(\alpha, n) \geq n \cdot (1 - \max_{m \leq n/x_0} F(x_0/n, m)).
\]

Function \(F(z, m)\) is increasing with \(m\), for \(m \leq 1/z\). Thus, for positive \(x_0\),

\[
(8) \quad \text{bal}(\alpha, n) \geq n \cdot (1 - F(x_0/n, \lfloor n/x_0 \rfloor)).
\]

4.2. **Upper bound.** Let \(m = \lfloor n/x_0 \rfloor\) and let \((y_1, \ldots, y_m)\) be the optimal rational solution of \(\Psi_{n,m}(x_0)\). Let \(v = (v_1, \ldots, v_m)\) be such that \(v_i = \lfloor y_i \rfloor\). Vector \(v\) contains only nonnegative, integer entries and \(v \leq y\). The shape of system \(\Psi_{n,m}(x_0)\) guarantees that \(v\) also satisfies \(\Psi_{n,m}(x_0)\). Finally we have

\[
x_0 + \sum_{i=1}^{m} (i-1)v_i > x_0 + \sum_{i=1}^{m} (i-1)y_i - \sum_{i=1}^{m-1} i.
\]
Therefore by Proposition 2.2 we get
\[ \text{bal}(\alpha, n) < n \cdot (1 - F(x_0/n, \lfloor n/x_0 \rfloor)) + m(m - 1)/2. \]

4.3. Finite \( \alpha \). Recall that \( x_0 = \lfloor n/(\alpha + 1) \rfloor \) and hence \( m = m(n) = \lfloor n/\lfloor n/(\alpha + 1) \rfloor \rfloor \). In particular \( \limsup_{n \to \infty} m(n) = \alpha + 1 \). Inequalities (8) and (9) imply the following theorem.

**Theorem 4.1.** The competitive ratio of \( \alpha \)-lazy on-line matching problem on bipartite graphs and \( \alpha \)-BALANCED algorithm equals
\[ \text{bal}(\alpha) = 1 - \frac{\alpha}{1 + \alpha} \prod_{i=1}^{\alpha-1} \frac{i + i^2}{1 + i + i^2}. \]

4.4. Infinite \( \alpha \). Note that
\[ \text{bal}(\alpha) \xrightarrow{\alpha \to \infty} 1 - \prod_{i=1}^{\infty} \frac{i + i^2}{1 + i + i^2} = 1 - \frac{\pi}{\cosh \frac{\sqrt{2}}{2} \pi}. \]

It immediately implies that the competitive ratio of \( \infty \)-BALANCED, which is an optimal algorithm, is at most \( 1 - \frac{\pi}{\cosh \frac{\sqrt{2}}{2} \pi} \). On the other hand we have \( \text{bal}(n, n) = \text{bal}(\infty, n) \). For \( \alpha = n \) we get \( x_0 = 0 \), hence positivity criterion \( n - m \cdot x_0 > 0 \) becomes void. Moreover, function \( F(0, m) \) increases indefinitely with \( m \). Then, inequality (8) becomes
\[ \text{bal}(n, n) \geq n \cdot (1 - \sup_{m} F(0, m)). \]

The supremum, however turns out to be \( \lim_{m \to \infty} F(0, m) = \prod_{i=1}^{\infty} \frac{i + i^2}{1 + i + i^2} = \frac{\pi}{\cosh \frac{\sqrt{3}}{2} \pi} \).

That justifies the following corollary.

**Corollary 4.2.** The competitive ratio of \( \infty \)-lazy on-line matching problem on bipartite graphs and \( \infty \)-BALANCED algorithm equals
\[ \text{bal}(\infty) = 1 - \prod_{i=1}^{\infty} \frac{i + i^2}{1 + i + i^2} = 1 - \frac{\pi}{\cosh \frac{\sqrt{3}}{2} \pi}. \]

5. Conclusion and remarks

In the classical on-line matching problem, randomized approach has a big advantage over deterministic one. This paper shows that the lazy method moves forward deterministic bounds. It is interesting to know what else can achieved with randomization of the lazy technique.

**Problem 1.** What is the competitive ratio of randomized version of the lazy matching problem?

The author of [9] consider a variant (called \( b \)-matching) where each server can realize up to \( b \) tasks. Competitive ratio of their optimal algorithm approaches \( 1 - \frac{1}{e} \) with \( b \to \infty \) which is a barrier for any randomized matching algorithm (see [11]). We expect that the lazy method is capable of breaking \( 1 - \frac{1}{e} \) limit in case of \( b \)-matching.

**Problem 2.** What is the competitive ratio of the lazy \( b \)-matching problem?
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