HILL’S EQUATION WITH RANDOM FORCING PARAMETERS:
THE LIMIT OF DELTA FUNCTION BARRIERS

Fred C. Adams$^{1,2}$ and Anthony M. Bloch$^{1,3}$

$^1$Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, MI 48109

$^2$Astronomy Department, University of Michigan, Ann Arbor, MI 48109

$^3$Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

Abstract

This paper considers random Hill’s equations in the limit where the periodic forcing function becomes a Dirac delta function. For this class of equations, the forcing strength $q_k$, the oscillation frequency $\lambda_k$, and the period $(\Delta \tau)_k$ are allowed to vary from cycle to cycle. Such equations arise in astrophysical orbital problems in extended mass distributions, in the reheating problem for inflationary cosmologies, and in periodic Schrödinger equations. The growth rates for solutions to the periodic differential equation can be described by a matrix transformation, where the matrix elements vary from cycle to cycle. Working in the delta function limit, this paper addresses several coupled issues: We find the growth rates for the $2 \times 2$ matrices that describe the solutions. This analysis is carried out in the limiting regimes of both large $q_k \gg 1$ and small $q_k \ll 1$ forcing strength parameters. For the latter case, we present an alternate treatment of the dynamics in terms of a Fokker-Planck equation, which allows for a comparison of the two approaches. Finally, we elucidate the relationship between the fundamental parameters $(\lambda_k, q_k)$ appearing in the stochastic differential equation and the matrix elements that specify the corresponding discrete map. This work provides analytic — and accurate — expressions for the growth rates of these stochastic differential equations in both the $q_k \gg 1$ and the $q_k \ll 1$ limits.
I. INTRODUCTION

This paper considers random Hill’s equations in the delta function limit. A random Hill’s equation can be written in the form

$$\frac{d^2 y}{dt^2} + [\lambda_k + q_k \hat{Q}(t)]y = 0,$$

where the barrier shape function $\hat{Q}(t)$ is periodic, so that $\hat{Q}(t + \Delta\tau) = \hat{Q}(t)$, where $\Delta\tau$ is the period (we generally take $\Delta\tau = \pi$). In the delta function limit, the periodic functions $\hat{Q}(t)$ become Dirac delta functions,

$$\hat{Q}(t) = \delta([t] - \pi/2),$$

where the square brackets indicate that time is measured mod-$\pi$. The parameter $q_k$ denotes the forcing strength, which is a random variable that takes on a new value every cycle (where the index $k$ determines the cycle). The parameter $\lambda_k$, which determines the natural oscillation frequency of the system, also varies from cycle to cycle. In this work the period $\Delta\tau$ is considered fixed; one can show that cycle to cycle variations in $\Delta\tau$ can be scaled out of the problem and included in the distributions of the $(\lambda_k, q_k)$ [2].

The original form of Hill’s equation [1] holds the values of the parameters constant [13], and such equations arise often in physics [19]. A straightforward generalization of this classic problem is to consider parameters that vary from cycle to cycle (according to a well-defined distribution). Further, the limit of delta function barriers (equation [2]) arises in many applications (see below) and thus provides a natural starting point for this analysis.

One specific motivation for considering random Hill’s equations arises from orbit problems in astrophysical settings, including dark matter halos, galactic bulges, tidal streams, and young embedded star clusters. These astrophysical systems generally have nonspherical, extended mass distributions, with corresponding potentials that are asymmetric. With this loss of symmetry, angular momentum is not conserved, orbits are not confined to particular planes, and orbital instabilities often arise. For example, if an orbit is initially confined to the principal plane of a dark matter halo (or any triaxial, extended mass distribution), the motion is unstable to perturbations out of the orbital plane (see Ref. [4] and Appendix A). The development of this instability [2] is described by a random Hill’s equation (as given by equation [1]), with sharply-peaked forcing barriers that can be described by delta functions (as given by equation [2]). This orbit instability arises in many other astrophysical systems, including embedded young star clusters, galactic bulges, and
tidal streams [4,6]. This instability produces a number of astrophysical effects, including changing the velocity distributions from radial to more isotropic, making highly flattened systems more rounded, and helping to disperse tidal streams.

In another application, the reheating epoch at the end of the inflationary phase in the early universe [12] is described by a “parametric resonance instability” [17]. During inflation, the potential of the inflaton field $\Phi$ dominates the energy density, which is primarily in the form of vacuum energy. After sufficient inflation has taken place, this energy must be converted into matter and radiation so that the universe can evolve into its present state. As a result, the inflaton field $\Phi$ must couple to matter fields $\chi$, and the subsequent conversion of energy is governed by a Hill’s equation, often the Mathieu equation [15]. Additional fluctuations [15,20] that are present during this process — due to thermal and quantum effects — convert the equation of motion for reheating into a random Hill’s equation (see Appendix B). During inflationary reheating, the fluctuations have small amplitudes and the forcing terms can be modeled as delta functions, so that the resulting problem is described by equations (1) and (2). The instability itself acts to rapidly convert the vacuum energy of the universe into matter and radiation. In the classical problem, however, the parameter space of Hill’s equation retains bands of stability that can inhibit this conversion. Random fluctuations tend to erase these bands of stability, as shown herein, and thereby increase the efficacy of the reheating process.

For completeness, we note that in quantum systems with periodic lattices and a source of noise, the corresponding Schrödinger equation takes the form of a random Hill’s equation [5,7]. This topic is relatively well developed [20], but the results presented in this paper can also be useful in this context.

Periodic differential equations in this class can be described by a discrete mapping of the coefficients of the principal solutions from one cycle to the next. The transformation matrix takes the form

$$
\mathcal{M}_k = \begin{bmatrix} h_k & (h_k^2 - 1)/g_k \\ g_k & h_k \end{bmatrix},
$$

where the subscript denotes the cycle. The matrix elements for the $k$th cycle are given by

$$
h_k = y_1(\pi) \quad \text{and} \quad g_k = \hat{y}_1(\pi),
$$

where $y_1$ and $y_2$ are the principal solutions for that cycle. For Hill’s equation with delta function barriers (2), the principal solutions have been found previously [4] and the matrix elements take the form

$$
h_k = \cos \varphi_k - \frac{q_k}{2\sqrt{\lambda_k}} \sin \varphi_k \quad \text{and} \quad g_k = -\sqrt{\lambda_k} \sin \varphi_k - q_k \cos^2(\varphi_k/2),
$$
where we have defined $\varphi_k = \sqrt{\lambda_k} \pi$. The index $k$ indicates that the quantities $(\lambda_k, q_k)$, and hence the solutions $(h_k, g_k)$, vary from cycle to cycle. Throughout this work, the random variables are taken to be independent and identically distributed (iid).

Note that one might expect the matrix in equation (3) to have four independent elements (instead of only two). In this paper, however, we specialize to the case where the periodic functions $\hat{Q}(t)$ are delta functions, which are symmetric about the midpoint of the period. This property implies that $y_1(\pi) = y_2(\pi)$, which eliminates one independent matrix element [1,19]. In addition, since the Wronskian of the original differential equation (1) is unity, the determinant of the matrix map must be unity, and this constraint eliminates another independent element.

The growth rates for Hill’s equation (1) are determined by the growth rates for matrix multiplication of the matrices $M_k$ given by equation (3). Here we denote the product of $N$ such matrices as $M^{(N)}$, and the growth rate $\gamma$ is defined by

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \log ||M^{(N)}||. \quad (6)$$

Previous work [10,11,18] shows that this result is independent of the choice of the norm $|| \cdot ||$.

This paper is organized as follows. To determine the growth rates for the differential equation (1), we consider the multiplication of infinite strings of random matrices of the form (3). Working in the delta function limit, this paper explores the regime where the forcing strengths are large $q_k \gg 1$ (Section II), and the opposite regime of small forcing parameters $q_k \ll 1$ (Section III). For the small $q_k$ limit, we develop an alternate treatment of the dynamics using the Fokker-Planck equation (Section IV). Next, we consider the relationship between the matrix elements $(h_k, g_k)$ that appear in the discrete map (3) and the random variables $(\lambda_k, q_k)$ that appear in the original differential equation (1). For the limiting case of delta functions barriers, we find this transformation explicitly (see equation [5]), and constrain the distributions of the matrix elements for given distributions of the input parameters in Section V. The paper concludes (in Section VI) with a summary of the results and a brief discussion of future applications. In addition to the appendices that outline the physical motivation for random Hill’s equations, we also present a simple iterative map (Appendix C) that reproduces our basic results for the growth rates.

II. THE LIMIT OF LARGE FORCING STRENGTH PARAMETER

This section considers the case of large forcing strengths $q_k$ for the problem with delta function barriers. This limit applies to Hill’s equations that govern the orbit instabilities
in dark matter halos, embedded young star clusters, and other extended mass distributions (Appendix A). For the triaxial orbits [4] that originally motivated this study, for example, the forcing strengths \( q_k \sim 1000 \) when the period \( \Delta \tau_k \) and oscillation parameter \( \lambda_k \) are of order unity.

In this limit, it is useful to factor the matrix \( M_k \) that connects solutions from cycle to cycle by writing it in the form

\[
M_k = h_k B_k
\]

where \( x_k = h_k/g_k \) and where \( \phi_k \equiv 1 - 1/h_k^2 \). The ansatz of equation (7) separates the growth rate for this problem into two parts: \( \gamma = \gamma_h + \gamma_B \). The first part \( \gamma_h \) of the growth rate is given by

\[
\gamma_h = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log |h_k|,
\]

while the remaining part \( \gamma_B \) is determined by matrix multiplication of the matrices \( B_k \). In the general problem [2,3], much of the work thus involves finding the growth rates \( \gamma_B \). In the limit of large \( q_k \), however, much simpler — but approximate — forms for the growth rates can be found, as shown below.

**Theorem 2.1:** Consider a random Hill’s equation with a delta function barrier. In the limit of large \( q_k \gg 1 \) and constant \( \lambda \), the growth rate has the form

\[
\gamma = \langle \log |2h_k| \rangle + \mathcal{O}(1/q_k),
\]

where the \( h_k \) are given by equation (4) and where the angular brackets denote the expectation value. This form is valid provided that \( \lambda \neq n^2 \), where \( n \) is an integer. The width of the zone for which equation (9) is not valid has order \( \delta \lambda = \mathcal{O}(1/q_k) \).

**Proof:** We separate the problem into two pieces according to equation (7). The quantities that appear in the matrix elements in the \( B_k \) are the ratios \( x_k = h_k/g_k \) and the correction factors \( \phi_k \equiv 1 - 1/h_k^2 \). For the case of delta function barriers, considered here, the quantities \( x_k \) and \( \phi_k \) can be written in the form

\[
x_k = \frac{q_k(\pi/\varphi) \sin \varphi - 2 \cos \varphi}{q_k(1 + \cos \varphi) + 2(\varphi/\pi) \sin \varphi} \quad \text{and} \quad \phi_k = 1 - \left( \frac{2 \varphi}{\pi q_k \sin \varphi - 2 \varphi \cos \varphi} \right)^2, \tag{10}
\]

where we have suppressed the subscripts on the angles \( \varphi = \sqrt{\lambda \pi} \). In the limit of large forcing strength \( q_k \), the ratios \( x_k \) become independent of the values of the \( q_k \). In particular, \( x_k \) and \( \phi_k \) take the asymptotic forms

\[
\lim_{q_k \to \infty} x_k = \frac{\pi \sin \varphi}{\varphi(1 + \cos \varphi)} \quad \text{and} \quad \lim_{q_k \to \infty} \phi_k = 1. \tag{11}
\]
For constant $\lambda$, the angles $\varphi = \sqrt{\lambda \pi}$ are also constant, and the $x_k$ are all the same in the limit $q_k \to \infty$. In this same limit, the matrices $B_k$ become constant from cycle to cycle and are denoted here as $B_0$. These matrices have the simple multiplication property

$$B_0^2 = 2B_0,$$  \hspace{1cm} (12)

so that the growth rate $\gamma_B = \ln 2$. The remaining part of the growth rate $\gamma_h$ is given by equation (8), where the $h_k$ are given by equation (4). Combining these two results yields the expression for the growth rate given in equation (9). The correction term is considered below.

The above derivation of the growth rate $\gamma_B$ is valid as long as the $x_k$ in equation (11) remain finite and nonzero. This requirement leads to the condition that $\lambda \neq n^2$, where $n$ is an integer. This result also follows from a previous theorem of Ishii [14,20]. We thus obtain the stated restriction on the range of validity of the growth rate. To constrain the width of the angular zone for which this result is not valid, we write $\varphi = n\pi + \delta\varphi$. Finding the condition for which $|h_k| < 1$, which is the condition for one cycle to be stable, we find that

$$\delta\varphi < \frac{4\sqrt{\lambda}}{q_k} = \frac{4n}{q_k}. \hspace{1cm} (13)$$

The width of the angular zone thus depends on $q_k$, which varies from cycle to cycle, but $\delta\varphi = O(1/q_k)$. The width of the stability zone for $\lambda$ is given by

$$\delta\lambda = \frac{2n}{\pi}\delta\varphi = \frac{8n^2}{\pi q_k}. \hspace{1cm} (14)$$

The width of the zone (in the parameter $\lambda$) for which the growth rate of equation (9) fails is thus of order $1/q_k$, as claimed.

In deriving the leading order term in equation (9), we have considered the variables $x_k$ to be constant in the limit of interest. In the more general case, the $x_k$ vary from cycle to cycle (note that varying $\lambda_k$, not considered here, would also contribute to variations in $x_k$). Including these variations leads to a correction $\Delta\gamma$ to the growth rate. As shown in Theorem 2 of Ref. [2], this correction can be written in the form

$$\Delta\gamma = \langle \log |1 + x_{k1}/x_{k2}| \rangle - \log 2, \hspace{1cm} (15)$$

where the $x_{k1}$ and $x_{k2}$ represent two independent samples of the variable $x_k$ (see equation [10]). Note that $\Delta\gamma \to 0$ in the limit where the $x_k$ are constant. In the limit where the $q_k$ are large, but not infinite, variations in the $x_k$ are small, and equation (15) can be expanded and written in the approximate form

$$\Delta\gamma = \frac{1}{\pi} \left\langle \frac{\varphi_{k1}}{q_{k1} \sin \varphi_{k1}} - \frac{\varphi_{k2}}{q_{k2} \sin \varphi_{k2}} \right\rangle + O(q_k^{-2}) \hspace{1cm} (16)$$
The first term is $O(1/q_k)$. If the distributions of $q_k$ and $\varphi_k$ are symmetric, the first term can vanish, and the correction $\Delta \gamma$ to the growth rate becomes second order in $1/q_k$.

The result given in equation (15) includes the variations of the $x_k$ but does not take into account possible deviations of the $\phi_k$ from unity. In order to determine how large these corrections can be, we consider the case where the correction factors $\phi_k$ are close to – but not exactly – unity. If $\gamma_B$ is the true growth rate for multiplication of the matrices $B_k$, and $\gamma_0$ is the growth rate obtained in the limit where $\phi_k \to 1$, then we denote the difference as $\delta \gamma \equiv \gamma_0 - \gamma_B$. For delta function barriers, we can use Theorem 2.3 of Ref. [3] to write this correction term in the form

$$
\delta \gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{x_k^2}{(x_k + x_{k+1})(x_k + x_{k-1})} \sin^2 \varphi_k \pi^2 q_k^2 ,
$$

where $\varphi_k = \sqrt{\lambda_k} \pi$. This expression is correct to leading order in $1/q_k$. As a result, we can use the asymptotic expressions for the $x_k$ to evaluate $\delta \gamma$ when the $q_k$ are large (but not infinite). In this case, the $x_k$ are independent of the $q_k$, and the correction $\delta \gamma$ has the order

$$
\delta \gamma = O\left(\frac{\lambda_k}{q_k^2}\right) ,
$$

where $\varphi_k = \sqrt{\lambda_k} \pi$. As the $q_k$ become large, this correction decreases, and we recover the growth rates given by equations (9) and (15). If the angle $\varphi$ is held fixed, then the $x_k$ in equation (17) are identical in the large $q_k$ limit, and this correction $\delta \gamma$ to the growth rate reduces to the form

$$
\delta \gamma = \left(\varphi \frac{1}{\pi \sin \varphi}\right)^2 \left\langle \frac{1}{q_k^2} \right\rangle = \frac{\lambda}{\sin^2 \varphi} \left\langle \frac{1}{q_k^2} \right\rangle .
$$

Note that these results are only valid when $\sin \varphi \neq 0$, which in turn requires $\lambda \neq n^2$ (where $n$ is an integer).

The above considerations provide corrections for the growth rate of Theorem 2.1 for cases where the $q_k$ are large, but corrections of order $1/q_k$ are still relevant. In the opposite limit where $q_k \to \infty$, the growth rate can be simplified further:

**Corollary 2.1:** For a random Hill’s equation with delta function barriers, in the limit $q_k \to \infty$, the growth rate approaches the form

$$
\gamma = \left\langle \log \left| \frac{q_k}{\sqrt{\lambda}} \sin \sqrt{\lambda} \pi \right| \right\rangle .
$$

This form is valid for $\lambda \neq n^2$, where $n$ is an integer.

**Proof:** This limit represents a stronger version of the conditions for which equation (9) is valid. Starting with equation (4), the matrix elements $h_k$ approach the following form in
the limit $q_k \to \infty$:

$$h_k \to -\frac{q_k}{2\sqrt{\lambda}} \sin \varphi.$$  \hspace{1cm} (21)

Using this expression in equation (9), we obtain the claimed form for the growth rate given in equation (20).

Notice that the difference between the asymptotic form of the growth rate from equation (20) and the approximation of equation (9) is first order in $1/q_k$. For comparison, the corrections due to variations in the $x_k$ are first order for asymmetric variations and second order for the symmetric case (see equation [16]). The corrections due to the departure of the $\phi_k$ from unity are second order (equations [17] and [18]).

Figure 1 illustrates the validity of the approximations derived in Theorem 2.1 and its corollary. Here we take Hill’s equation in the delta function limit, with fixed oscillation parameter $\lambda$. We then let the forcing strength vary according to $q_k = (1 + \xi_k)q_0$, where $\xi_k$ is a uniformly distributed random variable and the constant $q_0$ determines the amplitude of the fluctuations. The growth rates are calculated using three successive approximations: (a) the full matrix multiplication scheme from equation (3) with growth rate given by equation (6), (b) the approximation of equation (9) which is valid in the limit of large forcing strength $q_k$, and (c) the more extreme approximation of equation (20), which is valid in the limit $q_k \to \infty$. The upper curve in the figure shows the difference between the full growth rate of equation (6) and that of equation (20); the lower curve shows the difference between the full growth rate and that of equation (9). Both approximations work well for large $q_k$, measured here using $q_0$, where “large” means $q_0$ greater than $\sim 100$. Both curves approach power-law forms, with well-defined slopes, showing that the approximation of Theorem 2.1 is valid to second order in $1/q_0$, whereas the more extreme approximation of Corollary 2.1 is only accurate to first order in $1/q_0$.

III. THE LIMIT OF SMALL FORCING STRENGTH PARAMETER

This section considers the limiting regime where the forcing strength $q_k \ll 1$. This limit is expected to be applicable to the reheating problem after an inflationary epoch (Appendix B). The reheating phase takes place over many oscillations of the inflaton field and hence many cycles of the corresponding Hill’s equation. As a result, the fluctuations (given by the magnitude of the $q_k$) must be relatively small.

In general, when the forcing parameter $q_k$ is small, solutions to Hill’s equation tend to be stable in the classical regime, i.e., where the parameters do not vary from cycle to cycle. However, variations in the parameters $(\lambda_k, q_k)$ allow for unstable solutions, even
Figure 1: Degree of validity for the approximations to the growth rates for Hill’s equation in the delta function limit with large forcing parameters. The lower curve shows the difference between full growth rate calculated from matrix multiplication and that calculated from Theorem 2.1 (equation [9]). The upper curve shows the difference between the full growth rate and that calculated from the more extreme approximation of Corollary 2.1 (equation [20]). Here, the values of $\lambda$ are fixed and the values of $q_k$ fluctuate according to $q_k = (1 + \xi_k)q_0$, where $\xi_k$ is a uniform random variable and the constant $q_0$ provides a measure of the fluctuation amplitude (where $\langle q_k \rangle = 3q_0/2$).
if the growth rate would vanish in the absence of fluctuations. For the case of delta function barriers, the parameter \( q_k/\sqrt{\lambda_k} \) must be small for stability, so that classically stable solutions can also arise in the limit of large oscillation frequency \( \lambda_k \) (see equation [3]). Here we find the growth rate for a random Hill’s equation in the limit of small forcing strength for the case of delta function barriers:

**Theorem 3.1:** Consider a random Hill’s equation (1) with a delta function barrier so that \( \hat{Q}(t) \) is given by equation (2). In the limit of small \( q_k \ll 1 \), fixed \( \lambda \), and symmetric variations in the \( q_k \), the growth rate approaches the form:

\[
\gamma = \log \left[ 1 + \langle q_k^2 \rangle / 8\lambda \right],
\]

where the angular brackets denote expectation values. This form is valid for all \( \lambda > 0 \) except for narrow bands of stability centered on square integer values \( \lambda = n^2 \) (\( n \in \mathbb{Z} \)). The growth rate vanishes at these values of \( \lambda \) and the width \( \delta \lambda \) of the bands is given by

\[
\delta \lambda \approx 2q_k/\pi.
\]

**Proof:** In the limit of delta function barriers, the principal solutions are given by equation (5). For the case of small forcing parameters \( q_k \ll 1 \), the \( h_k \) are less than unity except for the narrow zones of parameter space defined by the condition \( \lambda = n^2 \) and by equation (23). As a result, we can rewrite the matrix elements \( h_k \) in the form

\[
h_k \equiv \cos \theta_k.
\]

The transformation matrix of equation (3) can be written in the form

\[
\mathcal{M}_k = \begin{bmatrix}
\cos \theta_k & -L_k \sin \theta_k \\
\sin \theta_k/L_k & \cos \theta_k
\end{bmatrix},
\]

where the parameter \( L_k \) is defined by

\[
L_k \equiv \sin \theta_k \frac{g_k}{q_k} = -\frac{\sin \varphi \left[ 1 + (q_k/\sqrt{\lambda}) \cot \varphi - (q_k^2/4\lambda) \sin^2 \varphi \right]^{1/2}}{\sqrt{\lambda} \sin \varphi + (q_k/2)(1 + \cos \varphi)}.
\]

To leading order in \( q_k \ll 1 \), \( L_k = L_0 = -1/\sqrt{\lambda} \), where \( L_0 \) is a constant. If we write \( L_k = L_0(1 + \eta_k) \), the perturbations can be written in the form

\[
\eta_k = \frac{\sin \theta_k}{\sin \varphi + (q_k/2\sqrt{\lambda})(1 + \cos \varphi)} - 1 = -\frac{q_k}{2\sqrt{\lambda} \sin \varphi} + \mathcal{O}(q_k^2),
\]

where the second equality defines the leading order expression.
As shown below, the product \( \eta_k \sin \theta_k \) appears in the expression for the growth rate and is thus the quantity of interest. Since \( \eta_k \) is first order in \( q_k \), we can use the leading order expression for \( \sin \theta_k \) to evaluate the product. For small \( q_k \ll 1 \) trigonometric identities imply the following transformation between the angle \( \varphi_k \) and the angle \( \theta_k \):

\[
\theta_k = \varphi + \frac{q_k}{2\sqrt{\lambda}}.
\]  

As a result, \( \sin \theta_k = \sin \varphi + \mathcal{O}(q_k) \), so that \( \eta_k \sin \theta_k = -q_k/(2\sqrt{\lambda}) \) to leading order.

Next we expand the transformation matrix of equation (25) into two parts,

\[
\mathcal{M}_k = \mathcal{M}_{0k}(\theta_k; L_0) + \mathcal{M}_{1k},
\]

where the first term is an elliptical rotation matrix with constant length parameter \( L_0 \) and where the second term has the form

\[
\mathcal{M}_{1k} = -\eta_k \sin \theta_k \begin{bmatrix} 0 & L_0 \\ 1/[L_0(1 + \eta_k)] & 0 \end{bmatrix}.
\]

Note that the first term in equation (29) is stable under matrix multiplication. Notice also that the second term \( \mathcal{M}_{1k} \), as written, includes the full correction (with no approximations).

As shown in the following analysis, the first non-vanishing contribution to the growth rate is second order in \( \eta_k \). As a result, we expand the product of \( N \) matrices \( \mathcal{M}_k \),

\[
\mathcal{M}_k^{(N)} = (\mathcal{M}_{0k} + \mathcal{M}_{1k})^N,
\]

including all terms to second order in the matrix \( \mathcal{M}_{1k} \),

\[
\mathcal{M}_k^{(N)} = \mathcal{M}_{0k}^{(N)} + \sum_{k=1}^{N} \mathcal{P}_k^{N} + \sum_{k,\ell}^{N} \mathcal{Q}_{k\ell}^{N}.
\]

The first sum includes partial product matrices of the form

\[
\mathcal{P}_k^{N} = \left\{ \prod_{j=k+1}^{N} \mathcal{M}_{j0} \right\} \mathcal{M}_{1k} \left\{ \prod_{j=1}^{k-1} \mathcal{M}_{j0} \right\} = \mathcal{E}_0(a_k; L_0) \mathcal{M}_{1k} \mathcal{E}_0(b_k; L_0).
\]

In the second equality we have evaluated the products using the properties of the elliptical rotation matrices, denoted here as \( \mathcal{E}_0 \), and we have defined the composite angles

\[
a_k \equiv \sum_{j=k+1}^{N} \theta_j \quad \text{and} \quad b_k \equiv \sum_{j=1}^{(k-1)} \theta_j.
\]

The second sum in the expansion of equation (32) involves partial product matrices with the form

\[
\mathcal{Q}_{k\ell}^{N} = \left\{ \prod_{j=k+1}^{N} \mathcal{M}_{0j} \right\} \mathcal{M}_{1k} \left\{ \prod_{j=\ell+1}^{k-1} \mathcal{M}_{0j} \right\} \mathcal{M}_{\ell1} \left\{ \prod_{j=1}^{\ell-1} \mathcal{M}_{0j} \right\}.
\]
This second sum includes all possible products of the above form, i.e., all possible locations of the two matrices that are of type $M_{1k}$ rather than $M_{0k}$. By construction, each matrix $Q_{k\ell}$ contains two factors of the random variable so that $Q_{k\ell} \propto \eta_k \eta_\ell$, where the $\eta_k$ and $\eta_\ell$ are independent realizations and hence are uncorrelated. The matrix elements from the third term in equation (32), the sum over the $Q_{k\ell}^N$, must thus vanish in the limit $N \to \infty$ (see also [3]). As a result, we only need to consider the contribution from the first sum in equation (32). In this sum, the first order terms, those proportional to $\eta_k$, will also vanish in the limit $N \to \infty$. We thus need to include the second order terms in the first sum. Using the result of equation (33) and expanding to second order in $\eta_j$, we thus obtain

$$\sum_{k=1}^N P_k^N = \sum_{k=1}^N \eta_k^2 \sin \theta_k \begin{bmatrix} -\sin a_k \cos b_k & L_0 \sin a_k \sin b_k \\ (1/L_0) \cos a_k \cos b_k & -\cos a_k \sin b_k \end{bmatrix}.$$  \hspace{1cm} (36)$$

After using this result in the expansion of equation (32), and evaluating the product $M_{0k}^N$, the eigenvalue $\Lambda$ for the full product matrix after $N$ steps is given by

$$\Lambda^2 - 2\Lambda \cos \theta_N + 1 + \sum_{k=1}^N \left\{ \eta_k^2 \sin \theta_k \left[ (\Lambda - \cos \theta_N) \sin \alpha_k + \sin \theta_N \cos \alpha_k \right] \right\} = 0,$$ \hspace{1cm} (37)

where we have defined $\alpha_k \equiv a_k + b_k$ (see equation [34]), and where

$$\theta_N \equiv \sum_{j=1}^N \theta_k.$$ \hspace{1cm} (38)

The zeroth order contribution to the eigenvalue is given by

$$\Lambda_0 = \cos \theta_N \pm i \sin \theta_N,$$ \hspace{1cm} (39)

and the leading order correction is given by

$$\Lambda_2 = \frac{\pm i}{2} \sum_{k=1}^N \left\{ \eta_k^2 \sin \theta_k (\cos \alpha_k \pm i \sin \alpha_k) \right\}.$$ \hspace{1cm} (40)

The magnitude of the full eigenvalue, $\Lambda = \Lambda_0 + \Lambda_2$, is then given by

$$|\Lambda| = 1 + \frac{1}{2} \sum_{k=1}^N \eta_k^2 \sin^2 \theta_k = 1 + \frac{1}{2} N \langle \eta_k^2 \sin^2 \theta_k \rangle,$$ \hspace{1cm} (41)

where the second equality is valid in the limit $N \to \infty$. To leading order in $\eta_k$, this expression can be rewritten in the form

$$|\Lambda| = \left[ 1 + \frac{1}{2} \langle \eta_k^2 \sin^2 \theta_k \rangle \right]^N.$$ \hspace{1cm} (42)
The corresponding growth rate thus becomes
\[ \gamma = \lim_{N \to \infty} \frac{1}{N} \log |\Lambda| = \log \left[ 1 + \frac{1}{2} \langle \eta_k^2 \sin^2 \theta_k \rangle \right]. \quad (43) \]

Using equation (27) to determine \( \eta_k \), we obtain the expression claimed in equation (22).

To prove the second part of this theorem, we note that the expansion of equation (27) is no longer valid when the second term in the denominator of equation (26) dominates the first. The condition for the expansion to fail can then be written in the form
\[ \left| \frac{q_k \cos^2(\varphi/2)}{\sqrt{\lambda} \sin \varphi} \right| \sim 1. \quad (44) \]
The left hand side of this equation blows up when \( \sin \varphi = 0 \), which occurs when \( \varphi = n\pi \) and \( n \) is an integer; equivalently, this singularity occurs when \( \sqrt{\lambda} \) is an integer (and \( \lambda \) is a square integer). Near these square integer values of \( \lambda \) of interest, we can write
\[ \varphi = \sqrt{\lambda}\pi \equiv n\pi + \delta \varphi, \quad (45) \]
where the second equality defines \( \delta \varphi \). Combining the above two results implies that
\[ \frac{\pi q_k/2}{n\pi + \delta \varphi} \left[ 1 + (-1)^n \cos(\delta \varphi) \right] \sim \delta \varphi. \quad (46) \]
For \( n \) even, we thus obtain \( \delta \varphi \sim q_k/n \) to leading order. By definition, \( \delta \lambda = 2n(\delta \varphi)/\pi \), so the width of the interval where equation (22) fails is given by \( \delta \lambda \sim 2q_k/\pi \), in agreement with equation (23). This derivation applies to even integers \( n \). For the case of odd \( n \), one can derive the analogous result.

To illustrate this set of results, we present the following numerical experiment: The natural oscillation frequency, as set by the parameter \( \lambda_k \), is fixed at a constant value. The forcing strength is then allowed to vary according to the ansatz
\[ q_k = q_0 \xi_k, \quad (47) \]
where \( q_0 \) is a fixed amplitude and \( \xi_k \) is a uniformly distributed random variable with \( -1 \leq \xi_k \leq 1 \). In the absence of the fluctuations, Hill’s equation would have bands of stability and bands of instability in the \((\lambda-q)\) plane of parameters [1,2,19]. However, with the cycle to cycle variations of the forcing strength given by equation (47), the bands of stability essentially disappear. Figure 2 shows the growth rate plotted as a function of \( \lambda \) for a collection of amplitudes \( q_0 \) (where the amplitudes are equally spaced logarithmically, so that \( q_0 = 10/2^\ell \) for \( \ell = 4,5,6,7,8 \)). Even for extremely small values of \( q_0 \) (and hence correspondingly small \( q_k \)), the growth rates are nonzero. The growth rate does vanish for
Figure 2: Growth rates for Hill’s equation in the delta function limit for fixed values of $\lambda$ and fluctuating values of $q_k$. The five curves shown here correspond to five values of the fluctuation amplitude $q_0$, where the $q_k = q_0 \xi_k$, where $\xi_k$ is a uniformly distributed random variable $-1 \leq \xi_k \leq 1$. For the five curves shown, the amplitudes are given by $q_0 = 10/2^\ell$ for $\ell = 4, 5, 6, 7, 8$. The dashed lines show the limiting form for the growth rate from equation (22).
Figure 3: Comparison of growth rates for Hill’s equation in the delta function limit for fixed $\lambda$ and fluctuating values of the forcing strength $q_k$. The solid curve corresponds to the exact result from matrix multiplication using a fluctuation amplitude $q_0 = 2.5$. The dot-dashed curve shows the approximation developed in this section (equation (22)). The dashed curve shows the growth rate $\gamma_\infty$ that results from an average of the growth rates for individual cycles (the asymptotic growth rate). Finally, the dotted curve shows the growth rate resulting from a constant value of the forcing strength $q = q_0/2$. 
particular values of the frequency parameter \( \lambda \), where \( \lambda = n^2 \) and \( n \) is an integer. At these particular frequencies, \( \sin \theta = 0 \) and \( h_k = \cos \theta = \pm 1 \) = constant. Notice also that in the limit \( q_0 \ll 1 \), this incarnation of the random Hill’s equation is equivalent to a simple harmonic oscillator with frequency \( \lambda \) and perturbative noise; this result – that noise leads to instability or can speed up instability – has analogs in previous work [9].

For the particular choice of \( q_k \) used in Figure 2, \( \langle q_k^2 \rangle = q_0^2/3 \). The dashed curves in Figure 2 show the approximation of equation (22) for the growth rate. In this case, the approximate form actually gives more accurate results than direct matrix multiplication (solid curves) due to incomplete sampling in the latter. However, the expression of equation (22) does not account for the vanishing of the growth rate for particular values of \( \lambda \).

As another illustration of how random variables change the landscape of parameter space, Figure 3 shows the growth rates as a function of (fixed) \( \lambda \) for several cases. The solid curve shows the result from full matrix multiplication, with the same sampling of \( q_k \) as used in Figure 2 with value \( q_0 = 2.5 \). The dot-dashed curve shows the prediction from the approximation of equation (22). The approximation works well except near the integer square values of the frequency \( \lambda \) where the growth rate vanishes. However, since the \( q_k \) are of order unity, rather than fully in the regime \( q_k \ll 1 \), the assumptions for the validity of equation (22) are not completely satisfied. As a result, the small amplitude oscillations of the approximate result (dot-dashed curve) about the true growth rate (solid curve) are real. In Ref. [2] we defined the asymptotic growth rate \( \gamma_\infty \) to be the growth rate for a random Hill’s equation resulting from an appropriate average of the growth rates for the individual cycles; this quantity is plotted as the dashed curve in Figure 3. For this set of \( q_k \) values, the amplitude is \( q_0 \), so the mean of the forcing parameter magnitude is \( q_0^2/2 \); for comparison, the dotted curve shows the growth rates for the classical problem (no random variables) with \( q = q_0/2 \). The dotted curves thus delineate the regions of stability and instability that characterize the parameter space of Hill’s equation. Note that the asymptotic growth rate \( \gamma_\infty \) is sometimes larger and sometimes smaller than the true growth rate. As a general rule, one finds \( \gamma_\infty > \gamma \) in or near the portions of parameter space for which the classical problem (fixed \( q \)) is unstable; for regimes in which the systems is classically stable, however, the opposite holds so that \( \gamma > \gamma_\infty \).

IV. FOKKER-PLANCK APPROACH

For the case of constant oscillation frequency parameter \( \lambda \) and sufficiently small forcing strengths \( q_k \), Hill’s equation in the limit of delta function barriers can be described by a Fokker-Planck equation of conventional form. Following standard methods, we define the
velocity \( V \) and diffusion constant \( D \) according to

\[
V \equiv \frac{dy}{dt} \quad \text{and} \quad D \equiv \frac{\langle q_k^2 \rangle}{\pi},
\]

(48)

where \( \pi \) is the period of the forcing intervals. The Fokker-Planck equation for the evolution of the distribution \( P(y, V, t) \) of phase space variables thus becomes

\[
\frac{\partial P}{\partial t} + V \frac{\partial P}{\partial y} - \lambda y \frac{\partial P}{\partial V} = \frac{D}{2} y^2 \frac{\partial^2 P}{\partial V^2}.
\]

(49)

Notice that including variations in the \( \lambda_k \), or working in the regime of large forcing parameters \( q_k \) (e.g., the highly unstable limit), would require additional terms in equation (49).

In order to reduce the complexity of equation (49), it would be useful to average over one of the independent variables. In this case, however, both \( V \) and \( y \) are on a nearly equal footing. In this problem, the system acts like a simple harmonic oscillator, except at the delta function barriers where its energy jumps due to the forcing. We thus transform into a type of “polar coordinates” [20] in which the energy plays the role of the radial coordinate; specifically we define

\[
E \equiv \frac{1}{2} \left( V^2 + \lambda y^2 \right) \quad \text{and} \quad \psi = \tan^{-1} \left( \sqrt{\lambda y/V} \right),
\]

(50)

where the corresponding inverse transformation takes the form

\[
V = 2\sqrt{E} \cos \psi \quad \text{and} \quad \sqrt{\lambda y} = 2\sqrt{E} \sin \psi.
\]

(51)

In terms of the new variables \((E, \psi)\), the Fokker-Planck equation becomes

\[
\frac{\partial P}{\partial t} + \sqrt{\lambda} \frac{\partial P}{\partial \psi} = \frac{2D}{\lambda} E \sin^2 \psi \left\{ \frac{\partial P}{\partial E} + 4E \cos^2 \psi \frac{\partial P^2}{\partial E^2} \right. \nonumber
\]

\[
+ \left. \frac{\sin 2\psi}{E} \frac{\partial P}{\partial \psi} + \frac{\sin^2 \psi}{E} \frac{\partial P^2}{\partial \psi^2} - 2 \sin 2\psi \frac{\partial^2 P}{\partial E \partial \psi} \right\}.
\]

(52)

If we now average over the angular variable \( \psi \), the partial derivative terms with respect to \( \psi \) vanish, and the Fokker-Planck equation simplifies to the form

\[
\frac{\partial P}{\partial t} = \frac{2D}{\lambda} E \left\{ \langle \sin^2 \psi \rangle \frac{\partial P}{\partial E} + 4\langle \sin^2 \psi \cos^2 \psi \rangle \frac{\partial P^2}{\partial E^2} \right\} = \frac{D}{\lambda} \left\{ E \frac{\partial P}{\partial E} + E^2 \frac{\partial P^2}{\partial E^2} \right\}.
\]

(53)

Next we change variables again, by defining

\[
\mu \equiv \log E.
\]

(54)
Note that we must take the logarithm of a dimensionless quantity. Given the form of equation (53), however, it is straightforward to introduce a dimensionless energy $\tilde{E} \equiv E/E_0$ before changing to the logarithmic form. The resulting Fokker-Planck equation thus becomes an ordinary diffusion equation

$$\frac{\partial P}{\partial t} = \frac{D}{\lambda} \frac{\partial^2 P}{\partial \mu^2}. \quad (55)$$

This diffusion equation has the normalized solution

$$P(\mu, t) = \left( \frac{\lambda}{4\pi Dt} \right)^{1/2} \exp \left[ -\frac{\mu^2 \lambda}{4Dt} \right], \quad (56)$$

which is appropriate for the boundary condition $P(\mu, t = 0) = \delta(\mu)$, i.e., the system starts out with $\mu = 0$ or energy $E = E_0$. The solutions $y(t)$ are oscillatory, but growing (in general). In order to extract a growth rate from this Fokker-Planck treatment of the problem, we first determine the expectation value of $y^2$, which takes the form

$$\langle y^2 \rangle = \int_{-\infty}^{\infty} P(\mu, t) d\mu \frac{4E^2 \sin^2 \psi}{\lambda} = \frac{2}{\lambda} \exp \left[ Dt/\lambda \right]. \quad (57)$$

To obtain the final expression, have averaged over the angle $\psi$, so that $2 \sin^2 \psi = 1$. Next we assume that the amplitude $|y|$ of the solution can be characterized by the relationship

$$|y| \approx \langle y^2 \rangle^{1/2} \propto \exp \left[ \gamma_{fp} t \right], \quad (58)$$

where $\gamma_{fp}$ is the growth rate resulting from this Fokker-Planck approach. The resulting estimate for the growth rate is

$$\gamma_{fp} = \frac{D}{2\lambda} = \frac{\langle q_k^2 \rangle}{2\pi \lambda}. \quad (59)$$

This growth rate is similar, but not identical to, that given by Theorem 3.1 for the case of constant frequency parameter $\lambda$, delta function barriers, and in the limit of small forcing parameters $q_k$. The functional dependence $\gamma \propto q_k^2/\lambda$ is the same, only the numerical coefficient differs. In order to derive the result (59), however, we have averaged the Fokker-Planck equation (52), and this procedure can produce such a numerical difference.

This treatment using the Fokker-Planck equation thus provides a good description of the problem for the case of small $q_k$ and constant $\lambda$. For varying values of $\lambda_k$, additional diffusive terms must be included. For the case of large $q_k$, however, the Fokker-Planck approach does not naturally reproduce the results obtained here using direct methods. The derivation of equation (49) involves truncating a series of terms in powers of $q_k^2$ [6], and such a truncation is only valid for sufficiently small forcing strengths. In the highly
unstable limit (large $q_k$), one must either use a highly modified form of the Fokker-Planck equation or abandon this approach altogether.

V. TRANSFORMATION FROM HILL’S EQUATION PARAMETERS TO MATRIX ELEMENTS

For Hill’s equations with delta function barriers, we can write the matrix elements $h_k$ and $g_k$ in the form given by equation (3) above. With these results in hand, we can directly construct the relationship between the fundamental parameters ($\lambda_k, q_k$) appearing in the original differential equation (1) and the moments of the distributions of the matrix elements.

We start by considering the case where the angle $\varphi_k$ is held fixed, but the forcing strength $q_k$ is allowed to vary. For ease of notation, we suppress the subscripts on the angle $\varphi$ and the forcing strength $q$. The mean (first moment) of the matrix element $h_k$ is then given by

$$\langle h_k \rangle = \cos \varphi - \frac{\pi}{2} \langle q \rangle \frac{\sin \varphi}{\varphi}.$$  \hspace{1cm} (60)

Similarly, the second moment takes the form

$$\langle h_k^2 \rangle = \cos^2 \varphi + \frac{\pi^2}{4} \langle q^2 \rangle \left( \frac{\sin \varphi}{\varphi} \right)^2 - \frac{\pi}{2} \frac{\cos \varphi \sin \varphi}{\varphi} \langle q \rangle.$$  \hspace{1cm} (61)

The variance is thus given by the expression

$$\sigma_h^2 = \frac{\pi^2}{4} \left( \frac{\sin \varphi}{\varphi} \right)^2 \left\{ \langle q^2 \rangle - \langle q \rangle^2 \right\} = \frac{\pi^2}{4} \left( \frac{\sin \varphi}{\varphi} \right)^2 \sigma_q^2,$$  \hspace{1cm} (62)

where the second equality defines the variance of the distribution of the forcing strength $q$. As a result, this limiting case provides a simple relationship between the width of the distribution of forcing strength $q$ and the width of the distribution of the resulting matrix element $h_k$, i.e.,

$$\sigma_h = \frac{\pi \sin \varphi}{2 \varphi} \sigma_q.$$  \hspace{1cm} (63)

Next we consider the opposite case in which the forcing strength $q$ is held fixed and the angle $\varphi$ varies over a range. Here we consider a range of angles so that expectation values are taken via the operator

$$\langle \ldots \rangle \equiv \frac{1}{\Gamma} \int_0^\Gamma d\varphi \ldots ,$$  \hspace{1cm} (64)

which holds for any given quantity in the brackets. Notice that we are using a distribution that is uniform in the variable $\varphi$. Since $\varphi \propto \sqrt{\lambda}$, this distribution is not uniform in the
variable $\lambda$, although an analogous analysis could be done for that case. For this choice of
distribution, the first moment of the matrix element is then given by the expression

$$
\langle h_k \rangle = \frac{\sin \Gamma}{\Gamma} - \frac{\pi q}{2\Gamma} \text{Si}(\Gamma),
$$

(65)

where $\text{Si}(\Gamma)$ is the sine integral [1]. The second moment is given by

$$
\langle h_k^2 \rangle = \frac{1}{2} + \frac{\sin 2\Gamma}{4\Gamma} + \left(\frac{\pi q}{2}\right)^2 \left\{ \frac{1}{\Gamma} \text{Si}(2\Gamma) - \frac{\sin^2 \Gamma}{\Gamma^2} \right\} - \frac{\pi q}{2\Gamma} \text{Si}(2\Gamma).
$$

(66)

These (general) expressions can be simplified by choosing the angle interval to have the
form $\Gamma = 2\pi m$, where $m$ is an integer. In this case, the corresponding variance reduces to
the form

$$
\sigma_h^2 = \frac{1}{2} + \frac{\pi q}{2\Gamma} \left(\frac{\pi q}{2} - 1\right) \text{Si}(2\Gamma) - \left(\frac{\pi q}{2\Gamma}\right)^2 \text{Si}^2(\Gamma).
$$

(67)

Next we note that these results simplify further in the limit $\Gamma \to \infty$, i.e., when we allow
the angle $\varphi$ to vary uniformly over the entire positive real line. In this limit we find

$$
\lim_{\Gamma \to \infty} \langle h_k \rangle = 0, \quad \lim_{\Gamma \to \infty} \langle h_k^2 \rangle = \frac{1}{2}, \quad \text{and} \quad \lim_{\Gamma \to \infty} \sigma_h = \frac{\sqrt{2}}{2}.
$$

(68)

As long as the distributions of the angle $\varphi$ and that of the forcing strength parameter
$q$ are independent, the expressions derived above for the moments can be generalized to
include both distributions in a straightforward manner. In particular, we continue to use
the distribution of equation (64) for the angle, and the same (unspecified) distribution
for the forcing strength as above. In this case, we must integrate over both the forcing
strength $q$ and the angle $\varphi$. By performing the integrals over $q$ first, we obtain expressions
analogous to equations (60) and (61) for the first two moments. With these results in
hand, we then average over the distribution of angle using equation (64). This procedure
produces expressions of the forms given by equations (65) and (66), with $q$ replaced by $\langle q \rangle$
and with $q^2$ replaced by $\langle q^2 \rangle$. For the case in which $\Gamma = 2\pi m$, the resulting expressions for
the first two moments take the form

$$
\langle h_k \rangle = -\frac{\pi}{2\Gamma} \text{Si}(\Gamma) \langle q \rangle \quad \text{and} \quad \langle h_k^2 \rangle = \frac{1}{2} + \frac{\pi^2}{4\Gamma} \text{Si}(2\Gamma) \langle q^2 \rangle - \frac{\pi}{2\Gamma} \text{Si}(2\Gamma) \langle q \rangle.
$$

(69)

The corresponding variance is thus given by

$$
\sigma_h^2 = \frac{1}{2} + \frac{\pi^2}{4\Gamma} \text{Si}(2\Gamma) \left\{ \langle q^2 \rangle - \frac{2}{\pi} \langle q \rangle \right\} - \frac{\pi^2}{4\Gamma^2} (\langle q \rangle)^2 \text{Si}^2(\Gamma).
$$

(70)

If we consider the limit where both $\langle q \rangle$ and $\Gamma$ are large, the variance of $h_k$ has the order

$$
\sigma_h^2 = \frac{1}{2} + \mathcal{O} \left( \frac{\sigma_q^2}{\Gamma} \right).
$$

(71)
Next we consider the analogous calculation for the matrix element $g_k$. For the limiting case in which the angle is held fixed and the forcing strength varies, the first two moments of the distribution are given by

$$\langle g_k \rangle = -\frac{1}{\pi} \varphi \sin \varphi - \frac{1}{2} (1 + \cos \varphi) \langle q \rangle ,$$  
(72)

and

$$\langle g_k^2 \rangle = \frac{1}{\pi^2} \varphi^2 \sin^2 \varphi + \frac{1}{4} (1 + \cos \varphi)^2 \langle q^2 \rangle - \frac{1}{\pi} \varphi \sin \varphi (1 + \cos \varphi) \langle q \rangle .$$

(73)

The corresponding variance thus takes the form

$$\sigma_g^2 = \frac{1}{4} (1 + \cos \varphi)^2 \left\{ \langle q^2 \rangle - \langle q \rangle^2 \right\} .$$

(74)

This result can be rewritten in a manner analogous to that found for the other principal solution (equation [63]), i.e.,

$$\sigma_g = \frac{1}{2} (1 + \cos \varphi) \sigma_q .$$

(75)

For fixed forcing strength $q$, and a distribution of angle given by equation (64), the first moment of the distribution becomes

$$\langle g_k \rangle = -\frac{1}{\pi} \left( \sin \frac{\Gamma}{\Gamma} - \cos \frac{\Gamma}{\Gamma} \right) - \frac{q}{2} \left( 1 + \sin \frac{\Gamma}{\Gamma} \right) ,$$

(76)

and the second moment is given by

$$\langle g_k^2 \rangle = \frac{1}{\pi^2} \left[ \frac{\Gamma^2}{6} - \frac{\cos(2\Gamma)}{4\Gamma} - \frac{2\Gamma^2 - 1}{8\Gamma} \sin(2\Gamma) \right] + \frac{q^2}{4} \left[ \frac{3}{2} + \frac{2\sin \Gamma}{\Gamma} + \frac{\sin(2\Gamma)}{4\Gamma} \right]$$

$$+ \frac{q}{\pi} \left[ - \cos \Gamma - \frac{1}{4} \cos(2\Gamma) + \frac{\sin \Gamma}{\Gamma} + \frac{\sin(2\Gamma)}{8\Gamma} \right] .$$

(77)

These expressions are somewhat cumbersome; if we take the angular interval to be $\Gamma = 2\pi m$, where $m$ is an integer as before, all of the sine terms vanish and the moments simplify to the forms

$$\langle g_k \rangle = \frac{1}{\pi} - \frac{q}{2} \quad \text{and} \quad \langle g_k^2 \rangle = \frac{1}{\pi^2} \left( \frac{\Gamma^2}{6} - \frac{1}{4\Gamma} \right) + \frac{3q^2}{8} - \frac{5q}{4\pi} .$$

(78)

In this case, the variance is given by

$$\sigma_g^2 = \frac{1}{\pi^2} \left( \frac{\Gamma^2}{6} - \frac{1}{4\Gamma} - 1 \right) + \frac{q^2}{8} - \frac{q}{4\pi} .$$

(79)

In the limit $\Gamma \to \infty$, the width of the distribution does not converge, but rather diverges linearly so that

$$\lim_{\Gamma \to \infty} \sigma_g \sim \frac{\sqrt{6}}{\pi 6} \Gamma .$$

(80)
As before, we can simultaneously include the distributions of angle and forcing parameter, provided that the variables are sampled in an independent manner. For the case where the angular interval is taken to be \( \Gamma = 2\pi m \), the moments reduce to the forms
\[
\langle g_k \rangle = \frac{1}{\pi} - \frac{1}{2} \langle q \rangle \quad \text{and} \quad \langle g_k^2 \rangle = \frac{1}{\pi^2} \left( \frac{\Gamma^2}{6} - \frac{1}{4\Gamma} \right) + \frac{3}{8}\langle q^2 \rangle - \frac{5}{4\pi} \langle q \rangle^2. \tag{81}
\]
The variance for this case is given by
\[
\sigma_g^2 = \frac{1}{\pi^2} \left( \frac{\Gamma^2}{6} - \frac{1}{4\Gamma} - 1 \right) + \frac{1}{4} \left\{ \frac{3}{2} \langle q^2 \rangle - \langle q \rangle^2 \right\} - \frac{1}{4\pi} \langle q \rangle. \tag{82}
\]
As a way to compare with the results found for the moments of the \( h_k \) distribution, we again consider the limit where both \( \langle q \rangle \) and \( \Gamma \) are large. In this case, the variance of the \( g_k \) has the order
\[
\sigma_g^2 = \mathcal{O}(\Gamma^2) + \mathcal{O}(\sigma_q^2). \tag{83}
\]
Comparing this result with equation (71), we find that when the angular interval \( \Gamma \) is not too large, the variance of the \( h_k \) and that of the \( g_k \) are of the same order. When the interval \( \Gamma \) is large, however, the variance of the \( g_k \) dominates.

Next we consider the limit of large forcing parameters \( q_k \). In this limit, the leading order growth rate is given by Theorem 2.1, and the corrections are determined by the variables \( x_k \) and \( \phi_k \) appearing in the matrix of equation (7). For the sake of definiteness, we define the mean \( q_0 = \langle q_k \rangle \), with \( q_0 \) large, and allow the \( q_k \) to have large variations about the mean, but not so large that \( q_k \to 0 \). Under these conditions, one can show that the variances of the variables \( x_k \) and \( \phi_k \) (from equation 7) are of the order
\[
\sigma_x^2 = \mathcal{O}(q_0^{-2}) \quad \text{and} \quad \sigma_\phi^2 = \mathcal{O}(q_0^{-4}). \tag{84}
\]
As a result, the variance is dominated by the \( x_k \) rather than by the \( \phi_k \). This result is consistent with the findings of Section II.

VI. CONCLUSIONS AND DISCUSSION

This paper has generalized and extended previous work concerning Hill’s equations (1) that contain random forcing parameters, with a focus on the case of delta function barriers (equation 2). In this formulation of the problem, both the natural oscillation frequency \( \lambda_k \) and the forcing strength \( q_k \) can vary from cycle to cycle. The development of the solutions to Hill’s equation, including the growth rates for instability, are given by the general problem of matrix multiplication (equation 3), where the matrix elements are
determined by the principal solutions for a given cycle. We have constructed the principal solutions for individual cycles using Dirac delta functions as the periodic barriers (equation 5). This construction allows us to explicitly show how both the matrix elements $h_k$ and $g_k$, and the growth rates $\gamma$, depend on the distributions of the original parameters $(\lambda_k, q_k)$ appearing in Hill’s equation (Section V).

In the limit of large forcing strength parameters $q_k$, the growth rates approach the form $\gamma \sim \langle \log |q_k| \rangle$ (see Theorem 2.1). In this limit, large $q_k$ values often lead to large values of the principal solutions at the end of the cycle; this result, in turn, demonstrates (by construction) that the highly unstable limit [2] can be realized using unremarkable values of the parameters. In the opposite limit of small forcing parameters, the growth rates approach the form $\gamma \sim \langle q_k^2 \rangle$ (Theorem 3.1). Our results in this limit show how the fluctuations in the Hill’s equation parameters act to fill in the bands of stability in the classic problem with fixed parameters (Figure 3). We have also found analytic results for the widths of the remaining bands of stability (where the growth rates vanish). In this same limit (small $q_k$), the Fokker-Planck equation provides an alternate description of the dynamics (Section IV) and consistent estimates for the growth rates. Finally, we have constructed a iterative map (Appendix C) that provides a heuristic argument for the general form of the growth rates in the limits of both large and small forcing strength $q_k$.

The results of this work can be used in a number of applications. For example, one motivation for considering random Hill’s equations was to study orbital instabilities in extended mass distributions, such as dark matter halos, galactic bulges, and young embedded star clusters (see Appendix A). The results presented herein show when the orbits are unstable and provide estimates for the corresponding growth rates. These results, in turn, help explain the observed dynamical structures in these astrophysical systems. Another important application involves the reheating problem at the end of the inflationary epoch in the early universe (see Appendix B). In this context, the introduction of stochastic perturbations (e.g., due to quantum fluctuations) leads to the disappearance of the bands of stability (see Figures 2 and 3). As result, fluctuations enhance the effectiveness of the reheating process. In addition to these motivating examples, random Hill’s equations arise in a wide variety of other physical problems [2–5,7,14,15,19–21].
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APPENDIX A:
RANDOM HILL’S EQUATION FROM ASTROPHYSICAL ORBITS

One application of Hill’s equation with random forcing terms involves the study of an instability that affects orbits in extended mass distributions, such as dark matter halos [4]. In this setting, the density profile \( \rho(\varpi) \) of the halo has the general form given by

\[
\rho(\varpi) = \rho_0 \frac{F(\varpi)}{\varpi},
\]  

where \( \rho_0 \) is a density scale and the variable \( \varpi \) is written in terms of the usual \((x, y, z)\) coordinates through the relation

\[
\varpi^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},
\]

where \( a > b > c > 0 \). The density field is thus constant on ellipsoids. The function \( F(\varpi) \) is approaches unity as \( \varpi \to 0 \) so that the density profile approaches the form \( \rho \sim 1/\varpi \) in the inner limit. For this regime, one can find analytic forms for both the potential and the force terms [4].

Further, when an orbit begins in any of the three principal planes, the motion can be unstable to perturbations in the perpendicular direction. Consider an orbit initially confined to the \( x - z \) plane, with a small perturbation in the perpendicular \( \hat{y} \) direction. In the limit \( |y| \ll 1 \), the equation of motion for the \( y \)-coordinate takes the form

\[
\frac{d^2 y}{dt^2} + \omega_y^2 y = 0 \quad \text{where} \quad \omega_y^2 = \frac{4/b}{\sqrt{c^2 x^2 + a^2 z^2 + b \sqrt{x^2 + z^2}}}. 
\]

(A3)

In this setting, the time evolution of the coordinates \((x, z)\) is determined by the original orbit. Since this orbital motion is nearly periodic, the \([x(t), z(t)]\) dependence of the parameter \( \omega_y^2 \) provides a periodic forcing term. The orbit has a maximum extent (outer turning points) which results in a minimum value for \( \omega_y^2 \), which in turn defines the natural oscillation frequency \( \lambda_k \). The parameter \( \omega_y^2 \) defined above can thus be written in the form

\[
\omega_y^2 = \frac{4/b}{\sqrt{c^2 x^2 + a^2 z^2 + b \sqrt{x^2 + z^2}}} = \lambda_k + Q_k(t),
\]

(A4)
where the index $k$ counts the number of orbit crossings, and the chaotic orbit in the original plane leads to different values of $\lambda_k$ and $Q_k(t)$ for each crossing. The shape of the functions $Q_k$ are nearly the same, however, so that one can write $Q_k(t) = q_k \hat{Q}(t)$, where the forcing strength parameters $q_k$ vary from cycle to cycle. These forcing strengths $q_k$ are determined by the inner turning points of the orbit (with appropriate weighting from the axis parameters $[a, b, c]$). Given the expansion of equation (A4), the equation of motion for the perpendicular coordinate becomes a random Hill’s equation, with the form of equation (II), as studied herein.

APPENDIX B:
RANDOM HILL’S EQUATION FROM REHEATING IN INFLATION

In the inflationary universe paradigm [12], the accelerated expansion of the universe is (usually) driven by the vacuum energy associated with a scalar field $\varphi$ (often called the inflaton). During the phase of accelerated expansion, the energy density of the universe itself decreases exponentially and the cosmos becomes relentlessly empty. This epoch is thought to take place when the universe is extremely young, with typical time scales of $\sim 10^{-36}$ sec.

In order for the inflationary epoch to solve the cosmological issues it was designed to alleviate, the end of inflation must involve a mechanism to fill the universe with energy (e.g., see the review in Ref. [17]). This process is called reheating. During the epoch of reheating, the equation of motion for the inflaton field displays oscillatory behavior about the minimum of its potential. Further, in order for the universe to become filled with energy (reheat), the inflaton field $\varphi$ must couple to matter or radiation fields. One simple type of interaction that is often considered uses an coupling term in the Lagrangian of the form

$$L_{\text{int}} = g\varphi \chi^2,$$

(B1)

where $\chi$ is a second scalar field that represents matter (radiation) and where the coupling constant $g$ sets the strength of the interaction. The field $\chi$ is generally expanded in terms of its Fourier modes $\chi_k$ since these quantities evolve independently. The resulting equation of motion for the matter field modes $\chi_k$ takes the form

$$\frac{d^2 \chi_k}{dt^2} + \left[ \omega_k^2 + p(t) + q(t) \right] \chi_k = 0,$$

(B2)

where $p(t)$ is a periodic function (given by the oscillatory behavior of the inflaton field) and $q(t)$ is a noise term that provides perturbations to the driving term $p(t)$ [21]. Note
that the index \( k \) refers here to the Fourier mode, although the forcing terms do vary from cycle to cycle. In the absence of fluctuations, the matter field modes \( \chi_k \) thus obey a type of Hill’s equation, which is subject to parametric instability [15,16]. The noise perturbations convert the equation into a random Hill’s equation [15,16,21], of the type studied herein. This type of equation was solved numerically using WKB methods [16], thereby finding the relevant physical solutions; nonetheless, the formulation of this paper can be applied to this class of reheating problems, and more general results can be obtained.

**APPENDIX C: AN ITERATIVE MAP**

As shown in the text, the growth rates for Hill’s equation depend on the forcing strength \( q_k \) according to \( \gamma \sim \langle q_k^2 \rangle \) in the limit of small symmetric \( q_k \), and \( \gamma \sim \langle \log |q_k| \rangle \) in the limit of large \( q_k \). These results hold both for the particular case of delta function barriers (considered here), and for the general problem [3]. In this Appendix, we construct a heuristic argument that reproduces these forms for the growth rate in the two limits. This treatment is highly approximate, by design, but allows for a simple interpretation of our previously obtained results.

Given the form of Hill’s equation in the delta function limit, the jump condition across the barrier takes the form

\[
\frac{dy}{dt} \bigg|_+ = \frac{dy}{dt} \bigg|_- - q_k y,
\]  

(C1)

where all of the functions are evaluated at the barrier. If we define \( V \equiv dy/dt \), and relabel the functions with an index \( k + 1 \) on the far side of the barrier, and an index \( k \) on the near side, we obtain an iterative map of the form

\[
V_{k+1} = V_k \left[ 1 - q_k \frac{y_k}{V_k} \right] = V_0 \prod_{k=1}^{N} \left[ 1 - q_k \frac{y_k}{V_k} \right],
\]  

(C2)

where we have continued the iteration back to the initial step to obtain the second equality. The growth rate \( \gamma \) for this map can then be defined according to

\[
\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left| 1 - q_k \frac{y_k}{V_k} \right|.
\]  

(C3)

Given the form of Hill’s equation away from the delta function barrier, the solutions are oscillatory with frequency \( \sqrt{\lambda_k} \), so that the function \( y_k \) and the velocity are related via

\[
\frac{y_k}{V_k} = \frac{1}{\sqrt{\lambda_k}} F(\sqrt{\lambda_k} t),
\]  

(C4)
where the function $F$ depends on the angle $\sqrt{\lambda_k} t$. If we use the ansatz implied by equation (C4), the growth rate takes the form

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left| 1 - \frac{q_k}{\sqrt{\lambda_k}} F \right|. \quad (C5)$$

In the limit of large forcing strength $q_k \gg 1$, the growth rate reduces to the form

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left( \frac{q_k}{\sqrt{\lambda_k}} F \right) \sim \log \left( \frac{q_k}{\sqrt{\lambda_k}} \right) \sim \log |h_k|, \quad (C6)$$

where we have ignored the function $F$ of the angle in obtaining the final approximate forms. This argument reduces the problem to a single (approximate) iterated jump condition, but still reproduces the proper dependence of the growth rate for the highly unstable limit ($\gamma \sim \langle \log |h_k| \rangle$). We note that ignoring the angular function is not valid when $F \to 0$. As shown above, this problem allows for narrow bands of stability where the growth rate can vanish even when the forcing strength is large. The presence of stable behavior ($\gamma \to 0$) can thus be accounted for through this heuristic argument (by allowing $F \to 0$).

In the opposite limit of small forcing strength $|q_k| \ll 1$, we can expand the logarithmic function in the expression for the growth rate to obtain

$$\gamma = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left( \frac{q_k}{\sqrt{\lambda_k}} F + \frac{q_k^2}{2\lambda_k} F^2 \right). \quad (C7)$$

For symmetric fluctuations, the first term vanishes in the limit, so that the quadratic term provides the leading order contribution to the growth rate. Ignoring the angular function $F$ as above, the growth rate becomes

$$\gamma \sim \langle q_k^2/\lambda_k \rangle. \quad (C8)$$

In this case, the iterated jump condition argument reproduces the proper dependence of the growth rate for the limit of symmetric and weak forcing (compare with Theorem 2.1).
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