THE CAUCHY PROBLEM OF THE WARD EQUATION

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Abstract
We generalize the results of [22], [11], [8] to study the inverse scattering problem of the Ward equation with non-small data and solve the Cauchy problem of the Ward equation with a non-small purely continuous scattering data.

Keywords: Self-dual Yang-Mills equation, Lax pair, inverse scattering problem, Riemann-Hilbert problem, Cauchy integral operator

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1. Introduction

The Ward equation (or the modified 2 + 1 chiral model)
\begin{equation}
\partial_t (J^{-1} \partial_t J) - \partial_x (J^{-1} \partial_x J) - \partial_y (J^{-1} \partial_y J) - [J^{-1} \partial_x J, J^{-1} \partial_y J] = 0,
\end{equation}
for \( J : \mathbb{R}^{2,1} \to SU(n) \), \( \partial_{w^a} = \partial / \partial w^a \), is obtained from a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation on \( \mathbb{R}^{2,2} \) [8], [19]. It is an integrable system which possesses the Lax pair [23]
\begin{equation}
[\lambda \partial_x - \partial_t - J^{-1} \partial_t J, \lambda \partial_y - \partial_x - J^{-1} \partial_x J] = 0
\end{equation}
with $\xi = \frac{t+y}{2}$, $\eta = \frac{t-y}{2}$. Note (1.2) implies that $J^{-1} \partial_\xi J = -\partial_x Q$, $J^{-1} \partial_\eta J = -\partial_y Q$.

Then by a change of variables $(\eta, x, \xi) \rightarrow (x, y, t)$, (1.2) is equivalent to

\begin{align}
(\partial_y - \lambda \partial_x) \Psi(x, y, t, \lambda) &= (\partial_x Q(x, y, t)) \Psi(x, y, t, \lambda), \\
(\partial_t - \lambda^2 \partial_x) \Psi(x, y, t, \lambda) &= (\lambda \partial_x Q + \partial_y Q) \Psi(x, y, t, \lambda)
\end{align}

(1), and the Ward equation (1.1) turns into:

\begin{align}
\partial_x \partial_t Q &= \partial_y^2 Q + [\partial_y Q, \partial_x Q].
\end{align}

The construction of solitons, the study of the scattering properties of solitons, and Darboux transformation of the Ward equation have been studied extensively by solving the degenerated Riemann-Hilbert problem and studying the limiting method [23], [24], [14], [2], [3], [15], [26]. In particular, Dai and Terng gave an explicit construction of all solitons of the Ward equation by establishing a theory of Backlund transformation [7].

For the investigation of the Cauchy problem of the Ward equation, Villarroel [22], Dai, Terng and Uhlenbeck [8] use Fourier analysis in the $x, y$-space to study the spectral theory of $L_\lambda = \partial_y - \lambda \partial_x$ in (1.3), whilst Fokas and Ioannidou [11] invert $L_\lambda$ by interpreting it as a 1-dimensional spectral operator with coefficients being the $x$-Fourier transform of functions. In both cases, small data conditions of $Q$ are required to ensure the invertibility of $L_\lambda$ and the solvability of the inverse problem. Under the small data condition, the eigenfunctions $\Psi$ possesses continuous scattering data only and therefore the solutions for the Ward equation do not include the solitons in previous study.

Nonetheless, the approach of Fokas and Ioannidou [11] shows that: after taking the Fourier transform in the $x$-space, (1.3) looks similar to the spectral problem of the AKNS system

\begin{align}
(\partial_x - \lambda J) \Psi(x, t, \lambda) &= q(x, t) \Psi(x, t, \lambda).
\end{align}

Where $J$ is a constant diagonal matrix with distinct eigenvalues. The solution of the forward and inverse scattering problem of the AKNS system is fairly complete, due to the work of Beals, Coifman, Deift, Tomei, Zhou [4], [6], [9]. In particular, the inverse scattering problem for the AKNS system and its associated nonlinear evolution equations is rigorously solved for generic $q \in L_1$ without small data condition [5].

The purpose of the present paper is to remove the small data condition in solving the scattering and inverse scattering problem of (1.3) and the Cauchy problem of the Ward equation (1.5) with a purely continuous scattering data. We summarize principal results as follows:

**Theorem 1.1.** Let $Q \in P_{\infty, 2, 0}$. Then there is a bounded set $Z \subset \mathbb{C}$ such that

- $Z \cap (\mathbb{C}\setminus\mathbb{R})$ is discrete in $\mathbb{C}\setminus\mathbb{R}$;
- For $\lambda \in \mathbb{C}\setminus(\mathbb{R} \cup Z)$, the problem (1.3) has a unique solution $\Psi$ and $\Psi^{-1} \in DH^2$;
- For $(x, y) \in \mathbb{R} \times \mathbb{R}$, the eigenfunction $\Psi(x, y, \cdot)$ is meromorphic in $\lambda \in \mathbb{C}\setminus\mathbb{R}$ with poles precisely at the points of $Z \cap (\mathbb{C}\setminus\mathbb{R})$;
- $\Psi(x, y, \lambda)$ satisfies:

\begin{align}
\lim_{|x| \to \infty} \Psi(\cdot, y, \lambda) &= 1, \quad \lim_{|y| \to \infty} \Psi(x, \cdot, \lambda) = 1, \quad \text{for } \lambda \in \mathbb{C}\setminus(\mathbb{R} \cup Z), \\
\Psi(x, y, \lambda) \text{ tends to 1 uniformly as } |\lambda| \to \infty;
\end{align}

(1.6) (1.7)
Theorem 1.2. Let \( L \) be the derivation properties of the spectral operator \( H \) of the
Hilbert problem with a non-small continuous data by the translating invariant and
on the real line, or the limit points will accumulate themselves. Assuming higher
regularities on the potential \( Q \) centered at \( \lambda \).

(1.1)

where \( \epsilon \geq \epsilon_j > 0 \) are any given constants, \( D_\epsilon(\lambda_j) \) denotes the disk of radius \( \epsilon \)
centered at \( \lambda_j \).

Here the function spaces \( P_{\infty,2,0} \) and \( D\mathbb{H}^2 \) are defined by

Definition 1.

\[
P_{\infty, k_1, k_2} = \{ q_\epsilon(x, y) : \mathbb{R} \times \mathbb{R} \to su(n) \}
\]

\[
L_1(d\xi dy), \quad ||x^k \partial_x^m q_\epsilon(\xi, y) ||_{L_1(d\xi)} \leq \epsilon \quad \text{for } 1 \leq i \leq \max\{5, k_1\}, \quad 0 \leq j, l \leq \max\{5, k_1\}, \quad 1 \leq h \leq k_1, \quad 0 \leq s \leq k_2 \}
\]

\[
D\mathbb{H}^k = \{ f | \partial_x^k f(x, y) \text{ are uniformly bounded in } L_2(\mathbb{R}, dx), \quad 0 \leq i \leq k \}
\]

To derive Theorem 1.1, we transform the existence problem of \( \Psi \) into a Riemann-
Hilbert problem with a non-small continuous data by the translating invariant and
the derivation properties of the spectral operator \( L_\lambda \), and an induction scheme.
Hence the scheme of Section 10 in [4] can be adapted to solve the Riemann-Hilbert
problem. That is, we first approximate the solution by a piecewise rational function.
Then the correction is made by a solution of a Riemann-Hilbert problem with small
data and a solution of a finite linear system. Since the eigenfunction obtained in
each induction step consists the data of the Riemann-Hilbert problem in the next
step, we need to obtain the \( H^2 \)-estimate \( (1.8) \) of the eigenfunction. Besides, the
boundary estimate \( (1.9) \) and the meromorphic property are derived in each step to
assure the solvability of the linear system.

In general, the points in \( Z \), i.e., poles of \( \Psi(x, y, \lambda) \), will occur or accumulate
on the real line, or the limit points will accumulate themselves. Assuming higher
regularities on the potential \( Q \) and \( Z = Z(\Psi) = \phi \) (there are no poles of \( \Psi(x, y, \lambda) \)),
we can extract the continuous scattering data:

Theorem 1.2. For \( Q \in P_{\infty, k, 1} \), \( k \geq 7 \), if \( Z = \phi \), then there exists uniquely a
function \( v(x, y, \lambda) \in \mathcal{G}_{c,k} \) which satisfies

\[
\Psi_+(x, y, \lambda) = \Psi_-(x, y, \lambda)v(x, y, \lambda), \quad \lambda \in R.
\]

Where the space \( \mathcal{G}_{c,k} \) is defined by

Definition 2. Let \( \mathcal{G}_{c,k}, k \geq 7 \), be the space consisting of continuous scattering
data \( v(x, y, \lambda), \lambda \in \mathbb{R} \), such that \( v \) satisfies the algebraic constraints:

\[
\det(v) = 1,
\]

\[
v = v^* > 0,
\]
and the analytic constraints: for \( i + j \leq k - 4 \),

\[
(1.12) \quad L_\lambda v = 0, \quad v(x, y, \lambda) = v(x + \lambda y, \lambda) \quad \text{for} \quad \forall x, y \in \mathbb{R},
\]

\[
(1.13) \quad \partial^j_x \partial^i_y (v - 1) \quad \text{are uniformly bounded in} \quad L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda);
\]

\[
(1.14) \quad \partial^j_x \partial^i_y (v - 1) \rightarrow 0 \quad \text{uniformly in} \quad L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda)
\]

\[ \text{as} \quad |x| \quad \text{or} \quad |y| \rightarrow \infty; \]

\[
(1.15) \quad \partial_\lambda v \quad \text{are in} \quad L_2(\mathbb{R}, d\lambda) \quad \text{and the norms depend continuously on} \quad x, y.
\]

Where \( L_\lambda = \partial_y - \lambda \partial_x \).

The characterization of the scattering data \( v \in \mathcal{S}_{c,k} \) is necessary. Since the Cauchy integral operator will play a key role in the inverse problem. The study of the asymptotic behavior of the scattering data \( v \) (hence the asymptotic behavior of the eigenfunctions \( \Psi \)) is important. Because the Cauchy operator is bounded in \( L_2 \) in general, an \( L_2 \)-estimate of \( \Psi \) and its derivatives will be good enough. However, a formal calculation will yield (1.19) if the inverse problem is solvable. Hence we provide the estimates (1.13)-(1.15).

The derivation of (1.13)-(1.15) basically relies on the \( L_2 \)-boundness of the Cauchy operator and the estimates obtained in the small-data problem. In particular, both of the 1-dimensional (Fokas and Ioannidou [11] or (2.7)) and the 2-dimensional formulation (Villarroel [22] or (3.1)) of the spectral problem are crucial in the derivation of the estimates with small data condition. That is, using (2.7), boundness or integrability in \( x \)-variable of the eigenfunctions \( \Psi \) comes first from the differentiability and integrability of the potentials \( Q \) via the Fourier transform. Then, strong asymptote in \( x, y \) or \( \lambda \)-variable of the eigenfunctions \( \Psi \) can be obtained by (3.1) and previous estimates. We lose some regularities in deriving strong asymptote. See the proof of Theorem 3.1 for example.

For the inverse problem, our results are:

**Theorem 1.3.** Given \( v(x, y, \lambda) \in \mathcal{S}_{c,k}, \ k \geq 7 \), there exists a unique solution \( \Psi(x, y, \cdot) \) for the Riemann-Hilbert problem \( (\lambda \in \mathbb{R}, v(x, y, \lambda)) \) such that

\[
(1.16) \quad \Psi - 1, \ \partial_x \Psi, \ \partial_y \Psi \quad \text{are uniformly bounded in} \quad L_2(\mathbb{R}, d\lambda).
\]

Moreover, for each fixed \( \lambda \not\in \mathbb{R} \), and \( i + j \leq k - 4 \),

\[
(1.17) \quad \partial^j_x \partial^i_y \Psi \in L_\infty(dx dy),
\]

\[
(1.18) \quad \partial^j_x \partial^i_y (\Psi - 1) \rightarrow 0 \quad \text{in} \quad L_\infty(dx dy), \quad \text{as} \quad x \quad \text{or} \quad y \rightarrow \infty.
\]

Theorem 1.3 is proved by a Riemann-Hilbert problem with a non-small purely continuous scattering data. Without uniform boundedness of \( \partial_\lambda v \), we need to handle separately the Riemann-Hilbert problem for \( |\lambda| > M >> 1 \) and \( |\lambda| \leq M \). For \( |\lambda| > M >> 1 \), the Riemann-Hilbert problem is a small-data problem and hence can be solved. For \( |\lambda| \leq M \), the Riemann-Hilbert problem is again factorized into a diagonal problem, a Riemann-Hilbert problem with small data, and a finite linear system. Note we obtain the globally solvability by applying the Fredholm property and the reality condition (1.11).

Moreover, good estimates for \( \Psi \) can be derived only for \( \lambda \not\in \mathbb{R} \). However, it is enough to imply satisfactory analytical properties of the potentials.
Theorem 1.4. Given \( v(x, y, \lambda) \in S_{c, k}, \ k \geq 7, \) the eigenfunction \( \Psi \) obtained by Theorem 1.3 satisfies (1.3) with

\[
Q(x, y) = \frac{1}{2\pi i} \int_{\mathbb{R}} \Psi_-(v - 1)d\zeta,
\]

and \( \Psi(x, \cdot, \lambda) \to 1 \) as \( y \to -\infty. \) Where \( \partial_x Q(x, y) \in su(n), \) and for \( i + j \leq k - 4, \ i > 0, \ \partial_i \partial_j Q, \ \partial_y Q, \ Q \in L_{\infty}, \ \partial_i \partial_j^h Q, \ \partial_y Q, \ Q \to 0 \) as \( x \) or \( y \to \infty. \)

Applying Theorem 1.1-1.4, we extend the results of [22], [11], [8] by:

Theorem 1.5. If \( Q_0 \in P_{\infty, k, 1}, \ k \geq 7, \) and there are no poles of the eigenfunction \( \Psi_0 \) of \( Q_0, \) then the Cauchy problem of the Ward equation (1.5) with initial condition

\[
Q(x, y, 0) = Q_0(x, y)
\]

admits a smooth global solution satisfying: for \( i + j + h \leq k - 4, \ i^2 + j^2 > 0, \)

\[
\partial_x Q(x, y, t) \in su(n),
\]

\[
\partial_i \partial^j Q, \ \partial_y Q, \ Q \in L_{\infty},
\]

\[
\partial_x \partial_i \partial^j Q, \ \partial_y Q, \ Q \to 0, \text{ as } x, y, t \to \infty.
\]

The paper is organized as: In Section 2, we review an existence theorem of Fokas and Ioannidou [11] by an analytical treatment. In Section 3, under the small-data constraint, we analyze the asymptotic behavior of the eigenfunctions. In Section 4 and 5, we solve the direct problem by justifying Theorem 1.1 and 1.2. The inverse problem is complete in Section 6 by proving Theorem 1.3 and 1.4. Finally, Theorem 1.5 is proved in Section 7.

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2. Direct problem I: Eigenfunctions with small data

Given a potential \( \partial_x Q(x, y) : \mathbb{R} \times \mathbb{R} \to su(n), \) and a constant \( \lambda \in \mathbb{C}, \) we consider the boundary value problem

\[
\begin{align*}
\partial_y \Psi(x, y, \lambda) - \lambda \partial_x \Psi(x, y, \lambda) - (\partial_x Q) \Psi(x, y, \lambda) &= 0, \\
\Psi(x, y, \lambda) &\to 1, \quad \text{as } y \to -\infty.
\end{align*}
\]

To investigate the problem, we denote throughout that

Definition 3.

\[
P_1 = \left\{ \partial_q g(x, y) : \mathbb{R} \times \mathbb{R} \to su(n) \mid |\xi g(\xi, y)|_{L_1(\xi dy)} < 1 \right\},
\]

\[
\mathbb{X} = \left\{ w(x, y) : \mathbb{R} \times \mathbb{R} \to M_n(\mathbb{C}) \mid \sup_y |\hat{w}(\xi, y)|_{L_1(\xi)} < \infty \right\},
\]

\[
\hat{\mathbb{X}} = \left\{ f(\xi, y) : \mathbb{R} \times \mathbb{R} \to M_n(\mathbb{C}) \mid \sup_y |f(\xi, y)|_{L_1(\xi)} < \infty \right\}.
\]
Thus we are led to consider the following integral equations

\[ |f| = \text{trace} (f^* f)^{\frac{1}{2}}, \quad f^* = \hat{f}. \]

\[ |f(\xi, y)|_{L_1(dy)} = \int_{\mathbb{R}} |f(\xi, y)|d\xi. \]

**Theorem 2.1.** Suppose \( Q \in \mathbb{P}_1 \). Then for all fixed \( \lambda \in \mathbb{C}^\pm \), there is uniquely a solution \( \Psi \) of (2.1) and (2.2) such that \( \Psi - 1 \in \mathbb{X} \). Moreover, for \( \lambda \in \mathbb{C}^\pm \),

\[ \lim_{|x| \to \infty} \Psi(\cdot, y, \lambda) = I, \quad \lim_{|y| \to \infty} \Psi(x, \cdot, \lambda) = I. \]

**Proof.** Write \( \Psi = 1 + W \). Then (2.1), (2.2) are transformed into

\[ \partial_{y} W - \lambda \partial_{x} W = (\partial_{x} Q) W + \partial_{x} Q, \]

\[ W(x, y, \lambda) \to 0 \text{ as } y \to -\infty. \]

Taking the Fourier transform with respect to the \( x \)-variable (in distribution sense), we obtain

\[ \partial_{y} \hat{W}(\xi, y, \lambda) - i \xi \lambda \hat{W}(\xi, y, \lambda) = (\partial_{x} Q) \hat{W}(\xi, y, \lambda) + \partial_{x} Q(\xi, y). \]

Thus we are led to consider the following integral equations

\[ \hat{W}(\xi, y, \lambda) = \begin{cases} 
\int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * \hat{W} + \partial_{x} Q \right) dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \geq 0; \\
- \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * \hat{W} + \partial_{x} Q \right) dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \leq 0; \\
- \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * \hat{W} + \partial_{x} Q \right) dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \geq 0; \\
\int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * \hat{W} + \partial_{x} Q \right) dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \leq 0.
\end{cases} \]

Where \( * \) is the convolution operator with respect to the \( \xi \)-variable. Define

\[ \mathcal{K}_{\lambda} f(\xi, y, \lambda) = \begin{cases} 
\int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * f \right) (\xi, y', \lambda) dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \geq 0; \\
- \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * f \right) (\xi, y', \lambda) dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \leq 0; \\
- \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * f \right) (\xi, y', \lambda) dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \geq 0; \\
\int_{y}^{\infty} e^{i \lambda \xi (y-y')} \left( \partial_{x} Q * f \right) (\xi, y', \lambda) dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \leq 0.
\end{cases} \]

Thus (2.4) turns into

\[ \hat{W} = \begin{cases} 
\mathcal{K}_{\lambda} \hat{W} + \int_{-\infty}^{y} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \geq 0; \\
\mathcal{K}_{\lambda} \hat{W} - \int_{-\infty}^{y} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^+, \xi \leq 0; \\
\mathcal{K}_{\lambda} \hat{W} - \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \geq 0; \\
\mathcal{K}_{\lambda} \hat{W} + \int_{-\infty}^{y} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^-, \xi \leq 0.
\end{cases} \]

where

\[ \int_{-\infty}^{y} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy', \quad \int_{y}^{\infty} e^{i \lambda \xi (y-y')} \partial_{x} Q(\xi, y') dy' \in \mathbb{X} \text{ by } Q \in \mathbb{P}_1. \]

Note

\[ |\mathcal{K}_{\lambda} f(\xi, y)|_{L_1(dy)} \leq \int_{-\infty}^{\infty} |\partial_{x} Q(\xi, y')|_{L_1(dy')} |f(\xi, y')|_{L_1(dy')} dy' \leq |\partial_{x} Q(\xi, y)|_{L_1(dy)} \sup_{y} |f|_{L_1(dy)}. \]
Hence
\[
K_{\lambda} : \mathcal{X} \to \mathcal{X}, \quad \|K_{\lambda}\| \leq |\partial_x Q(\xi, y)|_{L_1(d\xi dy)} < 1.
\]
So
\[
\bar{W} = \begin{cases}
(1 - K_{\lambda})^{-1} \int_{y}^{\infty} e^{\lambda \xi (y-y')} \partial_x Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^+, \; \xi \geq 0; \\
-(1 - K_{\lambda})^{-1} \int_{y}^{\infty} e^{\lambda \xi (y-y')} \partial_x Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^+, \; \xi \leq 0; \\
-(1 - K_{\lambda})^{-1} \int_{y}^{\infty} e^{\lambda \xi (y-y')} \partial_x Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^-, \; \xi \geq 0; \\
(1 - K_{\lambda})^{-1} \int_{y}^{\infty} e^{\lambda \xi (y-y')} \partial_x Q(\xi, y') dy', & \text{if } \lambda \in \mathbb{C}^-, \; \xi \leq 0.
\end{cases}
\]
Hence (2.7) is solvable if \(Q \in \mathbb{P}_1\). Furthermore, the eigenfunction of (2.1), (2.2) is given by:
\[
\Psi(x, y, \lambda) = \begin{cases}
1 + \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{y} dy' - \int_{-\infty}^{0} d\xi \int_{y}^{\infty} dy' \right) & \text{if } \lambda \in \mathbb{C}^+; \\
\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dy' & \text{if } \lambda \in \mathbb{C}^-.
\end{cases}
\]

The uniform boundedness of \(\Psi\) comes from Definition 3, (2.6) and the Riemann-Lebesgue Theorem, we obtain \(\Psi(x, y, \lambda) \to 1\) as \(|x| \to \infty\). On the other hand, (2.7), \(\partial_x Q \ast \bar{W}, \partial_x Q \in L_1(d\xi dy)\) and the Lebesgue Convergence Theorem imply that \(\Psi(x, y, \lambda) \to 1\) when \(|y| \to \infty\).

**Lemma 2.1.** Suppose \(\Psi\) satisfies (2.1), (2.2). Then for \(\lambda \not\in \mathbb{R}\),
\[
\det \Psi(x, y, \lambda) \equiv 1.
\]

**Proof.** Let \(e_1, \ldots, e_n\) denote the standard basis for \(\mathbb{C}^n\), \(\psi_k\) the \(k\)-th column vector of the matrix \(\Psi\). Let \(\Lambda^k(\mathbb{C}^n)\) denote the space of alternating \(k\)-forms on \(\mathbb{C}^n\). Hence \(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n = (\det \Psi)(e_1 \wedge e_2 \wedge \cdots \wedge e_n)\). Taking derivatives of both sides, we derive
\[
\{(\partial_y - \lambda \partial_x) (\det \Psi)\} (e_1 \wedge e_2 \wedge \cdots \wedge e_n) = (\det \Psi) \{(\partial_y - \lambda \partial_x) (e_1) \wedge e_2 \wedge \cdots \wedge e_n\} = (\partial_y - \lambda \partial_x) \{\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n\} = \{(\partial_y - \lambda \partial_x) \psi_1\} \wedge \cdots \wedge \psi_n + \cdots + \psi_1 \wedge \cdots \wedge (\partial_y - \lambda \partial_x) \psi_n = (\partial_x Q) \psi_1 \wedge \cdots \wedge \psi_n + \cdots + \psi_1 \wedge \cdots \wedge (\partial_x Q) \psi_n = (\text{trace } \partial_x Q) \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n.
\]
So
\[
(\partial_y - \lambda \partial_x) (\det \Psi) = 0
\]
by \(\partial_x Q \in su(n)\). Moreover, for \(\lambda \not\in \mathbb{R}\), the equation turns into the debar equation
\[
\partial_{\bar{z}} (\det \Psi) = 0, \quad x, \; y \in \mathbb{R},
\]
by the change of variables:
\[
x + \lambda y = \bar{x} + i\bar{y} = z, \quad \bar{x}, \; \bar{y} \in \mathbb{R}.
\]
Therefore the Liouville’s Theorem and (2.8) imply that \(\det \Psi \equiv 1\), for \(\lambda \not\in \mathbb{R}\). \(\square\)
Lemma 2.2. Suppose that \( Q \in \mathbb{P}_1 \). Then the reality condition
\[
\Psi(x, y, \lambda) \Psi(x, y, \bar{\lambda})^* = I
\]
holds for the eigenfunction \( \Psi \).

Proof. By Lemma 2.1, one derives
\[
(\partial_y - \lambda \partial_x) \Psi(x, y, \bar{\lambda})^* = -\Psi(x, y, \bar{\lambda})^* (\partial_y - \lambda \partial_x) \Psi(x, y, \bar{\lambda})^* + (\partial_x Q) \Psi(x, y, \bar{\lambda})^* - 1
\]
Besides, noting \( |\hat{f}_n|_{L^1(\xi)} \leq |\hat{f}_n|_{L^1(\xi)} \) and the boundary condition of \( \Psi \), we obtain
\[
\Psi^{-1} - 1 \in X.
\]
Hence the lemma follows from the uniqueness property in Theorem 2.1. □

3. Direct problem II: Asymptotic analysis with small data

The results and arguments will be applied or adapted in Section 4 and 5.

Denote
\[
(f *_{x,y} g)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - x', y - y') g(x', y') \, dx' \, dy',
\]
\[
(f *_{z,\bar{z}} g)(z, \bar{z}) = \int_C f(z - \zeta, \bar{z} - \bar{\zeta}) g(\zeta, \bar{\zeta}) \, d\zeta \, d\bar{\zeta}.
\]

By the change of variables (2.8), we then have
\[
(\partial_y - \lambda \partial_x)^{-1} = \frac{i}{2\lambda_I} \partial_{\bar{z}}^{-1} = -\frac{1}{4\pi \lambda_I} *_{z,\bar{z}} = -\frac{1}{2\pi i} \frac{\text{sgn}(\lambda_I)}{x + \lambda y} *_{x,y}
\]
with \( \lambda = \lambda_R + i\lambda_I \). Now let \( \mathcal{S} \) be the set of Schwartz functions. If \( Q \in \mathbb{P}_1 \cap \mathcal{S} \), then the eigenfunction \( \Psi \) obtained by Theorem 2.1 satisfies
\[
(3.1) \quad \Psi = 1 + G_\lambda((\partial_x Q) \Psi)
\]

Where
\[
(3.2) \quad G_\lambda f(x, y, \lambda) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\text{sgn}(\lambda_I) f(x - x', y - y', \lambda)}{x' + \lambda y'} \, dx' \, dy'
\]

The following lemma is due to Richard Beals.

Lemma 3.1. Suppose \( \varphi \in \mathcal{S} \). For \( |\lambda| \neq 0 \) and \(|\lambda_I| < 1\),
\[
|G_\lambda \varphi| \leq \frac{C}{|\lambda|} (\sup_y |\partial_y \varphi|_{L^1(\xi)} + \sup_y |\varphi|_{L^1(\xi)} + |\varphi|_{L^1(\xi \times dy)}).
\]

Where \( C \) is a constant.
Proof. Let \( \frac{1}{s} = \frac{\lambda_R}{\lambda_R + \lambda_I} \). So
\[
\frac{1}{\lambda} = \frac{1}{s} - i \frac{\lambda_I}{\lambda_R} \frac{1}{s}, \quad \left| \frac{1}{y + \frac{x}{\lambda}} \right| \leq \frac{1}{|y + \frac{x}{s}|}.
\]
Write
\[
G_{\lambda} \varphi = \frac{-1}{2\pi i \lambda} \int \int_{|y' + \frac{x'}{s}| < 1} \text{sgn}(\lambda_I) \frac{\varphi(x - x', y - y')}{y' + \frac{x'}{\lambda}} dx' dy' + \int \int_{|y' + \frac{x'}{s}| > 1} \text{sgn}(\lambda_I) \frac{\varphi(x - x', y - y')}{y' + \frac{x'}{\lambda}} dx' dy' + \int \int_{|y' + \frac{x'}{s}| < 1} \text{sgn}(\lambda_I) \frac{\varphi(x - x', y + \frac{x'}{s})}{y' + \frac{x'}{\lambda}} dx' dy' = I_1 + I_2 + I_3.
\]
In view of (3.3), it is easy to see that
\[
|I_1| \leq \frac{1}{2\pi |\lambda|} \int \sup_{z \in \mathbb{C}} \left| \frac{\partial_y \varphi(x - x', y - z)}{y' + \frac{x'}{\lambda}} \right| dx'
\]
(3.4)
\[
\leq \frac{C_1}{|\lambda|} \sup_y |\partial_y \varphi|_{L_1(dx)},
\]
(3.5)
\[
|I_2| \leq \frac{1}{2\pi |\lambda|} \int \int_{|y' + \frac{x'}{s}| > 1} \left| \frac{\varphi(x - x', y - y')}{y' + \frac{x'}{\lambda}} \right| dx' dy' \leq \frac{C_2}{|\lambda|} |\varphi|_{L_1(dx dy)}.
\]
Finally,
\[
|\text{sgn}(\lambda_I) \int_{|y' + \frac{x'}{s}| < 1} \frac{1}{y' + \frac{x'}{\lambda}} dy'|
\]
\[
= \left| \log \left[ \frac{1 - i \frac{\lambda_I x'}{\lambda_R s}}{1 - i \frac{\lambda_I x'}{\lambda_R s}} \right] \right|
\]
\[
= \left| \arg(1 - i \frac{\lambda_I x'}{\lambda_R s}) - \arg(-1 - i \frac{\lambda_I x'}{\lambda_R s}) \right| \leq \pi
\]
This yields
\[
I_3 \leq \frac{1}{2|\lambda|} \int |\varphi(x - x', y + \frac{x'}{s})| dx'
\]
(3.6)
\[
\leq \frac{C_3}{|\lambda|} \sup_y |\varphi|_{L_1(dx)}.
\]
Combining (3.4), (3.5), and (3.6), we prove the lemma. \( \square \)

**Lemma 3.2.** Suppose that \( Q \in \mathbb{P}_1 \cap S \). Then there exist a constant \( C_N \) such that
\[
|\partial^N_x \Psi| \leq C_N.
\]
Where \( C_N \) is a constant depending on \( Q \).
Proof. Since
\[
\xi^N \partial_x Q * \tilde{W} = \sum_{k=0}^N \binom{N}{k} (\xi - \xi')^k \xi^{N-k} \partial_x Q * \tilde{W}
\]
\[
= \sum_{k=0}^N \binom{N}{k} (\xi^k \partial_x Q) * (\xi^{N-k} \tilde{W}).
\]
It suffices to prove \(\xi^i \tilde{W} \in X\) for \(0 \leq k \leq N\). This can be proved by induction on \(k\) and using the same argument as in the proof of Theorem 2.1 if \(|\xi^N \partial_x Q|_{L_1(d\xi dy)} < \infty\).

Definition 4. Define
\[
P_{1,k} = \{\partial_x q(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow su(n) \mid \|\xi^i q(\xi, y)\|_{L_1(d\xi dy)} < 1, \text{ and } \|\xi^i q\|_{L_1(d\xi dy)} \text{ for } 1 \leq i \leq \max\{5, k\}, \text{ and } 0 \leq j, h \leq \max\{5, k\}\}
\]
Note that \(P_1 \in P_{1,k}\). For simplicity we abuse the notation \(\partial_x^i \partial_y^j Q, \partial_x^i \partial_y^j \Psi\) by \(Q_{x\cdots x, y\cdots y}\) and \(\Psi_{x\cdots x, y\cdots y}\) in the remaining part of this section.

Lemma 3.3. Suppose that \(Q \in P_{1,k}, k \leq 5\). Then
\[
|\partial_x^N \Psi| \leq C_N, 0 \leq N \leq 4.
\]
Moreover, as \(|\lambda| \to \infty\),
\[
|\partial_x \Psi|, |\partial_x^2 \Psi|, |\partial_x^3 \Psi| \leq \frac{C}{|\lambda|}.
\]
Where \(C_N, C\) is a constant depending on \(Q\).

Proof. The uniform boundedness of \(\partial_x^N \Psi, 0 \leq N \leq 4\) in Lemma 3.2 will be used in the proof. A direct computation yields
\[
\Psi - (1 - \frac{Q}{\lambda}) = \Psi \left(1 - \Psi^{-1}(1 - \frac{Q}{\lambda})\right),
\]
\[
(\partial_y - \lambda \partial_x) \left(\Psi^{-1}(1 - \frac{Q}{\lambda})\right) = -\frac{1}{\lambda} \Psi^{-1}(Q_y - Q_x Q).
\]
So
\[
\Psi_x + \frac{Q_x}{\lambda} = \Psi_x \left(1 - \Psi^{-1}(1 - \frac{Q}{\lambda})\right) - \Psi \left(\Psi^{-1}(1 - \frac{Q}{\lambda})\right)_x = I_1 + I_2
\]
by \((3.7)\). Therefore, inverting the operator \(\partial_y - \lambda \partial_x\) in \((3.8)\) and applying Lemma \(3.1\) and Lemma \(3.2\) we have

\[
|I_1| = \frac{1}{|\lambda|} |\Psi_x| |G_\lambda(\Psi^{-1}(Q_y - Q_xQ))|
\]

\[
\leq \frac{C}{|\lambda|^2} \left| \xi \hat{Q}_x \right|_{L_1(d\xi dy)} (\sup_y |\Psi^{-1}(Q_y - Q_xQ)|_{L_1(dx)}) + \sup_y |\Psi^{-1}(Q_y - Q_xQ)|_{L_1(dx)} + |\Psi^{-1}(Q_y - Q_xQ)|_{L_1(dx dy)}
\]

\[
\leq \frac{C}{|\lambda|^2} \left( \sum_{i=0}^1 \left| \xi \hat{Q}_x \right|_{L_1(d\xi dy)}^2 \right) \sum_{j,k=0}^2 \sup_y |\partial_x^{j+k} Q_x|_{L_1(dx)} + |\partial_x^{j+k} Q_x|_{L_1(dx dy)}
\]

\[
\leq \frac{C}{|\lambda|}
\]

as \(|\lambda| \to \infty\). Taking the \(x\)-derivatives of both sides of \((3.8)\), we derive

\[
|I_2| = \frac{1}{|\lambda|} \left| \Psi_x \left( \Psi^{-1}(Q_y - Q_xQ) \right)_x \right|
\]

\[
\leq \frac{C}{|\lambda|^2} \left( \sup_y |\Psi^{-1}(Q_y - Q_xQ)|_{xy} \right)_{L_1(dx)} + \sup_y |\Psi^{-1}(Q_y - Q_xQ)|_{xy} + \left| \left( \Psi^{-1}(Q_y - Q_xQ) \right)_x \right|_{L_1(dx dy)}
\]

\[
\leq \frac{C}{|\lambda|} \left( \sum_{i=0}^2 \left| \xi \hat{Q}_x \right|_{L_1(d\xi dy)}^2 \right) \sum_{j,k=0}^3 \sup_y |\partial_x^{j+k} Q_x|_{L_1(dx)} + |\partial_x^{j+k} Q_x|_{L_1(dx dy)}
\]

\[
\leq \frac{C}{|\lambda|}
\]

Here we have used that \((3.1)\) and Lemma \(3.2\).

By the same scheme as above and the following equalities

\[
\Psi_{xx} + \frac{Q_{xx}}{\lambda} = \Psi_{xx} \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right) + 2 \Psi_x \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right)_x
\]

\[
+ \Psi \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right)_{xx}
\]

\[
\Psi_{xxx} + \frac{Q_{xxx}}{\lambda} = \Psi_{xxx} \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right) + 3 \Psi_{xx} \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right)_x
\]

\[
+ 3 \Psi_x \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right) + \Psi \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right)_{xxx},
\]
Proof. Using the formula one derives

\[ |\Psi_{xx}| \leq \frac{C}{|\lambda|} \left( \sum_{i=0}^{3} |\xi^{i} \widehat{\xi x}|_{L_{1}(d\xi d\eta)}^2 \right) \sum_{j,k=0}^{4} \left[ \sup_{y} |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dx)}^2 + |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dxdy)}^2 \right] \]

\[ |\Psi_{xxx}| \leq \frac{C}{|\lambda|} \left( \sum_{i=0}^{4} |\xi^{i} \widehat{\xi x}|_{L_{1}(d\xi d\eta)}^2 \right) \sum_{j,k=0}^{5} \left[ \sup_{y} |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dx)}^2 + |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dxdy)}^2 \right] \]

Hence the estimates for \( \Psi_{xx} \) and \( \Psi_{xxx} \) follow. \( \square \)

**Lemma 3.4.** Suppose that \( Q \in \mathbb{P}_{1,k}, \ k \leq 5. \) Then

\[ |\partial_{y} \Psi| \leq \frac{C}{|\lambda|}, \quad (3.9) \]

\[ |\partial_{x} \partial_{y} \Psi| \leq \frac{C}{|\lambda|}, \quad (3.10) \]

as \( |\lambda| \to \infty. \) Where \( C \) is a constant depending on \( Q. \)

Proof. Using the formula

\[ \Psi_{y} + \frac{Q_{y}}{\lambda} = \Psi_{y} \left( 1 - \Psi^{-1}(1 - \frac{Q}{\lambda}) \right) - \Psi \left( \Psi^{-1}(1 - \frac{Q}{\lambda}) \right)_{y} = II_{1} + II_{2}. \]

and Lemma 3.3, one can derive

\[ |II_{1}| \]

\[ = \frac{1}{|\lambda|} |\Psi_{y} G_{\lambda} \left( \Psi^{-1}(Q_{y} - Q_{x}) \right)| \]

\[ \leq \frac{C}{|\lambda|^{2}} |\Psi_{y} (\sup_{y} |(\Psi^{-1}(Q_{y} - Q_{x})|_{L_{1}(dx)}) + \sup_{y} |\Psi^{-1}(Q_{y} - Q_{x})|_{L_{1}(dx)}} \]

\[ + |\Psi^{-1}(Q_{y} - Q_{x})|_{L_{1}(dxdy)}) \]

\[ \leq \frac{C}{|\lambda|^{2}} \left( \sum_{i=0}^{2} |\xi^{i} \widehat{\xi x}|_{L_{1}(d\xi d\eta)}^2 \right) \sum_{j,k=0}^{3} \left[ \sup_{y} |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dx)}^2 + |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dxdy)}^2 \right] \]

(by estimates of \( I_{1}, \) and \( I_{2} \) in Lemma 3.3).

\[ \leq \frac{C}{|\lambda|^{2}} \]

\[ |II_{2}| \]

\[ = \frac{1}{|\lambda|} |\Psi_{y} G_{\lambda} \left( \Psi^{-1}(Q_{y} - Q_{x}) \right)_{y}| \]

\[ \leq \frac{C}{|\lambda|^{2}} (\sup_{y} |(\Psi^{-1}(Q_{y} - Q_{x})|_{y} |_{L_{1}(dx)}) + \sup_{y} |(\Psi^{-1}(Q_{y} - Q_{x})|_{y} |_{L_{1}(dx)}} \]

\[ + |(\Psi^{-1}(Q_{y} - Q_{x})|_{y} |_{L_{1}(dxdy)}) \]

\[ \leq \frac{C}{|\lambda|} \left( \sum_{i=0}^{3} |\xi^{i} \widehat{\xi x}|_{L_{1}(d\xi d\eta)}^2 \right) \sum_{j,k=0}^{4} \left[ \sup_{y} |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dx)}^2 + |\partial^{j}_{x} \partial^{k}_{y} Q|_{L_{1}(dxdy)}^2 \right] \]

(by estimates of \( \Psi_{xx} \) in Lemma 3.3).

\[ \leq \frac{C}{|\lambda|}. \]
Theorem 3.1. If by Lemma 3.3, (3.9). Hence we prove (3.10).

(3.12)

Where the estimate $|\Psi_{yy}| = |\lambda^2 \Psi_{xx} + \lambda (Q_x \Psi)_x + (Q_x \Psi)_y|$ has been used. Thus (3.9) is proved. On the other hand, we write

\[
\Psi_{xy} + \frac{Q_{xy}}{\lambda} = \Psi_{xy} \left(1 - \Psi^{-1} \left(1 - \frac{Q}{\lambda} \right)\right) - \Psi_x \left(1 - \Psi^{-1} \left(1 - \frac{Q}{\lambda} \right)\right)_y + \Psi_y \left(1 - \Psi^{-1} \left(1 - \frac{Q}{\lambda} \right)\right)_x - \Psi \left(1 - \Psi^{-1} \left(1 - \frac{Q}{\lambda} \right)\right)_{xy}
\]

\[= III_1 + III_2 + III_3 + III_4.\]

Similarly, one can verify

\[
|III_1| \leq \frac{C}{|\lambda|^2} \sum_{i=0}^{3} |\xi_i \hat{Q}_x|_{L_1(d\xi dy)} \sum_{j,k=0}^{4} \left[ \sup_y |\partial_x^j \partial_y^k Q|_{L_1(dx)} + |\partial_x^j \partial_y^k Q|_{L_1(dx dy)} \right],
\]

\[
|III_2| \leq \frac{C}{|\lambda|^2} \sum_{i=0}^{3} |\xi_i \hat{Q}_x|_{L_1(d\xi dy)} \sum_{j,k=0}^{4} \left[ \sup_y |\partial_x^j \partial_y^k Q|_{L_1(dx)} + |\partial_x^j \partial_y^k Q|_{L_1(dx dy)} \right],
\]

\[
|III_3| \leq \frac{C}{|\lambda|^3} \sum_{i=0}^{3} |\xi_i \hat{Q}_x|_{L_1(d\xi dy)} \sum_{j,k=0}^{4} \left[ \sup_y |\partial_x^j \partial_y^k Q|_{L_1(dx)} + |\partial_x^j \partial_y^k Q|_{L_1(dx dy)} \right],
\]

\[
|III_4| \leq \frac{C}{|\lambda|^4} \sum_{i=0}^{4} |\xi_i \hat{Q}_x|_{L_1(d\xi dy)} \sum_{j,k=0}^{5} \left[ \sup_y |\partial_x^j \partial_y^k Q|_{L_1(dx)} + |\partial_x^j \partial_y^k Q|_{L_1(dx dy)} \right],
\]

by Lemma 3.3 3.4. Hence we prove 3.10. □

Theorem 3.1. If $Q \in \mathcal{P}_{1,k}$, $k \leq 5$, then as $|\lambda| \to \infty$,

(3.11) $|\Psi(x, y, \lambda) - \left(1 - \frac{Q}{\lambda}\right)| \leq \frac{C}{|\lambda|^2},$

(3.12) $|\partial_x \Psi(x, y, \lambda) + \frac{\partial_x Q}{\lambda}|, |\partial_y \Psi(x, y, \lambda) + \frac{\partial_y Q}{\lambda}| \leq \frac{C}{|\lambda|^2}.$

Where $C$ is a constant depending on $Q$.

Proof. Applying (3.8), Lemma 3.3 and 3.4 we obtain

\[
|\Psi - \left(1 - \frac{Q}{\lambda}\right)|
\]

\[
= |\Psi||1 - \Psi^{-1} \left(1 - \frac{Q}{\lambda}\right)|
\]

\[
= \left|\frac{\Psi}{|\lambda|}\right| G_{\lambda} \left(\Psi^{-1} (Q_y - Q_x Q)\right)
\]

\[
\leq \frac{C}{|\lambda|^2} |\hat{Q}_x|_{L_1(d\xi dy)} \left(\left|\sup_y (\Psi^{-1} (Q_y - Q_x Q))\right|_{L_1(dx)} + |\Psi^{-1} (Q_y - Q_x Q)|_{L_1(dx dy)} + |\Psi^{-1} (Q_y - Q_x Q)|_{L_1(dx dy)}\right)
\]

\[
\leq \frac{C}{|\lambda|^2} \sum_{i=0}^{3} |\xi_i \hat{Q}_x|_{L_1(d\xi dy)} \sum_{j,k=0}^{4} \left[ \sup_y |\partial_x^j \partial_y^k Q|_{L_1(dx)} + |\partial_x^j \partial_y^k Q|_{L_1(dx dy)} \right].
\]
as $|\lambda| \to \infty$. Therefore, (3.11) is proved.

To prove (3.12), we used the results of Lemma 3.3 and 3.4 to improve the estimates of $I_1$, $I_2$, $I_{11}$, and $I_{12}$ in the proof of Lemma 3.3, 3.4. More precisely,

$$|I_1| = \frac{1}{|\lambda|} |\Psi_x| |G_\lambda (\Psi^{-1}(Q_y - Q_x))|$$

$$\leq \frac{C}{|\lambda|^2} |\Psi_x| (\sup_y |(\Psi^{-1}(Q_y - Q_x))_y|_{L_1(dx)})$$

$$+ \sup_y |\Psi^{-1}(Q_y - Q_x)|_{L_1(dx)} + |\Psi^{-1}(Q_y - Q_x)|_{L_1(dx dy)})$$

$$\leq \frac{C}{|\lambda|^3} \sum_{i=0}^3 |\xi_i \widehat{Q_x}|_{L_1(d\xi)}^2 \sum_{j,k=0}^4 \left[ \sup_y |\partial_x^i \partial_y^k Q|_{L_1(dx)}^2 + |\partial_x^i \partial_y^k Q|_{L_1(dx dy)}^2 \right],$$

$$|I_2| = \frac{1}{|\lambda|} |\Psi G_\lambda (\Psi^{-1}(Q_y - Q_x))_x|$$

$$\leq \frac{C}{|\lambda|^2} (\sup_y |(\Psi^{-1}(Q_y - Q_x))_x|_{L_1(dx)}$$

$$+ \sup_y |(\Psi^{-1}(Q_y - Q_x))_x|_{L_1(dx)} + |(\Psi^{-1}(Q_y - Q_x))_x|_{L_1(dx dy)})$$

$$\leq \frac{C}{|\lambda|^3} \sum_{i=0}^3 |\xi_i \widehat{Q_x}|_{L_1(d\xi)}^2 \sum_{j,k=0}^4 \left[ \sup_y |\partial_x^i \partial_y^k Q|_{L_1(dx)}^2 + |\partial_x^i \partial_y^k Q|_{L_1(dx dy)}^2 \right],$$

$$|II_1| = \frac{1}{|\lambda|} |\Psi_x G_\lambda (\Psi^{-1}(Q_y - Q_x))|$$

$$\leq \frac{C}{|\lambda|^2} (\sup_y |(\Psi^{-1}(Q_y - Q_x))_y|_{L_1(dx)}$$

$$+ \sup_y |\Psi^{-1}(Q_y - Q_x)|_{L_1(dx)} + |\Psi^{-1}(Q_y - Q_x)|_{L_1(dx dy)})$$

$$\leq \frac{C}{|\lambda|^3} \sum_{i=0}^3 |\xi_i \widehat{Q_x}|_{L_1(d\xi)}^2 \sum_{j,k=0}^4 \left[ \sup_y |\partial_x^i \partial_y^k Q|_{L_1(dx)}^2 + |\partial_x^i \partial_y^k Q|_{L_1(dx dy)}^2 \right],$$

$$|II_2| = \frac{1}{|\lambda|} |\Psi G_\lambda (\Psi^{-1}(Q_y - Q_x))_y|$$

$$\leq \frac{C}{|\lambda|^2} (\sup_y |(\Psi^{-1}(Q_y - Q_x))_y|_{L_1(dx)}$$

$$+ \sup_y |(\Psi^{-1}(Q_y - Q_x))_y|_{L_1(dx)} + |(\Psi^{-1}(Q_y - Q_x)_y)|_{L_1(dx dy)})$$

$$\leq \frac{C}{|\lambda|^3} \sum_{i=0}^3 |\xi_i \widehat{Q_x}|_{L_1(d\xi)}^2 \sum_{j,k=0}^5 \left[ \sup_y |\partial_x^i \partial_y^k Q|_{L_1(dx)}^2 + |\partial_x^i \partial_y^k Q|_{L_1(dx dy)}^2 \right].$$

Here $|\Psi_y| = |\lambda \Psi_x + Q_{xy} \Psi + Q_x \Psi_y|$ and (3.10) have been used in the estimation of $II_2$. 

By induction, we can generalize the results of Lemma 3.3, 3.4 and Theorem 3.1 to

...
Corollary 3.1. Suppose that \( Q \in \mathbb{P}_{1,k} \). Then for \( i + h \leq \max\{k, 5\} - 4 \) and as \( |\lambda| \to \infty \),
\[
|\partial_x^j \partial_y^h \Psi(x, y, \lambda) - \partial_x^j \partial_y^h \left( 1 - \frac{Q}{\lambda} \right) | \leq \frac{C}{|\lambda|^2}.
\]

Remark 1. In general, the scattering transformation is a generalized Fourier transform. That is, it maps smooth potentials to decaying scattering data, and decaying potentials to smooth scattering data. As is known, the asymptotic expansion of eigenfunctions is related to the decayness of the scattering data. However, in the case of Ward equation, even for the Schwartz potentials, the second order asymptotic expansion of Theorem [3.7] seems difficult to be improved. To see it, the second order coefficient of the asymptotic expansion \( \Psi \), and an analogue of [3.8] need to be introduced. That is
\[
\Psi_2(x, y) = \int_{-\infty}^{x} (\Psi_y + Q) (x', y) dx',
\]
\[
c(y) = \int_{-\infty}^{\infty} (\Psi_y + Q) (x', y) dx',
\]
\[
\Phi(x) = \int_{-\infty}^{\infty} \phi(x') dx', \quad \int_{-\infty}^{\infty} \phi(x') dx' = 1,
\]
\[
f(x, y) = \Psi_2(x, y) - c(y) \Phi(x).
\]

and
\[
(\partial_y - \lambda \partial_x) \left( \mathbb{P}^{-1} \left( 1 - \frac{Q}{\lambda} + \frac{\Psi_2}{\lambda^2} \right) \right) = \frac{1}{\lambda^2} \mathbb{P}^{-1} (\partial_y \Psi_2 - Q \Psi_2)
\]
Where \( \phi \) is a Schwartz function. Then \( f(x, y) \), \( c(y) \) are Schwartz. It can be checked that \( \Psi_2 \) does not possess integrability in the \( x \)-variable. This causes troubles in estimating \( |\Psi - (1 - \frac{Q}{\lambda} + \frac{\Psi_2}{\lambda^2})| \) while inverting \( (3.13) \) to derive a higher order asymptotic expansion of \( \Psi \).

4. Direct problem III: Eigenfunctions with non-small data

First we introduce

Definition 5. The Cauchy operator \( \mathcal{C} \) and its limits \( \mathcal{C}_{\pm} \) are defined as:
\[
\mathcal{C} f(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - \lambda} d\zeta, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
\[
\mathcal{C}_{\pm} f(\lambda) = \lim_{\epsilon \to 0^\pm} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (\lambda \pm i\epsilon)} d\zeta, \quad \lambda \in \mathbb{R}.
\]
It is well-known that \( \mathcal{C}_{\pm} \) are bounded operators on \( L_p(\mathbb{R}) \) for \( 1 < p < \infty \), and \( \mathcal{C}_{\pm} f(\lambda) = \lim_{\lambda \to \lambda} \mathcal{C} f(\lambda), \lambda \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{0\} \).

Definition 6. Suppose \( v(\lambda) \) is defined on \( \mathbb{R} \). A function \( \Psi(\lambda) \) is called a solution of the Riemann-Hilbert problem (\( \lambda \in \mathbb{R}, v \)) if
\[
\Psi(\lambda) = 1 + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Psi_{\pm}(t) (v(t) - 1)}{t - \lambda} dt = 1 + C (\Psi_{\pm}(v - 1)).
\]
Where \( \Psi_{\pm}(\lambda) = \lim_{\lambda \to \lambda} \Psi(\lambda), \lambda \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{0\} \). Moreover, the function \( v(\lambda) \) is called the data of the Riemann-Hilbert problem (\( \lambda \in \mathbb{R}, v \)).
Suppose the data \( v(\lambda), \lambda \in \mathbb{R} \) satisfies \( \partial_\lambda^i (\Psi - 1) \in L_2(\mathbb{R}, d\lambda) \), for \( i = 0, 1, 2 \). It can be seen that \( \Psi \) is a solution of the Riemann-Hilbert problem \( (\lambda \in \mathbb{R}, v) \) if and only if
\[
\begin{align*}
    \partial_\lambda \Psi &= 0, \quad \lambda \in \mathbb{C}^+, \\
    \Psi_+ &= \Psi_-, \quad \lambda \in \mathbb{R}, \\
    \Psi &\to 1, \quad \text{as } |\lambda| \to \infty.
\end{align*}
\]

**Lemma 4.1.** Suppose the data \( v(\lambda), \lambda \in \mathbb{R} \), satisfies:
\[
v - 1 \in L_2(d\lambda), \\
|v - 1|_{L_\infty(d\lambda)} \|C\|_2 < 1.
\]
Then the Riemann-Hilbert problem \( (\lambda \in \mathbb{R}, v) \) has a unique solution \( \Psi \) such that \( \Psi - 1 \in L_\infty(d\lambda) \cap L_2(d\lambda) \). Moreover, if \( H^k = \{ f|\partial_\lambda^j f \in L_2(d\lambda), 0 \leq j \leq k \} \) and
\[
|v - 1|_{H^k(d\lambda)} \leq C|v - 1|_{H^k(d\lambda)}
\]
for some constant \( C \).

**Proof.** The proof can be driven by an adaptation of the proof of Theorem 8.9 and 9.20 in [4]. \( \square \)

**Lemma 4.2.** Suppose the data \( v(\lambda), \lambda \in \mathbb{R} \), is a scalar function satisfying:
- \( v(\lambda) \neq 0, \forall \lambda; \)
- \( \int_{-\infty}^{\infty} \text{darg} v(\lambda) = 0; \)
- \( v - 1, \partial_\lambda v \in L_2(d\lambda). \)
Then the Riemann-Hilbert problem \( (\lambda \in \mathbb{R}, v) \) has a unique solution \( \Psi \). Moreover, if \( v - 1 \in H_1^k(d\lambda) \)
then
\[
|\Psi_+ - 1|_{H^k(d\lambda)} \leq C|v - 1|_{H^k(d\lambda)}.
\]
Where \( H^k(d\lambda) = \{ f|\partial_\lambda^j f \in L_2(d\lambda), 0 \leq i \leq k \} \), and \( C \) is a constant depending on \( |v|_{L_\infty}, |1/v|_{L_\infty} \).

**Proof.** Note that by the Sobolev’s theorem, \( v - 1 \in C_0 \) by condition \( v - 1, \partial_\lambda v \in L_2(d\lambda) \). Here \( C_0 \) denotes continuous functions with limit 0 at \( \infty \). Hence the proof can be found in Appendix of [4]. \( \square \)

**Lemma 4.3.** Suppose \( Q \in \mathbb{P}_{\infty,2,0} \cap \mathbb{P}_1 \). Then the eigenfunction obtained in Theorem 2.4 satisfies:
1. \( \partial_\xi \sigma (\Psi, y, \lambda) - 1, i = 0, 1, 2 \), are uniformly bounded in \( L_2(dx) \);
2. \( \Psi(\cdot, y, \lambda) - 1, \partial_\xi \Psi(\cdot, y, \lambda) \to 0 \) uniformly in \( L_2(dx) \) as \( \lambda \to \infty \).

**Proof.** By noting that the Fourier transform is an isometry on the \( L_2 \) spaces, to prove (1), it suffices to show that \( \xi \hat{\Psi} \), \( i = 0, 1, 2 \), are uniformly bounded in \( L_2(d\xi) \). We will only treat the case of \( \lambda \in \mathbb{C}^+ \) and \( \xi \geq 0 \) for simplicity. Other cases can be handled similarly. Note
\[
|K_\lambda f(\xi, y, \lambda)|_{L_2(d\xi)} \leq |\partial_\xi Q(\xi, y)_{L_1(d\xi)} \sup_{y, \lambda} |f|_{L_2(d\xi)}.
\]
Denote $\hat{X}_2 = \{ f(\xi, y, \lambda) : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to M_n(\mathbb{C}) | \sup_{y, \lambda} |f(\xi, y, \lambda)|_{L_2(d\xi)} < \infty \}$. So
\[ K_\lambda : \hat{X} \cap \hat{X}_2 \to \hat{X} \cap \hat{X}_2, \quad ||K_\lambda|| \leq ||\hat{x}Q(\xi, y)||_{L_1(d\xi dy)}. \]

By the assumption $Q \in \mathbb{P}_{\infty, 2, 0}$, we have $\int_{-\infty}^{\infty} e^{i\lambda \xi(y-y')} \hat{\partial_x} Q(\xi, y')dy' \in \hat{X} \cap \hat{X}_2$. Therefore the solution $\hat{W}$ of (2.3) is in $\hat{X} \cap \hat{X}_2$. Moreover, one can derive
\[ \hat{\xi} \hat{W} = \int_{-\infty}^{\infty} e^{i\lambda \xi(y-y')} \left( \hat{\xi} \hat{Q}_x \right) * \hat{W} dy' + \int_{-\infty}^{\infty} e^{i\lambda \xi(y-y')} \hat{\xi} \hat{Q}_x * \left( \hat{\xi} \hat{W} \right) dy' \]
from (2.4). As a result, we have $\hat{\xi} \hat{W} \in \hat{X} \cap \hat{X}_2$, if $Q \in \mathbb{P}_{\infty, 2, 0} \cap \mathbb{P}_1$. The same argument can prove $\hat{\xi}^2 \hat{W}, \hat{\xi}^3 \hat{W} \in \hat{X} \cap \hat{X}_2$, if $Q \in \mathbb{P}_{\infty, 2, 0} \cap \mathbb{P}_1$. Hence (1) is justified.

To prove (2), by the definition of $\hat{X}$ and result of (1), the function $\hat{W}(\xi, y, \lambda)$ can be approximated uniformly by $g$ where
\[ |\hat{W} - g|_{L_2(d\xi) \cap L_1(d\xi)} < \epsilon, \]
and $g$ is a linear combination of step functions in $\xi$ with uniformly bounded coefficients in $y, \lambda$. Hence
\[ \left( \int_{-\infty}^{\infty} e^{i\xi \xi} g(\xi, y, \lambda)d\xi \right) \chi_{|\xi| > N} \to 0 \text{ uniformly in } L_2(dx) \text{ as } N \to \infty. \]
Where $\chi_{|\xi| > N}$ is the characteristic function of the set $\{|\xi| > N\}$. The above two inequalities imply that $(\Psi(x, y, \lambda) - 1) \chi_{|\xi| > N} \to 0$ uniformly in $L_2(dx)$ as $N \to \infty$.

We can prove the case of $(\hat{\xi} \Psi(x, y, \lambda)) \chi_{|\xi| > N}$ by the similar method. Combining with Theorem 3.1 and the Lebesque Convergence Theorem, one can prove (2). □

**Lemma 4.4.** Let $x + \lambda y = z$, $\partial_z = \frac{1}{2}(\partial_x + i \partial_y)$, and $f_{\pm, z}(x, \lambda) = \lim_{|y| \to 0 \pm} f(x, y, \lambda)$. If $f(x, y, \lambda)$ is the solution of the Riemann-Hilbert problem ($x \in \mathbb{R}, F(x, \lambda)$) and
\[ F(\cdot, \lambda) - 1, \quad \partial_x F(\cdot, \lambda), \quad f_{\pm, z}(\cdot, \lambda) - 1, \quad \partial_x f_{\pm, z}(\cdot, \lambda) \to 0 \]
in $L_2(dx)$ as $|\lambda| \to \infty$,

then
\[ f(x, y, \cdot) \text{ tends to } 1 \text{ uniformly as } |\lambda| \to \infty. \]

**Proof.** For $y = 0$, the lemma follows from the Sobolev’s Theorem, Lemma 4.1 and the assumption on $f_{\pm, z}, F$.

For simplicity, we omit the words "" for $|\lambda| > 1$"" in the following proof.

Decompose $f(x, y, \lambda)$ into
\[ f(x, y, \lambda) = 1 + \frac{1}{2\pi i} \int_{|t-x| < 1} \frac{f_-(F(t, \lambda) - 1)}{t-z} dt + \frac{1}{2\pi i} \int_{|t-x| > 1} \frac{f_-(F(t, \lambda) - 1)}{t-z} dt \]
\[ = 1 + I(x, y, \lambda) + II(x, y, \lambda) \]
Note $f_-(F - 1)(\cdot, \lambda)$ is uniformly Holder continuous by the assumption on $F, f_{\pm}$ and the imbedding theorem of Morrey [13]. Hence one has $I(x, y, \lambda) \to I_{\pm, z}(x, \lambda)$ uniformly as $y \to 0 \pm$ [12]. The uniform convergence of $II(x, y, \lambda) \to II_{\pm, z}(x, \lambda)$ as $y \to 0 \pm$ can be justified by the Holder inequality. Moreover, one can check that this convergence is independent of $x$. As a result, $f(x, y, \lambda) \to f_{\pm, z}(x, \lambda)$ uniformly as $y \to 0 \pm$. 

Since the lemma holds on the x-axis. The uniform convergence provided above implies that: for any $\epsilon > 0$, one can find $N_{\epsilon_1}$, $\delta_\epsilon$ such that $|f(x, y, \lambda) - 1| < \epsilon$ for $|\lambda| \geq N_{\epsilon_1}$, $|y| \leq \delta_\epsilon$. Besides, by the Holder inequality, we can find $N_{\epsilon_2}$ such that $|f(x, y, \lambda) - 1| < \epsilon$ for $|\lambda| > N_{\epsilon_2}$, $|y| \geq \delta_\epsilon$. Hence for any $\epsilon > 0$, we obtain

$$|f(x, y, \lambda) - 1| < \epsilon, \quad |\lambda| > \max\{N_{\epsilon_1}, N_{\epsilon_2}\}.$$  

\[
\square
\]

We can start to prove Theorem 1.1.

Proof. We will prove Theorem 1.1 by induction on the norm of $|\partial_x Q(\xi, y)|_{L_1(d\xi dy)}$.

Step 1: (The case of $n = 0$)

If $|\partial_x Q(\xi, y)|_{L_1(d\xi dy)} < (\frac{3}{2})^0$, the existence and (1.6) are proved by Theorem 2.1. The conditions (1.7), (1.8) and (1.9) are shown by Theorem 3.1 and Lemma 3.3. The holomorphic property comes from (2.7).

Step 2: (Transforming to a Riemann-Hilbert problem)

Suppose Theorem 1.1 holds for $|\partial_x Q(\xi, y)|_{L_1(d\xi dy)} < (\frac{3}{2})^n$. Note the eigenfunction corresponding to a y-translate of $Q$ is the y-translate of the eigenfunction. Thus after translation we may have

$$\int_R \int_{-\infty}^{\infty} |\partial_x Q(\xi, y)|dyd\xi = \int_R \int_{-\infty}^{\infty} |\partial_x Q(\xi, y)|dyd\xi < \left(\frac{3}{2}\right)^{n+1} < \left(\frac{3}{2}\right)^{n},$$

for a potential $\partial_x Q(x, y)$ with $|\partial_x Q(\xi, y)|_{L_1(d\xi dy)} < (\frac{3}{2})^{n+1}$. Let $\chi^\pm = \chi^\pm(y) \leq 1$ be smooth real-valued functions such that

$$\chi^- = \begin{cases} 1, & \text{for } y \leq 0, \\ 0, & \text{for } y \geq 1, \end{cases} \quad \partial_x Q^- = \partial_x Q(x, y)\chi^-(y), \quad |\partial_x Q^-|_{L_1(d\xi dy)} < (\frac{3}{2})^n,$$

$$\chi^+ = \begin{cases} 1, & \text{for } y \geq 0, \\ 0, & \text{for } y \leq -1, \end{cases} \quad \partial_x Q^+ = \partial_x Q(x, y)\chi^+(y), \quad |\partial_x Q^+|_{L_1(d\xi dy)} < (\frac{3}{2})^n.$$

So $Q^\pm \in \mathbb{P}_{\infty, 0}$ and $|\partial_x Q^\pm(\xi, y)|_{L_1(d\xi dy)} < (\frac{3}{2})^n$. By the induction hypothesis there exist bounded sets $Z^\pm$ such that $Z^\pm \cap (\mathbb{C}\setminus \mathbb{R})$ are discrete in $\mathbb{C}\setminus \mathbb{R}$ and for all $\lambda \in \mathbb{C}\setminus Z^\pm$, $Q^\pm$ have eigenfunctions $\Psi^\pm$ which fulfill the statements of Theorem 1.1. Here we remark that the meaning of the notation $\Psi^+$ is different from that of $\Psi_+$. The former is a function defined in the half plane $y \geq 0$, the latter means $\lim_{\lambda_1 \to -\infty} \Psi(x, y, \lambda)$. Hence any eigenfunction $\Psi$ for $Q$, whenever it exists, must be of the form

(4.1) $\Psi(x, y, \lambda) = \begin{cases} \Psi^-(x, y, \lambda)a^-(x + \lambda y, \lambda), & y \leq 0, \\ \Psi^+(x, y, \lambda)a^+(x + \lambda y, \lambda), & y \geq 0. \end{cases}$

Where for $y \in \mathbb{R}^\pm$,

(4.2) $\begin{cases} a^\pm(x + \lambda y, \lambda) \text{ is meromorphic in } \lambda \in \mathbb{C}\setminus \mathbb{R} \text{ with discrete poles}, \\ a^\pm(x, y, \lambda) \text{ satisfies (1.6), (1.7)}, \\ a^\pm_{x, x}(0, 0, \lambda) \text{ satisfies (1.8), (1.9)}. \end{cases}$

Conversely, if we can find $a^\pm$ such that $a^\pm$ satisfies (4.2) for $y \in \mathbb{R}^\pm$ and $a^+(a^-)^{-1}(x, 0, \lambda) = (\Psi^+)^{-1}\Psi^-(x, 0, \lambda)$ (The invertibility of $a^\pm$, $\Psi^\pm$ is implied by Lemma 2.1). Then we can define $\Psi(x, y, \lambda)$ by (4.1) and prove Theorem 1.1 in case of $|\partial_x Q(\xi, y)|_{L_1(d\xi dy)} < (\frac{3}{2})^{n+1}$. Therefore, we conclude this step by
Lemma 4.5. (Transforming into a Riemann-Hilbert problem) To prove Theorem 1.1, it is equivalent to solving: Find a bounded set $Z, f(\tilde{x}, \tilde{y}, \lambda)$, and $\tilde{f}(\tilde{x}, \tilde{y}, \lambda)$ such that $Z^\pm \subset Z$ and

- $Z \cap (\mathbb{C}\setminus\mathbb{R})$ is discrete in $\mathbb{C}\setminus\mathbb{R}$.
- For $\lambda \in \mathbb{C}^+ \setminus (\mathbb{R} \cup Z)$, $f$ is the unique solution of the Riemann-Hilbert problem ($\tilde{x} \in \mathbb{R}, F(\tilde{x}, \lambda)$).
- For $\lambda \in \mathbb{C}^- \setminus (\mathbb{R} \cup Z)$, $\tilde{f}$ is the unique solution of the Riemann-Hilbert problem ($\tilde{x} \in \mathbb{R}, F^{-1}(\tilde{x}, \lambda)$).
- $f, \tilde{f}$ are meromorphic in $\lambda \in \mathbb{C}\setminus\mathbb{R}$ with poles at the points of $Z \cap (\mathbb{C}\setminus\mathbb{R})$.
- $f, \tilde{f}$ satisfy (1.8), (1.9).

Where

\[
x + \lambda y = \tilde{x} + i\tilde{y} = z, \quad \tilde{x}, \tilde{y} \in \mathbb{R}
\]

and

\[
F(\tilde{x}, \lambda) = \Psi^-(\tilde{x}, 0, \lambda)^{-1}\Psi^+(\tilde{x}, 0, \lambda).
\]

Proof. Note that if $f, \tilde{f}$ exist for Lemma 4.5, then by Lemma 4.4 $f, \tilde{f}$ satisfy (1.6), (1.7) as well. Therefore, the lemma can be proved by the change of variables (4.3) (or (2.8)) and setting

\[
a_-(x + \lambda y, \lambda) = A^-(\tilde{x}, \tilde{y}, \lambda),
\]

\[
a_+(x + \lambda y, \lambda) = A^+(\tilde{x}, \tilde{y}, \lambda),
\]

with $\tilde{x}, \tilde{y} \in \mathbb{R}$.

\[
f(\tilde{x}, \tilde{y}, \lambda) = \begin{cases} (A^+)^{-1}(\tilde{x}, \tilde{y}, \lambda), & \text{for } \tilde{y} \geq 0, \lambda \in \mathbb{C}^+, \\ (A^-)^{-1}(\tilde{x}, \tilde{y}, \lambda), & \text{for } \tilde{y} \leq 0, \lambda \in \mathbb{C}^+.
\end{cases}
\]

\[
\tilde{f}(\tilde{x}, \tilde{y}, \lambda) = \begin{cases} (A^-)^{-1}(\tilde{x}, \tilde{y}, \lambda), & \text{for } \tilde{y} \geq 0, \lambda \in \mathbb{C}^-, \\ (A^+)^{-1}(\tilde{x}, \tilde{y}, \lambda), & \text{for } \tilde{y} \leq 0, \lambda \in \mathbb{C}^-.
\end{cases}
\]

in the above discussion. \hfill \square

Step 3: (Factorization: a diagonal problem, a Riemann-Hilbert problem with small data and a rational function)

For any square matrix $A$ we let $d_k^\pm(A)$ denote the upper ($k \times k$) principal minors. Also let $\beta_{ik}, i \leq k$ be the minor of $A$ formed of the first $i$ rows, the first $i-1$ columns, and the $k$th column, and $\gamma_{ki}$ be the minor of $A$ formed of the first $i$ columns, the first $i-1$ rows, and the $k$th row. The following factorization theorem can be found in [10].

Lemma 4.6. Suppose the principal minors $d_k^\pm(A) \neq 0$, for $1 \leq k \leq n$. Then the matrix $A$ can be represented as

\[
A = CSB,
\]
Proof. For \( \lambda \in \mathbb{C} \setminus [Z^+ \cup Z^-] \), the matrix \( \delta \) is a diagonal matrix with non-vanishing entries. So the winding number of \( \delta(x, \lambda) \) is well-defined by \( N(\lambda) = -\frac{1}{2\pi i} \int \arg \delta(t, \lambda) dt \). By (4.6) and (4.7), \( N(\lambda) \) is a continuous integer-valued function for \( x \in \mathbb{C} \setminus [Z^+ \cup Z^-] \). Thus \( N(\lambda) \equiv 0 \) by (4.8).

Lemma 4.7. For \( \lambda \in \mathbb{C} \setminus [Z^+ \cup Z^-] \), we have a factorization

\[
F(\bar{x}, \lambda) = (1 + g_1)^{-1} \delta (1 + g_u),
\]

where

(4.5) \( \delta \) is diagonal and \( g_u, g_1 \) is strictly upper (lower) triangular.

(4.6) \( g_u, g_1 \) are \( \lambda \)-meromorphic in \( \mathbb{C}^+ \) with poles at \( [Z^+ \cup Z^-] \).

(4.7) \( \partial_x^i (\delta - 1), \partial_x^i g_u, \partial_x^i g_1, i = 0, 1, 2 \) are uniformly bounded in \( L_2(dx) \) for \( \lambda \in \mathbb{C} \setminus \cup \lambda_j \in [Z^+ \cup Z^-] D, (\lambda_j) \). For any \( z_j \in \mathbb{C} \setminus \mathcal{R} \), fixing \( \epsilon_k \) for \( \forall k \neq j \) and letting \( \epsilon_j \to 0 \), these \( L_2(dx) \)-norms increase as \( C \nu_j \to h_j \) with uniform constants \( C \nu_j \), \( h_j > 0 \).

(4.8) \( \delta - 1, g_u, g_1, \partial_x \delta, \partial_x g_u, \partial_x g_1 \to 0 \) in \( L_2(dx) \) as \( \lambda \to \infty \).

Proof. By the same technique of the proof of Lemma 2.1 one proves det \( \Psi^\pm = 1 \) for \( \lambda \notin \mathcal{R} \). So det \( F \equiv 1 \). As a result, if \( d_1^\pm (F) (\bar{x}_0, \lambda_0) = 0 \) for some \( 1 \leq i < n \), then \( F \) must have a pole at \( (\bar{x}_0, \lambda_0) \). By det \( \Psi^\pm = 1 \) and (4.4), we obtain \( \lambda_0 \in [Z^+ \cup Z^-] \).

Therefore for \( \lambda \in \mathbb{C} \setminus [Z^+ \cup Z^-] \), we obtain a factorization by Lemma 4.6. The properties (4.9) - (4.12) are implied by

(4.9) \( F(\bar{x}, \lambda) \) is meromorphic in \( \lambda \in \mathbb{C}^+ \) with poles at \( [Z^+ \cup Z^-] \) at most;

(4.10) \( F(\bar{x}, \lambda) \) satisfies (1.8), (1.9)

which come from the induction hypothesis.

Lemma 4.8. (A diagonal Riemann-Hilbert problem) For \( \lambda \in \mathbb{C} \setminus [Z^+ \cup Z^-] \), the Riemann-Hilbert problem \( (\bar{x} \in \mathbb{R}, \delta(\bar{x}, \lambda)) \) has a solution \( \Delta(z, \lambda) \). Moreover,

- \( \Delta \) is \( \lambda \)-meromorphic in \( \mathbb{C}^+ \) with poles at \( [Z^+ \cup Z^-] \cap \mathbb{C}^+ \);

- \( \Delta_{z, z} \) satisfies (1.8), (1.9).

Proof. For \( \lambda \in \mathbb{C} \setminus [Z^+ \cup Z^-] \), the matrix \( \delta \) is a diagonal matrix with non-vanishing entries. So the winding number of \( \delta(x, \lambda) \) is well-defined by \( N(\lambda) = -\frac{1}{2\pi i} \int \arg \delta(t, \lambda) dt \). By (4.6) and (4.7), \( N(\lambda) \) is a continuous integer-valued function for \( x \in \mathbb{C} \setminus [Z^+ \cup Z^-] \). Thus \( N(\lambda) \equiv 0 \) by (4.8).
Combining with (4.7), and (4.8), Lemma 4.2 implies the existence of \( \Delta \) which satisfies the Riemann-Hilbert problem \( (\tilde{x} \in \mathbb{R}, \delta(\tilde{x}, \lambda)), \) (1.8), and (1.9).

The meromorphic property of \( \Psi(x, y, \cdot) \) is proved by (4.10), and
\[
\Delta(z, \lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \delta(t, \lambda)}{t - z} \, dt \right\}.
\]

\[\square\]

**Lemma 4.9.** For \( \lambda \in \mathbb{C}^- \cup \cup_{\lambda_j \in \{Z^+ \cup Z^\downarrow\}} D_\epsilon(\lambda_j) \), there exists
\[
R_\epsilon = \begin{cases} 
R_{\epsilon, u}(\tilde{x}, y, \lambda), & \text{for } y \geq 0, \\
R_{\epsilon, l}(\tilde{x}, y, \lambda), & \text{for } y \leq 0,
\end{cases}
\]
such that
\[
|\Delta_{-z}(1 + (R_\epsilon)_{-z})F(1 + (R_\epsilon)_{-z})^{-1}\Delta_{+z}^{-1}(\tilde{x}, \lambda) - 1|_{H^2(\mathbb{R}, d\tilde{x})} << 1.
\]
\[
|\Delta_{-z}(1 + (R_\epsilon)_{-z})F(1 + (R_\epsilon)_{-z})^{-1}\Delta_{+z}^{-1}(\tilde{x}, \lambda) - 1|_{L_\infty}\|C_\pm\| < 1.
\]
\[\text{(4.13)} \quad \text{\( (R_\epsilon)_u \) \( (R_\epsilon)_l \) is strictly upper (lower) triangular.}\]
\[\text{(4.14)} \quad \text{\( R_\epsilon \) can be meromorphically extended in } \lambda \in \mathbb{C}^- \text{ with poles at } Z^+ \cup Z^- \text{.}\]
\[\text{(4.15)} \quad \text{\( R_\epsilon \in H^2(\mathbb{R}, d\tilde{x}) \) and is rational in } \lambda \in \mathbb{C}^\times \text{, with finite simple poles (independent of } \lambda \text{ and each corresponding residue is an off diagonal matrix with only one nonzero entry. Moreover, the non-zero entry tends to 0 as } |\lambda| \to \infty.\}
\]

**Proof.** By the condition (4.8), there exists \( \delta_\epsilon \) such that \( |\chi_{\lambda}|_{\lambda|>\delta_\epsilon, H^2(\mathbb{R})} < \epsilon \). Moreover, by (4.7), for each \( \lambda_0 \in \mathbb{C}^- \cup \cup_{\lambda_j \in \{Z^+ \cup Z^\downarrow\}} D_\epsilon(\lambda_j), \) \( |\lambda_0| \leq \delta_\epsilon \), there exists \( N = N(\epsilon, \lambda_0) \) such that
\[
|u - p_{\epsilon, u}|_{H^2(\mathbb{R}, d\tilde{x})} < \epsilon \quad \text{for } \lambda \text{ in a small neighborhood of } \lambda_0.
\]

Where
\[
p_{\epsilon, u}(z, \lambda) = \sum_{j=-N^2}^{N^2} g_{u}(\frac{j}{N}, \lambda)P_\epsilon(z - \frac{j}{N}),
\]
\[
P_\epsilon(t) = \frac{1}{t - i\epsilon} - \frac{1}{t + i\epsilon} \text{ is the Poisson kernel \[\text{(Appendix A.2).}\]
\]

One can check that \( p_{\epsilon, u} \in H^2(\mathbb{R}, d\tilde{x}) \) satisfies (4.13), (4.14). Hence choosing a bigger \( N \) or \( \delta_\epsilon \), there exists a \( z \)-rational function, denoted as \( \tilde{p}_{\epsilon, u} \),
\[
|u - \tilde{p}_{\epsilon, u}|_{H^2(\mathbb{R}, d\tilde{x})} < \epsilon \quad \text{for } \forall \lambda \in \mathbb{C}^- \cup \cup_{\lambda_j \in \{Z^+ \cup Z^\downarrow\}} D_\epsilon(\lambda_j),
\]
and \( \tilde{p}_{\epsilon, u} \) satisfies (4.13), (4.14).

Consequently, using (4.9), (4.10), Lemma 4.7, (1.8) and the off-diagonal form of \( g_u \), one can find a \( z \)-rational function \( R_{\epsilon}(z, \lambda) \) which is an approximation of \( g_u \) on \( z \in \mathbb{R} \) and satisfies (4.11), (4.15).

The case of \( g_l \) can be done in analogy. \[\square\]

With Lemma 4.9, one can find a solution to the small-data Riemann-Hilbert problem \( (\tilde{x} \in \mathbb{R}, \Delta_{-z}(1 + (R_\epsilon)_{-z})F(1 + (R_\epsilon)_{-z})^{-1}\Delta_{+z}^{-1}) \). However, it is difficult to analyze the meromorphic property of the solution in a neighborhood of points in
Proof. One can multiply $g_u$ (or $g_l$ respectively) by product

$$P_{c,u} = \prod_{\lambda_j \in \{Z^+ \cup \mathbb{Z}^-\}} \frac{\lambda - \lambda_j}{\lambda + i}$$

so that $G_{c,u} = P_{c,u}g_u$ is holomorphic in $\lambda \in \mathbb{C}^+$. Then using (4.14) and the same argument as the proof of Lemma 4.9 one can approximate $G_{c,u}$ by a piecewise $z$-rational function $R'_{c,u}$. Let $\tilde{R}_{c,u} = P_{c,u}^{-1}R_{c,u}$. Let $\tilde{R}_{c,u}$. Then using (4.16), (4.17) and the same argument as the proof of Lemma 4.9 one can approximate $G_{c,u}$ by a piecewise $z$-rational function $R'_{c,u}$. Let $\tilde{R}_{c,u} = P_{c,u}^{-1}R_{c,u}$.

Next, choose $k_j$ sufficiently large in $\mathcal{U}_c(\lambda) = \prod_{\lambda_j \in \{Z^+ \cup \mathbb{Z}^-\}} \frac{\lambda - \lambda_j}{\lambda + i}^{k_j}$ to make $\mathcal{U}_c, \mathcal{U}_d, \Delta$ holomorphic in $\lambda \in \mathbb{C}^+$. Hence the lemma can be proved by an adaptation of the proof of Lemma 4.9. (Note the factors $\mathcal{U}_c, \mathcal{P}_{c,u}, \mathcal{P}_{c,l}$ are cancelled out.)

**Lemma 4.11.** (A Riemann-Hilbert problem with small data) The Riemann-Hilbert problem $(\tilde{x} \in \mathbb{R}, \Delta_{-,-}(1 + (\tilde{R}_{c,u})_{-,-})F(1 + (\tilde{R}_{c,u})_{+,z})^{-1}\Delta_{+,z})$ admits a solution $f_{c,s}(z, \lambda)$ for $\lambda \in \mathbb{C}^+ \setminus \{Z^+ \cup \mathbb{Z}^-\}$. Moreover,

- $f_{c,s}$ is meromorphic in $\lambda \in \mathbb{C}^+$ with poles at $\{Z^+ \cup \mathbb{Z}^-\}$.
- $(f_{c,s})_{\pm, z}$ satisfies (1.5), (1.6).

**Proof.** By the assumption (4.16), (4.17) one can apply Lemma 4.1 to find $f_{c,s}$ which satisfies (1.8) and the Riemann-Hilbert problem $(\tilde{x} \in \mathbb{R}, \Delta_{-,-}(1 + (\tilde{R}_{c,u})_{-,-})F(1 + (\tilde{R}_{c,u})_{+,z})^{-1}\Delta_{+,z})$.

Moreover, $f_{c,s}$ satisfies (1.9) by Lemma 4.1, (1.10). Lemma 4.8 and (4.20). Finally, $f_{c,s}$ is is meromorphic in $\lambda \in \mathbb{C}^+$ with poles at $\{Z^+ \cup \mathbb{Z}^-\}$ by (1.9), Lemma 4.8 and (1.19).

We conclude this step by a characterization of Lemma 4.5.

**Lemma 4.12.** (Factorization of the Riemann-Hilbert problem) Suppose $f(z, \lambda)$ fulfills the statement in Lemma 4.3. Then there exist a unique function $r_{c}(z, \lambda)$ and
a set $Z_\epsilon$, such that
\begin{equation}
(4.21)
 r_\epsilon(z, \lambda) = 1 + \sum_{k=1}^{N_\epsilon} (z - z_k)^{-1} c_{k,\epsilon}(\lambda),
\end{equation}
for some integer $N_\epsilon$, $Z_\epsilon \subset Z$, and for $\lambda \in C_\epsilon^+ \setminus Z$.\(\lambda\)
\begin{equation}
(4.22)
 c_{k,\epsilon} \text{ is meromorphic in } \lambda \in C_\epsilon^+ \text{ with poles at } Z_\epsilon,
\end{equation}
\begin{equation}
(4.23)
 c_{k,\epsilon}(\lambda) \to 0 \text{ as } |\lambda| \to \infty,
\end{equation}
\begin{equation}
(4.24)
 f = r_\epsilon f_{\epsilon,\omega} \Delta \left(1 + \tilde{R}_\epsilon \right).
\end{equation}

Conversely, suppose there are uniformly bounded sets $Z_\epsilon$, and functions $\{r_\epsilon\}$ which are $\lambda$-meromorphic in $C_\epsilon^+$ with poles at $Z_\epsilon$, satisfy (4.21) - (4.23), and
\begin{equation}
(4.25)
 r_\epsilon f_{\epsilon,\omega} \Delta \left(1 + \tilde{R}_\epsilon \right) \text{ is holomorphic in } z \in C^+\lambda.
\end{equation}
for $\lambda \in C_\epsilon^+ \setminus \left(Z_\epsilon \cup [Z^+ \cup Z^-]_\epsilon^+\lambda\right)$. Define $f_\epsilon = r_\epsilon f_{\epsilon,\omega} \Delta \left(1 + \tilde{R}_\epsilon \right)$ for $\lambda \in C_\epsilon^+$. Then we have
\begin{equation}
(4.26)
 f_\epsilon \text{ is meromorphic in } \lambda \in C_\epsilon^+ \text{ with poles at } Z_\epsilon \cup [Z^+ \cup Z^-]_\epsilon,
\end{equation}
\begin{equation}
(4.27)
 f_{\epsilon_1} = f_{\epsilon_2} \text{ for } \lambda \in C_\epsilon^+ \text{ if } \epsilon_1 > \epsilon_2.
\end{equation}
Hence $f = f_\epsilon$ is well-defined, and $f$ satisfies the statements in Lemma 4.7 with $Z = \cup Z(\lambda) \cup \{\lambda_j \in \mathbb{R} | \limsup_{r \to 0} |f_\epsilon(D_2\epsilon(\lambda_j) \cap C_\epsilon^+)| = \infty\}$. Here $Z(\lambda)$ denotes the poles of $f_\epsilon$.

Proof. First of all, by Lemma 2.1 det $r_{\epsilon}(z, \lambda) = \det \left(1 + \tilde{R}_\epsilon(z, \lambda)\right) = \det \Delta(z, \lambda) = 1$. So they are invertible at regular $\lambda$. Besides, $f(z, \lambda)$ and $r_{\epsilon}(z, \lambda)\Delta(z, \lambda)(1 + \tilde{R}_\epsilon(z, \lambda))$ are $z$-meromorphic, possess the same jump singularity across $z \in \mathbb{R}$, and tend to 1 at infinity. Therefore
\begin{equation}
(4.28)
 f \left[ f_{\epsilon,\omega}(z, \lambda)\Delta(z, \lambda)(1 + \tilde{R}_\epsilon(z, \lambda))\right]^{-1}
\end{equation}
is $z$-rational and (4.21) - (4.23) are satisfied by Lemma 4.8 and the assumption on $f$.

For the converse part, (4.26) comes immediately from the definition of $f_\epsilon$ and the meromorphic properties of $r_{\epsilon}, \Delta, \tilde{R}_\epsilon, f_{\epsilon,\omega}$ implied by assumption and Lemma 4.8.

Besides, by assumption, $f_{\epsilon_1}, f_{\epsilon_2}$ satisfy the same Riemann-Hilbert problem in Lemma 4.5 for $\lambda \in C_\epsilon^+ \setminus Z_{\epsilon_1}$. Thus (4.27) follows from the Liouville's theorem and the meromorphic properties. As a result, the well-defined property follows from (4.26) and (4.27).

The conditions (1.8), (1.9) can be proved by Lemma 4.8 and (4.21) - (4.23), $f = f_\epsilon$ (i.e., (4.21)), and $Z = \cup Z_{\epsilon} \cup \{\lambda_j \in \mathbb{R} | \limsup_{r \to 0} |f_{\epsilon}(D_2\epsilon(\lambda_j) \cap C_\epsilon^+)| = \infty\}$.\(\square\)

Step 4: (Solving the Riemann-Hilbert problem)

We complete the proof of Theorem 1.1 by finding a rational function $r_\epsilon$ in Lemma 4.12.

Lemma 4.13. (Existence of the rational function $r_\epsilon$) There exist a function $r_\epsilon$ and a uniformly bounded set $Z_\epsilon$ such that $r_\epsilon$ is $\lambda$-meromorphic in $C_\epsilon^+$ with poles at the points of $Z_\epsilon$ and satisfies (4.21) - (4.23), (4.26) for $\lambda \in C_\epsilon^+ \setminus \left(Z_\epsilon \cup [Z^+ \cup Z^-]^e\lambda\right)$.\(4.21\)
Proof. For simplicity, we drop $\epsilon$ in the notation $r_\epsilon$, $f_\epsilon$, $R_\epsilon$, · · · in the following proof.

(a) A linear system for $r(z, \lambda)$:

Let $\{z_k = \tilde{x}_k + i\tilde{y}_k\}$, $k = 1, \cdots, N$ be the simple poles of $R$ in $\mathbb{C}^\pm$ by (4.15). Denote
\begin{align*}
1 + R(z, \lambda) &= (z - z_j)^{-1}d_j + n_j + O(|z - z_j|), \\
(4.30) 
\end{align*}
\begin{align*}
f_\lambda \Delta(z, \lambda) &= \alpha_j + \beta_j(z - z_j) + O(|z - z_j|^2).
\end{align*}

at $z_j$. Thus
\begin{align*}
f_\lambda \Delta(1 + R)(z, \lambda) &= (z - z_j)^{-1}\alpha_j d_j + (\beta_j d_j + \alpha_j n_j) + O(|z - z_j|).
\end{align*}

Now let
\begin{align*}
(4.31) 
r(z, \lambda) &= 1 + \sum_{k=1}^{N} (z - z_k)^{-1}c_k.
\end{align*}

Hence at $z_j$,
\begin{align*}
r(z, \lambda) &= (z - z_j)^{-1}c_j + b_j + O(|z - z_j|),
\end{align*}
where
\begin{align*}
b_j &= 1 + \sum_{k \neq j} (z_j - z_k)^{-1}c_k.
\end{align*}

We then try to find $c_j$, such that $r(z, \lambda)f_\lambda(z, \lambda)\Delta(z, \lambda)(1 + R(z, \lambda))$ is holomorphic at $z_j$. This yields the linear system for $c_j$:
\begin{align*}
(4.32) 
c_j \alpha_j d_j &= 0, \\
(4.33) 
b_j \alpha_j d_j + c_j (\beta_j d_j + \alpha_j n_j) &= 0, 1 \leq j \leq N.
\end{align*}

(b) Solving the linear system (4.32), (4.33):

The properties (4.13), (4.15) imply that $n_j$ are invertible and $(d_j n_j^{-1})^2 = 0$. Therefore, it can be justified that (4.32) are consequences of (4.33).

Inserting (4.31) into (4.33), we obtain a system of $Nn^2$ linear equations in $Nn^2$ unknowns (the entries of $c_k$) with coefficients in entries of $d_j(\lambda)$, $n_j(\lambda)$, $\alpha_j(\lambda)$, $\beta_j(\lambda)$. Observing that as $|\lambda| \to \infty$,
\begin{align*}
d_j &\to 0, n_j \to 1, \alpha_j \to 1, \beta_j \to 0
\end{align*}
by Lemma 4.8(4.11) Therefore, (4.33) are solvable as $|\lambda| \to \infty$. Precisely, $c_k$ can be written in rational forms of $d_j$, $n_j$, $\alpha_j$, $\beta_j$ which are all holomorphic in $\lambda \in \mathbb{C}^+ \setminus [Z^+ \cup Z^-]$. Therefore, (4.33) are solvable for $\lambda \in \mathbb{C}^+ \setminus Z_\epsilon$ where $Z_\epsilon$ are uniformly bounded sets. Consequently, (4.21), (4.22), (4.23), and (4.24) are fulfilled.

By the same argument as the proof of Theorem 1.1, we have

Corollary 4.1. Suppose that $Q \in \mathbb{P}_{\infty, k, 0}$, $k \geq 2$ and $\Psi(x, y, \lambda)$ is the associated eigenfunction. Then

$\Psi - 1$ are uniformly bounded in $\mathbb{D}^k$ for $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup_{\lambda_\epsilon \in Z} D_\epsilon(\lambda_\epsilon))$.

In particular, if $\lambda_0$ is a removable singularity of $\Psi(x, y, \lambda)$, then

$\Psi - 1$ are uniformly bounded in $\mathbb{D}^k$ in a neighborhood of $\lambda_0$. 

By a similar argument as that in Lemma 2.1 and 2.2 and using the uniqueness property in Theorem 1.1, we can derive the same algebraic characterization of the eigenfunctions:

**Lemma 4.14.** Suppose that $Q \in \mathbb{P}_{\infty,k,0}$, $k \geq 2$. Then the eigenfunction $\Psi$ satisfies

\begin{align}
\text{(4.34)} & \quad \det \Psi(x, y, \lambda) = 1, \\
\text{(4.35)} & \quad \Psi(x, y, \lambda)\Psi(x, y, \lambda)^* = I.
\end{align}

for $\lambda \in \mathbb{C}\setminus \mathbb{R}$.

5. Direct problem IV: Asymptotic analysis with non-small data

We define the continuous scattering data and study its algebraic and analytic characteristics in this section. We first show that the existence of continuous scattering data for $Q \in \mathbb{P}_1$ is automatic.

**Lemma 5.1.** If $Q \in \mathbb{P}_1$, then the eigenfunction $\Psi(x, y, \cdot)$ obtained by Theorem 2.1 has limits $\Psi_{\pm}$ on $\mathbb{R}$.

Proof. Suppose $\{\lambda_k\} \subset \mathbb{C}^+$. Write $\hat{W}_k$ instead of $\hat{W}(\xi, y, \lambda_k)$ and

$$f_k = \begin{cases} 
\int_{-\infty}^y e^{i\lambda_k(\xi-(y'-\xi)\hat{\partial}_x Q(\xi, y'))dy'}, & \text{when } \xi \geq 0 \\
-\int_y^{\infty} e^{i\lambda_k(\xi-(y'-\xi)\hat{\partial}_x Q(\xi, y'))dy'}, & \text{when } \xi \leq 0.
\end{cases}$$

(5.1)

Then (2.4) and (2.5) imply

$$\hat{W}_k - \hat{W}_h = (1 - K_{\lambda_k})^{-1}(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h + (1 - K_{\lambda_h})^{-1}(f_k - f_h) = I_1 + I_2.$$

Now observing

$$I_1 = (1 - K_{\lambda_k})^{-1}(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h = \sum_{i=0}^N K_{\lambda_k}^i(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h + K_{\lambda_k}^{N+1}\sum_{i=0}^{\infty} K_{\lambda_h}^i(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h$$

(5.2)

$$I_1' = I_1''.$$

Note (2.6) and $\sup_y |\hat{W}_h|_{L_1(d\xi)} \leq (1 - |\hat{\partial_x Q(\xi, y)|_{L_1(d\xi dy)})^{-1}$ imply

$$\sup_y \left| \sum_{i=0}^{\infty} K_{\lambda_k}^i(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h |_{L_1(d\xi)} < C'$$

and

$$|I_1'|_{L_1(d\xi)} = \sup_y |K_{\lambda_k}^{N+1} \sum_{i=0}^{\infty} K_{\lambda_k}^i(K_{\lambda_k} - K_{\lambda_h})\hat{W}_h |_{L_1(d\xi)} \to 0, \text{ as } N \to \infty.$$
On the other hand,

\[
| (K_{\lambda_k} - K_{\lambda_h}) \hat{W}_h |_{L_1(d\xi)} \leq \int_{-\infty}^{y} |(e^{i\lambda_k \xi (y-y')} - e^{i\lambda_h \xi (y-y')})| |\hat{\partial_\xi Q}|_{L_1(d\xi)} |\hat{W}_h|_{L_1(d\xi)} dy' + \int_{y}^{\infty} |(e^{i\lambda_k \xi (y-y')} - e^{i\lambda_h \xi (y-y')})| |\hat{\partial_\xi Q}|_{L_1(d\xi)} |\hat{W}_h|_{L_1(d\xi)} dy' \to 0, \quad \text{as } k, h \to \infty.
\]

by the Lebesgue Convergence Theorem and \( Q \in \mathbb{P}_1 \). So

\[
(5.4) \quad |I_1'|_{L_1(d\xi)} = \left| \sum_{i=0}^{N} K_{\lambda k} (K_{\lambda_k} - K_{\lambda_h}) \hat{W}_\lambda |_{L_1(d\xi)} \to 0, \quad \text{as } k, h \to \infty.
\]

Hence \(|I_1|_{L_1(d\xi)} \to 0 \) as \( k, h \to \infty \) by \((5.2)-(5.4)\). A similar argument will induce

\[
|I_2|_{L_1(d\xi)} = \left| (1 - K_{\lambda_k})^{-1} (f_k - f_h) |_{L_1(d\xi)} \to 0 \right. \quad \text{as well. Therefore, we have } |\hat{W}_k - \hat{W}_h|_{L_1(d\xi)} \to 0 \quad \text{as } k, h \to \infty \text{ by } (5.1). \]

Taking the Fourier transform, we prove the lemma when \( \lambda \in \mathbb{C}^+ \).

The case of \( \lambda \in \mathbb{C}^- \) can be proved by analogy. \( \square \)

**Lemma 5.2.** Suppose that \( Q \in \mathbb{P}_1 \) and

\[
(5.5) \quad |\xi^2 \hat{Q}|_{L_1(d\xi dy)} < \infty.
\]

Then \( \Psi_+ \) and \( \Psi_- \) are continuously differentiable with respect to \( x \) and \( y \).

**Proof.** If \( \lambda_k \to \lambda_{\pm} \) and \( I_1, I_2 \) are closed intervals on \( \mathbb{R} \),

- \( \partial_\xi \Psi(x, y, \lambda_k) \) and \( \partial_\xi \Psi(x, y, \lambda_k) \) are Cauchy for each \((x, y) \in I_1 \times I_2\);
- \( \partial_\eta \Psi(x, y, \lambda_k) \) and \( \partial_\eta \Psi(x, y, \lambda_k) \) are uniformly bounded on \( I_1 \times I_2 \),

then \( \Psi_\pm \) is differentiable and \( \partial_\xi \Psi_\pm = (\partial_\xi \Psi)_\pm \) and \( \partial_\eta \Psi_\pm = (\partial_\eta \Psi)_\pm \) by the Lebesgue Convergence theorem. Therefore, the continuous differentiability will be implied by proving the uniform Cauchy property of \( \partial_\xi \Psi(x, y, \lambda_k) \) and \( \partial_\eta \Psi(x, y, \lambda_k) \) with respect to \( x, y \) in compact subsets.

Lemma \([5.1]\) and \((2.1)\) imply that the uniform convergence of \( \partial_\eta \Psi(x, y, \lambda_k) \) comes from that of \( \partial_\xi \Psi(x, y, \lambda_k) \). So it is sufficient to show

\[
|\xi \hat{W}(\xi, y, \lambda_k) - \xi \hat{W}(\xi, y, \lambda_h)|_{L_1(d\xi)} \to 0.
\]

By replacing \( \hat{\partial_\xi Q}(\xi, y') \) with \( \xi \hat{\partial_\xi Q}(\xi, y') \) in the representation of \( f_k \) in \((5.1)\), it can be shown by adopting a similar argument as that in the proof of Lemma \([5.1]\). \( \square \)

**Lemma 5.3.** For \( Q \in \mathbb{P}_1 \) and \( Q \) satisfies \((5.2)\), the eigenfunction \( \Psi(x, y, \cdot) \) is holomorphic in \( \mathbb{C}^\pm \) and has limits \( \Psi_\pm \) on \( \mathbb{R} \). Moreover, there exists a continuously differentiable function \( v(x + \lambda y, \lambda) \) such that

\[
\Psi_+(x, y, \lambda) = \Psi_-(x, y, \lambda) v(x + \lambda y, \lambda), \quad \mathcal{L}_x v = 0, \quad \lambda \in \mathbb{R},
\]

where \( \mathcal{L}_\lambda = \partial_\eta - \lambda \partial_\xi \).
By the results of Lemma 5.5, it is sufficient to prove this lemma for any fixed constant. However, for \( Q \in \mathbb{P}_{\infty,k,0}, k \geq 5 \), \( |\partial_x^i \partial_y^j (\Psi^\pm - 1)| \leq C \frac{1}{|\lambda|^2} \), as \( |\lambda| \to \infty \). Where \( C \) is a constant depending on \( Q \).

Proof. We follow the scheme in Section 3 to prove this lemma. Note that all of the arguments there can be repeated except the proof of Lemma 5.3. Where the small data condition has been used to assure the uniform boundedness of \( \partial_x^N \Psi \), \( 0 \leq N \leq k - 1 \). Hence to prove this lemma, one needs only to show

The uniform boundedness of \( \partial_x^N \Psi^\pm, 0 \leq N \leq k - 1 \) as \( |\lambda| \to \infty \).

However, since \( Q \in \mathbb{P}_{\infty,k,0}, k \geq 5 \), \( \Psi^\pm \) exists, the property (5.6) can be justified by Corollary 4.1 and the Sobolev’s theorem. □

We improve the boundary properties (1.6), (1.7) of Theorem 1.1 by:

Lemma 5.6. If \( Q \in \mathbb{P}_{\infty,k,0}, k \geq 5 \), \( \partial_x^i \partial_y^j (\Psi^\pm - 1) \to 0 \) uniformly in \( L_{\infty} \) as \( |x| \) or \( |y| \to \infty \).

Proof. By the results of Lemma 5.6 it is sufficient to prove this lemma for \( |\lambda| < c \) where \( c \) is any fixed constant. However, for \( |\lambda| < c, i + j \leq k - 4 \), \( \partial_x^i \partial_y^j (\Psi^\pm (x,0,\lambda) - 1) \to 0 \) uniformly in \( L_{\infty} \) as \( |x| \to \infty \) follow from (2.1), Corollary 4.1 and the Sobolev’s theorem. For \( y \neq 0 \), one can follow the argument of Lemma 4.4 to show the uniform convergence of \( \partial_x^i \partial_y^j \Psi \to \).
\[ \partial^2_x \partial_y \Psi_{\pm, z}. \]

Then the lemma is proved by the uniform convergence and applying Holder inequality to

\[ \Psi(x, y, \lambda) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Psi_{+}(t, 0, \lambda) - \Psi_{-}(t, 0, \lambda)}{t - (x + \lambda y)} dt. \]

\[ \square \]

**Lemma 5.7.** For \( Q \in \mathbb{P}_{k, 1} \cap \mathbb{P}_1, k \geq 7, \) we have

\[ |\partial_{\lambda} \Psi_{\pm}|, |\partial_{\lambda} \partial_{x} \Psi_{\pm}| < \frac{C}{|\lambda|}, \text{ as } |\lambda| \to \infty. \]

and \( C \) depends continuously on \( x, y. \)

**Proof.** By formula (2.7), we have

\[ \text{(5.7)} \quad \Psi(x, y, \lambda) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i\xi x \hat{W}(\xi, y, \lambda)} d\xi. \]

Write

\[ \hat{W}(\xi, y, \lambda) = \frac{1}{\lambda} A(\xi, y, \lambda). \]

Note \( \hat{W} \in \hat{X} \) with \( \hat{X} \) defined by Definition 3. Therefore Theorem 3.1 implies

\[ (5.8) \quad A \text{ is uniformly bounded in } \hat{X}. \]

Now we define

\[ \text{(5.9)} \quad \frac{B_1(\xi, y, \lambda)}{\lambda} = \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \partial_x Q(\xi, y') dy', \quad \text{if } \lambda \in \mathbb{C}^+, \xi \geq 0; \]

\[ \frac{B_2(\xi, y, \lambda)}{\lambda} = -\int_{y}^{\infty} e^{i\lambda \xi(y - y')} \partial_x Q(\xi, y') dy', \quad \text{if } \lambda \in \mathbb{C}^+, \xi \leq 0; \]

\[ \frac{B_3(\xi, y, \lambda)}{\lambda} = -\int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \partial_x Q(\xi, y') dy', \quad \text{if } \lambda \in \mathbb{C}^-, \xi \geq 0; \]

\[ \frac{B_4(\xi, y, \lambda)}{\lambda} = \int_{y}^{\infty} e^{i\lambda \xi(y - y')} \partial_x Q(\xi, y') dy', \quad \text{if } \lambda \in \mathbb{C}^-, \xi \leq 0. \]

By (2.5), (2.6), (5.8), (5.9), and Theorem 3.1 we obtain

\[ (5.10) \quad B_1, B_2, B_3, B_4 \text{ are uniformly bounded in } \hat{X}. \]

Differentiating both sides of (2.5), we obtain

\[ (5.11) \quad (1 - \mathcal{K}_\lambda) \partial_{\lambda} \hat{W} \]

\[ = \quad iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x Q \ast \hat{W} \right) dy' + \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \xi \partial_x Q dy' \]

\[- iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x Q \ast \hat{W} \right) dy' + \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \xi \partial_x Q dy' \]

\[ = \quad iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x^2 Q \ast \hat{W} \right) dy' + \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \partial_x^2 Q dy' \]

\[- iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x^2 Q \ast \hat{W} \right) dy' + \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \partial_x^2 Q dy' \]

\[ + iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x Q \ast \xi \hat{W} \right) dy' - iy \int_{-\infty}^{y} e^{i\lambda \xi(y - y')} \left( \partial_x Q \ast \xi \hat{W} \right) dy' \]
for \( \lambda \in \mathbb{C}^+ \), \( \xi \geq 0 \) (Other cases can be done similarly). Define
\[
\begin{align*}
C_1(\xi, y, \lambda) &= \int_{-\infty}^{y} e^{i\lambda\xi(y-y')} \frac{\partial^2 Q(\xi, y')}{\lambda} dy', \\
C_2(\xi, y, \lambda) &= \int_{-\infty}^{y} e^{i\lambda\xi(y-y')} y' \frac{\partial Q(\xi, y')}{\lambda} dy', \\
C_3(\xi, y, \lambda) &= \xi \hat{W}(\xi, y, \lambda).
\end{align*}
\]

Using the definition of \( P_{\infty,k,1} \), and following the way to prove (5.10), then one can show that
\[(5.12)\quad C_1, C_2, C_3 \text{ are uniformly bounded in } \hat{X} \]
if \( Q \in P_{\infty,k,1} \) and \( k \geq 6 \). Combining (5.7), (5.8), (5.11), (5.12), and (2.6), we prove \(|\partial_\lambda \Psi| < C|\lambda|\) as \(|\lambda| \to \infty\) and \( C \) depends continuously on \( x, y \).

\[\begin{align*}
\text{Lemma 5.8.} & \quad \text{If } Q \in P_{\infty,k,1}, k \geq 7, \text{ and } Z = \phi, \text{ then} \\
|\partial_\lambda \Psi| & < \frac{C}{|\lambda|}, \quad \text{as } |\lambda| \to \infty,
\end{align*}\]
and \( C \) depends continuously on \( x, y \).

Proof. Since the property we wish to justify is a local property. Without loss of generality, we need only to show
\[(5.13) \quad |\chi(x,y)\partial_\lambda \Psi| < \frac{C}{|\lambda|}, \quad \text{as } |\lambda| \to \infty.
\]

Where \( C \) depends continuously on \( x, y \), and \( \chi(x,y) \) is any fixed smooth function with compact support. Now by the induction scheme as the proof of Theorem 1.1 we have
\[
\Psi(x, y, \lambda) = \begin{cases} 
\Psi^-(x, y, \lambda) a^-(x, y, \lambda), & y \leq 0, \\
\Psi^+(x, y, \lambda) a^+(x, y, \lambda), & y \geq 0,
\end{cases}
\]
and
\[
\partial_\lambda \Psi = (\partial_\lambda \Psi^\pm) a^\pm + \Psi^\pm \partial_\lambda a^\pm.
\]
By induction and applying Lemma 5.6 and 5.7 it reduces to showing
\[
|\chi(x,y)\partial_\lambda a| < \frac{C}{|\lambda|} \text{ as } |\lambda| \to \infty.
\]

Where
\[
a(x, y, \lambda) = \begin{cases} 
a^-(x, y, \lambda), & y \leq 0, \\
a^+(x, y, \lambda), & y \geq 0,
\end{cases}
\]
By (5.14), one can derive the inhomogeneous Riemann-Hilbert problem
\[
(\chi \partial_\lambda a)_+, x, 0, \lambda = g(x, \lambda) + (\Psi^+)^{-1} \Psi^- (\chi \partial_\lambda a)_-, x, 0, \lambda,
\]
with
\[
g = [\partial_\lambda ((\Psi^+)^{-1} \Psi^-)] \chi a_-, x, 0, \lambda.
\]
Hence \[1\]
\[\chi \partial_x a(x, y, \lambda) = \Psi(x, y, \lambda)^{-1} \left[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(\Psi^+(t, 0, \lambda)g(t, \lambda)}{t - z} dt \right] \]
with \(x + \lambda y = z\), and
\[\Psi(x, y, \lambda) = \begin{cases} 
\Phi^-(x, y, \lambda), & y \leq 0, \\
\Phi^+(x, y, \lambda), & y \geq 0.
\end{cases}\]

Therefore by Lemma 5.7 and (5.15),
\[\left| \chi \partial_x a \right|_{L^2(\mathbb{R}, dx)} < C \left| \Psi^{-1} \right| \left| \int_{\mathbb{R}} \frac{\chi(t, 0)}{t - z} (\Psi^+(t, 0, \lambda)g(t, \lambda)} dt \right| \]
\(< C \left| \chi \right|_{L^2(\mathbb{R}, dx)} \]
\(< C \left| \chi \right|_{L^2(\mathbb{R}, dx)} \]
as \(|\lambda| \to \infty\). Furthermore, differentiating both sides of (5.15) and using Corollary 4.1, Lemma 5.7, we obtain
\[\left| \chi \partial_x \chi \partial_x a \right|_{L^2(\mathbb{R}, dx)} < C \left| \Psi^{-1} \right| \left| \int_{\mathbb{R}} \frac{\chi(t, 0)}{t - z} (\Psi^+(t, 0, \lambda)g(t, \lambda)} dt \right| \]
\(< C \left| \chi \right|_{L^2(\mathbb{R}, dx)} \]
\(< C \left| \chi \right|_{L^2(\mathbb{R}, dx)} \]
Hence the lemma follows from (5.16), (5.17), and Sobolev’s theorem.

We conclude this section by the proof of Theorem 1.2 and the definition of continuous scattering transformation.

**Proof.** The condition (1.12) follows from Lemma 5.4. The identity (1.10) comes from (4.34) and Lemma 5.4. Besides, (4.35) and Lemma 5.4 imply that for \(\lambda \in \mathbb{R}\)
\[v(x + \lambda y, \lambda) = \Psi_-(x + \lambda y, \lambda)^{-1} \Psi_+(x + \lambda y, \lambda) = \Psi_+(x + \lambda y, \lambda)^\ast \Psi_+(x + \lambda y, \lambda)\]
Therefore (1.11) follows.

Next note that Lemma 5.3 implies that
\[\partial_x \partial_y (\Psi^+ - 1) \text{ are uniformly bounded in } L^\infty \cap L^2(\mathbb{R}, d\lambda) \cap L^1(\mathbb{R}, d\lambda)\].

So (1.13) follows. Combining Lemma 5.6, (5.18), one obtains \(\partial_x \partial_y \left( v - 1 \right) \to 0 \) uniformly in \(L^\infty\). So condition (1.13) follows from (1.13), and the Lebesgue convergence theorem. Finally, condition (1.15) is derived by applying Lemma 5.8.

**Definition 7.** For \(Q \in \mathbb{P}_{\infty,k,1}, \ k \geq 7\), if the eigenfunction \(\Psi(x, y, \cdot)\) has limits \(\Psi^\pm\) on \(\mathbb{R}\), then we define the continuous scattering data of \(Q\) to be \(v \in S_{c,k}\) obtained by Theorem 1.2. Moreover, the continuous scattering transformation \(S_c\) on \(Q\) is defined by \(S_c(Q) = v\).
6. INVERSE PROBLEM: CONTINUOUS SCATTERING DATA

We first prove Theorem 1.3 by solving the Riemann-Hilbert problem via a modified scheme of Section 4.

**Proof.** First of all, (1.13), (1.14) and Lemma 4.1 imply that there exists a constant $M > 0$ such that, as $|x|$ or $|y| > M - 1$, the Riemann-Hilbert problem $(\lambda \in \mathbb{R}, v(x, y, \lambda))$ can be solved and

$$
|\partial_x^i \partial_y^j (\Psi - 1)|_{L_2(d\lambda)} \leq C |v - 1|_{L_2(d\lambda)}
$$

for a constant $C$. Hence (1.16) holds as $|x|$ or $|y| > M - 1$. Applying Holder inequality, (1.13), (1.14), and (6.1), we then derive:

For each fixed $\lambda \notin \mathbb{R}$, $v$ holds as $|x|$ or $|y| > M - 1$,

$$
\partial_x^i \partial_y^j (\Psi - 1) \in L_\infty(dx dy),
$$

$$
\partial_x^i \partial_y^j (\Psi - 1) \rightarrow 0 \text{ in } L_\infty(dx dy), \text{ as } x \text{ or } y \rightarrow \infty.
$$

Hence, to prove Theorem 1.3 it is sufficient to solve the Riemann-Hilbert problem $(\lambda \in \mathbb{R}, v(x, y, \lambda))$ and establish (1.16), (1.17) for $\max(|x|, |y|) < M$. The scheme in Section 4, in particular Lemma 4.7-4.13, can be adapted to the solving of this problem. More precisely,

**Lemma 6.1.** For $\lambda, x, y \in \mathbb{R}$, we have a factorization

$$
v(x, y, \lambda) = (1 + h_1)^{-1} (1 + h_a)(x, y, \lambda),
$$

and for $i + j \leq k - 4$,

- $\chi$ is diagonal and $h_a, h_l$ is strictly upper (lower) triangular.
- $\chi - 1, h_a, h_l \rightarrow 0$ uniformly in $L_\infty \cap L_2(\mathbb{R}, d\lambda)$ as $|x|$ or $|y| \rightarrow \infty$.

**Proof.** We use the positivity condition (1.11) to prove that $d_1^i, 1 \leq i \leq n$ vanishes nowhere for $\lambda \in \mathbb{R}$. Hence the statements can be proved by the same method as that in the proof of Lemma 4.7.

**Lemma 6.2.** (A diagonal Riemann-Hilbert problem) For $\max(|x|, |y|) < M$, there exists uniquely a solution $\Xi(x, y, \lambda)$ to the Riemann-Hilbert problem $(\lambda \in \mathbb{R}, \chi)$ such that

$$
\Xi - 1 \text{ are uniformly bounded in } H^1(\mathbb{R}, d\lambda);
$$

$$
\Xi - 1, \partial_x \Xi, \partial_y \Xi \text{ are uniformly bounded in } L_2(\mathbb{R}, d\lambda),
$$

and for each fixed $\lambda \notin \mathbb{R}$,

$$
\partial_x^i \partial_y^j \Xi \in L_\infty(dx dy) \text{ for } \max(|x|, |y|) < M.
$$

**Proof.** Applying (6.3), and (6.4), one obtains that

$$
\partial_x^i (\chi - 1) \text{ are uniformly bounded in } L_\infty \cap L_2(\mathbb{R}, d\lambda), i = 0, 1.
$$

Hence the winding number $N(x, y) = \frac{1}{2\pi i} \int \frac{d\arg \chi}{d\zeta}(x, y, \zeta) d\zeta$ is integer-valued. Moreover, the condition (6.4) implies that $N(x, y) \equiv 0$. 


Thus for $\max(|x|, |y|) < M$, the existence of $\Xi$, and (6.3) can be implied by (6.3), the Sobolev’s theorem, and Lemma 4.2. By (6.3), (6.5), and the formula

$$\Xi(x, y, \lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\},$$

$$\partial_x \Xi(x, y, \lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\} \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_x \chi(x, y, \zeta)}{\chi(x, y, \zeta)(\zeta - \lambda)} d\zeta \right),$$

$$\partial_y \Xi(x, y, \lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\} \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_y \chi(x, y, \zeta)}{\chi(x, y, \zeta)(\zeta - \lambda)} d\zeta \right),$$

$$\partial_x^2 \Xi(x, y, \lambda) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\} \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_x \chi(x, y, \zeta)}{\chi(x, y, \zeta)(\zeta - \lambda)} d\zeta \right)^2$$

$$- \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\} \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial^2 \chi(x, y, \zeta)}{\chi(x, y, \zeta)(\zeta - \lambda)} d\zeta \right)$$

$$+ \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\log \chi(x, y, \zeta)}{\zeta - \lambda} d\zeta \right\} \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_{xx} \chi(x, y, \zeta)}{\chi(x, y, \zeta)(\zeta - \lambda)} d\zeta \right).$$

we derive (6.6). Finally, we obtain (6.7) by H"older inequality.

Lemma 6.3. For $\max(|x|, |y|) < M$, there exists a function $H(x, y, \lambda)$ satisfying

$$H = \begin{cases} H_u(x, y, \lambda), & \text{for } \lambda \in \mathbb{C}^+, \\ H_l(x, y, \lambda), & \text{for } \lambda \in \mathbb{C}^-, \end{cases}$$

and

- $H(x, y, \lambda) \in L_\infty \cap H^1(\mathbb{R}, d\lambda)$, and $\partial_x^i \partial_y^j H(x, y, \lambda) \in L_\infty \cap L_2(\mathbb{R}, d\lambda)$
- $|\Xi\cdot(1+H_u)\cdot(1+H_+)^{-1}\Xi^{-1}(x, y, \lambda) - 1|_{H^1(\mathbb{R}, d\lambda)} \leq C < \infty$.
- $|\Xi\cdot(1+H_u)\cdot(1+H_-)^{-1}\Xi^{-1}(x, y, \lambda) - 1|_{L_\infty(\mathbb{C}^\pm)} \leq C < 1$.
- $H_u$ (H_l) is strictly upper (lower) triangular.
- $H$ is rational in $\lambda \in \mathbb{C}^\pm$, with only simple poles and each corresponding residue is off diagonal, with only one nonzero entry $\kappa$ and $\partial_x^i \partial_y^j \kappa \in L_\infty(\mathbb{R}^2, dxdy)$.

Proof. Combining (6.5) with the results of Lemma 6.1 and the same method as in the proof of Lemma 4.3, the lemma can be proved.

Lemma 6.4. (A Riemann-Hilbert problem with small data) For $\max(|x|, |y|) < M$, the Riemann-Hilbert problem $\lambda \in \mathbb{R}, \Xi\cdot(1+H_-)\cdot(1+H_+)^{-1}\Xi^{-1}(x, y, \lambda)$ admits a solution $\varphi_s(x, y, \lambda)$. Moreover,

$$\varphi_s - 1, \partial_x \varphi_s, \partial_y \varphi_s \text{ are uniformly bounded in } L_2(\mathbb{R}, d\lambda),$$

and for each fixed $\lambda \notin \mathbb{R}$,

$$\partial_x^i \partial_y^j (\varphi_s - 1) \in L_\infty(\mathbb{R}^2, dxdy).$$

Proof. The existence of the solution and its properties can be proved by Lemma 4.1, 6.1, 6.2, 6.3 the property of the Cauchy operator $\mathcal{C}$ and H"older inequality.
Lemma 6.5. (Factorization of the Riemann-Hilbert problem) Suppose \( \Psi(x, y, \lambda) \) satisfies Theorem 1.3. Then for \( \max(|x|, |y|) < M \), there exists a unique function \( u \),

\[
(6.8) \quad u(x, y, \lambda) = 1 + \sum_{k=1}^{N} (\lambda - \lambda_k)^{-1} a_k(x, y),
\]

and

\[
(6.9) \quad \partial_x^i \partial_y^j a_k \in L_\infty(dx dy),
\]

\[
(6.10) \quad \Psi(x, y, \lambda) = u \varphi_s \Xi(1 + H).
\]

Conversely, if for \( \max(|x|, |y|) < M \), \( \exists u(x, y, \lambda) \) satisfying (6.8), (6.9) and

\[
(6.11) \quad u \varphi_s \Xi(1 + H) \text{ is holomorphic for } \lambda \in \mathbb{C}^+, \]

Define \( \Psi = u \varphi_s \Xi(1 + H) \) for \( \max(|x|, |y|) < M \). Hence \( \Psi \) satisfies Theorem 1.3.

We then use Lemma 6.5 to prove Theorem 1.3:

(a) A linear system for \( u(x, y, \lambda) \):

Let

\[
(6.12) \quad u(x, y, \lambda) = 1 + \sum_{k=1}^{p} (\lambda - \lambda_k)^{-1} a_k.
\]

Then at \( \lambda_j \)

\[
(6.13) \quad u(x, y, \lambda) = (\lambda - \lambda_j)^{-1} a_j + b_j + O(|\lambda - \lambda_j|),
\]

with

\[
(6.14) \quad b_j = 1 + \sum_{k \neq j} (\lambda_j - \lambda_k)^{-1} a_k.
\]

Since \( \lambda_j \) is a simple pole of \( H \) and \( \varphi_s \Xi \) is regular at \( \lambda_j \). We can write

\[
(6.15) \quad 1 + H(x, y, \lambda) = (\lambda - \lambda_j)^{-1} h_j + n_j + O(|\lambda - \lambda_j|),
\]

\[
(6.16) \quad \varphi_s \Xi(x, y, \lambda) = \alpha_j + \beta_j (\lambda - \lambda_j) + O(|\lambda - \lambda_j|^2).
\]

We then try to find \( a_k \), such that \( u(x, y, \lambda) \varphi_s(x, y, \lambda) \Xi(x, y, \lambda)(1 + H(x, y, \lambda)) \) is holomorphic at \( \lambda_j \). This yields the linear system for \( a_k \):

\[
(6.17) \quad a_j \alpha_j h_j = 0, \quad 1 \leq j \leq p,
\]

\[
(6.18) \quad b_j \alpha_j h_j + a_j (\beta_j h_j + \alpha_j n_j) = 0, \quad 1 \leq j \leq p.
\]

(b) Solving the linear system (6.17)-(6.18):

Note by Lemma 6.3 one can conclude

\[
(6.19) \quad (h_j \alpha_j^{-1})^2 = 0.
\]

Therefore, it can be justified that (6.17) is a consequence of (6.18). Note the off-diagonal form of \( h_j \) in Lemma 6.1 is crucial here.

Inserting (6.14) into (6.18), we obtain a system of \( pn^2 \) linear equations in \( pn^2 \) unknowns (the entries of \( a_k \) with coefficients in entries of \( h_j(x, y), n_j(x, y), \alpha_j(x, y), \beta_j(x, y) \)). Therefore, we conclude the existence problem of \( \Psi \) is Fredholm.

(c) Solving the Riemann-Hilbert problem:

Using the Fredholm alteranalinity, we need only to show that: for any fixed \( x, y \) the homogeneous problem (with limit 0 rather than 1 as \( \lambda \to \infty \)) has only the trivial solution. Suppose \( f(x, y, \lambda) \) solves this homogeneous problem. Consider \( g(x, y, \lambda) = \)

\[
\]
Proof. From the holomorphic property in \( x, y, z \), we have
\[ (\psi_1(x, y, \lambda)) \equiv 1. \]

Hence we prove the solvability of the Riemann-Hilbert problem in Theorem 1.3.

Lemma 6.6. For the solution \( \psi \) of the Riemann-Hilbert problem obtained in Theorem 1.3, we have
\[ (6.20) \quad \det \psi(x, y, \lambda) \equiv 1, \]
\[ (6.21) \quad \psi(x, y, t, \lambda) \psi(x, y, t, \lambda)^* \equiv 1. \]

Proof. By (1.10), \( \det \psi(x, y, \cdot) \) has no jump across the real line. So applying the Liouville’s theorem, (6.20) follows from the holomorphic property in \( \mathbb{C}^\pm \). Thus the Cauchy’s theorem implies
\[
0 = \int_{\mathbb{R}} g_+(s) ds = \int_{\mathbb{R}} f_+(s) f_-(s)^* ds = \int_{\mathbb{R}} f_-(s) v(s) f_-(s)^* ds.
\]

Because of (1.11) we conclude \( f_- \equiv 0 \) on \( \mathbb{R} \), so also \( f_+ \equiv 0 \) and \( f \equiv 0 \).

Hence we prove the solvability of the Riemann-Hilbert problem in Theorem 1.3.

We conclude this section by the proof of Theorem 1.4 and the definition of inverse scattering transformation.

Proof. By (1.18), the boundary condition (2.2) is satisfied. Besides, the Cauchy integral formula, and Theorem 1.3 imply
\[ (6.23) \quad \psi(x, y, \lambda) = I + C\psi_-(v - 1). \]

For fixed \( x, y \in \mathbb{R} \), applying \( \mathcal{L}_\lambda \partial_y - \lambda \partial_x \) to (6.23) and using (1.10), (1.13), we obtain
\[
\mathcal{L}_\lambda \psi = \mathcal{L}_\lambda C\psi_-(v - 1)
\]
\[ = C(\mathcal{L}_\lambda \psi_-(v - 1) + [\mathcal{L}_\lambda, C] \psi_-(v - 1)) \]
\[ = \partial_x \left( \frac{1}{2\pi i} \int_{\mathbb{R}} \psi_-(x, y, \zeta) (v(x + \zeta y, \zeta) - 1) d\zeta \right) \]
\[ + C(\mathcal{L}_\lambda \psi_-(v - 1)). \]

(6.24)

with \( Q(x, y) \) given by (1.19). Hence comparing (6.23) and (6.24) and using the uniqueness result of Theorem 1.3, we obtain (2.1).

Besides, (1.13), (1.16), (1.19), and Holder inequality show that \( Q, \partial_x Q, \) and \( \partial_y Q \in L_\infty \). Furthermore, by (2.1), (1.17), (6.20), and the \( \lambda \)-independence of \( Q \), we
derive \( \partial_i^j \partial_y^Q \in L_\infty \) and \( \partial_x^i \partial_y^Q, \partial_y Q \to 0 \) as \( x \) or \( y \to \infty \), for \( i + j \leq k - 4, i > 0 \).

Finally, by (6.21) and (2.1), we have
\[
(\partial_x Q) \Psi(x, y, t, \lambda)^{*-1} = (\partial_y - \lambda \partial_x) \Psi(x, y, t, \lambda)^{*-1} = -\Psi(x, y, t, \lambda)^{*-1} ((\partial_y - \lambda \partial_x) \Psi(x, y, t, \lambda)^*) \Psi(x, y, t, \lambda)^{*-1} = -\Psi(x, y, t, \lambda)^{*-1} ((\partial_x Q) \Psi(x, y, t, \lambda)^*) \Psi(x, y, t, \lambda)^{*-1} = -(\partial_x Q)^* \Psi(x, y, t, \lambda)^{*-1}.
\]
Thus \( \partial_x Q(x, y) \in su(n) \).

**Definition 8.** For a function \( v \in \mathcal{S}_c \), we define the inverse scattering transformation \( \mathcal{S}_c^{-1} \) on \( v \) by \( \mathcal{S}_c^{-1}(v) = Q \), where \( Q \) is obtained by Theorem 1.3 and 1.4.

### 7. The Cauchy problem: Continuous scattering data

We prove Theorem 1.5 in this section.

**Proof.** We can apply Theorem 1.3 to find the eigenfunction \( \Psi(x, y, 0, \lambda) \). By assumption, and Theorem 1.2 \( \mathcal{S}_c(Q_0) \in \mathcal{S}_{c,k} \).

Now let us define \( v(t) \) by
\[
(7.1) \quad v(t) = \{ v(x, y, t, \lambda) = v(x + \lambda y + \lambda^2 t, \lambda) \}.
\]

For each \( t \in \mathbb{R} \), rewriting \( x + \lambda y + \lambda^2 t = x + \lambda(y + \lambda t) = x + \lambda^2(t + \frac{1}{2}y) \) and modifying the approach in proving lemmas in Section 3-5, one can justify that \( v(t) \in \mathcal{S}_{c,k} \) (see Definition 2). So \( v \) satisfies the algebraic constraints:
- \( \det(v) \equiv 1 \),
- \( \nu = v^* > 0 \),

and the analytic constraints: for \( i + j + h \leq k - 4 \),
- \( L_\lambda v = 0, M_\lambda v = 0; \)
- \( \partial_x^i \partial_y^j \partial_t^h (v - 1) \) are uniformly bounded in \( L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda); \)
- \( \partial_x^i \partial_y^j \partial_t^h (v - 1) \to 0 \) uniformly in \( L_\infty \cap L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda) \) as \( |x| \) or \( |y| \) or \( t \to \infty; \)
- \( \partial_v v \in L_2(\mathbb{R}, d\lambda) \) and the norms depend continuously on \( x, y \).

Where \( L_\lambda = \partial_y - \lambda \partial_x \), and \( M_\lambda = \partial_t - \lambda \partial_y \).

Now we apply Theorem 1.3 and 1.4 to show the existence of \( \Psi(x, y, t, \lambda) \) and \( Q(x, y, t) \) satisfying (2.1), and (2.2). More precisely,
\[
(7.2) \quad \Psi(x, y, t, \lambda) = I + \mathcal{C} \Psi_-(v - 1) = I + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Psi_-(x, y, \zeta)(v(x + \zeta y + \zeta^2 t, \zeta) - 1)}{\lambda - \zeta} d\zeta
\]
\[\Psi_\pm - 1, \partial_x \Psi_\pm, \partial_y \Psi_\pm, \partial_t \Psi_\pm \text{ are uniformly bounded in } L_2(\mathbb{d}\lambda),\]

and for each fixed \( \lambda \in \mathbb{R} \), \( i + j + h \leq k - 4 \)
\[
(7.3) \quad \partial_x^i \partial_y^j \partial_t^h \Psi \in L_\infty(dxdydt).
\]
In addition, 
\[ Q(x, y, t) = \frac{1}{2\pi i} \int_{\mathbb{R}} \Psi(x, y, t, \zeta)(v(x + \zeta y + \zeta^2 t, \zeta) - 1)d\zeta, \]
and for \( i + j + h \leq k - 4, i^2 + j^2 > 0, \)
\[ \partial_x^i \partial_y^j \partial_t^h Q, \partial_t Q, \in L_\infty, \tag{7.4} \]
\[ \partial_x^i \partial_y^j \partial_{\lambda}^h Q, \partial_{\lambda} Q, \to 0 \quad \text{in} \quad L_\infty. \tag{7.5} \]

To prove (1.4), we note it is equivalent to prove
\[ \mathcal{M}_\lambda \Psi = (\partial_\lambda Q)(x, y, t)\Psi(x, y, t, \lambda). \tag{7.6} \]
Applying \( \mathcal{M}_\lambda \) to both sides of (7.2) and using similar approach as that in the proof of Theorem 1.4, we obtain
\[ \mathcal{M}_\lambda \Psi = (\partial_\lambda Q)(x, y, t) + \mathcal{C}(\mathcal{M}_\xi \Psi) - (v - 1). \tag{7.7} \]
Comparing (7.2) and (7.7) and using the uniqueness result of Theorem 1.3 we obtain (7.6). The smooth and decay properties of \( Q \) can be derived by an argument similar to the proof of Theorem 1.4 and conditions (7.3)-(7.5).

We conclude this report by a brief remark on examples of \( Q_0 \in \mathbb{P}_{\infty, k, 1}, k \geq 7, \) and the corresponding eigenfunction \( \Psi_0 \) has no poles. The first class of examples is \( \mathbb{P}_1 \cap \mathcal{S} \) (\( \mathcal{S} \) is the set of Schwartz functions and \( \mathbb{P}_1 \) is defined by Definition 3). To construct an example with large norm, we let \( v(x, y, \lambda) = v(x + \lambda y, \lambda) \) satisfy
\[ \det(v) = 1, \quad v = v^* > 0, \quad v - 1 \in \mathcal{S}, \]
and for \( i, j, h \geq 0, \)
\[ \partial_x^i \partial_y^j \partial_{\lambda}^h (v - 1) \in L_2(\mathbb{R}, d\lambda) \cap L_1(\mathbb{R}, d\lambda) \quad \text{uniformly,} \]
\[ \partial_x^i \partial_y^j \partial_{\lambda}^h (v - 1) \to 0 \quad \text{in} \quad L_2(\mathbb{R}, d\lambda) \quad \text{uniformly, as} \quad |x|, |y| \to \infty. \]
We can solve the inverse problem and obtain \( \Psi_0 \in \mathcal{S} \) by the argument in proving Theorem 1.1. Note here we need to use the reality condition \( v = v^* > 0 \) to show the global solvability. Moreover, by using the formula \( Q_0(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi_0, - (v - 1)d\xi, \) one obtains that \( Q_0 \) is Schwartz and possesses purely continuous scattering data.

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