A Forward Quantum Markov Field on Graphs

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Abstract

In this paper, we propose a class of quantum Markov fields QMF on a graphs \( G = (V, E) \). The Markov structure of the considered QMF is investigated in the finer structure of a quasi-local algebra \( A_V \) of observables based over a graphs \( G \). Namely, the considered Markovian fields are infinite volume states defined through a generating couple \((\varphi^{(0)}, (\mathcal{E}_{\{y\} \cup N_y}))\) of a product state \( \varphi^{(0)} \) on \( A_V \) and a family of local transition expectations \( \mathcal{E}_{\{y\} \cup N_y} \) based on a vertex \( y \) and the set of its nearest-neighbors. The main result of the paper concerns the existence and the uniqueness of QMF associated with a couple \((\varphi^{(0)}, (\mathcal{E}_{\{y\} \cup N_y}))\) for on an important class of graphs including trees strictly.

Keywords: Quantum Markov property; graphs; transition expectations, Markov triplet.

1 Introduction

Markov random fields \([21, 24, 25]\) have become a standard tools in several areas such as classical probability, statistical physics, computer science, image segmentation. However, a satisfactory theory of quantum Markov fields is still missing.

The first attempts to construct such a theory are \([7, 8, 6]\). These papers extended the Dobrushin-Markov random fields on the integer lattice \( \mathbb{Z}^\nu \) (see \([21, 22]\)) to a quantum setting. In \([10, 11, 12] \) and \([13] \) quantum Markov chains on the Cayley were constructed and phase transitions were investigated through Ising type models. In \([10] \) some algebraic properties of
the disordered phase associated with an Ising type model were studied on
the same kind of trees.

From a quantum probabilistic viewpoint QMF are multi-dimensional ex-
tension of one-dimensional quantum Markov chains and states introduced by
L. Accardi in [1]. In [9] the notion of generalized quantum Markov states
has been extended to fields, i.e. to quasi-local algebras over graphs possessing
a hierarchy property. Namely, the notions of d-Markov chains and generalized Markov states on graphs have been investigated, as natural extension to
tables of one-dimensional quantum Markov chains.

The non-commutative Markov property [2] play a crucial role in the
description of quantum Markovian states. In [11]-[13] a multi-dimensional
Markov property was investigated on the fine structure of the Cayley trees
and in [10], [15] additional interactions between one-level nearest neighbors
vertices on the same kind of trees were investigated.

In [6] a backward quantum Markov fields was constructed on a tensor
algebra over arbitrary graph. Namely, the noncommutative Markov prop-
erty on the finer structure of the considered graph was investigated across
conditional expectations in the sense of [18] and [17].

In this paper, we investigate a tessellation on an infinite graph $G = (V, E)$
that splits the vertex set into two infinite subsets: $V_\infty$ and its complementary.
The quantum Markov prperty is investigated for local structure of a UHF
$\mathcal{A}_V$ of observables over the graph $G$. Namely, a non-commutative extension
of the notion of Markov triplet studied in [23] is carried out for transition
expectation. The construction of the forward Markov field is based on a
reference product state $\varphi(0) = \bigotimes_{x \in V} \varphi_x$ together with a famely of transition
expectation $\{E_y, y \in V_\infty\}$ acting on the observables localized on the local
algebra over the site $y$ and the its nearest-neighbors. Mainly, we show the
existence and the uniqueness of a forward quantum Markov field associated
with the couple $(\varphi(0), \{E_y, y \in V_\infty\})$. The noncommutative nature of the
considered transition expectations requires an enumeration, on the set $V_\infty$ on
which the resulting quantum Markov field is strongly attached. The provided
QMF are of great interest on tree-like graphs. We stress that a concrete
Ising and XY-type models in connections will be the purpose of a paper in
preparation aiming to apply the results of the present paper to the study of
phenomena of phase transitions for several graphs.

Let us briefly mention the organization of the paper, after preliminary
We construct, in section 3, a tessellation on an infinite, locally finite, connected undirected graph. In section 4, we investigate the quantum Markov property on fine structure of the graph for local transition expectations w.r.t. precise Markov triplets. Section 5 is devoted to the definition of forward quantum Markov fields and the formulation of the main result which concerns the existence and the uniqueness of quantum Markov field for a large class of graphs including the tree. Section 6 is devoted to prove the main results. In section 7 we prove some auxiliary results on graphs.

2 Preliminaries

Let $G = (V, E)$ be a (non-oriented simple) graph, where $V$ is a nonempty set and $E$ is identified as a subset of non-ordered pairs of $V$, i.e.

$$E \subset \{\{x, y\} \mid x, y \in E, x \neq y\} \setminus \{(x, x), x \in E\}.$$  

Elements of $V$ and $E$ are called, respectively, vertices and edges. Two vertices $x$ and $y$ are said to be nearest neighbors, written $x \sim y$, if and only if $\{x, y\} \in E$.

For $y \in V$ we denote its nearest neighbors by

$$N_y := \{x \in V \mid y \sim x\}$$  \hspace{1cm} (1)

Notice that $x \notin N_x$. The set $\{y\} \cup N_y$ is called interaction domain or plaquette at $y$. In the sequel, the graph $G$ is assumed to be locally finite in the sense that $|N_x| < \infty$ for each $x \in V$. An edge path joining two vertices $x$ and $y$ is a finite sequence of edges $x = x_0 \sim x_1 \sim \ldots x_{d-1} \sim x_d = y$, in this case $d$ is the length of the edge path. The graph is said to be connected if every two disjoint vertices can be joined by an edge path. In the sequel, we assume that the graph $G$ is infinite, connected and locally finite. Thus the set $V$ is automatically countable. For $\Lambda \subset V$ nonempty, define

- Internal boundary:
  $$\overset{\leftarrow}{\partial} \Lambda := \{x \in \Lambda \mid \exists y \in \Lambda^c; \ x \sim y\}$$  \hspace{1cm} (2)

- Interior:
  $$\overset{\circ}{\Lambda} := \Lambda \setminus \overset{\leftarrow}{\partial} \Lambda$$  \hspace{1cm} (3)

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• **External boundary:**

\[
\overrightarrow{\partial} \Lambda := \{ y \in \Lambda^c \mid \exists x \in \Lambda; \ x \sim y \} \tag{4}
\]

• **External closure:**

\[
\overline{\Lambda} := \Lambda \cup \overrightarrow{\partial} \Lambda \tag{5}
\]

Denote \( F \) the set of all finite subsets of the vertex \( V \) and \( \mathcal{F} \) the net generated by \( F \) ordered by the inclusion " \( \subset \) ".

To each site \( x \in V \), we associate a finite dimensional \( C^* \)-algebra of observable \( \mathcal{A}_x \). Put \( \mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_x \) the algebra of observables localized in a finite region \( \Lambda \subset V \), where \( \otimes \) denotes the algebraic tensor product.

Denote \( \mathcal{A} \) the quasi-local algebra (see section 2.6 of [20]) obtained by the \( C^* \)-inductive limit associated to the directed system of algebras \( \{ \mathcal{A}_\Lambda \}_{\Lambda \in \mathcal{F}} \) with the embedding

\[
j_{\Lambda, \tilde{\Lambda}} : a_\Lambda \in \mathcal{A}_\Lambda \mapsto a_\Lambda \otimes 1_{\tilde{\Lambda} \setminus \Lambda} \in \mathcal{A}_{\tilde{\Lambda}} \tag{6}
\]

In particular, if \( \tilde{\Lambda} = V \) one gets the following identification

\[
\mathcal{A}_\Lambda \cong j_{\Lambda}(\mathcal{A}_\Lambda) = \mathcal{A}_\Lambda \otimes 1_{\Lambda^c} \tag{7}
\]

this leads to the local algebra

\[
\mathcal{A}_{loc} = \bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda \tag{8}
\]

The quasi-local algebra \( \mathcal{A} \) is then the closure of \( \mathcal{A}_{loc} \)

\[
\mathcal{A} = \overline{\mathcal{A}_{loc}} = \bigcup_{\Lambda \in \mathcal{F}} \overline{\mathcal{A}_\Lambda}.
\]

The natural embedding of \( \mathcal{A}_x \) into \( \mathcal{A}_V \) will be denoted by

\[
j_x : a \in \mathcal{A}_x \mapsto a \otimes 1_{\{x\}^c}.
\]

Similarly, for \( \Lambda \in \mathcal{F} \) the embedding \( j_{\Lambda} \) can be written as follows

\[
j_{\Lambda} = \bigotimes_{x \in \Lambda} j_x.
\]


3 Tessellations on graphs

Let $G = (V, E)$ be an infinite connected graph which is locally finite (i.e. for each $x \in V$ the set $N_x$ if its nearest neighbors is finite.)
Fix a "root" $y_1 \in V$ and define by induction the following sets:

$$V_{0,1} := \{y_1\}; \quad V_1 = \{o\} \cup N_o$$  \hspace{1cm} (9)

Having defined $V_{0,n}$, put

$$V_n := V_{0,n} = \bigcup_{y \in V_{0,n}} (\{y\} \cup N_y)$$  \hspace{1cm} (10)

$$V_{0,n+1} := V_{0,n} \cup \partial V_n$$  \hspace{1cm} (11)

Put

$$V_\infty := \bigcup_{n \geq 1} V_{0,n}$$  \hspace{1cm} (12)

Each element of $V \setminus V_\infty$ belongs to the plaquette at a certain element of $V_\infty$

$$V \setminus V_\infty = \bigcup_{y \in V_\infty} \partial\{y\}$$

In the sequel we assume that the graph $G$ becomes totally disconnected by removing $V_\infty$ or $V \setminus V_\infty$. This means that every edge has an endpoint in $V_\infty$ and an endpoint in $V \setminus V_\infty$. Note that this property is satisfied by an important of graphs such as trees and the multi-dimensional integer lattice $\mathbb{Z}^d$.

**Proposition 1** For each $n \in \mathbb{N}^*$, let $E_n = \{\{x, y\} \in E; \quad x, y \in V_n\}$ then the following assertions holds

(i) The subgraph $G_n := (V_n, E_n)$ is finite and connected,

(ii) $V = \bigcup_{n \geq 1} V_n$ and $E = \bigcup_n E_n$.

It follows that the sequence $(V_n)_{n \in \mathbb{N}}$ is exhaustive for the vertex set $V$. i.e. $V_n \subset V_{n+1}$ for any finite subset $\Lambda$ of $V$ there exists an integer $n$ such that $\Lambda \subseteq V_n$. One gets

$$V_\infty := \bigcup_{n \geq 0} \partial V_n$$  \hspace{1cm} (13)
and
\[ V = \bigcup_{y \in V_{\infty}} \{ y \} \cup N_y \] (14)

In what follows, the family \( \{ V_{0,n}; \quad n = 1, 2, \cdots \} \) will be called a *tessellation* of the graph \( G \). For each \( n \in \mathbb{N} \) and \( y \in \partial V_n \), put
\[ N_y^{(p)} := N_y \cap \partial V_n \quad N_y^{(s)} := N_y \cap \partial V_{n+1}, \quad N_y^{(0)} := N_y \setminus (N_y^{(p)} \cup N_y^{(s)}) \] (15)

here \( N_y^{(s)} \) and \( N_y^{(p)} \) refer respectively to the sets of the *direct successors vertices* and *direct previous vertices* of the vertex \( y \).

**Proposition 2** For each integer \( n \geq 1 \), the following assertions holds

(i) \( \partial V_{n+1} = \bigcup_{y \in \partial V_n} N_y^{(s)} \),

(ii) \( \partial V_n = \bigcup_{y \in \partial V_n} N_y^{(p)} \).

**Remark 1** Note that generally the inclusion in (i) of Proposition 2 is strict. In fact the following case may occur: \( x \in V_{n+1} \setminus V_n \) with \( N_x = \{ y \} \) for some \( y \in \partial V_n \).

For the sake of simplicity, we assume, from now on, that the graph \( G = (V, E) \) satisfy the following conditions:
\[ N_y^{(0)} = \emptyset \] (16)
\[ N_y^{(s)} \cap N_z^{(s)} = \emptyset \] (17)

for every integer \( n \) and every \( y, z \in \partial V_n \) with \( y \neq z \).

The properties (16) and (17) are satisfied by an important class of graphs including trees (which are connected graphs with no cycles). Indeed, the property (17) is satisfied by any tree-like graph and (17) is satisfied for trees without degree-one vertices and in the case of the integer lattices \( \mathbb{Z}^d \).

**Proposition 3** Every tree enjoys the properties (16) and (17).
From Proposition ?? for each integer $n$ the subset $V_n$ given by (10) is finite then its external boundary $\partial V_n$ is also finite. The following enumeration arises naturally.
Let us fix an enumeration of the set $\partial V_n$

$$\partial V_n = \{ y_1^{(n)}, \ldots, y_{|\partial V_n|}^{(n)} \}$$  \hspace{1cm} (18)

According to (13), this leads to an enumeration on the set $V_\infty$ given by (12).

4 Quantum transition expectations and Markov triples

Recall that, a Umegaki conditional expectation is a norm one projection $E$ from a $C^*$-algebra $A$ into a $C^*$-subalgebra $B$.

**Definition 1** Let $A$ and $B$ be a $C^*$-subalgebra. A completely positive identity preserving map $E : A \to B$ is called transition expectation. If $C$ is a given $C^*$-subalgebra of $A$ such that

$$E(C' \cap A) \subseteq C' \cap B$$  \hspace{1cm} (19)

the map $E$ is called **Markov transition expectation (MTE)** with respect to the triplet $(A, B, C)$. Such a triplet will be referred as **Quantum Markov triplet** for the transition expectation $E$.

Note that, the Markov property (19) was first formulated in [1]. The notion of quantum triplet Markov consists a noncommutative extension of classical Markov triple studied in [23].

Let $\Lambda_1, \Lambda_2 \in \mathcal{F}$, with $\Lambda_2 \subseteq \Lambda_2$. Let $E_{\Lambda_1, \Lambda_2}$ be a Markov transition expectation w.r.t the triplet $(A_V, A_{\Lambda_1}, A_{\Lambda_2})$. Since $A'_{\Lambda_2} = A_{\Lambda_2}$ the Markov property (19) becomes

$$E_{\Lambda_1, \Lambda_2}(A_{\Lambda_2}) \subseteq A_{\Lambda_1 \setminus \Lambda_2}$$  \hspace{1cm} (20)

Now in order to investigate the Markov property on the finer structure of considered graph we consider for each $y \in V_\infty$ a Markov transition expectation $E_y$ w.r.t. the triplet $(A_{\{y\} \cup N_y}, A_N^{(p)}, A_N^{(s)})$. This leads to

$$E_y(A_{\{y\} \cup N_y}) \subseteq A_N^{(s)}.$$  \hspace{1cm} (21)
Lemma 1 Let $n$ be an integer and $\bar{\partial}V_n$ be enumerated as in (18). If for each $j = 1, \ldots, |\bar{\partial}V_n|$ a MTE $\mathcal{E}_{y_j^{(n)}}$ w.r.t. the triplet $(\mathcal{A}_{y_j^{(n)}} \cup N_{y_j^{(n)}}), \mathcal{A}_{y_j^{(n)}}$, then the map
\[ \mathcal{E}_{n,n+1} := \mathcal{E}_{y_1^{(n)}} \circ \cdots \circ \mathcal{E}_{y_j^{(n)}} \] (22)
is a Markov transition expectation w.r.t. the triplet $(\mathcal{A}_{V_{n+1} \setminus V_n}, \mathcal{A}_{V_{n+1} \setminus V_n}, \mathcal{A}_{\bar{\partial}V_n})$.

In particular, the equation
\[ \mathcal{E}_{n,n+1} (\mathcal{A}_{V_{n+1} \setminus V_n}) \subseteq \mathcal{A}_{V_{n+1} \setminus V_n} \] (23)
will be referred as the level Markov property.

Proof. Since each $\mathcal{E}_{y_j^{(n)}}$ is completely positive and identity preserving then, from (22), the map $\mathcal{E}_{n,n+1}$ is completely positive identity preserving. By Proposition 2 and (16) one has
\[ \bigcup_{y \in \bar{\partial}V_n} \{y\} \cup N_y = \bar{\partial}V_n \cup \bar{\partial}V_n \cup \bar{\partial}V_{n+1} = V_{n+1} \setminus V_n. \]
According to (16) and (17) for $y, z \in \bar{\partial}V_n$ with $x \neq y$, one gets
\[ (\{y\} \cup N_y) \cup (\{z\} \setminus N_z) = (\{y\} \cup N_y) \cup (\{z\} \cup (N_z \setminus (N_y \cap N_z))). \]

Then, one finds
\[ \mathcal{E}_z \circ \mathcal{E}_y (\mathcal{A}_{\{y\} \cup N_y} \vee \mathcal{A}_{\{z\} \cup N_z}) = \mathcal{E}_z (\mathcal{E}_y (\mathcal{A}_{\{y\} \cup N_y} \otimes \mathcal{A}_{\{z\} \cup N_z \setminus (N_y \cap N_z)})) \] (24)
\[ = \mathcal{E}_y (\mathcal{A}_{\{y\} \cup N_y} \otimes \mathcal{E}_z (\mathcal{A}_{\{z\} \cup N_z \setminus (N_y \cap N_z)})) \subseteq \mathcal{A}_{N_y} \otimes \mathcal{A}_{N_z}. \]

Iterating (24) on elements of $\{y_1^{(n)}, \ldots, y_j^{(n)}\}$, one sees that $\mathcal{E}_{n,n+1}$ maps $\mathcal{A}_{V_{n+1} \setminus V_n} \circ \mathcal{A}_{\bar{\partial}V_n}$ into $\mathcal{A}_{V_{n+1} \setminus V_n} = \mathcal{A}_{V_{n+1} \setminus V_n} \cap \mathcal{A}_{\bar{\partial}V_n}$. Therefore, $\mathcal{E}_{n,n+1}$ is a transition expectation w.r.t. the given triplet and it satisfies the Markov property (23). \(\square\)

Theorem 1 Let $\mathcal{E}_{n,n+1}$ be given by (22). Then
\[ \mathcal{E}_{V_{n+1}^c} := \mathcal{E}_{n,n+1} \circ \cdots \circ \mathcal{E}_{0,1} \] (25)
is a MTE w.r.t. the triplet $(\mathcal{A}_V, \mathcal{A}_{V_{n+1}^c}, \mathcal{A}_{V_{n+1}})$.

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Proof. From Lemma 1 for each \( k = 1, \cdots, n \) the map \( E_{k,k+1} \) is a MTE with respect to the triplet \( (A_{V_{k+1}\setminus V_k}, A_{V_{k+1}\setminus V_k}, A_{V_k}) \).

A simple induction shows that \( A_{V_{n+1}} \) is transition expectation with respect to the given triplet. □

5 Forward quantum Markov fields

This section will be devoted to the definition of forward quantum Markov fields on the local structure of the considered graph w.r.t. the considered tessellation in section 3. Consider a product state

\[
\varphi^{(0)} = \bigotimes_{x \in V} \varphi_x^{(0)} \in \mathcal{S}(A_V)
\]

where \( \varphi_x^{(0)} \) is a state on the algebra \( A_x \). If \( \Lambda \subseteq V \), we denote \( \varphi^{(0)}_{\Lambda} := \bigotimes_{x \in \Lambda} \varphi_x^{(0)} \).

Let \( E_y \) be a transition expectation w.r.t. the triplet \( (A_{\{y\} \cup N_y}, A_{N_y^{(s)}}, A_{\{y\} \cup N_y^{(p)}}) \) for each \( y \in V_{\infty} \) and \( (E_{n,n+1})_{1 \leq k \leq n} \) be given by (22). Define for each \( n \in \mathbb{N} \)

\[
\varphi_n = \varphi_{V_{n+1}}^{(0)} \circ E_{n,n+1} \circ \cdots \circ E_{0,1}
\]

From Theorem 1, the functional \( \varphi_n \) is a state on the algebra \( A_V \).

Definition 2 Any limit point \( \varphi \) (point–wise on \( A \)) of states of the form (27) is called a Forward quantum Markov fields on \( A \). The state \( \varphi^{(0)} \in \mathcal{S}(A_V) \) is called reference state and \( (E_y) \) is called sequence of Markov transition expectations associated with the Markov field \( \varphi \).

If there exist two or more limit points of (27), the pair \( \{\varphi^{(0)}, (E_y)_{y \in V_{\infty}}\} \) is said to admit phase transitions.

Remark 2 In Definition 2, the Markov property is investigated on the finer structure of the considered graph. Namely, the transition expectations \( E_y \) satisfies the local Markov property (27). Note that in [9], [5] the Markov property was expressed only w.r.t. levels of the considered graphs.

Definition 3 A sequence \( \{\varphi_n\} \) of states on \( A_V \) is called convergent in the strongly finite sense if, for any \( a \in A_V \), there exists \( n_a \in \mathbb{N} \) such that for any \( n \geq n_a \),

\[
\varphi_n(a) = \varphi_{n_a}(a)
\]
The following theorem is the main result of the paper.

**Theorem 2** Let \( G = (V, E) \) be a graph and \( V_\infty \) be given by (12) and satisfy (16) and (17). Let \( (\varphi^0, (E_y)_{y \in V_\infty}) \) be a couple satisfying (26) and (21). Assume that

\[
\varphi^{(0)}_{N_y} \circ E_y(a_{N_y} \otimes 1 \otimes I_{N_y^{(s)}}) = \varphi^{(0)}_{N_y}(a_{N_y}^{(p)}) \quad ; \quad a_{N_y}^{(p)} \in A_{N_y}^{(p)}
\]

for each \( y \in V_\infty \). Then there exists a unique forward quantum Markov field \( \varphi \) associated with the couple \( (\varphi^0, (E_y)_{y \in V_\infty}) \). Moreover, the sequence states \( (\varphi_n) \) given by (27) converges on the strongly finite sense into a unique state \( \varphi \) on \( A_V \).

6 Proof of theorem 2

**Lemma 2** Let \( k \) be a positive integer and \( D_1, D_2, \ldots, D_k \) be subsets of a given set \( X \). Denote

\[
\Delta_0 = \emptyset, \quad \Delta_j = D_j \setminus (D_1 \cup \cdots \cup D_{j-1}), \quad j = 1, \ldots, k
\]

then the parts \( \Delta_1, \Delta_2, \ldots, \Delta_k \) are pair-wise disjoint and

\[
\bigcup_{1 \leq j \leq k} D_j = \bigcup_{1 \leq j \leq k} \Delta_j
\]

**Lemma 3** In the notations of Theorem 2. Let \( b = \bigotimes_{x \in \partial V_n} b_x \in A_{\partial V_n} \). Then

\[
\mathcal{E}_{n,n+1}(b \otimes 1_{\partial V_n} \otimes 1_{\partial V_{n+1}}) = \bigotimes_{1 \leq j \leq |\partial V_n|} \mathcal{E}_{y_j}^{(n)} \left( b_{\Delta_j} \otimes 1_{y_j^{(n)}} \otimes 1_{N_y^{(s)}} \right)
\]

where \( \Delta_0 = \emptyset, \quad \Delta_j = N_{y_j^{(n)}}^{(p)} \setminus \left( N_{y_1^{(n)}}^{(p)} \cup \cdots \cup N_{y_{j-1}^{(n)}}^{(p)} \right), \quad j = 1, \ldots, |\partial V_n| \) and \( b_{\Delta_j} = \bigotimes_{x \in \Delta_j} b_x \).

**Proof.** From Lemma 2 (with \( D_j = N_{y_j^{(n)}}^{(p)} \)) the sets \( \Delta_j \) are pair-wise disjoint and \( \cup_j \Delta_j = \cup_j N_{y_j^{(n)}}^{(p)} = \partial V_n \). Without lose of generality, we assume that \( b \) is localized in the form

\[
b = \bigotimes_{1 \leq j \leq |\partial V_n|} b_{\Delta_j}; \quad b_{\Delta_j} = \bigotimes_{x \in \Delta_j} b_x
\]
Since $\bigcup_{2 \leq j \leq |\partial V_n|} \Delta_j \subset (\{y_1^{(n)}\} \cup N_{y_1^{(n)}})^c$, then, (17) leads to

$$E_{y_1^{(n)}}(b) = \left( \bigotimes_{2 \leq j \leq |\partial V_n|} b_{\Delta_j} \right) \otimes E_{y_1,n}(b_{\Delta_1} \otimes 1_{y_1,n} \otimes 1_{N_{y_1,n}})$$

and

$$E_{y_1^{(n)}}(b_{\Delta_1} \otimes 1_{y_1^{(n)}} \otimes 1_{N_{y_1^{(n)}}}) \in A_{N_{y_1^{(n)}}} \subset A_{y_1^{(n)}}^{(n)} N_{y_1^{(n)}}, \quad j = 2, \ldots , |\partial V_n|$$

Then

$$E_{n,n+1}(b) = E_{y_1^{(n)}} \circ \cdots \circ E_{y_2^{(n)}} \left( \bigotimes_{1 \leq j \leq |\partial V_n|} b_{\Delta_j} \right) \otimes E_{y_1^{(n)}}(b_{\Delta_1} \otimes 1_{y_1^{(n)}} \otimes 1_{N_{y_1^{(n)}}})$$

Iterating this procedure, one gets (29).

**Proof.** (proof of theorem 2)

From Theorem 25 and (26) the functional $\varphi_n$ given by (27) is a state on the algebra $A_{V}$. According to the linearity of the continuity of $(\varphi_n)_n$, it is enough to show the point-wise limit on the local algebra $A_{V,\text{loc}}$. Let $a \in A_{V,\text{loc}}$. By Proposition ?? there exists an integer $n$ such that $a \in A_{\partial V_n}$. One has

$$\varphi_n(a) = \varphi^{(0)}_{(V_n)^c} \circ E_{n,n+1} \circ \cdots \circ E_{0,1}(a),$$

$$= \varphi^{(0)}_{(\partial V_{n+1})} \circ E_{n,n+1}(b)$$

where $b = E_{n-1,n} \circ \cdots \circ E_{0,1}(a) \in A_{\partial V_{n+1}}$. Up to linear extension it is enough to consider the case where $b$ is localized in the form

$$b = \bigotimes_{x \in \partial V_n} b_x = \bigotimes_{x \in \partial V_n} b_x; \quad b_x = 1_{A_x} \quad \forall x \in \partial (p)V_n \setminus \partial V_n$$

applying above mentioned remark with $D_j = N_{y_j^{(n)}}^{(p)}$, $j = 1, \ldots , |\partial V_n|$ and $\Delta_j = D_j \setminus (D_1 \cup \cdots \cup D_{j-1}) \subseteq N_{y_j^{(n)}}^{(p)}$, $j = 1, \ldots , |\partial V_n|$ (with $X_0 = \emptyset$), thus again we can restrict to elements $b$ of the form:

$$b = \bigotimes_{j=1}^{|\partial V_n|} b_{Y_j}; \quad b_{Y_j} = \bigotimes_{x \in Y_j} b_x$$
Using Lemma 3, one obtains
\[
\varphi_n(a) = \varphi_{\overrightarrow{V_n+1}}(b) = \left( \bigotimes_{j=1}^{|[\overrightarrow{V_n}]|} \varphi_N^{(n)}(y_j) \right) \circ \mathcal{E}_{y_j^{(n)}} \circ \cdots \circ \mathcal{E}_{y_j^{(n)}} \left( \bigotimes_{j=1}^{|[\overrightarrow{V_n}]|} b_{y_j} \right)
\]
and using condition (28), one gets
\[
\varphi_{N_y}^{(n)}(y_j) \circ \mathcal{E}_{y_j^{(n)}} \left( b_{y_j} \otimes 1^{(n)} \otimes 1^{N_y^{(n)}} \right) = \varphi_{N_y}^{(p)}(b_{y_j})
\]
then
\[
\varphi_n(a) = \prod_{1 \leq j \leq |\overrightarrow{V_n}|} \varphi_{N_y}^{(p)}(b_{y_j}) = \varphi_{\overrightarrow{V_n}}(b) = \varphi_{(V_{n-1})_c} \circ \mathcal{E}_{\overrightarrow{V_{n-1}}} \circ \cdots \circ \mathcal{E}_{\overrightarrow{V_\infty}}(a) = \varphi_{n-1}(a)
\]
Iterating one gets, for each \( m \geq n \)
\[
\varphi_m(a) = \varphi_{m-1}(a) = \cdots = \varphi_{n-1}(a)
\]
therefore the sequence \( \{\varphi_n\}_n \) converges to a state \( \varphi \in S(A_V) \) for the weak-*-topology.

7 Annex

Proof of proposition. 1°

(i) By induction: for \( n = 1 \) one has \( V_1 = \{y\} \cup N_y \) connected and finite (because the graph \( G \) is locally finite). Assume that \( G_n \) is connected and finite. One has
\[
V_{n+1} = V_n \cup \overrightarrow{V_n} \cup \leftarrow V_{n+1} = V_n \cup \bigcup_{y \in \overrightarrow{V_n}} \{y\} \cup N_y.
\]
Since the graph $G$ is locally finite and $V_n$ is finite then $\partial V_n$ and $
abla_{y\in\partial V_n}\{y\} \cup N_y$ are also finite then $V_{n+1}$ is finite.

In addition, $V_n$ is connected, every element $y$ of $\partial V_n$ is joined through one edge to some vertex $x \in V_n$. For $z \in \partial V_{n+1}$ there exits $y \in \partial V_n$ such that $z \in N_y$ therefore $z \sim y \sim x$ for some $x \in V_n$. Therefore, $G_{n+1} = (V_{n+1}, E_{n+1})$ is also connected.

(ii) Let $x \in V$, denoting $d = d(x, y_1)$ and let $x_0 = y_1 \sim x_1 \sim \cdots \sim x_d = x$ one has

$$x_1 \sim y_1 \Rightarrow x_1 \in N_y \subset V_1, \cdots, x = x_d \sim x_{d-1} \in V_{d-1} \Rightarrow x_d \in \overline{V}_{d-1} \subset V_d$$

then $V \subset \bigcup_n V_n$, the second inclusion is obvious.

By construction the sequence $(V_n)_n$ is increasing and it absorbs all finite sets of $V$. We conclude that $V = \bigcup_n V_n$ and $E = \bigcup E_n$. □

**Proof of Proposition.**

(i) It’s clear that $\bigcup_{y\in\partial V_n} N_y^{(s)} \subset \partial V_{n+1}$.

Conversely, by construction $V_{n+1} = V_n \cup \partial V_n \cup \overleftarrow{\partial} V_{n+1}$, the plaquette at each element of $V_n \cup \partial V_n$ is included in $V_{n+1}$ thus

$$\overleftarrow{\partial} V_{n+1} \subset V_{n+1} \setminus (V_n \cup \partial V_n) = \bigcup_{y\in\partial V_n} N_y$$

then

$$\overleftarrow{\partial} V_{n+1} = \bigcup_{y\in\partial V_n} N_y \cap V_{n+1} = \bigcup_{y\in\partial V_n} N_y^{(s)}$$

(ii) By definition of $\partial V_n$ one has

$$(\forall x \in \overleftarrow{\partial} V_n, \exists y \in V_n^c, x \sim y) \Leftrightarrow \forall x \in \overleftarrow{\partial} V_n, \exists y \in \partial V_n; x \in N_y$$

and since $\overleftarrow{\partial} V_n \cap \overrightarrow{\partial} V_{n+1} = \emptyset$ and $N_y^{(s)} \subset \overrightarrow{\partial} V_{n+1}$, one get $x \in N_y^{(p)}$ therefore, $\partial V_n \subset \bigcup_{y\in\partial V_n} N_y^{(p)}$. 

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Proof of Proposition. Let $G = (V, E)$ be a tree occupied with the sets $\{V_{0,n}, n \in \mathbb{N}\}$ given by (11).

By Proposition II the graph $G_n = (V_n, E_n)$ is connected. and since each vertex from $\overrightarrow{\partial} V_n$ is related to some vertex from $V_n$ then $\overrightarrow{V}_n$ is connected, in particular every edge from $V_n$ is joined to the root $o$ through an edge path. Let $y$ and $z$ be two disjoint vertices of $\overrightarrow{\partial} V_n$, consider two edge-paths $\gamma_{y,y_1}: u_1 = y \sim u_2 \sim \cdots \sim u_k = y_1$ and $\gamma_{y_0,z}: v_1 = y_0 \sim v_2 \sim \cdots \sim v_m = z$ joining respectively $y$ with $y_0$ and $y_0$ with $z$. If $x \in N(s) \cap N(p)$ then

$$\gamma: x \sim (y = u_1) \sim \cdots \sim (u_k = o = v_1) \sim v_2 \sim \cdots \sim (v_m = z) \sim x$$

is a cycle on the tree $G$ is a tree. Then $N(s) \cap N(s) = \emptyset$ and the tree $G$ enjoys the (17).

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