Small groups of finite Morley rank with a tight automorphism

Ulla Karhumäki

Joint work with Pınar Üğurlu

Ranked Groups: The Return
23-24 September 2021 at Université Claude Bernard Lyon 1
The Cherlin-Zilber conjecture (Cherlin 1979 and Zilber 1977)

Infinite simple groups of finite Morley rank are isomorphic to Chevalley groups over an algebraically closed fields.

In any group $G$ of fRM, the Sylow 2-subgroups are conjugate [Borovik, Poizat 2007] and their structure is well-understood:

$\text{Syl}_G = U \times T$,

where $U$ is 2-unipotent and $T$ is 2-divisible. If the ambient group $G$ is infinite then:

Either $\text{Syl}_G = 1$ (degenerated type) or $\text{Syl}_G$ is infinite. [Borovik, Burdges, Cherlin 2007]

Either $U = 1$ (odd type) or $T = 1$ (even type). (No mixed type groups exist.) [Altınel, Borovik, Cherlin 2008]

If $1 \neq \text{Syl}_G = U$ then (C-Z) holds. Namely, $G \hookrightarrow X(K)$ for an a.c. field $K$ of $\text{char}(K) = 2$. [Altınel, Borovik and Cherlin 2008]
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Small groups
Let $H = X(K)$ be a Chevalley group for $K$ a.c. with $\text{char}(K) \neq 2$, and $T$ be a maximal algebraic torus of $H$. The ‘size’ of $H$ can be described in different ways, e.g. by $\dim_{\text{Zar}}(H)$ or by

$$\dim_{\text{Zar}}(T) = \text{pr}_2(H) = \# \text{ of copies of } \mathbb{Z}_{2\infty} := \{ x \in \mathbb{C}^\times : x^{2^n} = 1, n \in \mathbb{N} \}.$$
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By $\text{pr}_2(G)$? Makes sense...but we still don't know how to prove that if $\text{pr}_2(G) = 1$ then $G \subset = \text{PSL}_2(K)$ for $K$ a.c. So, one needs further assumptions to the identification of ‘small’ $G$. For example:

I Minimal simple groups: every proper definable connected subgroup is solvable.

[Jaligot, Cherlin, Deloro, Altinel, Frécon, Burdges...]

I The presence of a tight automorphism whose fixed-point subgroup is pseudofinite—this is our framework.
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  - *Minimal simple* groups: every proper definable connected subgroup is solvable.[Jaligot, Cherlin, Deloro, Altinel, Frécon, Burdges...]
  - The presence of a *tight* automorphism whose fixed-point subgroup is pseudofinite—this is our framework.
Pseudofinite fields and simple pseudofinite groups

Definition: An infinite structure is called pseudofinite if every first-order sentence true in it also holds in some finite structure or, equivalently, if it is elementarily equivalent to a non-principal ultraproduct of finite structures.
Pseudofinite fields and simple pseudofinite groups

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- $\mathbb{Z}, +$ is not pseudofinite.
- $\mathbb{Q}, + \equiv \prod_{p \in \mathbb{P}} \mathbb{C}_p / \mathcal{U}$ is pseudofinite.
- A (twisted) Chevalley group $X(F)$ is pseudofinite iff $F$ is pseudofinite.
- Algebraically closed fields are not pseudofinite.
- $F \equiv \prod_{p \in \mathbb{P}} \mathbb{F}_p / \mathcal{U}$ is pseudofinite of $\text{char}(F) = 0$. 

Theorem (Ax 1968) An infinite field is pseudofinite iff it is perfect, quasi-finite and PAC.

Theorem (Wilson 1995 and Ryten 2007) A simple group is pseudofinite iff it is isomorphic to a (twisted) Chevalley group over a pseudofinite field.
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A simple group is pseudofinite iff it is isomorphic to a (twisted) Chevalley group over a pseudofinite field.
The Principal conjecture

The theory ACFA of algebraically closed fields with generic automorphism is axiomatised [Chatzidakis, Hrushovski 1999] as follows: \((K, \sigma) \models \text{ACFA}\) iff:

- \(K \models \text{ACF}\) and \(\sigma \in \text{Aut}(F)\).

- Let \(V\) be an irreducible variety and let \(S\) be an irreducible subvariety of \(V \times \sigma(V)\) s.t. both \(\pi_1 : S \to V\) and \(\pi_2 : S \to \sigma(V)\) are dominant. Then there exists \(a \in V(K)\) s.t. \((a, \sigma(a)) \in S\).

If \((K, \sigma) \models \text{ACFA}\) then \(\text{Fix}_K(\sigma)\) is pseudofinite. [Macintyre1997/Chatzidakis, Hrushovski 1999]
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Fixed point subgroups of generic automorphisms of ‘structures with certain nice model-theoretic properties’ resemble pseudofinite groups. [Hrushovski 2002]

The Principal conjecture (Hrushovski 2002/Uğurlu 2009)

Let \(G\) be an infinite simple group of finite Morley rank with a generic automorphism \(\alpha\). Then the fixed point subgroup \(C_G(\alpha)\) is pseudofinite.
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The Principal conjecture (Hrushovski 2002/Uğurlu 2009)
Let \(G\) be an infinite simple group of finite Morley rank with a generic automorphism \(\alpha\). Then the fixed point subgroup \(C_G(\alpha)\) is pseudofinite.

- \((C-Z) \Rightarrow (PC)\). [Chatzidakis and Hrushovski 1999]

- We aim to prove that \((PC) \Rightarrow (C-Z)\).
A tight automorphism $\alpha$

From now on:
1. Groups (resp. fields) are considered in pure group (resp. field) language.
2. Given a subset $X$ of a group of fRM $G$, $\overline{X}$ is the definable closure of $X$ in $G$. 

Example: We have $(K, U) \models ACFA [Hrushovski 1996]$, where $U: K = \prod_{i} p^{i_2} P_{alg}^{p_{i_2}}/\mathcal{U}$ is the non-standard Frobenius automorphism of $K$.

Let $G = X(K)$ be a simple Chevalley group and $H$ be a definable, connected and $U$-invariant subgroup of $G$. Then $U$ induces an automorphism on $G$ s.t. $X(\text{Fix}_K(U)) = X(\mathcal{O}_p^{i_2} P_{alg}^{p_{i_2}}/\mathcal{U}) \hookrightarrow \mathcal{O}_p^{i_2} P_{alg}^{p_{i_2}}/\mathcal{U}$ is pseudofinite. $CH(U) = H(k)$, with $k$ pseudofinite. So, $CH(U) \subset \text{Zar} = H$. 

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**Definition (Üğurlu 2009)**

An automorphism $\alpha$ of an infinite simple group of fRM $G$ is called **tight** if, for any connected definable and $\alpha$-invariant subgroup $H \leq G$, $C_H(\alpha) = H$. 

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**Definition (Uğurlu 2009)**

An automorphism $\alpha$ of an infinite simple group of fRM $G$ is called **tight** if, for any connected definable and $\alpha$-invariant subgroup $H \leq G$, $\mathcal{C}_H(\alpha) = H$.

**Example:** We have $(K, \phi_U) \models \text{ACFA}[\text{Hrushovski 1996}]$, where

$$
\phi_U : K = \prod_{p_i \in P} \mathbb{F}_{p_i}^{\text{alg}} / U \longrightarrow \prod_{p_i \in P} \mathbb{F}_{p_i}^{\text{alg}} / U, \quad [x_i]_U \mapsto [x_i^{p_i}]_U
$$

is the **non-standard Frobenius automorphism** of $K$.

Let $G = X(K)$ be a simple Chevalley group and $H$ be a definable, connected and $\phi_U$-invariant subgroup of $G$. Then $\phi_U$ induces an automorphism on $G$ s.t.

- $X(\text{Fix}_K(\phi_U)) = X(\prod_{p_i \in P} \mathbb{F}_{p_i} / U) \cong \prod_{p_i \in P} X(\mathbb{F}_{p_i}) / U$ is pseudofinite.
- $\mathcal{C}_H(\phi_U) = H(k)$, with $k$ pseudofinite. So, $\mathcal{C}_H(\phi_U)^{\text{Zar}} = H$. 

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Tight $\alpha$ with pseudofinite fixed-point subgroup

The socle $\text{Soc}(H)$ of a group $H$ is the subgroup generated by all minimal normal non-trivial subgroups of a group $H$.

**Theorem (Uğurlu 2009)**

Let $G$ be an infinite simple group of fRM and $\alpha$ be a tight automorphism of $G$ s.t. $C_G(\alpha) = P \equiv \prod_{i \in I} P_i/\mathcal{U}$ is pseudofinite. Then there is a definable normal $S$ of $P$ s.t.

$$X(F) \cong \prod_{i \in I} \text{Soc}(P_i)/\mathcal{U} \equiv S \trianglelefteq P \leq \text{Aut}(S),$$

where $F$ is a pseudofinite field. Moreover, $\overline{S} = G$. 
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**Remarks:**

1. $G$ has involutions as the simple pseudofinite group $S$ has involutions.
2. For almost all $i$, The socle $\text{Soc}(P_i)$ is uniformly definable normal subgroup of $P_i$. So $P/S \equiv \prod_{i \in I} (P_i/\text{Soc}(P_i))/\mathcal{U}$. 
Let $G$ be an infinite simple group of fRM with a tight automorphism $\alpha$ whose fixed point subgroup $P = C_G(\alpha)$ is pseudofinite. We have $\text{pr}_2(G) = n \geq 1$. To prove that (C-Z) $\iff$ (PC) we need to prove the following two steps:

1. **Algebraic identification step**: Show that $S$ is of untwisted Lie type $X$ and of Lie rank $n$, and, that $\text{char}(F) \neq 2$. Then prove that this forces $G$ to be isomorphic to a Chevalley group $X(K)$, of the same untwisted Lie type $X$ and the same Lie rank $n$ as $S$, over an a.c. field $K$ of $\text{char}(K) \neq 2$.

2. **Model-theoretic step**: Prove that a generic automorphism of $G$ is tight.
Let $G$ be an infinite simple group of fRM with a tight automorphism $\alpha$ whose fixed point subgroup $P = C_G(\alpha)$ is pseudofinite. We have $\text{pr}_2(G) = n \geq 1$. To prove that $(C-Z) \iff (PC)$ we need to prove the following two steps:

1. **Algebraic identification step:** We know that there is a pseudofinite (twisted) Chevalley group $S = X(F)$ s.t. $\overline{S} = G$.
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Our results (K. and Uğurlu 2021)

From now on, $G$ is an infinite simple group of fRM with $\text{pr}_2(G) = 1$ admitting a tight automorphism $\alpha$ whose fixed-point subgroup $C_G(\alpha) = P \equiv \prod_{i \in I} P_i/\mathcal{U}$ is pseudofinite and $S \cong \chi(F) \cong \prod_{i \in I} \text{Soc}(P_i)/\mathcal{U}$. 

Proposition

$S \cong \chi(F) = \text{PSL}_2(F)$, where $F$ is a pseudofinite field of $\text{char}(F) \neq 2$.

Theorem (Version 1.)

If $1$ is a square in $F$ and $\text{char}(F) > 2$, then $G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) > 2$.

Theorem (Version 2.)

If $1$ is a square in $F$ and the Sylow 2-subgroups of $S$ are not Klein 4-groups, $G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) \neq 2$.

Almost an theorem

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**Theorem (Version 2.)**

*If $-1$ is a square in $F^\times$ and the Sylow 2-subgroups of $S$ are not Klein 4-groups, $G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) \neq 2$.***
Our results (K. and Uğurlu 2021)

From now on, $G$ is an infinite simple group of fRM with $\text{pr}_2(G) = 1$ admitting a tight automorphism $\alpha$ whose fixed-point subgroup $C_G(\alpha) = P \equiv \prod_{i \in I} P_i/\mathcal{U}$ is pseudofinite and $S \cong X(F) \cong \prod_{i \in I} \text{Soc}(P_i)/\mathcal{U}$.

**Proposition**

$S \cong \text{PSL}_2(F)$, where $F$ is a pseudofinite field of $\text{char}(F) \neq 2$.

**Theorem (Version 1.)**

*If $-1$ is a square in $F^\times$ and $\text{char}(F) > 2$, then $G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) > 2$.***

**Theorem (Version 2.)**

*If $-1$ is a square in $F^\times$ and the Sylow 2-subgroups of $S$ are not Klein 4-groups, $G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) \neq 2$.***

**Almost an theorem**

$G \cong \text{PSL}_2(K)$ for $K$ a.c. of $\text{char}(K) \neq 2$. 
Sylow 2-subgroups of $S$ and $G$, $S \cong \text{PSL}_2(F)$

**Theorem (Deloro and Jaligot 2010)**

*Let $H$ be an odd type connected group of fRM with $\text{pr}_2(H) = 1$. Then exactly one of the following holds.*

1. $\text{Syl}_H = \text{Syl}^o_H \cong \mathbb{Z}_{2\infty}$.
2. $\text{Syl}_H = \text{Syl}^o_H \rtimes \langle \omega \rangle$ *for an involution $\omega$ which inverts $\text{Syl}^o_H$.*
3. $\text{Syl}_H = \text{Syl}^o_H \cdot \langle \omega \rangle$ *for an element $\omega$ of order 4 which inverts $\text{Syl}^o_H$.*

Any Sylow 2-subgroup $\text{Syl}_G$ of $G$ must be of type (2) as for otherwise $S$ satisfies the FO-expressible statement 'Every subgroup of order 4 is cyclic'.

Sylow 2-subgroups of $S$ are either conjugate dihedral groups or as $\text{Syl}_G$.

In particular, the finite simple groups in the ultraproduct $S$ has dihedral Sylow 2-subgroups.

**Theorem (Gorenstein and Walter 1962)**

*Let $H$ be a finite simple group with dihedral Sylow 2-subgroups. Then either $H \cong \text{PSL}_2(q)$, $q > 5$ or $H \cong A_7$.*
Sylow 2-subgroups of $S$ and $G$, $S \cong \text{PSL}_2(F)$

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Ulla Karhumäki
Structures of $S \cong \text{PSL}_2(F), \text{PGL}_2(F)$ and $P$

1. $\text{PSL}_2(F) \cong S \leq C_G(\alpha) = P \leq G$, for $F$ pseudofinite of $\text{char}(F) \neq 2$.
2. $P \leq \text{PGL}_2(F) \rtimes \text{Aut}(F)$.

- Let $P = \prod_{i \in I} P_i / U$. Then $P/S \cong \prod_{i \in I} P_i / \text{soc}(P_i) / U = \prod_{i \in I} (P_i/\text{PSL}_2(\mathbb{F}_q)) / U$. $P_i \cong \text{PGL}_2(q_i) \rtimes \text{Aut}(q_i) \Rightarrow P_i/\text{PSL}_2(q_i)$ has an abelian subgroup of index 2.

- $x \in P \Rightarrow x = sdt$, $s \in S$, $d \in \text{Diag}(S)$, $t \in \text{Aut}(F)$.

- $\text{Diag}(S) \times \text{Aut}(F)$ leave invariant $U \cong F^+$ and $T \cong (F^x)^2$. $\Rightarrow P/S \cong \text{N}_P(T)/\text{N}_S(T) \cong \text{N}_P(\text{Aut}(F))/U$.
How to identify \( G \)

**Aim:** \( \overline{S} = G \) is a split Zassenhaus group, acting on the set of left cosets of \( \overline{B} \) in \( G \), with a one-point stabiliser \( \overline{B} \) and a two-point stabiliser \( \overline{T} \). This implies that \( G \cong \text{PSL}_2(K) = \text{PGL}_2(K) \) for \( K \) a.c. and of \( \text{char}(K) \neq 2 \).[Delahan, Nesin 1995]
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For above, we need to observe things as

- $G = \overline{B} \uplus \overline{U}\omega_0\overline{B}$.
- $\overline{B}^g \cap \overline{U} = 1$ for all $g \in G \setminus \overline{B}$ (in particular, $N_G(\overline{U}) = N_G(\overline{B}) = \overline{B}$).
- $C_G^o(u) = C_G^o(\overline{U}) = \overline{U}$ for all $u \in \overline{U}^*$.
- $N_G(\overline{T}) = C_G(i) = \langle \overline{T}, \omega_0 \rangle$.
- $\overline{B} = \overline{U} \bowtie \overline{T}$ is a split Frobenius group.
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- $\overline{B} = \overline{U} \times \overline{T}$ is a split Frobenius group.

1. Prove that $[\ell:S]<\infty$ and use twist to get information up to connected components:

   For $u \in U^*$: $C^G(u)^{(a)} \leq C^G(u)^{(d)} = C^G(u)^{(u)} \leq C^G(u)^{(a)}$, $C^G(u) = \overline{U} \implies C^G(\omega_0 u) = \overline{U}^*$.

2. Prove that there is an involution $\iota \in \overline{T}$. Then, for $\overline{T}^\iota \iota$, $C_G(\overline{T}^\iota \iota) = \overline{T}$.
We know that $P/S \cong N_P(U)/B$ and $P/S \cong N_P(T)/N_S(T)$ is abelian-by-finite. To prove that $[P : S] < \infty$, we observe the following things.
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1. $(N_P^0(U))' \leq U$ and $(N_P^0(T))' \leq N_S(T)$.
2. $[N_P(T) : N_P(T) \cap N_P(U)] < \infty$.
3. $[C_P(T) : T] < \infty$. 
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3. $[C_P(T) : T] < \infty$.

**Sketch of proof of (1).**

Clearly $(N_P^0(U))' \leq B$.
As $N_P^0(U)$ is connected and solvable, $N_P^0(U)'$ is nilpotent\[Nesin 1990\].
As $B^0 \leq N_P^0(U)$ we have $N_P^0(U)' \leq F(B^0)$.
It can be proven that $F(B^0) = \overline{U}$ which gives us $(N_P^0(U))' \leq U$. 
$i \in \overline{T}^\circ$

With assumptions in Theorem (Version 2):

- For the unique involution $i \in T$, we have

$$C_G^\circ(i) = C_{C_G(i)}(\alpha)^\circ = C_{C_G(i)}(\alpha) = C_{C_G^\circ(\alpha)}(i)^\circ = C_S(i)^\circ = \langle T, \omega_0 \rangle^\circ = \overline{T}^\circ.$$
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- Since \( i \in T \), we know that a Sylow 2-subgroup \( Syl_S \) of \( S \) is in \( N_S(T) \). Let \( Syl_G \) be a Sylow 2-subgroup of \( G \) containing the Klein 4-group \( \langle i \rangle \times \langle \omega_0 \rangle \). As \( Syl_S \) is not a Klein 4-group, \( \omega_0 \) inverts \( Syl_G^\circ \). So \( i \in Syl_G^\circ \).
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Without extra assumptions:

- Enough to prove that $C_G^\circ(t) = \overline{T}^\circ$ for all $t \in \overline{T}^\circ$: Then $\overline{T}$ is generous in $G$ and so there is $1 \neq x \in \overline{T} \cap C_G(H)$ for some maximal decent torus of $H$ of $G$. So $\mathbb{Z}_{2^{\infty}} \leq H \leq C_G^\circ(x) = \overline{T}^\circ$. 
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Without extra assumptions:

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Idea for doing above: Prove that $\bigcup_{t \in \overline{T}^*} C_G^\circ(t) \cup \omega_0^\circ$ is abelian by considering its intersection with the maximal subgroups of $S$. 