LOCATION OF NODAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH GRADIENT DEPENDENCE

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Abstract. Existence and regularity results for quasilinear elliptic equations driven by \((p,q)\)-Laplacian and with gradient dependence are presented. A location principle for nodal (i.e., sign-changing) solutions is obtained by means of constant-sign solutions whose existence is also derived. Criteria for the existence of extremal solutions are finally established.

1. Introduction. We consider the following quasilinear elliptic problem with gradient dependence

\[
\begin{aligned}
-\Delta_p u - \mu \Delta_q u &= f(x, u, \nabla u) \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

(1)

on a bounded domain \(\Omega\) in \(\mathbb{R}^N\) \((N \geq 2)\) with \(C^2\)-boundary \(\partial \Omega\), where \(\Delta_p\) and \(\Delta_q\) stand for the \(p\)-Laplacian and \(q\)-Laplacian on \(W^{1,p}_0(\Omega)\) and \(W^{1,q}_0(\Omega)\), respectively, with \(1 < q < p < +\infty\), and a constant \(\mu \geq 0\). If \(\mu = 0\), the equation in (1) is driven by the \(p\)-Laplacian, whereas for \(\mu = 1\) the leading operator is the \((p,q)\)-Laplacian \(\Delta_p + \Delta_q\), which is a nonhomogeneous operator. We suppose the \(C^2\) smoothness for the boundary \(\partial \Omega\) in order to make use of appropriate regularity results that will be needed later on. For the sake of brevity and in order to get more insight on the main ideas, we suppose that \(N > p\). The case \(N \leq p\) is actually simpler and consequently is omitted.

The nonlinearity \(f\) in (1) is a Carathéodory function \(f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\), that is, \(f(\cdot, s, \xi)\) is measurable for every \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\), and \(f(x, \cdot, \cdot)\) is continuous for a.e. \(x \in \Omega\), satisfying the growth condition

2010 Mathematics Subject Classification. Primary: 35J62; Secondary: 35J92.

Key words and phrases. Quasilinear elliptic equations, \((p,q)\)-Laplacian, gradient dependence, nodal solutions, constant-sign solutions, location, sub-supersolutions, extremal solutions.

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H(f)_1 there exist constants a_1, a_2 \geq 0, 0 \leq \alpha < p^*-1, 0 \leq \beta < \frac{(p^*-1)p}{p-1}, 1 \leq \gamma < p^*,
and a function \sigma \in L^{\gamma} (\Omega), \sigma \geq 0, such that

|f(x,s,\xi)| \leq \sigma(x) + a_1|s|^\alpha + a_2|\xi|^\beta \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \xi \in \mathbb{R}^N.\n
Here we denote \( p^* := \frac{Np}{N-p} \), which is the Sobolev critical exponent. Furthermore for every \( r \geq 1 \) we set \( r^* := \frac{r}{r-1} \), that is the conjugate exponent of \( r \).

For a later use, we pose an additional, this time unilateral condition:
H(f)_2 there exist constants b_1, b_2 \geq 0 such that \( b_1\lambda_{1,p}^{-1} + b_2 < 1 \), and a function \( \tau \in L^1(\Omega) \) such that

\[ f(x,s,\xi)s \leq \tau(x) + b_1|s|^p + b_2|\xi|^p \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \xi \in \mathbb{R}^N. \]

The notation \( \lambda_{1,p} \) stands for the first eigenvalue of \(-\Delta_p \) on \( W^{1,p}_0(\Omega) \), which is expressed by

\[ \lambda_{1,p} = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}, \tag{2} \]

where \( \| \cdot \|_p \) denotes the usual \( L^p \)-norm.

We point out that condition H(f)_2 is independent of condition H(f)_1. For instance, H(f)_2 is fulfilled if the sign requirement \( f(x,s,\xi)s \leq 0 \) holds, without any growth restriction for \( f(x,s,\xi) \).

A solution \( u \in W^{1,p}_0(\Omega) \) of problem (1) is understood in the weak sense meaning that

\[ \int_{\Omega} (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2})(\nabla u, \nabla h)_{\mathbb{R}^N} dx = \int_{\Omega} f(x,u,\nabla u)h dx \tag{3} \]

for all \( h \in W^{1,p}_0(\Omega) \). Hereafter, \( (\cdot,\cdot)_{\mathbb{R}^N} \) stands for the standard Euclidean scalar product in \( \mathbb{R}^N \). The definition makes sense in view of condition H(f)_1.

We emphasize that due to the presence of the gradient \( \nabla u \) in the right-hand side of the equation, the variational methods are not applicable to problem (1). This difficulty is overcome by using the theory of pseudomonotone operators. In further results we also employ the method of sub-supersolutions for quasilinear elliptic equations combined with comparison arguments.

In the present paper we first note that, under assumptions H(f)_1 and H(f)_2, a general existence result and a priori estimates for weak solutions to problem (1) hold. Then we turn to study the regularity up to the boundary for the solutions to problem (1). In this respect we need to strengthen the conditions H(f)_1 and H(f)_2 (see Lemma 2.1) as follows:

H(f) \: f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ is a Carathéodory function for which there exist constants } c_1 \geq 0, c_2 \geq 0, r > \max\{ \frac{p^*}{p-1}, N \}, \text{ and a function } \omega \in L^{\infty}(\Omega), \omega \geq 0, \text{ such that }

\[ |f(x,s,\xi)| \leq \omega(x) + c_1|s|^{r^*} + c_2|\xi|^\tilde{p} \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \xi \in \mathbb{R}^N. \]

Hypothesis H(f) enables us to apply recent estimates due to Cianchi and Maz’ya [4] ensuring the global boundedness of the gradient of any solution \( u \) to (1). Owing to the homogeneous boundary condition, it follows that \( u \in L^{\infty}(\Omega) \) and \( \|u\|_{\infty} \) is bounded above by a constant independent of \( u \). This is actually a substitute for the Moser iteration technique (see, e.g., [8, Theorem C]) and has to be regarded in conjunction with Lieberman’s estimates [6, 7] for the regularity up to the boundary. In particular, every solution of problem (1) belongs to the space \( C^{1}(\overline{\Omega}) = \{u \in C^{1}(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \} \). Notice that the strong maximum principle (see [11], [10]) is
closely related to the positive cone $\mathcal{C}^1_0(\Omega)_+ = \{ u \in \mathcal{C}^1_0(\Omega) : u(x) \geq 0 \text{ for all } x \in \Omega \}$ of $\mathcal{C}^1_0(\Omega)$, which has a nonempty interior given by

$$\text{int} (\mathcal{C}^1_0(\Omega)_+) = \{ u \in \mathcal{C}^1_0(\Omega) : u > 0 \text{ on } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial \Omega \},$$

where $\nu(x)$ stands for the unit outward normal to $\partial \Omega$.

Our first main objective is the location of nodal (that is, sign-changing) solutions for problem (1). Specifically, we prove that if assumptions $H(f)_1$, $H(f)_2$ and $f(x,0,0) = 0$ for a.e. $x \in \Omega$ are fulfilled, then every nodal (i.e., sign-changing) solution of problem (1) should be between two opposite constant-sign solutions. In particular, this provides the powerful fact that the existence of a nodal solution implies under the stated hypotheses that two opposite constant-sign solutions must exist. Moreover, this phenomenon occurs even unilaterally: if we only have $f(x,0,0) \geq 0$ (resp. $f(x,0,0) \leq 0$) for a.e. $x \in \Omega$, then every nodal solution to problem (1) is bounded above by a positive solution (resp. bounded below by a negative solution).

Our second main objective is to establish the existence of extremal (or barrier) solutions to problem (1), that is a biggest solution and a smallest solution. Taking into account what was said before about the location of nodal solutions, the biggest nontrivial solution is positive and the smallest nontrivial solution is negative, and they belong to $\mathcal{C}^1_0(\Omega)$. Our approach for obtaining the extremal solutions relies on the method of sub-supersolutions related to problem (1). We investigate separately the cases $\mu = 0$ and $\mu > 0$ because there are relevant differences in the construction of subsolutions and supersolutions corresponding to the two cases. It is worth mentioning that the problem with $\mu = 0$:

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases} \quad (4)$$

is a limiting case of problem (1) as $\mu \to 0^+$ (see [1]). When $\mu > 0$ we have to impose an additional condition describing the behavior of $f(x,s,0)$ as $|s| \to +\infty$. Related results on extremal solutions for variational inequalities, but without gradient dependence, can be found in [2].

The rest of the paper is organized as follows. Section 2 is devoted to preliminary results regarding the existence and regularity of solutions to problem (1). Section 3 sets forth our principle for locating nodal solutions. Section 4 discusses the presence of a positive solution and a negative solution accompanying every nodal solution. Section 5 presents our results on extremal solutions when $\mu = 0$, whereas Section 6 focuses on the case $\mu > 0$ in problem (1).

2. Preliminary results. In the sequel, the Banach spaces $W^{1,p}(\Omega)$ and $L^p(\Omega)$ are equipped with the usual norms $\| \cdot \|_{1,p}$ and $\| \cdot \|_p$, respectively, whereas the space $W^{1,p}_0(\Omega)$ is endowed with the norm

$$\|u\| = \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$  

The continuity of the embedding of $W^{1,p}_0(\Omega)$ in $L^r(\Omega)$ for $1 \leq r \leq p^*$ guarantees the existence of a constant $c_r > 0$ such that

$$\|u\|_r \leq c_r \|u\| \text{ for all } u \in W^{1,p}_0(\Omega). \quad (5)$$
In view of assumption $H(f)_1$, we can consider the Nemytskii operator $N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (where $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$) defined by
\[ N_f(u) = f(\cdot, u, \nabla u) \text{ for all } u \in W_0^{1,p}(\Omega). \] (6)

Then problem (1) can be equivalently written as
\[ -\Delta_p u - \mu \Delta_q u = N_f(u) \text{ in } W^{-1,p'}(\Omega). \]

First, we point out that condition $H(f)$ is stronger than both $H(f)_1$ and $H(f)_2$.

**Lemma 2.1.** Hypothesis $H(f)$ implies $H(f)_1$ and $H(f)_2$.

**Proof.** By the choice of $r$ in $H(f)$, we have that $\frac{p^*}{r} < p^* - 1$ and $\frac{p}{r} < \frac{(p^* - 1)p}{p^*}$. Consequently, $H(f)$ yields $H(f)_1$ by taking $\sigma(x) = \omega(x)$, $\alpha = \frac{p^*}{r}$ and $\beta = \frac{p}{r}$.

From $H(f)$ and Young’s inequality we obtain that
\[
|f(x, s, \xi)s| \leq |\omega(x)||s| + c_1|s|^\frac{p^*}{p} + c_2|\xi|^\frac{p}{p^*} \leq |\omega(x)||s| + c_1|s|^\frac{p^*}{p} + c(\varepsilon)|s|^{\frac{p}{p^*} + \varepsilon} + \varepsilon|\xi|^p
\]
for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, with an arbitrary $\varepsilon > 0$ and a corresponding constant $c(\varepsilon)$. We note that the choice of $r$ in $H(f)$ results in $\frac{p^*}{r} + 1 < p$ and $\frac{p}{r} - 1 < p$. A further application of Young’s inequality yields
\[
|f(x, s, \xi)s| \leq c(\delta)(|\omega(x)|^{p^*} + 1) + \delta|s|^p + \varepsilon|\xi|^p
\]
for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, with $\delta > 0$ arbitrary and some constant $c(\delta)$. Therefore, choosing $\delta, \varepsilon$ sufficiently small, we can infer that $H(f)_2$ holds true, which completes the proof. \hfill \Box

For easy reference we recall the notions of supersolution and of subsolution for problem (1). A function $u \in W^{1,p}(\Omega)$ is called a supersolution of problem (1) if it satisfies $u \geq 0$ on $\partial \Omega$ and
\[
\int_\Omega (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2})(\nabla u, \nabla h)_{\mathbb{R}^N} \, dx \geq \int_\Omega f(x, u, \nabla u)h \, dx
\]
for all $h \in W_0^{1,p}(\Omega)$, $h \geq 0$ a.e. in $\Omega$. A function $u \in W^{1,p}(\Omega)$ is called a subsolution of problem (1) if it satisfies $u \leq 0$ on $\partial \Omega$ and
\[
\int_\Omega (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2})(\nabla u, \nabla h)_{\mathbb{R}^N} \, dx \leq \int_\Omega f(x, u, \nabla u)h \, dx
\]
for all $h \in W_0^{1,p}(\Omega)$, $h \geq 0$ a.e. in $\Omega$. Owing to the growth condition $H(f)_1$, the notions of supersolution and subsolution of problem (1) are correctly defined. The following useful property of sub-supersolutions is shown in [3, Theorem 3.20].

**Lemma 2.2.** (a) If $u_1, u_2$ are subsolutions of problem (1), then so is $\max\{u_1, u_2\}$.
(b) If $u_1, u_2$ are supersolutions of problem (1), then so is $\min\{u_1, u_2\}$.

Later on we shall need the $L^p$-normalized, positive eigenfunction $\phi_{1,p}$ of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda_{1,p}$ (see (2)), so that
\[
-\Delta_p \phi_{1,p} = \lambda_{1,p} \phi_{1,p}^{p-1} \text{ in } \Omega, \ \phi_{1,p} = 0 \text{ on } \partial \Omega, \ \int_\Omega \phi_{1,p}^p \, dx = 1,
\]
and the $L^q$-normalized positive eigenfunction $\phi_{1,q}$ of $-\Delta_q$ on $W^{1,q}_0(\Omega)$ corresponding to $\lambda_{1,q}$, that is

$$-\Delta_q \phi_{1,q} = \lambda_{1,q} \phi_{1,q}^{-1} \text{ in } \Omega, \quad \phi_{1,q} = 0 \text{ on } \partial \Omega, \quad \text{with } \int_{\Omega} \phi_{1,q}^q \, dx = 1.$$ 

By the nonlinear regularity theory and strong maximum principle, we know that $\phi_{1,p}, \phi_{1,q} \in \text{int} (C^1_0(\overline{\Omega}))$ and, for positive constants $c_1$ and $c_2$, there holds

$$c_1 \phi_{1,p}(x) \leq \phi_{1,q}(x) \leq c_2 \phi_{1,p}(x) \quad \text{for all } x \in \Omega.$$

We also recall that an operator $A : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ is called pseudomonotone if for every sequence $(u_n)_{n \geq 1} \subset W^{1,p}_0(\Omega)$ such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}_0(\Omega)$ and

$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

it holds

$$\liminf_{n \to \infty} \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle \quad \text{for all } v \in W^{1,p}_0(\Omega).$$

Recall also that an operator $A : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ is coercive if for every sequence $(u_n)_{n \geq 1} \subset W^{1,p}_0(\Omega)$ such that $\|u_n\| \to +\infty$ as $n \to \infty$ we have

$$\lim_{n \to \infty} \frac{\langle A(u_n), u_n \rangle}{\|u_n\|} = +\infty.$$

We quote from [1, Theorem 1] the following existence result for problem (1).

**Lemma 2.3.** Assume that conditions $H(f)_1$ and $H(f)_2$ hold. Then problem (1) has at least a (weak) solution $u \in W^{1,p}_0(\Omega)$.

Next we set forth a result addressing the regularity of solutions and a priori estimates for problem (1).

**Lemma 2.4.** Assume that condition $H(f)$ holds. Then there are constants $R > 0$ and $\gamma \in (0, 1)$ such that every solution $u \in W^{1,p}_0(\Omega)$ of problem (1) belongs to $C^{1,\gamma}(\overline{\Omega})$ and satisfies the estimate

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})} \leq R.$$

**Proof.** The fact that the solution set of problem (1) is nonempty is ensured by Lemmas 2.1 and 2.3. Let $u \in W^{1,p}_0(\Omega)$ be any solution of (1). Inserting $h = u$ in (3) and using $H(f)_2$ and (2), we have

$$\int_{\Omega} (|\nabla u|^p + \mu |\nabla u|^q) \, dx = \int_{\Omega} f(x, u, \nabla u) u \, dx$$

$$\leq \int_{\Omega} \left( \tau(x) + b_1 |u|^p + b_2 |\nabla u|^p \right) \, dx$$

$$\leq \int_{\Omega} \tau(x) \, dx + \left( b_1 \lambda_{1,p}^{-1} + b_2 \right) \int_{\Omega} |\nabla u|^p \, dx.$$

Since $\mu \geq 0$ and $b_1 \lambda_{1,p}^{-1} + b_2 < 1$, there exists a constant $c > 0$ independent of the solution $u$ such that

$$\|u\| \leq c. \quad (7)$$

Then by (5) and (7) we infer that $\|u\|_{p'} \leq c_{p'} c$ and $\|\nabla u\|^p \leq c$ for every solution $u$ of problem (1). Using these estimates through the growth condition $H(f)$, it follows that

$$\|f(\cdot, u, \nabla u)\|_r \leq c_0, \quad (8)$$

where $c_0$ is a constant depending only on $\Omega$ and the data $f$. 

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Theorem 3.1. Assume that conditions location of nodal solutions for problem (1).

3. Location principle for nodal solutions.

In this section we focus on the location of nodal solutions for problem (1).

Theorem 3.1. Assume that conditions $H(f)_1$ and $H(f)_2$ are satisfied.

(i) If $f(x,0,0) \geq 0$ for a.e. $x \in \Omega$, then for every nodal solution $u_0$ of problem (1) there exists a nontrivial solution $u_+$ of (1) such that $u_0 \leq u_+$ and $u_+ \geq 0$ on $\Omega$.

(ii) If $f(x,0,0) \leq 0$ for a.e. $x \in \Omega$, then for every nodal solution $u_0$ of problem (1) there exists a nontrivial solution $u_-$ of (1) such that $u_0 \geq u_-$ and $u_- \leq 0$ on $\Omega$.

(iii) If $f(x,0,0) = 0$ for a.e. $x \in \Omega$, then for every nodal solution $u_0$ of problem (1) there exist two other nontrivial solutions $u_+$ and $u_-$ of (1) such that $u_- \leq u_0 \leq u_+$, $u_+ \geq 0$ and $u_- \leq 0$ on $\Omega$.

Proof. (i) Let $u_0$ be a nodal solution of problem (1). The assumption $f(x,0,0) \geq 0$ for a.e. $x \in \Omega$ ensures that 0 is a subsolution of problem (1). By Lemma 2.2 (a) we infer that

$$u := \max\{0, u_0\}$$

is a subsolution of problem (1). (11)

Let $T : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)$ be the truncation operator defined by

$$(Tu)(x) = \begin{cases} u(x) & \text{if } u(x) \geq u(x) \\ u(x) & \text{if } u(x) < u(x) \end{cases}$$

(12)

for a.e. $x \in \Omega$. It is clear that the operator $T : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)$ is bounded and continuous. We also consider the following cut-off function by setting for a.e. $x \in \Omega$, every $s \in \mathbb{R}$:

$$b(x,s) = \begin{cases} 0 & \text{if } s \geq u(x) \\ -(u(x) - s)^{p-1} & \text{if } s < u(x). \end{cases}$$

Let $B : W^{1,p}_0(\Omega) \to W^{-1,q'}(\Omega)$ denote the corresponding Nemytskii operator, that is $Bu(x) = b(x,u(x))$, which is completely continuous in view of the compact embedding of $W^{1,p}_0(\Omega)$ into $L^p(\Omega)$. 


Now we state the auxiliary problem
\[
\begin{cases}
  -\Delta_p u - \mu \Delta q u + Bu = f(x, Tu, \nabla (Tu)) \\
  u = 0
\end{cases}
\quad \text{in } \Omega.
\] (13)

Using the Nemitskii operator \( N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \) given by (6), problem (13) can be equivalently expressed as
\[
-\Delta_p u - \mu \Delta q u + Bu = N_f \circ T(u) \quad \text{in } W^{-1,p'}(\Omega).
\] (14)

By \( H(f)_1 \) it is clear that the operator \(-\Delta_p - \mu \Delta q + B - N_f \circ T : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)\) is bounded.

We claim that the operator \(-\Delta_p - \mu \Delta q + B - N_f \circ T\) is pseudomonotone on \( W_0^{1,p}(\Omega) \). In order to show this, let a sequence \((u_n)_{n \geq 1} \subset W_0^{1,p}(\Omega)\) be such that \( u_n \rightharpoonup u \) in \( W_0^{1,p}(\Omega) \) and
\[
\limsup_{n \to \infty} (-\Delta_p u_n - \mu \Delta q u_n + Bu_n - N_f(Tu_n), u_n - u) \leq 0.
\] (15)

By \( H(f)_1 \) we have that \( \frac{p}{p-\alpha} < p^* \). Invoking the boundedness of the operator \( T \) and Rellich-Kondrachov compact embedding theorem, it follows that the sequence \((Tu_n)^{\alpha}_{n \geq 1}\) is bounded in \( L^{\frac{p}{p-\alpha'}}(\Omega) = L^{\frac{p^*}{p^* - \alpha}}(\Omega) \) and that \( u_n \to u \) in \( L^{\frac{p^*}{p^* - \alpha}}(\Omega) \).

Therefore Hölder’s inequality implies
\[
\int_\Omega |Tu_n|^{\alpha} |u_n - u| \, dx \leq \|Tu_n\|_{p^*}^{\alpha} \|u_n - u\|_{\frac{p^*}{p^* - \alpha}} \to 0.
\] (16)

Also, by \( H(f)_1 \) we have that \( \frac{p}{p-\beta} < p^* \). Using again the boundedness of the operator \( T \) and Rellich-Kondrachov compact embedding theorem, we infer that the sequence \((\nabla (Tu_n))^{\beta}_{n \geq 1}\) is bounded in \( L^{\frac{p}{p-\beta'}}(\Omega) = L^{\frac{p^*}{p^* - \beta}}(\Omega) \) and that \( u_n \to u \) in \( L^{\frac{p^*}{p^* - \beta}}(\Omega) \).

Consequently, Hölder’s inequality yields
\[
\int_\Omega |\nabla (Tu_n)|^{\beta} |u_n - u| \, dx \leq \|\nabla (Tu_n)\|^{\beta}_{p} \|u_n - u\|_{\frac{p^*}{p^* - \beta}} \to 0.
\] (17)

Therefore, thanks to assumption \( H(f)_1 \) and (16), (17), we derive that
\[
\lim_{n \to \infty} \langle N_f(Tu_n), u_n - u \rangle = \lim_{n \to \infty} \int_\Omega f(x, Tu_n, \nabla (Tu_n))(u_n - u) \, dx = 0.
\] (18)

Combining (15), (18), and the complete continuity of \( B \), it turns out that
\[
\limsup_{n \to \infty} (-\Delta_p u_n - \mu \Delta q u_n, u_n - u) \leq 0.
\]

Using the \((S)_+\)-property of \(-\Delta_p - \mu \Delta q\) (see [10, Proposition 2.70]), we conclude that \( u_n \to u \) in \( W_0^{1,p}(\Omega) \). Thus it is true that
\[
\lim_{n \to \infty} \langle -\Delta_p u_n - \mu \Delta q u_n + Bu_n - N_f(Tu_n), u_n - v \rangle
\[
= \langle -\Delta_p u - \mu \Delta q u + Bu - N_f(Tu), u - v \rangle
\]
for all \( v \in W_0^{1,p}(\Omega) \), which proves that the operator \(-\Delta_p - \mu \Delta q + B - N_f \circ T\) is pseudomonotone.

Now we verify that the operator \(-\Delta_p - \mu \Delta q + B - N_f \circ T\) is coercive. Let \((u_n)_{n \geq 1}\) be a sequence in \( W_0^{1,p}(\Omega) \) such that \( \|u_n\| \to +\infty \) as \( n \to \infty \). By \( H(f)_2 \), Hölder’s
inequality and the definition of $T$, we get the estimate
\[
\langle (-\Delta_p - \mu \Delta_q + B - N_f \circ T)(u_n), u_n \rangle \\
= \|u_n\|^p + \mu \|\nabla u_n\|^q - \int_{\{u_n < u\}} (u - u_n)^{p-1} u_n \, dx \\
- \int_{\{u_n < u\}} f(x, u, \nabla u) u_n \, dx - \int_{\{u < u_n\}} f(x, u_n, \nabla u_n) u_n \, dx \\
\geq \|u_n\|^p - \ell_1 \|u_n - u\|^p - \ell_2 \|u_n\|^p - \int_{\Omega} (\tau(x) + b_1 |u_n|^p + b_2 \nabla u_n|^p) \, dx \\
\geq (1 - b_1 \lambda_1^{-1} - b_2) \|u_n\|^p - \ell_1 c_p^{p-1} \|u_n - u\|^p - \ell_2 c_p \|u_n\| - \|\tau\|_1,
\]
with positive constants $\ell_1, \ell_2$. In view of the condition $b_1 \lambda_1^{-1} + b_2 < 1$ supposed in $H(f)$, we obtain that the operator $-\Delta_p - \mu \Delta_q + B - N_f \circ T$ is coercive. According to the properties above, we are in a position to apply the main theorem for pseudomonotone operators (see [3, Theorem 2.99]) to the operator $-\Delta_p - \mu \Delta_q + B - N_f \circ T$. It entails the existence of $u_+ \in W_0^{1,p}(\Omega)$ solving (14). Therefore $u_+$ is a solution of the auxiliary problem (13).

Let us show the inequality
\[
u_+(x) \geq u(x) \text{ a.e. in } \Omega. \tag{19}
\]
Indeed, acting as test function on (14) (or (13)) with $(u_+ - u)^- = \max\{0, -(u_+ - u)\} \in W_0^{1,p}(\Omega)$ and using (12) and (11) lead to
\[
(-\Delta_p u_+ - \mu \Delta_q u_+, (u_+ - u)^-) = \int_{\Omega} (u - u_+)^{p-1} (u_+ - u)^- \, dx \\
+ \int_{\Omega} f(x, Tu_+, \nabla(Tu_+))(u_+ - u)^- \, dx \\
= \int_{\{u_+ < u\}} (u - u_+)^p \, dx + \int_{\Omega} f(x, u_+, \nabla u)(u_+ - u)^- \, dx \\
\geq \int_{\{u_+ < u\}} (u - u_+)^p \, dx + \langle (-\Delta_p u_+ - \mu \Delta_q u_+, (u_+ - u)^-), \nabla u \rangle \\
+ \mu \int_{\{u_+ < u\}} (|\nabla u_+|^p - |\nabla u|^p) \nabla u_+ \cdot \nabla u \, dx \\
\geq \int_{\{u_+ < u\}} (|\nabla u_+|^p - |\nabla u|^p) \nabla u_+ \cdot \nabla u \, dx \\
+ \int_{\{u_+ < u\}} (u - u_+)^p \, dx \leq 0.
\]
By means of the (strict) monotonicity of the map $\xi \mapsto (|\xi|^p + \mu |\xi|^q)\xi$ on $\mathbb{R}^N$, we infer that the set $\{u_+ < u\} = \{x \in \Omega : u_+(x) < u(x)\}$ has zero Lebesgue measure, so (19) holds true.

It is seen that (19) renders $Tu_+ = u_+$. This, in conjunction with the fact that $u_+$ is a solution of problem (13), guarantees that $u_+$ is a solution of problem (1), too.

Since the solution $u_0$ of (1) is nodal, its positive part $u_0^+ = \max\{0, u_0\}$ is strictly positive on a subset of $\Omega$ of positive measure, so from (19) and (11) we deduce that $u_+ \geq 0$ and $u_+ \neq 0$. 

Let $u_0$ be a nodal solution of problem (1). The inequality $f(x,0,0) \leq 0$ for a.e. $x \in \Omega$ expresses that 0 is a supersolution of problem (1). Then Lemma 2.2 (b) provides that
\[
\underline{\pi} := \min\{0, u_0\} \text{ is a supersolution of problem (1).}
\] (20)
We introduce the truncation operator $T : W^{1,p}_0(\Omega) \rightarrow W^{1,p}_0(\Omega)$ by
\[
(Tu)(x) = \begin{cases} u(x) & \text{if } u(x) \leq \underline{\pi}(x) \\ \underline{\pi}(x) & \text{if } u(x) > \underline{\pi}(x) \end{cases}
\] (21)
for a.e. $x \in \Omega$. Furthermore, we also consider the next cut-off function assigning for a.e. $x \in \Omega$, every $s \in \mathbb{R}$:
\[
b(x,s) = \begin{cases} 0 & \text{if } s \leq \underline{\pi}(x) \\ (s - \underline{\pi}(x))^{p-1} & \text{if } s > \underline{\pi}(x). \end{cases}
\]
As before denote by $B : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ the associated Nemytskii operator, namely $Bu(x) = b(x,u(x))$. Corresponding to these $T$ and $B$ we state the new auxiliary problem (13).

As in part (i) we prove that the operator $-\Delta_p - \mu \Delta_q + B - N_f \circ T : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is bounded, pseudomonotone, and coercive. Through the main theorem for pseudomonotone operators (see [3, Theorem 2.99]), this results in the existence of $u_- \in W^{1,p}_0(\Omega)$ satisfying
\[
-\Delta_p u_- - \mu \Delta_q u_- + Bu_- = N_f \circ T(u_-).
\] (22)
We claim that
\[
u_-(x) \leq \underline{\pi}(x) \text{ a.e. in } \Omega.
\] (23)
In order to prove the claim we act on (22) with the test function $(u_- - \underline{\pi})^+ = \max\{0, u_- - \underline{\pi}\} \in W^{1,p}_0(\Omega)$ and make use of (20) and (21). We are led to
\[
(-\Delta_p u_- - \mu \Delta_q u_-,(u_- - \underline{\pi})^+) + \int_{\Omega} (u_- - \underline{\pi})^{p-1}(u_- - \underline{\pi})^+\,dx = \int_{\Omega} f(x,\nabla u_-,\nabla(Tu_-))(u_- - \underline{\pi})^+\,dx \\
\leq (-\Delta_p \underline{\pi} - \mu \Delta_q \underline{\pi},(u_- - \underline{\pi})^+).
\]
Following the reasoning in part (a) we can show that the claim in (23) is valid.

From (22), (21) and (23) we infer that $u_- \in W^{1,p}_0(\Omega)$ is a solution of problem (1). Recalling that the solution $u_0$ of (1) is nodal, for its negative part $u^-_0 = \max\{0,-u_0\}$ we have that $-u^-_0$ is strictly negative on a subset of $\Omega$ of positive measure. Then (23) and (20) enable us to conclude that $u_- \leq 0$ and $u_- \not\equiv 0$.

(iii) If $f(x,0,0) = 0$ for a.e. $x \in \Omega$, then we can apply simultaneously assertions (i) and (ii), which gives rise to two nontrivial opposite constant-sign solutions $u_+$ and $u_-$ of problem (1) with the properties required in the statement. \qed

4. Positive and negative solutions generated by nodal solutions. The goal of this section is to provide a criterion for having in Theorem 3.1 the enclosure of the nodal solution $u_0$ with a positive solution $u_+$ and a negative solution $u_-$. This is achieved by requiring certain behavior near $(0,0)$ for $f(x,\cdot,\cdot)$ and through the strong maximum principle adapted to the presence of the gradient of solution in the right-hand side of equation (1). The approach is closely related to the regularity of the solutions as discussed in Lemma 2.4.
Theorem 4.1. Assume that condition $H(f)$ is satisfied.

(i) If there exist constants $\delta \in (0, 1)$, $k_0 > 0$ and $k_1 > 0$ such that
\[
f(x, s, \xi) \geq -k_0 s^{p-1} - k_1 (|\xi|^{p-1} + \mu |\xi|^{q-1})
\] (24)
for a.e. $x \in \Omega$, all $0 \leq s \leq \delta$, $|\xi| \leq \delta$, then for every nodal solution $u_0$ of problem (1) there exists a (positive) solution $u_+ \in \text{int}(C_0^1(\bar{\Omega}^*_+))$ of (1) such that $u_0 \leq u_+ \text{ on } \Omega$.

(ii) If there exist constants $\delta \in (0, 1)$, $k_0 > 0$ and $k_1 > 0$ such that
\[
f(x, s, \xi) \leq k_0 |s|^{p-1} + k_1 (|\xi|^{p-1} + \mu |\xi|^{q-1})
\] (25)
for a.e. $x \in \Omega$, all $-\delta \leq s \leq 0$, $|\xi| \leq \delta$, then for every nodal solution $u_0$ of problem (1) there exists a (negative) solution $u_- \in -\text{int}(C_0^1(\bar{\Omega}^*_+))$ of problem (1) such that $u_0 \geq u_- \text{ on } \Omega$.

(iii) If there exist constants $\delta \in (0, 1)$, $k_0 > 0$ and $k_1 > 0$ such that
\[
|f(x, s, \xi)| \leq k_0 |s|^{p-1} + k_1 (|\xi|^{p-1} + \mu |\xi|^{q-1})
\] (26)
for a.e. $x \in \Omega$, all $-\delta \leq s \leq 0$, $|\xi| \leq \delta$, then for every nodal solution $u_0$ of problem (1) there exist a (positive) solution $u_+ \in \text{int}(C_0^1(\bar{\Omega}^*_+))$ and a (negative) solution $u_- \in -\text{int}(C_0^1(\bar{\Omega}^*_+))$ of problem (1) such that holds $u_- \leq u_0 \leq u_+ \text{ on } \Omega$.

Proof. (i) Owing to Lemma 2.1 we know that the present hypotheses are stronger than those in Theorem 3.1. Since assumption (24) entails that $f(x, 0, 0) \geq 0$ for a.e. $x \in \Omega$, by Theorem 3.1 (i) there exists a nontrivial nonnegative solution $u_0$ of (1) with $u_0 \leq u_+ \text{ on } \Omega$. It suffices to prove that the solution $u_+$ fulfills the properties required in the statement.

Due to assumption $H(f)$ we have at our disposal the regularity result in Lemma 2.4, which ensures that $u_+ \in C_0^1(\bar{\Omega})$, so in particular $0 \leq u_+(x) \leq M$ and $|\nabla u_+(x)| \leq M$ for all $x \in \Omega$, with a constant $M > 1$.

Now we rely on the strong maximum principle in [11, Theorem 5.4.1] and on Hopf’s boundary point lemma in [11, Theorem 5.5.1] (see also [10, Section 8.1]) applied to the function $u_+ \in C_0^1(\bar{\Omega})$ that satisfies
\[-\Delta_p u_+ - \mu \Delta q u_+ = f(x, u_+, \nabla u_+) \text{ in } \Omega.
\]
In order to meet the needed requirements let us denote $A(s) := s^{p-2} + \mu s^{q-2}$ for $s > 0$. We find that
\[c := \lim_{t \to 0^+} \frac{t A'(t)}{A(t)} > -1,
\] since $c = q - 2$ if $\mu > 0$ and $c = p - 2$ if $\mu = 0$.

From assumption $H(f)$ and the inequality $\frac{p}{r} < p - 1$ we derive that there are constants $k_2 = k_2(\delta, M) > 0$ and $k_3 = k_3(\delta, M) > 0$ with
\[f(x, s, \xi) \geq -k_2 s^{p-1} \text{ for a.e. } x \in \Omega, \text{ all } s \geq \delta, |\xi| \leq M,
\]
\[f(x, s, \xi) \geq -k_3 (|\xi|^{p-1} + \mu |\xi|^{q-1}) \text{ for a.e. } x \in \Omega, \text{ all } 0 \leq s \leq \delta, \delta \leq |\xi| \leq M.
\]
Combining with (24) we arrive at
\[f(x, s, \xi) \geq -C_1 \Phi(|\xi|) - f(s) \text{ for a.e. } x \in \Omega, \text{ all } s \geq 0, |\xi| \leq M,
\] where $\Phi(s) := s A(s) = s^{p-1} + \mu s^{q-1}$ and $f(s) = C_2 s^{p-1}$, with constants $C_1 = C_1(\delta, M) > 0$ and $C_2 = C_2(\delta, M) > 0$. Taking into account that $M > 1$, conditions (B1) and (F2) in [11] are fulfilled.
We note that
\[ H(s) := s\Phi(s) - \int_0^s \Phi(t) \, dt = \frac{p-1}{p} s^p + \mu \frac{q-1}{q} s^q \]
and
\[ F(s) := \int_0^s f(t) \, dt = C_2 \frac{p}{s^p} \]
for all \( s \geq 0 \). The function \( H \) is increasing on \([0, +\infty)\) and we have
\[ F(s) \leq H(C_2^\frac{p}{p} (p-1)^{-\frac{q}{p}} s) \]
hence
\[ H^{-1}(F(s)) \leq C_2^\frac{1}{p} (p-1)^{-\frac{q}{p}} s \quad \text{for all } s > 0. \]
Thus the condition
\[ \int_{s_0}^{\infty} \frac{ds}{H^{-1}(F(s))} = +\infty \]
required in (1.1.5) of [11] holds true. All the hypotheses of [11, Theorem 5.4.1] and [11, Theorem 5.5.1] are satisfied (see also [11, Remark 3, p. 117]), thereby \( u_+ \in \text{int} \left( C_0^1(\Omega)_+ \right) \).

(ii) Observe that hypothesis (25) implies that \( f(x,0,0) \leq 0 \) for a.e. \( x \in \Omega \). Hence, by means of Lemma 2.1, we can refer to Theorem 3.1 (ii) for obtaining a nontrivial nonpositive solution \( u_- \) of (1) such that \( u_0 \geq u_- \) on \( \Omega \). Consequently, it is sufficient to show that the solution \( u_- \) has the properties required in the statement.

First, through assumption \( H(f) \) and Lemma 2.4, we know that \( u_- \in C_0^1(\Omega) \). Then we utilize the strong maximum principle in [11, Theorem 5.4.1] and Hopf’s boundary point lemma in [11, Theorem 5.5.1] applied to the nonnegative function \(-u_- \in C_0^1(\Omega)\) verifying
\[ -\Delta_p (-u_-) - \mu \Delta_q (-u_-) = \hat{f}(x,-(u_-), \nabla(-u_-)) \quad \text{in } \Omega, \]
where \( \hat{f}(x,s,\xi) := -f(x,-s,-\xi) \). From now on the proof proceeds in a manner similar to part (i) with \(-u_- \) in place of \( u_+ \).

(iii) We note that assumption (26) implies both (24) and (25). Then part (iii) can be readily obtained from the assertions (i) and (ii), which completes the proof. \( \square \)

**Remark 4.2.** Part (iii) of Theorem 4.1 holds more generally whenever both unilateral conditions (24) and (25) are satisfied, which is weaker than assuming the bilateral condition (26). Note that (24) and (25) are fulfilled by a function \( f \) that satisfies the local sign condition
\[ f(x,s,\xi)s \geq 0 \quad \text{for a.e. } x \in \Omega, \text{ all } -\delta \leq s \leq \delta, |\xi| \leq \delta, \]
for some \( \delta > 0 \).

**Remark 4.3.** Denoting by \( \lambda_{2,p} \) the second eigenvalue of \(-\Delta_p \) on \( W_0^{1,p}(\Omega) \), it is shown in [9, Theorem 2 (b)] the existence of a nodal solution \( u_0 \in C_0^1(\Omega) \) between a negative solution \( u_- \in \text{int} \left( C_0^1(\Omega)_+ \right) \) and a positive solution \( u_+ \in \text{int} \left( C_0^1(\Omega)_+ \right) \) of problem (1) provided \( f \) does not depend on \( \xi \in \mathbb{R}^N \) and there exist \( \eta_2, \theta \in L^\infty(\Omega) \), with \( \eta_2(x) \geq \lambda_{2,p} \) for a.e. \( x \in \Omega, \eta_2 \neq \lambda_{2,p} \) on a set of positive measure, such that
\[ \eta_2(x) \leq \liminf_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} \leq \theta(x) \quad \text{uniformly for a.e. } x \in \Omega. \]

Theorem 4.1 complements this result offering a verifiable criterion for the converse property, namely that every nodal solution of problem (1) is located within the
ordered interval determined by a negative solution and a positive solution of problem (1).

Remark 4.4. The result in Theorem 4.1 is sharp. For example, every nontrivial solution of the problem \(-\Delta_p u = \lambda |u|^{p-2}u\) in \(\Omega\), \(u = 0\) on \(\partial \Omega\), with \(\lambda > \lambda_{1,p}\), is nodal. Notice that in this case hypothesis \(H(f)\) is not fulfilled.

5. Extremal solutions for problem (4). The aim of this section is to prove the existence of extremal solutions, i.e., the biggest and smallest solutions for problem (4). This will permit to bound the whole set of nodal solutions by a positive solution and a negative solution, thus complementing the location principle in Theorem 3.1 (4). This will permit to bound the whole set of nodal solutions by a positive solution existence of extremal solutions, i.e., the biggest and smallest solutions for problem Extremal solutions for problem (4).

The variational characterizations of \(\lambda_{1,p,\delta}\) and \(\lambda_{1,p}\) (see (2)) give

\[
\lambda_{1,p,\delta} = \inf_{u \in W^{1,p}_0(\Omega_\delta) \setminus \{0\}} \frac{\|\Delta_p u\|_{L^p(\Omega_\delta)}}{\|u\|_{W^{1,p}_0(\Omega_\delta)}} \leq \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\|\Delta_p u\|_{L^p(\Omega)}}{\|u\|_{W^{1,p}_0(\Omega)}} = \lambda_{1,p},
\]

whence

\[
\limsup_{\delta \downarrow 0} \lambda_{1,p,\delta} \leq \lambda_{1,p}. \tag{28}
\]

Take a sequence \(\delta_n \downarrow 0\) as \(n \to \infty\). From (27) and (28) we derive

\[
\int_{\Omega} |\nabla \phi_{1,p,\delta_n}|^p dx = \int_{\Omega_\delta} |\nabla \phi_{1,p,\delta_n}|^p dx = \lambda_{1,p,\delta_n} \leq M
\]

with a constant \(M > 0\), so up to a subsequence \(\phi_{1,p,\delta_n} \rightharpoonup u\) in \(W^{1,p}_0(\Omega_\delta_n)\) and \(\phi_{1,p,\delta_n} \to u\) in \(L^p(\Omega_\delta_n)\) as \(n \to \infty\) with some \(u \in W^{1,p}_0(\Omega_\delta_n)\).

We have that \(u = 0\) a.e. in \(\Omega_\delta_n \setminus \Omega\) because for any \(v \in C_c^\infty(\Omega_\delta_n \setminus \Omega)\) it holds

\[
\supp v \subset \Omega_\delta_n \setminus \Omega_\delta_n \text{ provided } n \text{ is sufficiently large, thus}
\]

\[
\int_{\Omega_\delta_n \setminus \Omega} u v dx = \lim_{n \to \infty} \int_{\Omega_\delta_n} \phi_{1,p,\delta_n} v dx = 0.
\]

Consequently, since \(u \in W^{1,p}_0(\Omega)\) and \(\int_\Omega u^p dx = 1\), we find that

\[
\lambda_{1,p} \leq \int_{\Omega} |\nabla u|^p dx \leq \liminf_{n \to \infty} \int_{\Omega_\delta_n} |\nabla \phi_{1,p,\delta_n}|^p dx = \liminf_{n \to \infty} \lambda_{1,p,\delta_n}.
\]

In view of (28), the proof is complete. \(\square\)

The next result provides subsolutions and supersolutions for problem (4).
**Proposition 5.2.** Assume $H(f)_1$ with $\sigma \in L^\infty(\Omega)$, $\beta < p - 1$, and $(\alpha = p - 1$ and $a_1 < \lambda_{1,p})$ or $(\alpha < p - 1$ and $a_1 < \lambda_{1,p})$. Then there exists $\delta > 0$ with the property that, for every $\delta \in (0, \hat{\delta}]$, there is $t_0 = t_0(\delta) > 0$ such that $-t\phi_{1,p,\delta}$ and $t\phi_{1,p,\delta}$ are respectively subsolution and supersolution of problem (4) for all $t \geq t_0$.

**Proof.** Since $a_1 < \lambda_{1,p}$, Lemma 5.1 yields $\hat{\delta} > 0$ such that $a_1 < \lambda_{1,p,\delta}$ for all $\delta \in (0, \hat{\delta}]$. Because $\phi_{1,p,\delta} \in \text{int}(C^1_0(\hat{\Omega}_p^\delta))$, there exist constants $\rho = \rho(\delta) > 0$ and $t_0 = t_0(\delta) > 0$ with $\phi_{1,p,\delta}(x) \geq \rho$ and

$$
\frac{\sigma(x)}{(p\rho_0)^{p-1}} + \frac{a_1}{(p\rho_0)^{p-\alpha-1}} + \frac{a_2}{\rho^{p-1}p_0^{p-\beta-1}}|\nabla \phi_{1,p,\delta}(x)|^\beta \leq \lambda_{1,p,\delta}
$$

for all $x \in \Omega$. Combining with our hypothesis leads to

$$
f(x, t\phi_{1,p,\delta}(x), \nabla(t\phi_{1,p,\delta})(x)) \leq \sigma(x) + a_1(t\phi_{1,p,\delta}(x))^{\alpha} + a_2(t|\nabla \phi_{1,p,\delta}(x)|)^\beta
$$

$$
\leq \left( \frac{\sigma(x)}{(p\rho_0)^{p-1}} + \frac{a_1}{(p\rho_0)^{p-\alpha-1}} + \frac{a_2}{\rho^{p-1}p_0^{p-\beta-1}}|\nabla \phi_{1,p,\delta}(x)|^\beta \right) (t\phi_{1,p,\delta}(x))^{p-1}
$$

$$
\leq \lambda_{1,p,\delta}(t\phi_{1,p,\delta}(x))^{p-1} = -\Delta_p(t\phi_{1,p,\delta})(x)
$$

in $\Omega$, for all $t \geq t_0$. Therefore $t\phi_{1,p,\delta}$ is a supersolution of problem (4). The proof for the subsolution is similar. \hfill \square

Our existence result of extremal solutions for (4) is as follows.

**Theorem 5.3.** Assume $H(f)$. Then:

(i) Problem (4) has a biggest solution $\hat{u} \in C^1_0(\hat{\Omega})$ and a smallest solution $\hat{\nu} \in C^1_0(\hat{\Omega})$. In particular, every solution of (4) is contained in the ordered interval $[\hat{\nu}, \hat{u}]$.

(ii) If (24)–(25) hold and the set of nodal solutions of problem (4) is nonempty, then we have $\hat{u} \in \text{int}(C^1_0(\hat{\Omega}^p))$ and $\hat{\nu} \in -\text{int}(C^1_0(\hat{\Omega}^p))$.

**Proof.** Denote by $\mathcal{S}$ the set of solutions of problem (4) endowed with the pointwise partial order $\leq$. Lemmas 2.1 and 2.3 guarantee that $\mathcal{S}$ is nonempty.

Let us show that the ordered set $\mathcal{S}$ is upward directed, that is, if $u_1, u_2 \in \mathcal{S}$, then there is $u \in \mathcal{S}$ such that $\max\{u_1, u_2\} \leq u$. Indeed, if $u_1, u_2 \in \mathcal{S}$, by Lemma 2.2 we know that $u = \max\{u_1, u_2\}$ is a subsolution of problem (4), whereas Lemma 2.4 entails that $u_1, u_2 \in C^1_0(\hat{\Omega})$. Note that $H(f)$ allows us to apply Proposition 5.2, which ensures that there is $\delta > 0$ such that $t\phi_{1,p,\delta}$ is a supersolution of problem (4) for $t > 0$ large enough. Moreover, for a possibly larger $t$, we can suppose that $t\phi_{1,p,\delta} \geq u$ in $\Omega$ because $\hat{u}$ is bounded and $\phi_{1,p,\delta}$ is bounded away from zero on $\hat{\Omega}$ since $\phi_{1,p,\delta} \in \text{int}(C^1_0(\hat{\Omega}^p))$. Now invoking the general theory of sub-supersolutions (see [3, Theorem 3.17] or [10, Proposition 11.8]), it follows the existence of $u \in \mathcal{S}$ such that $u(x) \leq u(x) \leq t\phi_{1,p,\delta}(x)$ for all $x \in \Omega$, which proves our claim.

Next we show that each chain $\mathcal{C}$ has an upper bound in $\mathcal{S}$. On the basis of [5, p. 336], there is a sequence $(u_n)_{n \geq 1} \subset \mathcal{C}$ such that

$$
sup \mathcal{C} = sup u_n = \lim_{n \to \infty} u_n.
$$

(29)

According to Lemma 2.4, the sequence $(u_n)_{n \geq 1}$ is bounded in $C^{1,\gamma}(\hat{\Omega})$ for some $\gamma \in (0, 1)$. Since $C^{1,\gamma}(\hat{\Omega})$ is compactly embedded in $C^1(\hat{\Omega})$, along a relabeled subsequence we have that $u_n \to u$ in $C^1_0(\hat{\Omega})$ as $n \to \infty$. In view of (29) it turns out...
that $u = \sup \mathcal{C}$. In addition, we can pass to the limit in the equation $-\Delta_p u_n = f(x, u_n, \nabla u_n)$ in $W^{-1,p'}(\Omega)$, obtaining that $u \in \mathcal{S}$.

The preceding discussion enables us to apply Zorn’s lemma, which provides a maximal element $\hat{u} \in \mathcal{S}$. Actually, $\hat{u}$ is the biggest element in $\mathcal{S}$. Indeed, if $u \in \mathcal{S}$, as shown before we can find $v \in \mathcal{S}$ such that $v \geq \max\{\hat{u}, u\}$. Then the maximality of $\hat{u}$ renders $\hat{u} = v$, so $\hat{u} \geq u$ in $\Omega$.

Under the additional assumption (24), we may invoke Theorem 4.1 (i) ensuring that if problem (4) possesses nodal solutions, then $\hat{u} \in \interior(C^+_{0}(\overline{\Omega}))$.

Arguing in the same way, we can establish the existence of the smallest solution $\hat{v}$ of problem (4). Furthermore, through Theorem 4.1 (ii) we get that $\hat{v} \in -\interior(C^+_{0}(\overline{\Omega}))$ provided that (25) is valid and there are nodal solutions. \hfill \Box

6. Extremal solutions for problem (1). Now we focus on problem (1) with $\mu > 0$, which basically is the case driven by the $(p,q)$-Laplacian. Here we cannot proceed as for problem (4), i.e. when $\mu = 0$, because if $\mu > 0$, the nonhomogeneity of the operator $-\Delta_p - \mu \Delta_q$ prevents to construct subsolutions and supersolutions by means of the first eigenfunctions of $-\Delta_p$ or $-\Delta_q$. We obtain subsolutions and supersolutions for problem (1) with $\mu > 0$ by relying on a different idea based on the following assumption:

$$\liminf_{s \to +\infty} f(x, s, 0) \leq \gamma_1 < 0 < \gamma_2 \leq \limsup_{s \to -\infty} f(x, s, 0) \quad (30)$$

uniformly for a.e. $x \in \Omega$, with constants $\gamma_1$ and $\gamma_2$.

We formulate the following existence and enclosure result.

**Theorem 6.1.** Assume that $H(f)$ and (30) are verified, and that $\mu > 0$ in problem (1). Then:

(i) Problem (1) has a biggest solution $\hat{u} \in C^1_0(\overline{\Omega})$ and a smallest solution $\hat{v} \in C^1_0(\overline{\Omega})$. Thus, every solution of (1) is contained in the ordered interval $[\hat{v}, \hat{u}]$.

(ii) If (24)–(25) hold and the set of nodal solutions of problem (1) is nonempty, then $\hat{u} \in \interior(C^+_{0}(\overline{\Omega}))$ and $\hat{v} \in -\interior(C^+_{0}(\overline{\Omega}))$.

**Proof.** Denote by $\mathcal{S}$ the set of solutions of problem (1). As proven in Lemma 2.3, the set $\mathcal{S}$ is nonempty. Due to assumption $H(f)$ it is allowed to apply Lemma 2.4 for obtaining the uniform bound

$$M := \sup_{u \in \mathcal{S}} \|u\|_{\infty} < +\infty.$$ 

By virtue of (30), there exists $s_0 > M$ satisfying

$$f(x, s_0, 0) < 0 \text{ for a.e. } x \in \Omega.$$ 

It follows that $s_0$ is a constant supersolution of problem (1) such that $s_0 \geq u$ for all $u \in \mathcal{S}$.

We claim that the set $\mathcal{S}$ endowed with the pointwise order $\leq$ is upward directed. To this end let $u_1, u_2 \in \mathcal{S}$. From Lemma 2.2 we know that $\underline{u} := \max\{u_1, u_2\}$ is a subsolution for problem (1). Since $s_0 \geq \underline{u}$, the general theory of sub-supersolutions (see [3, Theorem 3.17] or [10, Proposition 11.8]) provides a solution $u \in \mathcal{S}$ within the ordered interval $[\underline{u}, s_0]$, which proves the claim.

Next, as in the proof of Theorem 5.3, we can check that Zorn’s lemma can be applied on the set $\mathcal{S}$, thus getting a maximal element $\hat{u} \in \mathcal{S}$. The fact that $\hat{u}$ is the biggest element of $\mathcal{S}$ can be seen following the pattern of the corresponding part in
the proof of Theorem 5.3, so we omit it. Similar arguments lead to the existence of the smallest solution of problem (1). Part (i) of the statement is established.

Finally, combining with Theorem 4.1, we deduce part (ii) of the statement of the theorem. The proof is thus complete.

\textbf{Remark 6.2.} Theorem 6.1 is also valid if, in place of (30), we assume the existence of constants $c, \bar{c}$ such that

$$c \leq -\sup_{u \in S} \|u\|_{\infty}, \quad \bar{c} \geq \sup_{u \in S} \|u\|_{\infty}, \quad f(x, \bar{c}, 0) \leq 0 \leq f(x, c, 0) \text{ for a.e. } x \in \Omega.$$ 

\textbf{Acknowledgments.} The authors are grateful to Lucas Fresse for important suggestions.

\textbf{REFERENCES}

[1] D. Averna, D. Motreanu and E. Tornatore, Existence and asymptotic properties for quasilinear elliptic equations with gradient dependence, \textit{Appl. Math. Lett.}, 61 (2016), 102–107.

[2] S. Carl, Barrier solutions of elliptic variational inequalities, \textit{Nonlinear Anal. Real World Appl.}, 26 (2015), 75–92.

[3] S. Carl, V. K. Le and D. Motreanu, \textit{Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications}, Springer, New York, 2007.

[4] A. Cianchi and V. Maz’ya, Global gradient estimates in elliptic problems under minimal data and domain regularity, \textit{Commun. Pure Appl. Anal.}, 14 (2015), 285–311.

[5] N. Dunford and J. T. Schwartz, \textit{Linear Operators. I. General Theory}, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[6] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, \textit{Nonlinear Anal.}, 12 (1988), 1203–1219.

[7] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations, \textit{Comm. Partial Differential Equations}, 16 (1991), 311–361.

[8] S. Miyajima, D. Motreanu and M. Tanaka, Multiple existence results of solutions for the Neumann problems via super- and sub-solutions, \textit{J. Funct. Anal.}, 262 (2012), 1921–1953.

[9] D. Motreanu, V. V. Motreanu and N. S. Papageorgiou, A unified approach for multiple constant sign and nodal solutions, \textit{Adv. Differential Equations}, 12 (2007), 1363–1392.

[10] D. Motreanu, V. V. Motreanu and N. S. Papageorgiou, \textit{Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems}, Springer, New York, 2014.

[11] P. Pucci and J. Serrin, \textit{The Maximum Principle}, Springer, New York, 2007.

Received August 2016; revised April 2017.

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