A MINIMUM PRINCIPLE FOR POTENTIALS WITH APPLICATION TO CHEBYSHEV CONSTANTS

A. REZNIKOV, E. B. SAFF, AND O. V. VLASIUK

ABSTRACT. For “Riesz-like” kernels $K(x, y) = f(|x - y|)$ on $A \times A$, where $A$ is a compact $d$-regular set $A \subset \mathbb{R}^p$, we prove a minimum principle for potentials $U^\mu_K = \int K(x, y) d\mu(x)$, where $\mu$ is a Borel measure supported on $A$. Setting $P_K(\mu) = \inf_{y \in A} U^\mu_K(y)$, the $K$-polarization of $\mu$, the principle is used to show that if $\{\nu_N\}$ is a sequence of measures on $A$ that converges in the weak-star sense to the measure $\nu$, then $P_K(\nu_N) \to P_K(\nu)$ as $N \to \infty$. The continuous Chebyshev (polarization) problem concerns maximizing $P_K(\mu)$ over all probability measures $\mu$ supported on $A$, while the $N$-point discrete Chebyshev problem maximizes $P_K(\mu)$ only over normalized counting measures for $N$-point multisets on $A$. We prove for such kernels and sets $A$, that if $\{\nu_N\}$ is a sequence of $N$-point measures solving the discrete problem, then every weak-star limit measure of $\nu_N$ as $N \to \infty$ is a solution to the continuous problem.

Keywords: Maximal Riesz polarization, Chebyshev constant, Hausdorff measure, Riesz potential, Minimum principle

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1. INTRODUCTION

For a nonempty compact set $A \subset \mathbb{R}^p$, a kernel $K: A \times A \to \mathbb{R} \cup \{\infty\}$ and a measure $\mu$ supported on $A$, the $K$-potential of $\mu$ is defined by

$$U^\mu_K(y) := \int_A K(x, y) d\mu(x), \ y \in \mathbb{R}^p.$$ 

Assuming that $K$ is lower semi-continuous, the Fatou lemma implies that if $y_n \to y$ as $n \to \infty$, we have

$$\liminf_{n} U^\mu_K(y_n) \geq U^\mu_K(y);$$

thus $U^\mu_K$ is a lower semi-continuous function on $\mathbb{R}^p$. We define the weak* topology on the space of positive Borel measures as follows.

Definition 1.1. Let $(\mu_n)_{n=1}^\infty$ be a sequence of positive Borel measures supported on a compact set $A$. We say that the measures $\mu_n$ converge to the measure $\mu$ in the weak* sense, $\mu_n \rightharpoonup \mu$, if for any function $\varphi$ continuous on $A$ we have

$$\int \varphi(x) d\mu_n(x) \to \int \varphi(x) d\mu(x), \ n \to \infty.$$ 

For a measure $\mu$ supported on $A$ its $K$-polarization is defined by

$$P_K(\mu) := \inf_{y \in A} U^\mu_K(y).$$
In the following definition we introduce two special constants which denote the maximum value of $P_K(\mu)$ when $\mu$ ranges over all probability measures and when $\mu$ ranges over all probability measures supported on finite sets.

**Definition 1.2.** For a positive integer $N$ the *discrete* $N$-th $K$-polarization (or Chebyshev) constant of $A$ is defined by

$$\mathcal{P}_K(A,N) := \sup_{\omega_N \subset A} \inf_{y \in A} U_{\mathcal{K}}^{\omega_N}(y),$$

where the supremum is taken over $N$-point multisets $\omega_N$; i.e., $N$-point sets counting multiplicities, and where $\nu_{\omega_N}$ is the normalized counting measure of $\omega_N$:

$$\nu_{\omega_N} := \frac{1}{N} \sum_{x \in \omega_N} \delta_x.$$

Moreover, we say that the probability measure $\nu$ supported on $A$ *solves the continuous $K$-polarization problem* if

$$\inf_{y \in A} U_{\mathcal{K}}^{\nu}(y) = \sup_{\mu \in A} \inf_{y \in A} U_{\mathcal{K}}^{\mu}(y) =: T_K(A),$$

where the supremum is taken over all probability measures $\mu$ supported on $A$.

The following result has been known since 1960’s; it relates the asymptotic behavior of $\mathcal{P}_K(A,N)$ as $N \to \infty$ with $T_K(A)$.

**Theorem 1.3** (Ohtsuka, [3]). Assume $A \subset \mathbb{R}^p$ is a compact set and $K: A \times A \to (-\infty, \infty]$ is a lower semi-continuous symmetric kernel bounded from below. Then

$$\mathcal{P}_K(A,N) \to T_K(A), \ N \to \infty.$$

What has been as yet unresolved for integrable kernels on sets $A$ of positive $K$-capacity is whether, under the mild assumptions of symmetry and lower semi-continuity of $K$, every limit measure (in the weak$^*$ sense) of a sequence of normalized counting measures $\nu_{\omega_N}$ associated with optimal $N$-th $K$-polarization constants attains $T_K(A)$. We remark that such a result does not necessarily hold for non-integrable kernels. Consider a two-point set $A = \{0, 1\}$ and any kernel $K$ with $K(0,0) = K(1,1) = \infty$ and $K(0,1) = K(1,0) < \infty$. Then, for any $N \geq 2$, the measure $\nu_N := (1/N) \delta_0 + ((N - 1)/N) \delta_1$ attains $\mathcal{P}_K(A,N) = \infty$. However, $\nu_N \not\to \delta_1$, which does not attain $T_K(A) = \infty$.

One case when such a result holds is for $K \in C(A \times A)$. Namely, the following is true, see [1], [4], [5] and [6].

**Theorem 1.4.** Let $A \subset \mathbb{R}^p$ be a compact set and $K \in C(A \times A)$ be a symmetric function. A sequence $(\omega_N)_{N=1}^{\infty}$ of $N$-point multisets on $A$ satisfies

$$\lim_{N \to \infty} P_K(\nu_{\omega_N}) = T_K(A)$$

if and only if every weak$^*$-limit measure $\nu^*$ of the sequence $(\nu_{\omega_N})_{N=1}^{\infty}$ attains $T_K(A)$.

Notice that this theorem does not cover cases when $K$ is unbounded along the diagonal of $A \times A$; in particular, Riesz kernels $K(x,y) = |x - y|^{-s}$ when $s > 0$. The following theorem by B. Simanek applies to Riesz kernels (as well as more general kernels) but under rather special conditions on the set $A$ and the Riesz parameter.
Remark. Examples of such functions for any \( d \)-Riesz-like function. If \( \nu \) is the unique measure that attains \( T_\nu(A) \). Furthermore, if \( \nu_N \) is an \( N \)-point normalized counting measure that attains \( \mathcal{P}_f(A,N) \), then \( \nu_N \rightarrow \nu_{eq} \) as \( N \rightarrow \infty \).

We remark that if \( A = \mathbb{B}^d \), the \( d \)-dimensional unit ball and \( f(t) = t^{-s} \) with \( d - 2 \leq s < d \), then Theorem 1.5 applies, while if \( 0 < s < d - 2 \) or \( f(t) = \log(2/t) \), the assumptions of this theorem are not satisfied. However, it was shown by Erdélyi and Saff [3] that for this case the only \( N \)-point normalized counting measure \( \nu_N \) that attains \( \mathcal{P}_f(\mathbb{B}^d,N) \) is \( \nu_N = \delta_0 \).

In this paper we obtain a convergence theorem that holds for all integrable Riesz kernels provided the set \( A \) is \( d \)-regular.

**Definition 1.6.** A compact set \( A \subset \mathbb{R}^d \) is called \( d \)-regular, \( 0 < d \leq \rho \), if there exist two positive constants \( c \) and \( C \) such that for any point \( y \in A \) and any \( r \) with \( 0 < r < \text{diam}(A) \), we have \( cr^d \leq \mathcal{H}_d(B(y,r) \cap A) \leq Cr^d \), where \( \mathcal{H}_d \) is the \( d \)-dimensional Hausdorff measure on \( \mathbb{R}^p \) normalized by \( \mathcal{H}_d([0,1]^d) = 1 \).

Further, we introduce a special family of kernels.

**Definition 1.7.** A function \( f : (0,\infty) \rightarrow (0,\infty) \) is called \( d \)-Riesz-like if it is continuous, strictly decreasing, and for some \( \varepsilon \) with \( 0 < \varepsilon < d \) and \( t_\varepsilon > 0 \) the function \( t \mapsto t^{d-\varepsilon} f(t) \) is increasing on \([0,t_\varepsilon]\); the value at zero is formally defined by

\[
\lim_{t \to 0^+} t^{d-\varepsilon} f(t).
\]

The kernel \( K \) is called \( d \)-Riesz-like if \( K(x,y) = f(|x-y|) \).

**Remark.** Examples of such functions \( f \) include \( s \)-Riesz potentials \( f(t) = t^{-s} \) for \( 0 < s < d \), as well as \( f(t) = \log(c/t) \), where the constant \( c \) is chosen so that \( \log(c/|x-y|) > 0 \) for any \( x,y \in A \). Further, we can consider \( f(t) := t^{-s} \cdot (\log(c/t))^\alpha \) for any \( \alpha > 0 \) and \( 0 < s < d \). We also do not exclude the case when \( f \) is bounded; e.g., \( f(t) = e^{-ct^2}, c > 0 \).

Under above assumptions on \( A \) and \( f \), we first study the behavior of \( P_K(\mu_N) \) as \( \mu_N \rightarrow \mu \). In what follows, when \( K(x,y) = f(|x-y|) \) we write \( U_f, P_f \) and \( T_f(A) \) instead of \( U_K, P_K \) and \( T_K(A) \). We prove the following.

**Theorem 1.8.** Let \( A \) be a \( d \)-regular compact set, and \( f \) be a \( d \)-Riesz-like function. If \( (\nu_N)_{N=1}^\infty \) is a sequence of measures on \( A \) with \( \nu_N \rightarrow \nu \), then \( P_f(\nu_N) \rightarrow P_f(\nu) \) as \( N \rightarrow \infty \).

This theorem is a direct consequence of a minimum principle for potentials, introduced below in Theorem 2.5 From Theorem 1.8 we derive the following result.

**Theorem 1.9.** Let \( A \) and \( f \) satisfy the conditions of Theorem 1.8. For each \( N \) let \( \nu_N \) be an \( N \)-point normalized counting measure that attains \( \mathcal{P}_f(A,N) \). If \( \nu^* \) is any weak*-limit measure of the sequence \( (\nu_N) \), then \( \nu^* \) solves the continuous \( f \)-polarization problem.

Notice that whenever there is a unique measure \( \nu \) that solves the continuous polarization problem on \( A \), then Theorem 1.9 implies that the whole sequence \( \{\nu_N\} \) converges to \( \nu \) in the weak* sense.
2. A MINIMUM PRINCIPLE FOR RIEZ-LIKE POTENTIALS

We begin this section with some known results from potential theory. In what follows, all measures will have support on $A$. We proceed with the following definition, important in potential theory.

**Definition 2.1.** A set $E \subset A$ is called $K$-negligible if for any compact set $E_1 \subset E$ and any measure $\mu$ such that $U_K^\mu$ is bounded on $E_1$, we have $\mu(E_1) = 0$.

The following definition describes a useful class of kernels.

**Definition 2.2.** The kernel $K$ is said to be regular if for any positive Borel measure $\mu$ the following is satisfied: if the potential $U_K^\mu$ is finite and continuous on $\text{supp} \mu$, then it is finite and continuous in the whole space $\mathbb{R}^p$.

It is known, see [7], that a kernel of the form $K(x,y) = f(|x-y|)$, where $f$ is a continuous non-negative strictly decreasing function, is regular. Regularity of a kernel implies the following two results.

**Theorem 2.3** *(Principle of descent, Lemma 2.2.1 in [7])* Assume $K$ is regular. If $\nu_n \Rightarrow \mu$ and $y_n \to y_\infty$ as $n \to \infty$, then

$$\liminf_n U_{K}^{\nu_n}(y_n) \geq U_{K}^{\mu}(y_\infty).$$

**Theorem 2.4** *(Lower envelope, Theorem 3 in [2])* Assume $K$ is regular. If $\nu_n \Rightarrow \mu$, then the set

$$E := \{ y \in A : \liminf_n U_{K}^{\nu_n}(y) > U_{K}^{\mu}(y) \},$$

is $K$-negligible.

The new minimum principle mentioned in the title is the following.

**Theorem 2.5.** Let $A$ be a $d$-regular set, and $f$ be a $d$-Riesz-like function on $(0, \infty)$. If for a measure $\mu$ on $A$ and a constant $M$,

$$U_f^\mu(y) \geq M, \quad y \in A \setminus E,$$

(3)

where $E$ is $f$-negligible, then $U_f^\mu(y) \geq M$ for every $y \in A$.

We proceed with a proposition that is an analog of the Lebesgue differentiation theorem for potentials.

**Definition 2.6.** For a function $\varphi : A \to \mathbb{R}$ and a point $y \in A$, we call $y$ a weak $d$-Lebesgue point of $\varphi$ if

$$\varphi(y) = \lim_{r \to 0^+} \frac{1}{\mathcal{H}_d(A \cap B(y,r))} \int_{A \cap B(y,r)} \varphi(z) \mathcal{H}_d(z),$$

where $B(y,r)$ denotes the open ball in $\mathbb{R}^p$ with center at $y$ and radius $r$.

**Proposition 2.7.** Suppose $A$ and $f$ satisfy conditions of Theorem 2.5 and $\mu$ is a measure supported on $A$. Then every point $y \in A$ is a weak $d$-Lebesgue point of $U_f^\mu$.

We start by proving the following technical lemma.
Lemma 2.8. There exist positive numbers $C_0$ and $r_0$ such that for any $x \in A$ and any $r < r_0$:

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{A \cap B(y, r)} f(|x-z|) d\mathcal{H}_d(z) \leq C_0. \tag{4}$$

Proof. Notice that the left-hand side of (4) is equal to

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{0}^{\infty} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du.$$  

Since $f$ is decreasing, we see that

$$\{z \in A \cap B(y, r) : f(|x-z|) > u\} = \{z \in A \cap B(y, r) \cap B(x, f^{-1}(u))\}.$$  

This set is empty when $f^{-1}(u) < |x-y| - r$ or $u > f(|x-y| - r)$.

We consider two cases.

Case 1: $|x-y| > 2r$. Then we obtain the estimate

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{0}^{\infty} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du = \frac{f(|x-y| - r)}{f(|x-y|)}.$$  

Since $|x-y| > 2r$, we have $|x-y| - r \geq |x-y|/2$; thus,

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{0}^{\infty} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du \leq \frac{f(|x-y|/2)}{f(|x-y|)}.$$  

Finally, if recall that the function $t \mapsto f(t) \cdot t^{d-\varepsilon}$ is increasing for $t \in [0, t_\varepsilon]$. If $|x-y| > t_\varepsilon$, we get

$$\frac{f(|x-y|/2)}{f(|x-y|)} \leq \frac{f(t_\varepsilon/2)}{f(\text{diam}(A))}.$$  

If $|x-y| \leq t_\varepsilon$, we use that

$$f(|x-y|/2) \cdot (|x-y|/2)^{d-\varepsilon} \leq f(|x-y|) \cdot (|x-y|)^{d-\varepsilon};$$

thus

$$\frac{f(|x-y|/2)}{f(|x-y|)} \leq 2^{d-\varepsilon}.$$  

Case 2: $|x-y| \leq 2r$. Again, we need only integrate for $u \leq f(|x-y| - r)$. Setting $f$ equal to $f(0)$ for any negative argument, we write

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{0}^{f(|x-y| - r)} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du =$$

$$\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \left( \int_{0}^{f(|x-y| + r)} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du + \int_{f(|x-y| - r)}^{f(|x-y| + r)} \mathcal{H}_d(z \in A \cap B(y, r) : f(|x-z|) > u) du \right)$$
Trivially,
\[
\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \cdot \int_0^{f(|x-y|+r)} \mathcal{H}_d(\{z \in A \cap B(y, r) : f(|x-z|) > u\})\,du
\]  
(7) \leq \frac{f(|x-y|+r)}{f(|x-y|)} \leq 1.

Furthermore, since $|x-y| \leq 2r$, we have $f(|x-y|) \geq f(2r)$; thus,
\[
\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{f(|x-y|)}^{f(|x-y|-r)} \mathcal{H}_d(\{z \in A \cap B(y, r) : f(|x-z|) > u\})\,du
\]  
(8) \leq \frac{1}{c \cdot r^d} \cdot \frac{1}{f(2r)} \int_{f(|x-y|+r)}^{\infty} \mathcal{H}_d(\{z \in A \cap B(x, f^{-1}(u))\})\,du
\leq \frac{C}{r^d f(2r)} \int_{f(|x-y|+r)}^{\infty} (f^{-1}(u))^d\,du.

Note that the assumption $|x-y| \leq 2r$ implies $f(|x-y|+r) \geq f(3r)$, and thus
\[
\frac{1}{\mathcal{H}_d(A \cap B(y, r))} \cdot \frac{1}{f(|x-y|)} \int_{f(|x-y|)}^{f(|x-y|-r)} \mathcal{H}_d(\{z \in A \cap B(y, r) : f(|x-z|) > u\})\,du
\]  
(9) \leq \frac{C}{c \cdot r^d f(2r)} \int_{f(3r)}^{\infty} (f^{-1}(u))^d\,du.

We now observe that our assumption that $f(t)t^{d-\varepsilon}$ is increasing on $[0, t_\varepsilon]$ implies that the function $u^{1/(d-\varepsilon)} f^{-1}(u)$ is decreasing on $[f(t_\varepsilon), \infty)$. Therefore, for $3r < t_\varepsilon$ we have
\[
\frac{C}{r^d f(2r)} \int_{f(3r)}^{\infty} (f^{-1}(u))^d\,du = \frac{C}{r^d f(2r)} \int_{f(3r)}^{\infty} (u^{1/(d-\varepsilon)} f^{-1}(u))^d \cdot u^{-d/(d-\varepsilon)}\,du
\leq \frac{C_1}{r^d f(2r)} \cdot f(3r)^{d/(d-\varepsilon)} \cdot (3r)^d \cdot f(3r)^{1-d/(d-\varepsilon)} \leq C_2.
\]

\[\square\]

Proof of Proposition 2.7 We formally define $f(0) := \lim_{t \to 0^+} f(t) \in (0, \infty]$. Without loss of generality, we consider the case $f(0) = \infty$; otherwise the potential $U_f^\mu$ is continuous on $\mathbb{R}^d$ and the proposition holds trivially.

Define
\[
\Phi_f^\mu(y) := \frac{1}{\mathcal{H}_d(A \cap B(y, r))} \int_{A \cap B(y, r)} U_f^\mu(u)\,d\mathcal{H}_d(u).
\]
Tonnelli’s theorem and Lemma 2.8 imply
\[
\Phi_f^\mu(y) = \frac{1}{\mathcal{H}_d(A \cap B(y, r))} \int_{A \cap B(y, r)} \int_A f(|x-z|)\,d\mu(u)\,d\mathcal{H}_d(z)
\]  
(11) \leq C_0 \int_A f(|x-y|)\,d\mu(x) = C_0 U_f^\mu(y).

We first suppose $U_f^\mu(y) = \infty$. Since $U_f^\mu$ is lower semi-continuous, we obtain that for any large number $N$ there is a positive number $r_N$, such that $U_f^\mu(x) > N$ in $B(y, r_N)$. Then
for any \( r < r_N \) we get
\[
\Phi_r^\mu(y) = \frac{1}{\mathcal{H}_d(A \cap B(y, r))} \int_{A \cap B(y, r)} U_f^\mu(z) d\mathcal{H}_d(z) > N.
\]
This implies that \( \Phi_r^\mu(y) \to \infty = U_f^\mu(y) \) as \( r \to 0^+ \).

Now assume \( U_f^\mu(y) = \int_A f(|x|) d\mu(x) < \infty \). Notice that since \( f(0) = \infty \), the measure \( \mu \) cannot have a mass point at \( y \). Consequently, for any \( \eta > 0 \) there exists a ball \( B(y, \delta) \), such that
\[
\int_{B(y, \delta)} f(|x|) d\mu(x) < \eta.
\]
Consider measures \( d\mu' := \mathbb{1}_{B(y, \delta)} d\mu \), and \( \mu_c := \mu - \mu' \).

Since \( y \not\in \text{supp}(\mu_c) \), the potential \( U_f^{\mu_c} \) is continuous at \( y \). This implies
\[
\Phi_r^{\mu_c}(y) = \frac{1}{\mathcal{H}_d(A \cap B(y, r))} \int_{A \cap B(y, r)} U_f^{\mu_c}(z) d\mathcal{H}_d(z) \to U_f^{\mu_c}(y), \quad r \to 0^+.
\]

Also, on applying (11) to \( \mu' \), it follows that
\[
\Phi_r^\mu(y) = \Phi_r^{\mu_c}(y) + \Phi_r^{\mu'}(y) \leq \Phi_r^{\mu_c}(y) + C_0 U_f^{\mu'}(y) \leq \Phi_r^{\mu_c}(y) + C_0 \eta.
\]
Taking the lim sup, we obtain
\[
\limsup_{r \to 0^+} \Phi_r^\mu(y) = U_f^\mu(y) + C_0 \eta \leq U_f^\mu(y) + C_0 \eta.
\]
On the other hand, we know that
\[
\Phi_\mu^\mu(y) = \Phi_r^{\mu_c}(y) + \Phi_r^{\mu'}(y) \geq \Phi_r^{\mu_c}(y),
\]
and thus
\[
\liminf_{r \to 0^+} \Phi_r^\mu(y) \geq U_f^\mu(y) = U_f^\mu(y) - U_f^{\mu'}(y) \geq U_f^\mu(y) - \eta.
\]
This, together with (12) and the arbitrariness of \( \eta \) implies the assertion of Proposition 2.7.

We are ready to deduce Theorem 2.5.

**Proof of Theorem 2.5** We first claim that any \( f \)-negligible subset \( E \) of \( A \) has \( \mathcal{H}_d \)-measure zero. Indeed, take any compact set \( E_1 \) inside our \( f \)-negligible set \( E \). Then for any \( y \in A \)
\[
U_f^\mathcal{H}_d(y) = \int_A f(|x-y|) d\mathcal{H}_d(x) = \int_0^\infty \mathcal{H}_d(x \in A \cap B(y, f^{-1}(u))) du.
\]
We notice that if \( u < f(\text{diam}(A)) =: u_0 \), then \( f^{-1}(u) > \text{diam}(A) \), and so \( A \cap B(y, f^{-1}(u)) = A \). Thus,
\[
U_f^\mathcal{H}_d(y) = \left( \int_0^{u_0} + \int_{u_0}^\infty \right) \mathcal{H}_d(x \in A \cap B(y, f^{-1}(u))) du
\]
\[
\leq u_0 \mathcal{H}_d(A) + C \int_{u_0}^\infty (f^{-1}(u))^d du,
\]
which is bounded by a constant that does not depend on \( y \), as proved in the Case 2 of Lemma 2.3 see inequality (10). Since the set \( E \) is negligible, we conclude that \( \mathcal{H}_d(E_1) = 0 \). Thus, for any compact subset \( E_1 \) of \( E \) we have \( \mathcal{H}_d(E_1) = 0 \) and so \( \mathcal{H}_d(E) = 0 \) as
claimed. Now, let the measure $\mu$ satisfy (3). Then for any $y \in A$, we deduce from Proposition 2.7 and the fact that $U^\mu_f(z) \geq M$ holds $\mathcal{H}_d$-a.e. on $A$ that $U^\mu_f(y) \geq M$. □

3. PROOF OF THEOREM 1.8 AND THEOREM 1.9

Proof of Theorem 1.8. For any increasing infinite subsequence $\mathcal{N} \subset \mathbb{N}$, choose a subsequence $\mathcal{N}_1$, such that

$$\liminf_{N \in \mathcal{N}} P_f(\nu_N) = \lim_{N \in \mathcal{N}_1} P_f(\nu_N).$$

For each $N \in \mathcal{N}_1$ take a point $y_N$, such that $P_f(\nu_N) = U_f^\nu(y_N)$. Passing to a further subsequence $\mathcal{N}_0 \subset \mathcal{N}_1$ we can assume $y_N \to y_\infty$ as $N \to \infty$, $N \in \mathcal{N}_0$. Then the principle of descent, Theorem 2.3, implies

$$\liminf_{N \in \mathcal{N}} P_f(\nu_N) = \lim_{N \in \mathcal{N}_0} U_f^\nu(y_N) \geq U_f^\nu(y_\infty) \geq P_f(\nu).$$

Furthermore, for any $y \in A$ we have

$$\liminf_{N \in \mathcal{N}} U_f^\nu(y) \geq \liminf_{N \in \mathcal{N}} P_f(\nu_N) =: M_f(\mathcal{N}), \quad y \in A.$$

By Theorem 2.4, $\liminf_{N \in \mathcal{N}} U_f^\nu(y) = U_f^\nu(y)$ for every $y \in A \setminus E$, where $E$ is an $f$-negligible set that can depend on $\mathcal{N}$. Therefore,

$$U_f^\nu(y) \geq M_f(\mathcal{N}), \quad y \in A \setminus E.$$

From the minimum principle, Theorem 2.5, we deduce that

$$U_f^\nu(y) \geq M_f(\mathcal{N}) = \liminf_{N \in \mathcal{N}} P_f(\nu_N), \quad \forall y \in A,$$

and therefore

$$(15) \quad P_f(v) \geq \liminf_{N \in \mathcal{N}} P_f(\nu_N).$$

Combining estimates (14) and (15), we deduce that for any subsequence $\mathcal{N}$ we have

$$\liminf_{N \in \mathcal{N}} P_f(\nu_N) = P_f(v).$$

This immediately implies

$$\lim_{N} P_f(\nu_N) = P_f(v),$$

which completes the proof. □

Proof of Theorem 1.9. Assume for a subsequence $\mathcal{N}$ we have $\nu_N \overset{\star}{\to} \nu^\star$ as $N \to \infty$, $N \in \mathcal{N}$. From [8] we know that

$$P_f(\nu_N) \to T_f(A), \quad N \to \infty.$$ 

On the other hand, from Theorem 1.8 we know that

$$P_f(\nu_N) \to P_f(\nu^\star), \quad N \to \infty, \quad N \in \mathcal{N}.$$ 

Therefore,

$$P_f(\nu^\star) = T_f(A),$$

which proves the theorem. □
Remark. Suppose \( A = \bigcup_{k=1}^{m} A_k \), where \( A_k \) is a \( d_k \)-regular compact set, and that, for some positive number \( \delta \), we have \( \text{dist}(A_i, A_j) \geq \delta \) for \( i \neq j \). Further assume that \( f \) is a \( d_k \)-Riesz-like kernel for every \( k = 1, \ldots, m \), and \( \mu \) is a measure supported on \( A \). Then the result of Theorem [2.5] and thus of Theorems [1.8] and [1.9] hold. To see this, we first show that every \( y \in A_k \) is a weak \( d_k \)-Lebesgue point of \( U_f^\mu \). Indeed, setting \( d_\mu_k := BD_{A_k} d_\mu \) yields

\[
U_f^\mu(y) = \sum_{k=1}^{m} U_{f_k}^\mu(y).
\]

Proposition [2.7] implies that if \( y \in A_k \), then \( y \) is a weak \( d_k \)-Lebesgue point of \( U_{f_k}^\mu \). Moreover, for any \( j \neq k \) we have \( y \not\in \text{supp}(\mu_j) \); thus \( U_{f_j}^\mu \) is continuous at \( y \), and our assertion about weak Lebesgue points of \( U_f^\mu \) follows. Similar to (13), we then deduce that if a set \( E \subset A \) is \( f \)-negligible, then \( \mathcal{H}_{d_k}(E \cap A_k) = 0 \) for every \( k = 1, \ldots, m \); therefore, the assertion of Theorem [2.5] remains true and the proofs of Theorems [1.8] and [1.9] go exactly as before.

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Center for Constructive Approximation, Department of Mathematics, Vanderbilt University

E-mail address: aleksandr.b.reznikov@vanderbilt.edu

E-mail address: edward.b.saff@vanderbilt.edu

E-mail address: oleksandr.v.vlasiuk@vanderbilt.edu