Light-cone M5 and multiple M2-branes

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Abstract
We present the light-cone gauge fixed Lagrangian for the M5-brane; it has a residual ‘exotic’ gauge invariance with the group of 5-volume preserving diffeomorphisms, SDiff\textsubscript{5}, as gauge group. For an M5-brane of topology $\mathbb{R}^2 \times M_3$, for closed 3-manifold $M_3$, we find an infinite tension limit that yields an $SO(8)$-invariant $(1+2)$-dimensional field theory with ‘exotic’ SDiff\textsubscript{3} gauge invariance. We show that this field theory is the Carrollian limit of the Nambu bracket realization of the ‘BLG’ model for multiple M2-branes.

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1. Introduction

A $(1+2)$-dimensional relativistic gauge theory based on a Filippov 3-algebra [1] (see also [2]) rather than a Lie algebra was proposed recently by Bagger and Lambert [3], and by Gustavsson [4], as a model of multiple M2-branes. The model has an $OSp(8|4)$ conformal symmetry [5] as expected for the infra-red fixed point of the Yang–Mills-type gauge theory on coincident $D2$-branes. The construction requires a metric on the 3-algebra and if this metric is positive definite then the structure constants of the 3-algebra define a totally-antisymmetric fourth-rank tensor satisfying a ‘fundamental’ identity\footnote{Let $(B, C)$ be anticommuting variables taking values in a Filippov $n$-algebra. Then the fundamental identity is equivalent to $\{B, \ldots, B, [C, \ldots, C]\} = n\{[B, \ldots, B, C], C, \ldots, C\}$.}. When the structure constants vanish one has a trivial 3-algebra and the model reduces to a free theory for the $N = 8$ scalar multiplet, as

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\textsuperscript{5} Since the original version of this paper was posted on the archives, it has been shown that ‘BLG-like’ models can be constructed from a class of ‘generalized’ 3-algebras for which this fourth-rank tensor need not be totally antisymmetric [6].

\textsuperscript{6} Let $(B, C)$ be anticommuting variables taking values in a Filippov $n$-algebra. Then the fundamental identity is equivalent to $\{B, \ldots, B, [C, \ldots, C]\} = n\{[B, \ldots, B, C], C, \ldots, C\}$.
expected for the conformal limit of a single planar M2-brane. A non-trivial realization based on the Lie algebra $so(4)$ was given by Bagger and Lambert [3], and it appears to describe two coincident M2-branes on an orbifold [7, 8]. It has since been shown that the only other finite-dimensional realizations are direct sums of copies of this ‘$so(4)$-based’ algebra with trivial Abelian 3-algebras [9, 10]. Other possibilities emerge when one allows for Lorentzian metrics on the 3-algebra [11, 12] but these models have ghosts; we refer to some very recent works for further discussion of this point [13, 14], and to [15] for a supergravity perspective. Various other facets of Bagger–Lambert–Gustavsson (BLG) models have been addressed in other papers; an incomplete list can be found in [16–19]. In the context of the original BLG model, with positive definite metric, there remains one other possibility: there is an infinite-dimensional realization of the 3-algebra in terms of the Nambu bracket on a three-dimensional space [20, 21, 12]. In this realization, the BLG model is essentially an exotic gauge theory for the group of volume preserving diffeomorphisms of this space, where by ‘exotic’ we mean that the gauge theory is not of Yang–Mills type.

This is not the first occasion on which exotic gauge theories based on volume-preserving diffeomorphisms have appeared. They also arise from light-cone gauge fixing of relativistic $p$-brane actions for $p > 2$; these are ‘exotic’ gauge theories with a group of $p$-volume preserving diffeomorphisms, $SDiff_p$, as the gauge group [22]. This generalizes the (dimensionally-reduced) Yang–Mills-type actions for $p = 2$ where the Yang–Mills gauge group is a group of area-preserving diffeomorphisms that may loosely be regarded as $SU(\infty)$ [23]. In particular, the light-cone gauge-fixed ten-dimensional ($\mathcal{N} = 1$) 5-brane is an exotic gauge theory with an $SDiff_5$ gauge group [22]. Clearly, a similar result should hold for the 11-dimensional M5-brane, and one purpose of this paper is to present this $SDiff_5$-invariant action.

Our starting point is the Hamiltonian M5-brane action for a general supergravity background [24], which can be deduced from the Lorentz-covariant M5-brane action [25, 26] after a ‘temporal gauge’ choice for the PST-gauge invariance [27]. One advantage of the Hamiltonian form is that the passage to the light-cone gauge-fixed theory is conceptually simpler. A further advantage, specific to the M5-brane, is that the nonlinear self-duality of its worldvolume 3-form field strength $H = dA$ is very simply incorporated, off-shell, by the disappearance from the action (excepting boundary terms) of the time components of the 2-form potential $A$. After partial fixing of the worldvolume reparametrization invariance by the choice of light-cone gauge, one is left with an $SO(9)$-invariant $SDiff_5$ exotic gauge theory.

As is well known, the flux of the 2-form potential on the M5-brane may be interpreted as M2-branes ‘dissolved’ in the M5-brane. Thus, a single M5-brane may contain multiple M2-branes and is therefore a promising starting point for a construction of the BLG model for multiple-M2-branes. Another indication of this is that the Nambu-bracket realization of the BLG theory introduces some ‘internal’ Riemannian 3-manifold $M_3$, so that the ‘total’ space dimension is $2 + 3 = 5$. In fact, it has been proposed in recent papers that the Nambu-bracket realization of the BLG model is equivalent to the M5-brane action [20, 21] (see also [28]). However, in the 11-dimensional Minkowski vacuum of M-theory considered in [20, 21] and here, the symmetry algebra of the M5-brane action is an 11-dimensional super-Poincaré algebra with tensor charges [29], so it would be remarkable if an $OSp(8|4)$ theory were to emerge.

Another purpose of this paper is to address this issue from the ‘opposite’ direction: starting from the light-cone gauge fixed M5-brane action, we consider an M5-brane of topology $\mathbb{R}^2 \times M_3$ and then consider how the BLG theory, in its Nambu-bracket realization, might emerge from it. It is natural to suppose that the $SDiff_3$ gauge group of the Nambu-
bracket BLG theory is a subgroup of the SDiff_5 gauge group of the M5 theory, so we propose a partial gauge fixing that identifies the $\mathbb{R}^2$ coordinates with two of the M5-brane coordinates. This breaks the manifest $SO(9)$ invariance to $SO(7)$, but we consider whether this could be enhanced to $SO(8)$ in some limit. At the same time, we expect to find some $(1 + 2)$-dimensional theory with an SDiff_3 residual gauge group. One obvious way that this could happen is if all fields are assumed to be independent of position in $\mathbb{R}^2$, but this amounts to a double-dimensional reduction and it yields an $SO(7)$-invariant non-conformal 3-brane action on $M_3$, rather than an $SO(8)$-invariant conformal theory on $\mathbb{R}^2$. Here, we keep the dependence on all worldvolume coordinates, including the $\mathbb{R}^2$ coordinates, but we rescale the worldvolume fields by a power of the M5-brane tension such that the rescaled fields have the dimensions expected of a conformal $(1 + 2)$-dimensional theory, and we also introduce rescaled dimensionless coordinates for $M_3$. We then show that the infinite tension limit yields an SDiff_3 invariant gauge theory in which $SO(7)$ is enhanced to $SO(8)$. In fact, the theory we get this way differs from the BLG theory only in the absence of space derivatives. Even though the fields depend on the $\mathbb{R}^2$ coordinates, there are no derivatives with respect to them; we thus find a 'Carrollian' limit [30, 31] of the BLG theory (see, e.g., [32] for a recent discussion of this limit).

We will begin with a summary of some essential details of the superspace geometry of 11-dimensional supergravity, and of the M5-brane action, and then proceed to our first result: the light-cone gauge fixed action for an M5-brane in the 11-dimensional Minkowski vacuum of M-theory. We then consider M5-branes of topology $\mathbb{R}^2 \times M_3$, partially gauge fix the SDiff_3 invariance, and show how a global $SO(8)$ and local SDiff_3 emerge in a $T \to \infty$ limit that involves rescaling fields and coordinates by powers of $T$ to have the dimensions expected of a BLG theory. We then summarize the results, explain their relation to the Carrollian limit of BLG theory, and speculate on possible extensions.

2. Superspace and M5 preliminaries

An (on-shell) supergravity background is determined by the supervielbein one-form $E^A = (E^\alpha, E^a)$ and the 3-form and 6-form potentials $C_3$ and $C_6$, subject to constraints on the torsion 2-form $T^a = DE^A$ and on the 4-form and 7-form field strengths $R_4$ and $R_7$. These constraints imply that the vector component of the torsion 2-form takes the form

$$T^a = -iE^\alpha \wedge E^\beta \Gamma_{a\beta},$$

and that

$$R_4 = dC_3 = E^\alpha \wedge E^\beta \wedge \bar{\Gamma}_{a}^{(2)} + \frac{1}{4!} E^\alpha \wedge \cdots \wedge E^{2\mu} F_{\mu_1 \cdots \mu_4},$$

$$R_7 = dC_6 + \frac{1}{2} C_3 \wedge dC_3 = iE^\alpha \wedge E^\beta \wedge \bar{\Gamma}_{a}^{(5)} + \frac{1}{7!} E^\alpha \wedge \cdots \wedge E^{2\mu} F_{\mu_1 \cdots \mu_7},$$

where

$$\bar{\Gamma}_{a}^{(n)} := \frac{1}{n!} E^\alpha \wedge \cdots \wedge E^{n\alpha} \Gamma_{a_1 \cdots a_n}.$$  (2.3)

Let $Z^M$ be local coordinates for the 11-dimensional superspace, and let $\xi^m$ be local coordinates for the M5-brane worldvolume. The embedding of the worldvolume in the superspace is described by coordinate functions $Z^M(\xi)$ that define a map from the worldvolume to the superspace. Differential forms on superspace may thereby be pulled back to the worldvolume. We will use the same notation for a superspace form and its pullback as the context should make it clear which is meant. Thus, the pullback of the supervielbein is

$$E^A = d\xi^m E^A_m, \quad E^A_m := \partial_m Z^M E^A_M.$$  (2.4)
and the induced worldvolume metric is \( g_{mn} = E_a E_b \eta_{ab} \), where \( \eta \) is the mostly minus Minkowski 11-metric.

As we will be using a Hamiltonian form of the M5-brane action, we set \( \xi^m = (t, \sigma^i) \) \((i = 1, 2, 3, 4, 5)\) and we write

\[
E^A = dt E_t^A + d\sigma^i E_i^A.
\]

The induced, positive definite metric on the five-dimensional ‘worldspace’ is

\[
5_{ij} = -E^a E^b \eta_{ab}.
\]

We denote by \( |5_g| \) the determinant of this metric. Similarly, the pullback of the 3-form potential is

\[
C_3 = \frac{1}{3!} d\xi^m \wedge d\xi^n \wedge d\xi^l C_{lnm} := \frac{1}{3!} dZ^M \wedge dZ^N \wedge dZ^K C_{KLM}(Z).
\]

This is used to construct the worldvolume 3-form field-strength \( H = dA - C_3 \) for the worldvolume 2-form potential \( A \) of the M5-brane.

We are now in a position to write down the Hamiltonian form of the M5-brane action. More precisely, we choose an intermediate form that requires only the introduction of a Lorentz-vector momentum variable \( P_a \), and a timespace split; for example,

\[
A = dt \wedge d\sigma^i A_0^i + \frac{1}{2} d\sigma^j \wedge d\sigma^i A_{ij}^M d\sigma^k \wedge d\sigma^j H_{ijk}.
\]

The feature of the action that results in the nonlinear self-duality of \( H \) is a constraint relating the variables canonically conjugate to \( A_{ij} \) to \( H_{ijk} \) [24], and then \( A_0^i \) appears in the action only through a surface term. The resulting Lagrangian density is

\[
\mathcal{L}_{M5} = P_a E^a + T \dot{Z}^M C_M - \frac{T}{8} \varepsilon^{ijklm} \dot{A}_{ij} \partial_k A_{lm} - \frac{\xi}{2} \left[ P_a \eta^{ab} P_b - T^2 |5_g| \left( 1 - \frac{1}{3!} H_{ijk} H^{ijk} \right) \right],
\]

where

\[
\mathcal{V}_I := \frac{1}{4! \sqrt{|5_g|}} \varepsilon^{ijklm} H_{ijk} H_{lmn},
\]

and

\[
C_M := \frac{1}{5!} \varepsilon^{ijklm} C_{ijklmM} + \frac{1}{4!} \varepsilon^{ijklm} (C_{ijkl} + 2 H_{ijk}) C_{lmM}.
\]

The variables \((t, \sigma^i)\) are the ‘lapse’ and ‘shift’ Lagrange multipliers for the Hamiltonian constraint and worldspace diffeomorphism constraints, respectively. Note that the ‘kinetic’ term for \( A \) is not manifestly gauge invariant but its gauge variation is a total derivative.

In this paper, we consider only the 11-dimensional Minkowski vacuum, for which

\[
E^a = dX^a - i d\Theta \Gamma^a \Theta,
\]

where \( \Theta \) is an \( SO(1, 10) \) Majorana spinor, so that

\[
\bar{\Theta} := \Theta^T \Gamma^0 = \Theta^T C,
\]

where \( \Theta^T \) is the transpose of \( \Theta \) (viewed as a column vector), and \( C \) is the (unitary) antisymmetric charge conjugation matrix. In a Majorana basis, the (unitary) Dirac matrices

\^8 In terms of the original covariant action [25], this is a consequence of the PST symmetry [27].
are pure imaginary; for example
\( \Gamma^0 = 1_{16} \otimes \sigma_2, \quad \Gamma^i = -1_{16} \otimes i \sigma_1, \quad \Gamma^I = -\gamma^I \otimes i \sigma_3 \quad (I = 1, \ldots, 9), \)

where \( 1_{16} \) is the \( 16 \times 16 \) identity matrix, and \( \gamma^I \) are the nine \( 16 \times 16 \) real symmetric \( SO(9) \) Dirac matrices, satisfying \( \{ \gamma^I, \gamma^J \} = 2\delta^{IJ} 1_{16} \). In this basis we may choose \( C = \Gamma^0 \), so that \( \Theta = 1_{32} \), is a real 32-component spinor.

3. Light-cone M5-brane

We choose coordinates such that the Minkowski 11-metric is
\[
\text{d}s^2_{11} = \text{d}X^{++} \text{d}X^{--} - \text{d}X^I \text{d}X^I, \quad (I = 1, \ldots, 9).
\]

The corresponding Dirac matrices, multiplied by the charge conjugation matrix, are
\[
C \Gamma^{++} = 2 \begin{pmatrix} 1_{16} & 0 \\ 0 & 0 \end{pmatrix}, \quad C \Gamma^{--} = 2 \begin{pmatrix} 0 & 0 \\ 0 & 1_{16} \end{pmatrix}, \quad C \Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \gamma^I & 0 \end{pmatrix}.
\]

In this basis,
\[
\Theta = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix},
\]

where \( \theta_{\pm} \) are 16 component real \( SO(9) \) spinors.

The light cone gauge is defined by
\[
X^{++} = t, \quad P_{--} = -\frac{T}{4} \tilde{e}, \quad \Gamma^{++} \Theta = 0,
\]

where \( \tilde{e} \) is the volume form for some (time-independent) ‘fiducial’ 5-metric admitted by whatever topology we choose for the M5-brane, and the factor of \( 1/4 \) is for later convenience.

The constraint on \( \Theta \) implies that \( \theta_- = 0 \), as a result of which
\[
E^{++} = 1, \quad E^{--} = \hat{X}^{--} - 2i\hat{\theta}_+^T \theta_+, \quad E^I = \hat{X}^I, \quad E^{++}_J = 0, \quad E^{--}_J = \partial_J \hat{X}^{--} - 2i\partial_J \theta_+^T \theta_+, \quad E^{++}_J = \partial_J \hat{X}^J,
\]

and the pullbacks of the superspace potentials are
\[
C_3 = -i dX^{++} \wedge dX^I \wedge d\theta^T_+ \wedge \gamma^I \theta_+, \quad C_6 = \frac{i}{4!} dX^{++} \wedge dX^I \wedge dX^K \wedge dX^L \wedge dX^M \wedge d\theta^T_+ \wedge \gamma^{IJKLM} \theta_+.
\]

Using these results, we find that \( C_M = \delta_M^+ C_+, \) where
\[
C_+ = \frac{i}{24} \varepsilon^{ijklm} \partial_i X^J \delta_j X^J \delta_k X^K \delta_l X^L (\delta_m \theta^T_+ \gamma^{IJKLM} \theta_+ + \frac{1}{2} \varepsilon^{ijklm} \partial_i X^J (\delta_j \theta^T_+ \gamma^{IJKLM} \theta_+) \partial_k A_{lm}).
\]

The M5 Lagrangian density now reads
\[
\mathcal{L}_{5} = \hat{X}^I P_I + \frac{i T}{2} \hat{e} \theta^T_+ \theta_+ - \frac{8 T}{8} \hat{A}_{ij} \varepsilon^{ijklm} \partial_k A_{lm} - \frac{T}{T} \frac{\mathcal{G}}{\tilde{e}} \left( 1 + \frac{1}{3!} H_{ijk} H^{ijk} \right) - \frac{P_I P_I}{T} + T C_+ + \tilde{e} s^I K_I + \frac{T}{4} X^{--} \partial_J (\tilde{e} s^I),
\]

\[\text{(3.1)}\]
where
\[ \tilde{e} K_i := \partial_i X^I P_I - \frac{T}{4!} \varepsilon_{ijklm} H_{ij} H_{lm} + \frac{iT}{2} \tilde{e} \partial_i \bar{\theta}_i \theta. \] (3.10)

The variable \( X^{-} \) is now a Lagrange multiplier imposing the constraint
\[ \partial_i (\bar{e} s') = 0. \] (3.11)

This condition is solved locally by
\[ \bar{e} s' = \varepsilon_{ijklm} \partial_j \Sigma_{klm}, \] (3.12)

where \( \Sigma_{klm} \) is the unconstrained Lagrange multiplier for the constraint \( \partial_i K_j = 0 \), defined up to an obvious Abelian gauge transformation. It imposes the vanishing of \( \partial_i K_j \), which generates (via Poisson brackets) the 5-volume-preserving diffeomorphisms of the canonical variables.

Rather than solving the constraint for \( s' \), we may proceed on the understanding that it is constrained by (3.11). We may then rewrite the Lagrangian density as
\[ \mathcal{L}_{M5} = D_t X^I P_I + \frac{iT}{2} \tilde{e} D_t \bar{\theta} \partial_e \theta - \frac{T}{4!} \varepsilon_{ijklm} (D_t A)_{ij} H_{klm} \]
\[ - \frac{P_I P_I}{T \tilde{e}} - \frac{T}{4!} \varepsilon_{ijklm} \left( 1 + \frac{1}{3!} H_{ij} H^{jk} \right) + T C_{++}, \] (3.13)

where \( D_t \) is a covariant time derivative:
\[ D_t X^i := \dot{X}^i + s^j \partial_j X^i, \quad D_t \theta = \dot{\theta} + s^i \partial_i \theta, \]
\[ (D_t A)_{ij} := \dot{A}_{ij} + s^k H_{kij}, \quad D_t P_I = \dot{P}_I + \partial_j (s^j P_I). \] (3.14)

The infinitesimal SDiff5 gauge transformations are
\[ \delta X^i = -\xi^i \partial_i X^t, \quad \delta \theta = -\xi^i \partial_i \theta, \quad \delta P_I = -\partial_i (\xi^i P), \]
\[ \delta A_{ij} = -\xi^k \partial_k A_{ij} + 2\partial_i \xi^k A_{jk}, \quad \delta s^i = \dot{\xi}^i + [s, \xi]^i, \] (3.15)

where \([,] \) is the Lie bracket of worldspace vector fields, and the vector parameter \( \xi \) satisfies
\[ \partial_i (\bar{e} \xi^i) = 0. \] (3.16)

Setting to zero all fermions and the gauge fields, for simplicity, and eliminating \( P_I \) as an auxiliary field, we arrive at an SDiff5-invariant Lagrangian density of the form
\[ \mathcal{L} = \frac{1}{4} T \tilde{e} |D_t X|^2 - \bar{e} V. \] (3.17)

The potential is
\[ V = \frac{T}{5!} \varepsilon_{ijklm} \left( \frac{T}{5!} \sum_{I,J,K,L,M} \{X^I, X^J, X^K, X^L, X^M\}^2, \right) \]
(3.18)

where
\[ \{X^I, X^J, X^K, X^L, X^M\} := \tilde{e}^{-1} \varepsilon_{ijklm} \partial_i X^I \partial_J X^L \partial_k X^K \partial_M X^L \partial_L X^L, \] (3.19)

which is a generalized Nambu bracket, itself a generalization of the Poisson bracket. Note that we define the bracket with the inverse of the ‘fiducial’ density \( \tilde{e} \) in order that it maps products of scalars to a scalar.\(^9\)

\(^9\) The analogous analysis for the M2-brane leads to a similar result but with a bilinear Poisson bracket instead of a multi-linear Nambu bracket. The Poisson bracket may again be defined with a factor of \( \tilde{e}^{-1} \), because this is consistent with the Jacobi identity, and it should be so defined in order that products of scalars get mapped to a scalar. To see the necessity of this factor, it suffices to consider a spherical M2-brane such that \( X^2 + Y^2 + Z^2 = 1 \); one finds that \( \{X, Y\}_{PB} = Z \), and cyclic permutations, only if \( \tilde{e} \) is the volume form on the unit sphere.
As a prelude to the procedure considered in the remainder of the paper, we now show how a rescaling of the variables by appropriate powers of the tension $T$ allows all dependence on $T$ to be factored out. Specifically, we set
\[ X^I = T^{-v} \tilde{X}^I, \quad P_I = T^{1-v} \tilde{P}_I, \quad A = T^{-v} \tilde{A}, \quad \theta_s = T^{-v} \tilde{\theta}_s, \] (3.20)
for arbitrary real constant $v$. The result is that
\[ I[X, P, A, \theta_s] = T^{1-2v} \tilde{I}[\tilde{X}, \tilde{P}, \tilde{A}, \tilde{\theta}_s], \] (3.21)
where $\tilde{I}$ is the action functional with $T$-dependent integrand $L_{MS}$ of (3.9) and $\tilde{I}$ is the same functional but with $T = 1$. For $v = 1/2$, the $T^{1-2v}$ prefactor is unity and the dimensions of the variables become the standard dimensions for fields in a six-dimensional spacetime.

4. Further gauge fixing and a hypertensile limit

We now suppose that the M5-brane has topology $\mathbb{R}^2 \times M_3$ for some compact 3-manifold $M_3$. This means that we may choose local transverse space coordinates $X^I$, and local worldspace coordinates $\sigma^\alpha$, such that
\[ X^I = (X^I, x^\alpha), \quad \hat{I} = 1, \ldots, 7, \quad \alpha = 1, 2, \] (4.1)
where $x^\alpha$ are Cartesian coordinates for $\mathbb{R}^2$. The fiducial worldspace density $\bar{e}$ should now be understood as a worldvolume density on $M_3$, independent of time and the $\mathbb{R}^2$ coordinates. We may also rewrite the invariant worldspace alternating tensor density:
\[ \epsilon^{ijklm} \rightarrow \epsilon^{\alpha\beta} \bar{\epsilon}^{\alpha\beta}. \] (4.2)

An implication of our partial gauge choice $X^\alpha = x^\alpha$ is that the manifest $SO(9)$ invariance is broken to a manifest $SO(7)$ invariance\(^{10}\). Accordingly, we split the $SO(9)$ Dirac matrices into reducible $SO(7)$ Dirac matrices $\gamma^I$ and the two matrices $\gamma^\alpha$, which are reducible (16 $\times$ 16) Dirac matrices for $\mathbb{R}^2$. These matrices have the anti-commutators
\[ [\gamma^I, \gamma^J]_s = \delta^{IJ} 1_{16}, \quad [\gamma^\alpha, \gamma^\beta]_s = \delta^{\alpha\beta} 1_{16}. \] (4.3)

Having eliminated the variables $X^\alpha$ by a gauge choice, we expect to be able to express the conjugate variables $P_\mu$ in terms of the remaining variables, and this will eventually be done. However, we postpone this step as it can be done more simply after we have settled other issues. One such issue is whether there is a ‘hidden’ linearly-realized $SO(8)$ invariance, as suggested by the fact that the $\mathbb{R}^2$ component of the worldvolume 2-form $A$ may be identified as an eighth scalar. As we shall see, there is an ‘enhancement’ of $SO(7)$ to $SO(8)$ but only in a particular infinite tension (hypertensile) limit that involves first rescaling the fields and coordinates. Another question is the nature of the residual gauge group. The problem is that $\bar{e}$ does not satisfy its own divergence-free condition, as would be expected for an SDiff\(_3\) gauge theory; instead, its divergence is related to the divergence of $s^\alpha$ by the constraint (3.11). As we shall see, this problem is resolved in the hypertensile limit.

We first consider a rescaling of the fields of the type (3.20). Leaving aside the 2-form potential for the moment, this means that
\[ X^I = T^{-v} \phi^I, \quad P_I = T^{1-v} \pi_I, \quad \theta_s = T^{-v} \Psi, \] (4.4)
for new variables $(\phi^I, \pi_I, \Psi)$. As a consequence of this rescaling,
\[ X^I P_I = \frac{iT}{2} \bar{e} \partial_s^T \theta_s = T^{1-2v} \left[ \phi^I \pi_I + \frac{i}{2} \bar{e} \Psi^T \Psi \right]. \] (4.5)

\(^{10}\) There is still an $SO(9)$ invariance, of course, but it becomes part of the nonlinearly realized $SO(1, 10)$ invariance.
Ultimately, we will choose
\[ \nu = 1/4 \] (4.6)
because this yields dimensions for the rescaled fields that are appropriate for a conformal theory in (1 + 2) dimensions; recall that \( T \) has mass dimension \( [T] = 6 \) in fundamental units. This choice leads to an overall factor of \( \sqrt{T} \) but this can be cancelled, in the action, by a rescaling of the \( M_3 \) coordinates. This requirement fixes the rescaling of the \( M_3 \) coordinates for the choice \( \nu = 1/4 \) but we will find it convenient to retain \( \nu \) as a free positive parameter on the understanding that \( \nu = 1/4 \) will be our ultimate choice. For other values of \( \nu \) the rescaling of the \( M_3 \) coordinates can be fixed by requiring that all leading terms in the Lagrangian density for large \( T \) appear with the same factor of \( T^{1-2\nu} \). As we shall verify, this happens when the \( M_3 \) coordinates are rescaled such that
\[ \partial_a = T^{2\nu/3} \partial_a. \] (4.7)
For \( \nu = 1/4 \) the rescaled coordinates are dimensionless and \( d^3\sigma = (1/\sqrt{T})d^3\tilde{\sigma} \), as required. If we now define a new pair of variables \( (\phi^5, \pi_8) \) by
\[ \frac{1}{2} \varepsilon^{\alpha\beta} \tilde{A}_{\alpha\beta} = T^{-\nu} \phi^5, \quad \frac{1}{3!} \varepsilon^{\alpha\beta\gamma} \tilde{H}_{\alpha\beta\gamma} = -T^{-\nu} \pi_8, \] (4.8)
and rescaled mixed and \( M_3 \) components of \( A \) by
\[ A_{\alpha\beta} = T^{-4\nu/3} b_{\alpha\beta}, \quad \tilde{A}_{\alpha\beta} = T^{-2\nu/3} \tilde{A}_{\alpha\beta}, \] (4.9)
then
\[ -\frac{T}{8} \varepsilon_{ijklm} \tilde{A}_{ij} \partial_k \tilde{A}_{lm} = T^{1-2\nu} \left[ \phi^5 \pi_8 - \frac{1}{2} \varepsilon^{\alpha\beta} \tilde{b}_{\alpha\beta} \varepsilon^{\alpha\beta\gamma} (\partial_\gamma \tilde{b}_{\beta\gamma} - \partial_\beta \tilde{A}_{\beta\gamma}) \right]. \] (4.10)
The \( \phi^5 \pi_8 \) term provides an \( SO(8) \) completion of (4.5). The second term transforms into a total derivative under the Abelian gauge transformation
\[ \delta b_{\alpha\beta} = \partial_\alpha \lambda_\beta, \quad \delta \tilde{A}_{\alpha\beta} = 2 \partial_\alpha \lambda_\beta, \] (4.11)
which is an invariance of the field strengths
\[ \tilde{H}_{\alpha\beta\gamma} := \tilde{\partial}_\alpha b_{\beta\gamma} - \tilde{\partial}_\beta b_{\gamma\alpha} - \tilde{\partial}_\gamma \tilde{A}_{\alpha\beta}, \quad \tilde{H}_{\alpha\beta\gamma} := 3 \tilde{\partial}_\alpha \tilde{A}_{\beta\gamma}. \] (4.12)
The rescalings conspire such that \( H_{\alpha\beta\gamma} = \tilde{H}_{\alpha\beta\gamma} \), from which it follows that
\[ \frac{1}{3!} \varepsilon^{\alpha\beta\gamma} \tilde{H}_{\alpha\beta\gamma} = -T^{-\nu} \pi_8. \] (4.13)
This tells us, first, that we may trade the gauge-invariant part of \( \tilde{A}_{\alpha\beta} \) for \( \pi_8 \) and, second, that this trade involves a factor of \( T^{-\nu} \); in other words, there is a 3-vector field \( \Lambda \) such that
\[ \tilde{A}_{\alpha\beta} = 2 \partial_\alpha \Lambda_\beta + O(T^{-\nu}). \] (4.14)
In principle, the \( O(T^{-\nu}) \) term can be expressed in terms of \( \pi_8 \), but this expression would be non-local on \( M_3 \). In addition, it is unclear how it could be part of some \( SO(8) \) invariant term. For this reason, among others that we will encounter later, we shall be considering a \( T \to \infty \) limit, and in this spirit we write
\[ \tilde{H}_{\alpha\beta\gamma} = \tilde{\partial}_\alpha (b_{\beta\gamma} - \partial_\gamma \Lambda_\beta) - \tilde{\partial}_\beta (b_{\gamma\alpha} - \partial_\gamma \Lambda_\beta) + O(T^{-\nu}). \] (4.15)
We see that \( \Lambda \) is a St"uckelberg field: invariance of \( \tilde{H}_{\alpha\beta\gamma} \) under the Abelian gauge transformation \( \delta b_{\alpha\beta} = \partial_\alpha \lambda_\beta \) is ensured by the ‘shift’ \( \delta \Lambda_\alpha = \lambda_\alpha \). This gauge invariance is therefore ‘spontaneously’ broken by the St"uckelberg mechanism; we may choose to set
\[ \Lambda = 0, \text{ thereby ‘fixing’ this gauge freedom}^{11}. \text{ Note, however, that there is still an unbroken Abelian gauge invariance of } b \text{ viewed as a } 2\text{-vector-valued gauge potential on } M_3. \]

With the rescalings as given, we also have
\[ H_{a \dot{b} \gamma} = T^{-2/3} \tilde{H}_{a \dot{b} \gamma}, \]
and hence the components of \( K_\alpha \), defined in (3.10), take the form
\[ K_\alpha = T^{1-v/3} [\tilde{K}_\alpha + O(T^{-v})], \]
\[ \dot{K}_\alpha = T^{1-v}[\tilde{\dot{K}}_\alpha + O(T^{-v})], \]
where, for the gauge choice \( \Lambda = 0 \),
\[ \tilde{K}_\alpha = \tilde{\tilde{K}}_\alpha = \tilde{\tilde{\dot{K}}}_\alpha = 0, \]
\[ \tilde{\dot{K}}_\alpha = \tilde{\dot{\tilde{K}}}_\alpha = T^{-1} \tilde{\dot{P}}_a. \]
Note the absence of fermionic terms in \( \tilde{K}_\alpha \); for finite \( T \) they appear with a \( \partial_\alpha \) derivative but are suppressed in the \( T \to \infty \) limit. As we discuss below, the disappearance of \( \Re^2 \) derivatives is a general effect of the limit we consider.

We are now in a position to see how an SDiff\(3 \) gauge group will emerge in the \( T \to \infty \) limit.

First, we define rescaled shift functions by
\[ s^a = T^{-2/3} \tilde{s}^a, \quad s^\alpha = T^{-v} \tilde{s}^\alpha. \]
As a consequence, we have
\[ s^a \tilde{K}_a = T^{1-2v/3} [\tilde{s}^a \tilde{\tilde{K}}_a + \tilde{s}^\alpha \tilde{\dot{K}}_\alpha + O(T^{-v})], \]
and
\[ \partial_\alpha (\tilde{s}^a \tilde{\tilde{K}}_a) = \tilde{\tilde{\partial}}_\alpha (\tilde{s}^a) + O(T^{-v}). \]
In the \( T \to \infty \) limit, the rescaled shift-function components \( \tilde{s}^a \) satisfy a divergence-free condition, and the components \( \tilde{s}^\alpha \) become unconstrained Lagrange multipliers that impose the constraint \( \tilde{K}_\alpha = 0 \), which is trivially solved for \( \tilde{\dot{P}}_a \). Thus,
\[ \tilde{\dot{P}}_a = -\tilde{\varepsilon}^{a \beta \gamma} \tilde{\partial}_\beta \phi^\gamma \tilde{\partial}_\gamma b_{a \nu} + O(T^{-v}). \]

Now we turn to the terms in the Hamiltonian. One such term is
\[ \frac{P_1 P_I}{T \tilde{e}} = T^{1-2v} [\tilde{e}^{-1} \pi_I \pi_I + \tilde{e}^{-1} \tilde{\dot{P}}_a \tilde{\dot{P}}_a], \]
Substitution for \( \tilde{\dot{P}}_a \) yields a term that is not \( SO(8) \) invariant but we still have many other terms to consider. For example,
\[ \tilde{e}^{-1} T^1 g^1 = T^{1-2v} [\tilde{e}^{-1} \det(\tilde{\partial}_a \phi^I \tilde{\partial}_I \phi^A)] + O(T^{-2v}) \]
\[ = T^{1-2v} \left[ \frac{1}{3!} \tilde{e}(\phi^I \phi^J \phi^K)^2 + O(T^{-2v}) \right], \]
where we have defined the Nambu bracket of functions \( (F, G, H) \) by
\[ \{F, G, H\} = \tilde{e}^{-1} \tilde{\varepsilon}^{a \beta \gamma} \tilde{\partial}_a F \tilde{\partial}_\beta G \tilde{\partial}_\gamma H. \]

Omitting the overall power of \( T \), which is the same as in (4.10), we see that the leading term in the \( T \to \infty \) limit is an \( SO(7) \)-invariant potential that can be expressed as a sum of squares of Nambu brackets for the seven scalar fields \( \phi^I \).

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11 Alternatively, one can just write all the formulae to follow in terms of \( \bar{b}_{a \dot{b}} := b_{a \dot{b}} - \partial_a \Lambda_{\dot{b}} \).
A similar computation yields
\[
\frac{T^5 g}{3! e^T} H_{i j k} H^{i j k} = T^{1 - 2 e} \left[ e^{-1} \pi^2 + e^{-1} (e^{\alpha \beta \gamma} \partial_\alpha \phi^I \partial_\beta \phi^J \partial_\gamma \phi^J) + \frac{1}{2} \bar{\psi} \{ \phi^I, \phi^J \}^2 + O(T^{-2}) \right].
\]
(4.27)

The first term on the right-hand side provides the \( SO(8) \) completion of the \( \pi^2 \) term in (4.24), and the second term provides the \( SO(8) \) completion of the \( \tilde{P}_{ij}^2 \) term in (4.24). The third term provides the \( SO(8) \) completion of the leading, potential, term of (4.25).

We have now found all terms of leading order in an expansion in inverse powers of \( T \) that survive the truncation in which all fermion terms are omitted. The \( SO(8) \) invariance of this bosonic truncation can be made manifest by defining an \( SO(8) \)-vector valued scalar field \( \Phi \), and its conjugate momentum \( \Pi \), by
\[
\Phi^I = (\phi^I, \phi^8), \quad \Pi_I = (\pi_I, \pi_8).
\]
(4.28)

We shall postpone a presentation of the manifestly \( SO(8) \) invariant results in this notation until we have dealt with the fermion terms.

### 4.1. Fermions

We have already seen in (4.5) that there is a fermion bilinear ‘kinetic’ term, and in (4.18) that there is a fermion bilinear contribution to the SDiff₃ constraint function. These fermion terms are manifestly \( SO(8) \), in fact \( SO(9) \), invariant. However, we still have to consider the fermion bilinear \( T \mathcal{C}_{++} \). In our rescaled variables this becomes
\[
T \mathcal{C}_{++} = T^{1 - 2 e} \left[ i e^{a \beta \gamma} \partial_\alpha \bar{\phi} \bar{\phi}^\alpha \bar{\phi} \psi_T \gamma^\beta \psi + \frac{1}{2} e^{a \beta \gamma} \partial_\alpha \bar{\phi} \bar{\phi}^\alpha \bar{\phi} \psi_T \gamma^\beta \psi \right.
\]
\[
+ i e^{a \beta \gamma} \partial_\alpha \bar{\phi} \bar{\phi}^\alpha \bar{\phi} \psi_T \gamma^\beta \psi + O(T^{-2}) \right],
\]
(4.29)

where
\[
\gamma^I = \left( \begin{array}{cc} 0 & \rho^I_{AB} \\ \bar{\rho}^I_{AB} & 0 \end{array} \right), \quad \gamma^9 = \left( \begin{array}{cc} \delta_{AB} & 0 \\ 0 & -\delta_{AB} \end{array} \right).
\]
(4.30)

This is not obviously \( SO(8) \) invariant. To show that it is \( SO(8) \)-invariant, we must first decompose the \( SO(9) \) spinor \( \psi \) into its irreducible \( SO(8) \) representations. To do this we choose the \( SO(9) \) gamma matrices \( \gamma^I = (\gamma^I, \gamma^9) \) to be
\[
\gamma^I = \left( \begin{array}{c} 0 \\ \rho^I_{AB} \end{array} \right), \quad \gamma^9 = \left( \begin{array}{cc} \delta_{AB} & 0 \\ 0 & -\delta_{AB} \end{array} \right),
\]
(4.31)

where \( \rho^I \) are \( SO(8) \) ’sigma’ matrices, with transpose \( \tilde{\rho}^I \); i.e. \( \tilde{\rho}^I_{AB} := \rho^I_{BA} \). These \( 8 \times 8 \) matrices satisfy
\[
\rho^I \tilde{\rho}^J + \rho^J \tilde{\rho}^I = 2 \delta^{IJ} 1_s, \quad \tilde{\rho}^I \rho^J + \rho^J \tilde{\rho}^I = 2 \delta^{IJ} 1_c,
\]
(4.32)

where \( 1_s \) and \( 1_c \) are the identity matrices acting on \( 8 \), and \( 8 \), spinors. The \( SO(8) \) generators acting on \( 8 \), and \( 8 \), spinors are
\[
\rho^{IJ}_{AB} := (\rho^I \rho^J)^{AB}, \quad \tilde{\rho}^{IJ}_{AB} := (\tilde{\rho}^I \rho^J)^{AB}.
\]
(4.33)

\(^{12}\) It is useful to keep in mind the \( SO(7) \) invariant representation for the \( SO(8) \) sigma matrices in terms of octonionic structure constants, for which \( \rho^8_{AB} = \delta_{AB} \) (see, e.g., [33]).
In this basis, an $SO(9)$ spinor $\Psi$ takes the form
\[ \Psi = \begin{pmatrix} \chi_A \\ \bar{\chi}_A \end{pmatrix}, \] (4.34)
where $\chi$ and $\bar{\chi}$ are $SO(8)$ spinors in, respectively, the $\bar{8}$ and $8$ representations. We may trade these spinors for a doublet of $8$-plet, which we may view as an $8$-plet of real $SO(1,2)$ spinors,
\[ \psi_A = \begin{pmatrix} \chi_A \\ (\rho^8 \chi)_A \end{pmatrix} \quad (A = 1, \ldots, 8). \] (4.35)

To facilitate this new interpretation, we introduce *irreducible* $2 \times 2$ Dirac matrices $\tilde{\gamma}^\mu$ satisfying
\[ [\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_\tau = \eta^{\mu\nu}, \quad (\mu = 0, 1, 2). \] (4.36)
For a ‘mostly minus’ signature convention (in accord with the choice made for the 11-dimensional Minkowski metric) these matrices are pure imaginary in a Majorana basis. A convenient choice is
\[ \tilde{\gamma}^0 = \tau_2, \quad \tilde{\gamma}^1 = i \tau_1, \quad \tilde{\gamma}^2 = i \tau_3 \] (4.37)
where $(\tau_1, \tau_2, \tau_3)$ are the Hermitian Pauli matrices. The $2 \times 2$ charge conjugation matrix $c$ can be chosen to be $\tilde{\gamma}^0$, in which case $c \tilde{\gamma}^0 = 1_2$ and both $c \tilde{\gamma}^1 = \tau_1$ and $c \tilde{\gamma}^2 = -\tau_1$ are real symmetric matrices. For this choice, the Dirac conjugate of an $SO(1,2)$ Majorana spinor $\psi$ is
\[ \bar{\psi} = \psi^t \tilde{\gamma}^0, \] (4.38)
where the superfix $t$ indicates the transpose of the 2-component spinor. As $\rho^8$ squares to the identity, the ‘kinetic’ term for $\Psi$ can now be written as
\[ \frac{1}{2} \Psi^T \Psi = -\frac{1}{2} \bar{\psi}_A \tilde{\gamma}^0 \psi_A. \] (4.39)

In this new notation, (4.29) becomes
\[ T_{\varepsilon^{+}} = T_{1^{-2 \varepsilon}} \varepsilon^{\alpha \beta} \left[ \tilde{\partial}_a b_{a\beta} (\tilde{\partial}_f \tilde{\psi}_A \tilde{\psi}^a \psi_A) - \frac{1}{2} \tilde{\partial}_a \phi^f \tilde{\partial}_f \phi^f \tilde{\partial}_f \tilde{\psi}_A \rho^f_{AB} \psi_B + O(T^{-\nu}) \right]. \] (4.40)

Finally, we can now rewrite the expression (4.18) for $\tilde{K}_a$ in manifestly $SO(8)$ invariant form as
\[ \tilde{e} \tilde{K}_a = \tilde{\partial}_a \phi^f \Pi_f - \frac{1}{2} \varepsilon^{\alpha \beta \gamma} \tilde{\partial}_a b_{a\beta} - \tilde{\partial}_f b_{f\alpha} \tilde{\partial}_f \tilde{\psi}_A \rho^f_{AB} \psi_B - \frac{1}{2} \tilde{\psi}_A \tilde{\psi}_A \tilde{\partial}_a \psi_A. \] (4.41)

5. Carrollian BLG

We have now shown that the Lagrangian density of the light-cone gauge fixed M5-brane can be written, after some further gauge fixing, in the form
\[ L_{M5} = T_{1^{-2 \varepsilon}} \left[ L + O(T^{-\nu}) \right] \] (5.1)
where $L$ is an $SO(8)$ invariant constructed from rescaled fields that are functions of coordinates $x^\mu = (t, x^a)$ of a three-dimensional Minkowski spacetime, and rescaled coordinates $\tilde{\sigma}^a$ for a compact 3-space. As we observed previously, the overall factor of $T_{1^{-2 \varepsilon}}$ cancels from the action $I_{M5}$ when $\nu = 1/4$, and in this case we can define
\[ I := \lim_{T \to \infty} I_{M5} = \int d^3x \left[ d^3 \tilde{\sigma} \tilde{L} \right]. \] (5.2)
Putting together the results of the previous section, we see that
\[ \mathcal{L} = \Phi^I \Pi_I - \frac{i}{2} \bar{\psi}_A \gamma^0 \psi_A - \frac{1}{2} e^{a\beta\gamma} \bar{\psi}_A \gamma^a \phi_{\alpha \beta} + \bar{\psi}_A \tilde{K}_a - \tilde{\mathcal{H}} \]  
(5.3)

where
\[ \tilde{\mathcal{H}} = \frac{1}{\bar{\epsilon}} \left[ \Pi_I + (e^{a\beta\gamma} \bar{\phi}_a \Phi^I \bar{\phi}_a \phi_{\alpha \beta})^2 \right] + \frac{1}{3!} \bar{\epsilon} \sum_{I,J,K} \{ \phi^I, \phi^J, \phi^K \}^2 \]
\[ + i e^{a\beta\gamma} \bar{\phi}_a b_{\alpha \beta} (\bar{\phi}_a \psi_A \gamma^a \psi_A) - \frac{1}{2} \bar{\epsilon} \left[ \phi^I, \phi^J, \psi_A \right] \rho_{AB} \psi_B. \]
(5.4)

The constraint function \( K_a \) is given by (4.41) and the Lagrange multiplier for this constraint satisfies
\[ \tilde{\phi}_a (\bar{\epsilon} \gamma^a) = 0, \]
(5.5)

which we may solve, locally, by writing
\[ \bar{\epsilon} \gamma^a = -2 B^a, \quad B^a := e^{a\beta\gamma} \tilde{\phi}_a b_{\alpha \beta}. \]
(5.6)

As we shall see shortly, the unconstrained Lagrange multiplier \( b_{\alpha \beta} \), which is only defined up to a gauge transformation with \( \delta b_{\alpha \beta} = \tilde{\lambda}_{\alpha \beta} \), may be combined with \( b_{\alpha \beta} \) to form a 3-vector valued \( SO(1,2) \)-vector \( b_{\mu \alpha \beta} \), and similarly for \( B^a \), which is the time component of a 3-vector valued \( SO(1,2) \)-vector \( B^\nu \).

We now rewrite the Lagrangian density as
\[ \mathcal{L} = D_t \Phi^I \Pi_I - \frac{i}{2} \bar{\psi}_A \gamma^0 D_t \psi_A - \tilde{\mathcal{H}} + \mathcal{L}_{CS}, \]
(5.7)

where
\[ \mathcal{L}_{CS} = -\frac{1}{\bar{\epsilon}} e^{a\beta\gamma} \tilde{\phi}_a [\bar{\phi}_a - 2 \bar{\epsilon} e^{a\beta\gamma} \tilde{\phi}_a \bar{\phi}_a b_{\alpha \beta} (\bar{\phi}_a b_{\alpha \beta} - \bar{\phi}_a b_{\alpha \beta})] \tilde{\phi}_a b_{\beta \gamma}. \]
(5.8)

The ‘CS’ subscript will be explained in the subsection to follow. The SDiff\( _3 \) covariant time derivative \( D_t \) is now defined, locally, on any of the fields (\( \Phi^I, \psi_A \)), which we denote collectively by \( \Xi \), as
\[ D_t \Xi := \partial_t \Xi + \tilde{\phi}_a \partial_a \Xi = \partial_t \Xi - 2 \bar{\epsilon} e^{a\beta\gamma} \tilde{\phi}_a \bar{\phi}_a b_{\alpha \beta} \Xi. \]
(5.9)

Conspicuous by their absence are any \( \mathbb{R}^2 \) derivatives of any of the fields, scalar, spinor or gauge. In the Nambu bracket realization of the BLG theory [3], these derivatives occur together with minimal coupling terms, such that both that are taken into account via the ‘covariant derivative’ [20]
\[ D_\alpha \Xi = \partial_\alpha \Xi - 2 \bar{\epsilon} e^{a\beta\gamma} \tilde{\phi}_a \bar{\phi}_a b_{\alpha \beta} \Xi = \partial_\alpha \Xi - 2 \bar{\epsilon} e^{a\beta\gamma} \tilde{\phi}_a \bar{\phi}_a b_{\alpha \beta} (\Xi, b_{\alpha \beta}, \sigma^a). \]
(5.10)

The ‘connection’ terms in this derivative yield the terms in (5.3) that couple \( b \) to \( \Phi \) and \( \psi \). As can be seen by comparison with (5.9), the covariant derivatives \( (D_t, D_\alpha) \) are the components of the 3-vector covariant derivative
\[ D_\mu \Xi = \partial_\mu \Xi - 2 \{ \Xi, b_{\mu \alpha}, \sigma^a \}, \quad b_{\mu \alpha} = (b_{t \alpha}, b_{\alpha \beta}). \]
(5.11)

If we now eliminate the 8-momentum, we arrive at the Lagrangian density
\[ \mathcal{L} = \frac{\bar{\epsilon}}{4} |D_t \Phi|^2 - |(D_\alpha - \partial_\alpha) \Phi|^2] + \mathcal{L}_{CS} - \frac{i}{2} \bar{\psi}_A \gamma^0 D_t \psi_A + \bar{\psi}_A \gamma^a (D_\alpha - \partial_\alpha) \psi_A \]
\[ - \frac{\bar{\epsilon}}{3!} \sum_{I,J,K} \{ \Phi^I, \Phi^J, \Phi^K \}^2 + \frac{i}{2} \bar{\psi}_A \gamma^a (\Phi^I, \Phi^J, \psi_A) \rho_{AB} \psi_B. \]
(5.12)
where we now recognize \( D_\mu = (D_t, D_\alpha) \) as an SDiff_3 covariant derivative (we elaborate on the SDiff_3 gauge invariance in the subsection to follow). Leaving aside the \( \mathcal{L}_{CS} \) term, it is now clear that we have found a Lagrangian density for a three-dimensional gauge theory in which the \( SO(1, 2) \) Lorentz invariance is broken by the subtraction of all terms with space derivatives. This is also true of the ‘CS’ term, as may be seen by considering the manifestly \( SO(1, 2) \)-invariant Lagrangian density [21]

\[
\mathcal{L}_{CS} = -\frac{1}{2} \varepsilon^{\mu\nu\rho} \left[ \partial_\mu b_{\nu\rho} + \frac{2}{3\varepsilon} \varepsilon_{\alpha\beta\gamma} B^{\alpha}_\mu B^\beta_\rho \right] B^\gamma_\rho, \tag{5.13}
\]

where, by definition,

\[
B^\alpha_\mu := \varepsilon^{\alpha\beta\gamma} \partial_\beta b_{\mu\gamma}, \quad \varepsilon^{\alpha\beta\gamma} \varepsilon_{\delta\epsilon\zeta} = 3! \delta^{\alpha\beta\gamma}_{\delta\epsilon\zeta}. \tag{5.14}
\]

It is straightforward to show, ignoring total space derivatives, that

\[
\mathcal{L}_{CS} = \tilde{\mathcal{L}}_{CS} - b_\alpha \varepsilon^{\alpha\beta\gamma} \partial_\beta B^\gamma_\rho. \tag{5.15}
\]

We have now shown that the dynamics of an M5-brane of topology \( \mathbb{R}^2 \times M_3 \) is governed, in the particular hypertensile limit that we have taken, by a \((1 + 2)\)-dimensional field theory. The fields of this theory, which are tensor-valued in an ‘auxiliary’ closed 3-manifold \( M_3 \), and multiplets of \( SO(8) \), consist of an \( \mathfrak{g}_8 \)-plet of scalar fields \( (\Phi) \), an \( \mathfrak{h}_8 \)-plet of two-component real \( \mathbb{H}(2, \mathbb{R}) \) spinor fields \( (\psi) \), both scalars on \( M_4 \), and an \( SO(8) \)-singlet vector field \( b \) that is also a vector potential on \( M_4 \). This field theory is precisely the BLG theory in its Nambu bracket realization except that all space derivatives are absent. The relativistic, \( SO(1, 2) \) invariance is broken down to \( SO(2) \) by this absence of spatial derivatives. It should not be thought that we have found a dimensional reduction of the BLG theory to one dimension (time) because the fields were never assumed to be independent of the \( \mathbb{R}^2 \) coordinates. Instead, what we have found is the Carrollian limit of the Nambu-bracket BLG theory, in which the speed of light has been taken to zero; this has precisely the effect of suppressing all spatial derivatives.

### 5.1. SDiff_3 gauge invariance

The BLG theory is an SDiff_3 gauge theory because it is invariant under the gauge transformations\(^{13}\)

\[
\delta \Xi = -\tilde{\zeta}^a \partial_\alpha \Xi, \quad \delta b_\alpha = d\omega_\alpha - \tilde{\zeta}^\beta \partial_\beta b_\alpha - \bar{\delta}_a \bar{\zeta}^\beta b_\beta, \tag{5.16}
\]

where

\[
\tilde{\zeta}^a = -2\varepsilon^{-1} \varepsilon^{\alpha\beta\gamma} \partial_\beta \omega_\gamma. \tag{5.17}
\]

This defines \( \omega_\beta \) in terms of \( \tilde{\zeta}^a \) only up to the addition of \( \bar{\delta}_a \zeta \), for any scalar \( \zeta \), but this addition leads to an SDiff_3 transformation of \( b_\alpha \) that is equivalent to the one given once account is taken of the unbroken Abelian gauge invariance of \( b_\alpha \). Note that

\[
\bar{\delta}_a (\bar{e} \tilde{\zeta}^a) = 0. \tag{5.18}
\]

Let \( B^a = dx^a B^a_\mu \), where \( B^a_\mu \) is as defined in (5.14); then

\[
\delta (-2\varepsilon^{-1} B^a) = d\tilde{\zeta}^a + (-2\varepsilon^{-1} B^\beta) \partial_\beta \tilde{\zeta}^a - \bar{\delta}_a \bar{\zeta}^\beta (-2\varepsilon^{-1} B^\beta). \tag{5.19}
\]

Recalling (5.6), we see that this includes the transformation

\[
\tilde{\zeta}^a = d\tilde{\zeta}^a + [\bar{\zeta}, \tilde{\zeta}]^a, \tag{5.20}
\]

\(^{13}\) We recall that \( \Xi \) stands collectively for the fields \( (\Phi^I, \psi_A) \).
where \([,]\) is the Lie bracket of vector fields on \(M_3\). This transformation follows from the \(\text{SDiff}_3\) transformation of \(s^i\) given in (3.15) after taking the hypertensile limit described in section 4 with

\[
\xi^\alpha = T^{-\frac{2}{3}} \tilde{\xi}^\alpha. \tag{5.21}
\]

The \(\text{SDiff}_3\) transformation of the space components \(b_{\alpha\dot{\alpha}}\) of \(b_{\dot{\alpha}}\) may be similarly deduced from those of \(A\) given in (3.15), and the result (in the gauge \(\Lambda_{\dot{a}} = 0\), agrees with (5.16)) if one drops the spatial derivative term \(\partial_\alpha \omega_{\dot{\alpha}}\).

With the exception of the \(\mathcal{L}_{\text{CS}}\) term, the \(\text{SDiff}_3\) gauge invariance of all terms of the BLG action is manifest (because it is constructed using the covariant derivative \(D\)). The \(\mathcal{L}_{\text{CS}}\) term is a type of Chern–Simons term in the sense that its variation is a total spacetime derivative\(^{14}\). Alternatively, one may verify the covariance of the functional derivative with respect to \(b_{\dot{\alpha}}\) of the ‘CS’ action functional obtained by integration of \(\mathcal{L}_{\text{CS}}\). This functional derivative is proportional to the field-strength 2-form\(^{15}\)

\[
F^\alpha := dB^\alpha + 2B^\beta \delta^\alpha_{\beta}(\tilde{e}^{-1} B^\beta). \tag{5.22}
\]

Note that this field-strength 2-form is not of Yang–Mills type, because we are dealing with an ‘exotic’ gauge theory, and \(\mathcal{L}_{\text{CS}}\) is therefore not a Chern–Simons term in the usual sense of this term. However, it has the properties required for \(\text{SDiff}_3\) invariance because the transformation of \(B\) induces the transformation\(^{16}\)

\[
\delta F^\alpha = 2\rho^\beta \delta^\alpha_{\beta}(\tilde{e}^{-1} F^\beta) - 2 F^\beta \delta^\alpha_{\beta}(\tilde{e}^{-1} \rho^\beta). \tag{5.23}
\]

In other words, \(F^\alpha\) transforms covariantly under an \(\text{SDiff}_3\) gauge-transformation, as claimed. In particular, the equation \(F = 0\) is \(\text{SDiff}_3\) invariant.

6. Conclusions

We have presented the light-cone gauge fixed action for the M5-brane in the 11-dimensional Minkowski vacuum of M-theory. As expected from earlier results, it has an ‘exotic’ \(\text{SDiff}_5\) gauge invariance. By considering an M5-brane of topology \(\mathbb{R}^2 \times M_3\), for some closed 3-manifold \(M_3\), we found a \((1 + 2)\)-dimensional Minkowski space field theory, which is plausibly related to the recent ‘BLG’ multiple M2-brane model because an M5-brane may contain ‘dissolved’ M2-branes. Crucially, the BLG model has an \(SO(8)\) invariance whereas only an \(SO(7)\) invariance is guaranteed by the M5 construction. We found a limit, formally one of infinite M5 tension \(T\) although the fields and coordinates were first scaled by powers of \(T\), in which the \(SO(7)\) invariance is enhanced to \(SO(8)\). In the same limit the partially gauge-fixed \(\text{SDiff}_5\) invariance is reduced to an \(\text{SDiff}_3\) invariance and a BLG-like theory emerges, complete with the expected potential term. However, the limit also suppresses \(\mathbb{R}^2\) derivatives. Starting from the BLG theory, one can achieve the same suppression of spatial derivatives by taking a ‘Carrollian’ limit, in which limit the speed of light is zero. We should point out that our Carrollian limit of the (super)conformal BLG theory is not itself conformal, although it is likely invariant under the contraction of the (super)conformal group that is implied by the contraction of its Lorentz subgroup to the Carroll group.

An interesting fact (which we passed over previously for the sake of simplicity of presentation) is that essentially the same results may be obtained by a zero tension limit if one chooses the parameter \(\nu\) defining the various rescalings to be negative. In this case, the

\(^{14}\) We may ignore total \(M_3\) derivatives since \(M_3\) has no boundary.

\(^{15}\) Here, as in (2.2), we use the differential form conventions with exterior derivative acting ‘from the right’.

\(^{16}\) Recall that \(\tilde{e}\) is a density on \(M_3\) which is constant in 2-space and time, so that \(d\tilde{e} = 0\).
fields and coordinates have ‘peculiar’ dimensions and the overall factor of $T_1^{-2\nu}$ multiplying the leading term in the Lagrangian density does not cancel in the action; one must rescale the action before taking the $T \to 0$ limit (as is done to define the Virasoro generators and BRST charge of a tensionless limit of the string in [34]). We do not know whether this fact is of any significance but, in light of it, it is worth recalling that the Carroll group arises naturally as the symmetry of a null brane in one higher dimension [32].

An obvious question is whether there is some other limit in which precisely the BLG theory emerges. We cannot say for sure but we consider this unlikely for various reasons. To start with, the symmetry algebra of the M5-brane in the Minkowski vacuum of M-theory is an 11-dimensional super-Poincaré symmetry with tensor charges, and neither this algebra nor any of its contractions contains the algebra of $OSp(8|4)$, which is the symmetry supergroup of the BLG theory. From this viewpoint, a better starting point might be an M5-brane in the $adS_4 \times S^7$ vacuum of M-theory, because an M5-brane in this background is $OSp(8|4)$ invariant [35], but it remains to be seen whether this will work. If it does, then it is likely that the limit of infinite $adS$ radius, in which the $adS_4 \times S^7$ vacuum degenerates to the 11-dimensional Minkowski vacuum, will correspond to the Carrollian limit of the ‘holographic’ BLG theory.

Another obvious question is whether analogous results might emerge by considering M5-branes of other topologies, for example $S^1 \times M_4$ for some closed 4-manifold $M_4$. One might imagine that this could be related to some ‘exotic’ $(1+1)$-dimensional gauge theory based on a Filippov 4-algebra. However, all we were able to find was a version of the $D4$-brane action in which all fields depend on a fifth space coordinate, but without derivatives with respect to it. Another possibility is an M5-brane of topology $R^3 \times M_2$; in this case there are many possibilities for rescaling fields and therefore, potentially, there are many possible limits. We hope to report on this case in a future publication.

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References

[1] Filippov V T 1985 N-Lie algebras Sib. Mat. Zh. 26 126–40
[2] Takhtajan L 1994 On foundation of the generalized Nambu mechanics (second version) Commun. Math. Phys. 160 295 (arXiv:hep-th/9301111)
[3] de Azcárraga J A, Izquierdo J M and Perez Bueno J C 1997 On the generalizations of Poisson structures J. Phys. A: Math. Gen. 30 669–81 (arXiv:hep-th/9605213)
[4] Gustavsson A 2008 Selfdual strings and loop space Nahm equations J. High Energy Phys. JHEP04(2008)083 (arXiv:0802.3456)
Class. Quantum Grav. 25 (2008) 245003  I A Bandos and P K Townsend

[6] Cherkis S and Saemann C 2008 Multiple M2-branes and generalized 3-Lie algebras Phys. Rev. D 78 066019 (arXiv:0807.0808)

[7] Lambert N and Tong D 2008 Membranes on an orbifold Phys. Rev. Lett. 101 041602 (arXiv:0804.1114)

[8] Distler J, Mukhi S, Papageorgakis C and van Raamsdonk M 2008 M2-branes on M-folds J. High Energy Phys. JHEP05(2008)038 (arXiv:0804.1256)

[9] Gauntlett J P and Gutowski J B 2008 Constraining maximally supersymmetric membrane actions arXiv:0804.3078

[10] Papadopoulos G 2008 M2-branes, 3-lie algebras and Plücker relations J. High Energy Phys. JHEP05(2008)105 (arXiv:0805.3803)

[11] Gomis J, Milanesi G and Russo J G 2008 Bagger–Lambert theory for general Lie algebras J. High Energy Phys. JHEP09(2008)113 (arXiv:0806.0738)

[12] Gomis J, Rodriguez-Gomez D, Van Raamsdonk M and Verlinde H 2008 Supersymmetric Yang–Mills theory from Lorentzian three-algebras J. High Energy Phys. JHEP08(2008)094 (arXiv:0806.0738)

[13] Bandres M A, Lipstein A E and Schwarz J H 2008 Ghost-free superconformal action for multiple M2-branes J. High Energy Phys. JHEP08(2008)117 (arXiv:0806.0054)

[14] Ezuthachan B, Mukhi S and Papageorgakis C 2008 D2 to D2 J. High Energy Phys. JHEP07(2008)012 (arXiv:0806.1639)

[15] Bergshoeff E A, de Roo M, Hohm O and Roest D 2008 Multiple membranes from gauged supergravity J. High Energy Phys. JHEP08(2008)091 (arXiv:0806.2584)

[16] van Raamsdonk M 2008 Comments on the Bagger–Lambert theory and multiple M2-branes J. High Energy Phys. JHEP05(2008)105 (arXiv:0803.3803)

[17] Ho P M, Imamura Y and Matsuo Y 2008 M2 to D2 revisited J. High Energy Phys. JHEP08(2008)103 (arXiv:0804.1202)

[18] Bandres M A, Lipstein A E and Schwarz J H 2008 Ghost-free superconformal action for multiple M2-branes J. High Energy Phys. JHEP08(2008)117 (arXiv:0806.0054)

[19] Morozov A 2008 On the problem of multiple M2 branes J. High Energy Phys. JHEP05(2008)076 (arXiv:0804.0913)

[20] Morozov A 2008 From simplified BLG action to the first-quantized M-theory JETP Lett. 7 659 (arXiv:0805.1703)

[21] Krishna C and Maccaferri C 2008 Membranes on calibrations J. High Energy Phys. JHEP07(2008)005 (arXiv:0803.1325)

[22] Gustavsson A 2008 One-loop corrections to Bagger–Lambert theory (arXiv:0805.4363)

[23] Furuuchi K, Shih S Y and Takimi T 2008 M-theory superalgebra from multiple membranes J. High Energy Phys. JHEP08(2008)072 (arXiv:0806.4044)

[24] Gustavsson A 2008 One-loop corrections to Bagger–Lambert theory (arXiv:0805.4443)

[25] Ho P M and Matsuo Y 2008 M5 from M2 J. High Energy Phys. JHEP06(2008)105 (arXiv:0804.3629)

[26] Ho P M, Imamura Y, Matsuo Y and Shiba S 2008 M5-brane in three-form flux and multiple M2-branes J. High Energy Phys. JHEP09(2008)014 (arXiv:0805.2898)

[27] Bergshoeff E, Sezgin E, Tani Y and Townsend P K 1990 Super P-branes as gauge theories of volume preserving diffeomorphisms Ann. Phys., NY 199 340–65
[23] Hoppe J 1982 Quantum theory of a massless relativistic surface and a two dimensional bound state problem PhD Thesis Massachusetts Institute of Technology (available at http://www.aei.mpg.de/jh-cgi-bin/viewit.cgi)
de Wit B, Hoppe J and Nicolai H 1988 On the quantum mechanics of supermembranes Nucl. Phys. B 305 545

[24] Bergshoeff E, Sorokin D P and Townsend P K 1998 The M5-brane Hamiltonian Nucl. Phys. B 533 303
(arXiv:hep-th/9805065)

[25] Bandos I A, Lechner K, Nurmagambetov A, Pasti P, Sorokin D P and Tonin M 1997 Covariant action for the super-five-brane of M-theory Phys. Rev. Lett. 78 4332 (arXiv:hep-th/9701149)

[26] Aganagic M, Park J, Popescu C and Schwarz J H 1997 World-volume action of the M-theory five-brane Nucl. Phys. B 496 191–214 (arXiv:hep-th/9701106)

[27] Pasti P, Sorokin D P and Tonin M 1997 Covariant action for a D = 11 five-brane with the chiral field Phys. Lett. B 398 41 (arXiv:hep-th/9701037)

[28] Park J H and Sochichiu C 2008 Single M5 to multiple M2: taking off the square root of Nambu–Goto action arXiv:0806.0335

[29] Sorokin D P and Townsend P K 1997 M-theory superalgebra from the M-5-brane Phys. Lett. B 412 265
(arXiv:hep-th/9708003)

[30] Lévy-Leblond J M 1965 Une nouvelle limite non-relativiste du group de Poincaré Ann. Inst. H. Poincaré 3 I

[31] Bacry H and Levy-Leblond J 1968 Possible kinematics J. Math. Phys. 9 1605

[32] Gibbons G W 2003 Thoughts on tachyon cosmology Class. Quantum Grav. 20 S321 (arXiv:hep-th/0301117)

[33] Jeon I, Kim J, Kim N, Kim S W and Park J H 2008 Classification of the BPS states in Bagger–Lambert theory J. High Energy Phys. JHEP07(2008)056 (arXiv:0805.3236)

[34] Sagnotti A and Tsulaaa M 2004 On higher spins and the tensionless limit of string theory Nucl. Phys. B 682 83
(arXiv:hep-th/0311257)

Bonelli G 2003 On the tensionless limit of bosonic strings, infinite symmetries and higher spins Nucl. Phys. B 669 159 (and references therein) (arXiv:hep-th/0305155)

[35] Claus P, Kallosh R, Kumar J, Townsend P K and van Proeyen A 1998 Conformal theory of M2, D3, M5 and D1+D5 branes J. High Energy Phys. JHEP06(1998)004 (arXiv:hep-th/9801206)