CONSTRANT ENERGY MINIMIZING GENERALIZED MULTISCALE
FINITE ELEMENT METHOD FOR INHOMOGENEOUS BOUNDARY
VALUE PROBLEMS WITH HIGH CONTRAST COEFFICIENTS

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ABSTRACT

In this article we develop the Constraint Energy Minimizing Generalized Multiscale Finite Element Method (CEM-GMsFEM) for elliptic partial differential equations with inhomogeneous Dirichlet, Neumann, and Robin boundary conditions, and the high contrast property emerges from the coefficients of elliptic operators and Robin boundary conditions. By careful construction of multiscale bases of the CEM-GMsFEM, we introduce two operators \( D^m \) and \( N^m \) which are used to handle inhomogeneous Dirichlet and Neumann boundary values and are also proved to converge independently of contrast ratios as enlarging oversampling regions. We provide a priori error estimate and show that oversampling layers are the key factor in controlling numerical errors. A series of experiments are conducted, and those results reflect the reliability of our methods even with high contrast ratios.

Keywords: Constraint energy minimization · multiscale finite element methods · high contrast problems · inhomogeneous boundary value problems

1 Introduction

Many practical problems drive us to study partial differential equations (PDE) with inhomogeneous coefficients. For example, Darcy's law in inhomogeneous or even fractured media, elasticity systems in composite materials. When coefficients show special structures, such as periodicity and stochasticity, a great number of mathematical theories have been established Bensoussan et al. [2011], Dal Maso [1993], Jikov et al. [1994], Pankov [1997], Conca and Vanninathan [1997], Cioranescu and Donato [1999], Cioranescu et al. [2008], Tartar [2009], Shen [2018], Armstrong et al. [2019], which have been cornerstones of multiscale modeling and simulations. As for general inhomogeneous coefficients which are usually accompanied by high contrast channels, it has been viewed as a long-standing challenge for traditional methods. The reason is in two aspects: channelized structures require fine meshes which dramatically increase freedom degrees, and high contrast ratios deteriorate convergences of solvers for final linear systems.

To handle those problems, many multiscale computational methods have been developed since the 1990s. To name a few, multiscale finite element methods Hou and Wu [1997], Hou et al. [1999], Chen and Hou [2003], Efendiev and Hou [2009], Heterogeneous Multiscale Methods (HMM) E and Engquist [2003], E et al. [2005], Abdulle et al. [2012], variational multiscale methods Hughes [1995], Brezzi et al. [1997], Hughes and Sangalli [2007], generalized finite element methods Babuška and Lipton [2011], Babuška et al. [2020], Generalized Multiscale Finite Element Methods (GMsFEM) Efendiev et al. [2013], Chung et al. [2014] 2016, and Localized Orthogonal Decomposition (LOD) methods Målqvist and Peterseim [2014], Henning and Målqvist [2014], Altmann et al. [2021], Hellman and Målqvist [2017], Målqvist and Peterseim [2021]. A universal thought in those methods (except HMMs) is encoding fine-scale information into basis functions of Finite Element Methods (FEM), then solving original problems on multiscale finite
element spaces whose dimensions have been greatly reduced compared to default FEMs. We also notice that most existing literature in multiscale computational methods chooses homogeneous Dirichlet or Neumann Boundary Value Problems (BVP) as model problems to study convergence theories and conduct numerical experiments, while extensions of those methods to inhomogeneous BVPs are sometimes nontrivial (e.g., Henning and Målqvist [2014]). Since solving inhomogeneous BVPs is a practical demand from the application side, it is necessary to examine the effectiveness of existing multiscale computational methods and extend them to complex BVPs.

This work is based on Constrained Energy Minimizing Generalized Multiscale Finite Element Methods (CEM-GMsFEM), which was originally proposed in Chung et al. [2018a] for high contrast problems and are applied to many applications [Vasilyeva et al. [2019], Wang et al. [2021], Chung and Pun [2020], Chung et al. [2018a]]. Note that there are two versions of CEM-GMsFEMs proposed in Chung et al. [2018a], and we focus on the modified one—relaxed CEM-GMsFEM, which shows advantage both in theories and implementations. However, according to the construction of multiscale bases in CEM-GMsFEMs—solving energy minimizing problems on oversampling regions, either pressures or flow rates (terms from Darcy’s law) of basis functions vanish on boundaries, which implies that it cannot be directly utilized in inhomogeneous BVPs. Moreover, additional physical coefficients will be introduced in Robin boundary conditions, and those coefficients may also be high contrast. Those reasons lead us to reconsider how to apply CEM-GMsFEMs to inhomogeneous BVPs in high contrast settings.

There are many common points in CEM-GMsFEMs and LOD methods, for example, both rely on exponential decay properties of bases and need mesh size-dependent oversampling regions to achieve an optimal convergence rate. We emphasize that large oversampling regions are essential here because there are theoretical evidences that reveal high contrast is strongly related to nonlocality [Bellieud and Bouchitte [1998], Briane [2002], Du et al. [2020]]. The major difference is that CEM-GMsFEMs solve element-wise eigenvalue problems to obtain auxiliary spaces and projection operators (such an idea is originated from GMsFEMs Efendiev et al. [2013]), while LOD methods adopt quasi-interpolation operators, such as the Scott-Zhang operator [Scott and Zhang [1990]]. Since eigenvalue problems have integrated coefficient information, exponential decay rates are now explicitly dependent on $\Lambda$, where $\Lambda$ is the minimal eigenvalue that the corresponding eigenvector is not included in auxiliary spaces. From numerical experiments, $\Lambda$ is quite stable with varying contrast ratios. A minor difference from implementations is that relaxed CEM-GMsFEMs deal with quadratic form minimization problems in constructing multiscale bases, while LOD methods need to solve saddle point problems.

The paper is organized as follows: in section 2 we introduce some preliminaries; in section 3 we present details of the methods, provide a rigorous analysis of numerical errors and extend the computational framework to inhomogeneous Robin BVPs; in section 4 we conduct a series of numerical experiments to verify accuracy of our methods in high contrast settings.

## 2 Preliminaries

Denote by $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $3$) a Lipschitz domain and $\mathbf{A}(x) \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ a matrix-valued function defined on $\Omega$, we consider the following model problem:

$$
\begin{aligned}
- \text{div} (\mathbf{A} \nabla u) &= f & \text{in } \Omega, \\
u \cdot \mathbf{A} \nabla u &= q & \text{on } \Gamma_N,
\end{aligned}
$$

where $\nu$ stands for outward unit normal vectors to $\partial \Omega$, $\Gamma_D$ and $\Gamma_N$ are two nonempty disjointed parts of $\partial \Omega$. In this paper, we present the following assumptions for the model problem:

**A1** There exist positive constants $0 < A_1 \leq A_2$ such that for a.e. $x \in \Omega$, $\mathbf{A}(x)$ is a positive define matrix with $A_1 \leq \lambda_{\text{min}}(\mathbf{A}(x)) \leq \lambda_{\text{max}}(\mathbf{A}(x)) \leq A_2$.

**A2** The source term $f \in L^2(\Omega)$, the Dirichlet boundary value term $g \in H^\frac{1}{2}(\Gamma_D)$ and the Neumann boundary value term $q \in L^2(\Gamma_N)$.

We rewrite the inhomogeneous BVP eq. (1) in a variational form for designing computational methods: find a solution $u_0 \in V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ such that for all $v \in V$,

$$
\int_{\Omega} \mathbf{A} \nabla u_0 \cdot \nabla v \, dx = \int_{\Omega} f v \, dx - \int_{\Omega} \mathbf{A} \nabla \tilde{g} \cdot \nabla v \, dx + \int_{\Gamma_N} qv \, d\sigma,
$$

where $\tilde{g} \in H^1(\Omega)$ with $\tilde{g} = g$ on $\Gamma_D$ in the trace sense. Obviously, $u_0$ and $u$ the solution of the original BVP have the relation $u = u_0 + \tilde{g}$.
To simply notations, we denote by $a(w,v)$ the bilinear form $\int_\Omega A \nabla w \cdot \nabla v \, dx$ on $V$, and $\|v\|_a = \sqrt{a(v,v)}$. For a subdomain $\omega \subset \Omega$, we also introduce a notation $\|v\|_{a(\omega)} = \sqrt{\int_\omega A \nabla v \cdot \nabla v \, dx}$.

Let $T^h$ be a conforming partition of $\Omega$ into elements, such as triangulations/quadrilaterals for 2D domains \cite{Brenner2008}, where $H$ is the coarse-mesh size to distinguish with another mesh $T^h$ which will be utilized to compute multiscale basis functions, also let $N$ be the number of elements. For each $K_i \in T^h$ with $1 \leq i \leq N$, we define an oversampled domain $K_i^m$ ($m \geq 1$) by an iterative approach:

$$K_i^m = \text{int} \left( \bigcup_{K \in T^h} \overline{\text{cl}(K)} \cup \overline{\text{cl}(K_i^{m-1})} \right) \cup \overline{\text{cl}(K_i^{m-1})} \right),$$

where $\text{int}(S)$ and $\overline{\text{cl}(S)}$ are the interior and the closure of a set $S$, and we also set $K_i^0 := K_i$ here for consistency purposes. Letting $N_v$ be the number of vertices contained in an element (i.e., $N_v = 3$ for a triangular mesh and $N_v = 4$ for a quadrilateral mesh), we can construct a set of Lagrange bases $\{\eta^1_i, \eta^2_i, \ldots, \eta^{N_v}_i\}$ of the element $K_i \in T^h$. Then we define $\tilde{\kappa}(x)$ piecewisely by

$$\tilde{\kappa}(x) := (N_v - 1) \sum_{j=1}^{N_v} A(\eta^j_i \nabla \eta^j_i \cdot \nabla \eta^j_i)$$

in $K_i$. An important concept in CEM-GMsFEM is the bilinear form $s(w,v) := \int_\Omega \tilde{\kappa} w v \, dx$, and note that $s(w,v)$ can be validly defined on $L^2(\Omega)$. Similary, we denote by $\|v\|_s := \sqrt{s(v,v)}$, and $\|v\|_{s(\omega)} := \sqrt{\int_\omega \tilde{\kappa} \|v\|^2 \, dx}$ for a subdomain $\omega \subset \Omega$.

The construction of the local auxiliary space $V^\text{aux}_i$ is by solving an eigenvalue problem in the element $K_i$: find $\lambda_i \geq 0$ and $\phi_i \in H^1(K_i)$ such that for all $v \in H^1(K_i)$,

$$\int_{K_i} A \nabla \phi_i \cdot \nabla v \, dx = \lambda_i \int_{K_i} \tilde{\kappa} \phi_i v \, dx.$$

We arrange the eigenvalues $\{\lambda_i\}_{i=0}^\infty$ in ascending order, and notice that $\lambda_i^0 = 0$ always holds. Denote by $V^\text{aux}_{i,1} := \text{span}\{\phi_i^0, \phi_i^1, \ldots, \phi_i^j\}$, we can show that the orthogonal projection $\pi_i$ (respect to the inner product $s(\cdot, \cdot)$) from $L^2(K_i)$ onto $V^\text{aux}_{i,1}$ is

$$\pi_i(v) := \sum_{j=0}^1 s(\phi_i^j, v) \phi_i^j.$$

We can immediately derive the following basic estimates: for all $v \in H^1(K_i)$,

$$\|v - \pi_i v\|_{s(K_i)}^2 \leq \frac{\|v\|_{s(K_i)}^2}{\lambda_i}, \quad \|\pi_i v\|_{s(K_i)}^2 = \|v\|_{s(K_i)}^2 - \|v - \pi_i v\|_{s(K_i)}^2 \leq \|v\|_{s(K_i)}^2.$$

The global auxiliary space $V^\text{aux}$ can be defined by taking a direct sum $V^\text{aux} := \bigoplus_i^N V^\text{aux}_i$, and the global projection is $\pi := \sum_i \pi_i$ accordingly.

Although eq. (4) shows that $V^\text{aux}$ can approximate $V$ satisfyingly and stably with respect to the contrast ratio $A_2/A_1$, functions in $V^\text{aux}$ may not be continuous in $\Omega$, and thus $V^\text{aux}$ cannot be used as a conforming finite element space. The essential thought in CEM-GMsFEMs is “extending” $\phi_i^j$ into $V$ by solving an energy minimization problem:

$$\psi_i^j = \arg\min \left\{ a(\psi, \psi) + s(\pi \psi - \phi_i^j, \pi \psi - \phi_i^j) : \psi \in V \right\},$$

which is a relaxed version of the following problem:

$$\arg\min \left\{ a(\psi, \psi) : \psi \in V, \pi \psi = \phi_i^j \right\},$$

\footnote{We implicitly utilize a zero-extension here, which extends $V^\text{aux}_i$ into $L^2(\Omega)$}
Moreover, it can be proved that $\psi_i^j$ decays exponentially fast away from $K_i$, which implies that solving eq. (5) on an oversampling domain is reasonable. Denote by $V_i^m$ the space

$$\{ v \in H^1(K_i^m) : v = 0 \text{ on } \Gamma_D \cap \partial K_i^m \text{ or } \Omega \cap \partial K_i^m \},$$

we can see that the zero-extension of a function in $V_i^m$ still belongs to $V$. Then the multiscale basis function $\psi_i^{j,m}$ is defined as follows:

$$\psi_i^{j,m} = \arg \min \{ a(\psi, \psi) + s(\pi \psi - \phi_i^{j,m}, \pi \psi - \phi_i^{j,m}) : \psi \in V_i^m \}.$$

It can be shown that $\psi_i^j$ and $\psi_i^{j,m}$ satisfy a variational form respectively:

$$a(\psi_i^j, v) + s(\pi \psi_i^j, \pi v) = s(\phi_i^j, \pi v) \quad \forall v \in V;$$

$$a(\psi_i^{j,m}, v) + s(\pi \psi_i^{j,m}, \pi v) = s(\phi_i^{j,m}, \pi v) \quad \forall v \in V_i^m. \tag{7}$$

We also introduce notations $V_{ms}^{glo} := \text{span} \{ \psi_i^j \}_{0 \leq j \leq l_i, 1 \leq i \leq N}$ and $V_{ms}^m := \text{span} \{ \psi_i^{j,m} \}_{0 \leq j \leq l_i, 1 \leq i \leq N}$. A basis property of $V_{ms}^{glo}$ is the orthogonality to the kernel of $\pi$ with respect to the inner product $a(\cdot, \cdot)$.

**Lemma 1 (Chung et al. [2018a])**. Let $v \in V_{ms}^{glo}$, then $a(v, v') = 0$ for any $v' \in V$ with $\pi v' = 0$. Moreover, if there exists $v \in V$ such that $a(v, v') = 0$ for any $v' \in V_{ms}^{glo}$, then $\pi v = 0$.

### 3 Computational method and analysis

#### 3.1 Method

The computational method for solving the model problem eq. (1) consists of four steps:

**Step1** Find $D_i^m \tilde{g} \in V_i^m$ and $N_i^m q \in V_i^m$ such that for all $v \in V_i^m$,

$$a(D_i^m \tilde{g}, v) + s(\pi D_i^m \tilde{g}, \pi v) = \int_{K_i} A \nabla \tilde{g} \cdot \nabla v \, dx, \tag{8}$$

$$a(N_i^m q, v) + s(\pi N_i^m q, \pi v) = \int_{\partial K_i \cap \Gamma_S} q v \, d\sigma. \tag{9}$$

Then take summations as $D^m \tilde{g} = \sum_{i=1}^N D_i^m \tilde{g}$ and $N^m q = \sum_{i=1}^N N_i^m q$.

**Step2** Prepare the multiscale function space $V_{ms}^m$ via eq. (7).

**Step3** Solve $w^m \in V_{ms}^m$ such that for all $v \in V_{ms}^m$,

$$a(w^m, v) = \int_{\Omega} f v \, dx - \int_{\Omega} A \nabla \tilde{g} \cdot \nabla v \, dx + \int_{\Gamma_S} q v \, d\sigma + a(D^m \tilde{g}, v) - a(N^m q, v). \tag{10}$$

**Step4** Construct the numerical solution to approximate the real solution of eq. (1) as

$$u \approx w^m - D^m \tilde{g} + N^m q + \tilde{g}.$$

Note that eqs. (7) to (9) are all solved on a fine mesh $T^h$, which is a refinement of $T^H$. For brevity, we will not explicitly point this out in here and following analysis parts. From the computational steps presented above, the multiscale finite element space $V_{ms}^m$ is reusable for different source terms, which will greatly accelerate computations in simulations. Moreover, if several particular boundary values (i.e., $g$ and $q$) that we are interested in admit a low-dimension structure, it is also possible to build abstract operators $D^m$ and $N^m$ to achieve an overall saving in computational resources.

#### 3.2 Analysis

We can also define “global” versions of $D^m$, $N^m$ and $w^m$ as $D^{glo} := \sum_{i=1}^N D_i^{glo}$, $N^{glo} := \sum_{i=1}^N N_i^{glo}$ and $w^{glo}$ respectively, where $D_i^{glo}$ satisfies for all $v \in V$

$$a(D_i^{glo} \tilde{g}, v) + s(\pi D_i^{glo} \tilde{g}, \pi v) = \int_{K_i} A \nabla \tilde{g} \cdot \nabla v \, dx \quad \forall; \tag{11}$$

$$a(D^{glo} \tilde{g}, v) + s(\pi D^{glo} \tilde{g}, \pi v) = \int_{\Omega} f v \, dx - \int_{\Omega} A \nabla \tilde{g} \cdot \nabla v \, dx + \int_{\Gamma_S} q v \, d\sigma + a(D^m \tilde{g}, v) - a(N^m q, v). \tag{12}$$

$$a(D^{glo} \tilde{g}, v) + s(\pi D^{glo} \tilde{g}, \pi v) = \int_{\Omega} f v \, dx - \int_{\Omega} A \nabla \tilde{g} \cdot \nabla v \, dx + \int_{\Gamma_S} q v \, d\sigma + a(D^m \tilde{g}, v) - a(N^m q, v). \tag{13}$$
\( \mathcal{N}_i^{glo} \) satisfies for all \( v \in V \)
\[
a(\mathcal{N}_i^{glo}, v) + s(\pi \mathcal{N}_i^{glo}, \pi v) = \int_{\Omega \cap \Gamma_N} qv \, d\sigma \quad \forall v \in V.
\] (12)

and \( w^{glo} \) satisfies for all \( v \in V^{glo}_{ms} \)
\[
a(w^{glo}, v) = \int_{\Omega} fv \, dx - \int_{\Omega} A \nabla \tilde{g} \cdot \nabla v \, dx + \int_{\Gamma_N} qu \, d\sigma + a(D^{glo} \tilde{g}, v) - a(\mathcal{N}^{glo}, v).
\] (13)

The starting point of analyzing CEM-GMsFEMs is that the “global” solution possess an optimal error estimate:

**Theorem 1.** Let \( D^{glo} \tilde{g}, N_i^{glo} q, w^{glo} \) be the solutions of eqs. (11) to (13) respectively, \( u \) be the real solution of the model problem eq. (1). Then
\[
\| w^{glo} - D^{glo} \tilde{g} + N_i^{glo} q + \tilde{g} - u \|_a \leq \frac{1}{\sqrt{A}} \| f \|_{s-1},
\] (14)

where
\[
\| f \|_{s-1} := \sup_{v \in L^2(\Omega)} \frac{\int_{\Omega} fv \, dx}{\| v \|_s}.
\]

\( D^{glo} := \sum_{i=1}^{N} D_i^{glo}, N^{glo} := \sum_{i=1}^{N} N_i^{glo} \) and \( A = \min \lambda_i^{s+1} \).

**Proof.** For simplicity, take \( e = w^{glo} - D^{glo} \tilde{g} + N_i^{glo} q + \tilde{g} - u \). By a direct computation, it gives for all \( v \in V^{glo}_{ms} \),
\[
a(e, v) = a(w^{glo}, v) - a(D^{glo} \tilde{g}, v) + a(N_i^{glo}, v) - a(u_0, v) \quad \text{eq. (10)}.
\]

We hence derive that \( \pi e = 0 \) via lemma [1]. Moreover, for all \( v \in V \) with \( \pi v = 0 \), we have \( a(w^{glo}, e) = 0 \) and \( a(e, v) = -a(D^{glo} \tilde{g}, v) + a(N_i^{glo}, v) - a(u_0, v) \).

From the definitions of \( D^{glo} \) and \( N^{glo} \) (see eqs. (11) and (12)) and recalling \( \pi v = 0 \), we can show
\[
a(D^{glo} \tilde{g}, v) = \int_{\Omega} A \nabla \tilde{g} \cdot \nabla v \, dx \quad \text{and} \quad a(N_i^{glo}, v) = \int_{\Gamma_N} qu \, d\sigma.
\]

Combining the variation form of \( u_0 \) eq. (2), we conclude that for \( v \in V \) with \( \pi v = 0 \)
\[
a(e, v) = \int_{\Omega} fv \, dx.
\]

It follows that
\[
\| e \|^2_a = \int_{\Omega} fe \, dx \leq \| f \|_{s-1} \| e \|_s = \| f \|_{s-1} \| e - \pi e \|_s \leq \frac{1}{\sqrt{A}} \| f \|_{s-1} \| e \|_a.
\]

The abstract problem Let \( K_i \in \mathcal{T}^H \) and \( t_i \in V' \) such that \( \langle t_i, v \rangle = 0 \) holds for any \( v \in V \) with \( \text{supp}(v) \subset \Omega \setminus K_i \); define an operator \( P_i : V' \rightarrow V \) with \( P_i t_i \) satisfying for all \( v \in V \),
\[
a(P_i t_i, v) + s(\pi P_i t_i, \pi v) = \langle t_i, v \rangle,
\] (15)
similarly, define \( P_i^m \) and \( P_i^m t_i \) such that for all \( v \in V_i^m \),
\[
a(P_i^m t_i, v) + s(\pi P_i^m t_i, \pi v) = \langle t_i, v \rangle.
\] (16)

The goal is deriving an estimate on
\[
\left\| \sum_{i=1}^{N} (P_i - P_i^m) t_i \right\|_a^2 + \left\| \pi \sum_{i=1}^{N} (P_i - P_i^m) t_i \right\|_s^2.
\]

We complete such an estimate by several lemmas. The first lemma shows that \( P_i^m t_i \) propose an exponentially decaying property.
Lemma 2. Let $A = \min_i \lambda_i^{1+1}$ and $m \geq 1$, there exists a positive constant $0 < \theta < 1$ such that
\[
\|P_i t_i\|_{a(\Omega, K_i^n)}^2 + \|\pi P_i t_i\|_{s(\Omega, K_i^n)}^2 \leq \theta^m \left( \|P_i t_i\|_a^2 + \|\pi P_i t_i\|_s^2 \right),
\]
where $\theta = \frac{c_A}{c_A^{1+1}}$ and
\[
c_A(A) = \max_{x \in [0, \frac{1}{2}]} \left( \frac{\cos(x) + \sin(x)}{\sqrt{A}} \right)^2.
\]
The next lemma bounds the error between $P_i t_i$ and $P_i^m t_i$.

Lemma 3. Keep the notations same as lemma 2 then
\[
\| (P_i - P_i^m) t_i \|_a^2 + \| \pi (P_i - P_i^m) t_i \|_s^2 \leq c_* \theta^{-m} \left( \|P_i t_i\|_a^2 + \|\pi P_i t_i\|_s^2 \right),
\]
where
\[
c_*(A) = \max_{x \in [0, \frac{1}{2}]} \left( \frac{1}{\sqrt{A}} + 1 \right) \cos(x) + \sin(x) \right)^2 + \left( \frac{\cos(x) + \sin(x)}{\sqrt{A}} \right)^2.
\]

We need a regularity assumption for $T^H$ before presenting the concluding lemma.

A3 There exists a positive constant $C_{ol}$ such that for all $K_i \in T^H$ and $m > 0$,\[
\# \{ K \in T^H : K \subset K_i^m \} \leq C_{ol} m^d.
\]

Lemma 4. Keep the notations same as lemmas 2 and 3 then
\[
\left\| \sum_{i=1}^N (P_i - P_i^m) t_i \right\|_a^2 + \left\| \pi \sum_{i=1}^N (P_i - P_i^m) t_i \right\|_s^2 \leq c_2^2 C_{ol} \theta^{-m} (m + 1)^d \sum_{i=1}^N \langle t_i, P_i t_i \rangle.
\]

A common trick in proving those lemmas is multiplying a cutoff function $\chi$ to $v$ and then inserting $\chi v$ into variational forms. Here is the definition of cutoff functions.

Definition 1. Let $V^H$ be the Lagrange basis function space of $T^H$. For an element $K_i \in T^H$, a cutoff function $\chi_i^{n,m} \in V^H$ satisfies properties:
\[
\chi_i^{n,m}(x) \equiv 1 \quad \text{in } K_i^n;
\]
\[
\chi_i^{n,m}(x) \equiv 0 \quad \text{in } \partial \Omega_i K_i^n;
\]
\[
0 \leq \chi_i^{n,m}(x) \leq 1 \quad \text{in } K_i^m \setminus K_i^n.
\]

We start to prove lemmas 2 to 4 now.

Proof of lemma 2 Replacing $v$ with $(1 - \chi_i^{m-1,m}) P_i t_i$ in eq. (15), recalling $1 - \chi_i^{m-1,m} \equiv 0$ in $K_i^{m-1}$ and $1 - \chi_i^{m-1,m} \equiv 1$ in $\partial \Omega_i K_i^m$, we then obtain
\[
\|P_i t_i\|_{a(\Omega_i, K_i^n)}^2 + \|\pi P_i t_i\|_{s(\Omega_i, K_i^n)}^2
\]
\[
= \int_{K_i^m \setminus K_i^{m-1}} \left( \chi_i^{m-1,m} - 1 \right) A \nabla P_i t_i \cdot \nabla P_i t_i \, dx
\]
\[
+ \int_{K_i^m \setminus K_i^{m-1}} P_i t_i A \nabla P_i t_i \cdot \nabla \chi_i^{m-1,m} \, dx
\]
\[
+ \int_{K_i^m \setminus K_i^{m-1}} \kappa \pi P_i t_i \cdot \pi \left( \chi_i^{m-1,m} - 1 \right) P_i t_i \, dx
\]
\[
:= I_1 + I_2 + I_3.
\]

According to definition 1 we have $\chi_i^{m-1,m} - 1 \leq 0$ in $K_i^m \setminus K_i^{m-1}$, which gives $I_1 \leq 0$. By the definition of $\kappa$ in eq. (3), it is easy to show
\[
A(x) \nabla \chi_i^{m-1,m} \cdot \nabla \chi_i^{m-1,m} \leq \kappa(x)
\]
in $K_i^m \backslash K_i^{m-1}$. Then, we could derive

$$I_2 \leq \|P_i t_i\|_a(K_i^m \backslash K_i^{m-1}) \|P_i t_i\|_s(K_i^m \backslash K_i^{m-1}).$$

For $I_3$, applying the Cauchy–Schwarz inequality and estimates [4], we have

$$I_3 \leq \|\pi P_i t_i\|_a(K_i^m \backslash K_i^{m-1}) \|\pi \left(\sum_{j=1}^m - 1\right) P_i t_i\|_s(K_i^m \backslash K_i^{m-1})
\leq \|\pi P_i t_i\|_a(K_i^m \backslash K_i^{m-1}) \|\pi \left(\sum_{j=1}^m - 1\right) P_i t_i\|_s(K_i^m \backslash K_i^{m-1})
\leq \|\pi P_i t_i\|_a(K_i^m \backslash K_i^{m-1}) \|\pi P_i t_i\|_s(K_i^m \backslash K_i^{m-1}).$$

Meanwhile, [4] provide an estimate for $\|P_i t_i\|_s(K_i^m \backslash K_i^{m-1})$ as

$$\|P_i t_i\|_s(K_i^m \backslash K_i^{m-1}) \leq \|P_i t_i - \pi P_i t_i\|_s(K_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(K_i^m \backslash K_i^{m-1})
\leq \frac{1}{\sqrt{A}} \|P_i t_i\|_a(K_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(K_i^m \backslash K_i^{m-1}).$$

Collecting all the estimates for $I_1$, $I_2$ and $I_3$, we arrive at

$$\|P_i t_i\|_a(D_i^m) + \|\pi P_i t_i\|_a(D_i^m)
\leq \left(\|P_i t_i\|_a(D_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(D_i^m \backslash K_i^{m-1})\right) \left(\frac{1}{\sqrt{A}} \|P_i t_i\|_a(D_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(D_i^m \backslash K_i^{m-1})\right)
\leq c_s(A) \left(\|P_i t_i\|_a(D_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(D_i^m \backslash K_i^{m-1})\right).$$

This yields an iterative relation

$$\|P_i t_i\|_a(D_i^m) + \|\pi P_i t_i\|_s(D_i^m)
\geq \left(1 + \frac{1}{c_s(A)}\right) \|P_i t_i\|_a(D_i^m \backslash K_i^{m-1}) + \|\pi P_i t_i\|_s(D_i^m \backslash K_i^{m-1}),$$

and also finishes the proof.

**Proof of lemma [3]** Let $z_i := (P_i - \pi_i^m) t_i$, and decompose $z_i$ as

$$z_i = \left\{\left(1 - \sum_{j=1}^m - 1\right) P_i t_i\right\} + \left\{\left(\sum_{j=1}^m - 1\right) P_i t_i + \sum_{j=1}^m - 1\right\} := z_i' + z_i''.$$

Recalling the definition of $\sum_{j=1}^m - 1$, we have $z_i'' \in V_i^m$. Then combining eqs. (15) and (16), we can obtain

$$a(z_i, z_i'') + s(\pi z_i, \pi z_i'') = 0.$$
Proof of lemma 4. Still take \( z_i := (P_i - P_i^m) t_i \) and \( z = \sum_{i=1}^{N} z_i \), and decompose \( z \) as
\[
z = \left\{ (1 - \chi_i^{m,m+1}) z \right\} + \left\{ \chi_i^{m,m+1} z \right\} := z' + z''.
\]
Noting that \( \text{supp}(z') \subset \Omega \setminus K_i^m \), \( \text{supp}(\pi z') \subset \Omega \setminus K_i^m \), \( \text{supp}(P_i^m t_i) \subset \text{cl}(K_i^m) \) and \( \text{supp}(\pi P_i^m t_i) \subset \text{cl}(K_i^m) \), we have
\[
a(P_i^m t_i, z') + s(\pi P_i^m t_i, \pi z') = 0
\]
and
\[
a(P_i t_i, z') + s(\pi P_i t_i, \pi z') = 0,
\]
which leads to
\[
a(z_i, z') + s(\pi z_i, \pi z') = 0.
\]
Use similar techniques in the proof of lemma 3.

Recalling the definition of \( C_{ol} \), it is easy to show
\[
\sum_{i=1}^{N} \left\| \frac{\|z\|_{a(K_i^{m+1})}^2 + \|\pi z\|_{s(K_i^{m+1})}^2}{\|z\|_{a(K_i^{m+1})} + \|\pi z\|_{s(K_i^{m+1})}} \right\|_{\mathcal{E}}^{1/2} \left\{ \|z_i\|_{a}^2 + \|\pi z_i\|_{s}^2 \right\}^{1/2} \leq C_{ol} (m + 1)^d \left( \|z\|_{a}^2 + \|\pi z\|_{s}^2 \right).
\]

Then by the Cauchy–Schwarz inequality, we get
\[
\|z\|_{a}^2 + \|\pi z\|_{s}^2 = \sum_{i=1}^{N} a(z_i, z) + s(\pi z_i, \pi z) \leq \left\{ c \cdot C_{ol} \left( \|z\|_{a}^2 + \|\pi z\|_{s}^2 \right) \right\}^{1/2} \left\{ \sum_{i=1}^{N} \|z_i\|_{a}^2 + \|\pi z_i\|_{s}^2 \right\}^{1/2}.
\]

Combining lemma 3 and
\[
\|P_i t_i\|_{a}^2 + \|\pi P_i t_i\|_{s}^2 = \langle t_i, P_i t_i \rangle
\]
prove the lemma.

A direct result of lemma 4 is the following corollary, which presents estimates of \( \langle \mathcal{D}^{\text{glo}} - \mathcal{D}^m \rangle \hat{g} \) and \( \langle \mathcal{N}^{\text{glo}} - \mathcal{N}^m \rangle q \).

Corollary 1. Let notations be the same as lemmas 2 to 4. Then
\[
\left\| \langle \mathcal{D}^{\text{glo}} - \mathcal{D}^m \rangle \hat{g} \right\|_{a}^2 + \left\| \pi \left( \langle \mathcal{D}^{\text{glo}} - \mathcal{D}^m \rangle \hat{g} \right) \right\|_{s}^2 \leq c_2^2 C_{ol} \theta^{m-1} (m + 1)^d \|\hat{g}\|_{a}^2,
\]
and
\[
\left\| \langle \mathcal{N}^{\text{glo}} - \mathcal{N}^m \rangle q \right\|_{a}^2 + \left\| \pi \left( \langle \mathcal{N}^{\text{glo}} - \mathcal{N}^m \rangle q \right) \right\|_{s}^2 \leq c_2^2 C_{ol} \theta^{m-1} (m + 1)^d \|q\|_{L^2(I_N)}^2,
\]
where \( C_{tr} \) is the modified norm of the trace operator \( V \rightarrow L^2(I_N) \)
\[
C_{tr} := \sup_{v \in V} \frac{\|v\|_{L^2(I_N)}}{\|v\|_{a}}.
\]
The main result of this subsection is the following theorem:

**Theorem 2.** Let \( D^m \tilde{g}, N^m q \) and \( w^m \) be the numerical solutions obtained in Step1-4, \( w^{\text{glo}} \) be the solution of eq. (13), and \( \Lambda, \theta, c, C_\alpha \) and \( C_{\text{inv}} \) be the constants defined in theorem 1, lemma 2, lemma 3, corollary 1 and lemma 5 respectively. Then

\[
\|w^m - D^m g + N^m q + \tilde{g} - u\|_a \leq \frac{1}{\sqrt{\Lambda}} \|f\|_{s-1} + c_\Lambda \sqrt{\frac{\theta^{1/2}}{C_\alpha}} (m + 1)^2 \left\{ \|\tilde{g}\|_a + C_\alpha \|q\|_{L^2(\Gamma_N)} + C_{\text{inv}} \|w^{\text{glo}}\|_a + \|\pi w^{\text{glo}}\|_s \right\},
\]

where

\[
\tilde{g} = \sum_{i=1}^N \int_{K_i} A \nabla \tilde{g} \cdot \nabla D_i^{\text{glo}} \tilde{g} \, dx + \sum_{i=1}^N \int_{\partial K_i \cap \Gamma_N} q \tilde{N}^{\text{glo}} \, d\sigma.
\]
Proof. Take \( e^m = w^m - D^m g + N^m q + \hat{g} - u \). According to variational forms eqs. (2) and (10), we have \( a(e^m, v) = 0 \) for all \( v \in V^m_{\text{ms}} \). Then for a \( w^* \in V^m_{\text{ms}} \) which will be chosen later, by the Galerkin orthogonality, it is easy to show
\[
\|e^m\|_a^2 = \|e^m - w^*_a + w^*_a\|^2 - \|w^*- w^m\|^2_a \leq \|w^m - D^m g + N^m q + \hat{g} - u\|^2_a .
\]
Splitting \( \|w^*_a - D^m g + N^m q + \hat{g} - u\|_a \) into four terms leads to an estimate
\[
\|w^*_a - D^m g + N^m q + \hat{g} - u\|_a \\
\leq \|w^\text{glo} - N^m q + \hat{g} - u\|_a + \|D^m g - D^m g\|_a + \|N^m q - N^m q\|_a + \|w^\text{glo} - w^m\|_a \\
\leq \frac{1}{\sqrt{A}} \|f\|_{s-1} + c\sqrt{C_q \theta^\frac{m+1}{2}} (m+1)^{\frac{d}{2}} \left( \|\hat{g}\|_a + C_u \|q\|_{L^2(V)} \right) + \|w^\text{glo} - w^m\|_a,
\]
where theorem [1] and corollary [1] are applied in the last line above. We are now left to estimate \( \|w^\text{glo} - w^m\|_a \).

The definition of \( R^\text{glo} \) in [20] implies that \( R^\text{glo} : L^2(\Omega) \to V^\text{glo} \) is a surjective map. We are hence able to find \( \varphi^* \in L^2(\Omega) \) such that \( w^\text{glo} = R^\text{glo} \varphi^* \). Meanwhile, setting \( w^*_a = R^m \varphi^* \), and the estimate of \( \|w^\text{glo} - w^m\|_a \) is transformed into \( \|\varphi^*\|_a \). Utilizing similar techniques in proving corollary [1] we can obtain
\[
\|\varphi^*\|^2_a \leq C^2 A \theta^m (m+1)^d \|\varphi^*\|^2_a .
\]

The variational form [20] yields a variational form for \( R^\text{glo} \), for all \( v \in V \),
\[
a(w^\text{glo}, v) + s(\pi w^\text{glo}, \pi v) = s(\pi \varphi^*, \pi v).
\]
According to lemma [5], it is possible to find \( \varphi^* \in V \) such that \( \varphi^* = \pi \varphi^* \) and \( \|\varphi^*\|_a \leq \|\pi \varphi^*\|_a \) . Replacing \( v \) with \( \varphi^* \) in the above variational form, we can obtain
\[
\|\varphi^*\|^2_a = a(w^\text{glo}, \varphi^*) + s(\pi w^\text{glo}, \pi \varphi^*) \leq \|\pi \varphi^*\|_a \left( C^2 A \theta^m (m+1)^d \|\varphi^*\|^2_a \right) ,
\]
and this finishes the proof.

We can dig more estimates from this theorem. It is easy to see that \( \|f\|_{s-1} = O(H) \). Recalling eqs. (11) and (12), we have \( \|D^m g\|_a = O(1) \) and \( \|N^m q\|_a = O(1) \). From the proof of theorem [1] it holds that
\[
\|w^\text{glo}\|_a = \|\pi (u_0 + D^m g - N^m q)\|_a \leq \|\pi u_0\|_a + \|\pi D^m g\|_a + \|\pi N^m q\|_a \\
\leq O(H^{-1}) + O(1) .
\]
Meanwhile, by eq. (13), it follows that \( \|w^\text{glo}\|_a = O(1) \). If assuming that \( C^2 A \theta^m (m+1)^d = O(H^2) \), we will have
\[
\|w^m - D^m g + N^m q + \hat{g} - u\|_a = O(H) .
\]

3.3 Extensions

In this subsection, we will extend the CEM-GMsFEM to inhomogeneous Robin BVPs, and the model problem is stated as follows:
\[
\begin{align*}
- \text{div} (A \nabla u) &= f & \text{in } \Omega, \\
\mathbf{b} u + \nabla u &= q & \text{on } \partial \Omega,
\end{align*}
\]
where \( \mathbf{b}(x) \in L^\infty(\partial \Omega) \) is a heterogeneous coefficient depends on certain physical laws. We propose an assumption for \( \mathbf{b} \) to make this problem uniquely solvable in \( V \):

A4 The function \( \mathbf{b}(x) \geq 0 \) for a.e. \( x \in \partial \Omega \), and there exists a positive constant \( b_0 > 0 \) and a subset \( \Gamma \subset \partial \Omega \) with \( \text{meas}(\Gamma) > 0 \), such that \( \mathbf{b}(x) \geq b_0 \) for a.e. \( x \in \Gamma \).

The bilinear form \( a(w, v) \) needs to be modified as \( \int_\Omega A \nabla w \cdot \nabla v \, dx + \int_\partial \mathbf{b} w v \, ds \), and for a subset \( \omega \subset \Omega \) the norm \( \|\cdot\|_{a(\omega)} \) is also redefined accordingly. Meanwhile, the eigenvalue problem for constructing the auxiliary space \( V^\text{aux} \) will change into
\[
\int_{K_i} A \nabla \phi_i \cdot \nabla v \, dx + \int_{\partial \Omega \cap \partial K_i} \mathbf{b} \phi_i v \, ds = \lambda_i \int_{K_i} \kappa \phi_i v \, dx .
\]

The computational method consists of three steps:
We first examine the exponential convergence of \( N^m q \in V_i^m \) such that for all \( v \in V_i^m \),

\[
a(N_i^m q, v) + s(\pi N_i^m q, \pi v) = \int_{\partial K_i \cap \partial \Omega} q v \, d\sigma \quad \forall v \in V_i^m.
\]

Then obtain \( N^m q = \sum_{i=1}^{N} N_i^m q \).

**Step 2** Prepare the multiscale function space \( V_{ms} \) via eq. (7) with a modified bilinear form \( a(\cdot, \cdot) \).

**Step 3** Solve \( w^m \in V_{ms}^m \) such that for all \( v \in V_{ms}^m \),

\[
a(w^m, v) = \int_{\Omega} f v \, dx + \int_{\Gamma_n} q v \, d\sigma - a(N^m q, v).
\]

**Step 4** Construct the numerical solution to approximate the real solution as

\[
u \approx w^m + N^m q.
\]

The detailed numerical analysis is similar with section 3.2, and we hence omit it here.

## 4 Numerical experiments

In this section, we will present several numerical experiments\(^2\) to emphasize that the method proposed can retain accuracy in a high contrast coefficient setting. For simplicity, we take the domain \( \Omega = (0, 1) \times (0, 1) \) and pointwise isotropic coefficients, i.e., \( \mathbf{A}(x) = \kappa(x) \mathbf{I} \). In all experiments, the medium has two phases, which means \( \kappa(x) \) only takes two values \( \kappa_m \) and \( \kappa_l \) with \( 1 = \kappa_m \ll \kappa_l \). We will calculate reference solutions on a \( 400 \times 400 \) mesh with the bilinear Lagrange FEM, and \( \kappa(x) \) is also generated from \( 400 \times 400 \) px figures. We display two different medium configurations in fig. 4 and denote the first one by \textbf{cfg-a}, the second one by \textbf{cfg-b}. The coarse mesh size \( H \) will be chosen from \( \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{60} \). For simplifying the implementation, we specially choose \( \bar{\kappa} = 24 \kappa / H^2 \) instead of the original definition eq. (3), and all theoretical results should still hold if we only require \( \bar{\kappa} \) satisfying \( A \nabla \chi \cdot \nabla \chi \leq \bar{\kappa} \) for any partition function \( \chi \). Moreover, we always set \( l_1 = l_2 = \cdots = l_N = l_m \), i.e., the number of eigenvectors used to construct auxiliary space \( V_{ms}^{aux} \) is fixed as \( l_m + 1 \).

### 4.1 Model problem 1

We consider the following model problem:

\[
\begin{align*}
- \text{div} (\kappa(x_1, x_2) \nabla u) &= f(x_1, x_2) \quad \forall (x_1, x_2) \in \Omega, \\
u(x_1, x_2) &= \tilde{g}(x_1, x_2) = x_1^2 + \exp(x_1 x_2) \quad \forall (x_1, x_2) \in \partial \Omega,
\end{align*}
\]

where \( \kappa \) is generated from \textbf{cfg-a} with various \( \kappa_1 \) and the source term \( f \) is a piecewise constant function whose value are taken via fig. 4.

We first examine the exponential convergence of \( (D^m - D^{\text{ela}}) \tilde{g} \) by setting \( H = 1/20 \) and \( l_m = 2 \). The results are reported in table 4 where we introduce notations

\[
D^m_a := \frac{\| (D^m - D^{\text{ela}}) \tilde{g} \|_a}{\| D^{\text{ela}} \tilde{g} \|_a} \quad \text{and} \quad D^m_L := \frac{\| (D^m - D^{\text{ela}}) \tilde{g} \|_{L^2(\Omega)}}{\| D^{\text{ela}} \tilde{g} \|_{L^2(\Omega)}}
\]

to measure errors, and \( \Lambda' = \max_k \lambda_k^m \) as a reference for \( \Lambda \). We can see that the fluctuation of \( \Lambda' \) with respect to the contrast ratio \( \kappa_1 / \kappa_m \) is almost unnoticeable in our test cases, which partly explains the effectiveness of the CEM-GMsFEM in high contrast problems. This observation is quite interesting, and we can transform it into a formal mathematical conjecture: it is possible to bound \( \lambda \) which is an eigenvalue of

\[
\int_{\Omega} \kappa \nabla u \cdot \nabla v \, dx = \lambda \int_{\Omega} \kappa uv \, dx \quad \forall v \in H^1(\Omega)
\]

\(^2\) All the experiments are conducted by using Python with Numpy\footnote{Harris et al. [2020]} and Scipy\footnote{Virtanen et al. [2020]} libraries, codes are hosted on Github (https://github.com/Laphet/CEM-GMsFEM.git).
with \( C \text{diam}(\Omega)^d \) while the constant \( C \) is independent with \( \sup_x \kappa / \inf_x \kappa \)? As predicted in corollary 1, the convergence behavior of \( D^m_{\kappa} \) should solely depend on \( \Lambda \), and our numerical results strongly support this argument. Comparing \( \|D^{\text{glo}}\|_a \) with \( \|D^{\text{glo}}\|_{L^2(\Omega)} \), we can find \( \|D^{\text{glo}}\|_a \) is almost linearly dependent on contrast ratios while such a relation is not significant on \( \|D^{\text{glo}}\|_{L^2(\Omega)} \). However, the exponential convergence property can allow us to compensate errors from \( \|D^{\text{glo}}\|_a \) through a modest larger oversampling region.

In the second experiment, we fix \( \kappa_I / \kappa_m = 10^4 \) and \( l_m = 2 \) to display how numerical solution errors, i.e.,

\[
E^m_a := \frac{\|w^m - D^m \hat{\gamma} - u_0\|_a}{\|u\|_a} \quad \text{and} \quad E^m := \frac{\|w^m - D^m \hat{\gamma} - u_0\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}
\]

change with coarse mesh sizes \( H \) and oversampling layers \( m \). The results are reported in table 2. An important observation is that errors will increase if we only reduce \( H \) while not enlarge oversampling layers \( m \), which is distinct from traditional finite element methods. Actually, if focusing on \( m = 1 \), we can see that \( E^1_a \) grows like \( O(H^{-1}) \). Recalling the error expression in theorem 2 such a numerical evidence supports the estimate \( \|\pi w^{\text{glo}}\|_a = O(H^{-1}) \). Due to the competence between \( \|\pi w^{\text{glo}}\|_a \) and \( m \), the minimal error occurs in the setting (\( H = 1/40, m = 4 \)) rather than (\( H = 1/80, m = 4 \)). Another interesting point is that, when \( m = 1 \), errors in the \( L^2 \) norm (\( \approx 8\%) \) are greatly
smaller than ones in the energy norm (70% ∼ 600%). Since $L^2$ norms can not reflect small oscillations of functions, we may imagine that by gradually increasing oversampling layers, numerical solutions obtained by the CEM-GMsFEM capture “macroscale” information first then resolve “finescale” details.

Table 2: The numerical errors of the model problem eq. (23) in the energy and $L^2$ norm with different coarse mesh sizes $H$ and oversampling layers $m$, while contrast ratios and eigenvector numbers are fixed as $\kappa_l/\kappa_m = 10^4$ and $l_m = 2$.

| $H$ | 1/10 | 1/20 | 1/40 | 1/80 |
|-----|------|------|------|------|
| $\|E\|_2$ | 7.792 $\times 10^{-4}$ | 1.455 | 3.065 | 6.029 |
| $\|E\|_2$ | 4.023 $\times 10^{-2}$ | 8.161 $\times 10^{-2}$ | 2.005 $\times 10^{-1}$ | 4.401 $\times 10^{-1}$ |
| $\|E\|_2$ | 2.662 $\times 10^{-3}$ | 2.632 $\times 10^{-3}$ | 7.753 $\times 10^{-4}$ | 2.304 $\times 10^{-4}$ |
| $\|E\|_2$ | 2.308 $\times 10^{-3}$ | 4.283 $\times 10^{-4}$ | 3.041 $\times 10^{-4}$ | 1.035 $\times 10^{-4}$ |
| $\|E\|_2$ | 6.957 $\times 10^{-2}$ | 6.603 $\times 10^{-2}$ | 7.445 $\times 10^{-2}$ | 8.062 $\times 10^{-2}$ |
| $\|E\|_2$ | 6.789 $\times 10^{-4}$ | 3.237 $\times 10^{-3}$ | 1.664 $\times 10^{-2}$ | 4.581 $\times 10^{-2}$ |
| $\|E\|_2$ | 7.070 $\times 10^{-5}$ | 7.016 $\times 10^{-5}$ | 2.860 $\times 10^{-4}$ | 2.315 $\times 10^{-4}$ |
| $\|E\|_2$ | 6.638 $\times 10^{-5}$ | 4.857 $\times 10^{-6}$ | 1.079 $\times 10^{-5}$ | 1.079 $\times 10^{-6}$ |

In the third experiment, we focus on numerical errors with different contrast ratios and oversampling layers, and record the results in table 3 where we set $H = 1/80$ and $l_m = 2$. It is not surprise that high contrast ratios will deteriorate numerical accuracy. However, the exponential convergence in $m$ alleviates this deterioration. Actually, the numerical accuracy of $(\kappa_l/\kappa_m = 10^6, m = 4)$ improves almost 20 times comparing to $(\kappa_l/\kappa_m = 10^6, m = 3)$. Similarly, as emphasized in the second experiment, the $L^2$ norm errors (∼ 8%) are significantly smaller than energy norm errors (190% ∼ 600%) when $m = 1$. This phenomenon reveals the potential of the CEM-GMsFEM in discovering homogenized surrogate models of high contrast problems [Chung et al., 2018b].

Table 3: The numerical errors the model problem eq. (23) in the energy and $L^2$ norm with different contrast ratios $\kappa_l/\kappa_m$ and oversampling layers $m$, while coarse mesh sizes and eigenvector numbers are fixed as $H = 1/80$ and $l_m = 2$.

| $\kappa_l/\kappa_m$ | $1.000 \times 10^2$ | $1.000 \times 10^3$ | $1.000 \times 10^4$ | $1.000 \times 10^5$ |
|---------------------|-------------------|-------------------|-------------------|-------------------|
| $\|E\|_2$ | 1.944 | 6.029 | 1.902 $\times 10^2$ | 6.013 $\times 10^2$ |
| $\|E\|_2$ | 1.790 $\times 10^{-1}$ | 4.401 $\times 10^{-1}$ | 1.061 | 3.002 |
| $\|E\|_2$ | 7.882 $\times 10^{-3}$ | 2.301 $\times 10^{-2}$ | 7.097 $\times 10^{-3}$ | 2.076 $\times 10^{-3}$ |
| $\|E\|_2$ | 3.943 $\times 10^{-4}$ | 1.035 $\times 10^{-3}$ | 3.141 $\times 10^{-4}$ | 9.882 $\times 10^{-4}$ |
| $\|u\|_2$ | 2.826 | 2.841 | 2.843 | 2.843 |
| $\|u\|_{L^2(D)}$ | 8.866 $\times 10^{-2}$ | 8.062 $\times 10^{-2}$ | 8.086 $\times 10^{-2}$ | 8.089 $\times 10^{-2}$ |
| $\|u\|_{L^2(D)}$ | 1.330 $\times 10^{-2}$ | 4.581 $\times 10^{-2}$ | 6.632 $\times 10^{-2}$ | 7.250 $\times 10^{-2}$ |
| $\|u\|_{L^2(D)}$ | 2.968 $\times 10^{-3}$ | 2.315 $\times 10^{-3}$ | 2.174 $\times 10^{-3}$ | 1.657 $\times 10^{-3}$ |
| $\|u\|_{L^2(D)}$ | 1.079 $\times 10^{-4}$ | 1.079 $\times 10^{-4}$ | 9.175 $\times 10^{-5}$ | 1.241 $\times 10^{-4}$ |
| $\|u\|_{L^2(D)}$ | 1.853 | 1.853 | 1.853 | 1.853 |
We test numerical errors with different $l_m$ in the fourth experiment, and the other parameters are set as $(H = 1/80, \kappa_1/\kappa_m = 10^3, m = 3)$. It is natural that providing more eigenvectors in constructing $V^{aux}$ performance will be better. However, the numerical report table 4 shows that error may not be reduced proportionally to eigenvector numbers. Because adding more eigenvectors improves convergence rates through increasing $\Lambda$, which we only know the asymptotic behavior when $l_m \to \infty$ (i.e., Weyl’s law (Ivrii [2016])). In practice, 3 or 4 eigenvectors is enough for obtaining satisfying accuracy, and there are several rules of thumb in determining how many eigenvectors should be applied when high conductivity channels appear (Chung et al. [2018a]).

Table 4: The numerical errors the model problem eq. (23) in the energy and $L^2$ norm with different numbers ($l_m + 1$) of eigenvectors applied in constructing $V^{aux}$, while other parameters are fixed as $(H = 1/80, \kappa_1/\kappa_m = 10^3, m = 3)$.

| $l_m$ | 0    | 1     | 2     | 3     |
|-------|------|-------|-------|-------|
| $E^e_i$ | 8.002 × 10^{-4} | 4.932 × 10^{-4} | 2.301 × 10^{-4} | 2.109 × 10^{-4} |
| $E^l_i$ | 6.297 × 10^{-4} | 3.589 × 10^{-4} | 2.315 × 10^{-4} | 2.002 × 10^{-4} |

4.2 Model problem 2

In this subsection, we study the following inhomogeneous Neumann BVP:

\[
\begin{align*}
- \text{div} (\kappa(x_1, x_2) \nabla u) &= f(x_1, x_2) \quad \forall (x_1, x_2) \in \Omega, \\
u \cdot \kappa \nabla u &= q(x_1, x_2) \quad \forall (x_1, x_2) \in (0, 1) \times \{1\}, \\
u \cdot \kappa \nabla u &= q(x_1, x_2) = 1 \quad \forall (x_1, x_2) \in \{1\} \times (0, 1), \\
u \cdot \kappa \nabla u &= q(x_1, x_2) = 0 \quad \forall (x_1, x_2) \in (0, 0.5) \times \{0\},
\end{align*}
\]  

(24)

where $\kappa$ is generated from $\text{cfg-b}$ with various $\kappa_1$ and the source term $f$ is a piecewise constant function whose value are taken via fig. 1.

We first plot $\mathcal{N}^{\text{glo}} q$ under different contrast ratios ($\kappa_1/\kappa_m = 1, 10, 10^2, 10^3$) in fig. 2 and note that when $\kappa_1/\kappa_m = 1$ is the medium is homogeneous. Due to high conductivity channels reacting with boundary, we can see from fig. 2 that on the four corners of $\Omega$, $\mathcal{N}^{\text{glo}} q$ of the test case ($\kappa_1/\kappa_m = 1$) on $\Gamma_N$ is different from rest cases. Moreover, all figures show that $\mathcal{N}^{\text{glo}} q$ decays rapidly from boundaries, which is already predicted by corollary 1. We also collect several numerical results table 5 to illustrate this point, where notations

\[ N^m_a := \frac{\| (\mathcal{N}^m - \mathcal{N}^{\text{glo}} q) \|_a}{\| \mathcal{N}^{\text{glo}} q \|_a} \quad \text{and} \quad N^m := \frac{\| (\mathcal{A}^m - \mathcal{N}^{\text{glo}} q) \|_{L^2(\Omega)}}{\| \mathcal{D}^{\text{glo}} q \|_{L^2(\Omega)}} \]

are adopted. We can read from table 5 that changes of $\mathcal{A}'$ is quite small, which has been observed in section 4.1. Owing to this fact, even we multiply by 10 on contrast ratios column by column, convergence histories of $N^m_a$ along with $m$ are still similar. The major difference with table 1 is that $\| \mathcal{N}^{\text{glo}} q \|_a$ does not grow linearly with respect to contrast ratios. This can be explained by the estimate $\| \mathcal{N}^{\text{glo}} q \|_a \leq C_{tr} \| q \|_{L^2(\Gamma_N)}$, and we may postulate that the influence of contrast ratios on $C_{tr}$ is limited.

Table 5: All the numerical tests are performed under $H = 1/20$ and $l_m = 2$. The results include relative errors between $\mathcal{N}^{\text{glo}} q$ and $\mathcal{N}^m q$ with respect to different oversampling layers $m$ and contrast ratios $\kappa_1/\kappa_m$, and the values of $\mathcal{A}' = \max \lambda^m_{a_i}$, $\| \mathcal{N}^{\text{glo}} q \|_a$ and $\| \mathcal{N}^{\text{glo}} q \|_{L^2(\Omega)}$ with different contrast ratios.

| $\kappa_1/\kappa_m$ | $1.0 \times 10^2$ | $1.0 \times 10^3$ | $1.0 \times 10^4$ | $1.0 \times 10^5$ | $1.0 \times 10^6$ |
|---------------------|------------------|------------------|------------------|------------------|------------------|
| $\mathcal{A}'$     | 8.726 × 10^{-4} | 8.802 × 10^{-4} | 8.810 × 10^{-4} | 8.811 × 10^{-4} | 8.811 × 10^{-4} |
| $N^m_a$            | 9.941 × 10^{-3} | 9.949 × 10^{-3} | 9.949 × 10^{-3} | 9.949 × 10^{-3} | 9.949 × 10^{-3} |
| $N^m_a$            | 3.133 × 10^{-4} | 1.911 × 10^{-4} | 1.760 × 10^{-4} | 1.709 × 10^{-4} | 1.709 × 10^{-4} |
| $\| \mathcal{N}^{\text{glo}} q \|_a$ | 1.979 × 10^{-4} | 1.988 × 10^{-4} | 1.989 × 10^{-4} | 1.989 × 10^{-4} | 1.989 × 10^{-4} |
| $\| \mathcal{N}^{\text{glo}} q \|_{L^2(\Omega)}$ | 8.127 × 10^{-3} | 8.475 × 10^{-3} | 8.467 × 10^{-3} | 8.467 × 10^{-3} | 8.467 × 10^{-3} |

The next experiment is parallel to the third experiment in section 4.1, and the main objective is verifying the effectiveness of the method proposed in high contrast settings. We choose $H = 1/80$ and $l_m = 2$, and the results are listed in
4.3 Model problem 3

In this subsection, we consider the following inhomogeneous Robin BVP:

\[
\begin{align*}
    - \nabla \cdot (\kappa(x_1, x_2) \nabla u) &= f(x_1, x_2) \quad \forall (x_1, x_2) \in \Omega, \\
    \nu \cdot \kappa(x_1, x_2) \nabla u + b(x_1, x_2) u &= q(x_1, x_2) \quad \forall (x_1, x_2) \in \partial \Omega,
\end{align*}
\]

(25)

where \(\kappa(x_1, x_2)\) is generated from \texttt{cfg-b} with \(b(x_1, x_2) = \kappa(x_1, x_2)\), and \(q(x_1, x_2)\) is defined as

\[
q(x_1, x_2) = \begin{cases} 
-1 & \text{on } \{0\} \times (0, 1), \\
1 & \text{on } \{1\} \times (0, 1), \\
1 & \text{on } (0, 0.5) \times \{0\}, \\
0 & \text{on } (0.5, 1) \times \{0\}, \\
0 & \text{on } (0, 0.5) \times \{1\}, \\
-1 & \text{on } (0.5, 1) \times \{1\}.
\end{cases}
\]
Table 6: The numerical errors of the model problem eq. (24) in the energy and $L^2$ norm with different contrast rations $\kappa_1/\kappa_m$ and oversampling layers $m$, while coarse mesh sizes and eigenvector numbers are fixed as $H = 1/80$ and $l_m = 2$.

| $\kappa_1/\kappa_m$ | $1.000 \times 10^1$ | $1.000 \times 10^4$ | $1.000 \times 10^7$ | $1.000 \times 10^9$ |
|----------------------|----------------------|----------------------|----------------------|----------------------|
| $E_{1}^E$            | $5.847 \times 10^{-4}$ | $4.263 \times 10^{-1}$ | $3.956 \times 10^{-4}$ | $3.922 \times 10^{-1}$ |
| $E_{2}^E$            | $3.597 \times 10^{-1}$ | $7.894 \times 10^{-1}$ | $9.280 \times 10^{-7}$ | $9.461 \times 10^{-1}$ |
| $E_{3}^E$            | $9.784 \times 10^{-4}$ | $6.477 \times 10^{-3}$ | $5.321 \times 10^{-2}$ | $3.491 \times 10^{-1}$ |
| $E_{4}^E$            | $<1.000 \times 10^{-6}$ | $<1.000 \times 10^{-6}$ | $6.489 \times 10^{-9}$ | $7.804 \times 10^{-4}$ |
| $\|u\|_{L^2(\Omega)}$ | $1.942 \times 10^{-2}$ | $1.575 \times 10^{-2}$ | $1.541 \times 10^{-2}$ | $1.538 \times 10^{-2}$ |

Note that $b(x_1, x_2)$ is also high contrast now, and traditional methods need a very fine mesh to resolve the channel structure and special treatments to solve final linear systems. The numerical results of our method are listed in table 7, which shows that the reliability of our method even with a $10^6$ contrast ratio.

Table 7: The numerical errors of the model problem eq. (25) in the energy and $L^2$ norm with different contrast rations $\kappa_1/\kappa_m$ and oversampling layers $m$, while coarse mesh sizes and eigenvector numbers are fixed as $H = 1/80$ and $l_m = 2$.

| $\kappa_1/\kappa_m$ | $1.000 \times 10^1$ | $1.000 \times 10^4$ | $1.000 \times 10^7$ | $1.000 \times 10^9$ |
|----------------------|----------------------|----------------------|----------------------|----------------------|
| $E_{1}^E$            | $5.393 \times 10^{-4}$ | $3.960 \times 10^{-1}$ | $3.711 \times 10^{-4}$ | $3.684 \times 10^{-1}$ |
| $E_{2}^E$            | $2.348 \times 10^{-4}$ | $3.248 \times 10^{-1}$ | $3.544 \times 10^{-4}$ | $3.581 \times 10^{-1}$ |
| $E_{3}^E$            | $2.433 \times 10^{-2}$ | $2.933 \times 10^{-2}$ | $2.847 \times 10^{-2}$ | $2.175 \times 10^{-1}$ |
| $E_{4}^E$            | $1.315 \times 10^{-3}$ | $1.315 \times 10^{-3}$ | $3.144 \times 10^{-3}$ | $1.046 \times 10^{-2}$ |
| $\|u\|_{L^2(\Omega)}$ | $2.384 \times 10^{-1}$ | $2.342 \times 10^{-1}$ | $2.311 \times 10^{-1}$ | $2.308 \times 10^{-1}$ |

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