N=1 de Sitter Supersymmetry Algebra

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Abstract

Recalling the universal covering group of de Sitter universe, the transformation properties of the spinor fields $\psi(x)$ and $\overline{\psi}(x)$, in the ambient space notation, are presented in this paper. The charge conjugation symmetry of the de Sitter spinor field is then discussed in the above notation. Using this spinor field and charge conjugation, de Sitter supersymmetry algebra in the ambient space notation has been established. It is shown that a novel de Sitter superalgebra can be attained by the use of spinor field and charge conjugation in the ambient space notation.

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1 Introduction

Recent astrophysical data received from type Ia supernovas indicate that our universe might currently be in a de Sitter (dS) phase. Therefore it is important to find a formulation of de Sitter quantum field theory with the same level of completeness and rigor as for its Minkowskian counterpart. Some questions, however, are usually put forth for the non-existence of supersymmetry models with a positive cosmological constant, i.e. supersymmetry in de Sitter space. Such arguments are often based on the non-existence of Majorana spinors for $O(4,1)$ [1, 2]. Pilch et al have shown that if for every spinor, its independent charge-conjugate could be defined, de Sitter supergravity can be established with even $N$ [1].

Bros et al. [3] presented a QFT of scalar free field in de Sitter space that closely mimics QFT in Minkowski space. We have generalized the Bros’s quantization of field, to quantization of fields with various spins in de Sitter space [4]. In continuation of previous works where the charge-conjugate spinor had been defined [5, 6], the supersymmetry in the ambient space notation has been studied in the present paper. Section two has been devoted to the discussion of the de Sitter group $SO(1,4)$, i.e. space-time symmetry of de Sitter space, and its universal

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covering group $Sp(2,2)$. Recalling the transformation properties of the spinor fields $\psi(x), \overline{\psi}(x)$, the charge conjugation symmetry of the de Sitter spinor field in ambient space notation is discussed in section 3. The general de Sitter superalgebra is presented in section 4. It is shown that a novel dS-superalgebra can be attained by the use of the spinor fields and charge conjugation in the ambient space notation. Finally, a brief conclusion and an outlook have been given in section 5. To illustrate the novel algebra, the generalized Jacobi identities are calculated in appendix A.

## 2 de Sitter group

The de Sitter space is an elementary solution of the positive cosmological Einstein equation in the vacuum. It is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4,$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The de Sitter metrics reads

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta |_{x^2 = -H^{-2}} = g^dS_{\mu\nu} dX^\mu dX^\nu, \quad \mu = 0, 1, 2, 3,$$

where the $X^\mu$'s are the 4 space-time intrinsic coordinates on dS hyperboloid. Different coordinate systems can be chosen for $X^\mu$. A 10-parameter group $SO_0(1,4)$ is the kinematical group of the de Sitter universe. In the limit $H = 0$, this reduces to the Poincaré group. There are two Casimir operators

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta},$$

$$Q^{(2)} = -W_\alpha W^\alpha, \quad W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^\beta\gamma L^\delta\eta,$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor and the $L_{\alpha\beta}$'s are the infinitesimal generators, which obey the commutation relations

$$[L_{\alpha\beta}, L_{\gamma\delta}] = -i(\eta_{\alpha\gamma} L_{\beta\delta} + \eta_{\beta\delta} L_{\alpha\gamma} - \eta_{\alpha\delta} L_{\beta\gamma} - \eta_{\beta\gamma} L_{\alpha\delta}).$$

$L_{\alpha\beta}$ can be represented as $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$, where $M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$ is the “orbital” part and $S_{\alpha\beta}$ is the “spinorial” part. The form of the $S_{\alpha\beta}$ depends on the spin of the field. For spin $\frac{1}{2}$ field, it can be defined as

$$S_{\alpha\beta} = -\frac{i}{4} [\gamma_\alpha, \gamma_\beta],$$

where the five $4 \times 4$ matrices $\gamma^\alpha$ are the generators of the Clifford algebra based on the metric $\eta_{\alpha\beta}$:

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta} \mathbb{1}, \quad \gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0.$$

An explicit and convenient representation is provided by [7, 8, 9]

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the $2 \times 2$ identity matrix.
\( \gamma^1 = \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \) \( (6) \)

where \( \Pi \) is the unit \( 2 \times 2 \) matrix and \( \sigma^i \) are the Pauli matrices. This representation had been proved to be useful in analysis of the physical relevance of the group representation \([8]\). In this representation,

\[ \gamma^\alpha T = \gamma^4 \gamma^2 \gamma^\alpha \gamma^4. \]

The spinor wave equation in de Sitter space-time has been originally deduced by Dirac in 1935 \([10]\), and can be obtained from the eigenvalue equation of the second order Casimir operator \([7, 8]\)

\[ (-i \not{x} \gamma \bar{\partial} + 2i + \nu) \psi(x) = 0, \] \( (7) \)

where \( \not{x} = \eta_{\alpha\beta} \gamma^\alpha x^\beta \) and \( \bar{\partial}_a = \partial_a + H^2 x_a x \cdot \partial. \) Due to covariance of the de Sitter group, the adjoint spinor is defined as follows \([8]\):

\[ \bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0 \gamma^4. \] \( (8) \)

Let us now recall the transformation properties of the spinor fields \( \psi(x) \) and \( \bar{\psi}(x) \). The two-fold, universal covering group of \( SO_0(1, 4) \), is the (pseudo-)symplectic group \( Sp(2, 2) \), \([9]\)

\[ Sp(2, 2) = \{ g \in Mat(2; \mathbb{H}) : \det g = 1, \ g^\dagger \gamma^0 g = \gamma^0 \}, \] \( (9) \)

where \( g^\dagger = g^T \tilde{g}, \tilde{g} \) is the transposed of \( g \) and \( \tilde{g} \) the quaternionic conjugate of \( g \). It should be noted that quaternionic \( P \) is

\[ P = (x^4, \bar{x}) = x^4 \Pi + ix^1 \sigma^1 - ix^2 \sigma^2 + ix^3 \sigma^3, \] \( (10) \)

where \( \sigma^i \) are the Pauli matrices and \( \tilde{P} = (x^4, -\bar{x}) \) is its conjugate. For obtaining the isomorphic relation between the two groups we define the matrices \( X \) associated with \( x \in X_H \) by:

\[ X = \begin{pmatrix} x^0 \\ \mathcal{P} \\ x^0 \end{pmatrix}. \] \( (11) \)

Through representation (6) of the \( \gamma \) matrices, \( X \) can be written in the following form:

\[ \not{x} = x \gamma = X \gamma^0 = \begin{pmatrix} x^0 \\ \mathcal{P} \\ -x^0 \end{pmatrix}. \] \( (12) \)

The transformation of \( X \), under the action of the group \( Sp(2, 2) \) is

\[ X' = gX\tilde{g}, \ \not{x}' = g \not{x}g^{-1}. \] \( (13) \)

For \( \Lambda \in SO_0(1, 4) \) and \( g \in Sp(2, 2) \) we have

\[
x'^\alpha = \eta^{\alpha\beta} x'_\beta = \frac{1}{4} tr(\gamma^\alpha \gamma^\beta) x'_\beta = \frac{1}{4} tr(\gamma^\alpha g \not{x} g^{-1})
\]

\[
= \frac{1}{4} tr(\gamma^\alpha g \gamma^\beta g^{-1}) x_\beta = \Lambda^{\alpha\beta}(g) x_\beta.
\] \( (14) \)
For every $g \in Sp(2, 2)$, there exists a transformation $\Lambda \in SO_0(1, 4)$, which satisfies the following relations
\[
\Lambda^\alpha_\beta(g) = \frac{1}{4} tr(\gamma^\alpha g \gamma_\beta g^{-1}), \quad \Lambda^\alpha_\beta g = g \gamma^\alpha g^{-1}.
\] (15)

The isomorphic relation between the two groups is
\[
SO_0(1, 4) \approx Sp(2, 2)/\mathbb{Z}_2.
\] (16)

The transformation laws for the $\psi(x)$ and its adjoint $\bar{\psi}(x)$, under which the de Sitter-Dirac equation is covariant, are :
\[
\psi(x) \rightarrow \psi'(x) = g^{-1}\psi(\Lambda(g)x),
\] (17)
\[
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(\Lambda(g)x)i(g),
\] (18)
where $i(g) \equiv -\gamma^4 g \gamma^4 [8, 9]$. Similar to the Minkowskian space, it is useful to define $g$ by
\[
g = \exp[-\frac{i}{2} \omega^{\alpha\beta} S_{\alpha\beta}], \quad \omega^{\alpha\beta} = -\omega^{\beta\alpha},
\] (19)
where $\gamma^0 g^\dagger \gamma^0 = g^{-1}$, i.e. $g \in Sp(2, 2)$.

## 3 Charge conjugation

Previously, the charge conjugation spinor $\psi^c$ was calculated in the ambient space notation [6]
\[
\psi^c = \eta_c C(\gamma^4)^{T}(\bar{\psi})^{T},
\] (20)
where $\eta_c$ is an arbitrary unobservable phase value, generally set to unity. In the present framework charge conjugation is an antilinear transformation. In the $\gamma$ representation (6) we had obtained [6]:
\[
C\gamma^0 C^{-1} = -\gamma^0, C\gamma^4 C^{-1} = -\gamma^4
\]
\[
C\gamma^1 C^{-1} = -\gamma^1, C\gamma^3 C^{-1} = -\gamma^3, C\gamma^2 C^{-1} = \gamma^2.
\] (21)

In this representation, $C$ commutes with $\gamma^2$, and anticommutes with other $\gamma$-matrices. Therefore the simple choice may be taken to be $C = \gamma^2$, where the following relation is satisfied
\[
C = -C^{-1} = -C^{T} = -C^{\dagger}.
\] (22)

This clearly illustrates the non-singularity of $C$.

The adjoint spinor, defined by $\bar{\psi}(x) \equiv \psi^{\dagger}(x)\gamma^0 \gamma^4$, transforms in a different way from $\psi$, under de Sitter transformation. It is easily shown that $\psi^c$ transforms in the same way as $\psi$ [6]
\[
\psi^c(x') = g(\Lambda)\psi^c(x).
\]

The wave equation of $\psi^c$ is different from the wave equation of $\psi$ by the sign of the ”charge” $q$ and $\nu$ [6]. Thus it follows that if $\psi$ describes the motion of a dS-Dirac ”particle” with the charge $q$, $\psi^c$ represents the motion of a dS-Dirac ”anti-particle” with the charge ($-q$). In other words $\psi$ and $\psi^c$ can describe ”particle” and ”antiparticle” wave functions. $\psi$ and $\psi^c$ are charge conjugation of each other
\[
(\psi^c)^c = C\gamma^0 \psi^c = C\gamma^0(C\gamma^0 \psi) = \psi.
\] (23)
4 N=1 Supersymmetry Algebra

Supersymmetry in de Sitter space has been studied by Pilch et al [1]. Recently, supersymmetry has been investigated in constant curvature space as well [11, 12]. In this section we have presented the supersymmetry algebra in ambient space notation. It is shown that if the spinor field and the charge conjugation operators are defined in the ambient space notation, a novel de Sitter superalgebra can be attained.

In order to extend the de Sitter group, the generators of supersymmetry transformation $Q_i^n$ are introduced. Here $i$ is the spinor index ($i = 1, 2, 3, 4$) and $n$ is the supersymmetry index $n = 1, ..., N$. $Q_i^n$'s are superalgebra spinor generators which transform as de Sitter group spinors. The generators $\tilde{Q}_i^n$ are defined by

$$\tilde{Q}_i = \left( Q^T \gamma^4 C \right)_i = \bar{Q}^c_i, \quad (24)$$

where $Q^T$ is the transpose of $Q$. It can be shown that $\tilde{Q} \gamma^4 Q$ is a scalar field under the de Sitter transformation [6].

For $N \neq 1$, closure of algebra requires extra bosonic generators. These do not necessarily commute with other generators and consequently are not central charges. They are internal symmetry generators [1]. These generators, shown by $T_{mn}$, commute with de Sitter generators.

Therefore the de Sitter superalgebra in four-dimensional space-time has the following generators:

- $M_{\alpha\beta}$, the generators of de Sitter group,
- the internal group generators $T_{nm}$, that are defined by the additional condition

$$T_{nm} = -T_{mn}; n, m = 1, 2, ..., N,$$

- the 4-component dS-Dirac spinor generators,

$$Q_i^n, \quad i = 1, 2, 3, 4, \quad n = 1, 2, ..., N.$$

To every generator $A$, a grade $p_a$ is assigned. For the fermionic generator $p_a = 1$, and for the bosonic generator $p_a = 0$. The bilinear product operator is defined by

$$[A, B] = -(-1)^{p_a p_b} [B, A]. \quad (25)$$

The generalized Jacobi identities is

$$(-1)^{p_a p_c} [A, [B, C]] + (-1)^{p_c p_b} [C, [A, B]] + (-1)^{p_b p_a} [B, [C, A]] = 0. \quad (26)$$

Using different generalized Jacobi identities, similar to the method presented by Pilch el al. [1], the full dS-superalgebra can be written in the following form:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\delta}),$$

$$[T_{rl}, T_{pm}] = -i(\omega_{rp} T_{lm} + \omega_{lm} T_{rp} - \omega_{rm} T_{lp} - \omega_{lp} T_{rm}),$$
\[ [M_{\alpha\beta}, T_{i \ell}] = 0, \]
\[ [Q_i^r, M_{\alpha\beta}] = (S_{\alpha\beta} Q^r)_i, \quad [\tilde{Q}_i^r, M_{\alpha\beta}] = -(\tilde{Q}^r, S_{\alpha\beta})_i \]
\[ [Q_i^r, T_ip] = -(\omega^r Q_p^r - \omega^p Q_i^r) \]
\[ \{Q_i^r, Q_j^l\} = \omega^{rt} \left( S^{\alpha\beta} \gamma^4 \gamma^2 \right)_{ij} M_{\alpha\beta} + \left( \gamma^4 \gamma^2 \right)_{ij} T^{rl}, \]

where \( S_{\alpha\beta} \) is defined by (4). The following relations are used to determine the above structure
\[
\left( S^{\alpha\beta} \gamma^4 \gamma^2 \right)^T = \left( S^{\alpha\beta} \gamma^4 \gamma^2 \right), \quad \left( \gamma^\alpha \gamma^4 \gamma^2 \right)^T = -\gamma^\alpha \gamma^4 \gamma^2, \quad \left( \gamma^4 \gamma^2 \right)^T = -\gamma^4 \gamma^2. \tag{27}
\]

It is necessary to obtain matrix \( \omega \), which determines the structure of the internal group. For even \( N \) de Sitter supersymmetry algebra, studied by Pilch et al [1], the matrix \( \omega \) has been obtained explicitly. A new dS-superalgebra, defined in the ambient space notation, is introduced in this stage for \( N = 1 \) case. In this case, \( T_{11} = 0, \omega = 1 \) and the simple de Sitter supersymmetry algebra is defined by the following relations:
\[
[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\alpha\delta} M_{\beta\gamma} - \eta_{\beta\gamma} M_{\alpha\delta}), \tag{28}
\]
\[
\{Q_i, Q_j\} = \left( S^{\alpha\beta} \gamma^4 \gamma^2 \right)_{ij} M_{\alpha\beta}, \tag{29}
\]
\[
[Q_i, M_{\alpha\beta}] = (S_{\alpha\beta} Q)_i, \quad [\tilde{Q}_i, M_{\alpha\beta}] = -(\tilde{Q} S_{\alpha\beta})_i. \tag{30}
\]

This can be proved by the use of generalized Jacobi identities (appendix).

Finally we present an explicit form of the supersymmetric generators which satisfy the above relations. We consider a superspace with bosonic coordinates \( x^\alpha \) and fermionic coordinates \( \theta_i \) where \( \theta_i \) is a four component de Sitter-Dirac Grassmann spinor in the ambient space notation. The suitable representation of these superalgebra generators in superspace are provided by
\[
\left\{ \begin{array}{l}
M_{\alpha\beta} = -i(x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha} + \frac{\partial}{\partial \theta_{\alpha}} S_{\alpha\beta} \theta), \\
Q = \gamma. \partial \theta + \frac{\partial}{\partial \theta},
\end{array} \right. \tag{31}
\]

where \( \frac{\partial}{\partial \theta_i} = (\gamma^2 \gamma^4)_{ik} \frac{\partial}{\partial \theta_k} \). Using the equation \( \{\theta_i, \frac{\partial}{\partial \theta_j}\} = \delta_{ij} \) and the following identities [11],
\[
(S^{\alpha\beta} \gamma^4 \gamma^2)_{ij} (S_{\alpha\beta})_{kl} + (S^{\alpha\beta} \gamma^4 \gamma^2)_{jk} (S_{\alpha\beta})_{il} + (S^{\alpha\beta} \gamma^4 \gamma^2)_{ki} (S_{\alpha\beta})_{jl} = 0, \]
\[
(S^{\alpha\beta})_{ij} (S_{\alpha\beta})_{kl} + (S^{\alpha\beta})_{il} (S_{\alpha\beta})_{kj} = (\gamma^\alpha)_{ij} (\gamma^\alpha)_{kl} + (\gamma^\alpha)_{il} (\gamma^\alpha)_{kj},
\]
it is straightforward to prove the above de Sitter supersymmetry algebra.

5 Conclusions

The formalism of the quantum field in de Sitter universe, in ambient space notation, is very similar to the quantum field formalism in Minkowski space. In this paper we present the de Sitter supersymmetry algebra in this notation, which is independent of the choice of the coordinate system. In addition, a novel superalgebra (28–30) has been obtained, which do not fall into the categories considered in previous works [1, 11, 13]. The importance of this formalism
may be shown further by the consideration of the linear quantum gravity and supergravity in
de Sitter space, which lays a firm ground for further study of the evolution of the universe.

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A \( N = 1 \) de Sitter superalgebra

The generalized Jacobi identities \((M, M, Q)\) and \((M, Q, Q)\) can be easily utilized to prove the
de Sitter supersymmetry algebra \((28 - 30)\). The generalized Jacobi identity \((M, M, Q)\) is

\[
[[M_{\alpha\beta}, M_{\gamma\delta}], Q_i] + [[Q_i, M_{\alpha\beta}], M_{\gamma\delta}] + [[M_{\gamma\delta}, Q_i], M_{\alpha\beta}] = 0. \tag{32}
\]

Using the equations \((28)\) and \((30)\), the following relations can be obtained:

\[
[[M_{\alpha\beta}, M_{\gamma\delta}], Q_i] = -i\eta_{\alpha\gamma}[M_{\beta\delta}, Q_i] - i\eta_{\beta\delta}[M_{\alpha\gamma}, Q_i] + i\eta_{\alpha\delta}[M_{\beta\gamma}, Q_i] + i\eta_{\beta\gamma}[M_{\alpha\delta}, Q_i],
\]

\[
[[Q_i, M_{\alpha\beta}], M_{\gamma\delta}] = [(S_{\alpha\beta})^i_j Q_j, M_{\gamma\delta}] = (S_{\alpha\beta})^i_j (S_{\gamma\delta})^k_l Q_k,
\]

\[
[[M_{\gamma\delta}, Q_i], M_{\alpha\beta}] = -(S_{\gamma\delta})^i_j Q_j, M_{\alpha\beta} = -(S_{\gamma\delta})^i_j (S_{\alpha\beta})^k_l Q_k.
\]

Implementing the above equations in the eq \((32)\),

\[-i\eta_{\alpha\gamma}[M_{\beta\delta}, Q_i] - i\eta_{\beta\delta}[M_{\alpha\gamma}, Q_i] + i\eta_{\alpha\delta}[M_{\beta\gamma}, Q_i] + i\eta_{\beta\gamma}[M_{\alpha\delta}, Q_i] + (S_{\alpha\beta})^i_j (S_{\gamma\delta})^k_l Q_k - (S_{\gamma\delta})^i_j (S_{\alpha\beta})^k_l Q_k = 0,
\]

\[i\eta_{\alpha\gamma}(S_{\beta\delta} Q_i) + i\eta_{\beta\delta}(S_{\alpha\gamma} Q_i) - i\eta_{\alpha\delta}(S_{\beta\gamma} Q_i) - i\eta_{\beta\gamma}(S_{\alpha\delta} Q_i) + [(S_{\alpha\beta})^i_j (S_{\gamma\delta})^k_l Q_k - (S_{\gamma\delta})^i_j (S_{\alpha\beta})^k_l Q_k = 0,
\]

\[i[(\eta_{\alpha\gamma} S_{\beta\delta} + \eta_{\beta\delta} S_{\alpha\gamma} - \eta_{\alpha\delta} S_{\beta\gamma} - \eta_{\beta\gamma} S_{\alpha\delta}) Q_i + [(S_{\alpha\beta} S_{\gamma\delta} - S_{\gamma\delta} S_{\alpha\beta}) Q_i = 0,
\]

it is shown that \(S_{\alpha\beta}\)'s satisfy the following commutation relation

\[[S_{\alpha\beta}, S_{\gamma\delta}] = -i(\eta_{\alpha\gamma} S_{\beta\delta} + \eta_{\beta\delta} S_{\alpha\gamma} - \eta_{\alpha\delta} S_{\beta\gamma} - \eta_{\beta\gamma} S_{\alpha\delta}).\]

This is none other than equation \((3)\). This generalized Jacobi identity verifies the relation \((30)\),
\textit{i.e.} commutation relation of \(Q\) and \(M\).

The generalized Jacobi identity \((M, Q, Q)\) is

\[
[[Q_i, Q_j], M_{\alpha\beta}] + [[M_{\alpha\beta}, Q_i], Q_j] - [[Q_j, M_{\alpha\beta}], Q_i] = 0. \tag{33}
\]

By the use of equations \((29)\) and \((30)\), we obtain

\[
(S^{\gamma\delta} \gamma^4 \gamma^2)_{ij} [M_{\gamma\delta}, M_{\alpha\beta}] - (S_{\alpha\beta})_{ik} Q_k, Q_j - (S_{\alpha\beta})_{ik} Q_k, Q_i] = 0,
\]

\[
(S^{\gamma\delta} \gamma^4 \gamma^2)_{ij} [M_{\gamma\delta}, M_{\alpha\beta}] - (S_{\alpha\beta})_{ik} (S^{\gamma\delta} \gamma^4 \gamma^2)_{kj} M_{\gamma\delta} - (S_{\alpha\beta})_{ik} (S^{\gamma\delta} \gamma^4 \gamma^2)_{ki} M_{\gamma\delta} = 0,
\]

\[
\]
\[
(S^\gamma_\delta \gamma^4 \gamma^2)_{ij} [M_{\gamma\delta}, M_{\alpha\beta}] - (S_{\alpha\beta} S^\gamma_\delta \gamma^4 \gamma^2)_{ij} M_{\gamma\delta} - (S^\gamma_\delta S_{\alpha\beta} \gamma^2 \gamma^4)_{ij} M_{\gamma\delta} = 0.
\]
Utilizing equation (27), we obtain
\[
(S^\gamma_\delta \gamma^4 \gamma^2)_{ij} [M_{\gamma\delta}, M_{\alpha\beta}] - (S_{\alpha\beta} S^\gamma_\delta \gamma^4 \gamma^2)_{ij} M_{\gamma\delta} - (S^\gamma_\delta S_{\alpha\beta} \gamma^2 \gamma^4)_{ij} M_{\gamma\delta} = 0.,
\]
\[
S^\gamma_\delta [M_{\gamma\delta}, M_{\alpha\beta}] - [S_{\alpha\beta}, S^\gamma_\delta] M_{\gamma\delta} = 0.
\]
This generalized Jacobi identity once again, verifies the anti-commutation relation of \(Q_i\) and \(Q_j\).

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