DIFFERENTIAL OPERATORS ON HITCHIN VARIETY

ANOOP SINGH

ABSTRACT. We introduce the notion of Hitchin variety over \( \mathbb{C} \). Let \( L \) be a holomorphic line bundle over a Hitchin variety \( X \). We investigate the space of all global sections of sheaf of differential operators \( D^k(L) \) and symmetric powers of sheaf of first order differential operators \( S^k(D^1(L)) \) over \( X \) and show that for a projective Hitchin variety both the spaces are one dimensional. As an application, we show that the space \( \mathcal{C}(L) \) of holomorphic connections on \( L \) does not admit any non-constant regular function.

1. Introduction

Let \( C \) be a smooth projective curve over \( \mathbb{C} \) of genus \( g \geq 2 \). Let \( \mathcal{M} := \mathcal{M}(r, d) \) be the moduli space of stable vector bundles of rank \( r \) and degree \( d \) on \( C \), where \( r \) and \( d \) are coprime. Then \( \mathcal{M} \) is a smooth projective variety of dimension \( r^2(g - 1) + 1 \). In [Hit87], Hitchin proved that the cotangent bundle \( T^* \mathcal{M} \) is an algebraically completely integrable Hamiltonian system, that is, we get a Hitchin fibration (for more details see [Hit87]). Let \( \mathcal{M}_{\text{Higgs}} \) denote the moduli space of stable Higgs bundles of rank \( r \) and degree \( d \) over \( C \). Then \( T^* \mathcal{M} \) is an open subset of \( \mathcal{M}_{\text{Higgs}} \) with codimension of \( \mathcal{M}_{\text{Higgs}} \setminus T^* \mathcal{M} \) in \( \mathcal{M}_{\text{Higgs}} \) is at least 2. Motivated by the properties of Hitchin fibration, we introduce the notion of Hitchin variety, which is defined in section 2. We give one more example of Hitchin variety in section 2.

The study of space of global sections of sheaves over an algebraic variety is always an interesting but a difficult problem. In [Bis02], Biswas studied the global sections of sheaves of differential operators on a polarised abelian variety. He showed that the space of global sections of sheaves of differential operators on an ample line bundle over an abelian variety is of dimension one. Motivated by this and some specific properties of a Hitchin variety, we shall study the space of global sections of sheaves of differential operators on any holomorphic line bundle and also on holomorphic vector bundle, using different techniques.

In [Hit87, Theorem 6.2, p.n. 110], Hitchin computed the global sections of the symmetric powers of tangent bundle of the moduli space of stable Higgs bundles of rank 2 and odd degree over a compact Riemann surface of genus \( g \geq 2 \).

In [Bis04, Lemma 4.1, p.n. 428], the global sections of symmetric powers of Atiyah algebra of a generalized theta line bundle \( \Theta \) over the moduli space \( \mathcal{N}_C^d(L) \) of stable vector bundles of rank \( r \) with fixed determinant \( L \) of degree \( d \), where \( r \) and \( d \) are coprime, have been studied. In [Sin20, Theorem 1.4], a similar method has been used to compute the algebraic functions on the moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface with fixed residues.

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To the best of our knowledge, the global sections of sheaves of differential operators on a holomorphic line bundle over the moduli space $\mathcal{M} = \mathcal{M}(r, d)$ have not been studied.

Let $X$ be a Hitchin variety (see the definition in the beginning of section 2) and $L$ be a holomorphic line bundle over $X$. For $k \geq 0$, we denote by $\mathcal{D}^k(L)$ the vector bundle over $X$ defined by the sheaf $\mathcal{D}iff^k_X(L, L)$ of differential operators on $L$. In Theorem 2.2, we show that

$$H^0(X, S^k(\mathcal{D}^1(L))) \cong H^0(X, \mathcal{O}_X)$$

for every $k \geq 0$, where $S^k(\mathcal{D}^1(L))$ denotes the $k$-th symmetric power of $\mathcal{D}^1(L)$. Moreover, if $X$ is a projective Hitchin variety, then we have (see Corollary (2.3))

1. $H^0(X, S^k(\mathcal{D}^1(L))) = \mathbb{C}$
2. $H^0(X, \mathcal{D}^k(L)) = \mathbb{C}$

Also, if $E$ is a holomorphic vector bundle over $X$, then in Theorem 2.4 we state the same result for vector bundle $\mathcal{B}(E)$ constructed in (2.36). As a consequence of Theorem 2.4 for a projective Hitchin variety $X$, we get that (see Corollary (2.5))

$$H^0(X, S^k(\mathcal{B}(E))) = \mathbb{C}.$$

In the last section 3, we consider the variety $\mathcal{C}(L)$ of holomorphic connections on $L$, and in Theorem 3.1 we prove that $H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \mathbb{C}$.

2. Differential operators on line bundles

A smooth algebraic variety $X$ over $\mathbb{C}$ of dimension $n$ is said to be Hitchin variety if there exist normal noetherian schemes $\mathcal{N}, V$ over $\mathbb{C}$, where $V$ is of dimension $n$, and a surjective proper morphism

$$\mathcal{H} : \mathcal{N} \rightarrow V$$

such that the following properties are satisfied.

1. The cotangent bundle $T^*X$ is an open subset of $\mathcal{N}$ with codimension of the complement $\mathcal{N} \setminus T^*X$ in $\mathcal{N}$ is at least 2.
2. A generic fibre of $\mathcal{H}$ is of the form

$$\mathcal{H}^{-1}(v) = A_v$$

such that

$$\mathcal{H}^{-1}(v) \cap T^*X = A_v \setminus F_v$$

where $v \in V$, $A_v$ is some abelian variety and $F_v$ is a closed subvariety of $A_v$ with $\text{codim}(F_v, A_v) \geq 2$.

Note that the tangent bundle of an abelian variety is trivial, hence the vector fields on generic fibres of $\mathcal{H}$ contained in $T^*X$ are constant, which follows from Hartog's theorem.

Such a phenomenon occurs naturally in the study of moduli space of stable vector bundles over a compact Riemann surface, where the cotangent bundle is an algebraically completely integrable Hamiltonian system, and is contained as an open subset in the moduli space of stable Higgs bundles with the codimension of its compliment in the moduli space of stable Higgs bundles is at least 2 (see [Hit87]). The property mentioned in the definition of Hitchin variety is known as algebraic complete integrability. See [Boa18].
Definition 1] for a general definition of finite-dimensional complex algebraic integrable Hamiltonian system.

Another example of Hitchin variety (see [BGH13 section 4, p.n. 1185]) is as follows.

Let $Y$ be a compact Riemann surface of genus $g \geq 3$. Let $G$ be a non-trivial connected semi-simple linear algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $\gamma \in \pi_1(G)$ be the topological type of a holomorphic principal $G$-bundles $E_G$ over $Y$. Let $\mathcal{M}^\gamma(Y, G)$ denote the moduli space of semi-stable holomorphic principal $G$-bundle $E_G$ over $Y$ of topological type $\gamma$. A stable principal $G$-bundle $E_G$ over $Y$ is called regularly stable if the automorphism group $\text{Aut}(E_G)$ is just the center of $G$. The regularly stable locus

$$\mathcal{M}^{\gamma, rs}(Y, G) \subseteq \mathcal{M}^\gamma(Y, G)$$

is open, and coincides with the smooth locus of $\mathcal{M}^\gamma(Y, G)$; see [BH12, Corollary 3.4]. Then the moduli space $\mathcal{M}^{\gamma, rs}(Y, G)$ is a Hitchin variety, because $\mathcal{M}^{\gamma, rs}(Y, G)$ is a smooth quasi-projective variety with $T^*\mathcal{M}^{\gamma, rs}(Y, G) \subseteq \mathcal{M}^{\gamma, rs}_{\text{Higgs}}(Y, G)$, where $\mathcal{M}^{\gamma, rs}_{\text{Higgs}}(Y, G)$ denotes the moduli space of regularly stable Higgs $G$ bundles $(E_G, \theta)$ over $Y$ with $E_G$ of topological type $\gamma \in \pi_1(G)$, and

$$\mathcal{H} : \mathcal{M}^{\gamma, rs}_{\text{Higgs}}(Y, G) \to V = \bigoplus_{i=1}^{\text{rank}(G)} \text{H}^0(Y, K_Y^{\otimes n_i})$$

is the Hitchin map, where $n_i$ are the degree of generators for the algebra $\text{Sym}(\mathfrak{g}^*)^G$; see [Hit87 section 4], [Lau88].

We state a version of Zariski’s main theorem, which can be proved using Stein factorisation. We use the following lemma to prove Theorem 2.2.

**Lemma 2.1.** Let $T, S$ be two noetherian schemes over $\mathbb{C}$. Let $\pi : T \to S$ be a surjective proper morphism such that the generic fibres are connected. Suppose that $S$ is normal. Then the natural morphism of sheaves $\pi^* : \mathcal{O}_S \to \pi_* \mathcal{O}_T$, is an isomorphism and every fibre of $\pi$ is connected.

Let $L$ be a holomorphic line bundle over $X$. We follow [GD67] and [Ram05] for the definition and properties of differential operators.

Let $k \geq 0$ be an integer. A differential operator of order $k$ is a $\mathbb{C}$-linear map

$$P : L \to L$$

such that for every open subset $U$ of $X$ and for every $f \in \mathcal{O}_X(U)$, the bracket

$$[P|_U, f] : L|_U \to L|_U$$

defined as

$$[P|_U, f]|_V(s) = P_V(f|_V s) - f|_V P_V(s)$$

is a differential operator of order $k - 1$, for every open subset $V$ of $U$, and for all $s \in L(V)$, where differential operator of order zero from $L$ to $L$ is just $\mathcal{O}_X$-module homomorphism.

Let $\text{Diff}^k_X(L, L)$ denote the set of all differential operator of order $k$. Then $\text{Diff}^k_X(L, L)$ is an $\mathcal{O}_X(X)$-module. For every open subset $U$ of $X$, $U \mapsto \text{Diff}^k_X(L|_U, L|_U)$ is a sheaf of differential operator of order $k$ from $L|_U$ to $L|_U$. This sheaf is denoted by $\mathcal{D}iff^k_X(L, L)$, which is locally free.
For $k \geq 0$, we denote by $\mathcal{D}^k(L)$ the vector bundle over $X$ defined by the sheaf $\text{Diff}^k_X(L, L)$. Note that $\mathcal{D}^0(L) = \mathcal{O}_X$ and we have following inclusion of vector bundles

$$\mathcal{O}_X = \mathcal{D}^0(L) \subset \cdots \subset \mathcal{D}^k(L) \subset \mathcal{D}^{k+1}(L) \subset \cdots \quad (2.3)$$

We have exact sequence of vector bundles

$$0 \rightarrow \mathcal{D}^{k-1}(L) \rightarrow \mathcal{D}^k(L) \xrightarrow{\sigma_k} S^k(TX) \rightarrow 0, \quad (2.4)$$

where $S^k(TX)$ is the $k$-th symmetric power of the tangent bundle of $X$ and $\sigma_k$ is the symbol homomorphism. It is called symbol exact sequence (see [Ram05], Chapter 2, for more details).

Consider the first order symbol operator

$$\sigma_1 : \mathcal{D}^1(L) \rightarrow TX$$

given in symbol exact sequence (2.4). This induces a morphism

$$S^k(\sigma_1) : S^k(\mathcal{D}^1(L)) \rightarrow S^k(TX)$$

of $k$-th symmetric powers. Now, because of the following composition

$$S^{k-1}(\mathcal{D}^1(L)) = \mathcal{O}_X \otimes S^{k-1}(\mathcal{D}^1(L)) \hookrightarrow \mathcal{D}^1(L) \otimes S^{k-1}(\mathcal{D}^1(L)) \rightarrow S^k(\mathcal{D}^1(L)),$$

we have

$$S^{k-1}(\mathcal{D}^1(L)) \subset S^k(\mathcal{D}^1(L))$$

for all $k \geq 1$.

Thus, we get a short exact sequence of vector bundles over $X$,

$$0 \rightarrow S^{k-1}(\mathcal{D}^1(L)) \rightarrow S^k(\mathcal{D}^1(L)) \xrightarrow{S^k(\sigma_1)} S^k(TX) \rightarrow 0. \quad (2.5)$$

In other words, we get a filtration

$$0 \subset S^0(\mathcal{D}^1(L)) \subset S^1(\mathcal{D}^1(L)) \subset \cdots \subset S^{k-1}(\mathcal{D}^1(L)) \subset S^k(\mathcal{D}^1(L)) \subset \cdots \quad (2.6)$$

such that

$$S^k(\mathcal{D}^1(L))/S^{k-1}(\mathcal{D}^1(L)) \cong S^k(TX)$$

for all $k \geq 1$. \quad (2.7)

Above filtration in (2.6) gives the following increasing chain of $\mathbb{C}$-vector spaces

$$H^0(X, \mathcal{O}_X) \subset H^0(X, S^1(\mathcal{D}^1(L))) \subset H^0(X, S^2(\mathcal{D}^1(L))) \subset \cdots \quad (2.8)$$

**Theorem 2.2.** Let $X$ be a Hitchin variety. Then, for $k \geq 0$, the inclusion homomorphism

$$H^0(X, \mathcal{O}_X) \rightarrow H^0(X, S^k(\mathcal{D}^1(L))) \quad (2.9)$$

is an isomorphism.

**Proof.** From isomorphism in (2.7), we have the following commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & S^{k-1}(\mathcal{D}^1(L)) & \rightarrow & S^k(\mathcal{D}^1(L)) & \xrightarrow{S^k(\sigma_1)} & S^k(TX) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S^{k-1}(TX) & \rightarrow & S^k(\mathcal{D}^1(L)) & \rightarrow & S^k(TX) & \rightarrow & 0
\end{array} \quad (2.10)$$
which gives rise to the following commutative diagram of long exact sequences

\[
\cdots \to \mathbb{H}^0(X, S^k TX) \xrightarrow{\delta_k} \mathbb{H}^1(X, S^{k-1}(D^1(L))) \to \cdots
\]

\[
\cdots \to \mathbb{H}^0(X, S^k TX) \xrightarrow{\delta_k} \mathbb{H}^1(X, S^{k-1} TX) \to \cdots
\]

(2.11)

Note that proving (2.9) is equivalent to show the following

\[
\mathbb{H}^0(X, S^{k-1}(D^1(L))) \cong \mathbb{H}^0(X, S^k(D^1(L)))
\]

for all \( k \geq 1 \).

Now, from above commutative diagram (2.11), it is enough to show that the connecting homomorphism \( \delta_k' \) is injective for all \( k \geq 1 \), which is equivalent to showing that the connecting homomorphism

\[
\delta_k : \mathbb{H}^0(X, S^k TX) \to \mathbb{H}^1(X, S^{k-1} TX)
\]

is injective for every \( k \geq 1 \).

Moreover, a connecting homomorphism can be expressed as the cup product by the extension class of the corresponding short exact sequence. We denote the extension class of the following short exact sequence

\[
0 \to S^{k-1}(TX) \xrightarrow{S^k(D^1(L))} S^k(TX) \to 0
\]

by \( \Gamma_k \).

Next, \( \Gamma_k \) can be described in terms of first Chern class \( c_1(L) \) of the line bundle \( L \), because the first Chern class of \( c_1(L) \) is nothing but the extension class of the following Atiyah exact sequence (see [Ati57])

\[
0 \to \mathcal{O}_X \to D^1(L) \xrightarrow{\sigma_1} TX \to 0,
\]

(2.14)

and the short exact sequence (2.5) is the \( k \)-th symmetric power of the Atiyah exact sequence (2.14).

Thus, the connecting homomorphism \( \delta_k \) can be described using the first Chern class \( c_1(L) \in H^1(X, T^*X) \) of the line bundle \( L \) over \( X \). Indeed, the cup product with \( c_1(L) \) gives rise to a homomorphism

\[
\mu : \mathbb{H}^0(X, S^k TX) \to \mathbb{H}^1(X, S^k TX \otimes T^*X)
\]

(2.15)

Also, we have a canonical homomorphism of vector bundles

\[
\beta : S^k TX \otimes T^*X \to S^{k-1} TX
\]

which induces a morphism of \( \mathbb{C} \)-vector spaces

\[
\beta^* : \mathbb{H}^1(X, S^k TX \otimes T^*X) \to \mathbb{H}^1(X, S^{k-1} TX).
\]

(2.16)

So, we get a morphism

\[
\tilde{\mu} = \beta^* \circ \mu : \mathbb{H}^0(X, S^k TX) \to \mathbb{H}^1(X, S^{k-1} TX).
\]

(2.17)

Then from the above observation we have \( \tilde{\mu} = \delta_k \). It is sufficient to show that \( \tilde{\mu} \) is injective.
Moreover, we have the natural projection
\[ \eta : T^*X \to X \]  
(2.18)
and
\[ \eta_*\eta^*\mathcal{O}_X = \oplus_{k \geq 0} S^kTX. \]  
(2.19)
Thus, we have
\[ H^j(T^*X, \mathcal{O}_{T^*X}) = \oplus_{k \geq 0} H^j(X, S^kTX) \text{ for all } j \geq 0. \]  
(2.20)
To compute \( H^j(T^*X, \mathcal{O}_{T^*X}) \), we will use the fact that \( X \) is a Hitchin variety.
Let \( g : T^*X \to \mathbb{C} \) be an algebraic function. Since the codimension of \( \mathcal{N} \setminus T^*X \) in \( \mathcal{N} \) is at least 2, and \( \mathcal{N} \) is normal noetherian scheme, \( g \) extends by Hartog’s theorem to an algebraic function \( \tilde{g} \) on \( \mathcal{N} \). Since \( H \) is surjective proper morphism, \( V \) is normal, and generic fibres of \( H \) are abelian varieties, from Lemma 2.1 we have an isomorphism
\[ \mathcal{O}_V \cong H_*\mathcal{O}_\mathcal{N}, \]
and hence there exists a unique algebraic function \( \hat{g} : V \to \mathbb{C} \) such that
\[ \hat{g} = \tilde{g} \circ H. \]
Set \( V = d(H^0(V, \mathcal{O}_V)) \subset H^0(V, \Omega^1_V) \) the space of all exact algebraic 1-form. Define a map
\[ \theta : H^0(T^*X, \mathcal{O}_{T^*X}) \to V \]  
(2.21)
by \( g \mapsto d\hat{g} \), where \( \hat{g} \) is the function on \( V \) which is defined by descent of \( g \) as above. Then \( \theta \) is an isomorphism.
From (2.20) and (2.21), we have
\[ \theta : \oplus_{k \geq 0} H^0(X, S^kTX) \to V \]  
(2.22)
which is an isomorphism.
Now, restrict \( H \) on \( T^*X \), and let \( T_H = T_{T^*X/V} = \text{Ker}(dH) \) be the relative tangent sheaf on \( T^*X \), where
\[ dH : T(T^*X) \to H^*TV \]
is a morphism of bundles. Now, we use the fact that the vector fields on \( H^{-1}(v) \) are constant, therefore, the pulled back bundle \( H^*TV \) is identified with \( T_H \), therefore, we have
\[ H^0(V, \Omega^1_V) \subseteq H^0(T^*X, T_H), \]
and hence from (2.22), we have an injective homomorphism
\[ \nu : V = \oplus_{k \geq 0} \theta(H^0(X, S^kTX)) \to H^0(T^*X, T_H). \]  
(2.23)
Consider the morphism
\[ H^0(T^*X, T_H) \to H^1(T^*X, T_H \otimes T^*T^*X) \]
defined by taking cup product with the first Chern class
\[ c_1(\eta^*L) \in H^1(T^*X, T^*T^*X). \]
Further using the pairing
\[ T_H \otimes T^*T^*X \to \mathcal{O}_{T^*X}, \]
we get a homomorphism
\[
\psi : H^0(T^*X, T_H) \rightarrow H^1(T^*X, O_{T^*X}).
\]
(2.24)
Since \(c_1(\eta^*L) = \eta^*(c_1L)\), we have
\[
\psi \circ \nu \circ \theta(\omega_k) = \tilde{\mu}(\omega_k),
\]
for all \(\omega_k \in H^0(X, S^kTX)\), because of (2.20). Since \(\nu\) and \(\theta\) are injective homomorphisms, it is enough to show that \(\psi|_{\nu(V)}\) is injective homomorphism. Let \(\omega \in V \setminus \{0\}\) be a non-zero exact 1-form. Choose \(u \in V\) such that \(\omega(u) \neq 0\). As previously discussed
\[
\mathcal{H}^{-1}(u) \cap T^*X = A_u \setminus F_u,
\]
where \(A_u\) is an abelian variety and \(F_u\) is a subvariety of \(A_u\) such that \(\text{codim}(F_u, A_u) \geq 2\). Now, \(\psi(\nu(\omega)) \in H^1(T^*X, O_{T^*X})\) and we have the restriction map
\[
H^1(T^*X, O_{T^*X}) \rightarrow H^1(\mathcal{H}^{-1}(u) \cap T^*X, O_{\mathcal{H}^{-1}(u) \cap T^*X}).
\]
Since \(\omega(u) \neq 0\), \(\psi(\nu(\omega)) \in H^1(\mathcal{H}^{-1}(u) \cap T^*X, O_{\mathcal{H}^{-1}(u) \cap T^*X})\). Because of the following isomorphisms
\[
H^1(\mathcal{H}^{-1}(u) \cap T^*X, O_{\mathcal{H}^{-1}(u) \cap T^*X}) \cong H^1(A_u, O_{A_u}) \cong H^0(A_u, T_{A_u}),
\]
it follows that \(\psi(\nu(\omega)) \neq 0\). This completes the proof.

\[\square\]

**Corollary 2.3.** Let \(X\) be a projective Hitchin variety. Then, for every line bundle \(L\) on \(X\) and for every \(k \geq 0\), we have
1. \(H^0(X, S^k(D^1(L))) = \mathbb{C}\)
2. \(H^0(X, D^k(L)) = \mathbb{C}\)

**Proof.** (1) follows immediately from Theorem 2.2
For (2), we need to show that the inclusions of \(\mathbb{C}\)-vector spaces
\[
H^0(X, O_X) \subset H^0(X, D^1(L)) \subset H^0(X, D^k(L)) \subset \ldots
\]
induced from (2.3), are equal, that is,
\[
H^0(X, D^k(L)) \cong H^0(X, D^k(L)) \text{ for all } k \geq 1.
\]
(2.27)
For that consider the following commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & D^{k-1}(L) & \rightarrow & D^k(L) & \rightarrow & S^k(TX) & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \downarrow & & \\
0 & \rightarrow & S^{k-1}(TX) & \rightarrow & D^k(L) & \rightarrow & S^k(TX) & \rightarrow & 0
\end{array}
\]
(2.28)
where the top short exact sequence is symbol exact sequence (2.24). The above commutative diagram gives the following commutative diagram of long exact sequences
\[
\begin{array}{cccccc}
\cdots & \rightarrow & H^0(X, S^kTX) & \rightarrow & H^1(X, D^{k-1}(L)) & \rightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & H^0(X, S^kTX) & \rightarrow & H^1(X, S^{k-1}TX) & \rightarrow & \cdots
\end{array}
\]
(2.29)
To show (2.27), it is enough to show that the connecting homomorphism $\beta_k$ is injective for all $k \geq 1$, which is equivalent to showing that $\delta_k$ is injective for all $k \geq 1$. Note that $\delta_k$ is the same connecting homomorphism arises in the proof of the above Theorem 2.2, which is injective. This completes the proof of (2). \hfill \Box

Let $E$ be a holomorphic vector bundle over a Hitchin variety $X$. We denote by $D^k(E)$ the vector bundle over $X$ associated to the sheaf $Diff_X^k(E, E)$ of $k$-th order differential operators on $E$. We have the **symbol exact sequence** of $E$.

$$0 \rightarrow \mathcal{E}nd(E) \xrightarrow{\mathcal{D}^1(E)} TX \otimes \mathcal{E}nd(E) \rightarrow 0,$$

which further reduces to the **Atiyah exact sequence** (see [Ati57]) of $E$

$$0 \rightarrow \mathcal{E}nd(E) \xrightarrow{\mathcal{A}t(E)} TX \rightarrow 0.$$  \tag{2.31}

Tensoring (2.31) with $\Omega^1_X$ produces a short exact sequence

$$0 \rightarrow \Omega^1_X \otimes \mathcal{E}nd(E) \xrightarrow{\mathcal{I}} \Omega^1_X \otimes \mathcal{A}t(E) \xrightarrow{\mathcal{I} \otimes \sigma_1} \Omega^1_X \otimes TX \rightarrow 0.$$  \tag{2.32}

Note that $\mathcal{O}_X \cdot \text{Id} \subset \mathcal{E}nd(TX) = \Omega^1_X \otimes TX$. Define

$$\Omega^1_X(\mathcal{A}t'(E)) := (\mathcal{I} \otimes \sigma_1)^{-1}(\mathcal{O}_X \cdot \text{Id}) \subset \Omega^1_X(\mathcal{A}t(E)),$$

where $\mathcal{I} \otimes \sigma_1$ is the projection in (2.32). So we have the short exact sequence of sheaves

$$0 \rightarrow \Omega^1_X(\mathcal{E}nd(E)) \rightarrow \Omega^1_X(\mathcal{A}t'(E)) \xrightarrow{\mathcal{I} \otimes \sigma_1} \mathcal{O}_X \rightarrow 0$$  \tag{2.33}

on $X$, where $\Omega^1_X(\mathcal{A}t'(E))$ is constructed above.

For simplicity, we shall denote $\mathcal{I} \otimes \sigma_1$ by $q$. Now, dualizing the above sequence (2.33), we get

$$0 \rightarrow \mathcal{O}_X \xrightarrow{q^*} (\Omega^1_X(\mathcal{A}t'(E)))^* \rightarrow (\Omega^1_X(\mathcal{E}nd(E)))^* \rightarrow 0$$  \tag{2.34}

which is nothing but the following short exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{q^*} TX \otimes (\mathcal{A}t'(E))^* \xrightarrow{\eta} TX \otimes (\mathcal{E}nd(E))^* \rightarrow 0.$$  \tag{2.35}

Further define a sheaf

$$\mathcal{B}(E) = \eta^{-1}(TX).$$  \tag{2.36}

Thus, we get the following exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{q^*} \mathcal{B}(E) \xrightarrow{\eta} TX \rightarrow 0$$  \tag{2.37}

which is similar to (2.14). Now for the vector bundle $\mathcal{B}(E)$ associated with $E$, we have the following theorem which admits an identical proof to the Theorem 2.2.

**Theorem 2.4.** For $k \geq 1$, and for every holomorphic vector bundle $E$ over the Hitchin variety $X$, we have

$$H^0(X, \mathcal{O}_X) \cong H^0(X, S^k(\mathcal{B}(E))).$$  \tag{2.38}

An immediate corollary of the above Theorem 2.4 is as follows

**Corollary 2.5.** Let $X$ be a projective Hitchin variety. Then, for every holomorphic vector bundle $E$ on $X$ and for every $k \geq 1$, we have

$$H^0(X, S^k(\mathcal{B}(E))) = \mathbb{C}.$$  \tag{2.39}
3. SHEAF OF HOLONOMIC CONNECTIONS ON A HOLOMORPHIC LINE BUNDLE

As above, let \( X \) be a Hitchin variety and \( L \) be a holomorphic line bundle over \( X \).

Take the dual of the Atiyah exact sequence (2.14)
\[
0 \to \Omega^1_X \xrightarrow{\sigma^*} D^1(L)^* \xrightarrow{\iota^*} \mathcal{O}_X \to 0.
\]

Consider \( \mathcal{O}_X \) as trivial line bundle \( X \times \mathbb{C} \). Let
\[
s : X \to X \times \mathbb{C}
\]
be a holomorphic section of the trivial line bundle defined by \( x \mapsto (x, 1) \).

Let \( S = \text{Im}(s) \subset X \times \mathbb{C} \) be the image of \( s \). Consider the inverse image
\[
i^{-1}S \subset D^1(L)^*,
\]
and denote it by \( \mathcal{C}(L) \). Then for every open subset \( U \subset X \), a holomorphic section of \( \mathcal{C}(L)|_U \) over \( U \) gives a holomorphic splitting of the Atiyah exact sequence (2.14), associated to the holomorphic vector bundle \( L|_U \to U \). For instance, suppose \( \tau : U \to \mathcal{C}(L)|_U \) is a holomorphic section. Then \( \tau \) will be a holomorphic section of \( D^1(L)^*|_U \) over \( U \), because \( \mathcal{C}(L) = i^{-1}S \subset D^1(L)^* \). Since \( \tau \circ i = i^*(\tau) = 1_U \), so we get a holomorphic splitting \( \tau \) of the Atiyah exact sequence (2.14) associated to \( L|_U \). Thus, \( L|_U \) admits a holomorphic connection. Conversely, given any splitting of Atiyah exact sequence (2.14) over an open subset \( U \subset X \), we get a holomorphic section of \( \mathcal{C}(L)|_U \).

**Theorem 3.1.** Let \( X \) be a projective Hitchin variety. Then, for every holomorphic line bundle \( L \), we have
\[
H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \mathbb{C}.
\]

**Proof.** Let \( \mathbb{P}(D^1(L)) \) be the projectivization of \( D^1(L) \), that is, \( \mathbb{P}(D^1(L)) \) parametrises hyperplanes in \( D^1(L) \). Let \( \mathbb{P}(TX) \) be the projectivization of the tangent bundle \( TX \). Notice that \( \mathbb{P}(TX) \) is a subvariety of \( \mathbb{P}(D^1(L)) \), and \( \mathbb{P}(TX) \) is the zero locus of the of a section of the tautological line bundle \( \mathcal{O}_{\mathbb{P}(D^1(L))}(1) \). Now, observe that \( \mathcal{C}(L) = \mathbb{P}(D^1(L)) \setminus \mathbb{P}(TX) \). Then we have
\[
H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \lim_{k \to \infty} H^0(\mathbb{P}(D^1(L)), \mathcal{O}_{\mathbb{P}(D^1(L))}(k)) = \lim_{k \to \infty} H^0(X, S^kD^1(L)) \quad (3.2)
\]

Now, from Corollary (2.3) (11), we have the result.

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**Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhusi, Prayagraj 211019, India**

*E-mail address: anoopsingh@hri.res.in*