Structures of boson and fermion Fock spaces in the space of symmetric functions

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We realize the Weil representation of infinite dimensional symplectic group and spinor representation of infinite-dimensional group GL by linear operators in the space of symmetric functions in infinite number of variables.

0.0. Purposes of this paper. Canonical unitary operators connecting a boson Fock space and a fermion Fock space with a space of symmetric functions are well known, see [PS], [MJD] and further references in these books.

The basic structure in the boson Fock space is a semigroup of Gauss operators. This semigroup contains the Friedrichs–Shale group of automorphisms of the canonical commutation relations; see [Ber1], [Ner2].

The basic structure in the fermion Fock space is a semigroup of Berezin operators (i.e., fermion analogs of Gauss operators). This semigroup contains the Friedrichs–Berezin group of automorphisms of the canonical anticommutation relations, see [Ber1], [Ner2].

The purpose of our paper is to transfer these structures to the space of symmetric functions. This problem is completely solved for the Gauss operators (see below Subsection 3.5) and partially solved for the Berezin operators (Subsection 5.3). Results of this paper were announced in [Ner4].

There are many problems of this kind, some of them are discussed in [Ner3], [Ner4], [TsV].

0.1. Operators in the space of symmetric functions. We consider a Hilbert space $S_{cl}$, whose elements are formal symmetric series $f(x_1, x_2, x_3, \ldots)$; $S_{cl}$ is equipped with a classical (Redfield) scalar product (see [Mac], I.4; see below 1.3). For each formal series $K(x_1, x_2, \ldots; y_1, y_2, \ldots)$ symmetric separately in $x_j$ and in $y_j$, we associate the linear operator $A$ in $S_{cl}$ given by

$$Af(x) = \langle K(x, y), f(y) \rangle_{S_{cl}};$$

(0.1)

here we consider $K(x, y)$ as a function in $y$ depending on the parameters $x$. We say that $K(x, y)$ is the kernel of the operator $A$.

0.2. the Weil representation. We realize the 'Weil’ representation of the Friedrichs–Shale symplectic group (see its definition below 2.10) by operators, whose kernels have the form

$$\prod_{k<l} \left\{1 + \sum_{i>0, j>0} a_{ij} x_k^i x_l^j \right\} \prod_{k<l} \left\{1 + \sum_{i>0, j>0} b_{ij} x_k^i y_l^j \right\} \prod_{k<l} \left\{1 + \sum_{i>0, j>0} c_{ij} y_k^i y_l^j \right\} \times$$

$$\times \prod_k \left\{1 + \sum_{i>0} \alpha_i x_k^i \right\} \prod_k \left\{1 + \sum_{i>0} \beta_i y_k^i \right\},$$

(0.2)

Supported by grant NWO-047-008-009
where $a_{ij} = a_{ji}$, $c_{ij} = c_{ji}$. Moreover, all the bounded operators in $S_{cl}$, whose kernels are given by expressions (0.2), form a semigroup; this semigroup is isomorphic to the semigroup of Gauss operators in the boson Fock space.

The correspondence (0.1) between the kernels $K(x, y)$ and linear operators depends on a scalar product. Many natural scalar products in the space of symmetric functions are known (see [Mac, I.4, III.4, VI.5, V.10; Ker, GR, Ner2, Section 10, Ner4]). Our construction literally survives for the Jack, Hall–Littlewood and Macdonald scalar products; maximal generality, then it exists, is a family of scalar product defined by Kerov in [Ker].

Nevertheless, in our context the classical case is distinguished, since it is related to the Virasoro algebra (see [PS]); for some formulae related to actions of the group of diffeomorphisms of the circle in $S_{cl}$, see [Ner4]. On explicit description of boson Fock spaces related to the Hall–Littlewood and Macdonald cases, see [Ner4].

0.3. The spinor representation. The natural group $O$ of symmetries of the fermion Fock space is the group of $(\infty + \infty) \times (\infty + \infty)$ complex invertible bounded matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $g$ is orthogonal

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and $B, C$ are Hilbert–Schmidt matrices (i.e., $\text{tr} B^* B, \text{tr} C^* C < \infty$); see [Ner2], IV.4, this group is larger than the 'group of automorphisms of canonical anti-commutation relations' (see [Ber1]) described by Friedrichs, Bogoluubov, Berezin and Shale–Stinespring.

The representation of $O$ in the fermion Fock space is an infinite dimensional variant of the spinor representation.

The space $S_{cl}$ is in a canonical one-to-one correspondence with some subspace in fermion Fock space that is called 'space of semi-infinite forms', see definitions below in 4.2-4.3. The natural group $GL$ of symmetries of the space of semiinfinite forms consists of $(\infty + \infty) \times (\infty + \infty)$ complex invertible bounded matrices $\begin{pmatrix} P & Q \\ R & T \end{pmatrix}$ such that $Q, R$ are Hilbert–Schmidt matrices and the Fredholm index of $P$ is zero (in our case, this is equivalent to the condition $\dim \ker P = \text{codim} \text{Im} P$), see [Ner2], IV.3–IV.4.

The group $GL$ is a subgroup in $O^2$.

Now consider a Laurent polynomial in two variables

$$S(u, v) = \sum_{-M<i<\infty, -M<j<\infty} a_{ij} u^i v^j$$

\footnote{In notation of [Ner2], our groups are the the following $(G, K)$-pairs: $GL = (GL(2\infty, \mathbb{C}), GL(\infty, \mathbb{C}) \times GL(\infty, \mathbb{C}))$ and $O = (O(2\infty, \mathbb{C}), GL(\infty, \mathbb{C}))$. For description of the embedding $GL \to O$, see [Ner2, IV.4]; to avoid misunderstanding, emphasis that $GL$ is a subgroup in the group $(O(2\infty, \mathbb{C}), GL(2\infty, \mathbb{C}))$ with the duplicated infinity.}
(only finite number of terms are nonzero). Define the kernel \( K(x, y) \) by the condition

\[
K(x, y) \big|_{x_N = x_{N+1} = \cdots = 0, y_N = y_{N+1} = \cdots = 0} = \\
\frac{\det_{1 \leq k, l \leq N} \{S(x_k, y_l) + \sum_{j=-M}^{\infty} x_k^j y_l^j \} \prod_{k=1}^{N} x_k^N y_k^N}{\prod_{1 \leq k < l \leq N} (x_k - x_l) \prod_{1 \leq k < l \leq N} (y_k - y_l)}
\]

for all \( N > M \).

We show that operators in \( S_{cl} \) with such kernels form an infinite-dimensional group \( GL_\infty \), whose elements \( g = \left( \begin{array}{cc} P & Q \\ R & T \end{array} \right) \) satisfy the condition: \( g - 1 \) has only finite number of nonzero matrix elements.

This is a partial answer to the question formulated above, since this group \( GL_\infty \) is a proper subgroup in the natural group of symmetries \( GL \).

This construction exists in the space \( S_{cl} \) and does not survive for general Kerov’s scalar products.

0.4. Structure of the paper. Sections 1–2 contain preliminaries on the space of symmetric functions and on the boson Fock space. In Section 3, we discuss the boson-symmetric correspondences.

In section 4, we introduce a space of semiinfinite forms and a space of skew-symmetric functions. In Section 5, we discuss a fermion-symmetric correspondence.

Acknowledgements. This work was done during my visit to Yale University in 1994. I thank prof. R. Howe for hospitality and discussions. I also thank the organizers of Russian–French workshop on combinatorics (Independent University of Moscow, May 2003), since this was an occasion for writing the present paper.

1. Symmetric functions

1.1. Symmetric functions. In this paper, \( x_1, x_2, \ldots \) is an infinite collection of formal variables.

We denote by \( \mathfrak{S} = \mathfrak{S}(x) \) the space of all formal series in the variables \( x_j \) symmetric with respect to permutations of \( x_j \). We call elements of \( \mathfrak{S} \) by symmetric functions.

By \( \mathfrak{S}^k \subset \mathfrak{S} \) we denote the space of symmetric formal series of degree \( k \) in the variables \( x_j \). This space is finite dimensional, its dimension equals the number \( p(k) \) of partitions of \( k \).

By \( \mathfrak{S} \subset \mathfrak{S} \) we denote the space of series of bounded degree.

1.2. Some bases in \( \mathfrak{S} \). For details, see [Mac], I.2-3.

a) Functions \( p_m \). A Newton sum \( p_m = p_m(x) \), where \( m = 1, 2, \ldots, \) is

\[
p_m(x) := \sum_{j=1}^{\infty} x_j^m.
\]
Denote by $\mathbf{m}$ a collection of nonnegative integers,

$$\mathbf{m} = (m_1, m_2, \ldots); \quad m_j = 0 \text{ for sufficiently large } j.$$

Denote

$$p_\mathbf{m}(x) := p_1(x)^{m_1} p_2(x)^{m_2} \ldots$$

The functions $p_\mathbf{m}$ form a basis in $\mathbb{S}$.

We also use another notation for the same functions. Let $\lambda$ be a sequence of integers

$$\lambda : \lambda_1 \geq \lambda_2 \geq \ldots, \quad \lambda_j = 0 \text{ for sufficiently large } j.$$

Then

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots.$$

In other words, $p_\lambda(x) = p_\mathbf{m}(x)$, where $m_j$ is the number of entries of $j$ into the collection $\lambda$.

b) **Monomial symmetric functions** $m_\lambda$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \ldots$ are integers and $\lambda_j = 0$ for sufficiently large $j$. Denote by $m_\lambda$ the sum of all distinct monomials of the form

$$x_{k_1}^{\lambda_1} x_{k_2}^{\lambda_2} x_{k_3}^{\lambda_3} \ldots \quad k_i \neq k_j.$$

Obviously, the system $m_\lambda$ is a basis in $\mathbb{S}$.

c) **Schur functions** $s_\lambda = s_\mu$. Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be nonnegative integers, and $\lambda_j = 0$ for sufficiently large $j$. The function $s_\lambda \in \mathbb{S}$ is defined by the rule

$$s_\lambda(x_1, x_2, \ldots) \biggr|_{x_{n+1}=x_{n+2}=\cdots=0} = \det \frac{x_k^{\lambda_i+n-j}}{\prod_{1 \leq k < l \leq n} (x_k - x_l)}.$$

We also use another notation for the same functions. Let $r$ be a sequence of integers such that

$$r : \quad r_1 > r_2 > \ldots \text{ and } r_j = -j \text{ for sufficiently large } j.$$

We assume

$$s_r := s_\lambda, \quad \text{where } \lambda_j = r_j + j.$$

**Remark.** Thus, we introduced 3 types of notation for Young diagrams; in these 3 cases we use respectively Greek letters ($\lambda$, $\mu$, etc.), bold Latin letters (m, n, etc.), and Gothic letters ($t$, $l$, etc.) as above.

1.3. **Scalar products in** $\mathbb{S}$. The classical scalar product (J.H.Redfield, 1927, see [Mac], I.4) in $\mathbb{S}$ is defined by the condition: the functions $p_\mathbf{m}$ are pairwise orthogonal and

$$\|p_\mathbf{m}\|^2 = \prod m_j! j^{m_j}.$$

The equivalent condition is: the Schur functions $s_\lambda$ form an orthonormal basis.
More generally, let us define *Kerov's family* $\langle \cdot, \cdot \rangle_\omega$ of scalar products in $\mathcal{S}$. Fix a sequence

$$\omega = (\omega_1, \omega_2, \ldots); \quad \omega_j > 0.$$  

We assume that the functions $p_m$ are pairwise orthogonal and

$$\|p_m\|^2 = \prod m_j! \omega_j^m.$$  

(1.1)

There are three following distinguished examples of such scalar products

$$\begin{align*}
\omega_j &= j\alpha \quad (Jack \ scalar \ products); \\
\omega_j &= j(1-t^j)^{-1} \quad (Hall-Littlewood \ scalar \ products); \\
\omega_j &= j \cdot \frac{1 - q_j}{1 - t^j} \quad (Macdonald \ scalar \ products). 
\end{align*}$$  

(1.2)

We denote by $\mathcal{S}_\omega$ the completion of the Euclidean space $\mathcal{S}$ equipped with the scalar product $\langle \cdot, \cdot \rangle_\omega$.

More carefully, denote by $\mathcal{S}^\tau_\omega$ the (finite dimensional) Euclidean space $\mathcal{S}^\tau$ equipped with the Kerov scalar product. Then $\mathcal{S}_\omega$ is a Hilbert direct sum of the Hilbert spaces $\mathcal{S}^\tau_\omega$,

$$\mathcal{S}_\omega = \bigoplus_{\tau=0}^\infty \mathcal{S}^\tau_\omega.$$

Each element $f \in \mathcal{S}_\omega$ can be represented as a sum of a series

$$f = \sum_{\tau=0}^\infty f_\tau; \quad \text{where } f_\tau \in \mathcal{S}^\tau, \text{ and } \sum \|f_\tau\|_{\mathcal{S}^\tau_\omega}^2 < \infty.$$

In particular, $\mathcal{S}_\omega \subset \overline{\mathcal{S}}$. The scalar product is given by

$$\langle f, g \rangle_\omega = \langle \sum f_\tau, \sum g_\tau \rangle_\omega := \sum \langle f_\tau, g_\tau \rangle_{\mathcal{S}^\tau_\omega}.$$

We emphasize that this expression has sense also for $f \in \overline{\mathcal{S}}, \ g \in \mathcal{S}$.

We also denote by $\mathcal{S}_\omega$ the space corresponding to the classical case $\omega_j = j$.

**1.4. Bisymmetric kernels.** Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two collections of formal variables. A *bisymmetric kernel* $K(x, y)$ is a formal series symmetric with respect to $x_j$ and symmetric with respect to $y_i$.

**1.5. Linear operators in $\mathcal{S}_\omega$.** Fix $\omega$. Represent a bisymmetric kernel $K(x, y)$ as a series in $x_j$ with coefficients depending on $y_i$,

$$K(x, y) = \sum x_1^{k_1} x_2^{k_2} x_3^{k_3} \ldots u_{k_1, k_2, k_3, \ldots}(y),$$

where $k_j$ are nonnegative integers, and all $k_j = 0$ except a finite number of $j$. Then $u_{k_1, k_2, k_3, \ldots}(y) \in \overline{\mathcal{S}}$, these expressions also are symmetric with respect to $k_j$. Hence

$$K(x, y) = \sum_{\lambda_1, \lambda_2, \ldots} m_\lambda(x) u_{\lambda_1, \lambda_2, \ldots}(y).$$
We define the linear operator $A_K$ in the space of symmetric functions by the formula

$$A_K f(x) = \langle K(x, y), \overline{f(y)} \rangle_\omega := \sum_{\lambda_1 \geq \lambda_2 \geq \ldots} m_{\lambda}(x) \langle u_{\lambda_1, \lambda_2, \ldots}, \overline{f(y)} \rangle_\omega,$$

(1.3)

where $\overline{f}$ denotes the usual complex conjugation.

We also represent this expression in the form

$$A_K f = \sum_{\sigma=0}^{\infty} [A_K f]_\sigma,$$

(1.4)

where

$$[A_K f]_\sigma := \sum_{\lambda_1 \geq \lambda_2 \geq \ldots, \sum \lambda_\sigma = \sigma} m_{\lambda}(x) \langle u_{\lambda_1, \lambda_2, \ldots}, \overline{f(y)} \rangle_\omega.$$

(1.5)

Obviously, $[A_K f]_\sigma \in S_\sigma$ and hence the summands of (1.4) are pairwise orthogonal.

**Proposition 1.1.** The map $K \mapsto A_K$ is a bijection of the space of all bisymmetric kernels to the space of all linear operators $S \to S$.

**Remark.** This bijection depends on $\omega$.

**Proposition 1.2.** Let $A : S_\omega \to S_\omega$ be a bounded operator. Then

a) There exists a bisymmetric kernel $K(x, y)$ such that $Af = A_K f$ for $f \in S$.

b) For $f \in S_\omega$, its image $Af$ equals (1.4)–(1.5); the series in the right hand side of (1.4) converges in the Hilbert space $S_\omega$.

**Proposition 1.3.** For each $\sigma = 0, 1, 2, \ldots$ choose an arbitrary (in general, nonorthogonal) basis $\xi_1^{(\sigma)}, \ldots, \xi_{p(\sigma)}^{(\sigma)}$ in $S^\sigma$. Represent a bisymmetric kernel $K(x, y)$ in the form

$$K(x, y) = \sum_{\sigma=0}^{\infty} \left[ \sum_{i=1}^{p(\sigma)} \xi_i^{(\sigma)}(x) v_i^{(\sigma)}(y) \right],$$

where $v_i^{(\sigma)}$ are some elements of $S$.

a) For each $f \in S$,

$$A_K f(x) = \sum_{\sigma} \left[ \sum_{i} \xi_i^{(\sigma)}(x) \langle v_i^{(\sigma)}(y), \overline{f(y)} \rangle_\omega \right].$$

(1.6)

Summands of the formal series $\sum_{\sigma}$ do not depend on a choice of a basis $\xi_i$. In particular, they coincide with summands of the series (1.4).

b) If $A_K$ is a bounded operator and $f \in S_\omega$, then $A_K f$ coincides with (1.6); the series $\sum_{\sigma}$ in the right hand side converges in the Hilbert space $S_\omega$.

**Proposition 1.4.** Let

$$K(x, y) = \sum_{m,n} \gamma_{m,n} p_m(x) p_n(y).$$
Then

\[ \langle A_{Kp_n}, p_m \rangle_\omega = \gamma_{m,n} \prod_j \omega_j^{m_j+n_j} m_j! n_j! . \]

**Proposition 1.5.** Let \( \omega_j = j \), i.e., we have the classical scalar product in \( S \). Let

\[ K(x, y) = \sum_{\lambda, \mu} \beta_{\lambda, \mu} s_\lambda(x) s_\mu(y) . \]

Then

\[ \langle A_{Ks_\mu}, s_\lambda \rangle_\omega = \beta_{\lambda, \mu} . \]

**Remark.** On the kernel of the identity operator, see [Mac], I.4, see also below 3.6.

**1.6. Proofs of Propositions 1.1–1.5.** We start from Proposition 1.3a. Represent \( K(x, y) \) as the series

\[ K(x, y) = \sum_{\tau, \sigma} K_{\tau, \sigma}(x, y) , \]

where \( K_{\tau, \sigma} \) has the degree \( \tau \) in the variables \( x_1, x_2, \ldots \) and the degree \( \sigma \) in the variables \( y_1, y_2, \ldots \).

Let \( A_{\tau, \sigma} \) be the operator with the kernel \( K_{\tau, \sigma}(x, y) \),

\[ A_{\tau, \sigma} : \oplus S^j \rightarrow \oplus S^j \]

Obviously, \( A_{\tau, \sigma} \) is zero on all \( S^j \) for \( j \neq \tau \), and \( A_{\tau, \sigma} : S^\tau \rightarrow S^\sigma \).

For the kernel \( K_{\tau, \sigma} \), the statement of Proposition 1.3a) is trivial. It is equivalent to the following obvious lemma.

**Lemma 1.6.** Let \( V, W \) be finite dimensional Euclidean spaces. Let \( e_j \) be a basis in \( V \). Let \( K \in V \otimes W \),

\[ K = \sum e_j \otimes h_j , \quad h_j \in W . \]

Then each linear operator \( W \rightarrow V \) has the form

\[ A_K w = \sum \langle h_j, w \rangle_W e_j \]

and \( A_K \) does not depend on a choice of the basis \( e_j \).

We apply this lemma to \( W = S_\omega^\sigma, V = S_\omega^\sigma \).

Thus, \( A_K \) is a block operator

\[ A_K = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{11} & A_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad (1.7) \]

in \( \oplus S^j \). Now Proposition 1.3a became obvious.
Let $A_K$ be bounded, let $f = \sum f_j$, $Af = \sum g_j$, where $f_j$, $g_j \in S^\prime$. Then

$g_\tau = \sum A_{\tau,\sigma} f_\sigma$, and this is the statement of Proposition 1.3b.

Thus Proposition 1.3 is proved.

If we assume $\xi = s_\lambda$, then we obtain Proposition 1.5.

If we assume $\xi = p_m$, then we obtain Proposition 1.4. This implies Proposition 1.1.

If we assume $\xi = m_\lambda$ in Proposition 1.3b, then we obtain Proposition 1.2.

1.7. Products of operators.

**Proposition 1.7.** Let for bisymmetric kernels $K$, $L$ the operators $A_K$, $A_L$ be bounded in $S_\omega$. Then $A_K A_L = A_M$, where

$$M(x, z) = \langle K(x, y), L(y, z) \rangle_{S_\omega(y)}.$$  \hspace{1cm} (1.8)

More exactly, we expand our kernels in $m_\lambda$

$$K(x, y) = \sum_\lambda m_\lambda(x) u_\lambda(y); \quad L(x, y) = \sum_\mu m_\mu(z) v_\mu(y)$$

and write

$$M(x, z) = \sum_{\lambda, \mu} \langle u_\lambda, v_\mu \rangle_{S_\omega} m_\lambda(x)m_\mu(z)$$

**Proof.** We represent $A_K$, $A_L$ as block matrices (1.7), and the statement becomes obvious. We also can refer to Proposition 1.4.

2. Boson Fock space. Semigroup of Gauss operators

Here we discuss some basic definitions related to the boson Fock space. For details, see [Ner2].

2.1. Boson Fock spaces with finite degrees of freedom. Let $n = 0, 1, 2, \ldots$. Consider the space $\mathbb{C}^n$ with the coordinates $z_1, z_2, \ldots, z_n$. Consider the space $P_{\omega}^n$ of polynomials in $z_j$; equip this space with the scalar product

$$\langle f, g \rangle = \pi^{-n} \int_{\mathbb{C}^n} f(z) \overline{g(z)} \exp\{-|z|^2\} |dz|,$$  \hspace{1cm} (2.1)

where $|dz|$ is the Lebesgue measure on $\mathbb{C}^n$.

The monomials $z_1^{m_1} \ldots z_n^{m_n}$ are pairwise orthogonal and

$$\|z_1^{m_1} \ldots z_n^{m_n}\|^2 = m_1! \ldots m_n!$$  \hspace{1cm} (2.2)

The boson Fock space $F_n$ with $n$ degrees of freedom is the completion of $P_{\omega}^n$ with respect to the scalar product (2.1). Elements of this space are entire functions $f(z)$ on $\mathbb{C}^n$ satisfying

$$\int_{\mathbb{C}^n} |f(z)|^2 \exp\{-|z|^2\} |dz| < \infty.$$
2.2. The boson Fock space with infinite number degrees of freedom. Let \( f = f(z_1, \ldots, z_n) \in \mathbb{F}_n \). We define the function \( I_n f \in \mathbb{F}_{n+1} \) as

\[
I_n f(z_1, \ldots, z_n, z_{n+1}) = f(z_1, \ldots, z_n)
\]

By (2.2), \( I_n \) is an isometric embedding \( \mathbb{F}_n \to \mathbb{F}_{n+1} \).

Consider the chain

\[
\mathbb{F}_0 \subset \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n \subset \cdots
\]

The boson Fock space \( \mathbb{F} = \mathbb{F}(z) \) with infinite degree of freedom is the completion of \( \bigcup_n \mathbb{F}_n \). The space \( \mathbb{F} \) is called the boson Fock space.

Consider the system of functions

\[
e_m(z) := \prod z_j^{m_j},
\]

where \( m_j \) are nonnegative integers and \( m_j = 0 \) for sufficiently large \( j \). Obviously, these functions form an orthogonal basis in \( \mathbb{F} \), and

\[
\|e_m\|^2 = \prod m_j!.
\]

Consider a series \( \sum c_m e_m(z) \) convergent in the Hilbert space \( \mathbb{F} \). It can easily be checked, that this series absolutely converges for each \( z \in l_2 \). Hence we can consider elements of \( \mathbb{F} \) as entire functions on \( l_2 \).

Let \( a = (a_1, a_2, \ldots) \in l_2 \). Denote by \( \varphi_a(z) \) the function\(^3\)

\[
\varphi_a(z) = \exp\left\{ \sum z_j a_j \right\}.
\]

For each \( f \in \mathbb{F} \), the following reproducing property holds

\[
\langle f, \varphi_a \rangle = f(a).
\]

2.3. Hilbert–Schmidt matrices. (see [DS], XI.6, XI.9) Recall that an infinite matrix \( A = \{a_{ij}\} \) is called a Hilbert–Schmidt matrix if

\[
\text{tr} A^* A = \sum_{ij} |a_{ij}|^2 < \infty.
\]

A matrix \( A \) belongs to the trace class if

\[
\text{tr} \sqrt{A^* A} < \infty.
\]

Recall, that for Hilbert–Schmidt matrices \( A, B \), the matrix \( AB \) belongs to the trace class.

Also, recall that for a matrix \( A \) of the trace class the determinant \( \det(1 + A) \) is well-defined.

\(^3\)There are many terms that are used for this system of functions: systems of coherent states, supercomplete basis, overfilled basis, delta-functions.
2.4. Gauss vectors. For a matrix $A$, the symbol $\|A\|$ below denotes the norm of the operator $l_2 \to l_2$, i.e., $\|A\|^2$ is the maximal eigenvalue of $A^*A$.

Let $A = \{a_{ij}\}$ be a symmetric (i.e., $a_{ij} = a_{ji}$) Hilbert–Schmidt matrix and $\|A\| < 1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in l_2$. The corresponding Gauss vector $b[A|\alpha]$ is a function in the variable $z \in l_2$ defined by

$$b[A|\alpha](z) = \exp\left\{ \frac{1}{2} \sum a_{ij}z_i z_j + \sum \alpha_j z_j \right\} = \exp\left\{ \frac{1}{2} zA z^t + az^t \right\}.$$

Here $^t$ denotes the transposition, $z$, $\alpha$ are considered as matrix–rows.

**Proposition 2.1.**

a) $b[A|\alpha] \in \mathbb{F}$.

b) $\langle b[A|\alpha], b[B|\beta] \rangle_{\mathbb{F}} = \det(1 - \overline{AB})^{-1/2} \exp\left\{ \frac{1}{2}(\alpha \overline{\beta}) \left( \begin{array}{cc} -A & 1 \\ 1 & -B \end{array} \right) \right\}$.

**Remark.**

$$\left( \begin{array}{cc} -A & 1 \\ 1 & -B \end{array} \right)^{-1} = \left( \begin{array}{cc} \overline{B}(1 - \overline{AB})^{-1} & (1 - \overline{BA})^{-1} \\ (1 - AB)^{-1} & A(1 - \overline{BA})^{-1} \end{array} \right).$$

**Remark.** Since $\|AB\| < 1$, we define

$$(1 - \overline{AB})^{-1/2} := 1 + \frac{1}{2} A\overline{B} + \frac{1}{8} (A\overline{B})^2 + \ldots$$

and $\det(1 - \overline{AB})^{-1/2} := \det[(1 - \overline{AB})^{-1/2}]$.

2.5. Operators. Let $H$ be a bounded operator $\mathbb{F} \to \mathbb{F}$. Let

$$H e_n = \sum_m h_{m,n} e_m.$$ 

Consider the function $K(z, u)$ (kernel of the operator) $H$ on $l_2 \times l_2$ given by

$$K(z, u) = \sum_{m,n} h_{m,n} n! e_m(z) e_n(u),$$

where $e_n$ is given by (2.3) and $n! = \prod n_j!$. This series converges on $l_2 \times l_2$, moreover for a fixed $z \in l_2$, the function $k_z(u) = K(z, u)$ as a function in $u$ belongs to $\mathbb{F}$.

We also have

$$\langle e_m, H e_n \rangle = h_{m,n} n! \quad (2.6)$$

The operator $H$ can be reconstructed from $K(z, u)$ by the formula

$$Hf(z) = \langle f(u), K(z, u) \rangle_{\mathbb{F}(u)} := \langle f(u), k_z(u) \rangle_{\mathbb{F}(u)}.$$

If $K_1$ is the kernel of $H_1$ and $K_2$ is the kernel of $H_2$, then the kernel of $H_1H_2$ is

$$L(z, w) = \langle K_1(z, u), K_2(u, w) \rangle_{\mathbb{F}(u)}.$$
2.6. Gauss operators. Let $A, B, C$ be infinite matrices. Let $S := \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$. Assume that these matrices satisfy the following conditions.

0°. $A, C$ are symmetric matrices, i.e., $A = A^t$, $C = C^t$. Equivalently, $S$ is symmetric.

1°. $\|S\| \leq 1$.

2°. $\|A\| < 1$, $\|C\| < 1$.

3°. The matrices $A, C$ are Hilbert–Schmidt.

Under these conditions, we define the Gauss operator

$$B \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} : \mathbb{F} \rightarrow \mathbb{F},$$

being the operator with the kernel

$$K(z, u) = \exp \left\{ \frac{1}{2} (z \mid u) \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} z^t \\ w^t \end{pmatrix} \right\}.$$

The conditions 1° – 3° are necessary but not sufficient for boundedness. There are two simple sufficient conditions of boundedness.

Theorem 2.2. ([Ner1]) a) If $A, C$ are operators of the trace class, then $B[\cdot]$ is bounded.

b) If $\left\| \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \right\| < 1$, then $B[\cdot]$ is bounded.

A product of two Gauss operators is well-defined and it is given by the following theorem.

Theorem 2.3. ([Ols], [NNO])

$$B \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} B \begin{bmatrix} U & V \\ V^t & W \end{bmatrix} =$$

$$= \det(1 - CU)^{-1/2} B \begin{bmatrix} A + BU(1 - CU)^{-1}B^t & B(1 - UC)^{-1}V \\ V^t(1 - CU)^{-1}B^t & W + V^t(1 - CU)^{-1}CV^t \end{bmatrix}.$$

2.7. Linear relations. Formula (2.7) hides a simple algebraic structure, namely a product of linear relations. Recall necessary definitions.

Let $V, W$ be linear spaces. A linear relation $P : V \rightrightarrows W$ is a subspace $P \subset V \oplus W$.

Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be linear relations. Their product $QP$ is a linear relation $V \rightrightarrows Y$ defined in the following way. A vector $v \oplus y \in V \oplus Y$ is an element of $QP$ if there exists $w \in W$ such that $v \oplus w \in P$, $w \oplus y \in Q$.

Example. Let $L : V \rightarrow W$ be a linear operator. Its graph is a linear relation. A product of operators corresponds to the product of their graphs in the sense of linear relations.

---

4for the case of unbounded operators, see [Ner2], VI.2
For a linear relation \( P : V \rightrightarrows W \), we define its \textit{kernel} \( \text{Ker} P \subset V \) and its \textit{indefiniteness} \( \text{Indef} P \subset W \) by
\[
\text{Ker} P = P \cap V; \quad \text{Indef} P = P \cap W.
\]

\textbf{2.8. Clarification of Theorem 2.3.} For a given matrix \( S := \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \) we consider the subspace \( \mathcal{P}[S] \subset [l_2 \oplus l_2] \oplus [l_2 \oplus l_2] \) consisting of vectors
\[
[v_+ \oplus v_-] \oplus [w_+ \oplus w_-]
\]
satisfying the equations
\[
\begin{pmatrix} v^+ \\ w^- \end{pmatrix} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} v^- \\ w^+ \end{pmatrix}.
\]
We consider this subspace as a linear relation \( l_2 \oplus l_2 \rightrightarrows l_2 \oplus l_2 \).

\textbf{Theorem 2.4.} \((\text{NNO}, \text{Ner1})\) The linear relations \( \mathcal{P}[S] \), where \( S \) satisfies conditions 1°–3°, form a semigroup with respect to the multiplication of linear relations. The equality
\[
\mathcal{B}[S_1] \mathcal{B}[S_2] = \text{const} \cdot \mathcal{B}[S_3].
\]
is equivalent to
\[
\mathcal{P}[S_1] \mathcal{P}[S_2] = \mathcal{P}[S_3].
\]
We call the semigroup of all the linear relations \( \mathcal{P}[S] \) by \textit{the symplectic semigroup}.

Now we intend to characterize linear relations of type \( \mathcal{P}[S] \).

\textbf{2.9. Geometric description of symplectic semigroup.} Denote by \( V^+, V^-, W^+, W^- \) four copies of the space \( l_2 \).

Let
\[
V = V^+ \oplus V^-, \quad W = W^+ \oplus W^-.
\]
We define in \( V \) the skew-symmetric bilinear form \( \{\cdot, \cdot\}_V \) and the Hermitian form \( [\cdot, \cdot]_V \) by
\[
\{\xi^+ \oplus \xi^-, \eta^+ \oplus \eta^-\}_V := \sum_j (\xi_j^+ \eta_j^- - \xi_j^- \eta_j^+);
\]
\[
[\xi^+ \oplus \xi^-, \eta^+ \oplus \eta^-]_V := \sum_j (\xi_j^+ \bar{\eta}_j^- - \xi_j^- \bar{\eta}_j^+).
\]
We define two forms in \( W \) by the same formulae. Further we define the skew symmetric form \( \{\cdot, \cdot\}_{V \oplus W} \) and the Hermitian form \( [\cdot, \cdot]_{V \oplus W} \) in \( V \oplus W \) by
\[
\{v \oplus w, v' \oplus w'\}_{V \oplus W} = \{v, v'\}_V - \{w, w'\}_W.
\]
\[
[v \oplus w, v' \oplus w']_{V \oplus W} = [v, v']_V - [w, w']_W.
\]
Our conditions for the matrix $S$ are equivalent to the following conditions for linear relations $P[S]$.

A) Our matrix $S = S^t$ is symmetric. This means that $P[S]$ is Lagrangian\(^5\) with respect to the skew-symmetric form $\{\cdot, \cdot\}_{V \oplus W}$.

B) The condition $\|S\| \leq 1$ is equivalent to the condition
\[
[v \oplus w, v \oplus w]_{V \oplus W} \geq 0 \quad \text{for all} \ v \oplus w \in P[S].
\]

C) The conditions $\|A\| \leq 1$, $\|C\| \leq 1$ follow from $\|S\| \leq 1$. The additional condition $\|A\| < 1$ is equivalent to positive definiteness of $[\cdot, \cdot]_V$ on Ker$P[S]$. The condition $\|C\| < 1$ is equivalent to negative definiteness of $[\cdot, \cdot]_W$ on Indef$P[S]$.

D) The condition 3° from 2.6 is not so transparent from geometrical point of view. The orthogonal projectors $P \cap (V \oplus W) \to W$ and $P \cap (V \oplus W_+) \to V_+$ must be Hilbert–Schmidt operators.

The unit element of the symplectic semigroup is the graph of the identity operator; the corresponding matrix $S$ is $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The group of automorphisms of the canonical commutation relations defined in the next subsection is the group of invertible elements of the symplectic semigroup.

2.10. The symplectic group. Consider the group $\text{Sp}(2\infty, \mathbb{R})$ of bounded invertible operators $g : l_2 \oplus l_2 \to l_2 \oplus l_2$ satisfying the following properties.

1) The operator $g$ commutes with the $\mathbb{R}$-linear transformation $v_1 \oplus v_2 \mapsto \overline{v_2} \oplus \overline{v_1}$, where $v \mapsto \overline{v}$ is the coordinate-wise complex conjugation. Equivalently, the matrix of $g$ has the following block form
\[
g = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}.
\]

Equivalently, $g$ preserves $\mathbb{R}$-linear subspace $V_\mathbb{R}$ consisting of all the vectors having the form $(v, \overline{v})$.

2) The operator $g$ preserves the skew-symmetric form $\{\cdot, \cdot\}$. Equivalently,
\[
g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

3) The operator $g$ preserves the Hermitian form $[\cdot, \cdot]$. Equivalently,
\[
g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Remark. Any two of conditions 1)–3) imply the third condition. The conditions 1) and 2) mean that $g$ preserves a skew-symmetric bilinear form in the real linear space $V_\mathbb{R}$.

\(^5\)A subspace $P$ is Lagrangian with respect to a form $\{\cdot, \cdot\}$ if $\{v, w\} = 0$ for each $v, w \in P$ and $P$ is a maximal among subspaces having this property.
The Friedrichs–Shale group $\text{Sp}$ of automorphisms of canonical commutation relations\(^6\) consists of elements $g \in \text{Sp}(2\infty, \mathbb{R})$, such that the block $Q$ is a Hilbert–Schmidt matrix.

**Proposition 2.5.** \cite{Ber1} For $g \in \text{Sp}$, define the operator

$$\rho(g) := B \begin{bmatrix} Q P^{-1} & (P^*)^{-1} \\ P^{-1} & -P^{-1} Q \end{bmatrix}$$

Then $\rho(g)$ is a projective representation of $\text{Sp}$ and the operators $\det(PP^*)^{-1/4} \rho(g)$ are unitary.

### 3. Boson-symmetric correspondences

In this section, we consider arbitrary spaces $S_\omega$.

#### 3.1. Definition of the correspondence.

For $f = f(z_1, z_2, \ldots) \in \mathbb{F}$, we define the element $\Theta f \in S_\omega$ by

$$\Theta f(x_1, x_2, x_3, \ldots) = f(\omega_1^{-1/2} p_1(x), \omega_2^{-1/2} p_2(x), \omega_3^{-1/2} p_3(x), \ldots). \quad (3.1)$$

**Theorem 3.1.** The map $\Theta$ is a unitary operator $\mathbb{F} \to S_\omega$.

**Proof.** This follow from formulae (1.1), (2.2). \Box

**Remark.** The square roots in formula (3.1) can be easily deleted after minor variation of definition of boson Fock space. This variant of a language is used in \cite{Ner4}. Also, the boson Fock spaces corresponding to classical scalar products (Redfield, Hall–Littlewood, Macdonald) are described in \cite{Ner4}.

**3.2. Inversion formula.**

**Theorem 3.2.** Let $g \in S_\omega$. Then

$$\Theta^{-1} g(z) = \langle g, \Phi_z \rangle_\omega,$$

where $\Phi_z(x) \in S_\omega$ is given by

$$\Phi_z(x) := \prod_k \exp \left\{ \sum_{j=1}^{\infty} \omega_j^{-1/2} x_j \right\}.$$  

**Proof.** The functions $\Phi_a(x) \in S_\omega$ correspond to the elements $\exp(\sum a_j z_j) \in \mathbb{F}$, see (2.4). It remains to apply the reproducing property (2.5). More carefully,

$$f(z) := \langle f, \varphi_z \rangle_\mathbb{F} = \langle \Theta f, \Theta \varphi_z \rangle_\omega = \langle \Theta f, \Phi_z \rangle_\omega.$$

**3.3. Correspondence of operators.**

**Proposition 3.3.** Let $A : \mathbb{F} \to \mathbb{F}$ be a bounded operator, let $K(z, u)$ be its kernel. Then the bisymmetric kernel of the operator

$$\Theta A \Theta^{-1} : S_\omega \to S_\omega$$

\footnote{It is the $(G,K)$-pair $(\text{Sp}(2\infty), U(\infty))$ in the notation of \cite{Ner2}.}
is given by
\[ K(\omega_1^{-1/2}p_1(x), \omega_2^{-1/2}p_2(x), \ldots; \omega_1^{-1/2}p_1(y), \omega_2^{-1/2}p_2(y), \ldots), \]

**Proof.** See Proposition 1.4 and formula (2.6).

### 3.4. Multiplicative vectors.

Let \( A = \{a_{ij}\} \) be a symmetric matrix \((i, j > 0)\), let \( \alpha = \{\alpha_j\} \) be a sequence. We define the formal series \( \Psi[A|\alpha] \in S \) by

\[
\Psi[A|\alpha](x) = \exp\left\{ \frac{1}{2} \sum_{1 \leq i < j < \infty} a_{ij} p_i(x) p_j(x) + \sum_{1 \leq j < \infty} \alpha_j p_j(x) \right\} = \prod_{1 \leq k < l < \infty} \exp\left\{ \sum_{1 \leq i < \infty} a_{ij} x_k^i x_l^j \right\}. \tag{3.2}
\]

**Remark.** Consider an arbitrary formal series having the form

\[
\prod_{1 \leq k < l < \infty} R(x_k, x_l) \prod_{1 \leq k < \infty} Q(x_k), \tag{3.3}
\]

where \( Q(x) = 1 + \sum_{j > 0} q_j x^j \) and

\[
R(x, y) = 1 + \sum_{i > 0, j > 0} r_{ij} x^i y^j; \quad r_{ij} = r_{ji}
\]

Each series (3.3) can be represented in the form (3.2).

Denote

\[
\Omega := \begin{pmatrix} \omega_1 & 0 & \ldots \\ 0 & \omega_2 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.4}
\]

**Theorem 3.4.**

a) Denote \( \tilde{A} := \Omega^{1/2} A \Omega^{1/2} \). Let the matrix \( \tilde{A} \) be Hilbert–Schmidt, and \( \|\tilde{A}\| < 1 \). Let \( \alpha \Omega^{1/2} \in l_2 \). Then \( \Psi[A|\alpha] \in S. \)

b) Let \( A, \alpha \) and \( B, \beta \) satisfy the conditions of statement a). Then

\[
\langle \Psi[A|\alpha], \Psi[B|\beta] \rangle_S = \det(1 - A \Omega B \Omega)^{-1/2} \det\left\{ \left( \begin{array}{c} \Omega^{-1} \\ -B \end{array} \right) \left( \begin{array}{c} -A \\ \Omega^{-1} \end{array} \right)^{-1} \left( \begin{array}{c} \alpha^t \\ -\beta^t \end{array} \right) \right\}
\]

**Proof.** The function \( \Psi[A|\alpha] \) is the image of the Gauss vector

\[
b[\Omega^{1/2} A \Omega^{1/2} | \alpha \Omega^{1/2}]
\]

under the map \( \Theta \). Thus the statement follows from Proposition 2.1.
3.5. Multiplicative bisymmetric kernels. We intend to discuss bisymmetric kernels of the type

$$\prod_{k<l} P(x_k, x_l) \prod_{k<l} Q(x_k, y_l) \prod_{k<l} R(y_k, y_l) \prod_{k} \pi(x_k) \prod_{k} \rho(y_k),$$

where $P$, $Q$, $R$, $\pi$, $\rho$ are formal series.

Fix symmetric matrices $A = \{a_{ij}\}$, $C = \{c_{ij}\}$ and a matrix $B = \{b_{ij}\}$, where $i, j > 0$. Assume that the matrix satisfies the conditions of Subsection 2.6, i.e.,

$$S := \begin{pmatrix} \Omega^{1/2} & 0 \\ 0 & \Omega^{1/2} \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} \Omega^{1/2} & 0 \\ 0 & \Omega^{1/2} \end{pmatrix}$$

satisfies the conditions of Subsection 2.6, i.e.,

1*. $\|S\| \leq 1$.

2*. $\|\Omega^{1/2} A \Omega^{1/2}\| < 1$, $\|\Omega^{1/2} C \Omega^{1/2}\| < 1$.

3*. $\Omega^{1/2} A \Omega^{1/2}$, $\Omega^{1/2} C \Omega^{1/2}$ are Hilbert–Schmidt matrices.

Then we define the bisymmetric kernel $R \left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right] (x, y)$ by

$$R \left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right] (x, y) = \prod_{k<l} \exp\left\{ \sum_{i>0, j>0} a_{ij} x_k^i x_l^j \right\} \prod_{k<l} \exp\left\{ \sum_{i>0, j>0} b_{ij} x_k^i y_l^j \right\} \times \prod_{k \geq 1} \exp\left\{ \sum_{j} \left( \frac{1}{2} \sum_{\sigma, \tau: \sigma + \tau = j} a_{\sigma \tau} x_k^j \right) \right\} \prod_{l \geq 1} \exp\left\{ \sum_{j} \left( \frac{1}{2} \sum_{\sigma, \tau: \sigma + \tau = j} c_{\sigma \tau} y_l^j \right) \right\} \quad (3.6)$$

The same kernel can be also represented in the form

$$R \left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right] = \exp\left\{ \sum_{i,j} \frac{1}{2} a_{ij} p_i(x) p_j(x) + \sum_{i,j} b_{ij} p_i(x) p_j(y) + \sum_{i,j} \frac{1}{2} c_{ij} p_i(y) p_j(y) \right\} \quad (3.7)$$

Denote by $S^\omega$ the operator $S \omega \to S \omega$ defined by the kernel $R \left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right]$. We have the following theorem:

**Theorem 3.5.** Let the conditions 1*–3* be satisfied.

a) Let $\|S\| < 1$. Then the operator $S^\omega$ is bounded.

b) Let $\Omega^{1/2} A \Omega^{1/2}$, $\Omega^{1/2} C \Omega^{1/2}$ be operators of the trace class. Then $S^\omega$ is bounded.

**Theorem 3.6.** Let $A \left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right] A \left[\begin{array}{cc} U & V \\ V^t & W \end{array} \right]$ be bounded operators. Then

$$\left[\begin{array}{cc} A & B \\ B^t & C \end{array} \right] \left[\begin{array}{cc} U & V \\ V^t & W \end{array} \right] = \det(1 - C \omega U \Omega)^{-1/2} \times$$

$$\times \left[\begin{array}{cc} A + B \omega U (\Omega^{-1} - C \omega U)^{-1} B^t & B (\Omega^{-1} - U \omega C)^{-1} V \\ V^t (\Omega^{-1} - C \omega U)^{-1} B^t & W + V^t (\Omega^{-1} - C \omega U)^{-1} C \omega V \end{array} \right].$$
The semigroup of operators $\mathfrak{A}[:]$ is isomorphic to the semigroup of Gauss operators $\mathfrak{B}[:]$.

**Proofs.** By Proposition 3.3,

$$\mathfrak{A} \left[ \begin{array}{cc} A & B \\ B^t & C \end{array} \right] = \Theta \circ \mathfrak{B} \left[ \begin{array}{cc} \Omega^{1/2} & 0 \\ 0 & \Omega^{1/2} \end{array} \right] \left( \begin{array}{cc} A & B \\ B^t & C \end{array} \right) \left( \begin{array}{cc} \Omega^{1/2} & 0 \\ 0 & \Omega^{1/2} \end{array} \right) \circ \Theta^{-1}.$$ 

Now our theorems follow from Theorems 2.2, 2.3.

3.6. **Example. The identity operator.** The identity operator in $\mathfrak{F}$ has the kernel $\exp\{\sum zju_j\}$.

Hence the bisymmetric kernel of the identity operator in $\mathfrak{S}_\omega$ is

$$K(x, y) = \prod_{k,l} \exp\left\{ \sum_{j>0} \omega_j^{-1} x^j_k y^j_l \right\}.$$ 

1) For the Redfield scalar product, i.e., $\omega_j = j$, we obtain

$$K(x, y) = \prod_{k,l} (1 - xky_l)^{-1}.$$ 

2) For the Jack scalar product, i.e., $\omega_j = j\alpha$, we have

$$K(x, y) = \prod_{k,l} (1 - xky_l)^{-1/\alpha}.$$ 

3) For the Macdonald scalar products (see (1.2)),

$$\sum_{j>0} \frac{x^j y^j}{\omega_j} = \sum_{j>0} \frac{1}{j} \frac{1 - t^j}{1 - q^j} x^j y^j.$$ 

Expanding

$$\frac{1 - \frac{t^j}{1 - q^j}}{1 - q^j} = \sum_{k \geq 0} q^{jk} - \frac{t^j}{1 - q^j} \sum_{k \geq 0} q^{jk},$$

we obtain

$$\sum_{j>0} \frac{x^j y^j}{\omega_j} = - \sum_{k \geq 0} \ln(1 - q^kxy) + \sum_{k \geq 0} \ln(1 - q^ktxy).$$

Finally,

$$K(x, y) = \prod_{k \geq 0} \frac{1 - txq^k}{1 - xq^k}.$$ 

All these formulae are well known, see [Mac], Sections I.4, III.4, VI.2, VI.10, see also [Ker].

3.7. **Example. The Heisenberg group.** Formula (3.5) is more general than (3.6). Our construction can be easily extended to this more general case,
it is sufficient to apply [Ner2], VI.4; the semigroup of all bounded operators in \( S_\omega \) with kernels (3.5) is isomorphic to the semigroup of all bounded operators in \( \mathbb{F} \) having kernels of the form

\[
\exp \left\{ \frac{1}{2} \left( z u \right) \left( \begin{array}{cc} K & L \\ L^t & M \end{array} \right) \left( \begin{array}{c} z^t \\ u^t \end{array} \right) + 2\alpha^t + u\beta^t \right\}
\]

We do not discuss the general case and consider an example.
Let \( \alpha, \beta \in l_2 \). Consider the operator \( T(\alpha, \beta) \) in \( \mathbb{F} \) whose the kernel is

\[
\exp \left\{ \sum_{j>0} \omega_j x_j \right\} \prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} x_k y_j \right\}. 
\]

Then (see [Ner2], Section VI.4)

\[
T(\alpha, \beta)T(\alpha', \beta') = \exp \left\{ \sum \alpha_j \beta_j \right\} T(\alpha + \alpha', \beta + \beta').
\]

Thus, the operators \( T(\alpha, \beta) \) form the complex Heisenberg group.

**Remark.** If \( \beta_j = -\pi_j \), then the operators \( T(\alpha, \beta) \) are unitary up to a scalar factor. Otherwise, they are unbounded. Nevertheless our operators and their products are well-defined; for details, see [Ner2], VI.4. □

Let us describe the corresponding construction in \( S_\omega \). Let \( a, b \in \Omega^{1/2}, b \in \Omega^{1/2}, \in l_2 \).
Consider the operator \( R(a, b) \) in \( S_\omega \), whose bisymmetric kernel is

\[
\prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} x_k y_j \right\} \prod_{k} \exp \left\{ \sum_{j>0} a_j x_k^j \right\} \prod_{l} \exp \left\{ \sum_{j>0} b_j y_l^j \right\}. 
\]

Then

\[
R(a, b)R(a', b') = \exp \left\{ \sum \omega_j b_j a'_j \right\} R(a + a', b + b'). 
\]

These operators form the complex Heisenberg group.

**Remark.** The Heisenberg algebra (creation-annihilation operators) consists of operators with kernels

\[
\prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} x_k^j y_l^j \right\} \cdot \left( r + \sum_k \sum_j s_j x_k^j + \sum_k \sum_j t_j y_k^j \right)
\]

**3.8. Example: a formula for kernels of operators.** We intend to write a formula reconstructing the bisymmetric kernel \( K(x, y) \) by the operator \( A \).

Let \( x_j, y_j, u_j \) be 3 collections of formal variables. Then

\[
K(x, y) = \prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} x_k^j u_l^j \right\}, A_u \prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} u_k^j y_l^j \right\} \}_{S_\omega [u]} (3.8)
\]

In this formula, \( A_u \) means that the operator \( A \) acts on symmetric functions in the variables \( u_j \) depending on the parameters \( y_j \). Then we consider scalar product of functions in the variables \( u_j \), depending on the parameters \( x_j, y_j \); see explanations to formula (1.8) above.
To prove (3.8), we expand
\[
\prod_{k,l} \exp \left\{ \sum_{j>0} \omega_j^{-1} u_k^j y_l^j \right\} = \sum_n \frac{\omega^n}{n!} p_n(u) p_n(y)
\]
and apply Proposition 1.4.

4. Space of semiinfinite forms \(\Lambda\) and space of skew-symmetric functions

By \(S_\infty\) (respectively \(S_{2\infty}\)) we denote the group of all finite permutations of the set \(\{1,2,3,\ldots\}\) (respectively \(\{-2,-1,0,1,2,3,\ldots\}\)).

4.1. Anticommuting variables. Let \(\xi_1, \ldots, \xi_n\) be anticommuting variables, i.e.,
\[
\xi_i \xi_j = -\xi_j \xi_i \quad (4.1)
\]
for all pairs \(i, j\); in particular \(\xi_j^2 = 0\). Denote by \(\Lambda_n\) the space of all polynomials in \(\xi_j\); evidently, \(\dim \Lambda_n = 2^n\). We assume that the monomials \(\xi_{i_1} \xi_{i_2} \ldots \xi_{i_k}\), where \(i_1 > i_2 > \cdots > i_k\), form an orthonormal basis in \(\Lambda_n\).

Consider a family of linear forms \(a_1(\xi), \ldots, a_m(\xi)\) and \(b_1(\xi), \ldots, b_m(\xi)\):
\[
a_k(\xi) = \sum_{j=1}^n a_{kj} \xi_j; \quad b_k(\xi) = \sum_{j=1}^n b_{kj} \xi_j.
\]

Denote by \(A\) and \(B\) the matrices \(\{a_{ij}\}, \{b_{ij}\}\).

**Lemma 4.1.** Let \(m \leq n\). Then
\[
\begin{align*}
a) \quad a_1(\xi) \ldots a_m(\xi) &= \sum_{j_1 > j_2 > \cdots > j_m} \det_{1 \leq i, k \leq m} \{a_{ij}, \xi_j\} \xi_{i_1} \ldots \xi_{i_m} \\
b) \quad \langle a_1(\xi), \ldots, a_m(\xi), b_1(\xi) \ldots b_m(\xi) \rangle &= \det \{\langle a_k, b_l \rangle\} = \det AB^* \\
c) \quad \prod_{l=1}^m \left( \sum_{k=1}^m b_{lk} a_k(\xi) \right) &= \det \{b_{kl}\} \prod_{l=1}^m a_l(\xi)
\end{align*}
\]

These statements are obvious.

4.2. The space \(\Lambda\). (see [Ber2], [FF]) Consider a countable collection
\[
\ldots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots
\]
of anticommuting variables,

A **semi-infinite monomial** is an infinite product of the type
\[
\Xi_t = \prod_{i=1}^{\infty} \xi_{k_i} := \xi_{k_1} \xi_{k_2} \xi_{k_3} \ldots, \quad \text{where } k_j = -j \text{ for sufficiently large } j.
\]

We allow to apply the rule (4.1) finite number of times. Hence, for \(\sigma \in S_{2\infty}\),
\[
\Xi_{\sigma t} = (-1)^{\sigma} \Xi_t.
\]
Modulo permutations, each collection \( \mathfrak{t} \) can be reduced to the form
\[
\mathfrak{t} : k_1 > k_2 > k_3 > \ldots, \quad \text{where } k_j = -j \text{ for sufficiently large } j. \tag{4.2}
\]

We emphasis that
\[
\{ \text{number of } k_j \geq 0 \} = \{ \text{number of negative } l \text{ such that } l \neq k_i \ \forall i \}.
\]

We define the space \( \Lambda \) as the Hilbert space whose orthonormal basis is \( \xi_{\mathfrak{t}}, k \), where \( k \) satisfies (4.2). We also define the space \( \bar{\Lambda} \) whose elements are formal series \( \sum_k c_k \xi_{\mathfrak{t}} \) with arbitrary \( c_k \).

4.3. Remark: the usual fermion Fock space. A fermion Fock space is a space of functions depending on a countable collection of anticommuting variables.

Consider an infinite collection \( \xi_0, \xi_1, \ldots, \eta_1, \eta_2, \ldots \) of anticommuting variables. Consider all possible finite monomials
\[
\xi_{u_1} \ldots \xi_{u_n} \eta_{v_1} \ldots \eta_{v_m}; \tag{4.3}
\]

Denote by \( \mathcal{L} \) the space (the fermion Fock space), whose orthonormal basis consists of such monomials.

To each vector \( \xi_{\mathfrak{t}} \), we assign the vector
\[
\prod_{k_j \geq 0} \xi_{k_j} \prod_{l : l < 0, l \neq k_i \ \forall i} \eta_{-l}.
\]

Hence we obtain a space, whose basis consists of products \( \xi_{u_1} \ldots \xi_{u_n} \eta_{v_1} \ldots \eta_{v_m} \). As we observed above, the number of \( \xi \) in this product equals the number of \( \eta \); i.e., we have \( m = n \) in (4.3).

Thus, \( \Lambda \) is a subspace in \( \mathcal{L} \).

4.4. Decomposable vectors in the Hilbert space \( \Lambda \). Consider two infinite matrices \( A = \{ a_{mi} \}, B = \{ b_{mj} \} \), where \( m > 0, 0 \leq i < \infty, -\infty < j < 0 \). Assume that \( A \) is a Hilbert–Schmidt matrix and \( B \) belongs to the trace class.

Consider the product
\[
\Xi[A; 1 + B] := \prod_{m=1}^{\infty} \left( \xi_{-m} + \sum_{i \geq 0} a_{mi} \xi_i + \sum_{j < 0} b_{mj} \xi_j \right);
\]

removing parentheses, we choose the summand \( \xi_{-m} \) from each parenthesis except a finite number of factors.

We call \( \Xi[A, 1 + B] \) by decomposable vectors. The vectors \( \xi_{\mathfrak{t}} \) defined above also have the form \( \Xi[A, 1 + B] \).

\(^7\)In particular, to each sequence \( \mathfrak{t} \) we associated a collection of integers \( u_1, \ldots, u_n, v_1, \ldots, v_n \). This is the Frobenius parameters of Young diagrams, see [Mac] I.1.

\(^8\)This condition can be replaced by other variants, see a discussion below and the next subsection.
Remark. Equivalently, we can consider one matrix $R := (A + B)$, whose matrix elements $r_{ml}$ are indexed by $m \geq 1$, $l \in \mathbb{Z}$. After this, we can write

$$\Xi[R] := \prod_{m=1}^{\infty} \left( \sum_{l} r_{ml} \xi_{l} \right)$$

We prefer $\Xi[A; 1 + B]$ as a basic notation, since it is more convenient for understanding of convergence/divergence of our series.

**Proposition 4.2.**

a) $\Xi[A, 1 + B] \in \Lambda$.

b) $\langle \Xi[A_{1}, 1 + B_{1}], \Xi[A_{2}, 1 + B_{2}] \rangle_{\Lambda} = \det (A_{1}A_{2}^{*} + (1 + B_{1})(1 + B_{2}^{*}))$.

c) $\Xi[A, B] = \sum_{k} \gamma_{k} \Xi_{k}$, where $\gamma_{k}$ is the minor of the matrix $(A + 1 + B)$ consisting of the columns with numbers $k_{1}, k_{2}, \ldots$.

d) Let $D$ be a matrix of the trace class. Then

$$\Xi[(1 + D)A, (1 + D)(1 + B)] = \det(1 + D) \Xi[A, B].$$

**Proof.** This statement is a variant of Lemma 4.1. The minors in Statement b) converge since $B$ is in the trace class. Thus we obtain a well-defined formal series in vectors $\Xi_{k}$. We have $\|\Xi[A; 1 + B]\|^{2} = \det(AA^{*} + (1 + B)(1 + B)^{*})$. This determinant is convergent since $AA^{*}$ and $B$ are in the trace class. \[\Box\]

Thus different matrices $(A + 1 + B)$ can give the same vector $\Xi[A, 1 + B]$. To reduce this freedom, we can consider the system of vectors $\Xi[D, 1]$

$$\prod_{m=1}^{\infty} \left( \xi_{-m} + \sum_{i=0}^{\infty} d_{mi} \xi_{i} \right) \quad (4.4)$$

A representation of a vector $\Xi[A, 1 + B]$ in the form $\text{const} \cdot \Xi[D, 1]$ is unique (indeed $D = (1 + B)^{-1}A$), but in the case $\det(1 + B) = 0$ such representation does not exist (it does not exist for all the vectors $\Xi_{k}$ except $\Xi_{-1, -2, \ldots}$).

In the notation of 4.3, element (4.4) corresponds to

$$\exp \left\{ \sum_{mi} d_{mi} \xi_{i} \eta_{m} \right\},$$

it is also a spinor function in the terminology of [Ner2].

**4.5. Decomposable vectors in the space of formal series $\Xi$.** Let $A$ be an arbitrary matrix and $B$ satisfies the condition

$$b_{mj} = 0 \text{ for } j \leq -m \quad (4.5)$$

Then the vector $\Xi[A; 1 + B]$ is a well-defined formal series in vectors $\Xi_{k}$, i.e., $\Xi[A; 1 + B] \in \Xi$. In other words, expressions of the following type are well defined

$$\prod_{m=1}^{\infty} (\xi_{-m} + c_{m1} \xi_{-m+1} + c_{m2} \xi_{-m+2} + \ldots) \quad (4.6)$$

for arbitrary $c_{mj}$.
Indeed, all the coefficients of its expansion in the basis $\Xi_k$ (see Proposition 4.2.c) are determinants having the following structure: diagonal elements are 1 starting some place, and only finite number of entries upper the diagonal are nonzero. Hence an evaluation of our coefficients is reduced to an evaluation of some determinants of finite size.

The expansion of (4.6) contains the term $\xi_{-1}\xi_{-2} \ldots$ and hence there are vectors $\Xi[A; 1 + B]$ that can not be represented in this form.

More generally, let $A$ be arbitrary and $B$ satisfy the condition

$$b_{mj} = 0 \text{ for } j \leq -m \text{ except finite number of entries} \quad (4.7)$$

In other words, we consider products of the form

$$\prod_{m=1}^{\infty} (\xi_m + c_{m1}\xi_{m+1} + c_{m2}\xi_{m+2} + \cdots + \sum_j \sigma_{mj}\xi_j),$$

where $\sigma_{mj} = 0$ except finite number of entries. Again, we obtain a well-defined element of the space $\Lambda$.

Below the both variants of the definition of decomposable vectors (in $\Lambda$ and in $\overline{\Lambda}$) are sufficient for our purposes.

**4.6. The action of $GL_{\infty}$ in $\Lambda$.** We consider infinite matrices $H = \{H_{ij}\}$, where $i, j$ range in $\mathbb{Z}$. Let $\delta_{ij}$ be the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

We define the group $GL_{\infty}$ as the group of all invertible matrices $H$ such that $h_{ij} - \delta_{ij} = 0$ for all pairs $(i, j)$ except finite number of entries.

We define the representation $\rho$ of $GL_{\infty}$ in $\Lambda$ by the assumption

$$\xi_i \mapsto \sum_i h_{ij}\xi_j.$$ 

The vectors $\Xi_t$ and their linear combinations are transformed according this rule.

We have

$$\rho(H)\Xi_t = \sum_{i: i_1 > i_2 > \ldots; i_j = -j \text{ for large } j}^{\text{det}} \{h_{ki,lj}\} \Xi_t. \quad (4.8)$$

This equality can be considered as a definition of the operators $\rho(H)$.

The operators $\rho(H)$ transform decomposable vectors to decomposable vectors:

$$\rho(H)\Xi[A, 1 + B] = \Xi[C, 1 + D], \quad \text{where } (C \ 1 + D) = (A \ 1 + B) H.$$ 

**4.7. The space of skew-symmetric functions.** Let $x_1, x_2, \ldots$ be formal variables. We say that a *long monomial* is an infinite product

$$x_1^{k_1} x_2^{k_2} x_3^{k_3} \ldots, \quad \text{where } k_j = -j \text{ for large } j.$$
We say that a formal linear combination of long monomials is a \textit{skew-symmetric function} if it is skew-symmetric with respect to finite permutations of variables.

Fix a collection
\[ t : \ k_1 > k_2 > k_3 > \ldots \quad \text{where } k_j = -j \text{ for large } j. \]

Define the skew-symmetric function
\[ \Omega_t = \sum_{\sigma \in S_{\infty}} (-1)^{\sigma} x_{\sigma(1)}^{k_1} x_{\sigma(2)}^{k_2} x_{\sigma(3)}^{k_3} \cdots \]

We introduce the scalar product in the space of skew-symmetric functions by the assumptions:
1. \( \Omega_t \) are pairwise orthogonal
2. \( \| \Omega_t \| = 1 \).

Denote by \( \mathcal{A} \) the Hilbert space of skew symmetric functions, i.e., the space, whose elements are series
\[ \sum_t c_t \Omega_t, \quad \text{where } \sum_t |c_t|^2 < \infty. \]

We also define the space \( \overline{\mathcal{A}} \), whose elements are formal series \( \sum_t c_t \Omega_t \) without any conditions for the coefficients \( c_t \).

4.8. Variant of definition. Consider a countable collection of formal variables \( x_1, x_2, \ldots \) We consider formal series (in the usual sense) in \( x_j^{\pm 1} \) satisfying the following condition of skew symmetry
\[ f(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots) = -\left( \frac{x_j}{x_i} \right)^{j-i} f(x_1, x_2, \ldots, x_j, \ldots, x_i, \ldots). \]

Then \( f(x) \cdot \prod_{j=1}^{\infty} x_j^{-j} \) is a skew symmetric function.

4.9. Identification of \( \Lambda \) and \( \mathcal{A} \). The canonical unitary operator \( \Lambda \rightarrow \mathcal{A} \) is defined by \( \Xi_t \rightarrow \Omega_t \).

4.10. Images of decomposable vectors. Fix matrices \( A, B \) as in 4.4 or in 4.5. For each \( m = 1, 2, \ldots \) we consider the formal Laurent series
\[ q_m(x) = \sum_{i \geq 0} a_{mi} x^i + \sum_{j < 0} b_{mj} x^j. \]

Also we will use an alternative notation for the same series
\[ q_m(x) = \sum_{-\infty < p < \infty} x^{mp} x^p. \]

Consider the determinant
\[ \Omega[A, 1 + B] = \det \begin{pmatrix} x_1^{1} + q_1(x_1) & x_2^{1} + q_1(x_2) & x_3^{1} + q_1(x_3) & \cdots \\ x_1^{2} + q_2(x_1) & x_2^{2} + q_2(x_2) & x_3^{2} + q_2(x_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
We understand this determinant as the following series in long monomials

\[
\sum_{n=0}^{\infty} \sum_{1 \leq k_1 < k_2 < \cdots < k_n} \left\{ \det \left( 1 \leq \alpha, \beta \leq n, \{ x_{k_\beta}^-(1 - \delta_{\alpha \beta}) + q_{k_\alpha}(x_{k_\beta}) \} \prod_{j:j \geq 1, j \neq k_\alpha} x_{j}^{-j} \right) \right\}. \tag{4.9}
\]

This means that we consider only summands of the determinant that differ from the diagonal product \( x_1^{-1} x_2^{-2} x_3^{-3} \ldots \) in finite number of terms. Under our conditions for \( A, B \), this series converges.

Another variant to understand this determinant is

\[
\lim_{N \to \infty} \left[ \det \left( 1 \leq j, k \leq N \right\{ x_{k}^{-m} + q_{m}(x_{k}) \} \cdot \prod_{l=N+1}^{\infty} x_{l}^{-l} \right] ; \tag{4.10}
\]

the limit means the coefficient-wise limit\(^9\).

If \( B = 0 \) and \( A \) is arbitrary, then our series is a well defined formal series, i.e., it is an element of \( \mathbb{K} \). The same is valid if \( B \) satisfies condition (4.5) or condition (4.7). For this case, the series for any coefficient in (4.9) is finite and each sequence of coefficients in (4.10) is stabilized starting some place.

In particular, the determinants of the form

\[
\det \left\{ x_{k}^{-m} \left( 1 + \sum_{j>0} c_{mj} x_{k}^{j} \right) \right\}
\]

are well-defined elements of \( \mathbb{K} \).

**Proposition 4.3.** The canonical map \( \Lambda \to \mathbb{K} \) takes a vector \( \Xi[A, 1+B] \in \Lambda \) to the vector \( \Omega[A, 1+B] \in \mathbb{K} \).

**Proof.** It is sufficient to expand \( \Omega[A, 1+B] \) in the functions \( \Xi_k \). The coefficient at \( \Xi_k \) is

\[
\det_{1 \leq m \leq \infty, 1 \leq \alpha \leq \infty} \left\{ x_{mk_\alpha}^- + \delta_{-m,k_\alpha} \right\}
\]

It coincides with the corresponding coefficient in Proposition 4.2c.

**Remark.** Also, we can write the following expression for \( \Omega[A, 1+B] \)

\[
x^{-1}_1 x^{-2}_2 x^{-3}_3 \ldots \det \begin{pmatrix}
1 + x_1 q_1(x_1) & x_2 + x_2^2 q_1(x_2) & x_3 + x_3^3 q_1(x_3) & \cdots \\
x_1^{-1} + x_1 q_2(x_1) & 1 + x_2^2 q_2(x_2) & x_3 + x_3^3 q_2(x_3) & \cdots \\
x_1^{-2} + x_1 q_3(x_1) & x_2^{-1} + x_2^2 q_3(x_2) & 1 + x_3^3 q_3(x_3) & \cdots \\
& \vdots & \vdots & \ddots
\end{pmatrix}
\]

and consider the determinant as a determinant \( \det(1+Z) \) in the sense of formal series in \( x_j^{\pm 1} \). This is equivalent to (4.9).

### 5. Fermion-symmetric correspondence

Here we consider only the Redfield scalar product in \( S \), i.e., we assume \( \omega_j = j \). We denote our space \( S_{\omega} \) by \( S_{cl} \).

\(^9\)The expression in brackets is a series in long monomials but not a skew symmetric function.
5.1. The canonical unitary operator $S_{cl} \to A$. Consider the element $\Delta \in A$ defined by

$$\Delta := \Omega - 1, -2, -3, \ldots = \sum_{\sigma \in S^\infty}(-1)^\sigma x_{\sigma(1)}^{-1}x_{\sigma(2)}^{-2}x_{\sigma(3)}^{-3}\cdots$$

For $f \in \mathbb{S}$, we consider the skew-symmetric function $\Delta \cdot f$.

**Proposition 5.1.** For the Schur functions $s_k$,

$$\Delta s_k = \Omega^k.$$

**Proof.** For fixed $N$, let us find terms (long monomials) of $\Delta \cdot s_k$ that contain the factor $\prod_{j>N} x_j^{-j}$. For this, we must follow terms of $s_k$ that do not contain $x_{N+1}, x_{N+2}, \ldots$. Hence it is sufficient to evaluate

$$s_k \bigg|_{x_{N+1}=x_{N+2}=\cdots=0} = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{j=1}^N x_{\sigma(j)}^{-j}.$$

But,

$$\sum_{\sigma \in S_N} (-1)^\sigma \prod_{j=1}^N x_{\sigma(j)}^{-j} = \prod_{1 \leq i < j \leq N} (x_i - x_j) \prod_{j=1}^N x_j^{-N}$$

and this implies the required statement. \[\Box\]

**Corollary 5.2.**

a) The map $f \mapsto \Delta f$ is a unitary operator $A \to S_{cl}$.

b) The map $f \mapsto \Delta f$ is a well-defined map of spaces of formal series $\mathbb{S} \to \bar{A}$.

5.2. Preimage of decomposable vectors.

**Lemma 5.3.** (Hua). Let $r_m(z) = c_0^{(m)} + c_1^{(m)} z + c_2^{(m)} z^2 + \ldots$ be formal series, $m = 1, \ldots, N$. Then

$$\det_{1 \leq m, p \leq N} \{r_m(z_p)\} = \sum_{l_1 < \cdots < l_N} \det_{1 \leq m \leq N, 1 \leq i \leq N} \{c_i^{(m)}\} \det_{1 \leq j \leq N, 1 \leq p \leq N} \{z_p^i\}.$$  

Let $A$ be arbitrary and $B$ satisfies the condition (4.7). Fix $M$ such that $b_{mi} = 0$ if $i \geq m, i \geq M$.

Consider the symmetric function $\Pi[A, 1 + B]$ defined by the condition

$$\Pi[A, 1 + B] \bigg|_{x_{N+1}=x_{N+2}=\cdots=0} = \prod_{p=1}^N x_p^{-N} \cdot \det_{1 \leq m, p \leq N} \{x_p^{-m} + \sum_{i \geq 0} a_{mi} x_p^i + \sum_{1 \geq j \geq -N} b_{mj} x_p^j\} / \prod_{1 \leq p < q \leq N} (x_p - x_q)$$

for all $N > M$.

**Corollary 5.4.**

$$\Delta \cdot \Pi[A, 1 + B] = \Omega[A, 1 + B].$$
The latter coefficient can be easily evaluated.

\[ \gamma \]

\[ a \] coefficient sufficiently large such that for \( i < j < k \), we have that \( \text{det}(\Omega[A, B]) \) skew symmetric with respect to \( y \).

Let \( x^i \) be the operator defined by the bisymmetric kernel \( H \).

The canonical unitary operator \( \Lambda \rightarrow \Lambda \rightarrow S_{\infty} \) intertwines the representation \( \rho(H) \) given by (4.8) and the representation \( A[H] \).
Proof. It is sufficient to verify b). We must check that the matrix elements of \( \rho[H] \) in the basis \( \Xi_k \in \Lambda \) are equal to the matrix elements of \( A[H] \) in the basis \( s_t \in S_{cl} \). The matrix elements of \( \rho[H] \) are given by (4.8). To find the matrix elements of \( A[H] \), we must expand the kernel \( L[H] \) in the Schur functions, see Proposition 1.5. This expansion is given by Corollary 5.6. □

Remark. Let extend the formalism of kernels to the space \( A \). Consider a series

\[
\sum_{k_1, k_2, \ldots ; l_1, l_2, \ldots} a_{k_1, k_2, \ldots ; l_1, l_2, \ldots} x_{l_1} y_{l_2} \ldots
\]

in long monomials. We say that it is a biskewsymmetric kernel if it is skew symmetric with respect to \( x_j \) and with respect to \( y_j \). For a biskewsymmetric kernel \( K(x, y) \), we define the operator \( A : \Lambda \to \Lambda \) by

\[
Af(x) = \langle K(x, y), f(y) \rangle_{A[y]}
\]

The group \( GL_\infty \) acts in \( A \) by operators whose biskewsymmetric kernels are

\[
K(x, y) = \det \left\{ \sum_{i,j} h_{ij} x_i y_j \right\},
\]

we consider only terms of the determinant that differ from the product \( \prod_k (x_k y_k)^{-k} \) in a finite number of places, see 4.10.

5.4. Images of some multiplicative vectors. Let \( r(x) = 1 + r_1 x + r_2 x^2 + \ldots \) Consider the vector

\[
R(x) = \prod_{k=1}^\infty r(x_k) \in \Xi.
\]

Proposition 5.8. a) The image of the vector \( R \) under the maps \( S_{cl} \to A \to \Lambda \) is

\[
\prod_{m=1}^\infty (\xi^{-m} + r_1 \xi^{-m+1} + r_2 \xi^{-m+2} + \ldots).
\]

If \( R(x) \in S_{cl} \), then (5.2) is an element of \( \Lambda \). Otherwise, (5.2) is in \( \overline{\Lambda} \).

b) The image of the vector \( R \) under the map \( S_{cl} \to \Xi \) is given by

\[
\det \{ x_k^{-m} r(x_k) \},
\]

where the determinant is defined in the same way as in 4.10.

Proof. The statement a) is a corollary of b). To obtain b), we write

\[
\prod_{k=1}^N r(x_k) = \frac{\{ \det x_k^{-m} r(x_k) \}}{\prod_{1 \leq i < k \leq N} (x_k - x_i)}
\]

and tend \( N \) to infinity. □
5.5. Another expressions for the same vectors. Let \( r(x) \) and \( R \) be the same as above (5.1). We intend to write the canonical form (4.4) for the image of \( R \) in \( \Lambda \). Let

\[
\begin{align*}
    r(x) &= 1 + r_1 x + r_2 x^2 + \cdots &= \exp a(x) &= \exp \{ a_1 x + a_2 x^2 + \ldots \}
\end{align*}
\]

By Proposition 3.1a, we have \( R \in S_{cl} \) iff \( \sum_j |a_j|^2 < \infty \). Therefore, \( a(x) \) is holomorphic in the disc \( |x| < 1 \), and hence \( r(x) \) is holomorphic in the same disc. Consider the function

\[
\frac{r(x)/r(u) - 1}{x - u}.
\]

This function is holomorphic in the bidisc \( |x| < 1, |u| < 1 \), and hence it can be expanded in a power series

\[
\frac{r(x)/r(u) - 1}{x - u} = \sum_{\alpha \geq 0, \beta \geq 0} \zeta_{\alpha \beta} x^\alpha u^\beta.
\]

**Proposition 5.9.**

a) The image of \( R \) under the map \( S_{cl} \to \mathbb{K} \to \Lambda \) is

\[
\prod_{m=1}^{\infty} (\xi - m + \sum_{j \geq 0} \zeta_{(m-1)j} \xi_j)
\]

b) The image of \( R \) under the map \( S_{cl} \to \mathbb{K} \) is

\[
det_{m \geq 1, k \geq 1} \{ x_k^{-m} + \sum_{j \geq 0} \zeta_{(m-1)j} x_k^j \}
\]

**Proof.** It is sufficient to prove b). By Proposition 5.8, the required vector is the determinant of the matrix

\[
\begin{pmatrix}
    \cdots + r_2 x_1 + r_1 x_1^{-1} & \cdots + r_2 x_2 + r_1 x_2^{-1} & \cdots + r_2 x_3 + r_1 x_3^{-1} & \cdots \\
    \cdots + r_2 + r_1 x_1^{-1} + x_1^{-2} & \cdots + r_2 + r_1 x_2^{-1} + x_2^{-2} & \cdots + r_2 + r_1 x_3^{-1} + x_3^{-2} & \cdots \\
    \cdots + r_2 x_1^{-1} + r_1 x_1^{-2} + x_1^{-3} & \cdots + r_2 x_2^{-1} + r_1 x_2^{-2} + x_2^{-3} & \cdots + r_2 x_3^{-1} + r_1 x_3^{-2} + x_3^{-3} & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We can replace \( n \)-th row by a linear combination of the first \((n-1)\) rows. Hence, we can replace the series in the \( n \)-th row by

\[
\mu_n(x) = x^{-n}(1 + r_1 x + r_2 x^2 + \cdots)(1 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}),
\]

where \( s_j \) are arbitrary.

Let us choose \( s_j \) defined by the rule

\[
(1 + r_1 x + r_2 x^2 + \cdots)(1 + s_1 x + s_2 x^2 + \cdots) = 1
\]

Then

\[
\mu_n(x) = x^{-n} + \sum_{j \geq 0} C_{nj} x^j
\]

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\[ C_{nj} = r_{n+j} + r_{n+j-1}s_1 + \cdots + r_{j+1}v_{n-1} \]

We write the generating function \( Q(x, y) = \sum C_{nj}x^{n-1}y^j \). A direct calculation gives \( Q(x, y)(x - y) = r(x)/r(y) - 1 \). \( \square \)

5.6. Example: inversion formula for boson-fermion correspondence. A **boson-fermion correspondence**\(^{10} \) is the composition of the canonical maps

\[ \mathbb{F} \to \mathbb{S}_{cl} \to \Lambda \to \Lambda. \]

Obviously, it is a unitary operator.

Let \( f(z) \) be an element of \( \mathbb{F} \), let \( g(\xi) \) be the corresponding element of \( \Lambda \). We intend to write a formula that reconstructs \( f \) by \( g \). For this, we find the images in \( \Lambda \) of the vectors \( \varphi_a \in \mathbb{F} \), see (2.4), and apply the reproducing property (2.5), see proof of Proposition 3.1.

Define the polynomials \( R_n(z) \) by

\[
\exp\left\{ z_1x + z_2x^2 + z_3x^3 + \ldots \right\} = 1 + R_1(z)x + R_2(z)x^2 + \ldots.
\]

In other words,

\[ R_n(z) = \sum_{s_1 + 2s_2 + 3s_3 + \cdots = n} \prod_j \frac{z_j}{s_j}. \quad (5.3) \]

It will be also convenient to assume

\[ R_0 = 1, \; R_{-1} = R_{-2} = \cdots = 0. \quad (5.4) \]

We evaluate the image of \( \varphi_a \) in \( \Lambda \) by Proposition 5.8 and obtain

\[
f(\sqrt{1}z_1, \sqrt{2}z_2, \sqrt{3}z_2, \ldots)) = \langle g(\xi), \prod_{m=1}^{\infty} (\xi_{-m} + R_1(\tau)\xi_{-m+1} + R_2(\tau)\xi_{-m+2} + \ldots) \rangle_{\Lambda}
\]

5.7. Example: another inversion formula for the boson-fermion correspondence. Proposition 5.9 allows to obtain another formula. Define the polynomials \( Q_{mn}(z) \) by

\[
\exp\left\{ \sum_{j>0} z_j(x_j - y^j) \right\} - 1 = \sum Q_{mn}(z)x^m y^n \quad (5.5)
\]

Then

\[
f(\sqrt{1}z_1, \sqrt{2}z_2, \sqrt{3}z_2, \ldots) = \langle g(\xi), \prod_{m=1}^{\infty} (\xi_{-m} + \sum_{j=0}^{\infty} Q_{m-1,j}(z)\xi_{j}) \rangle_{\Lambda}.
\]

\( ^{10} \)In this definition, the boson Fock space \( \mathbb{F} \) is identified with the subspace \( \Lambda \) in the fermion Fock space \( \mathcal{L} \). The whole space \( \mathcal{L} \) can be identified with \( \mathbb{F} \otimes C[t, t^{-1}] \), where \( C[t, t^{-1}] \) is space of Laurent series in formal variable \( t \), and \((t^k, t^l) = \delta_{kl} \). There are some other variants of this correspondence, see \( \text{Fre}, \text{PS}, \text{MJD}, \text{Ner2} \) for further discussions; discovering of the correspondence usually is attributed to Skyrme, 1971.
5.8. Example: generating functions for characters of the symmetric groups. Let $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_h > 0$ and $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots > 0$; let $\sum \mu_j = \sum \lambda_j = N$. Consider the irreducible representation of the symmetric group $S_N$ corresponding to the Young diagram $\mu$; denote by $\chi_\mu^\mu$ the value of its character on a permutation whose cycles has lengths $\lambda_j$. Then (see [Mac, I.7.7])

$$\chi_\mu^\mu = \langle p_\lambda, s_\mu \rangle$$

(in notation 1.2). Let $l_j$ be the number of entries of $j$ into the collection $\lambda$, let $m_j = \mu_j - j$, and let $m := (m_1, m_2, \ldots)$. We denote $\chi_m^m := \chi_\mu^\mu$ and write

$$\chi_m^m = \langle p_l, s_m \rangle$$

Hence

$$\exp \{ \sum_{j>0} a_j p_j(x) \} = \sum_{l} \left( \prod_j \frac{a_j^{l_j}}{l_j!} \right) p_l = \sum_{m} \left( \prod_j \frac{a_j^{l_j}}{l_j!} \right) \chi_m^m \cdot s_m \quad (5.6)$$

Then we transform the left-hand side to the form

$$\prod_k \exp \{ \sum_{j>0} a_j x_k^j \} = \prod_k \{ \sum_{j>0} R_j(a) x_k^j \}$$

Multiplying the left hand side and the right hand side of (5.6) by $\Delta$, we obtain

$$\det \{ x_k^{-m} (1 + \sum_{j>0} R_j(a) x_k^j) \} = \sum_{i} \sum_{m} \left( \prod_j \frac{a_j^{l_j}}{l_j!} \right) \chi_m^m \Omega_m \quad (5.7)$$

Now we are ready to write 3 formulae for generating functions.

1. Expanding the left hand side of the last equality in $\Omega_m$, and equating its coefficients, we obtain the following generating function for values of the character of a given representation of a symmetric group $S_N$

$$\sum_{i} \left( \chi_i^m \prod_j \frac{a_j^{l_j}}{l_j!} \right) = \det_{1 \leq i, j \leq h} \{ R_{m_i, -j}(a) \},$$

the polynomials $R$ are defined by (5.3)–(5.4) and $m_k = -k$ for $k > h$.

2. Applying the construction of 5.7, we obtain another expression for the same generating function. Let $u_1, \ldots, u_n, v_1, \ldots, v_n$ are the parameters for $m$ (or $\mu$) defined in 4.3. Then

$$\sum_{i} \left( \chi_i^m \prod_j \frac{a_j^{l_j}}{l_j!} \right) = \det_{1 \leq i, j \leq n} \{ Q_{e_j - 1 u_i}(a) \}$$

\[11\]In particular, this gives images of the functions $\epsilon_1(z) \in F$ in $\Lambda$ and preimages of the vectors $\Xi_m \in \Lambda$ in $F$. This was widely used in mathematical physics.

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where the polynomials $Q$ are defined by (5.5).

3. We also write identity (5.7) in the form

$$\left( \sum_{\sigma \in S_\infty} (-1)^\sigma \prod_{k=1}^\infty e^{-x_{\sigma(k)}^k} \right) \prod_{j=1}^\infty \exp \left\{ \sum_{m=1}^\infty \lambda_m \left( \sum_{\sigma \in S_\infty} (-1)^\sigma \prod_{k=1}^\infty x_{\sigma(k)}^m \right) \prod_j a_j^m \right\}$$

Thus, the coefficients of the formal series in the left hand side are all the values of all the characters of all the symmetric groups (up to signs and products of factorials).

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Math. Surveys

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