RADICAL TRANSVERSAL LIGHTLIKE HYPERSURFACES
OF ALMOST COMPLEX MANIFOLDS WITH NORDEN
METRIC

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Abstract. In this paper we introduce radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric. The study of these hypersurfaces is motivated by the fact that for indefinite almost Hermitian manifolds this class of lightlike hypersurfaces does not exist. We also establish that radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric have nice properties as a unique screen distribution and a symmetric Ricci tensor of the considered hypersurfaces of Kaehler manifolds with Norden metric. We obtain new results about lightlike hypersurfaces concerning to their relations with non-degenerate hypersurfaces of almost complex manifolds with Norden metric. Examples of the considered hypersurfaces are given.

1. Introduction

There exist two types submanifolds of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ with respect to the induced metric $g$ by $\overline{g}$ on the submanifold. If $g$ is non-degenerate or degenerate, the submanifold $(M, g)$ is non-degenerate or lightlike, respectively. In case $g$ is non-degenerate on $M$, both the tangent bundle $TM$ and the normal bundle $TM^\perp$ of $M$ are non-degenerate and $TM \cap TM^\perp = \{0\}$. However, in case $(M, g)$ is a lightlike submanifold of $\overline{M}$, a part of $TM^\perp$ lies in $TM$. Therefore the geometries of the non-degenerate and the lightlike submanifolds are different. The general theory of lightlike submanifolds has been developed in [2] by K. Duggal and A. Bejancu. The geometry of Cauchy-Riemann (CR) lightlike submanifolds of indefinite Kaehler manifolds was presented in [2], too. Some new classes of lightlike submanifolds of indefinite Kaehler, Sasakian and quaternion Kaehler manifolds were introduced in [3] by K. Duggal and B. Sahin. In [2, 3] many applications of lightlike geometry in the mathematical physics were given.

Lightlike hypersurfaces of indefinite Kaehler manifolds were studied in [2, 3]. In this paper we introduce radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric. Such class of lightlike hypersurfaces does not exist when the ambient manifold is an indefinite almost Hermitian manifold because its geometry is different from the geometry of an almost complex manifold with Norden metric. The difference arises due to the fact that the action of the almost complex structure $\overline{J}$ on the tangent space at each point of an almost complex manifold with Norden metric
\(\overline{M}\) is an anti-isometry with respect to the metric \(\overline{g}\). The metric \(\overline{g}\) on \(\overline{M}\) is called Norden metric (or B-metric). Moreover, the tensor field \(\overline{g}\) on \(\overline{M}\) defined by \(\overline{g}(X,Y) = \overline{g}(\mathcal{J}X,Y)\) is also Norden metric on \(\overline{M}\) while in the almost Hermitian case \(\overline{g}\) is a 2-form. Both metrics \(\overline{g}\) and \(\overline{g}\) on \(\overline{M}\) are of a neutral signature. The beginning of the investigations in the geometry of the almost complex manifolds with Norden metric was put by A. P. Norden \([11]\) and the researches have been continued by G. Ganchev, K. Gribachev, D. Mekerov, A. Borisov, V. Mihova \([7], [4], [5]\).

In Section 2 we recall some preliminaries about lightlike hypersurfaces of semi-Riemannian manifolds, almost complex manifolds with Norden metric and almost contact manifolds with B-metric. In Section 3 we define a radical transversal lightlike hypersurface of an almost complex manifold with Norden metric and prove that a lightlike hypersurface of such manifold is radical transversal if and only if the screen distribution of the lightlike hypersurface is holomorphic. In Section 4 we show that a radical transversal lightlike hypersurface of an almost complex manifold with Norden metric has a unique screen distribution up to a semi-orthogonal transformation. This property is important for the lightlike hypersurface because it guarantees that the induced geometrical objects on the hypersurface do not depend on the choice of the screen distribution. We establish that the Ricci tensor of radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric is symmetric, which is not true in general in the lightlike geometry. We close this section by some geometrical characterizations of the considered hypersurfaces. Since on an almost complex manifold with Norden metric there exist two Norden metrics, in \([10]\) we consider submanifolds which are non-degenerate with respect to the one Norden metric and lightlike with respect to the other one. Section 5 is devoted to the same topic. We prove that \((M, g)\) is a special non-degenerate hypersurface of an almost complex manifold with Norden metric \((\overline{M}, \mathcal{J}, \overline{g}, \overline{g})\) if and only if \((M, \overline{g})\) is a radical transversal lightlike hypersurface of \(\overline{M}\), where \(g\) and \(\overline{g}\) are the induced metrics by \(\overline{g}\) and \(\overline{g}\) on \(M\), respectively. We find relations between the induced geometrical objects on the hypersurfaces \((M, g)\) and \((M, \overline{g})\) of a Kaehler manifold with Norden metric and characterize both hypersurfaces. In the last section we give two examples of radical transversal lightlike hypersurfaces.

2. Preliminaries

2.1. Lightlike hypersurfaces of semi-Riemannian manifolds. Let \(M\) be a hypersurface of an \((m + 2)\)-dimensional semi-Riemannian manifold \((\overline{M}, \overline{g})\) of index \(q \in \{1, \ldots, m + 1\}\). \(M\) is a lightlike hypersurface of \(\overline{M}\) \([2]\) if at any \(u \in M\) \(\text{Rad}T_u M \neq \{0\}\), where \(\text{Rad}T_u M = T_u M \cap T_u M^\perp\). Because of for a hypersurface \(\dim(T_u M^\perp) = 1\) it follows that \(\dim(\text{Rad}T_u M) = 1\) and \(\text{Rad}T_u M = T_u M^\perp\). \(\text{Rad}T M\) is called a radical distribution on \(M\). Hence, the induced metric \(g\) by \(\overline{g}\) on a lightlike hypersurface \(M\) has a constant rank.
Moreover, there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^\perp$ in $TM$, which is called in [2] the screen distribution on $\overline{M}$. For any $S(TM)$ we have a unique transversal vector bundle $\text{tr}(TM)$ which is a lightlike complementary vector bundle (but not orthogonal) to $TM$ in $T\overline{M}$. So, the following decompositions of $T\overline{M}$ are valid:

(1) \[ T\overline{M} = S(TM) \perp (TM^\perp \oplus \text{tr}(TM)) = TM \oplus \text{tr}(TM), \]

where by $\perp$ (resp. $\oplus$) is denoted an orthogonal (resp. a non-orthogonal) direct sum. By $\Gamma(E)$ is denoted the $\mathcal{F}(M)$-module of smooth sections of a vector bundle $E$ over $M$, $\mathcal{F}(M)$ being the algebra of smooth functions on $M$. In ([2], Theorem 1.1, p. 79) it was proved if $(M, g, S(TM))$ is a lightlike hypersurface of $\overline{M}$, for any non-zero section $\xi$ of $TM^\perp$ on a coordinate neighbourhood $U \subset M$, there exists a unique section $N$ of $\text{tr}(TM)$ on $U$ satisfying:

(2) \[ \overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma(S(TM)). \]

The induced geometrical objects on a lightlike hypersurface $M$ of a semi-Riemannian manifold $\overline{M}$ have different properties from the properties of the ones on a non-degenerate hypersurface of $\overline{M}$. Therefore, follow [2], [3] we will recall basic formulas and facts about the induced geometrical objects on a lightlike hypersurface. Let $\nabla$ be the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$. The global Gauss and Weingarten formulas are

\[ \nabla_X Y = \nabla_X Y + h(X, Y), \]
\[ \nabla_X V = -A_V X + \nabla_X V, \quad \forall X, Y, V \in \Gamma(TM), \quad V \in \Gamma(\text{tr}(TM)), \]

where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while $h(X, Y)$ and $\nabla_X V$ belong to $\Gamma(\text{tr}(TM))$. The induced connection $\nabla$ on $M$ is a torsion-free linear connection and in general $\nabla$ is not metric connection. The linear connection $\nabla^i$ is called an induced linear connection on $\Gamma(\text{tr}(TM))$. The second fundamental form $h$ is symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$. The shape operator $A_V$ is $\Gamma(S(TM))$-valued and it is not self-conjugate with respect to $g$, i.e. $g(A_V X, Y) \neq g(X, A_V Y)$. The local Gauss and Weingarten formulas are

(3) \[ \nabla_X Y = \nabla_X Y + B(X, Y) N, \]
\[ \nabla_X N = -A_N X + \tau(X) N, \quad \forall X, Y \in \Gamma(TM_U), \]

where the pair of sections $\{\xi, N\}$ on $U \subset M$ satisfies (2), $B$ is a symmetric $\mathcal{F}(U)$-bilinear form which is called the local second fundamental form of $M$ and $\tau$ is a 1-form on $U$. We also have

(4) \[ A_N \xi = 0, \quad B(X, \xi) = 0, \quad \forall X \in \Gamma(TM_U). \]

Let $P$ denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. The following formulas are the Gauss and Weingarten equations for the screen distribution $S(TM)$

\[ \nabla_X P Y = \nabla_X^P P Y + h^*(X, PY), \]
\[ \nabla_X U = -A_U^X + \nabla_X^U U, \quad \forall X, Y \in \Gamma(TM), \quad U \in \Gamma(TM^\perp), \]
where $\nabla_X^*PY$ and $A^*_X X$ belong to $\Gamma(T(TM))$, $\nabla^*$ and $\nabla^{*\Gamma}$ are linear connections on $\Gamma(T(TM))$ and $\Gamma(TM^\perp)$, respectively; $h^*$ is a $\Gamma(TM^\perp)$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and $A'_U$ is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-bilinear operator on $\Gamma(TM)$. They are called the screen second fundamental form and screen shape operator of $S(TM)$, respectively. Locally for any $X, Y \in \Gamma(TM_U)$ we have

\[ \nabla_X^*PY = \nabla_X^*PY + C(X, PY)\xi, \quad \nabla_X^*X = -A^*_X X - \tau(X)\xi, \]

where $C(X, PY)$ is the local screen fundamental form of $S(TM)$. Both local second fundamental forms $B$ and $C$ are related to their shape operators by

\[ B(X, Y) = g(A^*_X X, Y), \quad C(X, PY) = g(A_N X, PY). \]

$\nabla^*$ is a metric connection, $A^*$ is self-conjugate with respect to $g$ and

\[ A^*_X \xi = 0, \quad h^*(\xi, PY) = 0. \]

As the screen distribution $S(TM)$ is not unique, the induced geometrical objects depend on the choice of $S(TM)$. Follow [2], [3] we will present their dependence (or otherwise) on the choice of a screen distribution. Let $F = \{\xi, N, W_i\}$, $i = 1, \ldots, m$ be a quasi-orthonormal basis of $\mathcal{M}$ along $M$, where $\{\xi\}$, $\{N\}$ and $\{W_i\}$ are the lightlike basis of $\Gamma(\text{Rad}(TM_U))$, $\Gamma(\text{tr}(TM_U))$ and the orthonormal basis of $\Gamma(S(TM)_U)$, respectively. Consider two quasi-orthonormal frames fields $F = \{\xi, N, W_i\}$ and $F' = \{\xi, N', W'_i\}$ induced on $U \subset M$ by $\{S(TM), \text{tr}(TM)\}$ and $\{S'(TM), \text{tr}'(TM)\}$, respectively for the same $\xi$. The following relationships between $F$ and $F'$ are valid

\[ W'_i = \sum_{j=1}^{m} W^j_i (W^j_i - \epsilon_j f_j \xi), \quad N' = N - \frac{1}{2} \left\{ \sum_{i=1}^{m} \epsilon_i (f_i)^2 \right\} \xi + \sum_{i=1}^{m} f_i W_i, \]

where $\{\epsilon_1, \ldots, \epsilon_m\}$ is the signature of the orthonormal basis $\{W_i\}$ and $W^j_i, f_i$ are smooth functions on $U$ such that $(W^j_i)$ are $m \times m$ semi-orthogonal matrices. It was proved (2, 3) that $B$ is independent of the choice of $S(TM)$, but both $B$ and $\tau$ depend on the choice of a section $\xi \in \Gamma(\text{Rad}(TM_U))$. Moreover, relationships between the induced objects $\{\nabla, \tau, A_N, A^*_\xi, C\}$ and $\{\nabla', \tau', A'_{N'}, A'^*_{\xi'}, C'\}$ by the screen distributions $S(TM)$ and $S'(TM)$, respectively, were given.

2.2. Almost complex manifolds with Norden metric. Let $(\mathcal{M}, \mathcal{J}, \mathcal{g})$ be a $2n$-dimensional almost complex manifold with Norden metric [4], i.e. $\mathcal{J}$ is an almost complex structure and $\mathcal{g}$ is a metric on $\mathcal{M}$ such that:

\[ \mathcal{J}^2 X = -X, \quad \mathcal{g}((\mathcal{J}X, \mathcal{J}Y)) = -\mathcal{g}(X, Y), \quad X, Y \in \Gamma(TM). \]

The tensor field $\mathcal{g}$ of type $(0, 2)$ on $\mathcal{M}$ defined by $\mathcal{g}(X, Y) = \mathcal{g}(\mathcal{J}X, Y)$ is a Norden metric on $\mathcal{M}$, too. Both metrics $\mathcal{g}$ and $\mathcal{g}$ are necessarily of signature $(n, n)$. The metric $\mathcal{g}$ is said to be an associated metric of $\mathcal{M}$. The Levi-Civita connection of $\mathcal{g}$ is denoted by $\nabla$. The tensor field $F$ of type $(0, 3)$ on $\mathcal{M}$ is defined by $F(X, Y, Z) = \mathcal{g}((\nabla_X \mathcal{J})Y, Z)$. Let $\nabla^*$ be the Levi-Civita
connection of $\vec{g}$. Then $\Phi(X, Y) = \nabla_X Y - \nabla_X Y$ is a tensor field of type $(1, 2)$ on $\bar{M}$. Since $\nabla$ and $\bar{\nabla}$ are torsion free we have $\Phi(X, Y) = \Phi(Y, X)$. A classification of the almost complex manifolds with Norden metric with respect to the tensor $F$ is given in [4] and eight classes are obtained. In [5] these classes are characterized by conditions for the tensor $\Phi$. The two types of characterization conditions for the class of the Kaehler manifolds with Norden metric are $F(X, Y, Z) = 0$ and $\Phi(X, Y) = 0$.

2.3. Almost contact manifolds with B-metric. Let $(\bar{M}, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact manifold with B-metric, i.e. $(\varphi, \xi, \eta)$ is an almost contact structure [1] and $g$ is a metric [6] on $\bar{M}$ such that

\[
\varphi^2 X = -\text{id} + \eta \otimes \xi, \quad \bar{\eta}(\xi) = 1,
\]

where $\text{id}$ denotes the identity transformation and $X, Y \in \Gamma(TM)$. Immediate consequences of the above conditions are:

\[
\bar{\eta} \circ \varphi = 0, \quad \varphi \xi = 0, \quad \text{rank} \varphi = 2n, \quad \bar{\eta}(X, Y) = g(X, \bar{\eta}(Y)),
\]

The 2$n$-dimensional distribution $\mathcal{D} : x \mapsto \mathcal{D}_x \subset T_x\bar{M}$ at each point $x \in \bar{M}$ defined by $\mathcal{D}_x = \text{Ker} \eta_x$ is called a contact distribution of $\bar{M}$. We have the following decomposition of $T_x\bar{M}$ which is orthogonal with respect to $\bar{g}$

\[
T_x\bar{M} = \mathcal{D}_x \perp \text{span}\{\xi_x\}.
\]

The tensor $\bar{g}$ given by $\bar{g}(X, Y) = g(X, \varphi Y) + \bar{\eta}(X)\bar{\eta}(Y)$ is a B-metric, too. Both metrics $\bar{g}$ and $\bar{g}$ are indefinite of signature $(n + 1, n)$. Let $\bar{\nabla}$ be the Levi-Civita connection of the metric $\bar{g}$. The tensor field $F$ of type $(0, 3)$ on $\bar{M}$ is defined by $F(X, Y, Z) = \bar{g}((\bar{\nabla}_X \varphi)Y, Z), \quad X, Y, Z \in \Gamma(TM)$. The following 1-forms are associated with $F$

\[
\theta(X) = \bar{\eta}^i F(e_i, e_j, X), \quad \theta^i(X) = \bar{g}^{ij} F(e_i, \varphi e_j, X), \quad \omega(X) = F(\xi, \xi, X),
\]

where $\{e_i, \xi\}, i = 1, \ldots, 2n$ is a basis of $T_u\bar{M}$ and $(\bar{g}^{ij})$ is the inverse matrix of $(\bar{g}_{ij})$. A classification of the almost contact manifolds with B-metric with respect to the tensor $F$ is given in [6] and eleven basic classes $\mathcal{F}_i (i = 1, 2, \ldots, 11)$ are obtained.

3. Radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric

First in this section we will show that there are lightlike hypersurfaces of an almost complex manifold with Norden metric which do not exist when the ambient manifold is an indefinite almost Hermitian manifold. This fact is a motivation for our researches in this paper.

Let $(\bar{M}, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \mathcal{J}, \overline{g})$, where $\overline{M}$ is an indefinite almost Hermitian manifold or an almost complex manifold with
Norden metric. Take $\xi \in \Gamma(TM^\perp)$ and according to (11) we can write $J\xi$ in the following manner

$$J\xi = \xi_1 + a\xi + bN,$$

where $\xi_1 \in \Gamma(S(TM))$, $N \in \Gamma(\text{tr}(TM))$ and $a, b$ are smooth functions on $M$. Since $J^2 = -id$, it is clear that the case $J\xi = a\xi$ is impossible. From (10), by using (2) we obtain $b = \overline{g}(J\xi, \xi)$. Now, if $M$ is an indefinite almost Hermitian manifold, then $b = 0$ and from (10) it follows that $J\xi$ is tangent to $M$. Thus, $J(TM^\perp)$ is always a distribution on $M$ of rank 1 such that $TM^\perp \cap J(TM^\perp) = \{0\}$.

Further, we assume that $M$ is an almost complex manifold with Norden metric. As $g$ is an anti-isometry with respect to $J$, the function $b$ is not zero, in general. In the case $b \neq 0$, the component of $J\xi$ with respect to $N$ does not vanish. Hence, we can consider lightlike hypersurfaces of $M$ such that $J(TM^\perp)$ does not belong to $TM$. Our aim in this section is to study one class of such lightlike hypersurfaces of $M$.

**Definition 3.1.** Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$. We say that $M$ is a **radical transversal lightlike hypersurface** of $M$ if $J(TM^\perp) = \text{tr}(TM)$.

**Remark 3.1.** Radical transversal lightlike submanifolds of indefinite Kaehler manifolds were introduced by B. Sahin in [12]. Note that the dimension $r$ of the radical distribution of these submanifolds is greater than one.

**Theorem 3.1.** Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$. $M$ is a radical transversal lightlike hypersurface of $\overline{M}$ if and only if the screen distribution $S(TM)$ is holomorphic with respect to $\overline{J}$.

**Proof.** Let $M$ be a radical transversal lightlike hypersurface of $\overline{M}$. Then $\overline{J}\xi = bN$, where the pair $\{\xi \in \Gamma(TM^\perp), N \in \Gamma(\text{tr}(TM))\}$ satisfies (2) and $b \in \mathcal{F}(\overline{M})$. Hence, for an arbitrary $W \in \Gamma(S(TM))$ we have $\overline{g}(\overline{J}W, \xi) = \overline{g}(W, bN) = 0$. Thus $\overline{J}W$ is tangent to $M$. Moreover, we have $\overline{g}(\overline{J}W, N) = \overline{g}\left(W, -\frac{1}{b}\xi\right) = 0$, which implies $\overline{J}W \in \Gamma(S(TM))$, i.e. $S(TM)$ is holomorphic. Conversely, let $S(TM)$ be holomorphic. Taking into account $\overline{g}(\overline{J}W, \xi) = 0$ and the decomposition (11), we obtain $\overline{g}(W, \xi_1) = 0$. Since $\overline{g}$ is non-degenerate on $S(TM)$, from the last equality it follows that $\xi_1 = 0$. Then (10) becomes

$$J\xi = a\xi + bN.$$

As $\overline{g}(\overline{J}\xi, J\xi) = 0$, by using (11) we have $2ab = 0$. The function $b$ in (11) is not zero because $TM^\perp \cap J(TM^\perp) = \{0\}$. Therefore $a = 0$ and $J\xi = bN$ which means that $M$ is a radical transversal lightlike hypersurface of $\overline{M}$. $\square$
In [2], p. 194 it was proved that a lightlike hypersurface of an indefinite Hermitian manifold is a CR-manifold. In order to prove our next result, analogously as in [2], we will use the following

**Theorem 3.2.** ([2], p. 193) A smooth manifold $L$ is a CR-manifold if and only if, it is endowed with an almost complex distribution $(D, J)$ (i.e. $J(D) = D$) such that

(12) $[JX, JY] - [X, Y] \in D$

and

(13) $N_J(X, Y) = 0,$

for all $X, Y \in D$.

**Theorem 3.3.** A radical transversal lightlike hypersurface $(M, g, S(TM))$ of a complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$ is a CR-manifold.

**Proof.** From Theorem 3.1 it follows that $S(TM)$ is an almost complex distribution on $M$ with an almost complex structure $J$ which is the restriction of $\overline{J}$ on $S(TM)$. Further, we will show that $S(TM)$ satisfies the conditions (12) and (13). Denote by $\overline{N}_J$ and $N_J$ the Nijenhuis tensors of $\overline{J}$ and $J$, respectively. As $M$ is a complex manifold, $\overline{N}_J = 0$ on $\overline{M}$, i.e.

\[
\overline{N}_J(X, Y) = [\overline{J}X, \overline{J}Y] - [X, Y] - \overline{J}([X, \overline{J}Y] + [\overline{J}X, Y]) = 0,
\]

for all $X, Y \in \Gamma(\overline{T M})$. Then for any $X, Y \in \Gamma(S(TM))$ we have

(14) $\overline{N}_J(X, Y) = [JX, JY] - [X, Y] - J([X, JY] + [JX, Y])$

\[
- \overline{J}(Q([X, JY] + [JX, Y])) = 0,
\]

where $P$ and $Q$ are the projection morphisms of $TM$ on $S(TM)$ and $TM^\perp$, respectively. Because of the second fundamental form $h$ is symmetric on $TM$, from the Gauss formula we obtain $[X, Y] \in \Gamma(TM)$ for any $X, Y \in \Gamma(TM)$. This fact and $S(TM) = S(TM)$ imply that for any $X, Y \in \Gamma(S(TM))$, the vector field $Z = [JX, JY] - [X, Y] - J(\overline{P}([X, JY] + [JX, Y]))$ is tangent to $\overline{M}$. Since $M$ is a radical transversal lightlike hypersurface of $\overline{M}$, we have that the vector field $V = \overline{J}(Q([X, JY] + [JX, Y]))$ belongs to $\Gamma(\text{tr}(TM))$. From the equality (14) it follows that the components $Z$ and $V$ of $\overline{N}_J$ with respect to $TM$ and $\text{tr}(TM)$, respectively, are both zero. The vanishing of $V$ shows that $Q([X, JY] + [JX, Y]) = 0$ and therefore $[X, JY] + [JX, Y] = P([X, JY] + [JX, Y])$. Hence, $Z$ becomes

(15) $Z = [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0,$

for any $X, Y \in \Gamma(S(TM))$. The condition (13) is valid because the expression for $Z$ from (15) is exactly $N_J$ on $S(TM)$. Moreover, by using of (15) we obtain $[JX, JY] - [X, Y] = J([X, JY] + [JX, Y])$, i.e. the condition (12) is true for any $X, Y \in \Gamma(S(TM))$. Then our assertion follows from Theorem 3.2. □
4. The induced geometrical objects on a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric

It is known that the induced geometrical objects on a lightlike hypersurface $M$ are well-defined if $M$ admits a unique or canonical screen distribution. Now we will investigate this important problem for the introduced lightlike hypersurfaces in the previous section. We state

**Theorem 4.1.** A radical transversal lightlike hypersurface $(M, g, S(TM))$ of a $2n$-dimensional almost complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$ has a unique screen distribution up to a semi-orthogonal transformation and a unique lightlike transversal vector bundle.

**Proof.** Let $S(TM)$ and $S'(TM)$ be two screen distributions on $M$, $\text{tr}(TM)$ and $\text{tr}'(TM)$ their lightlike transversal vector bundles, respectively. Take the quasi-orthonormal frames fields $F = \{\xi, N, W_i\}$ and $F' = \{\xi, N', W'_i\}$ induced on $U \subset M$ by $\{S(TM), \text{tr}(TM)\}$ and $\{S'(TM), \text{tr}'(TM)\}$, respectively. According to Theorem 3.1 $S(TM)$ and $S'(TM)$ are holomorphic. By using this fact and $N = \frac{1}{b} \overline{J} \xi$ we compute $g(W'_i, N) = 0$ for any $i \in \{1, \ldots, 2n - 2\}$. Thus, after multiplication by $N$ both sides of the first equality in (8) we get

$$2n - 2 \sum_{j=1}^{2n-2} W'_i \epsilon_j f_j = 0 \quad (i \in \{1, \ldots, 2n - 2\}),$$

where $(W'_i)$ is an matrix of $S(T_xM)$ at any point $x$ of $M$, belonging to $O(n - 1, n - 1)$. The determinant of the last homogeneous linear system does not vanish at any $x \in M$ and hence it has the unique solution $f_j = 0, j \in \{1, \ldots, 2n - 2\}$. Then (8) become $W'_i = \sum_{j=1}^{2n-2} W'_i W'_j \quad (i \in \{1, \ldots, 2n - 2\}), \quad N' = N$, which proves our assertion. □

Let $(M, g)$ be a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$. According to Definition 3.1 we have $\overline{J} \xi = bN$. By using (3) and the second equality in (5), for any $X \in \Gamma(TM)$ we compute

$$\nabla_X \overline{J} \ N = \frac{1}{b} A^*_N X + \overline{J}(A_N X) + \frac{1}{b} \left(2\tau(X) + \frac{1}{b} (X \circ b)\right) \xi. \quad (16)$$

As $\overline{M}$ is a Kaehler manifold with Norden metric, the left side of (16) vanishes. From $A_N X \in \Gamma(S(TM))$ and Theorem 3.1 we have $\overline{J}(A_N X) \in \Gamma(S(TM))$. Then (16) implies

$$A^*_N X = -b \overline{J}(A_N X), \quad (17)$$

$$\tau(X) = -\frac{1}{2b} (X \circ b). \quad (18)$$
An arbitrary \( Y \in \Gamma(TM) \) can be decomposed in the following manner
\[
Y = PY + QY = PY + \eta(Y)\xi,
\]
where \( \eta \) is a 1-form on \( M \) and \( \eta(Y) = \overline{g}(Y, N) \). Hence for \( \overline{J}Y \) we have
\[
\overline{J}Y = \overline{J}(PY) + \eta(Y)bN.
\]
By using (3), (5), (18) and (19) we obtain
\[
(\nabla_X \overline{J}) Y = \nabla^*_X \overline{J}(PY) - b\eta(Y)A_NX - \overline{J}(P(\nabla_X Y)) + \left( C(X, \overline{J}(PY)) + \frac{1}{b}B(X, Y) \right) \xi
\]
\[
+ \left( B(X, \overline{J}(PY)) + \frac{1}{2}\eta(Y)(X \circ b) + b(\nabla_X \eta)Y \right) N.
\]
Since \( (\nabla_X \overline{J}) Y = 0 \), the parts belonging to \( S(TM), TM^\perp \) and \( \text{tr}(TM) \) of the right side of (20) vanish and we have
\[
\nabla^*_X \overline{J}(PY) = b\eta(Y)A_NX + \overline{J}(P(\nabla_X Y)),
\]
\[
C(X, \overline{J}(PY)) = -\frac{1}{b}B(X, Y),
\]
\[
B(X, \overline{J}(PY)) = -\frac{1}{2}\eta(Y)(X \circ b) - b(\nabla_X \eta)Y.
\]
Substituting \( \overline{J}(PY) \) for \( Y \) in (21), (22) and taking into account that \( P(\overline{J}(PY)) = \overline{J}(PY), \eta(\overline{J}(PY)) = 0 \) and (23) we find
\[
\nabla^*_X PY = -\overline{J}(P(\nabla_X \overline{J}(PY))),
\]
\[
C(X, PY) = -\frac{1}{2b}\eta(Y)(X \circ b) - (\nabla_X \eta)Y.
\]
Having in mind (24), (25), (17) and (18), the formulas (5) become
\[
\nabla_X PY = -\overline{J}(P(\nabla_X \overline{J}(PY))) - \left( \frac{1}{2b}\eta(Y)(X \circ b) + (\nabla_X \eta)Y \right) \xi,
\]
\[
\nabla_X \xi = b\overline{J}(A_NX) + \frac{1}{2b}(X \circ b)\xi.
\]
From (24) by direct calculations we obtain
\[
(\nabla^*_X \overline{J}) PY = 0
\]
for any \( X, Y \in \Gamma(TM) \).

**Theorem 4.2.** Let \( (M, g, S(TM)) \) be a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric \((\overline{M}, \overline{J}, \overline{g})\). The shape operator \( A_N \) is self-conjugate with respect to \( g \) iff \( A_N \) commutes with the action of the almost complex structure \( \overline{J} \) on \( S(TM) \).
Proof. As $A^*$ is self-conjugate with respect to $g$, by using (17) we have

\begin{equation}
(28) \quad g(J(A_N X), Y) = g(X, J(A_N Y))
\end{equation}

for any $X, Y \in \Gamma(TM)$. Let $A_N$ be self-conjugate with respect to $g$ on $S(TM)$. Then for any $X, Y \in \Gamma(S(TM))$ we obtain

\[ g(J(A_N X), Y) = g(A_N X, JY) = g(X, A_N JY). \]

The last equality and (28) imply $g(X, A_N JY) = g(X, J(A_N Y))$. As $g$ is non-degenerate on $S(TM)$ it follows that $A_N \circ J = J \circ A_N$. Conversely, if $A_N \circ J = J \circ A_N$ on $S(TM)$, by using (28) we compute

\begin{equation}
(29) \quad g(A_N JX, Y) = g(X, J(A_N Y)), \quad X, Y \in \Gamma(S(TM)).
\end{equation}

Replacing $X$ from (29) by $JX$ we obtain $g(A_N X, Y) = g(X, A_N Y)$, i.e. $A_N$ is self-conjugate with respect to $g$. \qed

Further, using well known results for lightlike hypersurfaces from [2], [3] and the ones obtained in this section, we will give some geometrical characterizations of the considered hypersurfaces.

An immediate consequence from ([2], Theorem 2.3, p. 89) and Theorem 4.2 is the following

**Corollary 4.3.** Let $(M, g, S(TM))$ be a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric $(M, J, g)$. Then the following assertions are equivalent:

(i) $S(TM)$ is an integrable distribution.

(ii) $h^*(X, Y) = h^*(Y, X), \quad \forall X, Y \in \Gamma(S(TM))$.

(iii) $A_V \circ (JX) = J \circ (A_V X), \quad \forall X \in \Gamma(S(TM)), \quad \forall V \in \Gamma(tr(TM))$.

The Ricci tensor of the lightlike hypersurface $(M, g, S(TM))$ was defined in ([2], p. 95) by $Ric(X, Y) = \text{trace}\{Z \rightarrow R(X, Z)Y\}, \forall X, Y \in \Gamma(TM)$, where $R$ is the curvature tensor of $\nabla$. In general, Ricci tensor of $M$ is not symmetric because the induced connection $\nabla$ is not a metric connection. According to ([2], Theorem 3.2, p. 99) a necessary and sufficient condition the Ricci tensor of the induced connection $\nabla$ to be symmetric is each 1-form $\tau$ induced by $S(TM)$ to be closed, i.e. $d\tau = 0$ on $M$. Now, taking into account (18) we state

**Proposition 4.4.** The Ricci tensor of the induced connection $\nabla$ on a radical transversal lightlike hypersurface $(M, g, S(TM))$ of a Kaehler manifold with Norden metric $(M, J, g)$ is symmetric.

**Theorem 4.5.** Let $(M, g, S(TM))$ be a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric $(M, J, g)$. Then the following assertions are equivalent:

(i) $M$ is totally geodesic.

(ii) $S(TM)$ is totally geodesic.

(iii) $(\nabla_X \eta)Y = \eta(Y)\tau(X) = -\frac{1}{2b} \eta(Y)(X \circ b), \quad \forall X, Y \in \Gamma(TM)$.
Proof. According to ([2], Theorem 2.2, p. 88) we have that \( M \) is totally geodesic if and only if \( h \) vanishes identically on \( M \). From \( h(X, Y) = B(X, Y)N, X, Y \in \Gamma(TM) \) it follows that (i) is equivalent to \( B(X, Y) = 0 \). The screen distribution \( S(TM) \) is totally geodesic ([2], p. 110) if and only if \( C(X, PY) = 0, \forall X, Y \in \Gamma(TM) \). Then the equivalence of (i) and (ii) we obtain from (22). The equality (25) implies the equivalence of (ii) and (iii). □

Theorem 4.6. Let \((M, g, S(TM))\) be a radical transversal lightlike hypersurface of a Kaehler manifold with Norden metric \((\overline{M}, \overline{J}, \overline{g})\). Then

(i) \( M \) is totally umbilical iff \( A_N(PX) = \frac{\overline{b}}{b} \overline{J}(PX), \forall X \in \Gamma(TM) \) and \( \rho \in F(M) \).

(ii) \( S(TM) \) is totally umbilical iff \( A_\xi^*(PX) = -bk \overline{J}(PX), \forall X \in \Gamma(TM) \) and \( k \in F(M) \).

Proof. As \( A_\xi^* \xi = A_N \xi = 0 \), the equality (17) is equivalent to

(30) \[ A_\xi^*(PX) = \rho \overline{J}(A_N(PX)), \forall X \in \Gamma(TM). \]

According to ([2], p. 107, p. 110), \( M \) and \( S(TM) \) are totally umbilical if and only if, on each \( U \) of \( M \) there exists a smooth functions \( \rho \) and \( k \) such that

(31) \[ A_\xi^*(PX) = \rho PX \]

and

(32) \[ A_N X = kPX, \]

for any \( X \in \Gamma(TM) \), respectively. By using (30), (31) and (30), (32) we establish the truth of the assertions (i) and (ii), respectively. □

5. Hypersurfaces of an almost complex manifold with Norden metric which are non-degenerate with respect to the one Norden metric and lightlike with respect to the other one

Let \((\overline{M}, \overline{J}, \overline{g}, \overline{\g})\) be a \(2n\)-dimensional almost complex manifold with Norden metric and \( M \) be a \((2n-1)\)-dimensional hypersurface of \( \overline{M} \). An essential difference between an indefinite almost Hermitian manifold and an almost complex manifold with Norden metric is that there exist two Norden metrics \( \overline{g} \) and \( \overline{\g} \) on the manifold of the second type. Hence, we can consider two induced metrics \( g \) and \( \g \) on \( M \) by \( \overline{g} \) and \( \overline{\g} \), respectively. In [10] we have studied submanifolds of an almost complex manifold with Norden metric which are non-degenerate with respect to the one Norden metric and lightlike with respect to the other one. Our aim in this section is to show how the hypersurfaces \((M, g)\) and \((M, \g)\) of \( \overline{M} \) are related. We note that \( TM = \bigcup_{x \in M} T_xM \) is the tangent bundle of both \((M, g)\) and \((M, \g)\). We will denote: the normal bundle of \((M, g)\) and \((M, \g)\) by \( TM^\perp \) and \( TM^\perp \), respectively; an orthogonal
follows that both metrics \( g, S \) (resp. \( \tilde{g} \)) by \( \perp \) (resp. \( \perp \)) and a non-orthogonal direct sum by \( \oplus \) (resp. \( \oplus \)).

We consider a non-degenerate hypersurface \((M, g)\) of \( \overline{M} \) defined by the following conditions

\[
(33) \quad \overline{g}(\overline{N}, \overline{N}) = \epsilon, \ \epsilon = \pm 1; \quad \overline{g}(\overline{N}, \overline{JN}) = 0,
\]

where \( \overline{N} \) is the normal vector field to \( M \). In the case when \( \overline{N} \) is a time-like unit to \( M \) (\( \epsilon = -1 \)), the hypersurface \((M, g)\) was called in ([6], [8]) an isotropic hypersurface regarding the associated metric \( \overline{g} \) of \( \overline{M} \).

**Theorem 5.1.** Let \((\overline{M}, \overline{J}, \overline{g}, \overline{g})\) be an almost complex manifold with Norden metric and \( M \) be a hypersurface of \( \overline{M} \). \((M, g)\) is a non-degenerate hypersurface defined by \((33)\) iff \((M, \tilde{g})\) is a radical transversal lightlike hypersurface.

**Proof.** Let \((M, g)\) be a non-degenerate hypersurface of \( \overline{M} \) with normal vector field \( \overline{N} \) satisfying \((33)\). From \( g(\overline{N}, \overline{JN}) = 0 \) it follows \( \overline{JN} \in \Gamma(TM) \), i.e. \( \overline{J} \) transforms the normal bundle \( TM^\perp \) of \((M, g)\) in \( TM \) and \( \{\overline{JN}\} \) is a basis of \( \overline{J}(TM^\perp) \). Take \( X \in \Gamma(TM) \) and \( V = \lambda \overline{JN} \in \Gamma(\overline{J}(TM^\perp)) \), \( \lambda \in \mathcal{F}(M) \) we compute \( \overline{g}(X, V) = \lambda \overline{g}(\overline{JX}, \overline{JN}) = -\lambda \overline{g}(X, \overline{N}) = 0 \). The last equality implies \( V \) belongs to the normal bundle \( TM^\perp \) of \((M, \overline{g})\) and consequently \( \overline{J}(TM^\perp) \subseteq TM^\perp \). Now, if \( U \in \Gamma(TM^\perp) \) we have \( \overline{g}(X, U) = 0 \) for any \( X \in \Gamma(TM) \), which is equivalent to \( \overline{g}(X, U) = 0 \), \( \forall X \in \Gamma(TM) \). Hence, \( \overline{JU} \in \Gamma(TM^\perp) \) which implies \( TM^\perp \subseteq \overline{J}(TM^\perp) \). So, we obtain that \( TM^\perp = \overline{J}(TM^\perp) \). As \( \overline{J}(TM) \) is an 1-dimensional subbundle of \( TM \) it follows \( TM \cap TM^\perp = TM^\perp = \text{Rad}TM \), i.e. \((M, \tilde{g})\) is a lightlike hypersurface of \( \overline{M} \).

Because of \( \overline{J}(TM^\perp) \) is a non-degenerate subbundle of \( TM \) with respect to \( \overline{g} \), we put \( TM = \overline{J}(TM^\perp) \perp D \), where \( D \) is the complementary orthogonal with respect to \( \overline{g} \) vector subbundle of \( \overline{J}(TM^\perp) \) in \( TM \). Take \( X \in \Gamma(D) \) we compute \( \overline{g}(\overline{JX}, \overline{N}) = \overline{g}(X, \overline{JN}) = 0 \) which means that \( \overline{JX} \in \Gamma(TM) \). Moreover, \( \overline{g}(\overline{JX}, \overline{JN}) = 0 \) and consequently \( \overline{JX} \in \Gamma(D) \). So, we establish that \( D \) is holomorphic by the action of \( \overline{J} \). Then from ([10], Lemma 3.1) it follows that both metrics \( g \) and \( \tilde{g} \) are non-degenerate on \( D \). We also have \( \overline{g}(X, \overline{JN}) = \overline{g}(\overline{JX}, \overline{JN}) = 0 \) for any \( X \in \Gamma(D) \) which means \( D \perp TM^\perp \). Thus, we conclude the vector bundle \( D \) is a screen distribution of \((M, \tilde{g})\).

As \( D \) is holomorphic from Theorem 3.1 it follows that \((M, \tilde{g})\) is a radical transversal lightlike hypersurface of \( \overline{M} \) and \( \text{tr}(TM) = TM^\perp \). We note that for any \( \xi \in \Gamma(TM^\perp) \) and \( N \in \Gamma(\text{tr}(TM)) \) we have \( \xi = \lambda \overline{JN} \) and \( N = \mu \overline{N} \), where \( \lambda, \mu \in \mathcal{F}(M) \). The pair \( \{\xi, N\} \) on \((M, \tilde{g})\) satisfies the conditions \( \tilde{g}(\xi, \xi) = \tilde{g}(\overline{JN}, \overline{N}) = \tilde{g}(W, N) = 0, \forall W \in \Gamma(S(TM)) \). In the case \( \overline{N} \) is space-like (resp. time-like), the condition \( \tilde{g}(\xi, N) = 1 \) is fulfilled by \( \lambda \mu = -1 \) (resp. \( \lambda \mu = 1 \)). Conversely, let \((M, \tilde{g}, S(TM))\) be a radical transversal lightlike hypersurface of \( \overline{M} \). Hence, we have \( \overline{J}(TM^\perp) = \text{tr}(TM) \) and according to Theorem 3.1 \( S(TM) = S(TM) \). For any \( X \in \Gamma(S(TM)) \), \( \xi \in \Gamma(TM^\perp) \)
and \( N \in \Gamma(\text{tr}(T M)) \) we get
\[
\begin{align*}
\overline{g}(X, N) &= -\overline{g}(JX, N) = 0, \\
\overline{g}(\xi, N) &= -\overline{g}(J\xi, N) = 0, \\
\overline{g}(X, \xi) &= -\overline{g}(X, J\xi) = 0.
\end{align*}
\]

The above three equalities imply the vector bundles \( S(T M) \), \( T M^\perp \) and \( \text{tr}(T M) \) are mutually orthogonal with respect to \( \overline{g} \). Then the following decomposition of \( \overline{T M} \) is valid
\[
(34) \quad \overline{T M} = S(T M) \perp T M^\perp \perp \text{tr}(T M) = T M \perp \text{tr}(T M).
\]

From (34) it follows that the normal bundle \( T M^\perp \) of the hypersurface \( (M, g) \) coincides with the transversal vector bundle \( \text{tr}(T M) \) of \( (M, \overline{g}) \) and both \( T M \) and \( T M^\perp \) are non-degenerate with respect to \( g \) which means that \( (M, g) \) is a non-degenerate hypersurface of \( M \). Now, let \( \{\xi, N\} \) be a pair of sections on \( (M, \overline{g}) \) satisfying the conditions (2) and \( J\xi = bN, \ b \in \mathcal{F}(M) \). In the case \( b > 0 \) (resp. \( b < 0 \)) the vector field \( N = \pm \sqrt{b}N \) (resp. \( N = \pm \sqrt{-b}N \)) is a space-like (resp. time-like) normal unit to \( (M, g) \) and \( \overline{g}(N, JN) = 0 \) in both cases, which completes the proof.

An isotropic hypersurface \( (M, g) \) regarding the associated metric \( \overline{g} \) of an almost complex manifold with Norden metric \( (M, J, g, \overline{g}) \), equipped with the almost contact \( B \)-metric structure
\[
(35) \quad \varphi := J + \overline{g}(\cdot, JN)N, \quad \overline{\xi} := -JN, \quad \overline{\eta} := -\overline{g}(\cdot, JN), \quad g := \overline{g}|_M
\]
was called in [8] a hypersurface of second type of \( M \). In [8] it was proved that every hypersurface of second type \( (M, g) \) of a Kaehler manifold with Norden metric \( M \) is an almost contact manifold with \( B \)-metric belonging to the class \( \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \). Some classes of these hypersurfaces were characterized by the shape operator \( A \). Below we recall the characterization conditions of the following classes
\[
(36) \quad \mathcal{F}_0 : A = 0; \quad \mathcal{F}_4 : A = -\frac{\theta(\overline{\xi})}{2n-2} \varphi^2, \quad \theta(\overline{\xi}) = \text{tr} A;
\]
\[
\mathcal{F}_5 : A = -\frac{\theta^*(\overline{\xi})}{2n-2} \varphi, \quad \theta^*(\overline{\xi}) = \text{tr}(A \circ \varphi);
\]
\[
\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 : A \overline{\xi} = 0, \quad A \circ \varphi = \varphi \circ A.
\]

From now on in this section \((M, J, g, \overline{g})\) will stand for a \( 2n \)-dimensional Kaehler manifold with Norden metric, \((M, \varphi, \overline{\xi}, \overline{\eta}, g)\) and \((M, \overline{g})\) a hypersurface of second type of \( M \) and its corresponding radical transversal lightlike hypersurface from Theorem 5.1, respectively. Follow the proof of Theorem 5.1 and taking into account the definitions of \( \overline{\xi} \) and \( \overline{\eta} \) given in (35) we have \( \text{tr}(T M) = TM^\perp = \text{span}\{\overline{N}\}, \ TM^\perp = J(TM^\perp) = \text{span}\{\overline{\xi}\} \) and the contact distribution \( D \) of \((M, \varphi, \overline{\xi}, \overline{\eta}, g)\) coincides with the unique
(according to Theorem 4.2) screen distribution \( S(TM) \) of \((M, \bar{g})\). By using (9) and (1) we conclude that the tangent bundles \( TM \) and \( T\bar{M} \) can be decomposed in direct sums as follows

\[
(37) \quad TM = D \perp \text{span}\{\xi\}, \quad T\bar{M} = D \perp \text{span}\{\bar{\xi}\}
\]

and

\[
(38) \quad T\bar{M} = D \perp \text{span}\{\xi\} \perp \text{span}\{N\}, \quad T\bar{M} = D \perp (\text{span}\{\xi\} \oplus \text{span}\{N\}).
\]

We note that any \( X \in \Gamma(TM) \) can be written as \( X = PX + \eta(X)\xi \), where \( PX \in D \). Hence, \( \varphi X = \varphi(PX) \). On the other hand, by using (35) we compute \( \varphi(PX) = J(PX) \), i.e. we have

\[
(39) \quad \varphi X = \varphi(PX) = J(PX), \quad \forall X \in \Gamma(TM).
\]

Further, we will find relations between the induced geometrical objects on the hypersurfaces \((M, \varphi, \xi, \eta, \bar{g})\) and \((M, \bar{g})\) of \( M \). Let \( \nabla, \bar{\nabla} \) be the Levi-Civita connections of the metrics \( \bar{g}, \bar{g} \) on \( \bar{M} \), respectively, and \( \nabla, \bar{\nabla} \) be the induced linear connections on \((M, \varphi, \xi, \eta, \bar{g})\), \((M, \bar{g})\), respectively. According to [8], the formulas of Gauss and Weingarten for \((M, \varphi, \xi, \eta, \bar{g})\) are

\[
(40) \quad \nabla_X Y = \nabla_X Y - g(A_{\nabla}X, Y)\bar{N},
\]

\[
\nabla_X \bar{N} = -A_{\nabla}X, \quad \forall X, Y \in \Gamma(TM),
\]

where the shape operator \( A_{\nabla} \) satisfies

\[
(41) \quad \eta(A_{\nabla}X) = 0 \iff A_{\nabla}\xi = 0.
\]

As we have shown in the proof of Theorem 5.1 the pair of sections \( \{\xi, N\} \) on \((M, \bar{g})\) defined by

\[
(42) \quad \xi = \frac{1}{\lambda}J\bar{N} = -\frac{1}{\lambda}\xi, \quad N = \lambda\bar{N}, \quad \lambda \in \mathcal{F}(\bar{M})
\]

satisfies the conditions (2). By using (42) the formulas of Gauss and Weingarten (3) for \((M, \bar{g})\) become

\[
(43) \quad \bar{\nabla}_XY = \tilde{\nabla}_XY + \lambda B(X,Y)\bar{N},
\]

\[
\bar{\nabla}_X\bar{N} = -\tilde{A}_{\nabla}X + \left(\tau(X) - \frac{1}{\lambda}(X \circ \lambda)\right)\bar{N}, \quad \forall X, Y \in \Gamma(TM),
\]

where \( \tilde{A} \) is the shape operator of \((M, \bar{g})\). Taking into account that on \( \bar{M} \) the Levi-Civita connections \( \nabla \) and \( \bar{\nabla} \) coincide, the formulas (40), (43) and the decompositions (38) we get

\[
(44) \quad \tilde{\nabla}_XY = \nabla_X Y; \quad B(X,Y) = -\frac{1}{\lambda}g(A_{\nabla}X, Y);
\]

\[
\tilde{A}_{\nabla}X = A_{\nabla}X; \quad \tau(X) = \frac{1}{\lambda}(X \circ \lambda).
\]

The equality (41) implies \( A_{\nabla}X \in D \) and having in mind (39) we have

\[
(45) \quad \varphi(A_{\nabla}X) = J(A_{\nabla}X), \quad \forall X \in \Gamma(TM).
\]
From (17), (25) by using (42), (44) and (45) we obtain
\[
A^{*}_{\xi}X = -\varphi(A^{*}_N X); \quad C(X, PY) = \lambda g(A^{*}_N X, Y).
\]

We close this section by some geometrical characterizations of both \((M, \tilde{g})\) and \((M, \varphi, \xi, \eta, g)\). It is well known [9], if an almost contact manifold with \(B\)-metric belongs to the class \(F_{1} \oplus \ldots \oplus F_{4} \oplus F_{5} \oplus F_{6} \oplus F_{9} \oplus F_{10} \oplus F_{11}\), then the contact distribution \(\mathbb{D}\) of the manifold is an integrable distribution. Hence, if we suppose that the hypersurface \((M, \varphi, \xi, \eta, g)\) of \(\overline{M}\) belongs to the class \(F_{4} \oplus F_{5} \oplus F_{6} \), then its contact distribution \(\mathbb{D}\) is integrable. As \(\mathbb{D}\) is the screen distribution \(S(TM)\) of the corresponding hypersurface \((M, \tilde{g})\) to \((M, \varphi, \xi, \eta, g)\), we have that \(S(TM)\) is integrable, too. We will show that the converse statement is also true. Let \((M, \varphi, \xi, \eta, g)\) be a hypersurface of \(\overline{M}\) such that its contact distribution \(\mathbb{D}\) is integrable, which is equivalent to the assumption the screen distribution \(S(TM)\) of \((M, \tilde{g})\) is integrable. Then Corollary [11] implies \(A_V(J(PX)) = J(A_V PX)\) for any \(X \in \Gamma(TM)\). From the last equality, by using (39), (41), (45) and \(A_V = A_V\) on \(TM\), we obtain
\[
A_V(\varphi X) = \varphi(A_V X), \quad \forall X \in \Gamma(TM).
\]

Taking into account (41), (47) and (36) we conclude that \((M, \varphi, \xi, \eta, g)\) belongs to the class \(F_{4} \oplus F_{5} \oplus F_{6}\). So, we establish the truth of the following

**Proposition 5.2.** The assertion \((M, \varphi, \xi, \eta, g)\) belongs to the class \(F_{4} \oplus F_{5} \oplus F_{6}\) is equivalent to each of the following assertions:

(i) The contact distribution \(\mathbb{D}\) of \((M, \varphi, \xi, \eta, g)\) is integrable.

(ii) The screen distribution \(S(TM)\) of \((M, \tilde{g})\) is integrable.

Now, we prove

**Proposition 5.3.** The following assertions are equivalent:

(i) \((M, \varphi, \xi, \eta, g)\) is totally geodesic.

(ii) \((M, \tilde{g})\) is totally geodesic.

(iii) \((M, \varphi, \xi, \eta, g)\) belongs to the class \(F_{0}\).

*Proof.* As \((M, \varphi, \xi, \eta, g)\) is a non-degenerate hypersurface of \(\overline{M}\), it is totally geodesic if and only if the shape operator \(A\) vanishes identically. The hypersurface \((M, \tilde{g})\) is lightlike and it is totally geodesic \((2, 3)\) if and only if \(B(X, Y) = 0\) for any \(X, Y \in \Gamma(TM)\), which according to (44) is equivalent to \(g(A_N X, Y) = 0\) on \(TM\). Having in mind that \(g\) is non-degenerate on \(M\), the last equality is fulfilled if and only if \(A = 0\). Finally, from (39) we have that the characterization condition of the class \(F_{0}\) is \(A = 0\). Thus, each of the assertions (i), (ii) and (iii) is equivalent to the condition \(A = 0\), which completes the proof. \(\square\)

**Proposition 5.4.** \((M, \tilde{g})\) is totally umbilical iff \((M, \varphi, \xi, \eta, g)\) belongs to the class \(F_{5}\).
Proof. Let \((M, \overline{g})\) be totally umbilical. By using (31), (42), (16) and (11) we obtain \(A_{\overline{N}} X = -\lambda \rho_{\varphi} X\) for any \(X \in \Gamma(TM)\). From the last equality we find \(\lambda \rho = \frac{\text{tr}(A_{\overline{N}} \circ \varphi)}{2n-2}\) and hence

\begin{equation}
A_{\overline{N}} X = -\frac{\text{tr}(A_{\overline{N}} \circ \varphi)}{2n-2} \varphi X.
\end{equation}

According to (36) from (18) it follows \((M, \varphi, \overline{\xi}, \overline{\eta}, \overline{g}) \in \mathcal{F}_5\). Conversely, if \((M, \varphi, \overline{\xi}, \overline{\eta}, \overline{g}) \in \mathcal{F}_5\), then the equality (18) is valid. By using (16) and (42) we obtain \(A_{\xi}^* PX = \frac{\theta^*(\xi)}{\lambda(2n-2)} PX\), which implies \((M, \overline{g})\) is totally umbilical. □

Analogously we establish the truth of the following

**Proposition 5.5.** The screen distribution \(S(TM)\) of \((M, \overline{g})\) is totally umbilical iff \((M, \varphi, \overline{\xi}, \overline{\eta}, \overline{g})\) belongs to the class \(\mathcal{F}_4\).

6. Examples of radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric

**Example 1.** We consider \(\mathbb{R}^{2n+2} = \{ (u^1, \ldots, u^{n+1}; v^1, \ldots, v^{n+1}) \mid u^i, v^i \in \mathbb{R} \}\) as a complex Riemannian manifold with the canonical complex structure \(\mathcal{J}\). In [6] a metric \(\overline{g}\) on \((\mathbb{R}^{2n+2}, \mathcal{J})\) was defined by

\[ \overline{g}(X, X) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j, \]

where \(X = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i}\). It is easy to check that the canonical complex structure \(\mathcal{J}\) is an anti-isometry with respect to \(\overline{g}\) and hence \((\mathbb{R}^{2n+2}, \mathcal{J}, \overline{g})\) is a complex manifold with Norden metric. As usually, the associated metric to \(\overline{g}\) is denoted by \(\overline{g}\). Identifying the point \(p = (u^1, \ldots, u^{n+1}; v^1, \ldots, v^{n+1})\) in \(\mathbb{R}^{2n+2}\) with its position vector \(Z\), in [6] the following real hypersurface \(M\) of \(\mathbb{R}^{2n+2}\) was defined

\[ M : \overline{g}(Z, \mathcal{J}Z) = 0 ; \overline{g}(Z, Z) = \text{ch}^2 t , \quad t > 0. \]

It is clear that \(\mathcal{J}Z\) is orthogonal to \(TM\) with respect to \(\overline{g}\), i.e. \(\mathcal{J}Z \in TM^\perp\). For the vector field \(\overline{N} = (1/\text{ch})\mathcal{J}Z\) we have \(\overline{g}(\overline{N}, \overline{N}) = -1\), i.e. \(\overline{N}\) is a time-like unit normal to \((M, \overline{g})\). Since \(\overline{g}(\overline{N}, \mathcal{J}\overline{N}) = 0\), the vector field \(\mathcal{J}\overline{N}\) is a space-like unit, which belongs to \(TM\). Hence, \((M, \overline{g})\) is a non-degenerate hypersurface of \(\mathbb{R}^{2n+2}\) satisfying the conditions (35). Then from Theorem 5.4 it follows that \((M, \overline{g})\) is a radical transversal lightlike hypersurface of \(\mathbb{R}^{2n+2}\) such that \(TM^\perp = \text{span}\{\mathcal{J}\overline{N}\}\), \(\text{tr}(TM) = \text{span}\{\overline{N}\}\) and the screen distribution \(S(TM)\) coincides with the complementary orthogonal with respect to \(\overline{g}\) vector subbundle of \(\mathcal{J}(TM^\perp)\) in \(TM\). Taking into account that \((\mathbb{R}^{2n+2}, \mathcal{J}, \overline{g})\) is a complex manifold with Norden metric, from Theorem 3.3 we have that \((M, \overline{g})\) is a CR-manifold.

Moreover, in [6] we defined an almost contact structure \((\varphi, \overline{\xi}, \overline{\eta})\) and B-metric \(g\) on \(M\) by (35) and it was proved that \((M, \varphi, \overline{\xi}, \overline{\eta}, g)\) is an almost
contact manifold with $B$-metric in the class $\mathcal{F}_5$. According to Proposition ?? it follows that $(M, \overline{g})$ is totally umbilical.

**Example 2.** We consider the Lie group $GL(2; \mathbb{R})$ with a Lie algebra $\mathfrak{gl}(2; \mathbb{R})$. The real Lie algebra $\mathfrak{gl}(2; \mathbb{R})$ is spanned by the left invariant vector fields $\{X_1, X_2, X_3, X_4\}$, where we set

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

We define an almost complex structure $\overline{J}$ and a left invariant metric $\overline{g}$ on $\mathfrak{gl}(2; \mathbb{R})$ by

$$\overline{J}X_1 = X_4, \quad \overline{J}X_2 = X_3, \quad \overline{J}X_3 = -X_2, \quad \overline{J}X_4 = -X_1$$

and

$$\overline{g}(X_i, X_j) = -\overline{g}(X_j, X_i) = -1; \quad i = 1, 3; \quad j = 2, 4;$$

$$\overline{g}(X_i, X_j) = 0; \quad i \neq j; \quad i, j = 1, 2, 3, 4.$$ 

Using (49) and (50) we check that the metric $\overline{g}$ is a Norden metric and consequently $(GL(2; \mathbb{R}), \overline{J}, \overline{g}, \overline{g})$ is a 4-dimensional almost complex manifold with Norden metric.

The real special linear group $SL(2; \mathbb{R}) = \{A \in GL(2; \mathbb{R}) : \det(A) = 1\}$ is a Lie subgroup of $GL(2; \mathbb{R})$ with a Lie algebra $\mathfrak{sl}(2; \mathbb{R})$ of all $(2 \times 2)$ real traceless matrices. The Lie algebra $\mathfrak{sl}(2; \mathbb{R})$ is a 3-dimensional subalgebra of $\mathfrak{gl}(2; \mathbb{R})$, spanned by $\{X_1 - X_4, X_2, X_3\}$. Thus $SL(2; \mathbb{R})$ is a hypersurface of $GL(2; \mathbb{R})$. We find that the normal space (with respect to $\overline{g}$) $\mathfrak{sl}(2; \mathbb{R})^\perp$ is spanned by $\{X_1 - X_4\}$. Hence $\mathfrak{sl}(2; \mathbb{R}) \cap \mathfrak{sl}(2; \mathbb{R})^\perp = \mathfrak{sl}(2; \mathbb{R})^\perp = \text{span}\{\xi = X_1 - X_4\}$, i.e. $(SL(2; \mathbb{R}), g)$ is a lightlike hypersurface of $GL(2; \mathbb{R})$. We choose a holomorphic with respect to $\overline{J}$ screen distribution $S(\mathfrak{sl}(2; \mathbb{R}))$, spanned by $\{X_2, X_3\}$. From Theorem ?? it follows that $(SL(2; \mathbb{R}), g)$ is a radial transversal lightlike hypersurface and $\text{tr}(\mathfrak{sl}(2; \mathbb{R}))$ is spanned by $\left\{N = \frac{-X_1 - X_4}{2}\right\}$.

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