Asymptotically stable non-falling solutions of the Kapitza-Whitney pendulum

Ivan Polekhin

Abstract The planar inverted pendulum with a vibrating pivot point in the presence of an additional horizontal force field is studied. The horizontal force is not assumed to be small or rapidly oscillating. We assume that the pivot point of the pendulum rapidly oscillates in the vertical direction and the period of these oscillations is commensurable with the period of horizontal force. This system can be considered as a strong generalization of the Kapitza pendulum. Previously it was shown that for any horizontal force there always exists a non-falling periodic solution in the considered system. In particular, when there is no horizontal force, this periodic solution is the vertical upward position. In the paper we present analytical and numerical results concerning the existence of asymptotically stable non-falling periodic solutions in the system.

Keywords Forced oscillation · The Kapitza pendulum · The Whitney pendulum · Stabilization · Vibration

1 Introduction

The problem of motion of a pendulum with a vibrating pivot point is one of the well-studied non-linear dynamical systems. Like other classical systems—as examples we can mention here the Duffing equation or the Van der Pol oscillator—this system, firstly, is quite simple, which allows one to study this system analytically, and secondly, some non-trivial dynamical effects can be observed within the framework of this system.

Similarly to the the Duffing equation or the Van der Pol oscillator, the study of the pendulum with a vibrating pivot point goes beyond the boundaries of initial statements of the problems. Historically the problem of motion of a pendulum with a vibrating base goes back to A. Stephenson [1], N.N. Bogolyubov [2] and P.L. Kapitza [3, 4]. Let us also mention some recent papers on the considering problem [5–8]. The comprehensive bibliography can hardly be listed here. However, one can find a relatively full overview of the papers on the topic in [9, 10], including some references on the history of the problem.

In the presence of friction, for the pendulum with a vibrating base, one can show that the vertical upward position (usually unstable) becomes asymptotically
stable, provided that the parameters of the system satisfy some conditions. The proof follows from the classical method of averaging. If there is no friction in the system, the proof is more complicated and also involves KAM theory [11]. As a natural generalization of this system, one can consider a planar mathematical pendulum with a vibrating point of suspension and in a periodic in time horizontal force field.

If the horizontal force is rapidly oscillating, then we again can study this system by means of the averaging theory. This problem has been considered previously (see, for instance, [12]). In a more general statement of the problem, it is not assumed that the horizontal force has a small period. In [13] it was shown that for any horizontal periodic force, such that its period is commensurate to the period of the vertical oscillation, there always exists a periodic non-falling solution. Here we say that the considered solution is non-falling if the rod of the pendulum never becomes horizontal and remains in the upper half-plane. If the pivot point does not oscillate, we obtain the so-called Whitney pendulum (for details, see [13]). When there is no horizontal force, then we can consider the vertical upward equilibrium as a non-falling solution. Is it always possible to make this non-falling solution stable by changing the parameters of the vertical vibration? We will study this question in the paper.

In the first section we present the equations of motion and some analytical results on the problem. In particular, we will consider the case when the horizontal force is weak. The case when the horizontal force is not weak is considered numerically.

To be more precise, we consider the following two problems:

1. Given any 2π-periodic non-falling solution one can uniquely determine the corresponding horizontal force, which guarantees the existence of this solution. For which values of the amplitude of the vertical oscillation does this solution become stable?
2. Given any 2π-periodic horizontal force. Again, we find numerically the values of the amplitude such that there exists a periodic stable non-falling solution.

In the conclusion section we will briefly discuss some conjectures related to the problem.

2 Equations of motion
2.1 Planar pendulum

Let us consider a point of mass \( m \) moving along a circle of radius \( l \) in the presence of a gravitational field. We assume that there is a force of viscous friction acting on the point. If the circle is placed in a vertical plane, then this system is the classical planar mathematical pendulum (Fig. 1). If the pivot point of the pendulum moves along the vertical axis and the law of motion is given by a function \( h(t) \), then the position of the point is defined as follows:

\[
\begin{align*}
  x &= l \sin \varphi, \\
  y &= h(t) - l \cos \varphi.
\end{align*}
\]

Here \( \varphi \) is the angle between the vertical axis and the rod of the pendulum. The usual Lagrangian equations of motion for this system have the following form:

\[
ml^2 \ddot{\varphi} + mhl \sin \varphi = -mg l \sin \varphi.
\]

Here \( g \) is the acceleration of gravity. The usual Lagrangian equations of motion for this system (with the Lagrangian function \( L = T - \Pi \)) have the following form:

\[
\frac{m}{2} (ml^2 \dot{\varphi}^2 + 2h \dot{\varphi} l \sin \varphi), \quad \Pi = -mg l \cos \varphi.
\]

Fig. 1 Inverted pendulum with a vibrating pivot point
If we additionally consider viscous friction and a horizontal force acting on the system, then the above equation have to be rewritten as follows:

\[ ml^2 \ddot{\varphi} + mhl(t) \sin \varphi = -mg \sin \varphi - \mu l^2 \dot{\varphi} + F(t) l \cos \varphi. \]  

(1)

Here \( \mu \) is the coefficient of viscous friction and \( F(t) \) is a function of time. Without any loss of the generality, we can assume that \( m = g = l = 1 \). Then system (1) can be rewritten as follows:

\[ \ddot{\varphi} = p - h \sin \varphi, \]
\[ \dot{p} = -\sin \varphi - \mu p + \mu h \sin \varphi + h \cos \varphi + F \cos \varphi. \]  

(2)

Below we assume that \( h(t) \) is a rapidly oscillating function. To be more precise, we assume that this function has the form:

\[ h(t) = a \epsilon \sin \frac{t}{\epsilon}. \]

For small \( \epsilon > 0 \) it is possible to consider the averaged system. For this we replace functions \( h \) and \( h^2 \) with the averaged ones:

\[ \frac{1}{2\pi} \int_0^{2\pi} a \cos \tau d\tau = 0, \quad \frac{1}{2\pi} \int_0^{2\pi} a^2 \cos^2 \tau d\tau = \frac{a^2}{2} \]

Finally, we obtain the following system:

\[ \ddot{\varphi} = p, \]
\[ \dot{p} = -\sin \varphi - \mu p - \left( a^2 / 4 \right) \sin 2\varphi + F(s) \cos \varphi. \]  

(3)

\[ \dot{s} = 1. \]

Here \( \epsilon \) is an artificial time-like parameter (slow time) which is introduced in order to distinguish two types of time-dependent functions.

3 Analytical results

3.1 Periodic solutions of the averaged system

In this section we obtain some analytical results on the existence of a periodic solution (and its stability) for system (2). First, we will consider the same question for averaged system (3). For instance, by means of the classical Poincare method we will prove the existence of a periodic asymptotically stable solution, provided that the perturbation is small and non-autonomous. Therefore, from a result by N.N. Bogolyubov on the existence of an integral manifold, we obtain the existence of a periodic asymptotically stable solution in the original system.

We will also consider the case when the perturbation in (3) is not small. Here we will use a result by P. Torres on the existence and stability of periodic solutions for the Duffing equation [14].

Everywhere below we assume that all functions are real analytic, except for the cases when we explicitly assume that some function belongs to another class of regularity.

Proposition 1 Let function \( F(t) \) in (3) be as follows: \( F(t) = cf(t) \), where \( f \) is a 2\( \pi \)-periodic function. Let \( \mu > 0 \) and \( a^2 > 2 \), then for sufficiently small \( c > 0 \), for system (3), there exists a 2\( \pi \)-periodic asymptotically stable solution, and this solution coincides with the vertical upward equilibrium for \( c = 0 \).

Proof Let us denote the phase variable by \( x = (\varphi, p) \), and the right hand side by \( G(t, x, c) \). The standard method of proving the existence of a periodic solution for small \( c \) is based on the application of the implicit function theorem to the equation

\[ x(2\pi, x_0; c) - x_0 = 0, \]

here by \( x(t, x_0; c) \) we denote a the solution of the equation

\[ \dot{x} = G(t, x, c) \]

with initial condition \( x(0) = x_0 \). For \( c = 0 \) we have a trivial solution (the vertical equilibrium of the pendulum). For small \( c \neq 0 \) there exists a function \( x_0(c) \) such that \( x(2\pi, x_0(c), c) - x_0(c) = 0 \), provided that

\[ \det \left( \frac{\partial x(2\pi, x_0; 0)}{\partial x_0} - E \right) \neq 0. \]

Here \( E \) is the identity matrix. The determinant equals zero when the system in variations (w.r.t. solution \( \varphi = \pi, p = 0 \)) has nontrivial periodic solutions:

\[ \ddot{\varphi} = p, \]
\[ \dot{p} = -\varphi \left( 1 + \frac{a^2}{2} \right) - \mu p. \]

It can be easily shown that for function
we almost everywhere have $H = -\mu p^2 < 0$. Therefore, there exist no non-trivial periodic solutions for the system in variations.

The obtained solutions will be asymptotically stable. Indeed, the vertical position is asymptotically stable for $c = 0$. If we consider this equilibrium as a periodic solution, then we conclude that its characteristic multipliers are strictly inside the unit circle. From the continuous dependence of the characteristic multipliers on $c$, we obtain that for small $c$ the considered solution is also asymptotically stable. □

When the friction is also small, i.e. if we put $c = \mu$ in (3), then it is also possible to prove the existence of an asymptotically stable $2\pi$-periodic solution for small positive values of $\mu$.

This result follows from the following theorem proved by Lyapunov [15].

**Theorem 2** Let us have a non-autonomous Lyapunov system on a plane, i.e. a system of differential equations of the following form

$$\begin{align*}
\dot{x} &= -\lambda y + f_x(x,y) + \mu F_x(t,x,y,\mu), \\
\dot{y} &= \lambda x + f_y(x,y) + \mu F_y(t,x,y,\mu).
\end{align*}$$

Here $F_x, F_y$ are $2\pi$-periodic functions in $t$. Functions $f_x$ and $f_y$ are of order no less than two in $x, y$. We also assume that

$$f_x = -\frac{\partial S}{\partial y}, \quad f_y = \frac{\partial S}{\partial x}.$$ 

If $\lambda \notin \mathbb{Z}$, then there exists a unique $2\pi$-periodic solution of system (4). This solution depends analytically on $\mu$ for small $\mu$, and for $\mu = 0$ we obtain the trivial solution $x = y = 0$. Moreover, if

$$\mu \int_0^{2\pi} \left( \frac{\partial f}{\partial x} + \frac{\partial F}{\partial y} \right)|_{x=y=\mu=0} dt < 0,$$

then all solutions in the obtained one-parameter family of periodic solutions are asymptotically stable.

For instance, if we consider a Hamiltonian system, then $f$ corresponds to the terms of order no less than three in the Hamiltonian and $F$ is a small perturbation of the Hamiltonian.

If we apply this theorem to system (3), we obtain the following proposition.

**Proposition 3** Let $(1 + a^2/2)^{1/2} \notin \mathbb{Z}$, $F(t) = \mu f(t)$, where $f(t)$ is a $2\pi$-periodic function. Then for small $\mu > 0$ there is a $2\pi$-periodic asymptotically stable solution of (3).

**Proof** System (3) can be presented in the following form

$$\begin{align*}
\dot{x} &= -(1 + a^2/2)^{1/2} y + X(y) \\
\dot{y} &= -\mu y + \mu f(t) \cos(y/(1 + a^2/2)^{1/2}),
\end{align*}$$

where $X(y)$ is an analytic function of order no less than two in $y$. If $\mu = 0$, then this system is Hamiltonian. From Theorem 2 we obtain the existence of a periodic solution. Since $\mu > 0$, then $-2\pi \mu < 0$.

Therefore, this solution is asymptotically stable. □

Let us now consider the case when it is not assumed that $F(t)$ is infinitesimally small. We will use the following version of a result by P. Torres [14] (see also [16]).

Given a Duffing equation

$$\ddot{x} + cx + g(t,x) = 0,$$ 

where $c > 0$, $g : \mathbb{R}^2 \to \mathbb{R}$ is $2\pi$-periodic in $t$, by $\Omega_{k,c}$ we denote the following set

$$\Omega_{k,c} = \left\{ f \in L^1(0,2\pi) : f > 0, \right\}
\|f\|_k < \left( 1 + \frac{c^2}{4} \right) K\left( \frac{2k}{k-1} \right).$$

Here $f > 0$ means that $f \geq 0$ (a.e.) and $f > 0$ in a subset of positive measure.
\[ K(q) = \begin{cases} 
\frac{1}{q^{(2q)^{1/2}}} \left( \frac{2}{2+q} \right)^{1-2/q} & \text{if } 1 \leq q < +\infty, \\
\times \left( \frac{\Gamma(1/2)}{\Gamma(1/2+1/q)} \right)^2 & \text{if } q = +\infty.
\end{cases} \]

**Theorem 4** Let \( a \geq \beta \) be two numbers such that 
\( g(t, \alpha) > 0 \) and \( g(t, \beta) < 0 \) for all \( t \) and for some 
\( 1 \leq k \leq +\infty \) there exists \( f \in \Omega_{k,e} \) and for all \( x \in [\beta, \alpha] \)
\[
\frac{\partial g(t,x)}{\partial x} \leq f(t)(a.e.).
\]

Then system (5) has at least one asymptotically stable solution \( x(t) \) and \( x(t) \in (\beta, \alpha) \) for all \( t \).

As a corollary from this theorem we obtain the following result on the existence of an asymptotically stable \( 2\pi \)-periodic solution of (3). First, let us introduce the following notation

\[ k(\varphi) = \sin \varphi + \frac{a^2}{4} \sin 2\varphi. \]

Then

\[ g(t, \varphi) = k(\varphi) - F(t) \cos \varphi. \]

If \( a^2 > 2 \), then function \( k \) has one local minimum and one local maximum inside the interval \((\pi/2, 3\pi/2)\) (Fig. 2).

In these points \( \varphi_{\text{min}} \) and \( \varphi_{\text{max}} \) we have

\[ \cos \varphi_{\text{min}} = \cos \varphi_{\text{max}} = \frac{-1 - \sqrt{1 + 2a^2}}{2a^2}. \]

As \( a^2 \to \infty \), \( \varphi_{\text{min}} \to \frac{3}{4} \pi \) and \( \varphi_{\text{max}} \to \frac{5}{4} \pi \). Function \( k \) is monotonous on \((\varphi_{\text{min}}, \varphi_{\text{max}})\) and \( k(\varphi_{\text{min}}) < 0, k(\varphi_{\text{max}}) > 0 \). Therefore, if for some \( \varphi_1 \in (\varphi_{\text{min}}, \pi) \), \( \varphi_2 \in (\pi, \varphi_{\text{max}}) \) we have \( k(\varphi_1) < F(t) \cos(\varphi_1) \) and \( k(\varphi_2) > F(t) \cos(\varphi_2) \), then the first condition of Theorem 4 holds.

Function \( \frac{\partial k}{\partial \varphi} \) is strictly positive on \((\varphi_{\text{min}}, \varphi_{\text{max}})\). The maximum of this derivative equals \(-1 + a^2/2\).

We will consider \( k = \infty \) in Theorem 4. Therefore, \( K(2) = 1/4 \) and we can put \( f(t) = 1/4 \) (provided that \( \mu > 0 \)). We obtain the following result.

**Proposition 5** Let \( a^2 > 2 \). For any \( 2\pi \)-periodic \( F(t) \) and \( \varphi_1 \in [\varphi_{\text{min}}, \pi) \), \( \varphi_2 \in (\pi, \varphi_{\text{max}}] \) such that for all \( t \) we have \( k(\varphi_1) < F(t) \cos(\varphi_1) \), \( k(\varphi_2) > F(t) \cos(\varphi_2) \), and for all \( \varphi \in (\varphi_1, \varphi_2) \) we have

\[ \cos \varphi + \frac{a^2}{2} \cos 2\varphi + F(t) \sin \varphi < \frac{1}{4}, \]

there exists a \( 2\pi \)-periodic non-falling asymptotically stable solution of (3).

For instance, if \( a^2 = 2.4 \) and \( F(t) = A \sin t \), where \(|A| < 0.01\), then the conditions of the theorem hold, provided that \( \varphi_1 = \varphi_{\text{min}} \) and \( \varphi_2 = \varphi_{\text{max}} \).

### 3.2 Periodic solutions of the original system

Let us now discuss the connection between the existence of periodic solutions of the averaged (3) and the original (2) systems. The basic result which establishes this connection is a classical result by N.N. Bogolyubov on the existence of an integral manifold.

A more detailed explanation of the results presented below can be found in [17, 18].

Given a system of ordinary differential equations in the standard form of the method of averaging

\[ \frac{dx}{dt} = \varepsilon X(r, x), \]

where \( x \in \mathbb{R}^n \), \( r \in \mathbb{R} \), \( \varepsilon \) is a parameter, \( X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \). Let function \( X \) be a \( 2\pi \)-periodic in \( r \). We call system (6) the original system. Let us also consider the averaged system
\[
\frac{dx}{dt} = X_0(x),
\]

(7)

where function \( X_0 \) is the average of function \( X \) w.r.t. the periodic variable \( t \) and \( t = \varepsilon \tau \)

\[
X_0(x) = \frac{1}{2\pi} \int_0^{2\pi} X(\tau, x) d\tau.
\]

Let us assume that averaged system \( (7) \) has a periodic solution \( \tilde{x}(t) \) of period \( 2\pi \) and the characteristic multipliers of this solution are strictly inside the unit circle. Then for sufficiently small non-negative \( \varepsilon \) there exists an integral manifold close to solution \( \tilde{x}(t) \). Let us recall that we say that a manifold is integral if any solution starting at this manifold never leaves it.

To be more precise, if we have a \( 2\pi \)-periodic solution of the averaged system, then we have a one-parameter family of \( 2\pi \)-periodic solutions: if \( s \) is the initial time, then for any \( s \) we have a periodic solution. These solutions form an invariant manifold denoted by \( S_0 \). This manifold can be parameterized by \( s \) and \( \tau \). Moreover, \( S_0 \) is \( 2\pi \)-periodic in these variables (Fig. 3).

If \( \varepsilon > 0 \) is small, then we have another invariant manifold \( S_\varepsilon \) which is foliated into solutions of the original system. This manifold is also parameterized by \( s \) and \( \tau \) and is also \( 2\pi \)-periodic. For our system this manifold can be presented as follows:

\[
\varphi = f_\varphi(\tau, s), \quad p = f_p(\tau, s),
\]

where functions \( f_\varphi \) and \( f_p \) are \( 2\pi \)-periodic in both variables.

Note that the equation for \( s \) is the same for the original and the averaged system. If the periodic solution of the averaged system is asymptotically stable, then \( S_\varepsilon \) is also asymptotically stable. Therefore, the solutions of the original system is also asymptotically stable (w.r.t. to the perturbations of the phase variables \( \varphi \) and \( p \)).

Now let us remember that \( t \) and \( s \) have the same meaning originally. If \( \varepsilon \) is an arbitrary small number, then the right hand side of the original equation is not \( 2\pi \)-periodic: function \( F \) is \( 2\pi \)-periodic and function \( h \) is \( 2\pi\varepsilon \)-periodic. However, if \( \varepsilon = 1/n, n \in \mathbb{N} \), then the right hand side is \( 2\pi \)-periodic and solution has the form

\[
\varphi(t) = f_\varphi(nt, t), \quad p(t) = f_p(nt, t).
\]

These functions are periodic (not quasi-periodic).

In other words, if the periods of the vertical oscillation and the horizontal force are commensurable and the averaged system has an asymptotically stable \( 2\pi \)-periodic solution, then the original system also has an asymptotically stable \( 2\pi \)-periodic solution, provided that \( k \) is large.

4 Numerical results

4.1 Stability of a given periodic motion

The above analytical results hold only in the case when the external horizontal force is weak or some additional assumptions hold. Below we present numerical results for the cases when neither \( \mu \) nor \( F(t) \) in system \( (3) \) is small.

By a direct calculation, we obtain the following result showing that any given periodic non-falling solution can be obtained as a solution of our system for some periodic horizontal time-dependent force.

**Proposition 6** For any \( 2\pi \)-periodic function \( \varphi(t) \) such that \( \varphi(t) \in (\frac{\varepsilon}{2}, \frac{3\varepsilon}{2}) \) for all \( t \), there exists a unique \( 2\pi \)-periodic function \( \tilde{F}(t) \) such that \( \varphi(t) \) is a solution of \( (3) \). Function \( \tilde{F}(t) \) has the following form:
Let us assume that the values of parameters \( \mu \) and \( a \) in system (3) are given. Given any non-falling \( 2\pi \)-periodic trajectory of motion \( \varphi(t) \) \((\varphi(t) \in (\pi/2, 3\pi/2) \) for all \( t \)), we find \( F(t) \) such that \( \varphi(t) \) is a solution of (3). Here we put \( p(t) = \dot{\varphi}(t) \). For instance, if we put \( \varphi(t) = \pi - \pi/4 \cdot \cos t \), then we consider a periodic solution of amplitude \( \pi/4 \). For some values of \( a \) our solution may become asymptotically stable. In this case, in the original system we also have a non-falling \( 2\pi \)-periodic solution and these two solutions of systems (2) and (3) are close provided that \( \varepsilon = 1/k \) is small.

\[
F(t) = \frac{\ddot{\varphi}(t) + \sin \varphi(t) + \mu \dot{\varphi}(t) + (a^2/4) \sin 2\varphi(t)}{\cos \varphi(t)}.
\]

We will consider the following one-parameter family of solutions:

\[
\varphi_1(t) = \pi - A \cos t, \quad A \in [0, \pi/2).
\]

At the same time, we will consider various values of parameter \( \mu > 0 \). In Fig. 4 one can find the regions of asymptotic stability for this problem.

Note that some points of the region of asymptotic stability are arbitrarily close to vertical line \( A = \pi/2 \).

It means that for any amplitude \( A < \pi/2 \) we can stabilize the corresponding solution by choosing some \( a \) from a bounded interval.

Moreover, the distance between the consecutive points in which the region of asymptotic stability is

![Fig. 4](image.png)

*Fig. 4* Regions of asymptotic stability for various \( \mu \)
arbitrarily close to line $A = \pi/2$ equals $2\pi$. Based on the numerical results, it is also possible to conclude that for the first point in this sequence we have $a \to \pi$ as $\mu \to 0$.

The most interesting effect here is that, for a given value of $A$, i.e. for a given horizontal force, the corresponding unique periodic solution can be stable for some values of $a$ and can be unstable for larger $a$. To be more precise, for $A$ close to $\pi/2$, we have multiple switchings between stable and unstable regimes as $a$ grows to infinity.

### 4.2 Stability of a periodic non-falling solution for a given horizontal force

The second problem that we study numerically can be formulated as follows. For any $\mu > 0$ and $F \equiv 0$ there exists the vertical equilibrium of system (3) for which $\varphi = \pi$. This equilibrium is asymptotically stable when $a > \sqrt{2}$. It can be shown that for any $T$-periodic horizontal force $F$ there exists at least one non-falling $T$-periodic solution. If $F \equiv 0$ then the vertical equilibrium can be considered as the required periodic non-falling solution. We will numerically find the values of $a$ such that the corresponding periodic solution is asymptotically stable: we will start from known condition $a > \sqrt{2}$ (for $A = 0$) and will numerically continue this condition for larger $A$.

To be more precise, if our force $F$ depends on some parameter (denoted by $A$), then we will obtain a plot of $a$ as a function of $A$. For any given $A$, the corresponding value of $a$ can be described as follows: if the amplitude of the vertical oscillations is slightly less than $a$, then the corresponding periodic solution is non-stable, if the amplitude of the vertical oscillations is slightly greater than $a$, then the corresponding periodic solution is asymptotically stable. Everywhere below we assume that $T = 2\pi$.

Let us present a more detailed explanation of the algorithm. For any given force $F$ we have to find a periodic non-falling solution of system (3). Let us consider the following function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\Phi(\varphi_0, p_0) = \left( (\varphi(2\pi; \varphi_0, p_0) - \varphi_0)^2 + (p(2\pi; \varphi_0, p_0) - p_0)^2 \right)^{1/2}.$$  

Here $\varphi(t; \varphi_0, p_0)$ are $p(t; \varphi_0, p_0)$ the components of the solution of system (3) when $\varphi(0) = \varphi_0$ and $p(0) = p_0$. The periodic solutions correspond to the zeros of $\Phi$. We will use the method of numerical continuation and will try to find zeros close to the zeros obtained for close values of $A$. In other words, we restrict the search for periodic solutions to a neighborhood of the periodic solution obtained on the previous step (for the previous value of $A$).

When the zeros are found, we obtain numerically the monodromy matrices of the corresponding periodic solutions and check whether or not all characteristic multipliers belong to the unit circle. The required values of $a$ is determined by means of the bisection method.

Two level set plots of two different functions $\Phi$ and their zeros are shown in Fig. 5.

The resulting plots are shown in Fig. 6. In Fig. 7 one can find how an unstable periodic solution bifurcates and two unstable and one asymptotically stable periodic solutions appear as we cross the curve presented in Fig. 6 for $\mu = 0.1$.

The problem of finding of all periodic non-falling solutions for a given $A$ is more difficult from the computational point of view. Therefore, we mainly focus on the numerical continuation of a known solution. The following proposition allows one to simplify the search of a periodic solution for system (3). This proposition has been used to obtain Fig. 7.

**Proposition 7** A periodic non-falling solution of system (3) can exist only inside the rectangle $\varphi \in [\pi/2, 3\pi/2]$, $p \in [-P, P]$, where

$$P = \frac{1 + a^2/4 + \max |F|}{\mu}.$$  

**Proof** The first condition is the definition of a non-falling solution. Let us show that a periodic solution cannot exist in the region where $p > P$. Indeed, in all points of this region we have $\dot{p} < 0$ (it follows from system (3)). Therefore, we obtain that the solution cannot return to its initial condition. Similarly one can consider the case when $p < -P$.

5 Conclusion

A generalization of the well known Kapitza pendulum has been considered. In the case when there is no additional horizontal force the considered and system is obviously equivalent to the Kapitza pendulum.
Some results on the existence of asymptotically stable periodic non-falling solutions are presented. In particular, the obtained numerical results (Fig. 4) generalize classical conditions for the asymptotical stability of the upward vertical equilibrium of the Kapitza pendulum.

However, the considered system is far from being completely studied. In the conclusion section we would like to outline some open problem, which will be studied elsewhere.

First, it would be interesting to prove that for any periodic (and, therefore, bounded) force $F(t)$ there exists a sufficiently large $a$ such that system (3) has a periodic non-falling solution provided that $\varepsilon = 1/k$ is small. It is interesting to find the correspondence between the norm of $F(t)$ and the minimal possible value of $a$ such that this solution exists.

Second, it may be of some practical interest to prove results concerning the qualitative view of plots shown in Fig. 6. Is it true that these plots are always monotonously increasing? Is it possible to find several separated regions of stability for a given $A$ (similarly to the cases shown in Fig. 4)? It will be interesting to compare Fig. 4 with the well known Ince-Strutt diagram for the Mathieu equation (see, for instance, [19]).

Third, it is known that under some additional assumptions, there exist at least two periodic solutions of the Kapitza-Whitney pendulum [13]. Is it true that the existence of two unstable periodic non-falling solutions guarantees the existence of at least one stable periodic non-falling solution? Note that the mentioned two solutions are unstable if there is no friction in the system [20]. From the above results it follows that this statement holds if $F(t)$ is small.

As a last open problem in our short list we would like to mention the problem of existence of so-called subharmonic solutions (the solution is called subharmonic if this solution is $nT$-periodic and not $kT$-periodic for all $k < n$, $k, n \in \mathbb{N}$). It would be interesting to compare the minimal values of $a$ which stabilize the solution for $T$-periodic and $nT$-periodic solutions.
Finally, we can add that many questions that can be asked about the Kapitza pendulum can be easily reformulated for the Kapitza-Whitney pendulum.

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**Declarations**

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