Closed form solution of the maximum entropy equations with application to fast radio astronomical image formation

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ABSTRACT

In this paper we analyze the maximum entropy image deconvolution. We show that given the Lagrange multiplier a closed form can be obtained for the image parameters. Using this solution we are able to provide better understanding of some of the known behavior of the maximum entropy algorithm. The solution also yields a very efficient implementation of the maximum entropy deconvolution technique used in the AIPS package. It requires the computation of a single dirty image and inversion of an elementary function per pixel.

Subject headings: Radio interferometry, image formation, maximum entropy, deconvolution

1. Introduction

Radio astronomical imaging using earth rotation synthesis radio telescopes is an ill-posed problem due to the irregular sub-Nyquist sampling of the Fourier domain. During the last 40 years many deconvolution techniques have been developed to solve this problem. These algorithms are based on models for the radio image. Among these techniques we find the CLEAN method by [Hogbom (1974)], The maximum entropy algorithm (MEM) by [Frieden (1972), Gull and Daniell (1978), Ables (1974) and Wernecke (1977)], extensions of the CLEAN to support multi-resolution and wavelets by [Bhatnager and Cornwell (2004), Cornwell (2008) and Cornwell et al. (2008)], non-negative least squares by [Briggs (1995)], parametric based imaging by [Leshem and van der Veen (2000) and Ben-David and Leshem (2008)] and sparse $L_1$ reconstruction by [Levanda and Leshem (2008) and Wiaux et al. (Submitted 2009)]. While there is a major amount of experience in using these algorithms, we still lack a comprehensive theoretical analysis. This would become a more critical problem for the future generation of radio interferometers that will be built in the next two decades such as the square kilometer array (SKA), the Low Frequency Array (LOFAR), The Allen Telescope Array (ATA), the Long Wavelength Array (LWA) and the Atacama Large Millimeter Array (ALMA). These radio-telescopes will include many more stations, will have significantly increased sensitivity and some of them will operate at much lower frequencies than previous radio telescopes, and therefore would be more sensitive to modeling and calibration errors.

The maximum entropy image formation technique is one of the two most popular deconvolution techniques. The maximum entropy principle was first proposed by [Jaynes (1957)]. Jaynes (1982) provides a good overview of the philosophy behind the idea. Since then it has been used in a wide spectrum of imaging problems. The basic idea behind MEM is the following: Among all images which are consistent with the measured data and the noise distribution not all satisfy the positivity demand, i.e., the sky brightness is a positive function. Consider only those that satisfy the positivity demand. From these select the one that is most likely to have been created randomly. This idea has also been proposed by [Frieden (1972)] for optical images and applied to radio astronomical imaging in [Gull and Daniell (1978)]. Other ap-
proaches based on the differential entropy have also been proposed \cite{Ables1974} and \cite{Wernecke1977}. An extensive collection of papers discussing the various methods and aspects of maximum entropy can be found in the various papers in \cite{Roberts1984}. \cite{Narayan1986} provides an overview of various maximum entropy techniques and the use of the various options for choosing the entropy measure. Interestingly, in that paper, a closed form solution is given for the noiseless case, but not for the general case.

The approach of \cite{Gull1978} begins with a prior image and iterates between maximizing the entropy function and updating the $\chi^2$ fit to the data. The computation of the image based on a prior image is done analytically, but at each step the model visibilities are updated, through a two-dimensional Fourier transform. This type of algorithm is known as a fixed point algorithm, since it is based on iterating a function until it converges to a fixed point. While it is known that for the maximum entropy, this approach usually converges, it was recognized that the convergence can be slow (see \cite{Narayan1986}). Hence, improved methods based on Newton method and the Conjugate Gradient technique have been proposed by \cite{Cornwell1985,Sault1990,Skilling1984}. These methods perform direct optimization of the entropy function subject to the $\chi^2$ constraint.

In this paper we will provide a closed form solution for the maximum entropy image formation problem. This solution provides a novel short proof of the uniqueness of the maximum entropy solution and allows us to provide a theoretical explanation to the failure of the maximum entropy algorithm in cases of strong point sources. The explicit expressions for the solution allows us to quantify the effect of the free parameters involved in the maximum entropy algorithm. Using the closed form solution we will also develop a new technique for solving the maximum entropy deconvolution with a fixed number of operations per pixel, except an initial step of gridding, deconvolution and computation of a single dirty image as described in \cite{Taylor1999}. We believe that this paper is a significant step forward in understanding image deconvolution techniques.

2. The Maximum Entropy algorithm

We begin with a short description of the maximum entropy algorithm. Following the standard convention in this field \cite{Gull1978,Cornwell1985,Sault1990} we present the one dimensional case. Similar results for the two dimensional case are also valid (but require more complicated notation).

\cite{Gull1978} showed that the maximum entropy solution is given by solving the following Lagrangian optimization problem:

$$I^{MEM} = \arg \max_I \sum_\ell I_\ell \log I_\ell - \lambda \frac{1}{2} \chi^2(V), \quad (1)$$

where

$$\chi^2(V) = \sum_{k \in A} \frac{1}{\sigma_k^2} |V_k - V_k^{\text{model}}|^2, \quad (2)$$

and $\lambda$ is a Lagrange multiplier for the constraint that the model based visibilities should match the measured visibilities. Taking the derivative with respect to $I_\ell$ we obtain that the solution is given by:

$$I_\ell = \exp \left\{ -1 + \lambda \sum_{k \in A} \frac{\text{Re} \left( (V_k - V_k^{\text{model}}) e^{i 2 \pi \ell k} \right)}{\sigma_k^2} \right\}, \quad (3)$$

where the complex exponent comes from the Fourier transform relationship between $V_k^{\text{model}}$ and the intensity at direction $\ell$. The basic maximum entropy algorithm now proceeds by choosing an initial image model (typically flat image) computing the model based visibilities $V_k^{\text{model}}$ on the grid $A$. Using these visibilities a new model image is computed by equation \ref{eq:3}. New visibilities are computed from the new model and the process is iterated until convergence.

3. Closed form solution of the Maximum Entropy equations

We will now show that there is a much simpler technique to compute the maximum entropy image than the existing techniques. The discussion will also provide a novel simple proof for the uniqueness of the maximum entropy solution as
well as a very efficient solution. The solution depends on the Fourier inversion nature of the interferometric measurement equation, together with the common practice of gridding and interpolating the data to obtain simple fast Fourier transform between visibility domain and image domain. This approximation was used by Cornwell and Evans (1985) where the \( \chi^2 \) approximation was computed in the image domain using the gridded data. This will enable us to show that the unique maximum entropy function can be computed independently for each pixel, given a first step of data interpolation in the visibility domain. To that end assume that \( I = \langle I_\ell : \ell \in L \rangle \) is a fixed point of the maximum entropy iteration 4. Therefore, the visibilities \( V_{k} \) are just Fourier transform of \( I \) sampled at the points \( k \in A \), where \( A \) is the rectangular grid. To simplify the discussion we will assume that the measurement noise variance \( \sigma_k^2 = \sigma^2 \) is identical for all visibilities (an assumption that follows from standard noise calibration practice in radio interferometry) and that a uniform weighting is used. (we will comment in the end on other weighting schemes). The right hand side of 4 is just a Discrete Fourier Transform of the difference between the measured visibilities and the modeled visibilities. Hence the right hand side is actually the difference between the dirty image \( I_D^\ell \) based on the gridded visibilities and the model image at the point \( I_\ell \). Hence the fixed point of the maximum entropy procedure must satisfy for each pixel \( I_\ell \):

\[
I_\ell = \exp \left\{ -1 + \frac{\lambda}{\sigma^2} N \left( I_D^\ell - I_\ell \right) \right\}
\]

Alternatively this expression can be derived using Parseval identity, and deriving the \( \chi^2 \) term with respect to \( I_\ell \) in the image domain. Further simplification yields:

\[
\log I_\ell + \frac{N \lambda}{\sigma^2} I_\ell = \frac{N \lambda}{\sigma^2} I_D^\ell - 1
\]

Taking exponent of the two sides yields

\[
I_\ell \exp \left\{ \frac{N \lambda}{\sigma^2} I_\ell \right\} = \exp \left\{ \frac{N \lambda}{\sigma^2} I_D^\ell - 1 \right\}
\]

The function \( f_a(x) = xe^{ax} \) is monotonically increasing for \( a > 0 \) and all \( x \), non negative for all positive values of \( x \) and negative for negative values of \( x \). Since the right hand side of 6 is always positive every pixel in the reconstructed image is positive. By monotonicity of \( f_a(x) \) we obtain that \( f \) is invertible and the fixed point is uniquely determined given the Lagrange multiplier. The solution is now given by:

\[
I_\ell = f_a^{-1} \left( \exp \left\{ \frac{N \lambda}{\sigma^2} I_D^\ell - 1 \right\} \right).
\]

where \( a = N \lambda / \sigma^2 \). Equation 7 is very satisfying. It provides us with the desired understanding of the way the maximum entropy solution manipulates the gridded data. For each pixel of the dirty image \( I_D^\ell \) we obtain that if \( 1 << N \lambda I_D^\ell / \sigma^2 \) then

\[
\frac{\sigma^2}{N \lambda I_D^\ell} \log I_\ell + I_\ell \approx I_D^\ell.
\]

Hence whenever the dirty image is sufficiently strong we give preference to the dirty image. If on the other hand, the dirty image is weak, then there is higher risk of obtaining a sidelobe, so the prior is more significant, and sidelobes are more significantly suppressed. Note that if we have a very strong point source in the field of view, its sidelobes are also likely to be stronger than other emission in the image, and therefore the algorithm will not be able to suppress these sidelobes. This provides full explanation for the problems of maximum entropy deconvolution as described by Briggs et al. (see Taylor et al. [1999] chapter 8).

4. Extensions

The \( \chi^2 \) constraint is in many cases insufficient, and either a reference image or a constraint on the total flux is added in order to provide better fit is added. Our solution naturally extends to these cases. We will demonstrate it in the first case. Assume that we have a reference image. The problem 1 can be reformulated as

\[
I = \arg \max_{I} - \sum_{\ell} I_\ell \log \frac{I_\ell}{F_\ell} - \frac{1}{2} \sum_{k \in A} \frac{1}{\chi_k} \left| \tilde{V}_k - V_{k \text{model}} \right|^2,
\]

where \( F_\ell \) is a reference image (possibly low resolution image we already have). The ratio is taken from the expression for the relative entropy or the Kullback-Leibler divergence. Taking the derivative with respect to \( I_\ell \) and assuming again that
\( \sigma_k^2 = \sigma^2 \) results in the following solution

\[
I_t = \exp \left\{ -1 + F_t + \frac{N \lambda}{\sigma^2} (I_D^P - I_t) \right\}. \quad (10)
\]

Hence the solution is given by

\[
I_t = f_a^{-1} \left( \exp \left\{ \frac{N \lambda}{\sigma^2} I_D^P - 1 + F_t \right\} \right). \quad (11)
\]

where \( a = N \lambda I_t^D / \sigma^2 \) and \( f_a^{-1}(x) \) is computed as before.

Now that we have explicit expression of the reconstructed image given \( \lambda \) we can use either a bisection or Newton type technique to solve for the Lagrange multiplier that provides sufficiently good fit for the data. We will not provide explicit expressions, due to lack of space. The algorithm is described in Table [1].

Finally we comment that when non uniform weighting is used, closed form solution is harder, as already noted by Cornwell and Evans (1985). We adopt their solution and replace the \( \chi^2 \) term by the same image domain expression. The only thing that changes is the computation of the weighted dirty image. Since this technique has been used in the last three decades it is safe to believe that the impact on the ME algorithm is small.

5. Conclusions

In this paper we have demonstrated that the maximum entropy algorithm can be solved analytically. The solution is a two steps approach: Gridding of the visibilities into a rectangular grid and solution of the gridded maximum entropy problem in the image domain. We showed that the solution can be computed analytically requiring only an inversion of an elementary function for each pixel. The solution provides better theoretical understanding of the maximum entropy algorithm as well as significant acceleration of the algorithm implemented in existing radio astronomy packages, by reducing the number of Fourier transforms from 20-100 to a single dirty image computation. This enhancement will allow much faster maximum entropy deconvolution for very large images. We do not provide simulated comparison, since by our results, the proposed algorithm is identical to the maximum entropy algorithm as implemented in the VM and VTESS tasks in the AIPS package.

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| Table 1: Exact maximum entropy algorithm |
|-----------------------------------------|
| **Initialization:**                     |
| Choose a prior image $F_\ell$.          |
| Choose $\lambda$                        |
| Grid the data to obtain $V_k : k \in A$ |
| where $A$ is a uniform rectangular grid.|
| Compute the dirty image $I^D$.          |
| **Image deconvolution:**                |
| For each pixel $\ell$                   |
| Compute the closed form solution based on (11). |
| When $I^D_\ell$ is negative:           |
| choose the solution with maximum entropy.|
| End.                                    |
| Compute fit to data.                    |
| If fit is poor, increase $\lambda$ and redo deconvolution. |