Characterizations of derivations on spaces of smooth functions

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Dedicated to Professor Wojciech Kryszewski on the occasion of his 65-th birthday.

Abstract. We provide a list of equivalent conditions under which an additive operator acting on a space of smooth functions on a compact real interval is a multiple of the derivation.

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1. Introduction

By \( \mathbb{R} \) we denote the set of reals, \( \mathbb{Q} \) are rationals, \( \mathbb{Z} \) are integers, \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). If \( I \subseteq \mathbb{R} \) is an interval and \( k \in \mathbb{N}_0 \), then \( C^k(I) \) is the space of real-valued functions on \( I \) that are \( k \)-times continuously differentiable on the interior of \( I \). If \( k = 0 \), then we write simply \( C(I) \). The space \( C^k(I) \) is furnished with the standard pointwise algebraic operations and hence it is a real commutative algebra.

Definition. (e.g. Kuczma [12, page 391]) Assume that \( Q \) is a commutative ring and \( P \) is a subring of \( Q \). A function \( f: P \rightarrow Q \) is called derivation if it is additive:

\[
    f(x + y) = f(x) + f(y), \quad x, y \in P
\]

and it satisfies the Leibniz rule:

\[
    f(xy) = xf(y) + yf(x), \quad x, y \in P.
\]

The following theorem describes derivations over fields of characteristic zero.
Theorem 1. [12, Theorem 14.2.1] Let $K$ be a field of characteristic zero, $F$ be a subfield of $K$, $S$ be an algebraic base of $K$ over $F$ if it exists, and let $S = \emptyset$ otherwise. If $f : F \to K$ is a derivation, then, for every function $u : S \to K$ there exists a unique derivation $g : K \to K$ such that $g = f$ on $F$ and $g = u$ on $S$.

From this theorem it follows in particular that nonzero derivations $f : \mathbb{R} \to \mathbb{R}$ exist. It is well known they are discontinuous and very irregular mappings. For an exhaustive discussion of the notion of derivation and related functional equations the reader is referred to Gselmann [5,6], Gselmann, Kiss, Vincze [7] and the references therein. Recently Ebanks [2,3] studied derivations and derivations of higher order on rings.

The “model” example of a derivation is the operator of derivative on the space $C^k(I)$ for $k > 0$. Indeed, if we define $T : C^k(I) \to C(I)$ as $T(f) = f'$ for $f \in C^k(I)$, then clearly $C^k(I)$ is a subring of $C(I)$, $T$ is additive and it satisfies the Leibniz rule:

$$T(f \cdot g) = f \cdot T(g) + g \cdot T(f). \quad (3)$$

Crucial results about equation (3) on the space $C^k(I)$ are due to H. König and V. Milman. We refer the reader to their recent monograph [11]. They studied several operator equations and inequalities that are related to derivatives on the spaces of smooth functions. Later on, we will utilize their elegant result [11, Theorem 3.1] regarding (3). Briefly, if $I$ is an open set, then the general solution of (3) for all $f,g \in C^k(I)$ is of the form

$$T(f) = c \cdot f \cdot \ln |f| + d \cdot f', \quad f \in C^k(I) \quad (4)$$

for some continuous functions $c, d \in C(I)$, if $k > 0$, and

$$T(f) = c \cdot f \cdot \ln |f|, \quad f \in C^k(I) \quad (5)$$

if $k = 0$ (in formulas (4) and (5) the convention that $0 \cdot \ln 0 = 0$ is adopted). Note that no additivity is assumed.

It is a natural question to characterize real-to-real derivations among additive functions with the aid of a relation which is weaker than (2). In particular, the very first article published in the first volume of Aequationes Mathematicae by Nishiyama and Horinouchi [14] addresses this question. The authors studied the following relations, each of which is a direct consequence of (2) alone and together with (1) implies (2):

$$f(x^2) = 2xf(x), \quad x \in \mathbb{R}, \quad (6)$$

$$f(x^{-1}) = -x^{-2}f(x), \quad x \in \mathbb{R}, x \neq 0, \quad (7)$$

and

$$f(x^n) = ax^{n-m}f(x^m), \quad x \in \mathbb{R}, x \neq 0, \quad (8)$$
where $a \neq 1$ and $n, m$ are integers such that $am = n \neq 0$. Further similar results, as well as some generalizations, are due to Jurkat [8], Kannappan and Kurepa [9,10], Kurepa [13], among others. Ebanks [4] generalized and extended these results to arbitrary fields. A recent paper by Amou [1] provides some $n$-dimensional generalizations of the results of [8–10,13].

This paper provides versions of the above-mentioned results for operators $T: C^k(I) \rightarrow C(I)$. Therefore, we seek conditions which are equivalent to (3).

2. Main results

Throughout this section let us fix $k \in \mathbb{N}_0$ and an interval $I \subseteq \mathbb{R}$. We will study conditions upon an additive operator $T: C^k(I) \rightarrow C(I)$ which yield analogues to Eqs. (6), (14) and (8). Therefore, we will focus on the following operator relations:

$$T(f^2) = 2f \cdot T(f), \quad (9)$$
$$T(f) = -f^2 \cdot T\left(\frac{1}{f}\right), \quad (10)$$
$$T(f^n) = nf^{n-1} \cdot T(f). \quad (11)$$

Our first theorem is a simple observation that some reasonings concerning derivations from the real-to-real case can be extended to arbitrary commutative rings without substantial changes. We adopted parts of the proof of [12, Theorem 14.3.1].

**Theorem 2.** Assume that $Q$ is a commutative ring, $P$ is a subring of $Q$ and $T: P \rightarrow Q$ is an additive operator. Then, the following conditions are pairwise equivalent:

(i) $T$ satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in P$,
(ii) $T$ satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f, g \in P$,
(iii) $T$ satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in P$ and $n \in \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii). Fix arbitrarily $f, g \in P$. By (9) we get

$$T((f+g)^2) = 2(f+g) \cdot T(f+g).$$

Since $T$ is additive,

$$T(f^2) + 2T(f \cdot g) + T(g^2) = 2f \cdot T(f) + 2g \cdot T(f) + 2f \cdot T(g) + 2g \cdot T(g).$$

Using (9) again, after reductions we obtain (3).

(ii) $\Rightarrow$ (iii). If $n = 1$, then (11) reduces to an identity. Assume that (11) holds for some $n \in \mathbb{N}$ and all $f \in P$. Then, by (3) and the induction hypothesis we have

$$T(f^{n+1}) = T(f^n \cdot f) = f^n \cdot T(f) + f \cdot T(f^n) = f^n \cdot T(f) + nf^{n-1} \cdot T(f) = (n+1)f^n \cdot T(f).$$
(iii) ⇒ (i). Take \( n = 2 \).

The next corollary will be utilized later on.

**Corollary 1.** Assume that \( T : C^k(I) \to C(I) \) is an additive operator. Then, the following conditions are pairwise equivalent:

(i) \( T \) satisfies \( T(f^2) = 2f \cdot T(f) \) for all \( f \in C^k(I) \),
(ii) \( T \) satisfies \( T(f \cdot g) = f \cdot T(g) + g \cdot T(f) \) for all \( f, g \in C^k(I) \),
(iii) \( T \) satisfies \( T(f^n) = nf^{n-1} \cdot T(f) \) for all \( f \in C^k(I) \) and \( n \in \mathbb{N} \).

Our next result characterizes the Leibniz rule (3) on a domain restricted to functions separated from zero. Thus, we can consider conditions (10) and (11) for negative \( n \), which involve the function \( 1/f \). The situation is a bit more complicated, but Theorem 3 below has a mainly technical role.

**Theorem 3.** Assume that \( T : C^k(I) \to C(I) \) is an additive operator and \( \varepsilon_1 \in (0, 1) \), \( \varepsilon_2 \in (0, 1) \) and \( c \in (1, +\infty] \) are constants. Consider the following conditions:

(i) \( T \) satisfies \( T(f) = -f^2 \cdot T\left(\frac{1}{f}\right) \) for all \( f \in C^k(I) \), \( c > f > \varepsilon_1 \),
(ii) \( T \) satisfies \( T(f^2) = 2f \cdot T(f) \) for all \( f \in C^k(I) \), \( f > \varepsilon_2 \),
(iii) \( T \) satisfies \( T(f \cdot g) = f \cdot T(g) + g \cdot T(f) \) for all \( f, g \in C^k(I) \), \( f > \varepsilon_2 \), \( g > \varepsilon_2 \),
(iv) \( T \) satisfies \( T(f^n) = nf^{n-1} \cdot T(f) \) for all \( n \in \mathbb{Z} \) and all \( f \in C^k(I) \) such that \( \varepsilon_2 < f < 1/\varepsilon_2 \), and \( f^{n-1} > \varepsilon_2 \) if \( n > 0 \) and \( f^{n+1} > \varepsilon_2 \) if \( n < 0 \).

Then: (i) with \( c = +\infty \) implies (ii) with \( \varepsilon_2 > \sqrt{\varepsilon_1} \), (ii) and (iii) are equivalent, (iii) implies (iv), (iv) implies (i) with \( \varepsilon_1 = \varepsilon_2 \) and \( c = 1/\varepsilon_2 \).

**Proof.** (i) ⇒ (ii). First, note that by applying (10) for \( f = 1 \) and using the rational homogeneity of \( T \) we get that \( T \) vanishes on each constant function equal to a rational number. Observe that for an arbitrary rational \( \delta > 0 \) (which will be chosen later) the identity

\[
\frac{1}{f^2 - \delta^2} = \frac{1}{2\delta} \left( \frac{1}{f - \delta} - \frac{1}{f + \delta} \right)
\]

holds for \( f \in C^k(I) \) such that \( f > \delta \). Next, if \( \varepsilon_1 > 0 \) is given and \( \varepsilon_2 > \sqrt{\varepsilon_1} \), then we will find some rational \( \delta > 0 \) such that \( \varepsilon_2 > \varepsilon_1 + \delta \) and \( \varepsilon_2^2 > \varepsilon_1 + \delta^2 \). Consequently, if \( f \in C^k(I) \) and \( f > \varepsilon_2 \), then \( f \pm \delta > \varepsilon_1 \) and \( f^2 - \delta^2 > \varepsilon_1 \).
Using (i) three times together with (12) and the additivity of $T$ we obtain
\[
T(f^2) = T(f^2 - \delta^2) = -(f^2 - \delta^2)^2 T \left( \frac{1}{f^2 - \delta^2} \right)
\]
\[
= -\frac{1}{2\delta} (f^2 - \delta^2)^2 T \left( \frac{1}{f - \delta} - \frac{1}{f + \delta} \right)
\]
\[
= -\frac{1}{2\delta} (f + \delta)(f - \delta)^2 \left[ T \left( \frac{1}{f - \delta} \right) - T \left( \frac{1}{f + \delta} \right) \right]
\]
\[
= \frac{1}{2\delta} [(f + \delta)^2 T(f - \delta) - (f - \delta)^2 T(f + \delta)] = 2fT(f).
\]

(ii) $\Leftrightarrow$ (iii). Analogously as in Theorem 2 for $f > \varepsilon_2$ and $g > \varepsilon_2$. (iii) $\Rightarrow$ (iv). If $n = 1$, then (11) is trivially satisfied. Assume that $f$, $n$ and $\varepsilon_2$ satisfy the assumptions of (iv). For $n > 1$ we proceed like in Theorem 2. If $n = 0$, then (iv) reduces to $T(1) = 0$, which follows from (iii). If $n = -1$, then for $1/\varepsilon_2 > f > \varepsilon_2$ we have
\[
0 = T(1) = T \left( f \cdot \frac{1}{f} \right) = \frac{1}{f} \cdot T(f) + f \cdot T \left( \frac{1}{f} \right).
\]

Assume that $n < -1$. By downward induction, one can check that for $f^{n+1} > \varepsilon_2$ we have from (3)
\[
T(f^n) = T \left( f^{n+1} \cdot \frac{1}{f} \right) = f^{n+1} \cdot T \left( \frac{1}{f} \right) + \frac{1}{f} \cdot T(f^{n+1})
\]
\[
= -f^{n+1} \cdot f^{-2}T(f) + \frac{n+1}{f} \cdot f^n \cdot T(f) = nf^{n-1}T(f).
\]

(iv) $\Rightarrow$ (i). Take $n = -1$. $\square$

If we assume additionally that interval $I$ is compact, then the situation clarifies considerably.

**Theorem 4.** Assume that $I$ is compact and $T: C^k(I) \rightarrow C(I)$ is an additive operator. Then, the following conditions are pairwise equivalent:

(i) $T$ satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f,g \in C^k(I)$,

(ii) $T$ satisfies $T(f \cdot g) = f \cdot T(g) + g \cdot T(f)$ for all $f,g \in C^k(I), f > 0, g > 0$,

(iii) $T$ satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in C^k(I)$,

(iv) $T$ satisfies $T(f^2) = 2f \cdot T(f)$ for all $f \in C^k(I), f > 0$,

(v) $T$ satisfies $T(f) = -f^2 \cdot T \left( \frac{1}{f} \right)$ for all $f \in C^k(I), f > 0$,

(vi) $T$ satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in C^k(I)$ and $n \in \mathbb{N}$,

(vii) $T$ satisfies $T(f^n) = nf^{n-1} \cdot T(f)$ for all $f \in C^k(I), f > 0$ and $n \in \mathbb{N}$.

**Proof.** This statement is a consequence of Corollary 1 and Theorem 3. Since $I$ is compact, $f$ attains its global extrema. Thus, we will find some rational $r,q \in \mathbb{Q}$ such that $1/2 < rf + q < 2$. Moreover, as it was already observed in the proof of Theorem 3, each of the conditions of Theorem 4 implies that
$T(1) = 0$ and then $T$ vanishes on constant functions equal to a rational number. Consequently, we have $T(r f + q) = r T(f) + T(q) = r T(f)$ and therefore Theorem 3 applies to the conditions (ii), (iv), (v) and (vii) with appropriately chosen $\varepsilon_1$ and $\varepsilon_2$. The remaining conditions are equivalent by Corollary 1. Therefore, we are done if we prove for example the implication $(iv) \Rightarrow (iii)$.

Fix $f \in C^k(I)$ arbitrarily and choose $r, q \in \mathbb{Q}$ such that $1/2 < r f + q < 2$. By (iv) we get

$$T((r f + q)^2) = 2(r f + q)T(r f + q).$$

Then using additivity we obtain

$$r^2 T(f^2) + 2 r q T(f) + T(q^2) = 2 r^2 f T(f) + 2 r q T(f)$$

and after reduction

$$T(f^2) + 0 = 2 f T(f)$$

i.e. condition (iii).

One can join Corollary 1 and Theorem 4 with the mentioned result of H. König and V. Milman to obtain a corollary.

**Corollary 2.** Under the assumptions of Corollary 1 or Theorem 4, if $k > 0$, then each of the conditions listed there is equivalent to the following one:

(x) there exists some $d \in C(I)$ such that $T(f) = d \cdot f'$ for all $f \in C^k(I)$

and if $k = 0$, then $T = 0$ is the only additive operator that fulfils any of the equivalent conditions.

**Proof.** Consider $f(x) = x$ on $I$ and denote $\tilde{d} := T(f) \in C(I)$. Next, note that by [11, Theorem 3.1] the formulas (4) and (5), respectively hold on the interior of $I$ with some $c, d \in C(\text{int}I)$. The additivity of $T$ implies that $c = 0$. Therefore $\tilde{d}$ is a continuous extension of $d$ to the whole interval $I$. \qed

3. Final remarks

**Remark.** The inequalities between $f, g$ and constants $\varepsilon_1$ and $\varepsilon_2$ in Theorem 3 are not optimal. This however was not our goal since the role of this result is auxiliary only. Similarly, the inequality $f > 0$ in some of the conditions of Theorem 4 can be equivalently replaced by an estimate from above or from below by any other fixed constant.

Moreover, in the proof of Theorem 4 we showed more than is stated. Namely, it is equivalently enough to assume, instead of $f > 0$, that $f$ is bilaterally bounded by two rational numbers, like $1/2$ and $2$. However, since this generalization is only apparent and easy, we do not include it in the formulation of the theorem.
Example 1. Assume that \( \varphi : (1, \infty) \to \mathbb{R} \) is a smooth mapping that satisfies the equation
\[
\varphi(2x) = 2\varphi(x), \quad x \in (1, \infty).
\tag{13}
\]
Such mappings exist in abundance. In fact, every map \( \varphi_0 \) defined on \((1, 2]\) can be uniquely extended to a solution of (13). Next, let \( d : (e, \infty) \to \mathbb{R} \) be defined as
\[
d(x) = x \cdot \varphi(\ln x), \quad x \in (e, \infty).
\]
It is easy to see that
\[
d(x^2) = 2xd(x), \quad x \in (e, \infty)
\]
and
\[
d(xy) \neq xd(y) + yd(x)
\]
in general, unless \( \varphi \) is additive. Define \( T : C^1((e, \infty)) \to C((e, \infty)) \) as follows:
\[
T(f) = d \circ f, \quad f \in C^1((e, \infty)).
\]
One can see that \( T \) satisfies (9) for all \( f, g \in C((e, \infty)) \), but fails to satisfy the Leibniz rule (3). Thus, the assumption of additivity in all our results is essential. Observe also that \( T \) has the property that it vanishes on constant functions equal to a rational. This fact, as a consequence of additivity, was frequently used in the proofs of our Theorems 3 and 4. Therefore, the additivity assumption cannot be relaxed to this property.

Example 2. Assume that \( I \) is an interval and \( T \) is given by the formula
\[
T(f) = f'' - \frac{(f')^2}{f}, \quad f \in C^2(I), \; f > 0.
\]
Then \( T \) satisfies (3) for all \( f, g \in C^2(I) \) such that \( f > 0 \) and \( g > 0 \). This observation is a particular case of the second part of [11, Corollary 3.4]. Clearly, \( T \) is not additive. Moreover, \( T \) cannot be extended in such a way that it satisfies (3) on the whole space \( C^2(I) \).

The following examples show that if the domain of operator \( T \) is changed, then the conditions discussed in our results are no longer equivalent and various situations are possible.

Example 3. Let \( S \) be the space of all functions \( f \in C^1((0, \infty)) \) which satisfy the functional equation
\[
f(x + 1) = 2f(x), \quad x \in (0, \infty).
\tag{14}
\]
Note that \( S \) is not closed under multiplication. Moreover, each function \( f_0 : (0, 1] \to \mathbb{R} \) can be uniquely extended to a solution of (14). Therefore, \( S \)
is an infinite-dimensional subspace of $C^1((0, \infty))$. Define $T: C^1((0, \infty)) \to C^1((0, \infty))$ by the formula

$$T(f)(x) = f(x + 1), \quad f \in C^1((0, \infty)), \ x \in (0, \infty).$$

It is easy to check that $T$ is additive and satisfies (3) for $f, g \in S$. Thus, there are more solutions of (3) if the domain of $T$ is restricted to a particular subspace of $C^k(I)$.

**Example 4.** Let $P[x]$ be the space of all real polynomials of variable $x$. By $\text{deg}(f)$ we denote the degree of a polynomial $f \in P[x]$. Define $T: P[x] \to P[x]$ by

$$T(f)(x) = \text{deg}(f) \cdot f, \ f \in P[x].$$

Then $T$ is not additive, it satisfies (3) and there exists no extension of $T$ to the whole space $C^k(\mathbb{R})$ which is a solution of (3).

**Example 5.** Let $S := \{ f: (0, \infty) \to \mathbb{R} : f(x) = x^k \text{ for some } k \in \mathbb{Z} \text{ and } x \in (0, \infty) \}$. Note that $S$ is closed under multiplication but it is not a linear space. Next, let a double sequence $\varphi$ on $\mathbb{Z}$ of natural numbers be defined as follows: $\varphi(0) = 0$, $\varphi(k)$ is arbitrary but $\neq k$ if $k$ is odd, and if $k = 2^n \cdot m$ with some $n \in \mathbb{N}$ and odd $m \in \mathbb{Z}$, then

$$\varphi(k) := 2^{n^2-n} \cdot m^n \cdot \varphi(m).$$

Note that we have

$$\varphi(2k) = \varphi(2^{n+1} \cdot m) = 2^{n^2+n} \cdot m^{n+1} \cdot \varphi(m)$$

$$= 2^n \cdot m \cdot 2^{n^2-n} \cdot m^n \cdot \varphi(m) = k \cdot \varphi(k), \quad k \in \mathbb{Z}. \quad (15)$$

Define $T: S \to C((0, \infty))$ by

$$T(f)(x) := k \cdot x^{\varphi(k)}, \quad x \in (0, \infty) \quad (16)$$

if $f(x) = x^k$ for $x \in (0, \infty)$. One can see that if $f$ is of this form, then by (15)

$$T(f^2)(x) = 2k \cdot x^{\varphi(2k)} = 2k \cdot x^{k \cdot \varphi(k)} = 2f(x)T(f)(x)$$

for all $x \in (0, \infty)$, i.e. $T$ satisfies (9).

Moreover, one can see that (10) is equivalent to the equality

$$\varphi(k) - \varphi(-k) = 2k, \quad k \in \mathbb{Z}, \ k \neq 0.$$

Therefore, we can construct a sequence $\varphi$ such that $T$ defined by (16) satisfies (10) as well as another sequence $\varphi'$ for which $T$ does not satisfy (10). Finally, (3) is not true on $S$. Indeed, note that if (3) is satisfied by $T$ given by (16), then:
\[
\varphi(k + l) = \varphi(k) + l = \varphi(l) + k, \quad k, l \in \mathbb{Z}, k \neq 0, l \neq 0,
\]
which does not hold.

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**References**

[1] Amou, M.: Multiadditive functions satisfying certain functional equations. Aequ. Math. 93(2), 345–350 (2019)
[2] Ebanks, B.: Derivations and Leibniz differences on rings. Aequ. Math. 93(3), 629–640 (2019)
[3] Ebanks, B.: Derivations and Leibniz differences on rings: II. Aequ. Math. 93(6), 1127–1138 (2019)
[4] Ebanks, B.: Functional equations characterizing derivations and homomorphisms on fields. Results Math. 74(4), 1–12 (2019)
[5] Gselmann, E.: Notes on the characterization of derivations. Acta Sci. Math. (Szeged) 78(1–2), 137–145 (2012)
[6] Gselmann, E.: Characterizations of derivations. Diss. Math. 539, 65 (2019)
[7] Gselmann, E., Kiss, G., Vincze, C.: On functional equations characterizing derivations: methods and examples. Results Math. 73(2), 1–27 (2018)
[8] Jurkat, W.B.: On Cauchy’s functional equation. Proc. Am. Math. Soc. 16, 683–686 (1965)
[9] Kannappan, P., Kurepa, S.: Some relations between additive functions. I. Aequ. Math. 4, 163–175 (1970)
[10] Kannappan, P., Kurepa, S.: Some relations between additive functions. II. Aequ. Math. 6, 46–58 (1971)
[11] König, H., Milman, V.: Operator Relations Characterizing Derivatives. Birkhäuser, Cham (2018)
[12] Kuczma, M.: An introduction to the theory of functional equations and inequalities, 2nd edn. Birkhäuser Verlag, Basel. Cauchy’s equation and Jensen’s inequality; Edited and with a preface by Attila Gilányi (2009)
[13] Kurepa, S.: The Cauchy functional equation and scalar product in vector spaces. Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II 19, 23–36 (1964). (English, with Serbo-Croatian summary)
[14] Nishiyama, A., Horinouchi, S.: On a system of functional equations. Aequ. Math. 1, 1–5 (1968)

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