Recursive parameterisation and invariant phases of unitary matrices

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Abstract

We present further properties of a previously proposed recursive scheme for parameterisation of n-by-n unitary matrices. We show that the factors in the recursive formula may be introduced in any desired order. The method is used to study the invariant phases of unitary matrices. The case of four-by-four unitary matrices is investigated in detail. We also address the question of how to construct symmetric unitary matrices using the recursive approach.

1 Introduction

Unitary matrices play a central role in physics. For example, the Standard Model of particle physics is defined by a $SU(3) \times SU(2) \times U(1)$ symmetry group and many popular grand unified models are again based on unitary symmetries. Indeed explicit representations of unitary matrices are often so badly needed that there is already a vast literature on the subject (see, for example [1] and references therein).

Recently we have presented a simple-looking recursive parameterisation of general n-by-n unitary matrices [2], applicable also, of course, to subcategories such as special unitary matrices and orthogonal matrices which are of great importance in physics.

In recent publications Fujii and his collaborators [3] have found that the parameterisation in [2] looks interesting for constructing unitary gates for quantum computation but for that purpose more study is needed. In this paper we present further results on the structure of the recursive parameterisation hoping that it will be useful for future applications. You use the method to study the ”invariant phases” (to be defined below) of unitary matrices, by considering the symmetries of the recursive parameterisation. Subsequently, we give detailed attention to the case of four-by-four unitary matrices.

It should be emphasised that all parameterisation of a general n-by-n unitary matrix are equivalent to each another. However, for a specific application a certain parameterisation may be more convenient than others. Therefore, it is important to provide new parameterisations, a topic which has been addressed by other authors as well. See, for example, [4], a paper which contains an extended list of references and presents yet another representation of unitary matrices.

2 The parameterisation

A general n-by-n unitary matrix $X^{(n)}$ may be expressed as a product of three unitary matrices,

$$X^{(n)} = \Phi^{(n)}(\vec{\alpha})V^{(n)}\Phi^{(n)}(\vec{\beta})$$

(1)
where the matrices $\Phi$ are diagonal unitary matrices,

$$\Phi^{(n)}(\vec{\alpha}) = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_n})$$

(2)

$\Phi(\vec{\beta})$ is defined analogously, the $\alpha$’s and $\beta$’s being real. We shall refer to $\Phi$’s as external (pure phase) matrices.

The matrix $X^{(n)}$ has $n^2$ real parameters. The quantities $\vec{\alpha}$ and $\vec{\beta}$ take care of $2n - 1$ of these parameters because only the sums $\alpha_i + \beta_j$ enter, where $i$ and $j$ run from 1 to $n$. The remaining $(n - 1)^2$ real parameters reside in the non-trivial matrix $V^{(n)}$ which was the subject of the study presented in [2] and will be further investigated in this paper. For simplicity, whenever no confusion may arise, we refer to $V^{(n)}$ as the most general $n$-by-$n$ unitary matrix leaving out the qualifying statement that this is only true modulus the external matrices $\Phi(\vec{\alpha})$ and $\Phi(\vec{\beta})$. In [2] it was shown that the matrix $V^{(n)}$ may be written in the form

$$V^{(n)} = A_{n,2}A_{n,3} \ldots A_{n,n-1}A_{n,n}$$

(3)

where the $A_{n,k}$ are unitary matrices defined by

$$A_{n,k} = \begin{pmatrix}
A^{(k)} & 0 \\
0 & I_{n-k}
\end{pmatrix}$$

(4)

Here $I_{n-k}$ is the unit matrix of order $n - k$. For $k = n$ this unit matrix is absent. $A^{(k)}$ is a $k$-by-$k$ unitary matrix

$$A^{(k)} \equiv \begin{pmatrix}
1 - (1 - c_k)|A^{(k)}| & s_k|A^{(k)}| \\
-s_k < A^{(k)}| & c_k
\end{pmatrix}$$

(5)

Here $c_k$ and $s_k$ stand for cosine and sine of an angle denoted by $\theta_k$. Furthermore, $|A^{(k)}>$ is a $k - 1$ dimensional complex vector normalised to one,

$$|A^{(k)}> = \begin{pmatrix}
a_1^{(k)} \\
a_2^{(k)} \\
\vdots \\
a_{k-1}^{(k)}
\end{pmatrix}, \quad <A^{(k)}|A^{(k)}> = 1$$

(6)

and $(|A^{(k)}>< A^{(k)}|)_{ij} = a_i^{(k)}a_j^{(k)*}$. We shall refer to $|A^{(k)}>$ as the characteristic vector of order $k$.

The parameter counting was presented in [2] where it was shown that $V^{(n)}$, thus obtained, is the most general $n$-by-$n$ unitary matrix, again modulus the external matrices $\Phi$. The essential point is that $|A^{(k)}>$ introduces $2(k - 2)$ real parameters and not $2(k - 1)$, the reason being that it is normalised and its overall phase can be absorbed into the definition of the external matrices $\Phi$.

To summarise, in this recursive parameterisation the $n$-by-$n$ unitary matrix is represented by a product of $n-1$ unitary matrices, each with its own angle $\theta$ and characteristic vector $|A>$ while, for example, in the conventional approach in particle physics one would write the matrix as a product.

$^1$Note that the notations in this paper are a simplified version of that in [2]
of at least \( n(n-1)/2 \) matrices, these being Euler rotation matrices, \((n-1)(n-2)/2\) of them modified by phases (see, for example [5] and references therein). Note also that in Ref. [4] the matrix is parameterised by a product of \( n \) diagonal unitary matrices interlaced with \( n - 1 \) orthogonal matrices.

### 2.1 Reordering of the factors

At the first sight, the recursive parameterisation appears highly ordered and rigid. In Eq.\((3)\), the two-by-two structure is immediately followed by the three-by-three and so on. Actually, we may write these factors in any order we wish by observing that the order of factors in a given product, \( A_{n,r}A_{n,s} \), may be flipped as follows. For \( r < s \) we have

\[
A_{n,r}A_{n,s} = (A_{n,r}A_{n,s}A_{n,r}^\dagger)A_{n,r} \equiv A'_{n,s}A_{n,r}
\]

where \( A'_{n,s} \) has the same form as \( A_{n,s} \). The two characteristic vectors appearing in these matrices are related by a unitary rotation,

\[
|A'^{(s)}\rangle = \hat{A}^{(r)}|A^{(s)}\rangle
\]

\[
\hat{A}^{(r)} = \begin{pmatrix} A^{(r)} & 0 \\ 0 & I_{s-1-r} \end{pmatrix}
\]

For the case \( s < r \) we have

\[
A_{n,r}A_{n,s} = A_{n,s}(A_{n,s}^\dagger A_{n,r}A_{n,s}) \equiv A_{n,s}A''_{n,r}
\]

where now the two characteristic vectors are related by

\[
|A''^{(r)}\rangle = \hat{A}'^{(s)}|A^{(r)}\rangle
\]

Obviously, by inserting as many factors \( A_{n,j}^\dagger A_{n,j} = 1 \) as needed in the recursion formula, Eq.\((3)\), one may move the factors around as one wishes. The upshot is that in the reordering process a factor of lower rank simply "tunnels" through that of a higher rank without being affected but induces a unitary rotation of the characteristic vector of the latter. Thus the ensuing parameterisation remains the most general one. Note that the angles \( \theta_k \) remain invariant under reordering.

The recursive parameterisation looks highly asymmetric. However, using the above reordering procedure, one may construct manifestly symmetric unitary matrices (see Appendix).

### 2.2 Further properties of the recursive parameterisation

We wish to study, in more detail, the properties of the matrices \( A^{(k)} \) in Eq.\((5)\) as these are the building blocks of the recursive parameterisation.

To begin with we simplify the notation, to avoid indices, and introduce a generic matrix \( A \) defined by

\[
A \equiv \begin{pmatrix} 1 - (1 - c)|A > < A| & s|A > \\ -s < A| & c \end{pmatrix}
\]
Here, as usual, $c$ and $s$ stand for cosine and sine of an angle respectively. The angle itself will be denoted by $\theta$. Defining
\[ Y \equiv |A><A| \]  
we have that $Y$ is hermitian and satisfies
\[ Y|A> = |A>, \quad Y^2 = Y, \quad trY = 1, \quad detY = 0 \]  
where the vanishing of the determinant is, of course, only valid when $Y$ is a matrix and not just a number as is the case when $|A>$ is one dimensional. Following Fujii [3], we introduce a matrix $G$ which generates $A$,
\[ G \equiv \begin{pmatrix} 0 & -i|A> \\ i < A| & 0 \end{pmatrix} \]  
This matrix is hermitian and satisfies $G^3 = G$. A simple computation, using Eqs.(14), yields
\[ A = e^{i\theta G} = 1 + isG - (1 - c)G^2 \]  
This relation is reminiscent of the expansion of exponentials containing Pauli matrices $\sigma$ (in a short-hand notation, $e^{i\theta \sigma} = c + is\sigma$). The essential point here is that the series expansion of $e^{i\theta G}$, for arbitrarily $G$, terminates rapidly and does not continue for ever as the exponentials often tend to do. Furthermore we have
\[ trG = 0, \quad trG^2 = 2, \quad detA = 1 \]  
Note also that, for a fixed $G$, the matrix $A$ is Abelian with respect to $\theta$,
\[ A(\theta_i)A(\theta_j) = A(\theta_i + \theta_j) \]  
and
\[ A^{-1}(\theta) = A(-\theta) \]  
To rewrite the recursion formula, Eq.(3), in terms of $G$ and $A$ we must attach appropriate indices to our generic $G$ (or $A$) to distinguish the relevant factors. We introduce
\[ G_{n,k} = \begin{pmatrix} 0 & -i|A^{(k)}> \\ i < A^{(k)}| & 0 \end{pmatrix} \]  
where the required number of zeros have been added to make $G_{n,k}$ an n-by-n matrix. This yields that the factor $A_{n,k}$ in the recursion formula Eq.(3) is given by
\[ A_{n,k} \equiv e^{i\theta_k G_{n,k}} = 1 + is_kG_{n,k} - (1 - c_k)G_{n,k}^2 \]
3 Invariant phases of unitary matrices

The invariant phases of a unitary n-by-n matrix $V^n$ are defined as those phases of the matrix that cannot be "removed" with any choice of the external phase matrices $\Phi$ in Eq. (11). These phases play an important role in particle physics as they are measurable quantities related to CP violation (for a review see, for example, Ref. [6]).

Given a unitary matrix, the simplest way to detect the presence of invariant phases in it is to construct

$$\langle \alpha \beta; j k \rangle \equiv \text{Im}(V_{\alpha j} V_{\beta k} V_{*\alpha k} V_{*\beta j})$$

(22)

where we have suppressed the superscript $n$. The symbols $\alpha, \beta$ and $j, k$ now refer to rows and columns of the matrix and the indices are not summed. These imaginary parts are manifestly invariant under multiplication by the external phase matrices. Therefore if any of them is nonzero that would be a signal of the presence of a nonremovable phase in the matrix. We refer to these imaginary parts as invariant phases of the matrix instead of calling them invariants of the matrix that contain nonremovable phases. One may easily construct higher order invariants, containing properly chosen, six or more, elements of the matrix but these are in general reducible to the above set unless the matrix would have vanishing elements. For example, for $V_{\beta j} \neq 0$

$$\text{Im}(V_{\alpha j} V_{\beta k} V_{\gamma l} V_{\alpha k} V_{*\beta l} V_{*\gamma j}) = \frac{1}{|V_{\beta j}|^2} \text{Im} \left\{ (V_{\alpha j} V_{\beta k} V_{*\alpha k} V_{*\beta j} V_{\gamma l} V_{*\gamma j}) (V_{\beta j} V_{\gamma l} V_{*\beta l} V_{*\gamma j}) \right\}$$

(23)

$$= \frac{1}{|V_{\beta j}|^2} \{ \langle \alpha \beta, j k \rangle \langle \beta \gamma, j l \rangle + \langle \alpha \beta, j k \rangle \langle \beta \gamma, j l \rangle \}$$

(24)

where none of the indices is summed and

$$\langle \alpha \beta; j k \rangle \equiv \text{Re}[V_{\alpha j} V_{\beta k} V_{*\alpha k} V_{*\beta j}]$$

(25)

These real parts are also invariant under the action of the external matrices. The above reduction would not work if $V_{\alpha j} = 0$ but then the analysis is much simpler to begin with (see below) as the matrix contains fewer invariant phases.

Returning to the simplest invariant phases, there are altogether $[n(n-1)/2]^2$ quantities $\langle \alpha \beta; j k \rangle$, because these are antisymmetric under the interchange of the row indices, $\alpha \leftrightarrow \beta$, as well as under the interchange of the column indices, $j \leftrightarrow k$. However, we know that the most general $V^{(n)}$ has "only" $(n-1)(n-2))/2$ independent invariant phases. One may therefore look for $(n-1)(n-2))/2$ independent $\langle \alpha \beta; j k \rangle$’s and use them as a basis for expressing the remaining ones.

As mentioned above, the invariants in Eq. (22) play an essential role in the $n$-family version of the Standard Model of particle physics as they are measurable quantities related to CP-violation. For the case of $n = 3$ there is only one such quantity

$$\langle \alpha \beta; j k \rangle \equiv J \sum_{\gamma, i} \epsilon_{\gamma \alpha \beta} \epsilon_{ij k}$$

(26)

The row and column unitarity conditions for a three-by-three unitary matrix define six triangles. One may show that all these triangles have the same area and this unique area equals $J/2$. 


For \( n = 4 \) there are 36 possible invariants \((\alpha\beta; jk)\) but only three independent ones. In Eq. (3) an attempt was made to find an appropriate basis and carry through the above programme. The treatment of this issue is much simpler in the recursive parameterisation, as will be shown in the next section.

### 3.1 Invariant phases of four-by-four unitary matrices

For \( n = 4 \) we have from Eq. (3)

\[
V^{(4)} = A_{4,2} A_{4,3} A_{4,4}
\]

where each factor comes with its own \( \theta \) and characteristic vector \( |A>\). We denote the latter by

\[
|A^{(2)}> = 1, \quad |A^{(3)}> = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad |A^{(4)}> = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
\]

remembering that \( x \)'s and \( y \)'s are complex numbers and \( < A^{(k)}|A^{(k)}> = 1, \ k = 2, 3, 4 \). We shall now spell out this four-by-four matrix in order to exhibit its symmetries in a manifest fashion. This will also enable us to understand the general case of \( n \)-by-\( n \) matrices. Eq. (3) yields

\[
V^{(4)} = \left( \begin{array}{cccc}
  c_2 & s_2 & 0 & 0 \\
  -s_2 & c_2 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{array} \right) \left( \begin{array}{cccc}
  1 - (1 - c_3)x_1x_1^* & -(1 - c_3)x_1x_2^* & s_3x_1 & 0 \\
  -(1 - c_3)x_2x_1^* & 1 - (1 - c_3)x_2x_2^* & s_3x_2 & 0 \\
  -s_3x_1^* & -s_3x_2^* & c_3 & 0 \\
  0 & 0 & 0 & 1 \\
\end{array} \right)
\]

\[
\times \left( \begin{array}{cccc}
  1 - (1 - c_4)y_1y_1^* & -(1 - c_4)y_1y_2^* & -(1 - c_4)y_1y_3^* & s_4y_1 \\
  -(1 - c_4)y_2y_1^* & 1 - (1 - c_4)y_2y_2^* & -(1 - c_4)y_2y_3^* & s_4y_2 \\
  -(1 - c_4)y_3y_1^* & -(1 - c_4)y_3y_2^* & 1 - (1 - c_4)y_3y_3^* & s_4y_3 \\
  -s_4y_1^* & -s_4y_2^* & -s_4y_3^* & c_4 \\
\end{array} \right)
\]

We now focus on the symmetries of this matrix. By symmetries we mean transformations that leave \( V^{(4)} \) invariant modulus the external matrices \( \Phi \) in Eq. (1). A simple inspection shows that this matrix has two such symmetries, denoted by \( S_1 \) and \( S_2 \) and defined by

\[
S_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow e^{i\phi_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad y_3 \rightarrow e^{-i\phi_2}y_3
\]

\[
S_2 : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rightarrow e^{i\phi_3} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
\]

where \( \phi_2 \) and \( \phi_3 \) are arbitrary phases. The indices are to remind us of the dimension of the corresponding vector. Therefore the three independent phases in \( V^{(4)} \) can be chosen to be:

\[
\omega_1 = \phi(x_2) - \phi(x_1)
\]

\[
\omega_2 = \phi(y_2) - \phi(y_1)
\]

\[
\omega_3 = \phi(x_2) + \phi(y_3) - \phi(y_2)
\]
Here \( \phi(x_j) \) and \( \phi(y_k) \) denote the phases of the corresponding parameters. These phases are not invariant under the above symmetries and thus can’t appear as independent entities in computations of invariants of \( V^{(4)} \). Allowed to appear are the \( \omega \)’s or any combination of them because these are invariant under the action of both \( S_1 \) and \( S_2 \). We may, if we so wish, use the symmetry \( S_1 \) to rotate the phase of \( x_2 \) to zero, and then employ the symmetry \( S_2 \) to do the same with the ensuing \( y_3 \) whereby \( x_2 \) and \( y_3 \) may be taken to be real and say positive. The invariant phases in this "frame" are then \( \phi(x_1), \phi(y_1) \) and \( \phi(y_2) \). These constitute the maximum number of independent phases that \( V^{(4)} \) can possesses. By imposing further relations on the angles or the \( x \)’s and \( y \)’s this number could be smaller as shall be considered further below.

### 3.2 Generalisation to larger \( n \)

Going one order higher to \( n = 5 \), we have to introduce the relevant characteristic vector

\[
|A^{(5)}| \equiv \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{33}
\]

The corresponding matrix \( V^{(5)} \) has three symmetries given by

\[
S_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to e^{i\phi_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad y_3 \to e^{-i\phi_2} y_3, \quad z_3 \to e^{-i\phi_2} z_3 \tag{34}
\]

\[
S_2 : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \to e^{i\phi_3} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad z_4 \to e^{-i\phi_3} \tag{35}
\]

\[
S_3 : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \to e^{i\phi_4} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{36}
\]

In this case there are six invariant phases. These may be chosen as

\[
\begin{align*}
\omega_1 &= \phi(x_2) - \phi(x_1) \\
\omega_2 &= \phi(y_2) - \phi(y_1) \\
\omega_3 &= \phi(z_2) - \phi(z_1) \\
\omega_4 &= \phi(x_2) + \phi(y_3) - \phi(y_2) \\
\omega_5 &= \phi(x_2) + \phi(z_3) - \phi(z_2) \\
\omega_6 &= \phi(y_3) + \phi(z_4) - \phi(z_3) \tag{37}
\end{align*}
\]

As before, we may use \( S_1 \) to remove the phase of \( x_2 \) followed by \( S_2 \) and \( S_3 \) to rotate the ensuing \( y_3 \) and \( z_4 \) to be real and say positive. The invariant phases are then the phases of \( x_1, y_1, y_2, z_1, z_2 \) and \( z_3 \). Note that, from the very beginning we chose the angle \( \theta_2 \) not to be accompanied
by a phase, i.e., $|A^{(2)}| = 1$. One can, of course, leave the phase of $|A^{(2)}|$ arbitrary. This will introduce an extra symmetry which we have not bothered to write down as it is trivial.

The above procedure may be generalised to arbitrary order $n$. Without loss of generality, we may take, for example, the last component of all the characteristic vectors $|A^{(k)}|$ to be real. The invariant phases are then the phases of the remaining components. For an $n$-by-$n$ matrix there are then $1 + 2 + .. + (n-2) = (n-1)(n-2)/2$ such independent phases as expected.

3.3 The ”panel” approach to invariant phases

Another approach to constructing the invariant phases of a matrix is to consider the latter as a lattice. For the case of $n = 4$, considered from now on, the matrix can be visualised as shown

\begin{equation}
\begin{array}{ccc}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23} \\
P_{31} & P_{32} & P_{33} \\
\end{array}
\end{equation}

The bullets denote the sites where the matrix elements are situated. For example the bullets on the first row stand for $V_{11}$, $V_{12}$, $V_{13}$ and $V_{14}$ and so on. The $P$'s denote minipanels of the matrix, to be described here below. The invariants of interest to us are again

\begin{align}
(\alpha\beta; jk) & \equiv Im[V_{\alpha j}V_{\beta k}V_{\alpha k}^*V_{\beta j}^*] \\
\langle \alpha\beta; jk \rangle & \equiv Re[V_{\alpha j}V_{\beta k}V_{\alpha k}^*V_{\beta j}^*]
\end{align}

As mentioned before, there are 36 quantities $(\alpha\beta; jk)$ (six possible combinations of $\alpha$, $\beta$ multiplied by as many combinations of $j$, $k$) and we are looking for a set of three of them such that all the others can be expressed as functions of them and the real parts $\langle \alpha\beta; jk \rangle$. This problem was treated long time ago and was found to be rather involved. Here we provide some simplification. Nine of these invariants (the nearest neighbours) are explicitly exhibited on our lattice. Their analytic form may easily be read off from their locations. For example,

\begin{align*}
P_{11} & \equiv V_{11}V_{22}V_{12}^*V_{21}^* \\
P_{12} & \equiv V_{12}V_{23}V_{13}^*V_{22}^* \\
P_{22} & \equiv V_{22}V_{33}V_{23}^*V_{32}^* \\
P_{32} & \equiv V_{32}V_{43}V_{33}^*V_{42}^*
\end{align*}

and so on. Furthermore

\begin{equation}
P_{ab} = R_{ab} + iJ_{ab}
\end{equation}

where $R$ and $J$ denote the real and imaginary parts of the corresponding $P$. Thus the imaginary parts, in the notation employed earlier, are given by

\begin{align*}
J_{11} = (12; 12), & \quad J_{12} = (12; 23) & \quad J_{22} = (23; 23) & \quad J_{32} = (34; 23)
\end{align*}
and so forth. Similar expressions may be written down for the real parts.

Suppose that none of the matrix elements vanishes. Using unitarity conditions we then find

\[
\begin{align*}
J_{13} - (1 + \frac{R_{11}}{|V_{12}V_{22}|^2})J_{12} &= \frac{R_{12}}{|V_{12}V_{22}|^2}J_{11} \\
J_{13} - (1 + \frac{R_{33}}{|V_{33}V_{34}|^2})J_{23} &= \frac{R_{33}}{|V_{33}V_{34}|^2}J_{33} \\
J_{31} - (1 + \frac{R_{11}}{|V_{21}V_{22}|^2})J_{21} &= \frac{R_{21}}{|V_{21}V_{22}|^2}J_{11} \\
J_{31} - (1 + \frac{R_{33}}{|V_{33}V_{43}|^2})J_{32} &= \frac{R_{32}}{|V_{33}V_{43}|^2}J_{33} \\
J_{12} - \frac{R_{22}}{|V_{32}V_{33}|^2}J_{32} &= (1 + \frac{R_{32}}{|V_{32}V_{33}|^2})J_{22} \\
J_{21} - \frac{R_{22}}{|V_{23}V_{33}|^2}J_{23} &= (1 + \frac{R_{23}}{|V_{23}V_{33}|^2})J_{22}
\end{align*}
\]

In principle, we may take the set \(J_{11}, J_{22}, J_{33}\) to constitute a basis and determine the remaining six \(J\)'s in terms of them. These equations are rather complicated and need further thought concerning special cases. For example, if the matrix is symmetric we only have three equations but, of course, also only three unknown, say \(J_{12}, J_{13}\) and \(J_{23}\). We are not allowed to divide by vanishing matrix elements and so forth. Here below, we shall consider a simple and yet nontrivial example to demonstrate the technique and to compare it with the recursive approach which is much simpler and does not require thinking about the possible pitfalls.

### 3.4 A simple example

For simplicity we consider the case where two of the elements of the four-by-four matrix are zero, and where these elements are neither on the same row nor on the same column. All other elements of the matrix are assumed to be nonzero. Without loss of generality we may take the two vanishing elements to be \(V_{14}\) and \(V_{41}\) which in particle physics would correspond to the case where the mixing of the first and the fourth families is negligible. Our lattice, with its nine minipanels, now looks as follows

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \odot \\
J & J & 0 \\
\bullet & \bullet & \bullet & \bullet \\
J & J+J' & J' \\
\odot & 0 & J' & J' \\
\end{array}
\]

Here the \(\odot\)'s indicate where the vanishing matrix elements are situated and we have defined

\[
\begin{align*}
J &= J_{11} \equiv (12, 12), & J' &= J_{33} \equiv (34, 34)
\end{align*}
\]
A simple computation, using unitarity relations, gives the imaginary parts of the minipanels as marked in the lattice. Computing all the imaginary parts, we find that 19 of the 36 invariants \((\alpha\beta, jk)\) vanish. The nonvanishing ones, in addition to \(J\) and \(J'\) defined in Eq.\((45)\), are

\[
- (12, 13) = (12, 23) = -(13, 12) = (13, 13) = -(13, 23) = (23, 12) = -(23, 13) = J \\
-(23, 24) = (23, 34) = -(24, 23) = (24, 24) = -(24, 34) = (34, 23) = -(34, 24) = J' \quad (46)
\]

and

\[(23, 23) = J + J' \quad (47)\]

as exhibited in the corresponding panel. Moreover, we find

\[
\frac{J'^2}{J^2} = \frac{|V_{24}V_{34}|^2}{|V_{21}V_{31}|^2} = \frac{|V_{42}V_{43}|^2}{|V_{12}V_{13}|^2} \\
= \left( \frac{|V_{24}|^2 + |V_{34}|^2}{|V_{12}|^2 + |V_{13}|^2} \right)^2 = \left( \frac{|V_{42}|^2 + |V_{43}|^2}{|V_{21}|^2 + |V_{31}|^2} \right)^2 \quad (48)
\]

It is amusing to note that the unitarity relations for the above matrix define eight triangles. Using the method in \([7]\) one finds that four of these have each an area equal to \(J/2\) while the area of the other four is \(J'/2\).

We would now like to compute \(J\) and \(J'\). For this purpose we turn to the recursive parameterisation. It turns out that the calculations are simpler is we take \(V_{34} = V_{43} = 0\) instead of the above choice \(V_{14} = V_{41} = 0\). The two choices are equivalent as they are related to one another by interchanges in rows and columns. This amounts to a relabeling of the matrix elements which obviously can’t affect the results. After finishing the computations we can simply revert to the former case by interchanging rows one and three as well as columns one and three.

In the recursive parameterisation, Eq.\((29)\), the conditions \(V_{34} = V_{43} = 0\) give

\[
y_3 = x_1y_1^* + x_2y_2^* = 0 \quad (49)
\]

These condition tell us that \(|y_1| = |x_2|, |y_2| = |x_1|\) and that \(x_1\) and \(y_1\) are relatively real in the frame where \(x_2\) and \(y_2\) are taken to be real. Therefore, there is only one invariant phase, in this example. We introduce the lattice again

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\hat{J} & \hat{J} & 0 \\
\cdot & \cdot & \cdot & \cdot \\
\hat{J} & \hat{J} & 0 \\
\cdot & \cdot & \cdot & \circ \\
0 & 0 & 0 \\
\cdot & \cdot & \circ & \\
\end{array} \quad (50)
\]

\(\circ\)'s indicate where the vanishing matrix elements are situated. Furthermore, we have exhibited the imaginary parts of the minipanels starting with the definition

\[
\hat{J} = (12, 12) \quad (51)
\]
Taking into account the permutations, we find that $\hat{J}$ here is identical with our previous $J$ that we wanted to compute. Using the recursive parameterisation we find

\begin{align*}
J &= c_2c_3s_2s_3^2 Im(x_1^*x_2) \\
J' &= -c_2c_3s_2s_3^2 Im(x_1^*x_2)
\end{align*}

Thus

\begin{equation}
\frac{J'}{J} = -\frac{s_3^2}{s_4^2}
\end{equation}

For comparison, note that for a general three-by-three matrix (which we may obtain from Eq. (23) by putting $\theta_4 = 0$) the unique invariant is given by $J^{(3-fam)} = c_2c_3s_2s_3^2 Im(x_1^*x_2)$.

The recursive parameterisation allows us to compute all the imaginary parts for the most general case, i.e., irrespectively of whether the matrix has zeros or not. We find, for the general four-by-four matrix, parameterised as in Eq. (29)

\begin{align*}
(34,34) &= c_3c_4s_3s_4^2 |y_3| \{ |x_2y_2| sin(\omega_3) + |x_1y_1| sin(\omega_3 + \omega_2 - \omega_1) \} \\
(34,24) &= c_4s_3s_4^2 |x_2y_2| \{ s_3|x_1y_1| sin(\omega_3 - \omega_2) - c_3|y_3| sin\omega_3 \}
\end{align*}

where the angles $\omega_j$ are as defined in Eq. (32). We thus see that, as expected, only the invariant phases appear in these relations. We do not quote the remaining imaginary parts $(\alpha\beta, jk)$. The important point is that all of them are functions of $|x_j|$, $|y_j|$ and the three $\omega$’s, as expected.

### 4 Conclusions

In this paper, we have presented further properties of the recursive parameterisation of unitary matrices proposed in [2], where the matrix is written as a product of $n - 1$ matrices each with its own angle $\theta$ and characteristic vector $|A \rangle$. We have found that the factors in the recursive formula may be introduced in any desired order.

Encouraged by the convenience of the recursive method, we have taken a fresh look at the issue of invariant phases of unitary matrices. After having exhibited the symmetries of the parameterisation, we have shown how the invariant phases of $n$-by-$n$ matrices can be identified. Subsequently, we have paid particular attention to the case $n = 4$ and have compared the results with those of an earlier approach based on ”panels” of the matrix.

The recursive parameterisation has some really nice features because in some cases it allows the ”new physics” to be introduced in a gentle manner through the last factor in the recursion formula, a topic which we are currently studying.

In an earlier study [9], we found that there is a parameterisation that allows one to introduce, in a simple way, any desired angle of any of the so called unitarity triangles as one of the parameters in the quark mixing matrix for three families in the Standard Model of particle physics. The same parameterisation allowed us to choose the expansion parameter in this matrix to be $\lambda^2$ instead of $\lambda$ that one usually uses [5]. Indeed $\lambda$ is not so small ($\lambda = 0.2$). Therefore, the recursive parameterisation may be convenient whenever expansion in the above parameter is required, for example in model building or for construction of quark and lepton mass matrices. It turned out
that the parameterisation found in [9], with the above nice features, is indeed nothing but the order \( n = 3 \) version of the recursive parameterisation discussed in this paper and in [2].

Finally, in the appendix of this paper, we deal with the question of how to construct manifestly symmetric unitary matrices in the recursive framework.

5 Appendix: Symmetric unitary matrices

We write the symmetric unitary matrix in the form

\[
X^{\text{sym}}(n) = \Phi^{(n)}(\vec{a}) V^{\text{sym}}(n) \Phi^{(n)}(\vec{a})
\]

requiring \( V^{\text{sym}}(n) \) to be symmetric, as indicated by its superscript, and that the external matrices be the same (see Eq. (1)). A general symmetric unitary matrix has \( n(n + 1)/2 \) real parameters. The external matrix \( \Phi \) takes care of \( n \) of them. Thus \( n(n - 1)/2 \) real parameters reside in \( V^{\text{sym}}(n) \).

For a general \( V^{\text{sym}}(n) \), we have (see Section 2.2) that the factors in the recursion formula Eq. (3) may be written as

\[
A_{n,k} = e^{i\theta_k G_{n,k}}
\]

where the generating matrix \( G_{n,k} \) is hermitian. In order to obtain a symmetric \( A_{n,k} \) we must impose the additional requirement that the generating matrix be symmetric. This means that the corresponding characteristic vector is purely imaginary. For example, for \( k = 2 \) we obtain

\[
V^{\text{sym}}(2) = A_{n,2}^{\text{sym}} = \begin{pmatrix}
  c_2 & is_2 & 0 \\
  is_2 & c_2 & 0 \\
  0 & 0 & I_{n-2}
\end{pmatrix}
\]

and for \( k = 3 \)

\[
A_{n,3}^{\text{sym}} = \begin{pmatrix}
  1 - (1 - c_3)x_1^2 & -(1 - c_3)x_1x_2 & is_3x_1 & 0 \\
  -(1 - c_3)x_1x_2 & 1 - (1 - c_3)x_2^2 & is_3x_2 & 0 \\
  is_3x_1 & is_3x_2 & c_3 & 0 \\
  0 & 0 & 0 & I_{n-3}
\end{pmatrix}
\]

We have put \( |A| = i|x> \), where the \( x \)’s are real. Thus, the construction of symmetric factors in the recursion formula is a trivial task. But, of course, the product of these factors will not be symmetric. This defect is easily remedied by invoking the reordering procedure described in Section 2.1 in which we showed that the reordering of the factors in the recursion formula only amounts to a redefinition of the characteristic vectors. Therefore, we may write

\[
V^{\text{sym}}(n) = A_{n,2}^{\text{sym}} A_{n,3}^{\text{sym}} ... A_{n,n-1}^{\text{sym}} A_{n,n}^{\text{sym}} A_{n,n-1}^{\text{sym}} ... A_{n,3}^{\text{sym}} A_{n,2}^{\text{sym}}
\]

\( V^{\text{sym}}(n) \) thus obtained is manifestly unitary and symmetric. We must now count the number of its independent parameters. Each order \( k \) introduces \( k - 1 \) real parameters, these being the angle \( \theta_k \) and \( k - 2 \) components of the corresponding characteristic vector (one component being redundant because the vector is normalised). Therefore the total number of parameters in \( V^{\text{sym}}(n) \) is

\[
\sum_{k=2}^{n} (k - 1) = n(n - 1)/2
\]
as expected. Adding into this number the $n$ parameters coming from the external matrices amounts to the total of $n(n + 1)/2$ real parameters, as required.

Note that it would be somewhat more elegant to call the angles in the above factors $\theta_k/2$ instead of $\theta_k$, except the angle $\theta_n$ of the factor $A_{n,n}^{sym}$. The reason being that $A_{n,n}^{sym}$ appears only once while the others appear twice.

The chain in Eq. (39) looks long and perhaps a bit frightening. However, if needed for practical applications, it can be somewhat simplified as we shall now describe.

Consider the case $n = 3$, where we introduce

$$V^{(3)\text{sym}} = V^{(2)\text{sym}}(\theta_2/2)A^{(3)\text{sym}}V^{(2)\text{sym}}(\theta_2/2)$$  \hspace{1cm} (61)

where $V^{(2)\text{sym}}$ is as defined in Eq. (57), for $n = 3$. Multiplying the factors, we find

$$V^{(3)\text{sym}} = \begin{pmatrix}
    c_2 - (1 - c_3)u_1^2 & is_2 - (1 - c_3)u_1u_2 & is_3u_1 \\
    is_2 - (1 - c_3)u_1u_2 & c_2 - (1 - c_3)u_2^2 & is_3u_2 \\
    is_3u_1 & is_3u_2 & c_3
\end{pmatrix}$$ \hspace{1cm} (62)

Here

$$u_1 = c'_2x_1 + is'_2x_2, \quad u_2 = c'_2x_2 + is'_2x_1$$ \hspace{1cm} (63)

$c'_2 = \cos(\theta_2/2)$ and $s'_2 = \sin(\theta_2/2)$. Moreover, the $x$'s are as introduced in Eq. (58).

The essential point is that we may put aside the question of the origin of the $u$'s and their relationship with the $x$'s and simply consider them as our new variables, as two complex numbers that satisfy

$$|u_1|^2 + |u_2|^2 = 1$$ \hspace{1cm} (64)

Going to the next order, $n = 4$, we may use the identity

$$A_2^{sym}A_3^{sym}A_4^{sym}A_3^{sym}A_2^{sym} = A_2^{sym}A_3^{sym}A_2^{sym}[(A_2^{sym})^{-1}A_4^{sym}(A_2^{sym})^{-1}]A_2^{sym}A_3^{sym}A_2^{sym}$$

$$A_4^{sym} \equiv (A_2^{sym})^{-1}A_4^{sym}(A_2^{sym})^{-1}$$ \hspace{1cm} (65)

Here $V^{(3)\text{sym}}$ is as found in Eq. (32). From our earlier results, we have

$$A_4^{sym} = \begin{pmatrix}
    1 - (1 - c_4)y_1^2 & -(1 - c_4)y_1y_2 & -(1 - c_4)y_1y_3 & is_4y_1 \\
    -(1 - c_4)y_1y_2 & 1 - (1 - c_4)y_2^2 & -(1 - c_4)y_2y_3 & is_4y_2 \\
    -(1 - c_4)y_1y_3 & -(1 - c_4)y_2y_3 & 1 - (1 - c_4)y_3^2 & is_4y_3 \\
    is_4y_1 & is_4y_2 & is_4y_3 & c_4
\end{pmatrix}$$ \hspace{1cm} (66)

where $y$'s are real. Therefore, we may immediately write down the factor $A_4^{sym}$, without having to do any calculations. The first two components of the vector $y$ get "rotated" but $y_3$ is untouched. We find

$$A_4^{sym} = \begin{pmatrix}
    c_2 - (1 - c_4)v_1^2 & -is_2 - (1 - c_4)v_1v_2 & -(1 - c_4)v_1v_3 & is_4v_1 \\
    -is_2 - (1 - c_4)v_1v_2 & c_2 - (1 - c_4)v_2^2 & -(1 - c_4)v_2v_3 & is_4v_2 \\
    -(1 - c_4)v_1v_3 & -(1 - c_4)v_2v_3 & 1 - (1 - c_4)v_3^2 & is_4v_3 \\
    is_4v_1 & is_4v_2 & is_4v_3 & c_4
\end{pmatrix}$$ \hspace{1cm} (67)
where
\[ v_1 = c'_2 y_1 - i s'_2 y_2, \quad v_2 = c'_2 y_2 - i s'_2 y_1, \quad v_3 = y_3 \]
(68)
Again the vector \( v \) has unit norm and we may, as before, forget about the \( y \)'s and just use \( v \)'s, keeping in mind that \( v_1 \) and \( v_2 \) are complex numbers.

As a final example, we wish to compute the quantity \( J \) for the case of a three-by-three symmetric matrix, where
\[ (\alpha\beta; jk) \equiv J \sum_{\gamma, i} \epsilon_{\gamma\alpha\beta} \epsilon_{ijk} \]
(69)
and \( V^{(3)}_{\text{sym}} \) is as given in Eq.(62). A glance at this matrix yields
\[ J = c_2 c_3 s_3^2 Im(u_2)^2 = c_2 c_3 s_2 s_3^2 x_1 x_2 \]
(70)
This resembles our earlier result in Section 3.4 where we found \( J^{(3-fam)} = c_2 c_3 s_2 s_3^2 Im(x_1^* x_2) \). The meaning of the \( x \)'s in the two cases are, of course, different. Note that a general \( V^{(3)} \) has four parameters while \( V^{(3)}_{\text{sym}} \) has one less. Equation (70) is telling us that the symmetry requirement does not remove the invariant phase of the matrix. In the language of Euler rotations, where \( V^{(3)} \) would be parameterised with three rotation angles and one phase, the requirement that the matrix be symmetric keeps the phase but removes one of the rotation angles. Note that the new phase and angles will be functions of the former phase and angles.

References

[1] R. Gilmore, ’’Lie groups, Lie algebras, and some of their applications’’ (Wiley-Interscience, 1974);
H. Georgi, ’’Lie algebras in particle physics: from isospin to unified theories’’ (Reading, Mass., 1982)
[2] C. Jarlskog, math-ph/0504049, to be published in Jour. Math. Phys.
[3] K. Fujii, math-ph/0505047.
K. Fujii, K. Funahashi and T. Kobayashi quant-ph/0508006
[4] P. Dita, J. Phys. A: Math. Gen. 36 (2003) 2781; math-ph/0103005
[5] Review of Particle Physics, S. Eidelman et al., Phys. Lett B592 (2004) 1
[6] C. Jarlskog in CP Violation, Ed. C. Jarlskog (World Scientific, 1989) p. 3
[7] C. Jarlskog and R. Stora, Phys. Lett. B208 (1988) 268
[8] C. Jarlskog, Phys. Rev. D36 (1987) 2128
[9] C. Jarlskog, Phys.Lett. B615 (2005) 207-212 (hep-ph/0503199); see also C. Jarlskog, hep-ph/0504012