Fermi gases with generalized Rashba spin orbit coupling induced by a synthetic gauge field have the potential of realizing many interesting states such as rashbon condensates and topological phases. Here we develop a fluctuation theory of such systems and demonstrate that beyond-Gaussian effects are essential to capture the physics of such systems. We obtain their phase diagram by constructing an approximate non-Gaussian theory. We conclusively establish that spin-orbit coupling can enhance the exponentially small transition temperature \( T_c \) of a weakly attracting superfluid to the order of Fermi temperature, paving a pathway towards high \( T_c \) superfluids.

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Construction and study of model Hamiltonians with quantum gases has opened up the possibility of not only addressing long standing questions\[1\] but also creating systems that are not conventional. The recent advances in synthetic gauge fields\[2\],\[12\] have provided new impetus, motivating studies of interacting bosons and fermions in their presence that are of interest to a wide array of physicists (see review Ref.\[13\]).

A uniform non-Abelian gauge field results in a generalized Rashba spin orbit coupling (RSOC). Interacting fermions with RSOC have many interesting and novel features,\[14\],\[17\] while additional Zeeman fields can help realize topological states.\[15\],\[21\] Even in the presence of weak attractive interactions, a crossover from a BCS type superfluid to a rashbon-BEC can be achieved by increasing the strength of RSOC.\[14\],\[15\] Rashbon-BEC is a condensate of rashbons – a bosonic bound state of a fermion pair in presence of large RSOC – whose mass (between 2-2.5 fermion mass) determines its transition temperature which is of the order of the Fermi temperature. Indeed early studies\[16\],\[21\] suggest this enhancement of the transition temperatures of weakly attractive systems by \( t_\ell \approx \frac{\hbar^2}{m\kappa^2} \) as “free vacuum”.

A finite density \( \rho_0 \) of fermions determines a characteristic momentum scale \( k_F \) defined by \( \rho_0 = \frac{k_F^3}{3\pi^2} \) and an associated energy/temperature scale \( \gamma_F = \frac{k_F^2}{T_F} \). The singlet interaction (bare strength \( v \)) between the fermions is characterized by a scattering length \( a_\sigma \). Physics at temperature \( T \) and chemical potential \( \mu \) with volume \( V \) is studied using functional integral methods. After introducing pairing fields \( \eta(q) , \sqrt{F_G} = (iq, q) \) (boson–Bose-Matsubara frequency, \( q \) – wave vector) and integrating out the fermions, the action

\[
S[\eta , F] = -\ln \det[-G^{-1} - \frac{1}{v} \sum_q \eta^*(q) \eta(q)] - \frac{1}{\gamma} \sum_q \eta^*(q) \eta(q)(q) - \frac{1}{2} \sum_{1,2,3,4} \gamma^*(q_1) \gamma^*(q_2) K(q_1, q_2; q_3, q_4) \gamma(q_3) \gamma(q_4)
\]

is obtained, where \( F \) is a source field and \( G \) is the Greens function (functional of \( \eta \) and \( F \) ).\[23\] To study the normal state physics, we expand the exact action Eq.\[1\] about the saddle point where \( \eta(q) = 0 \).

\[
S \approx -\ln \det[-G_{0}^{-1} - \frac{1}{v} \sum_q \eta^*(q) \eta(q) + \sum_q \gamma^*(q) L(q) \gamma(q) + \sum_{1,2,3,4} \gamma^*(q_1) \gamma^*(q_2) K(q_1, q_2; q_3, q_4) \gamma(q_3) \gamma(q_4)]
\]

up to quartic order \( \gamma = \eta + F \); \( G_0 \) is the non-interacting Greens function. The quantities \( L \) and \( K \) are derivatives
of the action (Eq. (1)) to appropriate order in $\eta$. They are constrained by conservation laws, e.g., the arguments of $K$ have to satisfy momentum conservation.

**Gaussian Fluctuation Theory:** Retaining only the first three terms in Eq. (2) produces the Gaussian fluctuation theory\cite{21,26}, quadratic in $\eta$. Upon integration of the $\eta$ fields, we obtain

$$S[\eta] = \ln \det[-G_0^{-1}] + \sum_q \ln M(q) - \sum_q F^*(q)\chi(q)F(q),$$

where $M(q) = L(q) - \frac{1}{T} = L(q) + \frac{1}{\pi} \sum_k \frac{1}{|k|^2} - \frac{1}{4\pi\sigma_x}, L(q) = \frac{1}{\pi} \sum_k,_{\alpha,\beta} |A_{\alpha\beta}(q, k)|^2 \ln \left[\frac{x/T, \eta}{\xi} \left(\frac{2k + \beta}{2}\right)^{\alpha} \left(\frac{2k - \alpha}{2}\right)^{\beta}\right],$$

with $n_F(x) = 1/(e^{x/T} + 1)$ denoting the Fermi function, $A_{\alpha\beta}$ is the amplitude of the singlet in the two particle-state with momenta $q/2 \pm k$ and helicities $\alpha$ and $\beta$, and $\xi \equiv \varepsilon - \mu$. The analysis also produces the pairing susceptibility $\chi(q) = L(q) \left(\frac{L(q)}{2\pi^2} - 1\right)$, whose divergence from the positive side up on the reduction of temperature indicates a pairing instability. The first such divergence of $\chi(0, q)$ occurs at $q = 0$ (as we have verified), i.e., the system is most susceptible to homogeneous pairing. $T_c$ is then obtained via (the Thouless criterion\cite{20})

$$-\frac{1}{\pi\sigma_x} - \frac{1}{\pi} \sum_{k, \alpha} \left(\frac{1 - 2n_F(\xi k, \alpha)}{\xi k}\right) = 0.$$  

The equation of state of the system is determined from Eq. (3) as

$$\rho(T, \mu) = \frac{1}{\pi} \sum_{k, \alpha} n_F(\xi k, \alpha)$$

$$+ \frac{1}{\pi} \sum_q \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega n_B(\omega) \frac{\partial \arg(M^{+}(q, \omega))}{\partial \mu},$$

where $\arg(z)$ is the argument of the c-number $z$, $n_B(x) = 1/(e^{x/T} - 1)$ is the Bose function, and $M^{+}(q, \omega)$ is the analytic continuation of $M^{+}(i\omega \rightarrow z, q)$ evaluated just above the real axis in the $z$-plane, $\mu$ at a given $T$ is then determined from the solution of the equation $\rho(T, \mu) = \rho_0$. $M(z, q)$ may have an isolated zero below the scattering threshold $\omega_0(q)$ evaluated just above the real axis as $\lambda \rightarrow 0$. This suggests that the equilibrium state of the Gaussian theory with RSOC is not continuously connected to the free vacuum in the limit of $\lambda \rightarrow 0^+$ (see Fig. 1).

![Figure 1. (Color online) Dependence of chemical potential ($\mu$) on RSOC strength](image)

**Gaussian Fluctuation Theory:** The Gaussian theory, notably successful\cite{28} in the description of the interacting Fermi gas in free vacuum, has rather peculiar features in the presence of RSOC (non-Abelian gauge field) of the type $\lambda = (\lambda_x, \lambda_y, \lambda_z)$ where $\lambda_x \geq \lambda_y$. While we focus on the spherical gauge field, our discussion will be applicable to all such gauge fields.

Fig. 1 shows the dependence of $\mu$ on $\lambda$ at a fixed low $T$ ($T < T_F$) and negative $\alpha_x$. The remarkable feature is that the Gaussian theory has two solutions for $\mu$ for a given set of parameters. For $\lambda \ll k_F$, one of the solutions, called the “free vacuum branch” (FB) (see Fig. 1), is smoothly connected to that of the free vacuum found in previous works.\cite{25,24} The other solution always has $\mu < 0$, even for $\lambda \ll k_F$. For large $\lambda$, $\mu$ in this branch is determined by the threshold dispersion, and hence called the threshold branch (RB). Curiously $\mu$ along RB, which always has the lower free energy, approaches $0^-$ as $\lambda \rightarrow 0$. This suggests that the equilibrium state of the Gaussian theory with RSOC is not continuously connected to the free vacuum in the limit of $\lambda \rightarrow 0^+$ (see Fig. 1).

The physics of RB at small $\lambda$ can be traced to the contribution $\rho_b$ from the bound bosonic states to the total density (Eq. (5)). Fig. 2 shows the dispersion of such bosons as a function of their momentum $q$. Key points to be noted, whenever $\mu < 0$, are (i) even for negative scattering lengths, there are bound bosonic states in RB whenever $|q| < q_0 = \frac{2\pi\sigma_y}{\lambda_x}$, while they cease to exist at larger $|q|$. (ii) the binding energy of the bosons $E_b(q) - \omega_0(q) - \omega_0(q)$, even though significant for small $|q|$, is vanishingly small in the range $\frac{\lambda_x}{2} \lesssim |q| \lesssim q_0$. This physics is quite similar to what is found in the two-body problem.\cite{21} Such a bosonic dispersion can therefore accommodate a large number of particles forcing $\mu$ to be self-consistently negative. Although $E_b(q = 0) \rightarrow 0$ for vanishingly small $\lambda$ this phenomenon persists, resulting in RB not being smoothly connected to the free vacuum. Note also that FB does not have any contribution from $\rho_b$, since in this regime there is no bosonic bound state for $\mu > 0$. 

**Inadequacy of the Gaussian Theory:** The Gaussian theory, notably successful\cite{28} in the description of the interacting Fermi gas in free vacuum, has rather peculiar features in the presence of RSOC (non-Abelian gauge field) of the type $\lambda = (\lambda_x, \lambda_y, \lambda_z)$ where $\lambda_x \geq \lambda_y$. While
As is evident, the fully formed bound states in the range $\frac{2T}{V} \lesssim |q| \leq q_0$ with a vanishing binding energy can easily be destabilized. In particular, the quartic term in Eq. [2] with the coupling $K$ describes the interactions between the pairing fluctuations $\eta(q)$. A natural question is whether these weakly bound states are stable when the interactions between the bosonic fluctuations are taken into account. We address this question by quantifying the strength of these beyond-Gaussian effects by the parameter $b$ (proportional to $K(q_1, q_2; q_3, q_4)$ in the limit of zero momentum) [25, 27] obtained as

$$b = 1/4 \sum_{k, \alpha} \left( \frac{1 - 2n_F(\xi_{k, \alpha})}{\xi_{k, \alpha}^3} - \frac{2n_F(\xi_{k, \alpha}) (1 - n_F(\xi_{k, \alpha}))}{\xi_{k, \alpha}^2} \right).$$  

(6)

When $\mu$ is large and negative, as in a “boson dominated” state where the most prominent contribution arises from $\rho_0$, $b \approx \frac{\lambda^2 - 2\mu}{32\pi^2(\pi)^5/2}$. The physical meaning of $b$ can be made evident by noting that $b \sim a_{BB}^3$ when $\lambda = 0$ and the scattering length is small positive (free vacuum BEC side). Here $a_{BB}$ is the scattering length of two bosons (bound fermion pairs) and is proportional to $a_s$ [22]. Furthermore, for any $a_s$, as $\lambda \to \infty$, $b \to \lambda^{-3}$, which can be immediately identified with $a_{RR}^3$, where $a_{RR}$ is the rashbon-rashbon scattering length. [22] Therefore, $b^{1/3}$, is a length scale that characterizes the interactions among the pairing fields. Interestingly, this parameter is nonzero in the limit of $\lambda \to 0^+$ and grows with increasing $\lambda$ attaining a peak when $\lambda \approx k_F$ (see inset of Fig. 3), subsequently possessing the just discussed asymptotic behavior at large $\lambda$.

The effects of $b$ on the weakly bound states can now be estimated in a physical manner. The lowest order effect of $b$ would be to shift the energy of the bound state via a Hartree shift, i.e., $\omega_b(q) \to \omega_b(q) + \kappa b^{1/3} \rho_b(T, \mu)$ where $\kappa$ is a dimensionless number of order unity [24]. Clearly, the bound bosonic state will be unstable if the shift takes it into the scattering continuum, i.e., a necessary condition for the stability of the bound state is that

$$\omega_b(q) + \kappa b^{1/3} \rho_b(T, \mu) \leq \omega_0(q).$$

Thus RB can be stable only if

$$E_b(q) = 0 \geq \kappa b^{1/3} \rho_b(T, \mu).$$

Using this criterion we obtain the regime (see Fig. 3) where RB is eliminated by non-Gaussian effects i.e., $\lambda \lesssim \lambda_{ng}(T, a_s)$. For a given $a_s$, we find that $\lambda_{ng}$ increases with decreasing temperature [32]. These estimates provide a lower bound of $\lambda_{ng}$, which results in $\lambda_{ng} \approx k_F$ for temperatures $T \lesssim T_F$. Thus beyond-Gaussian effects are crucial in the most interesting regime of parameters.

**Approximate Non-Gaussian Theory:** Having firmly established that even a qualitatively correct description of spin-orbit coupled Fermi gases necessarily requires a beyond-Gaussian theory, we propose and discuss one such theory. A key desideratum of such a theory is the elimination of RB for $\lambda \lesssim \lambda_{ng}$, and a smooth evolution (at least $a_s$) from the free vacuum state at vanishing $\lambda$ to the rashbon gas at large $\lambda$. The implementation of such a theory is a formidable challenge, even as we note that Gaussian theory itself requires considerable calculational effort [33]. Faced with this reality, we develop an approximate non-Gaussian (ANG) theory by a suitable modification to the equation of state (Eq. [4]) that only entails the same calculational complexity as the Gaussian theory. The approximation follows the physical argument that the non-Gaussian term $b$ shifts $\omega_b(q)$ to $\omega_b(q) + \kappa b^{1/3} \rho_b^{GS}$ where $\rho_b^{GS}$ is the bound boson contribution calculated within the Gaussian approximation. Only those bosonic states that remain below the scattering continuum after this energy shift, i.e., the bosonic states for all $|q| \leq q_0$ obtained by $E_b(q_0) = \kappa b^{1/3} \rho_b^{GS}$, are stable to non-Gaussian effects. These arguments provide for the approximation to the contribution of the bosonic bound pairs to the equation of state (Eq. [4]) as $\rho_b^{ANG}(T, \mu) = -\frac{1}{V} \sum_{|q| \leq q_0} \eta_B(\omega_b(q)) \frac{\partial \omega_b(q)}{\partial \mu}$.  

Fig. 4 shows the results of this approximate non-Gaussian theory (see the curve marked ANG). The ap-
proximate theory does indeed possess the key features desired, notably the elimination of RB for $\lambda \ll k_F$. In this regime, the ANG chemical potential smoothly connects to free vacuum value. Furthermore, when $\lambda \gg k_F$, the ANG recovers the rashbon gas. In the ANG theory, RB appears only after a particular value of $\lambda$ which depends on $T$ and $a_s$ at which the solution switches from FB to RB. This evolution should be smooth in a detailed theory which also includes non-Gaussian effects in the free vacuum.

**Phase Diagram:** We now use the ANG theory to obtain the phase diagram shown in Fig. 4. For a small positive scattering length, which obtains a BEC of fermion pairs in free vacuum, increasing RSOC engenders a smooth crossover to the rashbon-BEC with the $T_c$ gradually changing from that set by the free vacuum boson mass (twice the fermion mass) to that set by the rashbon mass. At the resonant scattering length, the $T_c$ again evolves from that of the free vacuum unitary Fermi gas, to that of the rashbon-BEC. The scenario for a small negative scattering length is significantly different as discussed below.

One of the key aspects of the phase diagram Fig. 4 shown in detail in Fig. 5 is the large enhancement of $T_c$ for a system with a weak attractive interaction. For example, for $-\frac{1}{k_F a_s} = 2$, the $T_c$ is enhanced from $0.02T_F$ at $\lambda = 0$ to about $0.17T_F$ when $\lambda \approx k_F$. Further, there is a regime of $\lambda$ where $T_c$ decreases. Beyond this, $T_c$ is determined by two-body physics as shown by dashed-dot lines in Fig. 5. Fig. 5 also shows the mean field $T_c$ which makes fluctuation effects evident. For example, the $T_c$ from the ANG theory is about 85% of the mean field $T_c$ for $-\frac{1}{k_F a_s} = 2.0$, and it is reduced to about 60% of the mean field value when $-\frac{1}{k_F a_s} = 0.75$. The enhancement of $T_c$ for weak attractive interactions indicated by our ANG theory is a remarkable feature of spin orbit coupled systems (non-Abelian gauge fields). Finally, for any given $a_s$ (including $a_s < 0$), the $T_c$ at large $\lambda$ ($\lambda \gtrsim (k_F, 1/|a_s|)$) is independent of $a_s$, determined only by the rashbon mass.

As shown in Fig. 5, there is a much bigger regime of parameters (with weak interactions and RSOC) over which the crossover from a BCS like ground state to a rashbon-BEC occurs. The central point is that the superfluid with high $T_c$ occurs in this crossover regime. Indeed, it will be interesting to mimic this crucial finding in material systems to provide routes to making superconductors with high transition temperatures. On a different token, this physics can be uncovered in a cold atoms experiment at fixed negative $a_s$ and RSOC, by working with different trap centre densities, tracing out a path akin to the dotted line shown in Fig. 4. Another interesting point to note is that the enhanced binding induced by the RSOC will result in significant pseudogap features which could be observed even at higher temperatures.

In summary, we have shown the crucial role of beyond-Gaussian effects in spin orbit coupled Fermi gases. We have developed a simple theory that incorporates the beyond-Gaussian effects in an approximate fashion. Using this theory we obtain the phase diagram of the system. A key result of our calculation is the demonstration of the enhancement of the exponentially small superfluid transition temperature with weak attraction to values comparable to Fermi temperature. This important point provides clues to producing superconductors with high transition temperatures. Our approximate non-Gaussian theory uncovers the rich physics in spin-orbit coupled gases providing motivation for further de-
tailed theoretical considerations. Promising routes to treat beyond-Gaussian effects include the $G_0-G$ or $G-G$ schemes.\cite{35,36}

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*shenoy@physics.iisc.ernet.in*

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\[32\] It is worth mentioning that, unlike in free vacuum, an analytical expression for the imaginary part of $\mathcal{M}(\omega, \mathbf{q})$ is not available even for spherical gauge field. On top of this, the multiple integrals in evaluating $\rho_0(T, \mu)$ and $\rho_0(T, \mu)$ which are challenging for an efficient numerics even in free vacuum, are extremely involved in presence of gauge fields due to various factors including principal valued integrals whose locations also have to be obtained numerically, sharp features and divergences in the integrand, among many.

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The term RSOC induced by the non-Abelian gauge field, $G$, describes the kinetic energy including the chemical potential. The second term, $\xi$, describes the contact attraction among fermions as

$$S_0[\Psi] = \frac{1}{2} \sum_k \Psi^*(k)(-G_0^{-1}(k,k))\Psi(k').$$

where $G_0^{-1}(k,k') = \text{Diag}(ik_n - \xi_{k+}, ik_n + \xi_{k+}, ik_n - \xi_{k-}, ik_n + \xi_{k-})\delta_{k,k'}$ with $\xi_{k_\alpha} = \varepsilon_{k_\alpha} - \mu$ and $\mu$ is the chemical potential. The second term, $S_v$, in the action (Eq. (A1)) describes the contact attraction among fermions as

$$S_v[\Psi] = \frac{\upsilon T}{V} \sum_q S^*(q) S(q),$$

where $T$ and $V$ are respectively the temperature and volume of the system, $\upsilon$ is the bare interaction parameter. This last quantity is traded for the $s$-wave scattering length $a_s$ through regularization as $\frac{1}{4\pi a_s} = \frac{1}{\upsilon} + \Lambda$, where $\Lambda = \frac{1}{V} \sum k \frac{1}{\upsilon}$ denoting the ultraviolet cutoff. The quantity $S^*(q) = \sum_{k,\alpha\beta} A_{\alpha\beta}(q,k) c^*_\alpha \left( \frac{3}{2} + k \right) c^*_\beta \left( \frac{3}{2} - k \right)$ stands for the singlet pair density in Matsubara-Fourier space, $q = (i\upsilon q, q)$, where $i\upsilon q$ is the Bose-Matsubara frequency, $q$ is the center of mass momentum and $k$ is the relative momentum of a two-particle state with particles having helicities $\alpha$ and $\beta$. $A_{\alpha\beta}(q, k)$ is the weight of such a state in the singlet sector. The third term in Eq. (A1) contains external pairing source fields $F(q)$,

$$S_F[\Psi] = \sqrt{\frac{T}{V}} \sum_q F(q) S^*(q) + F^*(q) S(q).$$

This term anticipates a pairing instability in the system, and is added solely to aid the calculation of the pairing susceptibility (most of the formulae, therefore, will have $F = 0$).

We now perform a Hubbard-Stratanovich transformation on $S_v$ by introducing pairing fields $\eta(q)$,

$$S[\Psi, \eta, F] = \sum_{k,k'} \Psi^*(k)(-G^{-1}(k,k'))\Psi(k') - \frac{1}{\upsilon} \sum_q \eta^*(q) \eta(q),$$

where

$$G^{-1}(k,k') = G_0^{-1}(k,k') - \gamma(k,k'),$$

$$\gamma(k,k') = \left( \begin{array}{cccc} 0 & \gamma_+(k,k') & 0 & \gamma_+(k,k') \\ \tilde{\gamma}_+(k,k') & 0 & \tilde{\gamma}_+(k,k') & 0 \\ 0 & \gamma_-(k,k') & 0 & \gamma_-(k,k') \\ \tilde{\gamma}_-(k,k') & 0 & \tilde{\gamma}_-(k,k') & 0 \end{array} \right),$$

$$\gamma_{\alpha\beta}(q,k,k') = \sqrt{\frac{\upsilon}{\pi}} \sum_q \gamma(q) A_{\alpha\beta}(q,k - \frac{q}{2}) \delta_{q,k-k'},$$

$$\tilde{\gamma}_{\alpha\beta}(q,k,k') = \sqrt{\frac{\upsilon}{\pi}} \sum_q \gamma(q) A_{\alpha\beta}(-q,k - \frac{q}{2}) \delta_{q,k-k'},$$

and

$$\gamma(q) = \eta(q) + F(q).$$

The action is now quadratic in fermionic fields which can be integrated to yield

$$S[\eta, F] = - \ln \det[-G^{-1}] - \frac{1}{\upsilon} \sum_q \eta^*(q) \eta(q).$$

This is Eq. (1) in the main text.