Wavy spirals and their fractal connection with chirps

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Abstract

We study the fractal oscillatory of a class of real $C^1$ functions $x = x(t)$ near $t = \infty$. It is measured by oscillatory and phase dimensions, defined as box dimensions of the graph of $X(\tau) = x(\frac{1}{\tau})$ near $\tau = 0$ and the trajectory $(x, \dot{x})$ in $\mathbb{R}^2$, respectively, assuming that $(x, \dot{x})$ is a spiral converging to the origin. The relationship between these two dimensions has been established for a class of oscillatory functions using formulas for box dimensions of graphs of chirps and nonrectifiable wavy spirals, introduced in this paper. Wavy spirals are a specific type of spirals, given in polar coordinates by $r = f(\varphi)$, converging to the origin in non-monotone way as a function of $\varphi$. They emerged in our study of phase portraits associated to solutions of Bessel equations. Also, the rectifiable chirps and spirals have been studied.

Keywords: Wavy spiral, chirp, box dimension, Minkowski content, oscillatory dimension, phase dimension

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1. Introduction

Fractal analysis of differential equations has since the last decades emerged as an important tool in better understanding the behavior of their oscillatory solutions. The main focus of fractal analysis in dynamics is in fractal dimension theory. Its goal is in determining complexity of invariant sets and

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measures using fractal dimensions. The fractal dimension has been successfully used in studying, for instance, logistic map, Smale horseshoe, Lorenz attractor, Hénon Attractor, Julia and Mandelbrot sets, spiral trajectories, infinite-dimensional dynamical systems and even in probability theory, see [26].

In this work we are focused on studying the connection between the fractal dimension of graphs of oscillatory solutions, and the fractal dimension of the associated phase portraits. In particular, we use the box dimension, which we exploit instead of Hausdorff dimension. Due to the countable stability of Hausdorff dimension, its value is trivial on all smooth nonrectifiable curves, while the box dimension is nontrivial, that is, larger than 1. From the view of fractal analysis of trajectories and graphs of solutions of differential equations, the most interesting are solutions having phase plots and graphs of infinite length. The Hausdorff dimension is not suitable to classify these solutions, while the box dimension comes in handy.

Our work was initially inspired by Tricot [20], where box dimensions of graphs of a simple spiral ($r = \varphi^{-\alpha}$, $\alpha \in (0,1)$, in polar coordinates) and $(\alpha, \beta)$-chirp ($f(t) = t^\alpha \cos t^{-\beta}$, $0 < \alpha < \beta$) have been computed near the origin. Since then, these results have been generalized to some more general spiral trajectories of dynamical systems, and to chirp-like functions. Fractal properties of spiral trajectories of dynamical systems in the phase plane have been studied by Žubrinić and Županović, see e.g. [23]–[25]. An interesting behavior of the box dimension of spiral trajectories has been found and related to the bifurcation of a system, in particular to the Hopf bifurcation. On the other hand, the chirp-like behavior of solutions of different types of second-order linear differential equations is also of interest. It has been studied by Kwong, Pašić, Tanaka and Wong: Euler type equations are considered in [12, 13, 21], Hartman-Wintner type equations in [9], half linear equations in [16], and for the Bessel equation see [15]. More specifically, this work has been motivated by Pašić, Žubrinić and Županović [17], containing the first results connecting fractal properties of chirps and spirals, with applications to Liénard and Bessel equations.

All this encouraged us to study and analyze the connection between chirp-like functions and the corresponding spiral trajectories in the phase plane and vice versa. There are two possible ways of looking at the solutions: using the graph of a solution, or using the phase plot of the solution, and the latter was first theoretically developed by Poincaré. Our main results are obtained in Theorems 4 and 7. For some applications to differential equations and
A specific type of spirals associated to oscillatory solutions of Bessel equations, emerged in our study of phase portraits, converging to the origin in non-monotone way as a function of $\varphi$. We call them wavy spirals, see Definition 10. They also appear in the study of curves given by the parametrization of oscillatory integrals from Arnol’d, Gusein-Zade and Varchenko, [1, Part II]. These curves can exhibit even more complex behavior, having self-intersections. The oscillatory integrals from [1] are naturally related to generalized Fresnel integrals, and the fractal properties of associated spirals studied in [6].

Techniques of fractal analysis have also been successfully applied in the study of bifurcations: see e.g. Horvat Dmitrović [4], Li and Wu [22], Mardešić, Resman and Županović [10], Resman [19], as well as in the case of Hopf bifurcation at infinity considered in Radunović, Žubrinić and Županović [18], and for infinite-dimensional dynamical systems related to Schrödinger equation, see Milišić, Žubrinić and Županović [11].

2. Summary of results and definitions

Our results can be summarized in the following way. An $(\alpha, 1)$-chirp-like function near the origin, described in Theorem 4, is connected with a spiral “similar” to $r = \varphi^{-\alpha}$, defined in polar coordinates. If $\alpha \in (0, 1)$ then the box dimension of this spiral is equal to $\frac{2}{\alpha+1}$, see Theorems 2 and 4(i), and for $\alpha > 1$ this spiral is rectifiable, see Theorem 4(ii). Furthermore, we consider the opposite direction and generate a chirp from a given planar spiral. The obtained chirp is $(\alpha, 1)$-chirp-like function, $\alpha \in (0, 1)$, see Definition 8, and the box dimension of its graph is equal to $\frac{3-\alpha}{2}$, see Theorems 5 and 7. If the planar spiral is rectifiable and $\alpha > 1$, then the corresponding chirp-like function has rectifiable graph as well, see Theorem 8.

Let us introduce some definitions and notation. Given a bounded subset $A$ of $\mathbb{R}^N$, we define the $\varepsilon$-neighborhood of $A$ by $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$, where $d(y, A)$ denotes the Euclidean distance from $y$ to $A$. By lower $s$-dimensional Minkowski content of $A$, $s \geq 0$ we mean

$$\mathcal{M}^s_L(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

and analogously for the upper $s$-dimensional Minkowski content $\mathcal{M}^{+s}(A)$. If both these quantities coincide, the common value is called the $s$-dimensional

$$\mathcal{M}^s(A) := \lim_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$
Minkowski content of $A$, and is denoted by $\mathcal{M}^s(A)$. Now we can introduce the lower and upper box dimensions of $A$ by

$$\dim_B A := \inf \{ s \geq 0 : \mathcal{M}^s(A) = 0 \},$$

and analogously $\overline{\dim}_B A := \inf \{ s \geq 0 : \mathcal{M}^{s*}(A) = 0 \}$. If these two values coincide, we call it simply the box dimension of $A$, and denote it by $\dim_B A$.

If $0 < \mathcal{M}^d(A) \leq \mathcal{M}^d(A) < \infty$ for some $d$, then we say that $A$ is Minkowski nondegenerate. In this case obviously $d = \dim_B A$. In the case when lower or upper $d$-dimensional Minkowski contents of $A$ are 0 or $\infty$, where $d = \dim_B A$, or $\overline{\dim}_B A < \dim_B A$, we say that $A$ is degenerate.

More details on these definitions can be seen in Falconer [3] and Tricot [20]. Some generalizations can be seen in [10].

**Definition 1.** Let $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that $x$ is oscillatory function near $t = \infty$ if there exists a sequence $t_k \to \infty$ such that $x(t_k) = 0$, and the functions $x|_{(t_k, t_{k+1})}$ alternately change sign for $k \in \mathbb{N}$.

Analogously, let $u : (0, t_0] \to \mathbb{R}$, $t_0 > 0$, be a continuous function. We say that $u$ is oscillatory function near the origin if there exists a sequence $s_k$ such that $s_k \searrow 0$ as $k \to \infty$, $u(s_k) = 0$ and restrictions $u|_{(s_{k+1}, s_k)}$ alternately change sign for $k \in \mathbb{N}$.

**Definition 2.** (see Pašić [13]) Suppose that $v : I \to \mathbb{R}$, $I = (0, 1]$, is an oscillatory function near the origin, $d \in [1, 2)$. We say that $v$ is $d$-dimensional fractal oscillatory near the origin if $\dim_B G(v) = d$ and $0 < \mathcal{M}^d(G(v)) \leq \mathcal{M}^{d*}(G(v)) < \infty$, where $G(v)$ denotes the graph of $v$.

**Definition 3.** Assume that $x : [t_0, \infty) \to \mathbb{R}$ is oscillatory near $t = \infty$. Let us define $X : (0, 1/t_0] \to \mathbb{R}$ by $X(\tau) = x(1/\tau)$. It is clear that $X(\tau)$ is oscillatory near the origin. We measure the rate of oscillularity of $x(t)$ near $t = \infty$ by the rate of oscillularity of $X(\tau)$ near $\tau = 0$. More precisely, the oscillatory dimension $\dim_{osc}(x)$ (near $t = \infty$) is defined as box dimension of the graph of $X(\tau)$ near $\tau = 0$:

$$\dim_{osc}(x) = \dim_B G(X),$$

provided the box dimension exists.
Definition 4. Assume now that $x$ is of class $C^1$. We say that $x$ is a phase oscillatory function if the following condition holds: the set $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ in the plane is a spiral converging to the origin.

Definition 5. By a spiral here we mean the graph of a function $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, in polar coordinates, where

\[
\begin{cases}
  f : [\varphi_1, \infty) \to (0, \infty) \text{ such that } f(\varphi) \to 0 \text{ as } \varphi \to \infty, \\
  f \text{ is radially decreasing} \text{ (i.e., for any fixed } \varphi \geq \varphi_1 \text{ )} \\
  \text{the function } N \ni k \mapsto f(\varphi + 2k\pi) \text{ is decreasing).}
\end{cases}
\]

This definition appears in [23]. Depending on the context, by a spiral here we also mean the graph of a function $r = g(\varphi)$, $\varphi \leq \varphi' < 0$, in polar coordinates, such that the curve defined as the graph of $r = g(-\varphi)$, $\varphi \geq |\varphi'| > 0$, given in polar coordinates, satisfies [11]. It is easy to see that a spiral defined by a function $g(\varphi)$ is a mirror image of the spiral defined by $g(-\varphi)$, with respect to the $x$-axis. We also say that a graph of a function $r = f(\varphi)$, $\varphi \geq \varphi_2 > 0$, defined in polar coordinates, is a spiral near the origin if there exists $\varphi_2 \geq \varphi_1$ such that the graph of the function $r = f(\varphi)$, $\varphi \geq \varphi_2$, viewed in polar coordinates, is a spiral.

Definition 6. The phase dimension $\dim_{ph}(x)$ of a function $x : [t_0, \infty) \to \mathbb{R}$ of class $C^1$ is defined as the box dimension of the corresponding planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.

Oscillatory and phase dimensions are fractal dimensions, introduced in the study of chirp-like solutions of second order ODEs, see [17]. More about fractal dimensions in dynamics can be found in [26].

For two real functions $f(t)$ and $g(t)$ of a real variable we write $f(t) \simeq g(t)$ as $t \to 0$ (as $t \to \infty$) if there exist two positive constants $C$ and $D$ such that $C g(t) \leq f(t) \leq D g(t)$ for all $t$ sufficiently close to $t = 0$ (for all $t$ large enough). For a function $F : U \to V$, with $U, V \subset \mathbb{R}^2$, $V = F(U)$, we write $|F(x_1) - F(x_2)| \simeq |x_1 - x_2|$ if $F$ is a bi-Lipschitz mapping, i.e., both $F$ and $F^{-1}$ are Lipschitzian.

Definition 7. We write $f(t) \sim g(t)$ if $f(t)/g(t) \to 1$ as $t \to 0$ (as $t \to \infty$). Also, if $k$ is fixed positive integer, for two functions $f$ and $g$ of class $C^k$ we write,

\[
f(t) \sim_k g(t) \text{ as } t \to 0 \text{ (as } t \to \infty),
\]

if $f^{(j)}(t) \sim g^{(j)}(t)$ as $t \to 0$ (as $t \to \infty$) for all $j = 0, 1, \ldots, k$. 

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For example, \( \frac{(t-1)^{4-\alpha}}{t^3} \sim t^{-\alpha} \) as \( t \to \infty \), for \( \alpha \in (0,1) \).

Analogously, if \( k \) is fixed positive integer, for two functions \( f \) and \( g \) of class \( C^k \) we write

\[
  f(t) \asymp_k g(t) \text{ as } t \to 0 \quad (\text{as } t \to \infty),
\]

if \( f^{(j)}(t) \asymp g^{(j)}(t) \) as \( t \to 0 \) (as \( t \to \infty \)) for all \( j = 0, 1, ..., k \).

We write \( f(t) = O(g(t)) \) as \( t \to 0 \) (as \( t \to \infty \)) if there exists a positive constant \( C \) such that \( |f(t)| \leq C|g(t)| \) for all \( t \) sufficiently close to \( t = 0 \). (for all \( t \) large enough). Similarly, we write \( f(t) = o(g(t)) \) as \( t \to \infty \) if for every positive constant \( \varepsilon \) it holds \( |f(t)| \leq \varepsilon |g(t)| \) for all \( t \) sufficiently large.

**Definition 8.** Functions of the form

\[
  y = P(x) \sin(Q(x)) \text{ or } y = P(x) \cos(Q(x)),
\]

where \( P(x) \asymp x^\alpha, Q(x) \asymp_1 x^{-\beta} \) as \( x \to 0 \), are called \( (\alpha, \beta) \)-chirp-like function near \( x = 0 \).

### 3. Spirals generated by chirps

We study spirals generated by chirps in the sense of Theorem 4. To prove Theorem 4 about box dimension of a spiral generated by a chirp we need a new version of [23, Theorem 5]. Let us first recall [23, Theorem 5], cited here in a more condensed form, suitable for our purposes. The following theorem extends a result about box dimension of spiral from Dupain, Mendès France and Tricot, see [2, 20].

**Theorem 1 (Theorem 5 from [23]).** Let \( f \mid [\varphi_1, \infty) \to (0, \infty) \) be a decreasing function of class \( C^2 \), such that \( f(\varphi) \to 0 \) as \( \varphi \to \infty \). Let \( \alpha \in (0,1) \). Assume that there exist positive constants \( \underline{m}, \overline{m}, M_1, M_2 \) and \( M_3 \) such that for all \( \varphi \geq \varphi_1 > 0 \),

\[
  \underline{m} \varphi^{-\alpha} \leq f(\varphi) \leq \overline{m} \varphi^{-\alpha},
\]

\[
  M_1 \varphi^{-\alpha-1} \leq |f'(\varphi)| \leq M_2 \varphi^{-\alpha-1}, \quad |f''(\varphi)| \leq M_3 \varphi^{-\alpha}.
\]

Let \( \Gamma \) be the graph of \( r = f(\varphi) \) in polar coordinates. Then

\[
  \dim_B \Gamma = \frac{2}{1+\alpha}.
\]
Now we provide a new version of Theorem 1.

**Theorem 2 (Dimension of a piecewise smooth nonincreasing spiral).**

Let \( f : [\varphi_1, \infty) \to (0, \infty) \) be a nonincreasing and radially decreasing function, also a continuous and piecewise continuously differentiable. We assume that the number of smooth pieces of \( f \) in \([\varphi_1, \varphi_1]\) is finite, for any \( \varphi_1 > \varphi_1 \).

Assume that there exist positive constants \( m, \overline{m}, a, \) and \( M \) such that for all \( \varphi \geq \varphi_1 \),

\[
    m \varphi^{-\alpha} \leq f(\varphi) \leq \overline{m} \varphi^{-\alpha},
\]

\[
    a \varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + 2\pi),
\]

and for all \( \varphi \) where \( f(\varphi) \) is differentiable,

\[
    |f'(\varphi)| \leq M \varphi^{-\alpha-1}.
\]

Let \( \Gamma \) be the graph of \( r = f(\varphi) \) in polar coordinates. If \( \alpha \in (0, 1) \) then

\[
    \dim B \Gamma = \frac{2}{1 + \alpha}.
\]

For the proof of Theorems 2 and 4 below, we need the following Lemma 1 that is a generalization of [23, Lemma 1] dealing with smooth spirals.

**Lemma 1 (Excision property for piecewise smooth curves).** Let \( \Gamma \) be the image of a continuous and piecewise continuously differentiable function \( h : [\varphi_1, \infty) \to \mathbb{R}^2 \) (piecewise in the sense of Theorem 2). Assume that \( \dim B \Gamma > 1 \), \( \Gamma_1 := h(\varphi_1, \infty) \), for some fixed \( \varphi_1 > \varphi_1 \), and \( h([\varphi_1, \varphi_1]) \cap \Gamma_1 = \emptyset \). Then

\[
    \dim B \Gamma_1 = \dim B \Gamma, \quad \overline{\dim B} \Gamma_1 = \overline{\dim B} \Gamma.
\]

**Proof.** The proof is analogous to the proof of [23, Lemma 1], but with the following difference. Here, the curve \( \Gamma_2 := \Gamma \setminus \Gamma_1 = h([\varphi_1, \varphi_1]) \) is rectifiable due to piecewise rectifiability of \( h \) and due to the finite number of pieces in segment \( \varphi_1, \varphi_1 \). The function \( h \) is piecewise rectifiable due to its piecewise smoothness and continuity. Also, by careful examination of the proof of [23, Lemma 1], it follows that we can substitute the injectivity assumption on \( h \) with the weaker condition that \( h([\varphi_1, \varphi_1]) \cap \Gamma_1 = \emptyset \). (For more details see [23, Lemma 1].)

**Proof (Theorem 2).** The proof is analogous to the proof of [23, Theorem 5], but using the new Lemma 1.
Remark 1. Notice the difference between the assumptions of Theorem 1 and Theorem 2. In Theorem 1 the function \( f \) is decreasing and of class \( C^2 \). By careful examination of the proof of [23, Theorem 5], one can see that \( f \) being decreasing is used only in the sense of nonincreasing, that is, not strictly decreasing, hence in Theorem 1 we can assume that \( f \) is nonincreasing. Additional smoothness of \( f \) and additional conditions regarding constants \( M_1 \) and \( M_3 \) in Theorem 1 are used only in the calculations of Minkowski contents in [23, Theorem 5] which we exclude from our Theorem 2. Further reduction in smoothness of \( f \) from continuously differentiable to a piecewise continuously differentiable function can be found in Lemma 1.

Theorem 3 deals with a spiral \( \Gamma' \) described by \( r = f(\varphi) \), where \( f \) is increasing on some parts, see Definitions 9 and 10. We call this new property of \( \Gamma' \) spiral waviness.

Definition 9. Let \( r : [t_0, \infty) \to (0, \infty) \) be a \( C^1 \) function. Assume that \( r'(t_0) \leq 0 \). We say that \( r = r(t) \) is a wavy function if the sequence \((t_n)\) defined inductively by:

\[
\begin{align*}
t_{2k+1} & := \inf \{ t : t > t_{2k}, r'(t) > 0 \}, \quad k \in \mathbb{N}_0, \\
t_{2k+2} & := \inf \{ t : t > t_{2k+1}, r(t) = r(t_{2k+1}) \}, \quad k \in \mathbb{N}_0,
\end{align*}
\]

is well-defined, and satisfies the waviness condition:

\[
\begin{cases}
(i) & \text{The sequence } (t_n) \text{ is increasing and } t_n \to \infty \text{ as } n \to \infty. \\
(ii) & \text{There exists } \varepsilon > 0, \text{ such that for all } k \in \mathbb{N}_0 \text{ holds } t_{2k+1} - t_{2k} \geq \varepsilon. \\
(iii) & \text{For all } k \text{ sufficiently large it holds } \\
& \quad \text{osc}_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = o \left( t_{2k+1}^{-\alpha} \right), \quad \alpha \in (0, 1),
\end{cases}
\]

where \( \text{osc}_{t \in I} r(t) = \max_{t \in I} r(t) - \min_{t \in I} r(t) \).

Notice that \( \min_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = r(t_{2k+1}) \). Condition (i) means that the property of waviness of \( r(t) \) is global on the whole domain. Condition (ii) is connected to an assumption of Lemma 1. Condition (iii) is a condition
Figure 1: Function $r(t)$ for $p(t) = t^{-1/2}$, see Lemma $\text{[3]}$, $t_0 = 0.6$. This is a wavy function, see Definition $\text{[4]}$, with local minima at $t_{2k+1}$, $k = 0, 1, \ldots$

on a decay rate on the sequence of oscillations of $r$ on $I_k = [t_{2k+1}, t_{2k+2}]$, for $k$ sufficiently large. Also, notice that condition $r'(t_0) \leq 0$ assures that $t_1$ is well-defined. For an example of function $r(t)$, see Figure $\text{[1]}$.

**Remark 2.** Conditions (i) and (ii) in the waviness condition $\text{(2)}$ are not entirely independent. From (ii) and if $(t_n)$ is increasing follows that $t_n \to \infty$ as $n \to \infty$, but from (i) does not follow (ii). So, condition (ii) plus $(t_n)$ increasing is stronger than condition (i).

**Definition 10.** Let a spiral $\Gamma'$, given in polar coordinates by $r = f(\varphi)$, where $f$ is a given function. If there exists increasing or decreasing function of class $C^1$, $\varphi = \varphi(t)$, such that $r(t) = f(\varphi(t))$ is a wavy function, then we say $\Gamma'$ is a *wavy spiral*.

For an example of spiral $\Gamma'$, see Figure $\text{[2]}$. Now, using Theorem $\text{[2]}$ and Lemma $\text{[1]}$ we prove the following Theorem $\text{[3]}$. 

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Theorem 3 (Box dimension of a wavy spiral). Let $t_0 > 0$ and assume $r : [t_0, \infty) \to (0, \infty)$ is a wavy function. Assume that $\varphi : [t_0, \infty) \to [\varphi_0, \infty)$ is an increasing function of class $C^1$ such that $\varphi(t_0) = \varphi_0 > 0$ and there exists $\bar{\varphi}_0 \in \mathbb{R}$ such that

$$|\varphi(t) - \bar{\varphi}_0| - (t - t_0)| \to 0 \text{ as } t \to \infty.$$ (3)

Let $f : [\varphi_0, \infty) \to (0, \infty)$ be defined by $f(\varphi(t)) = r(t)$. Assume that $\Gamma'$ is a spiral defined in polar coordinates by $r = f(\varphi)$, satisfying (1). Let $\alpha \in (0, 1)$ is the same value as in (2)(iii) for wavy function $r$, and assume $\varepsilon'$ is such that $0 < \varepsilon' < \varepsilon$, where $\varepsilon$ is defined by (2)(ii) for wavy function $r$. Assume that there exist positive constants $\underline{m}$, $\underline{m}$, $\underline{a}'$ and $M$ such that for all $\varphi \geq \varphi_0$,

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\varphi^{-\alpha},$$ (4)

$$|f'(\varphi)| \leq M\varphi^{-\alpha - 1},$$ (5)

and for all $\Delta \varphi$, such that $\theta \leq \Delta \varphi \leq 2\pi + \theta$, there holds

$$\underline{a}' \varphi^{-\alpha - 1} \leq f(\varphi) - f(\varphi + \Delta \varphi),$$ (6)
where \( \theta := \min \{ \varepsilon', \pi \} \).

Then \( \Gamma' \) is a wavy spiral and

\[
\dim_B \Gamma' = \frac{2}{1 + \alpha}.
\]

The proof of Theorem 3 is given in [7].

Now, Theorem 3 enables us to calculate the box dimension of a spiral generated by a chirp, which is one of the main results of this paper.

**Theorem 4 (Chirp–spiral comparison).** Let \( \alpha > 0 \). Assume that \( X : (0, 1/\tau_0] \to \mathbb{R}, \tau_0 > 0, X(\tau) = P(\tau) \sin 1/\tau, \) where \( P(\tau) \) is a positive function such that \( P(\tau) \sim_3 \tau^\alpha \) as \( \tau \to 0 \). Define \( x(t) := X(1/t) \) and a continuous function \( \varphi(t) \) by \( \tan \varphi(t) = \dot{x}(t)/x(t) \).

(i) If \( \alpha \in (0, 1) \) then the planar curve \( \Gamma := \{(x(t), \dot{x}(t)) \in \mathbb{R} : t \in [\tau_0, \infty)\} \) generated by \( X \) is a wavy spiral near the origin. We have \( f(\varphi) \simeq |\varphi|^{-\alpha} \) as \( \varphi \to -\infty \), and

\[
\dim_{ph}(x) := \dim_B \Gamma = \frac{2}{1 + \alpha}.
\]

(ii) If \( \alpha > 1 \) then the planar curve \( \Gamma := \{(x(t), \dot{x}(t)) \in \mathbb{R} : t \in [\tau_0, \infty)\} \) is a rectifiable wavy spiral near the origin.

The proof of Theorem 4 consists of checking out the conditions of Theorem 3. The following lemmas make this checking easy.

**Lemma 2.** Let \( \alpha > 0 \) and assume that \( P(\tau), \tau \in (0, 1/t_0], t_0 > 0, \) be such that \( P(\tau) \sim_3 \tau^\alpha \) as \( \tau \to 0 \). Then \( p(t) := P(1/t) \sim_3 t^{-\alpha} \) as \( t \to \infty \) and vice versa. Furthermore, we have:

\[
\lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0, \quad \lim_{t \to \infty} \frac{p''(t)}{p(t)} = 0, \tag{7}
\]

\[
\frac{-p(t)}{p'(t)} \sim_3 \frac{t}{\alpha}, \quad \frac{2p'(t)}{p''(t)} \sim_3 \frac{2t}{\alpha + 1} \quad \text{as} \quad t \to \infty, \tag{8}
\]

\[
\sup_{t \in [t_0, \infty)} \left( -\frac{p(t)}{p'(t)} \right)' < \infty, \quad \sup_{t \in [t_0, \infty)} \left( \frac{2p'(t)}{p''(t)} \right)' < \infty. \tag{9}
\]

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The claims of Lemma 2 follow directly from the assumptions.

**Lemma 3.** Let $\alpha \in (0, 1)$ and

$$r(t) = p(t) \sqrt{1 + \frac{p^2(t)}{p^2(t)}} \sin^2 t + \frac{p'(t)}{p(t)} \sin 2t, \quad t \in [t_0, \infty), \ t_0 > 0,$$

where $p(t) \sim_1 t^{-\alpha}$ as $t \to \infty$.

Let $C \in \mathbb{R}$ and assume that $t(\varphi) = \varphi + C + O(\varphi^{-1})$ as $\varphi \to \infty$. Let $\Delta \varphi > 1$. Then there exists constant $k > 0$, independent of $\varphi$ and $\Delta \varphi$, such that for all $\varphi$ sufficiently large it holds

$$r(t(\varphi)) - r(t(\varphi + \Delta \varphi)) \geq k\varphi^{-\alpha - 1}(1 + O(\varphi^{-1})).$$

The proof of Lemma 3 can be seen in the Appendix.

**Proof (Theorem 4). (i) Step 1.** (The box dimension is invariant with respect to mirroring of a spiral.) We will prove the equivalent claim, that planar curve $\Gamma' = \{(x(t), -\dot{x}(t)) : t \in [\tau_0, \infty)\}$ is a wavy spiral defined by $r = f(\varphi)$, $\varphi \in [\phi_0, \infty)$, near the origin, satisfying $f(\varphi) \simeq \varphi^{-\alpha}$, in polar coordinates, near the origin, and $\dim_B \Gamma' = \frac{2}{1+\alpha}$. It is easy to see that curve $\Gamma$ is a mirror image of curve $\Gamma'$, with respect to the $x$-axis, hence $\Gamma$ is a wavy spiral. Reflecting with respect to the $x$-axis in the plane is an isometric map. As the isometric map is bi-Lipschitzian and therefore it preserves box dimensions, see [3, p. 44], we see that $\dim_B \Gamma = \dim_B \Gamma' = \frac{2}{1+\alpha}$.

**Step 2.** (Checking condition (4).) From

$$x(t) = p(t) \sin t,$$

$$\dot{x}(t) = p'(t) \sin t + p(t) \cos t,$$

where $p(t) := P(1/t)$, we compute

$$\tan \varphi(t) = \frac{\dot{x}(t)}{x(t)} = -\frac{p'(t)}{p(t)} \frac{1}{\tan t}. \quad (10)$$

By differentiating (10) we obtain

$$\frac{d\varphi}{dt}(t) = \cos^2 \varphi(t) \left[\frac{p^2(t) - p(t)p''(t)}{p^2(t)} + \frac{1}{\sin^2 t}\right]. \quad (11)$$
Using (10) again, we have
\[ \cos^2 \varphi(t) = \frac{1}{1 + \tan^2 \varphi(t)} = \frac{p^2(t) \sin^2 t}{p^2(t) + p^2(t) \sin^2 t + 2p(t)p'(t) \sin t \cos t}. \] (12)

Substituting into (11) and using (7) we get
\[ \lim_{t \to \infty} \frac{d\varphi}{dt}(t) = 1. \] (13)

From (13) it follows that \( \varphi \simeq t \) as \( t \to \infty \) and
\[ r^2(t) = (x(t))^2 + (-\dot{x}(t))^2 = p^2(t) + p^2(t) \sin^2 t + p(t)p'(t) \sin 2t \] (14)
implies
\[ f(\varphi(t)) = r(t) \simeq t^{-\alpha} \simeq \varphi^{-\alpha} \text{ as } t \to \infty. \] (15)

Notice that from (14) it follows that function \( r(t) \) is of class \( C^2 \) and by substituting (12) in (11), respecting (14), we see that function \( \varphi(t) \) is of class \( C^1 \).

**Step 3.** (Checking condition (5).) On the other hand, differentiating (14) we obtain that
\[ \frac{dr}{dt}(t) = \left[ 2p(t)p'(t) \cos^2 t + \frac{2p^2(t) + p(t)p''(t)}{2} \sin 2t \right] + \frac{p'(t)p''(t) \sin^2 t}{r(t)} \] (16)
Also, from (16) we have
\[ \frac{dr}{dt}(t) = \frac{2p(t)p'(t)}{r(t)} \cos^2 t + O(t^{-\alpha-2}) \text{ as } t \to \infty. \] (17)

Since \( \frac{dr}{dt}(t) = f'(\varphi) \cdot \frac{d\varphi}{dt}(t) \) and since by (13) we have \( \frac{d\varphi}{dt}(t) \simeq 1 \) as \( t \to \infty \), there exists \( C_0 > 0 \) and \( C_1 > C_0 \) such that
\[ |f'(\varphi)| \leq C_0 t^{-\alpha-1} \leq C_1 \varphi^{-\alpha-1} \text{ as } \varphi \to \infty. \]

**Step 4.** (Checking condition (3).) Using (10) and [7, Lemma 7], we obtain
\[ \tan \varphi(t) = -(\cot t + O(t^{-1})) = -\cot(t + O(t^{-1})) = \tan(t + \frac{\pi}{2} + O(t^{-1})) \]
as \( t \to \infty \). Since function \( \varphi(t) \) is continuous by the definition and \( O(t^{-1}) < \pi \) for \( t \) large enough, then there exists \( k \in \mathbb{Z} \) such that
\[
\varphi(t) = \left(t + \frac{\pi}{2} + k\pi\right) + O(t^{-1}) \text{ as } t \to \infty.
\]

From the definition of \( \varphi(t) \) we conclude that we may take without loss of generality \( k = 0 \). Finally, we get
\[
\varphi(t) = \left(t + \frac{\pi}{2}\right) + O(t^{-1}) \text{ as } t \to \infty.
\]

**Step 5.** (Checking condition (6).) From (13) it follows that there exists \( \tau_1 \geq \tau_0 \) such that \( \frac{d\varphi}{dt}(t) > 0 \) for all \( t \geq \tau_1 \) hence the function \( \varphi(t) \) is increasing for all \( t \) sufficiently large. As function \( \varphi(t) \) is continuous, we conclude that for all \( \varphi \) large enough there exists inverse function \( t = t(\varphi) \) of function \( \varphi = \varphi(t) \) and
\[
t(\varphi) = \left(\varphi - \frac{\pi}{2}\right) + O(\varphi^{-1}) \text{ as } \varphi \to \infty.
\]

Define value \( \phi_1 := \varphi(\tau_1) \) and notice that we can take \( \tau_1 \) sufficiently large such that \( \phi_1 \geq \phi_0 \).

From (14) we obtain
\[
r(t) = p(t) \sqrt{1 + \frac{p'^2(t)}{p^2(t)} \sin^2 t} + \frac{p'(t)}{p(t)} \sin 2t.
\]

By Lemma 3 we conclude that for fixed \( \Delta \varphi > 1 \) we have
\[
f(\varphi) - f(\varphi + \Delta \varphi) = r(t(\varphi)) - r(t(\varphi + \Delta \varphi)) \geq k_1 \varphi^{-\alpha - 1}, \tag{19}
\]
provided \( \varphi \) is large enough. Moreover, by careful examination of the proof of Lemma 3 we conclude that statement (19) uniformly holds for every \( \Delta \varphi \) from a bounded interval whose lower bound is greater than 1, also provided \( \varphi \) is large enough. (Notice we will have to take \( \theta \) from Theorem 3 to be larger than 1.)

**Step 6.** (\( \Gamma' \) is a spiral near the origin.) Now we can prove that \( \Gamma' \) is a spiral near the origin, that is, \( f(\varphi) \) satisfies condition \( \Pi \) near the origin. First, from (13) it follows that \( f(\varphi) \to 0 \) as \( \varphi \to \infty \). Second, from (19) it follows that \( f(\varphi) \) is radially decreasing for all \( \varphi \) large enough, that is, there exists \( \phi_2 \geq \phi_1 \) such that \( f|_{[\phi_2, \infty)} \) is radially decreasing.
Step 7. (The box dimension is invariant with respect to taking \( \tau_0 \) and \( \phi_0 \) sufficiently large.) First, we define \( \tau_2 \) to be such that \( \phi(\tau_2) = \phi_2 \). Notice that \( \tau_2 \) is well-defined and \( \tau_2 \geq \tau_1 \). As \( p(t) > 0 \), from (14) and the definition of \( x(t) \) and \( \dot{x}(t) \) it follows that \( r(t) > 0 \), that is, \( r(t) \) is strictly positive function. That means there exists constant \( m_1 > 0 \) such that for all \( t \in [\tau_0, \tau_2] \) it holds

\[
r(t) > m_1. \tag{20}
\]

Notice that \( \phi_2 \geq \phi_1 \geq \phi_0 \). From (15) it follows that \( r(t) \to 0 \) as \( t \to \infty \) so there exists \( \tau_3 \geq \tau_2 \) such that for all \( t \in [\tau_3, \infty) \) it holds

\[
r(t) < m_1. \tag{21}
\]

We define \( \phi_3 := \varphi(\tau_3) \). Notice that we could increase \( \tau_3 \) and \( \phi_3 \) to accommodate all requirements in different parts of the proof on \( t \) or \( \varphi \) being sufficiently large. Now, from (20) and (21) we conclude that

\[
\Gamma'|_{[\tau_0, \tau_2]} \cap \Gamma'|_{(\tau_3, \infty)} = \emptyset. \tag{22}
\]

As \( f|_{[\phi_2, \infty)} \) is radially decreasing and \( \varphi'(t) > 0 \) for all \( t \in [\tau_2, \infty) \) it follows that \( \Gamma'|_{(\tau_2, \infty)} \) does not have self intersections, so

\[
\Gamma'|_{[\tau_2, \tau_3]} \cap \Gamma'|_{(\tau_3, \infty)} = \emptyset. \tag{23}
\]

Finally, from (22) and (23) we have \( \Gamma'|_{[\tau_0, \tau_3]} \cap \Gamma'|_{(\tau_3, \infty)} = \emptyset \). Now, we can apply Lemma 11 on curve \( \Gamma' \).

Using Lemma 11 we see that without loss of generality we can assume that \( \tau_0 \) and \( \phi_0 \), in the assumptions of the theorem, are sufficiently large. Informally, we can always remove any rectifiable part from the beginning of \( \Gamma' \), without changing the box dimension of \( \Gamma' \).

Step 8. (Checking waviness condition (2).) By factoring (16), we get

\[
\frac{dr}{dt}(t) = \left( 1 + \frac{p''(t)}{p'(t)} \tan t \right) \left( 1 + \frac{p''(t)}{2p'(t)} \tan t \right) \frac{2p(t)p'(t)}{r(t)} \cos^2 t, \tag{24}
\]

for every \( t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \) (\( \cos t \neq 0 \)).

By Lemma 6 and Remark 7 see below, and using (8) and (9), there exists \( k_0 \in \mathbb{N}_0 \) such that the equations

\[
\tan t = -\frac{p(t)}{p'(t)}, \quad \tan t = -\frac{2p'(t)}{p''(t)}.
\]
have unique solutions \( \hat{t}_{2k} \) and \( t_{2k-1} \), respectively, in intervals \((k+k_0)\pi - \pi, (k+k_0)\pi - \frac{\pi}{2}\)), for each \( k \in \mathbb{N}_0 \), since

\[
- \frac{p(t)}{p'(t)} \sim \frac{t}{\alpha}, \quad - \frac{2p'(t)}{p''(t)} \sim \frac{2t}{\alpha + 1} \quad \text{as } t \to \infty.
\]

Moreover, by taking \( k_0 \) to be sufficiently large, from \( \text{(8)} \) and using inequalities \( 1 < 2/(\alpha + 1) < 1/\alpha \), we see that \( \hat{t}_{2k} \) and \( t_{2k-1} \) even lie in the smaller intervals

\[
((k+k_0)\pi - \frac{\pi}{2}, (k+k_0)\pi - \frac{\pi}{3}),
\]

for each \( k \in \mathbb{N}_0 \). (The statement is true for interval of any length provided upper bound is \((k+k_0)\pi - \frac{\pi}{2}\). We choose value \( \pi/3 \), because it is convenient later in the proof.)

Notice that because of \( \frac{1}{\alpha} \neq \frac{2}{\alpha + 1} \) we see that \(- \frac{p(t)}{p'(t)} \neq - \frac{2p'(t)}{p''(t)} \) for \( t \) sufficiently large, so \( \hat{t}_{2k} \neq t_{2k-1} \) for \( k_0 \) sufficiently large. Without loss of generality we can take \( t_{2k-1} < \hat{t}_{2k} \). So \( \hat{t}_{2k} - t_{2k-1} < \pi/3 \) for every \( k \in \mathbb{N} \), provided \( k_0 \) is sufficiently large. It is easy to see from \( \text{(24)} \) that \( \frac{d\varphi}{dt}(t) \) is positive between these solutions.

As \( \frac{d\varphi}{dt}(t) > 0 \) for all \( t \) sufficiently large, from \( \frac{d\varphi}{dt}(t) = f'(\varphi) \cdot \frac{d\varphi}{dt}(t) \) it follows that \( f'(\varphi) > 0 \) on set \( \bigcup_{k=1}^{\infty} (\varphi_{2k-1}, \varphi_{2k}) \) where \( \varphi_{2k-1} : = \varphi(t_{2k-1}) \) and \( \varphi_{2k} : = \varphi(\hat{t}_{2k}) \). This implies that function \( f(\varphi) \) is increasing for some \( \varphi \), so we can not apply Theorem \( \text{2} \) directly.

Notice that if \( t \in \bigcup_{k=0}^{\infty} (t_{2k-1}, \hat{t}_{2k}) \) then \( r'(t) > 0 \) and if \( t \in \bigcup_{k=0}^{\infty} (\hat{t}_{2k}, t_{2k+1}) \) then \( r'(t) < 0 \).

We would like to prove that for every \( k \in \mathbb{N}_0 \) there exists unique \( t_{2k} \in (\hat{t}_{2k}, t_{2k+1}) \) such that \( r(t_{2k}) = r(t_{2k-1}) \) and \( t_{2k} - t_{2k-1} < \pi/3 \) (where we will take \( k_0 \) from \( \text{(25)} \) to be sufficiently large). As \( r(\hat{t}_{2k}) > r(t_{2k-1}) \), and as function \( r(t) \) is continuous and strictly decreasing on interval \((\hat{t}_{2k}, t_{2k+1})\), it follows that, if such \( t_{2k} \) exists then it is necessary unique, so we only need to prove the existence.

For every \( k \in \mathbb{N}_0 \) we take \( \hat{t}_{2k} := t_{2k-1} + \pi/3 \). Notice that \( \hat{t}_{2k} \in (\hat{t}_{2k}, t_{2k+1}) \), because from \( \text{(25)} \) follows that \( t_{2k+1} - t_{2k-1} > 2\pi/3 \) and \( \hat{t}_{2k} - t_{2k-1} < \pi/3 \). Define \( \varphi_{2k} := \varphi(\hat{t}_{2k}) \) and take \( \varphi_{2k+1} \) as defined before. Using \( \text{(18)} \), we can take \( t \) or equivalently \( k_0 \) sufficiently large, such that \( (\pi/3 + 1)/2 \leq \varphi_{2k} - \varphi_{2k+1} \leq 2 \) for every \( k \in \mathbb{N}_0 \). (The exact value of the upper bound is not important. We just take some value larger than \( \pi/3 \). For lower bound, it is only important that it is larger than 1 and lower than \( \pi/3 \), so we take the mean value between these two.)
Now, using Lemma 3 analogously as in Step 5, we compute

\[
    r(t_{2k-1}) - r(\hat{t}_{2k}) = r(t(\varphi_{2k-1})) - r(t(\bar{\varphi}_{2k})) \\
    = r(t(\varphi_{2k-1})) - r(t(\varphi_{2k-1} + (\bar{\varphi}_{2k} - \varphi_{2k-1}))) \\
    \geq C_2\varphi_{2k-1}^-1 > 0,
\]

for some \( C_2 > 0 \), provided \( \varphi \) or equivalently \( k_0 \) is sufficiently large. From this follows \( r(\hat{t}_{2k}) < r(t_{2k-1}) \), and as function \( r(t) \) is of class \( C^1 \), strictly decreasing on interval \((\hat{t}_{2k}, \bar{t}_{2k})\) and \( r(\hat{t}_{2k}) > r(t_{2k-1}) \), we see that there exist \( t_{2k} \in (\hat{t}_{2k}, \bar{t}_{2k}) \) such that \( r(t_{2k}) = r(t_{2k-1}) \) and obviously \( t_{2k} - t_{2k-1} < \pi/3 \).

Using \( t_{2k+1} - t_{2k-1} > 2\pi/3 \), follows that \( t_{2k+1} - t_{2k} > 2\pi/3 - \pi/3 = \pi/3 \).

We established that for every \( k \in \mathbb{N}_0 \) holds \( t_{2k+1} > t_{2k} > t_{2k-1} \). Notice that \( r'(t_0) \leq 0 \) and that sequence \( (t_n), n \in \mathbb{N}_0 \), is the same as the sequence from Definition 9 defined for function \( r(t) \).

As \( t_{2k+1} - t_{2k-1} > 2\pi/3 \) for every \( k \in \mathbb{N}_0 \), we conclude that \( t_n \to \infty \) as \( n \to \infty \), which means that sequence \( (t_n) \) satisfies condition (2)(i).

As \( t_{2k+1} - t_{2k} > \pi/3 \) for every \( k \in \mathbb{N}_0 \), by taking \( \varepsilon = \pi/3 \), we see that sequence \( (t_n) \) satisfies condition (2)(ii).

Using (17) we conclude that there exist \( C_3, C_4 \in \mathbb{R} \), \( C_4 > C_3 > 0 \), such that

\[
    \text{osc}_{t \in [t_{2k+1}^-, t_{2k+2}^-]} r(t) = r(\hat{t}_{2k+2}) - r(t_{2k+1}) = \int_{t_{2k+1}}^{\hat{t}_{2k+2}} r'(t) \, dt \\
    \leq \frac{1}{3} \cdot \sup_{t \in [t_{2k+1}, t_{2k+2}]} r'(t) \leq C_3^\prime t_{2k+1}^- - \alpha - 2 \leq C_4^\prime t_{2k+2}^- - \alpha - 2,
\]

for every \( k \in \mathbb{N}_0 \), which means that sequence \( (t_n) \) satisfies condition (2)(iii).

Finally, we conclude that sequence \( (t_n) \) satisfies waviness condition (2), so \( r(t) \) is a wavy function and \( \Gamma' \) is a wavy spiral near the origin.

Step 9. (Final conclusion.) From the previous steps we see directly that all of the assumptions of Theorem 3 are fulfilled. We take \( \varepsilon' = (\pi/3+1)/2 < \varepsilon \) and \( \theta = \min\{\varepsilon', \pi\} = (\pi/3+1)/2 \). Using Theorem 3 we prove that \( \dim_B \Gamma' = \frac{2}{1+\alpha} \).

(ii) To prove that \( \Gamma \) is a wavy spiral near the origin notice that Steps 1–8 also hold for \( \alpha > 1 \).
To prove the rectifiability for $\alpha > 1$, from (15), (13) and (17) we have that there exist positive constants $C_5, M_1$ and $C_6$ such that for every $t \in [t_0, \infty)$ it holds

$$r(t) \leq C_5 t^{-\alpha}, \quad \varphi'(t) \leq M_1, \quad |r'(t)| \leq C_6 t^{-\alpha-1}.$$ 

Therefore

$$l(\Gamma) = l(\Gamma') = \int_{t_0}^{\infty} \sqrt{(r(t)\varphi'(t))^2 + (r'(t))^2} \, dt \leq \int_{t_0}^{\infty} \sqrt{M_1^2 C_5^2 t^{-2\alpha} + C_6^2 t^{-2\alpha-2}} \, dt \leq M_2(t_0) \int_{t_0}^{\infty} |t|^{-\alpha} \, dt < \infty.$$

\[\square\]

4. Chirps generated by spirals

Now we study some a converse of Theorem 4, where we obtain the box dimension of a chirp from the corresponding spiral. We begin with a theorem concerning the box dimension of the graph of a generalized $(\alpha, \beta)$-chirp.

**Theorem 5.** (Box dimension and Minkowski content of the graph of a generalized $(\alpha, \beta)$-chirp) Let $y(x) = p(x)S(q(x))$, $x \in I = (0, c]$, $c > 0$. Let the functions $p(x), q(x)$ and $S(t)$ satisfy the following assumptions:

$$p \in C(\bar{I}) \cap C^1(I), \quad q \in C^1(I), \quad S \in C^1(\mathbb{R}), \quad \text{ (26)}$$

The function $S(t)$ is assumed to be a $2T$-periodic real function defined on $\mathbb{R}$ such that

$$\begin{cases} 
S(a) = S(a + T) = 0 \text{ for some } a \in \mathbb{R}, \\
S(t) \neq 0 \text{ for all } t \in (a, a + T) \cup (a + T, a + 2T),
\end{cases} \quad \text{ (27)}$$

where $T$ is a positive real number and $S(t)$ alternately changes sign on intervals $(a + (k-1)T, a + kT)$, $k \in \mathbb{N}$. Without loss of generality, we take $a = 0$. Let us suppose that $0 < \alpha \leq \beta$ and:

$$p(x) \simeq_1 x^\alpha \quad \text{as} \quad x \to 0, \quad \text{ (28)}$$

$$q(x) \simeq_1 x^{-\beta} \quad \text{as} \quad x \to 0. \quad \text{ (29)}$$

Then $y(x)$ is $d$-dimensional fractal oscillatory near the origin, where $d = 2 - \frac{\alpha}{\beta+1}$. Moreover, $\dim_B(G(y)) = d$ and $G(y)$ is Minkowski nondegenerate.
Theorem 5 is a new version of [5, Theorems 5 and 6]. In Theorem 5 we do not need any assumptions on the curvature function of $y(x) = p(x)S(q(x))$, as it was needed in [5]. Before proving Theorem 5, we shall cite a new criterion for fractal oscillations of a bounded continuous function and after that we continue with two propositions concerning the properties of functions $p, q$ and $S$.

**Theorem 6 (Theorem 2.1. from [15]).** Let $y \in C^1((0, T])$ be a bounded function on $(0, T]$. Let $s \in [1, 2)$ be a real number and let $(a_n)$ be a decreasing sequence of consecutive zeros of $y(x)$ in $(0, T]$ such that $a_n \to 0$ when $n \to \infty$ and let there exist constants $c_1, c_2, \varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have:

$$c_1\varepsilon^{2-s} \leq \sum_{n \geq k(\varepsilon)} \max_{x \in [a_{n+1}, a_n]} |y(x)|(a_n - a_{n+1}),$$  \hspace{1cm} (30)

$$a_{k(\varepsilon)} \sup_{x \in (0, a_{k(\varepsilon)})} |y(x)| + \varepsilon \int_{a_{k(\varepsilon)}}^{a_1} |y'(x)|dx \leq c_2\varepsilon^{2-s},$$  \hspace{1cm} (31)

where $k(\varepsilon)$ is an index function on $(0, \varepsilon_0]$ such that

$$|a_n - a_{n+1}| \leq \varepsilon \text{ for all } n \geq k(\varepsilon) \text{ and } \varepsilon \in (0, \varepsilon_0).$$

Then $y(x)$ is fractal oscillatory near $x = 0$ with $\dim_B G(y) = s$.

We remark that the claim of Theorem 6 is true if we substitute $a_1$, appearing in (31) by $a_{k_0}$, where $k_0$ is a fixed natural number.

**Proposition 1.** Assume that the functions $p(x)$ and $q(x)$ satisfy conditions (26), (28) and (29). Then there exist $\delta_0 > 0$ and positive constants $C_1$ and $C_2$ such that:

$$C_1x^\alpha \leq p(x) \leq C_2x^\alpha, \quad C_1x^{\alpha-1} \leq p'(x) \leq C_2x^{\alpha-1},$$  \hspace{1cm} (32)

$$C_1x^{-\beta} \leq q(x) \leq C_2x^{-\beta}, \quad C_1x^{-\beta-1} \leq -q'(x) \leq C_2x^{-\beta-1},$$  \hspace{1cm} (33)

for all $x \in (0, \delta_0]$. Furthermore, there exists the inverse function $q^{-1}$ of the function $q$ defined on $[m_0, \infty)$, where $m_0 = q(\delta_0)$, and it holds:

$$q^{-1}(t) \asymp t^{-1/\beta} \text{ as } t \to \infty,$$  \hspace{1cm} (34)

$$C_1t^{-\frac{1}{\beta}}(t - s) \leq q^{-1}(s) - q^{-1}(t) \leq C_2s^{-\frac{1}{\beta}}(t - s), \quad m_0 \leq s < t.$$  \hspace{1cm} (35)
Proposition 2. For any function $S(t)$ satisfying (27), and for any function $q(x)$ with properties (26) and (29), we have:

(i) $S(kT) = 0$, $k \in \mathbb{N}$.

(ii) Let $a_k = q^{-1}(kT)$ and $s_k = q^{-1}(t_0 + kT)$, $k \in \mathbb{N}$, where $t_0 \in (0, T)$ is arbitrary. Then there exist $k_0 \in \mathbb{N}$ and $c_0 > 0$ such that $a_k \in (0, \delta_0]$, $y(a_k) = 0$, $s_k \in (a_{k+1}, a_k)$ for all $k \geq k_0$, $a_k \searrow 0$ as $k \to \infty$, $a_k \approx k^{-1/\beta}$ as $k \to \infty$, and

$$\max_{x \in [a_{k+1}, a_k]} |y(x)| \geq c_0(k + 1)^{-\alpha/\beta} \quad \text{for all } k \geq k_0, \ c_0 > 0. \tag{36}$$

(iii) There exists $\varepsilon_0 > 0$ and a function $k : (0, \varepsilon_0) \to \mathbb{N}$ such that

$$\frac{1}{T} \left( \frac{\varepsilon}{TC_2} \right)^{-\frac{\beta}{\alpha + \beta + 1}} \leq k(\varepsilon) \leq \frac{2}{T} \left( \frac{\varepsilon}{TC_2} \right)^{-\frac{\beta}{\alpha + \beta + 1}}. \tag{37}$$

In particular,

$$C_1 T((k + 1)T)^{-\frac{1}{\beta} - 1} \leq a_k - a_{k+1} \leq \varepsilon,$$

for all $k \geq k(\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proofs of Propositions 1 and 2 are provided in the Appendix.

**Proof (Theorem 5).** First we check inequality (30). By Proposition 2 we have:

$$\sum_{k \geq k(\varepsilon)} \max_{x \in [a_{k+1}, a_k]} |y(x)|(a_k - a_{k+1}) \geq c \sum_{k = k(\varepsilon) + 1}^{\infty} (k + 1)^{-\frac{\alpha + \beta + 1}{\beta}}$$

$$= c \sum_{k = k(\varepsilon)}^{\infty} (k)^{-\frac{\alpha + \beta + 1}{\beta}} = ca,$$

where the series $a = \sum_{k = k(\varepsilon)}^{\infty} \left( \frac{1}{k(\varepsilon)} \right)^{\frac{\alpha + \beta + 1}{\beta}}$ is convergent, because of $\frac{\alpha + \beta + 1}{\beta} > 1$. Then using the inequality $(\frac{1}{k(\varepsilon)})^{\frac{\alpha + \beta + 1}{\beta} - 1} < 1$ and (37) we obtain

$$\sum_{k \geq k(\varepsilon)} \max_{x \in [a_{k+1}, a_k]} |y(x)|(a_k - a_{k+1}) \geq ca \geq ca \left( \frac{1}{k(\varepsilon)} \right)^{\frac{\alpha + \beta + 1}{\beta} - 1}$$

$$\geq c_1 \varepsilon^{\frac{\alpha + 1}{\beta + 1}} = c_1 \varepsilon^{2 - (2 - \frac{\alpha + 1}{\beta + 1})},$$

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for all $\varepsilon \in (0, \varepsilon_0)$. By [15, Lemma 2.1.] this implies
\[ 0 < \mathcal{M}^d(G(y)) \quad \text{and} \quad \dim_B G(y) \geq d, \]
where $G(y)$ is the graph of $y$ and $d = 2 - \frac{\alpha + 1}{\beta + 1}$. Now we check inequality (31).

From (28) and (29) it follows that
\[ |y'(x)| = |p'(x)S(q(x)) + p(x)q'(x)S'(q(x))| \leq cx^{\alpha - \beta - 1}, \]
which holds near $x = 0$, where
\[ c = \max \{ \max_{x \in [0,2T]} |S(t)|, \max_{x \in [0,2T]} |S'(t)| \}. \]

By Proposition 2 we have:
\[ a_{k(\varepsilon)} \sup_{x \in (0,a_{k(\varepsilon)})} |y(x)| + \varepsilon \int_{a_{k(\varepsilon)}}^{a_{k(\varepsilon)+\varepsilon}} |y'(x)| dx \leq c\varepsilon^{\frac{\alpha + 1}{\beta + 1}} + \varepsilon[a^{\alpha - \beta}_{k_0} + a^{\alpha - \beta}_{k(\varepsilon)}] \leq c_2 \varepsilon^{\frac{\alpha + 1}{\beta + 1}}, \]
for all $\varepsilon \in (0, \varepsilon_0)$. By [15, Lemma 2.2.] it follows that
\[ \mathcal{M}^{ad}(G(y)) < \infty \quad \text{and} \quad \dim_B G(y) \leq d = 2 - \frac{\alpha + 1}{\beta + 1}. \]

Finally, combining the obtained results, we have that the graph $G(y)$ is Minkowski nondegenerate, and
\[ \dim_B G(y) = 2 - \frac{\alpha + 1}{\beta + 1} = d. \]

Now we can state a spiral-chirp comparison.

**Theorem 7 (Spiral-chirp comparison).** Let $\alpha \in (0, 1)$, and assume that $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, is a function of class $C^2$, such that the planar curve
\[ \Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\} \]
is a spiral $r = f(\varphi)$, $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$, in polar coordinates, near the origin, such that $f(\varphi) \approx \varphi^{-\alpha}$, as $\varphi \to \infty$, and $\dot{\varphi}(t) \approx 1$, as $t \to \infty$, where $\varphi(t)$ is a function of class $C^1$ defined by $\tan \varphi(t) = \frac{\dot{x}(t)}{\dot{x}(t)}$. Define $X(\tau) = x(1/\tau)$. Then $X = X(\tau)$ is $(\alpha, 1)$-chirp-like function, and
\[ \dim_{osc}(x) := \dim_B G(X) = \frac{3 - \alpha}{2}, \]
where $G(X)$ is graph of the function $X$. Furthermore, $G(X)$ is Minkowski nondegenerate.
Proof. Let us write the function $X(\tau)$ in the form

$$X(\tau) = p(\tau) \cos q(\tau), \quad \tau \in (0, \frac{1}{t_0}],$$

where

$$p(\tau) = f(\varphi(\frac{1}{\tau})), \quad q(\tau) = \varphi(\frac{1}{\tau}).$$

The function $p(\tau)$ is increasing near $\tau = 0$ since $\frac{1}{\tau}$ is decreasing, $\varphi(t)$ is increasing and $f(\varphi)$ is decreasing near $\varphi = \infty$. Furthermore, $p \in C([0, 1/t_0])$ since $p(0) = \lim_{\tau \to 0} f(\varphi(1/\tau)) = 0$, by noting that $\dot{\varphi} \simeq 1$ implies $\varphi(t) \to \infty$ as $t \to \infty$. Now, we shall exploit Theorem 5 by checking that its assumptions are satisfied with $S(q) = \cos q$ and $\beta = 1$. The functions $\varphi$, $p$ and $q$ have the following properties:

$$\varphi(t) \simeq t \quad \text{as} \quad t \to \infty \quad \text{or} \quad \varphi(\frac{1}{\tau}) \simeq \frac{1}{\tau} \quad \text{as} \quad \tau \to 0,$$

$$p(\tau) \simeq_1 \tau^\alpha \quad \text{as} \quad \tau \to 0,$$

$$q(\tau) \simeq_1 \frac{1}{\tau} \quad \text{as} \quad \tau \to 0,$$

$$q^{-1}(t) \simeq \frac{1}{t} \quad \text{as} \quad t \to \infty.$$

The function $q$ is decreasing near the origin, thus $q^{-1}$ exists for $t$ large enough. We see that all conditions of Theorem 5 are fulfilled. \qed

Remark 3. Theorem 7 is a new version of [17, Theorem 4]. If we compare Theorems 4 and 7 in terms of conditions, then we see that Theorem 7 requires derivatives of lower order than Theorem 4. Phase plane already gives us the information about the first derivative.

The following result shows that rectifiable spirals generate rectifiable chirp-like functions.

Theorem 8. (Rectifiability of a chirp generated by a rectifiable spiral) Let $\alpha > 1$, and assume that $x : [t_0, \infty) \to \mathbb{R}$, $t_0 > 0$, is a function of class $C^2$ such that the planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ is a rectifiable spiral $r = f(\varphi)$, $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$ in polar coordinates, near the origin, such that $f(\varphi) \simeq_1 \varphi^{-\alpha}$, as $\varphi \to \infty$, $|f''(\varphi)| \leq C \varphi^{-\alpha-2}$ and $\dot{\varphi}(t) \simeq 1$. 

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as \( t \to \infty \), where \( \varphi(t) \) is a function of class \( C^1 \) defined by \( \tan \varphi(t) = \frac{\dot{\chi}(t)}{x(t)} \). Define \( X(\tau) = x(1/\tau) \). Then \( X = X(\tau) \) is \((\alpha, 1)\)-chirp-like rectifiable function near the origin.

To prove the theorem we shall use the following two lemmas.

**Lemma 4.** Let \( F, G \in C^1(I) \), where \( I \) is an open interval in \( \mathbb{R} \), and assume that \( \inf F' > \sup G' \). Then the equation \( F(z) = G(z) \) has at most one solution.

**Proof.** Suppose that there are two different solutions \( z_1 \) and \( z_2 \). Then applying the mean-value theorem to \( F(z_1) - F(z_2) = G(z_1) - G(z_2) \), we obtain that there exist \( \tilde{z}_1 \) and \( \tilde{z}_2 \) such that \( F'(\tilde{z}_1) = G'(\tilde{z}_2) \). Therefore, \( \inf F' \leq \sup G' \). This contradicts the condition \( \inf F' > \sup G' \). \( \square \)

**Lemma 5.** Let \( F \in C^1(0, \infty) \) be such that \( F(z) \sim az \) as \( z \to \infty \) for some \( a < 0 \). Assume that \( \inf F' > -\infty \). Then there exists a nonnegative integer \( k_0 \) such that for each \( k \geq k_0 \) the equation \( \cot z = F(z) \) possesses the unique solution in \( J_k = (k\pi, (k + 1)\pi) \).

**Proof.** Since \( F(z) \) is continuous and \( F(z) \sim az \) as \( z \to \infty \), and \( \cot z \) restricted to \( J_k \) is continuous function onto \( \mathbb{R} \), it follows that the equation \( \cot z = F(z) \) possesses at least one solution \( z_k \) on each interval \( J_k \). We have to show that the solution is unique on each \( J_k \) for all \( k \) large enough.

Since \( m = \inf F' > -\infty \), there exists \( s_0 \in (\pi/2, \pi) \) sufficiently close to \( \pi \) such that \( \cot'(s_0) = -(\sin s_0)^{-2} < m \). The condition \( F(z) \sim az \) implies that, given any fixed \( b \in (a, 0) \), there exists \( M = M(b) > 0 \) such that \( F(z) < bz \) for all \( z \geq M \). Let us fix any such \( b \).

Let \( k_0 \) be a nonnegative integer such that \( b(0, k_0) < \cot s_0 \). It suffices to take \( k_0 > (b\pi)^{-1} \cot s_0 \). Taking \( k_0 \) still larger, we can achieve that \( k_0 \pi \geq M \). Hence, for \( z \geq k_0 \pi \) we have \( F(z) < bz \). In particular,

\[
F(z) < bz \leq b(0, k_0) < \cot s_0.
\]

Since for \( z \geq k_0 \pi \) we have \( F(z) < \cot s_0 \), while \( \cot z \geq \cot s_0 \) for each \( z \in J_k \setminus I_k \), where \( I_k = (k\pi + s_0, (k + 1)\pi) \), then all solutions of equation \( F(z) = \cot z \) for \( z \geq k_0 \pi \) are contained in \( \bigcup_{k \geq k_0} I_k \).

Let us define \( G(z) = \cot z \), and consider the equation \( F(z) = G(z) \) on \( I_k \) for any \( k \geq k_0 \). We have

\[
\sup_{I_k} G' = \cot'(k_0 \pi + s_0) = -(\sin s_0)^{-2} < \inf_{(0, \infty)} F' \leq \inf_{I_k} F'.
\]

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The unique solvability of \( F(z) = G(z) \) on \( I_k \) follows from Lemma 4. The equation is uniquely solvable on \( J_k \) as well, since there are no solutions in \( J_k \setminus I_k \).

**Remark 4.** The condition \( F(z) \sim az \) as \( z \to \infty \) in Lemma 5 can be weakened. It suffices to assume that \( F(z) < bz \) for some \( b < 0 \) and for all \( z \) large enough.

**Remark 5.** The condition \( \inf F' > -\infty \) in Lemma 5 cannot be dropped. To see this, we construct a function \( y = F(z) \) by means of a sequence of lines \( y = b_n z \), where \( a < b_n < 0 \) and \( b_n \to a \) as \( n \to \infty \). We first construct a continuous function \( F_0 \) such that on \( J'_k = (k\pi, (k+1)\pi] \),

\[
F_0(z) = \begin{cases} 
  b_k z, & \text{for } z \in (k\pi, z_k], \\
  \cot z, & \text{for } z \in (z_k, v_k], \\
  b_{k+1} z, & \text{for } z \in (v_k, (k+1)\pi],
\end{cases}
\]

where \( z_k \) and \( v_k \) are respective solutions of equations \( \cot z = b_k z \) and \( \cot b_{k+1} v = b_{k+1} v \) in \( J_k \). The function \( F_0 \) is of class \( C^1 \) everywhere in \((0, \infty)\) except at the points \( z_k \) and \( v_k \). We can perform its smoothing in sufficiently small neighborhoods of these points, in order to get a function \( F \in C^1(0, \infty) \). It is clear that \( F(z) \sim az \) as \( z \to \infty \) and \( \inf F' = -\infty \). But \( F(z) = \cot z \) possesses infinitely many solutions on each interval \( I_k \).

**Remark 6.** Assume that

\[ F(z) = \frac{f(z)}{f'(z)}, \]

where \( f \in C^2(0, \infty) \). (a) The condition \( \inf F' > -\infty \) is equivalent to \( f(z)f''(z) \leq C f'^2(z) \), where \( C \) is a positive constant. (b) The condition \( F(z) < bz \) for \( z \) sufficiently large, where \( b \) is a negative constant (see Remark 4), is satisfied if for all \( z \) large enough we have \( f(z) \geq az^{-a} \) and \( f'(z) \geq a_1 z^{-a-1} \), where \( a > 0 \) and \( a_1 < 0 \) are constants. It suffices to take \( b \in (a/a_1, 0) \).

A variation of Lemma 5 is the following lemma.

**Lemma 6.** Let \( F \in C^1(0, \infty) \) be such that \( F(z) \sim az \) as \( z \to \infty \) for some \( a > 0 \). Assume that \( \sup F' < \infty \). Then there exists a nonnegative integer \( k_0 \) such that for each \( k \geq k_0 \) the equation \( \tan z = F(z) \) possesses the unique solution in \( J_k = ((k - 1/2)\pi, (k + 1/2)\pi) \).
Remark 7. The condition \( F(z) \sim az \) as \( z \to \infty \) for \( a > 0 \) in Lemma 6 can be weakened by assuming that \( F(z) > az \) for some \( a > 0 \) and for all \( z \) large enough. If \( F(z) \) has the form \( F(z) = \frac{f(z)}{f'(z)} \), where \( f \in C^2(0, \infty) \), the condition \( \sup F' < \infty \) is equivalent to \( f(z)f''(z) \geq C f'^2(z) \), where \( C \) is positive constant. Also, in that case, the condition \( F(z) > az \) for \( z \) large enough is satisfied if for all \( z \) large enough we have \( f(z) \geq a_1 z^{-\alpha} \) and \( f'(z) \leq a_2 z^{-\alpha-1} \), where \( a_1 \) and \( a_2 \) are positive constants. It suffices to take \( a \in (0, \frac{a_1}{a_2}) \).

Proof (Theorem 8). We can write the function \( X(\tau) \) in the form \( X(\tau) = p(\tau) \cos q(\tau) \), where \( p(\tau) = f(\varphi(1/\tau)) \simeq \tau^\alpha, \ p'(\tau) \simeq \tau^{\alpha-1} \), \( q(\tau) = \varphi(1/\tau) \simeq \tau^{-1}, \ q'(\tau) \simeq -\tau^{-2} \) as \( \tau \to 0 \). It follows that \( X \) is an \((\alpha, 1)\)-chirp-like function. Using the assumptions of the theorem, for the function

\[
F(t) := \frac{pq'(t)}{p'(t)} (q^{-1}(t)) = \frac{f(t)}{f'(t)}
\]

we have \( F(t) \simeq -t \) as \( t \to \infty \), and \( \frac{f(t)f''(t)}{f'(t)^2} < C \), for \( t \) large enough, \( C > 0 \).

Then there exists \( k_0 \in \mathbb{N} \) such that the equation \( \cot q(t) = F(q(t)) = \frac{p(\tau)q'(\tau)}{p'(\tau)} \) has the unique solution \( s_k \in (a_{k+1}, a_k) \) where \( a_{k+1} = q^{-1}((2k+1)\frac{\pi}{2}) \) and \( a_k = q^{-1}((2k-1)\frac{\pi}{2}) \) for all \( k \geq k_0 \), see Lemma 5 and Remark 4. These solutions are just points of local extrema of \( X(\tau) \) on \((a_{k+1}, a_k), k \geq k_0 \). The sequence \((a_k)_{k \geq 1}\) of zero points of \( X \) on \((0, 1/t_0)\) is decreasing. Hence the sequence \((s_k)\) of consecutive points of local extrema of \( X \) is also decreasing. We have that \( a_k = q^{-1}((2k-1)\frac{\pi}{2}) \simeq k^{-1} \) as \( k \to \infty \). So the same is true also for \( s_k \), i.e., \( s_k \simeq k^{-1} \) as \( k \to \infty \), and we also have \( |X(s_k)| \leq p(s_k) \leq C s_k^\alpha \leq C_1 k^{-\alpha} \). This implies that

\[
\sum_{k=k_0}^{\infty} |X(s_k)| \leq C_1 \sum_{k=k_0}^{\infty} k^{-\alpha} < \infty
\]  

(38)

for \( \alpha > 1 \). The length of the graph \( G(X) \) is defined by

\[
\text{length}(G(X)) := \sup \sum_{i=1}^{m} \|(t_i, X(t_i)) - (t_{i-1}, X(t_{i-1}))\|_2,
\]

where the supremum is taken over all partitions \( 0 = t_0 < t_1 < \ldots < t_m = 1/t_0 \) of the interval \([0, 1/t_0]\) and where \( \|\cdot\|_2 \) denotes the Euclidean norm in \( \mathbb{R}^2 \). Using [14, Lemma 3.1.], it follows that \( \text{length}(G(X)) \leq 2 \sum_k |X(s_k)| + 1/t_0 \).

Then \( X \) is rectifiable due to (38). \( \square \)
Remark 8. Theorem \[\text{[4]}\] can be applied to planar systems with pure imaginary pair of eigenvalues, because the normal forms of such systems satisfy the conditions of Theorem \[\text{[7]}\] see \[\text{[23]}\], Theorem 9. Also, Theorem \[\text{[7]}\] can be applied to related second order differential equations.

Appendix A. Proofs of some technical results

Proof (Lemma \[\text{[3]}\]). From the assumption that \( p(t) \sim t^{-\alpha} \) as \( t \to \infty \) we have that for each \( \varepsilon > 0 \) there exist \( t_0 \geq t_0 \) such that for all \( t \geq t_0 \),

\[
(1 - \varepsilon)t^{-\alpha} < p(t) < (1 + \varepsilon)t^{-\alpha}, \quad (1 - \varepsilon)\alpha t^{-\alpha - 1} < -p'(t) < (1 + \varepsilon)\alpha t^{-\alpha - 1},
\]

where \( K_1 := \frac{1 + \varepsilon}{1 - \varepsilon} \). Then we have

\[
r_{\min}(t) \leq r(t) \leq r_{\max}(t), \quad \text{for all} \quad t \in [t_0, \infty),
\]

where

\[
r_{\min}(t) := \frac{p(t)}{t} \sqrt{1 - K_1 \alpha t^{-1}},
\]

\[
r_{\max}(t) := \frac{p(t)}{t} \sqrt{1 + K_1 \alpha t^{-1} + K_1^2 \alpha^2 t^{-2}},
\]

and without loss of generality we assume that \( t_0 > K_1 \alpha \). Therefore

\[
r(t(\varphi)) - r(t(\varphi + \Delta \varphi)) \geq r_{\min}(t_1) - r_{\max}(t_2) = \frac{p^2(t_1)(1 - K_1 \alpha t_1^{-1}) - p^2(t_2)(1 + K_1 \alpha t_2^{-1} + K_1^2 \alpha^2 t_2^{-2})}{p(t_1) \sqrt{1 - K_1 \alpha t_1^{-1} + p(t_2) \sqrt{1 + K_1 \alpha t_2^{-1} + K_1^2 \alpha^2 t_2^{-2}}}},
\]

where

\[
t_1 = t(\varphi) = \varphi(1 + O(\varphi^{-1})), \quad t_2 = t(\varphi + \Delta \varphi) = \varphi(1 + O(\varphi^{-1})) \quad \text{as} \quad t \to \infty.
\]

Now, using relations

\[
p(t_1) \sqrt{1 - K_1 \alpha t_1^{-1}} \simeq \varphi^{-\alpha}, \quad p(t_2) \sqrt{1 + K_1 \alpha t_2^{-1} + K_1^2 \alpha^2 t_2^{-2}} \simeq \varphi^{-\alpha},
\]

\[
t_1^{-\alpha} \simeq \varphi^{-\alpha}, \quad t_2^{-\alpha} \simeq \varphi^{-\alpha} \quad \text{as} \quad \varphi \to \infty,
\]

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If $\Delta \varphi > 1$, then choosing $\varepsilon < \frac{(\Delta \varphi)^{1/3} - 1}{(\Delta \varphi)^{1/3} + 1}$, we obtain the claim.

**Proof (Proposition 1).** From (28) and (29) inequalities (32) and (33) follow directly by definition. The function $q|_{(0, \delta_0]}$ is positive and decreasing, and its inverse function is defined on $[m_0, \infty)$. Relation (34) follows from (33) applying the well known formula for derivative of the inverse function. Then exploiting the mean value theorem and (34), we get (35).

**Proof (Proposition 2).** The claim in (i) is evident. To prove (ii), it suffices to take $k_0 \in \mathbb{N}$ such that $k_0 T \geq m_0$. We shall prove inequality (36) only, because other facts are easy consequences of Proposition 1. From (28) we obtain that $p(x)$ is positive and increasing function near $x = 0$, and we have

$$\max_{x \in [a_{k+1}, a_k]} |y(x)| \geq p(s_k)|S(q(s_k))| \geq cp(a_{k+1}) \geq c_1(a_{k+1})^\alpha \geq c_0(k + 1)^{-\frac{\beta}{\alpha}},$$

for all $k \geq k_0$, where $c = \min\{|S(t_0)|, |S(t_0 + T)|\}$, $c_1 = cC_1$ and $c_0 = cC_1^2$ are positive constants. Now we prove (iii). Let $\varepsilon > 0$ and let $k(\varepsilon) \in \mathbb{N}$ be such that

$$k(\varepsilon) \geq \frac{1}{T} \left( \frac{\varepsilon}{TC_2} \right)^{-\frac{\beta}{\alpha + 1}} = ce^{-\frac{\beta}{\alpha + 1}}, \quad c = T^{-1}(TC_2)^{-\frac{\beta}{\alpha + 1}}.$$

Let $\varepsilon'_0$ be such that for all $0 < \varepsilon \leq \varepsilon'_0$ it holds $k(\varepsilon) T \geq m_0 = q(\delta_0)$. Further, for all $\varepsilon < c^{-\frac{\beta}{\alpha + 1}}$ we have $2c\varepsilon^{-\frac{\beta}{\alpha + 1}} - c \varepsilon^{-\frac{\beta}{\alpha + 1}} > 1$. So, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$1 < c\varepsilon^{-\frac{\beta}{\alpha + 1}} \leq k(\varepsilon) \leq 2c\varepsilon^{-\frac{\beta}{\alpha + 1}}, \quad \text{for all } \varepsilon < c^{-\frac{\beta}{\alpha + 1}}.$$
Let us take $\varepsilon_0 = \min\{\varepsilon_0', c^{\frac{\beta+1}{\beta}}\}$. Then we can find $k(\varepsilon) \in \mathbb{N}$ such that
\[
c \varepsilon^{-\frac{\beta}{\beta+1}} \leq k(\varepsilon) \leq 2c \varepsilon^{-\frac{\beta}{\beta+1}}, \quad k(\varepsilon)T \geq m_0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0).
\]
Using (35), then for all $k \geq k(\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$ it holds
\[
C_1 T((k + 1)T)^{-\frac{1}{\beta}} \leq a_k - a_{k+1} \leq \varepsilon.
\]

References

[1] V. I. Arnol’d, S. M. Guseǐn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988. Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.

[2] Yves Dupain, Michel Mendès France, and Claude Tricot. Dimensions des spirales. *Bull. Soc. Math. France*, 111(2):193–201, 1983.

[3] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990. Mathematical foundations and applications.

[4] Lana Horvat Dmitrović. Box dimension and bifurcations of one-dimensional discrete dynamical systems. *Discrete Contin. Dyn. Syst.*, 32(4):1287–1307, 2012.

[5] Luka Korkut and Maja Resman. Oscillations of chirp-like functions. *Georgian Math. J.*, 19(4):705–720, 2012.

[6] Luka Korkut, Domagoj Vlah, Darko Žubrinić, and Vesna Županović. Generalized Fresnel integrals and fractal properties of related spirals. *Appl. Math. Comput.*, 206(1):236–244, 2008.

[7] Luka Korkut, Domagoj Vlah, and Vesna Županović. Fractal properties of Bessel functions. arXiv:1304.1762, Unpublished results.

[8] Luka Korkut, Domagoj Vlah, and Vesna Županović. Geometrical properties of a class of systems with spiral trajectories in $\mathbb{R}^3$. arXiv:1211.0918, Unpublished results.
[9] Man Kam Kwong, Mervan Pašić, and James S. W. Wong. Rectifiable oscillations in second-order linear differential equations. *J. Differential Equations*, 245(8):2333–2351, 2008.

[10] Pavao Mardešić, Maja Resman, and Vesna Županović. Multiplicity of fixed points and growth of \( \varepsilon \)-neighborhoods of orbits. *J. Differential Equations*, 253(8):2493–2514, 2012.

[11] Josipa Pina Milišić, Darko Žubrinić, and Vesna Županović. Fractal analysis of Hopf bifurcation for a class of completely integrable nonlinear Schrödinger Cauchy problems. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 60, 32, 2010.

[12] Mervan Pašić. Rectifiable and unrectifiable oscillations for a class of second-order linear differential equations of Euler type. *J. Math. Anal. Appl.*, 335(1):724–738, 2007.

[13] Mervan Pašić. Fractal oscillations for a class of second order linear differential equations of Euler type. *J. Math. Anal. Appl.*, 341(1):211–223, 2008.

[14] Mervan Pašić and Andrija Raguž. Rectifiable oscillations and singular behaviour of solutions of second-order linear differential equations. *Int. J. Math. Anal. (Ruse)*, 2(9-12):477–490, 2008.

[15] Mervan Pašić and Satoshi Tanaka. Fractal oscillations of self-adjoint and damped linear differential equations of second-order. *Appl. Math. Comput.*, 218(5):2281–2293, 2011.

[16] Mervan Pašić and James S. W. Wong. Rectifiable oscillations in second-order half-linear differential equations. *Ann. Mat. Pura Appl. (4)*, 188(3):517–541, 2009.

[17] Mervan Pašić, Darko Žubrinić, and Vesna Županović. Oscillatory and phase dimensions of solutions of some second-order differential equations. *Bull. Sci. Math.*, 133(8):859–874, 2009.

[18] Goran Radunović, Darko Žubrinić, and Vesna Županović. Fractal analysis of Hopf bifurcation at infinity. *International Journal of Bifurcation and Chaos*, 22(12):1230043–1–1230043–15, 2012.
[19] Maja Resman. Epsilon-neighborhoods of orbits and formal classification of parabolic diffeomorphisms. *Discrete Contin. Dyn. Syst.*, 33(8):3767–3790, 2013.

[20] Claude Tricot. *Curves and fractal dimension*. Springer-Verlag, New York, 1995. With a foreword by Michel Mendès France, Translated from the 1993 French original.

[21] James S. W. Wong. On rectifiable oscillation of Euler type second order linear differential equations. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 20, 12 pp. (electronic), 2007.

[22] Hao Wu and Weigu Li. Isochronous properties in fractal analysis of some planar vector fields. *Bull. Sci. Math.*, 134(8):857–873, 2010.

[23] Darko Žubrinić and Vesna Županović. Fractal analysis of spiral trajectories of some planar vector fields. *Bull. Sci. Math.*, 129(6):457–485, 2005.

[24] Darko Žubrinić and Vesna Županović. Fractal analysis of spiral trajectories of some vector fields in $\mathbb{R}^3$. *C. R. Math. Acad. Sci. Paris*, 342(12):959–963, 2006.

[25] Darko Žubrinić and Vesna Županović. Poincaré map in fractal analysis of spiral trajectories of planar vector fields. *Bull. Belg. Math. Soc. Simon Stevin*, 15(5, Dynamics in perturbations):947–960, 2008.

[26] Vesna Županović and Darko Žubrinić. Fractal dimensions in dynamics. In J.-P. Françoise, G.L. Naber, and S.T. Tsou, editors, *Encyclopedia of Mathematical Physics*, volume 2, pages 394–402. Elsevier, Oxford, 2006.