ENDOMORPHISMS OF POSITIVE CHARACTERISTIC TORI: ENTROPY AND ZETA FUNCTION

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Abstract. Let \( F \) be a finite field of order \( q \) and characteristic \( p \). Let \( \mathbb{Z}_F = F[t] \), \( \mathbb{Q}_F = F(t) \), \( \mathbb{R}_F = F((1/t)) \) equipped with the discrete valuation for which \( 1/t \) is a uniformizer, and let \( \mathbb{T}_F = \mathbb{R}_F/\mathbb{Z}_F \) which has the structure of a compact abelian group. Let \( d \) be a positive integer and let \( A \) be a \( d \times d \)-matrix with entries in \( \mathbb{Z}_F \) and non-zero determinant. The multiplication-by-\( A \) map is a surjective endomorphism on \( \mathbb{T}_d \). First, we compute the entropy of this endomorphism; the result and arguments are analogous to those for the classical case \( \mathbb{T}_d = \mathbb{R}_d/\mathbb{Z}_d \). Second and most importantly, we resolve the algebraicity problem for the Artin-Mazur zeta function of all such endomorphisms. As a consequence of our main result, we provide a complete characterization and an explicit formula related to the entropy when the zeta function is algebraic.

1. Positive characteristic tori and statements of the main results

The tori \( \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \) where \( d \) is a positive integer play an important role in number theory, dynamical systems, and many other areas of mathematics. In this paper, we study the entropy and algebraicity of the Artin-Mazur zeta function of a surjective endomorphism on the so-called positive characteristic tori.

Throughout this paper, let \( F \) be the finite field of order \( q \) and characteristic \( p \). Let \( \mathbb{Z}_F = F[t] \) be the polynomial ring over \( F \), \( \mathbb{Q}_F = F(t) \), and

\[
\mathbb{R}_F = F((1/t)) = \left\{ \sum_{i \leq m} a_i t^i : m \in \mathbb{Z}, a_i \in F \text{ for } i \leq m \right\}.
\]

The field \( \mathbb{R}_F \) is equipped with the discrete valuation

\[
v : \mathbb{R}_F \to \mathbb{Z} \cup \{\infty\}
\]
given by \( v(0) = \infty \) and \( v(x) = -m \) where \( x = \sum_{i \leq m} a_i t^i \) with \( a_m \neq 0 \); in fact \( \mathbb{R}_F \) is the completion of \( \mathbb{Q}_F \) with respect to this valuation. Let \( | \cdot | \) denote the non-archimedean absolute value \( |x| = q^{-v(x)} \) for \( x \in \mathbb{R}_F \). We fix an algebraic closure of \( \mathbb{R}_F \) and the absolute value \( | \cdot | \) can be extended uniquely to the algebraic closure (see Proposition 2.1). Let \( \mathbb{T}_F = \mathbb{R}_F/\mathbb{Z}_F \) and let \( \pi : \mathbb{R}_F \to \mathbb{T}_F \) be the quotient map. Every element \( \alpha \in \mathbb{T}_F \) has the unique preimage \( \tilde{\alpha} \in \mathbb{R}_F \) of the form

\[
\tilde{\alpha} = \sum_{i \leq -1} a_i t^i.
\]

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This yields a homeomorphism $\mathbb{T}_F \cong \prod_{i \leq -1} F$ of compact abelian groups. Let $\mu$ be the probability Haar measure on $\mathbb{T}_F$ and let $\rho$ be the metric on $\mathbb{T}_F$ given by $\rho(\alpha, \beta) := |\tilde{\alpha} - \tilde{\beta}|$. We fix a positive integer $d$ and let $\mu^d$ be the product measure on $\mathbb{T}^d_F$.

The analytic number theory, more specifically the theory of characters and $L$-functions, on $\mathbb{T}_F$ has been studied since at least 1965 in work of Hayes [Hay65]. Some relatively recent results include work of Liu-Wooley [LW10] on Waring’s problem and the circle method in function fields and work of Porritt [Por18] and Bienvenu-Lê [BL19] on correlation between the Möbius function and a character over $\mathbb{Z}_F$. For a recent work in the ergodic theory side, we refer the reader to the paper by Bergelson-Leibman [BL16] and its reference in which the authors establish a Weyl-type equidistribution theorem.

Let $A \in M_d(\mathbb{Z}_F)$ having non-zero discriminant. The multiplication-by-$A$ map yields a surjective endomorphism of $\mathbb{T}^d_F$ for which $\mu^d$ is an invariant measure, we abuse the notation by using $A$ to denote this endomorphism. Our first result is the following:

**Theorem 1.1.** Let $h(\mu^d, A)$ denote the entropy of $A$ with respect to $\mu^d$ and let $h(A)$ denote the topological entropy of $A$. Let $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of $A$. We have:

$$h(A) = h(\mu^d, A) = \sum_{i=1}^d \log \max\{|\lambda_i|, 1\}.$$ 

**Remark 1.2.** This is the same formula as the entropy of surjective endomorphisms of $\mathbb{T}^d$. The proof is not surprising either: we use similar arguments to the classical ones presented in the books by Walters [Wal82] and Viana-Oliveira [VO16] together with several adaptations to the non-archimedean setting of $\mathbb{R}^d_F$ and $\mathbb{T}^d_F$. What is important is the relationship between the entropy and the Artin-Mazur zeta function in the next main result.

Let $f : X \to X$ be a map from a topological space $X$ to itself. For each $k \geq 1$, let $N_k(f)$ denote the number of isolated fixed points of $f^k$. Assume that $N_k(f)$ is finite for every $k$, then one can define the Artin-Mazur zeta function [AM65]:

$$\zeta_f(z) = \exp \left( \sum_{k=1}^{\infty} \frac{N_k(f)}{k} z^k \right).$$

When $X$ is a compact differentiable manifold and $f$ is a smooth map such that $N_k(f)$ grows at most exponentially in $k$, the question of whether $\zeta_f(z)$ is algebraic is stated in [AM65]. The rationality of $\zeta_f(z)$ when $f$ is an Axiom A diffeomorphism is established by Manning [Man71] after earlier work by Guckenheimer [Guc70]. On the other hand, when $X$ is an algebraic variety defined over a finite field and $f$ is the Frobenius morphism, the function $\zeta_f(z)$ is precisely the classical zeta function of the variety $X$ and its rationality is conjectured by Weil [Wei49] and first established by Dwork [Dwo60]. For the dynamics of a univariate rational function, rationality of $\zeta_f(z)$ is established by Hinkkanen in characteristic zero [Hin94] while Bridy [Bri12, Bri16] obtains both rationality and transcendence results over positive characteristic when $f$ belongs to certain special families of rational functions. As before, let $A \in M_d(\mathbb{Z}_F)$ and we use $A$ to denote the induced endomorphism on $\mathbb{T}^d_F$. We will show that $N_k(A) < \infty$ for every $n$ and hence one can define the zeta function $\zeta_A(z)$. 
As a consequence of our next main result, we resolve the algebraicity problem for \( \zeta_A(z) \): we provide a complete characterization and an explicit formula when \( \zeta_A(z) \) is algebraic. We need a couple of definitions before stating our result.

Let \( K \) be a finite extension of \( \mathbb{R}_F \). Let

\[
\mathcal{O}_K := \{ \alpha \in K : |\alpha| \leq 1 \},
\]

\[
\mathcal{O}'_K = \{ \alpha \in K : |\alpha| = 1 \}, \quad \text{and}
\]

\[
p_K := \{ \alpha \in K : |\alpha| < 1 \}
\]

respectively denote the valuation ring, unit group, and maximal ideal. In particular:

\[
\mathcal{O} := \mathcal{O}_{\mathbb{R}_F} = F[[1/t]] \quad \text{and} \quad \mathfrak{p} := \mathfrak{p}_{\mathbb{R}_F} = \left\{ \sum_{i \geq -1} a_i t^i : a_i \in F \ \forall i \right\}.
\]

Note that \( \mathfrak{p} \) is the compact open subset of \( \mathbb{R}_F \) that is both the open ball of radius 1 and closed ball of radius \( 1/q \) centered at 0. The field \( \mathcal{O}_K/p_K \) is a finite extension of \( \mathcal{O}/\mathfrak{p} = F \) and the degree of this extension is called the inertia degree of \( K/\mathbb{R}_F \) \cite[p. 150]{Neu99}. Let \( \delta \) be this inertia degree, then \( \mathcal{O}_K/p_K \) is isomorphic to the finite field \( GF(q^\delta) \). By applying Hensel’s lemma \cite[pp. 129–131]{Neu99} for the polynomial \( X^{q^\delta-1} - 1 \), we have that \( K \) contains all the roots of \( X^{q^\delta-1} - 1 \). These roots together with 0 form a unique copy of \( GF(q^\delta) \) in \( K \) called the Teichmüller representatives. This allows us to regard \( GF(q^\delta) \) as a subfield of \( K \); in fact \( GF(q^\delta) \) is exactly the set of all the roots of unity in \( K \) together with 0. For every \( \alpha \in \mathcal{O}_K \), we can express uniquely:

\[
\alpha = \alpha(0) + \alpha(1)
\]

where \( \alpha(0) \in GF(q^\delta) \) and \( \alpha(1) \in p_K \).

**Definition 1.3.** Let \( \alpha \) be algebraic over \( \mathbb{R}_F \) such that \( |\alpha| \leq 1 \). Let \( K \) be a finite extension of \( \mathbb{R}_F \) containing \( \alpha \). We call \( \alpha(0) \) and \( \alpha(1) \) in \( \mathbb{R}_F \) respectively the constant term and \( p \)-term of \( \alpha \); they are independent of the choice of \( K \). When \( |\alpha| = 1 \), the order of \( \alpha \) modulo \( \mathfrak{p} \) means the order of \( \alpha(0) \) in the multiplicative group \( GF(q^\delta) \) where \( \delta \) is the inertia degree of \( K/\mathbb{R}_F \); this is independent of the choice of \( K \) as well. In fact, this order is the smallest positive integer \( n \) such that \( |\alpha^n - 1| < 1 \).

We identify the rational functions in \( \mathbb{C}(z) \) to the corresponding Laurent series in \( \mathbb{C}((z)) \).

**Definition 1.4.** A series \( f(z) \in \mathbb{C}((z)) \) is called D-finite if all of its formal derivatives \( f^{(n)}(z) \) for \( n = 0, 1, \ldots \) span a finite dimensional vectors space over \( \mathbb{C}(z) \). Equivalently, there exist an integer \( n \geq 0 \) and \( a_0(z), \ldots, a_n(z) \in \mathbb{C}[z] \) with \( a_n \neq 0 \) such that:

\[
a_n(z)f^{(n)}(z) + a_{n-1}f^{(n-1)}(z) + \ldots + a_0(z)f(z) = 0.
\]

**Remark 1.5.** Suppose that \( f(z) \in \mathbb{C}[[z]] \) is algebraic then \( f \) is D-finite, see [Sta80] Theorem 2.1.

Our next main result is the following:

**Theorem 1.6.** Let \( A \in M_d(\mathbb{Z}_F) \) and put \( r(A) = \prod_{\lambda} \max\{1, |\lambda|\} \) where \( \lambda \) ranges over all the \( d \) eigenvalues of \( A \); we have \( r(A) = e^{h(A)} \) when \( \det(A) \neq 0 \) thanks to Theorem \( [\text{[L]}] \). Among the \( d \) eigenvalues of \( A \), let \( \mu_1, \ldots, \mu_M \) be all the eigenvalues
that are roots of unity and let \( \eta_1, \ldots, \eta_N \) be all the eigenvalues that have absolute value 1 and are not roots of unity. For \( 1 \leq i \leq M \), let \( m_i \) denote the order of \( \mu_i \) modulo \( p \). For \( 1 \leq i \leq N \), let \( n_i \) denote the order of \( \eta_i \) modulo \( p \). We have:

(a) Suppose that for every \( j \in \{1, \ldots, N\} \), there exists \( i \in \{1, \ldots, M\} \) such that \( m_i | n_j \). Then \( \zeta_A(z) \) is algebraic and

\[
\zeta_A(z) = (1 - r(A)z)^{-1} \prod_{1 \leq i \leq M} \prod_{1 \leq i_1 < i_2 < \ldots < i_\ell \leq M} R_{A,i_1,\ldots,i_\ell}(z)
\]

where \( R_{A,i_1,\ldots,i_\ell}(z) := \left(1 - (r(A)z)^{\gcd(m_{i_1}, \ldots, m_{i_\ell})}\right)^{(-1)^{\ell+1}/\gcd(m_{i_1}, \ldots, m_{i_\ell})} \).

(b) Otherwise suppose there exists \( j \in \{1, \ldots, N\} \) such that for every \( i \in \{1, \ldots, M\} \), we have \( m_i \nmid n_j \). Then the series \( \sum_{k=1}^\infty N_k(A)z^k \) converges in the open disk \( \{ z \in \mathbb{C} : |z| < 1/r(A) \} \) and it is not \( D \)-finite. Consequently, the function \( \zeta_A(z) \) is transcendental.

Remark 1.7. We allow the possibility that any (or even both) of \( M \) and \( N \) to be 0. When \( N = 0 \), the condition in (a) is vacuously true and \( \zeta_A(z) \) is algebraic in this case. When \( N = 0 \) and \( M = 0 \) meaning that none of the eigenvalues of \( A \) has absolute value 1, the product in (a) is the empty product and

\[
\zeta_A(z) = \frac{1}{1 - r(A)z}.
\]

When \( M = 0 \) and \( N > 0 \), the condition in (b) is vacuously true and \( \zeta_A(z) \) is transcendental in this case.

Our results are quite different from results in work of Baake-Lau-Paskunas \[BLP10\]. In \[BLP10\], the authors prove that the zeta function of endomorphisms of the classical tori \( T^d \) are always rational. In our setting, we have cases when the zeta function is rational, transcendental, or algebraic irrational:

Example 1.8. Let \( F = \text{GF}(7) \) and let \( A \) be the diagonal matrix with diagonal entries \( \alpha, \beta \in \text{GF}(7)^* \) where \( \alpha \) has order 2 and \( \beta \) has order 3. Then

\[
\zeta_A(z) = \frac{(1 - z^2)^{1/2}(1 - z^3)^{1/3}}{(1 - z)(1 - z^6)^{1/6}}
\]

is algebraic irrational.

In work of Bell-Miles-Ward \[BMW14\], the authors conjecture and obtain some partial results concerning the following Pólya-Carlson type dichotomy \[Car21, Pöy28\] for a slightly different zeta function: it is either rational or admits a natural boundary at its radius of convergence.

Conjecture 1.9 (Bell-Miles-Ward, 2014). Let \( \theta : X \to X \) be an automorphism of a compact metric abelian group with the property that \( \tilde{N}_k(\theta) < \infty \) for every \( k \geq 1 \) where \( \tilde{N}_k(\theta) \) denotes the number of fixed points of \( \theta^k \). Then

\[
\tilde{\zeta}_\theta(z) := \exp\left(\sum_{k=1}^\infty \frac{\tilde{N}_k(\theta)}{k} z^k\right)
\]

is either a rational function or admits a natural boundary.
Remark 1.10. The difference between \( \tilde{\zeta} \) in \(1.9\) and the Artin-Mazur zeta function \( \zeta_f \) is that the latter involves the number of isolate fixed points. Example 1.8 is not included in Conjecture 1.9 since \( A_6 \) is the identity matrix and hence \( N_6(A) = \infty \) while we have \( N_6(A) = 0 \) (see Lemma 4.1). When \( A \in M_d(\mathbb{Z}) \) has the property that none of its eigenvalues is a root of unity, one can show that \( N_k(A) = \tilde{N}_k(A) \) and hence \( \zeta_A(z) = \tilde{\zeta}_A(z) \). Conjecture 1.9 predicts that when \( M = 0 \) and \( N > 0 \) in Theorem 1.6, the zeta function \( \zeta_A(z) = \tilde{\zeta}_A(z) \) admits the circle of radius \( 1/r(A) \) as a natural boundary. We can only prove this in some special cases and leave it for future work.

For the proof of Theorem 1.6, we first derive a formula for \( N_k(A) \) and it turns out that one needs to study \( |\lambda^k - 1| \) where \( \lambda \) is an eigenvalue of \( A \). When \( |\lambda| \neq 1 \), one immediately has \( |\lambda^k - 1| = \max\{1, |\lambda|^k\} \). However, when \( |\lambda| = 1 \) (i.e. \( \lambda \) is among the \( \mu_i \)'s and \( \eta_j \)'s), a more refined analysis is necessary to study \( |\lambda^k - 1| \).

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Notes added in May 2022. This paper is superseded by \[BGNS\] by Bell and the authors and no longer intended for publication. Inspired by the earlier work \[BNZ20, BNZ\], the paper \[BGNS\] establishes a general Pólya-Carlson criterion and applies this to confirm that the zeta function \( \zeta_A(z) \) admits the circle of radius \( 1/r(A) \) as a natural boundary in the transcendence case (see Remark 1.10).

2. Normed vector spaces and linear maps

Throughout this section, let \( K \) be a field that is complete with respect to a nontrivial absolute value \( | \cdot | \); nontriviality means that there exists \( x \in K^* \) such that \( |x| \neq 1 \). We have:

**Proposition 2.1.** Let \( E/K \) be a finite extension of degree \( n \). Then \( | \cdot | \) can be extended in a unique way to an absolute value on \( E \) and this extension is given by the formula:

\[
|\alpha| = |N_{E/K}(\alpha)|^{1/n} \text{ for every } \alpha \in E.
\]

The field \( E \) is complete with respect to this extended absolute value.

**Proof.** See \[Neu99\] pp. 131–132].

We now fix an algebraic closure of \( K \) and extend \( | \cdot | \) to an absolute value on this algebraic closure thanks to Proposition 2.1. For a vector space \( V \) over \( K \), a norm on \( V \) is a function \( \| \cdot \| : V \to \mathbb{R}_{\geq 0} \) such that:

- \( \|x\| = 0 \text{ iff } x = 0. \)
- \( \|cx\| = |c| \cdot \|x\| \text{ for every } c \in K \text{ and } v \in V. \)
- \( \|x + y\| \leq \|x\| + \|y\| \text{ for every } x, y \in V. \)
Two norms $\| \cdot \|$ and $\| \cdot \|'$ on $V$ are said to be equivalent if there exists a positive constant $C$ such that
\[
\frac{1}{C}\|x\| \leq \|x\|' \leq C\|x\|
\]
for every $x \in V$. It is well-known that any two norms on a finite dimensional vector space $V$ are equivalent to each other and $V$ is complete with respect to any norm, see [Neu99] pp. 132–133.

**Proposition 2.2.** Let $V$ be a vector space over $K$ of finite dimension $d > 0$. Let $\ell : V \to V$ be an invertible $K$-linear map such that there exist $\lambda \in K^*$ and a basis $x_1, \ldots, x_d$ of $V$ over $K$ with:
\[
\ell(x_1) = \lambda x_1 \text{ and } \ell(x_i) = \lambda x_i + x_{i-1} \text{ for } 2 \leq i \leq d;
\]
in other words, the matrix of $\ell$ with respect to $x_1, \ldots, x_d$ is one single Jordan block with eigenvalue $\lambda$. Let $\delta > 0$. Then there exists a norm $\| \cdot \|$ on $V$ such that:
\[
(1 - \delta)|\lambda| \cdot \|x\| \leq \|\ell(x)\| \leq (1 + \delta)|\lambda| \cdot \|x\|
\]
for every $x \in V$.

**Proof.** We proceed by induction on $d$. The case $d = 1$ is obvious since we can take $\| \cdot \|$ to be any norm and we have $\|\ell(x_1)\| = |\lambda|\|x_1\|$. Let $d \geq 2$ and suppose the proposition holds for any vector space of dimension at most $d - 1$. Let $V' = \text{Span}(x_1, \ldots, x_{d-1})$. By the induction hypothesis, there exists a norm $\| \cdot \|'$ on $V'$ such that:
\[
(1 - \delta)|\lambda| \cdot \|x'\|' \leq \|\ell(x')\|' \leq (1 + \delta)|\lambda| \cdot \|x'\|'
\]
for every $x' \in V'$.

Let $M$ be a positive number such that:
\[
\delta|\lambda|M \geq \|x_{d-1}\|'.
\]
Every $x \in V$ can be written uniquely as $x = ax_d + x'$ where $a \in K$ and $x' \in V'$, then we define the norm $\| \cdot \|$ on $V$ by the formula:
\[
\|x\| = |a|M + \|x'\|'.
\]
Note that $\ell(x) = a\lambda x_d + ax_{d-1} + \ell(x')$ and $\|\ell(x)\| = |\lambda|a|M + \|\ell(x') + ax_{d-1}\|'$. Therefore:
\[
\|\ell(x)\| \geq |\lambda|a|M + \|\ell(x')\|' - |a| \cdot \|x_{d-1}\|'
\]
\[
\geq (1 - \delta)|\lambda|a|M + (1 - \delta)|\lambda| \cdot \|x'\|' = (1 - \delta)|\lambda| \cdot \|x\|
\]
where the last inequality follows from (3) and (4). The desired upper bound on $\|\ell(x)\|$ is obtained in a similar way:
\[
\|\ell(x)\| \leq |\lambda|a|M + \|\ell(x')\|' + |a| \cdot \|x_{d-1}\|'
\]
\[
\leq (1 + \delta)|\lambda|a|M + (1 + \delta)|\lambda| \cdot \|x'\|' = (1 + \delta)|\lambda| \cdot \|x\|
\]
and we finish the proof. \qed

**Proposition 2.3.** Let $V$ be a vector space over $K$ of finite dimension $d > 0$. Let $\ell : V \to V$ be an invertible $K$-linear map such that the characteristic polynomial $P(X)$ of $\ell$ is the power of an irreducible polynomial in $K[X]$. By Proposition 2.2.
all the roots of $P$ have the same absolute value denoted by $\theta$. Let $\delta > 0$. Then there exists a norm $\| \cdot \|$ on $V$ such that $$ (1 - \delta)\|x\| \leq \|\ell(x)\| \leq (1 + \delta)\|x\| $$ for every $x \in V$.

**Proof.** Let $E$ be the splitting field of $P(X)$ over $K$. Let $V_E = E \otimes_K V$ and we still use $\ell$ to denote the induced linear operator on $V_E$. In the Jordan canonical form of $\ell$, let $s$ denote the number of Jordan blocks. Then we have a basis $x_{1,1}, \ldots, x_{1,d_1}, \ldots, x_{s,1}, \ldots, x_{s,d_s}$ of $V_E$ over $E$ such that for each $1 \leq i \leq s$, the map $\ell$ maps $V_{E,i} := \text{Span}_E(x_{i,1}, \ldots, x_{i,d_i})$ to itself and the matrix representation of $\ell$ with respect to $x_{i,1}, \ldots, x_{i,d_i}$ is the $i$-th Jordan block. By Proposition 2.2 there exists a norm $\| \cdot \|_i$ on $V_{E,i}$ such that $$ (1 - \delta)\|x\|_i \leq \|\ell(x)\|_i \leq (1 + \delta)\|x\|_i $$ for every $x \in V_{E,i}$. We can now define $\| \cdot \|$ on $V_E = V_{E,1} \oplus \cdots \oplus V_{E,s}$ as $\| \cdot \|_1 + \cdots + \| \cdot \|_s$. Then the restriction of $\| \cdot \|$ on $V$ is the desired norm. $\square$

**Corollary 2.4.** Let $V$ be a vector space over $K$ of finite dimension $d > 0$. Let $\ell : V \to V$ be an invertible $K$-linear map. Then there exist a positive integer $s$, subspaces $V_1, \ldots, V_s$ of $V$, and positive numbers $\theta_1, \ldots, \theta_s$ with the following properties:

(i) $\ell(V_i) \subseteq V_i$ for $1 \leq i \leq s$ and $V = V_1 \oplus \cdots \oplus V_s$.

(ii) The multiset $$ \{ |\lambda| : \text{eigenvalues } \lambda \text{ of } V \text{ counted with multiplicities} \} $$ of order $d$ is equal to the multiset $$ \{ \theta_1, \ldots, \theta_1, \theta_2, \ldots, \theta_2, \ldots, \theta_s, \ldots, \theta_s \} $$ in which the number of times $\theta_i$ appears is $\dim(V_i)$ for $1 \leq i \leq s$.

(iii) For every $\delta > 0$, for $1 \leq i \leq s$, there exists a norm $\| \cdot \|_i$ on $V_i$ such that $$ (1 - \delta)\theta_i\|x\|_i \leq \|\ell(x)\|_i \leq (1 + \delta)\theta_i\|x\|_i $$ for every $x \in V_i$.

**Proof.** By [DF04, p. 424], there exist $\ell$-invariant subspaces $V_1, \ldots, V_s$ of $V$ such that $V = V_1 \oplus \cdots \oplus V_s$ and for $1 \leq i \leq s$, the characteristic polynomial $P_i$ of the restriction of $\ell$ to $V_i$ is a power of an irreducible factor over $K$ of the characteristic polynomial of $\ell$. Let $\theta_i$ denote the common absolute value of the roots of $P_i$. Then we apply Proposition 2.2 and finish the proof. $\square$

### 3. The proof of Theorem 1.1

Recall from Section [1] that $\pi : \mathbb{R}_F \to \mathbb{P}_F$ denotes the quotient map,

$$ \mathfrak{p} := \mathfrak{p}_{\mathbb{R}_F} = \frac{1}{t}F[[1/t]] = \left\{ \sum_{i \leq -1} a_i t^i : a_i \in F \forall i \right\}, $$

every element $\alpha \in \mathbb{P}_F$ has the unique preimage $\tilde{\alpha} \in \mathbb{R}_F$ of the form

$$ \tilde{\alpha} = \sum_{i \leq -1} a_i t^i \in \mathfrak{p}, $$
µ denotes the probability Haar measure on \( T_F \), and \( \rho \) is the metric on \( T_F \) given by 
\[
\rho(\alpha, \beta) = |\tilde{\alpha} - \tilde{\beta}|.
\]
Let \( \tilde{\mu} \) be the Haar measure on \( \mathbb{R}_F \) normalized so that \( \tilde{\mu}(\mathbb{D}_F) = 1 \). Therefore, we have that \( \mathbb{D}_F \) and \( T_F \) are isometric as metric spaces and isomorphic as probability spaces.

Let \( d \) be a positive integer. On \( T_F^d \) and \( \mathbb{R}_F^d \) we have the respective product measures \( \mu^d \) and \( \tilde{\mu}^d \). Let \( \| \cdot \|_{(d)} \) be the norm on \( \mathbb{R}_F^d \) given by:
\[
|(x_1, \ldots, x_d)|_{(d)} = \max_{1 \leq i \leq d} |x_i|.
\]
Then the induced metric \( \rho_{(d)} \) on \( T_F^d \) is:
\[
\rho_{(d)}((\alpha_1, \ldots, \alpha_d), (\beta_1, \ldots, \beta_d)) = \max_{1 \leq i \leq d} |\tilde{\alpha}_i - \tilde{\beta}_i|.
\]

**Proposition 3.1.** Let \( V \) be a vector space over \( \mathbb{R}_F \) of dimension \( d \). Let \( \| \cdot \| \) be a norm on \( V \) and let \( \eta \) be a Haar measure on \( V \). There exist positive constants \( C_1 \) and \( C_2 \) such that the open ball
\[
B(r^-) := \{x \in V : \|x\| < r\}
\]
and the closed ball
\[
B(r) := \{x \in V : \|x\| \leq r\}
\]
satisfy
\[
C_1 r^d < \eta(B(r^-)), \eta(B(r)) < C_2 r^d
\]
for every \( r > 0 \).

**Proof.** After choosing a basis, we may identify \( V \) as \( \mathbb{R}_F^d \); recall the norm \( \| \cdot \|_{(d)} \) above. By uniqueness up to scaling of Haar measures, we may assume that \( \eta \) is the Haar measure normalized so that the set
\[
B' := \{(x_1, \ldots, x_d) \in \mathbb{R}_F^d : |(x_1, \ldots, x_d)|_{(d)} = \max_{1 \leq i \leq d} |x_i| \leq 1\}
\]
has \( \eta(B') = 1 \).

Since \( \| \cdot \| \) and \( \| \cdot \|_{(d)} \) are equivalent to each other, there exist positive \( C_3 \) and \( C_4 \) such that both \( B(r^-) \) and \( B(r) \) contain
\[
B'(C_3 r) := \{(x_1, \ldots, x_d) \in \mathbb{R}_F^d : |(x_1, \ldots, x_d)|_{(d)} = \max_{1 \leq i \leq d} |x_i| \leq C_3 r\}
\]
and are contained in
\[
B'(C_4 r) = \{(x_1, \ldots, x_d) \in \mathbb{R}_F^d : |(x_1, \ldots, x_d)|_{(d)} = \max_{1 \leq i \leq d} |x_i| \leq C_4 r\}.
\]

Let \( q^m \) (respectively \( q^n \)) be the largest (respectively smallest) power of \( q \) that is smaller than \( C_3 r \) (respectively larger than \( C_4 r \)). Then we have:
\[
\eta(B'(C_3 r)) \geq q^m d > (C_3 r/q)^d \quad \text{and} \quad \eta(B'(C_4 r)) \leq q^n d < (C_4 qr)^d.
\]
This finishes the proof. \( \Box \)

We apply Corollary 2.3 for the vector space \( \mathbb{R}_F^d \) and the multiplication-by-\( A \) map to get the invariant subspaces \( V_1, \ldots, V_s \) and positive numbers \( \theta_1, \ldots, \theta_s \). Fix a Haar measure \( \eta_S \) on \( V_i \) and let \( \eta := \eta_1 \times \cdots \times \eta_s \) which is a Haar measure on \( \mathbb{R}_F^d \). Let \( c > 0 \) such that \( \tilde{\mu}_d^c = c\eta \).

Fix \( \delta > 0 \), we assume that \( \delta \) is sufficiently small so that \( (1 + \delta)\theta_i < 1 \) whenever \( \theta_i < 1 \). For \( 1 \leq i \leq s \), let \( \| \cdot \|_i \) be a norm on \( V_i \) as given in Corollary 2.3. Every
We aim to obtain an upper bound on $\rho$ independent of $\epsilon$ thanks to equivalence of these norms. Hence for part (i), we can characterize the set $\| \cdot \|$ by:

Let $C$ be a positive constant for every non-zero $z \in \mathbb{Z}_d^d$ we define the norm $\| \cdot \|$ of the form $\| \cdot \|_{\beta}$ for some $\beta \in \mathbb{Z}_d^d$, let $d = (\tilde{\alpha} \cdot \tilde{A} \cdot y) = \|Ax - Ay\|$.

Lemma 3.2. We still use $\pi$ to denote the quotient map $\mathbb{R}_d^d \to \mathbb{T}_d^d$. There exists a positive constant $\alpha$ such that the following hold:

(i) For any $x \in \mathbb{P}^d$ and $y \in \mathbb{R}_d^d$, if $\|x - y\| \leq \alpha$ then $y \in \mathbb{P}^d$.

(ii) For any $x, y \in \mathbb{P}_d^d$ such that $\|x - y\| \leq \alpha$ and $\tau(\pi(\alpha x), \pi(\alpha y)) \leq \alpha$, we have $\tau(\pi(\alpha X), \pi(\alpha Y)) = \|Ax - Ay\|$.

Proof. For part (i), we can characterize the set $\mathbb{P}^d$ as the set of $x \in \mathbb{R}_d^d$ such that $\|x\| \leq 1/q$. Hence when $\|x - y\|$ is sufficiently small, we have that $\|x - y\| \leq 1/q$ thanks to equivalence of these norms. Hence $x - y \in \mathbb{P}^d$ and we have $y \in \mathbb{P}^d$.

We now consider part (ii). Since $\|z\| \geq 1$ for every non-zero $z \in \mathbb{Z}_d^d$ and since $\| \cdot \|$ and $\| \cdot \|_{\beta}$ are equivalent, there exists a positive constant $\alpha$ such that $\|z\| \geq \alpha$ for every non-zero $z \in \mathbb{Z}_d^d$.

There exists $\alpha$ such that $\|w\| \leq \alpha \|w\|$ for every $w \in \mathbb{R}_d^d$; for instance we may take $\alpha = (1 + \delta) \max_{1 \leq i \leq s} \theta_i$ thanks to the definition of $\| \cdot \|$ and properties of the $\| \cdot \|_{\beta}$s in Corollary 2.4.

We now choose $\alpha$ to be any positive constant such that $\alpha < \frac{\alpha}{\alpha + 1}$. Let $x, y \in \mathbb{R}_d^d$ satisfying conditions in the statement of the lemma. We have $\alpha \geq \tau(\pi(\alpha x), \pi(\alpha y)) = \|Ax - Ay\|$.

for some $z \in \mathbb{Z}_d^d$. If $z \neq 0$ then we have

$$\alpha \geq \tau(\pi(\alpha x), \pi(\alpha y)) = \|Ax - Ay\| \geq \|z\| - \|Ax - Ay\| + \|z\| \geq \alpha - \alpha,$$

contradicting the choice of $\alpha$. Hence $z = 0$ and we are done.

Proof of Theorem 1.1. Let $\alpha = (\alpha_1, \ldots, \alpha_\alpha) \in \mathbb{Z}_d^d$ and let $x = (\alpha_1, \ldots, \alpha_\alpha)$ which is the preimage of $\alpha$ in $\mathbb{P}^d$. Let $\epsilon > 0$ and $n \geq 1$. All the implicit constants below might depend on the choice of the norms $\| \cdot \|_{\beta}$s hence depending on $\delta$ but they are independent of $\epsilon$ and $n$.

Let $B(\alpha, \epsilon, n) := \{ \beta = (\beta_1, \ldots, \beta_\alpha) \in \mathbb{T}_d^d : \rho_{\beta}(A^j \alpha, A^j \beta) < \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \}.$

We aim to obtain an upper bound on $\mu^d(B(\alpha, \epsilon, n))$. Thanks to equivalence between $\rho_{\beta}$ and $\tau$, there exists a positive constant $\alpha$ such that $B(\alpha, \epsilon, n)$ is contained in

$B'(\alpha, \epsilon, n) := \{ \beta = (\beta_1, \ldots, \beta_\alpha) \in \mathbb{T}_d^d : \tau(A^j \alpha, A^j \beta) < \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \}.$

For $\beta = (\beta_1, \ldots, \beta_\alpha) \in B'(\alpha, \epsilon, n)$, let $y = (\tilde{\beta}_1, \ldots, \tilde{\beta}_\alpha)$ and we have $\|x - y\| = \tau(\alpha, A^j \beta) = \epsilon$. When $\epsilon$ is sufficiently small so that $\epsilon$ is smaller than the constant $\alpha$ in Lemma 3.2. we can apply this lemma repeatedly to get

$B'(\alpha, \epsilon, n) = \{ \pi(y) : y \in \mathbb{P}^d \text{ and } \|A^j x - A^j y\| < \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \}.$
By Lemma [3.2], the condition \( y \in \mathbb{R}^d \) is automatic once we have \( \|x - y\| < C_8 \epsilon < C_5 \) and \( x \in \mathbb{R}^d \). Let

\[
\tilde{B}'(x, \epsilon, n) := \{ y \in \mathbb{R}^d : \| A^j x - A^j y \| < C_8 \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \},
\]

we have \( \mu^d(\tilde{B}'(x, \epsilon, n)) = \mu^d(\tilde{B}'(x, \epsilon, n)) = c_\eta(\tilde{B}'(x, \epsilon, n)) \).

We express \( x = x_1 + \ldots + x_s \) and \( y = y_1 + \ldots + y_s \) where each \( x_i, y_i \in V_i \). The condition in the description of \( \tilde{B}'(x, \epsilon, n) \) is equivalent to \( \|x_i - y_i\| < C_8 \epsilon \) and \( \|A^j x_i - A^j y_i\| < C_8 \epsilon \) for every \( 1 \leq i \leq s \) and \( 1 \leq j \leq n - 1 \). We use Corollary [2.3] to have:

\[
(1 - \delta)\theta_i^2 \|x_i - y_i\| < \|A^j x_i - A^j y_i\| \leq ((1 + \delta)\theta_i)^2 \|x_i - y_i\|.
\]

Let \( I = \{ i \in \{1, \ldots, s\} : \theta_i \geq 1 \} \) and since we choose \( \delta \) sufficiently small so that \( (1 + \delta)\theta_i < 1 \) whenever \( \theta_i < 1 \), inequality (5) implies that the set \( \tilde{B}'(x, \epsilon, n) \) is contained in the set:

\[
\{ y = y_1 + \ldots + y_s : \|x_i - y_i\| < C_8 \epsilon((1 - \delta)\theta_i)^{-1} \text{ for } i \in I \\
\quad \quad \text{and } \|x_i - y_i\| < C_8 \epsilon \text{ for } i \notin I \}.
\]

Let \( d_i = \dim(V_i) \) for \( 1 \leq i \leq s \). By Proposition [3.1] there exists a constant \( C_9 \) such that:

\[
\mu^d(B'(\alpha, \epsilon, n)) = c_\eta(\tilde{B}'(x, \epsilon, n)) < C_9 \prod_{i \in I} (C_8 \epsilon)^{d_i}((1 - \delta)\theta_i)^{-d_i(n-1)}.
\]

Put \( h^+(\mu^d, A, x, \epsilon) = \limsup_{n \to \infty} -\log(\mu^d(B(\alpha, \epsilon, n))) \), then (6) implies:

\[
\sum_{i \in I} d_i \log(1 - \delta) + \sum_{i \in I} d_i \log \theta_i \leq h^+(\mu, A, x, \epsilon).
\]

Recall that our only assumption on \( \epsilon \) is that it is sufficiently small so that \( C_8 \epsilon < C_5 \).

For the other inequality, we argue in a similar way. There exists a constant \( C_{10} \) such that set \( B(\alpha, \epsilon, n) \) contains the set:

\[
B''(\alpha, \epsilon, n) := \{ \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{T}_d^d : \tau(A^j \alpha, A^j \beta) < C_{10} \epsilon \text{ for } 0 \leq j \leq n - 1 \}.
\]

And when \( \epsilon \) is sufficiently small so that \( C_{10} \epsilon < C_5 \), we apply Lemma [3.2] repeatedly to get

\[
B''(\alpha, \epsilon, n) = \{ \pi(y) : y \in \mathbb{R}^d \text{ and } \|A^j x - A^j y\| < C_{10} \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \}.
\]

Then consider

\[
\tilde{B}''(x, \epsilon, n) := \{ y \in \mathbb{R}^d : \| A^j x - A^j y \| < C_{10} \epsilon \text{ for } j = 0, 1, \ldots, n - 1 \},
\]

we have \( \mu^d(B''(\alpha, \epsilon, n)) = \tilde{\mu}^d(\tilde{B}''(x, \epsilon, n)) = c_\eta(\tilde{B}''(x, \epsilon, n)) \). Arguing as before, the set \( \tilde{B}''(x, \epsilon, n) \) contains the set:

\[
\{ y = y_1 + \ldots + y_s : \|x_i - y_i\| < C_{10} \epsilon((1 + \delta)\theta_i)^{-1} \text{ for } i \in I \\
\quad \quad \text{and } \|x_i - y_i\| < C_{10} \epsilon \text{ for } i \notin I \}.
\]

Then we can use Proposition [3.1] to get a constant \( C_{11} \) such that:

\[
C_{11} \prod_{i \in I} ((C_{10} \epsilon)^{d_i}((1 + \delta)\theta_i)^{-d_i(n-1)} < \eta(\tilde{B}''(x, \epsilon, n)).
\]
This implies

\[ h^+(\mu, A, x, \epsilon) \leq \sum_{i \in I} d_i \log(1 + \delta) + \sum_{i \in I} d_i \log \theta_i \]

when \( \epsilon \) is sufficiently small.

Therefore

\[ \sum_{i \in I} d_i \log(1 - \delta) + \sum_{i \in I} d_i \log \theta_i \leq \lim_{\epsilon \to 0^+} h^+(\mu, A, x, \epsilon) \leq \sum_{i \in I} d_i \log(1 + \delta) + \sum_{i \in I} d_i \log \theta_i. \]

Since \( \delta \) can be arbitrarily small, we conclude that

\[ \lim_{\epsilon \to 0^+} h^+(\mu, A, x, \epsilon) = \sum_{i \in I} d_i \log \theta_i = d \sum_{i=1} d_i \log \max \{|\lambda_i|, 1\} \]

where the last equality follows from Property (ii) in Corollary 2.4. By the Brin-Katok theorem (see [BK83] and [VO16, pp. 262–263]), we have:

\[ h(\mu^d, A) = d \sum_{i=1} d_i \log \max \{|\lambda_i|, 1\}. \]

It is well-known that \( h(\mu^d, A) = h(\mu^d, A) \) [Wal82, p. 197] and this finishes the proof. \( \square \)

4. The proof of Theorem 1.6

Throughout this section, we assume the notation in the statement of Theorem 1.6. Let \( I \) denote the identity matrix in \( M_d(\mathbb{Z}_F) \). The below formula for \( N_1(B) \) in the classical case is well-known [BLP10]:

**Lemma 4.1.** Let \( B \in M_d(\mathbb{Z}_F) \). The number of isolated fixed points \( N_1(B) \) of the multiplication-by-\( B \) map

\[ B: \mathbb{T}_F^d \to \mathbb{T}_F^d. \]

is \( |\det(B - I)| \). Consequently \( N_k(A) = |\det(A^k - I)| \) for every \( k \geq 1 \).

**Proof.** When \( \det(B - I) = 0 \), there is a non-zero \( x \in \mathbb{R}_F^d \) such that \( Bx = x \). Then for any fixed point \( y \in \mathbb{T}_F^d \), the points \( y + cx \) for \( c \in \mathbb{R}_F \) are fixed. By choosing \( c \) to be in an arbitrarily small neighborhood of 0, we have that \( y \) is not isolated. Hence \( N_1(B) = 0 \).

Suppose \( \det(B - I) \neq 0 \). There is a 1-1 correspondence between the set of fixed points of \( B \) and the set \( \mathbb{Z}_F^d/(B - I)\mathbb{Z}_F^d \). Since \( \mathbb{Z}_F \) is a PID, we obtain the Smith Normal Form of \( B - I \) that is a diagonal matrix with entries \( b_1, \ldots, b_d \in \mathbb{Z}_F \setminus \{0\} \) and a \( \mathbb{Z}_F \)-basis \( x_1, \ldots, x_d \) of \( \mathbb{Z}_F^d \) so that \( b_1x_1, \ldots, b_dx_d \) is a \( \mathbb{Z}_F \)-basis of \( (B - I)\mathbb{Z}_F \). Therefore the number of fixed points of \( B \) is:

\[ \prod_{i=1} d \text{card}(\mathbb{Z}_F/b_i\mathbb{Z}_F) = \prod_{i=1} d |b_i| = |\det(B - I)|. \]

\( \square \)

We fix once and for all a finite extension \( K \) of \( \mathbb{R}_F \) containing all the eigenvalues of \( A \) and let \( \delta \) be the inertia degree of \( K/\mathbb{R}_F \). For each \( \mu_i \) in the (possibly empty) multiset \( \{\mu_1, \ldots, \mu_M\} \) of eigenvalues of \( A \) that are roots of unity, we have the decomposition:

\[ \mu_i = \mu_i^{(0)} + \mu_i^{(1)} \]
with \( \mu_{i,(0)} \in \text{GF}(q^k)^* \) and \( \mu_{i,(1)} \in \mathfrak{p}_K \) as in (1); in fact \( \mu_{i,(1)} = 0 \) since \( \mu_i \) is a root of unity. Likewise, for each \( \eta_i \) in the (possibly empty) multiset \( \{\eta_1, \ldots, \eta_N\} \), we have:

\[
\eta_i = \eta_{i,(0)} + \eta_{i,(1)}
\]

with \( \eta_{i,(0)} \in \text{GF}(q^k)^* \) and \( \eta_{i,(1)} \in \mathfrak{p}_K \setminus \{0\} \).

**Proposition 4.2.** Let \( v_p \) denote the \( p \)-adic valuation on \( \mathbb{Z} \). Recall that the orders of \( \mu_{i,(0)} \) and \( \eta_{j,(0)} \) in \( \text{GF}(q^k)^* \) are respectively denoted \( m_i \) and \( n_j \) for \( 1 \leq i \leq M \) and \( 1 \leq j \leq N \); each of the \( m_i \)'s and \( n_j \)'s is coprime to \( p \). Let \( k \) be a positive integer, we have:

(i) For \( 1 \leq i \leq M \), \( |\mu_i^k - 1| = \begin{cases} 0 & \text{if } k \equiv 0 \mod m_i \\ 1 & \text{otherwise} \end{cases} \).

(ii) For \( 1 \leq j \leq N \), \( |\eta_j^k - 1| = \begin{cases} |\eta_{j,(1)}|^{|p^{v_p(k)}} & \text{if } k \equiv 0 \mod n_j \\ 1 & \text{otherwise} \end{cases} \).

(iii) \( N_k(A) = |\det(A^k - I)| = r(A)^k \left( \prod_{i=1}^{M} a_{i,k} \prod_{j=1}^{N} b_{j,k} \right)^{p^{v_p(k)}} \) where \( a_{i,k} = \begin{cases} 0 & \text{if } k \equiv 0 \mod m_i \\ 1 & \text{otherwise} \end{cases} \) and \( b_{j,k} = \begin{cases} |\eta_{j,(1)}| & \text{if } k \equiv 0 \mod n_j \\ 1 & \text{otherwise} \end{cases} \) for \( 1 \leq i \leq M \) and \( 1 \leq j \leq N \).

**Proof.** Part (i) is easy: \( \mu_i^k - 1 = \mu_{i,(0)}^k - 1 \) is an element of \( \text{GF}(q^k) \) and it is 0 exactly when \( k \equiv 0 \mod m_i \). For part (ii), when \( k \not\equiv 0 \mod n_j \), we have:

\[
\eta_j^k - 1 \equiv \eta_{j,(0)}^k - 1 \neq 0 \mod \mathfrak{p}_K,
\]

hence \( |\eta_j^k - 1| = 1 \). Now suppose \( k \equiv 0 \mod n_j \) but \( k \not\equiv 0 \mod p \), we have:

\[
\eta_j^k - 1 = (\eta_{j,(0)} + \eta_{j,(1)})^k - 1 = k\eta_{j,(0)}^{k-1}\eta_{j,(1)} + \sum_{\ell=2}^{k} \binom{k}{\ell} \eta_{j,(0)}^{k-\ell}\eta_{j,(1)}^\ell
\]

and since \( |k\eta_{j,(0)}^{k-1}\eta_{j,(1)}| = |\eta_{j,(1)}| \) is strictly larger than the absolute value of each of the remaining terms, we have:

\[
|\eta_j^k - 1| = |\eta_{j,(1)}|.
\]

Finally, suppose \( k \equiv 0 \mod n_j \). Since \( \gcd(n_j, p) = 1 \), we can write \( k = k_0 p^{v_p(k)} \) where \( k_0 \equiv 0 \mod n_j \) and \( k_0 \neq 0 \mod p \). We have:

\[
|\eta_j^k - 1| = |\eta_j^{k_0} - 1|^{p^{v_p(k)}} = |\eta_{j,(1)}|^{p^{v_p(k)}}
\]

and this finishes the proof of part (ii). Part (iii) follows from parts (i), (ii), and the definition of \( r(A) \).  \( \square \)

**Proof of Theorem 1.6.** First, we prove part (a). We are given that for every \( j \in \{1, \ldots, N\} \), there exists \( i \in \{1, \ldots, M\} \) such that \( m_i \mid n_j \).

Let \( k \geq 1 \). If \( m_i \mid k \) for some \( i \) then \( N_k(A) = 0 \) by part (c) of Proposition 4.2.

If \( m_i \mid k \) for every \( i \in \{1, \ldots, M\} \) then \( n_j \mid k \) for every \( j \in \{1, \ldots, N\} \) thanks to
the above assumption, then we have \( N_k(A) = r(A)^k \) by Proposition 4.2. Therefore

\[
\sum_{k=1}^{\infty} \frac{N_k(A)}{k} z^k
\]
is equal to:

\[
\sum_{k \geq 1} \frac{r(A)^k}{k} z^k - \sum_{k \geq 1} \frac{r(A)^k}{k} z^k - \sum_{\ell=1}^{M} \sum_{1 \leq i_1 < \ldots < i_{\ell} \leq M} (-1)^{\ell-1} \sum_{k \geq 1} \frac{r(A)^k}{k} z^k
\]

where the third “=” follows from the inclusion-exclusion principle. This finishes the proof of part (a).

For part (b), without loss of generality, we assume that \( m_i \nmid n_1 \) for \( 1 \leq i \leq M \). Put

\[
f(z) := \sum_{k=1}^{\infty} N_k(A) z^k.
\]

Proposition 4.2 gives that \( |N_k(A)| \leq r(A)^k \), hence \( f \) is convergent in the disk of radius \( 1/r(A) \). Assume that \( f \) is D-finite and we arrive at a contradiction. Consider

\[
c_k := \frac{N_k(A)}{r(A)^k} \quad \text{for } k = 1, 2, \ldots
\]

then the series

\[
\sum_{k=1}^{\infty} c_k z^k = f(z/r(A))
\]
is D-finite. Let \( \tau \) denote the ramification index of \( K/F \), then each \( |\eta_j|_1 \) has the form \( 1/q^{d_j/\tau} \) where \( d_j \) is a positive integer [Neu99, p. 150]. Combining this with (7) and Proposition 4.2, we have that the \( c_k \)'s belong to the number field \( E := \mathbb{Q}(p^{1/\tau}) \). Let \( | \cdot |_p \) denote the \( p \)-adic absolute value on \( \mathbb{Q} \), then \( | \cdot |_p \) extends uniquely to an absolute value on \( E \) since there is only one prime ideal of the ring of integers of \( E \) lying above \( p \). Put:

\[
Q = \prod_{1 \leq j \leq N} |\eta_j|_1 \quad \text{and} \quad Q_1 = \prod_{1 \leq j \leq N \atop n_j | n_1} |\eta_j|_1.
\]
Since both \( Q \) and \( Q_1 \) are powers of \( 1/q^{1/\tau} \) with positive integer exponents, we have:

\[
|Q|_p, |Q_1|_p > 1. \tag{8}
\]

Since \( m_i \nmid n_i \) for every \( i \), Proposition 4.2 and (7) yield:

\[
c_{n_1p^\ell} = Q_1^{p^\ell} \quad \text{for every integer } \ell \geq 0. \tag{9}
\]

On the other hand, Proposition 4.2 and (7) also yield:

\[
|c_k|_p \leq |Q|_p^{p^p(k)} \quad \text{for every integer } k > 1. \tag{10}
\]

The idea to finish the proof is as follows. D-finiteness of the series \( \sum_{k=1}^{\infty} c_k z^k \) implies a strong restriction on the “growth” of the coefficients \( c_k \)'s at least through a recurrence relation satisfied by the \( c_k \)'s. This growth could be in terms of local data such as absolute values of the \( c_k \)'s or global data such as Weil heights of the \( c_k \)'s [BNZ20]. It is indeed the \( |c_k|_p \)'s that will give us the desired contradiction.

The key observation is that when \( \ell \) is large \( |c_{n_1p^\ell}|_p = |Q_1|_p^{p^\ell} \) is exponential in \( p^\ell \) thanks to (8) and (9) while the “nearby” coefficients \( c_{n_1p^\ell-n} \) for a bounded positive integer \( n \) have small \( p \)-adic absolute values thanks to (10) since \( v_p(n_1p^\ell-n) \) is small compared to \( \ell \).

Since \( \sum_{k=1}^{\infty} c_k z^k \in E[[z]] \) is D-finite, there exist a positive integer \( s \) and polynomials \( P_0(z), \ldots, P_s(z) \in E[z] \) such that \( P_0 \neq 0 \) and

\[
P_0(k)c_k + P_1(k)c_{k-1} + \ldots + P_s(k)c_{k-s} = 0 \tag{11}
\]

for all sufficiently large \( k \) [Sta80]. In the following \( \ell \) denotes a large positive integer and the implied constants in the various estimates are independent of \( \ell \). Consider \( k = n_1p^\ell \), then the highest power of \( p \) dividing any of the \( k-i = n_1p^\ell-i \) for \( 1 \leq i \leq s \) is at most the largest power of \( p \) in \( \{1, 2, \ldots, s\} \). Combining this with (10), we have:

\[
|P_1(n_1p^\ell)c_{n_1p^\ell-i}|_p \ll 1 \quad \text{for } 1 \leq i \leq s. \tag{12}
\]

Now (9), (11), and (12) imply:

\[
|P_0(n_1p^\ell)|_p \ll |Q_1|_p^{-p^\ell}. \tag{13}
\]

This means for the infinitely many positive integers \( k \) of the form \( n_1p^\ell \), we have that \( |P_0(k)|_p \) is exponentially small in \( k \). This implies that \( k \) is unusually close to a root of \( P_0 \) with respect to the \( p \)-adic absolute value. One can use the product formula to arrive at a contradiction, as follows.

Let \( M_E = M_E^f \cup M_E^{\infty} \) be the set of all places of \( E \) where \( M_E^f \) consists of the finite places and \( M_E^{\infty} \) denotes the set of all the infinite places [BG06, Chapter 1]. For every \( w \in M_E \), we normalize \( \cdot \mid_w \) as in [BG06, Chapter 1] and the product formula holds. We still use \( p \) to denote the only place of \( E \) lying above \( p \) and the above \( \cdot \mid_p \) has already been normalized according to [BG06, Chapter 1]. We have:

\[
\prod_{w \in M_E^f} |P_0(n_1p^\ell)|_w \ll (n_1p^\ell)^\deg(P_0) \quad \text{and} \quad \prod_{w \in M_E^{\infty} \setminus \{p\}} |P_0(n_1p^\ell)|_w \ll 1. \tag{14}
\]
When $\ell$ is sufficiently large and $P_0(n_1 p^\ell) \neq 0$, we have that (8), (13) and (14) contradict the product formula:

$$\prod_{w \in M_K} |P_0(n_1 p^\ell)|_w = 1$$

and this finishes the proof that $f(z) = \sum_{k=1}^{\infty} N_k(A) z^k$ is not D-finite. The transcendence of $\zeta_A(z)$ follows immediately: if $\zeta_A(z)$ were algebraic then $f(z) = z \frac{\zeta_A'(z)}{\zeta_A(z)}$ would be algebraic and hence D-finite, see Remark 1.5.

\[ \Box \]

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