A HOPF TYPE LEMMA AND THE SYMMETRY OF
SOLUTIONS FOR A CLASS OF KIRCHHOFF EQUATIONS

YAHUI NIU

School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, China
Department of Mathematical Sciences, Yeshiva University, New York, NY, USA

(Communicated by Wenxiong Chen)

ABSTRACT. In this paper, we proved a fractional Kirchhoff version of Hopf lemma for anti-symmetry functions and applied it to prove the symmetry and monotonicity of solutions for fractional Kirchhoff equations in the whole space by method of moving planes. We also obtain radially symmetry and monotonicity of solutions for fractional Kirchhoff equations in the unit ball. As far as we know, this is the first time to apply direct method of moving planes to fractional Kirchhoff problems.

1. Introduction. In this paper, we consider fractional Kirchhoff equation

\[
(a + b \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx) (-\Delta)^{\frac{s}{2}} u = f(x, u), \quad u > 0,
\]

where \(a \geq 0, b > 0, \ s \in (0, 1)\) are constants,

\[
\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx
\]

and

\[
(-\Delta)^{\frac{s}{2}} u(x) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,
\]

\(C_{n,s}\) is a normalization constant, \(PV\) stands for Cauchy principal value. Under some assumptions on \(f\), we prove the symmetry and monotonicity of solutions for (1.1) both in bounded domain and in \(\mathbb{R}^n\).

Kirchhoff-type problems and its variants arise in various models of physical and biological systems and have been studied extensively in recent years. To extend the classical D’Alembert’s wave equations for free vibration of elastic strings, Kirchhoff in [11] proposed for the first time the following time dependent wave equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,
\]
where \( u = u(x, t) \) is the lateral displacement at the coordinate \( x \) and the time \( t \), \( \rho \) is the mass density, \( P_0 \) is the initial axial tension, \( h \) is the cross-section area, \( E \) is the Young modulus, \( L \) is the length. Pohozaev [16] studied the above type of Kirchhoff equations quite early. Since J. L. Lions [15] introduced an abstract functional framework to the following equation

\[
 u_{tt} + \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u),
\]
much attention has turned to this area, such as [1, 2, 13] and the reference therein.

In the last decade, a great attention has been focused on the study of nonlinear fractional Kirchhoff problem. In particular, Fiscella and Valdinoci in [9] proposed a stationary Kirchhoff-type equation which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. More precisely, they considered a model as follows:

\[
\begin{cases}
 M \left( \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dy \, dx \right) (-\Delta)^s u = \lambda f(x, u) + |u|^{2^*_s - 2} u, & \text{in } \Omega, \\
 u = 0, & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

(1.4)

where \( M(y) = a + by \) for all \( y \geq 0 \), \( a > 0 \), \( b \geq 0 \). Problem (1.4) is called non-degenerate if \( a > 0 \) and \( b \geq 0 \), while it is named degenerate if \( a = 0 \) and \( b > 0 \). For more details about the physical background of the fractional Kirchhoff model, we refer to Appendix A of [9]. Afterwards, the fractional Kirchhoff-type problems have been extensively investigated, for example, in [3], the authors investigated the existence and the asymptotic behavior of nonnegative solutions for a class of stationary Kirchhoff problems driven by a fractional integro-differential operator and involving a critical nonlinearity. Pucci and Saldi in [17] established the existence and multiplicity of nontrivial nonnegative entire solutions for a Kirchhoff type eigenvalue problem in \( \mathbb{R}^n \) involving a critical nonlinearity and the fractional Laplacian. We refer to such as [10, 18] for more recent results about fractional Kirchhoff-type problems.

As far as we know, the qualitative properties of fractional Kirchhoff problem has not been studied in the previous literatures.

In this paper, we devote to explore the symmetry and monotonicity of a class of fractional Kirchhoff equations.

Here, we recall the corresponding research methods for fractional Laplacian problems. The non-locality of the fractional Laplacian makes it difficult to be investigated. To circumvent this, Caffarelli and Silvestre in [4] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. One can also reduce a fractional equation into an integral equation as in [7], then apply the method of moving planes in integral forms or regularity lifting to obtain the monotonicity, symmetry and regularity of the solutions.

However, the applying of this two methods needs extra conditions to be imposed on the solutions. Moreover, they do not work for nonlinear nonlocal operators, such as the fractional \( p \)-Laplacian(see [8] for the details). Thanks to the works of Chen and Li et. in [5, 6], direct methods of moving planes are introduced for the fractional Laplacian and fractional \( p \)-Laplacian without going through extensions or integral equations, which have been applied to obtain symmetry, monotonicity, and non-existence of solutions for various semi-linear equations involving these nonlocal operators.
In this paper, inspired by [6], we intend to use the "direct" idea involving the moving plane methods to prove the symmetry and monotonicity of fractional Kirchhoff equations. While, for this problems, except for the nonlocal operator \((-\Delta)^s u\), there is also nonlocal term \(\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx\), which makes the research of this problem more interesting. As far as we know, this is the first time to apply direct method of moving planes to fractional Kirchhoff problems.

Before stating our results, we first give some notations, which will be used throughout this paper. Choose any ray from the origin as the positive \(x_1\) direction. Let
\[
T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \text{ for some } \lambda \in \mathbb{R} \}
\]
be the moving planes,
\[
\Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 > \lambda \}
\]
be the region to the right of the plane \(T_\lambda\) and
\[
x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]
be the reflection of \(x\) about the plane \(T_\lambda\).

In order to let the right hand of (1.2) and (1.3) make sense, we assume \(u \in H^s(\mathbb{R}^n) \cap L^2_{\text{loc}}(\mathbb{R}^n)\), where
\[
H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x - y|^{n+2s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \},
\]
and
\[
L^2_s = \{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} < +\infty \}.
\]

For any such \(u\) satisfying (1.1), set
\[
I(u) = a + b \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx.
\]
(1.5)

Denote
\[
u_\lambda(x) = u(x^\lambda).
\]

By a simple computation, we have
\[
I(u) = I(u_\lambda).
\]
(1.6)

To compare the values of \(u(x)\) with \(u_\lambda(x)\), we set
\[
w_\lambda(x) = u_\lambda(x) - u(x),
\]
which satisfies
\[
w_\lambda(x) = -w_\lambda(x^\lambda).
\]

Our main results are as follows: consider
\[
\begin{cases}
(a + b \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx)(-\Delta)^s u = f(x, u), \ u > 0, \text{ in } B_1(0), \\
u(x) = 0, \quad \text{in } B_1^c(0).
\end{cases}
\]
(1.7)

Assume
\((F_1)\): \(f(x, \cdot)\) is locally Lipschitz continuous, locally uniformly in \(x\): for any \(h > 0\) and any \(D \subset \subset B_1(0)\), there exists \(C_{h,D} > 0\), such that for any \(u, v \in [-h, h]\) and any \(x \in D\),
\[
|f(x, u) - f(x, v)| \leq C_{h,D}|u - v|.
\]
Later, Chen and Li in [5], generalized their result to the fractional $p$ solutions for $u$ under the same assumptions on $u$ and $u'$.

Remark 1. Depending on $n, s$, $c$ is the same with (1.9).

Theorem 1.1. Let $u \in C(\overline{B_1(0)}) \cap C^{1,1}_{loc}(B_1(0)) \cap H^s(\mathbb{R}^n)$ be a solution of (1.7) with $f(x, u) : B_1(0) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (F1)-(F2).

Then $u$ is radially symmetric and monotone decreasing about the origin.

We will see from the proof of Theorem 1.1, the following result in a general bounded domain is an immediately generalization of Theorem 1.1.

Corollary 1. Assume $\Omega \subset \mathbb{R}^n$ be a bounded domain which is convex in $x_1$-direction and symmetric with respect to $\{x \in \mathbb{R}^n \mid x_1 = 0\}$. Let $u \in C(\overline{\Omega}) \cap C^{1,1}_{loc}(\Omega) \cap H^s(\mathbb{R}^n)$ be a solution of

$$
\begin{cases}
(a + b \int_{\mathbb{R}^n} |(-\Delta)\frac{1}{2} u|^2 dx)(-\Delta)^s u = f(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \Omega^c,
\end{cases}
$$

with $f(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (F1) and

$$f(\tilde{x}_1, \tilde{x}', u) \geq f(x_1, x_1', u), \quad \forall x \in \Omega, |x_1| \geq |\tilde{x}_1|.$$

Then $u$ is symmetric with respect to $\{x \in \Omega \mid x_1 = 0\}$ and monotone decreasing in $x_1$-direction.

Next we investigate (1.1) in the whole space $\mathbb{R}^n$, we obtain

Theorem 1.2. Let $u \in L_{2,2} \cap C^{1,1}_{loc}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ be a bounded solution of (1.1) with $x \in \mathbb{R}^n$. Assume $f(x, \cdot)$ is lower semi-continuous uniformly with respect to $x$ and $f(x, u) = f(|x|, u)$ satisfies

$$f(x_0, u) > f(x, u), \quad \forall |x_0| < |x|, \quad x \in \mathbb{R}^n. \quad (1.8)$$

If for some constant $\alpha > 0$,

$$\frac{f(x, u) - f(x, v)}{u - v} \leq C(u^\alpha + v^\alpha) \text{ uniformly in } x, \quad \text{as } u, v \rightarrow 0. \quad (1.9)$$

and $u$ satisfies

$$\lim_{|x| \rightarrow \infty} |x|^{2s}[u(x)]^\alpha \leq \frac{I(u)\tilde{C}_{n, s}}{C}, \quad (1.10)$$

where $C$ is the same with (1.9), $I(u)$ is a constant defined in (1.5), $\tilde{C}_{n, s}$ is a constant depending on $n, s$.

Then $u$ is radially symmetric and monotone decreasing about the original point.

Remark 1. In [14], Li and Ni considered a semi-linear equation for regular Laplacian

$$-\Delta u(x) = f(u(x)), \quad x \in \mathbb{R}^n.$$

Under the assumption that

$$f'(s) \leq 0, \quad \text{for sufficiently small } s > 0, \quad (1.11)$$

and $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, they obtained the radial symmetry of positive solutions. Later, Chen and Li in [5], generalized their result to the fractional $p$–Laplacian, under the same assumptions on $u$ and $f(\cdot)$, they obtained the radial symmetry of solutions for

$$(-\Delta)_p^s u(x) = f(u(x)), \quad u > 0, \quad x \in \mathbb{R}^n.$$
In this paper, in the case \( f(x,u) = f(u) \), we assume \( f \) satisfies (1.9), which is much weaker than (1.11). For instance, \( f(s) = s^p, \ p > 1 + \alpha \), for sufficiently small \( s > 0 \), it satisfies (1.9), but doesn’t satisfy (1.11).

During the proof of Theorem 1.2, the following fractional Kirchhoff version Hopf lemma for anti-symmetric functions on half spaces is a powerful tool in carrying out the method of moving planes in \( \mathbb{R}^n \).

**Lemma 1.3 (Hopf lemma for anti-symmetry functions).** Let 

\[
\mathcal{L} \in \mathcal{L}_{2s} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)
\]

be a bounded solution of (1.1) with \( x \in \mathbb{R}^n \). Suppose \( w_\lambda \in C^3_{\text{loc}}(\Sigma_\lambda) \) satisfies

\[
\begin{cases}
I(u)(-\Delta)^s w_\lambda(x) + c_\lambda(x) w_\lambda(x) \geq 0, & \text{in } \Sigma_\lambda, \\
w_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda, \\
w_\lambda(x) = -w_\lambda(x^\lambda) & \text{in } \Sigma_\lambda,
\end{cases}
\]

where \( c_\lambda \) satisfies

\[
\lim_{x \to \partial \Sigma_\lambda} c_\lambda(x) = o\left(\frac{1}{|x_1|^{2}}\right).
\]

(1.12)

If there exists some point \( x \in \Sigma_\lambda \) satisfies

\[
w_\lambda(x) > 0,
\]

Then

\[
\frac{\partial w_\lambda}{\partial x_1}(x) > 0, \quad \forall \ x \in T_\lambda.
\]

(1.14)

**Remark 2.** When \( a = 1, \ b = 0 \), (1.1) is a fractional Laplacian equation, correspondingly, Theorem 1.3 is a Hopf type lemma for fractional Laplacian, which includes the result of Li and Chen in [12] as a special case. In fact, they assumed

\[
w_\lambda \in C^3_{\text{loc}}(\Sigma_\lambda) \quad \text{and} \quad \lim_{x \to \partial \Sigma_\lambda} c_\lambda(x) = o\left(\frac{1}{\text{dist}(x, \partial \Sigma_\lambda)^{2}}\right),
\]

(1.15)

which is stronger than (1.12). In order \( w_\lambda \in C^3_{\text{loc}}(\Sigma_\lambda) \), one requires \( f \) in equation (1.1) has the same regularity. In this paper, we can see from our proof of Lemma 1.3, if we assume \( c_\lambda \) is bounded, we only need the weaker condition \( w_\lambda \in \mathcal{L}_{2s} \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n) \), which only requires the Lipschitz continuity assumption on \( f \). Therefore, seen from the above two aspects, we obtain a more general result.

In Section 2, we develop some preliminary tools and the Hopf lemma for anti-symmetric functions, which play important roles in the proof of the moving plane process. In Section 3, we prove Theorem 1.1 and Theorem 1.2.

2. Various maximum principle and Hopf lemma for anti-symmetric functions. In this section, we will establish various maximum principles and Hopf lemma involving fractional Kirchhoff operators for anti-symmetric functions, which are key ingredients in applying the method of moving planes to fractional Kirchhoff equations.

To obtain the symmetry of solutions in the unit ball, the first step is to show that for \( \lambda \) sufficiently close to the right end of the domain, we have

\[
w_\lambda(x) \geq 0, \ x \in \Sigma_\lambda.
\]

This provides a starting position to move the plane. The following narrow region principle will serve for this purpose.
Theorem 2.1 (Narrow region principle). Let $\Omega$ be a bounded narrow region in 
$$\{ x \mid \lambda < x_1 < \lambda + l \} \subset \Sigma_{\lambda}$$
with $l$ small. Assume that $u(x) \in C^{1,1}_{loc}(\Omega) \cap L_{2s} \cap H^s(\mathbb{R}^n)$ is a bounded solution of (1.7), $w_\lambda$ is lower semi-continuous in $\bar{\Omega}$, and satisfies

$$\begin{cases}
I(u)(-\Delta)^s w_\lambda(x) + c_\lambda(x)w_\lambda(x) \geq 0, & x \in \Omega, \\
w_\lambda(x) \geq 0, & x \in \Sigma_{\lambda}\setminus\Omega, \\
w_\lambda(x) = -w_\lambda(x^\lambda), & x \in \Omega,
\end{cases}$$

(2.1)

where $c_\lambda(x)$ is bounded from below at the points where $w_\lambda(x) < 0.$

Then for sufficiently small $l$, we have

$$w_\lambda(x) \geq 0, \quad \forall \ x \in \Omega; \quad (2.2)$$

If $\Omega$ is unbounded, then the conclusion (2.2) still holds under the conditions

$$\lim_{|x| \to \infty} w_\lambda(x) \geq 0.$$

Proof. If (2.2) does not hold, by (2.1) and the lower semi-continuous of $w_\lambda$ in $\bar{\Omega}$, there exists $x_0 \in \Omega$ such that

$$w_\lambda(x_0) = \min_{\Sigma_{\lambda}} w_\lambda(x) < 0.$$

Then it follows that

$$I(u)(-\Delta)^s w_\lambda(x_0)$$

$$= I(u)C_{n,s}PV \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{n+2s}} dy$$

$$= I(u)C_{n,s}PV \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{n+2s}} dy + I(u)C_{n,s} \int_{\mathbb{R}^n \setminus \Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{n+2s}} dy$$

$$= I(u)C_{n,s}PV \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{n+2s}} dy + I(u)C_{n,s} \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y^\lambda)}{|x_0 - y^\lambda|^{n+2s}} dy$$

$$\leq I(u)C_{n,s}PV \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y)}{|x_0 - y|^{n+2s}} dy + I(u)C_{n,s} \int_{\Sigma_{\lambda}} \frac{w_\lambda(x_0) - w_\lambda(y^\lambda)}{|x_0 - y^\lambda|^{n+2s}} dy$$

$$= 2I(u)C_{n,s}w_\lambda(x_0) \int_{\Sigma_{\lambda}} \frac{1}{|x_0 - y|^{n+2s}} dy, \quad (2.3)$$

where the inequality holds since $|x_0 - y| < |x_0 - y^\lambda|$ in $\Sigma_{\lambda}.$

For $l > 0$ small and will be determined later, we set

$$H = \{ y = (y_1, y') \in \Sigma_{\lambda} \mid l < y_1 - (x_0)_1 < 2l, \ |y' - x_0| < l \},$$

then

$$\int_{\Sigma_{\lambda}} \frac{1}{|x_0 - y|^{n+2s}} dy \geq \int_H \frac{1}{|x_0 - y^\lambda|^{n+2s}} dy \geq \frac{C_{n,s}}{l^{n+2s}} |H| = \frac{C_{n,s}}{l^{2s}},$$

where $C_{n,s}$ is a constant, which may be different from line to line, depending on $n$, $s$, independent of $l$. Combine this with (2.3), we obtain

$$I(u)(-\Delta)^s w_\lambda(x_0) \leq \frac{I(u)C_{n,s}}{l^{2s}} w_\lambda(x_0).$$
Therefore, since $c_\lambda(x_0)$ is bounded from below, $l$ is small enough, we have

$$I(u)(-\Delta)^sw_\lambda(x_0) + c_\lambda(x_0)w_\lambda(x_0) \leq \left( \frac{I(u)\bar{c}_{n,s}}{l^2} + c_\lambda(x_0) \right)w_\lambda(x_0) < 0,$$

which contradicts (2.1). Thus (2.2) holds.

\[ \square \]

**Theorem 2.2 (Strong maximum principle).** Let $\Omega \subset \Sigma_\lambda$ be a bounded domain. Assume that $u(x) \in C^{1,1}_{loc}(\Omega) \cap L^2 \cap H^s(\mathbb{R}^n)$ is a bounded solution of (1.7), $w_\lambda$ is lower semi-continuous in $\bar{\Omega}$, and satisfies

$$\begin{cases}
I(u)(-\Delta)^sw_\lambda(x) \geq 0, & x \in \Omega, \\
w_\lambda(x) \geq 0, & x \in \Sigma_\lambda, \\
w_\lambda(x) = -w_\lambda(x^\lambda), & x \in \Omega.
\end{cases}$$

If there is a point $x \in \Omega$ such that $w_\lambda(x) > 0$, then we have

$$w_\lambda(x) > 0, \quad \forall \ x \in \Omega; \tag{2.5}$$

If $\Omega$ is unbounded, then the conclusion (2.5) still holds under the conditions

$$\lim_{|x| \to \infty} w_\lambda(x) \geq 0.$$

**Proof.** If there exists some point $x_0 \in \Omega$ such that

$$w_\lambda(x_0) = \min_{\Sigma_\lambda} w_\lambda(x) = 0,$$

It follows that

$$I(u)(-\Delta)^sw_\lambda(x_0)$$

$$= I(u)C_{n,s}PV \int_{\mathbb{R}^n} \frac{-w_\lambda(y)}{|x_0 - y|^{n+2s}} \, dy + I(u)C_{n,s}PV \int_{\Sigma_\lambda} \frac{-w_\lambda(y)}{|x_0 - y|^{n+2s}} \, dy$$

$$= I(u)C_{n,s}PV \int_{\Sigma_\lambda} \frac{-w_\lambda(y)}{|x_0 - y|^{n+2s}} \, dy + I(u)C_{n,s}PV \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{-w_\lambda(y)}{|x_0 - y|^{n+2s}} \, dy$$

$$= I(u)C_{n,s}PV \int_{\Sigma_\lambda} w_\lambda(y) \left( \frac{1}{|x_0 - y^\lambda|^{n+2s}} - \frac{1}{|x_0 - y|^{n+2s}} \right) \, dy < 0, \tag{2.6}$$

where the last inequality holds because

$$|x_0 - y| < |x_0 - y^\lambda| \quad \text{for} \ y \in \Sigma_\lambda,$$

and $w_\lambda(y) \neq 0$ in $\Sigma_\lambda$.

Since (2.6) contradicts (2.4), thus (2.5) holds.

\[ \square \]

**Remark 3.** From the proof of Theorem 2.2, we only need $I(u)(-\Delta)^sw_\lambda(x) \geq 0$ holds at the points where $w_\lambda(x) = 0$.

In the whole space $\mathbb{R}^n$, we start moving the plane from near either $+\infty$ or $-\infty$, and to this end, we employ the following
Theorem 2.3 (Decay at infinity). Let $\Omega \subset \Sigma_\lambda$ be an unbounded domain. Assume that $u(x) \in C^{1,1}_{\text{loc}}(\Omega) \cap L^2$ is a bounded solution of (1.7) and $w_\lambda$ satisfies
\[
\begin{cases}
I(u)(-\Delta)^s w_\lambda(x) + c_\lambda(x) w_\lambda(x) \geq 0, & x \in \Omega \\
w_\lambda(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \\
w_\lambda(x) = -w_\lambda(x^\lambda), & x \in \Omega,
\end{cases}
\] (2.7)
with
\[
\liminf_{x \to \infty, w_\lambda(x) < 0} |x|^{2s} c_\lambda(x) \geq -\frac{I(u) \bar{C}_{n,s}}{2},
\] (2.8)
where $\bar{C}_{n,s}$ is the same constant as in (1.10).

Then there exists a constant $R > 0$ (depending on $c_\lambda$ but independent of $w_\lambda$) such that if $\bar{x} \in \Omega$ satisfying
\[
w_\lambda(\bar{x}) = \min_{\bar{\Omega}} w_\lambda(x) < 0,
\] (2.9)
then
\[
|\bar{x}| \leq R.
\]

Proof. By the definition of $\Sigma_\lambda$ in Section 1, here we assume $\lambda \geq 0$. Otherwise, define $\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$. By (2.7) and (2.9), we use the similar computation to (2.3), we obtain
\[
I(u)(-\Delta)^s w_\lambda(\bar{x}) \leq I(u)C_{n,s} \int_{\Sigma_\lambda} \frac{2w_\lambda(\bar{x})}{|\bar{x} - y|^n + 2s} dy \leq \frac{I(u)\bar{C}_{n,s}}{|\bar{x}|^{2s}} w_\lambda(\bar{x}).
\]

Combining with (2.7), we have
\[
0 \leq I(u)(-\Delta)^s w_\lambda(\bar{x}) + c_\lambda(\bar{x}) w_\lambda(\bar{x}) \leq \left( \frac{I(u)\bar{C}_{n,s}}{|\bar{x}|^{2s}} + c_\lambda(\bar{x}) \right) w_\lambda(\bar{x}).
\]
Then, it follows from (2.9) that
\[
|\bar{x}|^{2s} c_\lambda(\bar{x}) \leq -I(u)\bar{C}_{n,s}.
\] (2.10)

By (2.8), there exists a constant $R > 0$ large enough, such that for any $|x| > R$, we have
\[
|x|^{2s} c_\lambda(x) \geq -\frac{I(u)\bar{C}_{n,s}}{2},
\]
which contradicts (2.10) as long as $|\bar{x}| \geq R$. Thus we complete the proof. \hfill \Box

Next, we prove the Hopf lemma

Proof of Lemma 1.3. For any $\bar{x} \in T_\lambda$, $r > 0$ will be determined later, denote
\[
B = B_r(\bar{x}), \quad B_\lambda = B \cap \Sigma_\lambda.
\]
By (1.13) and the continuity of $w_\lambda(x)$, there exists $D \subset \Sigma_\lambda$ and constant $C_1$ satisfy
\[
w_\lambda(x) > C_1, \quad x \in D.
\] (2.11)

Assume $D \cap B_\lambda = \emptyset$.

Next, we will construct a sub-solution of $w_\lambda$ in $B_\lambda$. Set
\[
g(x) = (x_1 - \bar{x}_1)\eta(|x|),
\]
where \( \eta(x) = \begin{cases} 1, & |x - \bar{x}| < \frac{r}{2}, \\ 0, & |x - \bar{x}| \geq r, \end{cases} \eta(x) \in C^\infty_0(B). \)

Denote
\[
\bar{w}(x) = w^D_\lambda(x) + \epsilon g(x),
\]
where \( w^D_\lambda(x) = \chi_{D \cup D^\lambda}(x)w_\lambda(x) \), \( D^\lambda \) is the symmetry domain of \( D \) with respect to \( T_\lambda \),
\[
\chi_{D \cup D^\lambda}(x) = \begin{cases} 1, & x \in D \cup D^\lambda, \\ 0, & x \in \mathbb{R}^n \setminus (D \cup D^\lambda). \end{cases}
\]

Note that \( w^D_\lambda \) is an anti-symmetry function. For \( x \in B_\lambda \), we have
\[
I(u)(-\Delta)^s w(x) = (u)_{C_{n,s}} PV \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x - y|^{n+2s}} dy
\]
\[
= (u)_{C_{n,s}} PV \int_{\mathbb{R}^n} \frac{\epsilon g(x) - \epsilon g(y)}{|x - y|^{n+2s}} dy
\]
\[
= (u)_{C_{n,s}} \int_{D \cup D^\lambda} \frac{-w_\lambda(y)}{|x - y|^{n+2s}} dy + (u)_{C_{n,s}} PV \int_{\mathbb{R}^n} \frac{\epsilon g(x) - \epsilon g(y)}{|x - y|^{n+2s}} dy
\]
\[
= (u)_{C_{n,s}} \int_{D} \frac{-w_\lambda(y)}{|x - y|^{n+2s}} dy + (u)_{C_{n,s}} \int_{D^\lambda} \frac{-w_\lambda(y)}{|x - y|^{n+2s}} dy + \epsilon I(u)(-\Delta)^s g(x)
\]
\[
= (u)_{C_{n,s}} \int_{D} \frac{w_\lambda(y)}{|x - y|^{n+2s}} dy + (u)_{C_{n,s}} \int_{D} \frac{1}{|x - y|^{n+2s}} dy + \epsilon I(u)(-\Delta)^s g(x), \qquad (2.12)
\]

By mean value theorem,
\[
\frac{1}{|x - y|^{n+2s}} - \frac{1}{|x - y|^{n+2s}} = \frac{-2(n + 2s)(x_1 - \lambda)(y_1 - \lambda)}{(\zeta(x, y, \lambda))^{n+2s}}, \qquad (2.13)
\]
where \( \zeta(x, y, \lambda) \) is between \( |x - y|^2 \) and \( |x - y|^2 \).

For \( x \in B_\lambda \),
\[
|x - y|^2 \leq |x - y|^2.
\]

So,
\[
\zeta(x, y, \lambda) \leq |x - y|^2 \leq |x - \bar{x}|^2 + |\bar{x} - y|^2 \leq r^2 + |\bar{x} - y|^2. \qquad (2.14)
\]

We also infer from \( g(x) \in C^\infty_0(B) \) that \( (-\Delta)^s g(x) \in C^\infty \), so,
\[
(-\Delta)^s g(x) = (-\Delta)^s g(\bar{x}) + \frac{\partial}{\partial x_1} ((-\Delta)^s g(\bar{x}))(x - \bar{x}) = C_2(x_1 - \bar{x}_1). \qquad (2.15)
\]

As a result, combining (2.13), (2.14), (2.15) with (2.12) and (2.11), we obtain that
\[
I(u)(-\Delta)^s w(x) \leq -2(n + 2s)(x_1 - \lambda) \int_{D} \frac{w_\lambda(y)(y_1 - \lambda)}{(r^2 + |\bar{x} - y|^2)^{n+2s/2}} dy + \epsilon C_2(x_1 - \bar{x}_1)
\]
\[
\leq -C_{n,s,r}(x_1 - \lambda) + \epsilon C_2(x_1 - \lambda), \qquad (2.16)
\]
where \( C_{n,s,r} > 0 \) is a constant depending on \( n, s, r \) and decreasing with respect to \( r \).
If (1.14) does not holds, then \( \frac{\partial w_\lambda}{\partial x_1}(\bar{x}) = 0 \). Then it follows from the the anti-symmetry of \( w_\lambda \), we also have \( \frac{\partial^2 w_\lambda}{\partial x_1^2}(\bar{x}) = 0 \). So, for \( r > 0 \) small, we have
\[
w_\lambda(x) = O(|x_1 - \bar{x}_1|^3), \quad x \in B_\lambda.
\]
As a result, Combining with (1.12), there exists some small \( \hat{\varepsilon} \) with \( \hat{\varepsilon} \to 0 \) as \( x_1 \to \lambda \), such that
\[
c_\lambda(x)w_\lambda(x) = o(|x_1 - \bar{x}_1|) = \hat{\varepsilon}|x_1 - \lambda|, \quad x \in B_\lambda. \quad (2.17)
\]
Then, choosing \( \varepsilon \) and \( r \) small enough, for \( x \in B_\lambda \), by (2.16) and (2.17), we obtain that
\[
I(u)(-\Delta)^s w(x) \leq (-C_{n,s,r} + \varepsilon C_2)|x_1 - \lambda| \leq -\hat{\varepsilon}|x_1 - \lambda| = -c_\lambda(x)w_\lambda(x) \leq I(u)(-\Delta)^s w_\lambda(x) \quad \text{in} \ B_\lambda. \quad (2.18)
\]
If our assumption are \( c_\lambda \) is bounded and \( w_\lambda \in L_{2s} \cap C^{1,1}_{loc}(\mathbb{R}^n) \), we can replace the above process of deriving (2.18) as follows: since \( w_\lambda(\bar{x}) = 0 \), for \( r > 0 \) small enough, then
\[
w_\lambda(x) = O(|x_1 - \lambda|).
\]
By the boundedness of \( c_\lambda \), we can choose \( \varepsilon, r \) small enough, such that
\[
I(u)(-\Delta)^s w(x) \leq (-C_{n,s,r} + \varepsilon C_2)|x_1 - \lambda| \leq -c_\lambda(x)O(|x_1 - \lambda|) = -c_\lambda(x)w_\lambda(x) \leq I(u)(-\Delta)^s w_\lambda(x) \quad \text{in} \ B_\lambda.
\]
As a result, set
\[
v(x) = w_\lambda(x) - \bar{w}(x),
\]
we obtain
\[
\begin{cases}
I(u)(-\Delta)^s v(x) \geq 0 & \text{in} \ B_\lambda, \\
v(x) \geq 0 & \text{in} \ \Sigma_\lambda \setminus B_\lambda.
\end{cases} \quad (2.19)
\]
We claim that
\[
v(x) \geq 0 \quad \text{in} \ B_\lambda. \quad (2.20)
\]
If (2.20) does not hold, there exists \( x^0 \in B_\lambda \) such that
\[
v(x^0) = \min_{\Sigma_\lambda} v(x) < 0,
\]
then combining with \( v(y) = -v(y^\lambda) \), we have
\[
I(u)(-\Delta)^s v(x^0)
= I(u)C_{n,s}PV \int_{\mathbb{R}^n} \frac{v(x^0) - v(y)}{|x^0 - y|^{n+2s}} dy
= I(u)C_{n,s}PV \int_{\Sigma_\lambda} \frac{v(x^0) - v(y)}{|x^0 - y|^{n+2s}} dy + I(u)C_{n,s}PV \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{v(x^0) - v(y)}{|x^0 - y|^{n+2s}} dy
= I(u)C_{n,s}PV \int_{\Sigma_\lambda} \frac{v(x^0) - v(y)}{|x^0 - y|^{n+2s}} dy + I(u)C_{n,s}PV \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{v(x^0) - v(y^\lambda)}{|x^0 - y^\lambda|^{n+2s}} dy
\leq I(u)C_{n,s}PV \int_{\Sigma_\lambda} \frac{2v(x^0)}{|x^0 - y^\lambda|^{n+2s}} dy < 0,
\]
which contradicts (2.19). Thus we proved (2.20). Consequently,
\[
w_\lambda(x) \geq \bar{w}(x) = \varepsilon(x_1 - \bar{x}_1)\eta(x) \quad \text{in} \ B_\lambda
\]
and
\[ w_\lambda(x) \geq \epsilon(x_1 - \bar{x}_1) \text{ in } B_\lambda \cap B_{\bar{r}}(\bar{x}). \]
For \( \bar{x} \in T_\lambda \), \( w_\lambda(\bar{x}) = 0 \), so
\[ w_\lambda(x) - w_\lambda(\bar{x}) \geq \epsilon(x_1 - \bar{x}_1) \text{ in } B_\lambda \cap B_{\bar{r}}(\bar{x}), \]
then
\[ \frac{\partial w_\lambda}{\partial x_1}(\bar{x}) = \lim_{x_1 \to \bar{x}_1} \frac{w_\lambda(x) - w_\lambda(\bar{x})}{x_1 - \bar{x}_1} \geq \epsilon > 0, \]
This is a contradiction. We finished the proof of Lemma 1.3. \( \square \)

3. Moving plane methods for fractional Kirchhoff operators and applications. In this section, we will prove Theorem 1.1 by moving plane method.

Set
\[ \Omega_\lambda = \Sigma_\lambda \cap B_1(0) = \{ x \in B_1(0) \mid x_1 > \lambda \}, \]
then by (1.1) and (1.6), \( w_\lambda \) satisfies
\[ I(u)(-\Delta)^s w_\lambda(x) + c_\lambda(x)w_\lambda(x) = 0 \text{ in } \Omega_\lambda, \]
where
\[ c_\lambda(x) = -\frac{f(x_\lambda, u_\lambda(x)) - f(x, u(x))}{u_\lambda(x) - u(x)}, \text{ if } u_\lambda(x) \neq u(x). \]

**Proof of Theorem 1.1.** We will carry on the proof in two steps.

**Step 1**: We prove that for \( \lambda < 1 \) sufficiently close to 1, we have
\[ w_\lambda(x) \geq 0 \text{ in } \Omega_\lambda. \]
Since
\[ |x^\lambda| < |x|, \text{ for } x \in \Omega_\lambda, \]
Then it follows from \((F_1)-(F_2)\) that
\[ c_\lambda(x) = -\frac{f(x, u_\lambda(x)) - f(x, u(x))}{u_\lambda(x) - u(x)} = -\frac{f(x\lambda, u_\lambda(x)) - f(x, u_\lambda(x))}{u_\lambda(x) - u(x)} = -\frac{f(x, u_\lambda(x)) - f(x, u(x))}{u_\lambda(x) - u(x)} \geq -C \]
holds at the points \( x \in \Omega_\lambda \) where \( w_\lambda(x) < 0 \). Now we can apply the narrow region principle Theorem 2.1 to conclude that (3.1) holds.

**Step 2**: Since (3.1) provides a starting point to move the plane. We move the plane \( T_\lambda \) toward the left as long as inequality (3.1) holds to its limiting position. Define
\[ \lambda_0 = \inf\{ \lambda \geq 0 \mid w_\mu(x) \geq 0, \ x \in \Omega_\mu, \ \mu \leq \lambda \}. \]
We will show that \( \lambda_0 = 0 \). Otherwise, the plane \( T_{\lambda_0} \) can be moved a little further to the left, which contradicts the definition of \( \lambda_0 \).

Suppose \( \lambda_0 > 0 \), by the definition of \( \lambda_0 \), we have
\[ w_{\lambda_0}(x) \geq 0, \ x \in \Omega_{\lambda_0} \]
and obviously, \( w_{\lambda_0}(x) \neq 0 \) in \( \Omega_{\lambda_0} \). Then by the strong maximum principle Theorem 2.2,
\[ w_{\lambda_0}(x) > 0, \ x \in \Omega_{\lambda_0}. \]
Therefore, by the continuity of \( w_{\lambda_0} \) with respect to \( x \), for \( \delta > 0 \) small, there exists \( c_0 > 0 \), such that
\[ w_{\lambda_0}(x) \geq c_0, \ x \in \Omega_{\lambda_0 + \delta}. \]
Furthermore, since $w_\lambda$ depends on $\lambda$ continuously, there exists $\varepsilon > 0$ small, such that $\lambda_0 - \varepsilon > 0$ and for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$,

$$w_\lambda(x) \geq 0, \ x \in \Omega_{\lambda_0 + \delta}.$$  \hfill (3.2)

For $x \in \Omega_\lambda \setminus \Omega_{\lambda_0 + \delta}$, which is a narrow region, similar to Step 1, applying narrow region principle Theorem 2.1 with $\Omega = \Omega_\lambda \setminus \Omega_{\lambda_0 + \delta}$, we can conclude that $w_\lambda(x) \geq 0, \ x \in \Omega_\lambda \setminus \Omega_{\lambda_0 + \delta}$.

Combining with (3.2), we obtain that for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$,

$$w_\lambda(x) \geq 0, \ x \in \Omega_\lambda,$$

which contradicts the definition of $\lambda_0$. Thus $\lambda_0 = 0$. That is, for $0 < x_1 < 1$, we have

$$u(-x_1, x_2, \cdots, x_n) \geq u(x_1, x_2, \cdots, x_n).$$

Then the arbitrariness of $x_1$ direction infers that $u$ is radially symmetric with respect to $0$. Also, we can infer from (3.1) and $\lambda_0 = 0$ that

$$w_\lambda(x) \geq 0 \text{ in } \Omega_\lambda, \ \forall \lambda \in [0, 1),$$

which means $u$ is monotone decreasing about the origin. This complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Since $u$ is a solution of (1.1) with $x \in \mathbb{R}^n$, then $w_\lambda$ satisfies

$$I(u)(-\Delta)^s w_\lambda(x) + c_\lambda(x) w_\lambda(x) = 0 \text{ in } \Sigma_\lambda,$$

where

$$c_\lambda(x) = \frac{\int f(x, u(x)) - f(x, u_\lambda(x)) w_\lambda(x)}{w_\lambda(x) - u(x)}, \text{ if } u_\lambda(x) \neq u(x).$$

We will prove $\lambda_0 = 0$. \hfill (3.4)

This means that $u$ is symmetry with respect to $x_1 = 0$ and monotone decreasing in $x_1 > 0$. Then, by the arbitrary of $x_1$ direction, we conclude that $u$ is symmetry and radially decreasing with respect to the original point. The details of this two steps are as follows:

**Step 1**: From (1.10), we have

$$\lim_{|x| \to \infty} u(x) = 0.$$

It follows that for each fixed $\lambda$

$$\lim_{|x| \to \infty} w_\lambda(x) = 0.$$

Therefore, if (3.3) does not hold, there exists $\hat{x} \in \Sigma_\lambda$ such that

$$w_\lambda(\hat{x}) = \min_{\Sigma_\lambda} w_\lambda(x) < 0.$$
Then by (1.9) and (1.8), for \( x \) satisfies \( w_\lambda(x) < 0 \),
\[
c_\lambda(x) = -\frac{f(x^\lambda, u_\lambda(x)) - f(x^\lambda, u(x))}{u_\lambda(x) - u(x)} - \frac{f(x^\lambda, u(x)) - f(x, u(x))}{u_\lambda(x) - u(x)} \geq -C(u'^2_\lambda(x) + u''_\lambda(x)) \geq -2C u''(x).
\]
Combining with (1.10), we have
\[
\lim_{|x| \to \infty} \frac{|x|^{2s}c_\lambda(x)}{u_\lambda(x) - u(x)} = -2C |x|^{2s}u''(x) \geq -2I(u) \bar{C}_{n,s}.
\]
Now by the Decay at infinity Theorem 2.3, there exists \( R > 0 \) such that
\[
|x| \leq R.
\]
As a result, for \( \lambda > R \), since \( \bar{x} \in \Sigma_\lambda \), this is a contradiction. Thus we obtain (3.3) and complete the proof of Step 1.

Step 2: Suppose \( \lambda_0 > 0 \), if \( w_{\lambda_0}(x) \equiv 0 \), since \( u \) and \( u_{\lambda_0} \) satisfy (1.1) and (1.6), by (1.8), we have
\[
I(u)(-\Delta)^s w_{\lambda_0}(x) = f(x^{\lambda_0}, u_{\lambda_0}(x)) - f(x, u(x)) = f(x^{\lambda_0}, u(x)) - f(x, u(x)) > 0 \quad \text{in} \Sigma_{\lambda_0},
\]
while \( I(u)(-\Delta)^s w_{\lambda_0}(x) \leq 0 \), this is a contradiction.

If \( w_{\lambda_0}(x) \neq 0 \), in \( \Sigma_{\lambda_0} \)
and there exists \( x \in \Sigma_{\lambda_0} \) such that \( w_{\lambda_0}(x) > 0 \). Then the strong maximum principle Theorem 2.2 infer that
\[
w_{\lambda_0}(x) > 0, \quad \forall \; x \in \Sigma_{\lambda_0}.
\]
By the lower semi-continuous assumption on \( f(x, \cdot) \) and (1.8),
\[
\sup_{\lambda_0} c_{\lambda_0}(x) (u_{\lambda_0}(x) - u(x)) = -\inf f(x^{\lambda_0}, u_{\lambda_0}) - f(x^{\lambda_0}, u) + f(x^{\lambda_0}, u) - f(x, u) \leq 0, \quad \text{as} \; \lambda_0 \to u.
\]
On the other hand, consider any \( \bar{x} \in T_{\lambda_0} \), there exists \( y = (y_1, \bar{x}) \in \mathbb{R}^n \) with \( x_1^{\lambda_0} < y_1 < x_1 \) such that
\[
c_{\lambda_0}(x)(u^{x_1^{\lambda_0}} - u(x)) = c_{\lambda_0}(x) \frac{\partial u}{\partial x_1}(y)(x_1^{\lambda_0} - x_1) = -2c_{\lambda_0}(x) \frac{\partial u}{\partial x_1}(y)(\bar{x} - x_1).
\]
If \( \text{dist}(x, T_{\lambda_0}) \to 0 \), which means
\[
x \to \bar{x}, \quad x^{\lambda_0} \to x, \quad y \to \bar{x} \in T_{\lambda_0}.
\]
It follows that
\[
u_{\lambda_0}(x) \to u(x), \quad \frac{\partial u}{\partial x_1}(y) \to \frac{\partial u}{\partial x_1}(\bar{x}).
\]
If \( \frac{\partial u}{\partial x_1}(\bar{x}) \neq 0 \), by (3.5), we have \( \frac{\partial u}{\partial x_1}(\bar{x}) < 0 \). It follows that
\[
\frac{\partial u_{\lambda_0}}{\partial x_1}(\bar{x}) = -2 \frac{\partial u}{\partial x_1}(\bar{x}) > 0.
\]
If \( \frac{\partial u}{\partial x_1}(\bar{x}) = 0 \), then for \( x, \; x^{\lambda_0} \) near \( \bar{x} \),
\[
u(x) = u(\bar{x}) + O(|x - \bar{x}|^2), \quad u(x^{\lambda_0}) = u(\bar{x}) + O(|x^{\lambda_0} - \bar{x}|^2).
\]
So,
\[
c_{\lambda_0}(x)(u^{x_1^{\lambda_0}} - u(x)) = c_{\lambda_0}(x) \cdot O(|x - \bar{x}|^2).
\]
If \(c_{\lambda_0} \leq 0\), it satisfies (1.12).
If \(c_{\lambda_0} > 0\), it follows from (3.5), (3.6) and (3.8) that
\[
0 \leq \sup_{\lambda_0} c_{\lambda_0}(x)(u(x^{\lambda_0}) - u(x)) = \sup_{\lambda_0} c_{\lambda_0}(x) \cdot O(|x - \bar{x}|^2) \leq 0,
\]
as \(u_{\lambda_0} \to u\), which implies that \(c_{\lambda_0}\) satisfies (1.12). Now, by Lemma 1.3, we obtain
\[
\frac{\partial w_{\lambda_0}}{\partial x_1}(\bar{x}) > 0, \quad \forall x \in \partial \Sigma_{\lambda_0}.
\]
(3.9)

On the other hand, by the definition of \(\lambda_0\), there exists a sequence \(\{\lambda_k\}, \lambda_k \nearrow \lambda_0\), and \(x_k \in \Sigma_{\lambda_k}\), such that
\[
w_{\lambda_k}(x_k) = \min_{\lambda_k} w_{\lambda_k} < 0 \quad \text{and} \quad \nabla w_{\lambda_k}(x_k) = 0.
\]
By (1.10), we have \(w_{\lambda_k}\) decay to 0 in the infinity, and thus there exists \(R\) sufficiently large such that \(x_k \in B_R(0) \cap \Sigma_{\lambda_k}\). As a result, up to a subsequence, there exists \(\hat{x}\) such that \(x_k \to \hat{x}\) as \(k \to \infty\), and
\[
w(\lambda_0)(\hat{x}) \leq 0, \quad \text{hence} \ \hat{x} \in \partial \Sigma_{\lambda_0}, \quad \nabla w(\lambda_0)(\hat{x}) = 0,
\]
by the arbitrariness of \(x \in T_{\lambda_0}\), this contradicts (3.7) and (3.9). Thus (3.4) is true and we finished the proof of Theorem 1.2.

\[\square\]

REFERENCES

[1] C. O. Alves and F. J. S. A. Correa, On existence of solutions for a class of problem involving a nonlinear operator, Commun. Appl. Nonlinear Anal., 8 (2001), 43–56.

[2] P. and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math., 108 (1992), 247–262.

[3] G. Autuori, A. Fiscella and P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, Nonlinear Anal., 125 (2015), 699–714.

[4] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equ., 32 (2007), 1245–1260.

[5] W. Chen and C. Li, Maximum principle for the fractional p-Laplacian and symmetry of solutions, Adv. Math., 335 (2018), 735–758.

[6] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math., 308 (2017), 404–437.

[7] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Commun. Pure Appl. Math., 59 (2006), 330–343.

[8] W. Chen and S. Qi, Direct methods on fractional equations, Discrete Contin. Dyn. S., 39 (2019), 1269–1310.

[9] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonlinear Anal., 94 (2014), 156–170.

[10] X. He and W. Zou, Ground state solutions for a class of fractional Kirchhoff equations with critical growth, Sci. China Math., 62 (2019), 853–890.

[11] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[12] C. Li and W. Chen, A Hopf type lemma for fractional equations, Proc. Amer. Math. Soc., 147 (2019), 1565–1575.

[13] G. Li and Y. Niu, The existence and local uniqueness of multi-peak positive solutions to a class of Kirchhoff type equations, Electron. J. Differ. Equ., 225 (2019), 1–19.

[14] Y. Li and W. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in \(\mathbb{R}^n\), Commun. Partial Differ. Equ., 18 (1993), 1043–1054.

[15] L. J. L, On some questions in boundary value problems of mathematical physics, North-Holland Math. Stud., 30 (1978), 284–346.

[16] S. I. Pohozaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N. S.), 96 (1975), 152–166.

[17] P. Pucci and S. Saldi, Critical stationary Kirchhoff equations in \(\mathbb{R}^n\) involving nonlocal operators, Rev. Mat. Iberoam., 32 (2016), 1–22.
[18] B. Zhang and L. Wang, Existence results for Kirchhoff-type superlinear problems involving the fractional Laplacian, Proc. Roy. Soc. Edinburgh Sect. A, 149 (2019), 1061–1081.

Received September 2020; revised December 2020.

E-mail address: yahuniu@163.com