Dirichlet Functors are Contravariant Polynomial Functors

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Abstract

Polynomial functors are sums of covariant representable functors from the category of sets to itself. They have a robust theory with many applications — from operads and opetopes to combinatorial species. In this paper, we define a contravariant analogue of polynomial functors: Dirichlet functors. We develop the basic theory of Dirichlet functors, and relate them to their covariant analogues.

1 Introduction

A polynomial functor \( P : \text{Set} \to \text{Set} \) is a sum of representables

\[
P(X) := \sum_{b \in B} X^{E_b}
\]

where the family of sets \( E_b \) depends on \( b \in B \). This data is known in the computer science literature as a “container” [2, 3, 1], but since such an indexed family of sets can be represented by as a function

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & B \\
\downarrow & & \\
B
\end{array}
\]

we will refer to it as a bundle.

Remarkably, all natural transformations between polynomial functors can be represented in terms of their associated bundles. A natural transformation \( P \to P' \) corresponds to a dependent pair of functions

\[
\left( f : B \to B', f^\sharp : (b : B) \to E_{f_b} \to E_b \right)
\]

noting that \( f \) acts covariantly on the base and \( f^\sharp \) acts contravariantly on fibers. In terms of bundles, a map \( \pi \to \pi' \) is a diagram of the following sort:

\[
\begin{array}{ccc}
E \xleftarrow{f_\sharp} & \bullet & \xrightarrow{\pi'} E' \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & B'
\end{array}
\]

We refer to these special spans as contravariant morphisms of bundles.
Theorem 1.1 (Theorem 2.17 of [4]). The category of polynomial functors and natural transformations is equivalent to the category of bundles and contravariant morphisms.

This begs the question: what, then, are we to make of the more obvious, covariant morphisms of bundles

\[
\begin{array}{ccc}
E & \xrightarrow{\text{tot}(f_\sharp)} & E' \\
\pi \downarrow & & \downarrow \pi' \\
B & \xrightarrow{f} & B'
\end{array}
\]

for which \( f_\sharp \) is covariant in each fibers?

It turns out that these covariant maps of bundles correspond to natural transformations between the appropriate sums of contravariant representables:

\[D(X) := \sum_{b \in B} E_b^X.\]

Polynomial functors as in (1) get their name from the case in which \( B \) and \( E \) are finite sets. Consider the function \( \text{card} \ E(-) : B \to \mathbb{N} \), which takes the cardinality of each fiber \( E_b \). Letting \( B_n := (\text{card}E(-))^{-1}(n) \), i.e. \( B_n \) is the set of elements whose fibers have \( n \) elements, we find that for any set \( X \), the cardinality

\[|P(X)| = \sum_{n \in \mathbb{N}} |B_n||X|^n\]

is a polynomial in the cardinality of \( X \). Similarly,

\[|D(X)| = \sum_{n \in \mathbb{N}} |B_n|n^{|X|}\]

resembles a Dirichlet series in the cardinality of \( X \) – without the usual negative sign.\(^1\) Accordingly, we call such sums of contravariant representables Dirichlet functors.

We will show in this paper that Dirichlet functors are, quite robustly, the contravariant analogue of polynomial functors. In particular, the many equivalent ways to say that a functor is polynomial have contravariant analogues.

Theorem 1.2 (See discussion in 1.18 of [4]). Let \( P : \text{Set} \to \text{Set} \) be a functor. Then the following are equivalent.

1. \( P \) is polynomial.
2. \( P \) is the sum of covariant representables.
3. There is a bundle \( \pi : E \to B \) together with a natural isomorphism

\[P(X) \cong \sum_{b \in B} X^{E_b} .\]

\(^1\)Future work of the first author will recover the negative sign by generalizing the theory of Dirichlet functors to homotopy types.
Or, equivalently, a natural isomorphism of $P$ with the composite

$$\text{Set} \xrightarrow{\Delta_i} \text{Set} \xrightarrow{\Pi_x} \text{Set} \xrightarrow{\Sigma_i} \text{Set}.$$ 

4. $P$ is accessible and preserves connected limits.

Analogously, we will prove the following theorem.

**Theorem 1.3.** Let $D : \text{Set}^{\text{op}} \to \text{Set}$ be a contravariant functor. Then the following are equivalent.

1. $D$ is Dirichlet.
2. $D$ is the sum of contravariant representables.
3. There is a bundle $\pi : E \to B$ together with a natural isomorphism

$$D(X) \cong \sum_{b \in B} E_b^X.$$ 

Or, equivalently, a natural isomorphism of $D$ with the composite

$$\text{Set}^{\text{op}} \xrightarrow{(\Delta_i)_{\text{op}}} \left(\text{Set}_{/B}\right)^{\text{op}} \xrightarrow{\Sigma_i} \text{Set}.$$ 

4. $D$ preserves connected limits.

Note that we no longer need to assume accessibility. This is a general feature of the theory of Dirichlet functors; it is a bit “smaller” and more manageable than that of polynomials. In particular, a Dirichlet functor is determined by its action on the terminal morphism $!_0 : 0 \to 1$ of the empty set. As a corollary, Dirichlet functors form a topos.

**Theorem 1.4.** The functor $\text{Set}^{\text{op}} \to \text{Fun}(\text{Set}^{\text{op}}, \text{Set})$, given by sending $\pi : E \to B$ to the induced Dirichlet functor $X \mapsto \sum_{b \in B} E_b^X$, is fully faithful, and so gives an equivalence

$$\text{Set}^{\text{op}} \simeq \text{Dir}$$

between the topos of bundles and the category of Dirichlet functors, with inverse given by evaluation at $!_0 : 0 \to 1$.

Now, object-wise, a Dirichlet functor and a polynomial functor are determined by the same data — a bundle $\pi : E \to B$ of sets. Accordingly, one would expect for any set $N$ a transformation

$$X^N \mapsto N^X$$

turning polynomial functors into Dirichlet functors, and vice versa. But the natural transformations between each sort of functor induce different morphisms between the bundles; natural transformations between polynomial functors induce contravariant bundle morphisms, while natural transformations between Dirichlet functors induce covariant bundle
morphisms. However, if we restrict to those morphisms of bundles which are isovariant on the fibers — that is, the pullback diagrams of the form

\[
\begin{array}{ccc}
E & \xrightarrow{\text{tot}(f_\sharp)} & E' \\
\pi \downarrow & & \downarrow \pi' \\
B & \xrightarrow{f} & B'
\end{array}
\]

which preserve the number of elements in each row — we find that such a morphism is both a co- and contravariant morphism of bundles. It is well known that such cartesian morphisms of bundles correspond to cartesian natural transformations between polynomial functors [4, Theorem 3.8] — those whose naturality squares are pullbacks. This is true as well for Dirichlet functors.

**Theorem 1.5.** A natural transformation \(D \to D'\) of Dirichlet functors is Cartesian if and only if the corresponding bundle map

\[
\begin{array}{ccc}
E & \xrightarrow{\text{tot}(f_\sharp)} & E' \\
\pi \downarrow & & \downarrow \pi' \\
B & \xrightarrow{f} & B'
\end{array}
\]

is a pullback. As a corollary, we have an equivalence of categories

\[\text{Poly} \simeq \text{Dir}\]

between polynomial functors with cartesian natural transformations and Dirichlet functors with cartesian natural transformations.

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## 2 Dirichlet Functors

Before diving into the theory of Dirichlet functors, let’s first consider the category \(\text{Set}\) of bundles of sets and covariant bundle maps. For our proofs to go smoothly, we will need to explicitly keep track of the self-dualizing isomorphism \(\downarrow \overset{\simeq}{\to} (\downarrow)^\text{op}\) on the walking arrow.

**Definition 2.1.** We let \(\downarrow\) be the walking arrow — the category \(\text{dom} \to \text{cod}\) consisting of a single morphism. We denote by \(\sigma : \downarrow \to (\downarrow)^\text{op}\) the self-dualizing isomorphism of \(\downarrow\), and note that \(\sigma^{-1} = \sigma^\text{op}\).
Proposition 2.2. There is an adjoint sextuple:

\[
\begin{array}{ccc}
\text{Set} & \overset{\text{const}}{\longrightarrow} & \text{Set}^+ \\
\downarrow & & \downarrow \\
\text{dom} & \overset{!(-)}{\longrightarrow} & \text{cod} \\
\end{array}
\]

All three functors \(\text{Set} \rightarrow \text{Set}^+\) are fully faithful.

Proof. There is an adjoint triple \(1 \overset{\text{cod}}{\twoheadleftarrow} \downarrow \). The middle three functors—\(\text{cod}, \text{const}, \text{dom}\)—are given by restricting along this adjoint triple. The next two, \(!(-)\) and \(!(-)_\downarrow\), are given by Kan extending, or more concretely:

\[
X \mapsto \begin{cases}
0 & \text{if } X \text{ is empty,} \\
1 & \text{otherwise.}
\end{cases}
\]

It is easy to see that the unit \(X \rightarrow \text{dom}!X\) is an isomorphism (it is in fact an identity), so \(!(-)_\downarrow\) is fully faithful. Therefore, all three functors going from \(\text{Set} \rightarrow \text{Set}^+\) are fully faithful.

The existence of the final adjoint can be deduced from the fact that \(!(-)_\downarrow\) preserves all limits. It sends a bundle \(\pi : E \rightarrow B\) to the largest subset \(ZC(\pi)\) of \(B\) for which the following square is a pullback

\[
\begin{array}{ccc}
0 & \overset{\gamma}{\longrightarrow} & E \\
\downarrow & & \downarrow \pi \\
ZC(\pi) & \overset{\pi}{\longrightarrow} & B
\end{array}
\]

We prove a few quick facts we will use later.

Lemma 2.3. The functor \(!(-)_\downarrow : \text{Set} \rightarrow \text{Set}^+\) from (2) preserves connected colimits.

Proof. To see that \(!(-)_\downarrow\) preserves connected colimits, recall that colimits in \(\text{Set}^+\) are calculated pointwise. It remains to show, then, that the map of bundles

\[
\begin{array}{ccc}
\text{colim } X_i & \longrightarrow & \text{colim } X_i \\
\downarrow & \downarrow & \downarrow \\
\text{colim } 1 & \longrightarrow & 1
\end{array}
\]

is an isomorphism in \(\text{Set}^+\), for which it suffices to show that \(\text{colim } 1\) is terminal. But the colimit of a diagram of terminal objects is the set of connected components of its indexing category. Since we assumed the indexing category was connected, this contains a single element.
Lemma 2.4. The Yoneda embedding $y : (\downarrow)^{op} \to \mathbf{Set}^\downarrow$ is equal to the composite

$$(\downarrow)^{op} \xrightarrow{\sigma^{op}} \downarrow^0 \xrightarrow{\downarrow^{(-)}} \mathbf{Set} \xrightarrow{\mathbf{Set}^\downarrow} \mathbf{Set}^\downarrow.$$ 

As a corollary, any $\pi : \downarrow \to \mathbf{Set}$ is naturally isomorphic to the composite

$$\downarrow \xrightarrow{\sigma} (\downarrow)^{op} \xrightarrow{\downarrow^0^{op}} \mathbf{Set}^{op} \xrightarrow{\mathbf{Set}^\downarrow^{(-)^{op}}} (\mathbf{Set}^\downarrow)^{op} \xrightarrow{\mathbf{Set}^\downarrow} \mathbf{Set}$$

by the Yoneda lemma.

Proof. One checks directly. 

Now we will define the extent of a bundle $\pi$ to be the Dirichlet functor $\text{ext}_\pi : \mathbf{Set}^{op} \to \mathbf{Set}$ that it corresponds to. This sends a bundle $\pi : E \to B$ to the functor

$$\text{ext}_\pi(X) := \sum_{b \in B} E^X_b.$$ 

We will, however, give a more abstract definition of the extent, and then calculate a number of presentations of it.

Definition 2.5. Consider the functor $\downarrow^0^{op} \circ \sigma : \downarrow \to \mathbf{Set}^{op}$ picking out the unique morphism $1 \to 0$ in $\mathbf{Set}^{op}$. Sending a functor $\mathbf{Set}^{op} \to \mathbf{Set}$ to its precomposition with $\downarrow^0^{op} \circ \sigma$ gives an evaluation functor

$$\mathbf{Fun}(\mathbf{Set}^{op}, \mathbf{Set}) \xrightarrow{\text{ev}_{\downarrow^0}} \mathbf{Set}^\downarrow.$$ 

This functor admits a right adjoint by right Kan extension along $\downarrow^0^{op} \circ \sigma : \downarrow \to \mathbf{Set}^{op}$; we define the Dirichlet extent functor $\text{ext} : \mathbf{Set}^\downarrow \to \mathbf{Fun}(\mathbf{Set}^{op}, \mathbf{Set})$ to be this right adjoint. It sends any bundle $\pi$ to

$$\text{ext}_\pi := \text{ran}_{\downarrow^0^{op} \circ \sigma} \pi.$$

Proposition 2.6. Let $\pi : E \to B$ be a bundle. The following are equivalent:

1. The extent $\text{ext}_\pi : \mathbf{Set}^{op} \to \mathbf{Set}$ of $\pi$ from Definition 2.5.

2. The functor

$$X \mapsto \sum_{b \in B} E^X_b,$$

or equivalently the composite

$$\mathbf{Set}^{op} \xrightarrow{(\Delta_B)^{op}} (\mathbf{Set}/B)^{op} \xrightarrow{\mathbf{Set}/B(-, \pi)} \mathbf{Set}/B \xrightarrow{\Sigma_B} \mathbf{Set}.$$ 

3. The pullback in $\mathbf{Fun}(\mathbf{Set}^{op}, \mathbf{Set})$:

\[
\begin{array}{ccc}
\text{ext}_\pi & \xrightarrow{\gamma} & B \\
\downarrow & & \downarrow \\
\mathbf{Set}(-, E) & \longrightarrow & \mathbf{Set}(-, B)
\end{array}
\]
4. The restricted representable functor \( \mathbf{Set}^+(!(-), \pi) \).

5. The functor
\[
X \mapsto \lim(\text{Hom}_{\mathbf{Set}}(l_0, X)^{\text{op}} \to \downarrow^{\text{op}} \sigma^{\text{op}} \to \downarrow^{\pi} \mathbf{Set})
\]
where \( \text{Hom}_{\mathbf{Set}}(l_0, X) \) is the comma category of \( l_0 : \downarrow \to \mathbf{Set} \) over \( X : * \to \mathbf{Set} \).

6. The functor
\[
X \mapsto \lim((X^\circ)^{\text{op}} \to (1^\circ)^{\text{op}} \to \downarrow^{\pi} \mathbf{Set})
\]
where \((-)^\circ\) is the left cone 2-functor, adjoining an initial object.

**Proof.** We have presented these results in order of most understandable to most computational; we will prove it a somewhat opposite order.

First, we note that the conical limit formula for \( \text{ext}_\pi \equiv \text{ran}_0 \pi \) as a right Kan extension says
\[
\text{ext}_\pi(X) = \lim(\text{Hom}_{\mathbf{Set}}^{\text{op}}(X, l_0^{\circ} \circ \sigma) \to \downarrow^{\pi} \mathbf{Set}).
\]

Now, \( \text{Hom}_{\mathbf{Set}}^{\text{op}}(X, l_0^{\circ} \circ \sigma) \simeq \text{Hom}_{\mathbf{Set}}(l_0 \circ \sigma^{\text{op}}, X)^{\text{op}} \) over \( \downarrow \). Furthermore, we have the following equivalence:
\[
\text{Hom}_{\mathbf{Set}}(l_0 \circ \sigma^{\text{op}}, X)^{\text{op}} \to \downarrow^{\text{op}} \sigma \sim \text{Hom}_{\mathbf{Set}}(l_0, X)^{\text{op}} \to \downarrow^{\text{op}}
\]
and therefore we may equivalently calculate this limit as
\[
\lim(\text{Hom}_{\mathbf{Set}}(l_0, X)^{\text{op}} \to \downarrow^{\text{op}} \sigma^{\text{op}} \to \downarrow^{\pi} \mathbf{Set}).
\]

This gives us the equivalence between (1) and (5).

The comma category \( \text{Hom}_{\mathbf{Set}}(l_0, X) \) simply adjoins an initial object to (the discrete category) \( X \). Therefore, we find that (5) and (6) are equivalent.

Every set \( X \) is the colimit of the diagram \( X^\circ \xrightarrow{1^\circ} l_0 \to \mathbf{Set} \), namely:

\[
\begin{array}{cccccc}
& X & \xrightarrow{1} & \cdots & \xrightarrow{1} & l_0 \\
1 & \downarrow & & \downarrow & & \downarrow \\
0 & \downarrow & & \downarrow & & \\
& 1 & & 1 & & \\
\end{array}
\]

Since, by Lemma 2.3, \( l(-) \) preserves connected colimits, we may make the following identification of (4) with (6) using Lemma 2.4:
\[
\text{Set}^+(!X, \pi) = \text{Set}^+(!\text{colim}(X^\circ \xrightarrow{1^\circ} l_0 \to \mathbf{Set}), \pi) \simeq \text{Set}^+(\text{colim}(X^\circ \xrightarrow{1^\circ} l_0 \to \mathbf{Set} \xrightarrow{l(-)} \mathbf{Set}^+), \pi) \simeq \text{lim}((X^\circ)^{\text{op}} \to (1^\circ)^{\text{op}} \to \text{Set}^{\text{op}} \to (\text{Set}^+)^{\text{op}} \to \mathbf{Set}) \simeq \text{lim}((X^\circ)^{\text{op}} \to (1^\circ)^{\text{op}} \to \sigma^{\text{op}} \to \downarrow^{\pi} \mathbf{Set}).
\]
We see that (3) is equivalent to (4) by noting that the following square of natural transformations is a pullback:

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{\gamma} & \text{Set}(\text{cod}(-), \text{cod}(-)) \\
\downarrow & & \downarrow \\
\text{Set}(\text{dom}(-), \text{dom}(-)) & \xrightarrow{\pi} & \text{Set}(\text{dom}(-), \text{cod}(-))
\end{array}
\]

and restricting the right side to $\pi$ and the left side to $\simeq_{\downarrow}$.

Finally, we note that the set $\text{Set}^\downarrow(\simeq_{\downarrow}, \pi)$ is naturally isomorphic to the set $\sum_{b \in B} P_b^X$, letting us identify (4) with (2).

Now we are ready to intrinsically characterize the Dirichlet functors.

**Definition 2.7.** A Dirichlet functor is a contravariant functor $D : \text{Set}^{\text{op}} \to \text{Set}$ which preserves connected limits. We denote by $\text{Dir}$ the category of Dirichlet functors and natural transformations.

**Theorem 2.8.** The functor $\text{ext}$ is fully faithful, and gives an equivalence

$$\text{Set}^\downarrow \simeq \text{Dir}.$$

As a corollary, the category of Dirichlet functors is a topos.

**Proof.** Since $\simeq_{\downarrow} : \downarrow \to \text{Set}^{\text{op}}$ is fully faithful, the counit of the $\text{ev} : \downarrow \vdash \text{ext}$ adjunction, the universal cell defining the right Kan extension $\text{ext}(-)$, is an isomorphism. Thus $\text{ext}$ is fully faithful. In what follows we show that a functor is Dirichlet—preserves connected limits—if and only if it is the extent of a bundle, proving the equivalence of $\text{Dir}$ with $\text{Set}^\downarrow$.

If $D$ is the extent of a bundle $\pi$, then by Prop 2.6, $D$ is naturally isomorphic to the restricted representable $\text{Set}^\downarrow(\simeq_{\downarrow}, \pi)$. By Lemma 2.3, this sends connected colimits in $\text{Set}$ to connected limits.

Now, we show that if $D$ is Dirichlet, then the unit $D \to \text{ext}_{\text{Dir}}$ is an isomorphism. By Prop 2.6,

$$\text{ext}_{\text{Dir}}(X) = \text{lim}(X^\alpha \xrightarrow{\simeq} 1^\alpha \xrightarrow{D_{\text{Dir}}} \text{Set}).$$

Every set $X$ is the connected colimit of the diagram $X^\alpha \to \downarrow \xrightarrow{\simeq} \text{Set}$, and therefore if $D$ preserves this limit, then $D(X)$ is precisely the above limit $\text{ext}_{\text{Dir}}(X)$.

**Remark 2.9.** Since polynomial functors preserve connected limits, the composite $P \circ D$ of a polynomial functor after a Dirichlet functor is Dirichlet. On the other hand, the composite $D' \circ D^{\text{op}}$ of two Dirichlet functors does not in general preserve connected limits, since $D^{\text{op}}$ sends connected colimits in $\text{Set}$ to connected colimits in $\text{Set}^{\text{op}}$, and $D'$ does not necessarily preserve these. Furthermore, composites of Dirichlet functors are not in general accessible.
Remark 2.10. The six adjoints of Proposition 2.2 correspond, under the equivalence of Theorem 2.8, to
\[
\begin{align*}
ZC(D|_0) & \cong \text{the coefficient of } 0^X \text{ in } D. \\
\text{ext}(!^C) & \cong X \mapsto C \times 0^X \\
cod(D|_0) & \cong D(0) \\
\text{ext}(\text{const}(C)) & \cong X \mapsto C \\
\text{dom}(D|_0) & \cong D(1) \\
\text{ext}(!_C) & \cong X \mapsto C^X
\end{align*}
\]
In particular, !(-) corresponds to the Yoneda embedding.

3 Cartesian Transformations between Dirichlet Functors

In this section, we turn to cartesian transformations between Dirichlet functors. We will show that the category of Dirichlet functors and cartesian transformations is equivalent to the category of polynomial functors and cartesian transformations.

**Proposition 3.1.** A natural transformation \( \phi : D \to D' \) between Dirichlet functors is cartesian if and only if the induced bundle map \( D|_0 \to D'|_0 \) is a pullback.

As a corollary, the equivalence \( \text{Dir} \simeq \text{Set}^{\dagger} \) restricts to an equivalence
\[
\text{Dir}^c \simeq \text{Set}^{\dagger}
\]
between Dirichlet functors with cartesian natural transformations and bundles with pullback squares.

**Proof.** We want to show that for any \( f : D \to D' \), the square
\[
\begin{array}{ccc}
D(1) & \xrightarrow{f_1} & D'(1) \\
\downarrow{\pi} & & \downarrow{\pi'} \\
D(0) & \xrightarrow{f_0} & D'(0)
\end{array}
\]
is a pullback in \( \text{Set} \) iff for all functions \( g : X \to X' \), the naturality square
\[
\begin{array}{ccc}
D(X') & \xrightarrow{f_{X'}} & D'(X') \\
\downarrow{D(g)} & & \downarrow{D'(g)} \\
D(X) & \xrightarrow{f_X} & D'(X)
\end{array}
\]
is a pullback in \( \text{Set} \). We will freely use the natural isomorphism \( D(X) \cong \text{Set}^{\dagger}(!_X, D|_0) \) from Proposition 2.6, which allows us to identify Diagram (4) with
\[
\begin{array}{ccc}
\text{Set}^{\dagger}(!_{X'}, D'|_0) & \xrightarrow{f_{0,1}} & \text{Set}^{\dagger}(!_{X'}, D'|_0) \\
\uparrow{!_{g,1}} & & \downarrow{!_{g,1}} \\
\text{Set}^{\dagger}(!_X, D|_0) & \xrightarrow{f_{0,1}} & \text{Set}^{\dagger}(!_X, D'|_0)
\end{array}
\]

The square in Diagram (3) is a special case of that in Diagram (4), namely for \( g := !_0 \); this establishes the only-if direction.

To complete the proof, suppose that Diagram (3) is a pullback, take an arbitrary \( g: X \to X' \), and suppose given a commutative solid-arrow diagram as shown:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \\
D(1) & \xrightarrow{D} & D'(1) \\
\downarrow & & \downarrow \\
1 & \xrightarrow{1} & 1 \\
\downarrow & & \downarrow \\
D(0) & \xrightarrow{D} & D'(0)
\end{array}
\]

We can interpret the statement that Diagram (5) is a pullback as saying that there are unique dotted arrows making the diagram commute. So, we need to show that if the front face is a pullback, then there are unique diagonal dotted arrows as shown, making the diagram commute. This follows quickly from the universal property of the pullback. □

**Theorem 3.2.** There is an equivalence of categories

\[ \text{Poly}_r \simeq \text{Dir}_r \]

between the category of polynomial functors and cartesian transformations and Dirichlet functors and cartesian transformations. This equivalence sends representables to representables

\[ (-)^N \mapsto N(-). \]

**Proof.** This follows by composing the equivalence of \( \text{Dir}_r \) with \( \text{Set}^\downarrow \) from Proposition 3.1 with that of [4, Proposition 3.14], noting that \( \text{Set}^\downarrow \) is the category of (1, 1)-polynomials. □

**Corollary 3.3.** Let \( D \) be a Dirichlet functor. Then the category \( \text{Dir}_r/D \) of Dirichlet functors with a cartesian map to \( D \) is a topos.

**Proof.** By Theorem 3.2, this category is equivalent to \( \text{Poly}_r/P \) for a polynomial \( P \). But this is a topos as observed in [5, Remark 2.6.2]. □

Now, since \( \text{Dir} \) is a topos, so is \( \text{Dir}_r/D \). And, as we saw above, \( \text{Dir}_r/D \) is a topos as well. What is the relationship between \( \text{Dir}_r/D \) and \( \text{Dir}_{r/D} \)?

We will show that \( \text{Dir}_{r/D} \) is a subtopos of \( \text{Dir}_r/D \) with the left exact left adjoint to the inclusion given by the vertical / cartesian factorization system on \( \text{Set}^\downarrow \).

**Theorem 3.4.** For any \( \pi: \downarrow \to \text{Set} \), we have a subtopos inclusion

\[ \text{Set}_{r/\pi}^\downarrow \hookrightarrow \text{Set}_{/\pi}^\downarrow \]

with left exact left adjoint given by the vertical / cartesian factorization system:

\[
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & \downarrow \pi & \rightarrow \\
B' & \longrightarrow & B
\end{array}
\quad
\begin{array}{ccc}
E' & \longrightarrow & \bullet & \longrightarrow & E \\
\downarrow & \downarrow \gamma & \downarrow & \downarrow \pi \\
B' & \longrightarrow & B' & \longrightarrow & B
\end{array}
\]

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As a corollary, $\text{Dir}_D \hookrightarrow \text{Dir}_D$ is a subtopos inclusion.

Proof. We have displayed both the action of the left adjoint and its unit — the universal map into the pullback. The counit is always an isomorphism since pullbacks are unique up to unique isomorphism.

That this is lex follows quickly from the fact that taking pullbacks commutes with taking (finite) limits.

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