Global Symmetries of Noncommutative Field Theory

C.Gonera\textsuperscript{a1}, P.Kosiński\textsuperscript{a1}, P.Maślanka\textsuperscript{a1} and S.Giller\textsuperscript{b1}

\textsuperscript{a}Department of Theoretical Physics II, Institute of Physics, University of Łódź, Pomorska 149/153, 90 - 236 Łódź Poland.
\textsuperscript{b}Institute of Physics, Jan Długosz Academy, Armii Krajowej 13/15, 42-200 Częstochowa, Poland

Abstract. It is shown how a construction of so called \( \theta \)-Poincare group - a global counterpart of twisted Poincare algebra (which can be considered as symmetry algebra of Noncommutative Field Theory (NCFT) with noncommutativity of coordinates being controlled by constant, antisymmetric matrix \( \theta \)) can be extended to others groups containing translations as abelian subgroup.

1. Introduction

In this talk it is presented how a construction of a global counterpart of twisted Poincare algebra can be extended to others groups containing translations as abelian subgroup. It starts with a brief recapitulation of twisted Poincare algebra approach to NCFT on space-time with commutation relation of coordinate operators being defined by constant, antisymmetric matrix \([1] - [4]\). Then a global counterpart of twisted Poincare algebra (which is a dual Hopf algebra) so called \( \theta \)-Poincare group is presented [5] - [7]. Finally, in the last point it is sketched how this global construction of \( \theta \)-Poincare group can be extended to others groups containing translations as abelian subgroup [8].

2. Twisted Poincare algebra

It is well known that due to Weyl-Moyal correspondence NCFT that is FT on space-time with noncommuting coordinate operators can be considered as FT on usual commutative one with pointwise product being replaced by appropriate star product. In particular, for NCFT on space-time with coordinate operators \( \hat{x}^\mu \) satisfying commutation relations being described by constant, antisymmetric matrix \( \theta^{\mu\nu} \):

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu},
\]

Weyl-Moyal correspondence maps field operators \( \hat{F}(\hat{x}) \) into their symbols \( F(x) \) (being functions on commutative space-time) and product of operators into a star product which

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can be written as
\[
(F_1 \star F_2)(x) = \mu \left( e^{\frac{i}{2} \Theta^\mu_\nu P_\mu \otimes P_\nu} F_1 \otimes F_2 \right)(x) \quad (2)
\]
where \( \mu \) denotes the pointwise multiplication and \( P_\mu = -i \frac{\partial}{\partial x^\mu} \).

Now, one of the major problems around NCFT considered here is that it is not Poincaré covariant. On the level of the operator symbols it is related to the fact that star product is not Lorentz covariant in the sense that star product given by eq. (2) does not commute with the action of Lorentz group. That is in general
\[
F(x) \star G(x) |_{x \to \Lambda x} \neq F(\Lambda x) \star G(\Lambda x). \quad (3)
\]

On the LHS the \( \star \) product is taken before the Lorentz group action while on the RHS Lorentz transformation is performed before the \( \star \) product operation.

On the infinitesimal level this lack of Lorentz covariance is reflected in the fact that the action of ordinary Lorentz transformation generator \( M^\mu_\nu \) on the \( \star \) product cannot be expressed (unlike in the commutative case) in terms of the action of its coproduct \( \Delta(M^\mu_\nu) \).

\[
M^\mu_\nu(F \star G)(x) \neq \mu(\exp\left(-\frac{i}{2} \Theta^\mu_\nu P_\mu \otimes P_\nu \right) \Delta(M^\mu_\nu)(F(x) \otimes G(x)))
\equiv M^\mu_\nu F(x) \star G(x) + F(x) \star M^\mu_\nu G(x) \quad (4)
\]

where
\[
M^\mu_\nu = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \text{ is the Lorentz transformation generator while}
\]
\[
\Delta(M^\mu_\nu) \equiv M^\mu_\nu \otimes I + I \otimes M^\mu_\nu \text{ is coproduct of } M^\mu_\nu.
\]
However, it has been observed [1], [2] that the action of \( M^\mu_\nu \) on \( \star \) product can be expressed in terms of its twisted coproduct \( \Delta_t(M^\mu_\nu) \) given by

\[
\Delta_t(M^\mu_\nu) = \exp\left(\frac{i}{2} \Theta^\mu_\nu P_\mu \otimes P_\nu \right) \Delta(M^\mu_\nu) \exp\left(-\frac{i}{2} \Theta^\mu_\nu P_\mu \otimes P_\nu \right)
\equiv \Delta(M^\mu_\nu) + \frac{1}{2} P_\mu \otimes (\Theta^\mu_\nu P_\nu - \Theta^\nu_\mu P_\mu) - \frac{1}{2} (\Theta^\mu_\rho P_\nu - \Theta^\nu_\rho P_\mu) \otimes P_\rho, \quad (5)
\]

indeed one can check that
\[
M^\mu_\nu(F \star G)(x) = \mu(\exp\left(-\frac{i}{2} \Theta^\alpha_\beta P_\alpha \otimes P_\beta \right) \Delta_t(M^\mu_\nu)(F(x) \otimes G(x))). \quad (6)
\]

This observation provides a nice reinterpretation of the usual Lorentz/Poincare covariance breaking by NCFT. It consists in promoting twisted Poincare algebra to symmetry algebra of NCFT with the \( \star \) product of coordinates given by \( \Theta \) matrix [1], [2].

The twisted Hopf algebras are well known in Quantum Group business [9]. Roughly speaking, they are constructed out of the cocommutative Hopf algebras by performing appropriate deformation of standard coproduct. In particular, twisted Poincare algebra is a Hopf algebra generated by ordinary Poincare algebra generators \( M^\mu_\nu, P_\nu \) satisfying the standard commutation relations with coproduct being twisted by the element \( \exp\left(\frac{i}{2} \Theta^\mu_\nu P_\mu \otimes P_\nu \right) \) [4].
3. \( \theta - \text{Poincare Group} \)

Having defined the twisted Poincare Hopf algebra in terms of generators one can construct its global counterpart that is a dual Hopf algebra of group elements functions with appropriate algebraic and coalgebraic structures defining a deformation \( P_{\Theta} \) of Poincare group so called \( \Theta - \text{Poincare group} \).

It appears that \( P_{\Theta} \) is generated by Hermitean elements \( \hat{\Lambda}^\mu_\nu, \hat{a}^\mu \) obeying the following relations [5],[6].

\[
\begin{align*}
\Delta \hat{\Lambda}^\mu_\nu &= \hat{\Lambda}^\mu_\alpha \otimes \hat{\Lambda}^\alpha_\nu \\
\Delta \hat{a}^\mu &= \hat{\Lambda}^\mu_\alpha \otimes \hat{a}^\alpha + \hat{a}^\mu \otimes 1 \\
\varepsilon(\hat{\Lambda}^\mu_\nu) &= \delta^\mu_\nu \\
\varepsilon(\hat{a}^\mu) &= 0 \\
S(\hat{\Lambda}^\mu_\nu) &= \hat{\Lambda}^\mu_\nu \\
S(\hat{a}^\mu) &= -\hat{\Lambda}^\mu_\alpha \hat{a}^\alpha \\
[\hat{\Lambda}^\mu_\nu, \cdot] &= 0 \\
[\hat{a}^\mu, \hat{a}^\nu] &= -i \Theta^{\mu\alpha}(\hat{\Lambda}^\mu_\rho \hat{\Lambda}^\rho_\nu - \delta^\mu_\rho \delta^\nu_\sigma)
\end{align*}
\]

Algebraic relations in \( P_{\Theta} \) group follow from FRT equation [10]

\[
R(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)R
\]

where \( T_{a b} \) is a 5 \( \times \) 5 matrix representing a Poincare transformation;

\[
T = \begin{bmatrix}
\hat{\Lambda}^\mu_\nu & \hat{a}^\mu \\
- & - & - \\
0 & - & - & - \\
\end{bmatrix};
\]

here \( \mu, \nu = 0, ..., 3 \), \( \hat{\Lambda}^\mu_\nu \) is Lorentz matrix, \( \hat{a}^\mu \) denotes translation while \( \exp(\frac{i}{2} \Theta^{\mu\nu} P^\mu_\rho \otimes P^\nu_\rho) \) is universal R matrix (for a given twist of classical group \( R = F_{21} F^{-1} \)).

The coalgebraic structure of \( \Theta - \text{Poincare group} \) is a result of standard coproduct definition

\[
\Delta T^a_b = T^a_c \otimes T^c_b
\]

and usual definitions of counit \( \varepsilon \) and antipod S maps.

The action of \( \Theta - \text{Poincare group} \) on noncommutative space-time is defined in the following standard way

\[
\begin{align*}
\hat{\delta}^\mu &\rightarrow \hat{\Lambda}^\mu_\nu \otimes \hat{\delta}^\nu + \hat{a}^\mu \otimes I \\
F(\hat{x}) &\rightarrow F(\hat{\Lambda}^\mu_\nu \otimes \hat{\delta}^\nu + \hat{a}^\mu \otimes I) = F(\hat{\Lambda}, \hat{a}, \hat{x}).
\end{align*}
\]

This action cannot be directly expressed in terms of field operators symbols and standard Moyal product because of noncommutative elements \( \hat{a}^\mu \) entering the action of \( P_{\Theta} \) group. However, one still can deal with field operator symbols defined in standard way and depending on classical i.e. commutative variables \( \Lambda, a, x \) provided one modifies usual star product given
by eq.(2). This can be done by taking into account that Lorentz elements $\Lambda$ commutes with $\hat{x}$ and $\hat{a}^{\mu}$ and by writting commutation relations for for $\hat{a}^{\mu}$’s in a form

$$[\hat{a}^{\mu}, \hat{a}^{\nu}] = i \Theta^{\mu\nu}$$

$$\Theta^{\mu\nu} = -\Theta^{\alpha\sigma} (\delta_{\mu}^{\alpha} \delta_{\nu}^{\sigma} - \delta_{\nu}^{\alpha} \delta_{\mu}^{\sigma})$$

(12)

Then, one can check that Weyl-Moyal correspondence maps the product of operators depending on noncommutative variables $\hat{x}, \hat{a}^{\mu}$ into a following modified $\star_{M}$ star product of these operators symbols $[7]$

$$(F \star_{M} G)(\Lambda, a, x) \equiv \mu(\exp\left[\frac{i}{2} (\Theta^{\mu\nu} \partial_{x^{\mu}} \otimes \partial_{x^{\nu}} - \Theta^{\mu\nu} \partial_{a^{\mu}} \otimes \partial_{a^{\nu}}) \right] F(\Lambda, a, x) \otimes G(\Lambda, a, x)).$$

(13)

Let us note that:

a) for functions depending on space-time variables only modified star product reduces to the standard Moyal one,

b) modified star product of functions on classical Poincare group encodes the algebraic structure of $\Theta$ - Poincare group. In particular

$$[A^{\mu} \cdot, \star_{M} = 0$$

$$[a^{\mu}, a^{\nu}]_{\star_{M}} = i \Theta^{\mu\nu}$$

(14)

where $[A, B]_{\star_{M}} \equiv A \star_{M} B - B \star_{M} A$ .

c) the new $\star_{M}$ product enjoys a very important property that it commutes with the action of the Poincare group. In other words although $\Theta$- Poincare group acts on operator symbols as the standard Poincare one does (i.e. $F(\hat{x}) \longrightarrow F(\Lambda \hat{x} + a)$) its action on $\star_{M}$ product of symbols is given by

$$(F \star_{M} G)(x) |_{x \rightarrow \Lambda x + a} = F(\Lambda x + a) \star_{M} G(\Lambda x + a).$$

(15)

On the LHS one takes first the star product (reducing here to the standard Moyal one) and then applies Poincare transformation while on the RHS Poincare transformation is taken before applying the $\star_{M}$ product. It means that the $\star_{M}$ product transforms in covariant way under $P_{\Theta}$ group which allows to consider $\Theta$-Poincare group as a global symmetry group of our NCFT.

Taking infinitesimal Poincare transformations $(I + \omega, a)$ with a and $\omega$ infinitesimal ones and writing $F((I + \omega)x + a)$ and $(F \star_{M} G)((I + \omega)x + a)$ up to terms linear in $\omega$ and $a$ gives the generators of $\Theta$ Poincare group.

$$\hat{P}_{\mu} F = i \partial_{\mu} F$$

$$\hat{M}_{\alpha \beta} F = -i (x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}) F$$

$$\hat{P}_{\mu} (F \star G) = (\hat{P}_{\mu} F) \star G + F \star (\hat{P}_{\mu} G)$$

$$\hat{M}_{\alpha \beta} (F \star G) = (\hat{M}_{\alpha \beta} F) \star G + F \star (\hat{M}_{\alpha \beta} G) +$$

$$\frac{1}{2} \Theta^{\mu\nu} \left[ \eta_{\mu \alpha} (\hat{P}_{\beta} F) \star (\hat{P}_{\nu} G) + \eta_{\nu \alpha} (\hat{P}_{\mu} F) \star (\hat{P}_{\beta} G) - \eta_{\nu \beta} (\hat{P}_{\mu} F) \star (\hat{P}_{\alpha} G) \right]$$

$$\mu(\exp(-\frac{i}{2} \Theta^{\alpha\beta} P_{\alpha} \otimes P_{\beta}) \Delta(M_{\mu\nu}) F \otimes G) = M_{\mu\nu} (F \star G).$$

(16)
As expected generators \( \hat{X} \) of \( \Theta \) Poincare group coincide with generators \( X^i \) of twisted Poincare algebra confirming that \( P_{\Theta} \) group is a global counterpart of the former.

### 4. Generalization of global twisted groups construction

Now, one can try to extend the above approach to construction of \( \Theta \)-Poincare group to others group containing translations as abelian subgroup [8]. A key idea is to generalize an observation that nontrivial algebraic structure of \( \Theta \) - Poincare group can be considered as result of demanding that group action commutes with star product. It suggests the following general procedure. Take a group \( G \) containing translation as abelian subgroup and define a star product \( (H_1 \ast H_2)(g) \) on group manifold in such a way that group action commutes with star product in the sense previously discussed i.e.

\[
(F_1 \ast F_2)(x) \big|_{x \rightarrow gx} = F_1(gx) \ast F_2(gx)
\]

where on the left hand side the standard star product on noncommutative space-time is taken prior to the group action while on the right hand side the star product, both in group and space-time variables is taken at the end.

To show that this strategy can, at least in principle, be implemented let us assume that the group \( G \) contains translations as abelian subgroup; assume further, that any element of \( G \) can be written as a product of translation and an element of some subgroup \( S \subset G \).

\[
g = g(\vec{a}, \vec{\zeta}) = e^{ia^\mu P_\mu} e^{i\zeta^a K_a}
\]

where \( g \in G, \ P_\mu \) are translation generators while \( K_a \) - the generators of \( S \). Then writing the product of two elements in the form

\[
g(\vec{a}', \vec{\zeta}') = g(\vec{a}', \vec{\zeta}) g(\vec{a}, \vec{\zeta}) = e^{ia'^\mu P_\mu} e^{i\zeta'^a K_a} e^{ia^\mu P_\mu} e^{i\zeta^a K_a}
\]

\[
e^{ia'^\mu P_\mu} (e^{i\zeta'^a K_a} e^{ia^\mu P_\mu} e^{i\zeta^a K_a}) = e^{ia'^\mu P_\mu} g(\vec{a}, \vec{\zeta}) (e^{i\zeta'^a K_a} e^{i\zeta^a K_a}) = e^{i(h^a(\vec{\zeta}, \vec{a}) + a'^a) P_\mu} e^{i\lambda^a(\vec{\zeta}, \vec{a})} K_a
\]

gives composition rules

\[
a'^a = h^a(\vec{\zeta}, \vec{a}) + a'^a; \quad h^a(\vec{\zeta}, 0) = 0
\]

\[
\zeta'^a = \lambda^a(\vec{\zeta}, \vec{a})
\]

The right/left action of translation subgroup elements \( g(\vec{a}, 0) \) generates left \( P_L \) / right \( P_R \) invariant vector fields.

\[
P_L^\mu(\vec{\zeta}, \vec{a}) = \frac{\partial h^\mu(\vec{\zeta}, \vec{a})}{\partial a^\mu} \big|_{a=0} \frac{\partial}{\partial a^\nu} + \frac{\partial h^\mu(\vec{\zeta}, 0, \vec{a})}{\partial a^\mu} \big|_{a=0} \frac{\partial}{\partial \zeta^a}
\]

\[
P_R^\mu(\vec{\zeta}, \vec{a}) = \frac{\partial}{\partial a^\mu}
\]

Both fields are related by the adjoint action of \( G \).

Now, one can identify space - time \( M \) with coset space \( G/S; \ M = G/S = (\vec{x}, (\vec{\zeta}, (\vec{x}, \vec{\zeta}), ..)) \) and functions on \( M \) with ones on \( G \) constant over \( S; \ F : M \rightarrow R \).
\( F(\vec{x}) = F(\vec{x}, \vec{\zeta}) = F(\vec{x}, \vec{\zeta}') \) for all \( \vec{\zeta}, \vec{\zeta}' \). The action of \( G \) on \( \mathcal{M} \) following from composition rules reads
\[
g(\vec{a}, \vec{\zeta}) : x^\mu \rightarrow x'^\mu = h^\mu(\vec{\zeta}, \vec{x}) + a^\mu \tag{22}\]
and \( S \) becomes the stability subgroup of the point \( x^\mu = 0 \).

The standard star product on \( \mathcal{M} \) can be written as
\[
(F_1 \ast F_2)(\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^R_{\mu} \otimes P^R_{\nu} F_1 \otimes F_2} \right)(\vec{x}) \tag{23}\]
where \( \mu \) denotes the pointwise multiplication and \( P^R_{\mu} = \frac{\partial}{\partial x^\mu} \).

In order to define the star product on group manifold in such a way that eq. (17) holds i.e.
\[
(F_1 \ast F_2)(\vec{x}) \mid_{\vec{x} \rightarrow g\vec{x}} = F_1(g\vec{x}) \ast F_2(g\vec{x}) \tag{24}\]
ote that due to the properties of invariant vector fields on Lie group the action of right invariant vector fields can be expressed in terms of action of left invariant ones \([8]\)

\[
P^R_{\mu}(\vec{x}) F(g\vec{x}) = P^L_{\mu}(g) F(\vec{x}) \tag{25}\]

Therefore, the right-hand side of eq. (24) can be rewritten as (where the star product is taken with respect to space-time coordinates)
\[
F_1(g\vec{x}) \ast F_2(g\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^R_{\mu} \otimes P^R_{\nu} F_1 \otimes F_2} \right)(\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^L_{\mu}(g) \otimes P^L_{\nu}(g) F_1 \otimes F_2} \right)(g\vec{x}) \tag{26}\]

On the other hand right invariance of \( P^R_{\mu} \) implies
\[
P^R_{\mu}(g\vec{x}) F(\vec{x}) = P^R_{\mu}(g) F(\vec{x}) = P^L_{\mu}(\vec{\zeta}, \vec{\alpha}) F(\vec{h}(\vec{\zeta}, \vec{x}) + \vec{\alpha}) = \frac{\partial F(\vec{h}(\vec{\zeta}, \vec{x}) + \vec{\alpha})}{\partial x^\mu} = \frac{\partial F(\vec{x})}{\partial x^\mu} \mid_{\vec{x} \rightarrow g\vec{x}} \tag{27}\]

and LHS of eq. (24) can be rewritten in the following form
\[
(F_1 \ast F_2)(g\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^R_{\mu}(\vec{x}) \otimes P^R_{\nu}(\vec{x}) F_1 \otimes F_2} \right)(\vec{x}) \mid_{\vec{x} \rightarrow g\vec{x}} = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^L_{\mu}(\vec{a}) \otimes P^L_{\nu}(\vec{a}) F_1 \otimes F_2} \right)(g\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} P^L_{\mu}(g) \otimes P^L_{\nu}(g) F_1 \otimes F_2} \right)(g\vec{x}) = \mu \left( e^{\frac{i}{2} \Theta^{\mu\nu} (P^R_{\mu}(\vec{a}) \otimes P^R_{\nu}(\vec{a}) - P^L_{\mu}(\vec{a}) \otimes P^L_{\nu}(\vec{a})) \cdot e^{\frac{i}{2} \Theta^{\mu\nu}(P^L_{\mu}(g) \otimes P^L_{\nu}(g)) F_1(g\vec{x}) \otimes F_2(g\vec{x})} \right) = \sum_{n=0}^{\infty} \left( \frac{i}{n!} \Theta^{\alpha_1 \sigma_1} \ldots \Theta^{\alpha_n \sigma_n} (P^R_{\alpha_1}(\vec{a}) P^R_{\sigma_1}(\vec{a}') - P^L_{\alpha_1}(\vec{a}) P^L_{\sigma_1}(\vec{a}')) \ldots \right) \left( (P^R_{\sigma_n}(\vec{a}') P^R_{\sigma_n}(\vec{a}')) - P^L_{\sigma_n}(\vec{a}) P^L_{\sigma_n}(\vec{a}')) \right) F_1(g\vec{x}) \ast F_2(g'\vec{x}) \big|_{g \rightarrow g'} \tag{28}\]
where the star product on the right-hand side is taken only with respect to the $x$-variables. So, one concludes that group action commutes with star product provided the star product on group manifold is defined by

$$
(H_1 \ast H_2)(g) = \mu \left( e^{i \frac{1}{2} \Theta^{\mu \nu} (P^R_\mu \otimes P^R_\nu - P^L_\mu \otimes P^L_\nu)} H_1 \otimes H_2 \right)(g)
$$

(29)

This star product on group manifold can be used to derive deformed group algebra and it provides the proper framework for discussing the space-time symmetries of NCFT. Any theory obtained from the commutative $G$-invariant one by replacing the pointwise product by the $\ast$ one will be invariant under the action of quantum counterpart of group $G$.

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