Sound, Precise, and Fast Abstract Interpretation with Tristate Numbers

Harishankar Vishwanathan*, Matan Shachnai†, Srinivas Narayana‡ and Santosh Nagarackatte§

Rutgers University, USA

Email: *harishankar.vishwanathan@rutgers.edu, †mny35@cs.rutgers.edu, ‡srinivas.narayana@rutgers.edu, §santosh.nagarackatte@cs.rutgers.edu

Abstract—Extended Berkeley Packet Filter (BPF) is a language and run-time system that allows non-supersusers to extend the Linux and Windows operating systems by downloading user code into the kernel. To ensure that user code is safe to run in kernel context, BPF relies on a static analyzer that proves properties about the code, such as bounded memory access and the absence of operations that crash. The BPF static analyzer checks safety using abstract interpretation with several abstract domains. Among these, the domain of tnums (tristate numbers) is a key domain used to reason about the bitwise uncertainty in program values. This paper formally specifies the tnum abstract domain and its arithmetic operators. We provide the first proofs of soundness and optimality of the abstract arithmetic operators for tnum addition and subtraction used in the BPF analyzer. Further, we describe a novel sound algorithm for multiplication of tnums that is more precise and efficient (runs 33\% faster on average) than the Linux kernel’s algorithm. Our tnum multiplication is now merged in the Linux kernel.

Index Terms—Abstract domains, Program verification, Static analysis, Kernel extensions, eBPF

I. INTRODUCTION

Static analysis is an integral part of compilers \cite{1, 2, 3, 4}, sandboxing technologies \cite{5, 6, 7}, and continuous integration testing \cite{8}. For example, static analysis may be used to prove that the value of a program variable will always be bounded by a known constant, allowing a compiler to eliminate dead code \cite{9} or a sandbox to remove an expensive run-time check \cite{5}.

Our work is motivated by static analysis in the context of Berkeley Packet Filter (BPF), a language and run-time system \cite{10, 11} that enables users to extend the functionality of the Linux and Windows operating systems without writing kernel code. BPF is widely deployed in production systems today \cite{12, 13, 14, 15, 16, 17, 18}. BPF uses a static analyzer to validate that user programs are safe before they are executed in kernel context \cite{11, 7}: the analyzer must be able to show that the program does not access unpermitted memory regions, does not leak privileged kernel data, and does not crash. If the analyzer is unable to prove these properties, the user program is rejected and cannot execute in kernel context.

BPF static analysis must be sound, precise, and fast.

- **Soundness**: Unsound analysis that accepts malicious code may result in arbitrary read-write capabilities for users in the kernel \cite{19}. Unfortunately, the Linux static analyzer has been a source of numerous such bugs in the past \cite{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33}.

- **Precision**: To provide a usable system, the analyzer must not reject safe programs due to imprecision in its analysis. Users often need to rewrite their programs to get their code past the analyzer \cite{7, 22, 55}.

- **Speed**: The analyzer must keep the time and overheads to load a BPF program minimal \cite{11, 36, 37}. Programs are often used to trace systems running heavy workloads.

The BPF static analyzer employs abstract interpretation \cite{38} with multiple abstract domains to track the types, liveness, and values of program variables across all executions. One of the key abstract domains, termed tristate numbers or tnums in the Linux kernel \cite{39}, tracks which bits of a value are known to be 0, known to be 1, or unknown (denoted \(\mu\)) across executions. For example, a 4-bit variable \(x\) abstracted to 01\(\mu\)0 can take on the binary values 0100 and 0110. The analyzer can infer that the expression \(x \leq 8\) will always return true, and use this fact later to show the safety of a memory access.

The kernel provides algorithms to implement bit-wise operations such as and (\&), or (\|), and shifts (\(<<, >>\)) over tnums. The kernel also provides efficient algorithms for arithmetic (addition, subtraction, and multiplication) over tnums. In particular, addition and subtraction run in \(O(1)\) time over \(n\)-bit program variables given \(n\)-bit machine arithmetic instructions.

Unfortunately, the kernel provides no formal reasoning or proofs of soundness or precision of its algorithms. Prior works that explored abstract domains for bit-level reasoning \cite{40, 41, 42, 43, 44} provide sound and precise abstract operators for bit-wise operations (\&, 1, \|, \(>>\), etc.). The only arithmetic algorithms we are aware of \cite{42} are much slower than the kernel’s algorithms \cite{11}. Arithmetic operations are tricky to reason about as they propagate uncertainty across bits in non-obvious ways. For example, suppose \(a\) is known to be the \(n\)-bit constant 11\(\cdots\)1 and \(b\) is either 0 or 1 across all executions. Only one bit is uncertain among the operands, yet all bits in \(a + b\) are unknown, since \(a + b\) can be either 11\(\cdots\)1 or 00\(\cdots\)0.

This paper makes the following contributions \cite{11}. We provide the first proofs of soundness and optimality (i.e., maximal precision \cite{3, 45}) of the kernel’s algorithms for addition and subtraction. We believe this result is remarkable for abstract operators exhibiting \(O(1)\) run time and reasoning about uncertainty across bits. We were unable to prove the soundness of the kernel’s tnum multiplication. Instead, we present a novel multiplication algorithm that is provably sound. It is also more precise and 32\% faster than prior implementations \cite{42, 39}. 
This algorithm is now merged into the latest Linux kernels. Our reproducible artifact is publicly available [46].

II. BACKGROUND

The BPF static analyzer in the kernel checks the safety of BPF programs by performing abstract interpretation using the tnum abstract domain (among others). In this section, we provide a primer on abstract interpretation and describe the tnum abstract domain and its operators.

A. Primer on Abstract Interpretation

Abstract interpretation [38] is a form of static analysis that captures the values of program variables in all executions of the program. Abstract interpretation employs abstract values and abstract operators. Abstract values are drawn from an abstract domain, each element of which is a concise representation of a set of concrete values that a variable may take across executions. For example, an abstract value from the interval abstract domain [47] \{[a, b] \mid a, b \in \mathbb{Z}, a \leq b\} models the set of all concrete integer values (i.e., \(x \in \mathbb{Z}\)) such that \(a \leq x \leq b\).

Abstraction and Concretization functions. An abstraction function \(\alpha\) takes a concrete set and produces an abstract value, while a concretization function \(\gamma\) produces a concrete set from an abstract value. For example, the abstraction of the set \(\{2, 4, 5\}\) in the interval domain is \([2, 5]\), which produces the set \(\{2, 3, 4, 5\}\) when concretized.

Formally, the domains of the abstraction and concretization functions are two partially-ordered sets (posets) that induce a lattice structure. We denote the concrete poset \(\mathbb{C}\) with the ordering relationship among elements \(\subseteq\). Similarly, we denote the abstract poset \(\mathbb{A}\) with the ordering relationship \(\subseteq\). For example, the interval domain employs the concrete poset \(\mathbb{C} \triangleq \mathbb{Z}\), the power set of \(\mathbb{Z}\), with the subset relation \(\subseteq\) (e.g., \(\{1, 2\} \subseteq \{1, 2, 3\}\)) as its ordering relation. The abstract poset is \(\mathbb{A} \triangleq \mathbb{Z} \times \mathbb{Z}\) with the ordering relation \([a, b] \subseteq [c, d] \iff (c \leq a) \land (d \geq b)\).

A value \(a \in \mathbb{A}\) is a sound abstraction of a value \(c \in \mathbb{C}\) if and only if \(c \subseteq \mathbb{C} \gamma(a)\). Moreover, \(a\) is an exact abstraction of \(c\) if \(c = \gamma(a)\). Abstractions are often not exact, over-approximating the concrete set to permit concise representation and efficient analysis in the abstract domain. For example, the interval \([2, 5]\) is a sound but inexact abstraction of the set \(\{2, 4, 5\}\).

Abstract operators are functions over abstract values which return abstract values. An abstract operator implements an “abstract version” of a concrete operation over concrete sets, hence enabling a static analysis to construct the abstract results of program execution. For example, abstract integer addition in the interval domain (denoted \(+\)) abstracts concrete integer addition (denoted \(+\)) as follows: \([a_1, b_1] +\mathbb{A}[a_2, b_2] \triangleq [a_1 + a_2, b_1 + b_2]\). Abstract operators typically over-approximate the resulting concrete set to enable decidable and fast analysis at the expense of precision. For a concrete set \(S \in \mathbb{C}\), suppose we use the shorthand \(f(S)\) to denote the set \(\{f(x) \mid x \in S\}\). An abstract operator \(g : \mathbb{A} \rightarrow \mathbb{A}\) is a sound abstraction of a concrete operator \(f : \mathbb{C} \rightarrow \mathbb{C}\) if \(\forall a \in \mathbb{A} : f(\gamma(a)) \subseteq \mathbb{C} g(a)\). Further, \(g\) is exact if \(\forall a \in \mathbb{A} : f(\gamma(a)) = \gamma(g(a))\).

Galois connection. Pairs of abstraction and concretization functions \((\alpha, \gamma)\) are said to form a Galois connection if [45]:
1) \(\alpha\) is monotonic, i.e., \(x \subseteq\mathbb{C} y \implies \alpha(x) \subseteq\mathbb{A} \alpha(y)\)
2) \(\gamma\) is monotonic, \(a \subseteq\mathbb{A} b \implies \gamma(a) \subseteq\mathbb{C} \gamma(b)\)
3) \(\gamma \circ \alpha\) is extensive, i.e., \(\forall c \in \mathbb{C} : c \subseteq\mathbb{C} \gamma(\alpha(c))\)
4) \(\alpha \circ \gamma\) is reductive, i.e., \(\forall a \in \mathbb{A} : \alpha(\gamma(a)) \subseteq\mathbb{A} a\).

The Galois connection is denoted as \((\mathbb{C}, \subseteq\mathbb{C}) \xrightarrow{\alpha, \gamma} (\mathbb{A}, \subseteq\mathbb{A})\). The existence of a Galois connection enables reasoning about the soundness and the precision of any abstract operator.

Optimality. Suppose \((\mathbb{C}, \subseteq\mathbb{C}) \xrightarrow{\alpha, \gamma} (\mathbb{A}, \subseteq\mathbb{A})\) is a Galois connection. Given a concrete operator \(f : \mathbb{C} \rightarrow \mathbb{C}\), the abstract operator \(\alpha \circ f \circ \gamma\) is the smallest sound abstraction of \(f\) that is, for any sound abstraction \(g : \mathbb{A} \rightarrow \mathbb{A}\) of \(f\), we have \(\forall a \in \mathbb{A} : \alpha(\gamma(f(a))) \subseteq\mathbb{A} g(a)\). We call \(\alpha \circ f \circ \gamma\) the optimal, or maximally precise abstraction, of \(f\).

B. The Tnum Abstract Domain

Tnums enable performing bit-level analysis by abstracting each bit of a program variable separately. Across executions, each bit is either known to be 0, known to be 1, or uncertain, denoted by \(\mu\). For an \(n\)-bit program variable, the abstract value corresponding to the variable has \(n\) ternary digits, or trits. Each trit has a value of 0, 1, or \(\mu\).

Bit-level abstract interpretation has been addressed in several prior works using the bitfield abstract domain [40] [41] [42] and the known bits abstract domain [43] [3] [44]. Abstraction and concretization functions forming a Galois connection already exist [41], as well as sound and optimal abstract operators for bit-level operations like bit-wise-and (\&), bit-wise-or (\|), and shifts (\langle, \rangle) [3] [41]. In contrast to prior work, this paper explores provably sound, optimal, and computationally-efficient abstract operators corresponding to arithmetic operations such as addition, subtraction, and multiplication. The Linux kernel analyzer, despite heavily leveraging this domain’s abstract operators, formally lays out neither the soundness nor optimality for the abstract arithmetic operations.

Abstract and Concrete Domains. Tnums track each bit of variables drawn from the set of \(n\)-bit integers \(\mathbb{Z}_n\). The ordering relation \(\subseteq\mathbb{C}\) is the subset relation:
\[ a \subseteq\mathbb{C} b \triangleq a \subseteq b \]

The abstract poset \(\mathbb{A}\) is the set of \(n\)-bit tnums \(\mathbb{T}_n\) (each trit is 0, 1, or \(\mu\)). Suppose we represent the trit in the \(i^{th}\) position of \(a\) by \(a[i]\). The ordering relation \(\subseteq\mathbb{A}\) between abstract elements is defined by:
\[ P \subseteq\mathbb{A} Q \triangleq \forall i, 0 \leq i \leq n - 1, \forall k \in \{0, 1\} : (P[i] = \mu \Rightarrow Q[i] = \mu) \land (Q[i] = k \Rightarrow P[i] = k) \]

Fig. 1 shows Hasse diagrams of the lattices induced by these posets for integers with bit width \(n\). The concrete domain consists of all elements of the power set of \(\{0, 1, 2, 3\}\) and the abstract domain consists of tnums of the form \(t_1 t_0\) where each
will use the notation \( (\cdot) \) we say that such a tnum is not
\[ \alpha_n \]
The abstraction function \( \alpha \) represents the abstract value
\[ P.v \]
\[ k \]
The values of the \( k \)th bits of \( P.v \) and \( P.m \) are used to inform the value of the \( k \)th trit of \( P \).
\[ (P.v[k] = 0 \land P.m[k] = 0) \triangleq P[k] = 0 \]
\[ (P.v[k] = 1 \land P.m[k] = 0) \triangleq P[k] = 1 \]
\[ (P.v[k] = 0 \land P.m[k] = 1) \triangleq P[k] = \mu \]
We define the domain of abstract values \( T_n \triangleq Z_n \times Z_n \).
If for a tnum \( P \), \( P.v[k] = P.m[k] = 1 \) at some position \( k \), we say that such a tnum is not well-formed. All such tnum
represent the abstract value \( \perp \) and the concrete empty set \( \emptyset \).
\[ \forall P : (\exists k : P.v[k] \land P.m[k] = 1) \iff P = \perp \]
A large fraction of random bit patterns \((v, m)\) aren’t well
formed: in particular, only \( 3^n \) among the \( 2^{2n} \) \( n \)-bit \((v, m)\) bit patterns correspond to well-formed tnum that are not \( \perp \).
We are now ready to define the Galois connection for the tnum abstraction domain using the above implementation of abstract values. These take a form similar to the functions defined in prior work [41]. In the discussion that follows we will use the notation \((\& , 1 , \oplus , \& , < , >)\) respectively for the bitwise and, or, exclusive-or, negation, left-shift, and right-shift operations over \( n \)-bit bit vectors.

Galois connection. Given a concrete set \( C \subseteq 2^{2^n} \). The abstraction function \( \alpha : 2^{2^n} \rightarrow Z_n \times Z_n \) is defined as follows.
\[ \alpha_k(C) \triangleq k\{c \mid c \in C\} \]
\[ \alpha_1(C) \triangleq \{c \mid c \in C\} \]
\[ \alpha(C) \triangleq (\alpha_k(C) , \alpha_k(C) \oplus \alpha_1(C)) \]
This abstraction function is sound. However, it is not exact, as
easily seen from the fact that there are \( 2^{3^n} \) elements in \( C \) but only \( 3^n \) well-formed tnums in \( T_n \). Many concrete sets will be
over-approximated. However, \( \alpha \) is a composition of functions
that abstract the domain exactly when each bit is considered separately [48]. Informally, given a concrete set \( C \subseteq \mathbb{C} \) and \( x, y \in C \), \( \alpha(C) \) contains an uncertain trit at position \( k \) iff \( C \)
contains \( x \) and \( y \) with bits differing at \( k \).
\[ \forall k \in \{0, 1\} : \alpha(C)[k] = b \iff \forall x \in C : x[k] = b \]
\[ \alpha(C)[k] = \mu \iff \exists x, y \in C : x[k] = 0 \land y[k] = 1 \]
This abstraction function \( \alpha \) is bitwise exact.
Further, consider a tnum \( P \in T_n \) implemented as
\[ (P.v, P.m) \in Z_n \times Z_n \]. Then the concretization function
\( \gamma : Z_n \times Z_n \rightarrow 2^{2^n} \) is defined as:
\[ \gamma(P) = \gamma((P.v, P.m)) \triangleq \{c \in \mathbb{Z} \mid c \& P.m = P.v\} \]
\[ \gamma(\perp) \triangleq \emptyset \]
Then \( \alpha \) and \( \gamma \) form a Galois connection. Informally, the tnum
obtained from applying \( \alpha \) on a set of concrete values always
soundly over-approximates the original set if concretized.
An illustration of this fact can be seen in Fig. 1. Please refer to the extended technical report [49] for the (standard) proof. The existence of the Galois connection enables, in principle, constructing sound and optimal abstract operators
over tnums. The abstraction of the concrete set \( \{1, 2, 3\} \) soundly
over-approximates it: \( \{1, 2, 3\} \subseteq \gamma(\alpha(\{1, 2, 3\})) \).

Abstract operators on tnums. The BPF instruction set supports
the following (typical) concrete operations over 64-bit registers: add, sub, mul, div, or, and, leh, rsh, neg, mod, xor and arsh. To soundly analyze general BPF
programs, the BPF static analyzer requires abstract operators corresponding to all the above concrete operations. For some operators, notably div and mod, defining a precise abstract
operator is challenging. In such cases, the BPF static analyzer
conservatively and soundly sets all the output trits to unknown.

Challenges. Despite enjoying a Galois connection, constructing efficient optimal abstractions for arithmetic operators is non-trivial. Given a concrete operator \( f \), the optimal abstract
operator \( \alpha \circ f \circ \gamma \) is infeasible to compute in practice [50][51][52]. For example, if \( f \) is a concrete operator of arity 2, there may be \( 2^{2^n} \) computations of \( f \) after the first concretization \( \gamma(.) \) in
the worst case (the average case is not much better).

Prior work on the bitfield domain [41], a domain similar
to tnums \( 2^{2^n} \rightarrow Z_n \times Z_n \), presents abstract operators for
bitwise or, and, exclusive-or, left and right shift operations that are optimal. However, most prior works on the bitfield and
known bits abstract domains [43][3][4][40][41] fail to provide
abstract arithmetic operators for addition, subtraction, and multiplication. To our knowledge, Regehr and Duongsa [42] provide the only known abstract operators for arithmetic in this domain, based on ripple-carry logic and composition of abstract operators. These operators are sound but not optimal. Further, they have a runtime of $O(n)$ for $n$-bit abstract addition and subtraction, and $O(n^2)$ for abstract multiplication.

In the next section, we present proofs of soundness and optimality for abstract operators for addition and subtraction originally developed (without formal proof) in the Linux kernel. These operators run in $O(1)$ time given $n$-bit machine arithmetic instructions ($n = 64$ in the kernel). Such efficiency is remarkable, given that in general addition and subtraction use ripple-carry operations creating dependencies between the bits. We also present an abstract multiplication operator that is provably sound, empirically more precise, and faster than the abstract multiplication in [42] and the Linux kernel. Notably, none of the algorithms in this paper use the composition structure $\alpha \circ f \circ \gamma$ or “merely” compose existing sound abstract operators. This motivated us to develop dedicated proof techniques.

### III. Soundness and Optimality of Abstract Arithmetic over Tnums

We explore the soundness and optimality of tnum arithmetic operators, specifically addition, subtraction, and multiplication. The kernel proposes abstract operators for each of them, but lacks any proof of soundness. Hence, we perform an automated (bounded bitwidth) verification of the soundness of the kernel’s tnum abstract operators (§III-A) using SMT solvers. We were able to prove the soundness of the kernel’s abstract addition, subtraction, and all other bitwise operators up to 64-bits, and soundness of the kernel’s multiplication up to 8-bits. Motivated by these results, we undertook an analytical study of these algorithms, which led us to paper-and-pen proofs of both soundness and optimality of the kernel’s abstract operators for addition and subtraction over unbounded bitwidths (§III-B). We were unable to analytically prove the soundness of the kernel’s tnum multiplication for unbounded bitwidths. Hence, we developed a new algorithm for tnum multiplication that is provably sound, empirically more precise, and faster than the kernel’s tnum multiplication for unbounded bitwidths. Hence, we perform an automated proof techniques.

#### A. Automatic Bounded Verification of Kernel Tnum Arithmetic

We encode verification conditions corresponding to the soundness of tnum abstract arithmetic operators in first order logic and discharge them to a solver. We use the theory of bitvectors. Our verification conditions are specific to a particular bitwidth ($n$). We use 64-bit bitvectors to encode the tnum operations wherever feasible ($n = 64$ in the kernel). For a tnum $P$ drawn from the set of $n$-tirit tnums $\mathbb{T}_n$, we denote its kernel implementation by $(P.v, P.m) \in \mathbb{Z}_n \times \mathbb{Z}_n$.

**Soundness of 2-ary operators.** Recall from Section §II the notion of soundness of an abstract operator. We can generalize this notion to 2-ary operators $\text{op}_\gamma : \mathbb{T}_n \times \mathbb{T}_n \rightarrow \mathbb{T}_n$ and $\text{op}_\gamma : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n$. We say that $\text{op}_\gamma$ is a sound abstraction of $\text{op}_\gamma$ iff the following condition (Eqn. 8) holds.

$$\forall P, Q \in \mathbb{T}_n : \{ \text{op}_\gamma(x, y) \mid x \in \gamma(P), y \in \gamma(Q) \} \subseteq \gamma(\text{op}_\gamma(P, Q))$$

To encode (8) in first-order logic, recall that the concrete order $\subseteq$ is just the subset relationship between the two sets. At a high level, the subset relationship $S_1 \subseteq S_2$ in (8) can be encoded by universally quantifying over the members of $S_1$ and writing down the query $\forall x \in S_1 : x \in S_2$. The formula $x \in S_1$ is easy to encode given the left-hand side of (8). To encode $x \in S_2$ from the right-hand side of (8), we define a membership predicate. This predicate asserts that $x \in \gamma(R)$ where $R \triangleq \text{op}_\gamma(P, Q)$. Finally, we ensure that the universally quantified tnums $P$ and $Q$ are non-empty, and encode the action of the concrete and abstract operators $\text{op}_\gamma$ and $\text{op}_\gamma$ in logic. The details follow.

**Membership predicate** $x \in \gamma(P)$. Consider a concrete value $x$ that is contained in the concretization of tnum $P$. Using the definition of the concretization function in (7), we write the predicate member:

$$\text{member}(x, P) \triangleq x \& P.m = P.v$$

**Quantifying over well-formed tnums.** To ensure that (8) only quantifies over non-empty tnums, we encode one more predicate, wellformed, based on (4):

$$\text{wellformed}(P) \triangleq P.v \& P.m = 0$$

**Putting it all together.** The soundness predicate for a given pair of abstract and concrete operators $\text{op}_\gamma$, $\text{op}_\gamma$ is

$$\forall P, Q \in \mathbb{T}_n, x, y \in \mathbb{Z}_n : \text{wellformed}(P) \& \text{wellformed}(Q) \& \text{member}(x, P)$$

$$\& \text{member}(y, Q) \& z = \text{op}_\gamma(x, y) \& R = \text{op}_\gamma(P, Q)$$

$$\Rightarrow \text{member}(z, R)$$

An SMT solver can show the validity of this formula by proving that the negation of this formula is unsatisfiable.

**Example: encoding abstract tnum addition.** We show how to encode the soundness of the abstract addition operator over tnums. The kernel uses the algorithm tnum_add from Listing 1 to perform abstract addition over two tnums. The predicate add below captures the result of abstract addition of $P$ and $Q$ into $R$.

$$\text{add}(P, Q, R) \triangleq$$

$$(sv = P.v + Q.v) \& (sm = P.m + Q.m) \& (\Sigma = sv + sm)$$

$$\& (x = \Sigma \oplus sv) \& (\eta = \chi \lor P.m \lor Q.m) \& (R.v = sv \& \neg \eta)$$

$$\& (R.m = \eta)$$

We can plug in the add predicate in place of $\text{op}_\gamma$ in Eqn. 11. The function $\text{op}_\gamma$ is just $n$-bit bitvector addition.
\[\Sigma := sv + sm = 10101\]
\[sm := P.m + Q.m = 00100\]
\[\eta := \chi \mid P.m \mid Q.m = 00110\]
\[sv := P.v + Q.v = 10001\]
\[\chi := \Sigma\]

**Listing 1:** Linux kernel’s implementation of tnum addition (tnum_add)

```python
def tnum_add(tnum P, tnum Q):
    u64 sv := P.v + Q.v
    u64 sm := P.m + Q.m
    u64 \Sigma := sv + sm
    u64 \chi := \Sigma \oplus sv
    u64 \eta := \chi \mid P.m \mid Q.m
    tnum R := tnum(sv \& \neg \eta, \eta)
    return R
```

**Observations from bounded verification.** We encoded the first-order logic formulas to perform bounded verification of the soundness of the following tnum operators defined in the Linux kernel: addition, subtraction, multiplication, bitwise or, bitwise and, bitwise exclusive-or, left-shift, right-shift, and arithmetic right-shift. We have spot-checked the correctness of our encodings with respect to the kernel source code using randomly-drawn tnum inputs; the details of this testing harness are in our extended technical report [49].

For all operators except multiplication, verification succeeded for bitvectors of width 64 in just a few seconds. In contrast, verification of multiplication (kern_mul), shown in Listing 2 succeeds quickly at bitwidth \(n = 8\), but does not complete even after 24 hours with bitwidth \(n = 16\). This is due to the presence of non-linear operations and large unrolled loops. This observation motivated us to develop a new, provably sound algorithm for tnum multiplication (SIII-C).

Further, our bounded verification efforts helped us uncover non-obvious properties of tnum arithmetic: (1) tnum addition is not associative, (2) tnum addition and subtraction are not inverse operations, and (3) tnum multiplication is not commutative.

B. Soundness and Optimality of Tnum Abstract Addition

We present an analytical proof of the soundness and optimality of the kernel’s abstract addition operator for unbounded bitwidths. The proof for subtraction, which is very similar in structure, is in our extended technical report [49].

**An example.** The source code for abstract addition (tnum_add) is shown in Listing 1. Figure 2 illustrates tnum addition with an example. In particular, adding two tnums “by hand”, as shown in Fig. 2(a), propagates uncertainty explicitly in the carries, rippling the carry bits through the tnums one bit position at a time. However, as seen in Fig. 2(b), tnum_add does not use any such ripple-carry structure in its computations. Yet, as we show later (and illustrated in Fig. 2(c)), tnum_add implicitly reasons about the unknown bits in the sequence of carries produced during the addition.

**Definition 1. Full adder equations.** When adding two concrete binary numbers \(p\) and \(q\), each bit of the addition result \(r\) is set according to the following:

\[r[i] = p[i] \oplus q[i] \oplus c_{in}[i]\]

where \(\oplus\) is the exclusive-or operation and \(c_{in}[i] = c_{out}[i-1]\) and \(c_{out}[i-1]\) is the carry-out from the addition in bit position \(i-1\). The carry-out bit at the \(i^{th}\) position is given by

\[c_{out}[i] = (p[i] \& q[i]) \mid (c_{in}[i] \& (p[i] \oplus q[i]))\]

**Key proof technique.** We show the soundness and optimality of tnum_add by reasoning about the set of all possible concrete outputs, i.e., the results of executions of concrete additions over elements of the input tnums \(P, Q \in \mathbb{T}\). If we denote \(+\) the concrete addition operator over \(\mathbb{Z}_n\), this is the set \(\{p + q \mid p \in \gamma(P) \land q \in \gamma(Q)\}\) or \(+\gamma(P), \gamma(Q)\) in short. The proof proceeds by finding bit positions in the output set that can be shown to be either a 1 or a 0 in all members of that set (respectively lemmas 2 and 3). Every other bit position is such that there are elements in the concrete output set that differ at that bit position. Lemma 4 invokes the bitwise-exactness (Eqn. 6) of the abstraction function \(\alpha\), and along with Lemma 5 shows that tnum_add is a sound and optimal abstraction for \(+\) (i.e., the same as \(\alpha \circ + \circ \gamma\)).

Consider the addition that occurs “by hand” in Fig. 2(a). Intuitively, at a given bit position of the output tnum, the result...
will be unknown if either of the operand bits \( p[i] \) or \( q[i] \) is unknown, or if the carry-in bit \( c_m[i] \) (generated from less-significant bit positions) is unknown. Note that these three bits may be (un)known independent of each other since they depend on different parts of the input tnums. The crux of the proof lies in identifying which carry-in bit positions vary across different concrete additions. This is done by distinguishing the carries generated due to the unknown bits in the operands from the carries that will be present or absent in any concrete addition drawn from the input tnums. In the example in Fig. 2(a), the sequence of carries is 10\( _m \)00, with the middle carry-in bit being uncertain and all others known to be 0s or 1s in all concrete additions from the input tnums.

Suppose \( p \) and \( q \) are two concrete values in tnum \( P \) and tnum \( Q \), respectively, i.e., \( p \in \gamma(P), q \in \gamma(Q) \).

**Lemma 2. Minimum carries lemma.** The addition \( sv = P.v + Q.v \) will produce a sequence of carry bits that has the least number of 1s out of all possible additions \( p + q \).

The consequence of this lemma is that any concrete addition \( p + q \) will produce a sequence of carry bits with 1s in at least those positions where the \( sv \) addition produced carry bits set to 1 (the extended technical report provides a proof of this lemma). Fig. 2(c) shows the set of carries produced in \( sv \) (i.e., \( sv_c \triangleq 10100 \)). Any addition \( p + q \) will produce a 1-bit carry in the same positions as the 1 bits in \( sv_c \).

**Lemma 3. Maximum carries lemma.** The addition \( \Sigma = (P.v + P.m) + (Q.v + Q.m) \) will produce the sequence of carry bits with the most number of 1s out of all possible additions \( p + q \).

The consequence of this lemma is that any concrete addition \( p + q \) will produce a sequence of carry bits with 1s in at least those positions where the \( \Sigma \) addition produced carry bits set to 0 (proof is available in our extended technical report). Fig. 2(c) shows the set of carries produced in \( \Sigma \) (i.e., \( \Sigma_c \triangleq 10100 \)). Any addition \( p + q \) will produce a 0-bit carry in the same positions as the 0 bits in \( \Sigma_c \).

**Lemma 4. Capture uncertainty lemma.** Let \( sv_c \) and \( \Sigma_c \) be the sequence of carry-in bits from the additions in \( sv \) and \( \Sigma \), respectively. Suppose \( \chi_c[k] = 0 \) have carry bits fixed in all concrete additions \( p + q \) from \( +(\gamma(P), \gamma(Q)) \). The bit positions \( k \) where \( \chi_c[k] = 1 \) vary depending on the concrete addition: i.e., \( \exists p_1, p_2 \in \gamma(P), q_1, q_2 \in \gamma(Q) \) such that \( p_1 + q_1 \) has its carry bit set at position \( k \) but \( p_2 + q_2 \) has that bit unset.

Intuitively, from the minimum carries lemma, any carry bit that is set in \( sv_c \) must be set in the sequence of carry bits in any concrete addition \( p + q \). Similarly, from the maximum carries lemma, any carry bit that is unset in \( \Sigma_c \) must be unset in the sequence of carry bits in any concrete addition \( p + q \). Hence, \( sv_c \oplus \Sigma_c \) represents the carries that may arise purely from the uncertainty in the concrete operands picked from \( P \) and \( Q \). Further, these carry do in fact differ in two concrete additions \( sv \) and \( \Sigma \). From the bit-wise exactness of the tnum abstraction function \( \alpha \) (Eqn. 6), it follows that these are precisely the bits that must be unknown in the resulting tnum due to the carries. See the extended technical report for a detailed proof.

Hence, the mask in the resulting tnum must be \( (sv_c \oplus \Sigma_c) \| P.m \| Q.m \). However, \( \text{tnum\_add} \) uses the final mask \( (sv \oplus \Sigma) \| P.m \| Q.m \) (see Listing 1). Lemma 5 shows that these two quantities are, in fact, always the same.

**Lemma 5. Equivalence of mask expressions.** The expressions \( (sv \oplus \Sigma) \| P.m \| Q.m \) and \( (sv_c \oplus \Sigma_c) \| P.m \| Q.m \) compute the same result.

We prove this lemma using the rules of propositional logic in our extended technical report. Together, these lemmas allow us to show the soundness and optimality of \( \text{tnum\_add} \) below.

**Theorem 6. Soundness and optimality of tnum\_add** The algorithm \( \text{tnum\_add} \) shown in Listing 1 is a sound and optimal abstraction of concrete addition over \( n \)-bit bitvectors for unbounded \( n \).

### C. Sound and Efficient Tnum Abstract Multiplication

This section describes a novel algorithm for tnum multiplication and a proof that it is a sound abstraction of multiplication of \( n \)-bit concrete values for unbounded \( n \). Our algorithm has \( O(n) \) run time. It is not an optimal abstraction of concrete multiplication. However, as we show later (SIV), our algorithm is empirically more precise and faster than all known prior implementations of multiplication in this abstract domain. We were able to contribute our algorithm to the tnum implementation in the latest Linux kernel.

```python
1. def our_mul_simplified(tnum P, tnum Q):
2.     ACCv := tnum(0, 0)
3.     ACCw := tnum(0, 0)
4. # loop runs bitwidth times:
5.     for i in range(0, bitwidth):
6.         # LSB of tnum P is a certain 1
7.         if (P.v[0] == 1) and (P.m[0] == 0):
8.             ACCv := tnum_add(ACCv, tnum(Q.v, 0))
9.             ACCw := tnum_add(ACCw, tnum(0, Q.m))
10.            # LSB of tnum P is uncertain
11.       else if (P.m[0] == 1):
12.           ACCw := tnum_add(ACCw, tnum(0, Q.v|Q.m))
13.           # Note: no case for LSB is certain 0
14.     P := tnum_lshift(P, 1)
15.     Q := tnum_lshift(Q, 1)
16.     tnum R := tnum_add(ACCv, ACCw)
17.     return R
```

Listing 3: A simplified implementation of our tnum multiplication algorithm (our_mul_simplified).

**Our algorithm our_mul through an example.** Our tnum abstract multiplication algorithm is shown in Listing 4. The algorithm in Listing 3 is semantically equivalent to it, but easier to understand, so we explain the algorithm and its proof primarily using the algorithm in Listing 3.

Similar to the prior multiplication algorithms proposed in bit-level reasoning domains [42, 39], our algorithm is inspired
by the long multiplication method to generate the product of two binary values. The algorithm proceeds in a single loop iterating over the bitwidth of the input tnums. Our \texttt{our\_mul} takes two input tnums \( P \) and \( Q \), and returns a result \( R \).

Fig. 3(a) shows an example. Suppose we are given tnums \( P = \mu01 = (P.v = 001, P.m = 100) \) and \( Q = \mu10 = (Q.v = 010, Q.m = 100) \) to multiply. Two fully concrete \( n \)-bit binary numbers may be multiplied in two steps: (i) by computing the products of each bit in the multiplier \( P \) with the multiplicand \( Q \), to generate \( n \) partial products, and (ii) adding the \( n \) partial products after appropriately bit-shifting them. To generalize long multiplication to \( n \)-bit tnums which contain unknown (\( \mu \)) trits, we add new rules: 0) \( * \mu = 0 \); 1) if the least significant bit of any concrete value (\( 0 \) or \( 1 \)) is uncertain 1, then \( \gamma \) computes the addition of the partial products must occur through the abstract addition operator \( \texttt{tnum\_add} \).

Our tnum multiplication algorithm for the same pair of inputs is shown in Fig. 3(b). The algorithm uses two tnums, \( ACC_V \) and \( ACC_M \), which are initialized \( (v, m) = (0, 0) \). The tnums \( ACC_V \) and \( ACC_M \) accumulate abstract partial products generated in each iteration using tnum abstract additions (\( \texttt{tnum\_add} \)). The algorithm proceeds as follows:

1) If the least significant bit of any concrete value \( x \in \gamma(P) \) is known to be 1, then \( ACC_V \) (resp. \( ACC_M \)) accumulates the known bits (resp. unknown bits) in \( Q \) (e.g., iteration 1);
2) If the least significant bit of any \( x \in \gamma(P) \) is known to be 0, \( ACC_V \) and \( ACC_M \) remain unchanged (e.g., iteration 2);
3) If the least significant bit of \( P \) is unknown (\( \mu \)), then \( ACC_V \) is unchanged, but \( ACC_M \) accumulates a tnum with a mask such that all possible bits that may be set in any \( x \in \gamma(P) \) are also set in the mask (e.g., iteration 3).

At the end of each iteration, \( P \) (resp. \( Q \)) is bit-shifted to the right (resp. left) by 1 position to ensure that the next partial product is appropriately shifted before addition. The specific methods of updating \( ACC_V \) and \( ACC_M \) in each iteration make our \texttt{our\_mul} distinct from prior multiplication algorithms [42] [39]. In particular, our \texttt{our\_mul} decomposes the accumulation of partial products into two tnums and uses just a single loop over the bitwidth. These modifications are crucial to the precision and efficiency of our \texttt{our\_mul} (§IV).

\textbf{Key proof techniques.} Recall that a (unary) abstract operator \( g \) is a sound abstraction of a concrete operator \( f \) if \( \forall a \in A : f(\gamma(a)) \subseteq \gamma(g(a)) \). We show that our abstract multiplication algorithm \texttt{our\_mul} is sound by showing that \( \{ x \in \gamma(P) \land y \in \gamma(Q) \} \subseteq \gamma(\texttt{our\_mul}(P, Q)) \) for any tnums \( P, Q \in T_n \). We denote the former set \( \ast(\gamma(P), \gamma(Q)) \) in short. The \( \ast \) is the concrete multiplication over \( n \)-bit bitvectors.

All the known abstract multiplication algorithms in this domain are composed of abstract additions and abstract shifts. A typical approach to prove soundness of such operators is to invoke the result that when sound abstract operators are composed soundly, i.e., in the same way as the corresponding concrete operators are composed, the result is a sound abstraction of the composed concrete operator [45] Theorem 2.6. The soundness of the abstract multiplication from Regehr and Duongsaa [42] may be proved as a special case of this general result. However, this approach is not applicable to proving the soundness of our \texttt{our\_mul}, since our \texttt{our\_mul}’s composition does not mirror any composition of (concrete) additions and shifts to produce a product. Instead, we are forced to develop a proof specifically for our \texttt{our\_mul} by observing, through two intermediate lemmas (Lemma 9 and Lemma 8) that the concrete products in \( \ast(\gamma(P), \gamma(Q)) \in \gamma(\texttt{our\_mul}(P, Q)) \).

Below, we show a sketch of the proof of the soundness of our \texttt{our\_mul\_simplified}, and argue (Lemma 11) that our \texttt{our\_mul} is equivalent to our \texttt{our\_mul\_simplified}.

\textbf{Observation 7.} For two concrete bitvectors \( x \) and \( y \) of width \( n \) bits, the result of multiplication \( y \ast x \) is just

\[ y \ast x = \sum_{k=0}^{n-1} x[k] \ast (y \ll k) \]
We call each term \( x[k] \ast (y << k) \) a partial product.

**Lemma 8. Tnum set union with zero.** Given a non-empty tnum \( P \in \mathbb{Z}_n \times \mathbb{Z}_n \), define \( Q \triangleq \text{tnum}(0,P.v \mid P.m) \). Then, (i) \( x \in \gamma(P) \Rightarrow x \in \gamma(Q) \), and (ii) \( 0 \in \gamma(Q) \).

Intuitively, any tnum \((0, 0n)\) when concretized contains the value 0. Further, building a new tnum \(Q\) whose mask has set all the bits corresponding to the set value or mask bits of \(P\) ensures that \(\gamma(P) \subseteq \gamma(Q)\). The full proof is in our extended technical report [49]. For example, given \( P = 0101 = (001, 010) \) and \( Q = (010, 011) \), we have \(\gamma(P) \subseteq \gamma(Q)\) and \(0 \in \gamma(Q)\).

For the next lemma, we define a variable-arity version of \(\text{tnum_add}\) as follows: \(\text{tnum_add}_{j=0}^{n-1}(T_j)\) is evaluated by folding the \(\text{tnum_add}\) operator over the list of tnum operands \(T_0, T_1, \ldots, T_{n-1}\) from left to right.

**Lemma 9. Value-mask-decomposed tnum summations.** Given \(n\) non-empty tnums \(T_0, T_1, \ldots, T_{n-1} \in \mathbb{T}_n\). Suppose we pick \(n\) values \(z_0, z_1, z_2, \ldots, z_{n-1} \in \mathbb{Z}_n\) such that \(\forall 0 \leq j \leq n-1: z_j \in \gamma(T_j)\). Define tnum

\[
S \triangleq \text{tnum_add}_{j=0}^{n-1}(\text{tnum_add}_{j=0}^{n-1}(\text{tnum}(T_j, v, 0)), \text{tnum_add}_{j=0}^{n-1}(\text{tnum}(0, T_j, m)))
\]

where \(\text{tnum_add}_{j=0}^{n-1}(\cdot, \cdot)\) is a variable-arity version of \(\text{tnum_add}\) defined above. Then: \(\sum_{j=0}^{n-1} z_j \in \gamma(S)\).

Intuitively, suppose we had \(n\) tnums \(T_i, 0 \leq i \leq n-1\) and we seek to construct a new tnum \(S\) whose concretization \(\gamma(S)\) contains all possible concrete sums from the \(T_i\), i.e., such that \(\{\sum_{j=0}^{n-1} x_j \mid x_i \in \gamma(T_i)\} \subseteq \gamma(S)\). The most natural method to construct such a tnum \(S\) is to use the sound abstract addition operator \(\text{tnum_add}\) over the \(T_i\), i.e., \(\text{tnum_add}_{j=0}^{n-1}(T_j)\). This lemma provides another method of constructing such a tnum \(S\): decompose the tnums \(T_i\) each into two tnums, consisting of the values and the masks separately. Use \(\text{tnum_add}\) to separately add the value tnums, add the mask tnums, and finally add the two resulting tnums from the value-sum and mask-sum to produce \(S\). Then \(S\) contains all concrete sums. The full proof of this lemma is in the extended technical report. For example, suppose \(T_1 = 1000 = (101, 000), T_2 = 0100 = (011, 000)\). Then \(\forall x_1 \in \gamma(T_1), x_2 \in \gamma(T_2) : x_1 + x_2 \in \gamma(\text{tnum_add}((110, 0), (0, 11)))\).

**Theorem 10. Soundness of our_mul.** \(\forall x \in \gamma(P), y \in \gamma(Q)\) the result \(R\) returned by our\_mul\_simplified (Listing 4) is such that \(x \ast y \in \gamma(R)\), assuming that abstract tnum addition (\(\text{tnum_add}\)) and abstract tnum shifts (\(\text{tnum_lshift}, \text{tnum_rshift}\)) are sound.

We prove this theorem by showing three properties, whose full proofs are in the extended technical report. Below, \(P_{in}\) and \(Q_{in}\) are the formal parameters to our\_mul.

**Property P1.** \(P\) and \(Q\) are bit-shifted versions of \(P_{in}\) and \(Q_{in}\). This property follows naturally from the algorithm, which only updates the tnums \(P\) and \(Q\) in the code using tnum bit-shift operations (\(\text{tnum_lshift, tnum_rshift}\)).

**Property P2.** \(ACC_{V}\) and \(ACC_{M}\) are value-mask-decomposed summations of partial products. There exist tnums \(T_0, T_1, \ldots, T_{n-1}\) such that (i) any concrete \(j^{th}\) partial product, \(z_j \triangleq x[j] \ast (y << j) \in \gamma(T_j)\), for \(0 \leq j \leq n - 1\); (ii) at the end of the \(k^{th}\) iteration of the loop, \(ACC_{V} = \text{tnum_add}_{j=0}^{k-1}(\text{tnum}(T_j, v, 0))\), and (iii) at the end of the \(k^{th}\) iteration of the loop, \(ACC_{M} = \text{tnum_add}_{j=0}^{k-1}(\text{tnum}(0, T_j, m))\).

At a high level, this property states that there is a set of tnums \(T_j\), where \(\gamma(T_j)\) contains all possible concrete values of the \(j^{th}\) partial product term \(z_j \triangleq x[j] \ast (y << j)\) (Observation 9). In the example in Fig. 3(b), \(T_0 = 10, T_1 = 0000, T_2 = \mu_000\). In the case where \(P[0] = 0\ mu_0\), we use Lemma 9 to show that the \(T_j\) constructed by our\_mul\_simplified is such that \(\gamma(Q) \subseteq \gamma(T_j)\) and \(0 \in \gamma(T_j)\). We also show that \(ACC_{V}\) is the value-sum of the \(T_j\) (see Lemma 9) while \(ACC_{M}\) is the mask-sum. In Fig. 3(b), \(ACC_{V} = \text{tnum_add}((010, 0000, 000000)\) and \(ACC_{M} = \text{tnum_add}((000, 0000, 000000)\).

**Property P3.** (Product containment) \(\sum_{j=0}^{n-1} z_j \in \gamma(\text{tnum_add}(ACC_{V}, ACC_{M}))\). That is, \(\forall x \in \gamma(P), y \in \gamma(Q) : x \ast y \in \gamma(\text{tnum_add}(ACC_{V}, ACC_{M}))\).

This result follows from property P2 and Lemma 9. Property P3 concludes a proof of soundness of our\_mul\_simplified: \(\forall x \in \gamma(P), y \in \gamma(Q) : x \ast y \in \gamma(R)\).

**Lemma 11. Correctness of strength reductions.** Our\_mul (Listing 4) is equivalent to our\_mul\_simplified (Listing 3) in terms of its input-output behavior.

The existence of two accumulating tnums \(ACC_{V}\) and \(ACC_{M}\) in our\_mul\_simplified allows us to use Lemma 9 to prove soundness. However, it is unnecessary to construct \(ACC_{V}\) iteration by iteration. We observe that \(ACC_{V}\) is merely accumulating \((Q, v, 0)\) whenever \(P[0]\) is known to be 1. All bits in each tnum accumulated into \(ACC_{V}\) are known. When \(\text{tnum_add}\) is used to add \(n\) tnums \((v_i, 0), 0 \leq i \leq n - 1\) it is easy to see that the result is \((\sum_{i=0}^{n-1} v_i, 0)\). Using Observation 7 we see that at end of the loop, \(ACC_{V} = \text{tnum}(P.v \ast Q.v, 0)\). As an added optimization, our\_mul soundly terminates the loop early if \(P = (0, 0)\) at the beginning of any iteration. Since our\_mul and our\_mul\_simplified are equivalent, our\_mul is also a sound abstraction of \(*\).

While our\_mul is sound, it is not optimal. Key questions remain in designing a sound and optimal algorithm with \(O(n)\) or faster run time. (1) How can we incorporate correlation in unknown bits across partial products? For example, multiplying \(P = 11, Q = \mu_1\) produces the partial products \(T_1 = 11, T_2 = \mu_0\). However, the two \(\mu\) trits in \(T_2\) are concretely either both 0 or both 1, resulting from the same \(\mu\) trit in \(Q\). Failing to consider this in the addition makes the result imprecise. (2) Can we design a sound, precise, and fast tnum addition operator of arity \(n\)? (3) Escaping long multiplication, is it possible to use concrete multiplication \(*\) over tnum masks to determine the unknown bits in the result?

**IV. EXPERIMENTAL EVALUATION**

Tnum operations are only one component of the Linux kernel’s BPF analyzer. To keep our measurement and contributions focused, our evaluation focuses on the precision and speed of our tnum multiplication operation relative to prior algorithms.
Prior algorithms for abstract multiplication. Apart from Linux kernel’s multiplication \cite{39}, Regehr and Duongsa\cite{42} also provide an algorithm for abstract multiplication in a domain similar to tnums, which is called the bitwise domain. Listing \ref{lst:bitwiseMul} presents their multiplication algorithm, which we call bitwise_mul. We experimentally compare the precision and performance of our tnum multiplication with both bitwise_mul and the Linux kernel version (kern_mul). We have performed bounded verification of the soundness of both kern_mul and bitwise_mul at bitwidth $n = 8$.

Similar to our_mul, bitwise_mul is also inspired from long multiplication. It generates partial products that are subsequently added after appropriately bit-shifting them. We observed that bitwise_mul needs to be carefully implemented with machine arithmetic operations to have reasonable performance. In bitwise_mul, the function multiply_bit modifies the second operand $Q$ based on the $i^{th}$ bit of the first operand $P$. If the $i^{th}$ bit of $P$ is $\mu$, this modification is done in a trit-by-trit fashion (i.e., by iterating over the $n$ trits of $Q$ and setting them to $\mu$). To improve bitwise_mul’s performance with tnums, we substituted this loop with a single tnum operation \texttt{tnum_add}. The full results are compared in Fig. \ref{fig:bitwise业绩}.


def bitwise_mul(tnum P, tnum Q):
    tnum sum = tnum(0, 0)
    # loop runs bitwidth times
    for i in range(0, bitwidth):
        sum = tnum_add(sum, tnum_lshift(product, i))
    return sum


def multiply_bit(tnum P, tnum Q, u8 i):
    # Bit position i of tnum P is a certain 0
    if (P.v[i] == 0 and P.m[i] == 0):
        return tnum(0, 0)
    # Bit position i of tnum P is a certain 1
    elif (P.v[i] == 1 and P.m[i] == 0):
        return Q
    # Bit position i of tnum P is uncertain
    else:
        # "kill" all bits of Q that are a certain 1,
        # i.e. set them to uncertain
        for j in range(0, bitwidth):
            if (Q.v[j] == 1 and Q.m[j] == 0):
                Q.v[j] = 0
                Q.m[j] = 1
        return Q

Listing 5: Bitwise multiplication (\textit{i.e., bitwise\_mul}) by Regehr et al. \cite{42}

Setup. We performed all our experiments on the Cloudlab \cite{53} testbed. We used two 10-core Intel Skylake CPUs running at 2.20 GHz for a total of 20 cores, with 192GB of memory.

A. Evaluation of Precision of our\_mul

We evaluate the precision of our_mul compared to bitwise_mul and kern_mul by exhaustively evaluating all pairs of tnum inputs at a given bitwidth $n$. We set $n = 8$ to keep the running times tractable.

Consider two abstract tnum multiplication operations $\text{op}_1$ and $\text{op}_2$. Given two tnums $P$ and $Q$, suppose $R_1 \equiv \text{op}_1(P, Q)$ and $R_2 \equiv \text{op}_2(P, Q)$. For fixed $P$ and $Q$, $\text{op}_1$ is more precise than $\text{op}_2$ if $R_1 \neq R_2 \land R_1 \sqsubseteq A R_2$, or equivalently, $R_1 \neq R_2 \land \gamma(R_1) \sqsubseteq \gamma(R_2)$, where $\subseteq$ is the subset relation $\subseteq$. In general, two such output tnums $R_1$ and $R_2$ need not be comparable using the abstract order $\sqsubseteq A$. For example, at bitwidth $n = 9$, with input tnums $P = 0000000011, Q = 011p011p\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\mu\μ
B. Performance evaluation of our_mul

We compare the performance (in CPU cycles measured using the RDTSC time stamp counter) of all the tnum multiplication algorithms discussed in this paper: kern_mul [39], bitwise_mul [42], and our new algorithm our_mul. We perform the experiment using 40 million randomly sampled tnum pairs (64-bit), repeating each input pair 10 times and measuring the minimum number of cycles across these trials. Figure 5 reports the cumulative distribution of this cycle count across all the sampled inputs. All multiplication algorithms have a loop, and for some algorithms, the number of iterations of the loop depend on number of unknown bits in the input operands. Hence, the number of cycles varies across inputs.

We observe that our_mul is faster (i.e., fewer CPU cycles taken) than all the other versions of tnum multiplication. On average, kern_mul takes around 393 cycles, our optimized version of bitwise_mul takes 387 cycles, and our_mul takes 262 cycles (when we take the average of the minimum of 10 trials for each input tnum pair). Our optimizations to bitwise_mul to use machine arithmetic were important as it improved the performance significantly (i.e., from 4921 cycles to 387 cycles). In summary, efficient use of machine arithmetic and the novel computation and summation of partial products makes our_mul 33% (resp. 32%) faster on average than kern_mul (resp. our optimized version of bitwise_mul).

V. RELATED WORK

BPF safety. Given the widespread use of BPF, recent efforts have explored building safe JIT compilers and interpreters [54, 55, 56, 57]. These works assume the correctness of the in-kernel static checker and the JIT translation happens after the BPF code passes the static checker. Prevail [7] proposes an offline BPF analyzer using abstract interpretation with the zone abstract domain and supports programs with loops. In contrast to this paper, these prior efforts have not looked at verifying the tnum operations in the Linux kernel’s static analyzer or explored the tnum domain specifically.

Abstract interpretation. Many static analyses use an abstract domain for abstract interpretation [38, 58, 59]. Abstract domains like intervals, octagons [45], and polyhedra-based domains [60] enhance the precision and efficiency of the underlying operations. Unlike the Linux kernel’s tnums, their intended use is in offline settings. Of particular relevance to our work is the known-bits domain from LLVM [44, 3, 43], which, like tnums, is used to reason about the certainty of individual bits in a value. Our work on verifying tnums will be likely useful to LLVM’s known-bits analysis, as prior work does not provide proofs of precision and soundness for arithmetic operations such as addition and multiplication.

Safety of static analyzers. One way to check for soundness and precision bugs in static analyzers is to use automated random testing [61, 62]. Recently, Taneja et al. [3] test dataflow analyses in LLVM to find soundness and precision bugs by using an SMT-based test oracle [63]. Bugariu et al. [64] test the soundness and precision of widely-used numerical abstract domains [65, 60]. They use mathematical properties of such domains as test oracles while comparing against a set of representative domain elements. They assume that the oracle specification of operations is correct and precise. This paper differs from these approaches in that we formalize and construct analytical proofs for the abstract operations.

VI. CONCLUSION

Abstract domains like tnums are widely used to track register values in the Linux kernel and in various compilers. This paper performs verification of tnum arithmetic operations, and develops a new implementation for tnum multiplication. Our algorithm for tnum multiplication is sound, precise, and faster than Linux’s kernel multiplication. Our new multiplication algorithm is now part of the Linux kernel.

ACKNOWLEDGMENTS

This paper is based upon work supported in part by the National Science Foundation under FMITF-Track I Grant No. 2019302 and the Facebook’s Networking Systems Research Award. We thank Edward Cree for his feedback on strength reductions and precision improvements for tnum operations.
A. Proofs for Auxiliary Lemmas for Tnum Addition

We provide proofs for the additional lemma required to prove the soundness and maximal precision of tnum addition. Let \( x[i] \) denote the \( i \)-th bit of an integer \( x \in \mathbb{Z}_n \).

Note that by our definition of \( \alpha \) and \( \gamma \), given a non-empty tnum \( T \in \mathbb{Z}_n \times \mathbb{Z}_n \), the following holds:

\[
T.v[i] = 1 \iff \forall c \in \gamma(T) : c[i] = 1
\]

\[
T.v[i] = 0 \land T.m[i] = 0 \iff \forall c \in \gamma(T) : c[i] = 0
\]

\[
T.v[i] = 0 \land T.m[i] = 1 \iff \forall c \in \gamma(T) : c[i] = 0 \lor c[i] = 1
\]

(13)

Suppose \( p \) and \( q \) are two concrete values in tnum \( P \) and tnum \( Q \), respectively, i.e., \( p \in \gamma(P), q \in \gamma(Q) \).

**Lemma 12.** (Lemma 2 in the main text). **Minimum carries lemma.** The addition \( sv = p.v + q.v \) will produce a sequence of carry bits that has the least number of 1s out of all possible additions \( p + q \).

**Proof.** We prove any concrete addition \( p + q \) will produce a sequence of carry bits with 1s in at least those positions where \( sv \) produced carry bits set to 1.

Let \( c_{sv}[i] \) denote the carry out bit at the \( i \)-th position produced by \( P.v + Q.v \) and \( c_{out}[i] \) denote the carry out bit at the \( i \)-th position produced by \( p + q \). We will prove by induction that if \( p.v[i] + Q.v[i] \) produces a carry bit then so must \( p[i] + q[i] \).

**Base case:** At bit position \( i = 0 \), \( p.v[0] + Q.v[0] \) produces a carry bit only when \( P.v[0] = 1 \) and \( Q.v[0] = 1 \). Since \( P.v[0] = 1 \) and \( Q.v[0] = 1 \), by Eqn. 13 this implies that \( \forall p : p[0] = 1 \) and that \( \forall q : q[0] = 1 \). Hence, \( p[i] + q[i] \) must also produce a carry bit. Hence, \( c_{sv}[0] = 1 \rightarrow c_{out}[0] = 1 \).

**Induction step:** Assume that for all \( 0 \leq j \leq i-1 \), \( c_{sv}[j] = 1 \rightarrow c_{out}[j] = 1 \). Now, we show that \( c_{sv}[i] = 1 \rightarrow c_{out}[i] = 1 \) by considering all possible cases:

1) \( P.v[i] = 0 \) and \( Q.v[i] = 0 \). Irrespective of any carry-in \( c_{in}[i] \), \( 0 + 0 + c_{in}[i] \) can never produce a carry-out. Hence, \( c_{sv}[i] = 1 \rightarrow c_{out}[i] = 1 \) holds vacuously.

2) \( P.v[i] = 1 \) and \( Q.v[i] = 1 \). Irrespective of any carry-in \( c_{in}[i] \), \( 1 + 1 + c_{in}[i] \) will always produce a carry-out. Now, since is \( P.v[i] = 1 \) and \( Q.v[i] = 0 \), by Eqn. 13 implies that \( \forall p : p[i] = 1 \) and \( \forall q : q[i] = 1 \). Hence all concrete additions \( p[i] + q[i] \) must produce a carry-out. Hence, \( c_{sv}[i] = 1 \rightarrow c_{out}[i] = 1 \) holds.

3) \( P.v[i] = 1 \wedge Q.v[i] = 0 \) \( \lor \) \( (P.v[i] = 0 \wedge Q.v[i] = 1) \). From Eqn. 13 this implies that \( \forall p, q : (p[i] = 1 \wedge q[i] = 1) \lor (p[i] = 1 \lor q[i] = 0) \lor (p[i] = 0 \wedge q[i] = 1) \). That is, either \( p[i] = 1 \) or \( q[i] \) is unknown, or \( q[i] = 1 \) and \( p[i] \) is unknown. This case entails two possibilities:

a) \( c_{sv}[i] = 0 \) is the case when there is no carry-out produced by the additions \( 1 + 0 + 0 \) or \( 0 + 1 + 0 \). \( c_{sv}[i] = 1 \rightarrow c_{out}[i] = 1 \) holds vacuously.

b) \( c_{sv}[i] = 1 \) is the case when there is a carry-out produced by \( 1 + 0 + 1 \) or \( 0 + 1 + 1 \), which always evaluates to 1. Now, by the
Lemma 13. (Lemma 3 in the main text). Maximum carries lemma. The addition $\Sigma = (P.v + P.m) + (Q.v + Q.m)$ will produce the sequence of carry bits with the most number of 1s out of all possible additions $p + q$.

Proof. We prove that any $p + q$ will produce a sequence of carry bits with 1s in at most those positions where $\Sigma$ produced carry bits set to 1.

Let $c_{2^i}[i]$ denote the carry out bit at the $i^{th}$ position produced by $P.v + P.m$ and $c_{out}[i]$ denote the carry out bit at the $i^{th}$ position produced by $p + q$. We will prove by induction that if $c_{2^i}[i]$ does not produce carry bit then neither will $c_{out}[i]$.

Base case: At bit position $i = 0$, $c_{2^i}[0] = (P.v[0] + P.m[0]) + (Q.v[0] + Q.m[0])$ does not produce carry in any position only when $P.v[0] = 0 \land P.m[0] = 0$ or $Q.v[0] = 0 \land Q.m[0] = 0$. Hence, $p[i] + q[i] = 0$. Therefore, $c_{2^i}[0] = 0 \rightarrow c_{out}[0] = 0$.

Induction step: Assume that for all $0 \leq j < i$, $c_{2^j}[j] = 0 \rightarrow c_{out}[j] = 0$. Now, let $c_{2^i}[i] = 0 \rightarrow c_{out}[i] = 0$ by considering all possible cases:

1) $P.v[i] + P.m[i] = 0$ and $Q.v[i] + Q.m[i] = 0$. Irrespective of any carry-in $c_{in}[i]$, $0 + 0 + c_{in}[i]$ can never produce a carry-output. Now, $P.v[i] + P.m[i] = 0$ holds only when $P.v[i] = 0 \land P.m[i] = 0$. By Eqn. 13, $p[i] = 0 \land q[i] = 0$. Hence, irrespective of any carry-in $p[i] + q[i]$ will never produce carry-out. Hence, $c_{2^i}[i] = 0 \rightarrow c_{out}[i] = 0$.

2) $P.v[i] + P.m[i] = 1$ and $Q.v[i] + Q.m[i] = 1$. Irrespective of any carry-in $c_{in}[i]$, this always produces a carry-out. Hence, $c_{2^i}[i] = 0 \rightarrow c_{out}[i] = 0$.

3) $P.v[i] + P.m[i] = 0$ and $Q.v[i] + Q.m[i] = 1$. This case entails two possibilities:
   a) $c_{2^i}[i - 1] = 1$. This implies that $c_{2^i}[i] = 1$. Hence, $c_{2^i}[i] = 0 \rightarrow c_{out}[i] = 0$ holds vacuously.
   b) $c_{2^i}[i - 1] = 0$. This implies that $c_{2^i}[i] = 0$. Note because $P.v[i] + P.m[i] = 0$ we have $P.v[i] = 0$ and $P.m[i] = 0$, and by Eqn. 13 this implies $p[i] = 0$. Now, by the induction hypothesis $c_{out}[i - 1] = 0$. Hence, irrespective of the value of $q[i]$, $c_{out}[i]$ does not produce a carry. Hence, $c_{2^i}[i] = 0 \rightarrow c_{out}[i] = 0$ holds.

4) $P.v[i] + P.m[i] = 1$ and $Q.v[i] + Q.m[i] = 0$. This case is completely symmetrical to case 3.

Hence, it follows that if $(P.v + P.m) + (Q.v + Q.m)$ does not produce a carry, then neither will $p[i] + q[i]$, for all $p \in \gamma(P)$ and $q \in \gamma(Q)$.

Lemma 14. (Lemma 4 in the main text). Capture uncertainty lemma. Let $sv_c$ and $\Sigma_c$ be the sequence of carry-in bits from the additions in $sv$ and $\Sigma$, respectively. Suppose $\chi_c \triangleq sv_c \oplus \Sigma_c$. The bit positions $k$ where $\chi_c[k] = 0$ have carry bits fixed in all concrete additions $p + q$ from $+\gamma(P), \gamma(Q))$. The bit positions $k$ where $\chi_c[k] = 1$ vary depending on the concrete addition $i.e., \exists p_1, p_2 \in \gamma(P), q_1, q_2 \in \gamma(Q)$ such that $p_1 + q_1$ has its carry bit set at position $k$ but $p_2 + q_2$ has that bit unset.

Proof. Consider the carry sequences $sv_c$ and $\Sigma_c$. At any given bit position $i$, we consider the following cases:

1) $\Sigma_c[i] = 0$, and $sv_c[i] = 0$. From the maximum carry lemma (Lemma 13), $\Sigma_c[i] = 0$ implies that no addition $p[i - 1] + q[i - 1] + c_{in}[i - 1]$ (where $p \in \gamma(P)$ and $q \in \gamma(Q)$, and $c_{in}[i]$ is the carry-out from bit position $i - 2$) produced a carry-out from position $i - 1$. Equivalently, the carry-in at bit position $i$ is always unset. That is, all concrete additions $p + q$ have a fixed carry-in bit of 0 at position $i$.

2) $\Sigma_c[i] = 1$, and $sv_c[i] = 1$. From the minimum carries lemma, (Lemma 12), $sv_c[i] = 1$ implies that all additions $p[i - 1] + q[i - 1] + c_{in}[i - 1]$ (where $p \in \gamma(P)$ and $q \in \gamma(Q)$, and $c_{in}[i - 1]$ is the carry-out from bit position $i - 1$) produced a carry-out from position $i - 1$. Equivalently, the carry-in at bit position $i$ is always set. That is, all concrete additions $p + q$ have a fixed carry-in bit of 1 at position $i$.

3) $\Sigma_c[i] = 1$, and $sv_c[i] = 0$. Note that it always holds that $P.v \in \gamma(P)$ and $Q.v \in \gamma(Q)$. Since $sv_c[i] = 0$, we know one concrete addition $p + q$ that definitely does not produce a carry-out from bit position $i - 1$: setting $p = P.v$ and $q = Q.v$. Now note also that it always holds that $P.v + P.m \in \gamma(P)$ and $Q.v + Q.m \in \gamma(Q)$. Since $\Sigma_c[i] = 1$, we know one concrete addition $p + q$ that definitely produces a carry-out from bit position $i - 1$: setting $p = P.v + P.m$ and $q = Q.v + Q.m$.

4) $\Sigma_c[i] = 0$, and $sv_c[i] = 1$. This case is never possible, because if the additions in $sv[i - 1]$ produce a carry out (it is given that $sv_c[i] = 1$), then so must the additions in $\Sigma[i - 1]$ (but it is given that $\Sigma_c[i] = 0$).

From cases 1) and 2) above, it is clear that if $\Sigma_c[i]$ and $sv_c[i]$ are the same i.e., if $\chi_c[i] = 0$, then all the additions at position $i - 1$ exclusively (i) produce a carry-out, or (ii) do not produce a carry-out. From case 3) it is clear that if $\Sigma_c[i]$ and $sv_c[i]$ differ i.e., if $\chi_c[i] = 1$, then some additions at position $i - 1$ produce a carry-out and some additions do not produce a carry-out. This proves our result.

For the next lemma, recall the definitions of the full adder equations.

Definition 15. (Definition 7 in the main text). Full adder equations. When adding two concrete binary numbers $p$ and $q$, each bit of the addition result $r$ is set according to the following:

$$r[i] = p[i] \oplus q[i] \oplus c_{in}[i]$$
where $\oplus$ is the exclusive-or operation and $c_{in}[i] = c_{out}[i - 1]$ and $c_{out}[i - 1]$ is the carry-out from the addition in bit position $i - 1$. The carry-out bit at the $i$th position is given by

$$c_{out}[i] = (p[i] & q[i]) \oplus (c_{in}[i] \oplus (p[i] \oplus q[i]))$$

**Lemma 16. (Lemma 3 in the main text). Equivalence of mask expressions.** Tnum addition ($\text{tnum\_add}$) uses $(sv \oplus \Sigma) \mid P.m \mid Q.m$ to compute the mask of the result. It is in fact the case that $(sv \oplus \Sigma) \mid P.m \mid Q.m$ and $(sv \oplus \Sigma_1) \mid P.m \mid Q.m$ compute the same result.

**Proof.** We have to prove that given $sv_c[i+1] \oplus \Sigma_c[i+1] = 1$, this implies that there exist $p_1, p_2, q_1, q_2$ such that $p_1[i] + q_1[i] + c_{in}[i]$ produces a carry out, and $p_2[i] + q_2[i] + c_{in}[i]$ does not produce a carry out.

Recall that $sv = P.v + Q.v$ and $\Sigma = (P.v + P.m) + (Q.v + Q.m)$ and $sv_c$ and $\Sigma_c$ be the sequence of carry-in bits from the additions in $sv$ and $\Sigma$, respectively. We prove that the terms $(sv \oplus \Sigma_1) \mid P.m \mid Q.m$ and $(sv \oplus \Sigma) \mid P.m \mid Q.m$ are exactly the same.

From Definition 15 of the full-adder equation, we have that the carry-in bit of $sv[i] = p[i] \oplus q[i] \oplus c_{in}[i]$. Hence, the $i$th bit of $sv[i] = P.v[i] \oplus Q.v[i] \oplus sv_c[i]$. Similarly, the $i$th bit of $\Sigma[i]$ can be written as $(P.v[i] + P.m[i]) \oplus (Q.v[i] + Q.m[i]) \oplus \Sigma_c[i]$. Consider $sv[i] \oplus \Sigma[i]$.

$$sv[i] \oplus \Sigma[i] = P.v[i] \oplus Q.v[i] \oplus sv_c[i]$$

Now $sv[i] \oplus \Sigma[i]$ can differ from $sv_c[i] \oplus \Sigma_c[i]$, if their exclusive-or is 1, i.e., $(sv[i] \oplus \Sigma[i]) \oplus (sv_c[i] \oplus \Sigma_c[i])$ is 1.

$$P.v[i] \oplus Q.v[i] \oplus sv_c[i]$$

Given a non-empty tnum $P \in \mathbb{Z}_n \times \mathbb{Z}_n$, define $Q \triangleq \text{tnum}(0, P.v \mid P.m)$. Then, $i \in \gamma(P) \Rightarrow x \in \gamma(Q)$, and $(ii) \ 0 \in \gamma(Q)$.

**Proof.** By the definition of $\gamma$ in Eqn. 14 for a tnum $T$ such that $T.v = 0$, $\gamma(0, T.m) = \{c \mid c \& T.m = 0\}$. Since $c = 0$ satisfies the condition, it is true that $0 \in \gamma(Q)$. Additionally, we can see intuitively that building a new tnum $Q$ whose mask has set all the bits corresponding to the set value or mask bits of $P$ ensures that $\gamma(P) \subseteq \gamma(Q)$.

More formally, consider any concrete value $x \in \gamma(P)$, we observe that it is sufficient to prove the membership of $x$ in tnum $\gamma(Q)$ bit-by-bit, since $\gamma$ is bitwise exact. Note that $Q.v = 0$. For each bit position $k$, there are three possible cases.

1) $P.v[k] = 1, P.m[k] = 0$. This implies that $x[k] = 1$, due to the bitwise exactness of $\gamma$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 1$. $x[k] = 1$ satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q)$.

2) $P.v[k] = 0, P.m[k] = 0$. This implies that $x[k] = 0$, due to the bitwise exactness of $\gamma$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 0$. $x[k] = 0$ satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q)$.

3) $P.v[k] = 0, P.m[k] = 1$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 1$. Whatever the value of $x[k]$, it satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q)$.

Hence, we can say that without loss of generality, for all bit positions $k$, $x[k] \in \gamma(P[k]) \Rightarrow x[k] \in \gamma(Q[k])$. Hence $x \in \gamma(P) \Rightarrow x \in \gamma(Q)$.

**Lemma 17. (Lemma 6 in the main text). Soundness and optimality of tnum_add.** The algorithm tnum_add shown in Listing 1 is a sound and optimal abstraction of concrete addition over $n$-bit bitvectors.

**Proof.** Since tnum_add captures all, and only, the uncertainty in the concrete results of tnum addition, it is sound and optimal.

**B. Proof of our new algorithm for tnum multiplication**

In this section, we present the complete proof of our tnum multiplication algorithm in Listing 3. Recall first, the definition of our concretization function $\gamma$.

$$\gamma(t) = \gamma((t.v, t.m)) \triangleq \{ c \in \mathbb{Z} \mid c \& -t.m = t.v \}$$

**Observation 18. (Observation 2 in the main text).** For two concrete bitvectors $x$ and $y$ of width $n$ bits, the result of multiplication $y \times x$ is just

$$y \times x = \sum_{k=0}^{n-1} x[k] \times (y \ll k)$$

We call each term in the summation $x[k] \times (y \ll k)$ a partial product.

**Lemma 19. (Lemma 8 in the main text). Tnum set union with zero.** Given a non-empty tnum $P \in \mathbb{Z}_n \times \mathbb{Z}_n$, define $Q \triangleq \text{tnum}(0, P.v \mid P.m)$. Then, $i \in \gamma(P) \Rightarrow x \in \gamma(Q)$, and $(ii) \ 0 \in \gamma(Q)$.

**Proof.** By the definition of $\gamma$ in Eqn. 14 for a tnum $T$ such that $T.v = 0$, $\gamma(0, T.m) = \{c \mid c \& -T.m = 0\}$. Since $c = 0$ satisfies the condition, it is true that $0 \in \gamma(Q)$. Additionally, we can see intuitively that building a new tnum $Q$ whose mask has set all the bits corresponding to the set value or mask bits of $P$ ensures that $\gamma(P) \subseteq \gamma(Q)$.

More formally, consider any concrete value $x \in \gamma(P)$, we observe that it is sufficient to prove the membership of $x$ in tnum $\gamma(Q)$ bit-by-bit, since $\gamma$ is bitwise exact. Note that $Q.v = 0$. For each bit position $k$, there are three possible cases.

1) $P.v[k] = 1, P.m[k] = 0$. This implies that $x[k] = 1$, due to the bitwise exactness of $\gamma$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 1$. $x[k] = 1$ satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q[k])$.

2) $P.v[k] = 0, P.m[k] = 0$. This implies that $x[k] = 0$, due to the bitwise exactness of $\gamma$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 0$. $x[k] = 0$ satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q[k])$.

3) $P.v[k] = 0, P.m[k] = 1$. By the way of constructing $Q$, we have $Q.v[k] = 0$, and $Q.m[k] = 1$. Whatever the value of $x[k]$, it satisfies $\gamma(Q.v[k], Q.m[k])$. Hence $x[k] \in \gamma(Q[k])$.

Hence, we can say that without loss of generality, for all bit positions $k$, $x[k] \in \gamma(P[k]) \Rightarrow x[k] \in \gamma(Q[k])$. Hence $x \in \gamma(P) \Rightarrow x \in \gamma(Q)$.\]
where $\text{tnum}_\text{add}(\gamma)$ is a variable-arity version of $\text{tnum}_\text{add}$ defined above. Then, $\sum_{j=0}^{n-1} z_j \in \gamma(S)$.

Proof. Intuitively, suppose we had $n$ tnums $T_i, 0 \leq i \leq n-1$ and we seek to construct a new tnum $S$ whose concreteization $\gamma(S)$ contains all possible concrete sums from the $T_i$, i.e., such that $\{\sum_{j=0}^{n-1} x_j \mid x_j \in \gamma(T_i)\} \subseteq \gamma(S)$. The most natural method to construct such a tnum $S$ is to use the sound abstract addition operator $\text{tnum}_\text{add}$ over the $T_i$, i.e., $\text{tnum}_\text{add}_{\sum_{j=0}^{n-1} (T_j)}$. This lemma provides another method of constructing such a tnum $S$: decompose the tnums $T_i$ each into two tnums, consisting of the values and the masks separately. Use $\text{tnum}_\text{add}$ to separately add the value tnums, add the mask tnums, and finally add the two resulting tnums from the value-sum and mask-sum to produce $S$.

Note that given a non-empty tnum $T$, its concreteization $\gamma(T)$ contains $T.v$ and also, $T.v$ is the smallest element in $\gamma(T)$.

\[
x \in \gamma(T.v) \land \forall x \in \gamma(T), x \geq T.v
\]

(16)

For proving Lemma 20 we start by proving the simpler property below.

Property P0 (value-mask decomposition of a single tnum).

Given a non-empty tnum $T$. For any $x \in \gamma(T)$, we can decompose the concrete value $x$ as the sum of two concrete values, i.e., $x = x' + x''$ where $x' \in \gamma(T.v, 0)$ and $x'' \in \gamma(0, T.m)$.

Proof. For any $x \in \gamma(T)$, from Eqn. 16 it must be that $x \geq T.v$. Additionally, consider $(x - T.v)$. The only bits that may be set in $(x - T.v)$ are those that correspond to unknown trits in $T$, which implies that $\gamma(T.m) = 1$, i.e., $\forall k \in \{0, 1\}, (x - T.v) = 1 \rightarrow T[i] = 0 \rightarrow T.m[i] = 1$. Hence, it is true that $(x - T.v) \& T.m = 0$.

In particular, setting $y = x - T.v$ and rewriting the equation above, $(y - 0) \& -T.m = T.v$. By definition of $\gamma$ (Eqn. 14), this implies that $y \in \gamma(\text{tnum}(0, T.m))$. Since $x = y + T.v$, we can set $x' = T.v$ and $x'' = y$ to prove the result.

Using the value-mask decomposition property above, we can rewrite all the concrete $z_j = z'_j + z''_j$, such that $z'_j \in \gamma(\text{tnum}(T_j, v, 0))$ and $z''_j \in \gamma(\text{tnum}(0, T_j, m))$. Assuming that tnum addition is sound, it is the case that

1) $\sum_{j=0}^{n-1} z'_j \in \gamma(\text{tnum}_\text{add}_{\sum_{j=0}^{n-1} (T_j)}(\text{tnum}(T_j, v, 0)))$,

2) $\sum_{j=0}^{n-1} z''_j \in \gamma(\text{tnum}_\text{add}_{\sum_{j=0}^{n-1} (T_j)}(\text{tnum}(0, T_j, m)))$,

3) Hence, it follows that

\[
\sum_{j=0}^{n-1} z'_j + \sum_{j=0}^{n-1} z''_j \in \gamma(\text{tnum}_\text{add}_{\sum_{j=0}^{n-1} (T_j)}(\text{tnum}(T_j, v, 0)) \land \text{tnum}_\text{add}_{\sum_{j=0}^{n-1} (T_j)}(\text{tnum}(0, T_j, m)))
\]

Moreover, by the associativity of arithmetic addition, $\sum_{j=0}^{n-1} z'_j + \sum_{j=0}^{n-1} z''_j$ is just $\sum_{j=0}^{n-1} z_j$, since each $z_j = z'_j + z''_j$. This proves our result.

---

Theorem 21. (Theorem 10 in the main text). **Soundness of our_mul.** $\forall x \in \gamma(P), y \in \gamma(Q)$ the result $R$ returned by our_mul_simplified (Listing 3) is such that $x \ast y \in \gamma(R)$, assuming that abstract tnum addition ($\text{tnum}_\text{add}$) and abstract tnum shifts ($\text{tnum}_\text{lshift}, \text{tnum}_\text{rshift}$) are sound.

Proof. At a high level, the algorithm constructs two tnums at each iteration of the loop. Intuitively, the first tnum $ACC_V$ sums the concrete bits of each partial product, while the second tnum $ACC_M$ sums the uncertain bits of each partial product using tnum addition. Then, $\text{tnum}_\text{add}(ACC_V, ACC_M)$ is a tnum which contains all possible sums of the partial products, i.e., all possible products $x \ast y$ where $x \in \gamma(P)$ and $y \in \gamma(Q)$.

More formally, our proof proceeds by establishing three properties described below. Suppose the formal parameters to our_mul are denoted $P_{in}$ and $Q_{in}$ respectively. We assume that the loop counter $i$ is initialized to 0 and is incremented by 1 at the end of each iteration up to the bit length $n$ of the input tnums. When we say “the $k^{th}$ iteration” for some fixed $k$, we mean the loop iteration where the value of the loop counter $i$ at the end of the loop body is $k$, $1 \leq k \leq n$.

**Property P1.** $P$ and $Q$ are bit-shifted versions of $P_{in}$ and $Q_{in}$. At the end of the $k^{th}$ iteration, we have that $P = \text{tnum}_\text{lshift}(P_{in}, k)$ and $Q = \text{tnum}_\text{lshift}(Q_{in}, k)$. Further, within the body of iteration $k$ of the loop, $P = \text{tnum}_\text{rshift}(P_{in}, k-1)$ and $Q = \text{tnum}_\text{lshift}(Q_{in}, k-1)$.

In particular, within the body of the $k^{th}$ iteration, $P.v[0] = P_{in}.v[k-1]$ and $P.m[0] = P_{in}.m[k-1]$. $Q.v = Q_{in}.v << (k-1)$ and $Q.m = Q_{in}.m << (k-1)$.

Proof. The algorithm only modifies $P$ and $Q$ via tnum bitshifting at the end of each iteration. It is straightforward to show the properties above by induction on the loop counter $i$ and the semantics of the tnum bit shifting algorithms.

**Property P2.** $ACC_V$ and $ACC_M$ are value-mask-decomposed summations of partial products. There exist tnums $T_0, T_1, \ldots, T_{n-1}$ such that (i) any concrete $j^{th}$ partial product, $z_j \triangleq x[j] \ast y \ll j$ is a member of $\gamma(T_j)$, i.e., $z_j \in \gamma(T_j)$ for $0 \leq j \leq n-1$; (ii) at the end of the $k^{th}$ iteration of the loop, $ACC_V = \text{tnum}_\text{add}_{\sum_{i=0}^{k-1} \text{tnum}(T_i, v, 0)}$, and (iii) at the end of the $k^{th}$ iteration of the loop, $ACC_M = \text{tnum}_\text{add}_{\sum_{i=0}^{k-1} \text{tnum}(0, T_i, m)}$.

Proof. Consider the $j^{th}$ concrete partial product $z_j \triangleq x[j] \ast y \ll j$, where $P_{in} \in \gamma(y), Q_{in} \in \gamma(0)$, and $0 \leq j \leq n-1$. In the algorithm, the consideration of the concrete $j^{th}$ partial product $z_j$ occurs within the code body of the $j^{th}$ iteration, where $k = j+1$. By Property P1 and the soundness of tnum bitshifting, at the beginning of the $k^{th}$ iteration, $y \ll j \in \gamma(Q)$. Further, by Property P1, the soundness of tnum bitshifting, and the definition of $\gamma$, it is also the case that $x[j] \& P_{in}.m[0] = P.v[0]$.

(i) **Construction of $T_j$.** Given $x \in \gamma(P)$, there are three possibilities for the bit $x[j]$.  

1) $x[j]$ is a known 1: Suppose $x[j]$ is known to be 1 (i.e., $x[j] = 1$, $P_{in}.v[j] = 1$, and $P_{in}.m[j] = 0$). Then, the
concrete partial product \( z_j = y << j \). Since \( y << j \in \gamma(Q) \), we set \( T_j \triangleq Q \). Hence, \( z_j \in \gamma(T_j) \).

2) \( x[j] \) is uncertain: Suppose \( x[j] \) is uncertain (i.e., \( P_m, m[j] = 1 \) and \( P_n, v[j] = 0 \)). The partial product \( z_j \) can take on a value 0 if \( x[j] = 0 \), else the value of the partial product is \( y << j \). Since \( y << j \in \gamma(Q) \), we can construct \( T_j \) so that it contains the concrete value 0 as well as all concrete values in \( Q \). Invoking Lemma 19, setting \( T_j \triangleq tnum(0, Q.v | Q.m) \) will ensure that \( z_j \in \gamma(T_j) \).

3) \( x[j] \) is a known 0: Suppose \( x[j] \) is known to be 0 (i.e., \( x[j] = 0, P_n.v[j] = 0 \), and \( P_n.m[j] = 0 \)). Then, the partial product \( z_j = 0 \). We set \( T_j \triangleq 0 \). Hence, \( z_j \in \gamma(T_j) \).

(ii) \( \text{ACC}_V = tnum_{\text{add}}^{-1}(\gamma(T_j), v, 0) \). Within the body of the loop, \( \text{ACC}_V \) is only updated using tnum addition operations, and in at most one line of code per loop iteration. Further, it can be seen easily that this update \( \text{ACC}_V \) occurs using the value component of the corresponding partial product tnum \( T_j \) (including the possibility of making no updates to \( \text{ACC}_V \), when \( T_j.v = 0 \)). It is straightforward to construct an inductive proof that \( \text{ACC}_V = tnum_{\text{add}}^{-1}(\gamma(T_j), v, 0) \) after the \( i \)th iteration.

(iii) \( \text{ACC}_M = tnum_{\text{add}}^{-1}(0, T_j.m) \). The proof is very similar to Property P2 (ii) above.

\[ \]

Property P3 (Product containment). At the end of the \( n \)th iteration, \( \text{tnum_{add}}(\text{ACC}_V, \text{ACC}_M) \) contains the sum of all concrete partial products \( z_j \) for \( j = 0, \ldots, n - 1 \). That is, \( \text{tnum_{add}}(\text{ACC}_V, \text{ACC}_M) \) contains all concrete values of \( x \cdot y \).

Proof. We apply Lemma 20 where the constituent tnums are \( T_0, T_1, \ldots, T_{n-1} \) and the concrete values considered are the possible partial products \( z_j \). Combined with Property P2, we have that for all possible partial products \( z_j, \sum_{j=0}^{n-1} z_j < \gamma(\text{tnum_{add}}(\text{ACC}_V, \text{ACC}_M)) \). Further, from Observation 18 we know that \( \sum_{j=0}^{n-1} z_j = \sum_{j=0}^{n-1} x[j] \cdot y << j = x \cdot y \). Hence, \( \text{tnum_{add}}(\text{ACC}_V, \text{ACC}_M) \) contains all concrete products \( x \cdot y \).

Property P3 concludes the proof of soundness of our_mul1. The returned tnum \( R \) contains all products of concrete values \( x \cdot y \) where \( x \in \gamma(P) \) and \( y \in \gamma(Q) \).

\[ \]

C. Proof for Tnum Subtraction

```c
    def tnum_sub(tnum P, tnum Q):
        u64 dv := P.v - Q.v
        u64 α := dv + P.m
        u64 β := dv - Q.m
        u64 χ := α & β
        u64 η := χ | P.m & Q.m
        tnum R := tnum(dv & ~η, η)
        return R
```

Listing 6: Linux kernel’s implementation of tnum subtraction (tnum_sub)

**Theorem 22.** Tnum subtraction as defined by the Linux kernel Listing 6 is sound and optimal.

**Proof.** We first look at the equation for the subtraction of two binary numbers using the full subtractor equation.

**Definition 23.** When subtracting two concrete binary numbers \( p \) and \( q \), each bit of the subtraction result \( r \) is set according to the following:

\[
    r[i] = p[i] \oplus q[i] \oplus b_{in}[i]
\]

where \( \oplus \) is the exclusive-or operation and \( b_{in}[i] = b_{out}[i - 1] \) and \( b_{out}[i - 1] \) is the borrow out from the subtraction in bit position \( i - 1 \). The borrow out bit at the \( i \)th position is defined by the full subtractor equation:

\[
    b_{out}[i] = (\neg p[i] \& q[i]) \lor (b_{in}[i] \& \neg (p[i] \& q[i]))
\]

At a given bit position in the tnum, the result of a tnum subtraction will be uncertain if either of the operand bits \( p[i] \) or \( q[i] \) is uncertain, or if the borrow-in bit \( b_{in}[i] \) generated from more-significant bit positions is uncertain. The crux of the proof lies in identifying when borrow-in bits are uncertain, by distinguishing the borrow generated due to the uncertainty in the operands from the borrow present in any concrete subtraction from the input tnums.

Given two tnums \( P \) and \( Q \), the uncertainty contributed by the operands to the result is \( P.v \perp Q.v \). To account for uncertainty that arises in \( b_{in}[i] \), we need to consider the sequence of borrows that can result from the subtraction. Let \( p \) and \( q \) be any concrete value in tnum \( P \) and tnum \( Q \), respectively.

Suppose \( p \) and \( q \) are two concrete values in tnum \( P \) and tnum \( Q \), respectively, i.e., \( p \in \gamma(P), q \in \gamma(Q) \).

**Lemma 24.** Minimum Borrows Lemma. The subtraction \( \alpha = (P.v + P.m) - Q.v \) will produce the sequence of borrow bits with the least number of 1s out of all possible subtractions \( p - q \).

**Proof.** Let \( b_{out}[i] \) denote the borrow out bit at the \( i \)th position produced by \( p - q \) and \( b_{in}[i] \) denote the borrow out bit at the \( i \)th position produced by \( (P.v + P.m) - Q.v \). We will prove by induction that if \( (P.v[i] + P.m[i]) - Q.v[i] \) produces a borrow bit then so must \( p[i] - q[i] \):

**Base case:** At bit position \( i = 0 \), \( (P.v[i] + P.m[i]) - Q.v[i] \) produces a borrow out bit only when \( P.v[i] + P.m[i] = 0 \) and \( Q.v[i] = 1 \). Since a certain bit has the same value across all members of the set then by Eqn. 13 \( p[i] = 0 \) and \( q[i] = 1 \), which must also produce a borrow bit. Hence, \( b_{in}[i] = 1 \rightarrow b_{out}[i] = 1 \) holds.

**Induction step:** Assume that for all \( 0 \leq j \leq i - 1 \), \( b_{out}[j] \geq b_{in}[j] \). Now, we show that \( b_{out}[i] \geq b_{in}[i] \) by considering all possible cases:

1) \( P.v[i] + P.m[i] = 1 \) and \( Q.v[i] = 0 \), \( b_{in}[i] \) will never produce a borrow which means \( b_{in}[i] = 1 \rightarrow b_{out}[i] = 1 \) holds.

2) \( P.v[i] + P.m[i] = 0 \) and \( Q.v[i] = 1 \), \( b_{in}[i] \) will always produce a borrow bit. Since a certain bit has the same
Lemma 25. Maximum Borrows Lemma. The subtraction \( \beta = P.v - (Q.v + Q.m) \) will produce the sequence of borrow bits with the most number of 1s out of all possible subtractions \( p - q \).

We prove that That is, any \( p - q \) will produce a sequence of borrow bits with 1s in at most those positions where \( \beta \) produced borrow bits set to 1.

Proof. Let \( b_{\text{out}}[i] \) denote the borrow out bit at the \( i \)th position produced by \( p - q \) and \( b_{\alpha}[i] \) denote the borrow out bit at the \( i \)th position produced by \( P.v - (Q.v + Q.m) \). We will prove by induction that if \( P.v - (Q.v + Q.m) \) does not produce a borrow bit then neither will \( p[i] - q[i] \):

Base case: At bit position \( i = 0 \), a concrete subtraction \( p[0] - q[0] \) produces a borrow bit only when \( p[0] = 0 \) and \( q[0] = 1 \). If \( p[0] = 0 \) then it must mean that \( P.v[0] = 0 \) (regardless if the bit is certain or not). Similarly, \( q[0] = 1 \) implies \( Q.v[0] = 1 \) (when the bit is certain). Otherwise, \( Q.m[0] = 1 \). Thus, \( P.v - (Q.v + Q.m) \) will produce a borrow bit whenever any \( p[0] + q[0] \) produces a borrow, and not more. Hence, \( b_{\beta}[i] = 0 \) holds.

Induction step: Assume that for all \( 0 \leq j \leq i - 1 \), \( b_{\beta}[j] \geq b_{\text{out}}[j] \). Now, we show that \( b_{\beta}[i] = 0 \rightarrow b_{\text{out}}[i] = 0 \) holds. By considering all possible cases:

1) \( P.v[i] = 1 \) and \( Q.v[i] + Q.m[i] = 0 \). This will never produce a borrow bit. Since \( P.v = 1 \) implies \( p = 1 \) and \( Q.v[i] + Q.m[i] = 0 \) implies \( q = 0 \) (by Eqn. 13) then \( p[i] - q[i] \) will not produce a borrow bit either. Hence, \( b_{\beta}[i] = 0 \rightarrow b_{\text{out}}[i] = 0 \) holds.

2) \( P.v[i] = 0 \) and \( Q.v[i] + Q.m[i] = 1 \). This will always produce a borrow bit which means, \( b_{\beta}[i] = 0 \rightarrow b_{\text{out}}[i] = 0 \) holds vacuously.

3) \( P.v[i] = 0 \) and \( Q.v[i] + Q.m[i] = 0 \). \( b_{\beta}[i] \) will not produce a borrow only when position \( i - 1 \) does not produce a borrow. This case implies that \( p[i] = 0 \) and \( q[i] = 0 \) by Eqn. 13 which will not produce a borrow only when bit position \( i - 1 \) does not produces a borrow. Thus, by the induction hypothesis, \( b_{\beta}[i] = 0 \rightarrow b_{\text{out}}[i] = 0 \) holds.

Hence, if \( P.v[i] - (Q.v[i] + Q.m[i]) \) does not produce a borrow bit then neither will concrete subtraction \( p[i] - q[i] \).

Lemma 26. Capture uncertainty lemma. Let \( \alpha_b \) and \( \beta_b \) be the sequence of borrow-in bits from the subtractions in \( \alpha \) and \( \beta \), respectively. Suppose \( \chi_b \equiv \alpha_b \oplus \beta_b \). The bit positions \( k \) where \( \chi_c[k] = 0 \) have borrow bits fixed in all concrete subtractions \( p - q \) from \( \gamma(P), \gamma(Q) \). The bit positions \( k \) where \( \chi_c[k] = 1 \) vary depending on the concrete subtraction: i.e., \( \exists p_1, p_2 \in \gamma(P), q_1, q_2 \in \gamma(Q) \) such that \( p_1 - q_1 \) has its borrow bit set at position \( k \) but \( p_2 - q_2 \) has that bit unset.

Intuitively, from the minimum borrows lemma, any borrow bit that is produced by \( \alpha_b \) must also be produced by any concrete subtraction \( p - q \) because \( \alpha_b \) contains the minimal amount of borrow bits. From the maximum borrows lemma, no concrete subtraction \( p - q \) may have more borrow bits than \( \beta_b \). Hence, \( \alpha_b \oplus \beta_b \) represents the borrow that arise purely from the uncertainty in the concrete operands picked from \( P \) and \( Q \).

Soundness and optimality. Tnum subtraction (\texttt{tnum_sub}) uses \( (\alpha \oplus \beta) \mid P.m \mid Q.m \) to compute the uncertainty of the result. To show soundness, we prove that \( (\alpha \oplus \beta) \mid P.m \mid Q.m \) and \( (\alpha_b \oplus \beta_b) \mid P.m \mid Q.m \) compute the same result.

From Definition 23 we have \( r[i] = p[i] \oplus q[i] \oplus b_{\alpha}[i] \). Hence, the \( i \)th-bit of \( r[i] \) is \( (P.v[i] + P.m[i]) \oplus Q.v[i] \oplus b_{\alpha}[i] \). Similarly, the \( i \)th-bit of \( \beta[i] \) is \( P.v[i] \oplus (Q.v[i] + Q.m[i]) \oplus b_{\beta}[i] \).

To identify where \( \alpha \oplus \beta \) can differ from \( \alpha_b \oplus \beta_b \), consider the exclusive-or of them (i.e., \( (\alpha \oplus \beta) \oplus (\alpha_b \oplus \beta_b) \)). If they are identical, the result is 0. They are different when \( (\alpha \oplus \beta) \oplus (\alpha_b \oplus \beta_b) \) is 1.
which allows quantifying over non-empty tnums

\[(\alpha[i] \land \beta[i]) \lor (\alpha_0[i] \land \beta_0[i]) \]

\[= (P.v[i] + P.m[i]) \land Q.v[i] \land P.v[i] \land Q.v[i] + Q.m[i]
\]

Now consider the sub-term in the above formula

\[\text{Lemma 27. Soundness and optimality of tnum sub. The}
\text{algorithm tnum sub shown in Listing 1 is a sound and optimal}
\text{abstraction of concrete subtraction over n-bit bitvectors.}

\[\text{Proof. Since tnum sub captures all, and only, the uncertainty}
\text{in the concrete results of tnum subtraction, it is sound and}
\text{optimal.} \]

\[\text{D. Automated verification of tnum operations using SMT solvers}
\]

We previously discussed our SMT encoding of the implementation of

\[\text{tainum of operators using theory of fixed-size bitvectors in }\S\text{III-A}
\text{We use the theory of fixed-size bitvectors, and encode the implementation using the Python bindings of the}
\text{Z3 SMT solver [66]. The size of the bitvector is a parameter for performing bounded automated verification. Given that the Linux kernel uses machine arithmetic with 64-bits, we use bitvectors of width 64 wherever feasible.}

\[\text{Encoding abstract tnum multiplication. The Linux kernel’s}
\text{implementation for abstract multiplication of tnums, kern_mul, is shown in Listing 2 Among the kernel’s abstract operators,}
\text{multiplication is the most challenging to verify automatically,}
\text{because it involves a call to a function hma that contains
\text{a loop. Fortunately, the loop test is simple to encode (the loop runs at most n times for n-bit bitvectors), hence it is}
\text{possible to unroll the loop and rewrite the code in static single assignment form [67]. The rest of the encoding of multiplication is straightforward. The details follow.}

\[\text{Recall that we defined the following predicates: (1) the}
\text{member predicate which asserts for a concrete value x and}
\text{tnum P that } x \in \gamma(P), \text{and (2) the well formed predicate,}
\text{which allows quantifying over non-empty tnums P.}
\]

\[\text{member}(x, P) \triangleq x \land \neg P.m = P.v
\]

\[\text{well formed}(P) \triangleq P.v \land P.m = 0 \]

\[\forall P, Q \in T_n, x, y \in \mathbb{Z}_n:
\text{well formed}(P) \land \text{well formed}(Q) \land \text{member}(x, P)
\]

\[\land \text{member}(y, Q) \land z = \text{op}_C(x, y) \land R = \text{op}_T(P, Q)
\]

\[\Rightarrow \text{member}(z, R) \]

\[\text{Equation (19) below shows the predicate hma which encodes the}
\text{hma function from Listing 2 with the loop shown unrolled}
\text{2 times. The if-then-else term encodes if-then-else and add is}
\text{the predicate from Eqn. (12).}
\]

\[\text{hma}(ACC_{in}, x_{in}, y_{in}, R) \triangleq
\]

\[\text{t}
\]

\[\text{\text{well formed}(P) \land \text{well formed}(Q) \land \text{member}(x, P)
\]

\[\land \text{member}(y, Q) \land z = \text{op}_C(x, y) \land R = \text{op}_T(P, Q)
\]

\[\Rightarrow \text{member}(z, R) \]

\[\text{Equation (19) below shows the predicate hma which encodes the}
\text{hma function from Listing 2 with the loop shown unrolled}
\text{2 times. The if-then-else term encodes if-then-else and add is}
\text{the predicate from Eqn. (12).}
\]

\[\text{hma}(ACC_{in}, x_{in}, y_{in}, R) \triangleq
\]

\[\text{add}(ACC_0, \text{tnum}(0, x_0, ACC_1),
\]

\[\text{ACC}_1 = \text{ACC}_0
\]

\[\land (y_1 = y_0 >> 1) \land (x_1 = x_0 \land 1)
\]

\[\land \text{ite}(y_1 & 1 = 1,
\]

\[\text{add}(ACC_1, \text{tnum}(0, x_1, ACC_2),
\]

\[\text{ACC}_2 = \text{ACC}_1
\]

\[\land (y_2 = y_1 >> 1) \land (x_2 = x_1 \land 1)
\]

\[\ldots
\]

\[\land R = ACC_{64}
\]

\[\text{Spot-checking the correctness of our SMT encodings. To}
\text{ensure that our encodings of the tnum operators in first-order}
\text{logic are accurate, we developed a test harness. Consider a}
\text{tnum abstract operator op(tnum P, tnum Q) with a first order}
\text{logic formula f_op encoding its action symbolically. f_op}
\text{contains bitvector variables corresponding to values and masks for input}
\text{tnums P and Q and output tnum R. We tested our encoding}
\text{as follows. First, we produced many random input tnum pairs}
\text{(X, Y). Next, for each pair (X, Y), we executed the C code}
\text{of the kernel tnum operator to produce an output tnum Z.}
\text{Finally, we use Z3 [66] and obtain a model for the formula}
\text{P = X \land Q = Y \land f_op. The test passes only if the formula}
\text{above is satisfiable and its model interpretation of output R is}
\text{the same as tnum Z from the C execution.}
\]

\[\text{E. Comparing the precision of our mul versus kern_mul with}
\text{increasing bitwidth}
\]

\[\text{Setup. We evaluate the precision of our mul compared to}
\text{bitwise_mul and kern_mul with increasing bitwidth. For a}
\text{particular bitwidth, we provide as input all possible tnum}
\text{pairs at that bitwidth to both these algorithms. We perform this}
\text{experiments for bitwidths 5 through 10 (we stop at n = 10}
\text{to keep running times tractable). We then try to answer the}
\text{questions: What percentage of input pairs lead to a better}
\text{precision in the result produced by our_mul? How does this}
\text{percentage change with increasing bitwidth?}
\]

\[\text{Results. We make the following observations (see Table 1).}
\]

\[\text{(1) The percentage of inputs where our_mul produces a}
\text{different output that kern_mul increases. (2) Output tnums}
\text{are always comparable for bitwidths n = 5 though n = 8. For}
\]
higher bitwidths, although less than 100% of differing output tnums are comparable, a high fraction (> 99%) still remain comparable. (3) The tnums produced by our_mul are more precise than kern_mul for a higher fraction of the comparable output tnums. (4) Considering comparable output tnums for a given input, the fraction of output tnums produced by our_mul which are more precise than kern_mul increases as bitwidth increases. This trend is quite remarkable: it goes to show that our_mul produces increasingly more precise output tnums than kern_mul as the bitwidth (n) of the tnums increases.

| bitwidth | 5 | 6 | 7 | 8 | 9 | 10 |
|----------|---|---|---|---|---|----|
| total tnum pairs | 59049 | 531441 | 4782969 | 43046721 | 38742048 | 3486784401 |
| output kern_mul to output our_mul | | | | | | 99.895% | | 0.105% | 19.773% | 80.225% |
| output our_mul comparable to more precise | | | | | | 99.966% | 0.034% | 100% | 22.778% | 77.222% |
| kern_mul | | | | | | 99.964% | 0.036% | 100% | 21.537% | 78.463% |
| output kern_mul to output our_mul | | | | | | 99.923% | 0.077% | 100% | 20.744% | 79.256% |
| output our_mul comparable to more precise | | | | | | 99.907% | 0.093% | 100% | 20.191% | 79.808% |
| kern_mul | | | | | | 99.838% | 0.162% | 100% | 21.222% | 78.778% |
| output kern_mul to output our_mul | | | | | | 99.895% | 0.105% | 100% | 22.778% | 77.222% |
| output our_mul comparable to more precise | | | | | | 99.966% | 0.034% | 100% | 22.778% | 77.222% |
| kern_mul | | | | | | 99.964% | 0.036% | 100% | 21.537% | 78.463% |
| output kern_mul to output our_mul | | | | | | 99.923% | 0.077% | 100% | 20.744% | 79.256% |
| output our_mul comparable to more precise | | | | | | 99.907% | 0.093% | 100% | 20.191% | 79.808% |
| kern_mul | | | | | | 99.838% | 0.162% | 100% | 21.222% | 78.778% |

Table 1: Table comparing outputs from our_mul and kern_mul, when they are given inputs drawn from all possible tnum pairs of bit of width 5 through 10. The table shows (L - R, starting from column 3): (i) Percentage of inputs where outputs from both the algorithms are exactly the same. (ii) Percentage of inputs where outputs from both algorithms differ (iii) Of those outputs which differ, the percentage of outputs which are comparable. (iv) Of the outputs that differ but are still comparable, the percentage where kern_mul is more precise than our_mul. (v) Of the outputs that differ but are still comparable, the percentage where our_mul is more precise than kern_mul.

F. Proof of the Galios connection

**Theorem 28.** \(2^{2n} \cong Z_n \times Z_n\) is a Galios connection.

We prove that our abstraction function \(\alpha : 2^{2n} \rightarrow Z_n \times Z_n\), and the concretization function \(\gamma : Z_n \times Z_n \rightarrow 2^{2n}\) form a Galois connection by proving the following properties:

1. \(\gamma\) is monotonic
2. \(\alpha\) is monotonic
3. \(\alpha \circ \gamma\) is extensive
4. \(\alpha \circ \gamma\) is reductive

**Property G1.** \(\alpha\) is monotonic, i.e., \(\forall C_1, C_2 \in 2^{2n}, C_1 \subseteq C_2 \rightarrow \alpha(C_1) \subseteq \alpha(C_2)\).

**Proof.** We have \(C_1 \subseteq C_2\), Hence, by the definition of \(\alpha\) in Eqn. [5] the following properties hold:

\[
\begin{align*}
\alpha_k(C_2) &= \alpha_k(C_1) \& \alpha_k(C_2 - C_1) \\
\alpha_i(C_2) &= \alpha_i(C_1) \& \alpha_i(C_2 - C_1) \\
\forall k \in 0, 1: \alpha_k(C_2) &= k \& \alpha_1(C_1) = k \\
\forall k \in 0, 1: \alpha_k(C_2) &= k \& \alpha_k(C_1) = k
\end{align*}
\]

(20)

Here, \(C_2 - C_1\) is the set difference, which includes all the elements in \(C_2\) that are not in \(C_1\). Now, For a concrete set \(C\) and for some \(c \in C\) we use the notation \(c[i]\) to denote the \(i^{th}\) bit of \(c\). Similarly, for a tnum \(T\), we use the notation \(T[i]\) to denote the \(i^{th}\) bit of \(T\).

We have to prove that \(\alpha(C_1) \subseteq \alpha(C_2)\). By the definition of the partial order of the abstract domain, this means:

\[
\forall i, 0 \leq i \leq n - 1, \forall k \in \{0, 1\}:
\]

\[
\begin{align*}
(\alpha(C_1)[i] = \mu \Rightarrow \alpha(C_2)[i] = \mu) \land \\
(\alpha(C_2)[i] = k \Rightarrow \alpha(C_1)[i] = k)
\end{align*}
\]

(21)

Take a particular bit position \(i\). We consider each clause of the conjunction one after the other.

1. \(\alpha(C_1)[i] = \mu = \text{tnum}(0, 1)\). This implies, that \(\alpha_k(C_1[i] = 0) \& \alpha_1(C_1[i] = 1)\). It follows from Eqn. [20] that \(\alpha_k(C_2[i] = 0) \& \alpha_1(C_2[i] = 1)\). Hence \(\alpha(C_2[i] = \text{tnum}(0, 1) = \mu\). Hence, \(\alpha(C_1[i] = k \& \alpha(C_2[i] = k\).

2) We consider two cases:

a) \(\alpha(C_2[i] = 0 = \text{tnum}(0, 0)\). This implies, that \(\alpha_k(C_1[i] = 0) \& \alpha_1(C_2[i] = 1)\). It follows from Eqn. [20] that \(\alpha_k(C_1[i] = 0) \& \alpha_1(C_2[i] = 1)\). Hence \(\alpha(C_1[i] = 0 \& \alpha(C_2[i] = 1)\). From both the above cases, we get: \(\forall k \in 0, 1, \alpha(C_2[i] = k \& \alpha(C_1[i] = k\).

Thus we see that, for a particular \(i\), \(\forall k \in \{0, 1\}\) \(\alpha(C_2[i] = k \& \alpha(C_1[i] = k\). Without loss of generality this holds for all bit positions \(0 < i < n\). Hence Eqn. [21] holds.

**Property G2.** \(\gamma\) is monotonic, i.e., \(\forall T_1, T_2 \in \text{T}_n, T_1 \subseteq \text{T}_2 \rightarrow \gamma(T_1) \subseteq \gamma(C_2)\).

**Proof.** From the definition of the partial order on the abstract domain (Eqn. [2] in the main text), we have for two tnums \(T_1\) and \(T_2\).

\[
T_1 \subseteq \text{T}_2 \equiv \forall i, 0 \leq i \leq n - 1, \forall k \in \{0, 1\}:
\]

\[
\begin{align*}
(T_1[i] = \mu \Rightarrow T_2[i] = \mu) \\
\land (T_1[i] = k \Rightarrow T_2[i] = k) \forall T_2[i] = \mu \\
\land (T_2[i] = k \Rightarrow T_1[i] = k)
\end{align*}
\]

(22)

We will prove \(\gamma(T_1) \subseteq \gamma(T_2)\) by showing that \(\forall x : x \in \gamma(T_1) \rightarrow x \in \gamma(T_2)\). Consider a particular \(x \in \gamma(T_1)\). By the definition of \(\gamma\), we have, \(x \sim T_1, m = T_1, v\). Or alternatively, we have, where \(x[i]\) represents the \(i^{th}\) bit of a value: \(\forall i : x[i] \& T_1, m[i] = T_1, v[i]\).

We consider the following 3 cases:

1) \(T_1, v[i] = 0, \text{and } T_1, m[i] = 1, i.e., T_1, i = \mu\). From Eqn. [22] this implies that \(T_2, v[i] = 0, \text{and } T_2, m[i] = 1\). Hence every \(x[i]\) that satisfies \(x[i] \& T_1, m[i] = T_1, v[i]\) also satisfies \(x[i] \& T_2, m[i] = T_2, v[i]\).

2) \(T_1, v[i] = 0, \text{and } T_1, m[i] = 0, i.e., T_1, i = 0\). By Eqn. [22] we can have the following cases:
a) \((T_2[i] = 0)\) i.e., \(T_2.v[i] = 0\), and \(T_2.m[i] = 0\). Here \(T_2.v[i] = T_1.v[i]\), and \(T_2.m[i] = T_1.m[i]\). Hence every \(x[i]\) that satisfies \(x[i] \& T_1.m[i] = T_1.v[i]\) also satisfies \(x[i] \& T_2.m[i] = T_2.v[i]\).

b) \((T_2[i] = \mu)\) i.e., \(T_2.v[i] = 0\), and \(T_2.m[i] = 1\). The \(x[i]\) that satisfies \(x[i] \& T_1.m[i] = T_1.v[i]\) is only 0, but the \(x[i]\) that satisfies \(x[i] \& T_2.m[i] = T_2.v[i]\) are \((0, 1)\). Again, every \(x[i]\) that satisfies \(x[i] \& T_1.m[i] = T_1.v[i]\) also satisfies \(x[i] \& T_2.m[i] = T_2.v[i]\).

3) \(T_1.v[i] = 1\), and \(T_1.m[i] = 1\), i.e., \((T_1[i] = 1)\). This case is similar to the above case. Again, we have that every \(x[i]\) that satisfies \(x[i] \& T_1.m[i] = T_1.v[i]\) also satisfies \(x[i] \& T_2.m[i] = T_2.v[i]\).

From the above case analysis, we have that for a particular bit position \(i\), and \(x \in \gamma(T_1)\), \(x[i] \& T_1.m[i] = T_1.v[i] \rightarrow x[i] \& T_2.m[i] = T_2.v[i]\). Since we are dealing with only bitwise operations, we can say that this property holds for all bit positions \(i\). Hence, for some \(x \in \gamma(T_1)\), \(x[i] \& T_1.m[i] = T_1.v[i] \rightarrow x[i] \& T_2.m[i] = T_2.v[i]\). Without loss of generality, this holds for all \(x \in \gamma(T_1)\). By the definition of \(\gamma\), we have \(\forall x : x \in \gamma(T_1) \rightarrow x \in \gamma(T_2)\). This proves the result.

Property G3. \(\gamma \circ \alpha\) is extensive, i.e., \(C \in 2^{Z} : C \subseteq \gamma(\alpha(C))\).

Proof. Consider a particular \(C \in 2^{Z}\). This set is constructed by elements drawn from \(Z\), i.e., \(C = \{c | c \in Z\}\). Let \(T = \alpha(C)\). To show extensivity, we have to show that: \(\forall c \in C : c \in \gamma(T)\).

Now, by the definition of \(\gamma\) in Eqn. 7, we have: \(c \in \gamma(T) \Leftrightarrow \forall i, c[k] \& \neg(T.m[i]) = T.v[i]\).

We consider the following cases:

1) \(\forall c_a, c_b \in C, k \in \{0, 1\} : c_a[i] = c_b[i] = k\). This means that all the elements in the set \(C\) share the same value at bit position \(i\). This gives us two cases:
   a) \(\forall c_a, c_b \in C : c_a[i] = c_b[i] = 0\) Consider the tnum \(T = \alpha(C)\). By the definition of \(\alpha\) (Eqn. 5), \(T.v[i] = 0, T.m[i] = 0\). Thus it holds that \(\forall c \in C, c[i] \& \neg T.m[i] = T.v[i]\), since \(0 \& \neg 0 = 0\).
   b) \(\forall c_a, c_b \in C : c_a[i] = c_b[i] = 1\) Consider the tnum \(T = \alpha(C)\). By the definition of \(\alpha\) (Eqn. 5).

Since both \(\alpha\) and \(\gamma\) are bitwise exact, we can say that for a particular \(c \in C, \forall i : c[i] \& \neg T.m[i] = T.v[i]\). This implies \(c \& \neg T.m = T.v\). Without loss of generality, we can say \(T.v[i] = 1, T.m[i] = 0\). Thus it holds that \(\forall c \in C, c[i] \& \neg T.m[i] = T.v[i]\), since \(1 \& \neg 0 = 1\).

2) \(\exists c_a, c_b \in C, k \in \{0, 1\} : c_a[i] \neq c_b[i] = k\). This means that there are two elements drawn from the set \(C\) that do not share the same value at bit position \(i\). Consider the tnum \(T = \alpha(C)\). From the definition of \(\alpha\) (Eqn. 5), \(T.v[i] = 0, T.m[i] = 1\). Regardless of whether \(c[i] = 1\) or \(c[i] = 0\), it satisfies holds that \(\forall c \in C, c[i] \& \neg T.m[i] = T.v[i]\), since both \(1 \& \neg 1 = 0, \text{ and } 0 \& \neg 1 = 0\).

That \(\forall c \in C, c \& \neg T.m = T.v\). This gives us the result: \(\forall c \in C : c \in \gamma(\alpha(C))\).