Hadamard type fuzzy inequality for \((s, m)\)-convex function in second sense

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Abstract
In this paper we prove a Hadamard type fuzzy inequality for \((s, m)\)-convex function in second sense and some examples are given.

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1 Introduction
Fuzzy measure and fuzzy integral first introduced by Sugeno [24]. It can be used for modelling problems in non-deterministic environment. The use of the Sugeno integral can be envisaged from two points of view, decision under uncertainty and multi-criteria decision making [8].

The integral inequalities are significant mathematical tools both in theory and applications. The integral inequalities such as Jensen, Holder, Chebyshev and Minkowski are widely used in various fields of mathematics including forecasting of time-series, information science, decision making under risk and probability theory, differential equations.

Hanson [13] gave the notion of invexity as a significant generalization of classical convexity. Ben-Israel and Mond [14] introduced the preinvex functions a special case of invex functions. Latif
and Shoaib [15] discussed the concept of \( m \)-preinvex functions and \((\alpha, m)\)-preinvex functions. In [7, 9, 10, 11] author studied Hermite-Hadamard type inequalities for \( r \)-convex function.

The study of inequalities for Sugeno integral was initiated by Roman-Flores et.al. [5], [6]. Since many authors have studied the different types of integral inequalities for fuzzy integral see [1]-[5], [20, 22, 23].

In this paper we give the Hadamard type inequality for \((s, m)\)-convex functions in second sense with respect to Sugeno integral.

\section{Preliminary}

The definitions and basic properties of fuzzy measures and fuzzy integrals that will be used in the next sections and can be found in [26], [24].

Suppose that \( \varphi \) is a \( \sigma \)-algebra of subsets of \( X \) and \( \mu : \varphi \rightarrow [0, \infty) \) be a non-negative, extended real valued set function. We say that \( \mu \) is a fuzzy measure if

1. \( \mu(\emptyset) = 0; \)
2. \( E, F \in \varphi \) and \( E \subset F \) imply \( \mu(E) \leq \mu(F); \)
3. \( \{E_n\} \subset \varphi, E_1 \subset E_2 \subset \ldots \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n); \)
4. \( \{E_n\} \subset \varphi, E_1 \supset E_2 \supset \ldots \), \( \mu(E_1) < \infty \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n). \)

If \( f \) is non-negative real-valued function defined on \( X \), we denote the set \( \{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\} \) by \( F_{\alpha} \) for \( \alpha \geq 0 \), where if \( \alpha \leq \beta \) then \( F_{\beta} \subset F_{\alpha}. \)

Let \( (X, \varphi, \mu) \) be a fuzzy measure space, we denote by \( M^+ \) the set of all non-negative measurable functions with respect to \( \varphi. \)

**Definition 2.1.** (Sugeno [24]). Let \( (X, \varphi, \mu) \) be a fuzzy measure space, \( f \in M^+ \) and \( A \in \varphi. \) the Sugeno integral (or fuzzy integral) of \( f \) on \( A. \) with respect to the fuzzy measure \( \mu, \) is defined as

\[ (s) \int_A f \, d\mu = \bigvee_{\alpha \geq 0} \left[ \alpha \wedge \mu(A \cap F_{\alpha}) \right], \]
when $A = X$,

$$(s) \int_X f \, d\mu = \bigvee_{\alpha \geq 0} [\alpha \land \mu(F_{\alpha})],$$

where $\bigvee$ and $\land$ denote the operations sup and inf on $[0, \infty)$, respectively.

Some of the properties of fuzzy integrals are as follows.

**Proposition 2.1.** Let $(X, \emptyset, \mu)$ be fuzzy measure space, $A, B \in \mathcal{P}$ and $f, g \in M^+$ then:

1. $(s) \int_A f \, d\mu \leq \mu(A);$  
2. $(s) \int_A k d\mu = k \land \mu(A)$, $k$ for non-negative constant;
3. $(s) \int_A f \, d\mu \leq (s) \int_A g \, d\mu$, for $f \leq g$;
4. $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \implies (s) \int_A f \, d\mu \geq \alpha$;
5. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \implies (s) \int_A f \, d\mu \leq \alpha$;
6. $(s) \int_A f \, d\mu > \alpha \iff$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$;
7. $(s) \int_A f \, d\mu < \alpha \iff$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

**Remark 2.1.** Consider the distribution function $F$ associated to $f$ on $A$, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then due to (4) and (5) of Proposition 2.1, we have $F(\alpha) = \alpha \implies (s) \int_A f \, d\mu = \alpha$. Fuzzy integral can be obtained by solving the equation $F(\alpha) = \alpha$.

**Definition 2.2.** Let $(s, m) \in (0, 1]^2$ be a pair of real numbers. A function $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be $(s, m)$-convex function in second sense if

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda^s f(x) + m(1 - \lambda)^s f(y),\quad (1)$$

holds for all $(x, y) \in I$ and $\lambda \in [0, 1]$.

Some inequalities for $(s, m)$-convex functions in second sense are obtained in [10]-[19]. If $(s, m) = (1, 1)$, then we obtain the definition of convex function. If $(s, m) = (s, 1)$, then we obtain definition of $s$-convex function in the second sense. It is denoted by $K^2_{s,m}$, the set of all $(s, m)$-convex functions in the second sense.

Now we give the Lemma proved in [20].
Lemma 2.1. \[20] Let \( x \in [0, 1] \), then the inequality

\[(1 - x)^s \leq 2^{1-s} - x^s, \]

holds for \( s \in (0, 1] \).

3 Main Results

In \[21\] U. S. Kirmaci et.al. proved the following Hadamard type inequalities for product of convex function and \( s \)-convex functions.

Theorem 3.1. Let \( f, g : [a, b] \rightarrow \mathbb{R}, a, b \in [0, \infty), a < b \) be functions such that \( g \) and \( fg \) are in \( L^1([a, b]) \). If \( f \) is convex and non-negative on \([a, b]\) and \( g \) is \( s \)-convex function on \([a, b]\) for some fixed \( s \in (0, 1) \) then

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a, b) + \frac{1}{(s+1)(s+2)}N(a, b),
\]

where, \( M(a, b) = f(a)g(a) + f(b)g(b) \) and \( N(a, b) = f(a)g(b) + f(b)g(a) \).

Now consider an example.

Example 3.1. Consider \( X = [0, 1] \) and let \( \mu \) be the Lebesgue measure on \( X \). If we take the function \( f(x) = \frac{x^2}{2} \) and \( g(x) = \frac{x^3}{2} \), \( f(x), g(x) \in K_{s,1}^2 \) for \( s \in (0, 1/3) \). Let \( s = 1/3 \) the Sugeno integral

\[(s) \int_0^1 \frac{x^5}{4} d\mu = 0.1269. \]

Also, \( \frac{1}{s+2}M(a, b) + \frac{1}{(s+1)(s+2)}N(a, b) = 0.1071 \).

This proves that the right hand side (3) of Hadamard type inequalities for \((s, m)\)-convex functions in second sense is not satisfied for Sugeno integral.

In this section we give an Hadamard type inequalities for product of \((s, m)\)-convex function in second sense which is based on Sugeno integral.
Theorem 3.2. Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Let $(s, m) \in (0, 1]^2$ and $f, g : [a, b] \rightarrow [0, \infty)$ are $(s, m)$-convex functions in second sense, such that $f(b) > mf(a)$ and $g(b) > mg(a)$ then

$$(s) \int_a^b fg \, d\mu \leq \min\{\beta, b - a\},$$

where $\beta$ is given by

$$(b - ma)^2 - (b - ma)^2 \left(\frac{\beta - m2^{1-s}g(a)}{g(b) - mg(a)}\right)^{\frac{s}{s - 1}} - (b - ma)^2 \left(\frac{\beta - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{\frac{s}{s - 1}}$$

$$+ (b - ma)^2 \left(\frac{\beta - m2^{1-s}f(a)}{f(b) - mf(a)}\right)^{\frac{s}{s - 1}} \cdot \left(\frac{\beta - m2^{1-s}g(a)}{g(b) - mg(a)}\right)^{\frac{s}{s - 1}} = \beta. \quad (4)$$

Proof. Let $f(x), g(x) \in K_{s,m}^2$ for $x \in [a, b]$, we have

$$f(x) = f\left(m\left(1 - \frac{x - ma}{b - ma}\right) a + \left(\frac{x - ma}{b - ma}\right) b\right)$$

$$\leq m\left(1 - \frac{x - ma}{b - ma}\right)^s f(a) + \left(\frac{x - ma}{b - ma}\right) f(b). \quad (5)$$

$$g(x) = g\left(m\left(1 - \frac{x - ma}{b - ma}\right) a + \left(\frac{x - ma}{b - ma}\right) b\right)$$

$$\leq m\left(1 - \frac{x - ma}{b - ma}\right)^s g(a) + \left(\frac{x - ma}{b - ma}\right) g(b). \quad (6)$$

Form Lemma 2.1, we have

$$\left(1 - \frac{x - a}{b - a}\right)^s \leq 2^{1-s} - \left(\frac{x - a}{b - a}\right)^s. \quad (7)$$

Thus, from (5), (6) and (7), we have

$$f(x) \leq m2^{1-s}f(a) + \left(\frac{x - ma}{b - ma}\right)^s [f(b) - mf(a)] = p_1(x).$$

$$g(x) \leq m2^{1-s}g(a) + \left(\frac{x - ma}{b - ma}\right)^s [g(b) - mg(a)] = p_2(x). \quad (8)$$
By Proposition 2.1, we have

\[
\int_a^b fg \, d\mu \leq \int_a^b \left( m2^{1-s}f(a) + \left( \frac{x-ma}{b-ma} \right)^s [f(b) - mf(a)] \right) \left( m2^{1-s}g(a) + \left( \frac{x-ma}{b-ma} \right)^s [g(b) - mg(a)] \right) d\mu
\]

\[
= \int_a^b p_1(x)p_2(x) \, d\mu. \tag{9}
\]

To calculate Sugeno integral, we consider the distribution function \( F \) given by

\[
F(\beta) = \mu([a, b] \cap \{ x | p_1(x)p_2(x) \geq \beta \})
\]

\[
= \mu([a, b] \cap \{ x | p_1(x) \geq \beta \}) \cdot \mu([a, b] \cap \{ x | p_2(x) \geq \beta \})
\]

\[
= \mu\left( [a, b] \cap \left\{ x \mid m2^{1-s}f(a) + \left( \frac{x-ma}{b-ma} \right)^s [f(b) - mf(a)] \geq \beta \right\} \right)
\]

\[
\mu\left( [a, b] \cap \left\{ x \mid m2^{1-s}g(a) + \left( \frac{x-ma}{b-ma} \right)^s [g(b) - mg(a)] \geq \beta \right\} \right)
\]

\[
= \mu\left( [a, b] \cap \left\{ x \mid x \geq (b-ma) \left( \frac{\beta - m2^{1-s}f(a)}{f(b) - mf(a)} \right)^{\frac{s}{2}} + ma \right\} \right)
\]

\[
\mu\left( [a, b] \cap \left\{ x \mid x \geq (b-ma) \left( \frac{\beta - m2^{1-s}g(a)}{g(b) - mg(a)} \right)^{\frac{s}{2}} + ma \right\} \right)
\]

\[
= \left( b-ma \right) - \left( b-ma \right) \left( \frac{\beta - m2^{1-s}f(a)}{f(b) - mf(a)} \right)^{\frac{s}{2}}
\]

\[
\left( b-ma \right) - \left( b-ma \right) \left( \frac{\beta - m2^{1-s}g(a)}{g(b) - mg(a)} \right)^{\frac{s}{2}}
\]

\[
\tag{10}
\]

Let \( F(\beta) = \beta \) and solution of (10) is given by (4). By Proposition 2.1 and Remark 2.1, we have

\[
\int_a^b fg \, d\mu \leq \min\{ \beta, b-a \}.
\]

\[
(9)
\]

\[
\]

**Remark 3.1.** Let \( s = m = 1 \), \((s, m) \in (0, 1]^2 \) and \( f, g : [a, b] \rightarrow [0, \infty) \) are convex functions such that \( f(b) > f(a) \) and \( g(b) > g(a) \).
Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Then

$$ (s) \int_a^b f g d\mu \leq \min\{\beta, b - a\}, $$

where $\beta$ is given as

$$ \left( b - a \left( 1 - \frac{\beta - f(a)}{f(b) - f(a)} \right) \right) \left( b - a \left( 1 - \frac{\beta - g(a)}{g(b) - g(a)} \right) \right) = \beta. \quad (11) $$

**Theorem 3.3.** Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Let $(s, m) \in (0, 1]^2$ and $f, g : [a, b] \to [0, \infty)$ are $(s, m)$-convex functions in second sense, such that $f(b) < mf(a)$ and $g(b) < mg(a)$ then

$$ (s) \int_a^b f g d\mu \leq \min\{\beta, b - a\}, $$

where $\beta$ is given by

$$ (b - ma)^2 \left( \frac{\beta - m2^{1-s} f(a)}{f(b) - mf(a)} \right)^{\frac{1}{s}} \left( \frac{\beta - m2^{1-s} g(a)}{g(b) - mg(a)} \right)^{\frac{1}{s}} \right) \right) \right) + (b - ma)(ma - a) \left( \frac{\beta - m2^{1-s} f(a)}{f(b) - mf(a)} \right)^{\frac{1}{s}} \right) \right) + (b - ma)(ma - a) \left( \frac{\beta - m2^{1-s} g(a)}{g(b) - mg(a)} \right)^{\frac{1}{s}} \right) + (ma - a)^2 = \beta. \quad (12) $$

**Proof.** Similar to the Theorem (3.2), consider the functions

$$ p_1(x) = m2^{1-s} f(a) + \left( \frac{x - ma}{b - ma} \right)^s [f(b) - mf(a)], \quad (13) $$

$$ p_2(x) = m2^{1-s} g(a) + \left( \frac{x - ma}{b - ma} \right)^s [g(b) - mg(a)]. \quad (14) $$
Now consider the distribution function $F$ given as

$$F(\beta) = \mu([a, b] \cap \{ x | \beta = \mu([a, b] \cap \{ x | p_1(x) p_2(x) \geq \beta \})$$

$$= \mu([a, b] \cap \{ x | p_1(x) \geq \beta \} \cup [a, b] \cap \{ x | p_2(x) \geq \beta \})$$

$$= \mu\left( [a, b] \cap \left\{ x | m 2^{1-s} f(a) + \left( \frac{x - ma}{b - ma} \right)^{\hat{s}} [f(b) - mf(a)] \geq \beta \right\} \right)$$

$$\mu\left( [a, b] \cap \left\{ x | m 2^{1-s} g(a) + \left( \frac{x - ma}{b - ma} \right)^{\hat{s}} [g(b) - mg(a)] \geq \beta \right\} \right)$$

$$= \mu\left( [a, b] \cap \left\{ x | x \leq (b - ma) \left( \frac{\beta - m 2^{1-s} f(a)}{f(b) - mf(a)} \right)^{\frac{1}{\hat{s}}} + ma \right\} \right).$$

$$\mu\left( [a, b] \cap \left\{ x | x \leq (b - ma) \left( \frac{\beta - m 2^{1-s} g(a)}{g(b) - mg(a)} \right)^{\frac{1}{\hat{s}}} + ma \right\} \right).$$

$$= \left( b - ma \right) \left( \frac{\beta - m 2^{1-s} f(a)}{f(b) - mf(a)} \right)^{\frac{1}{\hat{s}}} + ma - a.$$  \hspace{1cm} (15)

Let $F(\beta) = \beta$ and solution of (15) is given by (12). By Proposition 2.1 and Remark 2.1, we have

$$(s) \int_a^b f gd\mu \leq \min\{ \beta, b - a \}. \hspace{1cm} \square$$

**Remark 3.2.** Let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Let $s = 1, m = 1$ and $(s, m) \in (0, 1]^2$, $f, g : [a, b] \to [0, \infty)$ are convex functions such that $f(b) < f(a)$ and $g(b) < g(a)$. Then

$$(s) \int_a^b f gd\mu \leq \min\{ \beta, b - a \},$$

where $\beta$ is given as

$$(b - a)^2 \left( \frac{\beta - f(a)}{f(b) - f(a)} \right) \cdot \left( \frac{\beta - g(a)}{g(b) - g(a)} \right) = \beta. \hspace{1cm} (16)$$

**Remark 3.3.** If $f(b) = mf(a)$ and $g(b) = mg(a)$, from (13) and (14), we have $p_1(x) = m 2^{1-s}$ and $p_2(x) = m 2^{1-s}$ and Proposition 2.1, we have

$$(s) \int_a^b f gd\mu \leq \{ m^2 2^{2-2s} f(a) g(a), b - a \}. \hspace{1cm} (17)$$
Example 3.2. Consider the function $f(x) = x^{3/2}$ and $g(x) = x^{1/2}$ then $f(x)$, $g(x)$ are convex functions i.e. $f(x), g(x) \in K_{s,m}^{2}$, where $s = m = 1$. Let $\mu$ be the Lebesgue measure on $x = [1, 4]$. Thus $f(4) > mf(1)$ and $g(4) > mg(1)$. By Theorem 3.2 we have

$$2.4384 = (s) \int_{1}^{4} x^{2} d\mu \leq \min\{2.5302, 4 - 1\} = 2.5302.$$  \hspace{1cm} (18)

Example 3.3. Consider the function $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$, then $f(x)$, $g(x)$ are convex functions i.e. $f(x), g(x) \in K_{s,m}^{2}$, where $s = m = 1$. Let $\mu$ be the Lebesgue measure on $X = [1, 2]$. Thus $f(2) < mf(1)$ and $g(2) < mg(1)$. By Theorem 3.3 we have

$$0.3247 = (s) \int_{1}^{2} \frac{1}{x^4} d\mu \leq \min\{0.4802, 2 - 1\} = 0.4802.$$  \hspace{1cm} (19)

References

[1] Caballero, J. and Sadarangani, K., Chebyshev type inequality for Sugeno integrals, Fuzzy Sets Syst., 161(2010), 1480-1487.

[2] Agahi, H., Mesiar R. and Ouyang, Y., General Minkowski type inequalities for Sugeno integral, Fuzzy Sets Syst., 161(2010), 708-715.

[3] Li, D., Cheng, Y., Wang, X. and Zang, S., Barnes-Godunova-Levin type inequalities of Sugeno integral for an $(\alpha, m)$-concave function. J. Inequ. Appl., (2015).

[4] Roman-Flores, H., Flores-Franulic A. and Chalco-Cano, Y., A Convolution type inequality for fuzzy integrals, Appl. Math. Comput., 195(2008), 94-99.

[5] Roman-Flores, H., Flores-Franulic A. and Chalco-Cano, Y., A Jensen type inequality for fuzzy integrals Inform. Sci., 177(2007), 3192-3201.

[6] Flores-Franulic, A. and Roman-Flores, H., A Chebyshev type inequality for fuzzy integrals, Appl. Maths. Comp., 190(2007), 1178-1184.
[7] Gill, P., Pearce, C. and Peccaric, J., Hadamard’s types inequality for \( r \)-convex functions, J. Math. Anal.Appl., 215(1997), 461-470.

[8] Dubois, D., Prade, H. and Sabbadin, R., Qualitative decision theory with Sugeno integrals, Pro. of UAI, 98(1998), 121-128.

[9] Sarikaya, M. and Kiris, M., Some new inequalities of Hermite-Hadamard type for \( s \)-convex functions, Miskolc Math. Note 16(2015), 491-501.

[10] Ngoc, N., Vinh, N. and Hien, P., Integral inequalities of Hadamard type for \( r \)-convex functions. Int. Math. Forum., 4(2009), 1723-1728.

[11] Zabandan, G., Hermite-Hadamard type inequality for \( r \)-convex functions, J. Inequ. Appl., 215(2012), 1-8.

[12] Park, J., Some Hadamard’s type inequalities for co-ordinated \((s, m)\)-convex mapplings in the second sens, Far. East. J. Math. Sci.,15 (2011), 205-216.

[13] Hanson, M., On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl., 80(1981), 545-550.

[14] Ben-Israel, A. and Mond, B., What is invexity? J. Aust. Math. Soc. Ser. B, Appl. Math., 28(1986), 1-9.

[15] Latif, M. and Shoaib, M., Hermite-Hadamard type integral inequalities for differentiable \( m \)-preinvex and \((\alpha, m)\)-preinvex functions, J. Egypt. Math. Soc., 23(2015),236-241.

[16] Eftekhari, N., Some remarks on \((s, m)\)-convexity in second sense. J. Math. Ineq., 8(2014), 489-495.

[17] Vivas, M., Fejer type inequalities for \((s, m)\)-convex function in second sense, Appl. Math. Inf. Sci., 10(2016), 1689-1696.

[18] Du, T., Li, Y. and Yang, Z., A generalization of Simpson’s inequality via differentible mapping using extened \((s, m)\)-convex functions, Appl. Math. Comput., 293(2015), 358-369.
[19] Yang, Z., Li, Y. and Du, T., A generalization of Simpson type inequality via differentiable function using $(s, m)$-convex functions, Ital. J. Pure Appl. Math., 35(2015), 327-338.

[20] Ren, H., Wang, G. and Luo, L., Sandor type fuzzy inequality Based on the $(s, m)$-convex function in the second sense., Symmetry, 9(2017), 1-10.

[21] Kirmaci, U., Ozdemir, M. and Pecaric, J., Hadamard type inequalities for $s$-convex function, Appl. Maths. Comp., 193(2007), 26-35.

[22] Li, L., Hermite-Hadamard type fuzzy inequality based on $s$-convex function in the second sense, Mathematics Letters, 3(2017), 77-82.

[23] Pachpatte, D. and Shinde, K., Hermite-Hadamard type inequality for $r_1$-convex function and $r_2$-convex function using Sugeno integral, @FMI, 14(2017), 613-620.

[24] Sugeno, M., Theory of fuzzy integrals and its applications (Ph.D Thesis), Tokyo Institute of Technology, (1974).

[25] Wang, Z. and Klir, G., Fuzzy Measures Theory, Plenum press, New York, (1992).

[26] Wang, Z. and Klir, G., Generalized Measure Theory, Springer, New York, (2008).