Quantitative limit theorems and bootstrap approximations for empirical spectral projectors

Moritz Jirak∗ Martin Wahl†

Abstract

Given finite i.i.d. samples in a Hilbert space with zero mean and trace-class covariance operator Σ, the problem of recovering the spectral projectors of Σ naturally arises in many applications. In this paper, we consider the problem of finding distributional approximations of the spectral projectors of the empirical covariance operator ˆΣ, and offer a dimension-free framework where the complexity is characterized by the so-called relative rank of Σ. In this setting, novel quantitative limit theorems and bootstrap approximations are presented subject only to mild conditions in terms of moments and spectral decay. In many cases, these even improve upon existing results in a Gaussian setting.

1 Introduction

Let X be a random variable in a separable Hilbert space H with expectation zero and covariance operator Σ = EX ⊗ X. A fundamental problem in high-dimensional statistics and statistical learning is dimension reduction, that is one seeks to reduce the dimension of X, while keeping as much information as possible. Letting (λj) be the sequence of positive eigenvalues of Σ (in non-increasing order) and (uj) be a corresponding orthonormal system of eigenvectors, solutions to this problem are given by the projections PJX with spectral projector PJ =  j∈J uj ⊗ uj and index set J.

In statistical applications, the distribution of X and in particular its covariance structure are unknown. Instead, one often observes a sample X1, . . . , Xn of n independent copies of X, and the problem now consists of

∗Universität Wien, Austria. E-mail: moritz.jirak@univie.ac.at
†Humboldt-Universität zu Berlin, Germany. E-mail: martin.wahl@math.hu-berlin.de

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finding an estimator of $P_J$. The idea of PCA is to solve this problem by first estimating $\Sigma$ by the empirical covariance operator $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} X_i \otimes X_i$, and subsequently constructing the empirical counterpart $\hat{P}_J$ of $P_J$ based on $\hat{\Sigma}$ (see Section 2.2.1 for a precise definition). Hence, a key problem is to control and quantify the distance between $\hat{P}_J$ and $P_J$.

Over the past decades, an extensive body of literature has evolved around this problem, see e.g. [11], [23], [16], [41], [24] for some overviews. A traditional approach for studying the distance between $\hat{P}_J$ and $P_J$ is to control a norm measuring the distance between the empirical covariance operator $\hat{\Sigma}$ and the population covariance operator $\Sigma$. Once this has been established, one may then deduce bounds for $\hat{P}_J - P_J$ by inequalities such as Davis-Kahan’s inequality, see for instance [17], [48], and [7], [21] for some recent results and extensions. However, for a more precise statistical analysis, fluctuation results like limit theorems or bootstrap approximations are much more desirable.

The more recent works by [27], [28], [29] (and related) are of particular interest here. Among other things, they provide the leading order of $\mathbb{E}\|\hat{P}_J - P_J\|_2^2$ and a precise, non-asymptotic analysis of distributional approximations of $\|\hat{P}_J - P_J\|_2^2$ in terms of Berry-Esseen type bounds, given a Gaussian setup. Some extensions and related questions are discussed in [31], [25], [26]. However, as noted in [36], these results have some limitations, and bootstrap approximations may be more desirable and flexible. Again, in a purely Gaussian setup, [36] succeeds in presenting a bootstrap procedure with accompanying bounds that alleviates some of the problems attached to limit distribution for inferential purposes. Let us point out though that, from a mathematical point of view, the results of [29] and [36] are somewhat complementary. More precisely, there are scenarios where the bound of the bootstrap approximation of Theorem 2.1 in [36] fails (meaning that it only yields a triviality), whereas the bound in Theorem 6 of [29] does not, and vice versa, see Section 5 for some examples and further discussions. The topic of limit theorems and bootstrap approximations has also been broadly investigated for eigenvalues and related quantities, see for instance [8], [47], [32], [18], [30].

The aim of this work is to provide quantitative bounds for both distributional (e.g. CLTs) and bootstrap approximations, subject to comparatively mild conditions in terms of moments and spectral decay. Concerning the latter, our results display a certain invariance, being largely unaffected by polynomial, exponential (or even faster) decay.

To be more specific, let us briefly discuss the two main previous ap-
proaches used in [29] and [36]. The Berry-Essen type bounds of [29] rely on classical perturbation (Neumann) series, an application of the isoperimetric inequality for (Hilbert space-valued) Gaussian random variables, and the classical Berry-Esseen bound for independent real-valued random variables. As a consequence, Theorem 6 in [29] yields a Berry-Essen type bound for distributional closeness of $\|\hat{P}_J - P_J\|_2^2$ (appropriately normalized) and a standard Gaussian random variable. In contrast, [36] derive novel comparison and anti-concentration results for certain quadratic Gaussian forms, allowing them to directly compare the random variables in question, by-passing any argument involving a limit distribution. They do, however, rely on similar perturbation and concentration results as in [29]. Finally, let us mention that both only consider the case of single spectral projectors (meaning that $J$ is the index set of one eigenvalue), and heavily rely on the assumption of Gaussianity for $X$.

Removing (Sub)-Gaussianity also means that the isoperimetric inequality is no longer available, which, however, is the key ingredient of the above approach. Therefore, we opted for a different route, which is more inspired by the relative rank approach developed in [19], [20], [21], [39], [45], and consists of the following three key ingredients:

- **Relative perturbation bounds**: Employing the aforementioned concept of the relative rank, we establish general perturbation results for $\|\hat{P}_J - P_J\|_2$. These results may be of independent interest.

- **Fuk-Nagaev type inequalities**: We formulate general Fuk-Nagaev type inequalities for independent sums of random operators that allow us to replace the assumption of (sub-)Gaussianity by weaker moment assumptions. Together with the relative perturbation bounds, this leads to approximations for $\|\hat{P}_J - P_J\|_2$ (valid with high probability) in regimes that appeared to be out of reach by previous approaches based on absolute perturbation theory.

- **Bounds for Wasserstein and uniform distance for Hilbert space-valued random variables**: The last key point are general bounds for differences of probability measures on Hilbert spaces. Together with the above expansions and concentration inequalities, these permit us to obtain novel quantitative limit theorems and bootstrap approximations. Moment conditions are expressed in terms of the Karhunen–Loève coefficients of $X$.

To briefly demonstrate the flavour of our results, let us present two examples where the eigenvalues $\lambda_j$ of $\Sigma$ have a rather opposing behaviour. The
first result deals with polynomially decaying eigenvalues, a common situation in statistics and machine learning (cf. [3], [13], [15]). For single projectors ($\mathcal{J} = \{ J \}$), subject to mild moment conditions for the Karhunen–Loève coefficients of $X$, we show that

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( n \left\| \hat{P}_J - P_J \right\|_2^2 \leq x \right) - \mathbb{P} \left( \left\| L_{\mathcal{J}} Z \right\|_2^2 \leq x \right) \right| \lesssim \frac{J}{n^{1/2}} (\log J \log n)^{3/2}$$

for all $J \geq 1$, where $L_{\mathcal{J}} Z$ is an appropriate Gaussian random variable. Key points here are the mild moment conditions and the explicit dependence on $J$, which is optimal up to log-factors. This result, among others, is novel even for the Gaussian case. For a precise statement, see Section 5.1.2.

The second case deals with a pervasive factor like structure, a popular model in econometrics and finance (cf. [11], [1], [43]). In this case, again subject to mild moment conditions for the Karhunen–Loève coefficients of $X$, we have

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( n \left\| \hat{P}_J - P_J \right\|_2^2 \leq x \right) - \mathbb{P} \left( \left\| L_{\mathcal{J}} Z \right\|_2^2 \leq x \right) \right| \lesssim \frac{J^3}{n^{1/2}} (\log n)^{3/2} + J^{5/2}$$

for all $J \geq 1$, where $L_{\mathcal{J}} Z$ is an appropriate Gaussian random variable. As before, we are not aware of a similar result even in a Gaussian setup. For a precise statement, see Section 5.2.1.

This work is structured as follows. We first introduce some notation and establish a number of preliminary results in Sections 2.1–2.3. Bounds for quantitative limit theorems are then presented in Section 3 for both the uniform and the Wasserstein distance, whereas Section 4 contains accompanying results for a suitable bootstrap approximation. Finally, we discuss some key models from the literature and provide a comparison to previous results in Section 5. Proofs are given in the remaining Section 6.

## 2 Preliminaries

### 2.1 Basic notation

We write $\lesssim$, $\gtrsim$ and $\asymp$ to denote (two-sided) inequalities involving a multiplicative constant. If the involved constant depends on some parameters, say $p$, then we write $\lesssim_p$, $\gtrsim_p$ and $\asymp_p$. For $a, b \in \mathbb{R}$, we write $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Given a subset $\mathcal{J}$ of the index set (in most cases $\mathbb{N}$), we denote with $\mathcal{J}^c$ its complement in the index set. Given a (real) Hilbert space $\mathcal{H}$, we always write $\| \cdot \| = \| \cdot \|_{\mathcal{H}}$ for the corresponding norm. For a
bounded linear operator $A$ and $q \in [1, \infty]$, we denote with $\|A\|_q$ the Schatten $q$-norm. In particular, $\|A\|_2$ denotes the Hilbert-Schmidt norm, $\|A\|_1$ the nuclear norm and $\|A\|_\infty$ the operator norm. For a self-adjoint compact operator $A$ mapping $\mathcal{H}$ into itself, we write $|A|$ for the absolute value of $A$ and $|A|^{1/2}$ for the positive self-adjoint square-root of $|A|$. For a random variable $X$, we write $\overline{X} = X - \mathbb{E}X$.

2.2 Tools from perturbation theory

In this section, we present our underlying perturbation bounds. Instead of applying standard perturbation theory, we make use of recent improvements from [21], [45] adapted for our purposes. Proofs are presented in Section 2.2.

2.2.1 Further notation

Throughout this section, $\Sigma$ denotes a positive self-adjoint compact operator on the separable Hilbert space $\mathcal{H}$ (in applications, it will be the covariance operator of the random vector $X$ with values in $\mathcal{H}$). By the spectral theorem there exists a sequence $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ of positive eigenvalues (which is either finite or converges to zero) together with an orthonormal system of eigenvectors $u_1, u_2, \ldots$ such that $\Sigma$ has spectral representation

$$\Sigma = \sum_{j \geq 1} \lambda_j P_j,$$

with rank-one projectors $P_j = u_j \otimes u_j$, where $(u \otimes v)x = (v, x)u$, $x \in \mathcal{H}$. Without loss of generality we shall assume that the eigenvectors $u_1, u_2, \ldots$ form an orthonormal basis of $\mathcal{H}$ such that $\sum_{j \geq 1} P_j = I$. For $1 \leq j_1 \leq j_2 \leq \infty$, we consider an interval of the form $J = \{j_1, \ldots, j_2\}$. We write

$$P_J = \sum_{j \in J} P_j, \quad P_{J^c} = \sum_{k \in J^c} P_k$$

for the orthogonal projection on the direct sum of the eigenspaces of $\Sigma$ corresponding to the eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_2}$, and onto its orthogonal complement. Moreover, let

$$R_{J^c} = \sum_{k < j_1} \frac{P_k}{\lambda_k - \lambda_{j_1}} + \sum_{k > j_2} \frac{P_k}{\lambda_k - \lambda_{j_2}}$$

be the reduced ‘outer’ resolvent of $J$. Finally, let

$$g_J = \min(\lambda_{j_1-1} - \lambda_{j_1}, \lambda_{j_2} - \lambda_{j_2+1})$$
be the gap between the eigenvalues with indices in \( J \) and the eigenvalues with indices not in \( J \). If \( J = \{ j \} \) is a singleton, then we also write \( g_j \) instead of \( g_J \).

Let \( \hat{\Sigma} \) be another positive self-adjoint compact operator on \( H \) (in applications, it will be the empirical covariance operator). We consider \( \hat{\Sigma} \) as a perturbed version of \( \Sigma \) and write \( E = \hat{\Sigma} - \Sigma \) for the (additive) perturbation. Again, there exists a sequence \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq 0 \) of eigenvalues together with an orthonormal system of eigenvectors \( \hat{u}_1, \hat{u}_2, \ldots \) such that we can write

\[
\hat{\Sigma} = \sum_{j \geq 1} \hat{\lambda}_j \hat{P}_j
\]

with \( \hat{P}_j = \hat{u}_j \otimes \hat{u}_j \). We shall also assume that the eigenvectors \( \hat{u}_1, \hat{u}_2, \ldots \) form an orthonormal basis of \( H \) such that \( \sum_{j \geq 1} \hat{P}_j = I \). We write

\[
\hat{P}_J = \sum_{j \in J} \hat{P}_j, \quad \hat{P}_{J^c} = \sum_{k \in J^c} \hat{P}_k
\]

for the orthogonal projection on the direct sum of the eigenspaces of \( \hat{\Sigma} \) corresponding to the eigenvalues \( \hat{\lambda}_1, \ldots, \hat{\lambda}_2 \), and onto its orthogonal complement.

### 2.2.2 Main perturbation bounds

Let us now present our main perturbation bound. The following quantity will play a crucial role

\[
\delta_J = \delta_J(E) := \left\| \left( |R_{J^c}|^{1/2} + g_J^{-1/2} P_J \right) E \left( |R_{J^c}|^{1/2} + g_J^{-1/2} P_J \right) \right\|_{\infty}, \tag{2.1}
\]

where

\[
|R_{J^c}|^{1/2} = \sum_{k<j_1} \frac{P_k}{|\lambda_k - \lambda_{j_1}|^{1/2}} + \sum_{k>j_2} \frac{P_k}{|\lambda_k - \lambda_{j_2}|^{1/2}}
\]

is the positive self-adjoint square-root of \( |R_{J^c}| \). In the special case that \( J = \{ j \} \) is a singleton, it has been introduced in [45]. Moreover, for a Hilbert-Schmidt operator \( A \) on \( H \) we define

\[
L_J A = \sum_{j \in J} \sum_{k \in J^c} \frac{1}{\lambda_j - \lambda_k} (P_k A P_j + P_j A P_k).
\tag{2.2}
\]

The following result presents a linear perturbation expansion with remainder expressed in terms of \( \delta_J(E) \).
Proposition 1. We have

\[ \|P_J - \hat{P}_J\|_2 \leq 4\sqrt{2} \min(|J|, |J^c|)^{1/2} \delta_J \]  

(2.3)

and

\[ \|\hat{P}_J - P_J - L_J E\|_2 \leq 20\sqrt{2} \min(|J|, |J^c|) \delta_J^2. \]  

(2.4)

Using the identities \( P_J - \hat{P}_J = \hat{P}_{J^c} - P_{J^c} \) and \( L_J E = -L_{J^c} E \), it is possible to replace \( \delta_J \) by \( \min(\delta_J, \delta_{J^c}) \) in Proposition 1. We implicitly assume that \( J \) is chosen such that \( \delta_J = \min(\delta_J, \delta_{J^c}) \) in what follows.

Clearly, we can bound \( \delta_J \leq \|E\|_\infty / g_J \), in which case the inequality (2.3) turns into a standard perturbation bound, cf. Lemma 2 in [21] or Equation (5.17) in [39].

The dependence on \( \min(|J|, |J^c|) \) in (2.4) can be further improved, by using the first inequality in Lemma 11 below (instead of the second one). Since this leads to additional quantities that have to be controlled, such improvements are not pursued here.

Corollary 1. We have

\[ \|P_J - \hat{P}_J\|^2_2 - \|L_J E\|^2_2 \lesssim \min(|J|, |J^c|)^{3/2} \delta_J^3 + \min(|J|, |J^c|)^2 \delta_J^4. \]

For our next consequence (used in the bootstrap world), let \( \hat{\Sigma} \) be a third positive self-adjoint compact operator on \( \mathcal{H} \). Similarly as above we will write \( \hat{E} = \hat{\Sigma} - \Sigma \) and \( \hat{P}_J \) for the orthogonal projection on the direct sum of the eigenspaces of \( \hat{\Sigma} \) corresponding to the eigenvalues with indices in \( J \).

Corollary 2. We have

\[ \|P_J - \hat{P}_J\|^2_2 - \|L_J (\hat{E} - E)\|^2_2 \]

\[ \lesssim \min(|J|, |J^c|)^{3/2} \delta_J^3 (E) + \delta_J^3 (\hat{E})) + \min(|J|, |J^c|)^2 (\delta_J^4 (E) + \delta_J^4 (\hat{E})). \]

2.3 Tools from probability

2.3.1 Metrics for convergence of laws

In this section, we provide several upper bounds for the uniform metric and the Wasserstein metric. The propositions are proved in Section 6.2. For two real-valued random variables \( X, Y \), the uniform (or Kolmogorov) metric is defined by

\[ U(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|. \]
Moreover, for two (induced) probability measures $P_X, P_Y$, let $L(P_X, P_Y)$ be the set of probability with marginals $P_X, P_Y$. Then the Wasserstein metric (of order $p$) is defined as the minimal coupling in $L^p$-distance, that is,

$$W^p_p(X, Y) = \inf \left\{ \int \|x - y\|^p P(dx, dy) : P \in L(P_X, P_Y) \right\}.$$ 

Since we only consider the case $p = 1$, we abbreviate $W = W_1$.

**Setting 1.** Let $T$ be a real-valued random variable. Moreover, let $Y_1, \ldots, Y_n$ be independent random variables taking values in a separable Hilbert space $\mathcal{H}$ satisfying $E Y_i = 0$ and $E \|Y_i\|^q < \infty$ for some $q > 2$. Set

$$S = n^{-1} \sum_{i=1}^n \|Y_i\|^2 \quad \text{and} \quad \Psi = \frac{1}{n} \sum_{i=1}^n E Y_i \otimes Y_i.$$ 

Let $\lambda_1(\Psi) \geq \lambda_2(\Psi) \geq \cdots > 0$ be the positive eigenvalues of $\Psi$ (each repeated a number of times equal to its multiplicity). Moreover, let $Z$ be a centered Gaussian random variable in $\mathcal{H}$ with covariance operator $\Psi$, that is $E Z = 0$ and $E Z \otimes Z = n^{-1} \sum_{i=1}^n E Y_i \otimes Y_i$.

In applications $T$ and $S$ will correspond to (scaled or bootstrap versions of) $\|\hat{P}_J - P_J\|^2_2$ and its approximation $\|L_J E\|^2_2$, respectively. Our first result deals with the 1-Wasserstein distance between $T$ and $\|Z\|^2$.

**Proposition 2.** Consider Setting 1 with $q \in (2, 3]$. Assume $|T| \leq C_T$ almost surely, and $\sum_{i=1}^n E \|Y_i\|^r \leq n^{r/2}$ for $r \in \{2, q\}$. Then, for all $s > 0$ and $u > 0$,

$$W(T, \|Z\|^2) \lesssim_q, s \left( n^{-q/2} \sum_{i=1}^n E \|Y_i\|^q \right)^{\frac{q-2}{2(q-3)}} + C_T P(|T - S| > u) + u + \frac{1}{C_T^s}.$$ 

In connection to the uniform metric, the following quantities and relations will be crucial:

$$A := E \|Z\|^2 = ES = \|\Psi\|_1,$$

$$B := E^{1/2}(\|Z\|^2 - \|\Psi\|_1) = \sqrt{2} \|\Psi\|_2,$$

$$C := E^{1/3}(\|Z\|^2 - \|\Psi\|_1)^3 = 2 \|\Psi\|_3. \quad (2.5)$$

where we also used that $E(G^2 - 1)^2 = 2$ and $E(G^2 - 1)^3 = 8$ for $G \sim \mathcal{N}(0, 1)$. By monotonicity of the Schatten norms, we have $C \lesssim B \lesssim A$. 


Proposition 3. Consider Setting 1 with $q = 3$. Then, for all $u > 0$,
\[
U(T, \|Z\|^2) \lesssim_p n^{-3/5} \left( \sum_{i=1}^{n} \frac{E\|Y_i\|^3}{\lambda_{3/4}^1(\Psi)} \right)^{2/5} + P(|T - S| > u) + \frac{u}{\sqrt{\lambda_{1,2}(\Psi)}},
\]
as well as
\[
U(T, \|Z\|^2) \lesssim_p \left( n^{-1/2} \lambda_0^{-3}(\Psi) \frac{(AC)^3}{B^3} \right)^{1+1/10} + n^{-1/2} \lambda_0^{-1}(\Psi) \frac{(A^2C)^3}{B^3} + P(|T - S| > u) + \frac{u}{\sqrt{\lambda_{1,2}(\Psi)}},
\]
with
\[
\lambda_{1,j}(\Psi) = \prod_{i=1}^{j} \lambda_i(\Psi) \quad \text{for} \; j \geq 1.
\] (2.6)

Let $G \sim \mathcal{N}(0,1)$, Then, using invariance properties of the uniform metric, Lemma 16 below and (2.5), we get
\[
U\left(\frac{T - A}{B}, G\right) \lesssim_p \left( \frac{C}{B} \right)^3 + \text{bound of Proposition 3}.
\]
However, in this case one may dispose of any conditions on the eigenvalues of $\Psi$ altogether, as is demonstrated by our next result below.

Proposition 4. Consider Setting 1 with $q \in (2, 3]$. Then, for all $u > 0$,
\[
U\left(\frac{T - A}{B}, G\right) \lesssim_q n^{-q/8} \left( \frac{A^2}{B^3} \sum_{i=1}^{n} E\|Y_i\|^{q/2} \right)^{1/4} + \left( \frac{C}{B} \right)^3 + P(|T - S| > u) + \frac{u}{B}.
\]

2.3.2 Concentration inequalities

In this section, we recall several useful concentration inequalities that we will need in the proofs below. Additional results are given in Section 6.3. We will make frequent use of the classical Fuk-Nagaev inequality for real-valued random variables (cf. [35]).

Lemma 1. Let $Z_1, \ldots, Z_n$ be independent real-valued random variables. Suppose that $EZ_i = 0$ and $E|Z_i|^p < \infty$ for some $p > 2$ and all $i = 1, \ldots, n$. Then, it holds that
\[
P\left(\left| \sum_{i=1}^{n} Z_i \right| \geq x \right) \leq \left( 2 + \frac{2}{p} \right) \frac{\mu_{n,p} x}{x^p} + 2 \exp\left( - \frac{a_p x^2}{\mu_{n,2}} \right),
\]
with $a_p = 2e^{-p(p + 2)^{-2}}$ and $\mu_{n,p} = \sum_{i=1}^{n} E|Z_i|^p$. 

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In particular, if $Z_1, \ldots, Z_n$ are independent copies of a real-valued random variable $Z$ with $\mathbb{E}Z = 0$ and $\mathbb{E}|Z|^p < \infty$, then there is a constant $C > 0$ such that

$$
P\left(\left|\sum_{i=1}^{n} Z_i\right| \geq C(\mathbb{E}|Z|^p)^{1/p} \sqrt{n \log n}\right) \leq n^{1-p/2}. \tag{2.7}$$

We will also apply Banach space versions of the Fuk-Nagaev inequality. The following general result is from [10].

**Lemma 2 ([10]).** Let $(B, \|\cdot\|)$ be a real separable Banach space with dual $B^*$ and let $B^*_1$ be the unit ball of $B^*$. Let $Z_1, \ldots, Z_n$ be independent $B$-valued random variables with mean zero such that for some $s > 2$, $\mathbb{E}\|Z_i\|^s < \infty$, $1 \leq i \leq n$. Then we have for $0 < \nu \leq 1$, $\delta > 0$ and any $t > 0$,

$$
P\left(\left\|\sum_{i=1}^{n} Z_i\right\| \geq (1+\nu)\mathbb{E}\left\|\sum_{i=1}^{n} Z_i\right\| + t\right)
\leq C \sum_{i=1}^{n} \mathbb{E}\|Z_i\|^s / t^s + \exp\left(-\frac{t^2}{(2 + \delta)\varpi_n}\right),$$

where $\varpi_n = \sup\{\sum_{i=1}^{n} \mathbb{E}f^2(Z_i) : f \in B^*_1\}$ and $C$ is a positive constant depending on $\nu$, $\delta$ and $s$.

Applying Lemma 2 to the Hilbert space of all Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, we get the following corollary (see also Lemma 1 in [21]).

**Corollary 3.** Let $Y = \sum_{j \geq 1} \vartheta_j^{1/2} \zeta_j u_j$ be a random variable taking values in a separable Hilbert space $\mathcal{H}$, where $\vartheta = (\vartheta_1, \vartheta_2, \ldots)$ is a sequence of positive numbers with $\|\vartheta\|_1 := \sum_{j \geq 1} \vartheta_j < \infty$, $u_1, u_2, \ldots$ is an orthonormal system in $\mathcal{H}$, and $\zeta_1, \zeta_2, \ldots$ is a sequence of centered random variables satisfying $\sup_{j \geq 1} \mathbb{E}|\zeta_j|^{2p} \leq C_1$ for some $p > 2$ and $C_1 > 0$. Let $Y_1, \ldots, Y_n$ be $n$ independent copies of $Y$. Then, we have

$$
P\left(\frac{1}{n} \sum_{i=1}^{n} (Y_i \otimes Y_i - \mathbb{E}Y_i \otimes Y_i)\right)_2 > \frac{C_2 t \|\vartheta\|_1}{\sqrt{n}} \leq \frac{n^{1-p/2}}{t^p} + e^{-t^2}$$

for all $t \geq 1$, where $C_2 > 0$ is a constant depending only on $C_1$ and $p$.

We will also need a concentration inequality for the nuclear norm.
Corollary 4. In the setting of Corollary 3, we have
\[
P\left(\left\| \frac{1}{n} \sum_{i=1}^{n} (Y_i \otimes Y_i - EY_i \otimes Y_i) \right\|_{1} \geq C_2 t \sqrt{n} \right) \leq \frac{n^{1-p/2}}{t^p} \|\vartheta\|_p^p + e^{-t^2/\omega_n^2},
\]
for all \( t \geq \|E(Y \otimes Y - \Sigma)^2\|_1 \), where \( \omega_n^2 \geq \sup \|S\|_{\infty} \leq 1 \) and \( C_2 > 0 \) is a constant depending only on \( C_1 \) and \( p \).

It is also possible to apply Lemma 2 to the operator norm. Yet, since the involved expectation term is difficult to bound in this case, we proceed differently and apply techniques from [34] instead.

Lemma 3. In the setting of Corollary 3, assume additionally that \( \|E(Y \otimes Y - EY \otimes Y)^2\|_\infty = 1 \). Then we have
\[
P\left(\left\| \frac{1}{n} \sum_{i=1}^{n} (Y_i \otimes Y_i - EY_i \otimes Y_i) \right\|_{\infty} \geq C_2 t \sqrt{n} \right) \leq C_2 n^{1-p/2} \|\vartheta\|_p^p + \|\vartheta\|_2^2 e^{-t^2},
\]
for all \( t \geq 1 \), where \( C_2 > 0 \) is a constant depending only on \( C_1 \) and \( p \).

Lemma 3 is not optimal in terms of \( t \) but sufficient for the choice \( t \asymp \sqrt{\log n} \).

An important special case is given if the expansion for \( Y \) in Corollary 3 coincides with the Karhunen-Loève expansion. For this, suppose that \( Y \) is centered and strongly square-integrable, meaning that \( EY = 0 \) and \( E\|Y\|^2 < \infty \). Let \( \Sigma = EY \otimes Y \) be the covariance operator of \( Y \), which is a positive, self-adjoint trace class operator, see e.g. [17, Theorem 7.2.5]. Let \( Y_1, \ldots, Y_n \) be independent copies of \( Y \) and let \( \hat{\Sigma} = n^{-1} \sum_{i=1}^{n} Y_i \otimes Y_i \) be the empirical covariance operator. Let \( \lambda_1, \lambda_2, \ldots \) and \( u_1, u_2, \ldots \) be the eigenvalues and eigenvectors of \( \Sigma \) as introduced in Section 2.1. Then we can write \( Y = \sum_{j \geq 1} \lambda_j^{1/2} \eta_j u_j \), where \( \eta_j = \lambda_j^{-1/2} \langle u_j, Y \rangle \) are the Karhunen-Loève coefficients of \( Y \). By construction, the \( \eta_j \) are centered, uncorrelated and satisfy \( E\eta_j^2 = 1 \). Now, if the Karhunen-Loève coefficients satisfy \( \sup_{j \geq 1} |E\eta_j|^{2p} \leq C_1 \) for some \( p > 2 \) and \( C_1 > 0 \), then Corollary 3 implies
\[
P\left(\|\hat{\Sigma} - \Sigma\|_2 > C_2 \text{tr}(\Sigma)t/\sqrt{n} \right) \leq n^{1-p/2} t^{-p} + e^{-t^2}
\]
for all \( t \geq 1 \), while Corollary 4 implies
\[
P\left(\|\hat{\Sigma} - \Sigma\|_1 \geq C_2 t/\sqrt{n} \right) \leq n^{1-p/2} t^{-p} \text{tr}^p(\Sigma) + e^{-t^2/\omega_n^2}
\]
(2.8)
for all \( t \geq \|(E(Y \otimes Y - \Sigma)^2)^{1/2}\|_1 \), where \( \sigma_n^2 \geq \sup_{\|S\|_\infty \leq 1} \text{tr}^2(S(Y \otimes Y - \Sigma)) \).

If additionally \( \|E(Y \otimes Y - \Sigma)^2\|_\infty = 1 \) holds, then Lemma 3 implies that, for all \( t \geq 1 \),

\[
P(\|\hat{\Sigma} - \Sigma\|_\infty \geq C_2t/\sqrt{n}) \leq n^{1-p/2}t^p \text{tr}^p(\Sigma) + \text{tr}^2(\Sigma)e^{-t^2}. \tag{2.9}
\]

Finally, in order to apply the above concentration inequalities, we will make frequent use of the following moment computations based on properties of the Karhunen-Loève coefficients.

**Lemma 4.** Let \( Y = \sum_{j \geq 1} \vartheta_j^{1/2} \zeta_j u_j \) be an \( \mathcal{H} \)-valued random variable, where \( \vartheta = (\vartheta_1, \vartheta_2, \ldots) \) is a sequence of positive numbers with \( \|\vartheta\|_1 < \infty, u_1, u_2, \ldots \) is an orthonormal system in \( \mathcal{H} \), and \( \zeta_1, \zeta_2, \ldots \) is a sequence of centered random variables satisfying \( \sup_{j \geq 1} E|\zeta_j|^{2p} \leq C \) for some \( p \geq 2 \) and \( C > 0 \). Then

(i) \( E\|Y\|^{2r} \leq C^{r/p}\|\vartheta\|_1^r \) for all \( 1 \leq r \leq p \).

(ii) \( \text{tr}(E(Y \otimes Y - EY \otimes Y)^2) \leq C^{2/p}\|\vartheta\|_1^2 \).

Suppose additionally that the \( \zeta_j \) are uncorrelated and satisfy \( E\zeta_j \zeta_k^2 \zeta_s = 0 \) for all \( j, k, s \) such that \( j \neq s \). Then

(iii) \( \|E(Y \otimes Y - EY \otimes Y)^2\|_\infty \leq C^{2/p}\|\vartheta\|_1\|\vartheta\|_1 \).

(iv) \( \|(E(Y \otimes Y - EY \otimes Y)^2)^{1/2}\|_1 \leq C^{1/p}(\sum_{j \geq 1} \vartheta_j^{1/2})\|\vartheta\|_1^{1/2} \).

In Claim (i), it suffices to assume that \( p \geq 1 \).

## 3 Quantitative limit theorems

### 3.1 Assumptions and main quantities

Throughout this section, let \( X \) be a random variable taking values in \( \mathcal{H} \). We suppose that \( X \) is centered and strongly square-integrable, meaning that \( E X = 0 \) and \( E\|X\|^2 < \infty \). Let \( \Sigma = EX \otimes X \) be the covariance operator of \( X \), which is a positive self-adjoint trace class operator. Let \( X_1, \ldots, X_n \) be independent copies of \( X \) and let

\[
\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n X_i \otimes X_i
\]

be the empirical covariance operator.

Our main assumptions are expressed in terms of the Karhunen-Loève coefficients of \( X \), defined by \( \eta_j = \lambda_j^{-1/2} \langle X, u_j \rangle, \ j \geq 1 \).
**Assumption 1.** Suppose that for some \( p > 2 \)

\[
\sup_{j \geq 1} \mathbb{E}|\eta_j|^{2p} \leq C_\eta.
\]

The actual condition on the number of moments \( p \) will be mild, and depends on the desired rate of convergence, we refer to our results for more details. Apart from a relative rank condition, Assumption 1 is essentially all we need. In order to simplify the bounds (and proofs), we also demand a non degeneracy condition.

**Assumption 2.** There is a constant \( c_\eta > 0 \), such that for every \( j \neq k \),

\[
\mathbb{E}\eta_j^2 \eta_k^2 \geq c_\eta.
\]

In the special case of bootstrap approximations, the situation is more complicated, and thus our next condition is more restrictive. It allows us to explicitly compute moments of certain, rather complicated random variables in connection with some of our bounds, see Lemma 7 below for more details.

**Assumption 3** (\( m \)-th cumulant uncorrelatedness). If any of the indices \( i_1, \ldots, i_m \in \mathbb{N} \) is unequal to any of the others, then

\[
\mathbb{E}\eta_{i_1} \eta_{i_2} \cdots \eta_{i_m} = 0.
\]

Let us point out that both Assumptions 2 and 3 are trivially true if the sequence \((\eta_k)\) is independent. Apart from results concerning the bootstrap, Assumption 3 will be particularly convenient for discussing our examples in Section 5, allowing us in particular to relate the variance-type term \( \sigma_J^2 \) (cf. Definition 1 below) to higher order cumulants. This leads to very simple and explicit bounds, mirroring previous Gaussian results, and is in line with our recent findings in [20], where it is shown that moment conditions alone are not enough to get Gaussian type behaviour in general in this context, and an entirely different behaviour is possible. Also note that related assumptions, like the \( L_1 \rightarrow L_2 \) condition (see for instance [33], [46]), have already been used in the literature for such a purpose.

**Definition 1.** For \( J = \{1, \ldots, j_2\} \), we define

\[
\sigma_J = \sigma_J(\Sigma) = \left\| \mathbb{E}\left( (|R_J|^{1/2} + g_{J_e}^{-1/2}P_{J_e})X \otimes X (|R_J|^{1/2} + g_{J_e}^{-1/2}P_{J_e}) \right)^2 \right\|_\infty^{1/2}.
\]

On the other hand, for \( J = \{j_1, \ldots, j_2\} \) with \( j_1 > 1 \), we define

\[
\sigma_J = \sigma_J(\Sigma) = \left\| \mathbb{E}\left( (|R_{J_e}|^{1/2} + g_{J_e}^{-1/2}P_{J_e})X \otimes X (|R_{J_e}|^{1/2} + g_{J_e}^{-1/2}P_{J_e}) \right)^2 \right\|_\infty^{1/2}.
\]
The quantity $\sigma_J$ plays an important role in the analysis of $\delta_J$ (resp. $\delta_{J^c}$) defined in (2.1). The size of $\sigma_J$ can be characterized by the so-called relative ranks defined as follows.

**Definition 2.** For $J = \{1, \ldots, j\}$, we define

$$r_J = r_J(\Sigma) = \sum_{k \leq j} \frac{\lambda_k}{\lambda_k - \lambda_{j+1}} + \frac{1}{\lambda_{j+1}} \sum_{k>j} \lambda_k.$$ 

On the other hand, for $J = \{j_1, \ldots, j\}$ with $j_1 > 1$, we define

$$r_J = r_J(\Sigma) = \sum_{k<j} \frac{\lambda_k}{\lambda_k - \lambda_{j_1}} + \sum_{k>j} \frac{\lambda_k}{\lambda_{j_2} - \lambda_k} + \frac{1}{g_J} \sum_{k \in J} \lambda_k.$$ 

For ease of exposition, we distinguish the above two cases rather than defining the universal quantity $\min(\sigma_J, \sigma_{J^c})$ and $\min(r_J, r_{J^c})$, respectively. Note that in the special case of $J = \{j\}$, our definition coincides with the notion of the relative rank given in [20]. In the general case, we use a slightly weaker version of [21].

The quantities $r_J$ and $\sigma_J$ are related as follows.

**Lemma 5.** Suppose that Assumption 1 holds. Then

$$\sigma_J \lesssim r_J.$$ 

If additionally Assumptions 2 and 3 hold with $m = 4$, then

$$\sigma_J^2 \gtrsim \frac{\lambda_{j_2}}{\lambda_{j_2} - \lambda_{j+1}} r_J = \frac{\lambda_{j_2}}{g_J} r_J \quad \text{if} \quad J = \{1, \ldots, j\},$$ 

and

$$\sigma_J^2 \gtrsim \max \left( \frac{\lambda_{j_1}-1}{\lambda_{j_1-1} - \lambda_{j_1}}, \frac{\lambda_{j_1}}{g_J} \right) r_J \quad \text{if} \quad J = \{j_1, \ldots, j\} \quad \text{with} \quad j_1 > 1.$$ 

**Proof of Lemma 5.** If $J = \{j_1, \ldots, j\}$ with $j_1 > 1$, then Lemma 5 follows from Lemma 4 applied to the transformed data $X' = (|R_{J^c}|^{1/2} + g_J^{-1/2} P_J) X$. The main observation is that $X'$ has the same Karhunen-Loève coefficients as $X$ and thus satisfies Assumption 1 with the same constant $C_\eta$. Moreover, the eigenvalues of the covariance operator $\Sigma' = E X' \otimes X'$ are transformed in such a way that the eigenvalues are $\lambda_k/(\lambda_k - \lambda_{j_1})$ for $k < j_1$, $\lambda_k/g_J$ for $k \in J$, and $\lambda_k/(\lambda_{j_2} - \lambda_k)$ for $k > j_2$. Hence, the two claims follow from Lemma 4 (ii) and (iii), noting that the bound in Lemma 4 (iii) is sharp under the additional assumptions. If $J = \{1, \ldots, j_1\}$, then the claim follows similarly by changing the role of $J$ and $J^c$. \qed
The relative rank $r_J$ and $\sigma_J^2$ allow to characterize the behavior of $\delta_J$, given in (2.1).

**Lemma 6.** If Assumption 1 holds, then

$$
\mathbb{P} \left( \delta_J(E) > C \sqrt{\frac{\sigma_J^2 \log n}{n}} \right) \lesssim_p \mathbb{P}_{J,n,p}
$$

with

$$
\mathbb{P}_{J,n,p} := n^{1-p/2}(\log n)^{p/2} \left( \frac{r_J}{\sigma_J} \right)^p.
$$

(3.1)

**Proof of Lemma 6.** If $J = \{j_1, \ldots, j_2\}$ with $j_1 > 1$, then Lemma 6 follows from (2.9) applied to the transformed and scaled data $X' = \sigma_J^{-1}(|R_J|^{1/2} + g_J^{-1/2} P_J)X$, using again that $X'$ has the same Karhunen-Loève coefficients as $X$ and thus satisfies Assumption 1 with the same constant $C_\eta$. Moreover, the eigenvalues of the covariance operator $\Sigma' = \mathbb{E} X' \otimes X'$ are transformed in such a way that $\text{tr}(\Sigma') = r_J$. Hence, an application of (2.9) with $t = C \sqrt{\log n}$ yields

$$
\mathbb{P} \left( \delta_J > C \sqrt{\frac{\sigma_J^2 \log n}{n}} \right) \lesssim_p n^{1-p/2}(\log n)^{p/2} \left( \frac{r_J}{\sigma_J} \right)^p,
$$

where we also used the first claim in Lemma 5 and $p > 2$. If $J = \{1, \ldots, j_1\}$, then the claim follows similarly by changing the role of $J$ and $J^c$. \qed

We continue by recalling the following asymptotic result from multivariate analysis. By [9, Proposition 5], we have

$$
\sqrt{n}(\hat{\Sigma} - \Sigma) \overset{d}{\to} Z,
$$

(3.2)

where $Z$ is a Gaussian random variable in the Hilbert space of all Hilbert-Schmidt operators (endowed with trace-inner product) with mean zero and covariance operator $\text{Cov}(Z) = \text{Cov}(X \otimes X)$. More concretely, we have

$$
Z = \sum_{j \geq 1} \sum_{k \geq j} \sqrt{\lambda_j \lambda_k} \xi_{jk}(u_j \otimes u_k),
$$

(3.3)

where the upper triangular coefficients $\xi_{jk}$, $k \geq j$ are Gaussian random variables with

$$
\mathbb{E} \xi_{jk} = 0, \quad \mathbb{E} \xi_{jk} \xi_{rs} = \mathbb{E}(\eta_j \eta_k - \delta_{jk})(\eta_r \eta_s - \delta_{rs}),
$$
and the lower triangular coefficients $\xi_{jk}$, $k < j$, are determined by $\xi_{jk} = \xi_{kj}$. In the analysis of $\|P_J - P_J\|_2$, the random variable $L_JZ$ defined in (2.2) plays a central role. It is a centered Gaussian random variable taking values in the separable Hilbert space of all (self-adjoint) Hilbert-Schmidt operators on $H$. Let

$$\Psi_J = \Psi_J(\Sigma) = E(L_JZ \otimes L_JZ)$$

be the covariance operator of $L_JZ$. We will make frequent use of the following quantities and relations (cf. (2.5)).

$$A_J = A_J(\Sigma) = E\|L_JZ\|_2^2 = \|\Psi_J\|_1,$$
$$B_J = B_J(\Sigma) = E^{1/2}(\|L_JZ\|_2^2 - A_J(\Sigma))^2 = \sqrt{2}\|\Psi_J\|_2,$$
$$C_J = C_J(\Sigma) = E^{1/3}(\|L_JZ\|_2^2 - A_J(\Sigma))^3 = 2\|\Psi_J\|_3. \quad (3.4)$$

The following lemma provides the connection of the quantities $A_J$, $B_J$ and $C_J$ with the eigenvalues of $\Sigma$.

**Lemma 7.** Suppose that Assumptions 1 and 2 are satisfied. Then

$$A_J \asymp \sum_{j \in J} \sum_{k \notin J} \frac{\lambda_j \lambda_k}{(\lambda_k - \lambda_j)^2}.$$

If additionally Assumption 3 holds with $m = 4$, then the eigenvalues of $\Psi_J$ are given by

$$2\alpha_{jk}^2 \frac{\lambda_j \lambda_k}{(\lambda_k - \lambda_j)^2}, \quad j \in J, k \notin J,$$

with $\alpha_{jk} = E\eta_j^2 \eta_k^2$. In particular, we have

$$B_J^2 \asymp \sum_{j \in J} \sum_{k \notin J} \left(\frac{\lambda_j \lambda_k}{(\lambda_k - \lambda_j)^2}\right)^2, \quad C_J^3 \asymp \sum_{j \in J} \sum_{k \notin J} \left(\frac{\lambda_j \lambda_k}{(\lambda_k - \lambda_j)^2}\right)^3.$$

**Proof of Lemma 7.** Using the representation in (3.3), we get

$$\|L_JZ\|_2^2 = 2 \sum_{j \in J} \sum_{k \notin J} \frac{\lambda_j \lambda_k}{(\lambda_k - \lambda_j)^2} \xi_{jk}^2$$

Hence, the lower bound follows from inserting Assumption 2, while the upper bounds follows from Assumption 1 and Hölder’s inequality. Moreover,
subject to Assumption 3 with \( m = 4 \), the random variable \( L_J Z \) has the Karhunen-Loève expansion

\[
L_J Z = \sum_{j \in J} \sum_{k \in J} \sqrt{2} \alpha_{jk} \frac{\lambda_j \lambda_k}{\lambda_j - \lambda_k} \alpha_{jk} \frac{\xi_{jk}}{\sqrt{2}} (u_j \otimes u_k + u_k \otimes u_j),
\]

and the last two claims follow from the last relations given in (3.4).

### 3.2 Main results

This section is devoted to quantitative limit theorems for \( \| \hat{P}_J - P_J \|_2 \). We state several results subject to different conditions, all having a different range of application, and refer to Section 5 for examples and illustrations. Proofs are given in Section 6.4. We assume throughout this section that \( n \geq 2 \). Our first result concerns estimates based on the Wasserstein distance \( W \), given below.

**Theorem 1.** Suppose that Assumptions 1 and 2 hold and that

\[
\sigma_J^2 \frac{\log n}{n} |J| \leq c
\]

for some sufficiently small constant \( c > 0 \). Then, for any \( s > 0 \),

\[
W\left( \frac{n}{A_J} \left\| \hat{P}_J - P_J \right\|_2^2, \frac{1}{A_J} \left\| L_J Z \right\|_2^2 \right) \lesssim_{p,s} n^{-\epsilon_p} + \frac{(\log n)^{3/2} \sigma_J^3 |J|^{3/2}}{n^{1/2} A_J} + \frac{n^2 |J|}{A_J} p_{J,n,p} + \left( \frac{A_J}{n |J|} \right)^s,
\]

with \( p_{J,n,p} \) from (3.1) and \( \epsilon_p = (2 - q)^2 / 4(2q - 3) \), \( q = p \land 3 \).

**Remark 1.** We require Condition (3.5) for all of our results. It is, however, redundant in most cases (see for instance Section 5) in the following sense: Upper bounds are nontrivial only if (3.5) holds any way.

The large generality of Theorem 1 has a price: Note that \( (A_J/n)^2 \) may scale as the variance of \( \| \hat{P}_J - P_J \|_2^2 \) or not. In particular, if the expectation \( E \| \hat{P}_J - P_J \|_2^2 \) significantly dominates the square root of the variance, finer approximation results can be obtained by studying the appropriately centred and scaled version. As a first result in this direction, we present the following.
Theorem 2. Suppose that Assumptions 1 and 2 and (3.5) hold with $p \geq 3$. Then we have

$$U\left(n \left\| \hat{P}_J - P_J \right\|_2^2, \left\| L_J Z \right\|_2^2 \right) \lesssim_p n^{-1/5} \left( \frac{A_J}{\sqrt{\lambda_{1,2}(\Psi_J)}} \right)^{3/5}$$

$$+ \frac{(\log n)^{3/2}}{n^{1/2}} \frac{\sigma_J^3 |J|^{3/2}}{\sqrt{\lambda_{1,2}(\Psi_J)}} + p_{J,n,p}$$

as well as

$$U\left(n \left\| \hat{P}_J - P_J \right\|_2^2, \left\| L_J Z \right\|_2^2 \right) \lesssim_p \left( n^{-1/2} \frac{A_J^3 C_J^3}{\lambda_0^6(\Psi_J) B_J^2} \right)^{1+1/10}$$

$$+ n^{-1/2} \frac{A_J^6 C_J^3}{\lambda_{1,6}(\Psi_J) B_J^3}$$

$$+ \frac{(\log n)^{3/2}}{n^{1/2}} \frac{\sigma_J^3 |J|^{3/2}}{\sqrt{\lambda_{1,2}(\Psi_J)}} + p_{J,n,p}.$$
An important feature of both Corollary 5 and Theorem 3 is the error term \((C/J/B_J)^{3}\). As can be deduced from [2], it is, in general, necessary for a central limit theorem to have
\[
\left(\frac{C_J}{B_J}\right)^{3} \to 0 \quad \text{as } n \text{ increases,} \tag{3.6}
\]
otherwise it cannot hold.

So far, we did not formulate any kind of dependence or particular relation between \(n\), \(B_J\) and \(C_J\). However, in Section 5, we will present specific examples to illustrate and further discuss condition (3.6).

4 Bootstrap approximations

Bootstrap methods are nowadays one of the most popular ways for measuring the significance of a test or building confidence sets. Among others, their superiority compared to (conventional) limit theorems stems from the fact that they offer finite sample approximations. Some bootstrap methods even outperform typical Berry-Esseen bounds attached to corresponding limit distributions, see for instance [14]. As is apparent from our results in Section 3, our (unknown) normalising sequences are quite complicated. Hence, as is discussed intensively in [36], bootstrap methods are a very attractive alternative in the present context.

4.1 Assumptions and main quantities

We first require some additional notation. We denote with \(\mathcal{X} = \sigma(X_i, i \in \mathbb{N})\) the \(\sigma\)-algebra generated by the whole sample. We further denote with \((X_i')\) an independent copy of \((X_i)\). Next, we introduce the conditional expectations and probabilities
\[
\tilde{E}[\cdot] = E[\cdot | \mathcal{X}], \quad \tilde{P}(\cdot) = \tilde{E}1_{\{\cdot\}}.
\]
The (conditional) measure \(\tilde{P}\) will act as our probability measure in the bootstrap-world. Finally, recall the uniform metric \(U(A, B)\) for real-valued random variables \(A, B\), and denote with
\[
\tilde{U}(A, B) = \sup_{x \in \mathbb{R}} \left| \tilde{P}(A \leq x) - \tilde{P}(B \leq x) \right| \tag{4.1}
\]
the corresponding conditional version. As before, we assume throughout that \(n \geq 2\).
There are many ways to design bootstrap approximations. A popular and powerful method are multiplier methods, which we also employ here. To this end, let \((w_i)\) be an i.i.d. sequence with the following properties.

\[ Ew_i^2 = 1, \quad Ew_i^{2p} < \infty, \quad (4.2) \]

where \(p\) corresponds to the same value as in Assumption 1. Moreover, denote with \(\sigma_w^2 = E(w_i^2 - 1)^2\). One may additionally demand \(Ew_i = 0\), but this is not necessary. Throughout this section (and the corresponding proofs), we always assume the validity of (4.2) without mentioning it any further.

Our bootstrap method is quite simple and given below.

**Algorithm 1** (Bootstrap). Given \((X_i)\) and \((w_i)\), construct the sequence \((\tilde{X}_i) = (w_iX_i)\). Treat \((\tilde{X}_i)\) as new sample, and compute the corresponding empirical covariance matrix \(\tilde{\Sigma}\) and projection \(\tilde{P}_J\) accordingly.

Note that our multiplier bootstrap is slightly different from the one of \([36]\), but a bit more convenient to analyze. On the other hand, by passing to the complex domain \(\mathbb{C}\) as underlying field of our Hilbert space, it is not hard to show that Theorem 4, Theorem 5 and the attached corollaries below are equally valid for the multiplier bootstrap employed in \([36]\).

### 4.2 Main results

Recall that \((X'_i)\) is an independent copy of \((X_i)\), and thus \(\hat{P}'_J\), the empirical projection based on \((X'_i)\), is independent of \(X\). Throughout this section, we entirely focus on the uniform metric \(\tilde{U}\). One may derive analogous results for the Wasserstein distance, but we feel that the (statistical) value of such results is much less important here, see in particular Corollaries 6 and 7 below.

**Theorem 4.** Suppose that Assumptions 1, 2 hold and that (3.5) is satisfied. Moreover, suppose that

\[ \frac{A_J}{C_J} \sqrt{\frac{\log n}{n}} \leq c \quad (4.3) \]

is satisfied for some sufficiently small constant \(c > 0\). Let \(q \in (2, 3]\) and \(s \in (0, 1)\) and assume that \(p > 2q\). Then, with probability at least \(1 - C_pP_{J, n,p}^{1-s} - C_p n^{1-p/(2q)}\), \(C_p > 0\), we have

\[ \tilde{U} \left( \sigma_w^{-2} \| \hat{P}_J - \hat{P}'_J \|_2^2, \| \hat{P}'_J - P_J \|_2^2 \right) \preceq_p A_{J, n,p,s}, \]
where

$$A_{J,n,p,s} = n^{1/4-q/8} \left( \frac{A_J}{B_J} \right)^{3/4} + \left( \frac{C_J}{B_J} \right)^3 + \frac{\sqrt{\log n A_J}}{\sqrt{n B_J}}$$

$$+ n^{-1/2} \log^{3/2} n \frac{\sigma_J^3 |J|^3/2}{B_J} + \mathbf{p}^s_{J,n,p}. \quad (4.4)$$

The bound $A_{J,n,p,s}$ above is based on Theorem 3 (among other things), which explains the origin of $(C_J/B_J)^3$. Loosely speaking, this means that we approximate with a standard Gaussian distribution to show closeness in Theorem 4.

The uniform metric is strong enough to deduce quantitative statements for empirical quantiles, which is more useful in statistical applications. To exemplify this further, we state the following result.

**Corollary 6.** Given the assumptions and conditions of Theorem 4 and

$$\hat{q}_\alpha = \inf \left\{ x : \hat{P} \left( \frac{n}{\sigma^2_w} \| \hat{P}_J - \hat{P}_J \|_2^2 \leq x \right) \geq 1 - \alpha \right\}, \quad (4.5)$$

we have

$$\left| \alpha - \mathbb{P} \left( n \| \hat{P}_J - P_J \|_2^2 > \hat{q}_\alpha \right) \right| \lesssim_p p^{1-s} + n^{1-p/2} + A_{J,n,p,s},$$

where $A_{J,n,p,s}$ is defined in (4.4).

Next, we present our second bound main result in this section.

**Theorem 5.** Suppose that Assumptions 1 and 2 hold with $p > 6$, Assumption 3 holds with $m \in \{4, 8\}$, and that (3.5) and (4.3) are satisfied. Finally, suppose that

$$\sqrt{\frac{\log n}{n}} A_J \lesssim \lambda_2(\Psi_J). \quad (4.6)$$

Let $s \in (0, 1)$. Then, with probability at least $1 - C_p p^{1-s} - C_p n^{1-p/6}$, $C_p > 0$, we have

$$\hat{U} \left( \sigma_w^{-2} \| \hat{P}_J - \hat{P}_J \|_2^2, \| \hat{P}_J' - P_J \|_2^2 \right) \lesssim_p B_{J,n,p,s},$$

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where

\[
B_{J,n,p,s} = n^{-1/5} \left( \frac{A_J}{\sqrt{\lambda_{1,2}(\Psi_J)}} \right)^{3/5} + \frac{ \log^{3/2} n}{n^{1/2}} \frac{\sigma_J^3 |J|^{3/2}}{\sqrt{\lambda_{1,2}(\Psi_J)}} + \log^{3/2} n \frac{\sigma_J^3 |J|^{3/2}}{\sqrt{\lambda_{1,2}(\Psi_J)}} + \left( \frac{p_{J,n,p} + n^{1-p/6}}{s} \right),
\]

(4.7)

and \( p_{J,n,p} \) is given in (3.1). In all the results above, we require \( J = \{j_1, \ldots, j_2\} \) and \( I = \{1, \ldots, i_2\} \) with \( i_2 > j_2 + 2 \).

Remark 2. As in Theorem 2, it is possible to establish a \( \sqrt{n} \) rate at the cost of additional factors involving various eigenvalues. Since the proof is very similar, we omit any further details here.

The quantities \( A_{J,I} \) and \( \Psi_{J,I} \) that appear in the above bound are not defined yet. In essence, the set \( I \) is used to truncate some lower degree indices. The exact definition requires some preparation, and is given in Section 6.6.1. In contrast to Theorem 4, Theorem 5 above is built around Theorem 2, and thus avoids the error term \( (C_J/B_J)^3 \). However, apart from the eigenvalues, this comes at (other) additional costs, in particular the expressions \( \text{tr}(\Psi_{J,I}^{1/2}) \) and \( A_{J,I} \) require a careful selection of the set \( I \). As before, we have the following corollary.

Corollary 7. Given the assumptions and conditions of Theorem 5 and \( \hat{q}_a \) defined as in (4.5), we have for \( s \in (0,1) \)

\[
\left| \alpha - \mathbb{P}\left( n\|\hat{P}_J - P_J\|^2 > \hat{q}_a \right) \right| \lesssim p_{J,n,p}^{1-s} + B_{J,n,p,s},
\]

where \( B_{J,n,p,s} \) is given in (4.7).

5 Applications: Specific models and computations

In this Section, we discuss specific models to illustrate our results with explicit bounds and compare them to previous results.

To this end, we discuss two basic, fundamental models omnipresent in the literature. In our first model, we consider the case where the eigenvalues \( \lambda_j \) decay at a certain rate (polynomial or exponential). This behaviour is typically encountered in functional data analysis or in a machine learning context involving kernels, see for instance [15], [3], [13]. Our second model is the classical spiked covariance model (factor model), which is more popular in high dimensional statistics, econometrics and probability theory, see [22], [11], [1].
5.1 Model I

Throughout this section, we assume that Assumptions 1, 2 and 3 \((m = 4)\) hold with \(p > 3\).

For \(J \geq 1\), we consider the set \(\mathcal{J} = \{1, \ldots, J\}\) and the singleton \(\mathcal{J}' = \{J\}\). For simplicity, we assume that \(n \geq 2\), so \(\log n\) is always well-defined. Throughout this Section, all constants depend on the parameter \(a\). To simplify notation, we will not indicate this explicitly.

5.1.1 Exponential decay

We first consider the case of exponential decay, that is, we suppose that for some \(a > 0\), we have \(\lambda_j = e^{-aj}\) for all \(j \geq 1\).

In this setup, the relative rank \(r_{\mathcal{J}}, A_{\mathcal{J}}\) and related quantities have already been computed in the literature, see for instance [19], [20], [39]. By similar (straightforward) computations together with Lemmas 5 and 7, we have

\[
\begin{align*}
  r_{\mathcal{J}} &\asymp \sigma_{\mathcal{J}}^2 \asymp J, & p_{\mathcal{J}, n, p} &\asymp n^{1-p/2} J^{p/2} \log^{p/2} n, \\
  A_{\mathcal{J}} &\asymp B_{\mathcal{J}} \asymp C_{\mathcal{J}} \asymp 1,
\end{align*}
\]

as well as

\[
\begin{align*}
  r_{\mathcal{J}'} &\asymp \sigma_{\mathcal{J}'}^2 \asymp J, & p_{\mathcal{J}', n, p} &\asymp n^{1-p/2} J^{p/2} \log^{p/2} n, \\
  A_{\mathcal{J}'} &\asymp B_{\mathcal{J}'} \asymp C_{\mathcal{J}'} \asymp 1.
\end{align*}
\]

Moreover, using again Lemma 7, we have

\[
\lambda_j(\Psi_{\mathcal{J}}) \asymp \lambda_j(\Psi_{\mathcal{J}'}) \asymp 1, \quad j = 1, \ldots, 6.
\]

It is now easy to apply the results. For example, Theorem 2 yields

\[
U\left( n\|P_{\mathcal{J}} - P_{\mathcal{J}}\|_2^2,\|L_{\mathcal{J}} Z\|_2^2 \right) \lesssim_p \left( \frac{J^6 \log^3 n}{n} \right)^{1/2}
\]

for cumulated projectors, and

\[
U\left( n\|P_{\mathcal{J}'} - P_{\mathcal{J}'}\|_2^2,\|L_{\mathcal{J}'}(Z)\|_2^2 \right) \lesssim_p \left( \frac{J^3 \log^3 n}{n} \right)^{1/2}
\]

for the special case of single projectors. Here, we used that \(p_{\mathcal{J}, n, p}\) is negligible due to the fact that \(p > 3\), and that condition (3.5) can be dropped due to the fact that \(p > 3\).
to the fact that the uniform distance is bounded by 1, while the right-hand side of these bounds exceeds 1 whenever (3.5) does not hold, that is, when \( n^{-1/2} J (\log n)^{1/2} \gtrsim 1 \).

**Comparison and discussion:** The results of [29] are not applicable in this setup and only provide the trivial bound \( \leq 1 \). More precisely, Theorem 6 in [29] only yields a non-trivial result if

\[
    n^2 \text{Var} \| \hat{P}_J - P_J \|_2^2 \to \infty.
\]

However, this is not the case here. In fact, a normal approximation by a standard Gaussian random variable \( G \) is not possible at all in this setup. This follows from (3.6) and the fact that \( B_J \approx C_J \).

In stark contrast, our bounds above in (5.2) and (5.3) provide non-trivial results even for moderately large \( J \). This is a consequence of the relative approach that we employ here. In addition, our probabilistic assumptions are much weaker compared to [29]. Hence, even for the Gaussian case, our results are new.

Next, regarding bootstrap approximations, if \( p \geq 9 \), then an application of Corollary 7 (with \( s = 1/2, I = \mathbb{N} \)) yields

\[
    \left| \alpha - \mathbb{P} \left( n \| \hat{P}_J - P_J \|_2^2 > \hat{q}_\alpha \right) \right| \lesssim_p \frac{1}{n^{1/5}} + \left( \frac{J^6 \log^3 n}{n} \right)^{1/2}.
\]

**Comparison and discussion:** In contrast to [29], the bootstrap approximation of [36], Theorem 2.1, is applicable if we additionally assume Gaussianity. For finite \( J \), it appears that the bound provided by Theorem 2.1 is better than ours for single projectors. However, this drastically changes if \( J \) increases: Simple computations show that their rate is lower bounded by

\[
    \left( \frac{e^{6aJ} \log^3 n}{n} \right)^{1/2},
\]

which is useless already for \( J \geq (\log n)/6a \). On the other hand, our bound 5.4 yields useful results even for \( J \leq n^{1/6-\delta} \), \( \delta > 0 \). Thus, despite having much weaker assumptions, our results extend and improve upon those of [36] even in the Gaussian case.

### 5.1.2 Polynomial decay

We next consider the case of polynomial decay, that is, we suppose that there is a constant \( a > 1 \) such that \( \lambda_j = j^{-a} \) for all \( j \geq 1 \). For simplicity, we assume that \( J \geq 2 \).
As in the previous Section 5.1.1, straightforward computations, together with Lemmas 5 and 7, yield

\[
\begin{align*}
\mathbf{r}_J \asymp J \log J, & \quad \mathbf{p}_{J,n,p} \asymp n^{1-p/2} \left( \log J \log n \right)^{p/2}, \\
\sigma_J^2 \asymp A_J \asymp J^2 \log J, & \quad B_J \asymp C_J \asymp J^2.
\end{align*}
\] (5.6)

as well as

\[
\begin{align*}
\mathbf{r}_{J'} \asymp J \log J, & \quad \mathbf{p}_{J,n,p} \asymp n^{1-p/2} \left( \log J \log n \right)^{p/2}, \\
\sigma_{J'}^2 \asymp J' \log J, & \quad A_{J'} \asymp B_{J'} \asymp C_{J'} \asymp J'.
\end{align*}
\] (5.7)

Moreover, using again Lemma 7, it follows that

\[
\lambda_j(\Psi_J) \asymp \lambda_j(\Psi_{J'}) \asymp J^2, \quad j = 1, \ldots, 6.
\] (5.8)

It is now again easy to apply the results. For example, Theorem 2 yields

\[
U \left( n \left\| \hat{P}_J - P_J \right\|_2^2 \left\| L_J Z \right\|_2^2 \right) \lesssim_p n^{-1/2} (\log J)^6 + n^{-1/2} J^{3/2} (\log J \log n)^{3/2}
\]

for general projectors \( P_J \), and

\[
U \left( n \left\| \hat{P}_{J'} - P_{J'} \right\|_2^2 \left\| L_{J'} Z \right\|_2^2 \right) \lesssim_p n^{-1/2} J (\log J \log n)^{3/2}
\]

for the special case of single projectors. Here, we again exploited that condition (3.5) can be dropped due to the fact that the uniform distance is bounded by 1, while the right-hand side of these bounds exceeds 1 whenever (3.5) does not hold, that is, when \( n^{-1/2} J^{3/2} (\log J \log n)^{1/2} \gtrsim 1 \) and \( n^{-1/2} J (\log J \log n)^{1/2} \gtrsim 1 \), respectively. Observe that in case of single projectors, the dependence on \( J \) is optimal up to log-factors, we refer to [20], Example 2 for a more detailed discussion.

**Comparison and discussion:** As in the exponential case, the results of [29] are not applicable here. On the other hand, again due to the relative nature of our approach, our bounds in (5.2) are quite general, simple, and new even in for the Gaussian case. We emphasize in particular that our bounds are independent of \( a \) up to constants.

Next, we want to apply Corollary 7. To this end, we need to strengthen Assumption 3 to \( m = 8 \). First, if \( a > 2 \), we choose \( \mathcal{I} = \mathbb{N} \), in which case we have \( \Psi_{J,\mathcal{I}} = \Psi_J \) (using Lemma 7) and thus

\[
\text{tr}(\sqrt{\Psi_{J,\mathcal{I}}}) = \text{tr}(\sqrt{\Psi_J}) \asymp J^2.
\]
Hence, if \( p \geq 9 \), an application of Corollary 7 (with \( s = 1/2 \)) yields
\[
\left| \alpha - P\left(n\|\hat{P}_J - P_J\|_2^2 > \hat{q}_\alpha \right) \right| \lesssim_p \frac{1}{n^{1/5}} (\log J)^{3/5} + \frac{J^{5/2}}{n^{1/2}} (\log n \log J)^{3/2}.
\]

Second, if \( 1 < a \leq 2 \), let \( \mathcal{I} = \{I, I+1, \ldots\} \) with \( I \geq 2J \), in which case
\[
\text{tr}(\sqrt{\Psi_{J,I}}) \asymp J^{1+a/2} I^{1-a/2},
\]
\[
A_{J,I,c} \asymp J^{1+a} I^{1-a}.
\]

Thus, if \( p \geq 9 \), an application of Corollary 7 (with \( s = 1/2 \)) yields
\[
\left| \alpha - P\left(n\|\hat{P}_J - P_J\|_2^2 > \hat{q}_\alpha \right) \right| \lesssim_p \frac{1}{n^{1/5}} (\log J)^{3/5} + \frac{J^{5/2}}{n^{1/2}} (\log n \log J)^{3/2}
\]
\[
+ \frac{J^{a/2} I^{1-a/2}}{n^{1/2}} (\log n \log J)^{1/2} + J^{a-1} I^{1-a} \log n.
\]

Balancing with respect to \( I \) leads to
\[
\left| \alpha - P\left(n\|\hat{P}_J - P_J\|_2^2 > \hat{q}_\alpha \right) \right| \lesssim_p \frac{1}{n^{1/5}} (\log J)^{3/5} + \frac{J^{5/2}}{n^{1/2}} (\log n \log J)^{3/2}
\]
\[
+ \left( \frac{J^2 \log J}{n} \right)^{1-1/a} (\log n)^{1/a}.
\]

\( 5.9 \)

**Comparison and discussion:** The situation is similarly as before in the exponential case: For finite \( J \), Theorem 2.1 in [36] gives superior results for single projectors, if the underlying distribution is Gaussian. However, their rate is lower bounded by
\[
\left( \frac{J^{6a+2} \log^3 n}{n} \right)^{1/2},
\]
and thus leads to a much smaller range for \( J \), particularly for larger \( a \). In contrast, the range for \( J \) for our results in (5.9) is invariant in \( a \), and the overall bound even slightly improves as \( a \) gets larger.

5.2 Model II

For simplicity, we only consider the (more important) case \( J = \{1, \ldots, J\} \) in this Section. We note, however, that single projectors can readily be handled in a similar manner. Throughout this Section, we assume the validity of Assumptions 1, 2 and 3 with \( p \geq 3 \) and \( m = 4 \).
5.2.1 Factor models - pervasive case

Recall $\mathcal{J} = \{1, \ldots, J\}$, where we assume $J \geq 6$. The literature on (approximate, pervasive) factor models typically assumes that the first $J$ eigenvalues diverge at rate $\asymp d$ (with $d = \dim \mathcal{H}$), whereas all the remaining eigenvalues are bounded and do not, in total, have significantly more 'energy' than any of the first $J$ eigenvalues. This assumption can be expressed in terms of pervasive factors, see for instance [1], [43], which are particularly relevant in econometrics. In the language of statistics, this means the remaining $d - J$ components do not explain significantly more (variance) than any of the first $J$.

Here, we generalize the above conditions as follows. We assume that there exist constants $0 < c \leq C < \infty$, such that

$$
\lambda_1 \leq C \lambda_J, \quad \lambda_J - \lambda_{J+1} \geq c \lambda_J, \quad \frac{\operatorname{tr} \mathcal{J} \Sigma}{\lambda_1} \leq C. \quad (5.11)
$$

Observe that this implies

$$
\frac{\operatorname{tr} \Sigma}{\lambda_1} \asymp J,
$$

which is the desired feature of pervasive factor models. It is convenient to introduce a notion of a subset trace by

$$
\operatorname{tr}_I \Sigma = \sum_{i \in I} \lambda_i, \quad I \subseteq \mathbb{N}. \quad (5.12)
$$

We then have the following relations.

$$
\mathcal{r}_J \asymp \sigma_J^2 \asymp J + \frac{\operatorname{tr} \mathcal{J} \Sigma}{\lambda_J}, \quad \mathcal{p}_{J,n,p} \asymp n \left( \frac{\log n \operatorname{tr}_J \Sigma}{n} \right)^{p/2},
$$

$$
A_J \asymp J \frac{\operatorname{tr} \mathcal{J} \Sigma}{\lambda_J} \quad B_J^2 \asymp J \frac{\operatorname{tr} \mathcal{J} \Sigma^2}{\lambda_J^2}, \quad C_J^3 \asymp J \frac{\operatorname{tr} \mathcal{J} \Sigma^3}{\lambda_J^3}. \quad (5.13)
$$

It turns out that pervasive factor models lead to particularly simple results. Indeed, an application of Theorem 2 yields

$$
\mathcal{U} \left( n \| \hat{P}_J - P_J \|_2^2, \| L_J Z \|_2^2 \right) \lesssim_p \left( \frac{J^6}{n} \right)^{1/2} \left( (\log n)^{3/2} + J^{5/2} \right), \quad (5.14)
$$

where we used that $(C_J / B_J)^3 \lesssim 1/\sqrt{J}$. 

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We note that there is no restriction on the dimension $d$ of the underlying Hilbert space in this case.

**Comparison and discussion:** Our bound in (5.14) is easy to use and fairly general in terms of underlying assumptions. The literature does not appear to have a comparable result, even in the Gaussian case. The single projector results of Theorem 6 in [29] only yield the trivial bound $\leq 1/\sqrt{J}$. Hence, if we consider the general projectors $P_J, J = \{1, \ldots, J\}$ with $J \to \infty$, one may also use a standard Gaussian approximation as in Theorem 3.

Next, we turn to the bootstrap. Using Corollary 7 ($p > 6$, $s = 1/2$, $I = \{1, \ldots, d\}$, $m = 8$ in Assumption 3), we arrive at

$$
\left| \alpha - \mathbb{P} \left( n \| \hat{P}_J - P_J \|_2^2 > q_\alpha \right) \right| \lesssim \left( \frac{J^3}{n} \right)^{1/5} + \left( \frac{J^6 \log^3 n}{n} \right)^{1/2} + \left( \frac{J \log n}{n} \right)^{1/2} + n^{(6-p)/12},
$$

(5.15)

where we also used $\text{tr} (\sqrt{\Psi_J}) \lesssim \sum_{1 \leq k \leq d} \sqrt{\lambda_k / \lambda_1} \lesssim \sqrt{d}$.

**Comparison and discussion:** The situation is related to Model I. If the dimension $d$ is small and the setup is purely Gaussian, the results of [36] are superior\(^1\). However, for larger $d$, this changes, as can be seen as follows. In the present context, their rate is lower bounded by

$$
d \sqrt{\frac{\log n}{n}}.
$$

(5.16)

This implies in particular $d = o(\sqrt{n / \log n})$ for a non trivial result. In contrast, our bound (5.15) is valid also for $d$ as large as $d = o(n / \log n)$. For a sake of better comparison, we assumed here that $J$ is fixed, since the results of [36] only apply to single projectors. Among others, a key reason for our improvement compared to [36] is the usage of our concentration inequality in Corollary 4.

### 5.2.2 Spiked covariance

We next consider a simple spiked covariance model of the form

$$
\lambda_{J+1} = \cdots = \lambda_d = \sigma^2 \quad \text{and} \quad \sigma^2 + g_J = \lambda_J \leq \cdots \leq \lambda_1 \leq \sigma^2 + C g_J,
$$

\(^1\)As mentioned in Remark 2, one can improve our result in this case.
where $\sigma^2 > 0$ is the level of noise, $g_J = \lambda_J - \lambda_{J+1} > 0$ is the relevant spectral gap, and $C > 0$ is a constant. For simplicity, we assume that $\sigma^2 = 1$ such that $J$ and $g_J$ are the only remaining parameters. Moreover, we assume that $d \geq 6$, $J \leq d - J$ and $g_J \in (0,1]$. In particular, all eigenvalues have the same magnitude up to multiplicative constants and we have the following relations

$$r_J \asymp \frac{d}{g_J}, \quad \sigma_J^2 \asymp \frac{d}{g_J^2}, \quad P_{J,n,p} \asymp n \left( \frac{d \log n}{n} \right)^{p/2},$$

$$A_J \asymp \frac{dJ}{g_J^2}, \quad B_J^2 \asymp \frac{dJ}{g_J^4}, \quad C_J^3 \asymp \frac{dJ}{g_J^6}. \quad (5.17)$$

Moreover, using again Lemma 7 and the fact that $J > 6$, it follows that $\lambda_j (\Psi^J) \asymp 1/g_J^2$, $j = 1, \ldots, 6$. It is now again easy to apply the results. For example, Theorem 1 yields

$$W \left( \frac{n}{A_J} \left\| \hat{P}_J - P_J \right\|_2^2, \frac{1}{A_J} \left\| L_J Z \right\|_2^2 \right) \lesssim_p n^{-1/12} + \left( \log n \right)^{3/2} \left( \frac{Jd}{ng_J^2} \right)^{1/2}$$

$$+ n^2 \left( \frac{d \log n}{n} \right)^{p/2}, \quad (5.18)$$

provided that $\log(n)dJ/(ng_J^2) \lesssim 1$, which amounts to condition (3.5). Note that if we demand that the bound in (5.18) is of magnitude $o(1)$, (3.5) is trivially true. Moreover, an application of Theorem 3 yields

$$U \left( \frac{n}{B_J} \left\| \hat{P}_J - P_J \right\|_2^2 - A_J, G \right) \lesssim_p \left( \frac{dJ}{n^{1/3}} \right)^{3/8} + \frac{1}{\sqrt{dJ}}$$

$$+ \frac{dJ \log^{3/2} n}{\sqrt{ng_J}} + n \left( \frac{d \log n}{n} \right)^{p/2}, \quad (5.19)$$

where we again exploited that condition (3.5) can be dropped due to the fact that the uniform distance is bounded by 1, while the right-hand side of these bounds exceeds 1 whenever (3.5) does not hold, that is, when $dJ \log(n)/(ng_J^2) \gtrsim 1$. Note that the bound in (5.19) is only nontrivial if $d \to \infty$ due to the error term $(dJ)^{-1/2}$. However, if $d$ is bounded, one may (again) apply Theorem 2 (which also applies for $d \to \infty$), we omit the details.

**Comparison and discussion:** We first observe that for $(Jd)/(ng_J^2) \leq n^{-\delta}$, $\delta > 0$ and $p$ large enough, (5.18) is bounded by $n^{-1/12+\delta/3}$. It follows,
in particular, that
\[ \text{En}[\|\hat{P}_J - P_J\|_2^2 = E\|L_J Z\|_2^2(1 + O(n^{-1/12\sqrt{3}}))). \] (5.20)

This should be contrasted with previous results in the purely Gaussian setting, see for instance [6], and [5] for a closely related result. Regarding (5.19), the situation is a bit more complex, as, even for large enough \( p \), three terms may be dominant. Following the discussion in [29] below Theorem 6, let us fix \( J \) (and hence \( g_J \)), and assume \( d = d_n \to \infty \) as \( n \) increases, and that \( p \) is large enough. Then (5.19) and Theorem 6 in [29] amount to
\[
\left( \frac{d}{n^{1/3}} \right)^{3/8} + \frac{1}{\sqrt{d}} \quad (5.19), \quad \sqrt{\frac{d}{n}} \log \left( \frac{n}{d} \lor 2 \right) + \frac{1}{\sqrt{d}} \quad (29), \text{Theorem 6}.
\]

So in this particular case, our bound is inferior compared to the Gaussian case treated in [29], only allowing a range for \( d = o(n^{1/3}) \) compared to (almost) \( d = o(n) \). On the other hand, [29] only treats \( J = \{ J \} \), and our distributional assumptions are much weaker. In particular, no Sub-Gaussianity or even Gaussianity is necessary for (5.19) to hold.

Next, we consider bootstrap approximations. Assuming \( p > 6 \), Corollary 6 \((q = 3, s = 1/2)\), yields
\[
\left| \alpha - \mathbb{P} \left( n \|\hat{P}_J - P_J\|_2^2 > \hat{q}_0 \right) \right| \lesssim_p \left( \frac{d}{n^{1/3}} \right)^{3/8} + \frac{1}{\sqrt{dJ}} + \frac{d J \log^{3/2} n}{\sqrt{n g_J}} + n^{1/2} \left( \frac{d \log n}{n} \right)^{p/4}. \] (5.21)

For a comparison with Theorem 2.1 in [36], we follow the discussion above and reconsider the case where \( J, g_J \) are finite, \( d = d_n \to \infty \) and \( p \) is large enough. Our bound (5.21), and the one of Theorem 2.1 in [36] (purely Gaussian setup), are of magnitude
\[
\left( \frac{d}{n^{1/3}} \right)^{3/8} + \frac{1}{\sqrt{d}} \quad (5.21), \quad \frac{(d \sqrt{n} \log n)^3}{\sqrt{n}} \quad ([36], \text{Theorem 2.1}).
\]

We see that for smaller \( d \), the bound of [36] is superior compared to ours. However, for larger \( d \), our bound prevails. In particular, our range of applicability \( d = o(n^{1/3}) \) is larger compared to \( d = o(n^{1/6}/(\log n)^{3/2}) \), despite not demanding any Sub-Gaussianity or even Gaussianity as in [36].

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6 Proofs

6.1 Proofs for Section 2.2

The proofs will be based on a series of lemmas. We start with an eigenvalue separation property.

**Lemma 8.** If \(\delta_J \leq 1\), then we have

\[
\hat{\lambda}_{j_1} - \lambda_{j_1} \leq \delta_J (\lambda_{j_1-1} - \lambda_{j_1}), \quad \hat{\lambda}_{j_2} - \lambda_{j_2} \geq -\delta_J (\lambda_{j_2} - \lambda_{j_2+1}).
\]

**Proof.** The claim follows from [20, Proposition 1], using the same line of arguments as in the proof of Lemma 2 in [45]. For completeness, we repeat the proof. Set

\[
T_{\geq j_1} = \sum_{k \geq j_1} \frac{1}{\lambda_{j_1} + \delta_J (\lambda_{j_1-1} - \lambda_{j_1}) - \lambda_k} P_k,
\]

\[
T_{\leq j_2} = \sum_{k \leq j_2} \frac{1}{\lambda_k + \delta_J (\lambda_{j_2} - \lambda_{j_2+1}) - \lambda_{j_2}} P_k.
\]

Then [20, Proposition 1] states that \(\hat{\lambda}_{j_1} - \lambda_{j_1} \leq \delta_J (\lambda_{j_1-1} - \lambda_{j_1})\) (resp. \(\hat{\lambda}_{j_2} - \lambda_{j_2} \geq -\delta_J (\lambda_{j_2} - \lambda_{j_2+1})\)), provided that \(\|T_{\geq j_1} E T_{\geq j_1}\|_\infty \leq 1\) (resp. \(\|T_{\leq j_2} E T_{\leq j_2}\|_\infty \leq 1\)). Now, by simple properties of the operator norm, using that

\[
\lambda_{j_1} + \delta_J (\lambda_{j_1-1} - \lambda_{j_1}) - \lambda_k \geq \begin{cases} 
\delta_J g_J, & k = j_1, \ldots, j_2, \\
\lambda_{j_2} - \lambda_k, & k > j_2,
\end{cases}
\]

we get (recall that \(\delta_J \leq 1\))

\[
\|T_{\geq j_1} E T_{\geq j_1}\|_\infty \leq \|(|R_{\mathcal{J}}|^{-1/2} + (\delta_J g_J)^{-1/2} P_{\mathcal{J}}) E(|R_{\mathcal{J}}|^{-1/2} + (\delta_J g_J)^{-1/2} P_{\mathcal{J}})|\|_\infty \\
\leq \delta_J^{-1} \|(|R_{\mathcal{J}}|^{-1/2} + g_J^{-1/2} P_{\mathcal{J}}) E(|R_{\mathcal{J}}|^{-1/2} + g_J^{-1/2} P_{\mathcal{J}})|\|_\infty \leq 1.
\]

Similarly, we have \(\|T_{\leq j_2} E T_{\leq j_2}\|_\infty \leq 1\), and the claim follows.

The following lemma follows from simple properties of the operator norm.

**Lemma 9.** We have

\[
\max \left(\|R_{\mathcal{J}}|^{-1/2} E |R_{\mathcal{J}}|^{-1/2}\|_\infty, g_J^{-1/2} \||R_{\mathcal{J}}|^{-1/2} E P_{\mathcal{J}}|\|_\infty, g_J^{-1} \|P_{\mathcal{J}} E P_{\mathcal{J}}\|_\infty\right) \leq \delta_J.
\]
Lemma 10. If \( \delta_J < 1/2 \), then we have
\[
g^{-1/2}_J \| |R_{J^c}|^{-1/2} \hat{P}_J \|_2 \leq \sqrt{\min(|J|, |J^c|)} \frac{\delta_J}{1 - 2\delta_J}
\]
and
\[
\| P_{J^c} \hat{P}_J \|_2 \leq \sqrt{\min(|J|, |J^c|)} \frac{\delta_J}{1 - 2\delta_J},
\]
where \( |R_{J^c}|^{-1/2} \) is the inverse of \( |R_{J^c}|^{1/2} \) on the range of \( P_{J^c} \).

Proof. By Lemma 8, we have for every \( k < j_1 \),
\[
\lambda_k - \hat{\lambda}_{j_1} = \lambda_k - \lambda_{j_1} - (\hat{\lambda}_{j_1} - \lambda_{j_1}) \geq \lambda_k - \lambda_{j_1} - \delta_J (\lambda_{j_1-1} - \lambda_{j_1}) \geq (1 - \delta_J) (\lambda_k - \lambda_{j_1})
\]
and thus also
\[
\lambda_k - \lambda_j \geq (1 - \delta_J) (\lambda_k - \lambda_{j_1}), \quad \forall k < j_1, j \in J. \tag{6.1}
\]
Similarly, we have
\[
\hat{\lambda}_j - \lambda_k \geq (1 - \delta_J) (\lambda_{j_2} - \lambda_k), \quad \forall k > j_2, j \in J. \tag{6.2}
\]
By the identity \((\hat{\lambda}_j - \lambda_k) P_k \hat{P}_j = P_k E \hat{P}_j\), valid for every \( j, k \geq 1 \), we get
\[
\| |R_{J^c}|^{-1/2} \hat{P}_J \|_2^2 = \sum_{k < j_1, j \in J} \frac{\lambda_k - \lambda_{j_1}}{\lambda_k - \lambda_j}^2 \| P_k E \hat{P}_j \|_2^2 + \sum_{k > j_2, j \in J} \frac{\lambda_{j_2} - \lambda_k}{\lambda_j - \lambda_k}^2 \| P_k E \hat{P}_j \|_2^2. \tag{6.3}
\]
Inserting (6.1) and (6.2) into (6.3), we arrive at
\[
\| |R_{J^c}|^{-1/2} \hat{P}_J \|_2 \leq \frac{\| |R_{J^c}|^{1/2} E \hat{P}_J \|_2}{1 - \delta_J}.
\]
Moreover, we have
\[
\| |R_{J^c}|^{1/2} E \hat{P}_J \|_2 \leq \| |R_{J^c}|^{1/2} E P_J \hat{P}_J \|_2 + \| |R_{J^c}|^{1/2} E P_{J^c} \hat{P}_J \|_2
\]
\[
= \| |R_{J^c}|^{1/2} E P_J \hat{P}_J \|_2 + \| |R_{J^c}|^{1/2} E |R_{J^c}|^{1/2} |R_{J^c}|^{-1/2} \hat{P}_J \|_2 \leq \| |R_{J^c}|^{1/2} E P_J \|_2 + \| |R_{J^c}|^{1/2} E |R_{J^c}|^{1/2} \|_\infty \| |R_{J^c}|^{-1/2} \hat{P}_J \|_2.
\]

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Combining these two estimates with Lemma 9, we arrive at

\[ \|R_{\mathcal{J}^c}|^{-1/2}\hat{P}_{\mathcal{J}}\|_2 \leq \frac{\|R_{\mathcal{J}^c}|^{1/2}EP_{\mathcal{J}}\|_2}{1-\delta_{\mathcal{J}}} + \frac{\delta_{\mathcal{J}}\|R_{\mathcal{J}^c}|^{-1/2}\hat{P}_{\mathcal{J}}\|_2}{1-\delta_{\mathcal{J}}}, \]

and the first claim follows inserting

\[ g_{\mathcal{J}}^{-1/2}\|R_{\mathcal{J}^c}|^{1/2}EP_{\mathcal{J}}\|_2 \leq \sqrt{\min(|\mathcal{J}|, |\mathcal{J}^c|)}g_{\mathcal{J}}^{-1/2}\|R_{\mathcal{J}^c}|^{1/2}EP_{\mathcal{J}}\|_{\infty} \]

\[ \leq \sqrt{\min(|\mathcal{J}|, |\mathcal{J}^c|)}\delta_{\mathcal{J}}, \tag{6.4} \]

where we again applied Lemma 9 in the second inequality. To get the second claim, note that

\[ \|P_{\mathcal{J}^c}\hat{P}_{\mathcal{J}}\|_2 = \|R_{\mathcal{J}^c}|^{1/2}|R_{\mathcal{J}^c}|^{-1/2}\hat{P}_{\mathcal{J}}\|_2 \]

\[ \leq \|R_{\mathcal{J}^c}|^{1/2}\|_{\infty}\|R_{\mathcal{J}^c}|^{-1/2}\hat{P}_{\mathcal{J}}\|_2 \leq g_{\mathcal{J}}^{-1/2}\|R_{\mathcal{J}^c}|^{-1/2}\hat{P}_{\mathcal{J}}\|_2 \]

and the second claim follows from inserting the first claim.

In what follows, we abbreviate

\[ \gamma_{\mathcal{J}} := \left\| \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j} \right\|_{\infty}. \tag{6.5} \]

Lemma 11. Suppose that \( \delta_{\mathcal{J}} < 1/2 \). Then we have

\[ \left\| P_{\mathcal{J}^c}\hat{P}_{\mathcal{J}} + \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j} \right\|_2 \]

\[ \leq \sqrt{\min(|\mathcal{J}|, |\mathcal{J}^c|)} \left( \frac{2\delta_{\mathcal{J}}^2}{1-2\delta_{\mathcal{J}}} + \frac{\delta_{\mathcal{J}}\gamma_{\mathcal{J}}}{(1-2\delta_{\mathcal{J}})(1-\delta_{\mathcal{J}})} \right). \]

In particular, we have

\[ \left\| P_{\mathcal{J}^c}\hat{P}_{\mathcal{J}} + \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j} \right\|_2 \]

\[ \leq \min(|\mathcal{J}|, |\mathcal{J}^c|) \left( \frac{2\delta_{\mathcal{J}}^2}{1-2\delta_{\mathcal{J}}} + \frac{\delta_{\mathcal{J}}^2}{(1-2\delta_{\mathcal{J}})(1-\delta_{\mathcal{J}})} \right). \]

Proof. Proceeding as in (4.10)–(4.11) in [20], we have

\[ P_{\mathcal{J}^c}\hat{P}_{\mathcal{J}} + \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j} = \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \hat{P}_{\mathcal{J}^c} - \sum_{k\in\mathcal{J}^c} \sum_{j\in\mathcal{J}} P_kEP_k \hat{P}_{\mathcal{J}} \]

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\[ + \sum_{k \in J^c} \sum_{j \in J} \sum_{l \in J} \frac{P_k E P_j E \hat{P}_l}{(\lambda_k - \lambda_j)(\lambda_k - \hat{\lambda}_l)}. \]

Hence, by the triangular inequality, we get

\[
\left\| P_{J^c} \hat{P}_J + \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_j}{\lambda_k - \lambda_j} P_{J^c} \hat{P}_J \right\|_2 \\
\leq \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_j \hat{P}_J}{\lambda_k - \lambda_j} \right\|_2 + \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_{J^c} \hat{P}_j}{\lambda_k - \lambda_j} \right\|_2 \\
+ \left\| \sum_{k \in J^c} \sum_{j \in J} \sum_{l \in J} \frac{P_k E P_j E \hat{P}_l}{(\lambda_k - \hat{\lambda}_l)(\lambda_k - \lambda_j)} \right\|_2 =: I_1 + I_2 + I_3. \]

First, we have

\[
I_1 = \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_j}{\lambda_k - \lambda_j} P_{J^c} \hat{P}_J \right\|_2 \leq \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_j}{\lambda_k - \lambda_j} \right\|_\infty \| P_{J^c} \hat{P}_J \|_2 \]

Inserting \( \| P_{J^c} \hat{P}_J \|_2 = \| P_{J^c} \hat{P}_J \|_2 \) and the second bound in Lemma 10, we get

\[
I_1 \leq \sqrt{\min(|J|, |J^c|)} \frac{\delta_J \gamma_J}{1 - 2\delta_J}, \]

where \( \gamma_J \) is defined in (6.5). Second, by Lemma 8, proceeding as in the proof of Lemma 10, we get

\[
I_2 \leq \frac{g_{J^c}}{1 - \delta_J} \| R_{J^c}^{1/2} E P_{J^c} \hat{P}_J \|_2 = \frac{g_{J^c}}{1 - \delta_J} \| R_{J^c}^{1/2} E |R_{J^c}|^{1/2} |R_{J^c}|^{-1/2} \hat{P}_J \|_2 \\
\leq \frac{g_{J^c}}{1 - \delta_J} \| R_{J^c}^{1/2} E |R_{J^c}|^{1/2} \infty \| |R_{J^c}|^{-1/2} \hat{P}_J \|_2. \]

Inserting Lemmas 9 and 10, we get

\[
I_2 \leq \sqrt{\min(|J|, |J^c|)} \frac{\delta_J^2}{(1 - 2\delta_J)(1 - \delta_J)}. \]

Finally,

\[
I_3 \leq \frac{g_{J^c}}{1 - \delta_J} \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k E P_j E \hat{P}_J}{\lambda_k - \lambda_j} \right\|_2. \]

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\[
\leq \frac{g_{Jc}^{-1}}{1 - \delta_J} \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} P_J E P_J \right\|_2 \\
+ \frac{g_{Jc}^{-1}}{1 - \delta_J} \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} P_J E P_J \hat{P}_J \right\|_2 \\
\leq \frac{g_{Jc}^{-1}}{1 - \delta_J} \left( \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} \right]\|P_J E P_J\|_\infty \\
+ \frac{g_{Jc}^{-1}}{1 - \delta_J} \left( \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} \right]\|P_J E R_{J^c}^{-1/2} \|_\infty \|R_{J^c}^{-1/2} \hat{P}_J\|_2.
\]

Applying Lemmas 9 and 10 and the inequality
\[
\left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} \right\|_2 \leq g_{Jc}^{-1/2}\|R_{J^c}^{-1/2} EP_J\|_2 \leq \sqrt{\min(|J|, |J^c|) \delta_J},
\]
we get
\[
I_3 \leq \sqrt{\min(|J|, |J^c|)} \left( \frac{\delta_J^2}{1 - \delta_J} + \frac{\delta_J^2 \gamma_J}{(1 - 2\delta_J)(1 - \delta_J)} \right).
\]

Collecting these bounds for \(I_1, I_2, I_3,\) we get
\[
\left\| P_{J^c} \hat{P}_J + \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} \right\|_2 \\
\leq \sqrt{\min(|J|, |J^c|)} \left( \frac{\delta_J^2}{1 - \delta_J} + \frac{\delta_J^2 + \delta_J^2 \gamma_J}{(1 - 2\delta_J)(1 - \delta_J)} + \frac{\delta_J \gamma_J}{1 - 2\delta_J} \right) \\
= \sqrt{\min(|J|, |J^c|)} \left( \frac{2\delta_J^2}{1 - 2\delta_J} + \frac{\delta_J \gamma_J}{(1 - 2\delta_J)(1 - \delta_J)} \right).
\]

By (6.6), we have
\[
\gamma_J \leq \left\| \sum_{k \in J^c} \sum_{j \in J} \frac{P_k EP_j}{\lambda_k - \lambda_j} \right\|_2 \leq \sqrt{\min(|J|, |J^c|) \delta_J},
\]
and the second claim follows from inserting this into the first one. 

**Proof of Proposition 1.** In order to prove (2.3), assume first that \(\delta_J < 1/4\). Then it follows from \(\|P_J - \hat{P}_J\|^2 = 2\|P_{J^c} \hat{P}_J\|^2\) and the second inequality in Lemma 10 that (2.3) holds. On the other hand, we always have \(\|P_J - \hat{P}_J\|^2 \leq 2\min(|J|, |J^c|)\), implying (2.3) also in the case that \(\delta_J \geq 1/4\).
It remains to prove (2.4). Applying the identities $I = P_\mathcal{J} + P_{\mathcal{J}_c} = \hat{P}_\mathcal{J} + \hat{P}_{\mathcal{J}_c}$ and $P_\mathcal{J} P_{\mathcal{J}_c} = 0$, we have

\[ \hat{P}_\mathcal{J} - P_\mathcal{J} = \hat{P}_\mathcal{J} P_{\mathcal{J}_c} - \hat{P}_{\mathcal{J}_c} P_\mathcal{J} \]

\[ = \hat{P}_\mathcal{J} P_{\mathcal{J}_c} - P_{\mathcal{J}_c} \hat{P}_\mathcal{J} - P_\mathcal{J} \hat{P}_{\mathcal{J}_c} \]

\[ = \hat{P}_\mathcal{J} P_{\mathcal{J}_c} + P_{\mathcal{J}_c} \hat{P}_\mathcal{J} - P_\mathcal{J} \hat{P}_{\mathcal{J}_c} \]

\[ = \hat{P}_\mathcal{J} P_{\mathcal{J}_c} + P_{\mathcal{J}_c} \hat{P}_\mathcal{J} - P_\mathcal{J} \hat{P}_{\mathcal{J}_c} - P_\mathcal{J} \hat{P}_{\mathcal{J}_c} P_\mathcal{J}. \]

Combining this with the triangle inequality, we obtain

\[ \| \hat{P}_\mathcal{J} - P_\mathcal{J} - L_\mathcal{J} E \|_2 \]

\[ \leq \| \hat{P}_\mathcal{J} P_{\mathcal{J}_c} + P_{\mathcal{J}_c} \hat{P}_\mathcal{J} - L_\mathcal{J} E \|_2 + \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} P_\mathcal{J} \|_2 + \| P_\mathcal{J} \hat{P}_{\mathcal{J}_c} P_\mathcal{J} \|_2. \]

Inserting the definition of the linear term in (2.2), we get

\[ \| \hat{P}_\mathcal{J} P_{\mathcal{J}_c} + P_{\mathcal{J}_c} \hat{P}_\mathcal{J} - L_\mathcal{J} E \|_2 \leq \| \hat{P}_\mathcal{J} P_{\mathcal{J}_c} + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}_c} P_j E P_k \|_2 + \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}_c} P_k E P_j \|_2 \]

\[ = 2 \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}_c} P_k E P_j \|_2. \]

We again start with the case $\delta_\mathcal{J} < 1/4$. First, by Lemma 10 and the identity $\| \hat{P}_\mathcal{J} P_{\mathcal{J}_c} \|_2 = \| \hat{P}_{\mathcal{J}_c} P_\mathcal{J} \|_2$, we get

\[ \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} P_{\mathcal{J}_c} \|_2 \leq \| \hat{P}_\mathcal{J} P_{\mathcal{J}_c} \|_2^2 \leq 4 \min(|\mathcal{J}|, |\mathcal{J}_c|) \delta_\mathcal{J}^2 \]

and

\[ \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} P_{\mathcal{J}_c} \|_2 \leq \| P_{\mathcal{J}_c} P_\mathcal{J} \|_2^2 \leq 4 \min(|\mathcal{J}|, |\mathcal{J}_c|) \delta_\mathcal{J}^2. \]

Second, by Lemma 11, we have

\[ 2 \| P_{\mathcal{J}_c} \hat{P}_\mathcal{J} + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}_c} P_k E P_j \|_2 \leq 2 \left( 4 + \frac{8}{3} \right) \min(|\mathcal{J}|, |\mathcal{J}_c|) \delta_\mathcal{J}^2. \]

Hence, in the case $\delta_\mathcal{J} < 1/4$, we arrive at

\[ \| \hat{P}_\mathcal{J} - P_\mathcal{J} - L_\mathcal{J} E \|_2 \leq \left( 8 + \frac{16}{3} + 8 \right) \min(|\mathcal{J}|, |\mathcal{J}_c|) \delta_\mathcal{J}^2. \]
Finally, if $\delta_{\mathcal{J}} \geq 1/4$, then we can use $\|P_{\mathcal{J}} - \hat{P}_{\mathcal{J}}\|_2 \leq \sqrt{2} \min(|\mathcal{J}|, |\mathcal{J}^c|)^{1/2}$ and (6.6) to obtain
\[
\|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|_2 \leq \|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}}\|_2 + \|L_{\mathcal{J}E}\|_2
\leq \min(|\mathcal{J}|, |\mathcal{J}^c|)^{1/2}(\sqrt{2} + \sqrt{2}\delta_{\mathcal{J}})
\leq \min(|\mathcal{J}|, |\mathcal{J}^c|)(16\sqrt{2} + 4\sqrt{2})\delta_{\mathcal{J}}^2.
\]
This completes the proof. \qed

**Proof of Corollary 1.** By the Cauchy-Schwarz inequality, we have
\[
\|P_{\mathcal{J}} - \hat{P}_{\mathcal{J}}\|^2 - \|L_{\mathcal{J}E}\|^2
= \|P_{\mathcal{J}} - \hat{P}_{\mathcal{J}} - L_{\mathcal{J}E}\|^2 - \|L_{\mathcal{J}E}\|^2
\leq \|P_{\mathcal{J}} - \hat{P}_{\mathcal{J}} - L_{\mathcal{J}E}\|^2 + 2\|P_{\mathcal{J}} - \hat{P}_{\mathcal{J}} - L_{\mathcal{J}E}\|_2\|L_{\mathcal{J}E}\|_2.
\]
Hence, the claim follows from inserting the second claim of Proposition 1 and (6.6). \qed

**Proof of Corollary 2.** By the triangular inequality, the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ and the Cauchy-Schwarz inequality, it follows that
\[
\|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|^2 - \|L_{\mathcal{J}(E - \tilde{E})}\|^2
\leq \|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E - \tilde{E}} - (\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E})\|^2
+ 2 \langle \hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E - \tilde{E}} - (\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}), L_{\mathcal{J}(E - \tilde{E})} \rangle
\leq 2\|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|^2 + 2\|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|_2^2
+ 2\|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|_2 + \|\hat{P}_{\mathcal{J}} - P_{\mathcal{J}} - L_{\mathcal{J}E}\|_2(\|L_{\mathcal{J}E}\|_2 + \|L_{\mathcal{J}\tilde{E}}\|_2).
\]
Hence, the claim follows from inserting the second claim of Proposition 1 and (6.6) applied to both $\hat{\Sigma}$ and $\tilde{\Sigma}$. \qed

### 6.2 Proofs for Section 2.3.1

**Lemma 12.** Consider Setting 1. Then we have
\[
U \left( n^{-1} \| \sum_{i=1}^{n} Y_i \|^2 + V, \| Z \|^2 + V \right) \preceq v^{-3} n^{-q/2} \sum_{i=1}^{n} E \| Y_i \|^q.
\]
Here, $V \sim \mathcal{N}(0, v^2)$, $v \in (0, 1]$, is independent of the $Y_i$ and $Z$. 37
Proof of Lemma 12. The proof is mainly relying on methods given in [4]. For a random variable $Y$ in $H$, let $Y^\delta = Y 1_{\|Y\| \leq \delta \sqrt{n}}$. Then by properties of the uniform distance (or a simple conditioning argument), we have

$$U \left( n^{-1} \| \sum_{i=1}^n Y_i \|^2 + V, n^{-1} \| \sum_{i=1}^n Y_i \|^2 + V \right) \leq U \left( n^{-1} \| \sum_{i=1}^n Y_i \|^2, n^{-1} \| \sum_{i=1}^n Y_i \|^2 \right)$$

$$\leq \sum_{i=1}^n P(\|Y_i\| > \sqrt{n}) \leq n^{-q/2} \sum_{i=1}^n \mathbb{E}\|Y_i\|^q. \quad (6.7)$$

An application of Esseens smoothing inequality (cf. [12, Lemma 1, XVI.3]) yields

$$U \left( n^{-1} \| \sum_{i=1}^n Y_i \|^2 + V, n^{-1} \| \sum_{i=1}^n Y_i \|^2 + V \right) \leq n^{-1} \sum_{i=1}^n P(\|Y_i\| > \sqrt{n}) \leq n^{-q/2} \sum_{i=1}^n \mathbb{E}\|Y_i\|^q. \quad (6.8)$$

Next, proceeding as in [44] (see Equation (86), resp. Theorem 4.6 in [4]), one derives that

$$\left| \mathbb{E}e^{i \xi n^{-1} \| \sum_{i=1}^n Y_i \|^2} - \mathbb{E}e^{i \xi \|Z\|^2} \right| \lesssim \sum_{i=1}^n |\xi| + |\xi|^3 \left( \mathbb{E}n^{-1} \|Y_i\|^2 1_{\|Y_i\| \geq \sqrt{n}} + \mathbb{E}n^{-3/2} \|Y_i\|^3 1_{\|Y_i\| \leq \sqrt{n}} \right). \quad (6.9)$$

Since $q \in (2, 3]$, we have

$$\mathbb{E}n^{-1} \|Y_i\|^2 1_{\|Y_i\| \geq \sqrt{n}} \leq n^{-q/2} \mathbb{E}\|Y_i\|^q$$

and

$$\mathbb{E}n^{-3/2} \|Y_i\|^3 1_{\|Y_i\| \leq \sqrt{n}} \leq n^{-q/2} \mathbb{E}\|Y_i\|^q.$$

It follows that (recall $v \in (0, 1]$)

$$\int_{\mathbb{R}} \frac{\left| \mathbb{E}e^{i \xi n^{-1} \| \sum_{i=1}^n Y_i \|^2} - \mathbb{E}e^{i \xi \|Z\|^2} \right|}{\xi} e^{-v^2 \xi^2/2} d\xi \lesssim v^{-3} \int_{\mathbb{R}} (1 + |\xi|^2)n^{-q/2} \sum_{i=1}^n \mathbb{E}\|Y_i\|^q e^{-v^2 \xi^2/2} d\xi$$

$$\lesssim v^{-3} n^{-q/2} \sum_{i=1}^n \mathbb{E}\|Y_i\|^q.$$

This completes the proof. \qed
Lemma 13 ([38], Theorem 4.1). Let \((Y_i) \in \mathcal{H}\) be an independent sequence with \(EY_i = 0\) and \(E\|Y_i\|^q < \infty\) for all \(i\) and \(q \geq 1\). Then,

\[
E\|\sum_{i=1}^{n} Y_i\|^q \lesssim \left( \sum_{i=1}^{n} E\|Y_i\|^2 \right)^{q/2} + \sum_{i=1}^{n} E\|Y_i\|^q.
\]

Lemma 14. Consider Setting 1 and assume in addition that \(\sum_{i=1}^{n} E\|Y_i\|^r \leq n^{r/2}\) for \(r \in \{2, q\}\). Then

\[
W\left(n^{-1}\|\sum_{i=1}^{n} Y_i\|^2 + V, \|Z\|^2 + V\right) \lesssim q \sqrt{\frac{q}{2}} \left(n^{-q/2} \sum_{i=1}^{n} E\|Y_i\|^q\right)^{1-2/q}.
\]

Here, \(V \sim \mathcal{N}(0, v^2), \quad v \in (0, 1]\), is independent of \(Y_1, \ldots, Y_n\) and \(Z\).

Proof of Lemma 14. We use the well-known fact that for real-valued random variables \(X, Y\), we have the dual representation

\[
W(X, Y) = \int_{\mathbb{R}} |P(X \leq x) - P(Y \leq x)| \, dx.
\]

We conclude that for any \(a > 0\) and \(s > 1\), Markov’s inequality gives

\[
W(X, Y) \leq 2aU(X, Y) + 2 \int_{a}^{\infty} P(|X| \geq x) + P(|Y| \geq x) \, dx
\]

\[
\leq 2aU(X, Y) + 2s^{-1}a^{1-s}(E|X|^s + E|Y|^s).
\]

We apply this inequality with \(s = q/2 > 1\). By assumption, we have \(E\|Z\|^2 \leq 1\) and thus \(E\|Z\|^q \lesssim q 1\) (cf. Lemma 4). Lemma 12, the inequality \((x + y)^{q/2} \leq 2^{q/2-1}(x^{q/2} + y^{q/2}), \quad x, y \geq 0\), together with Lemma 13 then yield

\[
W\left(n^{-1}\|\sum_{i=1}^{n} Y_i\|^2 + V, \|Z\|^2 + V\right)
\]

\[
\lesssim q aU\left(n^{-1}\|\sum_{i=1}^{n} Y_i\|^2 + V, \|Z\|^2 + V\right)
\]

\[
+ a^{1-q/2}(E\|n^{-1/2} \sum_{i=1}^{n} Y_i\|^q + E\|Z\|^q + E|V|^{q/2})
\]

\[
\lesssim q a\sqrt{\frac{q}{2}} n^{-q/2} \sum_{i=1}^{n} E\|Y_i\|^q + a^{1-q/2}.
\]

Selecting \(a = \sqrt{q} n^{-q/2} \sum_{i=1}^{n} E\|Y_i\|^q\), the claim follows. \(\square\)
Lemma 15. Consider Setting 1. Suppose in addition that $\mathbb{E}\|Z\|^2 \leq 1$ and $|T| \leq C_T$ almost surely. Then for any $u, s > 0$, we have

$$W(T, \|Z\|^2) \lesssim_s W(S + V, \|Z\|^2 + V) + v + C_T \mathbb{P}(|T - S| > u) + u + C_T^{1-s}.$$ 

Here, $V \sim \mathcal{N}(0, v^2)$, $v \in (0, 1]$, is independent of $S, T, Z$.

Proof of Lemma 15. Let $\mathcal{E} = \{|T - S| \leq u\}$. By the triangle inequality and $1_\mathcal{E} \leq 1$, we have

$$W(T, \|Z\|^2) \leq W(T, 1_\mathcal{E}) + W(\|Z\|^2, \|Z\|^2 1_\mathcal{E}) + W(T 1_\mathcal{E}, \|Z\|^2 1_\mathcal{E})$$

$$\leq \mathbb{E}|T| 1_{\mathcal{E}} + \mathbb{E}\|Z\|^2 1_{\mathcal{E}} + W(S 1_\mathcal{E}, \|Z\|^2 1_\mathcal{E}) + W(S 1_\mathcal{E}, T 1_\mathcal{E})$$

$$\leq \mathbb{E}|T| 1_{\mathcal{E}} + \mathbb{E}\|Z\|^2 1_{\mathcal{E}} + W(S, \|Z\|^2) + u.$$

Since again by the triangle inequality

$$W(S, \|Z\|^2) \leq W(S + V, \|Z\|^2 + V) + 2\mathbb{E}|V|,$$

we get

$$W(T, \|Z\|^2) \leq \mathbb{E}|T| 1_{\mathcal{E}} + \mathbb{E}\|Z\|^2 1_{\mathcal{E}} + u + W(S + V, \|Z\|^2 + V) + 2\mathbb{E}|V|.$$ 

Since $\mathbb{E}\|Z\|^2 \leq 1$, it follows that $\mathbb{E}\|Z\|^{2s + 2} \lesssim_s 1$ for all $s > 0$ (cf. Lemma 4) and thus

$$\mathbb{E}\|Z\|^2 1_{\|Z\|^2 \geq x} \leq C_T^{-s} \mathbb{E}\|Z\|^{2s + 2} \lesssim_s C_T^{-s}.$$ 

Using this bound, we arrive at

$$\mathbb{E}|T| 1_{\mathcal{E}} + \mathbb{E}\|Z\|^2 1_{\mathcal{E}} \leq 2C_T \mathbb{P}(\mathcal{E}^c) + \mathbb{E}\|Z\|^2 1_{\|Z\|^2 \geq C_T}$$

$$\lesssim q C_T \mathbb{P}(\mathcal{E}^c) + C_T^{-s},$$

and the claim follows from $\mathbb{E}|V| \leq v$.

Proof of Proposition 2. The claim follows by combining Lemmas 14 and 15 and balancing with respect to $v$.

Lemma 16. Let $Y_1, \ldots, Y_n$ be independent random variables satisfying $\mathbb{E}Y_i = 0$, $\mathbb{E}Y^2_i = 1$ and $\mathbb{E}|Y_i|^3 < \infty$. For $a_i \in \mathbb{R}$, let $A_n = \sum_{i=1}^n a_i^2$. Then

$$\mathbb{U}\left(A_n^{-1/2} \sum_{i=1}^n a_i Y_i, G\right) \lesssim A_n^{-3/2} \sum_{i=1}^n |a_i|^3 \mathbb{E}|Y_i|^3.$$
Proof of Lemma 16. This immediately follows from a general version of the Berry-Esseen Theorem for non-identically distributed random variables, see e.g. [37].

Lemma 17. Let $G \sim \mathcal{N}(0,1)$ and $T, S$ be real-valued random variables. Then, for any $a, b > 0, c \in \mathbb{R}$,

$$U((T - c)/b, G) \lesssim U((S - c)/b, G) + \mathbb{P}(|T - S| > a) + a/b.$$  

Proof of Lemma 17. Recall first that the uniform metric $U$ is invariant with respect to affine transformations of the underlying random variables. Next, note that

$$\mathbb{P}(T \leq x) \leq \mathbb{P}(S \leq x + a) + \mathbb{P}(|T - S| > a).$$

A corresponding lower bound is also valid. Therefore, using the above mentioned affine invariance and the fact that the distribution function of $G$ is Lipschitz continuous, the claim follows.

Next, we recall the following smoothing inequality (cf. [42], Lemma 4.2.1).

Lemma 18. Let $X$ be a real-valued random variable, $G \sim \mathcal{N}(0, \sigma^2)$, $\sigma > 0$, and $G_\epsilon \sim \mathcal{N}(0, \epsilon^2)$, $\epsilon > 0$, independent of $X, G$. Then there exists an absolute constant $C$, such that

$$U(X, G) \leq U(X + G_\epsilon, G + G_\epsilon) + C\frac{\epsilon}{\sigma}.$$  

Proof of Proposition 4. Due to Lemma 17 (with $c = A, b = B$), it suffices to derive a bound for

$$U((S - A)/B, G).$$

For any $\epsilon > 0$, let $G_\epsilon \sim \mathcal{N}(0, \epsilon^2)$. By Lemma 18 and the triangle inequality, we get

$$U(S - A)/B, G)$$
$$\lesssim U((S - A)/B + G_\epsilon, G + G_\epsilon) + \epsilon$$
$$\lesssim U((S - A)/B + G_\epsilon, (\|Z\|^2 - A)/B + G_\epsilon) + \epsilon.$$
By the affine invariance of $U$, we have

$$U\left((S - A)/B + G, (\|Z\|^2 - A)/B + G\right) = U(S/A) + G, B/A, \|Z\|^2/A + G, B/A).$$

Setting $V = G, B/A$ implies $v^2 = EV^2 = (B\epsilon/A)^2$. Hence an application of Lemma 12 yields

$$U(S/A + G, B/A, \|Z\|^2/A + G, B/A) \lesssim \left(\frac{A}{\epsilon B}\right)^{3/2} n^{-q/2} \sum_{i=1}^{n} \frac{E\|Y_i\|^q}{A^{q/2}}.$$

On the other hand, by the regularity of $U$ (or explicitly by a simple conditioning argument) and Lemma 16 (applied with $n = 1$), we obtain the bound

$$U((\|Z\|^2 - A)/B + G, G + G) \lesssim C^3 B^3.$$

Piecing everything together, the claim follows.

For the next result, recall the definition of $\lambda_{1,j}(\Psi)$, given in (2.6).

**Lemma 19** (Lemma 3 in [36]). Let $Z \in \mathcal{H}$ be a Gaussian random variable with $EZ = 0$ and covariance operator $\Psi$. Then, for any $a > 0$,

$$\sup_{x > 0} P(x < \|Z\|^2 < x + a) \leq \frac{a}{\sqrt{\lambda_{1,2}(\Psi)}}.$$

In particular, for any $x \geq 0$, we have

$$\limsup_{a \to 0} \frac{\left| P(\|Z\|^2 < x + a) - P(\|Z\|^2 < x) \right|}{a} \leq \frac{1}{\sqrt{\lambda_{1,2}(\Psi)}},$$

that is, we have an uniform upper bound for the density of $\|Z\|^2$.

**Lemma 20** (Lemma 2 in [36]). For $i \in \{1, 2\}$, let $Z_i$ be Gaussian with $EZ_i = 0$ and covariance operator $\Psi_i$. Then

$$U(\|Z_1\|^2, \|Z_2\|^2) \lesssim \left(\frac{1}{\sqrt{\lambda_{1,2}(\Psi_1)}} + \frac{1}{\sqrt{\lambda_{1,2}(\Psi_2)}}\right)\|\Psi_1 - \Psi_2\|_1.$$

**Lemma 21.** Consider Setting 1 with $q = 3$. Then, we have

$$U\left(\frac{1}{\sqrt{\lambda_{1,2}(\Psi)}} \sum_{i=1}^{n} \frac{E\|Y_i\|^3}{\lambda_{1,2}(\Psi)}\right)^{2/5} \leq n^{-3/5}\left(\sum_{i=1}^{n} \frac{E\|Y_i\|^3}{\lambda_{1,2}(\Psi)}\right)^{2/5}.$$
Proof of Lemma 21. We argue very similarly as in the proof of Lemma 12, using also the notation therein. By Esseen’s smoothing inequality (cf. [12, Lemma 1, XVI.3]) together with Lemma 19, we have for any $a > 0$

\[
\begin{align*}
\mathbf{U}\left(\sum_{i=1}^{n} (Y_i/a)^2, \|Z/a\|^2\right) &\leq \int_{-t}^{t} \frac{\mathbb{E} e^{i\xi n^{-1} \sum_{i=1}^{n} Y_i/a^2} - \mathbb{E} e^{i\xi Z/a^2}}{\sqrt{\lambda_{1,2}(\Psi)t}} d\xi + \frac{a^2}{\sqrt{\lambda_{1,2}(\Psi)t}} \\
&\lesssim t^3 n^{-3/2} \sum_{i=1}^{n} \frac{\mathbb{E} \|Y_i\|^3}{a^3} + \frac{a^2}{\sqrt{\lambda_{1,2}(\Psi)t}}.
\end{align*}
\]

Setting $t = 1$ and using also (6.7), the claim follows from the (optimal) choice $a = (n^{-3/2} \sum_{i=1}^{n} \mathbb{E} \|Y_i\|^3 \sqrt{\lambda_{1,2}(\Psi)})^{1/5}$.

Finally, let us mention the following result of [40].

**Lemma 22.** Consider Setting 1 with $q = 3$. Then

\[
\mathbf{U}\left(\sum_{i=1}^{n} Y_i^2, \|Z\|^2\right) \lesssim \left(\frac{A C^3}{B^3}\right)^{1+1/10} \lambda_{6}^{-1}(\Psi) + n^{-1/2} \lambda_{1,6}^{-1}(\Psi) \frac{(A^2 C)^3}{B^3},
\]

Proof of Proposition 3. There exists a constant $C > 0$ such that, for all $x \geq 0$,

\[
\begin{align*}
P(T \leq x) &\leq P(S \leq x + u) + P(|T - S| > u) \\
&\leq P(\|Z\|^2 \leq x + u) + \mathbf{U}(S, \|Z\|^2) + P(|T - S| > u) \\
&\leq P(\|Z\|^2 \leq x) + C u \sqrt{\lambda_{1,2}(\Psi)} + \mathbf{U}(S, \|Z\|^2) + P(|T - S| > u),
\end{align*}
\]

where we used Lemma 19 in the last step. In the same way, one derives a corresponding lower bound. Hence, the first claim now follows from Lemma 21. Using Lemma 22 instead of Lemma 21, the second claim follows.

6.3 Proofs for Section 2.3.2

Proof of Corollary 3. The claim follows from Lemma 1 in [21].

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Proof of Corollary 4. Let $B$ be the Banach space of all nuclear operators on $\mathcal{H}$ equipped with the nuclear norm. It is well-known that the dual space $B^*$ of $B$ is the Banach space of all bounded linear operators equipped with the operator norm $\| \cdot \|_\infty$. Moreover, for a bounded linear operator $S$, the corresponding functional is given by $T \mapsto \text{tr}(TS)$. In order to apply Lemma 2 with $Z_i = Y_i \otimes Y_i$ and $s = p > 2$, it remains to bound all involved quantities. First, since the map $A \mapsto \text{tr}(A^{1/2})$ is concave on the set of all positive self-adjoint trace-class operators, Jensen’s inequality yields

$$E\|\frac{1}{n} \sum_{i=1}^{n} Y_i \otimes Y_i \|_1 \leq \text{tr} \left( \text{E}^{1/2} \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \otimes Y_i \right)^2 \right) = \frac{1}{\sqrt{n}} \text{tr} \left( \text{E}^{1/2} (Y \otimes Y)^2 \right).$$

Next, since $Y \otimes Y$ is rank-one, we have $\| Y \otimes Y \|_1 = \| Y \|_2^2$. Hence, by the triangle inequality and Jensen’s inequality, we have

$$E\| Y \otimes Y \|_1^p \leq 2^{p-1} E\| Y \otimes Y \|_1^p + 2^{p-1} E\| Y \otimes Y \|_1^p \leq 2^p E\| Y \otimes Y \|_1^p \leq 2^p \text{tr}(Y \otimes Y)^2.$$ 

Combining this with Lemma 4, we get

$$E\| Y \otimes Y \|_1^p \leq 2^p C \| \vartheta \|_1^p.$$ 

(6.11)

Finally, by the above discussion, we have

$$\omega_n^2 = \sup_{\| S \|_\infty \leq 1} E \text{tr}^2(S Y \otimes Y).$$

This completes the proof.

Proof of Lemma 3. The proof is based on (a variant of) Theorem 3.2 in [34], see Lemma 23 below, which follows from a standard approximation result. To this end, for any $\theta > 0$, let the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\theta^{-1} \psi(\theta x) = \text{sign}(x)(|x| \wedge \theta^{-1}).$$

For a compact, symmetric operator $A$ and function $f : \mathbb{R} \rightarrow \mathbb{R}$, we denote the usual functional transformation $f(A)$, that is, function $f$ acts only on the spectrum of $A$.

Lemma 23. Let $(Z_i)$ with $Z_i \overset{d}{=} Z$ be a sequence of i.i.d. compact self-adjoint random operators on $\mathcal{H}$, and let $\tau_n^2 \geq n\|E Z^2\|_\infty$. Then

$$\mathbb{P} \left( \left\| \sum_{i=1}^{n} \left( \frac{1}{\theta} \psi(\theta Z_i) - E Z_i \right) \right\|_\infty \geq t \sqrt{n} \right) \leq 2\bar{d} \left( 1 + \frac{1}{\theta t \sqrt{n}} \right) \exp \left( - \theta t \sqrt{n} + \frac{\theta^2 \tau_n^2}{2} \right),$$

where $\bar{d} = \text{tr}(E Z^2)/\|E Z^2\|_\infty$. 

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We now turn to the proof of Lemma 3, combining Lemma 23 with a truncation argument. To this end, set $\theta = t/\sqrt{n}$ and $\tau_n^2 = n\|E(Y \otimes Y)^2\|_\infty = n$, recalling that $\|E(Y \otimes Y)^2\|_\infty = 1$. First, we have

$$P\left(\frac{1}{\theta} \psi(\theta Y_i \otimes Y_i) \neq Y_i \otimes Y_i \text{ for at least one } 1 \leq i \leq n\right)$$

$$\leq P\left(\max_{1 \leq i \leq n} \|Y_i \otimes Y_i\|_\infty \geq \sqrt{n}/t\right)$$

$$\leq \sum_{i=1}^{n} n^{-p/2} t^p E\|Y_i \otimes Y_i\|^p_\infty = n^{1-p/2} t^p E\|Y \otimes Y\|^p_\infty.$$ 

Combining this with (6.11), it follows that

$$P\left(\left\|\sum_{i=1}^{n} Y_i \otimes Y_i\right\|_\infty \geq t\sqrt{n}\right) \leq n^{1-q/2} \|\vartheta\|_\| + P\left(\left\|\sum_{i=1}^{n} \frac{1}{\theta} \psi(\theta Y_i \otimes Y_i)\right\|_\infty \geq t\sqrt{n}\right).$$

An application of Lemma 23 yields

$$P\left(\left\|\sum_{i=1}^{n} \frac{1}{\theta} \psi(\theta Y_i \otimes Y_i)\right\|_\infty \geq t\sqrt{n}\right) \leq 4 \text{ tr } (E(Y \otimes Y)^2) \exp\left(-\frac{t^2}{2}\right),$$

The claim now follows from Lemma 4 (ii). \[\square\]

**Proof of Lemma 4.** For $1 \leq r \leq p$, we have

$$\|Y\|^{2r} = \left(\sum_{j \geq 1} \vartheta_j \zeta_j^2\right)^r.$$

Hence, by the triangle inequality, the Hölder inequality and the moment assumption, we have

$$(E\|Y\|^{2r})^{1/r} \leq \sum_{j \geq 1} \vartheta_j (E \varphi_j^{2r})^{1/r} \leq \sum_{j \geq 1} \vartheta_j (E \varphi_j^{2p})^{1/p} \leq C^{1/p} \sum_{j \geq 1} \vartheta_j = C^{1/p} \vartheta_1.$$ 

This gives claim (i). In order to prove (ii), let us write

$$(Y \otimes Y - EY \otimes Y)^2 = \left(\sum_{j,k \geq 1} \vartheta_j \vartheta_k \zeta_j \zeta_k (u_j \otimes u_k)\right)^2$$

$$= \sum_{j,k,s \geq 1} \vartheta_j \vartheta_k \vartheta_s \zeta_j \zeta_k \zeta_s (u_j \otimes u_k).$$
where the second equality follows from the fact that \((u_j \otimes u_k)(u_r \otimes u_s)\) is equal to \(u_j \otimes u_s\) if \(k = r\) and equal to 0 otherwise. Hence,
\[
\text{tr}(Y \otimes Y - EY \otimes Y)^2 = \sum_{j,k \geq 1} \vartheta_j \vartheta_k (\zeta_j \zeta_k)^2
\]
and thus
\[
\text{tr}(E(Y \otimes Y - EY \otimes Y)^2) \leq C^{2/p} \sum_{j,k \geq 1} \vartheta_j \vartheta_k,
\]
as can be seen from inserting \(E(\zeta_j \zeta_k)^2 \leq (E\zeta_j^4)^{1/2}(E\zeta_k^4)^{1/2} \leq C^{2/p}\). To see the improvements of (iii)–(iv), let us note that under the additional assumptions, we have \(E\zeta_j \zeta_k \zeta_s = 0\) for \(j \neq s\). Indeed, if \(j \neq s\), then either \(j \neq k\) and \(k \neq s\) in which case \(E\zeta_j \zeta_k \zeta_s = E\zeta_j \zeta_s = 0\), or \(j = k\) and \(k \neq s\) in which case \(E\zeta_j \zeta_k \zeta_s = E\zeta^3_k\zeta_s - (E\zeta_j^2)(E\zeta_k\zeta_s) = 0\), or \(j \neq k\) and \(s = k\) in which case \(E\zeta_j \zeta_k \zeta_s = E\zeta_j \zeta_j - (E\zeta_j^2)(E\zeta_k\zeta_j) = 0\). Hence, under the additional assumptions, we have
\[
E(Y \otimes Y - EY \otimes Y)^2 = \sum_{j,k \geq 1} \vartheta_j \vartheta_k E(\zeta_j \zeta_k)^2 (u_j \otimes u_j),
\]
leading to
\[
\|E(Y \otimes Y - EY \otimes Y)^2\|_\infty = \max_{j \geq 1} \sum_{k \geq 1} \vartheta_j \vartheta_k E(\zeta_j \zeta_k)^2 \leq C^{2/p} \|\vartheta\|_\infty \|\vartheta\|_1
\]
and
\[
\text{tr}((E(Y \otimes Y - EY \otimes Y)^2)^{1/2}) = \sum_{j \geq 1} \left( \sum_{k \geq 1} \vartheta_j \vartheta_k E(\zeta_j \zeta_k)^2 \right)^{1/2} \leq C^{1/p} \left( \sum_{j \geq 1} \sqrt{\vartheta_j} \right) \|\vartheta\|_1^{1/2}.
\]
This gives claims (iii) and (iv).

6.4 CLTs: Proofs for Section 3

Proof of Theorem 1. We want to apply Proposition 2 with the choices
\[
T = \frac{n}{A_J} \|\hat{P}_J - P_J\|_2^2 \quad \text{and} \quad S = \frac{n}{A_J} \|L_J E\|_2^2,
\]
For this, let us write
\[
S = \frac{1}{n} \left\| \sum_{i=1}^{n} Y_i \right\|_2^2 \quad \text{with} \quad Y_i = \frac{1}{\sqrt{A_J}} L_J(X_i \otimes X_i).
\]

The random variable \( L_J X \otimes X \) takes values in the separable Hilbert space of all (self-adjoint) Hilbert-Schmidt operators on \( \mathcal{H} \) (endowed with trace-inner product) and has the decomposition
\[
L_J(X \otimes X) = \sum_{j \in \mathcal{J}} \sum_{k \notin \mathcal{J}} \frac{\sqrt{\lambda_j \lambda_k}}{\lambda_j - \lambda_k} \zeta_{jk}(u_j \otimes u_k + u_k \otimes u_j) \tag{6.12}
\]
with \( \zeta_{jk} = \eta_j \eta_k \), as can be seen from inserting the Karhunen-Loève expansion of \( X \) and the definition of \( L_J \). Using Assumption 1, the Cauchy-Schwarz inequality and the fact that the summation is over different indices, we have
\[
E \zeta_{jk} = 0 \quad \text{and} \quad E|\zeta_{jk}|^p \leq C_{\eta}.
\]
Hence, by Lemma 4(i) and Lemma 7, we have for every \( 1 \leq r \leq p \),
\[
E\|L_J(X \otimes X)\|_r^r \lesssim \left( \sum_{j \in \mathcal{J}} \sum_{k \notin \mathcal{J}} \frac{\lambda_k \lambda_j}{(\lambda_k - \lambda_j)^2} \right)^{r/2} \asymp A_J^{r/2}. \tag{6.13}
\]
In particular, setting \( q = p \wedge 3 \), we get
\[
E\|Y_i\|_2^p \lesssim 1
\]
for all \( r \in [2, q] \). By Corollary 1, we have
\[
|T - S| = \frac{n}{A_J} \left( \|\hat{P}_J - P_J\|_2^2 - \|L_J E\|_2^2 \right) \leq \frac{n}{A_J} (|\mathcal{J}|^{3/2} \delta_{\mathcal{J}}^{3/2}(E) + |\mathcal{J}|^2 \delta_{\mathcal{J}}^{4}(E)).
\]
Thus, by (3.5) and Lemma 6, we get
\[
P(|T - S| > u) \lesssim_p P_{\mathcal{J}, n, p}, \tag{6.14}
\]
as long as
\[
u = Cn^{-1/2} \log^{3/2} \frac{n \sigma^3_{\mathcal{J}}|\mathcal{J}|^{3/2}}{A_J} \tag{6.15}
\]
Finally, we have
\[
\frac{n}{A_J} \left\| \hat{P}_J - P_J \right\|_2^2 \leq \frac{2|\mathcal{J}|n}{A_J} =: C_T. \tag{6.16}
\]
Inserting these choices for \( T, S, u \) and \( C_T \) into Proposition 2 (applied with \( q = p \wedge 3 \)), the claim follows.
Proof of Theorem 3. We want to apply Proposition 4 with the choices \( A = A_J, B = B_J, C = C_J \) (recall (3.4)),

\[
T = n\|\hat{P}_J - P_J\|_2^2, \quad S = n\|L_JE\|_2^2,
\]
and

\[
u = Cn^{-1/2}(\log n)^{3/2}\sigma_J^3 |J|^{3/2}.
\]

For this, let us write

\[
S = \frac{1}{n}\left\| \sum_{i=1}^{n} Y_i \right\|_2^2 \text{ with } Y_i = L_J (X_i \otimes X_i).
\]

Inserting these choices for \( A, B, C, T, S, \nu \) into Proposition 4 (applied with \( q = \min(3, p) \)), the claim follows from inserting (6.13) and (6.14). \( \square \)

Proof of Corollary 5. Using Lemma 16, we may argue as in the proof of Proposition 4. \( \square \)

Proof of Theorem 2. We proceed exactly as in the proof of Theorem 3, using Proposition 3 instead of Proposition 4. \( \square \)

6.5 Bootstrap I: Proof of Theorem 4

We start by applying Proposition 4 with respect to \( \tilde{U} \), and

\[
\tilde{T} = \frac{n}{\sigma_J^2} \|\hat{P}_J - P_J\|_2^2 \quad \text{and} \quad \tilde{S} = \frac{n}{\sigma_J^2} \|L_J(\hat{E} - E)\|_2^2.
\] (6.17)

In doing so, we consider Setting 1 with

\[
\tilde{S} = \frac{1}{n}\left\| \sum_{i=1}^{n} \tilde{Y}_i \right\|_2^2, \text{ where } \tilde{Y}_i = \frac{w_i^2 - 1}{\sigma_w} Y_i, \quad Y_i = L_J (X_i \otimes X_i).
\]

Note that we use an additional tilde to indicate that we are dealing with the bootstrap quantities. Hence, we have

\[
\hat{\Psi}_J = \frac{1}{n} \sum_{i=1}^{n} \hat{E}\hat{Y}_i \otimes \hat{Y}_i = \frac{1}{n} \sum_{i=1}^{n} \hat{Y}_i \otimes \hat{Y}_i,
\]

as well as

\[
\hat{A}_J = \|\hat{\Psi}_J\|_1 = \hat{E}\hat{S},
\]

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$$\bar{B}_J = \sqrt{2} \| \tilde{\Psi}_J \|_2,$$
$$\bar{C}_J = 2 \| \tilde{\Psi}_J \|_3.$$

The following lemma shows that the quantities $\tilde{A}_J$, $\bar{B}_J$, $\bar{C}_J$ are concentrated around their corresponding quantities $A_J$, $B_J$, $C_J$ from (3.4).

**Lemma 24.** Suppose that (4.3) holds. Then with probability at least $1 - 2n^{1-p/4}$, we have

$$|\tilde{A}_J - A_J| \lesssim_p A_J \frac{\sqrt{\log n}}{\sqrt{n}},$$
$$|\bar{B}_J - B_J| \lesssim_p A_J \frac{\sqrt{\log n}}{\sqrt{n}},$$
$$|\bar{C}_J - C_J| \lesssim_p A_J \frac{\sqrt{\log n}}{\sqrt{n}}.$$

In particular, for $n$ large enough, we have with the same probability,

(i) $A_J \leq 2 \tilde{A}_J \leq 4 A_J$,
(ii) $B_J \leq 2 \bar{B}_J \leq 4 B_J$,
(iii) $C_J \leq 2 \bar{C}_J \leq 4 C_J$.

**Proof.** First, (i)–(iii) are consequences of (4.3) and the first three claims (note that $C_J \lesssim B_J \lesssim A_J$ by properties of the Schatten norms). Let us start proving the first claim. By independence of $(w_i)$ and $(Y_i)$, we have

$$\tilde{A}_J = \| \tilde{\Psi} \|_1 = \tilde{E} S = \frac{1}{n} \sum_{i=1}^n \tilde{E} \| Y_i \|_2^2 = \frac{1}{n} \sum_{i=1}^n \| Y_i \|_2^2.$$

By (6.13), we have $(E(\|Y\|_2^2)^{p/2})^{1/p} \lesssim_p A_J$. Since $E \tilde{A}_J = A_J$, (2.7) yields that for some constant $C > 0$,

$$\mathbb{P} \left( |\tilde{A}_J - A_J| \geq CA_J \frac{\sqrt{\log n}}{\sqrt{n}} \right) \leq n^{1-p/4}.$$

Using that $\bar{B}_J = \sqrt{2} \| \tilde{\Psi}_J \|_2$ and $B_J = \sqrt{2} \| \Psi_J \|_2$, the triangle inequality and $\| \tilde{\Psi}_J \|_2 - \| \Psi_J \|_2 \leq \| \tilde{\Psi}_J - \Psi_J \|_2$, the second claim follows if we can show that, with probability at least $1 - n^{1-p/4}$,

$$\| \tilde{\Psi}_J - \Psi_J \|_2 = \left\| \frac{1}{n} \sum_{i=1}^n Y_i \otimes Y_i - \Psi \right\|_2 \lesssim_p A_J \frac{\sqrt{\log n}}{\sqrt{n}}. \quad (6.18)$$
In order to get (6.18), we apply Corollary 3. For this, recall from (6.12) that we have

\[ Y_i = Y = L_j X \otimes X = \sum_{j \in \mathcal{J}} \sum_{k \notin \mathcal{J}} \sqrt{\lambda_j \lambda_k} \zeta_{jk} (u_j \otimes u_k + u_k \otimes u_j), \]

with \( \zeta_{jk} = n_j \eta_k \), where the \( \zeta_{jk} \) satisfy \( \mathbb{E} \zeta_{jk} = 0 \) and \( \mathbb{E} |\zeta_{jk}|^p \leq C_n \) for all \( j \in \mathcal{J} \) and \( k \notin \mathcal{J} \). Hence, Corollary 3 and Lemma 7 yield that for some \( C > 0 \),

\[ \mathbb{P} \left( \| \tilde{\Psi} - \Psi \|_2^2 \geq C \frac{t}{\sqrt{n}} A_{\mathcal{J}} \right) \leq n^{1-p/4} \frac{t^p}{t^p/2} + e^{-t^2}, \]

where \( t \geq 1 \), and (6.18) follows by setting \( t = C \sqrt{\log n} \). Finally, using that \( \tilde{C}_{\mathcal{J}} = 2 \| \tilde{\Psi} \|_3 \) and \( C_\mathcal{J} = 2 \| \Psi \|_3 \) and the inequality \( \| \tilde{\Psi} \|_3 - |\Psi \|_3 | \leq \| \tilde{\Psi} - \Psi \|_3 \leq \| \tilde{\Psi} - \Psi \|_2 \), the third claim follows from (6.18). Applying the union bound completes the proof.

Corollary 8. Let \( q \in (2, 3] \) with \( q \leq p \) and \( s \in (0, 1) \). Then, with probability at least \( 1 - C_p \mathbb{P}_{\mathcal{J}, n, p}^{1-s} - C_p n^{1-p/(2q)} \), \( C_p > 0 \), we have

\[ \mathcal{U} \left( \frac{\tilde{T} - \tilde{A}_\mathcal{J}}{B_\mathcal{J}}, G \right) \lesssim_p n^{1/4-q/8} \frac{A_{\mathcal{J}}^{3/4}}{B_{\mathcal{J}}^{3/4}} + \frac{C_{\mathcal{J}}^3}{B_{\mathcal{J}}^3} + n^{-1/2} (\log n)^{3/2} \frac{C_{\mathcal{J}}^3}{B_{\mathcal{J}}} + \mathbb{P}_{\mathcal{J}, n, p}^s. \]

Proof. Applying Proposition 4 to the choices (6.17), we get

\[ \mathcal{U} \left( \frac{\tilde{T} - \tilde{A}_\mathcal{J}}{B_\mathcal{J}}, G \right) \lesssim \frac{1}{n^{q/8}} \frac{\tilde{A}_{\mathcal{J}}^3}{B_{\mathcal{J}}^3} \sum_{i=1}^n \frac{\|Y_i\|_2^q}{\tilde{A}_{\mathcal{J}}^q} \left( \sum_{i=1}^n \frac{\|Y_i\|_2^q}{\tilde{A}_{\mathcal{J}}^q} \right)^{1/4} + \frac{C_{\mathcal{J}}^3}{B_{\mathcal{J}}^3} + \mathbb{P}(|\tilde{S} - \tilde{T}| > u) + \frac{u}{B_{\mathcal{J}}}. \]

By (6.13), we have \( (\mathbb{E} \|Y\|_2^q)^{q/p} \lesssim A_{\mathcal{J}}^{q/2} \). Hence, (2.7) yields that for some constant \( C > 0 \),

\[ \mathbb{P} \left( \sum_{i=1}^n \left( \|Y_i\|_2^q - \mathbb{E} \|Y_i\|_2^q \right) \geq C A_{\mathcal{J}}^{q/2} \sqrt{n \log n} \right) \leq n^{1-p/(2q)}. \]

Hence, with probability at least \( 1 - n^{1-p/(2q)} \),

\[ \sum_{i=1}^n \|Y_i\|_2^q \lesssim n \mathbb{E} \|Y\|_2^q + \sqrt{n \log n} A_{\mathcal{J}}^{q/2} \lesssim n A_{\mathcal{J}}^{q/2}. \]
By Corollary 2, we have
\[
|\tilde{T} - \tilde{S}| = \frac{n}{\sigma_w^2} \left\| \tilde{P}_J - \tilde{P}_J \right\|_2^2 - \left\| L_J(\tilde{E} - E) \right\|_2^2
\leq \frac{n}{\sigma_w^2} \left( |J|^{3/2}(\delta_J^2(E) + \delta_J^2(\tilde{E})) + |\hat{J}|^2 (\delta_J^2(E) + \delta_J^2(\tilde{E})) \right).
\]

Similarly as in Lemma 6, we have
\[
P\left( \delta_J(\tilde{E}) > C \sqrt{\frac{\sigma_J^2 \log n}{n}} \right) \leq_p P_{J,n,p} \tag{6.20}
\]

In fact, since \( w \) is independent of \( X \) and satisfies \( Ew^2 = 1 \), the random variable \( w(|R_{J^c}|^{1/2} + g_J^{-1/2}P_J)X \) has the same covariance operator as \( X' = (|R_{J^c}| + g_J^{-1/2}P_J)X \) and its Karhunen-Loève coefficients are given by \( w\eta_j \), where \( \eta_j \) are the Karhunen-Loève coefficients of \( X' \) and \( X \). Hence, (6.20) follows by the same line of arguments as Lemma 6. Combining (6.20) with (3.5), we thus obtain
\[
P(|\tilde{T} - \tilde{S}| > u) \leq_p P_{J,n,p} \text{ with } u = Cn^{-1/2}(\log n)^{3/2}\sigma_J^2 |\hat{J}|^{3/2} \tag{6.21}
\]
for \( C > 0 \) sufficiently large. By Markov’s inequality, we have
\[
P(|\tilde{S} - \tilde{T}| > u) \geq P_{J,n,p} \leq \frac{1}{P_{J,n,p}} P(|\tilde{S} - \tilde{T}| > u) \leq_p P_{J,n,p}^{1-s}
\]
Due to Lemma 24, we can replace \( \hat{A}_J, \tilde{B}_J, \tilde{C}_J \) by its counterparts \( A_J, B_J, C_J \) with probability at least \( 1 - Cn^{-\gamma/4}, C > 0 \). Piecing everything together, the claim follows.

Proof of Theorem 4. To simplify the notation, we assume w.l.o.g. \( \sigma_w^2 = 1 \). Then by affine invariance and the triangle inequality
\[
\mathbb{U} \left( \left\| \tilde{P}_J - \tilde{P}_J \right\|_2^2, \left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 \right)
= \mathbb{U} \left( \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J}, \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J} \right)
\leq \mathbb{U} \left( \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J}, G \right) + \mathbb{U} \left( G, \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J} \right).
\]
The first term on the right-hand side is treated in Corollary 8. In order to deal with the second term, we can directly apply Theorem 3 as follows. By affine invariance, the triangle inequality and independence
\[
\mathbb{U} \left( G, \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J} \right) \leq \mathbb{U} \left( G, \frac{n\left\| \tilde{P}_J' - \tilde{P}_J \right\|_2^2 - \hat{A}_J}{B_J} \right)
\]

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For the first term on the right-hand-side, we can apply Theorem 3. Note that the corresponding bound is dominated by the one provided by Corollary 8.

For the second term, using that the standard Gaussian distribution function \( \Phi \) is Lipschitz continuous and satisfies for \( a \geq 1 \) and \( x > 0 \) that \( \Phi(ax) - \Phi(x) \leq (a - 1)x\Phi'(x) \lesssim a - 1 \), we get

\[
\tilde{U}
\begin{pmatrix}
G, \tilde{B}JG + \tilde{A}J - A_J
\end{pmatrix}
\lesssim \left| \tilde{A}J - A_J \right| + \left| \frac{B_J}{B_J} - 1 \right| + \left| \frac{\tilde{B}_J}{B_J} - 1 \right|.
\]

Due to Lemma 24, we have

\[
\left| \tilde{A}J - A_J \right| + \left| \frac{B_J}{B_J} - 1 \right| + \left| \frac{\tilde{B}_J}{B_J} - 1 \right| \lesssim_p \frac{\sqrt{\log n} A_J}{\sqrt{n} B_J},
\]

with probability at least \( 1 - 2n^{1-p/4} \). Combining all bounds together with the union bound for the involved probabilities, the claim follows.

**Proof of Corollary 6.** Denote with \( \hat{E} \) the event where Theorem 4 applies, hence

\[
P(\hat{E}) \geq 1 - C p^{1-p} \sqrt{n^{1-p/2}}.
\]

We now slightly reverse the argument used to prove Theorem 4, where we again assume \( \sigma_w^2 = 1 \) for simplicity. The triangle inequality and the affine invariance gives

\[
\tilde{U}
\begin{pmatrix}
n\|\hat{P}_J - \hat{P}_J\|_2^2, B_JG + A_J
\end{pmatrix}
\leq \tilde{U}
\begin{pmatrix}
\|\hat{P}_J - \hat{P}_J\|_2^2, \|P'_J - P_J\|_2^2
\end{pmatrix}
+ \tilde{U}
\begin{pmatrix}
B_JG + A_J, n\|\hat{P}'_J - P_J\|_2^2
\end{pmatrix},
\]

where we recall that \( X_i' \) denotes independent copies of \( X_i \). By Theorems 3 and 4 and again the affine invariance, we get that on the event \( \hat{E} \)

\[
\tilde{U}
\begin{pmatrix}
n\sigma_w^{-1}\|\hat{P}_J - \hat{P}_J\|_2^2, \sqrt{2}B_JG + A_J
\end{pmatrix} \lesssim \tilde{A}_{J,n,p,s}.
\]

Let \( q^{\hat{G}}_\alpha \) be the quantile of the distribution function of \( B_JG + A_J \). The above and Lemma 25 (see below) yields that on the event \( \hat{E} \), there exists a constant \( C_1 > 0 \) such that for any \( \alpha \in [0, 1] \) and \( \delta > C_1 \tilde{A}_{J,n,p,s} \)

\[
\hat{q}_{\alpha + \delta} \leq q^{\hat{G}}_\alpha \leq \hat{q}_{\alpha - \delta}.
\]

(6.23)
Consequently, for $\delta > C_1 A_{J,n,p,s}$, it follows that for some $C_2 > 0$

$$\mathbb{P}\left(n\|\hat{P}_J - P_J\|_2^2 \leq \hat{q}_\alpha\right) \leq \mathbb{P}\left(n\|\hat{P}_J - P_J\|_2^2 \leq \hat{q}_\alpha, \hat{\mathcal{E}}\right) + \mathbb{P}(\hat{\mathcal{E}}^c)$$

$$\leq \mathbb{P}\left(n\|\hat{P}_J - P_J\|_2^2 \leq q_{\alpha - \delta}^G, \hat{\mathcal{E}}\right) + \mathbb{P}(\hat{\mathcal{E}}^c)$$

$$\leq \mathbb{P}\left(n\|\hat{P}_J - P_J\|_2^2 \leq q_{\alpha - \delta}^G\right) + 2\mathbb{P}(\hat{\mathcal{E}}^c)$$

$$\leq \mathbb{P}\left(B_J G + A_J \leq q_{\alpha - \delta}^G\right) + C_2 A_{J,n,p} + 2\mathbb{P}(\hat{\mathcal{E}}^c)$$

$$\leq 1 - \alpha + (C_1 + C_2) A_{J,n,p,s} + 2\mathbb{P}(\hat{\mathcal{E}}^c),$$

where we also used Theorem 3. In the same manner, one obtains the corresponding lower bound, and the claim follows together with (6.22).

**Lemma 25.** Let $F, G$ be distribution functions such that

$$\sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq \delta.$$

If $G$ is continuous, then for any $\delta' > 3\delta \geq 0$

$$q_{\alpha + \delta'}(F) \leq q_{\alpha}(G) \leq q_{\alpha - \delta'}(F).$$

Here, $q_{\alpha}(H) = \inf\{x : H(x) \geq 1 - \alpha\}, H \in \{F, G\}$.

**Proof of Lemma 25.** Suppose first that $q_{\alpha}(G) > q_{\alpha - \delta'}(F), \delta' > \delta$. Then

$$1 - \alpha = G(q_{\alpha}(G)) \geq F(q_{\alpha}(G)) - \delta \geq F(q_{\alpha - \delta'}(F)) - \delta \geq 1 - (\alpha - \delta') - \delta,$$

which yields a contradiction. Observe next that due to the continuity of $G$, we have

$$F(x) - F(x_\pm) \leq 2\delta.$$

Hence, if $q_{\alpha}(G) < q_{\alpha + \delta'}(F), \delta' > 3\delta$, then

$$1 - \alpha = G(q_{\alpha}(G)) \leq F(q_{\alpha}(G)) + \delta \leq F(q_{\alpha + \delta'}(F)) + \delta \leq 1 - \alpha - \delta' + 3\delta,$$

which again yields a contradiction. Hence the claim follows. 

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6.6 Bootstrap II: Proof of Theorem 5

6.6.1 Additional notation

Let us first introduce the additional key quantities, also used in the formulation of Theorem 5. Recall that \( J = \{ j_1, \ldots, j_2 \} \) and \( I = \{ 1, \ldots, i_2 \} \) with \( i_2 > j_2 + 2 \). We use the same notation as in the proof of Theorem 4 in Section 6.5. Let

\[
\tilde{T} = \frac{n}{\sigma_w} \| \tilde{P}_J - \hat{P}_J \|_2^2 \quad \text{and} \quad \tilde{S}_I = \frac{n}{\sigma_w} \| L_{J,I}(\tilde{E} - E) \|_2^2,
\]

where

\[
L_{J,I} \Psi = \sum_{j \in J} \sum_{k \in J \cap I} \frac{1}{\lambda_j - \lambda_k} (P_k A P_j + P_j A P_k)
\]

for a Hilbert-Schmidt operator \( A \) on \( H \). For this, let us write

\[
\tilde{S}_I = \frac{1}{n} \left\| \sum_{i=1}^n \tilde{Y}_{iI} \right\|^2_2 \quad \text{with} \quad \tilde{Y}_{iI} = \frac{w^2 - 1}{\sigma_w} Y_{iI}, \quad Y_{iI} = L_{J,I}(X_i \otimes X_i),
\]

(6.24)

and let

\[
\Psi_{J,I} = E Y_I \otimes Y_I \quad \text{and} \quad \tilde{\Psi}_{J,I} = \frac{1}{n} \sum_{i=1}^n \tilde{E} \tilde{Y}_{iI} \otimes \tilde{Y}_{iI} = \frac{1}{n} \sum_{i=1}^n Y_{iI} \otimes Y_{iI}
\]

with \( Y_I = L_{J,I} X \otimes X \). Moreover, let \( Z_{J,I} \) be Gaussian with \( E Z_{J,I} = 0 \) and covariance operator \( \Psi_{J,I} \), independent of \( X \), and likewise \( \tilde{Z}_{J,I} \) be Gaussian with \( E \tilde{Z}_{J,I} = 0 \) and covariance operator \( \tilde{\Psi}_{J,I} \). Finally, we set

\[
A_{J,I} = \| \Psi_{J,I} \|_1, \quad \tilde{A}_{J,I} = \| \tilde{\Psi}_{J,I} \|_1,
\]

\[
B_{J,I} = \sqrt{2} \| \Psi_{J,I} \|_2, \quad \tilde{B}_{J,I} = \sqrt{2} \| \tilde{\Psi}_{J,I} \|_2,
\]

\[
C_{J,I} = 2 \| \Psi_{J,I} \|_3, \quad \tilde{C}_{J,I} = 2 \| \tilde{\Psi}_{J,I} \|_3.
\]

A more explicit expression of \( A_{J,I} \) (resp. \( A_{J,I} \)) is given in (6.26) below.

6.6.2 Proofs

We first apply Proposition 3 to \( \tilde{U} \) and establish the following result.
Corollary 9. Let $s \in (0, 1)$. Then, with probability at least $1 - C_p(\mathbf{p}_{\mathcal{J}, n, p} + n^{1-p/6})^{1-s}$, $C_p > 0$ we have

$$
\tilde{U}\left(\tilde{T}, \|\tilde{\zeta}_{\mathcal{J}, \mathcal{I}}\|_2^2\right) \lesssim_p n^{-1/5} \left(\frac{A_{\mathcal{J}}}{\sqrt{\lambda_{1,2}(\Psi_{\mathcal{J}})}}\right)^{3/5} + n^{-1/2} \log^{3/2} n \frac{\sigma_{\mathcal{J}}^3 |\mathcal{J}|^{3/2}}{\sqrt{\lambda_{1,2}(\Psi_{\mathcal{J}})}} + \frac{A_{\mathcal{J}, \mathcal{I}^c} \log n}{\sqrt{\lambda_{1,2}(\Psi_{\mathcal{J}})}} + (\mathbf{p}_{\mathcal{J}, n, p} + n^{1-p/6})^s.
$$

The following lemma is needed in the proof of Corollary 9 and extends some basic inequalities from Section 6.5 to the truncated setting.

Lemma 26. We have, with probability at least $1 - 3n^{1-p/(2q)}$,

(i) $\|\Psi_{\mathcal{J}} - \Psi_{\mathcal{J}, \mathcal{I}}\|_1 \lesssim_p A_{\mathcal{J}, \mathcal{I}^c}$.

(ii) $|\bar{S} - \bar{S}_{\mathcal{I}}| \lesssim_p A_{\mathcal{J}, \mathcal{I}^c} \log n$.

(iii) $\left|\sum_{i=1}^n (\|\bar{Y}_i\|_2^2 - E[\|\bar{Y}_i\|_2^2])\right| \lesssim_p A_{\mathcal{J}, \mathcal{I}^c}^2 \sqrt{n \log n} \lesssim_p A_{\mathcal{J}, \mathcal{I}^c}^2 \sqrt{n \log n}$.

(iv) $\left|\tilde{\Psi}_{\mathcal{J}, \mathcal{I}} - \Psi_{\mathcal{J}, \mathcal{I}}\right|_2 \lesssim_p A_{\mathcal{J}, \mathcal{I}^c} \sqrt{n^{-1/2} \log n} \lesssim_p A_{\mathcal{J}} \sqrt{n^{-1/2} \log n}$.

Proof of Lemma 26. Inserting the Karhunen-Loève expansion of $X$ into the definition of $Y_{\mathcal{I}}$, we have

$$
Y_{\mathcal{I}} = \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J} \cap \mathcal{I}} \sqrt{\frac{2\lambda_k \lambda_j}{\lambda_j - \lambda_k}} \zeta_{jk} (u_j \otimes u_k + u_k \otimes u_j) / \sqrt{2} \tag{6.25}
$$

with $\zeta_{jk} = \eta_j \eta_k$. Moreover, by Assumptions 1, 2 and 3 with $m = 4$, the Cauchy-Schwarz inequality and the fact that summation is over different indices, we have that the $\zeta_{jk}$ are centered, uncorrelated and satisfy $E[|\zeta_{jk}|^p] \leq C_\eta$ and $\alpha_{jk} = E[\zeta_{jk}^2] \leq c_\eta$ for all $j \in \mathcal{J}, k \in \mathcal{J}^c \cap \mathcal{I}$. Hence, scaling the $\zeta_{jk}$ appropriately, (6.25) yields the Karhunen-Loève expansion of $Y_{\mathcal{I}}$ and the eigenpairs of $\Psi_{\mathcal{J}, \mathcal{I}}$ are given by

$$
\alpha_{jk} \frac{2\lambda_k \lambda_j}{(\lambda_k - \lambda_j)^2} \quad \text{and} \quad \frac{1}{\sqrt{2}} (u_j \otimes u_k + u_k \otimes u_j), \quad j \in \mathcal{J}, k \in \mathcal{J}^c \cap \mathcal{I}.
$$

In order to obtain the first claim, observe that $\Psi_{\mathcal{J}}$ has the same eigenvalues and eigenvectors, but with indices $k \in \mathcal{J}^c$ instead of $k \in \mathcal{J}^c \cap \mathcal{I}$. Hence

$$
\|\Psi_{\mathcal{J}} - \Psi_{\mathcal{J}, \mathcal{I}}\|_1 = \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{I}^c} \alpha_{jk} \frac{2\lambda_k \lambda_j}{(\lambda_k - \lambda_j)^2} = A_{\mathcal{J}, \mathcal{I}^c}. \tag{6.26}
$$
By properties of the Hilbert-Schmidt norm, we have due to the orthogonality of $P_P$ and $P_P^c$
\[\tilde{S} - \tilde{S}_I = \tilde{S} - \frac{n}{\sigma_w^2} \| L_{J,I}(\tilde{E} - E) \|_2^2 = \frac{n}{\sigma_w^2} \| L_{J,I}^c(\tilde{E} - E) \|_2^2.\]

The latter can be written as
\[\frac{1}{n} \left\| \sum_{i=1}^n \tilde{Y}_{i,Ic} \right\|_2^2 \text{ with } \tilde{Y}_{i,Ic} = \frac{w_i}{\sigma_w} Y_{i,Ic} - \tilde{w}_i, \quad Y_{i,Ic} = L_{J,Ic} X_i \otimes X_i.

Now, using that $E \| \tilde{Y}_{i,Ic} \|_2^p \lesssim A_{J,Ic}^{p/2}$ by Lemma 4, claim (ii) can be deduced from Lemma 2, setting $t = CA_{J,Ic}^{1/2} \sqrt{\log n}$. Claims (iii) and (iv) follow by the same arguments as in (6.18) and (6.19).

**Proof of Corollary 9.** Since $i_2 > j_2 + 2$, we have $\lambda_l(\tilde{\Psi}_{J,I}) = \lambda_l(\tilde{\Psi}_J)$ for $l = 1, 2$ (cf. the begin of the proof of Lemma 26). By Weyl’s inequality, $\| \cdot \|_\infty \leq \| \cdot \|_2$ and Lemma 26, we get for $l = 1, 2$,
\[\left| \lambda_l(\tilde{\Psi}_{J,I}) - \lambda_l(\tilde{\Psi}_J) \right| = \left| \lambda_l(\tilde{\Psi}_{J,I}) - \lambda_l(\tilde{\Psi}_J) \right|
\leq \left\| \tilde{\Psi}_{J,I} - \tilde{\Psi}_{J,I} \right\|_2 \lesssim_p \frac{\sqrt{\log n}}{\sqrt{n}} A_J \quad (6.27)
\]

with probability at least $1 - 3n^{1-p/4}$. Invoking (4.6), we obtain that on this event
\[2\lambda_l(\tilde{\Psi}_{J,I}) \geq \lambda_l(\tilde{\Psi}_J), \quad l = 1, 2.
\]

Next, observe $E \| \tilde{Y}_{i,Ic} \|_2^3 \lesssim_q A_{J,Ic}^{3/2} \lesssim_q A_{J}^{3/2}$ by Lemma 4. Combining this with Lemma 26(iii), we arrive at
\[\sum_{i=1}^n \| \tilde{Y}_{i,I} \|_2^3 \lesssim_q n A_{J,Ic}^{3/2} \lesssim_q n A_{J}^{3/2}, \quad (6.28)
\]

with probability at least $1 - 2n^{1-p/4}$. Proposition 3, applied to the setting in (6.24), then yields, with probability at least $1 - 2n^{1-p/4}$,
\[\bar{U}\left(\tilde{T}, \| \tilde{Z}_{J,I} \|_2^2 \right) \lesssim_p \frac{1}{n^{1/5}} \left( \frac{A_J}{\sqrt{\lambda_{1,2}(\tilde{\Psi}_J)}} \right)^{3/5} + \bar{P}(\| \tilde{T} - \tilde{S}_I \| > u) + \frac{u}{\sqrt{\lambda_{1,2}(\tilde{\Psi}_J)}}.
\]

By Corollary 2, we have
\[|\tilde{T} - \tilde{S}_I| \leq |\tilde{T} - \tilde{S}| + |\tilde{S} - \tilde{S}_I|\]
\[ \leq \frac{n}{\sigma_w} \left( |J|^{3/2} (\delta_3^3(E) + \delta_3^3(\tilde{E})) + |J|^2 (\delta_4^4(E) + \delta_4^4(\tilde{E})) + |\tilde{S} - \tilde{S}_I| \right). \]

Letting
\[ u = Cn^{-1/2}(\log n)^{3/2}\sigma^3_J|J|^{3/2} + CA_{J,I}c \log n, \]
it follows from (6.21) and Lemma 26 that
\[ P(|\tilde{T} - \tilde{S}_I| > u) \leq P(|\tilde{T} - \tilde{S}| > u/2) + P(|\tilde{S} - \tilde{S}_I| > u/2) \lesssim_p P_{J,n,p} + n^{1-p/6}. \]

By Markov’s inequality, we get for \( s \in (0,1) \)
\[ P(\tilde{S}_I - T > u) \geq (P_{J,n,p} + n^{1-p/6})^{1-s}. \]

Piecing all bounds together, the claim follows from the union bound. \hfill \square

**Lemma 27.** With probability at least \( 1 - 4n^{1-p/4} \), we have
\[ \hat{U}(\|L_JZ\|_2, \|\tilde{Z}_{J,I}\|_2) \lesssim_p \frac{\sqrt{A_{J,I} \text{tr}(\sqrt{\Psi_{J,I}}) \sqrt{\log n}}}{\sqrt{\lambda_{1,2}(\Psi)}} + \frac{A_{J,I,c}}{\sqrt{\lambda_{1,2}(\Psi)}}. \]

**Proof of Lemma 27.** By the triangle inequality and independence, we have
\[ \hat{U}(\|L_JZ\|_2, \|\tilde{Z}_{J,I}\|_2) \leq \hat{U}(\|Z_{J,I}\|_2, \|\tilde{Z}_{J,I}\|_2) + U(\|Z_{J,I}\|_2, \|L_JZ\|_2) \]
(6.29)

We start with the first term on the right-hand side. An application of Lemma 20 gives
\[ \hat{U}(\|Z_{J,I}\|_2, \|\tilde{Z}_{J,I}\|_2) \lesssim \left( \frac{1}{\sqrt{\lambda_{1,2}(\Psi_{J,I})}} + \frac{1}{\sqrt{\lambda_{1,2}(\tilde{\Psi}_{J,I})}} \right) \|\Psi_{J,I} - \tilde{\Psi}_I\|_1. \]

In the proof of Corollary 9, we have shown that with probability at least \( 1 - 2n^{1-p/4} \), we have \( 4\lambda_{1,2}(\Psi_{J,I}) \geq \lambda_{1,2}(\Psi) \). In order to control \( \|\Psi_{J,I} - \tilde{\Psi}_I\|_1 \), we will apply Corollary 4. To this end, by Lemma 4 (i) and (iv) (note that the 8-th cumulant uncorrelatedness assumption implies that the \( \zeta_{jk} = \eta_j\eta_k \) satisfy the additional conditions from Lemma 4, and that the additional weights \( (w_i^2 - 1)/\sigma_w \) do not affect these assumptions), we get
\[ (\mathbb{E}||\tilde{Y}_I||_2^{2/p})^{2/p} \lesssim_p A_{J,I} \]
and
\[ \text{tr} \left( \sqrt{E(Y_{\mathcal{I}} \otimes Y_{\mathcal{I}} - \mathbf{1}_2)} \right) \leq_p \sqrt{\text{tr} (\Psi_{\mathcal{I}, \mathcal{I}}) \text{tr} (\Psi_{\mathcal{I}, \mathcal{I}}^{1/2})} = \text{tr} (\Psi_{\mathcal{I}, \mathcal{I}}^{1/2}) \sqrt{A_{\mathcal{J}, \mathcal{I}}}. \]
Similarly, as shown below, for an operator \( S \) on \( \text{HS}(\mathcal{H}) \) with \( \|S\|_{\infty} \leq 1 \), we have under the 8-th cumulant uncorrelatedness assumption that
\[ \sqrt{E \text{tr}^2 (S(Y_{\mathcal{I}} \otimes Y_{\mathcal{I}} - \mathbf{1}_2))} \leq_p \text{tr} (\Psi_{\mathcal{I}, \mathcal{I}}) = A_{\mathcal{J}, \mathcal{I}}. \]  
(6.30)
Hence, by Corollary 4, we have
\[ \|\Psi_{\mathcal{J}, \mathcal{I}} - \mathbf{1}_2\|_1 \leq_p \sqrt{\log n} \sqrt{A_{\mathcal{J}, \mathcal{I}} \text{tr} (\Psi_{\mathcal{J}, \mathcal{I}}^{1/2})}/\sqrt{n}, \]  
(6.31)
with probability at least \( 1 - 2n^{-p/4} \). By the union bound and the inequality \( A_{\mathcal{J}, \mathcal{I}} \leq A_\mathcal{J} \), we conclude that
\[ \tilde{U} (\|Z_{\mathcal{J}, \mathcal{I}}\|_2, \|Z_{\mathcal{J}, \mathcal{I}}\|_2) \leq_p \frac{\sqrt{\log n} \sqrt{A_{\mathcal{J}, \mathcal{I}} \text{tr} (\Psi_{\mathcal{J}, \mathcal{I}}^{1/2})}}{\sqrt{n} \lambda_{1,2} (\Psi)} \]  
(6.32)
with probability at least \( 1 - 4n^{-p/4} \). It remains to consider the second term on the right-hand side of (6.29). By Lemma 20, Lemma 26(i), and the fact that \( \lambda_i (\Psi_{\mathcal{J}, \mathcal{I}}) = \lambda_i (\Psi), \) \( i = 1, 2, \) we have
\[ \tilde{U} (\|L_{\mathcal{J}} Z\|_2, \|Z_{\mathcal{J}, \mathcal{I}}\|_2) \leq \frac{1}{\sqrt{\lambda_{1,2} (\Psi)}} \|\Psi - \Psi_{\mathcal{J}, \mathcal{I}}\|_1 \leq_p \frac{A_{\mathcal{J}, \mathcal{I}}}{\sqrt{\lambda_{1,2} (\Psi)}}. \]
This completes the proof. \( \square \)

Proof of Theorem 5. The basic idea is the same as in the proof of Theorem 4. By the triangle inequality
\[ \tilde{U} (\sigma_w^{-2} \|\tilde{P}_J - \tilde{P}_J\|_2^2, \|\tilde{P}_J' - P_J\|_2^2) \leq \tilde{U} (n \sigma_w^{-2} \|\tilde{P}_J - \tilde{P}_J\|_2^2, \|\tilde{Z}_{\mathcal{J}, \mathcal{I}}\|_2^2) + \tilde{U} (\|\tilde{Z}_{\mathcal{J}, \mathcal{I}}\|_2^2, n \|\tilde{P}_J' - P_J\|_2^2). \]
For the first term on the left-hand side, we can apply Corollary 9. By the triangle inequality and independence
\[ \tilde{U} (\|\tilde{Z}_{\mathcal{J}, \mathcal{I}}\|_2^2, n \|\tilde{P}_J' - P_J\|_2^2) \leq \tilde{U} (\|\tilde{Z}_{\mathcal{J}, \mathcal{I}}\|_2^2, \|L_{\mathcal{J}} Z\|_2^2) \]
\[ + \tilde{U} (\|L_{\mathcal{J}} Z\|_2^2, n \|\tilde{P}_J' - P_J\|_2^2). \]  
(6.33)
Lemma 27 deals with \( \tilde{U} (\|\tilde{Z}_{\mathcal{J}, \mathcal{I}}\|_2^2, \|L_{\mathcal{J}} Z\|_2^2) \), while Theorem 2 deals with \( \tilde{U} (\|L_{\mathcal{J}} Z\|_2^2, n \|\tilde{P}_J' - P_J\|_2^2) \). Piecing everything together, the claim follows. \( \square \)
Proof of Corollary 7. Using Theorems 2 and 5 instead of Theorems 3 and 4, we may proceed as in the proof of Corollary 6.

Proof of Equation (6.30). Using the notation \( \mathbf{j} = (j, k) \) for \( j \in \mathcal{J} \) and \( k \in \mathcal{J}^c \cap \mathcal{I} \) and \( \mathcal{J} = \mathcal{J} \times (\mathcal{J}^c \cap \mathcal{I}) \), the Karhunen-Loève expansion of \( \tilde{Y}_\mathcal{I} \) is given by

\[
\tilde{Y}_\mathcal{I} = \sum_{j \in \mathcal{J}} \tilde{\vartheta}_{j, \mathcal{I}} \tilde{\zeta}_j \tilde{u}_j
\]

with Karhunen-Loève coefficients

\[
\tilde{\vartheta}_j = \frac{w^2 - 1}{\eta_j \eta_k} \alpha_{jk}^{1/2} \tilde{\zeta}_j \tilde{u}_j
\]

and eigenpairs of \( \Psi_{\mathcal{J}, \mathcal{I}} = \mathbb{E} \tilde{Y}_\mathcal{I} \otimes \tilde{Y}_\mathcal{I} \) given by

\[
\tilde{\vartheta}_j = 2 \alpha_{jk} \frac{\lambda_k \lambda_j}{(\lambda_k - \lambda_j)^2} \quad \text{and} \quad \tilde{u}_j = \frac{\beta_{jk}}{\sqrt{2}} (u_j \otimes u_k + u_k \otimes u_j), \quad j = (j, k) \in \mathcal{J}
\]

with \( \alpha_{jk} = \mathbb{E} \eta_j^2 \eta_k^2 \) and \( \beta_{jk} = \text{sgn}(\lambda_k \lambda_j/(\lambda_j - \lambda_k)) \). This can be seen by multiplying (6.25) with \((w^2 - 1)/\sigma_w\). Here, the uncorrelatedness of \( \tilde{\zeta}_j \) follows from the 4-th cumulant uncorrelatedness assumption. Moreover, by Assumptions 1 and 2, we have

\[
c_\eta \leq \alpha_{jk} \leq C_\eta^2 / p \quad \text{and} \quad \mathbb{E}|\tilde{\vartheta}_j|^p \leq \frac{C_\eta}{\tilde{C}_\eta}, \quad j = (j, k) \in \mathcal{J}.
\]

We now write

\[
\text{tr}^2(S(\tilde{Y}_\mathcal{I} \otimes \tilde{Y}_\mathcal{I} - \mathbb{E} \tilde{Y}_\mathcal{I} \otimes \tilde{Y}_\mathcal{I})) = \left( \sum_{j_1, j_2 \in \mathcal{J}} \sqrt{\tilde{\vartheta}_{j_1, \mathcal{I}} \tilde{\vartheta}_{j_2, \mathcal{I}} \tilde{\vartheta}_{j_1, \mathcal{I}} \tilde{\vartheta}_{j_2, \mathcal{I}} \langle u_{j_1}, S u_{j_2} \rangle} \right)^2 = \sum_{j_1, j_2, j_3, j_4 \in \mathcal{J}} \sqrt{\tilde{\vartheta}_{j_1, \mathcal{I}} \tilde{\vartheta}_{j_2, \mathcal{I}} \tilde{\vartheta}_{j_3, \mathcal{I}} \tilde{\vartheta}_{j_4, \mathcal{I}} \langle u_{j_1}, S u_{j_2} \rangle \langle u_{j_3}, S u_{j_4} \rangle}.
\]

On the one hand, we have \( \mathbb{E}|\tilde{\vartheta}_{j_1, \mathcal{I}} \tilde{\vartheta}_{j_2, \mathcal{I}} \tilde{\vartheta}_{j_3, \mathcal{I}} | \leq \tilde{C}_\eta^4 / p \) for all indices \( j_1, j_2, j_3, j_4 \in \mathcal{J} \). On the other hand, if \( \mathbb{E}|\tilde{\vartheta}_{j_1, \mathcal{I}} \tilde{\vartheta}_{j_2, \mathcal{I}} \tilde{\vartheta}_{j_3, \mathcal{I}} \tilde{\vartheta}_{j_4, \mathcal{I}} | \neq 0 \), then the 8-th cumulant uncorrelatedness implies that the two sets \( \{j_1, j_2, j_3, j_4\} \) and \( \{k_1, k_2, k_3, k_4\} \) both contain at most 2 elements (different indices). Combining these facts with the bound \( |\langle u_{j_1}, S u_{j_2} \rangle \langle u_{j_3}, S u_{j_4} \rangle| \leq ||S||_\infty^2 \leq 1 \), we conclude that

\[
\mathbb{E} \text{tr}^2(S(\tilde{Y}_\mathcal{I} \otimes \tilde{Y}_\mathcal{I} - \mathbb{E} \tilde{Y}_\mathcal{I} \otimes \tilde{Y}_\mathcal{I})) \leq 59
\]
\begin{align*}
\leq C_{\eta}^{4/p}C_{\eta}^{4/p} \sum_{j_1, j_2 \in J, k_1, k_2 \in J_c \cap I} \left( \frac{3\lambda_{j_1} \lambda_{j_2} \lambda_{k_1} \lambda_{k_2}}{(\lambda_{j_1} - \lambda_{k_1})^2(\lambda_{j_2} - \lambda_{k_2})^2} \right) \\
+ C_{\eta}^{4/p}C_{\eta}^{4/p} \sum_{j_1, j_2 \in J, k_1, k_2 \in J_c \cap I} \left( \frac{6\lambda_{j_1} \lambda_{j_2} \lambda_{k_1} \lambda_{k_2}}{|\lambda_{j_1} - \lambda_{k_1}| |\lambda_{j_1} - \lambda_{k_2}| |\lambda_{j_2} - \lambda_{k_2}| |\lambda_{j_2} - \lambda_{k_1}|} \right) \\
\leq 9C_{\eta}^{4/p}C_{\eta}^{4/p} \left( \sum_{j \in J, k \in J_c \cap I} \left( \frac{\lambda_j \lambda_k}{(\lambda_j - \lambda_k)^2} \right)^2 \right).
\end{align*}

Using again Assumption 2 and the definition of $\Psi_{J,I}$, the claim follows. \qed

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