I. INTRODUCTION

The state of a quantum system is represented by a vector in Hilbert space, and its dynamics can be described by a unitary time evolution preserving the inner or scalar product of two vectors in the Hilbert space. A notion of distance between quantum states and Riemannian structure of Hilbert space are important geometrical concepts, however until the discovery of Berry phase in 1984 physicists generally did not show interest in the geometry of quantum theory [1]. Experimental observations of Berry phase led to important mathematical elucidations of the geometry of quantum mechanics - in particular the holonomy of fiber bundles [2]. Physical state space is isomorphic to complex projective space which is Kahler, and U(1) part of its holonomy can be interpreted as the geometric or Berry phase as shown by Page [2]. Another geometric effect was discovered in 1990 in the adiabatic transition probability for a two level quantum system [3]; a preliminary announcement of Weyl - Kahler space was made in 1992 [4] as a possible explanation of this effect. The aim of this paper is to give an exposition of Weyl - Kahler space and to show that it is a natural geometry of quantum theory.

Weyl’s unified theory aims at a single action principle to describe electromagnetic and gravitational fields in a generalised geometry postulating the principle of Gauge invariance [5]. In the preface to the first American printing his book, Weyl writes: “This attempt has failed”. Development in quantum theory led him to reject his original idea of Gauge invariance though he envisaged “a deep modification of the foundations of quantum mechanics”. Dirac in 1973 revived Weyl’s theory [6], however physicists did not show the interest in this. A comprehensive discussion on Weyl’s theory and a radically new approach to the meaning of electronic charge are given in a recent monograph [7]. Physics does not preclude Weyl’s original gauge transformations since there is a well defined concepts of the length of a complex vector in the Hilbert space which may be postulated to acquire length holonomy under parallel transport. Mathematically it is natural to seek Weyl structure in complex manifolds. In the next section a generalised complex space with hermitian matrix and Weyl gauge transformations is discussed. Physically plausible arguments are presented in Section 3 to suggest Weyl - Kahler space as a geometrical framework for quantum theory. Prospective application and open problems constitute the last section.

II. WEL - KAHLER SPACE

Levi - Civita parallelism is a powerful idea in differential geometry. A vector parallel transported round some closed curve in a complex manifold gets rotated by some matrix when it returns to its starting point. The set of matrices for all curves in the manifold at a given point constitute the holonomy group of the manifold. For a Kahler space, the holonomy is SU(n) if the Ricci tensor vanishes (here n is the dimension of the manifold). What happens if the length of a vector under parallel transport also changes ? Weyl space admits such a vectorlength holonomy. Let us consider a 2n-dimensional real manifold covered by a system of complex coordinate neighbourhoods such that the transition functions in the intersection of two neighbourhoods are holomorphic. Such a manifold is said to admit a complex structure [8]. We follow Yano’s convention [9] and use Latin indices to run over 1, 2, ...n, 1, 2, ...n for a complex manifold \( C_n \). A metric in \( C_n \) is defined as

\[
ds^2 = g_{ij}dz^i dz^j
\]

and a linear ground one form is

\[
A = A_idz^i
\]

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The fundamental metric tensor $g_{ij}$ is assumed to be self-conjugate and hybrid, and the covariant vector $A_i$ is self conjugate. Weyl gauge transformations are given by

$$ds \rightarrow ds' = \lambda ds$$  \hspace{1cm} (3)

$$A_i \rightarrow A'_i = A_i + \partial_i (\ln \lambda)$$  \hspace{1cm} (4)

Here $\lambda$ is a real scalar function. A complex space with hermitian metric and gauge field one-form is defined to be a Weyl-Hermite space.

A tensor $T$ transforming to $\lambda^N T$ under the gauge transformations (3) is defined to be co-tensor of power $N$: $g_{ij}$ is a co-tensor of power 2. The self conjugate mixed tensor

$$F_{ij} = \begin{bmatrix} i\delta^\mu_\nu & 0 \\ 0 & -i\delta^{\bar{\mu}}_{\bar{\nu}} \end{bmatrix}$$  \hspace{1cm} (5)

is in-tensor. A skew symmetric self conjugate tensor

$$F_{ij} = F^k_i g_{kj}$$  \hspace{1cm} (6)

is a co-tensor of power 2. In the Weyl-Hermite space generalised gauge invariant Christoffel symbols $\Gamma^i_{ij}$ can be derived following Eddington [4], specifically we have

$$^*\Gamma^X_{\mu\lambda} = \Gamma^X_{\mu\lambda} - g^{\bar{\nu}\rho}(g_{\lambda\bar{\nu}}A_{\mu} + g_{\mu\bar{\nu}}A_{\lambda})$$  \hspace{1cm} (7)

$$^*\Gamma^X_{\mu\lambda} = \Gamma^X_{\mu\lambda} - g^{\bar{\nu}\rho}(g_{\mu\bar{\nu}}A_{\lambda} - g_{\mu\lambda}A_{\bar{\nu}})$$  \hspace{1cm} (8)

$$^*\Gamma^X_{\mu\lambda} = 0$$  \hspace{1cm} (9)

and complex conjugates of these. One can calculate curvature tensors in the usual way. We now define a Weyl-Kahler space either by deducing a necessary and sufficient condition such that $C_n$ and $\bar{C}_n$ are always parallel or postulating that the covariant derivative of $F_{ij}$ vanishes. Equivalent conditions are

$$^*\Gamma^X_{\mu\lambda} = 0$$  \hspace{1cm} (10)

$$\nabla_k F^i_j = 0$$  \hspace{1cm} (11)

It can be proved that covariant derivative of $F_{ij}$ does not vanish, and the two form

$$F = 2ig_{\mu\bar{\nu}}dz^\mu \wedge d\bar{z}^\bar{\nu}$$  \hspace{1cm} (12)

is not closed. Thus unlike Kahler space in which $dF = 0$, a three form $H$ is closed in Weyl-Kahler space

$$dH = 0$$  \hspace{1cm} (13)

$$H = dF = 2A\nabla F$$  \hspace{1cm} (14)

A semi Weyl-Kahler space is defined such that

$$\nabla_k F^i_j = 0$$  \hspace{1cm} (15)

$$d\tilde{F} = 0$$  \hspace{1cm} (16)

where $\tilde{F}$ is a two form $\phi F$; such $\phi$ is a co-scalar of power -2. An interesting choice is to take scalar curvature for $\phi$. 
A. Remarks

In mathematical literature conformal class of metrics on complex manifolds have been studied by Vaisman and Einstein - Weyl structure structure by some groups, see references cited in [9]. Hermitian - Weyl manifold of Pedersen et al is not same as Weyl - Hermite manifold defined here, and seems similar to Weyl - Kahler. Semi Weyl - Kahler manifold introduced for the first time in 1966 [10] has not noticed by mathematicians. Further our main interest is in the change of length of a vector under parallel transport.

III. QUANTUM STATE SPACE

A quantum state is a vector in a Hilbert space which for simplicity is assumed finite dimensional. In Dirac's bra and Ket notation, let \( |\psi(t)\rangle \) be an element of an \( (N+1) \) dimensional Hilbert space \( H \) i.e. the null vector is substracted out. In this complex space one has a scalar product and a hermitian metric. In component form,

\[
|\psi(t)\rangle = |Z_0, Z_1, \ldots Z_N\rangle
\]  

Using scalar product, the length of this vector is

\[
<\psi|\psi> = \Delta_{\alpha\beta} \bar{Z}^\alpha Z^\beta
\]

A ray is an equivalence class of status for \( |\psi\rangle = c|\psi\rangle \) where \( c \) is a non-zero complex number. The space isometric to complex projective space \( CP^N \). The complex coordinates in \( CP^N \) are given by

\[
w^i = \frac{Z^i}{Z^0}
\]

For the normalised states \( <\psi|\psi> \) is equal to one, thus the states lies on the unit sphere \( S^{2N+1} \). The complex manifold \( CP^N \) is evolved with the Fubini-Study metric and the Kahler form (closed) i.e. it is a Kahler manifold. The Kahler potential is given by

\[
K = \frac{1}{2} \ln(1 + \omega_i \omega^i)
\]

So far we have summarized the standard mathematical structure of quantum mechanics. It is non-controversial, and together with the self-adjoint linear operators representing physical quantities completes the quantum mechanics. Physical interpretation of the state vector and quantum measurement are the most debated, and as yet unresolved problems. In this paper we do not intend to address the epistemological and interpretational questions, however plausible arguments to seek a generalisation of the Hilbert space are outlined. As pointed out in the preceeding section it is mathematically natural to allow Weyl structure to complex manifold, but that may not necessarily be useful for physics. There are atleast two quantum physics phenomena indicating that Weyl - Kahler space deserves serious consideration as a geometry of quantum mechanics: 1) geometric adiabatic transition amplitude, and 2) quantum measurements.

Let \( |\psi_i\rangle \) and \( |\psi_f\rangle \) be two states in the Hilbert space \( H \), then \( <\psi_f|\psi_i> \) is defined to be the transition probability amplitude from \( |\psi_i\rangle \) to \( |\psi_f\rangle \). For a physical observable the expectation value of the corresponding hermitian operator \( \hat{S} \) is

\[
s = \frac{<\psi|\hat{S}|\psi>}{<\psi|\psi>}
\]

The expectation value is a real number, and the state vector in a ray possess the same expectation value. We now examine the first phenomenon, namely the geometric adiabatic amplitude. For a two-level quantum systems with a complex hermitian Hamiltonian, it was shown by Berry that a geometric factor occurs in the transition amplitude [3]. Joye et al [11] consider the geometric phase for a closed path in the complex plane around the eigenvalue crossing. The
eigenvalues of the Hamiltonian using Teichmuller spaces are used to define a metric for the study of the eigenvalues crossing. Recalling the geometric phase as a holonomy arising form the parallel transport of a vector around a closed curve, in $CP^N$, it appears logical to interpret the geometric transition amplitude as a vectorlength holonomy, see [12] for further discussion on the theoretical explanation of experiments in optics.

Theory of measurement is a complex issue; orthodox Copenhagers believe that by denying the objective reality to the wavefunction its collapse or reduction upon measurement becomes an empty question. Anandan in the last section of his paper [13] argues that one may ascribe ontological or statistical reality to the wavefunction, and that protective measurement favours the former as opposed to Einstein’s ensemble one. A compreheensive critique of Copenhagen interpretation by post [14] and the most unsatisfactory treatment of single object e.g. an electron or a photon, in quantum theory [15] show that quantum theory may not be a fundamental theory. In recent years, new ideas like quantum - nondemolition measurement and interaction free measurement have been proposed; perhaps(!) a variant of many-world- interpretation (MWI) in the form of consistent histories approach has also emerged as an alternative to orthodoxy. In the interpretation due to Omnes [14] the four basic ingredients are consistent histories decoherence, logic and semi-classical physics. According to him the last two distinguish his theory from that of others and branching of the wavefunction of universe a la Everett’s MWI refers to non-issue. Ironically Griffiths does not find any commonality between MWI and the consistent histories approach while Gell-Mann identifies 'many-worlds' with 'many-histories' and considers himself to be 'poat- Everett investigators' see discussion following De Witt’s talk in [17]. Significant contribution in this approach has been made by Isham [18] towards the mathematical structure for a space of generalised histories and decoherence functionals. According to Griffiths [19] a fundamental principal of quantum reasoning may be stated: A meaningful description of a (closed) quantum mechanical system, including its time development, must employ a single framework. For a quantum system a sequence of events is represented by projectors $E_1, E_2, ..., E_n$ on the Hilbert space at a succession of times $t_1 < t_2, ..., < t_n$. A history is represented by a projector

$$Y = E_1 \otimes E_2 \otimes E_n$$

on the history space

$$\bar{H} = H \otimes H \otimes H \otimes ... H$$

of the tensor product of n copies of H. A consistent Boolean algebra of history projectors is called a framework. Isham considers a set of histories and an associated set of decoherence functions; this pair replaces the lattice of propositions and the space of states in orthodox quantum mechanics. In his framework, an n-time homogeneous history proposition $(\alpha t_1, \alpha t_2, ... \alpha t_n)$ can be associated with a projection operator

$$Y_1 = \alpha t_1 \alpha t_2 ... \alpha t_n$$

on the n-fold tensor product of n copies of the Hilbert space An attractive feature of the consistent histories approach is that there is no need for a priori time order, and its potential to address the problem of time in quantum gravity. Splitting of wavefunction upon measurement and role of time in quantum mechanics are key factors pointing to the physical relevance of Weyl - Kahler space: tensor product space Eq. (23) can be replaced by a multiple-connected space and topological index may define a time parameter if quantum-state space is generalised to Weyl-Kahler.

The novel property of Weyl-Kahler space is obtained from Eq.(14). Since H is closed, using Eq. (14) we get

$$dA = 0$$

Thus A is closed, and the line integral of A round a loop is zero unless $A_z$ is multivalued. Locally $A = df$, therefore nontrivial multiply-connected geometry results if A is not exact. Assuming topological obstruction leading to the nonvanishing of the line integral of A, we have

$$\oint A_i dz^i = n\Lambda$$

Here $\nabla$ is some constant, and the coordinates $Z^i$ are those given in Eq.(17) i.e. these are the coordinates on the Hilbert space not the physical space, and the curve on this space is parameterised by $t$. A vector $|\psi>$ parallel -transported along a curve enclosing the topological obstruction (or a branching point) on a Hilbert space $H$ will jump
on to the other connected Hilbert space; let it be denoted by $|\psi\rangle$. We postulate the principle that the length change under gauge transformation is the transition probability $<\phi|\psi>$. Explicitly we have

$$<\phi|\phi> = \exp(n\nabla) <\psi|\psi>$$

(27)

Length holonomy in Eq.(27) for a two-level quantum system can be interpreted as geometric amplitude factor [3]. In the problem of measurement, we interpret each successive measuring act characterized by $n$, and the exponential factor as a weight for multiply connected Hilbert spaces replacing Eq.(23). The norm of a vector is preserved since each Hilbert space is assigned a different scale or gauge. Multiply connected Hilbert spaces have the same dimension, and integer $n$ can be used to define a time ordering of the measuring events. Since $A$ is closed, a natural physical interpretation of the measurement is 'interaction free process' akin to the Aharonov-Bohm effect. Collapse of wavefunction is interpreted as a transition of the state vector to another Hilbert space. A vector which remains on one Hilbert space may acquire geometric phase; it is only for multiply-connected geometry that geometric amplitude and wavefunction collapse can be accounted for in this approach.

IV. PROSPECTIVES AND PROBLEMS

Geometric approach to quantum theory does not imply physical reality of wavefunction; in fact there is no evidence to ascribe wavefunction to a single object without facing counter-intuitive conclusions or paradoxes. Weyl-Kahler space for quantum mechanics requires new principles embodied in Eq.(26) and (27). There is a need to build concrete illustrative examples for geometric amplitude and measurement process since the present paper is limited to conceptual development of the idea. Tentatively, consistent histories approach may find the idea of multiply-connected spaces.

Interesting problems from mathematics point of view also exists; we list some of them. Semi Weyl-Kahler space defined by Eqs.(15) and (16) is a new geometry, and almost everything is open for study. The most important problem is, of course, the holonomy group classifications of Weyl-Kahler manifolds. For non-exact $A$, Hodge decomposition theorem allows non-trivial global properties. The nature of topological obstructions, and examples of multiply-connected Weyl-Kahler spaces are open problems. In superstring-theories SU(3) holonomy of Calabi-Yao folds has been found to be inspired by physical arguments; obviously Weyl-Kahler spaces hold promise for exploring new avenues in this area.

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