Dynamics of Ising models near zero temperature: real-space renormalization approach

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Abstract. We consider the stochastic dynamics of Ising ferromagnets (either pure or random) near zero temperature. The master equation satisfying detailed balance can be mapped onto a quantum Hamiltonian which has an exact zero-energy ground state representing the thermal equilibrium. The largest relaxation time $t_{eq}$ governing the convergence towards this Boltzmann equilibrium in finite-size systems is determined by the lowest non-vanishing eigenvalue $E_1 = 1/t_{eq}$ of the quantum Hamiltonian $H$. We introduce and study a real-space renormalization procedure for the quantum Hamiltonian associated to the single-spin-flip dynamics of Ising ferromagnets near zero temperature. We solve explicitly the renormalization flow for two cases. (i) For the one-dimensional random ferromagnetic chain with free boundary conditions, the largest relaxation time $t_{eq}$ can be expressed in terms of the set of random couplings for various choices of the dynamical transition rates. The validity of these RG results in $d = 1$ is checked by comparison with another approach. (ii) For the pure Ising model on a Cayley tree of branching ratio $K$, we compute the exponential growth of $t_{eq}(N)$ with the number $N$ of generations.

Keywords: renormalization group, disordered systems (theory), stochastic processes (theory)

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References

1. Introduction

The stochastic dynamics of classical Ising ferromagnets has been much studied for fifty years [1, 2]. In particular, many works have been devoted to the domain growth dynamics at low temperature $T < T_c$ (or at zero temperature $T = 0$ in $d = 1$, where the critical temperature vanishes, $T_c = 0$) when the initial condition is random (see the review on phase ordering dynamics [3]).

In this paper, we do not consider the dynamics starting from a random initial condition, but focus instead on the largest relaxation time $t_{\text{eq}}$ needed for a finite system to converge towards thermal equilibrium. This largest relaxation time $t_{\text{eq}}$ is defined as the inverse of the smallest non-vanishing eigenvalue $E_1$ of the time-evolution operator. Near zero temperature, more precisely when the temperature is much smaller than any
ferromagnetic coupling $J_{ij}$,

$$0 < T \ll J_{ij},$$

(1)

the thermal equilibrium is dominated by the two ferromagnetic ground states where all spins take the same value, and the largest relaxation time $t_{eq}$ corresponds to the time needed to go from one ground state (where all spins take the value $+1$) to the opposite ground state (where all spins take the value $-1$). We should stress that we consider that the temperature is arbitrarily small, but does not vanish, so that the transition between the two ground states is possible and the final state of the dynamics is unique (for studies on the zero-temperature dynamics, where the spin-flips corresponding to an energy-increase become impossible, we refer to the recent works [4] and references therein).

Of course, near zero temperature, the equilibration time $t_{eq}$ becomes extremely large, and numerical simulations of the microscopic dynamics become inefficient. Here, we introduce and study a real-space renormalization procedure valid near zero temperature for the dynamics of pure or random ferromagnets. The paper is organized as follows. In section 2, we recall the standard mapping between the master equation describing the stochastic dynamics of classical systems at temperature $T$ and a special type of quantum Hamiltonian that has an exact zero-energy eigenstate. In section 3, we describe the various choices of dynamical transition rates for single-spin-flip dynamics of the classical Ising model, and the corresponding quantum Hamiltonians. In section 4, we introduce the real-space renormalization procedure for these general quantum Hamiltonians. In section 5, we show how a closed RG procedure can be defined and exactly solved for the random ferromagnetic chain. In section 6, we solve the RG flow for the pure Ising model on a Cayley tree. Section 7 summarizes our conclusions. In appendix A, we describe another approach that allows us to check the validity of the RG results of section 5. Finally, in appendix B, we discuss the contributions that can depend on the choice of transition rates satisfying detailed balance.

2. Relaxation of classical models towards thermal equilibrium

2.1. Master equation satisfying detailed balance

To define the stochastic dynamics of a classical system, it is convenient to consider the master equation

$$\frac{dP_t(C)}{dt} = \sum_{C'} P_t(C')W(C' \rightarrow C) - P_t(C)W_{out}(C)$$

(2)

that describes the time evolution of the probability $P_t(C)$ to be in configuration $C$ at time $t$. The notation $W(C' \rightarrow C)$ represents the transition rate per unit time from configuration $C'$ to $C$, and

$$W_{out}(C) \equiv \sum_{C'} W(C \rightarrow C')$$

(3)

represents the total exit rate out of configuration $C$. 

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For a classical system where each configuration $C$ has some energy $U(C)$, the convergence towards Boltzmann equilibrium at temperature $T = 1/\beta$ in any finite system

$$P_{eq}(C) = \frac{e^{-\beta U(C)}}{Z},$$

where $Z$ is the partition function

$$Z = \sum_{C} e^{-\beta U(C)},$$

can be ensured by imposing the detailed balance property

$$e^{-\beta U(C)} W(C \rightarrow C') = e^{-\beta U(C')} W(C' \rightarrow C).$$

2.2. Mapping onto a Schrödinger equation in configuration space

As is well known (see for instance the textbooks [5]–[7]), the non-symmetric operator describing the stochastic dynamics of a classical model at temperature $T$ can be transformed into a symmetric quantum Hamiltonian problem. In the field of disordered systems, this mapping has been much used for one-dimensional continuous models [8]–[11], and more recently for many-body spin systems like the Sherrington–Kirkpatrick model ([12] and appendix B of [13]). In the field of pure spin models, this mapping has been used for more than fifty years [2], [14]–[17].

In the present context, this standard mapping consists in the change of variable

$$P_{t}(C) \equiv e^{-\frac{(\beta/2)U(C)}{2}} \psi_{t}(C) = e^{-\frac{\beta}{2} U(C)} \langle C | \psi_{t} \rangle.$$

Then, the master equation of equation (2) becomes the imaginary-time Schrödinger equation for the ket $|\psi_{t}\rangle$,

$$\frac{d}{dt} |\psi_{t}\rangle = -H |\psi_{t}\rangle,$$

where the quantum Hamiltonian

$$H = \sum_{C} \epsilon(C) |C\rangle \langle C| + \sum_{C,C'} V(C,C') |C\rangle \langle C'|$$

contains the on-site energies

$$\epsilon(C) = W_{out}(C) \equiv \sum_{C'} W(C \rightarrow C')$$

and the hoppings (using equation (6))

$$V(C,C') = -e^{-\frac{\beta}{2}(U(C')-U(C))} W(C' \rightarrow C) = -e^{-\frac{\beta}{2}(U(C)-U(C'))} W(C \rightarrow C')$$

$$= -\sqrt{W(C \rightarrow C')} W(C' \rightarrow C).$$

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2.3. Properties of the quantum Hamiltonian $\mathcal{H}$

Let us denote by $E_n$ the eigenvalues of the quantum Hamiltonian $\mathcal{H}$ and by $|\psi_n\rangle$ the associated normalized eigenvectors,

$$\mathcal{H}|\psi_n\rangle = E_n|\psi_n\rangle,$$  \hspace{1cm} (12)

$$\sum_C |\psi_n(C)|^2 = 1.$$  \hspace{1cm} (13)

The evolution operator $e^{-t\mathcal{H}}$ can be expanded on the eigenstates,

$$e^{-t\mathcal{H}} = \sum_n e^{-E_n t}|\psi_n\rangle\langle\psi_n|.$$  \hspace{1cm} (14)

The conditional probability $P_t(C|C_0)$ to be in configuration $C$ at $t$ if one starts from the configuration $C_0$ at time $t = 0$ can be written as

$$P_t(C|C_0) = e^{-\beta(\langle U(C)\rangle - \langle U(C_0)\rangle)} \sum_n e^{-E_n t}\psi_n(C)\psi_n^*|C_0\rangle.$$  \hspace{1cm} (15)

The quantum Hamiltonian $\mathcal{H}$ has special properties that come from its relation to the dynamical master equation.

(i) The ground state energy is $E_0 = 0$, and the corresponding eigenvector is given by

$$|\psi_0\rangle = \frac{\sum_C e^{-\beta/2 U(C)}}{\sqrt{Z}} |C\rangle.$$  \hspace{1cm} (16)

The normalization $1/\sqrt{Z}$ comes from the quantum normalization of equation (13).

This property ensures the convergence towards the Boltzmann equilibrium in equation (7) for any initial condition $C_0$,

$$P_t(C|C_0) \approx \frac{e^{-\beta/2 U(C)} - e^{-\beta/2 U(C_0)}}{Z} = P_{eq}(C).$$  \hspace{1cm} (17)

(ii) The other energies $E_n > 0$ determine the relaxation towards equilibrium. In particular, the lowest non-vanishing energy $E_1$ determines the largest relaxation time ($1/E_1$) of the system,

$$P_t(C|C_0) - P_{eq}(C) \approx \frac{e^{-E_1 t} e^{-\beta/2 U(C)} - e^{-\beta/2 U(C_0)} \psi_1(C)\psi_1^*|C_0\rangle}{Z}.$$  \hspace{1cm} (18)

Since this largest relaxation time represents the ‘equilibrium time’, i.e. the characteristic time needed to converge towards equilibrium, we will use the following notation:

$$t_{eq} \equiv \frac{1}{E_1}.$$  \hspace{1cm} (19)

In summary, the relaxation time $t_{eq}$ can be computed without simulating the dynamics by any method able to compute the first excited energy $E_1$ of the quantum Hamiltonian $\mathcal{H}$ (where the ground state is given by equation (16) and has for eigenvalue $E_0 = 0$). For instance, in [12], the ‘conjugate gradient’ method has been used to study numerically the statistics of the largest relaxation time in various disordered models. Let us now describe more precisely how this general framework applies to single-spin-flip dynamics of Ising models.
3. The quantum Hamiltonian associated to single-spin-flip dynamics

3.1. Single-spin-flip dynamics

We consider a system of classical spins $S_i = \pm 1$, where each configuration $C = S_1, S_2, \ldots$ has for energy

$$U(C) = -\sum_{i<j} J_{ij} S_i S_j. \quad (20)$$

The couplings $J_{ij}$ may be random.

Within a single-spin-flip dynamics, the configuration $|C⟩ = |S_1⟩|S_2⟩ \cdots |S_N⟩$ containing $N$ spins is connected via the transition rates $W(C \to C')$ to the $N$ configurations obtained by the flip of a single spin $S_k \to -S_k$ denoted by

$$|C_k⟩ = σ^x_k |C⟩ \quad (21)$$

in terms of the Pauli matrix $σ_x$. The energy difference between the two configurations reads

$$U(C_k) - U(C) = 2 S_k \sum_i J_{ki} S_i. \quad (22)$$

The quantum Hamiltonian of equations (9), (10) and (11) can thus be rewritten as

$$\mathcal{H} = \sum_C \epsilon(C)|C⟩⟨C| + \sum_C \sum_{k=1}^N V(C, σ^x_k C)σ^x_k |C⟩⟨C|$$

$$= \sum_C \sum_{k=1}^N W(C \to σ^x_k C) e^{(β/2)[U(C_k) - U(C)]} \left[ e^{-(β/2)[U(C_k) - U(C)]} - σ^x_k \right] |C⟩⟨C|. \quad (23)$$

3.2. Simplest choice of the transition rates

It is clear from equation (23) that the simplest quantum Hamiltonian corresponds to the following choice of the dynamical transition rate:

$$W(C = \{S_i\} \to C_k = σ^x_k C) = e^{-(β/2)[U(C_k) - U(C)]} = e^{-β S_k (\sum_{i\neq k} J_{ik} S_i)}. \quad (24)$$

The quantum Hamiltonian of equation (23) then reads in terms of Pauli matrices

$$\mathcal{H}^\text{simple} = \sum_C \sum_{k=1}^N \left[ e^{-(β/2)[U(C_k) - U(C)]} - σ^x_k \right] |C⟩⟨C|$$

$$= \sum_C \sum_{k=1}^N e^{-β S_k (\sum_{i\neq k} J_{ik} S_i)} - σ^x_k |C⟩⟨C|$$

$$= \sum_{k=1}^N \left[ e^{-βσ^x_k (\sum_{i\neq k} J_{ik} σ^z_i)} - σ^x_k \right], \quad (25)$$

where we have used the identity $1 = \sum_C |C⟩⟨C|$. The quantum Hamiltonian of equation (25) has been mentioned as the simplest one for the one-dimensional ferromagnetic chain in equation (4) of [15].
3.3. Glauber choice

The Glauber choice for the transition rates [2],

\[ W(C = \{S_i\} \to C_k = \sigma^x_k C) = \frac{e^{-(\beta/2)[U(C_k) - U(C)]}}{2 \cosh ((\beta/2) [U(C_k) - U(C)])} = \frac{e^{-\beta S_k \sum_{i\neq k} J_{ik} S_i}}{2 \cosh(\beta \sum_{i\neq k} J_{ik} S_i)}, \]  

(26)
corresponds to the more complicated quantum Hamiltonian

\[ H_{\text{Glauber}} = \sum_{k=1}^{N} \frac{1}{2 \cosh \left( \beta \left( \sum_{i\neq k} J_{ik} \sigma_i^z \right) \right)} \left( e^{-\beta \sigma^x_k (\sum_{i\neq k} J_{ik} \sigma_i^z)} - \sigma^x_k \right), \]  

(27)

where we have used the facts that \( \sigma^z_k \) has for eigenvalues (\( \pm1 \)) and that \( \cosh \) is an even function.

This quantum Hamiltonian of equation (27) has been used already for the Sherrington–Kirkpatrick spin-glass model and for the finite dimensional ferromagnetic Ising model in [13] (see appendices B and C respectively). For the one-dimensional pure ferromagnetic chain, where each spin \( S_k \) has only two neighbors, the local field \( B_k = \sum_{i} J_{ik} S_i = J(S_{k-1} + S_{k+1}) \) can take only the three values \( h_k = -2J, 0, 2J \), so that one may replace the exponential factors using projection operators to recover the forms given in [15]–[17]. This type of ‘first quantized’ quantum spin Hamiltonian can be transformed further into a ‘second quantized’ Hamiltonian involving annihilation/creation operators or Fermi operators using Jordan–Wigner transformation (see the review [18]): this method has been followed in particular in [14] for the Glauber dynamics of the pure Ising chain. However, in the present paper, we will work directly on the ‘first quantized’ form of the quantum spin Hamiltonian of equation (25) or equation (27), with the aim of defining a real-space renormalization approach, in analogy with the strong disorder RG rules introduced for the random transverse field Ising model in its first-quantized form (see the discussion in section 4.1 below).

3.4. Most general choice

To better understand the structure of the renormalized Hamiltonian that will be generated by the real-space RG procedure introduced in the following section, it is useful to consider the most general choice satisfying the detailed balance equation of (6),

\[ W(C = \{S_i\} \to C_k = \sigma^x_k C) = G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N) e^{-\beta S_k \sum_{i\neq k} J_{ik} S_i}, \]  

(28)

where \( G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N) \) is an arbitrary positive function of the \( (N - 1) \) spins \( i \neq k \) that may depend on the index \( k \). The positivity requirement

\[ G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N) > 0 \]  

(29)
ensures that all elementary single flips are possible with a non-vanishing rate, so that the ground state of equation (16) is unique, and the dynamics converges towards thermal equilibrium. Since the reversed transition of equation (28) has the following rate:

\[ W(C_k = \sigma^x_k C \to C = \{S_i\}) = G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N) e^{+\beta S_k \sum_{i\neq k} J_{ik} S_i}, \]  

(30)
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the amplitude $G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N)$ represents the ‘symmetric part’ of the two opposite transitions involving the flip of the spin $k$ (when all other spins remain fixed),

$$G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N) = \sqrt{W(C \rightarrow \sigma_k^z C)} W(\sigma_k^z C \rightarrow C).$$

The corresponding quantum Hamiltonian of equation (23) reads

$$\mathcal{H}_{\text{general}} = \sum_{k=1}^{N} G_k(\sigma_1^z, \ldots, \sigma_{k-1}^z, \sigma_{k+1}^z, \ldots, \sigma_N^z) \left( e^{-\beta \sigma_k^z (\sum_{i \neq k} J_{ik} \sigma_i^z) - \sigma_k^z} \right).$$

In finite dimensions with only nearest neighbor interactions, it seems natural to consider local rates where the amplitude only involves the spins $i$ that are neighbors of $k$ (i.e. such that $J_{ki} \neq 0$),

$$G_k(\sigma_1^z, \ldots, \sigma_{k-1}^z, \sigma_{k+1}^z, \ldots, \sigma_N^z) \rightarrow G_k^{\text{local}}(\{\sigma_i^z\}_{J_{ik} \neq 0}).$$

To respect the symmetries of the classical energy of equation (20), a further requirement is usually that the amplitude should only be an even function $G(x) = G(-x)$ of the local field

$$G_k^{\text{local}}(\{\sigma_i^z\}_{J_{ik} \neq 0}) \rightarrow G \left( \sum_i J_{ki} \sigma_i^z \right).$$

In the next section, we introduce a renormalization procedure for the amplitude $G_k$ and we will see that even if one starts with a function of the local field (equation (34)), this form is not stable via renormalization and will generally lead to a local function of the neighboring spins (equation (33)).

4. Renormalization rules near zero temperature

In this section, we derive the appropriate renormalization rules for the quantum Hamiltonian introduced in the previous section in relation with the single-spin-flip dynamics of classical ferromagnets.

4.1. Differences from the disordered quantum Ising model

Let us first mention that in the high-temperature limit $\beta J_{i,j} = J_{i,j}/T \ll 1$, the quantum Hamiltonian of equation (25) or (27) reduces to the standard transverse-field Ising model

$$H_{\text{simple,Glauber}} \sim N + \sum_{k=1}^{N} \left( -\beta \sigma_k^z \left( \sum_{i \neq k} J_{ik} \sigma_i^z \right) - \sigma_k^z \right).$$

When the couplings $J_{ij}$ are random, the low-energy physics is then well described by the strong disorder RG procedure valid both in one dimension [19] and in higher dimensions $d > 1$ [20]–[22] (see [23] for a review). However, here we are interested in the opposite limit of very low temperature where $\beta J_{i,j} = J_{i,j}/T \gg 1$ (equation (1)), where one cannot linearize the exponentials in the quantum Hamiltonians of equations (25) and (27). In the following, we thus derive appropriate RG rules in this opposite regime.
4.2. Analysis of an elementary operator

The general quantum Hamiltonian of equation (32) can be considered as a sum

$$\mathcal{H}^{\text{general}} = \sum_{k=1}^{N} h_k$$  \hspace{1cm} (36)

of elementary operators

$$h_k \equiv G_k(\sigma^z_1, \ldots, \sigma^z_{k-1}, \sigma^z_{k+1}, \ldots, \sigma^z_N) \left( e^{-\beta \sigma^z_k (\sum_j J_{kj} \sigma^j_k)} - \sigma^z_k \right).$$  \hspace{1cm} (37)

The corresponding matrix elements in the \( \sigma^z \) basis

$$\langle S'_1, \ldots, S'_N | h_k | S_1, \ldots, S_N \rangle = \left( \prod_{j \neq k} \delta_{S'_j,S_j} \right) G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_N)$$

$$\times \langle S'_k | e^{-\beta \sigma^z_k (\sum_j J_{kj} S_j)} - \sigma^z_k | S_k \rangle$$  \hspace{1cm} (38)

are diagonal for all spins \( j \neq k \).

4.2.1. Effective problem for a single spin. For each fixed value of the local field \( B_k = \sum_i J_{ki} S_i \), we may diagonalize the effective problem for the single spin \( k \),

$$h^\text{eff}_k(B_k) \equiv e^{-\beta \sigma^z_k B_k} \sigma^z_k.$$  \hspace{1cm} (39)

The two normalized eigenstates are respectively

$$|v_k(B_k)\rangle \equiv \frac{1}{\sqrt{2 \cosh(\beta B_k)}} \left[ e^{(\beta/2)B_k} |S_k = +1\rangle + e^{-(\beta/2)B_k} |S_k = -1\rangle \right]$$  \hspace{1cm} (40)

and

$$|w_k(B_k)\rangle \equiv \frac{1}{\sqrt{2 \cosh(\beta B_k)}} \left[ e^{-(\beta/2)B_k} |S_k = +1\rangle - e^{(\beta/2)B_k} |S_k = -1\rangle \right]$$  \hspace{1cm} (41)

with eigenvalues

$$h^\text{eff}_k(B_k) |v_k(B_k)\rangle = 0,$$

$$h^\text{eff}_k(B_k) |w_k(B_k)\rangle = 2 \cosh(\beta B_k) |w_k(B_k)\rangle,$$  \hspace{1cm} (42)

so that the single spin Hamiltonian of equation (39) can be rewritten as the projector

$$h^\text{eff}_k(B_k) = 2 \cosh(\beta B_k) |w_k(B_k)\rangle \langle w_k(B_k) |$$

$$= \left[ e^{-(\beta/2)B_k} |S_k = +1\rangle - e^{(\beta/2)B_k} |S_k = -1\rangle \right] \left[ e^{(\beta/2)B_k} \langle S_k = +1| - e^{-(\beta/2)B_k} \langle S_k = -1| \right]$$

$$= \sigma^z_k e^{-(\beta/2)\sigma^z_k B_k} [ |S_k = +1\rangle + |S_k = -1\rangle ] [ \langle S_k = +1| + \langle S_k = -1| \sigma^z_k e^{-(\beta/2)\sigma^z_k B_k}. $$  \hspace{1cm} (43)

In terms of operators, the elementary operator of equation (37) may thus be rewritten as

$$h_k = G_k(\sigma^z_1, \ldots, \sigma^z_{k-1}, \sigma^z_{k+1}, \ldots, \sigma^z_N) \sigma^z_k e^{-(\beta/2)\sigma^z_k \sum_j J_{kj} \sigma^j_k}$$

$$\times [ |S_k = +1\rangle + |S_k = -1\rangle ] [ \langle S_k = +1| + \langle S_k = -1| \sigma^z_k e^{-(\beta/2)\sigma^z_k \sum_j J_{kj} \sigma^j_k}. $$  \hspace{1cm} (44)
4.2.2. Properties of elementary operators. Equation (44) is convenient to see explicitly the positivity property for any ket $|\psi\rangle$,

$$
\langle\psi|h_k|\psi\rangle = \sum_{S_1,\ldots,S_{k-1},S_{k+1},\ldots,S_N} G_k(S_1,\ldots,S_{k-1},S_{k+1},\ldots,S_N)
\times \left| e^{-\beta/2}\sum_{i} J_{ki} S_i \psi(S_1,\ldots,S_{k-1},S_k = +1, S_{k+1},\ldots,S_N) \right|^2
\geq 0,
$$

(45)
as a consequence of the positivity of the $G_k$ (equation (29)). Moreover, it is clear that the exactly known ground state of zero energy of equation (16),

$$
|\psi_0\rangle = \sum_{S_1,\ldots,S_N} \langle S_1,\ldots,S_N|\psi_0\rangle |S_1,\ldots,S_N\rangle
= \frac{1}{\sqrt{Z_N(\beta)}} \sum_{S_1,\ldots,S_N} e^{\beta/2\sum_{i<j\leq N} J_{ij} S_i S_j} |S_1,\ldots,S_N\rangle,
$$

(46)
satisfies

$$
\psi_0(S_1,\ldots,S_{k-1},S_k = +1, S_{k+1},\ldots,S_N) = e^{\beta\sum_{i} J_{ki} S_i},
\psi_0(S_1,\ldots,S_{k-1},S_k = -1, S_{k+1},\ldots,S_N) = e^{\beta\sum_{i} J_{ki} S_i},
$$

(47)
and is thus annihilated by all elementary operators $h_k$ for $k = 1,\ldots,N$,

$$
h_k|\psi_0\rangle = 0,
$$

(48)
as it should be.

4.3. Renormalization of the sum of two neighboring elementary operators

4.3.1. Sum of two neighboring elementary operators. The sum of two neighboring local operators of equation (37) of index $k$ and $l$ with $J_{kl} \neq 0$ reads

$$
h_k + h_l = G_k(\sigma^z_1,\ldots,\sigma^z_{k-1},\sigma^z_{k+1},\ldots,\sigma^z_N) \left( e^{-\beta\sigma^z_k(\sum_{i\neq k} J_{ki} \sigma^z_i) - \sigma^z_k} \right)
+ G_l(\sigma^z_1,\ldots,\sigma^z_{l-1},\sigma^z_{l+1},\ldots,\sigma^z_N) \left( e^{-\beta\sigma^z_l(\sum_{j\neq l} J_{lj} \sigma^z_j) - \sigma^z_l} \right),
$$

(49)
The corresponding matrix elements in the $\sigma^z$ basis,

$$
\langle S'_1,\ldots,S'_N|h_k + h_l|S_1,\ldots,S_N\rangle = \left( \prod_{j \neq (k,l)} \delta_{S'_j,S_j} \right)
\times \left[ G_k(S_1,\ldots,S_{k-1},S_{k+1},\ldots,S_N)\langle S'_k|e^{-\beta\sigma^z_k(\sum_{i\neq k} J_{ki} S_i) - \sigma^z_k}|S_k\rangle
+ G_l(S_1,\ldots,S_{l-1},S_{l+1},\ldots,S_N)\langle S'_l|e^{-\beta\sigma^z_l(\sum_{j\neq l} J_{lj} S_i) - \sigma^z_l}|S_l\rangle \right],
$$

(50)
are diagonal for all spins $j \neq (k,l)$.
4.3.2. Effective two-spin problem. For each fixed value of all the other external spins \( S_{j \neq (k, l)} \), using the notations

\[
B_k \equiv \sum_{i \neq l} J_{ki} S_i, \\
B_l \equiv \sum_{j \neq k} J_{lj} S_j
\]  
(51)

and

\[
g^S_k \equiv G_k(S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_l, \ldots, S_N), \\
g^S_l \equiv G_l(S_1, \ldots, S_k, \ldots, S_{l-1}, S_{l+1}, \ldots, S_N),
\]  
(52)

we have to diagonalize the effective problem for the two spins \((k, l)\),

\[
h_{k,l}^{\text{eff}} = g^S_k \left( e^{-\beta J_{kl} \sigma_k^+ \sigma_l^+} - \sigma_k^+ \right) + g^S_l \left( e^{-\beta J_{kl} \sigma_l^+ \sigma_k^+} - \sigma_l^+ \right).
\]  
(53)

The four-dimensional vector

\[
|u_\lambda\rangle = \sum_{S_k = \pm, S_l = \pm} c^{S_k \pm S_l}_\lambda |S_k, S_l\rangle
\]  
(54)

is an eigenvector of the operator \( h_{k,l}^{\text{eff}} \) of equation (53) with eigenvalue \( \lambda \) if

\[
0 = \left[ e^{-\beta J_{kl} \left( g^+_k e^{-\beta B_k} + g^+_l e^{-\beta B_l} \right) - \lambda} \right] c^{++}_\lambda - g^+_l c^{+-}_\lambda - g^+_k c^{++}_\lambda, \\
0 = \left[ e^{-\beta J_{kl} \left( g^-_k e^{\beta B_k} + g^-_l e^{\beta B_l} \right) - \lambda} \right] c^{-+}_\lambda - g^+_k c^{-+}_\lambda - g^-_k c^{++}_\lambda, \\
0 = \left[ e^{\beta J_{kl} \left( g^-_k e^{-\beta B_k} + g^+_l e^{\beta B_l} \right) - \lambda} \right] c^{+-}_\lambda - g^-_k c^{+-}_\lambda - g^-_k c^{-+}_\lambda, \\
0 = \left[ e^{\beta J_{kl} \left( g^+_k e^{\beta B_k} + g^-_l e^{-\beta B_l} \right) - \lambda} \right] c^{++}_\lambda - g^-_k c^{++}_\lambda - g^-_k c^{-+}_\lambda.
\]  
(55)

4.3.3. Finding the lowest non-zero eigenvalue \( \lambda_1 \). We already know the exact ground state \(|u_{\lambda=0}\rangle\) of eigenvalue \( \lambda = 0 \), with components (not normalized here)

\[
c^{S_k S_l}_{\lambda=0} = e^{(\beta/2) J_{kl} S_k S_l + (\beta/2) B_k S_k + (\beta/2) B_l S_l}.
\]  
(56)

To find the other three exact eigenvalues, one has to solve the remaining cubic equation. Here, since we are interested in the other small eigenvalue \( \lambda_1 \ll e^{\beta J_{kl}} \), we will neglect \( \lambda_1 \) in the two last equations of equation (55) to obtain

\[
c^{+-}_\lambda = e^{-\beta J_{kl}} \frac{g^+_l c^{++}_\lambda + g^-_l c^{-+}_\lambda}{g^-_k e^{\beta B_k} + g^+_l e^{\beta B_l}}, \\
c^{-+}_\lambda = e^{-\beta J_{kl}} \frac{g^-_l c^{++}_\lambda + g^+_l c^{-+}_\lambda}{g^-_k e^{\beta B_k} + g^+_l e^{\beta B_l}},
\]  
(57)
which we may replace in the two first equations of equation (55),

\[
0 = \left[ g_k e^{-\beta B_k} + g_i e^{-\beta B_i} - \lambda_1 e^{\beta J_{kl}} \right] c_{\lambda}^{++} \\
- g_l \left( g_k^{++} + g_k^{--} \right) + g_k^{--} + g_l^{--} - g_k^{++} c_{\lambda}^{++} + g_k^{--} c_{\lambda}^{--} \\
0 = \left[ g_k e^{-\beta B_k} + g_i e^{-\beta B_i} - \lambda_1 e^{\beta J_{kl}} \right] c_{\lambda}^{--} \\
- g_l \left( g_k^{++} + g_k^{--} \right) + g_k^{--} + g_l^{--} - g_k^{++} c_{\lambda}^{--} + g_k^{--} c_{\lambda}^{++} \\
\]

Equation (58).

For \( \lambda_1 = 0 \), we recover the exact solution of equation (56) as we should. The other eigenvalue reads

\[
\lambda_1 = e^{-\beta J_{kl}} 2 \cosh[\beta(B_k + B_i)] \frac{g_k^{++} + g_k^{--} + e^{-\beta B_k} + g_i^{++} + g_i^{--} + e^{-\beta B_i}}{g_k^{++} + g_k^{--} + g_i^{++} + g_i^{--} + e^{\beta(B_k + B_i)} + g_k^{++} e^{-\beta B_k} + g_i^{++} e^{-\beta B_i}},
\]

with the corresponding components of the eigenvector \( |u_{\lambda_1}\rangle \) (not normalized here)

\[
c_{\lambda_1}^{++} = e^{(\beta/2)J_{kl}} \left( e^{-\beta(B_k + B_i)} |++\rangle - e^{(\beta/2)(B_k + B_i)} |--\rangle \right),
\]

\[
c_{\lambda_1}^{--} = -e^{(\beta/2)J_{kl}} \left( e^{-\beta(B_k + B_i)} |--\rangle - e^{(\beta/2)(B_k + B_i)} |++\rangle \right).
\]

The other two components \( c_{\lambda_1}^{+-} \) and \( c_{\lambda_1}^{-+} \) are given by equation (57).

### 4.3.4. Projection onto the two ferromagnetic states

The projection of the operator of equation (53) onto its two lowest states of eigenvalues \( \lambda_0 = 0 \) and \( \lambda_1 \) reads

\[
h_{k,l}^{\text{eff}} \simeq \frac{\lambda_1}{\langle u_{\lambda_1} | u_{\lambda_1} \rangle} |u_{\lambda_1}\rangle \langle u_{\lambda_1}|.
\]

At the level of approximation we are working with, we wish to keep only the two ferromagnetic states ++ and -- (the two other states +- and -+ have been taken into account in equation (57) to produce renormalized rates between ++ and -- in equation (58)). Therefore, we keep only the two following leading components in the eigenvector:

\[
|u_{\lambda_1}\rangle \simeq e^{(\beta/2)J_{kl}} \left( e^{-\beta(B_k + B_i)} |++\rangle - e^{(\beta/2)(B_k + B_i)} |--\rangle \right),
\]

with the corresponding normalization

\[
\langle u_{\lambda_1} | u_{\lambda_1} \rangle \simeq e^{\beta J_{kl}} 2 \cosh[\beta(B_k + B_i)].
\]

Equation (61) becomes

\[
h_{k,l}^{\text{eff}} \simeq \frac{\lambda_1}{2 \cosh[\beta(B_k + B_i)]} \left( e^{-\beta(B_k + B_i)} |++\rangle - e^{(\beta/2)(B_k + B_i)} |--\rangle \right)
\times \left( e^{-\beta(B_k + B_i)} |++\rangle - e^{(\beta/2)(B_k + B_i)} |--\rangle \right).
\]

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Therefore, the two spins $S_k$ and $S_l$ now form a single renormalized ferromagnetic cluster, which we may represent by a single spin with the two states

$$|+\rangle_R = |++\rangle, \quad |-\rangle_R = |-\rangle,$$

with the renormalized external field (see equation (51))

$$B_R = B_k + B_l = \sum_{i \neq (k,l)} (J_{ki} + J_{li})S_i,$$

which corresponds to the natural renormalization of ferromagnetic coupling between the new cluster $(kl)$ and $i$,

$$J_{(kl)i}^R = J_{ki} + J_{li}.$$

In terms of this renormalized cluster, equation (64) reads

$$h_{k,l}^{\text{eff}} \simeq G_R \left( e^{-(\beta/2)B_R} |+\rangle_R - e^{(\beta/2)B_R} |-\rangle_R \right) \left( e^{-(\beta/2)B_R} \langle + |_R - e^{(\beta/2)B_R} \langle - |_R \right).$$

It has the same form as an elementary operator of equation (44) with the renormalized amplitude (using equation (59))

$$G_R = \frac{\lambda_1}{2 \cosh[\beta(B_k + B_l)]} = \frac{e^{-\beta J_{kl} g_k^+ g_k^- + g_l^+ g_l^- e^{-\beta B_k} + g_l^+ g_l^- (g_k^+ e^{\beta B_l} + g_k^- e^{-\beta B_l})}}{g_k^+ g_k^- + g_l^+ g_l^- + g_k^+ g_l^+ e^{\beta B_k} + g_k^- g_l^- e^{-\beta B_k}}$$

in terms of the variables defined in equation (52) for a given value of the external spins $S_i \neq (k,l)$.

**4.3.5. Renormalization rules for operators.** At the level of operators in the whole Hilbert space, the renormalized Hamiltonian of equation (64) reads (using the same notation as in equation (37))

$$h_{(k,l)}^R \equiv G_{kl}^R(\sigma_1^z, \ldots, \sigma_{k-1}^z, \sigma_{k+1}^z, \sigma_{l-1}^z, \sigma_{l+1}^z, \sigma_N^z) \left( e^{-\beta \sigma_k B_R - \sigma_R^z} \right)$$

where the amplitude of equation (69) is the renormalized operator

$$G_{kl}^R(\sigma_1^z, \ldots, \sigma_{k-1}^z, \sigma_{k+1}^z, \sigma_{l-1}^z, \sigma_{l+1}^z, \sigma_N^z) = \frac{e^{-\beta J_{kl} g_k^+ g_k^- + g_l^+ g_l^- e^{-\beta B_k} + g_l^+ g_l^- (g_k^+ e^{\beta B_l} + g_k^- e^{-\beta B_l})}}{g_k^+ g_k^- + g_l^+ g_l^- + g_k^+ g_l^+ e^{\beta (B_k + B_l)} + g_k^- g_l^- e^{-\beta (B_k + B_l)}}.$$
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in terms of the operators

\[ g^\pm_k \equiv G_k(\sigma^z_1, \ldots, \sigma^z_{k-1}, \sigma^z_{k+1}, \ldots, S_l = \pm, \ldots, \sigma^z_N), \]

\[ g^\pm_l \equiv G_l(\sigma^z_1, \ldots, S_k = \pm, \ldots, \sigma^z_{l-1}, \sigma^z_{l+1}, \ldots, \sigma^z_N), \]

\[ B_k \equiv \sum_{i \neq l} J_{ki} \sigma^z_i, \]

\[ B_l \equiv \sum_{i \neq k} J_{li} \sigma^z_i. \]  

(72)

4.3.6. Final state of the RG procedure. For a system of \(N\) spins, we will obtain after \((N - 1)\) RG steps a single renormalized spin \(S_R = \pm 1\) representing the two ground states of the whole sample, where all spins take the same value \(S_R\). Since there is no renormalized external field left, \(B_R = 0\), the final effective Hamiltonian for the single spin \(S_R\) simply reads

\[ \mathcal{H}^\text{final}_N = G^\text{final}(N) (1 - \sigma^x_R), \]

(73)

where \(G^\text{final}(N)\) is a numerical amplitude. The quantum ground state of zero energy is the symmetric combination of the two classical ferromagnetic ground states, as it should be,

\[ |\psi^{\text{(final)}}_0\rangle = \frac{|S_R = +\rangle + |S_R = -\rangle}{\sqrt{2}}, \]

(74)

whereas the excited quantum eigenstate is the antisymmetric combination of the two classical ferromagnetic ground states,

\[ |\psi^{\text{(final)}}_1\rangle = \frac{|S_R = +\rangle - |S_R = -\rangle}{\sqrt{2}}, \]

(75)

with the eigenvalue

\[ E^{\text{final}}_1(N) = 2G^\text{final}(N). \]

(76)

The conclusion is that the equilibrium time of equation (19) can thus be obtained from the final renormalization amplitude \(G^\text{final}_N\) as

\[ t_{eq}(N) \equiv \frac{1}{E^{\text{final}}_1(N)} = \frac{1}{2G^\text{final}(N)}. \]

(77)

In summary, we have thus defined a renormalized procedure for the amplitudes \(G\) that allows us to obtain in the end the equilibration time \(t_{eq}(N)\). Now, to better understand the meaning of the RG rules of equation (71), let us give some explicit examples.

4.3.7. Example: first renormalization step starting from the simple Hamiltonian of equation (25). The quantum Hamiltonian of equation (25) corresponds to the initial simple case where all amplitudes are unity,

\[ G^\text{ini}_k(\sigma^z_1, \ldots, \sigma^z_{k-1}, \sigma^z_{k+1}, \ldots, \sigma^z_N) = 1. \]

(78)
The first RG step where \( k \) and \( l \) are grouped into a single renormalized cluster yields the following amplitude (equation (71)) with \( g_k^\pm = 1 = g_l^\pm \):

\[
G_{kl}^R = e^{-\beta J_{kl} \cosh(\beta B_k) + \cosh(\beta B_l)} \frac{1}{1 + \cosh(\beta (B_k + B_l))} \\
= e^{-\beta J_{kl} \cosh(\beta \sum_{i \neq l} J_{kl} \sigma_i^z) + \cosh(\beta \sum_{i \neq k} J_{il} \sigma_i^z)} \frac{1}{1 + \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z)}. \tag{79}
\]

Therefore, this renormalized amplitude does not remain a number as in the initial condition of equation (73), but becomes an operator that involves the neighboring spins of \( k \) and \( l \).

4.3.8. Example: first renormalization step starting from the Glauber Hamiltonian of equation (27). The quantum Hamiltonian of equation (27) corresponds to the initial case

\[
G_k^{\text{ini}}(\sigma_1^z, \ldots, \sigma_{k-1}^z, \sigma_{k+1}^z, \ldots, \sigma_N^z) = \frac{1}{2 \cosh(\beta \sum_i J_{ki} \sigma_i^z)}, \tag{80}
\]

so that the operators of equation (72) read

\[
g_k^\pm = \frac{1}{2 \cosh \beta (B_k \pm J_{kl})}, \quad g_l^\pm = \frac{1}{2 \cosh \beta (B_l \pm J_{kl})}. \tag{81}
\]

Equation (71) for the renormalized amplitude then becomes

\[
G_{kl}^R = \frac{1}{2 \cosh \beta (B_k + B_l) + 2e^{2J_{kl} + B_k - B_l}} + \frac{1}{2 \cosh \beta (B_k + B_l) + 2e^{2J_{kl} - B_k + B_l}} \\
= \frac{1}{2 \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z) + 2e^{2J_{kl} + \sum_{i \neq (k,l)} (J_{ki} - J_{li}) \sigma_i^z}} \\
+ \frac{1}{2 \cosh(\beta \sum_{i \neq (k,l)} (J_{ki} + J_{li}) \sigma_i^z) + 2e^{2J_{kl} - \sum_{i \neq (k,l)} (J_{ki} - J_{li}) \sigma_i^z}). \tag{82}
\]

Therefore, this renormalized amplitude does not remain of the Glauber form of equation (80) in terms of the renormalized local field \( B_R = B_k + B_l \). Actually, it does not even remain a function of the single renormalized local field \( B_R \).

4.3.9. Discussion. These two examples show that the renormalized amplitudes \( G_R \) become generically a function of the \( \sigma^z \) operators of the neighboring spins, i.e. of the form of equation (33). It is not clear to us at this stage how to determine the operator form that would remain stable upon the general RG rule of equation (71). In the following, we will thus concentrate on two geometries where we can obtain closed RG rules, namely the one-dimensional case and the Cayley tree.

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5. Boundary renormalization for the random ferromagnetic chain

In this section, we consider the random ferromagnetic chain of \( N \) spins with the classical energy (equation (20))

\[
U(S_1, \ldots, S_N) = - \sum_{i=1}^{N-1} J_{i,i+1} S_i S_{i+1},
\]

with free boundary conditions for the two boundary spins \( S_1 \) and \( S_N \). The couplings \( J_{i,i+1} \) are positive random variables.

5.1. Closed RG for the simple Hamiltonian of equation (25) in one dimension

5.1.1. First step of the RG rules in the bulk. The first RG step where the two neighboring sites \((k, k+1)\) are grouped into a single ferromagnetic cluster leads to the following renormalized amplitude (equation (79)) for \( 1 < k < N - 1 \):

\[
G^R_{k,k+1} = e^{-\beta J_{k,k+1}} \frac{\cosh(\beta J_{k-1,k}) + \cosh(\beta J_{k+1,k+2})}{1 + \cosh \left( \beta \left[ J_{k-1,k} \sigma^z_{k-1} + J_{k+1,k+2} \sigma^z_{k+2} \right] \right)}.
\]

(84)

Since there remains the renormalized local field operator \( [J_{k-1,k} \sigma^z_{k-1} + J_{k+1,k+2} \sigma^z_{k+2}] \) in the denominator, this renormalized amplitude is not just a number as the initial condition. To avoid this difficulty, we will now consider what happens near one boundary.

5.1.2. First step of the RG rules near the boundary. We now consider the case \( k = 1 \), where we make a ferromagnetic cluster out of the two sites \((1, 2)\) near the boundary. Since the spin zero does not exist, we have \( J_{k-1,k} = J_{0,1} = 0 \), so that equation (84) reduces to a renormalized number (without any operator anymore)

\[
G^R_{1,2} = e^{-\beta J_{1,2}} \frac{1 + \cosh(\beta J_{2,3})}{1 + \cosh(\beta J_{2,3})} = e^{-\beta J_{1,2}}.
\]

(85)

This shows that we may define a simple closed boundary RG procedure as follows.

5.1.3. Closed boundary RG procedure. If we iterate the renormalization near the boundary, the renormalized state after \((n - 1)\) RG steps will be as follows. The spins \((1, 2, \ldots, n)\) have been grouped together into a single renormalized spin with some associated renormalized amplitude \( G^R_{[1,n]} \). The other spins \((n + 1, \ldots, N)\) of the chain are still in their initial form with amplitude unity, \( G_k = 1 \), for \( n + 1 \leq k \leq N \). Let us now perform the \( n \) RG step where the spin \((n + 1)\) is included in the boundary cluster. The RG rule of equation (71) yields the recurrence

\[
G^R_{[1,n+1]} = e^{-\beta J_{n,n+1}} G^R_{[1,n]} \frac{2G^R_{[1,n]} + 2 \cosh(\beta J_{n+1,n+2})}{\left( G^R_{[1,n]} \right)^2 + 1 + G^R_{[1,n]} \left( 2 \cosh(\beta J_{n+1,n+2}) \right)}.
\]

(86)

Within the bulk, i.e. for \( J_{n+1,n+2} > 0 \), in order to be consistent with our previous approximations in the low-temperature limit (equation (1)), we may replace

\[
2 \cosh(\beta J_{n+1,n+2}) \quad \Rightarrow \quad e^{\beta J_{n+1,n+2}}.
\]

(87)
Moreover, we expect that the renormalized amplitudes $G^R$ only decay upon renormalization, and are thus smaller than their initial unity value (see the first step of equation (85)),

$$G^R_{[1,n]} \leq 1.$$  

(88)

Therefore, equation (86) reduces to

$$G^R_{[1,n+1]} \simeq e^{-\beta J_{n,n+1}} G^R_{[1,n]} \left[ \frac{e^{\beta J_{n+1,n+2}}}{1 + G^R_{[1,n]} e^{\beta J_{n+1,n+2}}} \right],$$

(89)

i.e. this recurrence becomes simpler in terms of inverse variables,

$$\frac{e^{\beta J_{n+1,n+2}}}{G^R_{[1,n+1]}} \simeq e^{\beta J_{n,n+1} + \beta J_{n+1,n+2}} + \frac{e^{\beta J_{n,n+1}}}{G^R_{[1,n]}}.$$  

(90)

5.1.4. Final result for a chain of $N$ spins. For a finite chain of $N$ spins where $J_{N,N+1} = 0$, the recurrence of equation (86) yields for the last step with $n + 1 = N$

$$G^R_{[1,N]} = e^{-\beta J_{N-1,N}} G^R_{[1,N-1]} \left[ \frac{2G^R_{[1,N-1]} + 2}{(G^R_{[1,N-1]})^2 + 1 + G^R_{[1,N-1]}^2} \right],$$

(91)

i.e. using inverse variables

$$\frac{1}{G^R_{[1,N]}^{\text{final}}} = \frac{1}{2} \left[ e^{\beta J_{N-1,N}} + \frac{e^{\beta J_{N-1,N}}}{G^R_{[1,N-1]}} \right].$$

(92)

We now use iteratively the recurrence of equation (90) valid in the bulk to obtain

$$\frac{1}{G^R_{[1,N]}} = \frac{1}{2} \left[ e^{\beta J_{N-1,N}} + e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}} + \frac{e^{\beta J_{N-2,N-1}}}{G^R_{[1,N-2]}} \right],$$

$$= \frac{1}{2} \left[ e^{\beta J_{N-1,N}} + e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}} + e^{\beta J_{N-3,N-2} + \beta J_{N-2,N-1}} + \frac{e^{\beta J_{N-3,N-2}}}{G^R_{[1,N-3]}} \right],$$

$$\simeq \frac{1}{2} \sum_{k=1}^{N} e^{\beta (J_{k-1,k} + J_{k,k+1})}$$

(93)

(with the notations $J_{0,1} = 0 = J_{N,N+1}$).

The conclusion of this RG procedure is thus that the equilibration time of the finite random chain of $N$ spins reads for the ‘simple’ dynamics (equation (77))

$$t_{\text{eq}}^{\text{simple}}(N) = \frac{1}{E_1^{\text{final}}(N)} = \frac{1}{2G^R_{[1,N]}} = \frac{1}{4} \sum_{k=1}^{N} e^{\beta (J_{k-1,k} + J_{k,k+1})}.$$  

(94)

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5.2. Closed RG for dynamics depending only on the local field in one dimension

We now consider the more general case where the amplitude \( G_k \) is a single even function \( G(x) = G(-x) \) of the local field (see equation (34)),

\[
G_k = G \left( J_{k-1,k} \sigma_{k-1}^z + J_{k,k+1} \sigma_{k+1}^z \right).
\]

The Glauber Hamiltonian of equation (27) corresponds to the special function

\[
G(x) = \frac{1}{2 \cosh(\beta x)}.
\]

5.2.1. Closed boundary RG procedure. If we iterate the renormalization near the boundary as in section 5.1.3, we obtain finally the following recurrence (equation (71)):

\[
G_R^{[1,n+1]} = e^{-\beta J_{n+1,n}^2} G_R^{[1,n]} \frac{f_{n+1}^+ f_{n+1}^- + f_{n+1}^+ f_{n+1}^- (2 \cosh(\beta J_{n+1,n+2}))}{(f_{n+1}^+ + f_{n+1}^-)^2 + G_R^{[1,n]}(f_{n+1}^+ e^{\beta J_{n+1,n+2}} + f_{n+1}^- e^{-\beta J_{n+1,n+2}})},
\]

in terms of the numbers

\[
f_{n+1}^\pm \equiv G(J_{n,n+1} \pm J_{n,n+2}).
\]

Within the bulk, i.e. for \( J_{n+1,n+2} > 0 \), in order to be consistent with our previous approximations in the low-temperature limit (equation (1)), we may use equation (87) and the fact that the renormalized amplitudes \( G_R \) only decay upon renormalization to simplify equation (97) to

\[
G_R^{[1,n+1]} \approx e^{-\beta J_{n+1,n}} G_R^{[1,n]} \frac{f_{n+1}^+ f_{n+1}^- e^{3\beta J_{n+1,n+2}}}{f_{n+1}^+ f_{n+1}^- + G_R^{[1,n]}(f_{n+1}^+ e^{3\beta J_{n+1,n+2}} + f_{n+1}^- e^{-3\beta J_{n+1,n+2}})},
\]

i.e. this recurrence becomes simpler in terms of inverse variables,

\[
e^{-3\beta J_{n+1,n+2}} \approx \frac{e^{3\beta J_{n+1,n+2}} + e^{-3\beta J_{n+1,n+2}}}{G_R^{[1,n]}},
\]

5.2.2. Final result for a chain of \( N \) spins. For a finite chain of \( N \) spins where \( J_{N,N+1} = 0 \), equation (98) reads

\[
f_N^\pm \equiv G(J_{N-1,N} \pm 0) = f_N^-,
\]

and the recurrence of equation (97) yields for the last step with \( n+1 = N \)

\[
G_R^{[1,N]} = e^{-\beta J_{N-1,N}} G_R^{[1,N-1]} \frac{2f_N^- (G_R^{[1,N-1]} + f_N^-)}{(G_R^{[1,N-1]} + f_N^-)^2}
\]

\[
= e^{-\beta J_{N-1,N}} G_R^{[1,N-1]} \frac{2f_N^-}{G_R^{[1,N-1]} + f_N^-},
\]

doi:10.1088/1742-5468/2013/02/P02037
i.e. using inverse variables
\[
\frac{1}{G_{[1,N]}^{R}} = \frac{1}{2} \left[ \frac{e^{\beta J_{N-1,N}}}{f_{N}} + \frac{e^{\beta J_{N-1,N}}}{G_{[1,N-1]}^{R}} \right].
\] (103)

We now use iteratively the recurrence of equation (100) valid in the bulk to obtain
\[
\frac{1}{G_{[1,N]}^{R}} = \frac{1}{2} \left[ \frac{e^{\beta J_{N-1,N}}}{f_{N}} + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}} + \frac{e^{\beta J_{N-1,N}}}{G_{[1,N-2]}^{R}} \right]
= \frac{1}{2} \left[ \frac{e^{\beta J_{N-1,N}}}{f_{N}} + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}} + \frac{e^{\beta J_{N-3,N-2} + \beta J_{N-2,N-1}}}{f_{N-2}} + \frac{e^{\beta J_{N-3,N-2}}}{G_{[1,N-3]}^{R}} \right]
\geq \frac{1}{2} \sum_{k=1}^{N} \frac{e^{\beta (J_{k-1,k} + J_{k,k+1})}}{f_{k}}.
\] (104)

(with the notations $J_{0,1} = 0 = J_{N,N+1}$).

Using equation (98), the conclusion of this RG procedure is thus that the equilibration time of the finite random chain of $N$ spins reads for the dynamics defined by the amplitudes of equation (97)
\[
t_{\text{eq}}(N) = \frac{1}{E_{1}^{\text{final}}(N)} = \frac{1}{2G_{[1,N]}^{R}} = \frac{1}{4} \sum_{k=1}^{N} \frac{e^{\beta (J_{k-1,k} + J_{k,k+1})}}{G(J_{k-1,k} - J_{k,k+1})}.
\] (105)

In particular, for the Glauber dynamics corresponding to equation (96), the equilibration time reads
\[
t_{\text{Glauber}}(N) = \frac{1}{4} \sum_{k=1}^{N} e^{\beta (J_{k-1,k} + J_{k,k+1})} 2 \cosh (\beta J_{k-1,k} - \beta J_{k,k+1}) = \frac{1}{4} \sum_{k=1}^{N} [e^{2\beta J_{k-1,k}} + e^{2\beta J_{k,k+1}}]
= \frac{1}{2} \left[ 1 + \sum_{k=1}^{N-1} e^{2\beta J_{k,k+1}} \right].
\] (106)

5.3. Discussion

In this section, we have obtained a closed RG procedure in dimension $d = 1$ that yields the explicit result of equation (105) for the equilibration time of a finite chain. The validity of this RG result is checked in appendix A using another method. Since we compare the explicit expression in terms of the set of all random couplings, for various choices of the transition rates, this agreement shows the exactness of the RG procedure near zero temperature.

The physical meaning of equation (105) is that the equilibration time is the sum of $N$ random variables, possibly slightly correlated (the same random coupling $J_{n,n+1}$ appears in the two terms corresponding to $k = n$ and $n+1$). For the Glauber case of equation (106), even these slight correlations disappear and the equilibration time reduces to the sum of independent random variables $e^{2\beta J_{k,k+1}}$, whose distribution can be computed from the distribution of the random couplings $J_{k,k+1}$. 

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Note that in the limit of a pure chain where all ferromagnetic couplings have the same value $J$, equation (105) reduces to

$$ t_{\text{pure}}^\text{eq}(N) = \frac{1}{4} \left[ 2 e^{\beta J} G(J) + (N - 2) e^{2\beta J} G(0) \right] \propto \frac{N e^{2\beta J}}{G(0)}; \quad (107) $$

as expected, the Arrhenius factor $e^{2\beta J}$ comes from the barrier $(2J)$ to create a domain-wall at one boundary and the prefactor $N$ comes from the small probability of order $1/N$ that this domain-wall crosses the whole system of size $N$ instead of returning back. For the Glauber case where $G(0) = 2$, equation (107) is in agreement with the leading term near zero temperature of the open pure chain discussed in [24].

6. Boundary renormalization for the Ising model on the Cayley tree

In this section, we consider the pure ferromagnetic Ising model with the classical energy of equation (20) defined on a Cayley tree of branching ratio $K$ (coordination $(K + 1)$) with $N$ generations and with free boundary conditions on all the boundary spins. We focus on the dynamics corresponding to the simplest choice of equation (24), where the corresponding quantum Hamiltonian of equation (25) has initial amplitudes $G_0$ all equal to unity.

We wish to define a closed boundary RG procedure that preserves the symmetry between the $K$ offspring of a given site. Therefore, the basic RG step concerns $K$ renormalized boundary spins $(S_1, S_2, \ldots, S_K)$ whose renormalized dynamics is described by some renormalized amplitude $G$ (which is a number and not an operator) and their common ancestor spin $S$ whose dynamics is still described by the initial amplitude $g = 1$ and by the external field $B_a = J_a \sigma^z$ induced by its next ancestor spin $S_a$. The ferromagnetic couplings still have their initial value $J$ so that we have to study the following effective Hamiltonian for these $(K + 1)$ spins $(S_1, \ldots, S_K, S)$:

$$ H_{K+1}^\text{simple} \equiv \left( e^{-\beta \sigma^z (\sum_{i=1}^K J_S i + B_a)} - \sigma^x \right) + G \sum_{i=1}^K \left( e^{-\beta \sigma^z J_S i - \sigma^x} \right). \quad (108) $$

After the first RG step, the renormalized amplitude $G$ is expected to become smaller and smaller, so the most appropriate approach is a perturbative analysis in the parameter $G$.

6.1. Properties of the Hamiltonian $H_{K+1}^\text{simple}$ for $G = 0$

For $G = 0$, the spins $(S_1, \ldots, S_K)$ cannot flip and are thus frozen, so the problem reduces to the single spin $S$ in the external field $B = \left( \sum_{i=1}^K J_S i + B_a \right)$ that we have already studied in section 4.2.1. Therefore the $2^{K+1}$ states can be decomposed into the following states.

(i) The $2^K$ states corresponding to equation (40),

$$ |v_{S_1, S_2, \ldots, S_K}^0 S_j, S_j \rangle \equiv \prod_{j=1}^K |S_j \rangle \sum_{S=\pm} \frac{e^{(\beta/2) S (\sum_{i=1}^K J_S i + B_a)}}{\sqrt{2 \cosh \beta (\sum_{i=1}^K J_S i + B_a)}} |S \rangle, \quad (109) $$

which have a vanishing eigenvalue for $G = 0$,

$$ \left( e^{-\beta \sigma^z (\sum_{i=1}^K J_S i + B_a)} - \sigma^x \right) |v_{S_1, S_2, \ldots, S_K}^0 S_j, S_j \rangle = 0. \quad (110) $$

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The physical interpretation is that the spin $S$ is at equilibrium with respect to the frozen spins $(S_1, \ldots, S_K)$.

(ii) The $2^K$ states corresponding to equation (41),
\[ |u_0^{S_1, S_2, \ldots, S_K} \rangle \equiv \prod_{j=1}^{K} |S_j \rangle \sum_{S=\pm} \frac{S e^{-(\beta/2)S(\sum_{i=1}^{K} J S_i + B_a)}}{\sqrt{2 \cosh \beta (\sum_{i=1}^{K} J S_i + B_a)}} |S \rangle, \tag{111} \]
which have a finite eigenvalue for $G = 0,$
\[ (e^{-\beta \sigma^z (\sum_{i=1}^{K} J \sigma^z_i + B_a)} - \sigma^z) |u_0^{S_1, S_2, \ldots, S_K} \rangle = 2 \cosh \beta \left( \sum_{i=1}^{K} J S_i + B_a \right) |u_0^{S_1, S_2, \ldots, S_K} \rangle. \tag{112} \]

Note that the exact ground state (equation (16)) of the Hamiltonian of equation (108),
\[ |\psi_0 \rangle = \frac{1}{\sqrt{Z_{K+1}}} \sum_{S_1=\pm, S_2=\pm} \cdots \sum_{S_K=\pm} e^{\beta S(\sum_{i=1}^{K} J S_i + B_a)} |S_1 \rangle |S_2 \rangle \cdots |S_K \rangle |S \rangle, \]
\[ Z_{K+1} = \sum_{S_1=\pm, S_2=\pm} \cdots \sum_{S_K=\pm} e^{\beta S(\sum_{i=1}^{K} J S_i + B_a)} \]
\[ = \sum_{S_1=\pm, S_2=\pm} \cdots \sum_{S_K=\pm} 2 \cosh \beta \left( \sum_{i=1}^{K} J S_i + B_a \right), \tag{113} \]
belongs to the subspace spanned by the $2^K$ states $|u_0^{S_1, S_2, \ldots, S_K} \rangle$ (equation (110))
\[ |\psi_0 \rangle = \frac{1}{\sqrt{Z_{K+1}}} \sum_{S_1=\pm, S_2=\pm} \cdots \sum_{S_K=\pm} \sqrt{2 \cosh \beta \left( \sum_{i=1}^{K} J S_i + B_a \right)} |v_0^{S_1, S_2, \ldots, S_K} \rangle. \tag{114} \]

### 6.2. Perturbation in the parameter $G$

We have seen that for $G = 0$ there are $2^K$ states (equation (110)) that have zero energy. For small $G > 0$, the perturbation will lift this degeneracy: the exact ground state of equation (114) will keep its zero energy for arbitrary $G$, but the other $(2^K - 1)$ eigenvalues will become positive as soon as $G > 0$. To determine them, we need to diagonalize the perturbation within the subspace spanned by the $2^K$ vectors $|v_0^{S_1, S_2, \ldots, S_K} \rangle$, i.e. we look for an eigenstate via the linear combination
\[ |u_\lambda \rangle = \sum_{S_1=\pm, S_2=\pm, \ldots, S_K=\pm} T^{S_1, S_2, \ldots, S_K} |v_0^{S_1, S_2, \ldots, S_K} \rangle. \tag{115} \]

The eigenvalue equation
\[ 0 = (H_{K+1}^{\text{simple}} - \lambda) |u_\lambda \rangle \]
\[ = \sum_{S_1=\pm, S_2=\pm, \ldots, S_K=\pm} T^{S_1, S_2, \ldots, S_K} \left[ \sum_{i=1}^{K} G (e^{-\beta \sigma^z_i J \sigma^z_i} - \sigma_i^z) - \lambda \right] |v_0^{S_1, S_2, \ldots, S_K} \rangle \tag{116} \]
can be projected onto the $2^K$ bra $\langle v_0^{S_1',...,S_K'} | (H_{K+1}^{\text{simple}} - \lambda) | v_0^{S_1,...,S_K} \rangle$

$$0 = \langle v_0^{S_1',...,S_K'} | (H_{K+1}^{\text{simple}} - \lambda) | v_0^{S_1,...,S_K} \rangle$$

$$= \langle v_0^{S_1',...,S_K'} | \sum_{S_1=\pm,...,S_K=\pm} T^{S_1,...,S_K} \left[ \sum_{i=1}^{K} G \left( e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^z \right) - \lambda \right] | v_0^{S_1,...,S_K} \rangle. \quad (117)$$

From the matrix elements

$$\langle v_0^{S_1',S_2',...,S_K'} | e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^z | v_0^{S_1,...,S_K} \rangle$$

$$= \left( \prod_j \delta_{S_j',S_j} \right) \frac{2 \cosh(\sum_{j \neq i} J S_j + B_a)}{2 \cosh(\sum_{j=1}^{K} J S_j + B_a)} - \left( \prod_j \delta_{S_j',S_j} \right) \delta_{S_i',-S_i} \times \frac{2 \cosh(\sum_{j \neq i} J S_j + B_a)}{2 \cosh(\sum_{j \neq i} J S_j - J S_i + B_a)} \right) \frac{2 \cosh(\sum_{j \neq i} J S_j + B_a)}{2 \cosh(\sum_{j \neq i} J S_j + J S_i + B_a)}$$

we obtain that equation (117) yields

$$0 = \sum_{S_1=\pm,...,S_K=\pm} T^{S_1,...,S_K} \times \left[ \sum_{i=1}^{K} G \langle v_0^{S_1',...,S_K'} | \left( e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^z \right) | v_0^{S_1,...,S_K} \rangle - \lambda \langle v_0^{S_1',...,S_K'} | v_0^{S_1,...,S_K} \rangle \right] \left[ \sum_{i=1}^{K} G \left( e^{-\beta \sigma_i^z J \sigma^z} - \sigma_i^z \right) - \lambda \right]$$

$$= \left[ \sum_{i=1}^{K} G \frac{2 \cosh(\sum_{j \neq i} J S'_j + B_a)}{2 \cosh(\sum_{j=1}^{K} J S'_j + B_a)} - \lambda \right] T^{S_1',...,S_K'}$$

$$- G \sum_{i=1}^{K} \left[ \frac{2 \cosh(\sum_{j \neq i} J S'_j + B_a)}{2 \cosh(\sum_{j \neq i} J S'_j - J S'_i + B_a)} \right] \frac{2 \cosh(\sum_{j \neq i} J S'_j + B_a)}{2 \cosh(\sum_{j \neq i} J S'_j + J S'_i + B_a)}$$

$$T^{S_1',S_2',...,S_K'} = t \left( k = \sum_{i=1}^{K} \frac{1 + S_i'}{2} \right). \quad (120)$$

For the two extremal cases $t(K)$ (all spins are (+)) and $t(0)$ (all spins are (−)), equation (119) becomes

$$0 = \left[ G K \frac{2 \cosh(\sum_{j=1}^{K} J + B_a)}{2 \cosh(\sum_{j \neq i} J + B_a)} - \lambda \right] t(K)$$

$$- G K \frac{2 \cosh(\sum_{j \neq i} J + B_a)}{\sqrt{2 \cosh(\sum_{j \neq i} J + B_a) \sqrt{2 \cosh(\sum_{j \neq i} J + B_a)}}} t(K - 1),$$

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\[ 0 = \left[ GK \frac{2 \cosh \beta \((K - 1)J - B_a\)}{2 \cosh \beta(KJ - B_a)} - \lambda \right] t(0) \]

\[ - GK \frac{2 \cosh \beta \((K - 1)J - B_a\)}{\sqrt{2} \cosh \beta((K - 2)J - B_a) \sqrt{2} \cosh \beta(KJ - B_a)} t(1), \]

whereas for the non-extremal case, \(0 < k < K\), equation (119) reads

\[ 0 = \left[ Gk \frac{2 \cosh \beta(2k - K - 1)J + B_a}{2 \cosh \beta(2k - K)J + B_a} \right. \]

\[ + G(K - k) \frac{2 \cosh \beta(2k - K + 1)J + B_a}{2 \cosh \beta(2k - K)J + B_a} - \lambda \left] t(k) \right. \]

\[ - Gk \frac{2 \cosh \beta(2k - K - 1)J + B_a}{\sqrt{2} \cosh \beta((2k - K)J - B_a) \sqrt{2} \cosh \beta((2k - K)J + B_a)} t(k - 1) \]

\[ - G(K - k) \frac{2 \cosh \beta(2k - K + 1)J + B_a}{\sqrt{2} \cosh \beta((2k - K)J + B_a) \sqrt{2} \cosh \beta((2k - K)J + B_a)} t(k + 1). \]

It is easy to check that the components (not normalized here) of the exact ground state of equation (113),

\[ t_0(k) = \sqrt{2 \cosh \beta((2k - K)J + B_a)}, \]

satisfy equations (121) and (122) for \(\lambda = 0\), as they should.

To compute the lowest non-vanishing eigenvalue \(\lambda_1\), it is consistent to set \(\lambda_1 = 0\) in equation (122) for all non-extremal cases \(0 < k < K\), leading to

\[ 0 = \left[ k \frac{2 \cosh \beta(2k - K - 1) + B_a}{\sqrt{2} \cosh \beta((2k - K)J + B_a)} + (K - k) \frac{2 \cosh \beta(2k - K + 1) + B_a}{\sqrt{2} \cosh \beta((2k - K)J + B_a)} \right] t(k) \]

\[ - k \frac{2 \cosh \beta((2k - K - 1) + B_a)}{\sqrt{2} \cosh \beta((2k - K - 2) + B_a)} t(k - 1) \]

\[ - (K - k) \frac{2 \cosh \beta((2k - K + 1) + B_a)}{\sqrt{2} \cosh \beta((2k - K + 2) + B_a)} t(k + 1), \]

which may be solved in terms of the boundary conditions \(t(K)\) and \(t(0)\).

To obtain the explicit solution, it is convenient to introduce the amplitudes \(A(k)\) with respect to the ground state components of equation (123),

\[ t(k) \equiv A(k)t_0(k) = A(k)\sqrt{2 \cosh \beta((2k - K)J + B_a)}, \]

so that equation (124) takes the simpler form

\[ A(k) = p_-(k)A(k - 1) + p_+(k)A(k + 1), \]
with the notation

\[ p_-(k) \equiv \frac{k \cosh \beta ((2k - K - 1) + B_a)}{(k \cosh \beta ((2k - K - 1) + B_a) + (K - k) \cosh \beta ((2k - K + 1) + B_a))}, \]

\[ p_+(k) \equiv \frac{(K - k) \cosh \beta ((2k - K + 1) + B_a)}{(k \cosh \beta ((2k - K - 1) + B_a) + (K - k) \cosh \beta ((2k - K + 1) + B_a))} \]

\[ = 1 - p_-(k). \]  

(127)

Let us introduce two linearly independent solutions. The solution corresponding to the boundary conditions

\[ Q_K(0) = 0, \]
\[ Q_K(K) = 1 \]  

(128)

can be obtained by recurrence [25] and reads

\[ Q_K(k) = \frac{R_K(1, k)}{R_K(1, K)} \]  

(129)

using Kesten variables [26]

\[ R_K(1, 0) = 0, \]
\[ R_K(1, 1) = 1, \]
\[ R_K(1, k \geq 2) = 1 + \sum_{m=1}^{k-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)}, \]

\[ R_K(1, K) = 1 + \sum_{m=1}^{K-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)} \]

\[ = 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \cdots + \frac{p_-(1)p_-(2)\cdots p_-(K - 1)}{p_+(1)p_+(2)\cdots p_+(K - 1)}. \]  

(130)

Similarly, the solution corresponding to the boundary conditions

\[ Q_0(0) = 1, \]
\[ Q_0(K) = 0 \]  

(131)

reads

\[ Q_0(k) = \frac{R_0(k, K - 1)}{R_0(0, K - 1)}. \]  

(132)
with
\[ R_0(K, K - 1) = 0, \]
\[ R_0(K - 1, K - 1) = 1, \]
\[ R_0(k \leq K - 2, K - 1) = 1 + \sum_{m=k+1}^{K-1} \prod_{n=m}^{K-1} p_+(n) p_-(n), \]
\[ R_0(0, K - 1) = 1 + \sum_{m=1}^{K-1} \prod_{n=m}^{K-1} p_+(n) p_-(n), \]
\[ = 1 + \frac{p_+(K - 1)}{p_-(K - 1)} + \cdots + \frac{p_+(K - 1)p_+(K - 2) \cdots p_+(1)}{p_-(K - 1)p_-(K - 2) \cdots p_-(1)}. \]

It is useful to introduce the continuation of the ground state components of equation (123) to half-integers to rewrite the ratios
\[ \frac{p_-(k)}{p_+(k)} = \frac{k2 \cosh \beta((2k - K - 1) + B_a)}{(K - k)2 \cosh \beta((2k - K + 1) + B_a)} = \frac{kt_0^2(k - 1/2)}{(K - k)t_0^2(k + 1/2)}, \]
and the products
\[ \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)} = \prod_{n=1}^{m} \left[ \frac{n t_0^2(n - 1/2)}{(K - n)t_0^2(n + 1/2)} \right] = \frac{m!(K - 1 - m)!}{(K - 1)!} \frac{t_0^2(1/2)}{t_0^2(m + 1/2)}, \]
\[ \prod_{m=n}^{K-1} \frac{p_+(n)}{p_-(n)} = \prod_{n=m}^{K-1} \left[ \frac{(K - n)t_0^2(n + 1/2)}{n t_0^2(n - 1/2)} \right] = \frac{(m - 1)!(K - m)!}{(K - 1)!} \frac{t_0^2(1/2)}{t_0^2(m - 1/2)}. \]

In particular, in the following, we will need the two denominators
\[ R_K(1, K) = 1 + \sum_{m=1}^{K-1} \frac{m!(K - 1 - m)!}{(K - 1)!} \frac{t_0^2(1/2)}{t_0^2(m + 1/2)} = t_0^2(1/2) \sum_{m=0}^{K-1} c_K^{-m} \frac{1}{t_0^2(m + 1/2)} \]
and
\[ R_0(0, K - 1) = 1 + \sum_{m=1}^{K-1} \frac{(m - 1)!(K - m)!}{(K - 1)!} \frac{t_0^2(K - 1/2)}{t_0^2(m - 1/2)} \]
\[ = t_0^2 \left( K - \frac{1}{2} \right) \sum_{m=0}^{K-1} c_K^{-m} \frac{1}{t_0^2(m + 1/2)} \]
that determine the solutions near the boundaries of \( Q_K \),
\[ Q_K(1) = \frac{R_K(1, 1)}{R_K(1, K)} = \frac{1}{R_K(1, K)}; \]
\[ 1 - Q_K(K - 1) = \frac{R_K(1, K) - R_K(1, K - 1)}{R_K(1, K)} = \frac{1}{R_0(0, K - 1)}. \]
and of \( Q_0 \),
\[
Q_0(K - 1) = \frac{R_0(K - 1, K - 1)}{R_0(0, K - 1)} = \frac{1}{R_0(0, K - 1)},
\]
\[
1 - Q_0(1) = \frac{R_0(0, K - 1) - R_0(1, K - 1)}{R_0(0, K - 1)} = \frac{1}{R_K(1, K)}.
\]

The solution of the system (126) that satisfies the boundary conditions (equation (125))
\[
A(0) = \frac{t(0)}{t_0(0)},
\]
\[
A(K) = \frac{t(K)}{t_0(K)}
\]
can be obtained by the linear combination
\[
A(k) = A(0)Q_0(k) + A(K)Q_K(k) = \frac{t(0)}{t_0(0)} Q_0(k) + \frac{t(K)}{t_0(K)} Q_K(k),
\]
so that the solution of the system (124) reads
\[
t(k) = A(k)t_0(k) = \left[ \frac{t(0)}{t_0(0)} Q_0(k) + \frac{t(K)}{t_0(K)} Q_K(k) \right] t_0(k).
\]

To determine \( \lambda_1 \), we just need to replace
\[
t(1) = \left[ \frac{t_0(1)}{t_0(0)} Q_0(1) \right] t(0) + \left[ \frac{t_0(1)}{t_0(K)} Q_K(1) \right] t(K),
\]
\[
t(K - 1) = \left[ \frac{t_0(K - 1)}{t_0(0)} Q_0(K - 1) \right] t(0) + \left[ \frac{t_0(K - 1)}{t_0(K)} Q_K(K - 1) \right] t(K)
\]
in equation (121) to obtain the following system of two linear equations for the two components \( t_{\lambda_1}(0) \) and \( t_{\lambda_1}(K) \):
\[
0 = \left[ 1 - Q_K(K - 1) - \lambda_1 \frac{t_0^2(K)}{G K t_0^2(K - 1/2)} \right] t_{\lambda_1}(K) - \frac{t_0(K)}{t_0(0)} Q_0(K - 1)t_{\lambda_1}(0),
\]
\[
0 = \left[ 1 - Q_0(1) - \lambda_1 \frac{t_0^2(0)}{G K t_0^2(1/2)} \right] t_{\lambda_1}(0) - \frac{t_0(0)}{t_0(K)} Q_K(1)t_{\lambda_1}(K).
\]

The two components (not normalized here) are orthogonal to equation (123) as they should be and read
\[
t_{\lambda_1}(0) = t_0(K) = \sqrt{2 \cosh \beta(KJ + B_a)},
\]
\[
t_{\lambda_1}(K) = -t_0(0) = -\sqrt{2 \cosh \beta(KJ - B_a)}.
\]

The corresponding eigenvalue \( \lambda_1 \) reads using equations (136), (137), (138) and (139)
\[
\lambda_1 = \frac{G K}{\sum_{m=0}^{K-1} c_{K-1} \cosh(\beta(2m+1-K)J+B_a)} \left[ \frac{1}{2 \cosh \beta(KJ - B_a)} + \frac{1}{2 \cosh \beta(KJ + B_a)} \right].
\]
Since this expression is unchanged via the transformation \( B_a = JS_a \rightarrow -B_a \), we may replace \( B_a \) by its absolute value \( |B_a| = J \) to obtain the final expression for the lowest non-vanishing eigenvalue \( \lambda_1 \) at first order in perturbation with respect to the parameter \( G \),

\[
\lambda_1 = \frac{GK}{\sum_{m=0}^{K-1} \frac{1}{2 \cosh \beta J (2m+2-K)}} \left[ \frac{1}{2 \cosh \beta J (K-1)} + \frac{1}{2 \cosh \beta J (K+1)} \right] + O(G^2). \tag{147}
\]

### 6.3. Renormalization rule for the amplitude \( G^R \)

Let us now project the Hamiltonian of equation (108) onto its two lowest eigenvalues \( \lambda_0 = 0 \) and \( \lambda_1 \),

\[
H_{K+1}^{\text{simple}} \simeq \frac{\lambda_1}{\langle u_{\lambda_1} | u_{\lambda_1} \rangle} |u_{\lambda_1}\rangle \langle u_{\lambda_1}|, \tag{148}
\]

where the eigenvector of equation (115) can be approximated at low temperature by its two components onto fully ferromagnetic states,

\[
|u_{\lambda_1}\rangle \simeq t_{\lambda_1}(K) \left( \prod_{j=1}^{K} |S_j = 1\rangle \right) \sum_{S = \pm} \frac{e^{(\beta/2)S(KJ + B_a)}}{\sqrt{2 \cosh \beta (KJ + B_a)}} |S\rangle
\]

\[
+ t_{\lambda_1}(0) \left( \prod_{j=1}^{K} |S_j = -1\rangle \right) \sum_{S = \pm} \frac{e^{(\beta/2)S(-KJ + B_a)}}{\sqrt{2 \cosh \beta (KJ - B_a)}} |S\rangle
\]

\[
\simeq t_{\lambda_1}(K) \left( \prod_{j=1}^{K} |S_j = 1\rangle \right) |S = +1\rangle + t_{\lambda_1}(0) \left( \prod_{j=1}^{K} |S_j = -1\rangle \right) |S = -1\rangle, \tag{149}
\]

with the coefficients (equation (145))

\[
t_{\lambda_1}(0) = \sqrt{2 \cosh \beta (KJ + B_a)} \simeq e^{(\beta/2)(KJ + B_a)}, \]

\[
t_{\lambda_1}(K) = -\sqrt{2 \cosh \beta (KJ - B_a)} \simeq -e^{(\beta/2)(KJ - B_a)}. \tag{150}
\]

Finally, at leading order near zero temperature, one obtains

\[
|u_{\lambda_1}\rangle \simeq e^{(\beta/2)K} \left[ e^{(\beta/2)B_a} \left( \prod_{j=1}^{K} |S_j = 1\rangle \right) |S = +1\rangle - e^{-(\beta/2)B_a} \left( \prod_{j=1}^{K} |S_j = -1\rangle \right) |S = -1\rangle \right], \tag{151}
\]

with the corresponding normalization (using \( B_a = JS_a \))

\[
\langle u_{\lambda_1} | u_{\lambda_1}\rangle \simeq e^{\beta(KJ + B_a)} + e^{\beta(KJ - B_a)} = e^{\beta(KJ)2 \cosh(\beta B_a)} = e^{\beta(KJ)2 \cosh(\beta J)}. \tag{152}
\]

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In terms of the renormalized spin
\[ |S_R = +\rangle \equiv \left( \prod_{j=1}^{K} |S_j = 1\rangle \right) |S = +\rangle, \tag{153} \]
\[ |S_R = -\rangle \equiv \left( \prod_{j=1}^{K} |S_j = -1\rangle \right) |S = -\rangle, \]
and of the external local field \( B_R = B_a = JS_a \), the effective Hamiltonian of equation (148) can be rewritten as an elementary operator (equation (44))
\[
H_{\text{simple}}^{K+1} \simeq G_R \left( e^{-(\beta/2)B_R} |S_R = +\rangle - e^{(\beta/2)B_R} |S_R = -\rangle \right) \times \left( e^{-(\beta/2)B_R} \langle S_R = + | - e^{(\beta/2)B_R} \langle S_R = - | \right), \tag{154} \]
with the renormalized amplitude (using equation (147))
\[
G_R = \frac{\lambda_1}{2 \cosh \beta J} = \frac{G K}{2 \cosh \beta J \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J(2m+2-K)}} \times \left[ \frac{1}{2 \cosh \beta J(K-1)} + \frac{1}{2 \cosh \beta J(K+1)} \right]. \tag{155} \]

To be consistent with the previous low-temperature approximations, we now should evaluate the leading behavior of equation (155) near zero temperature, i.e. we should replace hyperbolic functions by exponentials. In particular, one has
\[
\frac{1}{2 \cosh \beta J(2m+2-K)} = \frac{1}{e^{\beta J(2m+2-K)} + e^{-\beta J(2m+2-K)}} \simeq \frac{1}{2} \quad \text{if } m = \frac{K}{2} - 1 \]
\[
\simeq e^{-\beta J[2m+2-K]} \quad \text{if } m \neq \frac{K}{2} - 1, \tag{156} \]
so that the leading term near low temperature of equation (155) depends on the parity of \( K \).

**6.3.1. Leading behavior near zero temperature for even \( K \).** When the branching ratio \( K \) is even, then \((K/2 - 1)\) is an integer, so that the integer \( m \) can take this value, and the sum in the denominator of equation (155) is dominated by this contribution,
\[
\sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J(2m+2-K)} \simeq \frac{1}{2C_{K-1}^{(K/2)-1}}, \tag{157} \]
so that equation (155) reads at leading order
\[
K \text{ even: } G^R \simeq G e^{-\beta JK} 2KC_{K-1}^{(K/2)-1} = G e^{-\beta JK} 2 \frac{K!}{(K/2)! (K/2 - 1)!}. \tag{158} \]

For instance, for \( K = 2 \), one obtains
\[ K = 2 : G^R \simeq 4e^{-2\beta J} G. \tag{159} \]
6.3.2. Leading behavior near zero temperature for odd $K$. When the branching ratio $K$ is odd, then $(K/2 - 1)$ is not an integer so that $m$ cannot take this value, and the sum in the denominator is dominated by the contributions of the two closest integers $m = (K - 3)/2$ and $m = (K - 1)/2$, 

$$
\sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m 2 \cosh \beta J (2m + 2 - K)} \approx \sum_{m=0}^{K-1} \frac{e^{-\beta J |2m+2-K|}}{C_{K-1}^m} \approx \frac{e^{-\beta J}}{C_{K-1}^{(K-3)/2}} + \frac{e^{-\beta J}}{C_{K-1}^{(K-1)/2}}
$$

$$
= e^{-\beta J} \left[ \frac{((K + 1)/2)!((K - 3)/2)!}{(K - 1)!} + \frac{((K - 1)/2)((K - 1)/2)!}{(K - 1)!} \right]
$$

$$
= e^{-\beta J} K \frac{((K - 1)/2)!((K - 3)/2)!}{(K - 1)!},
$$

so that equation (155) reads at leading order

$$
K \text{ odd: } G_R \approx Ge^{-\beta J(K-1)} \frac{(K - 1)!}{((K - 1)/2)!((K - 3)/2)!} = Ge^{-\beta J(K-1)}(K - 1)C_{K-2}^{(K-1)/2}. \quad (161)
$$

For instance, for $K = 3$, one obtains

$$
K = 3 : G^R \approx 2e^{-2\beta J}G. \quad (162)
$$

6.4. Conclusion for the equilibrium time $t_{eq}^\text{simple}(N)$ of a Cayley tree with $N$ generations

Let us now consider a finite Cayley tree of branching ratio $K$ with $N$ generations. For the first RG step where $G_0 = 1$, we cannot use the perturbative analysis presented above to obtain $G_1$. However, since $G_1 \ll 1$ at low temperature, we may use the perturbative analysis given above to obtain the recursion

$$
G_{n+1} \approx \frac{G_n}{\rho(K^n)} \approx \frac{G_1}{[\rho(K)]^n} \quad (163)
$$

for all RG steps corresponding $1 \leq n \leq N - 1$, where the factor $\rho(K)$ has been evaluated at low temperature (equations (158) and (161)),

$$
\rho(K) = \begin{cases} 
  e^{\beta J K} & \text{for even } K \\
  2KC_{K-1}^{K/2-1} \frac{e^{\beta J K}}{e^{\beta J K(K-1)}} & \text{for odd } K \\
  \frac{(K - 1)C_{K-1}^{K-1/2}}{2(K!)} & \text{for odd } K.
\end{cases} \quad (164)
$$

Finally, at the last RG step, we could take into account that the center has $(K + 1)$ neighbors instead of $K$, and has no further ancestor, $B_a = 0$. However, this anomalous last step is only a boundary multiplicative contribution, as is $G_1$, and cannot change the dependence upon the number $N$ of generations for large $N$ coming from equation (163),

$$
G_{\text{final}}^N \propto \frac{1}{[\rho(K)]^N}. \quad (165)
$$

In summary, we obtain that the equilibrium time $t_{eq}^\text{simple}(N) = 1/(2G_{\text{final}}^N)$ (equation (77)) of a Cayley tree of branching ratio $K$ grows exponentially with the number
\( N \) of generations,
\[
  t_{\text{eq}}^{\text{simple}}(N) \propto [\rho(K)]^N,
\]  
(166)

where the growth factor \( \rho(K) \) is given explicitly by equation (164) for any \( K \).

### 6.5. Equilibrium time \( t_{\text{eq}}^{\text{Glauber}}(N) \) for the Glauber dynamics

The above results concerning the simple dynamics can be extended to the Glauber dynamics as follows. The Hamiltonian \( H_{K+1}^{\text{simple}} \) of equation (108) has to be replaced for the first step by
\[
H_{K+1}^{\text{Glauber}} \equiv \frac{1}{2 \cosh (\beta \sum_{i=1}^{K} J \sigma_i^z + B_a)} \left( e^{-\beta \sigma_i^z \sum_{i=1}^{K} J \sigma_i^z + B_a} - \sigma^x \right) \]
\[
+ \frac{1}{2 \cosh (\beta J \sigma^z)} \sum_{i=1}^{K} (e^{-\beta \sigma_i^z J \sigma^z} - \sigma^x).
\]  
(167)

Since the \( K \) leaves have no external field and are just linked to \( \sigma^z \), the amplitude \( 1/2 \cosh (\beta J \sigma^z) \) reduces to the number \( 1/2 \cosh (\beta J) \). The remaining non-trivial amplitude \( 1/2 \cosh (\beta \sum_{i=1}^{K} J \sigma_i^z + B_a) \) will disappear when we apply the perturbation method within the subspace annihilating the corresponding operator \( (e^{-\beta \sigma_i^z \sum_{i=1}^{K} J \sigma_i^z + B_a} - \sigma^x) \). Our conclusion is, thus, that the equilibrium time \( t_{\text{eq}}^{\text{Glauber}}(N) \) for the Glauber dynamics will have exactly the same leading exponential behavior in \( N \) as the result of equation (166) derived for the simple dynamics
\[
  t_{\text{eq}}^{\text{Glauber}}(N) \propto [\rho(K)]^N,
\]  
(168)
even if the prefactor can differ (see the discussion on the differences between the equilibrium times of the simple and Glauber dynamics in appendix B).

### 6.6. Comparison with previous results on dynamical barriers

From equations (166) and (168), we obtain that the energetic barrier \( B_K(N) \) defined as the coefficient of \( \beta \) in \( \ln t_{\text{eq}}(N) \),
\[
B_K(N) = \lim_{\beta \to +\infty} \frac{\ln t_{\text{eq}}^{\text{simple}}(N)}{\beta} \left\{ \begin{array}{ll}
NJK + O(1) & \text{for even } K \\
NJ(K-1) + O(1) & \text{for odd } K
\end{array} \right.
\]  
(169)
grows linearly with the number \( N \) of generations (i.e. logarithmically with the number of sites \( N \sim K^N \)) in agreement with previous works of physicists [27]–[29] and mathematicians [30]–[33]. Besides this correct scaling with \( N \), it appears that the slope \( (K-1)J \) for odd \( K \) of equation (169) coincides with the slope obtained in [28], where a so-called ‘disjoint strategy’ is optimal, whereas the slope \( KJ \) for even \( K \) of equation (169) differs from the slope \( J(K-1) \) obtained in [27, 28], where a so-called ‘non-disjoint strategy’ is optimal. We refer to [27]–[31] for more explanation on the differences between disjoint and non-disjoint strategies. For the present work, it is clear that the renormalization procedure making coherent clusters of spins within sub-trees corresponds to the disjoint strategy.

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Finally, besides the Arrhenius factor involving the energetic barrier of equation (169), the present renormalization procedure predicts explicit combinatorial prefactors for the exponential growth factor $\rho(K)$ (equation (164)) that have not been previously discussed in the literature, to the best of our knowledge.

7. Conclusion

In this paper, we have introduced a real-space RG procedure valid near zero temperature to evaluate the largest relaxation time of classical random ferromagnets. We have used the standard mapping between the master equations satisfying detailed balanced and quantum Hamiltonians having an exact zero-energy ground state. The largest relaxation time $t_{eq}$ governing the convergence of the dynamics towards the Boltzmann equilibrium is determined by the lowest non-vanishing eigenvalue $E_1 = 1/t_{eq}$ of the quantum Hamiltonian $H$. We have thus defined appropriate real-space RG rules for the quantum Hamiltonian to evaluate $E_1$ for finite systems. We have described how the renormalization flow can be explicitly solved for the following two cases.

(i) For the one-dimensional random ferromagnetic chain with free boundary conditions, the largest relaxation time $t_{eq}$ can be expressed in terms of the set of random couplings for various choices of the dynamical transition rates. The validity of these RG results in $d = 1$ has been checked by comparison with another approach in appendix A.

(ii) For the pure Ising model on a Cayley tree of branching ratio $K$ (coordinance $(K + 1)$), we have computed the exponential growth of $t_{eq}(N)$ with the number $N$ of generations.

In a companion paper [34], we explain how the renormalization flow can also be explicitly solved for the hierarchical Dyson Ising model. In the future, we hope to obtain numerical results for the RG flow in finite dimensions, $d > 1$.

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Appendix A. Check of the validity of the RG procedure in $d = 1$

In this appendix, we present another approach to check the results of the RG procedure obtained in section 5 for the random ferromagnetic chain.

A.1. Ansatz for the first excited quantum state in terms of exit probabilities

A.1.1. Eigenequation for $\psi_1$. For the quantum Hamiltonian $\mathcal{H}$ corresponding to the spin-flip dynamics of classical spin models with the energy of equation (20), the exact ground state of equation (16),

$$
\psi_0(C) = \frac{e^{-(\beta/2)U(C)}}{\sqrt{Z}} = \frac{e^{-(\beta/2)\sum_{i<j} J_{ij} S_i S_j}}{\sqrt{Z}},
$$

(A.1)

$$
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$$
is invariant under a global flip of all the spins,
\[ \psi_0(-C) = \psi_0(C). \] (A.2)
In contrast, the first excited state will be antisymmetric under a global flip of all the spins,
\[ \psi_1(-C) = -\psi_1(C), \] (A.3)
but its modulus is expected to nearly coincide with \( \psi_0(C) \) in the two valleys around the two classical ferromagnetic ground states. It is thus convenient to set
\[ \psi_1(C) = \psi_0(C) A(C) \] (A.4)
and to look for the antisymmetric amplitude \( A(C) \) (antisymmetric under a global flip of all the spins),
\[ A(-C) = A(C). \] (A.5)
The eigenvalue equation for the quantum Hamiltonian of equations (9), (10) and (11),
\[ \mathcal{H}\psi_1 = E_1 \psi_1, \] (A.6)
becomes via the change of variables of equation (A.4)
\[ [W_{\text{out}}(C) - E_1] A(C) = \sum_{C'} W(C \to C') A(C'). \] (A.7)
For a large system where \( E_1 \) is small, we expect that \( E_1 \) can be neglected with respect to \( W_{\text{out}}(C) \) for all configurations different from the two classical ground states, so that one obtains the approximate equation
\[ A(C) \approx \sum_{C'} \frac{W(C \to C')}{W_{\text{out}}(C)} A(C') = \sum_{C'} \pi_C(C') A(C'), \] (A.8)
where
\[ \pi_C(C') \equiv \frac{W(C \to C')}{W_{\text{out}}(C)} = \frac{W(C \to C')}{\sum_{C''} W(C \to C'')} \] (A.9)
represents the probability that the first exit from configuration \( C \) leads to \( C' \) for the master equation of equation (2), with the normalization
\[ \sum_{C'} \pi_C(C') = 1. \] (A.10)

A.1.2. Relation with exit probabilities. Exit probabilities are known to satisfy a backward master equation similar to equation (A.8) (see, for instance, the textbooks [5]–[7]). More precisely, in a ferromagnet, one may introduce the probability \( Q_+(C) \) that the dynamics starting in configuration \( C \) reaches first the configuration \( C_+ \) (all spins plus) then the configuration \( C_- \) (all spins minus). The complementary probability \( Q_-(C) = 1 - Q_+(C) \) represents the probability that the dynamics starting in configuration \( C \) reaches first the configuration \( C_- \) then the configuration \( C_+ \). The escape probability satisfies the backward master equation
\[ Q_+(C) = \sum_{C'} \pi_C(C') Q_+(C') \] (A.11)
for all configurations $C$ different from the two ground states, and the boundary conditions
\begin{align}
Q_+(C_+) &= 1, \\
Q_+(C_-) &= 0.
\end{align}
(A.12)

This suggests the following ansatz for the antisymmetric $A(C)$ satisfying equation (A.8) up to a normalization factor $\mathcal{N}$:
\begin{align}
A^{\text{ansatz}}(C) &= \mathcal{N}(2Q_+(C) - 1) = \mathcal{N}(1 - 2Q_-(C)); \\
\text{using } Q_+(C) &= Q_-(C) \text{ one obtains } A(-C) = -A(C).
\end{align}
(A.13)

The only points where $Q_+(C)$ does not satisfy equation (A.11) are the two boundaries $C_+$ and $C_-$ where $Q_+$ is given by the boundary conditions corresponding to
\begin{align}
\frac{A^{\text{ansatz}}(C_+)}{\mathcal{N}} &= 2Q_+(C_+) - 1 = 1, \\
\frac{A^{\text{ansatz}}(C_-)}{\mathcal{N}} &= 2Q_+(C_-) - 1 = -1.
\end{align}
(A.14)

Let us now estimate $E_1$ for the ansatz of equation (A.13) corresponding to
\begin{align}
\psi_1^{\text{ansatz}}(C) &= \psi_0(C)A^{\text{ansatz}}(C) = \mathcal{N}\psi_0^{\text{ansatz}}(C)(2Q_+(C) - 1)
\end{align}
(A.15)

via
\begin{align}
E_1 &= \frac{\langle \psi_1^{\text{ansatz}} | H_Q | \psi_1^{\text{ansatz}} \rangle}{\langle \psi_1^{\text{ansatz}} | \psi_1^{\text{ansatz}} \rangle} \\
&= \frac{\sum_C \psi_0^2(C)A^{\text{ansatz}}(C) [W_{\text{out}}(C)A^{\text{ansatz}}(C) - \sum_{C'} W(C \rightarrow C')A^{\text{ansatz}}(C')]^{\text{2}}}{\sum_C \psi_0^2(C)(A^{\text{ansatz}}(C))^{\text{2}}}.
\end{align}
(A.16)

In the numerator, all configurations $C$ different from $C_+$ and $C_-$ give zero contributions as a consequence of equation (A.11). Therefore, the only contributions in the numerator come from $C = C_+$ and from $C = C_-$, where we may use the boundary conditions of equation (A.14) to obtain
\begin{align}
E_1 &= \left\{ \psi_0^2(C_+) \left[ W_{\text{out}}(C_+) - \sum_{C'} W(C_+ \rightarrow C')(2Q_+(C') - 1) \right] \\
&+ \psi_0^2(C_-) \left[ W_{\text{out}}(C_-) - \sum_{C'} W(C_- \rightarrow C')(1 - 2Q_+(C')) \right] \right\} \\
&\times \left\{ \sum_C \psi_0^2(C)(2Q_+(C) - 1)^{\text{2}} \right\}^{-1} \\
&= 2\psi_0^2(C_+) [\sum_{C'} W(C_+ \rightarrow C')Q_-(C')] + \psi_0^2(C_-) [\sum_{C'} W(C_- \rightarrow C')Q_+(C')] \\
&\sum_C \psi_0^2(C)(2Q_+(C) - 1)^{\text{2}}. 
\end{align}
(A.17)

The numerator involves the probability of reaching first $C_-$ before returning to $C_+$ when one leaves $C_+$, and the probability of reaching first $C_-$ before returning to $C_+$ when one leaves $C_-$, which are the same by symmetry.
A.2. Application to the random ferromagnetic chain near zero temperature

We now focus on the random ferromagnetic chain of $N$ spins of equation (83) with free boundary conditions for the two boundary spins $S_1$ and $S_N$. Near zero temperature (equation (1)), we may neglect the configurations containing more than one domain-wall, and work within the space of the following $(2N)$ configurations:

$$|k\rangle_N^{\text{sym}} = \frac{1}{\sqrt{2}} \left[ |S_1 = \cdots = S_k = -1; S_{k+1} = \cdots = S_N = +1\rangle + |S_1 = \cdots = S_k = 1; S_{k+1} = \cdots = S_N = -1\rangle \right],$$

$$|k\rangle_N^{\text{asym}} = \frac{1}{\sqrt{2}} \left[ |S_1 = \cdots = S_k = -1; S_{k+1} = \cdots = S_N = +1\rangle - |S_1 = \cdots = S_k = 1; S_{k+1} = \cdots = S_N = -1\rangle \right],$$

(A.18)

where $k = 0, 1, \ldots, N - 1$. In physical terms, $|0\rangle^{\text{sym}}$ and $|0\rangle^{\text{asym}}$ are the symmetric and antisymmetric combinations of the two ferromagnetic ground states where all spin have the same signs, whereas $|k\rangle_N^{\text{sym}}$ and $|k\rangle_N^{\text{asym}}$ with $1 \leq k \leq N - 1$ are the symmetric and antisymmetric combinations of the states where there exists a single domain-wall between the sites $(k, k+1)$.

We consider the quantum Hamiltonian

$$\mathcal{H}_N = \sum_{2 \leq k \leq N-1} G (J_{k-1,k} \sigma^z_{k-1} + J_{k,k+1} \sigma^z_{k+1}) \left[ e^{-\beta \sigma^z_k (J_{k-1,k} \sigma^z_{k-1} + J_{k,k+1} \sigma^z_{k+1})} - \sigma^z_k \right]$$

$$+ G (J_{1,2}) \left[ e^{-\beta \sigma^z_1 J_{1,2} \sigma^z_2} - \sigma^z_1 \right] + G (J_{N-1,N} \sigma^z_{N-1}) \left[ e^{-\beta \sigma^z_N J_{N-1,N} \sigma^z_{N-1}} - \sigma^z_N \right],$$

(A.19)

where the amplitudes $G_k$ are given by a single even function $G(x) = G(-x)$ of the local field (see equation (34)).

A.2.1. The first two eigenvectors within the single domain-wall approximation. The ground state $|\psi_0\rangle$ of zero energy is exactly known from equation (A.1),

$$|\psi_0\rangle_N = \frac{1}{Z_N} \sum_{S_1, \ldots, S_N} e^{(\beta/2) \sum_{i=1}^{N-1} J_{i,i+1} S_i S_{i+1}} |S_1, \ldots, S_N\rangle.$$  

(A.20)

Within the reduced space of configurations containing no more than one domain-wall (equation (A.18)), the ground state reduces to

$$|\psi_0\rangle_N \simeq |0\rangle_N^{\text{sym}} + \sum_{k=1}^{N-1} e^{-\beta J_{k,k+1}} |k\rangle_N^{\text{sym}}$$

(A.21)

near zero temperature.

To respect the antisymmetry of equation (A.3), the first excited state will be a linear combination of the antisymmetric states of equation (A.18),

$$|\psi_1\rangle_N \simeq |0\rangle_N^{\text{asym}} + \sum_{k=1}^{N-1} e^{-\beta J_{k,k+1}} A_N(k) |k\rangle_N^{\text{asym}},$$

(A.22)

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with some amplitudes $A_N(k)$ that we wish to determine. The eigenvalue equation for this first excited state of the Hamiltonian of equation (A.19) reads

$$0 = (\mathcal{H}_N - E_1) |\psi_1\rangle_N$$

$$= |0\rangle_N^\text{asym} \left[-E_1 + f_1 e^{-\beta J_{1,2}} (1 - A_N(1)) + f_N e^{-\beta J_{N-1,N}} (1 + A_N(N - 1))\right]$$

$$+ |1\rangle_N^\text{asym} \left[-E_1 e^{-\beta J_{1,2}} A_N(1) + f_1^- (A_N(1) - 1) + f_2^- e^{-\beta J_{2,3}} (A_N(1) - A_N(2))\right]$$

$$+ \sum_{k=2}^{N-2} |k\rangle_N^\text{asym} \left[-E_1 e^{-\beta J_{k,k+1}} A_N(k) + f_k^- e^{-\beta J_{k-1,k}} (A_N(k) - A_N(k - 1))\right]$$

$$+ f_{k+1}^- e^{-\beta J_{k+1,k+2}} (A_N(k) - A_N(k + 1))$$

$$+ |N - 1\rangle_N^\text{asym} \left[-E_1 e^{-\beta J_{N-1,N}} A_N(N - 1)\right]$$

$$+ f_N^- e^{-\beta J_{N-2,N-1}} (A_N(N - 1) - A_N(N - 2)) + f_N^- (A_N(N - 1) + 1)\right]$$

(A.23)

in terms of the numbers

$$f_k^- \equiv G(J_{k-1,k} - J_{k,k+1}).$$

(A.24)

A.2.2. Ansatz with exit probabilities. Instead of solving exactly the eigenvalue problem of an $N \times N$ matrix of equation (A.23), we have proposed in section A.1 the following approximation: in all coefficients involving $|k\rangle_N^\text{asym}$ with $k = 1, \ldots, N - 1$, we may neglect the term containing $E_1$ with respect to the others to obtain the $(N - 1)$ equations for $k = 1, \ldots, N - 1$,

$$f_k^- e^{-\beta J_{k-1,k}} \left(A_N^{\text{ansatz}}(k) - A_N^{\text{ansatz}}(k - 1)\right)$$

$$+ f_{k+1}^- e^{-\beta J_{k+1,k+2}} \left(A_N^{\text{ansatz}}(k) - A_N^{\text{ansatz}}(k + 1)\right) = 0,$$  

(A.25)

with the following boundary conditions:

$$A_N^{\text{ansatz}}(0) = 1, \quad A_N^{\text{ansatz}}(N) = -1.$$  

(A.26)

The only remaining term in equation (A.23) is then the first line involving $|0\rangle_N^\text{asym}$ that determines the value of the energy $E_1$ as

$$E_1^{\text{ansatz}}(N) = f_1^- e^{-\beta J_{1,2}} \left(A_N^{\text{ansatz}}(0) - A_N^{\text{ansatz}}(1)\right)$$

$$+ f_N^- e^{-\beta J_{N-1,N}} \left(A_N^{\text{ansatz}}(N - 1) - A_N^{\text{ansatz}}(N)\right).$$

(A.27)

Therefore, we have replaced the eigenvalue problem of equation (A.23) by a simpler homogeneous recurrence equation (equation (A.25)) with the boundary equations of (A.26), which can be solved as follows.

A.2.3. Exact solution for exit probabilities in one dimension. It is convenient to set, as in equation (A.13),

$$A_N^{\text{ansatz}}(k) = 2Q_0(k) - 1 = 1 - 2Q_N(k),$$  

(A.28)

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where \( Q_N(k) \) satisfies
\[
Q_N(k) = p_+(k)Q_N(k+1) + p_-(k)Q_N(k-1),
\]
with the respective probabilities
\[
p_+(k) \equiv \frac{f_{k+1}^- e^{-\beta J_{k+1,k+2}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}} + f_k^- e^{-\beta J_{k-1,k}}},
\]
\[
p_-(k) \equiv \frac{f_k^- e^{-\beta J_{k-1,k}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}} + f_k^- e^{-\beta J_{k-1,k}}} = 1 - p_+(k)
\]
and the boundary conditions
\[
Q_N(0) = 0,
Q_N(N) = 1.
\]
Then, \( Q_N(k) \) represents the probability of reaching first the boundary \( k = N \) rather than the boundary \( k = 0 \) for a random walker starting at \( k \) and moving with the probabilities of equation (A.30). The well-known solution of this standard problem can be obtained by recurrence [25] using Kesten variables [26] and reads
\[
Q_N(k) = \frac{R(1,k)}{R(1,N)}.
\]
with
\[
R(1,0) = 0,
R(1,1) = 1,
R(1,k \geq 2) = 1 + \sum_{m=1}^{k-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)},
\]
\[
R(1,N) = 1 + \sum_{m=1}^{N-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)}
= 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \cdots + \frac{p_-(1)p_-(2)\cdots p_-(N-1)}{p_+(1)p_+(2)\cdots p_+(N-1)}.
\]
The corresponding estimate of the energy of equation (A.27) reads using equation (A.28)
\[
E_1^{\text{ansatz}}(N) = f_1^- e^{-\beta J_{1,2}} \left( (1 - 2Q_N(0)) - (1 - 2Q_N(1)) \right)
+ f_N^- e^{-\beta J_{N-1,N}} \left( (1 - 2Q_N(N-1)) - (1 - 2Q_N(N)) \right)
= 2f_1^- e^{-\beta J_{1,2}} (Q_N(1) - Q_N(0)) + 2f_N^- e^{-\beta J_{N-1,N}} (Q_N(N) - Q_N(N-1))
= 2f_1^- e^{-\beta J_{1,2}} R(1,1) + f_N^- e^{-\beta J_{N-1,N}} \left[ R(1,N) - R(1,N-1) \right]
= 2f_1^- e^{-\beta J_{1,2}} + f_N^- e^{-\beta J_{N-1,N}} \left[ \frac{p_-(1)p_-(2)\cdots p_-(N-1)}{p_+(1)p_+(2)\cdots p_+(N-1)} \right]
\]
\[
= 2 \frac{1 + \sum_{m=1}^{N-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)}}{1 + \sum_{m=1}^{N-1} \prod_{n=1}^{m} \frac{p_-(n)}{p_+(n)}}
\]
\[
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\]
in terms of the ratios (equation (A.30))

\[
\frac{p_-(k)}{p_+(k)} = \frac{f_k^- e^{-\beta J_{k-1,k}}}{f_{k+1}^- e^{-\beta J_{k+1,k+2}}} = \frac{f_k^-}{f_{k+1}^-} e^{\beta(J_{k+1,k+2} - J_{k-1,k})}. \tag{A.35}
\]

Taking into account that absent links correspond to vanishing coupling, \( J_{0,1} = 0 = J_{N,N+1} \), one obtains

\[
\frac{p_-(1)p_-(2) \cdots p_-(N-1)}{p_+(1)p_+(2) \cdots p_+(N-1)} = \frac{f_1^-}{f_N^-} e^{-\beta J_{1,2} + \beta J_{N-1,N}} \tag{A.36}
\]

and

\[
R(1, N) = 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \cdots + \frac{p_-(1)p_-(2) \cdots p_-(N-1)}{p_+(1)p_+(2) \cdots p_+(N-1)}
\]

\[
= 1 + \frac{f_1^-}{f_2^-} e^{\beta J_{2,3}} + \frac{f_1^-}{f_3^-} e^{-\beta J_{1,2} + \beta J_{2,3} + \beta J_{3,4}} + \frac{f_1^-}{f_4^-} e^{-\beta J_{1,2} + \beta J_{3,4} + \beta J_{4,5}}
\]

\[
+ \frac{f_1^-}{f_5^-} e^{-\beta J_{1,2} + \beta J_{5,6}} + \cdots + \frac{f_1^-}{f_{N-1}^-} e^{-\beta J_{1,2} + \beta J_{N-2,N-1} + \beta J_{N-1,N}} + \frac{f_1^-}{f_N^-} e^{-\beta J_{1,2} + \beta J_{N-1,N}}
\]

\[
= f_1^- e^{-\beta J_{1,2}} \left[ \frac{e^{\beta J_{1,2}}}{f_1^-} + \frac{e^{\beta J_{1,2} + \beta J_{2,3}}}{f_2^-} \right.
\]

\[
+ \frac{e^{\beta J_{2,3} + \beta J_{3,4}}}{f_3^-} + \frac{e^{\beta J_{3,4} + \beta J_{4,5}}}{f_4^-} + \cdots \left. + \frac{e^{\beta J_{N-2,N-1} + \beta J_{N-1,N}}}{f_{N-1}^-} + \frac{e^{\beta J_{N-1,N}}}{f_N^-} \right]
\]

\[
= f_1^- e^{-\beta J_{1,2}} \sum_{k=1}^{N} \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-}, \tag{A.37}
\]

so that the energy of equation (A.34) reads

\[
E_{\text{ansatz}}(N) = 2f_1^- e^{-\beta J_{1,2}} + \frac{f_N^- e^{-\beta J_{N-1,N}} [(f_1^- / f_N^-) e^{-\beta J_{1,2} + \beta J_{N-1,N}}]}{R(1, N)} = \frac{4f_1^- e^{-\beta J_{1,2}}}{R(1, N)} \tag{A.38}
\]

\[
= \sum_{k=1}^{N} \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-},
\]

i.e. the equilibrium time reads using equation (A.24)

\[
t_{\text{eq}}(N) = \frac{1}{E_{\text{ansatz}}(N)} = \frac{1}{4} \sum_{k=1}^{N} \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{f_k^-} = \frac{1}{4} \sum_{k=1}^{N} \frac{e^{\beta J_{k-1,k} + \beta J_{k,k+1}}}{G(J_{k-1,k} - J_{k,k+1})} \tag{A.39}
\]

in agreement with the RG result of equation (105) derived in the text.

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A.3. Exact renormalization rules in configuration space for escape probabilities $Q_{\pm}(C)$

As a final remark, let us mention the link with previous works concerning renormalization rules in configuration space. As explained in [35], backward master equations satisfy exact renormalization rules in configuration space. Upon the elimination of the configuration $C_0$, the surviving configurations $C$ satisfy the same equation as before (equation (A.11)),

$$W^R_{\text{out}}(C)Q_{\pm}(C) = \sum_{C'} W^R(C \rightarrow C')Q_{\pm}(C'),$$  \hspace{1cm} (A.40)

where the renormalized transition rates $W^R$ evolve with the RG equations,

$$W^R_{\text{new}}(C \rightarrow C') = W^R(C \rightarrow C') + \frac{W^R(C \rightarrow C_0)W^R(C_0 \rightarrow C')}{W^R_{\text{out}}(C_0)},$$ \hspace{1cm} (A.41)

These RG rules for backward master equations are exact and can be used [35], but only for small sizes as a consequence of the exponential growth of configurations. The RG rules of equation (A.41) were first derived via a strong disorder RG approach [36].

Note that the RG rules of equation (A.41) can be rewritten directly for the renormalized probabilities

$$\pi^R_C(C') \equiv \frac{W^R(C \rightarrow C')}{W^R_{\text{out}}(C)},$$  \hspace{1cm} (A.42)

which evolve according to

$$\pi^R_{\text{new}}(C') = \frac{W^R_{\text{new}}(C \rightarrow C')}{W^R_{\text{out}}'(C)} = \frac{\pi^R_C(C') + \pi^R_C(C_0)\pi^R_{C_0}(C')}{1 - \pi^R_C(C_0)\pi^R_{C_0}(C)}. \hspace{1cm} (A.43)$$

Appendix B. Dependence on the choice of the dynamics

In this appendix, we describe how the equilibrium time depends on the choice of the dynamics satisfying detailed balance.

B.1. The case of the Glauber dynamics

For the Glauber dynamics, one expects that the dynamical barrier will coincide with the maximal energy cost on the optimal path between the two ground states. For instance, for the one-dimensional random ferromagnetic chain, the result of equation (106) satisfies

$$\frac{1}{\beta} \ln \left[ t^{\text{Glauber}}_{\text{eq}}(N) \right] = \max_{0 \leq k \leq N-1} \left[ 2J_{k,k+1} \right] = \max_{0 \leq k \leq N} \left( U^{(k,N-k)}_N - U^{\text{GS}}_N \right), \hspace{1cm} (B.1)$$

where $U^{(k,N-k)}_N$ represents the energy of the configuration where the first $k$ spins are $(-1)$, whereas all other spins are $(+1)$.

To better understand the differences from the simple dynamics described below, it is useful to write the result of equation (106) for the two smallest sizes, with $N = 2$ and $3$.
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spins,

\[ t_{\text{Glauber}}^\text{simple}(N = 2) = \frac{1}{2} \left[ 1 + e^{2\beta J_{1,2}} \right], \]
\[ t_{\text{Glauber}}^\text{simple}(N = 3) = \frac{1}{2} \left[ 1 + e^{2\beta J_{1,2}} + e^{2\beta J_{2,3}} \right]. \]  
(B.2)

B.2. The case of the simple dynamics

For the ‘simple’ dynamics, the correspondence of equation (B.1) between the dynamical barrier and the maximal energy cost of a single domain-wall does not hold, as can be seen already for the one-dimensional case with \( N = 2 \) and 3 spins since equation (94) reads

\[ t_{\text{simple}}^\text{simple}(N = 2) = \frac{1}{2} e^{\beta J_{1,2}}, \]
\[ t_{\text{simple}}^\text{simple}(N = 3) = \frac{1}{4} \left[ e^{\beta J_{1,2}} + e^{\beta (J_{1,2} + J_{2,3})} + e^{\beta J_{2,3}} \right]. \]  
(B.3)

For \( N = 2 \), the difference by a factor of 2 between the dynamical barriers can be understood from the differences between the transition rates for the simple dynamics, (equation (24))

\[ W_{\text{simple}}^\text{simple}(++) \rightarrow (+-) = W_{\text{simple}}^\text{simple}(++) \rightarrow (-+) = e^{-\beta J_{1,2}}, \]
\[ W_{\text{simple}}^\text{simple}(+-) \rightarrow (++) = W_{\text{simple}}^\text{simple}(+-) \rightarrow (--) = e^{+\beta J_{1,2}}. \]  
(B.4)

and for the Glauber dynamics (equation (26))

\[ W_{\text{Glauber}}^\text{Glauber}(++) \rightarrow (+-) = W_{\text{Glauber}}^\text{Glauber}(++) \rightarrow (-+) = \frac{e^{-\beta J_{1,2}}}{1 + e^{-2\beta J_{1,2}}}, \]
\[ W_{\text{Glauber}}^\text{Glauber}(+-) \rightarrow (++) = W_{\text{Glauber}}^\text{Glauber}(+-) \rightarrow (--) = \frac{1}{1 + e^{-2\beta J_{1,2}}}. \]  
(B.5)

For \( N = 2 \) spins, the equilibrium time is determined by the rate \( W(++) \rightarrow (+-) \) to create a domain-wall when starting from one ground state (the time to eliminate the domain-wall is then negligible), and these two rates are respectively of order \( e^{-\beta J_{1,2}} \) for the simple dynamics and of order \( e^{-2\beta J_{1,2}} \) for the Glauber dynamics. One could argue that the Glauber dynamics is more ‘physical’, in the sense that all transition rates remain bounded near zero temperature, whereas in the ‘simple’ dynamics transition rates corresponding to a decrease of the energy diverge near zero temperature. Nevertheless, one would expect on physical grounds that the difference between the dynamical barriers of the two dynamics would remain of order \( O(1) \), as found in this paper for the one-dimensional case and the tree case, and as found in the companion paper [34] for the Dyson hierarchical Ising model.

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