Solitons in the Horava-Witten supergravity

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Abstract

We study classical BPS five-brane solutions in the Horava-Witten supergravity. The presence of the eleventh dimension add a new feature, namely the dependence of the solution on this new coordinate. For gauge five-branes with an instanton size less than the eleventh radius and in the neighborhood of the center of the neutral five-brane, important corrections to the ten-dimensional solution appear for all values of the string coupling constant. We compute the mass and magnetic charge of the five-brane solitons and the result is shown to agree with the membrane and five-brane quantization conditions. Compactified to four dimensions, our solutions are interpreted as axionic strings.

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1 Introduction

Solitonic states play an important rôle in understanding the nonperturbative dynamics in quantum field theory and string theory (for a review, see for ex. [1]). In supersymmetric theories, BPS states are believed to be stable under quantum corrections [2]. Their presence is very important in checking various duality conjectures, where solitons in one theory are mapped into elementary particles in the dual theory.

It was shown by Strominger in 1990 [3] that the low-energy limit of the ten-dimensional (10d) heterotic strings have classical solutions coming from instantons in the gauge group dressed with background values for the gravitational multiplet. This solution turned out to be crucial for the conjectured string-five brane duality in 10d [4].

Recently, it was argued [5] that the strong coupling limit of the $E_8 \times E_8$ heterotic string is the eleven-dimensional supergravity, the eleventh coordinate being defined on the interval $0 \leq x^{11} \leq l$, with the two gauge groups sitting on the boundaries $x^{11} = 0$ and $x^{11} = l$, respectively. The length $l$ is related to the string coupling constant $e^{\phi_0}$ by

$$l = \frac{1}{2} (4\pi k_{11}^2)^{1/9} e^{2\phi_0}.$$  

This conjecture seems to be free of gauge and gravitational anomalies [6] and is in agreement with constraints coming from membrane-five brane Dirac quantization condition [6], [7].

The purpose of this paper is the study of classical five-brane solutions in the Horava-Witten theory, thus generalizing the heterotic solutions [8] (this was also studied recently in [9]; our results of Section 2 partly overlap with this paper). A particular emphasis will be put on the dependence of the solutions on the eleventh coordinate. In particular, we will find here a new manifestation of the phenomenon observed in [8], namely that by compactification, if we put an instanton on one boundary, the volume of the compact space on the other boundary becomes smaller, with a quantity related in our case to the instanton size. We study this phenomenon in more detail and show that even at weak coupling, for very small instanton size, the Strominger solution has important corrections. We also consider the neutral five-brane solution and find that spacetime is effectively eleven-dimensional near the center of the five-brane even at very weak coupling.
2 Supersymmetric five-brane solutions in Horava-Witten Lagrangian

We look for a solution of the Horava-Witten supergravity with six dimensional Poincaré and four dimensional Lorentz invariances and which preserves half of the supersymmetries. The six dimensional coordinates parametrising the world-volume of the five-brane are denoted by $y^a$, $a = 0, \ldots, 5$, the four dimensional ones by $x^\mu$ and the eleventh coordinate parametrizing $S^1/Z^2$ by $x^{11}$. In the following, we shall be interested in five-branes that are transverse to the eleventh dimension.

The most general metric with the mentioned isometries is of the form

$$g = e^{2A}dy^a dy^b \eta_{ab} + e^{2B}dx^\mu dx^\nu \delta_{\mu\nu} + e^{2C}dx^{11} dx^{11},$$

where $A$, $B$ and $C$ are functions of $x^2 = x^\mu x^\nu \delta_{\mu\nu}$ and $x^{11}$. In the following all indices are raised and lowered with the Minkowski and Euclidean metric. The moving basis is given by

$$\theta^a = e^{A}dy^a, \quad \theta^\mu = e^{B}dx^\mu, \quad \theta^{11} = e^{C}dx^{11}.$$  

The four-form field strength, in order to be invariant under the isometries, has to have vanishing components where at least one of the indices is $a$. The equation of motion for this field reads $d^* G = 0$, because $G \wedge G = 0$. This equation may be solved as $G = \ast d\Lambda$ where $\Lambda$ is a six-form. The most general six-form compatible with the isometries can be written as $\Lambda = D\epsilon_6$, where $D$ is a function of $x^2$ and $x^{11}$ and $\epsilon_6$ is given by $dy^0 \wedge \ldots \wedge dy^5$. So $G$ is determined by a single function $D$ and is given by

$$G = e^{-6A+2B+C} \frac{\epsilon_{\nu\alpha\beta}}{6} \partial_\mu Ddx^\nu \wedge dx^\alpha \wedge dx^\beta \wedge dx^{11} + e^{-6A-C+4B} \partial_{11} Ddx^1 \wedge \ldots \wedge dx^4,$$

where $\epsilon_{\nu\alpha\beta}$ is the totally antisymmetric tensor with values $\pm 1$ and 0. In the moving basis we have

$$G = e^{-6A-B} \frac{\epsilon_{\nu\alpha\beta}}{6} \partial_\mu D\theta^\nu \wedge \theta^\alpha \wedge \theta^\beta \wedge \theta^{11} + e^{-6A-C} \frac{\epsilon_{\nu\alpha\beta}}{24} \partial_{11} D\theta^\mu \wedge \ldots \wedge \theta^\beta.$$  

We now demand that the solution preserve a fraction of the eleven-dimensional supersymmetries. We use conventions where $\{\Gamma^I, \Gamma^J\} = 2\eta^{IJ}$ and $\Gamma^{I_1 I_2 \ldots I_n} = \frac{1}{n!} \{\Gamma^{I_1}, \Gamma^{I_2} \ldots \Gamma^{I_n} \pm \text{permutations}\}$. The supersymmetry transformation rule of the eleven dimensional gravitino is given by

$$\delta \Psi_I = D_I \epsilon + \frac{\sqrt{2}}{288} (\Gamma_I^{JKLM} - 8\delta_I^{JKLM}) G^{JKLM} \epsilon,$$

where $\partial_I$ is the partial derivative, $\Gamma^I$ is the gamma matrix, $D_I$ is the covariant derivative, $\epsilon$ is the gravitino, and $G$ is the four-form field strength. The term $\int C \wedge X_8$ where $X_8 = dX_7$ is a polynomial in the curvature of order 8 modifies the equation of motion by a term in $X_8$. This term turns out to be vanishing for the five-brane solution as can be checked by calculating the curvature of the the metric (1).
where $G_{JKLM}$ are the components of $G$ in the moving basis.

The covariant derivative when acting on spinors reads explicitly

$$D = d + \frac{1}{2} [\Gamma^a \Gamma^{11} e^{-C} \partial_1 A + \Gamma^a \Gamma^\mu e^{-B} \partial_\mu A] \theta_a$$

$$+ \frac{1}{2} [\Gamma^{\mu \nu} e^{-B} \partial_\nu B + \Gamma^\mu \Gamma^{11} e^{-C} \partial_1 B] \theta_\mu - \frac{1}{2} \Gamma^\mu \Gamma^{11} e^{-B} \partial_\mu C \theta_{11} \tag{7}$$

In order to impose that the solution preserve some supersymmetries we have to look for a solution to

$$\delta \Psi_I = 0 \tag{8}.$$ 

A straightforward calculation gives

$$\Gamma^{JKLM} a G_{JKLM} = 24 \Gamma^a \Gamma^5 e^{-6A-C} \partial_1 D + 24 \Gamma^a \Gamma^\mu \Gamma^{11} e^{-6A-B} \partial_\mu D \tag{9},$$

$$\Gamma^{JKLM} a G_{JKLM} = 24 \Gamma^5 \Gamma^{11} e^{-6A-B} \partial_1 D \tag{9},$$

$$\Gamma^{JKLM} G_{JKLM} = 24 \Gamma^5 \Gamma^{11} e^{-6A-C} \partial_1 D \tag{9},$$

$$\Gamma^{JKLM} G_{JKLM} = -6 \Gamma^A \Gamma^5 e^{-6A-B} \partial_1 D \tag{9},$$

$$\Gamma^{JKLM} G_{JKLM} = -6 \Gamma^B \Gamma^{11} e^{-6A-B} \partial_1 D \tag{9},$$

$$\Gamma^{JKLM} G_{JKLM} = -6 \Gamma^A \Gamma^5 e^{-6A-C} \partial_1 D \tag{9},$$

$$\Gamma^{JKLM} G_{JKLM} = -6 \Gamma^B \Gamma^{11} e^{-6A-C} \partial_1 D \tag{9}.$$ 

In order to obtain these relations we have used

$$\Gamma^5 = \frac{1}{24} \epsilon_{\mu \nu \alpha \beta} \Gamma^{\mu \nu \alpha \beta}, \quad \Gamma^{\mu \nu} = -e^{\mu \nu \alpha \beta} \Gamma^\alpha \Gamma^5, \quad \epsilon_{\alpha \beta \mu \nu} \Gamma^{\mu \nu} = -2 \Gamma^5 \Gamma_{\alpha \beta} \tag{10}.$$ 

We look for solutions to (8) of the form $\epsilon = e^{-E} \eta$, with $\eta$ an arbitrary constant spinor of positive eleven dimensional chirality $\Gamma^{11} \eta = \eta$ and $E$ a function of $x^2$ and $x^{11}$. The a component of the condition (8) is

$$\frac{1}{2} [\Gamma_a \Gamma^{11} e^{-C} \partial_1 A + \Gamma_a \Gamma^\mu e^{-B} \partial_\mu A] \eta + \frac{\sqrt{2}}{12} \Gamma_a \Gamma_5 \Gamma^{11} e^{-6A-C} \partial_1 D + \Gamma_a \Gamma^\mu \Gamma^5 \Gamma^{11} e^{-6A-B} \partial_\mu D] \eta = 0 \tag{11}.$$ 

which gives

$$\left(\frac{1}{2} \Gamma_a \Gamma^{11} e^{-C} \partial_1 A + \frac{\sqrt{2}}{12} \Gamma_a \Gamma_5 \Gamma^{11} e^{-6A-C} \partial_1 D \right) \eta = 0 \tag{12},$$

$$\left(\frac{1}{2} \Gamma_a \Gamma^\mu e^{-B} \partial_\mu A + \frac{\sqrt{2}}{12} \Gamma_a \Gamma^\mu \Gamma^5 \Gamma^{11} e^{-6A-B} \partial_\mu D \right) \eta = 0 \tag{13}.$$ 

The first equation implies that $\eta$ is chiral in the four dimensional space, say $\Gamma^5 \eta = \alpha \eta$, where $\alpha = \pm 1$, and

$$e^{6A} \partial_1 A + \alpha \frac{\sqrt{2}}{6} \partial_1 D = 0 \tag{14}.$$ 

The solution reads

$$e^{6A} = -\alpha \sqrt{2} D + a \tag{15}.$$
where \( a \) does not depend on \( x^{11} \). The second equation gives
\[
e^{6A} \partial_\mu A + \alpha \frac{\sqrt{2}}{6} \partial_\mu D = 0 ,
\] (16)
which implies that \( a \) is a constant.

The \( \mu \) component of the equation (8) gives
\[
- e^{-B} \partial_\mu E \eta + \frac{1}{2} \left[ \Gamma^{\mu \nu} e^{-B} \partial_\nu B + \Gamma^\mu \Gamma^{11} e^{-C} \partial_{11} B \right] \eta \\
+ \frac{\sqrt{2}}{12} [\Gamma^5 \Gamma^{11} e^{-6A - B} \partial_\nu D - 2 \Gamma^\mu \Gamma^5 e^{-6A - C} \partial_{11} D + 2 \Gamma^\lambda \Gamma^5 \Gamma^{11} e^{-6A - B} \partial_\lambda D] \eta = 0 .
\] (17)

Setting all the terms to zero gives
\[
- \partial_\mu E + \frac{\sqrt{2}}{12} \alpha e^{-6A} \partial_\mu D = 0 , \quad \partial_\nu B - \frac{\sqrt{2}}{3} \alpha e^{-6A} \partial_\nu D = 0 , \\
\partial_{11} B - \frac{\sqrt{2}}{3} \alpha e^{-6A} \partial_{11} D = 0 .
\] (18)

The last two equations combined with (15) give
\[
\partial_I B = \frac{\sqrt{2}}{3} \alpha \frac{\partial_I D}{-\alpha \sqrt{2} D + a} .
\] (19)

The solution of (18) is
\[
e^{3B} = \frac{b}{-\alpha \sqrt{2} D + a} ,
\] (20)
where \( b \) is a constant which can be set to one by a rescaling of the \( x^\mu \), so we get \( e^{3B} = e^{-6A} \).

The 11 component of the equation (3) reads
\[
- e^{-C} \partial_{11} E \eta - \frac{1}{2} \Gamma^\mu \Gamma^{11} e^{-B} \partial_\mu C \eta \\
+ \frac{\sqrt{2}}{12} \left[ \Gamma^{11} \Gamma^5 e^{-6A - C} \partial_{11} D + 2 \Gamma^\lambda \Gamma^5 e^{-6A - B} \partial_\lambda D \right] \eta = 0 .
\] (21)

Setting all the terms to zero, we get
\[
- \partial_{11} E + \frac{\sqrt{2}}{12} \alpha e^{-6A} \partial_{11} D = 0 , \quad \partial_\mu C - \frac{\sqrt{2}}{3} \alpha e^{-6A} \partial_\mu D = 0 .
\] (22)

The last equation combined with (15) gives
\[
e^{3C} = \frac{c(x^{11})}{-\alpha \sqrt{2} D + a} ,
\] (23)
where \( c(x^{11}) \) depends only on \( x^{11} \). By reparametrisation of \( x^{11} \), \( c \) can be set to one so we get \( C = B \). Finally, from equations (18) and (22) we get \( E = -A/2 \).

In summary, for the solution to be supersymmetric the metric has to be of the form
\[
g = e^{2A} dy^a dy^b \eta_{ab} + e^{-4A} (dx^\mu dx^\nu \delta_{\mu\nu} + dx^{11} dx^{11}) ,
\]
and the four-form \( G \) be given by
\[
G = \frac{1}{(-\alpha \sqrt{2D + a})^2} \left[ \frac{e^{\mu\nu\alpha\beta}}{6} \partial_\mu D dx^\nu dx^\alpha dx^\beta dx^{11} + \partial_{11} D dx^1 \wedge \ldots \wedge dx^4 \right].
\]
The two function \( A \) and \( D \) are related by (14).

It is convenient to define \( \tilde{D} \) by
\[
\tilde{D} = \frac{1}{-\alpha \sqrt{2D + a}} ,
\]
so the expression of \( G \) simplifies to
\[
\alpha \sqrt{2} G = \frac{e^{\mu\nu\alpha\beta}}{6} \partial_\mu \tilde{D} dx^\nu \wedge dx^\alpha \wedge dx^\beta \wedge dx^{11} + \partial_{11} \tilde{D} dx^1 \wedge \ldots \wedge dx^4 .
\]
\( A \) is given by \( e^{6A} = \tilde{D}^{-1} \), so finally the metric is given by
\[
g = \tilde{D}^{-\frac{2}{3}} dy^a dy^b \eta_{ab} + \tilde{D}^{\frac{2}{3}} (dx^\mu dx^\nu \delta_{\mu\nu} + dx^{11} dx^{11}) .
\]
The Bianchi identity is also simply expressed with the aid of \( \tilde{D} \) since
\[
\alpha \sqrt{2} dG = (\partial^\mu \partial_\mu \tilde{D} + \partial^2_{11} \tilde{D}) dx^1 \wedge \ldots dx^4 \wedge dx^{11} ,
\]
so the Bianchi identity in the bulk reads
\[
\Delta_5 \tilde{D} = 0 ,
\]
where \( \Delta_5 \) is the five-dimensional Euclidean Laplacian.

The gaugino transformation rule under supersymmetry is given by \( \delta \chi = \Gamma^{IJ} F_{IJ} \epsilon \). Since the solution we are seeking is invariant under the six-dimensional Poincaré group we have \( F_{aI} = 0 \), so the transformation rule can be written as \( \delta \chi = \Gamma^{\mu\nu} F_{\mu\nu} \epsilon \).

The spinorial parameter is chiral in four dimensions, so
\[
\delta \chi = \Gamma^{\mu\nu} F_{\mu\nu} \frac{1 + \alpha \Gamma^5}{2} \epsilon = \frac{1}{2} F_{\mu\nu} (\Gamma^{\mu\nu} - \frac{\alpha}{2} \epsilon^{\mu\nu}_{\alpha\beta} \Gamma_\alpha \Gamma_\beta) \epsilon .
\]
The vanishing of \( \delta \chi \) implies that the field strength must be self or anti-self-dual depending on the four dimensional chirality
\[
F_{\alpha\beta} = \frac{\alpha}{2} \epsilon^{\mu\nu}_{\alpha\beta} F_{\mu\nu} .
\]
In the following sections we shall examine various solutions to the self-duality equations and the Bianchi identity subject to the Horava-Witten boundary conditions for $G$. We end this section with some general remarks.

There is a class of solutions we can get by using a spinor $\epsilon$ which vary along $x^{11}$ such that it is chiral, but of opposite chirality, on the two boundaries and non-chiral for $x^{11} \neq 0, l$. The spinor must satisfy the Horava-Witten projection condition $\epsilon(-x^{11}) = \Gamma_{11} \epsilon(x^{11})$. Then, in the Weyl representation, the spinor we are searching corresponds to the following ansatz

$$\epsilon = e^{-E} \left( \frac{\cos \frac{x^{11}}{2l} \epsilon_1}{\sin \frac{x^{11}}{2l} \epsilon_2} \right),$$  

(33)

where $\epsilon_i$ are Majorana-Weyl spinors. Inserting this ansatz into (6), as before, it turns out that the former solution (27), (28) still satisfies the supersymmetry conditions $\delta \Psi_a = \delta \Psi_\mu = 0$. However, the supersymmetry is necessarily broken along the eleventh dimension

$$\delta \Psi_{11} = e^{-E} \left( -\frac{1}{2l} \sin \frac{x^{11}}{2l} \epsilon_1 \cos \frac{x^{11}}{2l} \epsilon_2 \right)$$

(34)

and for every value of $x^{11}$. Notice also that the spinor (33) has a discontinuity $\epsilon(-l) = -\epsilon(l)$, which is also present in supersymmetry breaking driven by a gaugino condensation [10] or in the Scherk-Schwarz mechanism [11], [12].

It is of some interest to rewrite the solution above (27), (28), after the Wick rotation, from the point of view of the 4$d$–5$d$ theory obtained after the compactification of the six coordinates $y^a$, in the simplest Calabi-Yau like truncation [13], where the truncated fields are invariant under the $SU(3)$ holonomy group of the compactified space. The Einstein 5$d$ metric can be obtained through the Weyl rescaling

$$g^{(11)}_{ab} = \tilde{D}^{-1/3} \delta_{ab}, g^{(11)}_{\mu\nu} = \tilde{D}^{2/3} g^{(5)}_{\mu\nu},$$

(35)

where here $\mu, \nu = 1 \cdots 5$. The solution (28) becomes in the 5$d$ units simply

$$g = \tilde{D}^{-1/2} dy^a dy^b \eta_{ab} + dx^\mu dx^\nu \delta_{\mu\nu} + dx^5 dx^5,$$

(36)

hence the gravitational part of the solution becomes trivial. The 5$d$ theory contains a (universal) hypermultiplet, which in 4$d$ gives, after the Horava-Witten projection, the dilaton $S$ containing as the real part the volume $V_6$ and as imaginary part the Hodge dual of the antisymmetric tensor field $C_{5\mu\nu}$. A second complex modulus $T$ appears by the 5$d$–4$d$ compactification, the real part of which is the fifth radius $R_5$ and the imaginary part the pseudoscalar defined by

$$C_{5ij} = \epsilon_{ij} b$$

(14), [11], where $i, j = 1, 2, 3$ are complex indices in compactified space.

Then we can reexpress the solution (27), (28) as the background functions

$$S = \tilde{D}^{-1} + i\sqrt{2}(c - \tilde{D}^{-1}), T = 1,$$

(37)
where \( c \) is an arbitrary real constant corresponding to the Peccei-Quinn symmetry \( S \rightarrow S + i \alpha \). So our solution corresponds by compactification to an axionic string, generalizing the ones discussed in the weakly-coupled case in [15].

2.1 Heterotic gauge five-brane

The simplest solution to the self-duality equations is obtained by first choosing a \( SU(2) \) subgroup of \( E_8 \) and putting the t’Hooft solution in this \( SU(2) \). In order to write the solution define \( \sigma^\mu \) by \( (1, i \alpha \vec{\sigma}) \), where \( \vec{\sigma} \) are the Pauli matrices. Then the solution may be written as

\[
F = \left( \frac{\rho}{\rho^2 + x^2} \right)^2 \sigma^\mu dx^\mu \sigma^\nu dx^\nu,
\]

where \( \rho \) is the size of the instanton and

\[
\int \text{tr} F \wedge F = 16 \alpha \pi^2.
\]

Our definition is such that

\[
\text{tr} F \wedge F = \frac{1}{30} \text{Tr} F \wedge F,
\]

where \( \text{Tr} \) is the trace in the adjoint of \( E_8 \).

Supersymmetry invariance as well as anomaly cancellation require that the restriction of \( G \) to the boundaries be given by [5]

\[
G|_{x^{11}=0} = -\frac{1}{\sqrt{2}} \frac{k^{2/3}}{(4\pi)^{5/3}} [\text{tr}(F \wedge F) - \frac{1}{2} \text{tr}(R \wedge R)] \equiv \frac{1}{\sqrt{2\alpha}} f(x^2) \epsilon_4,
\]

\[
G|_{x^{11}=l} = \frac{1}{\sqrt{2}} \frac{k^{2/3}}{(4\pi)^{5/3}} [\text{tr}(F \wedge F) - \frac{1}{2} \text{tr}(R \wedge R)] \equiv -\frac{1}{\sqrt{2\alpha}} g(x^2) \epsilon_4,
\]

where \( \epsilon_4 = dx^1 \wedge \cdots \wedge dx^4 \) and for the one instanton solution given above we have

\[
\text{tr}(F \wedge F) = 96 \alpha \left( \frac{\rho^4}{\rho^2 + x^2} \right) \epsilon_4.
\]

Finally, the soliton is determined by \( \tilde{D} \) which verifies the Laplace equation (30) and is subject to the boundary conditions

\[
\partial_{11} \tilde{D}|_{x^{11}=0} = f(x^2), \quad \partial_{11} \tilde{D}|_{x^{11}=l} = -g(x^2).
\]

In order to solve the equation (31) subject to the boundary conditions (32) it is very convenient to consider \( \tilde{D} \) to be the restriction of an even and periodic function in \( x^{11} \) with period \( 2l \). Then \( \tilde{D} \) with the given boundary conditions satisfies the following equation

\[
(\Delta_4 + \partial_{11}^2) \tilde{D} = 2f \delta(x^{11}) + 2g \delta(x^{11} - l),
\]

where \( \delta(x^{11}) \) is the delta distribution periodic with period \( 2l \):

\[
\delta(x^{11}) = \frac{1}{2l} + \frac{1}{l} \sum_{n=1}^{\infty} \cos \frac{n\pi x^{11}}{l}.
\]

In order to solve the equation (33), introduce the Green function \( G(x^2, x^{11}) \) defined by

\[
(\Delta_4 + \partial_{11}^2) G = \delta^4(x) \delta(x^{11}).
\]
From the Fourier transform of the above equation we get

$$G(x, x^{11}) = -\frac{1}{2l} \sum_{n=-\infty}^{\infty} \int \frac{d^5 k}{(2\pi)^4} \frac{e^{ikx}}{k^2} \delta(k_5 - \frac{n\pi}{l}) , \quad (44)$$

where $x^5 = x^{11}$. By performing a Poisson resummation on $n$, we can recast the result in the form

$$G = -\frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{|x^2 + (x^{11} + 2nl)^2|^{3/2}} . \quad (45)$$

This form of $G$ is particularly suitable to examine the behavior for large $l$ since only the term $n = 0$ contributes to the sum and the Green function is that of $R^5$. The Fourier expansion in $x^{11}$ can be obtained by calculating the integral in (44); it is given by

$$G = -\frac{1}{2l(2\pi)^2} \left( \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2n\pi}{lx} K_1(nx\pi/l) \cos \frac{n\pi x^{11}}{l} \right) , \quad (46)$$

where $x = \sqrt{x^2}$ and $K_1$ is the McDonald function of order one. This form of $G$ is convenient to study the limit where $x$ is much larger than $l$; the asymptotic behavior of $G$ in this case is given by

$$G = -\frac{1}{2l(2\pi)^2} \left( \frac{1}{x^2} + \frac{\sqrt{2\pi}}{\sqrt{lx^3}} e^{-x\pi/l} \cos \frac{\pi x^{11}}{l} \right) + \ldots \quad (47)$$

The first term represents the Green function on $R^4$.

Taking into account the condition for asymptotic flatness, the solution to equation (41) is given by

$$\tilde{D} = 1 + 2 \int d^4 y G(x - y, x^{11}) f(y) + 2 \int d^4 y G(x - y, x^{11} - l) g(y) . \quad (48)$$

The Fourier expansion in $x^{11}$ of $\tilde{D}$ can be obtained by using the form (46) of $G$. The gauge five-brane is obtained by neglecting the term $trR^2$, which is legitimate to the first order in $k^{2/3}_{11}$. In particular, the zero mode can be explicitly calculated and is given for $g = 0$ and $f$ given by the instanton solution (38), by

$$\tilde{D}_{10d} = 1 + \frac{4k_{11}^{2/3}}{l(4\pi)^{5/3}} \frac{x^2 + 2\rho^2}{(x^2 + \rho^2)^2} . \quad (49)$$

This is precisely the 10d Strominger solution expressed in M-theory units, as it should be. The relation between $\tilde{D}$ and the 10d dilaton is $e^{2\phi} = ((2l)^3/(4\pi k_{11}^{2/3})) \tilde{D}$.

5Expressed in string units, the coefficient multiplying the instanton factor in the right-hand side is equal to $\alpha'$, in agreement with ref. [1].
The exact solution (48) gives however, even in weak-coupling regime $l \to 0$, important corrections to (49) coming from instantons of small size. In order to study this quantitatively, notice that, by using (45), for $\rho \ll l$, we can expand the expressions

$$
\tilde{D}(0,0) = 1 - \frac{\zeta}{4\pi^2 \rho^3} \int d^4y \frac{1}{(1+y^2)^{4/3}} - \frac{\zeta}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2nl)^3} \int d^4y \left( \frac{1}{1+y^2} \right)^2 \frac{1}{\left[ \left( \frac{\rho^2}{(2nl)^2} y^2 + 1 \right)^{3/2} \right]},
$$

$$
\tilde{D}(0,l) = 1 - \frac{\zeta}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3 l^3} \int d^4y \left( \frac{1}{1+y^2} \right)^2 \frac{1}{\left[ \left( \frac{\rho^2}{(2n+1)^2} y^2 + 1 \right)^{3/2} \right]},
$$

(50)

where $\zeta = -192k_{11}^{2/3}/(4\pi)^{5/3}$ reads from (39). We get

$$
\tilde{D}(0,0) = 1 - \frac{5\pi\zeta}{64\rho^3} - \frac{\zeta}{96l^3} \left( \sum_{n=1}^{\infty} \frac{1}{n^3} \right) + \frac{3\zeta\rho^2}{16l^5} \left( \sum_{n=1}^{\infty} \frac{1}{n^5} \right) + \cdots,
$$

$$
\tilde{D}(0,l) = 1 - \frac{\zeta}{12l^3} \left( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \right) + \frac{\zeta\rho^2}{4l^5} \left( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \right) + \cdots.
$$

(51)

By comparing (51) with (49), it is clear that we get important corrections to the Strominger solution for $x = 0$ and if $\rho \leq l$, i.e. in the center of the instanton and for instantons of a size smaller than the eleventh dimension. The leading term of the Strominger solution is $1/l\rho^2$, to be compared to (51), which shows in addition that the leading terms have different behaviour on the two boundaries. This means that even if the ten-dimensional solution gives the average in $x^{11}$ of the exact eleven-dimensional solution the latter has important fluctuations around this average. Notice that the space-time near the center of the instanton is eleven-dimensional even in the limit of small $l$.

In the $l >> x >> \rho$ limit, the leading term in the solution behaves as

$$
\tilde{D} = 1 + \frac{(4\pi k_{11}^2)^{1/3}}{4\pi^2 (x^2 + x_{11}^2)^{3/2}},
$$

(52)

which is essentially the eleven-dimensional solution, too.

If we compactify the soliton solution to 4d, by considering the $y_a$ coordinates as compact coordinates and performing a Wick rotation we get an instantonic four-dimensional solution. We can define the volume of the compactified space

$$
V_6(x^2, x^{11}) = \int d^6y \sqrt{g_6} = \tilde{D}^{-1}(x^2, x^{11})V_0,
$$

(53)

where $V_0$ is a reference volume. It is well known that in explicit compactifications, the volume of compact 6-manifold defines the (inverse of) gauge couplings. It is
therefore of interest to see the dependence of the volume on \(x^{11}\) and in particular the values on the two boundaries. By a straightforward computation we get

\[
\dot{D}(x^2, l) - \dot{D}(x^2, 0) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \int d^4 y (f - g)(y) \frac{(-1)^n}{[(x - y)^2 + n^2 l^2]^{3/2}} = \frac{1}{2\pi^{5/2}} \int d^4 y (f - g)(y) \int_0^\infty dt e^{-(x - y)^2 t^2} \theta_4(i l^2 t^2, \pi),
\]

(54)

where \(\theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi n^2 \tau}\) is the Jacobi function. We find therefore a dependence on \(x^{11}\) of the volume, in analogy with the dependence found in [8]. If we put, for example, an instanton in \(x^{11} = 0\), we find \(V_6(l) < V_6(0)\) and the gauge coupling becomes stronger on the boundary without the instanton.

2.2 Neutral five-brane

Consider the limit \(\rho \to 0\) of the gauge five-brane. Then then gauge potential tends to zero but

\[
tr(F \wedge F) \to 16\alpha^2 \delta^4(x) \epsilon_4
\]

(55)

and if \(g = 0\), the equation for \(\dot{D}\) becomes

\[
\Delta_5 \dot{D} = -2(4\pi k_{11}^2)^{1/3} \delta^5(x),
\]

(56)

which represents a neutral five-brane localised at the boundary in \(x^{11} = 0\) and \(x^\mu = 0\).

Neutral five-branes localised in the bulk are characterized by

\[
f(x^2) = g(x^2) = 0.
\]

(57)

In this case the function \(\dot{D}\) satisfies the equation

\[
(\Delta_4 + \partial_{11}^2)\dot{D} = q \delta^4(x - x_0) \left[ \delta(x^{11} - x_{0}^{11}) + \delta(x^{11} + x_{0}^{11}) \right],
\]

(58)

where a source in \(-x_{0}^{11}\) has been added because from its construction \(\dot{D}\) must be even. On the other hand, the Bianchi identity is modified by a five-brane in a way \(dG = -\sqrt{2} k_{11}^2 \alpha T_6 \delta\) that fixes the charge to be

\[
q = -2k_{11}^2 T_6.
\]

(59)

Note that \(\alpha = 1\) corresponds to a five-brane and \(\alpha = -1\) to an anti-five-brane. We will see in the next paragraph that this is consistent with the five-brane Dirac quantization condition.

The solution of equation (58) is

\[
\dot{D} = 1 + q \left[ G(x - x_0, x^{11} - x_{0}^{11}) + G(x - x_0, x^{11} + x_{0}^{11}) \right].
\]

(60)
In the decompactified limit \( l \to \infty \), this becomes the (symmetric under \( x^{11} \to -x^{11} \)) solution found in [13] in the context of 11d SUGRA. For \( l \to 0 \), we recover the 10d neutral solution [3]. For \( x - x_0, x^{11} - x_0^{11} << l \),

\[
\tilde{D} = 1 + \frac{k_{11}^2 T_6}{4\pi^2} \left( \frac{1}{[(x - x_0)^2 + (x^{11} - x_0^{11})^2]^{3/2}} + \frac{1}{[(x - x_0)^2 + (x^{11} + x_0^{11})^2]^{3/2}} \right). 
\]

(61)

As for the gauge five-brane this means that near the center the solution differs from its ten-dimensional limit and is basically eleven-dimensional. In analogy with the computation we did for the heterotic five-brane, we can compute the difference of the compactified space volume for the two boundaries. The result is

\[
\tilde{D}(x^2, l) - \tilde{D}(x^2, 0) = \frac{q}{8\pi^2} \sum_{a=\pm 1} \sum_{n=-\infty}^{\infty} \frac{1}{[(x-x_0)^2 + (n l + a x_0^{11})^2]^{3/2}} 
\]

\[
\quad = \frac{q}{4\pi^{5/2}} \sum_{a=\pm 1} \int_0^\infty dt \frac{t}{2} e^{-(x-x_0)^2 t^2 + i \pi a \frac{x_0^{11}}{l}} \left[ \frac{a x_0^{11}}{l^{3/2}} \right] \left( \frac{i t^2}{\pi} \right). 
\]

(62)

Notice that for \( x_0^{11} = l/2 \) we get \( \tilde{D}(x^2, l) = \tilde{D}(x^2, 0) \), so the difference in the two volumes really comes here from the asymmetric position of the center of the soliton with respect to the two boundaries.

3 Mass and charges of solitons

We compute in the following the mass and the magnetic charge of the soliton solutions, checking that they are compatible with the Dirac quantization condition of the fundamental membrane-five-brane pair. We can consider more general solutions of type \((n_1, n_2)\), in an obvious notation. As an example, let us explicitly write the heterotic five-brane soliton in the particular case of one instanton on each boundary, denoted by the \( (1, 1) \) case. The solution (61) reads

\[
\tilde{D} = 1 + \frac{4 k_{11}^{2/3}}{l (4\pi)^{5/3}} \left\{ \frac{(x - x_1)^2 + 2 \rho_1^2}{[(x - x_1)^2 + \rho_1^2]^{3/2}} + \frac{(x - x_2)^2 + 2 \rho_2^2}{[(x - x_2)^2 + \rho_2^2]^{3/2}} \right\} + \cdots , 
\]

(63)

where \( x_1, x_2 \) are the positions of the two instantons and the dots are the contributions of the Kaluza-Klein modes in the expansion (61), which will turn out to give no contribution in computing the mass and the charges of the soliton solution. In the following, we use the orbifold picture of \( S^1/Z_2 \) and the integrals over the eleventh dimension are half of the integrals on the corresponding circle \( S^1 \). For the Strominger \((1, 0)\) case, we get in a straightforward way

\[
Q = \frac{1}{k_{11}} \int_{S^3 \times S^1/Z_2} G = \frac{1}{2k_{11}} \int_{S^3 \times S^1} G = -\frac{1}{\sqrt{2} \alpha} \left( \frac{4\pi}{k_{11}} \right)^{1/3} . 
\]

(64)
We therefore find that the magnetic charge is the same as in the Strominger solution [3], which is expected for a BPS soliton. For a more general solution \((n_1, n_2)\), the magnetic charge is simply multiplied by \(n_1 + n_2\). Notice that the quantization condition for a four-cycle discussed in [17] is here automatically satisfied.

The ADM mass per unit volume [18] is given here by

\[
M = \frac{1}{4k_{11}^2} \int_{S^3 \times S^1} dx^{11} \sqrt{g_{11,11}} d\Omega_3 x^3 n^\mu \left( \frac{\partial h_{\mu\nu}}{\partial x^\nu} - \frac{\partial h_{\nu\nu}}{\partial x^\mu} - \sum_{a=1}^5 \frac{\partial h_{a\mu}}{\partial x^\mu} \right),
\]

(65)

where \(\mu, \nu\) are indices on \(S^3 \times S^1\), \(\Omega_3 = 2\pi^2\) and \(x \to \infty\). The result is

\[
M = \frac{1}{2} \left( \frac{4\pi}{k_{11}^4} \right)^{1/3} = T_6,
\]

(66)

where \(T_6\) is the five-brane tension. We must check that this result is compatible with the membrane-five-brane quantization condition

\[
k_{11}^2 T_3 T_6 = n\pi, \quad \frac{T_3^2}{T_6} = m\pi,
\]

(67)

where \(T_3\) is the membrane tension. It is readily seen that the above expression (66) is indeed in agreement with (67). From equations (64) and (67) we get the following relation between the charge and the mass of the soliton expressing its BPS saturation [18]

\[
M = -\frac{\alpha}{\sqrt{2k_{11}}} Q.
\]

(68)

For the neutral five-brane with \(x_{11}^0\), the quantization condition on \(\int G\) over a four-cycle [14] implies that the charge of the soliton \(q\) must be quantized:

\[
q = \frac{\alpha}{\sqrt{2}} \int_{S^3 \times S^1} G = \alpha\sqrt{2} k_{11} Q = -\left(4\pi k_{11}^2\right)^{1/3} n,
\]

(69)

where \(n\) is an integer. For \(n = 1\) we find that \(q = -\left(4\pi k_{11}^2\right)^{1/3}\), which is identical with the result we would get from eq. (69). The corresponding mass, computed with (63) is

\[
M = -\frac{q}{2k_{11}^2} = T_6 = -\frac{\alpha}{\sqrt{2k_{11}}} Q.
\]

(70)

The relation between the mass and the charge is the same as that of the gauge five-brane.
4 Conclusions

The purpose of this letter is to find classical solutions in the M-theory context of Horava-Witten and study the phenomena arising as a consequence of the dependence of the solutions on the eleventh coordinate. By interpreting the world-volume of the five-brane as the 6d compactified space, we find that the compactified space volume depends on $x^{11}$. In the compactified theory, our classical solutions are interpreted as axionic strings depending on the fifth coordinate. It was shown that, for heterotic five-brane solutions, new features appear for small (compared to the eleventh radius) instanton size, changing significantly the Strominger solution. We shown by an explicit computation that the mass of the heterotic and neutral five-brane solutions are in agreement with the membrane-five-brane quantization condition and derive the quantized charge for the neutral, five-brane solution.

We would like to make one comment concerning the $tr R^2$ part of the boundary conditions (39). It can be easily verified that the Riemannian curvature of our solution (48) has vanishing $tr (R^2)$. However from the derivation of the boundary conditions from ten-dimensional anomalies it seems that one has to use in $tr (R^2)$ not the spin connection but a connection which is the analog of the generalised connection $\Omega_-$ which contains the spin connection plus terms depending on $G_{11\mu\nu\lambda}$. Moreover, the spin connection is the one calculated in the string metric and not in the M-theory metric. In any case, the gauge solution is obtained as in the ten-dimensional case by neglecting the terms $tr (R^2)$, in order to obtain a solution to first order in $k_{11}^{2/3}$. The discussion concerning $tr (R^2)$ is relevant for generalising the symmetric five-brane [3]. No obvious generalization is possible, however, so this issue is not important for our purposes.

The perspectives of the strongly-coupled $E_8 \times E_8$ heterotic string for particle physics phenomenology seem to be promising [19], [14], [11], [12]. The classical solutions we found in this paper lead, by compactification, to classical solutions in 4d which may be relevant for cosmological issues and for understanding non-perturbative aspects of the compactified theory.
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