DIRICHLET IMPROVABILITY FOR $S$-NUMBERS

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Abstract. We study the problem of improving Dirichlet’s theorem of metric Diophantine approximation in the most general and multiplicative form in $S$-adic setting. Our approach is based on translation of the problem related to Dirichlet improvability into a dynamical one, and the main technique of our proof is the $S$-adic version of quantitative nondivergence estimate due to D.Y. Kleinbock and G. Tomanov. The main result of this paper can be regarded as the number field version of earlier works of D.Y. Kleinbock and B. Weiss [34], and of the second named author and Anish Ghosh [20]. Also this in turn generalises a result of Shreyasi Datta and M. M. Radhika [10] on singularity of vectors to any number field $K$ and $S$ containing all archimedian places.

1. Introduction

In this paper we study the improvement of Dirichlet’s theorem of Diophantine approximation in the $S$-adic setting. The branch of Number theory called ‘Diophantine approximation’ deals with approximation of real numbers by rationals and its higher dimensional analogues, and the Dirichlet’s theorem is one of its foundational results. It is indeed quite natural to ask how much improvement the above mentioned theorem of Dirichlet undergoes for a given real number, vector, matrix etc. Recall that given $A \in M_{m \times n}(\mathbb{R})$ and $0 < \varepsilon < 1$, we say that Dirichlet’s theorem can be $\varepsilon$-improved for $A$, and write $A \in DI_{\varepsilon}(m, n)$, if for any $t \gg 1$, there exists $q \in \mathbb{Z}^n \setminus \{0\}$ and $p \in \mathbb{Z}^m$ satisfying

$$\|Aq - p\| < \varepsilon e^{-t/m} \quad \text{and} \quad \|q\| < \varepsilon e^{t/n},$$

where $\|\cdot\|$ denotes the sup norm on $\mathbb{R}^n$. Also we say that $A \in M_{m \times n}(\mathbb{R})$ is singular if $A \in DI_{\varepsilon}(m, n)$ for any $\varepsilon > 0$. In the pioneering works [13, 14], H. Davenport and W. Schmidt showed that the irrational numbers that are Dirichlet improvable are precisely the ones which are badly approximable ([13]). Furthermore, they have also established the first metrical result in the complete generality which says that, $DI_{\varepsilon}(m, n)$ has zero Lebesgue measure for any $\varepsilon \in (0, 1)$.

In 1964, V. Sprindžuk developed a new avenue, usually referred to as ‘Metric Diophantine approximation on manifolds’, while resolving a long standing conjecture of K. Mahler that says, almost every point on the curve $(x, x^2, \ldots, x^n)$ is not ‘very well approximable’ by rationals with respect to the induced one dimensional Lebesgue measure. The central problem of this subject is to investigate the extent the properties of a generic point in $\mathbb{R}^n$ with respect to Lebesgue measure or some nice measures are inherited by embedded submanifolds. Later on V. Sprindžuk himself conjectured a more general form of that of Mahler. Several partial progress on this conjecture had been made, yet the original conjecture seemed to remain beyond one’s reach. In 1998, D.Y. Kleinbock and G.A. Margulis proved the above mentioned conjecture of Sprindžuk in their landmark work [28] using technique from homogeneous dynamics. Undoubtedly it was a stroke of genius for them to observe a Dani type correspondence between very well approximability of a vector and cusp excursion behaviour of the trajectory of certain unimodular lattice. At the core of their argument there is a sharp quantitative estimate, known as the ‘Quantitative nondivergence’ estimate, on nondivergence of trajectories under multidimensional diagonal flows that shows trajectories with such a nature are rare. Soon this Quantitative nondivergence estimate has been generalized, extended to many directions. To mention a few:

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• In [29], D.Y. Kleinbock, E. Lindenstrauss and B. Weiss generalize the results of [28] to more general class of measures, namely ‘friendly measures’.

• D.Y. Kleinbock and G. Tomanov established the quantitative nondivergence estimate for flows on homogeneous spaces of products of real and $p$-adic Lie groups in [32]. Using this, they proved the $S$-adic analogue of [28].

The reader is also suggested to see [6, 26, 19] for other generalizations and extensions of that estimate which have number theoretic implications.

The question of Dirichlet improvability for points lying in manifolds has been first studied by Davenport and Schmidt, in fact they showed in [13] that, the set of all $x \in \mathbb{R}$ for which $(x, x^2) \in DL_1(1, 2)$ has zero Lebesgue measure for any $\varepsilon < 4^{-1/3}$. Further developments in this direction were made by R.C. Baker [4, 5], Y. Bugeaud [8], and M. Dodson, B. Rynne and J. Vickers [16, 17]. In [34] Kleinbock and Weiss significantly generalize earlier works of Davenport, Schmidt, Baker and Bugeaud on Dirichlet improvability. One of the main results of [34] is the existence of a constant $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $f, \nu(DL_1(1, n)) = 0$, for continuous, good and nonplanar maps $f$ and Radon, Federer measure $\nu$. Note that the above mentioned $\varepsilon_0$ is quite far from 1 and getting the same result as above for all $\varepsilon < 1$ is open and seems to be quite challenging. For analytic curves not contained in any affine hyperplane, this has been resolved by N. Shah in [47]. See also [48, 49] for some more results related to this.

Sooner or later, there has been a drive to generalize classical results in Diophantine approximation to the setting of an arbitrary number field. The adelic successive minima theorem has been established independently by McFeat [41] and Bombieri and Vaaler [7]. See also [9, 31], and the references therein for $S$-adic version of other foundational results of geometry of numbers. There seem to exist multiple versions, although with very little variance, of Dirichlet’s theorem over number fields in the literature ([9, 24, 44, 45, 46]). In the recent work [1], Mahbub Alam and Anish Ghosh obtained several quantitative improvements to Dirichlet’s theorem in number fields generalizing an weighted spiralling of approximates. The metric theory on manifolds in this context has germinated some time ago. The $S$-arithmetic version of Sprindžuk’s conjecture has been proved by Kleinbock and Tomanov in [32] modifying the technique of [28]. Subsequently using the results of [32], A. Mohammadi and A. Salehi Golsefidy have settled the convergence case of $S$-arithmetic Khintchine-type theorem for $S$-adic non-degenerate analytic manifolds in [42] and [43]. An inhomogeneous theory on manifolds of the same, both convergence and divergence case, is developed in the $S$-arithmetic setting in [11] by Shreyasi Datta and Anish Ghosh. In a later work [12], the authors have also proved the $p$-adic analogue of inheritance of Diophantine exponents for affine subspaces, in a stronger form indeed. The badly approximable vectors have also been studied extensively in number fields in several works (see [18, 27, 2]).

Since together with the euclidean and $S$-adic ones function field counterparts complete the general theory of Diophantine approximation over local fields, it is worth mentioning some of developments that took place of late in the setting of function fields. The reader should note that there are many interesting parallels, i.e., many results of usual euclidean Diophantine approximation also hold over function fields, however there are certain striking exceptions as well. For example, there is no analogue of Roth’s theorem for function fields over a finite field. The geometry of numbers in the context of function fields was developed by K. Mahler ([39]) in 1940, and using results analogous to that of the euclidean case, one can obtain Dirichlet type theorem in positive characteristic (see [20] for an elementary proof of the most general form of Dirichlet’s theorem in this setting). V. Sprindžuk established the positive characteristic analogue of Mahler’s conjecture in [50]. We refer the reader to [15, 37] for general surveys and to [3, 19, 21, 23, 25, 35, 36, 38] for some of the recent developments. Sprindžuk’s conjecture over a local field of positive characteristic was settled by Anish Ghosh in [22]. The improvability of Dirichlet’s theorem has been first studied in this context by Arijit Ganguly and Anish Ghosh [20, 21]. Indeed, the main result of [20] can be regarded as the function field analogue of that of one of the main results of [34]. On the other
hand, much to one’s surprise, it has been shown in [21, Theorem 2.4] that the Laurent series that are Dirichlet improvable are precisely the rational functions, which is completely opposite to the euclidean scenario.

It is quite expected to be inquisitive about Dirichlet improbability over number fields. Notwithstanding that in this setting there have been multiple versions of Dirichlet’s theorem in the literature as mentioned earlier, the problem of improbability of the same does not seem to have been pursued much to the best of author’s knowledge; although a more restrictive notion, namely singular vectors, for totally real number fields $K$ and $S$ being the collection of all archimedean places, has been coined and discussed at length in the very recent paper [10] by Shreyasi Datta and M. M. Radhika. We take up this as the theme of this project. The main result of this paper, namely Theorem 3.13, similar to those of [34, Theorem 1.5] and [20, Theorem 3.7], deals with the Dirichlet improbability of a generic point, with respect to some nice class of measures, lying in the image of a good and nonplanar map in the $S$-adic setting. Thus Theorem 3.13 of this paper can be regarded as the number field version of [34, Theorem 1.5] and [20, Theorem 3.7].

Further to the above, the notion singularity of vectors is now legitimately extended to any arbitrary number field $K$ and $S$ containing all archimedean places, and that, too, in any direction (see Definition 3.2). This is indeed equivalent to that of [10] (see Lemma 3.4), when $K$ is totally real and $S$ consists of all archimedean places. From whence our Theorem 3.13 can be regarded as a generalization of [10, Theorem 2.3].

We first establish a multiplicative version (see Theorem 2.1) of Dirichlet’s theorem for $S$-adic matrices. Although it can be proved in the exactly similar manner to that of [9, Lemma 5.1], it is clearly the most general version of Dirichlet’s theorem over number fields. In order to provide a brief overview of our result to the reader, here we first state a simplified version of the above mentioned Theorem 2.1 as follows. To begin with, we suppose $K$ is a number field of degree $d$ over $\mathbb{Q}$. The completion of $K$ with respect to the place $v$ is denoted by $K_v$. Let $S$ be a finite collection of pairwise nonequivalent valuations of $K$ containing all archimedean places. By $K_S$ we denote the direct product of all $K_v$ for $v \in S$. Let $\mathcal{O}_S = \{x \in K : |x|_v \leq 1 \ \forall \ v \notin S\}$ denotes the ring of $S$-integers of $K$, where $|\cdot|_v$ denotes the normalized absolute value in $K_v$. We define the field constant

$$\text{const}_K = \left(\frac{2}{\pi}\right)^s |D_K|^{1/2}, \quad (1.1)$$

where $s$ is the number of complex places of $K$ and $D_K$ is the discriminant of $K$.

**Theorem 1.1.** For each $v \in S$, let $y^{(v)} = (y_1^{(v)}, \ldots, y_n^{(v)}) \in K_v^n$. Also let $\varepsilon, \delta \in K_v \setminus \{0\}$ be given for each $v \in S$ with $|\varepsilon|_v < 1$ and $|\delta|_v \geq 1$ for each $v \in S$ and

$$\prod_{v \in S} |\varepsilon|_v |\delta|_v^{n_v} = (\text{const}_K)^{n+1}.$$  

Then there exist $\mathbf{q} = (q_1, \ldots, q_n) \in \mathcal{O}_S^n \setminus \{0\}$ and $p \in \mathcal{O}_S$ satisfying

$$|q_1 y_1^{(v)} + q_2 y_2^{(v)} + \cdots + q_n y_n^{(v)} - p|_v \leq |\varepsilon|_v \quad \text{and} \quad \max_{1 \leq j \leq n} |q_j|_v \leq |\delta|_v,$$

for all $v \in S$.

For proof, we refer to [9, Lemma 5.1]. Now the notion of Dirichlet improbablity can be defined as follows:

**Definition 1.2.** Given a vector $y = (y^{(v)})_{v} \in K_v^n$, where $y^{(v)} = (y_1^{(v)}, \ldots, y_n^{(v)}) \in K_v^n$ for each $v \in S$ and $0 < \varepsilon < 1$. We say that Dirichlet’s theorem can be $\varepsilon$-improved for $y$ or use the notation $y \in DI_{\varepsilon}(1, n, K_S)$ if given any $\varepsilon, \delta \in K_v \setminus \{0\}$ for each $v \in S$ with $|\varepsilon|_v < 1$ and $|\delta|_v \geq M$ for each $v \in S$, where $M \geq 1$ is sufficiently large and

$$\prod_{v \in S} |\varepsilon|_v |\delta|_v^{n_v} = (\text{const}_K)^{n+1},$$

for each
there exist \( \mathbf{q} = (q_1, \ldots, q_n) \in \mathcal{O}_S^S \setminus \{0\} \) and \( p \in \mathcal{O}_S \) satisfying
\[
|q_1 y_1^{(v)} + q_2 y_2^{(v)} + \cdots + q_n y_n^{(v)} - p|_v \leq \varepsilon|\varepsilon|_v \quad \text{and} \quad \max_{1 \leq j \leq n} |q_j|_v \leq \varepsilon|\delta_v|_v,
\]
for all \( v \in S \).

In the next couple of sections we will prove multiplicative version of Dirichlet’s theorem over number fields (Theorem 2.1) and define a more general version of Dirichlet improvability (Definition 3.1). Now we state a special case of our Theorem 3.13.

\[ \text{Theorem 1.3.} \quad \text{Let } X = \prod_{v \in S} K_v^{l_v}, \text{ where } l_v \in \mathbb{N} \text{ and } U = \prod_{v \in S} U_v \subseteq X \text{ be open, and let } f = (f^{(v)})_v : U \to K_v^n \text{ be a map, where } f^{(v)} = \left( f_1^{(v)}, \ldots, f_n^{(v)} \right) \text{ are continuous maps from } U_v \text{ to } K_v^n \text{ such that } f_1^{(v)}, \ldots, f_n^{(v)} \text{ are polynomials and } 1, f_1^{(v)}, \ldots, f_n^{(v)} \text{ are linearly independent over } K_v \text{ for each } v \in S. \text{ Then there exists } \varepsilon_0 > 0 \text{ such that for all } \varepsilon < \varepsilon_0, f(x) \text{ is not Dirichlet } \varepsilon \text{-improvable for } \lambda \text{ almost every } x \in U, \text{ where } \lambda = \prod_{v \in S} \lambda_v \text{ such that } \lambda_v \text{ is a Haar measure on } K_v^n. \]

Note that one can obtain an explicit estimate of the constant \( \varepsilon_0 \) appearing in the above theorem; indeed it is not difficult to compute that in terms of known quantities for \( n = 2, 3 \) and nice maps etc. So in that sense our proof can be regarded as ‘effective’. However optimality for that value of \( \varepsilon_0 \), computed as above, cannot be expected; although at the same time the set up of Theorem 3.13 is very much general. Our proof is based on the \( S \)-arithmetic version of the quantitative nondivergence estimate (see §6.6 [33]).

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\[ \text{2. Dirichlet’s theorem over number fields} \]

In this section, we first state Dirichlet’s theorem over number fields, which is a slight modification of [9, Lemma 5.1]. Before stating Dirichlet’s theorem, we recall some notations and terminologies from [9].

We take \( K \) to be an algebraic number field of degree \( d \) over \( \mathbb{Q} \). The collection of all nontrivial places of \( K \) is denoted by \( P_K \). For any \( v \in P_K \), \( K_v \) will denote the completion of \( K \) with respect to the place \( v \). We know that for any \( v \in P_K \), \( K_v \) is isomorphic to a finite extension of either \( \mathbb{R} \) or \( \mathbb{Q}_p \) for some prime \( p \). Now for any given \( v \in P_K \), if \( v \) is an archimedean place, we say that \( v \) lies over infinity, denoted by \( v|_{\infty} \). If \( v \) is a nonarchimdean place then \( v \) is the extension of some \( p \)-adic valuation. In this case we say \( v \) lies over finite prime \( p \), denoted by \( v|_p \).

- \( d_v = [K_v : \mathbb{Q}_v] \) denotes the local degree of \( K_v \) for each place \( v \in P_K \).
- \( | \cdot |_v \) denotes the absolute value on \( K_v \) for each place \( v \in P_K \).

For each place \( v \in P_K \), we normalize the absolute value \( | \cdot |_v \) as follows:

1. if \( v|_p \), then \( |p|_v = p^{-1} \).
2. if \( v|_{\infty} \), then for any \( x \in K_v, |x|_v = |x| \), where \( | \cdot | \) is the Euclidean absolute value on \( \mathbb{R} \) or \( \mathbb{C} \).

So \( | \cdot |_v \) extends the usual \( p \)-adic absolute value if \( v|_p \) and the Euclidean absolute value if \( v|_{\infty} \).

Our second normalized absolute value \( | \cdot |_{v}^{d_v} \) on \( K_v \) is defined by
\[
|x|_{v} = |x|^{d_v}_v.
\]

Due to this normalization, we have the product formula:
\[
\prod_{v \in P_K} |x|_v = 1 \quad (2.1)
\]
for all \( x \in K, x \neq 0 \).
Then there exists $\overline{x} = (x_1, x_2, \ldots, x_n) \in K^n_v$, we extend our absolute values as follows:

$$|\overline{x}|_v = \max_{1 \leq i \leq n} \{|x_i|_v\},$$

$$\|\overline{x}\|_v = \max_{1 \leq i \leq n} \{|x_i|_v\}.$$

- Let $v$ be a finite place of $K$. We denote the maximal compact (open) subring of $K_v$ by $O_v$, that is,

$$O_v = \{x \in K_v : |x|_v \leq 1\}.$$

- A subset $R_v$ in $K^n_v$ is a $K_v$-lattice if it is a compact open $O_v$-module in $K^n_v$. It is easy to observe that, $O^n_v$ is a $K_v$-lattice in $K^n_v$.

- Let $S$ be a finite collection of places of $K$ containing all archimedean places. We define the ring of $S$-integers as

$$O_S = \{x \in K : x \in O_v, \text{ for all } v \not\in S\}.$$

- The adele ring of $K$ is denoted by $K_\mathbb{A}$. $K$ is diagonally embedded inside $K_\mathbb{A}$, so that we may view $K \subseteq K_\mathbb{A}$ by the natural diagonal map $\rho : K \to K_\mathbb{A}$ defined by

$$\rho(\alpha) = (\alpha, \alpha, \alpha, \ldots)$$

for $\alpha \in K$.

- We say that a nonempty subset $R_v \subseteq K^n_v$ a regular set if it has one of the following forms:

  1. If $v \nmid \infty$ then $R_v$ is a bounded, convex, closed, symmetric subset of $K^n_v$ with nonzero volume.

  2. If $v \mid \infty$ then $R_v$ is a $K_v$-lattice in $K^n_v$.

- We say that a subset $R$ of $K^n_\mathbb{A}$ is admissible if $R = \prod_{v \in P_K} R_v$, where $R_v$ is a regular set in $K^n_v$ for each $v \in P_K$ and $R_v = O^n_v$ for almost all places $v$.

- We define the field constant for the number field $K$ as:

$$\text{const}_K = \left(\frac{2}{\pi}\right)^s |D_K|^{1/2}, \quad (2.2)$$

where $D_K$ is the discriminant of $K$ and $s$ is the number of complex places of $K$.

We are now ready to state and prove a version of Dirichlet’s theorem over number fields. The following theorem is a slight modification of [9, Lemma 5.1].

**Theorem 2.1.** Let $A_v$ be an $m \times n$ matrix over $K_v$ for each $v \in S$. Also let

$$\varepsilon^{(1)}_v, \ldots, \varepsilon^{(m)}_v, \delta^{(1)}_v, \ldots, \delta^{(n)}_v \in K_v \setminus \{0\}$$

be given for each $v \in S$ so that $|\varepsilon^{(i)}_v|_v < 1, \forall i = 1, \ldots, m$ and $|\delta^{(j)}_v|_v \geq 1, \forall j = 1, \ldots, n$ for each $v \in S$ and

$$\prod_{v \in S} \{|\varepsilon^{(1)}_v|_v \cdots |\varepsilon^{(m)}_v|_v |\delta^{(1)}_v|_v \cdots |\delta^{(n)}_v|_v| = \text{const}_K^{m+n}. \quad (2.3)$$

Then there exists $\overline{x} = (x_1, \ldots, x_n) \in O^n_S \setminus \{0\}$ and $\overline{y} = (y_1, \ldots, y_m) \in O^n_B$ satisfying

$$\|A_v^{(i)} \overline{x} - y_i\|_v \leq \|\varepsilon^{(i)}_v\|_v \quad \text{for} \quad i = 1, 2, \ldots, m$$

$$\|x_j\|_v \leq \|\delta^{(j)}_v\|_v \quad \text{for} \quad j = 1, 2, \ldots, n, \quad (2.4)$$

for all $v \in S$, where $A_v^{(i)}$ is the $i$th row of $A_v$ for $i = 1, 2, \ldots, m$.

**Proof.** The proof of this theorem goes along the lines of the original proof by Burger with slight modification. For each place $v \in P_K$, we define a $(m+n) \times (m+n)$ matrix $B_v$ over $K_v$ by:

$$B_v = \begin{pmatrix} \delta_v^{-1}I_n & \mathbf{0} \\ \varepsilon_v^{-1}A_v & \varepsilon_v^{-1}I_m \end{pmatrix},$$

for all $v \in S$, where $\varepsilon_v^{-1}A_v$ is the $m \times n$ matrix over $K_v$, whose $i$th row is $\varepsilon_v^{(i)}A_v^{(i)}$ for $i = 1, \ldots, m$; $\delta_v^{-1}I_n = \text{diag} (\delta_v^{(1)}^{-1}, \delta_v^{(2)}^{-1}, \ldots, \delta_v^{(n)}^{-1})$ and $\varepsilon_v^{-1}I_m = \text{diag} (\varepsilon_v^{(1)}^{-1}, \varepsilon_v^{(2)}^{-1}, \ldots, \varepsilon_v^{(m)}^{-1})$. Also define $B_v = I_{m+n}$ for all $v \not\in S$. 
Define $R_v \subseteq K_v^{m+n}$ by
$$R_v = \{ \tilde{z} \in K_v^{m+n} : \|B_v \tilde{z}\|_v \leq 1 \}$$
and let $\mathcal{R} = \prod_v R_v$. Now it is easy to see that $\mathcal{R} \subseteq K^m$ and $\mathcal{R}$ is admissible. Observe that
$$V(\mathcal{R}) = 2^{d(m+n)} \left( \frac{\pi}{2} \right)^{s(m+n)} |D_K|^{-(m+n)/2} \left( \prod_v |\det B_v|_v \right)^{-1}$$
$$= 2^{d(m+n)} \left( \left( \frac{\pi}{2} \right)^s |D_K|^{-1/2} \right)^{m+n} \left( \prod_v |\varepsilon_v(1)|_v \ldots |\varepsilon_v(m)|_v \ldots |\rho_v(1)|_v \ldots |\rho_v(n)|_v \right)$$
$$= 2^{d(m+n)}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{(m+n)}$ be the successive minima of $\mathcal{R} \subseteq K^m$ with respect to $K^{m+n}$. Then by the adelic successive minima theorem [7, Theorem 3], we know that
$$\lambda_1 \leq \lambda_2 \ldots \lambda_{(m+n)} = \lambda_{(m+n)}.$$ 
Thus we have $\lambda_1 \leq 1$, since $\lambda_v$'s are increasing and $V(\mathcal{R}) = 2^{d(m+n)}$. Hence there exists a point
$$\left( \tilde{x}, \tilde{y} \right) \in K^{m+n} \setminus \{0\} \cap \mathcal{R}.$$ 
By our definition of $R_v$ for $v \notin S$, we have
$$\left( \tilde{x}, \tilde{y} \right) \in \mathcal{O}_S^{m+n}.$$ 
Also for $v \in S$, we have
$$\|A_v(i) \tilde{x} - y_i\|_v \leq \|\varepsilon_v(i)\|_v$$
and
$$\|x_j\|_v \leq \|\rho_v(j)\|_v$$
for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. And it is easy to see that $\tilde{x} \neq \tilde{0}$ due to the product formula (2.1).

3. Dirichlet improvability and the main theorem

Before introducing the notion of “Dirichlet improvability”, we first recall few notations from [33].

- Throughout this article $K$ will represent a number field of degree $d$.
- $S$ will denote a finite set of places of $K$ containing all the archimedean places. Precisely, let $S = \{ v_1, \ldots, v_\ell \}$, where $v_i \in P_K$ for $i = 1, \ldots, \ell$, such that $S$ contains all archimedean places.
- We will denote the normalized absolute value on $K_v$ by $| \cdot |_v$, which is same as earlier defined normalization $| \cdot |_v$ on $K_v$.
- $K_S$ is the direct product of all the completions $K_v$, $v \in S$, that is
$$K_S = \prod_{v \in S} K_v.$$ 
$K$ is imbedded inside $K_S$ via the diagonal imbedding. The set of all archimedean places of $K$ is denoted by $S_a \subseteq S$ and we denote $S \setminus S_a$ by $S_f$. Let $S_c$ and $S_r$ respectively denote the set of all complex places and the set of all real places of $K$.
- The elements of $K_S$ will be denoted as $x = (x^{(v)})_{v \in S}$, or simply $x = (x^{(v)})_v$, where $x^{(v)} \in K_v$. For $x \in K_S$ we define the content $c(x)$ of $x$ as follows:
$$c(x) = \prod_{v \in S} |x^{(v)}|_v.$$ 
- As usual the ring of integers of $K$ is denoted by $\mathcal{O}_K$.
- $\mathcal{O}_S$, i.e., the ring of $S$-integers of $K$, is also diagonally embedded inside $K_S$. We will use this identification conveniently.
We will denote the elements of $K_S^n$ by bold alphabets as $\mathbf{x} = (x^{(v)})_v \in K_S^n$, where $x^{(v)} = (x_1^{(v)}, \ldots, x_m^{(v)}) \in K_v^m$.

We denote by $\| \cdot \|_{v,2}$ the square of the usual Hermitian norm on $K_v^m$ if $v$ is complex, the usual Euclidean norm on $K_v^m$ if $v$ is real, and the sup norm defined by

$$
\| (x_1^{(v)}, \ldots, x_m^{(v)}) \|_{v,2} = \max_i |x_i^{(v)}|
$$

if $v$ is nonarchimedean.

Note that if $v$ is nonarchimedean and $\vec{x} = (x_1, \ldots, x_m) \in K_v^m$, then $\| \vec{x} \|_{v,2} = |\vec{x}|_v$.

We will endow both $K_S$ and $K_S^n$ with the product metric on them respectively. More precisely for any $x = (x^{(v)})_v \in K_S$ and $x = (x^{(v)})_v \in K_S^n$ the norms are given by:

$$
\| x \| := \sup_{v \in S} |x^{(v)}|_v
$$

and

$$
\| x \| := \sup_{v \in S} \| x^{(v)} \|_{v,2}
$$

respectively.

For $x = (x^{(v)})_v \in K_S^n$ we define the content $c(x)$ of $x$ by:

$$
c(x) = \prod_{v \in S} \| x^{(v)} \|_{v,2}.
$$

For the sake of convenience, we will keep the notations $\| \cdot \|_{v,2}$ and $c(\cdot)$ to denote the norms and the content on the exterior powers $\bigwedge^n(K_v^m)$ and $\bigwedge^n(K_S^n)$, respectively.

Given a field $F$, the set of all $m \times n$ matrices over $F$ is denoted by $M_{m \times n}(F)$.

We define $M_{m \times n}(K_S)$ as:

$$
M_{m \times n}(K_S) := \prod_{v \in S} M_{m \times n}(K_v).
$$

Elements of $M_{m \times n}(K_S)$ will be denoted by $g = (g^{(v)})_v$, where $g^{(v)} \in M_{m \times n}(K_v)$.

Similarly,

$$
GL(m, K_S) := \prod_{v \in S} GL(m, K_v)
$$

and

$$
GL^1(m, K_S) := \left\{ (g^{(v)})_v \in GL(m, K_S) : \prod_{v \in S} \left| \det (g^{(v)})_v \right| = 1 \right\}.
$$

$$
a^+ := \{ t := (t_1^{(1)}, t_2^{(2)}, \ldots, t_1^{(m+n)}, \ldots, t_2^{(1)}, t_2^{(2)}, \ldots, t_1^{(m+n)}) \in \mathbb{R}_+^{m+n} \times \cdots \times \mathbb{R}_+^{m+n} : t_{v, i} < 1, t_{v, r} \geq 1 \text{ for } i = 1, \ldots, m; j = m+1, \ldots, m+n; r = 1, \ldots, \ell \text{ and } \prod_{v \in S} t_v^{(1)} \cdots t_v^{(m+n)} = (\text{const}_K)^{m+n} \},
$$

where $\mathbb{R}_+$ is the set of positive real numbers.

Now let $\mathcal{J}$ be a subset of $a^+$ defined by

$$
\mathcal{J} := \{ t \in a^+ : t_v^{(1)} = \cdots = t_v^{(m)} = \cdots = t_v^{(m+n)} \text{ for } r = 1, \ldots, \ell \}.
$$

Let $S$ contain precisely all archimedean places of $K$. Given $t \in \mathcal{J}$, let $t_v^{(1)} := t_v^{(m)} = \cdots = t_v^{(m+n)}$ and $t_{v, r}^{(m+n)} = t_{v, r}^{(m+n)}$ for $r = 1, \ldots, \ell$. Keeping this in mind, let $\mathcal{J}_0$ be the subset of $\mathcal{J}$ defined as follows:

$$
\mathcal{J}_0 := \{ t \in \mathcal{J} : t_{v, s} = t_{v, r} = t_{v}^{(s, r)} \text{ for } r, s = 1, \ldots, \ell \}.
$$

$\mathcal{J}_0$ will be called the central ray of $a^+$.

For $t \in a^+$, define $\| t \|_\infty = \max_{1 \leq i \leq m+n; 1 \leq r \leq \ell} t_{v, r}^{(i)}$. 
For $i = 1, \ldots, \ell$, let $p_i : \mathbb{A}^+ \rightarrow \mathbb{R}^{m+n}$ be the projection of $\mathbb{A}^+$ on the $i$-th factor $\mathbb{R}^{m+n}$ defined by

$$p_i(t) = (t^{(1)}, t^{(2)}, \ldots, t^{(m+n)}) \quad \forall \ t \in \mathbb{A}^+.$$  

We will say that a subset $\mathcal{T}$ of $\mathbb{A}^+$ is $S$-unbounded if $p_i|_{\mathcal{T}}$ is unbounded for each $i = 1, \ldots, \ell$.

Now we define Dirichlet improvability in this context. Let $\mathbb{A}^+$ be as above and $\mathcal{T}$ be a $S$-unbounded subset of $\mathbb{A}^+$.

**Definition 3.1.** Given a $A = (A_v)_{v} \in M_{m \times n}(K_S)$, where $A_v$ is a $m \times n$ matrix over $K_v$ for each $v \in S$, and $0 < \varepsilon < 1$. We say that Dirichlet’s theorem can be $\varepsilon$-improved for $A$ along $\mathcal{T}$ or use the notation $A \in D_{\mathcal{T}}(m, n, K_S, \mathcal{T})$ if there is $t_0 > 0$ such that given any $\varepsilon^{(1)}_v, \varepsilon^{(2)}_v, \ldots, \varepsilon^{(m)}_v, \delta^{(1)}_v, \delta^{(2)}_v, \ldots, \delta^{(n)}_v \in K_v \setminus \{0\}$ for each $v \in S$ with

$$t := (|\varepsilon^{(1)}_{v_1}|_{v_1}, \ldots, |\varepsilon^{(m)}_{v_1}|_{v_1}, |\delta^{(1)}_{v_1}|_{v_1}, \ldots, |\delta^{(m)}_{v_1}|_{v_1}),$$

and $\|p_i|_{\mathcal{T}}(t)\|_{\infty} > t_0$ for each $i = 1, \ldots, \ell$, there exists $\vec{x} = (x_1, \ldots, x_n) \in O_S^n \setminus \{0\}$ and $\vec{y} = (y_1, \ldots, y_m) \in O_S^n$ satisfying

$$\begin{align*}
\|A^{(i)}_v \vec{x} - \vec{y}\|_v &\leq \varepsilon \|\varepsilon^{(i)}_v\|_v \quad \text{for} \ i = 1, 2, \ldots, m \\
\|x_j\|_v &\leq \varepsilon \|\delta^{(j)}_v\|_v \quad \text{for} \ j = 1, 2, \ldots, n,
\end{align*}$$

for all $v \in S$, where $A^{(i)}_v$ is the $i$th row of $A_v$ for $i = 1, 2, \ldots, m$. In particular, a vector $\vec{x} = (x^{(v)}) \in K_S^n$, where $x^{(v)} = (x_1^{(v)}, \ldots, x_n^{(v)}) \in K_v^n$ is said to be Dirichlet $\varepsilon$-improvable along $\mathcal{T}$ if the corresponding matrix $A = ([x_1^{(v)} \ldots x_n^{(v)}])_{v \in S} \in D_{\mathcal{T}}(1, n, K_S, \mathcal{T})$. When $\mathcal{T} = \emptyset$, then simply we will say $A$ is Dirichlet $\varepsilon$-improvable.

Singualr vectors can be defined in the $S$-adic context in the exactly similar way to that of euclidean.

**Definition 3.2.** Let $m, n \in \mathbb{N}$ and $\mathcal{T}$ be a $S$-unbounded subset of $\mathbb{A}^+$. We say $A = (A_v)_{v} \in M_{m \times n}(K_S)$, where $A_v \in M_{m \times n}(K_v)$, for each $v \in S$, is singular along $\mathcal{T}$ if for all $\varepsilon \in (0,1)$, $A$ is Dirichlet $\varepsilon$-improvable along $\mathcal{T}$. In particular, a vector $\vec{x} = (x^{(v)}) \in K_S^n$, where $x^{(v)} = (x_1^{(v)}, \ldots, x_n^{(v)}) \in K_v^n$ for each $v \in S$ is said to be singular along $\mathcal{T}$ if $A = ([x_1^{(v)} \ldots x_n^{(v)}])_{v \in S}$ is singular along $\mathcal{T}$. When $S$ consists of precisely all archimedian places of $K$ and $\mathcal{T} = \emptyset$, then simply we will say $A$ is singular. We denote the set of all vectors of $K_S^n$ that are singular along $\mathcal{T}$ by $Sing(K_S, n, \mathcal{T})$.

When $K$ is totally real and $S = \{v\}$ is the collection of all archimedian places of $K$, the notion of singular vectors has been introduced in [10].

**Definition 3.3.** [10, Definition 2.2] Let $K$ be a totally real number field and $S$ is the collection of all archimedian places of $K$. We say $\vec{x} = (x_1, \ldots, x_n) \in K^n_S$ is singular if for every $c > 0$, for all sufficiently large $Q > 0$ there exists $0 \neq q \in O^n_K, q_0 \in O_K$ satisfying the following system

$$\begin{align*}
\|q \cdot \vec{x} + q_0\| &< \frac{c}{Q}, \\
||q\| &\leq Q.
\end{align*}$$

The set of all singular vectors in $K^n_S$, in this sense, is denoted by $Sing^n_S$.

Indeed, when $S$ is the set of all archimedian places of $K$, Definition 3.3 is equivalent to our definition. The following proposition establishes that.

**Lemma 3.4.** $Sing(K_S, n, \emptyset) = Sing^n_S$, if $S$ contains precisely all normalized archimedean places of $K$.

**Proof.** By a standard algebraic argument using fractional ideals etc., one can easily see that in this case $O_S = O_K$. Let $\vec{x} \in Sing^n_S$. We will show that $\vec{x} \in Sing(K_S, n, \emptyset)$. For that let us
take $0 < \varepsilon < 1$. Suppose that $\varepsilon_v, \delta_v \in K_v \setminus \{0\}$, for each $v \in S$ with $|\varepsilon_v|_v = |\varepsilon_v'|_v = \varepsilon_S$, $|\delta_v|_v = |\delta_v'|_v = \delta_S$ for all $v, v' \in S$ and $\varepsilon_S < 1, \delta_S > 1$ with 

$$
\varepsilon_S \delta_S^n = (\text{const}_K)^{n+1}.
$$

We choose $c > 0$ and $Q$ sufficiently large so that

$$
\begin{cases}
\frac{c}{Q} < \varepsilon_S \\
Q \leq \varepsilon \delta_S,
\end{cases}
$$

holds. Since $\mathbf{x}$ is singular, for this chosen $c$ and sufficiently large $Q$ there exists $0 \neq \mathbf{q} \in \mathcal{O}_K^n$, $q_0 \in \mathcal{O}_K$ such that (3.6) holds. Observe that if $\mathbf{x} = (x_1, \ldots, x_n) \in K^n_\alpha$ and $x_i = (x_i^{(v)}) \in K_v$ for $i = 1, \ldots, n$, then (3.6) can be rewritten as

$$
|q_1 x_1^{(v)} + q_2 x_2^{(v)} + \cdots + q_n x_n^{(v)} + q_0|_v \leq \frac{c}{Q^n} \quad \text{and} \quad \max_{1 \leq j \leq n} |q_j|_v \leq Q,
$$

for all $v \in S$.

Now combining (3.7) and (3.8), we get that $\mathbf{x} \in K^n_\alpha$ is Dirichlet $\varepsilon$-improvable along $J_0$. Therefore we have

$$
\text{Sing}_n^\alpha \subseteq \text{Sing}(K_\alpha, n, J_0).
$$

The other containment $\text{Sing}(K_\alpha, n, J_0) \subseteq \text{Sing}_n^\alpha$ is obvious. \hfill \Box

**Remark 3.5.** In the one dimensional case, singular points are precisely the elements of $K$ ([10, Theorem 2.6]). In fact, when $K = \mathbb{Q}$, one can adopt the strategy of the proof of [21, Theorem 2.4] and easily see that, if $0 < \varepsilon < \frac{1}{2}$, the real numbers for which the Dirichlet’s theorem can be improved by $\varepsilon$ amount are precisely the rationals. The authors are grateful to Sanju Velani for drawing the attention of the second named author to this fact in an informal discussion.

Unless specified otherwise, we let $t$ stand for (3.4) now onward. Before going to our main result, let us recall some terminology from [29], [32] and [33].

**Definition 3.6.** Let $X$ be a metric space. We say that $X$ is Besicovitch if for any bounded subset $A$ of $X$ and for any family $\mathcal{B}$ of nonempty open balls in $X$ such that every $x \in A$ is a center of some open ball of $\mathcal{B}$, there exists a constant $N_X$ (called Besicovitch constant of $X$) and there is a finite or countable subfamily $\{B_i\}$ of $\mathcal{B}$ such that

$$
1_A \leq \sum_i 1_{B_i} \leq N_X,
$$

i.e. $A \subset \bigcup_i B_i$, and the multiplicity of that subcovering is at most $N_X$.

Throughout this article, $N_X$ will always represent Besicovitch constant of the metric space $X$.

**Example 3.7.**

1. By Besicovitch’s Covering Theorem [40, Theorem 2.7], the Euclidean spaces $\mathbb{R}^n$ are Besicovitch spaces.

2. The space of our interest $X = \prod_{v \in S} K_v\ell_v$, where $l_v \in \mathbb{N}$, is also Besicovitch [33, Section 1.6].

Let $X$ be a Besicovitch metric space and $(F, | \cdot |)$ is a valued field. Now for a subset $B$ of $X$ and a given function $f : X \to F$, we set

$$
\|f\|_B := \sup_{x \in B} |f(x)|.
$$

Now if $f : X \to F$ is a measurable function, $\mu$ is a Radon measure on $X$ and $B$ is a subset of $X$ with $\mu(B) > 0$, we define

$$
\|f\|_{\mu, B} := \|f\|_{B \cap \text{supp} (\mu)}.
$$

**Definition 3.8.** Given $C, \alpha > 0$, open $U \subset X$ and a Radon measure $\mu$ on $X$, $f : X \to F$ is said to be $(C, \alpha)$-good on $U$ with respect to $\mu$ if for every ball $B \subset U$ with center in $\text{supp} (\mu)$ and any $\varepsilon > 0$ one has

$$
\mu(\{x \in B : |f(x)| < \varepsilon\}) \leq C \left( \frac{\varepsilon}{\|f\|_{\mu, B}} \right)^\alpha \mu(B).
$$
The properties listed below follow immediately from Definition 3.8.

**Lemma 3.9.** Let \(X, U, \mu, F, f, C, \alpha\) be as given above. Then we have

(a) \(f\) is \((C, \alpha)\)-good on \(U\) with respect to \(\mu \iff |f|\).

(b) \(f\) is \((C, \alpha)\)-good on \(U\) with respect to \(\mu \iff \) so is \(cf\) for all \(c \in F\).

(c) \(\forall i \in I, f_i\) are \((C, \alpha)\)-good on \(U\) with respect to \(\mu \iff \sup_{i \in I} |f_i|\).

(d) \(f\) is \((C, \alpha)\)-good on \(U\) with respect to \(\mu\), and \(c_1 \leq \frac{|f(x)|}{g(x)} \leq c_2\) for all \(x \in U \implies g\) is \((C^{(c_2/c_1)}\alpha)\)-good on \(U\) with respect to \(\mu\).

(e) Let \(C_2 > 1\) and \(\alpha_2 > 0\). \(f\) is \((C_1, \alpha_1)\)-good on \(U\) with respect to \(\mu\) and \(C_1 \leq C_2, \alpha_2 \leq \alpha_1 \implies f\) is \((C_2, \alpha_2)\)-good on \(U\) with respect to \(\mu\).

We recall an important lemma from [33, Section 2.3], which will be useful later in our discussion.

**Lemma 3.10.** For each \(i = 1, \ldots, m\), let \(X_i\) be a metric space, \(\mu_i\) be a measure on \(X_i\), \(U_i \subseteq X_i\) open, \(f_i : X_i \to \mathbb{R}\) be a function which is \((C_i, \alpha_i)\)-good on \(U_i\) with respect to \(\mu_i\). Then the function \(f : \prod_{i=1}^m X_i \to \mathbb{R}\) defined by \(f(x_1, \ldots, x_m) = f_1(x_1)f_2(x_2)\ldots f_m(x_m)\) is \((C, \alpha)\)-good on \(U_1 \times \cdots \times U_m\) with respect to \(\mu_1 \times \cdots \times \mu_m\), with \(C \) and \(\alpha \) explicitly computed in terms of \(C_i, \alpha_i\) for \(i = 1, \ldots, m\).

Now given a Random measure \(\mu\) on \(X\), an open subset \(U\) of \(X\) with \(\mu(U) > 0\) and a map \(f = (f_1, f_2, \ldots, f_m) : U \to F^m\), say that the pair \((f, \mu)\) is \((C, \alpha)\)-good on \(U\) if any \(F\)-linear combination of \(1, f_1, \ldots, f_m\) is \((C, \alpha)\) good on \(U\) with respect to \(\mu\).

**Definition 3.11.** Let \(f = (f_1, f_2, \ldots, f_m)\) be a map from \(U\) to \(F^m\), where \(m \in \mathbb{N}\). We say that \((f, \mu)\) is nonplanar on \(U\) if for any ball \(B \subseteq U\) centered in \(\text{supp}(\mu)\), the restrictions of \(1, f_1, f_2, \ldots, f_m\) to \(B \cap \text{supp}(\mu)\) are linearly independent over \(F\); in other words, if \(f(B \cap \text{supp}(\mu))\) is not contained in any proper affine subspace of \(F^m\).

If \(B = B(x, r)\), where \(x \in X\) and \(r > 0\), is a ball in \(X\) and \(c > 0\), we will use the notation \(cB\) to denote the ball \(B(x, cr)\).

**Definition 3.12.** Let \(\mu\) be a Radon measure on \(X\) and \(D > 0\). \(\mu\) is said to be \(D\)-Federer on an open subset \(U\) of \(X\) if for all balls \(B\) centered at \(\text{supp}(\mu)\) with \(3B \subseteq U\), one has

\[
\frac{\mu(3B)}{\mu(B)} \leq D.
\]

We finish this section by stating the main result of this paper, which says that for sufficiently small \(\varepsilon > 0\), \(\mu\)-almost every vector lying in the image of a good and nonplanar map is not Dirichlet \(\varepsilon\)-improvable along \(T\) for a large class of measures \(\mu\) in this \(S\)-adic set-up.

**Theorem 3.13.** Let \(X = \prod_{v \in S} X_v\), where \(X_v = K_v^{l_v}\) with \(l_v \in \mathbb{N}\) for \(v \in S\), which is a Besicovitch metric space, \(\mu_v\) be a measure on \(X_v\) \((v \in S)\) so that \(\mu = \prod_{v \in S} \mu_v\) is \(D\)-Federer measure on \(X\), and \(U = \prod_{v \in S} U_v \subseteq X\) be open with \(\mu(U) > 0\) and let \(f = (f(v))_v : U \to K_S^n\) be a map, where \(f(v)\) are continuous maps from \(U_v \subseteq K_v^{l_v}\) to \(K_v^n\) such that \((f(v), \mu_v)\) is nonplanar on \(U_v\) for all \(v \in S\) and for each \(v \in S\), \((f(v), \mu_v)\) is \((C_v, \alpha_v)\)-good on \(U_v\). Then there exists \(\varepsilon_0 = \varepsilon_0(n, C_v, \ldots, C_v, \alpha_v)\) such that for any \(\varepsilon < \varepsilon_0\)

\[
f_v(\mu(DL(1, n, K_S, T))) = 0 \quad \text{for any S-unbounded } T \subseteq a^+.
\]

**Corollary 3.14.** Let the hypothesis be same as that of Theorem 3.13. Then we have,

\[
f_v(\mu(Sing(K_S, n, T))) = 0, \quad \text{for any S-unbounded } T \subseteq a^+.
\]

In particular, \(f_v(\mu(Sing(K_S, n, J_0))) = 0\).

In view of Lemma 3.4, it is now clear the above-mentioned Theorem 3.13 generalizes the following:

**Theorem 3.15.** [10, Theorem 2.3] Suppose \(X = \prod_{v \in S} X_v\) is a Besicovitch space and \(\mu = \prod_{v \in S} \mu_v\) is a Federer measure on \(X\) and let \(f : X \to K_S^n\), \(f(x) = (f_v(x_v))_{v \in S}\) be a continuous map such that \((f, \mu)\) is nonplanar for \(\mu\)-almost every point of \(X\) and for each \(v \in S\), \((f_v, \mu_v)\) is good for \(\mu_v\)-almost every point of \(X_v\). Then \(f_v(\mu(Sing_{K_S}^2)) = 0\).
The key idea to prove Theorem 3.13 is the so-called ‘quantitative nondivergence’ technique, a generalization of non-divergence of unipotent flows on homogeneous spaces. To apply this technique first we need to translate our problem related to Dirichlet $\varepsilon$-improvability to a problem regarding Dirichlet improvability in certain properties of flows on some homogeneous spaces, which is the main theme of our next section.

4. The correspondence

Define

$$\Omega_{S,m} := \{g\mathcal{O}_S^m : g \in \text{GL}(m, K_S)\}$$

and

$$\Omega_{S,m}^\varepsilon := \{\Lambda \in \Omega_{S,m} : \text{cov}(\Lambda) = (\sqrt{\mathcal{D}_K})^m\}.$$  

Let $G := \text{GL}^1 (m, K_S)$ and $\Gamma := \text{GL}(m, \mathcal{O}_S)$, and let $\pi : G \to G/\Gamma$ be the quotient map. Then $G$ acts on $G/\Gamma$ by left translation via the rule $g\pi(h) = \pi(gh)$ for $g, h \in G$. We define a map

$$\tau : M_{m \times n}(K_S) \to \text{GL}^1 (m + n, K_S)$$

given by

$$\tau ((A_v)_v) = (a_{A_v})_v,$$

where

$$g_{A_v} := \begin{bmatrix} I_m & A_v \\ 0 & I_n \end{bmatrix}.$$  

We put $\tau := \pi \circ \tau$. Now the homogeneous space $\text{GL}^1 (m, K_S) / \text{GL}(m, \mathcal{O}_S)$ is naturally identified with $\Omega_{S,m}^\varepsilon$, so that we have $\tau ((A_v)_v) = (A_v)_v \mathcal{O}_S^{m+n}$. Define

$$\Omega_{\varepsilon} := \{\Lambda \in \Omega_{S,m+n} : \delta(\Lambda) \geq \varepsilon\},$$

where $\delta : \Omega_{S,m+n} \to \mathbb{R}_+$ defined by

$$\delta(\Lambda) := \min \{c(\mathbf{x}) : \mathbf{x} \in \Lambda \setminus \{0\}\},$$

for all $\Lambda \in \Omega_{S,m+n}$.

Again we let $\varepsilon_v^{\varepsilon(1)}, \varepsilon_v^{\varepsilon(2)}, \ldots, \varepsilon_v^{\varepsilon(m)}, \delta_v^{\delta(1)}, \delta_v^{\delta(2)}, \ldots, \delta_v^{\delta(n)} \in K_v \setminus \{0\}$ for each $v \in S$ and define $g_{\varepsilon,\delta_v}$ by

$$g_{\varepsilon,\delta_v} := \text{diag} \left( (\varepsilon_v^{\varepsilon(1)})^{-1}, \ldots, (\varepsilon_v^{\varepsilon(m)})^{-1}, (\delta_v^{\delta(1)})^{-1}, \ldots, (\delta_v^{\delta(n)})^{-1} \right).$$

Then clearly $g_{\varepsilon,\delta_v} \in \text{GL}(m + n, K_S)$.

Now the following proposition, which may be regarded as an analogue of ‘Dani correspondence’ over number fields, helps us to convert our problem regarding Dirichlet improvability to a problem of homogeneous dynamics.

**Proposition 4.1.**

$$A = (A_v)_v \in \text{DI}_e (m, n, K_S, \mathcal{T}) \implies (g_{\varepsilon,\delta_v})_v (g_{A_v})_v \mathcal{O}_S^{m+n} \notin \Omega_{(m+n)|S|/2+|\mathcal{T}|/\varepsilon},$$

for all $\varepsilon_v^{\varepsilon(1)}, \varepsilon_v^{\varepsilon(2)}, \ldots, \varepsilon_v^{\varepsilon(m)}, \delta_v^{\delta(1)}, \delta_v^{\delta(2)}, \ldots, \delta_v^{\delta(n)} \in K_v \setminus \{0\}$ (for each $v \in S$) with $t \in \mathcal{T}$ and $\|p_t\mathcal{T}(t)\|_\infty >> 1$ for each $i = 1, \ldots, \ell$, where $t$ is given by (3.4).

**Proof.** Let $A = (A_v)_v \in \text{DI}_e (m, n, K_S, \mathcal{T})$. Then given any $\varepsilon_v^{\varepsilon(1)}, \varepsilon_v^{\varepsilon(2)}, \ldots, \varepsilon_v^{\varepsilon(m)}, \delta_v^{\delta(1)}, \delta_v^{\delta(2)}, \ldots, \delta_v^{\delta(n)} \in K_v \setminus \{0\}$, for each $i = 1, \ldots, \ell$, there exists $\bar{x} = (x_1, \ldots, x_n) \in \mathcal{O}_S^n \setminus \{0\}$ and $\bar{y} = (y_1, \ldots, y_m) \in \mathcal{O}_S^m$ such that (3.5) holds. Now observe that

$$(g_{\varepsilon,\delta_v})_v (g_{A_v})_v \mathcal{O}_S^{m+n} = \left( (\varepsilon_v^{\varepsilon(1)})^{-1} (A_v^{\varepsilon(1)} \tilde{b} - a_1), \ldots, (\varepsilon_v^{\varepsilon(m)}(1))^{-1} (A_v^{\varepsilon(m)} \tilde{b} - a_m), \delta_v^{\delta(1)} b_1, \ldots, \delta_v^{\delta(n)} b_n \right)_v \quad (4.1)$$

where $\bar{z}^{(v)} = (\bar{a}, \bar{b}) \in K^{m+n}$ for each $v \in S$, $\bar{a} = (a_1, \ldots, a_m) \in \mathcal{O}_S^n$ and $\bar{b} = (b_1, \ldots, b_n) \in \mathcal{O}_S^m$.

Now consider $\bar{x} \in \mathcal{O}_S^n \setminus \{0\}$ and $\bar{y} \in \mathcal{O}_S^m$ satisfying (3.5) as elements of $K_S^n \setminus \{0\}$ and $K_S^m$ respectively, and denote them as $\mathbf{x} = (\mathbf{x}^{(v)})_v \in \mathcal{O}_S^n \setminus \{0\}$ and $\mathbf{y} = (\mathbf{y}^{(v)})_v \in \mathcal{O}_S^m$ respectively.
Then \((x, y) \in \Omega_S^{m+n}\), and for this point let \(w = (w^{(v)})_v\) be the corresponding point on the lattice \((g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n}\). Now using (3.4) and (3.5) we get that

\[
c(w) = \prod_{v \in S} \|w^{(v)}\|_{v, 2} = \left( \prod_{v \in S} \|w^{(v)}\|_{v, 2} \right) \left( \prod_{v \in S} \|w^{(v)}\|_{v, 2} \right) \left( \prod_{v \in S} \|w^{(v)}\|_{v, 2} \right) < \left( \prod_{v \in S} (m + n) \varepsilon^2 \right) \left( \prod_{v \in S} (m + n) \varepsilon^2 \right) \left( \prod_{v \in S} \varepsilon \right) = \left( m + n \right)^{|S_v|/2} \varepsilon^{|S_v|} \left( m + n \right)^{|S_{v'}|/2} \varepsilon^{|S_{v'}|} \varepsilon,
\]

since \(d_v \geq 1, |S_v| + 2|S_v| + |S_{v'}| \geq |S| \geq 1 \) and \(0 < \varepsilon < 1\), where for any finite set \(A, |A|\) denotes the number of elements of \(A\).

Therefore \(\delta(A) < (m + n)^{|S_v|/2 + |S_{v'}|} \varepsilon\), where \(A = (g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n}\) and so we finally get our desired result.

Now using the above proposition we can say that given a Besicovitch metric space \(X\), a Federer measure \(\mu\) on \(X\), a open subset \(U\) of \(X\) with \(\mu(U) > 0\) and a map \(F : U \rightarrow M_{m \times n}(K_S)\) to prove \(F, \mu(DL_m, \mu(U), \mathcal{T})) = \mu(F^{-1}(DL_m, \mu(U), \mathcal{T})) = 0\) it is enough to show that

\[
\mu(F^{-1}\left( \{ (A_v)_v \in M_{m \times n}(K_S) : (g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n} \notin \Omega_{(m+n)}^{S_v^2 + |S_v|} \varepsilon \} \right) = 0, \tag{4.2}
\]

for all \(v^{(1)}_v, v^{(2)}_v, \ldots, v^{(m)}_v, \delta^{(1)}_v, \delta^{(2)}_v, \ldots, \delta^{(n)}_v \in K_v \setminus \{0\}\) (for each \(v \in S\)) with \(t \in \mathcal{T}\) and \(\|p_t\|_{\mathcal{T}(t)}\|_{\infty} \gg 1\) for each \(i = 1, \ldots, \ell\), where \(t\) is given by (3.4).

Now we suppose that for any ball \(B \subseteq U\) centered in \(\text{supp}(\mu)\) there exists a \(c \in (0, 1)\) such that

\[
\mu(B \cap F^{-1}\left( \{ (A_v)_v \in M_{m \times n}(K_S) : (g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n} \notin \Omega_{(m+n)}^{S_v^2 + |S_v|} \varepsilon \} \right) \leq c \mu(B), \tag{4.3}
\]

holds for all \(v^{(1)}_v\)’s and \(\delta^{(j)}_v\)’s satisfying the aforementioned conditions.

If (4.3) holds true, then for any ball \(B \subseteq U\) centered in \(\text{supp}(\mu)\) we have

\[
\mu(B \cap F^{-1}\left( \{ (A_v)_v \in M_{m \times n}(K_S) : (g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n} \notin \Omega_{(m+n)}^{S_v^2 + |S_v|} \varepsilon \} \right) \leq c \mu(B) \tag{4.4}
\]

for all \(v^{(1)}_v, v^{(2)}_v, \ldots, v^{(m)}_v, \delta^{(1)}_v, \delta^{(2)}_v, \ldots, \delta^{(n)}_v \in K_v \setminus \{0\}\) (for each \(v \in S\)) with \(t \in \mathcal{T}\) and \(\|p_t\|_{\mathcal{T}(t)}\|_{\infty} \gg 1\) for each \(i = 1, \ldots, \ell\). Since (4.4) holds for any ball \(B \subseteq U\), it follows that no point \(x \in U \cap \text{supp}(\mu)\) is a point of density of the sets

\[
F^{-1}\left( \{ (A_v)_v \in M_{m \times n}(K_S) : (g_{v, \delta_v})_v (g_{A_v})_v \Omega_S^{m+n} \notin \Omega_{(m+n)}^{S_v^2 + |S_v|} \varepsilon \} \right),
\]

for all \(v^{(1)}_v, v^{(2)}_v, \ldots, v^{(m)}_v, \delta^{(1)}_v, \delta^{(2)}_v, \ldots, \delta^{(n)}_v \in K_v \setminus \{0\}\) (for each \(v \in S\)) with \(t \in \mathcal{T}\) and \(\|p_t\|_{\mathcal{T}(t)}\|_{\infty} \gg 1\) for each \(i = 1, \ldots, \ell\), which gives us (4.2), because of the following version of the Lebesgue density theorem for our concerned metric space \(X = \prod_{v \in S} K_v^l\), where \(l_v \in \mathbb{N}\), endowed with the product metric.

**Lemma 4.2.** If \(X = \prod_{v \in S} K_v^l\), where \(l_v \in \mathbb{N}\), endowed with the product metric is our concerned metric space, \(\mu\) be a Radon measure on \(X\) and \(\Omega\) is any measurable subset of \(X\). Then almost
every point \( x \in \Omega \cap \text{supp} \ (\mu) \) is a point of density of \( \Omega \), that is,
\[
\lim_{\mu (B) \to 0} \frac{\mu (B \cap \Omega)}{\mu (B)} = 1.
\]

The proof of Lemma 4.2 goes along the same line as the Lebesgue Density Theorem for Euclidean spaces with some appropriate modifications. Hence we leave it to the reader.

As the space of our interest \( X \) is locally compact, Hausdorff and second countable, to prove Theorem 3.13, it is enough to prove the theorem locally, that is, after choosing a suitable \( \varepsilon_0 > 0 \) it is enough to show that for all \( w \in U \cap \text{supp} \ (\mu) \), there exists a ball \( B' \subseteq U \) containing \( w \) such that
\[
\mu (B' \cap f^{-1} (DI_\varepsilon (1, n, K_S, \mathcal{T}))) = \mu (\{ x \in B' : f(x) \in DI_\varepsilon (1, n, K_S, \mathcal{T}) \}) = 0 \quad (4.5)
\]
for all \( \varepsilon < \varepsilon_0 \). And now from the above discussion it follows that Theorem 3.13 is an immediate consequence of the following proposition.

**Proposition 4.3.** For any \( l, n \in \mathbb{N} \), there exists \( \tilde{C} \) with the following property: whenever a ball \( B = \prod_{v \in S} B (x_v, r) \) centered in \( \text{supp} (\mu) \), a Radon measure \( \mu = \prod_{v \in S} \mu_v \) on \( X = \prod_{v \in S} K_{v}^{l_v} \) which is D-Federer on \( \tilde{B} := 3^{n+1} B = \prod_{v \in S} B (x_v, 3^{n+1} r) \) with \( \mu_v \) is a measure on \( K_v^{l_v} \) and \( f = (f_v) : \tilde{B} \rightarrow K_n^S \), be a given map, where \( f_v = (f_v^{(1)}, \ldots, f_v^{(n)}) \) are continuous maps from \( \tilde{B}_v := B (x_v, 3^{n+1} r) \subseteq K_v^{l_v} \) to \( K_n^S \) for each \( v \in S \), so that

(i) for some \( C_v, \alpha_v > 0 \), any linear combination of \( 1, f_v^{(1)}, \ldots, f_v^{(n)} \) is \((C_v, \alpha_v)\)-good on \( \tilde{B}_v \) with respect to \( \mu_v \) for all \( v \in S \) and

(ii) the restrictions of \( 1, f_1^{(v)}, \ldots, f_n^{(v)} \) to \( B (x_v, r) \cap \text{supp} (\mu_v) \) are linearly independent over \( K_v \) for all \( v \in S \).

Then for any \( 0 < \varepsilon \leq \frac{\tilde{C}}{\sqrt{D_K}} \) we have
\[
\mu (\left\{ x \in B : (g_{v_1}, \ldots, g_{v_\ell}) \in \mathcal{T} (f (x)) \notin \Omega (n+1)^{\left| S_r \cup \left| S_{\rho} \right| \right|} \right\}) \leq \tilde{C} \varepsilon^n \mu (B), \quad (4.6)
\]
for all \( \varepsilon \in (1), \varepsilon \in (2), \ldots, \varepsilon \in (m), g_1, \ldots, g_{\ell} \in K_n \setminus \{ 0 \} \) (for each \( v \in S \)) with \( t \in \mathcal{T} \) and \( \|p \|_{\mathcal{T} (t)} \|_{\infty} \gg 1 \) for each \( i = 1, \ldots, \ell \), where \( t \) is given by (3.4), and \( \rho \) is given by (5.10). \( \tilde{C} \) depends only on \((n, C_v, \ldots, C_v, \alpha_v, \ldots, \alpha_v, D, N_X, K_S)\) and the expression for \( \tilde{C} \) is given by (5.11).

Now Theorem 3.13 follows immediately from Proposition 4.3. We choose \( 0 < \varepsilon_0 \leq \frac{\tilde{C}}{\sqrt{D_K}} \) so that \( \tilde{C} \varepsilon_0^n < 1 \). In view of the expression of \( \tilde{C} \) (5.11), it is easy to see that \( \varepsilon_0 \) depends only on \((n, C_v, \ldots, C_v, \alpha_v, \ldots, \alpha_v, D, N_X, K_S)\). Now consider \( x \in U \cap \text{supp} (\mu) \) and choose a ball \( B' \) containing \( x \) such that \( B' \subseteq B' := 3^{n+1} B' \subseteq U \). Next choose any ball \( B \subseteq B' \) with centre in \( \text{supp} (\mu) \) and consider \( \tilde{B} := 3^{n+1} B \). Since by the hypothesis of Theorem 3.13, for each \( v \in S \), \( f_v^{(v)}, \mu_v \) is nonplanar and \((C_v, \alpha_v)\)-good on \( U_v \) with respect to \( \mu_v \), conditions (i) and (ii) of Proposition 4.3 holds immediately. Now we set \( c = \tilde{C} \varepsilon_0^n < 1 \), and observe that (4.3) holds whenever \( 0 < \varepsilon < \varepsilon_0 \), in view of Proposition 4.3. This completes the proof of Theorem 3.13, assuming Proposition 4.3.

Now we need to prove Proposition 4.3. We will do it using the so called ‘Quantitative nondivergence’ technique and our next section is all about that.

5. Quantitative Nondivergence and the Proof of Proposition 4.3

We first recall the \( S \)-arithmetic quantitative nondivergence theorem from [33].

### 5.1. Quantitative Nondivergence

We denote the set of all primitive submodules of \( O_S^m \) by \( \Psi (O_S, m) \). We recall the following theorem from §6.3 of [33].

**Theorem 5.1.** Let \( X \) be a Besicovitch metric space, \( \mu \) be a D-Federer measure on \( X \). For \( m \in \mathbb{N} \), let a ball \( B \subseteq X \) and a continuous map \( h : \tilde{B} \rightarrow \text{GL}(m, K_S) \) be given, where \( \tilde{B} := 3^m B \). Suppose that X is a \( C, \alpha > 0 \) and \( 0 < \rho < 1 \) one has

(C1) for every \( \Delta \in \Psi (O_S, m) \), the function \( \text{cov} (h (\cdot) \Delta) \) is \((C, \alpha)\)-good on \( B \) with respect to \( \mu; \)
(C2) for every $\Delta \in \mathcal{P}(\mathcal{O}_S, m)$, $\|\text{cov}(h(\cdot))\Delta\|_{\mu, B} \geq \rho$. Then for any positive $\varepsilon \leq \rho^2/\sqrt{|D_K|}$ one has

$$
\mu \left( \{ x \in B : \delta(h(x)\mathcal{O}_S^m) < \varepsilon \} \right) \leq mC \left( N_X D^2 \right)^m \left( \frac{\varepsilon \sqrt{|D_K|}}{\rho} \right)^n \mu(B).
$$

(5.1)

For more details and the proof of Theorem 5.1, see ([33], §6, Theorem 6.3).

5.2. The proof of Proposition 4.3. From the definition of $\Omega_\varepsilon$, it follows that

$$
(g_{\varepsilon, \delta_v}, \delta_\varepsilon, \varepsilon, \rho) \notin \Omega_{\rho^2/2 + |\varepsilon|} \varepsilon \iff \delta \left( (g_{\varepsilon, \delta_v}, \varepsilon, (g_{\varepsilon}(\varepsilon(x_v)))_v \mathcal{O}_S^{n+1} \right) < (n + 1)^{|S_v|/2 + |\varepsilon|} \varepsilon.
$$

We will use Theorem 5.1 in the setting

$$
X = \prod_{v \in S} X_v = \prod_{v \in S} K_v^I, m = n + 1;
$$

$$
\mu, B, C_v, \alpha_v, \delta_v \text{ and } D \text{ as in Proposition 4.3};
$$

$$
h(x) = (g_{\varepsilon, \delta_v}, \varepsilon, (g_{\varepsilon}(\varepsilon(x_v)))_v \forall x = (x_v)_v \in B,
$$

where

$$
g_{\varepsilon, \delta_v} := \text{diag} \left( \left( \varepsilon_v \right)^{-1}, \left( \delta_v \right)^{-1}, \left( \delta_v \right)^{-1}, \ldots, \left( \delta_v \right)^{-1} \right) \in \text{GL}(n + 1, K_v),
$$

with $\varepsilon_v$’s and $\delta_v$’s satisfying the aforementioned conditions and

$$
g_{\varepsilon}(\varepsilon(x_v)) := \left[ \begin{array}{cc} 1 & \mathbf{f}(\varepsilon(x_v)) \\ 0 & I_n \end{array} \right] \in \text{GL}(n + 1, K_v).
$$

Now to get our desired result equation (4.6), we just need to verify the two conditions (C1) and (C2) of Theorem 5.1. Observe that to check (C1) and (C2), we first need explicit expressions for functions $x \mapsto \text{cov}(h(x)\Delta)$ in our setting.

We recall that given any $\Delta \in \mathcal{P}(\mathcal{O}_S, n + 1)$ of rank $j$ there exists an element $\mathbf{w} \in \Lambda^j(\mathcal{O}_S^{n+1})$ such that

$$
\text{cov}(\Delta) = a_K c(\mathbf{w}),
$$

where $a_K \in \mathbb{R}$ is a constant depending on $K$. Therefore, we have $\text{cov}(h(x)\Delta) = a_K c(h(x)\mathbf{w})$.

Before proceeding further let us fix some notations. We take $e_0, e_1, \ldots, e_n \in K_S^{n+1}$ as the standard basis of $K_S^{n+1}$ over $K_S$, where

$$
e_i := \left( e_i^{(v)} \right)_v, \ e_i^{(v)} \in K_v^{n+1} \text{ for each } i = 0, \ldots, n;
$$

and $\{e_i^{(v)}\}^n_{i=0}$ forms the standard basis of $K_v^{n+1}$ over $K_v$ for each $v \in S$. Now let $I = \{i_1, i_2, \ldots, i_j\} \subset \{0, 1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_j$ and $\mathbf{e}_I := e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_j} \in \Lambda^j(K_S^{n+1})$. Then the standard basis of $\Lambda^j(K_S^{n+1})$ is given by

$$
\{ \mathbf{e}_I : I = \{i_1, i_2, \ldots, i_j\} \subset \{0, 1, \ldots, n\} \text{ with } i_1 < i_2 < \cdots < i_j \},
$$

and for any element $\mathbf{b} = \sum_{I \subseteq \{0, 1, \ldots, n\} \#I = j} b_I \mathbf{e}_I$, we define $\|\mathbf{b}\| = \max_I \|b_I\|$.

So we can write the earlier mentioned $\mathbf{w} \in \Lambda^j(\mathcal{O}_S^{n+1})$ as $\mathbf{w} = \sum_I w_I \mathbf{e}_I$, where $w_I \in \mathcal{O}_S$. Now we will find the expression of $(g_{\varepsilon, \delta_v}, \varepsilon, (g_{\varepsilon}(\varepsilon(x_v)))_v \mathbf{w}$ componentwise. Note that $g_{\varepsilon}(\varepsilon(x_v))$ leaves $e_0^{(v)}$ invariant and sends $e_i^{(v)}$ to $f_i^{(v)}(x_v)e_0^{(v)} + e_i^{(v)}$ for all $i = 1, \ldots, n$. Therefore, we have

$$
(g_{\varepsilon}(\varepsilon(x_v)))_v \left( \mathbf{e}_I \right) = \begin{cases} 
\mathbf{e}_I \vspace{0.5cm} & \text{if } 0 \in I \\
\mathbf{e}_I + \sum_{i \in I} \pm f_i^{(v)}(x_v)e_0^{(v)} & \text{otherwise}. 
\end{cases}
$$

and
Now we will check the conditions (C1) and (C2).

where

Now using (5.4) and (5.5), we get that

In view of (5.4), it is easy to observe that the components of all the coordinates of

Thus \((g_{\varepsilon, \delta_v}) \circ (g_{f_v(x_v)}) \circ w\) is given by

Now we will check the conditions (C1) and (C2).

- **Checking (C1):** Due to (5.4), we can have the explicit expression for \(\text{cov}(h(x)\Delta)\) and that is given by

In view of (5.4), it is easy to observe that the components of all the coordinates of

are linear combinations of 1, \(f_1^{(v)}, \ldots, f_n^{(v)}\). Then the condition (C1) immediately follows from hypothesis (i) of Proposition 4.3, Lemma 3.9(b,c) and Lemma 3.10. So there exists \(C, \alpha > 0\) such that \(x \mapsto \text{cov}(h(x)\Delta)\) is \((C, \alpha)\)-good on \(B\) with respect to \(\mu\), where \(C\) and \(\alpha\) explicitly computed in terms of \(C_{\varepsilon_1}, \ldots, C_{\varepsilon_n}, \alpha_{\varepsilon_1}, \ldots, \alpha_{\varepsilon_n}\).

- **Checking (C2):** Now by the compactness of the unit sphere in \(K_0^{n+1}\) and hypothesis (ii) of Proposition 4.3, there exists \(\rho_v > 0\) such that for any \(a = (a_0, a_1, \ldots, a_n) \in K_0^{n+1}\) we have

where \(B = \prod_{v \in S} B_v\).

Now using (5.4) and (5.5), we get that

since \(|\delta_v^{(i)}|_v \geq 1\) for all \(i = 1, \ldots, n\) and each \(v \in S\). Therefore, we have

and hence

\[
\|\text{cov}(h(x)\Delta)\|_{\mu, B} \geq \left(\sqrt{D_K}\right)^j \left(\prod_{v \in S} |\varepsilon_v^{(1)}|_v |\delta_v^{(1)}|_v \ldots |\delta_v^{(n)}|_v \right)^{-1} \left(\prod_{v \in S} \rho_v\right),
\]

(5.8)
since $a_K \geq (\sqrt{D_K})^j$ and $w_I \in \mathcal{O}_S$ for each $I$.

Let

$$\tilde{\rho} \overset{\text{def}}{=} (\sqrt{D_K})^j \left( \prod_{v \in S} [\kappa_v^{(1)}_v |\delta_v^{(1)}_v| \cdots |\delta_v^{(n)}_v|] \right)^{-1} \left( \prod_{v \in S} \rho_v \right)$$

$$= (\sqrt{D_K})^j (\text{const}_K)^{-(n+1)} \left( \prod_{v \in S} \rho_v \right). \quad (5.9)$$

So condition (C2) holds with

$$\rho = \min\{1, \tilde{\rho}\}. \quad (5.10)$$

Now as a consequence of Theorem (5.1), the Proposition (4.3) follows immediately with

$$\tilde{C} = C \left( N_X D^2 \right)^{n+1} (n+1)^{1+\frac{\alpha|S_j|}{2}} + \max_{1 \leq v \leq n} \left( \sqrt{D_K} \rho \right)^{\alpha}. \quad (5.11)$$

\[\blacklozenge\]

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