Abstract. We study connections between recollements of the derived category $D(\text{Mod} R)$ of a ring $R$ and tilting theory. We first provide constructions of tilting objects from given recollements, recovering several different results from the literature. Secondly, we show how to construct a recollement from a tilting module of projective dimension one. By [31], every recollement of $D(\text{Mod} R)$ is associated to a differential graded homological epimorphism $\lambda : R \to S$. We will focus on the case where $\lambda$ is a homological ring epimorphism or even a universal localization. Our results will be employed in a forthcoming paper in order to investigate stratifications of $D(\text{Mod} R)$.

Introduction

Recollements of triangulated categories are 'exact sequences' of triangulated categories, which describe the middle term by a triangulated subcategory and a triangulated quotient category. Recollements have first been defined by Beilinson, Bernstein and Deligne [7] in a geometric context, where stratifications of spaces imply recollements of derived categories of sheaves, by using derived versions of Grothendieck’s six functors (which conveniently get axiomatized by the concept of recollement). As certain derived categories of perverse sheaves are equivalent to derived categories of modules over blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, recollements do exist for the corresponding algebras as well. Here, the stratification provided by iterated recollements, is by derived categories of vector spaces. This is one of the fundamental, and motivating, properties of quasi-hereditary algebras, introduced by Cline, Parshall and Scott (see [33]).

The first examples of recollements of derived categories of rings have been produced by direct constructions, using derived functors of known functors on abelian level. Subsequently, a necessary and sufficient criterion has been given [23] for a (bounded) derived module category of an algebra to admit a recollement, with subcategory and quotient category again being derived module categories of rings. This criterion is formulated in terms of two exceptional objects that fully describe the recollement. Later on, the criterion has been extended and modified so as to cover derived categories of differential graded algebras and unbounded derived categories as well and to work for any differential graded ring [19, 31]. All these results characterize the existence of a recollement in terms of two exceptional objects. In the special case of the quotient or the subcategory being zero, one exceptional object is zero and the other is a tilting complex, that is, one recovers Morita theory of derived categories. While in this special case, the role of the tilting complex is very natural in the context of tilting theory, little is known about connections between recollements of derived module categories and tilting theory. The aim of this article is to start exploring such potential connections. We will first provide constructions of tilting objects from given recollements. Our constructions will be general enough to cover quite a few, and rather diverse, situations studied in the literature (usually without mentioning recollements). Conversely, we will show how to construct a recollement from a classical tilting module (of projective dimension one); in this way we will extend results in [3] and put them into a general framework.
In the first section we will collect existence results and categorical methods to construct recollements. The second section leads to the first main result, Theorem 2.4 and its variation Theorem 2.5 (for a situation satisfying some finiteness conditions), which construct a tilting object from the two exceptional objects describing a recollement; the axioms of a recollement imply that there are no morphisms between the two exceptional objects in one direction, and we also assume that morphism in the opposite direction are concentrated in at most two degrees. The subsequent section three applies the first main result in quite diverse situations, thus recovering and re-interpreting various results from the literature. In the fourth section we start with a classical or a large tilting module of projective dimension one over any ring, and construct a recollement from it. The main result, Theorem 4.8 describes both the subcategory and the quotient category in such a recollement. The latter is a derived module category in the classical case; the former is shown to be equivalent to a derived module category if and only if a certain universal localization is a homological epimorphism. Examples of such situations are given in the final section; some of these examples also illustrate differences between various technical terms used in developing the theory. In an appendix, we provide a construction for reflections in triangulated categories.

In the subsequent article [4], we will be strongly using the results of the present article to address a basic and so far completely open question about recollements: Is there a Jordan-Hölder theorem for derived categories? In other words, is there an existence and uniqueness result for iterated recollements (that is, for stratifications of derived categories)? We will show by various examples of 'exotic stratifications' that the answer (and the validity of such a Jordan Hölder theorem) depends very much on the choice of triangulated categories (such as derived categories of algebras or of differential graded algebras or other triangulated categories). Moreover, we will provide positive answers; in particular, we will prove a Jordan-Hölder theorem for bounded derived categories of artinian hereditary rings and thus also for all piecewise hereditary algebras. Here, crucial use will be made in particular of Theorem 4.8 which will allow to identify the end terms of certain recollement situations as derived module categories. We will also discuss when hereditary rings are derived simple.

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1. Recollements and localizations

In this section, recollements are defined and various criteria for the existence of recollements are discussed.

Throughout this paper, \( \mathcal{D} \) denotes a triangulated category with small coproducts (that is, coproducts indexed over a set), and \([1]\) denotes the shift functor.

1.1. Recollements. Let \( \mathcal{X}, \mathcal{Y} \) be triangulated categories. \( \mathcal{D} \) is said to be a recollement of \( \mathcal{X} \) and \( \mathcal{Y} \) if there are six triangle functors as in the following diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{i^*} & \mathcal{D} \\
\downarrow{j^*} & & \downarrow{j^!} \\
\mathcal{X} & \xleftarrow{i_*} & \mathcal{Y}
\end{array}
\]

such that

1. \( (i^*, i_*), (i^!, i!), (j^!, j^!), (j^*, j_*) \) are adjoint pairs;
2. \( i_*, j_*, j^! \) are full embeddings;
3. \( i^! \circ j_* = 0 \) (and thus also \( j^! \circ i^* = 0 \) and \( i^* \circ j^! = 0 \)).
(4) for each \( C \in \mathcal{D} \) there are triangles
\[
\begin{align*}
\text{inc}_\mathcal{Y}^b(C) & \to C \to j_a(C) \\
\text{inc}_\mathcal{X}^a(C) & \to C \to \text{inc}_\mathcal{Y}^q(C)
\end{align*}
\]

Recollements are closely related to localization, which will be discussed below.

1.2. Bousfield localization. A triangle functor \( L : \mathcal{D} \to \mathcal{D} \) is said to be a localization functor if there is a natural transformation \( \eta : \text{Id} \to L \) such that for all \( X \in \mathcal{D} \)

(i) \( L \circ \eta_X = \eta_{L(X)} \), and

(ii) \( \eta \) induces an isomorphism \( L(X) \cong L^2(X) \).

Such a localization functor determines a full subcategory \( \mathcal{X} \) of \( \mathcal{D} \) whose objects are precisely the \( X \in \mathcal{D} \) such that \( L(X) = 0 \). Subcategories of \( \mathcal{D} \) arising in this way are called localizing subcategories.

Note that \( \mathcal{X} \) is a thick subcategory of \( \mathcal{D} \), so we can form the quotient category \( \mathcal{D}/\mathcal{X} \), see [37]. We consider the quotient functor \( \pi : \mathcal{D} \to \mathcal{D}/\mathcal{X} \) and we denote by \( \mathcal{Y} \) the right orthogonal class of \( \mathcal{X} \) given by all objects \( Y \in \mathcal{D} \) such that \( \text{Hom}_{\mathcal{D}}(X,Y) = 0 \) for all \( X \in \mathcal{X} \).

The following statements hold true (see e.g. [1, 1.6])

1. The functor \( \pi : \mathcal{D} \to \mathcal{D}/\mathcal{X} \) induces an equivalence \( \pi \circ \text{inc}_\mathcal{Y} : \mathcal{Y} \to \mathcal{D}/\mathcal{X} \) with inverse \( \rho \).
2. The functor \( \text{inc}_\mathcal{Y} \) has a left adjoint \( q = \rho \circ \pi \).
3. The functor \( \text{inc}_\mathcal{X} \) has a right adjoint \( a \).

We thus obtain triangle functors as in the following diagram:

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\text{inc}} & \mathcal{D} \\
\text{inc} & & \text{inc}
\end{array}
\]

where \( q \circ \text{inc}_\mathcal{X} = 0 \) and \( a \circ \text{inc}_\mathcal{Y} = 0 \).

Note that the localization functor \( L \) preserves small coproducts if and only if the category \( \mathcal{Y} \) is closed under small coproducts. In this case the localizing subcategory \( \mathcal{X} \) is said to be a smashing subcategory, and there even is a recollement

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\text{inc}_\mathcal{Y}} & \mathcal{D} \\
\text{inc}_\mathcal{X} & & \text{inc}_\mathcal{X}
\end{array}
\]

More precisely,

1. the functor \( \text{inc}_\mathcal{Y} \) has a right adjoint \( b \),
2. the functor \( a \) has a right adjoint \( j \),
3. \( j \) is a full embedding, and \( b \circ j = 0 \);
4. for each \( C \in \mathcal{D} \) there are triangles
\[
\begin{align*}
\text{inc}_\mathcal{Y}^b(C) & \to C \to j_a(C) \\
\text{inc}_\mathcal{X}^a(C) & \to C \to \text{inc}_\mathcal{Y}^q(C)
\end{align*}
\]

For details on the correspondence between smashing subcategories and recollements we refer to [30, 4.4.14, 4.2.4, 4.2.5], [31].

Let us now turn to our main example.
1.3. The derived category of a ring. Let $R$ be a ring, and let $\text{Mod-}R$ be the category of all right $R$-modules. We denote by $\mathcal{D}(R)$ the unbounded derived category of $\text{Mod-}R$. The category $\text{Mod-}R$ is identified with the subcategory of $\mathcal{D}(R)$ consisting of the stalk complexes concentrated in degree zero. Of course, every module $M$ is quasi-isomorphic to the complex given by a projective resolution of $M$.

1.4. Generators, compact objects, tilting objects. Given a class of objects $Q$ in $\mathcal{D}$, the smallest full triangulated subcategory of $\mathcal{D}$ which contains $Q$ and is closed under small coproducts is denoted by $\text{Tria} Q$ (note that some authors use the notation $\text{Tria}^{\oplus}_{n} Q$). If $Q$ consists just of one object $Q$, we write $\text{Tria} Q$.

The triangulated category $\mathcal{D}$ satisfies the principle of infinite dévissage (with respect to $Q$) if $\mathcal{D} = \text{Tria} Q$. In this case, $\mathcal{D}$ is generated by $Q$, that is, an object $M$ of $\mathcal{D}$ is zero whenever $\text{Hom}_{\mathcal{D}}(Q[n], M) = 0$ for every object $Q$ of $Q$ and every $n \in \mathbb{Z}$. Sometimes also the converse holds true. For example, if $\mathcal{Y}$ is a full triangulated subcategory of $\mathcal{D}$ generated by $Q$ and $\text{Tria} Q$ is an aisle in $\mathcal{D}$ contained in $\mathcal{Y}$, then $\mathcal{Y} = \text{Tria} Q$, see [30, 4.3.5 and 4.3.6], [91].

An object $P$ of $\mathcal{D}$ is said to be compact if the functor $\text{Hom}_{\mathcal{D}}(P, -)$ preserves small coproducts. Furthermore, $P$ is said to be self-compact if the restricted functor $\text{Hom}_{\mathcal{D}}(P, -) |_{\text{Tria} P}$ preserves small coproducts.

It is well known that a complex $P \in \mathcal{D}(R)$ is compact if and only if it is quasi-isomorphic to a bounded complex consisting of finitely generated projective modules. In particular, the compact objects of $\text{Mod-}R$ are precisely the modules in $\text{mod-}R$ of finite projective dimension. Here $\text{mod-}R$ denotes the subcategory of $\text{Mod-}R$ given by all modules possessing a projective resolution consisting of finitely generated modules.

An object $T$ in $\mathcal{D}$ is called exceptional (or a partial tilting object) if $T$ has no self extensions, i.e. $\text{Hom}_{\mathcal{D}}(T, T[k]) = 0$ for all nonzero integers $k$. Furthermore, $T$ is called a tilting object if it is compact, exceptional, and $\mathcal{D}$ is generated by $T$.

We will frequently use the following result due to Keller.

Theorem. [20, Theorem 8.5], [21] Let $R$ be a ring, and let $\mathcal{D}$ be a full triangulated subcategory of $\mathcal{D}(R)$ closed under coproducts. If $T$ is a compact generator of $\mathcal{D}$, then there is a differential graded algebra $E = \mathbb{R}\text{Hom}(T, T)$ with homology $H^{*}(E) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(T, T[i])$ such that the functor $- \otimes_{E}^{L} T : \mathcal{D}(E) \to \mathcal{D}$ is a triangle equivalence.

1.5. Localizing subcategories generated by a set. By results of Bousfield and Neeman, every set $Q$ of compact objects in $\mathcal{D}(R)$ defines a smashing subcategory $\text{Tria} Q$ and therefore a recollement of $\mathcal{D}(R)$ (see e.g. [30, 4.4.16 and 4.4.3]). We will often work under weaker assumptions and will need a result from [11] stating that any set of objects in $\mathcal{D}(R)$ gives rise to a localizing subcategory.

Theorem [11, 4.5] Let $Q$ be a set of objects in $\mathcal{D}(R)$. Set $X = \text{Tria} Q$, and let $\mathcal{Y} = \ker \text{Hom}_{\mathcal{D}}(X, -)$ be the right orthogonal class. Then $X$ is a localizing subcategory of $\mathcal{D}(R)$, and $\mathcal{Y}$ consists of the objects $Y \in \mathcal{D}(R)$ such that $\text{Hom}_{\mathcal{D}(R)}(Q[n], Y) = 0$ for all $Q \in Q$ and $n \in \mathbb{Z}$. If $Q$ consists of compact objects, then $X$ is even a smashing subcategory.
1.6. Recollements induced by single objects. The following result was first proved by the second named author for bounded derived categories [23], and it was then further developed by several authors [19, 30] (note that in [23] a condition has been misstated, see [32] for a discussion). The versions of this result in [23] and in [19] are assuming that all triangulated categories are derived categories of (differential graded) rings; therefore, the exceptional objects that appear there are images of two of the rings. The exceptional objects appearing in the following version are, in general different, even if all categories are derived categories of rings.

**Theorem.** ([30], 5.2.9, [31]) The derived category $\mathcal{D}(R)$ of a ring $R$ is a recollement of derived categories of rings if and only if there are objects $T_1, T_2 \in \mathcal{D}(R)$ such that

(i) $T_1$ is compact and exceptional,
(ii) $T_2$ is self-compact and exceptional,
(iii) $\text{Hom}_\mathcal{D}(T_1[n], T_2) = 0$ for all $n \in \mathbb{Z}$,
(iv) $\{T_1, T_2\}$ generates $\mathcal{D}(R)$.

We will need the following “non-compact version” of this theorem.

**Theorem.** Assume that $\mathcal{D}$ has a compact generator $R$. Then the following statements are equivalent.

1. $\mathcal{D}$ is a recollement of triangulated categories generated by a single object.
2. There is an object $T_1 \in \mathcal{D}$ such that $\text{Tria} T_1$ is a smashing subcategory of $\mathcal{D}$.
3. There is an object $T_1 \in \mathcal{D}$ such that $\text{Ker Hom}_\mathcal{D}(\text{Tria} T_1, -)$ is closed under coproducts.
4. There are objects $T_1, T_2 \in \mathcal{D}$ such that
   (i) $\text{Ker Hom}_\mathcal{D}(\text{Tria} T_1, -)$ is closed under coproducts,
   (ii) $T_2$ is self-compact,
   (iii) $\text{Hom}_\mathcal{D}(T_1[n], T_2) = 0$ for all $n \in \mathbb{Z}$,
   (iv) $\{T_1, T_2\}$ generates $\mathcal{D}$.

**Proof.** $(1) \Rightarrow (2)$: Condition (1) implies the existence of a smashing subcategory $\mathcal{X}$ generated by an object $T_1$. We have just seen in [15] that $\text{Tria} T_1$ is a localizing subcategory of $\mathcal{D}$ (i.e. an aisle in $\mathcal{D}$) which is contained in $\mathcal{X}$. So, we infer from [30], 4.3.6 that $\mathcal{X} = \text{Tria} T_1$.

By [12] and [15] the conditions (2) and (3) are equivalent. $(4) \Rightarrow (3)$ is clear.

It remains to show $(3) \Rightarrow (4), (1)$: It follows from condition (3) that there is a recollement

$$
\mathcal{Y} \xrightarrow{\text{inc}} \mathcal{D} \xleftarrow{\text{inc}} \mathcal{X} = \text{Tria} T_1
$$

and by [30], 4.3.6, 4.4.8, [31], the compact generator $R$ of $\mathcal{D}$ is mapped by $q$ to a compact generator $T_2 = q(R)$ of $\mathcal{Y}$. As above, we infer $\mathcal{Y} = \text{Tria} T_2$, and we immediately verify (ii) and (iii). Finally, condition (iv) follows from the existence of triangles inc, $a(C) \rightarrow C \rightarrow \text{inc}_\mathcal{Y} q(C) \rightarrow$ with $q(C) \in \text{Tria} T_2$ and $a(C) \in \text{Tria} T_1$ for each object $C \in \mathcal{D}$.

In the case when $\mathcal{D} = \mathcal{D}(R)$ and $T_1$ is compact and exceptional, we provide a construction of the object $T_2 = q(R)$ in the Appendix. More precisely, we construct the $\mathcal{Y}$-reflection $M \rightarrow q(M)$ of $M$ for those $M \in \mathcal{D}$ such that $\text{Hom}_\mathcal{D}(T_1, M[i]) = 0$ for sufficiently large $i$.

Here is another source of examples for recollements.

1.7. Homological ring epimorphisms. Let $\lambda : R \rightarrow S$ be a ring epimorphism, that is, an epimorphism in the category of rings. Following Geigle and Lenzing [17], we say that $\lambda$ is a homological ring epimorphism if $\text{Tor}_i^R(S, S) = 0$ for all $i > 0$. Note that this holds true if and only if the restriction functor $\lambda_* : \mathcal{D}(S) \rightarrow \mathcal{D}(R)$ induced by $\lambda$ is fully faithful [17], 4.4, [30], 5.3.1. As shown in [30], Section 5.3, [31], we then obtain a recollement.
where \( F = - \otimes_R S \) is the derived tensor product, \( G = R \text{Hom}_R(S, -) \) is the derived Hom-functor, \( X \) is the object occurring in the triangle
\[
X \to R \xrightarrow{\lambda} S \to \tau
\]
and \( \tau = - \otimes_R L X \). This also follows from \cite{33} Theorem 2.4 (1)] (which proves that a ‘partial’ recollement can be completed).

There is also a converse result: by \cite{30} 5.4.4, \cite{31}, every recollement of \( D(R) \) is associated to a differential graded homological epimorphism \( \lambda : R \to S \). In this paper, we will focus on the case of \( \lambda \) being a homological ring epimorphism.

Following \cite{16}, we will say that two ring epimorphisms \( \lambda : R \to S \) and \( \lambda' : R \to S' \) are equivalent if there is a ring isomorphism \( \psi : S \to S' \) such that \( \lambda' = \psi \lambda \). The equivalence classes with respect to this equivalence relation are called epiclasses.

Moreover, we will say that two recollements
\[
\mathcal{Y} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j_*} \mathcal{X} \quad \text{and} \quad \mathcal{Y}' \xrightarrow{i'_*} \mathcal{D} \xrightarrow{j'_*} \mathcal{X}'
\]
are equivalent if the essential images of \( i_* \) and \( i'_* \), of \( j_* \) and \( j'_* \), and of \( j_! \) and \( j'_! \) coincide, respectively.

The following observation is implicit in \cite{30} 31.

**Proposition.** Let \( R \) be a ring and \( \mathcal{D} = D(R) \) its derived category. Then there is a bijection between the epiclasses of homological ring epimorphisms starting in \( R \) and the equivalence classes of those recollements
\[
\mathcal{Y} \xrightarrow{i^*} \mathcal{D} \xrightarrow{j^*} \mathcal{X}
\]
for which \( i^*(R) \) is an exceptional object of \( \mathcal{Y} \).

**Proof.** Let \( \lambda : R \to S \) be a homological ring epimorphism, and consider the recollement induced by \( \lambda \) as above. Then the image of \( R \) under the functor \( F = - \otimes_R S \) is isomorphic to \( S \) and thus an exceptional object of \( D(S) \).

Conversely, take a recollement as in the Proposition, for which \( Q = i^*(R) \) is an exceptional object of \( \mathcal{Y} \). Note that \( Q \) is a compact generator of \( \mathcal{Y} \) by \cite{30} 4.3.6 and 4.4.8, whence a compact tilting object in \( \mathcal{Y} \). By Keller’s theorem in \cite{14} there is a differential graded algebra \( E = R \text{Hom}(Q, Q) \) having homology concentrated in zero and \( H^0(E) \cong \text{Hom}_\mathcal{Y}(Q, Q) \cong \text{End}_{D(R)}(i_*(Q)) \), such that the functor \( - \otimes_E Q \) defines a triangle equivalence between the derived category of \( E \) and \( \mathcal{Y} \). Setting \( S = \text{End}_{D(R)}(i_*(Q)) \) we obtain a full embedding \( i : D(S) \to D(R) \).

Note that \( H^n(i_*(Q)) \cong \text{Hom}_{D(R)}(R, i_*(Q)[n]) \cong \text{Hom}_\mathcal{Y}(Q, Q[n]) \), hence \( i_*(Q) \) has homology concentrated in zero, and \( H^0(i_*(Q)) \cong S \).

Moreover, the unit \( \eta \) of the adjoint pair \((i^*, i_*)\) yields a \( \mathcal{Y} \)-reflection
\[
\eta_R : R \to i_* i^*(R) = i_*(Q),
\]
that is, \( \text{Hom}_{D(R)}(\eta_R, i_*(Y)) : \text{Hom}_{D(R)}(i_*(Q), i_*(Y)) \to \text{Hom}_{D(R)}(R, i_*(Y)) \) is a bijection for every \( Y \in \mathcal{Y} \). This allows to define a ring homomorphism \( \lambda : R \to S \) by associating to any element \( r \in R \) the left multiplication \( m_r : R \to R, x \mapsto rx \) and setting \( \lambda(r) = i_*(i^*(m_r)) : i_*(Q) \to i_*(Q) \).
In this way, $S$ becomes a right $R$-module, that is, a complex concentrated in zero, which is quasi-isomorphic to $i_*(Q)$. It follows that the restriction functor $\lambda_* : \mathcal{D}(S) \to \mathcal{D}(R)$ induced by $\lambda$ coincides with the full embedding $i : \mathcal{D}(S) \to \mathcal{D}(R)$, showing that $\lambda$ is a homological ring epimorphism.

Now it is clear how to define the stated bijective correspondence. 

\[ \square \]

1.8. Universal localization. Finally, we focus on a special kind of homological ring epimorphisms.

**Theorem.** [36, Theorem 4.1] Let $\Sigma$ be a set of morphisms between finitely generated projective right $R$-modules. Then there are a ring $R_\Sigma$ and a morphism of rings $\lambda : R \to R_\Sigma$ such that

1. $\lambda$ is $\Sigma$-inverting, i.e., if $\alpha : P \to Q$ belongs to $\Sigma$, then $\alpha \otimes_R 1_{R_\Sigma} : P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism of right $R_\Sigma$-modules, and

2. $\lambda$ is universal $\Sigma$-inverting, i.e., if $S$ is a ring such that there exists a $\Sigma$-inverting morphism $\psi : R \to S$, then there exists a unique morphism of rings $\psi : R_\Sigma \to S$ such that $\psi \lambda = \psi$.

The morphism $\lambda : R \to R_\Sigma$ is a ring epimorphism with $\text{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the universal localization of $R$ at $\Sigma$.

Let now $\mathcal{U}$ be a set of finitely presented right $R$-modules of projective dimension one. For each $U \in \mathcal{U}$, consider a morphism $\alpha_U$ between finitely generated projective right $R$-modules such that

\[ 0 \to P \xrightarrow{\alpha_U} Q \to U \to 0 \]

We will denote by $R_\mathcal{U}$ the universal localization of $R$ at $\Sigma = \{ \alpha_U \mid U \in \mathcal{U} \}$. In fact, $R_\mathcal{U}$ does not depend on the class $\Sigma$ chosen, cf. [10, Theorem 0.6.2], and we will also call it the universal localization of $R$ at $\mathcal{U}$.

In general, a universal localization need not be a homological ring epimorphism, see [29] and Example [5.4]. Universal localizations with this stronger homological property were studied by Neeman and Ranicki. We will need the following result, which is a combination of some of their results in [28].

**Theorem.** Let $\mathcal{U}$ be a set of finitely presented right $R$-modules of projective dimension one. Assume that the universal localization $\lambda : R \to R_\mathcal{U}$ is a homological ring epimorphism. Then there is a recollement

\[ \mathcal{D}(R_\mathcal{U}) \xrightarrow{F} \mathcal{D}(R) \xleftarrow{G} \mathcal{D}(R) / \text{inc} \mathcal{U} \]

where $F = - \otimes_R^L R_\mathcal{U}$ is the derived tensor product, and $G = R \text{Hom}_R(R_\mathcal{U}, -)$ is the derived Hom-functor.

**Proof.** By [1.5] and [1.2] $\mathcal{X} = \text{Tri} \mathcal{U}$ is a smashing subcategory of $\mathcal{D}(R)$ which gives rise to a recollement

\[ \mathcal{D}(R) / \mathcal{X} \xrightarrow{\pi} \mathcal{D}(R) \xleftarrow{\text{inc}} \mathcal{X} \]

where $\pi : \mathcal{D}(R) \to \mathcal{D}(R) / \mathcal{X}$ is the quotient functor onto the Verdier quotient. It is shown in [28, 5.3] that there is a (unique) functor $T : \mathcal{D}(R) / \mathcal{X} \to \mathcal{D}(R_\mathcal{U})$ such that the derived tensor product $F$ factors through $\pi$ as $F = T \circ \pi$. Moreover, combining [28, 7.4, 6.5, 8.7] one obtains that $Q = \pi(R)$ satisfies $\text{Hom}_{\mathcal{D}(R) / \mathcal{X}}(Q', Q) = R_\mathcal{U}$ and $\text{Hom}_{\mathcal{D}(R) / \mathcal{X}}(Q', Q[n]) = 0$ for all integers $n \in \mathbb{Z}$. 

\[ \square \]
2. Constructing tilting objects from recollements.

In this section we start with two exceptional objects coming from the two end terms of a recollement and construct a tilting object from them.

Recall that \( D \) denotes a triangulated category with small coproducts. Let \( T_1, T_2 \) be two exceptional objects in \( D \) such that

\[
(A1) \quad \text{Hom}_D(T_1, T_2[k]) = 0 \text{ for all } k \in \mathbb{Z},
\]

\[
(A2) \quad \text{Hom}_D(T_2, T_1[k]) = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0,1\}.
\]

Assumption (A2) generalizes the familiar condition on (exceptional) modules to have projective dimension at most one.

Choose any morphism \( \alpha : T_2 \to T_1[1] \) and consider the triangle determined by \( \alpha \):

\[
T_1 \to T \xrightarrow{\gamma} T_2 \xrightarrow{\alpha} T_1[1].
\]

The next Proposition gives a necessary and sufficient condition for when \( T \) is exceptional.

**Proposition 2.1.** With the notations above, \( T \) is an exceptional object if and only if the homomorphism \( \text{End}_D(T_2) \oplus \text{End}_D(T_1[1]) \to \text{Hom}_D(T_2, T_1[1]) \) induced by \( \alpha \), mapping \( (f, g) \) to \( \alpha \circ f + g[1] \circ \alpha \), is surjective.

**Proof.** Applying \( \text{Hom}_D(-, T_2[k]) \) to the triangle determined by \( \alpha \) one obtains a long exact sequence

\[
\ldots \to \text{Hom}_D(T_2, T_2[k]) \to \text{Hom}_D(T, T_2[k]) \to \text{Hom}_D(T_1, T_2[k]) \to \ldots
\]

By assumption \( \text{Hom}_D(T_1, T_2[k]) = 0 \) for all integers \( k \), and \( \text{Hom}_D(T_2, T_2[k]) = 0 \) for all nonzero integers \( k \). Hence \( \text{Hom}_D(T, T_2[k]) = 0 \) for all nonzero integers \( k \). Applying \( \text{Hom}_D(-, T_1[k]) \) one obtains

\[
\ldots \to \text{Hom}_D(T_2, T_1[k]) \to \text{Hom}_D(T, T_1[k]) \to \text{Hom}_D(T_1, T_1[k]) \to \ldots
\]

By assumption \( \text{Hom}_D(T_1, T_1[k]) = 0 \) for all \( k \neq 0 \), and \( \text{Hom}_D(T_2, T_1[k]) = 0 \) for all \( k \neq 0, 1 \). Hence \( \text{Hom}_D(T, T_1[k]) = 0 \) for all \( k \neq 0, 1 \).

Applying \( \text{Hom}_D(T, -) \) to the triangle one obtains

\[
\ldots \to \text{Hom}_D(T, T_1[k]) \to \text{Hom}_D(T, T[1]) \to \text{Hom}_D(T, T_2[k]) \to \ldots
\]

It follows that \( \text{Hom}_D(T, T[k]) = 0 \) for all \( k \neq 0, 1 \), and that \( \text{Hom}_D(T, T[1]) = 0 \) if and only if the map \( (T, \alpha) : \text{Hom}_D(T, T_2) \to \text{Hom}_D(T, T_1[1]) \) induced by \( \alpha \) is surjective.

Now consider the following commutative diagram

\[
\begin{array}{ccc}
0 = \text{Hom}_D(T_1[1], T_2) & \xrightarrow{(T_1, \alpha)} & \text{Hom}_D(T_1[1], T_1[1]) \\
\text{Hom}_D(T_2, T_2) & \xrightarrow{(T_2, \alpha)} & \text{Hom}_D(T_2, T_1[1]) \\
\text{Hom}_D(T, T_2) & \xrightarrow{(T, \alpha)} & \text{Hom}_D(T, T_1[1]) \\
0 = \text{Hom}_D(T_1, T_2) & \xrightarrow{(T_1, \alpha)} & \text{Hom}_D(T_1, T_1[1]) = 0
\end{array}
\]

\( n \neq 0 \). By [28, 5.6] it follows that the functor \( T \) is an equivalence, so the recollement above is equivalent to the one in the statement. \( \square \)
It is clear that $(\gamma, T_1[1])$ is surjective. Hence $(T, \alpha)$ is surjective if and only if the morphism
\[(T_2, \alpha) \oplus (\alpha, T_1[1]) : \text{End}_D(T_2) \oplus \text{End}_D(T_1[1]) \to \text{Hom}_D(T_2, T_1[1])\]
is surjective.

An alternative proof can be based on Lemma 2.1 in [24].

A morphism $\alpha : M \to N$ in $\mathcal{D}$ is called left-universal if for any morphism $f : M \to N$ there exists $f_M : M \to M$ such that $f = \alpha \circ f_M$, yielding the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{f_M} & M \\
\downarrow{f} & & \downarrow{\alpha} \\
N & & N
\end{array}
\]

In other words, $\alpha$ is left universal if and only if the map $\text{End}_D(M) \to \text{Hom}_D(M, N)$ induced by $\alpha$ is surjective.

Dually one defines right-universal morphisms: $\alpha$ is right universal if and only if the map $\text{End}_D(N) \to \text{Hom}_D(M, N)$ induced by $\alpha$ is surjective.

**Proposition 2.2.** Let $T_1$ and $T_2$ be two exceptional objects in $\mathcal{D}$ satisfying conditions (A1) and (A2). Then the following statements hold true.

1. The object $T \oplus T_2$ is exceptional if and only if the morphism $\alpha : T_2 \to T_1[1]$ is left-universal.
2. The object $T \oplus T_1$ is exceptional if and only if the morphism $\alpha : T_2 \to T_1[1]$ is right-universal.

**Proof.** (1) By assumption, $T_2$ has no self extensions, and as in the proof of Proposition 2.1 one verifies $\text{Hom}_D(T, T_2[k]) = 0$ for all $k \neq 0$. Applying $\text{Hom}_D(T_2, -)$ to the triangle
\[T_1 \to T \xrightarrow{\alpha} T_2 \xrightarrow{\alpha} T_1[1]\]
one obtains a long exact sequence
\[\ldots \to \text{Hom}_D(T_2, T_2[k-1]) \to \text{Hom}_D(T_2, T_1[k]) \to \text{Hom}_D(T_2, T_2[k]) \to \text{Hom}_D(T_2, T_2[k]) \to \ldots\]

The assumptions (A1) and (A2) imply that $\text{Hom}_D(T_2, T_2[k]) = 0$ for all $k \neq 0, 1$. Moreover, $\text{Hom}_D(T_2, T_1[1]) = 0$ if and only if the map $\text{Hom}_D(T_2, T_2) \to \text{Hom}_D(T_2, T_1[1])$ induced by $\alpha$ is surjective, which is equivalent to the left universality of $\alpha$. This completes the ‘only if’ part. For the ‘if’ part notice further that by Proposition 2.1 the object $T$ is exceptional if $\alpha$ is left universal.

(2) follows by similar arguments: Applying the functor $\text{Hom}(T_1, -)$ we see that $\text{Hom}_D(T_1, T_2[k]) \cong \text{Hom}_D(T_1, T_1[k])$ vanishes for all $k \neq 0$. Next, applying $\text{Hom}_D(-, T_1[1])$ we get as in the proof of Proposition 2.1 that $\text{Hom}_D(T, T_1[1])$ vanishes for all $k \neq 0, 1$. Finally, we observe that $\text{Hom}_D(T, T_1[1]) = 0$ if and only if $\alpha$ is right universal. \hfill \Box

**Corollary 2.3.** Let $T_1$, $T_2$ be exceptional objects in $\mathcal{D}$ satisfying (A1). If $\text{Hom}_D(T_2, T_1[k]) = 0$ for all but one integer $k = n$, then $T_1[n] \oplus T_2$ is an exceptional object in $\mathcal{D}$.

Let us now assume that $\mathcal{D}$ admits a recollement
\[
\begin{array}{c}
\mathcal{Y} \xleftarrow{i^*} & \mathcal{D} \xrightarrow{j_*} & \mathcal{X}
\end{array}
\]

Since $i_!$ and $J_!$ are full embeddings, we identify $\mathcal{Y}$ and $\mathcal{X}$ with their images under $i_!$ and $j_!$ respectively.
Furthermore, suppose that \( \text{Hom}(T_2, T_1[k]) = 0 \) for all \( k \in \mathbb{Z} \setminus \{0, 1\} \), and \( I = \text{Hom}(T_2, T_1[1]) \) is a set. Consider the morphism \( \alpha : T_2^{(I)} \to T_1[1] \) induced by all elements of \( I \), and let
\[
T_1 \to T \to T_2^{(I)} \xrightarrow{\alpha} T_1[1]
\]
be the triangle determined by \( \alpha \). Then \( T \oplus T_2 \) is an exceptional generator of \( \mathcal{D} \).

**Proof.** First of all, note that the morphism \( \alpha : T_2^{(I)} \to T_1[1] \) is left-universal. Indeed, every map \( f \in \text{Hom}_{\mathcal{D}}(T_2, T_1[1]) \) factors through \( \alpha \) by construction, and so does every map \( f \in \text{Hom}_{\mathcal{D}}(T_2^{(I)}, T_1[1]) \) by the universal property of coproducts:

\[
\begin{array}{ccc}
T_2^{(I)} & \xrightarrow{f} & T_1[1] \\
\downarrow \alpha & & \\
T_2^{(I)} & \xrightarrow{\alpha} & T_1
\end{array}
\]

Next, we verify that the objects \( T_1 \) and \( T_2^{(I)} \) in \( \mathcal{D} \) satisfy the assumptions of Proposition 2.2 (1). Of course, \( T_1 \) is an exceptional object. Also \( T_2^{(I)} \) is an exceptional object. In fact, by the self-compactness of \( T_2 \), we have \( \text{Hom}_{\mathcal{D}}(T_2^{(I)}, T_2^{(I)}) \cong \text{Hom}_{\mathcal{D}}(T_2^{(I)}, T_2^{(I)}) \cong \text{Hom}_{\mathcal{D}}(T_2, T_2^{(I)}) = 0 \) for all \( n \neq 0 \). Further, for all \( n \in \mathbb{Z} \) we have \( T_2^{(I)}[n] \in \mathcal{Y} \), and we infer from the orthogonality in the recollement that \( \text{Hom}_{\mathcal{D}}(T_1, T_2^{(I)}) = 0 \), proving condition (A1). Condition (A2) holds by assumption, because \( \text{Hom}_{\mathcal{D}}(T_2^{(I)}, T_1[1]) \cong \text{Hom}_{\mathcal{D}}(T_2, T_1[1]) \).

Now Proposition 2.2 (1) yields that \( T \oplus T_2^{(I)} \), and thus also \( T \oplus T_2 \), is an exceptional object.

So, it remains to show that \( T \oplus T_2 \) generates \( \mathcal{D} \), or equivalently, that \( T_1 \oplus T_2 \) generates \( \mathcal{D} \). Assume that \( M \in \mathcal{D} \) satisfies \( \text{Hom}_{\mathcal{D}}(T_1 \oplus T_2, M[n]) = 0 \) for all \( n \), and take the canonical triangle defined by the recollement of \( \mathcal{D} \)
\[
M_\mathcal{Y} \to M \to M_\mathcal{X} \to M_\mathcal{Y}[1]
\]
where \( M_\mathcal{X} \in \mathcal{X} \) and \( M_\mathcal{Y} \in \mathcal{Y} \). Applying \( \text{Hom}_{\mathcal{D}}(T_1, -) \) we have
\[
\text{Hom}_{\mathcal{D}}(T_1, M_\mathcal{X}[n]) = 0 \text{ for all } n.
\]

Since \( \mathcal{X} \) is generated by \( T_1 \), we deduce \( M_\mathcal{X} = 0 \), whence \( M \cong M_\mathcal{Y} \in \mathcal{Y} \). Since \( \mathcal{Y} \) is generated by \( T_2 \), and
\[
\text{Hom}_{\mathcal{D}}(T_2, M[n]) = 0 \text{ for all } n,
\]
we conclude that \( M = 0 \). Now the proof is complete. \( \square \)

A particularly nice situation arises by adding some finiteness conditions.

**Theorem 2.5.** Assume that \( \mathcal{D} \) is \( K \)-linear over a field \( K \). Let \( \mathcal{X} \) be a localizing subcategory of \( \mathcal{D} \), and \( \mathcal{Y} = \text{Ker} \text{Hom}_{\mathcal{D}}(\mathcal{X}, -) \). Let further \( T_1, T_2 \in \mathcal{D} \) be compact objects such that \( T_1 \) is a tilting object in \( \mathcal{X} \), and \( T_2 \) is a tilting object in \( \mathcal{Y} \). Assume that
\[
\text{Hom}_{\mathcal{D}}(T_2, T_1[k]) = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0, 1\}.
\]

Furthermore, suppose that \( \text{Hom}_{\mathcal{D}}(T_2, T_1[1]) \) is a finite dimensional \( K \)-vector space with basis \( \alpha_1, \ldots, \alpha_m \). Consider the canonical maps
\[
\alpha : T_2^m \to T_1[1], \quad \beta : T_2 \to T_1[1]^\oplus m
\]
Refine a construction by a 'Bongartz complement' [8].

3. Some examples

Now we apply the previous results to various situations in the literature. In all cases, recollements come up naturally. These recollements then produce exceptional objects or tilting objects previously constructed in different ways. Moreover, the recollements may be used to give new proofs of some known results; we refrain from giving details and instead just provide references.

Example 3.1. Injective ring epimorphisms have been studied in [3] in order to construct tilting modules of projective dimension one. We recover this construction by showing that the recollement induced by an injective homological epimorphism produces the tilting object found in [3]. We have seen in [17] that every homological epimorphism \( \lambda : R \to S \) gives rise to a recollement of \( \mathcal{D}(R) \).

Let \( A \) be a finite dimensional algebra, and \( M \) a finite dimensional right \( A \)-module with projective dimension 0 or 1. Suppose \( M \) is an exceptional module (that is, \( \text{Ext}^1_A(M, M) = 0 \)) such that \( \text{Hom}(M, M) = K \) is a skew field. Write \( m \) for the dimension of \( \text{Ext}^1_A(M, A) \) over \( K \), and construct the universal extension

\[
0 \to A \to N \to M^\oplus m \to 0.
\]

Indeed, \( N \) is the Bongartz complement of \( M \).

On the other hand, by assumption \( M \) is a compact exceptional object in the derived module category \( \mathcal{D}(A) \). By [1.5] and [1.6] it generates a smashing subcategory \( \text{Tria} M \) and a recollement of the form
In fact, we know from [16] that \( \text{Ker} \text{Hom}_D(\text{Tria} M, -) = \text{Tria} i^*(A) \), and we will see in Proposition 6.1 in the Appendix that \( i^*(A) = N \). In particular, \( i^*(A) \) is exceptional. Hence by Proposition 1.7 the recollement is induced by a homological ring epimorphism \( \lambda : A \to B \), where \( B = \text{End}(N) \) is the endomorphism ring of \( N \), and \( \text{Tria} N \) is equivalent to the derived category \( D(B) \). Since \( M \) is a compact exceptional generator of \( \text{Tria} M \), we infer from Keller’s theorem in [14] that \( \text{Tria} M \) is equivalent to \( D(K) \). Thus the recollement has the form

\[
\begin{array}{ccc}
D(B) & \xrightarrow{i^*} & D(A) \\
\downarrow \text{proj} & \\
D(K) & \xleftarrow{j^*} & D(M)
\end{array}
\]

We will see in Lemma 4.1 and 4.2 that the essential image of the restriction functor \( \lambda_* : \text{Mod-B} \to \text{Mod-A} \) coincides with the perpendicular category

\[
\hat{M} = \{ X \in \text{Mod-A} \mid \text{Hom}_A(M, X) = \text{Ext}^1_A(M, X) = 0 \}
\]

and that \( \lambda \) can be chosen as universal localization at \( \mathcal{U} = \{ M \} \). Moreover, \( \lambda : A \to B \), when viewed as an \( A \)-module homomorphism, coincides up to isomorphism with the map \( A \to N \) in the universal extension (this can also be deduced from the adjointness of \( (i^*, i_* \) ), and it is therefore injective. By induction we recover [17] Theorem 4.16.

For example, take \( A \) to be a canonical algebra of weight type \((p_1, p_2, \cdots, p_n)\), and \( M \) an exceptional simple regular module corresponding to the weight \( p_i \). By [17] Theorem 10.3 we obtain that the algebra \( B \) is Morita equivalent to the canonical algebra of weight type \((p_1, \cdots, p_{i-1}, p_i - 1, p_{i+1}, \cdots, p_n)\).

**Example 3.3.** Ladkani has constructed and studied derived equivalences for incidence algebras of partially ordered sets. The exceptional objects he considered in this context [25] are also produced by our construction in Section 2.

Let \( X \) be a finite poset, \( i : Y \to X \) a closed subset, and \( j : U \to X \) the open complement. Following Ladkani’s notation, we let \( \text{Sh}(X) \) be the category of sheaves over \( X \) with values in the category of finite dimensional vector spaces over a field \( K \). By [25] this is equivalent to the category \( \text{mod}(KX) \) of finite dimensional modules over the incidence algebra \( KX \). Let \( D^b(X) = D^b(\text{Sh}(X)) \cong D^b(\text{mod}(KX)) \) be the bounded derived category. By [25], there exists a ‘left’ recollement of \( D^b(X) \) built up by \( D^b(Y) \) and \( D^b(U) \)

\[
\begin{array}{ccc}
D^b(Y) & \xrightarrow{i^*} & D^b(X) \\
\downarrow \text{proj} & \\
D^b(U) & \xleftarrow{j^*} & D^b(U)
\end{array}
\]

Take \( T_2 \) to be the direct sum of indecomposable projective modules of \( KY \), and \( T_1 \) the direct sum of indecomposable injective modules of \( KU \). One checks directly, as in [25] Proposition 4.5], that \( \text{Hom}(i_*(T_2), j_!(T_1)[k]) \neq 0 \) if and only if \( k = 1 \). Hence by Corollary 2.3 \( i_*(T_2) \oplus j_!(T_1)[1] \) is a tilting object in \( D^b(X) \) (as shown in [25] Proposition 4.5]).

**Lemma 3.4.** Let \( A \) be a finite dimensional algebra over a field \( K \), \( e \in A \) an idempotent. Assume that the global dimension of \( eAe \) is finite, and that \( A \cong \bigoplus_{eAe} eAe \). Then there exists a recollement of the form

\[
\begin{array}{ccc}
D^b(A/eAe) & \xrightarrow{i^*} & D^b(A) \\
\downarrow \text{proj} & \\
D^b(eAe) & \xleftarrow{j^*} & D^b(A)
\end{array}
\]
This follows from \cite{33} Theorem 2.7 (b). The recollement is the derived version of the following recollement of abelian categories

\[
\begin{array}{c}
\text{mod}(A/AeA) \xrightarrow{i^*} \text{mod}(A) \xrightarrow{j_*} \text{mod}(eAe)
\end{array}
\]

where \( i^* = - \otimes_A A/AeA \), \( i^* = \text{Hom}_A(A/AeA, -) \), \( j_* = j_! \), \( j^* = j^! = \text{Hom}_A(eAe, -) = - \otimes_A eAe \), and \( j_* = \text{Hom}_{eAe}(A, -) \).

**Example 3.5.** Let \( A \) be a finite dimensional quasi-hereditary algebra and \( e \in A \) a maximal idempotent. Then the conditions in Lemma \cite{5} are fullfilled, and the regular module can be constructed from the recollement.

In this situation, the ideal \( AeA \) generated by \( e \) is a heredity ideal. In particular it is projective as \( A \)-module, and the quotient \( A/AeA \) is again quasi-hereditary. Take \( T_2 \) to be the characteristic tilting module of \( A/AeA \), and \( T_1 = eAe \). Then \( T_2 := i_*(T_2) \) is the characteristic tilting module of \( A \) associated to \( 1-e \), and \( T_1 := j_!(T_1) = eAe \) is the projective standard module of \( A \) associated to \( e \). Since \( T_2 \) has projective dimension at most 1, \( \text{Hom}(T_2, T_1[k]) \neq 0 \) implies \( k = 0, 1 \). Consider the right universal map \( T_2 \rightarrow T_1[1]^{\oplus m} \) where \( m = \dim \text{Hom}(T_2, T_1[1]) \) and the corresponding triangle

\[
T_1^{\oplus m} \rightarrow C_2 \rightarrow T_2 \rightarrow T_1[1]^{\oplus m}.
\]

We infer from Theorem \cite{2.5} that \( C_2 \oplus T_1 \) is a tilting object. In fact, \( C_2 \) is the projective module corresponding to \( 1 - e \), hence \( C_2 \oplus T_1 \) is the regular module \( A \).

**Example 3.6.** Assem, Happel and Trepode \cite{5} construct tilting modules for a one-point extension algebra from tilting modules over the given algebras. We recover their construction.

Let \( B \) be a finite dimensional algebra over an algebraically closed field \( K \), and \( P_0 \) a fixed projective right \( B \)-module (in \cite{5} left modules are used). Denote by \( A = B[P_0] \) the one-point extension of \( B \) by \( P_0 \), that is, the matrix algebra

\[
A = \begin{bmatrix} B & 0 \\ P_0 & K \end{bmatrix}
\]

with ordinary matrix addition and multiplication induced from the module structure of \( P_0 \).

Write \( e = e_B \) for the identity of \( B \), viewed as an idempotent in \( A \) satisfying that \( B = eAe = Ae \) and \( A/AeA \cong K \). We assume the algebra \( B \) has finite global dimension. Then by Lemma \cite{3.4} there exists a recollement of the following form

\[
\begin{array}{c}
\text{D}^b(K) \xrightarrow{i^*} \text{D}^b(A) \xrightarrow{j_*} \text{D}^b(B)
\end{array}
\]

Take \( \tilde{T}_2 = K \in \text{mod}(A/AeA) \) to be the simple module, and \( \tilde{T}_1 \in \text{mod}(B) \) to be any tilting \( B \)-module. Define \( T_2 = i_*(\tilde{T}_2) \) and \( T_1 = j_!(\tilde{T}_1) \). By the definition of recollement \( \text{Hom}_D(T_2, T_1[k]) = 0 \) for all \( k \). Notice that \( T_2 \) is an injective \( A \)-module, hence \( \text{Hom}_D(T_1, T_2[k]) = 0 \) for all nonzero \( k \). We conclude from Corollary \cite{2.3} that \( T_1 \oplus T_2 \) is a tilting \( A \)-module (as shown in \cite{5} Proposition 4.1(b)).

### 4. Constructing recollements from tilting objects

We have seen in Section \cite{2} that recollements of the derived category can be used to construct tilting objects or large tilting modules. We are now interested in the opposite direction: using tilting theory to produce recollements. This will be achieved in the special case of tilting modules of projective dimension one. Let us start with some preliminaries.
Notation. We fix a ring $R$ and work in the category $\text{Mod-}R$ of all right $R$-modules. For a class of modules $\mathcal{C}$ we denote
\[ \mathcal{C}^o = \{ M \in \text{Mod-}R \mid \text{Hom}_R(C, M) = 0 \text{ for all } C \in \mathcal{C} \}, \]
\[ \mathcal{C}^\perp = \{ M \in \text{Mod-}R \mid \text{Ext}^i_R(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i > 0 \}. \]
The (right) perpendicular category of $\mathcal{C}$ is denoted by
\[ \widehat{\mathcal{C}} = \mathcal{C}^o \cap \mathcal{C}^\perp. \]
Furthermore, we denote by $\text{Add}\mathcal{C}$ the class consisting of all modules isomorphic to direct summands of direct sums of modules of $\mathcal{C}$. Finally, $\text{Gen}\mathcal{C}$ denotes the class of modules generated by modules of $\mathcal{C}$.

Recall that a subcategory $\mathcal{Y}$ of $\text{Mod-}R$ is said to be reflective if the inclusion $\text{inc}_\mathcal{Y} : \mathcal{Y} \to \text{Mod-}R$ has a left adjoint. This means that every module $M \in \text{Mod-}R$ admits a $\mathcal{Y}$-reflection, that is, a morphism $\eta_M : M \to B$ such that $B \in \mathcal{Y}$ and $\text{Hom}_R(\eta_M, Y) : \text{Hom}_R(B, Y) \to \text{Hom}_R(M, Y)$ is bijective for all $Y \in \mathcal{Y}$. Of course, $\mathcal{Y}$-relections are uniquely determined up to isomorphism.

Lemma 4.1. Let $\mathcal{U} \subset \text{Mod-}R$ be a class of modules of projective dimension at most one such that the class $\mathcal{U}^\perp$ is closed under coproducts. Then the following statements hold true.

1. The perpendicular category $\widehat{\mathcal{U}}$ is closed under products, coproducts, kernels, and cokernels. In particular, $\widehat{\mathcal{U}}$ is a reflective subcategory of $\text{Mod-}R$.
2. There is a ring epimorphism $\lambda : R \to S$, which is uniquely determined up to equivalence, such that $\widehat{\mathcal{U}}$ coincides with the essential image of the restriction functor $\lambda_* : \text{Mod-}S \to \text{Mod-}R$.
3. The map $\lambda : R \to S_R$, when viewed as an $R$-module homomorphism, is the $\widehat{\mathcal{U}}$-reflection of $R$.
4. If $\mathcal{U}$ consists of finitely presented modules, then $\lambda$ can be chosen as universal localization at $\mathcal{U}$.

Proof. (1) Clearly, $\widehat{\mathcal{U}}$ is closed under direct products, and $\mathcal{U}^0$ is closed under direct products and submodules, hence also under direct sums. Moreover, note that the assumptions on $\mathcal{U}$ imply that $\mathcal{U}^\perp$ is a torsion class, that is, it is closed under epimorphic images and direct sums. So, we deduce that $\widehat{\mathcal{U}}$ is closed under direct sums.

We now verify that $\widehat{\mathcal{U}}$ is closed under kernels. Consider
\[ 0 \to \text{Ker} f \to Y \xrightarrow{f} Z \xrightarrow{\text{Im} f} \]
with $Y, Z \in \widehat{\mathcal{U}}$. Since $\mathcal{U}^0$ is closed under submodules and $\mathcal{U}^\perp$ is a torsion class, we get $\text{Im} f \in \mathcal{U}^0 \cap \mathcal{U}^\perp = \widehat{\mathcal{U}}$. Now, for $U \in \mathcal{U}$, applying $\text{Hom}_R(U, -)$ to the short exact sequence $0 \to \text{Ker} f \to Y \to \text{Im} f \to 0$, we get $\text{Ext}_R^1(U, \text{Ker} f) = 0$. This shows that $\text{Ker} f \in \widehat{\mathcal{U}}$.

The closure under cokernels is proved by similar arguments.

Statements (2) and (3) now follow from [16, 1.2], and statement (4) is proven in [21, 1.7].

We now generalize the construction of the recollement given in Example 3.1. Let us fix a module $M \in \text{Mod-}R$ of projective dimension at most one such that $M^\perp$ is closed under coproducts. Set $\mathcal{X} = \text{Tria} M$
and consider the orthogonal class
\[ \mathcal{Y} = \text{Ker} \text{Hom}_{D(R)}(\mathcal{X}, -) \]
of all objects \( Y \in D(R) \) such that \( \text{Hom}_{D(R)}(X, Y) = 0 \) for all \( X \in \mathcal{X} \).

By Theorem 1.5, the category \( \mathcal{X} \) is a localizing subcategory of \( D(R) \). Actually, it is even a smashing subcategory due to the following observation.

**Lemma 4.2.** Let \( M \) be a module of projective dimension at most one with corresponding stalk complex \( M' \), and let \( Y \in D(R) \) be a complex. The following statements are equivalent.

1. \( \text{Hom}_{D(R)}(M'[n], Y) = 0 \) for all \( n \in \mathbb{Z} \).
2. All homologies \( H^n(Y') \), \( n \in \mathbb{Z} \), belong to \( \widehat{M} \).

**Proof.** Note that for a projective module \( P \) and a complex \( Y \), there is a natural isomorphism
\[ \text{Hom}_{D(R)}(P, Y'[n]) \sim \text{Hom}_R(P, H^n(Y')) \], \( \forall n \in \mathbb{Z} \).

If the projective dimension of \( M \) is zero, i.e. \( M = P \) is projective, then \( \text{Hom}_{D(R)}(P, Y'[n]) = 0 \) for all \( n \in \mathbb{Z} \) if and only if \( \text{Hom}_R(P, H^n(Y')) = 0 \) for all \( n \in \mathbb{Z} \), and this is equivalent to \( H^n(Y') \in \widehat{P} \) for all \( n \in \mathbb{Z} \).

Now suppose the projective dimension of \( M \) is one. Let \( 0 \to P_1 \xrightarrow{\alpha} P_0 \to M \to 0 \) be a projective resolution of \( M \). Applying the functor \( \text{Hom}_{D(R)}(-, Y') \) to the triangle \( P_1 \to P_0 \xrightarrow{\alpha} M \to 0 \), we find that \( \text{Hom}_{D(R)}(M, Y'[n]) = 0 \) for all \( n \in \mathbb{Z} \) if and only if \( \text{Hom}_{D(R)}(P_0, Y'[n]) \xrightarrow{\alpha^*} \text{Hom}_{D(R)}(P_1, Y'[n]) \) is an isomorphism for all \( n \in \mathbb{Z} \), if and only if \( \text{Hom}_R(P_0, H^n(Y')) \xrightarrow{\alpha^*} \text{Hom}_R(P_1, H^n(Y')) \) is an isomorphism for all \( n \in \mathbb{Z} \). Again this is equivalent to \( H^n(Y') \in \widehat{M} \) for all \( n \in \mathbb{Z} \), by applying the functor \( \text{Hom}_R(-, H^n(Y')) \) to the short exact sequence \( 0 \to P_1 \xrightarrow{\alpha} P_0 \to M \to 0 \).

**Proposition 4.3.** Let \( M \in \text{Mod-}R \) be a module of projective dimension at most one such that \( M^\perp \) is closed under coproducts, and denote \( \mathcal{X} = \text{Tria} M \). Then the orthogonal class \( \mathcal{Y} = \text{Ker} \text{Hom}_{D(R)}(\mathcal{X}, -) \) is closed under small coproducts, and \( \mathcal{X} \) is a smashing subcategory of \( D(R) \).

**Proof.** We know from Theorem 1.5 that \( \mathcal{Y} \) is the category of all complexes \( Y' \) such that \( \text{Hom}_{D(R)}(M'[n], Y') = 0 \) for all \( n \in \mathbb{Z} \), which means by Lemma 4.2 that all homologies \( H^n(Y'), n \in \mathbb{Z} \), belong to the perpendicular category \( \widehat{M} \). Now if \( (Y'_i)_{i \in I} \) is a family of complexes in \( \mathcal{Y} \), then the \( n - \text{th} \) homology of its coproduct is isomorphic to the coproduct of the \( n - \text{th} \) homologies \( \bigoplus_{i \in I} H^n(Y'_i) \) and thus belongs to \( \widehat{M} \) by Lemma 1.1(3). This shows that \( \mathcal{Y} \) is closed under coproducts, and thus \( \mathcal{X} \) is a smashing subcategory of \( D(R) \).

**Corollary 4.4.** Every module \( M \in \text{Mod-}R \) of projective dimension at most one such that \( M^\perp \) is closed under coproducts induces a recollement
\[
\mathcal{Y} \quad \xlongrightarrow{\text{D(R)}} \quad \text{Tria} M
\]

**Example 4.5.** Let \( P \) be a finitely generated projective \( R \)-module. Write \( \tau_P(R) \) for the trace of \( P \) in \( R \) and set \( E = \text{End}_{D(R)} P \). Then
1. \( P \) is a compact exceptional object, so it induces a recollement
\[
\mathcal{Y} \quad \xlongrightarrow{\text{D(R)}} \quad \text{Tria} P \sim D(E)
\]
In fact, $P$ is a tilting object in $\text{Tria} P$. So, $\text{Tria} P \sim \mathcal{D}(E)$ by Keller’s theorem in [1,4]

(2) By Lemma [5,1] the perpendicular category $\tilde{P}$ is a reflective subcategory of $\text{Mod}-R$. As shown in [11, Section 1], the $\tilde{P}$-reflection of $R$ is $R/\tau P(R)$, so there is a ring epimorphism $\lambda : R \rightarrow S$ such that $\tilde{P}$ is the essential image of the restriction functor $\lambda_*$, and $S_R$ as a right $R$-module is isomorphic to $R/\tau P(R)$. Moreover, $\lambda : R \rightarrow S$ can be chosen as universal localization at $P$, or equivalently, at the zero map $\Sigma = \{ \sigma : 0 \rightarrow P \}$. We can also prove this directly. Indeed, $\lambda$ is $\Sigma$-inverting since $P \otimes_R S$ becomes zero, and it is universal with this property, because for any $\Sigma$-inverting ring homomorphism $\mu : R \rightarrow S'$ we have $P \otimes_R S' = 0$, hence $\tau P(R) \otimes_R S' = 0$ and therefore $\mu(\tau P(R)) = 0$.

(3) If $\lambda : R \rightarrow S$ is a homological epimorphism, then by using the triangle $\tau P(R) \rightarrow R \xrightarrow{\lambda} S \rightarrow$ we infer from [1,7] that we have a recollement

$$
\mathcal{D}(R_P) \xrightarrow{\sim} \mathcal{D}(R) \xrightarrow{\sim} \text{Tria} \tau P(R)
$$

This is equivalent to the recollement in (1). Indeed, we know by Lemma [1,2] that $\mathcal{Y}$ is the full triangulated subcategory of $\mathcal{D}(R)$ consisting of the complexes with all homologies in $\tilde{P}$, which is identified with $\text{Mod}-S$ by (2). The following Lemma [4,6] will show that $\mathcal{Y} = \mathcal{D}(S)$.

(4) If $P$ is generated by an idempotent $e \in R$, then the trace of $P$ in $R$ is the two-sided ideal $ReR$. Hence the ring $S$ is the quotient ring $R/ReR$, and $\lambda$ is the natural projection $R \rightarrow R/ReR$.

Note that the latter is a homological epimorphism if and only if $Re \otimes_{Re} Re = Re$, and such an ideal $Re$ is called a stratifying ideal (see [9, Section 2]). In this case we obtain a recollement

$$
\mathcal{D}(R/ReR) \xrightarrow{\sim} \mathcal{D}(R) \xrightarrow{\sim} \mathcal{D}(eRe) \sim \text{Tria} ReR
$$

This is the unbounded version of Lemma [3,4].

**Lemma 4.6.** Let $\lambda : R \rightarrow S$ be a homological ring epimorphism. Then the full triangulated subcategory $\mathcal{Y}$ of $\mathcal{D}(R)$ consisting of those complexes whose cohomologies are $S$-modules coincides with the essential image of the restriction functor $\lambda_* : \mathcal{D}(S) \rightarrow \mathcal{D}(R)$.

**Proof.** We identify $\mathcal{D}(S)$ with its image under $\lambda_*$. It is clear that $\mathcal{D}(S) \subset \mathcal{Y}$. Conversely we need to show any complex in $\mathcal{Y}$ is contained in $\mathcal{D}(S)$. Since the restriction functor $\lambda_* : \mathcal{D}(S) \rightarrow \mathcal{D}(R)$ has both a left adjoint and a right adjoint, the subcategory $\mathcal{D}(S)$ of $\mathcal{D}(R)$ is closed under both small products and small coproducts. Therefore it is closed under taking homotopy limits and colimits (for a definition see the Appendix).

By using the canonical truncation we see that any bounded complex $M'$ is generated by its cohomology, in the sense that $M' \in \text{Tria} (\oplus_n H^n(M'))$. Any bounded above complex in $\mathcal{Y}$ can be expressed as the homotopy limit of its ‘quotient’ complexes. These ‘quotient’ complexes are obtained from the canonical truncation, and hence are bounded and generated by their cohomologies. Since canonical truncation preserves cohomology, the ‘quotient’ complexes are generated by $S_R$ in the sense that they belong to $\text{Tria} S_R$. Thus they belong to $\mathcal{D}(S)$. It follows that any bounded above complex in $\mathcal{Y}$ belongs to $\mathcal{D}(S)$. Dually, we express a bounded below complex in $\mathcal{Y}$ as the homotopy colimit of its ‘sub’-complexes, which are also obtained from the canonical truncation and thus bounded and belong to $\text{Tria} S_R$. Since $\text{Tria} S_R$ is closed under small coproducts and hence closed under homotopy colimits, we see that any bounded below complex in $\mathcal{Y}$ actually belongs to $\text{Tria} S_R$, which is contained in $\mathcal{D}(S)$. Finally since
any complex is generated by a bounded above complex and a bounded below complex by the
canonical truncation, we conclude that any complex in \( Y \) belongs to \( \mathcal{D}(S) \).

Next, we consider recollements related to tilting modules. Recall that a module \( T \) is said to
be a \textit{tilting module (of projective dimension at most one)} if \( \text{Gen} T = T^\perp \), or equivalently, if the following conditions are satisfied:

\begin{itemize}
  \item [(T1)] \( \text{proj.dim}(T) \leq 1 \);
  \item [(T2)] \( \text{Ext}^1_R(T,T(I)) = 0 \) for each set \( I \); and
  \item [(T3)] there is an exact sequence \( 0 \to R \to T_0 \to T_1 \to 0 \) where \( T_0,T_1 \) belong to \( \text{Add}T \).
\end{itemize}

The class \( T^\perp \) is then called a \textit{tilting class}. We say that two tilting modules \( T \) and \( T' \) are \textit{equivalent} if their tilting classes coincide.

\textbf{Remark.} (1) Note that, in contrast to the definition of a tilting object, a tilting module need
not be compact. This is the reason why one has to require the property “exceptional” in the
stronger form of condition (T2).

(2) Suppose that a module \( T_1 \in \text{Mod-}R \) satisfies conditions (T1) and (T2). Then \( T_1^\perp \) is closed
under coproducts if and only if there are a set \( I \) and a short exact sequence \( 0 \to R \to T_0 \to T_1 \to 0 \) such that \( T_0 \oplus T_1 \) is a tilting module \cite[1.8 and 1.9]{12}. So, the (strongly) exceptional
modules satisfying the assumptions of Corollary \cite[4.4]{12} are precisely the modules \( T_1 \) that are direct
summands of a tilting module \( T \) with \( T^\perp = T_1^\perp \).

Every tilting module is associated to a class of finitely presented modules of projective dimension
one \cite{6} and thus to universal localization.

\textbf{Theorem 4.7.} \cite{2} For every tilting module \( T \) of projective dimension one there exist an exact
sequence
\[
0 \to R \to T_0 \to T_1 \to 0
\]

and a set \( U \) of finitely presented modules of projective dimension one such that
\[
\begin{itemize}
  \item [(1)] \( T_0,T_1 \in \text{Add}T \),
  \item [(2)] \( U^\perp = \text{Gen}T = T_1^\perp \),
  \item [(3)] \( \overset{\text{Ess}}{U} = \overset{\text{Ess}}{T_1} \) coincides with the essential image of the restriction functor
  \( \text{Mod-}R_U \to \text{Mod-}R \)
  induced by the universal localization \( \lambda_U : R \to R_U \).
\end{itemize}

We are now ready for the main result of this section. It associates a recollement to every tilting
module, and it discusses when this recollement has the properties considered in Theorem \cite[2.4]{2}.

\textbf{Theorem 4.8.} Every tilting module \( T \) of projective dimension one gives rise to a recollement
\[
\begin{array}{ccc}
Y & \xrightarrow{\text{inc}} & \mathcal{D}(R) \\
\text{inc} & & \text{inc} \\
\mathcal{X} & & \mathcal{X}
\end{array}
\]

with the following properties.

\begin{itemize}
  \item [(1)] There is a set \( U \) of finitely presented modules of projective dimension one such that
    \( \text{Gen}T = U^\perp \), and \( \mathcal{X} = \text{Tria}U \).
  \item [(2)] There is a module \( T_1 \in \text{Add}T \) such that \( \text{Gen}T = T_1^\perp \), and \( \mathcal{X} = \text{Tria}T_1 \). In particular,
    \( T_1 \) is an exceptional generator of \( \mathcal{X} \).
\end{itemize}
(3) $T_2 = q(R)$ is a compact generator of $\mathcal{Y}$. Moreover, $T_2$ is a tilting object in $\mathcal{Y}$ if and only if the universal localization $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$ of $R$ at $\mathcal{U}$ is a homological epimorphism. In this case, there is an equivalence $\mathcal{D}(R_{\mathcal{U}}) \to \mathcal{Y}$, and the recollement above is equivalent to the one induced by $\lambda_{\mathcal{U}}$.

If, in addition, the $R$-module $R_{\mathcal{U}}$ has projective dimension at most one, then $\text{Hom}_{\mathcal{D}(R)}(T_2, T_1[n]) = 0$ for all $n \neq 0, 1$.

(4) $T \in \text{mod-}R$ if and only if there is a ring $E$ and an equivalence $\mu : \mathcal{X} \to \mathcal{D}(E)$ such that $\mu(T_1) = E_E$. In this case, we can choose $\mathcal{U} = \{T_1\}$.

**Proof.** Choose $T_1$ and $\mathcal{U}$ as in Theorem 4.7. Then $\mathcal{D}(R)$ is a recollement of $\text{Tria}T_1$ and $\mathcal{Y} = \text{Ker Hom}_{\mathcal{D}(R)}(\text{Tria}T_1, -)$ by 4.4 and it is also a recollement $\text{Tria}\mathcal{U}$ and $\mathcal{Y}' = \text{Ker Hom}_{\mathcal{D}(R)}(\text{Tria}\mathcal{U}, -)$. Recall that $\mathcal{Y}$ is the category of all complexes $Y$ such that $\text{Hom}_{\mathcal{D}(R)}(T_1[n], Y) = 0$ for all $n \in \mathbb{Z}$, which means by Lemma 4.2 that all homologies $H^n(Y), n \in \mathbb{Z}$, belong to the perpendicular category $T_1$. Similarly, $\mathcal{Y}'$ consists of all complexes $Y$ such that all homologies $H^n(Y), n \in \mathbb{Z}$, belong to the perpendicular category $\mathcal{U}$. But $T_1 = \mathcal{U}$, thus $\mathcal{Y} = \mathcal{Y}'$, and the two recollements coincide. This proves (1) and (2).

(3) First of all, note that the compact generator $R$ of $\mathcal{D}(R)$ is mapped by $q$ to a compact generator $T_2 = q(R)$ of $\mathcal{Y}$, see [30, 4.3.6, 4.4.8].

If $T_2$ is a tilting object, then we know from 1.7 that our recollement is equivalent to the one induced by a homological ring epimorphism $\lambda : R \to S$. That means that $\mathcal{Y}$ coincides with the essential image of the restriction functor $\lambda_* : \mathcal{D}(S) \to \mathcal{D}(R)$. But then, using the description of $\mathcal{Y}$ given in Lemma 4.2, we see that $\mathcal{T}_1$ coincides with the essential image of the restriction functor $\text{Mod-S} \to \text{Mod-R}$ induced by $\lambda$. On the other hand, we know from Theorem 4.7 that $\mathcal{T}_1 = \mathcal{U}$ coincides with the essential image of the restriction functor $\text{Mod-R}_{\mathcal{U}} \to \text{Mod-R}$ induced by the universal localization $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$. By the uniqueness of the ring epimorphism in Lemma 4.1(2) we conclude that $\lambda$ and $\lambda_{\mathcal{U}}$ are in the same epiclass, and thus also $\lambda_{\mathcal{U}}$ is a homological epimorphism.

Conversely, if $\lambda_{\mathcal{U}}$ is a homological epimorphism, then we know from 1.8 that our recollement is equivalent to the one induced by $\lambda_{\mathcal{U}}$. In particular, it follows from 1.7 that $T_2 = q(R)$ is an exceptional object, hence a compact tilting object in $\mathcal{Y}$. Moreover, $T_2$ is quasi-isomorphic to the stalk complex given by the $R$-module $R_{\mathcal{U}}$. Thus $\text{Hom}_{\mathcal{D}(R)}(T_2, T_1[n]) \cong \text{Ext}^n_R(R_{\mathcal{U}}, T_1)$ vanishes for all $n \neq 0, 1$ if $\text{pdim} R_{\mathcal{U}} \leq 1$.

(4) If $T \in \text{mod-}R$, then $T_1$ is compact, hence a tilting object in $\mathcal{X}$. So, by Keller’s theorem in 1.4 there is a differential graded algebra $E = \mathbb{R}
\text{Hom}(T_1, T_1)$ having homology concentrated in zero and $H^0(E) \cong \text{End}_{\mathcal{D}(R)}(T_1)$ with an equivalence $\mu : \mathcal{X} \to \mathcal{D}(E)$ such that $\mu(T_1) = E_E$.

Conversely, if we have an equivalence $\mu : \mathcal{X} \to \mathcal{D}(E)$ such that $\mu(T_1) = E_E$, then there is a fully faithful functor $\mathcal{D}(E) \to \mathcal{D}(R)$ mapping $E_E$ onto $T_1$. By [19, 1.7] it follows that $T_1$ is compact in $\mathcal{X}$, and we infer from [30, 4.4.8] that $T_1$ is even compact in $\mathcal{D}(R)$. But this means that $T_1$, and therefore also $T$, is in $\text{mod-}R$.

**Remark.** Let the assumptions and notations be as in Theorem 4.8.

(1) $R_{\mathcal{U}} \cong \mathcal{T}_0/\mathcal{T}_1(\mathcal{T}_0)$ where $\mathcal{T}_1(\mathcal{T}_0)$ denotes the trace of $T_1$ in $T_0$. This follows from [11 Section 1], since we know from Lemma 4.1(3) that the universal localization $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$, when viewed as an $R$-module homomorphism, is the $\mathcal{U}$-reflection of $R$.

(2) $\mathcal{Y} = \text{Tria}T_2 = \text{Tria}R_{\mathcal{U}}$. In fact, by definition the module perpendicular category $\mathcal{U}$ is a subcategory of the triangular perpendicular category $\mathcal{Y} = \text{Ker Hom}_{\mathcal{D}(R)}(\mathcal{U}, -) = \text{Tria}T_2$. Hence $R_{\mathcal{U}}$ belongs to $\text{Tria}T_2$ and $\text{Tria}R_{\mathcal{U}} \subset \text{Tria}T_2$. Conversely, $\text{Tria}T_2$ is closed both under small coproduct by definition and under small product since it is the right perpendicular category of
Using the argument of the proof of Lemma 4.6, any complex in Trias $T_2$ is generated by its cohomology, and hence contained in Trias $R_{\mathcal{U}}$ by Lemma 4.2 and Theorem 4.7.

(3) If $T_2$ is exceptional, then $R_{\mathcal{U}} \cong T_2$. In fact, we see as in the proof of the Proposition in 1.7 that $T_2$ has homology concentrated in zero, and yields a $\mathcal{Y}$-reflection $\eta_{\mathcal{V}} : R \to T_2$. Then $T_2 \in \hat{\mathcal{U}}$ is also a $\hat{\mathcal{U}}$-reflection of $R_{\mathcal{U}}$, and by the uniqueness of reflections, it must be isomorphic to $R_{\mathcal{U}}$. In [4] we will see examples where $T_2$ and $R_{\mathcal{U}}$ are not isomorphic.

Corollary 4.9. With the assumptions and notations of Theorem 4.7, the following statements are equivalent.

1. $\text{Hom}_R(T_1, T_0) = 0$.
2. There is a recollement

$$
\text{Trias }T_0 \xleftarrow{\text{inc}} D(R) \xrightarrow{a} \text{Trias }T_1
$$

In this case, the recollement above is equivalent to the one induced by $\lambda_{\mathcal{U}}$, and $T$ is equivalent to the tilting module $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$.

Proof. The implication (2) $\Rightarrow$ (1) follows from the definition of recollement. (1) $\Rightarrow$ (2): We know that $D(R)$ is a recollement of $\mathcal{X} = \text{Trias }T_1$ and $\mathcal{Y} = \text{Ker }\text{Hom}_{D(R)}(\text{Trias }T_1, -)$. Condition (1) means that $T_0 \in \hat{T}_1$, whence the stalk complex $T_0$ belongs to $\mathcal{Y}$. So the exact sequence $0 \to R \to T_0 \to T_1 \to 0$ gives rise to a triangle

$$
T_1 [-1] \to R \to T_0 \to
$$

where $T_1 [-1] \in \mathcal{X}$ and $T_0 \in \mathcal{Y}$. Now apply Theorem 4.8 using that $q(R) = T_0$ is exceptional. For the last statement apply [3, 2.10], see also [2, 2.5].

5. Examples of recollements induced by tilting modules

We now provide some new examples of recollements, illustrating particular features of our results and serving as counterexamples to some questions that suggest themselves.

Example 5.1. Our results go beyond classical (finitely presented) tilting modules. Here we present two recollements induced by 'large' tilting modules over the Kronecker algebra. The first one uses the Lukas tilting module; this recollement turns out to be a hidden derived equivalence. The second example uses divisible modules; here the resulting recollement is non-trivial, and it is not equivalent to the standard recollement by derived categories of vector spaces, so it becomes a counterexample in the context of a Jordan-Hölder theorem for derived categories (see [4]).

Let $R$ be the Kronecker algebra, and consider the preprojective component $p^\perp$. By the Auslander-Reiten formula

$$
p^\perp = \circ p
$$

so $p^\perp$ is the class of all right modules having no non-zero homomorphism to $p$, or in other words, the class of all modules that have no non-zero finitely generated preprojective direct summand (see [35, Corollary 2.2]). There is an infinite dimensional tilting module $L$ generating $p^\perp$. Its construction goes back to work by Lukas, cf. [26, 22].

The recollement of $D(R)$ induced by $L$ is trivial. In fact, let us take an exact sequence

$$
0 \to R \to L_0 \to L_1 \to 0
$$
and a set $\mathcal{U}$ of finitely presented indecomposable modules as in Theorem 4.7 that is, $L_0, L_1 \in \text{Add} L, \mathcal{U}^\perp = \text{Gen} L = \mathfrak{p}^\perp$, and $\mathcal{U} = \hat{\mathcal{L}}$. Then $\mathcal{U}$ is contained in $^\perp (\mathfrak{p}^\perp)$ and therefore in $\mathfrak{p}$. Observe that the indecomposable preprojective $R$-modules, up to isomorphism, form a countable family $(P_n)_{n \in \mathbb{N}}$ where $P_1$ is simple projective, and each $P_n$ with $n > 1$ generates all modules having no direct summands isomorphic to one of $P_1, \ldots, P_{n-1}$, hence in particular every module in $\mathfrak{p}^\perp$. From this we deduce that every module $X \in \mathfrak{p}^\perp$ is generated by $U$, and thus cannot belong to $\mathcal{U}^\circ$, unless $X = 0$. Thus $L_1 = \mathcal{U} = \mathcal{U}^\circ \cap \mathfrak{p}^\perp = 0$. But since $\mathcal{U}$ consists of the complexes with all homologies in $\hat{\mathcal{L}}_1$ by Lemma 4.2, this implies that $\mathcal{U} = 0$ and $X = D(R)$.

Let us now consider the class of indecomposable regular right $R$-modules $t$. Again by the Auslander-Reiten formula the tilting class $t^\perp = \mathcal{O}_t$ is the torsion class of all divisible modules, see [35]. We fix a tilting module $W$ which generates $t^\perp$. It is shown in [34] that $W$ can be chosen as the direct sum of a set of representatives of the Prüfer $R$-modules and the generic $R$-module $G$. Moreover, there is an exact sequence $0 \to R \to W_0 \to W_1 \to 0$

where $W_0 \cong G^d$, and $W_1$ is a direct sum of Prüfer modules. Note that $\text{Hom}_R(W_1, W_0) = 0$, so we are in the situation of Corollary 4.9. Thus $W$ is equivalent to the tilting module $R_t \oplus R_t/R$, and there is a recollement

$$
\begin{tikzcd}
\mathcal{D}(R_t) \arrow{r}{\text{inc}} & \mathcal{D}(R) \arrow{r}{\text{inc}} & \text{Tri} \mathcal{O}_t
\end{tikzcd}
$$

where $R_t \cong \text{End}_R W_0 \cong (\text{End}_R G)^{d \times d}$, see also [13], [3, 4.7].

**Example 5.2.** The following example shows that the recollement constructed as in Theorem 4.8 from a tilting module $T$ can be induced by an injective homological epimorphism $\lambda : R \to Q$ despite the fact that $T$ is not of the form $Q \oplus Q/R$ as in [3] or in Example 3.1.

Let $R$ be a commutative domain, and $Q$ its quotient field. Denote by $\mathcal{D}$ the class of all divisible modules. It was shown by Facchini [14] that there is a tilting module of projective dimension one generating $\mathcal{D}$, namely the Fuchs’ divisible module $\delta$, cf. [15, §VII.1]. Recall further that $\mathcal{D} = U^\perp$ where

$$
U = \{ R/rR \mid r \in R \}
$$

denotes a set of representatives of all cyclically presented modules. Moreover, the exact sequence $0 \to R \to \delta \to \delta/R \to 0$ has the properties stated in Theorem 4.7. In particular, the perpendicular category $\hat{\delta}/R = \hat{U}$ is the class of all divisible torsion-free modules.

Note that the universal localization of $R$ at $U$ is given by the injective flat epimorphism $\lambda : R \to Q$, see [3, 3.7]. So, we obtain a recollement of the form

$$
\begin{tikzcd}
\mathcal{D}(Q) \arrow{r}{\text{inc}} & \mathcal{D}(R) \arrow{r}{\text{inc}} & \text{Tri} \mathcal{U} = \text{Tri} \delta/R
\end{tikzcd}
$$

On the other hand, $\delta$ is not equivalent to a tilting module of the form $S \oplus S/R$ as in Example 3.1 unless $R$ is a Matlis domain, see [3, 2.11 (4)].
Example 5.3. In the next example, we start with a tilting object, assign a recollement to it as in Theorem 4.8, and then construct a tilting object from the recollement as in Theorem 2.5. The resulting tilting object is different from the tilting object we started with.

Let $K$ be a field, and let $R$ be the $K$-algebra given the quiver

$$
\begin{array}{c}
1 & \alpha & \beta \\
\alpha & \beta & 2 \\
\end{array}
$$

with the relation $\beta \alpha = 0$. Denote by $P_i, I_i, S_i, i = 1, 2$, the indecomposable projective, injective, and the simple right $R$-modules, and set $T = P_2 \oplus S_2$. The minimal left add-$T$-approximation of $R$ is given by the exact sequence

$$
0 \to R \to (P_2)^2 \to S_2 \to 0
$$

Note that $S_2$ is the socle of $P_2$, hence $\text{Hom}_R(S_2, P_2) \neq 0$, and $T$ is not equivalent to a tilting module of the form $S \oplus S/R$ as in Example 3.1, see [3, 2.10]. Setting $U = \{S_2\}$, one easily verifies that $\text{Gen}_T = \text{Add}\{P_2, I_1, S_2\} = U^\perp$, and that the perpendicular category $\widehat{U} = \text{Add}I_1$. Using that the universal localization $\lambda_U : R \to R_U$, when viewed as an $R$-module homomorphism, is the $\widehat{U}$-reflection of $R$, we obtain $R_U \cong I_1^2$ as $R$-modules, and $R_U \cong \text{End}I_1^2 \cong K^{2 \times 2}$ as rings. In particular, it follows that $\text{Ext}_R^i(R_U, R_U) = 0$ for all $i > 0$, so $\lambda$ is a homological epimorphism by [17, 4.9], and we obtain a recollement of the form

$$
\mathcal{D}(K^{2 \times 2}) \sim \mathcal{D}(R_U) \quad \xrightarrow{\text{inc}} \mathcal{D}(R) \quad \xrightarrow{\text{inc}} \text{Tria} S_2 \sim \mathcal{D}(K)
$$

Moreover

$$
T_1 = S_2, \quad T_2 = R_U \cong I_1
$$

satisfy the assumptions of Theorem 2.5 because $T_1$ has injective dimension one and therefore $\text{Hom}_{\mathcal{D}(R)}(T_2, T_1[n]) \cong \text{Ext}_R^n(T_2, T_1)$ vanishes for $n > 1$. For $n = 1$ we have a one-dimensional space $\text{Hom}_{\mathcal{D}(R)}(T_2, T_1[1]) \cong \text{Ext}_R^1(I_1, S_2)$ with basis given by the almost split sequence

$$
0 \to S_2 \to I_2 \to I_1 \to 0
$$

which yields the triangle

$$
I_2 \to I_1 \to S_2[1] \to
$$

Note that applying Theorem 2.5 we don’t get the original tilting module $T$, but a new tilting object, namely $I_2 \oplus I_1$.

Example 5.4. We close with an example where the universal localization $\lambda_U$ is not a homological epimorphism. Let $K$ be a field, and let $R$ be the $K$-algebra given the quiver

$$
\begin{array}{c}
1 & \alpha & \beta \\
\alpha & \gamma & \beta \\
\beta & \delta & 3 \\
\end{array}
$$

with the relations $\alpha \gamma = \delta \gamma = \delta \beta = 0$ and $\beta \alpha = \gamma \delta$. Denote by $P_i, I_i, S_i, i = 1, 2, 3$, the indecomposable projective, injective, and the simple right $R$-modules. Indeed $P_1 = 1^2, P_2 = 1^2 2^3$, and $P_3 = 2^3$.
$R$ is quasi-hereditary with characteristic tilting module $T' = P_1 \oplus P_2 \oplus S_1$. The minimal left \text{add}$T'$-approximation of $R$ is given by the exact sequence

$$0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow 0$$

where $T_0 = P_1 \oplus (P_2)^2$ and $T_1 = \frac{2}{1}$. We consider the tilting module $T = T_0 \oplus T_1$ and set $\mathcal{U} = \{T_1\}$. By Remark on page 13 the $R$-module $R_\mathcal{U}$ can be computed as $T_0/\tau T_1(T_0)$ where $\tau T_1(T_0)$ denotes the trace of $T_1$ in $T_0$. It follows that $R_\mathcal{U} \cong S_1 \oplus (P_2/S_2)^2$, which has non-trivial self-extensions. We conclude that the universal localization at $\mathcal{U} = \{T_1\}$ is not a homological epimorphism.

Another example for a universal localization that is not a homological epimorphism is given in 29, where a ring of global dimension $\leq 2$ with universal localization of global dimension $\geq 3$ is constructed. The present example is quite different since $R_\mathcal{U}$ is hereditary and $\text{gldim}R = 4$.

6. Appendix: Construction of the triangulated reflection

Let $\mathcal{D} = \mathcal{D}(R)$ be the derived module category of a ring $R$, and $T_1$ an exceptional object in $\mathcal{D}$. Set $\mathcal{Y} = \text{Ker} \text{Hom}_{\mathcal{D}}(\text{Tria} T_1, -)$. We know from 1.2 and 1.5 that $\mathcal{X} = \text{Tria} T_1$ is a localizing subcategory of $\mathcal{D}$, thus the inclusion $\text{inc}: \mathcal{Y} \rightarrow \mathcal{D}$ has a left adjoint $q$. We want to calculate the $\mathcal{Y}$-reflection of $R$, that is, a morphism $R \rightarrow q(R)$ such that $q(R)$ lies in $\mathcal{Y}$ and the induced map $\text{Hom}_{\mathcal{D}}(q(R), \mathcal{Y}) \rightarrow \text{Hom}_{\mathcal{D}}(R, \mathcal{Y})$ is an isomorphism for any $Y \in \mathcal{Y}$.

First assume the endomorphism ring of $T_1$ is a skew-field and the extensions between $T_1$ and $R$ are free of finite rank over the skew-field.

**Proposition 6.1.** Suppose $T_1 \in \mathcal{D}$ is self-compact exceptional with endomorphism ring $k$ being a skew-field. Suppose further that the morphism spaces $\text{Hom}_{\mathcal{D}}(T_1, R[i])$, $i \in \mathbb{Z}$, are finite dimensional over $k$, and let $n_i = \text{dim}_k \text{Hom}_{\mathcal{D}}(T_1, R[i])$. Consider the canonical map $\alpha : S = \oplus_i T_1[-i] \oplus n_i \rightarrow R$ given by basis elements of these spaces. Then the cone of $\alpha$ is a $\mathcal{Y}$-reflection of $R$.

**Proof.** The triangle $S \xrightarrow{\alpha} R \rightarrow C \rightarrow$ gives rise to the long exact cohomology sequence

$$\cdots \rightarrow \text{Hom}_{\mathcal{D}}(T_1[-i], S) \xrightarrow{f} \text{Hom}_{\mathcal{D}}(T_1[-i], R) \rightarrow \text{Hom}_{\mathcal{D}}(T_1[-i], C) \rightarrow \cdots$$

Here, $\text{Hom}_{\mathcal{D}}(T_1[-i], S) = \text{Hom}_{\mathcal{D}}(T_1, S[i]) = \text{Hom}_{\mathcal{D}}(T_1, \oplus_j T_1[i - j] \oplus n_j)$. Since $T_1$ is self-compact and exceptional, $\text{Hom}_{\mathcal{D}}(T_1, \oplus_{j \neq i} T_1[i - j] \oplus n_j) = 0$. Hence $\text{Hom}_{\mathcal{D}}(T_1[-i], S) = \text{Hom}_{\mathcal{D}}(T_1, T_1 \oplus n_i)$ has dimension $n_i$ over $k$. Moreover, $\text{Hom}_{\mathcal{D}}(T_1[-i], R) = \text{Hom}_{\mathcal{D}}(T_1, R[i])$ also has dimension $n_i$ over $k$. By construction, $f$ is an isomorphism. Therefore, $\text{Hom}_{\mathcal{D}}(T_1[-i], C) = \text{Hom}_{\mathcal{D}}(T_1, C[i])$ vanishes for all $i$, which shows $C \in \mathcal{Y}$.

Given $Y \in \mathcal{Y}$, apply $\text{Hom}_{\mathcal{D}}(-, Y)$ to the triangle $S \rightarrow R \rightarrow C \rightarrow$. Since $\text{Hom}_{\mathcal{D}}(S[i], Y) = 0$ for all $i$, the induced map $\text{Hom}_{\mathcal{D}}(C, Y) \rightarrow \text{Hom}_{\mathcal{D}}(R, Y)$ is an isomorphism.

In general, we have the following method.

**Lemma 6.2.** Let $T_1 \in \mathcal{D}$ be a self-compact exceptional object and $M \in \mathcal{D}$. Suppose that there is $N \in \mathbb{Z}$ such that $\text{Hom}_{\mathcal{D}}(T_1, M[i]) = 0$ for all $i > N$. Then there exists a complex $M_1 \in \mathcal{D}$ and a map $M \rightarrow M_1$ such that the following holds:

(i) $\text{Hom}_{\mathcal{D}}(T_1, M_1[i]) = 0$ for all $i > N - 1$;

(ii) The induced map $\text{Hom}_{\mathcal{D}}(T_1, M[i]) \rightarrow \text{Hom}_{\mathcal{D}}(T_1, M_1[i])$ is an isomorphism for all $i \leq N - 2$,

and is injective for $i = N - 1$;

(iii) The induced map $\text{Hom}_{\mathcal{D}}(M_1, Y) \rightarrow \text{Hom}_{\mathcal{D}}(M, Y)$ is an isomorphism for all $Y \in \mathcal{Y}$. 
Proof. Consider the universal triangle \( T_1[-N][i] \overset{\alpha}{\to} M \to M_1 \) where \( \alpha \) is the canonical map induced by all elements of \( I = \text{Hom}_D(T_1, M[N]) = 0 \). Applying \( \text{Hom}_D(T_1, -) \) to the triangle, \( M_1 \) is seen to be as desired.

Without loss of generality we assume that \( N = 0 \). In this way we get a sequence of maps of complexes \( M = M_0 \overset{\sigma_0}{\to} M_1 \overset{\sigma_1}{\to} M_2 \overset{\sigma_2}{\to} \cdots \to M_n \overset{\sigma_n}{\to} \cdots \) such that

(i) \( \text{Hom}_D(T_1, M_n[0]) = 0 \) for all \( i > -n \);

(ii) The induced map \( \text{Hom}_D(T_1[i], M_n) \overset{(\sigma_n)_*}{\to} \text{Hom}_D(T_1[i], M_{n+1}) \) is an isomorphism for \( i \geq n + 2 \), and is injective for \( i = n + 1 \);

(iii) The induced map \( \text{Hom}_D(M_{n+1}, Y) \overset{(\sigma_n)^*}{\to} \text{Hom}_D(M_n, Y) \) is an isomorphism for all \( Y \in \mathcal{Y} \).

By definition, the homotopy colimit (see for example [30, 4.4.9]), here denoted by \( M_\infty \), is given (up to non-unique isomorphism) by the triangle

\[ \bigoplus_{n \geq 0} M_n \overset{1-\sigma}{\to} \bigoplus_{n \geq 0} M_n \overset{\pi}{\to} M_\infty \to \]

where \( 1-\sigma \) is defined by \((1, -\sigma_n)^*\) on the \( n \)-th component \( M_n \). The homotopy limit is defined dually by using direct products.

**Theorem 6.3.** Let \( T_1 \in \mathcal{D} \) be a compact exceptional object and \( M \in \mathcal{D} \). Suppose that there is \( N \in \mathbb{Z} \) such that \( \text{Hom}_D(T_1, M[i]) = 0 \) for all \( i > N \). Define \( M_n \) as above. Let \( \iota : \bigoplus_{n \geq 0} M_n \to M_\infty \) be the canonical embedding. Then \( \pi \circ \iota : M \to M_\infty \) is the \( \mathcal{Y} \)-reflection of \( M \).

**Proof.** The colimit \( M_\infty \) lies in \( \mathcal{Y} \) iff for each integer \( i \), the map

\[ (1-\sigma)_* : \text{Hom}_D(T_1[i], \bigoplus_{n \geq 0} M_n) \to \text{Hom}_D(T_1[i], \bigoplus_{n \geq 0} M_n) \]

is bijective. If \( i < 0 \) then by construction \( \text{Hom}_D(T_1[i], M_n) = 0 \) for all \( n \geq 0 \), and hence \( \text{Hom}_D(T_1[i], \bigoplus_{n \geq 0} M_n) = 0 \) is zero as \( T \) is compact. Now assume \( i \geq 0 \). It follows from the construction that \( \text{Hom}_D(T_1[i], M_n) = 0 \) for all \( n > i \). Hence \( \text{Hom}_D(T_1[i], \bigoplus_{n \geq 0} M_n) = 0 \)

\[ \text{Hom}_D(T_1[i], \bigoplus_{n \geq 0} M_n) = \text{Hom}_D(T_1[i], \bigoplus_{n = 0}^i M_n) = 0. \]

The map \( (1-\sigma)_* \) is given by

\[ (1-\sigma)_*(f_0, f_1, \ldots, f_i) = (f_0, f_1 - \sigma_0 \circ f_0, \ldots, f_i - \sigma_{i-1} \circ f_{i-1}). \]

It is straightforward now to see the bijectivity.

It remains to prove that \( \pi \circ \iota : M \to M_\infty \) is the reflection of \( M \), that is, for any \( Y \in \mathcal{Y} \), the induced map \( (\pi \circ \iota)^* : \text{Hom}_D(M_\infty, Y) \to \text{Hom}_D(M, Y) \), sending \( f \) to \( f \circ \pi \circ \iota \), is bijective.

Take any map \( g_0 : M = M_0 \to Y \). By the construction of \( M_n \), there exists uniquely for each \( n \geq 0 \) a map \( g_n : M_n \to Y \) such that \( g_{n-1} = g_n \circ \sigma_{n-1} \). Write \( g \) for the map \((g_n) : \bigoplus_{n \geq 0} M_n \to Y \). Then \( g_0 = \iota^*(g) = g \circ \iota \). Apply \( \text{Hom}_D(-, Y) \) to the triangle

\[ \bigoplus_{n \geq 0} M_n \overset{1-\sigma}{\to} \bigoplus_{n \geq 0} M_n \overset{\iota}{\to} M_\infty \to \]

to obtain a long exact sequence

\[ \cdots \to \text{Hom}_D(\bigoplus_{n \geq 0} M_n, Y[-1]) (1-\sigma)^* \to \text{Hom}_D(\bigoplus_{n \geq 0} M_n, Y[-1]) \to \text{Hom}_D(M_\infty, Y) \to \]

\[ \text{Hom}_D(\bigoplus_{n \geq 0} M_n, Y) (1-\sigma)^* \to \text{Hom}_D(\bigoplus_{n \geq 0} M_n, Y) \to \cdots \]

where \( (1-\sigma)^*(h_n) = (h_n - \iota_{n+1} \circ \sigma_n)_n \). It is clear that the map \( g \) constructed above lies in the kernel of \( (1-\sigma)^* \), and \( \iota^* : \text{Hom}_D(\bigoplus_{n \geq 0} M_n, Y) \to \text{Hom}_D(M, Y) \), when restricted on \( \text{Ker}(1-\sigma)^* = \text{Im} \pi^* \), becomes a bijection onto \( \text{Hom}_D(M, Y) \).
Note that the map \((1 - \sigma)^* : \text{Hom}_D(\oplus_{n \geq 0} M_n, Y[-1]) \to \text{Hom}_D(\oplus_{n \geq 0} M_n, Y[-1])\) is surjective, because all \((\sigma_n)^* : \text{Hom}_D(M_{n+1}, Y[-1]) \to \text{Hom}_D(M_n, Y[-1]), n \geq 0,\) are isomorphisms. Hence \(\pi^* : \text{Hom}_D(M_\infty, Y) \to \text{Hom}_D(\oplus_{n \geq 0} M_n, Y)\) is an injection. Combining this with the arguments above, we obtain the bijectivity of \((\pi \circ \iota)^* : \text{Hom}_D(M_\infty, Y) \to \text{Hom}_D(M, Y)\). \(\square\)

In the situation of Theorem [4.8], this method can be used for computing \(q(R)\), the \(\mathcal{Y}\)-reflection of \(R\). In particular, it follows immediately that \(q(R)\) is right bounded, namely it belongs to \(\mathcal{D}^-(R)\).

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