Some New Harmonically Convex Function Type Generalized Fractional Integral Inequalities

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1. Introduction

In the present era, fractional integral operators involving inequalities are widely derived by [1–4]. These fractional integral operators of any arbitrary real or complex order involve a different type of kernel. The field of fractional calculus has gained considerable importance among mathematicians and scientists due to its wide applications in sciences, engineering, and many other fields [5–9]. Hadamard and Fejér–Hadamard type inequalities have been discussed for many functions using different fractional operators with different kernels. Abbas and Farid [10] proposed the Hadamard and Fejér–Hadamard type inequalities for harmonically convex functions using the two-sided generalized fractional integral operator. Farid et al. [11,12] discussed these results in generalized form with an extended generalized Mittag–Leffler function. Hadamard and Fejér–Hadamard type inequalities are widely studied by the researchers [12–19]. The objective of this paper is to derive Hadamard, Fejér–Hadamard, and some other related type inequalities for the harmonically convex function via a generalized fractional operator with a nonsingular function as its kernel, which involves a multi-index Bessel function. For a recent related weighted fractional generalized approach, we refer to [20].

Hermite–Hadamard inequality and Fejér–Hadamard inequality are given by
Theorem 1 ([21–23]). The inequality derived on the interval $I = [u, v] \subseteq \mathbb{R}$ called Hermite Hadamard inequality is given by

$$\rho \left[ \frac{u + v}{2} \right] \leq \frac{1}{v - u} \int_u^v \rho(t) dt \leq \frac{\rho(u) + \rho(v)}{2},$$

(1)

where $u, v \in I$, with $u \neq v$ and $\rho : I \rightarrow \mathbb{R}$ is a convex function.

Theorem 2 ([21, 24, 25]). The Fejér–Hadamard inequality is defined for a convex function $\rho : I \rightarrow \mathbb{R}$ and for a function $\mu : I \rightarrow \mathbb{R}$, which is non-negative, integrable, and symmetric about $\frac{u + v}{2}$, defined by

$$\rho \left[ \frac{u + v}{2} \right] \int_u^v \mu(t) dt \leq \int_u^v \rho(t) \mu(t) dt \leq \left[ \frac{\rho(u) + \rho(v)}{2} \right] \int_u^v \mu(t) dt,$$

(2)

where $u, v \in I$, with $u \neq v$.

Definition 1 ([21, 26]). A function $\rho : [u, v] \rightarrow \mathbb{R}$ is said to be convex if

$$\rho \left[ tx + (1 - t)y \right] \leq t \rho(x) + (1 - t) \rho(y)$$

(3)

holds for all $x, y \in [u, v]$ and $t \in [0, 1]$.

Definition 2 ([21, 22]). Let $I$ be an interval of nonzero real numbers. Then a function $\rho : I \rightarrow \mathbb{R}$ is said to be harmonically convex if

$$\rho \left[ \frac{u + v}{2} \right] \int_u^v \mu(t) dt \leq \int_u^v \rho(t) \mu(t) dt \leq \left[ \frac{\rho(u) + \rho(v)}{2} \right] \int_u^v \mu(t) dt,$$

(4)

holds for all $u, v \in I$ and $t \in [0, 1]$.

Definition 3 ([21, 27]). A function $\rho : [u, v] \rightarrow \mathbb{R}$ where $I \subset \mathbb{R}$ contains nonzero real numbers is said to be harmonically symmetric about $\frac{u + v}{2}$ if

$$\rho \left[ \frac{1}{t} \right] = \rho \left[ \frac{1}{\frac{u + (1 - t)v}{v}} \right].$$

(5)

t \in [u, v]

Definition 4 ([28, 29]). The Pochhammer’s symbol is defined for $s \in \mathbb{N}$ as

$$(\mu)_s = \begin{cases} 1, & \text{for } s = 0, \mu \neq 0, \\ \mu(\mu + 1) \cdots (\mu + s - 1), & \text{for } s \geq 1, \end{cases}$$

(6)

where $\mu \in \mathbb{C}$.

Definition 5 ([30]). The generalized multi-index Bessel function defined by Choi et al. as follows;

$$J_{(\gamma)_{\lambda_{\sigma}}}^{(\gamma)_{\lambda_{\sigma}}} (t) = \sum_{s=0}^{\infty} \frac{(\lambda)_s}{(s!) \prod_{j=1}^{m} \Gamma(\gamma_j s + \tau_j + 1)} (-t)^s,$$

(7)

where $\gamma_j, \tau_j, \lambda \in \mathbb{C}, j = 1, 2, 3 \cdots m, \Re(\lambda) > 0, \Re(\tau_j) > -1, \sum_{j=1}^{m} \Re(\gamma_j) > \max(0 : \Re(\sigma) - 1), \sigma > 0.$
We define the following generalized fractional integral with a nonsingular function (generalized multi-index Bessel function) as a kernel.

**Definition 6.** The generalized fractional integral operators (left and right-sided) containing the multi-index Bessel function in its kernel are, respectively, defined by

\[
\left(\mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}\right)(z) = \int_{a}^{b} (z-t)^{\gamma_j} f(t)^{\tau_j} \rho(t) dt
\]

and

\[
\left(\mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}\right)(z) = \int_{z}^{b} (t-z)^{\gamma_j} f(t)^{\tau_j} \rho(t) dt,
\]

where \(\gamma_j, \tau_j, \lambda, \xi \in \mathbb{C}, j = 1, 2, 3 \cdots m, \Re(\lambda) > 0, \Re(\tau_j) > -1, \sum_{j=1}^{m} \Re(\gamma_j) > \max(0 : \Re(\sigma) - 1), \sigma > 0 \) and \(\rho \in L[u, v], t \in [u, v]\).

**Remark 1.** If we put \(\zeta = 0, m = 1\) and replace \(\tau_j\) by \(\tau_j - 1\), it reduces to left and right-sided Riemann–Liouville fractional integral operator.

**2. Main Results**

In this section, we present Hadamard, and Fejér–Hadamard type inequalities for harmonically convex functions by employing the new generalized fractional integral operators with a multi-index Bessel function as its kernel. We also establish a new version of inequalities by expressing the generalized fractional integral operator as the sum of two fractional integrals.

**Theorem 3.** Let \(\theta, \psi : [a, b] \rightarrow \mathbb{R}, (0 < a < b, \text{range}(\psi) \subset [a, b])\) be functions such that \(\theta \in L_{1}[a, b]\) is a positive and harmonically convex function and \(\psi\) is differentiable and strictly increasing on \([a, b]\), then for the integral operators defined in Definition 6, we have

\[
\theta \left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \left(\frac{1}{\psi(b)}\right) \left(\mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}\right) \leq \frac{1}{2} \left\{ \left(\frac{1}{\psi(a)}\right) + \left(\frac{1}{\psi(b)}\right) \right\}
\]

where \(\theta(x) = \frac{1}{x}\) for all \(x \in \left[\frac{1}{b}, \frac{1}{a}\right]\).

**Proof.** If \(\theta\) is harmonically convex on \([a, b]\), for every \(x, y \in [a, b]\), the following inequality holds

\[
\theta \left(\frac{2\psi(x)\psi(y)}{\psi(x) + \psi(y)}\right) \leq \theta(\psi(x)) + \theta(\psi(y)).
\]

Now, taking \(\psi(x) = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\) and \(\psi(y) = \frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\) in Equation (11), we have

\[
2\theta \left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \leq \theta \left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right) + \theta \left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\right).
\]
By multiplying by \((1 - t)^{\gamma_j}\) and then integrating over \([0, 1]\), we get
\[
2\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \int_0^1 (1 - t)^{\gamma_j} \int (1 - t)^{\gamma_j} \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} dt
\leq \int_0^1 (1 - t)^{\gamma_j} \int (1 - t)^{\gamma_j} \left\{ \theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1 - t)\psi(a)} \right) \right\} dt
\]

\[
2\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \left[ \frac{1}{\psi(a)} - \frac{1}{\psi(b)} \right] \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(b)} \right)
\leq \sum_{n=0}^{\infty} \frac{(\lambda)^{\sigma} (-\zeta)^{\xi}}{n! \prod_{j=1}^{n} \Gamma (\gamma_j + \tau_j + 1)} \left[ \int_0^1 (1 - t)^{\tau_j + \gamma_j} \psi \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) dt \right]
\]

\[
\int_0^1 (1 - t)^{\gamma_j} \psi \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) dt.
\]

Solving the integrals involved in right side of inequality (13) by making substitution
\[
\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \text{ in first integral and } \frac{1}{v} = \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \text{ in the second integral, we have}
\]

\[
\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(b)} \right)
\leq \frac{1}{2} \left[ \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(b)} \right) + \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(a)} \right) \right].
\]

To obtain the second part of the inequality, the harmonic convexity of \(\theta\), we have the following relation
\[
\theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) \leq \theta(\psi(a)) + \theta(\psi(b)).
\]

Multiplying by \((1 - t)^{\gamma_j}\) and integrating over \([0, 1]\) in Equation (15), we have
\[
\int_0^1 (1 - t)^{\gamma_j} \psi \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) \psi 
\leq \int_0^1 (1 - t)^{\gamma_j} \psi \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \right) \psi dt
\]

\[
\leq \left[ \theta(\psi(a)) + \theta(\psi(b)) \right] \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(a)} \right).
\]

Solving the integrals involved in left side of inequality (16) by making substitution
\[
\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(a)} \text{ in first integral and } \frac{1}{v} = \frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)} \text{ in the second integral, we obtain}
\]

\[
\frac{1}{2} \left[ \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(b)} \right) + \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(a)} \right) \right]
\leq \left[ \theta(\psi(a)) + \theta(\psi(b)) \right] \mathcal{F}^{(\gamma_j, \gamma_j)}_{\lambda, \nu, \zeta, \xi, \mu} \left( \frac{1}{\psi(a)} \right).
\]

Combining (14) and (17), we get the desired result. □
Corollary 1. If $\psi(x) = \frac{1}{x}$ in Theorem 3 then the following inequality holds

$$\theta \left[ \frac{2}{a + b} \left( \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{x} \right) \right) \right] \leq \frac{1}{2} \left[ \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{a} \right) + \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{b} \right) \right]$$

$$\leq \left(\frac{\theta(\frac{a}{x}) + \theta(\frac{b}{x})}{2}\right) \left[ \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{x} \right) \right].$$

(18)

Now, we derive the following Lemma before giving the next result.

Lemma 1. Let $\theta, \psi : [a, b] \rightarrow \mathbb{R}, 0 < a < b, \text{range}(\psi) \subset [a, b]$ be functions such that $\theta$ is positive, $\theta \in L_1[a, b]$, and $\psi$ is differentiable and strictly increasing. If $\theta$ is a harmonically convex function on $[a, b]$ and satisfies $\theta \left( \frac{1}{\psi(x)} \right) = \theta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$, we have

$$\mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(a)} \right) \right) = \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(b)} \right) \right),$$

(19)

where $\mu(x) = \frac{1}{x}, \forall x \in \left[ \frac{1}{a}, \frac{1}{b} \right]$.

Proof. Consider

$$\mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(a)} \right) \right) = \int \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(a)} \right) \right) \frac{1}{\psi(x)} \, du.$$

(20)

Putting $u = \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x)$ and using $\theta \left( \frac{1}{\psi(x)} \right) = \theta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$ in Equation (20), we have

$$\mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(a)} \right) \right) = \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(b)} \right) \right).$$

(21)

By the addition of $\mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \theta \circ \mu \left( \frac{1}{\psi(a)} \right) \right)$ in Equation (21) on both sides, we have the required result. \(\square\)

Theorem 4. Let $\theta, \psi, \eta : [a, b] \rightarrow \mathbb{R}, 0 < a < b, \text{range}(\psi), \text{range}(\eta) \subset [a, b]$, be functions such that $\theta \in L_1[a, b]$ is a positive function, $\psi$ is a differentiable and strictly increasing function and $\eta$ is nonnegative and integrable and satisfies $\eta \left( \frac{1}{\psi(x)} \right) = \eta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$, then the following inequality holds

$$\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \left[ \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{\psi(a)} \right) + \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{\psi(b)} \right) \right]$$

$$\leq \left[ \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{\psi(a)} \right) + \mathcal{F}_{\lambda, \sigma, \xi}^{(\gamma)} \left( \frac{1}{\psi(b)} \right) \right] \eta \circ \mu \left( \frac{1}{\psi(a)} \right)$$

(22)
where $\mu(x) = \frac{1}{x}, \forall x \in [\frac{1}{\alpha}, 1], \theta \eta \circ \mu = (\theta \circ \mu)(\eta \circ \mu)$.

**Proof.** By using the harmonic convexity of $\theta$, we have

$$
2\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \leq \theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)} \right). 
$$

(23)

By multiplying by $(1-t)^{\gamma_j + \tau_j} \eta\left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right)$ in Equation (23) and then integrating over the closed interval $[0, 1]$, we have

$$
2\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \int_0^1 (1-t)^{\gamma_j + \tau_j} \eta\left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) dt \leq \int_0^1 (1-t)^{\gamma_j + \tau_j} \eta\left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) dt 
$$

(24)

$$
\times \left[ \theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)} \right) \right] dt 
$$

(25)

By making a substitution of \( \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \) in the first integral and \( \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)} \) in second integrals occurring at right side and \( \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \) in the integral occurring at left side of inequality (25) and using \( \eta\left( \frac{1}{\psi(x)} \right) = \eta\left( \frac{1}{\psi(x)} \right) \), we have

$$
\theta \left( \frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)} \right) \left[ \mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}(\frac{1}{\psi(x)}) - \eta \circ \mu \left( \frac{1}{\psi(b)} \right) + \mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}(\frac{1}{\psi(a)}) - \eta \circ \mu \left( \frac{1}{\psi(a)} \right) \right] 
$$

(26)

$$
\leq \left[ \mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}(\frac{1}{\psi(b)}) - \eta \circ \mu \left( \frac{1}{\psi(b)} \right) + \mathcal{J}_{\lambda,\sigma}^{(\gamma_j,\tau_j)}(\frac{1}{\psi(a)}) - \eta \circ \mu \left( \frac{1}{\psi(a)} \right) \right]. 
$$

Now, we take

$$
\theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)} \right) \leq \theta(\psi(a)) + \theta(\psi(b)). 
$$

(27)
By multiplying \((1 - t)^{\tau_j}f(x)\psi_n(x)\) in Equation (27) and then integrating over \([0, 1]\), we get

\[
\begin{align*}
\int_0^1 (1 - t)^{\tau_j}f(x)\psi_n(x) \eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1 - t)\psi(a)}\right) dt
+ \int_0^1 (1 - t)^{\tau_j}f(x)\psi_n(x) \eta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)}\right) dt
\leq (\theta(\psi(a)) + \theta(\psi(b))) \int_0^1 (1 - t)^{\tau_j}f(x)\psi_n(x) \eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1 - t)\psi(a)}\right) dt.
\end{align*}
\] (28)

Solving the integrals involved in left side of inequality (28) by making substitution

\[
\int_0^1 (1 - t)^{\tau_j}f(x)\psi_n(x) \eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1 - t)\psi(a)}\right) dt = \frac{1}{u} \psi(a)\psi(b)\left(\frac{1}{t\psi(b) + (1 - t)\psi(a)}\right)
\] in the first integral and

\[
\int_0^1 (1 - t)^{\tau_j}f(x)\psi_n(x) \eta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)}\right) dt = \frac{1}{u} \psi(a)\psi(b)\left(\frac{1}{t\psi(a) + (1 - t)\psi(b)}\right)
\] in the integral on the right side of the inequality and using \(\eta\left(\frac{1}{\psi(x)}\right) = \eta\left(\frac{1}{\psi_0(x) + \frac{1}{\psi_0(x)} - \psi(x)}\right)\), we have

\[
\left[\mathcal{J}_{\lambda, \psi_n(z)}(\frac{1}{\psi_0})\right] \theta \eta \circ \mu \left(\frac{1}{\psi(b)}\right) + \left[\mathcal{J}_{\lambda, \psi_n(z)}(\frac{1}{\psi_0})\right] \theta \eta \circ \mu \left(\frac{1}{\psi(a)}\right)
\] (29)

Combining (26) and (29), we have the required result. \(\square\)

**Theorem 5.** Let \(\theta, \psi : [a, b] \to \mathbb{R}, (0 < a < b, \text{range}(\psi) \subset [a, b])\) be functions, such that \(\theta \in L_1[a, b]\) is a positive and harmonically convex function and \(\psi\) is differentiable and strictly increasing, then the following inequality holds for the operators defined in Definition 6

\[
\theta\left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right)\left(\mathcal{J}_{\lambda, \psi_n(z)}(\frac{\psi(a) + \psi(b)}{2\psi(1)})\right) - 1\left(\frac{1}{\psi(b)}\right)
\leq \frac{1}{2}\left[\left(\mathcal{J}_{\lambda, \psi_n(z)}(\frac{\psi(a) + \psi(b)}{2\psi(1)})\right) \theta \circ \mu \left(\frac{1}{\psi(b)}\right) + \left(\mathcal{J}_{\lambda, \psi_n(z)}(\frac{\psi(a) + \psi(b)}{2\psi(1)})\right) \theta \circ \mu \left(\frac{1}{\psi(a)}\right)\right]
\] (30)

where \(\mu(x) = \frac{1}{2} \forall x \in [\frac{1}{2}, \frac{3}{2}]\)

**Proof.** We have

\[
2\theta\left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \leq \theta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1 - t)\psi(a)}\right) + \theta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1 - t)\psi(b)}\right).
\] (31)
By multiplying \((1-t)^{\gamma_j} f_{(\xi_j^m a)} (\zeta (1-t)^{\gamma_i})\) on both sides and then integrating over \([\frac{1}{2}, 1]\), we get

\[
\int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j} f_{(\xi_j^m a)} (\zeta (1-t)^{\gamma_i}) dt \leq \int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j} \left\{ \theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)} \right) \right\} dt
\]

Solving the integrals involved in the right side of inequality (32) by making a substitution of \(\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\) in the first integral and \(\frac{1}{v} = \frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)}\) in the second integral as well as in the integral occurring at the left side of inequality (32), we have

\[
\theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) \leq \frac{1}{2} \left[ \mathcal{G}_{\lambda, \sigma, \zeta}^{(\gamma_j, \gamma_i)} (\frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)}) + \mathcal{G}_{\lambda, \sigma, \zeta}^{(\gamma_j, \gamma_i)} (\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}) \right].
\]

To obtain the second part of inequality, the harmonic convexity of \(\theta\) gives the following relation

\[
\theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) + \theta \left( \frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)} \right) \leq \theta(\psi(a)) + \theta(\psi(b)).
\]

Multiplying by \((1-t)^{\gamma_j} f_{(\xi_j^m a)} (\zeta (1-t)^{\gamma_i})\) and integrating over \([\frac{1}{2}, 1]\), we get

\[
\int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j} \left( \psi(a)\psi(b) \right) \theta \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) dt + \int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j} \left( \psi(a)\psi(b) \right) \theta \left( \frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)} \right) dt \leq \left[ \theta(\psi(a)) + \theta(\psi(b)) \right] \mathcal{G}_{\lambda, \sigma, \zeta}^{(\gamma_j, \gamma_i)} (\frac{\psi(a)\psi(b)}{2\psi(a) + (1-t)\psi(b)}).
\]

Combining (33) and (36), we have the result. \(\square\)
Remark 2. 1. If $\psi(x) = x$, $m = 1$, $\xi = 0$ and $\tau_j$ is replaced by $\tau_j - 1$, it reduces to the result produced by Mehmet et al. [31]

$$\theta \left( \frac{2ab}{a+b} \right) \leq \frac{1}{2} \left( \frac{b}{a} \right) \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} $$

where $\mu(x) = \frac{1}{x}$, $\forall x \in \left[ \frac{1}{2}, \frac{3}{2} \right]$

Lemma 2. Let $\theta, \psi : [a, b] \to \mathbb{R}$, $0 < a < b$, $\text{range}(\psi) \subset [a, b]$ be functions such that $\theta > 0$, $\theta \in L_1[a, b]$, and $\psi$ is differentiable and strictly increasing. If $\theta$ is a harmonically convex function on $[a, b]$ and satisfies $\theta \left( \frac{1}{\psi(x)} \right) = \theta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$, we have

$$\mathcal{J}_{\lambda, \sigma, \xi, \zeta}^{(\gamma, \tau)}_{\lambda, \sigma, \xi, \zeta} \left( \frac{\psi(a) + \psi(b) + \psi(x)}{\psi(a) + \psi(b) + \psi(x)} \right) = \frac{1}{2} \left[ \mathcal{J}_{\lambda, \sigma, \xi, \zeta}^{(\gamma, \tau)}_{\lambda, \sigma, \xi, \zeta} \left( \frac{\psi(a) + \psi(b) + \psi(x)}{\psi(a) + \psi(b) + \psi(x)} \right) \right] \right]$$

Substituting $u = \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x)$ and using $\theta \left( \frac{1}{\psi(x)} \right) = \theta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$ in Equation (38), we have

$$\mathcal{J}_{\lambda, \sigma, \xi, \zeta}^{(\gamma, \tau)}_{\lambda, \sigma, \xi, \zeta} \left( \frac{\psi(a) + \psi(b) + \psi(x)}{\psi(a) + \psi(b) + \psi(x)} \right) = \mathcal{J}_{\lambda, \sigma, \xi, \zeta}^{(\gamma, \tau)}_{\lambda, \sigma, \xi, \zeta} \left( \frac{\psi(a) + \psi(b) + \psi(x)}{\psi(a) + \psi(b) + \psi(x)} \right)$$

By the addition of $\mathcal{J}_{\lambda, \sigma, \xi, \zeta}^{(\gamma, \tau)}_{\lambda, \sigma, \xi, \zeta} \left( \frac{\psi(a) + \psi(b) + \psi(x)}{\psi(a) + \psi(b) + \psi(x)} \right)$ in Equation (39) on both sides, we have the required result. □

Theorem 6. Let $\theta, \psi, \eta : [a, b] \to \mathbb{R}$, $0 < a < b$, $\text{range}(\psi), \text{range}(\eta) \subset [a, b]$ be functions such that $\theta \in L_1[a, b]$ is a positive function, $\psi$ is a differentiable, strictly increasing function and $\eta$ is nonnegative and integrable and satisfies $\eta \left( \frac{1}{\psi(x)} \right) = \eta \left( \frac{1}{\psi(a)} + \frac{1}{\psi(b)} - \psi(x) \right)$, then the following inequality holds for the operators defined in Definition 6.

$$\theta \left( \frac{2ab}{a+b} \right) \leq \frac{1}{2} \left( \frac{b}{a} \right) \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} \left( \frac{b}{a} \right)^{-1} $$

where $\mu(x) = \frac{1}{x}$, $\forall x \in \left[ \frac{1}{2}, \frac{3}{2} \right]$, $\theta \eta \circ \mu = (\theta \circ \mu) (\eta \circ \mu)$. 

Proof. By the harmonic convexity of $\theta$, we have

$$2\theta\left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \leq \theta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right) + \theta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\right). \tag{41}$$

By multiplying $(1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)$ in the equation (41) and then integrating over the closed interval $[\frac{1}{2}, 1]$, we get

$$2\theta\left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)dt \leq \int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)dt \times \left[\theta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right) + \theta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\right)\right] \int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)dt \times \left[\theta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right) + \theta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\right)\right]. \tag{42}$$

By substituting $\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}$ in the first integral and $\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}$ in the second integrals occurring at the right side and $\frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}$ in the integral occurring at left side of inequality (42), we have

$$\theta\left(\frac{2\psi(a)\psi(b)}{\psi(a) + \psi(b)}\right) \nabla_{\lambda, \varphi, \xi, \eta, \gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{1}{\psi(a)}\right) + \nabla_{\lambda, \varphi, \xi, \eta, \gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{1}{\psi(b)}\right) \leq \left[\nabla_{\lambda, \varphi, \xi, \eta, \gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{1}{\psi(a)}\right) + \nabla_{\lambda, \varphi, \xi, \eta, \gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{1}{\psi(b)}\right)\right]. \tag{43}$$

Now, we take

$$\theta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right) + \theta\left(\frac{\psi(a)\psi(b)}{t\psi(a) + (1-t)\psi(b)}\right) \leq \theta(\psi(a)) + \theta(\psi(b)). \tag{44}$$

By multiplying $(1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)$ in Equation (44) and then integrating over $[\frac{1}{2}, 1]$, we get

$$\int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)dt \leq \theta(\psi(a)) + \theta(\psi(b)) \tag{45}$$

$$\int_{\frac{1}{2}}^{1} (1-t)^{\gamma_j}_{(\gamma_t)_{F_{\lambda\gamma}}}(\zeta(1-t)^{\gamma_1})\eta\left(\frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)}\right)dt \leq \theta(\psi(a)) + \theta(\psi(b)) \tag{45}$$
Solving the integrals involved in left side of inequality (45) by making substitution 
\[ \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \]
in the first integral and 
\[ \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \]
in the second integral and 
\[ \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \]
in the integral on the right side of the inequality and using the above lemma and the condition 
\[ \frac{1}{u} = \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \]
we have

\[ \left[ J^{(\gamma;\tau)}_{\lambda;x} \left( \frac{\psi(a)\psi(b)}{t\psi(b) + (1-t)\psi(a)} \right) \right] + \left( \theta(\psi(a)) + \theta(\psi(b)) \right) \frac{1}{2} \left[ \frac{1}{t\psi(b) + (1-t)\psi(a)} \right] \frac{1}{(\psi(a))} \ \eta \circ \mu \left( \frac{1}{(\psi(a))} \right) \]

Combining (43) and (46), we have the required result. □

3. Conclusion Remarks

In this article, we established Hadamard and Fejér–Hadamard type inequalities via a new generation of the generalized fractional integral operators (8) and (9) with a non-singular function (multi-index Bessel function) as its kernel for harmonically convex functions. It is concluded that many classical inequalities cited in the literature can be easily derived by employing certain conditions on generalized fractional integral operators (8) and (9). We believe that our formulated inequalities will be useful to investigate the stability of certain fractional controlled systems.

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