Numerical analysis of one dimensional heat transfer on varying metal

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Abstract. Heat is an energy that we often encounter in our daily lives. Heat can be transferred from one medium to another, such as a solid medium. Heat transfer in a solid is called heat conduction or diffusion. Numerical analysis is one approach used to solve differential diffusion in many cases. In this paper, the concern is the proposed finite difference method to simulate one-dimensional heat transfer on varying metals. This numerical method utilizes the Neumann boundary conditions as well as the Taylor series in finding differential diffusion solutions. The solution obtained was applied and simulated in the case of heat transfer by conduction on various metals. From this simulation, we can obtain data in the form of temperature distribution across various metals by adjusting to boundary conditions. Then, the distribution is used to predict when various metals reach their equilibrium temperature. The final equilibrium temperature on varying metal must satisfy Thermodynamics Law. In order to illustrate the accuracy, the varying boundary conditions are presented. The results obtained in the form of temperature distribution will be simulated with the help of the MATLAB program to obtain conclusions from the objects of this paper. The conclusion indicates that using finite difference is accurate in some boundary conditions.

1. Introduction

Heat is a phenomenon that is inseparable from daily life. Heat is included in the energy category, so heat can move from one medium to another. Heat transfer is a transfer of heat energy that can occur due to temperature differences in a system. This heat transfer occurs in a high-temperature system to a low-temperature system. This heat transfer can occur in a medium, one of which is a solid medium. Heat transfer in a solid medium can be called conduction or diffusion. One of the solid media that is good at delivering heat energy is metal. Some materials including metal, including iron, aluminum, manganese, and so on. The fundamental form of the diffusion equation can be written by,

$$\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

The diffusion is an interesting phenomenon to be studied, where diffusion is related to many events in science[1–5]. Based on the importance of the diffusion, the solution of a diffusion equation is widely studied by several researchers[1–8]. The method that can be used to solve a diffusion equation is a numerical method, such as the finite difference method. The finite difference method uses the Taylor series approach to find solutions for diffusion differential equations. This method is assisted with
Neumann boundary conditions [6−8] which state the value of the diffusion differential at the limits of a system is zero, it can help it.

The purpose of this article is to discuss the application of finite difference methods to metal materials, such as metal. This example presents the solutions in the form of a graph of temperature distribution over time. Then, from the graph, we can predict when the metal material reaches its equilibrium point with a finite difference method.

2. Diffusion Equation with Finite Difference Method

The fundamental form of the diffusion equation (1) can be written by,

\[
\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \quad t \geq 0
\]  

(1)

where \( u = u(x, t) \) is the dependent variable, and \( \alpha \) is a constant coefficient. Equation (1) shows the phenomenon of diffusion in the objects that are reviewed in one dimension or slab. Where the slab has a length from 0 to \( L \) and takes place in the span of 0 to \( t \).

Equation (1) can also be expressed as a simple function of being,

\[ u_t = K u_{xx} \]  

(2)

Using the transformation of finite difference method accuracy up to the Taylor approach, it can be expressed by,

\[
\frac{u(x,t+\Delta t)−u(x,t)}{\Delta t} = K \frac{u(x+\Delta x,t)−2u(x,t)+u(x−\Delta x,t)}{\Delta x^2}
\]  

(3)

Explicitly equation (3) can also be written as,

\[
\frac{u(x_i+t_{n+1})−u(x_i,t_n)}{\Delta t} = K \frac{u(x_{i+1},t_{n})−2u(x_i,t_n)+u(x_{i−1},t_n)}{\Delta x^2}
\]  

(4)

Then, substitute \( x_i + \Delta x = x_{i+1} \) and \( t_n + \Delta t = t_{n+1} \),

\[
\frac{1}{\Delta t}[u_{i+1}^{n+1} − u_{i}^{n}] = K \frac{1}{\Delta x^2}[u_{i+1}^{n} − 2u_{i}^{n} + u_{i−1}^{n}]
\]  

(5)

\[
u_{i}^{n+1} − u_{i}^{n} = \frac{s}{\Delta x^2}[u_{i+1}^{n} − 2u_{i}^{n} + u_{i−1}^{n}]
\]  

(6)

Where \( s = \frac{K \Delta t}{\Delta x^2} \), then the solution can be written as,

\[
u_{i}^{n+1} = s[u_{i+1}^{n} − 2u_{i}^{n} + u_{i−1}^{n}] + u_{i}^{n}
\]  

(7)

3. Stability Requirements for the Finite Difference Method to the Heat Equation

3.1. Analysis of Fourier-von Neumann stability

Based on equation (1) which is written as a diffusion differential equation solution,

\[
u_{i}^{n+1} = s[u_{i+1}^{n} − 2u_{i}^{n} + u_{i−1}^{n}] + u_{i}^{n}
\]

where the initial conditions are written with,

\[
u_{i}^{0} = u(x, 0) = f(x_i)
\]

and the boundary conditions are,

\[
u_{0}^{n} = u(0, t) = 0
\]

\[
u_{N}^{n} = u(N, t) = 0
\]

Where \( s = \frac{K \Delta t}{\Delta x^2} \), \( x_i = \Delta x \), \( t = n\Delta t \).

If we suppose that,

\[
u_{i}^{n} = e^{i\alpha x}Q_{i}^{n} = e^{i\alpha \Delta x}Q_{i}^{n}
\]

(8)

By substituting equation (8) for equation (7), we get that

\[
e^{i\alpha \Delta x}Q_{i+1}^{n+1} − e^{i\alpha \Delta x}Q_{i}^{n} = s[e^{i\alpha (i+1)\Delta x}Q_{i}^{n} − 2e^{i\alpha \Delta x}Q_{i}^{n} + e^{i\alpha (i−1)\Delta x}Q_{i}^{n}]
\]
\[ e^{i\alpha\Delta x}(Q - 1) = s\left[e^{i\alpha\Delta x} Q^n (e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x})\right] \]

Then both segments are multiplied by \( \frac{1}{e^{i\alpha\Delta x} Q^n} \),

\[
(Q - 1) = s\left(e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x}\right)
\]

\[
(Q - 1) = s\left((e^{i\alpha\Delta x} + e^{-i\alpha\Delta x}) - 2\right)
\]

\[
(Q - 1) = s(2\cos(\alpha\Delta x) - 2)
\]

\[
Q = 1 - 2s(1 - \cos(\alpha\Delta x))
\]

The solution of equation (7) with the boundary conditions \( u_0^n = 0 \) and \( u_N^n = 0 \) is

\[
u_i^n = \sin \frac{m\pi x}{L} \frac{t}{Q^n}
\]

Where,

\[
Q = 1 - 2s(1 - \cos(\alpha\Delta x))
\]

and \( \alpha = \frac{m\pi}{L} \), \( m = 1,2,3 \ldots \)

3.2. Provisions of Solution Stability

The solution is said to be stable if,

\[
|Q| \leq 1
\]

\[
|1 - 2s(1 - \cos(\alpha\Delta x))| \leq 1
\]

Then,

\[
-1 \leq 1 - 2s(1 - \cos(\alpha\Delta x)) \leq 1
\]

\[
-2 \leq -2s(1 - \cos(\alpha\Delta x)) \leq 0
\]

\[
0 \leq s(1 - \cos(\alpha\Delta x)) \leq 1
\]

Note that,

\[
0 \leq 1 - \cos(\alpha\Delta x) \leq 1
\]

Then,

\[
0 \leq s \leq \frac{1}{2}
\]

So, the stability of the solution requirements for finite difference methods must comply with the equation,

\[
\frac{1}{2} \leq \frac{K.\Delta t}{\Delta x^2}
\]

4. Analysis of Temperature Distribution in Metals

In this study, several metals are used as independent variables. We know that metal has different thermal conductivity values. By applying the different methods up to this point, we can find out the temperature distribution in each metal that is used as the object of research. From this temperature distribution, we can obtain a variety of desired data, one of which is the time of each heat transfer in the metal to reach its equilibrium temperature.

From equation (7), we will be used that equation to solve the temperature distribution problems in several metals. The solution of this settlement will be reviewed in one dimension. By using the Neumann boundary conditions, the limits of the temperature distribution are determined. Where the Neumann boundary states that,

\[
\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(l,t)}{\partial x} = 0
\]

for each \( t > 0 \).

Then, based on table 1 [9,10], it can be seen that each metal has a different thermal conductivity value will affect the temperature distribution produced, either graphically or data. The thermal conductivity of each metal will be substituted into equation (7) in the finite difference method.
Table 1. Thermal Conductivity of Metals at 200K

| Metal      | Thermal Conductivity, Watt m⁻¹ K⁻¹ |
|------------|-----------------------------------|
| Aluminum   | 237                               |
| Copper     | 413                               |
| Iron       | 94                                |
| Manganin   | 17.2                              |

Then, the solution of the diffusion differential equation for each metal in table 1 can be presented in graphical form according to equation (7),

In this case, we assume that \( \Delta x = 2 \) and \( \Delta t = (500)^{-1} \). In this case, the graph shows an area, where the area formed is the temperature distribution per unit time. From the graphs shown above, we can conclude that when the value of the thermal conductivity becomes greater, then the extent of the shading is getting tighter.

By considering the Neumann boundary conditions, we can determine the boundary temperatures of 50 at \( x = 0 \) and \( x = L \). And then, between \( x = 0 \) and \( x = L \) will form a straight line that indicating the metal has reached equilibrium temperature. In the process of analyzing the results of this method, we can
use MATLAB programs to help us to find out the data that we need it. Table 2 shows the time taken for each metal to reach its equilibrium point.

| Table 2. Time is taken to reach its equilibrium point for a metal |
|-------------------|-------------------|
| Alumin            | 0.5560            |
| Copper            | 0.3140            |
| Iron              | 1.4760            |
| Manganin          | 7.7820            |

In the results obtained above, it seems that the value of thermal conductivity affects the time value required by metal to reach its equilibrium point. In the example of the case above, manganin is faster to reach the equilibrium point than aluminum.

5. Conclusion
The finite difference method is one method in finding solutions to diffusion problems, for example, diffusion in metal. This numerical method utilizes the Neumann boundary conditions as well as the Taylor series in finding differential diffusion solutions. The solution obtained was applied and simulated in the case of heat transfer by conduction on various metals. From this simulation, we can obtain data in the form of temperature distribution across various metals by adjusting to boundary conditions. The results of this method can be presented in the graphical form of the temperature distribution per unit time of the metal. Then, the distribution is used to predict when various metals reach their equilibrium temperature. In the future, this research can be used to assist in solving the heat transfer phenomenon in terms of several dimensions of material by taking into account the boundary conditions used. So it is expected that it will be easy to determine a desired physical quantity for this phenomenon, as one of them in this paper is the time of metal to reach thermal equilibrium.

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