FLOER THEORY, FROBENIUS MANIFOLDS
AND INTEGRABLE SYSTEMS

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Abstract. Following the work of Piunikhin-Salamon-Schwarz, the Floer cohomology for Hamiltonian symplectomorphisms with its pair-of-pants product is ring isomorphic to the small quantum cohomology ring of the underlying symplectic manifold. Employing the rich algebraic structures of rational symplectic field theory, we show how the Frobenius manifolds and the resulting infinite-dimensional integrable systems of Gromov-Witten theory translate to Hamiltonian Floer theory. The main application of our results is the generalization of the classical mirror symmetry conjecture from closed to open Calabi-Yau manifolds worked out in [13].

Summary

The Floer theory of Hamiltonian symplectomorphisms is an important tool in symplectic geometry. Floer cohomology was invented by A. Floer to prove the Arnold conjecture about the number of symplectic fixed points and since then was improved to answer many other questions in symplectic geometry. Following M. Schwarz and P. Seidel ([21]), there exists the so-called pair-of-pants product in Floer cohomology. Apart from the Arnold conjecture for degenerate Hamiltonians, it is used in proofs of the Conley conjecture and plays a crucial role in the definition of symplectic quasimorphisms. While the pair-of-pants product involves the

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Floer cohomology groups of different Hamiltonian symplectomorphisms, it in particular defines a graded commutative and associative product on the sum of the Floer cohomologies of all powers of a given Hamiltonian symplectomorphism $\phi$.

The above mentioned applications of the pair-of-pants product build on its relation with the small quantum product of the underlying symplectic manifold. Following Piunikhin-Salamon-Schwarz ([19]) there exists an ring isomorphism between the (sum of the) Floer cohomology groups with its pair-of-pants product and the small quantum cohomology ring. On the other hand, the small quantum product only involves a very small part of rational Gromov-Witten theory, since it just counts holomorphic spheres with three marked points. In order to use the geometric information of all rational Gromov-Witten invariants, it is known that there also exists a big version of the quantum cup product, which recovers the full rational Gromov-Witten potential ([17]). Following B. Dubrovin, it equips (together with the canonical Euler vector field, the canonical unit vector field and the canonical flat metric) the affine space of quantum cohomology with the structure of a Frobenius manifold.

In this paper we show how the big quantum product and the resulting Frobenius manifold translate to the Floer theory of a Hamiltonian symplectomorphism $\phi$, extending the above relation between the small quantum product and the pair-of-pants product. In particular, in this paper we define a big pair-of-pants product which counts, in contrast to the TQFT structures on Floer cohomology introduced in [21] and [18], Floer solutions (in the sense of [18]) with varying conformal structure and fixed asymptotic markers. Since the pair-of-pants product satisfies properties like commutativity and associativity only after passing to cohomology, the same is clearly true for the big pair-of-pants product. In order to define the corresponding new algebraic and geometric structures in Floer theory, we make use of the rich algebraic structures of the rational symplectic field theory of the mapping torus of $\phi$ ([9],[10]).

While we show that the cylindrical contact cohomology of the corresponding mapping torus $M_{\phi}$ agrees with the sum of the Floer cohomologies of all powers of $\phi$, one can show that the differential in full contact homology equips the chain space of cylindrical contact cohomology with the structure of a (infinite-dimensional) differential graded manifold. Apart from giving a nice interpretation as an extension of the Lie algebra structure in Hamiltonian Floer theory, we show

**Theorem 0.1.** The big pair-of-pants product equips the differential graded manifold of full contact homology with the structure of a cohomology $F$-manifold in such a way that, at the tangent space at zero, we recover the (small) pair-of-pants product on Floer cohomology.

In order to explain our result, let us recall the definition of a cohomology $F$-manifold from [16], see also [15] and [13].
Definition 0.2. A cohomology F-manifold is a formal pointed differential graded manifold $Q_X$ equipped with a graded commutative and associative product for vector fields

$$\star : T^{(1,0)} Q_X \otimes T^{(1,0)} Q_X \to T^{(1,0)} Q_X.$$  

Note that, in his papers, Merkulov is working with different definitions of cohomology F-manifolds and $F_\infty$-manifolds. Note that Merkulov allows $Q$ to be any formal pointed graded manifold, which is more general in the sense that each (graded) vector space naturally carries the structure of a formal pointed (graded) manifold with the special point being the origin, see [14]. Furthermore, we remark that our cohomology F-manifolds are indeed infinite-dimensional and formal in the sense that we do not specify a topology on them.

By extending the isomorphism proof for contact homology from [9] and [10], we show that, for different choices of auxiliary data like almost complex structures and Hamiltonian perturbations, the resulting cohomology F-manifolds are isomorphic in a canonical way. In particular, in the case when the Hamiltonian is equal to zero, we show that we recover the cohomology F-manifold structure on the quantum cohomology of $M$ given by the big quantum product. On the other hand, the main application of our result is given in our subsequent paper [13]. By passing from closed to open symplectic manifolds, we show that our results to define a cohomology F-manifold structure on the symplectic cohomology of $M$, which indeed fits nicely with the expectations of mirror symmetry.

Conjecture 0.3. ([13]) If $M$ and $M^\vee$ are two open Calabi-Yau manifolds which are mirror to each other (in the sense of homological mirror symmetry), then the ring isomorphism between $SH^*(M)$ and $H^*(M^\vee, \wedge^* T_{M^\vee})$ can be lifted to an isomorphism of cohomology F-manifolds.

As for the application to spectral invariants discussed in the appendix, the crucial observation is that the resulting Floer curves automatically satisfy the required monotonicity condition, that is, they are indeed Floer solution in the sense of [18]. Note that we can still keep the main features like the maximums principle in the case of open manifolds when we work with domain-dependent Hamiltonian perturbations for transversality, see the appendix. In contrast to [18], we are however able to define moduli spaces of Floer solutions with varying conformal structure, which contain more geometric information.

Note that cohomology F-manifolds are generalizations of Dubrovin’s Frobenius manifolds, dropping the request for an underlying potential as well as for a flat structure. While the classical approach to integrable systems hence does not generalize immediately from Gromov-Witten theory to Floer theory, we instead show how rational symplectic field theory provides us with the desired Floer generalization of the integrable system from Gromov-Witten theory. For this we show how the resulting commuting Hamiltonian systems on the rational SFT homology of $M_\phi$ from [8] and [12] are related to the moduli spaces studied in the Floer theory of
ϕ, generalizing the relation between the rational symplectic field theory of $S^1 \times M$ and the Gromov-Witten potential of $M$ described in [9].

In order to turn our results into mathematical theorems in the strict sense, in the appendix we prove transversality for all occurring moduli space under the assumption that the symplectic manifold is semipositive. Indeed, instead of just referring to the polyfold theory of Hofer and his collaborators, we show how the transversality results for Hamiltonian mapping tori established by the author in [10] can be modified to cover the case of general symplectic mapping tori. While the latter shows how to deal with moduli spaces of holomorphic curves with three or more punctures using domain-dependent Hamiltonian perturbations, note that for holomorphic spheres and holomorphic cylinders (there are no holomorphic planes!) we essentially make use of the regularity (and nondegeneracy) result for Gromov-Witten theory and Floer cohomology for semipositive symplectic manifolds.

This paper is organized as follows: While in 1.1 we review the definition of the pair-of-pants product in the Floer theory of symplectomorphisms, in 1.2 we summarize the definition of the big quantum product and its relation to Frobenius manifolds. After showing in 2.1 how the contact homology complex leads to a differential graded manifold, in 2.2. we define the big pair-of-pants product and show that it turns the differential graded manifold into a cohomology F-manifold. Finally, in 2.3, we show how the integrable systems generalize from Gromov-Witten theory to Floer theory using a bypass obtained from symplectic field theory and end with establishing the analytical foundations and discussing some possible applications in the appendix.

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1. Introduction

1.1. Floer theory for symplectomorphisms. Let $(M, ω)$ be a closed symplectic manifold and let $H : S^1 \times M \to \mathbb{R}$ be a time-dependent Hamiltonian. The resulting Hamiltonian symplectomorphism is the time-one map $ϕ = ϕ^1_H$ of the flow of the time-dependent symplectic gradient $X^H_t$ of $H_t = H(t, ·)$. In order to be able to prove transversality for all occurring
moduli spaces, see the generalization of the results from [10] in the appendix, we assume that \((M, \omega)\) is semipositive in the sense that
\[
\omega(A) > 0, \quad c_1(A) \geq 3 - n \Rightarrow c_1(A) \geq 0
\]
for all \(A \in \pi_2(M)\), see [17]. Note that this includes the case of monotone symplectic manifolds as well as the case of symplectic manifolds \((M, \omega)\) with \(c_1(A) = 0\) for all \(A \in \pi_2(M)\), which contains the important class of Calabi-Yau manifolds. Furthermore we assume that, after choosing Hamiltonian perturbations as in the appendix, all fixed points of the Hamiltonian symplectomorphism \(\phi\) are nondegenerate, in particular, isolated. We first briefly review the definition of the Floer cohomology groups \(H^*_F(\phi)\) of the Hamiltonian symplectomorphism \(\phi = \phi^1_H\).

Floer cohomology groups. Let \(\mathcal{P}(\phi)\) denote the set of contractible one-periodic orbits of the flow of \(X^H_t\). Using the evaluation at \(0 \in S^1\), note that the one-periodic orbits \(x : S^1 \to M\) are in one-to-one correspondence with fixed points \(p = \phi(p)\) of the Hamiltonian symplectomorphism \(\phi\) via evaluation at the base point \(0 \in S^1\), \(p = x(0)\). Unambiguously we will not distinguish between one-periodic orbits and the corresponding fixed point and we will assume without mentioning that the underlying one-periodic orbit for each fixed point is contractible. Using the Conley-Zehnder index \(\text{CZ}(x)\) of \(x\), we can view \(x\) as a \(\mathbb{Z}_2\)-graded object with grading \(|x| = \text{CZ}(x) + 2(\dim M - 2) \mod 2\). For the definition of the Conley-Zehnder index, assume that we have chosen for every contractible closed orbit \(x\) a disk \(u : D^2 \to M\) with \(u(e^{2\pi it}) = x(t)\) which defines a unique unitary trivialization of the pullback bundle \(x^*TM\); since we only work with a \(\mathbb{Z}_2\)-grading, our choice of the spanning disk does not affect the grading. Further note that the additional summand does not appear in the original definition of Floer cohomology, but will become natural later on. Following ([17], section 11.1) we let \(\Lambda\) denote the universal Novikov ring of all formal power series in the (even) formal variable \(t\) with rational coefficients,
\[
\Lambda \ni \lambda = \sum_{\epsilon \in \mathbb{R}} n_\epsilon t^\epsilon : \#\{\epsilon \leq c : n_\epsilon \neq 0\} < \infty \text{ for all } c \in \mathbb{Q}.
\]
Note that our choice of coefficients ensures that \(\Lambda\) is indeed a field. With this we introduce the Floer cochain groups \(\text{CF}^*(\phi)\) to be the \(\mathbb{Z}_2\)-graded vector space spanned by all fixed points \(x \in \mathcal{P}(\phi)\) with coefficients in the field \(\Lambda\).

In order to define the coboundary operator \(\partial : \text{CF}^*(\phi) \to \text{CF}^{*+1}(\phi)\), we start with choosing an \(\omega\)-compatible almost complex structure \(J\) on \(M\). Note that for the necessary regularity result in Floer cohomology it is sufficient to work with fixed \(J\) as long as one is allowed to perturb the \(S^1\)-dependent Hamiltonian function; for details see the appendix. For two given fixed points \(x^-, x^+ \in \mathcal{P}(\phi)\), let \(\mathcal{M}^{x^+}_{x^-}(A)\) denote the moduli space of cylinders \(u : \mathbb{R} \times S^1 \to M\) satisfying Floer’s perturbed Cauchy-Riemann equation
\[
\bar{\partial}_J Hu = \partial_s u + J(u) \cdot (\partial_t u - X^H_t(u)) = 0,
\]
connecting the corresponding two one-periodic orbits in the sense that \(u(s, t) \to x^\pm(t)\) as \(s \to \pm \infty\) and representing the absolute homology class
$A \in \pi_0(M)$. Note that for the latter we use that for the definition of the Conley-Zehnder index we have already chosen a spanning disk for every contractible closed orbit $x$. Furthermore it is important to observe that the moduli space carries a natural $\mathbb{R}$-action. With this we define the coboundary operator $\partial : \text{CF}^*(\phi) \to \text{CF}^{*+1}(\phi)$ as

$$\partial x^- = \sum_{x^+, A} \mathcal{M}^{x^+}_x(A)/\mathbb{R} \cdot x^+ t^\omega(A),$$

where $\mathcal{M}^{x^+}_x(A)/\mathbb{R}$ denotes the algebraic count of elements in the moduli space of cylinders modulo $\mathbb{R}$-shift in the case when $\text{ind}(u^-) = \text{CZ}(x^+) - \text{CZ}(x^-) + 2c_1(A) = |x^+| - |x^-| + |t^\omega(A)| = 1$ and is equal to zero else.

In order to ensure that we always get a finite count, we use that $\mathcal{M}^{x^+}_x(A)/\mathbb{R}$ is compact when $\text{ind}(u) = 1$. On the other hand, when $\text{ind}(u) = 2$, $\mathcal{M}^{x^+}_x(A)$ can be compactified to a one-dimensional moduli space with boundary given by

$$\partial^1 \mathcal{M}^{x^+}_x(A)/\mathbb{R} = \bigcup \mathcal{M}^{x^+}_x(A^+)/\mathbb{R} \times \mathcal{M}^{x^+}_x(A^-)/\mathbb{R},$$

where the union runs over all fixed points $x \in \mathcal{P}(\phi)$ with $\text{ind}(u^+) = \text{ind}(u^-) = 1$ for $(u^+, u^-) \in \mathcal{M}^{x^+}_x(A^+) \times \mathcal{M}^{x^+}_x(A^-)$ and $A^+ + A^- = A$. Note that here we implicitly use our assumption that the symplectic manifold is semi-positive. Translating the above compactness result into algebra, we have shown that we indeed have $\partial \circ \partial = 0$, so that we can define the Floer cohomology groups as

$$HF^*(\phi) = H^*(\text{CF}^*(\phi), \partial).$$

Furthermore it can be shown that the cohomology groups for different choices of almost complex structures and Hamiltonian symplectomorphisms $\phi$ are isomorphic. In particular, when $\phi$ is Hamiltonian, then the Floer cohomology groups $HF^*(\phi)$ are isomorphic to the quantum cohomology groups $QH^*(M)$, which here are defined as the singular cohomology groups of $M$ with coefficients in the universal Novikov ring $\Lambda$ from above.

**Pair-of-pants product.** The important difference between the quantum cohomology groups and the singular cohomology groups is that there exists a quantum cup product on $QH^*(M)$, which is a deformation of the classical cup product on $H^*(M)$. It is defined by counting holomorphic spheres in the symplectic manifold $(M, \omega)$ equipped with a compatible almost complex structure $J$ with three marked points,

$$u : \hat{S} = S^2 \setminus \{z_0, z_1, z_\infty\} \to M, \quad \bar{\partial}_J(u) = du + J(u) \cdot du \cdot j = 0.$$ 

Note that each such map indeed extends smoothly over the removed points using finiteness of energy. After applying an appropriate Moebius transform we can assume that the three points are $z_0 = 0$, $z_1 = 1$, $z_\infty = \infty$, which equips $\hat{S}$ with unique coordinates, i.e., kills all the automorphisms of the domain.
While this leads to a commutative and associative product on $QH^*(M)$ (in the graded sense), see the next section, one can also define a product involving different Floer cohomology groups, the so-called pair-of-pants product, see [21]. It is compatible with the small quantum product via the isomorphism between Floer cohomology and quantum cohomology. On the other hand, for reasons that will become later clear, see also the introduction, we want to restrict ourselves to the case where all appearing Hamiltonian symplectomorphisms are multiples of a single one. On the chain level we define the pair-of-pants product $\ast_0 : \text{CF}(\phi^{k_0}) \otimes \text{CF}(\phi^{k_1}) \to \text{CF}(\phi^{k_0+k_1})$ by

$$x_0 \ast_0 x_1 = \sum_{x_{\infty}, A} \# M_{x_0,x_1}^{x_{\infty}}(A) \cdot x_{\infty}^{\omega(A)},$$

where $x_0, x_1, x_{\infty}$ is a fixed point of $\phi^{k_0}, \phi^{k_1}, \phi^{k_0+k_1}$, that is, a $k_0$-, $k_1$- and $k_0 + k_1$-periodic orbit of $H$, respectively.

The corresponding moduli space $M_{x_0,x_1}^{x_{\infty}} = M_{x_0,x_1}^{x_{\infty}}(A)$ consists of maps $u : \mathring{S} \to M$ converging to the periodic orbit $x_i$ near $z_i$, $i = 0, 1, \infty$ and satisfying the Floer equation $\partial J H u = A^{(0,1)}(du + X^H \otimes \beta)$, see also [18]. Note that, following [21] and [18], we assume that we have additionally chosen fixed asymptotic markers (directions) at each puncture. In compatible cylindrical coordinates $(s_i, t_i)$ near each of the punctures $z_0, z_1, z_\infty$ (the asymptotic marker agrees with $0 \in S^1$), we require that $u(s_i, t_i) \to x_i(t_i)$ as $s_0, s_1 \to -\infty$, $s_\infty \to +\infty$. Furthermore, as in [18], we assume $\beta$ is a one-form on $\mathring{S}$ which agrees with $-k_0 dt_0, -k_1 dt_1$ and $(k_0 + k_1) dt_\infty$ in the cylindrical coordinates $(s_i, t_i)$ around $z_i$, and $X^H = X^H_z$ is the symplectic gradient of a domain-dependent Hamiltonian function $H : \mathring{S} \times M \to \mathbb{R}$, $H_z := H(z, \cdot)$ which agrees with $H_{t_i} := k_i H_{t_i}$, $(\phi^{1}_{H^k} = \phi^{k}_{H})$ in the cylindrical ends. Finally note that we can again assign an absolute homology class $A \in \pi_2(M)$ to each $u \in M_{x_0,x_1}^{x_{\infty}}$ assuming that we have again chosen spanning surfaces for all (contractible) orbits.

As before we note that regularity can be proven for all relevant moduli spaces, see the appendix. Similar as for the moduli spaces of holomorphic sections in $\mathbb{R} \times M \phi$ one can prove that the moduli spaces $M_{x_0,x_1}^{x_{\infty}} = M_{x_0,x_1}^{x_{\infty}}(A)$ are compact when the Fredholm index is equal to zero and can be compactified to a moduli space with boundary

$$\partial^1 M_{x_0,x_1}^{x_{\infty}} = \bigcup M_x^{x_{\infty}} / \mathbb{R} \times M_x^{x_{\infty}} \cup M_{x_0,x}^{x_{\infty}} \times M_{x_1}^{x_{\infty}} / \mathbb{R} \cup M_{x,x_1}^{x_{\infty}} \times M_{x_0}^{x_{\infty}} / \mathbb{R}$$

when the Fredholm index is one, which again uses that $(M, \omega)$ is assumed to be semi-positive. Note that the above statement holds only for fixed $A \in \pi_2(M)$, but for notional simplicity we will often drop the homotopy classes from our formulas. While the first compactness implies that we get a finite count in the definition of the pair-of-pants product, the second compactness result translates into the algebraic result that $\ast_0$ commutes with the boundary operators,

$$\partial \circ \ast_0 = \ast_0 \circ (\partial \otimes 1 + 1 \otimes \partial),$$
which in turn proves that $\star_0$ descends to a map on cohomology,
$$\star_0 : \text{HF}(\varphi^{k_0}) \otimes \text{HF}(\varphi^{k_1}) \to \text{HF}(\varphi^{k_0+k_1}).$$

1.2. **Big quantum product and Frobenius manifolds.** In this subsection we review the small and the big quantum cup product as well as Frobenius manifolds, which translate the axioms of Gromov-Witten theory from algebra into geometry. For further details we refer to [17].

**Small quantum product and Gromov-Witten potential.** As mentioned above, the quantum cup product $\star : \text{QH}^*(M) \otimes \text{QH}^*(M) \to \text{QH}^*(M)$ agrees with the pair-of-pants product in the Floer theory of symplectomorphisms in the special case when the symplectomorphism is the identity. Since in this case there are no isolated sets of fixed points, we use the natural evaluation map on the (compactified) moduli space $\mathcal{M}_3(A)$ of $J$-holomorphic spheres in $M$ with three additional marked points,

$$\text{ev} = (\text{ev}_0, \text{ev}_1, \text{ev}_\infty) : \mathcal{M}_3(A) \to M \times M \times M, \ u \mapsto (u(0), u(1), u(\infty))$$
to pullback classes from the target manifold.

For a basis of cohomology classes $\theta_\alpha \in H^*(M)$, $\alpha = 1, \ldots, K$, which are again viewed as graded objects with grading given by $|\theta_\alpha| = \deg \theta_\alpha - 2$, the quantum cup product $\star : \text{QH}^*(M) \otimes \text{QH}^*(M) \to \text{QH}^*(M)$ is defined by

$$\theta_{\alpha_0} \star \theta_{\alpha_1} = \sum_{\alpha_\infty, \beta, A} \eta_{\alpha_\infty, \beta} \int_{\mathcal{M}_3(A)} \text{ev}_0^* \theta_{\alpha_0} \wedge \text{ev}_1^* \theta_{\alpha_1} \wedge \text{ev}_\infty^* \theta_{\beta} \cdot \theta_{\alpha_\infty} t^{\omega(A)},$$

where $\eta_{\alpha_\beta}$ denotes the Poincare pairing on $H^*(M)$.

While the quantum cup product just involves moduli spaces of holomorphic spheres with three marked points, the rational Gromov-Witten potential $F$ of $(M, \omega)$ also takes into account spheres with more than three marked points. Here the moduli space $\mathcal{M}_{r+1}(A)$ of holomorphic spheres with $r + 1$ additional marked points consists of tuples $(u, z_0, \ldots, z_{r-1}, z_\infty)$, where $u : S^2 \to M$ is a holomorphic sphere and $z_0, \ldots, z_{r-1}, z_\infty$ are marked points on $S^2$. As before we assume that $(z_0, z_1, z_\infty) = (0, 1, \infty)$ which ensures that there are no nontrivial automorphisms of the sphere $S^2 \setminus \{z_0, z_1, \ldots, z_{r-1}, z_\infty\}$ and write $(u, z_2, \ldots, z_{r-1}) \in \mathcal{M}_r(A)$.

Using the evaluation map $\text{ev} = (\text{ev}_0, \ldots, \text{ev}_{r-1}, \text{ev}_\infty) : \mathcal{M}_{r+1}(A) \to M^{r+1}$ given by

$$\text{ev}(u, z_2, \ldots, z_{r-1}) = (u(0), u(1), u(z_2), \ldots, u(z_{r-1}), u(\infty)),$$

the Gromov-Witten potential of $(M, \omega)$ is defined as the generating function $F = F(q)$, $q = (q_1, \ldots, q_K)$ given by

$$\sum_{r=1}^\infty \frac{1}{(r+1)!} \sum_{\alpha_0, \ldots, \alpha_\infty} \int_{\mathcal{M}_{r+1}(A)} \text{ev}_0^* \theta_{\alpha_0} \wedge \ldots \wedge \text{ev}_{r-1}^* \theta_{\alpha_{r-1}} \wedge \text{ev}_\infty^* \theta_{\alpha_\infty} \cdot q_{\alpha_0} \cdot \ldots \cdot q_{\alpha_\infty} t^{\omega(A)}.$$

Here $(q_1, \ldots, q_K)$ are formal variables assigned to the basis of cohomology classes $\theta_1, \ldots, \theta_K \in H^*(M)$ with grading given by $|q_\alpha| = -|\theta_\alpha|$. Note that
they can be viewed as coordinates of a linear space $Q$ over the field of Laurent polynomials in $z_1, \ldots, z_N$, which is canonically isomorphic to $Q\mathbb{H}^*(M)$ by identifying $\theta_\alpha \in Q\mathbb{H}^*(M)$ with the unit vector $e_\alpha = (0, \ldots, 1, \ldots, 0) \in Q$.

**Big quantum product and Frobenius manifolds.** Employing gluing of holomorphic spheres one can show that the Gromov-Witten potential satisfies the WDVV-equations given (up to sign due to the integer grading of the cohomology classes) by

$$
\frac{\partial^3 F}{\partial q_{\alpha_0} \partial q_{\alpha_1} \partial q_{\alpha_\infty}} \frac{\eta_{\alpha\infty, \beta_0}}{\partial q_{\beta_0} \partial q_{\beta_1} \partial q_{\beta_\infty}} = \frac{\partial^3 F}{\partial q_{\alpha_0} \partial q_{\beta_0} \partial q_{\alpha_\infty}} \frac{\eta_{\alpha\infty, \beta_0}}{\partial q_{\beta_0} \partial q_{\alpha_1} \partial q_{\beta_\infty}},
$$

where on both sides we sum over the indices $\alpha_\infty, \beta_0 = 1, \ldots, K$. They can be interpreted as associativity equation for a family of new products.

The idea is to use the above triple derivatives of the Gromov-Witten potential to define a product $\star_q : T_q Q \otimes T_q Q \to T_q Q$ on the tangent space at each $q \in Q$ by

$$
\frac{\partial}{\partial q_{\alpha_0}} \star_q \frac{\partial}{\partial q_{\alpha_1}} = \sum_{\alpha_\infty, \beta} \eta_{\alpha\infty, \beta} \cdot \left( \frac{\partial^3 F}{\partial q_{\alpha_0} \partial q_{\alpha_1} \partial q_{\beta_0}} \right)(q) \cdot \frac{\partial}{\partial q_{\beta_\infty}}.
$$

Here observe that the tangent space $T_q Q$ at each $q = (q_1, \ldots, q_K)$ is canonically isomorphic to the original space $Q\mathbb{H}^*(M)$ by identifying $\theta_\alpha$ with $\partial/\partial q_\alpha$, where $|\partial/\partial q_\alpha| = |\theta_\alpha|$. The coefficient $(\partial^3 F/\partial q_{\alpha_0} \partial q_{\alpha_1} \partial q_{\beta_0})(q)$ is given by

$$
\frac{1}{(r-2)!} \sum_{\alpha_2, \ldots, \alpha_{r-1} A} \int_{M_{r+1}(A)} ev_0^* \theta_{\alpha_0} \wedge \ldots \wedge ev_\infty^* \theta_{\beta} \cdot q_{\alpha_2} \cdot \ldots \cdot q_{\alpha_{r-1}} t^{e(A)}.
$$

The new product is called the **big quantum cup product** and is indeed a deformation of the quantum product $\star$ in the sense that the latter agrees with the product $\star_q$ at $q = (q_1, \ldots, q_K) = 0$,

$$
\star = \star_0 : T_0 Q \otimes T_0 Q \to T_0 Q.
$$

For this observe that

$$
\left( \frac{\partial^3 F}{\partial q_{\alpha_0} \partial q_{\alpha_1} \partial q_{\beta}} \right)(0) = \sum_A \int_{M_3(A)} ev_0^* \theta_{\alpha_0} \wedge ev_1^* \theta_{\alpha_1} \wedge ev_\infty^* \theta_{\beta} \cdot t^{e(A)}.
$$

However note that the big quantum product now involves moduli spaces of holomorphic spheres with an arbitrary number of additional marked points, that is, the full rational Gromov-Witten potential of $(M, \omega)$.

Following Dubrovin, see [IS], the Gromov-Witten potential $F \in T^{(0,0)} Q$, viewed as a function (= $(0,0)$-tensorfield) on $Q$, endows the space $Q \cong Q\mathbb{H}^*(M)$ with the structure of a Frobenius manifold. Apart from the fact that $F$ satisfies the WDVV-equations, it is important that $F$ is homogeneous in the sense that $L_E F = 0$, where $L_E : T^{(r,s)} Q \to T^{(r,s)} Q$ denotes the grading operator given by the Lie derivative with respect to the Euler vector field

$$
E = \sum_\alpha (2 - \deg \theta_\alpha) \cdot q_\alpha \frac{\partial}{\partial q_\alpha} \in T^{(1,0)} Q.
$$
Here we assume for simplicity that the first Chern class of \((M, \omega)\) indeed vanishes, that is, \(c_1(A) = 0\) for all \(A \in \pi_2(M)\).

The big pair-of-pants product \(\star\) can be viewed as a \((1, 2)\)-tensorfield on \(Q\). Together with the Poincare pairing, \(*_q\) turns each tangent space \(T_q Q\), \(q \in Q\) into a Frobenius algebra. Here the unit element is \(\partial/\partial q_1\), where \(q_1\) is the formal graded variable for the canonical zero-form \(\theta_1 = 1\).

2. Floer theory, Frobenius manifolds and integrable systems

2.1. Contact homology and Hamiltonian Floer theory. The goal of this paper is to show how to define a big version of the pair-of-pants product in Floer theory which generalizes the (small) pair-of-pants product in the same way as the big quantum product extends the small quantum product in Gromov-Witten theory. The resulting new algebraic structures will be defined in an extension of Eliashberg-Givental-Hofer’s symplectic field theory for mapping tori.

Floer cohomology and cylindrical contact cohomology. In this first part we review how Floer cohomology can be embedded into the framework of symplectic field theory, for further details we refer to subsection 2.1 in [10]. We start with the observation that (parametrized) one-periodic Hamiltonian orbits \(x : S^1 \to M\) are in one-to-one correspondence with unparametrized one-periodic orbits \(\gamma\) of the canonical vector field \(\partial_t\) on the corresponding mapping torus \(M_\phi = \mathbb{R} \times M/\{(t, p) \sim (t + 1, \phi(x))\}\) by setting \(\gamma : S^1 \to M_\phi\), \(\gamma(t) = (t, x)\) where \(x\) is viewed as the corresponding fixed point. Following [8], example 1.2), see also [10], note that \(M_\phi\) naturally carries a stable Hamiltonian structure in the sense of [2] given by \((\tilde{\omega} = \omega, \tilde{\lambda} = dt)\) with Reeb vector field \(\tilde{R} = \partial_t\). As described in [10], the stable Hamiltonian manifold \(M_\phi\) can be identified with \(S^1 \times M\) equipped with the \(H\)-dependent stable Hamiltonian structure \((\tilde{\omega}^H = \omega + dH_t \wedge dt, \tilde{\lambda}^H = dt)\) with Reeb vector field \(\tilde{R}^H = \partial_t + X^H\), where the underlying diffeomorphism between \(M_\phi\) and \(S^1 \times M\) is given by the Hamiltonian flow, \(S^1 \times M \to M_\phi, (t, p) \mapsto (t, \phi_t^H(p))\).

Generalizing the one-to-one correspondence between (parametrized) orbits \(x^\pm\) in \(M\) and unparametrized orbits \(\gamma^\pm\) in \(M_\phi = S^1 \times M\), one can show that the moduli space of (parametrized) Floer cylinders \(\mathcal{M}_x^{c+}\) connecting \(x^+\) and \(x^-\) can be identified with the moduli space of unparametrized \(J\)-holomorphic cylinders in \(\mathbb{R} \times M_\phi \cong \mathbb{R} \times S^1 \times M\) converging to \(\{+\infty\} \times \gamma^+\) and \(\{-\infty\} \times \gamma^-\) in the cylindrical ends. For this observe that the \(\omega\)-compatible almost complex structure \(J\) on \((M, \omega)\) and the \(S^1\)-dependent Hamiltonian \(H_t\) naturally defines a cylindrical almost complex structure \(\tilde{J} = \tilde{J}^H\) on \(\mathbb{R} \times S^1 \times M\) in the sense of [2], compatible with the stable Hamiltonian structure, by setting \(\tilde{J}\partial_s = \partial_t + X^H\) and requiring that \(\tilde{J}\) agrees with \(J\) on \(TM\), see [10] and [8], example 1.2). Then an easy computation shows, see also ([10], proposition 2.2 and 2.4), that unparametrized \(\tilde{J}\)-holomorphic maps \(\tilde{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \times M\) with \(\tilde{u}(s, t) \to (\pm\infty, \gamma^\pm(t))\) as \(s \to \pm\infty\) are in one-to-one correspondence with connecting Floer cylinders \(u \in \mathcal{M}_x^{c-} x^+\). For this observe that \(\tilde{u}\)
can be written as a tuple $\tilde{u} = (h, u)$, where $u$ satisfies Floer’s perturbed Cauchy-Riemann equation and $h$ is an automorphism of the cylinder which, after applying the inverse automorphism, we can always assume to be the identity. Note that the natural $\mathbb{R}$-action on $\mathcal{M}_\gamma(A)$ corresponds to the natural $\mathbb{R}$-symmetry on the space of $J$-holomorphic maps to the cylindrical almost complex manifold $\mathbb{R} \times M_\phi$.

Following [9], the cylindrical contact cohomology $HC^*_\text{cyl} = HC^*_\text{cyl}(M_\phi)$ of the mapping torus $M_\phi$ is the cohomology of a cochain complex, $HC^*_\text{cyl} = H^*(C^*, \partial)$, where the cochain space $C^*$ is now defined to be the linear space generated by the closed unparametrized orbits $\gamma$ of the Reeb vector field with coefficients in the universal Novikov ring $\Lambda$ from before. Note that the period of each closed orbit in $M_\phi \cong S^1 \times M$ agrees with the degree of the map to the base circle and hence the cochain space naturally splits, $C^* = \bigoplus_k C^*_k$, where $C^*_k$ is generated by the orbits of period $k \in \mathbb{N}$. As before we work with a $\mathbb{Z}_2$-grading given by $|\gamma| = CZ(\gamma) + 2(\dim M - 2) \mod 2$, where $CZ(\gamma)$ denotes the Conley-Zehnder index for closed Reeb orbits defined in [9]. The coboundary operator $\partial : C^* \to C^*$ is defined as

$$\partial\gamma^− = \frac{1}{\kappa_\gamma} \cdot \sum_{\gamma^+, A} \# \mathcal{M}_{\gamma^−}(A)/\mathbb{R} \cdot \gamma^+ \omega(A),$$

where $\kappa_\gamma$ denotes the multiplicity of the closed orbit $\gamma$, see [9]. Note that, as in the definition of cylindrical contact homology in [22], still the multiplicity $\kappa_\gamma$—and not $\kappa_\gamma^+$—appears, since for the passing from homology to cohomology we just need to interchange the roles of $\gamma^+$ and $\gamma^−$.

Here $\mathcal{M}_{\gamma^−} = \mathcal{M}_{\gamma^−}(A)$ denotes the moduli space of unparametrized $\tilde{J}$-holomorphic cylinders $\tilde{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times M_\phi$ converging to $\gamma^+$ and $\gamma^−$ near the cylindrical ends, $\tilde{u}(s, t+\tau^\pm) \to (\pm \infty, \gamma^\pm(kt))$ as $s \to \pm \infty$ for some $\tau^\pm \in S^1$. For the latter observe that, although we now want to consider the orbits as unparametrized objects, in the original definition from [9] one arbitrarily fixes a parametrization by choosing a special point on each closed Reeb orbit $\gamma$. In order to provide a natural link between the cochain complexes of cylindrical contact cohomology and Floer cohomology, we will modify the original definition and choose on each $k$-periodic orbit not one but $k$ special points naturally given by the intersection of the orbit with the fibre over $\pi^{-1}(0) \subset M_\phi$ of the projection $M_\phi \to S^1$. In order to cure for the resulting overcounting, we assume that every special point comes with the rational weight $1/k$. Note that when $\gamma$ is multiply-covered then some of these special points might coincide and we sum the weights correspondingly; in particular, when $\gamma$ is a $k$-fold cover of a one-periodic orbit, then we agree with the original definition in [9]. The $k$ special points in turn define $k$ asymptotic markers (directions) at each cylindrical end and we follow [9] and assume that the moduli spaces $\mathcal{M}_{\gamma^−}$ are made up of maps $\tilde{J}$-holomorphic maps $\tilde{u}$ as above together with asymptotic markers at each cylindrical end up to reparametrization of the underlying cylinder. Note that, in contrast to the original definition in [9], note that our choices of special
points on $\gamma^+$ (and $\gamma^-$) lead to a natural $\mathbb{Z}_k(\times \mathbb{Z}_k)$-action on $\mathcal{M}_{\gamma^\pm}$ and we assume that every unparametrized holomorphic cylinder with asymptotic markers in $\mathcal{M}_{\gamma^\pm}$ comes equipped with the weight given by the product of the weights assigned to the special points defining the asymptotic markers.

While the closed orbits of period one are in bijection with the fixed points $x$ in $\mathcal{P}(\phi)$, note that for general $k \in \mathbb{N}$ the fixed points in $\mathcal{P}(\phi^k)$ are in $k$-to-one-correspondence with closed orbits of period $k$ when the underlying orbit is simple. For this observe that for every fixed point $x \in \mathcal{P}(\phi^k)$ the points $\phi(x), \ldots, \phi^{k-1}(x)$ are also fixed points of $\phi^k$ which induces a natural $\mathbb{Z}_k$-action on the cochain space $\text{CF}^*(\phi^k)$. Note that the Conley-Zehnder indices (mod 2) of $\phi^i(x)$ agree for all $i = 0, \ldots, k - 1$ by symmetry reasons, since the corresponding one-periodic orbits just differ by reparametrization and the spanning surface $u$ for $x$ naturally defines spanning surfaces for all $\phi^i(x)$. On the other hand, $x, \phi(x), \ldots, \phi^{k-1}(x)$ all represent the same unparametrized $k$-periodic Reeb orbit $\gamma$. Without further mentioning, we will only consider closed Reeb orbits where the underlying parametrized orbits in $M$ are contractible. Then the Conley-Zehnder index of $\gamma$ defined in [9] agrees with the Conley-Zehnder index of $x$ (we can use the same spanning surfaces to define the index for $\gamma$). More precisely, there is indeed a one-to-one correspondence between the $k$ fixed points and the $k$ special points that we have chosen on $\gamma$ above. While it is not hard to see from our discussion above that the Floer cochain complex for $\phi$ is contained in the cochain complex of the cylindrical contact cohomology of $M_\phi$, we now show that the full contact cohomology as an interpretation in terms of the Floer cohomologies of all powers $\phi^k$ of the underlying Hamiltonian symplectomorphism $\phi = \phi_H^1$ (defined using the same $\omega$-compatible almost complex structure). Let $\text{CF}^*(\phi^k)_{\mathbb{Z}_k} \subset \text{CF}^*(\phi^k)$ of $\mathbb{Z}_k$-invariant elements. By symmetry reason it follows that the coboundary operator restricts to a coboundary operator $\partial : \text{CF}^*(\phi^k)_{\mathbb{Z}_k} \rightarrow \text{CF}^{*+1}(\phi^k)_{\mathbb{Z}_k}$.

**Proposition 2.1.** For every $k \in \mathbb{N}$ the natural identification between the cochain subspace $C^*_k \subset C^*$ generated by the $k$-periodic Reeb orbits and the subspace of $\mathbb{Z}_k$-invariant elements in $\text{CF}^*(\phi^k)$ given by

$$C^*_k \rightarrow \text{CF}^*(\phi^k)_{\mathbb{Z}_k}, \gamma \mapsto \frac{1}{k}(x + \ldots + \phi^{k-1}(x))$$

is compatible with the coboundary operators in cylindrical contact cohomology and Floer cohomology. Together with $HF^*(\phi^k)_{\mathbb{Z}_k} = HF^*(\phi^k)$, it follows that the cylindrical contact cohomology of the mapping torus $M_\phi$ is naturally isomorphic to the sum of the Floer cohomologies of all powers of $\phi$,

$$\text{HC}^*_{cyl}(M_\phi) \cong \bigoplus_k HF^*(\phi^k).$$

**Proof.** The proof for $k = 1$ is already given above, since have shown that connecting Floer cylinders in $\mathcal{M}^{x^+}$ are in one-to-one correspondence with unparametrized cylinders in $\mathcal{M}^{y^+}_{\gamma^\pm}$. For the case when $k$ is an arbitrary natural number, observe first that the moduli space $\mathcal{M}^{x^+}$ is only non-empty when $\gamma^+$ and $\gamma^-$ have the same period $k$ by homological reasons. It again
follows from ([10], proposition 2.2), see also ([10], proposition 2.4), that
\[ \tilde{u} = (h, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M \cong \mathbb{R} \times S^1 \times M \] is a \( J \)-holomorphic cylinder precisely when \( h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \) is holomorphic and \( u : \mathbb{R} \times S^1 \to M \) satisfies the Floer equation
\[ \partial_{J, h}(u) = \Lambda^{0,1}(du + X^H \otimes dh) = 0. \]
When \( \tilde{u} \) represents an element in \( \mathcal{M}^{\tau^+}_{\gamma^-} \) with \( k \)-periodic orbits, then it follows that \( h \) is a \( k \)-fold unbranched covering map from the cylinder to itself, which in turn implies that \( u \) satisfies the Floer equation for the pair \((J, H^k)\) with the \( 1/k \)-periodic Hamiltonian \( H^k \). After applying an automorphism of the domain, note that for every \( \tilde{u} \in \mathcal{M}^{\tau^+}_{\gamma^-} \) we can always assume that the induced covering map \( h : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1 \) is given by \( h(s, t) = (ks, kt) \).

After fixing the \( k \)-fold covering map \( h \) using the action of the automorphism group, note that there still remains a \( \mathbb{Z}_k \)-action. In analogy to the relation between closed orbits and fixed points, it follows that there is a \( k \)-to-one correspondence between unparametrized \( \tilde{u} \)-holomorphic cylinders and cylinders \( u : \mathbb{R} \times S^1 \to M \) satisfying the Floer equation for \((J, H^k)\) given by reparametrization of the underlying cylinder. In particular, we have

\[ \# \mathcal{M}^{\tau^+}_{\gamma^-} / \mathbb{R} = \frac{1}{k} \sum_{i^+ = 0}^{k-1} \mathcal{M}^{\phi^{-1}(x^+)}_{\phi^-(x^-)} / \mathbb{R} \]

in case that \( x^+, \ldots, \phi^{-1}(x^+) \) represents the orbit \( \gamma^+ \). Note that when \( \gamma^+ \) or \( \gamma^- \) is multiply-covered with multiplicity \( \kappa_{\gamma^\pm} \) and hence some of the fixed points \( \gamma^+, \ldots, \phi^{-1}(x^+) \) agree, we still need to count them as different, since for each holomorphic cylinder in \( \mathcal{M}^{\tau^+}_{\gamma^-} \) there are now \( \kappa_{\gamma} \) possible directions for the asymptotic marker. On the other hand, in the same way as the closed one-periodic orbits \( x_0, \ldots, x_{k-1} : S^1 \to M \) corresponding to fixed points \( x^+, \ldots, \phi^{-1}(x^+) \) are obtained by reparametrization, \( x_i(t) = x(t+i/k) \), the moduli space \( \mathcal{M}^{\tau^+}_{\gamma^-} \) from Floer cohomology is naturally isomorphic to the moduli space \( \mathcal{M}^{\phi^{-1}(x^+)}_{\phi^-(x^-)} \) for all \( 0 \leq i \leq k-1 \) via reparametrization. It follows that

\[ \sum_{i^- = 0}^{k-1} \# \mathcal{M}^{\tau^+}_{\phi^-(x^-)} / \mathbb{R} = \sum_{i^- = 0}^{k-1} \# \mathcal{M}^{\phi^{-1}(x^+)}_{\phi^{-1}(x^-)} / \mathbb{R} \]

for all \( 0 \leq i^+ \leq k-1 \). Together with the above identity we find that

\[ \# \mathcal{M}^{\tau^+}_{\gamma^-} / \mathbb{R} = \sum_{i^- = 0}^{k-1} \# \mathcal{M}^{\tau^+}_{\phi^-(x^-)} / \mathbb{R}. \]

Using this we can show that the chain map \( C^*_{k} \to \oplus_k \text{CF}^*(\phi^k)^{\mathbb{Z}_k}, \quad \gamma \mapsto \frac{1}{\kappa_{\gamma}}(x + \ldots + \phi^{k-1}(x)) \) has the desired property. Indeed it follows that with respect to the above identification the differential \( \partial : \oplus_k \text{CF}^*(\phi^k)^{\mathbb{Z}_k} \to \oplus_k \text{CF}^*(\phi^k)^{\mathbb{Z}_k} \) in Floer cohomology agrees with the differential in cylindrical
contact cohomology,
\[
\partial \left( \frac{1}{\kappa_{\gamma^-}} (x^- + \ldots + \phi^{k-1}(x^-)) \right)
\]
\[
= \frac{1}{\kappa_{\gamma^-}} \cdot \sum_{x^+} \left( \sum_{i=0}^{k-1} \# \mathcal{M}^+_{\phi^i(x^-)}(A) / \mathbb{R} \right) \cdot x^+ t^\omega(A)
\]
\[
= \frac{1}{\kappa_{\gamma^-}} \cdot \sum_{x^+} \# \mathcal{M}^+_{\gamma^-}(A) / \mathbb{R} \cdot x^+ t^\omega(A)
\]
\[
= \frac{1}{\kappa_{\gamma^-}} \cdot \sum_{\gamma^+, A} \# \mathcal{M}^+_{\gamma^-}(A) / \mathbb{R} \cdot \frac{1}{\kappa_{\gamma^+}} (x^+ + \ldots + \phi^{k-1}(x^+)) t^\omega(A).
\]

Note that when \( \gamma \) corresponds to a bad orbit, then its contributions to the Floer differential are zero by symmetry reasons. Finally, for Hamiltonian symplectomorphisms with sufficiently \( C^2 \)-small Hamiltonian (depending on \( k \)) note that all fixed points of \( \phi^k \) correspond to critical points of the underlying Hamiltonian and hence are already fixed points of \( \phi \). It follows that \( \text{HF}^*(\phi^k)^2 \mathbb{Z} = \text{HF}^*(\phi^k) = \text{QH}^*(M) \) for such small Hamiltonian symplectomorphisms. From the invariance properties of Floer homology we then get \( \text{HF}^*(\phi^k)^2 \mathbb{Z} \cong \text{QH}^*(M) \cong \text{HF}^*(\phi^k) \) and hence \( \text{HF}^*(\phi^k)^2 \mathbb{Z} = \text{HF}^*(\phi^k) \) for all Hamiltonian symplectomorphisms. \(
\)

Recall that the pair-of-pants product defines a product \( \star_0 : \text{HF}^*(\phi^{k_0}) \otimes \text{HF}^*(\phi^{k_1}) \rightarrow \text{HF}^*(\phi^{k_0+k_1}) \) for all natural numbers \( k_0, k_1 \). All together, it follows that it defines a product on the direct sum of the Floer cohomologies of all different powers of \( \phi \) and hence on the cylindrical contact cohomology,
\[
\star_0 : \text{HC}^*_\text{cyl}(M_\phi) \otimes \text{HC}^*_\text{cyl}(M_\phi) \rightarrow \text{HC}^*_\text{cyl}(M_\phi).
\]

Contact homology and differential graded manifolds. Introducing again formal variables \( q_\gamma \) for each closed Reeb orbit \( \gamma \) (with underlying one-periodic orbits of some \( H^k \) which are contractible) with the same \( \mathbb{Z}_2 \)-grading \( |q_\gamma| = -|\gamma| \mod 2 \) such that \( \sum q_\gamma \gamma \) is pure of degree zero, observe that the cochain space \( C^* = \bigoplus_k C^*_\text{cyl}(\phi^k) \) of the cylindrical contact cohomology can be identified with the tangent space \( T_0 \mathbb{Q} \) at zero of an infinite-dimensional linear coordinate space \( \mathbb{Q} \) by identifying \( \gamma \in C^* \) with \( \partial/\partial q_\gamma \in T_0 \mathbb{Q} \). Since \( C^* \) is a vector space over the universal Novikov ring \( \Lambda \) (with rational coefficients), the corresponding coordinates \( q_\gamma \) take values in \( \Lambda \). While in the next subsection we will show that the pair-of-pants product can be defined directly on the cochain space of cylindrical contact cohomology and hence provides us with a map \( \star : T_0 \mathbb{Q} \otimes T_0 \mathbb{Q} \rightarrow T_0 \mathbb{Q} \), the desired deformed version of the pair-of-pants product should then provide us with a family of maps \( \star_q : T_q \mathbb{Q} \otimes T_q \mathbb{Q} \rightarrow T_q \mathbb{Q} \) on each tangent space, that is, defines a \((1,2)\)-tensor field \( \star \in \mathcal{T}^{(1,2)} \mathbb{Q} \) on the coordinate space \( \mathbb{Q} \).

In same way as the big quantum product counts holomorphic spheres with an arbitrary number of marked points, the big pair-of-pants product will count maps from punctured spheres with an arbitrary number of marked points to the symplectic manifold satisfying a perturbed Cauchy-Riemann
equation. Similar as for the Gromov-Witten invariants and different from the Floer curves considered in [21] and [18], we will consider Floer curves with a varying conformal structure. It is well-known that the pair-of-pants product satisfies relevant properties like associativity and commutativity only after passing to Floer cohomology, i.e., cylindrical contact homology. Since the big pair-of-pants product should count holomorphic curves approaching an arbitrary number of fixed points, it is natural to assume that the differential of the correct homology theory also counts holomorphic curves with an arbitrary number of cylindrical ends. The central idea of the paper is that, by passing from the small to the big pair-of-pants product, on the underlying space of formal variables \(q\) paper is that, by passing from the small to the big pair-of-pants product, on the underlying space of formal variables \(q\) the cylindrical contact homology differential needs to be replaced by differential of full contact homology of the mapping torus.

Following [9] and [10], the full contact cohomology of \(M\phi\) is defined as the cohomology of the cochain complex

\[
\text{HC}^\ast(M\phi) = H^\ast(\mathfrak{A}, \partial),
\]

where the chain space \(\mathfrak{A}\) consists of (polynomial) functions in the variables \((q_i)\) with coefficients in the universal Novikov ring \(\Lambda\). Recall that the latter is indeed a field since we work with rational coefficients and we continue to work with a \(\mathbb{Z}_2\)-grading. The differential \(\partial : \mathfrak{A}\to \mathfrak{A}_{-1}\) is then defined by counting unparametrized punctured \(J\)-holomorphic curves with one positive cylindrical end but an arbitrary number of negative cylindrical ends. Using the Leibniz rule it is given by

\[
\partial q_{\gamma^+} = \sum_{\Gamma, A} \frac{1}{r!} \kappa^\Gamma q^\Gamma \mathcal{M}^\gamma^+(\Gamma; A) / \mathbb{R} \cdot q^0 \mu(A)
\]

with \(q^\Gamma = q_{\gamma_0} \cdots q_{\gamma_{r-1}}\) and \(\kappa^\Gamma = \kappa_{\gamma_0} \cdots \kappa_{\gamma_{r-1}}\).

For every closed unparametrized orbit \(\gamma^+\) (of period \(k \in \mathbb{N}\)) and every ordered set of closed unparametrized orbits \(\Gamma = (\gamma_0, \ldots, \gamma_{r-1})\) (of periods \(k_0, \ldots, k_{r-1}\) with \(k_0 + \ldots + k_{r-1} = k\)) of the Reeb vector field on \(M\phi \cong S^1 \times M\), the moduli space \(\mathcal{M}^\gamma^+(\Gamma; A)\) consists of equivalence classes of tuples \((\tilde{u}, z_0, \ldots, z_{r-1})\) together with an asymptotic marker (direction) at each \(z_i\), where \(z_0, \ldots, z_{r-1}\) is a collection of marked points on \(C = S^2 \setminus \{\infty\}\) and \(\tilde{u} = (h, u) : \tilde{S} \to \mathbb{R} \times M\phi \cong \mathbb{R} \times S^1 \times M\) is a \(J\)-holomorphic map from the resulting punctured sphere \(\tilde{S} = C \setminus \{z_0, \ldots, z_{r-1}\} = S^2 \setminus \{z_0, \ldots, z_{r-1}, z_\infty = \infty\}\) to the cylindrical almost complex manifold \(\mathbb{R} \times M\phi\). As in the definition of the pair-of-pants product we require that in compatible cylindrical coordinates \((s^+, t^+)\) near \(z_\infty\) and \((s_i, t_i)\) near \(z_i\) that \(\tilde{u}(s^+, t^+) \to (+\infty, \gamma^+(kt^+))\) as \(s^+ \to +\infty\) and \(\tilde{u}(s_i, t_i) \to (-\infty, \gamma_i(k_i t_i))\) as \(s_i \to -\infty\) for all \(i = 0, \ldots, r - 1\).

Note that the parametrization on the orbit is again given by the choice of one of the special points on the orbits. On the other hand, in contrast to the pair-of-pants product, the asymptotic markers (and hence the cylindrical coordinates) at each puncture are not fixed, but, as in the
definition of cylindrical contact cohomology, the asymptotic markers are
fixed by the special points. As for cylindrical contact cohomology we
consider unparametrized $\tilde{J}$-holomorphic curves and assume that elements
in the moduli space $\mathcal{M}^\gamma_+$ are equivalence classes under the obvious action
of the group of Moebius transformations on $\mathbb{C} = \mathbb{S}^2 \setminus \{\infty\}$. Furthermore we
assume, as before, that they are equipped with a rational weight given by
the product of the rational weights of the special marked points defining
the asymptotic markers. Finally it can be shown, see the proof of the next
proposition, that to each $\tilde{J}$-holomorphic curve in $\mathcal{M}^\gamma_+$ one can still assign
a class $A \in \pi_2(M)$.

It is shown in [2], see also [10], that the moduli space $\mathcal{M}^\gamma_+(\Gamma)/\mathbb{R} = \mathcal{M}^\gamma_+(\Gamma, A)/\mathbb{R}$ is compact when the index is one and, when the index is two,
can be compactified to a one-dimensional moduli space with boundary

$$\partial^1 \mathcal{M}^\gamma_+(\Gamma)/\mathbb{R} = \bigcup \mathcal{M}^\gamma_+(\Gamma')/\mathbb{R} \times \mathcal{M}^\gamma_+(\Gamma'')/\mathbb{R}$$

formed by moduli spaces of the same type. For the latter we again use
that $(M, \omega)$, see the proof of the next proposition. The above compactness
result for one-dimensional moduli spaces translates into $\partial \circ \partial = 0$, so that $HC^\omega(M_\phi) = H^*(\mathfrak{A}^r, \partial)$ is well-defined. By combining the invariance proof
for contact homology in [9] (where it is shown that it is independent of
the choice of contact form) with the invariance proof in Floer cohomology
and hence in cylindrical contact cohomology, it can further be shown, see
the next subsection, that $HC^\omega(M_\phi)$ is independent of the choice of the $\omega$
-compatible almost complex structure $J$ and the underlying time-dependent
Hamiltonian $H$.

When $\Gamma$ consists of a single orbit $\gamma^-$, then we just get back the moduli
spaces of cylindrical contact homology from before, $\mathcal{M}^\gamma_+(\gamma^-) = \mathcal{M}^\gamma_+$. In
the same way as the cylindrical contact (co)homology has an immediate
interpretation in Hamiltonian Floer theory, the same is indeed true for
the new algebraic structures arising from the more general moduli spaces
$\mathcal{M}^\gamma_+(\Gamma)$. For this we want to work for the moment with a different
algebraic setup, see [9] and [3].

Instead of using the information of all moduli spaces $\mathcal{M}^\gamma_+(\Gamma)$ to define the
chain complex $(\mathfrak{A}^\gamma, \partial)$ of full contact homology, one can use it to define an
$L_\infty$-structure on the cylindrical contact cohomology. For this observe that the identity $\partial \circ \partial = 0$ for the boundary operator of full contact homology
immediately shows that the moduli spaces $\mathcal{M}^\gamma_+(\gamma_0, \gamma_1)$ can be used to define
a Lie bracket $[\cdot, \cdot] : HC^\gamma_{cy}(M_\phi) \otimes HC^\gamma_{cy}(M_\phi) \rightarrow HC^\gamma_{cy}(M_\phi)$ given on the
cochain space $C^\gamma$ of cylindrical contact cohomology by

$$[\gamma_0, \gamma_1] = \frac{1}{k_{\gamma_0, \gamma_1}} \sum_{\gamma^+} \# \mathcal{M}^\gamma_+(\gamma_0, \gamma_1; A)/\mathbb{R} \cdot \gamma^+ t^{\omega(A)}.$$ 

In the same way, the moduli spaces $\mathcal{M}^\gamma_+(\Gamma)$ with $\# \Gamma > 2$ can be used
to define an infinite-sequence of higher bracket operations on $C^\gamma$ satisfying
(together with the coboundary and the bracket from before) the infinite sequence of $L_\infty$-relations.

**Proposition 2.2.** The coefficients $1/\kappa^1 \cdot \#M^r(\Gamma; A)/\mathbb{R}$ appearing in the definition of the contact homology differential and the $L_\infty$-structure count Floer solutions $u : \hat{S} \to M$ in the sense ([18], 6.1) with one positive puncture **with varying conformal structure and simultaneously rotating asymptotic markers**. In particular, the above $L_\infty$-structure on $HC^*_{cyl}(M_\phi) = \bigoplus_k HF^*(\phi^k)$ extends the Lie bracket on Floer cohomology defined in ([1], 2.5.1)

**Proof.** As in the case of cylinders, following ([10], proposition 2.2), the map $\tilde{u} = (h, u) : \hat{S} \to \mathbb{R} \times M_\phi \cong \mathbb{R} \times S^1 \times M$ is $J$-holomorphic precisely when $h : \hat{S} \to \mathbb{R} \times S^1$ is holomorphic and $u : \hat{S} \to M$ satisfies the Floer equation $\tilde{\partial}_{J,h}(u) = \Lambda^{0,1}(du + X_{h_2}^H \otimes dh_2)$. Forgetting the map $u$ and hence mapping $(\tilde{u}, z_0, \ldots, z_{r-1})$ to $(h, z_0, \ldots, z_{r-1})$ defines a projection from $M^r(\Gamma)/\mathbb{R}$ to the moduli space $M^r(k_0, \ldots, k_{r-1}) = M(k_0, \ldots, k_{r-1})$ of holomorphic functions $h$ on $\mathbb{C}$ with $r$ zeroes $z_1, \ldots, z_{r-1}$ of predescribed orders $k_0, \ldots, k_{r-1}$ up to Moebius transformations of $\mathbb{C}$ and multiplication by a real number, see the proof of ([10], lemma 2.3), which can be identified with $M_{r+1} \times S^1$ with $M_{r+1}$ denoting the moduli space of conformal structures on the $r+1$-punctured sphere. On the other hand, the fibre of this projection over each point in $M_{r+1} \times S^1$ is precisely a moduli space of maps $u : \hat{S} \to M$ from a punctured Riemann surface of fixed conformal structure and fixed asymptotic markers (fixed by the map $h$, see discussion at the end) considered in ([18], 6.1). In particular, note that the special Floer equation $\tilde{\partial}_{J,h}(u) = 0$ from above indeed satisfies the monotonicity assumption in [18] since $\beta = dh_2$ immediately gives $d\beta \leq 0$. On the other hand, as in the case of the pair-of-pants product described before, we can still assign a class $A \in \pi_2(M)$ to each map $u : \hat{S} \to M$ (and hence every element in $M^r(\Gamma)$) by closing the punctured surface using the spanning surfaces chosen for the one-periodic orbits in $M$ corresponding to $\gamma^+$ and $\gamma_0, \ldots, \gamma_{r-1}$. Concerning the relation between the holomorphic map $h$ and the asymptotic markers, we remark that the projection $M^r(\Gamma) \to M(k_0, \ldots, k_{r-1})$ is indeed supposed to remember the asymptotic markers, that is, we want to think of $M(k_0, \ldots, k_{r-1})$ as the moduli spaces of full contact homology when $(M, \omega)$ is the point. With this in mind, one shall think of the factor $S^1$ as the space of asymptotic markers at the positive puncture which determines $h$ as well as $k_i$ asymptotic markers at the negative puncture $z_i$ for each $i = 0, \ldots, r-1$, where the combinatorical factor $1/\kappa^1$ takes care of the multiplicity and weights.

**Remark 2.3.** We emphasize that the monotonicity property of the Floer equation $\tilde{\partial}_{J,h}(u) = 0$ which we obtained naturally will be crucial in the case when we generalize the results of this paper from closed symplectic manifolds to (open) Liouville manifolds. Indeed, as it is discussed in ([18], appendix 4) and [1], the monotonicity is needed to establish a maximums principle for Floer solutions in Liouville manifolds which in turn is the crucial ingredient for compactness of the moduli spaces.
For the geometrical interpretation of the big pair-of-pants product, which we will define in the next section, we need a more geometrical view of contact homology. For this observe that the chain algebra $\mathcal{A}$ can be identified with the algebra $\mathcal{T}^{(0,0)} \mathcal{Q}$ of (polynomial) $(0,0)$-tensor fields (=functions) on the underlying coordinate super space $\mathcal{Q}$. On the other hand, the differential $\partial : \mathcal{T}^{(0,0)} \mathcal{Q} \to \mathcal{T}^{(0,0)} \mathcal{Q}$ is given by the vector field

$$X = \sum_{\gamma} \left(\sum_{\Gamma, \lambda} \frac{1}{r!} \kappa_{2} M^{r}_{\lambda \Gamma, \lambda} : q_{\Gamma}^{r} \omega^{\lambda}(\lambda) \right) \frac{\partial}{\partial q_{\gamma}} \in \mathcal{T}^{(1,0)} \mathcal{Q}.$$ 

**Proposition 2.4.** The pair $(\mathcal{Q}, X)$ defines a (infinite-dimensional) differential graded manifold $\mathcal{Q}_{X}$.

**Proof.** For the definition of a differential graded manifold we refer to [5]. The master equation $\partial \circ \partial = 0$ for the differential $\partial : \mathcal{A} \to \mathcal{A}$ translates for the corresponding vector field $X \in \mathcal{T}^{(1,0)} \mathcal{Q}$ into the identity

$$[X, X] = 2X^{2} = 0.$$ 

Here $[,]$ denotes the Lie bracket on vector fields and the first equality follows from the fact that $X$ is homogeneous of degree $+1$, $|X| = +1$. In other words, $X$ is an odd cohomological vector field (in the sense of [5]) and hence defines a differential graded manifold $\mathcal{Q}_{X}$. □

Note that the differential graded manifold $\mathcal{Q}_{X}$ is formal in the sense that we do not specify a topology on $\mathcal{Q}$ and hence on $\mathcal{Q}_{X}$. Nevertheless, the important property for us is that on the differential graded manifold $\mathcal{Q}_{X}$ one still has functions $\mathcal{T}^{(0,0)} \mathcal{Q}_{X}$ and vector fields $\mathcal{T}^{(1,0)} \mathcal{Q}_{X}$, which in turn can be used to define arbitrary tensor fields $\mathcal{T}^{(r,s)} \mathcal{Q}_{X}$.

First we use the fact (already observed in [9] in the section on satellites) that the above identity for $X$ implies that the Lie derivative defines a differential on arbitrary tensor fields $\mathcal{T}^{(r,s)} \mathcal{Q}$,

$$\mathcal{L}_{X} : \mathcal{T}^{(r,s)} \mathcal{Q} \to \mathcal{T}^{(r,s)} \mathcal{Q}, \mathcal{L}_{X} \circ \mathcal{L}_{X} = 0.$$ 

While $\mathcal{T}^{(0,0)} \mathcal{Q}_{X}$ agrees with contact homology, the space of vector fields $\mathcal{T}^{(1,0)} \mathcal{Q}_{X}$ is defined using the Lie derivative,

$$\mathcal{T}^{(0,0)} \mathcal{Q}_{X} := H^{s}(\mathcal{T}^{(0,0)} \mathcal{Q}, X), \mathcal{T}^{(1,0)} \mathcal{Q}_{X} := H^{s}(\mathcal{T}^{(1,0)} \mathcal{Q}, [X, \cdot]).$$

On the other hand, since the Lie derivative commutes with the contraction of tensors, we find that this also holds for arbitrary tensor fields,

$$\mathcal{T}^{(r,s)} \mathcal{Q}_{X} = H^{s}(\mathcal{T}^{(r,s)} \mathcal{Q}, \mathcal{L}_{X}).$$

We end this subsection with discussing the relation between the differential graded manifold from contact homology and the affine manifold structure of cylindrical cohomology. In order to clarify the role of homology and cohomology, we refer to the remark at the end of the proof of the theorem below.

Observe that the contact homology differential naturally can be written as an infinite sum, $X = \sum_{r=1}^{\infty} X_{r}$, where $X_{r} \in \mathcal{T}^{(1,0)} \mathcal{Q}$ contains only those summands with $q_{\gamma}$-monomials of length $r$. It is an important observation
that, in our case of Hamiltonian mapping tori, this sum indeed starts with
\( r = 1 \), since there obviously are no \( J \)-holomorphic disks in \( \mathbb{R} \times S^1 \times M \).
On the other hand, the first summand \( X_1 \) agrees with the differential \( \partial \) in
cylindrical homology,

\[
X_1 = \sum_{\gamma^+} \left( \sum_{\gamma^-} \frac{1}{\kappa_{\gamma^-}} \mathcal{M}^+_{\gamma^-}(A) \cdot q_{\gamma^-}^{-\ell(A)} \right) \frac{\partial}{\partial q_{\gamma^+}} \in T^{(1,0)} Q. 
\]
Furthermore, the first equation \([X_1, X_1] = 0\), obtained by expanding the
master equation \([X, X] = 0\) with respect to the sum \( X = X_1 + X_2 + \ldots \), just
reproduces the master equation \( \partial \circ \partial = 0 \) for the cylindrical theory. Using
this we can prove the following

**Proposition 2.5.** For each \((r, s) \in \mathbb{N} \times \mathbb{N}\) there exists a spectral sequence
computing the tensor field homology \( E_\infty = T^{(r,s)} Q_X \) with \( E_2 \)-page given by
the corresponding space of tensor fields on the affine manifold of cylindrical
contact cohomology, \( E_2 = T^{(r,s)} Q_{X_1} = T^{(r,s)} HC^*_{cyl} \).

**Proof.** The result uses the spectral sequence for filtered complexes. The
corresponding filtration subspaces \( T^{(r,s)}_{\geq \ell} Q \subset T^{(r,s)} Q \) are spanned by
\( q^\ell \partial / \partial q^\ell \otimes dq^{r-\ell} \) with \( \# \Gamma \geq \ell \). While it is easy to see that the
Lie derivative \( \mathcal{L}_X \) respects this filtration, we furthermore have that \( \mathcal{L}_X \)
maps \( T^{(r,s)}_{\geq \ell} Q \) to \( T^{(r,s)}_{\geq \ell + 1} Q \). On the other hand, the statement that
\( T^{(r,s)} Q_{X_1} = T^{(r,s)} HC^*_{cyl} \), that is, the differential graded manifold \( Q_{X_1} \)
agrees with the affine manifold of cylindrical contact cohomology, follows
immediately from the definition of the Lie derivative with respect to
\( X_1 \) using \( X_1(q_{\gamma^+}) = \sum_{\gamma^-} \frac{1}{\kappa_{\gamma^-}} \mathcal{M}^+_{\gamma^-}(A) \cdot q_{\gamma^-}^{-\ell(A)} \), \([X_1, \partial / \partial q_{\gamma^-}] = \sum_{\gamma^-} \frac{1}{\kappa_{\gamma^-}} \mathcal{M}^+_{\gamma^-}(A) \cdot \partial / \partial q_{\gamma^+} q_{\gamma^-}^{-\ell(A)} \). \( \square \)

Note that while the first formula for the coordinates \( q_{\gamma^-} \) is the differential
in cylindrical contact homology, the second formula for the tangent vectors
\( \partial / \partial q_{\gamma^-} \) (corresponding to points \( \gamma \)) is the differential in cylindrical contact
cohomology from before. Since the grading of \( q_{\gamma^-} \) and \( \partial / \partial q_{\gamma^-} \) differs by sign,
both differentials raise the grading by one.

Indeed the fact that, in the case of (Hamiltonian) mapping tori, the co-
homological vector field \( X \in T^{1,0} Q \) from full contact homology does not
have a constant term, can also be used in another direction. Since the latter
automatically implies that the cohomological vector field vanishes at zero,
\( X(0) = 0 \) for \( q = 0 \), we can define the tangent space of \( Q_X \) to be

\[
T_0 Q_X := Q_{X_1} \overset{1}{=} H^*(T_0 Q, \partial) = HC^*_{cyl}(M_0) = \bigoplus_k HF^*(\phi^k),
\]
where \( \partial : T_0 Q \to T_0 Q \) denotes the coboundary of cylindrical contact
cohomology. This is motivated by the following

**Proposition 2.6.** Since \( X(0) = 0 \), there exists a natural restriction (or
evaluation) map for tensor fields at \( q = 0 \),

\[
T^{(r,s)} Q_X \to (T_0 Q_X^{\otimes s})^* \otimes T_0 Q_X^{\otimes r}, \ [\alpha] \mapsto [\alpha|_{q=0}].
\]
Proof. For the proof we show that the Lie derivative of a tensor field \( \alpha \in T^{(r,s)}Q \) in the direction of \( X \) at \( q = 0 \) can be computed from the restriction of the tensor \( \alpha_0 = \alpha|_{q=0} \) at \( q = 0 \) and the cylindrical contact cohomology differential \( \partial : T_0 Q \to T_0 Q \) by

\[
(\mathcal{L}_X \alpha)|_{q=0} = \alpha_0 \circ \partial - \partial \circ \alpha_0,
\]

where here \( \partial \) denotes the obvious extension (using Leibniz rule) to \( T_0 Q^\otimes r \), \( T_0 Q^\otimes s \).

First, using the definition of the Lie derivative for tensor fields, we find that

\[
(\mathcal{L}_X \alpha)|_{q=0}(\frac{\partial}{\partial q^0}|_q \otimes \ldots \otimes dq^+_\gamma)|_{q=0}
\]

is given by

\[
\left(X\left(\alpha\left(\frac{\partial}{\partial q^0} \otimes \ldots \otimes dq^+_\gamma\right)\right)\right)|_{q=0} - \left(\alpha\left(\mathcal{L}_X \frac{\partial}{\partial q^0} \otimes \ldots \otimes dq^+_\gamma\right)\right)|_{q=0} - (\ldots) - (-1)^{\sum q^0 + \ldots + q^+ - 1} \alpha_0\left(\frac{\partial}{\partial q^0} \otimes \ldots \otimes (\mathcal{L}_X dq^+_\gamma)|_{q=0}\right).
\]

Now employing that \( X|_{q=0} = 0 \), we find that the first summand involving the derivative of \( \alpha \) vanishes and only the other summands involving only the value of the tensor field at the point zero remain,

\[
- \alpha_0\left(\left(\mathcal{L}_X \frac{\partial}{\partial q^0} \right)|_{q=0} \otimes \ldots \otimes dq^+_\gamma|_{q=0}\right) - (\ldots) - (-1)^{\sum q^0 + \ldots + q^+ - 1} \alpha_0\left(\frac{\partial}{\partial q^0} \otimes \ldots \otimes (\mathcal{L}_X dq^+_\gamma)|_{q=0}\right).
\]

With the observation that the cylindrical contact cohomology differential \( \partial \) is given by \( \partial : T_0 Q \to T_0 Q \),

\[
\frac{\partial}{\partial q^-|_{q=0}} \mapsto \left(\mathcal{L}_X \frac{\partial}{\partial q^-}\right)|_{q=0} = \frac{\partial X}{\partial q^-|_{q=0}} = \sum_{\gamma^+:A} 1_{\kappa^-} \mathcal{M}_{\gamma^+}^\gamma(A) t^{\omega(A)} \cdot \frac{\partial X}{\partial q^+|_{q=0}},
\]

the claim follows. \( \square \)

Note that the same proof shows that one can define a restriction at all points \( q_0 \in Q \) with \( X(q_0) = 0 \). It maps to tensor products of the deformed cylindrical contact homology \( HC_{cyl,q_0} = H^*(T_{q_0} Q, \partial_{q_0}) \) with the same chain space \( T_{q_0} Q \cong T_0 Q \) but deformed differential given by

\[
\partial_{q_0} : T_{q_0} Q \to T_{q_0} Q, \quad \frac{\partial}{\partial q^-|_{q=q_0} \mapsto \frac{\partial X}{\partial q^-|_{q=q_0}}.}
\]
2.2. **Big pair-of-pants product and cohomology F-manifolds.** In this section we define the big version of the pair-of-pants product in Hamiltonian Floer theory by building on the relation between Hamiltonian Floer theory and Eliashberg-Givental-Hofer’s symplectic field theory discussed above. More precisely we show that, using the special geometry of Hamiltonian mapping tori, one can enrich the algebraic framework of symplectic field theory by product structures.

*New moduli spaces of Floer solutions.* Recall from proposition 2.2 above that the moduli spaces $\mathcal{M}^\gamma(\Gamma)/\mathbb{R}$ of $J$-holomorphic curves $\tilde{u} = (h, u) : \hat{S} \to \mathbb{R} \times S^1 \times M$, used in the definition of the full contact homology of the Hamiltonian mapping torus $M_\phi \cong S^1 \times M$, can be identified with moduli spaces of Floer solutions $u : \hat{S} \to M$ (in the sense of [18], 6.1) starting from a Riemann sphere with one positive and many negative punctures. In contrast to the moduli spaces considered in [18], we allow the conformal structure on the punctured Riemann sphere to vary and allow the asymptotic markers above and below to rotate simultaneously. In particular, note that the moduli space used in subsection 1.1 for the definition of the Lie bracket $[\cdot, \cdot]$ and the moduli space used in subsection 1.1 for defining the pair-of-pants product $\star_0$ on $\text{HC}^*(M_\phi) = \bigoplus_k \text{HF}^*(\phi^k)$ are closely related. Indeed, while in the first case (after counting with the right combinatorial factors) the asymptotic marker at the positive puncture is unconstrained but fixes the asymptotic markers at the negative punctures, in the second case the all asymptotic markers are fixed right away.

Generalizing this, the new moduli spaces used to define the big pair-of-pants product will consist of Floer solutions $u : \hat{S} \to M$ with an arbitrary number of negative punctures and varying conformal structure but with fixed asymptotic markers at all punctures (depending on the underlying conformal structure). Note that they can be considered as an intermediate case between the moduli spaces $\mathcal{M}^\gamma(\Gamma)/\mathbb{R}$ (where the asymptotic markers as well as the conformal structure are not fixed) used to define the $L_\infty$-structure and the moduli spaces of Floer solutions considered in [18] and [21] (where the asymptotic markers as well as the conformal structure are fixed) used to define the TQFT structure in Floer theory. In particular, while the geometric information of the TQFT structure can be recovered from the small pair-of-pants product (and the unit), this will in general not be the case for the big pair-of-pants product. This should be compared to the difference between the small and the big quantum product: While the first sees just the geometric information of the 3-point Gromov-Witten invariants, the latter uses the information of all rational Gromov-Witten invariants.

Reduced to the essence, for the definition of the new moduli spaces for the big pair-of-pants product we use that the fact that, in contrast the well-known case of contact manifolds, there exists a natural projection from the moduli space $\mathcal{M}^\gamma(\Gamma)$ of $J$-holomorphic curves $\tilde{u} = (h, u) : \hat{S} \to \mathbb{R} \times M_\phi \cong \mathbb{R} \times S^1 \times M$ to a moduli space $\mathcal{M}(k_1, \ldots, k_{r-1})$ of holomorphic functions $h : \hat{S} \to \mathbb{R}$.
$\hat{S} \to \mathbb{R} \times S^1$, see the proof of proposition 2.2. In order to fix the asymptotic markers and hence the holomorphic map $h : \hat{S} \to M$, we introduce an additional marked point $z^*$ on the underlying punctured Riemann sphere $\hat{S} = S^2 \setminus \{z_0, \ldots, z_{r-1}, \infty\}$. As in [9] we denote by $\mathcal{M}^+_1(\Gamma)$ the moduli space of $J$-holomorphic curves $(\hat{u}, z_0, \ldots, z_{r-1}, z^*)$ in $\mathbb{R} \times S^1 \times M$ with one (unconstrained) additional marked point. Using the natural projection map $M_\phi \cong S^1 \times M \to S^1$, note that there exists an evaluation map

$$\text{ev} : \mathcal{M}^+_1(\Gamma) \to \mathbb{R} \times S^1, \quad (h, u, z_0, \ldots, z_{r-1}, z^*) \mapsto h(z^*)$$

and we denote by $\mathcal{M}^+_2(\Gamma) = \text{ev}^{-1}(0) \subset \mathcal{M}^+_1(\Gamma)$ the submoduli space of $J$-holomorphic curves where the additional marked point gets mapped to the special point $0 \in \mathbb{R} \times S^1$. Since $h : \hat{S} \to \mathbb{R} \times S^1$ is a $k$-fold covering map ($k = k_0 + \ldots + k_{r-1}$), it however follows that the natural map $\mathcal{M}^+_2(\Gamma) \to \mathcal{M}^+_1(\Gamma)$ (given by forgetting $z^*$) is just a $k$-fold covering (away from the codimension-two-locus where $0 \in \mathbb{R} \times S^1$ is a critical value of $h$), similar as in the proof of the divisor equation in Gromov-Witten theory.

In order to fix the asymptotic markers and hence the holomorphic map $h$, we have to constrain the additional marked point (a priori) without using the map. Note first that, in analogy to the moduli spaces used in the definition of the pair-of-pants product in 1.1, the moduli space $\mathcal{M}^+_1(\gamma_0, \gamma_1)$ can equivalently be defined as the set of $J$-holomorphic maps $\hat{u} : \hat{S} \to M$ starting from the three punctured sphere $\hat{S} = S^2 \setminus \{0, 1, \infty\}$. Here we use that we can kill the automorphisms of the domain by setting $z_0 = 0$, $z_1 = 1$ (and $z_\infty = \infty$), which equivalently provides us with unique coordinates on the punctured sphere.

Using these unique coordinates, we can fix the position of the additional marked $z^*$ on $\hat{S}$ a priori for each moduli space $\mathcal{M}^+_1(\gamma_0, \gamma_1)$, where we assume that $z^*$ does not accidently coincides with the branch point of the induced map $h : \hat{S} \to \mathbb{R} \times S^1$. With this we define the submoduli space $\mathcal{M}^+_{\gamma_0, \gamma_1} \subset \mathcal{M}^+_1(\gamma_0, \gamma_1) \subset \mathcal{M}^+_1(\gamma_0, \gamma_1)$ of $J$-holomorphic maps $(h, u, z^*) \in \mathcal{M}^+_1(\gamma_0, \gamma_1)$ where the additional marked $z^*$ is constrained using the unique coordinates given by the three punctures $z_0 = 0$, $z_1 = 1$ (and $z_\infty = \infty$) and required to get mapped to $0 \in \mathbb{R} \times S^1$ under the map $h$.

**Proposition 2.7.** Using the new moduli spaces $\mathcal{M}^+_1(\gamma_0, \gamma_1)$ and natural identification of the cochain space $C^*$ of cylindrical contact cohomology with the sum $\bigoplus_k \text{CF}^*(\phi^k)$ of the cochain spaces of the Floer cohomologies, the pair-of-pants product can be defined on the chain level on cylindrical contact cohomology by

$$\gamma_0 *_0 \gamma_1 = \sum_{\gamma^+, A, \gamma^{-}} \frac{1}{k_{\gamma_0} k_{\gamma_1}} \cdot \frac{1}{k} \cdot \# \mathcal{M}^+_1(\gamma_0, \gamma_1) \cdot \gamma^+ \cdot \nu(A).$$

**Proof.** Following proposition 2.2, the moduli space $\mathcal{M}^+_1(\Gamma) / \mathbb{R}$ can be identified with the moduli space of Floer solutions $u : \hat{S} \to M$ starting from the three-punctured sphere with unconstrained asymptotic marker at the
positive puncture which in turn constrains the asymptotic markers at the two negative punctures via the induced map $h$. While the submoduli space $M_{\gamma_0,\gamma_1}^{+} \subset M^+_{\gamma_0,\gamma_1}$ is precisely characterized by the fact that the map $h$ is fixed (note that the requirement that the additional marked point gets mapped to $0 \in \mathbb{R} \times S^1$ automatically kills the $\mathbb{R}$-symmetry on the moduli space), note that there is just a $k$-to-one correspondence between $\tilde{J}$-holomorphic curves in $M_{\gamma_0,\gamma_1}^{+}$ and Floer solutions $u : \hat{S} \to M$ with fixed asymptotic markers at all punctures, since there is just a $k$-to-one correspondence between asymptotic markers at the positive puncture and corresponding maps $h$ to the cylinder. Note that coefficient $1/k$ can be equivalently be interpreted by saying that we do not count elements in the submoduli space of $M_{\gamma_0,\gamma_1}^{+}$, but in the corresponding submoduli space of the quotient $M_{\gamma_0,\gamma_1}^{+}/\mathbb{Z}_k$ under the natural $\mathbb{Z}_k$-action given by rotating the asymptotic marker at the positive puncture by $1/k \in S^1 = \mathbb{R}/\mathbb{Z}$. □

Note that the above definition of the moduli space $M_{\gamma_0,\gamma_1}^{+} \subset M_{\gamma_0,\gamma_1}^{+}$ does not immediately generalize to the case when the number of punctures is greater than three. For this recall that we fixed the position of the additional marked point $z^*$ using the canonical coordinates determined by the three punctures. When there are more than three marked points on the sphere, such canonical coordinates only exist whenever one selects three marked points from the given $r$ marked points. It is the key observation for our definition of the big pair-of-pants product that, as for the big quantum product but unlike for the Gromov-Witten potential, such choice of three special punctures (the positive puncture and two of the negative punctures) is natural.

Without loss of generality, let us assume that we use the first two and the last marked point to define coordinates by setting $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. While in the case of three punctures from before we could directly use the resulting coordinates to fix the position of the additional marked point for every moduli space $M_{\gamma_0,\gamma_1}^{+}$, we now need to be more careful than before. The reason is that, in contrast to before, it might happen that the additional marked point might coincide with one of the remaining punctures as the $\tilde{J}$-holomorphic curve $(\tilde{u}, 0, 1, z_2, \ldots, z_{r-1}, z^*)$ varies inside the moduli space $M_{\gamma_0,\gamma_1}^{+}$. In order to solve this problem, we make use of the language of coherent collections of sections in tautological line bundles developed in [12].

Observing that there are again no nontrivial automorphisms of the domain, note that there exists a forgetful map,

$$\text{ft} : M_{\gamma_0,\gamma_1}^{+} / \mathbb{R} \to S^2, \ (\tilde{u}, 0, 1, z_2, \ldots, z_{r-1}, z^*) \mapsto z^*,$$

which extends naturally extends over the compactification of the moduli space. Note that $S^2$ can be identified with the (compactified) moduli space $M_4$ of Riemann spheres with four marked points $(z_0, z_1, z_\infty, z^*)$, where the identification is precisely given by setting $(z_0, z_1, z_\infty) = (0, 1, \infty)$. Instead of fixing the position of $z^*$, we can equivalently think about integrating (the
pull back) of the Poincare dual of the point class in the cohomology ring of \( \mathcal{M}_4 \) over the moduli space. As it commonly known in Gromov-Witten theory, the relevant cohomology class agrees with the first Chern class (and hence the Euler class) of the tautological line bundle \( \mathcal{L} = \mathcal{L}_4 \) over \( \mathcal{M}_4 \) whose fibre over \( z^* \in S^2 \cong \mathcal{M}_4 \) by definition is the cotangent space \( T^*_z S^2 \). Indeed, we see that \( \mathcal{L} \to \mathcal{M}_4 \) simply agrees with the cotangent bundle \( T^* S^2 \to S^2 \) which is a line bundle of degree one.

Instead of integrating the first Chern class of \( \mathcal{L} \) over the moduli space \( \mathcal{M}_4 \), we can equivalently count the zeroes of a generic (and hence transversal) section \( s \) in \( \mathcal{L} \to \mathcal{M}_4 \cong S^2 \). In view of the above picture, the unique zero of \( s, s^{-1}(0) = \{ z^* \} \subset S^2 \), fixes the position of additional marked point. For fixing the position of the additional marked point on every \( J \)-holomorphic curve \( (\tilde{u} = (h, u), 0, 1, z_1, \ldots, z_j, 1, z^*) \) in \( \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \), we study the pullback bundle

\[
\mathcal{L}^* := \text{ft}^* \mathcal{L} \to \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R}
\]

under the forgetful map \( \text{ft} : \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \to \mathcal{M}_4 \). For this recall that we want to ensure that the position of the additional marked point is varying generically with the position of the punctures, and that on \( \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \), in contrast to \( \mathcal{M}_4 \), we now see the positions of the remaining punctures, see also the remark below.

Recall that on the (compactified) moduli space \( \mathcal{M}_4 \cong S^2 \) of four-punctured spheres the count of zeroes is independent of the choice of the generic section \( s \) in \( \mathcal{L} \to \mathcal{M}_4 \) and given by the integral of the first Chern class over the fundamental class. However, this is no longer true for sections in the pullback bundles \( \mathcal{L}^* \to \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \), since the compactified moduli space is no longer closed but has codimension one boundary \( \partial^1 \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \) of the form

\[
\mathcal{M}_{1, J}^+ (\Gamma)/\mathbb{R} \times \mathcal{M}_{1, J}^C (\Gamma^0)/\mathbb{R} \text{ and } \mathcal{M}_{1, J}^+ (\Gamma)/\mathbb{R} \times \mathcal{M}_{1, J}^C (\Gamma^0)/\mathbb{R},
\]

see the last subsection. Note that we have to allow the additional marked point to sit either on the upper or the lower level of the broken \( J \)-holomorphic curve (in the sense of [2]). We emphasize that by the naturality of the map \( \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \to \mathcal{M}_4 \cong S^2 \), the pullback tautological bundle \( \mathcal{L}^* = \text{ft}^* \mathcal{L} \) extends smoothly over the codimension-one boundary.

Following [12], we now consider coherent collections of sections \( (s) \) in the pullback tautological bundles \( \mathcal{L}^* \) over all moduli spaces \( \mathcal{M}_{1, J}^+ (\Gamma)/\mathbb{R} \). Roughly speaking, coherency means that the sections chosen for the moduli spaces appearing in the codimension-one boundary of a given moduli space determine the chosen section in the tautological line bundle of the given moduli space on its codimension one-boundary. Furthermore we assume, as in [12], that all sections are generic in the sense that they meet the zero section transversally. More precisely, since here we are not working with one of the classical tautological line bundles over \( \mathcal{M}_{1, J}^+ (\gamma_0, \gamma_1, \Gamma) \) but with the pullback of the tautological line bundle over \( \mathcal{M}_4 \) under the forgetful
map, in contrast to the definition in [12] we need to make a small case distinction.

Indeed, we see that the forgetful map from $\mathcal{M}_1^+(\Gamma')$ or $\mathcal{M}_1^+(\Gamma'')$ to $\mathcal{M}_4$ is only nontrivial in the case that $\Gamma'$ contains $\gamma_0$ or $\gamma_1$, or $\Gamma''$ contains $\gamma_0$ and $\gamma_1$, respectively. In these cases (only) we require that the chosen section $s$ for $\mathcal{M}_1^+(\gamma_0, \gamma_1, \Gamma)$ agrees with the chosen section $s'$ for $\mathcal{M}_1^+(\Gamma')$ or the chosen section $s''$ for $\mathcal{M}_1^+(\Gamma'')$, respectively. Note that in the other cases the forgetful map to $\mathcal{M}_4$ maps the whole component carrying the additional marked point to a single point.

**Definition 2.8.** For a given collection $(s)$ of sections in the pullback bundles $L^* = ft^* L$ over all moduli spaces $\mathcal{M}^+(\gamma_0, \gamma_1, \Gamma)/\mathbb{R}$, we define the new moduli spaces $\mathcal{M}_{\gamma_0, \gamma_1}^+(\Gamma)$ by

$$\mathcal{M}_{\gamma_0, \gamma_1}^+(\Gamma) := s^{-1}(0) \cap \text{ev}^{-1}(0) \subset \mathcal{M}_1^+(\gamma_0, \gamma_1, \Gamma)/\mathbb{R},$$

where $\text{ev} : \mathcal{M}_1^+(\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \to S^1$, $(\bar{u}, 0, 1, z_2, \ldots, z_{r-1}, z^*) \mapsto h_2(z^*)$ is the natural evaluation map to the circle with $\bar{u} = (h_1, h_2, u) : \tilde{S} \to \mathbb{R} \times S^1 \times M$.

Note that equivalently we can define $\mathcal{M}_{\gamma_0, \gamma_1}^+(\Gamma)$ to be the zero set of $s$, viewed as a section in the pullback bundle $L^*$ over $\mathcal{M}_s^+(\gamma_0, \gamma_1, \Gamma) \subset \mathcal{M}_1^+(\gamma_0, \gamma_1, \Gamma)$. We assume that the genericity of section includes the fact that even the intersections $s^{-1}(0) \cap \text{ev}^{-1}(0)$ are transversal. On the other hand, we furthermore assume, as in [12], that the coherent collections are chosen symmetric with respect to reordering of the punctures. More precisely, since the first two punctures $z_0 = 0$, $z_1 = 1$ play a different role than the remaining punctures $z_2, \ldots, z_{r-1}$, in contrast to [12], we only require symmetry with respect to reordering the ordered tuples $(z_0, z_1)$ and $(z_2, \ldots, z_{r-1})$. Recall from [12] that the simultaneous requirement of symmetry and genericity requires that our coherent sections need to be multi-sections in the sense of [4].

**Remark 2.9.** Since we just need to ensure that the position of the additional marked point is generically varying with the position of the punctures is independent of the map $u$ to $M$, for our construction is sufficient to study the (intermediate) pullback bundle $ft^* L \to \mathcal{M}_1(k_1, \ldots, k_{r-1})$ under the second forgetful map (again denoted by $ft$) in $\mathcal{M}_1^+(\gamma_0, \gamma_1, \Gamma) \to \mathcal{M}_1(k_1, \ldots, k_{r-1}) \to \mathcal{M}_4$, where $\mathcal{M}_1(k_1, \ldots, k_{r-1})$ denotes the underlying moduli space of holomorphic functions $h$ (with one additional marked point) from the proof of proposition 2.2. In particular, our construction just uses additional geometric structures that exist on the moduli spaces of symplectic field theory for the target manifold $S^1$.

**Big pair-of-pants product.** Note that, in view of our discussion above, the new moduli spaces $\mathcal{M}_{\gamma_0, \gamma_1}^+(\Gamma)$ are indeed generalizations of the moduli spaces $\mathcal{M}_{\gamma_0, \gamma_1}^+$ in the sense that $\mathcal{M}_{\gamma_0, \gamma_1}^+ = \mathcal{M}_{\gamma_0, \gamma_1}(\emptyset)$. Together with proposition 2.6 this motivates the following definition.
Definition 2.10. On the chain level, the big pair-of-pants product is defined to be the $(1,2)$-tensor field $\star \in T^{(1,2)} Q$ on the super space $Q$ given at each point $q = (q_\gamma) \in Q$ by $\star_q : T_q Q \otimes T_q Q \to T_q Q$ with
\[
\frac{\partial}{\partial q_\gamma} \star q \frac{\partial}{\partial q_\gamma}
\]
given by
\[
\sum_{\gamma, A} \frac{1}{(r-2)!} \kappa_{\gamma_0} \kappa_{\gamma_1} \kappa_1 \cdot \frac{1}{k} \# \mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma, A) \cdot q^{r_\omega(A)} \frac{\partial}{\partial q_\gamma}
\]
where $k = k_0 + \ldots + k_{r-1}$ is the period of the closed orbit $\gamma^+.

In analogy to Gromov-Witten theory, from $\mathcal{M}_{\gamma_0,\gamma_1}^+ = \mathcal{M}_{\gamma_0,\gamma_1}^+(\emptyset)$ it follows that the big pair-of-pants product is indeed a deformation of the classical pair-of-pants product in the sense that at $q = (q_\gamma) = 0$ it agrees with the small product,
\[
\frac{\partial}{\partial q_0} \star_0 \frac{\partial}{\partial q_1} = \sum_{\gamma} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1}} \cdot \left( \sum_A \# \mathcal{M}_{\gamma_0,\gamma_1}^+(A) \cdot q^{r_\omega(A)} \right) \frac{\partial}{\partial q_\gamma}.
\]

For this recall that we again identify the linear coordinate space $Q$ with the tangent space $T_0 Q$ at zero by identifying each closed Reeb orbit $\gamma$ with the tangent vector $\partial/\partial q_\gamma$.

Like for the small pair-of-pants product, we can only expect the big pair-of-pants product to satisfy algebraic properties like associativity and commutativity when viewing it as an element in some cohomology. The main step is to show that the big pair-of-pants product indeed defines an element on cohomology, that is, descends from the super space $Q$ to the the differential graded manifold $Q_X$ from contact homology.

Proposition 2.11. The big pair-of-pants product $\star \in T^{(1,2)} Q$ and the homological vector field $X \in T^{(1,0)} Q$ from contact homology satisfy
\[
\mathcal{L}_X \star = 0.
\]

Of course, the proof of the theorem relies on the translation from geometry into algebra of a compactness result for the new moduli spaces. This is the content of the following lemma. We emphasize that all occurring products of moduli spaces are to be understood as direct products as in [9], see also the appearance of combinatorical factors in the subsequent proof of the theorem.

Lemma 2.12. By counting broken $\tilde{J}$-holomorphic curves (with signs) in the codimension-one boundary of the moduli space $\mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma)$ one obtains that the sum of the following terms is equal to zero,

\[
(1) \sum_{\gamma \in \Gamma'} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa_{\Gamma'}} \frac{1}{k} \# \mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \cdot \frac{1}{\kappa_{\Gamma''}} \# \mathcal{M}^+(\Gamma'')
\]
\[
(2) \sum_{\gamma_0 \in \Gamma''} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa_{\Gamma'}} \frac{1}{k} \# \mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \cdot \frac{1}{\kappa_{\Gamma''}} \# \mathcal{M}^+(\Gamma'')
\]
\[
(3) \sum_{\gamma_1 \in \Gamma''} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa_{\Gamma'}} \frac{1}{k} \# \mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \cdot \frac{1}{\kappa_{\Gamma''}} \# \mathcal{M}^+(\Gamma'')
\]
the codimension one boundary of the moduli space.

Pactness result for the moduli space for contact homology stated above that

First, it is just a combinatorial exercise to deduce from the co-

Proof. First, it is just a combinatorial exercise to deduce from the compactness result for the moduli space for contact homology stated above that the codimension one boundary of the moduli space $\mathcal{M}^{\gamma^+}(\gamma_0, \gamma_1, \Gamma)$ has the corresponding components

1. $\mathcal{M}^{\gamma^+}(\gamma_0, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$ with $\gamma \in \Gamma'$,
2. $\mathcal{M}^{\gamma^+}(\gamma, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$ with $\gamma_0 \in \Gamma''$,
3. $\mathcal{M}^{\gamma^+}(\gamma_0, \gamma, \Gamma') \times \mathcal{M}(\Gamma'')$ with $\gamma_1 \in \Gamma''$,
4. $\mathcal{M}^{\gamma^+}(\Gamma') \times \mathcal{M}(\gamma_0, \gamma_1, \Gamma'')$ with $\gamma \in \Gamma'$.

For this observe that after splitting up into a two-level holomorphic curve either the three special punctures still lie on the same component (which leads to components of type 1) or there are two special punctures on one component and one special puncture on the other component (which leads to components of type 2, 3 and 4). After introducing an unconstrained additional marked point, it follows that each of the boundary components above gives two boundary components, depending on whether the additional marked point sits on the upper or the lower level of the broken holomorphic curve.

Let us first consider the case where the additional marked point sits on the unique component which carries two or three of the special punctures. Denoting by $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma') \subset \mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma')$ (and so on) the zero set of the (coherently) chosen section in the corresponding tautological line bundle, it follows from the coherency that the codimension-one boundary of the zero set $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma) \subset \mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma)$ has the analogous components

1. $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
2. $\mathcal{M}_1^{\gamma^+}(\gamma, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
3. $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma, \Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
4. $\mathcal{M}(\Gamma') \times \mathcal{M}(\gamma_0, \gamma_1, \Gamma'')/\mathbb{R}$.

On the other hand, after employing the evaluation map to $S^1$ given by the additional marked point, we get precisely the components that we want to count for our statement,

1. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
2. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
3. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
4. $\mathcal{M}(\Gamma') \times \mathcal{M}(\gamma_0, \gamma_1, \Gamma'')/\mathbb{R}$.

Concerning the combinatorial factors in the statement, observe that the multiplicities $k(k')$ of $\gamma^+ (\gamma')$ show up because induced holomorphic maps to the cylinders are $k$- ($k'$) fold coverings. More precisely, after defining

\[
\sum_{\gamma \in \Gamma'} \frac{1}{k \Gamma^+} \# \mathcal{M}(\Gamma') \cdot \frac{1}{k_0 \Gamma^+} \frac{1}{k_0 \Gamma^+} \# \mathcal{M}_{\gamma_0, \gamma_1}^{\gamma'}(\Gamma''),
\]

where $k (k')$ is the period of $\gamma^+ (\gamma')$ and we take the union over all $\Gamma', \Gamma''$ whose union (apart from the special orbits explicitly mentioned) is $\Gamma$. 

For this observe that after splitting up into a two-level holomorphic curve either the three special punctures still lie on the same component (which leads to components of type 1) or there are two special punctures on one component and one special puncture on the other component (which leads to components of type 2, 3 and 4). After introducing an unconstrained additional marked point, it follows that each of the boundary components above gives two boundary components, depending on whether the additional marked point sits on the upper or the lower level of the broken holomorphic curve. 

Let us first consider the case where the additional marked point sits on the unique component which carries two or three of the special punctures. Denoting by $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma') \subset \mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma')$ (and so on) the zero set of the (coherently) chosen section in the corresponding tautological line bundle, it follows from the coherency that the codimension-one boundary of the zero set $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma) \subset \mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma)$ has the analogous components

1. $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
2. $\mathcal{M}_1^{\gamma^+}(\gamma, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
3. $\mathcal{M}_1^{\gamma^+}(\gamma_0, \gamma, \Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
4. $\mathcal{M}(\Gamma') \times \mathcal{M}(\gamma_0, \gamma_1, \Gamma'')/\mathbb{R}$.

On the other hand, after employing the evaluation map to $S^1$ given by the additional marked point, we get precisely the components that we want to count for our statement,

1. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
2. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
3. $\mathcal{M}_0^{\gamma^+}(\Gamma') \times \mathcal{M}(\Gamma'')/\mathbb{R}$,
4. $\mathcal{M}(\Gamma') \times \mathcal{M}(\gamma_0, \gamma_1, \Gamma'')/\mathbb{R}$.

Concerning the combinatorial factors in the statement, observe that the multiplicities $k(k')$ of $\gamma^+ (\gamma')$ show up because induced holomorphic maps to the cylinders are $k$- ($k'$) fold coverings. More precisely, after defining
\( \mathcal{M}_{\gamma_0, \gamma_1}^+ (\Gamma) \subset \mathcal{M}_{1}^{\gamma_1+1, \gamma_1}(\gamma_0, \gamma_1, \Gamma)/\mathbb{R} \) by applying the evaluation map to the circle, we still have \( k \) possible choices for the asymptotic markers at the positive punctures. Finally, concerning the remaining codimension-one boundary components of \( \mathcal{M}_{1}^{\gamma_1+1} (\gamma_0, \gamma_1, \Gamma) \),

1. \( \mathcal{M}_{\gamma_0, \gamma_1}^+ (\gamma_0, \gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}_{1}^{\gamma_1}(\Gamma'')/\mathbb{R} \),
2. \( \mathcal{M}_{\gamma_0, \gamma_1}^+ (\gamma_1, \Gamma')/\mathbb{R} \times \mathcal{M}_{1}^{\gamma_1}(\Gamma'')/\mathbb{R} \),
3. \( \mathcal{M}_{\gamma_0, \gamma_1}^+ (\gamma_0, \Gamma')/\mathbb{R} \times \mathcal{M}_{1}^{\gamma_1}(\Gamma'')/\mathbb{R} \),
4. \( \mathcal{M}_{1}^{\gamma_1}(\Gamma')/\mathbb{R} \times \mathcal{M}_{1}^{\gamma_1}(\gamma_0, \gamma_1, \Gamma'')/\mathbb{R} \),

it suffices to observe that these boundary components do not contribute to the codimension-one boundary of \( \mathcal{M}_{\gamma_0, \gamma_1}^+ (\Gamma) \). The reason is that, since the additional marked points on each holomorphic curve get mapped to a single point under extension of the forgetful map to the boundary of \( \mathcal{M}_{1}^{\gamma_1+1} (\gamma_0, \gamma_1, \Gamma) \), the pullback tautological line bundle is indeed trivial on these components and we can always assume that the coherent section has no zeroes over these components.

\[ \square \]

For the proof of the theorem it remains to translate the geometrical result of the lemma into algebra.

**Proof. (of the theorem)** Using the definition of the Lie derivative of higher tensors, we obtain for any choice of basis vectors \( \partial/\partial q_{\gamma_0}, \partial/\partial q_{\gamma_1} \in \mathcal{T}(1,0) \mathbb{Q} \) and forms \( dq_{\gamma_+} \in \mathcal{T}^{(0,1)} \mathbb{Q} \) that

\[
(\mathcal{L}_X \ast) \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes dq_{\gamma_+} \right) = \mathcal{L}_X \ast \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes dq_{\gamma_+} \right) \\
- \ast \left( \mathcal{L}_X \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes dq_{\gamma_+} \right) \right) \\
- (-1)^{|q_{\gamma_0}|} \ast \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \mathcal{L}_X \frac{\partial}{\partial q_{\gamma_1}} \otimes dq_{\gamma_+} \right) \\
- (-1)^{|q_{\gamma_0}| + |q_{\gamma_1}|} \ast \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes \mathcal{L}_X dq_{\gamma_+} \right).
\]

Now using the definition of \( \ast \in \mathcal{T}^{(1,2)} \mathbb{Q} \),

\[
\ast \left( \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes dq_{\gamma_+} \right) = \sum_{\Gamma, A} \frac{1}{(r-2)!} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa^+} \frac{1}{k} \# \mathcal{M}_{\gamma_0, \gamma_1}^+ (\Gamma) \cdot q^\Gamma t^{\omega(A)}
\]

we find that the first summand is given by

\[
\mathcal{L}_X \left( \sum_{\Gamma, A} \frac{1}{(r-2)!} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa^+} \frac{1}{k} \# \mathcal{M}_{\gamma_0, \gamma_1}^+ (\Gamma) \cdot q^\Gamma t^{\omega(A)} \right) \\
= \sum_{\Gamma, A} \frac{1}{(r-2)!} \frac{1}{\kappa_{\gamma_0} \kappa_{\gamma_1} \kappa^+} \frac{1}{k} \# \mathcal{M}_{\gamma_0, \gamma_1}^+ (\Gamma) \cdot \mathcal{L}_X q^\Gamma \cdot t^{\omega(A)}.
\]

Together with

\[
\mathcal{L}_X q_\gamma = X(q_\gamma) = \sum_{\Gamma, A} \frac{1}{r!} \frac{1}{\kappa^+} \# \mathcal{M}_\gamma (\Gamma, A) \cdot q^\Gamma t^{\omega(A)}
\]
and using the Leibniz rule, we find that the first summand is precisely counting the boundary components of type 1, $\mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \times \mathcal{M}_\gamma^+(\Gamma'')$. For the combinatorical factors, observe that there are $\kappa_\gamma$ ways to glue two multiply-covered orbit $\gamma$.

For the other summands, we can show in the same way that they correspond to the other boundary components.

Indeed, using
\[
\mathcal{L}_X \frac{\partial}{\partial q_{\gamma_0}} = \frac{\partial X}{\partial q_{\gamma_0}} = \sum_{\gamma} \sum_{\Gamma, A} \frac{1}{r!} \kappa_{\gamma_0, \gamma}^\Gamma \# M^\gamma_{\gamma_0, (\gamma_0, \Gamma), A} q_{\gamma}^\Gamma \omega(A) \frac{\partial}{\partial q_{\gamma}},
\]
(and similar for $\gamma_1$), it follows that the second and the third summand correspond to boundary components $\mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \times \mathcal{M}_\gamma^+(\Gamma'')$ and $\mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma') \times \mathcal{M}_\gamma^+(\Gamma'')$ with $\gamma_0, \gamma_1 \in \Gamma''$ of type 2. For the combinatorical factors we refer to the remark above.

Finally, using
\[
(L_X dq_{\gamma_0} + (\frac{\partial}{\partial q_{\gamma_0}})) = -(\frac{\partial}{\partial q_{\gamma}}) \sum_{\Gamma, A} \frac{1}{r!} \kappa_{\gamma_0, \gamma}^\Gamma \# M^\gamma_{\gamma_0, (\gamma, \Gamma), A} q_{\gamma}^\Gamma \omega(A)
\]
we find that the last summand corresponds to boundary components $\mathcal{M}_\gamma^+(\Gamma') \times \mathcal{M}_{\gamma_0,\gamma_1}^+(\Gamma'')$ with $\gamma \in \Gamma'$ of type 3, where the combinatorical factors are treated as above.

We now give the main definition of this paper.

**Definition 2.13.** A cohomology $F$-manifold is a formal pointed differential graded manifold $Q_X$ equipped with a graded commutative and associative product for vector fields
\[
* : T^{(1,0)} Q_X \otimes T^{(1,0)} Q_X \rightarrow T^{(1,0)} Q_X.
\]

Note that, in his papers, Merkulov is working with different definitions of cohomology $F$-manifolds and $F_\infty$-manifolds, a generalization of cohomology $F$-manifold making use of the higher homotopies of the product. The definition used in this paper is an adaption of the definition of cohomology $F$-manifolds from [16] in view of the original definition of $F_\infty$-manifolds from the first paper [15]. Note that Merkulov allows $Q$ to be any formal pointed graded manifold, which is more general in the sense that each (graded) vector space naturally carries the structure of a formal pointed (graded) manifold with the special point being the origin, see [14]. Furthermore, we remark that our cohomology $F$-manifolds are indeed infinite-dimensional and formal in the sense that we do not specify a topology on them.

With this we can now state main theorem of this paper.
Theorem 2.14. The big pair-of-pants product equips the formal pointed differential graded manifold of full contact homology (from proposition 2.4) with the structure of a cohomology F-manifold in such a way that, at the tangent space at zero, we recover the (small) pair-of-pants product on Floer cohomology.

Proof. The proof splits up into three parts.

Vector field product: Obviously the main ingredient for the proof is proposition 2.11. There we have shown that big pair-of-pants product $\star \in T^{(1,2)} Q$ and the cohomological vector field $X \in T^{(1,0)} Q$ from full contact homology satisfy the master equation $L_X \star = 0$. This should be seen as generalization of the master equation relating the small pair-of-pants product and the coboundary operator of Floer cohomology, that is, cylindrical contact cohomology, at the end of subsection 1.1. In the same way as the latter proves that the small pair-of-pants product descends to a product on cylindrical contact cohomology, the master equation of proposition 2.11 shows that the big pair-of-pants product defines an element in the tensor homology $H^*(T^{(1,2)} Q, \mathcal{L}_X)$. Following the discussion of the concept of differential graded manifolds in subsection 2.1, it follows from the compatibility of the Lie derivative with the contraction of tensors that the big pair-of-pants product defines a $(1,2)$-tensor field $\star \in T^{(1,2)} Q_X$, that is, a product of vector fields on the differential graded manifold $Q_X$.

Commutativity: In order to prove the main theorem, it remains to show that this product is commutative and associative (in the graded sense). In order to see that the big pair-of-pants product is already commutative on the chain level, it suffices to observe that the coherent collections of sections $(s)$ (used to define the new moduli spaces $\mathcal{M}^{1,2}_{(\gamma_0,\gamma_1)}(\Gamma)$) are chosen such that they are invariant under reordering of the ordered tuples of punctures $(z_0, z_1)$ and $(z_2, \ldots, z_{r-1})$. Hence it directly follows that the count of elements in $\mathcal{M}^{1,2}_{(\gamma_0,\gamma_1)}(\Gamma)$ and $\mathcal{M}^{1,2}_{(\gamma_1,\gamma_0)}(\Gamma)$ agree and hence also

$$\frac{\partial}{\partial q_{\gamma_0}} \star \frac{\partial}{\partial q_{\gamma_1}} = \frac{\partial}{\partial q_{\gamma_1}} \star \frac{\partial}{\partial q_{\gamma_0}}$$

on the chain level, up to a sign determined by the $\mathbb{Z}_2$-grading of the formal variables $q_{\gamma_0}, q_{\gamma_1}$.

Associativity: In contrast to commutativity, we cannot expect to have associativity for the big pair-of-pants on the chain level, but only after viewing it as a vector field product on the differential graded manifold $Q_X$. More precisely, we can show that the resulting $(1,3)$-tensors $\star_{10,2}$ and $\star_{1,02}$ on $Q$ defined by

\begin{align*}
\star_{10,2} : \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes \frac{\partial}{\partial q_{\gamma_2}} \rightarrow & \left( \frac{\partial}{\partial q_{\gamma_1}} \star \frac{\partial}{\partial q_{\gamma_0}} \right) \star \frac{\partial}{\partial q_{\gamma_2}}, \\
\star_{1,02} : \frac{\partial}{\partial q_{\gamma_0}} \otimes \frac{\partial}{\partial q_{\gamma_1}} \otimes \frac{\partial}{\partial q_{\gamma_2}} \rightarrow & \frac{\partial}{\partial q_{\gamma_1}} \star \left( \frac{\partial}{\partial q_{\gamma_0}} \star \frac{\partial}{\partial q_{\gamma_2}} \right),
\end{align*}
do not agree, but only up to some \( L_X \)-exact term which only vanishes after passing to the differential graded manifold \( Q_X \). Due to its importance, this is the content of the following lemma, which finishes the proof of the main theorem.

**Lemma 2.15.** We have \( *_{10,2} - *_{1,02} = L_X \alpha \) with some tensor field \( \alpha \in \mathcal{T}^{(1,0)} Q \).

**Proof.** The corresponding \( (1,3) \)-tensor field \( \alpha \) is again defined by counting certain submoduli spaces \( \mathcal{M}^{\gamma^+}_{\gamma_0,\gamma_1,\gamma_2}(\Gamma) \subset \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \) of Floer solutions,

\[
\alpha : \frac{\partial}{\partial q^{\gamma_0}} \otimes \frac{\partial}{\partial q^{\gamma_1}} \otimes \frac{\partial}{\partial q^{\gamma_2}} \mapsto \sum_{\kappa_{\gamma_0}\kappa_{\gamma_1}\kappa_{\gamma_2}} \frac{1}{k!) \# \mathcal{M}^{\gamma^+}_{\gamma_0,\gamma_1,\gamma_2}(\Gamma; A) \cdot q^{\Gamma} \omega(A) \cdot \frac{\partial}{\partial q_{\gamma_2}}.
\]

Here \( \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \) denotes the moduli space of \( J \)-holomorphic curves \( ((h,u), z_0, z_1, z_2, z_3, \ldots, z_{r-1}, z_{\infty}, z_1^+, z_2^+) \) as used in the differential of full contact homology (and for the definition of the submoduli spaces \( \mathcal{M}^+_2(\Gamma) \)), but now equipped with two unconstrained additional marked points \( z_1^+, z_2^+ \). Using these two additional marked points, we can now not only define two evaluation maps \( ev_{1,2} : \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \rightarrow S^1 \) given by

\[
((h,u), z_0, z_1, z_2, z_3, \ldots, z_{r-1}, z_{\infty}, z_1^+, z_2^+) \mapsto h_2(z_1^+),
\]

but also two forgetful maps \( ft_{1,2} : \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \rightarrow \mathcal{M}_4 \cong S^2 \) given by

\[
((h,u), z_0, z_1, z_2, z_3, \ldots, z_{r-1}, z_{\infty}, z_1^+, z_2^+) \mapsto (z_0, z_{1,2}, z_{\infty}, z_1^{1,2}).
\]

The two forgetful maps in turn can be used to define two pullback bundles

\[
\mathcal{L}^+_1 : ft_{1,2} \mathcal{L} \rightarrow \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R},
\]

with the tautological line bundle \( \mathcal{L} \rightarrow \mathcal{M}_4 \) (which naturally can be identified with the cotangent bundle over \( S^2 \)) from before.

Assuming that we have chosen collections of sections \( (s_{1,2}) \) in the pullback bundles \( \mathcal{L}^+_1 \) over all moduli spaces \( \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \), we define the new moduli spaces, similar as before, as common zero locus

\[
\mathcal{M}^{\gamma^+}_{\gamma_0,\gamma_1,\gamma_2}(\Gamma) := s_{1}^{-1}(0) \cap ev_1^{-1}(0) \cap s_{2}^{-1}(0) \cap ev_2^{-1}(0)
\]

in the moduli space \( \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \) from before. As in the definition of the new moduli spaces for the big pair-of-pants product, we however again need to impose a coherency condition, which (again in contrast to \cite{[12]} is now only needed for those boundary components, where now both additional marked points do not sit on the component of the broken curve which gets contracted to a point under the forgetful map. For this observe that, as in the proof of lemma 2.12, the remaining boundary components will not contribute to the resulting master equation.

Now observe that the codimension-one boundary of the moduli space \( \mathcal{M}^+_2(\gamma_0,\gamma_1,\gamma_2,\Gamma) / \mathbb{R} \) consists of components of the form

\[
\mathcal{M}^{\gamma^+}_{1,0}(\Gamma') / \mathbb{R} \times \mathcal{M}^{\gamma^+}_{0,1}(\Gamma'') / \mathbb{R}
\]
and
\[ \mathcal{M}^{\gamma+}_{0,0}(\Gamma')/\mathbb{R} \times \mathcal{M}^{\gamma'}_{1,0}(\Gamma'')/\mathbb{R} \]
as well as of components of the form
\[ \mathcal{M}^{\gamma+}_{2,0}(\Gamma')/\mathbb{R} \times \mathcal{M}^{\gamma'}_{2,0}(\Gamma'')/\mathbb{R} \]
and
\[ \mathcal{M}^{\gamma+}(\Gamma')/\mathbb{R} \times \mathcal{M}^{\gamma'}(\Gamma'')/\mathbb{R}. \]

Note that the difference between the first and the second case lies in the fact that in the first case the first marked point lies in the upper component (and the second marked point on the lower component), while in the second case the second marked points sits on the upper component (and the first marked point on the lower component). While in the third case it follows that we only need to consider those components where \( \Gamma' \) contains \( \gamma_0 \), or \( \gamma_1 \) and \( \gamma_2 \), in the fourth case we only need to consider those components where \( \Gamma'' \) contains \( \gamma_0 \), \( \gamma_1 \) and \( \gamma_2 \). On the other hand, in the first case it follows that we only need to consider the case when \( \Gamma' \) contains \( \gamma_1 \) (and hence \( \Gamma'' \) contains \( \gamma_0 \) and \( \gamma_2 \)), while in the second case we need that \( \Gamma' \) contains \( \gamma_2 \) (and hence \( \Gamma'' \) contains \( \gamma_0 \) and \( \gamma_1 \)).

While in the third and in the fourth case we are again dealing with moduli spaces with two additional marked points, note that in the first and in the second case we are dealing with moduli spaces with one additional marked point, for which we have already chosen (coherent collections of) sections for the definition of the new moduli spaces for the big pair-of-pants product. In order to find the big pair-of-pants product in the resulting master equation, we clearly assume that we are employing the choices of sections that we have already made. Arguing as in the proof of the proposition 2.11, it then follows that by counting the holomorphic curves in codimension-one boundary of each moduli space \( \mathcal{M}^{\gamma+}_{0,0,\gamma_1,\gamma_2}(\Gamma) \), we obtain the master equation of the statement. Indeed, while the boundary components of the first and of the second type lead to the appearance of \( \star_{10,2} \) and \( \star_{1,02} \), the remaining boundary components of the third and the fourth type show that associativity does not hold on the chain level but only up to the exact term \( \mathcal{L}_X \alpha \). \( \square \)

We now turn to the invariance properties of the new objects. For this we show that for different choices of auxiliary data like almost complex structure and (domain-dependent) Hamiltonian perturbations, see also the appendix, we obtain cohomology F-manifolds which are isomorphic in the natural sense.

First, observe that the invariance properties of contact homology proven in [9], see [10] for the case of Hamiltonian mapping tori, lead to an isomorphism of the corresponding differential graded manifolds in the following sense. Let \((Q^+, X^+)\) and \((Q^-, X^-)\) be pairs of chain spaces and cohomological vector fields for full contact homology, obtained using two different choices of cylindrical almost complex structures \( J^\pm \) on \( \mathbb{R} \times S^1 \times M \) defined using two different choices of \( S^1 \)-dependent Hamiltonian functions \( H^\pm \) and \( \omega \)-compatible almost complex structures \( J^\pm \). Following [10], by choosing a smooth family \((H_s, J_s)\) of \( S^1 \)-dependent Hamiltonians and \( \omega \)-compatible
almost complex structures interpolating between \((H^+, J^+)\) and \((H^-, J^-)\), we can equip the cylindrical manifold \(\mathbb{R} \times S^1 \times M\) with the structure of an almost complex manifold with cylindrical ends in the sense of \([2]\). By counting elements in moduli spaces \(\tilde{\mathcal{M}}^{+\pm}(\Gamma) = \tilde{\mathcal{M}}^{+\pm}(\Gamma; A)\) of \(J\)-holomorphic curves, it is shown in \([9]\), see also \([10]\), that we can define a chain map 

\[
\varphi_{(0,0)} : T^{(0,0)} Q^+ \rightarrow T^{(0,0)} Q^- \quad \text{for the full contact homology by defining}
\]

\[
\varphi_{(0,0)}(q_{\gamma^+}) = \sum_{\Gamma, A} \frac{1}{k!} \cdot \# \tilde{\mathcal{M}}^{+\pm}(\Gamma; A) \cdot q^\Gamma t^{\omega(A)}
\]

and \(\varphi_{(0,0)}(q_{\gamma_1^+} \cdots q_{\gamma_s^+}) := \varphi_{(0,0)}(q_{\gamma_1^+}) \cdots \varphi_{(0,0)}(q_{\gamma_s^+})\), which is indeed compatible with the differential in full contact homology in the sense that \(X^- \circ \varphi_{(0,0)} = \varphi_{(0,0)} \circ X^+\). After passing to homology, it can be shown, see \([9]\) and \([10]\), that the map \(\varphi_{(0,0)}\) indeed defines an isomorphism of the full contact homology algebras,

\[
\varphi_{(0,0)} : T^{(0,0)} Q^+_{X^+} \xrightarrow{\cong} T^{(0,0)} Q^-_{X^-}.
\]

Following \((14)\), subsection 4.1) this defines an isomorphism of differential graded manifolds \(Q^+_{X^+}\) and \(Q^-_{X^-}\). In particular, note the map \(\varphi_{(0,0)}\) on the space of functions on \(Q\) uniquely defines the corresponding map \(\varphi_{(0,1)} : T^{(0,1)} Q^+ \rightarrow T^{(0,1)} Q^-\) on the space of forms by the requirement that \(d \circ \varphi_{(0,0)}(q_{\gamma^+}) = \varphi_{(0,1)} \circ d\) with the exterior derivative \(d : T^{(0,0)} Q^\pm \rightarrow T^{(0,1)} Q^\pm\). It is given by

\[
\varphi_{(0,1)}(q^{\Gamma^+} d q_{\gamma^+}) = \varphi_{(0,0)}(q^{\Gamma^+}) \varphi_{(0,1)}(dq_{\gamma^+})
\]

with \(\varphi_{(0,1)}(dq_{\gamma^+}) = d(\varphi_{(0,0)}(q_{\gamma^+}))\). On the other hand, the compatibility of the Lie derivative \(L_X\) with the exterior derivative directly proves that \(L_{X^-} \circ \varphi_{(0,1)} = \varphi_{(0,1)} \circ L_{X^+}\) and we hence obtain a map on the spaces of forms on the differential graded manifolds, \(\varphi_{(0,1)} : T^{(0,1)} Q^+_{X^+} \rightarrow T^{(0,1)} Q^-_{X^-}\) which, by the same arguments as for the spaces of functions, is indeed an isomorphism. On the other hand, having shown that the spaces of functions and one-forms on \(Q^\pm\) are isomorphic, this automatically implies that the same is true for all the other tensor fields \(T^{(r,s)} Q^\pm_{X^\pm}\). We end by remarking that the above results about the higher tensor fields for contact homology indeed can be deduced from the discussions in the final subsection of \([9]\) in the context of rational symplectic field theory.

**Theorem 2.16.** For different choices of auxiliary data like \(S^1\)-dependent Hamiltonian functions \(H^\pm\) and \(\omega\)-compatible almost complex structures \(J^\pm\), the isomorphism between the resulting differential graded manifolds \(Q^\pm_{X^\pm}\) and \(Q^\pm_{X^-}\) extends to an isomorphism of cohomology \(F\)-manifolds. In particular, after setting \(H = 0\), we recover the cohomology \(F\)-manifold structure on \(\Lambda^+ QH^\ast(M) = \bigoplus_{k \in \mathbb{N}} QH^\ast(M)\) given by the big quantum product.

**Proof.** We prove that the isomorphism of tensor fields respects the product structure by showing that

\[
\varphi_{(0,2)} \circ * \ast * \circ \varphi_{(0,1)}
\]

where the isomorphism \(\varphi_{(0,2)} : T^{(0,2)} Q^+_{X^+} \rightarrow T^{(0,2)} Q^-_{X^-}\) is on the chain level given by \(\varphi_{(0,2)}(q^{\Gamma^+} \cdot dq_{\gamma_1^+} \otimes dq_{\gamma_2^+}) = \varphi_{(0,0)}(q^{\Gamma^+}) \cdot \varphi_{(0,1)}(dq_{\gamma_1^+}) \otimes \varphi_{(0,1)}(dq_{\gamma_2^+})\).
Denote by $\widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma}$ the moduli space of $\tilde{J}$-holomorphic curves $(\tilde{u}, z_0, z_1, z_2, \ldots, z_{r-1}, z^*)$ in the topologically trivial symplectic cobordism $\mathbb{R} \times S^1 \times M$ interpolating between the two choices of auxiliary data $(H^\pm, J^\pm)$ and with an additional unconstrained marked point.

As in the cylindrical case, we can define a pullback line bundle $L^* = f^* L$ over $\widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma}$ using the forgetful map $f_\Gamma : \widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma} \to \mathcal{M}_4$ defined by mapping $(\tilde{u}, z_0, z_1, z_2, \ldots, z_{r-1}, z^*, z^*)$ to $(z_0, z_1, z_{\infty}, z^*)$. Furthermore there still exists an evaluation map to the circle, $ev : \widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma} \to S^1$ given by $ev((\tilde{u}, z_0, z_1, z_2, \ldots, z_{r-1}, z_{\infty}, z^*) = h_2(z^*)$ with $\tilde{u} = (h_1, h_2, u) : \hat{S} \to M$. Summarizing we can again define new moduli spaces $\widehat{\mathcal{M}}_{\gamma_0,\gamma_1}^{\gamma_0,\gamma_1,\Gamma}$ as common zero locus $s_\Gamma^{-1}(0) \cap ev^{-1}(0)$ using a section $s$ in the pullback line bundle $L^* \to \widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma}$.

For the proof we now have to consider the codimension one-boundary of the submoduli space $\widehat{\mathcal{M}}_{\gamma_0,\gamma_1}^{\gamma_0,\gamma_1,\Gamma}$. Instead of four types of two-level curves we now have eight types of two-level curves, where the factor two just results from the fact that we have to distinguish which level is cylindrical and which is non-cylindrical. In order to get the required compactness result involving the new moduli spaces defined for the big pair-of-pants product for the two different choices of auxiliary data $(H^\pm, J^\pm)$, note that we additionally need to interpolate between two different choices of coherent collections of sections $(s^\pm)$. In other words, for all moduli spaces $\widehat{\mathcal{M}}_1^{\gamma_0,\gamma_1,\Gamma}$ we have to follow [12] and choose coherent collections of sections which are connecting the two choices of coherent collections of sections $(s^\pm)$ for the moduli spaces appearing in the boundary, see [12] for a precise definition. As in the corresponding result for satellites in [9], we obtain that the morphism and the product commute up to terms which are exact for the Lie derivatives with respect to $X^\pm$.

Finally, note that in the case when $H = 0$, the new moduli spaces used for the definition of the big pair-of-pants product count Floer solutions $u : \hat{S} \to M$ satisfying the Cauchy-Riemann equation $\bar{\partial}_J(u) = 0$ with varying conformal structure and fixed asymptotic markers. Since each such solution is indeed a map from the closed sphere to $M$ and we can hence also the asymptotic markers are not needed anymore, we precisely end up with the moduli spaces of $J$-holomorphic spheres in $M$ defining the big quantum product. On the other hand, when $H = 0$, the cohomological vector field $X$ is vanishing. For this observe that, in contrast to the moduli spaces for the big pair-of-pants product, we now count Floer solutions with (simultaneously) rotating asymptotic markers. After setting $H = 0$ it follows that we still arrive at moduli spaces of $J$-holomorphic spheres but with a free $S^1$-symmetry given by the unconstrained rotating asymptotic markers. □
We end this section with a discussion how our definition of a cohomology F-manifold can be upgraded by incorporating further geometric structures.

First, note that a unit can be added in by allowing additional marked points on the holomorphic curves which are unconstrained (in the sense that one integrates over the moduli space the pullback under the evaluation map of the canonical zero-form \(1 \in H^0(M) = H^0(S^1, H^0(M))\), and enlarging the coordinate space \(Q\) by a new coordinate \(q_1\) keeping track of these additional points. Since this does not add new contributions to the contact homology vector field \(X\), we have
\[
\mathcal{L}_X \frac{\partial}{\partial q_1} = 0
\]
and hence \(\partial/\partial q_1 \in T^{(1,0)} Q\). On the other hand, the big pair-of-pants product tensor \(\ast\) now additionally counts rigid orbit cylinders with one marked point, which shows that \(\partial/\partial q_1\) is indeed the unit for the \(\ast_q\)-product on each tangent space \(T_q Q\).

Furthermore we observe that, the \(\mathbb{Z}_2\)-grading for the formal variables can indeed be upgraded to an integer grading when the symplectic manifold \((M, \omega)\) is Calabi-Yau in the sense that \(c_1(A) = 0\) for all \(A \in \pi_2(M)\). As for the Frobenius manifolds in Gromov-Witten theory, we can then further define a so-called Euler vector field \(E\) on \(Q\). It is given by
\[
E = \sum \gamma (CZ(\gamma) - 2(\dim M - 2)) \cdot q_{\gamma} \frac{\partial}{\partial q_{\gamma}} + 2 \cdot q_1 \frac{\partial}{\partial q_1} \in T^{(1,0)} Q.
\]
The fact that the big pair-of-pants product counts holomorphic curves with Fredholm index two (they are virtually rigid after dividing out the \(\mathbb{R}\)- as well as the \(S^1\)-symmetry on the map to the cylinder) translates directly into the equation \(\mathcal{L}_E \ast = 2\ast\). In the same way it follows from the fact that the contact homology vector field is counting holomorphic curves with Fredholm index one that \(\mathcal{L}_E X = [E, X] = X\). Now it follows from the Jacobi identity and the definition of Lie bracket for arbitrary tensor fields that
\[
[\mathcal{L}_E, \mathcal{L}_X] = \mathcal{L}_{[E,X]} = \mathcal{L}_X.
\]
For every tensor \(\alpha \in T^{(r,s)} Q\) hence \(\mathcal{L}_X \alpha = 0\) implies \(\mathcal{L}_X \mathcal{L}_E \alpha = 0\) and \(\mathcal{L}_E \mathcal{L}_X \alpha\) is \(\mathcal{L}_X\)-exact. While the Euler vector \(E\) itself due to \(\mathcal{L}_X E = -\mathcal{L}_E X = -X \neq 0\) does not descend to cohomology, it follows that the Lie derivative with respect to \(E\) still descends to an operator \(\mathcal{L}_E : T^{(r,s)} Q_X \to T^{(r,s)} Q_X\).

After adding a unit and restricting the case of Calabi-Yau manifolds, it follows that the tuple \((Q, X, \ast, e, E)\), consisting of the coordinate super space \(Q\) and the differential vector field \(X\) of contact homology, the big pair-of-pants product \(\ast\), the unit vector field \(e = \partial/\partial q_1\) and the Euler vector field \(E\), satisfies the following list of axioms:

- \(Q\) is a \(\mathbb{Z}\)-graded vector space,
• $E \in \mathcal{T}^{(1,0)} Q$ is an Euler vector field in the sense that $Ef = |f| f$ for all homogeneous functions $f \in \mathcal{T}^{(0,0)} Q$.

• $X \in \mathcal{T}^{(1,0)} Q$ satisfies $[X, X] = 2X^2 = 0$, $[E, X] = X$ and $X(0) = 0$, so that $(Q, X)$ defines a differential graded manifold $Q_X$ with a grading operator $\mathcal{L}_E : \mathcal{T}^{(r,s)} Q_X \to \mathcal{T}^{(r,s)} Q_X$ given by $E$ and a restriction map at the origin $0 \in Q$ for all tensor fields,

• $\ast \in \mathcal{T}^{(1,2)} Q$ satisfies $\mathcal{L}_X \ast = 0$ and $\mathcal{L}_E \ast = 2\ast$, and the induced map $\ast : \mathcal{T}^{(1,0)} Q_X \otimes \mathcal{T}^{(1,0)} Q_X \to \mathcal{T}^{(1,0)} Q_X$ is a graded commutative and associative product,

• $e = \partial/\partial q_1 \in \mathcal{T}^{(1,0)} Q$ satisfies $[X, e] = 0$ and $e \in \mathcal{T}^{(1,0)} Q_X$ is a unit for the multiplication given by $\ast$.

Note that cohomology F-manifolds are generalizations of Hertling-Manin’s F-manifolds, which themselves are generalizations of Dubrovin’s Frobenius manifolds. While the generalization from F-manifolds to cohomology F-manifolds is needed since we can only expect the big-pair-of-pants product to be well-behaved after passing to cohomology, note that the generalization from Frobenius manifolds to F-manifolds results from the observation that there is no underlying potential. Indeed, since with just one formal variable for each fixed point one cannot distinguish between inputs and outputs, it is immediately clear that there is no Floer potential $F \in \mathcal{T}^{(0,0)} Q_X$ such that the big quantum product $\ast \in \mathcal{T}^{(1,2)} Q_X$ is given by its triple derivatives.

2.3. Floer theory and commuting Hamiltonian systems in SFT. It was shown by B. Dubrovin, see [7], that to every Frobenius manifold $Q$ one can assign an infinite-dimensional integrable system on the loop space $\Lambda Q$ of $Q$. By definition, it consists of an infinite sequence of linearly independent commuting Hamiltonian functions which span the space of symmetries of the first Hamiltonian, see [7] for the precise definition. For this recall that in Gromov-Witten theory the integrable system appears as flat coordinates for the deformed flat connection $\tilde{\nabla}$ on the cotangent bundle to the Frobenius manifold $Q$ times $\mathbb{C}^*$ given by the flat metric on $Q$, the big quantum product $\ast$ and the Euler vector field $E$.

While we have shown that the big pair-of-pants product defines a $(1,2)$-tensor field $\ast \in \mathcal{T}^{(1,2)} Q_X$ on the differential graded manifold of contact homology, note that the Euler vector field $E \in \mathcal{T}^{(1,0)} Q$ as well as the canonical flat structure on $Q$ do not descend to well-defined structures on the differential graded manifold $Q_X$ in general. Indeed, recall from the definition of the cohomology F-manifold that the flat structure is not part of the definition, and only the grading operator given by the Lie derivative with respect to the Euler vector field, but not the Euler vector field itself, gives a well-defined geometrical object on the differential graded manifold.
While it seems that the classical approach to integrable systems does not generalize immediately from Gromov-Witten theory to Floer theory, it was outlined by Y. Eliashberg in his ICM plenary talk, see [8], that the integrable system of the Gromov-Witten theory of \((M, \omega)\) naturally arises in the rational symplectic field theory (SFT) of the trivial mapping torus \(S^1 \times M\). Instead of trying to generalize the classical approach starting from the cohomology F-manifold of Floer theory introduced before, the SFT approach leads in a much more natural way to the desired generalization of the integrable systems from Gromov-Witten theory to the Floer theory of symplectomorphisms.

Indeed, using symplectic field theory one gets an infinite system of commuting Hamiltonians on the rational SFT of the mapping torus of every symplectomorphisms on a closed symplectic manifold, which agrees with the integrable system from Gromov-Witten theory in the case when the symplectomorphism is the identity. We emphasize that this observation was the starting point for our project of relating Floer theory, Frobenius manifolds and integrable systems, and even guided us to our definition of the big pair-of-pants product and its relation to cohomology F-manifolds. While for the definition of the cohomology F-manifold we used the full contact homology of mapping tori, the commuting Hamiltonian systems naturally live on their rational SFT homology, whose differential now counts holomorphic curves with an arbitrary number of positive (and negative) cylindrical ends and whose chain complex contains the full contact homology complex as a subcomplex. The reason is that the rational SFT homology naturally carries a Poisson bracket, which in turn leads to the natural appearance of commuting Hamiltonian systems in SFT, see [8] and [12]. Apart from the fact that we claim that the cohomology F-manifold structure on contact homology indeed can be extended to rational SFT after introducing additional marked points on the curves (in order to model nodal breakings which now need to be included), the system of commuting Hamiltonian functions on rational SFT still restricts to a system of commuting vector fields on the differential graded manifold of full contact homology.

**Commuting Hamiltonians on rational SFT homology.** Following [9], [12] we start with reviewing the appearance of the commuting Hamiltonian systems in the rational SFT.

Rational SFT is a generalization of contact (co)homology in the sense that its definition involves moduli spaces of holomorphic maps with not just arbitrary many negative ends, but also arbitrary many positive ends, see [9] for details. For two ordered sets \(\Gamma^+ = (\gamma_0^+, \ldots, \gamma_r^+)\) and \(\Gamma^- = (\gamma_0^-, \ldots, \gamma_r^-)\) of closed orbits \(\gamma_0^+, \ldots, \gamma_r^+ \in \bigcup_{k \in \mathbb{N}} \hat{\mathcal{P}}(\phi^k)\) the moduli space \(\mathcal{M}_r(\Gamma^+, \Gamma^-)\) consists of equivalence classes of tuples \((u, (z_1^+, \ldots, z_r^+)), (\gamma_1^+, \ldots, \gamma_r^-), (z_1^-, \ldots, z_r^-))\) with equivalence relation given by the action of the automorphisms of the domain and the \(\mathbb{R}\)-action on the target, see [9] for details. Here \((z_1^+, \ldots, z_r^+), (\gamma_1^-, \ldots, \gamma_r^-)\) and \((z_1^-, \ldots, z_r^-)\)
are disjoint collections of marked points on $S^2$ and $u$ is a $J$-holomorphic map from the punctured sphere $\tilde{S} = S^2 \setminus \{z^1_1, \ldots, z^r_r\}$ to $\mathbb{R} \times M_\phi$ converging to the closed orbits $\gamma^\pm_i \in \tilde{\mathcal{P}}(\phi^k)$ in the positive/negative cylindrical end near $z^\pm_i, i = 1, \ldots, r$. In particular, the induced map $h = \pi \circ u : \tilde{S} \to \mathbb{R} \times S^1$ again defines a branched covering from $S^2$ to itself with branch points $z^\pm_i$ of order $k^\pm_i, i = 1, \ldots, r$ over $\infty$ and $0$, respectively. Note that when $\Gamma^+$ consists of a single orbit $\gamma^+$ and there no additional marked points ($r = 0$), then we just get back the moduli spaces of contact homology from before, $\mathcal{M}^{\gamma^+}(\Gamma^-) = \mathcal{M}_0(\Gamma^+, \Gamma^-)$.

As in Gromov-Witten theory we can use the additional marked points $z_1, \ldots, z_r$ to define evaluation maps

$$
ev = (ev_1, \ldots, ev_r) : \mathcal{M}_r(\Gamma^+, \Gamma^-) \to M^r_\phi$$

given by

$$
ev_i(u, (z^1_1, \ldots, z^r_r), (z_1, \ldots, z_r)) \mapsto u(z_i), \ i = 1, \ldots, r,$$

which can be used to pullback differential forms from the target. Using the spectral sequence for fibre bundles we find that the cohomology ring of the mapping torus $M_\phi$ is given by the cohomology of the circle with twisted coefficients,

$$
H^*(M_\phi) = H^0(S^1, \mathcal{H}^*(M)) \oplus H^1(S^1, \mathcal{H}^{*-1}(M)),
$$

since it converges after the first page, $E_2^{p,q} = E_{\infty}^{p,q}$. We now choose a string of differential forms $\theta_1, \ldots, \theta_K$ on $M_\phi$, where we will assume that the forms represent a basis of the first summand, $\theta_\alpha \in H^0(S^1, \mathcal{H}^*(M))$.

In contrast to the case of contact homology described before, we assign to each closed orbit $\gamma \in \bigcup_{k \in \mathbb{N}} \tilde{\mathcal{P}}(\phi^k)$ now two formal graded variables $p_\gamma$ and $q_\gamma$ with $|p_\gamma| = + \text{CZ}(\gamma) - 2(\dim M - 2)$. Note that the gradings of both formal variables here differs by sign from the original definition in [9], in order to be consistent with other definitions. As in Gromov-Witten theory we further assign to each cohomology class $\theta_\alpha, \alpha = 1, \ldots, K$ a formal graded variable $t_\alpha$ with $|t_\alpha| = 2 - \deg \theta_\alpha$. They can again be viewed as coordinates of a super space $\mathbf{V}$, which contains the coordinate super space $\mathbf{Q}$ of contact homology after setting $p = 0 = t$ for $p = (p_\gamma), t = (t_\alpha)$, but now carries a natural symplectic super-form $\sum_{\gamma} dp_\gamma \wedge dq_\gamma$ (in the formal sense). In particular, the space of functions $\mathfrak{F} = \mathcal{T}^{(0,0)} \mathbf{V}$ now carries a (graded) Poisson bracket $\{\cdot, \cdot\} : \mathfrak{F} \otimes \mathfrak{F} \to \mathfrak{F}$.

Using the moduli spaces defined above we can define the rational SFT Hamiltonian $h \in \mathfrak{F} = \mathcal{T}^{(0,0)} \mathbf{V}$ of (rational) SFT as the sum over all $\Gamma^+, \Gamma^-$, $I$, where the coefficient in front of the monomial $t^I p^{\Gamma^+_I} q^{\Gamma^-_I} t^{I(A)}$ with $p^{\Gamma^+_I} = p_{\gamma^+_1} \cdots p_{\gamma^+_l}, q^{\Gamma^-_I} = q_{\gamma^-_1} \cdots q_{\gamma^-_l}, t^I = t_{\alpha_1} \cdots t_{\alpha_r}$ is given by

$$
\frac{1}{\Gamma^+_I p^{\Gamma^+_I} q^{\Gamma^-_I} t^{I(A)}} \int_{\mathcal{M}_r(\Gamma^+, \Gamma^-, A)} ev^*_I \theta_{\alpha_1} \wedge \ldots \wedge ev^*_r \theta_{\alpha_r}.
$$
It was shown in [12] that for each differential form one can define an infinite sequence of commuting Hamiltonians $h_{\alpha,j} \in T^{(0,0)} V$. After introducing a special additional marked point $z_0$, we use that as well-known in Gromov-Witten theory there is a tautological line bundle $L$ over each moduli space $\mathcal{M}_{r+1}(\Gamma^+, \Gamma^-)$, whose fibre over $(u,z_0) = (u,(z_1,\ldots,z_r))$ is given by the cotangent line $L_{(u,z_0)} = T^*_uS$, and which extends smoothly over the compactified moduli space. Since the SFT moduli spaces have codimension-one boundary, it however does not make sense to integrate powers of the Euler class (= first Chern class) over the moduli space. In contrast, we have introduced in [12] the notion of (generic) coherent collections of (multi-valued) sections ($s$) in the tautological line bundles over all moduli spaces, whose zero sets $\mathcal{M}_{r+1}(\Gamma^+, \Gamma^-) = s^{-1}(0)$ in all $\mathcal{M}_{r+1}(\Gamma^+, \Gamma^-)$ can be viewed as (Poincare dual) of a coherent Euler class involving all moduli spaces at once, see [12] for details.

Choosing $j$ generic coherent collections of sections ($s_j$) and defining
\[
\mathcal{M}_{r+1}^j(\Gamma^+, \Gamma^-) = s_1^{-1}(0) \cap \ldots \cap s_j^{-1}(0) \subset \mathcal{M}_{r+1}(\Gamma^+, \Gamma^-),
\]
we define the desired sequence of Hamiltonians $h_{\alpha,j} \in T^{(0,0)} V$ again as the sum over all $\Gamma^+$, $\Gamma^-$, $I$, where the coefficient in front of the monomial $t^lp^{r^+}q^{r^-}w^{(A)}$ is now given by
\[
\frac{1}{r!r^+l^+ l^+ r^- l^-} \int_{\mathcal{M}_{r+1}^j(\Gamma^+, \Gamma^-, A)} ev^*_j(\theta_\alpha \wedge dt) \wedge ev^*_1 \theta_{\alpha_1} \wedge \ldots \wedge ev^*_r \theta_{\alpha_r},
\]
with the canonical one-form $dt \in H^1(S^1, H^0(M)) \subset H^1(M_p)$ given by the $S^1$-coordinate on $M_p$.

It was shown in [9] that the rational SFT Hamiltonian $h \in \mathfrak{P}$ satisfies the master equation $\{h,h\} = 0$. Similar as in contact homology this is equivalent to the fact that the symplectic gradient $X = X^h \in \mathcal{T}^{(1,0)} V$ of $h$ with respect to the above formal symplectic super-form is an odd homological vector field on $V$ and hence again defines a differential graded manifold $V_X = (V,X)$ with tensor fields $\mathcal{T}^{(r,s)} V_X := H^s(\mathcal{T}^{(r,s)} V, L_X)$. Furthermore, as in contact homology, for two different choices of auxiliary data like almost complex structure and (domain-dependent) Hamiltonian perturbations, the resulting differential graded manifolds $(V^+,X^+)$ and $(V^-,X^-)$ are isomorphic. The following generalisation of this result for the commuting Hamiltonians was shown in [12].

**Theorem 2.17.** Since $\{h,h\} = 0$, the Hamiltonians $h_{\alpha,j}$ descend to a sequence of functions on the differential graded manifold given by $(V,X)$, which pairwise commute with respect to the Poisson bracket on $T^{(0,0)} V_X$,
\[
\{h_{\alpha,j},h_{\beta,k}\} = 0 \in T^{(0,0)} V_X.
\]
Furthermore they are independent under choices of auxiliary data like almost complex structure and Hamiltonian perturbations in the sense that the under
the isomorphism of rational SFT established in \[9\] they get mapped to each other,
\[ T^{(0,0)} V^+_{X^+} \xrightarrow{\cong} T^{(0,0)} V^-_{X^+}, \ h_{\alpha,j}^+ \mapsto h_{\alpha,j}^-, \ \alpha = 1, \ldots, K, j \in \mathbb{N}. \]

Relation with Floer theory of symplectomorphisms. We now want to discuss the relation of the commuting Hamiltonian systems of the SFT of \(M_\phi\) with the Floer theory of the underlying symplectomorphism \(\phi\).

Note that in the case when the symplectomorphism \(\phi\) is the identity, it was already observed in \[9\] that the SFT of \(S^1 \times M\) (more in general, of every circle bundle \(S^1 \to V \to (M, \omega)\)) is determined by the Gromov-Witten theory of the underlying symplectic manifold \((M, \omega)\). This in turn was used in \[8\], see also \[18\], to prove that the commuting Hamiltonian system determined by the Gromov-Witten potential of \((M, \omega)\) for \(S^1 \times M\) indeed agrees with the integrable system obtained from the Frobenius manifold determined by the Gromov-Witten potential of \((M, \omega)\).

In view of the fact that the Floer theory of a symplectomorphism \(\phi\) generalizes the Gromov-Witten theory of \(M\) in the same way as the SFT of its mapping torus \(M_\phi\) generalizes the SFT of \(S^1 \times M\), we obtain following Floer generalization of this result.

With the above choice of differential forms \(\theta_1, \ldots, \theta_K \in H^0(S^1, \mathcal{H}^*(M)) \subset H^*(M_\phi)\) and making use of the canonical one-form \(dt \in H^1(M_\phi)\) as in the definition, every commuting Hamiltonian \(h_{\alpha,j}\) is counting holomorphic sections in symplectic fibre bundle with fibre \((M, \omega)\) over punctured spheres. For the following statement we restrict to the case \(j = 0\) as in \[9\], the generalization to arbitrary \(j \in \mathbb{N}\) is then obvious.

**Theorem 2.18.** The coefficient of the monomial \(t^p q^r q^{-\nu(A)}\) in \(h_{\alpha,0}\) is given by the integral of the pullback \(\text{ev}^*_\phi \theta_\alpha \wedge \text{ev}^*_\phi \theta_\alpha \wedge \ldots \wedge \text{ev}^*_\phi \theta_\alpha\) (no \(dt \)!) of forms over the moduli space of holomorphic sections \((u, (z_1^+, \ldots, z_r^\pm), (z_0, z_1, \ldots, z_r))\). Here \(u\) is a holomorphic section in the pullback bundle \(h_0^*(\mathbb{R} \times M_\phi)\), where the preferred holomorphic map \(h_0 : \hat{S} = S^2 \setminus \{z_1^+, \ldots, z_r^\pm\} \to \mathbb{R} \times S^1\) is now determined by the requirement \(h_0(z_0) = (0, 0) \in \mathbb{R} \times S^1\).

Instead of selecting three marked points as in the definition of the big pair-of-pants product, we now use that integrating the pullback \(\text{ev}^*_\phi dt\) of the canonical one-form on the mapping torus is equivalent to requiring that for every element \(u = (u, (z_1^+, \ldots, z_r^\pm), (z_0, \ldots, z_r))\) the special marked point \(z_0\) (used to define \(\text{ev}_0\)) gets mapped to a fixed point on \(S^1\) under the induced map \(h = \pi \circ u : \hat{S} = S^2 \setminus \{z_1^+, \ldots, z_r^\pm\} \to \mathbb{R} \times S^1\). Note that when \(\phi\) is the identity map, then \(h_0(\mathbb{R} \times M_\phi) = \hat{S} \times M\) and \(u : \hat{S} \to M\) is a holomorphic sphere in \(M\) with additional marked points \(z_1^+, \ldots, z_r^\pm, z_0, z_1, \ldots, z_r\). For the other cohomology classes, we use that \(H^0(\hat{S}, \mathcal{H}^*(M))\) can be canonically identified using the bundle map with \(H^0(S^1, \mathcal{H}^*(M))\), which in the case of
\( \phi \) being the identity just agrees with \( H^*(M) \).

While we have seen that the commuting Hamiltonian systems have a natural geometrical interpretation in Floer theory, we emphasize that we have the following immediate consequence of the invariance properties of the commuting Hamiltonian systems under choices of auxiliary data.

**Corollary 2.19.** In the case when the symplectomorphism \( \phi \) is the time-one map of a Hamiltonian flow, then the system of commuting Hamiltonians \( h_{\alpha,j} \in \mathcal{T}^{(0,0)} \mathbf{V}_X \) is isomorphic to the integrable system of Gromov-Witten theory of the underlying symplectic manifold.

We end this section with a short discussion about how the commuting Hamiltonians on rational SFT help us to find a substitute for the rational Gromov-Witten potential in the Floer theory of a symplectomorphism.

For this let \( F \in \mathcal{T}^{(0,0)} \mathbf{V} \) be the generating function, whose coefficient in front of the monomial \( t^I p^{q^+} q^{q^-} t^e(A) \) is given by the integral of the pullback \( \text{ev}^*_i \theta_{\alpha_1} \wedge \ldots \wedge \text{ev}^*_r \theta_{\alpha_r} \) of forms over the moduli space of holomorphic sections \( (u, (z_1^{\pm_1}, \ldots, z_r^{\pm_r})), (z_0, z_1, \ldots, z_r) \) determined by \( \Gamma^+ \) and \( \Gamma^- \). While we claim that \( F \) does not lie in the kernel of the symplectic vector field \( X \in \mathcal{T}^{(1,0)} \mathbf{V} \) of rational SFT and hence has no chance to define an invariant of the symplectomorphism \( \phi \), we claim that the above generating function agrees with the first descendant Hamiltonian \( h_{0,1} \) for the canonical zero-form up to a natural weighting factor in front of its summands.

Indeed, since by the above theorem \( h_{0,1} \) counts holomorphic sections with one additional marked point carrying one psi class, it follows from (an analogue of) the dilaton equation that the coefficient of \( h_{0,1} \) in front of each monomial agrees with the coefficient of \( F \) multiplied with the Euler characteristic of the underlying punctured sphere. While \( h_{0,1} \) and \( F \) hence carry the same geometrical information (since the Euler characteristic is nonzero for spheres with three or more marked points), the first descendant Hamiltonian \( h_{0,1} \) (in contrast to \( F \)) indeed defines an invariant of the symplectomorphism \( \phi \).

### 3. Appendix

#### 3.1. Transversality using domain-dependent Hamiltonians.

In this appendix we will show how to adapt the results of the author in [10] to establish the necessary nondegeneracy of orbits and transversality for all appearing moduli spaces. For simplicity we restrict ourselves to the case of full contact homology. In particular, we do not need to employ the polyfold theory of Hofer-Wysocki-Zehnder, but show that we can still prove the desired transversality using the classical approach of domain-dependent almost complex structures, so that our results are already rigorous in the strict mathematical sense. Since everything is only a mild generalization of the results from [10], we only focus on the changes that need to be made, and refer for details to our detailed paper [10].
We want to emphasize that, even after employing domain-dependent Hamiltonian perturbations, we still keep the monotonicity features for the Floer curves. While for the spectral invariants we just need to see that we can choose the perturbations arbitrary small, for the definition of the cohomology $F$-manifold on the symplectic cohomology of a Liouville manifold it is sufficient to assume that the Hamiltonian perturbations are in fact independent of points on the domain outside a compact region containing the closed Hamiltonian orbits. Hence a maximum principle still exists which ensures that the curves are not leaving the compact region. On the other hand, in the compact region (in which they hence stay) the Hamiltonian is allowed to be domain-dependent and we can hence achieve transversality as required. Note that it is not possible to achieve transversality using domain-dependent almost complex structures $J$, since the latter do not affect the orbit curves studied in \[11\].

For the discussion we have to distinguish between domain-stable holomorphic curves (the underlying punctured sphere is already stable in the sense that it has no nontrivial automorphisms, which means that it carries at least three punctures) and domain-unstable holomorphic curves like holomorphic spheres, holomorphic planes and holomorphic cylinders.

**Holomorphic spheres, planes and cylinders.** First, it is a well-known result from Gromov-Witten theory, see \[17\], that one can prove regularity for all appearing moduli spaces of holomorphic spheres when the underlying symplectic is semipositive.

As in \[10\] we observe next that there exist no holomorphic planes in $\mathbb{R} \times M_\phi$ for every symplectomorphism $\phi$, which simply follows from the fact that there is no branched covering map from the plane to the cylinder.

On the other hand, we have seen that the cylindrical contact homology complex of $M_\phi$ (the subcomplex of contact homology/rational SFT whose differential just counts cylinders) can be naturally identified with the sum of the Floer cohomology complexes for all powers $\phi^k$ of the symplectomorphism $\phi$. It follows that the transversality problem for domain-unstable curves in SFT of mapping tori reduces to the transversality results for symplectic Floer cohomology.

Apart from assuming monotonicity in order to be able to deal with bubbling-off of holomorphic spheres as described above, it is a classical result (see \[6\] and \[17\]), that one can prove nondegeneracy for all fixed points and transversality for all moduli spaces of cylinders when one considers a sufficiently generic time-dependent Hamiltonian function $H = H^k : S^1 \to M$. Similar as in \[10\], we however cannot assume that for arbitrary $k \in \mathbb{N}$ we can work with a single-valued Hamiltonian function $H_k$ given by the Hamiltonian function $H = H^1$ for $k = 1$ by $H^k_t := H_{kt}, t \in \mathbb{R}$. The problem is that the resulting function $H^k$ additionally satisfies $H^k_{t+1/k} = H^k_t$, which contradicts the request for genericity of the Hamiltonian and leads to
multiply-covered cylinders. In order to have both the symmetry condition as well as regularity, we again need to consider *multi-valued* Hamiltonian perturbations which destroy all multiply-covered cylinders, see [4] for the precise definitions.

*Domain-stable holomorphic curves.* It remains to prove transversality for holomorphic curves with three or more punctures. While these curves lead to the involved algebraic structures discussed in [9] and in this paper, from the point of transversality they actually cause less problems (up to the compatibility problem with the choices for the other moduli spaces) than the domain-unstable holomorphic curves. Indeed, the latter are the reason why transversality is not proved for symplectic field theory in general, which in turn was the starting point for the polyfold project of Hofer, Wysocki and Zehnder.

Indeed it was shown in [10] that one can prove transversality for all moduli spaces of domain-stable holomorphic curves when one introduces domain-dependent Hamiltonian perturbations, generalizing the Hamiltonian perturbations used for the moduli spaces of holomorphic cylinders discussed above. Here one uses that the underlying punctured sphere has no nontrivial automorphisms, so that one can allow the Hamiltonian to depend on points of the punctured sphere, see [10] for details. Furthermore it was shown in [10] that the resulting class of perturbations is indeed large enough to prove transversality for generic choices and that all choices can be made coherent in the sense that they are compatible with compactness and gluing of moduli spaces, which also involves the moduli spaces of domain-unstable holomorphic curves.

While we claim that the main results carry over naturally, here is a short discussion of how the setup of [10] needs to be improved to cover the case of general Hamiltonian mapping tori.

First, since we now need to employ time-dependent Hamiltonians for the cylinders, we now can no longer work with the moduli space $\mathcal{M}_s$ of punctured Riemann spheres. Instead we want to use that for every moduli space $\mathcal{M}^\gamma(\Gamma)$ there exists a natural map to the moduli space $\mathcal{M}(\vec{k})$ of holomorphic maps from $r + 1$-punctured sphere to the cylinder, see the proof of proposition 2.2. Here $\vec{k} = (k_0, \ldots, k_{r-1})$ $k_0, \ldots, k_{r-1}$ is the ordered set of periods of the orbits in $\Gamma$ and the map is defined by forgetting the map $u : \hat{S} \to M$. In other words, it only remembers the conformal structure and, in contrast to the construction in [10], also the map $h$ to the cylinder and hence the asymptotic markers.

After introducing an unconstrained additional marked point, we obtain the corresponding universal curve $\mathcal{M}_1(\vec{k}) \to \mathcal{M}(\vec{k})$. Generalizing the setup in [10], we now define a domain-dependent Hamiltonian perturbation as a map $H(\vec{k}) : \mathcal{M}_1(\vec{k}) \to C^\infty(M)$. The fibre over each point $j \in \mathcal{M}(\vec{k})$ (which now stands for the conformal structure and the asymptotic markers) defines a Hamiltonian function which depends on points on the corresponding
Riemann surface with cylindrical ends.

After extending the universal curve to the compactification of \( \mathcal{M}(\vec{k}) \), note that the fibre is a compact Riemann surface with boundary circles. The resulting \( S^1 \)-parametrization near each puncture will be viewed as time coordinate for the time-dependent Hamiltonian perturbation used to prove transversality for the corresponding cylinder. Note that every end automatically has a period assigned to it. In order to keep the \( \mathbb{Z}_k \)-symmetry on our moduli spaces (\( k = k_0 + \ldots + k_{r-1} \)) is the period of \( \gamma^+ \), similar for the other punctures), the idea is to forget the asymptotic markers but indeed only to remember the holomorphic map \( h \) in \( \mathcal{M}(k_0, \ldots, k_{r-1}) \), which provides us with multi-valued Hamiltonian perturbations. Apart from the fact that the multi-valued Hamiltonian perturbations fix the domain-dependent Hamiltonian perturbations in the cylindrical ends (see [10]), for the rest we claim that the geometrical setup to define coherent domain-dependent Hamiltonians naturally extends from the classical Deligne-Mumford moduli space of punctured spheres \( \mathcal{M}_s \) to the new moduli space \( \mathcal{M}(\vec{k}) \).

Indeed note that \( \mathcal{M}(\vec{k}) \) is still a smooth manifold (with boundary) and its boundary strata are moduli spaces of curves have a less number of punctures, so the definition of coherent Hamiltonian perturbations generalizes in the obvious way. On the other hand, we claim that the resulting class of Hamiltonian perturbations is still large enough to prove transversality for a generic choice, as can be seen easily from the proof in [10]. Note that in the case when we use \( H = 0 \) in order to relate our big pair-of-pants product to the big quantum product, we actually use the \( C^2 \)-small Hamiltonian perturbations constructed in [10].

### 3.2. Spectral invariants and the big pair-of-pants product.

We end this paper by discussing some possible applications to the theory of spectral invariants. To be more precise, we will that the big pair-of-pants leads again to a triangle inequality for a novel class of spectral invariants.

We begin with quickly recalling the definition of spectral invariants in Floer cohomology due to Oh and Schwarz, see [21].

For each Hamiltonian symplectomorphism \( \phi \) let \( \text{CF}^*_A(\phi) \subset \text{CF}^*_A(\phi) \) denote the subspace spanned by \( \tilde{x} = xt^w(A) \) with action \( -\mathcal{A}(\tilde{x}) = -\mathcal{A}(x) - \omega(A) \leq \lambda \), where the action \( \mathcal{A}(x) \) of the symplectic fixed point \( x \) is defined again making use of the spanning surfaces introduced earlier in order to define the Conley-Zehnder index. Note that since we assume that all fixed points represent contractible orbits in \( M \), we can assume that the spanning surface is given by the spanning disk from the original definition by Oh-Schwarz.

As in [21] it then follows that the action and the resulting definitions are well-defined after choosing a lift \( \tilde{\phi} \in \widetilde{\text{Ham}} \) of \( \phi \in \text{Ham} \subset \text{Symp} \). Further note that since \( H_2(M_\phi) = \pi_1(\Omega_\phi M) \), \( \tilde{\gamma} \) can be viewed as a point in the
universal covering \( \tilde{\Omega}_\phi M \to \Omega_\phi M \).

Finally note that the actions (as well as the indices) of the fixed points \( x, \ldots, \phi^{k-1}(x) \) representing the same closed orbit \( \gamma \) indeed agree, so that we can define \( \mathcal{A}(\gamma) = \mathcal{A}(x) = \ldots = \mathcal{A}(\phi^{k-1}(x)) \) and can essentially forget about the distinction between fixed points and closed orbits. In particular, recall that for Hamiltonian symplectomorphisms \( \phi \) we have \( \mathrm{HF}^*(\phi^k) = \mathrm{HF}^*(\phi) \) for all \( k \in \mathbb{N} \) and hence \( \mathrm{HC}_{cy}^*(M_\phi) = \bigoplus_k \mathrm{HF}^*(\phi^k) \).

Since the differential in Floer cohomology decreases the action, it restricts to a differential \( \partial \) on \( CF_\lambda^*(\tilde{\phi}) \) and one can define the filtered Floer cohomology \( \mathrm{HF}_\lambda^*(\tilde{\phi}) \). Using the inclusion map on cohomology, \( \mathrm{HF}_\lambda^*(\tilde{\phi}) \to \mathrm{HF}^*(\phi) \), and the isomorphism \( \mathrm{HF}^*(\phi) \cong \mathrm{QH}^* \), the Oh-Schwarz spectral invariant is defined by

\[
\rho_{\tilde{\phi}}(\tilde{\theta}) := \inf\{ \lambda \in \mathbb{R} : \tilde{\theta} \in \mathrm{im}(\mathrm{HF}_\lambda^*(\tilde{\phi}) \to \mathrm{QH}^*) \}, \quad \tilde{\theta} = \theta t^{\omega(A)} \in \mathrm{QH}^*.
\]

The most important property of the spectral invariants for applications is that they satisfy the product rule

\[
\rho_{\phi^k}(\tilde{\theta}_0 \ast \tilde{\theta}_1) \leq \rho_{\phi^{k_0}}(\tilde{\theta}_0) + \rho_{\phi^{k_1}}(\tilde{\theta}_1), \quad k = k_0 + k_1,
\]

where \( \ast : \mathrm{QH}^* \otimes \mathrm{QH}^* \to \mathrm{QH}^* \) denotes the small quantum product. Note that the spectral invariants for the different powers \( \phi^k \) of the Hamiltonian symplectomorphism \( \phi \) can be reassembled into a spectral map \( \rho_{\tilde{\phi}} : \bigoplus_{k \in \mathbb{N}} \mathrm{QH}^* \cong \Lambda^+ \mathrm{QH}^* \to \mathbb{R} \), where \( \Lambda^+ \mathrm{QH}^* \) again denotes the positive loop space of \( \mathrm{QH}^* \).

In analogy to above, we now give a novel definition of big spectral invariants for vector fields, which should be compared with the definition of the action filtration in contact homology in [11].

To this end let \( \mathcal{T}_\lambda^{(1,0)} \mathbb{Q} \) be the subspace of \( \mathcal{T}^{(1,0)} \mathbb{Q} \) spanned by \( q_{\gamma_1} \ldots q_{\gamma_r} \frac{\partial}{\partial q_{\gamma_r}} t^{\omega(A)} \) with \( \mathcal{A}(\gamma_1) + \ldots + \mathcal{A}(\gamma_r) - \mathcal{A}(\gamma) - \omega(A) \leq \lambda \). As in [11] it follows from the action-energy relation for holomorphic curves in cylindrical almost complex manifolds from [2] that the differential \( \mathcal{L}_X = [X, \cdot] \) respects this filtration and hence can be used to define \( \mathcal{T}_\lambda^{(1,0)} \mathbb{Q}_X := H^*(\mathcal{T}_\lambda^{(1,0)} \mathbb{Q}, [X, \cdot]) \). Assuming for notational simplicity that the unit element is removed such that \( \mathcal{T}^{(1,0)} \mathbb{Q}_X \cong \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \), in analogy to above definition we define the big spectral invariant \( \rho_{\tilde{\phi}} : \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \to \mathbb{R} \) by

\[
\tilde{\rho}_{\tilde{\phi}}(Y) := \inf\{ \lambda \in \mathbb{R} : Y \in \mathrm{im}(\mathcal{T}_\lambda^{(1,0)} \mathbb{Q}_X \to \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^*) \}
\]

for all \( Y \in \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \). As in the small case, we can prove the following big product axiom for the big spectral invariants.

**Theorem 3.1.** For all \( Y, Z \in \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \) we have that

\[
\tilde{\rho}_{\tilde{\phi}}(Y \ast Z) \leq \tilde{\rho}_{\tilde{\phi}}(Y) + \tilde{\rho}_{\tilde{\phi}}(Z),
\]

where \( \ast : \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \to \mathcal{T}^{(1,0)} \Lambda^+ \mathrm{QH}^* \) denotes the big quantum product lifted componentwise to \( \bigoplus_{k \in \mathbb{N}} \mathrm{QH}^* \cong \Lambda^+ \mathrm{QH}^* \).
Proof. In analogy to the proof of the small product axiom this follows from the fact that the big pair-of-pants product is compatible with the action filtration on $T^{(1,0)} Q$. While in \cite{21} this is the reason why the author uses the bundle definition introduced at the beginning rather than the original definition from its Ph.D. thesis, here we can elegantly deduce the compatibility from the aforementioned action-energy relation from \cite{2}. Recalling the definition of the big pair-of-pants product on the chain level,

$$\frac{\partial}{\partial q_{\gamma_0}} \star \frac{\partial}{\partial q_{\gamma_1}} := \sum_{\gamma^+} \left( \sum_{\Gamma, A} \frac{1}{(r-2)!} \frac{1}{k!} \# M_{\gamma^+}^{\gamma_0, \gamma_1} (\Gamma, A) \cdot q^{\tau} t^{\omega(A)} \right) \frac{\partial}{\partial q_{\gamma^+}},$$

the coefficient in front of $q^{\tau} \frac{\partial}{\partial q_{\gamma^+}} + t^{\omega(A)}$ can only be nonzero whenever $M_{\gamma^+}^{\gamma_0, \gamma_1} (\Gamma)$ and hence $M_{\gamma^+}^{\gamma_0, \gamma_1, \Gamma}$ is non-empty. Using the action-energy relation from \cite{2} we can now deduce that $A(\gamma^+) - A(\gamma_0) - A(\gamma_1) - A(\Gamma) + \omega(A) \geq 0$, which in turn is equivalent to

$$A(\Gamma) - A(\gamma^+) - \omega(A) \leq - A(\gamma_0) - A(\gamma_1),$$

which proves the statement as in the classical product formula. \hfill \square

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