A simple theory of the quasi-equilibrium regime of atmospheric circulation, using the concept of a modified potential vorticity

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Abstract. The application of the general Ertel’s vorticity theorem to the diagnosis of general atmospheric circulation processes is discussed. The concepts of a modified potential vorticity (PV) and the "PV of the second generation" are used; the latter quantity is obtained by substituting a PV into Ertel's vorticity theorem. The requirement that the "second-generation PV" vanishes when the PV is properly modified leads to the condition for local compensation of the action of heat sources and frictional stresses on the Earth's surface: namely, to the condition of surface air heating / cooling under anticyclonic / cyclonic circulation conditions. The situation with locally uncompensated anomalous wintertime near-surface air heating over the Barents Sea is discussed in the context of the current reduction in the ice cover over the Arctic, which at the global level should be compensated by the intensification of anticyclonic circulation systems over the continents and the corresponding surface air cooling.

1. Preliminaries

This note draws attention to the properties and possible applications in the theory of the general atmospheric circulation of a "twice potential vorticity", or a "potential vorticity of the second generation," defined by the formula

\[ P_\theta = \frac{(\omega_a \cdot \nabla \Pi)}{\rho} \]

where \( P_\theta = \frac{(\omega_a \cdot \nabla \Pi)}{\rho} \) is Ertel's potential vorticity [1]. Here, \( \omega_a \) is the absolute vorticity vector, \( \theta \) is the potential temperature and \( \rho \) is the air density. Ertel's general vorticity theorem

\[ \frac{D}{Dt} \omega_a \cdot \nabla \Lambda = \frac{1}{\rho} \nabla \Lambda \cdot (\nabla \Pi \times \nabla \theta) + \frac{\omega_a \cdot \nabla \Lambda}{\rho} + \frac{\nabla \Lambda \cdot \nabla \times F}{\rho} \]  \( \text{(1)} \)

is valid for any function \( \Lambda \) of time \( t \) and coordinates \( x \). Here, \( \Pi \) is the Exner function and \( F \) stands for the non-potential external forces, including the frictional force; the dot above variables denotes the individual time derivative \( D/Dt \). Assuming \( \Lambda = \theta \), we obtain the evolution equation of the Ertel potential vorticity (PV)

\[ \frac{D}{Dt} P_\theta = \frac{D}{Dt} \frac{\omega_a \cdot \nabla \theta}{\rho} = \frac{\omega_a \cdot \nabla \dot{\theta}}{\rho} + \frac{\nabla \theta \cdot \nabla \times F}{\rho} \]  \( \text{(2)} \)

whereas, assuming \( \Lambda = P_\theta \), we arrive at the equation...
with a non-zero baroclinic term on the right-hand side of (3). In the adiabatic approximation and in the absence of non-potential forces, both terms on the right-hand side of (2) and the last two terms on the right-hand side of (3) vanish, but the PV of the second generation can evolve owing to the first term on the right-hand side of (3). The latter property ensures applications of a "second-generation PV" in the problem of determining the "absolute velocity" of the sea currents [2].

2. Modified PV
As follows from (1), equations (2) and (3) retain their form, or in other words, are covariant, under the replacement \( \theta \rightarrow \chi(\theta) \) everywhere except for the vector product \( \nabla \times \nabla \nabla \nabla \times \) on the right-hand side of (3). Here, \( \chi \) is a monotonically increasing, but in all other respects arbitrary function of its argument. The class of monotonic functions is convenient in that a one-to-one transition is possible between the \( \theta \)- and \( \chi \)-coordinate systems.

Consider two such "coordinate systems" in which the corresponding PVs are related by the functional relations 

\[
P_\theta = \chi P_\theta \quad \text{and} \quad P_\chi = \chi' P_\theta + \chi'' P_\theta^2.
\]

A prime on top denotes the derivative with respect to \( \theta \). Individual time derivatives of \( P_\theta \) and \( P_\chi \) are related by the formula 

\[
P_\theta = \chi P_\theta + \chi' \dot{P}_\theta.
\]

Formally, along with the Eulerian (individual) time-derivative, \( \frac{\partial}{\partial t} \), one can introduce an operator 

\[
\hat{D}[\phi] = \rho \cdot \nabla \phi,
\]

which we may call the Ertel derivative. Then, by definition, 

\[
P_\theta = \hat{D}[\theta], \quad P_\theta = \hat{D}[P_\theta] = \hat{D}(\theta), \quad \text{and} \quad P_\chi = \hat{D}[P_\chi] = \hat{D}[\chi].
\]

We supplement the definition of the function \( \chi(\theta) \) by the condition that it vanishes at the upper boundary of the atmosphere; then, it follows from the monotonic increase property that \( \chi(\theta) \) is negative everywhere. Following the ideas of [3], we assume that 

\[
\chi(\theta) = -p'(\theta)/g,
\]

where \( p'(\theta) \) is the standard pressure distribution on isentropic surfaces and \( g \) is the acceleration due to gravity.

3. Atmospheric PV charge conservation
A.M. Obukhov [4] drew attention to the fact that by virtue of the very construction of equation (2), although diabatic heating and turbulent friction are present separately in the atmosphere, but their combined effect on the temporal evolution of a PV may turn out to be zero. In principle, this idea needs some correction, since such compensation most likely occurs not in the original \( \theta \)-coordinates, but in the transformed \( \chi \)-coordinate system and, at the same time, only locally. Thus, the function \( \chi(\theta) \) is a functional of the considered region \( \Omega \) of the atmosphere and, strictly speaking, this function must be provided with a corresponding index, \( \chi = \chi_\Omega(\theta) \). For other regions of the atmosphere, \( \Omega^* \), \( \Omega^{**} \), etc., there will be slightly different functions, \( \chi_\Omega^*(\theta) \), \( \chi_\Omega^{**}(\theta) \), etc. We can hypothesize that for the whole atmosphere there exists such a universal function \( \chi(\theta) \) that provides compensation of the diabatic heating and friction on average throughout the atmosphere, i.e. explains the constancy in time of the PV charge (see [4]) of the atmosphere in the transformed coordinates,

\[
\frac{D}{Dt} \iiint P_\rho \rho d r = 0.
\]

Using equation (1) with \( \Lambda = \chi(\theta) = -p'(\theta)/g \), substituting it into (4) and applying the Gauss–Ostrogradsky theorem, we arrive at the following equation (cf. [5, 6])
\[
\frac{D}{Dt} \int \int \int P_r \rho d\tau = \int \int \int \left[ \omega_{zz} \frac{1}{g} \frac{d\varphi}{d\theta} + (\nabla \times F)_z \right] d\sigma = 0, \tag{5}
\]
where the integration in the right-hand side integral is over the Earth surface. The subscript "z" below variables denotes the vertical component of the corresponding vector. In deriving (5), it is used that diabatic heating and frictional forces vanish at the upper boundary of the atmosphere. A certain disadvantage of (5) is that the curl of the vertical component of the frictional force enters the second term in the surface integral, but not the curl of the frictional stress on the Earth surface does, which has a more immediate physical meaning. Nevertheless, using a generalization of the Ekman boundary layer model \[7\], it is possible to relate \((\nabla \times F)_z\) to the curl of the geostrophic wind velocity \(v\) by means of the formula \((\nabla \times F)_z = -\lambda (\nabla \times v)_z\) \[5, 6\], where the constant \(\lambda\) has the dimension of the inverse time. It turns out that to fulfill (5) a certain correlation between surface heating and vertical relative vorticity must be observed everywhere: namely, air heating in anticyclones and cooling in cyclones should take place. Such a correlation is generally observed in the extratropical latitudes of both hemispheres: namely, surface heating of air prevails in the region of subtropical anticyclones and surface cooling dominates poleward of the latitude 40° in the region predominantly occupied by cyclonic eddies.

4. Second-generation PV applications

An independent derivation of a relation analogous to (5) is of interest, where, however, the curl of the turbulent frictional stresses on the Earth surface will directly appear. We will take into account that the geometrical meaning of the transformation \(\theta \rightarrow \chi(\theta)\) consists in changing the angle of inclination of the surfaces of constant PV values to the horizontal. By an appropriate choice of \(\chi(\theta)\), this angle can be made much larger than for the classical Ertel PV (figure 1) and ideally it can become close to 90°, at least locally (cf. \[5, 6\]). This means that the isoscalar surfaces \(P_z = \text{const}\) are almost vertical and practically not permeated by vortex lines of the absolute vorticity, the main contribution to which is made by the planetary vorticity in the quasi-geostrophic approximation (see also Appendix). We require that the PV of the second generation, \(P_{\chi}\), is close to zero (small) in each of the selected atmospheric regions, \(\Omega,\ \Omega',\ \Omega''\), etc., under a proper choice of functions \(\chi = \tilde{\chi}_\alpha(\theta),\tilde{\chi}_\alpha'(\theta),\tilde{\chi}_{\alpha''}(\theta),\ldots\), which, by the way, may differ somewhat from the functions \(\chi_\alpha(\theta),\chi_{\alpha'}(\theta),\chi_{\alpha''}(\theta),\ldots\) of Section 3. Hence, we arrive at the relation \(\chi' P_{\chi} + \chi'' P_{\chi} = 0\).

Substituting this into the equation \(\dot{P}_z = \chi' \dot{P}_{\chi} + \chi'' \dot{P}_{\chi}\), we find that
\[
\dot{P}_z = \left[ \dot{P}_0 - \left( \frac{P_{\chi}}{P_0} \right) \dot{\theta} \right] \dot{\chi}'. \tag{6}
\]
Let us find the consequences of equation (6). Applying (2), after simple transformations, when multiplying and simultaneously dividing the right-hand side of (6) by \(P_0\), we have
\[
\dot{P}_z = \left[ \frac{\omega_\alpha}{\rho} \cdot \nabla \left( \frac{\dot{\theta}}{P_0} \right) + \frac{\nabla \theta \cdot \nabla \times F}{\omega_\alpha \cdot \nabla \theta} \right] \dot{P}_{\chi} = \left[ \frac{\omega_\alpha}{\rho} \cdot \nabla \left( \frac{\dot{\theta}}{P_0} \right) + \frac{\nabla \theta \cdot \nabla \times F}{\omega_\alpha \cdot \nabla \theta} \right] P_z,
\]
for any function \(\chi\) that ensures the fulfillment of the condition \(\chi' P_{\chi} + \chi'' P_{\chi} = 0\), or identically
\[
\frac{D}{Dt} \ln P_z = \frac{\omega_\alpha}{\rho} \cdot \nabla \left( \frac{\dot{\theta}}{P_0} \right) + \frac{\nabla \theta \cdot \nabla \times F}{\omega_\alpha \cdot \nabla \theta}.
\]
It is noteworthy that the right-hand side of the last equation does not depend on the choice of the function $\chi$ and this equation is valid for a set of functions $\chi = \tilde{z}_0(\theta), \tilde{z}_1(\theta), \tilde{z}_2(\theta), \ldots$. Integrating over the mass of the atmosphere northward of the latitude circle at 20°, where the PV is non-zero and the quasi-geostrophic approximation is valid, and using the fact that the diabatic heating vanishes at the upper boundary of the atmosphere, we have from the Gauss–Ostrogradsky theorem that

$$\frac{d}{dt} \int \ln P_x \rho d\tau = -\int \omega_\omega \left( \frac{\partial}{\partial \rho} \right) d\sigma + \int \nabla \cdot \nabla \times F \cdot \rho d\tau.$$

Integration in the surface integral extends over the Earth surface northward of latitude 20°. To simplify the integral, we use with good approximation that $\omega_\omega \cdot \nabla \theta \approx \partial \omega_\omega / \partial \theta \partial \zeta$. Here, $f$ is the Coriolis parameter. We also use that approximately $\nabla \theta \cdot \nabla \times F \approx (\nabla \times F) \cdot (\partial \theta / \partial \zeta)$ and introduce the vector $\tau$ of horizontal turbulent frictional stresses by the formula $F = (1/\rho)(\partial \tau / \partial z) \approx -g (\partial \tau / \partial P)$. Integrating over height from the Earth surface to the top of the atmosphere and assuming that the frictional stress vanishes at the upper boundary of the atmosphere, we finally get

$$\frac{d}{dt} \int \ln P_x \rho d\tau \approx \int \left[ g^{-1} \frac{\partial}{\partial \rho} \partial \theta / \partial P - f^{-1} (\nabla \times \tau) \right] d\sigma. \quad (7)$$

We assume that in the quasi-equilibrium climate regime the left-hand side of (7) is approximately equal to zero. Consequently, there should be approximately zero the right-hand side of (7), which is more directly computed based on observational / modeling data than the right-hand side of (5). In the context of equation (7), it can be assumed that $\tau = c_\rho \rho |\mathbf{v}_s| \mathbf{v}_s$, where $c_\rho = O(10^{-3})$ and $\mathbf{v}_s$ is the wind.
velocity at a 10-m height (surface wind velocity). It is also used that
\[
\frac{\partial \theta}{\partial p} = -\frac{\theta}{\rho g T} (\gamma_a - \gamma),
\]
where \( \gamma \) is the actual temperature lapse rate, \( \gamma_a \) is the dry-adiabatic temperature lapse rate and \( T \) is the
temperature. As a result, we arrive at the relation
\[
-\iint \left[ \frac{T}{\gamma_a - \gamma} \frac{\partial \theta}{\partial \theta} + f^{-1} c_D \left( \nabla \times \left( |v_s| \nabla \theta \right) \right)_z \right] \rho d\sigma \approx 0.
\]
(8)

As in Section 3, it turns out that surface heating of air should predominate in the quasi-equilibrium
climate regime in the region of anticyclonic vorticity, with stable vertical stratification of the
atmosphere. On the other hand, in the region of cyclonic vorticity, air cooling should prevail.
Alternatively, it can be written that
\[
1 \sin \cos \sin \rho \phi \phi \phi = -\tau,
\]
where \( E_h \) is the thickness of the
Ekman boundary layer and \( \phi \) is the angle of deviation of the 10-m wind velocity from the geostrophic
wind velocity (see [5, 6]). In this case we have
\[
-\iint \left[ \frac{T}{\gamma_a - \gamma} \frac{\partial \theta}{\partial \theta} + h_E \frac{\sin \phi}{\cos \phi - \sin \phi} \left( \nabla \times v_s \right)_z \right] \rho d\sigma \approx 0.
\]
(9)
Choosing the values \( \phi = \pi/8, \ h_E = 400\text{ m}, \ \gamma_a - \gamma = 1/300\text{ K m}^{-1}, \ \left( \nabla \times v_s \right)_z = -10^{-5}\text{ s}^{-1} \) and putting
\( \theta \approx T \), we find that \( \theta \approx 0.8\text{ K day}^{-1} \). Similar values, only characterizing surface air cooling, are
obtained for the area occupied by cyclonic circulation.

5. Discussion and conclusions

However, there are regions on the Earth where local compensation for the action of diabatic heating
and surface friction is not fulfilled, but on the contrary, both terms in the integrands in (8) and / or (9)
have the same sign. If they are both positive, which takes place in the region of storm tracks over the
oceans, then according to (7), this leads to a decrease in the integral \( \iiint \ln P_x \rho d\tau \). In a quasi-
equilibrium regime, this decrease should be compensated by an increase in the integral \( \iiint \ln P_x \rho d\tau \)
due to the region where both terms in the integrands in (8) and / or (9) are negative, i.e. where surface
cooling of the air takes place in the presence of anticyclonic circulation. In winter, this occurs in areas
occupied by the Siberian anticyclone and its spurs, as well as in the area of blocking anticyclones.
Reduction in sea ice over the Arctic during the past two decades and the non-freezing wintertime
Barents Sea, lead to an increase in the area characterized by ground-level air heating during pre-winter
and wintertime. The total flux of sensible and latent heat from the open surface of the sea to the
atmosphere can reach a value of 200 W m\(^{-2}\) there, which corresponds to
\( \dot{\theta} \approx 1.3 \times 10^{4} \text{ K s}^{-1} \approx 11.5 \text{ K day}^{-1} \) if we assume that the heat that enters the atmosphere is released in
the lower 1.5 km-thick layer. The resulting first term in the integrand in (9),
\[
\frac{T}{\gamma_a - \gamma} \frac{\partial \theta}{\partial \theta} \approx 8 \times 10^{-2}\text{ ms}^{-1}
\]
(we assume that the square of the Brunt-Väisälä frequency in the polar atmosphere has the magnitude
of 5 \times 10^{3}s^{-2} [8]), cannot be compensated by anticyclonic circulation over the Barents Sea itself.
In fact, it follows from the condition of inertial stability that the magnitude of the second term in the
integrand in (9) cannot exceed \( h_E \frac{\sin \phi}{\cos \phi - \sin \phi} f \approx 4.4 \times 10^{-2}\text{ ms}^{-1} \), where it was again used that
\( \phi = \pi/8, \ h_E = 400\text{ m} \) and also the value of the Coriolis parameter \( f = 1.46 \times 10^{-4}\text{ s}^{-1} \) in the polar region
was taken. On the other hand, cyclonic circulation is usually observed observed in the heating region in
the Barents Sea, which adds to the term \( \frac{T}{\gamma_a - \gamma} \frac{\partial \theta}{\partial \theta} \) and increases the imbalance. Therefore, as a
compensation, i.e. in order that there is a quasi-equilibrium climate regime over the Northern Hemisphere that corresponds to the current state of the atmosphere over the Arctic, this anomalous heating over the Barents Sea should be accompanied by an intensification of the wintertime blocking (and surface cooling of the air) over the continents and, accordingly, by colder winters there (cf. Petoukhov, Semenov, 2010). If we take into account that the rate of diabatic cooling over continents is less in magnitude than the heating rate over the Barents Sea and that the relative vorticity in anticyclones has substantially smaller magnitude than the Coriolis parameter, then for such compensation the area occupied by anticyclones should significantly exceed the area of the anomalous heating over the Arctic.

The proposed simple theoretical scheme hints on the possibility for a global compensation of the anomalous wintertime heating of the atmosphere over the Arctic, but does not indicate on the specific geographical location of the compensatory winter anticyclones. The latter can be obtained as a result of more complete model calculations and / or detailed numerical simulations with global / regional atmospheric models.

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Appendix

Under the quasi-geostrophic approximation, the potential vorticities of the second generation take the form

$$\frac{\omega_u \cdot \nabla P_\theta}{\rho} \approx g^2 f^2 \frac{\partial^2 \theta}{\partial p^2}, \quad \frac{\omega_u \cdot \nabla P_\eta}{\rho} \approx g^2 f^2 \frac{\partial^2 \eta}{\partial p^2}.$$  

Here, $f$ is the Coriolis parameter; see other notations in the main text. The following relation takes place,

$$\frac{\partial^2 \eta}{\partial p^2} = \frac{\partial}{\partial p} \left( \eta \frac{\partial \theta}{\partial p} \right) = \eta^\ast \left( \frac{\partial \theta}{\partial p} \right)^2 + \eta \frac{\partial^2 \theta}{\partial p^2}.$$  

The condition that the modified second-generation potential vorticity is close to zero reads

$$\frac{\chi'}{\chi} \approx \frac{\partial^2 \theta}{\partial p^2},$$

or

$$\frac{d}{d \theta} \ln \chi' \approx \frac{\partial}{\partial p} \left( \frac{1}{\partial \theta/\partial p} \right) \equiv \frac{\partial}{\partial \theta} \ln \left( \frac{1}{\partial \theta/\partial p} \right).$$

The last relation is, indeed, approximately satisfied since $\chi' g \frac{\partial \theta}{\partial p} \approx \frac{d \chi^\ast}{d \eta} \frac{\partial \eta}{\partial p} \approx \eta^\ast \partial^2 \theta/\partial p^2 \approx -1$.

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