Limitations of the Standard Gravitational Perfect Fluid Paradigm

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Abstract

We show that the standard perfect fluid paradigm is not necessarily a valid description of a steady state gravitational source unless the background geometry has very high symmetry. By incoherently averaging over a compete set of modes of a scalar field propagating in a curved background, we show that for a static, spherically symmetric geometry the energy-momentum tensor that ensues will in general be of the form $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} + qV_\mu V_\nu$ where $V_\mu$ is a spacelike vector. Such a $qV_\mu V_\nu$ type term is absent for an incoherent fluid in flat spacetime since a Minkowski geometry has much higher symmetry than a spherically symmetric one, and would thus be missed in a covariantizing of a flat spacetime $T_{\mu\nu}$. However, in a general relativistic treatment of systems such as stars, such $qV_\mu V_\nu$ type terms cannot automatically be ignored, with their contribution in the strong gravity limit potentially being quite significant.
I. INTRODUCTION

In gravitational theory it is standard to take the energy-momentum tensor of a steady state gravitational fluid source to be in the form of a perfect fluid. As such, a steady state fluid consists of a set of particles which are all in free fall or a set of wave modes whose contributions to the energy-momentum tensor are added incoherently. In flat spacetime the energy-momentum tensor of such a fluid will have the perfect fluid form

\[ T_{\mu\nu} = \left( \rho_f + p_f \right) U_{\mu} U_{\nu} + p_f g_{\mu\nu}, \]

with fluid energy density \( \rho_f \) and fluid pressure \( p_f \), and it is conventional to covariantize this expression in order to obtain the energy-momentum tensor which is to describe a steady state fluid in a curved geometry. The covariantization prescription is to replace all flat space tensors by curved space ones and all flat space derivatives by covariant ones, so that a curved space perfect fluid is to be described by (1) as written in a non-flat background, with no other terms with the requisite tensor structure being deemed relevant. Remarkably, no justification for the appropriateness of such a procedure appears to have been given in the literature, and its use is simply taken as being self-evident.

In this paper we examine this prescription and find that it does not in fact hold in general. In particular, we find that it does not necessarily hold in a situation where it is commonly used, namely in a spherically symmetric system such as a star. To understand why there might even be a concern for spherical systems, we note that spherical systems are only isotropic about a single point (the center of the system). Such a symmetry only requires for any given tensor \( A_{\mu\nu} \), that its \( A_{\theta\theta} \) and \( A_{\phi\phi} \) components be equal, and imposes no restriction on \( A_{rr} \). A familiar example of this is the form of the Einstein tensor \( G_{\mu\nu} \) in the static, spherical geometry

\[ ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]

where the non-zero components of \( G_{\mu\nu} \) are given by

\[
G_{00} = - \frac{B}{r^2} + \frac{B}{r^2 A} - \frac{BA'}{rA^2},
\]

\[
G_{rr} = \frac{A}{r^2} - \frac{1}{r^2} - \frac{B'}{rB},
\]

\[
G_{\theta\theta} = \frac{G_{\phi\phi}}{\sin^2 \theta} = - \frac{r^2 B''}{2AB} + \frac{r^2 A'B'}{4A^2 B} + \frac{r^2 B^2}{4AB^2} - \frac{rB'}{2AB} + \frac{rA'}{2A^2}.
\]
As we see, there no relation of the form $G_{r}^{r} = G_{\theta}^{\theta}$ in (3), and not only that, there could not be since the Bianchi identity $G_{\mu\nu}^{\nu} = 0$ relates the radial derivative of $G_{r}^{r}$ to $G_{\theta\theta}^{\theta}$, to thereby force $G_{r}^{r}$ to be a first-derivative function of the metric coefficients and $G_{\theta\theta}^{\theta}$ to be a second-derivative one. This same problem is not evaded if one works in isotropic coordinates, where, because of coordinate invariance, the metric can actually be brought to a form

$$ds^2 = -H(\rho)dt^2 + J(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2)$$

(4)

that does have the generic perfect fluid form $g_{\rho}^{\rho} = g_{\theta}^{\theta} = g_{\phi}^{\phi}$. Specifically, even in this coordinate system the Einstein tensor

$$G_{00} = \frac{2HJ'}{\rho J^2} - \frac{3HJ'^2}{4J^3} + \frac{HJ''}{J^2},$$

$$G_{\rho\rho} = -\frac{J'}{\rho J} - \frac{H'J'}{2HJ} - \frac{J'^2}{4J^2} - \frac{H'}{\rho H},$$

$$G_{\theta\theta} = -\frac{\rho^2 J''}{2J} - \frac{\rho J'}{2J} + \frac{\rho^2 J'^2}{2J^2} - \frac{\rho H'}{2H} + \frac{\rho^2 H'^2}{4H^2} - \frac{\rho^2 H''}{2H}$$

(5)

is still not of a perfect fluid form, and indeed must still not be since now the Bianchi identity relates the radial derivative of $G_{\rho\rho}$ to $G_{\theta\theta}$. Since the gravitational equations of motion would relate the Einstein tensor (or some equally covariant alternate gravity equivalent) to the energy-momentum tensor, even in isotropic coordinates we should not expect the components of $T_{\mu\nu}$ to necessarily obey $T_{\rho}^{\rho} = T_{\theta}^{\theta} = T_{\phi}^{\phi}$.

To determine what should be the case, and to understand why one is tempted to use (1) in the curved space case, it is instructive to ask how (1) came about in the flat space case in the first place. Specifically, to establish a flat space perfect fluid form, one has to evaluate the partition function for the system, and to do so one incoherently averages over all of the modes of the system. When this is done for the modes of a scalar field or of a Maxwell field propagating in a flat background, one does indeed recover the perfect fluid form [1, 2], just as one should since it is precisely such an incoherent sum over modes which produces the energy and pressure of flat space black-body radiation and the familiar $p_f = \rho_f / 3$ associated with a traceless $T_{\mu\nu}$. The interest of [1] and [2] (and also [3]) was to extend the incoherent averaging calculation to a background Robertson-Walker geometry, with (1) indeed then being recovered, just as one would want. However, both a Minkowski background and a Robertson-Walker background have very high symmetry, Robertson-Walker being maximally 3-symmetric and Minkowski being maximally 4-symmetric, with both of these geometries
being isotropic about every spatial point. In contrast, unlike these particular geometries, the geometry associated with (2) or (4) has much lower symmetry, being isotropic only about one point, the origin of coordinates. We thus cannot immediately anticipate that an incoherent averaging over modes propagating in the geometry associated with (2) or (4) will necessarily lead to (1). In fact, we will show that in general it does not necessarily do so. Specifically, to show that we do not always recover (1) in general, we only need to find one counterexample. We thus only need to find one appropriate choice of metric coefficients for which one can do the incoherent summation over an infinite set of modes analytically and then fail to recover (1). It is precisely such a calculation that we provide in this paper.

In Sec. II we introduce the incoherent averaging procedure. In Sec. III we apply it to a fluid in a specific, exactly solvable, static, spherically symmetric geometry, to find that we do not obtain a perfect fluid form. In Sec. IV we discuss the difference between treating fluids as incoherently averaged waves or treating fluids as incoherently averaged particles that are in geodesic free fall. In Sec. V we show that through the imposition of boundary conditions there can be departures from a perfect fluid form for finite-sized systems even in flat spacetime. Finally, in Sec. VI we discuss the implications of kinetic theory for the structure of fluids.

II. THE CURVED SPACE ENERGY-MOMENTUM TENSOR

For our purposes here an appropriate model to study is a minimally coupled scalar field $S(x)$ with action

$$ I = -\int d^4 x (-g)^{1/2} \frac{1}{2} \nabla_\mu S \nabla^\mu S $$

propagating in the background associated with the metric of (4). For the scalar field the equation of motion is given by $\nabla_\mu \nabla^\mu S = 0$, i.e. by

$$ -\frac{1}{H} \frac{\partial^2 S}{\partial t^2} + \frac{1}{H^{1/2} J^{3/2} \rho^2} \frac{\partial}{\partial \rho} \left[ H^{1/2} J^{1/2} \rho^2 \frac{\partial S}{\partial \rho} \right] + \frac{1}{J \rho^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 S}{\partial \phi^2} \right] = 0, $$

and the energy-momentum tensor is of the form

$$ T_{\mu\nu} = \nabla_\mu S \nabla_\nu S - \frac{1}{2} g_{\mu\nu} \nabla_\alpha S \nabla^\alpha S. $$
The equation of motion can be separated, and for a real scalar field the general solution can be expressed in terms of four characteristic solutions

\[ S_1(x) = N^m_\ell \sin(\omega t) S_{\omega, \ell}(\rho) P^m_\ell(\theta) \sin(m\phi), \quad S_2(x) = N^m_\ell \sin(\omega t) S_{\omega, \ell}(\rho) P^m_\ell(\theta) \cos(m\phi), \]
\[ S_3(x) = N^m_\ell \cos(\omega t) S_{\omega, \ell}(\rho) P^m_\ell(\theta) \sin(m\phi), \quad S_4(x) = N^m_\ell \cos(\omega t) S_{\omega, \ell}(\rho) P^m_\ell(\theta) \cos(m\phi), \]

where \( N^m_\ell \) is given by

\[ N^m_\ell = (-1)^m \left[ \frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!} \right]^{1/2}, \]

and where the radial term obeys

\[ \left[ \frac{\omega^2}{H} + \frac{1}{H^{1/2} J^{3/2} \rho^2} \frac{d}{d\rho} \left( H^{1/2} J^{1/2} \rho^2 \frac{d}{d\rho} \right) - \frac{\ell(\ell + 1)}{J \rho^2} \right] S_{\omega, \ell}(\rho) = 0. \]

To illustrate the incoherent averaging procedure, we note that in a given mode such as \( S_1(x) \) a quantity such as \( \nabla_\alpha S \nabla^\alpha S \) evaluates to

\[ \nabla_\alpha S_1 \nabla^\alpha S_1 = -\frac{\omega^2}{H} \cos^2(\omega t) |S_{\omega, \ell} N^m_\ell P^m_\ell|^2 \sin^2(m\phi) + \frac{1}{J} \sin^2(\omega t) \left[ \frac{dS_{\omega, \ell}}{d\rho} \right]^2 \left[ N^m_\ell P^m_\ell \right]^2 \sin^2(m\phi) \]
\[ + \frac{1}{J \rho^2} \sin^2(\omega t) |S_{\omega, \ell}|^2 \left[ \left[ \frac{dP^m_\ell}{d\theta} \right]^2 \sin^2(m\phi) + \frac{m^2}{\sin^2\theta} \right] N^m_\ell P^m_\ell \cos^2(m\phi). \]

On now adding to this expression those obtained when one uses the other three above solutions, one obtains

\[ \sum_{i=1}^{i=4} \nabla_\alpha S_i \nabla^\alpha S_i = -\frac{\omega^2}{H} |S_{\omega, \ell} N^m_\ell P^m_\ell|^2 + \frac{1}{J} \left[ \frac{dS_{\omega, \ell}}{d\rho} \right]^2 \left[ N^m_\ell P^m_\ell \right]^2 \]
\[ + \frac{1}{J \rho^2} |S_{\omega, \ell}|^2 \left[ \left[ \frac{dP^m_\ell}{d\theta} \right]^2 + \frac{m^2}{\sin^2\theta} \right] N^m_\ell P^m_\ell. \]

Using standard properties of the spherical harmonics,

\[ \sum_m [N^m_\ell]^2 [P^m_\ell]^2 = \frac{(2\ell + 1)}{4\pi}, \]
\[ \sum_m [N^m_\ell]^2 m^2 [P^m_\ell]^2 = \frac{(2\ell + 1)\ell(\ell + 1) \sin^2\theta}{8\pi}, \]
\[ \sum_m [N^m_\ell]^2 \left[ \frac{dP^m_\ell}{d\theta} \right]^2 = \frac{(2\ell + 1)\ell(\ell + 1)}{8\pi}, \]

just as in \cite{1} one sums over the azimuthal quantum number \( m \) to obtain

\[ \sum_m \sum_{i=1}^{i=4} \nabla_\alpha S_i \nabla^\alpha S_i = \frac{(2\ell + 1)}{4\pi} \left[ \left( -\frac{\omega^2}{H} + \frac{\ell(\ell + 1)}{J \rho^2} \right) |S_{\omega, \ell}|^2 + \frac{1}{J} \left[ \frac{dS_{\omega, \ell}}{d\rho} \right]^2 \right] = K(\omega, \rho, \ell). \]
with (15) serving to define $K(\omega, \rho, \ell)$. With regard to (15), we note that because the angular part of the metric is maximally 2-symmetric, the sum on the azimuthal $m$ removes any dependence on the angle $\theta$. Repeating this same procedure for the rest of $T_{\mu\nu}$ of (8) then yields (again following [1])

\begin{align*}
T_{00}(\omega, \rho, \ell) &= \frac{(2\ell + 1)}{4\pi} \omega^2 [S_{\omega,\ell}]^2 + \frac{HK}{2}, \\
T_{\rho\rho}(\omega, \rho, \ell) &= \frac{(2\ell + 1)}{4\pi} \left[ \frac{dS_{\omega,\ell}}{d\rho} \right]^2 - \frac{JK}{2}, \\
T_{\theta\theta}(\omega, \rho, \ell) &= \frac{1}{\sin^2 \theta} T_{\phi\phi}(\omega, \rho, \ell) = \frac{(2\ell + 1)(\ell + 1)}{8\pi} [S_{\omega,\ell}]^2 - \frac{\rho^2 J K}{2},
\end{align*}

with all other components of $T_{\mu\nu}$ being zero.

To complete the incoherent averaging we need to sum over all $\ell$ values as well, an infinite summation. However, if we revert back to flat space where $H(\rho) = J(\rho) = 1$, the radial equation is then solved by the spherical Bessel functions $j_\ell(\omega \rho)$, and the sum on $\ell$ can then be performed analytically using the completeness relation for Bessel functions:

\begin{align*}
\sum_{\ell} (2\ell + 1) j_\ell^2 &= 1, &\sum_{\ell} (2\ell + 1) j_\ell j_\ell \frac{d}{d\rho} &= 0, &\sum_{\ell} (2\ell + 1) \left[ \frac{d j_\ell}{d\rho} \right]^2 &= \frac{\omega^2}{3}, \\
\sum_{\ell} (2\ell + 1) j_\ell \frac{d^2 j_\ell}{d\rho^2} &= -\frac{\omega^2}{3}, &\sum_{\ell} (2\ell + 1) \ell(\ell + 1) j_\ell^2 &= \frac{2\omega^2 \rho^2}{3}.
\end{align*}

On defining $K(\omega, \rho) = \sum_{\ell} K(\omega, \rho, \ell)$ and $T_{\mu\nu}(\omega, \rho) = \sum_{\ell} T_{\mu\nu}(\omega, \rho, \ell)$, one obtains $K(\omega, \rho) = 0$ and

\begin{align*}
T_{00}(\omega, \rho) &= \frac{\omega^2}{4\pi}, & T_{\rho\rho}(\omega, \rho) &= \frac{\omega^2}{12\pi}, & T_{\theta\theta}(\omega, \rho) &= \frac{1}{\sin^2 \theta} T_{\phi\phi}(\omega, \rho) = \frac{\omega^2 \rho^2}{12\pi},
\end{align*}

One thus obtains none other than the perfect fluid form given in (11), with the averaging on $\ell$ removing any dependence on the coordinate $\rho$ from $T_{\mu\nu}$ because of the maximal 3-symmetry of the spatial part of a Minkowski metric. The incoherent averaging prescription thus gives a perfect fluid in flat spacetime, just as one would want.

### III. INCOHERENT AVERAGING IN CURVED SPACETIME

To follow this same procedure in curved space is not at all as straightforward, since for an arbitrary choice of $H(\rho)$ and $J(\rho)$ the radial equation will not necessarily be solvable in terms of named functions, and the needed completeness relation for the ensuing modes may
not even be known at all. Moreover, in the dynamical case where the energy-momentum
tensor is used as the source of tensors such as the Einstein tensor given in (5), one has
to solve for $H(\rho)$ and $J(\rho)$ self-consistently, a procedure that would not only not appear
at all likely to yield an analytic result, since it involves an infinite summation, it does not
immediately lend itself to numerical approximation either. However, to test the viability
of the perfect fluid assumption, one only has to seek an appropriate choice of $H(\rho)$ and
$J(\rho)$ for which one can do the $\ell$ summation analytically, to see whether or not the relation
$T_{\rho\rho} = T_{\theta\theta}/\rho^2$ then does in fact ensue.

Since we are able to do the $\ell$ summation analytically for Bessel functions, we shall seek a
form for $H(\rho)$ and $J(\rho)$ for which the radial equation can be reduced to the Bessel equation.
To this end we set $J(\rho) = H(\rho) = f^2(\rho)$. For this choice, the metric in (4) reduces to
\[ ds^2 = f^2(\rho)[-dt^2 + d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2], \]
(19)
to thus be conformal to flat. However, it is not flat since in it the Einstein tensor in (5)
takes the non-vanishing form
\[ G_{00} = \frac{4f'f}{2f^2 + 2f''}, \quad G_{\rho\rho} = -\frac{4f'f}{2f^2 + 2f''} - \frac{3f'^2}{f^2}, \quad G_{\theta\theta} = -\frac{2\rho f'}{f^2} + \frac{\rho^2 f'^2}{f^2} - \frac{2\rho^2 f''}{f}, \]
(20)
a form which is still not a perfect fluid unless $f(\rho)$ just happens to obey
\[ \frac{f''}{f} - \frac{f'}{\rho f} - \frac{2f'^2}{f^2} = 0. \]
(21)

For the metric of (19), on substituting $S_{\omega,\ell}(\rho) = Q_{\omega,\ell}(\rho)/f(\rho)$ the radial equation (11)
reduces to
\[ \left[ \omega^2 + \frac{d^2}{d\rho^2} + 2 \frac{d}{d\rho} - \frac{\ell(\ell + 1)}{\rho^2} - \frac{1}{f} \left( f'' + \frac{2f'}{\rho} \right) \right] Q_{\omega,\ell}(\rho) = 0. \]
(22)
Thus, for any choice of $f(\rho)$ for which
\[ -\frac{1}{f} \left( f'' + \frac{2f'}{\rho} \right) = \kappa^2 \]
(23)
where $\kappa^2$ is positive, one finds that (22) reduces to none other than the Bessel function
equation
\[ \left[ \omega^2 + \kappa^2 + \frac{d^2}{d\rho^2} + 2 \frac{d}{\rho d\rho} - \frac{\ell(\ell + 1)}{\rho^2} \right] Q_{\omega,\ell}(\rho) = 0, \]
(24)
with immediate solution $Q_{\omega,\ell}(\rho) = j_{\ell}(\lambda \rho)$ where $\lambda = (\omega^2 + \kappa^2)^{1/2}$. For (23) solutions are
readily given as
\[ f(\rho) = \frac{[\alpha \sin(\kappa \rho) + \beta \cos(\kappa \rho)]}{\rho}, \]
(25)
and thus form a whole family of solutions labeled by all real values of the parameters $\kappa$, $\alpha$ and $\beta$. For any $f(\rho)$ which obeys (23), the Einstein tensor reduces to

$$G_{00} = -\frac{f'^2}{f^2} - 2\kappa^2, \quad G_{\rho\rho} = \frac{4f'}{\rho f} - \frac{3f'^2}{f^2}, \quad G_{\theta\theta} = \frac{2\rho f'}{f} + \frac{\rho^2 f'}{f} + 2\rho^2 \kappa^2,$$

(26)

and in solutions of the form given in (25) is still not in the form of a perfect fluid, with the solutions in (25) not obeying (21).

To now evaluate the energy-momentum tensor when (23) is imposed, on summing over $\ell$ as before, one obtains

$$K(\omega, \rho) = \frac{(f'^2 + \kappa^2)}{4\pi f^4},$$

(27)

with the incoherently-averaged energy-momentum tensor itself being given by

$$T_{00}(\omega, \rho) = \frac{1}{4\pi} \left[ \frac{\omega^2}{f^2} + \frac{f'^2}{2f^2} + \frac{\kappa^2}{2f^2} \right],$$

$$T_{\rho\rho}(\omega, \rho) = \frac{1}{4\pi} \left[ \frac{\omega^2}{3f^2} + \frac{f'^2}{2f^2} - \frac{\kappa^2}{6f^2} \right],$$

$$T_{\theta\theta}(\omega, \rho) = \frac{1}{4\pi} \left[ -\frac{\rho^2 \omega^2}{3f^2} - \frac{\rho^2 f'^2}{2f^4} - \frac{\rho^2 \kappa^2}{6f^2} \right].$$

(28)

As we see, $T_{\mu\nu}(\omega, \rho)$ is not in the form of a perfect fluid since $T_{\rho}^\rho - T_{\theta}^\theta = f'^2/4\pi f^6$ is not zero. Since our analysis is valid for any $f(\rho)$ which obeys (25), we recognize a whole family of metrics for which a perfect fluid is not obtained. In general then, the use of perfect fluid sources for spherically symmetric gravitational systems must be regarded as open to question.

**IV. CONTRASTING AVERAGING OVER MODES WITH AVERAGING OVER PARTICLES**

To characterize the tensor structure of the $T_{\mu\nu}$ that incoherent averaging has led us to, we need to augment the perfect fluid energy-momentum tensor $T_{\mu\nu}$ of (1) with a further term of the form $q_f V_\mu V_\nu$ where $V_\mu$ is a unit spacelike vector in the radial direction, and just like $\rho_f$ and $p_f$, $q_f$ is equally a general coordinate scalar. With reference to the metric of (4), we define $V_\mu = (0, J^{1/2}, 0, 0)$, and with $U_\mu = (H^{1/2}, 0, 0, 0)$ as usual, find that we can write the energy-momentum tensor as

$$T_{\mu\nu} = (\rho_f + p_f)U_\mu U_\nu + p_f g_{\mu\nu} + q_f V_\mu V_\nu.$$  

(29)
For the energy-momentum tensor of (28) we identify
\[
\rho_f = \frac{1}{4\pi} \left[ \frac{\omega^2}{f^4} + \frac{f'^2}{2f^6} + \frac{\kappa^2}{2f^4} \right], \quad p_f = \frac{1}{4\pi} \left[ \frac{\omega^2}{3f^4} - \frac{f'^2}{2f^6} - \frac{\kappa^2}{6f^4} \right], \quad q_f = \frac{f'^2}{4\pi f^6},
\] (30)
with a summation over \( \omega \) being implicit. For the form of \( T_{\mu\nu} \) given in (29), in the geometry associated with the metric of (4) the covariant conservation condition \( T^\mu_{\ ;\nu} = 0 \) yields
\[
p'_f + q'_f + \left( \frac{J'}{J} + \frac{2}{\rho} \right) q_f + \frac{H'}{2H} (\rho_f + p_f + q_f) = 0,
\] (31)
to thus show the explicit role played by the \( q_f \) term in the energy and momentum balance.

Not only does \( q_f \) act analogously to \( p_f \) in the covariant conservation condition, it even does so in dynamical equations such as the Einstein equations \( G_{\mu\nu} = -8\pi GT_{\mu\nu} \). Specifically, if we work in the weak gravity limit where we can set \( H(\rho) = 1 + h(\rho), \ J(\rho) = 1 + j(\rho) \) with \( h(\rho) \) and \( j(\rho) \) both being of order \( G \), we can consistently find solutions in which \( \rho_f \) is of order one and \( p_f \) and \( q_f \) are both of order \( G \). In the weak gravity limit the Einstein equations and the conservation condition associated with the metric of (4) reduce to
\[
\frac{2j'}{\rho} + j'' = -8\pi G \rho_f, \quad -\frac{j'}{\rho} - \frac{h'}{\rho} = -8\pi G (p_f + q_f),
-\frac{j''}{2} - \frac{j'}{2\rho} - \frac{h'}{2\rho} - \frac{h''}{2} = -8\pi G p_f, \quad p'_f + q'_f + \frac{2}{\rho} q_f + \frac{h'}{2\rho} = 0,
\] (32)
and to order \( G \) thus have a consistent weak gravity limit in which \( j + h = 0 \). In and of themselves then, the dynamical equations do not require \( q_f \) to be at least one order in \( G \) smaller than \( p_f \).

In discussing fluids one can consider them to be composed of either particles or waves. To effect the incoherent averaging in the above we averaged over waves rather than particles. However, for a set of non-interacting particles, if one takes them to be in free fall and averages over them, one does get a perfect fluid with no \( q_f \) type term. Specifically, since no such term is present for such a fluid in flat spacetime where all the particles move on flat space geodesics, in the presence of gravity, free-falling particles will continue to move on geodesics and thus still behave as a perfect fluid. We thus need to ask why it was that the averaging over waves did not give a perfect fluid, and how one can then reconcile the free-falling particle and the incoherent-averaged wave descriptions. To this end we need to recall the conditions under which rays normal to wavefronts are geodesic.

As noted in [4], if we set \( S(x) = \exp(iT(x)) \), from the wave equation \( \nabla^\mu \nabla_\mu S = 0 \) we obtain
\[
\nabla^\mu T \nabla_\mu T - i\nabla^\mu \nabla_\mu T = 0.
\] (33)
For wavelengths that are short with respect to the typical scale of the problem, (33) reduces to \( \nabla^\mu T \nabla_\mu T = 0 \), to yield first \( \nabla^\mu T \nabla_\nu \nabla_\mu T = 0 \), and then

\[
\nabla^\mu T \nabla_\mu \nabla_\nu T = 0
\]

(34)
since \( \nabla_\mu \nabla_\nu T = \nabla_\nu \nabla_\mu T \). Since normals to the wavefronts obey the eikonal relation

\[
\nabla^\mu T = \frac{dx^\mu}{dq} = k^\mu
\]

(35)
where \( q \) measures distance along the normal and \( k^\mu \) is the wave vector of the wave, on noting that \( d/dq = (dx^\mu/dq)(\partial/\partial x^\mu) \), from (34) we thus obtain

\[
k^\mu \nabla_\mu k^\nu = \frac{d^2x^\nu}{dq^2} + \Gamma^\nu_{\mu\lambda} \frac{dx^\mu}{dq} \frac{dx^\lambda}{dq} = 0.
\]

(36)
Recognizing (36) as the massless geodesic equation, we see that in the short wavelength limit, rays move on geodesics. Moreover, exactly the same result can be obtained \([4]\) for massive scalar fields as well, with it being a general rule that short wavelength rays are geodesic. Referring now to the incoherently averaged \( T_{\mu\nu} \) given in (28), we see that in the large \( \omega \) limit \( T_{\mu\nu} \) does indeed reduce to a perfect fluid with the \( q_f \) term becoming negligible. Perfect fluids are thus to be associated with short wavelength modes alone, with departures from a perfect fluid form being expected to occur at longer wavelengths where one can no longer ignore the \( \nabla^\mu \nabla_\mu T \) term in (33). The potential importance of such long wavelength modes thus depends on the scale of spatial variation of the system of interest, a dynamical rather than a kinematical issue. Moreover, the relative importance of the longer wavelength modes will increase as the size of a system is decreased, to potentially give a departure from a perfect fluid form that could be substantial when a system such as star collapses to a size of order its Schwarzschild radius \([5]\).

V. IMPLICATIONS OF BOUNDARY CONDITIONS

While we have discussed possible departures from a perfect fluid form in curved space, such departures can even occur in flat space due to boundary effects of finite sized systems. Such concerns do not arise in flat space when we consider plane waves, since they fill all space. However, suppose we consider a finite-sized cavity which is filled with massless scalar modes in thermal equilibrium (a spinless black body). Now it is long known that if we take the
cavity to be a cubical cavity of side of length \( a \) and take the modes to obey periodic boundary conditions, while we would find corrections of order \( hc/kT a \) to the familiar \( T_{00} \sim T^4 \) black-body formula, because of the cubic symmetry these corrections would treat all components of the pressure tensor equivalently and leave the perfect fluid form \( T_{xx} = T_{yy} = T_{zz} = T_{00}/3 \) (i.e. \( T_{rr} = T_{00}/r^2 = T_{\phi\phi}/r^2 \sin^2 \theta = T_{00}/3 \)) intact. These finite size effects would only be significant for wavelengths of order \( a \), and would become inconsequential as the size of the cavity is increased.

However, suppose we consider a spherical cavity of radius \( a \), and rather than periodic boundary conditions \([1]\), instead require that the modes have radial wave functions which vanish at the surface of the cavity. The radial functions would then obey \( j_\ell(\omega a) = 0 \), with the allowed frequencies then being given by the zeroes of the Bessel functions. Each Bessel function has its own infinite set of discrete zeroes \([e.g. the zeroes of \( j_0(x) = (\sin x)/x \) obey \( \sin x = 0 \), and those of \( j_1(x) = (\sin x)/x^2 - (\cos x)/x \) obey \( \sin x = x \cos x \)].

On labeling these zeroes as \( j_\ell^\ell \), inside a flat space spherical cavity the allowed frequencies are given by the discrete \( \omega_n^\ell = j_\ell^\ell/a \), and the radial wave functions themselves are given by \( j_\ell(j_\ell^\ell r/a) \). 

In flat space there is no distinction between the radial coordinates \( r \) and \( \rho \) of \((2)\) and \((4)\). The sum over modes proceeds initially as previously, with \([16]\) being replaced by

\[
T_{00}(\omega_n^\ell, r, \ell) = \frac{(2\ell + 1)}{8\pi} \left[ (\omega_n^\ell)^2 + \frac{\ell(\ell + 1)}{r^2} \right] [j_\ell(j_\ell^\ell r/a)^2] + \left( \frac{dj_\ell(j_\ell^\ell r/a)}{dr} \right)^2,
\]

\[
T_{rr}(\omega_n^\ell, r, \ell) = \frac{(2\ell + 1)}{8\pi} \left[ \omega_n^\ell)^2 - \frac{\ell(\ell + 1)}{r^2} \right] [j_\ell(j_\ell^\ell r/a)^2] + \left( \frac{dj_\ell(j_\ell^\ell r/a)}{dr} \right)^2,
\]

\[
\frac{1}{r^2}T_{\theta\theta}(\omega_n^\ell, r, \ell) = \frac{(2\ell + 1)}{8\pi} \left[ (\omega_n^\ell)^2 [j_\ell(j_\ell^\ell r/a)^2] - \left( \frac{dj_\ell(j_\ell^\ell r/a)}{dr} \right)^2 \right].
\]

(37)

To complete the incoherent averaging we would need to sum over all \( \omega_n^\ell \) for a fixed \( \ell \) and then sum over all \( \ell \). However, to see whether or not a perfect fluid form might emerge, we only need to look near \( r = a \). Noting that every Bessel function obeys the recursion relation

\[
\frac{dj_\ell(\omega r)}{dr} = \frac{\ell}{r} j_\ell(\omega r) - \omega j_{\ell+1}(\omega r),
\]

and recalling that the zeroes of \( j_\ell(x) \) are distinct from those of \( j_{\ell+1}(x) \), we see that at \( r = a \) \( j_\ell(j_\ell^\ell r/a) \) vanishes but \( dj_\ell(j_\ell^\ell r/a)/dr \) does not. Thus at \( r = a \) we have

\[
T_{00}(\omega_n^\ell, a, \ell) = T_{rr}(\omega_n^\ell, a, \ell) = -\frac{1}{a^2} T_{\theta\theta}(\omega_n^\ell, a, \ell) = \frac{(2\ell + 1)}{8\pi} \left( \frac{dj_\ell(j_\ell^\ell r/a)}{dr} \right)^2 \bigg|_{r=a}.
\]

(39)
Since $T_{rr}(\omega_n^\ell, a, \ell)$ is positive definite and equal and opposite to the negative definite $T_{\theta\theta}(\omega_n^\ell, a, \ell)/a^2$, and since the sum over $\omega_n^\ell$ and $\ell$ does not change this, we see that we do not recover the perfect fluid form at $r = a$. A straightforward Taylor series expansion of the form $r = a - \epsilon$ shows that this result remains true to second order in $\epsilon$. To get a sense of how the incoherent sum at arbitrary $r$ might look, we have numerically summed over the first 100 zeroes of each of the first 100 $\ell$ values of $j_\ell(j_n^\ell r/a)$ (i.e. 10,000 terms in total), and as we see in Fig. (1), there is no sign of a perfect fluid form. Even in flat spacetime then, boundary conditions can prevent a fluid from being of the prefect fluid form when the fluid is in thermal equilibrium inside a spherical cavity of finite radius.
VI. IMPLICATIONS OF KINETIC THEORY

While the gravitational equations themselves provide no specific basis for leaving out any $q_f$ type term from the fluid $T_{\mu\nu}$ in general, in the weak gravity limit one might still be able to exclude such terms via the use of kinetic theory. In applying kinetic theory to gravitational systems there are two approaches, one based on the Boltzmann equation, and the other based on the Liouville equation. Since there are some differences between these two approaches [7] with the former being more appropriate for stars and the latter more appropriate for systems such as a cluster of galaxies, we describe them both.

Consider first a set of particles each of mass $m$ in some long-range (typically gravitational) external potential $V_{\text{ext}}(x)$ that are undergoing rapid (typically atomic) momentum conserving two-body collisions $v + v_2 \rightarrow v' + v'_2$ through a scattering angle $\Omega$ with differential collision cross section $\sigma(\Omega)$. In the absence of two-body correlations the one-particle distribution function $f(x, v, t)$ obeys the Boltzmann equation (see e.g. [8])

$$\frac{\partial f(x, v, t)}{\partial t} + v \cdot \frac{\partial f(x, v, t)}{\partial x} - \frac{\partial V_{\text{ext}}(x)}{\partial x} \cdot \frac{\partial f(x, v, t)}{\partial v} = \int |v - v_2| \sigma(\Omega) [f(x, v', t) f(x, v'_2, t) - f(x, v, t) f(x, v_2, t)] d^3v_2 d\Omega. \tag{40}$$

As such, (40) admits of an exact time-independent Maxwell-Boltzmann type solution

$$f_{\text{MB}}(x, p, t) = C \exp \left[ -\frac{mv^2}{2kT} - \frac{mV_{\text{ext}}(x)}{kT} \right], \tag{41}$$

where the temperature $T$ and the coefficient $C$ are independent of $x$. (That $f_{\text{MB}}$ is an exact solution is because it makes both sides of (40) vanish separately.) However, since averages with this distribution function are given by $\langle A \rangle = \int d^3p A f_{\text{MB}} / \int d^3p f_{\text{MB}}$, for any dynamical variable $A(x, v, t)$, the $V_{\text{ext}}(x)$ term would drop out and all averages evaluated with this $f_{\text{MB}}$ would be spatially independent. Moreover, the number density itself would behave as $n(x) = \int d^3p f_{\text{MB}}(x, p, t) \sim \exp(-mV_{\text{ext}}(x)/kT)$, and give a spatial dependence that is nothing like that required of the density of a star of finite size.

Now as such, use of the above Boltzmann equation presupposes that the only collisions of relevance are atomic type ones, and that those collisions dominate the relaxation of the system to the distribution function $f_{\text{MB}}(x, p, t)$. However, in stars gravitational collisions also play a role, and their effect cannot be isolated solely in the $\partial V_{\text{ext}}(x)/\partial x$ term in (40). Rather, they serve to modify the right-hand side of (40) as well. However, since the cross-section for gravitational scattering is infinite, the effect of gravity cannot be accounted for by
a collision integral term in which one simply includes a gravitational contribution to $\sigma(\Omega)$. Rather, one should work with the Liouville equation as it is better suited to handle long range forces, and we will discuss this below.

However, before doing so, we note that in kinetic theory, through use of the method of the most probable distribution, it is possible to determine the distribution function without needing to actually solve or even construct the Boltzmann equation at all. This method does not determine the approach to equilibrium (i.e. the temporal behavior of the distribution function), but does give the equilibrium configuration that results, and it is valid even if the distribution function does not obey an equation such as (40) at all. Thus we can use the method of the most probable distribution to get a sense of how gravitational interactions might modify the distribution function (41). Ordinarily one applies the method by taking the system of interest to be confined to a region of phase space with a fixed total energy and fixed volume and introduces spatially-independent Lagrange multipliers for the total number of particles and the total energy [8]. In the absence of gravity this leads to the Maxwell-Boltzmann distribution being the overwhelmingly likely one. In the presence of non-relativistic gravity we can adapt this method to concentric shells within a star, and take the temperature and density to be a constant within any given shell but to vary from one shell to the next. The approximation here is that particles stay within shells and do not exchange energy or momentum with any particles except those in their own shell. (Since gravity is a long range force, at best this assumption could only be valid when gravity is weak.) In this approximation the Lagrange multipliers for the total number of particles and the total energy within a given shell are taken to be uniform throughout the shell but to depend on the location of the shell, with the most probable (mp) distribution function throughout the star then being given by

$$f_{\text{mp}}(x, p, t) = \frac{n}{(2\pi m\theta)^{3/2}} \exp \left[ -\frac{m(v - u)^2}{2\theta} \right].$$ (42)

In (42) the particle number density $n(x, t) = \int d^3p f_{\text{mp}}$, the average velocity $u(x, t) = \int d^3p f_{\text{mp}}p/mn(x, t)$ and the temperature $\theta(x, t) = (m/3) \int d^3p f_{\text{mp}}|v - u(x, t)|^2/n(x, t)$ are now all spatially dependent.

While such a distribution function would not be an exact solution to the Boltzmann equation, in the event that $n(x, t)$, $u(x, t)$ and $\theta(x, t)$ are all slowly spatially varying, $f_{\text{mp}}(x, p, t)$ would be a good first-order approximation to it. For such a distribution function the pressure
tensor evaluates to

\[ P_{ij} = m n(x, t) \langle (v_i - u_i)(v_j - u_j) \rangle = \delta_{ij} n(x, t) \theta(x, t), \]  

(43)

to thus be of none other than of perfect fluid form. If the full distribution function is taken to obey the Boltzmann equation, then in the next order the pressure tensor evaluates to

\[ P_{ij} = \delta_{ij} n \theta - \tau n \theta \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right] \]  

(44)

where \( \tau \) is the mean free time between particle collisions. With \( n(x, t) \) depending on position, the \( \partial_j u_i + \partial_i u_j \) term would not be proportional to \( \delta_{ij} \). When \( n(x, t) \) is slowly varying then, the pressure only begins to depart from \( \delta_{ij} \) in corrections to first order. To the extent then that weak gravity would cause a star to bind with a slowly varying density and that \( \tau \) would be small (i.e. a small mean free time between atomic collisions), the \( q_f \) type term could be neglected in lowest order. Thus our ability to ignore \( q_f \) in a weak gravity star depends on how good a dynamical approximation the above \( f^{mp}(x, p, t) \) is to the full \( f(x, p, t) \). The use of perfect fluids as gravitational sources for weak gravity stars is thus equivalent to using \( f^{mp}(x, p, t) \) as the one-particle distribution function. However, once one has to go beyond \( f^{mp}(x, p, t) \) as would be the case for stronger gravity, there would no longer appear to be any immediately apparent justification for ignoring \( q_f \) type terms.

In terms of the geodesic equation discussion given earlier, we now see that short wavelength for eikonal purposes means short with respect to the distance scale on which the particle number density \( n(x, t) \) varies. The slower the spatial variation of \( n(x, t) \) then, the fewer the number of modes that will not incoherently average to a perfect fluid.

In the above discussion it is the non-gravitational collisions which dominate, with the discussion being given for a weak gravity star in which there are atomic collisions between the atoms in the star. However, for a cluster of galaxies, it is gravity itself which has to establish an equilibrium distribution of galaxies. In clusters there is a lot of X-ray producing plasma located in the region between the individual galaxies in the cluster. Collisions between the atomic particles in the plasma can readily bring the plasma to thermal equilibrium, but unless they can bring the galaxy distribution to equilibrium too, it would be up to gravity to do so.

To describe the role that gravity plays in a cluster of galaxies we turn to the Liouville equation. We treat each of the \( N \) galaxies in the cluster as a non-relativistic point particle
of mass \( m \) with position \( \mathbf{x}_i \) and velocity \( \mathbf{v}_i \). Each galaxy moves in the gravitational field \( \phi(\mathbf{x}_i) \) produced by the other galaxies in the cluster, and obeys
\[
\frac{d^2 \mathbf{x}_i}{dt^2} = -\frac{\partial \phi(\mathbf{x}_i)}{\partial \mathbf{x}_i}. \tag{45}
\]

One introduces the normalized (to one) \( 6N \)-dimensional distribution function \( f^{(N)}(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N, t) \), and finds that it obeys the Liouville equation
\[
\frac{df^{(N)}}{dt} = \sum_{\alpha=1}^{N} \left[ \mathbf{v}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \mathbf{x}_\alpha} - \frac{\partial \phi_\alpha}{\partial \mathbf{x}_\alpha} \cdot \frac{\partial f^{(N)}}{\partial \mathbf{v}_\alpha} \right] = 0, \tag{46}
\]
where
\[
\phi_\alpha(\mathbf{x}_\alpha) = \sum_{\beta \neq \alpha} \phi(\mathbf{x}_\alpha, \mathbf{x}_\beta). \tag{47}
\]

If the distribution function is symmetric under the interchange of any pair of particles and sufficiently convergent asymptotically, upon integrating (46), one finds (see e.g. [9]) that for large \( N \), the one- and two-particle distribution functions
\[
f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) = \int d^3 \mathbf{x}_2 d^3 \mathbf{v}_2 \ldots d^3 \mathbf{x}_N d^3 \mathbf{v}_N f^{(N)},
\]
\[
f^{(2)}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = \int d^3 \mathbf{x}_3 d^3 \mathbf{v}_3 \ldots d^3 \mathbf{x}_N d^3 \mathbf{v}_N f^{(N)}, \tag{48}
\]
are related by
\[
\frac{\partial f^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}_1} = N \int \frac{\partial \phi(\mathbf{x}_1, \mathbf{x}_2)}{\partial \mathbf{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \mathbf{v}_1} d^3 \mathbf{x}_2 d^3 \mathbf{v}_2. \tag{49}
\]

In terms of the two-body correlation function defined by
\[
g(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f^{(2)}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) - f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t)f^{(1)}(\mathbf{x}_2, \mathbf{v}_2, t) \tag{50}
\]
and the kinetic theory distribution \( f(\mathbf{x}, \mathbf{v}, t) = N f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) \) that is normalized to
\[
\int d^3 \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) = n(\mathbf{x}, t), \quad \int d^3 \mathbf{x} n(\mathbf{x}, t) = N, \tag{51}
\]
we find that (49) takes the form
\[
\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} - \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}}
= N^2 \frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{\partial \phi(\mathbf{x}, \mathbf{x}_2)}{\partial \mathbf{x}} g(\mathbf{x}, \mathbf{v}, \mathbf{x}_2, \mathbf{v}_2, t) d^3 \mathbf{x}_2 d^3 \mathbf{v}_2 \tag{52}
\]
where
\[
V(\mathbf{x}) = \int d^3 \mathbf{x}_2 d^3 \mathbf{v}_2 f(\mathbf{x}_2, \mathbf{v}_2, t) \phi(\mathbf{x}, \mathbf{x}_2) = \int d^3 \mathbf{x}_2 n(\mathbf{x}_2, t) \phi(\mathbf{x}, \mathbf{x}_2). \tag{53}
\]
As such, the potential \( V(x) \) introduced in (53) serves as a mean-field potential, and should the system relax so that the two-body correlation function becomes negligible, the left-hand side of (52) would then become equal to zero. At first glance the left-hand side of (52) looks quite like the left-hand side of the Boltzmann equation (40). However, in (40) \( V_{\text{ext}}(x) \) is an external one-body potential, while in (52) \( V(x) \) is a statistically averaged two-body potential. Moreover, while the vanishing of \( g(x_1, v_1, x_2, v_2, t) \) would cause the right-hand side of (52) to vanish, it would not have the same effect on the right-hand side of the Boltzmann equation of (40). In fact, it was the very requirement that \( g(x_1, v_1, x_2, v_2, t) \) vanish that led to the explicit form for the collision integral term given on the right-hand side of (40) in the first place, with the vanishing of \( g(x_1, v_1, x_2, v_2, t) \) not requiring the vanishing of the Boltzmann equation collision integral term. When the right-hand side of (52) does vanish, all one can say is that the steady state solution to (52) simply requires that \( f(x, v, t) \) be an arbitrary function of the quantity \( v^2/2 + V(x) \). With (52) possessing no analog of the collision integral term in (40) (gravity being long range), there is nothing to force \( f(x, v, t) \) to be an exponential function of \( v^2/2 + V(x) \). The specific dependence on \( v^2/2 + V(x) \) that \( f(x, v, t) \) would acquire in equilibrium would depend entirely on how the two-body \( g(x_1, v_1, x_2, v_2, t) \) would behave before it becomes negligible. In the absence of knowing how \( g(x_1, v_1, x_2, v_2, t) \) behaves, one is not able to recover any analog of (43). Consequently, without detailed information on the two-body correlation function, for clusters, even weak gravity ones, one is not able to anticipate that in steady state the galaxies in a cluster would behave like a perfect fluid [11].

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[2] Y. Deng and P. D. Mannheim, Gen. Rel. Gravit. 20, 969 (1988).
[3] Y. Deng and P. D. Mannheim, Astrophys. Sp. Sci. 135, 261 (1987).
[4] P. D. Mannheim, Prog. Part. Nucl. Phys. 56, 340 (2006).
[5] With the vector \( n_\mu = (U_\mu, \cos \theta V_\mu, 0, 0) \) being timelike, and with \( n_\mu n_\nu T^{\mu\nu} \) evaluating to \( \rho_f - \cos^2 \theta (q_f + p_f) \), for large enough \( q_f \) it might be possible to violate the weak energy condition \( n_\mu n_\nu T^{\mu\nu} > 0 \). For strong gravity systems then, by generating a large enough \( q_f \) term, it might
be possible for a collapsing star to evade the Hawking-Penrose singularity theorems.

[6] For modes of the form $j_\ell(\omega r)P^m_\ell(\theta)$, periodic boundary conditions require that $\ell$ be even.

[7] P. D. Mannheim, *Linear potentials in galaxies and clusters of galaxies*, astro-ph/9504022, April 1995.

[8] K. Huang *Statistical Mechanics*, Second Edition, J. Wiley, New York, N. Y. (1987).

[9] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton University Press, Princeton, N. J. (1987).

[10] The use of the method of the most probable distribution as used above for stars is not readily applicable here, since for particles whose motions are controlled by long range forces alone (i.e. in the absence of short range forces), the cluster cannot readily be approximated by shells of particles that do not exchange energy and momentum with particles in other shells.

[11] As noted in [9], by integrating (52) one can obtain a virial relation $\int d^3x \langle v^2 \rangle - \int d^3x \langle x \cdot \partial V(x)/\partial x \rangle = \partial (\int d^3x \langle x \cdot v \rangle) / \partial t + N^2 \int d^3x d^3v d^3x_2 d^3v_2 g(x, v, x_2, v_2, t) x \cdot \partial \phi(x, x_2) / \partial x$ in which the pressure tensor does not appear. Thus the virial relation and its steady state limit $\int d^3x \langle v^2 \rangle - \int d^3x \langle x \cdot \partial V(x)/\partial x \rangle = 0$ are not sensitive to whether or not the galaxies behave as a perfect fluid.