Abstract

We give a geometric interpretation of the Khovanov complex for virtual links. Geometric interpretation means that we use a cobordism structure like D. Bar-Natan, but we allow non orientable cobordisms. Like D. Bar-Natans geometric complex our construction should work for virtual tangles too.

This geometric complex allows, in contrast to the geometric version of V. Turaev and P. Turner, a direct extension of the classical Khovanov complex \((h = t = 0)\) and of the variant of Lee \((h = 0, t = 1)\).

Furthermore we give a classification of all unoriented TQFTs which can be used to define virtual link homologies with this geometric construction.
Figure 1: The virtual Khovanov complex of the unknot.

| sign | string | comultiplication | string | multiplication |
|------|--------|------------------|--------|----------------|
| +    | \( \rightarrow \nabla \) | \( \Delta^+_\pm \) | \( \rightarrow \nabla \) | \( m^+_\pm \) |
|      | \( \rightarrow \nabla \) | \( \Delta^+_\pm = \Phi_1 \circ \Delta^+_\pm \) | \( \rightarrow \nabla \) | \( m^+_\pm = m^+_\pm \circ \Phi_1 \) |
|      | \( \rightarrow \nabla \) | \( \Delta^-_\pm = \Phi_2 \circ \Delta^-_\pm \) | \( \rightarrow \nabla \) | \( m^-_\pm = m^+_\pm \circ \Phi_2 \) |
|      | \( \rightarrow \nabla \) | \( \Delta^+_\pm = \Phi_{12} \circ \Delta^+_\pm \) | \( \rightarrow \nabla \) | \( m^+_\pm = m^+_\pm \circ \Phi_{12} \) |
|      | \( \rightarrow \nabla \) | \( \Delta^-_\pm = \Phi_1 \circ \Delta^-_\pm \circ \Phi_1^* \) | \( \rightarrow \nabla \) | \( m^-_\pm = \Phi_1 \circ m^+_\pm \circ \Phi_1 \) |
|      | \( \rightarrow \nabla \) | \( \Delta^-_\pm = \Phi_2 \circ \Delta^-_\pm \circ \Phi_2^* \) | \( \rightarrow \nabla \) | \( m^-_\pm = \Phi_2 \circ m^+_\pm \circ \Phi_2 \) |
|      | \( \rightarrow \nabla \) | \( \Delta^-_\pm = \Phi_{12} \circ \Delta^-_\pm \circ \Phi_{12}^* \) | \( \rightarrow \nabla \) | \( m^-_\pm = \Phi_{12} \circ m^+_\pm \circ \Phi_{12} \) |

Figure 2: The assignment which decorations we use. If the sign is negative, then the saddle should carry an extra minus sign.
Introduction

In this paper we consider virtual link diagrams $L_D$, i.e. planar graphs of valency four where every vertex is either an overcrossing $\times$ or an undercrossing $\times$ or a virtual crossing, which is marked with a circle. We also allow circles, i.e closed edges without any vertices.

We call the crossings $\times$ and $\times$ classical crossings or just crossings. For a virtual link diagram $L_D$ we define the mirror image $\overline{L_D}$ of $L_D$ by switching all classical crossings from an overcrossing to an undercrossing and vice versa.

A virtual link $L$ is an equivalence class of virtual link diagrams modulo isotopies and generalised Reidemeister moves (see Figure 3). We call the moves RM1, RM2 and RM3 the classical Reidemeister moves, the moves vRM1, vRM2 and vRM3 the virtual Reidemeister moves and the move mRM the mixed Reidemeister move.

We call a virtual link diagram $L_D$ classical if all crossings of $L_D$ are classical crossings. Furthermore we say a virtual link $L$ is classical if the set $L$ contains a classical link diagram. We use the notions c-link (diagram) and v-link (diagram) as an abbreviated version of classical link (diagram) and virtual link (diagram).

The notions of an oriented virtual link diagram and of an oriented virtual link are defined analogue. The latter modulo isotopies and oriented generalised Reidemeister moves.

![Figure 3: The generalised Reidemeister moves are the moves pictured plus their mirror image.](image-url)

Virtual links are an essential part of modern knot theory and were proposed
by L. Kauffman in [7]. They arise from the study of links which are embedded in $\Sigma \times [0, 1]$ for an orientable surface $\Sigma$. These links were studied by F. Jaeger, L. Kauffman and H. Saleur in [6].

From this perception v-links are a combinatorial interpretation of projections of such links on $\Sigma$. It is well-known that two v-link diagrams are equivalent iff their corresponding surface embeddings are stable equivalent, i.e. equal modulo

- the Reidemeister moves RM1, RM2 and RM3 and isotopies;
- adding/removing handles which does not affect the link diagram;
- homeomorphisms of surfaces.

For a sketch of the proof see L. Kauffman [8]. For an example see Figure 4.

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The reader who is unfamiliar with virtual knot theory may for example check the nice introduction papers by L. Kauffman and V. Manturov (see [10]) or another paper by L. Kauffman (see [9]).

Suppose one has a crossing $c$ in a diagram of a v-link (or an oriented v-link). We call a substitution of a crossing like in Figure 5 a resolution of the crossing $c$. Furthermore if we have a v-link diagram $L_D$, a resolution of the v-link diagram $L_D$ is a diagram where all crossings of $L_D$ are replaced by one of the two resolutions from Figure 5.

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Figure 4: Two knot diagrams on a torus. The first v-knot is called the virtual trefoil.

Figure 5: The two possible resolutions of a crossing. The left one is called 0-resolution and the right one is called 1-resolution.
Let $a$ be a word in the alphabet $\{0, 1\}$. We denote with $\gamma_a$ the resolution of a v-link diagram $L_D$ with $|a|$ crossings, where the $i$-th crossing of $L_D$ is resolved $a_i \in \{0, 1\}$. We denote the number of v-circles in the resolution $\gamma_a$ as $|\gamma_a|$.

Moreover suppose we have two words $a, b$ with $a_k = b_k$ for $k = 1, \ldots, |a| = |b|, k \neq i$ and $a_i = 0, b_i = 1$. Then we call $S: \gamma_a \to \gamma_b$ a (formal) saddle between the resolutions.

Furthermore suppose we have a v-link diagram $L_D$ with at least two crossings $c_1, c_2$. We call a quadruple $F = (\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11})$ of four resolutions of the v-diagram $L_D$ a face of the diagram $L_D$ if in all four resolutions $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ all crossings of $L_D$ are resolved in the same way except that $c_1$ in resolved $i$ and $c_2$ is resolved $j$ in $\gamma_{ij}$ (with $i,j \in \{0, 1\}$). Furthermore there should be an oriented arrow from $\gamma_{ij}$ to $\gamma_{kl}$ if $i = j = 0$ and $k = 0, l = 1$ or $k = 1, l = 0$ or if $i = 0, j = 1$ and $k = l = 1$ or if $i = 1, j = 0$ and $k = l = 1$.

We also consider algebraic faces of a resolution. That is the same as above, but we replace $F_{ij}$ with $\otimes_a \ A$ if $F_{ij}$ has $n$ components. Here $A$ is an $R$-module and $R$ is a commutative, unital ring.

One of the greatest development in modern knot theory was the discovery of Khovanov homology by M. Khovanov in his famous paper [11] (D. Bar-Natan gave an exposition of Khovanov’s construction in [1]).

He defined a chain complex which we call the classical Khovanov complex. It is a categorification of the Jones polynomial for c-links, i.e. the graded Euler characteristic of the complex is the Jones polynomial, and it is a strong invariant of c-links itself (it is strict stronger then the Jones polynomial). So it is only natural to look for such a categorification of the Jones polynomial for v-links.

To construct this complex M. Khovanov associate a graded vector space (or $R$-module) to each resolution. Furthermore he uses a certain TQFT between two resolutions. With this TQFT he has defined a differential between the graded vector spaces.

This differential is made of a multiplication $m: A \otimes A \to A$ and a comultiplication $\Delta: A \to A \otimes A$ for the $R$-module $A = R[ X ]/(X^2)$ with gradings $\text{deg } 1 = 1, \text{deg } X = -1$. The comultiplication $\Delta$ is

$$\Delta: A \to A \otimes A:\begin{cases}1 \mapsto 1 \otimes X + X \otimes 1, \\X \mapsto X \otimes X.\end{cases}$$

The problem in the case of v-links is the emergence of a new map. This happens, because for v-links it is possible that a saddle $S: \gamma_a \to \gamma_b$ between two resolutions does not change the number of v-circles, i.e. $|\gamma_a| = |\gamma_b|$. This is a difference between c-links and v-links, i.e. in the first cases one always has $|\gamma_a| = |\gamma_b| + 1$ or $|\gamma_b| + 1 = |\gamma_a|$.

So in the algebraic complex we need a new map $\theta: A \to A$ together with the classical multiplication and comultiplication $m: A \otimes A \to A$ and $\Delta: A \to A \otimes A$. As we see later the only possible way to extend the classical Khovanov complex to
v-links is to set $\theta = 0$. But then a face could look like (maybe with extra signs)

$$A \otimes A \xrightarrow{\Delta} A \xrightarrow{m} A.$$  \hspace{1cm} (1)

We call such a face a problematic face.

With $\theta = 0$ and the classical $\Delta, m$, this faces does not commute for $R = \mathbb{Z}$. Therefore there is no straightforward extension of the Khovanov complex to v-links. An extension of the classical Khovanov complex to v-links was defined by V. Manturov in the $\mathbb{Z}/2$-case in [18] and in the $\mathbb{Z}$-case in [19].

Another great development was the geometric interpretation of the Khovanov complex by D. Bar-Natan in [2]. This geometric interpretation is a generalisation of the classical Khovanov complex for c-links and has functorial properties. He constructed a geometric complex whose chain groups are formal direct sums of c-link resolutions and whose differentials are formal matrices of cobordisms between these resolutions.

His construction is an invariant of c-links modulo chain homotopy and the local relations $S, T, 4Tu$, also called Bar-Natan relations (see Figure 6).

![Figure 6: The local relations](image)

With this construction it is possible to classify all TQFTs which can be used to define c-link homologies from this approach. See [13].

In his paper he proved that the $R$-module $A$ is of the form $R[X]/(X^2 - hX - t)$ and the comultiplication $\Delta$ is of the form

$$\Delta: A \to A \otimes A; \begin{cases} 1 \mapsto -h \cdot 1 \otimes 1 + 1 \otimes X + X \otimes 1, \\ X \mapsto t \cdot 1 \otimes 1 + X \otimes X. \end{cases}$$

One of these variants of the Khovanov complex is the version of E.S. Lee (see in her paper [13]) with $h = 0, t = 1$. This variant has many nice properties. For example
it was used by J. Rasmussen in [21] to give a new, purely combinatorical, proof of the Milnor conjecture. We call her variant the Khovanov-Lee complex.

Furthermore the construction of D. Bar-Natan is completely local, i.e. it is also an invariant of classical tangles. An application of this local behaviour is for example a calculation method for the classical Khovanov complex which is much faster than the direct calculation. The method was used to write a computer program which can calculate the Khovanov homology quite fast (see D. Bar-Natans paper [3]).

Therefore it is also natural to look for an extension of this geometric interpretation from c-links to v-links. This was first done by V. Turaev and P. Turner in their paper [23].

But their extension to v-links is not a generalisation of the classical Khovanov complex, i.e. the functors they have defined can not be used to extend the classical Khovanov complex \( (h = t = 0) \) or the Khovanov-Lee complex \( (h = 0, t = 1) \) from c-links to v-links.

In this paper we give a construction in the spirit of D. Bar-Natan which can be used to extend the classical Khovanov complex \( (h = t = 0) \) and the Khovanov-Lee complex \( (h = 0, t = 1) \) from c-links to v-links.

Furthermore a slight change of the definition of our complex leads directly to the extension of V. Turaev and P. Turner from [23]. We shortly mention this, but it is a straightforward change of the relation in our category (see the remarks 2.17 and 4.7). The main difference is that we have two different cup-cobordisms in our category.

We show in Theorem 2.15 that our construction is the same (up to chain isomorphisms) as the construction of D. Bar-Natan in [2] if one considers c-links. Hence one can say that our construction is the extension of the classical Khovanov complex to v-links.

**A brief summary**

We give a brief summary of the main ideas of our construction. We define the virtual Khovanov complex \([L_D]\) in the spirit of D. Bar-Natan (see his paper [2]). Hence the objects at the vertices of the virtual Khovanov complex \([L_D]\) are resolutions of a v-link diagram \(L_D\) and the morphisms are (possible unorientable) cobordisms between these resolutions. As describe before this alone does not lead to a well-defined chain complex, because the problematic face from 1 would not commute.

The main idea to solve this problem is to decorate the (possible unorientable) cobordisms. Then we glue the cobordisms together with an orientation preserving homeomorphism for every boundary component where the decorations are equal and with an orientation retaining homeomorphism for every boundary component where the decorations are different. Hence in our category, which we call \(u\text{Cob}^2_R(\emptyset)\), we have different (co)multiplications, depending on the different decorations.

We describe this category and some basic relations in the first section. The pantsup-morphism \(m^+\) and the pantsdown-morphism \(\Delta^+\) behave like the standard (co)multiplication in the category \(\text{Cob}^2(\emptyset)\) (the reader not familiar with this category should look in the nice book of J. Kock from [14]).
All other decorations of these two cobordisms are obtained by composition with an isomorphism $\Phi^{-}$. See Table 2.

In our pictures the source of a cobordism is always the top and the target is the bottom.

Moreover our cobordisms should be surfaces between resolutions of v-link diagrams. This is the reason why we do not embed them into $\mathbb{R}^{2} \times [-1,1]$. We use immersed surfaces. As an example we pictured the virtual Reidemeister cobordisms in Figure 7. They are all isomorphisms in our category, i.e. their inverses are the cobordisms from bottom to top rather than from top to bottom. We call them vRM-cobordisms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The virtual Reidemeister cobordisms. The red (vertical) lines means that the rest and not pictured part of the surfaces should be the identity.}
\end{figure}

In the second section we define a complex which we call the virtual Khovanov complex (see Definition 2.3). This complex is an invariant of v-links modulo chain homotopy and modulo the Bar-Natan relations from Figure 6 (see Theorem 2.14).

Thus a crucial question is how we spread the different decorations to the cobordisms. We do this in the following way. We choose orientations for every resolution of the v-link diagram. This induces local orientations for the saddles. Then we use Table 2 to spread the different decorations.

Furthermore we use the numbering of the different v-circles in the resolutions to define the saddle sign (see Figure 8), i.e. the X-marker assignment induces a numbering (red or grey if one do not use colours) in the following way: the not affected v-circles should be numbered like in the fixed numbering (black). The string with the X-marker should be the lower v-circle and the other string of Figure 8 should be the bigger v-circle (in the resolution where both strings belong to different v-circles are two numbers left after the identification from before). Then the saddle sign should be positive if the permutation between the fixed numbering and the induced numbering is even and negative otherwise.

This assignment differs between multiplication and comultiplication. The only extra signs in the complex should be the saddle signs.

The complex $J \cdot K$ of a virtual diagram of the unknot is pictured in Figure 9. This is the faces from 1. We mention that the two upper cobordisms in the figure are...
Figure 8: The assignment of the sign. The string with the X-marker should be part the lower and the other of the higher numbered v-circle. The corresponding numbering (red or grey) is compared to a predetermined numbering (black).

glued together with an orientation preserving and an orientation reserving homeomorphism. That is the reason why this faces is anticommutative in our category, because the two lower cobordisms are the two times punctured projective plane immersed into $\mathbb{R}^2 \times [-1,1]$.

There are three main points about the construction. We have to show that the complex is a well-defined chain complex, i.e. that all faces anticommute, we have to show that the complex is independent of the different choices one could make, i.e. if we change the orientation for the resolutions or the numbering of crossings or the numbering of components, then the two (well-defined) chain complexes $J_1$ and $J_2$ should be isomorphic, and we have to show that the complex is invariant under the generalised RM moves (up to chain homotopies and the Bar-Natan relations).

That all faces anticommute is a non-trivial point of the whole construction. To see this we have to proof that the anticommutativity of the faces is invariant under different choices and vRM1, vRM2, vRM3 and mRM moves and virtualisations. This is much easier in the construction of V. Turaev and P. Turner in [23]. That is why there construction does not lead to an extension of the classical Khovanov complex to v-links.

To see that this is a well-defined chain complex we use a trick of V. Manturov, i.e. we reduce the question of anticommutativity to so-called basic faces (see Figures 25, 26 and [19]). We call a face basic if it is of the form from Figures 25, 26 (or a mirror image of one of these faces).

A main trick is that the orientable faces of type 1a and 1b can have an even or an odd number of extra signs. The faces of type 2a and 2b and all disjoint faces always
have an odd number of extra signs. This works because the two comultiplications in the faces of type 1a and 1b will have different signs iff the number of signs is even (so we have an odd number of extra signs after considering the signs from the saddles). See Figure 9.

Figure 9: A face of type 1a can have an even or an odd number of extra signs. But the comultiplications have also different signs iff the number of extra signs is even.

With this trick we are able to prove that all faces are anticommutative (see Theorem 2.8). A main point for this is the Lemma 2.5, i.e. we have to ensure that our construction sends anticommutative faces to anticommutative faces under the vRM1, vRM2, vRM3 and vRM moves and virtualisations (see Figure 23).

We proof that the construction is invariant under all possible choices in Lemma 2.4. The same lemma ensures that the anticommutativity of the faces is invariant under the different choices.

Because all three relations do not contain any boundary components, they are equal (in our category) to the local relations of Bar-Natan in his paper [2]. We
call \( \text{uCob}^2_{R}(\mathcal{O}) \), the category from above modulo the Bar-Natan relations. With this observation we are able to use the main ideas of the proof of the invariance of D. Bar-Natan from [2].

In the third section we describe a method how one gets an algebraic complex from our geometric construction. This can be done by using an unoriented TQFT (which we call uTQFT), i.e. a functor from the category \( \text{uCob}^2_{R}(\mathcal{O}) \) to the category of \( R \)-modules.

This functor should satisfy several axioms (see Definition 3.1). We follow the construction of V. Turaev and P. Turner from [23]. The main difference is that our morphisms are cobordisms together with decorated boundary components.

In the fourth section we introduce the notion of a skew-extended Frobenius algebra, i.e. a Frobenius algebra \( A \) together with extra structure, namely a \( R \)-linear map \( \Phi : A \to A \) and an element \( \theta \in A \). The map \( \Phi \) is not an involution of Frobenius algebras like in [23], but a skew-involution, i.e. \( \varepsilon \circ \Phi = -\varepsilon \) rather then \( \varepsilon \circ \Phi = \varepsilon \).

We show that the isomorphism classes of skew-extended Frobenius algebras is in 1:1-correspondence to the isomorphism classes of uTQFTs. This is done in the Theorem 4.4.

At the end of the section we classify all possible aspherical uTQFTs which can be used to define v-link homology. This is the main part of section five and is done in Lemma 4.5 and Theorem 4.15.

In the last section we do some calculation with a MATHEMATICA program, called vKh.m, that we programmed. One may look in this section for more examples.

The first example of two v-links with equal virtual Jones polynomial but different virtual Khovanov homology appears already for v-links with seven crossings. After this ‘wall’ our calculations suggest that this will appear frequently.

**Notation**

For a v-link diagram \( L_D \) we call \( \check{\times} \) the 0- and \( \check{\times} \) the 1-resolution of the crossing \( \times \). For an oriented v-link diagram \( L_D \) we call \( \check{\times} \) a positive and \( \check{\times} \) a negative crossing. The number of positive crossings is called \( n^+ \) and the number of negative crossings is called \( n^- \).

For a given v-link diagram \( L_D \) with \( n \)-numbered crossings we define a collection of closed (maybe virtual) curves \( \gamma_a \) in the following way: Let \( a \) be a word of length \( n \) in the alphabet \( \{0, 1\} \). Then \( \gamma_a \) is the collection of closed (maybe virtual) curves which arise when one makes a \( a_i \)-resolution at the \( i \)-th crossing of \( L_D \) for all \( i = 1, \ldots, n \). We call such a collection \( \gamma_a \) the \( a \)-th resolution of \( L_D \).

We can choose an orientation for the different components of \( \gamma_a \). We call such a \( \gamma_a \) an oriented resolution.

If we ignore orientations then there are \( 2^n \) different resolutions \( \gamma_a \) of \( L_D \). We say a resolution has length \( m \) if it contains exactly \( m \) 1-letters. That is \( m = \sum_{i=1}^{n} a_i \).

For two resolutions \( \gamma_a \) and \( \gamma_{a'} \) with \( a_r = 0 \) and \( a'_r = 1 \) for one fixed \( r \) and \( a_i = a'_i \) for \( i \neq r \) we define a saddle between the resolutions \( S \). This means: choose a small (no other crossing, classical and virtual, should be involved) neighbourhood \( N \) of the \( r \)-th crossing and define a cobordism between \( \gamma_a \) and \( \gamma_{a'} \) to be the identity outside of \( N \) and a saddle inside of \( N \).
We denote with $|\gamma_a|$ the number of disjoint v-circles of the (un-)orientated resolution $\gamma_a$.

$R$ will always denote a commutative and unital ring.

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1 The geometric category

In the first section we describe our geometric category which we call $u\text{Cob}^2_R(\varnothing)$. This is a category of cobordisms between v-link resolutions in the spirit of D. Bar-Natan (see [2]), but we admit that the cobordisms are non orientable as in [23].

Let us call $m$ a saddle cobordism from two circles to one (pantsup) and $\Delta$ a saddle cobordism between one circle and two circles (pantsdown). Furthermore let us call $\theta$ a two times punctured projective plane $\mathbb{RP}^2$. This is a punctured Möbius strip, i.e. a cobordism from one circle to one circle. See Figure 10.

The basic idea of the construction is that the normal pantsup- and pantsdown-cobordisms in the variant of [23] do not satisfy the relation $m \circ \Delta = \theta^2$. But we need this relation for the face from 1. This is the case because for v-links we need an extra information, namely the information how two cobordisms are glued together.

To deal with this problem we decorate the boundary components of a cobordism with a formal sign $+, -$. With this construction $m_i \circ \Delta_j$ is sometimes $= \theta^2$ and sometimes $\neq \theta^2$, depending on $i, j = 1, \ldots, 8$. The first case will occur iff $m_i \circ \Delta_j$ is a non orientable surface.

If one considers a cylinder as a cobordism, then there are four different ways to decorate the boundary as indicated in Figure 10. The main idea of this construction is the usage of a cobordism $F_\pm: \mathcal{O} \to \mathcal{O}$ between two circles. This cobordism can not be embedded in $\mathbb{R}^2 \times [-1, 1]$ (as an cobordism) if one considers the decorations of the boundary as the orientation of the circles at the boundary. In this picture it is well-known as a part of the most common immersion of the Klein bottle.

Furthermore we need relations on the decorated cobordisms. One of these relations identifies all boundary preserving homeomorphic cobordisms if their boundary decorations are all equal or are all different (up to a sign). Furthermore some of the standard relations of the category $\text{Cob}^2_R(\varnothing)$ (see for example in the book of J. Kock [14]) should hold. We call the category with the extra signs $u\text{Cob}^2_R(\varnothing)$ and the category $u\text{Cob}^2_R(\varnothing)^*$ without the extra signs.

Therefore there will be two different cylinders (cobordisms of genus one between two circles) in the category $u\text{Cob}^2_R(\varnothing)$. The same is true for the category $u\text{Cob}^2_R(\varnothing)^*$.

At the end of this section we will proof some basic relations (Lemma 1.7) between the generators of our category. We also characterise the cobordisms of the face 1 (Proposition 1.9). At the end of this section we will be able to define the virtual Khovanov complex. This is the main part of section two.
Analogue to the paper of V. Turaev and P. Turner we make the following definition but beware that we consider v-circles as objects and cobordisms together with decorations. We call the decorations $+,-$ of the boundary component $k$ the \textit{gluing number} of the boundary component $k$.

\begin{definition}
(The category of cobordisms with boundary decorations). We describe the category $u\text{Cob}^2_R(\emptyset)$ in six steps. We describe the objects first. Then we define the morphisms as cobordisms generated by seven generators and we describe the morphisms and their boundary decorations. After this we introduce a short hand notation for the morphisms and we state the relations.

Our category should be $R$-pre-additive. The symbol $\sqcup$ should denote the disjoint union (the coproduct in our category).

The objects:

The objects $\operatorname{Ob}(u\text{Cob}^2_R(\emptyset))$ are disjoint unions of numbered $v$-circles (circles with virtual crossings but without classical crossings). We denote the objects as $O = \sqcup_{i \in I} O_i$. Here $O_i$ are the v-circles and $I$ is a finite, ordered index set.

We call a v-circle without virtual crossings a $c$-circle. The objects of the category should be unique up to \textit{planar isotopies} on four valent graphs.

The generators:

The generators of $\operatorname{Mor}(u\text{Cob}^2_R(\emptyset))$ are the eight cobordisms from Figure 10. The cobordisms pictured are all between $c$-circles. Every orientable generator has a decoration from the set $\{+,-\}$ at every boundary component.

![Figure 10: The generators of the set of morphisms. The cobordism on the right is the Möbius cobordism, i.e. a two times punctured projective plane.](image)

We consider these cobordisms up to boundary preserving homeomorphisms (as abstract surfaces). Hence between circles with v-crossings the generators are the same up to boundary preserving homeomorphisms, but immersed into $\mathbb{R}^2 \times [-1,1]$.

The eight cobordisms are (from left to right): a \textit{cap-cobordism} and a \textit{cup-cobordism} between the emptyset and one circle and vice versa. Both are homeomorphic to a disc $D^2$ and both should have a positive gluing number. We denote them as $\iota^+$ and $\varepsilon^+$.

Two \textit{cylinders} from one circle to one circle. The first should have two positive gluing numbers and we denote this cobordism as $\operatorname{id}_+^+$. The second should have a negative upper gluing number and a positive lower gluing number and we denote this as $\Phi^-_+^+$.

A \textit{multiplication-} and a \textit{comultiplication-cobordism} with only positive gluing numbers. Both are homeomorphic to a three times punctured $D^2$. We denote them as $m_+^+$ and $\Delta_+^+$.

A \textit{permutation-cobordism} between two upper and two lower boundary circles with only positive gluing numbers. We denote this as $\tau_+^+$. 

13
A two times punctured projective plane, also called Möbius cobordism. This cobordism is not orientable, hence it has no gluing numbers. We denote this as $\theta$.

The morphisms:
The morphisms $\text{Mor}(\mathcal{uCob}^2_R(\emptyset))$ are cobordisms between the objects in the following way. First we identify the collection of numbered v-circles with circles immersed into $\mathbb{R}^2$.

Given two objects $O_1, O_2$ with $k_1, k_2$ numbered v-circles a morphism $C: O_1 \to O_2$ is a surface immersed in $\mathbb{R}^2 \times [-1, 1]$ whose boundary lies only in $\mathbb{R}^2 \times \{-1, 1\}$ and is the disjoint union of the $k_1$ numbered v-circles from $O_1$ in $\mathbb{R}^2 \times \{1\}$ and the disjoint union of the $k_2$ numbered v-circles from $O_2$ in $\mathbb{R}^2 \times \{-1\}$.

The morphisms should be generated (as abstract surfaces) by the generators from above (see Figure 10).

The decorations:
Every morphism $C: O_1 \to O_2$ in $\text{Mor}(\mathcal{uCob}^2_R(\emptyset))$ is a cobordism between the numbered v-circles of $O_1$ and $O_2$. Let us say that the v-circles of $O_1$ are numbered from $1, \ldots, k$ and the v-circles from $O_2$ are numbered from $k+1, \ldots, l$.

Every orientable cobordism should have a decoration on the $i$-th boundary circle. This decoration is an element of the set $\{+, -\}$. We call this decoration of the $i$-th boundary component the $i$-th gluing number of the cobordism.

Hence the morphisms of the category are pairs $(C, w)$. Here $C: O_1 \to O_2$ is a cobordism from $O_1$ to $O_2$ immersed in $\mathbb{R}^2 \times [-1, 1]$ and $w$ is a string of length $l$ in such a way that the $i$-th letter of $w$ is the $i$-th gluing number of the cobordism or $w = 0$ if the cobordism is non orientable.

Short hand notation:
We denote a morphism $C$ which is a connected surfaces as $C^u_l$. Here $u, l$ are words in the alphabet $\{+,-\}$ in such a way that the $i$-th character of $u$ (of $l$) is the gluing number of the $i$-th circle of the upper (of the lower) boundary.

The construction above ensures that this notation is always possible. Therefore we denote an arbitrary orientable morphism $(C, w)$ as

$$C = C^u_{i_1} \sqcup \cdots \sqcup C^u_{i_k}. $$

Here $C^u_{i_j}$ are its connected components and $u_i, l_i$ are words in $\{+,-\}$.

For a non orientable morphism we do not need any boundary decorations.

The relations:
There are different relations on the cobordisms, namely topological relations and three combinatorial relations. The latter relations are describe by the gluing numbers of the cobordisms and the gluing of the cobordisms.

The relations on the morphisms should be the relations pictured below, i.e. the three combinatorial $(1)$ and $(2)$ for the orientable and $(3)$ for non orientable cobordisms, commutativity and cocommutativity relations, associativity and coassociativity relations, a Frobenius relation, unit and counit relations, permutation relations, the torus and Möbius relations and different commutation relations. Latter ones are not pictured, but all of them should hold with a plus sign. If the reader is unfamiliar with these relations we refer to the book of J. Kock [14]. It should be clear how to translate the commutation relations from this book to our concept (adding gluing numbers and some extra relations for the $\theta$ cobordism).
Figure 11: (1) The first combinatorical relations.

Figure 12: (2) The second combinatorical relations.

Figure 13: (3) The third combinatorical relations.

Figure 14: (4) Commutativity and co-commutativity relations.

Figure 15: (5) The associativity relations.

Figure 16: (6) The Frobenius relation.

Figure 17: (7) Unit relations.

Figure 18: (8) Counit relations.

Figure 19: (9) The first permutation relation.

Figure 20: (10) The second permutation relations.
In the following pictures the gluing numbers are shown at the right side of the boundary components. An \( u \) or a \( l \) means an arbitrary gluing number and \(-u, -l\) should be the gluing numbers \( u \) or \( l \) multiplied by \(-1\). Furthermore the bolt should represent a non orientable surfaces and not pictured parts should be arbitrary.

From this relations it is clear that the cobordism \( u \times l \id^* \) is the identity morphism between \(|I|\) v-circles. The cobordism \( \Phi^- \) changes the boundary decoration of a morphism after composition. Hence the category above contains all possibilities for the decorations of the boundary components.

The category \( \mathbf{uCob}^2_R(\emptyset)^* \) should be the same as above, but without all minus signs in the relations (we mean honest minus signs, i.e. the minus-decorations should still be in use).

Both categories are semi-strict monoidal categories. The monoidal structure should be induced by the disjoint union \( u \). Moreover both categories are symmetric. Hence by Mac Lane’s coherence theorem (see [22]) we can assume that the categories are strict, symmetric monoidal categories.

The rest of the section can also be done with the category \( \mathbf{uCob}^2_R(\emptyset)^* \) by dropping all the corresponding minus signs.

As in [2] we define the category \( \mathbf{Mat}(\mathcal{C}) \) as the category of formal matrices over a pre-additive category \( \mathcal{C} \), i.e. the objects \( \text{Ob}(\mathbf{Mat}(\mathcal{C})) \) are ordered, formal direct sums of the objects \( \text{Ob}(\mathcal{C}) \) and the morphisms \( \text{Mor}(\mathbf{Mat}(\mathcal{C})) \) are matrices of morphisms \( \text{Mor}(\mathcal{C}) \). The composition is defined through the standard matrix multiplication. This category is also called the additive closure of the pre-additive category \( \mathcal{C} \).

Furthermore we define the category \( \mathbf{Kom}(\mathcal{C}) \) as the category of formal chain complexes over a pre-additive category \( \mathcal{C} \), i.e. the objects \( \text{Ob}(\mathbf{Kom}(\mathcal{C})) \) are formal chain complexes whose chain groups are objects from \( \text{Ob}(\mathcal{C}) \) and whose differentials \( d_i \) are morphisms from \( \text{Mor}(\mathcal{C}) \). The morphisms \( \text{Mor}(\mathbf{Mat}(\mathcal{C})) \) are chain maps between two formal chain complexes. The category \( \mathcal{C} \) is pre-additive. Therefore the notion \( d_i \circ d_i = 0 \) makes sense. The category \( \mathbf{Kom}(\mathcal{C})_{/h} \) is the category modulo
formal chain homotopy.

Furthermore we define $u\text{Cob}^2_R(\emptyset)_l$, which has the same objects as the category $u\text{Cob}^2_R(\emptyset)$, but modulo the local relations from Figure 6 or from 2. So we get the following definition:

**Definition 1.2.** We call $u\text{Kob}_R(\emptyset)$ the category $\text{Kom}(\text{Mat}(u\text{Cob}^2_R(\emptyset)))$. Here our objects are formal chain complexes of formal direct sums of the category of (possible non orientable) cobordisms with boundary decorations. Then we define $u\text{Kob}_R(\emptyset)_h$ as the category $u\text{Kob}_R(\emptyset)$ modulo formal chain homotopy.

Furthermore we define $u\text{Kob}_R(\emptyset)_l$ and $u\text{Kob}_R(\emptyset)_hl$ in the obvious sense. The notations $u\text{Cob}^2_R(\emptyset)_h(l)$ mean that we consider all possible cases, namely with or without a $h$ and with or without a $l$.

One effective way of calculation in $u\text{Cob}^2_R(\emptyset)$ is the usage of the *Euler characteristic*. We define the Euler characteristic in the usual way:

**Definition 1.3** (Euler characteristic). Let $C: \mathcal{O}_1 \to \mathcal{O}_2$ be a morphism in the category $u\text{Cob}^2_R(\emptyset)$. The Euler characteristic $\chi$ of $C$ is defined as $\chi(C) = V - E + F$. Here $V$ is the number of 0-cells, $E$ is the number of 1-cells and $F$ is the number of 2-cells of $C$ in an arbitrary CW-decomposition of the cobordism.

It is well-known that the Euler characteristic is invariant under homeomorphisms (this makes the definition well-defined) and that it satisfies

$$\chi(C_1 \circ C_2) = \chi(C_1) + \chi(C_2) - \chi(O_2)$$

for any two cobordism $C_1: \mathcal{O}_1 \to \mathcal{O}_2$ and $C_2: \mathcal{O}_2 \to \mathcal{O}_3$. Also it satisfies

$$\chi(C_1 \cup C_2) = \chi(C_1) + \chi(C_2).$$

Because the objects from $u\text{Cob}^2_R(\emptyset)$ are disjoint unions of v-circles, we get the following corollaries immediately from the Definition 1.1 and 1.3.

**Corollary 1.4.** The Euler characteristic satisfies $\chi(C_1 \circ C_2) = \chi(C_1) + \chi(C_2)$ for all morphisms $C_1, C_2$ from $u\text{Cob}^2_R(\emptyset)$.

**Corollary 1.5.** The generators from $u\text{Cob}^2_R(\emptyset)$ satisfy $\chi(\text{id}_+) = \chi(\text{id}_-) = 0$ and $\chi(\Phi^+_m) = \chi(\Phi^-_m) = 0$, and $\chi(\Delta^+_{m^+}) = \chi(\Delta^-_{m^+}) = \chi(\theta) = -1$. The composition of a cobordism $C$ with $\text{id}_+^m$ or $\Phi^+_m$ does not change $\chi(C)$.

Recall that a saddle between v-circles is a cobordism which looks like a saddle for a certain neighbourhood and is the identity outside of this neighbourhood.

**Corollary 1.6.** All saddles are homeomorphic to the following three cobordism. Hence after decorate the boundary components we get nine different possibilities.

(a) A two times punctured projective plane $\theta = \mathbb{R}P^2_2$, iff the saddle is a cobordism between two circles;

(b) a pantsup-morphism $m$, iff the saddle is a cobordism from two circles to one circle;
(c) a pantsdown-morphism $\Delta$, iff the saddle is a cobordism from one circle to two circles.

**Proof.** One can use the Euler characteristic here. A saddle $S$ has $\chi(S) = -1$. Hence we get the statement. \hfill \Box

At the end of this section we deduce some basic properties of the relations between the basic cobordisms. After this little lemma we proof a proposition which is a key point for the understanding of the problematic face from [1]. We mention the difference between the relations (b),(c) and (d),(e).

**Lemma 1.7.** Let $R$ a commutative, unital ring. Then the generators satisfy the following rules:

(a) $\Phi_+ \circ \Phi_- = \id_+ \circ \id_+ = \tau_+^+ \circ \tau_-^+ = \id_+;

(b) (\Phi_+ \cup \Phi_-) \circ \Delta_{++} = \Delta_{+-} = -\Delta_{-+} = -\Delta_{++} \circ \Phi_+;

(c) (\Phi_+ \cup \id_+) \circ \Delta_{++} = \Delta_{-+} = -(\id_+ \cup \Phi_-) \circ \Delta_{++} \circ \Phi_-;

(d) m_+^+ \circ (\Phi_+ \cup \Phi_-) = m_-^+ = m_+^+ = m_+^+ \circ \Phi_+;

(e) m_+^+ \circ (\Phi_+ \cup \id_+) = m_-^+ = m_-^+ = \Phi_+ \circ m_+^+ \circ (\id_+ \cup \Phi_-);

(f) m_+^+ \circ \Delta_{++} = (\id_+ \cup \Delta_{++}) \circ (m_+^+ \cup \id_+)$ (Frobenius relation);

(g) $m_+^+ \circ (m_+^+ \cup \id_+) = m_-^+ \circ (\id_+ \cup m_+^+)$ (associativity relation);

(h) $(\Delta_{++} \cup \id_+) \circ \Delta_{++} = (\id_+ \cup \Delta_{++}) \circ \Delta_{++}$ (associativity relation);

(i) $m_+^+ \circ \tau_+^+ \circ (\Phi_+ \cup \id_+) = m_-^+$ (first permutation-phi relation);

(j) $(\Phi_+ \cup \id_+) \circ \tau_+^+ \circ \Delta_{++} = \Delta_{-+}$ (second permutation-phi relation);

(k) $\theta \circ \Phi_+ = \Phi_+ \circ \theta = \theta;

(l) $\mathcal{K} = \emptyset^2$. Here $\mathcal{K}$ is the two times punctured Klein bottle;

**Proof.** Most of the equations follows directly from the relations above. The rest are easy to check and therefore omitted. \hfill \Box

**Example 1.8.** The two cylinders $\id_+^+, \Phi_-^-$ are the only isomorphisms between two equal objects. Let us denote $\mathcal{O}_1$ and $\mathcal{O}_2$ two objects which differs only though a finite sequence of the virtual Reidemeister moves.

Then the vRM-cobordisms from Figure [7] induces isomorphisms $\mathcal{C} : \mathcal{O}_1 \rightarrow \mathcal{O}_2$. To see this we mention that the three cobordisms are isomorphisms themselves. There inverses are the cobordisms which we obtain, if we go from bottom to top rather then from top to bottom.

It is easy to show that these cobordisms are inverses (use statement (a) from Lemma 1.7).

**Proposition 1.9** (Non orientable faces). Let $\Delta_{12}^{u_1 u_2}$ and $m_{u_1^+}^{u_2^+}$ be the surfaces from Figure [10]. Then the following is equivalent:

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18
(a) \( m_{u_1'} u_2' \circ \Delta_{l_1 l_2}^u = \mathcal{K}; \)

(b) \( l_1 = u_1' \) and \( l_2 = -u_2' \) or \( l_1 = -u_1' \) and \( l_2 = u_2' \). Otherwise \( m_{u_1'} u_2' \circ \Delta_{l_1 l_2}^u \) is a two times punctured torus \( T \).

This relation is also called the Möbius relation.

Proof. Let us call the composition \( C = m_{u_1'} u_2' \circ \Delta_{l_1 l_2}^u \). A quick computation shows that \( \chi(C) = -2 \). Because \( C \) has two boundary components, \( C \) is either a 2-times punctured torus or a 2-times punctured Klein bottle.

Then the statement follows from the torus and Möbius relations in Figure 22.

2 The geometric complex

In the present section we define the geometric complex which we call the virtual Khovanov complex \([L_D]\) of an oriented v-link diagram \( L_D \). All v-link diagrams should be oriented from now on. We do not mention this any more.

This complex is an element of our category \( \mathbf{uKob}_R(\mathcal{O}) \). Here \( R \) is a commutative, unital ring again.

We will define the virtual Khovanov complex analogue to the definition of the classical complex from M. Khovanov (see [11] or D. Bar-Natans exposition in [1]). The main difference is that our complex has decorated saddle cobordisms between the resolution and we do not need to spread extra signs in a different way, i.e. based on the saddles itself and not based on its position in the complex. This saddle sign is defined in 2.1.

Moreover we define the saddle decorations in Definition 2.2. The main part of this definition is the Table 2. For this we need to choose an orientation for every resolution of the diagram.

Thus we have to ensure that our complex is a well-defined chain complex which is independent of the different choices, i.e. we have to ensure that all faces anticommute and that different choices lead to isomorphic complexes.

This is done in the Lemma 2.4 and the Theorem 2.8.

Because we define the complex purely geometrical, we could use D. Bar-Natans arguments (see [2], [23]) to show that the complex is an invariant of v-links modulo chain homotopy and the Bar-Natan relations. This is the third point and is be done in Theorem 2.14.

The whole construction in this section works in the category \( \mathbf{uKob}_R(\mathcal{O})^* \) too. There are only the following differences:

- we do not need the saddle signs (see Definition 2.1);
- we need extra signs for the complex like in the classical case, because faces commute without them. Hence they anticommute after adding these extra signs;
- faces in the complex will be invariant under virtualisations (see Definition 2.1) without any further work.
Definition 2.1 (Virtualisations and the sign of a saddle). We call the substitution of Figure 23 a virtualisation of a classical crossing. We call a virtual diagram $L'_D$ which is obtained from a v-link diagram $L_D$ through a finite sequence of virtualisations a virtualisation of $L_D$.

![Figure 23: The virtualisation of a classical crossing.](image)

By the Corollary 1.6 we know that every saddle cobordism $S$ is homeomorphic to $\theta$, $m$ or $\Delta$. We need an extra information for the last two cobordisms. We call this extra information the sign of the saddle.

Recall that the v-circles in the resolutions of the v-link diagram $L_D$ are numbered. This numbering is fixed and we have to choose it before calculating the saddle signs. The numbers should all be numbers from the set $\{1, \ldots, k\}$. Here $k$ is the number of the v-circles in the corresponding resolution. We call this the fixed numbering.

Now every crossing can be rotated until it look like $\times$. There are two different of these rotations for every crossing. Fix one of these two for every crossing and thus for every saddle which belongs to the crossing.

After fixing this position every saddle can be viewed as a formal symbol $\rightarrow \times$ together with a X-marker as pictured in Figure 8.

The X-marker induces a numbering of the v-circles in the following way:

- There are two fixed numberings for every saddle, i.e. one for the top and one for the bottom resolution;
- the bottom numbering should induce a numbering on the top resolution if the saddle is a multiplication;
- the top numbering should induce a numbering on the bottom resolution if the saddle is a comultiplication;
- in both cases there could be v-circles in both resolutions which are unaffected by the saddle. These v-circles have a number in the first resolution and a (maybe different) number in the second;
- the unaffected v-circles should carry the same number in the induced numbering compared to the numbering of the bottom (multiplication) or the top (comultiplication);
• because an orientable saddle always merges or splits two v-circles, there are exactly two v-circles and two numbers in the set \{1, \ldots, k\} left;

• the v-circle with the X-marker should carry the lower of the two numbers and the v-circle without the marker should carry the bigger number.

We call this numbering *induced (by the saddle)*. In Figure 8 the fixed numbering is the black and the induced numbering is the red (grey) numbering. Then the *saddle sign* should be positive if the permutation between the two numberings, i.e. between the fixed numbering and the induced numbering, is even and negative if it is odd.

The following argument ensures that the number of signs in a face does not change modulo two under rotation: the two possible rotations of a crossing (in such a way that the crossing looks like \(\times\)) differs by a rotation \(\pi\). Hence a saddle with a positive sign changes to its sign to negative and vice versa. Thus if two saddles which belongs to the same crossing have the same signs before the rotation, then they have the same after the rotation. The same holds if the two saddles have different signs.

With this we are able to define a decoration for every boundary component of the saddles which we will call the *saddle decorations*.

**Definition 2.2** (The saddle decorations). Let \(L_D\) be a v-link diagram. Moreover let \(S: \gamma_a \to \gamma_b\) be a saddle between the orientated resolutions \(\gamma_a, \gamma_b\). Then we have the cases (a),(b),(c) from Corollary 1.6. Hence there are three different cases:

(a) in this case the saddles is the cobordism \(\theta\) from Figure 10. The gluing numbers are not important because \(\theta\) is non orientable;

(b,c) in this case the saddles are the cobordisms \(m_{l_1 l_2}^{u_1 u_2}\) (for the case (b)) or \(\Delta_{l_1 l_2}^{u_1}\) (for the case (c)) from Figure 10.

We need to define the gluing numbers for the cases (b) and (c). We define this decorations depending on the orientations of the resolutions \(\gamma_a, \gamma_b\) and the sign of the saddle. So we choose an orientation for every resolution.

The resolutions \(\gamma_a, \gamma_b\) are disjoint v-circles. This means that \(\gamma_a, \gamma_b\) are diagrams of an orientated v-link diagram with no c-crossings.

Every saddle can be viewed as an formal symbol \(\gamma_a \to \gamma_b\) together with a formal local orientation of these strings which is equal to the orientations of \(\gamma_a, \gamma_b\). Then the decorations on \(m_{l_1 l_2}^{u_1 u_2}\) or \(\Delta_{l_1 l_2}^{u_1}\) are given by Table 2. Here the numbering of the circles is important. The circle with the lower number should be the left (for \(m\)) or the lower (for \(\Delta\)).

If the saddle is not of this form we must rotate it to match the form in the table. For example a \(\Delta\)-saddle of the form \(\gamma_a \to \gamma_b\) with a lower number for the right string would be \(\gamma_a \to \gamma_b\), i.e. the cobordism \(\Delta_{-}\).

Furthermore the saddle should carry a formal extra sign if the saddle sign from Definition 2.1 is negative.

We need decorations for the cylinders of \(S\) too. This cylinders are either \(\text{id}^+_x\), if the corresponding boundary circles have the same orientations in \(\gamma_a, \gamma_b\), or \(\Phi_x\), if the corresponding boundary circles have different orientations in \(\gamma_a, \gamma_b\).
So these decorations on the boundary of $S: \gamma_a \to \gamma_b$ are only depending on the orientations of the v-circles from $\gamma_a$ and $\gamma_b$. Hence between two oriented resolutions there exists exactly one of this decorations. This is the one we call the saddle decoration of $S$. This decoration depends on the chosen orientations of the v-circles in the resolutions.

A saddle $S: \gamma_a \to \gamma_b$ together with decorations is a morphism in the category $\mathbf{uCob}^2_{R(\mathcal{O})}$ between the objects $\gamma_a, \gamma_b$.

At this point we are finally able to define the virtual Khovanov complex. We call this complex geometric.

**Definition 2.3** (Geometric complex). For a v-link diagram $L_D$ with $n$ ordered crossings we define the geometric complex $\mathbf{J}_{L_D}$ as follows:

- for $i = 0, \ldots, n$ the $i-n$ chain module is the formal direct sum of all possible oriented $\gamma_a$ of length $i$;
- there are only morphisms between the chain modules of length $i$ and $i + 1$;
- if two words $a, a'$ differ only in exactly one letter and $a_r = 0$ and $a'_r = 1$ then there is a morphism between $\gamma_a$ and $\gamma_a'$. Otherwise all morphisms between components of length $i$ and $i + 1$ are zero;
- this morphism is a saddle between $\gamma_a$ and $\gamma_a'$;
- we consider oriented resolutions (we choose them) and the saddles should carry the saddle decorations from Definition 2.2.

Given words $a_1, a_2, a_3, a_4$ together with four saddles between $\gamma_{a_1}$ and $\gamma_{a_2}$, $\gamma_{a_3}$ and between $\gamma_{a_2}$ and $\gamma_{a_4}$ and between $\gamma_{a_3}$ and $\gamma_{a_4}$. Then we call the diagram

$\begin{tikzpicture}
\node (a) at (0,0) {$\gamma_{a_1}$};
\node (b) at (1,1) {$\gamma_{a_2}$};
\node (c) at (2,0) {$\gamma_{a_3}$};
\node (d) at (1,-1) {$\gamma_{a_4}$};
\draw[->] (a) to (b);
\draw[->] (b) to (c);
\draw[->] (c) to (d);
\draw[->] (d) to (a);
\end{tikzpicture}$

a (un-)orientated face $F$ of the complex $[L_D]$. We call $\gamma_{a_1}$ the first, $\gamma_{a_2}$ and $\gamma_{a_3}$ the second and $\gamma_{a_4}$ the third part of the face.

At this point it is not clear why we can choose the numbering of the crossings, the numbering of the v-circles and the orientation of the resolutions. Furthermore it is not clear why this complex is a well-defined chain complex.

But we show in Lemma 2.4 that the complex is independent of these choices, i.e. if $[L_D]_1$ and $[L_D]_2$ are well-defined chain complexes with different choices, then they are equal up to chain isomorphisms.

Moreover we show in Theorem 2.8 and Corollary 2.9 that the complex is indeed a well-defined chain complex.

For an example see Figure 1. This figure shows the virtual Khovanov complex of a v-diagram of the unknot.
The main problem now is to show that the faces of the complex are anticommutative and that the complex is independent of the different choices (up to chain isomorphisms). After this is done we can show that the complex is an v-link invariant (up to chain homotopy and the Bar-Natan relations).

**Lemma 2.4.** Let $L_D$ be a v-link diagram and let $[L_D]_1$ be its geometric complex from Definition 2.3 with arbitrary orientations for the resolutions. Let $[L_D]_2$ be the complex with the same orientations for the resolutions except for one circle $c$ in one resolution $\gamma_a$. If a face $F_1$ from $[L_D]_1$ is anticommutative, then the corresponding face $F_2$ from $[L_D]_2$ is also anticommutative.

Moreover if $[L_D]_1$ is a well-defined chain complex, then it is isomorphic to $[L_D]_2$, which is also a well-defined chain complex.

The same statement is true if the difference between the two complexes is the numbering of the crossings, the choice of the rotation for the calculation of the saddle signs, rotations/isotopies of the v-link diagram or the fixed numbering of the v-circles.

**Proof.** Assume that the face $F_1$ is anticommutative. Then the different orientation of the circle $c$ corresponds to a composition of all morphisms of the face $F_2$ with this circle as a boundary component with $\Phi^{-} \Phi^{+}$. Hence the face $F_2$ is also anticommutative, because both outgoing (or incoming) morphism of $F_2$ are composed with an extra $\Phi^{-}$ if the circle is in the first (or last) resolution of the faces. If it is in one of the second resolutions, then we have to use the relation $\Phi^{-} \circ \Phi^{-} = \text{id}^{-}$ from Lemma 1.7.

Thus if the first complex is a well-defined chain complex, then the same is true for the second. The isomorphism is induced by the isomorphism $\Phi^{-}$. The second statement is true because the numbering of the crossings does not affect the cobordisms at all. Hence the argument can be shown analogous to the classical case (see for example [11]).

On the third point: That anticommutative faces stay anticommutative if one changes between the two possible choices is discussed in Definition 2.1. The chain isomorphism is induced by a sign permutation.

The penultimate statement follows directly by the definition of the saddle sign and decorations.

To see the latter statement we observe that a change of the numbering influences the saddle signs and the decorations. But the decorations can be adjusted in such a way that they do not change by a different choice of orientations (without any changes for the anticommutativity because of the first statement). Thus we only need to consider the saddle signs.

To see that the anticommutativity still holds after renumbering we assume that only two v-circles in one resolution switch their number. The statement is true if the corresponding resolution is not a part of the face $F_2$. Moreover because of the relation (3) from Figure 13 the statement is true for a face $F_2$ with unorientable saddles.

Thus we have to show that every orientated saddle effected by the renumbering changes its sign. Hence there are four cases, i.e. the renumbering can take place.
in the top or bottom resolution of the saddle and the saddle can be a pantsup- or pantsdown-morphism.

Assume that the saddle is a multiplication and the renumbering is in the top resolution. Then one needs an extra transposition for the permutation, because the induced numbering stays the same as before, but the fixed numbering changes by a transposition. If the resolution is at the bottom, then one needs an extra permutation too, because this time the induced numbering changes by a transposition, but the fixed numbering at the top stays the same. An analogue argument works for the comultiplication. See Figure 24.

Figure 24: A renumbering of the v-circles changes the sign.

Hence the number of negative saddle signs does not change modulo two. Thus the face $F_2$ remains anticommutative. The chain isomorphism is induced by a permutation with extra signs.

For the next lemma it is necessary to use the saddle signs for the saddles. Otherwise the lemma would be false.

**Lemma 2.5.** Let $L_D, L'_D$ be v-link diagrams which differs only by a virtualisation of one crossing $c$. If a face $F$ is anticommutative in $[L_D]$, then the corresponding face $F'$ is anticommutative in $[L'_D]$.

Moreover if $[L_D]$ is a well-defined chain complex, then it is isomorphic to $[L'_D]$, which is also a well-defined chain complex.

The same statement is true if $L_D$ and $L'_D$ differs only by a vRM1, vRM2, vRM3 or mRM move.

**Proof.** The statement about anticommutativity is clear if one of the saddles which belongs to the crossing $c$ is non orientable. This is true because of the relations
from Figure 13 and Proposition 1.9. Thus we can assume that both saddles are orientable.

Furthermore it is clear that the two composition of the saddles are boundary preserving homeomorphic after the virtualisation. Hence the only thing we have to ensure is that the decorations and signs work out correctly.

We use the Lemma 2.4 here, i.e. we can choose the orientations and the numberings in such a way that the saddles which do not belong to the crossing $c$ have the same local orientations and numberings.

We observe the following: the sign and the local orientations of a saddle can only changes if the saddle belongs to the crossing $c$, i.e. the local orientations always changes (see Figure 23) and the sign only changes if the two strings in the bottom picture of Figure 23 are part of two different v-circles.

A change of the local orientations multiplies an extra sign for comultiplication, but no extra sign for multiplication. This follows from the Table 2 and the relations in (1) of Figure 11.

Hence the anticommutativity still holds if the two saddles which belong to the crossing $c$ are both multiplications or comultiplications, because their decorations and signs change in the the same way.

If one is a multiplication and one is a comultiplication, then we have two cases, i.e. the multiplication gets an extra sign or not. The comultiplication always gets an extra sign because the local orientations change. But the multiplication will change its saddle sign if the comultiplication does not change its saddle sign. Hence the number of extra signs does not change modulo two. This ensures that the faces stays anticommutative (see lower part of Figure 23).

That the face $F'$ stays commutative after a vRM1, vRM2, vRM3 or mRM move follows because neither the local orientations nor the signs of any cobordism change. Thus all decorations and signs are the same.

The chain isomorphisms are induced by the vRM-cobordisms from Figure 7, morphisms of type $\Phi^+$ and identity morphisms. Recall that all this cobordisms are isomorphisms in our category.

For the proof of the next lemma we refer the reader to the paper [19] of V. Manturov.

Lemma 2.6. Let $L_D$ be a v-link diagram. Then $L_D$ can be reduced by a finite sequence of isotopies, vRM1, vRM2, vRM3, mRM moves and virtualisations to a v-link diagram $L'_D$ in such a way that a fixed connected face of $L'_D$ is isotopic to one of the basic faces from the Figures 25, 26 (or to one of their mirror images) up to vRM1, vRM2, vRM3 moves on the face itself.

These lemmata allow us to check an arbitrary orientations on the basic faces with arbitrary numbering of crossings/components. That we only need to check these basic faces is a trick of V. Manturov (see [19]).

Proposition 2.7. Let $L_D$ be a v-link diagram with a diagram which is isotopic to one of the projections from Figure 25 or 26. Then $[L_D]$ is a chain complex, i.e. the basic faces are anticommutative.

Moreover disjoint faces are always anticommutative.
Figure 25: The four orientable basic faces.

Figure 26: The six non orientable basic faces.
Proof. Because of Lemma 2.4, we only need to check that the faces are anticommutative for orientations of the resolutions of our choice with an arbitrary numbering. Then we are left with three different cases, i.e. the v-link diagram of $L_D$ is orientable, i.e. all saddles are orientable, or the face is non orientable, i.e. two or four of the saddles are non orientable or the face is disjoint.

For the first case we see that every resolution contains only c-circles. We proof the anticommutativity of the corresponding face for the following orientations of the resolutions. All appearing circles should be numbered in ascending order from left to right or outside to inside.

Because every resolution contains only c-circles, we choose a negative orientation for the circles except for the two nested circles that appear in two resolution of a face of type 1a or 1b. This is a counterclockwise orientation for all the not nested circles and a clockwise orientation for the two nested circles. Hence all appearing cylinders are id-morphisms.

It follows from this convention that every 0-resolution (or 1-resolution) $\times$ of a crossing $\times$ (or a crossing $\times$) is of the form $\l$ and every 1-resolution (or 0-resolution) $\times$ of a crossing $\times$ (or a crossing $\times$) is of the form $\rightarrow$.

Moreover the only face with an even number of saddle signs is of type 1a. All we need to do is compare these local orientations with the ones from Table 2. We see that we have to check the following equations:

1. $\Delta_+^+ \circ m_-^- \frac{1}{2} = -\Delta_-^- \circ m_+^+$ (face of type 1a);
2. $m_+^+ \circ \Delta_+^+ = m_+^+ \circ \Delta_+^+$ (face of type 1b);
3. $(\Delta_+^+ \cup \id_+^+) \circ (\id_+^+ \cup m_+^+) = m_+^+ \circ \Delta_+^+$ (face of type 2a);
4. $m_+^+ \circ (m_+^+ \cup \id_+^+) = m_+^+ \circ (\id_+^+ \cup m_+^+)$ (face of type 2b).

Most of these equations are easy to calculate. The reader should check that the cobordisms on the left and the right side of every equation are homeomorphic (using Proposition 1.9 and Lemma 1.7).

Furthermore the second equation is clear and the other three follows easy using the result of Lemma 1.7. Hence they are all anticommutative because only the first face has an even number of saddle signs.

The non orientable faces of type 1b, 2a, 2b, 3a and 3b are easy to check. One can use the Euler characteristic here and the relation (3) of Figure 13.

The non orientable face of type 1a is the face from 1. Here we have to use Proposition 1.9. We get two $\theta$ cobordisms and a $\Delta$- and a $m$-cobordism. Because of the relation (3) of Figure 13 we can ignore the saddle signs.

Again we can choose an orientation for the resolutions. We can do this for example in the following way (like in Figure 1):

- the first Möbius strips should be $\theta_2 \rightarrow \rightarrow$ and $\theta_2 \rightarrow$;
- the pantsdown should be $\Delta_-^- \rightarrow \rightarrow$ and the pantsup should be $m_-^- \rightarrow \rightarrow$.

We use Proposition 1.9 to see that this face is anticommutative.
The reader should check that all disjoint faces with only orientable saddles have an odd number of saddle signs. The disjoint faces with two or four non orientable saddles anticommute because of the relation (3) of Figure 13.

This proposition leads us to an important theorem and an easy corollary.

**Theorem 2.8 (Faces commute).** Let \( L_D \) be a \( v \)-link diagram. Let \[ [L_D] \] be the geometric complex from Definition 2.3 with arbitrary possible choices. Then every face of the complex \([L_D]\) is anticommutative.

**Proof.** This is a direct consequence of the Proposition 2.7 and the three Lemmata 2.4, 2.5 and 2.6.

**Corollary 2.9.** The complex \([L_D]\) is a chain complex. Thus it is an object in the category \( uKob_\mathcal{R}(\emptyset) \).

There is a way to represent the geometric complex of a \( v \)-link diagram \( L_D \) as a cone of two \( v \)-links diagrams \( L_D^0, L_D^1 \). Here one crossing of \( L_D \) is resolved 0 in \( L_D^0 \) and 1 in \( L_D^1 \). The cone construction works in the case of pre-additive categories.

We note a lemma which follows directly from the definition of the geometric complex. There is a saddle between every resolution which are resolved equal at the other crossings of \( L_D \). There is a way to represent the geometric complex of a \( v \)-link diagram as a cone of two \( v \)-links diagrams \( L_D^0, L_D^1 \). Here one crossing of \( L_D \) is resolved 0 in \( L_D^0 \) and 1 in \( L_D^1 \). The cone construction works in the case of pre-additive categories.

A cone of two chain complexes is defined below. We also note the notion of a double cone (a cone of two cones). The whole construction can be done repetitive for the \( n \)-th cone. But we only need the first two, so we write them down concrete.

**Definition 2.10 (The cone).** Let \( C, D \) be two chain complexes with chain groups \( C_i, D_i \) and differentials \( c_i, d_i \). Let \( \varphi: C \to D \) be a chain map. The cone of \( C, D \) along \( \varphi \) is the chain complex \( \Gamma(\varphi:C \to D) \) with the chain groups and differentials

\[
\Gamma_i = C_i \oplus D_{i-1} \quad \text{and} \quad \gamma_i = \begin{pmatrix} -c_i & 0 \\ \varphi_i & d_{i-1} \end{pmatrix},
\]

i.e. if the two chain complexes \( C, D \) looks like

\[
C, D: \cdots \xrightarrow{c_{i-1},d_{i-1}} C_i, D_i \xrightarrow{c_i,d_i} C_{i+1}, D_{i+1} \xrightarrow{c_{i+1},d_{i+1}} C_{i+2}, D_{i+2} \xrightarrow{c_{i+2},d_{i+2}} \cdots
\]

then the cone along \( \varphi \) is generated by direct sums over the diagonal like below.

\[
\cdots \xrightarrow{c_{i-1}} C_i \xrightarrow{-c_i} C_{i+1} \xrightarrow{-c_{i+1}} C_{i+2} \xrightarrow{-c_{i+2}} \cdots
\]

\[
\cdots \xrightarrow{d_{i-1}} D_i \xrightarrow{\varphi_i} D_{i+1} \xrightarrow{\varphi_{i+1}} D_{i+2} \xrightarrow{\varphi_{i+2}} \cdots
\]

Furthermore let \( C, D, E, F \) be chain complexes with chain groups \( C_i, D_i, E_i, F_i \) and differentials \( c_i, d_i, e_i, f_i \). Let \( \varphi: C \to D, \varphi': E \to F, \psi: C \to E \) and \( \psi': D \to F \) be chain maps. The maps should satisfy \( \psi' \circ \varphi = \varphi' \circ \psi \).
The double cone (cone of two cones) of $C, D, E, F$ along $\varphi, \varphi', \psi, \psi'$ is the chain complex $\Gamma(C \to D \oplus E \to F)$ with the chain groups and differentials

$$\Gamma_i = C_i \oplus D_{i-1} \oplus E_{i-1} \oplus F_{i-2} \text{ and } \gamma_i = \begin{pmatrix} -c_i & 0 & 0 & 0 \\ \varphi_i & d_{i-1} & 0 & 0 \\ \psi & 0 & -e_{i-1} & 0 \\ 0 & -\psi' & \varphi' & f_{i-2} \end{pmatrix},$$

i.e. this is the cone of the two cones $\Gamma(\varphi: C \to D) \to \Gamma(\varphi': E \to F)$ where $\Psi$ is the map $\Psi = \left( \begin{array}{cc} \psi & 0 \\ 0 & -\psi' \end{array} \right)$.

**Lemma 2.11.** Let $\mathcal{C}$ be a category of chain complexes, i.e. the objects are chain complexes and the maps are chain maps. Let $C, D, E, F$ and $\varphi, \varphi', \psi$ and $\psi'$ be like above.

Then $C \xrightarrow{\varphi} D$ and $\Gamma(\varphi: C \to D) \to \Gamma(\varphi': E \to F)$ are chain complexes of chain complexes, hence objects in the category $\text{Kom}(\mathcal{C})$.

If $\varphi: C \to D$ and $\varphi': E \to F$ are chain homotopic in $\text{Kom}(\mathcal{C})$, then $\Gamma(\varphi: C \to D)$ and $\Gamma(\varphi': E \to F)$ are chain homotopy in $\mathcal{C}$.

An analogue statement holds for the double cone.

**Proof.** A straightforward calculation.

**Lemma 2.12.** Let $L_D$ be a v-link diagram and let $c_1, c_2$ be two crossings of $L_D$. Let $L_D^0$ be the v-link where the crossing $c$ is resolved 0 and let $L_D^1$ be the v-link where the crossing $c$ is resolved 1. Then we have

$$[L_D] = \Gamma([L_D^0] \xrightarrow{\varphi} [L_D^1]).$$

Moreover let $L_D^{00}, L_D^{10}, L_D^{01}$ and $L_D^{11}$ be the obvious diagrams. Then we have

$$[L_D] = \Gamma([L_D^{00}] \to [L_D^{01}] \oplus [L_D^{10}] \to [L_D^{11}]).$$

**Proof.** The proof is analogous to the proof for the classical Khovanov complex.

The only thing to prove is the fact that the map $\varphi$, which resolves the crossing, induces a chain map. This is true because we can take the induced orientation (from the orientations of the resolutions of $L_D^0$ and $L_D^1$) of the strings of $\varphi$. This gives us the gluing numbers for the morphisms of $\varphi$.

Here we need the Lemma 2.4 again to ensure that all faces anticommute.

The proof for the double cone follows from the statement for the cone.

**Example 2.13.** Let $L_D$ be the v-diagram of the unknot from Figure 1. Then we have

$$[L_D] = \Gamma(\varphi: [\text{unknot}] \to [\text{unknot}]) = \Gamma(\varphi: L_D^0 \to L_D^1).$$

If we choose the orientation for the resolutions for the chain complexes $L_D^0, L_D^1$ to be the ones from Figure 1, then the map $\varphi$ is of the form $\varphi = (0, m_{+})$.

As a short hand notation we only picture a certain part of a v-link diagram. The rest of the diagram can be arbitrary. Now we state the following theorem about the geometrical complex.
Theorem 2.14 (The geometric complex is an invariant). Let $L_D, L'_D$ be two v-link diagrams which differs only through a finite sequence of isotopies and generalised Reidemeister moves. Then the complexes $[L_D]$ and $[L'_D]$ are equal in $uKob(\varnothing)_{hi}$.

Proof. We have to check invariance under the generalised Reidemeister moves from Figure 3.

The usual Reidemeister moves RM1,RM2 and RM3 can be proven analogue to the original proof of D. Bar-Natan in [2]. Here we must use the Bar-Natan relations $S,T,4TU$.

The Bar-Natan relations does not contain any boundary components. Therefore we do not need extra decorations for them.

Because of this we can take the same chain maps as D. Bar-Natan in [2] (the cobordisms should be the identity outside of the pictures). The only thing we have to ensure is that our cobordisms have the adequate decorations. For this we number the v-circles in a way that the pictured v-circles have the lowest numbers and we use the orientations given below (see Lemma 2.4).

Furthermore the whole construction is in the category $uKob(\varnothing)$, i.e. the complexes are chain complexes of chain complexes and the chain maps are sequences of chain maps.

Therefore we give the gluing numbers of the cobordisms now. We follow the proof of D. Bar-Natan from [2], i.e. for the RM1 and RM2 moves one has to show that the given maps induces chain homotopies, using the rules from Definition 1.1 and Lemma 1.7 and the cone construction from Definition 2.10. We use the Lemma 2.11 to get the statement for the RM1 and RM2 move.

Then the RM3 move follows with the cone construction.

We mention that we do not care about the saddle signs at this point because they only affect the anticommutativity of the faces. Hence, after adding some extra signs, the entire arguments work analogue.

We consider oriented v-link diagrams. Thus there are a lot of cases to check. But all cases for the RM1 and RM2 moves are analogous to the cases from below, i.e. one case for the RM1 move and three cases for the RM2 move.

The case for the RM1 move is pictured in Figure 3. For the RM2 move we show that the virtual Khovanov complexes of $[\includegraphics{figure3}]$ and $[\includegraphics{figure4}]$, $[\includegraphics{figure5}]$ and $[\includegraphics{figure6}]$ are chain homotopic. Here both cases contains two different cases them self. For the left case the upper left string can be connected to the upper right or to the lower left. For the other case the upper left string can be connected to the lower right or to the upper right. But the last case is analogous to the first. So we only consider the first three cases.

For the RM1 move we only have to resolve one crossing in the left picture and no crossing in the right. We choose the orientation in such a way that the saddle is a multiplication of the form $\includegraphics{figure7}$. Thus it is the multiplication $m^- = m^+$. For the RM2 move we have to resolve two crossings in the left picture and no crossing in the right.
For the first two cases we choose the orientation in such a way that the corresponding saddles are of the form $\mathcal{J} \rightarrow \mathcal{K}$ for the left crossing and of the form $\mathcal{K} \rightarrow \mathcal{J}$ for the right crossing.

Hence we only have $\Delta_{+}^{+} = -\Delta_{-}^{+}$ and $m_{-}^{++} = m_{+}^{++}$ saddles in the possible complexes.

For the third case we choose the orientation in such a way that the corresponding saddles are of the form $\mathcal{J} \rightarrow \mathcal{K}$ or $\mathcal{K} \rightarrow \mathcal{J}$ for the left crossing and of the form $\mathcal{K} \rightarrow \mathcal{J}$ or $\mathcal{J} \rightarrow \mathcal{K}$ for the right crossing.

Hence we only have $m_{-}^{++}$, $\Delta_{-}^{+}$ and $\theta$ saddles in the possible complexes.

We give the required chain maps $F, G$ and the homotopy $h$. These cobordisms are the same as in [2] with extra decorations. One can proof that these maps are chain maps and that $F \circ G$ and $G \circ F$ are homotopic to the identity using the same arguments as D. Bar-Natan in [2] and the relations from Lemma 1.7.

We repress the notation $\Gamma(\cdot)$ in the following.

For the RM1 move we have:

$$\begin{array}{cccc}
\begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & 0 & 0 \\
\begin{array}{c}
G;
\end{array} & \begin{array}{c}
F;
\end{array} & \begin{array}{c}
0
\end{array} & \begin{array}{c}
0
\end{array} \\
\begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & \begin{array}{c}
m_{-}^{++}\end{array} & \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array}
\end{array}$$

The maps $F, G$ are the same as in [2] with the decorations from above. With this maps we can follow the proof of D. Bar-Natan. Hence we also need to give an extra chain homotopy $h$. It should be the one from below.

$h: \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} \rightarrow \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array}; h = \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K}
\end{array}$

For the RM2 move the first two cases are:

$$\begin{array}{cccc}
\begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & 0 & 0 \\
\begin{array}{c}
0
\end{array} & \begin{array}{c}
G;
\end{array} & \begin{array}{c}
\operatorname{id}_{+}^{+}
\end{array} & \begin{array}{c}
\operatorname{id}_{+}^{+}
\end{array} \\
\begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & \begin{array}{c}
d^{-1}
\end{array} & \begin{array}{c}
d^{0}
\end{array} \\
\begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & \begin{array}{c}
\mathcal{K} \rightarrow \mathcal{K}
\end{array} & \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{J}
\end{array}
\end{array}$$

Here the differentials are either $d^{-1} = (\Delta_{+}^{+} \Delta_{-}^{+})^{T}$ and $d^{0} = (m_{+}^{++} m_{+}^{++})^{T}$ in the first case or $d^{-1} = (\operatorname{id}_{+}^{+} \Delta_{+}^{+} m_{+}^{++})^{T}$ and $d^{0} = (m_{+}^{++} \operatorname{id}_{+}^{+} \Delta_{+}^{+})$ in the second case.

We can follow the proof of D. Bar-Natan again. Therefore we need to give a chain homotopy again. This chain homotopy should be

$h^{-1}: \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K} \oplus \mathcal{J} \rightarrow \mathcal{K}
\end{array} \rightarrow \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K}
\end{array}; h^{-1} = \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K}
\end{array}$

$h^{0}: \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K} \oplus \mathcal{J} \rightarrow \mathcal{K}
\end{array} \rightarrow \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K}
\end{array}; h^{0} = \begin{array}{c}
\mathcal{J} \rightarrow \mathcal{K}
\end{array}$.
For the RM2 move the last case is:

\[
\begin{array}{ccc}
\text{[}\begin{array}{c}
\includegraphics{case1.png}
\end{array}\text{]} & \xrightarrow{0} & \text{[}\begin{array}{c}
\includegraphics{case2.png}
\end{array}\text{]} \\
\text{[}\begin{array}{c}
\includegraphics{case3.png}
\end{array}\text{]} & \xrightarrow{d^{-1}} & \text{[}\begin{array}{c}
\includegraphics{case4.png}
\end{array}\text{]}
\end{array}
\]

Here the differentials are either \( d^{-1} = \left( \Delta_{\theta} \right)^T \) and \( d^0 = (m^\theta - \theta) \). Furthermore saddles of the maps \( F, G \) are also \( \theta \) saddles. Hence we do not need any decorations for them.

The chain homotopy should be

\[
\begin{align*}
&h^{-1} \colon \text{[}\begin{array}{c}
\includegraphics{case1.png}
\end{array}\text{]} \rightarrow \text{[}\begin{array}{c}
\includegraphics{case2.png}
\end{array}\text{]}; h^{-1} = \left( \begin{array}{c}
\includegraphics{homotopy1.png}
\end{array}\right) \\
&h^0 \colon \text{[}\begin{array}{c}
\includegraphics{case3.png}
\end{array}\text{]} \rightarrow \text{[}\begin{array}{c}
\includegraphics{case4.png}
\end{array}\text{]}; h^0 = \left( \begin{array}{c}
\includegraphics{homotopy2.png}
\end{array}\right).
\end{align*}
\]

In all there cases it is easy to check (following the proof of D. Bar-Natan) that the given maps \( F, G \) are chain homotopies. Furthermore \( G \) satisfies the conditions of a strong deformation retract, i.e. \( G \circ F = \text{id}, F \circ G = h^0 \circ d^0 + d^{-1} \circ h^{-1} \) and \( h \circ F = 0 \).

Because of this we can follow the proof of D. Bar-Natan again to show the invariance under the RM3 move. Here we use the notion of the cone from Definition 2.10 and Lemma 2.12. We skip this because it is analogous to the proof of D. Bar-Natan (with the maps from above).

The invariance under the virtual Reidemeister moves vRM1, vRM2 and vRM3 follow from Lemma 2.5.

Therefore the only move left is the mixed Reidemeister move mRM. We have

\[
\text{[}\begin{array}{c}
\includegraphics{case1.png}
\end{array}\text{]} = \Gamma(\text{[}\begin{array}{c}
\includegraphics{case5.png}
\end{array}\text{]})
\]

and

\[
\text{[}\begin{array}{c}
\includegraphics{case2.png}
\end{array}\text{]} = \Gamma(\text{[}\begin{array}{c}
\includegraphics{case6.png}
\end{array}\text{]}).
\]

There is a vRM2 move in both right parts of the cones. This move can be resolved. Hence the geometric complex changes only up to an isomorphism (see Lemma 2.5). Therefore we have

\[
\text{[}\begin{array}{c}
\includegraphics{case3.png}
\end{array}\text{]} \simeq \Gamma(\text{[}\begin{array}{c}
\includegraphics{case7.png}
\end{array}\text{]})
\]

and

\[
\text{[}\begin{array}{c}
\includegraphics{case4.png}
\end{array}\text{]} \simeq \Gamma(\text{[}\begin{array}{c}
\includegraphics{case8.png}
\end{array}\text{]}).
\]

Therefore we see that the left and right parts of the cones are equal geometric complexes. Hence the geometric complexes of two v-links which differ only through a mRM move are isomorphic.

This finish the proof. \( \square \)
A question which arises from Theorem 2.14 is if the geometric complex yields any new information for c-links (compared to the classical Khovanov complex constructed by D. Bar-Natan in [2]). The following theorem answers this question negative, i.e. the complex from Definition 2.3 is the classical complex up to chain isomorphisms.

To see this we mention that the cobordisms $m^{++}, \Delta^{++}$ have the same behaviour as the classical (co)multiplications. Therefore let $[L_D]_c$ denote the classical Khovanov complex, i.e. every pantsup- or pantsdown-cobordisms should be of the form $m^{++}, \Delta^{++}$ and we add the extra signs (see [11],[2]). Beware that this complex is not a chain complex for an arbitrary v-link diagram $L_D$. But it is indeed a chain complex for any c-link diagram, i.e. a diagram without v-crossings.

**Theorem 2.15.** Let $L_D$ be a c-link diagram. Then $[L_D]$ and $[L_D]_c$ are chain isomorphic.

**Proof.** Because $L_D$ does not contain any v-crossing, the complex has no $\theta$-saddles. Moreover every circle is a c-circle. Hence we can orient them + or –, i.e. counterclockwise or clockwise. We choose a numbering for the circles.

Because every circle is oriented clockwise or counterclockwise every saddle is of the form $\frac{1}{2} \to \frac{1}{2}$. Hence every saddle is of the form $m_+^{++} = m_-^{--}$ or $\Delta^{++}$ or $\Delta^{--}$. Thus up to a sign these maps are the classical maps.

We proof the theorem by a spanning tree argument. We choose such a spanning tree. Start at the source and reorient the circles in such a way that the maps which belongs to the edges in the tree are the classical maps $m_+^{++}$ or $\Delta^{++}$. This is possible because we can use $m_+^{++} = m_-^{--}$ here. We do this until we reach the end.

We repeat the process rearranging the numbering in such a way that the corresponding maps have the same sign as in the classical Khovanov complex. This is possible because every face has an odd number of minus signs (if we count the sign from the relation $\Delta^{--} = -\Delta^{++}$).

Hence after we reach the end every saddle is the classical saddle together with the classical sign. The change of orientations/numberings does not change the complex because of Lemma 2.4. Thus this finish the proof.

**Remark 2.16.** We could use the Euler characteristic to introduce the structure of a graded category (see [2]) on $u\text{Cob}^2_R(\varnothing)$ (and hence on $Kob_R(\varnothing)$).

The differentials in the geometric complex from Definition 2.3 have all $\deg = 0$ (after a grade shift), because their Euler-characteristic is -1 (see Corollary 1.5).

Then it is easy to proof that the geometric complex is a v-link invariant under graded homotopy. The proof of this is analogous to the one of D. Bar-Natan.

**Remark 2.17.** If one does the same construction as above in the category, then the whole construction (recall that we have to add formal extra signs like in the classical complex now) becomes easier in the following sense:

We do not need to calculate saddle signs. We just use the Table 2. This is the case because the Lemma 2.5 is true without the saddle signs.

The rest of this section can be proven completely analogue.

This construction leads us to an equivalent of the construction of V. Turaev and P. Turner from [23].
3 The algebraic complex

In this short section we construct the algebraic complex of a v-link diagram $L_D$. It is an invariant of virtual links $L$, i.e. modulo the generalised Reidemeister moves from Figure 3.

We follow the construction of V. Turaev and P. Turner for extended Frobenius systems and uTQFTs. But our uTQFT correspond to skew-extended Frobenius algebras, i.e. the map $\Phi$ is a skew involution rather than an involution like in [23].

We get an invariant for v-links which is an extension of the Khovanov complex for $R = \mathbb{Z}$ or $R = \mathbb{Q}$.

We denote any v-link diagram of the unknot with the symbol $\bigcirc$. Furthermore we view v-circles, i.e. v-links without classical crossings, as disjoint circles immersed into $\mathbb{R}^2$.

Definition 3.1 (uTQFT). An $(1+1)$-dimensional unoriented TQFT $F$ (we call this an uTQFT) is a semi-strict, symmetric, covariant functor $F : uCob^2_R(\emptyset) \to R\text{-MOD}$.

Here $F(\bigcirc)$ should be a finite generated and free $R$-module. Let $O, O'$ be two objects in $uCob^2_R(\emptyset)$ which are homeomorphic. Then $F(O) = F(O')$.

Also the functor $F$ should satisfy the following axioms:

1. Let $O, O'$ be two disjoint objects in $\text{Ob}(uCob^2_R(\emptyset))$. Then there exists a natural (with respect to homeomorphisms) isomorphism between $F(O \sqcup O')$ and $F(O) \otimes F(O')$.

2. The functor should satisfy $F(\emptyset) = R$.

3. For a cobordism $C : O \to O' \in \text{Mor}(uCob^2_R(\emptyset))$ the homomorphism $F(C)$ is natural with respect to homeomorphisms of cobordisms.

4. Let the cobordisms $C : O \to O' \in \text{Mor}(uCob^2_R(\emptyset))$ be a disjoint union of the two cobordism $C_{1, 2}$. Then $F(C) = F(C_1) \otimes F(C_2)$ under the identification from axiom (1).

Two uTQFTs $F, F'$ are called isomorphic if for each object of $O \in \text{Ob}(uCob^2_R(\emptyset))$ there is an isomorphism $F(O) \to F'(O)$, natural with respect to homeomorphisms of the objects and homeomorphisms of cobordisms, multiplicative with respect to disjoint union and the isomorphism assigned to $\emptyset$ is the id-morphism.

Remark 3.2. There are several things about the definition:

- Recall that our category is pre-additive. An uTQFT should be an additive functor. So we can extend this to a functor $F : uKob_R(\emptyset) \to \text{Kom(Mat}(R\text{-MOD}))$,

i.e. for every formal chain complex $CH(O)$ of objects of $uCob^2_R(\emptyset)$, i.e. v-circles, the object $F(CH(O))$ is a chain complex of $R$-modules and for every formal chain map $f: CH(O) \to CH(O')$ of possible non orientable decorated cobordisms the morphism $F(f)$ is a chain map of $R$-module homomorphisms.
An uTQFT $\mathcal{F}$ should be a covariant functor. Hence $\mathcal{F}(\text{id}^+) = \text{id}$. Furthermore it is symmetric and hence $\mathcal{F}(\tau^{++}) = \tau$. Here $\tau$ denotes the canonical permutation.

The permutation $\tau^{++}$ is natural. So we can assume that $A \otimes B$ and $B \otimes A$ are equal and not merely isomorphic.

For the exact definition of natural we refer the reader to [24].

**Definition 3.3** (Algebraic complex). Let $L_D$ be a v-link diagram. Then the algebraic complex of $L_D$ which belongs to the uTQFT $\mathcal{F}$ is the complex $\mathcal{F}(\|L\|)$.

We prove the following important result. Here $L_D, L_D'$ are a v-link diagrams. The proof is a direct consequence of Theorem 2.14.

**Theorem 3.4** (The algebraic complex is an invariant). For every uTQFT $\mathcal{F}$ which satisfies the Bar-Natan-relations (see [2] or Figure 2) the algebraic complex $\mathcal{F}(\|L_D\|)$ is a v-link invariant in the following sense:

For two equivalent (up to the generalised Reidemeister moves) v-link diagrams $L_D, L_D'$ the two chain complexes $\mathcal{F}(\|L_D\|)$ and $\mathcal{F}(\|L_D'\|)$ are equal up to chain homotopy.

This theorem allows us to speak of the algebraic complex $\mathcal{F}(\|L\|)$ of any oriented v-link $L$. Furthermore the category $\textbf{R-MOD}$ is abelian. Hence the category $\text{Kom}(\text{Mat}(\textbf{R-MOD}))$ is also an abelian category. So unlike in the just pre-additive category $\text{uKob}_R(\mathcal{O})$ we have the notion of homology. We denote the homology of the algebraic chain complex as $H(\mathcal{F}(\|L\|))$.

### 4 Skew-extended Frobenius algebras

In this section we describe the relation between uTQFTs and skew-extended Frobenius algebras. A relation of this kind was discovered by V. Turaev and P. Turner in [23] for extended Frobenius algebras and the functors they use. Even though our construction is different, their ideas can be used in our case too. This is the main part of Theorem 4.4.

With their work we can give a bunch of v-link invariant uTQFTs. And unlike their classification we get a skew-extended Frobenius algebra which is an extension for the classical Khovanov complex ($h = t = 0$) over $\mathbb{Q}$ (or $\mathbb{Z}$) and allows gradings.

Furthermore we get an extension of the Khovanov-Lee complex ($h = 0, t = 1$) too (see Lemma 4.5 and the Corollaries 4.10 and 4.12).

At the end of the section we are able to classify all aspherical uTQFTs which can be used to define v-links invariants (see Theorem 4.15).

We start with the definition of a skew-extended Frobenius algebra. For a $R$-bialgebra $A$ with comultiplication $\Delta$ and counit $\varepsilon$ we call a $R$-algebra homomorphism $\Phi: A \to A$ a skew-involution if $\Phi^2 = \text{id}$, $(\Phi \otimes \Phi) \circ \Delta \circ \Phi = -\Delta$ and $\varepsilon \circ \Phi = -\varepsilon$.

**Definition 4.1** (Skew-extended Frobenius algebras). A Frobenius algebra $A$ over $R$ is an unital, commutative algebra over $R$ which is projective and of finite type.
(as an $R$-module), together with a module homomorphism $\varepsilon : A \to R$ such that the bilinear form $\langle \cdot , \cdot \rangle$ defined by $\langle a, b \rangle = \varepsilon(ab)$ for all $a, b \in A$ is non degenerate.

An skew-extended Frobenius algebra $A$ over $R$ is a Frobenius algebra together with a skew-involution of Frobenius algebras $\Phi : A \to A$ and an element $\theta \in A$ which satisfy the equations

1. $\Phi(\theta a) = \theta a = \theta \Phi(a)$ for all $a \in A$,
2. $(m \circ (\Phi \otimes \text{id}) \circ \Delta)(1) = \theta^2$.

**Notation.** Because of $1 \in A$, there is a copy of $R$ in $A$. We call $\iota : R \to A$ the canonical inclusion. We can write a Frobenius algebra uniquely as $F = (R, A, \varepsilon, \Delta)$.

Moreover we can write such a skew-extended Frobenius algebra $F$ uniquely as $F = (R, A, \varepsilon, \Delta, \Phi, \theta)$.

**Definition 4.2.** Two skew-extended Frobenius algebras $F_1 = (R, A, \varepsilon, \Delta, \Phi, \theta)$ and $F_2 = (R, A', \varepsilon', \Delta', \Phi', \theta')$ are called isomorphic if there exists an isomorphism of Frobenius algebras $f : A \to A'$ which satisfies $f(\theta) = \theta'$ and $f \circ \Phi = \Phi' \circ f$.

We call a Frobenius algebra aspherical if $\varepsilon(\iota(1)) = 0$.

We say it is a rank2-Frobenius algebra if $A \simeq 1 \cdot R \oplus X \cdot R$.

**Recall 4.3.** The map $\varepsilon$ is called the counit of $A$. It can be used to define a multiplication $\Delta : A \to A \otimes A$. We will call $m : A \otimes A \to A$ the multiplication of $A$. The coproduct and the product make the two diagrams commutative. In a skew-extended Frobenius algebra the skew-involution $\Phi$ and the element $\theta$ makes the two diagrams

commutative (it is easy to check that the two equations from Definition 4.1 already imply $(m \circ (\Phi \otimes \text{id}) \circ \Delta)(a) = \theta^2 a$ for all $a \in A$). Here the map $\cdot \theta : A \to A$ is the multiplication with $\theta$ and the map $m' : A \otimes A \to A$ is the map $(\Phi \otimes \text{id}) \circ m$.

We recognise that the lower right diagram is the problematic face from 1. So the second equation from definition 4.1 is a key point in the definition.

**Theorem 4.4 (Turaev/Turner).** The isomorphism classes of $(1+1)$-dimensional untQFTs over $R$ are in bijective correspondence with the isomorphism classes of skew-extended Frobenius algebras over $R$. 36
Proof. First let us consider an uTQFT $\mathcal{F}$ over $R$. We describe a way to get a skew-
extended Frobenius algebra from it. Let us denote this algebra as $(R, A, \varepsilon, \Delta, \Phi, \theta)$. Therefore we get a TQFT $\mathcal{F}(\varnothing)$ as our underlying $R$-module.

Next we need a skew-involution $\Phi : A \to A$. We take the cylinder from Figure 10. Therefore $\Phi = \mathcal{F}(\Phi_{\varnothing})$.

The unit $\iota$ should be $\mathcal{F}(\iota_{+})$. Their is no further choice because $\iota_{+} = \iota_{-}$. The counit should be $\mathcal{F}(\varepsilon^{+})$. Here we have a choice because $\varepsilon^{+} \neq \varepsilon^{-}$. But because of $\varepsilon^{+} = -\varepsilon^{-}$, both choices lead to isomorphic algebras.

Now we need a multiplication $m$ and a comultiplication $\Delta$. One may suspect, that we have different choices for either of them, namely the eight $m_{\pm \pm}, \Delta_{\pm \pm}$. But the relations of a Frobenius algebra only allow one option. We discuss this now:

- The lower boundary components of $\Delta_{l_1 l_2}^{u}$ must have the same gluing numbers as the boundary component of $\varepsilon^{+}$ because $\mathcal{F}((\varepsilon^{+})^{\circ})$ should be the counit;
- because of the relation $\varepsilon \circ m \circ (\text{id} \otimes \varepsilon) = \varepsilon = \varepsilon \circ m \circ (\varepsilon \otimes \text{id})$, the lower boundary of $m_{1}^{u 1 u_2}$ must have the same gluing number as the boundary component $\varepsilon^{+}$. The same is true for the upper boundary (this means we need $m_{0}^{+} = m_{-}^{+}$);
- because of the relation $(\text{id} \otimes m) \circ (\Delta \otimes \text{id}) = \Delta \circ m = (m \otimes \text{id}) \circ (\text{id} \otimes \Delta)$, the $m_{1}^{u 1 u_2}$ must have the same gluing number on the lower boundary as the upper boundary of $\Delta_{l_1 l_2}^{u}$ (the reader should check that this is the only possible choice for the gluing numbers for $m_{1}^{u 1 u_2}$ and $\Delta_{l_1 l_2}^{u}$).

Therefore we have $\mathcal{F}(\iota_{+}) = \iota$, $\mathcal{F}(\varepsilon^{+}) = \varepsilon$, $\mathcal{F}(m_{0}^{+}) = m$ and $\mathcal{F}(\Delta_{++}) = \Delta$.

The last piece missing is the element $\theta \in A$. Consider a two times punctured projective plane $\mathbb{R}P_{2}$ (a punctured Möbius strip). This is $\theta$ in our notation.

Then $\theta \circ \iota_{+} : \mathcal{O} \to \mathcal{O}$ is a punctured projective plane (hence a Möbius strip). Set $\theta = \mathcal{F}(\theta \circ \iota_{+})(1)$. Because of the definition, this is an element of $\mathcal{F}(\varnothing) = A$.

We have to prove the equations needed for a skew-extended Frobenius algebra, i.e. that $\iota$ is a unit, $\varepsilon$ is a counit, $\Phi$ is a skew-involution, $m((\Delta))$ is a (co)multiplication and the commutativity of the faces from Recall 1.7

This is a straightforward verification bases on the relations from Lemma 1.7

This shows that every uTQFT has an underlying skew-extended Frobenius algebra.

For the other direction, i.e. if we assume that we have a skew-extended Frobenius algebra, we notice that this algebra has an underlying classical Frobenius algebra. Therefore we get a TQFT $\mathcal{F}'$ from this underlying Frobenius algebra. We want to use this TQFT to define an uTQFT $\mathcal{F}$. The TQFT $\mathcal{F}'$ is a covariant functor

$$\mathcal{F}' : \text{Cob}_{R}^{2}(\varnothing) \to R\text{-MOD}.$$ 

Let $\mathcal{O}$ be an object in $\text{uCob}_{R}^{2}(\varnothing)$. This object gives (modulo homeomorphisms) us a corresponding object $\mathcal{O}'$ in $\text{Cob}_{R}^{2}(\varnothing)$. We set $\mathcal{F}(\mathcal{O}) = \mathcal{F}'(\mathcal{O}')$.

This assignment clearly satisfies that $\mathcal{F}(\varnothing)$ is a finite generated, free $R$-module and $\mathcal{F}(\mathcal{O}_1) = \mathcal{F}(\mathcal{O}_2)$ for two homeomorphic objects $\mathcal{O}_1, \mathcal{O}_2$.

Moreover, because $\mathcal{F}'$ is a TQFT, this satisfies the first two axioms from our Definition 3.1.

Now we need to define $\mathcal{F}(\mathcal{C})$ for every morphisms from $\text{uCob}_{R}^{2}(\varnothing)$.
First we assume that $\mathcal{C}: \mathcal{O}_1 \to \mathcal{O}_2$ is orientable and connected. Then we have a corresponding morphism in $\text{Cob}^2_R(\varnothing)$, i.e. the same without the boundary decorations, which we call $\mathcal{C}': \mathcal{O}_1' \to \mathcal{O}_2'$. We denote the classical cap-, cup-, pantsup- and pantsdown-cobordisms in the category $\text{Cob}^2_R(\varnothing)$ with $\iota, \varepsilon, m, \Delta$. Let us define

$$\mathcal{F}(\iota_+) = \mathcal{F}'(\iota), \mathcal{F}(\varepsilon^+) = \mathcal{F}'(\varepsilon), \mathcal{F}(m^+) = \mathcal{F}'(m), \mathcal{F}(\Delta^+) = \mathcal{F}'(\Delta) \text{ and } \mathcal{F}(\Phi^-) = \Phi.$$  

The map $\Phi$ is the skew-involution in the skew-extended Frobenius algebra. Thus we can define $\mathcal{F}(\mathcal{C})$ in the following way. We decompose $\mathcal{C}'$ into the basis pieces $\iota, \varepsilon, m, \Delta$. Then $\mathcal{F}'(\mathcal{C}')$ is independent of this decomposition because $\mathcal{F}'$ is a TQFT. If we use the same decomposition for $\mathcal{C}$ (under the identification from above), we get a cobordism $\tilde{\mathcal{C}}$. For this cobordism we can define $\mathcal{F}(\tilde{\mathcal{C}})$. We see that we only have to change some of the boundary decoration of $\tilde{\mathcal{C}}$ to obtain $\mathcal{C}$. Hence we have

$$\mathcal{C} = C_1 \circ \tilde{\mathcal{C}} \circ C_2,$$

where $C_1, C_2$ are cylinders of the form $\text{id}_+^*$ or $\Phi^-$. Hence we can define

$$\mathcal{F}(\mathcal{C}) = \mathcal{F}(C_1) \circ \mathcal{F}(\tilde{\mathcal{C}}) \circ \mathcal{F}(C_2).$$

That this is also independent of the decomposition follows from the fact that $\text{id}_+^*, \Phi^-$ and the corresponding maps in the skew-extended Frobenius algebra are (skew-)involutions (see (a) from Lemma 1.7).

For a non connected, orientable cobordism $\mathcal{C}$ we extend the definition from above multiplicatively.

For a non orientable, connected cobordisms $\mathcal{C}$ we define $\mathcal{F}(\theta) = \theta$ first. Here the map $\cdot \theta: A \to A$ is the multiplication with the element $\theta$ in our skew-extended Frobenius algebra. Hence if we decompose $\mathcal{C} = C_{or} \# n \mathbb{RP}^2$ into an orientable part $C_{or}$ and $n$-times a projective plane we define

$$\mathcal{F}(\mathcal{C}) = \theta^n \mathcal{F}(C_{or}).$$

This is again independent from the decomposition of $C_{or}$ because of the first relation in a skew-extended Frobenius algebra, namely $\Phi(\theta a) = \theta a = \theta \Phi(a)$ for all $a \in A$. Furthermore it is independent from the decomposition $\mathcal{C} = C_{or} \# n \mathbb{RP}^2$ because if we replace a $2 - \mathbb{RP}^2$ with a torus $\mathcal{T}$ we see that $\mathcal{F}(\mathcal{O}_{or})$ is multiplied by a factor $(m \circ (\Phi \otimes \text{id}) \circ (m \circ (\Phi \otimes \text{id}) \circ \Delta)(1)\theta^{n-2}$. Hence, using the second relation of the skew-extended Frobenius algebra, we get

$$\mathcal{F}(C_{or} \# n \mathbb{RP}^2) = (\theta^n)\mathcal{F}(\mathcal{O}_{or}) = \theta^{n-2}(m \circ (\Phi \otimes \text{id}) \circ \Delta)(1)\mathcal{F}(\mathcal{O}_{or})$$

$$= \mathcal{F}(C_{or} \# \mathcal{T} \# (n - 2) \mathbb{RP}^2).$$

For a non connected, non orientable cobordisms $\mathcal{C}$ we extend the definition from above multiplicatively. Hence we only have to show the remaining axioms from the Definition 3.1. The reader should check this axioms. Here one could follow the end of the proof of V. Turaev and P. Turner from [23].

From now on we use the notions uTQFT and skew-extended Frobenius algebra simultaneous.
Lemma 4.5. Let \( \mathcal{F} \) be an aspherical rank 2-uTQFT. Then we have the following table for the images of the generators (Figure 10) as \( R \)-homomorphisms with values in the corresponding skew-extended Frobenius algebra. The elements \( a, h, t, \alpha, \beta \in R \) satisfy the relations

- \( a \in R \) is invertible;
- \( \alpha \gamma = \beta \gamma = 2 \alpha = 2 \beta = 0; \)
- \( ah = \gamma - a \alpha^2 - a \beta^2 t \) and \( a^2 \beta^2 h = 0. \)

Therefore the table will be:

| \( \mathcal{F}(\iota_+): R \to A; 1 \mapsto 1 \) | \( \mathcal{F}(\Phi^-): A \to A; \begin{cases} 1 \mapsto 1, \\ X \mapsto \gamma - X \end{cases} \) |
| --- | --- |
| \( \mathcal{F}(\varepsilon^+): A \to R; 1 \mapsto 0, X \mapsto 1, \) | \( \mathcal{F}(\theta): A \to A; \begin{cases} 1 \mapsto \alpha + \beta \cdot X, \\ X \mapsto \beta t + (\alpha + a \beta h) \cdot X \end{cases} \) |
| \( \mathcal{F}(m_{++}^-): A \otimes A \to A; \begin{cases} 1 \otimes 1 \mapsto 1, 1 \otimes X \mapsto X, \\ X \otimes 1 \mapsto X, X \otimes X \mapsto t + h \cdot X \end{cases} \) | \( \mathcal{F}(\Delta_{++}^-): A \to A \otimes A; \begin{cases} 1 \mapsto -h \cdot 1 \otimes 1 + a^{-1}(1 \otimes X + X \otimes 1), \\ X \mapsto a^{-1} t \cdot 1 \otimes 1 + a^{-1} \cdot X \otimes X. \end{cases} \) |

Table 1: The maps for the generators from Figure 10.

Proof. From Theorem 4.3 above we know that the unit is \( \iota = \mathcal{F}(\iota_+) \), the counit is \( \varepsilon = \mathcal{F}(\varepsilon^+) \), the multiplication is \( m = \mathcal{F}(m_{++}^-) \), the comultiplication is \( \Delta = \mathcal{F}(\Delta_{++}^-) \), the skew-involution is \( \Phi = \mathcal{F}(\Phi^-) \) and the element is \( \theta = \mathcal{F}(\theta)(1) \).

First we observe that the skew-extended Frobenius algebra \( A \) has an underlying Frobenius algebra of rank two, hence \( \iota \) has to be of the given form. Because it is also aspherical, i.e. \( \varepsilon(\iota(1)) = 1 \), we see that \( \varepsilon(1) = 1 \) and \( \varepsilon(X) = a \). The element \( a \in R \) is invertible because of the relation

\[(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta.\]

It is well-known (see for example M. Khovanov’s paper [13]) that such an algebra is of the form \( A = R[X]/(X^2 = t + ahX) \) with multiplication \( m = \mathcal{F}(m_{++}^-) \) and comultiplication \( \Delta = \mathcal{F}(\Delta_{++}^-) \) from above.

Next we look at the new structure. Because \( \theta \) is an element of \( A = 1 \cdot R \oplus X \cdot R \) we find \( \alpha, \beta \in R \) such that \( \theta = \alpha + \beta X \). Using the multiplication we see that \( X^2 = t + ah \cdot X \).

So an easy calculation shows that \( \theta \cdot X = \beta t + (\alpha + a \beta h)X \) which gives us the map \( \cdot \theta \) like above.

Because the map \( \Phi: A \to A \) is not only \( R \)-linear, but also a skew-involution, we get \( \Phi(1) = 1 \) and with \( \varepsilon \circ \Phi = -\varepsilon \) we get \( \Phi = \gamma - X \). Using the first relation of a skew-extended Frobenius algebra we get the relations \( \alpha \gamma = \beta \gamma = 2 \beta = 0 \) and \( 2(\alpha + a \beta h) = 2 \alpha = 0. \)
Using the second relation of a skew-extended Frobenius algebra, namely
\[ F(m^+_{i,j}) \circ F((\Phi_+ \cup \text{id}_+)) \circ F(\Delta^+_{i,j}) = (\theta)^2, \]
we get the last two relations \( ah = \gamma - a\alpha^2 - a\beta^2 t \) and \( a^2\beta^2 h = 0. \)
These are all relations we get, i.e. any other missing relation will also lead to one of these relations. \( \square \)

With this lemma we get the following five corollaries:

**Corollary 4.6** (The universal skew-extended Frobenius algebra). Every aspherical rank2-\( u \)TQFT comes from the rank2-\( u \)TQFT \( F_U = (R_U, A_U, \varepsilon_U, \Delta_U, \Phi_U, \theta_U) \) through base change. Here the ring \( R_U \) is \( R_U = \mathbb{Z}[a, \alpha, \beta, \gamma, t]/I \) with \( I \) is the ideal generated by (define \( h := \alpha^{-1}\gamma - \alpha^2 - \beta^2 t \)) the relations
\[ \alpha\gamma = \beta\alpha = 2\alpha = 2\beta = a^2\beta^2 h = 0 \]
and invertible \( a \in R_U \). Furthermore the algebra is \( A_U = R_U[X]/(X^2 = t + ahX) \) and the maps will be the ones from Table 4. \( \square \)

**Remark 4.7.** The reader familiar with the paper of V. Turaev and P. Turner from \[23\] will recognise that our universal skew-extended Frobenius algebra \( F_U \) is different from the one from V. Turaev and P. Turner. But this is an advantage (see Corollary 4.10).

As mentioned before in the Remark 2.17 the version of V. Turaev and P. Turner can be obtained from our concept too. The difference again are the relations \( \varepsilon^+ = -\varepsilon^- \) and \( \Delta^+ = -\Delta^- \). This forces \( F(\Phi^-) \) from the proof above to send \( X \mapsto \gamma - X \) instead of \( X \mapsto \gamma + X \).

The next corollary allows us to characterise the \( u \)TQFT which leads to \( v \)-link homology.

**Corollary 4.8.** Every aspherical rank2-\( u \)TQFT \( F \) satisfy the Bar-Natan relations from [3] or Figure 4.

**Proof.** View a sphere \( S^2 \) as a cobordism \( S^2: \emptyset \rightarrow \emptyset \). Then \( F(S^2) = F(\varepsilon^+) \circ F(\iota_+) \).
So we calculate \( F(S^2) = 0 \). Because of the axiom (4) from Definition 3.1 this is true for every cobordism with a sphere.

Analogue view a torus \( T \) as a cobordism \( T: \emptyset \rightarrow \emptyset \). Thus it is of the form \( F(T) = F(\varepsilon^+) \circ F(m^+_{1,2}) \circ F(\Delta^+_{1,2}) \circ F(\iota_+). \) An easy calculation with the maps of Table 4 shows, that \( F(T) = 2 \).

Because of the axiom (4), this is true for every cobordism with a torus.

The \( 4Tu \)-relation is algebraical just the formula
\[ \Delta_{12} \circ \iota + \Delta_{34} \circ \iota = \Delta_{13} \circ \iota + \Delta_{24} \circ \iota. \]
Here \( \Delta_{ij} : A \rightarrow A \otimes A \otimes A \otimes A \) is the map which sends an element \( a \in A \) to an element \( a_1 \otimes a_2 \otimes a_3 \otimes a_4 \) with \( a_k = a \) for \( k \neq i, j \) and \( a_i, a_j \) the first respectively the second tensor factor of \( \Delta(a) \) (see Figure 27).

That this relation is true is also an easy calculation. Again axiom (4) gives us the global statement.

Because this is true for the universal skew-extended Frobenius algebra \( F_U \), we get the statement for all aspherical rank2-\( u \)TQFTs from the Corollary 4.6. \( \square \)
Because with an aspherical rank2 skew-extended Frobenius algebra we can define a rank2 uTQFT which satisfies the Bar-Natan relations, we get:

**Corollary 4.9.** Every aspherical rank2 uTQFT can be used to define an invariant for v-links.

**Corollary 4.10** (The virtual Khovanov complex). There is a method to extend the Khovanov complex ($R_{Kh} = \mathbb{Z}, A_{Kh} = \mathbb{Z}[X]/(X^2 = 0, t = h = 0)$) from c-links to v-links by setting $\alpha = \beta = \gamma = 0$ and $a = 1$.

From now on we denote $\text{Kh}(L) = \mathcal{F}_{Kh}([L])$ the virtual Khovanov complex of a v-link $L$ and $H(\text{Kh}(L))$ for its homology.

**Remark 4.11.** Of course it is possible to introduce gradings (in the usual way by setting $\deg 1 = 1$ and $\deg X = -1$) for the complex from the Corollary 4.10. This is true because the map $\cdot \theta = 0$. In fact this is the only possibility where we can introduce gradings because all maps in the Khovanov complex must decrease the grading by one. And this is only possible if $\cdot \theta : A \to A$ is equal zero.

There is also an extension for the Khovanov-Lee complex (see her paper [15]) and two different extensions of D. Bar-Natan's variant ($R = \mathbb{Z}/2, h = 1, t = 0$) of the Khovanov complex (see his paper [2]).

**Corollary 4.12** (The virtual Khovanov-Lee complex). There is a method to extend the Khovanov-Lee complex ($R_{Lee} = \mathbb{Z}, A_{Lee} = \mathbb{Z}[X]/(X^2 = 0, t = 1, h = 0)$) from c-links to v-links by setting $\alpha = \beta = \gamma = 0$ and $a = 1$.

**Corollary 4.13** (The virtual Khovanov-Bar-Natan complex). There are two methods to extend Bar-Natan's variant of Khovanov homology (this is the Frobenius algebra $R_{BN} = \mathbb{Z}/2, A_{BN} = \mathbb{Z}/2[X]/(X^2 = 0, t = 0, h = 1)$) from c-links to v-links by setting $\alpha = \beta = 0$ and $\gamma = a = 1$ or by setting $\beta = \gamma = 0$ and $\alpha = a = 1$. The two extensions are not isomorphic skew-extended Frobenius algebras.

**Proof.** That this two skew-extended Frobenius algebras can be used as v-link homologies follows from Corollary 4.9. To see that the are not isomorphic skew-extended Frobenius algebras we notice that $\theta = 0$ in the first case and $\theta = 1$ in the second case. Because any isomorphism of skew-extended Frobenius algebras satisfies $f(1) = 1$ and $f(\theta) = \theta'$, they are not isomorphic.
Proposition 4.14. Let $L$ be a c-link and let $\mathcal{F}$ be a aspherical rank2-uTQFT. Then the complex $\mathcal{F}(\llbracket L \rrbracket)$ is the classical complex (up to chain isomorphisms) which is obtained by using the underlying TQFT $\mathcal{F}'$ of $\mathcal{F}$.

Proof. This is just the algebraic version of Theorem 2.15.

We denote these as $\mathcal{F}_{\text{Lee}}(L) = \mathcal{F}_{\text{Lee}}(\llbracket L \rrbracket)$ and $\mathcal{F}_{\text{BN}1}(L) = \mathcal{F}_{\text{BN}1}(\llbracket L \rrbracket)$ and $\mathcal{F}_{\text{BN}2}(L) = \mathcal{F}_{\text{BN}2}(\llbracket L \rrbracket)$.

If $L$ is a c-link, then these three (and any other of the possibilities) are the classical complexes.

Because M. Khovanov showed (see [13]) that every TQFT which respects the first Reidemeister move must have an underlying algebra $A \simeq R \oplus X \cdot R$ for an element $X \in A$, we also get the following theorem:

Theorem 4.15 (Classification of aspherical uTQFTs). For an aspherical uTQFT the following points are equivalent:

(a) It respects the first Reidemeister move $R_{M1}$;

(b) it is a rank2-uTQFT;

(c) it is of the form of Lemma 4.5;

(d) it can be used as a v-link invariant.

With the work already done the proof is simple:

Proof. (a)$\Rightarrow$(b): This was done by M. Khovanov and stays true.

(b)$\Rightarrow$(c): This is just the Lemma 4.5.

(c)$\Rightarrow$(d): This is the Corollary 4.9.

(d)$\Rightarrow$(a): This is clear.

Remark 4.16. At this state it is a fair question to ask why we use the relations (1) from Figure 11 (or the one without the sign for the variant of V. Turaev and P. Turner) for our cobordisms, i.e. why do we assume that $\Delta^+_{-\alpha}$ changes its sign under composition with $\Phi^+$ (or neither of them changes its sign for the variant of V. Turaev and P. Turner).

So what happens if we assume that $m^+_{\alpha}$ changes its sign under composition with $\Phi^+$ (or both)? One can repeat the whole construction from the sections one to four for these cases too. But this do not lead to something new, i.e. if we assume that $m^+_{\alpha}$ changes its sign we get an equivalent to the construction above and if we assume that both of them changes their signs we get an equivalent to the variant of V. Turaev and P. Turner again.

5 Some calculations

In this section we show some basic calculations with a computer program we have written. The program is a MATHEMATICA (see [24]) package called $vKh.m$. There is also a notebook called $vKh.nb$. 42
The input data is a v-link diagram in a circuit notation, i.e. the classical planar diagram notation, but we allow v-crossings. Hence the input data is a string of labelled X, i.e crossings are presented as symbols $X_{ijkl}$ where the numbers are increasing as we go around each v-link component and the edges around the crossing start counting from the lower incoming and proceeding counterclockwise. We denote such a diagram with $CD[X_{i,j,k,l},...,X_{m,n,o,p}]$.

After starting MATHEMATICA and loading our package vKh.m, we type in the unknot from Figure 1, the classical and virtual Trefoil. Our notation follows the notation of J. Green in his nice table of virtual knots (see [5]).

In[1]:= Unknot := CD[X[1,3,2,4], X[2,1,3,4]]; Knot21 := CD[X[1,3,2,4], X[4,2,1,3]]; Knot36 := CD[X[1,5,2,4], X[5,3,6,2], X[3,1,4,6]];

Let us denote the elements $1, X \in \mathbb{A} = \mathbb{Z}[X]/\langle X^2 \rangle = 0$ as $1 = vp[i]$ and $X = vm[i]$. Here the module $\mathbb{A}$ should belong to the $i$-th v-circle. Moreover we denote with the word $a$, which letters are from the alphabet $\{0, 1, \ast\}$ with exactly one $\ast$-entry, the cobordism starting at the resolution $\gamma_{\ast=0}$ and going to the resolution $\gamma_{\ast=1}$. Let us check the different morphisms.

In[2]:= d2[Unknot, "0*"], d2[Unknot, "*0"], d2[Unknot, "1*"], d2[Unknot, "*1"]
Out[2]= {{vp[1] -> vm[2] vp[1] - vm[1] vp[2], vm[1] -> vm[1] vm[2]}, {vp[1] -> 0, vm[1] -> 0}, {vp[1] -> 0, vm[1] -> 0}, {vp[1] vp[2] -> -vp[1], vm[2] vp[1] -> -vm[1], vm[1] vp[2] -> -vm[1], vm[1] vm[2] -> 0}}

We see that the two orientable morphisms are $\Delta_{+}^+$ and $-m_{-}^{--} = -m_{+}^{++}$. With the command KhBracket[Knot, r] we generate the $r$-th module of the complex (here for simplicity without gradings). Moreover with $d[Knot][KhBracket[Knot, r]]$ we calculate the image of the $r$-th differential for the whole module. Let us check the output.

In[3]:= KhBracket[Unknot, 0], KhBracket[Unknot, 1], KhBracket[Unknot, 2]
Out[3]= {{v[0, 0] vm[1], v[0, 0] vp[1]}, {v[0, 1] vm[1] vm[2], v[0, 1] vm[2] vp[1], v[0, 1] vm[1] vp[2], v[0, 1] vp[1] vp[2], v[1, 0] vm[1], v[1, 0] vp[1]}, {v[1, 1] vm[1], v[1, 1] vp[1]}}

In[4]:= d[Unknot][KhBracket[Unknot, 0]], d[Unknot][KhBracket[Unknot, 1]]
Out[4]= {{v[0, 1] vm[1] vm[2], v[0, 1] vm[2] vp[1] - v[0, 1] vm[1] vp[2]}, {0, -v[1, 1] vm[1], -v[1, 1] vm[1] - v[1, 1] vp[1], 0, 0}}

It is easy to check that the composition $d_1 \circ d_0$ is indeed zero.

In[5]:= d[Unknot][d[Unknot][KhBracket[Unknot, 0]]]
Out[5]= {0, 0}

Let us check this for the other two knots too.

In[6]:= d[Knot21][d[Knot21][KhBracket[Knot21, 0]]]
Out[6]= {0, 0, 0}

In[7]:= d[Knot36][d[Knot36][KhBracket[Knot36, 0]]], d[Knot36][d[Knot36]
Now let us check for the trefoil how the signs of the morphisms work out.

\[
\text{Out}[8]= \{1, -1, 1\}
\]

\[
\text{Out}[9]= \{1, 1, 1, -1, -1, -1\}
\]

\[
\text{Out}[10]= \{1, 1, 1\}
\]

We observe that all of the six different faces have an odd number of signs. For example the face \(F_1 = (\gamma_{000}, \gamma_{001} \oplus \gamma_{100}, \gamma_{010})\) gets a sign from the morphism \(d_{0,*0}\).

The face \(F_2 = (\gamma_{100}, \gamma_{101} \oplus \gamma_{110}, \gamma_{111})\) gets a sign from the morphism \(d_{1,*0}\).

The first face is of type 2b and the second is of type 1b. Hence after a virtualisation the latter should have an even number of signs, but the first should have an odd number signs. Lets check this. First we define a new knot diagram which we obtain by doing virtualisation on the second crossing of the trefoil.

\[
\text{Out}[12]= \{1, -1, 1\}
\]

\[
\text{Out}[13]= \{1, 1, 1, -1, -1, -1\}
\]

\[
\text{Out}[14]= \{1, -1, 1\}
\]

Indeed only the sign of the morphism \(d_{1,*1}\) is different now. Hence the face \(F_1\) still has an odd number, but the face \(F_2\) has an even number off signs. This should cancel with the extra sign of the pantsdown morphism \(d_{1,*1}\).

\[
\text{Out}[15]= \{1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}
\]

Let us look at some calculation results for the four knots. The output is Betti\([q,t]\), i.e. the Dimension of the homology group in quantum degree \(q\) and homology degree \(t\). The unknot should have trivial homology.

\[
\text{Out}[16]= \{\text{Betti}[1,-2] = 0, \text{Betti}[-1,0] = 0, \text{Betti}[0,-2] = 0, \text{Betti}[0,-1] = 1, \text{Betti}[0,0] = 0, \text{Betti}[0,1] = 0, \text{Betti}[0,1] = 0, \text{Betti}[1,0] = 0, \text{Betti}[1,2] = 0\}\]
For the other outputs we skip the Betti numbers. One can read them from the polynomial. The trefoil and its virtualisation have the same output (as they should).

\[ \text{Out[16]} = \frac{1}{q^3} + \frac{1}{q} + \frac{1}{q^9 \ t^3} + \frac{1}{q^5 \ t^2} \]

Let us check that the graded Euler characteristic is the Jones polynomial (up to shifts).

\[ \text{In[20]} = \text{Factor}\left(\frac{\text{vKh}[\text{Knot21}] / . \ t \rightarrow -1}{q + q^{-1}}\right) \]
\[ \text{Out[20]} = \frac{1 - q^2 + q^3}{q^5} \]

Another important observation is the following. The map $\Phi_+^{-}$ sends $1$ to itself, but $X$ to $-X$. Hence there is a good chance for 2-torsion. Let us check. Here Tor[q,t] denotes the torsion in quantum degree $q$ and homology degree $t$. Even the virtual trefoil has 2-torsion, but no 3-torsion.

\[ \text{In[22]} = \text{vKh}[\text{Knot21},2] \]
\[ \text{Out[22]} = \{\text{Tor[-2,-6]} = 0, \text{Tor[-2,-4]} = 0, \text{Tor[-2,-2]} = 0, \text{Tor[-1,-4]} = 1, \text{Tor[-1,-2]} = 0, \text{Tor[0,-3]} = 0, \text{Tor[0,-1]} = 0\} \]
\[ \text{Out[22]} = \frac{1}{q^4 \ t} \]

There seems to be a lot of 2-torsion!

\[ \text{In[24]} = \text{Knot32} := \text{CD}[X[2, 6, 3, 1], X[4, 2, 5, 1], X[5, 3, 6, 4]] ; \]
\[ \text{In[25]} = \text{vKh}[\text{Knot32}] \]
\[ \text{Out[25]} = \frac{1}{q^2} + \frac{1}{q} + \frac{1}{q^5 \ t^2} + \frac{1}{q \ t} + q^2 \ t \]
\[ \text{Out[26]} = \frac{1}{q^3 \ t} + t \]

Because the virtual Khovanov complex is invariant under virtualisation, there are many examples of non trivial v-knots with trivial Khovanov complex.
In[27]:= Knot459 := CD[X[2, 8, 3, 1], X[4, 2, 5, 1], X[3, 6, 4, 7], X[5, 8, 6, 7]];  

In[28]:= vKh[Knot32]  
Out[28]= 1/q + q  

Let us try an harder example. We mention that the faces are all anticommutative, hence the composition of the differentials is zero.

In[29]:= Knot53 := CD[X[1, 9, 2, 10], X[2, 10, 3, 1], X[5, 4, 6, 3], X[7, 4, 8, 5], X[8, 7, 9, 6]];  

In[30]:= vKh[Knot53]  
Out[30]= 1/q + q + q + 2 q^2 + 1/(q^3 t^2) + 2/(q^2 t) + q/t + 2 q t + 2 q^4 t + q^3 t^2 + 2 q^5 t^2 + q^7 t^3 + 2 + 1/q + q + 2 q^2 + q^3 + 1/(q^3 t^2) + 2/(q^2 t^2) + q/t + 2 q t + q^2 t + q^3 t + 2 q^4 t + q^2 t^2 + q^3 t^2 + 2 q^5 t^2 + q^6 t^2 + q^6 t^3 + q^7 t^3 + 2/q^2 + 1/q + 3 q + 1/(q^6 t^3) + 2/(q^5 t^2) + 1/(q^2 t^2) + 2/(q^3 t) + 2/(q t) + t + q^2 t + q^4 t + 1/q + 2 q^2 t + q^4 t^2 + 1 + 2/q^2 + 2 + q + 1/(q^6 t^3) + 2/(q^5 t^2) + 1/(q^4 t^2) + 1/(q^2 t^2) + 1/q + 2/q^2 + 2/q^4 t + 2/(q^3 t) + 2/(q t) + t + q^2 t + q^3 t + q^3 t^2 + q^4 t^2]  

In[31]:= Out[31] = vKh[Knot53, 2]  
Out[31] = 2/q^2 + 1/(q^5 t^2) + 1/(q^4 t) + 1/(q^3 t) + t + q t^2  

In[32]:= {d[Knot53][d[Knot53][KhBracket[Knot53, 0]]], d[Knot53][d[Knot53][KhBracket[Knot53, 1]]], d[Knot53][d[Knot53][KhBracket[Knot53, 2]]], d[Knot53][d[Knot53][KhBracket[Knot53, 3]]]}  
Out[32] = {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}  

The virtual Khovanov complex is strict stronger then the virtual Jones polynomial. The first example appears for v-links with seven crossings. Let us check two examples.

In[33]:= Example1 := CD[X[1, 4, 2, 3], X[2, 10, 3, 11], X[4, 9, 5, 10], X[11, 5, 12, 6], X[6, 1, 7, 14], X[12, 8, 13, 7], X[13, 9, 14, 8]]; Example2 := CD[X[1, 4, 2, 3], X[2, 10, 3, 11], X[4, 9, 5, 10], X[11, 5, 12, 6], X[6, 1, 7, 14], X[12, 8, 13, 7], X[13, 9, 14, 8]]; Example3 := CD[X[1, 4, 2, 3], X[2, 10, 3, 11], X[4, 9, 5, 10], X[11, 5, 12, 6], X[6, 1, 7, 14], X[12, 8, 13, 7], X[13, 9, 14, 8]]; Example4 := CD[X[1, 4, 2, 3], X[2, 10, 3, 11], X[4, 9, 5, 10], X[11, 5, 12, 6], X[6, 1, 7, 14], X[12, 8, 13, 7], X[13, 9, 14, 8]];  

So let us see what our program calculates.

In[34]:= {vKh[Example1], vKh[Example2], vKh[Example3], vKh[Example4]}  
Out[34] = {2 + 1/q + q + 2 q^2 + 1/(q^3 t^2) + 2/(q^2 t) + q/t + 2 q t + 2 q^4 t + q^3 t^2 + 2 q^5 t^2 + q^7 t^3 + 2 + 1/q + q + 2 q^2 + q^3 + 1/(q^3 t^2) + 2/(q^2 t^2) + q/t + 2 q t + q^2 t + q^3 t + 2 q^4 t + q^2 t^2 + q^3 t^2 + 2 q^5 t^2 + q^6 t^2 + q^6 t^3 + q^7 t^3 + 2/q^2 + 1/q + 3 q + 1/(q^6 t^3) + 2/(q^5 t^2) + 1/(q^2 t^2) + 2/(q^3 t) + 2/(q t) + t + 2 q^2 t + q^4 t^2 + 1 + 2/q^2 + 2 + q + 1/(q^6 t^3) + 2/(q^5 t^2) + 1/(q^4 t^2) + 1/(q^2 t^2) + 1/t + 1/(q^4 t) + 2/(q^3 t) + 2/(q t) + t + q^2 t + q^3 t + q^3 t^2 + q^4 t^2}]  

Good news: Example1 and Example2 have the same virtual Jones polynomial.
(\(t = -1\)), but different virtual Khovanov homology, i.e. Example2 has the six extra terms (compared to Example1) \(q^2 t, q^2 t^2, q^3, q^6 t^2\) and \(q^6 t^3\). They all cancel if we substitute \(t = -1\). An analogous effect happens for Example3 and Example4.

Furthermore our calculations suggest that this repeats frequently for v-knots with seven or more crossings.

The command line GausstoCD converts signed Gauss Code to a CD representation. The signed Gauss code has to start with the first overcrossing. To get the mirror image we can use the rule from below. For example the virtual trefoil and its mirror are not equivalent.

![Figure 28: Homology of the v-trefoil.](image1)

![Figure 29: Homology of the v-knot 4.1.](image2)

\[
\begin{align*}
\text{In}[35]:= & \quad \text{Knot21gauss} := \text{Ö}1-\text{O}2-\text{U}1-\text{U}2-; \\
\text{In}[36]:= & \quad \text{GausstoCD}[\text{Knot21gauss}] \\
\text{Out}[36]= & \quad \text{CD}[\text{X}[1, 4, 2, 3], \text{X}[2, 1, 3, 4]] \\
\text{In}[37]:= & \quad \text{GausstoCD}[\text{GausstoCD}[\text{Knot21gauss}]] /\text{.} \quad \text{X}[\text{i}, \text{j}, \text{k}, \text{l}] \text{;} \quad \text{X}[\text{i}, \text{l}, \text{k}, \text{j}] \\
\text{Out}[37]= & \quad \text{CD}[\text{X}[1, 3, 2, 4], \text{X}[2, 4, 3, 1]] \\
\text{In}[38]:= & \quad \{\text{vKh}[\text{GausstoCD}[\text{Knot21gauss}]], \\
& \quad \text{vKh}[\text{GausstoCD}[\text{Knot21gauss}]] /\text{.} \quad \text{X}[\text{i}, \text{j}, \text{k}, \text{l}] \text{;} \quad \text{X}[\text{i}, \text{l}, \text{k}, \text{j}]\} \\
\text{Out}[38]= & \quad \{q + q^{-3} + q^{-2} t + q^{6} t^{-2}, 1/q^{-3} + 1/q + 1/(q^{-6} t^{-2}) + 1/(q^{-2} t)\}
\end{align*}
\]

We used this to calculate the virtual Khovanov homology for all different v-knots with less or equal five crossings. The input was the list of v-knots from the virtual knot table (see [3]). The results are available in the data on the arXiv. One could visualise the polynomial with the function \texttt{Ployplot}. It creates an output like in the figures 28, 29.
The output of this v-knot and the output for the mirror of the virtual trefoil is pictured in the figures 28, 29. In these pictures the quantum grade is on the y-axis and the homology grade is on the x-axis.

Open issues

Here are two open problems which we observed:

• Our complex is an extension of the classical (even) Khovanov complex. We shortly discuss a method which should lead to an extension of odd Khovanov homology (with the structure of an exterior algebra, see P. Ozsvath and J. Rasmussen and Z. Szabo article [20]). Even and odd Khovanov homology differ over $\mathbb{Q}$ but are equal over $\mathbb{Z}/2$;

• secondly we discuss the relationship between the virtual Khovanov complex and the categorification of the higher quantum polynomials ($n \geq 3$) from M. Khovanov in [12] and M. Mackaay and P. Vaz in [17] and M. Mackaay, M. Stosic and P. Vaz in [16].

On the first point: the reader familiar with the paper from P. Ozsvath and J. Rasmussen and Z. Szabo (see [20]) may have already identified our map $F_{Kh}(\Phi^+_\pm \circ \Delta^+_{\pm}) : \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$ to be the comultiplication which they use.

One main difference between the even and odd Khovanov complex is the usage of this map instead of the standard map $F_{Kh}(\Delta^+_{\pm})$ and the structure of an exterior algebra instead of direct sums. Furthermore there are commutative and anticommutative faces in the odd Khovanov complex. But because every cube has an even number off both types of faces, there is a sign assignment which makes every face anticommute.

There is a concept of A. Beliakova and E. Wagner (see [4]). They use a category called $\text{OddCob}^2$ to describe the odd Khovanov complex in terms of cobordisms. This category is an extension of the category $\text{Cob}^2$ and they show how to construct an extended TQFT which leads exactly to the odd Khovanov complex.

It should be possible to do an analogue construction in our case. The starting category should be our category $\text{uCob}^2$.

One major problem is the question how to handle unorientable faces, because these faces can be counted as commutative or anticommutative. Furthermore one should admit that faces of type 1a and 1b can be commutative or anticommutative. Hence there is still much work to do.

One the second point: the key idea in the categorification of the $\mathfrak{sl}(n)$-polynomial for $n \geq 3$ is the usage of so-called foams, i.e. cobordisms with singularities between trivalent graphs. This very interesting approach is from M. Khovanov, M. Mackaay, M. Stosic and P. Vaz (see in their papers [12], [17] and [16]).

So in the virtual case one should use of a geometrical construction with virtual webs and decorated, possible non orientable foams (immersed rather then embedded). So their concept to categorify the $\mathfrak{sl}(n)$-polynomials for $n = 3$ should lift for
the case of v-links. This needs further work (the sign assignment seem to be the main point), but seems to be very interesting.

The $n > 3$ case is indeed more complicated. In their paper M. Mackaay, M. Stosic and P. Vaz (see [11]) use a special formula, the so-called Kapustin-Li formula, to find the adapted relations. But this formula only works in the orientable case and it has no straightforward extension to the non orientable case.

But hopefully the collection of relations they use is already enough to show invariance under the vRM1, vRM2, vRM3 and the mRM moves. At least in the case $n = 2$ the local relations are enough to show the invariance.

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49
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