Random Multiple Operator Integrals

Shih Yu Chang *

October 19, 2022

Abstract

The introduction of Schur multipliers into the context of Double Operator Integrals (DOIs) was proposed by V. V. Peller in 1985. This work extends theorem on Schur multipliers from measurable functions to their closure space and generalizes the definition of DOIs to Multiple Operator Integrals (MOIs) for integrand functions as Schur multipliers constructible by taking the limit of projective tensor product and by taking the limit of integral projective tensor product. According to such closure space construction for integrand functions, we demonstrate that any function defined on a compact set of a Euclidean space can be expressed by taking the limit of the projective tensor product of linear functions. We also generalize previous works about random DOIs with respect to finite dimensional operators, tensors, to MOIs with respect to random operators, which are defined from spectral decomposition perspectives. Based on random MOIs definitions and their properties, we derive several tail bounds for norms of higher random operator derivatives, higher random operator difference and Taylor remainder of random operator-valued functions.

Index terms— Schur multiplier, Double Operator Integrals (DOIs), Multiple Operator Integrals (MOIs), random operators, tail bounds.

1 Introduction

Basic concepts about Double Operator Integrals (DOIs) were first mentioned in [1] by considering integration and differentiation of functions of Hermitian operators and their applications to the theory of perturbations. It was Birman and Solomyak who established later the theory of double operator integrals [2,4]. Let $(\Lambda_1, A_1)$ and $(\Lambda_2, A_2)$ be spaces with spectral measures $A_1$ and $A_2$ on a Hilbert space $\mathcal{H}$. Given a bounded measurable function $\psi$ and a operator $X$ on Hilbert space $\mathcal{H}$, the DOI is defined as

$$\int_{\Lambda_1} \int_{\Lambda_2} \psi(\lambda_1, \lambda_2) dA_1(\lambda_1) X dA_2(\lambda_2),$$

where $\psi$ is named as an integrand function. We use conventional notation $\mathcal{S}_p$ to represent the $p$-th Schatten ideal. For every $X \in \mathcal{S}_1$, we say that the integrand function $\psi$ is a Schur multiplier of $\mathcal{S}_1$ associated with the spectral measures $\Lambda_1$ and $\Lambda_2$ if we have

$$\int_{\Lambda_1} \int_{\Lambda_2} \psi(\lambda_1, \lambda_2) dA_1(\lambda_1) X dA_2(\lambda_2) \in \mathcal{S}_1.$$
Note that the introduction of Schur multipliers into the context of double operator integrals was proposed by [5]. This is an extension of the notion of matrix Schur multipliers. The first contribution of this work is to extend the Schur multipliers [6] from measurable functions to their closure space. We also extend the definition of DOIs to Multiple Operator Integrals (MOIs) for integrand as Schur multipliers constructible by the \textit{limit of projective tensor product}, \( \hat{\otimes} \) and by the \textit{limit of integral projective tensor product}, \( \hat{\otimes} \). Therefore, several Lemmas discussed in this work about MOIs basic properties, e.g., continuity, perturbation, and higher order derivative representations by MOIs, will be generalized to functions constructible by taking the \textit{limit of projective tensor product} (\( \hat{\otimes} \)) of measurable functions, or by taking the \textit{limit of integral projective tensor product} (\( \hat{\otimes} \)) of measurable function [7]. According to such closure space construction for integrand functions, we will be able to represent any function \( f(\lambda_1, \lambda_2, \cdots, \lambda_m) \) defined on a compact set of a Euclidean space \( \mathbb{R}^m \) by the \textit{limit of projective tensor product} of linear functions.

The linear operators \( A_1, A_2 \) and \( X \) adopted by Eq. (1) are assumed to be deterministic. The consideration of linear operators under randomness settings is discussed by Skorohod in [8]. Two senses of linear random operators are defined there: strong random operators and weak random operators. Hackenbroch generalizes the concept of a strong random operator by replacing the index Hilbert space by a fixed dense subspace [9]. In this work, he proves that the operators with a densely defined adjoint have a unique closed extension with a measurable selection, particularly, including symmetric operators. The classes of random operators considered are all restricted by assuming that there is a dense nonrandom subspace, but the results are extensions of the results on self-adjoint extensions obtained by [8]. Thang and Quy present results on strongly random operators and bounded strongly random operators between separable Banach spaces [10]. They show that a bounded strongly random operator can be extended to a continuous linear operator from the set of Banach valued random variables to the set of Banach valued random operators equipped with the topology from convergence in probability. Thang and Quy prove several versions of the spectral theorem, including bounded self adjoint, and more generally bounded normal strongly random operators. These results amplify the results obtained by [8]. Although several existing works about random DOIs for finite dimensional operators, e.g., tensors, have been studied [11,12], we first attempt to consider random operators with MOIs in this work. However, the randomness of linear operators is equipped with a different randomness structure compared to aforementioned random linear operators. All operators considered in this work are assumed to have spectral decomposition characterized by eigenvalues and projector spaces (unitary operators formed by eigenspaces), then the randomness of a given linear operator is determined by the random variables of eigenvalues and random unitary operators with Haar measure.

Besides completeness for theorem on Schur multipliers, the extension definition of MOIs, integrand approximation by linear functions, our other contributions include following: derivation properties of MOIs when the integrand functions constructible by the \textit{limit of projective tensor product}, \( \hat{\otimes} \) and by the \textit{limit of integral projective tensor product}, \( \hat{\otimes} \); definition about random MOIs from spectral decomposition perspectives; application MOIs to derive tail bounds for higher random operator derivatives, higher random operator difference and Taylor remainder of random operator-valued functions.

The remainder of this paper is organized as follows. In Section 2 we discuss the completeness for Theorem on Schur multipliers and define MOIs with the completeness of integrand functions. We represent any function \( f(\lambda_1, \lambda_2, \cdots, \lambda_m) \) defined on a compact set of a Euclidean space \( \mathbb{R}^m \) by \textit{limit of projective tensor product} of linear functions in Section 3. We will discuss MOI properties and random MOI for the integrand function constructible by \textit{limit of integral projective tensor product} in Section 4. Finally, we will apply MOIs to derive tail bounds for higher random operator derivatives, higher random operator difference and Taylor remainder of random operator-valued functions in Section 5.
2 Completeness for Theorem on Schur multipliers

The purpose of this section is to define MOI with the completeness of integrand functions.

2.1 Double Operator Integrals

We use $\Gamma(A_1, A_2)$ to represent the space of Schur multipliers of $\mathcal{S}_1$ associated with the spectral measures $\Lambda_1$ and $\Lambda_2$. If the integrand function $\psi$ in Eq. (1) belongs to the projective tensor product as $L^\infty(A_1) \otimes L^\infty(A_2)$, i.e., the integrand function $\psi$ can be expressed as

$$\psi(\lambda_1, \lambda_2) = \sum_{n \geq 0} f_{1,n}(\lambda_1)f_{2,n}(\lambda_2),$$

where $f_{1,n} \in L^\infty(A_1)$, $f_{2,n} \in L^\infty(A_2)$ and $\sum_{n \geq 0} \|f_{1,n}(\lambda_1)\|_{L^\infty} \|f_{2,n}(\lambda_2)\|_{L^\infty} < \infty$; we have $\psi \in \Gamma(A_1, A_2)$. Then, for such integrand function $\psi$, we have

$$\int_{\Lambda_1} \int_{\Lambda_2} \psi(\lambda_1, \lambda_2)dA_1(\lambda_1)dA_2(\lambda_2) \in \mathcal{S}_1 = \sum_{n \geq 0} \left( \int_{\Lambda_1} f_{1,n}(\lambda_1)dA_1(\lambda_1) \right) X \left( \int_{\Lambda_2} f_{2,n}(\lambda_2)dA_2(\lambda_2) \right).$$

(4)

Instead the summation form provided by Eq. (3), the integrand function $\psi$ belongs to the integral projective tensor product as $L^\infty(A_1) \otimes L^\infty(A_2)$, i.e., the integrand function $\psi$ can be expressed as

$$\psi(\lambda_1, \lambda_2) = \int_{\Xi} f_1(\lambda_1, x)f_2(\lambda_2, x)d\mu(x),$$

(5)

where $(\Xi, \mu)$ is a measure space, $f_1$ is a measurable function on $\Lambda_1 \times \Xi$, $f_2$ is a measurable function on $\Lambda_2 \times \Xi$, and

$$\int_{\Xi} \|f_1(\cdot, x)\|_{L^\infty(A_1)} \|f_2(\cdot, x)\|_{L^\infty(A_2)} d\mu(x) < \infty.$$  

(6)

Then, for such integrand function $\psi$, we have

$$\int_{\Lambda_1} \int_{\Lambda_2} \psi(\lambda_1, \lambda_2)dA_1(\lambda_1)dA_2(\lambda_2) \in \mathcal{S}_1 = \int_{\Xi} \left( \int_{\Lambda_1} f_1(\lambda_1, x)dA_1(\lambda_1) \right) X \left( \int_{\Lambda_2} f_2(\lambda_2, x)dA_2(\lambda_2) \right) d\mu(x).$$

(7)

Let $\Lambda_1, \Lambda_2$ be compact sets, and, for any $x$, suppose we have

$$\mathcal{F}_1(\lambda_1, x) = \lim_{n \to \infty} f_{1,n}(\lambda_1, x)$$

$$\mathcal{F}_2(\lambda_2, x) = \lim_{n \to \infty} f_{2,n}(\lambda_2, x)$$

(8)

where $f_{1,n}$ is a measurable function on $\Lambda_1 \times \Xi$, and $f_{2,n}$ is a measurable function on $\Lambda_2 \times \Xi$. We also assume that

$$\int_{\Xi} \|\mathcal{F}_1(\cdot, x)\|_{L^\infty(A_1)} \|\mathcal{F}_2(\cdot, x)\|_{L^\infty(A_2)} d\mu(x) < \infty.$$  

(9)
We define the function \( \psi_n(\lambda_1, \lambda_2) \) as
\[
\psi_n(\lambda_1, \lambda_2) = \int_{\Xi} f_{1,n}(\lambda_1, x) f_{2,n}(\lambda_2, x) d\mu(x).
\] (10)

Then, the limiting function of \( \psi_n(\lambda_1, \lambda_2) \), denoted by \( \bar{\psi}(\lambda_1, \lambda_2) \), can be expressed as
\[
\bar{\psi}(\lambda_1, \lambda_2) = \lim_{n \to \infty} \psi_n(\lambda_1, \lambda_2).
\] (11)

If functions \( f_{1,n}, f_{2,n} \) are increasing with \( n \), or they are dominated by some integrable functions, we have
\[
\bar{\psi}(\lambda_1, \lambda_2) = \lim_{n \to \infty} \psi_n(\lambda_1, \lambda_2) = \lim_{n \to \infty} \int_{\Xi} f_{1,n}(\lambda_1, x) f_{2,n}(\lambda_2, x) d\mu(x)
\]
\[
= \int_{\Xi} \lim_{n \to \infty} (f_{1,n}(\lambda_1, x) f_{2,n}(\lambda_2, x)) d\mu(x)
\]
\[
= \int_{\Xi} \bar{f}_1(\lambda_1, x) \bar{f}_2(\lambda_2, x) d\mu(x).
\] (12)

We can summarize the above completeness arguments with Theorem on Schur multipliers given by [6] to have the following new theorem after taking the limit of the integrand function.

**Theorem 1** Let \( \bar{\psi} \) be a measurable function of \( \Lambda_1 \times \Lambda_2 \), the following statements are equivalent:

1. \( \bar{\psi} \in \Gamma(A_1, A_2) \);
2. \( \bar{\psi} \in L^\infty(A_1) \hat{\otimes} L^\infty(A_2) \);
3. there exist measurable functions \( \bar{f}_1 = \lim_{n \to \infty} f_{1,n}(\lambda_1, x) \) is a measurable function on \( \Lambda_1 \times \Xi \), and \( \bar{f}_2 = \lim_{n \to \infty} f_{2,n}(\lambda_2, x) \) is a measurable function on \( \Lambda_2 \times \Xi \) such that Eq. (12) holds and
\[ \int_{\Xi} \| \bar{f}_1(\cdot, x) \|_{L^\infty(A_1)} \| \bar{f}_2(\cdot, x) \|_{L^\infty(A_2)} d\mu(x) < \infty. \]

**Remark 1** If we can find functions \( \bar{f}_{1,n}(\lambda_1) = \lim_{m \to \infty} f_{1,n,m}(\lambda_1) \) and \( \bar{f}_{2,n}(\lambda_2) = \lim_{m \to \infty} f_{2,n,m}(\lambda_2) \) such that functions \( f_{1,n,m}, f_{2,n,m} \) are increasing with \( m \), or they are dominated by some integrable functions, Theorem 1 will still be true by replacing the statement 3 with
\[
\bar{\psi}(\lambda_1, \lambda_2) = \sum_{n \geq 0} \bar{f}_{1,n}(\lambda_1) \bar{f}_{2,n}(\lambda_2),
\] (13)
and
\[
\sum_{n \geq 0} \| \bar{f}_{1,n} \|_{L^\infty(A_1)} \| \bar{f}_{2,n} \|_{L^\infty(A_2)} < \infty.
\]

If a given bivariate function can be expressed by the right-hand side of Eq. (13), this function is said constructible by the *limit of projective tensor product*, \( \hat{\otimes} \). On the other hand, if a given bivariate function can be expressed by the right-hand side of Eq. (12), this function is said constructible by the *limit of integral projective tensor product*, \( \hat{\otimes} \).
2.2 Multiple Operator Integrals

We can easily extend the definition of the limit of projective tensor product and the limit of integral projective tensor product to three or more function spaces. We have following two multiple operator integrals for the integrand function constructible by limit of projective tensor product, \( \otimes \) and by limit of integral projective tensor product, \( \overline{\otimes} \). We use \( X^j \) to represent the sequence of operators \( A_1, A_{i+1}, \ldots, A_j \), similarly, we apply \( X^j \) to represent the sequence of operators \( X_i, X_{i+1}, \ldots, X_j \).

**Definition 1** Let \((\Lambda_1, A_1), (\Lambda_2, A_2), \ldots, (\Lambda_m, A_m)\) be spaces with spectral measures \( A^m_1 \) on a Hilbert space \( \mathcal{H} \). Given a bounded measurable function \( \psi_\otimes(\lambda_1, \lambda_2, \ldots, \lambda_m) \) \((m\text{-variables})\) constructible by the limit of projective tensor product, and a set of \((m-1)\) operators \( X^m_{i=1} \) on Hilbert space \( \mathcal{H} \), the Multiple Operator Integrals(MOIs), denoted as \( T^{X^m_{i=1}}_{\psi_\otimes} \), is defined as

\[
T^{X^m_{i=1}}_{\psi_\otimes} (X^m_{i=1}) \overset{def}{=} \int_{\Lambda_1} \cdots \int_{\Lambda_m} \psi_\otimes(\lambda_1, \lambda_2, \ldots, \lambda_m) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m),
\]

where \( \psi_\otimes(\lambda_1, \lambda_2, \ldots, \lambda_m) \) can be expressed as

\[
\psi_\otimes(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{n\geq 0} T_{1,n}(\lambda_1) T_{2,n}(\lambda_2) \cdots T_{m,n}(\lambda_m).
\]

**Definition 2** Let \((\Lambda_1, A_1), (\Lambda_2, A_2), \ldots, (\Lambda_m, A_m)\) be spaces with spectral measures \( A^m_1 \) on a Hilbert space \( \mathcal{H} \). Given a bounded measurable function \( \psi_\overline{\otimes}(\lambda_1, \lambda_2, \ldots, \lambda_m) \) \((m\text{-variables})\) constructible by the limit of integral projective tensor product, and a set of \((m-1)\) operators \( X^m_{i=1} \) on Hilbert space \( \mathcal{H} \), the Multiple Operator Integrals(MOIs), denoted as \( T^{X^m_{i=1}}_{\psi_\overline{\otimes}} \), is defined as

\[
T^{X^m_{i=1}}_{\psi_\overline{\otimes}} (X^m_{i=1}) \overset{def}{=} \int_{\Lambda_1} \cdots \int_{\Lambda_m} \psi_\overline{\otimes}(\lambda_1, \lambda_2, \ldots, \lambda_m) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m),
\]

where \( \psi_\overline{\otimes}(\lambda_1, \lambda_2, \ldots, \lambda_m) \) can be expressed as

\[
\psi_\overline{\otimes}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \int_{\mathcal{X}} T_{1}(\lambda_1, x) T_{2}(\lambda_2, x) \cdots T_{m}(\lambda_m, x) d\mu(x).
\]

Below, we will extend our previous work about DOI definition of Hermitian tensors to MOI definition based on Definitions 1 or 2. We use the symbol \( \lambda \) to represent the multiplication operation between two tensors with \( k \) common indices \( \{13\} \). From Theorem 3.2 in \( \{13\} \), every Hermitian tensor \( \mathcal{H} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) has the following decomposition

\[
\mathcal{H} = \sum_{i=1}^{I^N} \lambda_i \mathcal{U}_i \mathcal{U}_i^H \quad \text{with} \quad \langle \mathcal{U}_i, \mathcal{U}_i \rangle = 1 \quad \text{and} \quad \langle \mathcal{U}_i, \mathcal{U}_j \rangle = 0 \quad \text{for} \quad i \neq j,
\]

\[
\mathcal{H} \overset{def}{=} \sum_{i=1}^{I^N} \lambda_i \mathcal{P}_{\mathcal{U}_i}
\]
where $\mathcal{U}_i \in \mathbb{C}^{I_1 \times \cdots \times I_N \times 1}$, and the tensor $\mathcal{P}_{i}$ is defined as $\mathcal{U}_i \ast_1 \mathcal{U}_i^H$. The values $\lambda_i$ are named as eigenvalues. A Hermitian tensor with the decomposition shown by Eq. (18) is named as eigen-decomposition. A Hermitian tensor $\mathcal{H}$ is a positive definite (or positive semi-definite) tensor if all its eigenvalues are positive (or nonnegative).

Let $\mathcal{A}_i \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ for $i = 1, 2, \cdots, m$ be $m$ Hermitian tensors with the following eigen-decompositions:

$$A_i = \sum_{j=1}^{\|N\|} \lambda_{i,j} \mathcal{U}_{i,j} \ast_1 \mathcal{U}_{i,j}^H \equiv \sum_{j=1}^{\|N\|} \lambda_{i,j} \mathcal{P}_{U_{i,j}}. \tag{19}$$

We define multiple tensor integrals (MTIs) with respect to tensors $\mathcal{A}_i$ and the function $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$ which is either constructible by the limit of projective tensor product or by the limit of integral projective tensor product, denoted as $T_{\psi}^{-\mathcal{A}_i^m} (\mathcal{A}_i^{m-1})$, which can be expressed as

$$T_{\psi}^{-\mathcal{A}_i^m} (\mathcal{A}_i^{m-1}) = \sum_{i_1=1}^{\|N\|} \sum_{i_2=1}^{\|N\|} \cdots \sum_{i_m=1}^{\|N\|} \psi(\lambda_{1,1}, \lambda_{1,2}, \cdots, \lambda_{1,m}) \mathcal{P}_{U_{i_1,1}} \ast_1 \mathcal{A}_1 \ast_2 \mathcal{P}_{U_{i_2,1}} \ast_1 \mathcal{A}_2 \cdots \mathcal{P}_{U_{i_m,1}} \ast_1 \mathcal{A}_{m-1} \ast_2 \mathcal{P}_{U_{i_m,m}}, \tag{20}$$

where $\mathcal{A}_i \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ for $i = 1, 2, \cdots, m - 1$.

### 3 Integrand Approximation By Linear Functions

In this section, we will show that it is possible to represent any function $f(\lambda_1, \lambda_2, \cdots, \lambda_m)$ defined on a compact set of a Euclidean space $\mathbb{R}^m$ by the limit of projective tensor product of linear functions.

We begin with the following theorem from [14] which says that any multivariate polynomial in $\mathbb{R}^m$ can be expressed by a polynomial of inner product forms.

**Theorem 2** Let $\mathcal{P}_k^m$ denote the set of all polynomials in $\mathbb{R}^m$ with polynomial degree at most $k$, and let $x^{j_{i,d}} = x_1^{j_{i,d,1}} x_2^{j_{i,d,2}} \cdots x_m^{j_{i,d,m}}$ such that $j_{i,d,1} + j_{i,d,2} + \cdots + j_{i,d,m} = i$ for any $d$. For any polynomial in $\mathcal{P}_k^m$, we can express it as

$$p(x) = \sum_{i=0}^{k} \sum_{d=1}^{n_i} a_{i,d} x^{j_{i,d}}, \tag{21}$$

where coefficients $a_{i,d} \in \mathbb{R}$ and $n_i = \binom{m+i-1}{i}$. Note that the number $n_i$ indicates the total number of $i$-th degree polynomials in $m$ variables. Then, there exist a set of coefficients $c_{i,d} \in \mathbb{R}$ and a set of real vectors $v_{i,d} \in \mathbb{R}^m$ such that Eq. (21) can be expressed as:

$$p(x) = \sum_{i=0}^{k} \sum_{d=1}^{n_i} c_{i,d} \langle x, v_{i,d} \rangle^i. \tag{22}$$

Following theorem will apply Theorem 2 to show that any multivariate function $f(\lambda_1, \lambda_2, \cdots, \lambda_m)$ defined on a compact set of a Euclidean space $\mathbb{R}^m$ can be approximated by the limit of projective tensor product of linear functions.
Theorem 3  Given a continuous function \( f(x) \) with \( x \in \mathbb{R}^m \) defined on a compact set \( C \subset \mathbb{R}^m \) and any \( \epsilon > 0 \), we can find the following polynomial \( p(x) \) for all \( x \in C \) expressed as

\[
p(x) = \sum_{i=1}^{N} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle,
\]

where \( u_{i,j} \in \mathbb{R}^{m+1} \) and \( \bar{x} = [x, 1] \); such that

\[
|f(x) - p(x)| < \epsilon.
\]

(24)

Note that \( N = \sum_{i=0}^{k} n_i \), where \( n_i \) comes from the expression for \( p(x) \) provided by Eq. (21).

Proof:

Our first goal is to show that \( p(x) \in \mathcal{P}_k^m \) shown by Eq. (21) can be expressed as

\[
p(x) = \sum_{i=1}^{N} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle,
\]

(25)

where \( N = \sum_{i=0}^{k} n_i \) and \( u_{i,j} \in \mathbb{R}^{m+1} \).

From Theorem 2, we have

\[
p(x) = \sum_{i=0}^{k} n_i \sum_{d=1}^{i} c_{i,d}(x, v_{i,d})^i
\]

\[
= \sum_{i=1}^{n_0} (\langle x, 0 \rangle + c_{0,i})(\langle x, 0 \rangle + 1)^{i-1}
\]

\[+ \sum_{i=n_0+1}^{n_0+n_1} \langle x, c_{1,i-n_0}v_{1,i-n_0} \rangle + 0)(\langle x, 0 \rangle + 1)^{i-1}
\]

\[+ \cdots
\]

\[+ \sum_{i=n_0+\cdots+n_k} \left[ \langle x, c_{k,i-(n_0+\cdots+n_{k-1})}v_{k,i-(n_0+\cdots+n_{k-1})} \rangle + 0 \right]
\]

\[\cdot (\langle x, v_{k,i-(n_0+\cdots+n_{k-1})} \rangle + 0)^{k-1}(\langle x, 0 \rangle + 1)^{i-k} \].

(26)

On the other hand, from Eq. (25), we can express \( p(x) \) as

\[
p(x) = \sum_{i=1}^{N} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle
\]

\[
= \sum_{i=1}^{n_0} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle
\]

\[+ \sum_{i=n_0+1}^{n_0+n_1} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle + \cdots
\]

\[+ \sum_{i=n_0+\cdots+n_k} \prod_{j=1}^{i} \langle \bar{x}, u_{i,j} \rangle.
\]

(27)
Because the expression in Eq. (26) is a special case of Eq. (27), we have the expression of polynomial $p(x) \in \mathbb{R}[x]$ indicated by Eq. (22). Finally, this theorem is proved from generalized Weierstrass polynomial approximation Theorem [15].

## 4 MOIs Properties and Random MOIs

In this section, we will discuss MOI properties and random MOI for the integrand function constructible by the limit of integral projective tensor product, $\hat{\otimes}$. Same arguments will also be applied to integrand function constructible by the limit of projective tensor product, $\otimes$.

### 4.1 Properties of MOI

#### 4.1.1 Algebraic Properties of MOI

**Theorem 4** Let $(A_1, A_1), (A_1, A_2), \ldots, (A_m, A_m)$ be spaces with spectral measures $A_i^n$ on a Hilbert space $\mathcal{H}$.

(a) Let $\overline{\phi}_1, \overline{\psi}_1 \in L^\infty(A_1) \cdots \otimes L^\infty(A_m)$ and $\alpha, \beta \in \mathbb{C}$, then we have

$$T_{\phi_1}^{A_1^n} (X_1^{m-1}) = \alpha T_{\phi_1}^{A_1^n} (X_1^{m-1}) + \beta T_{\phi_1}^{A_1^n} (X_2^{m-1}).$$

(b) Given $\overline{\psi}_1, \overline{\psi}_2 \in L^\infty(A_1) \otimes L^\infty(A_k)$ and $\overline{\psi}_{1,2} \in L^\infty(A_k) \otimes L^\infty(A_m)$, we define $\overline{\psi}_{1,1} \otimes \overline{\psi}_{1,2} = \overline{\psi}_{1,1} \otimes \overline{\psi}_{1,2} \otimes \overline{\psi}_{1,2} \otimes \overline{\psi}_{1,2}$ then we have

$$\overline{\psi}_{1,1} \otimes \overline{\psi}_{1,2} \in L^\infty(A_1) \otimes \cdots \otimes L^\infty(A_m)$$

and

$$T_{\phi_1}^{A_1^n} (X_1^{m-1}) = T_{\phi_1}^{A_1^n} (X_1^{k-1}) L_k T_{\phi_1}^{A_1^n} (X_2^{m-1}).$$

(c) Let $\overline{\phi}_i \in L^\infty(A_1) \otimes \cdots \otimes L^\infty(A_m)$, and $\overline{\psi}_{i,j} \in L^\infty(A_1) \otimes \cdots \otimes L^\infty(A_j)$ for $i = 1, 2, \ldots, \ell$, be integrand functions composed by the $i$-th partition from $A_i^n$ into $\ell$ segments such that each segment will have at least one element. For example, $(A_1, A_2, A_3, A_4, A_5)$ is partitioned into $(A_1, A_2)$, $(A_3)$, $(A_4, A_5)$. Suppose we have the following relationship between the function $\overline{\phi}_i$ and functions $\overline{\psi}_{i,j}$ for $i = 1, 2, \ldots, \ell$:

$$\overline{\phi}_i (\lambda_1, \ldots, \lambda_m) = \prod_{i=1}^\ell \overline{\psi}_{i,j} (\lambda_i),$$

where $\lambda_i$ are those $j_i$ eigenvalues corresponding to spectral measures $A_{i,1}, \ldots, A_{i,j_i}$. Then, we have

$$T_{\phi_1}^{A_1^n} (X_1^{m-1}) = \left( \prod_{i=1}^{\ell-1} T_{\psi_{i,j_i}}^{A_{i,j_i}} (X_{j_i-1}^{j_i}) \right) T_{\psi_{j_0}}^{A_{j_0+1}} (X_{j_0}^{j_1}) \prod_{i=1}^{\ell-1} \overline{\psi}_{i,j_i} (X_{j_i}^{j_i+1}).$$

where $j_0 = 0$ and $j_\ell = m$. 

Proof:
The proof for part (a) is trivial from linearity of integration.

For part (b), because we have
\[
\psi_1 \otimes \lambda_1, \lambda_2, \ldots, \lambda_k = \int_{\Xi_1} f_1(\lambda_1, x) f_2(\lambda_2, x) \cdots f_k(\lambda_k, x) d\mu_1(x),
\]
and
\[
\psi_2(\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m) = \int_{\Xi_2} f_{k+1}(\lambda_{k+1}, x) \cdots f_m(\lambda_m, x) d\mu_2(x),
\]
then,
\[
\psi_1 \oplus \psi_2 = \left( \int_{\Xi_1} f_1(\lambda_1, x) f_2(\lambda_2, x) \cdots f_k(\lambda_k, x) d\mu_1(x) \right) \psi_1 \oplus \psi_2
\]
\[
\int_{\Xi_2} f_{k+1}(\lambda_{k+1}, x) \cdots f_m(\lambda_m, x) d\mu_2(x)
\]

Eq. (29) is true from the product measure space.

Since we have
\[
T_{\psi_1}^{\Delta_m}(X_{k}^{m-1}) = \int \cdots \int \psi_1(\lambda_1, x_1) \cdots \psi_2(\lambda_m, x_m) d\mu_1(\lambda_1) d\mu_2(\lambda_m)
\]
\[
= \left( \int \psi_1(\lambda_1, x) d\mu_1(\lambda_1) \right) \cdots \left( \int \psi_2(\lambda_m, x) d\mu_2(\lambda_m) \right) X_k
\]
\[
= \left( \int \psi_1(\lambda_1, x) d\mu_1(\lambda_1) \right) \cdots \left( \int \psi_2(\lambda_m, x) d\mu_2(\lambda_m) \right) X_k
\]
\[
= T_{\psi_1}(X_{k}^{m-1}) X_k T_{\psi_2}^{\Delta_{m+1}}(X_{k+1}^{m-1}).
\]
Therefore, we have Eq. (30).
Lemma 1

Following lemma will provide two norm estimations of MOIs.

\[ T_{\Phi}^m(A_1^{m-1}) = \int_{A_1} \cdots \int_{A_m} \bar{\psi}_{\otimes} dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m) \]

\[ = \int_{A_1} \cdots \int_{A_m} \left( \prod_{i=1}^{\ell} \bar{\psi}_{\otimes,i}(\Delta_i) \right) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m) \]

\[ = \left( \int_{A_{j_1+1}} \cdots \int_{A_j} \bar{\psi}_{\otimes,1}(\Delta_1) dA_1(\lambda_1) X_1 \cdots X_{j_1-1} dA_j(\lambda_{j_1}) \right) X_{j_1} \]

\[ \cdot \left( \int_{A_{j_1+1}} \cdots \int_{A_j} \bar{\psi}_{\otimes,2}(\Delta_2) dA_{j+1}(\lambda_{j_1+1}) X_{j_1+1} \cdots X_{j_2-1} dA_{j_2}(\lambda_{j_2}) \right) X_{j_2} \]

\[ \cdots \left( \int_{A_{j_{\ell-1}+1}} \cdots \int_{A_j} \bar{\psi}_{\otimes,\ell}(\Delta_{\ell}) dA_{j_{\ell-1}+1}(\lambda_{j_{\ell-1}+1}) X_{j_{\ell-1}+1} \cdots X_{j_\ell-1} dA_{j_\ell}(\lambda_{j_\ell}) \right) \]

\[ = \left( \prod_{i=1}^{\ell-1} T_{\Phi,i}^{A_{j_{i+1}}}(X_{j_{i+1}}^{j_i}) \right) T_{\Phi,\ell}^{A_{j_\ell}}(X_{j_\ell-1}^{j_\ell}) \]

where we apply Eq. (31) to \( = 1 \) and set \( j_0 = 1 \). Note that \( j_\ell = m \). \( \square \)

4.1.2 Norm Estimations for MOIs

Following lemma will provide two norm estimations of MOIs.

**Lemma 1** According to Definition [2] we have the following norm estimations for Eq. (16):

(a) Given a bounded measurable function \( \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) \) \((m\text{-variables})\) constructible by limit of integral projective tensor product, and a set of \((m-1)\) bounded linear operators \( X_1, X_2, \cdots, X_{m-1} \) on Hilber space \( \mathcal{H} \), we have

\[ \left\| \int_{A_1} \cdots \int_{A_m} \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m) \right\| \]

\[ \leq \left\| \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \prod_{i=1}^{m-1} \| X_i \| . \] (38)

(b) Given a bounded measurable function \( \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) \) \((m\text{-variables})\) constructible by limit of integral projective tensor product, and a set of \((m-1)\) bounded linear operators \( X_1, X_2, \cdots, X_{m-1} \) on Hilber space \( \mathcal{H} \), we have

\[ \left\| \int_{A_1} \cdots \int_{A_m} \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m) \right\| \]

\[ \leq \left\| \bar{\psi}_{\otimes}(\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \prod_{i=1}^{m-1} \| X_i \| . \] (38)
on Hilbert space $H$ such that $X_i \in S_p$, where $S_p$ is a Schatten–von Neumann class with parameter $p$, we also require that $\sum_{i=1}^{m} \frac{1}{p_i} \leq 1$. Then, we have

\[
\left\| \int_{\lambda_1}^{\lambda_2} \cdots \int_{\lambda_m}^{\lambda_m} \nu_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m) dA_1(\lambda_1) X_1 dA_2(\lambda_2) X_2 \cdots X_{m-1} dA_m(\lambda_m) \right\|_{S_q} \\
\leq \left\| \nu_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \prod_{i=1}^{m-1} \left\| X_i \right\|_{S_{p_i}},
\]

(39)

where $\frac{1}{q} = 1 - \sum_{i=1}^{m} \frac{1}{p_i}$.

**Proof:** By applying triangle inequality for integrals and Hölder’s inequality to Eq. (16). □

### 4.1.3 Perturbation Formula for MOI

Before presenting our perturbation formula for MOI, we have to define divided differences for function $\nu_\otimes(\lambda)$, which is a function constructible by limit of integral projective tensor product, $\otimes$. We assume this function can have $n$ times continuously differentiable on $\mathbb{R}$. The divided difference $\frac{\nu_\otimes^{[n]}}{\nu_\otimes^{[n+1]}}$ of order $n$ with respect to the $j$-th variable is defined by

\[
\frac{\nu_\otimes^{[n+1]}}{\nu_\otimes^{[n]}}(\lambda_0, \cdots, \lambda_{n+1}) \equiv \frac{\nu_\otimes^{[n]}(\lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_{n+1}) - \nu_\otimes^{[n]}(\lambda_0, \cdots, \lambda_j, \cdots, \lambda_{n+1})}{\lambda_j - \lambda_{j-1}}.
\]

(40)

We have the following perturbation formula for MOIs.

**Lemma 2** Let $\nu_\otimes$ be $n$ times differentiable on $\mathbb{R}$ and $C, D, A_1, \cdots, A_{n-1}$ be operators. Then, we have

\[
T^{A_1, C, A_1, \cdots, A_1}_m(X^m) - T^{A_1, D, A_1, \cdots, A_1}_m(X^m) = T^{A_1, C, D, A_1, \cdots, A_1}_m(X^m),
\]

(41)

where $m \leq n - 1$.

**Proof:** For $1 \leq j \leq m + 1$, we define

\[
f_j(\lambda_0, \cdots, \lambda_{m+1}) \equiv \lambda_j \nu_\otimes^{[m+1]}(\lambda_0, \cdots, \lambda_{m+1}),
\]

\[
g_j(\lambda_0, \cdots, \lambda_{m+1}) \equiv \nu_\otimes^{[m]}(\lambda_0, \cdots, \lambda_{j-1}, \lambda_{j+1}, \cdots, \lambda_{m+1}).
\]

(42)

From Eq. (40), we have

\[
f_j - f_{j-1} = g_j - g_{j-1}.
\]

(43)
Then, we have
\[
T_{\psi[m+1]} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, C - D, X_{i}^m)
\]
\[
= T_{\psi[m+1]} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, C, X_{i}^m) - T_{\psi[m+1]} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, D, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m) - T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]
\[
= T_{f_j} A_{\psi[m]}^{-1} \cdot C \cdot D \cdot X_{i}^m (X_{i}^{j-1}, Id, X_{i}^m)
\]

where \( Id \) is the identity operator and \( =_1 \) comes from Eq. (43).

4.1.4 Continuity for MOIs

In the next lemma, we establish continuity of a MOI defined by Definition 2.

**Lemma 3** Let \( \psi^{[n+1]} \) be \( n \) times differentiable on \( \mathbb{R} \) and \( A_i^{(m)} \rightarrow A_i \) as \( m \rightarrow \infty \), i.e., \( \| A_i^{(m)} - A_i \| \rightarrow 0 \) as \( m \rightarrow \infty \). We use \( \overline{A}^{(m)} \) to represent the sequence \( A_i^{(m)}, A_i^{(m)}[1], \cdots, A_i^{(m)}[n] \). Then, we have

\[
T_{\psi^{[n+1]}} A_i^{(m)} \rightarrow T_{\psi^{[n+1]}} A_i \quad \text{as} \quad m \rightarrow \infty,
\]

where \( X_1, X_2, \cdots, X_n \) are bounded operators by assuming that the norm for the function \( \psi^{[n+1]} \) and norms for the operators \( X_1, \cdots, X_n \) are bounded.

**Proof:** Because we have

\[
\| T_{\psi^{[n+1]}} A_i^{(m)} - T_{\psi^{[n+1]}} A_i \| = \sum_{i=1}^{n+1} \left( T_{\psi^{[n+1]}} A_i^{(m)} - T_{\psi^{[n+1]}} A_i \right) \left( X_{i}^{j-1}, A_i^{(m)} - A_i \right) \| X_{i}^{j-1} \| \| X_{i}^{j} \|
\]

where \( =_1 \) comes from telescoping summation and \( =_1 \) comes from Lemma 2

Applying norm triangle inequality and Lemma 1, we can upper bound Eq. (46) as

\[
\| T_{\psi^{[n+1]}} A_i^{(m)} - T_{\psi^{[n+1]}} A_i \| \leq \| \psi^{[n+1]} \| \sum_{i=1}^{n+1} \left( A_i^{(m)} - A_i \right) \| \prod_{j=1}^{n} X_j \|
\]

This theorem is provied when \( m \rightarrow \infty \).
4.2 Random MOIs

From the MOI definition provided by Definition 1 and Definition 2, the random MOI considered in this work is to assume random operators $A_i$ having spectrum decomposition as $\int \lambda_i dA_i(\lambda_i)$, where $i = 1, 2, \cdots, m$.

The randomness of the operator $A_i$ is determined by random variables $\lambda_i$ and random unitary operators $dA_i(\lambda_i)$ via Haar measure. The remaining parameters like the integrand function $\psi_\otimes$ or $\psi_\otimes'$, and operators $X_1, \cdots, X_m-1$ are assumed to be deterministic. The purpose of this section is to derive tail bounds for random MOI norms and establish the continuity property of random MOI.

**Theorem 5** We have the following tail bound for the norm estimations:

(a) Given a random MOI, $T^{m+1}_{\sqrt{m}} (\tilde{X}_m)$, we have

$$\text{Pr} \left( \left\| T^{m+1}_{\sqrt{m}} (\tilde{X}_m) \right\| \geq \theta \right) \leq \frac{m-1}{\theta} \prod_{i=1}^{m-1} \|X_i\| \mathbb{E} \left[ \|\psi_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m)\| \right],$$

where $\theta > 0$.

(b) Given a random MOI, $T^{m+1}_{\sqrt{m}} (X_m)$, we have

$$\text{Pr} \left( \left\| T^{m+1}_{\sqrt{m}} (X_m) \right\| \geq \theta \right) \leq \frac{1}{\theta} \prod_{i=1}^{m-1} \|X_i\| \mathbb{E} \left[ \|\psi_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m)\| \right],$$

where $\frac{1}{q} = 1 - \sum_{i=1}^{m} \frac{1}{p_i}$ and $\theta > 0$.

**Proof:**

For part (a), since we have

$$\text{Pr} \left( \left\| T^{m+1}_{\sqrt{m}} (\tilde{X}_m) \right\| \geq \theta \right) \leq \text{Pr} \left( \|\psi_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m)\| \prod_{i=1}^{m-1} \|X_i\| \geq \theta \right)$$

$$= \text{Pr} \left( \|\psi_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m)\| \geq \frac{\theta}{\prod_{i=1}^{m-1} \|X_i\|} \right)$$

$$\leq 2 \prod_{i=1}^{m-1} \|X_i\| \mathbb{E} \left[ \|\psi_\otimes(\lambda_1, \lambda_2, \cdots, \lambda_m)\| \right],$$

where $\leq_1$ is due to part (a) of Lemma 3 and $\leq_2$ comes from Markov inequality.
For part (b), since we have
\[
\Pr \left( \left\| \frac{T^{m+1}}{\psi_{[m]}} (X) \right\|_{\mathcal{F}} \geq \theta \right) \leq_1 \Pr \left( \left\| \overline{\psi}_\otimes (\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \prod_{i=1}^{m-1} \left\| X_i \right\|_{\mathcal{F}_{pi}} \geq \theta \right)
\]
\[= \Pr \left( \left\| \overline{\psi}_\otimes (\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \geq \frac{\theta}{\prod_{i=1}^{m-1} \left\| X_i \right\|_{\mathcal{F}_{pi}}} \right)
\leq_2 \frac{\prod_{i=1}^{m-1} \left\| X_i \right\|_{\mathcal{F}_{pi}}}{\theta} \mathbb{E} \left[ \left\| \overline{\psi}_\otimes (\lambda_1, \lambda_2, \cdots, \lambda_m) \right\| \right],
\]
(51)
where \(\leq_1\) is due to part (b) of Lemma [II] and \(\leq_2\) comes from Markov inequality.

In the remainder of this section, we will establish continuity of random MOI. We need the following definition about the convergence in mean for random operators.

**Definition 3** We say that a sequence of random operators \(X^{(m)}\) converges in the \(r\)-th mean towards the random operator \(X\) with respect to the operator norm \(\| \cdot \|\), if we have
\[
\mathbb{E} \left( \left\| X^{(m)} \right\|^r \right) \quad \text{exists},
\]
(52)
and
\[
\mathbb{E} \left( \| X \|^r \right) \quad \text{exists},
\]
(53)
and
\[
\lim_{m \to \infty} \mathbb{E} \left( \left\| X^{(m)} - X \right\|^r \right) = 0.
\]
(54)
We adopt the notation \(X^{(m)} \xrightarrow{r} X\) to represent that random operators \(X^{(m)}\) converges in the \(r\)-th mean to the random operator \(X\) with respect to the norm \(\| \cdot \|\).

Following theorem is about the continuity in the \(r\)-th mean of random MOI.

**Theorem 6** For \(i = 1, 2, \cdots, n+1\), we have random self-adjoint operators \(A_{[m]}^{(m)}\), \(A_i\) such that
\[
A_{[m]}^{(m)} \xrightarrow{r} A_i \quad \text{as} \quad m \to \infty,
\]
(55)
where \(1 \leq r < \infty\). Moreover, the norm for the real valued function \(\overline{\psi}_{[n+1]}\) and the norm for \(X_j\) for \(j = 1, 2, \cdots, n\), are assumed bounded. Then, we have
\[
\frac{T^{A_{[m]}^{(m)}n+1}}{\psi_{[n]+1}} (X) \xrightarrow{r} \frac{T^{A^{n+1}}}{\psi_{[n]+1}} (X).
\]
(56)

**Proof:**

From the proof in Lemma [3], we have
\[
\left\| \frac{T^{A_{[m]}^{(m)}n+1}}{\psi_{[n]+1}} (X) - \frac{T^{A^{n+1}}}{\psi_{[n]+1}} (X) \right\| \leq \left\| \overline{\psi}_{[n+1]} \right\| \sum_{i=1}^{n+1} \left( \left\| A_{[m]}^{(m)} - A_i \right\| \prod_{j=1}^{n} \left\| X_j \right\| \right).
\]
(57)
By raising the power $r$ and taking the expectation at the both sides of the inequality provided by Eq. (57), we have proved this theorem by conditions given by Eq. (55) and the following inequality:

$$\left(\sum_{i=1}^{n+1} a_i \right)^r \leq (n + 1)^{r-1} \left(\sum_{i=1}^{n+1} a_i^r \right) \text{ given } a_i \geq 0. \quad (58)$$

5 Random MOIs Applications

In this section, we will apply random MOIs to obtain several tail bounds related to several applications of MOIs. We will focus on functions constructible by the limit of integral projective tensor product, $\hat{T}$. Same arguments will also be applied to functions constructible by the limit of projective tensor product, $\hat{T}^\otimes$.

5.1 Tail Bound for Higher Random Operator Derivative

In this section, we will derive tail bounds for higher random operator derivatives. We begin with a lemma to express the first derivative of an operator-valued function by double operator integrals.

**Lemma 4** Let $X(t)$ be self-adjoint operators indexed by $t$ with bounded norm, i.e., $\|X(t)\| < \infty$, and $\overline{\psi}_\otimes$ be differentiable on $\mathbb{R}$. Then, we have

$$\frac{d}{dt} \overline{\psi}_\otimes(X(t)) = T^{X(t),X(t)}_{\overline{\psi}_\otimes} \left( \frac{dX(t)}{dt} \right). \quad (59)$$

**Proof:** Suppose the function $\overline{\psi}_\otimes(x)$ has the polynomial form as $\overline{\psi}_\otimes(x) = x^m$, by the chain rule of derivative, we have

$$\frac{d}{dt} \overline{\psi}_\otimes(X(t)) = \sum_{i=0}^{m-1} X^i(t) \frac{dX(t)}{dt} X^{m-i-1}(t). \quad (60)$$

Since the operator $X(t)$ can be represented by its spectrum as

$$X(t) = \int_{\lambda_t} \lambda dA_t(\lambda), \quad (61)$$

where $(\lambda_t, A_t)$ is the spectral measure at the instant $t$; by the spectral theorem, we can express Eq. (60) as

$$\frac{d}{dt} \overline{\psi}_\otimes(X(t)) = \sum_{i=0}^{m-1} \int_{\lambda_t} \lambda_1^i dA_t(\lambda_1) \frac{dX(t)}{dt} \int_{\lambda_t} \lambda_2^{m-i-1} dA_t(\lambda_2) \lambda_1^i \lambda_2^{m-i-1} \lambda_1 \frac{dX(t)}{dt} dA_t(\lambda_2)$$

$$= \int_{\lambda_t} \int_{\lambda_t} \left( \sum_{i=0}^{m-1} \lambda_1^i \lambda_2^{m-i-1} \right) dA_t(\lambda_1) \frac{dX(t)}{dt} dA_t(\lambda_2) \lambda_1^i \lambda_2^{m-i-1} \lambda_1 \frac{dX(t)}{dt} dA_t(\lambda_2) \lambda_1 \lambda_2^{m-i-1}$$

$$= \int_{\lambda_t} \int_{\lambda_t} \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} dA_t(\lambda_1) \frac{dX(t)}{dt} dA_t(\lambda_2). \quad (62)$$
From MOIs defined by Eq. (2) as \( m = 2 \) (DOIs), we have Eq. (59) for \( \overline{\psi}_\otimes(x) = x^m \). By linearity, this lemma is also true when \( \overline{\psi}_\otimes(x) \) is a polynomial function.

Suppose we can approximate \( \overline{\psi}_\otimes \) by a sequence of polynomials \( p_n(x) \), i.e., \( \| \overline{\psi}_\otimes - p_n \| \to 0 \) as \( n \to \infty \). Then, we have

\[
\left\| T_{X(t),X(t)} \left( \frac{dX(t)}{dt} \right) \right\| \leq \text{Const} \| \overline{\psi}_\otimes - p_n \| \| X(t) \| .
\]

This lemma is proved due to we can approximate \( \overline{\psi}_\otimes \) by polynomials. □

From Lemma 4, we are ready to have the following theorem about the tail bound for \( \left\| \frac{d\overline{\psi}_\otimes(X(t))}{dt} \right\| \).

**Theorem 7** Let \( X(t) \) be self-adjoint operators indexed by \( t \) with bounded norm, i.e., \( \| X(t) \| < \infty \), and \( \overline{\psi}_\otimes \) be differentiable on \( \mathbb{R} \). We also assume that \( \left\| \frac{dX(t)}{dt} \right\| < \Upsilon_X \). Then,

\[
\Pr \left( \left\| \frac{d\overline{\psi}_\otimes(X(t))}{dt} \right\| > \theta \right) \leq \frac{\Upsilon_X}{\theta} \mathbb{E} \left[ \left\| \overline{\psi}_\otimes \right\| \right].
\]

**Proof:** Because we have

\[
\Pr \left( \left\| \frac{d\overline{\psi}_\otimes(X(t))}{dt} \right\| > \theta \right) = \Pr \left( \left\| T_{X(t),X(t)} \left( \frac{dX(t)}{dt} \right) \right\| > \theta \right)
\]

\[
\leq \Pr \left( \left\| \overline{\psi}_\otimes \right\| \left\| \frac{dX(t)}{dt} \right\| > \theta \right)
\]

\[
= \Pr \left( \left\| \overline{\psi}_\otimes \right\| > \frac{\theta}{\Upsilon_X} \right)
\]

\[
\leq \Pr \left( \left\| \overline{\psi}_\otimes \right\| > \frac{\theta}{\Upsilon_X} \right),
\]

where \( \leq \) comes from Lemma 4. This lemma follows by Markov inequality. □

Following lemma is about the expression of the \( k \)-th operator-valued function by MOIs.

**Lemma 5** Let \( A, B \) be two operators, and \( \overline{\psi}_\otimes \) be \( k \in \mathbb{N} \) order differentiable function, then we have

\[
\frac{d^k \overline{\psi}_\otimes(A + tB)}{dt^k} \bigg|_{t=0} = k! T^{A + tB, \ldots, A + tB}_{\overline{\psi}_\otimes} \left( B, \ldots, B \right).
\]

**Proof:** We will prove this lemma by induction. The base case is proved by Lemma 4 for \( k = 1 \). Suppose
we have Eq. (66) for the $k - 1$-th derivative, then

$$
\left. \frac{d^k \psi_\otimes(A + tB)}{dt^k} \right|_{t=0} = \lim_{t \to 0} \frac{(k - 1)!}{t} \left( T_{\psi_\otimes[k-1]}^A + tB, \cdots, A + tB \right) - \left( T_{\psi_\otimes[k-1]}^A, B, \cdots, B \right) \frac{k}{k-1}
$$

$$
= \lim_{t \to 0} \frac{(k - 1)!}{t} \sum_{i=0}^{k-1} \left( T_{\psi_\otimes[k-1]}^A + tB, \cdots, A + tB, A, \cdots, A \right) - \left( T_{\psi_\otimes[k-1]}^A, B, \cdots, B \right) \frac{k}{k-1}
$$

$$
= 1 \ k! T_{\psi_\otimes[k]}^A + tB, \cdots, A + tB, A, \cdots, A \left( B, \cdots, B \right),
$$

where we apply perturbation formula, Lemma 2 in $= 1$.

From continuity property given by Lemma 3 we have

$$
\lim_{t \to 0} T_{\psi_\otimes[k]}^A + tB, \cdots, A + tB, A, \cdots, A \left( B, \cdots, B \right) = T_{\psi_\otimes[k]}^A, A \left( B, \cdots, B \right).
$$

This lemma is proved from Eq. (67) and Eq. (68).

From Lemma 5 we are ready to have the following theorem about the tail bound for

$$
\left. \frac{d^k \psi_\otimes(A + tB)}{dt^k} \right|_{t=0}.
$$

**Theorem 8** Let $A$ be random operators, $B$ be a determinist operator, and $\psi_\otimes$ be $k \in \mathbb{N}$ order differentiable function, then we have

$$
\Pr \left( \left\| \frac{d^k \psi_\otimes(A + tB)}{dt^k} \right\|_{t=0} > \theta \right) \leq \frac{k! \|B\|^k}{\theta} \mathbb{E} \left[ \|\psi_\otimes[k]\| \right].
$$

**Proof:** Because we have

$$
\Pr \left( \left\| \frac{d^k \psi_\otimes(A + tB)}{dt^k} \right\|_{t=0} > \theta \right) =_1 \Pr \left( \left\| k! T_{\psi_\otimes[k]}^A + tB, \cdots, A + tB \left( B, \cdots, B \right) \right\| > \theta \right)
$$

$$
\leq_2 \Pr \left( k! \|\psi_\otimes[k]\| \|B\|^k > \theta \right)
$$

$$
\leq_3 \frac{k! \|B\|^k}{\theta} \mathbb{E} \left[ \|\psi_\otimes[k]\| \right],
$$

where $= _1$ comes from Lemma 5 $\leq_2$ comes from Lemma 1 and $\leq_3$ comes from Markov inequality again.

$\square$
5.2 Tail Bound for Higher Random Operator Difference

In this section, we will extend the operator first order difference, i.e., $\overline{\psi}_{\otimes}(A) - \overline{\psi}_{\otimes}(A + B)$, to higher order difference. The higher operator difference is defined as

$$\Delta^k_B \overline{\psi}_{\otimes}(A) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \overline{\psi}_{\otimes}(A + iB),$$

(71)

where $A, B$ are self-adjoint operators on Hilbert space. We will have the following theorem about the tail bound for the higher operator difference given by Eq. (71).

**Theorem 9** Let $A$ be a self-adjoint random operator, $B$ be a deterministic self-adjoint operator and the eigenvalue for the operator $A + iB$ is represented by $\lambda_{i+1}$ for $i = 0, 1, \ldots, k$. We assume that $\max(\lambda_{j+1} - \lambda_j) < \kappa$ for $j = 1, 2, \ldots, k$. Then, we have

$$\Pr \left( \| \Delta^k_B \overline{\psi}_{\otimes}(A) \| > \theta \right) \leq \frac{k\kappa \|B\|^k}{\theta} \mathbb{E}(\overline{\psi}_{\otimes}).$$

(72)

**Proof:** From the higher operator difference provided by Eq. (71) and the divide difference definition, we have

$$\Delta^k_B \overline{\psi}_{\otimes}(A) = (-1)^k \Delta^{k-1}_B \overline{\psi}_{\otimes}(A) + (-1)^{k-1} \Delta^{k-1}_B \overline{\psi}_{\otimes}(A + B)$$

$$= \sum_{j=1}^{k} \int \cdots \int (\lambda_{j+1} - \lambda_j) \overline{\psi}^{[k]}_{\otimes}(\lambda_1, \ldots, \lambda_{k+1}) dP_A(\lambda_1)BdP_{A+B}(\lambda_2)B$$

$$\cdots BdP_{A+kB}(\lambda_{k+1}),$$

(73)

where $P_A(\lambda_1), \ldots, dP_{A+kB}(\lambda_{k+1})$ are spectral measures.

Then, we have

$$\Pr \left( \| \Delta^k_B \overline{\psi}_{\otimes}(A) \| > \theta \right)$$

$$= 1 \Pr \left( \sum_{j=1}^{k} \int \cdots \int (\lambda_{j+1} - \lambda_j) \overline{\psi}^{[k]}_{\otimes}(\lambda_1, \ldots, \lambda_{k+1}) dP_A(\lambda_1)B \cdots BdP_{A+kB}(\lambda_{k+1}) \right) > \theta \right)$$

$$\leq 2 \Pr \left( \sum_{j=1}^{k} \max(\lambda_{j+1} - \lambda_j) \| \overline{\psi}^{[k]}_{\otimes} \| \|B\|^k > \theta \right)$$

$$\leq 3 \Pr \left( k\kappa \| \overline{\psi}^{[k]}_{\otimes} \| \|B\|^k > \theta \right),$$

(74)

where $=1$ comes from Eq. (73), $=2$ comes from Lemma 1 and $\leq 3$ comes from assumptions about $\max(\lambda_{j+1} - \lambda_j)$. The proof of this theorem is complete from Markov inequality.

5.3 Tail Bound for Taylor Remainder of Random Operator-valued Functions

In this section, we will derive tail bounds for Taylor remainder of random operator-valued functions. We will consider two types of random operators: self-adjoint and unitary. Let us begin with *multi-index* notation definitions.
Multi-indices are represented by vector symbols as \( a \) or \( b \):

\[
a = (a_1, a_2, \cdots, a_n), \quad b = (b_1, b_2, \cdots, b_n),
\]

where \( a_i, b_i \in \{0, 1, 2, \cdots\} \). If \( a \) is a multi-index, we have the following notation definitions:

\[
|a| = a_1 + a_2 + \cdots + a_n,
\]

\[
a! = a_1!a_2!\cdots a_n!,
\]

\[
X^a = X_1^{a_1}X_2^{a_2}\cdots X_n^{a_n} \text{ where } x = (X_1, X_2, \cdots, X_n).
\]

5.3.1 Random Self-Adjoint Operators

All operators discussed in this section are assumed self-adjoint. Given a multi-variate operator-valued function \( f(X_1, X_2, \cdots, X_n) \) and perturbed operators \( (H_1, H_2, \cdots, H_n) \), we define the partial derivative for \( \partial^a f(X + t \circ H) \) as

\[
\partial^a f(X + t \circ H) = \frac{\partial|a| f(X + t \circ H)}{\partial t_1^{a_1} \partial t_2^{a_2} \cdots \partial t_n^{a_n}}
\]

where \( t \) is the vector of real variables \( (t_1, t_2, \cdots, t_n) \) and \( \circ \) is the Hadamard product. Then, we define the Taylor remainder for \( f(X_1, X_2, \cdots, X_n) \) with perturbed operators \( (H_1, H_2, \cdots, H_n) \) as

\[
R_{k,f,x}(H) = f(X + H) - \sum_{|a|<k} \frac{\partial^a f(X + t \circ H)}{a!} \bigg|_{t=0},
\]

where \( t \) is the vector of real variables \( (t_1, t_2, \cdots, t_n) \) and \( \circ \) is the Hadamard product.

If the function \( f(X_1, X_2, \cdots, X_n) \) can be decomposed as the summation of function constructible by the limit of integral projective tensor product \( \otimes \), i.e.,

\[
f(X_1, X_2, \cdots, X_n) = \overrightarrow{\phi}_{\otimes,1}(X_1) + \overrightarrow{\phi}_{\otimes,2}(X_2) + \cdots + \overrightarrow{\phi}_{\otimes,n}(X_n),
\]

the expression given by Eq. (78) can be further expressed as

\[
R_{k,f,x}(H) = \sum_{j=1}^{n} \left( \frac{\phi_{\otimes}(X_j + H_j)}{\phi_{\otimes,j}} - 1 \sum_{\ell=0}^{k-1} \frac{1}{\ell!} \frac{d^\ell}{dt_j^\ell} \frac{\phi_{\otimes,j}}{\phi_{\otimes,j}}(X_j + t_j H_j) \bigg|_{t_j=0} \right).
\]

Following lemma is given to express \( R_{k,f,x}(H) \) by MOIs.

**Lemma 6** Let the function \( f : \mathbb{R}^n \to \mathbb{R} \) can be decomposed as \( f(X) = \sum_{j=1}^{n} \overrightarrow{\phi}_{\otimes,j}(X_j) \), we have

\[
R_{k,f,x}(H) = \sum_{j=1}^{n} T_{\overrightarrow{\phi}_{\otimes,j}^{X_j+H_j,\cdots,X_j}}(H_j, \cdots, H_j).
\]

**Proof:** From Lemma 5 we have

\[
R_{k,\overrightarrow{\phi}_{\otimes,j},X_j}(H_j) = T_{\overrightarrow{\phi}_{\otimes,j}^{X_j+H_j,\cdots,X_j}}(H_j) - \sum_{\ell=1}^{k-1} T_{\overrightarrow{\phi}_{\otimes,j}^{X_j,\cdots,X_j}}(H_j, \cdots, H_j).
\]
By applying Lemma 2 to \( T_{\phi_{\otimes,j}}^{X_j+H_j,X_j}(H_j) \) at the R.H.S. of Eq. (82), we have

\[
R_{k,\phi_{\otimes,j},X_j}(H_j) = T_{\phi_{\otimes,j}}^{X_j+H_j,X_j}(H_j) - \sum_{\ell=2}^{k-1} T_{\phi_{\otimes,j}}^{X_j,\cdots,X_j}(H_j,\cdots,H_j).
\] (83)

This lemma is proved by repeating this procedure \( k-1 \) times and the decomposition relation given by Eq. (80).

Following theorem is about the tail bound for \( R_{k,f,X}(H) \).

**Theorem 10** For \( j = 1, 2, \cdots, n \), let \( X_j \) be a self-adjoint random operator, \( H_j \) be a deterministic self-adjoint operator. The function \( f : \mathbb{R}^n \to \mathbb{R} \) is assumed to be decomposed as \( f(X) = \sum_{j=1}^{n} \phi_{\otimes,j}(X_j) \). Then, we have

\[
\Pr (\|R_{k,f,X}(H)\| > \theta) \leq \sum_{j=1}^{n} \frac{n \|H_j\|^k}{\theta} \mathbb{E}[\psi_{\otimes,j}^{(k)}].
\] (84)

**Proof**: Because we have

\[
\Pr (\|R_{k,f,X}(H)\| > \theta) = \Pr (\left\| \sum_{j=1}^{n} T_{\phi_{\otimes,j}}^{X_j+H_j,X_j}(H_j,\cdots,H_j) \right\| > \theta) \\
\leq \Pr (\left\| \sum_{j=1}^{n} T_{\phi_{\otimes,j}}^{X_j,H_j,X_j}(H_j,\cdots,H_j) \right\| > \theta) \\
\leq \sum_{j=1}^{n} \Pr (\left\| T_{\phi_{\otimes,j}}^{X_j,H_j,X_j}(H_j,\cdots,H_j) \right\| > \theta) \\
\leq \sum_{j=1}^{n} \frac{n \|H_j\|^k}{\theta} \mathbb{E}[\psi_{\otimes,j}^{(k)}],
\] (85)

where \( =_1 \) comes from Lemma 6, \( \leq_2 \) is due to norm triangle inequality, and \( \leq_3 \) comes from Lemma 7 and Markov inequality again. \( \square \)

### 5.3.2 Random Unitary Operators

Given a multi-variate operator-valued function \( f(X_1, X_2, \cdots, X_n) \) and perturbed operators \( (H_1, H_2, \cdots, H_n) \), which are assumed to be bounded self-adjoint operators, we use \( e^{\epsilon_{t_{\epsilon}} H} \) to represent the vector \( (e^{\epsilon_{t_1} H_1}, e^{\epsilon_{t_2} H_2}, \cdots, e^{\epsilon_{t_n} H_n}) \), where \( t \overset{\text{def}}{=} \sqrt{-1} \). We define the partial derivative for \( \partial^a f(e^{\epsilon_{t_{\epsilon}} H} \circ X) \) as

\[
\partial^a f(e^{\epsilon_{t_{\epsilon}} H} \circ X) = \frac{\partial|a| f(e^{\epsilon_{t_{\epsilon}} H} \circ X)}{\partial t_1^{a_1} \partial t_2^{a_2} \cdots \partial t_n^{a_n}}.
\] (86)

We first define the Taylor remainder for \( f(X_1, X_2, \cdots, X_n) \) with perturbed operators \( (H_1, H_2, \cdots, H_n) \) as

\[
Q_{k,f,X}(H) = f(e^{H} \circ X) - \sum_{|a|<k} \frac{\partial^a f(e^{H} \circ X)}{a!} \bigg|_{t=0}.
\] (87)
If the function \( f \) can also be decomposed as Eq. (79), we can rewrite Eq. (87) as

\[
Q_{k,f,X}(H) = \sum_{j=1}^{n} \left( \overline{\phi}_{j}\left( e^{iH_j} X_j \right) - \sum_{\ell=0}^{k-1} \frac{1}{\ell!} d^{\ell}_j \overline{\phi}_{j}\left( e^{it_j H_j} X_j \right) \bigg|_{t_j=0} \right).
\]  

(88)

Following lemma is given to express \( R_{k,f,X}(H) \) by MOIs.

**Lemma 7** Let the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) can be decomposed as \( f(X) = \sum_{j=1}^{n} \overline{\phi}_{j}(X_j) \), we have

\[
Q_{k,f,X}(H) = \left( \sum_{j=1}^{n} \sum_{\ell=1}^{k} \sum_{|i|=1}^{\ell} \sum_{i_{\ell} \geq 1} \frac{1}{i_1! \cdots i_\ell!} \cdot \frac{(i H_j)^m}{m!} X_j, \left( \sum_{m=1}^{\infty} \frac{(i H_j)^m}{m!} X_j, \cdots, \left( \frac{(i H_j)^i_\ell}{i_\ell!} X_j \right) \right) \right).
\]  

(89)

**Proof:** We will prove this lemma by induction. For basic case with \( k = 1 \) and any \( j \) between 1 and \( n \), we have

\[
Q_{1,\overline{\phi}_{j},X_j}(H) = \overline{\phi}_{j}\left( e^{iH_j} X_j \right) - \overline{\phi}_{j}(X_j) = T_{e^{iH_j}X_j,X_j} \left( \sum_{m=1}^{\infty} \frac{(i H_j)^m}{m!} X_j \right).
\]  

(90)

Therefore, by summing over \( j \) with respect to Eq. (90), this lemma is valid for the basic case. Suppose that Eq. (89) is valid for \( k = q \) and any \( j \) between 1 and \( n \), we have

\[
Q_{q+1,\overline{\phi}_{j},X_j}(H) = \left( \sum_{\ell=1}^{q} \sum_{i_{\ell} \geq 1} \frac{1}{i_1! \cdots i_\ell!} \cdot \frac{(i H_j)^m}{m!} X_j, \cdots, \left( \frac{(i H_j)^i_\ell}{i_\ell!} X_j \right) \right).
\]  

21
\[
\begin{align*}
&= \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{e H_j X_j, X_j, \ldots, X_j} \left( \sum_{m=1}^{\infty} \frac{(i H_j)^m}{m!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right) \\
&\quad - \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{X_j, X_j, \ldots, X_j} \left( \frac{(i H_j)^{i_1}}{i_1!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right) \\
&\quad + \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{\phi q_{H_j}} X_j, X_j, \ldots, X_j \left( \frac{(i H_j)^{i_1}}{i_1!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right) \\
&\quad - \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{X_j, X_j, \ldots, X_j} \left( \frac{(i H_j)^{i_1}}{i_1!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right)
\end{align*}
\]

Sum 1

\[
= \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{e H_j X_j, X_j, \ldots, X_j} \left( \sum_{m=1}^{\infty} \frac{(i H_j)^m}{m!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right)
\]

Sum 2

\[
+ \sum_{\ell=1}^{q} \sum_{i_1, \ldots, i_\ell \geq 1} T_{\phi^{(\ell q)}_{\phi_{\ell q}}}^{e H_j X_j, X_j, \ldots, X_j} \left( \sum_{m=1}^{\infty} \frac{(i H_j)^m}{m!} X_j, \frac{(i H_j)^{i_1}}{i_1!} X_j, \frac{(i H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(i H_j)^{i_\ell}}{i_\ell!} X_j \right)
\]

(91)

where \(=1\) comes from Lemma 2.

By changing the summation index range as \(\alpha_1 = i_1 + 1\) and \(\alpha_p = i_p\) for \(2 \leq p \leq \ell\), we can express the
summation of $\text{Sum}_1$ as

$$\text{Sum}_1 = \sum_{\ell=1}^{q} \left( \sum_{\alpha_1 \geq 2, \alpha_2, \ldots, \alpha_\ell \geq 1} \frac{(\ell H_j)^m}{m!} X_j \right)$$

Similarly, by changing the summation index range from $\ell = 1, 2, \ldots, q$ to $\ell = 2, \ldots, q + 1, \alpha_1 = 1$ and $\alpha_p = i_{p-1}$ for $2 \leq p \leq \ell$, we can express the summation of $\text{Sum}_2$ as

$$\text{Sum}_2 = \sum_{\ell=2}^{q+1} \left( \sum_{\alpha_1 = 1, \alpha_2, \ldots, \alpha_\ell \geq 1} \frac{(\ell H_j)^m}{m!} X_j \right)$$

From Eqs. (91), (92) and (93), we have

$$Q_{q+1, \phi, \ldots, j}(H_j)$$

$$= T_{\phi[1]}(\sum_{m=q+1}^{\infty} \frac{(\ell H_j)^m}{m!} X_j)$$

$$+ \sum_{\ell=2}^{q+1} \left( \sum_{\alpha_1 = 1, \alpha_2, \ldots, \alpha_\ell \geq 1} \frac{(\ell H_j)^m}{m!} X_j \right)$$

$$+ T_{\phi[q+1]}(\sum_{m=1}^{\infty} \frac{(\ell H_j)^m}{m!} X_j, \ldots, \ell H_j X_j)$$

$$= \sum_{\ell=1}^{q+1} \sum_{i_1, i_2, \ldots, i_\ell \geq 1} \frac{(\ell H_j)^m}{m!} X_j \left( \sum_{i_{\ell+1}}^{\infty} \frac{(\ell H_j)^{i_{\ell+1}}}{i_{\ell+1}!} X_j \right) \left( \sum_{i_{\ell+2}}^{\infty} \frac{(\ell H_j)^{i_{\ell+2}}}{i_{\ell+2}!} X_j \right) \ldots \left( \sum_{i_{\ell+\ell}}^{\infty} \frac{(\ell H_j)^{i_{\ell+\ell}}}{i_{\ell+\ell}!} X_j \right).$$

Then, this lemma is proved by induction and summing Eq. (94) with respect to the variable $j$. $\square$

We require another lemma to bound MOIs associated to $Q_{k,f,X}(H)$. 23
Lemma 8 Let $X_j$ be a random unitary operators, $H_j$ be an operator with bounded norm and $\phi_{\otimes,j}$ be a function constructible by the limit of integral projective tensor product, then we have

$$\| T_{e^{iH_j}} X_j, X_j, \ldots, X_j \left( \sum_{m=1}^{\infty} \frac{(\mu H_j)^m}{m!} X_j, \frac{(\mu H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(\mu H_j)^{i_\ell}}{i_\ell!} X_j \right) \| \leq \left\| \phi e^{iH_j} \right\| \rho(H_j, i, \ell), \quad (95)$$

where $\rho(H_j, i, \ell)$ is defined as

$$\rho(H_j, i, \ell) \triangleq \left( \sum_{m=1}^{\infty} \frac{\|H_j\|^m}{m!} \right) \prod_{p=2}^{\ell} \frac{\|H_j\|^{i_p}}{i_p!}. \quad (96)$$

Proof: Since we have

$$\| T_{e^{iH_j}} X_j, X_j, \ldots, X_j \left( \sum_{m=1}^{\infty} \frac{(\mu H_j)^m}{m!} X_j, \frac{(\mu H_j)^{i_2}}{i_2!} X_j, \ldots, \frac{(\mu H_j)^{i_\ell}}{i_\ell!} X_j \right) \| \leq 1 \left( \sum_{m=1}^{\infty} \frac{\|H_j\|^m}{m!} \right) \prod_{p=2}^{\ell} \frac{\|H_j\|^{i_p}}{i_p!} \rho(H_j, i, \ell), \quad (97)$$

where $\leq_1$ comes from Lemma 1 and $\leq_2$ comes from that the operator norm is submultiplicative and triangle inequality, this lemma is proved by Eq. (96).

We are ready to present the following theorem about the tail bound for $Q_{k,f}(X, H)$.

Theorem 11 For $j = 1, 2, \ldots, n$, let $X_j$ be a random unitary operator, $H_j$ be a deterministic operator with bounded norm. The function $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be decomposed as $f(X) = \sum_{j=1}^{n} \phi_{\otimes,j}(X_j)$. Then, we have

$$\Pr (\|Q_{k,f}(X, H)\| > \theta) \leq \sum_{j=1}^{n} \sum_{\ell=1}^{k} \frac{k n \Theta(j, k, \ell)}{\theta} \mathbb{E} \left[ \left\| \phi_{\otimes,j} \right\| \right], \quad (98)$$

where $\Theta(j, k, \ell) \triangleq \sum_{i_1, \ldots, i_\ell \geq 1} \rho(H_j, i, \ell)$. Note that $\rho(H_j, i, \ell)$ is defined by Eq. (96).
Proof: Because we have

\[ \Pr \left( \|Q_{k,f,x}(H)\| > \theta \right) = 1 \]

\[
\begin{align*}
\Pr \left( \sum_{j=1}^{n} \sum_{\ell=1}^{k} \sum_{i_1, \ldots, i_\ell \geq 1} \left\| \frac{\psi_{\Theta(j,k,\ell)}^{[\ell]}(i_1, \ldots, i_\ell)}{\theta} \right\| X_j, \ldots, X_j \right) > \theta \\
\leq \sum_{j=1}^{n} \sum_{\ell=1}^{k} \Pr \left( \frac{\psi_{\Theta(j,k,\ell)}^{[\ell]}(i_1, \ldots, i_\ell)}{\theta} \right) > \theta \\
\leq \sum_{j=1}^{n} \sum_{\ell=1}^{k} \frac{\theta}{kn \Theta(j,k,\ell)} \quad (99)
\end{align*}
\]

where \( = 1 \) comes from Lemma 7, \( \leq 2 \) is due to Lemma 8, and \( \leq 3 \) Markov inequality again; this theorem is proved.

References

[1] J. L. Daletskii and S. G. Krein, “Integration and differentiation of functions of hermitian operators and applications to the theory of perturbations,” AMS Translations (2), vol. 47, no. 1-30, pp. 10–1090, 1965.
[2] M. S. Birman and M. Z. Solomyak, “Double stieltjes operator integrals and problems on multipliers,” in *Doklady Akademii Nauk*, vol. 171, no. 6. Russian Academy of Sciences, 1966, pp. 1251–1254.

[3] M. S. Birman and M. Solomyak, “Double stieltjes operator integrals, problems math. phys. leningrad. univ. 1 (1966) 33–67,” *English transl.: Topics Math. Physics*, vol. 1, pp. 25–54, 1967.

[4] ———, “Double operator integrals in a hilbert space,” *Integral equations and operator theory*, vol. 47, no. 2, pp. 131–168, 2003.

[5] V. V. Peller, “Hankel operators in the perturbation theory of unitary and self-adjoint operators,” *Functional Analysis and Its Applications*, vol. 19, no. 2, pp. 111–123, 1985.

[6] ———, “Multiple operator integrals and higher operator derivatives,” *Journal of Functional Analysis*, vol. 233, no. 2, pp. 515–544, 2006.

[7] A. Skripka and A. Tomskova, *Multilinear Operator Integrals*. Springer, 2019.

[8] A. V. Skorohod, *Random linear operators*. Springer Science & Business Media, 2001, vol. 1.

[9] W. Hackenbroch, “Point localization and spectral theory for symmetric random operators,” *Archiv der Mathematik*, vol. 92, no. 5, pp. 485–492, 2009.

[10] D. H. Thang and T. X. Quy, “On the spectral theorem for random operators,” *Southeast Asian Bulletin of Mathematics*, vol. 41, no. 2, 2017.

[11] S. Y. Chang, “Random double tensors integrals,” *arXiv preprint arXiv:2204.01927*, 2022.

[12] ———, “Random parametrization double tensors integrals and their applications,” *arXiv preprint arXiv:2205.03523*, 2022.

[13] M. Liang and B. Zheng, “Further results on Moore-Penrose inverses of tensors with application to tensor nearness problems,” *Computers and Mathematics with Applications*, vol. 77, no. 5, pp. 1282–1293, March 2019.

[14] C. K. Chui and X. Li, “Realization of neural networks with one hidden layer,” in *Multivariate approximation: From CAGD to wavelets*. World Scientific, 1993, pp. 77–89.

[15] D. Pérez and Y. Quintana, “A survey on the weierstrass approximation theorem,” *arXiv preprint math/0611038*, 2006.