Estimate of Minimal Noise in a Quantum Conductor

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Abstract

We study zero temperature fluctuations of charge flow in a metallic loop induced by time dependent magnetic flux, and solve for the optimal way of varying flux in order to minimize noise. Optimal time dependence of the flux is a sum of “solitons,” each corresponding to quantized flux change. Minimal noise coincides with that for the binomial distribution with the number of attempts equal to the number of solitons and with the probabilities defined by the scattering matrix of the system.

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In the theory of noise in quantum conductors it is usually regarded as a characteristic of transport complementary to conductance. During last years the literature on quantum noise was mostly concentrated on expressing it through transmission coefficients of conduction channel. Having the noise written as a sum of contributions from different channels allows one to relate it with the conductance for which such representation is known for a while, and also to compare noise in the two limits, quantum and classical. Comparison with classical shot noise leads to the understanding of quantum reduction of the noise, an important effect of Fermi statistics.

Beyond this there are several other interesting properties of quantum noise. One is the issue of probability distribution of charge fluctuations. The question arises naturally from the comparison with the Poisson statistics of classical shot noise. In the quantum case the statistics was found to be binomial with probabilities of outcomes related with transmission coefficients of elastic scattering in the system, and with the number of attempts weakly fluctuating near $eVt/h$, where $V$ is voltage applied to the system and $t$ is the time of measurement.

Another curious property of quantum noise described recently is its phase sensitivity named ‘non-stationary Aharonov-Bohm effect’. As its simplest realization one can consider metallic wire bent in a loop with magnetic flux applied to it. In this setup, when the flux is changed from one constant value to another during fixed time interval $\tau$, the fluctuations of charge measured during much longer time $t$ contain a term proportional to $\sin^2(\pi\Delta\Phi/\Phi_0) \ln t/\tau$, where $\Delta\Phi$ is total change of flux and $\Phi_0 = hc/e$. Both properties of this term, the logarithmic divergence and the periodicity in $\Delta\Phi$ can be put in connection with the orthogonality catastrophe theory by relating $\Phi(t)$ with the phase of forward scattering of the part of the wire threaded by the flux. In a different setup, involving same loop geometry but the flux $\Phi(t)$ varying periodically with time, the phase sensitivity leads to singularities of the noise at integer $eV/\hbar\Omega$, where $V$ is dc voltage and $\Omega$ is the frequency of the ac flux. The strengths of the singularities are oscillating functions of the flux amplitude and are independent of the frequency $\Omega$.

The question we address in this paper is about optimal way of changing flux that minimizes induced noise. It is clear from what has been said that for achieving minimum of the noise one should change the flux by integer amount,

$$\Delta\Phi = \Phi_{t=\infty} - \Phi_{t=-\infty} = n\Phi_0,$$

in order to suppress the logarithmic term. However, since for a given $\Delta\Phi$ the noise depends on the actual function $\Phi(t)$, not just on $\Delta\Phi$, we have a variational problem to solve for the noise as a functional of the time dependence of the flux. This functional was derived for a single channel ideal conductor with one localized scatterer. For $T = 0$ it is given by

$$\langle Q^2 \rangle = \frac{ge^2}{2\pi} D(1-D) \int \left| \int e^{i\phi(t)+i\omega t} dt \right|^2 |\omega|^2 \frac{d\omega}{2\pi},$$

where $\phi(t) = 2\pi\Phi(t)/\Phi_0$, $D$ is transmission coefficient, and $g$ is spin degeneracy. We shall study the variational problem (2) with the boundary condition (1) and show that its general solution has the form of a sum of ‘solitons’:

$$\Phi(t) = \pm \frac{\Phi_0}{\pi} \sum_{k=1}^{n} \tan^{-1} \left( \frac{t - t_k}{\tau_k} \right), \tau_k > 0,$$
where $t_k$ and $\tau_k$ are arbitrary constants. Under condition (1) any time dependence of the form (3) gives absolute minimum to the noise:

$$\min \left[ \langle \langle Q^2 \rangle \rangle \right] = g e^2 D (1 - D) |n|,$$

(4)

For an optimal time dependence of the voltage $V = -\partial \Phi / c \partial t$ therefore one has a sum of Lorentzian peaks:

$$V(t) = \mp \frac{\Phi_0}{c \pi} \sum_{k=1}^{n} \frac{\tau_k}{(t - t_k)^2 + \tau_k^2}.$$

In order to compare quantum noise with conductance, let us mention that average transmitted charge $\langle \langle Q \rangle \rangle$ equals $geD\Delta\Phi/\Phi_0 = ge^2D \int V(t)dt$, in accordance with Ohm’s law, i.e., there is no particular dependence on the way the flux change $\Delta\Phi$ is realized.

The result (3),(4) has a simple interpretation in terms of the binomial statistics picture of charge fluctuations. For the binomial distribution with probabilities of outcomes $p$ and $q$, $p + q = 1$, and with the number of attempts $N$, second moment is known to be equal to $pqN$. The comparison with Expr.(4) suggests to attribute to $n = \Delta\Phi/\Phi_0$ the meaning of the number of attempts. This interpretation is supported by the structure of the function (3) consisting of $n$ terms, each corresponding to unit change of flux. A remarkable property of the function (3) is its separability, expressed clearly both in the form of the terms and in the way the parameters $t_k, \tau_k$ enter the expression. Let us note that by making some of the $t_k$’s close to each other one can have an overlap in time of the ‘attempts’. The overlap, however, does not change the fluctuations (4). The situation reminds the one with solitons in integrable non-linear systems, or with non-interacting instantons in integrable field theories. Also, the absence of interference is interesting in the context of coherent nature of transport in this system: after all, we are just talking about scattering off of a time-dependent potential. Perhaps, proper interpretation of this effect should be sought in establishing relation with the theory of coherent states that have the property to eliminate to some extent the quantum mechanical interference.

Let us now turn to the variational problem. It is convenient to do the integral over $\omega$ first and to rewrite (2) as

$$\langle \langle Q^2 \rangle \rangle = -\frac{A}{\pi} \int \int \frac{e^{i\phi(t) - i\phi(t')}}{(t - t')^2} dt dt',$$

(5)

where $A = \frac{ge^2}{2\pi}D(1 - D)$. In order to avoid divergence at $t = t'$ the denominator in (5) should be understood as

$$\frac{1}{2} \left[ \frac{1}{(t - t' + i\delta)^2} + \frac{1}{(t - t' - i\delta)^2} \right],$$

(6)

the condition that one obtains by introducing regularization in (2): $|\omega| \to |\omega|e^{-|\omega|\delta}$. By considering variation of the functional (5) we have the equation for an extremum:

$$\text{Im} \left[ e^{i\phi(t)} \int \frac{e^{-i\phi(t')}}{(t - t')^2} dt \right] = 0.$$

(7)
By using Cauchy formula one checks that the functions

\[ e^{i\phi(t)} = \prod_{k=1}^{n} \frac{t - \lambda_k}{t - \bar{\lambda}_k}, \quad \lambda_k = t_k + i\tau_k, \quad (8) \]

satisfy (7) provided all \( \tau_k \) are of same sign. Obviously, the functions (8) are just another form of (3).

It remains to be shown that the functional reaches its minimum on the solutions (8). To prove it we proceed in the following steps. Let us write

\[ e^{i\phi(t)} = f_+(t) + f_-(t), \quad (9) \]

where \( f_+(t) \) and \( f_-(t) \) are bounded analytic functions of complex \( t \) in the upper and lower half plane, respectively. Representation (9) exists for any non-singular function and defines the functions \( f_+ \) and \( f_- \) up to a constant. Then we substitute Expr.(9) in (5) and apply Cauchy formula for the derivative,

\[ \dot{f}_\pm(t) = \pm \frac{i}{2\pi} \oint \frac{f_\pm(t')dt'}{(t - t')^2}, \]

where the contour of integration is chosen in the halfplane of analyticity of \( f_+ \) or \( f_- \), respectively. Thus one gets

\[ \langle \langle Q^2 \rangle \rangle = -iA \int (\bar{f}_+ \dot{f}_+ - \bar{f}_- \dot{f}_-)dt. \quad (10) \]

On the other hand,

\[ n = \frac{1}{2\pi i} \int e^{-i\phi(t)} \frac{d}{dt} e^{i\phi(t)} dt = -\frac{i}{2\pi} \int (\bar{f}_+ \dot{f}_+ + \bar{f}_- \dot{f}_-) dt, \quad (11) \]

where the last equality is a result of substituting (9) and using \( \int f_+ \dot{f}_- = \int f_- \dot{f}_+ = 0 \) that follows from Cauchy theorem. Now, Expr.(10) can be rewritten through Fourier components of \( f_+ \) and \( f_- \) as

\[ \langle \langle Q^2 \rangle \rangle = A \int_0^\infty (|f_+(\omega)|^2 + |f_-(-\omega)|^2) \omega \frac{d\omega}{2\pi}, \]

thus demonstrating positivity of both terms in (10). (It is used that \( f_+(\omega) = f_-(-\omega) = 0 \) for \( \omega < 0 \).) With this, by comparing (10) and (11) we obtain

\[ \langle \langle Q^2 \rangle \rangle \geq 2\pi A |n|. \quad (12) \]

Equality in (12) is reached only when either \( f_+(t) \) or \( f_-(t) \) vanishes. Therefore, to obtain the minimum one has to take the functions \( e^{i\phi(t)} \) that are regular in one of the half planes. This remark is sufficient to see that the functions (8) form a complete family of solutions.

It is worth mentioning that the method used to derive (12) copies almost entirely the procedure of derivation of the ‘duality’ condition in the theory of instantons. Like in other situations where the duality condition holds our ‘solitons’ do not interact: \( \langle \langle Q^2 \rangle \rangle \) shows no
dependence on the parameters $\lambda_k$ of the solution (8). Among numerous field theories that allow for exact solution of the instanton problem the one most similar to our case is the theory of classical Heisenberg ferromagnet in $D = 2$. For this case the instantons were found by mapping the order parameter space (i.e., the unit sphere) on the complex plane$^{10}$. Duality condition was shown to take the form of the constraint of analyticity or anti-analyticity of the mapped order parameter function (compare with the above derived condition $f + \frac{i}{\pi} \phi = 0$ or $f_- = 0$). Multi-instanton solutions were given as products of single instanton solutions (cf. Expr.(8)). This analogy obviously deserves more attention.

At this point let us examine an interesting non-optimal time dependence of the flux, the sum of two solitons with opposite charge:

$$\Phi(t) = \frac{\Phi_0}{\pi} \left[ \tan^{-1} \left( \frac{t - t_1}{\tau_1} \right) - \tan^{-1} \left( \frac{t - t_2}{\tau_2} \right) \right], \quad (13)$$

$\tau_{1,2} > 0$. This function corresponds to $e^{i\phi(t)}$ of the form (8) but with the poles in both half planes. In this case $\Delta \Phi = 0$ and thus $\langle \langle Q \rangle \rangle = 0$, so $\min[\langle \langle Q^2 \rangle \rangle] = 0$. With the function (13), however, one finds

$$\langle \langle Q^2 \rangle \rangle = 4\pi A \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2^*} \right|^2, \quad (14)$$

where $\lambda_{1,2} = t_{1,2} + i\tau_{1,2}$. For different values of the parameters $t_{1,2}$, $\tau_{1,2}$ Expr.(14) interpolates between two trivial limiting cases: (i) $\langle \langle Q^2 \rangle \rangle \to 0$ when the two flux steps in (13) have almost equal duration and almost overlap; (ii) $\langle \langle Q^2 \rangle \rangle \to 4\pi A$, when the flux steps either differ strongly in their duration or do not overlap. In the case (ii) the noise is two times bigger than the noise due to a single step, as it should be.

We see that when $\Delta \Phi/\Phi_0$ is of the order of one a non-optimal time dependence $\Phi(t)$ can considerably enhance the noise. It is not the case, however, for $\Delta \Phi/\Phi_0 \gg 1$. This limit was analyzed in our previous paper and it was found that when $\Phi(t)$ is a monotonous function the result

$$\langle \langle Q^2 \rangle \rangle = g e^{2(D(1 - D)||\Delta \Phi/\Phi_0||} \quad (15)$$

is quite accurate$^9$. By studying several examples we concluded that relative correction is small. However, there is more to say about how big the correction can be. Let us derive the result (15) by another method that allows to trace out the order of magnitude of higher order terms. For that let us take the flux in the form $\Phi(t) = \frac{\Delta \Phi}{\pi} \phi(t)$, where $\phi(t)$ is a smooth monotonous function, $\phi(-\infty) = 0$, $\phi(\infty) = 2\pi$. For integer $N = \Delta \Phi/\Phi_0 \gg 1$ the Fourier component of $e^{iN\phi(t)}$ entering Expr.(2) in the stationary phase approximation is given by

$$\int_{-\infty}^{\infty} e^{iN\phi(t)+i\omega t} dt = \sum_k \sqrt{\frac{2\pi i}{N\phi(t_k)}} e^{iN\phi(t_k)+i\omega t_k} + O(N^{-3/2}), \quad (16)$$

where $t_k$’s are real solutions of the equation $N\phi(t) + \omega = 0$. Then we can write

$$\left| \int_{-\infty}^{\infty} e^{iN\phi(t)+i\omega t} dt \right|^2 = \sum_k \frac{2\pi i}{N\phi(t_k)} + O(N^{-2}),$$

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and thus obtain

\[ \langle Q^2 \rangle = A \int_{-\infty}^{\infty} \sum_k \frac{|\omega|d\omega}{N\dot{\phi}(t_k)} + ... , \] (17)

where dots represent higher order terms. By differentiating both sides of the equation \( N\dot{\phi}(t) = -\omega \) one finds the relation \( d\omega = -N\ddot{\phi}(t_k)dt_k \), which means that \( |\omega|d\omega/\dot{\phi}(t_k) = -|\dot{\phi}(t_k)|dt_k \), and therefore the integral in Expr.(17) equals \( N \int_{-\infty}^{\infty} d\phi = 2\pi\Delta\Phi/\Phi_0 \). Since \( |\omega|d\omega \) scales as \( N^2 \), the correction to Expr.(17) can be evaluated as \( O(1) \), i.e., it is of the order of one for any \( N \). This means that relative accuracy of Expr.(15) is \( O(1/N) \).

A more intuitive way to understand the accuracy of Expr.(15) is to note that for a given \( n \) the number of parameters in the optimal flux dependence (3) is \( 2n \), i.e., half of them are ‘redundant’. Because of that any smooth monotonic function with sufficiently large variation \( \Delta\Phi \) can be rather accurately approximated by a function of the form (3), and therefore the noise exceeds the lower bound just slightly.

An implication of this result for the binomial statistics picture is as follows. As it was discussed above there is a (conjectured) correspondence of the terms of Expr.(3) and of the attempts. The deviation from the binomial distribution, that of course should exist for a non-optimal flux function, in the case of a smooth \( \Phi(t) \) will remain bounded as \( \Delta\Phi \) increases taking integer values. In more precise terms, accurate distribution will be written as a mixture of binomial distributions with different numbers \( N \) of attempts, \( P(m) = \sum_N \rho_N P_N(m) \), where \( P_N(m) = p^m q^{N-m} C^m_N \). The estimated correction implies that the distribution of attempts \( \rho_N \) has finite variance in the limit \( N = \Delta\Phi/\Phi_0 \to \infty \).

Before we close let us mention that in order to apply our results in the case of a mesoscopic metallic conductor with disorder, which is described by many conducting channels, one just needs to replace \( D(1-D) \) by \( \sum_n T_n(1-T_n) \), since different scattering channels contribute to the noise independently. The condition of validity of our treatment then is that the variation of the flux is sufficiently slow, so that \( \min[\tau_k] \gg h/E_c \), the time of diffusion across the sample. However, at non-zero temperature one also has to satisfy the condition \( \tau_k \ll h/T \), the time of phase breaking. So, the temperature interval where our estimate of the noise holds is \( T \leq E_c \).

In conclusion, we studied dependence of the noise in a quantum conductor on the shape of voltage pulse applied to it and found optimal time dependence that provides minimum of the noise for given average transmitted charge. Solution displays interesting analogy with the problem of instantons in the field theories obeying ‘duality’ condition. Optimal time dependence is a sum of Lorentzian peaks of voltage, each corresponding to a ‘soliton’ of flux. The change of flux for a soliton is equal to the flux quantum \( \Phi_0 \). The solitons are interpreted in terms of the binomial statistics picture of charge fluctuations as attempts to transmit electrons, one electron per each soliton.

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