ON REAL LOG CANONICAL THRESHOLDS

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Abstract. We introduce real log canonical threshold and real jumping numbers for real algebraic functions. A real jumping number is a root of the $b$-function up to a sign if its difference with the minimal one is less than 1. The real log canonical threshold, which is the minimal real jumping number, coincides up to a sign with the maximal pole of the distribution defined by the complex power of the absolute value of the function. However, this number may be greater than 1 if the codimension of the real zero locus of the function is greater than 1. So it does not necessarily coincide with the maximal root of the $b$-function up to a sign, nor with the log canonical threshold of the complexification. In fact, the real jumping numbers can be even disjoint from the non-integral jumping numbers of the complexification.

Introduction

Let $f_C$ be a nonconstant holomorphic function on a complex manifold $X_C$, and $\omega$ be a $C^\infty$ form of the highest degree with compact support on $X_C$. Then the integral $\int_{X_C} |f_C|^2 \omega$ is extended to a meromorphic function in $s$ on the entire complex plane (using a resolution of singularities [5] together with a partition of unity, see [1], [2].) Moreover, the largest pole of $\int_{X_C} |f_C|^2 \omega$ coincides up to a sign with the log canonical threshold of $f_C$ if $\omega$ is nonnegative and does not vanish on a point $x$ of $D_C := f_C^{-1}(0)$ where the log canonical threshold of $(f_C, x)$ attains the minimal. (This follows from the definition by using a resolution of singularities, see [7].)

Let $f$ be a nonconstant real algebraic function on a real algebraic manifold $X_R$, and $\omega$ be a $C^\infty$ form of the highest degree with compact support on $X_R$ such that the open subset $\{x \in X_R \mid \omega(x) \neq 0\}$ is oriented and $\omega(x)$ is positive on this subset. Then $\int_{X_R} |f|^s \omega$ is similarly extended to a meromorphic function in $s$ on the entire complex plane. But the largest pole of $\int_{X_R} |f|^s \omega$ does not necessarily coincide up to a sign with the log canonical threshold of the complexification $f_C : X_C \to \mathbb{C}$ of $f : X_R \to \mathbb{R}$, see Corollary 2 and Theorem 1 below.

Let $\mathcal{O}_{X_R}$ denote the sheaf of real analytic functions on $X_R$. We define the real multiplier ideals $\mathcal{J}(X_R, f^\alpha) \subset \mathcal{O}_{X_R}$ for $\alpha \in \mathbb{Q}_{>0}$ by the local integrability of $|g|/|f|^{\alpha}$ for $g \in \mathcal{O}_{X_R}$. (Here coherence of $\mathcal{J}(X_R, f^\alpha)$ is unclear.) We have $\mathcal{J}(X_R, f^\alpha) = \mathcal{O}_{X_R}$ for $0 < \alpha \ll 1$, but not necessarily $f^s \mathcal{J}(X_R, f^\alpha) = \mathcal{J}(X_R, f^{\alpha+1})$ for $\alpha > 0$, unless $f$ is of ordinary type. Here we say that $f$ is of ordinary type if $\text{codim} \, D_R = 1$ where $D_R = f^{-1}(0) \subset X_R$, and of exceptional type otherwise. Note that the above equality always holds in the complex case.

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By Hironaka [5], there is a resolution of singularities as real algebraic manifolds \( \pi_R : X'_R \to X_R \) which is a composition of blowing-ups along smooth centers over \( R \) and such that \( \pi^* f \) and \( \pi^* dx_1 \cdots dx_n \) are locally of the form \( u \prod_{i=1}^{r} x_i^{m_i} \) and \( u' \prod_{i=1}^{r} x_i'^{n_i} dx'_1 \cdots dx'_n \), respectively, where \( m_i \geq 1 \) for \( i \in [1, r] \). Here \( x_1, \ldots, x_n \) and \( x'_1, \ldots, x'_n \) are local coordinates of \( X_R \) and \( X'_R \) respectively, and \( u, u' \) are nowhere vanishing. So \( \pi^* f \) defines a divisor with normal crossings \( D'_R = \sum_{j \in J_R} m_j D'_{j,R} \), and we may assume that each \( D'_{j,R} \) is smooth by local cit. Let \( a_j \) be the multiplicity of the Jacobian of \( \pi_R \) along \( D'_{j,R} \). Note that \( m_j \) and \( a_j \) are given by the above \( m_i \) and \( a_i \) respectively if \( D'_{j,R} \) is locally defined by \( y'_i = 0 \).

**Proposition 1.** For \( g \in \mathcal{O}_{X_R,x} \) we have

\[
g \in \mathcal{J}(X_R, f^\alpha)_x \iff \pi^* g dx_1 \cdots dx_n \in (\pi_* \Omega^n_{X_R} - \sum_{j}[am_j]D'_{j,R})_x.
\]

However, \( \pi_* \Omega^n_{X_R} - \sum_{j}[am_j]D'_{j,R} \) may be larger than \( \mathcal{J}(X_R, f^\alpha)_{X_R} \) in general (even for \( 0 < \alpha \ll 1 \)), and coherence of these sheaves are unclear. By Proposition 1 there are increasing rational numbers \( 0 < \alpha_1 < \alpha_2 < \cdots \) such that

\[
\mathcal{J}(X_R, f^{\alpha_j}) = \mathcal{J}(X_R, f^{\alpha}) \supset \mathcal{J}(X_R, f^{\alpha_{j+1}}) \text{ if } \alpha_j < \alpha < \alpha_{j+1} \text{ } (j \geq 1),
\]

and \( \mathcal{O}_{X_R} = \mathcal{J}(X_R, f^\alpha) \supset \mathcal{J}(X_R, f^1) \text{ if } 0 < \alpha < \alpha_1 \). These numbers \( \alpha_j \) are called **real jumping numbers** of \( f \). (Here we add “real” since the complexification \( f_C \) of \( f \) can be identified with \( f \) in case \( f \in \mathbb{R}[x] \subset \mathbb{C}[x] \).) The minimal real jumping number \( \alpha_1 \) is called the **real log canonical threshold**, and is denoted by \( \text{lct}(f) \). This is the smallest number such that \( |f|^{-\alpha} \) is not locally integrable on \( X_R \). It may be strictly greater than 1 in case of exceptional type, see Theorem 1 below. We define the graded pieces by

\[
\mathcal{G}(X_R, f^\alpha) = \mathcal{J}(X_R, f^{\alpha-\varepsilon})/\mathcal{J}(X_R, f^\alpha) \text{ for } 0 < \varepsilon \ll 1,
\]

so that \( \alpha \) is a real jumping number of \( f \) if and only if \( \mathcal{G}(X_R, f^\alpha) \neq 0 \). Proposition 1 implies

**Corollary 1.** We have

\[
\text{lct}(f) = \min_{j \in J_R} \left\{ \frac{a_j + 1}{m_j} \right\}.
\]

A similar assertion holds for the log canonical threshold \( \text{lct}(f_C) \) by applying the same argument to the resolution of singularities of the complexification \( f_C \), and \( -\text{lct}(f_C) \) coincides with the largest root of \( b_{f_C}(s) \), see [7]. Let \( -p(f, \omega) \) denote the maximal pole of \( \int_{X_R} |f|^s \omega \). Then

**Corollary 2.** We have in general

\[
p(f, \omega) \geq \text{lct}(f) \geq \text{lct}(f_C),
\]

and \( p(f, \omega) = \text{lct}(f) \) if \( \omega(x) \neq 0 \) for some \( x \in X_R \) such that \( \mathcal{G}(X_R, f^\alpha)_x \neq 0 \) with \( \alpha = \text{lct}(f) \).

For the corresponding assertion in the complex case, see [7]. The relation with the complexification is quite complicated as is shown by the following.
Theorem 1. There are cases where \( \text{rlct}(f) > \text{lct}(f_C) \), and even \( \text{rlct}(f) > 1 \) in case of exceptional type. Moreover the real jumping numbers of \( f \) can be disjoint from the non-integral jumping numbers of \( f_C \) even in the case \( f_C \) has only an isolated singularity at a real point \( x \in X_R \subset X_C \).

This kind of phenomena may happen in case \( f \) has an isolated zero of simple type, see (3.3). Let \( b_f(s) \) be the \( b \)-function of \( f \) which is by definition the least common multiple of the local \( b \)-functions \( b_{f,x}(s) \) for \( x \in X_R \). Note that \( b_{f,x}(s) \) coincides with the local \( b \)-function \( b_{f_C,x}(s) \) of \( f_C \), since \( b_{f_C,x}(s) \in \mathbb{Q}[s] \) by Kashiwara [6]. So \( b_f(s) = b_{f_C}(s) \) in case \( \text{Sing} f \subset X_R \).

Theorem 2. Any real jumping number of \( f \) which is smaller than \( \text{rlct}(f) + 1 \) is a root of \( b_f(-s) \).

For the corresponding assertion in the complex case, see [4]. It seems that the case of an ideal generated by \( f_1, \ldots, f_r \) is reduced to the case \( r = 1 \) by considering \( f = \sum_{i=1}^{r} f_i^2 \) in the real case.

This note is written to answer questions of Professor S. Watanabe which are closely related to problems in the theory of learning machines (see e.g. [10]). I would like to thank him for interesting questions.

In Section 1 we recall some facts from the theory of resolutions of singularities due to Hironaka [5]. In Section 2 we prove Proposition 1 and Theorem 2. In Section 3 we prove Theorem 1 by constructing examples.

1. Resolution of singularities

In this section we recall some facts from the theory of resolutions of singularities due to Hironaka [5].

1.1. Analytic spaces associated to R-schemes. Let \( X \) be a scheme of finite type over \( \mathbb{R} \). We denote the associated real analytic space by \( X_{\mathbb{R}} \). The underlying topological space of \( X_{\mathbb{R}} \) is the set of \( \mathbb{R} \)-valued points \( X(\mathbb{R}) \) with the classical topology. The sheaf of real analytic functions on \( X_{\mathbb{R}} \) is defined by taking local embeddings of \( X \) into affine spaces and dividing the sheaf of real analytic functions on the affine spaces by the corresponding ideal.

We define \( X_{\mathbb{C}} \) similarly for a scheme \( X \) of finite type over \( \mathbb{C} \). In case \( X \) is a scheme of finite type over \( \mathbb{R} \), \( X_{\mathbb{C}} \) means the complex algebraic variety associated to the base change of \( X \) by \( \mathbb{R} \to \mathbb{C} \). So the underlying topological space of \( X_{\mathbb{C}} \) coincides with \( X(\mathbb{C}) \).

1.2. Hironaka’s resolution of singularities. Let \( X \) be a smooth scheme over \( \mathbb{R} \), and \( D \) an effective divisor on \( D \). By Hironaka [5] we have a resolution of singularities \( \pi : (X', D') \to (X, D) \) which is a composition of blowing-ups along smooth centers defined over \( \mathbb{R} \) and such that \( D' \) is a divisor with normal crossings which is locally defined by algebraic local coordinates defined over \( \mathbb{R} \), see loc. cit., Cor. 3 in p. 146 and also Def. 2 in p. 141. (Note that the last condition implies that the irreducible components \( D'_j \) of \( D' (j \in J) \) are smooth over \( \mathbb{R} \) by taking a point of \( \text{Sing} D'_j \)).
This induces a resolution of singularities \( \pi_\mathbb{R} : (X'_\mathbb{R}, D'_\mathbb{R}) \to (X_\mathbb{R}, D_\mathbb{R}) \) as in Introduction, and
\[
J_\mathbb{R} = \{ j \in J \mid D'_j(\mathbb{R}) \neq \emptyset \}.
\]
Note that if a smooth center \( C \) of a blow-up has a real point \( x \), then \( C \) is defined locally by using local algebraic coordinates over \( \mathbb{R} \), and hence \( C_\mathbb{R} \) is a smooth subvariety.

2. Proofs of Proposition 1 and Theorem 2

In this section we prove Proposition 1 and Theorem 2.

2.1. Proof of Proposition 1. With the notation of Introduction, we have locally
\[
\pi^* g f^{-\alpha} dx_1 \cdots dx_n = v \prod_{i=1}^r x_i'^{a_i + b_i - \alpha m_i} dx_1' \cdots dx'_n,
\]
if \( \pi^* g = u'' \prod_i x_i'^{b_i} \) locally, where \( v, u'' \) are nondivisible by \( x_i' (1 \leq i \leq r) \). For \( \gamma, c > 0 \), we have
\[
\int_0^c x^{\gamma-1} dx = \frac{c^\gamma}{\gamma},
\]
where \( x \) means \( x_i' \). Moreover, we have for \( \beta = \alpha m_i \) and \( p = a_i + b_i \)
\[
(2.1.1) \quad p \geq \lfloor \beta \rfloor \iff p > \beta - 1.
\]
So the implication \( \Leftarrow \) in Proposition 1 follows. For the converse, assume the right-hand side does not hold. Then the left-hand side does not hold by restricting to a neighborhood of a sufficiently general point of \( D'_j(\mathbb{R}) \) which is defined locally by \( x_i' = 0 \) and such that \( a_i + b_i - \alpha m_i \leq -1 \) (using positivity). So the assertion follows.

2.2. Proof of Corollary 1. By definition the minimal real jumping number is the smallest number \( \alpha \) such that \( 1 \notin \mathcal{J}(X_\mathbb{R}, f^\alpha) \), i.e. \( |f|^{-\alpha} \) is not locally integrable on \( X_\mathbb{R} \). By Proposition 1, this condition is equivalent to that \( a_j < \lfloor \alpha m_j \rfloor \) (i.e. \( a_j \leq \alpha m_j - 1 \), see (2.1.1)) for some \( j \in J_\mathbb{R} \). So the assertion follows.

2.3. Proof of Corollary 2. We take a resolution of singularities as in (1.2). This gives a resolution of singularities of the complexification. We define similarly \( a_j, m_j \) for any irreducible components \( D'_j \) of \( D' (j \in J) \), and we have as in [7]
\[
\text{lct}(f_\mathbb{C}) = \min_{i \in J} \left\{ \frac{a_j + 1}{m_j} \right\}.
\]
So the last inequality follows. Since \( \text{rlct}(f) \) is the smallest number \( \alpha \) such that \( |f|^{-\alpha} \) is not locally integrable on \( X_\mathbb{R} \), the first inequality and the last assertion follow.

2.4. Proof of Theorem 2. Let \( f_+(x) = f(x) \) if \( f(x) > 0 \) and \( f_+(x) = 0 \) otherwise. Set \( f_- = (-f)_+ \). Since \( |f|^s = (f_+)^s + (f_-)^s \), we consider
\[
I(\omega, s) = \int_{X_\mathbb{R}} (f_+)^s \omega,
\]
where \( \omega \) is a \( C^\infty \) form of the highest degree whose support is compact and is contained in a sufficiently small open subset \( U_\mathbb{R} \) of \( X_\mathbb{R} \) with local coordinates.
$x_1, \ldots, x_n$ giving an orientation of $U_R$. Then $I(\omega, s)$ is a holomorphic function on $\{ s \in \mathbb{C} \mid \text{Re } s > 0 \}$, and it is extended to a meromorphic function on the entire complex plane using a resolution of singularities, see \cite{1}, \cite{2}.

Let $x$ be a point of $D_R := f^{-1}(0) \subset X_R$, and $b_f(s)$ be the $b$-function of $f$ at $x$. We assume that $U_R$ is a sufficiently small open neighborhood of $x$ in $X_R$ so that we have the relation

\begin{equation}
(2.4.1) \quad b_f(s)f^s = Pf^{s+1} \text{ in } (\mathcal{O}_U[\frac{1}{f}])[s], \quad P \in \Gamma(U_R, \mathcal{D}_U[s]).
\end{equation}

Here $P$ is replaced by $-P$ if $f_+$ is replaced by $f_-$ (and $f$ by $-f$). Note that (2.4.1) holds in $\mathcal{O}_U[\frac{1}{f}]$ when $s$ is specialized to any complex number.

Let * be the involution of $\mathcal{D}_U$ such that $g^* = g$ for $g \in \mathcal{O}_U$, $(\partial/\partial x_i)^* = -\partial/\partial x_i$, and $(Q_1Q_2)^* = Q_2^*Q_1^*$ for $Q_1, Q_2 \in \mathcal{D}_U$, fixing the local coordinates $x_1, \ldots, x_n$ on $U$. This gives a right $\mathcal{D}_U$-module structure on $\Omega^n_U$ using a basis $dx_1 \wedge \cdots \wedge dx_n$.

Write $P = \sum_j P_j s^j$ with $P_j \in \mathcal{D}_U$, and set $P^* = \sum_j P^*_j s^j$. Let $r = \max\{ \text{ord } P_j \}$. Then, for any complex number $s$ with $\text{Re } s > r$, we have by (2.4.1) together with integration by parts

\begin{equation}
(2.4.2) \quad b_f(s)I(\omega, s) = \int_{U_R} b_f(s)(f_+)^*\omega = \int_{U_R} (f_+)^{s+1}(P^*\omega) = \sum_j I(P_j^*\omega, s+1)s^j,
\end{equation}

since $\prod((\partial/\partial x_i)^{\nu_i}(f_+)^s)$ is a continuous function on $U$ if $\text{Re } s > \sum_i \nu_i$. Here $P_j^*\omega$ is defined by trivializing $\Omega^n_U$ by $dx_1 \wedge \cdots \wedge dx_n$, and it may be written as $\omega P_j$ using the right $\mathcal{D}$-module structure explained above. By analytic continuation, (2.4.2) holds as meromorphic functions in $s$ on the entire complex plane.

Let $\alpha$ be a real jumping number of $f$ which is smaller than $\text{rlct}(f) + 1$. Assume that the above $x$ belongs to the support of $G(X_R, f^\alpha)$, and $\omega(x) \neq 0$. There is $g \in \Gamma(U_R, \mathcal{O}_U)$ such that $g \in \mathcal{J}(U_R, f^\alpha-\varepsilon)_x$ for $\varepsilon > 0$ and $g \notin \mathcal{J}(U_R, f^\alpha)_x$ (shrinking $U_R$ if necessary). Then $I(g\omega, s)$ is a holomorphic function in $s$ for $\text{Re } s > -\alpha$ using a resolution of singularities as in (2.1), and

\[ I(g\omega, s) \to +\infty \quad \text{as } s \to -\alpha, \]

(replacing $f_+$ with $f_-$ if necessary). On the other hand, the $I(P_j^*(g\omega), s+1)$ are holomorphic functions in $s$ for $\text{Re } s+1 > -\text{rlct}(f)$. Thus, replacing $\omega$ with $g\omega$ in (2.4.2), we get $b_f(-\alpha) = 0$ since $-\alpha + 1 > -\text{rlct}(f)$. So the assertion follows.

Remark. This argument shows that the order of pole of $I(\omega, s)$ at $-\text{rlct}(f)$ is at most the multiplicity of $-\text{rlct}(f)$ as a root of $b_f(s)$.

2.5. $b$-Function of the complexification. For $f \in \mathbb{R}\{\{x\}\}$, the $b$-function $b_f(s)$ of $f$ coincides with the $b$-function $b_{fc}(s)$ of the complexification $fc$ (which is identified with $f$ by $\mathbb{R}\{\{x\}\} \subset \mathbb{C}\{\{x\}\}$), since $b_{fc}(s) \in \mathbb{Q}[s]$ by Kashiwara \cite{6}.

Indeed, if there is $P = \sum_{\nu, \mu, k} a_{\nu, \mu, k} x^\nu \partial^\mu s^k$ with $a_{\nu, \mu, k} \in \mathbb{C}$ and satisfying

\[ b_{fc}(s)f^s = Pf^{s+1}, \]

then the same equation holds with $P$ replaced by $\sum_{\nu, \mu, k}(\text{Re } a_{\nu, \mu, k}) x^\nu \partial^\mu s^k$. 

2.6. Case of ideals. For an ideal $\mathcal{I}$ generated by $f_1, \ldots, f_r$, we may define the multiplier ideals $\mathcal{J}(X_R, \mathcal{I}^\alpha)$ by local integrability of \[ |g|/\left(\sum_i |f_i|^\alpha\right). \]
However, this is calculated by $\mathcal{J}(X_R, f^{\alpha/2})$ with $f = \sum_i f_i^2$, using \[ \sum_i |f_i|^2 \leq \left(\sum_i |f_i^\alpha|\right)^2 \leq r \sum_i |f_i|^\alpha. \]

3. Proof of Theorem 1

In this section we prove Theorem 1 by constructing examples.

3.1. Definition. We say that $f$ is of ordinary type if $\text{codim } D_R = 1$, and of exceptional type otherwise. Here $D_R = f^{-1}(0) \subset X_R$.

Write $f = \sum_{k \geq d} f_k \in R\{(x_1, \ldots, x_n)\}$ with $f_k$ homogeneous of degree $k$ and $f_d \neq 0$. We say that $f$ has an isolated zero of simple type if the equation $f_d = 0$ has no solution in $R^n \setminus \{0\}$ (e.g. if $f_d = \sum_{i=1}^n x_i^d$ with $d$ even).

3.2. Remarks. (i) The function $f$ is of ordinary type if and only if the reduced complex zero locus $(D_C)_{\text{red}}$ has a smooth real point. Note that \[ \dim_R(D_R \cap \text{Sing } (D_C)_{\text{red}}) < n-1, \] since $\text{Sing } (D_C)_{\text{red}}$ is defined over $R$ and has dimension $< n-1$ where $n = \dim X_R$.

(ii) In the case of exceptional type, the $D'_j$ for $j \in J_R$ are all exceptional divisors.

(iii) In the case of ordinary type, we have $\mathcal{J}(X_R, f^\alpha) \subset fO_{X_R}$ for $\alpha \geq 1$, and hence \[ (3.2.1) \quad f\mathcal{J}(X_R, f^\alpha) = \mathcal{J}(X_R, f^{\alpha+1}) \text{ for } \alpha > 0, \] shrinking $X_R$ to an open neighborhood of the points where the dimension of $D_R$ is $n-1$.

(iv) The above equality (3.2.1) always holds in the complex case, and \[ (3.2.2) \quad \text{JN}(f_C) = (\text{JN}(f_C) \cup \{0, 1\}) + N, \] where JN($f_C$) is the set of jumping numbers of $f_C$.

The following Proposition implies the first and second assertions of Theorem 1 in the case $n > d$, since we have always lct($f_C$) $\leq 1$.

3.3. Proposition. If $f$ has only an isolated zero of simple type (3.1), then \[ \mathcal{J}(X_R, f^\alpha_0) = m_0^{[\alpha d-n+1]} \quad \text{RJN}(f) = \{k/d \mid k \geq n\}, \quad \text{rlct}(f) = n/d, \] where $m_0$ be the maximal ideal of $O_{X_R, 0}$ and RJN($f$) denote the set of real jumping numbers of $f$.

Proof. In this case, we get a real resolution of singularities by the blow-up along the origin, and $D_R = \{0\}$ since the exceptional divisor is the total transform of $D_R$. In particular, $f$ is of exceptional type, see (3.1). Then $J_R = \{1\}$ and $(m_1, a_1) = (d, n-1)$. So the assertion follows from Proposition 1.
3.4 Example. Assume we have an expansion
\[ f = f_{d_1} + \sum_{k \geq d_2} f_k \in \mathbb{R}\{x_1, \ldots, x_n\}, \]
with \( f_k \) homogeneous of degree \( k \), \( f_{d_1} = g^e \) with \( g \) irreducible, \( h := f - g^e = \sum_{k \geq d_2} f_k \) is nondivisible by \( f_{d_1} \), and \( d_1 < d_2 \). Let \( Y \subset \mathbb{P}^{n-1} \) be the projective hypersurfaces defined by \( g \). Assume
\[ c := d_2 - d_1 \geq e \geq 2, \quad n > d := d_1/e, \]
and \( Y_\mathbb{R} \) is empty in the notation of (1.1). Then \( f \) has an isolated zero of simple type at the origin, and
\[ \text{rlct}(f) > \text{lct}(f_\mathbb{C}), \]
restricting \( f \) to a sufficiently small Zariski-open subset \( X \) of the affine space \( \mathbb{A}^n \) containing the origin and such that it is the only singular point of \( f \).

Indeed, let \( \pi : X' \rightarrow X \) be a resolution of singularities as in (1.2). Here we may blow-up along the origin first. Let \( D'_1 \subset X' \) denote the proper transform of the exceptional divisor of this blow-up. The pull-back of \( f \) by the blow-up along the origin is locally given by \( x_{d_1}^d(y^e + x^ez) \), where the exceptional divisor is locally defined by \( x = 0 \), and the proper transforms of \( g \) and \( h \) are locally given by \( y \) and \( z \) respectively. So the intersection of the proper transform of \( D \) and the exceptional divisor by the blow-up along the origin is identified with \( Y \), and the total transform of \( D \) is not a divisor with normal crossings at the generic point of \( Y \) since \( c \geq e \geq 2 \) and \( h \) is nondivisible by \( f_{d_1} \). So we have to blow-up along the proper transform of \( Y \) (after making it smooth). Let \( D'_2 \subset X' \) denote the proper transform of the exceptional divisor of this blow-up. Then we have
\[ \text{rlct}(f) = \frac{a_1 + 1}{m_1} = \frac{n}{d_1} > \frac{a_2 + 1}{m_2} = \frac{n + 1}{d_1 + e} \geq \text{rlct}(f_\mathbb{C}). \]

This also implies the first assertion of Theorem 1 with \( \text{rlct}(f) < 1 \) if \( n < d_1 \).

3.5 Example. With the above notation and assumptions, assume further
\[ h = f_{d_2}, \quad n = 3, \quad c = d = e = 2, \]
and \( Y_\mathbb{C} \) is smooth and intersects \( Z_\mathbb{C} \) at smooth points of \( Z_\mathbb{C} \), where \( Z \) is the hypersurface defined by \( h \). Then the resolution \( \pi : X' \rightarrow X \) is obtained by the two blowing-ups in Example (3.4), and we have \( J = \{1, 2\} \), \( J_\mathbb{R} = \{1\} \), \( m_1 = 4 \), \( m_2 = 6 \). So \( f \) has an isolated singularity at the origin, and the eigenvalues \( \lambda \) of the Milnor monodromy on \( H^2(F_0, \mathbb{C}) \) satisfy \( \lambda^4 = 1 \) or \( \lambda^6 = 1 \), where \( F_0 \) denotes the Milnor fiber.

For \( \lambda = i \), the \( \lambda \)-eigenspace of the Milnor cohomology \( H^2(F_0, \mathbb{C})_\lambda \) is calculated by the filtered de Rham complex of a filtered simple regular holonomic \( \mathcal{D} \)-module \((M, F)\) on \( \mathbb{P}_\mathbb{C}^2 \) whose restriction to the complement of \( Y_\mathbb{C} \) is a complex variation of Hodge structure of type \((0, 0)\) and rank 1, and whose local monodromy around \( Y_\mathbb{C} \) is \(-1\), see [9] (or [8], 3.3 and 3.5). Since \( F_0M \) is a line bundle such that \( \otimes^2 F_0 M = \mathcal{O}_{\mathbb{P}_2}(Y) \), we have \( F_0 M = \mathcal{O}_{\mathbb{P}_2}(1) \). Since \( \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)) = 0 \), this implies
\[ F^2 H^2(F_0, \mathbb{C})_\lambda = 0. \]
Thus \( \text{rlct}(f) (= 3/4) \) does not appear in the spectrum \([9]\) of \( f_C \). Then \( 3/4 \) is not a jumping number of \( f_C \) by \([3]\) in the isolated singularity case. Since the minimal jumping number of \( f_C \) is \( 2/3 \) by (3.4.1), we get by (3.2.2)

\[
\text{JN}(f_C) \subset \left\{ \frac{k}{6} + j \mid k = 4, 5, 6; j \in \mathbb{N} \right\}.
\]

(In fact, we can show the equality.) On the other hand, we have by Proposition (3.3)

\[
\text{RJN}(f) = \left\{ \frac{k}{4} \mid k \geq 3 \right\}.
\]

This implies the last assertion of Theorem 1.

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