Sampling colorings almost uniformly in sparse random graphs

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Abstract

The problem of sampling proper $q$-colorings from uniform distribution has been extensively studied. Most of existing samplers require $q \geq \alpha \Delta + \beta$ for some constants $\alpha$ and $\beta$, where $\Delta$ is the maximum degree of the graph. The problem becomes more challenging when the underlying graph has unbounded degree since even the decision of $q$-colorability becomes nontrivial in this situation. The Erdős-Rényi random graph $G(n, d/n)$ is a typical class of such graphs and has received a lot of recent attention. In this case, the performance of a sampler is usually measured by the relation between $q$ and the average degree $d$. We are interested in the fully polynomial-time almost uniform sampler (FPAUS) and the state-of-the-art with such sampler for proper $q$-coloring on $G(n, d/n)$ requires that $q \geq 5.5d$.

In this paper, we design an FPAUS for proper $q$-colorings on $G(n, d/n)$ by requiring that $q \geq 3d + O(1)$, which improves the best bound for the problem so far. Our sampler is based on the spatial mixing property of $q$-coloring on random graphs. The core of the sampler is a deterministic algorithm to estimate the marginal probability on blocks, which is computed by a novel block version of recursion for $q$-coloring on unbounded degree graphs.

1 Introduction

The problem of sampling from Gibbs measure has received extensive studies in both theoretical computer science and statistical physics. One notable example is the problem of sampling proper $q$-colorings from the uniform distribution, where a proper $q$-coloring on a graph $G(V, E)$ is a function $\sigma : V \rightarrow [q]$ such $\sigma(u) \neq \sigma(v)$ for every $(u, v) \in E$. We are interested in sampling algorithms which are fully polynomial-time almost uniform samplers (FPAUS), such that for any $\delta > 0$, it outputs a random proper $q$-coloring with total variation distance $\delta$ from the uniform distribution over all proper $q$-colorings in time polynomial in $\log \frac{1}{\delta}$ and the size of the graph.

The performances of algorithms for sampling proper $q$-colorings are compared by the bound on $q$ in terms of the maximum degree $d$ of the graph, usually represented in the form of $q \geq \alpha d + \beta$. We are fine with a constant $\beta = O(1)$, since it is same to say the result holds for sufficiently large constant $d$, and hence the performance of the algorithm is represented more critically by the bound on $\alpha$. The current best result is that $\alpha = \frac{1}{10}$ [Vig00], which is based on the method of Markov chain Monte-Carlo (MCMC). See [FV07] for a survey.

In recent years, much attention has been focused on the spatial mixing property (decay of correlation) of the Gibbs measure. The spatial mixing property says that when a random solution

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is sampled, the correlation between a variable and a boundary condition decays exponentially as the distance between them grows. In [GK12], this property was used to support deterministic approximate counting of the number of proper $q$-colorings or list-colorings, and the best current bound for deterministic approximate counting was achieved in [LY13], with $\alpha \approx 2.58$. The spatial mixing is already a widely studied topic in computer science. Besides the algorithmic implications, the “spatial mixing only” results [GMP05, GKM13, Yin14] are interesting in their own rights.

The problem of sampling proper $q$-colorings is substantially more complicated when the graph has unbounded degree, since in such case even the decision of $q$-colorability is nontrivial. A typical model for graphs with bounded average degree is the Erdős-Rényi random graph $G(n, d/n)$ with constant $d$, whose average degree is bounded by $(1 + \epsilon)d$ but the maximum degree is $\Theta\left(\log \log n\right)$ with high probability. The study was initiated in [DFFV06] that an MCMC based algorithm was given for sampling $q$-colorings in sparse random graphs with $q = \Theta\left(\frac{\log \log n}{\log \log \log n}\right)$, which is less than the maximum degree. In subsequent works [ES08, MS10], the bound on $q$ was decreased to a constant which is independent of $n$ but still grows fast in $d$. In a recent work [Eft14a], the bound was greatly improved to $q \geq 5.5d$, which is the best bound achieved so far for FPAUS.

Better bounds on $q$ can be achieved if we relax the requirement on the quality of sampling. In [Eft12, Eft14b], a sampling algorithm for proper $q$-colorings of $G(n, d/n)$ approaching the uniqueness threshold $q \geq (1 + \epsilon)d$ was introduced. The sampler is weaker than FPAUS: it samples proper $q$-colorings with a fixed inverse-polynomial total variation distance.

Sampling from Gibbs measure of other models on sparse graphs has also been well-studied. For example, in [MS09] an FPAUS for Ising model was presented for “locally tree-like” graphs which include $G(n, d/n)$ with constant $d$ as a special case. The result is stated in terms of the maximum interaction strengths $\beta_d$ which only depends on the average degree $d$. Recently, a tight relation between the decay of correlation and the notion of connective constant, a well-studied measure of average degree of a graph, has been established [SSY13, SSY15]. In this line of work, FPAUSes for hard core model and monomer-dimer model with optimal bounds in terms of connective constants are obtained. These results also apply to $G(n, d/n)$ with constant $d$ since graphs drawn from $G(n, d/n)$ are of connective constant $(1 + o(1))d$ with high probability.

1.1 Our contribution

We give an FPAUS for sampling proper $q$-colorings of sparse random graph $G(n, d/n)$ with constant $d$ when $q \geq \alpha d + \beta$ for $\alpha = 3$ and $\beta = O(1)$. By a loose estimation, $\beta$ can be bounded as $\beta < 250$. We remark that our focus is to improve $\alpha$. The effect of $\beta$ is the same as to say the result hold for sufficiently large constant $d$.

Theorem 1. There is a constant $\beta$, such that for any $d > 1$, if $q \geq 3d + \beta$, there exists an algorithm $S$, such that for $G \sim G(n, d/n)$, with probability $1 - O\left(\frac{1}{n}\right)$, $G$ is $q$-colorable and $S$ is an FPAUS for proper $q$-colorings of $G$.

In fact, our sampling algorithm works for a family of instances of the list-coloring problem, defined on sparse graphs. The general characterization is stated by Theorem 5 in next section.

Technique-wise, our sampling algorithm is based on spatial mixing, while almost all other FPAUSes for the problem relies on the path coupling of (block) Glauber dynamics, with only one exception [ES08] which also used spatial mixing. Our algorithm not only greatly improves the spatial mixing bound in [ES08] but also improves the state-of-the-arts for the problem.
Very interestingly, although our algorithm samples random solutions, it does not involve any randomness until the last step, which is a block-variant of the standard routine of Jerrum-Vazirani. Our sampling algorithm can be seen as a combination of a block version of the correlation decay based algorithm for approximate counting of list-colorings, and a block version of JVV sampler. This block view was generic in Glauber dynamics, and was adopted in a previous work [Yin14] to analyze spatial mixing of colorings of sparse graphs with unbounded degree. Compared with [Yin14], the decay of correlation established in the current paper emphasizes its algorithmic implications.

A comparison of our result with related results is listed in Table 1.1.

| Dependence on $q$ | Quality of the Sampler | Technique | Reference |
|--------------------|------------------------|-----------|-----------|
| $q \geq \Theta \left( \frac{\log \log n}{\log \log \log n} \right)$ | FPAUS | MCMC | [DFFV06] |
| $q \geq \text{poly}(d)$ | FPAUS | spatial mixing | [ES08] |
| $q \geq f(d)$ for some $f(\cdot)$ | FPAUS | MCMC | [MS10] |
| $q \geq 5.5d$ | FPAUS | MCMC | [Eft14a] |
| $q \geq (1 + \varepsilon)d$ | weaker sampler | combinatorial approach | [Eft12, Eft14b] |
| $q \geq (2 + \varepsilon)d$ | no sampler | spatial mixing | [Yin14] |
| $q \geq 3d$ | FPAUS | spatial mixing | this work |

Table 1: A comparison of results on sampling $q$-colorings in $G(n, d/n)$. All the bounds mentioned here hold for sufficiently large constant $d$.

2 Statement of the Main Result

Our main result holds more generally for a family of instances for the list-coloring defined on sparse graphs. The random graph $G(n, d/n)$ along with the homogeneous color-list $[q]$ falls into this family with high probability.

Let $G = (V, E)$ be a graph, $q > 0$ be an integer and denote $[q] = \{1, 2, \ldots, q\}$. A list-coloring instance is a pair $(G, L)$ where $L = \{L(v) : v \in V\}$ such that each $L(v) \subseteq [q]$ specifies a list of colors for vertex $v$. A proper coloring of $(G, L)$ is a assignment $\sigma : V \rightarrow [q]$ of colors to vertices such that $\sigma(v) \in L(v)$ and $\sigma(u) \neq \sigma(v)$ for every $(u, v) \in E$. A list-coloring instance $(G, L)$ is feasible if a proper coloring $\sigma$ exists. In the case that $L(v) = [q]$ for every $v \in V$, $(G, L)$ is an instance of $q$-coloring and we write it as $(G, [q])$.

**Definition 2.** Given a list-coloring instance $(G, L)$, a vertex $v$ is said to be permissive if for all neighbors $u$ of $v$ and $u = v$, it holds that $|L(u)| \geq \deg(u) + 5$.

**Definition 3.** Given a list-coloring instance $(G, L)$, a vertex set $B \subseteq V$ is a permissive block if for every $u \in \partial B$, it holds that $|L(u)| \geq \deg(u) + 5$, where $\partial B = \{u \in V \setminus B \mid \exists w \in B, (u, w) \in E\}$ is the vertex boundary of $B$. We denote by $B(v) = B_{G,L}(v)$ the minimal permissive block containing $v$.

Given a graph $G = (V, E)$ and a vertex $v \in V$, a rooted tree $T$ can be naturally constructed from all self-avoiding walks starting from $v$ as follows: Each vertex in $T$ corresponds to a self-avoiding walk (simple path in $G$) $P = (v, v_1, v_2, \ldots, v_k)$ starting from $v$, whose children correspond to all self-avoiding walks $(v, v_1, v_2, \ldots, v_k, v_{k+1})$ extending $P$, and the root of $T$ corresponds to the trivial
walk \(v\). The resulting tree, denoted by \(T_{SAW}(G,v)\), is called the \textit{self-avoiding walk (SAW) tree} constructed from vertex \(v\) in graph \(G\).

From this construction, every vertex in \(T = T_{SAW}(G,v)\) can be naturally identified with the vertex in \(V\) (many-to-one) at which the corresponding self-avoiding walk ends.

**Definition 4.** Let \((G,L)\) be a list-coloring instance, \(G = (V,E)\) and \(v \in V\). Let \(T = T_{SAW}(G,v)\). For \(\ell > 1\), a set \(S\) of vertices in \(T\) is called a permissive \(\ell\)-cutset, if: (1) the depths of all vertices in \(S\) are in \([\ell, 2\ell]\); (2) every self-avoiding walk from \(v\) of length \(\geq 2\ell\) must intersect with \(S\) in \(T\); and (3) every vertex in \(S\) is identified to a permissive vertex in \(G\) by \(T_{SAW}(G,v)\).

We then define two kinds of weight for paths.

Given an instance of list-coloring \((G,L)\), the piecewise function \(\delta_{G,L} : V \rightarrow \mathbb{R}^+\) is defined as follows:

\[
\delta_{G,L}(v) \triangleq \begin{cases} 
\frac{2}{|L(v)|-\deg_G(v) - 2} & \text{if } \deg_G(v) \leq |L(v)| - 5, \\
1 & \text{otherwise}.
\end{cases}
\tag{1}
\]

Let \(P = (v,v_1,\ldots,v_k)\) be a path in \(G\) from \(v\). The \textit{weight} and \textit{effective length} of \(P\), denoted respectively by \(w_{G,L}(P)\) and \(W_{G,L}(P)\), are defined as follows:

\[
w_{G,L}(P) \triangleq 2q \prod_{i=1}^{k} \delta_{G,L}(v_i) \quad \text{and} \quad W_{G,L}(P) \triangleq \exp \left( \left| \bigcup_{u \in P} B_G,L(u) \right| \ln q \right) = q^{\left| \bigcup_{u \in P} B_G,L(u) \right|}.
\]

Given \(\ell > 1\), for every vertex \(v \in V\), we define the following quantities:

\[
\mathcal{E}_\ell(v) \triangleq \sum_{k=\ell}^{2\ell-1} \sum_{\text{self-avoiding walk } P=(v,v_1,\ldots,v_k) \text{ in } G} w_{G,L}(P) \quad \text{and} \quad \tau_\ell(v) \triangleq \sum_{k=1}^{2\ell-1} \sum_{\text{self-avoiding walk } P=(v,v_1,\ldots,v_k) \text{ in } G} W_{G,L}(P).
\]

Our main theorem gives a sufficient condition for the existence of a fully polynomial-time almost uniform sampler (FPAUS) for proper list-colorings in terms of these parameters.

**Theorem 5.** There exists an FPAUS for proper list-coloring for a family of list-coloring instances if the following conditions are satisfied: For some \(\ell(n) = O(\log n)\), it holds that for every instance \((G,L)\) in the family, where \(G = (V,E)\) and \(|V| = n\), for every \(v \in V\),

1. there exists a permissive \(\ell(n)\)-cutset \(S\) in \(T_{SAW}(G,v)\);
2. \(\mathcal{E}_\ell(n)(v) = O\left(\frac{1}{n^\alpha}\right)\);
3. \(\tau_\ell(n)(v) = n^{O(1)}\).

The first two conditions guarantee a decay of correlation at exponential rate from a permissive boundary to a vertex, and have appeared in a previous work \cite{Yin14}. The third condition, which is new, is for bounding running time, such that it guarantees the number of self-avoiding walks of length \(\ell(n)\) to be polynomially bounded, and every such walk to be contained by a permissive block of size \(O(\log n)\).
3 Preliminaries

Graphs. Let \( G(V,E) \) be a graph. Let \( S \subseteq V \) be a set of vertices. We use \( G[S] \) to denote the subgraph of \( G \) induced by \( S \). We use \( \partial S \) and \( \delta S \) to indicate the vertex boundary and the edge boundary of \( S \) in \( G \) respectively, i.e., \( \partial S \triangleq \{ v \in V \setminus S \mid \exists u \in S, (u,v) \in E \} \) and \( \delta S \triangleq \{(u,v) \in E \mid u \in S \text{ and } v \in V \setminus S \} \).

Feasibility and Local Feasibility. Let \( (G(V,E), \mathcal{L}) \) be a feasible instance of list-coloring with \( L(v) \subseteq [q] \) for all \( v \in V \). Let \( S \subseteq V \) and \((G[S], \mathcal{L}_S)\) be the instance where \( \mathcal{L}_S = \{ L(v) \mid v \in S \} \). A coloring \( \sigma : S \rightarrow [q] \) is called (globally) feasible if there exists a proper coloring \( \pi : V \rightarrow [q] \) of \( (G,L) \) such that \( \rho \) and \( \pi \) are consistent on \( S \). We use \( L(S) \) to denote the set of all such feasible colorings. A coloring \( \sigma : S \rightarrow [q] \) is called locally feasible if it is a proper coloring of \((G[S], \mathcal{L}_S)\).

For permissive blocks, local feasibility implies global feasibility.

**Proposition 6.** Let \( (G(V,E), \mathcal{L}) \) be a feasible list-coloring instance. Let \( B \subseteq V \) be a permissive block and \( \pi \) be a locally feasible coloring on \( B \). Then \( \pi \) is feasible, i.e., \( \pi \in L(B) \).

**Proof.** Consider the instance \( (G', \mathcal{L}') \) where \( G' = G[V \setminus B] \) and \( \mathcal{L}' = \{ L'(v) \mid v \in V \setminus B \} \) such that \( L'(v) = L(v) \) for all \( v \notin \partial B \) and \( L'(v) = L(v) \setminus \pi(u) \) for all \( v \in \partial B \). This instance is equivalent to the instance of \((G,L)\) after pinning \( S \) with coloring \( \pi \). Since \((G,L)\) is feasible, there is a proper coloring for \( G[V \setminus (B \cup \partial B)] \). By our construction, it is clear that for every \( v \in \partial B \), \( |L'(v)| - \deg_G(v) \geq |L(v)| - \deg_G(v) \). We can therefore extend \( \sigma \) to a proper coloring of \((G', \mathcal{L}')\) in a greedy fashion and this proves that \( \pi \) is feasible.

With this proposition, we do not distinguish between locally and globally feasible colorings of a permissive block, and we just refer them as proper colorings of \( B \).

Gibbs Measure for Graph Coloring. Let \( (G(V,E), \mathcal{L}) \) be a feasible instance of list-coloring. The uniform distribution over all proper colorings of \((G,L)\) is called the Gibbs measure of list-coloring and is denoted by \( \mu \). For a vertex \( v \in V \) and \( c \in L(v) \), we use \( \Pr_{G,L}[\sigma(v) = c] \) to denote the marginal probability that \( v \) is assigned color \( c \) by \( \sigma \) when \( \sigma \) is sampled according to the Gibbs measure on \((G,L)\). For a set \( S \subseteq V \) and \( \pi \in L(S) \), we use \( \Pr_{G,L}[\sigma(S) = \pi] \) to denote the marginal probability that the set \( S \) is assigned coloring \( \pi \) by \( \sigma \).

Recursion for List Coloring. Let \((G(V,E), \mathcal{L})\) be a feasible instance of list-coloring, \( v \in V \) be a vertex and \( c \in L(v) \) a color in the list of \( v \). Denote \( B = B(v) \) the minimal permissive block containing \( v \), and we enumerate the boundary edges in \( \partial B \) by \( e_i = (u_i, v_i) \) for \( i = 1, 2, \ldots, m \), where \( v_i \notin B \). Note that with this notation more than one \( u_i \) or \( v_i \) may refer to the same vertex.

We fix some notations. Let \( G_B = G[V \setminus B] \). For a coloring \( \rho \in L(B) \) and every \( i \in [m] \), we denote \( \rho_i = \rho(u_i) \), and define \( \mathcal{L}_i^\rho = \{ L_i^\rho(v) : v \in V \setminus B \} \) as that \( L_i^\rho(w) = \begin{cases} L(w) \setminus \{\rho_{j}\}, & \text{if } w = v_j \text{ for some } j < i \\ L(w), & \text{otherwise.} \end{cases} \)

**Lemma 7.** Let \((G(V,E), \mathcal{L})\) be a feasible list-coloring instance. Let \( v \in V \) be an arbitrary vertex and \( B = B(v) \) where \( \partial B = \{(u_i, v_i) \mid i \in [m]\} \). Then for every \( i \in [m] \) and \( \rho \in L(B) \),

- if a vertex \( u \notin B \) is permissive in \((G(V,E), \mathcal{L})\), then it is permissive in \((G_B, \mathcal{L}_i^\rho)\);

- the instance \((G_B, \mathcal{L}_i^\rho)\) is feasible.
For every coloring \( \pi \in L(B) \), it holds that

\[
\Pr_{G,L}[\sigma(B) = \pi] = \frac{\prod_{i \in [m]} (1 - \Pr_{G_B,L_i^\pi}[\sigma(v_i) = \pi_i])}{\sum_{\rho \in L(B)} \prod_{i \in [m]} (1 - \Pr_{G_B,L_i^\rho}[\sigma(v_i) = \rho_i])}.
\]

Proof. In the construction, we never increase the degree of a vertex \( v_j \) and we decrease its degree by one as long as we remove a color from \( L(v_j) \). This fact implies that \(|L^\rho(u)| - \deg_G(u) \geq |L(u)| - \deg_G(u)\) for every \( u \in V \setminus B \). Thus if \( u \) is permissive in \((G,L)\), then it is permissive in \((G_B,L_i^\rho)\).

To verify the feasibility of \((G_B,L_i^\rho)\), note that the instance \((G[V \setminus (B \cup \partial B)], L)\) is feasible since our construction of \((G_B,L_i^\rho)\) only changes the color list and the degree of vertices in \( \partial B \). Since for every \( j \in [m] \), \(|L^\rho(v_j)| - \deg_G(v_j) \geq |L(v_j)| - \deg_G(v_j) \geq 5\), we can extend a proper coloring of \( V \setminus (B \cup \partial B) \) to a proper coloring of \( V \setminus B \) in a greedy fashion.

We then have

\[
\Pr_{G,L}[\sigma(B) = \pi] \overset{\bullet}{=} \frac{\Pr_{G_B,L^\pi} \left[ \bigwedge_{i \in [m]} \sigma(v_i) \neq \pi_i \right]}{\sum_{\rho \in L(B)} \Pr_{G_B,L^\rho} \left[ \bigwedge_{i \in [m]} \sigma(v_i) \neq \rho_i \right]} \overset{\bigodot}{=} \frac{\prod_{i \in [m]} \Pr_{G_B,L^\pi} \left[ \sigma(v_i) \neq \pi_i \right]}{\sum_{\rho \in L(B)} \prod_{i \in [m]} \Pr_{G_B,L^\rho} \left[ \sigma(v_i) \neq \rho_i \right]} \overset{\bigodot}{=} \frac{\prod_{i \in [m]} \Pr_{G_B,L_i^\pi} \left[ \sigma(v_i) \neq \pi_i \right]}{\sum_{\rho \in L(B)} \prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\rho} \left[ \sigma(v_i) = \pi_i \right] \right)} \overset{\bullet}{=} \frac{\prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\rho} \left[ \sigma(v_i) = \pi_i \right] \right)}{\sum_{\rho \in L(B)} \prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\rho} \left[ \sigma(v_i) = \rho_i \right] \right)}.
\]

(\(\bigodot\)) follows from the definition of marginal probability.

(\(\bigoplus\)) writes a probability as a telescopic product.

(\(\bigodot\)) is due the following fact: condition on the event that a vertex \( w \) does not take some color \( c \) is equivalent to removing \( c \) from the color list of \( w \).

This recursion can be seen as a blocked version for the computation tree recursion introduced in [GKT12] for list-colorings of graphs of bounded degree.

Corollary 8. For every \( c \in L(v) \), it holds that

\[
\Pr_{G,L}[\sigma(v) = c] = \sum_{\pi \in L(B) \text{ s.t. } \pi(v) = c} \prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\pi} \left[ \sigma(v_i) = \pi_i \right] \right) \frac{\prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\rho} \left[ \sigma(v_i) = \rho_i \right] \right)}{\sum_{\rho \in L(B)} \prod_{i \in [m]} \left( 1 - \Pr_{G_B,L_i^\rho} \left[ \sigma(v_i) = \rho_i \right] \right)}.
\]
4 Algorithms for estimating marginal probabilities

We give algorithms for estimating marginal probabilities according to the above recursions. The algorithms $\text{marg}(G, \mathcal{L}, S, v, c)$ and $\text{marg-block}(G, \mathcal{L}, S, B(v), \pi)$ are two recursive procedures calling each other, with inputs as follows:

- $(G, \mathcal{L})$ a feasible list-coloring instance;
- $v$ a vertex in $G$ and $B(v)$ the minimal permissive block containing $v$;
- $c \in L(v)$ and $\pi \in L(B(v))$ a proper coloring of $B(v)$;
- $S$ a set of self-avoiding walks from $v$.

The set of self-avoiding walks $S$ describes the stopping condition for the algorithms: The algorithms stop if the computation path appears in $S$. We are then ready to describe our algorithm for estimating the marginal probability at a vertex.

**Algorithm 1: marg($G, \mathcal{L}, S, v, c$)**

1. If the single vertex path $(v) \in S$, then return $1/|L(v)|$;
2. Compute $B(v)$;
3. For every $\rho \in L(B(v))$, let $\hat{p}_\rho \leftarrow \text{marg-block}(G, \mathcal{L}, S, B(v), \rho)$;
4. Return $\min \left\{ \sum_{\pi \in L(B(v))} \hat{p}_\pi \left( \max \left\{ \frac{1}{|L(v)|}, \deg_G(v) \right\} \right)^{-1} \right\}$

To describe the algorithm for estimating the block marginals, we need to introduce some notations. Let $B = B(v)$, and as before we enumerate the boundary edges in $\delta B$ by $e_i = (u_i, v_i)$ for $i = 1, 2, \ldots, m$, where $v_i \notin B$. With this notation more than one $u_i$ or $v_i$ may refer to the same vertex, which is fine. For every $i \in [m]$ and $\rho \in L(B)$, define $\mathcal{L}_\rho^i$ as in Lemma 7 Then for every $i \in [m]$, we construct the following objects:

- Let $P_i = (v, w_1, w_2, \ldots, w_k, v_i)$ be a self-avoiding walk from $v$ to $v_i$ such that all intermediate vertices $w_i$ are in $B(v)$. Since $B(v)$ is a minimal permissive block, such walk always exists, and let $P_i$ be an arbitrary one of them if there are multiple ones.
- Let $S_i$ be the set of self-avoiding walks $P$ in $G_B = G[V \setminus B(v)]$ starting from $v_i$ such that the concatenated walk $P_i P$ is in $S$.

Note that $P_i$ can be constructed in time linear to the size of $B(v)$.

**Algorithm 2: marg-block($G, \mathcal{L}, S, B(v), \pi$)**

1. Compute $P_i$ and $S_i$ for every $i \in [m]$;
2. $\hat{p}_{i,\rho} \leftarrow \text{marg}(G_B, \mathcal{L}_\rho^i, S_i, v_i, \rho_i)$ for every $i \in [m]$ and $\rho \in L(B(v))$;
3. Return $\sum_{\rho \in L(B(v))} \Pi_{i \in [m]} (1 - \hat{p}_{i,\rho})$.

Assuming the existence of permissive $\ell$-cutset for $T_{\text{RAW}}(G, v)$, we can use the set of self-avoiding walks represented by the vertices in this cutset as our $S$. The construction of such $S$ can be done efficiently and implicitly: by maintaining a counter $\ell$, such that each $S_i$ is replaced by an $\ell_i = \ell - |P_i|$, and Algorithm 2 stops at the first permissive vertex encountered after $\ell$ becoming negative. We run $\text{marg-block}(G, \mathcal{L}, S, B(v), \pi)$ with this implementation and denote the output as $\hat{P}_{G,\mathcal{L},\pi}$. 

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Theorem 9. For a family of feasible list-coloring instances satisfying the conditions in Theorem 3 for any instance \((G, L)\) from the family where \(G = (V, E)\) and \(|V| = n\), for any \(v \in V\) and any proper coloring \(\pi \in L(B)\) of the minimal permissive block \(B = B(v)\) containing \(v\), the above algorithm returns a \(\hat{P}_{G, L, \pi}\) in time polynomial in \(n\), satisfying

\[
1 - O\left(\frac{1}{n^3}\right) \leq \frac{\hat{P}_{G, L, \pi}}{\Pr_{G, L}[\sigma(B(v)) = \pi]} \leq 1 + O\left(\frac{1}{n^3}\right).
\]

Note that our error bound is multiplicative, which is stronger than the standard additive errors for estimating marginal probabilities. This is due to our choice of potential function for measuring the errors, and is critical to the accuracy of our sampling algorithm which progressively sampling proper coloring of blocks instead of vertices.

4.1 Upper bound for errors

Let \(\hat{P}_{G, L, S}(v, c) = \text{marg}(G, L, S, v, c)\) and \(\hat{P}_{G, L, S}(B(v), \pi) = \text{marg-block}(G, L, S, B(v), \pi)\). We define the error functions:

\[
\mathcal{E}_{G, L, S}(v) \triangleq \max_{c \in L(v)} \left| \log \left( \Pr_{G, L}[\sigma(v) = c] \right) - \log \left( \hat{P}_{G, L, S}(v, c) \right) \right|,
\]

\[
\mathcal{E}_{G, L, S}(B(v)) \triangleq \max_{\pi \in L(B)} \left| \log \left( \Pr_{G, L}[\sigma(B(v)) = \pi] \right) - \log \left( \hat{P}_{G, L, S}(B(v), \pi) \right) \right|.
\]

We then calculate an upper bound for the errors.

Definition 10. Let \((G(V, E), L)\) be a feasible list-coloring instance, and \(v \in V\). Let \(T = T_{SAW}(G, v)\). Assume that the root \(v\) has \(d\) children \(v_1, v_2, \ldots, v_d\) in \(T\), and let \(T_i\) denote the subtree rooted by \(v_i\). Let \(S\) be a subset of vertices in \(T\). The quantity \(\mathcal{E}_{T, L, S}\) is recursively defined as

\[
\mathcal{E}_{T, L, S} = \begin{cases} 
\sum_{i \in [d]} \delta_{G, L}(v_i) \cdot \mathcal{E}_{T_i, L, S} & \text{if } v \notin S, \\
\frac{\delta_{G, L}(v)}{2q} & \text{otherwise},
\end{cases}
\]

where \(\delta_{G, L}(\cdot)\) is defined as \([1]\), and we slightly abuse the notation such that \(\delta_{G, L}(v_i)\) is evaluated for the vertex in \(G\) to which \(v_i\) is identified by \(T_{SAW}(G, v)\).

Recall that every vertex in the SAW tree \(T = T_{SAW}(G, v)\) corresponds to a self-avoiding walk starting from \(v\). For a set \(S\) of vertices in \(T\), we abuse the notation and denote also by \(S\) the set of self-avoiding walks in \(G\) represented by these tree-vertices, so that \(\mathcal{E}_{G, L, S}(v), \mathcal{E}_{G, L, S}(B(v))\) and \(\mathcal{E}_{T, L, S}\) are all well defined.

Lemma 11. Let \((G(V, E), L)\) be a feasible list-coloring instance, and \(v \in V\). Let \(S\) be a subset of vertices in \(T = T_{SAW}(G, v)\) such that every element of \(S\) corresponds to a self-avoiding walk in \(G\) which ends at a permissive vertex. It holds that

\[
\mathcal{E}_{G, L, S}(v) \leq \mathcal{E}_{G, L, S}(B(v)) \leq \mathcal{E}_{T, L, S}.
\]

We need a few auxiliary lemmas to prove Lemma [11].
Lemma 12. Let \((G(V,E),L)\) be a feasible list-coloring instance, and \(v \in V\). If \(\deg_G(v) \leq |L(v)| - 1\), then for every \(c \in L(v)\), it holds that

\[
\Pr_{G,L}[\sigma(v) = c] \leq \frac{1}{|L(v)| - \deg_G(v)}.
\]

If \(v\) is a permissive vertex, then it holds that

\[
\Pr_{G,L}[\sigma(v) = c] \geq \frac{1}{|L(v)|^{2\deg_G(v)}}.
\]

Proof. The upper bound is easy: Given an arbitrary coloring of \(v\)'s neighbors, at least \(|L(v)| - \deg_G(v)\) colors remain. To prove the lower bound when \(v\) is a permissive vertex, we apply the recursion in Corollary 8. Note that \(v\) is a permissive vertex and thus \(B(v) = \{v\}\). Assume \(\delta B(v) = \{(v,v_i) \mid i \in [m]\}\), we have

\[
\Pr_{G,L}[\sigma(v) = c] = \frac{\prod_{i \in [m]} (1 - \Pr_{G_{B(v)},L_\tau}[\sigma(v_i) = c])}{\sum_{x \in L(v)} \prod_{i \in [m]} (1 - \Pr_{G_{B(v)},L_\tau}[\sigma(v_i) = x])}.
\]

Since each \(v_i\) satisfies that \(\deg_{G_i}(v_i) \leq |L(v)| - 5\), we can apply the upper bound just proved and then the lower bound follows. \(\square\)

We now introduce a few notations and prove a technical lemma.

Let \(x = (x_{c,i})_{c \in [L], i \in [m]}\) be a tuple of \(L \times m\) variables where \(x_{c,i} \in [0, \beta_i]\) for some positive real \(\beta_i < 1\). Define

\[
h(x) \triangleq \frac{\prod_{i \in [m]} (1 - x_{1,i})}{\sum_{k \in [L]} \prod_{i \in [m]} (1 - x_{k,i})},
\]

\[
\phi(x) \triangleq \log x,
\]

\[
\Phi(x) \triangleq \frac{d\phi(x)}{dx} = \frac{1}{x}.
\]

Lemma 13. Let \(x = (x_{c,i})_{c \in [L], i \in [m]}\) and \(\tilde{x} = (\tilde{x}_{c,i})_{c \in [L], i \in [m]}\). Assume \(x_{c,i}, \tilde{x}_{c,i} \in [0, \beta_i]\) for every \(c \in [L]\) and \(i \in [m]\). It holds that

\[
|\phi(h(x)) - \phi(h(\tilde{x}))| \leq \sum_{i \in [m]} \frac{2\beta_i}{1 - \beta_i} \cdot \max_{c \in [L]} |\phi(x_{c,i}) - \phi(\tilde{x}_{c,i})|.
\]

Proof. For every \(i \in [m]\), it holds that

\[
\left| \frac{\partial h}{\partial x_{1,i}} \right| = h \cdot \frac{1}{1 - x_{1,i}} \cdot \frac{\sum_{c=2}^{L} \prod_{i \in [m]} (1 - x_{c,i})}{\sum_{c \in [L]} \prod_{i \in [m]} (1 - x_{c,i})}.
\]

For every \(2 \leq c \leq L\) and \(i \in [m]\), it holds that

\[
\left| \frac{\partial h}{\partial x_{c,i}} \right| = h \cdot \frac{1}{1 - x_{c,i}} \cdot \frac{\prod_{i \in [m]} (1 - x_{c,i})}{\sum_{c \in [L]} \prod_{i \in [m]} (1 - x_{c,i})}.
\]
Thus for every $i \in [m]$,
\[
\sum_{c \in [L]} \frac{\Phi(h)}{\Phi(x_{c,i})} \left| \frac{\partial h}{\partial x_{c,i}} \right| = \frac{x_{1,i}}{1-x_{1,i}} \sum_{c=2}^{L} \frac{\prod_{i \in [m]}(1 - x_{c,i})}{\prod_{i \in [m]}(1 - x_{c,i})} + \sum_{c=2}^{L} \frac{\prod_{i \in [m]}(1 - x_{c,i})}{\prod_{i \in [m]}(1 - x_{c,i})} x_{c,i} \leq \frac{x_{1,i}}{1-x_{1,i}} + \max_{2 \leq c \leq L} \frac{x_{c,i}}{1-x_{c,i}} \leq \frac{2\beta_i}{1-2\beta_i}
\]

By mean-value theorem, there exists some $\tilde{x} = (\tilde{x}_{c,i})_{c \in [L], i \in [m]}$ with each $\tilde{x}_{c,i} \in [0, \beta_i]$ such that
\[
|\phi(h(x)) - \phi(h(\tilde{x}))| \leq \sum_{i \in [m]} \sum_{c \in [L]} \frac{\Phi(h(\tilde{x}))}{\Phi(x_{c,i})} \left| \frac{\partial h}{\partial x_{c,i}} \right| |\phi(x_{c,i}) - \phi(\tilde{x}_{c,i})| \leq \sum_{i \in [m]} \frac{2\beta_i}{1-2\beta_i} \max_{c \in [L]} |\phi(x_{c,i}) - \phi(\tilde{x}_{c,i})|
\]

Given a feasible list-coloring instance $(G(V,E), \mathcal{L})$ and any vertex $v \in V$, for the minimal permissible block $B = B(v)$ containing $v$, as before we enumerate the boundary edges in $\delta B$ by $e_i = (u_i, v_i)$ for $i = 1, 2, \ldots, m$, where $v_i \not\in B$. For any proper coloring $\rho \in L(B)$ of $B$, and any $i \in [m]$, the new instance $(G_B, \mathcal{L}'_i)$ is as defined in Section 3. And given a subset $S$ of self-avoiding walks starting from $v$, for any $i \in S$, the new set $S_i$ of self-avoiding walks starting from $v_i$ is as constructed in Algorithm 2.

**Lemma 14.** Assume the above notations. Let $(G(V,E), \mathcal{L})$ be a feasible list-coloring instance, $v \in V$ and $B = B(v)$. Let $S$ be a set of self-avoiding walks starting from $v$ which does not contain the trivial walk ($v$). It holds that
\[
\mathcal{E}_{G,\mathcal{L},S}(v) \leq \mathcal{E}_{G,\mathcal{L},S}(B(v)) \leq \sum_{v_i \in [n]} \frac{2}{|L(v_i)| - \deg_G(v_i) - 2} \max_{\rho \in L(B)} \mathcal{E}_{G_B,\mathcal{L}'_i,S}(v_i).
\]

**Proof.** If the trivial walk ($v$) is not contained in $S$, recall that in Algorithm 1 the estimation of marginal is computed as:
\[
\hat{P}_{G,\mathcal{L},S}(v, c) = \min \left\{ \sum_{\pi \in L(B(v)) \atop \text{s.t. } \pi(v) = c} \hat{P}_{G,\mathcal{L},S}(B(v), \pi), \frac{1}{\max \{1, |L(v)| - \deg_G(v)\}} \right\}
\]

By Lemma 12 it always holds that $\Pr_{G,\mathcal{L}}[\sigma(v) = c] \leq \frac{1}{\max \{1, |L(v)| - \deg_G(v)\}}$. Thus assuming $\hat{P}_{G,\mathcal{L},S}(v, c) = \sum_{\pi \in L(B(v)) \atop \text{s.t. } \pi(v) = c} \hat{P}_{G,\mathcal{L},S}(B(v), \pi)$ will not make the error $\mathcal{E}_{G,\mathcal{L},S}(v)$ smaller, and hence we
have

\[ \mathcal{E}_{G,L,S}(v) \leq \max_{c \in E(v)} \left| \log \left( \frac{\Pr_{G,L}[\sigma(v) = c]}{\hat{P}_{G,L,S}(B(v), \pi)} \right) \right| \]

\[ = \max_{c \in E(v)} \left| \log \left( \frac{\sum_{\pi \in L(B(v)) \text{ s.t. } \pi(v) = \epsilon} \Pr_{G,L}[\sigma(B(v)) = \pi]}{\hat{P}_{G,L,S}(B(v), \pi)} \right) \right| \]

\[ \leq \max_{\pi \in L(B)} \left| \log \left( \frac{\Pr_{G,L}[\sigma(B(v)) = \pi]}{\hat{P}_{G,L,S}(B(v), \pi)} \right) \right| \]

\[ = \mathcal{E}_{G,L,S}(B(v)), \]

where the last inequality is due to that for every positive \(a_1, \ldots, a_n, b_1, \ldots, b_n\), \(\sum_{i=1}^{n} a_i \leq \max_{i \in [n]} \frac{a_i}{b_i}\).

Since \((v) \notin S\), the value of \(\hat{P}_{G,L,S}(v, c)\) returned at step 4 of Algorithm 1 is computed from the recursion in Algorithm 2. We claim that Lemma 13 implies

\[ \mathcal{E}_{G,L,S}(B(v)) \leq \sum_{i \in [m]} \frac{2}{|L(v_i)| - \deg_G(v_i) - 2} \cdot \max_{\rho \in L(B)} \left| \log \left( \frac{\Pr_{G_B,L^G_i}[\sigma(v_i) = \rho_i]}{\hat{P}_{G_B,L^G_i,S}(v_i, \rho_i)} \right) \right| \]

\[ = \sum_{i \in [m]} \frac{2}{|L(v_i)| - \deg_G(v_i) - 2} \cdot \max_{\rho \in L(B)} \mathcal{E}_{G_B,L^G_i,S}(v_i). \]

To see this, observe that as a boundary vertex at a permisive block, \(v_i\) has that \(|L(v_i)| \geq \deg_G(v_i) + 5\). Note that the the \((G_B, L^G_i)\) obtained from \((G, L)\) by removing \(B(v)\) from \(G\) and by removing \(\rho_j\) from \(L(v_j)\) for all \(j < i\), can never make the gap \(|L(v_i)| - \deg_G(v_i)\) become smaller. Thus, by Lemma 12 we have

\[ \hat{P}_{G_B,L^G_i,S}(v_i, \rho_i) \leq \frac{1}{|L^G_i(v_i)| - \deg_{G_B}(v_i)} \leq \frac{1}{|L(v_i)| - \deg_G(v_i)}. \]

Also from step 4 of Algorithm 1 we have

\[ \hat{P}_{G_B,L^G_i,S}(v_i, \rho_i) \leq \frac{1}{|L^G_i(v_i)| - \deg_{G_B}(v_i)} \leq \frac{1}{|L(v_i)| - \deg_G(v_i)}. \]

Then Lemma 13 can be applied to complete the proof.

\[ \square \]

**Proof of Lemma 11.** We prove the theorem by applying induction on the depth of \(T(S)\), which is defined as the subtree of \(T\) obtained by removing the descendants of vertices in \(S\).

The base case is that \(T(S)\) contains only \(v\). If \((v) \notin S\), then our algorithm returns in step 4 without further recursive call to **marg**. In this case, we have \(\mathcal{E}_{G,L,S}(v) = 0\) since our algorithm gives a correct estimate. If \((v) \in S\), then it follows from Lemma 12 that \(\mathcal{E}_{G,L,S}(v) \leq (q - 1) \log 2 + \log q < 2q\).

Now assume the lemma holds for \(T(S)\) with smaller depth. Recall that \(\delta B(v) = \{(u_i, v_i) \mid i \in [m]\}\) and we specified a self-avoiding walk \(P_i\) from \(v\) to \(v_i\) in \(G[B(v) \cup \{v_i\}]\) for every \(i \in [m]\). Then we
have
\[
\mathcal{E}_{G,\mathcal{L},S}(v) \leq \mathcal{E}_{G,\mathcal{L},S}(B(v)) \\
\leq \sum_{i \in [m]} \frac{2}{|L(v_i)| - \deg_G(v_i) - 2} \cdot \max_{\rho \in L(B(v_i))} \mathcal{E}_{B(v_i),\mathcal{L}_i,\mathcal{S}_i}(v_i) \\
\leq \sum_{i \in [m]} \prod_{u \in P_i, u \neq v} \delta_{G,\mathcal{L}}(u) \cdot \max_{\rho \in L(B(v_i))} \mathcal{E}_{B(v_i),\mathcal{L}_i,\mathcal{S}_i}(v_i) \\
\leq \sum_{i \in [m]} \prod_{u \in P_i, u \neq v} \delta_{G,\mathcal{L}}(u) \cdot \mathcal{E}_{\text{SAW}(B(v_i),v_i),\mathcal{L}_i,\mathcal{S}_i} \\
\leq \mathcal{E}_{T,\mathcal{L},\mathcal{S}}.
\]

(*) is to apply Lemma \[14\]

(\[\heartsuit\]) is due to the fact that every vertex in \(u \in P_i\) except two ends satisfies \(\deg_G(u) > |L(v)| - 5\) and contributes 1 in the product.

(\[\spadesuit\]) follows from the induction hypothesis. This is applicable since: first, due to lemma \[7\] every \((G_{B(v_i)}, \mathcal{L}^\rho_i)\) is feasible and every permissive vertex in \((G, \mathcal{L})\) is also permissive in \((G_{B(v_i)}, \mathcal{L}^\rho_i)\); and second, by our construction, \(S_i\) is a set of self-avoiding walks from \(v_i\) ending at permissive vertices in \((G_{B(v_i)}, \mathcal{L}^\rho_i)\).

(*) is because for every \(i \in [m]\), \(P_i\) concatenated with \(T_{\text{SAW}}(G_{B(v_i)}, v_i)\) is a subtree of \(T\) and \(P_i \neq P_j\) for all \(i \neq j\).

\[\square\]

4.2 Upper bound for the running time

We give an upper bound to the running time of the algorithms in terms of effective lengths of paths.

**Lemma 15.** Let \((G(V,E), \mathcal{L})\) be a feasible list-coloring instance such that \(|L(v)| \leq q\) for all \(v \in V\). Let \(v \in V\) and \(T = T_{\text{SAW}}(G,v)\). Let \(S\) be a set of self-avoiding walks starting from \(v\), and \(T(S)\) the subtree of \(T\) obtained by removing the descendants of vertices in \(T\) corresponding to the walks in \(S\). Let \(P\) be the set of self-avoiding walks corresponding to the leaves in \(T(S)\). The running time of \(\text{marg}(G, \mathcal{L}, S, v, c)\) is \(O\left(\sum_{P \in P} W_G(P)\right)\).

**Proof.** We use \(\tau_{T,\mathcal{L},S}\) to denote the maximum running time of \(\text{marg}(G, \mathcal{L}, S, v, c)\) for all \(c \in L(v)\). We apply induction on the depth of \(T(S)\). If the depth of \(T(S)\) is one, the upper bound is trivial. Now assume the theorem holds for smaller depth. Denote \(B = B_{G,\mathcal{L}}(v)\) and assume \(\delta B = \{(u_i, v_i) \mid i \in [m]\}\). We use \(P_i\) to denote the set of walks from \(v_i\) to leaves in \(T_{\text{SAW}}(G_{B(v)}, v_i)\).
Then it holds that for some constant $C > 0$

$$\tau_{T, \mathcal{L}, S} = \sum_{\pi \in \mathcal{L}(B)} \sum_{i \in [m]} \tau_{T_{\text{SAW}}(G_{B, v_i}, \mathcal{L}_i^\pi, S_i)} + O\left(q|B|\right)$$

$$\leq q|B| \sum_{i \in [m]} \tau_{T_{\text{SAW}}(G_{B, v_i}, \mathcal{L}_i^\pi, S_i)} + O\left(q|B|\right)$$

$$\leq q|B| \sum_{i \in [m]} \left(C \cdot \sum_{P \in P_i} q|B G, \mathcal{L}(u)| + O\left(q|B|\right)\right)$$

$$= O\left(\sum_{P \in P} W_{G, \mathcal{L}}(P)\right).$$

(*) is due the fact that every instance we created during the computation is feasible (Lemma 7), and every permissive vertex in $(G_{B, \mathcal{L}})$ is also permissive in $(G_B, \mathcal{L})$. Thus replacing $\mathcal{L}_i^\pi$ by $\mathcal{L}$ does not change the running time of the algorithm.

(♦) is to apply induction hypothesis. This is applicable since a permissive vertex in $(G, \mathcal{L})$ is also permissive in $(G_B, \mathcal{L})$, and thus each $S_i$ is a set of self-avoiding walks from $v_i$ in $G_B$ that ends at permissive vertices. By our construction, the depth of $T_{\text{SAW}}(G_{B, v_i}) (S_i)$ is therefore smaller than the depth of $T_{\text{SAW}}(G, v) S$.

(◊) follows from the following two facts:

1. every walk in $P_i$ is part of a walk in $P$;
2. for every $i \in [m]$ and every $P \in P_i$, $B_G(v) \cap (\bigcup_{u \in P} B_{G_B, \mathcal{L}}(u)) = \emptyset$ and $B_{G_B, \mathcal{L}}(u) \subseteq B_G(u)$.

4.3 Proof of Theorem 9

Let $\ell = \ell(n)$. We let $S$ be the permissive $\ell(n)$-cutset for $T = T_{\text{SAW}}(G, v)$ and also the set of self-avoiding walks it represents, which is found by the algorithm, and hence $\hat{P}_{G, \mathcal{L}, \pi} = P_{G, \mathcal{L}, S}(B(v), \pi)$. By the conditions in Theorem 8 such cutset exists and can always be found by the algorithm.

It follows from Lemma 11 that

$$\mathcal{E}_{G, \mathcal{L}, S}(B(v)) \leq \mathcal{E}_{T, \mathcal{L}, S},$$

where

$$\mathcal{E}_{G, \mathcal{L}, S}(B(v)) = \max_{\pi \in \mathcal{L}(B(v))} \left| \log \left( \frac{\Pr_{G, \mathcal{L}}[\sigma(B(v)) = \pi]}{\hat{P}_{G, \mathcal{L}, \pi}} \right) - \log \left( \frac{\hat{P}_{G, \mathcal{L}, S}(B(v), \pi)}{\hat{P}_{G, \mathcal{L}, \pi}} \right) \right|.$$

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Since \( S \) is an \( \ell \)-cutset, it is clear that
\[
    \mathcal{E}_{T,L,S} \leq \sum_{k=\ell}^{2\ell-1} \sum_{\substack{u \text{ self-avoiding walk} \\ P = (u,v_1,\ldots,v_k) \text{ in } G}} w_{G,L}(P) = \mathcal{E}_\ell(v) = O\left(\frac{1}{n^3}\right).
\]
This implies
\[
1 - O\left(\frac{1}{n^3}\right) \leq \frac{\hat{P}_{G,L,\pi}}{\Pr_{G,L}[\sigma(B(v)) = \pi]} \leq 1 + O\left(\frac{1}{n^3}\right).
\]
Since \( S \) is a permissive \( \ell \)-cutset, every walk in \( T \) from \( v \) with length at least \( 2\ell \) must intersect with \( S \), it follows from Lemma\footnote{Lemma} that the running time of the algorithm is bounded by
\[
    O(\tau_\ell(v)) = n^{O(1)}.
\]

5 The sampling algorithm

Due to a seminal work of Jerrum, Valiant, and Vazirani\footnote{Jerrum, Valiant, and Vazirani\cite{JerrumValiantVazirani86},}, the efficient approximation of marginal probabilities as stated by Theorem\footnote{Proposition 6} would be sufficient to imply an FPAUS as long as the family of instances is self-reducible, that is, closed under the action of fixing (also called pinning) a subset of variables to arbitrary feasible values. However, the family of list-coloring instances satisfying the conditions in Theorem\footnote{Proposition 16} is not self-reducible in general. Nevertheless, we show that the family is closed under pinning permissible blocks, which is sufficient to support our FPAUS.

Given a feasible list-coloring instance \((G(V,E),L)\), for a subset \( B \subset V \) of vertices and any proper coloring \( \pi \in L(B) \), let \((G_B,L_\pi)\) be the new list-coloring instance obtained from pinning the coloring of \( B \) to be \( \pi \), where \( G_B = G[V \setminus B] \) and \( L_\pi = \{L(v) : v \in V \setminus B\} \) such that every \( L_\pi(v) \) is the same as \( L(v) \) except that all colors \( \pi(u) \) for such \( u \in B \) adjacent to \( v \) are removed from \( L(v) \).

**Proposition 16.** Let \((G(V,E),L)\) be from a family of feasible list-coloring instances satisfying the conditions with \( \ell(n) \) in Theorem\footnote{Proposition 16}. Then for any permissible block \( B \subset V \) and any proper coloring \( \pi \in L(B) \), the new instance \((G_B,L_\pi)\) is still feasible and satisfies the conditions in Theorem\footnote{Proposition 16} with the same value of \( \ell = \ell(n) \) where \( n = |V| \) is the number of vertices of the original graph.

**Proof.** Since \( B \) is a permissible block, the feasibility of \((G_B,L_\pi)\) follows from Proposition\footnote{Proposition 6}.

For any vertex \( u \in V \setminus B \), our construction guarantees that \(|L_\pi(u)| - \deg_{G_B}(u) \geq |L(u)| - \deg_{G}(u)\), thus if a vertex \( u \) is permissible in \((G,L)\), it is also permissible in \((G_B,L_\pi)\). Note that \( T_{\text{SAW}}(G_B,v) \) is a subtree of \( T_{\text{SAW}}(G,v) \), thus if \( S \) is a permissible \( \ell \)-cutset in \( T_{\text{SAW}}(G,u) \), then it is a permissible \( \ell \)-cutset in \( T_{\text{SAW}}(G_B,u) \).

For every \( u \in V \setminus B(v) \) and \( \ell > 0 \), we define
\[
    \mathcal{E}_\ell'(u) \triangleq \sum_{k=\ell}^{2\ell-1} \sum_{\substack{u \text{ self-avoiding walk} \\ P = (u,v_1,\ldots,v_k) \text{ in } G_B}} w_{G,B,L_\pi}(P), \quad \tau_\ell'(u) \triangleq \sum_{k=1}^{2\ell-1} \sum_{\substack{u \text{ self-avoiding walk} \\ P = (u,v_1,\ldots,v_k) \text{ in } G_B}} W_{G,B,L_\pi}(P)
\]
to be the respective \( \mathcal{E}_\ell(u) \) and \( \tau_\ell(u) \) evaluated on the new instance \((G_B,L_\pi)\).

Every self-avoiding walk \( P = (u,u_1,\ldots,u_k) \) in \( G_B \) is also a self-avoiding walk in \( G \). Moreover, \( w_{G,B,L}(P) = 2q\prod_{i=1}^{k} \delta_{G,B,L_\pi}(u_i) \leq 2q\prod_{i=1}^{k} \delta_{G,L}(u_i) = w_{G,L}(P) \) since \(|L_\pi(u_i)| - \deg_{G_B}(u_i) \geq |L(u_i)| - \deg_{G}(u_i)\) for every \( i \in [k] \). This proves that \( \mathcal{E}_\ell'(u) \leq \mathcal{E}_\ell(u) \).
Similarly, it holds that $W_{G,B,L_n}(P) \leq W_{G,L}(P)$ since for every $u \in V \setminus B$, the minimal permissive block in the new instance $B_{G,B,L_n}(u) \subseteq B_{G,L}(u)$. Therefore we have $\tau'_i(u) \leq \tau_i(u)$. \hfill \square

With this self-reducibility in terms of permissive blocks, the FPAUS can be constructed by following the routine of Jerrum, Valiant and Vazirani.

For a feasible list-coloring instance $(G, L)$, we use $Z(G, L)$ to denote the number of proper colorings. It is well known that an efficient approximation of $Z(G, L)$ can be implied by efficient approximation of marginal probabilities.

**Lemma 17.** For any family of feasible list-coloring instances satisfying the conditions in Theorem 3 there exists a polynomial-time algorithm which given as input a feasible list-coloring instance $(G, L)$ from the family, outputs a number $\hat{Z}(G, L)$ satisfying for some constant $c > 0$ that

$$1 - \frac{c}{n^2} \leq \frac{\hat{Z}(G, L)}{Z(G, L)} \leq 1 + \frac{c}{n^2}.$$

We describe the approximate counting algorithm. Assume $G = (V, E)$ and $n = |V|$. The algorithm constructs a sequence $(G^i, L^i)$ of list-coloring instances as follows:

- $(G^0, L^0) = (G, L)$;
- For every $i \geq 1$, let $v_i$ be an arbitrary vertex in $G^{i-1}$, $B^i = B_{G^{i-1}, L^{i-1}}(v_i)$, and $\pi_i \in L^{i-1}(B^i)$ an arbitrary proper coloring of $B^i$ in the instance $(G^{i-1}, L^{i-1})$. And let $(G^i, L^i) = (G^i_{B^i}, L^i_{\pi^i})$ be the instance resulting from pinning.

It follows from Proposition 16 that every $(G^i, L^i)$ is feasible, thus the sequence is well defined. Moreover, the algorithm can find each $\pi^i$ by exhaustive enumeration in time $q |B_{G^{i-1}, L^{i-1}}|$, which is polynomial in $n$ by Condition 3 in Theorem 3.

Suppose the above procedure gives us a sequence of $(G^i, L^i)$ and $B^i$ for $i = 1, 2, \ldots, t$. It is obvious that $\{B^i\}_{i=1}^t$ forms a partition of $V$. We denote by $\pi$ the concatenation of all $\pi^i$. It must be a proper coloring of all vertices in the original instance $(G, L)$.

Proposition 16 guarantees that the conditions in Theorem 3 is satisfied for every instance $(G^i, L^i)$ with the value of $\ell = \ell(n)$ where $n = |V|$ is fixed at the beginning. For each $i = 1, 2, \ldots, t$, the algorithm for estimating marginal probabilities in Theorem 3 is applied to compute an estimation $\hat{p}_i$ of $\Pr_{G^{i-1}, L^{i-1}}(\sigma(B^i) = \pi^i)$ in time polynomial in $n$. By Theorem 3 we have the following bound on multiplicative errors on marginal probabilities:

$$1 - O\left(\frac{1}{n^3}\right) \leq \hat{p}_i \leq \Pr_{G^{i-1}, L^{i-1}}(\sigma(B^i) = \pi^i) \leq 1 + O\left(\frac{1}{n^3}\right).$$

The estimation $\hat{Z}(G, L)$ is then computed as $\hat{Z}(G, L) = 1/\prod_{i=1}^t \hat{p}_i$.

The error bound for approximate counting in Lemma 17 is a consequence to the following probability identity:

$$\frac{1}{Z(G, L)} = \Pr_{G, L}[\sigma(V) = \pi] = \prod_{i=1}^t \Pr_{G^{i-1}, L^{i-1}}[\sigma(B^i) = \pi^i],$$

and the bound on the multiplicative errors for marginal probabilities.
With the above notations, the sampling algorithm can be described as follows by a standard routine of JVV. A calling of the algorithm \texttt{sample}(i; p) with parameters \(i = 0\) and \(p = 1\) samples exactly uniform proper list-coloring of \((G, \mathcal{L})\), conditioning on that it does not fail.

\begin{algorithm}
\caption{\texttt{sample}(i; \pi^1, \ldots, \pi^i; p)}
\begin{algorithmic}[1]
  \If {\(i = t\)}
    \State Compute \(\hat{Z} = \hat{Z}(G, \mathcal{L});\)
    \State With probability \(1 - \frac{1}{\hat{Z}(1 + c/n^2)^\cdot p}\) the algorithm fails;
    \State Output the coloring \(\pi\) of \(V\) which is a concatenation of all \(\pi^i\) for \(i \in [t]\);
  \Else
    \State Construct the instance \((G^i, \mathcal{L}^i)\) according to \(\pi^1, \ldots, \pi^i;\)
    \State Sample a coloring \(\pi^{i+1}\) on \(B^{i+1}\), according to the estimations \(\hat{p}_{i+1}\) of marginal probability \(\Pr_{G^i, \mathcal{L}^i}[\sigma(B^{i+1}) = \pi^{i+1}]\) for every \(\pi^{i+1} \in L(B^{i+1});\)
    \State Call \texttt{sample}(\(i + 1; \pi^1, \ldots, \pi^{i+1}; p \cdot \hat{p}_{i+1}\));
  \EndIf
\end{algorithmic}
\end{algorithm}

The FPAUS is given by repeatedly running the above sampling algorithm for many times to make the failure probability arbitrarily small.

\textbf{Proof of Theorem 5.} Proposition 16 ensures that all \((G^i, \mathcal{L}^i)\) are feasible and satisfies the conditions in Theorem 9 with the same value \(\ell = \ell(n)\) where \(n = |V|\) fixed at the beginning.

Thus we can apply the algorithm in Theorem 9 to compute \(\hat{p}_{i+1}\) in time polynomial in \(n\), which satisfies

\[
1 - O\left(\frac{1}{n^3}\right) \leq \Pr_{G^i, \mathcal{L}^i}[\sigma(B^{i+1}) = \pi^{i+1}] \leq 1 + O\left(\frac{1}{n^3}\right)
\]

If the sampling algorithm does not fail, it outputs every coloring \(\pi\) with the same probability, thus it is a uniform sampler.

It follows from Theorem 17 that

\[
1 - \frac{c}{n^2} \leq \frac{\hat{Z}(G, \mathcal{L})}{Z(G, \mathcal{L})} \leq 1 + \frac{c}{n^2}
\]

and

\[
\frac{1}{Z(G, \mathcal{L})(1 + \frac{c}{n^2})} \leq p \leq \frac{1}{Z(G, \mathcal{L})(1 - \frac{c}{n^2})}
\]

for some constant \(c > 0\). Therefore the probability that the algorithm does not fail is

\[
\frac{1}{\hat{Z} \cdot (1 + c/n^2) \cdot p} = \Omega(1).
\]

Thus for every \(\delta \in (0, 1)\), we repeat the algorithm \(O\left(\log \frac{1}{\delta}\right)\) times. Then with probability at least \(1 - \delta\) the algorithm uniformly outputs a proper coloring. With remaining probability \(\delta\), we output an arbitrary proper coloring. \(\blacksquare\)
6 Sampling $q$-colorings of random graphs

Let $d > 1$ and $q \geq 3d + 250$ be constants. Let $G$ be a random graph drawn from $\mathcal{G}(n, d/n)$. It is well-known that with this choice of $d$ and $q$, the random graph $G$ is $q$-colorable with high probability \cite{AN05}, thus as a list-coloring instance $(G, [q])$ is feasible with high probability.

This entire section is dedicated to verifying the following random graph property defined on the list-coloring instance $(G, [q])$: for $\ell(n) = \max \left\{ \frac{6 \log n + 3(d+1) \log 2}{d \log 2 - 6 \log d} \right\} = O(\log n)$, with probability $1 - O(\frac{1}{n})$, the followings hold for every vertex $v$ in $G$:

1. there exists a permissive $\ell(n)$-cutset $S$ in $T_{\text{SAW}}(G, v)$ (Lemma 18);
2. $\mathcal{E}_{\ell(n)}(v) = O\left(\frac{1}{n}\right)$ (Lemma 20);
3. $\zeta_{\ell(n)}(v) = n^{O(1)}$ (Lemma 22).

Theorem 1 is an easy consequence of this and Theorem 5.

Our algorithm requires that $q \geq \alpha d + \beta$ for $\alpha = 3$ and $\beta = 250$. Intuitively, since each vertex is of small degree with constant probability, Condition 1 and Condition 3 are easy to hold even with smaller $q$. However, Condition 2 critically requires that $\alpha \geq 3$, in order to guarantee the expected error contraction rate on a vertex is less than $\frac{1}{d}$. The requirement $\beta = 250$ is due to a loose estimate in the analysis of Condition 3 and it is an optimizable constant.

6.1 Permissive cutsets in random graphs

**Lemma 18.** Let $d > 1$ and $q \geq 3d + 8$ be constants. Let $(G, [q])$ be an instance of $q$-coloring, where $G(V, E) \sim \mathcal{G}(n, d/n)$. Let $L$ be such that $L \geq \frac{6 \log n + 3(d+1) \log 2}{d \log 2 - 6 \log d}$ and $L = o((\log n)^2)$. With probability $1 - O\left(\frac{1}{n}\right)$, there exists a permissive $L$-cutset in $T_{\text{SAW}}(G, v)$ for every vertex $v \in V$.

**Proof.** In the proof, we always assume that $n$ is sufficiently large. We assume $q = 3d + 8$ as for larger $q$, vertices are “more” permissive. It is sufficient to prove that, for every simple path $P = (v_0, v_1, \ldots, v_{2L-1})$ in $G$, there exists some $L \leq i < 2L$ such that $v_i$ is permissive in $(G, [q])$. We prove this by showing that

$$\Pr[\forall L \leq i < 2L, v_i \text{ is not permissive} \mid P \text{ is a path}] \leq 2 \cdot 2^{-d\ell}. \quad (2)$$

where $\ell = \left[\frac{L-1}{2}\right]$. This implies the statement of the lemma since the probability that there exists a $P = (v_0, v_1, \ldots, v_{2L-1})$ such that $P$ is a path and no $v_i \in P$ with $L \leq i < 2L$ is permissive is bounded by

$$\sum_{P=(v_0, v_1, \ldots, v_{2L-1})} \Pr[\forall L \leq i < 2L, v_i \text{ is not permissive} \wedge P \text{ is a path}]$$

$$= n^{2L}\left(\frac{d}{n}\right)^{2L-1} \Pr[\forall L \leq i < 2L, v_i \text{ is not permissive} \mid P \text{ is a path}]$$

$$< \frac{1}{n}.$$
Assume $P = (v_0, v_1, \ldots, v_{2L-1})$, let $G_P$ be the random graph drawn from $G(n, d/n)$ conditioning on that $P$ is a path. We let $V(P) = \{v_i \mid i = 0, \ldots, 2L-1\}$ denote vertices in $P$ and $E(P)$ denote edges in $P$. Let $d'(w)$ denote the degree of $u$ in $G_P$ contributed by edges not in $P$. We override the definition of permissiveness so that $u$ is permissive if $d'(w) \leq q - 7$ for all neighbors $w$ of $u$ and $w = u$. Clearly a vertex is permissive in the original sense if it is permissive in the new definition.

If the property of being permissive for every vertex $v_i$ is independent, then (2) holds immediately. However, this is not the case. We bypass the difficulty by considering vertices in $P$ that are “far away” with each other. Specifically, the argument consists of following two steps.

(I) We construct a subgraph $G'$ of $G_P$ and a set of vertices $V' \subseteq V(P)$ such that the permissiveness of vertices in $V'$ in $G'$ is independent. Since the permissiveness of a vertex is a local property, this only requires vertices in $V'$ are far away with each other in $G'$.

(II) We show that, with high probability, many vertices in $P$ are far away with others in $G$.

The property guarantees that the permissiveness of these vertices in $G_P$ is the same as its permissiveness in $G'$.

We first establish (I).

Let $A$ denote the event that every vertex in $G_P$ has degree at most $(\log n)^2$. We assume Lemma 19 and thus $\Pr[A] = 1 - n^{-\Omega(\log n \log \log n)}$. For every $v \in V$, let $N^2(v) = \{v\} \cup N(v) \cup \bigcup_{u \in N(v)} N(u)$ denote the set of vertices whose distance to $v$ is at most 2.

Let $G_0(V_0, E_0), G_1(V_1, E_1), \ldots, G_{L-1}(V_{L-1}, E_{L-1})$ be a sequence of random graphs defined as follows:

- $V_0 = (V \setminus V(P)) \cup \{v_L\}$ and $G_0 = G_P[V_0]$.

For $i = 1, 2, \ldots, L - 1$,

- $V_i = (V_{i-1} \setminus N^2(v_{L+i-1})) \cup \{v_{L+i}\}, G_i = G_P[V_i]$.

In fact, each $G_i$ is a random graph with vertex set $V_i$ distributed according to $G(|V_i|, d/n)$.

Define a graph $G'(V', E')$ as such that $V' = V$ and $E' = \bigcup_{i=0}^{L-1} E_i \cup E(P)$. Then by our construction, $G'$ is a subgraph of $G_P$ and $\text{dist}_{G' \setminus E(P)}(v_{L+i}, v_{L+j}) \geq 4$ where $G' \setminus E(P)$ is the graph obtained from $G'$ by removing edges $E(P)$.

Let $\ell = \lceil (L - 1)/3 \rceil$ and denote $V' = \{V_{L+3j} \mid j \in [\ell]\}$. For every $j \in [\ell]$, define the indicator function

$$X_j = \begin{cases} 1 & \text{if } v_{L+3j} \text{ is not permissive in } (G', [q]), \\ 0 & \text{otherwise}. \end{cases}$$

We now prove that for every $j \in [\ell]$ and any $(x_k)_{k \in [j-1]} \in \{0, 1\}^{j-1}$

$$\Pr \left[ X_j = 1 \mid A \land \bigwedge_{k \in [j-1]} X_k = x_k \right] < 0.04^d + n^{-\Omega(\log n \log \log n)}. \quad (3)$$

Since for every $k < j$, $\text{dist}_{G' \setminus E(P)}(v_{L+3k}, v_{L+3j}) \geq 4$, the permissiveness of $v_{L+3j}$ in $G'$ is independent of the permissiveness of $v_{L+3k}$.

For every $j \in [\ell]$, we define $H_j(U_j, F_j)$ as the “union” of $G_{L+3j-1}, G_{L+3j}, G_{L+3j+1}$:

- $U_j = V_{L+3j-1} \cup V_{L+3j} \cup V_{L+3j+1}$.
• \( F_j = E_{L+3j-1} \cup E_{L+3j} \cup E_{L+3j+1} \cup \{(v_{L+3j-1}, v_{L+3j}), (v_{L+3j}, v_{L+3j+1})\} \).

It is clear that \( v_{L+3j} \) is permissive in \( G' \) if and only if it is permissive in \( H_j \). Conditioning on \( \mathcal{A} \), we have that the number of neighbors of each vertex in \( U_j \) is distributed according to \( \text{Bin}((1-o(1))n, d/n) \).

\[
\Pr \left[ X_j = 1 \mid \mathcal{A} \land \bigwedge_{k \in [j-1]} X_k = x_k \right] = \Pr [X_j = 1 \mid \mathcal{A}]
\]

\[
= \Pr [v_{L+3j} \text{ is not permissive in } (H_j, [q]) \mid \mathcal{A}]
\]

\[
= \Pr [v_{L+3j} \text{ is not permissive in } (H_j, [q])] + n^{-\Omega(\log n \log \log n)}.
\]

Denote \( d'(u) \) the degree of vertex \( u \) in \( H_j \) contributed by edges not in \( P \). Note that if \( v_{L+3j} \) is not permissive in \( (H_j, [q]) \), then either \( d'(u) \geq 3d + 4 \) or at least one of its at most 3d + 5 neighbors has degree \( d'(u) \geq 3d + 4 \). Let \( X \sim \text{Bin}((1-o(1))n, d/n) \), by union bound and the Chernoff bound we have

\[
\Pr [v_{L+3j} \text{ is not permissive in } (H_j, [q])] \leq (3d + 6) \Pr [X \geq 3d + 4] \leq 0.04^d.
\]

This establishes \( (3) \). Let \( X = \sum_{j \in [\ell]} X_j \). Applying Chernoff bound\(^1\) we have

\[
\Pr [X \geq \ell/2 - 1] < 2^{-\ell t}.
\]

We now proceed to establish (II). For every \( 0 \leq i \leq L - 1 \), we say \( v_{L+i} \) is bad if for some other \( 0 \leq j \leq L - 1 \), \( \text{dist}_{G \setminus E(P)}(v_{L+i}, L+j) < 4 \). We show that there are not too many bad vertices:

\[
\Pr [\text{there are at least } \ell/2 \text{ bad vertices in } \{v_L, \ldots, v_{2L-1}\} \mid \mathcal{A}]
\]

\[
\leq 17^\ell \left( \frac{6(\log n)^7}{n} \right)^{\sqrt{7}} + \left( \frac{15(\log n)^{13} \sqrt{7}}{n} \right)^{\sqrt{7}/2}. \tag{5}
\]

The probability above is upper bounded by

\[
\left( \frac{L}{\ell/2} \right) \cdot \Pr [w_1, \ldots, w_{\ell/2} \text{ are all bad } \mid \mathcal{A}]
\]

where \( w_1, \ldots, w_{\ell/2} \) are vertices in \( \{v_L, \ldots, v_{2L-1}\} \).

We use \( B \) to denote the event that for some \( w_t \) there are at least \( \sqrt{7} w_j \)s satisfying \( \text{dist}_{G \setminus E(P)}(w_j, w_j) < 4 \). Conditioning on \( \mathcal{A} \), the degree of every vertex is at most \( (\log n)^2 \) and thus there are at most \( t \triangleq (\log n)^2 + (\log n)^4 + (\log n)^6 \leq 3(\log n)^6 \) vertices within distance 3 to \( w_t \). It follows from the union bound that

\[
\Pr [B \mid \mathcal{A}] \leq \frac{\ell}{2} \left( \frac{\ell/2}{\sqrt{7}} \right) \left( \frac{n^{-1} - \sqrt{7}}{n^{-1}} \right)^{\sqrt{7}} \leq \left( \frac{6(\log n)^7}{n} \right)^{\sqrt{7}}.
\]

\(^1\)Although \( X_j \)s are not independent here, condition \( (3) \) is strong enough to obtain the same bound when computing the moment generating function in the proof of Chernoff bound.
Thus
\[
\mathbb{P} \left[ w_1, \ldots, w_{\ell/2} \text{ are all bad } \mid \mathcal{A} \right] \\
\leq \mathbb{P} \left[ w_1, \ldots, w_{\ell/2} \text{ are all bad } \mid \mathcal{A} \land \overline{\mathcal{E}} \right] + \left( \frac{6 (\log n)^7}{n} \right)^{\sqrt{7}/2}.
\]

Conditioning on \( \overline{\mathcal{E}} \), we can find \( U = \left\{ w_{k_1}, \ldots, w_{k_{\sqrt{\ell/2}}} \right\} \subseteq \left\{ w_1, \ldots, w_{\ell/2} \right\} \) such that \( k_1 < k_2 < \cdots < k_{\sqrt{\ell/2}} \) and \( \text{dist}_{G_P \setminus E(P)}(w_{k_i}, w_{k_j}) \geq 4 \) for every \( i \neq j \). It is sufficient to bound
\[
\mathbb{P} \left[ w_{k_1}, \ldots, w_{k_{\sqrt{\ell/2}}} \text{ are all bad } \mid \mathcal{A} \land \overline{\mathcal{E}} \right] = \prod_{i=1}^{\sqrt{7}/2} \mathbb{P} \left[ w_{k_i} \text{ is bad } \mid (\forall j < i, w_{k_j} \text{ is bad}) \land \mathcal{A} \land \overline{\mathcal{E}} \right].
\]

For a vertex \( v \in V \), we use \( N^3(v) \) to denote the set of vertices within distance 3 to \( v \) (without using \( E(P) \)). Fix \( i \in [\sqrt{\ell/2}] \). Let \( H \) be a random graph conditioning on a fixed \( G[\bigcup_{j<i} N^3(w_{k_j})] \).

Consider the following two events:
\[
\mathcal{E}_1 : w_{k_i} \text{ is bad in } G_P;
\]
\[
\mathcal{E}_2 : N^3(w_{k_i}) \cap \bigcup_{j<i} N^3(w_{k_j}) \neq \emptyset.
\]

Then
\[
\mathbb{P}_H \left[ w_{k_i} \text{ is bad} \right] \leq \mathbb{P}_H \left[ \mathcal{E}_1 \lor \mathcal{E}_2 \right].
\]

This is because the existence of the edges in \( G[\bigcup_{j<i} N^3(w_{k_j})] \) increases the chance of \( w_{k_i} \) being bad only if \( \mathcal{E}_2 \) happens. Since conditioning on \( \mathcal{A} \), \( \left| \bigcup_{1 \leq j < i} N^3(w_{k_j}) \right| < 2 \sqrt{7} (\log n)^6 \) and the condition \( \overline{\mathcal{E}} \) decreases the probability that a vertex is bad, we have
\[
\prod_{i=1}^{\sqrt{7}/2} \mathbb{P} \left[ w_{k_i} \text{ is bad } \mid (\forall j < i, w_{k_j} \text{ is bad}) \land \mathcal{A} \land \overline{\mathcal{E}} \right] \leq \left( \frac{15 (\log n)^{13}}{n} \right)^{\sqrt{7}/2}.
\]

Combining with above, we obtain (5).

The lemma then follows from (I) and (II): By our construction, if a vertex \( v_{L+i} \) is not bad and it is permissive in \( G' \), then it is permissive in \( G_P \). It follows from (II) that with probability \( 1 - 2^{-dL} \), there are at least \( \ell/2 + 1 \) permissive vertices in \( V' \) in \( G' \). By (I), with probability \( 1 - 17^L \left( \left( \frac{6 (\log n)^7}{n} \right)^{\sqrt{7}} + \left( \frac{15 (\log n)^{13}}{n} \right)^{\sqrt{7}/2} \right) \), there are at least \( L - \ell/2 \) vertices in \( \left\{ v_{L+i} \mid 0 \leq i \leq L - 1 \right\} \) that are not bad. Thus by union bound, there exists at least one vertex in \( V' \) that is permissive in \( G_P \), with probability \( 1 - 2^{-dL} - 17^L \left( \left( \frac{6 (\log n)^7}{n} \right)^{\sqrt{7}} + \left( \frac{15 (\log n)^{13}}{n} \right)^{\sqrt{7}/2} \right) \). This implies
\[
\mathbb{P} \left[ \forall L \leq i < 2L, v_i \text{ is not permissive } \mid P \text{ is a path} \right] \\
\leq 2^{-dL} + 17^L \left( \left( \frac{6 (\log n)^7}{n} \right)^{\sqrt{7}} + \left( \frac{15 (\log n)^{13}}{n} \right)^{\sqrt{7}/2} \right) \\
< 2 \cdot 2^{-dL}.
\]

The last inequality is due to the fact that \( \ell = o((\log n)^2) \). \( \square \)
It remains to bound $\Pr[A]$.

**Lemma 19.**

$$\Pr[A] = n^{-\Omega(\log n \log \log n)}.$$

**Proof.**

$$\Pr[\exists v \in V \text{ s.t. } \deg_G(v) > (\log n)^2] \leq n \left( \frac{n}{(\log n)^2} \right)^2 \left( \frac{d}{n} \right)^{(\log n)^2} \leq n \left( \frac{ed}{(\log n)^2} \right)^{(\log n)^2} = n^{-\Omega(\log n \log \log n)}.$$

\[\square\]

### 6.2 Correlation decay in random graphs

**Lemma 20.** Let $d > 1$ and $q \geq 3d + 8$ be constants. Let $(G, [q])$ be an instance of $q$-coloring where $G(V, E) \sim G(n, d/n)$. Let $L$ be such that $L \geq 6 \log n \log \left( \frac{q - 4}{3d} \right)$ and $L = o\left( \sqrt{n} \right)$. With probability $1 - O\left( \frac{1}{n} \right)$, it holds that $E_L(v) = O\left( \frac{1}{n^3} \right)$ for every $v \in V$.

We first prove a technical lemma.

**Lemma 21.** Let $f_a(x)$ be a piecewise function defined as

$$f_a(x) = \begin{cases} \frac{2}{a-x} & \text{if } x \leq a-2 \\ 1 & \text{o.w.} \end{cases}$$

Let $X$ be a random variable distributed according to binomial distribution $\text{Bin} \left( n, \frac{d}{n} \right)$ where $d > 1$ is a constant. Then for $a \geq 3d + 3$ and all sufficiently large $n$, it holds that $E[f_a(X)] < \frac{2}{a}$.

**Proof.** Denote $p(k) = \binom{n}{k} \left( \frac{d}{n} \right)^k \left( 1 - \frac{d}{n} \right)^{n-k}$ the probability density function of $\text{Bin} \left( n, \frac{d}{n} \right)$. Then

$$1 - E[f_a(x)] = \sum_{k=0}^{\lfloor a-2 \rfloor} \left( 1 - \frac{2}{a-x} \right) p(k).$$

Define $g(x) = 1 - \frac{2}{a-x}$ and we now prove that

$$\sum_{k=0}^{\lfloor a-2 \rfloor} g(k)p(k) > 1 - \frac{3}{a}.$$

$g(x)$ can be approximated by the polynomial

$$\tilde{g}(x) \triangleq \frac{a-d-2}{a-d} - \frac{2(x-d)}{(a-d)^2} - \frac{2(x-d)^2}{(a-d)^3} - \frac{2(x-d)^3}{(a-d)^4} - \frac{2(x-d)^4}{(a-d)^5} - \frac{2(x-d)^5}{(a-d)^6} - \frac{2(x-d)^6}{(a-d)^6}.$$
It is easy to verify that
\[ g(x) - \tilde{g}(x) = \frac{2(a - 1 - x)(x - d)^6}{(a - d)^6(a - x)}, \]
which is positive for all \(0 \leq x \leq a - 2\). Thus it is sufficient to show that
\[ \sum_{k=0}^{[a-2]} \tilde{g}(k)p(k) > 1 - \frac{3}{a}. \]

Assume \(a = 3d + b\) for some \(b \geq 3\). Since \(\tilde{g}(x)\) is a polynomial with degree 6, we can directly compute its expectation:
\[ \mathbf{E}[\tilde{g}(x)] = \frac{1}{(2d + b)^6n^5} \left( C_5n^5 + C_4n^4 + O(n^3) \right). \]

where
\[
C_5 = -2b^5 + 6b^4 + (-4 - 2b - 2b^2 - 2b^3 - 20b^4 + 12b^5)d \\
\quad + (-74 - 4b - 12b^2 - 80b^3 + 60b^4)d^2 + (-50 - 24b - 160b^2 + 160b^3)d^3 \\
\quad + (-16 - 160b + 240b^2)d^4 + (-64 + 192b)d^5 + 64d^6; \\
C_4 = 2(46 + 7b + 3b^2 + b^3)d^2 + 2(234 + 18b + 6b^3)d^3 + 2(69 + 12b)d^4 + 16d^5.
\]

Note that \(C_4 > 0\), thus for sufficiently large \(n\), it holds that
\[ \mathbf{E}[\tilde{g}(x)] \geq \frac{C_5}{(2d + b)^6n^5}. \]

On the other hand, \(\mathbf{E}[\tilde{g}(x)]\) can be decomposed as
\[ \mathbf{E}[\tilde{g}(x)] = \sum_{k=0}^{[a-2]} \tilde{g}(k)p(k) + \sum_{k=[a-1]}^{n} \tilde{g}(k)p(k). \]

It can be verified that \(\tilde{g}(x)\) is monotonically decreasing when \(x \geq a - 2\) and \(\tilde{g}(a - 2) = \frac{(2-2d-b)^6}{(2d+b)^6} < 0\). Therefore, \(\sum_{k=[a-1]}^{n} \tilde{g}(k)p(k) < 0\) and
\[
\sum_{k=0}^{[a-2]} \tilde{g}(k)p(k) \geq \mathbf{E}[\tilde{g}(x)] \geq \frac{C_5}{(2d + b)^6n^5} \left( 1 - \frac{3}{3d + b} \right) + \frac{C}{(b + 2d)^6(b + 3d)}
\]
where
\[
C = b^6 + (-4b - 2b^2 - 2b^3 - 2b^4 + 10b^5)d + (-12 - 80b - 20b^2 - 18b^3 + 40b^4)d^2 \\
\quad + (-222 - 92b - 60b^2 + 80b^3)d^3 + (-150 - 88b + 80b^2)d^4 \\
\quad + (-48 + 32b)d^5
\]
and it is positive for \(b \geq 3\).
Proof of Lemma 20. Let $v \in V$ be arbitrary fixed. By linearity of expectation, we have

$$E \left[ \mathcal{E}_L \right] \leq \sum_{k=L}^{2L-1} n^k \left( \frac{d}{n} \right)^k \cdot E \left[ w_{G_i[n]}(P) \big| P = (v, v_1, \ldots, v_k) \text{ is a path} \right]$$

$$= 2q \sum_{k=L}^{2L-1} d^k \cdot E \left[ \prod_{i=1}^k \delta_{G_i[n]}(v_i) \big| P = (v, v_1, \ldots, v_k) \text{ is a path} \right].$$

Fix a tuple $P = (v, v_1, \ldots, v_k)$. To calculate the expectation, we construct an independent sequence whose product dominates the $\prod_{i=1}^k \delta_{G_i[n]}(v_i)$ as follows.

Conditioning on $P = (v, v_1, \ldots, v_k)$ being a path in $G$, let $X_1, X_2, \ldots, X_k$ be random variables such that each $X_i$ represents the number of edges between $v_i$ and vertices in $V \setminus \{v_1, \ldots, v_k\}$; and let $Y$ be a random variable representing the number of edges between vertices in $\{v_1, \ldots, v_k\}$ except for the edges in the path $P = (v, v_1, \ldots, v_k)$. Then $X_1, \ldots, X_k, Y$ are mutually independent binomial random variables with each $X_i$ distributed according to $\text{Bin}(n - k, \frac{d}{n})$ and $Y$ distributed according to $\text{Bin}(\binom{k}{2} - k + 1, \frac{d}{n})$, and for each $v_i$ in the path we have $d(v_i) = X_i + 2 + Y_i$ with some $Y_1 + \ldots + Y_k = 2Y$.

Note that $\delta_{G_i[n]}(v_i) = f_q(\text{deg}_G(v_i) + 2)$ where function $f_q(x)$ is as defined in Lemma 21. Note that the ratio $f_q(x)/f_q(x-1)$ is always upper bounded by 2, and we have the identity $f_q(x+1) = f_q-1(x)$. Thus, conditioning on that $P = (v, v_1, \ldots, v_k)$ is a path, the product $\prod_{i=1}^k \delta(v_i)$ can be bounded as follows:

$$\prod_{i=1}^k \delta(v_i) = \prod_{i=1}^k f_q(X_i + 2 + Y_i + 2) \leq 2^{2Y} \prod_{i=1}^k f_{q-4}(X_i).$$

Let $d' = \frac{d - 7}{3}$, thus we have $d' > d$. Let $X$ be a binomial random variable distributed according to $\text{Bin}(n, \frac{d'}{n})$, thus $X$ probabilistically dominates every $X_i$ whose distribution is $\text{Bin}(n - k, \frac{d'}{n})$. Since $X_1, \ldots, X_k, Y$ are mutually independent conditioning on $P = (v, v_1, \ldots, v_k)$ being a path in $G$, for any $P = (v, v_1, \ldots, v_k)$ we have

$$E \left[ \prod_{i=1}^k \delta(v_i) \big| P \text{ is a path} \right] \leq E \left[ 4^Y \prod_{i=1}^k f_{q-4}(X_i) \right] \leq E \left[ 4^Y \right] E \left[ f_{q-2}(X) \right]^k.$$

Recall that $Y \sim \text{Bin} \left( \binom{k}{2} - k + 1, \frac{d'}{n} \right)$, the expectation $E \left[ 4^Y \right]$ can be bounded as

$$E \left[ 4^Y \right] \leq \sum_{\ell=0}^{k^2} 4^\ell \left( \frac{k^2}{\ell} \right) \left( \frac{d'}{n} \right)^\ell \left( 1 - \frac{d'}{n} \right)^{k^2 - \ell} = \left( 1 + \frac{3d}{n} \right)^k \leq \exp \left( \frac{3dk^2}{n} \right).$$

Since $q - 4 \geq 3d' + 3$, it follows from Lemma 21 that $E \left[ f_{q-2}(X) \right] < \frac{3}{q - 4}$. Therefore,

$$E \left[ \prod_{i=1}^k \delta(v_i) \big| P \text{ is a path} \right] \leq \exp \left( \frac{3dk^2}{n} \right) \left( \frac{3}{q - 4} \right)^k \leq \frac{1}{d^k} \exp \left( -k \log \left( \frac{q - 4}{3d} \right) + \frac{3dk^2}{n} \right).$$

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Thus,
\[
\mathbb{E} [\mathcal{E}_L] \leq 2q \sum_{k=L}^{2L-1} \left( -k \log \left( \frac{q - 4}{3d} \right) + \frac{3dk^2}{n} \right)
\]
\[
\leq 2qL \exp \left( -L \log \left( \frac{q - 4}{3d} \right) + o(1) \right)
\]
\[
\leq \frac{c_d}{n^3}
\]
for some constant \(c_d > 0\) that may depend on \(d\).

By Markov’s inequality,
\[
\Pr \left[ \mathcal{E}_L \geq \frac{1}{n^3} \right] = O \left( \frac{1}{n^2} \right).
\]

Then the Lemma follows from the union bound.

\[\square\]

6.3 Running time in random graphs

Lemma 22. Let \(d > 1\) and \(q \geq 3d + 250\) be constants. Let \((G, [q])\) be an instance of \(q\)-coloring where \(G(V, E) \sim \mathcal{G}(n, d/n)\). For \(L > 1\), with probability \(1 - O \left( \frac{1}{n^2} \right)\), it holds that for every \(v \in V\),
\[
\tau_L(v) \leq 2^{O(L)} n^{O(1)}.
\]

Given \(P = (v_1, \ldots, v_L)\), we are going to upper bound the probability
\[
\Pr \left[ \left| \bigcup_{i \in [L]} B(v_i) \right| \geq t \bigg| P \text{ is a path} \right]
\]
for every \(t > 0\). Recall the definition of \(B(v)\), a vertex \(v \neq u \in B(v)\) is required to satisfy that \(\deg_G(u) > q - 5\). In this subsection, we relax the requirement to \(\deg_G(u) > q - 7\). Since the existence of a fixed path can contribute at most two degrees for a vertex, it is clear that with the new definition, we can drop the condition that \(P\) is a path.

Let \(v \in V\), we use \(B^*(v)\) to denote \(B(v) \cup \partial B(v)\). Since \(|B^*(v)| \geq |B(v)|\), it suffices to bound
\[
\Pr \left[ \left| \bigcup_{i \in [L]} B^*(v_i) \right| \geq t \right].
\]

In the following, we use \(\text{Bin}(n, p)\) to denote the binomial distribution. For a set \(S\), we use \(\text{Bin}(S, p)\) to denote the distribution of functions \(\rho : S \to \{0, 1\}\) such that \(\rho(x) = 1\) with probability \(p\) independently for every \(x \in S\). We use \(|\rho|\) to denote \(|\{x \in S \mid \rho(x) = 1\}|\). We now analyze a binomial branching model.

1. \(X_1, X_2, \ldots\) is an infinite sequence of i.i.d. random variables where each \(X_t\) is of following distribution:
   - Let \(X \sim \text{Bin}(n, d/n)\).
   - If \(X \leq (q - 7)/2\), \(X_t = 0\), otherwise \(X_t = X\).

2. \(Y_1, Y_2, \ldots\) is an infinite sequence of random variables that \(Y_0 = L\) and \(Y_i = Y_{i-1} + X_i - 1\) for every \(i \geq 1\).
Let $\Omega$ denote the above random process, we use the hitting time of $\Omega$ to bound $|\bigcup_{i \in [L]} B^*(v_i)|$.

**Lemma 23.** Let $v$ be a vertex in $V$ and $G(V, E) \sim \mathcal{G}(n, d/n)$. We use $Z$ to denote the least $t$ such that $Y_t = 0$. For every $t > 0$ and $i \in V$, it holds that

$$\Pr \left[ \bigcup_{i \in [L]} B^*(v_i) \geq t \right] \leq \Pr [Z \geq t].$$

**Proof.** We now describe a coupling for $\mathcal{G}(n, d/n)$ and $\Omega$. The following procedure, based on BFS from $v$, samples a joint distribution of $\bigcup_{i \in [L]} B^*(v_i)$ and $Y_0, Y_1, \ldots$

The main idea of the coupling is to use the binomial branching model to simulate BFS on random graphs. A vertex in the random graph stops to branch new vertices as long as its degree is at most $q - 7$. However, its degree is contributed by two parts: edges sampled from itself and edges sampled by previous vertices. In the branching model, we weaken the stopping condition for a vertex to that the vertex branches at most $\frac{q - 7}{2}$ new vertices. In this way, we always have more active vertices in the binomial branching model.

In each step, we have an active set $A_t$ that is the current queue for BFS and $B_t$ is the current permissive block found. Each time we choose a new vertex in $A_t$ and sample its neighborhood. $\deg_t(u)$ is the degree of vertex $u$ contributed by vertices whose neighbors has been sampled before or at stage $t$. We also have an active set $A_t^\Omega$ for $\Omega$ and we maintain a surjective mapping $f_t : A_t^\Omega \to A_t$ during the process.

1. Let $t = L$ and do following.

1.1. Let $A_t = \{v_1, \ldots, v_L\}$ be the initial active set for $\mathcal{G}(n, d/n)$ and $B_0 = \emptyset$ be the initial block. $\deg_t(u) = 0$ for all $u \in V$.

1.2. Let $A_t^\Omega = \{u_1, \ldots, u_L\}$ be the initial active set for $\Omega$ and let $Y_0 = L$.

1.3. Let $f_t : A_t^\Omega \to A_t$ be the function that $f_t(u_i) = v_i$ for all $i \in [L]$.

2. If $A_t = \emptyset$, the procedure halts. Otherwise, increase $t$ by 1 and do following:

2.1. Choose some $v \in A_{t-1}$. Let $B_t = B_{t-1} \cup \{v\}$.

2.2. Choose some $u \in A_{t-1}^\Omega$ such that $f_{t-1}(u) = v$.

3. Let $\rho \sim \text{Bin}(V \setminus B_t, d/n)$.

3.1. If $\deg_{t-1}(v) + |\rho| > q - 7$: Let $S = \{w \in V \setminus B_t \mid \rho(w) = 1\}$ and $A_t = (A_{t-1} \setminus \{v\}) \cup S$.

Let $\deg_t(w) = \deg_{t-1}(w) + 1$ for every $w \in S$. $\deg_t(u) = \deg_{t-1}(u)$ for every $u \in V \setminus S$.

If $\deg_{t-1}(v) + |\rho| \leq q - 7$, do nothing.

3.2. Let $\tilde{\rho} \sim \text{Bin}(B_t, d/n)$ and $\rho' = \rho \cup \tilde{\rho}$ denote the function on $V$ that is consistent with both $\rho$ and $\tilde{\rho}$. If $|\rho'| > (q - 7)/2$, let $X_t = |\rho'|$; otherwise, let $X_t = 0$. Let $Y_t = Y_{t-1} + X_t - 1$.

Let $S' = \{v_1, \ldots, v_{X_t}\}$ be a set of $X_t$ new vertices and let $A_t^\Omega = (A_{t-1}^\Omega \setminus \{u\}) \cup S'$.

3.3. We distinguish between four cases.

3.3.1 (If $|\rho'| > (q - 7)/2$ and $\deg_{t-1}(v) + |\rho| > q - 7$) Let $f' : S' \to S$ be an arbitrary surjective mapping and let $f_t : \text{dom}(f_{t-1}) \cup S' \to \text{ran}(f_{t-1}) \cup S$ be the function that is consistent with $f_{t-1}$ on the domain of $f_{t-1}$ and is consistent with $f'$ on $S'$.
3.3.2 (If $|\rho'| > (q - 7)/2$ and $\deg_{t-1}(v) + |\rho| \leq q - 7$) Do nothing.

3.3.3 (If $|\rho'| \leq (q - 7)/2$ and $\deg_{t-1}(v) + |\rho| > q - 7$) Define $T \triangleq \{ w \in A^0_t \mid f_{t-1}(w) = v \}$. Let $f' : T \to S$ be an arbitrary surjective mapping and let $f_1 : \text{dom}(f_{t-1}) \to \text{ran}(f_{t-1}) \cup S$ be the function that is consistent with $f'$ on $T$ and consistent with $f_{t-1}$ on other domain of $f_{t-1}$.

3.3.4 (If $|\rho'| \leq (q - 7)/2$ and $\deg_{t-1}(v) + |\rho| \leq q - 7$) Do nothing.

4. Goto 2.

We first verify that the procedure is well-defined. The only problem is from step 3.3.1 and step 3.3.3 where we have to ensure that the surjective mapping $f'$ exists.

In step 3.3.1, this is obvious since $|S'| = |\rho'| \geq |\rho| = |S|$. We need more effort to verify the existence of $f'$ in step 3.3.3. In this step,

$$\deg_{t-1}(v) > q - 7 - |\rho| > q - 7 - |\rho'| \geq (q - 7)/2 \geq |\rho'|.$$

For every $t > 0$, define the property $P_t$ as:

$$P_t : \{|w \in A^0_t \mid f_{t-1}(w) = v\} \geq \deg_{t-1}(v) - 1.$$

Then $P_t$ implies the existence of $f'$ in stage $t$. We prove $P_t$ by induction on $t$. The $t = L$ case is trivial and assume $P_t$ holds for smaller $t$. Thus the algorithm is well-defined up to stage $t$. We increase the degree of vertex $v$ by one only if at some stage $t' < t$, it holds that in step 3.1, $v \in S$. By induction hypothesis, we extend $f_{t'-1}$ by some surjective mapping $f'$ that maps at least one new vertex from $A^0_t$ to $v$. The new vertex remains in $A^0_{t-1}$ and is still mapped to $v$ under $f_{t-1}$ by our construction. Only one such vertex is removed at step 2.2 in stage $t$, so $P_t$ holds.

Assume the procedure halts at stage $\tau$. Then it is clear that $B_\tau = \bigcup_{i \in [L]} B^*(v_i)$ and the procedure is a coupling. Since for every $t \leq \tau$, $f_t$ is a surjective mapping from $A^0_t \to A_t$, we have $|A^0_t| \geq |A_t|$. By our construction, $|A^0_t| = Y_t$ and $|B_t| = t$. Thus for all $t < \tau$, $Y_t > 0$. This proves the lemma.

**Lemma 24.** For every $t > 0$,

$$\Pr[Z \geq t] \leq e^{-\frac{2L}{\sqrt{2m}t} + \frac{1}{2m}L}.$$

**Proof.** It holds that $\Pr[Z \geq t] \leq \Pr[Y_t \geq 0]$ where $Y_t = L - t + \sum_{i=1}^t X_i$ is the sum of $t$ i.i.d. variables $X_i$. Since each $X_i$ is a “cut-off” version of binomial random variable, we can compute its moment generating function directly. For every $s > 0$, it holds that

$$\Pr[Y_t \geq 0] = \Pr[e^{sY_t} \geq 1] \leq E[e^{sY_t}] = e^{s(L-t)} \prod_{i=1}^t E[e^{sX_i}] = e^{s(L-t)} (E[e^{sX_1}])^t.$$
Recall that $X \sim \text{Bin}(n, d/n)$. Let $p = (q - 7)/2$, we have

$$
E \left[ e^{sX_1} \right] = \Pr [X \leq p] + \sum_{k=p+1}^{n} e^{sk} \cdot \Pr [X = k]
$$

$$
\leq 1 + \sum_{k=p+1}^{n} e^{sk} \cdot \Pr [X \geq k]
$$

$$
\leq 1 + \sum_{k=p+1}^{\infty} \left( \frac{2}{3} e^{s+\frac{1}{2}} \right)^k
$$

$$
\leq e^{\frac{\alpha(s)}{1 - \alpha(s)}}
$$

where $\alpha(s) = \frac{2}{3} e^{s+\frac{1}{2}}$. Let $s = \frac{1}{100}$, we have

$$
\Pr [Y_t \geq 0] \leq e^{\frac{-2500}{2500} t + \frac{2500}{100} L}.
$$

**Proof of Lemma 22.** By union bound, we have

$$
\Pr \left[ \exists P = (v_1, \ldots, v_L), W_{G, \mathcal{L}}(P) \geq q^{2500 (L \log d + \frac{d}{100} + \log n)} \right]
$$

$$
\leq \sum_{P=(v_1, \ldots, v_L)} \Pr \left[ W_{G, \mathcal{L}}(P) \geq q^{2500 (L \log d + \frac{d}{100} + \log n)} \mid P \text{ is a path} \right]
$$

$$
\leq n^L \left( \frac{d}{n} \right)^L \Pr \left[ W_{G, \mathcal{L}}(P) \geq q^{2500 (L \log d + \frac{d}{100} + \log n)} \mid P \text{ is a path} \right]
$$

$$
= d^L \Pr \left[ W_{G, \mathcal{L}}(P) \geq q^{2500 (L \log d + \frac{d}{100} + \log n)} \mid P \text{ is a path} \right]
$$

$$
\leq \frac{1}{n}.
$$

The last inequality follows from Lemma 23 and Lemma 24.

On the other hand, let $P_L(v)$ denote the set of self-avoiding walks from $v$ with length at most $2L - 1$. The expected number of $|P_L(v)|$ is bounded by

$$
\sum_{k=1}^{2L-1} n^k \left( \frac{d}{n} \right)^k \leq \frac{d}{d-1} d^{2L}.
$$

By Markov inequality,

$$
\Pr \left[ |P_L(v)| > \frac{d}{d-1} d^{2L} n^2 \right] \leq \frac{1}{n^2}.
$$

Then by union bound,

$$
\Pr \left[ \exists v \in V, |P_L(v)| > \frac{d}{d-1} d^{2L} n^2 \right] \leq \frac{1}{n}.
$$

Again by union bound, with probability $1 - O \left( \frac{1}{n} \right)$, for every $v \in V$,

$$
\tau_L(v) \leq q^{2500 (L \log d + \frac{d}{100} + \log n)} \cdot \frac{d}{d-1} d^{2L} n^2 = 2^{O(L)} n^{O(1)}.
$$

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