Measurement and Probability in Relativistic Quantum Mechanics

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Abstract

Ultimately, any explanation of quantum measurement must be extendable to relativistic quantum mechanics (RQM), since many precisely confirmed experimental results follow from quantum field theory (QFT), which is based on RQM. Certainly, the traditional “collapse” postulate for quantum measurement is problematic in a relativistic context, at the very least because, as usually formulated, it violates the relativity of simultaneity. Among alternatives to the traditional collapse interpretation, the Everettian approach of an unmodified, unitary quantum formalism is the only one that has been clearly extended to RQM and QFT. However, the usual “many worlds” interpretation of such an approach leads to difficulty in how to even define probabilities over different possible “worlds”. The present paper addresses this difficulty by providing a relativistic model of measurement, in which the state of the universe is decomposed into decoherent histories of measurements recorded within it. Zurek’s concept of envariance can be generalized to this context of relativistic spacetime, giving an objective definition of the probability of any one of these quantum histories, consistent with Born’s rule. This then leads to the statistics of any repeated experiment also tending to follow the Born rule as the number of repetitions increases. The wave functions that we actually use for such experiments are local reductions of very coarse-grained superpositions of universal eigenstates, and their “collapse” can be re-interpreted as simply an update based on additional incremental knowledge gained from a measurement about the “real” eigenstate of our universe.

Keywords: relativistic quantum mechanics; measurements; probability; Born’s rule; envariance

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I. INTRODUCTION

A. Measurement

The concept of probability traditionally enters the formalism of quantum mechanics through a projection or “collapse” postulate [1] and Born’s rule [2]. However, this poses a particular problem for relativistic quantum mechanics (RQM), because any universal collapse process clearly violates the relativity of simultaneity. (And the problem is even worse when considering measurable effects on space-time curvature in general relativity [3].) For this reason, relativistic interpretations of quantum theory often take an Everettian or “many worlds” approach [4, 5].

For example, Hartle and colleagues have addressed interpretational issues relativistically in terms of decoherence and spacetime path integrals [6–12]. In such an approach, rather than collapsing, a quantum state decoheres, usually through some dynamic process, into parallel branches that are effectively orthogonal to each other. Since there is no interference between these branches, each can be given a classical probability for being the “true” state.

Indeed, Wallace pointedly argues that the possible alternatives to a unitary Everettian interpretation of quantum mechanics, Bohmian mechanics and dynamical-collapse theories, have so far not been shown to be extendable to RQM and quantum field theory (QFT). And, since many of the most accurate predictions of quantum mechanics are based on QFT, this calls into question any fundamental approach other than unitary mechanics [13].

Building on this, Adlam asks if we have any viable solution to the measurement problem, particularly when this is put into the context of QFT [14]. She concludes “that a successful interpretation or modificatory strategy will have the following four features:

- It makes no substantial change to the formalism of unitary quantum mechanics (at least at the microscopic level).
- Decoherence plays a significant role in the emergence of classical reality.
- Observers (approximately) see a unique outcome to each measurement and are able to (approximately) establish a shared observable reality.
- This shared observable reality supervenes on beables which are approximate and emergent, and/or non-dynamical, and/or non-microscopically defined.”

In this paper, I propose a mathematically consistent framework for interpreting measurement and probability in RQM that meets the above criteria. This framework requires no change at all to the formalism of unitary RQM, adopting the modern Everettian view of decoherence. Further, it is natural to also adopt a “timeless” four-dimensional spacetime formalism, which, in Adlam’s terms, makes the proposal here a “non-dynamical” approach.

B. Probability

It is not enough, however, to simply avoid the issues of collapse by taking an Everettian view of the unitary evolution and branching of the quantum state. One still needs to explain how a specific branch is selected based on the classical probability assigned to it. Moreover, this raises the further question of how to even define what “probability” means.
when all possible alternatives actually exist in one branch or another. And, if one does find a reasonable definition of “probability”, it is still necessary to explain why a decohered branch should be assigned a probability based on the usual Born rule.

Of course, it is well-known that Gleason’s theorem requires the Born rule when defining an additive probability measure on a Hilbert space (of dimension at least three) [15]. And Hossenfelder has recently shown the rule can be derived even more simply assuming only a distribution that is “continuous, independent of [dimension], and invariant under unitary operations” [16]. But, as Zurek remarks, mathematical derivations such as these give “no physical insight into why the result should be regarded as probability” [17].

Everett essentially applied a relative frequency approach to define probability [4], as have subsequent authors (such as [18, 19]). However, this approach has often been criticized as being circular or inconsistent in the end (see, for example, [20–22]). More recently, an alternative has been to apply Bayes’ Theorem [23], so that “probability quantifies a degree of belief for a single trial, without any a priori connection to limiting frequencies” [24, 25]. The cost, however, is that such a definition of probability is inherently subjective, relying on an understanding of how the “level of ignorance” or “degree of belief” of a rational agent should be adjusted based on observation.

Wallace further considers the “preferences of a rational agent” in the context of the Everett interpretation, using a decision-theoretic approach to “prove that such an agent acts exactly as if he assigns probabilities to the outcomes of future events in accordance with the Born rule” [26]. (This also seems to be in line with how Everett actually thought of quantum probabilities, consistent with his later work in operations research after leaving physics [27].) Wallace takes some pains to carefully develop this as an objective definition of probability, showing that there is no “residual problem of probability” for the Everett interpretation. However, he also admits that these decision-theoretic arguments cannot really be applied in the cosmological case, since “they would require the agent to have the power to rearrange the cosmos on a vast scale.”

Essentially underlying both the frequentist and Bayesian approaches to probability is a simple conception, as stated by Laplace (though Bayes’ work actually preceded Laplace’s publication): probability is “the ratio of the number of favorable cases to that of all the cases possible” [28], and the “cases favorable to the event being sought” are considered to be all equally likely. This leads to a principle of indifference, such that, if a change to the system under consideration simply moves it from one “favorable case” to another, then an observer will be indifferent to such a change in computing probabilities.

In this regard, Zurek notes that, if we can define a physical system symmetry among “favorable cases”, then we could make the subjective principle of indifference into a definition of objective probabilities. He proposes entanglement-assisted invariance (“envariance”) as the symmetry to do this [17, 22, 29]. Zurek’s work, however, was in the context of non-relativistic quantum mechanics.

In previous papers, I have discussed how Zurek’s ideas can be applied in a relativistic and cosmological context to formulate a probability interpretation for selecting among eigenstates of the universe [30, 31]. The result is an objective definition of probability that can be applied at the scale of the universe as a whole, but which also accounts for how quantum mechanics makes successful probabilistic predictions for observations of systems within the universe. That work, however, was based on a specific spacetime path formalism for relativistic quantum mechanics [32, 33]. The present paper addresses similar issues, but generalizes the results by using only a minimal relativistic quantum formalism that does not
require a detailed model of spacetime paths.

C. Overview

The goal of this paper is to present a mathematically consistent argument for why the observed statistical distribution of quantum measurement results follows Born’s rule, without recourse to any concept of “collapse” of the wave function, consistent with RQM. After the mathematical development of this argument, I will return to whether this can be considered a “successful” interpretation.

Section II outlines the traditional formalism of Hilbert space and projection operators, in the context of RQM, as used in this paper. Section III then addresses how to model measurements using this formalism. Since the formalism being used is for a “timeless” relativistic universe, the model does not describe a measurement “process” over time, but, rather, treats measurement as the result of correlations due to interactions between a measured system, a measuring apparatus and the environment. The result is a decomposition of the state of the universe into orthogonal eigenstates (i.e., branches) corresponding to each of the possible measurement outcomes.

The next step is then to provide a probability interpretation for selecting the “actual” measurement result. Section IV tackles this using Zurek’s concept of envariance to provide an objective basis for Laplace’s principle of indifference. Given this conception of probability, Sec. V presents a derivation of Born’s rule (also adapted from Zurek), providing the desired interpretation of the amplitudes of the branch states of the universe as probabilities. However, real experiments are carried out on localized systems within the universe. So, the final step in this section is to show that, if Born’s rule applies to states of the universe as a whole, this implies that the rule is also valid for predicting the statistical results of repeated experiments.

Section VI then discusses the interpretational implications of the results presented in the previous sections, considering the features of a “successful interpretation” listed in Sec. I A.

II. RELATIVISTIC QUANTUM MECHANICS

Take the following as postulates for relativistic quantum mechanics.

1. A state $|\Psi\rangle$ is a normalizable vector in a separable Hilbert space $\mathcal{K}$ (usually taken to be normalized to $|\Psi|^2 = 1$).

2. There is a unitary representation $\{\Delta x, \Lambda\} \rightarrow \hat{U}(\Delta x, \Lambda)$ of the Poincaré group, defined on $\mathcal{K}$.

3. A physical state satisfies $\hat{H}|\Psi\rangle = 0$, for some self-adjoint Hamiltonian operator $\hat{H}$ that commutes with $\hat{U}$, forming a subspace $\mathcal{H}$ of $\mathcal{K}$.

Note that states, as defined above, are states in spacetime. That is, each state is taken to be effectively the state of the entire universe over all space and time. The Hamiltonian as defined here is not a time evolution operator on states but, rather, a constraint on which states are physical.
For a free theory, $\hat{H} = \hat{P}^2 + m^2$ (where $\hat{P}$ is the momentum operator defined by $\hat{U}(\Delta x, 0) = \exp(-i\hat{P} \cdot \Delta x)$), essentially enforcing an on-shell mass condition. For an interacting theory, $\hat{H}$ introduces all the dynamics of the theory. Particularly for a theory that includes gravity, the constraint $\hat{H}\langle \Psi \rangle = 0$ is the Wheeler-DeWitt equation [34]. (See also [35, 36] on Dirac’s theory of constraints in Hamiltonian mechanics and [37] on Hamiltonian constraint mechanics and solving the Wheeler-DeWitt equation in the context of quantum gravity.)

The concept of a “state of the universe” goes back at least to Everett’s original formulation of the universal wave function (“state function of the whole universe”) [38], though this explicit concept was removed from his thesis as first published [4]. However, Hartle and Hawking [39] later proposed a ground-state wave function of the universe that gives “the amplitude for the Universe to appear from nothing”. Hartle and Hawking make no reference to the earlier work of Everett in their paper, because it deals with technical issues of how to formulate such a wave function for quantum gravity, not questions of interpretation. But further work of Hartle and colleagues can be seen exactly as addressing such questions [6–12].

Of course, given a state of the universe, one still needs to have some way to discuss the interaction of systems within the universe, where a system is any part of the universe that can be delineated from the rest of the universe. For any specific system, some propositions about the universe as a whole will concern the condition of that system, while some will not. We will characterize a system by a set of propositions that can be made concerning it, which essentially defines what we are interested in “knowing” about it. For simplicity, assume that any system can be characterized by a discrete set of conditions.

To formalize this, first define an observable as a self-adjoint operator $\hat{A}$ on $\mathcal{H}$. For the purposes of this paper, it will be sufficient to assume an observable $\hat{A}$ has a discrete set of normalized eigenstates $|a_i\rangle$ such that

$$\hat{A} = \sum_i a_i \hat{P}_A^i,$$

where the $a_i$ are complex eigenvalues and the $\hat{P}_A^i$ are projection operators defined by

$$\hat{P}_A^i \equiv |a_i\rangle\langle a_i|.$$

A projection operator $\hat{P} \chi \alpha$ characterizes a subspace $\hat{P} \mathcal{H}$ of $\mathcal{H}$ such that $\hat{P} |\Psi\rangle \in \hat{P} \mathcal{H}$ for any $|\Psi\rangle \in \mathcal{H}$. Such a projection operator can then be interpreted as representing a logical proposition that is true for the states $|\Psi\rangle \in \hat{P} \mathcal{H}$ (i.e., for which $|\Psi\rangle$ is a unit eigenvector of $\hat{P}$), false for the states for which $\hat{P} |\Psi\rangle = 0$ and undetermined otherwise [1]. Since $|\Psi\rangle$ is a state of the universe, we can also say that $\hat{P}$ describes a condition of the universe that holds for those $|\Psi\rangle \in \hat{P} \mathcal{H}$.

So, suppose $\{\hat{P}_\alpha^S\}$ is a set of projection operators representing the conditions characterizing a system $S$. Then each $\hat{P}_\alpha^S$ represents the proposition that system $S$ is in some condition $\alpha$. Requiring that these conditions be independent means that the subspaces so identified for each $\alpha$ must be mutually orthogonal, giving

$$\hat{P}_\alpha^S \hat{P}_\beta^S = \delta_{\alpha\beta} \hat{P}_\alpha^S.$$

Further, for a system to be completely characterized, it must be in one of the possible conditions for it, so

$$\sum_\alpha \hat{P}_\alpha^S = 1.$$
It should be kept in mind that such operators define propositions on all of spacetime, within a particular state of the universe. Some care must be taken in considering the completeness of the propositions associated with a system, in that there will be many states of the universe in which the system will essentially not exist. For example, suppose that the system of interest is a measuring instrument and that the projection operators characterizing it represent pointer states of the instrument. But this presumes that the instrument is actually there (and turned on, and operating, etc.). There will be many states of the universe in which this is simply not the case.

By convention, for any system $S$, take the condition $\alpha = 0$ to represent the system “not existing”. The actually interesting conditions of the system are indexed by $\alpha > 0$. By definition, the non-existence assertion $\hat{P}_S^0$ is never truly interesting (at least for the cases considered here), but it is only when this operator is included that the set of $\hat{P}_S^\alpha$ is complete.

III. MEASUREMENTS AND RECORDS

In general, the actual dynamics of the theory will not be important for this paper. However, as a result of the constraint $\hat{H}\Psi = 0$, two systems $A$ and $S$ may interact such that the condition of $A$ is fully or partially determined by the condition of $S$. In particular, if $A$ is intended as a measuring apparatus, then its conditions will be adjusted so that they are perfectly correlated with the conditions of interest of $S$, with $\alpha$ being the pointer conditions corresponding to the measured quantities $\beta$:

$$
\hat{P}_A^\alpha \hat{P}_S^\beta |\Psi\rangle = \delta_{\alpha\beta} \hat{P}_S^\beta |\Psi\rangle .
$$

(Note that, in this case, $A$ might actually “exist” even in the condition $\alpha = 0$ that corresponds to $\beta = 0$, in which $S$ does not exist. But this is still an uninteresting condition, since no measurement can be made anyway when $S$ does not exist.)

For a proper measurement, simple correlation is not enough, though. $A$ must also leave a record of its result. This record is left through interaction of $A$ the environment $E$, which is the rest of the universe other than $A$ and $S$. And this record must be independent of any interaction of the environment with $S$. (Indeed, to ensure classicality it should be redundant and accessible to observers, though these details will not be critical here; see also [40, 41].)

In effect, the environment is another (ever present) system whose conditions of interest (in this case) are the records correlated with the measurement results of $A$:

$$
\hat{P}_E^\alpha \hat{P}_A^\beta |\Psi\rangle = \delta_{\alpha\beta} \hat{P}_A^\beta |\Psi\rangle .
$$

Using the completeness of the $\hat{P}_A^\gamma$ with Eqs. (1) and (2) then gives:

$$
\hat{P}_E^\alpha \hat{P}_S^\beta |\Psi\rangle = \sum_\gamma \hat{P}_E^\alpha \hat{P}_A^\gamma \hat{P}_S^\beta |\Psi\rangle
= \sum_\gamma \delta_{\alpha\gamma} \delta_{\gamma\beta} \hat{P}_S^\beta |\Psi\rangle
= \delta_{\alpha\beta} \hat{P}_S^\beta |\Psi\rangle .
$$

It is not particularly important here what basis is chosen for the measurement in Eq. (1). However, it is the three-way correlation between the environment, the apparatus and the
system that ensures that the chosen basis is unambiguous and physical for the recorded measurement [42]. For the rest of the discussion here, though, it will be sufficient to assume the correlation between the condition of a system and a record of its condition in the environment, as in Eq. (3), without explicitly including the measuring apparatus that led to that record. (For more on the issue of basis selection, in a non-relativistic context, see the description of einselection and quantum Darwinism by Zurek et al. [17, 40, 41, 43–45].)

So, consider that

\[ \hat{P}_S^\beta |\Psi\rangle = \psi_S^\beta(\Psi) |s^\beta(\Psi)\rangle, \]

where \( |s^\beta(\Psi)\rangle \) is a unit eigenstate of \( \hat{P}_S^\beta \) and \( \psi_S^\beta(\Psi) \) is the magnitude of \( \hat{P}_S^\beta |\Psi\rangle \). Further, since the record in the environment is independent of interaction with \( S \), we can take \( \hat{P}_E^\alpha \) and \( \hat{P}_S^\beta \) to commute, so \( |s^\beta(\Psi)\rangle \) is actually a joint eigenstate of \( \hat{P}_E^\alpha \) and \( \hat{P}_S^\beta \). Therefore:

\[ \hat{P}_E^\alpha \hat{P}_S^\beta |\Psi\rangle = \delta_{\alpha\beta} \psi_S^\beta(\Psi) |e^\alpha(\Psi)s^\beta(\Psi)\rangle. \]

Since the \( \hat{P}_E^\alpha \) and \( \hat{P}_S^\beta \) are complete sets,

\[ |\Psi\rangle = (\sum_\alpha \hat{P}_E^\alpha)(\sum_\beta \hat{P}_S^\beta)|\Psi\rangle \]

\[ = \sum_{\alpha\beta} \hat{P}_E^\alpha \hat{P}_S^\beta |\Psi\rangle \]

\[ = \sum_\alpha \psi_S^\alpha |e^\alpha s^\alpha\rangle \]

(where the notation has been simplified by eliding explicit functional dependence on the state \( \Psi \)).

Because of the required orthogonality of the set of projection operators for a system, Eq. (4) is a complete orthogonal decomposition of the state \( |\Psi\rangle \). Each of the eigenstates \( |e^\alpha s^\alpha\rangle \) represents a branch of the universe in which \( E \) records that \( S \) is in condition \( \alpha \). This is similar to the decomposition into branch states in consistent or decoherent histories formalisms [46–49], but, rather than representing a time-ordered history of quantum propositions, the branch states here are eigenstates of the universe for all spacetime.

If we were to now apply Born’s rule directly to Eq. (4), then we would interpret the squared amplitudes \( |\psi_S^\alpha|^2 \) as the probabilities for the universe to be in the branch \( |e^\alpha s^\alpha\rangle \)—that is, for the system \( S \) to be in the condition \( \alpha \). But how do we understand what a probability even is when we are considering different branches of the entire universe? We turn to this issue next.

IV. PROBABILITY

Note that the decomposition in Eq. (4) is an entanglement between the system \( S \) and its environment \( E \). This allows for what Zurek calls envariance, short for entanglement-assisted invariance [22] (originally termed “environment-assisted invariance” [29, 43, 45]). Envariance is a symmetry of an entangled state in which an operation on one of the two entangled systems can be undone by an operation solely on the other.

For example, define the unitary operator

\[ \hat{U}_S(\sigma_\alpha) = \sum_\alpha e^{i\sigma_\alpha} \hat{P}_S^\alpha \].

7
This operator acts solely on the subspaces of $H$ that correspond to the conditions of $S$. It results in a new state of the universe $|\Psi\rangle$ with the same branch states, but changes in the phases of the coefficients $\psi_{S}^{\alpha}$:

$$|\Psi\rangle = \hat{U}_{S}(\sigma_{a})|\Psi\rangle = \sum_{\alpha} e^{i\sigma_{a}\psi_{S}^{\alpha}}|e^{\alpha}s^{\alpha}\rangle.$$ 

Now define a similar operator $\hat{U}_{E}(\sigma_{a})$ for $E$. This operator acts solely on the subspaces of $H$ that correspond to the conditions of $E$. However, because of the entanglement of $S$ and $E$, $\hat{U}_{E}$ can be used to reverse the effect of $\hat{U}_{S}$:

$$\hat{U}_{E}(-\sigma_{a})\hat{U}_{S}(\sigma_{a})|\Psi\rangle = |\Psi\rangle.$$ 

Thus, the state $|\Psi\rangle$ as given in Eq. (4) is invariant under the action of $\hat{U}_{E}(-\sigma_{a})\hat{U}_{S}(\sigma_{a})$. That is, the effect on the universe of an operation on the system $S$ can be undone by an operation on the environment alone—effectively just a compensating adjustment in how the condition of the system is recorded. As argued by Zurek, this implies that the original operation on $S$ cannot be considered physically significant. This means that the phases of the coefficients $\psi_{S}^{\alpha}$ cannot have physical significance.

Next, define $\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}$ to be a unitary operator that swaps the conditions $\beta \leftrightarrow \gamma$ in $S$:

$$\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}|\Psi\rangle = \psi_{S}^{\beta}|e^{\beta},s^{\gamma}\rangle + \psi_{S}^{\gamma}|e^{\gamma},s^{\beta}\rangle + \sum_{\alpha\neq\beta,\gamma} |e^{\alpha},s^{\alpha}\rangle.$$ 

Let $\hat{U}_{E}^{(\beta\leftrightarrow\gamma)}$ similarly swap conditions in $E$. Then

$$|\Psi\rangle = \hat{U}_{E}^{(\beta\leftrightarrow\gamma)}\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}|\Psi\rangle = \psi_{S}^{\beta}|e^{\gamma},s^{\gamma}\rangle + \psi_{S}^{\gamma}|e^{\beta},s^{\beta}\rangle + \sum_{\alpha\neq\beta,\gamma} |e^{\alpha},s^{\alpha}\rangle.$$ 

If it happens that $\psi_{S}^{\beta} = \psi_{S}^{\gamma}$, then $|\Psi\rangle = |\Psi\rangle$. That is, the action of $\hat{U}_{E}^{(\beta\leftrightarrow\gamma)}\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}$ is invariant—the swap operation $\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}$ on the system is undone by the operation $\hat{U}_{E}^{(\beta\leftrightarrow\gamma)}$ on the environment. In this case, we have essentially just relabelled the states for the $\beta$ and $\gamma$ conditions in both the system and the environment. This implies that the action of $\hat{U}_{S}^{(\beta\leftrightarrow\gamma)}$ should not be considered physical when $\psi_{S}^{\beta} = \psi_{S}^{\gamma}$—or, since we can disregard phases, even when just $|\psi_{S}^{\beta}| = |\psi_{S}^{\gamma}|$.

Since swapping the states $|e^{\beta},s^{\beta}\rangle$ and $|e^{\gamma},s^{\gamma}\rangle$ is not physically significant when $|\psi_{S}^{\beta}| = |\psi_{S}^{\gamma}|$, these states can be considered physically indistinguishable. This provides the desired physical symmetry for defining objective probabilities (as discussed in Sec. 1): states that are physically indistinguishable when swapped should be considered equally likely. In particular, suppose all the coefficients in Eq. (4) have equal magnitude. Then all the branch states represent equally likely branches of the universe, since swapping any one of them with any other leaves the overall state of the universe unchanged, with the swapped conditions simply relabelled.

Now, the condition of having all coefficients of equal magnitude may seem rather restrictive, but it is enough to allow us to derive the Born rule [22].
V. BORN’S RULE

Begin by assuming that the $|\psi_S^\alpha|^2$ are rational, that is, $\psi_S^\alpha = \sqrt{m/\alpha}/M$, for some natural numbers $m_\alpha$ and $M$. The normalization $\sum_\alpha |\psi_S^\alpha|^2 = 1$ implies $M = \sum_\alpha m_\alpha$. Introduce an ancillary system $C$ with doubly-indexed projection operators $\hat{P}_{C \alpha \beta}$ such that

$$\hat{P}_{C \alpha \beta} |e^\alpha, s^\alpha, \alpha\rangle = |c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle / \sqrt{m_\alpha}, \quad \beta = 1, \ldots, m_\alpha,$$

for unit eigenstates $|c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle$ of $\hat{P}_{C \alpha \beta}$. This is essentially a subdivision of the environment to allow a finer-grained recording of the condition of $S$. Then

$$|\Psi\rangle = \sum_\alpha \sqrt{m_\alpha / M} \sum_\beta |c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle / \sqrt{m_\alpha} = \sum_{\alpha \beta} \sqrt{1 / M} |c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle.$$  \hspace{1cm} (5)

The coefficients in Eq. (5) are now all the same, meeting the condition stated at the end of Sec. IV. Therefore, as argued in that section, each of the states $|c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle$ can be considered to be equally likely. Since there are a total of $M = \sum_\alpha m_\alpha$ terms,

$$\text{Prob}(|c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle) = \frac{1}{M}.$$  \hspace{1cm} (6)

For each $\alpha$, $m_\alpha$ of the overall system/environment/ancilla states correspond to the system outcome $\alpha$, so the probability for this outcome is

$$\text{Prob}(|e^\alpha, s^\alpha, \alpha\rangle) = m_\alpha \text{Prob}(|c_{\alpha \beta}, e^\alpha, s^\alpha, \alpha\rangle) = m_\alpha / M = |\psi_S^\alpha|^2,$$

which is just the Born rule. (This assumes the additivity of probabilities, but it is possible to come to the same conclusion without making this assumption [22].) The derivation can then be extended to irrational $|\psi_S^\alpha|^2$ by continuity.

Of course, this only establishes the probability interpretation for the coefficients of branch states such as $|e^\alpha, s^\alpha, \alpha\rangle$ in the expansion of the state of the universe. It is not immediately clear how such an assignment of probabilities, essentially for a state of the entire universe, relates to the statistics of the physical results of measurement processes occurring within the universe. To clearly establish this relationship, let us consider how such statistics are computed.

Suppose the same experiment is repeated, independently, $n$ times. The “same experiment” means identical experimental setups, but each run of the experiment is still a separate system in the universe, as defined in Sec. II. Let $S_i$ for $i = 1, \ldots, n$, be the systems representing the repeated runs of the experiment, giving the results $m_i$.

Since the systems are independent, we can define joint eigenstates of the projection operators for the experimental results for each of the systems:

$$\hat{P}_{S_i}^m |m_1, \ldots, m_n\rangle = \delta_{m_i m_{m_i}}, |m_1, \ldots, m_n\rangle.$$  \hspace{1cm} (7)

Keeping the common environment of all the systems implicit, we can then decompose the state of the universe $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{m_i} \psi(m_1) \cdots \psi(m_n) |m_1, \ldots, m_n\rangle,$$  \hspace{1cm} (6)
where the summation is over all possible experimental results. Since the experimental setups are all identical, the coefficient function $\psi(m)$ is the same for all the systems, with
\[ \sum_m |\psi(m)|^2 = 1, \]

Each of the states $|m_1, \ldots, m_n\rangle$ in the expansion of $|\Psi\rangle$ in Eq. (6) represents a branch of the universe in which the specific measurement results $m_1, \ldots, m_n$ are obtained for the $n$ repetitions of the experiment. Applying the Born rule, as already established for branch states, the probability that the universe is in the state $|m_1, \ldots, m_n\rangle$ is
\[ \text{Prob}(|m_1, \ldots, m_n\rangle) = |\psi(m_1)|^2 \cdots |\psi(m_n)|^2. \]

The question to be asked is then how the relative frequency of any given result $\ell$ in the set $\{m_i\}$ compares to the probability $|\psi(\ell)|^2$ predicted by the Born rule for that individual result.

Of course, as noted in Sec. I, the use of relative frequencies to define probability is problematic [20–22]. However, these problems relate to attempts to fundamentally justify the Born probability rule itself using a relative frequency approach. What is being done here is different. Given that the Born probability rule applies for branches of the state of the universe, we are exploring whether the statistics of repeated measurement results within any such branch would be expected to follow a similar rule. In this regard, criticisms of, e.g., circularity and the need for additional assumptions, do not apply.

So, we can take the relative frequency for a specific measurement result $\ell$ within the set $\{m_i\}$ to be given by the function
\[ f_\ell(m_1, \ldots, m_n) = \frac{1}{n} \sum_{i=1}^n \delta_{m_i, \ell}. \]

This quantity is itself an observable, for the operator
\[ \hat{F}_\ell|m_1, \ldots, m_n\rangle = f_\ell(m_1, \ldots, m_n)|m_1, \ldots, m_n\rangle. \]

The expected value of $\hat{F}_\ell$ is then
\[ \langle \hat{F}_\ell \rangle \equiv \sum_{m_1, \ldots, m_n} f_\ell(m_1, \ldots, m_n) \text{Prob}(m_1, \ldots, m_n) \]
\[ = \sum_{m_1, \ldots, m_n} f_\ell(m_1, \ldots, m_n) |\psi(m_1)|^2 \cdots |\psi(m_n)|^2 \]
\[ = \frac{1}{n} \sum_{m_1, \ldots, m_n} \sum_{i=1}^n \delta_{m_i, \ell} |\psi(m_1)|^2 \cdots |\psi(m_n)|^2 \]
\[ = \frac{1}{n} \sum_{i=1}^n \sum_{m_1, m_{i-1}, m_i+1, \ldots, m_n} |\psi(m_1)|^2 \cdots |\psi(m_{i-1})|^2 |\psi(m_i)|^2 \cdots |\psi(m_n)|^2 \]
\[ = \frac{1}{n} \sum_{i=1}^n |\psi(m_i)|^2 = |\psi(m_\ell)|^2. \]
We can now consider the states $|f_\ell, m_1, \ldots, m_n\rangle$, which represent branches of the universe in which a specific relative frequency is measured for a specific set of experimental results. The total probability for measuring a certain $f_\ell$ is the sum of the probabilities for each of the states for which $nf_\ell$ of the $m_i$ have the value $\ell$:

$$p(f_\ell) = \binom{n}{nf_\ell} \langle \hat{F}_\ell \rangle^{nf_\ell} (1 - \langle \hat{F}_\ell \rangle)^{n(1-f_\ell)}.$$ 

The probability $p(f_\ell)$ is a Bernoulli distribution. By the de Moivre-Laplace theorem, for large $n$, this distribution is sharply peaked at the mean $f_\ell = \langle \hat{F}_\ell \rangle = |\psi(\ell)|^2$. Thus, the probability becomes almost certain that a choice of one of the states $|f_\ell, m_1, \ldots, m_n\rangle$ will be a history in which the observed relative frequency will be near the prediction given by the usual Born probability interpretation. Of course, for finite $n$, there is still the possibility of a “maverick” universe in which $f_\ell$ is arbitrarily far from the expected value—but this is statistically possible for any probability interpretation over a finite population. It is not a fundamental problem here, though, because we are not using relative frequencies to define probability itself, only to interpret experimental results.

VI. CONCLUSION

The mathematical development in this paper can be informally summarized as follows.

1. Presume that every physical state in relativistic quantum mechanics is essentially a state of the universe over all spacetime.

2. Any observable represented as an operator on such a state will then induce a decomposition into orthogonal eigenstates, each of which represents a potential result of the observable.

3. Taking Laplace’s “principle of indifference” as the basis for defining “probability”, and using Zurek’s concept of envariance, each such eigenstate can be considered to have a probability of being the actual result given by the square of the corresponding eigenvalue, that is, by Born’s rule.

4. Given Born’s rule for the probability of eigenstates of the universe, the statistics of experimental measurements of an observable will also tend to also follow Born’s rule (as the number of repeated measurements increases).

The result is a mathematically consistent argument for why the observed statistical distribution of quantum measurement results follows Born’s rule, without recourse to any concept of “collapse” of the wave function. Indeed, since a timeless relativistic formalism is used, there is no conception of dynamic evolution of the state, let alone its “collapse” at any point in time. Instead, measurement statistics are seen to be simply the consequence of an objective probability distribution over a population of alternate eigenstates of the universe. (While I have not explicitly computed this distribution, it is essentially the joint probability density of a “non-commutative model” given in [50]. See also [51].)

We can now consider how this mathematical approach exhibits Adlam’s features of a “successful interpretation”, enumerated in Sec. I A. To start, the approach makes no change
at all to the formalism of unitary quantum mechanics, per Adlam’s first point. And decoherence has a fundamental role in it, per Adlam’s second point. But it is interesting to consider this role in a bit more detail.

Key to the mathematical approach presented here is that measurements are recorded in the environment of the measuring apparatus, independently of the system that was measured. This means that such records are represented by commuting observables on the state of the universe. Therefore, in principle at least, it is possible to construct joint eigenstates of all the measurement records that could possibly ever be made in the universe (consistent with the full state of the universe and the Hamiltonian constraint representing the physics in it). These eigenstates form the complete set of most fine-grained decoherent histories of the entire universe that can be given classical probabilities.

And, obviously, if one parameterizes the state of the universe with the result of every possible measurement, then the result of every possible measurement will be completely determined in a universe so parameterized. What has been shown in this paper is simply that such a parameterization can actually be used to label an orthogonal decomposition of any state of the universe, and that a consistent Born-rule probability interpretation can be given for the eigenstates in such a decomposition.

Even so, since such a fine-grained history consists of records it is, by definition, just the history of what becomes known in the universe. If something is not measured and recorded, then there is simply no way to know within the universe whether it happened one way or another. Thus, recorded measurements effectively result in decoherence, while lack of measurement allows for interference.

For example, consider the traditional two-slit experiment. If the experiment is performed without measuring which slit each particle goes through, then there will not be separate, decoherent history eigenstates based on this choice, and the experiment results will reflect interference. On the other hand, if the transit of particles is observed and recorded for even just one of the slits, then there will be separate history eigenstates based on this observation and no interference in the experiment results.

Addressing this more explicitly gets to Adlam’s third point, how observers see a unique, shared outcome to each measurement. Let \( \{S_i\} \) be the complete set of systems for which measurements can be recorded in the universe with state \( |\Psi\rangle \). The decomposition into fine-grained histories is then

\[
|\Psi\rangle = \sum_{\alpha_i} |s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots\rangle,
\]

where the \( |s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots\rangle \) are orthogonal eigenstates in which each \( S_i \) has the recorded measurement \( \alpha_i \) (recall that this includes the possibility of \( \alpha_i = 0 \), meaning that \( S_i \) does not exist or is not measured in that history of the universe).

The \( s_i^{\alpha_i} \) completely enumerate the fine-grained history eigenstates of \( |\psi\rangle \). In each such eigenstate, each \( S_i \) is known to be in a specific condition with probability 1. That is,

\[
\hat{P}_{S_i} |s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots\rangle = \delta_{\alpha_i,\beta} |s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots\rangle.
\]

If one could truly know all the \( s_i^{\alpha_i} \), then the corresponding eigenstate \( |s_1^{\alpha_1}, s_2^{\alpha_2}, \ldots\rangle \) would, in fact, represent the “real” history of the universe. (This is similar to the idea of “one real fine-grained history” proposed by Gell-Mann and Hartle in a subjective probabilistic context [52].)

Of course, the “real” history \( s_i^{\alpha_i} \) is not completely knowable, even in principle. Not only would it likely require infinite knowledge, but it would lead to the paradox that, to “know”
it within the universe, it would itself have to be recorded! Instead, all the measurement records made so far are only a small subset of the $S_i$, determining a vast, coarse-grained superposition of the fine-grained history eigenstates consistent with those records:

$$|\Psi_K\rangle = \sum_{\alpha_i} \left( \prod_{i \in K} \delta_{\alpha_i,\kappa_i} \right) |s_{1}^{\alpha_1}, s_{2}^{\alpha_2}, \ldots\rangle,$$

where $K$ is the (relatively small) set of indices of systems with known measurement records, and the $\kappa_i$ are the recorded measurements for those systems.

Moreover, most of what happens in the expanse of the cosmos has little or no relevance to the experiments we carry out in our corner of the universe. So, the quantum state that we actually use for any specific system under consideration—$S_1$, say—is reduced over everything else in the universe:

$$\rho_{S_1} = \text{Tr}_{S_i \neq S_1} |\Psi_K\rangle\langle \Psi_K|.$$

Now, suppose $1 \notin K$, and then a measurement of $S_1$ is performed and recorded (obviously involving some other system as the measurement apparatus, the details of which are not important here). The result is that we now, in principle, know the state of the universe a little more precisely, as $\psi_K'$, where $K' = K \cup \{1\}$ and $S_1$ has the known condition $\kappa_1$. But $\psi_{K'}$ is then an eigenstate of $\hat{P}_{S_1}^{\kappa_1}$ and the corresponding reduced state $\rho'_{S_1}$ is similarly an eigenstate for the reduced state space of $S_1$, with eigenvalue $\kappa_1$.

This is the effective “collapse” of the state of $S_1$. Living our lives forward in time as we do, we accumulate new measurement records over time. The so-called “collapse” of a quantum state is then simply the update of a local reduced state when we get additional knowledge of a new record after carrying out a measurement. This update seems discontinuous, but that is only because we have ignored the implicit parts of the earlier state that integrated over the possible choices of the future measurement. An update exactly projects the original state into a more constrained Hilbert subspace consistent with the new measurement result, and the new state is then a bit “closer” to the actual “real” history of the universe.

Essentially, this can be considered a relativistic generalization of a consistent histories interpretation of quantum mechanics [46, 49]. However, instead of a “history” being a sequence of measurement projections made over time, a history is a set of projections representing conditions of the universe across spacetime [30, 31]. Of course, it is still possible to choose observables that represent conditions at specific points in time, and then organize them in a time-ordered history, in which case the relativistic generalization reduces to the traditional non-relativistic approach, as would be expected.

And this gets to Adlam’s final point, on the nature of shared observable reality. As noted in Sec. IA, the fundamental “beables” here are essentially “non-dynamical”. Indeed, one can consider the $s_{i}^{\alpha_i}$ to act much like “hidden variables”, since they completely determine the results of every measurement made in the universe with eigenstate $|s_{1}^{\alpha_1}, s_{2}^{\alpha_2}, \ldots\rangle$. However, they are truly “hidden”, since each value $\alpha_i$ can literally only be determined within the universe by actually making a measurement and recording that value.

So, on the one hand, such states $|s_{1}^{\alpha_1}, s_{2}^{\alpha_2}, \ldots\rangle$ can be considered an ontic representation of the “real” history of the universe. On the other hand, quantum states as we typically use them are pragmatically epistemic. They record our knowledge on some small parts of the universe, based on the measurements that we have recorded so far, and they get updated as we get new knowledge.
It is worth noting that I am using the terms “ontic” and “epistemic” here not in the technical sense of “ψ-ontic” and “ψ-epistemic” as defined by Harrigan and Spekkens [53], but rather in the traditional philosophical sense (as discussed in [54]). Indeed, by Harrigan and Spekkens’ definition, the state ρ_{S_1} is ψ-ontic, since it can be deterministically and uniquely computed from the underlying |s_1^{1}, s_2^{2}, ...⟩ states. However, the point is that we cannot ever really know, even in principle, how to carry out this computation without actually making measurements of the system. As a result we are forced to consider ρ_{S_1} as simply representing our “best knowledge” of S_1, defining a probability distribution for the result of measuring the system, over the underlying “hidden” fine-grained states of the universe.

This viewpoint on states such as ρ_{S_1} is thus similar to the ψ-ensemble interpretation of [55], in which the wave function is interpreted as representing a true ensemble over hidden variable states. Nevertheless, the quantum formalism used here is entirely orthodox, so, while the interpretation explains what “collapse” means, it sheds little further light on other particularly quantum effects such as interference. Interpretationally, this may be the best we can do without truly introducing additional “hidden variables” into the formalism.
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