Slow-roll corrections to inflaton fluctuations on a brane

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Abstract. Quantum fluctuations of an inflaton field, slow rolling during inflation are coupled to metric fluctuations. In conventional four-dimensional cosmology one can calculate the effect of scalar metric perturbations as slow-roll corrections to the evolution of a massless free field in de Sitter spacetime. This gives the well-known first-order corrections to the field perturbations after horizon exit. If inflaton fluctuations on a four-dimensional brane embedded in a five-dimensional bulk spacetime are studied to first order in slow roll, then we recover the usual conserved curvature perturbation on superhorizon scales. But on small scales, at high energies, we find that the coupling to the bulk metric perturbations cannot be neglected, leading to a modified amplitude of vacuum oscillations on small scales. This is a large effect which casts doubt on the reliability of the usual calculation of inflaton fluctuations on the brane neglecting their gravitational coupling.

Keywords: cosmology with extra dimensions, cosmological perturbation theory, inflation

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1. Introduction

Inflation is probably the simplest scenario for the origin of primordial fluctuations in our Universe [1]. Small scale vacuum fluctuations can be stretched to astrophysical scales by a period of accelerated expansion. Inflation provides a test of high energy physics because the perturbations are generated from very short scales at high energies in the very early universe. These perturbations carry signatures from high energy physics, which can be tested by means of astronomical observations.

The slow-roll approximation [2] is a useful tool for studying the fluctuations generated during inflation. If we can neglect the coupling to metric perturbations and the effective mass of the field, then the perturbations are described by the fluctuations of a free scalar field in de Sitter spacetime. This gives the familiar result that the power spectrum of scalar field perturbations at horizon crossing is given by \((H/2\pi)^2\). One can then calculate the comoving curvature perturbation which is conserved on superhorizon scales for adiabatic perturbations.

However inflaton perturbations will be coupled to gravity (metric perturbations) at first order in the slow-roll parameters. In four-dimensional general relativity it is known how one can consistently include linear metric perturbations by working in terms of the...
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gauge invariant combination of scalar field and curvature perturbations, the so-called Mukhanov–Sasaki variable, which obeys a simple wave equation [3]. Gravitational effects are negligible at small scales and high energies, where perturbations can be normalized to the usual Bunch–Davies vacuum state. On large scales (superhorizon scales) the comoving curvature perturbation is conserved, allowing one to relate observations of temperature anisotropies in the cosmic microwave background to high energy vacuum fluctuations during inflation. Exact solutions are known for the special case of power-law inflation in general relativity which generalize the de Sitter result and have been used to calculate first-order slow-roll corrections in more general inflation models [4].

In this paper, we develop a new way to derive slow-roll corrections based on a slow-roll expansion about de Sitter spacetime. In four-dimensional general relativity we show how to recover the usual first-order slow-roll corrections. Our method may be useful when one cannot derive an exact solution and the background spacetime is given as a perturbation about de Sitter spacetime.

We then apply our method to inflation in the brane world model. New ideas in the string theory suggest that our observable universe is a four-dimensional hypersurface, or brane, in a higher dimensional bulk spacetime [5]. The simplest example of this model is the Randall–Sundrum model where there is a brane embedded in a five-dimensional anti-de Sitter (AdS) spacetime [6]. An AdS spacetime has a characteristic curvature scale $\mu$ associated with the negative cosmological constant in the bulk. The spacetime shrinks exponentially away from the brane and this geometry effectively compactifies the five-dimensional spacetime with the effective size $\mu^{-1}$. On large length scales $L > \mu^{-1}$, four-dimensional Einstein gravity is recovered, while on small scales, the gravity becomes five-dimensional [7]. In the early universe when the Hubble horizon is smaller than $\mu^{-1}$, we expect to find significant effects from higher dimensional bulk spacetime. Indeed, the Friedmann equation is modified from the conventional four-dimensional theory for $H\mu \gg 1$ [8]. This modification of Friedmann can provide a novel model for inflation [9]–[11].

In [9], the amplitude of the curvature perturbation is calculated by taking into account the modification of the Friedmann equation. This work has been extended to include higher order corrections in slow-roll parameters [12] and the formula has been widely used to confront this model with the observations [13]. But to derive these formulae for the spectrum of the primordial curvature perturbations the effect of coupling to five-dimensional gravity has been neglected and in particular it is assumed that the power spectrum of inflaton perturbations at horizon crossing is given by $(H/2\pi)^3$. This assumption is only valid to zeroth order in slow-roll parameters. At first order the inflaton perturbations will be coupled to metric perturbations. In the brane world, metric perturbations live in the five-dimensional spacetime, and thus we must check whether five-dimensional effects change the result from conventional four-dimensional theory. In particular, at small scales/high energies, the five-dimensional effects could be large.

The first attempt to study the back-reaction due to metric perturbations was made in [14]. There, perturbations are solved perturbatively in slow-roll parameters. We should emphasize that this is the only possible way to perform the calculations analytically. If the background spacetime of the brane deviates from de Sitter spacetime, we cannot solve the bulk metric perturbations analytically. In contrast to the case for four-dimensional general relativity, there are no other exact solutions known for the perturbation equations.
Thus we must develop a new approach to calculate the effect of slow-roll corrections. In this paper, we extend earlier studies and investigate the back-reaction due to higher dimensional perturbations using a slow-roll expansion.

The structure of the rest of the paper is as follows. In section 2, we describe our new approach for deriving first-order slow-roll corrections in a conventional four-dimensional cosmology. In section 3, we review an inflation model in the Randall–Sundrum brane world driven by an inflaton field on the brane. In section 4, we derive the equations that govern the coupled system of inflaton fluctuations on the brane and metric perturbations in the bulk. In section 5, the first-order corrections to the inflaton fluctuations on the brane are solved. In section 6, we discuss the implications of our result for the brane world inflation model.

2. Slow-roll expansion of scalar perturbations in 4D cosmology

2.1. Background spacetime

We consider an inflaton $\phi$ whose potential energy density $V(\phi)$ drives inflation. In the conventional four-dimensional general relativity described by the metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j,$$

the Friedmann equation and the equation of motion for the homogeneous field, $\phi$, are given by

$$H^2 = \frac{\kappa_4^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi},$$

where $H = \dot{a}/a$, $\kappa_4 = 8\pi G_4$ and $G_4$ is the 4D gravitational coupling constant. A dot indicates a derivative with respect to cosmic time, $t$. Slow-roll parameters are defined by

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}.$$ 

Slow-roll inflation is described by small values of $\epsilon$ and $\eta$.

2.2. Slow-roll corrections to inflaton fluctuations

The inhomogeneous inflaton fluctuation, $\delta\phi$, is coupled to the metric perturbations. In the longitudinal gauge, the perturbed metric is written as

$$ds^2 = -(1 + 2\Psi)dt^2 + a(t)^2(1 + 2\Phi)\delta_{ij}dx^i dx^j.$$

The coupled equations for $\delta\phi$, $\Psi$ and $\Phi$ can be simplified by using Mukhanov–Sasaki variable defined by [3]

$$u = a \left( \delta\phi - \dot{\phi} \frac{\Psi}{H} \right).$$
Expanding $u$ in Fourier modes, the wave equation for $u$ is given by
\begin{equation}
\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2}\right) u_k = 0,
\end{equation}
where $z \equiv (a\dot{\phi})/H$ and $\tau$ is a conformal time defined as
\begin{equation}
\tau = \int \frac{dt}{a(t)}.
\end{equation}

In the case of the slow-roll inflation, the mass term in the Mukhanov–Sasaki equation (7) can be approximated as
\begin{equation}
\frac{1}{z} \frac{d^2 z}{d\tau^2} = \frac{1}{\tau^2} \left(2 + 6\epsilon - 3\eta + \mathcal{O}(\eta^2, \epsilon^2)\right),
\end{equation}
up to first order of the slow-roll parameters. Then equation (7) can be expressed as
\begin{equation}
\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{\tau^2}(2 + 6\epsilon - 3\eta)\right) u_k = 0.
\end{equation}

Usually, the appropriately normalized solution with the correct asymptotic behaviour at small scales is obtained by solving equation (10) directly as
\begin{equation}
u = \frac{3}{2} + 2\epsilon - \eta
\end{equation}

\begin{equation}
H_\nu(1)(-k\tau),
\end{equation}
where $\nu$ is the Hankel function of the first kind of order $\nu$. Here we assumed the Bunch–Davies vacuum state where perturbations stay in the Minkowski vacuum at small scales. Equation (11) is an exact solution of the perturbation equation (10) only if the slow-roll parameters $\epsilon$ and $\eta$ are constant. However, their variation in a Hubble time is second order and hence of higher order in the slow-roll expansion. Thus we can take $\epsilon$ and $\eta$ to be evaluated around the time of horizon crossing.

We are interested in the asymptotic form of the solution well outside the horizon. Taking the limit $-k\tau \to 0$ yields the asymptotic form of $u_k$:
\begin{equation}
\lim_{-k\tau \to 0} u_k \sim \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2} (-\tau)^{1/2} H_\nu(1)(-k\tau),
\end{equation}
where we have used the formula for the poly-Gamma function
\begin{equation}
\psi(3/2) \equiv \frac{\Gamma'(3/2)}{\Gamma(3/2)} = 2 - \gamma - 2 \ln 2,
\end{equation}
where $\gamma \simeq 0.58$ is Euler’s constant.

The quantity that is related to observables today is the power spectrum of the curvature perturbation given by
\begin{equation}
P_{R}^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}} \left| \frac{u_k}{z} \right|.
\end{equation}
From equations (9) and (10), it can be shown that, at large scale, the time dependences of \( u_k \) and \( z \) are the same, that is, \( P^{1/2}_R \) is constant. Note that the constancy of \( R \) in the large scale limit does not depend on the slow-roll approximation, but holds for any adiabatic perturbation. Thus this comoving curvature perturbation can be related to the perturbation in the radiation density on large scales long after inflation has ended.

For the model with a monotonic potential, the following relation holds:

\[
|z| = \frac{a|\dot{\phi}|}{H} = \frac{2}{\kappa^2} \frac{a}{H} \left| \frac{dH}{d\phi} \right|, \tag{16}
\]

and conformal time can be evaluated up to the first order in slow-roll parameters as

\[
\tau = -\frac{1}{aH}(1 + \epsilon). \tag{17}
\]

Thus the power spectrum of the curvature perturbation is given by

\[
P^{1/2}_R = \left[ 1 - (2C + 1)\epsilon + C\eta \right] \kappa^2 \frac{H^2}{4\pi} \left( \frac{H^2}{[dH/d\phi]} \right)_{k=aH}, \tag{18}
\]

where \( C = -2 + \ln 2 + \gamma \simeq -0.73 \). The terms proportional to the slow-roll parameters are called the Stewart–Lyth correction [4].

### 2.3. Perturbing about de Sitter spacetime

In this subsection, we reproduce the usual slow-roll corrections in a perturbative approach which does not require any exact solution of the perturbation equation other than that in a de Sitter spacetime. This will be more suited to extension to the case of brane world gravity.

At zeroth order in slow-roll parameters, the spacetime is described by the de Sitter spacetime. Thus we can expand the spacetime from de Sitter spacetime. The scale factor is expanded as

\[
a(t) = a^{(0)}(t) + a^{(1)}(t) + \mathcal{O}(\epsilon^2), \quad a^{(0)}(t) = \exp(Ht). \tag{19}
\]

Accordingly, the Mukhanov–Sasaki variable is expanded as

\[
u_k(\tau) = u_k^{(0)}(\tau) + u_k^{(1)}(\tau) + \mathcal{O}(\epsilon^2), \tag{20}
\]

where \( u_k^{(0)} \equiv a\delta\phi^{(0)} \) and \( u_k^{(1)} \equiv a(\delta\phi^{(1)} - (\dot{\phi}/H)\Psi) \). Substituting equation (20) into equation (10), the zeroth-order equation is given by

\[
\frac{d^2u_k^{(0)}}{d\tau^2} + \left( k^2 - \frac{2}{\tau^2} \right) u_k^{(0)} = 0. \tag{21}
\]

Since we expect the effects of the deviation from de Sitter spacetime to be insignificant at small scales, the form of \( u_k^{(0)} \) is determined by demanding a Bunch–Davies vacuum

\[
u_k^{(0)}(\tau) = A(-\tau)^{1/2}H_{3/2}^{(1)}(-k\tau), \tag{22}
\]

where \( A = (\sqrt{\pi}/2)e^{i\theta} \) and the phase \( \theta \) is fixed so that \( u_k(\tau) \rightarrow (1/\sqrt{2k})e^{-ik\tau} \).
Next, we must solve \( u_k^{(1)} \) sourced from this zeroth-order solution:
\[
\frac{d^2 u_k^{(1)}}{d\tau^2} + \left( k^2 - \frac{2}{\tau^2} \right) u_k^{(1)} - \frac{1}{\tau^2} (6\epsilon - 3\eta) u_k^{(0)} = 0. 
\]
\[ (23) \]

If we impose the boundary conditions (i) \( u_k^{(1)}(\tau) \) is negligible in the limit \( \tau \to -\infty \) and (ii) \( u_k^{(1)}(\tau) \) does not diverge faster than \( u_k^{(0)}(\tau) \) in the limit \( \tau \to 0 \), then we find that the solution is given by
\[
\begin{align*}
  u_k^{(1)} &= C_1(-k\tau)^{1/2} J_{3/2}(-k\tau) + C_2(-k\tau)^{1/2} H_{3/2}^{(1)}(-k\tau), \\
  C_1 &= \frac{\pi i}{2} (6\epsilon - 3\eta) A \int_{-\infty}^{\tau} \frac{d\tau'}{\tau'} \left\{ H_{3/2}^{(1)}(-k\tau') \right\}^{2}, \\
  C_2 &= \frac{-\pi i}{2} (6\epsilon - 3\eta) A \int_{-\infty}^{\tau} \frac{d\tau'}{\tau'} H_{3/2}^{(1)}(-k\tau') J_{3/2}(-k\tau'), 
\end{align*}
\]
\[ (24) \]

where \( J_\nu \) is the Bessel function of the order \( \nu \).

We take the limit \( -k\tau \to 0 \) and compare the asymptotic form with equation (13). Using the small arguments limit of the Bessel functions
\[
J_{3/2}(x) \sim \left( \frac{x}{2} \right)^{3/2} \frac{1}{\Gamma(5/2)}, \quad H_{3/2}^{(1)}(x) \sim -i \frac{\Gamma(3/2)}{\pi} \left( \frac{2}{x} \right)^{3/2},
\]
\[ (25) \]
we can show that the zeroth-order Mukhanov–Sasaki variable approaches
\[
\begin{align*}
  u_k^{(0)}(\tau) &\to -4i \frac{\Gamma(3/2)}{\pi} A(2k)^{-1/2}(-k\tau)^{-1}. \\
\end{align*}
\]
\[ (26) \]

Next, we must evaluate the asymptotic form of the first-order Mukhanov–Sasaki variable. Using equation (25), \( C_1 \) is evaluated as
\[
C_1 \to 4i(2\epsilon - \eta) A \frac{\Gamma(3/2)^2}{\pi} \frac{1}{(-k\tau)^3}. 
\]
\[ (27) \]
We should be careful in evaluating the asymptotic behaviour of \( C_2 \) because subleading terms are comparable to the contribution from \( C_1 \). Using
\[
J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left( \sin x \frac{x}{x} - \cos x \right), \quad J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left( \sin x + \cos x \frac{x}{x} \right),
\]
\[ (28) \]
the integral for \( C_2 \) in equation (24) can be evaluated as
\[
\int_{-\infty}^{\tau} d\tau' \frac{1}{\tau'} H_{3/2}^{(1)}(-k\tau') J_{3/2}(-k\tau') \sim -\frac{2i}{3\pi} C_i(-2k\tau) + \frac{14i}{9\pi}, 
\]
\[ (29) \]
where \( C_i \) is the integrated cosine function defined as
\[
C_i(x) \equiv -\int_{x}^{\infty} \frac{\cos t}{t} dt. 
\]
\[ (30) \]
For small \( -k\tau \), the integrated cosine function can be expressed as
\[
C_i(-2k\tau) \to \gamma + \ln 2 + \ln(-k\tau). 
\]
\[ (31) \]
Therefore, the asymptotic form of $C_2$ is given by
\[ C_2 \to (2\epsilon - \eta) A \left( \gamma + \ln 2 + \ln(-k\tau) + \frac{\gamma}{2} \right). \] (32)

Then we obtain the asymptotic form of $u_k$ for $-k\tau \to 0$ up to the first order in the slow-roll parameters:
\[ u_k(\tau) \to (-i) e^{i\theta} \left\{ 1 + (2\epsilon - \eta)(2 - \gamma - \ln 2) \right\} \left\{ 1 - (2\epsilon - \eta) \ln(-k\tau) \right\} \frac{1}{\sqrt{2k}} (-k\tau)^{-1}, \] (33)

where we have used the fact that $A = (\sqrt{\pi}/2) e^{i\theta}$. This should be compared with equation (13). There appears a logarithmic term which diverges for $-k\tau \to 0$. However, if we can renormalize this divergence by rewriting the logarithmic term as
\[ 1 - (2\epsilon - \eta) \ln(-k\tau) \simeq (-k\tau)^{-2\epsilon + \eta}, \] (34)

we see that equation (33) is consistent with equation (13).

Indeed, the logarithmic divergence for $-k\tau \to 0$ in $u_k$ does not show up in the spectrum of the curvature perturbation. In order to see this, we expand the curvature perturbation as
\[ \mathcal{P}_{\mathcal{R}}^{1/2} = \mathcal{P}_{\mathcal{R}}^{1/2}(0) + \mathcal{P}_{\mathcal{R}}^{1/2}(1) + \mathcal{O}(\epsilon^2). \] (35)

On the other hand, by the definition of the curvature perturbation (15), we can write the spectrum of the curvature perturbation up to the first order in slow-roll parameters as
\[ \mathcal{P}_{\mathcal{R}}^{1/2} \simeq \sqrt{\frac{k^3}{2\pi}} \left| \frac{u_k(0)}{z(0)} + \frac{u_k(1)}{z(0)} \right|, \] (36)

where we also expanded $z \equiv (a\dot{\phi})/H$ as
\[ z = z^{(0)} + z^{(1)} + \mathcal{O}(\epsilon^2). \] (37)

Since there is a difficulty in defining the curvature perturbation in de Sitter spacetime, we concentrate on the ratio between the zeroth order and the first order of the curvature perturbation. From comparing equation (35) to (36), the ratio is given by
\[ \frac{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}(1)}{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}(0)} = \frac{u_k^{(1)}}{u_k^{(0)}} - \frac{z^{(1)}}{z^{(0)}}. \] (38)

In order to evaluate equation (38), we must obtain $z^{(0)}$ and $z^{(1)}$, that is, we must solve equation (9) perturbatively. Substituting equation (37) into equation (9), the equation for $z$ at zeroth order is given by
\[ \frac{d^2 z^{(0)}}{d\tau^2} = \frac{2}{\tau^2} z^{(0)}. \] (39)

If we consider only the growing mode, the zeroth-order solution for $z^{(1)}$ can be obtained as
\[ z^{(0)} = B\tau^{-1}, \] (40)

where $B$ is an integration constant. This zeroth-order solution gives a source term in the equation for $z$ at first order:
\[ \frac{d^2 z^{(1)}}{d\tau^2} = \frac{2}{\tau^2} z^{(1)} + \frac{(6\epsilon - 3\eta)z^{(0)}}{\tau^2}. \] (41)
The growing mode solution for the first order, \( z^{(1)} \), is given by
\[
z^{(1)} = -(2\epsilon - \eta)B\tau^{-1}\ln(-k\tau) + BD\tau^{-1},
\] (42)
where \( D \) is another integration constant. Then we get
\[
\frac{z^{(1)}}{z^{(0)}} = -(2\epsilon - \eta)\ln(-k\tau) + D.
\] (43)
This logarithmic divergence term exactly cancels the logarithmic divergence term in \( u_k \):
\[
\frac{u_k^{(1)}}{u_k^{(0)}} = (2 - \gamma - \ln 2 - \ln(-k\tau))(2\epsilon - \eta).
\] (44)
From equations (43) and (44) we obtain
\[
\frac{\{P_{1/2}^R\}^{(1)}}{\{P_{1/2}^R\}^{(0)}} = -C(2\epsilon - \eta) - D,
\] (45)
where \( C = -2 + \ln 2 + \gamma \simeq -0.73 \) is again a numerical constant. We cannot determine \( D \) in this approach, because of the difficulty of defining the comoving curvature perturbation in pure de Sitter spacetime. However, we can still fix \( D \) as follows. Neglecting the logarithmic term, which is cancelled by the contribution from \( u_k \), the solution for \( z \) is written as
\[
z = B(1 + D)\tau^{-1}.
\] (46)
This must be compared with the definition of \( z \):
\[
z = \frac{a|\dot{\phi}|}{H} \sim -\frac{|\dot{\phi}|}{H^2}(1 + \epsilon)\tau^{-1},
\] (47)
where the solution for \( a \) up to the first order was used. Then we can identify \( B = -|\dot{\phi}|/H^2 \) and \( D = \epsilon \). Then equation (45) agrees with the Stewart–Lyth correction given by equation (18).

### 3. Slow-roll inflation in the Randall–Sundrum brane world

In this section, we apply our perturbative approach to the brane world model. We consider the simplest version of the brane world inflation model based on the Randall–Sundrum model. We will consider a single brane embedded in a five-dimensional AdS spacetime. We assume that the inflaton \( \phi \) is confined to the brane while gravity can propagate in the whole five-dimensional spacetime [9].

The five-dimensional metric describing this model is given by [8]
\[
ds^2 = dy^2 - N(y, t)^2 dt^2 + A(y, t)^2 \delta_{ij} dx^i dx^j,
\] (48)
where
\[
A(y, t) = a(t) \left[ \cosh \mu y - \left( 1 + \frac{k^2 \rho}{6\mu} \right) \sinh \mu y \right],
\] (49)
\[
N(y, t) = \cosh \mu y - \left( 1 - \frac{k^2 \rho}{6\mu} (2 + 3w) \right) \sinh \mu y,
\] (49)
\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\] (50)
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and \( w = P/\rho \). The brane is located at \( y = 0 \) and the inflaton is confined to this hypersurface. On the brane, the Friedmann equation and the equation of motion for the scalar field are given by

\[
H^2 = \frac{k^2}{3} \rho + \frac{k^4}{36} \rho^2, \\
\ddot{\phi} + 3H \dot{\phi} = -\frac{dV}{d\phi},
\]

where \( k^2 = \kappa^2 \mu \), \( \kappa^2 = 8\pi G_5 \) and \( G_5 \) is 5D gravitational coupling. We can define slow-roll parameters in the same way as for the conventional cosmology, equation (4).

Unfortunately, the background metric (48) is not in general a separable function with respect to \( y \) and \( t \). Thus we cannot solve the metric perturbations analytically. In order to solve for the \( y \) dependence of the bulk gravitons and to study the time dependence of the perturbations on the brane, we will expand about the special case of a de Sitter spacetime on the brane. This corresponds to the background solution to zeroth order in slow-roll expansion. For a de Sitter brane, the AdS bulk gives a separable form for the bulk metric [15]:

\[
ds^2 = dy^2 + N^2(y) \left[-dt^2 + a^2(t)\delta_{ij} dx^i dx^j \right],
\]

where

\[
a(t) = e^{Ht},
\]

\[
N(y) = \frac{H}{\mu} \sinh \mu (y_h - |y|),
\]

and \( y = \pm y_h \) are Cauchy horizons [15], with

\[
y_h = \frac{1}{\mu} \coth^{-1} \left( \sqrt{1 + \left( \frac{H}{\mu} \right)^2} \right).
\]

It is often useful to work in terms of the conformal bulk coordinate \( z = \int dy/N(y) \):

\[
z = \text{sgn}(y) H^{-1}_0 \ln \left[ \coth \frac{1}{2} \mu (y_h - |y|) \right].
\]

The Cauchy horizon is now at \( |z| = \infty \), and the brane is located at \( z = \pm z_b \), with

\[
z_b = \frac{1}{H} \sinh^{-1} \left( \frac{H}{\mu} \right).
\]

The line element, equation (53), becomes

\[
ds^2 = N^2(z) \left[-dt^2 + dz^2 + e^{2Ht} d\vec{x}^2 \right],
\]

where

\[
N(z) = \frac{H}{\mu \sinh(H|z|)}.
\]
4. Equations for bulk metric perturbations and inflaton perturbations on the brane

In this section, we derive the basic equations for the coupled Mukhanov–Sasaki variable on the brane and bulk metric perturbations following [14].

4.1. Master variable for perturbations in the AdS bulk

In the background spacetime given by equation (59) bulk metric perturbations can be solved using the master variable [16,17]. The perturbed metric is given by

\[ ds^2 = N(z)^2[(1 + 2A_{yy}) dz^2 + 2A_y dt dz - (1 + 2A_y) dt^2 + a^2(1 + 2\mathcal{R})\delta_{ij} dx^i dx^j]. \]  

(61)

In the special case of a de Sitter brane in the AdS bulk, the metric variables are written by the master variable \( \Omega \) as

\[
\begin{align*}
A &= -\frac{a^{-1}N^{-3}}{6}\left(2\Omega'' - 3\frac{N'}{N}\Omega' + \ddot{\Omega} - \mu^2 N^2 \Omega\right), \\
A_y &= a^{-1}N^{-3}\left(\frac{\Omega'}{N}\right), \\
A_{yy} &= a^{-1}N^{-3}\left(\Omega'' - 3\frac{N'}{N}\Omega' + 2\ddot{\Omega} + \mu^2 N^2 \Omega\right), \\
R &= a^{-1}N^{-3}\left(\Omega'' - \ddot{\Omega} - 2\mu^2 N^2 \Omega\right).
\end{align*}
\]

(62), (63), (64), (65)

From the perturbed five-dimensional Einstein equation, we can derive the equation for \( \Omega \):

\[
\ddot{\Omega} - 3H\dot{\Omega} - \left(\Omega'' - 3\frac{N'}{N}\Omega'\right) + \frac{k^2}{a^2}\Omega - \mu^2 N^2 \Omega = 0.
\]

(66)

Solutions of the master equation can be separated into eigenmodes of the time-dependent equation on the brane and bulk mode equation:

\[
\Omega(t, y; \vec{x}) = \int d^3\vec{k} \, dm \, \alpha_m(t) u_m(z) e^{i\vec{k}.\vec{x}},
\]

where

\[
\begin{align*}
\ddot{\alpha}_m - 3H\dot{\alpha}_m + \left[m^2 + \frac{k^2}{a^2}\right] \alpha_m &= 0, \\
u'' - 3\frac{N'}{N} \nu' + \mu^2 N^2 \nu + m^2 \nu &= 0.
\end{align*}
\]

(67), (68)

Note that the Hubble damping term \(-3H\dot{\alpha}_m\) has the ‘wrong sign’, i.e., this is not the standard wave equation for a scalar field in four dimensions.

If we write \( \alpha_m = a^2 \varphi_m \) and work in terms of the conformal time \( \tau = -1/(aH) \), the time-dependent part of the wave equation (67) can be rewritten as

\[
\frac{d^2 \varphi_m}{d\tau^2} + \left[k^2 - 2 - \frac{(m^2/H^2)}{\tau^2}\right] \varphi_m = 0.
\]

This is the same form as for the time-dependent mode equation commonly given for a massive scalar field in de Sitter spacetime. The general solution is given by

\[
\varphi_m(\eta; \vec{k}) = \sqrt{-k\tau} B_{\nu}(-k\tau), \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2},
\]

(69)
where $B_\nu$ is a linear combination of Bessel functions of order $\nu$. The solutions oscillate at early times/small scales for all $m$, with an approximately constant amplitude while they remain within the de Sitter event horizon ($k \gg aH$). ‘Heavy modes’, with $m > (3/2)H$, continue to oscillate as they are stretched to superhorizon scales, but their amplitude rapidly decays away, $|u_m| \propto a^{-3}$. But for ‘light modes’ with $m < (3/2)H$, the perturbations become overdamped at late times/large scales ($k \ll aH$), and decay more slowly: $|u_m| \propto a^{2\nu-3}$.

### 4.2. Mukhanov–Sasaki equation on the brane

Now we introduce a scalar field fluctuation on the brane. We expand the scalar field perturbation in terms of slow-roll parameters:

$$\delta \phi = \delta \phi^{(0)} + \delta \phi^{(1)} + \ldots.$$  \hfill (70)

The zeroth order of the scalar field fluctuation obeys the following equation of motion:

$$\ddot{\delta \phi}^{(0)} + 3H \dot{\delta \phi}^{(0)} + \frac{k^2}{a^2} \delta \phi^{(0)} = 0.$$  \hfill (71)

The metric perturbations are generated by the zeroth-order fluctuation of the scalar field through the induced Einstein equations on the brane \[18\]:

\begin{align}
3H \dot{\Psi} - 3H^2 \Phi + \frac{k^2}{a^2} \Psi &= \frac{\kappa^2_{4,\text{eff}}}{2} (\dot{\phi} \dot{\phi}_0 + V' \delta \phi^{(0)}) + \frac{\kappa^2_4}{2} \delta \rho_E, \\
H \Phi - \dot{\Psi} &= \frac{\kappa^2_{4,\text{eff}}}{2} \phi \delta \phi^{(0)} - \frac{\kappa^2_4}{2} \delta q_E, \\
- \ddot{\Psi} - 3H \dot{\Psi} + H \dot{\Phi} + 3H^2 \Phi - \frac{1}{3} \frac{k^2}{a^2} (\Psi + \Phi) &= \frac{\kappa^2_{4,\text{eff}}}{2} (\dot{\phi} \dot{\phi}_0 - V' \delta \phi^{(0)}) + \frac{\kappa^2_4}{6} \delta \rho_E, \\
-a^2 (\Psi + \Phi) &= \kappa^2_4 \delta \pi_E, \\
\end{align}

where

$$A(y=0,t) = \Phi(t), \quad R(y=0,t) = \Psi(t),$$

$$\kappa^2_4 \delta \rho_E = \frac{k^4 a^{-5}}{3} \Omega,$$

$$\kappa^2_4 \delta q_E = \frac{k^2 a^{-3}}{3} \left( \ddot{\Omega} - H \dot{\Omega} \right),$$

$$\kappa^2_4 \delta \pi_E = \frac{a^{-3}}{2} \left( \ddot{\Omega} - H \dot{\Omega} + \frac{k^2 a^{-2}}{3} \right),$$

and

$$\kappa_{4,\text{eff}} = - \kappa_4 \left. \frac{N'}{N} \right|_{y=0}.$$  \hfill (80)

The contributions $\delta \rho_E, \delta q_E$ and $\delta \pi_E$ come from the projected 5D Weyl tensor and these describe the effect of the bulk gravitational perturbations \[19\]. The metric fluctuations in turn affect the dynamics of the first-order scalar field perturbation

$$\ddot{\delta \phi}^{(1)} + 3H \dot{\delta \phi}^{(1)} + \frac{k^2}{a^2} \delta \phi^{(1)} = -V'' \delta \phi^{(0)} - 3 \dot{\phi} \Psi + \dot{\phi} \dot{\Phi} - 2V' \Phi.$$  \hfill (81)
In order to evaluate the effect from metric perturbations, it is useful to use the Mukhanov–Sasaki variable $Q$ as in the conventional cosmology:

$$Q = \delta \phi - \frac{\dot{\phi}}{H} \Psi.$$  \hspace{1cm} (82)

In terms of slow-roll expansion, we have $Q^{(0)} = \delta \phi^{(0)}$ and $Q^{(1)} = \delta \phi^{(1)} - (\dot{\phi}/H) \Psi$. Then using the induced Einstein equations, equations (72), (74) and (75), we can derive the equation for $Q^{(1)}$:

$$\ddot{Q}^{(1)} + 3H\dot{Q}^{(1)} + \frac{k^2}{a^2} Q^{(1)} = -V'' Q^{(0)} - 6\dot{H} Q^{(0)} + J,$$  \hspace{1cm} (83)

where

$$J = -\frac{\kappa^2}{3H} \left( k^2 \delta \pi_E + \delta \rho_E \right)$$

$$= -\frac{\dot{\phi}}{H} \frac{k^2 a^{-3}}{6} \left( \ddot{\Omega} - H \dot{\Omega} + \frac{k^2}{a^2} \Omega \right).$$  \hspace{1cm} (84)

This equation is the same as the standard four-dimensional cosmology except for the term $J$, which describes the corrections from the five-dimensional bulk perturbations. Because $J$ contains the five-dimensional quantity $\Omega$, we must solve the bulk equation for $\Omega$ to evaluate the effects.

### 4.3. Boundary condition for $\Omega$

In order to solve $\Omega$, we must specify the boundary condition for $\Omega$. We rewrite the expressions of $\Phi$ and $\Psi$, equation (62) and (65), as [20]

$$\Psi = \frac{a^{-1} N^{-3}}{6} \left[ 3 \frac{N'}{N} \mathcal{F} - 3H(\dot{\Omega} - H\dot{\Omega}) - a^{-2} \Delta \Omega \right],$$  \hspace{1cm} (85)

$$\Phi = \frac{a^{-1} N^{-3}}{6} \left[ -3 \frac{N'}{N} \mathcal{F} - 3\ddot{\Omega} + 6H\dot{\Omega} - 3H^2 \Omega + 2a^{-2} \Delta \Omega \right],$$  \hspace{1cm} (86)

where

$$\mathcal{F} = \Omega' - \frac{N'}{N} \Omega.$$  \hspace{1cm} (87)

Substituting these expressions into the induced Einstein equations (72)–(75), we obtain the equations written only in $\mathcal{F}$ and $\delta \phi^{(0)}$:

$$-3H\dot{\mathcal{F}} - k^2 a^{-2} \mathcal{F} = \kappa^2 a (\ddot{\phi} \delta \phi^{(0)} + V'(\phi) \delta \phi^{(0)}),$$  \hspace{1cm} (88)

$$\dot{\mathcal{F}} = \kappa^2 a \ddot{\phi} \delta \phi^{(0)},$$  \hspace{1cm} (89)

$$\ddot{\mathcal{F}} + 2H\dot{\mathcal{F}} = \kappa^2 a (\ddot{\phi} \delta \phi^{(0)} - V'(\phi) \delta \phi^{(0)}).$$  \hspace{1cm} (90)

These equations can be thought as the boundary conditions for $\Omega$. Combining the junction conditions, equations (88)–(90), we get an evolution equation for $\mathcal{F}$:

$$\ddot{\mathcal{F}} - \left( H + 2\frac{\dot{\phi}}{\phi} \right) \dot{\mathcal{F}} + k^2 a^{-2} \mathcal{F} = 0.$$  \hspace{1cm} (91)

This is consistent with the equation for the scalar field, equation (71).
5. Perturbative solutions

We must solve the coupled equations (equations (66)) for $\Omega$ and equation (83) for $Q$. Introducing the dimensionless quantities

\[ Q(t) = Ha(t)^{-1}u(\tau), \quad \Omega(z, t) = \kappa^2 \dot{\phi}H^{-1}\omega(z, \tau), \quad (92) \]

the coupled equations are written as

\[ k^2 \tau^2 \left( \ddot{\omega} + \frac{4}{\tau} \dot{\omega} + k^2 \omega \right) = \omega'' + 3 \cosh H \frac{z \omega'}{\sinh H z} + \frac{1}{\sinh^2 H z} \omega', \quad (93) \]

\[ \dot{\mathcal{F}}_\omega = aH^2 u, \quad \mathcal{F}_\omega = \left( \omega' + \frac{\cosh H z \omega}{\sinh H z} \right)_{z=\tau_b}, \quad (94) \]

\[ \ddot{u} + k^2 u - \frac{1}{\tau^2} (2 + 6 \epsilon - 3 \eta) u = J_u, \quad J_u = -\beta^2 k^2 \tau^2 \left( \ddot{\omega} + \frac{2}{\tau} \dot{\omega} + k^2 \omega \right), \quad (95) \]

where a dot denotes a derivative with respect to $\tau$ and

\[ \beta^2 = \frac{\kappa^2 \dot{\phi}^2}{6H}. \quad (96) \]

At the leading order in slow-roll parameters, $\beta^2$ can be written as

\[ \beta^2 = \frac{1}{3} \frac{H}{\mu} \left( 1 + \left( \frac{H}{\mu} \right)^2 \right)^{-1/2}. \quad (97) \]

Thus $\beta^2$ is essentially the slow-rolling parameter and it controls the strength of coupling between inflaton perturbation and gravitational perturbations in the bulk. We solve the coupled equations perturbatively in terms of small $\beta^2$.

5.1. Zeroth-order solutions

At the zeroth order where $\beta^2 = 0$, the solution for $u$ is given by

\[ u^{(0)} = C_1 (-k\tau)^{1/2} J_{-3/2}(-k\tau) + C_2 (-k\tau)^{1/2} J_{3/2}(-k\tau). \quad (98) \]

Then $\mathcal{F}_\omega$ becomes

\[ \mathcal{F}(\tau) = -C_1 H \sqrt{\frac{2}{\pi}} \frac{\cos(-k\tau)}{-k\tau} + C_2 H \sqrt{\frac{2}{\pi}} \frac{\sin(-k\tau)}{-k\tau}. \quad (99) \]

This gives the boundary condition for $\omega$. The solution for $\omega$ in the bulk subject to this condition is obtained as [14]

\[ \omega^{(0)}(z, \tau) = -2C_1 \sum_{\ell=0}^{\infty} (-1)^{\ell} \left( 2\ell + \frac{1}{2} \right) \frac{\sinh H \tau_{2\ell}}{\sinh H \tau_{2\ell+1}} \frac{Q_{2\ell}(\cosh H z)}{Q_{2\ell+1}(\cosh H z)} (-k\tau)^{-3/2} J_{2\ell+1/2}(-k\tau) \]

\[ + \ 2C_2 \sum_{\ell=0}^{\infty} (-1)^{\ell} \left( 2\ell + \frac{3}{2} \right) \frac{\sinh H \tau_{2\ell+1}}{\sinh H \tau_{2\ell+2}} \frac{Q_{2\ell+1}(\cosh H z)}{Q_{2\ell+2}(\cosh H z)} (-k\tau)^{-3/2} J_{2\ell+3/2}(-k\tau), \quad (100) \]
where the identities
\begin{align}
\cos(x) &= \sqrt{2\pi} \sum_{\ell=0}^{\infty} (-1)^\ell \left(2\ell + \frac{1}{2}\right) x^{-1/2} J_{2\ell+1/2}(x), \tag{101}
\sin(x) &= \sqrt{2\pi} \sum_{\ell=0}^{\infty} (-1)^\ell \left(2\ell + \frac{3}{2}\right) x^{-1/2} J_{2\ell+3/2}(x), \tag{102}
\end{align}
were used.

At large scales $-k\tau \to 0$, the dominant contribution comes from the $\ell = 0$ mode. On the other hand, on small scales $-k\tau \to \infty$, all modes becomes comparable and we need to take into account an infinite ladder of the modes. This means that gravity becomes five-dimensional at small scales.

In practice, we must approximate the infinite sum to proceed with the calculations. We first check the identity equations (101) and (102) to see whether we can approximate the infinite summation by introducing a cut-off $\ell_c$ into the summation. From figure 1, we can see that if we increase the cut-off $\ell_c$, the identity is satisfied for large $-k\eta$, i.e. on small scales. This implies that as long as we start from a finite time $-k\eta$, we can approximate the infinite ladder of the modes by introducing sufficiently large $\ell_c$.

Figure 2 shows the bulk solution for $\omega(z, t)$ with introduction of a sufficiently large cut-off $\ell_c$. The solution is localized near the brane and decays towards the horizon, $z \to \infty$. This is a bound state that is supported by an oscillation of the inflaton fluctuation on the brane. This kind of bound state generally appears in coupled brane and bulk...
oscillators [23]. A toy example is shown in the appendix. A key point here is that, in this case, the bound state is a summation of many different eigenstates of different eigenvalues (equation (100)). This fact becomes crucial in the analysis of the next order solution.

5.2. First-order solutions

Now it is possible to calculate the next order equation for \( u^{(1)} \):

\[
\frac{d^2 u_k}{d\tau^2} + \left( k^2 - \frac{1}{\tau^2}(2 + 6\epsilon - 3\eta) \right) u_k = J_u, \tag{103}
\]

where \( J_u \) describes the effect of the back reaction from the bulk perturbations. We can use the zeroth-order solution to evaluate \( J_u \) as

\[
J_u = \frac{2}{3} \epsilon k^2 C_1 \sum_{\ell=0}^{\infty} (-1)^{\ell} \left(2\ell + \frac{1}{2}\right) \triangle (2\ell; H\mu) \\
\times \left(2\ell(2\ell - 1)(-k\tau)^{-3/2} J_{2\ell+1/2} + 2(-k\tau)^{-1/2} J_{2\ell+3/2}(-k\tau) \right) \\
- \frac{2}{3} \epsilon k^2 C_2 \sum_{\ell=0}^{\infty} (-1)^{\ell} \left(2\ell + \frac{3}{2}\right) \triangle (2\ell + 1; H\mu) \\
\times \left(2\ell(2\ell + 1)(-k\tau)^{-3/2} J_{2\ell+3/2} + 2(-k\tau)^{-1/2} J_{2\ell+5/2}(-k\tau) \right), \tag{104}
\]
Figure 3. Source term $J_u(-z)$ as a function of $z$ with cut-offs $\ell_c = 20$ and 30. Here we take $H\mu \gg 1$.

where

$$\Delta(n; H\mu) = \frac{H}{\mu} \left( 1 + \left( \frac{H}{\mu} \right)^2 \right)^{-1/2} \frac{Q_n(cosh Hz_b)}{Q_1(cosh Hz_b)}.$$  \hspace{1cm} (105)

The quantity $\Delta(n; H\mu)$ controls the amplitude of the corrections to the Mukhanov–Sasaki equations from the bulk over the change of the energy scales of the inflation.

In order to evaluate $J_u$, we need to introduce a cut-off in the summation at sufficiently large $\ell$. Figure 3 shows the behaviour of $J_u$ against the change of the cut-off $\ell_c$. A good feature here is that the behaviour of $J_u$ for small $-k\tau$ does not change even if we increase the cut-off. Thus we can reproduce the correct behaviour of $J_u$ by means of a finite summation of modes as long as we are considering a finite time interval.

5.2.1. Large scales. On large scales $-k\tau \to 0$, the $\ell = 0$ mode in the $C_1$ mode dominates, which corresponds to an $m^2 = 2H^2$ mode. Thus we can approximate the infinite ladder of the modes by a single mode on superhorizon scales. This indicates that, at large scales, gravity looks four dimensional. Then we can easily show that $J_u$ is suppressed for $-k\tau \to 0$ and the Mukhanov–Sasaki equation becomes completely the same as in conventional cosmology. Thus we can show the conservation of the curvature perturbation $\mathcal{R}$ on large scales in the same way as in conventional cosmology [21].

5.2.2. Small scales. At low energies $H/\mu \ll 1$, $\Delta(n; H/\mu)$ can be approximated as

$$\Delta(n; H/\mu) = \left( \frac{H}{\mu} \right)^2 \left( \gamma + \psi(n + 1) + \log(H/\mu) - \log 2 \right),$$  \hspace{1cm} (106)
where we have assumed that \( n \) is not large. Thus, the source term is well suppressed by the term \( \Delta(n; H/\mu) \) at low energies. However, at sufficiently small scales, large \( \ell \) modes become important and the approximation (106) does not hold. Thus we could still get an effect on very much subhorizon scales (\( k \gg \mu^{-1} \gg H \)). In this case, we need to introduce a large cut-off in the summation of \( \ell \) and it is technically difficult to perform a calculation.

At high energies, the amplitude of \( \Delta(n; H\mu) \) becomes large as \( H\mu \) becomes large, but, at sufficiently high energies \( H\mu \rightarrow \infty \), \( \Delta(n; H\mu) \) becomes independent of \( H\mu \) as seen from figure 4. Indeed, we can obtain the asymptotic form of \( \Delta(l; H/\mu) \) for \( H/\mu \rightarrow \infty \) as

\[
\Delta(l; H\mu) \rightarrow -\frac{1}{n + 1}.
\]

(107)

In the following, we consider this limit. In this high energy limit, \( J_u \) is well fitted as

\[
J_u \sim \frac{2\epsilon}{3} k^2 A \left[ C_1(-k\tau)^{-1/2}\cos(-k\tau + \varphi) - C_2(-k\tau)^{-1/2}\sin(-k\tau + \varphi) \right],
\]

(108)

for \(-140 < k\tau < -40 \) where \( A = 0.4 \) and \( \varphi = 0.9 \). Then, the equation of motion for the first-order Mukhanov variable is given as

\[
\frac{d^2 u_k^{(1)}}{d\tau^2} + \left( k^2 - \frac{2}{\tau^2} \right) u_k^{(1)} - \frac{1}{\tau^2} (6\epsilon - 3\eta) u_k^{(0)} - J_u(\tau) = 0.
\]

(109)

On using the asymptotic behaviour of the Bessel function at small scales (large \(-k\tau\)), the third term behaves like \((-k\tau)^{-2}\sin(-k\tau)\), while the fourth term behaves as \((-k\tau)^{-1/2}\sin(-k\tau)\). Therefore, at least at small scales, the effect from the bulk metric perturbations dominates over the effect from the standard corrections to the de Sitter geometry. Thus we will neglect the third term. The general solutions are given by the

Figure 4. \( \Delta(n; H/\mu) \) as a function of \( H/\mu \).
linear combination of $(-k\tau)^{1/2}J_{3/2}(-k\tau)$ and $(-k\tau)^{1/2}J_{-3/2}(-k\tau)$. On choosing the initial conditions so that $u_k(\tau_i) = u_k^{(0)}(\tau_i)$ at $\tau = \tau_i$, we find the following form of the solution:

$$u_k^{(1)} = D_1(-k\tau)^{1/2}J_{3/2}(-k\tau) + D_2(-k\tau)^{1/2}J_{-3/2}(-k\tau),$$

(110)

where $D_1$ and $D_2$ are given by

$$D_1 = \frac{\pi}{2} \int_{k\tau_i}^{k\tau} \mathrm{d}(k\tau')(-k\tau')^{1/2}J_{3/2}(-k\tau')J_u(\tau'),$$

$$D_2 = -\frac{\pi}{2} \int_{k\tau_i}^{k\tau} \mathrm{d}(k\tau')(-k\tau')^{1/2}J_{-3/2}(-k\tau')J_u(\tau').$$

(111)

For specifying the behaviour of the first-order Mukhanov variable, we must evaluate $D_1$ and $D_2$. Using the asymptotic form for Bessel functions at small scales, $D_1$ and $D_2$ are well approximated as

$$D_1 \approx -\frac{2\epsilon}{3} A \sqrt{\frac{\pi}{2}} \int_{k\tau_i}^{k\tau} \mathrm{d}(k\tau')(-k\tau')^{-1/2}\sin(-k\tau')$$

$$\times \left[ C_1 \cos(-k\tau' + \varphi) - C_2 \sin(-k\tau' + \varphi) \right],$$

$$D_2 \approx \frac{2\epsilon}{3} A \sqrt{\frac{\pi}{2}} \int_{k\tau_i}^{k\tau} \mathrm{d}(k\tau')(-k\tau')^{-1/2}\cos(-k\tau')$$

$$\times \left[ C_1 \cos(-k\tau' + \varphi) - C_2 \sin(-k\tau' + \varphi) \right].$$

(112)

Then, on small scales, the first-order solution is given by

$$u_k^{(1)} \to ((F(\tau) - F(\tau_i))\cos(-k\tau) + (G(\tau) - G(\tau_i))\sin(-k\tau),$$

(113)

where

$$F(\tau) = \frac{\epsilon AC_1 \sqrt{\pi}}{3} \left[ -S \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) \cos \varphi - \left( C \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) - \frac{2}{\sqrt{\pi}} \sqrt{-k\tau} \right) \sin \varphi \right]$$

$$+ \frac{\epsilon AC_2 \sqrt{\pi}}{3} \left[ S \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) \sin \varphi - \left( C \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) - \frac{2}{\sqrt{\pi}} \sqrt{-k\tau} \right) \cos \varphi \right],$$

$$G(\tau) = \frac{\epsilon AC_1 \sqrt{\pi}}{3} \left[ \left( C \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) + \frac{2}{\sqrt{\pi}} \sqrt{-k\tau} \right) \cos \varphi - S \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) \sin \varphi \right]$$

$$+ \frac{\epsilon AC_2 \sqrt{\pi}}{3} \left[ - \left( C \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) + \frac{2}{\sqrt{\pi}} \sqrt{-k\tau} \right) \sin \varphi - S \left( \frac{2\sqrt{-k\tau}}{\sqrt{\pi}} \right) \cos \varphi \right],$$

(114)

where $S$ and $C$ are Fresnel functions.

We see that the first-order perturbations grow like $\sqrt{-k\tau} - \sqrt{-k\tau_i}$. Then if we formally take the limit $-k\tau_i \to \infty$, the first-order corrections diverge. Thus our perturbative approach breaks down. The amplitudes of the zeroth-order oscillations of inflaton fluctuations are significantly affected by the first-order corrections.

We should take care in interpreting this result for the amplitude. In a toy model of a coupled boundary and bulk oscillators described in the appendix, this change of amplitude due to the first-order perturbations is merely caused by the breakdown of the perturbative
Slow-roll corrections to inflaton fluctuations on a brane

expansion. In the toy model, the coupling to the bulk oscillator just changes the phase of the brane oscillator. In that case we can renormalize the first-order perturbation so that the first-order corrections appear only in the phase of the oscillations and do not have a large effect on the amplitude. However, in the case of inflaton fluctuations, we cannot do this kind of renormalization. This is due to the phase $\phi$ in the source term of the first-order equation (see the appendix). The phase originates from the fact that the zeroth-order oscillation cannot be matched by a single bulk eigenmode with the same frequency as the brane oscillator and we need an infinite ladder of modes. Thus we can say that the effects on the amplitude from first-order corrections are not artificial effects of our perturbative approach.

In conventional cosmology, the amplitude of the inflaton oscillations $u$ remains constant, so we can impose initial conditions on any scale far inside the horizon. However, in the brane world case, the coupling to the bulk metric perturbations changes the amplitude of the zeroth-order inflaton oscillation $u$, so the effect crucially depends on the initial conditions. In general, classically, we can also impose arbitrary initial conditions for $\Omega$. Indeed, it is always possible to add homogeneous solutions which satisfy the boundary condition given by

$$\mathcal{F} = 0.$$  

(116)

Then we find an infinite tower of massive modes starting from $m^2 = 9H^2/4$. Arbitrary initial conditions for $\Omega$ can be satisfied by an appropriate summation of these massive modes. These massive modes also affect the evolution of inflaton fluctuations $u$.

We have tried to solve the coupled equations for inflaton fluctuations and the master variable directly using a numerical method [22]. If we begin with the initial condition for $\omega$ given by equation (100), the numerical solution for $u$ agrees well with our perturbative solutions as long as perturbations remain valid. We have also tried using different initial conditions for $\Omega$ and find that the effects on the amplitude of $u$ depend on the initial conditions for $\omega$ in the bulk.

The initial conditions for $u$ and $\omega$ must be determined by quantum theory on small scales. Thus we must quantize the coupled system of the inflaton fluctuations $u$ and the master variable $\omega$ consistently. This is in contrast to the conventional cosmology case where we can specify the vacuum for $u$ by neglecting the gravitational effects far inside the horizon. This means that the assumption that the power spectrum of inflaton perturbations at horizon crossing is given by $(H/2\pi)^2$ could be invalid and we may have significant effects on the amplitude of perturbations from the back-reaction due to the bulk metric perturbations.

6. Conclusion

In this paper we have studied the effect of metric perturbations upon inflaton fluctuations during inflation, at first order in slow-roll parameters $\epsilon$ and $\eta$, which describe the dimensionless slope and curvature of the potential. If we neglect the slope and curvature of the inflaton potential then we obtain the familiar results for free field fluctuations in de Sitter spacetime, with a scale invariant power spectrum on large (superhorizon) scales. We take this as our zeroth-order result in a slow-roll expansion.

In four-dimensional general relativity we were able to calculate corrections to the field evolution perturbatively to first order in a slow-roll expansion, including linear metric
perturbations. As far as we are aware this is the first time the slow-roll corrections have been calculated in the manner. We reproduce the familiar slow-roll corrections usually derived from Lyth and Stewart’s exact solution to the linear perturbation equations in power-law inflation.

On a four-dimensional brane world, embedded in a five-dimensional bulk, there are no exact solutions for cosmological perturbations (for a vacuum bulk described by Einstein gravity) except for the case of an exactly de Sitter brane. Thus the only way to calculate slow-roll corrections is perturbatively in a slow-roll expansion. We have calculated the leading order bulk metric perturbations sourced from the zeroth-order inflaton fluctuations on the brane. We find that inflaton fluctuations support an infinite tower of discrete bulk perturbations, with negative effective mass squared.

Including the effect of the metric perturbations as an inhomogeneous source term in the wave equation for the first-order inflaton fluctuations, we find that the effect of bulk metric perturbations becomes small on large scales, and we recover the usual result that the comoving curvature perturbation becomes constant outside the horizon.

However, at small scales (or early times for a given comoving wavelength) the effect of bulk metric perturbations cannot be neglected. We are able to give an approximate solution for inflaton fluctuations at high energies and on subhorizon scales using a truncated tower of bulk modes. This shows that the bulk metric perturbations change the amplitude of inflaton field fluctuations on the brane. By including a large number of bulk modes we can model this effect for many oscillations, but ultimately this change of amplitude becomes a large effect leading to a breakdown of our perturbative analysis.

It is not surprising in some ways that we see a large effect at small scales, as these are high momentum modes which are expected to be strongly coupled to the bulk. Nonetheless, this invalidates the usual assumption that gravitational effects are small far inside the cosmological horizon. It seems necessary to consistently solve for the coupled evolution of brane and bulk modes. We tried to solve this problem numerically and verified the validity of our perturbative approach as long as perturbations remain good. But it was also found that the change of the amplitude depends on the initial conditions for bulk metric perturbations. Detailed analysis of numerical solutions goes beyond the scope of the present paper and they will be presented in a separate paper [22]. In order to give definite predictions for the amplitudes of scalar perturbations in high energy inflation, we must specify the quantum vacuum state for coupled inflaton fluctuations and metric perturbations consistently and determine initial conditions. For this purpose, it would be useful to study the quantum theory of the toy model for coupled bulk–brane oscillators in more detail, where we can consistently quantize a coupled system [23].

Our result implies the possibility that the assumption that the power spectrum of inflaton perturbations at horizon crossing on a brane is given by \( (H/2\pi)^2 \) could be invalid and we may have significant effects on the amplitudes of perturbations from the back-reaction due to the bulk metric perturbations.

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Appendix: A toy model for a coupled bulk–brane system

In this appendix, we present a simple toy model for coupled brane and bulk oscillators. Let us consider a toy model for a brane field \( q(t) \) and a bulk field \( \phi \) in the Minkowski bulk, which satisfy

\[
\ddot{q} + \mu^2 q = -\beta \phi, \\
\ddot{\phi} = \phi'' - m^2 \phi, \quad \phi'(y = 0) = \frac{\beta}{2} q.
\]

We solve the equations perturbatively in terms of small \( \beta \). Without coupling, the zeroth-order solution for \( q \) is given by

\[
q^{(0)}(t) = C_1 \cos(\mu t) + C_2 \sin(\mu t).
\]

If we assume \( m > \mu \), the zeroth-order solution for \( \phi \) is obtained as

\[
\phi^{(0)} = -\frac{\beta}{2 \sqrt{m^2 - \mu^2}} (C_1 \cos(\mu t) + C_2 \sin(\mu t)) e^{-\sqrt{m^2 - \mu^2} y}.
\]

Note that the bulk field has a negative effective mass squared and decays towards \( y \to \infty \). This is a normalizable bound state supported by an oscillation of \( q(t) \) on the brane. The equation for the next order \( q^{(1)}(t) \) is given by

\[
\ddot{q}^{(1)} = -\mu^2 q^{(1)} + \frac{\beta^2}{2 \sqrt{m^2 - \mu^2}} (C_1 \cos(\mu t) + C_2 \sin(\mu t)).
\]

Including the zeroth-order solution, the solution for \( q(t) \) is given by

\[
q^{(1)}(t) = C_1 \left( \cos \mu t + \frac{\beta^2}{4 \mu \sqrt{m^2 - \mu^2}} t \sin \mu t \right) + C_2 \left( \sin \mu t - \frac{\beta^2}{4 \mu \sqrt{m^2 - \mu^2}} t \cos \mu t \right)
\]

where we impose the initial condition so that \( q(0) = q^{(0)}(0) \).

A problem is that the perturbation grows linearly in time. However, we need to be careful to interpret this growth of perturbations. In this toy model, we can easily find an exact solution. The corresponding exact solution becomes

\[
q(t) = C_1 \cos \left( \mu - \frac{\beta^2}{4 \mu \sqrt{m^2 - \mu^2}} t \right) + C_2 \sin \left( \mu - \frac{\beta^2}{4 \mu \sqrt{m^2 - \mu^2}} t \right),
\]

\[
\phi(y, t) = -\frac{\beta}{2 \sqrt{m^2 - \mu^2}} q(t) e^{-\sqrt{m^2 - \mu^2} y},
\]

for \( \beta \ll 1 \). The effect of the coupling merely changes the frequency of the brane oscillator. The origin of the linear instability is that the naive expansion in terms of \( \beta \) is not efficient.
Indeed, we can use
\[
\cos(A + B) = \cos A \cos B - \sin A \sin B \sim \cos A - B \sin A,
\]
\[
\sin(A + B) = \sin A \cos B + \sin B \cos A \sim \sin A + B \cos A,
\] (A.8)
for $B \ll 1$ and expand the exact solution into equation (A.5). However, this perturbation breaks down for large $t$. A crucial difference of the inflaton fluctuations case from the toy model is that the source term for the first-order equation for $q$ contains a phase $\varphi$ (compare equations (108) and (109) to equation (A.4)). Then a perturbative solution cannot be written in a form like equation (A.6) using equation (A.8). This indicates that there could be a modification of the amplitude as well as the phase shift. The phase $\varphi$ originates from the feature that the brane oscillation cannot be matched by a single bound state (compare equations (100) and (A.3)). Thus this is an essential difference between the toy model and the inflaton fluctuations case.

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