**Abstract**
In this paper, we present a straightforward systematic method for the exact and approximate calculation of integrals that appear in formulae for the period of anharmonic oscillators and other problems of interest in classical mechanics.

1. Introduction

Many physical phenomena can be described in terms of vibrational motion of particles [1, 2]. For pedagogical reasons one commonly starts the discussion of the problem by means of the simplest case of one degree of freedom described by a time-dependent coordinate $x(t)$. A simple example is a particle of mass $m$ attached to a wall by a spring or a string (moving on a surface without friction or just hanging from the ceiling) [1, 2]. The trajectory $x(t)$ is a solution of Newton’s equation of motion $m \ddot{x} = f(x)$ for a conservative force $f(x) = -V'(x)$, where $V(x)$ is the potential–energy function. Unless stated otherwise, dots and primes indicate differentiation with respect to time and coordinate, respectively. The amplitude of the bounded motion is the largest distance from the equilibrium position that, without loss of generality, we assume to take place at $x = 0$. A relevant feature of such oscillatory motion is the period $T$ given by the condition $x(t + T) = x(t)$.

A common approximation in the case of small displacements from equilibrium consists of the expansion of the potential–energy function in a Taylor series: $V(x) = v_2 x^2/2 + v_3 x^3/3 + v_4 x^4/4 + \cdots$, where we choose $V(0) = 0$ without loss of generality. The simplest vibrational model is a harmonic oscillator given by $V(x) = v_2 x^2/2$, where $v_2 = k > 0$ is the force constant. In this case one solves the equation of motion exactly and finds that the period is independent of the amplitude of the motion $T = 2\pi \sqrt{m/k}$ [1, 2]. This potential–energy function is symmetrical about the origin: $V(-x) = V(x)$.

The simple harmonic oscillator may be unsatisfactory to model vibrational motion in the real world [1, 2]. In order to illustrate the main features of the more realistic anharmonic
motion we commonly resort to polynomial potentials [1–3]. For example, \( V(x) = v_2x^2/2 + v_4x^4/4 \) and \( V(x) = v_2x^2/2 + v_3x^3/3 \) give rise to the simplest symmetrical and nonsymmetrical anharmonic oscillators. In such cases the period of the motion depends on the amplitude [1, 3].

The discussion of periodic motion in one dimension is important in most introductory courses on classical mechanics [1, 2]. Some problems can be solved exactly, but in most cases one has to resort to approximate solutions [1–3]. Simple but sufficiently accurate approximate solutions for such problems are very important in understanding relevant features of the phenomena that arise from anharmonic vibrations (for example, the thermal expansion of solids [1, 2]). In addition to it, in some cases one is simply satisfied with accurate numerical results, and expressions suitable for computation are most welcome.

The purpose of this paper is to discuss the exact and approximate calculation of the period of a particle that moves in one dimension under the effect of a polynomial anharmonic potential like those discussed above.

2. Oscillatory motion in one dimension

Consider a particle of mass \( m \) moving in one dimension under a potential–energy function \( V(x) \). Without loss of generality we assume that \( V(x) \) has a minimum at \( x = 0 \); more precisely, we assume that \( V(0) = 0, V'(0) = 0 \) and \( V''(0) > 0 \), where the prime indicates differentiation with respect to \( x \). Following the standard notation in classical mechanics, we use a dot to indicate differentiation with respect to time; for example, \( \dot{x} = \frac{dx}{dt} \) is the velocity of the particle and \( \ddot{x} = \frac{d^2x}{dt^2} = a \) its acceleration.

From the equation of motion

\[
m\ddot{x} = -V'(x)
\]  

we easily obtain an integral of the motion

\[
E = \frac{m\dot{x}^2}{2} + V(x)
\]

which is the total energy. The motion of the particle is restricted to the interval \( x_− < x < x_+ \), where the turning points \( x_± \) satisfy \( V(x_±) = E \); that is to say, \( \dot{x} = 0 \) at those points.

It is well known that the period of the motion is given by [1, 3]

\[
T = \sqrt{2m} \int_{x_−}^{x_+} \frac{dx}{\sqrt{E - V(x)}}
\]

from which we obtain the frequency \( \Omega = 2\pi/T \).

We can simplify the equations of motion by the introduction of a dimensionless time \( \tau = \omega_0 t \), where \( \omega_0 \) is an arbitrary frequency. If we define

\[
\mathcal{E} = \frac{E}{m\omega_0^2}, \quad U(x) = \frac{V(x)}{m\omega_0^2}
\]

then we obtain the equations of motion for a particle of unit mass; for example:

\[
\mathcal{E} = \frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 + U(x).
\]

The period reads

\[
T = \frac{\sqrt{2}}{\omega_0} \int_{x_−}^{x_+} \frac{dx}{\sqrt{\mathcal{E} - U(x)}}.
\]
It is worth noting that equation (5) is not dimensionless because \( \mathcal{E} \) and \( U(x) \) have units of length squared. In order to get a truly dimensionless equation we should define a dimensionless coordinate \( q = x/L \), where \( L \) has units of length. Thus \( E/(m\omega_0^2L^2) \) and \( V(Lq)/(m\omega_0^2L^2) \) are the dimensionless counterparts of the total and potential energies, respectively. In this paper, however, we have opted for equations that are similar to those often found in current literature although they are not dimensionless.

3. The main integral

It follows from the above discussion that the period is proportional to an integral of the form

\[
I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q(x)}},
\]

where \( Q(x) \) exhibits simple zeros at \( x_- \) and \( x_+ \) and is positive definite for all \( x_- < x < x_+ \). That is to say, we can write

\[
Q(x) = (x_+ - x)(x - x_-)R(x)
\]

where \( R(x) > 0 \) for all \( x_- \leq x \leq x_+ \).

The reason for rewriting our problem in this somewhat abstract way is that we can then use the solutions to integral (7) for problems other than the period of a motion in one dimension. We will discuss an example later on.

In order to develop suitable exact and approximate expressions for integral (7) we define the reference function

\[
Q_0(x) = \frac{\omega^2}{2} (x_+ - x)(x - x_-)
\]

that satisfies the appropriate boundary conditions at the turning points. It is clear that \( Q_0(x) \) is the function that would appear in the treatment of a harmonic oscillator, except that the arbitrary frequency \( \omega \) is dimensionless. Then we rewrite (7) as

\[
I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q_0(x)\sqrt{1 + \Delta(x)}}}
\]

where

\[
\Delta(x) \equiv \frac{Q(x) - Q_0(x)}{Q_0(x)} = \frac{2R(x) - \omega^2}{\omega^2}.
\]

The change of variables

\[
x = \frac{x_+ + x_-}{2} + \frac{x_+ - x_-}{2} \cos \theta
\]

makes integral (10) much simpler:

\[
I = \frac{\sqrt{2}}{\omega} \int_0^\pi \frac{d\theta}{\sqrt{1 + \Delta}}.
\]

In this way we obtain an exact expression for the period, which in most cases one has to calculate numerically. In order to derive simple analytical formulae we expand

\[
\frac{1}{\sqrt{1 + \Delta}} = \sum_{j=0}^\infty (-1/2)^j \Delta^j
\]
where \( \binom{n}{a} = a!/[(a-b)!b!] \) is a combinatorial number. Note that this series converges for all \( x \) such that \( |\Delta| < 1 \). We thus obtain a series for integral (13):

\[
I = \sum_{j=0}^{\infty} I_j, \quad I_j = \frac{\sqrt{2}}{\omega} \left( -1/j^2 \right) \int_{0}^{\pi} \Delta^j d\theta.
\]

(15)

Clearly, we can derive approximate expressions for integral (13) by means of the partial sums:

\[
I^{(N)} = \sum_{j=0}^{N} I_j, \quad N = 0, 1, \ldots
\]

(16)

The dimensionless frequency \( \omega \) is arbitrary and we may choose it conveniently in order to improve the rate of convergence of the partial sums (16) as shown in what follows.

Let \( R_M \) and \( R_m \) be the maximum and minimum values of \( R(x) \) in the interval \([x_-, x_+]\) and \( \Delta_M \) and \( \Delta_m \) the corresponding values of \( \Delta(x) \). Since \( R(x) \) is positive definite we know that \( R_M \geq R(x) \geq R_m > 0 \). If we choose the value of the adjustable parameter \( \omega \) so that \( \Delta_M = -\Delta_m \) we obtain

\[
\omega_b^2 = R_M + R_m > 0
\]

(17)

and

\[
\Delta_b(x) = \frac{2R(x) - R_M - R_m}{R_M + R_m}.
\]

(18)

The subscript \( b \) indicates that this particular value of \( \omega \) ‘balances’ the maximum and minimum values of \( \Delta(x) \). Note that \( |\Delta(x)| < 1 \) for all \( x_- \leq x \leq x_+ \) because \( \Delta_M = (R_M - R_m)/(R_M + R_m) < 1 \). In the next section we will illustrate the effect of this choice of optimum frequency with a particular example.

4. The Duffing oscillator

Our first example is the symmetrical anharmonic oscillator

\[
V(x) = \frac{v_2 x^2}{2} + \frac{v_4 x^4}{4}.
\]

(19)

We have four distinct cases: first, if both \( v_2 \) and \( v_4 \) are positive the potential is a simple well that supports periodic motion for all \( E > 0 \) (figure 1). Second, if \( v_2 \) is positive and \( v_4 \) negative, then \( V(x) \) is a well at \( x = 0 \) with two finite barriers at \( x = \pm \sqrt{v_2/|v_4|} \), and periodic motion is possible only for \( 0 < E < -v_2^2/(4v_4) \) (figure 2). These two cases appear in the approximate treatment of a particle of mass \( m \) attached to two points by elastic strings [1]. Third, if \( v_2 \) is negative and \( v_4 \) positive, then the potential exhibits two wells of depth \( V_w = -v_2^2/(4v_4) \) at \( x = \pm \sqrt{|v_2|/v_4} \), and a barrier \( V_b = 0 \) at \( x = 0 \). In this case there is oscillatory motion for all \( E > V_w \) (figure 3). Finally, if both \( v_2 \) and \( v_4 \) are negative, then the potential does not support bounded motion disregarding the value of the energy (figure 4). In what follows we consider just the two cases with \( v_2 > 0 \).

If we apply the transformation outlined in section 2 with \( \omega_0 = \sqrt{v_2/m} \) we obtain the potential–energy function for the well-known Duffing oscillator [3]

\[
U(x) = \frac{x^2}{2} + \frac{\lambda x^4}{4}
\]

(20)

where \( \lambda = v_4/v_2 \). Since it is parity invariant \( (U(-x) = U(x)) \) then \( x_+ = -x_- = A \) is the amplitude of the oscillations.
When $\lambda > 0$ the amplitude of the motion increases unboundedly with the energy; on the other hand, when $\lambda < 0$ the potential exhibits two barriers of height $1/(4\lambda)$ at $x = \pm 1/\sqrt{-\lambda}$ and therefore the amplitude of the periodic motion cannot be greater than $A_L = 1/\sqrt{-\lambda}$.

According to the general discussion of the preceding section, it follows from

$$Q(x) = \mathcal{E} - U(x) = (A^2 - x^2) \left[ \frac{1}{2} + \frac{\lambda}{4} (A^2 + x^2) \right]$$

that

$$R(x) = \frac{1}{2} + \frac{\lambda}{4} (A^2 + x^2)$$

and

$$\Delta = \frac{1 + \lambda A^2 - \omega^2 - \frac{\lambda A^2}{2} \sin^2 \theta}{\omega^2}$$

(21) (22) (23)
where we have substituted $x = A \cos \theta$ according to the transformation (12). We conclude that the period depends on the dimensionless parameter $\rho = \lambda A^2$ that is the ratio of $v_4 A^4$ and $v_2 A^2$, both having units of energy.

When $\lambda < 0$ there is a minimum value of $\rho$ for which there is periodic motion: $\rho_L = \frac{\lambda A^2}{\xi} = -1$; on the other hand, for $\lambda > 0$ there is periodic motion for all $\rho > 0$. All in all, there is periodic motion if $\rho > -1$.

If we choose $\omega = \sqrt{1 + \rho}$ then we obtain an already known suitable compact expression for the integral [3]

$$I = \frac{\sqrt{2}}{\sqrt{1 + \rho}} \int_0^\pi \frac{d\theta}{\sqrt{1 - \xi \sin^2 \theta}}, \quad \xi = \frac{\rho}{2\rho + 2}.$$  

Figure 3. Anharmonic oscillator (19) with $v_2 < 0$ and $v_4 > 0$.

Figure 4. Anharmonic oscillator (19) with $v_2 < 0$ and $v_4 < 0$. 
This equation yields the series
\[ I = \frac{\sqrt{2}\pi}{\sqrt{1 + \rho}} \sum_{j=0}^{\infty} \left(\frac{-1/2}{j}\right)^2 \xi^j \] (25)
that converges for all \(|\xi| < 1\). Note that this series does not converge in the physical subregion \(-1 < \rho < -2/3\) and, consequently, the analytical expressions that we may derive from it will not be valid for all the values of the energy that give rise to periodic motion.

The results of section 3 enable us to improve the approximation just discussed. For the particular case of the Duffing oscillator we have \(R_m = R(0) = 1/2 + \rho/4\) and \(R_M = R(\pm A) = 1/2 + \rho/2\) so that the optimum frequency is
\[ \omega_b^2 = \frac{4 + 3\rho}{4} \] (26)
and
\[ \Delta_b(\theta) = \frac{\rho}{4 + 3\rho} \cos(2\theta). \] (27)

The resulting alternative integral
\[ I = \frac{2\sqrt{2}}{\sqrt{4 + 3\rho}} \int_0^{\pi} \frac{d\theta}{\sqrt{1 + \xi \cos(2\theta)}} \] \[ \xi = \frac{\rho}{4 + 3\rho} \] (28)
gives rise to the series
\[ I = \frac{2\sqrt{2}\pi}{\sqrt{4 + 3\rho}} \sum_{j=0}^{\infty} \left(\frac{-1}{j}\right)^2 \left(\frac{-1/2}{2j}\right) \xi^{2j} \] (29)
which converges for all \(|\xi| < 1\) and includes all the physical values of \(\rho\) because \(\xi(\rho = -1) = -1\) and \(\xi(\rho \to \infty) = 1/3\). In this way we may obtain simple analytical expressions for the period valid for all values of the energy consistent with periodic motion. In order to make this discussion clearer, figure 5 shows the two expansion variables \(\xi\) as functions of \(\rho\).
According to equation (6) the period is given by
\[ T = \frac{\sqrt{2}}{\omega_0} I. \] (30)

Equation (29) enables us to derive simple analytical approximate expressions for the period. For concreteness and simplicity we choose \( \omega_0 = 1 \) in what follows. For example, the first two approximations are
\[ T^{(0)} = \frac{4\pi}{\sqrt{4 + 3\rho}} \] (31)
and
\[ T^{(1)} = \frac{\pi(147\rho^2 + 384\rho + 256)}{4(4 + 3\rho)^{3/2}}. \] (32)

These expressions are expected to be accurate for small values of \( \rho \), and in fact they give the exact result for \( \rho = 0 \). However, they are also accurate for extremely great values of \( \rho \). Note, for example, that our approximate expressions yield the singularity at infinity
\[ \lim_{\rho \to \infty} \sqrt{\rho}T = 4 \int_0^{\pi} \frac{d\theta}{\sqrt{3 + \cos(2\theta)}} \approx 7.4162987 \] (33)
with surprising accuracy:
\[ \lim_{\rho \to \infty} \sqrt{\rho}T^{(0)} = \frac{4\pi}{\sqrt{3}} \approx 7.26 \quad \lim_{\rho \to \infty} \sqrt{\rho}T^{(1)} = \frac{49\sqrt{3}\pi}{36} \approx 7.406. \] (34)

We conclude that they are suitable for most purposes because their accuracy is even greater for all \( \rho < \infty \). If necessary, one easily improves them by simply adding more terms of the series (29).

It is worth comparing our proposed series with other known approximations. For example, at first order series (25) gives us
\[ T^{(1)}_{\text{ser}} = \frac{\pi(9\rho + 8)}{4(\rho + 1)^{3/2}} \] (35)
and the straightforward Taylor expansion about \( \rho = 0 \) reads [3]:
\[ T^{(1)}_{\text{Taylor}} = 2\pi \left(1 - \frac{3\pi\rho}{8}\right). \] (36)

Figure 6 clearly shows that our approach is more accurate than those well-known alternative ones.

5. Quadratic–cubic oscillator

Parity-invariant oscillators exhibit symmetric turning points; if the potential is nonsymmetrical so are the turning points. The simplest example of the latter is
\[ V(x) = \frac{v_2}{2} x^2 + \frac{v_3}{3} x^3, \] (37)
where \( v_2 > 0 \). If we again choose \( \omega_0 = \sqrt{v_2/m} \) then we obtain
\[ U(x) = \frac{x^2}{2} + \frac{\lambda}{3} x^3, \quad \lambda = \frac{v_3}{v_2}. \] (38)

To begin with, note that \( U(-x, -\lambda) = U(x, \lambda) \) so that we consider only the case \( \lambda > 0 \) without loss of generality. The potential–energy function \( U(x) \) shows a barrier of height
Figure 6. Period of the Duffing oscillator calculated exactly, by means of our series (32), by a standard series (35), and by the Taylor expansion (36).

Figure 7. Quadratic–cubic oscillator (38).

\[ U(x_M) = \frac{1}{6\lambda^2} \] at \( x_M = -\frac{1}{\lambda} \) and a minimum at \( x = 0 \). Consequently, there is bounded periodic motion for all \( 0 \leq E < U(x_M) \). As shown in figure 7, the polynomial \( Q(x) = E - U(x) \) has three real roots \( x_3 < x_- < 0 < x_+ \), where \( x_- \) and \( x_+ \) are the turning points.

If we write
\[ Q(x) = (x - x_-)(x_+ - x)(b_0 + b_1x) \]
then we obtain
\[ b_0 = -\frac{x_+x_-}{2(x_+^2 + x_-x_+ + x_-^2)}, \quad b_1 = \lambda = -\frac{x_+ + x_-}{2(x_+^2 + x_- x_+ + x_-^2)} \]
and
\[ \lambda = -\frac{3}{2} \frac{x_- + x_+}{x_+^2 + x_+ x_- + x_-^2}. \]  
(41)

Since \( \lambda > 0 \) and \( x_+^2 + x_+ x_- + x_-^2 = (x_+ + x_-)^2 - x_+ x_- > 0 \) then \( x_+ + x_- < 0 \). Taking into account that \( b_0 > 0 \) and \( b_1 > 0 \) we conclude that \( R_m \) and \( R_M \) take place at the turning points; therefore,
\[ \omega_b^2 = R(x_+) + R(x_-) = -\frac{x_-^2 + 4x_+ x_- + x_-^2}{2(x_+^2 + x_+ x_- + x_-^2)}. \]  
(42)

In order to verify that \( \omega_b \) is real for all values of \( E \) below the barrier, note that
\[ x_3 = -\frac{b_1}{b_0} = -\frac{x_+ x_-}{x_+ + x_-}. \]  
(43)

leads to
\[ x_+^2 + 4x_+ x_- + x_-^2 = (x_+ + x_-)^2 + 2x_+ x_- = (x_+ + x_-)(x_+ + x_- - 2x_3) < 0. \]  
(44)

After the change of variables (12) the function \( \Delta_1(\theta) \) takes a particularly simple form
\[ \Delta_1(\theta) = \xi \cos \theta, \]
\[ \xi = \frac{(x_+^2 - x_-^2)}{x_+^2 + 4x_+ x_- + x_-^2} = \frac{x_+ - x_-}{x_+ + x_- - 2x_3}. \]  
(45)

Since \( x_+ - x_- > 0 \), and \( x_+ + x_- - 2x_3 > x_+ - x_- \), we conclude that \( 0 < \xi < 1 \) for all bounded motion.

The resulting integral
\[ I = \frac{\sqrt{2}}{\omega_b} \int_0^\pi \frac{d\theta}{\sqrt{1 + \xi \cos \theta}} \]  
(46)

gives rise to the series
\[ I = \frac{\sqrt{2} \pi}{\omega_b} \sum_{j=0}^{\infty} (-1)^j \left( -\frac{1}{2} \right)_j \left( -\frac{1}{2} \right)_j \xi^{2j}, \]  
(47)

which is similar to the one derived above for the Duffing oscillator and converges for all \( |\xi| < 1 \). The greatest value of the energy consistent with periodic motion corresponds to the particle in unstable equilibrium at the top of the barrier \( E = U(x_M) \), where \( x_3 = x_- \) and \( \xi = 1 \). In other words, series (47) converges for all periodic motion.

Equation (46) gives us a simple and exact expression for the period of the anharmonic oscillator (38) that one may calculate numerically for a given set of potential parameters. On the other hand, equation (47) provides approximate analytical expressions that one makes as accurate as desired by simply adding a sufficiently large number of terms. The choice of one or another depends on the particular application.

Following a different procedure Apostol [4] derived the exact expression for the period of the quadratic–cubic oscillator
\[ T = \frac{3}{2(k_x \omega_0 \sqrt{x_+ - x_3})} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}, \]
\[ k^2 = \frac{x_+ - x_-}{x_+ - x_3}. \]  
(48)

The expansion of this equation in powers of \( k^2 \) is also convergent for all values of the energy consistent with periodic motion because \( k^2 < 1 \). It is worth mentioning that our approach applies to any anharmonic oscillator and is therefore more general than the one developed by Apostol for the quadratic–cubic oscillator [4].

In the appendix, we show an interesting application of the results of this section to the calculation of the precession of the perihelion of a planet orbiting around the sun.
6. Conclusions

In this paper, we present a straightforward systematic procedure for constructing exact and approximate expressions for the period of anharmonic oscillators. The recipe is simple: first, we factor the function $Q(x)$ and obtain the turning points and the function $R(x)$ as in equation (8). Second, we obtain the maximum and minimum values of $R(x)$ in the interval between the turning points which determine the optimum value of $\omega$. Thus we are left with an exact expression for the period that we may use in numerical applications. In addition to it, we may expand this exact expression in a Taylor series in order to obtain partial sums that become analytical expressions for the period. We can increase the accuracy of the approach because the series converges to the exact result for all values of the energy that give rise to periodic motion.

The method proposed in this paper is not restricted to the period of anharmonic oscillators with polynomial potentials. We may, for example, expand a given arbitrary potential $U(x)$ about its minimum to any desired degree and then apply the approach developed above. Moreover, some other problems have been expressed in terms of integrals of the form (7), such as, for example, the deflection of light by a massive body or the precession of a planet orbiting around a star [5]. In the appendix we show an application of the equations for the quadratic–cubic oscillator to the precession of the perihelion of a planet orbiting around the sun.

There is a wide range of interesting applications for the present method and for that reason we believe that it is suitable for teaching in advanced undergraduate courses on classical mechanics.

Appendix. The precession of the perihelion of a planet

The Lagrangian for the relative motion of two bodies that interact through a central potential–energy function $V(r)$ is [8]

$$L = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) \quad (A.1)$$

where $r$ is the distance between the bodies, $\phi$ is the polar angle and $m$ is the reduced mass of the system. There are two constants of the motion, the angular momentum $M = mr^2 \dot{\phi}$, and the total energy:

$$E = \frac{mr^2}{2} + \frac{M^2}{2mr^2} + V(r). \quad (A.2)$$

This equation resembles the energy of the motion of a particle of mass $m$ in one dimension under the effect of the effective potential $W(r) = V(r) + M^2/(2mr^2)$. The time spent by the particle passing from a distance $r_1$ to a distance $r_2$ is given by [8]

$$t_{12} = \int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{m}(E - V(r)) - \frac{M^2}{mr^2}}} \quad (A.3)$$

It follows from the conservation of angular momentum, rewritten as $d\phi = \frac{M}{mr^2} dt$, that the angle spanned by the orbit during the time $t_{12}$ is

$$\phi_{12} = \int_{r_1}^{r_2} \frac{M \, dr}{r^2 \sqrt{2m(E - V(r)) - \frac{M^2}{r^2}}} \quad (A.4)$$

We now assume that $r_1 = r_{\text{min}}$ and $r_2 = r_{\text{max}}$ are the perihelion and aphelion of the orbit, respectively. If $\phi_{12} \neq \pi$ the orbit described by the body will perform a precession, i.e. the
main axis of the orbit will move by an amount \(2 \phi_1 - \pi\) each time the body passes through the perihelion [8]. The angle of precession will therefore be given by

\[
\Delta \phi = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{M \, dr}{r^2 \sqrt{2m \left[ E - V(r) \right] - \frac{M^2}{r^2}}} - 2\pi. \tag{A.5}
\]

It is well known that no precession is observed \((\Delta \phi = 0)\) in the case of the Newtonian potential \(V(r) = -\frac{k}{r}\), where the orbits for negative energy are simple ellipses. It is much more interesting to calculate the precession of the orbit of a body moving in a Newtonian potential with a perturbation \(\delta V = \gamma/r^3\), which is considered as a problem in [8]. In this case the angle of precession is given by

\[
\Delta \phi = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{M \, dr}{r^2 \sqrt{2m (E + \frac{k}{r} - \frac{\gamma}{r^3}) - \frac{M^2}{r^2}}} - 2\pi. \tag{A.6}
\]

In order to obtain a simpler expression for \(\Delta \phi\) we change the variable \(r = 1/(a + z)\), where \(a = (-M^2 + \sqrt{12\gamma km^2 + M^4})/(6\gamma m)\) removes the linear term in the resulting polynomial. Thus we rewrite integral (A.6) as

\[
\Delta \phi = 2 \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{M \, dz}{\sqrt{2m (E - \tilde{V}(z))}} - 2\pi, \tag{A.7}
\]

where

\[
\tilde{V}(z) = W \left( \frac{1}{a + z} \right) - W_{\text{min}} = \frac{12\gamma km^2 + M^4}{2m} z^2 + \gamma z^3. \tag{A.8}
\]

\(W_{\text{min}}\) is the minimum of the effective potential, \(\tilde{E} = E - W_{\text{min}}\), and \(\tilde{V}(z_{\text{min}}) = \tilde{V}(z_{\text{max}})\).

Note that \(\tilde{V}(z)\) is exactly of the form (37) with \(v_2 = \sqrt{12\gamma km^2 + M^4}/m\) and \(v_3 = 3\gamma\). We can therefore apply equation (47) and obtain

\[
\Delta \phi = -2\pi + \frac{2\pi M}{\sqrt{m} \omega_b (12\gamma km^2 + M^4)^{1/4}} \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} -1/2 \\ j \end{array} \right) \left( \begin{array}{c} -1/2 \\ 2j \end{array} \right) \xi^{2j}. \tag{A.9}
\]

Although we have derived explicit expressions for \(\omega_b\) and \(\xi\) from the general results of section 5, we do not show them here because they are rather complicated. However, if \(\tilde{E}\) is small we can simplify equation (A.9) by expansion of \(\omega_b\) and \(\xi\) as follows:

\[
\omega_b \approx 1 - \frac{6\gamma^2 m^3}{(12\gamma km^2 + M^4)^{3/4}} \tilde{E} - \frac{114\gamma^4 m^6}{(12\gamma km^2 + M^4)^{3/4}} \tilde{E}^2 + O(\tilde{E}^3), \tag{A.10}
\]

\[
\xi \approx \frac{2\sqrt{2\gamma} m^{3/2}}{(12\gamma km^2 + M^4)^{3/4}} \tilde{E}^{1/2} + O(\tilde{E}^{3/2}). \tag{A.11}
\]

To leading order we obtain the approximate expression for the precession:

\[
\Delta \phi_{LO} = -2\pi + \frac{2\pi M}{(12\gamma km^2 + M^4)^{1/4}} \tag{A.12}
\]

which can be further expanded in powers of \(|\gamma m^2 k/M^4| \ll 1\) to give

\[
\Delta \phi_{LO} \approx \frac{6\gamma km^2 \pi}{M^4} + \frac{45\gamma^2 k^2 m^4 \pi}{M^8} - \frac{405\gamma^3 k^3 m^6 \pi}{M^{12}} + \ldots. \tag{A.13}
\]

Note that the first term in this expansion is exactly the result of [8].

Equation (A.12) provides an accurate approximation to the precession angle for arbitrary values of \(\gamma\), in contrast with the approximation of [8] which is acceptable only for sufficiently
small $\gamma$. Note that equation (A.12) is independent of $\bar{E}$ and is valid only in a neighbourhood of $\bar{E} = 0$. This limitation comes from keeping just the leading term of the expansion of $\omega_b$ and $\xi$ in series of $\bar{E}$. Equation (A.9) gives us suitable expressions for greater values of $\bar{E}$ which we will not consider in this discussion.

Figure 8 shows $\Delta \phi$ for the energy $E = 0.5W_{\text{min}}$ calculated numerically from the integral, from the first term of equation (A.13) [8] (approximation a), from equation (A.9) to lowest order (approximation b) and from equation (A.12) (approximation c). We have arbitrarily chosen $M = m = k = 1$. We appreciate that even our simplest expressions are more accurate than those commonly found in some well-known textbooks [8].

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