Complexity of Scott Sentences

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Abstract

We give effective versions of some results on Scott sentences. We show that if \( \mathcal{A} \) has a computable \( \Pi_\alpha \) Scott sentence, then the orbits of all tuples are defined by formulas that are computable \( \Sigma_\beta \) for some \( \beta < \alpha \). (This is an effective version of a result of Montalbán \[\Pi\].) We show that if a countable structure \( \mathcal{A} \) has a computable \( \Sigma_\alpha \)

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Scott sentence and one that is computable $\Pi_\alpha$, then it has one that is computable $d\Sigma_\beta$ for some $\beta < \alpha$. (This is an effective version of a result of A. Miller [9].) We also give an effective version of a result of D. Miller [10]. Using the non-effective results of Montalbán and A. Miller, we show that a finitely generated group has a $d\Sigma_2$ Scott sentence iff the orbit of some (or every) generating tuple is defined by a $\Pi_1$ formula. Using our effective results, we show that for a computable finitely generated group, there is a computable $d\Sigma_2$ Scott sentence iff the orbit of some (every) generating tuple is defined by a computable $\Pi_1$ formula.

1 Introduction

The $L_{\omega_1\omega}$-formulas are infinitary formulas in which the disjunctions and conjunctions are over countable sets, and the strings of quantifiers are finite. We consider $L_{\omega_1\omega}$-formulas with only finitely many free variables. There is no prenex normal form for $L_{\omega_1\omega}$-formulas. We cannot, in general, bring the quantifiers to the front. However, we can bring the negations inside, and this gives a kind of normal form. Formulas in this normal form are classified as $\Sigma_\alpha$ or $\Pi_\alpha$ for countable ordinals $\alpha$.

1. A formula $\varphi(\bar{x})$ is $\Sigma_0$ and $\Pi_0$ if it is finitary quantifier-free.

2. Let $\alpha > 0$.

   (a) $\varphi(\bar{x})$ is $\Sigma_\alpha$ if it is a countable disjunction of formulas $(\exists \bar{u})\psi(\bar{x}, \bar{u})$, where $\psi$ is $\Pi_\beta$ for some $\beta < \alpha$.

   (b) $\varphi(\bar{x})$ is $\Pi_\alpha$ if it is a countable conjunction of formulas $(\forall \bar{u})\psi(\bar{x}, \bar{u})$, where $\psi$ is $\Sigma_\beta$ for some $\beta < \alpha$.

We use special notation for some further classes of formulas.

1. A formula is $\Sigma_{< \alpha}$ (resp. $\Pi_{< \alpha}$) if it is $\Sigma_\beta$ (resp. $\Pi_\beta$) for some $\beta < \alpha$.

2. A formula is $d\Sigma_\alpha$ if it is the conjunction of a formula that is $\Sigma_\alpha$ and one that is $\Pi_\alpha$.

**Negations.** For a $\Sigma_\alpha$ (resp. $\Pi_\alpha$) formula $\varphi$, in normal form, we write $\text{neg}(\varphi)$ for the natural $\Pi_\alpha$ (resp. $\Sigma_\alpha$) formula in normal form that is logically equivalent to the negation of $\varphi$.

**Computable infinitary formulas.** Roughly speaking, the computable infinitary formulas are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and
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conjunctions are over c.e. sets. For more about computable infinitary formulas, see [1]. We classify the computable infinitary formulas as computable $\Sigma_\alpha$ or computable $\Pi_\alpha$ for computable ordinals $\alpha$. We may refer to computable $\Sigma_{<\alpha}$ formulas, or computable $d$-$\Sigma_\alpha$ formulas.

Scott [13] proved the following.

**Theorem 1.1** (Scott Isomorphism Theorem). Let $A$ be a countable structure for a countable language $L$. Then there is a sentence of $L_{\omega_1\omega}$ whose countable models are just the isomorphic copies of $A$.

A sentence with the property above is called a *Scott sentence* for $A$. The complexity of an optimal Scott sentence for a structure $A$ measures the internal complexity of $A$.

For a countable language $L$, let $C$ be a countably infinite set of new constants. Identifying the constants with natural numbers, we suppose that $C = \omega$. Let $Mod(L)$ be the class of $L$-structures with universe $\omega$, and for a computable infinitary sentence $\psi$ in the language $L \cup C$, let $Mod(\psi)$ be the class of $L$-structures with universe $\omega$ that satisfy $\psi$. There is a natural topology on $Mod(L)$, generated by basic open (actually clopen) neighborhoods of the form $Mod(\varphi)$, for $\varphi$ a finitary quantifier-free sentence in the language $L \cup C$. We define the Borel hierarchy of classes $K \subseteq Mod(L)$ as follows.

1. $K$ is $\Sigma_0$ and $\Pi_0$ if it is a basic clopen neighborhood.

2. For $0 < \alpha < \omega_1$,

   (a) $K$ is $\Sigma_\alpha$ if it is a countable union of sets each of which is $\Pi_\beta$ for some $\beta < \alpha$.

   (b) $K$ is $\Pi_\alpha$ if it is a countable intersection of sets each of which is $\Sigma_\beta$ for some $\beta < \alpha$.

   (c) $K$ is $d$-$\Sigma_\alpha$ if it is a difference of $\Sigma_\alpha$ sets.

Vaught [15] proved the following.

**Theorem 1.2** (Vaught). For a set $K \subseteq Mod(L)$, closed under isomorphism, $K$ is $\Pi_\alpha$ in the Borel hierarchy iff there is a $\Pi_\alpha$ sentence $\varphi$ of $L_{\omega_1\omega}$ such that $K = Mod(\varphi)$. 
It is easy to see, as a corollary of Vaught’s Theorem, that the same holds for $\Sigma_\alpha$ and $d$-$\Sigma_\alpha$ sets and sentences.

If $L$ is a computable language, then we have also the effective Borel hierarchy. Let $K \subseteq Mod(L)$.

1. $K$ is effective $\Sigma_0$ and effective $\Pi_0$ if it is a basic clopen neighborhood.

2. For $0 < \alpha < \omega_1^{CK}$,
   
   (a) $K$ is effective $\Sigma_\alpha$ if it is a c.e. union of sets each of which is effective $\Pi_\beta$ for some $\beta < \alpha$,

   (b) $K$ is effective $\Pi_\alpha$ if it is a c.e. intersection of sets each of which is effective $\Sigma_\beta$ for some $\beta < \alpha$.

We may effectively identify elements of $Mod(L)$ with elements of $2^\omega$, and then the effective Borel sets are exactly the hyperarithmetic sets of functions in $2^\omega$. Vanden Boom [14] proved the effective analogue of Vaught’s Theorem.

**Theorem 1.3** (Vanden Boom). For a set $K \subseteq Mod(L)$, closed under isomorphism, $K$ is $\Pi_\alpha$ in the effective Borel hierarchy iff there is a computable $\Pi_\alpha$ sentence $\varphi$ of $L_{\omega_1^{CK}}$ such that $K = Mod(\varphi)$.

Montalbán [11] proved that for a countable ordinal $\alpha \geq 1$, a countable structure $A$ has a $\Pi_{\alpha+1}$ Scott sentence iff the orbits of all tuples are defined by $\Sigma_\alpha$ formulas. The implication $\Leftarrow$ is as in the proof of Scott’s Theorem. For the implication $\Rightarrow$, Montalbán’s proof was clever. We shall use the ideas from his proof to obtain further results.

In Section 2, we show that for a countable ordinal $\alpha \geq 2$, if $A$ has a $\Pi_\alpha$ Scott sentence, then the orbits of all tuples in $A$ are defined by $\Sigma_{\alpha}$ formulas. (Montalbán’s Theorem gives this in the case where $\alpha$ is a successor ordinal.) For limit $\alpha$, the implication $\Leftarrow$ fails. There are familiar structures $A$ such that the orbits of all tuples are defined by $\Sigma_{\alpha}$ formulas but there is no $\Pi_\alpha$ Scott sentence. Next, we give an effective version of Montalbán’s theorem, saying that for a computable ordinal $\alpha \geq 2$, if $A$ has a computable $\Pi_\alpha$ Scott sentence, then the orbits of all tuples are defined by computable $\Sigma_{\alpha}$ formulas. Even for successor ordinals $\alpha$, the implication $\Leftarrow$ fails. We construct an example of a computable structure such that the orbits of all tuples are defined by finitary quantifier-free formulas, but there is no computable $\Pi_2$ Scott sentence.
In Section 3, we consider further results that can be proved using ideas from Section 2. A. Miller showed that if \( A \) has a \( \Pi_\alpha \) Scott sentence and a \( \Sigma_\alpha \) Scott sentence, then it has a Scott sentence that is \( d\Sigma_{<\alpha} \). The proof was based on a result of D. Miller [10] on separators for disjoint sets axiomatized by \( \Pi_\alpha \) sentences. Our Theorem 3.2 is an effective version of the result of A. Miller, saying that if \( A \) has a computable \( \Pi_\alpha \) Scott sentence and a computable \( \Sigma_\alpha \) Scott sentence, then it has a Scott sentence that is computable \( d\Sigma_{<\alpha} \). In [10], D. Miller gave an effective version of his result, which, unfortunately, was not sufficient to prove Theorem 3.2. We give a direct proof of Theorem 3.2. Then, in Theorem 3.4, we give an effective version of the result of D. Miller that would have served to prove Theorem 3.2.

In Section 4, we consider finitely generated groups. For such a group, there is always a \( \Sigma_3 \) Scott sentence, and if the group is computable, then there is a computable \( \Sigma_3 \) Scott sentence (see [8]). Often, however, there is a simpler Scott sentence. In fact, the second author had conjectured that every finitely generated group has a \( d\Sigma_2 \) Scott sentence, and every computable finitely generated group has a computable \( d\Sigma_2 \) Scott sentence. Recently, Harrison-Trainor and Ho [4] gave an example disproving both conjectures. We show that if \( G \) is a finitely generated group, then \( G \) has a \( d\Sigma_2 \) Scott sentence iff there is a generating tuple whose orbit is defined by a \( \Pi_1 \) formula, and if \( G \) is a computable finitely generated group, then \( G \) has a computable \( d\Sigma_2 \) Scott sentence iff there is a generating tuple whose orbit is defined by a computable \( \Pi_1 \)-formula.

Recall that the definition above of \( \text{Mod}(L) \), all \( L \)-structures have universe \( \omega \). Throughout this paper, all structures given are assumed to be countably infinite with universe \( \omega \), and all structures we build are guaranteed by our techniques to have universe \( \omega \).

## 2 Varying Montalbán’s Theorem

Montalbán [11] proved the following.

**Theorem 2.1** (Montalbán). *Suppose \( \alpha \geq 1 \) is a countable ordinal, and let \( \mathcal{A} \) be a countable structure for a countable language \( L \). Then \( \mathcal{A} \) has a \( \Pi_{\alpha+1} \) Scott sentence iff the automorphism orbit of each tuple is defined by a \( \Sigma_\alpha \) formula.*

For the implication \( \Leftarrow \), the proof is as for Scott’s Isomorphism Theorem.
Proof of $\Leftarrow$. For each $\bar{a}$, let $\varphi_{\bar{a}}(\bar{x})$ be a $\Sigma_\alpha$ formula that defines the orbit of $\bar{a}$. For each $\bar{a}$, we determine a sentence $\rho_{\bar{a}}$ as follows.

- $\rho_{\emptyset}: \bigwedge_b (\exists x) \varphi_b(x) \land (\forall x) \bigvee_b \varphi_b(x)$
- $\rho_{\bar{a}}: (\forall \bar{u}) [\varphi_{\bar{a}}(\bar{u}) \rightarrow (\bigwedge_b (\exists x) \varphi_{\bar{a},b}(\bar{u}, x) \land (\forall x) \bigvee_b \varphi_{\bar{a},b}(x))]$

Our Scott sentence is the conjunction of the sentences $\rho_{\bar{a}}$. It is not difficult to see that this is $\Pi_{\alpha+1}$. \hfill $\square$

The implication $\Rightarrow$ in Montalbán’s result also holds for limit ordinals, with no change in the proof. Here is the statement.

**Theorem 2.2.** Let $\mathcal{A}$ be a countable structure for a countable language $L$. Let $\alpha \geq 2$. If $\mathcal{A}$ has a $\Pi_\alpha$ Scott sentence, then the orbit of each tuple is defined by a $\Sigma_{<\alpha}$ formula.

In our account of Montalbán’s proof, we use a “consistency property.” This is a family of sets of sentences arising in a kind of Henkin construction, developed by Makkai for producing models of $L_{\omega_1\omega}$ sentences. See [6] for a discussion of consistency properties. The definition that we give below is not standard, but it suits our needs.

**Definition 1.** Let $L$ be a countable language, and let $C$ be a countably infinite set of new constants. A consistency property is a non-empty set $\mathcal{C}$ of finite sets $S$ of sentences, each obtained by substituting constants from $C$ for the free variables in an $L_{\omega_1\omega}$ formula in normal form, such that the following conditions hold:

1. for $S \in \mathcal{C}$, if $\varphi \in S$, where $\varphi = \bigwedge_i (\forall \bar{u}_i) \varphi_i(\bar{u}_i)$, then for each $i$ and each appropriate tuple of constants $\bar{c}$, there exists $S' \supseteq S$ in $\mathcal{C}$ with $\varphi_i(\bar{c}) \in S'$,

2. for $S \in \mathcal{C}$, if $\varphi \in S$, where $\varphi = \bigvee_i (\exists \bar{u}_i) \varphi_i(\bar{u}_i)$, then for some $i$ and $\bar{c}$, there exists $S' \supseteq S$ in $\mathcal{C}$ with $\varphi_i(\bar{c}) \in S'$,

3. for $S \in \mathcal{C}$, for each finitary quantifier-free $L$-formula $\varphi(\bar{x})$ and appropriate tuple $\bar{c}$, there exists $S' \supseteq S$ in $\mathcal{C}$ with $\pm \varphi(\bar{c}) \in S'$,

4. for $S \in \mathcal{C}$, if $F$ is an $n$-place function symbol, and $c_1, \ldots, c_n \in C$, there is a constant $d \in C$ such that for some $S' \supseteq S$ in $\mathcal{C}$, the sentence $F(c_1, \ldots, c_n) = d$ is in $S'$,

5. for $S \in \mathcal{C}$, and distinct $c, c' \in C$, the sentence $c = c'$ is not in $S$. 
6. for $S \in C$, the set of finitary quantifier-free sentences in $S$ is consistent.

**Lemma 2.3.** If $C$ is a consistency property, then we can form a countable chain $(S_n)_{n \in \omega}$ of elements of $C$ such that the set $\{S_n : n \in \omega\}$ is also a consistency property.

Proof sketch. This is clear just from the fact that the sets $S$ in $C$ are finite, and there are only finitely many clauses (in the definition of consistency property) asking for extensions. \qed

**Proposition 2.4.** Let $(S_n)_{n \in \omega}$ be a countable chain such that $\{S_n : n \in \omega\}$ is a consistency property. Then the set of atomic sentences and negations of atomic sentences in $\bigcup_n S_n$ is the atomic diagram of a well defined structure $B$, with universe equal to $C$. Moreover, all sentences $\varphi \in \bigcup_n S_n$ are true of the appropriate elements in $B$.

Proof. For an $n$-placed relation symbol $R$, we let $R^B$ be the set of $(c_1, \ldots, c_n) \in \bigcup_n S_n$. Conditions (3) and (6) guarantee that $R^B$ is a well-defined relation on $C^n$. Suppose $F$ is an $n$-placed function symbol in $L$. Then $F^B(c_1, \ldots, c_n) = d$ if the sentence $F(c_1, \ldots, c_n) = d$ is in some $S_n$. Conditions (4), (5), and (6) guarantee that $F^B$ is well-defined. For $c, d \in C$, the sentence $c = d$ is in $\bigcup_n S_n$ iff the constants $c$ and $d$ are actually the same. An easy induction on terms $\tau(x)$ shows that $\tau^B(c) = d$ iff the sentence $\tau(c) = d$ is in $\bigcup_n S_n$. Then an easy induction on finitary quantifier-free formulas $\varphi$ shows that $B \models \varphi(c)$ iff the sentence $\varphi(c)$ is in $\bigcup_n S_n$. Finally, an easy induction on sentences shows that if $\varphi \in \bigcup_n S_n$, then $B \models \varphi$. \qed

We want a consistency property $C$ that produces models of the $\Pi_\alpha$ Scott sentence $\varphi = \bigwedge_i (\forall \bar{u}_i) \varphi_i(\bar{u}_i)$, where $\varphi_i(\bar{u}_i) = \bigvee_j (\exists \bar{v}_{i,j}) \psi_{i,j}(\bar{u}_i, \bar{v}_{i,j})$. We also want to control the complexity of the sentences that appear in $S \in C$. Instead of putting $\varphi$ into various sets $S \in C$, we add to the six conditions in the definition of consistency property a seventh condition guaranteeing that $\varphi$ is witnessed.

7. for $S \in C$, for each $i$ and each appropriate $\bar{c}$, there exist $j$, and an appropriate $\bar{d}$ such that for some $S' \supseteq S$ in $C$, $\psi_{i,j}(\bar{c}, \bar{d}) \in S'$.

If $C$ is a consistency property satisfying Conditions (1)–(7), then there is a chain $(S_n)_{n \in \omega}$ of elements of $C$ such that $\{S_n : n \in \omega\}$ also satisfies Conditions (1)–(7). For any such chain $(S_n)_{n \in \omega}$ the resulting structure is a model of $\varphi$. Our consistency property $C$ will be the set of finite sets $S$ of
sentences in the language \( L \cup C \), each \( \Sigma_\beta \) or \( \Pi_\beta \) for some \( \beta \) such that \( \beta + 1 < \alpha \) (and each, recall, obtained by substituting constants from \( C \) for the free variables in an \( L_{\omega_1\omega} \) formula in normal form), where some interpretation of the constants from \( C \) appearing in the sentences of \( S \), mapping distinct constants to distinct elements of \( A \), makes all of these sentences true. (Since the set of sentences is finite, there are only finitely many constants from \( C \) to assign to elements of \( A \).) It is easy to see that this satisfies Conditions (1)–(7).

**Claim:** For each tuple \( \bar{a} \) of distinct elements, our \( C \) fails to satisfy the following further condition.

8. for each \( S \in C \), and for each \( \bar{c} \), a tuple of distinct constants of the same length as \( \bar{a} \), there is some \( \Pi_{< \alpha} \) formula \( \psi(\bar{x}) = \bigwedge_i (\forall \bar{u}_i) \psi_i(\bar{x}, \bar{u}_i) \), true of \( \bar{a} \), and some \( S' \supseteq S \) in \( C \) such that for some \( i \) and some \( \bar{d} \), \( \text{neg}(\psi_i(\bar{c}, \bar{d})) \in S' \). (Note that for some \( \beta \) such that \( \beta + 1 < \alpha \), \( \psi_i(\bar{c}, \bar{d}) \) is \( \Sigma_\beta \), and \( \text{neg}(\psi_i(\bar{c}, \bar{d})) \) is \( \Pi_\beta \).)

**Proof of Claim.** If our \( C \) satisfied Conditions (1)–(8), then, we would have countable chains \( (S_n)_{n \in \omega} \) yielding models of \( \varphi \) with no tuple satisfying all of the \( \Pi_{< \alpha} \) formulas true of \( \bar{a} \). Since \( \varphi \) is a Scott sentence for \( A \), this is impossible. \( \square \)

By the Claim, there must be some set of sentences \( S \in C \) and some tuple of distinct constants \( \bar{c} \) from \( C \), of the same length as \( \bar{a} \), that witness the failure of Condition (8). Let \( \bar{c}' \) be the tuple of all constants from \( C \), other than \( \bar{c} \), appearing in \( S \), and let \( \chi(\bar{c}, \bar{c}') \) be the finite conjunction of the sentences in \( S \) and sentences expressing that the elements of \( \bar{c}, \bar{c}' \) are pairwise distinct. Then \( A \) satisfies the \( \Sigma_{< \alpha} \) sentence \( (\exists \bar{x}) \chi(\bar{a}, \bar{x}) \); and for all \( \Pi_{< \alpha} \) formulas \( \psi \) true of \( \bar{a} \), \( A \) satisfies the \( \Pi_{< \alpha} \) sentences logically equivalent to \( (\forall \bar{u}) [(\exists \bar{x}) \chi(\bar{u}, \bar{x}) \rightarrow \psi(\bar{u})] \). Thus, we have a \( \Sigma_{< \alpha} \) formula \( (\exists \bar{x}) \chi(\bar{u}, \bar{x}) \) that generates the complete \( \Pi_{< \alpha} \) type of \( \bar{a} \). For each \( \bar{a} \), let \( \varphi_{\bar{a}}(\bar{u}) \) be a \( \Sigma_{< \alpha} \) formula that generates the complete \( \Pi_{< \alpha} \) type of \( \bar{a} \). We claim that these formulas define the orbits. To show this, it is enough to prove the following lemma.

**Lemma 2.5.** The family \( F \) of finite functions taking \( \bar{a} \) to a tuple \( \bar{b} \) satisfying \( \varphi_{\bar{a}} \) has the back-and-forth property.

**Proof of Lemma.** We first show that for any \( \bar{a} \) and \( \bar{b} \), if \( \bar{b} \) satisfies \( \varphi_{\bar{a}} \), then \( \bar{a} \) satisfies \( \varphi_{\bar{b}} \). To see this, note that \( \text{neg}(\varphi_{\bar{b}}(\bar{x})) \) is \( \Pi_{< \alpha} \). If this were true of \( \bar{a} \), then it would be true of \( \bar{b} \), a contradiction. Suppose \( \bar{b} \) satisfies \( \varphi_{\bar{a}}(\bar{x}) \). For
any $d$, there exists $c$ such that $\bar{a}, c$ satisfies $\varphi_{\bar{b}, d}(\bar{x}, y)$. To see this, note that $(\forall y)\neg(\varphi_{\bar{b}, d}(\bar{x}, y))$ is $\Pi_{<\alpha}$, so if it were true of $\bar{a}$, then it would also be true of $\bar{b}$, a contradiction. If $\bar{a}, c$ satisfies $\varphi_{\bar{b}, d}(\bar{x}, y)$, then $\bar{b}, d$ satisfies $\varphi_{\bar{a}, c}(\bar{x}, y)$, so we can go back. Now, suppose that $\bar{b}$ satisfies $\varphi_{\bar{a}}(\bar{x})$, and take $c$. Since $\bar{a}$ satisfies $\varphi_{\bar{a}}(\bar{x})$, the argument above says that there exists $d$ such that $\bar{b}, d$ satisfies $\varphi_{\bar{a}, c}(\bar{x}, y)$, and then $\bar{a}, c$ satisfies $\varphi_{\bar{b}, d}(\bar{x}, y)$. Therefore, we can go forth. Hence, for each $\bar{a}$, $\varphi_{\bar{a}}(\bar{x})$ is a $\Sigma_{<\alpha}$ formula that defines the orbit of $\bar{a}$. This completes the proof of Theorem 2.2.

Below, we give a pair of examples.

**Example 1.** Let $\mathcal{A}$ be an ordering of type $\omega^\omega$. Then the orbits of all tuples in $\mathcal{A}$ are defined by $\Sigma_{<\omega}$ formulas (in fact, the natural defining formulas are computable $\Sigma_{<\omega}$). However, $\mathcal{A}$ has no $\Pi_\omega$ Scott sentence.

**Proof.** We use the following familiar results (see [1]).

**Facts:**

1. $\omega^\omega \leq_\omega \omega^{\omega+1}$,

2. for each $\beta < \omega^\omega$, there are computable $\Sigma_{<\omega}$ formulas $\lambda(x)$ and $\mu(x, y)$ such that $\lambda(x)$ holds iff the interval to the left of $x$ has order type $\beta$ and $\mu(x, y)$ holds iff the interval between $x$ and $y$ has order type $\beta$.

It follows from Fact 1 and a well-known result of Karp (see [6] or [7]) that every $\Pi_\omega$ sentence true of $\omega^\omega$ is true of $\omega^{\omega+1}$. Therefore, $\omega^\omega$ has no $\Pi_\omega$ Scott sentence. Take a tuple $\bar{a} = (a_1, \ldots, a_n)$. Ordinals are rigid, so to define the orbit of $\bar{a}$, we define the tuple itself. Say that the interval to the left of $a_i$ has type $\beta_i$. Applying Fact 2, we get a computable $\Sigma_{<\omega}$ formula $\lambda_i(x_i)$ saying that the interval to the left of $x_i$ has type $\beta_i$. The conjunction of the formulas $\lambda_i(x_i)$ defines the tuple $\bar{a}$.

**Example 2.** Let $\mathcal{A}$ be an expansion of the ordering of type $\omega^\omega$ with a unary predicate $U_0$ for the interval $[0, \omega)$ and unary predicates $U_n$ for the interval $[\omega^n, \omega^{n+1})$, for $n \geq 1$. Again, we have computable $\Sigma_{<\omega}$ formulas defining the orbits of all tuples. We have a computable $\Pi_\omega$ Scott sentence. This is the conjunction of a computable $\Pi_2$ sentence saying $(\forall x) \bigvee_n U_n(x)$, a finitary $\Pi_2$ sentence saying that $<$ is a linear ordering of the universe, with all elements of $U_n$ before all elements of $U_{n+1}$, and computable $\Sigma_{<\omega}$ sentences saying what is the order type of $U_n$.

We turn to the effective version of Theorem 2.2.
**Theorem 2.6.** Let $\mathcal{A}$ be a structure for a computable language $L$ (the structure need not have a computable copy). Suppose $\alpha \geq 2$. If $\mathcal{A}$ has a computable $\Pi_\alpha$ Scott sentence, then the orbit of each tuple is defined by a computable $\Sigma_{<\alpha}$ formula.

**Proof.** The proof essentially the same as that for Theorem 2.2. The corresponding rules for a consistency property involve computable infinitary formulas, so the conjunctions and disjunctions are over c.e. sets of indices. Our particular consistency property $\mathcal{C}$ will consist of the finite sets of computable $\Pi_\beta$ and computable $\Sigma_\beta$ sentences, for $\beta + 1 < \alpha$, such that some interpretation of the constants, mapping distinct constants to distinct elements of $\mathcal{A}$, makes all of the sentences true. These technical changes necessitate no significant change in the argument that constructs, for each tuple $\bar{a}$, a computable $\Sigma_{<\alpha}$ formula true of $\bar{a}$ that implies all computable $\Pi_{<\alpha}$ formulas true of $\bar{a}$.

For each $\bar{a}$, let $\varphi_{\bar{a}}(\bar{x})$ be a $\Sigma_{<\alpha}$ formula that implies all computable $\Pi_{<\alpha}$-formulas true of $\bar{a}$. To show that for each $\bar{a}$, the orbit is defined by $\varphi_{\bar{a}}$, the following analogue of Lemma 2.5 suffices; the proof is exactly the same as above.

**Lemma 2.7.** The family $\mathcal{F}$ of finite functions taking a tuple $\bar{a}$ to a tuple $\bar{b}$ satisfying $\varphi_{\bar{a}}$ has the back-and-forth property.

We do not have the effective version for the other implication even in the case where $\alpha$ is a computable successor ordinal.

**Proposition 2.8.** There is a computable structure $\mathcal{A}$ such that the orbits of all tuples are defined by computable $\Sigma_1$ (even finitary quantifier-free) formulas, but there is no computable $\Pi_2$ Scott sentence.

**Proof.** The proof owes much to that of Badaev [2], showing that there is a computable enumeration of a “discrete” set of functions that is not “effectively discrete”. We start with a computable subtree $T$ of $2^{<\omega}$ with the following features.

1. There are no terminal nodes.

2. There is just one non-isolated path $p$, where this is non-computable.

We may construct $T$ such that for all $\sigma \in T$, $\sigma 0 \in T$, and the only non-isolated path has the form $0^{s_0}1^{k_0}0^{s_1}1^{k_1} \ldots$, where $(k_n)_{n \in \omega}$ is a list of
the elements of the halting set, in increasing order, and $s_n$ is the number of steps in the halting computation of $\varphi_{k_n}(k_n)$. At stage 0, we put $\emptyset$ into $T$. Suppose that we have determined $T_s$ at stage $s$, where $T_s$ is the set of nodes in $T$ of length at most $s$. At stage $s + 1$, we add $\sigma 0$ for all $\sigma$ of length $s$. In addition, we consider $\varphi_{k,s+1}(k)$ for all $k \leq s + 1$. For the computations that halt, we arrange the $k$’s to form $k_0 < k_1 < \ldots < k_r$, and determine the appropriate halting times $s_0, \ldots, s_r$. We put into $T_{s+1}$ the appropriate initial segment of the sequence $0^{s_0}1^{k_0} \ldots 0^{s_r}1^{k_r}$. Note that if we are inserting a new $k_i$, it is because $s_i = s + 1$, and we already had the appropriate initial segment in $T_s$. At stage $s + 1$, the nodes just described are the only ones that we add to $T_{s+1}$. Therefore, the tree $T = \bigcup_{s \in \omega} T_s$ is computable.

We turn the tree $T$ into a class of structures. The language $\mathcal{L}$ consists of unary predicates $U_n$ for $n \in \omega$. In each $\mathcal{L}$-structure $A$, we have infinitely many elements $a$ representing each isolated path $f$ in $T$, in that if $f(n) = 0$, then $A \models \neg U_n a$ and if $f(n) = 1$, then $A \models U_n a$.

We can give a computable set of axioms for the elementary first order theory of these structures. For $\sigma \in T$, we have a finitary quantifier-free formula $\sigma(x)$ that is the conjunction of $U_n x$ for $\sigma(n) = 1$ and $\neg U_n x$ for $\sigma(n) = 0$. For each $n$, we have a finitary quantifier-free formula $T_n(x)$ that is the disjunction over $\sigma \in T \cap 2^n$ of the formulas $\sigma(x)$. Consider the axioms $(\forall x) T_n(x)$ for all $n$ and $(\exists z^n x) \sigma(x)$ for $\sigma \in T$. Let $T^*$ be the theory generated by these axioms.

The countable models of $T^*$ all have infinitely many elements representing each isolated path. In addition, they may have one or more elements representing the non-isolated path. To see that the axioms generate a complete theory, we note that any finitary sentence mentions only finitely many $U_m$, say for $m < n$. The reducts of the various countable models of $T^*$ to this smaller language are all isomorphic.

Consider the model of $T^*$ with no elements representing the non-isolated path. Clearly, this model has a computable copy $A$ with universe $A = \omega$. To see this, computably partition the set $A$ into infinitely many infinite, computable sets. For each $\sigma \in T$, consider the infinite path composed of $\sigma$ followed by all 0’s; assign all of the elements $a$ from one of the infinite sets in the partition to this path. Note that if $\sigma_i$ isolates the path represented by $a_i$, then the conjunction of the formulas giving the equalities on the $a_i$ and the formulas $\sigma_i$ generates the complete elementary first order type of $\bar{a}$. Moreover, since the language has only unary predicates, this formula actually defines the orbit of $\bar{a}$. Similarly, for any model $D$ of $T^*$, $A$ is isomorphic
to an elementary substructure of $D$, and any mapping that sends each element $a \in A$ to an element $d \in D$ representing the same isolated path is an elementary embedding.

Let $B$ be the model of $T^*$ with just one element $b$ representing the non-isolated path. We write $A$ for the substructure of $B$ isomorphic to the structure $A$ above. We want to show that any computable $\Pi_2^0$ sentence true of $A$ is true of $B$. It is enough to show that any computable $\Sigma_2^0$ sentence true of $B$ is true of $A$. Take a computable $\Sigma_2^0$ sentence $\varphi = \bigvee (\exists \vec{u}_i) \varphi_i(\vec{u}_i)$ true of $B$. Say $B \models \varphi_i(b, \vec{a})$, where $\varphi_i(x, \vec{v})$ is computable $\Pi_1$, $\vec{u}_i = x, \vec{v}$. Let $\delta(\vec{v})$ be a finitary quantifier-free formula generating the type of $\vec{a}$. Let $p(x, \vec{v})$ be the c.e. set of finitary universal conjuncts of $\varphi_i(\vec{u}_i)$. Let $\Gamma$ consist of the computable set of axioms for $T^*$, plus $\delta(\vec{c})$, plus $p(d, \vec{c})$.

Let $f$ be the non-isolated path through $T$. We cannot have $\Gamma \vdash \sigma(d)$, for all finite $\sigma \subseteq f$, since then $f$ would be computable. So, for some $\sigma \subseteq f$, $\Gamma \cup \{\neg \sigma(d)\}$ has a model $C$, where the elements of $\vec{c}$ and $d$ necessarily all represent isolated paths. Now, by what we noted above, the reduct of $C$ to the language $L$ has an elementary substructure isomorphic to $A$, and we may suppose that $\vec{c}$ and $d$ are in this elementary substructure, and that $\vec{a}$ is mapped to $\vec{c}$. Therefore, for the element $a'$ mapped to $d$, $A \models \varphi_i(a', \vec{a})$, as required. \hfill $\square$

The next result gives conditions sufficient to guarantee that a computable structure has a computable $\Pi_{\alpha+1}$ Scott sentence.

**Proposition 2.9.** Let $\alpha \geq 1$ be a computable ordinal. Suppose $A$ is a computable structure, and there is a $\Sigma_\alpha$ Scott family $\Phi$ consisting of computable $\Sigma_\alpha$ formulas, with no parameters. Then $A$ has a computable $\Pi_{\alpha+1}$ Scott sentence.

**Proof.** We can prove the following.

**Claim:** There is a computable function taking each tuple $\vec{c}$ to a computable $\Sigma_\alpha$ formula $\varphi_{\vec{c}}(\vec{x})$ that defines the orbit of $\vec{c}$.

**Proof of Claim.** Let $a$ be a notation for $\alpha$ (see [1] for a technical exposition of Kleene’s system of notations for computable ordinals). We may suppose that all elements of $\Phi$ have indices of the form $(\Sigma, a, e)$. Let $R$ be the $\Sigma_\alpha$ relation consisting of pairs $(\vec{c}, e)$ such that the formula $\psi_{\vec{c}, e}$ with index $(\Sigma, a, e)$ is an element of $\Phi$ that is true of $\vec{c}$. We can construct a computable sequence of computable $\Sigma_\alpha$ formulas $\tau_{\vec{c}, e}$, defined for all $\vec{c}$ and all $e$, built up out of $\top$ and $\bot$, such that $\tau_{\vec{c}, e}$ is logically equivalent to $\top$ if $(\vec{c}, e) \in R$ and
\[ \bot \] otherwise. We let \( \varphi_{\bar{c}}(\bar{x}) \) be the disjunction over all \( e \), of the formulas \( \tau_{\bar{c},e} \land \psi_{\bar{c},e}(\bar{x}) \). \hfill \square

Using the formulas \( \varphi_{\bar{c}}(\bar{x}) \) from the Claim, we can build a computable \( \Pi_{\alpha+1} \) Scott sentence as Scott did. We take the conjunction of the computable \( \Sigma_{\alpha} \) sentence \( \varphi_{\emptyset} \) and the computable \( \Pi_{\alpha+1} \) sentences \( \rho_{\bar{a}} \) saying

\[
(\forall \bar{x})[\varphi_{\bar{a}}(\bar{x}) \rightarrow ((\forall y) \bigvee b \varphi_{a,b}(\bar{x},y) \land (\exists y)\varphi_{a,b}(\bar{x},y))] .
\]

\hfill \square

Recall that in an example above, we showed that for a computable ordering of type \( \omega^\omega \), the orbits of all tuples are defined by computable \( \Sigma_{<\omega} \) formulas, but there is no \( \Pi_{\omega} \) Scott sentence.

3 Varying the results of A. Miller and D. Miller

A. Miller [9] proved that for a countable ordinal \( \alpha \geq 2 \), if \( \mathcal{A} \) has a \( \Sigma_{\alpha} \) Scott sentence and a \( \Pi_{\alpha} \) Scott sentence, then it has one that is \( d-\Sigma_{\alpha} \). We say a little about A. Miller’s proof. The case where \( \alpha \) is a limit ordinal is trivial; in fact, if \( \mathcal{A} \) has a \( \Sigma_{\alpha} \) Scott sentence \( \varphi \), then there is a \( \Sigma_{<\alpha} \) Scott sentence. The sentence \( \varphi \) is a countable disjunction of formulas \( \varphi_i \), each of which is \( \Sigma_{<\alpha} \). One of the disjuncts \( \varphi_i \) is true in \( \mathcal{A} \), and this is a \( \Sigma_{<\alpha} \) Scott sentence.

The interesting case is as follows.

**Theorem 3.1** (A. Miller). *For a countable ordinal \( \alpha \geq 1 \), if \( \mathcal{A} \) has a Scott sentence that is \( \Pi_{\alpha+1} \) and one that is \( \Sigma_{\alpha+1} \), then it has one that is \( d-\Sigma_{\alpha} \).*

**Sketch of proof.** A. Miller used a result of D. Miller [10] saying that for disjoint sets \( A, B \subseteq \text{Mod}(L) \) both axiomatized by \( \Pi_{\alpha+1} \) sentences, there is a separator (i.e., a set containing \( A \) and disjoint from \( B \)) that is a countable union of sets axiomatized by \( d-\Sigma_{\alpha} \) sentences. Suppose that \( \mathcal{A} \) has Scott sentences \( \varphi \) and \( \psi \), where \( \varphi \) is \( \Pi_{\alpha+1} \) and \( \psi \) is \( \Sigma_{\alpha+1} \). Applying the result of D. Miller to the disjoint sets \( A = \text{Mod}(\varphi) \) and \( B = \text{Mod}(\text{neg}(\psi)) \), we get a separator which is \( \text{Mod}(\gamma) \) for some sentence \( \gamma \) which is a countable disjunction of \( d-\Sigma_{\alpha} \) sentences. Since \( \text{Mod}(\text{neg}(\psi)) \) is the complement of \( \text{Mod}(\varphi) \) in \( \text{Mod}(L) \), \( \text{Mod}(\gamma) = \text{Mod}(\varphi) \); so \( \gamma \) is a Scott sentence for \( \mathcal{A} \). Thus, \( \mathcal{A} \) satisfies one of the disjuncts of \( \gamma \), and this is also a Scott sentence for \( \mathcal{A} \). \hfill \square
Our goal is to prove the following.

**Theorem 3.2.** For a computable ordinal \( \alpha \geq 2 \), if \( A \) has a Scott sentence that is computable \( \Pi_\alpha \) and one that is computable \( \Sigma_\alpha \), then there is one that is computable \( d-\Sigma_{<\alpha} \).

Again, the case where \( \alpha \) is a limit ordinal is trivial. We want to prove that if \( A \) has one Scott sentence that is computable \( \Pi_{\alpha+1} \) and another that is computable \( \Sigma_{\alpha+1} \), then there is one that is computable \( d-\Sigma_{\alpha} \).

D. Miller gave an effective version of his separation theorem, saying that if \( A \) and \( B \) are disjoint subsets of \( \text{Mod}(L) \), axiomatized by \( \Pi_{\alpha+1} \) sentences in the admissible fragment \( L_{\omega_1^{CK}} \), then there is a separator that is a disjoint union of sets axiomatized by \( d-\Sigma_{\alpha} \) formulas in \( L_{\omega_1^{CK}} \). This is not good enough for our purposes. We give a direct proof of the following.

**Lemma 3.3 (Main Lemma).** Let \( \alpha \geq 1 \) be a computable ordinal. If \( A \) has a computable \( \Sigma_{\alpha+1} \) Scott sentence \( \varphi \) and a computable \( \Pi_{\alpha+1} \) Scott sentence \( \psi \), then there is a computable \( d-\Sigma_{\alpha} \) Scott sentence.

**Proof.** The sentence \( \varphi \) has the form \( \bigvee_{i \in W} (\exists \bar{u}_i) \varphi_i(\bar{u}_i) \), where each \( \varphi_i \) is computable \( \Pi_\alpha \), and \( W \) is a c.e. set. For some \( i \) and some \( \bar{a} \), we have \( A \models \varphi_i(\bar{a}) \). By Theorem 2.6, the orbit of \( \bar{a} \) is defined by a computable \( \Sigma_\alpha \) formula \( \gamma(\bar{u}) \). Note that \( (\exists \bar{u}) \gamma(\bar{u}) \) is computable \( \Sigma_\alpha \), and \( (\forall \bar{u})(\gamma(\bar{u}) \rightarrow \varphi_i(\bar{u})) \) is logically equivalent to a computable \( \Pi_\alpha \) sentence. The conjunction of these is a Scott sentence for \( A \). \( \square \)

Below, we give an effective version of D. Miller’s result, which would suffice to prove Theorem 3.2.

**Theorem 3.4.** Let \( L \) be a computable language. For a computable ordinal \( \alpha \geq 2 \), suppose \( A \) and \( B \) are disjoint subsets of \( \text{Mod}(L) \) axiomatized by computable \( \Pi_\alpha \) sentences. Then there is a separator that is the union of a countable family of sets each of which is axiomatized by a computable \( d-\Sigma_{<\alpha} \) sentence.

**Proof.** Let \( A = \text{Mod}(\varphi) \), and let \( B = \text{Mod}(\psi) \), where \( \varphi \) and \( \psi \) are computable \( \Pi_\alpha \) sentences. Say that \( \varphi = \bigwedge_{i \in W} (\forall \bar{u}_i) \varphi_i(\bar{u}_i) \), where \( W \) is a c.e. set, and each \( \varphi_i(\bar{u}_i) \) has the form \( \bigvee_{j \in W_{e_i}} (\exists v_{i,j}) \xi_{i,j}(\bar{u}_i, v_{i,j}) \), where each \( \xi_{i,j} \) is \( \Pi_\beta \) for some \( \beta \) such that \( \beta + 1 < \alpha \), and each \( W_{e_i} \) is a c.e. set whose index \( e_i \) (according to some canonical indexing of the c.e. sets) is determined computably from the index \( i \) of \( \varphi_i \).
Let $C$ be an infinite computable set of new Henkin constants, corresponding to the natural numbers. For each $A \in A$, let $C_A$ be the consistency property consisting of the finite sets $S$ of sentences in the language $L \cup C$, each computable $\Sigma_\beta$ or $\Pi_\beta$ for some $\beta$ such that $\beta + 1 < \alpha$ (and each, recall, obtained by substituting constants from $C$ for the free variables in a computable $L_{\omega_1}$ formula in normal form), where some interpretation of the constants from $C$ appearing in the sentences of $S$, mapping distinct constants to distinct elements of $A$, makes all of these sentences true. Recall the special computable $\Pi_\alpha$ sentence $\varphi$ and its sub-formulas, $\varphi_i$ and $\xi_{i,j}$. The set $C_A$ is a consistency property $C$ satisfying the following condition:

\[ \forall S \in C, \text{for each } i \in W \text{ and each appropriate } \bar{c}, \text{there exist } j \in W_{e_i} \text{ and an appropriate } \bar{d} \text{ such that for some } S' \supseteq S \in C, \xi_{i,j}(\bar{c}, \bar{d}) \in S'. \]

For any consistency property $C$ satisfying $\star$, there is a chain $(S_n)_{n \in \omega}$ of elements of $C$ such that $\{ S_n : n \in \omega \}$ is also a consistency property satisfying $\star$. For any such chain, the resulting structure is a model of $\varphi$.

We now consider the other special computable $\Pi_\alpha$ sentence $\psi$. Say that $\psi = \bigwedge_{i \in W'} (\forall \bar{u}_i) \psi_i(\bar{u}_i)$, where $W'$ is a c.e. set, and for each $i$, $\psi_i(\bar{u}_i) = \bigvee_{j \in W_{e_i}} (\exists \bar{v}_j) \theta_{i,j}(\bar{u}_i, \bar{v}_j)$, where $\theta_{i,j}$ is $\Pi_\beta$ for some $\beta$ such that $\beta + 1 < \alpha$, and each $W_{e_i}$ is a c.e. set whose index $e_i$ (according to some canonical indexing of the c.e. sets) is determined computably by the index $i$ of $\varphi_i$.

Since $A = \text{Mod}(\varphi)$ and $B = \text{Mod}(\psi)$ are disjoint subsets of $\text{Mod}(L)$, for any $A$ satisfying $\varphi$, $C_A$ cannot satisfy the following added condition, which would witness the truth of $\psi$:

\[ \forall S \in C, \text{for all } i \in W' \text{ and all } \bar{c} \text{ appropriate for } \bar{u}_i, \text{there exists } S' \supseteq S \text{ such that for some } j \in W_{e_i} \text{ and } \bar{d}, \text{we have } \theta_{i,j}(\bar{c}, \bar{d}) \in S'. \]

It follows that there must exist $S \in C_A$, and some $i$ and $\bar{c}$ appropriate for $\bar{u}_i$, such that for all $j$ in the c.e. set $W_{e_i}$ and all $\bar{d}$, $S \cup \{ \theta_{i,j}(\bar{c}, \bar{d}) \}$ is not satisfied by any assignment in $A$. Let $\bar{c}'$ be the tuple of constants from $C$, other than $\bar{c}$, that appear in $S$, and let $\chi(\bar{c}, \bar{c}')$ be the conjunction of $S$ and sentences expressing that the tuples $\bar{c}, \bar{c}'$ are disjoint and the elements of $\bar{c}'$ are distinct.

Now, although $\bar{c}$ is appropriate for the tuple of variables $\bar{u}_i$, it might be that the tuple $\bar{c}$ assigns the same constant to multiple variables; i.e, the tuple $\bar{c}$ could list the same constant multiple times. Therefore, to define $\chi(\bar{u}_i, \bar{x})$ unambiguously, for a given constant in $\bar{c}$, substitute the variable from $\bar{u}_i$ with least index to which this element of $\bar{c}$ was assigned. Finally,
let \( \rho(\bar{u}_i, \bar{x}) \) be the conjunction of \( \chi(\bar{u}_i, \bar{x}) \) and formulas that express the pairwise equality of any elements of \( \bar{u}_i \) to which the same constant in \( \bar{c} \) was assigned, and the pairwise inequality of any elements of \( \bar{u}_i \) to which different constants in \( \bar{c} \) were assigned. Then

\[
\mathcal{A} \models (\forall \bar{u}_i)[((\exists \bar{x})\rho(\bar{u}_i, \bar{x})) \to (\bigwedge_{j \in W_{ei}} (\forall \bar{v}_{i,j})(\neg \theta_{i,j}(\bar{u}_i, \bar{v}_{i,j})))]
\]

Note that \( (\exists \bar{u}_i, \bar{x})\rho(\bar{u}_i, \bar{x}) \) is computable \( \Sigma^{<\alpha} \), and

\[
(\forall \bar{u}_i)[((\exists \bar{x})\rho(\bar{u}_i, \bar{x})) \to (\bigwedge_{j \in W_{ei}} (\forall \bar{v}_{i,j})(\neg \theta_{i,j}(\bar{u}_i, \bar{v}_{i,j})))]
\]

is logically equivalent to a computable \( \Pi^{<\alpha} \) sentence. Both sentences are true in \( \mathcal{A} \). They cannot both be true in any model of \( \psi \), for then there would be a tuple satisfying \( \neg \psi_i(\bar{u}_i) \). The conjunction gives a computable \( d-\Sigma^{<\alpha} \) sentence that is true in \( \mathcal{A} \) and not true in any model of \( \psi \). Let \( M_A \) be the class of models for this sentence. As our separator, we take the union of the sets \( M_A \). While there may be uncountably many models \( \mathcal{A} \) of \( \varphi \), there are only countably many pairs of computable infinitary sentences. Hence, our separator is the union of a countable family of sets \( S_A \).

4 Finitely generated groups

Knight and Saraph \cite{8} observed that every computable finitely generated group has a computable \( \Sigma_3 \) Scott sentence. However, for many kinds of computable finitely generated groups, there is a computable \( d-\Sigma_2 \) Scott sentence. In particular, this is so for finitely generated free groups \cite{3}, finitely generated Abelian groups, the infinite dihedral group of rank 2 \cite{8}, further variants of the dihedral group \cite{12}, polycyclic groups, and certain groups of interest in geometric group theory (lamplighter and Baumslag-Solitar groups) \cite{5}. Based on the known examples, Ho and Knight had conjectured that every computable finitely generated group has a computable \( d-\Sigma_2 \) Scott sentence. Knight also conjectured that every finitely generated group (not necessarily computable) has a \( d-\Sigma_2 \) Scott sentence (not necessarily computable \( d-\Sigma_2 \)). Recently, Harrison-Trainor and Ho \cite{4} gave an example of a computable finitely generated group that does not have a \( d-\Sigma_2 \) Scott sentence, thereby disproving both conjectures.

Here we give necessary and sufficient conditions for a finitely generated group to have a \( d-\Sigma_2 \) Scott sentence. We also give necessary and sufficient
conditions for a computable finitely generated group to have a computable $d$-$\Sigma_2$ Scott sentence. We show that for a finitely generated group, there is a $d$-$\Sigma_2$ Scott sentence iff for some generating tuple, the orbit is defined by a $\Pi_1$ formula iff for each generating tuple, the orbit is defined by a $\Pi_1$ formula. For a computable finitely generated group, there is a computable $d$-$\Sigma_2$ Scott sentence iff for some generating tuple, the orbit is defined by a computable $\Pi_1$ formula iff for each generating tuple, the orbit is defined by a computable $\Pi_1$ formula.

4.1 Finitely generated groups with a $d$-$\Sigma_2$ Scott sentence

In [8], it is observed that a computable finitely generated group has a computable $\Sigma_3$ Scott sentence. Throughout the rest of this section, we use the notation $\langle \bar{x} \rangle \simeq \langle \bar{y} \rangle$ to represent the computable $\Pi_1$ formula that says $\bar{x}$ and $\bar{y}$ satisfy the exact same relators and non-relators. Note that if $\bar{a}$ is a fixed tuple of a group $G$, then the set of relators and non-relators satisfied by $\bar{a}$ is a set computable in $G$. The notation $\langle \bar{x} \rangle \equiv \langle \bar{a} \rangle$ represents the conjunction of the formulas of the forms $\bar{w}(\bar{x}) = 1$ and $\bar{w}(\bar{x}) \neq 1$ that are true of $\bar{a}$ in $A$. This formula should not be thought of as including $\bar{a}$ as parameters; instead, it represents a $\Pi_1$ formula in $\bar{x}$.

**Proposition 4.1.** Every finitely generated group has a $\Sigma_3$ Scott sentence.

**Proof.** Let $G$ be a group with generating tuple $\bar{a}$. As in [8], we get a Scott sentence saying that $(\exists \bar{x})(\langle \bar{x} \rangle \equiv \langle \bar{a} \rangle \& (\forall y)(\bigvee_w w(\bar{x}) = y)]$. □

We can prove the following.

**Theorem 4.2.** For a finitely generated group $G$, the following are equivalent:

1. $G$ has a $\Pi_3$ Scott sentence,
2. $G$ has a $d$-$\Sigma_2$ Scott sentence
3. for some generating tuple, the orbit is defined by a $\Pi_1$ formula
4. for each generating tuple, the orbit is defined by a $\Pi_1$ formula.

**Proof.** Clearly, (2) $\Rightarrow$ (1). Using the result of A. Miller, together with the fact that $G$ has a $\Sigma_3$ Scott sentence, we get (1) $\Rightarrow$ (2). To complete the proof, we will show that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1). For (1) $\Rightarrow$ (4), suppose
G has a Π₃ Scott sentence, and let \( \bar{a} \) be a generating tuple. By the result of Montalbán, the orbit of \( \bar{a} \) is defined by a computable Σ₂ formula \( \varphi(\bar{x}) = \bigvee_i (\exists \bar{u}_i) \varphi_i(\bar{x}, \bar{u}_i) \), where \( \varphi_i \) is Π₁. Take \( i \) and \( \bar{b} \) such that \( G \models \varphi_i(\bar{a}, \bar{b}) \). For some tuple of words \( \bar{w} \), \( G \models \bar{w}(\bar{a}) = \bar{b} \). Then the orbit of \( \bar{a} \) is defined by the Π₁ formula \( \psi(\bar{u}) \). We show that for all tuples \( \bar{b} \), the orbit is defined by a Σ₂ formula. Suppose \( G \models \bar{b} = \bar{w}(\bar{a}) \). Then the orbit of \( \bar{b} \) is defined by the Σ₂-formula \( \varphi(\bar{x}) = (\exists \bar{u})(\psi(\bar{u}) \land \bar{x} = \bar{w}(\bar{u})) \). Then by the result of Montalbán, \( G \) has a Π₃ Scott sentence.

In this proof of Theorem 4.2 above, we used Miller’s result to show that if \( G \) has a Π₃ Scott sentence, then it has a \( d^{-}\Sigma₂ \) Scott sentence. There is an alternative proof, using the following result of Ho [5].

**Lemma 4.3** (Generating Set Lemma). Let \( G \) be a computable finitely generated group, and suppose \( \varphi(\bar{x}) \) is a computable Σ₂ formula, satisfied in \( G \), such that all tuples satisfying \( \varphi(\bar{x}) \) generate \( G \). Then \( G \) has a computable \( d^{-}\Sigma₂ \) Scott sentence.

**Proof.** Let \( G = \langle \bar{a} \rangle \), where \( \bar{a} \) satisfies the formula \( \varphi(\bar{x}) \).

Ho’s Scott sentence is the conjunction of the following:

1. the computable Π₂ sentence saying \( (\forall \bar{x})[\varphi(\bar{x}) \rightarrow (\forall y) \bigvee_w w(\bar{x}) = y] \),

2. the computable Σ₂ sentence saying \( (\exists \bar{x})[\varphi(\bar{x}) \land \langle \bar{x} \rangle \cong \langle \bar{a} \rangle] \).

We automatically have the following non-effective analogue of Ho’s result.

**Lemma 4.4.** Suppose \( G \) is a finitely generated group, and there is a Σ₂ formula \( \varphi(\bar{x}) \), satisfied in \( G \), and such that all tuples satisfying \( \varphi(\bar{x}) \) generate \( G \). Then \( G \) has a \( d^{-}\Sigma₂ \) Scott sentence.

**Alternative proof of Theorem 4.2.** If there is a Π₃ Scott sentence, then by the result of Montalbán, there is a Σ₂ formula defining the orbit of a generating tuple. Then by the analogue of the result of Ho, there is a \( d^{-}\Sigma₂ \) Scott sentence.
4.2 Computable finitely generated groups with a computable $d$-$\Sigma_2$ Scott sentence

In [8], it is observed that every computable group has a computable $\Sigma_3$ Scott sentence. We can prove the following.

**Theorem 4.5.** For a computable finitely generated group $G$, the following are equivalent:

1. there is a computable $\Pi_3$ Scott sentence,
2. there is a computable $d$-$\Sigma_2$ Scott sentence,
3. for some generating tuple, the orbit is defined by a computable $\Pi_1$ formula,
4. for each generating tuple, the orbit is defined by a computable $\Pi_1$ formula.

**Proof.** We show that $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2)$. For $(2) \Rightarrow (1)$, we note that a computable $d$-$\Sigma_2$ Scott sentence may be regarded as a computable $\Pi_3$ Scott sentence. For $(1) \Rightarrow (4)$, if $G$ has a computable $\Pi_3$ Scott sentence, and $\bar{a}$ is a generating tuple, then by Theorem 2.6 the orbit of $\bar{a}$ is defined by a computable $\Sigma_2$ formula $\psi(\bar{x}) = \bigvee_{i \in W} (\exists \bar{u}_i) \varphi_i(\bar{x}, \bar{u}_i)$, where each $\varphi_i$ is computable $\Pi_1$, and $W$ is some c.e. set. For some $i \in W$ and some $\bar{b}$, $G \models \varphi_i(\bar{a}, \bar{b})$. For some tuple of words $\bar{w}$, $G \models \bar{w}(\bar{a}) = \bar{b}$. Then the orbit of $\bar{a}$ is defined by the computable $\Pi_1$ formula $\psi_i(\bar{x}, \bar{w}(\bar{x}))$. Clearly, $(4) \Rightarrow (3)$. For $(3) \Rightarrow (2)$, let $\bar{a}$ be a generating tuple whose orbit is defined by a computable $\Pi_1$ formula. We may regard this as a computable $\Sigma_2$ formula. By the result of Ho, $G$ has a computable $d$-$\Sigma_2$ Scott sentence. \[\qed\]

4.3 Example of Harrison-Trainor and Ho

Ho and Harrison-Trainor [4] gave the definition below. They considered not just finitely generated groups, but more general finitely generated structures. We consider only groups.

**Definition 2** (Harrison-Trainor-Ho). A finitely generated group $G$ is self-reflective if there is a generating tuple $\bar{a}$ and a tuple $\bar{b}$ generating a proper subgroup $H$, such that

$$(G, \bar{a}) \cong (H, \bar{b})$$

and every existential formula true of $\bar{b}$ in $G$ is true of $\bar{b}$ in $H$. 

Proposition 4.6. Let $G$ be a finitely generated group. Then $G$ is self-reflective iff there is a generating tuple $\bar{a}$ whose orbit is not defined by a $\Pi_1$ formula.

Proof. $\Rightarrow$ Suppose that $G$ is self-reflective, witnessed by $\bar{a}$ and $\bar{b}$, where $\bar{a}$ generates $G$, $\bar{b}$ generates a proper subgroup $H$, and all existential formulas true of $\bar{b}$ in $G$ are also true of $\bar{b}$ in $H$. Since $(G, \bar{a}) \cong (H, \bar{b})$, these existential formulas are also true of $\bar{a}$ in $G$. Therefore, the universal formulas true of $\bar{a}$ in $G$ are also true of $\bar{b}$ in $G$. This means that any $\Pi_1$ formula true of $\bar{a}$ is also true of $\bar{b}$. Then the orbit of $\bar{a}$ is not defined by a $\Pi_1$ formula.

$\Leftarrow$ Suppose that $G$ is generated by a tuple $\bar{a}$ whose orbit is not defined by a $\Pi_1$ formula. Let $\varphi(\bar{x})$ be the $\Pi_1$ formula obtained as the conjunction of the universal formulas true of $\bar{a}$. There must be some $\bar{b}$ satisfying $\varphi(\bar{x})$ and not generating $G$. Let $H$ be the subgroup generated by $\bar{b}$. Then $(G, \bar{a}) \cong (H, \bar{b})$, and all existential formulas true of $\bar{b}$ in $G$ are true of $\bar{a}$ in $G$ and hence true of $\bar{b}$ in $H$. This means that $G$ is self-reflective. \qed

Harrison-Trainor and Ho constructed a computable finitely generated group $G$ that is self-reflective. They also showed that for any such group, the index set is $m$-complete $\Sigma^0_3$. In this way, they arrived at the fact that their group $G$ has no computable $d$-$\Sigma_2$ Scott sentence. Relativizing, they got the fact that for any set $X$, the set of $X$-computable indices for copies of $G$ is $m$-complete $\Sigma^0_3$ relative to $X$, so there is no $X$-computable $d$-$\Sigma_2$ Scott sentence. It follows that the group $G$ has no $d$-$\Sigma_2$ Scott sentence.

5 Problems

1. Is there a finitely presented group that is self-reflective?

2. Is there a precise sense in which most finitely generated groups are not self-reflective? Can we say in terms of limiting density that the typical finitely generated group has a $d$-$\Sigma_2$ Scott sentence?

3. Is there a computable finitely generated group with a $d$-$\Sigma_2$ Scott sentence but no computable $d$-$\Sigma_2$ Scott sentence?

4. Give necessary and sufficient conditions for a computable structure $A$ to have a computable $\Pi_{\alpha+1}$ Scott sentence.
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