IMPULSES IN DRIVING SEMIGROUPS OF NONAUTONOMOUS
DYNAMICAL SYSTEMS: APPLICATION TO CASCADE
SYSTEMS

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Abstract. In this paper we investigate the long time behavior of a nonauto-
nomous dynamical system (cocycle) when its driving semigroup is subjected to
impulses. We provide conditions to ensure the existence of global attractors for
the associated impulsive skew-product semigroups, uniform attractors for the
coupled impulsive cocycle and pullback attractors for the associated evolution
processes. Finally, we illustrate the theory with an application to cascade
systems.

1. Introduction. The theory of impulsive differential equations began in the 1970’s
with the works of the Russian physicist V. F. Rožko in [23, 24, 25], and it has been
proven to be very efficient in describing evolutions of many real world phenomena.
The reader may consult, for instance, [1, 2, 11, 16, 18]. Recently, the theory of

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impulsive dynamical systems has regained the attention of some authors, with several papers concerning the long-time behavior of solutions, specially the existence of attractors for such systems (see, for instance, [3, 4, 5, 8, 12, 20, 22]).

In the nonautonomous framework, the theory developed so far is dedicated to the study of impulsive nonautonomous dynamical systems (see, for instance, [3, 4]), and it is pursued in the following manner: given metric spaces $Z$ and $\Sigma$, and a continuous function $\varphi : \mathbb{R}^+ \times \Sigma \times Z \to Z$, called cocycle, with driving semigroup $\theta$, satisfying some additional conditions (see Section 2), one introduces an impulse set $M$ in $Z$ such that when the orbit $t \mapsto \varphi(t, \sigma, z)$ of $z$ at the fiber $\sigma$ intercepts $M$ at a time $t_0$, it receives an impulse, through the action of a continuous function $I$, and the process is reset, starting now at $I(\varphi(t_0, \sigma, z))$ at the fiber $\theta t_0 \sigma$. In this setting, there are results concerning the existence of the so called cocycle attractor for $\varphi$ and of the uniform attractor for $\varphi$, as well as the relationship between them.

In this paper, we pursue a different approach: the impulse now is considered on $\Sigma$ rather than on $Z$. The motivation for this change is that, when we deal with a model that is governed by a separate equation (in this case, the cocycle $\varphi$ is governed by the driving semigroup $\theta$), it is useful to be able to make corrections on the governing equation (impulses on $\theta$) in order to control the behavior of the solutions of the model (the cocycle $\varphi$). We point out that, in this paper, the impulses depend on the state (and not on time). That means that there is a set in the phase space $\Sigma$ which is responsible by the discontinuities of the solutions of $\theta$, which provides a different approach from the theory presented in [19], where the author studies differential equations $x' = f(t, x)$, where the function $f$ is discontinuous. Also, solutions in Filippov’s framework have to be absolutely continuous, which is another relevant detail that makes it different from the theory presented here, where the solutions can be (and usually are) discontinuous. Also, here we allow infinitely many distinct driving vector fields, which differs from the theory presented in [17].

We divide our paper as follows: Section 2 contains a brief summary of the already solidified theories of dynamical systems in the continuous framework (evolution processes, semigroups, cocycles, and their respective attractors), as well as the theory of impulsive autonomous dynamical systems and impulsive semigroups, with a collection of known results concerning the existence of global attractors for the impulsive case. The single new result in this section is Proposition 5, which will be used in the application, and provides sufficient conditions to obtain a global attractor for an impulsive semigroup.

In Section 3, the main theoretical section of this paper, we present the construction and definition of the key object of study, the coupled impulsive cocycle (see Definition 3.1), denoted by $\varphi_c$. Differently from the case where the impulses occur on the phase space $Z$ rather than only on the driving space $\Sigma$, through Proposition 6 we are able to show that the resulting impulsive trajectories $[0, \infty) \times Z \ni (t, z) \rightarrow \varphi_c(t, \sigma)z \in Z$ are continuous for each fixed $\sigma \in \Sigma$. This feature is expected since, roughly speaking, the impulse here only changes the direction of the semiflow without breaking it. We also present conditions to ensure the existence of the uniform attractor for $\varphi_c$ (Theorem 3.2), we relate the solutions in the global attractor of the impulsive driving semigroup $\tilde{\theta}$ with evolution processes $S_{\eta}$ in $Z$, and we present a result concerning the existence of a pullback attractor for $S_{\eta}$ (Theorem 3.3). To end this section, we present a simple application of this theory with a one-dimensional nonautonomous ordinary differential equation.
Lastly, we exhibit an application of this theory to cascade systems in Section 4. Such systems represent the motivation to our approach: it consists of two equations, where the first depends on the second, but the second stands on its own. Here, the behavior of the second equation governs the behavior of the first, and applying an impulsive solely on the second one, we study the long-time behavior of the solutions of the first.

2. Preliminaries.

2.1. Continuous framework. This topic concerns the theory of continuous dynamical systems. The reader may consult [13, 14] for more details.

Let \((Z,d)\) be a metric space and \(C(Z)\) be the set of all continuous maps from \(Z\) into itself. A family \(S = \{S(t,s) : t \geq s\} \subset C(Z)\), with \(t,s \in \mathbb{R}\), is called an evolution process if it satisfies:

(i) \(S(t,t)z = z\) for all \(t \in \mathbb{R}\) and \(z \in Z\);
(ii) \(S(t,r)S(r,s) = S(t,s)\) for all \(t \geq r \geq s\), \(t,r,s \in \mathbb{R}\).

A family \(\hat{A} = \{A_t\}_{t \in \mathbb{R}}\) of subsets of \(Z\) is said to be \(S\)-invariant if

\[ S(t,s)A_t = A_t \quad \text{for all} \quad t \geq s. \]

We say that a family \(\hat{A}\) \(S\)-pullback attracts \(B \subset Z\) if

\[ \lim_{s \to -\infty} d_H(S(t,s)B, A_t) = 0 \quad \text{for each} \quad t \in \mathbb{R}, \]

where \(d_H(C,D)\) denotes the Hausdorff semidistance between two nonempty subsets of \(C\) and \(D\) of \(Z\), that is, \(d_H(C,D) = \sup_{x \in C} \inf_{y \in D} d(x,y)\). A family \(\hat{A}\) is a pullback attractor for \(S\) if \(A_t\) is compact in \(Z\) for each \(t \in \mathbb{R}\), \(\hat{A}\) is \(S\)-invariant, \(\hat{A}\) \(S\)-pullback attracts each \(B \subset Z\) bounded, and \(\bigcup_{t \in \mathbb{R}} A_t\) is bounded. Clearly, when a pullback attractor for \(S\) exists, it is unique.

An evolution process is said to be:

(a) pullback asymptotically compact if given \(t \in \mathbb{R}\) and sequences \(\{s_n\} \subset \mathbb{R}\) and \(\{z_n\} \subset Z\), with \(s_n \to -\infty\) and \(\{z_n\}\) bounded in \(Z\), then the sequence \(\{S(t,s_n)z_n\}\) has a convergent subsequence in \(Z\);
(b) pullback dissipative if there exists a bounded set \(B_0 \subset Z\) such that given \(B \subset Z\) bounded and \(t \in \mathbb{R}\), then there exists \(s_0 = s_0(B,t) \leq t\) such that

\[ S(t,s)B \subset B_0 \quad \text{for all} \quad s \leq s_0. \]

The set \(B_0\) is called a pullback dissipative set for \(S\).

Given \(B \subset Z\) and \(t \in \mathbb{R}\), its pullback \(\omega\)-limit at \(t\) is given by

\[ \omega_t(B) = \bigcap_{\tau \leq t} \bigcup_{s \leq \tau} S(t,s)B. \]

The pullback \(\omega\)-limit of \(B\) is the family \(\tilde{\omega}(B) = \{\omega_t(B)\}_{t \in \mathbb{R}}\), and has a simple characterization in terms of sequences: given \(B \subset Z\) and \(t \in \mathbb{R}\), we have \(z \in \omega_t(B)\) if and only if there exists \(s_n \to -\infty\) and \(\{z_n\} \subset B\) with \(S(t,s_n)z_n \to z\).

The most used result to obtain pullback attractors is the following:

**Proposition 1.** An evolution process \(S\) is pullback asymptotically compact and pullback dissipative if and only if it has a pullback attractor. When the pullback attractor \(\hat{A}\) for \(S\) exists, it is given by \(A_t = \omega_t(B_0)\) for each \(t \in \mathbb{R}\), where \(B_0\) is a pullback dissipative set for \(S\).
When \( S(t, s) = T(t - s) \) for all \( t \geq s \), the family \( T = \{ T(t) : t \geq 0 \} \) is called a semigroup, and satisfies \( T(0)z = z \) for all \( z \in Z \) and \( T(t + s) = T(t)T(s) \) for all \( t, s \geq 0 \). In the case of semigroups we have the following definitions:

(a) \( A \subset Z \) is a \( T \)-invariant set if \( T(t)A = A \) for all \( t \geq 0 \);

(b) \( A \subset Z \) \( T \)-attracts \( B \subset Z \) if \( d_H(T(t)B, A) \to 0 \) as \( t \to \infty \);

(c) given \( B \subset Z \), its \( \omega \)-limit set is defined by \( \omega(B) = \bigcap_{t \geq 0} \bigcup_{\tau \geq t} T(\tau)B \);

(d) \( A \subset Z \) is a \emph{global attractor} for \( T \) if it is compact, \( T \)-invariant and \( T \)-attracts each bounded subset of \( Z \);

(e) a semigroup \( T \) is \emph{asymptotically compact} if given sequences \( t_n \to \infty \) and \( \{z_n\} \) bounded in \( Z \), then the sequence \( \{T(t_n)z_n\} \) has a convergent subsequence;

(f) a semigroup \( T \) is \emph{dissipative} if there exists a bounded subset \( B_0 \) of \( Z \) with the property that given \( B \subset Z \) bounded there exists \( t_0 = t_0(B) \geq 0 \) such that \( T(t)B \subset B_0 \) for \( t \geq t_0 \). A set \( B_0 \) satisfying this condition is a \emph{dissipative set} for \( T \).

A global attractor for a semigroup, when it exists, is unique. Also, the analogous result to Proposition 1 holds for semigroups, that is, a semigroup \( T \) has a global attractor if and only if it is asymptotically compact and dissipative, and when it exists, it is the \( \omega \)-limit set of a dissipative set for \( T \).

To end this section we present the concept of cocycles and skew-product semigroups. We consider a semigroup \( \theta \) in a metric space \((\Sigma, d_\Sigma)\) and to simplify the notation, we set \( \theta_t = \theta(t) \) for \( t \geq 0 \).

We consider for each pair \((t, \sigma) \in \mathbb{R}_+ \times \Sigma\) a map \( \varphi(t, \sigma) \in C(Z) \), satisfying:

(i) \( \varphi(0, \sigma)z = z \) for all \( \sigma \in \Sigma \) and \( z \in Z \);

(ii) \( \varphi(t + s, \sigma) = \varphi(t, \theta_t \sigma)\varphi(s, \sigma) \) for all \( t, s \geq 0 \) and \( \sigma \in \Sigma \).

Condition (ii) is called \emph{cocycle property}, and the map \( \varphi: \mathbb{R}_+ \times \Sigma \times Z \to Z \) given by \( \varphi(t, \sigma, z) = \varphi(t, \sigma)z \) is called a cocycle. The semigroup \( \theta \) in this context is called the \emph{driving semigroup} of \( \varphi \).

A cocycle \( \varphi \) is \emph{uniformly asymptotically compact} if given sequences \( \{z_n\} \subset Z \), \( \{\sigma_n\} \subset \Sigma \), both bounded, and \( t_n \to \infty \), then the sequence \( \{\varphi(t_n, \sigma_n)z_n\} \) has a convergent subsequence, and \emph{uniformly dissipative} if there exists a bounded set \( B_0 \subset Z \), called a \emph{uniformly dissipative set} for \( \varphi \), such that given \( B \subset Z \) and \( \Gamma \subset \Sigma \), both bounded, then there exists \( t_0 = t_0(B, \Gamma) \geq 0 \) such that

\[ \varphi(t, \sigma)B \subset B_0 \quad \text{for all} \quad t \geq t_0 \quad \text{and} \quad \sigma \in \Gamma. \]

**Remark 1.** We point out that, in every setting (evolution processes, semigroups or cocycles) \emph{dissipativity implies asymptotical compactness} when the space is a finite dimensional vector space.

A \emph{uniform attractor} for \( \varphi \) is a compact and minimal closed set \( \mathcal{A} \subset Z \) such that for each \( B \subset Z \) and \( \Gamma \subset \Sigma \), both bounded, we have

\[ \lim_{t \to \infty} \sup_{\sigma \in \Gamma} d_H(\varphi(t, \sigma)B, \mathcal{A}) = 0, \]

where, \emph{minimality} means that if \( C \) is a closed set verifying (1) then \( \mathcal{A} \subset C \).

This definition of \emph{uniform attractor} is analogous to [9, Definition 3.3], and differs slightly from [3, Definition 5.3] (for the impulsive case) and [13, Definitions 16.7 and 16.8] (for the continuous case), since we are not assuming that \( \Sigma \) is compact and \( \theta \)-invariant. When that is the case, then \( \Xi = \Sigma \) is the global attractor for \( \theta \) and these definitions coincide.
Using \( \varphi \), we can construct a semigroup \( \Pi \), called the *skew-product semigroup* of \( \varphi \), in \( Z \times \Sigma \) by
\[
\Pi(t)(z, \sigma) = (\varphi(t, \sigma)z, \theta(t, \sigma)) \quad \text{for } t \geq 0 \text{ and } (z, \sigma) \in Z \times \Sigma,
\]
where in \( Z \times \Sigma \) we consider the product metric.

2.2. **Impulsive framework.** We begin with a semigroup \( T \) on \( Z \) satisfying:
\[
\mathbb{R}_+ \times Z \ni (t, z) \mapsto T(t)z \in Z \text{ is a continuous map.} \tag{2}
\]
An *impulsive dynamical system* \((Z, T, M, I)\) consists of the semigroup \( T \) on \( Z \), a nonempty closed subset \( M \subset Z \) such that for every \( z \in M \) there exists \( \epsilon = \epsilon(z) > 0 \) satisfying
\[
\bigcup_{\ell \in (0, \epsilon)} \{T(t)z\} \cap M = \emptyset, \tag{3}
\]
and a continuous function \( I : M \to Z \). We call \( M \) the *impulsive set* and \( I \) the *impulse function* of \((Z, T, M, I)\).

**Remark 2.** Condition (3) is based on [15], and differs from the usual condition presented for impulsive dynamical systems (cf. [5, Definition 2.7] or [10, Definition 2.6]). However, the results of [10] remain true when considering the definition of impulsive dynamical systems presented in this paper.

Since the semigroup \( T \) satisfies (2), condition (3) ensures that \( M \) has empty interior in \( Z \).

Given an impulsive dynamical system \((Z, T, M, I)\) we define
\[
M_T^+(z) = \left( \bigcup_{t \geq 0} T(t)z \right) \cap M \quad \text{for each } z \in Z.
\]
Using either [8, Proposition 1.4] or [21], we obtain if \( M_T^+(z) \neq \emptyset \) then there exists a unique \( s > 0 \) such that \( T(s)z \in M \) and \( T(t)z \notin M \) for \( 0 < t < s \), and we are able to define the function \( \phi_T : Z \to (0, \infty) \) by
\[
\phi_T(z) = \begin{cases} 
  s, & \text{if } T(s)z \in M \text{ and } T(t)z \notin M \text{ for } 0 < t < s, \\
  \infty, & \text{if } M_T^+(z) = \emptyset.
\end{cases}
\]
If \( M_T^+(z) \neq \emptyset \), then the value \( \phi_T(z) \) represents the smallest positive time such that the positive semitrajectory of \( z \) meets \( M \), and thus \( \phi_T \) will be called the *impact time map* of \( T \).

The *positive impulsive semitrajectory* of \( z \in Z \) under \((Z, T, M, I)\) is a map \( \tilde{T}(\cdot)z \) defined on an interval \([0, \ell_2)\), \( 0 < \ell_2 \leq \infty \), with values in \( Z \) given by the following inductive rule:

**Step 1:** if \( M_T^+(z) = \emptyset \), then \( \phi_T(z) = \infty \) and we define \( \tilde{T}(t)z = T(t)z \) for all \( t \geq 0 \).

In this first case, the process ends here. However, if \( M_T^+(z) \neq \emptyset \), then we denote \( z_0^+ = z, s_0 = t_0 = \phi_T(z_0^+) < \infty, z_1^+ = I(T(s_0)z_0^+) \) and define \( \tilde{T}(\cdot)z \) on \([0, t_0]\) by
\[
\tilde{T}(t)z = \begin{cases} 
  T(t)z_0^+, & \text{if } 0 \leq t < t_0, \\
  z_1^+, & \text{if } t = t_0,
\end{cases}
\]
and in this second case we move to Step 2.

**Step 2:** If \( M_T^+(z_1^+) = \emptyset \), then \( \phi_T(z_1^+) = \infty \) and we define \( \tilde{T}(t)z = T(t - t_0)z_1^+ \) for all \( t \geq t_0 \). In this case, the process ends here. However, if \( M_T^+(z_1^+) \neq \emptyset \), then we
denote \( s_1 = \phi_T(z^+_n) < \infty \), \( t_1 = s_1 + t_0 \), \( z^+_2 = I(T(s_1)z^+_1) \) and define \( \tilde{T}(\cdot) \) on \([t_0, t_1]\) by
\[
\tilde{T}(t) = \begin{cases} 
T(t - t_0)z^+_1, & \text{if } t_0 \leq t < t_1, \\
z^+_2, & \text{if } t = t_1,
\end{cases}
\]
and in this case, we move to the next step.

**Step n:** Assume that \( \tilde{T}(\cdot) \) is defined on the interval \([t_{n-1}, t_n]\) and that \( \tilde{T}(t_n) = z^+_{n+1} \) (setting \( t_{-1} = 0 \)) where \( s_n = \phi_T(z^+_n) \) and \( t_n = s_n + t_{n-1} \). If \( M^+_T(z^+_{n+1}) = \emptyset \), then \( \phi_T(z^+_{n+1}) = \infty \) and we define \( \tilde{T}(t) = T(t - t_n)z^+_{n+1} \) for \( t \geq t_n \). In this case, the process ends here. However, if \( M^+_T(z^+_{n+1}) \neq \emptyset \), then we denote \( s_{n+1} = \phi_T(z^+_{n+1}) < \infty \), \( t_{n+1} = s_{n+1} + t_n \), \( z^+_{n+2} = I(T(s_{n+1})z^+_{n+1}) \) and define \( \tilde{T}(\cdot) \) on \([t_n, t_{n+1}]\) by
\[
\tilde{T}(t) = \begin{cases} 
T(t - t_n)z^+_{n+1}, & \text{if } t_n \leq t < t_{n+1}, \\
z^+_{n+2}, & \text{if } t = t_{n+1}.
\end{cases}
\]

This process can either end after a finite number of steps, namely if \( M^+_T(z^+_{n+1}) = \emptyset \) for some \( n \in \mathbb{N} \), or continue indefinitely, when \( M^+_T(z^+_{n+1}) \neq \emptyset \) for all \( n \in \mathbb{N} \). In the former case \( \tilde{T}(\cdot) \) is defined on \( \mathbb{R}_+ \) (and \( \ell_z = \infty \)), but in the latter it is defined on the interval \([0, \ell_z]\), where \( \ell_z = \lim_{n \to \infty} t_n = \sum_{i=0}^{\infty} s_i \) can be either finite or infinity.

From [6, Proposition 2.1] or [15, Lemma 2.4], it follows for each \( z \in Z \) that:

(i) \( \tilde{T}(0)z = z \);

(ii) \( \tilde{T}(t+s)z = \tilde{T}(t)\tilde{T}(s)z \) for \( t, s \in [0, \ell_z] \) such that \( t + s \in [0, \ell_z] \).

Since we are concerned about long time behavior of solutions, we will assume that
\[
\ell_z = \infty \quad \text{for all } z \in Z, \tag{4}
\]
and in this case, the impulsive dynamical system \((Z, T, M, I)\) will be called an *impulsive semigroup*, and will be denoted simply by \( \tilde{T} \). The definitions of invariance, attraction, \( \omega \)-limit set, asymptotical compactness, dissipativity and dissipative set for an impulsive semigroup \( \tilde{T} \) are analogous to the ones for a semigroup \( T \), just replacing \( T \) by \( \tilde{T} \). We point out that the definition of global attractor for \( \tilde{T} \) is *not the same*.

**Definition 2.1.** Given an impulsive semigroup \( \tilde{T} \), we say that \( A \subset Z \) is a *semi attractor* for \( \tilde{T} \) if \( A \) is compact and \( \tilde{T} \)-attracts all bounded subsets of \( Z \). In addition, when \( A \setminus M \) is \( \tilde{T} \)-invariant, we say that \( A \) is a *global attractor* for \( \tilde{T} \).

Global attractors may not be (and usually are not) unique. Nevertheless, if \( A_1 \) and \( A_2 \) are global attractors for \( \tilde{T} \), then \( A_1 \setminus M = A_2 \setminus M \). Now, analogously to the continuous case, we obtain the following:

**Proposition 2.** An impulsive semigroup \( \tilde{T} \) has a semi attractor if \( \tilde{T} \) is asymptotically compact and dissipative. This semi attractor, when it exists, is given by \( A = \tilde{\omega}(B_0) \), where \( B_0 \) is a dissipative set for \( \tilde{T} \).

The proof of a more general version of this last proposition (in the case of *impulsive evolution processes*) rather than semigroups) can be found in [10], and for that reason we have decided to omit it from this paper.

In [5, Example 3.11] one can see that these two conditions (asymptotic compactness and dissipativity) alone are not enough to prove the existence of a global attractor for \( \tilde{T} \). To deal with the problem of \( \tilde{T} \)-invariance of \( A \setminus M \), as in [5, Lemma 4.4], we have the following:
Proposition 3. Let \( \tilde{T} \) be an impulsive semigroup with a semi attractor \( A \). If \( I(M) \cap M = \emptyset \) and there exists \( s > 0 \) such that \( A \setminus M \subset \tilde{T}(s)(A \setminus M) \), then \( A \setminus M \) is \( \tilde{T} \)-invariant.

To obtain a global attractor, as in [5], we require the following conditions:

(T) if \( z \in M, t > 0 \) and \( \{z_n\} \subset Z \) is such that \( \{z_n\} \) is convergent and \( T(t)z_n \to z \), then there exists a sequence \( \{\alpha_n\} \subset \mathbb{R} \) with \( \alpha_n \to 0 \) such that \( t + \alpha_n \geq 0 \) and

up to a subsequence, \( T(t + \alpha_n)z_n \in M \);

(H) there exists \( \xi > 0 \) such that \( \phi_T(y) \geq 2\xi \) for all \( y \in I(M) \).

Note that, in particular, (H) implies (4). Following [5, Theorem 4.7], if \( \tilde{T} \) has a semi attractor \( A, I(M) \cap M = \emptyset \) and (T) and (H) hold, then \( A \setminus M \subset \tilde{T}(\xi)(A \setminus M) \), which implies that \( A \) is, in fact, a global attractor for \( \tilde{T} \). These considerations are summarized in:

Proposition 4. Let \( \tilde{T} \) be an asymptotically compact and dissipative impulsive semigroup such that \( I(M) \cap M = \emptyset \) and both (T) and (H) hold. Then the semi attractor \( A = \tilde{\omega}(B_0) \), where \( B_0 \) is a dissipative set for \( \tilde{T} \), is also a global attractor for \( \tilde{T} \).

Remark 3. We note that hypothesis (H) is required to ensure that the last condition of Proposition 3 is satisfied, which concludes the invariance of the semi attractor in Proposition 4. Condition (H) holds, for instance, when \( I(M) \cap M = \emptyset \) and \( I(M) \) is compact (see [8, Remark 1.6]).

We will present a result regarding the dissipativity of an impulsive semigroup \( \tilde{T} \), which can be useful in applications.

Proposition 5. Let \( \tilde{T} \) be an impulsive semigroup in a Banach space \( Z \) with \( I(M) \) bounded. If there exists \( h > 0 \) such that for each \( B \subset Z \) bounded, there exists a bounded function \( h_B : [0, \infty) \to [0, \infty) \) with \( h_B(t) \to 0 \) as \( t \to \infty \) and

\[
\sup_{z \in B} \|T(t)z\|^2 \leq h_B(t) + k \quad \text{for all } t \geq 0, \tag{5}
\]

then \( \tilde{T} \) is dissipative.

Proof. Define \( K = \{T(t)z : z \in I(M) \} \) and \( 0 \leq t < \phi_T(z) \}, \) which is bounded since \( I(M) \) is bounded and (5) holds.

Let \( t \geq 0 \) and \( z \in K \), that is, \( z = T(s)x \) for some \( x \in I(M) \) and \( 0 \leq s < \phi_T(x) \). Note that \( \tilde{T}(t)z = \tilde{T}(t)T(s)x = \tilde{T}(t)\tilde{T}(s)x = \tilde{T}(t+s)x \). If \( t+s < \phi_T(x) \) then we obtain \( \tilde{T}(t)z = \tilde{T}(t+s)x = \tilde{T}(t+s)x \in K \). If \( t+s = \phi_T(x) \), then \( \tilde{T}(t)z = \tilde{T}(t+s)x = \tilde{T}(r)x^+ \in I(M) \subset K \). Lastly, if \( t+s > \phi_T(x) \) then \( \tilde{T}(t)z = \tilde{T}(r)x^+ \), for some \( 0 \leq r \leq \phi_T(x^+) \) and \( x^+ \in I(M) \). In this last case, if \( 0 \leq r < \phi_T(x^+) \) then \( \tilde{T}(t)z = \tilde{T}(r)x^+ + T(r)x^+ \in K \), and if \( r = \phi_T(x^+) \) then \( \tilde{T}(t)z = \tilde{T}(r)x^+ = I(T(r)x^+) \in I(M) \subset K \). Thus, in any case, \( \tilde{T}(t)K \subset K \).

Define \( B_0 = \{z \in Z : \|z\|^2 \leq 2k\} \cup K \), take \( B \subset Z \) bounded and choose \( t_0 = t_0(B) \geq 0 \) such that \( h_B(t) < k \) for \( t \geq t_0 \). Set

\( B_1 = \{z \in B : \phi_T(z) \leq t_0\} \) and \( B_2 = \{z \in B : \phi_T(z) > t_0\} \).

For \( t \geq t_0 \) and \( z \in B_1 \), we have \( \phi_T(z) < t \) and thus

\( \tilde{T}(t)z = \tilde{T}(t - \phi_T(z))\sum_{\xi \in I(M) \cap K} (\tilde{T}(\phi_T(z)))z \in K \).
On the other hand, if \( t \geq t_0 \), \( z \in B_2 \) and \( \phi_T(z) \leq t \) then using the same reasoning as above we conclude that \( \tilde{T}(t)z \in K \). If \( t \leq t < \phi_T(z) \) then \( \tilde{T}(t)z = T(t)z \in \{ z \in Z : \|z\|^2 \leq 2k \} \) from the choice of \( t_0 \). Therefore, \( \tilde{T}(t)B \subset B_0 \) for all \( t \geq t_0 \).

\[ \square \]

3. **Coupled impulsive cocycles.** We consider a continuous cocycle, that is, a continuous map \( \varphi : \mathbb{R}_+ \times \Sigma \times Z \rightarrow Z \) and its driving semigroup \( \theta \) satisfies (2), and we construct the skew-product semigroup \( \Pi \) of \( \varphi \) in \( Z \times \Sigma \). With \( \theta \) we construct an impulsive dynamical system \( (\Sigma, \theta, M, I) \), \( M \subset \Sigma \) is the impulsive set and \( I : M \rightarrow \Sigma \) is the impulse function, and we assume that (4) holds, which gives us the impulsive semigroup \( \tilde{\theta} \).

Define \( \tilde{\pi} = \pi \times \pi \), \( I : \tilde{\pi}(x, \sigma) = (x, I(\sigma)) \). Note that given \( \sigma \in M \), since \( M \) is an impulsive set for \( \theta \), we can choose \( \epsilon = \epsilon(\sigma) > 0 \) such that

\[ \left( \bigcup_{t \in (0, \epsilon)} \theta_t(\sigma) \right) \cap M = \emptyset. \]

It is simple to see that \( (\bigcup_{t \in (0, \epsilon)} \Pi(t)(x, \sigma)) \cap \tilde{\pi} = \emptyset \), and hence \( (Z \times \Sigma, \Pi, M, I) \) is an impulsive dynamical system in \( Z \times \Sigma \) satisfying (4). Thus \( \tilde{\Pi} \) is an impulsive semigroup, with impact time map \( \phi_{\Pi}(z, \sigma) \) depending only on \( \sigma \).

**Definition 3.1.** For each \( (t, \sigma) \in \mathbb{R}_+ \times \Sigma \), we define the map \( \varphi_c(t, \sigma) : Z \rightarrow Z \) by

\[ \varphi_c(t, \sigma)z = \pi_Z(\tilde{\Pi}(t)(z, \sigma)), \quad z \in Z, \]

where \( \pi_Z : Z \times \Sigma \rightarrow Z \) is the projection in the first coordinate, that is, \( \varphi_c \) is such that \( \tilde{\Pi}(t)(z, \sigma) = (\varphi_c(t, \sigma)z, \theta_t(\sigma)) \) for all \( t \geq 0 \), \( \sigma \in \Sigma \) and \( z \in Z \). This map can also be seen as a map \( \varphi_c : \mathbb{R}_+ \times \Sigma \times Z \rightarrow Z \) and it is called coupled impulsive cocycle.

We can also give a description of \( \varphi_c \) in terms of \( \varphi \), as follows: given \( z \in Z \) and \( \sigma \in \Sigma \), set \( s_0 = t_0 = 0, \sigma^+_0 = \sigma \) and \( z_0 = z \). If \( s_1 = \phi_{\theta}(\sigma) = \infty \), then \( \varphi_c(t, \sigma)z = \varphi(t, \sigma)z = \varphi(t - t_0, \sigma^+_0)z_0 \) for all \( t \geq t_0 = 0 \). If \( s_1 < \infty \), setting \( t_1 = t_0 + s_1 \), then we obtain

\[ \varphi_c(t, \sigma)z = \varphi(t - t_0, \sigma^+_0)z_0 \quad \text{for} \quad t_0 \leq t \leq t_1. \]

Now for \( \sigma^+_1 = \tilde{\theta}_{t_1} \sigma^+_0 \) and \( z_1 = \varphi(t_1 - t_0, \sigma^+_0)z_0 \), if \( s_2 = \phi_{\theta}(\sigma^+_1) = \infty \) then \( \varphi_c(t, \sigma)z = \varphi(t - t_1, \sigma^+_1)z_1 \) for all \( t \geq t_1 \). But if \( s_2 < \infty \), setting \( t_2 = t_1 + s_2 \), then

\[ \varphi_c(t, \sigma)z = \varphi(t - t_1, \sigma^+_1)z_1 \quad \text{for} \quad t_1 \leq t \leq t_2. \]

Inductively, this process can end after a finite number of steps, if \( \phi_{\theta}(\sigma^n_+) = \infty \) for some \( n \in \mathbb{N} \), in which case \( \varphi_c(t, \sigma)z = \varphi(t - t_n, \sigma^n_+)z_n \) for all \( t \geq t_n \), or can continue indefinitely, generating sequences \( \{ \sigma^n_+ \} \subset I(M) \), \( \{ z_n \} \subset Z \) and \( t_n \nearrow \infty \) (since \( \theta \) satisfies (4)) such that

\[ \varphi_c(t, \sigma)z = \varphi(t - t_n, \sigma^n_+)z_n \quad \text{for} \quad t_n \leq t \leq t_{n+1}. \]

Since \( \tilde{\Pi} \) is an impulsive semigroup, it is clear that the coupled impulsive cocycle \( \varphi_c \) satisfies:

(i) \( \varphi_c(0, \sigma)z = z \) for all \( z \in Z \) and \( \sigma \in \Sigma \);

(ii) \( \varphi_c(t + s, \sigma) = \varphi_c(t, \theta_s(\sigma)) \varphi_c(s, \sigma) \), for all \( t, s \geq 0 \) and \( \sigma \in \Sigma \).

In other words, \( \varphi_c \) is a cocycle, and its driving semigroup is the impulsive semigroup \( \tilde{\theta} \). The definition of uniform attractor for \( \varphi_c \) is analogous as for a cocycle \( \varphi \), replacing \( \varphi \) by \( \varphi_c \).
Impulsive semigroups do not satisfy condition (2), but since the impulses in this case occur only on the second variable of $\theta$, one can expect to obtain continuity for $\varphi_c$ for each fixed $\sigma \in \Sigma$.

**Proposition 6.** For each fixed $\sigma \in \Sigma$, the map $\mathbb{R}_+ \times Z \ni (t, z) \mapsto \varphi_c(t, \sigma)z \in Z$ is continuous.

**Proof.** Fix $\sigma \in \Sigma$ and set $s = \phi_0(\sigma)$. If $s = \infty$ then $\varphi_c(t, \sigma)z = \varphi(t, \sigma)z$ for all $t \geq 0$ and $z \in Z$. The continuity follows from the continuity of $\varphi$.

Now, assume that $s < \infty$ and consider sequences $t_n \to t$ and $z_n \to z$.

**Case 1:** $t < s$.
Since $t_n \to t$, we may assume that $t_n < s$ for all $n \in \mathbb{N}$. Hence, using the continuity properties of $\varphi$, we have $\varphi_c(t_n, \sigma)z_n = \varphi(t_n, \sigma)z_n \to \varphi(t, \sigma)z = \varphi_c(t, \sigma)z$.

**Case 2:** $t = s$.
We have $\alpha_n = t_n - s \to 0$ and we can assume that $|\alpha_n| < \phi_0(\tilde{\theta}_s \sigma)$ for all $n \in \mathbb{N}$. It is enough to consider the cases when $\alpha_n < 0$ for all $n \in \mathbb{N}$ or $\alpha_n \geq 0$ for all $n \in \mathbb{N}$.

If $\alpha_n < 0$ for all $n \in \mathbb{N}$, then $t_n < s$ and $\varphi_c(t_n, \sigma)z_n = \varphi(t_n, \sigma)z_n \to \varphi(s, \sigma)z = \varphi_c(s, \sigma)z$.

Now, if $\alpha_n \geq 0$ for all $n \in \mathbb{N}$, then
$$
\varphi_c(t_n, \sigma)z_n = \varphi_c(\alpha_n + s, \sigma)z_n = \varphi(\alpha_n, \tilde{\theta}_s \sigma)\varphi_c(s, \sigma)z_n = \varphi(\alpha_n, \tilde{\theta}_s \sigma)\varphi(s, \sigma)z_n,
$$
where the last equality follows from the fact that $\alpha_n < \phi_0(\tilde{\theta}_s \sigma)$ and that $\varphi_c(s, \sigma)z_n = \varphi(s, \sigma)z_n$ for all $n \in \mathbb{N}$. Hence, with the continuity of $\varphi$, we conclude that $\varphi_c(t_n, \sigma)z_n \to \varphi(s, \sigma)z = \varphi_c(s, \sigma)z$.

**Case 3:** $t > s$.
This case follows from Cases 1 and 2, with an induction argument on the number of impulses (of the impulsive semitrajectory $\tilde{\theta}$) when going from 0 to $t$.

If $\tilde{\theta}$ satisfies (T), then given $(z, \sigma) \in M$, $t > 0$ and $\{(z_n, \sigma_n)\} \subset Z \times \Sigma$ such that $\{(z_n, \sigma_n)\}$ is convergent and $\Pi(t)(z_n, \sigma_n) \to (z, \sigma)$, we have $\sigma \in M$ and $\theta_t \sigma_n \to \sigma$. Thus there exists $\alpha_n \to 0$ with $\theta_{t+\alpha_n} \sigma_n \in M$, up to a subsequence. Hence, along this subsequence, we have
$$
\Pi(t + \alpha_n)(z_n, \sigma_n) = (\varphi(t + \alpha_n, \sigma_n)z_n, \theta_{t+\alpha_n} \sigma_n) \in M,
$$
that is, $\Pi$ satisfies (T). Clearly ($H$) holds for $(Z \times \Sigma, \Pi, M, 1)$, provided it holds for $(\Sigma, \theta, M, I)$. Thus, we have the following result.

**Theorem 3.2.** Assume that $\tilde{\theta}$ is asymptotically compact and dissipative, $I(M) \cap M = \emptyset$, $\theta$ satisfies (T) and ($H$) holds for $(\Sigma, \theta, M, I)$. Suppose also that $\varphi_c$ is uniformly asymptotically compact and uniformly dissipative. Then $\hat{\Pi}$ has a global attractor $\mathbb{A}$ in $Z \times \Sigma$ and $\mathbb{A} = \pi_Z(\hat{\mathbb{A}})$ is the uniform attractor of $\varphi_c$.

**Proof.** If $\Gamma_0$ is a dissipative set for $\tilde{\theta}$ and $B_0$ is a uniformly dissipative set for $\varphi_c$, then $B_0 \times \Gamma_0$ is a dissipative set for $\hat{\Pi}$. Moreover, if $\{(z_n, \sigma_n)\}$ is bounded in $Z \times \Sigma$ and $t_n \to \infty$, then $\varphi_c(t_n, \sigma_n)z_n$ has a convergent subsequence in $Z$ and $\{\tilde{\theta}_n \sigma_n\}$ also has a convergent subsequence in $\Sigma$, so $\{\hat{\Pi}(t_n)(z_n, \sigma_n)\}$ has a convergent subsequence in $Z \times \Sigma$. From Proposition 4 and our previous considerations, $\hat{\Pi}$ has a global attractor $\hat{\mathbb{A}}$ in $Z \times \Sigma$, and clearly $\mathbb{A} = \pi_Z(\hat{\mathbb{A}})$ is the uniform attractor for $\varphi_c$. 

\qed
3.1. Associated evolution processes. From Proposition 4, when \( \theta \) is an asymptotically compact and dissipative impulsive semigroup satisfying \( I(M) \cap M = \emptyset \), \((T)\) and \((H)\), it possesses a global attractor \( \Xi \). A function \( \eta : \mathbb{R} \to \Sigma \) is a global solution of \( \tilde{\theta} \) if
\[
\tilde{\theta}_\eta(s) = \eta(t + s) \quad \text{for all } t \geq 0 \text{ and } s \in \mathbb{R}.
\]
Using [8, Proposition 4.3], we have
\[
\Xi \backslash M = \{ \eta(0) : \eta \text{ is a bounded global solution of } \tilde{\theta} \}.
\] (6)
Since \( \Xi \backslash M \) is \( \tilde{\theta} \)-invariant, we can consider the restriction of \( \tilde{\theta} \) to \( \Xi \backslash M \), denoted again as \( \tilde{\theta} \), and for each global solution \( \eta \) of \( \tilde{\theta} \) in \( \Xi \backslash M \), we can consider the evolution process
\[
S_\eta(t, s) = \varphi_c(t - s, \eta(s)) \quad \text{for all } t \geq s.
\]
Thus, we may present the following result.

**Theorem 3.3.** If \( \varphi_c \) is uniformly asymptotically compact and uniformly dissipative, then for each global solution \( \eta \) of \( \tilde{\theta} \) in \( \Xi \backslash M \), the associated evolution process \( S_\eta \) has a pullback attractor \( \hat{A}_\eta \).

**Proof.** Fix a bounded global solution \( \eta \) of \( \tilde{\theta} \) in \( \Xi \backslash M \). If \( t \in \mathbb{R}, s_n \to -\infty \) and \( \{ z_n \} \) is bounded in \( Z \), then \( S_\eta(t, s_n)z_n = \varphi_c(t - s_n, \eta(s_n))z_n \) for each \( n \in \mathbb{N} \), and from the uniform asymptotical compactness of \( \varphi_c \), \( \{ S_\eta(t, s_n)z_n \} \) has a convergent subsequence in \( Z \).

Now, if \( B_0 \) is a uniformly dissipative set for \( \varphi_c \) and \( B \subset Z \) is bounded, then we can choose \( t_0 = t_0(B, \eta(\mathbb{R})) \geq 0 \) such that \( S_\eta(t, s)B = \varphi_c(t - s, \eta(s))B \subset B_0 \), for all \( t - s \geq t_0 \), that is, for \( s \leq t - t_0 \). Hence, Proposition 1 implies that \( S_\eta \) has a pullback attractor \( \hat{A}_\eta \) in \( Z \).

**Example 1.** Consider \( Z = \Sigma = [0, \infty) \) and the cocycle
\[
\varphi(t, \sigma)z = z e^{-t} + te^{-(t + \sigma)} \quad \text{for } t, \sigma, z \geq 0,
\]
with driving semigroup \( \theta, \sigma = t + \sigma \) on \( \Sigma \). In this case, \( \mathbb{R}_+ \ni t \mapsto \varphi(t, \sigma)z \in \mathbb{R} \) is the solution of the following ordinary differential equation:
\[
\begin{cases}
\dot{u} = -u + e^{-(t + \sigma)}, & t > 0, \\
u(0) = z.
\end{cases}
\]
Let \( M \) be the set of positive integers and \( I(n) = 0 \) for all \( n \in M \). Then \( (\Sigma, \theta, M, I) \) is an impulsive dynamical system on \( \Sigma \). We have \( \phi_\theta(\sigma) = [\sigma] - \sigma \) for each \( \sigma \in \Sigma \backslash \{ M \cup \{0\} \} \), where \([\sigma]\) denotes the smallest integer greater than or equal \( \sigma \), and \( \phi_\theta(\sigma) = 1 \) for \( \sigma \in M \cup \{0\} \), in particular, \((\Sigma, \theta, M, I) \) satisfies condition \((H)\). It is also simple to see that \( \theta \) satisfies \((T)\) and that \( \Xi = [0, 1] \) is the global attractor for the impulsive semigroup \( \tilde{\theta} \). The positive impulsive semitrajectories of \( \tilde{\theta} \) are given by
\[
\tilde{\theta}_\eta(t, \sigma) = \begin{cases}
t + \sigma, & 0 \leq t < \phi_\theta(\sigma), \\
0, & t = \phi_\theta(\sigma), \\
t - \phi_\theta(\sigma) - n, & t > \phi_\theta(\sigma), n \leq t - \phi_\theta(\sigma) < n + 1,
\end{cases}
\]
and we note that \( 0 < \phi_\theta(\sigma) \leq 1 \) for all \( \sigma \in \Sigma \). For \( s = \phi_\theta(\sigma) \) we have \( \tilde{\Pi}(t)(z, \sigma) = (\varphi(t, \sigma)z, \tilde{\theta}_\sigma) \) for \( 0 \leq t \leq s \) and
\[
\tilde{\Pi}(t)(z, \sigma) = (\varphi(t - s - n, 0)\varphi(1, 0)^n\varphi(s, \sigma)z, \tilde{\theta}_{t-s-n}0),
\]
for $n < t - s \leq n + 1$. But $\varphi(1,0)^{n}z = ze^{-n} + \frac{1-e^{-n}}{e-1}$, hence we have $\varphi_c(t,\sigma)z = ze^{-t} + te^{-(t+\sigma)}$ if $0 \leq t \leq s$ and

$$\varphi_c(t,\sigma)z = ze^{-t} + se^{-(t+\sigma)} + e^{-(t-s-n)}\frac{1-e^{-n}}{e-1} + (t-s-n)e^{-(t-s-n)},$$

if $n < t - s \leq n + 1$, for all $z \in \mathcal{Z}$ and $\sigma \in \Sigma$.

If $B \subset \mathcal{Z}$ is bounded, with $M = \sup_{z \in B} |z|$, we have

$$|\varphi_c(t,\sigma)z| \leq (M + 1)e^{-t} + \frac{1-e^{-n}}{e-1} + 1,$$

for all $\sigma \in \Sigma$. Since $n \to \infty$ when $t \to \infty$, we can see that for $t$ sufficiently large

$$|\varphi_c(t,\sigma)z| \leq 4 \quad \text{for all } z \in B \text{ and } \sigma \in \Sigma,$$

consequently, $\varphi_c$ is uniformly dissipative (therefore, uniformly asymptotically compact). Using Theorem 3.2, $\tilde{\Pi}$ has a global attractor $\hat{A}$ in $\mathcal{Z} \times \Sigma$ and $\mathcal{A} = \pi_{\Sigma} (\hat{A})$ is the uniform attractor of $\varphi_c$. In this particular case, $\mathcal{A} \subset \{z \in \mathcal{Z} : |z| \leq 4\}$. Moreover, for a global solution $\eta$ of $\theta$ in $\mathcal{Z}\backslash M = [0,1)$, the evolution process $S_{\eta}(t,r) = \varphi_c(t-r,\eta(r))$ has a pullback attractor $\hat{A}_{\eta}$. Therefore, if $s(r) = \theta_{\eta}(\eta(r))$ and $n < t - r - s(r) \leq n + 1$ then

$$S_{\eta}(t,r)z = ze^{-(t-r)} + s(r)e^{-(t-r+s(r))} + e^{-(t-r-s(r)-n)}$$

$$\times \frac{1-e^{-n}}{e-1} + (t-r-s(r)-n)e^{-(t-r-s(t)-n)},$$

for $t \geq r$ and $z \in \mathcal{Z}$.

4. Application to cascade systems. Let $X,Y$ be Banach spaces and consider a cascade system of the form

$$\begin{align*}
\dot{x} &= f(x,y), \quad t > 0, \\
\dot{y} &= g(y), \quad t > 0, \\
x(0), y(0) &= (x_0, y_0) \in X \times Y.
\end{align*}$$

We assume that for each $(x_0,y_0) \in X \times Y$, this problem has a unique solution $\Pi(\cdot)(x,y)$ defined for all $t \geq 0$, such that the map $\mathbb{R}_+ \times X \times Y \ni (t,x_0,y_0) \mapsto \Pi(t)(x_0,y_0) \in X \times Y$ is continuous. Clearly, with these conditions, the problem

$$\begin{align*}
\dot{y} &= g(y), \quad t > 0 \\
y(0) &= y_0 \in Y,
\end{align*}$$

generates a semigroup $\theta$ on $Y$, and for each $y_0 \in Y$, we can consider the problem

$$\begin{align*}
\dot{x} &= f(x,\theta t y_0), \quad t > 0, \\
x(0) &= x_0 \in X.
\end{align*}$$

Denoting by $\varphi(\cdot,y_0)x_0$ the solution to this problem, which is defined for all $t \geq 0$, we obtain $\Pi(t)(x_0,y_0) = (\varphi(t,y_0)x_0,\theta t y_0)$ for all $t \geq 0$, and $\varphi$ is a cocycle.

Now, let $M$ be an impulsive set and $I : M \to Y$ be an impulse function for $\theta$ in $Y$, so that $(Y,\theta,M,I)$ is an impulsive dynamical system. Assuming that (4) holds for $\theta$, we have the impulsive semigroup $\theta$ and we can construct the impulsive semigroup $\tilde{\Pi}$ in $X \times Y$ and its associated coupled impulsive cocycle $\varphi_c$.

Moreover, if $\theta$ satisfies (T) and (H), $I(M) \cap M = \emptyset$, and $\tilde{\theta}$ is dissipative and asymptotically compact, then $\tilde{\theta}$ has a global attractor $\Xi$. If $\varphi_c$ is uniformly asymptotically compact and uniformly dissipative, Theorem 3.3 guarantees the existence
of a pullback attractor \( \hat{A} \) for the evolution process \( S_\eta(t, s) = \varphi_\eta(t - s, \eta(s)) \) for each global solution \( \eta \) of \( \tilde{\theta} \) in \( \mathbb{E} \setminus M \). The evolution process \( S_\eta \) can be viewed as the solution of the problem

\[
\dot{x} = f(x, \eta(t)), \ t \geq s,
\]

in some weak sense, since the right hand side of the above equation is no longer continuous in the variable \( t \) (\( \eta \) is not continuous). Note that equation \( \dot{x} = f(x, \eta(t)) \) makes sense except for the times where \( \eta \) suffers a jump through the function \( I \), and hence, apart from these points, \( S_\eta(t, s)x_0 \) is a solution of \( \dot{x} = f(x, \eta(t)) \) for \( t \geq s \) with \( x(s) = x_0 \).

### 4.1. An ODE in finite dimension.
Consider a finite-dimensional cascade system of the form

\[
\begin{cases}
\dot{x} = f(x, y), \ t > 0, \\
\dot{y} = g(y), \ t > 0, \\
(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^{m+n},
\end{cases}
\]

where \( g: \mathbb{R}^n \to \mathbb{R}^n \) and \( f: \mathbb{R}^{m+n} \to \mathbb{R}^m \) are locally Lipschitz functions such that there exist constants \( \alpha, \beta > 0 \) such that for \( \|x\|, \|y\| \geq \beta \) we have

\[
\langle f(x, y), x \rangle \leq -\alpha \|x\|^2 \quad \text{and} \quad \langle g(y), y \rangle \leq -\alpha \|y\|^2,
\]

where \( \langle \cdot, \cdot \rangle \) is the usual scalar product in either \( \mathbb{R}^m \) or \( \mathbb{R}^n \). Then for each \( (x_0, y_0) \in \mathbb{R}^{m+n} \), there exists a unique solution \( \mathbb{R}_+ \ni t \mapsto \Pi(t)(x_0, y_0) \in \mathbb{R}^{m+n} \), and the map \( \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \ni (t, x_0, y_0) \mapsto \Pi(t)(x_0, y_0) \in \mathbb{R}^{m+n} \) is continuous. Thus \( \{\Pi(t): t \geq 0\} \) defines a semigroup in \( \mathbb{R}^{m+n} \).

In what follows, we write \( \langle x(t), y(t) \rangle = \Pi(t)(x_0, y_0) \) for all \( t \geq 0 \).

**Lemma 4.1.** If \( \|\Pi(t)(x_0, y_0)\| = \beta \) for some \( t \geq 0 \) then there exists \( \epsilon > 0 \) such that for all \( t \in (\tilde{t}, \tilde{t} + \epsilon) \) we have \( \|\Pi(t)(x_0, y_0)\| < \beta \).  

**Proof.** Since for all \( t \geq 0 \) we have

\[
\frac{d}{dt}\|\Pi(t)(x_0, y_0)\|^2 = \frac{d}{dt}\|\langle x(t), y(t) \rangle \|^2 = 2\langle f(x(t), y(t)), x(t) \rangle + 2\langle g(y(t)), y(t) \rangle,
\]

we obtain

\[
\frac{d}{dt}\|\Pi(t)(\tilde{x}_0, \tilde{y}_0)\|^2 |_{t = \tilde{t}} \leq -2\alpha\|\Pi(\tilde{t})(\tilde{x}_0, \tilde{y}_0)\|^2 = -2\alpha\beta < 0.
\]

Consequently, there exists \( \epsilon > 0 \) such that for \( t \in (\tilde{t}, \tilde{t} + \epsilon) \) we have

\[
\|\Pi(t)(x_0, y_0)\| < \|\Pi(\tilde{t})(x_0, y_0)\| = \beta.
\]

**Proposition 7.** The ball \( \{|x|^2 + \|y\|^2 \leq \beta^2\} \) in \( \mathbb{R}^{m+n} \) is positively invariant for \( \Pi \).

**Proof.** In fact, from Lemma 4.1, if \( \|\Pi(t)(x_0, y_0)\| \leq \beta \) and for some \( t \geq 0 \) we have \( \|\Pi(t)(x_0, y_0)\| = \beta \) then there exists \( \epsilon > 0 \) such that \( \|\Pi(s)(x_0, y_0)\| \leq \beta \) for \( s \in (t, t + \epsilon) \). This shows that \( \|\Pi(t)(x_0, y_0)\| \leq \beta \) for all \( t \geq 0 \).

**Lemma 4.2.** If \( \|\Pi(t)(x_0, y_0)\| \geq \beta \) then \( \Pi(t)(x_0, y_0)\) is \( e^{-\alpha t}\|x_0, y_0\| \) for as long as \( \|\Pi(t)(x_0, y_0)\| \geq \beta \).
Proof. As in the proof of Lemma 4.1 we have

$$\frac{d}{dt} \|\Pi(t)(x_0, y_0)\|^2 = \frac{d}{dt} \|(x(t), y(t))\|^2$$

$$= 2\langle f(x(t), y(t)), x(t) \rangle + 2\langle g(y(t)), y(t) \rangle \leq -2\alpha \|\Pi(t)(x_0, y_0)\|^2$$

for all $t \geq 0$ whenever $\|\Pi(t)(x_0, y_0)\| \geq \beta$. This implies that

$$\|\Pi(t)(x_0, y_0)\|^2 \leq e^{-2\alpha t} \|(x_0, y_0)\|^2,$$

for all $t \geq 0$ whenever $\|\Pi(t)(x_0, y_0)\| \geq \beta.$ \hfill \Box

These results show that

$$\|\Pi(t)(x_0, y_0)\|^2 \leq e^{-2\alpha t} \|(x_0, y_0)\|^2 + \beta^2,$$

for all $(x_0, y_0) \in \mathbb{R}^{m+n}$ and all $t \geq 0$. Hence, $\Pi$ has (a connected) global attractor $\hat{A}$ in $\mathbb{R}^{m+n}$.

Thus, as explained before, we obtain a semigroup $\theta$ in $\mathbb{R}^n$ for the second equation in (8), and for each $y_0 \in \mathbb{R}^n$ and $t \geq 0$ we have

$$\|\theta_t y_0\|^2 \leq e^{-2\alpha t} \|y_0\|^2 + c,$$

where $c > 0$ is a constant. This semigroup $\theta$ has (a connected) global attractor $\Sigma = \pi_{\mathbb{R}^n}(\hat{A})$. We use now the results of [7] to choose an impulsive set $M$, and to that end we need to assume that this semigroup $\theta$ is, in fact, a group in $\mathbb{R}^n$, that is, $\theta_t$ is defined for all $t \in \mathbb{R}$ (this is the case when $g$ has linear growth, for instance).

We consider $M \subset \mathbb{R}^n$ a hypersurface on $\mathbb{R}^n$ such that

$$\langle g(y), \tilde{n_y} \rangle \neq 0$$

for all $y \in M$, \hspace{1cm} (11)

where $\tilde{n_y}$ denotes the normal vector of $M$ at $y$. Using [7, Theorem 9] and [5, Theorem 7.5], we obtain $M$ is an impulsive set for $\theta$ that satisfies condition (T). For instance, we could take $M$ as an $(n-1)$-dimensional sphere with radius $\gamma \geq \beta$ (and condition (9) implies (11)).

Defining any continuous function $I: M \to \mathbb{R}^n$ with $I(M)$ compact and $I(M) \cap M = \emptyset$, condition (H) is also verified. Hence, we have defined the impulsive semigroup $\dot{\theta}$ in $\Sigma$. From (10) and Proposition 5, $\dot{\theta}$ is dissipative, and hence, asymptotically compact (see Remark 1). Thus $\dot{\theta}$ has a global attractor $\Xi \subset \Sigma$ in $\mathbb{R}^n$.

We can construct the impulsive dynamical system $(\mathbb{R}^{m+n}, \Pi, \mathbb{R}^n \times M, \Pi)$, where $\Pi(x, y) = (x, I(y))$ for $y \in M$. Consequently, we obtain the impulsive semigroup $\Pi$ and the associated coupled impulsive cocycle $\varphi$. If $\varphi$ is uniformly dissipative (hence, uniformly asymptotically compact), then the semigroup $\Pi$ has a global attractor $\hat{A}$, the coupled impulsive cocycle has a uniform attractor $\hat{A} = \pi_{\mathbb{R}^n}(\hat{A})$, and for each global solution $\eta \in \Xi \setminus M$, the evolution process $S_\eta(t, s)$ associated (in some sense) to the problem

$$\dot{x} = f(x, \eta(t)), \; t \geq s,$$

given by $S_\eta(t, s) = \varphi_c(t-s, \eta(s))$, has a pullback attractor $\hat{A}_\eta$ in $\mathbb{R}^m$.

One way to obtain the uniform dissipativity of $\varphi$ in this case is the following (see [8, Example 4.9]): assume that there exists a function $V \in C^1(\mathbb{R}^{m+n}, \mathbb{R})$ and constants $\alpha_1, \alpha_2 > 0$ and $\mu \in \mathbb{R}$ satisfying

(i) $\nabla V(x, y) \cdot (f(x, y), g(y)) \leq \alpha_1 - \alpha_2 V(x, y)$ for all $(x, y) \in \mathbb{R}^{m+n}$;

(ii) $V(x, I(y)) \leq \mu$ for all $x \in \mathbb{R}^n$ and $y \in M$;

1In this case, the solutions of both $\dot{y} = g(y)$ and $\dot{y} = -g(y)$ do not blow-up in finite time.
(iii) $\pi_\mathbb{R}^{-1}(V^{-1}(-\infty, \mu + \frac{\alpha_2}{\alpha_1})]$ is bounded.

With these assumptions, if $K = V^{-1}(-\infty, \mu + \frac{\alpha_2}{\alpha_1}]$, then given $B \times C \subset \mathbb{R}^{m+n}$ bounded there exists $t_0 \geq 0$ such that $\tilde{\Pi}(t)(x, y) \in K$ for all $(x, y) \in B \times C$ and $t \geq t_0$. Therefore,

$$\varphi_c(t, y)B \subset \pi_\mathbb{R}^{-1}(K) \quad \text{for all } y \in C \text{ and } t \geq t_0,$$

which proves the uniform dissipativity of $\varphi_c$.

For each $\gamma \geq \beta$, the ball $|x|^2 + |y|^2 \leq \gamma^2$ in $\mathbb{R}^{m+n}$ is positively invariant for $\Pi$, that is, $\|\Pi(t)(x, y)\| \leq \gamma$ for all $t \geq 0$ and $|x|^2 + |y|^2 \leq \gamma^2$. If $M$ is a compact hypersurface in $\mathbb{R}^n$, then we choose $\gamma > 0$ such that both $M$ and $I(M)$ are inside the ball $|y| \leq \gamma$ in $\mathbb{R}^n$, and we can restrict ourselves with the study of $\Pi$ in $|x|^2 + |y|^2 \leq \gamma^2$, and, inside this ball, one possible choice of $V$ is $V(x, y) = |x|^2 + |y|^2$, with $\mu = \gamma^2$, $\alpha_2 = 1$ and $\alpha_1 = 2\gamma^2 + \sup\{ |f(x, y)|^2 + ||y||^2 : |x|^2 + |y|^2 \leq \gamma^2 \}$, since

$$\nabla V(x, y) \cdot (f(x, y), g(y)) = 2\langle f(x, y), x \rangle + 2\langle g(y), y \rangle \leq ||f(x, y)||^2 + ||g(y)||^2 + |x|^2 + |y|^2$$

$$= ||f(x, y)||^2 + ||g(y)||^2 + 2V(x, y) - V(x, y) \leq 2\gamma^2 + \sup_{|x|^2 + |y|^2 \leq \gamma^2} \{ ||f(x, y)||^2 + ||g(y)||^2 \} - V(x, y),$$

for every $(x, y) \in \mathbb{R}^{m+n}$ such that $|x|^2 + |y|^2 \leq \gamma^2$.

4.2. A forced heat equation coupled with an one-dimensional ODE. This example is an adaption of the problem presented in [26]. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following coupled problem:

$$\begin{align*}
\frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) &= g(y(t)), \quad (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) &= 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \\
u(0, x) &= u_0(x), \quad x \in \Omega, \\
\frac{dy}{dt} &= f(y), \quad t > 0, \\
y(0) &= y_0 \in \mathbb{R},
\end{align*}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function. Assume that the problem

$$\begin{align*}
\frac{dy}{dt} &= f(y), \quad t > 0, \\
y(0) &= y_0,
\end{align*}$$

has a unique solution $y(\cdot, y_0) : [0, \infty) \to \mathbb{R}$ defined for each $y_0 \in \mathbb{R}$, and such that $(t, y_0) \in [0, \infty) \times \mathbb{R} \mapsto y(t, y_0) \in \mathbb{R}$ is a continuous map. Then the family $T$ given by $\theta(t)y_0 = y(t, y_0)$ is a semigroup in $\mathbb{R}$ which satisfies (2). Assume that $M \subset \mathbb{R}$ is a closed set and $I : M \to \mathbb{R}$ is a continuous function such that $(\mathbb{R}, \theta, M, I)$ is an impulsive dynamical system satisfying (H), with associated impulsive semigroup $\tilde{\theta}$ in $\mathbb{R}$. Lastly, assume that $\bar{\theta}$ has a global attractor $\Xi$ in $\mathbb{R}$. We restrict our impulsive semigroup $\tilde{\theta}$ to $\Sigma = \Xi$.

**Lemma 4.3.** For each global solution $\eta$ of $\bar{\theta}$ in $\Xi$, define $g_\eta(t) = g(\eta(t))$. Then $g_\eta$ is measurable and there exists $k \geq 0$, independent of $\eta$, such that

$$|g_\eta(t)| \leq k \quad \text{for all } t \in \mathbb{R} \quad (12)$$
and each global solution $\eta$ of $\tilde{\theta}$ in $\Xi$. In particular, $g_\eta \in L^2_{l.o.c}(\mathbb{R})$ and

$$\sup_{h \in \mathbb{R}} \int_h^{h+1} |g_\eta(s)|^2 ds \leq k^2,$$

for each global solution $\eta$ of $\tilde{\theta}$ in $\Xi$, that is, each $g_\eta$ is translation bounded.

**Proof.** From the definition of impulsive trajectories, each global solution $\eta$ of $\tilde{\theta}$ in $\Xi$ is piecewise continuous in $\mathbb{R}$. Since $g$ is continuous, their composition is measurable. Since $\Xi$ is compact in $\mathbb{R}$ and $g$ is continuous, (12) and the last assertion are trivial.

Thus, from [26, Theorem 3.1 and Theorem 3.4], for each global solution $\eta$ of $\tilde{\theta}$ in $\Xi$, $s \in \mathbb{R}$, $T > 0$ and $p \geq 2$, the problem

$$\begin{cases}
    u_t(t, x) - \Delta u(t, x) = g(\eta(t)), & (t, x) \in (s, s + T) \times \Omega, \\
    u(t, x) = 0, & (t, x) \in (s, s + T) \times \partial \Omega, \\
    u(s, x) = u_0(x), & x \in \Omega,
\end{cases}$$

has a unique solution $u(\cdot, s, u_0, \eta) \in C([s, s + T], L^2(\Omega)) \cap L^2(s, s + T; H^0_0(\Omega)) \cap L^p(s, s + T; L^p(\Omega))$ and for each $u_0, v_0 \in L^2(\Omega)$ and $t \in [s, s + T]$ we have

$$\|u(t, s, u_0, \eta) - v(t, s, v_0, \eta)\|_{L^2(\Omega)} \leq \|u_0 - v_0\|_{L^2(\Omega)}.$$ 

Moreover, there exists an uniform absorbing set $B_0$ in $H^0_0(\Omega)$ (w.r.t. $\eta$), that is, for each bounded subset $B$ of $L^2(\Omega)$ there exists $T(B) \geq 0$ such that

$$\varphi_c(t - s, \eta(s))B \subset B_0 \quad \text{for} \quad t - s \geq T(B).$$

This shows that the associated coupled impulsive cocycle $\varphi_c$ is uniformly dissipative and uniformly asymptotically compact. In this way, it possesses a uniform attractor $\mathcal{A}$ in $L^2(\Omega)$. Moreover, for each $\eta$, the associated evolution process $S_\eta$ in $L^2(\Omega)$ given by

$$S_\eta(t, s)u_0 = \varphi_c(t - s, \eta(s)) = u(t, s, u_0, \eta) \quad \text{for} \quad t \geq s,$$

possesses a pullback attractor $\hat{A}_\eta$.

**Final comments.** We point out that the method to ensure the existence of attractors in this paper is based on [9] and [3], which requires stronger assumptions but gives stronger results. We are able to find not only the uniform attractor, but also pullback attractors for $\varphi_c$ and the global attractors for the associated impulsive skew-product semiflow $\Pi$.

Following [15], we could consider a slightly modified definition of an impulsive dynamical system. Using the ideas presented in [15], it is possible to prove the existence of a uniform attractor for $\varphi_c$ without assuming condition (H), but assuming (4) and that $I(M) \cap M = \emptyset$. We remark, however, that condition (H) implies condition (4), and in certain cases it is simple to obtain it (see Remark 3).

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