The Level Crossing Phenomenon with Yukawa Interactions

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Abstract

In the minimal model of electroweak interactions we carefully investigate the spectrum of the massive euclidean Dirac operator in three, four and five dimensions ($D$) in the presence of topologically nontrivial external fields. More specifically we study the cases of the instanton ($D = 4$), sphaleron ($D = 3$) as well as that of the ($D = 5$) Dirac operator that pertains to the existence of the global $SU(2)$ anomaly. We establish the existence of normalizable massive fermion zero modes in all three cases. We give closed form expressions which relate the massive with the massless zero modes. As a consequence the level crossing phenomenon is shown to be manifest and generic in the presence of Yukawa interactions.
1 Introduction

The phenomenon of fermion level crossing and its relation to the existence of normalizable fermionic zero modes of the Dirac operator in the presence of topologically non-trivial gauge fields is well known [1]. It has been extensively studied and firmly established for the case of massless fermions. To be more precise let us discuss it in three superficially different physical circumstances. Firstly, the normalizable zero mode of the $D = 4$ euclidean Dirac operator in the weak instanton field. Secondly, the normalizable solution of the zero energy Dirac equation in the sphaleron field and thirdly the global anomaly for an $SU(2)$ gauge theory coupled to an odd number of Weyl fermion doublets. The first two examples are related to the problem of baryon number violation in the standard electro-weak model [2]. In this case the baryon ($B$) and lepton ($L$) number violations are given by

$$\Delta B = \Delta L = -n_g \Delta N_{CS} \quad (1.1)$$

This effect is due to the fluctuations of the gauge and Higgs fields over the barrier between topologically inequivalent vacua that correspond to different Chern-Simons numbers $N_{CS}$. In eq.(1.1) $n_g$ is the number of generations. More importantly it can be equivalent seen as the fermionic energy level crossing in the presence of the time dependent external gauge and Higgs fields with a variable Chern-Simons characteristic $\frac{1}{2}, 1, \frac{3}{2}, 2$. In this process the occupied positive energy level appear or disappear according to the value of $\Delta N_{CS}$. In turn the level crossing for fermionic Hamiltonian is equivalent to the existence of normalizable zero modes of the $D = 4$ Dirac operator in a euclidean topologically non-trivial external field. The sphaleron configuration on the other hand is a saddle point of the static energy functional which has $N_{CS} = 1/2$. It has been found $\frac{1}{2}, 1, \frac{3}{2}, 2$ that the zero-energy Dirac equation in the sphaleron field has a normalizable solution that corresponds exactly to the crossing point of a zero energy level.

The $SU(2)$ global anomaly [10] is also related to the phenomenon of level crossing of the $D = 4$ Dirac operator. In analogy with the case of instantons and sphalerons this eigenvalue flow corresponds to the existence of a zero mode for an appropriately defined $D = 5$ Dirac operator in the presence of an external topologically non-trivial gauge field. As a result the fermionic path integral is not gauge invariant and the theory is not self-consistent.

The problem that we address in the present work is common to all the above mentioned physical realizations of the level crossing phenomenon. In the realistic electro-weak theory fermions become massive due to the Higgs mechanism. In the presence of Yukawa interactions the Higgs field is a non-vanishing constant at spatial infinity. In this paper we consider the effect of non-vanishing Yukawa couplings and fermion masses on the problem of existence of fermionic zero modes in the presence of topologically non-trivial external fields. While some preliminary work for the instanton [11, 12] and sphaleron [13, 14] cases has already been done we clarify some of the conceptual issues that arise in these works. Moreover we attempt to reach a single physical point of view on the question of existence of massive zero modes and level crossing in the presence of a fermion mass matrix $M$ for all three cases.
At first it could appear surprising the occurrence of level crossing in the spectrum of massive fermions. We present a heuristic physical picture that makes it plausible. To that end it is sufficient to establish at some point in $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5$ respectively vanishing fermion masses with singular external gauge fields. For the instanton case ($D = 4$) this is trivially true. In the singular gauge the gauge field falls off to zero at $\mathbb{R}^4$ infinity while it is singular at the instanton center where the mass matrix is zero. This is also guaranteed by the well-known topological theorem $\pi^3(SU(2)) = \mathbb{Z}$. As a result the zero mode wave function is concentrated at the center of the instanton. Indeed this fact is confirmed by an explicit construction of massive zero modes for some special cases \[1, 2\]. Though there exists no index theorem for the $D = 3$ sphaleron case as $\pi^2(SU(2)) = 0$ \[4\] there still exists a point in $\mathbb{R}^3$ where fermion masses are zero \[13\].

For the case of the $SU(2)$ anomaly ($D = 5$) it is somewhat difficult to demonstrate our claims as there are no explicit instanton-like configurations for $D = 5$. We may however offer an intuitive argument for the existence of the level crossing in this case as well. Let us firstly consider the $D = 5$ Dirac operator in an external topologically nontrivial gauge field. Such a configuration is topologically guaranteed by the fact that $\pi^4(SU(2)) = \mathbb{Z}_2$ \[14\]. In our case where Yukawa interactions are present it is tacitly taken that Higgs fields are suitably included as external fields. As a consequence of our construction we take at $\mathbb{R}^5$ infinity our gauge fields to be a pure gauge $A_{\mu} \propto U^{-1} \partial_{\mu} U$ with $M \propto U$ being the fermion mass matrix. Here $U$ is a unitary matrix, a noncontractible map from $S^4$ into $SU(2)$ on a sphere at $\mathbb{R}^5$ infinity. It means that one can continuously deform it to the identity everywhere on the sphere except for the region near a single point where $U \neq 1$ (a singularity). As the radius $R$ of the sphere takes value from infinity to zero we form a line where the topological non-triviality is concentrated in its neighbourhood and the external field changes very rapidly. Therefore at $R = 0$ we should have either zero or singularity. By continuity of the mass matrix the latter is impossible, and hence we should expect the existence of zero in $M$ with level crossing to occur. In the main part of our work we will explicitly demonstrate the existence of the massive fermion zero mode along with the spectral flow of the appropriate Dirac operators for $D = 3, 4, 5$. In each case we examine their integrability property and give closed form expressions that relate the zero mode wave functions for the massive fermions with those of the massless ones.

The paper is organized as follows. In sect.2 the case of zero modes for the massive fermions in the weak instanton field is worked out. Sect.3 is devoted to the case of a sphaleron. The generalization of Witten’s $SU(2)$ anomaly for the case of massive fermions is carried through in sect.4. There we prove that the Atiyah-Singer mod 2 index theorem is sufficient to guarantee the existence of an odd number of zero modes for the massive $D = 5$ Dirac operator in the presence of topologically non-trivial external fields. We finally close with some final comments and speculations on unanswered questions.
2 Massive Fermions in a Weak Instanton field

The properties of the massive euclidean Dirac operator \( (D = 4) \) in external weak instanton fields has been considered in refs. \[11, 12\]. Many of our conclusions are therefore likely to overlap with those of the above mentioned authors but we include them here for completeness. The instanton configuration contains both non-trivial configurations of gauge and Higgs fields. Let us consider the functional integral over fermionic fields in a topologically non-trivial external gauge and Higgs fields. Though the explicit form of this configuration is not essential, in what follows we will make use of the ones given by t’Hooft \[2\]. For the case of an anti-instanton field:

\[
W_\mu^\alpha = \frac{2}{x^2 + \rho^2} \bar{\eta}_{\alpha\mu\nu} x_\nu.
\]  

(2.1)

The Higgs field is given by

\[
\varphi(x) = -i(\tau_\mu^a x_\mu) \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{x^2 + \rho^2}}
\]

(2.2)

Here \( \rho \) is the scale of the anti-instanton, and \( v \) is the vacuum expectation value of the field \( \varphi^0 \). The t’Hooft’s symbols \( \bar{\eta}_{abc} \) are defined as \( \bar{\eta}_{abc} = \varepsilon_{abc} \) with \( a, b, c = 1, 2, 3 \) and \( \bar{\eta}_{a4b} = -\bar{\eta}_{ab4} = \delta_{ab}, \eta_{a44} = 0; \tau_\mu^a = (\vec{\tau}, +i) \). The expressions (2.1) and (2.2) are given in the regular gauge.

As to the fermions, we consider for simplicity two weak doublets only. For instance, these may be two quark doublets with different colour indices, or one quark and one lepton doublet. For definiteness they are:

\[
q_L = \begin{pmatrix} u \\ d \end{pmatrix}, \quad q'_L = \begin{pmatrix} u' \\ d' \end{pmatrix}
\]

(2.3)

with associated weak singlet right-handed states as \( u_R, d_R, u'_R, d'_R \). The generalization to the case of the twelve existing doublets is trivial.

When the weak doublets are considered in pairs it is possible to pass to the pure vector interaction of fermions with \( W \) bosons. To achieve this we introduce instead of fermion fields \( q' \) the charge-conjugated fields:

\[
\bar{q}_R = \varepsilon C q'_L = \begin{pmatrix} C\bar{d}_L^R \\ -C\bar{u}'_L \end{pmatrix}, \quad \bar{q}'_L = \varepsilon C q'_R = \begin{pmatrix} C\bar{d}_R^L \\ -C\bar{u}'_R \end{pmatrix}.
\]  

(2.4)

Here \( \varepsilon = i\sigma_2 \) acts on the isotopic indices, \( C \) is the charge-conjugation matrix. By introducing

\[
\psi = \psi_L + \psi_R, \quad \psi_L = q_L, \quad \psi_R = C\varepsilon q'_L \\
\eta = \eta_R + \eta'_L, \quad \eta_R = q'_R, \quad \eta'_L = C\varepsilon q'_R,
\]

(2.5)

it becomes obvious that both components of \( \psi, \psi_L \) and \( \psi_R \), are weak doublets while \( \eta_R \) and \( \eta'_L \) are singlets. Therefore only the \( \psi \) field has a vector gauge interaction:

\[
L_W = i\bar{\psi} \gamma \psi + i\bar{\eta} \gamma \eta, \quad \gamma \psi = \gamma_\mu D_\mu = \gamma_\mu(\partial_\mu - iW_\mu).
\]  

(2.6)
Clearly the mixing of the quark and antiquark fields in eq. (2.5) is very unnatural with respect to colour and electric charge (weak hypercharge). As those interactions are irrelevant to our problem at hand from now on we only keep the Yukawa couplings:

$$-L_Y = h_u \bar{q}_L \varepsilon_{ij} u_R \varphi_j^i + h_d \bar{q}_L d_R \varphi_j^i + H.c. + (u, d, h_u, h_d \rightarrow u', d', h_u', h_d').$$  \hspace{1cm} (2.7)

Here the Higgs field is \( \varphi^i = (\varphi^+, \varphi^0) \) and the fermion masses are given by \( m_u = h_u v/\sqrt{2}, \ m_d = h_d v/\sqrt{2}, \ m'_u = h'_u v/\sqrt{2}, \ m'_d = h'_d v/\sqrt{2} \).

By using the fields \( \psi \) and \( \eta \) of eq. (2.5) one can rewrite the Lagrangian (2.7) as follows:

$$-L_Y = \bar{\psi}_L M \eta_R + \bar{\eta}_R M^+ \psi_L - \bar{\psi}_R \varepsilon M^T \varepsilon \eta_R = -\bar{\eta}_L \varepsilon M' \varepsilon \psi_R,$$  \hspace{1cm} (2.8)

where the mass matrix \( M(x) \) is given by

$$M(x) = \begin{pmatrix}
    h_u \varphi^0(x)^*, & h_d \varphi^+(x) \\
    -h_u \varphi^+(x)^*, & h_d \varphi^0(x)
\end{pmatrix}. \hspace{1cm} (2.9)$$

\( M' \) is taken from \( M \) by substituting \( h_u, h_d \rightarrow h'_u h'_d \).

We now make a Euclidean rotation upon which the fermion fields rotate into

$$\psi, \eta \rightarrow \psi, \eta, \hspace{0.5cm} \bar{\psi}, \bar{\eta} \rightarrow -i \psi^+, -i \eta^+.$$  \hspace{1cm} (2.10)

The mass terms in \( L_Y \) contain now the bilinear combinations of the fields with the same chirality. The Lagrangian reads now as follows

$$L = L_W + L_Y,$$  \hspace{1cm} (2.11)

\hspace{1cm} $$L_W = -i \bar{\psi} D /\psi - i \bar{\eta} D /\eta,$$  \hspace{1cm} (2.12)

\hspace{1cm} $$L_Y = -i \bar{\psi}_R M \eta_R - i \bar{\eta}_L M^+ \psi_L + i \bar{\psi}_L \varepsilon M' \varepsilon \eta_L + i \bar{\eta}_R \varepsilon M'^T \varepsilon \psi_R.$$  

It is well known that in the absence of the Yukawa couplings a massless fermion has a zero-mode in the (anti)instanton field. At first glance one would expect that when fermions acquire masses in the presence of the Yukawa couplings the zero mode disappears. One can see this not to be true. In order to look for the wave function for the fermion zero mode one should consider the classical equations of motion for the fields \( \psi \) and \( \eta \) in the external gauge and Higgs fields. In the Euclidean version of the theory they are obtained by a variation of the Lagrangian and they are given by:

$$\bar{\D} \psi_L = -M \eta_R, \hspace{0.5cm} \bar{\D} \eta_R = -M^+ \psi_L,$$  \hspace{1cm} (2.13)

$$\bar{\D} \psi_R = \varepsilon M'^T \varepsilon \eta_L, \hspace{0.5cm} \bar{\D} \eta_L = \varepsilon M'^T \varepsilon \psi_R.$$  \hspace{1cm} (2.14)

It is known \[2\] that for a massless fermion there is a zero mode of right-handed chirality in the instanton field as well as one of a left-handed chirality for the anti-instanton field (2.1). As it was pointed out in ref. \[12\] there exists the generalization
of the usual $\gamma_5$-chirality for the case of massive fermions, namely that of the $\Gamma_5$-chirality. It is defined to be $\Gamma_5 = (-1)^{2T+1}\gamma_5$ where $T$ is the weak isospin. The solutions to eqs.(2.13) and (2.14) are classified by the $\Gamma_5 = \pm 1$ eigenvalues. For massive leptons or quark fields $\Gamma_5 = +1$ (left handed fermions have $T = 1/2$ while right handed ones $T = 0$). For antileptons and antiquarks $\Gamma_5 = -1$. The index theorem that relates the number of left ($n_L$) and right ($n_R$) handed fermion zero modes to the topological charge of an external gauge field $Q_T$ given by $n_L - n_R = -Q_T$ generalizes for the case of massive fermion zero modes.

In what follows we will present a closed form expression for such a massive zero mode in the model under consideration. At first let us consider an anti-instanton configuration. Since for $M = M' = 0$ there is a left-handed $\psi$ zero-mode we expect that $\Gamma_5 = +1$, i.e. we look for solutions $\psi_R = 0$, $\eta_L = 0$ but with $\psi_L \neq 0$, $\eta_R \neq 0$. In the instanton field one should solve eqs.(2.14). Let us denote the wave function of the massless Dirac operator as $\psi_{0L}$ with

$$\mathcal{D}\psi_{0L} = 0. \quad (2.15)$$

We are looking for a solution to eqs.(2.13) with the following features:

i) it coincides with $\psi_{0L}$ in the $M \to 0$ limit,

ii) it is normalizable.

We try to find such a solution iteratively. To this end we express $\eta_R$ in terms of $\psi_L$ and substitute it back to eq.(2.13). We get

$$\mathcal{D}\psi_L = M \frac{1}{\partial} M^+ \psi_L. \quad (2.16)$$

If we start with $\psi_L^{(0)} = \psi_{0L}$ as the zero level approximation one then gets the following equation for the first level approximation $\psi_L^{(1)}$:

$$\mathcal{D}\psi_L^{(1)} = M \frac{1}{\partial} M^+ \psi_{0L} \quad (2.17)$$

where $\psi_L = \psi_{0L} + \psi_L^{(1)} + \ldots$. We assume that $\psi_L^{(1)}$ is orthogonal to $\psi_{0L}$. At this stage of the iteration we must formally check the integrability of this equation. This is a necessary condition for the consistency of the chosen zero level approximation. Indeed the left hand side of eq.(2.17) is orthogonal to $\psi_{0L}$ since the operator $\mathcal{D}$ is antihermitean and annihilates the $\psi_{0L}$ outstate when we integrate it by parts. Fortunately the integrability condition is automatically satisfied here because the massless zero mode wave function $\psi_{0L}$ is a Weyl spinor. However this is not the case with the sphaleron field and the $SU(2)$ global anomaly. From eq.(2.17) we may easily find

$$\psi_L^{(1)} = \frac{1}{\mathcal{D}} M \frac{1}{\partial} M^+ \psi_{0L}. \quad (2.18)$$

Note that this expression is well defined since the operator $\mathcal{D}$ has no normalizable zero modes in the right handed spinor subspace. However in what follows it is
convenient that we use the alternative form
\[ \psi^{(1)}_L = \bar{D} \frac{1}{D^2} \frac{1}{\Theta} M^+ \psi_0 L. \] (2.19)

Due to the antiselfduality property of the anti-instanton field we have implicitly made use of the following identity for the Dirac operator in the anti-instanton external field:
\[ \bar{D}^2 (1 - \gamma_5) = D^2 (1 - \gamma_5). \] (2.20)

It is easy to verify that at each step of the iterative procedure the integrability condition is trivially satisfied. The solution is found as an expansion in powers of the mass matrix \( M \). In a closed form it is given by
\[ \psi_L = \psi_0 L + \bar{D} \frac{1}{D^2 - M (1/\Theta) M^+ \bar{D}} \frac{1}{\Theta} M^+ \psi_0 L, \quad \eta_R = - \frac{1}{\Theta} M^+ \psi_L. \] (2.21)

By a substitution of the expressions (2.21) into (2.16) one can check directly its validity. Moreover we may note that the operator \( D^2 - M (1/\Theta) M^+ \bar{D} \) has no normalizable zero modes, at least for values of \( M \) not too big, in order that the expression (2.21) be well defined. Indeed the scalar operator \(-D^2\) is positively definite in the appropriate space of functions and has no normalizable zero modes. On the other hand the correction \( M (1/\Theta) M^+ \bar{D} \) is small at least for small values of the Yukawa couplings and hence of the matrix \( M \). We therefore may assume that the expression (2.21) has no singularities at least for small values of the Yukawa couplings. Nevertheless we have no arguments against the existence of singularities for large ones. Such singularities would probably imply that discrete zero modes appear or disappear at large Yukawa couplings while the number of normalizable zero modes is a topological invariant only under ‘small’ deformations of the external fields, and hence of the mass matrix \( M \).

Let us now demonstrate that the wave function (2.21) is normalizable. For that it is sufficient to check that expression (2.21) decreases rapidly enough at large distances \( x \to \infty \). This we can more conveniently do in the regular gauge for the instanton configuration. At infinity we have that
\[ D_\mu \to U \partial_\mu U^{-1}, \quad M \to U M_0, \] (2.22)

where \( U \) is an element of the \( SU(2) \) group that corresponds to the instanton configuration and \( M_0 \) is a constant matrix. By a direct substitution of expressions (2.22) back into eq.(2.21) we easily find that
\[ \psi_L \propto U \frac{1}{\Theta^2 - M_0^2} U^{-1} \psi_0 L. \] (2.23)

It becomes obvious from the above that \( \psi_L \) behaves better at infinity than \( \psi_0 L \) itself a normalizable wave function. Indeed we trivially deduce from eq.(2.23) that
\[ \psi_L \leq U \frac{1}{x^2 M_0^2} U^{-1} \psi_0 L. \] (2.24)
Hence $\psi_L$ is normalizable. Our general arguments are complimentary to the explicit construction of the massive fermion zero mode wave function in ref.\cite{11, 12}. There for the special case of equal fermionic masses ($h_u = h_d$) the wave function $(\psi_L, \eta_R)$ in the anti-instanton field is found to be exponentially suppressed at $x \to \infty$ like

$$\psi_L, \eta_R \propto e^{-mx} x^{3/2}$$

whereas for the massless zero mode wave function $\psi_{0L} \propto (x^2 + \rho^2)^{-3/2}$. We must emphasize here that our proof of the existence of a normalizable zero mode relies heavily on the hypothesis of the behaviour (2.22) of the mass matrix at infinity. As a consequence the number of normalizable zero modes is a topological invariant under smooth deformations of the mass matrix $M$ that preserve condition (2.22). Our assumption of (anti)self-duality of the gauge field was necessary for the demonstration of absence of singularities in expression (2.21). We could have formally considered cases where the mass matrix does not obey condition (2.22). In this case the standard index theorem does not hold as we deal with a non-compact $D = 4$ manifold. In fact the external fields do not rapidly decrease at infinity. As it has been shown the index theorem for massless fermions should in this case be generalized in such a way so as to include "phase shifts"\cite{13}. We believe that such a generalization carries through for the case of massive fermions as well. The phase shifts in such a case will receive contributions both from gauge fields and the mass matrix.

### 3 Massive Fermion in a Sphaleron Field

It is well known that the fermionic Hamiltonian in a sphaleron field possesses a normalizable zero mode $\mathbb{F}$. In the standard electro-weak model where the fermions acquire their masses via the Higgs mechanism this was also shown to be true for the special case of degenerate in mass fermion doublets $\mathbb{F}$. In particular by the use of the variational ansatz the author demonstrated the existence of a normalizable solution of the zero-energy Dirac equation in the sphaleron field. In this section we show the presence of such a massive fermion zero mode for the general non-degenerate case. More specifically we obtain a closed form expression which relates the massive with the massless zero mode.

At first let us consider the case of massless fermions in an external sphaleron field. The $D = 3$ Dirac operator $\sigma_i D_i$ is antihermitean and purely imaginary since

$$i\sigma_2 \epsilon (\sigma D)^* i\sigma_2 \epsilon = -\sigma D,$$

where $\epsilon = i\tau_2$ acts on the $SU(2)$ indices and $i\sigma_2$ acts on the spinor ones. Consequently all eigenvalues $i\lambda$ are purely imaginary and all the wave functions can be chosen real

$$i\sigma_2 \epsilon \psi_{\lambda}^* = \psi_{\lambda}.$$  

This is in contrast for example to the case of Witten’s anomaly where an appropriate anti-Hermitean Dirac operator is purely real $\mathbb{F}$. There as a consequence its non-zero levels are paired $(i\lambda, -i\lambda)$ and cross zero only in pairs. In turn this implies
that the number of zero modes of the $D = 5$ Dirac operator is invariant under smooth deformations of the external field in agreement with the Atiyah-Singer index theorem modulo 2 [16]. This is due to the non-triviality of the fourth homotopy group $\pi^4(SU(2)) = \mathbb{Z}_2$. The situation is similar for a different reason in the case of the sphaleron. As this configuration is contractible ($\pi^2(SU(2)) = 0$) one should not expect the number of zero modes of the $D = 3$ Dirac operator to be a topological invariant. Alas this is not the whole story. The sphaleron configuration (see below) is Parity odd. The same is true for the Dirac operator in the presence of the sphaleron gauge field. As a consequence there exists a pairing up of the nonzero eigenvalues ($i\lambda$, $-i\lambda$). The number of normalizable zero modes modulo 2 of the $D = 3$ Dirac operator in the presence of a P-odd external gauge field is a topological invariant under smooth P-odd deformations of the gauge field. This property also generalizes for the case of nonvanishing Yukawa interactions as we will demonstrate shortly.

The sphaleron configuration in the temporal gauge $A_0^a = 0$ reads as

$$A_i^a = \frac{1}{g} \frac{f(\xi)}{x^2} \epsilon_{iaj} x_j, \quad \varphi = \frac{v}{\sqrt{2}} \frac{h(\xi)}{\sqrt{x^2}} i(x_i \tau_i) \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$

(3.3)

where $\varphi$ is the Higgs doublet, $v$ is its vacuum expectation value, $a$ is an $SU(2)$ index, $g$ is a gauge coupling constant and $\xi = g v |x|$. The dimensionless functions $f(\xi)$ and $h(\xi)$ have the following asymptotic behaviour:

$$f(\xi) = \alpha \xi^2 \quad \text{for} \quad \xi \to 0,$$

(3.4)

$$f(\xi) = 1 - \gamma \exp(-\xi/2) \quad \text{for} \quad \xi \to \infty,$$

$$h(\xi) = \beta \xi \quad \text{for} \quad \xi \to 0,$$

$$h(\xi) = 1 - \delta \exp(-\sqrt{2\lambda/g^2} \xi) \quad \text{for} \quad \xi \to \infty.$$

Here $\lambda$ is the Higgs self coupling constant, and $\alpha, \beta, \gamma$ and $\delta$ are fixed numbers.

The $D = 3$ Dirac operator in this external field has a normalizable zero mode

$$(\sigma D)\psi_0 = 0,$$

(3.5)

where

$$\psi_0^a = \epsilon_{ai} u(x^2).$$

(3.6)

Here $\alpha$ is a spinor index with $i$ being an $SU(2)$ doublet index. The function $u$ has the following behaviour

$$u(x) \propto \exp(-\alpha x^2) \quad \text{for} \quad \xi \to o,$$

(3.7)

$$u(x^2) \propto 1/x^2 \quad \text{for} \quad \xi \to \infty.$$
We use the notations of sect.2 with the appropriate dimensional reduction. Here \( \psi \) is a two component Weyl spinor. By eliminating \( \eta \) in eq.(3.8) we get
\[
(\sigma D)\psi = M(\sigma \partial)^{-1}M^+\psi. \tag{3.9}
\]
Here it should be noted that as \( M \) is completely general the operator \((\sigma D) - M(\sigma \partial)^{-1}M^+\) does not satisfy relation (3.1) and therefore is not purely imaginary. This is true only in the "symmetric" case where \( h_u = h_d \). There all eigenfunctions can be chosen to be real just as in eq.(3.2). In this limit the approximate solution of eq.(3.8) is possible as it was actually done in ref.[9]. In our approach the Yukawa couplings are taken to be arbitrary. We may note that the operator \( \sigma D - M(\sigma \partial)^{-1}M^+ \), as well as the massless Dirac operator, is \( P \)-odd. Hence the number of its zero modes is an invariant modulo 2 under smooth deformations of the gauge and Higgs fields which preserve their \( P \)-odd character. Since the massless Dirac operator has exactly one normalizable zero mode, this is also the case for at least small values of the Yukawa couplings. Below we give an explicit construction of a fermionic zero mode for nonvanishing Yukawa interactions. We now try to solve eq.(3.9) iteratively starting with \( \psi_0 \) as our zero level approximation. The solution \( \psi \) is in this sense expanded in powers of the mass matrix \( M \). At the first step of the iteration procedure one gets
\[
(\sigma D)\psi^{(1)} = M(\sigma \partial)^{-1}M^+\psi_0. \tag{3.10}
\]
Let us firstly check the integrability of the above equation. By this we mean that
\[
a_0 = \int d^3x \psi_0^+ M(\sigma \partial)^{-1}M^+\psi_0 = 0. \tag{3.11}
\]
It is easy to see that this is true. Indeed we note that the sphaleron configuration (3.3) is parity \((P)\) odd, i.e. under the transformation \( x_i \rightarrow -x_i \). The integrand therefore in eq.(3.11) is \( P \)-odd and hence the integral over all directions of \( x_i \) vanishes. Actually one should be careful here since the integral (3.11) is formally logarithmically divergent. In order to make our claims more rigorous we introduce a dimensional regularization. We do our computation in \( D = 3 - \epsilon \) dimensions and then take the limit \( \epsilon \rightarrow 0 \). Our final expressions in eq.(3.11) are \( \epsilon \) independent in this limit.

We must emphasize at this point that we can write down the integrability condition for the exact solution to eq.(3.9). By noting that the left hand side of eq.(3.9) is orthogonal to \( \psi_0 \) we get
\[
\int d^3x \psi_0^+ M(\sigma \partial)^{-1}M^+\psi = 0. \tag{3.12}
\]
It is worthy to emphasize at this point that there is no problem of integrability in the case of the \( D = 4 \) Dirac operator \([12]\) in the weak instanton configuration. There integrability is automatic as a consequence of the anti-selfduality of the anti-instanton gauge field. We find the exact solution to eq.(3.9) to be given by
\[
\psi = \psi_0 + (\sigma D)\frac{1}{((\sigma D)^2 + \alpha P - M(\sigma \partial)^{-1}M^+(\sigma D))}M \frac{1}{(\sigma \partial)}M^+\psi_0. \tag{3.13}
\]
$P$ is the projector onto the zero mode subspace and $\alpha$ is a regularizing parameter which is introduced here to make the expression well defined. One can straightforwardly check that our solution is independent of $\alpha$ and obeys eq. (3.9). To this end we act on the wave function given by (3.13) by the operator $(\sigma D - M(1/\sigma\partial))M^+$. We find an expression that is proportional to the following one

$$P
\frac{1}{(\sigma D)^2 + \alpha P - M(\sigma\partial)^{-1}M^+\sigma D}M \frac{1}{(\sigma\partial)} M^+ \psi_0 = 0,$$

where $P = \psi_0\psi_0^+$. In order to compute the above expression it is convenient that we explicitly split the associated functional space into two subspaces $F = F_\perp \oplus F_0$, where $F_0$ is the one dimensional one which is generated by $\psi_0$ while $F_\perp$ is orthogonal to $\psi_0$. The non-local operator in eq. (3.14) can be represented as a block matrix. More graphically

$$((\sigma D)^2 + \alpha P - M(\sigma\partial)^{-1}M^+\sigma D)^{-1} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where the operator $A = ((\sigma D)^2 - (1 - P)M(\sigma\partial)^{-1}M^+\sigma D)^{-1}$ maps $F_\perp$ into $F_\perp$, $B = (1/\alpha)PM(\sigma\partial)^{-1}M^+\sigma D)A$ maps $F_\perp$ into $F_0$ and finally $C = (1/\alpha)P$ maps $F_0$ onto $F_0$. In such a formulation of the problem we have

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

while $M(\sigma\partial)^{-1}M^+\psi_0$ is now identified by the vector

$$(1 - P)M(\sigma\partial)^{-1}M^+\psi_0, \quad PM(\sigma\partial)^{-1}M^+\psi_0).$$

By a direct substitution of eqs. (3.15) and (3.16) into eq. (3.14) we easily get

$$PM \frac{1}{(\sigma\partial)} M^+ \psi_0 + PM \frac{1}{(\sigma\partial)} M^+ \sigma D \frac{1}{(\sigma D)^2 - M(\sigma\partial)^{-1}M^+\sigma D} \times \frac{1}{(\sigma\partial)} M^+ \psi_0 = 0$$

Equivalently

$$\int d^3x \psi_0^+ \left( M \frac{1}{(\sigma\partial)} M^+ + M \frac{1}{(\sigma\partial)} M^+ (\sigma D - (1 - P)M(\sigma\partial)^{-1}M^+ (1 - P) \times M \frac{1}{(\sigma\partial)} M^+ \right) \psi_0 = 0.$$
wave function (3.13) is normalizable. The argument here is similar to the instanton one \((D = 4)\). At spatial infinity the gauge field is a pure gauge and along with the mass matrix \(M\) are given by

\[
A_i = U \partial_i U^+, \quad M = M_0 U.
\]  

Here \(i = 1, 2, 3\) and \(M_0\) is a constant, whereas \(U\) is an element of \(SU(2)\). For large distances the normalization integral is given by

\[
\int d^3 x \psi^+ \psi \propto \int d^3 x \xi_0^+ \partial^4/\left(\partial^2 - M_0^2\right)^2 \xi_0,
\]

where \(\xi_0 = U M_0\). As this expression is manifestly finite the massive zero mode \(\psi\) is normalizable if we take into account eq.(3.6) for the massless zero mode. At this point we believe that our arguments with regard to the existence of a massive fermion zero mode in the background of a sphaleron field become complete. Level crossing is hence automatically obtained for the general case of fermions getting their masses from Yukawa interactions.

4 Global Anomaly for Massive Fermions

It is well known that \(SP(n)\) gauge theories have no local anomalies. Moreover when coupled with an odd number of Weyl fermions they possess a global anomaly and become self inconsistent. The simplest such example is an \(SU(2)\) gauge theory with one fermionic Weyl doublet. This model was previously considered in some detail for the case of massless fermions \([10]\). In this section we extent Witten’s arguments in order to take into account the presence of Yukawa interactions. We will show that the Atiyah-Singer index theorem mod 2 is sufficient for the presence of the anomaly as a consequence of the existence of a massive normalizable zero mode of the \(D = 5\) Dirac operator.

Let us first sketch Witten’s arguments \([10]\). The partition function of the euclidean version of the model of a doublet of massless fermions coupled to an \(SU(2)\) gauge field reads as follows

\[
Z = \int D\psi_L D\psi_L^+ \int D A_\mu \exp\left(-\int d^4 x \left[ (1/2g^2) Tr F^2_{\mu\nu} + \psi_L^+ i \slashed{D} \psi_L \right] \right).
\]

There \(A_\mu\) is an \(SU(2)\) gauge field, \(\psi_L\) is a left-handed Weyl fermion doublet, \(g\) is the gauge coupling constant, \(\slashed{D} = D_\mu \gamma_\mu\) is a Dirac operator restricted to act on a Weyl doublet. The fermionic part of the integral eq.(4.1) is ill defined. However it can be formally integrated as the square root of a functional integral over Dirac fermions. As such it implies the doubling of the fermionic degrees of freedom from one to two Weyl left-handed doublets. Because the 1/2 representation of \(SU(2)\) is pseudoreal a left-handed doublet can be mapped to the right-handed one. A theory with two left-handed Weyl doublets is thus equivalent to a vector-like one with a single Dirac doublet. The fermionic functional integral is given by \(det(i \slashed{D})\) and it is well defined. Then we formally have that

\[
\int D\psi_L D\psi_L^+ \exp\int \psi_L^+ i \slashed{D} \psi_L = (det i \slashed{D})^{1/2}.
\]
The sign of the square root is ill defined. As a way out Witten defines the root in eq.(4.2) as the product of all positive eigenvalues of a Dirac operator.

If for a given $A_\mu$ the sign in eq.(4.2) is arbitrarily fixed as Witten showed there always exists a configuration $A^U_\mu$ that can be reached continuously from $A_\mu$ for which the fermionic determinant has an opposite sign, i.e.

$$(\det iD(A_\mu))^{1/2} = - (\det iD(A^U_\mu))^{1/2}. \quad (4.3)$$

Here $A_\mu$ is taken to be the gauge transformed configuration of $A_\mu$. This means that the partition function

$$Z = \int DA_\mu (\det iD)^{1/2} \exp(-1/2g^2 \int d^4x Tr F_{\mu\nu}^2) \quad (4.4)$$

vanishes due to the contribution with an opposite sign of $A_\mu$ and $A^U_\mu$. This occurs in all topologically distinct sectors of the field configurations $A_\mu$ independently.

As we continuously vary the external field value from $A_\mu$ to $A^U_\mu$ an odd number of these eigenvalues flow through zero switching their sign. Such a level crossing effect is a reflection of the existence of an odd number of normalizable zero modes for a properly defined $D = 5$ Dirac operator in a topologically nontrivial gauge field. It is a result of the nontrivial fourth homotopy group of $SU(2)$

$$\pi^4(SU(2)) = \mathbb{Z}_2. \quad (4.5)$$

The $D=5$ topologically non-trivial gauge field must accordingly belong to the non-trivial homotopy class in $\mathbb{Z}_2$ and it is the interpolating configuration between $A_\mu$ and $A^U_\mu$. An appropriate $D = 5$ Dirac operator is actually a generalization of the $D = 4$ one that includes evolution in the fifth coordinate $t$. In what follows we generalize Witten’s argument for the case of Yukawa interactions and the presence of fermion mass through the Higgs mechanism. In this context we will prove Witten’s conjecture of the validity of his arguments for this case too. It is worthwhile to also stress that the $SU(2)$ global anomaly can also be understood as a manifestation of the existence of a local anomaly for an $SU(3)$ gauge theory \[18\]. From this point of view the generalization to the case of massive fermions is of course straightforward. This is due to the fact that the local anomaly is independent of Yukawa couplings which are $SU(3)$ invariant \[19\]. It seems nevertheless interesting to try to understand the issue in terms of a level crossing phenomenon for the $D = 4$ Dirac operator. This is the aim of what is to follow. We do it by proving a generalization of the index theorem mod 2 for the massive $D = 5$ Dirac operator.

We first consider the index theorem mod 2 for a massless $D = 5$ Dirac operator. Instead of making a unitary transformation to a real representation for the $D = 5$ fermions which transform under the $O(4) \times SU(2)$ group \[10\] we introduce

$$D = (\gamma_5 \nabla_t + \nabla). \quad (4.6)$$

Here $\nabla_\mu$ is the usual covariant derivative, $\nabla = \nabla_\mu \gamma_\mu$, whereas $\nabla_t$ is the covariant derivative for the fifth coordinate. Eigenvalues and eigenfunctions are defined by the following equation

$$D\psi = i\lambda \psi, \quad (4.7)$$
where \(\lambda\)'s are real since \(\mathcal{D}\) is antihermitean. The operator \(\mathcal{D}\) is real in the following sense

\[ C\gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}. \]  

(4.8)

Here \(C = i\gamma_2\gamma_0\) and \(C\gamma_5\) are the usual \(D = 4\) and \(D = 5\) charge conjugation matrices, \(\epsilon\) is a \(2 \times 2\) antisymmetric matrix that acts on \(SU(2)\) indices. The reality condition implies the pairing up of all the non-zero eigenvalues. More precisely if \(\mathcal{D}\psi = i\lambda\psi\) then

\[ \mathcal{D}(C\gamma_5\psi^*) = -i\lambda C\gamma_5\psi. \]  

(4.9)

The zero mode wave functions \(\psi_0\) (if they exist) can be chosen real satisfying a Majorana-like condition:

\[ C\gamma_5\psi_0^* = \psi_0. \]  

(4.10)

This reality condition means that the number of zero modes is a topological invariant modulo 2 since non-zero modes can cross the zero level only in pairs when external fields vary smoothly. The crucial point is that such a zero mode does exist in the external field due to the index theorem modulo 2 \[16\] if the external field is topologically nontrivial. The reality of the Dirac operator \(\mathcal{D}\) guarantees a pairing of its eigenvalues. They correspond to the existence of the nontrivial homotopy class that arises in the nontrivial topology \(\pi^4(SU(2)) = \mathbb{Z}_2\).

We now proceed to examine massive fermions in such an external field. In this model we have a left handed chiral fermion \(SU(2)\) doublet \(q_L\) and a pair of right handed singlet fermions which can be combined into a doublet \(q_R\). This is necessary for the introduction of fermionic masses. The problem at hand now is how to generalize the definition of the chiral fermionic determinant for the massive case. We find it convenient and natural to define it as a square root of the fermionic functional integral for two fermionic \(q_i^L, q_i^R\), \(i = 1, 2\), multiplets with the same mass matrix \(M\). We follow, to this end, the recipe of sect.2. We combine them into the Dirac spinors \(\psi\) and \(\eta\) whereas \(\psi\) is an \(SU(2)\) doublet and \(\eta\) is a couple of \(SU(2)\) singlet spinors. As a result we get the operator in the fermionic kinetic term as before for the particular case where \(M = M'\). This operator acts in the space of pairs \((\psi, \eta)\) and reads as

\[ T(M) = \begin{pmatrix} \nabla & M\epsilon + L - \epsilon\epsilon L \\ MR - \epsilon M^*\epsilon L \end{pmatrix}, \]  

(4.11)

where \(L(R) = (1 + (-)\gamma_5)/2\). We restrict ourselves to the case of equal fermion masses \((h_u = h_d)\). In this case \(M = -\epsilon M^*\epsilon\) with \(M\) being proportional to an element of \(SU(2)\) group. This assumption will appear necessary with regard to a definition of an appropriate chiral fermionic determinant and a \(D = 5\) antihermitian euclidean Dirac operator. Indeed we note that \(PT(M = -\epsilon M^*\epsilon)\) is antihermitian where \(P\) is given by

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

We thus reach to the following definition for the chiral fermionic determinant for the massive case

\[ \Delta_{ch} = (det T(M))^{1/2}. \]  

(4.12)
In order to generalize our discussion of level crossing we now define an appropriate $D = 5$ operator (for $M = -\epsilon M^* \epsilon$) as
\[
\hat{D} = \gamma_5 \begin{pmatrix} \nabla_t & 0 \\ 0 & -\partial_t \end{pmatrix} + P \hat{T} = \begin{pmatrix} \gamma_5 \nabla_t + \nabla & M \\ -M^* & -\gamma_5 \partial_t - \phi \end{pmatrix}.
\] (4.13)

We observe that $\hat{D}$ satisfies the following reality condition
\[
C \epsilon \gamma_5 \hat{D}^* C \epsilon \gamma_5 = \hat{D}.
\] (4.14)

This is an important property that our Dirac operator ($D = 5$) must satisfy in complete analogy with the massless fermion case. In effect there is a pairing up of all of the non-zero eigenvalues in its spectrum. This furthermore implies that the number of its zero modes, if they exist, is a topological invariant mod 2.

The solution $\Psi_0$ to the zero mode equation
\[
\hat{D} \Psi_0 = 0
\] (4.15)
can be chosen real such that
\[
C \epsilon \gamma_5 \Psi_0^* = \Psi_0.
\] (4.16)

Notice that such a solution is not of any definite chirality. We may now represent $\Psi_0$ as a superposition of an $SU(2)$ doublet $\psi$ and two $SU(2)$ singlets $\chi_1$ and $\chi_2$, i.e. $\Psi_0 = (\psi, \chi)$ with $\chi = (\chi_1, \chi_2)$. The zero mode equations then read as
\[
\hat{D} \psi + M \chi = 0, \quad M^* \psi + \hat{D} \chi_0 = 0,
\] (4.17)
where
\[
(\gamma_5 \nabla_t + \nabla) = \hat{D}, \quad (\gamma_5 \partial_t + \phi) = \hat{D}_0.
\] (4.18)

By eliminating $\chi$ from eqs.(4.17) we get
\[
\hat{D} \psi = M(1/\hat{D}_0) M^* \psi.
\] (4.19)

Let us assume that there is exactly one zero mode of the operator $\hat{D}$ such as in the case of eq.(4.15). We want to find the massive zero mode wave function, and moreover express the solution of the above equation in terms of the zero mode wave function $\psi_0$ for the massless case. This state of affairs is similar to that of the $D = 3$ spheralon example.

We search for a solution iteratively by starting with $\psi_0$ as a zero level approximation. The first level correction satisfies the following equation
\[
\hat{D} \psi^{(1)} = M \frac{1}{\hat{D}_0} M^* \psi_0.
\] (4.20)

The integrability condition is deduced from the orthogonality of the left hand side of eq.(4.20) to $\psi_0$
\[
a_0 = \int d^5 x \psi_0^+ M \frac{1}{\hat{D}_0} M^* \psi_0 = 0.
\] (4.21)
Indeed if we make a transposition in eq.(4.21) and subsequently take into account the reality conditions (4.14,4.16) for the anti-hermitean operator $D_0$ we get that $a_0 = -a_0 = 0$.

It is amusing to note that we did not use so far any properties of the external fields in contrast to the sphaleron case. This is due to the reality property of the $D = 5$ operator $D$ whereas the $D = 3$ Dirac operator is purely imaginary. We may now solve eq.(4.20)

$$
\psi^{(1)} = \frac{1}{D} M \frac{1}{D_0} M^+ \psi_0.
$$

This is a well defined expression on the basis of eq.(4.21). The exact solution to eq.(4.19) is given by

$$
\psi = \psi_0 + \frac{1}{D^2 + \alpha P - M D_0^{-1}} \frac{1}{M} \frac{1}{D_0} M^+ \psi_0,
$$

where $P$ is the zero mode subspace projector. This expression is well defined and does not depend on a regularizing parameter $\alpha$. By substitution of (4.23) for $\psi$ into eq.(4.19) we get the following equation

$$
P \frac{1}{D^2 + \alpha P - M D_0 M^+ D_0^{-1}} M \frac{1}{D_0} M^+ \psi_0 = 0.
$$

By using the block matrix representation similar to the case of the sphaleron we find an equivalent form of eq.(4.24)

$$
\int d^3 x \psi_0^+ \left( \frac{1}{D_0} M^+ M + \frac{1}{D_0} M^+ (1 - P) \frac{1}{D - (1 - P)M D_0^{-1}} \frac{1}{M} \frac{1}{D_0} M^+ (1 - P) \right) \times \\
\times (1 - P) M \frac{1}{D_0} M^+ \psi_0 = 0.
$$

The above expression is well defined. It is certainly satisfied as a consequence of the reality conditions (4.14,4.16) and the mass degeneracy condition $M = -\epsilon M^* \epsilon$. This can be checked by a transposition similar to the case of eq.(4.21).

Let us now turn to the issue of normalizability of $\psi$ in eq.(4.23). It is easy to see that this wave function decreases more rapidly at infinity provided that the mass matrix $M$ is an asymptotically covariant constant similar to the case of the $D = 4$ instanton + Higgs configurations. This necessitates that the gauge field is asymptotically a pure gauge $A_\mu \propto U \partial_\mu U^{-1}$ with the mass matrix being $M \propto UM_0$ at infinity where $U$ is an element of the $SU(2)$ group and $M_0$ is a constant. In this case the wave function (4.23) is normalizable. It is to be emphasized at this point that by changing $M$ from zero to a nonzero value we can obtain additional zero modes but always in pairs. It is of some significance to note that this condition implies that in the presence of Yukawa interactions the interpolation between two $D = 4$ configurations of gauge fields $A_\mu$ and $A_\mu^U$ which is given by a $D = 5$ gauge field configuration should also include Higgs fields.
We now turn to the case where the operator $\mathcal{D}$ has a number of normalizable zero modes $\psi_{0i}$, $i = 1, ..., N$. These zero modes can be chosen to obey the reality condition

$$C\gamma_5 \epsilon \psi^{*}_{0i} = \psi_{0i}. \quad (4.26)$$

In order to solve eq.(4.19) we start the iteration procedure with a linear combination

$$\psi_0 = c_i \psi_{0i} \quad (4.27)$$
as a zero level approximation. At the first step we get eq.(4.20). The integrability condition reads now as follows for all $i$

$$\int d^5x \psi^+_{0i} M D_0^{-1} M^+ \psi_0 = 0. \quad (4.28)$$

This means that the vector $c_i$ should be annihilated by the matrix

$$a_{ij}^{(0)} = \int d^5x \psi^+_{0i} M D_0^{-1} M^+ \psi_{0j}. \quad (4.29)$$

A solution of eq.(4.20) and hence of eq.(4.19) exists only if the matrix $a_{ij}^{(0)}$ has zero eigenvalues. By considering the Hermitian conjugate and the complex one of $a_{ij}$ we can easily check that this matrix is anti-hermitian and purely real

$$a_{ij}^{(0)+} = -a_{ji}^{(0)}, \quad a_{ij}^{(0)*} = a_{ij}^{(0)}. \quad (4.30)$$

In general this matrix is non-zero and belongs to the $O(N)$ algebra. Let us first consider the case of an odd number of $\psi_0$'s, i.e. $N = 2n + 1$. It is easy to see that the matrix $a_{ij}^{(0)}$ has at least one zero eigenvalue. Indeed it is clear that $O(N)$ transformations preserve the reality conditions (4.26) and (4.16). By making successive $O(N)$ rotations we can bring the matrix $a_{ij}^{(0)}$ in the form for which all non-vanishing elements are organized into cells proportional to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.31)$$

which are located along the diagonal. These cells act onto $n$ orthogonal subspaces of the space of $O(2n+1)$ vectors. Since $N$ is odd at least one of eigenvalues of the matrix $a_{ij}^{(0)}$ is zero. The other eigenvalues are combined into pairs $(i\lambda, -i\lambda)$ corresponding to cells given above. In general they do not vanish. However when the mass matrix $M$ varies the zero eigenvalues of the matrix $a_{ij}^{(0)}$ can appear or disappear in pairs.

In the case of even $N$, i.e. $N = 2n$, all eigenvalues of $a_{ij}^{(0)}$ are generically non-zero and are arranged in pairs $(i\lambda, -i\lambda)$ as above.

We get the zero level approximate solution to eq.(4.19) by picking the eigenvector of $a_{ij}^{(0)}$ that corresponds to the zero eigenvalue for the vector $c_i$ in eq.(4.27). It is obvious from the above construction that the 'oddness' of the number of the fermionic zero modes does not change. In particular there is at least one normalizable zero mode of massive fermion in the $D = 5$ topologically nontrivial configuration since it has exactly one zero mode for massless ones.
The exact solution to eq.(4.19) is given by (4.23) where \( P \) is the projector onto the total zero mode subspace. We must be more careful here than with the case of the one massless zero mode \((N = 1)\) since in general the function \( M \tilde{\mathcal{D}}_0^{-1} M^+ \psi_0 \) is not orthogonal to the zero mode subspace. By the use of the block matrix representation it is easy to see that the expression (4.23) obeys eq.(4.19) provided that the following holds true for all \( i \)

\[
\int d^5 x \psi^+_0 \left( M \frac{1}{\tilde{\mathcal{D}}_0} M^+ + M \frac{1}{\tilde{\mathcal{D}}_0} (1 - P) \times \frac{1}{\tilde{\mathcal{D}} - (1 - P) M \tilde{\mathcal{D}}_0^{-1} M^+ (1 - P) M \frac{1}{\tilde{\mathcal{D}}_0} M^+ \right) \psi_0 = 0.
\]

The above expression is well defined and implies that the vector \( c_j \) should be the eigenvector that corresponds to the zero eigenvalue of the following matrix

\[
a_{ij} = \int d^5 x \psi^+_0 \left( M \frac{1}{\tilde{\mathcal{D}}_0} M^+ + M \frac{1}{\tilde{\mathcal{D}}_0} (1 - P) \times \frac{1}{\tilde{\mathcal{D}} - (1 - P) M \tilde{\mathcal{D}}_0^{-1} M^+ (1 - P) M \frac{1}{\tilde{\mathcal{D}}_0} M^+ \right) \psi_0^j,
\]

i.e.

\[
\sum_j a_{ij} c_j = 0.
\]

In the leading order in the Yukawa couplings this condition reproduces eq. (4.28). It can be easily seen that it is equivalent to the integrability condition analogous to eq.(4.21). One may observe that the matrix \( a_{ij} \) is purely real and antisymmetric by using the reality condition (4.26). By repeating here the procedure we followed for the leading approximation we find that the matrix \( a_{ij} \) has exactly an odd number of zero eigenvalues for odd \( N = 2n + 1 \) and correspondingly an even number of zero eigenvalues for even \( N = 2n \). In the latter case the zero eigenvalues can be absent. Hence there is an invariance for the odd/even number of fermionic zero modes under smooth deformations of the mass matrix \( M \) (see eq.(4.19)).

At this point our arguments that generalize Witten’s observation of an \( SU(2) \) anomaly for odd number of massless fermion doublets in the presence of Yukawa interactions are complete. The existence of the zero mode for massive \( D = 5 \) Dirac operator was demonstrated for the case of degenerate mass matrix. The treatment of the general non-symmetric \( (h_u \neq h_d) \) case will be present elsewhere.

**Conclusions**

In the present work we considered the effect of Yukawa couplings and fermion masses on the problem of existence of fermionic zero modes in the presence of topologically nontrivial fields for an \( SU(2) \) gauge theory. We demonstrated the existence of normalizable zero energy solutions of the Dirac operator in the presence of a sphaleron the instanton and that of the \( SU(2) \) global anomaly. As such we equivalently confirmed the existence of the fermionic energy level crossing phenomenon in
the presence of time dependent external gauge and Higgs fields in all three physical situations. While the first two cases are relevant to the Baryon and Lepton quantum number violations in the standard electroweak theory the latter is associated with the global $SU(2)$ anomaly in the presence of an odd number of Weyl doublets.

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