Extremal Kerr black hole entropy in Poincaré gauge theory

B. Cvetković and D. Rakonjac*
Institute of Physics, University of Belgrade,
Pregrevica 118, 11080 Belgrade-Zemun, Serbia

Abstract

We analyze the near horizon symmetry of the extremal Kerr black hole within the framework of Poincaré gauge theory (PG) for two important limiting cases: Riemannian and teleparallel solution. We show that the algebra of canonical generators is realized by Virasoro algebra, with central charge which depends on the black hole horizon radius. The conformal entropy of the black hole is obtained via Cardy formula.

1 Introduction

Recently a new Hamiltonian method [1] for the computation of black hole entropy within the framework of Riemann-Cartan geometry has been proposed. The method has been verified for a number of vacuum solutions such as Schwarzschild(-AdS), Kerr(-AdS) solution as well as a solution coupled to electromagnetic field, Kerr-Newmann-AdS solution, [1, 2, 3, 4, 5].

The method [1] is based on a variational principle, originally proposed by Regge and Teitelboim, see [6]. The black hole entropy is obtained from the variation of the boundary term on the black hole horizon, i.e. $T\delta S = \delta \Gamma_H$. In the framework of Riemanninan geometry this method was established and developed by Wald [7]. Moreover, the differentiability of the canonical generator is closely related to the validity of the first law of the black hole mechanics.

The method [1] is inapplicable in the case of the extremal black holes. Namely, in that case black hole temperature vanishes, $T = 0$, and the equation $T\delta S = \delta \Gamma_H$ cannot be solved for the black hole entropy. For extremal black holes the first law is satisfied disregarding the value of the black hole entropy. However in general relativity (GR), there is another way of computing black hole entropy of the extremal Kerr black holes based on near horizon conformal symmetry, in regard of the recently established Kerr/CFT correspondence [8], see also [9].

The subject of the present paper is the computation of the black hole entropy for the extremal Kerr black holes in the framework of PG, where both curvature and torsion influence the gravitational dynamics [10, 11, 12], by analyzing the near horizon conformal symmetry.
symmetry. Let us note that near horizon structure of black holes with torsion has already been examined within three-dimensional gravity [13].

After introducing the suitable set of consistent near horizon boundary conditions for extremal Kerr black holes in PG, we obtain that asymptotic symmetry group has a conformal subgroup, realized by Virasoro algebra. We shall show that the first order formulation of the generator of the local symmetry derived in [1], can be used to compute the near horizon algebra of the improved generators, as well as the corresponding central charge. These results are used to compute conformal entropy of the extremal Kerr black hole via Cardy’s formula. The result for the entropy represents a smooth limit of the result for the gravitational black hole entropy of the generic (non-extremal) Kerr black hole. Thus, we demonstrated the full power of the Nester’s covariant Hamiltonian approach [14], and contributed to the better understanding of the equality of gravitational and conformal entropy.

The paper is organized as follows. In section 2 we shall introduce the tetrad formulation of the extremal Kerr black hole solution and near horizon geometry (NHEK) in the framework of PG. The suitable set of consistent asymptotic conditions for near horizon geometry in tetrad formalism of PG is established in section 3. Inspection of the symmetries that preserve these boundary conditions leads to conformal symmetry of NHEK geometry. In the section 4 we shall consider the canonical realization of the near horizon conformal symmetry for Riemannian solution in PG. We shall make use of the canonical generator from the first order formulation obtained in [1] to compute the conserved and central charge of the conformal near horizon symmetry, which depends on black hole horizon radius. The conserved and central charge are going to be used in the Cardy’s formula to compute the conformal black hole entropy. Another important limiting case of PG, teleparallel gravity, where gravitational dynamics is characterized by vanishing curvature and non-vanishing torsion, is analyzed in section 5. Section 6 is devoted to concluding remarks, while appendices contain some technical details.

Our conventions are the same as in ref. [5]. The Latin indices \((i, j, \ldots)\) are the local Lorentz indices, the Greek indices \((\mu, \nu, \ldots)\) are the coordinate indices, and both run over \(0, 1, 2, 3\). The orthonormal coframe (tetrad) \(\psi^i\) and the metric compatible (Lorentz) connection \(\omega^{ij} = -\omega^{ji}\) are 1-forms, the dual basis (frame) is \(e_i = e_i^\mu \partial_\mu\). The metric components in the local Lorentz and coordinate basis are \(\eta_{ij} = (1, -1, -1, -1)\) and \(g_{\mu\nu} = \eta_{ij} \partial^i_\mu \partial^j_\nu\), respectively, and \(\varepsilon_{ijmn}\) is the totally antisymmetric symbol with \(\varepsilon_{0123} = 1\). The Hodge dual of a form \(\alpha\) is denoted by \(*\alpha\), and the wedge product of forms is implicit.

2 Tetrad formulation of extremal Kerr black hole geometry

In this section we shall introduce the tetrad formulation of the extremal Kerr black hole geometry. We shall introduce the near horizon geometry (NHEK), which represents our starting point in the study of the near horizon structure of the extremal Kerr black holes within Riemann-Cartan geometry.
2.1 Metric, conserved charges and the first law

Let us now give a brief overview of the basic features of the extremal Kerr black holes. We use the same notation as in [2]. A “diagonal” form of the extremal Kerr metric \((m = a)\) in Boyer-Linquist coordinates [15]

\[
ds^2 = N^2 \left( dt + m \sin^2 \theta d\phi \right)^2 - \frac{dr^2}{N^2} - \rho^2 d\theta^2 - \frac{\sin^2 \theta}{\rho^2} \left[ m dt + (r^2 + m^2) d\phi \right]^2,
\]

(2.1a)

where

\[
N = \frac{r - m}{\rho}, \quad \rho^2 := r^2 + m^2 \cos^2 \theta.
\]

(2.1b)

The equation \(N = 0\) defines the extremal black hole horizon, which radius \(r_+\) is given by

\[
r_+ = m.
\]

(2.2)

The two horizons that exist in Kerr solution coincide in the extremal case. The value of the angular velocity \(\Omega_+\) on the horizon takes the simple form

\[
\Omega_+ = \frac{1}{2r_+} \equiv \frac{1}{2m},
\]

(2.3)

while the surface gravity and temperature vanish

\[
\kappa = \frac{r_+ - m}{2mr_+} = 0, \quad T = \frac{\kappa}{2\pi} = 0.
\]

(2.4)

The conserved charges energy and angular momentum of Kerr black hole in PG take the form [2]

\[
E = m, \quad J = m^2.
\]

(2.5)

Let us note that the first law of black hole thermodynamics reads

\[
0 = T \delta S \equiv \delta E - \Omega_+ \delta J = \delta m - \frac{1}{2m} \delta m^2.
\]

(2.6)

It is satisfied in the of the extremal Kerr black hole disregarding the value of the black hole entropy.

2.2 Near horizon extremal Kerr geometry

Let us now introduce the following coordinate transformations [8, 9]

\[
\tilde{t} = \frac{\varepsilon t}{2r_+}, \quad y = \frac{\varepsilon r_+}{r - r_+}, \quad \varphi = \phi + \Omega_+ t,
\]

(2.7)

along with the limit \(\varepsilon \to 0\). The metric takes the following form

\[
ds^2 = r_+^2 (1 + \cos^2 \theta) \left[ \frac{d\tilde{t}^2}{y^2} - \frac{dy^2}{y^2} - d\theta^2 - \left( \frac{2 \sin \theta}{1 + \cos^2 \theta} \right)^2 \left( d\varphi - \frac{d\tilde{t}}{y} \right)^2 \right].
\]

(2.8)
Let us note that the above transformation is not a simple coordinate transformation, in that the resulting geometry is not equivalent to the starting one. That can be readily seen by noticing that the above metric is not asymptotically flat. The resulting geometry has been extensively studied in [8, 16] Tetrads. The form of the metric implies the following "diagonal" choice of the vielbein $\vartheta^i$

\[
\vartheta^0 = \frac{\rho_+}{y} d\tilde{t}, \quad \vartheta^1 = -\frac{\rho_+}{y} dy, \\
\vartheta^2 = \rho_+ d\theta, \quad \vartheta^3 = \frac{2 \sin \theta}{\sqrt{1 + \cos^2 \theta}} r_+ \left( d\varphi - \frac{d\tilde{t}}{y} \right),
\]

(2.9)

where $\rho_+ = r_+ \sqrt{1 + \cos^2 \theta}$

For later convenience let us now also display the form of the Riemannian connection and curvature.

Riemannian connection. From the relation $d\vartheta^i + \tilde{\omega}^i_{\;k} \vartheta^k = 0$, we get that the nonzero components of the Riemannian connection are

\[
\tilde{\omega}^{01} = -\frac{\vartheta^0}{\rho_+} \frac{r_+ \sin \theta}{\rho_+^3} \vartheta^3, \quad \tilde{\omega}^{02} = \frac{r_+ \sin 2\theta}{2\rho_+^3} \vartheta^0, \quad \tilde{\omega}^{12} = \frac{r_+ \sin 2\theta}{2\rho_+^3} \vartheta^1,
\]

\[
\tilde{\omega}^{03} = -\frac{r_+^2 \sin \theta}{\rho_+^3} \vartheta^1, \quad \tilde{\omega}^{13} = -\frac{r_+ \sin \theta}{\rho_+^3} \vartheta^0, \quad \tilde{\omega}^{23} = \frac{2r_+^2 \cot \theta}{\rho_+^3} \vartheta^3.
\]

(2.10)

Riemannian curvature. The Riemannian curvature reads

\[
\tilde{R}^{01} = -2C \vartheta^0 \vartheta^1 - 2D \vartheta^2 \vartheta^3, \quad \tilde{R}^{02} = C \vartheta^0 \vartheta^2 - D \vartheta^1 \vartheta^3,
\]

\[
\tilde{R}^{03} = C \vartheta^0 \vartheta^3 + D \vartheta^1 \vartheta^2, \quad \tilde{R}^{12} = C \vartheta^1 \vartheta^2 - D \vartheta^0 \vartheta^3,
\]

\[
\tilde{R}^{13} = C \vartheta^1 \vartheta^3 + D \vartheta^0 \vartheta^2, \quad \tilde{R}^{23} = -2C \vartheta^2 \vartheta^3 + 2D \vartheta^0 \vartheta^1,
\]

(2.11)

where

\[
C = \frac{r_+^4}{\rho_+^4} (1 - 3 \cos^2 \theta), \quad D = \frac{r_+^4 \cos \theta}{\rho_+^6} (3 - \cos^2 \theta).
\]

### 3 Asymptotic conditions and asymptotic symmetry

In this section we shall first introduce the asymptotic conditions for the metric of the extremal Kerr black hole solution in the near horizon region. Due to the fact that NHEK is not asymptotically flat, finding consistent asymptotic boundary conditions is not a priori an obvious task. This result has been established in GR [8], however, care should be taken, as asymptotic boundary conditions of the metric do not precisely dictate the asymptotic boundary conditions for the tetrad. Therefore the consistent choice for the tetrad, as well as near horizon conformal symmetry will be provided in this section.

Metric asymptotics. In accordance with [8] we introduce the following set of consistent asymptotic conditions for the metric near the asymptotic boundary $y = 0$

\[
\tilde{g}_{\mu \nu} \sim \begin{pmatrix}
\mathcal{O}_{-2} & \mathcal{O}_0 & \mathcal{O}_1 & \bar{g}_{i\varphi} + \mathcal{O}_0 \\
\mathcal{O}_0 & \bar{g}_{yy} + \mathcal{O}_{-1} & \mathcal{O}_0 & \mathcal{O}_{-1} \\
\mathcal{O}_1 & \mathcal{O}_0 & \bar{g}_{\theta\varphi} + \mathcal{O}_1 & \mathcal{O}_1 \\
\bar{g}_{i\varphi} + \mathcal{O}_0 & \mathcal{O}_{-1} & \mathcal{O}_1 & \mathcal{O}_0
\end{pmatrix},
\]

(3.1)
where

\[ \bar{g}_{tt} = \frac{r_{+}^{2}(1 + \cos^{2} \theta)}{y^{2}} - \frac{4\sin^{2} \theta}{1 + \cos^{2} \theta} \frac{r_{+}^{2}}{y^{2}}, \]
\[ \bar{g}_{t\phi} = \frac{4\sin^{2} \theta}{1 + \cos^{2} \theta} \frac{r_{+}}{y}, \]
\[ \bar{g}_{yy} = -\frac{r_{+}^{2}(1 + \cos^{2} \theta)}{y^{2}}, \]
\[ \bar{g}_{\theta\theta} = -r_{+}^{2}(1 + \cos^{2} \theta), \]

are background metric components and we use the notation \( O_{n} := O(y^{n}) \).

**Tetrad fields.** The asymptotic form of the vielbein is given by

\[
\bar{\vartheta}^{i}_{\mu} \sim \begin{pmatrix}
O_{-1} & O_{1} & O_{2} & O_{1} \\
O_{1} & \bar{\vartheta}^{1}_{y} + O_{0} & O_{1} & O_{0} \\
O_{1} & O_{0} & \bar{\vartheta}^{2}_{\theta} + O_{1} & O_{1} \\
\bar{\vartheta}^{3}_{i} f(\varphi) + O_{0} & O_{1} & O_{2} & \frac{\bar{\vartheta}^{3}_{\phi}}{f(\varphi)} + O_{1}
\end{pmatrix},
\]

where background tetrad fields are given by

\[
\bar{\vartheta}^{i}_{\mu} = \begin{pmatrix}
\rho_{+} \\
y \\
0 \\
0 \\
-\frac{2\sin \theta r_{+}}{\sqrt{1 + \cos^{2} \theta}} \\
0 \\
\rho_{+} \\
0 \\
\rho_{+} \\
0 \\
\frac{2\sin \theta r_{+}}{\sqrt{1 + \cos^{2} \theta}}
\end{pmatrix},
\]

where \( f(\varphi) = 1 + h(\varphi) \) is an arbitrary function of \( \varphi \), such that \( h(\varphi) \ll 1 \).

**Asymptotic symmetry.** The transformation law of \( \bar{\vartheta}^{i}_{\mu} \) under PG transformations reads

\[ \delta_{0}\bar{\vartheta}^{i}_{\mu} = \theta^{i}_{k}\bar{\vartheta}^{k}_{\mu} - (\partial_{\mu} \xi^{\rho})\bar{\vartheta}^{i}_{\rho} - \xi^{\rho}\partial_{\rho}\bar{\vartheta}^{i}_{\mu}, \]

where \( \xi^{\mu} \) and \( \theta^{ij} \) are parameters of local translations and local Lorentz rotations, respectively.

The asymptotic form of the metric is preserved by asymptotic Killing vector \( \xi^{\mu} \) of the following form

\[ \xi^{t} = T + O_{3}, \quad \xi^{y} = y \partial_{\phi} \epsilon(\varphi) + O_{2}, \quad \xi^{\theta} = O_{1}, \quad \xi^{\phi} = \epsilon(\varphi) + O_{2}. \]

The transformation corresponding to \( T \) is a constant time translation, and we are able to restrict our attention to the conformal group disregarding this transformation, due to the fact that its generator commutes with the generator of the conformal symmetry. The subdominant terms correspond to trivial diffeomorphisms, and they be disregarded, so that the final form of the asymptotic Killing vector reads

\[ \xi = (y \partial_{\phi} \epsilon(\varphi)) \partial_{y} + \epsilon(\varphi) \partial_{\phi}. \]
All the parameters of Lorentz rotations obtained from the invariance of the tetrad fields are asymptotically vanishing
\[
\theta^{01} = O_2, \quad \theta^{02} = O_2, \quad \theta^{03} = O_1, \\
\theta^{12} = O_1, \quad \theta^{13} = O_2, \quad \theta^{23} = O_2.
\] (3.5c)

The Riemannian connection can be expressed in terms of tetrad fields and therefore its asymptotic form is invariant under transformations (3.5).

The transformations with \( \epsilon = 0 \) represent \textit{residual gauge transformations} which give trivial contribution to the conserved charge. Therefore, the asymptotic symmetry group is defined as a factor group with respect to residual transformations. From the general algebra of PG we get the composition rule for the asymptotic transformations
\[
[\delta_0(\epsilon_1), \delta_0(\epsilon_2)] = \delta_0(\epsilon_3), \\
\epsilon_3 = \epsilon_1 \epsilon'_2 - \epsilon_2 \epsilon'_1,
\] (3.6)
where \( \epsilon' := \partial_\varphi \epsilon \). In terms of Fourier modes
\[
\ell_n := \delta_0(\epsilon = e^{in\varphi}),
\]
the algebra of the asymptotic symmetry takes the Virasoro form
\[
[\ell_n, \ell_m] = i(m - n) \ell_{m+n}.
\] (3.7)

In what follows we shall analyze the canonical realization of the asymptotic symmetry in the two important cases – Riemannian PG solution and teleparallel solution.

4 Riemannian extremal Kerr black hole in PG

In this section we shall analyze the Riemannian extremal Kerr black hole within the framework of PG. It is well known that Kerr black hole is a solution of the GR field equations with vanishing cosmological constant \( \Lambda = 0 \), and so is its extremal case. Existence of the corresponding near horizon geometry is a property of the extremal Kerr, and it should be noted that this near horizon geometry is the solution of GR field equations as well [17]. From the general theorem which states that GR solutions also represent solutions of PG, one can conclude that the Kerr black hole, as well as NHEK also satisfy, the PG field equations for \( \Lambda = 0 \). There is also a direct proof based on the form of the effective PG Lagrangian [2]
\[
L_G = -\ast(a_0 R + 2\Lambda) + \frac{1}{2} b_1 R^{ij} \ast R_{ij},
\] (4.1)
which defines the corresponding covariant momenta as \( H_i = 0 \) and
\[
H_{ij} = -2a_0 \ast(\partial_i \varphi) + b_1 \ast R_{ij},
\] (4.2a)
or in more detail

\begin{align*}
H_{01} &= -2a_0 \vartheta^2 \vartheta^3 + 2b_1 (-2C \vartheta^2 \vartheta^3 + 2D \vartheta^0 \vartheta^1), \\
H_{02} &= 2a_0 \vartheta^1 \vartheta^3 + 2b_1 (-C \vartheta^1 \vartheta^3 - D \vartheta^0 \vartheta^2), \\
H_{03} &= -2a_0 \vartheta^1 \vartheta^2 + 2b_1 (C \vartheta^0 \vartheta^3 + D \vartheta^1 \vartheta^2), \\
H_{12} &= -2a_0 \vartheta^0 \vartheta^3 + 2b_1 (-C \vartheta^0 \vartheta^2 + D \vartheta^1 \vartheta^3), \\
H_{13} &= 2a_0 \vartheta^0 \vartheta^1 + 2b_1 (-2C \vartheta^0 \vartheta^1 - 2D \vartheta^2 \vartheta^3), \\
H_{23} &= -2a_0 \vartheta^0 \vartheta^1 + 2b_1 (-2C \vartheta^0 \vartheta^1 - 2D \vartheta^2 \vartheta^3).
\end{align*}

(4.2b)

In the following subsections, the conserved and central charge on the horizon will be computed. We shall make use of the general expression for the variation of the canonical generator on the horizon [1]

\begin{equation}
\delta \Gamma_H = \oint_{S_H} \delta B(\xi),
\end{equation}

\begin{equation}
\delta B(\xi) := (\xi,J J^j)\delta H_i + \delta \vartheta^i (\xi,J H_i) + \frac{1}{2}(\xi,J \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta \omega^{ij}(\xi,J \delta H_{ij}).
\end{equation}

where \(\xi\) is either exact or an asymptotic Killing vector.

### 4.1 Conserved charge

Let us now compute the conserved charge on the horizon. It is obtained for \(\xi = \partial_\varphi\). Since \(H_i = 0\) the variation of the canonical generator \([4.3]\) reduces now to:

\begin{align*}
\delta \Gamma_H := \oint_{S_H} \delta B, \\
\delta B = \frac{1}{2}\omega_{ij}\varphi \delta H_{ij} + \frac{1}{2}(\delta \omega^{ij})H_{ij}\varphi.
\end{align*}

The non-vanishing contribution to the conserved charge stems from:

\begin{equation}
\tilde{\omega}^{01}\varphi \delta H_{01} = \frac{4a_0 \sin^2 \theta}{(1 + \cos^2 \theta)^2} \delta (2 \sin \theta r^2_+) d\theta d\varphi.
\end{equation}

Now we get that the conserved charge reads:

\begin{equation}
J = \oint_{S_H} \tilde{\omega}^{01}\varphi \delta H_{01} = 16\pi a_0 r^2_+ \equiv r^2_+,
\end{equation}

where we used

\[\int_0^\pi \frac{\sin^3 \theta}{(1 + \cos^2 \theta)^2} d\theta = 1.\]

### 4.2 Central charge and black hole entropy

We shall compute the central charge from the algebra of the improved canonical generators, which has the following form:

\[\{\tilde{G}(\epsilon_1), \tilde{G}(\epsilon_2)\} = \tilde{G}(\epsilon_3) + C,\]

(4.7)
where $\epsilon_3$ is defined by the composition rule (3.6) and $C$ is the central term of the algebra.

After using the main result of the seminal Brown-Henneaux paper [18], the canonical algebra (4.7) can be simplified and it takes the form of the following weak equality:

$$\{\tilde{G}(\epsilon_1), \tilde{G}(\epsilon_2)\} \approx \delta_0(\epsilon_1)\Gamma_H(\epsilon_2) \approx \Gamma_H(\epsilon_3) + C,$$

(4.8)

The central term is a constant functional and therefore it can be computed by varying the background configuration. Non-zero contributions to the above variation are given by

$$\frac{1}{2} \oint_{S_H} (\xi_2 \bar{\omega}^{ij})\delta_0(\xi_1)\bar{H}_{ij} + \delta_0(\xi_1)\bar{\omega}^{ij}(\xi_2 \bar{H}_{ij}) = 8a_0 r_+^2 \int_0^{2\pi} (\epsilon_1 \epsilon_2' - \epsilon_2 \epsilon_1')d\varphi$$

$$- 4a_0 r_+^2 \int_0^{2\pi} (\epsilon_1' \epsilon_2'' - \epsilon_2' \epsilon_1'')d\varphi$$

(4.9)

The first term in the equation above can be identified with the surface term with parameter $\epsilon_3 = \epsilon_1 \epsilon_2' - \epsilon_2 \epsilon_1'$, while the second one gives the central charge

$$C = -4a_0 r_+^2 \int_0^{2\pi} (\epsilon_1' \epsilon_2'' - \epsilon_2' \epsilon_1'')d\varphi.$$  

(4.10)

For the computational details see appendix A.

In terms of Fourier modes the canonical algebra of the improved generators reads:

$$\{L_n, L_m\} = -i(n - m)L_{m+n} - \frac{c}{12}i n^3\delta_{n,-m},$$

(4.11)

where in the string theory normalization

$$c = 12 \cdot 16 \pi a_0 r^2_+ = 12 r^2_+ \equiv 12 J.$$  

(4.12)

Let us note that central charge does not depend on action parameter $b_1$, but does depend on the parameter of the horizon radius $r_+$.

Now the entropy can be calculated via Cardy’s formula:

$$S = 2\pi \sqrt{\frac{c}{6} \left( J - \frac{c}{24} \right)} = 2\pi r^2_+.$$  

(4.13)

The result for the conformal entropy of the extremal Riemannian Kerr black hole in PG represents a smooth limit obtained from the expression for gravitational entropy of the generic Kerr black hole in the same theory [2].

5 Extremal Kerr black hole in TG

Teleparallel gravity is a special case of PG, which is defined by the condition of vanishing Riemann-Cartan curvature, $R^{ij} = 0$ [19]. The Kerr solution does indeed solve the equations of motion of teleparallel gravity [2].

Let us note that from the condition $R^{ij} = 0$ it does not follow that the connection $\omega^{ij}$ (which is a "pure gauge") vanishes. Since, connection does not influence the PG dynamics,
we shall adopt the simplest choice $\omega^{ij} = 0$. Thus, the tetrad field remains the only dynamical variable, and torsion takes the form $T^i = d\vartheta^i$. For the spacetime with tetrad (2.9), the nonvanishing components of torsion are given by

\[
T^0 = -\frac{1}{\rho_+} \vartheta^0 \vartheta^1 + \frac{r_+^2 \sin 2\theta}{2\rho_+^3} \vartheta^0 \vartheta^2, \quad T^1 = \frac{r_+^2 \sin 2\theta}{2\rho_+^3} \vartheta^1 \vartheta^2, \\
T^3 = \frac{2r_+^2 \sin \theta}{\rho_+^3} \vartheta^0 \vartheta^1 + \frac{2r_+^2 \cos \theta}{\rho_+^3 \sin \theta} \vartheta^2 \vartheta^3.
\]

All three irreducible parts of $T^i$ are nonvanishing.

The Lagrangian of the teleparallel equivalent of GR, so called $\text{GR}_\parallel$, is given by

\[
L_T := a_0 T^i \star \left( (1) T^i - 2^{(2)} T^i - \frac{1}{2} (3) T^i \right). \tag{5.2}
\]

This equivalence ensures that every vacuum solution of GR is also a solution of $\text{GR}_\parallel$ and in particular, this is true for the extremal Kerr spacetime. Though the two theories are dynamically equivalent, their geometric content is quite different: GR is characterized by a Riemannian curvature and vanishing torsion, whereas the teleparallel geometry of $\text{GR}_\parallel$ has a nontrivial torsion but vanishing curvature.

The covariant momentum is given by

\[
H^i = 2a_0 \star \left( (1) T^i - 2^{(2)} T^i - \frac{1}{2} (3) T^i \right), \tag{5.3a}
\]

and its explicit form reads

\[
H^0 = 2a_0 \left( -\frac{r_+^3 \sin \theta}{\rho_+^3} \vartheta^0 \vartheta^2 + \frac{\cos \theta}{\rho_+ \sin \theta} \vartheta^1 \vartheta^3 \right), \\
H^1 = 2a_0 \left( \frac{\cos \theta}{\rho_+ \sin \theta} \vartheta^0 \vartheta^3 - \frac{r_+^2 \sin \theta}{\rho_+^3} \vartheta^1 \vartheta^2 \right), \\
H^2 = -2a_0 \frac{\vartheta^0 \vartheta^3}{\rho_+}, \\
H^3 = 2a_0 \left( \frac{r_+^2 \sin 2\theta}{\rho_+^3} \vartheta^0 \vartheta^1 + \frac{1}{\rho_+} \vartheta^0 \vartheta^2 - \frac{r_+^2 \sin \theta}{\rho_+^3} \vartheta^2 \vartheta^3 \right). \tag{5.3b}
\]

### 5.1 Conserved charge

We shall now compute the conserved charge on the horizon. It is obtained for $\xi = \partial_\varphi$. Since $H_{ij} = 0$ the variation of the canonical generator (4.3) reduces now to

\[
\delta \Gamma_H := \oint_{S_H} \delta B, \\
\delta B = b_{i\varphi} \delta H^i + (\delta \vartheta^i) H_{i\varphi} \tag{5.4}.
\]

The non-vanishing contribution to the conserved charge stems from

\[
\vartheta^3 \varphi \delta H_3 + (\delta \vartheta^3) H_{3\varphi} = \frac{4a_0 \sin^2 \theta}{(1 + \cos^2 \theta)^2} \delta (2 \sin \theta r_+^2) d\theta d\varphi. \tag{5.5}
\]
Now we get that the conserved charge reads
\[ J = \oint_{S_H} \vartheta^3 \phi \delta H_3 + (\delta \vartheta^3)H_{3\varphi} = 16\pi a_0 r_+^2 \equiv r_+^2. \] (5.6)

### 5.2 Central charge and black hole entropy

The central charge can again be obtained from the algebra of the improved canonical generators as in the previous section. The central term, computed by varying the background configuration, (for details see appendix B) is given by
\[ C = -4a_0 r_+^2 \int_0^{2\pi} (\epsilon_1' \epsilon_2'' - \epsilon_2' \epsilon_1'') d\varphi. \] (5.7)

In terms of Fourier modes the canonical algebra of the improved generators reads:
\[ \{L_n, L_m\} = -i(n - m)L_{m+n} - \frac{c}{12} i n^3 \delta_{n,-m}, \] (5.8)
where in the string theory normalization
\[ c = 12 \cdot 16\pi a_0 r_+^2 = 12r_+^2 \equiv 12J. \] (5.9)

The entropy of the extremal Kerr black hole in GR\_\parallel can be calculated via Cardy’s formula:
\[ S = 2\pi \sqrt{\frac{c}{6} \left( J - \frac{c}{24} \right)} = 2\pi r_+^2. \] (5.10)

The result for the conformal entropy of the extremal Kerr black hole in GR\_\parallel represents a smooth limit obtained from the expression for gravitational entropy of the generic Kerr black hole in the same theory [2].

### 6 Concluding remarks

We analyzed the near horizon symmetry for the extremal Kerr black hole in the framework of PG by using the Hamiltonian approach in the first order formulation of the theory. We have shown, considering two important limits of PG, namely Riemannian and teleparallel solution, that the algebra of improved canonical generators for the extremal Kerr black hole takes the form of Virasoro algebra with classical central charge which depends on the black hole horizon radius. We computed the extremal Kerr black hole entropy via Cardy’s formula, finding that conformal entropy calculated this way equals the smooth limit of the non-extremal gravitational entropy. The method we developed can be extended to the extremal Kerr black hole with torsion in the generic case. Also it would be interesting to examine near horizon structure of the extremal Reissner-Nordström-like black hole solutions with torsion [20] [21].

### Acknowledgments

This work was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.
A Central charge for extremal Riemannian black hole in PGT

The central charge stems from the variation of the surface term $\delta_0(\epsilon_1)\Gamma_H(\epsilon_2)$ on the background configuration.

Asymptotic Killing vector, after disregarding residual gauge transformations reads

$$\xi = ye'^y + \epsilon \partial_\varphi.$$  \hfill (A.1)

We shall make use of the following non-vanishing internal products

$$\xi \vartheta^1 = -r_+ \sqrt{1 + \cos^2 \theta} \epsilon', \quad \xi \vartheta^3 = \frac{2r_+ \sin \theta}{\sqrt{1 + \cos^2 \theta}} \epsilon,$$

$$\xi \bar{\omega}^{01} = -\frac{2 \sin^2 \theta}{(1 + \cos^2 \theta)^2} \epsilon, \quad \xi \bar{\omega}^{03} = \frac{\sin \theta}{1 + \cos^2 \theta} \epsilon',$$

$$\xi \bar{\omega}^{12} = -\frac{\sin \theta \cos \theta}{1 + \cos^2 \theta} \epsilon', \quad \xi \bar{\omega}^{23} = \frac{4 \cos \theta}{(1 + \cos^2 \theta)^2} \epsilon.$$ \hfill (A.2b)

The non-vanishing terms in the variation of the background configuration of tetrad fields (on the boundary defined by $\tilde{t} = \text{const}$ and $y \to 0$) are given by

$$\delta_0 \bar{\vartheta}^1 = r_+ \sqrt{1 + \cos^2 \theta} \epsilon'' d\varphi,$$

$$\delta_0 \bar{\vartheta}^3 = -\frac{2r_+ \sin \theta}{\sqrt{1 + \cos^2 \theta}} \epsilon' d\varphi.$$ \hfill (A.3b)

Consequently, for the canonical momenta $\bar{H}_{ij}$ we obtain

$$\delta_0 \bar{H}_{01} = 4(a_0 + 2b_1 C)r_+^2 \sin \theta \epsilon' d\theta d\varphi,$$

$$\delta_0 \bar{H}_{02} = 2(a_0 - b_1 C)r_+^2 (1 + \cos^2 \theta) \epsilon'' d\theta d\varphi,$$

$$\delta_0 \bar{H}_{12} = -2b_1 Dr_+^2 (1 + \cos^2 \theta) \epsilon'' d\theta d\varphi,$$

$$\delta_0 \bar{H}_{23} = 8b_1 Dr_+^2 \sin \theta \epsilon' d\theta d\varphi.$$ \hfill (A.4d)

After term by term integration we get

$$\oint_{S_H} (\xi_2 \bar{\omega}^{01}) \delta_0(\xi_1) \bar{H}_{01} = -\left(8a_0 r_+^2 + \frac{1}{2} b_1 (8 + 3\pi)\right) \int_0^{2\pi} \epsilon_2 \epsilon_1' d\varphi,$$ \hfill (A.5a)

$$\oint_{S_H} (\xi_2 \bar{\omega}^{03}) \delta_0(\xi_1) \bar{H}_{03} = (4a_0 r_+^2 - b_1) \int_0^{2\pi} \epsilon_2 \epsilon_1'' \epsilon_1' d\varphi,$$ \hfill (A.5b)

$$\oint_{S_H} (\xi_2 \bar{\omega}^{12}) \delta_0(\xi_1) \bar{H}_{12} = b_1 \int_0^{2\pi} \epsilon_2 \epsilon_1'' d\varphi,$$ \hfill (A.5c)

$$\oint_{S_H} (\xi_2 \bar{\omega}^{23}) \delta_0(\xi_1) \bar{H}_{23} = \frac{1}{2} b_1 (8 + 3\pi) \int_0^{2\pi} \epsilon_2 \epsilon_1' d\varphi.$$ \hfill (A.5d)
In the above expression we made use of the following definite integrals

\[
\int_0^\pi \frac{\sin^3 \theta}{(1 + \cos^2 \theta)^2} = 1, \quad \int_0^\pi \frac{\sin^2 \theta(1 - 3 \cos^2 \theta)}{(1 + \cos^2 \theta)^5} = \frac{1}{32}(8 + 3\pi),
\]

\[
\int_0^\pi \frac{\sin \theta(1 - 3 \cos^2 \theta)}{(1 + \cos^2 \theta)^3} = \frac{1}{2}, \quad \int_0^\pi \frac{\sin \theta \cos \theta(3 - \cos^2 \theta)}{(1 + \cos^2 \theta)^3} = \frac{1}{2},
\]

\[
\int_0^\pi \frac{\sin \theta \cos^2 \theta}{(1 + \cos^2 \theta)^5} = \frac{1}{64}(8 + 3\pi).
\]

After summing up all the contributions we get the first term

\[
\frac{1}{2} \oint_{S_H} (\xi_2 \bar{\omega}^{ij}) \delta_0 (\xi_1) H_{ij} = -8a_0 r_+^2 \int_0^{2\pi} \epsilon_2 \epsilon'_1 d\varphi + 4a_0 r_+^2 \int_0^{2\pi} \epsilon'_2 \epsilon''_1 d\varphi.
\]

The non-vanishing variations of the connection are given by

\[
\delta_0 \bar{\omega}^{01} = \frac{2 \sin^2 \theta}{(1 + \cos^2 \theta)^2} \epsilon' d\varphi, \quad (A.8a)
\]

\[
\delta_0 \bar{\omega}^{03} = -\frac{\sin \theta}{1 + \cos^2 \theta} \epsilon'' d\varphi, \quad (A.8b)
\]

\[
\delta_0 \bar{\omega}^{12} = \frac{\sin \theta \cos \theta}{1 + \cos^2 \theta} \epsilon'' d\varphi, \quad (A.8c)
\]

\[
\delta_0 \bar{\omega}^{23} = -\frac{4 \cos \theta}{(1 + \cos^2 \theta)^2} \epsilon' d\varphi. \quad (A.8d)
\]

The internal products with canonical momenta are given by

\[
\xi \bar{H}_{01} = 4(a_0 + 2b_1C)r_+^2 \sin \theta d\theta, \quad (A.9a)
\]

\[
\xi \bar{H}_{03} = 2(a_0 - b_1C)(1 + \cos^2 \theta)r_+^2 \epsilon' d\theta, \quad (A.9b)
\]

\[
\xi \bar{H}_{12} = -2b_1 D(1 + \cos^2 \theta)r_+^2 \epsilon' d\theta, \quad (A.9c)
\]

\[
\xi \bar{H}_{23} = 8b_1 D r_+^2 \sin \theta d\theta. \quad (A.9d)
\]

Since all the integrals over \( \theta \) are identical the second term takes the following form

\[
\frac{1}{2} \delta_0 (\xi_1) \omega^{ij} (\xi_2 \bar{H}_{ij}) = 8a_0 r_+^2 \int_0^{2\pi} \epsilon_2 \epsilon'_1 d\varphi - 4a_0 r_+^2 \int_0^{2\pi} \epsilon'_2 \epsilon''_1 d\varphi.
\]

\[
\text{(A.10)}
\]

**B Central charge of the extremal Kerr black hole in TG**

In TG the curvature equals zero and therefore we have to compute the variation the variation of the background canonical covariant momenta \( \bar{H}_i \)

\[
\delta_0 \bar{H}_1 = -2a_0 \frac{r_+ \sin \theta}{\sqrt{1 + \cos^2 \theta}} \epsilon'' d\theta d\varphi, \quad (B.1a)
\]

\[
\delta_0 \bar{H}_3 = -4a_0 \frac{\sin^2 \theta}{r_+(1 + \cos^2 \theta)^{3/2}} \epsilon' d\theta d\varphi. \quad (B.1b)
\]
As in the Riemannian case after performing integration $\theta$, we directly obtain

$$
(\xi_2 \partial^i)\delta_0(\xi_1) \bar{H}_i = 4a_0 r_+^2 \int_0^{2\pi} \epsilon'_1 \epsilon''_2 d\varphi - 8a_0 r_+^2 \int_0^{2\pi} \epsilon_2 \epsilon'_1 d\varphi. \quad (B.2)
$$

The non-trivial internal products of the canonical momenta read

$$
\xi_1 \bar{H}_1 = -2a_0 \frac{r_+ \sin \theta}{\sqrt{1 + \cos^2 \theta}} \epsilon' d\theta, \quad (B.3a)
$$

$$
\xi_1 \bar{H}_3 = -4a_0 \frac{\sin^2 \theta}{r_+ (1 + \cos^2 \theta)^{3/2}} \epsilon d\theta. \quad (B.3b)
$$

The second term takes the following form

$$
\delta_0(\xi_1) \partial^i (\xi_2 \partial^j \bar{H}_i) = -4a_0 r_+^2 \int_0^{2\pi} \epsilon'_1 \epsilon''_2 d\varphi + 8a_0 r_+^2 \int_0^{2\pi} \epsilon_1 \epsilon'_2 d\varphi. \quad (B.4)
$$

References

[1] M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D 99 (2019) 10, 104058 [arXiv:1903.02263].

[2] M. Blagojević and B. Cvetković, Hamiltonian approach to black hole entropy: Kerr-like spacetimes, Phys. Rev. 100 (2019) 4, 044029 [arXiv: 1905.04928].

[3] M. Blagojević and B. Cvetković, Entropy in general relativity: Kerr-AdS black hole, Phys. Rev D 101 (2020) 084023 [arXiv:2002.05029]; Thermodynamics of Riemannian Kerr-AdS black holes in Poincaré gauge theory, Phys. Lett. B 816 (2021) 136242 (5 pages) [arXiv:2103.00330].

[4] M. Blagojević and B. Cvetković, Entropy of Reissner-Nordström-like black holes, Phys. Lett. B 824 (2022) 136815 (5 pages) [arXiv:2112.02099].

[5] M. Blagojević and B. Cvetković, Entropy of Kerr-Newman-AdS black holes with torsion, Phys.Rev.D 105 (2022) 10, 104014 [arXiv:2203.14696].

[6] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) 88 (1974) 286-318.

[7] R. Wald, Black hole entropy is Noether charge, arXiv:gr-qc/9307038.

V. Iyer and R. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, arXiv:gr-qc/9403028; A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes, arXiv:gr-qc/9503052.

[8] M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. D 80 (2009) 124008 [arXiv:0809.4266].
[9] S. Carlip, Effective conformal descriptions of black hole entropy, arXiv:1107.2678. Black hole thermodynamics, in: One hundred years of general relativity, edited by W. T. Ni (World Scientific, Singapore, 2017), chapter 22.

[10] F. W. Hehl, Four lectures on Poincaré gauge theory, in Proc. 6th Course of the International School of Cosmology and Gravitation on Spin Torsion and Supergravity, edited by P. G. Bergmann and V. de Sabbatta (Plenum, New York, 1980); F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, Metric-affine gauge theory of gravity: Feld equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. 258 (1995) 1-177.

[11] Gauge Theories of Gravitation, A Reader with Commentaries, edited by M. Blagojević and F. W. Hehl (Imperial College Press, London, 2013).

[12] M. Blagojević, Gravitation and Gauge Symmetries (IoP, Bristol, 2002).

[13] B. Cvetković and D. Simić, Near-horizon geometry with torsion, Phys.Rev.D 99 63 (2019) 2, 024032 [arXiv: 1809.00555].

[14] J. M. Nester, A covariant Hamiltonian for gravity theories, Mod. Phys. Lett. 06 (1991) 2655-2661; in Directions in General Relativity, Vol. I, edited by B. L. Hu, M. P. Ryan and C. V. Vishveshwara (Cambridge University Press, Cambridge, England, 1993) pp. 245-260.

[15] J. B. Griffiths and J. Podolsky, Exact Space-Times in Einstein’s General Relativity (Cambridge University Press, Cambridge, England, 2009).

[16] J. Bardeen and G. T. Horowitz, The Extreme Kerr Throat Geometry: A Vacuum Analog of AdS$_2$ × S$^2$, Phys. Rev. D 60, 104030 (1999) [arXiv: hep-th/9905099].

[17] H. K. Kunduri and J. Lucietti, Classification of near-horizon geometries of extremal black holes, Living Rev.Rel. 16 (2013) 8, [arXiv: 1306.2517].

[18] J.D. Brown and M. Henneaux, On the Poisson Brackets of Differentiable Generators in Classical Field Theory, J.Math.Phys. 27 (1986) 489-491.

[19] F. Müller-Hoissen and J. Nitsch, Teleparallelism — A viable theory of gravity?, Phys. Rev. D 28, 718-728 (1983).

[20] J. A. R. Cembranos and J. G. Valcarcel, New torsion black hole solutions in Poincaré gauge theory, J. Cosmol. Astropart. Phys. 01, 014 (2017) [arXiv:1608.00062].

[21] J. A. R. Cembranos and J. G. Valcarcel, Extended Reissner–Nordström solutions sourced by dynamical torsion, Phys. Lett. B 779, 143–150 (2018) [arXiv:1708.00374].