Labeled Sequent Calculus and Countermodel Construction for Justification Logics

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Abstract. Justification logics are modal-like logics that provide a framework for reasoning about epistemic justifications. This paper introduces labeled sequent calculi for justification logics, as well as for hybrid modal-justification logics. Using the method due to Sara Negri, we internalize the Kripke-style semantics of justification logics, known as Fitting models, within the syntax of the sequent calculus to produce labeled sequent calculus. We show that our labeled sequent calculi enjoy a weak subformula property, all of the rules are invertible and the structural rules (weakening and contraction) and cut are admissible. Finally soundness and completeness are established, and termination of proof search for some of the labeled systems are shown. We describe a procedure, for some of the labeled systems, which produces a derivation for valid sequents and a countermodel for non-valid sequents. We also show a frame correspondence for justification logics in the context of labeled sequent calculus.

Keywords: Justification logic, Modal logic, Fitting model, Labeled sequent calculus, Analyticity

1 Introduction

Artemov in [1, 2] proposed the Logic of Proofs, LP, to present a provability interpretation for the modal logic S4 and the intuitionistic propositional logic. LP extends the language of propositional logic by proof terms and expressions of the form $t : A$, with the intended meaning “term $t$ is a proof of $A$”. Terms are constructed from variables and constants by means of operations on proofs. LP can be also viewed as a refinement of the epistemic logic S4, in which knowability operator $\Box A$ (A is known) is replaced by explicit knowledge operators $t : A$ ($t$ is a justification for $A$). The exact correspondence between LP and S4 is given by the Realization Theorem: all occurrences of $\Box$ in a theorem of S4 can be replaced by suitable terms to produce a theorem of LP, and vice versa. Regarding this theorem, LP is called the justification (or explicit) counterpart of S4.

The justification counterpart of other modal logics were developed in [7, 10, 11, 21, 23, 34, 37]. Various proof methods for the realization theorem are known, such as the syntactic and constructive proofs (see e.g. [1, 2]), semantic and non-constructive proofs (see e.g. [16]), and indirect proofs using embedding (see e.g. [21, 23]). We give a proof of realization for modal logic KB and its extensions using embedding of justification logics.

Combination of modal and justification logics, aka logics of justifications and belief, were introduced in [6, 8, 19, 27]. In this paper, we introduce modal-justification logics which include the previous logics of justifications and belief from [27], and some new combinations. A modal-justification logic MLJL is a combination of a modal logic ML and a justification logic JL such that JL is the justification counterpart of ML.
Various proof systems have been developed for LP (see [2, 14, 16, 20, 36]), for intuitionistic logic of proofs [3, 35], for S4LP and S4LPN (see [15, 20, 25]), and for justification logics of belief (see [22]). All aforementioned proof systems are cut-free. However, the only known analytic proof method is Finger’s tableau system for LP [14]. Moreover, most justification logics still lack cut-free proof systems. The aim of this paper will be to present labeled sequent calculus for justification logics, which enjoy the subformula property and cut elimination.

In a labeled sequent calculus some additional information, such as possible worlds and accessibility relation of Kripke models, from semantics of the logic are internalized into the syntax. Thus sequents in these systems are expressions about semantics of the logic. We employ Kripke-style models of justification logics, called Fitting models (cf. [5, 16]), and a method due to Negri [30] to present G3-style labeled sequent calculi for justification logics. Thus the syntax of the labeled systems of justification logics also contains atoms for representing evidence function of Fitting models.

Further, we present Fitting models and labeled sequent calculi based on Fitting models for modal-justification logics MLJL. For S4LPN (and S4LP), we present two labeled sequent calculi one based on Fitting models (Fitting models for S4LPN will be presented in Section 10.3) and the other based on Artemov-Fitting models [9]. The latter system has the subformula and subterm properties, whereas these properties do not hold in the former.

In all labeled systems the rules of weakening, contraction, and cut are admissible, and all rules are invertible. Soundness and completeness of the labeled systems with respect to Fitting models are also shown. The method used in the proof of completeness theorem (Theorem 8.2) gives a procedure to produce countermodels for non-valid sequents, and helps us to prove the termination of proof search for some of the labeled systems. Termination of proof search is shown using the analyticity of the labeled systems, i.e. the subformula, sublabel, and subterm properties. Thus decidability results for some justification and modal-justification logics are achieved, in the case that finite constant specifications are used.

The paper is organized as follows. In Section 2, we introduce the axiomatic formulation of modal and justification logics, and show the correspondence between them. We also generalize the Fitting’s embedding theorem to all justification logics and show how it can be used to prove the realization theorem. In Section 3, we describe Fitting models for justification logics and show how a possible evidence function can be extended to an admissible one. In Section 4, we present labeled sequent calculi for justification logics, and in Section 5 we establish the basic properties of these systems. In Section 6 we show the analyticity of some of the labeled systems. In Section 7 we prove the admissibility of structural rules. Then, in Section 8 we prove soundness and completeness of the labeled systems and give a procedure to construct a proof tree or a countermodel for a given sequent. We also show a frame correspondence for justification logics in the context of labeled sequent calculus. In Section 9 we establish the termination of proof search for some of the labeled systems. Finally, in Section 10 we present Fitting models and labeled sequent calculus for modal-justification logics and for S4LPN, and also a labeled sequent calculus based on Artemov-Fitting models for S4LPN.

2 Modal and justification logics

In this section, we recall the axiomatic formulation of modal and justification logics, and explain the correspondence between them.
2.1 Modal logics

Modal formulas are constructed by the following grammar:

\[ A ::= P \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid \Box A, \]

where \( P \) is a propositional variable, \( \bot \) is a propositional constant for falsity. The basic modal logic \( K \) has the following axiom schemes and rules:

**Taut.** All propositional tautologies,

**K.** \( \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \),

The rules of inference are *Modus Ponens* and *Necessitation rule*:

**MP.** from \( \vdash A \) and \( \vdash A \rightarrow B \), infer \( \vdash B \).

**Nec.** from \( \vdash A \), infer \( \vdash \Box A \).

Other modal logics are obtained by adding the following axiom schemes to \( K \) in various combinations:

**T.** \( \Box A \rightarrow A \).

**D.** \( \Box \bot \rightarrow \bot \).

**4.** \( \Box A \rightarrow \Box \Box A \).

**B.** \( \neg A \rightarrow \Box \neg \Box A \).

**5.** \( \neg \Box A \rightarrow \Box \neg \Box A \).

In this paper we consider the following 15 normal modal logics: \( K, T, D, K4, KB, K5, KB5, K45, D5, DB, D4, D45, TB, S4, S5 \). The name of each modal logic indicates the list of its axioms, except \( S4 \) and \( S5 \) which can be named \( KT4 \) and \( KT45 \), respectively.

2.2 Justification logics

The language of justification logics is an extension of the language of propositional logic by the formulas of the form \( t : F \), where \( F \) is a formula and \( t \) is a justification term. *Justification terms (or terms for short)* are built up from (justification) variables \( x, y, z, \ldots \) (possibly with sub- or superscript) and (justification) constants \( a, b, c, \ldots \) (possibly with subscript) using several operations depending on the logic: application ’\( \cdot \)’, sum ’\( + \)’, verifier ’\( ! \)’, negative verifier ’\( ? \)’, and weak negative verifier ’\( \overline{?} \)’. A term is called *ground* if it does not contain any justification variable. The definition of subterm is in the usual way: \( s \) is a subterm of \( s, s + t, t + s, s \cdot t, !s, ?s, \) and \( ?s \). Justification formulas (JL-formulas) are constructed by the following grammar:

\[ A ::= P \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid t : A, \]

where \( P \) is a propositional variable, \( \bot \) is a propositional constant for falsity, and \( t \) is a justification term.

For a JL-formula \( A \), the set of all subformulas of \( A \), denoted by \( \text{Sub}(A) \), is defined inductively as follows: \( \text{Sub}(P) = \{P\} \), for propositional variable \( P \); \( \text{Sub}(\bot) = \{\bot\} \); \( \text{Sub}(A \rightarrow B) = \{A \rightarrow B\} \cup \text{Sub}(A) \cup \text{Sub}(B) \); \( \text{Sub}(t : A) = \{t : A\} \cup \text{Sub}(A) \). For a set \( S \) of JL-formulas, \( \text{Sub}(S) \) denotes the set of all subformulas of the formulas from \( S \).

We now begin with describing the axiom schemes and rules of the basic justification logic \( J \), and continue with other justification logics. The basic justification logic \( J \) is the weakest justification logic we shall be discussing. Other justification logics are obtained by adding certain axiom schemes to \( J \).
Definition 2.1. The language of basic justification logic $J$ contains the binary operations $\cdot$ and $+$ on terms. Axioms schemes of $J$ are:

**Taut.** All propositional tautologies,

**Sum.** Sum axiom, $s : A \rightarrow (s + t) : A$, $s : A \rightarrow (t + s) : A$,

**jK.** Application axiom, $s : (A \rightarrow B) \rightarrow (t : A \rightarrow (s \cdot t) : B)$,

Justification logic $JT$ and its extensions $LP$, $JTB$, $JT5$, $JT45$, $JTB4$, $JTB45$ contain in addition the following axiom scheme:

**jT.** Factivity axiom, $t : A \rightarrow A$.

Justification logic $JD$ and its extensions $JD4$, $JD5$, $JDB$, $JDB4$, $JDB5$, $JDB45$ contain in addition the following axiom scheme:

**jD.** Consistency, $t : \bot \rightarrow \bot$.

The language of justification logic $J4$ and its extensions $LP$, $J4$, $JT45$, $J45$, $JTB4$, $JTB45$, $JDB4$, $JDB45$ contains in addition the unary operation $!$ on terms, and these logics contain the following axiom scheme:

**j4.** Positive introspection axiom, $t : A \rightarrow !t : t : A$.

The language of justification logic $J5$ and its extensions $JD5$, $JT5$, $J45$, $J5$, $JTB4$, $JTB5$, $JDB5$, $JDB45$, $JTB45$, $JDB45$ contains in addition the unary operation $? on terms, and these logics contain the following axiom scheme:

**j5.** Negative introspection axiom, $\neg t : A \rightarrow ?t : \neg t : A$.

The language of justification logic $JB$ and its extensions $JDB$, $JT$, $JB4$, $J4$, $JTB4$, $JTB5$, $JDB5$, $JDB45$, $JTB45$, $JDB45$ contains in addition the unary operation $\bar{?}$ on terms, and these logics contain the following axiom scheme:

**jB.** Weak negative introspection axiom, $\neg A \rightarrow \bar{?}t : \neg t : A$.

All justification logics have the inference rule Modus Ponens. Moreover, if $j4$ is not an axiom of the justification logic it has the Iterated Axiom Necessitation rule:

**IAN.** $\vdash c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A$, where $A$ is an axiom instance of the logic, $c_{i_j}$’s are arbitrary justification constants and $n \geq 1$.

If $j4$ is an axiom of the justification logic it has the Axiom Necessitation rule:

**AN.** $\vdash c : A$, where $A$ is an axiom instance of the logic and $c$ is an arbitrary justification constant.

As it is clear from the above definition, the name of each justification logic is indicated by the list of its axioms. For example, $JT45$ is the extension of $J$ by axioms $jT$, $j4$, and $j5$ (and moreover, it contains the rule $AN$). An exception is the logic of proofs $LP$ which can be named as $JT4$. 

4
Definition 2.3. Let $JL$ be a justification logic which does not contain axiom $j4$ in its axiomatization.

1. A constant specification $CS$ for $JL$ is a set of formulas of the form $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A$, where $n \geq 1$, $c_{i_j}$'s are justification constants and $A$ is an axiom instance of $JL$, such that it is downward closed: if $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS$, then $c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS$.
2. A constant specification $CS$ is axiomatically appropriate if for each axiom instance $A$ of $JL$ there is a constant $c$ such that $c : A \in CS$, and if $F \in CS$ then $c : F \in CS$ for some constant $c$.
3. A constant specification $CS$ is schematic if whenever $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS$ for some axiom instance $A$, then for every instance $B$ of the same axiom scheme $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : B \in CS$.

Definition 2.4. Let $CS$ be a constant specification. If $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS (n \geq 1)$, then the sequent $c_{i_n}, c_{i_{n-1}}, \ldots, c_{i_1}$ of justification constants are called $CS$-associated with the axiom instance $A$. A constant is $CS$-associated if it occurs in a sequence of constants which is $CS$-associated with some axiom instance. A constant specification $CS$ is injective if each justification constant occurs at most once in the formulas of $CS$ as a $CS$-associated constant.

Let $JL_{CS}$ (or $JL(CS)$) be the fragment of $JL$ where the (Iterated) Axiom Necessitation rule only produces formulas from the given $CS$. In the rest of the paper whenever we use $JL_{CS}$ it is assumed that $CS$ is a constant specification for $JL$. By $JL_{CS}, S \vdash F$ we mean that formula $F$ is derivable in $JL_{CS}$ from the set

Remark 2.1. Goetschi and Kuznets in [23] considered $A \to ?t : \neg t : \neg A$ (which was called axiom $jB$ there) instead of axiom $jB$ in justification logic $JB$ and its extensions. Moreover, they assign levels to justification constants, and consider the following iterated axiom necessitation rule (instead of $IAN$) for all justification logics:

$$A \text{ is an axiom instance } \frac{c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A}{IAN}$$

where $c^j_i$ is a constant of level $j$. Let us denote Goetschi and Kuznets' justification logics containing axiom $jB$ in the above mentioned formalization by $JB', JDB', JTB', JB4', JDB4', JTB4', JTB5', JDB5', JB45', JTB45', JDB45'$. Note that axiom $jB$ is provable in $JB$ and its extensions.

In what follows, $JL$ (possibly with subscript) denotes any of the justification logics defined in Definition 2.1, unless stated otherwise. $Tm_{JL}$ and $Fm_{JL}$ denote the set of all terms and all formulas of $JL$, respectively. Note that for justification logics $JL_1$ and $JL_2$, if $Tm_{JL_1} \subseteq Tm_{JL_2}$ then $Fm_{JL_1} \subseteq Fm_{JL_2}$.

We now proceed to the definition of Constant Specifications.
of formulas $S$. Every proof in JL generates a finite (injective) constant specification $CS$, which contains those formulas which are introduced by $IAN/AN$. The total constant specification for JL, denoted $TCS_{JL}$, is the largest constant specification which is defined as follows:

- if $JL$ contains axiom $j4$:
  
  $$TCS_{JL} = \{ c : A \mid c \text{ is a justification constant, } A \text{ is an axiom instance of } JL \}$$

- if $JL$ does not contain axiom $j4$:
  
  $$TCS_{JL} = \{ c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \mid c_{i_n}, c_{i_{n-1}}, \ldots, c_{i_1} \text{ are justification constants, } A \text{ is an axiom instance of } JL \}$$

By JL we mean $JL_{TCS_A}$. It is immediate that $TCS_{JL}$ is axiomatically appropriate and schematic, but it is not injective.

**Remark 2.2.** In some literature on justification logics (e.g. [26, 27]) constant specifications for all justification logics are defined as in Definition 2.3. In this case all justification logics contain the term operation $!$, and the Axiom Necessitation Rule for $JL_{CS}$ is formulated as follows:

$$
\frac{c : A \in CS}{! \ldots !c : \ldots : !c : A} \text{ AN!}
$$

All the results obtained in this formulation can be obtained in our formulation with small modifications in the proofs.

**Theorem 2.1 (Deduction Theorem).** $JL_{CS}, S, A \vdash B$ if and only if $JL_{CS}, S \vdash A \rightarrow B$.

For a given justification formula $F$, we write $F[t/x]$ for the result of simultaneously replacing all occurrences of variable $x$ in $F$ by term $t$. Substitution lemma holds in all justification logics JL.

**Lemma 2.1 (Substitution Lemma).**

1. Given a schematic constant specification $CS$ for JL, if

   $$JL_{CS}, S \vdash F,$$

   then for every justification variable $x$ and justification term $t$, we have

   $$JL_{CS}, S[t/x] \vdash F[t/x].$$

2. Given an axiomatically appropriate constant specification $CS$ for JL, if

   $$JL_{CS}, S \vdash F,$$

   then for every justification variable $x$ and justification term $t$, we have

   $$JL_{CS}, S[t/x] \vdash F[t/x],$$

where $\bar{F}$ is obtained from formula $F$ by (possibly) replacing some justification constants with other constants.
The following lemma is standard in justification logics.

**Lemma 2.2 (Internalization Lemma).** Given axiomatically appropriate constant specification $CS$ for $JL$, if

$$JL_{CS}, A_1, \ldots, A_n \vdash F,$$

then there is a justification term $t(x_1, \ldots, x_n)$, for variables $x_1, \ldots, x_n$, such that

$$JL_{CS}, x_1 : A_1, \ldots, x_n : A_n \vdash t(x_1, \ldots, x_n) : F.$$

In particular, if $JL_{CS} \vdash F$, then there is a ground justification term $t$ such that $JL_{CS} \vdash t : F$.

### 2.3 Correspondence theorem

In order to show the correspondence between justification logics (particularly $JB$ and its extensions) and their modal counterparts, it is helpful to recall the definition of embedding. There are various kinds of embedding of justification logics defined in the literature (cf. [17, 18, 22, 23]). We first recall the definition of embedding of Goetschi and Kuznet [22, 23] which is more general than the others.

**Definition 2.5.** Let $JL_1$ and $JL_2$ be two justification logics over languages $L_1$ and $L_2$ respectively.

1. An operation translation $\omega$ from $L_1$ to $L_2$ is a total function that for each $n \geq 0$, maps every $n$-ary operation $\ast$ of $L_1$ to an $L_2$-term $\omega(\ast) = \omega_\ast(x_1, \ldots, x_n)$, which do not contain variables other than $x_1, \ldots, x_n$.
2. Justification logic $JL_1$ embeds in $JL_2$, denoted by $JL_1 \sqsubseteq JL_2$, if there is an operation translation $\omega$ from $L_1$ to $L_2$ such that $JL_1 \vdash F$ implies $JL_2 \vdash F\omega$ for any $L_1$-formula $F$, where $F\omega$ results from $F$ by replacing each $n$-ary operation $\ast$ by the $L_2$-term $\omega_\ast$.
3. Two justification logics $JL_1$ and $JL_2$ are equivalent, denoted by $JL_1 \equiv JL_2$, if each embeds in the other.

Recall that our justification logics axiomatized using axiom schemes. Following [22, 23], the formula representation of a scheme $X$ is defined as a formula of the form $A(x_1, \ldots, x_n, P_1, \ldots, P_m)$, with $n, m \geq 0$, in which all terms in $X$ replaced by variables $x_1, \ldots, x_n$ and all formulas in $X$ replaced by propositional variables $P_1, \ldots, P_m$. The following theorem from [22, 23] is a useful tool for showing the embedding of justification logics.

**Theorem 2.2 (Embedding $\sqsubseteq$).** Let $JL_1$ and $JL_2$ be two justification logics as defined in Definition 2.1 or Remark 2.1 over languages $L_1$ and $L_2$ respectively. Let $L_1$ and $JL_1$ have the following conditions:

1. the set of constants of $L_1$ is divided into levels,
2. MP and iAN are the only rules of $JL_1$,
3. $JL_1$ is axiomatized using axiom schemes,
4. the formula representations of the axioms of $JL_1$ do not contain constants,

and $L_2$ and $JL_2$ have the following conditions:

1. $JL_2$ satisfies the substitution closure, i.e. if $JL_2 \vdash F$, then $JL_2 \vdash F[t/x]$ for every variable $x$ and term $t \in Tm_{JL_2}$,
2. $JL_2$ satisfies the internalization property, i.e. if $JL_2 \vdash F$, then there is a ground justification term $t \in Tm_{JL_2}$ such that $JL_2 \vdash t : F$. 


If there exists an operation translation \( \omega \) from \( \mathcal{L}_1 \) to \( \mathcal{L}_2 \) such that for every axiom \( X \) of \( \mathcal{J}_1 \) with formula representation \( A(x_1, \ldots, x_n, P_1, \ldots, P_m) \), the \( \mathcal{L}_2 \)-formula \( A(x_1, \ldots, x_n, P_1, \ldots, P_m) \omega \) is provable in \( \mathcal{J}_2 \), then \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \).

**Example 2.1.** Let \( \mathcal{J}_1 \) be one of the following justification logics

\[
\mathcal{J}_B', \mathcal{J}_DB', \mathcal{J}_TB', \mathcal{J}_B4', \mathcal{J}_B5', \mathcal{J}_DB4', \mathcal{J}_TB4', \mathcal{J}_TB5', \mathcal{J}_DB5', \mathcal{J}_TB45', \mathcal{J}_DB45'
\]

from Remark 2.1, and \( \mathcal{J}_2 \) be the corresponding justification logic

\[
\mathcal{J}_B, \mathcal{J}_DB, \mathcal{J}_TB, \mathcal{J}_B4, \mathcal{J}_B5, \mathcal{J}_DB4, \mathcal{J}_TB4, \mathcal{J}_TB5, \mathcal{J}_DB5, \mathcal{J}_TB45, \mathcal{J}_DB45
\]

from Definition 2.1. Note that \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) satisfy all the conditions of Theorem 2.4, and moreover \( \mathcal{J}_2 \vdash P \rightarrow \exists x : \neg A \rightarrow \neg P \). Consider the identity operation translation \( \omega^{id} \) which is defined as \( \omega^{id}(\ast) := \ast(x_1, \ldots, x_n) \) for each \( n \)-ary operation \( \ast \). Now it is easy to show that for every axiom \( X \) of \( \mathcal{J}_1 \) with formula representation \( A(x_1, \ldots, x_n, P_1, \ldots, P_m) \), the formula \( A(x_1, \ldots, x_n, P_1, \ldots, P_m) \omega^{id} \) is provable in \( \mathcal{J}_2 \), and hence \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \).

In the following we will state the connection between justification logics and modal logics.

**Definition 2.6.** For a justification formula \( F \), the forgetful projection of \( F \), denoted by \( F^\circ \), is a modal formula defined inductively as follows: \( P^\circ := P \) (\( P \) is a propositional variable), \( \bot^\circ := \bot \), \( (\neg A)^\circ := \neg A^\circ \), \( (A \ast B)^\circ := A^\circ \ast B^\circ \) (where \( \ast \) is a propositional connective), and \( (t : A)^\circ := \Box A^\circ \). For a set \( S \) of justification formulas let \( S^\circ = \{ F^\circ \mid F \in S \} \).

**Definition 2.7.** A \( \mathcal{J} \)-realization \( r \) of a modal formula \( F \) is a formula \( F^r \) in the language of \( \mathcal{J} \) that is obtained by replacing all occurrences of \( \Box \) with justification terms from \( \Gamma m_{\mathcal{J}} \). A realization \( r \) of \( F \) is called normal if all negative occurrences of \( \Box \) in \( F \) are replaced by distinct justification variables.

The realization theorem for modal logic \( \mathcal{M}L \) and justification logic \( \mathcal{J}L \) states that theorems of \( \mathcal{M}L \) can be realized into the theorems of \( \mathcal{J}L \); in other words \( \mathcal{M}L \subseteq \mathcal{J}L^\circ \).

It is proved in [22, 23] that if \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \) then \( \mathcal{J}_1^\circ \subseteq \mathcal{J}_2^\circ \). Thus, if \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \) and \( \mathcal{M}L \) is a modal logic such that \( \mathcal{M}L \subseteq \mathcal{J}_1^\circ \), then \( \mathcal{M}L \subseteq \mathcal{J}_2^\circ \). This shows that how embedding of justification logics can be used to prove the realization theorem.

As axiomatic formulation of justification logics suggests, modal logics \( \mathcal{M}L \) are forgetful projections of their corresponding justification logics \( \mathcal{J}L \).

**Theorem 2.3 (Correspondence Theorem).** The following correspondences hold:

\[
\begin{align*}
\mathcal{J}^\circ &= \mathcal{K}, & \mathcal{J}T^\circ &= \mathcal{T}, & \mathcal{J}D^\circ &= \mathcal{D}, & \mathcal{J}4^\circ &= \mathcal{K}4, & \mathcal{J}5^\circ &= \mathcal{K}5, \\
\mathcal{J}D5^\circ &= \mathcal{D}5, & \mathcal{J}D4^\circ &= \mathcal{D}4, \mathcal{J}45^\circ &= \mathcal{K}45, & \mathcal{J}D45^\circ &= \mathcal{D}45, \mathcal{L}P^\circ &= \mathcal{S}4, & \mathcal{J}T5^\circ &= \mathcal{J}T45^\circ &= \mathcal{S}5. \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{J}B^\circ &= \mathcal{K}B, & \mathcal{J}TB^\circ &= \mathcal{T}B, & \mathcal{J}DB^\circ &= \mathcal{DB}, & \mathcal{J}B4^\circ &= \mathcal{B}5^\circ &= \mathcal{K}B45^\circ &= \mathcal{B}5, \\
\mathcal{J}TB5^\circ &= \mathcal{J}DB5^\circ &= \mathcal{J}TB45^\circ &= \mathcal{J}DB45^\circ &= \mathcal{J}TB4^\circ &= \mathcal{J}DB4^\circ &= \mathcal{S}5.
\end{align*}
\]

Moreover, all the realization results of (1) and (2) are normal realizations.
Proof. The proof of the correspondences in (1) can be found in [2, 5, 10, 11, 23, 37]. For the correspondences in (2), we use Example 2.1 and the following correspondences from [23]:

\[ (J B')^o = K B, \quad (J T B')^o = T B, \quad (J T B B')^o = D B, \quad (J B 4')^o = (J B 5')^o = (J B 45')^o = K B 5, \]

\[ (J T B B')^o = (J D B 5')^o = (J D B 45')^o = (J D B 4')^o = (J D B 4')^o = S 5. \]  

(3)

We only prove that \( J B^o = K B \), the proof of the remaining cases is similar.

If \( J B \vdash F \), then by induction on the proof of \( F \) it is easy to show that \( K B \vdash F^o \). Hence \( J B^o \subseteq K B \).

For the opposite direction, that is the realization theorem, suppose that \( K B \vdash F \). By Example 2.1, since \( J B^o \subseteq K B \), we have \( (J B')^o \subseteq J B^o \). Then, by (3) we have \( (J B')^o = K B \), and thus \( K B \subseteq J B^o \). \( \dashv \)

Finally, we recall the definition of Fitting’s embeddings and show how it can be used to prove the realization theorem.

**Definition 2.8.** Let \( J L_1(C S_1) \) and \( J L_2(C S_2) \) be two justification logics such that \( T m_{J L_1} \subseteq T m_{J L_2} \). Justification logic \( J L_1(C S_1) \) embeds in \( J L_2(C S_2) \), denoted by \( J L_1(C S_1) \rightarrow J L_2(C S_2) \), if there is a mapping \( m \) from constants of \( J L_1 \) to ground terms of \( J L_2 \) such that \( J L_1(C S_1) \vdash F \) implies \( J L_2(C S_2) \vdash m(F) \) for any formula \( F \in F m_{J L_1} \), where \( m(F) \) results from \( F \) by replacing each constant \( c \) by the justification term \( m(c) \).

**Definition 2.9.** Two justification logics \( J L_1(C S_1) \) and \( J L_2(C S_2) \), in the same language, are equivalent, denoted by \( J L_1(C S_1) \equiv J L_2(C S_2) \), if \( J L_1(C S_1) \rightarrow J L_2(C S_2) \) and \( J L_2(C S_2) \rightarrow J L_1(C S_1) \).

Since justification constants are 0-ary operations, operation translations in Definition 2.5 are extensions of mappings considered by Fitting. Therefore, if \( J L_1 \subseteq J L_2 \) then \( J L_1 \rightarrow J L_2 \). The following theorem is a generalization of Fitting’s result (Theorem 9.2 in [18]) on the embedding of justification logics.

**Theorem 2.4 (Embedding \( \rightarrow \)).** Let \( J L_1(C S_1) \) and \( J L_2(C S_2) \) be two justification logics as defined in Definition 2.1 or Remark 2.1 such that \( T m_{J L_1} \subseteq T m_{J L_2} \). Assume

1. \( C S_1 \) is a finite constant specification for \( J L_1 \).
2. \( J L_1(C S_1) \) is axiomatized using axiom schemes.
3. \( J L_2(C S_2) \) satisfies the substitution closure, i.e. if \( J L_2(C S_2) \vdash F \), then \( J L_2(C S_2) \vdash F[t/x] \) for every variable \( x \) and term \( t \in T m_{J L_2} \).
4. \( J L_2(C S_2) \) satisfies the internalization property, i.e. if \( J L_2(C S_2) \vdash F \), then there is a ground justification term \( t \in T m_{J L_2} \) such that \( J L_2(C S_2) \vdash t : F \).
5. Every axiom instance of \( J L_1(C S_1) \) is a theorem of \( J L_2(C S_2) \).

Then \( J L_1(C S) \rightarrow J L_2(C S) \).

**Proof.** Suppose \( J L_1 \) and \( J L_2 \) are two justification logics such that \( T m_{J L_1} \subseteq T m_{J L_2} \). If \( C S_1 \) is empty, then let \( m \) be the identity function. Obviously \( J L_1(\emptyset) \rightarrow J L_2(C S_2) \).

Now suppose \( C S_1 \) is not empty. In the rest of the proof it is helpful to consider a fix list of all terms of \( T m_{J L_1} \) and \( T m_{J L_2} \), e.g. \( T m_{J L_1} = \{ s_1^1, s_2^1, \ldots \} \) and \( T m_{J L_2} = \{ s_1^2, s_2^2, \ldots \} \). In the rest of the proof we write \( A(d_1, \ldots, d_m) \) to denote that \( d_1, \ldots, d_m \) are all the constants occurring in the \( J L_1 \)-formula \( A \) in the order of the list \( T m_{J L_1} \). We detail the proof for justification logics defined in Definition 2.1. The case of justification logics defined in Remark 2.1 is similar. We first define mapping \( m \) from constants to ground terms of \( T m_{J L_2} \).

If \( c \) is not a \( C S_1 \)-associated constant, then define \( m(c) = c \). Next suppose \( c \) is a \( C S_1 \)-associated constant.

Suppose \( J L_1 \) contains the Axiom Necessitation rule \( A N \). We distinguish two cases:
(a) If constant $c$ occurs only once as a $\text{CS}_1$-associated constant, then the proof is similar to the proof of Theorem 9.2 in [18], however we repeat it here for convenience. Assume $c : A(d_1, \ldots, d_m) \in \text{CS}_1$. We shall define the mapping $m$ on $c$. Let $x_1, x_2, \ldots, x_m$ be distinct justification variables not occurring in $A$ (in the order of the list $Tm_{\text{JL}_1}$). Since $A$ is an axiom instance of $\text{JL}_1(\text{CS}_1)$, which is axiomatized using axiom schemes, $A(x_1, x_2, \ldots, x_m)$ will also be an axiom instance. Then by hypothesis 5, $A(x_1, x_2, \ldots, x_m)$ is a theorem of $\text{JL}_2(\text{CS}_2)$. Since $\text{JL}_2(\text{CS}_2)$ satisfies the internalization property, there is a ground justification term $t \in Tm_{\text{JL}_2}$ such that $\text{JL}_2(\text{CS}_2) \vdash t : A(x_1, x_2, \ldots, x_m)$ (if there is more than one such term, we choose the first in the order of the list $Tm_{\text{JL}_2}$). Now, let $m(c) = t$.

(b) If constant $c$ occurs more than once as a $\text{CS}_1$-associated constant, then suppose

$$c : A_1(d_{11}, \ldots, d_{1_{m_1}}), \ldots, c : A_k(d_{k_1}, \ldots, d_{k_{m_k}}),$$

for some $k \geq 2$, are all formulas of $\text{CS}_1$ with $c$ as a $\text{CS}_1$-associated constant. Similar to the case (a), we find ground justification terms $t_1, \ldots, t_k \in Tm_{\text{JL}_2}$ such that

$$\text{JL}_2(\text{CS}_2) \vdash t_j : A_j(x_{j1}^1, x_{j2}^1, \ldots, x_{jm_j}^1),$$

for $j = 1, 2, \ldots, k$, where $x_{j1}^1, x_{j2}^1, \ldots, x_{jm_j}^1$ are distinct justification variables not occurring in $A_j$. Letting $t = t_1 + \ldots + t_k$, by axiom $\text{Sum}$ and $\text{MP}$, we have $\text{JL}_2(\text{CS}_2) \vdash t : A(x_1, x_2, \ldots, x_m)$, for $j = 1, 2, \ldots, k$. Now, let $m(c) = t$.

Suppose $\text{JL}_1$ contains the Iterated Axiom Necessitation rule $\text{IAN}$. Similar to the previous argument, We distinguish two cases:

(a') If constant $c$ occurs only once as a $\text{CS}_1$-associated constant, then suppose $c : c_{i_n} : \ldots : c_i : A(d_1, \ldots, d_m) \in \text{CS}_1$. We shall define the mapping $m$ on $c$. Similar to case (a), $A(x_1, x_2, \ldots, x_m)$ is a theorem of $\text{JL}_2(\text{CS}_2)$, where $x_1, x_2, \ldots, x_m$ are distinct justification variables not occurring in $A$ (again in the order of the list $Tm_{\text{JL}_1}$). Since $\text{JL}_2(\text{CS}_2)$ satisfies the internalization property, by applying the internalization lemma $i_n + 1$ times, we obtain ground justification terms $t, t_i, t_{i_2}, \ldots, t_{i_n} \in Tm_{\text{JL}_2}$ such that $\text{JL}_2(\text{CS}_2) \vdash t : A_{i_1}(x_1, x_2, \ldots, x_m)$. Now, let $m(c) = t$.

(b') If constant $c$ occurs more than once as a $\text{CS}_1$-associated constant in sequences of constants, then suppose

$$c : c_{i_1} : \ldots : c_{i_n} : A_1(d_{11}, \ldots, d_{1_{m_1}}), \ldots, c : c_{k_n} : \ldots : c_{k_1} : A_k(d_{k_1}, \ldots, d_{k_{m_k}}),$$

for some $k \geq 2$, are all formulas of $\text{CS}_1$ where $c$ occurs in the sequences of constants as a $\text{CS}_1$-associated constant (since $\text{CS}_1$ is downward closed, we can consider only those formulas of $\text{CS}_1$ which begins with $c$). Similar to the case (a'), we find ground justification terms $t_j, t_{i_1}, \ldots, t_{i_n} \in Tm_{\text{JL}_2}$, for $j = 1, 2, \ldots, k$, such that

$$\text{JL}_2(\text{CS}_2) \vdash t_j : t_{i_1} : \ldots : t_{i_n} : A_j(x_1^j, x_2^j, \ldots, x_{m_j}^j),$$

where $x_1^j, x_2^j, \ldots, x_{m_j}^j$ are distinct justification variables not occurring in $A_j$. Letting $t = t_1 + \ldots + t_k$, by axiom $\text{Sum}$ and $\text{MP}$, we have $\text{JL}_2(\text{CS}_2) \vdash t : A_j(x_1^j, x_2^j, \ldots, x_{m_j}^j)$, for $j = 1, 2, \ldots, k$. Now, let $m(c) = t$. 

10
Now suppose \( F \) is any theorem of \( \text{JL}_1(\text{CS}_1) \). By induction on the proof of \( F \), we show that \( m(F) \) is a theorem of \( \text{JL}_2(\text{CS}_2) \).

If \( F \) is an axiom instance of \( \text{JL}_1(\text{CS}_1) \), since \( \text{JL}_1(\text{CS}_1) \) is axiomatized by schemes, \( m(F) \) will also be an axiom instance of \( \text{JL}_1 \), and hence a theorem of \( \text{JL}_2(\text{CS}_2) \) by hypothesis 5.

If \( F \) follows from \( G \) and \( G \rightarrow F \) by Modus Ponens, then by the induction hypothesis \( m(G) \) and \( m(G \rightarrow F) = m(G) \rightarrow m(F) \) are theorems of \( \text{JL}_2(\text{CS}_2) \). Then by Modus Ponens \( m(F) \) is a theorem of \( \text{JL}_2(\text{CS}_2) \).

Finally suppose \( F = c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A(d_1, \ldots , d_m) \) is obtained by \( \text{IAN} \) (when \( i_n \geq 1 \)) or by \( \text{AN} \) (when \( i_n = 1 \)), i.e., \( c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in \text{CS}_1 \), where \( A = A(d_1, \ldots , d_m) \) is a \( \text{JL}_1(\text{CS}_1) \) axiom. If \( x_1, x_2, \ldots , x_m \) are justification variables not occurring in \( A \), as above, there are ground terms \( t_{i_1}, t_{i_2}, \ldots , t_{i_n} \in Tw_{\text{JL}_2} \) such that

\[ t_{i_1} = m(c_{i_1}), t_{i_2} = m(c_{i_2}), \ldots , t_{i_n} = m(c_{i_n}) \]

and

\[ \text{JL}_2(\text{CS}_2) \vdash t_{i_1} : \ldots : t_{i_2} : t_{i_1} : A(x_1 , x_2, \ldots , x_m) . \]

Since \( \text{JL}_2(\text{CS}_2) \) satisfies the substitution closure, we obtain

\[ \text{JL}_2(\text{CS}_2) \vdash t_{i_1} : \ldots : t_{i_2} : t_{i_1} : A(m(d_1), m(d_2), \ldots , m(d_m)) , \]

but we have

\[ t_{i_n} : \ldots : t_{i_1} : A(m(d_1), \ldots , m(d_m)) = m(c_{i_n}) : \ldots : m(c_1) : m(A) = m(c_{i_n} : \ldots : c_1 : A) = m(F) . \]

Thus \( \text{JL}_2(\text{CS}_2) \vdash m(F) . \)

**Theorem 2.5.** Suppose \( \text{JL}_1(\text{CS}_1) \) and \( \text{JL}_2(\text{CS}_2) \) satisfy all the conditions of Theorem 2.4 except that \( \text{CS}_1 \) is not required to be finite, instead it is an injective constant specification for \( \text{JL}_1 \). Then \( \text{JL}_1(\text{CS}_1) \leftrightarrow \text{JL}_2(\text{CS}_2) \).

**Proof.** Proceed just as in the proof of Theorem 2.4, except that now cases (b) and (b’) cannot happen. \( \dashv \)

The above theorems give criterions to showing \( \text{JL}_1(\text{CS}_1) \leftrightarrow \text{JL}_2(\text{CS}_2) \), where \( \text{CS}_1 \) is finite or injective and \( \text{CS}_2 \) is the total constant specification \( T_{\text{CS}_{JL_2}} \) or an arbitrary schematic constant specification. Note that all justification logics \( \text{JL} \) of Definition 2.1 and Remark 2.1 satisfy hypothesis 2 of Theorem 2.4. Thus we have

**Corollary 2.1.** Let \( \text{JL}_1 \) and \( \text{JL}_2 \) be two justification logics as defined in Definition 2.1 or Remark 2.1 such that \( Tw_{\text{JL}_1} \subseteq Tw_{\text{JL}_2} \), \( \text{CS}_1 \) be a finite (or an injective) constant specification for \( \text{JL}_1 \), and \( \text{CS}_2 \) be an axiomatically appropriate schematic constant specification for \( \text{JL}_2 \), and every axiom of \( \text{JL}_1(\text{CS}_1) \) is a theorem of \( \text{JL}_2(\text{CS}_2) \). Then \( \text{JL}_1(\text{CS}_1) \leftrightarrow \text{JL}_2(\text{CS}_2) \).

Using the above corollary we give some examples here.

**Example 2.2.** Define the justification logic \( \text{JT}45’ \) in the language of \( \text{JT}45 \) by replacing the axiom scheme \( \text{j5} \) by \( !t : t \leftrightarrow A \vee ?t : \neg t : A \). Let \( \text{CS}_1 \) and \( \text{CS}_1’ \) be finite (or injective) constant specifications for \( \text{JT}45 \) and \( \text{JT}45’ \), respectively, and \( \text{CS}_2 \) and \( \text{CS}_2’ \) be axiomatically appropriate schematic constant specifications for \( \text{JT}45 \) and \( \text{JT}45’ \), respectively. It is not difficult to show that the substitution closure and internalization property hold for \( \text{JT}45’(\text{CS}_2’), \) and hence \( \text{JT}45(\text{CS}_1) \leftrightarrow \text{JT}45’(\text{CS}_2’), \) and \( \text{JT}45’(\text{CS}_1’) \leftrightarrow \text{JT}45(\text{CS}_2). \)
Example 2.3. Let JL₁ be one of the following justification logics

\[ JB', JDB', JTB', JB4', JB5', JDB4', JTB4', JTB5', JDB5', JB45', JTB45', JDB45' \]

from Remark 2.1, and JL₂ be the corresponding justification logic

\[ JB, JDB, JTB, JB4, JB5, JDB4, JTB4, JTB5, JDB5, JB45, JTB45, JDB45 \]

from Definition 2.1. If CS is a finite (or an injective) constant specifications for JL₁, and CS₂ be an axiomatically appropriate schematic constant specification for JL₂, then it is easy to prove that JL₁(CS) \(\hookrightarrow\) JL₂(CS₂).

The following lemma shows how Fitting’s embedding can be used to reduce the realization theorem from one justification logic to another.

Lemma 2.3. Let JL₁(CS₁) and JL₂(CS₂) be two justification logics such that Tm_JL₁ \(\subseteq\) Tm_JL₂.

1. if JL₁(CS₁) \(\hookrightarrow\) JL₂(CS₂) then JL₁(CS₁)° \(\subseteq\) JL₂(CS₂)°.
2. If JL₁(CS₁) \(\hookrightarrow\) JL₂(CS₂) and ML \(\subseteq\) JL₁(CS₁)°, then ML \(\subseteq\) JL₂(CS₂)°.

Proof. For (1), suppose JL₁(CS₁) \(\hookrightarrow\) JL₂(CS₂) and F° \(\in\) JL₁(CS₁)°. Hence JL₁(CS₁) \(\vdash\) F. Thus, for the embedding m witnesses JL₁(CS₁) \(\hookrightarrow\) JL₂(CS₂), we have JL₂(CS₂) \(\vdash\) m(F). By induction on the complexity of the formula F, it is easy to prove that (m(F))° = F°. Therefore, F° \(\in\) JL₂(CS₂)°. Clause 2 follows easily from 1.

Remark 2.3. In [11] and [21], authors use operation ? instead of \(\bar{?}\) in the language of JB and its extensions, and axioms \(A \rightarrow?t : \neg t : \neg A \rightarrow?t : \neg t : A\) are used instead of \(\text{jb}\) and \(\text{jB}\), respectively (moreover in [11] rule \(\text{AN!}\) is used for all justification logics). For what follows, let JB₂, JTB₂, JDB₂, JB5₂, JB45₂ denote justification logics with axiom \(A \rightarrow?t : \neg t : \neg A\) instead of \(\text{jB}\) and \(\text{jB}\), hence \(\text{jB}₂, \text{JTB}₂, \text{JDB}₂, \text{JB5}₂, \text{JB45}₂\) denote justification logics with axiom \(A \rightarrow?t : \neg t : \neg A\) instead of \(\text{jB}\). Similar to Example 2.3, one can show that

\[ JB₂' \hookrightarrow JB₁, JTB₂' \hookrightarrow JTB₁, JDB₂' \hookrightarrow JDB₁, JB5₂' \hookrightarrow JB5₁, JB45₂' \hookrightarrow JB45₁. \]

(4)

where for simplicity we omit the constant specifications in (4). The correspondence theorems

\[ JB₂° = KB, JTB₂° = TB, JDB₂° = DB, JB5₂° = JB45₂° = KB5 \]

are proved in [21] using Theorem 2.5, Lemma 2.3, embeddings in (4), and the fact that the correspondences

\( (JB₂')° = KB, (JTB₂')° = TB, (JDB₂')° = DB, (JB5₂')° = (JB45₂')° = KB, \)

which are proved in [11] uses injective constant specifications (this assumption is claimed in Remark 5.10 in [11]).
3 Semantics of justification logics

In this section, we recall the definitions of Fitting models for justification logics. Fitting models (or F-models for short) are Kripke-style models, were first developed by Fitting in [16] for LP. In Section 4, we internalize these models within the syntax of sequent calculus to produce labeled sequent calculi for justification logics. First we recall a definition from [26].

Definition 3.1. Let JL be a justification logic, CS be a constant specification for JL.

1. A possible evidence function on a nonempty set W for \( JL_{CS} \) is any function \( A : Tm_{JL} \times Fm_{JL} \to 2^W \) such that if \( c : F \in CS \), then \( A(c, F) = W \).
2. For two possible evidence functions \( A_1, A_2 \) on W for \( JL_{CS} \), we say that \( A_2 \) is based on \( A_1 \), and write \( A_1 \subseteq A_2 \), if \( A_1(t, F) \subseteq A_2(t, F) \) for any \( t \in Tm_{JL} \) and \( F \in Fm_{JL} \).
3. A Kripke frame is a pair \( (W, R) \), where \( W \) is a non-empty set (of possible worlds) and accessibility relation \( R \) is a binary relation on \( W \).
4. A Fitting frame for \( JL_{CS} \) is a triple \( (W, R, A) \), where \( (W, R) \) is a Kripke frame and \( A \) is a possible evidence function on \( W \) for \( JL_{CS} \).
5. A possible Fitting model for \( JL_{CS} \) is a quadruple \( (W, R, A, V) \) where \( (W, R, A) \) is a Fitting frame for \( JL_{CS} \) and \( V \) is a valuation (that is a function from the set of propositional variables to subsets of \( W \)).

Definition 3.2. For a possible Fitting model \( M = (W, R, A, V) \) the forcing relation \( \models \) defined as follows:

1. \( (M, w) \models P \) iff \( w \in V(P) \), for propositional variable \( P \),
2. \( \models \) respects classical Boolean connectives,
3. \( (M, w) \models t : F \) iff \( w \in A(t, F) \) and for every \( v \in W \) with \( wRv \), \( (M, v) \models F \).

Definition 3.3. A Fitting model \( M = (W, R, E, V) \) for justification logic \( JL_{CS} \) is a possible Fitting model in which \( E \) is a possible evidence function on \( W \) for \( JL_{CS} \), meeting the following conditions:

\( E_1 \). Application: \( E(s, A \rightarrow B) \cap E(t, A) \subseteq E(s \cdot t, B) \),
\( E_2 \). Sum: \( E(s, A) \cup E(t, A) \subseteq E(s + t, A) \).

\( E \) is called admissible evidence function for \( JL_{CS} \).

In order to define F-models for other justification logics of Definition 2.1 certain additional conditions should be imposed on the accessibility relation \( R \) and possible evidence function \( E \).

Definition 3.4. A Fitting model \( M = (W, R, E, V) \) for justification logic \( JL_{CS} \) is a Fitting model for \( JL_{CS} \), in which the admissible evidence function \( E \) is now a possible evidence function on \( W \) for \( JL_{CS} \), and \( R, E \) satisfy the following conditions:

- if \( JL_{CS} \) contains axiom \( jT \), then \( R \) is reflexive.
- if \( JL_{CS} \) contains axiom \( jD \), then \( R \) is serial.
- if \( JL_{CS} \) contains axiom \( j4 \), then \( R \) is transitive, and \( E \) satisfies:
  \( E_3 \). Monotonicity: If \( w \in E(t, A) \) and \( wRv \), then \( v \in E(t, A) \).
  \( E_4 \). Positive introspection: \( E(t, A) \subseteq E(t, t : A) \).
- if \( JL_{CS} \) contains axiom \( jB \), then \( R \) is symmetric, and \( E \) satisfies:
  \( E_5 \). Weak negative introspection: If \( (M, w) \models \neg A \), then \( w \in E(?t, \neg t : A) \), for all terms \( t \).
- if $\mathcal{J}_\mathcal{CS}$ contains axiom $j_5$, then $\mathcal{E}$ satisfies:

$\mathcal{E}6$. Negative introspection: $[\mathcal{E}(t, A)]^c \subseteq \mathcal{E}(\neg t : A)$, where $S^c$ means the complement of the set $S$ relative to the set of worlds $W$.

$\mathcal{E}7$. Strong evidence: if $w \in \mathcal{E}(t, A)$, then $(M, w) \vDash t : A$.

We say that a formula $F$ is true in a model $M$ (denoted by $M \vDash F$), if $(M, w) \vDash F$ for all $w \in W$. For a set $S$ of formulas, $M \vDash S$ provided that $M \vDash F$ for all formulas $F$ in $S$. Note that given a constant specification $\mathcal{CS}$ for $\mathcal{J}_L$, and a model $M$ of $\mathcal{J}_\mathcal{CS}$ we have $M \vDash \mathcal{CS}$ (in this case it is said that $M$ respects $\mathcal{CS}$). By $\mathcal{J}_\mathcal{CS}$-model we mean Fitting model for justification logic $\mathcal{J}_\mathcal{CS}$.

In modal logic, each modal axiom $T$, $D$, $B$, 4, and 5 characterizes a class of Kripke frames (see columns 1 and 2 of Table 1). Informally, each justification axiom characterizes a class of Fitting frames (see columns 3, 4 and 5 of Table 1).

| Modal axiom | $\mathcal{R}$ is ... | Justification axiom | $\mathcal{R}$ is ... | $\mathcal{E}$ satisfies ... |
|-------------|----------------------|---------------------|----------------------|-----------------------------|
| $T$: $\Box A \rightarrow A$ | Reflexive | $jT$: $t : A \rightarrow A$ | Reflexive | $-$ |
| $D$: $\Box \bot \rightarrow \bot$ | Serial | $jD$: $t : \bot \rightarrow \bot$ | Serial | $-$ |
| $4$: $\Box A \rightarrow \Box \Box A$ | Transitive | $j4$: $t : A \rightarrow !t : t : A$ | Transitive | $\mathcal{E}3, \mathcal{E}4$ |
| $B$: $\neg A \rightarrow \Box \neg A$ | Symmetric | $jB$: $\neg A \rightarrow ?t : \neg t : A$ | Symmetric | $\mathcal{E}5$ |
| $5$: $\neg \Box A \rightarrow \Box \neg \Box A$ | Euclidean | $j5$: $\neg t : A \rightarrow ?t : \neg t : A$ | $-$ | $\mathcal{E}6, \mathcal{E}7$ |

Table 1. Modal and justification axioms with corresponding frame properties.

**Theorem 3.1 (Completeness)**. Let $\mathcal{J}_L$ be one of the justification logics of Definition 2.1. Let $\mathcal{CS}$ be a constant specification for $\mathcal{J}_L$, with the requirement that if $\mathcal{J}_L$ contains axiom scheme $jD$ then $\mathcal{CS}$ should be axiomatically appropriate. Then justification logics $\mathcal{J}_\mathcal{CS}$ are sound and complete with respect to their $\mathcal{J}_\mathcal{CS}$-models.

**Proof**. The first detailed presentation of the proof of completeness theorem was presented in [16] by Fitting for LP. The proof of completeness for other justification logics is similar to those given in [5, 21, 26, 27, 37]. Completeness of justification logics $\mathcal{JB}$, $\mathcal{JTB}$, $\mathcal{JDB}$, $\mathcal{JB5}$, $\mathcal{JB45}$ are proved in [21], and of justification logic $\mathcal{JB}_{\mathcal{CS}}$ and all its extensions are proved in [27].

Soundness is straightforward. Let us only check the truth of axiom $\neg A \rightarrow ?t : \neg t : A$ in a $\mathcal{JB}_{\mathcal{CS}}$-model $M = (W, \mathcal{R}, \mathcal{E}, \mathcal{V})$. Suppose $(M, w) \vDash \neg A$. By ($\mathcal{E}5$) we have $w \in \mathcal{E}(?t, \neg t : A)$, for any term $t$. Now suppose $v$ is an arbitrary world such that $wRv$. Since $\mathcal{R}$ is symmetric we have $vRw$. We claim that $(M, v) \vDash \neg t : A$. Indeed, otherwise from $(M, v) \vDash t : A$ and $vRw$, we get $(M, w) \vDash A$, which is a contradiction. Hence $(M, w) \vDash ?t : \neg t : A$.

For completeness, we construct a canonical model $M = (W, \mathcal{R}, \mathcal{E}, \mathcal{V})$ for each $\mathcal{J}_L$. Let $W$ be the set of all maximal consistent sets in $\mathcal{J}_\mathcal{CS}$, and define other components of $M$ as follows. For all $\Gamma, \Delta$ in $W$:

$$\Gamma R \Delta \iff \Gamma^b \subseteq \Delta,$$

$$\mathcal{E}(t, A) = \{ \Gamma \in W \mid t : A \in \Gamma \},$$
\[ \Gamma \in V(P) \iff P \in \Gamma \]

where \( \Gamma^o = \{ A \mid t : A \in \Gamma, \text{ for some term } t \} \) and \( P \) is a propositional variable. We define \( \vdash \) on arbitrary formulas as in Definition 3.2. Since \( CS \subseteq \Gamma \) for each \( \Gamma \in W, E \) is a possible evidence function on \( W \) for \( JL_{CS} \). The Truth Lemma can be shown for \( JL_{CS} \): for each formula \( F \) and each \( \Gamma \in W, \)

\[(M, \Gamma) \vdash F \iff F \in \Gamma.\]

The base case follows from the definition of \( V \) in the canonical model, and the cases of Boolean connectives are standard. We only verify the case in which \( F \) is of the form \( t : A \). If \( (M, \Gamma) \vdash t : A \), then \( \Gamma \in E(t, A) \), and therefore \( t : A \in \Gamma \). Conversely, suppose \( t : A \in \Gamma \). Then \( \Gamma \in E(t, A) \), and by the definition of \( R \), \( A \in \Delta \) for each \( \Delta \in W \) such that \( \Gamma \Delta \). By the induction hypothesis, \( (M, \Delta) \vdash A \). Therefore, \( (M, \Gamma) \vdash t : A \).

For each justification logic \( JL \), it suffices to show that the canonical model \( M \) is an \( JL \)-model or, equivalently, \( E \) is an admissible evidence function for \( JL_{CS} \). We only verify it for justification logic \( JB \) and its extensions. For the canonical model of \( JB \) we establish that the accessibility relation \( R \) is symmetric and the possible evidence function \( E \) satisfies \( E5 \).

\( R \) is symmetric: Assume that \( \Gamma \Delta \) for \( \Gamma, \Delta \in W \), and further \( t : A \in \Delta \). It suffices to prove that \( A \in \Gamma \). Suppose, in contrary, that \( A \) is not in \( \Gamma \). Since \( \Gamma \) is a \( \text{JB}_{CS} \)-maximally consistent set, \( \neg A \) and \( \neg t : A \notin \Gamma \). Hence \( \neg t : A \notin \Gamma \). Now, by \( \Gamma \Delta \), we conclude that \( \neg t : A \in \Delta \), which is a contradiction. Thus \( A \) is in \( \Gamma \), and therefore \( \Delta R \Gamma \).

\( E \) satisfies \( E5 \): Suppose that \( (M, \Gamma) \vdash \neg A \). Then by the Truth Lemma \( \neg A \in \Gamma \). On the other hand, \( \neg t : A \notin \Gamma \), for every term \( t \). Hence \( \neg t : A \notin \Gamma \), and therefore by the definition of evidence function \( E \) we have \( \Gamma \in E(?t, \neg t : A) \).

Finally, suppose \( JL_{CS} \nvdash F \). Then the set \( \{ \neg F \} \) is \( JL_{CS} \)-consistent, and it can be extended to a maximal consistent set \( \Gamma \). Hence \( F \notin \Gamma \), and by the Truth Lemma \( (M, \Gamma) \nvdash F \). Thus \( M \nvdash F \). \hfill \( \Box \)

Remark 3.1. One can verify that the canonical model of all justification logics defined in the proof of Theorems 3.1 is fully explanatory. A Fitting model \( M = (W, R, E, V) \) is fully explanatory if for any world \( w \in W \) and any formula \( F \), whenever \( w \vdash F \) for every \( v \in W \) such that \( wRv \), then for some justification term \( t \) we have \( w \vdash t : F \). This fact was first proved by Fitting in [16] for the logic of proofs LP. Artemov [5] established it for \( J \), and his proof can be adapted for other justification logics.

Remark 3.2. It is easy to show that the canonical model of all justification logics defined in the proof of Theorems 3.1 enjoys the strong evidence property (\( E7 \)). To this end, suppose \( \Gamma \in E(t, A) \). By the definition of \( E \) in the canonical model \( M \), we have \( t : A \in \Gamma \). Then by the Truth Lemma, \( (M, \Gamma) \vdash t : A \), as desired.

Remark 3.3. It is useful to show that in the canonical model of \( JS \) and its extensions the accessibility relation is Euclidean and also we have: if \( \Gamma \Delta \) and \( \Delta \in E(t, A) \), then \( \Gamma \in E(t, A) \). The latter is called anti-monotonicity condition. The proof is as follows:

**Anti monotonicity.** Suppose \( \Gamma \Delta \) and \( \Delta \in E(t, A) \). By the definition of \( E \), we have \( t : A \in \Delta \). Let us suppose \( \Gamma \notin E(t, A) \), or equivalently \( t : A \notin \Gamma \), and derive a contradiction. Since \( t : A \notin \Gamma \), and \( \Gamma \) is a maximal consistent set, we have \( \neg t : A \in \Gamma \). Since \( \neg t : A \in \Gamma \), we have \( t : A \notin \Gamma \). Now \( \Gamma \Delta \) yields \( \neg t : A \in \Delta \), which is a contradiction.

**Euclideanness of \( R \).** Suppose \( \Gamma \Delta, \Gamma \Sigma, \) and \( t : A \in \Delta \). We need to show that \( A \in \Sigma \). To this end it suffices to show that \( A \in \Gamma \) (this together with \( \Gamma \Sigma \) implies that \( A \in \Sigma \)). Suppose towards a contradiction that \( t : A \notin \Gamma \). Thus \( \neg t : A \in \Gamma \), and hence \( ?t : \neg t : A \in \Gamma \). Now \( \Gamma \Delta \) yields \( \neg t : A \in \Delta \), which is a contradiction.
In the rest of this section we show that how possible evidence functions can be extended to admissible evidence functions. The following definitions are inspired by the work of Studer [39] which introduced *inductively generated evidence functions* for singleton Fitting models (although our definition of inductively generated evidence function for J5 and its extension is not quite the same as Studer’s).

**Definition 3.5.** The rank of a justification term is defined as follows:
1. \( rk(x) = rk(c) = 0 \), for any justification variable \( x \) and justification constant \( c \),
2. \( rk(s + t) = rk(s \cdot t) = \max(rk(s), rk(t)) + 1 \),
3. \( rk(?t) = rk(?!t) = rk(t) + 1 \).

In the following definition and lemmas the accessibility relation of Kripke frames for J5 and its extension is required to be Euclidean. The properties of the accessibility relation of Kripke frames for other justification logics are as before (see Definition 3.4 or Table 1).

**Definition 3.6.** Let \( \mathcal{JL}_{\mathbf{CS}} \) be one of the justification logics which do not contain axiom \( \text{jB} \), \((\mathcal{W}, \mathcal{R})\) be a Kripke frame for \( \mathcal{JL}_{\mathbf{CS}} \), and \( \mathcal{E}_0 \) be a possible evidence function on \( \mathcal{W} \) for \( \mathcal{JL}_{\mathbf{CS}} \). We construct possible evidence functions \( \mathcal{E}_i \) \((i \geq 1)\) inductively as follows:
1. \( \mathcal{E}_{i+1}(x, F) = \mathcal{E}_i(x, F) \), for any justification variable \( x \).
2. \( \mathcal{E}_{i+1}(c, F) = \mathcal{E}_i(c, F) \), for any justification constant \( c \).
3. \( \mathcal{E}_{i+1}(s \cdot t, F) = \mathcal{E}_i(s \cdot t, F) \cup \{ w \in \mathcal{W} \mid w \in \mathcal{E}_i(s, G) \cap \mathcal{E}_i(t, G) \text{, for some formula } G \text{, and } rk(s \cdot t) = i + 1 \} \).
4. \( \mathcal{E}_{i+1}(s + t, F) = \mathcal{E}_i(s + t, F) \cup \{ w \in \mathcal{W} \mid w \in \mathcal{E}_i(s, F) \text{ and } rk(s + t) = i + 1 \} \cup \{ w \in \mathcal{W} \mid w \in \mathcal{E}_i(t, F) \text{ and } rk(s + t) = i + 1 \} \).
5. \( \mathcal{E}_{i+1}(?, t, F) = \mathcal{E}_i(?t, F) \cup \{ w \in \mathcal{W} \mid w \in \mathcal{E}_i(t, G) \text{, for some formula } G \text{ such that } F = t : G \text{, and } rk(?t) = i + 1 \} \), if \( \text{j4} \) is an axiom of \( \mathcal{JL}_{\mathbf{CS}} \).
6. \( \mathcal{E}_{i+1}(!, t, F) = \mathcal{E}_i(!t, F) \cup \{ w \in \mathcal{W} \mid v \in \mathcal{E}_i(t, F) \text{, with } rk(t) = i + 1 \} \), if \( \text{j4} \) is an axiom of \( \mathcal{JL}_{\mathbf{CS}} \).
7. \( \mathcal{E}_{i+1}(t, F) = \mathcal{E}_i(t, F) \cup \{ w \in \mathcal{W} \mid v \in \mathcal{E}_i(t, F) \text{, with } rk(t) = i + 1 \} \), if \( \text{j5} \) is an axiom of \( \mathcal{JL}_{\mathbf{CS}} \).
8. \( \mathcal{E}_{i+1}(?!t, F) = \mathcal{E}_i(?!t, F) \cup \{ w \in \mathcal{W} \mid w \notin \mathcal{E}_i(t, G) \text{, for some formula } G \text{ such that } F = ?!t : G \text{, and } rk(?!t) = i + 1 \} \), if \( \text{j5} \) is an axiom of \( \mathcal{JL}_{\mathbf{CS}} \).

Inductively generated evidence function \( \mathcal{E} \) based on \( \mathcal{E}_0 \) is defined as follows: \( \mathcal{E}(t, F) = \bigcup_{i=0}^{\infty} \mathcal{E}_i(t, F) \), for any term \( t \) and formula \( F \).

Note that \( \mathcal{E}_i(t, F) \subseteq \mathcal{E}_{i+1}(t, F) \), for any term \( t \), formula \( F \), and \( i \geq 0 \). Using this fact it is easy to see that if \( w \in \mathcal{E}_i(t, F) \) and \( rk(t) \leq i \), then \( w \in \mathcal{E}_i(t, F) \). It is easy to show the following.

**Lemma 3.1.** Let \( \mathcal{JL}_{\mathbf{CS}} \) be one of the justification logics which do not contain axiom \( \text{jB} \), \((\mathcal{W}, \mathcal{R})\) be a Kripke frame for \( \mathcal{JL}_{\mathbf{CS}} \), \( \mathcal{E}_0 \) be a possible evidence function on \( \mathcal{W} \) for \( \mathcal{JL}_{\mathbf{CS}} \), and \( \mathcal{E} \) be the inductively generated evidence function based on \( \mathcal{E}_0 \). Then \( \mathcal{E} \) satisfies the application (\( \mathcal{E}1 \)) and sum (\( \mathcal{E}2 \)) conditions. If \( \mathcal{JL}_{\mathbf{CS}} \) contains axiom \( \text{j4} \), then \( \mathcal{E} \) also satisfies the monotonicity (\( \mathcal{E}3 \)) and positive introspection (\( \mathcal{E}4 \)) conditions. If \( \mathcal{JL}_{\mathbf{CS}} \) contains axiom \( \text{j5} \), then \( \mathcal{E} \) also satisfies the negative introspection (\( \mathcal{E}6 \)) and anti-monotonicity (\( \mathcal{E}8 \)) conditions.

**Lemma 3.2.** Let \( \mathcal{JL}_{\mathbf{CS}} \) be one of the justification logics that contains axiom \( \text{j5} \) and do not contain axiom \( \text{jB} \), \((\mathcal{W}, \mathcal{R})\) be a Kripke frame for \( \mathcal{JL}_{\mathbf{CS}} \), \( \mathcal{E}_0 \) be a possible evidence function on \( \mathcal{W} \) for \( \mathcal{JL}_{\mathbf{CS}} \), \( \mathcal{E} \) be the inductively generated evidence function based on \( \mathcal{E}_0 \), and \( \mathcal{V} \) be a valuation. Consider the possible Fitting model \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \). If for all terms \( r \in T_{\mathcal{JL}} \), formulas \( F \in F_{\mathcal{JL}} \), and \( w \in \mathcal{W} \) we have \( w \in \mathcal{E}_0(r, F) \Rightarrow (\mathcal{M}, w) \models r : F \),
then \( \mathcal{E} \) satisfies the strong evidence condition (\( \mathcal{E}7 \)).
Proof. By induction on \( i \) we show that for all \( w \in W \)

\[ w \in \mathcal{E}_i(r, F) \Rightarrow (M, w) \models r : F. \]

The base case, \( i = 0 \), follows from the assumption. For the induction hypothesis, suppose that for all \( w \in W, w \in \mathcal{E}_i(r, F) \) implies \((M, w) \models r : F\). For any \( w \in W \), if \( w \in \mathcal{E}_{i+1}(r, F) \) for \( i \geq 0 \), and \( w \in \mathcal{E}_j(r, F) \) for \( j < i + 1 \), then by the induction hypothesis \((M, w) \models r : F\). Assume now that for an arbitrary \( w \in W \) we have \( w \in \mathcal{E}_{i+1}(r, F) \) for \( i \geq 0 \), and \( w \notin \mathcal{E}_j(r, F) \) for any \( j < i + 1 \). We have the following cases:

1. \( r = s \cdot t \) and \( w \in \mathcal{E}_{i+1}(s \cdot t, F) \). Then \( w \in \mathcal{E}_i(s, G \rightarrow F) \cap \mathcal{E}_i(t, G) \), for some formula \( G \), and \( \text{rk}(s \cdot t) = i + 1 \). Thus, by the induction hypothesis, \((M, w) \models s : (G \rightarrow F) \) and \((M, w) \models t : G\). Therefore, \( w \in \mathcal{E}(s, G \rightarrow F) \) and \( w \in \mathcal{E}(t, G) \). By Lemma 3.1, \( w \in \mathcal{E}(s \cdot t, F) \). Now it is easy to show that for arbitrary world \( v \in W \) such that \( w \mathcal{R} v \) we have \((M, v) \models F\). Hence \((M, w) \models s \cdot t : F\), which is what we wished to show.

2. \( r = s + t \) and \( w \in \mathcal{E}_{i+1}(s + t, F) \). Thus \( w \in \mathcal{E}_i(t, F) \) or \( w \in \mathcal{E}_i(s, F) \), and \( \text{rk}(s + t) = i + 1 \). If \( w \in \mathcal{E}_i(t, F) \), then by the induction hypothesis, \((M, w) \models t : F\). Therefore, \( w \in \mathcal{E}(t, F) \). By Lemma 3.1, \( w \in \mathcal{E}(s + t, F) \). On the other hand, for arbitrary world \( v \in W \) such that \( w \mathcal{R} v \) we have \((M, v) \models F\). Hence \((M, w) \models s + t : F\). Proceed similarly if \( w \in \mathcal{E}_i(s, F) \).

3. \( j4 \) is an axiom of \( \mathcal{J}_{\mathcal{CS}} \), \( r = \neg t \), and \( w \in \mathcal{E}_{i+1}(!t, F) \). Then \( w \in \mathcal{E}_i(t, G) \), for some formula \( G \) such that \( F = t : G \), and \( \text{rk}(!t) = i + 1 \). Thus, by the induction hypothesis, \((M, w) \models t : G\). Therefore, \( w \in \mathcal{E}(t, G) \). By Lemma 3.1, \( w \in \mathcal{E}(!t, F) \). Since \( \mathcal{R} \) is transitive, it is easy to show that for arbitrary world \( v \in W \) such that \( w \mathcal{R} v \) we have \((M, v) \models F\). Hence \((M, w) \models \neg t : F\).

4. \( j4 \) is an axiom of \( \mathcal{J}_{\mathcal{CS}} \), \( w \in \mathcal{E}_{i+1}(r, F) \), \( \mathcal{R} w, v \in \mathcal{E}_i(r, F) \), and \( \text{rk}(r) = i + 1 \). Thus, by the induction hypothesis, \((M, v) \models r : F\). Since \( \mathcal{R} \) is transitive, it is easy to show that for arbitrary world \( u \in W \) such that \( w \mathcal{R} u \) we have \((M, u) \models F\). Hence, since \( w \in \mathcal{E}(r, F) \), we have \((M, w) \models r : F\).

5. \( j5 \) is an axiom of \( \mathcal{J}_{\mathcal{CS}} \), \( w \in \mathcal{E}_{i+1}(r, F) \), \( \mathcal{R} w, v \in \mathcal{E}_i(r, F) \), and \( \text{rk}(r) = i + 1 \). Thus, by the induction hypothesis, \((M, v) \models r : F\). Since \( \mathcal{R} \) is Euclidean, it is easy to show that for arbitrary world \( u \in W \) such that \( w \mathcal{R} u \) we have \((M, u) \models F\). Hence, since \( w \in \mathcal{E}(r, F) \), we have \((M, w) \models r : F\).

6. \( j5 \) is an axiom of \( \mathcal{J}_{\mathcal{CS}} \), \( r = ?t \), and \( w \in \mathcal{E}_{i+1}(?t, F) \). Then \( w \notin \mathcal{E}_i(t, G) \), for some formula \( G \) such that \( F = \neg t : G \), and \( \text{rk}(?t) = i + 1 \). For arbitrary \( v \in W \) such that \( w \mathcal{R} v \), we need to show that \((M, v) \models \neg t : G\). Suppose towards a contradiction that \( v \in \mathcal{E}(t, G) \). By anti-monotonicity, \( w \in \mathcal{E}(t, G) \). This, together with \( \text{rk}(t) = i \), implies that \( w \in \mathcal{E}_i(t, G) \), which is a contradiction. Thus \( v \notin \mathcal{E}(t, G) \), and hence \((M, v) \models \neg t : G\). Therefore, \((M, w) \models ?t : \neg t : G\). ⊢

4 Labeled sequent calculus

Negri and von Plato in [32] proposed a method to transform universal axioms into rules of Gentzen system. Universal axioms are first transformed into conjunctions of formulas of the form \( P_1 \land \ldots \land P_m \rightarrow Q_1 \lor \ldots \lor Q_n \), where \( P_i \) and \( Q_j \) are atomic formulas. Then each conjunct is converted into the regular rule scheme:

\[
\frac{Q_1, P_1, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_n, \Gamma \Rightarrow \Delta}{P_i, \Gamma \Rightarrow \Delta} \quad \text{Reg}
\]

in which \( P \) abbreviates \( P_1, \ldots, P_m \). In [29] the method was extended to transform geometric axioms of the form \( \forall z(P_1 \land \ldots \land P_m \rightarrow \exists x_1 M_1 \lor \ldots \lor \exists x_n M_n) \), where each \( M_i \) is a conjunction of atomic
Initial sequents:

\[ w \vdash P, \Gamma \Rightarrow \Delta, w \vdash P \quad (Ax) \]
\[ w \vdash \bot, \Gamma \Rightarrow \Delta \quad (Ax \bot) \]

\( P \) is a propositional variable.

Propositional rules:

\[
\begin{array}{c}
\frac{\Gamma \Rightarrow \Delta, w \vdash A}{w \vdash \neg A, \Gamma \Rightarrow \Delta} \quad (L \neg) \\
\frac{w \vdash A, w \vdash B, \Gamma \Rightarrow \Delta}{w \vdash A \land B, \Gamma \Rightarrow \Delta} \quad (L \land) \\
\frac{w \vdash A, \Gamma \Rightarrow \Delta \quad w \vdash B, \Gamma \Rightarrow \Delta}{w \vdash A \lor B, \Gamma \Rightarrow \Delta} \quad (L \lor) \\
\frac{\Gamma \Rightarrow \Delta, w \vdash A, \Gamma \Rightarrow \Delta}{w \vdash A \rightarrow B, \Gamma \Rightarrow \Delta} \quad (L \rightarrow) \\
\frac{\Gamma \Rightarrow \Delta, w \vdash A, \Gamma \Rightarrow \Delta, w \vdash B}{w \vdash A \rightarrow B, \Gamma \Rightarrow \Delta} \quad (R \rightarrow) \\
\end{array}
\]

Table 2. Labeled sequent calculus G3c for propositional logic.

Formulas \( Q_{i_1}, \ldots, Q_{i_k} \), to rules by geometric rule scheme:

\[
\frac{Q_1(y_1/x_1), P_1, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n(y_n/x_n), P_2, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}^{GRS}
\]

where \( Q_i \) abbreviates \( Q_{i_1}, \ldots, Q_{i_k} \), and the eigenvariables \( y_1, \ldots, y_n \) of the premises are not free in the conclusion. \( A(y/x) \) indicates \( A \) after the substitution of the variable \( y \) for the variable \( x \).

Using \( \text{Reg} \) and \( GRS \), Sara Negri in [30] proposed cut-free labeled sequent calculi for a wide variety of modal logics characterized by Kripke models. By adopting a labeled language including possible worlds as labels and the accessibility relation, she presented G3-style sequent calculi (i.e. sequent calculi without structural rules) for normal modal logics. In this section, we extend this method to present labeled sequent calculi for justification logics.

Let us first define the language of labeled sequent calculus, called labeled language, for propositional logic. This language will be extended later in sections 4.1, 4.2 and 4.3. We need a countably infinite set \( L \) of labels \( w, v, u, \ldots \) which are used in the labeled systems as possible worlds of Kripke style models. The labeled language for propositional logic consists of labeled formulas (or forcing formulas) \( w \vdash A \), where \( A \) is a formula of propositional logic. Sequents are expressions of the form \( \Gamma \Rightarrow \Delta \), where \( \Gamma \) and \( \Delta \) are multisets of formulas in the labeled language. Initial sequents (or axioms) and rules for the labeled sequent calculus G3c of propositional logic are similar to the ordinary sequent calculus which are augmented by labels, see Table 2.

In Sections 4.1–4.3, in order to construct labeled sequent calculi for modal and justification logics, we will extend the underlying language of the formula \( A \) in \( w \vdash A \) to formulas in the language of modal and justification logics. Moreover, we will extend the labeled language of G3c to an extended labeled language with new atoms (e.g. relational, evidence). When we extend a labeled sequent calculus G3L of logic \( L \) to G3L’ of logic \( L’ \) (i.e. G3L’ contains all initial sequents and rules of G3L), all formulas \( A \) in
labeled formulas \( w \vdash A \), in initial sequents and rules of \( G3L' \), should be considered in the language of \( L' \), and also \( \Gamma, \Delta \) may now contain the new atoms of the extended labeled language.

In all of the rules and initial sequents of this paper, \( \Gamma, \Delta \) are called the side formulas. The formula(s) in the premise(s) and conclusion of a rule not in the side formulas are called active and principal, respectively. Also, the formula(s) in an initial sequent not in the side formulas are called principal.

Negri [30] considered a zero-premise rule \( L\bot \) instead of initial sequent \( (Ax\bot) \). She used \( w : A \) to denote labeled formulas in the labeled language. To avoid confusion with justification formulas of the form \( t : A \), we replace it by \( w \vdash A \). In addition, compare to [30] we use \( \to \) in place of \( \supset \) for implication, \( \land \) in place of \& for conjunction, and for simplicity, we deal only with \( \Box \) and take \( \lozenge \) as a definable modality in the modal language. We also add the rules \( (L\Box) \) and \( (R\Box) \) to \( G3c \). The name of rules given here is also different from Negri’s.

### 4.1 Labeled sequent calculi of modal logics

Extended labeled language for modal logics consists of labeled formulas \( w \vdash A \), in which \( A \) is a modal formula, and relational atoms (or accessibility atoms) \( wRv \). Negri in [30] presented the modal rules \( (L\Box) \) and \( (R\Box) \) for modal logics (see Table 3). Rule \( (R\Box) \) has the restriction that the label \( v \) (called eigenlabel) must not occur in the conclusion. To derive the properties of the accessibility relation (such as reflexivity, symmetric, transitivity, etc.) initial sequents for \( R \) (denoted by \( (AxR) \)) are added. All the axioms and rules in Table 2 and 3 constitute a G3-style labeled system for the basic modal logic \( K \), denoted by \( G3K \). Labeled systems for other modal logics are obtained by adding rules, that correspond to the properties of the accessibility relation in their Kripke models from Table 4, according to Table 5. For example,

\[
\begin{align*}
G3DB & = G3K + (Ser) + (Sym). \\
G3S4 & = G3K + (Ref) + (Trans). \\
G3S5 & = G3K + (Ref) + (Trans) + (Eucl) + (Eucl_\ast).
\end{align*}
\]

Note that contracted instances of rule \( (Eucl) \):

\[
\frac{vRv, wRv, \Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta} (Eucl_\ast)
\]

should be added to those labeled systems that contain rule \( (Eucl) \). In the seriality rule \( (Ser) \) the eigenlabel \( v \) must not occur in the conclusion (this rule is obtained by the geometric rule scheme \( GRS \)). All fifteen modal logics of the modal cube have labeled sequent calculus. Negri in [30, 31] shows that all these labeled systems are sound and complete with respect to their Hilbert systems, all of
the rules in the modal labeled systems are invertible and structural rules (weakening and contraction rules) and cut are admissible. The termination of proof search was proved for G3K, G3T, G3KB, G3TB, G3S4, and G3S5.

4.2 Labeled sequent calculus based on F-models

Extended labeled language for justification logics consist of labeled formulas \( w \vdash A \), relational atoms \( wRv \), and evidence atoms \( wE(t, A) \), where \( w \) and \( v \) are labels from \( L \), \( t \) is a justification term and \( A \) is a JL-formula. These atoms respectively denote the statements \((M, w) \vdash A, wRv \) and \( w \in E(t, A) \) in Fitting models. Thus in this language we are able to give a symbolic presentation of semantical elements of Fitting models. For example, application \((E1)\) and negative introspection \((E6)\) conditions could be expressed in the extended labeled language as follows:

\[
\begin{align*}
&wE(s, A \to B) \land wE(t, A) \to wE(s \cdot t, B), \\
&wE(t, A) \lor wE(?t, \neg t : A).
\end{align*}
\]

By the definition of forcing relation for formulas of the form \( t : A \) in F-models:

\[
(M, w) \vdash t : A \text{ iff } w \in E(t, A) \text{ and for every } v \in W \text{ with } wRv, (M, v) \vdash A,
\]

and the regular rule scheme \((Reg)\), we obtain the rules \((L :)\) and \((E)\) for justification logics, see Table 6. Rule \((R :)\) is obtained similar to rule \((R\square)\) in modal labeled systems, and likewise the eigenlabel \( v \) in the premise of the rule \((R :)\) must not occur in the conclusion. Again to derive the properties of admissible evidence function, we should add initial sequents for evidence atoms \((AxE)\) from Table 6.
Initial sequents:

\[ wRv, \Gamma \Rightarrow \Delta, wRv \quad (AxR) \]
\[ wE(t, A), \Gamma \Rightarrow \Delta, wE(t, A) \quad (AxE) \]

JL rules:

\[
\begin{align*}
  \vdash A, & \vdash t : A, wRv, \Gamma \Rightarrow \Delta \\
  \vdash A, & \vdash t : A, wRv, \Gamma \Rightarrow \Delta \\
  \vdash A, & \vdash t : A, wRv, \Gamma \Rightarrow \Delta \\
  \vdash A, & \vdash t : A, wRv, \Gamma \Rightarrow \Delta
\end{align*}
\]

In \((R:)\) the eigenlabel \(v\) must not occur in the conclusion of rule.

Rules for evidence atoms:

\[
\begin{align*}
  wE(s + t, A), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE(s + t, A), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE(s \cdot t, B), wE(s, A \rightarrow B), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE(s, A \rightarrow B), w \in E(t, A), \Gamma \Rightarrow \Delta
\end{align*}
\]

Iterated axiom necessitation rule:

\[
\begin{align*}
  wE(c_{i_n}, c_{i_{n-1}} : \ldots : c_1 : A), \Gamma \Rightarrow \Delta \\
  \Gamma \Rightarrow \Delta
\end{align*}
\]

Table 6. Initial sequents and rules for labeled sequent calculus G3J to be added to G3c.

Rules for evidence atoms:

\[
\begin{align*}
  wE((t, t : A), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE((t, t : A), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE((t, t : A), wE(t, A), \Gamma \Rightarrow \Delta \\
  wE((t, t : A), wE(t, A), \Gamma \Rightarrow \Delta
\end{align*}
\]

Axiom necessitation rule:

\[
\begin{align*}
  wE(c, A), \Gamma \Rightarrow \Delta \\
  \Gamma \Rightarrow \Delta
\end{align*}
\]

Table 7. Rules for labeled sequent calculi based on F-models.
| Justification axiom | Corresponding rule |
|---------------------|--------------------|
| jT                  | (Ref)              |
| jD                  | (Ser)              |
| jB                  | (E?), (Sym)        |
| j4                  | (E!), (Mon), (Trans) |
| j5                  | (SE), (E?)         |

Table 8. Corresponding rules to be added to G3J for labeled sequent calculi of justification logics.

Now we define labeled sequent calculi for various justification logics according to properties of R and E in their F-models. Labeled system G3J is the extension of G3c, Table 2, by the initial sequents and rules from Table 6. Table 8 specifies which rules must be added to G3J from Table 7 to get labeled systems G3JL for various justification logics. All rules of Tables 6, 7 are obtained from regular rule scheme Reg.

Labeled system G3J4 and its extensions possesses rule (AN) and other labeled systems rule (IAN). In the rules (AN) and (IAN), the formula A is an axiom of JL, c and c_i’s are justification constants.

The labeled sequent calculus G3JL_CS is obtained from G3JL by restricting rule (IAN)/(AN) as follows: for each evidence atom wE(c_{i_n}, c_{i_{n-1}}, \ldots, c_{i_1} : A) (or wE(c, A)) in the premise of the rule (IAN) (or (AN)) we should have c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS (or c : A \in CS). G3JL denotes G3JL_{TCS_JL}. In the rest of the paper whenever we use G3JL_CS it is assumed that CS is a constant specification for JL.

5 Basic properties

In the rest of the paper, for a justification logic JL defined in Definition 2.1, let G3JL denote its corresponding labeled sequent calculus based on F-models. Let G3JL^- denote those labeled sequent calculi which do not contain rules (SE) or (E?) in its formulation. Thus G3JL^- could be each of the labeled sequent calculi G3J, G3J4, G3JD, G3JD4, G3JT, G3LP.

The height of a derivation is the maximum number of successive applications of rules in it. Given a constant specification CS, we say that a rule is height-preserving CS-admissible if whenever an instance of its premise(s) is derivable in G3JL_CS with height n, then so is the corresponding instance of its conclusion. In the rest of the paper, \varphi stands for one of the formulas w \vDash A, wRv, or wE(t, A). We begin with the following simple observation:

**Lemma 5.1.** Sequents of the form w \vDash A, \Gamma \Rightarrow \Delta, w \vDash A, with A an arbitrary JL-formula, are derivable in G3JL_CS.

**Proof.** The proof involves a routine induction on the complexity of the formula A. ⊥

**Lemma 5.2.** The following rule is CS-admissible in G3JL_CS:

\[
\begin{align*}
\frac{wE(s + t, A), wE(t + s, A), wE(t, A), \Gamma \Rightarrow \Delta}{wE(t, A), \Gamma \Rightarrow \Delta} \quad (E+) 
\end{align*}
\]

**Proof.** We have the derivation
\[
\begin{align*}
\frac{\text{w}E(s + t, A), \text{w}E(t + s, A), \text{w}E(t, A), \Gamma \Rightarrow \Delta}{\text{w}E(t + s, A), \text{w}E(t, A), \Gamma \Rightarrow \Delta} & \quad \text{(El+)} \\
\frac{\text{w}E(t, A), \Gamma \Rightarrow \Delta}{\text{w}E(t, A), \Gamma \Rightarrow \Delta} & \quad \text{(Er+)}
\end{align*}
\]

\[\text{Lemma 5.3.} \quad \text{The following rule is } \text{CS}\text{-admissible in } \text{J5}_\text{CS} \text{ and its extensions:}\]

\[
\frac{\text{w}E(t, A), \text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta}{\text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta} \quad \text{(Anti-Mon)}
\]

\[\text{Proof.} \quad \text{We have the derivation}\]

\[
\begin{align*}
\text{v} \vdash t : A, \text{w} \vdash \neg t : A, \text{w}E(\neg t, A), \text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta, \text{v} \vdash t : A & \quad \text{(L-)} \\
\text{v} \vdash \neg t : A, \text{w} \vdash \neg t : A, \text{w}E(\neg t, A), \text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta & \quad \text{(L :)} \\
\text{w} \vdash \neg t : A, \text{w}E(\neg t, A), \text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta & \quad \text{(SE)} \\
\text{w}E(t, A), \text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta & \quad \text{(SE)} \\
\text{v}E(t, A), \text{w}Rv, \Gamma \Rightarrow \Delta & \quad \text{(E?)}
\end{align*}
\]

\[\text{with top-sequent in the right branch is derivable by Lemma 5.1.}\]

Let \(\varphi(v/w)\) denote the result of simultaneously substituting label \(v\) for all occurrences of label \(w\) in \(\varphi\). For a multiset of labeled formulas \(\Gamma\), let \(\Gamma(v/w) = \{\varphi(v/w) \mid \varphi \in \Gamma\}\). For a derivation \(D\), let \(D(v/w)\) denote the result of simultaneously substituting label \(v\) for all occurrences of label \(w\) in \(D\). Next, we prove the substitution lemma for labels.

\[\text{Lemma 5.4 (Substitution of Labels).} \quad \text{The rule of substitution}\]

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma(v/w) \Rightarrow \Delta(v/w)} \quad \text{Subs}
\]

\[\text{is height-preserving } \text{CS}\text{-admissible in } \text{G3JL}_\text{CS}.\]

\[\text{Proof.} \quad \text{The proof is by induction on the height } n \text{ of the derivation of } \Gamma \Rightarrow \Delta \text{ in } \text{G3JL}_\text{CS}. \text{ If } n = 0, \text{ then } \Gamma \Rightarrow \Delta \text{ is an initial sequent, and so is } \Gamma(v/w) \Rightarrow \Delta(v/w). \text{ If } n > 0, \text{ then suppose the last rule of the derivation is } (R). \text{ If } (R) \text{ is any rule without label condition, then use the induction hypothesis and then apply the rule } (R). \text{ Now assume } (R) \text{ is } (R:) \text{ or } (Ser); \text{ we only sketch the proof for } (R:), \text{ the case for } (Ser) \text{ is similar. Suppose } (R:) \text{ is the last rule:}\]

\[
\frac{\text{w}R\text{u}, \text{w}E(t, A), \Gamma' \Rightarrow \Delta', \text{u} \vdash A}{\text{w}E(t, A), \Gamma' \Rightarrow \Delta', \text{w} \vdash t : A} \quad \text{(R:)}
\]

\[\text{If the eigenlabel } \text{u} \text{ is not } v, \text{ then using the induction hypothesis (substitute } v \text{ for } \text{w}) \text{ and rule } (R:) \text{ we obtain a derivation of height } n\]

\[
\frac{\text{v}R\text{u}, \text{v}E(t, A), \Gamma'(v/w) \Rightarrow \Delta'(v/w), \text{u} \vdash A}{\text{v}E(t, A), \Gamma'(v/w) \Rightarrow \Delta'(v/w), \text{v} \vdash t : A} \quad \text{(R:)}
\]

\]

23
If \( u = v \), then by the induction hypothesis (substitute \( v' \) for \( u \)) we obtain a derivation of height \( n - 1 \)
of

\[
\text{wRv', wE}(t, A), \Gamma' \Rightarrow \Delta', v' \vdash A,
\]

where \( v' \) is a fresh label, i.e. \( v' \neq w \) and \( v' \) does not occur in \( \Gamma' \cup \Delta' \). Then using the induction hypothesis (substitute \( v \) for \( w \)) we obtain a derivation of height \( n \)

\[
\text{vRv', vE}(t, A), \Gamma'(v/w) \Rightarrow \Delta'(v/w), v' \vdash A \quad (R:).
\]

Finally, suppose the rule \((R)\) is the axiom necessitation rule \((AN)\):

\[
\frac{\text{uE}(c, A), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (AN),
\]

where \( c : A \in \text{CS} \). If \( u \neq w \) then using the induction hypothesis (substitute \( v \) for \( w \)) and rule \((AN)\) we obtain a derivation of height \( n \)

\[
\frac{\text{uE}(c, A), \Gamma'(v/w) \Rightarrow \Delta'(v/w)}{\Gamma'(v/w) \Rightarrow \Delta'(v/w)} \quad (AN).
\]

If \( u = w \) then using the induction hypothesis (substitute \( v \) for \( w \)) and rule \((AN)\) we obtain a derivation of height \( n \)

\[
\frac{\text{vE}(c, A), \Gamma'(v/w) \Rightarrow \Delta'(v/w)}{\Gamma'(v/w) \Rightarrow \Delta'(v/w)} \quad (AN).
\]

Proceed similarly if \((R)\) is \((IAN)\). In the induction step, a cases-by-case analysis shows that no step (specially \((AN)\) and \((IAN)\)) change the underlying constant specification \( \text{CS} \), and hence the rule \( \text{Subs} \) is \( \text{CS} \)-admissible.

\[\]
Proposition 6.2 (Weak Subformula Property). All formulas in a derivation in $G3Jd_{CS}$ are either labeled-subformulas of labeled formulas in the endsequent or atomic formulas of the form $wE(t, A)$ or $wRv$.

The following examples show that the labeled-subformula property does not hold in $G3J5$, $G3JB$ and their extensions.

Example 6.1. Consider the following derivation in $G3J5d$:

$(Ax)$

$\frac{v \Vdash P, w \Vdash t : P, wRv, wE(t, P) \Rightarrow v \Vdash P}{(L :)}$

$w \Vdash t : P, wRv, wE(t, P) \Rightarrow v \Vdash P$

$(SE)$

$wRv, wE(t, P) \Rightarrow v \Vdash P$

where $P$ is a propositional variable. Note that $w \Vdash t : P$ is not a labeled-subformula of any labeled formula in the endsequent.

Example 6.2. Consider the following derivation in $G3JBd$:

$(Ax)$

$\frac{v \Vdash P, w \Vdash s : P, wRv \Rightarrow v \Vdash P, wE(\overline{t}, -t : s : P)}{(L :)}$

$\frac{w \Vdash s : P, wRv \Rightarrow v \Vdash P, wE(\overline{t}, -t : s : P)}{w \Vdash s : P, wRv \Rightarrow v \Vdash P, wE(s, c \cdot \overline{t}, Q \rightarrow Q)}$

$(E?)$

Note that $w \Vdash s : P$ is not a labeled-subformula of any labeled formula in the endsequent.

Example 6.3. Let $CS$ be a constant specification for $JB$ which contains $c : (-t : s : P \rightarrow (Q \rightarrow Q))$. Consider the following derivation $D$ in $G3JB_{CS}$:

$(Ax)$

$\frac{v \Vdash P, w \Vdash s : P, wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}{(L :)}$

$\frac{w \Vdash s : P, wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}{wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}$

$(E?)$

in which the derivation $D'$ is as follows:

$(AxE)$

$\frac{wE(c \cdot \overline{t}, Q \rightarrow Q), wE(c \cdot -t : s : P \rightarrow (Q \rightarrow Q)), wE(\overline{t}, -t : s : P), wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}{wE(c \cdot \overline{t}, -t : s : P), wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}$

$(E)$

$\frac{wE(\overline{t}, -t : s : P), wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}{wE(\overline{t}, -t : s : P), wRv \Rightarrow v \Vdash P, wE(c \cdot \overline{t}, Q \rightarrow Q)}$

$(IAN)$

Note that $w \Vdash s : P$ is not a labeled-subformula of any labeled formula in the endsequent.

Since the regular rule scheme ($Reg$) and geometric rule scheme ($GRS$) produce new atoms in their premises, when the rules are read upwardly, those rules that are instances of ($Reg$) and ($GRS$) may violate the analyticity and termination of proof search. For example, in labeled systems $G3JB$, $G3J5$, 

25
and their extensions the labeled-subformula property does not hold, as Examples 6.1-6.3 show. The reason is that the rules \((E?)\) and \((SE)\) are obtained by \((Reg)\) from

\[
\text{w} \models A \lor \text{w}E(?t, \neg t : A),
\]

and

\[
\text{w}E(t, A) \rightarrow \text{w} \models t : A,
\]

where (6) and (7) are formal expressions of weak negative introspection \((E5)\) and strong evidence \((E7)\) conditions in the extended labeled language. Moreover, in contrast to other conditions on evidence function, conditions \((E5)\) and \((E7)\) involve forcing assertion. In other words, both (6) and (7) contain labeled formulas, and these labeled formulas are appeared only in the premise(s) of the rules \((E?)\) and \((SE)\).

Although \(\text{G3JL}^-\) enjoy the labeled-subformula property, since the rules \((E-), (El+), (Er+), (El), (IAN)/(AN)\) produce new justification terms in evidence atoms in their premises, when the rules are read upwardly, they may violate the analyticity. Since some rules, like \((Ref), (Ser), (E?)\), produce (new) labels in their premise(s), when the rules are read upwardly, they may also violate the analyticity. In the following we shall show the subterm and sublabel property which are needed to prove the analyticity for \(\text{G3JL}^-\). First, similar to [12], we show the sublabel property for all our labeled sequent calculi \(\text{G3JL}\).

**Definition 6.2.** A rule instance has the sublabel property if every label occurring in any premise of the rule is either an eigenlabel or occurs in the conclusion. A derivation has the sublabel property if all rule instances occurring in it has the sublabel property.

In a derivation with sublabel property all labels are either an eigenlabel or labels in the endsequent. This property is called the sublabel property.\(^2\) Instances of the rules \((Ref), (Ser), (E?), (IAN)/(AN)\) in derivations may produce a derivation which does not have the subterm property. In the following we try to show the sublabel property. First we need a lemma.

**Lemma 6.1.** Suppose \(D\) is a derivation with the sublabel property for \(\Gamma \Rightarrow \Delta\) in \(\text{G3JL}_{CS}\), and labels \(w, v\) occur in \(D\) such that \(w\) occurs in \(\Gamma \cup \Delta\) and \(v\) is not an eigenlabel. Then the derivation \(D(w/v)\) is a derivation with the sublabel property for \(\Gamma(w/v) \Rightarrow \Delta(w/v)\) in \(\text{G3JL}_{CS}\).

**Proof.** The proof involves a routine induction on the height of the derivation \(D\) of \(\Gamma \Rightarrow \Delta\) in \(\text{G3JL}_{CS}\).

\(\Box\)

**Proposition 6.3 (Sublabel Property).** Every sequent derivable in \(\text{G3JL}_{CS}\) has a derivation with the sublabel property; in other words, every derivable sequent \(\Gamma \Rightarrow \Delta\) has a derivation in which all labels are eigenlabels or labels in \(\Gamma \cup \Delta\).

**Proof.** By induction on the height of the derivation we transform every derivation into one with sublabel property. The base case is trivial, since every initial sequent has the sublabel property. For the induction step, suppose \((R)\) is a topmost rule which does not have the sublabel property. For each label, say \(v\), in any premise of \((R)\) that is not in the conclusion and is not an eigenlabel, we substitute for it (in the derivations of all the premises) any label, say \(w\), that is in the endsequent. Note

\(^2\) It is called the subterm property in [30], and does not to be confused with the subterm property stated in Proposition 6.4.
that in any derivation in \( G3JL_{CS} \) all labels in a rule’s conclusion are in all its premises. Thus \( w \) is already in the conclusion of each premise, and therefore by Lemma 6.1 the derivations of the premise(s) have still the sublabel property. Moreover, the conclusion of \((R)\) is unchanged under this substitution, and the resulting rule instance has now the sublabel property. By repeating the above argument for all rules which do not have the sublabel property we finally obtain a derivation with sublabel property. ∎

Example 6.4. Here is a derivation of \( w \vdash x : P \Rightarrow w \vdash P \) in \( G3JT \) which does not have the sublabel property:

\[
(Ax) \\
\begin{array}{c}
w \vdash P, v \vdash P, w \vdash P, x : P \\
\end{array} \\
\begin{array}{c}
v \vdash P, w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \tag{L :}
\]

\[
(Ref) \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \tag{Ref}
\]

The topmost instance of rule \((Ref)\) does not have the sublabel property, because the label \( v \) is not in the conclusion of the rule. By the substitution described in the proof of Proposition 6.3, substitute \( w \) for \( v \), the derivation is transformed to one with the sublabel property:

\[
(Ax) \\
\begin{array}{c}
w \vdash P, w \vdash P, w \vdash P, x : P \\
\end{array} \\
\begin{array}{c}
w \vdash P, w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \tag{L :}
\]

\[
(Ref) \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \\
\begin{array}{c}
w \vdash P, x : P \Rightarrow w \vdash P \\
\end{array} \tag{Ref}
\]

Now we prove the subterm property for \( G3J, G3J4, G3J4, G3J4D, G3JT, G3LP \). We use the following notations in the rest of this paper. For an arbitrary sequent \( \Gamma \Rightarrow \Delta \) in the language of \( G3JL \), and constant specification \( CS \) for \( JL \), let

- \( Tm(\Gamma \Rightarrow \Delta) \) denote the set of all terms which occur in a labeled formula or an evidence atom in \( \Gamma \cup \Delta \),
- \( Fm(\Gamma \Rightarrow \Delta) \) denote the set of all JL-formulas which occur in a labeled formula or an evidence atom in \( \Gamma \cup \Delta \),
- \( Sub_{Tm}(\Gamma \Rightarrow \Delta) \) denote the set of all subterms of the terms from \( Tm(\Gamma \Rightarrow \Delta) \),
- \( Sub_{Fm}(\Gamma \Rightarrow \Delta) \) denote the set of all JL-subformulas of the formulas from \( Fm(\Gamma \Rightarrow \Delta) \),

**Definition 6.3.** By E-rules we mean the following rules: \((E-)\), \((El+)\), \((Er+)\), \((E!)\), \((IAN)/(AN)\).

1. \((R)-thread\). Suppose \((R)\) is an instance of an E-rule in a derivation \( D \) in \( G3JL_{CS} \). A sequence of sequents in \( D \) is called an \((R)-thread\) if the sequence begins with an initial sequent and ends with the premise of \((R)\), and every sequent in the sequence (except the last one) is the premise of a rule of \( G3JL_{CS} \), and is immediately followed by the conclusion of this rule.

2. Related evidence atoms. Related evidence atoms in a derivation in \( G3JL_{CS} \) are defined as follows:

   (a) Corresponding evidence atoms in side formulas in premise(s) and conclusion of rules are related.
(b) Active and principal evidence atoms in the rules \((\text{Mon}), (R:)\), \((\text{El}+)\), \((\text{Er}+)\), \((\text{E}!)\) are related.
(c) In an instance of rule \((E:+)\):

\[
\frac{wE(s \cdot t, B), wE(s, A \rightarrow B), wE(t, A), \Gamma \Rightarrow \Delta}{wE(s, A \rightarrow B), w \in E(t, A), \Gamma \Rightarrow \Delta} (E:+)
\]

evidence atoms \(wE(s \cdot t, B)\) and \(wE(t, A)\) in the premise are related to \(wE(t, A)\) in the conclusion, and evidence atoms \(wE(s \cdot t, B)\) and \(wE(s, A \rightarrow B)\) in the premise are related to \(wE(s, A \rightarrow B)\) in the conclusion of the rule.
(d) The relation ‘related evidence atoms’, defined in the above clauses, is extended by transitivity.

3. Family of evidence atoms. Suppose \((R)\) is an instance of an E-rule in a derivation \(D\) in \(\text{G3JL}_{-CS}\), and \(e\) is an evidence atom in the premise of \((R)\). The set of all evidence atoms related to \(e\) in all \((R)\)-threads in \(D\) is called the family of \(e\) in \(D\).

4. Subterm property of E-rules.
   (a) An instance of rule \((\text{El}+)\) with active evidence atom \(wE(s + t, A)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(s + t \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).
   (b) An instance of rule \((\text{Er}+)\) with active evidence atom \(wE(t + s, A)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(t + s \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).
   (c) An instance of rule \((E:)\) with active evidence atom \(wE(s \cdot t, B)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(s \cdot t \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).
   (d) An instance of rule \((E!)\) with active evidence atom \(wE(!t, t : A)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(!t \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).
   (e) An instance of rule \((\text{AN})\) with active evidence atom \(wE(c, A)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(c \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).
   (f) An instance of rule \((\text{IAN})\) with active evidence atom \(wE(c_{i_n}, c_{i_{n-1}}, \ldots, c_{i_1} : A)\) in a derivation with endsequent \(\Gamma \Rightarrow \Delta\) has the subterm property if \(c_{i_n}, c_{i_{n-1}}, \ldots, c_{i_1} \in \text{Sub}_{Tm}(\Gamma \Rightarrow \Delta)\).

It is easy to verify that for an evidence atom \(wE(t, A)\) in the premise of an E-rule in a derivation in \(\text{G3JL}_{-CS}\), the typical form of evidence atoms in its family is \(vE(r, B)\), where \(t\) is a subterm of \(r\), \(v\) is a label, and \(B\) is a formula.

Lemma 6.2. Every sequent derivable in \(\text{G3JL}_{-CS}\) has a derivation in which all instances of E-rules has the subterm property.

Proof. Suppose \(D\) is a derivation of a sequent in \(\text{G3JL}_{-CS}\). If all instances of E-rules in \(D\) has the subterm property, then \(D\) is the desired derivation. Otherwise, consider an application of an E-rule in \(D\) which does not have the subterm property. We detail the proof only in the case of rule \((\text{El}+)\), the cases of other E-rules are handled in a similar way. Consider an application of \((\text{El}+)\) in \(D\):

\[
\frac{wE(s + t, A), wE(t, A), \Gamma \Rightarrow \Delta}{wE(t, A), \Gamma \Rightarrow \Delta} (\text{El}+)
\]

which does not have the subterm property. Consider the family of the evidence atom \(wE(s + t, A)\) (in the premise of \((\text{El}+)\)) in \(D\). We have the following possibilities:

1. An evidence atom \(vE(r, B)\) in the family of \(wE(s + t, A)\) is principal in an initial sequent \((\text{Ax}E)\).
2. An evidence atom \(vE(r, B)\) in the family of \(wE(s + t, A)\) is principal in a rule \((R:)\).
3. No evidence atom in the family of \(wE(s + t, A)\) is principal in an initial sequent \((\text{Ax}E)\) or in a rule \((R:)\).
We first show that the first and second possibilities cannot happen.

In the first case, the initial sequent is of the form \( \nu E(r, B), \Gamma' \Rightarrow \Delta, \nu E(r, B) \). Since the rules of \( \text{G3JL}_{\text{CS}} \) do not remove any evidence atom from succedent of sequents, the evidence atom \( \nu E(r, B) \) appears in the endsequent. Note that \( s + t \) is a subterm of \( r \). Therefore, \( s + t \) occurs as the subterm of \( r \) in the endsequent, which would contradict to the assumption that \( (El+) \) does not have the subterm property. Thus, this case cannot happen.

In the second case, the rule \((R :)\) is of the form:

\[
\frac{\nu Ru, \nu E(r, B), \Gamma' \Rightarrow \Delta', \nu \models B}{\nu E(r, B), \Gamma' \Rightarrow \Delta', \nu \models r : B} (R :)
\]

By the labeled-subformula property (Proposition 6.1) for \( \text{G3JL}_{\text{CS}} \), \( \nu \models r : B \) is a subformula of a formula in the endsequent. Therefore, \( s + t \) occurs as the subterm of \( r \) in the endsequent, which would contradict to the assumption that \( (El+) \) does not have the subterm property. Thus, this case cannot happen.

Thus the only possible case is: no evidence atom in the family of \( wE(s + t, A) \) is principal in an initial sequent \((AxE)\) or in a rule \((R :)\). We call such applications of rule \((El+)\) superfluous.3 In the following we show how to remove the superfluous instances of rule \((El+)\) from the derivation \( D \).

Consider the bottommost superfluous instances of \((El+)\) in the derivation \( D \), with active evidence atom \( wE(s + t, A) \). Find the family of \( wE(s + t, A) \) in the derivation, and all rules which has an active evidence atom from this family. Remove all occurrences of evidence atoms in the family of \( wE(s + t, A) \) from the derivation, and also remove all rules which have these evidence atoms as active formulas (by removing a rule we mean removing the premise of that rule from the derivation). Note that, since rule \((El+)\) is superfluous, evidence atoms in the family of \( wE(s + t, A) \) can only be active in \((Mon)\) and \( E \)-rules, and they are side formulas in the other rules of the derivation. Thus, removing these evidence atoms only affect on the rule \((Mon)\) and \( E \)-rules, and do not affect the validity of the remaining rules in the derivation. Therefore, the result of removing those rules which has an active evidence atom from the family of \( wE(s + t, A) \) is still a derivation. Note that, the superfluous application of \((El+)\) is also removed from the derivation, and the new derivation produce no new superfluous application of \((El+)\) (or other rules). By repeating the above argument for all bottommost superfluous applications4 of rules \((El+)\) in the derivation, we finally find a derivation in which all instances of rules \((El+)\) has the subterm property, and moreover this procedure will finally terminate.

For the rules \((E\bar{r}+)\), \((E-)\), \((E!)\), \((IAN)\), \((AN)\) with active evidence atoms \( wE(t + s, A) \), \( wE(s \cdot t, A) \), \( wE(t, t : A) \), \( wE(c_i, c_{i-1} \ldots : c_1 : A) \), \( wE(c, A) \), respectively, proceed just as in the case of \((El+)\) (i.e. remove all superfluous applications of these rules), except that now consider the family of the evidence atom \( wE(t + s, A) \), \( wE(s \cdot t, A) \), \( wE(t, t : A) \), \( wE(c_i, c_{i-1} \ldots : c_1 : A) \), \( wE(c, A) \), respectively.

**Example 6.5.** Here is an example of a derivation of \( w \models x : P \Rightarrow w \models (x + y) : P \) in \( \text{G3J4} \) with a superfluous application of rule \((E!)\):

3 Note that any instance of an E-rule which does not have the subterm property is superfluous, but not vice versa.

4 It is easy to see that for a superfluous application of \((El+)\) with active evidence atom \( wE(s + t, A) \) in a derivation, all applications of E-rules that are above \((El+)\) and have an active evidence atom from the family of \( wE(s + t, A) \) are superfluous. Thus removing the bottommost superfluous application of \((El+)\), instead of an arbitrary one, will remove all other superfluous applications of E-rules that are above it and have an active evidence atom from the family of \( wE(s + t, A) \).
Proposition 6.4 (Subterm Property). Every sequent $\Gamma \Rightarrow \Delta$ derivable in $\text{G3JL}_{CS}$ has a derivation in which all terms in the derivation are in $\text{Sub}_{\Gamma m}(\Gamma \Rightarrow \Delta)$.
Definition 6.4. A derivation is called analytic if all labeled formulas in the derivation are labeled-subformulas of labeled formulas in the endsequent, all terms in the derivation are subterms of terms in the endsequent, and all labels are eigenlabels or labels in the endsequent.

Now from Propositions 6.1, 6.3, 6.4 it follows that:

Corollary 6.1 (Analyticity). Every sequent \( \Gamma \Rightarrow \Delta \) derivable in \( \text{G3JL} \subset \text{CS} \) has an analytic derivation.

The analyticity will be used in the proof search procedure described in the proof of Theorem 8.2 (in fact we search an analytic derivation for a derivable sequent).

7 Structural properties

In this section, we show that all structural rules (weakening and contraction) and cut are admissible in \( \text{G3JL} \). All proofs are similar to those of modal logic [30] adapted for justification logics, and so details are omitted safely. Again in this section, \( \varphi \) stands for one of the formulas \( w \models A \), \( wRv \), or \( wE(t, A) \).

Theorem 7.1. The rules of weakening

\[
\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \quad \text{(LW)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \quad \text{(RW)}
\]

are height-preserving CS-admissible in \( \text{G3JL}_{\text{CS}} \).

Proof. By induction on the height \( n \) of the derivation of the premise. If \( n = 0 \), then \( \Gamma \Rightarrow \Delta \) is an initial sequent, and so are \( \varphi, \Gamma \Rightarrow \Delta \) and \( \Gamma \Rightarrow \Delta, \varphi \). If \( n > 0 \), then suppose the last rule of the derivation is \( (R) \). If \( (R) \) is any rule without label condition or \( \varphi \) does not contain any eigenlabel of \( (R) \), then use the induction hypothesis and apply rule \( (R) \). Now suppose \( (R) \) is \( (R : \wedge) \) or \( (\text{Ser}) \), and \( \varphi \) contains the eigenlabel of \( (R) \). We only sketch the proof for \( (R : \wedge) \):

\[
\frac{wRv, wE(t, A), \Gamma' \Rightarrow \Delta', v \models A}{wE(t, A), \Gamma' \Rightarrow \Delta', w \models t : A} \quad \text{(R : \wedge)}
\]

By height-preserving substitution (substitute \( u \) for \( v \)) we obtain a derivation of height \( n - 1 \) of

\[
wRu, wE(t, A), \Gamma' \Rightarrow \Delta', u \models A,
\]

where \( u \neq w \), \( u \) does not occur in \( \Gamma' \cup \Delta' \), and \( u \) does not occur in \( \varphi \). Then by the induction hypothesis we obtain a derivation of height \( n - 1 \) of

\[
\varphi, wRu, wE(t, A), \Gamma' \Rightarrow \Delta', u \models A,
\]

or of

\[
wRu, wE(t, A), \Gamma' \Rightarrow \Delta', u \models A, \varphi.
\]

Then by \( (R : \wedge) \) we obtain a derivation of height \( n \) of

\[
\varphi, wE(t, A), \Gamma' \Rightarrow \Delta', w \models t : A,
\]

or of

\[
wE(t, A), \Gamma' \Rightarrow \Delta', w \models t : A, \varphi.
\]

The case for \( (\text{Ser}) \) is similar. \( \dashv \)
A rule is said to be height-preserving $\text{CS}$-invertible if whenever an instance of its conclusion is derivable in $\text{G3JL}_{\text{CS}}$ with height $n$, then so is the corresponding instance of its premise(s).

**Proposition 7.1.** All the rules of $\text{G3JL}_{\text{CS}}$ are height-preserving $\text{CS}$-invertible.

**Proof.** The invertibility of propositional rules is proved similar to that in the ordinary sequent calculus $\text{G3c}$ in [40]. The invertibility of other rules, except rule $(R :)$, are obtained by admissible weakening. We establish the height-preserving invertibility of $(R :)$, by induction on the height $n$ of the derivation $wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A$. If $n = 0$, then $wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A$, and consequently $wRv, wE(t, A), \Gamma \Rightarrow \Delta, v \vdash A$ are initial sequents (for any fresh label $v$). If $n > 0$, then suppose the last rule of the derivation is $(R)$. If $(R)$ is any rule without label condition, then use the induction hypothesis and apply rule $(R)$. For example, suppose the last rule is $(AN)$

$$
\frac{uE(c, B), wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A}{wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A} (AN)
$$

where $c : B \in \text{CS}$, and possibly $u = w$. By the induction hypothesis we obtain a derivation of height $n - 1$ of

$$
uE(c, B), wRv, wE(t, A), \Gamma \Rightarrow \Delta, v \vdash A,
$$

for any fresh label $v$. Then by applying the rule $(AN)$ we obtain a derivation of height $n$ of

$$
wRv, wE(t, A), \Gamma \Rightarrow \Delta, v \vdash A.
$$

The case for $(IAN)$ is similar. We now check the case in which $wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A$ is the conclusion of rule $(Ser)$.

$$
uRv, wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A (Ser)
$$

where the eigenlabel $v$ is not in the conclusion. By the induction hypothesis we obtain a derivation of height $n - 1$ of

$$
wRw', uRv, wE(t, A), \Gamma \Rightarrow \Delta, w' \vdash A,
$$

for any fresh label $w'$ (specially $w' \neq v$). Then by applying the rule $(Ser)$ we obtain a derivation of height $n$ of

$$
wRw', wE(t, A), \Gamma \Rightarrow \Delta, w' \vdash A,
$$

as desire. $\dashv$

In order to show the admissibility of contraction we need to show that contracted instances of $(Trans)$ are admissible (the proof is similar to the proof of Proposition 3 in [24]).

**Lemma 7.1.** The rule

$$
\frac{wRw, wRw, \Gamma \Rightarrow \Delta}{wRw, \Gamma \Rightarrow \Delta} (Trans_{\ast})
$$

is height-preserving $\text{CS}$-admissible in all labeled systems $\text{G3JL}_{\text{CS}}$ which contain $(Trans)$.
Theorem 7.2. The proof is by simultaneous induction on the height of derivation for left and right contraction.

Proof. The rule \((\text{Trans})\) is obviously admissible in \(G3LP, G3T45, G3TB45, G3TB4\) due to the fact that they contain rule \((\text{Ref})\). Now by induction on the height of the derivation of the premise of \((\text{Trans})\), \(wRw, wRw, \Gamma \Rightarrow \Delta\), we show that \((\text{Trans})\) is admissible in \(G3J4, G3JD4, G3JB4, G3J45, G3JB45, G3JD45, G3JDB45,\) and \(G3JDB4\).

If the premise is an initial sequent, then so is the conclusion. If the premise is obtained by a rule \((R)\), we have two cases: (i) None of the occurrences of \(wRw\) is principal in \((R)\). For example, suppose \((R)\) is a one premise rule and consider the following derivation of height \(n\)

\[
\frac{D}{wRw, wRw, \Gamma' \Rightarrow \Delta'} (R)
\]

By the induction hypothesis we have a derivation of height \(n - 1\) of \(wRw, \Gamma' \Rightarrow \Delta'\). Then by applying the rule \((R)\) we obtain a derivation of height \(n\) of \(wRw, \Gamma \Rightarrow \Delta\). (ii) One of the occurrences of \(wRw\) is principal in \((R)\). Then \((R)\) may be \((\text{Trans})\), \((\text{Sym})\), \((L:)\), or \((\text{Mon})\). Suppose \((R)\) is \((\text{Trans})\) and consider the following derivation of height \(n\)

\[
\frac{D}{wRw, wRw, wRw, \Gamma \Rightarrow \Delta} (\text{Trans})
\]

Then by the induction hypothesis (applied twice) we get a derivation of height \(n - 1\) of \(wRw, \Gamma \Rightarrow \Delta\). The case for \((\text{Sym})\) is similar. Suppose \((R)\) is \((L:)\) and consider the following derivation of height \(n\)

\[
\frac{D}{w \vdash A, w \vdash t : A, wRw, wRw, \Gamma \Rightarrow \Delta} (L:)
\]

By the induction hypothesis we get a derivation of height \(n - 1\) of \(w \vdash A, w \vdash t : A, wRw, \Gamma \Rightarrow \Delta\). Then by applying \((L:)\) we obtain a derivation of height \(n\) of \(w \vdash t : A, wRw, \Gamma \Rightarrow \Delta\). Suppose \((R)\) is \((\text{Mon})\) and consider the following derivation of height \(n\)

\[
\frac{D}{wE(t, A), wE(t, A), wRw, wRw, \Gamma \Rightarrow \Delta} (\text{Mon})
\]

By the induction hypothesis we get a derivation of height \(n - 1\) of \(wE(t, A), wE(t, A), wRw, \Gamma \Rightarrow \Delta\). Then by applying \((\text{Mon})\) we obtain a derivation of height \(n\) of \(wE(t, A), wRw, \Gamma \Rightarrow \Delta\).
Consider the following derivation of height $n$
\[
\frac{D}{wE(t, A), wRv, \Gamma \Rightarrow \Delta, w \parallel t : A, v \parallel A} \quad (R :).
\]
By Proposition 7.1, we have a derivation of height $n - 1$ of
\[
wE(t, A), wRv, wRu, \Gamma \Rightarrow \Delta, u \parallel A, v \parallel A,
\]
for any fresh label $u$. Then by height-preserving substitution, we obtain a derivation of height $n - 1$ of
\[
wE(t, A), wRv, wRu, \Gamma \Rightarrow \Delta, v \parallel A, v \parallel A.
\]
By the induction hypothesis for left and right contraction, we get\[
wE(t, A), wRv, wRu, \Gamma \Rightarrow \Delta, w \parallel t : A.
\]
Finally, by applying the rule $(R :)$, we obtain a derivation of height $n$ for $wE(t, A), \Gamma \Rightarrow \Delta, w \parallel t : A$.

If the last rule is $(Trans)$ and $w = v = u$ then use Lemma 7.1. The other cases of the proof is similar to the proof of Theorem 4.12 in [30], and is omitted here. 

---

**Theorem 7.3.** The Cut rule

\[
\begin{array}{c}
\Gamma \Rightarrow \Delta, \varphi \\
\Gamma, \Gamma' \Rightarrow \Delta, \Delta'
\end{array}
\]

\[\text{Cut}
\]

is CS-admissible in $\mathbf{G3JL}_{CS}$. 

**Proof.** The proof is by induction on the size of cut formula $\varphi$ with a subinduction on the level of Cut. The size of $w \parallel A$ is defined as the size of the formula $A$, and formulas $wE(t, A)$ and $wRv$ is considered to be of size 1. The level of a Cut is the sum of the heights of the derivations of the premises. Let us first consider the cases that the cut formula is of the form $w \parallel t : A$. Suppose $w \parallel t : A$ is principal in both premises of the cut $(R :) - (L :)$:

\[
\begin{array}{c}
D_1 \\
\frac{wRv, wE(t, A), \Gamma \Rightarrow \Delta, v \parallel A, \Gamma \Rightarrow \Delta, w \parallel t : A}{} \quad (R :)
\end{array}
\]

\[
\begin{array}{c}
D_2 \\
\frac{u \parallel A, w \parallel t : A, wRv, \Gamma' \Rightarrow \Delta'}{}
\end{array}
\]

\[
\frac{}{wE(t, A), wRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{Cut}
\]

where $\varphi$ is not in the conclusion of $(R :)$.

First we construct the derivation $D_3$ as follows:

\[
\begin{array}{c}
D_1 \\
\frac{wRv, wE(t, A), \Gamma \Rightarrow \Delta, v \parallel A, \Gamma \Rightarrow \Delta, w \parallel t : A, u \parallel A, \Gamma \Rightarrow \Delta, \Delta'}{} \quad (R :)
\end{array}
\]

\[
\begin{array}{c}
D_2 \\
\frac{u \parallel A, w \parallel t : A, wRv, \Gamma' \Rightarrow \Delta'}{}
\end{array}
\]

\[
\frac{}{wE(t, A), u \parallel A, wRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{Cut}_1
\]

And then we have

\[
\begin{array}{c}
D_1 (u/v) \\
\frac{wRv, wE(t, A), \Gamma \Rightarrow \Delta, u \parallel A, wE(t, A), u \parallel A, wRv, \Gamma, \Gamma' \Rightarrow \Delta, w \parallel t : A, u \parallel A, \Gamma \Rightarrow \Delta, \Delta'}{} \quad (R :)
\end{array}
\]

\[
\begin{array}{c}
D_2 \\
\frac{u \parallel A, w \parallel t : A, wRv, \Gamma' \Rightarrow \Delta'}{}
\end{array}
\]

\[
\frac{}{wE(t, A), wE(t, A), wRv, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'} \quad \text{Cut}_2
\]

\[
\frac{}{wE(t, A), wRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{Cut}_3
\]

\[
\frac{}{wE(t, A), wRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{Cut}_4
\]

34
where $Ctr$ denotes repeated applications of left and right contraction rules, and the premise
\[ wRu, wE(t, A), \Gamma \Rightarrow \Delta, u \models A \]
of $Cut_2$ is obtained by the substitution lemma (Lemma 5.4) from $wRu, wE(t, A), \Gamma \Rightarrow \Delta, v \models A$. The $Cut_1$ has smaller level and $Cut_2$ has smaller cut formula. Hence, by the induction hypothesis, they are admissible.

Now, consider the cut $(R : ) - (E)$ as follows:
\[
\begin{array}{c}
\Gamma \Rightarrow \Delta, v \models A \\
\hline
\Gamma \Rightarrow \Delta, w \models t : A
\end{array}
\]
\[
\begin{array}{c}
wE(t, A), \Gamma \Rightarrow \Delta, w \models t : A \\
\hline
wE(t, A), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'
\end{array}
\]

where $v$ is not in the conclusion of $(R : )$. This derivation is transformed into
\[
\begin{array}{c}
wEu, wE(t, A), \Gamma \Rightarrow \Delta, v \models A \\
\hline
wE(t, A), \Gamma \Rightarrow \Delta, w \models t : A
\end{array}
\]
\[
\begin{array}{c}
wE(t, A), wE(t, A), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'
\end{array}
\]

where the above $Cut$ is of smaller level, and thus by the induction hypothesis is admissible.

Now suppose the cut formula is of the form $wE(t, A)$ or $wEu$. We study two cases here. First note that, by a simple inspection of all rules, we find out that no relational atoms $wEu$ and evidence atoms $wE(t, A)$ can be principal in the succeedent of sequents of rules (so as a cut formula they might be a side formula in a rule, or a principal formula in an initial sequent). Let us consider the following $Cut$ with cut formula $wE(t, A)$:
\[
\begin{array}{c}
\Gamma' \Rightarrow \Delta', wE(t, A) \\
\hline
\Gamma \Rightarrow \Delta, wE(t : A)
\end{array}
\]
\[
\begin{array}{c}
wEu, wE(t, A), \Gamma' \Rightarrow \Delta', v \models A \\
\hline
wE(t, A), \Gamma' \Rightarrow \Delta', \Delta', w \models t : A
\end{array}
\]

where the rule $(R : )$ in the left premise of the $Cut$ can be any rule. We permute the cut upward to obtain a $Cut$ with lower level, and then we apply the rule $(R : )$:
\[
\begin{array}{c}
\Gamma'' \Rightarrow \Delta'', wE(t : A) \\
\hline
\Gamma' \Rightarrow \Delta', \Delta', w \models t : A
\end{array}
\]

In the case that the rule $(R : )$ is a rule with label condition, such as $(R : )$, first we apply a suitable substitution on labels (substitute the eigenlabel of $(R : )$ with a fresh label not used in the derivation), and then we proceed as the above argument. For example, consider the following $Cut$ on $wEu$:
\[
\begin{array}{c}
uEu', uE(s, B), \Gamma \Rightarrow \Delta, wE(t, A) \\
\hline
uE(s, B), \Gamma \Rightarrow \Delta, wEu', u \models s : B
\end{array}
\]
\[
\begin{array}{c}
v \models A, wEu, w \models t : A, \Gamma' \Rightarrow \Delta' \\
\hline
wEu, w \models t : A, \Gamma' \Rightarrow \Delta'
\end{array}
\]

35
It is transformed into the following Cut with lower level:

\[
\begin{align*}
\frac{D_1(u''/u')}{uRu''', uE(s, B), \Gamma \Rightarrow \Delta, wRu', u'' \Vdash B} & \quad D_2 \quad \frac{\nu \Vdash A, wRv, w \Vdash t : A, \Gamma' \Rightarrow \Delta'}{wRu, w \Vdash t : A, \Gamma' \Rightarrow \Delta'} (L :) \\
\frac{uRu'', uE(s, B), w \Vdash t : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', u'' \Vdash B}{uE(s, B), w \Vdash t : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', u \Vdash s : B} (R :)
\end{align*}
\]

where \(u''\) is a fresh label not used in the derivation.

\[\Box\]

8 Soundness and completeness

In this section, we shall prove the soundness and completeness of our labeled sequent calculi with respect to F-models. The proofs of soundness (Theorem 8.1) and completeness (Theorem 8.2) are similar to those of modal logics given by Negri in [31]. The proof of completeness presents a derivation of a given valid sequent or a countermodel in the case of failure of proof search. Hence, it helps us to give a proof search for sequents. We start with the definition of an interpretation (offered by Negri in [31]) which provides a translation between labels used in derivations of a labeled sequent calculus and possible worlds in F-models.

**Definition 8.1.** Let \(\mathcal{M} = (W, R, E, \mathcal{V})\) be a \(\mathcal{JL}_{\text{CS}}\)-model and \(L\) be the set of all labels. An interpretation \([\cdot]\) of the labels \(L\) in model \(\mathcal{M}\), or simply an \(\mathcal{M}\)-interpretation, is a function \([\cdot] : L \rightarrow \mathcal{V}\). An \(\mathcal{M}\)-interpretation \([\cdot]\) validates a formula of the extended labeled language in the following sense:

- \([\cdot]\) validates the labeled formula \(w \Vdash A\), provided that \((\mathcal{M}, [w]) \Vdash A\),
- \([\cdot]\) validates the relational atom \(wRv\), provided that \([w]R[v]\),
- \([\cdot]\) validates the evidence atom \(wE(t, A)\), provided that \([w] \in E(t, A)\).

A sequent \(\Gamma \Rightarrow \Delta\) is valid for an \(\mathcal{M}\)-interpretation \([\cdot]\), if whenever \([\cdot]\) validates all the formulas in \(\Gamma\) then it validates at least one formula in \(\Delta\). A sequent is valid in a model \(\mathcal{M}\) if it is valid for every \(\mathcal{M}\)-interpretation.

The following lemma is helpful in the rest of the paper.

**Lemma 8.1.** Given a \(\mathcal{JL}\)-formula \(A\) and a \(\mathcal{JL}_{\text{CS}}\)-model \(\mathcal{M}\), the formula \(A\) is true in \(\mathcal{M}\) (i.e., \(\mathcal{M} \Vdash A\)) if and only if the sequent \(\Gamma \Rightarrow w \Vdash A\) is valid in \(\mathcal{M}\), for arbitrary label \(w\).

**Proof.** Suppose \(A\) is true in the model \(\mathcal{M} = (W, R, E, \mathcal{V})\). Then for every \(\mathcal{M}\)-interpretation \([\cdot]\), and every label \(w\) we have \([w] \in W\), and therefore \((\mathcal{M}, [w]) \Vdash A\). Thus, the interpretation \([\cdot]\) validates the sequent \(\Rightarrow w \Vdash A\). Since the interpretation \([\cdot]\) is arbitrary, the sequent \(\Rightarrow w \Vdash A\) is valid in \(\mathcal{M}\).

Conversely, suppose the sequent \(\Rightarrow w \Vdash A\) is valid in \(\mathcal{M}\), i.e., it is valid for every \(\mathcal{M}\)-interpretation. For an arbitrary world \(w \in W\), define the interpretation \([\cdot]\) on \(\mathcal{M}\) such that \([w] = w\). Since \([\cdot]\) validates \(\Rightarrow w \Vdash A\), we have \((\mathcal{M}, w) \Vdash A\). Thus, \(A\) is true in \(\mathcal{M}\).

Now we show the soundness of labeled sequent calculi with respect to F-models.

**Theorem 8.1 (Soundness).** If the sequent \(\Gamma \Rightarrow \Delta\) is derivable in \(G3\mathcal{JL}_{\text{CS}}\), then it is valid in every \(\mathcal{JL}_{\text{CS}}\)-model.
Proof. By induction on the height of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3JL}_{\mathsf{CS}}$. Initial sequents are obviously valid in every $\mathsf{JL}_{\mathsf{CS}}$-model. We only check the induction step for the rules $(E)$, $(R :)$, $(E?)$, $(AN)$, and $(\mathbb{S}er)$. For the case of propositional rules and the rules of the accessibility relation, refer to Theorem 5.3 in [31]. The case of other rules are similar or simpler.

Suppose $\Gamma \Rightarrow \Delta$ is $w \Vdash A, \Gamma' \Rightarrow \Delta$, the conclusion of rule $(E)$, with the premise $wE(t, A), w \Vdash t : A, \Gamma' \Rightarrow \Delta$, and assume by the induction hypothesis that the premise is valid in every $\mathsf{JL}_{\mathsf{CS}}$-model. Let $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$ be a $\mathsf{JL}_{\mathsf{CS}}$-model and $[\cdot]$ be an arbitrary $\mathcal{M}$-interpretation which validates $w \Vdash t : A$ and all the formulas in $\Gamma'$. In particular, $(\mathcal{M}, [w]) \Vdash t : A$. We have to prove that this interpretation validates one of the formulas in $\Delta$. Since $(\mathcal{M}, [w]) \Vdash t : A$, by the definition of forcing relation $\Vdash$, we have $[w] \in \mathcal{E}(t, A)$. Thus, $[\cdot]$ validates all the formulas in the antecedent of the premise. Hence by the induction hypothesis, it validates one of the formulas in $\Delta$, as desired.

Suppose $\Gamma \Rightarrow \Delta$ is $wE(t, A), \Gamma' \Rightarrow \Delta', w \Vdash t : A$, the conclusion of rule $(R :)$, with the premise $wRv, wE(t, A), \Gamma' \Rightarrow \Delta', v \Vdash A$, and assume by the induction hypothesis that the premise is valid in every $\mathsf{JL}_{\mathsf{CS}}$-model. Let $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$ be a $\mathsf{JL}_{\mathsf{CS}}$-model and $[\cdot]$ be an arbitrary $\mathcal{M}$-interpretation which validates $w \Vdash t : A$, and all the formulas in $\Gamma'$. In particular, $[w] \in \mathcal{E}(t, A)$. We have to prove that this interpretation validates one of the formulas in $\Delta'$ or validates $w \Vdash t : A$. Suppose $w$ is an arbitrary element of $W$ such that $[w]Rw$ (if such a world $w$ does not exist, then obviously we have $(\mathcal{M}, [w]) \Vdash t : A$) and $[\cdot]'$ be the interpretation identical to $[\cdot]$ except possibly on $v$, where we put $[v]' = w$. Clearly, $[\cdot]'$ validates all the formulas in the antecedent of the premise, so it validates a formula in $\Delta'$ or validates $v \Vdash A$. In the former case, since $v$ is not in $\Delta'$, $[\cdot]$ validates a formula in $\Delta'$. In the latter case, we have $(\mathcal{M}, w) \Vdash A$. Since by the assumption $[w] \in \mathcal{E}(t, A)$ and $w$ is arbitrary, $[\cdot]$ validates $w \Vdash t : A$.

Among rules for evidence atoms we only check the induction step for $(E?)$ in $\mathbf{G3J5}$ and its extensions. Let $\mathsf{JL}$ be $\mathbf{J5}$ or one of its extensions. Suppose $\Gamma \Rightarrow \Delta$ is the conclusion of the rule $(E?)$, with the premises $wE(t, A), \Gamma \Rightarrow \Delta$ and $wE(?t, \neg t : A), \Gamma \Rightarrow \Delta$, and assume by the induction hypothesis that the premises are valid in every $\mathsf{JL}_{\mathsf{CS}}$-model. Let $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$ be a $\mathsf{JL}_{\mathsf{CS}}$-model and $[\cdot]$ be an arbitrary $\mathcal{M}$-interpretation which validates all the formulas in $\Gamma$. We have to prove that this interpretation validates one of the formulas in $\Delta$. There are two cases: (i) If $[w] \in \mathcal{E}(t, A)$ then the antecedent of the premise $wE(t, A), \Gamma \Rightarrow \Delta$ is validated in $[\cdot]$, and hence one of the formulas in $\Delta$ is validated. (ii) If $[w] \notin \mathcal{E}(t, A)$, then by negative introspection condition $(E6)$ of $\mathsf{JL}_{\mathsf{CS}}$-models we have $[w] \in \mathcal{E}(?t, \neg t : A)$. Thus the antecedent of the premise $wE(?t, \neg t : A), \Gamma \Rightarrow \Delta$ is validated in $[\cdot]$. Hence, $[\cdot]$ validates a formula in $\Delta$. Therefore the conclusion is valid in $\mathcal{M}$.

For axiom necessitation rules we only consider $(AN)$ (rule $(IAN)$ is treated similarly). Suppose $\Gamma \Rightarrow \Delta$ is the conclusion of the rule $(AN)$, with the premise $wE(c, A), \Gamma \Rightarrow \Delta,$ and assume by the induction hypothesis that the premise is valid in every $\mathsf{JL}_{\mathsf{CS}}$-model, and $c : A \in \mathsf{CS}$. Let $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$ be a $\mathsf{JL}_{\mathsf{CS}}$-model and $[\cdot]$ be an arbitrary $\mathcal{M}$-interpretation which validates all the formulas in $\Gamma$. We have to prove that this interpretation validates one of the formulas in $\Delta$. Since $\mathcal{M}$ is a $\mathsf{JL}_{\mathsf{CS}}$-model, we have $\mathcal{E}(c, A) = W$. Thus $[w] \in \mathcal{E}(c, A)$, and $[\cdot]$ validates the antecedent of the premise. Hence $[\cdot]$ validates a formula in $\Delta$, as desired.

Let $\mathsf{JL}$ be $\mathbf{JD}$ or one of its extensions. Suppose $\Gamma \Rightarrow \Delta$ is the conclusion of rule $(\mathbb{S}er)$, with the premise $wRv, \Gamma \Rightarrow \Delta$, and assume by the induction hypothesis that the premise is valid in every $\mathsf{JL}_{\mathsf{CS}}$-model. Let $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$ be a $\mathsf{JL}_{\mathsf{CS}}$-model and $[\cdot]$ be an arbitrary $\mathcal{M}$-interpretation which validates all the formulas in $\Gamma$. We have to prove that $[\cdot]$ validates one of the formulas in $\Delta$. Since $[w] \in W$ and $R$ is serial, there exists $w \in W$ such that $[w]Rw$. Let $[\cdot]'$ be the interpretation identical to $[\cdot]$ except possibly on $v$, where we put $[v]' = w$. Since the eigenlabel $v$ is not in $\Gamma$, $[\cdot]'$ validates all the formulas in $\Gamma$. Thus, $[\cdot]'$ validates all the formulas in the antecedent of the premise, so it validates...
a formula in $\Delta$. Since the eigenlabel $v$ is not in $\Delta$, $[\cdot]$ validates all the formulas in $\Delta$.

Now we show that theorems of Hilbert systems of justification logics are derivable in their labeled systems.

**Proposition 8.1.** If $A$ is a theorem of $\text{JL}_{\text{CS}}$, then $\Rightarrow w \Vdash A$ is derivable in $\text{G3JL}_{\text{CS}}$, for arbitrary label $w$.

**Proof.** By induction on the derivation of $A$ in $\text{JL}_{\text{CS}}$. If $A$ is an axiom of $\text{JL}_{\text{CS}}$, then apply root-first the rules of $\text{G3JL}_{\text{CS}}$ and possibly use Lemma 5.1. For example, we derive the axiom $\text{jD}$, $t : \bot \rightarrow \bot$, in $\text{G3JD}_{\text{CS}}$:

\[(Ax, \bot)\]
\[
\forall v : \bot, w R v, w \Vdash t : \bot \Rightarrow w \Vdash \bot \quad (L :)
\]
\[
w R v, w \Vdash t : \bot \Rightarrow w \Vdash \bot \quad (\text{Ser})
\]
\[
w \Vdash t : \bot \Rightarrow w \Vdash \bot \quad (R \rightarrow)
\]

If $A$ is obtained by Modus Ponens from $B \rightarrow A$ and $B$, then by the induction hypothesis and admissibility of rule $\text{Cut}$, the sequent $\Rightarrow w \Vdash A$ is derivable in $\text{G3JL}$. If $A = c : B \in \text{CS}$ is obtained by Axiom Necessitation rule, then we have:

\[
\begin{array}{c}
\text{D} \\
\text{E}
\end{array}
\]

\[
\Rightarrow w \Vdash c : B \quad (\text{AN})
\]

where $D$ is the standard derivation of axiom $B$ in $\text{G3JL}_{\text{CS}}$. Now, suppose $A = c_{i_n} : c_{i_{n-1}} : \ldots : c_1 : B \in \text{CS}$ is obtained by Iterated Axiom Necessitation rule. Note that constant specifications are downward closed. Thus, in order to prove $\Rightarrow w \Vdash c_{i_n} : c_{i_{n-1}} : \ldots : c_1 : B$, we can add $w E(c_{i_m} : c_{i_{m-1}} : \ldots : c_1 : B)$, for every $1 \leq m \leq n$, to the antecedent of the premise of $(\text{IAN})$, when we apply it upward. We have:

\[
\begin{array}{c}
\text{D} \\
\text{E}
\end{array}
\]

\[
\Rightarrow w_n \Vdash B
\]

\[
\begin{array}{c}
w_1 E(c_{i_{n-1}}, c_{i_{n-2}} : \ldots : c_1 : B), w R w_1, w E(c_{i_{n-1}} : \ldots : c_1 : B) \Rightarrow w_2 \Vdash c_{i_{n-2}} : \ldots : c_1 : B \quad (R :)
\end{array}
\]

\[
\Rightarrow w_1 \Vdash c_{i_{n-1}} : \ldots : c_1 : B \quad (\text{IAN})
\]

\[
\begin{array}{c}
w R w_1, w E(c_{i_n}, c_{i_{n-1}} : \ldots : c_1 : B) \Rightarrow w \Vdash c_{i_n} : c_{i_{n-1}} : \ldots : c_1 : B \quad (R :) \Rightarrow w \Vdash c_{i_n} : c_{i_{n-1}} : \ldots : c_1 : B
\end{array}
\]

where

\[
\begin{array}{c}
\Gamma = \{w_{n-1}, w_{n-2} R w_{n-1}, \ldots, w R w_1, w_{n-1} \in E(c_1, B),
\end{array}
\]

\[
w_{n-2} E(c_{i_2}, c_1 : B), \ldots, w_1 \in E(c_{i_{n-1}}, c_{i_{n-2}} : \ldots : c_1 : B), w E(c_{i_n}, c_{i_{n-1}} : \ldots : c_1 : B)\}
\]

and $D$ is the standard derivation of axiom $B$ in $\text{G3JL}_{\text{CS}}$. 

---

38
Now we prove that the labeled sequent calculi of justification logics are equivalent to their Hilbert systems.

**Corollary 8.1.** Let $JL$ be a justification logic, and $CS$ be a constant specification for $JL$, with the requirement that if $JL$ contains axiom scheme $jD$ then $CS$ should be axiomatically appropriate. Then $A$ is true in every $JL_{CS}$-model if and only if the sequent $\Rightarrow w \vdash A$ is provable in $G3JL_{CS}$, for arbitrary label $w$.

**Proof.** Suppose that the sequent $\Rightarrow w \vdash A$ is provable in $G3JL_{CS}$. Then by Soundness Theorem 8.1, $\Rightarrow w \vdash A$ is valid in every $JL_{CS}$-model, and therefore by Lemma 8.1, the formula $A$ is true in every $JL_{CS}$-model.

Conversely, suppose that the formula $A$ is true in every $JL_{CS}$-model. Then by Completeness Theorem 3.1, the formula $A$ is provable in $JL_{CS}$, and therefore by Proposition 8.1, the sequent $\Rightarrow w \vdash A$ is provable in $G3JL_{CS}$.

For completeness of labeled sequent calculi with respect to F-models, we will describe a procedure which produces a derivation for valid (derivable) sequents and a countermodel for non-valid (undervivable) sequents. In backward proof search of a sequent $\Gamma \Rightarrow \Delta$, some rules repeat the main formula in their premise(s), and therefore they can be applied infinitely many times. For example, applying rule $(L :)$ backwardly on the sequent $w \vdash t : A, wRv, \Gamma' \Rightarrow \Delta'$

we get $v \vdash A, w \vdash t : A, wRv, \Gamma'' \Rightarrow \Delta''$

Because of the formulas $w \vdash t : A$ and $wRv$ in the antecedent of the premise, we can apply this rule backwardly infinitely many times. Since the rules of contraction are height-preserving admissible, it seems that the rule $(L :)$ does not need to apply on each pair of formulas $w \vdash t : A$ and $wRv$ more than once. To show this fact, we first show that applications of $(L :)$ on the same pair of principal formulas can be made consecutive by the permutation of rule $(L :)$ over other rules.

**Lemma 8.2.** Rule $(L :)$ permutes down with respect to all rules of $G3JL_{CS}$. The permutability with respect to $(R :)$, and rules for relational atoms (Table 4) have the condition that the principal formulas of $(L :)$ are not active in them.

**Proof.** The proof is similar to that in [30] (rule $(L :)$ is treated like rule $(L \Box)$ in modal logics). We only show the permutation with respect to $(E)$ and $(R :)$.

$(L :)$ can permute down as follows

\[
\frac{v \vdash A, uE(s, B), u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta}{uE(s, B), u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta} (L :)
\]

\[
\frac{v \vdash A, u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta}{u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta} (E)
\]

\[
\frac{v \vdash A, u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta}{u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta} (L :)
\]

\[
\frac{v \vdash A, u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta}{u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta} (E)
\]

\[
\frac{v \vdash A, u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta}{u \vdash s : B, w \vdash t : A, wRv, \Gamma \Rightarrow \Delta} (L :)
\]
Note that in this case the principal formula of \( (L:) \) could be active in \( (E) \), i.e. in the above derivations \( u \vDash s : B \) can be equal to \( w \vDash t : A \). For \( (R:) \), in the derivation

\[
\begin{align*}
\text{D} \\
v \vDash A, uRu', uE(s, B), w \vDash t : A, wRv, \Gamma \Rightarrow \Delta, u' \vDash B \\
\text{(L:)} \\
uE(s, B), w \vDash t : A, wRv, \Gamma \Rightarrow \Delta, u \vDash s : B \\
\text{(R:)}
\end{align*}
\]

where principal formulas of \( (L:) \), i.e. \( w \vDash t : A, wRv \), are not active in \( (R:) \), the permutation of \( (L:) \) over \( (R:) \) is as follows

\[
\begin{align*}
\text{D} \\
v \vDash A, uE(s, B), w \vDash t : A, wRv, \Gamma \Rightarrow \Delta, u' \vDash B \\
\text{(R:)} \\
uE(s, B), w \vDash t : A, wRv, \Gamma \Rightarrow \Delta, u \vDash s : B \\
\text{(L:)}
\end{align*}
\]

\[\text{Corollary 8.2. In a branch of a derivation in } \text{G3JL}_c, \text{ it is enough to apply rule } (L:) \text{ only once on the same pair of principal formulas.}\]

\[\text{Proof. Suppose } (L:) \text{ is applied twice on the same principal formula, for example}\]

\[
\begin{align*}
\text{D} \\
v \vDash A, w \vDash t : A, wRv, \Gamma' \Rightarrow \Delta' \\
\text{(L:)} \\
w \vDash t : A, wRv, \Gamma' \Rightarrow \Delta' \\
\text{: D'} \\
v \vDash A, w \vDash t : A, wRv, \Gamma \Rightarrow \Delta \\
\text{(L:)}
\end{align*}
\]

Then, since the upper application of \( (L:) \) is applied on the same principal formulas \( w \vDash t : A, wRv \), in the part \( \text{D}' \) of the derivation there is no rule \( (R:) \), or rules for relational atoms with active formulas \( w \vDash t : A, wRv \). Therefore, by Lemma 8.2, by permuting down the upper \( (L:) \) we obtain

\[
\begin{align*}
\text{D} \\
v \vDash A, w \vDash t : A, wRv, \Gamma' \Rightarrow \Delta' \\
\text{: D'} \\
v \vDash A, v \vDash A, w \vDash t : A, wRv, \Gamma \Rightarrow \Delta \\
\text{(L:)} \\
w \vDash t : A, wRv, \Gamma \Rightarrow \Delta \\
\text{(L:)}
\end{align*}
\]

By applying height-preserving contraction, we see that the upper application of \( (L:) \) is redundant. \( \dagger \)

Since the rules of contraction are height-preserving admissible, we consider the following requirement for all rules:

40
Theorem 8.2. Let $\mathcal{JL}$ be one of the justification logics such that do not contain axiom $j\mathbf{B}$, and $\mathcal{CS}$ be a finite constant specification for $\mathcal{JL}$. Every sequent $\Gamma \Rightarrow \Delta$ in the language of $\mathcal{G}^{3}\mathcal{JL}_{\mathcal{CS}}$ is either derivable in $\mathcal{G}^{3}\mathcal{JL}_{\mathcal{CS}}$ or it has a $\mathcal{JL}_{\mathcal{CS}}$-countermodel.

Proof. Following the proof of completeness of labeled systems of modal logics, Theorem 5.4 in [31], we present a procedure which constructs a finitely branching reduction tree with the root $\Gamma \Rightarrow \Delta$ by applying the rules of $\mathcal{G}^{3}\mathcal{JL}_{\mathcal{CS}}$ in all possible ways. If all branches of the reduction tree reach initial sequents we obtain a proof for $\Gamma \Rightarrow \Delta$. Otherwise we have a branch in which its topmost sequent is not an initial sequent and no reduction step can be applied or it is an infinite branch. In this case, we construct a countermodel by means of this branch. We begin with some general remarks, and continue with the construction of the reduction tree.

In constructing the reduction tree the procedure always obey the condition ($\dagger$). Moreover, by Corollary 6.1, the procedure obeys the subterm and sublabel properties.

If $\mathcal{JL}$ contains axiom $j\mathbf{S}$, then we fix an enumeration of all terms of $\mathcal{JL}$:

$$T_{m\mathcal{JL}} = \{t_{1}, t_{2}, \ldots\},$$

which begins with terms of $\text{Sub}_{T_{m}}(\Gamma \Rightarrow \Delta)$, and an enumeration of all formulas of $\mathcal{JL}$:

$$F_{m\mathcal{JL}} = \{F_{1}, F_{2}, \ldots\},$$

which begins with formulas of $\text{Sub}_{F_{m}}(\Gamma \Rightarrow \Delta)$. Finally, using $T_{m\mathcal{JL}}$ and $F_{m\mathcal{JL}}$, we give an enumeration of all pairs of the form $(?t_{1}, \neg t_{1} : A_{1}),$ for some $t_{1} \in T_{m\mathcal{JL}}$ and $A_{1} \in F_{m\mathcal{JL}}$, as follows:

$$\Pi = \{(?t_{1}, \neg t_{1} : A_{1}), (?t_{2}, \neg t_{2} : A_{2}), \ldots\}.$$

Since by Lemma 5.3 rule $\text{(Anti-Mon)}$ is admissible in $\mathcal{G}^{3}\mathcal{J5}$ and its extensions, we add it to the list of rules of these labeled systems (we need this rule in the construction of countermodel).

**Reduction tree:** In stage 0, we put $\Gamma \Rightarrow \Delta$ at the root of the tree. For each branch, in stage $n > 0$, if the top-sequent of the branch is an initial sequent, i.e. $(Ax), (Ax\bot), (AxR), (AxE)$, then we terminate the construction of the branch. Otherwise, we continue the construction of the branch by writing, above its top-sequent, other sequent(s) that are obtained by applying the following stages, depend on the rules of the logic. The number of stages in the construction of the reduction tree depends on the number of rules which $\mathcal{G}^{3}\mathcal{JL}_{\mathcal{CS}}$ contains. In general, the following list (which determines an order on the rules of the system) can be used for various systems dealt here:

$$(L\neg), (R\neg), (L\land), (R\land), (L\lor), (R\lor), (L\rightarrow), (R\rightarrow), (L::), (R::), (E), (AN)/(\text{IAN}), (El+), (Er+), (E\cdot), (E\dagger), (\text{Mon}), (\text{E'?}), (SE), (\text{Anti-Mon}), (\text{Re.f}), (\text{Ser}), (\text{Trans}).$$

(8)

There are 15 rules common in all labeled systems in the list (8): $(L\neg), (R\neg), \ldots, (E\cdot)$. Thus each $\mathcal{G}^{3}\mathcal{JL}_{\mathcal{CS}}$ system has $15 + r$ stages, for $r \geq 0$. At stage $15 + r + 1$ we come back to stage 1, and continue until an initial sequent is found or the branch becomes saturated. A branch is called saturated if it is an

---

5 Another alternative to define the reduction tree is to stipulate that, instead of applying consecutively the stages according to list (8), at each stage apply one of the stages non-deterministically.
infinite branch or if its top-sequent is not an initial sequent and no more stage of the reduction tree can be applied. Otherwise it is said to be unsaturated.\footnote{The terminology is due to Dyckhoff and Negri [13], but our definition is a bit different. In the reduction tree described in [13] all branches are finite.} We first consider 15 common stages in all of the justification logics.

Stage $n = 1$ of rule $(L \neg)$: If the top-sequent is of the form:

$$w_1 \vdash \neg A_1, \ldots, w_m \vdash \neg A_m, \Gamma' \Rightarrow \Delta'$$

where $w_1 \vdash \neg A_1, \ldots, w_m \vdash \neg A_m$ are all the formulas in the antecedent with a negation as the outermost logical connective, we write

$$\Gamma' \Rightarrow \Delta', w_1 \vdash A_1, \ldots, w_m \vdash A_m$$

on top of it. This stage corresponds to applying $m$ times rule $(L \neg)$.

Stage $n = 2$ of rule $(R \neg)$: If the top-sequent is of the form:

$$\Gamma' \Rightarrow \Delta', w_1 \vdash \neg A_1, \ldots, w_m \vdash \neg A_m$$

where $w_1 \vdash \neg A_1, \ldots, w_m \vdash \neg A_m$ are all the formulas in the succedent with a negation as the outermost logical connective, we write

$$w_1 \vdash A_1, \ldots, w_m \vdash A_m, \Gamma' \Rightarrow \Delta'$$

on top of it. This stage corresponds to applying $m$ times rule $(R \neg)$. For the stages $n = 3, 4, \ldots, 8$, correspond to other propositional rules in the list (8), refer to the proof of Theorem 5.4 in [31].

Stage $n = 9$ of rule $(L :)$: If the top-sequent is of the form:

$$w_1 \vdash t_1 : A_1, \ldots, w_m \vdash t_m : A_m, w_1 R v_1, \ldots, w_m R v_m, \Gamma' \Rightarrow \Delta'$$

where all pairs $w_i \vdash t_i : A_i$ and $w_i R v_i$ from the antecedent of the topmost sequent are listed, we write the following node on top of it (regarding condition (\dagger)):

$$v_1 \vdash A_1, \ldots, v_m \vdash A_m, w_1 \vdash t_1 : A_1, \ldots, w_m \vdash t_m : A_m, w_1 R v_1, \ldots, w_m R v_m, \Gamma' \Rightarrow \Delta'.$$

This stage corresponds to applying $m$ times rule $(L :)$.

Stage $n = 10$ of rule $(R :)$: If the top-sequent is of the form:

$$w_1 E(t_1, A_1), \ldots, w_m E(t_m, A_m), \Gamma' \Rightarrow \Delta', w_1 \vdash t_1 : A_1, \ldots, w_m \vdash t_m : A_m$$

where all pairs $w_i E(t_i, A_i)$ and $w_i \vdash t_i : A_i$ from the topmost sequent are listed, we write the following node on top of it:

$$w_1 R v_1, \ldots, w_m R v_m, w_1 E(t_1, A_1), \ldots, w_m E(t_m, A_m), \Gamma' \Rightarrow \Delta', v_1 \vdash A_1, \ldots, v_m \vdash A_m$$

where $v_1, \ldots, v_m$ are fresh labels, not yet used in the reduction tree. This stage corresponds to applying $m$ times rule $(R :)$.

Stage $n = 11$ of rule $(E)$: If the top-sequent is of the form:

$$w_1 \vdash t_1 : A_1, \ldots, w_m \vdash t_m : A_m, \Gamma' \Rightarrow \Delta'$$
where all labeled formulas \( w_i \parallel t_i : A_i \) from the topmost sequent are listed, we write the following node on top of it (regarding condition (†)):

\[
w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), w_1 \parallel t_1 : A_1, \ldots, w_m \parallel t_m : A_m, \Gamma' \Rightarrow \Delta'
\]

This stage corresponds to applying \( m \) times rule \((E)\).

Stage \( n = 12 \) of rule \((IAN)/(AN)\): If \( \Gamma' \Rightarrow \Delta' \) is the top-sequent of the branch, then write a similar sequent on top of it (regarding condition (†)) with additional evidence atoms of the form \( wE(c, F) \) in the antecedent, for every formula \( c : F \) in \( CS \) and every \( w \) in \( \Gamma' \cup \Delta' \).

Stage \( n = 13 \) of rule \((El+)\): If the top-sequent is of the form:

\[
w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

where \( w_i E(t_i, A_i) \) are all evidence atoms for which \( s_i + t_i \in Sub_{Tm}(\Gamma' \Rightarrow \Delta') \) for some term \( s_i \) (Proposition 6.4) and \( w_i E(s_i + t_i, A_i) \) is not in \( \Gamma' \) (condition (†)), then we add the following node on top of it:

\[
w_1 E(s_1 + t_1, A_1), \ldots, w_mE(s_m + t_m, A_m), w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

This corresponds to applying \( m \) times rule \((El+)\). The stage \( n = 14 \) of rule \((Er+)\) is similar.

Stage \( n = 15 \) of rule \((E-)\): If the top-sequent is of the form:

\[
w_1 E(s_1, A_1 \rightarrow B_1), \ldots, w_mE(s_m, A_m \rightarrow B_m), w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

where \( w_i E(s_i, A_i \rightarrow B_i) \) and \( w_i E(t_i, A_i) \) are all pairs of evidence atoms for which \( s_i \cdot t_i \in Sub_{Tm}(\Gamma' \Rightarrow \Delta') \) and \( w_i E(s_i \cdot t_i, B_i) \) is not in \( \Gamma' \), then we add the following node on top of it:

\[
w_1 E(s_1 \cdot t_1, B_1), \ldots, w_mE(s_m \cdot t_m, B_m), w_1 E(s_1, A_1 \rightarrow B_1), \ldots, w_mE(s_m, A_m \rightarrow B_m), \Gamma' \Rightarrow \Delta'
\]

This corresponds to applying \( m \) times rule \((E-)\).

Stage of rule \((E!)\): If the labeled system \( G3JL \) contains rule \((E!)\), and the top-sequent is of the form:

\[
w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

where \( w_i E(t_i, A_i) \) are all evidence atoms for which \( t_i \in Sub_{Tm}(\Gamma' \Rightarrow \Delta') \) and \( w_i E(!s_i, A_i) \) is not in \( \Gamma' \), then we add the following node on top of it:

\[
w_1 E(!t_1, A_1), \ldots, w_mE(!t_m, A_m), w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

This corresponds to applying \( m \) times rule \((E!)\).

If the labeled system \( G3JL \) contains rule \((Mon)\), and the top-sequent is of the form:

\[
w_1 Rv_1, \ldots, w_mE(t_m, A_m), w_1 E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]

where all pairs \( w_i Rv_i \) and \( w_i E(t_i, A_i) \) from the antecedent of the topmost sequent are listed, then, regarding condition (†), we write the following node on top of it:

\[
v_1 E(t_1, A_1), \ldots, v_mE(t_m, A_m), w_1 Rv_1, \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'
\]
This stage corresponds to applying $m$ times rule $(Mon)$.

Stage of rule $(E?)$: Suppose the labeled system $G3JL$ contains rule $(E?)$. For these systems we need to save the number of times we apply the stage of $(E?)$ in a counter. Suppose this is the $m$th time we apply this stage. Consider the initial segment

$$\Pi_m = \{(?t_1, \neg t_1 : A_1), \ldots, (?t_m, \neg t_m : A_m)\},$$

of $\Pi$. Write the following $2^k$ sequents on top of the top-sequent $\Gamma' \Rightarrow \Delta'$:

$$D_1, \ldots, D_k, \Gamma' \Rightarrow \Delta'$$

where $D_j$ ($1 \leq j \leq k$) is either $wE(t_i, A_i)$ or $wE(?t_i, \neg t_i : A_i)$, for $w \in \Gamma' \cup \Delta'$ and $1 \leq i \leq m$, such that $D_j$ has not been added already to the antecedent of sequents in the previous stages of $(E?)$ (condition $†$). This stage corresponds to applying $k$ times rule $(E?)$.

Stage of rule $(SE)$: If the labeled system $G3JL$ contains rule $(SE)$, then if the top-sequent is of the form:

$$w_1E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'$$

where all evidence atoms $w_iE(t_i, A_i)$ from the topmost sequent are listed, we write the following node on top of it (regarding condition $†$)):

$$w_1 \models t_1 : A_1, \ldots, w_m \models t_m : A_m, w_1E(t_1, A_1), \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'$$

This stage corresponds to applying $m$ times rule $(SE)$.

Stage of rule $(Anti-Mon)$: If the labeled system $G3JL$ contains rules $(E?)$ and $(SE)$ (and so rule $(Anti-Mon)$ is admissible), and the top-sequent is of the form:

$$w_1Rv_1, \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'$$

where all pairs $w_iRv_i$ and $v_iE(t_i, A_i)$ from the antecedent of the topmost sequent are listed, then, regarding condition $†$, we write the following node on top of it:

$$w_1E(t_1, A_1), \ldots, w_mE(t_m, A_m), w_1Rv_1, \ldots, w_mE(t_m, A_m), w_1Rv_1, \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'.$$

This stage corresponds to applying $m$ times rule $(Anti-Mon)$.

If the labeled system $G3JL$ contains rule $(Ref)$, add to the antecedent of the top-sequent $\Gamma' \Rightarrow \Delta'$ all the relational atoms $wRw$, for $w$ in $\Gamma' \cup \Delta'$, that are not in $\Gamma'$ yet (condition $†$).

If the labeled system $G3JL$ contains rule $(Ser)$, add to the antecedent of the top-sequent $\Gamma' \Rightarrow \Delta'$ all the relational atoms $wRv$, for $w$ in $\Gamma' \cup \Delta'$ and fresh label $v$.

If the labeled system $G3JL$ contains rule $(Trans)$, and the top-sequent is of the form:

$$w_1Rv_1, \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'$$

where all pairs $w_iRv_i$ and $v_iRu_i$ from the antecedent of the topmost sequent are listed, then, regarding condition $†$, we write the following node on top of it:

$$w_1Ru_1, \ldots, w_mE(t_m, A_m), w_1Ru_1, \ldots, w_mE(t_m, A_m), \Gamma' \Rightarrow \Delta'.$$

If all the topmost sequents of the reduction tree are initial sequents, then we terminate the construction of the reduction tree. In this case by transforming each stage which is applied in the reduction
tree to (possibly more than one application of) the corresponding rule, we can write a derivation for \( \Gamma \Rightarrow \Delta \). Otherwise, the reduction tree has at least one saturated branch.

**Countermodel:** Suppose the reduction tree has a (finite or infinite) saturated branch, say \( \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots \) where \( \Gamma_0 \Rightarrow \Delta_0 \) is the root sequent \( \Gamma \Rightarrow \Delta \). Let

\[
\mathcal{T} = \bigcup_{i \geq 0} \Gamma_i, \quad \mathcal{A} = \bigcup_{i \geq 0} \Delta_i.
\]

We shall define a model \( \mathcal{M} \) and an \( \mathcal{M} \)-interpretation \([\cdot]\) in which \([\cdot]\) validates all the formulas in \( \mathcal{T} \) and no formulas in \( \mathcal{A} \). The construction of the Fitting countermodel \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \) for \( \Gamma \Rightarrow \Delta \) is as follows:

1. The set of possible worlds \( \mathcal{W} \) is all the labels that occur in \( \mathcal{T} \cup \mathcal{A} \).
2. The accessibility relation \( \mathcal{R} \) is determined by relational atoms in \( \mathcal{T} \) as follows: if \( wRv \) is in \( \mathcal{T} \), then \( wRv \) (otherwise \( wRv \) does not hold). Moreover, for \( J5 \) and its extensions extend \( \mathcal{R} \) to its Euclidean closure.
3. For the evidence function \( \mathcal{E} \), we first construct a possible evidence function \( \mathcal{E}_0 \) on \( \mathcal{W} \) for \( J\mathcal{L}_{CS} \) as follows. \( \mathcal{E}_0 \) is determined by evidence atoms in \( \mathcal{T} \): if \( wE(t, A) \) is in \( \mathcal{T} \), then \( w \in \mathcal{E}_0(t, A) \) (otherwise \( w \not\in \mathcal{E}_0(t, A) \)). Since in stage 12 we add evidence atoms \( wE(c, c) \), for \( c : F \in \mathcal{CS} \) and label \( w \) in the reduction tree, to the antecedent of sequents, we have \( w \in \mathcal{E}_0(c, c) \) for every \( c : F \in \mathcal{CS} \) and \( w \in \mathcal{W} \). Therefore, \( \mathcal{E}_0 \) is a possible evidence function on \( \mathcal{W} \) for \( J\mathcal{L}_{CS} \). Now let \( \mathcal{E} \) be the inductively generated admissible evidence function based on \( \mathcal{E}_0 \) (as defined in Definition 3.6).
4. The valuation \( \mathcal{V} \) is determined by labeled formulas in \( \mathcal{T} \) and \( \mathcal{A} \) as follows: if \( w \Vdash P \) is in \( \mathcal{T} \), then \( w \in \mathcal{V}(P) \), and if \( w \not\Vdash P \) is in \( \mathcal{A} \), then \( w \not\in \mathcal{V}(P) \) (where \( P \) is a propositional variable).

In order to show that \( \mathcal{M} \) is the desired countermodel we need the following lemmas.

**Lemma 8.3.** If \( w \in \mathcal{E}(r, F) \) and \( r \in \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \), then \( wE(r, F) \) is in \( \mathcal{T} \).

*Proof.* Suppose \( w \in \mathcal{E}(r, F) \) and \( r \in \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \). Then \( w \in \mathcal{E}_i(r, F) \), for some \( i \geq 0 \). By induction on \( i \) we show that if \( w \in \mathcal{E}_i(r, F) \), then \( wE(r, F) \) is in \( \mathcal{T} \).

The base case, \( i = 0 \), follows from the definition of \( \mathcal{E}_0 \). For the induction hypothesis, suppose that if \( w \in \mathcal{E}_i(r, F) \), then \( wE(r, F) \) is in \( \mathcal{T} \). If \( w \in \mathcal{E}_{i+1}(r, F) \) for some \( i \geq 0 \), and \( w \in \mathcal{E}_j(r, F) \) for \( j < i + 1 \), then by the induction hypothesis \( wE(r, F) \) is in \( \mathcal{T} \). Assume now that we have \( w \in \mathcal{E}_{i+1}(r, F) \) for \( i \geq 0 \), and \( w \not\in \mathcal{E}_j(r, F) \) for any \( j < i + 1 \). We have the following cases:

1. \( r = s \cdot t \), and \( w \in \mathcal{E}_{i+1}(s \cdot t, F) \). Then \( w \in \mathcal{E}_i(s, G \Rightarrow F) \cap \mathcal{E}_i(t, G) \), for some formula \( G \), and \( rk(s \cdot t) = i + 1 \). Thus, by the induction hypothesis, \( wE(s, G \Rightarrow F) \) and \( wE(t, G) \) are in \( \mathcal{T} \). Therefore, \( wE(s, G \Rightarrow F) \) and \( wE(t, G) \) are in \( \Gamma_k \), for some \( k \geq 0 \). Since \( r \in \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \), after the stage of rule \( (E-) \), evidence atom \( wE(s \cdot t, F) \) is added to the antecedent of the sequent. Hence \( wE(s \cdot t, F) \) is in \( \Gamma_i \), for some \( l \geq k \). Therefore, \( wE(s \cdot t, F) \) is in \( \mathcal{T} \).
2. \( r = s + t \), and \( w \in \mathcal{E}_{i+1}(s + t, F) \). Thus \( w \in \mathcal{E}_i(t, F) \) or \( w \in \mathcal{E}_i(s, F) \), and \( rk(s + t) = i + 1 \). If \( w \in \mathcal{E}_i(t, F) \), then by the induction hypothesis, \( wE(t, F) \) is in \( \mathcal{T} \). Therefore, \( wE(t, F) \) is in \( \mathcal{T} \). Hence \( wE(s + t, F) \) is in \( \mathcal{T} \). Proceed similarly if \( w \in \mathcal{E}_i(s, F) \).
3. \( j_4 \) is an axiom of \( \mathcal{L}_{CS} \), \( r \vdash t \), and \( w \in \mathcal{E}_{i+1}(t, F) \). Then \( w \in \mathcal{E}_i(t, G) \), for some formula \( G \) such that \( F = t : G \), and \( rk(l) = i + 1 \). Thus, by the induction hypothesis, \( wE(t, G) \) is in \( \mathcal{T} \). Therefore, \( wE(t, G) \) is in \( \Gamma_k \), for some \( k \geq 0 \). Since \( r \in \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \), after the stage of rule \( (E!) \), evidence atom \( wE(t, F) \) is added to the antecedent of the sequent. Hence \( wE(t, F) \) is in \( \Gamma_l \), for some \( l \geq k \). Therefore, \( wE(t, F) \) is in \( \mathcal{T} \).

4. \( j_4 \) is an axiom of \( \mathcal{L}_{CS} \), \( w \in \mathcal{E}_{i+1}(r, F), vRw, v \in \mathcal{E}_i(r, F), \) and \( rk(r) = i + 1 \). Thus, by the induction hypothesis, \( vE(r, F) \) is in \( \mathcal{T} \). By definition of \( \mathcal{R} \) for countermodel \( M \), \( vRw \) is in \( \mathcal{T} \). Therefore, \( vE(r, F) \) and \( vRw \) are in \( \Gamma_k \), for some \( k \geq 0 \). After the stage of rule \( (\text{Mon}) \), evidence atom \( wE(r, F) \) is added to the antecedent of the sequent. Hence \( wE(r, F) \) is in \( \Gamma_l \), for some \( l \geq k \). Therefore, \( wE(r, F) \) is in \( \mathcal{T} \).

5. \( j_5 \) is an axiom of \( \mathcal{L}_{CS} \), \( r \vdash ?s \), and \( w \in \mathcal{E}_{i+1}(?s, F) \). Then \( w \notin \mathcal{E}_i(s, G) \), for some formula \( G \) such that \( F = \neg s : G \), and \( rk(s) = i + 1 \). Since \( ?s, \neg s : G \) \( \in \mathcal{P} \), we have \( ?s, \neg s : G = (?m, \neg t_m : A_m) \) for some \( m \). Thus, in the stage of rule \( (E?) \) whose counter’s value is greater than or equal to \( m \), evidence atom \( wE(?s, \neg s : G) \) is added to the antecedent of a sequent. Hence \( wE(?s, \neg s : G) \) is in \( \mathcal{T} \).

6. \( j_5 \) is an axiom of \( \mathcal{L}_{CS} \), \( w \in \mathcal{E}_{i+1}(r, F), wRv, v \in \mathcal{E}_i(r, F), \) and \( rk(r) = i + 1 \). Thus, by the induction hypothesis, \( vE(r, F) \) is in \( \mathcal{T} \). By definition of \( \mathcal{R} \) for countermodel \( M \), \( wRv \) is in \( \mathcal{T} \). Therefore, \( vE(r, F) \) and \( wRv \) are in \( \Gamma_k \), for some \( k \geq 0 \). After the stage of rule \( (\text{Anti-Mon}) \), evidence atom \( wE(r, F) \) is added to the antecedent of the sequent. Hence \( wE(r, F) \) is in \( \Gamma_l \), for some \( l \geq k \). Therefore, \( wE(r, F) \) is in \( \mathcal{T} \).

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**Lemma 8.4.** Given an arbitrary \( \mathcal{L} \)-formula \( A \), if \( w \models A \) is in \( \mathcal{T} \), then \( (\mathcal{M}, w) \models A \), and if \( w \notmodels A \) is in \( \overline{\mathcal{T}} \), then \( (\mathcal{M}, w) \notmodels A \).

**Proof.** By induction on the complexity of \( A \). The base case follows from the definition of \( \mathcal{V} \). We consider in details only the case that \( A \) is of the form \( t : B \) (the proof of other cases are stated in details in [31]).

Suppose \( w \models t : B \) is in \( \mathcal{T} \). Then, there is some \( i \) such that \( w \notmodels t : B \) appears first in \( \Gamma_i \). Therefore, after the stage of rule \( (E) \), \( wE(t, B) \) is added to \( \Gamma_j \) for some \( j > i \). Hence, by definition of evidence function \( \mathcal{E} \) for countermodel \( M \), we have \( w \not\in \mathcal{E}_i(t, B) \), and hence \( w \in \mathcal{E}_i(t, B) \). Now consider an arbitrary world \( v \) in \( \mathcal{W} \) such that \( wRv \). By definition of accessibility relation \( \mathcal{R} \) for \( M \), \( wRv \) should be in \( \Gamma_k \), for some \( k \). Then we can find \( k \) such that both \( wRv \) and \( w \models t : B \) are in \( \Gamma_k \). Hence, in the stage of rule \( (L) \), we add \( v \models B \) to \( \mathcal{T} \). Thus, by the induction hypothesis, we can conclude that \( (\mathcal{M}, v) \models B \). Therefore, \( (\mathcal{M}, w) \models t : B \).

If \( w \models t : B \) is in \( \overline{\mathcal{T}} \). Then \( w \notmodels t : B \) is in \( \overline{\Delta_i} \), for some \( i \). We have two cases:

(i) \( wE(t, B) \) is in \( \mathcal{T} \). Then \( wE(t, B) \) is in \( \Gamma_j \), for some \( j \). Find minimum index \( k \) such that \( \Gamma_k \Rightarrow \Delta_k \) is of the form \( wE(t, B), l' \Rightarrow \Delta', w \models t : B \). After the stage of rule \( (R') \), we obtain a sequent \( \Gamma_l \Rightarrow \Delta_l \) of the form \( wRv, wE(t, B), l'' \Rightarrow \Delta'' \), \( v \notmodels B \), for some fresh label \( v \) and \( l > k \). Thus, by the induction hypothesis, \( (\mathcal{M}, v) \notmodels B \) and, by the definition of \( \mathcal{R} \), \( wRv \). Hence, \( (\mathcal{M}, w) \notmodels t : B \).

(ii) \( wE(t, B) \) is not in \( \mathcal{T} \) in \( \mathcal{G}_{\mathcal{L}} \). Since \( w \models t : B \) is in \( \overline{\mathcal{T}} \), by the labeled-subformula property, we have \( t : B \in \text{Sub}_{\mathcal{E}_m}(\Gamma \Rightarrow \Delta) \), and hence \( t \) is in \( \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \). For labeled systems which contains rule \( (SE) \), by tracing the labeled formula \( w \models t : B \) downward, inspection of all rules shows that no labeled formula related to \( w \models t : B \) can be active in an application of rule \( (SE) \). Therefore, \( t : B \in \text{Sub}_{\mathcal{E}_m}(\Gamma \Rightarrow \Delta) \), and hence \( t \) is in \( \text{Sub}_{\mathcal{T}}(\Gamma \Rightarrow \Delta) \). By Lemma 8.3, we have \( w \not\in \mathcal{E}(t, B) \). Hence \( (\mathcal{M}, w) \notmodels t : B \).

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46
The stages used in the reduction tree ensure that the accessibility relation $R$ for $\mathcal{M}$ satisfies those conditions needed for $\mathcal{JL}$-models. Moreover, if $\mathcal{JL}_{CS}$ does not contain axiom $j5$, then by Lemma 3.1, $E$ is an admissible evidence function for $\mathcal{JL}_{CS}$.

If $\mathcal{JL}_{CS}$ does contain axiom $j5$, then by Lemma 3.1 $E$ satisfies the negative introspection condition (and also anti-monotonicity). In order to show that $E$ satisfies the strong evidence condition, we need to verify that for all terms $r$, formulas $F$, and $w \in W$ if $w \in E_0(r, F)$, then $(\mathcal{M}, w) \models r : F$. By the definition of $E_0$, if $w \in E_0(r, F)$, then $wE(t, A)$ is in $\mathcal{T}$. Thus $wE(t, A)$ is in $I_i$, for some $i \geq 0$. After the stage of rule $(SE)$, the labeled formula $w \models r : F$ is added to the antecedent of the sequents. Now by Lemma 8.4, $(\mathcal{M}, w) \models r : F$. Thus, by Lemma 3.2, $E$ satisfies the strong evidence condition, and hence in this case $\mathcal{M}$ is a $\mathcal{JL}_{CS}$-model.

Finally, let $\mathcal{M}$-interpretation $[\cdot]$ be the identity function. Note that if $wE(r, F)$ is in $\mathcal{T}$; then $w \in E_0(r, F)$, and hence $w \in E(r, F)$.

For these labeled systems, if the reduction tree of a sequent has a saturated branch, then the sequent has a countermodel, and hence, by Theorem 8.1, it is not derivable.

Corollary 8.4. Let $\mathcal{JL}$ be one of the justification logics such that do not contain axiom $jB$, and $\mathcal{CS}$ be a finite constant specification for $\mathcal{JL}$. If sequent $\Gamma \Rightarrow \Delta$ is valid in every $\mathcal{JL}_{CS}$-model, then it is derivable in $\mathcal{G3JL}_{CS}$.

In the next section, using the reduction tree and above corollary, we show how to decide whether a sequent is derivable. Let us now give some examples. The first example shows that reduction tree of each sequent (except the empty sequent) in $\mathcal{G3JD}_{CS}$ and its extensions (always) contains an infinite saturated branch.

Example 8.1. Let us try to construct a proof tree for the sequent $\models w \models P$ in $\mathcal{G3JD}_0$, where $P$ is a propositional variable.

\begin{verbatim}
  : 
  \uparrow 
  wRv_3, v_1 Rv_3, v_2 Rv_3, wRv_2, v_1 Rv_2, wRv_1 \Rightarrow w \models P 
  \uparrow 
  wRv_2, v_1 Rv_2, wRv_1 \Rightarrow w \models P 
  \uparrow 
  wRv_1 \Rightarrow w \models P 
  \uparrow 
  \Rightarrow w \models P
\end{verbatim}
where all sequents (except the root sequent) are obtained by stages of rule (Ser), and \(v_1, v_2, \ldots\) are distinct labels different from \(w\). Since the branch is infinite, it is saturated. Thus, we can construct a JD\(_3\)-countermodel \(\mathcal{M} = (W, R, E, V)\) for the root sequent as follows:

- \(W = \{w, v_1, v_2, \ldots\}\).
- \(R = \{(w, v_i) \mid i \geq 1\} \cup \{(v_i, v_j) \mid i < j\}\).
- \(E(t, A) = \emptyset\), for any term \(t\) and formula \(A\).
- \(V(Q) = \emptyset\), for any propositional variable \(Q\).

Obviously, \(R\) is serial and \(E\) satisfies application (\(E1\)) and sum (\(E2\)) conditions. Moreover, by Lemma 8.4, we have \((\mathcal{M}, w) \not\models P\). By letting \([\cdot]\) be the identity \(\mathcal{M}\)-interpretation, we conclude that \(\mathcal{M}\) is a JD\(_3\)-countermodel of \(\Rightarrow w \models P\).

**Example 8.2.** Let us try to construct a proof tree for a justification version of Löb principle, \(x : (y : A \rightarrow A) \rightarrow z : A\), in G3\(\mathbb{I}_g\). Here, to have a simple reduction tree we replace occurrences of \(\Box\) in the Löb principle \(\Box(\Box A \rightarrow A) \rightarrow \Box A\) by justification variables \(x, y\) and \(z\), respectively.

\[
\begin{align*}
&wE(x, y : A \rightarrow A), w \models x : (y : A \rightarrow A) \Rightarrow w \models z : A \\
&w \models x : (y : A \rightarrow A) \Rightarrow w \models z : A \\
&\Rightarrow w \models x : (y : A \rightarrow A) \rightarrow z : A
\end{align*}
\]

The second and third sequent in the reduction tree are obtained from the root sequent by applying \((R \rightarrow)\) and \((E)\), respectively. Since no other rules can be applied on the third sequent, and it is not an initial sequent, the branch is saturated. Thus, we can construct a J\(\mathbb{I}\)-countermodel \(\mathcal{M}\) for the root sequent. Let \(W = \{w\}\) and \(R\) be the empty set. For the evidence function \(E\), we have \(w \in E_0(x, y : A \rightarrow A)\). Now let \(E\) be the inductively generated evidence function based on \(E_0\). Finally let \(V(P) = \emptyset\), for any propositional variable \(P\). By Lemma 8.4 we obtain \((\mathcal{M}, w) \not\models x : (y : A \rightarrow A)\) and \((\mathcal{M}, w) \not\models z : A\) (they can also be proved directly in the model \(\mathcal{M}\)). Thus \((\mathcal{M}, w) \not\models x : (y : A \rightarrow A) \rightarrow z : A\).

We close this section with a discussion on the correspondence theory in justification logics. Following the correspondence theory of modal logics, one can expect that justification axioms defines classes of suitable frames. For simplicity, in the rest of this section, we assume that the language of all justification logics contain all term operations \(+, -, \cdot, !, ?\). The set of all terms and formulas are denoted by \(Tm\) and \(Fm\) respectively.

**Definition 8.2.** 1. A Fitting frame is a triple \(\mathcal{F} = (W, R, E)\) such that \(W\) is a non-empty set, \(R\) is the accessibility relation on \(W\), and \(E\) is a possible evidence function on \(W\), i.e. \(E : Tm \times Fm \rightarrow 2^W\).

2. A possible Fitting model is an Fitting frame enriched by a valuation \(V\). For a possible Fitting model \(\mathcal{M} = (W, R, A, V)\), where \(\mathcal{F} = (W, R, A)\) is a Fitting frame, we say that \(\mathcal{M}\) is based on \(\mathcal{F}\).

3. A J\(\mathbb{I}\)-formula \(A\) is valid in a possible Fitting model \(\mathcal{M}\), denoted \(\mathcal{M} \models A\), if \((\mathcal{M}, w) \models A\) for every \(w \in W\).

4. A J\(\mathbb{I}\)-formula \(A\) is valid in a class of possible Fitting models \(\mathcal{M}\), denoted \(\mathcal{M} \models A\), if \(\mathcal{M} \models A\) for every possible Fitting model \(\mathcal{M} \in \mathcal{M}\).

5. A J\(\mathbb{I}\)-formula \(A\) characterizes a class of possible Fitting model \(\mathcal{M}\): \[\mathcal{M} \in \mathcal{M} \iff \mathcal{M} \models A.\]
6. A $A$-formula $A$ is valid in a world $w$ in an Fitting frame $F$, denoted $(F, w) \models A$, if $(M, w) \models A$ for every possible Fitting model $M = (F, V)$ based on $F$.

7. A $A$-formula $A$ is valid in a Fitting frame $F$, denoted $F \models A$, if $(F, w) \models A$ for every $w \in W$.

8. A $A$-formula $A$ is valid in a class of Fitting frames $F$, denoted $F \models A$, if $F \models A$ for every Fitting frame $F \in F$.

9. A $A$-formula $A$ characterizes a class of Fitting frames $F$, if for all Fitting frames $F$:

$$F \in F \iff F \models A.$$ 

Soundness and completeness of justification logics with respect to F-models (Theorem 3.1) naturally propose the following characterizations (cf. also Table 1):

1. $x : (P \to Q) \to (y : P \to x \cdot y : P)$ characterizes the class of Fitting frames in which the evidence function satisfies the application condition ($E1$).

2. $x : P \lor y : P \to x + y : P$ characterizes the class of Fitting frames in which the evidence function satisfies the sum condition ($E2$).

3. $x : P \to P$ characterizes the class of Fitting frames in which the accessibility relation is reflexive.

4. $x : \perp \to \perp$ characterizes the class of Fitting frames in which the accessibility relation is serial.

5. $x : P \to !x : P$ characterizes the class of Fitting frames in which the accessibility relation is transitive and the evidence function satisfies the monotonicity ($E3$) and positive introspection ($E4$) conditions.

6. $\neg P \to ?x : \neg x : P$ characterizes the class of possible Fitting models in which the accessibility relation is symmetric and the evidence function satisfies the weak negative introspection condition ($E5$).

7. $x : P \to ?x : \neg x : P$ characterizes the class of possible Fitting models in which the evidence function satisfies the negative introspection ($E6$) and strong evidence ($E7$) conditions.

The above characterizations are used implicitly in the literature of justification logics, but it seems there are no proofs for them. In the following theorem we give a similar characterization in the context of labeled systems.

**Theorem 8.3 (Fitting Frame Correspondence).** Suppose $x$ is a justification variable and $P$ is a propositional variable.

1. The sequent $\Rightarrow w \models x : P \to P$ is derivable in $G3JL_{CS}$ if and only if $G3JL_{CS}$ contains ($Ref$).

2. The sequent $\Rightarrow w \models x : \perp \to \perp$ is derivable in $G3JL_{CS}$ if and only if $G3JL_{CS}$ contains ($Ser$).

3. The sequent $\Rightarrow w \models x : P \to !x : x : P$ is derivable in $G3JL_{CS}$ if and only if $G3JL_{CS}$ contains ($EI$) + ($Mon$) + ($Trans$).

4. The sequent $\Rightarrow w \models \neg P \to ?x : \neg x : P$ is derivable in $G3JL_{CS}$ if and only if $G3JL_{CS}$ contains ($E?\overline{\text{E}}$) + ($Sym$).

5. The sequent $\Rightarrow w \models \neg x : P \to ?x : \neg x : P$ is derivable in $G3JL_{CS}$ if and only if $G3JL_{CS}$ contains ($SE$) + ($E?\overline{\text{E}}$).

*Proof.* The if directions are easy. Let us prove the only if directions, or rather their contrapositives. We detail the proof only for clause (1). Suppose $G3JL_{\emptyset}$ does not contain rule ($Ref$). If $G3JL_{\emptyset}$ does not contain rules ($Ser$) or ($E?$), then the reduction tree of $\Rightarrow w \models x : P \to P$ in $G3JL_{\emptyset}$ is as follows:
\[ w E(x, P), w \vdash x : P \Rightarrow w \vdash P \]
\[ \uparrow \]
\[ w \vdash x : P \Rightarrow w \vdash P \]
\[ \uparrow \]
\[ \Rightarrow w \vdash x : P \rightarrow P \]

where the second and third sequents are obtained by the stages of rules \((R \rightarrow)\) and \((E)\) respectively. Since this branch is saturated, by Corollary 8.4, the sequent \(\Rightarrow w \vdash x : P \rightarrow P\) is not derivable in \(G3J_\emptyset\). For labeled systems that contain \((Ser)\) or \((E?)\) we obtain an infinite saturated branch, which again shows that the sequent is not derivable. The proof for other clauses, except (4), are similar.

For clause (4), Suppose \(G3J_\emptyset\) does not contain rules \((E?)\) and \((Sym)\). If we try to find a derivation for the sequent \(\Rightarrow w \vdash \neg P \rightarrow ?x : \neg x : P\) in \(G3J_\emptyset\), which does not contain rules \((Ref)\), \((Ser)\), \((E?)\), we see that this sequent only fit into the conclusion of rule \((R \rightarrow)\):

\[ w \vdash \neg P \Rightarrow w \vdash ?x : \neg x : P \quad (R \rightarrow) \]

and the sequent \(w \vdash \neg P \Rightarrow w \vdash ?x : \neg x : P\) only fit into the conclusion of rule \((L \neg)\):

\[ \Rightarrow w \vdash ?x : \neg x : P, w \vdash P \quad (L \neg) \]
\[ \Rightarrow w \vdash \neg P \Rightarrow w \vdash ?x : \neg x : P \quad (R \rightarrow) \]

Now the the sequent \(\Rightarrow w \vdash ?x : \neg x : P, w \vdash P\) do not fit into the conclusion of any rules. Hence \(\Rightarrow w \vdash \neg P \rightarrow ?x : \neg x : P\) is not derivable in \(G3J_\emptyset\). If \(G3J_\emptyset\) contains the rules \((Ref)\), \((Ser)\), \((E?)\), upwardly, in any step of the previous argument do not yield to an initial sequent. Thus again \(\Rightarrow w \vdash \neg P \rightarrow ?x : \neg x : P\) is not derivable in \(G3J_\emptyset\).

The above results can be extended to the case that \(CS\) is not empty (we leave it to the reader to verify the details of this).

\[ \square \]

9 Termination of proof search

In this section, we establish the termination of proof search for labeled systems \(G3J_{CS}, G3JT_{CS}, G3LP_{CS}\), for finite \(CS\). In this respect, it is useful to consider the reduction tree of a sequent, which was constructed in the proof of Theorem 8.2.\(^7\) We show that the reduction tree of every sequent in \(G3J_{CS}, G3JT_{CS}, G3LP_{CS}\), for finite \(CS\), is finite. Thus, it is decidable whether the reduction tree has a saturated branch, and hence it is decidable whether the sequent is derivable. In order to show that the reduction tree of a sequent is finite, we find bounds on the number of applications of rules. First we define the negative and positive parts of a sequent \(\Gamma \Rightarrow \Delta\). Let

\[ \Gamma^f = \{ A \mid w \vdash A \text{ occurs in } \Gamma \}. \]

\(^7\) In [21] termination of proof search for these systems are proved using minimal derivations (defined by Negri in [30]).
Theorem 9.2. Given any finite constant specification $CS$, and any sequent $\Gamma \Rightarrow \Delta$ in the language of $G3JT$, it is decidable whether the sequent is derivable in $G3JT_{CS}$.

Proof. Suppose $\Gamma \Rightarrow \Delta$ is any sequent to be shown derivable. Construct the reduction tree with the root $\Gamma \Rightarrow \Delta$. In the reduction tree the stages of propositional rules reduce the complexity of formulas in the sequents (propositional rules have premises in which the active formulas are strictly simpler than the principal formula). In fact, in each branch of the reduction tree the number of applications of propositional rules are bounded by the number of the corresponding connective in the endsequent. Rule $(R : \cdot)$ decreases the complexity of its principal labeled formula but adds a relational atom. The number of applications of this rule (and therefore the number of eigenlabels introducing by this rule) is bounded by $p(\cdot)$. Rule $(L : \cdot)$ adds a new labeled formula to the sequent. Regarding condition $(\dagger)$, rule $(L : \cdot)$ in the reduction tree could apply only once on a pair of formulas $w \vDash t : A$ and $wRv$. Thus the number of applications of $(L : \cdot)$ with principal formula $w \vDash t : A$ is bounded by the number of relational atoms of the form $wRv$, which can be found in the antecedent of the root sequent or may be introduced by rule $(R : \cdot)$ in the antecedent of sequents. Hence the number of applications of $(L : \cdot)$ is bounded by $n(\cdot)(p(\cdot) + r)$.

Rules $(E)$, $(IAN)$, $(El+)$, $(Er+)$ and $(E\cdot)$ add one new evidence atom to the sequent, and increase the size of the sequent. Thus, we have to find bounds on the number of applications of each one. For $(E)$, since moving upward no rule omit any evidence atom from the antecedent of sequents, the number of applications of $(E)$ is bounded by $n(\cdot)$. For $(IAN)$, since constant specification $CS$ is finite, by the sublabel property, the number of applications of $(IAN)$ is bounded by $|CS|(p(\cdot) + l)$ (by $|CS|$ we mean the number of elements of $CS$). By the construction of the reduction tree, we add an evidence atom $wE(t + s, A)$ to the antecedent of the topsequent provided that $t + s \in Sub_{\tau, m}(\Gamma \Rightarrow \Delta)$. Therefore, the number of applications of $(El+)$ (and similarly $(Er+)$) is bounded by the number of evidence atoms that are in the antecedent of the root sequent or may be introduced by the rules $(E)$ or $(IAN)$. Hence, the number of applications of $(El+)$ or $(Er+)$ is bounded by $n(+) (e + n(\cdot) + |CS|(p(\cdot) + l))$. By a similar argument, we get a bound on the number of applications of rule $(E\cdot)$ by $n(\cdot)(e + n(\cdot) + |CS|(p(\cdot) + l)).$\(\dagger\)

Note that, as the above proof shows, the number of applications of $(L : \cdot)$ and $(IAN)$ (or $(AN)$) depends on the number of applications of $(R : \cdot)$ and the number of applications of $(El+)$, $(Er+)$, and $(E\cdot)$ depends on the number of applications of $(E)$ and $(IAN)$.

Theorem 9.2. Given any finite constant specification $CS$, and any sequent $\Gamma \Rightarrow \Delta$ in the language of $G3JT$, it is decidable whether the sequent is derivable in $G3JT_{CS}$. 

51
Proof. Rule (Ref) add a relational atom to the sequent. By the construction of the reduction tree, (Ref) only introduces atoms \( wRw \) in which \( w \) is a label in the root sequent or is an eigenlabel. Thus, the number of applications of (Ref) is bounded by \( 1 + p(\cdot) \). Since the atoms \( wRw \) introduced by (Ref), may produce new applications of the rule \( (L:\cdot) \), in this case, the number of applications of \( (L:\cdot) \) is bounded by \( n(\cdot)(2p(\cdot) + r + l) \). The other bounds remain unchanged. 

Rules (Trans) and (Mon) in G3LP may produce infinitely many applications of rules \( (L:\cdot) \) and \( (R:\cdot) \). For example, suppose we try to find a derivation for sequent \( vRw, wE(t, A) \Rightarrow v \Vdash s : \neg t : A \) in G3LP:

\[
\begin{align*}
&wRw_1, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w_1 \Vdash t : A \quad (R:\cdot) \\
vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w \Vdash t : A \quad (L:\cdot) \\
w \Vdash \neg t : A, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow (L) \\
vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow (L) \\
\end{align*}
\]

Now, by applying (Trans) and (Mon) upwardly on the topsequent we have:

\[
vRw_1, w_1E(t, A), wRw_1, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w_1 \Vdash t : A. \quad (9)
\]

Next, by applying \( (L:\cdot) \) and then \( (L) \) upwardly we obtain:

\[
vRw_1, w_1E(t, A), wRw_1, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w_1 \Vdash t : A. \quad (10)
\]

Again, by applying \( (R:\cdot) \) upwardly we have:

\[
w_1Rw_2, vRw_1, w_1E(t, A), wRw_1, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w_1 \Vdash t : A, w_2 \Vdash t : A. \quad (11)
\]

We can repeat the steps (9)-(11) and the applications of \( (R:\cdot) \) infinitely many times. But these stages are redundant. Indeed, steps (9)-(11) can be shortened in the following way. By applying the substitution \( (w_1/w_2) \) on sequent (11), we obtain a derivation of the same height of

\[
w_1Rw_1, vRw_1, w_1E(t, A), wRw_1, vRw, wE(t, A), v \Vdash s : \neg t : A \Rightarrow w_1 \Vdash t : A, w_1 \Vdash t : A.
\]

Then, by height-preserving contraction and (Ref) we obtain a derivation of

\[
vRw_1, w_1E(t, A), wRw_1, vRw, wE(t, A), s : \neg t : A \Rightarrow w_1 \Vdash t : A.
\]

But this sequent is the same as sequent (9), which has been obtained in two steps shorter than (9). In fact, we can bound the number of applications of \( (R:\cdot) \) in derivations, as follows:

**Proposition 9.1.** In a derivation of a sequent in G3LP\(_{CS} \) for each formula of the form \( t : A \) in the positive part of the sequent, it is enough to have at most \( n(\cdot) \) applications of \( (R:\cdot) \) iterated on a chain of accessible worlds \( wRw_1, w_1Rw_2, \ldots \) with principal formula \( w_1 \Vdash t : A \).

**Proof.** Let \( n(\cdot) = m \), and assume that \( t : A \) is a formula in the positive part of the endsequent of the derivation. Consider the worst case, in which all of the \( m \) negative occurrences of \( : \) appear in a block in a formula of the form \( w \Vdash t_1 : t_2 : \ldots : t_m : B \) in the antecedent of the endsequent (in which \( B \) contains no negative occurrence of \( : \)). After the first application of \( (R:\cdot) \) on \( t : A \), we have the accessibility atom
\( wRw_1 \) in the antecedent. Then an application of \((L : )\) produces the formula \( w_1 \vdash t_2 : \ldots : t_m : B \). After the second application of \((R : )\), we have the new accessibility atom \( w_1Rw_2 \), and by \((\text{Trans})\) we get \( wRw_2 \). Then applications of \((L : )\) add to the antecedent the formulas \( w_2 \vdash t_2 : \ldots : t_m : B \) and \( w_2 \vdash t_3 : \ldots : t_m : B \). After \( m \) applications of \((R : )\), the antecedent contains in addition:

\[
\begin{align*}
w_m & \vdash t_2 : \ldots : t_m : B, w_m \vdash t_3 : \ldots : t_m : B, \ldots, w_m \vdash B.
\end{align*}
\]

If we apply \((R : )\) one more time, and then apply \((L : )\), we have also in the antecedent the formulas

\[
\begin{align*}
w_{m+1} & \vdash t_2 : \ldots : t_m : B, w_{m+1} \vdash t_3 : \ldots : t_m : B, \ldots, w_{m+1} \vdash B.
\end{align*}
\]

Similar to the example discussed before the proposition, by the substitution \((w_m/w_{m+1})\) and then applying height-preserving contraction and \((\text{Ref}f)\), we can shorten the derivation. Thus, the last step of \((R : )\) is superfluous.

Thus, if \( n(\cdot) > 0 \) then the number of applications of \((R : )\) is bounded by \( p(\cdot)n(\cdot) \), and if \( n(\cdot) = 0 \) then the number of applications of \((R : )\) is bounded by \( p(\cdot) \). Hence, in general, the number of applications of \((R : )\) is bounded by \( p(\cdot)(n(\cdot) + 1) \).

Regarding Proposition 9.1, in the construction of the reduction tree in Theorem 8.2, we need to restrict the number of applications of \((R : )\) on each subformula \( t : A \) in the positive part of the root sequent. To this end, let \( p(\cdot) = p \) and \( t_1 : A_1, \ldots, t_p : A_p \) be all subformulas of the form \( t : A \) in the positive part of the endsequent \( \Gamma \Rightarrow \Delta \). In order to calculate the number of applications of \((R : )\), we employ counters \( k_1, \ldots, k_p \) for the formulas \( t_1 : A_1, \ldots, t_p : A_p \), respectively. In stage \( n = 0 \) of the construction of the reduction tree, let \( k_i = 0 \) for each \( i = 1, \ldots, p \). Each time we apply rule \((R : )\) on the formula \( t_i : A_i \), we update the value of \( k_i \) by one, i.e. \( k_i = k_i + 1 \). Rule \((R : )\) can be applied on \( t_i : A_i \) provided that \( k_i \leq n(\cdot) \).

**Theorem 9.3.** Given any finite constant specification \( \text{CS} \), and any sequent \( \Gamma \Rightarrow \Delta \) in the language of \( \text{G3LP} \), it is decidable whether the sequent is derivable in \( \text{G3LP}_{\text{CS}} \).

**Proof.** Since the number of applications of \((R : )\) is bounded by \( p(\cdot)(n(\cdot) + 1) \), the number of applications of rules \((L : )\) and \((\text{AN})\) are bounded by \( n(\cdot)p(\cdot)(n(\cdot) + 1) + r + l + p(\cdot) \) and \( |\text{CS}|p(\cdot)(n(\cdot) + 1) + l \), respectively. Rule \((\text{Mon})\) may produce new applications of rules \((El+)\), \((Er+)\), \((E')\) and \((El)\), and vice versa. However, the number of applications of these rules are bounded, since there are a finite number of relational and evidence atoms in the antecedent of the endsequent and the reduction tree of a sequent complying condition \((\dagger)\). The number of applications of \((\text{Trans})\) depends on \( r \) and the number of applications of \((R : )\), and thus is bounded. The number of applications of \((E)\) is bounded by \( n(\cdot) \) as before. \( \dagger \)

### 10 Other labeled systems

In this section we will briefly introduce other variants of labeled sequent calculus for justification logics. We first introduce some admissible rules.

**Lemma 10.1.** The following rules are \( \text{CS}-\)admissible in \( \text{G3J5}_{\text{CS}} \) and its extensions:

\[
\begin{align*}
\text{(R1)} & \quad \frac{\Gamma \Rightarrow \Delta, wE(t, A)}{\Gamma \Rightarrow \Delta, w \vdash t : A} \quad \frac{wE(?t, -t : A), \Gamma \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash t : A} \quad \text{(R2)}
\end{align*}
\]

53
\[
\frac{w \vdash t : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w \vdash t : A} \quad (R3) \quad \frac{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{wE(t, A), wRv, \Gamma \Rightarrow \Delta} \quad (RA)
\]

**Proof.** For \( (R1) \), using admissible \text{Cut}, we have:

\[
\begin{align*}
\mathcal{D} \\
\frac{w \vdash t : A, wE(t, A) \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash t : A} \\
\frac{wE(t, A) \Rightarrow \Delta}{w \vdash t : A} \\
\frac{w \vdash t : A, v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\frac{wE(?t, \neg t : A) \Rightarrow \Delta}{w \vdash t : A} \\
\frac{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\end{align*}
\]

where \( \mathcal{D} \) is the standard derivation of \( w \vdash t : A, wE(t, A) \Rightarrow \Delta, w \vdash t : A \) from Lemma 5.1. For \( (R2) \) we have:

\[
\begin{align*}
\mathcal{D} \\
\frac{w \vdash t : A, wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash t : A} \\
\frac{wE(t, A) \Rightarrow \Delta}{w \vdash t : A} \\
\frac{w \vdash t : A, v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\frac{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\end{align*}
\]

where \( \mathcal{D} \) is the standard derivation of \( w \vdash t : A, wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A \) from Lemma 5.1. For \( (R3) \), using admissible \text{Cut}, we have:

\[
\begin{align*}
\mathcal{D}' \\
\frac{w \vdash t : A, wE(t, A), \Gamma \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash ?t : t : A} \\
\frac{w \vdash t : A, \Gamma \Rightarrow \Delta}{w \vdash t : A} \\
\frac{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\frac{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\end{align*}
\]

where \( \mathcal{D}' \) is the standard derivation of \( \Rightarrow w \vdash ?t : t : A, w \vdash t : A \) in G3J5. For \( (R4) \), using admissible \( \text{LW} \), we have:

\[
\begin{align*}
\frac{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{LW} \\
\frac{w \vdash t : A, v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{L :} \\
\frac{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{SE} \\
\end{align*}
\]

### Lemma 10.2.

The following rule is CS-admissible in G3JB\textsubscript{CS} and its extensions:

\[
\frac{wE(?t, \neg t : A), \Gamma \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash t : A} \quad (Ew?)
\]

**Proof.** We have:

\[
\begin{align*}
\mathcal{D} \\
\frac{w \vdash t : A, \Gamma \Rightarrow \Delta, w \vdash t : A}{\Gamma \Rightarrow \Delta, w \vdash t : A} \\
\frac{w \vdash t : A, v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{v \vdash A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\frac{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta}{w \vdash t : A, wE(t, A), wRv, \Gamma \Rightarrow \Delta} \\
\end{align*}
\]

where \( \mathcal{D} \) is the standard derivation of \( w \vdash t : A, \Gamma \Rightarrow \Delta, w \vdash t : A \) from Lemma 5.1. \( \dashv \)
Labeled sequent calculi $G3JB$ and its extensions can be formulated by rule $(Ew\vec{?})$ instead of $(E\vec{?})$. If $G3JL_{CS}$ contains $(E\vec{?})$, then let $G3JL_{CS}^{w}$ denote the resulting system where the rule $(E\vec{?})$ is replaced by the rule $(Ew\vec{?})$ (these systems were studied in [21]). Although these systems enjoy the labeled-subformula property, we fail to show the admissibility of $Cut$. Here are some counterexamples.

**Example 10.1.** The sequent $\Rightarrow w \Vdash A \rightarrow ?t : \neg t : \neg A$ can only be proved by rule $Cut$ (e.g., a $Cut$ on the formula $w \Vdash \neg\neg A$ in $G3JL_{CS}$ and its extensions.

**Example 10.2.** The sequent $\Rightarrow wE(x, A), wE(?s, \neg s : x : A)$, where $x$ is a justification variable, can only be proved by rule $Cut$ (e.g., a $Cut$ on the formula $w \Vdash x : A$) in $G3JL_{CS}^{w}$ and its extensions.

### 10.1 Labeled sequent calculus based on $Fp$-models

Pacuit in [34] show that the negative introspection axiom $j5$ can be characterized by the following conditions:

$E9$. Anti-monotonicity: If $v \in E(t, A)$ and $wRv$, then $w \in E(t, A)$.

$E10$. Negative proof checker: If $(M, w) \Vdash \neg t : A$, then $w \in E(?t, \neg t : A)$.

and Euclideanness of $R$. We call these models $Fp$-models (for more details cf. [34]).

**Theorem 10.1.** ([34]) Let $JL$ be $J5$ or one of its extensions, and $CS$ be a constant specification for $JL$. Then justification logics $JL_{CS}$ are sound and complete with respect to their $JL_{CS}$-$Fp$-models.

Regarding $Fp$-models, there is another formulation of labeled systems for $J5$ and its extensions by replacing the rules $(E\vec{?})$ and $(SE)$ with the following rules:

$$
\frac{wE(?t, \neg t : A), w \Vdash \neg t : A, \Gamma \Rightarrow \Delta}{w \Vdash \neg t : A, \Gamma \Rightarrow \Delta} \quad (E') \quad \frac{wE(t, A), vE(t, A), wRv, \Gamma \Rightarrow \Delta}{v \in E(t, A), wRv, \Gamma \Rightarrow \Delta} \quad (\text{Anti-Mon})
$$

$$
\frac{vRu, wRv, wRu, \Gamma \Rightarrow \Delta}{wRv, wRu, \Gamma \Rightarrow \Delta} \quad (Eucl)
$$

If $G3JL_{CS}$ contains $(E\vec{?})$, $(SE)$, then let $G3JL_{CS}^{Fp}$ (the labeled sequent calculus based on $Fp$-models) denote the resulting system where the rules $(E\vec{?})$, $(SE)$ are replaced by the rules $(E')$, $(\text{Anti-Mon})$, $(Eucl)$. Thus labeled systems $G3JL_{CS}^{Fp}$ are defined only for justification logics $JL$ that contain axiom $j5$. All the definitions of Sections 4-8 can be adopted for labeled systems $G3JL_{CS}^{Fp}$ (these systems first appeared in [21]). Main properties of labeled systems $G3JL_{CS}^{Fp}$ are listed in the following theorem (the proof is similar to that of labeled systems based on $F$-models).

**Theorem 10.2.** Suppose $JL$ is a justification logic with axiom $j5$, $CS$ is a constant specification for $JL$, and $G3JL_{CS}^{Fp}$ is its labeled sequent calculus based on $Fp$-models.

1. All sequents of the form $w \Vdash A, \Gamma \Rightarrow \Delta, w \Vdash A$, with $A$ an arbitrary $JL$-formula, are derivable in $G3JL_{CS}^{Fp}$.
2. All labeled formulas in a derivation in $G3JL_{CS}^{Fp}$ are labeled-subformulas of labeled formulas in the endsequent.
3. The rules of substitution $(Subs)$, and weakening are height-preserving $CS$-admissible in $G3JL_{CS}^{Fp}$.
4. If the sequent $\Gamma \Rightarrow \Delta$ is derivable in $G3JL_{CS}^{Fp}$, then it is valid in every $JL_{CS}$-$Fp$-model.
It seems the rules of $G_{3JL}^{Fp}$ are not invertible, and the rules of contraction are not admissible. Although the systems $G_{3JL}^{Fp}$ enjoys the subformula property, we fail to show the admissibility of $Cut$. Here are some counterexamples.

**Example 10.3.** It is not hard to verify that in the system $G_{3JT5}^{Fp}$ (and its extensions) the sequent $\Rightarrow w \Vdash \neg A \rightarrow t : \neg t : A$ can only be proved by rule $Cut$ (e.g., a $Cut$ on the formula $w \Vdash t : A$).

**Example 10.4.** The sequent $\Rightarrow wE(t, A), wE(?t, \neg t : A)$ can only be proved by rule $Cut$ (e.g., a $Cut$ on the formula $w \Vdash t : A$) in $G_{3J5}^{Fp}$ and its extensions.

With regard to the above facts for the completeness we have:

**Theorem 10.3.** Suppose $JL$ is a justification logic with axiom $j5$, $CS$ is a constant specification for $JL$, and $G_{3JL}^{Fp}$ is its labeled sequent calculus based on $Fp$-models. A $JL$-formula $A$ is provable in $JL_{CS}$ iff $\Rightarrow w \Vdash A$ is provable in $G_{3JL}^{Fp} + (LC) + (RC) + Cut$.

**10.2 Labeled sequent calculus based on Fk-models**

Models for $JD$ and its extensions can be also characterized by the class of F-models with the following condition instead of seriality of $R$:

$E8$. Consistent evidence: $E(t, \bot) = \emptyset$, for all terms $t$.

These models are introduced by Kuznets in [26], and were called Fk-models there. Completeness of $JD$ (and its extensions) with respect to Fk-models can be proved without the requirement that constant specifications should be axiomatically appropriate.

**Theorem 10.4.** ([26]) Let $JL$ be $JD$ or one of its extensions, and $CS$ be a constant specification for $JL$. Then justification logics $JL_{CS}$ are sound and complete with respect to their $JL_{CS}$-Fk-models.

Regarding Fk-models, there is another formulation of labeled systems for $JD$ and its extensions, by replacing the rule for seriality ($Ser$) with the following initial sequent ($AxE\bot$):

$$wE(t, \bot), \Gamma \Rightarrow \Delta \quad (AxE\bot).$$

If $G_{3JLCS}$ contains ($Ser$), then let $G_{3JLCS}^{Fk}$ (the labeled sequent calculus based on Fk-models) denote the resulting system where the rule ($Ser$) is replaced by the initial sequent ($AxE\bot$). All the definitions of Sections 4-8 can be adopted for labeled systems $G_{3JLCS}^{Fk}$ (these systems first appeared in [21]). Main properties of labeled systems $G_{3JLCS}^{Fk}$ are listed in the following theorem (the proof is similar to that for labeled systems based on F-models).

**Theorem 10.5.** Suppose $JL$ is a justification logic with axiom $jD$, $CS$ is a constant specification for $JL$, and $G_{3JLCS}^{Fk}$ is its labeled sequent calculus based on Fk-models.

1. All sequents of the form $w \Vdash A, \Gamma \Rightarrow \Delta, w \Vdash A$, with $A$ an arbitrary $JL$-formula, are derivable in $G_{3JLCS}^{Fk}$.
2. All rules of $G_{3JLCS}^{Fk}$ are height-preserving $CS$-invertible.
3. The rules of substitution ($Subs$), weakening, contraction and $Cut$ are height-preserving $CS$-admissible in $G_{3JLCS}^{Fk}$.
4. If the sequent $\Gamma \Rightarrow \Delta$ is derivable in $G_{3JLCS}^{Fk}$, then it is valid in every $JL_{CS}$-Fk-model.
5. A \( JL \)-formula \( A \) is provable in \( JL_{CS} \) iff \( w \parallel \vdash A \) is provable in \( G3JL_{CS}^{F_k} \).

**Theorem 10.6.**
1. All labeled formulas in a derivation in \( G3JD_{CS}^{F_k} \) (or in \( G3JD_{CS}^{F_k}4 \)) are labeled-subformulas of labeled formulas in the endsequent.
2. Every sequent derivable in \( G3JD_{CS}^{F_k} \) (or in \( G3JD_{CS}^{F_k}4 \)) has a derivation with the sublabel property.
3. The rule \( (Trans_\ast) \) is admissible in \( G3JD_{CS}^{F_k} \) and its extensions.

The subterm property of E-rules (Lemma 6.2), and consequently the subterm property (Proposition 6.4), does not hold for these systems. For example, the sequent \( wE(x, P \rightarrow \bot), wE(y, P) \Rightarrow \) is derivable in \( G3JD_{\emptyset} \):

\[
(AxE \bot) \\
(\ldots) \\
\vdash (E^\ast)
\]

but has no derivation in which the rule \( (E^\ast) \) has the subterm property. Also the countermodel construction of Theorem 8.2 cannot be used to produce \( F_k \)-models (in this respect, see Note 3.3.35 and Example 3.3.36 of [26]).

### 10.3 Labeled sequent calculus for modal-justification logics

Modal-justification logics are combinations of modal and justification logics. We combine the modal logic \( ML \) and justification logic \( JL \) provided that \( JL \circ = ML \), and in this case the respective modal-justification logic is denoted by \( MLJL \). The language of \( MLJL \) is an extension of the language of \( JL \) and \( ML \). Thus formulas of \( MLJL \) are constructed by the following grammar:

\[
A ::= P | \bot | \neg A | A \land A | A \lor A | A \rightarrow A | t : A | \Box A,
\]

where \( t \in Tm_{MLJL} \). Axioms and rules of \( MLJL \) are the axioms and rules of \( ML \) and \( JL \) and the connection axiom “\( t : A \rightarrow \Box A \)”, i.e.

\[
MLJL = ML + JL + (t : A \rightarrow \Box A).
\]

We consider also the logic \( S4LPN \) which has the same language, axioms, and rules as \( S4LP \) with one additional axiom “\( \neg t : A \rightarrow \Box \neg t : A \)”, implicit-explicit negative introspection axiom, i.e.

\[
S4LPN = S4LP + (\neg t : A \rightarrow \Box \neg t : A).
\]

Constant specifications for \( MLJL \) and \( S4LPN \), and the system \( MLJL_{CS} \) and \( S4LPN_{CS} \) are defined in the usual way.

The first modal-justification logic \( S4LP \) was introduced by Artemov and Nogina in [8, 9], where \( S4LPN \) was also introduced there. The system \( KJ \) was considered by Fitting in [19]\(^8\), and by Artemov in [6]. Many of the modal-justification logics introduced here, are already considered in [27], with the name logics of justifications and belief. However, the following modal-justification logics are new:

\[
S5JT5, KB5JB4, KB5JB45, S5JTB5, S5JDB5, S5JTB45, S5JDB45, S5JTB4, S5JDB4.
\]

\(^8\) He also introduced the logic \( S5LPN \) which is obtained from \( S4LPN \) by adding the modal axiom \( 5 \).
In fact we use the requirement \( \text{JL}^0 = \text{ML} \) in the definition of \( \text{MLJL} \) which is more general than the definition of logics of justifications and belief in [27].\(^9\) The following lemma from [23] is our justification to define the modal-justification logics in (12).

**Lemma 10.3.** ([23]) There exist terms \( t_1(x), t'_1(x), t_2(x) \) such that for any term \( s \) and formula \( A \) we have

1. \( \text{JT5} \vdash s : A \rightarrow t_1(s) : s : A \).
2. \( \text{JB5} \vdash s : A \rightarrow t'_1(s) : s : A \).
3. \( \text{JB4} \vdash \neg s : A \rightarrow t_2(s) : \neg s : A \).
4. \( \text{JDB4} \vdash s : A \rightarrow A \).
5. \( \text{JDB5} \vdash s : A \rightarrow A \).

For example, although the logic \( \text{S5JT5} \) has the modal axiom \( 4, \Box A \rightarrow \Box \Box A \), and does not have the justification axiom \( j4 \), by the above lemma \( s : A \rightarrow t_1(s) : s : A \) is a theorem of \( \text{S5JT5} \).

**Example 10.5.** If finite or injective constant specifications are used in proofs, it is easy to show that

\[
\text{S4LP} = \text{S4} + \text{LP} + t : A \rightarrow \Box t : A,
\]

\[
\text{S4LPN} = \text{S4} + \text{LP} + \Box t : A \lor \Box \neg t : A.
\]

In [20] it is shown that theorems of \( \text{S4LP} \) and \( \text{S4LPN} \) can be realized respectively in \( \text{LP} \) and \( \text{JT45} \).

**F-models for \( \text{MLJL} \) (except those listed in (12))** were presented in [15, 27]. We introduce F-models for others, as well as for \( \text{S4LPN} \), and show their completeness.

**Definition 10.1.** An \( \text{MLJL}_{\text{CS}} \)-model \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \) is an \( \text{JL}^0 \)-model, where now \( \mathcal{E} \) is a possible evidence function on \( \mathcal{W} \) for \( \text{MLJL}_{\text{CS}} \). In addition, if \( \text{MLJL} \) contains the modal axiom \( 5 \) then the accessibility relation \( \mathcal{R} \) should be Euclidean. A \( \text{S4LPN}_{\text{CS}} \)-model is a \( \text{S4LP}_{\text{0}} \)-model such that \( \mathcal{E} \) is a possible evidence function on \( \mathcal{W} \) for \( \text{S4LPN}_{\text{CS}} \) which meets the strong evidence (\( \mathcal{E}8 \)) and anti-monotonicity (\( \mathcal{E}9 \)) conditions. The definition of forcing relation (Definition 3.2) for these models has the following additional clause:

\[
(\mathcal{M}, w) \models \Box A \iff \text{for every } v \in \mathcal{W} \text{ with } w \mathcal{R} v, \ (\mathcal{M}, v) \models A.
\]

We first show soundness and completeness of \( \text{S4LPN} \).

**Theorem 10.7.** \( \text{S4LPN}_{\text{CS}} \) are sound and complete with respect to their \( \text{S4LPN}_{\text{CS}} \)-models.

**Proof.** For soundness direction of \( \text{S4LPN} \), let us show the validity of implicit-explicit negative introspection axiom \( \neg t : A \rightarrow \Box \neg t : A \). Suppose \( (\mathcal{M}, w) \models \neg t : A \) and \( w \mathcal{R} v \). By the strong evidence condition, we have \( w \not\in \mathcal{E}(t, A) \). Hence by the anti-monotonicity condition, \( v \not\in \mathcal{E}(t, A) \). Thus \( (\mathcal{M}, v) \models \neg t : A \), and therefore \( (\mathcal{M}, w) \models \Box \neg t : A \).

The proof of completeness is similar to that of \( \text{S4LP} \) in [15]. Given a constant specification \( \text{CS} \) for \( \text{S4LPN} \), we construct a canonical F-model \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \) for \( \text{S4LPN}_{\text{CS}} \) as follows: Let \( \mathcal{W} \) be all \( \text{S4LPN}_{\text{CS}} \)-maximally consistent sets, and define the accessibility relation \( \mathcal{R} \) on \( \mathcal{W} \) by, \( \Gamma \mathcal{R} \Delta \iff \Gamma \subseteq \Delta \),

\(^9\) It is worth noting that the logics \( \text{TLP}, \text{S4LP}, \text{S5LP} \), and the logic \( \text{S5JT45} \) were introduced respectively by Artemov in [4] and by Rubtsova in [38], with the difference that they consider multi-agent modal logics for its modal part, and thus these logics cannot be considered neither as modal-justification logics nor as logics of justifications and belief.

58
where $I^\square = \{ A \mid \square A \in I \}$. The evidence function $\mathcal{E}$ and the valuation $\mathcal{V}$ are defined similar to the canonical model of $\mathcal{JL}$ in the proof of Theorem 3.1. Truth Lemma can be proved easily: for each formula $F$ and each $I \in \mathcal{W}$,

$$(M, I) \models F \iff F \in I.$$ 

We only verify the case in which $F$ is of the form $t : A$ and $\square A$.

If $(M, I) \models t : A$, then $I \in \mathcal{E}(t, A)$, and therefore $t : A \in I$. Conversely, suppose $t : A \in I$. Then $I \in \mathcal{E}(t, A)$. Since $t : A \rightarrow \square A \in I$, we have $\square A \in I$. By the definition of $\mathcal{R}$, $A \in \Delta$ for each $\Delta$ such that $I \mathcal{R} \Delta$. By the induction hypothesis, $(M, \Delta) \models A$. Therefore, $(M, I) \models t : A$.

If $\square A \in I$, then $A \in \Delta$ for each $\Delta$ such that $I \mathcal{R} \Delta$. By the induction hypothesis, $(M, \Delta) \models A$. Therefore, $(M, I) \models \square A$. Conversely, suppose $\square A \notin I$. Then $I^\square \cup \neg \{A\}$ is a consistent set. If it were not consistent, then $S4LPN_{CS} \vdash B_1 \land B_2 \land \ldots \land B_n \rightarrow A$ for some $\square B_1, \square B_2, \ldots, \square B_n \in I$. By reasoning in $S4LPN$ we have $S4LPN_{CS} \vdash \square B_1 \land \square B_2 \land \ldots \land \square B_n \rightarrow \square A$, hence $\square A \in I$ a contradiction. Thus $I^\square \cup \neg \{A\}$ is consistent. Extend it to a maximal consistent $\Delta$. It is obvious that $\Delta \in \mathcal{W}$, $I \mathcal{R} \Delta$ and $A \notin \Delta$. By the induction hypothesis, $(M, \Delta) \not\models A$. Therefore, $(M, I) \not\models \square A$.

Obviously, $\mathcal{R}$ is reflexive and transitive, and $\mathcal{E}$ satisfies $\mathcal{E}1$-$\mathcal{E}4$. The proof of the strong evidence condition ($E8$) for $E$ is similar to that given in Remark 3.2. We verify the anti-monotonicity condition ($E9$).

$E$ satisfies $E9$: suppose $I \mathcal{R} \Delta$ and $\Delta \in \mathcal{E}(t, A)$. By the definition of $\mathcal{E}$, we have $t : A \in \Delta$. Let us suppose $I \notin \mathcal{E}(t, A)$, or equivalently $t : A \notin I$, and derive a contradiction. Since $t : A$ is not in $I$, and $I$ is a maximal consistent set, we have $\neg t : A \in I$. By implicit-explicit negative introspection axiom, $\square \neg t : A \in I$, and so $\neg t : A \in I^\square$. Now $I \mathcal{R} \Delta$ yields $\neg t : A \in \Delta$, which is a contradiction. \hfill \dag

**Theorem 10.8.** Let MLJL be a modal-justification logic, and CS be a constant specification for MLJL, with the requirement that if JL includes axiom scheme $jd$ then CS should be also axiomatically appropriate. MLJL$_{CS}$ are sound and complete with respect to their F-models.

**Proof.** The proof is similar to the proof of Theorem 3.1. Soundness follows from soundness of modal and justification logics. Soundness of logics listed in (12) needs attention. For example, since reflexivity and Euclideanness of the accessibility relation of S5JT5$_{CS}$-models implies transitivity, axiom 4 is valid in S5JT5$_{CS}$-models.

For completeness, we construct a canonical model $\mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V})$ for each MLJL$_{CS}$. Let $\mathcal{W}$ be the set of all maximal consistent sets in MLJL$_{CS}$, and define the accessibility relation $\mathcal{R}$, the evidence function $\mathcal{E}$ and the valuation $\mathcal{V}$ are defined similar to the canonical model of S4LPN in the proof of Theorem 10.7. The forcing relation $\models$ on arbitrary formulas are defined as Definitions 3.2 and 10.1. The Truth Lemma can be shown for the canonical model of MLJL$_{CS}$ similar to the proof of Theorem 10.7. For each modal-justification logic MLJL$_{CS}$, it is easy to see that the canonical model $\mathcal{M}$ of MLJL$_{CS}$ is an MLJL$_{CS}$-model. \hfill \dag

In order to develop labeled systems for MLJL and S4LPN, we extend the extended labeled language of justification logics to include labeled formulas $\mathcal{w} \models A$, in which $A$ is a formula in the language of MLJL and S4LPN, respectively. Now, labeled systems for MLJL and S4LPN based on F-models are defined as follows:

\[
\begin{align*}
G3MLJL &= G3ML + G3JL. \\
G3S4LPN &= G3S4LP + (\text{Anti-Mon}) + (SE),
\end{align*}
\]
Put differently, if $\mathsf{5}$ is not an axiom of ML, then $\mathsf{G3MLJL}$ is obtained by adding the rules $(\mathcal{L} \Box), (\mathcal{R} \Box)$ to $\mathsf{G3JL}$. Otherwise, if $\mathsf{5}$ is an axiom of ML, $\mathsf{G3MLJL}$ is obtained by adding the rules $(\mathcal{L} \Box), (\mathcal{R} \Box), (\mathcal{Eucl})$, and $(\mathcal{Eucl}_*)$ to $\mathsf{G3JL}$. Labeled systems $\mathsf{G3MLJLCS}$ and $\mathsf{G3S4LPNCS}$ are defined in the usual way.

We now state the main results of the labeled systems of $\mathsf{MLJL}$ and $\mathsf{S4LPN}$. First note that the definition of a labeled-subformula of a labeled formula (Definition 6.1) can be extended as follows: the labeled-subformulas of $w \vDash \Box A$ are $w \vDash \Box A$ and all labeled-subformulas of $v \vDash A$ for arbitrary label $v$.

**Theorem 10.9.** Let $\mathsf{G3MLJL}^-$ denote any of the labeled systems $\mathsf{G3KJ}$, $\mathsf{G3K4J4}$, $\mathsf{G3JD}$, $\mathsf{G3DJ4JD}$, $\mathsf{G3TJT}$, $\mathsf{G3S4LP}$. 

1. All sequents of the form $w \vDash A, \Gamma \Rightarrow \Delta, w \vDash A$, with $A$ an arbitrary MLJL-formula, are derivable in $\mathsf{G3MLJLCS}$.
2. All labeled formulas in a derivation in $\mathsf{G3MLJLCS}$ are labeled-subformulas of labeled formulas in the endsequent.
3. Every sequent derivable in $\mathsf{G3MLJLCS}$ has an analytic derivation.
4. All rules of $\mathsf{G3MLJLCS}$ are height-preserving CS-invertible.
5. The rule $(\mathcal{Trans}_*)$ is admissible in those systems $\mathsf{G3MLJLCS}$ which contain $(\mathcal{Trans})$.
6. The rules of substitution $(\mathcal{Subs})$, weakening, contraction and $(\mathcal{Cut})$ are CS-admissible in $\mathsf{G3MLJLCS}$.
7. If the sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G3MLJLCS}$, then it is valid in every $\mathsf{MLJLCS}$-model.
8. Let $A$ be a formula in the language of MLJL, and CS be a constant specification for MLJL with the requirement that if MLJL contains axiom scheme jD then CS should be also axiomatically appropriate. Then $A$ is provable in $\mathsf{MLJLC}$ if $\Rightarrow w \vDash A$ is provable in $\mathsf{G3MLJLCS}$.
9. Suppose MLJL is a modal-justification logic that does not contain axiom jB, and CS is a finite constant specification for MLJL. Then every sequent $\Gamma \Rightarrow \Delta$ in the language of $\mathsf{G3MLJL}$ is either derivable in $\mathsf{G3MLJLCS}$ or it has a $\mathsf{MLJLC}$-countermodel.

All the above results, except clauses 2, 3, hold if we replace MLJL and $\mathsf{G3MLJL}$ with $\mathsf{S4LPN}$ and $\mathsf{G3S4LP}$ respectively.

**Proof.** The proofs are similar to the respective proofs in Sections 5-9, and the proofs for modal logics in [30]. In fact, the cases for rules $(\mathcal{L} \Box)$ and $(\mathcal{R} \Box)$ are similar to rules $(\mathcal{L} :)$ and $(\mathcal{R} :)$, respectively.

1. Obvious.
2. Obvious.
3. We first need to show that every sequent derivable in $\mathsf{G3MLJLCS}$ has a derivation with the subterm and sublabel property. The proof of the subterm and sublabel property is similar to that given for Propositions 6.3, 6.4.
4. The proof is similar to that of Proposition 7.1. We only check the invertibility of rule $(\mathcal{R} :)$ where in the induction step $wE(t, A), \Gamma \Rightarrow \Delta, w \vDash t : A$ is the conclusion of the rule $(\mathcal{R} \Box)$:

\[
\frac{uRu', wE(t, A), \Gamma \Rightarrow \Delta', u' \vDash A, w \vDash t : A}{wE(t, A), \Gamma \Rightarrow \Delta', u \vDash \Box A, w \vDash t : A} (\mathcal{R} \Box)
\]

where the eigenlabel $u'$ is not in the conclusion. By the induction hypothesis we obtain a derivation of height $n - 1$ of $wRv, uRu', wE(t, A), \Gamma \Rightarrow \Delta', u' \vDash A, v \vDash A$, for any fresh label $v$. Then by applying the rule $(\mathcal{R} \Box)$ we obtain a derivation of height $n$ of $wRv, wE(t, A), \Gamma \Rightarrow \Delta, u \vDash \Box A, v \vDash A$, as desired. The proof of invertibility of rule $(\mathcal{R} \Box)$ is similar to that for $(\mathcal{R} :)$.
5. The proof is by induction on the height of the derivation of \( wRw, wRw, \Gamma \Rightarrow \Delta \) in MLJLCS. In the induction step, \( wRw \) may be principal in the rules (Eucl), (Eucl\(_*\)), (L\( \Box \)). The proof for cases (Eucl) and (Eucl\(_*\)) is similar to the case of rule (Trans), and for the case (L\( \Box \)) is similar to the case of rule (L\( \Box \)) in the proof of Lemma 7.1. The case where \( wRw \) is principal in (Anti-Mon) in G3S4LP\(_{\text{CS}}\) is similar to the case of rule (Mon) in the proof of Lemma 7.1.

6. The proofs are similar to the respective proofs in Sections 5-9, and the proofs for modal logics in [30].

7. Similar to the proof of Theorem 8.1. The validity-preserving of rules (L\( \Box \)) and (R\( \Box \)) have already been shown in Theorem 5.3 in [31], and that of rules (Eucl), (Eucl\(_*\)) (and (Anti-Mon) in G3S4LP\(_{\text{CS}}\)) are obvious.

8. Similar to the proof of Corollary 8.1. As an example, we prove the connection principle:

\[
\begin{align*}
\mathcal{D} \\
\frac{v \Vdash A, wRv, w \Vdash t : A \Rightarrow v \Vdash A}{wRv, w \Vdash t : A \Rightarrow v \Vdash A} \text{ (L\( : \))} \\
\frac{w \Vdash t : A \Rightarrow w \Vdash \Box A}{\Rightarrow w \Vdash t : A \Rightarrow \Box A} \text{ (R\( \Box \))}
\end{align*}
\]

where the eigenlabel \( v \) is different from \( w \) and \( \mathcal{D} \) is the derivation of the topmost sequent by clause 1.

9. Similar to the proof of Theorem 8.2. Additional stages for rules (L\( \Box \)), (R\( \Box \)), (Eucl), (Eucl\(_*\)) (and (Anti-Mon) for G3S4LP\(_{\text{CS}}\)) should be added to the construction of the reduction tree.

We close this section by showing the termination of proof search for G3KJCS, G3TJTCS, G3S4LP\(_{\text{CS}}\), for finite CS. For any given sequent, let \( n(\Box) \) and \( p(\Box) \) be the number of occurrences of \( \Box \) in the negative and positive part of the sequent.

Termination of proof search for G3KJCS and G3TJTCS follows from Theorems 9.1, 9.2

**Theorem 10.10.** Given any finite constant specification CS, and any sequent \( \Gamma \Rightarrow \Delta \) in the language of G3KJ, it is decidable whether the sequent is derivable in G3KJCS.

**Theorem 10.11.** Given any finite constant specification CS, and any sequent \( \Gamma \Rightarrow \Delta \) in the language of G3TJT, it is decidable whether the sequent is derivable in G3TJTCS.

For G3S4LP, we need bounds on the number of applications of (R\( : \)) and (R\( \Box \)), similar to that given for (R\( : \)) in G3LP in Proposition 9.1. For G3S4LP, since rule (L\( \Box \)), as well as rule (L\( : \)), can be used in the argument given in the proof of Proposition 9.1, we have:

**Proposition 10.1.** In a derivation of a sequent in G3S4LP for each formula of the form \( t : A \) in its positive part, it is enough to have at most \( n(\Box) + n(\Box) \) applications of (R\( : \)) iterated on a chain of accessible worlds \( wRw_1, w_1Rw_2, \ldots \), with principal formula \( w_i \Vdash t : A \).

Since the proof of the following proposition repeats the proof of Proposition 6.9 in [30], we omit it here.

**Proposition 10.2.** In a derivation of a sequent in G3S4LP, for each formula of the form \( \Box A \) in its positive part, it is enough to have at most \( n(\Box) + n(\Box) \) applications of (R\( \Box \)) iterated on a chain of accessible worlds \( wRw_1, w_1Rw_2, \ldots \), with principal formula \( w_i \Vdash \Box A \).
The system G3S4LP combines G3LP and G3S4. As you can see from the proof of connection principle in G3S4LP (see the proof of Theorem 10.9(8)), in the backward proof search applications of (R□) introduce relational atoms, and therefore can increase the number of applications of (L□). Similarly, applications of (R:) can produce new applications of (L□). In spite of this fact, by Propositions 10.1 and 10.2 the number of applications of (R:) and (R□), and consequently of (L:) and (L□) are bounded. Thus, by Theorem 9.3 and termination of proof search of G3S4 (Corollary 6.10 in [30]), we have

**Theorem 10.12.** Given any finite constant specification CS, and any sequent \( \Gamma \Rightarrow \Delta \) in the language of G3S4LP, it is decidable whether the sequent is derivable in G3S4LP_{CS}.

### 10.4 Labeled sequent calculus based on AF-models

In this section we recall Artemov-Fitting models (or AF-models) for S4LP and S4LPN, which were first introduced in [9], and then we introduce labeled sequent calculus based on AF-models for S4LP and S4LPN.

**Definition 10.2.** A structure \( M = (W, R, R^e, \mathcal{E}, V) \) is an S4LP_{CS}-AF-model if \( (W, R, \mathcal{E}, V) \) is an S4LP_{CS}-model and \( R^e \) is a reflexive and transitive evidence accessibility relation, such that \( R \subseteq R^e \). Here, the Monotonicity property should be read as follows:

- **Monotonicity:** If \( w \in \mathcal{E}(t, A) \) and \( w R^e v \), then \( v \in \mathcal{E}(t, A) \).

Moreover, the forcing relation on formulas of the form \( \Box A \) and \( t : A \) are defined as follows:

- \( (M, w) \models \Box A \) iff for every \( v \in W \) with \( w R v \), \( (M, v) \models A \),
- \( (M, w) \models t : A \) iff \( w \in \mathcal{E}(t, A) \) and for every \( v \in W \) with \( w R^e v \), \( (M, v) \models A \).

S4LP_{CS}-AF-models are S4LP-AF-models where \( \mathcal{E} \) is a possible evidence function on \( W \) for S4LPN_{CS} and \( R^e \) is also symmetric.

**Theorem 10.13.** ([9]) S4LP_{CS} and S4LPN_{CS} are sound and complete with respect to their AF-models.

By internalizing AF-models of S4LP and S4LPN into the syntax of labeled systems, we will obtain a labeled system with the labeled-subformula property for S4LPN (as Example 6.1 shows the labeled-subformula property does not hold for G3S4LPN). In the following, we will define labeled systems G3S4LP_{e} and G3S4LPN_{e}.

Let us first extend the extended labeled language of justification logics by labeled formulas \( w \models A \), in which \( A \) is a formula in the language of S4LPN, and **evidence relational atoms** \( w R^e v \) (which presents the evidence relation \( w R^e v \) in AF-models). System G3S4LPN_{e} is an extension of the labeled sequent calculus G3S4 with initial sequents and rules from Table 9. Rule \((R^e)\) reflects the condition \( R \subseteq R^e \) in AF-models. Again, the initial sequent \((Ax R^e)\) is added to conclude the properties of the evidence accessibility relation. System G3S4LP_{e} is obtained from G3S4LPN_{e} by removing the rule \((Sym^e)\). In other words,

\[
G3S4LP_{e} = G3S4LP_{e} + (Sym^e).
\]

Labeled systems G3S4LP_{e,CS} and G3S4LPN_{e,CS} are defined in the usual way. All the results of Sections 5-9 can be extended to the systems G3S4LP_{e,CS} and G3S4LPN_{e,CS}. Note that Definition 8.1 is extended to the labeled systems G3S4LP_{e,CS} and G3S4LPN_{e,CS} as follows. For AF-model \( M = (W, R, R^e, \mathcal{E}, V) \) and \( M \)-interpretation \([\cdot] : L \rightarrow W \), we add the following clause to Definition 8.1:
Theorem 10.14. 1. All sequents of the form
\[ wR^v \Gamma \Rightarrow \Delta, wR^v \quad (AxR^v) \]
\[ wE(t, A) \Rightarrow \Delta, wE(t, A) \quad (AxE) \]

Rules:
\[ \frac{\forall t \vdash x : A, wR^v, \Gamma \Rightarrow \Delta}{w \vdash t : A, wR^v, \Gamma \Rightarrow \Delta} \quad (L:\forall) \]
\[ w \vdash t : A, wR^v, \Gamma \Rightarrow \Delta \quad (R:\forall) \]
\[ \frac{wE(t, A), w \vdash t : A, \Gamma \Rightarrow \Delta}{wE(t, A), \Gamma \Rightarrow \Delta} \quad (E) \]

In (R:\forall) the eigenlabel \( v \) must not occur in the conclusion of rule.

Rules for evidence atoms:
\[ \frac{wE(s \cdot t, B), wE(s, A \rightarrow B), wE(t, A), \Gamma \Rightarrow \Delta}{wE(s \cdot t, B), wE(s, A \rightarrow B), wE(t, A), \Gamma \Rightarrow \Delta} \quad (El+) \]
\[ \frac{wE(t + s, A), wE(t, A), \Gamma \Rightarrow \Delta}{wE(t, A), \Gamma \Rightarrow \Delta} \quad (Er+) \]
\[ \frac{wE(t, A), \Gamma \Rightarrow \Delta}{wE(t, A), \Gamma \Rightarrow \Delta} \quad (E!) \]

Axiom necessitation rule:
\[ \frac{wE(c, A), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (AN) \]

Rules for evidence relational atoms:
\[ \frac{wR^v, wR^v, bR^v, \Gamma \Rightarrow \Delta}{wR^v, \Gamma \Rightarrow \Delta} \quad (R:\forall) \]
\[ \frac{wR^v, \Gamma \Rightarrow \Delta}{wR^v, \Gamma \Rightarrow \Delta} \quad (Ref:\forall) \]
\[ \frac{vR^v, wR^v, \Gamma \Rightarrow \Delta}{vR^v, \Gamma \Rightarrow \Delta} \quad (Sym:\forall) \]
\[ \frac{wR^v, wR^v, \Gamma \Rightarrow \Delta}{wR^v, \Gamma \Rightarrow \Delta} \quad (Trans:\forall) \]

Table 9. Initial sequent and rules which should be added to G3S4 to obtain G3S4LP^v.

\[ \] validates the evidence relational atom \( wR^v \), provided that \( [w]R^v[v] \).

Inductively generated evidence functions for systems S4LP^vCS and S4LPN^vCS based on a possible evidence function (on Kripke frame \((W, R, R^v))\), is defined similar to the one for LP with the difference that we replace clause (6) in Definition 3.6 by the following one:

6. \( E_i(t, F) = E_i(t, F) \cup \{w \in W \mid v \in E_i(t, F), vR^v w, rk(t) = i + 1\} \).

We now state the main properties of G3S4LP^vCS and G3S4LPN^vCS.

Theorem 10.14. 1. All sequents of the form \( w \vdash A, \Gamma \Rightarrow \Delta, w \vdash A \), with \( A \) arbitrary S4LPN-formula, are derivable in G3S4LPN^vCS.
2. All labeled formulas in a derivation in G3S4LPN^vCS are labeled-subformulas of labeled formulas in the endsequent.
3. All formulas in a derivation in G3S4LPN^vCS are either labeled-subformulas of a labeled formula in the endsequent or atomic formulas of the form \( wE(t, A), wR^v, \) or \( wR^v \).
4. Every sequent derivable in $G_3S4LPN_{CS}^e$ has an analytic derivation.
5. All rules of $G_3S4LPN_{CS}^e$ are height-preserving $CS$-invertible.
6. The rule
   \[
   \frac{wR^e w, wR^e w, \Gamma \Rightarrow \Delta}{wR^e w, \Gamma \Rightarrow \Delta} \quad (Trans^e_1)
   \]
   is height-preserving $CS$-admissible in $G_3S4LPN_{CS}^e$.
7. The rules of substitution ($\text{Subs}$), ($Trans^e_1$), weakening, contraction and $Cut$ are $CS$-admissible in $G_3S4LPN_{CS}^e$.
8. If the sequent $\Gamma \Rightarrow \Delta$ is derivable in $G_3S4LPN_{CS}^e$, then it is valid in every $S4LPN_{CS}$-$AF$-model.
9. Let $A$ be a formula in the language of $S4LPN$, and $CS$ be a constant specification for $S4LPN$. Then $A$ is provable in $S4LPN_{CS}$ iff $w \Vdash A$ is provable in $G_3S4LPN_{CS}^e$.
10. Given any finite constant specification $CS$ for $S4LPN$, every sequent $\Gamma \Rightarrow \Delta$ in the language of $G_3S4LPN$ is either derivable in $G_3S4LPN_{CS}^e$ or it has a $S4LPN_{CS}$-$AF$-countermodel.

All the above results hold if we replace $S4LPN$ and $G_3S4LPN_{CS}^e$ respectively with $S4LP$ and $G_3S4LP^e$.

Proof. The proof of these clauses are similar to those for $G3JL$. Only note that item 6 follows easily from the fact that $G_3S4LP^e$ and $G_3S4LPN^e$ contain the rule ($Ref^e$). For clause 10, we construct the reduction tree similar to $G3JL$ with stages correspond to rules of $G_3S4LPN_{CS}^e$ (and $G3S4LP^e_{CS}$). The stages of new rules ($L^e$), ($R^e$), ($Mon^e$), ($Trans^e$), and ($Ref^e$) are similar to the stages of ($L$), ($R$), ($Mon$), ($Trans$), and ($Ref$) respectively.

For stage of rule ($R^e$), if the top-sequent is of the form:
\[
w_1Rv_1, \ldots, w_mRv_m, \Gamma' \Rightarrow \Delta'
\]
where all relational atoms $w_iRv_i$ from the antecedent of the topmost sequent are listed. Then, regarding condition ($\dagger$), we write the following node on top of it:
\[
w_1R^e v_1, \ldots, w_mR^e v_m, w_1Rv_1, \ldots, w_mRv_m, \Gamma' \Rightarrow \Delta'
\]

For stage of rule ($Sym^e$), if the top-sequent is of the form:
\[
w_1R^e v_1, \ldots, w_mR^e v_m, \Gamma' \Rightarrow \Delta'
\]
where all relational atoms $w_iR^e v_i$ from the antecedent of the topmost sequent are listed, then, regarding condition ($\dagger$), we write the following node on top of it:
\[
v_1R^e w_1, \ldots, v_mR^e w_m, w_1R^e v_1, \ldots, w_mR^e v_m, \Gamma' \Rightarrow \Delta'
\]

If the reduction tree has a saturated branch, in order to construct an Artemov-Fitting countermodel $\mathcal{M} = (W, R, R^e, E, V)$, we add the following clause to the definition of countermodel in the proof of Theorem 8.2:

5. The evidence accessibility relation $R^e$ is determined by evidence relational atoms in $\mathcal{T}$ as follows:
   if $wR^e v$ is in $\mathcal{T}$, then $wR^e v$ (otherwise $wR^e v$ does not hold).

The rest of the proof is similar to that in Theorem 8.2. ⊣

Terminating of proof search of $G_3S4LPN_{CS}^e$ (and $G3S4LP_{CS}^e$) for finite $CS$ follows from the following propositions.
Proposition 10.3. In a derivation of a sequent in $G3S4LP\mathcal{CS}$, for each formula of the form $t : A$ in its positive part, it is enough to have at most $n(\cdot)$ applications of $(R^e)$ iterated on a chain of accessible worlds $wR^e w_1, w_1 R^e w_2, \ldots$, with principal formula $w_i \models t : A$. The same holds for $G3S4LP\mathcal{CS}$.

Proposition 10.4. In a derivation of a sequent in $G3S4LP\mathcal{CS}$, for each formula of the form $\Box A$ in its positive part, it is enough to have at most $n(\Box)$ applications of $(R\Box)$ iterated on a chain of accessible worlds $wR\Box w_1, w_1 R\Box w_2, \ldots$, with principal formula $w_i \models \Box A$. The same holds for $G3S4LP\mathcal{CS}$.

Thus, similar to the proof of termination of proof search of $G3LP$, restrictions on the number of applications of $(R^e)$ and $(R\Box)$ should be imposed in the reduction tree construction for the system $G3S4LP\mathcal{CS}$ (and $G3S4LP\mathcal{CS}$).

Theorem 10.15. Given any finite constant specification $\mathcal{CS}$, and any sequent $\Gamma \Rightarrow \Delta$ in the language of $G3S4LP\mathcal{CS}$, it is decidable whether the sequent is derivable in $G3S4LP\mathcal{CS}$. The same holds for $G3S4LP\mathcal{CS}$.

Proof. The termination of proof search for $G3S4LP\mathcal{CS}$ follows from an argument similar to those of $G3S4$ and $G3LP$, and the fact that the number of applications of rule $(R^e)$ is bounded by a function of $r$ and the number of applications of rules $(R\Box)$, $(Ref)$, $(Trans)$. Rules $(Ref^e)$ and $(Trans^e)$ are treated similar to $(Ref)$ and $(Trans)$, respectively, in $G3LP$. For $G3S4LP\mathcal{CS}$, the rule $(Sym^e)$ increase the number of applications of $(L^e)$. The number of applications of $(Sym^e)$ is bounded by a function of the number of evidence relational atoms in the antecedent of the root sequent and of the number of applications of rules $(R^e)$, $(Ref^e)$, $(Trans^e)$, $(R^e)$.

11 Conclusion

The main achievement of this paper has been to provide a modular approach to the proof theory of justification logics. We have presented contraction- and cut-free labeled sequent calculus for some justification logics. These systems enjoy the labeled-subformula property and the sublabel property. Termination of proof search were proved for some of the labeled systems. The method described in this paper can be used to provide labeled sequent calculus for other justification logics that have Kripke-Fitting-style models, such as multi-agent logics $T_n\mathcal{LP}$, $S4_n\mathcal{LP}$ and $S5_n\mathcal{LP}$ (cf. [4]).

It is also possible to internalize Mkrtchyan models [5, 28], which are singleton Fitting models, within the syntax of sequent calculus to produce sequent systems for justification logics. Moreover, using Mkrtchyan models we may create label-free sequent systems. To this end replace $w \models A$ and $wE(t, A)$ with $A$ and $E(t, A)$, respectively, and omit relational atoms $wRv$ from the labeled language, and then change initial sequents and rules of Tables 6,7 accordingly. We leave the precise formulation of these systems for another work.

There remain still some questions. How could one extend these results to find labeled systems based on F-models for $JB$ and $J5$ and their extensions such that termination of proof search is proved? In other words, does the subterm property hold for the labeled systems $G3JB$, $G3J5$ and their extensions? In this paper, our approach was to internalize the known Fitting models of the justification logics. Of course one could try to give labeled systems based on other semantics.

Negri presented a cut-free labeled system for provability logic $GL$ (cf. [30]). Is it possible to extend it to the logic of proofs and provability $GLA$ ([8, 33])?
All the results of this paper originate from [21], except results on labeled systems G3MLJL (other than G3S4LP) in Section 10.3.

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